A Mixed Linear Quadratic Optimal Control Problem with a Controlled Time Horizon

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Abstract

A mixed linear quadratic (MLQ, for short) optimal control problem is considered. The controlled stochastic system consists of two diffusion processes which are in different time horizons. There are two control actions: a standard control action $u(\cdot)$ enters the drift and diffusion coefficients of both state equations, and a stopping time $\tau$, a possible later time after the first part of the state starts, at which the second part of the state is initialized with initial condition depending on the first state. A motivation of MLQ problem from a two-stage project management is presented. It turns out that solving MLQ problem is equivalent to sequentially solve a random-duration linear quadratic (RLQ, for short) problem and an optimal time (OT, for short) problem associated with Riccati equations. In particular, the optimal cost functional can be represented via two coupled stochastic Riccati equations. Some optimality conditions for MLQ problem is therefore obtained using the equivalence among MLQ, RLQ and OT problems.

Keywords. Mixed linear-quadratic optimal control, optimal stopping, maximum principle, Riccati equation.

1 Preliminary and Problem Formulation

Let $T > 0$ be given and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space on which a one dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. We consider the following stochastic controlled system:

$$
\begin{align*}
&\begin{cases}
    dX_1(t) = \left[A_1(t)X_1(t) + B_1(t)u(t)\right]dt + \left[C_1(t)X_1(t) + D_1(t)u(t)\right]dW(t), & t \in [0, \tau), \\
    dX(t) = \left[A(t)X(t) + B(t)u(t)\right]dt + \left[C(t)X(t) + D(t)u(t)\right]dW(t), & t \in [\tau, T], \\
    X_1(0) = x, & X(\tau) = K(\tau)X_1(\tau - 0),
\end{cases} \\
&X_1(\cdot) \text{ is a basic project, } X(\cdot) \text{ is a total system.}
\end{align*}
$$

(1.1)

where $A_1(\cdot), B_1(\cdot), C_1(\cdot), D_1(\cdot), A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ and $K(\cdot)$ are given matrix-valued functions of compatible sizes. In the above, $X(\cdot) = (X_1(\cdot), X_2(\cdot))$ is the state process, taking values in $\mathbb{R}^n$, which is decomposed into two parts, $X_i(\cdot)$ is valued in $\mathbb{R}^{n_i} (i = 1, 2, n_1 + n_2 = n)$, $u(\cdot)$ is a (usual) control process taking values in some set $U \subseteq \mathbb{R}^m$, and $\tau$ is an $\mathbb{F}$-stopping time. From the above, we see that the part $X_1(\cdot)$ of the state process $X(\cdot)$ starts to run from $x_1 \in \mathbb{R}^{n_1}$ at $t = 0$. The total system will start to run at a later time $t = \tau$, with the initial state $X(\tau)$ depending on $X_1(\tau - 0)$. Besides the usual control $u(\cdot)$, the stopping time $\tau$ will also be taken as a control. The above state equation can be interpreted as follows: we let $X_1(\cdot)$ represent the dynamics of some basic project whereas $X_2(\cdot)$, initialized at the time

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\(\tau\), represents an additional or an auxiliary project. It is notable that the initial value of \(X_2(\cdot)\) depends on \(X_1(\tau)\), the value of the first component of the state at time \(\tau\). Some real examples are as follows.

**Example 1.1.** (Urban Planning) Let \(X_1(t)\) denote some quantity of the dynamic value of some basic infrastructure investment in urban planning (for example, the transportation network, systematic pollution protection, and so on) at time \(t\), while \(X_2(t)\) denotes the quantity of the real-estate property in urban planning, at time \(t \geq \tau\), where \(\tau \in (0, T)\) is the time moment at which some basic infrastructure has been set. Note that the construction of basic infrastructure will still be continued (although it will be less intensive) after \(\tau\). It follows naturally the real estate property should depend closely on \(X_1(\cdot)\).

**Example 1.2.** (Applied Technology) Let \(X_1(\cdot)\) represent the capital investment of some high-tech company in the phrase of primary research and development (R&D) while let \(X_2(\cdot)\) denote the capital investment in the phrase of technology marketing and product promotion, etc. Of course, \(X_2(\cdot)\) will depend on the competitive ability of the product which in turn depends on the technology ability in basic research \(X_1(\cdot)\).

Note that if \(A(\cdot), B(\cdot), C(\cdot), D(\cdot)\) are of the following form:

\[
A(\cdot) = \begin{pmatrix} 0 & 0 \\ 0 & A_2(\cdot) \end{pmatrix}, \quad B(\cdot) = \begin{pmatrix} 0 \\ B_2(\cdot) \end{pmatrix}, \quad C(\cdot) = \begin{pmatrix} 0 & 0 \\ 0 & C_2(\cdot) \end{pmatrix}, \quad D(\cdot) = \begin{pmatrix} 0 \\ D_2(\cdot) \end{pmatrix},
\]

then on the time period \([\tau, T]\), the part \(X_1(\cdot)\) will be completely stopped and only the part \(X_2(\cdot)\) will be running.

Now we introduce the following quadratic cost functional:

\[
J(x_1; u(\cdot), \tau) = \frac{1}{2} \mathbb{E} \left\{ \int_{0}^{T} \left[ \langle Q_1(t)X_1(t), X_1(t) \rangle + \langle R_1(t)u(t), u(t) \rangle \right] dt + \langle G_1(\tau)X_1(\tau), X_1(\tau) \rangle \right. \\
\left. + \int_{\tau}^{T} \left[ \langle Q(t)X(t), X(t) \rangle + \langle R(t)u(t), u(t) \rangle \right] dt + \langle GX(T), X(T) \rangle \right\},
\]

(1.2)

where \(Q_1(\cdot), R_1(\cdot), Q(\cdot), R(\cdot)\), and \(G_1(\cdot)\) are symmetric matrix-valued functions, and \(G\) is a symmetric matrix, of suitable sizes. Roughly speaking, our optimal control problem is to minimize \(J(x_1; u(\cdot), \tau)\) over the set of all admissible controls \((u(\cdot), \tau)\). We now make our problem formulation more precise.

For Euclidean space \(\mathbb{R}^n\), we denote by \(\langle \cdot, \cdot \rangle\) its inner product and \(|\cdot|\) the induced norm. Next, let \(\mathbb{R}^{m \times n}\) be the set of all \(m \times n\) real matrices, \(S^n\) be the set of all \(n \times n\) symmetric real matrices, and for any \(M = (m_{ij}) \in \mathbb{R}^{m \times n}\), \(M^T\) stands for its transpose and let

\[
|M| = \left( \sum_{i,j} m_{ij}^2 \right)^{\frac{1}{2}}, \quad \forall M \in \mathbb{R}^{m \times n}.
\]

For \(X = \mathbb{R}^n, \mathbb{R}^{n \times m}\), etc., let

\[
L^2(a, b; X) = \left\{ f : [a, b] \to X \mid f(\cdot) \text{ is measurable, } \int_a^b |f(t)|^2 dt < +\infty \right\},
\]

\[
L^\infty(a, b; X) = \left\{ f : [a, b] \to X \mid f(\cdot) \text{ is measurable, } \sup_{t \in [a, b]} |f(t)| < \infty \right\},
\]

\[
C([a, b]; X) = \left\{ f : [a, b] \to X \mid f(\cdot) \text{ is continuous} \right\}.
\]
and

\[ L^p_F(\Omega; X) = \left\{ \xi : \Omega \to X \mid \xi \text{ is } F_t\text{-measurable, } \mathbb{E}[|\xi|^p] < \infty \right\}, \quad p \geq 1, \]

\[ L^\infty_F(\Omega; X) = \left\{ \xi \in L^1_F(\Omega; X) \mid \text{esssup}_{\omega \in \Omega} |\xi(\omega)| < \infty \right\}, \]

\[ L^p_F(a, b; X) = \left\{ f : [a, b] \times \Omega \to X \mid f(\cdot) \text{ is } \mathbb{F}\text{-adapted, } \mathbb{E} \int_a^b |f(t)|^p dt < +\infty \right\}, \quad p \geq 1, \]

\[ L^\infty_F(a, b; X) = \left\{ f(\cdot) \in L^1_F(a, b; X) \mid \text{esssup}_{(t, \omega) \in [a, b] \times \Omega} |f(t, \omega)| < \infty \right\}, \]

\[ C_p([a, b]; L^p_F(\Omega; X)) = \left\{ f(\cdot) \in L^p_F(a, b; X) \mid t \mapsto f(t) \text{ is continuous, } \sup_{t \in [a, b]} \mathbb{E}[|f(t)|^p] < \infty \right\}, \quad p \geq 1. \]

In the definition of \( C_p([a, b]; L^p_F(\Omega; X)) \), \( t \mapsto f(t) \) is continuous means that
\[ \lim_{t \to t_0} \mathbb{E}[|f(t) - f(t_0)|^p] = 0, \quad \forall t_0 \in [a, b]. \]

Now, let us introduce the following sets:

\[ U[0, T] = \left\{ u : [0, T] \times \Omega \to U \mid u(\cdot) \in L^2_F(0, T; \mathbb{R}^m) \right\}, \]

\[ T[0, T] = \left\{ \tau : \Omega \to [0, T] \mid \tau \text{ is an } \mathbb{F}\text{-stopping time} \right\}. \]

Hereafter, \( U \subseteq \mathbb{R}^m \) is assumed to be convex and closed. Any \( u(\cdot) \in U[0, T] \) is called a \textit{regular admissible control} while \( \tau \in T[0, T] \) is called an \textit{admissible stopping time}. Under some mild conditions, for any \( x_1 \in \mathbb{R}^{n_1} \), and \( (u(\cdot), \tau) \in U[0, T] \times T[0, T] \), \( 1.1 \) admits a unique strong solution \( X(\cdot) \equiv X(\cdot; x_1, u(\cdot), \tau) \), and the cost functional \( 1.2 \) is well-defined. Having this, we can pose the following problem.

**Problem (MLQ)** For given \( x_1 \in \mathbb{R}^{n_1} \), find a \( (\bar{u}(\cdot), \bar{\tau}) \in U[0, T] \times T[0, T] \) such that
\[ J(x_1; \bar{u}(\cdot), \bar{\tau}) = \inf_{(u(\cdot), \tau) \in U[0, T] \times T[0, T]} J(x_1; u(\cdot), \tau) \equiv V(x_1). \]

Any \( (\bar{u}(\cdot), \bar{\tau}) \in U[0, T] \times T[0, T] \) satisfying the above is called an optimal control pair, \( \bar{X}(\cdot) \equiv X(\cdot; x_1, \bar{u}(\cdot), \bar{\tau}) \) is called the corresponding optimal trajectory, \( (\bar{X}(\cdot), \bar{u}(\cdot), \bar{\tau}) \) is called an \textit{optimal triple}. In the above, MLQ problem stands for \textit{mixed linear-quadratic} problem, in which, one has a usual control \( u(\cdot) \) mixed with a control \( \tau \) of stopping time. We have the following points to the above MLQ problem formulation.

- A special feature of Problem (MLQ) is that in minimizing the cost functional, one needs to select a regular control \( u(\cdot) \) from \( U[0, T] \) and at the same time, one has to find the best time \( \bar{\tau} \) to initiate or trigger the whole system. Note that the initial value \( X(\tau) \) of \( X(\cdot) \) at \( \tau \) depends on the value of \( X_1(\tau - 0) \), which in turn depends on the regular control on \( [0, \tau) \). From this viewpoint, the Problem (MLQ) is some kind of combination of a usual stochastic optimal control and an optimal stopping time problems. Similar problems have been investigated in literature, including Øksendal and Sulem \footnote{8} where some optimal resource extraction optimization problem was addressed. By applying the dynamic programming method, the optimal policy can be characterized by some Hamilton-Jacobi-Bellman variational inequalities, see Krylov \footnote{9}, for relevant treatment. In contrast, here, we aim to investigate the problem by the variational method which could lead to Pontryagin type maximum principle.
• Consider the following simple but illustrating example, from which we can see the significant difference between Problem (MLQ) and other relevant ones when applying the possible perturbation method. Suppose the controlled state equation is given by

\[
\begin{align*}
    dX_1(t) &= a_1 X_1(t) dW(t), \quad t \in [0, T], \\
    dX_2(t) &= a_2 X_2(t) dW(t), \quad t \in [\tau, T], \\
    X_2(\tau) &= K X_1(\tau),
\end{align*}
\]

where \(a_1, a_2, K\) are some constants. Let \(\bar{X}(\cdot) \equiv (\bar{X}_1(\cdot), \bar{X}_2(\cdot))\) be the solution corresponding to \(\bar{\tau}\). Introduce a perturbation on \(\tau\) of the form: \(\tau^\rho = \bar{\tau} + \rho \tau, \rho > 0\), with \(\tau\) being another stopping time. Let the solution corresponding to \(\tau^\rho\) be \(X^\rho(\cdot) \equiv (X_1^\rho(\cdot), X_2^\rho(\cdot))\). Then \(X_1^\rho(\cdot) = \bar{X}_1(\cdot)\) is independent of \(\rho\), and for \(t \in [\tau^\rho, T]\),

\[
    X_2^\rho(t) - \bar{X}_2(t) = K \left( \bar{X}_1(\tau^\rho) - \bar{X}_1(\bar{\tau}) \right) + \int_{\tau^\rho}^{t} a_2 \left( X_2^\rho(s) - \bar{X}_2(s) \right) dW(s) - \int_{\tau^\rho}^{\tau^\rho} a_2 \bar{X}_2(s) dW(s).
\]

The presence of the term \(\int_{\tau^\rho}^{\tau^\rho} a_2 \bar{X}_2(s) dW(s)\) makes the first-order Taylor expansion in convex variation failed to work. This is mainly due to the fact that

\[
    \mathbb{E} \left( \int_{\tau^\rho}^{\tau^\rho} a_2 \bar{X}_2(s) dW(s) \right)^2 = \mathbb{E} \int_{\tau^\rho}^{\tau^\rho} (a_2 \bar{X}_2(s))^2 ds,
\]

which, in rough sense, suggests \(\int_{\tau^\rho}^{\tau^\rho} a_2 \bar{X}_2(s) dW(s)\) be of order \(\sqrt{\rho}\) instead of \(\rho\). On the other hand, it is not feasible to apply the second-order Taylor expansion to introduce the second-order variational equation (as suggested by Peng [9], Yong and Zhou [13] etc.). This is mainly because the time horizon on which \(X_2(\cdot)\) is defined depends on the selection of \(\tau\), thus the spike variation method cannot be applied here either.

• Problem (MLQ) also differs from the well-studied stochastic impulse control problem, since as the time passes \(\tau\), instead of having a jump for the state as a usual impulse control does, our controlled system changes the dimension of the state (from \(X_1(\cdot)\) to \(X(\cdot)\)).

In summary, the involvement of \(\tau\) into the control variable makes the Problem (MLQ) essentially different from other classical optimal control problems, and the standard perturbation jointly on \((\bar{u}(\cdot), \bar{\tau})\) is not workable directly. Keep this in mind, in this paper, we take the following strategy to study Problem (MLQ): we first connect the Problem (MLQ) into some random-duration linear quadratic (RLQ, for short) optimal control problem, and an optimal time (OT, for short) problem to the associated Riccati equations. By Problem (RLQ), we can obtain some necessary condition for the regular optimal control \(\bar{u}(\cdot)\); by Problem (OT), we can obtain some necessary condition satisfied by the optimal time \(\bar{\tau}\). Next, by solving Problem (RLQ) and Problem (OT) consecutively, we can solve the original Problem (MLQ).

The rest of this paper is organized as follows. In Section 2, we get a stochastic maximum principle for a little more general two-stage random-duration optimal control problems. Based on it, Section 3 is devoted to a study of the random-duration linear quadratic optimal control problems. The state feedback optimal control is derived via some stochastic Riccati-type equations and the optimal cost functional is also calculated explicitly. In Section 4, an equivalence between Problem (MLQ) and Problems (RLQ)–(OT) is established. In Section 5, for the case of one-dimension with constant coefficients, we characterize the optimal time \(\bar{\tau}\).
2 Random-Duration Optimal Control Problem

In this section, we consider the following controlled stochastic differential equation (SDE, for short)

\[
\begin{array}{ll}
    dX_1(t) = b^1(t, X_1(t), u(t))dt + \sigma^1(t, X_1(t), u(t))dW(t), & t \in [0, \tau), \\
    dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), & t \in [\tau, T], \\
    X_1(0) = x_1, & X(\tau) = \Phi(\tau, X_1(\tau)),
\end{array}
\]

(2.1)

where \( \tau \in \mathcal{T}(0, T) \) is some fixed stopping time and \( u(\cdot) \in \mathcal{U}[0, T] \) is an admissible control. The cost functional is

\[
J^*(x_1; u(\cdot)) = \mathbb{E}\left[ \int_0^\tau g^1(t, X_1(t), u(t))dt + \int_\tau^T g(t, X(t), u(t))dt + h^1(\tau, X_1(\tau)) + h(X(T)) \right].
\]

(2.2)

Consider the following random-duration optimal control (ROC, for short) problem:

**Problem (ROC)** For \( x_1 \in \mathbb{R}^n \) and \( \tau \in \mathcal{T}[0, T] \), find a \( \bar{u}(\cdot) \in \mathcal{U}[0, T] \) such that

\[
J^*(x_1; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J^*(x_1; u(\cdot)) \equiv V^*(x_1).
\]

(2.3)

The following basic assumptions will be in force:

(H2.1) Let

\[
b, \sigma : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n, \quad b^1, \sigma^1 : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n,
\]

\[
\Phi : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n
\]

be measurable such that

\[
(t, \omega) \mapsto (b(t, x, u, \omega), \sigma(t, x, u, \omega)),
\]

\[
(t, \omega) \mapsto (b^1(t, x, u, \omega), \sigma^1(t, x, u, \omega), \Phi(t, x, \omega)),
\]

are progressively measurable,

\[
(x, u) \mapsto (b(t, x, u, \omega), \sigma(t, x, u, \omega)),
\]

\[
(x_1, u) \mapsto (b^1(t, x_1, u, \omega), \sigma^1(t, x_1, u, \omega), \Phi(t, x_1, \omega)),
\]

are continuously differentiable, and for some constant \( L > 0 \),

\[
|b_x(t, x, u)| + |b_u(t, x, u)| + |b^1_x(t, x_1, u)| + |\sigma^1_x(t, x_1, u)| + |\Phi_x(t, x)| \leq L,
\]

\[
\forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times U, \text{ a.s. ,}
\]

and

\[
|b(t, 0, u)| + |\sigma(t, 0, u)| + |b^1(t, 0, u)| + |\sigma^1(t, 0, u)| + |\Phi(t, 0)| \leq L, \quad (t, u) \in [0, T] \times U, \text{ a.s.}
\]

(H2.2) Let

\[
g : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}, \quad g^1 : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R},
\]

\[
h : \mathbb{R}^n \times \Omega \to \mathbb{R}, \quad h^1 : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}
\]
be measurable such that
\[(t, \omega) \mapsto g(t, x, u, \omega), \quad (t, \omega) \mapsto g^1(t, x, u, \omega)\]
are progressively measurable,
\[(x, u) \mapsto g(t, x, u, \omega), \quad (x_1, u) \mapsto g^1(t, x_1, u, \omega),\]
\[(x_1, u) \mapsto (b^1(t, x_1, u, \omega), \sigma^1(t, x_1, u, \omega), \Phi(t, x_1, \omega)),\]
are continuously differentiable, and for some constant \(L > 0\),
\[|g_x(t, x, u)| + |g_u(t, x, u)| + |h_x(x)| \leq L(1 + |x| + |u|),\]
\[|g^1_x(t, x_1, u)| + |g^1_u(t, x_1, u)| + |h^1_x(t, x_1)| \leq L(1 + |x_1| + |u|),\]
\[\forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times U,\]
and
\[|g(t, 0, u)| + |g^1(t, 0, u)| + |h^1(t, 0)| \leq L, \quad \forall (t, u) \in [0, T] \times U.\]

By some standard arguments, we see that under assumptions (H2.1)–(H2.2), for any \(x_1 \in \mathbb{R}^n\), and \((u(\cdot), \tau) \in U[0, T] \times \mathcal{T}[0, T]\), the state equation (H2.1) admits a unique solution \(X(\cdot; x_1, u(\cdot), \tau)\) and the cost functional \(J^*(x_1; u(\cdot))\) is well-defined. Therefore, Problem (ROC) makes sense. Suppose \((\bar{X}(\cdot), \bar{u}(\cdot))\) is an optimal pair of Problem (ROC), depending on \((x_1, \tau)\). For any \(v(\cdot) \in U[0, T]\), denote
\[u(\cdot) = v(\cdot) - \bar{u}(\cdot).\]

By the convexity of \(U\), we know that
\[u^\rho(\cdot) = \bar{u}(\cdot) + \rho u(\cdot) = (1 - \rho)\bar{u}(\cdot) + \rho v(\cdot) \in U[0, T], \quad \forall \rho \in (0, 1).\]

By the optimality of \((\bar{X}(\cdot), \bar{u}(\cdot))\), we have
\[\lim_{\rho \to 0} \frac{J^*(x_1; u^\rho(\cdot)) - J^*(x_1; \bar{u}(\cdot))}{\rho} \geq 0.\]

Making use of some similar arguments in [7], we have the following result.

**Lemma 2.1.** Suppose (H2.1)–(H2.2) hold. Let \((\bar{X}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (ROC). Then
\[\mathbb{E} \left\{ \int_0^\tau \left[ g^1_x(t, \bar{X}(t), \bar{u}(t))\xi_1(t) + g^1_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dt + h^1_x(t, \bar{X}(t), \bar{u}(t))\xi_1(t) \right. \]
\[\left. + \int_\tau^T \left[ g_x(t, \bar{X}(t), \bar{u}(t))\xi(t) + g_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dt + h_x(\bar{X}(T), \bar{u}(t))\xi(T) \right\} \geq 0,\]
where \(\xi(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))\) is the solution to the following variational system:
\[
\begin{aligned}
d\xi_1(t) &= \left[ b^1_x(t, \bar{X}(t), \bar{u}(t))\xi_1(t) + b^1_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dt \\
&\quad + \left[ \sigma^1_x(t, \bar{X}(t), \bar{u}(t))\xi_1(t) + \sigma^1_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dW(t), \quad t \in [0, \tau), \\
d\xi(t) &= \left[ b_x(t, \bar{X}(t), \bar{u}(t))\xi(t) + b_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dt \\
&\quad + \left[ \sigma_x(t, \bar{X}(t), \bar{u}(t))\xi(t) + \sigma_u(t, \bar{X}(t), \bar{u}(t))u(t) \right] dW(t), \quad t \in [\tau, T], \\
\xi_1(0) &= 0, \quad \xi(\tau) = \Phi_{x_1}(\tau, \bar{X}(\tau))\xi_1(\tau).
\end{aligned}
\]

The following is a Pontryagin type maximum principle.
Theorem 2.2. Suppose (H2.1)–(H2.2) hold. Let \((\bar{X}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (ROC). Then the following two backward stochastic differential equations (BSDEs, for short) admit unique adapted solutions \((p(\cdot), q(\cdot))\) and \((p_1(\cdot), q_1(\cdot))\):

\[
\begin{cases}
 dp(t) = \left[-b_x(t, \bar{X}(t), \bar{u}(t))^T p(t) + \sigma_x(t, \bar{X}(t), \bar{u}(t))^T q(t) \\
- g_x(t, \bar{X}(t), \bar{u}(t))^T\right] dt + q(t) dW(t), & t \in [\tau, T], \\
p(T) = -h_x(\bar{X}(T))^T,
\end{cases}
\]

(2.5)

\[
\begin{cases}
 dp_1(t) = \left[-b_{x_1}(t, \bar{X}_1(t), \bar{u}(t))^T p_1(t) + \sigma_{x_1}(t, \bar{X}_1(t), \bar{u}(t))^T q_1(t) \\
- g_{x_1}(t, \bar{X}_1(t), \bar{u}(t))^T\right] dt + q_1(t) dW(t), & t \in [0, \tau], \\
p_1(\tau) = -h_{x_1}(\tau, \bar{X}_1(\tau))^T + \Phi_{x_1}(\tau, \bar{X}_1(\tau))^T p(\tau).
\end{cases}
\]

(2.6)

Moreover, the following variational inequalities hold

\[
\begin{cases}
 [p_1(t)^T b_{x_1}(t, \bar{X}_1(t), \bar{u}(t)) + q_1(t)^T \sigma_{x_1}(t, \bar{X}_1(t), \bar{u}(t)) - g_{x_1}(t, \bar{X}_1(t), \bar{u}(t))] [v - \bar{u}(t)] \leq 0,
\end{cases}
\]

(2.7)

\[t \in [0, \tau], \quad v \in U, \quad \text{a.s.} \]

\[t \in [\tau, T], \quad v \in U, \quad \text{a.s.}\]

Proof. Applying Itô formula to \(\langle p_1(\cdot), \xi_1(\cdot)\rangle\) and \(\langle p(\cdot), \xi(\cdot)\rangle\), respectively, we have

\[
\mathbb{E} \left[ \langle -h_{x_1}(\tau, \bar{X}_1(\tau))^T + \Phi_{x_1}(\tau, \bar{X}_1(\tau))^T p(\tau), \xi_1(\tau) \rangle \right] = \mathbb{E} \left\{ p_1(\tau) \xi_1(\tau) \right\}
\]

\[
= \mathbb{E} \int_{\tau}^{T} \left[ \langle -[b_{x_1}(t, \bar{X}_1, \bar{u})^T p_1 + \sigma_{x_1}(t, \bar{X}_1, \bar{u})^T q_1 - g_{x_1}(t, \bar{X}_1, \bar{u})^T], \xi_1 \rangle \right. \\
\left. + \langle p_1, b_{x_1}(t, \bar{X}_1, \bar{u}) \rangle \xi_1 + b_{x_1}(t, \bar{X}_1, \bar{u}) \rangle + \langle q_1, \sigma_{x_1}(t, \bar{X}_1, \bar{u}) \rangle \xi_1 + \langle g_{x_1}(t, \bar{X}_1, \bar{u}) u \rangle \right] dt
\]

\[
= \mathbb{E} \int_{\tau}^{T} \left[ b_{x_1}(t, \bar{X}_1, \bar{u}) \xi_1 + \langle b_{x_1}(t, \bar{X}_1, \bar{u})^T p_1 + \sigma_{x_1}(t, \bar{X}_1, \bar{u})^T q_1, u \rangle \right] dt,
\]

and

\[
\mathbb{E} \langle -h_x(\bar{X}(T))^T, \xi(T) \rangle = \mathbb{E} \langle p(T), \xi(T) \rangle
\]

\[
= \mathbb{E} \left\{ \langle p(\tau), \xi(\tau) \rangle + \int_{\tau}^{T} \left[ \langle -[b_x(t, \bar{X}, \bar{u})^T p + \sigma_x(t, \bar{X}, \bar{u})^T q - g_x(t, \bar{X}, \bar{u})^T], \xi \rangle \right. \\
\left. + \langle p, b_x(t, \bar{X}, \bar{u}) \rangle \xi + b_x(t, \bar{X}, \bar{u}) u \rangle + \langle q, \sigma_x(t, \bar{X}, \bar{u}) \rangle \xi + \langle g_x(t, \bar{X}, \bar{u}) u \rangle \right] dt \right\}
\]

\[
= \mathbb{E} \left\{ \langle p(\tau), \Phi_x(\tau, \bar{X}_1(\tau)) \xi(\tau) \rangle + \int_{\tau}^{T} \left[ g_x(t, \bar{X}, \bar{u}) \xi + \langle b_x(t, \bar{X}, \bar{u})^T p + \sigma_x(t, \bar{X}, \bar{u})^T q, u \rangle \right] dt \right\}.
\]

Then, we obtain

\[
0 \leq \mathbb{E} \left\{ \int_{\tau}^{T} \left[ b_{x_1}(t, \bar{X}_1, \bar{u}) \xi_1 + g_{x_1}(t, \bar{X}_1, \bar{u}) u \right] dt + h_{x_1}(\tau, \bar{X}_1(\tau)) \xi_1(\tau) \right.
\]

\[
+ \int_{\tau}^{T} \left[ g_x(t, \bar{X}, \bar{u}) \xi + g_x(t, \bar{X}, \bar{u}) u \right] dt + h_x(\bar{X}(T)) \xi(T) \right\}
\]

\[
= \mathbb{E} \left\{ \int_{\tau}^{T} \left[ -b_{x_1}(t, \bar{X}_1, \bar{u})^T p_1 + \sigma_{x_1}(t, \bar{X}_1, \bar{u})^T q_1 + g_{x_1}(t, \bar{X}_1, \bar{u})^T u \right] dt + \langle \Phi_{x_1}(\tau, \bar{X}_1(\tau))^T p(\tau), \xi_1(\tau) \rangle \right.
\]

\[
+ \int_{\tau}^{T} \left[ -b_x(t, \bar{X}, \bar{u})^T p + \sigma_x(t, \bar{X}, \bar{u})^T q + g_x(t, \bar{X}, \bar{u})^T u \right] dt - \langle p(\tau), \Phi_{x_1}(\tau, \bar{X}_1(\tau))^T \xi(\tau) \rangle \right\}
\]
\[
\begin{align*}
\mathbb{E}\left\{ & \int_0^T \left( -b^1_u(t, \bar{x}_1, \bar{u})^T p_1 - \sigma^1_u(t, \bar{x}_1, \bar{u})^T q_1 + g^1_u(t, \bar{x}_1, \bar{u})^T v - v \bar{u} \right) dt \\
& + \int_\tau^T \left( -b_u(t, \bar{x}, \bar{u})^T p - \sigma_u(t, \bar{x}, \bar{u})^T q + g_u(t, \bar{x}, \bar{u})^T v - v \bar{u} \right) dt \right\}.
\end{align*}
\]
Note that \(v(\cdot) \in \mathcal{U}[0, T]\) is arbitrary, we therefore have \(2.7\).

3 Random-Duration Linear Quadratic Problem

For any \((x_1, \tau) \in \mathbb{R}^{n_1} \times \mathcal{T}[0, T]\), we consider the following controlled linear system:

\[
\begin{align*}
\begin{cases}
dX_1(t) &= [A_1(t)X_1(t) + B_1(t)u(t)] dt + [C_1(t)X_1(t) + D_1(t)u(t)] dW(t), & t \in [0, \tau], \\
dX(t) &= [A(t)X(t) + B(t)u(t)] dt + [C(t)X(t) + D(t)u(t)] dW(t), & t \in [\tau, T], \\
X_1(0) &= x_1, & X(\tau) = K(\tau)X_1(\tau - 0),
\end{cases}
\end{align*}
\]
with quadratic cost functional as follows:

\[
\begin{align*}
J^*(x_1; u(\cdot)) &= \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left[ \langle Q_1(t)X_1(t), X_1(t) \rangle + \langle R_1(t)u(t), u(t) \rangle \right] dt + \langle G_1(\tau)X_1(\tau), X_1(\tau) \rangle \\
& \quad + \int_\tau^T \left[ \langle Q(t)X(t), X(t) \rangle + \langle R(t)u(t), u(t) \rangle \right] dt + \langle GX(T), X(T) \rangle \right\}.
\end{align*}
\]

Now we pose the following problem.

**Problem (RLQ)** For given \((x_1, \tau) \in \mathbb{R}^{n_1} \times \mathcal{T}[0, T]\), find a \(\bar{u}(\cdot) \in \mathcal{U}[0, T]\) such that

\[
J^*(x_1; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J^*(x_1; u(\cdot)) \equiv V^*(x_1).
\]

We call the above a **random-duration linear quadratic** (RLQ, for short) problem. For the above problem, we introduce the following hypothesis.

**(H3.1)** The following holds:

\[
\begin{align*}
\begin{cases}
A_1(\cdot), C_1(\cdot) & \in L^\infty_T(0; \mathbb{R}^{n_1 \times n_1}), & A(\cdot), C(\cdot) & \in L^\infty_T(0; \mathbb{R}^{n \times n}), \\
B_1(\cdot), D_1(\cdot) & \in L^\infty_T(0; \mathbb{R}^{n_1 \times m}), & B(\cdot), D(\cdot) & \in L^\infty_T(0; \mathbb{R}^{n \times m}), \\
Q_1(\cdot) & \in L^\infty_T(0; \mathbb{S}^{n_1}), & Q(\cdot) & \in L^\infty_T(0; \mathbb{S}^n), \\
R_1(\cdot), R(\cdot) & \in L^\infty_T(0; \mathbb{S}^m), & G(\cdot) & \in L^\infty_T(\Omega; \mathbb{S}^n), \\
G(\cdot) & \in C_b([0, T]; L^\infty_T(\Omega; \mathbb{S}^n)), & K(\cdot) & \in C_b([0, T]; L^\infty_T(\Omega; \mathbb{R}^{n \times n_1})).
\end{cases}
\end{align*}
\]

Moreover, for some \(\delta > 0\),

\[
\begin{align*}
\begin{cases}
R_1(t) & \geq \delta I_{n_1}, & R(t) & \geq \delta I_n, \\
Q_1(t), G_1(t) & \geq 0, & Q(t) & \geq 0, & G & \geq 0,
\end{cases} & \quad t \in [0, T], \text{ a.s.}
\end{align*}
\]

It is clear that under (H3.1), for any given \((x_1, \tau) \in \mathbb{R}^{n_1} \times \mathcal{T}[0, T]\), the map \(u(\cdot) \mapsto J(x_1; u(\cdot), \tau)\) is convex and coercive. Therefore, Problem (RLQ) admits a unique optimal control \(\bar{u}(\cdot) \in \mathcal{U}[0, T]\). Now, let \((\bar{X}(\cdot), \bar{u}(\cdot))\) be the optimal pair of Problem (RLQ), depending on \((x_1, \tau) \in \mathbb{R}^{n_1} \times \mathcal{T}[0, T]\). By Theorem 2.2, on \([\tau, T]\) the optimal pair \((\bar{X}(\cdot), \bar{u}(\cdot))\) satisfies the following:

\[
\begin{align*}
\begin{cases}
d\bar{X}(t) &= [A(t)\bar{X}(t) + B(t)\bar{u}(t)] dt + [C(t)\bar{X}(t) + D(t)\bar{u}(t)] dW(t), \\
dp(t) &= -[A(t)^T p(t) + C(t)^T q(t) - Q(t)\bar{X}(t)] dt + q(t) dW(t), \\
p(T) &= -G\bar{X}(T), \\
\bar{X}(\tau) &= K(\tau)\bar{X}_1(\tau), \\
B(t)^T p(t) + D(t)^T q(t) - R(t)\bar{u}(t) &= 0.
\end{cases}
\end{align*}
\]
The above is a coupled FBSDE in random duration. To solve it, we let
\[ p(t) = -P(t)\dot{X}(t), \quad t \in [0, T], \]
for some \( P(\cdot) \) satisfying
\[
\begin{cases}
  dP(t) = \Gamma(t)dt + \Lambda(t)dW(t), & t \in [0, T], \\
  P(T) = G.
\end{cases}
\]
Then apply Itô's formula, we have
\[
( - A^T P \dot{X} + C^T q - Q \dot{X} ) dt - q dW = -dp = d(P \dot{X})
\]
\[
= \left\{ \Gamma \dot{X} + P(A \dot{X} + B \dot{u}) + \Lambda(C \dot{X} + D \dot{u}) \right\} dt + \left\{ \Lambda \dot{X} + P(C \dot{X} + D \dot{u}) \right\} dW
\]\
\[
= \left\{ (\Gamma + PA + \Lambda C) \dot{X} + (PB + \Lambda D) \dot{u} \right\} dt + \left\{ (\Lambda + PC) \dot{X} + PD \dot{u} \right\} dW.
\]
Hence,
\[
q = -(\Lambda + PC) \dot{X} - PD \dot{u}.
\]
Then
\[
R \ddot{u} = B^T P + D^T q = -B^T P \ddot{X} - D^T (\Lambda + PC) \ddot{X} - D^T P \ddot{u}.
\]
This implies
\[
\ddot{u} = -(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \ddot{X}.
\]
Substituting \( \ddot{u} \) into the expression of the above \( q \) yields
\[
q = -(\Lambda + PC) \ddot{X} + PD(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \ddot{X}.
\]
Moreover,
\[
0 = (A^T P + Q) \ddot{X} - C^T q + (\Gamma + PA + \Lambda C) \ddot{X} + (PB + \Lambda D) \ddot{u}
\]
\[
= (\Gamma + PA + A^T P + \Lambda C + Q) \ddot{X}
\]
\[
- C^T \left[ -(\Lambda + PC) \ddot{X} + PD(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \ddot{X} \right]
\]
\[
-(PB + \Lambda D)(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \ddot{X}
\]
\[
= \left( \Gamma + PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q \right)
\]
\[
-(PB + \Lambda D + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \ddot{X}.
\]
Therefore, we take
\[
\Gamma = - \left[ PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q \right.
\]
\[
\left. -(PB + \Lambda D + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \right].
\]
Consequently, the corresponding Riccati equation reads
\[
\begin{cases}
  dP = \left[ PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q \\
  \quad - (PB + \Lambda D + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T \Lambda + D^T PC) \right] dt \\
  \quad + \Lambda dW, & t \in [0, T], \\
  P(T) = G, & R + D^T PD > 0.
\end{cases}
\]
Moreover, the optimal value of the cost functional is given by
\[
\langle K(\tau)^TP(\tau)K(\tau) + G_1(\tau) \rangle \tilde{X}_1(\tau),
\]
where solutions on \([0, T]\).

\[ \begin{aligned}
    d\tilde{X}_1 &= [A_1\tilde{X}_1 + B_1R_1^{-1}B_1^Tp_1 + B_1R_1^{-1}D_1^Tq_1]dt + [C_1\tilde{X}_1 + D_1R_1^{-1}B_1^Tp_1 + D_1R_1^{-1}D_1^Tq_1]dW(t), \\
    dp_1 &= -[A_1^T p_1 + C_1^T q_1 - Q_1\tilde{X}_1]dt + q_1dW(t), \\
    \tilde{X}_1(0) &= x_1, \quad p_1(\tau) = -[K(\tau)^TP(\tau)K(\tau) + G_1(\tau)]\tilde{X}_1(\tau), \\
    R_1\tilde{u}_1 &= B_1^Tp_1 + D_1^Tq_1.
\end{aligned} \]  

\[(3.8)\]

Similar to the above, the corresponding Riccati equation is
\[
\begin{aligned}
    dP_1 &= -[P_1A_1 + A_1^TP_1 + C_1^TP_1C_1 + \Lambda_1C_1 + C_1^T\Lambda_1 + Q_1 \\
    &\quad - (P_1B_1 + \Lambda_1D_1 + C_1^TP_1D_1)(R_1 + D_1^TP_1D_1)^{-1}(B_1^TP_1 + D_1^T\Lambda_1 + D_1^TP_1C_1)]dt \\
    &\quad + \Lambda_1dW, \quad t \in [0, \tau], \\
    P_1(\tau) &= K(\tau)^TP(\tau)K(\tau) + G_1(\tau), \quad R_1 + D_1^TP_1D_1 > 0.
\end{aligned} \]

Following [10], we know that under (H3.1), Riccati equations (3.7) and (3.9) admit unique adapted solutions on \([0, T]\) and \([0, \tau]\) respectively. The following is a verification theorem.

**Theorem 3.1.** Let (H3.1) hold. Let \(P(\cdot)\) and \(P_1(\cdot)\) be the solutions of Riccati equations (3.7) and (3.9) respectively. Then Problem (RLQ) has an optimal control with state feedback form as follows:

\[(3.10)\]

\[
\tilde{u}(t) = \begin{cases} 
-\Psi(t)\tilde{X}(t), & t \in [\tau, T], \\
-\Psi_1(t)\tilde{X}_1(t), & t \in [0, \tau), 
\end{cases}
\]

where

\[(3.11)\]

\[
\Psi(t) = \left[R(t) + D(t)^TP(t)D(t)\right]^{-1}\left[B(t)^TP(t) + D(t)^TP(t)C(t) + D(t)^T\Lambda(t)\right], \\
\Psi_1(t) = \left[R_1(t) + D_1(t)^TP_1(t)D_1(t)\right]^{-1}\left[B_1(t)^TP_1(t) + D_1(t)^TP_1(t)C_1(t) + D_1(t)^T\Lambda_1(t)\right],
\]

Moreover, the optimal value of the cost functional is given by

\[(3.12)\]

\[J^*(x_1; \tilde{u}(\cdot)) = \frac{1}{2}\langle P_1(0)x_1, x_1 \rangle = V^*(x_1).\]

**Proof.** For any \(u(\cdot) \in U[0, T]\), let \(X(\cdot)\) be the corresponding state process. Applying Itô formula to \(\langle P(\cdot)X(\cdot), X(\cdot) \rangle\) on the interval \([\tau, T]\), we obtain (let \(\Gamma\) be defined by (3.6))

\[
\mathbb{E}[\langle GX(T), X(T) \rangle - \langle P(\tau)X(\tau), X(\tau) \rangle] = \mathbb{E}\left[\int_{\tau}^{T} (\langle (\Gamma + PA + A^TP + C^TPC + AC + C^T\Lambda)X, X \rangle \\
+ 2\langle (B^TP + D^T\Lambda + D^TPC)X, u \rangle + \langle D^TPDu, u \rangle)dt \right] = \mathbb{E}\left\{\int_{\tau}^{T} [2\langle (B^TP + D^TPC + D^T\Lambda)X, u \rangle - \langle QX, X \rangle + \langle D^TPDu, u \rangle \\
+ \langle (PB + C^TPD + \Lambda D)(R + D^TPD)^{-1}(B^TP + D^TPC + D^T\Lambda)X, X \rangle]dt \right\}.
\]
It follows that:

\[ J^*(x_1; u(\cdot)) - \frac{1}{2} \mathbb{E} \left( P(\tau)X(\tau), X(\tau) \right) \]
\[ = \frac{1}{2} \mathbb{E} \left\{ \int_0^\tau \left( \langle Q_1X_1, X_1 \rangle + \langle R_1u, u \rangle \right) dt + \langle G_1(\tau)X_1(\tau), X_1(\tau) \rangle \right. \\
\left. + \int_\tau^T \left( \langle QX, X \rangle + \langle Ru, u \rangle \right) dt + \langle GX(T), X(T) \rangle \right. \\
\left. - \langle GX(T), X(T) \rangle + \int_\tau^T \left[ 2 \langle (B^T P + D^T PC + D^T \Lambda) X, u \rangle - \langle QX, X \rangle + \langle D^T PDu, u \rangle \right. \\
\left. + \langle (PB + CT PD + \Lambda D)(P + D^T PD)^{-1}(B^T P + D^T PC + D^T \Lambda) X, X \rangle \right] dt \right\} \]

\[ = \frac{1}{2} \mathbb{E} \left\{ \int_0^\tau \left( \langle Q_1X_1, X_1 \rangle + \langle R_1u, u \rangle \right) dt + \langle G_1(\tau)X_1(\tau), X_1(\tau) \rangle \right. \\
\left. + \int_\tau^T \left( \langle R + D^T PD \rangle^{-1}(B^T P + D^T PC + D^T \Lambda) X \right. \\
\left. + 2 \langle (B^T P + D^T PC + D\Lambda) X, u \rangle + \langle (R + D^T PD) u, u \rangle \right] dt \right\} \]

\[ = \frac{1}{2} \mathbb{E} \left\{ \int_0^\tau \left( \langle Q_1X_1, X_1 \rangle + \langle R_1u, u \rangle \right) dt + \langle G_1(\tau)X_1(\tau), X_1(\tau) \rangle \right. \\
\left. + \int_\tau^T \left( R + D^T PD \right)^\frac{1}{2} \left| u + (R + D^T PD)^{-1}(B^T P + D^T PC + D^T \Lambda) X \right|^2 dt \right\}. \]

Therefore, we have

\[ J^*(x_1; u(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \langle P_1(0)x_1, x_1 \rangle \right. \\
\left. + \int_0^\tau \left| (R_1 + D_1^T P_1 D_1)^\frac{1}{2} \left| u + (R_1 + D_1^T P_1 D_1)^{-1}(B_1^T P_1 + D_1^T P_1 C_1 + D_1^T \Lambda_1) X_1 \right|^2 \right. \\
\left. + \int_\tau^T \left| (R + D^T PD)^\frac{1}{2} \left| u + (R + D^T PD)^{-1}(B^T P + D^T PC + D^T \Lambda) X \right|^2 dt \right\} \]

\[ \equiv \frac{1}{2} \mathbb{E} \left\{ \langle P_1(0)x_1, x_1 \rangle + \int_0^\tau \left| (R_1 + D_1^T P_1 D_1)^{-1}(u + \Psi_1 X_1) \right|^2 dt + \int_\tau^T \left| (R + D^T PD)^{-1}(u + \Psi X) \right|^2 dt \right\}. \]

Then our conclusion follows.

4 The Equivalence of Control Problems

Under (H3.1), Problem (RLQ) admits a unique optimal pair which can be represented by \(P^*_1(\cdot), \Lambda^*_1(\cdot)\). In this section, we will establish some connection between Problem (MLQ) and Problem (RLQ). To this end, we denote the solution to Riccati equation (3.3) on \([0, \tau]\) by \((P^*_1(\cdot), \Lambda^*_1(\cdot))\), emphasizing its dependence on \(\tau\) via the terminal condition. It is clear that \(\tau \mapsto P^*_1(\tau)\) is continuous. Therefore, the following problem makes sense:
**Problem (OT)** For given \( x_1 \in \mathbb{R}^{n_1} \), find a \( \bar{\tau} \in \mathcal{T}[0, T] \) such that

\[
(4.1) \quad (P^\tau_1(0)x_1, x_1) = \inf_{\tau \in \mathcal{T}[0, T]} (P^\tau_1(0)x_1, x_1).
\]

Any \( \bar{\tau} \in \mathcal{T}[0, T] \) satisfying the above is called an optimal time of Problem (OT). Note that in the case \( n_1 = 1 \), if \( \bar{\tau} \) is an optimal time for some \( x_1 \in \mathbb{R} \setminus \{0\} \), then it is an optimal time for all \( x_1 \in \mathbb{R} \setminus \{0\} \). On the other hand, since there might not be some kind of monotonicity of the map \( \tau \mapsto P^\tau_1(s) \), optimal time \( \bar{\tau} \) may not be unique. Now we establish the equivalence between Problem (MLQ) and Problems (RLQ) and (OT).

**Theorem 4.1.** Suppose (H3.1) holds and \((\bar{u}(), \bar{\tau})\) is an optimal control pair of Problem (MLQ). Then the optimal control \( \bar{u}() \) can be represented by

\[
(4.2) \quad \bar{u}(t) = \begin{cases} 
-\Psi(t)\bar{X}(t), & t \in [\bar{\tau}, T], \\
-\Psi_1^\tau(t)\bar{X}_1(t), & t \in [0, \bar{\tau}), 
\end{cases}
\]

where

\[
(4.3) \quad \begin{cases} 
\Psi(t) = \left[ R(t) + D(t)^TP(t)D(t) \right]^{-1} \left[ B(t)^TP(t) + D(t)^TP(t)C(t) + D(t)^T\Lambda(t) \right], \\
\Psi_1^\tau(t) = \left[ R_1(t) + D_1(t)^TP_1^\tau(t)D_1(t) \right]^{-1} \left[ B_1(t)^TP_1^\tau(t) + D_1(t)^TP_1^\tau(t)C_1(t) + D_1(t)^T\Lambda_1^\tau(t) \right].
\end{cases}
\]

Moreover, the optimal value of the cost functional for Problem (MLQ) is given by

\[
(4.4) \quad J(x_1; \bar{u}(), \bar{\tau}) = \frac{1}{2} (P^\tau_1(0)x_1, x_1),
\]

with \( \bar{\tau} \) being an optimal time for Problem (OT) to Riccati equation (3.7) and (3.9).

**Proof.** Suppose \((\bar{u}(), \bar{\tau})\) is an optimal pair of Problem (MLQ), that is

\[
J(x_1; \bar{u}(), \bar{\tau}) = \inf_{(u(), \tau) \in \mathcal{U}[0, T] \times \mathcal{T}[0, T]} J(x_1; u(), \tau).
\]

We fix \( \bar{\tau} \). Then the above implies

\[
J^\bar{\tau}(x_1; \bar{u}()) = \inf_{u() \in \mathcal{U}[0, T]} J^\bar{\tau}(x_1; u()).
\]

Then it follows from Theorem 3.1 that \( \bar{u}() \) admits representation (4.2) with \( \Psi() \) and \( \Psi_1^\tau() \) given by (4.3), and

\[
J^\bar{\tau}(x_1; \bar{u}()) = \frac{1}{2} (P^\bar{\tau}_1(0)x_1, x_1).
\]

Next, for any stopping time \( \tau \in \mathcal{T}[0, T] \), we can construct a control \( \bar{u} \in \mathcal{U}[0, T] \) satisfying

\[
\bar{u}(t) = \begin{cases} 
-\Psi(t)\bar{X}(t), & t \in [\tau, T], \\
-\Psi_1^\tau(t)\bar{X}_1(t), & t \in [0, \tau),
\end{cases}
\]

where \( \bar{X}(), \bar{X}_1() \) are defined in similar way to \( X(), \bar{X}_1() \) but replacing \( \bar{\tau} \) by \( \tau \). Following the similar arguments, we can prove

\[
J^\tau(x_1; \bar{u}()) = \inf_{u() \in \mathcal{U}[0, T]} J^\tau(x_1; u()) = \frac{1}{2} (P^\tau_1(0)x_1, x_1).
\]
Moreover, for some \( \delta > 0 \),
\[
\frac{1}{2} \left< \mathcal{P}^\tau(x_1; x_1) \right> = J^\tau(x_1; \bar{u}(\cdot)) = J(x_1; \bar{u}(\cdot), \bar{\tau})
\]
\[
= \inf_{(u(\cdot), \tau) \in \mathcal{U}[0, T] \times \mathcal{T}[0, T]} J(x_1; u(\cdot), \tau) \leq J(x_1; u(\cdot), \tau) = \frac{1}{2} \left< \mathcal{P}^\tau(x_1; x_1) \right>.
\]
That is, \( \bar{\tau} \) solves the Problem (OT). Hence the results.

Next, we consider the case of deterministic coefficients. More precisely, we introduce the following assumption.

\[\text{(H3.1')}\] The following holds:
\[
\begin{align*}
A_1(\cdot), C_1(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_1 \times n_1}), & A(\cdot), C(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n \times n}), \\
B_1(\cdot), D_1(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n_1 \times m}), & B(\cdot), D(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n \times m}), \\
Q_1(\cdot) &\in L^\infty(0, T; \mathbb{S}^{n_1}), & Q(\cdot) &\in L^\infty(0, T; \mathbb{S}^n), \\
R(\cdot) &\in L^\infty(0, T; \mathbb{S}^m), & R_1(\cdot) &\in L^\infty(0, T; \mathbb{S}^m), \\
K(\cdot) &\in C([0, T]; \mathbb{S}^n),
\end{align*}
\]
Moreover, for some \( \delta > 0 \),
\[
\begin{align*}
R_1(t) &\geq \delta I_{n_1}, & R(t) &\geq \delta I_n, \\
Q_1(t), G_1(t) &\geq 0, & Q(t) &\geq 0, & G &\geq 0,
\end{align*}
\]
\( t \in [0, T] \).

We have the following result.

**Proposition 4.2.** Let (H3.1') hold. Let \( P(\cdot) \) solves
\[
\begin{align*}
\dot{P} + PA + A^TP + C^TPC + Q - (PB + C^TPD)(R + D^TPD)^{-1}(B^TP + D^TPC) &= 0, \\
P(T) &= G, & R + D^TPD &> 0,
\end{align*}
\]
and \( P_1(\cdot) \) solves
\[
\begin{align*}
\dot{P}_1 + PA_1 + A^T_1P_1 + C^T_1P_1C_1 + Q_1 \\
- (P_1^T B_1 + C_1^T P_1^T D_1)(R_1 + D_1^T P_1^T D_1)^{-1}(B_1^T P_1 + D_1^T P_1 C_1) &= 0, & t &\in [0, r], \\
P_1(r) &= K(r)^T P(r) K(r) + G_1(r), & R_1 + D_1^T P_1 D_1 &> 0,
\end{align*}
\]
with \( r \in (0, T) \). Then Problem (MLQ) has an optimal control pair \( (\bar{u}(\cdot), \bar{r}) \in \mathcal{U}[0, T] \times [0, T] \) where
\[
\bar{u}(t) = \begin{cases} -\Psi(t)\bar{X}(t), & t \in [\bar{r}, T], \\ -\Psi_1(t)\bar{X}_1(t), & t \in [0, \bar{r}], \end{cases}
\]
with \( \bar{r} \) being deterministic and
\[
\begin{align*}
\Psi(t) &= \left[ R(t) + D(t)^TP(t)D(t) \right]^{-1} \left[ B(t)^TP(t) + D(t)^TP(t)C(t) \right], \\
\Psi_1(t) &= \left[ R_1(t) + D_1(t)^TP_1(t)D_1(t) \right]^{-1} \left[ B_1(t)^TP_1(t) + D_1(t)^TP_1(t)C_1(t) \right].
\end{align*}
\]
Moreover,
\[
\left< \mathcal{P}^\tau(x_1; x_1) \right> = \inf_{r \in [0, T]} \left< \mathcal{P}_1^\tau(x_1; x_1) \right>.
\]
The cost functional reads

\[ \frac{1}{2} \langle P_1^x(0)x_1, x_1 \rangle = \inf_{\tau \in \mathbb{T}[0,T]} \frac{1}{2} \langle P_1^x(0)x_1, x_1 \rangle = V(x_1). \]

This proves our result.

For the convenience below, we state the following problem.

**Problem (DOT)** Find \( \bar{r} \in [0,T] \) such that

\[ \langle P_1^x(0)x_1, x_1 \rangle = \inf_{r \in [0,T]} \langle P_1^x(0)x_1, x_1 \rangle. \]

By Proposition 4.2, we see that the optimal time \( \bar{r} \) in Problem (DOT) solves Problem (MLQ). Moreover, in general, the optimal time for Problem (DOT) depends on the Riccati equation \( P_1^x(\cdot) \) and the initial condition \( x_1 \). The following example makes this clear.

**Example 4.3.** Let \( n_1 = 2, n_2 = 1 \), and let

\[
\begin{align*}
A &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix}, \\
B &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
C &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
D &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
C_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
D_1 &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
Q &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
R &= 1, \\
G &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
Q_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
R_1 &= 1, \\
G_1 &= \begin{pmatrix} g_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
K &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{align*}
\]

with \( a \in \mathbb{R}, g, g_1 \in (0, \infty) \). Then for any \( r \in (0,T) \), we have the state equation

\[
\begin{align*}
\dot{X}_1(t) &= X_1^2(t), & t \in [0,r), \\
\dot{X}_2(t) &= u(t), & t \in [0,r), \\
\dot{X}_2(t) &= aX_2(t) + u(t), & t \in [r,T], \\
X_1(0) &= x_1, \\
X_2(r) &= X_1^2(r).
\end{align*}
\]

The cost functional reads

\[
J(x_1; u(\cdot), r) = \frac{1}{2} \left\{ \int_0^r |u(t)|^2 dt + g_1|X_1^2(r)|^2 + \int_r^T |u(t)|^2 dt + g|X_2(T)|^2 \right\}.
\]

In this case, we have

\[
P(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & P_2(t) & 0 \end{pmatrix}, \quad t \in [r,T],
\]
with \( P_2(\cdot) \) solves the following Riccati equation:

\[
\begin{aligned}
\begin{cases}
\dot{P}_2(t) + 2aP_2(t) - P_2(t)^2 = 0, & t \in [r, T], \\
P_2(T) = g.
\end{cases}
\end{aligned}
\]

This equation admits a unique solution \( P_2(\cdot) \). We claim that,

\[
P_2(t) = \begin{cases} 
\frac{2age^{2a(T-t)}}{g(e^{2a(T-t)} - 1) + 2a}, & t \in [r, T], \quad a \neq 0, \\
\frac{g}{1 + g(T-t)}, & t \in [r, T], \quad a = 0.
\end{cases}
\]

Note that if \( a > 0 \), then \( g(e^{2a(T-t)} - 1) + 2a \geq 2a > 0 \), and if \( a < 0 \), then \( g(e^{2a(T-t)} - 1) + 2a \leq 2a < 0 \). Hence,

\[
P_2(t) > 0, \quad t \in [r, T].
\]

Next, the Riccati equation for \( P_1(\cdot) \) reads

\[
\begin{aligned}
\begin{cases}
\dot{P}_1(t) + P_1(t)A_1 + A_1^TP_1(t) - P_1(t)MP_1(t) = 0, & t \in [0, r], \\
P_1(r) = \bar{G}_1(r),
\end{cases}
\end{aligned}
\]

where

\[
M = B_1B_1^T = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \quad \bar{G}_1(r) = K^TP(r)K + G_1 = \left( \begin{array}{cc} g_1 + P_2(r) & 0 \\ 0 & 0 \end{array} \right).
\]

We now solve the above Riccati equation by a method found in [6]. To this end, let

\[
\tilde{P}_1(t) = P_1(t) - \bar{G}_1(r), \quad t \in [0, r].
\]

Then (suppressing \( t \) and \( r \))

\[
\begin{aligned}
\dot{\tilde{P}}_1 &= \tilde{P}_1 = -\left[ P_1A_1 + A_1^TP_1 - P_1MP_1 \right] \\
&= -\left[ (\tilde{P}_1 + \bar{G}_1)A_1 + A_1^T(\tilde{P}_1 + \bar{G}_1) - (\tilde{P}_1 + \bar{G}_1)M(\tilde{P}_1 + \bar{G}_1) \right] \\
&= -\left[ \tilde{P}_1(A_1 - M\bar{G}_1) + (A_1 - M\bar{G}_1)^T\tilde{P}_1 - \tilde{P}_1M\tilde{P}_1 + \bar{G}_1A_1 + A_1^T\bar{G}_1 - \bar{G}_1M\bar{G}_1 \right].
\end{aligned}
\]

Note that

\[
M\bar{G}_1 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} g_1 + P_2(r) & 0 \\ 0 & 0 \end{array} \right) = 0,
\]

\[
\bar{G}_1A_1 = \left( \begin{array}{cc} g_1 + P_2(r) & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & g_1 + P_2(r) \\ 0 & 0 \end{array} \right).
\]

Hence, \( \tilde{P}_1(\cdot) \) should be the solution to the following Riccati equation:

\[
\begin{aligned}
\begin{cases}
\dot{\tilde{P}}_1(t) + \tilde{P}_1(t)A_1 + A_1^T\tilde{P}_1(t) - \tilde{P}_1(t)M\tilde{P}_1(t) + \bar{g}_1J = 0, & t \in [0, r], \\
\tilde{P}_1(r) = 0,
\end{cases}
\end{aligned}
\]
where
\[
J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{g}_1 = g_1 + P_2(r).
\]

Next, we let
\[
\mathcal{A} = \begin{pmatrix} A_1 & -M \\ -\tilde{g}_1 J & -A_1^T \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & \tilde{g}_1 & 0 & 0 \\ -\tilde{g}_1 & 0 & -1 & 0 \end{pmatrix}.
\]

Then according to [3], we have the following representation of the solution \( P_1(\cdot) \):
\[
\tilde{P}_1(t) = -\left[ (0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} (0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t \in [0, r],
\]
as long as the involved inverse exists. We now calculate \( e^{\mathcal{A}(r-t)} \). Direct computation show that
\[
\mathcal{A}^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ \tilde{g}_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tilde{g}_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}^3 = \begin{pmatrix} \tilde{g}_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\tilde{g}_1^2 & -\tilde{g}_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}^4 = 0.
\]

Hence,
\[
e^{\mathcal{A}(r-t)} = I + (r-t)\mathcal{A} + \frac{(r-t)^2}{2} \mathcal{A}^2 + \frac{(r-t)^3}{6} \mathcal{A}^3
\]
\[
= \begin{pmatrix} 1 + \frac{\bar{g}_1 (r-t)^3}{6} & (r-t) & \frac{(r-t)^3}{6} & \frac{(r-t)^2}{2} \\ \frac{\bar{g}_1 (r-t)^2}{2} & 1 & \frac{(r-t)^2}{2} & \frac{-r(t)}{3} \\ -\tilde{g}_1 (r-t) & 1 - \frac{\bar{g}_1 (r-t)^3}{6} & \frac{-r(t)}{3} & 1 \\ \frac{-\bar{g}_1 (r-t)}{2} & 0 & -\frac{-r(t)}{3} & 1 \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix},
\]

Then
\[
(0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} = (0, I) \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = \Phi_{22} = \begin{pmatrix} 1 - \frac{\bar{g}_1 (r-t)^3}{6} & \frac{\bar{g}_1 (r-t)^2}{2} \\ -\frac{-r(t)}{3} & 1 \end{pmatrix}.
\]

Since
\[
det \left( \begin{array}{cc} 1 - \frac{\bar{g}_1 (r-t)^3}{6} & \frac{\bar{g}_1 (r-t)^2}{2} \\ -\frac{-r(t)}{3} & 1 \end{array} \right) = 1 + \frac{\bar{g}_1 (r-t)^3}{3} > 0, \quad \forall t \in [0, r],
\]
we have
\[
\begin{pmatrix} 1 - \frac{\bar{g}_1 (r-t)^3}{6} & \frac{\bar{g}_1 (r-t)^2}{2} \\ -\frac{-r(t)}{3} & 1 \end{pmatrix}^{-1} = \frac{3}{3 + \bar{g}_1 (r-t)^3} \begin{pmatrix} 1 & \frac{-\bar{g}_1 (r-t)^2}{6} \\ \frac{-r(t)}{3} & \frac{-\bar{g}_1 (r-t)}{2} \end{pmatrix}.
\]

On the other hand,
\[
(0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} I \\ 0 \end{pmatrix} = (0, I) \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = \Phi_{21} = \begin{pmatrix} -\frac{\bar{g}_1 (r-t)^3}{6} & -\bar{g}_1 (r-t) \\ \frac{-\bar{g}_1 (r-t)}{2} & 0 \end{pmatrix}.
\]

Hence,
\[
\tilde{P}_1(t) = -\left[ (0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} (0, I) e^{\mathcal{A}(r-t)} \begin{pmatrix} I \\ 0 \end{pmatrix} = -\Phi_{22}^{-1} \Phi_{21}
\]
\[
= -\frac{3}{3 + \bar{g}_1 (r-t)^3} \begin{pmatrix} 1 & \frac{-\bar{g}_1 (r-t)^2}{6} \\ \frac{-r(t)}{3} & \frac{-\bar{g}_1 (r-t)}{2} \end{pmatrix} \begin{pmatrix} \bar{g}_1 (r-t) \\ -\bar{g}_1 (r-t)^2 \end{pmatrix} = \frac{3}{3 + \bar{g}_1 (r-t)^3} \begin{pmatrix} \bar{g}_1 (r-t) \\ -\bar{g}_1 (r-t)^2 \end{pmatrix}.
\]
Consequently,

\[ P_1(t) = \bar{P}_1(t) + \bar{G}_1(r) = \frac{3[g_1 + P_2(r)]}{3 + [g_1 + P_2(r)](r - t)}\begin{pmatrix} 1 & r - t \\ r - t & (r - t)^2 \end{pmatrix}, \quad t \in [0, r], \]

which is positive definite on \([0, r]\). Hence,

\[ V(r, x_1) = (P'_1(0)x_1, x_1) = \frac{3[g_1 + P_2(r)]}{3 + [g_1 + P_2(r)]r^3}[x_1^1]^2 + 2rx_1^1x_1^2 + r^2(x_1^1)^2 \]

\[ = \frac{3[g_1 + P_2(r)]}{3 + [g_1 + P_2(r)]r^3}[x_1^2 + rx_1^1]^2. \]

Clearly, for different \(x_1 \in \mathbb{R}^2\), the optimal \(\bar{r}\) will be different in general.

## 5 One-Dimensional Cases with Constant Coefficients

In this section, we make the following assumption

\[
\begin{cases}
    n_1 = n_2 = m = 1, \\
    A(t) = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad C(t) = \begin{pmatrix} 0 & 0 \\ 0 & C_2 \end{pmatrix}, \quad D(\cdot) = \begin{pmatrix} 0 \\ D_2 \end{pmatrix}, \\
    Q(t) = \begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad R(t) = R_2, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}, \quad K(\cdot) = \begin{pmatrix} 1 \\ K \end{pmatrix}, \\
    A_1(t) = A_1, \quad B_1(t) = B_1, \quad C_1(t) = C_1, \quad D_1(t) = D_1, \\
    Q_1(t) = Q_1, \quad R_1(t) = R_1, \quad G_1(t) = G_1,
\end{cases}
\]

where \(A_i, B_i, C_i, D_i, Q_i, R_i, G_i, K\) are all constants \((i = 1, 2)\), and

\[
R_1, R_2 > 0, \quad Q_1, Q_2, G_1, G_2 \geq 0, \quad K \neq 0.
\]

Then the controlled system becomes

\[
\begin{aligned}
    &dX_1(t) = [A_1X_1(t) + B_1u(t)]dt + [C_1X_1(t) + D_1u(t)]dW(t), \quad t \in [0, \tau], \\
    &dX_2(t) = [A_2X_2(t) + B_2u(t)]dt + [C_2X_2(t) + D_2u(t)]dW(t), \quad t \in [\tau, T], \\
    &X_1(0) = x_1, \quad X_2(\tau) = KX_1(\tau - 0),
\end{aligned}
\]

and the cost functional is

\[
J(x_1; u(\cdot), \tau) = \frac{1}{2}\mathbb{E}\left\{ \int_0^{\tau} [Q_1X_1(t)^2 + R_1u(t)^2]dt + G_1X_1(\tau)^2 + \int_{\tau}^{T} [Q_2X_2(t)^2 + R_2u(t)^2]dt + G_2X_2(T)^2 \right\}.
\]

Thus, the first component \(X_1(\cdot)\) of the state process will be completely terminated from \(\tau\) on. In this case, we have

\[
P(t) = \begin{pmatrix} 0 & 0 \\ 0 & P_2(t) \end{pmatrix}, \quad t \in [\tau, T],
\]

where \(P_2(\cdot)\) satisfies

\[
\begin{cases}
    \dot{P}_2(t) + (2A_2 + C_2^2)P_2(t) + Q_2 - \frac{(B_2 + C_2D_2)^2P_2(t)^2}{R_2 + D_2^2P_2(t)} = 0, \quad t \in [0, T], \\
    P_2(T) = G_2, \quad R_2 + D_2^2P_2(t) > 0,
\end{cases}
\]
and \( P_1^r(\cdot) \) satisfies
\[
(5.6) \begin{cases}
\dot{P}_2(t) + F_2(P_2(t)) = 0, & t \in [0, T], \\
P_2(T) = G_2,
\end{cases}
\]
and
\[
(5.8) \begin{cases}
\dot{P}_1^r(t) + F_1(P_1^r(t)) = 0, & t \in [0, r], \\
P_1^r(r) = K^2P_2(r) + G_1.
\end{cases}
\]

Note that since \( n_1 = 1 \), the optimal time \( \bar{r} \) of Problem (DOT) is independent of the initial state \( x_1 \). Thus, the optimal time \( \bar{r} \) satisfies
\[
(5.9) \quad P_1^\bar{r}(0) = \inf_{r \in [0, T]} P_1^r(0).
\]

The following gives a necessary condition for \( \bar{r} \).

**Proposition 5.2.** Let (5.1) and (5.2) hold. Then the optimal time \( \bar{r} \) to Problem (DOT) satisfies the following condition:

\[
(5.10) \quad F_1(K^2P_2(\bar{r}) + G_1) - K^2F_2(P_2(\bar{r})) \begin{cases}
\geq 0, & \bar{r} = 0, \\
= 0, & \bar{r} \in (0, T), \\
\leq 0, & \bar{r} = T.
\end{cases}
\]

**Proof.** We first claim that
\[
\Pi^r(t) = \frac{d}{dr} P_1^r(t), \quad t \in [0, T],
\]
exists for any \( r \in (0, T) \) and \( \Pi^r(\cdot) \) solves
\[
(5.11) \begin{cases}
\Pi^r(t) + F_1^r(P_1^r(t)) \Pi^r(t) = 0, & t \in [0, r], \\
\Pi^r(r) = F_1(K^2P_2(r) + G_1) - K^2F_2(P_2(r)).
\end{cases}
\]

To see this, let \( r \in (0, T) \) and \( \varepsilon > 0 \) small so that \( r \pm \varepsilon \in (0, T) \). Consider the following: for any \( t \in [0, r] \),

\[
P_1^{r+\varepsilon}(t) - P_1^r(t) = K^2 \left[ P_2(r + \varepsilon) - P_2(r) \right] + \int_r^{r+\varepsilon} F_1(P_1^{r+\varepsilon}(s)) \, ds \\
+ \int_t^r \left[ F_1(P_1^{r+\varepsilon}(s)) - F_1(P_1^r(s)) \right] \, ds.
\]

Then by a standard argument, we have the existence of the following limit:
\[
\Pi_1^r(t) \equiv \lim_{\varepsilon \downarrow 0} \frac{P_1^{r+\varepsilon}(t) - P_1^r(t)}{\varepsilon}, \quad t \in [0, r],
\]

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and
\[ \Pi^r_+(t) = K^2 \dot{P}_2(r) + F_1(P_1^r(r)) + \int_t^r F'(P_1^r(s)) \Pi^r_+(s) ds \]
\[ = -K^2 F_2(P_2(r)) + F_1(K^2 P_2(r) + G_1) + \int_t^r F'(P_1^r(s)) \Pi^r_+(s) ds, \quad t \in [0, r]. \]

On the other hand,
\[ P^{r-\varepsilon}_1(t) - P^r_1(t) = K^2 \left[ P_2(r - \varepsilon) - P_2(r) \right] - \int_{r-\varepsilon}^r F_1(P_1^r(s)) ds \]
\[ + \int_{r-\varepsilon}^r \left[ F_1(P_1^{r-\varepsilon}(s)) - F_1(P_1^r(s)) \right] ds. \]

Again, by a standard argument, we have the existence of the following limit:
\[ \Pi^r_-(t) = \lim_{\varepsilon \downarrow 0} \frac{P^{r-\varepsilon}_1(t) - P^r_1(t)}{-\varepsilon}, \quad t \in [0, r], \]
and
\[ \Pi^r_-(t) = K^2 \dot{P}_2(r) + F_1(P_1^r(r)) + \int_t^r F'_1(P_1^r(s)) \Pi^r_-(s) ds \]
\[ = -K^2 F_2(P_2(r)) + F_1(K^2 P_2(r) + G_1) + \int_t^r F'_1(P_1^r(s)) \Pi^r_-(s) ds. \]

Thus, \( r \mapsto P^r_1(t) \) is differentiable with derivative
\[ \frac{d}{dr} P^r_1(t) = \Pi^r(t) \equiv \Pi^r_\pm(t), \quad t \in [0, T], \]
and \( \Pi(\cdot) \) satisfies (5.11). Clearly,
\[ \Pi^r(t) = e^{\int_t^r F'_1(P_1^r(s)) ds} \left[ F_1(K^2 P_2(r) + G_1) - K^2 F_2(P_2(r)) \right], \quad t \in [0, r]. \]

Now, since \( r \mapsto P^r_1(0) \) attains a minimum at \( \bar{r} \in (0, T) \), we have
\[ \Pi^\bar{r}(0) = 0, \]
which leads to our conclusion for \( \bar{r} \in (0, T) \). Now, if \( \bar{r} = 0 \), then
\[ \Pi^0(0) = \lim_{\varepsilon \downarrow 0} \frac{P^\varepsilon_1(0) - P^0_1(0)}{-\varepsilon} \geq 0. \]
Finally, if \( \bar{r} = T \), then
\[ \Pi^T(0) = \lim_{\varepsilon \downarrow 0} \frac{P^{T-\varepsilon}_1(0) - P^T_1(0)}{-\varepsilon} \leq 0. \]

This completes the proof.

If the optimal time \( \bar{r} \) is either 0 or \( T \), our Problem (MLQ) becomes less interesting. Therefore, the optimal time \( \bar{r} \) is said to be non-trivial if \( \bar{r} \in (0, T) \). From the above, one has the following corollary.

**Corollary 5.2.** Let (5.1) and (5.2) hold. Then
\[
\left\{ \begin{array}{l}
F_1(K^2 G_2 + G_1) - K^2 F_2(G_2) > 0 \quad \Rightarrow \quad \bar{r} < T, \\
F_1(K^2 P_2(0) + G_1) - K^2 F_2(P_2(0)) < 0 \quad \Rightarrow \quad \bar{r} > 0.
\end{array} \right.
\]

(5.12)
Therefore, \(\bar{r}\) is non-trivial if that
\[(5.13) \quad F_1(K^2G_2 + G_1) - K^2F_2(G_2) > 0 > F_1(K^2P_2(0) + G_1) - K^2F_2(P_2(0)).\]

Note that in principle, conditions in (5.13) are checkable. Let us now look at some special cases for which we can say something about the optimal time \(\bar{r}\). Let
\[(5.14) \quad D_2 = G_1 = 0, \quad R_2 = K = 1, \quad B_2 \neq 0.\]

In this case, we observe the following:
\[
F_1(K^2P + G_1) - K^2F_2(P) = F_1(P) - F_2(P)
\]
\[
= (2A_1 + C_1^2)P + Q_1 - \left(\frac{(B_1 + C_1D_1)^2P^2}{R_1 + D_1^2P}\right) - \left(2A_2 + C_2^2\right)P + Q_2 - B_2^2P^2
\]
\[
= \left[2(A_1 - A_2) + C_1^2 - C_2^2\right]P + Q_1 - Q_2 + B_2^2P^2 - \left(\frac{(B_1 + C_1D_1)^2P^2}{R_1 + D_1^2P}\right) = \frac{\Theta(P)}{R_1 + D_1^2P},
\]
where
\[
\Theta(P) = \frac{D_1^2B_2^2P^3}{R_1 + D_1^2P} + \left[2(A_1 - A_2) + C_1^2 - C_2^2\right]D_1^2 + R_1B_2^2 - \left(2(A_1 - A_2) + C_1^2 - C_2^2\right)R_1 + \left[(Q_1 - Q_2)D_1^2\right]P + (Q_1 - Q_2)R_1.
\]

Then (5.13) is implied by
\[(5.15) \quad \Theta(G_2) > 0 > \Theta(P_2(0)).\]

On the other hand, in the current case, the Riccati equation for \(P_2(\cdot)\) becomes
\[(5.16) \quad \begin{cases} \dot{P}_2(t) + (2A_2 + C_2^2)P_2(t) + Q_2 - B_2^2P_2(t)^2 = 0, & t \in [0, T], \\ P_2(T) = G_2. \end{cases}\]

Denote
\[\lambda_{\pm} = \frac{2A_2 + C_2^2 \pm \sqrt{(2A_2 + C_2^2)^2 + 4B_2^2Q_2}}{2B_2^2}.\]

Then we may rewrite the above Riccati equation as
\[(5.17) \quad \begin{cases} \dot{P}_2(t) - B_2^2[P_2(t) - \lambda_+]P_2(t) - \lambda_+ = 0, & t \in [0, T], \\ P_2(T) = G_2. \end{cases}\]

If
\[G_2 = \lambda_{\pm},\]
then the unique solution \(P_2(\cdot)\) is given by
\[P_2(t) = G_2, \quad t \in [0, T],\]
for which, (5.13) cannot be true. Therefore, in what follows, we assume that
\[(5.18) \quad G_2 \neq \lambda_{\pm}.\]

Then the solution \(P_2(\cdot)\) is given by
\[P_2(t) = \frac{\lambda_+(G_2 - \lambda_-)e^{B_2^2(\lambda_+ - \lambda_-)(T-t)} - \lambda_- (G_2 - \lambda_+)}{(G_2 - \lambda_-)e^{B_2^2(\lambda_+ - \lambda_-)(T-t)} - (G_2 - \lambda_+)}, \quad t \in [0, T].\]
We claim that
\[ P_2(0) = \frac{\lambda_+(G_2 - \lambda_-)e^{B_2^2(\lambda_+ - \lambda_-)T} - \lambda_-(G_2 - \lambda_+)}{(G_2 - \lambda_-)e^{B_2^2(\lambda_+ - \lambda_-)T} - (G_2 - \lambda_+)} \neq G_2. \]

In fact, it is easy to see that there is no \( t_0 \in (0, T) \) such that
\[ B_2^2[P_2(t_0) - \lambda_+][P_2(t_0) - \lambda_-] = \dot{P}_2(t_0) = 0. \]

Otherwise, by the uniqueness of solutions to ODEs, we must have
\[ P_2(t) = \begin{cases} \lambda_+, & t \in [0, T], \quad \text{if } P_2(t_0) - \lambda_+ = 0, \\ \lambda_-, & t \in [0, T], \quad \text{if } P_2(t_0) - \lambda_- = 0, \end{cases} \]
both of which contradict (5.18). Actually, when (5.18) holds, by observing the sign of \( \dot{P}_2(\cdot) \), one has the following:
\[ (5.19) \quad P_2(0) \begin{cases} > G_2, & \text{if } G_2 \in (\lambda_-, \lambda_+), \\ < G_2, & \text{if } G_2 \notin [\lambda_-, \lambda_+]. \end{cases} \]

In particular, if \( Q_2 = 0 \), then
\[ \lambda_+ = \frac{2A_2 + C_2^2}{B_2^2}, \quad \lambda_- = 0. \]

Consequently,
\[ P_2(0) = \frac{G_2(2A_2 + C_2^2)e^{(2A_2 + C_2^2)T}}{G_2B_2^2e^{(2A_2 + C_2^2)T} + 2A_2 + C_2^2 - G_2B_2^2} \begin{cases} > G_2, & 2A_2 + C_2^2 > G_2B_2^2, \\ < G_2, & 2A_2 + C_2^2 < G_2B_2^2. \end{cases} \]

Further, let \( D_1 = 0 \) and \( R_1 = 1 \). Then
\[ \Theta(P) = (B_2^2 - B_1^2)P^2 + \left(2A_1 + C_1^2 - (2A_2 + C_2^2)\right)P + Q_1. \]

Let
\[ B_2^2 < B_1^2. \]

Then \( \Theta(\cdot) \) has a unique positive root:
\[ P_+ = \frac{(2A_1 + C_1^2) - (2A_2 + C_2^2) + \sqrt{[(2A_1 + C_1^2) - (2A_2 + C_2^2)]^2 + 4(B_1^2 - B_2^2)Q_1}}{2(B_1^2 - B_2^2)}. \]

Hence, if
\[ 2A_2 + C_2^2 > G_2B_2^2, \]
and
\[ 0 < G_2 < P_+ < P_2(0), \]
then (5.15) holds and \( \tilde{r} \) is non-trivial.

It is clear that many other cases for which \( \tilde{r} \) is non-trivial can be discussed in the similar fashion. However, we prefer to omit the details here.
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