A Conjecture Involving Positive Solutions of a Simple Scalar Linear Time-Varying State Equation with Delay

Chung-Han Hsieh,∗ B. Ross Barmish,** and John A. Gubner***

Abstract—A simple conjecture is presented concerning positive solutions of a scalar, time-varying, linear state equation with delay. Although the equation arises in the context of stock trading, no knowledge of finance is needed in the analysis to follow. Starting from positive initial conditions, the state $X(k)$ is governed by a nonnegative constant parameter $\alpha$ and a time-varying parameter $v(k)$ with known bounds but otherwise arbitrarily time-varying. We conjecture that if the state $X(k)$ is positive for $k \leq N$ in response to a “distinguished path” $v^*(k)$ which we define, then for all admissible paths $v(k)$, positivity of the state is also guaranteed. The conjecture is supported by theoretical analysis for some special cases and simulations.

I. INTRODUCTION

The motivation for this paper is derived from an emerging line of research involving the use of system-theoretic ideas to buy and sell stock; e.g., see [1]-[7]. Under some assumptions arising in the finance literature in the context of “frictionless” markets; e.g., see [1]. To make our work accessible to the control community, this paper is written so that no expertise in finance is needed.

For such a market, there are no transaction costs such as brokerage commission or fees. In addition, perfect liquidity is assumed. That is, there is no gap between the bid and ask prices and the trader can buy or sell any number of shares, including fractions, at the traded price. These assumptions arise in the finance literature in the context of “frictionless” markets; e.g., see [1]. To make our work accessible to the control community, this paper is written so that no expertise in finance is needed.

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well-known fact that minimization of a multilinear function over a hypercube is attained at one of the extreme points; see [8] for a discussion related to this topic, it follows that \( X(v, k) \) is minimized at one of \( v^0 \). Hence, \( X(v, k) \) is positive for all \( v \in V^N \) and all \( k \leq N \) if and only if for each \( k = 0, \ldots, N \),

\[
X(v^0, k) > 0
\]

for \( i = 1, 2, \ldots, 2^N \). The key point to note is that this positivity test for large \( N \) is computationally prohibitive. For example, in the stock market, one can easily have \( N > 100 \). This computational intractability consideration motivates our conjecture; i.e., in lieu of \( 2^N \) paths, it suffices to check positivity of the state just along the distinguished path \( v^* \). Besides the elimination of the computational complexity issue afforded by the conjecture, what is also intriguing is the fact that the distinguish path \( v^* \) is not necessarily a state-minimizing path. That is, for some values of \( k \), it is possible to have

\[
X(v^*, k) > \min_{v \in V^N} X(v, k).
\]

An example to this effect is given in Section II. Finally, it should also be noted that the minimum value on the right-hand side above is non-increasing with respect to \( k \). To prove this, suppose a path \( v^0 = (v^0(0), \ldots, v^0(k-1)) \) minimizes \( X(v, k) \), then it suffices to show that there is an admissible path \( v^0 = (\tilde{v}^0(0), \ldots, \tilde{v}^0(k)) \) such that

\[
X(\tilde{v}^0, k+1) \leq X(v^0, k).
\]

Indeed, letting \( \tilde{v}^0(k) = v^\min \) if \( X(v^0, k-1) > 0 \) and \( \tilde{v}^0(k) = v^\max \) if \( X(v^0, k-1) < 0 \), with \( \tilde{v}^0 \) defined, we obtain

\[
X(\tilde{v}^0, k+1) = X(v^0, k) + \alpha(1 + v^\max)X(v^0, k-1) - X(v^0, k) = X(v^0, k) + \alpha(1 + v^\max)X(v^0, k-1).
\]

Since the second term on the right-hand side above is nonpositive, we obtain the desired inequality.

**Plan for Remainder of Paper:** Section II provides some technical results and simulations which support the State Positivity Conjecture. Most notable is the so-called State Positivity Lemma which provides a sufficient condition under which the conjecture is true. That is, for \( \alpha \) less than the bound given in the lemma, the time horizon \( N \) is no longer relevant and we obtain \( X(v, k) > 0 \) for all \( k \) and for all admissible paths \( v \). In the sequel, since the horizon \( N \) is not fixed, we refer to this as the *all-time state positivity* condition. In Section III, we provide a closed form expression for \( X(v^*, k) \) along the distinguished path \( v^* \) which is used to study the asymptotic behavior of \( X(v^*, k) \). Subsequently, in Section IV, some ramifications of the conjecture are discussed. Specifically, if the conjecture is true for all \( N \geq 0 \) and all \( \alpha \) in a restricted range, we improve upon the condition for all-time state positivity given in Section II. In Section V, we provide some additional remarks related to the state positivity issue. Finally, in Section VI, some concluding remarks are given and possible directions for future research are indicated.

**II. Support for The Conjecture**

In this section we provide both theory and simulation to support the conjecture. To this end, in the lemma to follow, we first consider the special case when \( \alpha \) satisfies a simple priori bound. As seen below, for this case, the conjecture is seen to be true for all \( N \geq 0 \) and all \( \alpha \geq 0 \) satisfying the bound given in the lemma below.

**State Positivity Lemma:** Suppose the parameter \( \alpha \) satisfies

\[
0 \leq \alpha < \frac{1}{1 + v^\max}.
\]

Then, for any admissible \( v \), the all-time state positivity condition

\[
X(v, k) > 0
\]

is satisfied for all \( k \geq 0 \).

**Proof:** To establish the desired result, instead of using \( v^\min \), it suffices to prove that the state positivity condition holds with the larger bound \( v^\min = -1 \). It is obvious that for \( k = 0, 1 \), the desired result holds. We now proceed using induction. Fix any \( k \geq 1 \), and suppose that for all admissible paths

\[
v = (v(0), v(1), \ldots, v(k))
\]

we have \( X(v, i) > 0 \) for \( i \leq k \). We must show \( X(v, k+1) > 0 \). Indeed, beginning with

\[
X(v, k+1) = X(v, k) + \alpha(1 + v(k-1))X(v, k-1)
\]

\[
\geq \min_{v(k-1)} (X(v, k) + \alpha(1 + v(k-1))v(k)X(v, k-1)),
\]

since \( X(v, k+1) \) is affine linear in \( v(k-1) \), the minimum occurs at either \( v(k-1) = -1 \) or \( v(k-1) = v^\max \). For the case \( v(k-1) = -1 \), we have \( X(v, k+1) \geq X(v, k) > 0 \) where the last inequality holds because of the induction hypothesis. For the second case \( v(k-1) = v^\max \), we have

\[
X(v, k+1) \geq X(v, k) + \alpha(1 + v^\max)X(v, k-1)
\]

\[
= X(v, k-1) + \alpha(1 + v(k-2))v^\max X(v, k-2) + \alpha(1 + v^\max)v(k)X(v, k-1).
\]

Noting that \( \alpha(1 + v(k-2))v^\max X(v, k-2) \geq 0 \) because of the induction hypothesis and the facts that \( 1 + v(k-2) \geq 0 \) and \( \alpha \geq 0 \), it follows that

\[
X(v, k+1) \geq X(v, k-1) + \alpha(1 + v^\max)v(k)X(v, k-1)
\]

\[
\geq (1 - \alpha(1 + v^\max))X(v, k-1)
\]

which, in view of the assumed bound on \( \alpha \), implies \( X(v, k+1) > 0 \). □

**Remarks:** When \(-1 < v^\min < 0\), the upper bound on \( \alpha \) in the lemma above can be improved if the State Positivity Conjecture assumed to be true for all \( N \geq 0 \) and all
 Validity of the Conjecture When \( v_{\min} = -1 \): To see this, suppose \( X(v^*,k) > 0 \) for \( k = 2, \ldots, N \). Since \( v_{\min} = -1 \), we readily obtain
\[
X(v^*,2) = (1 - \alpha(1 + v_{max}))X_0
\]
whose assumed positivity implies \( \alpha < 1/(1 + v_{max}) \). Then the State Positivity Lemma shows that \( X(v,k) > 0 \) for all admissible \( v \) and all \( k \). Furthermore, if \( \alpha \) equals or exceeds this upper bound, the solution above has \( X(v^*,2) \leq 0 \). Hence, when \( v_{\min} = -1 \), the lemma gives the best possible bound for all-time state positivity.

 Validity of the Conjecture for Small \( N \): The cases \( N = 0 \) and \( N = 1 \) are immediate because \( X(0) = X(1) = X_0 > 0 \) are the initial conditions. The case \( N = 2 \) follows by first observing that
\[
X(v,2) = X_0[1 + \alpha(1 + v(0))v(1)]
\]
and then noting that \( 1 + v(0) \geq 0 \), guarantees
\[
X(v,2) \geq X_0[1 + \alpha(1 + v_{max})v_{\min}] = X(v^*,2).
\]
Hence, if \( X(v^*,2) > 0 \), the other three admissible paths \( v \) satisfy \( X(v,2) > 0 \). The case \( N = 3 \) is similar, but a proof of the conjecture requires more work. Indeed, we first note that for all admissible \( v(0) \) and \( v(1) \),
\[
X(v,3) = X(v,2) + \alpha(1 + v(1))v(2)X_0
\]
is minimized with \( v(2) = v_{\min} \). Hence, it suffices to show that the quantity
\[
f(v(0),v(1)) = (1 + v_{\min} + v(0))v(1)
\]
is minimized by \( (v(0),v(1)) = (v_{max},v_{\min}) \). Now, in view of the multilinearity of \( f(v(0),v(1)) \), the minimum is obtained with \( (v(0),v(1)) \) being one of the four extreme combinations for \( v(k) \). For notational convenience let \( \Delta = v_{max} - v_{\min} \). Then noting that \( \Delta > 0 \), we compare \( f(v_{\max},v_{\min}) \) with the other three candidates for minimality and obtain
\[
f(v_{\max},v_{\min}) - f(v_{\min},v_{max}) = -(1 + v_{\min})\Delta;
\]
\[
f(v_{\max},v_{\min}) - f(v_{\min},v_{\min}) = v_{\min}\Delta;
\]
\[
f(v_{\max},v_{\min}) - f(v_{max},v_{max}) = -(1 + v_{\min} + v_{max})\Delta
\]
which are all negative. Therefore, since \( v^* \) is the minimizer of \( X(v,3) \), positivity of \( X(v^*,3) \) implies \( X(v,3) > 0 \) for all admissible \( v \). While the analysis above indicates that the distinguished path \( v^* \) leads to the minimum state trajectory up to \( N = 3 \), the example below shows that the minimizing path is not always the distinguished path \( v^* \).

Simulation Supporting the Conjecture for \( N = 15 \): Consider \( X_0 = 1 \), \( v_{max} = 0.9 \), \( v_{\min} = -0.8 \), \( N = 15 \), and \( \alpha = 0.53 \). Note that the choice of \( \alpha \) exceeds the upper bound on \( \alpha < 0.5263 \) required in the State Positivity Lemma. Hence, all-time state positivity is not guaranteed and, as explained in Section I, it suffices to check positivity of the state \( X(v^*,k) \) resulting from each of the \( 2^N \) extreme paths \( v^* \) obtained using \( v_{\min} \) and \( v_{\max} \). To this end, we ran a Matlab script to generate all possible \( 2^{15} = 32,768 \) paths and corresponding trajectories \( X(v^*,k) \). In Figure 1, the critical trajectories which determine positivity of \( X(v,k) \) are shown. Most intriguing is the fact that for some values of \( k \geq 7 \), the distinguished path, shown in black, has values \( X(v^*,k) \) which are greater than those of the minimizing state trajectory; i.e.,
\[
X(v^*,k) > \min_{v \in V^N} X(v^*,k).
\]
Nevertheless, consistent with the conjecture, \( X(v^*,k) \) is positive and the other state \( X(v^*,k) \) are positive too.

Although the constant parameter \( \alpha = 0.53 \) used in the simulations exceeds the upper bound \( 1/(1 + v_{max}) \approx 0.5263 \) required in the State Positivity Lemma, as seen in Section IV, if the conjecture is assumed to be true, it is not the largest possible \( \alpha \) assuring \( X(v^*,k) > 0 \). In fact using the result in Section IV, state positivity can be guaranteed for \( \alpha < \alpha^* \approx 0.5886 \).

III. State Along the Distinguished Path
Since the conjecture involves the state along the distinguished path \( v^* \), it is of interest to study the behavior of \( X(v^*,k) \) in its own right. In the lemma below, a closed form for \( X(v^*,k) \) is given. This result proves to be useful when analyzing asymptotic behavior.

Lemma 1: Assume \( -1 < v_{\min} < 0 \). If
\[
\alpha \neq \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]
then the state solution along the distinguished path \( v^* \) is given by \( X(v^*, 0) = X(v^*, 1) = X_0 \),

\[
X(v^*, k) = \frac{X_0}{2\sqrt{\theta}} (\lambda^k g_+ + \lambda^{-k} g_-), \quad k \geq 2
\]

where

\[
\begin{align*}
\theta &= 4\alpha v_{\min}(1 + v_{\min}) + 1, \\
g_+ &= \sqrt{\theta} \pm (2\alpha(v_{\max} + 1)v_{\min} + 1)
\end{align*}
\]

and

\[
\lambda_\pm = \frac{1}{2} \left( 1 \pm \sqrt{\theta} \right)
\]

are the eigenvalues of \( A(v^*, k) \). For the singular case

\[
\alpha = \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]

the state solution is given by

\[
X(v^*, k) = \frac{2^{-k}X_0((1 - v_{\max} + 2\max)v_{\min} + 1 + v_{\max})}{1 + v_{\min}}.
\]

**Proof:** Starting with the initial conditions described in Section 1; i.e., \( X(v^*, 0) = X(v^*, 1) = X_0 \), the next step is \( X(v^*, 2) = (1 + \alpha(1 + v_{\max})v_{\min})X_0 \). Then for \( k \geq 2 \), we have

\[
X(v^*, k + 1) = X(v^*, k) + \alpha(1 + v_{\min})v_{\min}X(v^*, k - 1).
\]

Now using this recursion, beginning with the augmented system, we generate

\[
\begin{bmatrix} X(v^*, k + 1) \\ X(v^*, k) \end{bmatrix} = \begin{bmatrix} 1 & \alpha(1 + v_{\min})v_{\min} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X(v^*, k) \\ X(v^*, k - 1) \end{bmatrix}
\]

introduced in Section 1, we obtain

\[
X(v^*, k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha(1 + v_{\min})v_{\min} \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} X(v^*, 2) \\ X(v^*, 1) \end{bmatrix}.
\]

Now, we consider two cases: For the generic case

\[
\alpha \neq \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]

we have

\[
X(v^*, k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^{k-1}(v^*, k) \begin{bmatrix} X(v^*, 2) \\ X(v^*, 1) \end{bmatrix}
\]

which, after a lengthy but straightforward computation reduces to

\[
X(v^*, k) = \frac{2^{-k}X_0((1 + \sqrt{\theta})^{k-1}g_+ + (1 - \sqrt{\theta})^{k-1}g_-)}{\sqrt{\theta}}
\]

\[
= \frac{X_0}{2\sqrt{\theta}} (\lambda^k g_+ + \lambda^{-k} g_-).
\]

Another lengthy but straightforward computation shows that \( \lambda_{\pm} \) are the eigenvalues of \( A(v^*, k) \). For the singular case

\[
\alpha = \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]

the state equation above reduces to

\[
X(v^*, k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1/4 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} X(v^*, 2) \\ X(v^*, 1) \end{bmatrix}
\]

which, again, upon a third lengthy but straightforward computation, we obtain

\[
X(v^*, k) = \frac{2^{-k}X_0((1 - v_{\max} + 2\max)v_{\min} + 1 + v_{\max})}{1 + v_{\min}}.
\]

**Asymptotic Behavior of \( X(v^*, k) \):** In the lemma below, we characterize the asymptotic behavior of \( X(v^*, k) \). Since \( \alpha = 0 \) trivially leads to \( X(k) = X_0 \) for all \( k \), we restrict our attention to \( \alpha > 0 \).

**Lemma 2:** Along the distinguished path \( v^* \), if

\[
0 < \alpha < \frac{1}{|v_{\min}|(1 + v_{\min})}
\]

and \(-1 < v_{\min} < 0\), then

\[
\lim_{k \to \infty} X(v^*, k) = 0.
\]

**Proof:** Along \( v^* \), for the subinterval described by

\[
0 < \alpha < \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]

using the formula

\[
X(v^*, k) = \frac{X_0}{2\sqrt{\theta}} (\lambda^k g_+ + \lambda^{-k} g_-)
\]

given in Lemma 1, it is readily shown that \( 0 < \theta < 1 \). This forces \( |\lambda_+| < 1 \) and \( |\lambda_-| < 1 \). Thus, by the well-known unit circle stability criterion, for example see [9], it follows that \( X(v^*, k) \to 0 \) as \( k \to \infty \). For the singular case

\[
\alpha = \frac{1}{4|v_{\min}|(1 + v_{\min})},
\]

which is the end point of the subinterval above, by Lemma 1, we have

\[
X(v^*, k) = \frac{2^{-k}X_0((1 - v_{\max} + 2\max)v_{\min} + 1 + v_{\max})}{1 + v_{\min}}.
\]

Thus, when \( k \to \infty \), it is easy to see that the \( 2^{-k} \) multiplier forces the linear term in \( k \) and forces \( X(v^*, k) \to 0 \). Finally, to complete the proof, we consider the remaining portion of the interval of interest

\[
\frac{1}{4|v_{\min}|(1 + v_{\min})} < \alpha < \frac{1}{|v_{\min}|(1 + v_{\min})},
\]
In this case, the eigenvalues $\lambda_{\pm} = (1 \pm \sqrt{\theta})/2$ are complex-valued because $\theta < 0$. To establish the desired convergence result, we observe that
$$|\lambda_{\pm}| = |\lambda_{-}| = \sqrt{\alpha v_{\text{min}}(1 + v_{\text{min}})} < 1$$
where the last inequality holds by using the upper bound on $\alpha$. Since the modulus of both eigenvalues are strictly less than one, the unit circle stability criterion again leads to $X(v^*, k) \to 0$ as $k \to \infty$. \quirentag{□}

IV. RAMIFICATIONS OF THE CONJECTURE
Some ramifications of the State Positivity Conjecture are discussed in this section.

**Improved $\alpha$ Bound for All-Time State Positivity:** Recall that the State Positivity Lemma applies when
$$0 \leq \alpha < \frac{1}{1 + v_{\text{max}}}.$$  
The following result shows that if the conjecture is true and $v_{\text{min}} > -1$, then the State Positivity Lemma can be improved upon.

**Improved State Positivity Lemma:** For $-1 < v_{\text{min}} < 0$, let
$$\alpha^* = \frac{v_{\text{max}} - v_{\text{min}}}{|v_{\text{min}}|(1 + v_{\text{max}})^2},$$
and assume $0 \leq \alpha \leq \alpha^*$. Then $X(v^*, k) > 0$ for all $k$. Hence, if the conjecture holds for some $N$, then for every admissible $v$, it follows that
$$X(v, k) > 0$$
for $k \leq N$. Furthermore, if the conjecture holds for all $N \geq 0$, then all-time state positivity is assured.

**Proof:** Since $\alpha = 0$ implies $X(v, k) = X_0 > 0$ for all $k$, we assume $\alpha > 0$. To establish positivity of $X(v^*, k)$, we proceed as follows. First note that
$$\frac{1}{4|v_{\text{min}}|(1 + v_{\text{min}})} \geq \frac{v_{\text{max}} - v_{\text{min}}}{|v_{\text{min}}|(1 + v_{\text{max}})^2}$$
on account of the fact that the above display is equivalent to
$$((1 + v_{\text{max}}) - 2(1 + v_{\text{min}}))^2 \geq 0.$$  
Furthermore the above inequalities are both strict if and only if $v_{\text{max}} \neq 1 + 2v_{\text{min}}$. Suppose $v_{\text{max}} \neq 1 + 2v_{\text{min}}$. Then $0 < \alpha \leq \alpha^*$ implies
$$\alpha < \frac{1}{4|v_{\text{min}}|(1 + v_{\text{min}})},$$
and so in Lemma 1 we have $0 < \theta < 1$ and $\lambda_{\pm} > 0$. It suffices to prove that $g_+ \geq 0$ and that at least one of them is positive. In the formula for $g_+$, the quantity $1 + 2\alpha(1 + v_{\text{max}})v_{\text{min}}$ is either negative or nonnegative. If it is nonnegative, then $g_+ > 0$ while $g_- \geq 0$ if and only if
$$\theta \geq [1 + 2\alpha(1 + v_{\text{max}})v_{\text{min}}]^2,$$
which is equivalent to $\alpha \leq \alpha^*$. Similarly, if the quantity is negative, then $g_- > 0$, while $g_+ \geq 0$ if and only if $\alpha \leq \alpha^*$.

Suppose $v_{\text{max}} = 1 + 2v_{\text{min}}$. Then for
$$\alpha = \alpha^* = \frac{1}{4|v_{\text{min}}|(1 + v_{\text{min}})},$$
the $X(v^*, k)$ for this singular case given in Lemma 1 applies and is clearly positive for all time. For
$$0 < \alpha < \alpha^* = \frac{1}{4|v_{\text{min}}|(1 + v_{\text{min}})},$$
we argue as in the preceding paragraph and obtain $0 < \theta < 1$, $\lambda_{\pm} > 0$, and $g_\pm = \sqrt{\theta}\pm\theta > 0$. \quirentag{□}

**Remark:** When $v_{\text{min}} \to -1$, we see that $\alpha^*$ converges down to the upper bound for $\alpha$ given in the State Positivity Lemma; i.e.,
$$\lim_{v_{\text{min}} \to -1} \alpha^* = \frac{1}{1 + v_{\text{max}}}.$$

V. SOME REMARKS RELATED TO STATE POSITIVITY
Some additional remarks related to the state positivity problem are provided in this section.

**The State Positivity Parameter Set:** Since positivity of the state depends on the constant parameter $\alpha$ and the number of stages $N$, it is natural to study the set, call it $A(N)$, consisting of all possible parameters $\alpha$ assuring positivity of the state up to stage $N$. By the State Positivity Lemma, we already have
$$[0, \frac{1}{1 + v_{\text{max}}}] \subseteq A(N)$$
for all $N$. Moreover, by definition of $A(N)$, it is easy to see that these sets are intervals and that
$$A(N + 1) \subseteq A(N)$$
for all $N$. That is, these sets $A(N)$ are nested. Now, with
$$\alpha_{\text{max}}(N) \doteq \sup\{\alpha : \alpha \in A(N)\}$$
being the largest value of $\alpha$ assuring positivity of state up to stage $N$, we define
$$\alpha_{\infty}^* \doteq \lim_{N \to \infty} \alpha_{\text{max}}(N)$$
and note that this limit exists because $\alpha_{\text{max}}(N)$ is non-increasing in $N$ with lower bound $\alpha_{\text{max}}(N) > 1/(1+v_{\text{max}})$.

For small values of $N$, it is easy to obtain $\alpha_{\text{max}}(N)$ in closed form. To illustrate, for $N = 2$, since
$$X(v, 2) \geq (1 + \alpha(1 + v_{\text{max}})v_{\text{min}})X_0 = X(v^*, 2),$$
it is straightforward to see that
$$\alpha_{\text{max}}(2) = \frac{1}{|v_{\text{min}}|(1 + v_{\text{max}})}.$$
Subsequently, for $N = 3$, lower bounding $X(v, 3)$ by $X(v^*, 3)$, we obtain
\[
\alpha_{\text{max}}(3) = \frac{1}{v_{\text{min}}[(1 + v_{\text{max}} + 1 + v_{\text{min}})].
\]

**Revisit of the Improved State Positivity Lemma:** If the conjecture is true for all $N \geq 0$ and all $\alpha \geq 0$, then the Improved State Positivity Lemma in Section IV leads to an important consequence regarding the asymptotic behavior of $\alpha_{\text{max}}(N)$. That is, with $\alpha^*$ in the improved lemma, noting that
\[
\frac{1}{1 + v_{\text{max}}} < \alpha^* \leq \alpha_{\text{max}}(N)
\]
for all $N$ and taking the limit as $N \to \infty$, we have
\[
\lim_{N \to \infty} \alpha_{\text{max}}(N) > \frac{1}{1 + v_{\text{max}}}.
\]
Said another way, if conjecture is true, then $\alpha_{\text{max}}$, the largest constant parameter assuring all-time state positivity, is strictly larger than the upper bound given in the initial State Positivity Lemma.

**Relation to Positive System Theory:** Considering the augmented two-state linear time-varying system
\[
x(k+1) = A(v(k))x(k)
\]
with
\[
A(v, k) = \begin{bmatrix} 1 & \alpha_i(1 + v_i(k - 1)v_i(k)) \\ 1 & 0 \end{bmatrix}
\]
as described in Section I, if the parameter $v(k)$ above is nonnegative, then, since $\alpha \geq 0$, it follows that the entries of the system matrix $A(v, k)$ has all of its entries nonnegative. In this case, with nonnegative initial condition, consistent with results in positive system theory, for example, see [10]-[14], the state $x(k)$ is componentwise nonnegative. More generally, the applicability of results from positive systems appears to be rather limited to the problem at hand since the time-varying parameter $v(k)$ is allowed to be negative.

**VI. CONCLUSION AND FUTURE WORK**
The focal point of this paper is a conjecture involving state positivity which is motivated a class of problems which arise in finance. To guarantee the desired positivity of the state for all admissible paths $v(k)$ and all stages $k \leq N$, it suffices to check positivity of the state just along the distinguished path $v^*$. This conjecture, if true, is useful to mitigate some potential computational complexity which would otherwise arise if one needs to check all $2^N$ extreme paths. To support the conjecture, we proved a result called the State Positivity Lemma which provides a sufficient condition under which the conjecture is true for all $N$ and all $\alpha$ in a restricted range. Then, a closed form expression for $X(v^*, k)$ along the distinguished path $v^*$ was given and it was used to study the asymptotic behavior of $X(v^*, k)$ and the related state positivity issue.

Regarding further research, we mention two attractive directions for future work. The first is obviously to pursue a proof of the conjecture. The second possible direction involves studying the state positivity problem when $v(k)$ is a vector-valued rather than a scalar; e.g., say $v(k) \in \mathbb{R}^m$ with $v_i(k)$ being the $i$-th component. This formulation, motivated by portfolio rebalancing problems involving latency, leads to the more general state equation
\[
X(k+1) = X(k) + \sum_{i=1}^m \alpha_i(1 + v_i(k - 1))v_i(k)X(k - 1)
\]
where $\alpha_i \geq 0$ are scalar constant parameters. In the vector case, the generalization of distinguished path would be important to define as the starting point for a more general theory.

Finally it should be mentioned that there is an alternative point of view which involves treating $v(k)$ as a disturbance. Then our formulation can be viewed as a robustness problem. That is, we want to know if robust positivity is guaranteed against a fixed constant parameter $\alpha \geq 0$, time horizon $N$, and all admissible $v \in V^N$.

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