On the Computation of Multivariate Scenario Sets for the Skew-t and Generalized Hyperbolic Families

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Abstract

We examine the problem of computing multivariate scenarios sets for skewed distributions. Our interest is motivated by the potential use of such sets in the stress testing of insurance companies and banks whose solvency is dependent on changes in a set of financial risk factors. We define multivariate scenario sets based on the notion of half-space depth (HD) and also introduce the notion of expectile depth (ED) where half-spaces are defined by expectiles rather than quantiles. We then use the HD and ED functions to define convex scenario sets that generalize the concepts of quantile and expectile to higher dimensions. In the case of elliptical distributions these sets coincide with the regions encompassed by the contours of the density function. In the context of multivariate skewed distributions, the equivalence of depth contours and density contours does not hold in general. We consider two parametric families that account for skewness and heavy tails: the generalized hyperbolic and the skew-t distributions. By making use of a canonical form representation, where skewness is completely absorbed by one component, we show that the HD contours of these distributions are near-elliptical and, in the case of the skew-Cauchy distribution, we prove that the HD contours are exactly elliptical. We propose a measure of multivariate skewness as a deviation from angular symmetry and show that it can explain the quality of the elliptical approximation for the HD contours.

Keywords: angular symmetry; expectile depth; generalized hyperbolic distribution; half-space depth; multivariate scenario sets; skew-t distribution.

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1 Introduction

While the topic of this paper is of independent computational statistical interest, the original motivation for studying these issues comes from applications in financial risk management. Let $X$ be a random vector representing changes in a set of so-called financial risk factors, such as equity indexes, interest rates, foreign exchange rates, etc. These risk factors impact the value of a financial portfolio and lead to a random loss given by $L = \ell(X)$. The portfolio might be a derivatives desk at a bank or a product book (e.g. annuity book) of an insurer.

We will assume that: (1) we have data that permit the statistical estimation of a model for $X$; (2) the function $\ell$ is known to us. That is, for any value $x$ we are able to compute the resulting loss $\ell(x)$. The function $\ell$ contains information about the size of the positions in the portfolio and encapsulates the valuation formulas necessary to quantify the effect of changes in the risk factors on the values of the positions.

A first question of possible interest is: how do we construct a scenario set $S$ based on the probability distribution of $X$ that includes plausible scenarios?

In our opinion there are advantages to using scenario sets that are based on the idea of half-space depth rather than sets that are based on density. In the presence of multivariate skewness, density sets tend to be dragged towards the shorter tails of the distribution and exclude too many extreme scenarios in the outer tail.

An example is given in Figure 1. We have plotted typical one-year changes in yield for a 3-year government bond and a 10-year government bond. These are the kinds of risk factors that would be considered in quantifying, for example, the change in the value of a portfolio of government bonds, or a portfolio of annuity liabilities. A bivariate normal inverse Gaussian (NIG) distribution has been fitted to the data (see Example 4.12) and, on the basis of the fitted model, density contours have been plotted and lines are shown that divide the plane into two half-spaces with probabilities $\alpha = 0.005$ and $1 - \alpha = 0.995$. The set formed by the intersection of closed half-spaces with probability $1 - \alpha$ is denoted $Q_\alpha$. Points on the boundary $\delta Q_\alpha$ are said to have depth $\alpha$ so that $Q_\alpha$ is the set of points with depth at least $\alpha$.

If we construct a scenario set $Q_\alpha$ based on depth we want to be able to say whether $x \in Q_\alpha$ for some arbitrary point $x$. It is often the case that a regulator or manager asks the risk modeller to consider a particular extreme scenario and work out how costly it might be. In order to weigh the importance or plausibility of this scenario we would like to know its depth $\alpha$, i.e. the largest value of $\alpha$ for which $x \in Q_\alpha$.

Suppose we are given a whole series of scenarios to consider and for each scenario we calculate the associated loss. Let $R = \{ x : \ell(x) > \ell_0 \}$ for some $\ell_0$ denote a ruin set, i.e. a set of scenarios that lead to an unacceptably large loss. We would like to identify the ruin scenario that is most plausible, in the sense that it has the maximum depth $\alpha$. This is known as the reverse stress testing problem.

In this paper we will consider the properties of $Q_\alpha$ and related scenario sets when the distribution of $X$ is in either the skew-$t$ (ST) or generalised hyperbolic (GH) family. These are flexible families of skewed and potentially heavy-tailed distributions that are useful for modelling financial risk-factor changes. The NIG distribution fitted to the data in Figure 1 belongs to the latter family and we see that, in this case, the set $\delta Q_\alpha$ is very close to (but
Figure 1: Picture shows change in yields for 3-year and 10-year government bond over 1 year. A bivariate NIG distribution has been fitted and corresponding density contours. The dashed line corresponds to the approximating ellipsoid for the $\alpha$-depth set $Q_\alpha$ for $\alpha = 0.005$; the grey lines show the boundaries of half spaces with probabilities $\alpha$ and $(1 - \alpha)$.
not exactly) an ellipse. This “near ellipsoidal” behaviour is true of other distributions in these families.

The paper is structured as follows. In Section 2 we define multivariate scenario sets based on half-space depth (HD) and briefly describe some important properties of the half-space median that are relevant for the purposes of this paper. In Section 3 we introduce the related notion of expectile depth (ED) which generalizes the univariate concept of expectile to higher dimensions.

In Section 4 we examine the problem of computing depth sets based on HD and ED for the ST (Section 4.1) and GH (Section 4.2) families of distributions. We show that the computation of depth sets can be simplified by making use of a canonical form representation, where skewness is completely absorbed by one component. We also prove that the HD contours for the skew-Cauchy (SC) distribution are elliptical, giving a further simplification in the computation.

We provide an algorithm for the construction of approximating ellipsoids to depth sets based on HD and investigate the quality of such an approximation. In Section 4.3 we examine the relationship between ellipsoidal depth sets and angular symmetry. We propose a multivariate measure of skewness as a deviation from angular symmetry and show that this measure can explain the quality of the ellipsoidal approximation. Numerical results, in Section 4.4 show its concordance with the probability of misclassification. Although the probability of misclassification is a direct and more interpretable measure of the error we incur when using ellipsoidal approximations, it is much more difficult to compute. We conclude with a discussion in Section 5.

2 Half-Space Depth

2.1 Half-Space Depth and its Relation to Quantiles

Let \((\Omega ,\mathcal{F},P)\) be a probability space and \(X : \Omega \to \mathbb{R}^d\) be a given random vector. For any \(y \in \mathbb{R}^d\) and any directional vector \(u \in \mathbb{R}^d \setminus \{0\}\), let
\[
H_{y,u} = \{x \in \mathbb{R}^d : u^\top x \leq u^\top y\}
\]
denote the closed half-space bounded by the hyperplane through \(y\). We write the probability that \(X\) lies in the half-space \(H_{y,u}\) as
\[
P_X(H_{y,u}) = P(u^\top X \leq u^\top y).
\]
The half-space depth of the point \(x \in \mathbb{R}^d\) with respect to the probability distribution of \(X\) is given by
\[
\text{HD}_X(x) = \inf_{\|u\|=1} P_X(H_{x,u}),
\]
where \(\|\cdot\|\) is the Euclidean norm. We note that HD is an affine invariant measure, meaning that if \(A \in \mathbb{R}^{d \times d}\) is a non-singular matrix and \(b \in \mathbb{R}^d\) is a vector, then \(\text{HD}_{AX+b}(AX+b) = \text{HD}_X(x)\).

Let \(\alpha \in (0,0.5]\) be a probability value. The main definition of a scenario set that we use is
\[
Q_\alpha = \bigcap\{H_{y,u} : P_X(H_{y,u}) \geq 1 - \alpha\}
\]
which is the intersection of all closed half-spaces with probability at least \((1 - \alpha)\). Sets of this kind are considered by many authors including Massé & Theodorescu (1994), Rousseeuw & Ruts (1999) and McNeil & Smith (2012). The construction is sometimes referred to as half-space trimming.

For \(\theta \in (0, 1)\) we also define a \(\theta\)-quantile function on \(\mathbb{R}^d \setminus \{0\}\) by writing \(q_\theta(u)\) for the \(\theta\)-quantile of \(u^\top X\). Then, \((2)\) can be expressed in terms of \(q_\theta(u)\) by

\[
Q_\alpha = \{ x : u^\top x \leq q_{1-\alpha}(u), \forall u \}\,.
\]

We will make the assumption that \(X\) has a strictly positive probability density on \(\mathbb{R}^d\). This assumption is satisfied by the distributions that interest us in this paper and allows us to pass easily between the concepts of quantiles and depth. Under this assumption, we have

\[
\{ x : u^\top x \leq q_{1-\alpha}(u) \} = \{ x : P_X(H_{x,u}) \leq 1 - \alpha \}
\]

from which it can be easily deduced that

\[
Q_\alpha = \{ x : HD_X(x) \geq \alpha \}.
\] (3)

We thus refer to \(Q_\alpha\) as a depth set and we refer to \(\partial Q_\alpha\), the boundary of \(Q_\alpha\), as the \(\alpha\) depth contour; the contour consists of the points with depth exactly equal to \(\alpha\).

In Figure 1, we show an example of the half-space trimming construction for \(\alpha = 0.005\) (the mesh of grey straight lines) as well as the depth contour \(\partial Q_{0.005}\) (the dashed curve).

### 2.2 The half-space median and angular symmetry

The function \(HD_X(x)\) can be used to define an affine equivariant median known as the half-space median (or Tukey median) of \(X\). This is the set of maximal \(HD\) given by

\[
\beta_X = \arg \max_{x \in \mathbb{R}^d} HD_X(x).
\] (4)

By affine equivariant we mean that, if \(A \in \mathbb{R}^{d \times d}\) is a non-singular matrix and \(b \in \mathbb{R}^d\) is a vector, then \(\beta_{AX+b} = A\beta_X + b\).

The half-space median is, in general, not unique unless \(X\) is symmetric according to some notion of multivariate symmetry; see Serfling (2006) for a survey of multivariate concepts of symmetry. More general conditions for uniqueness are given in Small (1987) who shows that a sufficient condition for uniqueness in the bivariate case is the strict positivity of the density function. The least restrictive definition of multivariate symmetry, under which the half-space median is unique, is angular symmetry. The random variable \(X\) is angularly symmetric about a point \(\eta\) if

\[
\frac{X - \eta}{\|X - \eta\|} \overset{d}{=} - \frac{X - \eta}{\|X - \eta\|},
\] (5)

where \(\overset{d}{=}\) indicates equality in distribution. Dutta, Ghosh & Chaudhuri (2011) show that \(HD_X(\eta) = 1/2\) if and only if \(\eta\) is the center of angular symmetry. Thus, if \(\eta\) is the centre of angular symmetry, then \(\beta_X = \eta\). This property is used in Section 4.3 to define two different measures of multivariate skewness as deviations from angular symmetry.
Note that if \( \beta_X \) is the center of angular symmetry, then \( \beta_X \) also corresponds to the component-wise median. This is clear, since if \( \text{HD}_X(\beta_X) = 1/2 \) then
\[
P_X(H\beta_X, u) = 1/2, \quad \forall u \in \mathbb{R}^d.
\]
If we set \( u = e_i \), the \( i \)th unit vector, we can infer that the \( i \)th element of \( \beta_X \) is the univariate median of the marginal distribution of the \( i \)th component of \( X \).

3 Expectile Depth

3.1 Definitions

The notion of an expectile was introduced by Newey & Powell [1987] as the solution of an asymmetric least squares regression problem, analogous to quantile regression. Given an integrable random variable \( Y \) in \( \mathbb{R} \) and \( \theta \in (0, 1) \), the \( \theta \)-expectile of the distribution function \( F_Y \) of \( Y \) is the unique solution \( y \) of the equation
\[
\theta \mathbb{E}((Y - y)^+) = (1 - \theta) \mathbb{E}((Y - y)^-)
\]
where \( x^+ = \max(x, 0) \), \( x^- = \max(-x, 0) \) and \( \mathbb{E}(\cdot) \) is the expectation with respect to the distribution of \( Y \).

It was shown by Jones [1994] that the \( \theta \)-expectile of \( F_Y \) can be expressed as the \( \theta \)-quantile of the related distribution function
\[
\tilde{F}_Y(y) = \frac{yF_Y(y) - \mu(y)}{2(yF_Y(y) - \mu(y)) + \mathbb{E}(Y) - y}
\]
where \( \mu(y) := \int_{-\infty}^y xdF_Y(x) \) is the lower partial moment of \( F_Y \). For any random variable \( Y \) the distribution function \( \tilde{F}_Y(y) \) is continuous and strictly increasing on its support, implying that the \( \theta \)-expectile is uniquely defined for all \( \theta \) in \( (0, 1) \).

In our application, for a fixed random vector \( X \in \mathbb{R}^d \), we define the \( \theta \)-expectile function \( e_\theta(u) \) to be the \( \theta \)-expectile of the distribution \( u^\top X \) for \( u \in \mathbb{R}^d \) and, for \( \alpha \in (0, 0.5] \) we consider a scenario set of the form
\[
E_\alpha = \{x : u^\top x \leq e_{1-\alpha}(u), \forall u\}.
\]
Let
\[
\text{ED}_X(x) = \inf_{\|u\|=1} \tilde{P}_X(H_{x,u})
\]
denote the smallest probability of a half-space \( H_{x,u} \) when probabilities are calculated according to \( \tilde{P}_X(H_{x,u}) = \tilde{F}_{u^\top X}(u^\top x) \). We refer to \( \text{ED}_X(x) \) as the expectile depth of \( x \) with respect to the distribution of \( X \) and note that it is also an affine invariant measure. The scenario set may also be expressed as
\[
E_\alpha = \{x : \text{ED}_X(x) \geq \alpha\}.
\]

With obvious notation, we use \( \partial E_\alpha \) to indicate the boundary of \( E_\alpha \), and refer to this as the \( \alpha \)-expectile depth contour.
3.2 Properties of expectile depth

There are both practical and theoretical reasons for considering expectile depth as an alternative to standard (quantile) depth. On the one hand the expectile can simply be viewed as a kind of generalized quantile; see Bellini et al. (2013). Using the techniques of McNeil & Smith (2012) it is straightforward to show that, when $X$ has an elliptical distribution, the set $E_\alpha$ is an ellipsoidal set like $Q_\alpha$, with axis lengths in identical proportions. However, for general distributions expectile depth sets will have different shapes to (quantile) depth sets.

At a more theoretical level, both the quantile function $q_\theta(u)$ and the expectile function $e_\theta(u)$ are positive-homogeneous functions on $\mathbb{R}^d$, meaning that they are functions $r: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $r(ku) = kr(u)$ for $k > 0$.

A fundamental result in convex analysis can be used to show that a positive-homogeneous function $r$ has a representation as the so-called support function $r(u) = \sup\{u^\top x : x \in S\}$ (7) of the convex set $S = \{x \in \mathbb{R}^d : v^\top x \leq r(v) \text{ for all } v \in \mathbb{R}^d\}$ (8) if and only if the function $r$ is also subadditive on $\mathbb{R}^d$; see Rockafellar (1970) or McNeil & Smith (2012) for technical details and recall that a subadditive function satisfies $r(u_1 + u_2) \leq r(u_1) + r(u_2)$ for all $u_1$ and $u_2$ in $\mathbb{R}^d$.

The set $S$ in (8) is $Q_\alpha$ when $r = q_{1-\alpha}$ and it is $E_\alpha$ when $r = e_{1-\alpha}$. However, the quantile function $q_\theta$ is not subadditive in general, but only for certain underlying random vectors $X$ and certain values of $\theta$. For example, for elliptically distributed random vectors $q_\theta$ is subadditive for $\theta > 0.5$. But when $X = (X_1, X_2)$ is a vector comprising two independent standard exponential random variables, McNeil & Smith (2012) show that $q_\theta$ is not subadditive for $\theta = 0.72$. In contrast, the expectile function $e_\theta$ is subadditive for any random vector $X$ with finite mean and $\theta > 0.5$.

Let $\alpha$ satisfy $0 < \alpha < 0.5$ and assume that $E_\alpha$ and $Q_\alpha$ are non-empty. The implication of (7) is that for any $u \in \mathbb{R}^d$ there exists a scenario $x(u) \in \partial E_\alpha$, i.e. a scenario on the boundary of the expectile depth set, such that $e_{1-\alpha}(u) = u^\top x(u)$. However, there may exist values $u \in \mathbb{R}^d$ such that $q_{1-\alpha}(u) > u^\top x$ for all $x \in Q_\alpha$. In this sense $\partial E_\alpha$ is more satisfactory as a multivariate analogue of the expectile than is $\partial Q_\alpha$ as a multivariate analogue of the quantile.

We note that scenario sets of the form (8) can be based on other positive-homogeneous and subadditive functions. In McNeil & Smith (2012) a scenario set based on the function $e_\theta(u) = E(u^\top X \mid u^\top X \geq q_\theta(u))$ is proposed; this is related to the so-called expected shortfall risk measure.

4 Depth Sets for Skewed Distributions

We now consider two families of multivariate skewed distributions, both of which have a canonical form, obtained by an affine transformation, in which all of the skewness is absorbed by one of the marginal distributions. In view of the affine invariance of HD and
ED, it suffices to be able to calculate these quantities for random vectors in their canonical form.

### 4.1 Skew-t distribution

The skew-t (ST) distribution is a flexible model for skewed and heavy-tailed multivariate data [Azzalini & Capitanio 2003; Azzalini & Genton 2008]. Let \( \xi, \gamma \in \mathbb{R}^d \) and \( \nu \in \mathbb{R}^+ \) denote the parameters of location, skewness and the degrees of freedom; let \( \Omega \in \mathbb{R}^{d \times d} \) be a symmetric, positive-definite dispersion matrix. A random variable \( X \) in \( \mathbb{R}^d \) is distributed according to a \( \text{ST}_d(\xi, \Omega, \gamma, \nu) \) distribution if it has density function

\[
f_{\text{ST}_d}(x) = 2t_{d}(\nu)T_1\left(\gamma^\top \omega^{-1}(x - \xi) \left(\frac{\nu + d}{Q_x + \nu}\right)^{1/2}; \nu + d\right), \quad x \in \mathbb{R}^d,
\]

where

\[
t_d(x; \nu) = \frac{\Gamma((\nu + d)/2)}{|\Omega|^{1/2}(\pi\nu)^{d/2}\Gamma(\nu/2)}(1 + Q_x/\nu)^{(\nu+d)/2}, \quad Q_x = (x - \xi)^\top \Omega^{-1}(x - \xi),
\]

\[\omega = \text{diag}(\omega_1, \ldots, \omega_d) = \text{diag}(\omega_{11}, \ldots, \omega_{dd})^{1/2}\]

and \( T_1(\cdot; \nu) \) denotes the univariate Student \( t \) distribution function with \( \nu \) degrees of freedom.

For later use, we also define the following correlation matrix

\[
\Omega = \omega^{-1} \Omega \omega^{-1}.
\]

Skewness and tails heaviness are regulated by the parameters \( \gamma \) and \( \nu \), respectively; these two parameters jointly characterize the shape of the distribution. If \( \gamma = 0 \) the Student \( t \) distribution is recovered; if \( \nu \to \infty \) we obtain the skew-normal (SN) distribution [Azzalini 2005; Azzalini & Capitanio 1999]; if \( \nu \to \infty \) and \( \gamma = 0 \) we obtain the multivariate normal distribution.

The next result, which follows from the linear transformation result given in Appendix A.1, introduces the canonical form of the ST distribution (CST). The importance of the canonical form in summarizing important features related to the location, skewness and kurtosis of the ST family is discussed in an unpublished paper by Capitanio (2012).

**Theorem 4.1.** Let \( X \sim \text{ST}_d(\xi, \Omega, \gamma, \nu) \) where \( \Omega = BB^\top \) and \( B \in \mathbb{R}^{d \times d} \). Let \( \gamma_s = (\gamma^\top \Omega \gamma)^{1/2} \) and define \( X^* = P^\top B^{-1}(X - \xi) \), where \( P \) is an orthonormal matrix with first column equal to \( \gamma_s^{-1}B^\top\omega^{-1} \) and \( T_1(\cdot; \nu) \) denotes the univariate Student \( t \) distribution function with \( \nu \) degrees of freedom. Then \( X^* \sim \text{ST}_d(0, I_d, \gamma_s e_1, \nu) \).

If \( X \) is in canonical form, we simply write \( X \sim \text{CST}_d(\gamma, \nu) \), where \( \gamma \) is the scalar parameter of skewness; if \( \nu = 1 \), the case of the skew-Cauchy (SC) distribution, then \( X \sim \text{CSC}_d(\gamma) \); if \( \nu = \infty \), then \( X \sim \text{CSN}_d(\gamma) \).

Without loss of generality we have assumed that for a random vector \( X \) in canonical form all of the asymmetry is absorbed in the first component and we have \( X_i \overset{d}{=} -X_i \) for \( i \neq 1 \). We now show that the HD and ED contours of the CST distribution are symmetric with respect to the axis of the unique asymmetric component. This property is useful in the construction of scenario sets based on either HD or ED of any ST distribution, as shown later in some examples.
Theorem 4.2. Let \( x \) and \( x' \) denote two points in \( \mathbb{R}^d \) whose first elements are equal and the others differ in sign at most. If \( X \sim \text{CST}_d(\gamma, \nu) \), then \( \text{HD}_X(x) = \text{HD}_X(x') \) and \( \text{ED}_X(x) = \text{ED}_X(x') \).

Proof. For a given normalized vector \( u \in \mathbb{R}^d \), consider the half-space \( H_{x,u} \).

Let \( J \subseteq \{2, \ldots, d\} \) be a set of indices indicating the elements of \( x' \) that have opposite sign with respect to the corresponding elements of \( x \). Define the random vector \( X' \) so that \( X'_i = -X_i \) if \( i \in J \) and \( X'_i = X_i \) if \( i \in J^C \), where \( J^C \) is the complement of \( J \); similarly define \( u' \) so that \( u'_i = -u_i \) if \( i \in J \) and \( u'_i = u_i \) if \( i \in J^C \). Since \( X' \overset{d}{=} X \) it follows that

\[
P(\mathbf{u}^\top X \leq \mathbf{u}^\top x) = P(\mathbf{u}^\top X' \leq \mathbf{u}^\top x) = P(\mathbf{u}^\top X \leq \mathbf{u}^\top x')
\]

and hence that

\[
P_X(H_{x,u}) = P_{X'}(H_{x',u}) \quad \text{and} \quad \tilde{P}_X(H_{x,u}) = \tilde{P}_{X'}(H_{x',u}).
\]

We obtain \( \text{HD}_X(x) = \text{HD}_X(x') \) and \( \text{ED}_X(x) = \text{ED}_X(x') \) when we take the infimum over all \( ||u|| = 1 \).

In the special case of the SC distribution, the computation of HD contours is further simplified. As shown in the next theorem, the HD contours of the CSC distribution are circular, and hence ellipsoidal for the general SC distribution; we use \( \sec(\cdot) \) and \( \tan(\cdot) \) to indicate the secant and tangent functions, respectively.

Theorem 4.3. If \( X \sim \text{CST}_d(\gamma) \), then

\[
Q_\alpha = \left\{ x : (x_1 - s(\alpha))^2 + \sum_{i=2}^{d} x_i^2 \leq t(\alpha)^2 \right\}
\]

where

\[
s(\alpha) = \frac{\gamma}{\sqrt{1 + \gamma^2}} \sec \left\{ \left( \frac{1}{2} - \alpha \right) \pi \right\} \quad \text{and} \quad t(\alpha) = \tan \left\{ \left( \frac{1}{2} - \alpha \right) \pi \right\}, \alpha \in (0, 0.5].
\]

Proof. From the expression of the univariate quantile function of the SC distribution [Beboodian, Jamalizadeh & Balakrishnan 2006], for any directional vector \( u \in \mathbb{R}^d \), we can write

\[
q_{1-\alpha}(u) = u_1 s(\alpha) + t(\alpha),
\]

where \( u_1 \) is the first element of \( u \). It follows that

\[
Q_\alpha = \left\{ x : u_1^\top x \leq u_1 s(\alpha) + t(\alpha), \forall u \right\} \quad \Rightarrow \quad \left\{ x : \left( u_1 \frac{(x_1 - s(\alpha))}{t(\alpha)} \right) + \sum_{i=2}^{d} u_i \frac{x_i}{t(\alpha)} \leq 1, \forall u \right\}.
\]

By observing that the Euclidean unit ball \( \{ y : y^\top y \leq 1 \} \) can be written as \( \{ y : u_1^\top y \leq 1, \forall u \} \), we conclude that for \( x \in Q_\alpha \), the vectors \( y = [x - (s(\alpha), 0, \ldots, 0)^\top] / t(\alpha) \) describe the unit ball and therefore

\[
Q_\alpha = \left\{ x : (x_1 - s(\alpha))^2 + \sum_{i=2}^{d} x_i^2 \leq t(\alpha)^2 \right\}.
\]
Figure 2: Half-space depth contours $\partial Q_{0.1}$, $\partial Q_{0.2}$ and $\partial Q_{0.3}$ in the case of the two bivariate variables $X$ (left panels) and $Y$ (right panels) as defined in Example 4.4, with $\nu = 1$ (top panels) and $\nu = 5$ (lower panels).
Figure 3: Expectile depth contours $\partial E_{0.1}$, $\partial E_{0.2}$ and $\partial E_{0.3}$ in the case of the two bivariate variables $X$ (left panel) and $Y$ (right panel) as defined in Example 4.5.

We now give some examples to illustrate the depth contours and expectile depth contours of certain special cases of the ST distribution. While the skew-Cauchy has elliptical depth contours, other cases have contours that are near-elliptical; the quality of an elliptical approximation will be investigated further in Section 4.4. Algorithm 1 is used to calculate half-space depth and a similar approach can be used for expectile depth, as indicated in Example 4.5.

**Example 4.4.** Let $X \sim \text{CST}_2(3, \nu)$ and define

$$A = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 & -2 \\ 1/2 & -1/2 \end{pmatrix}$$

and $b^\top = (-2, 1)$. The linear transformation $Y = AX + b$ has distribution $\text{ST}_2(b, C, d, \nu)$, where

$$C = \begin{pmatrix} 5/2 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

and $d^\top \approx (-2.336, 2.828)$. Figure 3 shows the construction of $\partial Q_\alpha$, for $\alpha \in \{0.1, 0.2, 0.3\}$, for the random variables $X$ and $Y$, letting $\nu$ vary over the values $\{1, 5\}$. An efficient computation of the depth contours for $Y$ is given by applying an affine transformation to the depth contours computed for $X$, where skewness is completely absorbed by the first component. Note that from Theorem 4.3, we only need to compute half of the depth contour and obtain the other half by symmetry. In the case of $\nu = 1$ the computation is further simplified by the circular shape of the depth contours of the canonical form.

**Example 4.5.** Let $\phi(\cdot; a)$ and $\Phi(\cdot; a)$ be the density and distribution functions of a univariate SN distribution with skewness parameter $a$, respectively. If $X \sim \text{CSN}_2(\gamma)$ and $y = u^\top x$,
then
\[ \tilde{P}_X(h;\mathbf{x}) = \frac{p(y)}{2p(y) + \delta \sqrt{2/\pi} - y} \]
where
\[ p(y) = 2\sqrt{\frac{1 + \gamma^2}{2\pi}} \Phi (\delta y; 0) - \phi (\delta y; \tilde{\delta}) - y \Phi (y; \tilde{\delta}) , \]
with
\[ \delta = \frac{\gamma}{\sqrt{1 + \gamma^2}} \quad \text{and} \quad \tilde{\delta} = \frac{u_1 \gamma}{\sqrt{1 + \gamma^2 (1 - u_1^2)}}, u_1 \in (-1, 1). \]

Figure 3 shows the expectile depth contours for the random variables \( \mathbf{X} \) and \( \mathbf{Y} = A \mathbf{X} + \mathbf{b} \) for \( \gamma = 3 \), where \( A \) and \( \mathbf{b} \) are defined in Example 4.4.

**Algorithm 1** Computation of HD for the ST distribution

Let \( \mathbf{X} \sim \text{ST}_d(\xi, \Omega, \gamma, \nu) \) and \( \mathbf{x} \in \mathbb{R}^d \):

1. compute \( A = P^\top B^{-1} \) where \( P \) and \( B \) are given in Theorem 4.1;
2. compute \( \gamma = (\gamma^\top \Omega \gamma)^{1/2} \);
3. transform \( \mathbf{x} \) in \( \mathbf{x}^* = A(\mathbf{x} - \xi) \);
4. Use numerical optimization to minimize
\[ \min \{ F_Y(u^\top \mathbf{x}^*), 1 - F_Y(u^\top \mathbf{x}^*) \} \]
with respect to the directional vector \( u \), where \( Y \sim \text{CST}_1(\gamma_*, \nu) \) and \( \gamma_* \) given by (16).

### Generalized hyperbolic distribution

A class of multivariate skewed distributions that has received a lot of attention in the financial literature is the class of generalized hyperbolic (GH) distributions; see McNeil, Frey & Embrechts (2005) and Eberlein (2010). Let \( \mu, \kappa \in \mathbb{R}^d \) denote the parameters of location and skewness, let \( \Sigma \in \mathbb{R}^{d \times d} \) be a symmetric, positive-definite dispersion matrix and let \( \lambda \in \mathbb{R} \), \( \chi, \psi \in \mathbb{R}^+ \) be scalars. \( \mathbf{X} \) has a generalized hyperbolic distribution, written \( \mathbf{X} \sim \text{GH}_d(\mu, \Sigma, \kappa, \lambda, \chi, \psi) \), if it has density
\[ f_{\text{GH}_d}(\mathbf{x}) = c \frac{K_{\lambda-d/2} \left( \sqrt{(\chi + Q_x)(\psi + \kappa^\top \Sigma^{-1} \kappa)} \right) \exp \left\{ (\mathbf{x} - \mu)^\top \Sigma^{-1} \mathbf{x} \right\}}{((\chi + Q_x)(\psi + \kappa^\top \Sigma^{-1} \kappa))^{(d/2-\lambda)/2}}, \quad \mathbf{x} \in \mathbb{R}^d, \]
where
\[ c = \frac{(\chi \psi)^{-\lambda/2} \psi^\lambda (\psi + \kappa^\top \Sigma^{-1} \kappa)^{d/2-\lambda}}{(2\pi)^{d/2} |\Sigma|^{1/2} K_{\lambda}(\sqrt{\chi \psi})} \quad \text{and} \quad Q_x = (\mathbf{x} - \mu)^\top \Sigma^{-1} (\mathbf{x} - \mu), \]
and where \( K_{\lambda}(\cdot) \) denotes the modified Bessel function of third kind. This class of distributions can be stochastically represented as mean-variance mixtures of normal distributions using the representation
\[ \mathbf{X} \stackrel{d}{=} \mu + W \kappa + \sqrt{W} \mathbf{Z}, \quad (10) \]
where
(i) $Z \sim N_d(0, I_d)$;

(ii) $A$ is a $d \times d$ matrix such that $\Sigma = AA^\top$;

(iii) $W$ has a generalized inverse Gaussian (GIG), denoted by $W \sim \text{GIG}(\lambda, \chi, \psi)$, with density function (17) in the Appendix.

Note that $\text{GH}_d(\mu, a\Sigma, a\kappa, \lambda, \chi/a, a\psi)$ and $\text{GH}_d(\mu, \Sigma, \kappa, \lambda, \chi, \psi)$ are equal in distribution for $a > 0$, which causes an identifiability problem. This problem can be solved by imposing a constraint on the model parameters; see the NIG case below.

An important feature of the GH distribution is its flexibility. It also contains several special cases and, in particular, we consider the following.

1. **Normal-inverse-Gaussian (NIG) distribution**: $\lambda = -1/2$ and $\chi = \psi$ (our choice of identifiability constraint).

2. **Skewed-t (St) distribution**: $\lambda = -\nu/2$, $\chi = \nu$ and $\psi = 0$.

Other special cases and a more detailed discussion of the GH family of distributions are given in McNeil, Frey & Embrechts (2005).

We now introduce the canonical form of the GH distribution. The following result follows from the general result on linear transformations in Appendix A.2.

**Theorem 4.6.** Let $X \sim \text{GH}_d(\mu, \Sigma, \kappa, \lambda, \chi, \psi)$ where $\Sigma = BB^\top$ for $B \in \mathbb{R}^{d \times d}$. Let $\kappa_* = (\kappa^\top \Sigma^{-1} \kappa)^{1/2}$ and define $X^* = P^\top B^{-1}(X - \mu)$ where $P$ is an orthonormal matrix having the first column equal to $\kappa_*^{-1} B^{-1} \kappa$. Then $X^* \sim \text{GH}_d(0, I_d, \kappa_* e_1, \lambda, \chi, \psi)$.

If $X$ is in canonical form, then we write $X \sim \text{CGH}_d(\kappa, \lambda, \chi, \psi)$ where $\kappa$ is the scalar skewness parameter; in the case of the NIG distribution, we write $X \sim \text{CNIG}_d(\kappa, \psi)$; and in the case of the St distribution, $X \sim \text{CSt}_d(\kappa, \nu)$.

Theorem 4.2 in Section 4.1 can be easily extended to the GH family using a similar argument that makes use of the canonical form, thus we omit it.

In the following example we calculate half-space depth contours for special cases of the GH distribution. A similar approach to Algorithm 1 is used to compute half-space depth at points $x \in \mathbb{R}$. Since the GH family is closed under linear operations, the probabilities of half spaces are simply computed from the distribution function of univariate GH distributions. The example suggests that the depth contours are particularly close to elliptical for many GH distributions and, in view of this, we also calculate an approximating ellipsoid using Algorithm 2. Expectile depth is also straightforward to compute, as we illustrate.

**Example 4.7.** Let $X \sim \text{CSt}_2(3, \nu)$ and $Y \sim \text{CNIG}_2(3, \psi)$. Figure 4 shows some HD contours for $X$ (top panels), letting $\nu$ vary over the set $\{3, 10\}$, and $Y$ (lower panels), with $\psi \in \{1/10, 1\}$. Dotted lines correspond to approximating ellipsoids obtained from Algorithm 2. As shown in Figure 4, larger values of $\nu$ for $X$ and larger values of $\psi$ for $Y$ result in a better ellipsoidal approximation of the depth contours (see Section 4.3). Figure 5 shows ED contours for $X$ with $\nu = 10$ (left panel) and $Y$ with $\psi = 1$ (right panel).
Figure 4: Half-space depth contours $\partial Q_{0.1}$, $\partial Q_{0.2}$ and $\partial Q_{0.3}$ in the case of the random variables $X$ (top panels), with $\nu = 3$ (left panel) and $\nu = 10$ (right panel), and $Y$ (lower panels), as defined in Example 4.7; dotted lines correspond to approximating ellipsoids to each of the depth contours, obtained using Algorithm 2.
Figure 5: Expectile depth contours $\partial E_{0.1}$, $\partial E_{0.2}$ and $\partial E_{0.3}$ for the bivariate variables $X$ and $Y$ in Example 4.7.

Algorithm 2 Computation of the approximating ellipsoid of $Q_\alpha$ for the GH distribution

Let $X \sim \text{GH}_d(\mu, \Sigma, \kappa, \lambda, \chi, \psi)$:

1. Compute $\kappa_\ast = (\kappa^\top \Sigma^{-1} \kappa)^{1/2}$.
2. Compute $A = P^\top B^{-1}$ where $P$ and $B$ are given in Theorem 4.6.
3. For each component of $X^\ast \sim \text{CGH}_d(\kappa_\ast, \nu)$ compute the $\alpha$-quantile $a_i$, the $(1 - \alpha)$-quantile $b_i$, and set $c_i = (a_i + b_i)/2$ and $d_i = |a_i - b_i|/2$, for $i = 1, \ldots, d$.
4. Approximate $Q_\alpha$ with the ellipsoid

$$
\tilde{Q}_\alpha = \left\{ x : (x - A^{-1}c - \mu)^\top A^\top DA(x - A^{-1}c - \mu) \leq 1 \right\},
$$

where $D = \text{diag}(d_1^{-2}, \ldots, d_d^{-2})$. 

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4.3 Relationship between angular symmetry and ellipsoidal depth sets

A natural measure of skewness that quantifies the deviation of a random variable \( X \) in \( \mathbb{R}^d \) from angular symmetry is

\[
d_1(X) = \frac{1}{2} - \text{HD}_X(\beta_X),
\]

where \( \beta_X \) is the half-space median. The affine equivariance of the median \( \beta_X \) implies that \( d_1(X) \) is an affine invariant measure of skewness. This can be seen by letting \( c(X) = AX + b \) denote the affine transformation that puts \( X \) into its canonical form and observing that

\[
d_1(c(X)) = \frac{1}{2} - \text{HD}_{c(X)}(\beta_{c(X)}) = \frac{1}{2} - \text{HD}_{AX+b}(A\beta_X + b) = \frac{1}{2} - \text{HD}_X(\beta_X) = d_1(X).
\]

For the ST and GH distributions we also define an alternative measure of skewness by

\[
d_2(X) = \frac{1}{2} - \text{HD}_{c(X)}(\eta_{c(X)}),
\]

where \( \eta_{c(X)} \) denotes the component-wise median of \( c(X) \); this measure is simpler to calculate. Note that \( d_2(X) \geq d_1(X) \) since \( \text{HD}_{c(X)}(\beta_{c(X)}) \geq \text{HD}_{c(X)}(\eta_{c(X)}) \) and if \( X \) is angularly symmetric then \( d_1(X) = d_2(X) = 0 \). Alternative measures of multivariate skewness are discussed in a non-parametric context in Liu, Prelius & Singh (1999).

Figure 6 shows some curves of \( d_2 \) for different CST distributions against \( 1/\nu \) and letting \( \gamma \) vary over the set \( \{ \pm 1, \pm 2, \pm 3, \pm 5, \pm 10, \pm \infty \} \). Here, we only focused on \( \nu \geq 1 \) in order to highlight the different values taken by \( d_2 \) in this interval; however, for \( \nu < 1 \), the different curves monotonically increase towards \( 1/2 \). Deviation from angular symmetry appears to reach its maximum at about 0.035, that corresponds to the case of the independent pair of variables \( (Z_1, Z_2)^\top \), where \( Z_2 \) is a standard normal variable and \( Z_1 \) is a half-normal variable. However, closeness of the ST distribution to angular symmetry is indicated for values of \( \nu \) close to 1. Indeed, the following results prove that in the case of the SC distribution, angular symmetry holds exactly.

**Theorem 4.8.** If \( (X, Z)^\top \sim \text{CSC}_2(\gamma) \), then for any \( a \in \mathbb{R} \) the median of \( Y_a = X + aZ \) is

\[
\eta = \frac{\gamma}{\sqrt{1 + \gamma^2}}.
\]

**Proof.** From the linear forms of the ST distribution (see Appendix A.1)

\[
Y_a \sim \text{SC}_1\left(0, 1 + a^2, \frac{\gamma_a}{\sqrt{1 + a^2}}\right),
\]

where \( \gamma_a = \gamma/\sqrt{1 + a^2(1 + \gamma^2)} \). The median of \( Y_a \) is then (see Section 2.3 in Behboodian, et al. (2006))

\[
\frac{\sqrt{1 + a^2}}{\sqrt{1 + \gamma_a^2}} \gamma_a = \frac{\gamma}{\sqrt{1 + \gamma^2}}.
\]

**Corollary 4.9.** If \( X \sim \text{CSC}_d(\gamma) \), then \( X \) is angularly symmetric about \( \eta^\top = (\eta, 0 \ldots, 0) \) where \( \eta \) is given by (13).
Proof. It follows from Theorem 4.8 that the component-wise median of $X$ is \( \eta^\top = (\eta, 0 \ldots, 0) \) where \( \eta \) is given by (13). For any directional vector \( u \in \mathbb{R}^d \), we can write

\[
P_X(H_{\eta,u}) = P \left( u_1 X_1 + \sum_{i=2}^d u_i X_i \leq u_1 \eta \right),
\]

(14)

If \( u_1 = 0 \) the above equation equals 1/2; let \( u_1 > 0 \) and set \( a = \sqrt{1 - u_2^2}/u_1 \). Using Appendix A.1 it may be easily shown that \( X_1 + \sum_{i=2}^d u_i X_i/u_1 = Y_a \), where \( Y_a \) is defined in Theorem 4.8. It follows that (14) can be expressed as

\[
P \left( X_1 + \frac{1}{u_1} \sum_{i=2}^d u_i X_i \leq \eta \right) = P(Y_a \leq \eta) = 1/2.
\]

Since \( u_1 \) is arbitrary we conclude that \( \text{HD}_X(\eta) = 1/2 \) and hence that \( \eta \) is the half-space median, as well as the center of angular symmetry of \( X \). \( \square \)

Corollary 4.10. If \( X \sim \text{SC}_d(\xi, \Omega, \gamma) \), then \( X \) is angularly symmetric about \( \xi + \omega \delta \) where \( \omega = \text{diag}(\omega_{11}, \ldots, \omega_{dd})^{1/2}, \ \Omega = \omega^{-1} \Omega \omega^{-1} \) and

\[
\delta = \frac{\Omega \gamma}{\sqrt{1 + \gamma^\top \Omega \gamma}}.
\]

Proof. Let \( X^* = c(X) = P^\top B^{-1}(X - \xi) \) where \( P \) and \( B \) are defined in Theorem 4.1. Since \( X^* \) is angularly symmetric about its median \( \beta_{X^*} = (\eta, 0 \ldots, 0) \) where \( \eta \) is given in (13), it follows easily that \( X \) is angularly symmetric about its median \( \beta_X = \xi + BP \beta_{X^*} = \xi + \omega \delta \). \( \square \)

Figure 7 is an analogous plot to Figure 6 for the case of the CSt2 distribution. In this case the degrees of freedom play an opposite role with respect to the CST2 case: \( d_2 \to 0 \) as \( \nu \to \infty \). This can be explained as follows. Using the stochastic representation in (10), as \( \nu \to \infty \), \( W \) tends to a degenerate distribution which is constant in 1. From (10), we have that \( X \) tends in distribution to \( N_d(\mu + \kappa, \Sigma) \) which is elliptically (hence also angularly) symmetric.

A similar argument applies to the NIG distribution. As \( \psi \to \infty \), \( W \) tends in distribution to the constant 1 and the same conclusions are drawn as in the previous case. This is reflected in Figure 8, where \( d_2 \to 0 \) for decreasing values of \( 1/\psi \). Hence large values of \( \psi \) results in a better ellipsoidal approximation.

4.4 Probability of misclassification using ellipsoidal approximations

Let \( X \) denote a random variable in \( \mathbb{R}^d \), belonging either to the ST or GH family. If \( \hat{Q}_\alpha \) is the approximating ellipsoid of \( Q_\alpha \) then the misclassification set is given by

\[
M = M_1 \cup M_2,
\]

where \( M_1 = \{ x : x \in Q_\alpha \text{ and } x \notin \hat{Q}_\alpha \} \) is the set of false negatives and \( M_2 = \{ x : x \notin Q_\alpha \text{ and } x \in \hat{Q}_\alpha \} \) is the set of false positives. The probability of misclassification is

\[
P(X \in M) = \int_M f_X(x) \, dx,
\]

(15)
Figure 6: Deviation from angular symmetry for the ST family.
Figure 7: Deviation from angular symmetry for the St family.
Figure 8: Deviation from angular symmetry for the NIG family.
where \( f_X \) is the density function of \( X \). The above integral is intractable, in general, so we use numerical quadrature to approximate the integral as a sum over a fine grid. Letting \( \tilde{x}_1, \ldots, \tilde{x}_n \in M \) be a regular grid covering \( M \) with cell area \( \Delta \), we have

\[
\int_M f_X(x) \, dx \approx \Delta \sum_{i=1}^{n} f_X(\tilde{x}_i).
\]

For small values of \( \alpha \) (for example \( \alpha = 0.05 \)) we find that, in general, the misclassification probability is relatively low with values exceeding 0.1 only in very extreme cases with very strong asymmetry. Additionally, we find that the main component in the misclassification probability is generally given by \( P(X \in M_1) \) while \( P(X \in M_2) \) is often negligible. The next example illustrate these observations.

**Example 4.11.** Let \( X \sim \text{CNIG}_2(\kappa, 1/10) \), \( Y \sim \text{CSN}_2(\gamma) \) and \( Z \sim \text{CSt}_2(\kappa, 5) \). In this example, we consider the approximation of \( Q_{0.05} \) with the ellipse \( \tilde{Q}_{0.05} \). In Tables 1-3 the probability mass of the sets \( M_1 \) and \( M_2 \), and the index \( d_2 \) are reported for each of the three random random vectors while the skewness parameters \( \gamma \) and \( \kappa \) are allowed to vary. In all three cases, the misclassification probability increases for increasing values of the skewness parameter. The random variable \( X \) appears to have the highest misclassification probability of about 0.095 for \( \kappa = 30 \), a case of extremely high asymmetry. Only in the case of \( Y \)

| \( \kappa \) | \( P(X \in M_1) \) | \( P(X \in M_2) \) | \( d_2(X) \) |
|---|---|---|---|
| 1 | 0.008 | 0.000 | 0.009 |
| 2 | 0.028 | 0.000 | 0.018 |
| 5 | 0.069 | 0.000 | 0.026 |
| 15 | 0.089 | 0.000 | 0.029 |
| 30 | 0.095 | 0.000 | 0.029 |

| \( \gamma \) | \( P(Y \in M_1) \) | \( P(Y \in M_2) \) | \( d_2(Y) \) |
|---|---|---|---|
| 1 | 0.002 | 0.002 | 0.004 |
| 2 | 0.008 | 0.005 | 0.013 |
| 5 | 0.025 | 0.009 | 0.029 |
| 10 | 0.035 | 0.010 | 0.033 |
| 50 | 0.036 | 0.010 | 0.035 |

| \( \kappa \) | \( P(Z \in M_1) \) | \( P(Z \in M_2) \) | \( d_2(Z) \) |
|---|---|---|---|
| 1 | 0.001 | 0.000 | 0.005 |
| 3 | 0.002 | 0.001 | 0.013 |
| 5 | 0.003 | 0.001 | 0.015 |
| 10 | 0.003 | 0.001 | 0.017 |
| 20 | 0.003 | 0.001 | 0.017 |

is the probability mass on \( M_2 \) not negligible, with a maximum value of about 0.010. The random variable \( Z \) has the lowest misclassification probability even for very high values of \( \kappa \), reaching a maximum value of about 0.004. Note that the measure of skewness \( d_2 \) is also a
good indicator of the quality of the ellipsoidal approximation, although it is not comparable between different families of distributions.

Example 4.12. We now consider the yield data for 3-year and 10-year government bonds, plotted in Figure [4]. We fitted a bivariate NIG and obtained the following maximum likelihood estimates

\[
\hat{\mu}^\top = (-0.017, -0.016), \\
\hat{\Sigma} = \begin{pmatrix}
8.860 \times 10^{-6} & -5.350 \times 10^{-6} \\
-5.350 \times 10^{-6} & 2.844 \times 10^{-5}
\end{pmatrix}, \\
\hat{\kappa}^\top = (0.026, 0.020), \\
\hat{\psi} = 5.527.
\]

The resulting index of skewness is \( \hat{d}_2 = 0.001 \) which indicates a low level of skewness (as deviation from angular symmetry) of the estimated NIG distribution. Indeed, the misclassification probability of the ellipsoidal approximation is extremely low as is also evident in Figure [4].

5 Discussion

In this paper we have shown how multivariate scenario sets based on HD and ED can be efficiently computed in the case of the ST and GH distributions. Computation can be simplified by making use of a canonical form representation where only one component is asymmetric at most. Additionally, in the case of multivariate sets based on HD, ellipsoids represents a good approximation with a probability of misclassification exceeding 0.1 only in cases of extremely high skewness. We have demonstrated that the quality of the ellipsoidal approximations can be explained in terms of the closeness of the ST and GH distributions to angular symmetry. We proposed a measure of the departure from angular symmetry which is easy to compute and concordant with the misclassification probability.

In our examples of ellipsoidal approximations, we only considered bivariate cases. However, the affine invariance property and the availability of a canonical form, with all the asymmetry absorbed in the marginal distribution of the first component, means that this is sufficient to gain an understanding of the quality of the approximation. Indeed, the skewness index \( d_2 \), introduced in Section 4.3 and the curves shown in Figures 6, 7 and 8 are independent of the dimension of the underlying vector. Although the probability of misclassification will change with dimension, the geometry of these distributions means that the changes will remain modest.

The near-elliptical shape of the depth sets for ST and GH distributions, and the availability of a simple method of constructing an elliptical approximation, makes these distributions attractive for modelling the behaviour of financial risk factors in the stress testing applications mentioned in the Introduction.

Although we only considered the ST and GH distributions, we believe that some of the results can be easily extended to other multivariate skewed distributions which are closed under affine transformations and which admit a canonical form representation. An example
is given by skew scale mixtures of normal variates \cite{Branco2001} of which the multivariate skew-slash distribution \cite{Wang2006} is one of the many special cases.

Finally, some of the presented material might also be useful to address the related problem of defining multivariate measures of skewed distributions that do not depend on the existence of the moments of the distribution. For example, Theorem 4.3 and Algorithm 2 also allow for an exact and approximate computation, respectively, of a multivariate measure of kurtosis proposed by \cite{Wang2005}, who only considered elliptical distributions as parametric examples. Additionally, Corollary 4.10 provides a measure of location for the SC distribution, namely its center of angular symmetry, that should be preferred to the location parameter $\xi$ which usually lies in regions far from the “center” of the distribution. It would also be interesting to investigate whether the half-space median of the ST and GH distributions is unique or whether a specific point of maximum depth can be identified by making use of the canonical form.

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**References**

AZZALINI, A. (2005). The skew-normal distribution and related multivariate families. *Scandinavian Journal of Statistics* **32**, 159–188.

AZZALINI, A. & CAPITANIO, A. (1999). Statistical applications of the multivariate skew-normal distribution. *Journal of Royal Statistics. Series B* **61**, 579–602. The full article is available at arXiv.org:0911.2093v1.

AZZALINI, A. & CAPITANIO, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t distribution. *Journal of Royal Statistical Society. Series B* **65**, 367–389. The full article is available at arXiv.org:0911.2342v1.

AZZALINI, A. & GENTON, M. G. (2008). Robust likelihood methods based on the skew-t and related distributions. *International Statistical Review* **76**, 106–129.

BEHBOODIAN, J., JAMALIZADEH, A. & BALAKRISHNAN, N. (2006). A new class of skew-Cauchy distributions. *Statistics and Probability Letters* **76**, 1488–1493.

BELLINI, F., KLAS, B., MÜLLER, A. & GIANIN, M. (2013). Generalized quantiles as risk measures. Preprint.

BRANCO, M. D. & DEY, D. K. (2001). A general class of multivariate skew-elliptical distributions. *Journal of Multivariate Analysis* **79**, 99 – 113.

CAPITANIO, A. (2012). On the canonical form of scale mixtures of skew-normal distributions. Available at arXiv.org:1207.0797v1.
A Linear forms

Let $A$ be a non-singular $k \times d$ matrix with rank $k \leq d$ and let $b \in \mathbb{R}^k$. 
A.1 Skew-$t$ distribution

If $X \sim ST_d(\xi, \Omega, \gamma, \nu)$, then $Y = AX + b \sim ST_k(\xi_Y, \Omega_Y, \gamma_Y, \nu)$, where

$$
\begin{align*}
\xi_Y &= A\xi + b, \\
\Omega_Y &= A\Omega A^\top, \\
\gamma_Y &= \frac{\omega_Y \Omega_Y^{-1} C^\top \gamma}{\sqrt{1 + \gamma^\top (\Omega - C\Omega_Y^{-1} C^\top) \gamma}} \\
\end{align*}
$$

with $C = \omega^{-1} \Omega A^\top$.

If $X \sim CST_d(\gamma, \nu)$ and $u$ is a directional vector in $\mathbb{R}^d$, then $u^\top X \sim CST_1(\gamma_*, \nu)$, where

$$
\gamma_* = \frac{u_1 \gamma}{\sqrt{1 + \gamma^2 (1 - u_1^2)}}, u_1 \in (-1, 1). \quad (16)
$$

A.2 Generalized hyperbolic distribution

If $X \sim GH_d(\mu, \Sigma, \kappa, \lambda, \chi, \psi)$, then $Y = AX + b \sim ST_k(\mu_Y, \Sigma_Y, \kappa_Y, \lambda, \chi, \psi)$, where

$$
\begin{align*}
\mu_Y &= A\mu + b, \\
\Sigma_Y &= A\Sigma A^\top, \\
\kappa_Y &= A\kappa.
\end{align*}
$$

If $X \sim CGH_d(\kappa, \lambda, \chi, \psi)$ and $u$ is a directional vector in $\mathbb{R}^d$, then $u^\top X \sim CGH_1(u_1 \kappa, \lambda, \chi, \psi)$.

B Generalized Inverse Gaussian distribution

If $X$ has a generalized inverse Gaussian (GIG) distribution, written as $X \sim GIG(\lambda, \chi, \psi)$, then its density is

$$
f_{GIG}(x) = \frac{\chi^{-\lambda} \left(\sqrt{\chi \psi}\right)^\lambda}{2K_\lambda\left(\sqrt{\chi \psi}\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2} (\chi x^{-1} + \psi x)\right\}, \quad x > 0, \quad (17)
$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind of order $\lambda$ and with the following constraints on the other parameters: $\chi > 0$, $\psi \geq 0$ if $\lambda < 0$; $\chi > 0$, $\psi > 0$ if $\lambda = 0$; $\chi \geq 0$, $\psi > 0$ if $\lambda > 0$. 

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