Resonance $Y$-type soliton, hybrid and quasi-periodic wave solutions of a generalized (2 + 1)-dimensional nonlinear wave equation

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Abstract In this paper, we consider a generalized (2 + 1)-dimensional nonlinear wave equation. Based on the bilinear method, the $N$-soliton solutions are obtained. The resonance $Y$-type soliton, which is similar to the capital letter $Y$ in the spatial structure, and the interaction solutions between different types of resonance solitons are constructed by adding some new constraints to the parameters of the $N$-soliton solutions. The new type of two-opening resonance $Y$-type soliton solutions is presented by choosing some appropriate parameters in 3-soliton solutions. The hybrid solutions consisting of resonance $Y$-type solitons, breathers and lumps are investigated. The trajectories of the lump waves before and after the collision with the resonance $Y$-type solitons are analyzed from the perspective of mathematical mechanism. Furthermore, the multi-dimensional Riemann-theta function is employed to investigate the quasi-periodic wave solutions. The one-periodic and two-periodic wave solutions are obtained. The asymptotic properties are systematically analyzed, which establish the relations between the quasi-periodic wave solutions and the soliton solutions. The results may be helpful to provide some effective information to analyze the dynamical behaviors of solitons, fluid mechanics, shallow water waves and optical solitons.

Keywords Generalized (2 + 1)-dimensional nonlinear wave equation · Resonance $Y$-type solitons · Hybrid solutions · Quasi-periodic wave solutions

1 Introduction

The construction of nonlinear localized waves of the integrable systems is one of the most important topics of the nonlinear sciences. The nonlinear localized waves appear in many fields of sciences and technology, such as fluids, Bose–Einstein condensation, shallow water waves and nonlinear optics. Solitons, lumps, breathers and rogues waves, well known to us, are all the localized waves [1–9]. Lump solution is a rational solution localized in all directions in space, which can be seen limit of the infinite period of the breather wave [10–14]. The long-wave limit method is one of the effective methods to construct the multiple lump solutions and the hybrid solutions of the integrable systems that can be transformed into bilinear equations [15–17]. The resonance soliton is a special kind of soliton. The resonant phenomena exist in
many integrable systems. The resonance of the solitons occurs when the phase shift experienced by the colliding solitons becomes infinity or tends to become infinity [18]. The fission and fusion of the solitons are all the resonant phenomena [19–21]. Fusional soliton can describe the dynamical behavior that multiple solitons converge into one soliton. On the contrary, the fissionable solitons can be regarded as a single soliton dispersing into multiple branches [22]. Both of the interactions are inelastic. Long-wave–short-wave resonance interaction arises during the nonlinear collision between low-frequency long waves and high-frequency short waves [18]. The results show that the resonance of solitons is one of the reasons of the instability of solitons. Tajiri et al. found that the existence of the periodic soliton resonance may be related to the instabilities [23,24]. Soliton molecules are the bound states of solitons which have been experimentally discovered on optical systems [25–28]. Velocity resonance of the solitons can give rise to a soliton molecule, and the interaction between two soliton molecules is elastic [29–31]. The difference between the soliton molecules and resonance solitons is that the former appear in the situation that phase velocities of the interacting waves match each other while the later arises under the condition of special restrictions on the parameters of multiple soliton solutions. The linear superposition principle can be used for constructing the resonance solutions [32–34]. The hybrid solutions consisting of soliton molecules and lump waves were investigated by partial velocity resonance and partial long-wave limits [35,36]. It is interesting that Li et al. considered a more generalized constraint of the parameters in N-solitons to construct the resonance Y-type soliton solutions and the hybrid solutions among the resonance Y-type solitons, breathers, soliton molecules and lumps [37,38].

In the last decade, the quasi-periodic wave solutions have attracted the attention of many scholars. Fan et al. developed the multi-dimensional Riemann–theta functions to the bilinear forms to construct the quasi-periodic wave solutions [39–42]. The main inspiration for their work comes from Nakamura’s work. In 1980s, Nakamura first proposed a convenient way to construct the quasi-periodic wave solution with the aid of Hirota’s bilinear method [43,44]. It is convenient to analyze the frequencies, wave numbers, phase shifts and amplitudes and all the parameters in the Riemann matrix are free. Then, Tian et al. presented the Riemann–Hirota method to investigate the solvability of the quasi-periodic wave solutions of many integrable systems [45,46]. Chen et al. investigated the quasi-periodic wave solutions and discussed their asymptotic behaviors of some high-dimensional nonlinear equations [47,48]. The one-periodic wave solutions of the modified generalized Vakhnenko equation and a higher-order KdV-type equation were derived by Wang and Chen [49,50]. Furthermore, the Bäcklund transformations were employed to establish a unifying scheme to construct the quasi-periodic wave solutions of the integrable equations which possess two or more equations in bilinear forms [51]. This scheme is called Riemann–Bäcklund method. It proved that the Riemann–Bäcklund method is an effective method to construct the quasi-periodic wave solutions of some integrable systems with constant and variable coefficients [52,53].

In 2016, Gao et al. construct a new Hirota bilinear equation via which has the form

\[
\left( D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_y^2 \right) f \cdot f = 0, \tag{1}
\]

with \(D_t, D_x, D_y, \text{ and } D_z\) are the bilinear derivative operators [54–56], which can be defined by

\[
D_x^N D_y^M D_z^N D_t^M (f \cdot g) = (\partial_x - \partial_{x'})^N (\partial_y - \partial_{y'})^N (\partial_z - \partial_{z'})^N (\partial_t - \partial_{t'})^N \left( f(x, y, z, t) \cdot g(x', y', z', t') \right)_{x=x', y=y, z=z, t=t}. \tag{2}
\]

Under the variable transformation \(u = 2[\ln f(x, y, z, t)]_x\), Eq. (1) can be transformed into a nonlinear evolution equation

\[
u_{yt} - u_{xxxx} - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz} = 0. \tag{3}
\]

It is similar to the Korteweg–de Vries (KdV) equation, Eq. (3) has the dynamical characteristics of resonant multiple waves [57]. Assuming \(z = y\), the equation of dimensional reduction can be obtained as

\[
u_{yt} - u_{xxxx} - 3(u_x u_y)_x + 3u_{xx} + 3u_{yy} = 0, \tag{4}
\]

whose bilinear form is

\[
\left( D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_y^2 \right) f \cdot f = 0, \tag{5}
\]

by means of the transformation \(u = 2(\ln f)_x\). The lump solutions and properties of the localization of the lump wave were investigated by Lü and Ma [58].

Equation (4) can be regarded as the generalization of the KdV equation. Equation (4) is reduced into the KdV equation \(U_T - 6UU_X + U_{XXX} = 0\) under the condition \(t = -T, x = X, y = X\) and \(-u_x = U\).
KdV equation can describe various kinds of nonlinear waves, such as weakly nonlinear water waves, hydro-magnetic waves in a cold plasma, ion-acoustic waves and acoustic waves in harmonic crystals [59]. Since different coefficients in nonlinear equations contain different physical backgrounds, such as medium inhomogeneity, speed of the infinitesimal gravity waves, different boundary conditions or different external forces, it is meaningful to study the equation with generalized constant coefficients [60]. In order to study the influence of the coefficients on the wave interaction, Hu et al. considered the $(2+1)$-dimensional nonlinear wave equation
\[
u_{xt} + c_1 \left[u_{xxxxy} + 3 \left(2u_xu_y + uu_{xy}\right)\right] + 3u_{xx} \int_{-\infty}^{x} u_y \, dx' + c_2 u_{yy} = 0, \tag{6}
\]
and investigated the inelastic interaction solutions of lump–kink type and lump–soliton type. Compared with Eqs. (4), (6) is a generalized model to investigate nonlinear dynamical phenomena in shallow water, plasma and nonlinear optics [61]. By using the variable transformation $u = 2 \ln f_{xx}$, one obtains the generalized bilinear form
\[
\left(D_t D_y + c_1 D_x^3 D_y + c_2 D_y^2 \right) f \cdot f = 0. \tag{7}
\]
To study the nonlinear model with more complex dynamical behavior, we consider a generalized $(2+1)$-dimensional nonlinear wave (2DNW) equation
\[
u_{xt} + c_1 \left[u_{xxxxy} + 3 \left(2u_xu_y + uu_{xy}\right)\right] + 3u_{xx} \int_{-\infty}^{x} u_y \, dx' + c_2 u_{yy} + c_3 u_{xx} = 0, \tag{8}
\]
where $c_1, c_2$ and $c_3$ are arbitrary constants, which can be used to describe the shallow water wave and the small-amplitude surface wave with nonlinearity and weak perturbation in fluid mechanics. The last two terms of the equation constitute the classical wave equation in space. In essence, the 2DNW equation is a generalization of KdV equation in algebraic mechanism in the sense of the bilinear integrability. Physically, constant $c_1$ represents the coefficients of dispersion, nonlinearity and integration, while constants $c_2$ and $c_3$ stand for the disturbed wave velocity effects in the directions of $x$ and $y$. The 2DNW equation can describe the water wave model with more complex dispersion effect and fluctuation in $y$ direction. Due to the physical nature of the KdV equation, as a dimensional generalization of KdV equation, Eq. (8) may have wide applications in fluid mechanics. We investigated the $M$-lump, high-order breather and hybrid solutions of the 2DNW equation [7]. Zhao et al. studied the integrability and some mixed solutions of Eq. (8) [62]. This paper focuses on investigating the resonance $Y$-type soliton solutions, some new types of hybrid solutions and the quasi-periodic wave solutions of the 2DNW equation and analyzing dynamical characteristics of each kind of solutions.

The organization of the paper is as follows. In Sect. 2, the $N$-soliton solutions of Eq. (8) are obtained by means of the bilinear method. In Sect. 3, resonance $Y$-type soliton solutions and interaction solutions between resonance solitons and breathers are obtained by taking some constraints to the parameters of the $N$-soliton solutions. In Sect. 4, the hybrid solutions consisting of the resonance $Y$-type solitons and the lumps are constructed. The trajectories of the multiple lump waves before and after the interaction are analyzed from the perspective of mathematical mechanism. In Sect. 5, the multi-dimensional Riemann-theta function is used to construct the quasi-periodic wave solutions. We show the quasi-periodic wave solutions convergent to the soliton solutions under a small amplitude limit. Finally, some conclusions are given in the last section.

## 2 $N$-soliton solutions

Equation (8) can be transformed into a bilinear equation
\[
\left(D_t D_y + c_1 D_x^3 D_y + c_2 D_y^2 + c_3 D_x^2 \right) f \cdot f = 0, \tag{9}
\]
by means of the variable transformation
\[
u = 2 \ln f_{xx}, \tag{10}
\]
where the $f(x, y, t)$ is a function about variables $x$, $y$ and $t$. It is a fact that $u = u(x, y, t)$ is a solution of Eq. (8) if and only if $f$ is a solution of the bilinear equation (9). In order to construct the $N$-soliton solutions, the auxiliary function $f$ can be taken as
\[
f = f_N = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \eta_j + \sum_{j<s} \mu_j \mu_s A_{js} \right), \tag{11}
\]
where
\[
\eta_j = k_j x + l_j y + w_j t + \eta_{0j},
\]
\[
w_j = - \left( c_1 k_j^2 + c_2 l_j + \frac{c_3 k_j^2}{l_j} \right).
\]
ls = \frac{-3c_1k_jk_sl_j + 3l_jk_j^2c_1 + 2c_3k_j \pm \sqrt{9c_1k_j (c_1k_jl_j + \frac{4}{3}c_3) (k_j - k_j)^2 l_j}}{2k_j (3c_1k_jk_sl_j - 3c_1k_j^2l_j + c_3k_j)}.

(16)

exp(x) = 0 is true if and only if \( x = \ln(0) \). If one takes all the \( A_{ij} \) to 0, the \( N \)-soliton solutions can be reduced to the resonance \( Y \)-type soliton solutions with the form

\[ u = 2 \left( \ln \left( 1 + \sum_{j=1}^{N} \exp \left( \sum_{\mu=0}^{j-1} \mu_j \eta_j + A_{1j} \mu_1 \eta_j + \sum_{j=1}^{N} \left( A_{j2j} \mu_2 \eta_j \right) \right) \right) \right)_{xx} \]

(13)

Since the structure of this resonance soliton is simple, it is difficult to further construct the hybrid solutions between \( Y \)-type soliton and other types of solitons. Chen et al. considered a generalized form of the auxiliary function

\[ f = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \eta_j + \sum_{j>1}^{N} A_{1j} \mu_1 \eta_j \right) + \sum_{j=2}^{N} A_{j2j} \mu_2 \eta_j \]

(14)

to construct the interaction solutions between a lump and \((N - 2)\)-fissionable wave of the \((2 + 1)\)-dimensional Sawada–Kotera equation \[63\]. Li et al. established a more general relation among the parameters of the \( N \)-soliton solutions to obtain the resonance \( Y \)-type solitons of some \((2 + 1)\)-dimensional integrable systems \[37,38\]. Inspired by their work, we shall study the constraint of the parameters of function \( f \) (11) to investigate the resonance \( Y \)-type solitons and the hybrid solutions between resonance \( Y \)-type solitons and other type of localized waves.

### 3 Resonance \( Y \)-type solitons

**Proposition 1** The nonlinear superposition of \( M \)-resonance \( Y \)-type solitons and \( P \)-resonance \( Y \)-type soliton solutions for Eq. (8) can be obtained from the \( N \)-soliton solutions by taking the parameters as

\[ e^{A_{ij}} = 0, \]

\[ (1 \leq j < s \leq M, M < j < s \leq N, N = M + P), \]

(15)

where

The explicit mathematical form of the resonance \( Y \)-type soliton solution is given as

\[ u = 2 \left( \ln f_R \right), \]

\[ f_R = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \eta_j \right) + \sum_{j<s}^{N} \mu_j \mu_s A_{js} - \sum_{1<s<\mu}^{M} \mu_j \mu_s A_{js} - \sum_{M<s<N}^{N} \mu_j \mu_s A_{js} \]

(17)

where

\[ \eta_j = k_j x + l_j y + \left( w_j + \eta_j^0 \right), \]

\[ w_j = -\left( c_1k_j^2 + c_2l_j + \frac{c_3k_j^2}{l_j} \right), \]
Resonance $Y$-type soliton, hybrid

Fig. 1 (Color online) A collision between two solitons of $u$ with $k_1 = \frac{1}{2}, l_1 = \frac{1}{2}, k_2 = \frac{1}{2}, l_2 = -\frac{1}{2}, c_1 = c_2 = c_3 = 1, \eta_1^0 = \eta_2^0 = 0$. $a \ t = -20, b \ t = 0, c \ t = 20$

Fig. 2 (Color online) A collision among three solitons of $u$ with $k_1 = \frac{1}{2}, l_1 = \frac{1}{2}, k_2 = \frac{1}{2}, l_2 = -\frac{1}{2}, k_3 = \frac{1}{2}, l_3 = \frac{1}{2}, c_1 = c_2 = c_3 = 1, \eta_1^0 = \eta_2^0 = \eta_3^0 = 0$. $a \ t = -20, b \ t = 0, c \ t = 20$

Fig. 3 (Color online) A collision among four solitons of $u$ with $k_1 = \frac{1}{2}, l_1 = \frac{1}{2}, k_2 = \frac{1}{2}, l_2 = -\frac{1}{2}, k_3 = \frac{1}{2}, l_3 = \frac{1}{2}, k_4 = \frac{1}{2}, l_4 = -\frac{1}{2}, c_1 = c_2 = c_3 = c_4 = 1, \eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0$. $a \ t = -20, b \ t = 0, c \ t = 20$

If one takes $M = 2, P = 2$ in proposition 1, the interaction solutions between two 2-resonance $Y$-type solitons can be given as

$$u = 2(\ln f_4)_{xx},$$

with

$$f_4 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + e^{\eta_1 + \eta_2 + A_{12}} + e^{\eta_1 + \eta_3 + A_{13}} + e^{\eta_1 + \eta_4 + A_{14}} + e^{\eta_2 + \eta_3 + A_{23}} + e^{\eta_2 + \eta_4 + A_{24}},$$

where the relevant parameters $\eta_j$ and $A_{js}$ are determined by (12), (15) and (16).

To describe the evolutionary dynamical behaviors of the three different types of interaction between 2-resonance $Y$-type solitons, three different figures are plotted by choosing appropriate parameters. (I) Fig-
Fig. 4 (Color online) The dynamical characteristics of the 2-soliton solution with $k_1 = \frac{1}{3}, l_1 = \frac{1}{2}, k_2 = \frac{1}{3}, l_2 = -\frac{1}{4}, \eta_1^0 = \eta_2^0 = 0$. a $u(x, y = 30, t = -4)$ (brown), $u(x, y = 30, t = 0)$ (blue), $u(x, y = 30, t = 4)$ (green) and $c_1 = c_2 = c_3 = 1$. b $u(x, y = 30, t = -4)$ (brown), $u(x, y = 30, t = 0)$ (blue), $u(x, y = 30, t = 4)$ (green) and $c_1 = 2, c_2 = 3, c_3 = 4$. c $u(x, y = 0, t = 0)$ and $c_1 = 1$ (brown), $c_1 = 20$ (blue), $c_1 = 500$ (green) and $c_2 = c_3 = 1$, respectively.

Fig. 5 (Color online) An interaction between two fission resonance $Y$-type solitons of $u$ determined by Eq. (19) with $k_1 = \frac{1}{3}, l_1 = \frac{1}{2}, k_2 = \frac{1}{3}, l_2 = -\frac{1}{4}, k_3 = \frac{3}{17}, l_3 = \frac{1}{2}, k_4 = \frac{1}{3}, l_4 = \frac{109}{1340} + \frac{3\sqrt{129}}{1340}, c_1 = c_2 = c_3 = 1, \eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0$. a $t = -10$, b $t = 0$, c $t = 10$.

Fig. 6 (Color online) An interaction between two fusion resonance $Y$-type solitons of $u$ determined by Eq. (19) with $k_1 = \frac{1}{3}, l_1 = \frac{1}{2}, k_2 = \frac{1}{3}, l_2 = -\frac{1}{4}, k_3 = \frac{3}{17}, l_3 = \frac{1}{2}, k_4 = \frac{1}{3}, l_4 = \frac{109}{1340} - \frac{3\sqrt{129}}{1340}, c_1 = c_2 = c_3 = 1, \eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0$. a $t = -10$, b $t = 0$, c $t = 10$.

In order to exhibit characteristics of the phase-shift with explicit mathematical form, the explicit expression of function $f_4$ in the interaction solution between one fission and one fusion 2-resonance $Y$-type solitons can be written as
Fig. 7 (Color online) An interaction between one fission and one fusion 2-resonance Y-type solitons of $u$ determined by Eq. (19) with $k_1 = \frac{1}{4}, l_1 = \frac{1}{2}, k_2 = \frac{3}{11}, l_2 = \frac{1}{3}, k_3 = \frac{1}{2}, l_3 = \frac{1}{4}, k_4 = \frac{1}{2}, l_4 = \frac{109}{129}, c_1 = c_2 = c_3 = 1, n_4^0 = n_2^0 = n_3^0 = n_4^a = 0$. a $t = -10, b$ $t = 0, c$ $t = 10, d$ the contourplots of a, e the contourplots of b, f the contourplots of c.

$$f_4 = \frac{2}{227} \left( 571193 + 37959 \sqrt{129} \right)^{-1} \left( 2607 \sqrt{129} + 162559 \right)^{-1} \left( 16205806136195e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}} \right)$$

$$+ 869373365832 \sqrt{129} e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 869373365832 \sqrt{129} e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 91371618132 \sqrt{129} e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 1175281725285 \sqrt{129} e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 738173846682 \sqrt{129} e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 1292629879864 e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 1198767540664 e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 1198767540664 e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ 19323391703114 e^{\frac{5}{18} x + \frac{26}{55} y - \frac{161101}{184800} \sqrt{129}}$$

$$+ \eta_0^a,$$
Fig. 8 (Color online) An interaction between one fission and one fusion 2-resonance Y-type solitons of \( u \) determined by Eq. (19).

\[
a u (x = -100, y, t = -10), \quad b u (x = -100, y, t = 0), \quad c u (x = -100, y, t = 10), \quad d u (x = 50, y, t = -10), \quad e u (x = 50, y, t = 0), \quad f u (x = 100, y, t = 10)
\]

when the parameters are selected as \( k_1 = \frac{1}{3}, l_1 = \frac{1}{2}, k_2 = \frac{1}{8}, l_2 = \frac{3}{11}, k_3 = \frac{1}{2}, l_3 = \frac{1}{5}, k_4 = \frac{1}{5}, l_4 = \frac{109}{1340} - \frac{3 \sqrt{129}}{1340}, c_1 = c_2 = c_3 = 1, \eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0 \). Substituting the function \( f_4 (21) \) into the variable transformation \( u = 2(\ln f)_{xx} \), one can obtain the interaction solution presented in Figs. 7 and 8.

To analyze the nature of the collision between the resonance solitons, Figs. 7 and 8 are plotted. The contour images clearly show the interaction between two different resonance Y-type solitons. The structure of the resonance Y-type soliton just like the capital letter \( Y \). We analyze the variations of the amplitudes of each branch of the mixed wave consisting of a fusion and a fission resonance Y-type soliton with time when \( x \) value is fixed by means of the results of the numerical simulation in Fig. 8. Figure 8 shows that the interaction between the resonance Y-type solitons has no effect on the amplitude of each branch, which indicates that the interaction between the resonance Y-type solitons is elastic.

If the function \( f_3 \) is selected as

\[
f_3 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_2+\eta_1} + A_{23},
\]

and the parameters \( l_2 \) and \( l_3 \) satisfy the condition (16), and \( A_{23} \) is determined by Eq. (12), we can obtain a new type of two-opening resonance Y-type soliton by means of the transformation (10). The resonance soliton with two openings is composed of two resonance Y-type solitons with opposite openings. From Fig. 9, we can see that the wave fuses first and then fissions. The two-

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opening resonance Y-type soliton is a combination of a fusion and a fission resonance Y-type soliton. The density map show that two solitons form a virtual soliton, which is same as the virtual soliton of the Sawada–Kotera equation with a nonvanishing boundary condition [64]. Due to the structure of this resonance soliton is similar to the capital letter X, it can also be called X-type resonance soliton. Furthermore, if we choose the parameters in the N-soliton solutions as

\[ e^{A_{js}} = 0, \eta_{M+2P-1} = \eta^*_M, \quad (1 \leq j < s \leq M), \]

\[ 1 \leq p \leq P, N = M + 2P, \] (23)

the interaction solutions between a M-resonance Y-type soliton and P-order breather wave can be constructed. In order to construct the interaction solution between a fission 2-resonance Y-type soliton and a breather wave, the function \( f \) should be chosen as

\[ f_4 = 1 + e^{01} + e^{02} + e^{03} + e^{04} + e^{01} + e^{03} + A_{14} + e^{02} + e^{04} + A_{24} + e^{03} + e^{04} + A_{34} + e^{02} + e^{03} + e^{04} + A_{23} + A_{24} + A_{34}, \] (24)

where \( \eta_3 = \eta^*_s \) and \( A_{js} \) is determined by Eq. (12). Figure 10 is presented to show the interaction between a fission 2-resonance Y-type soliton and a breather wave. Physically, breather solutions are relevant to localization-type phenomena in optics, condensed matter physics and biophysics [65]. This kind of interaction solutions can provide some new insights in related fields.

4 Hybrid solutions consisting of the resonance Y-type solitons and lump waves

Proposition 2 The hybrid solutions consisting of a P-resonance Y-type soliton and M-order lump wave can be obtained by establishing the relations among the parameters of N-soliton solutions as

\[ k_{2m-1}^s = k_{2m}^s = K_{2m-1}^s, \]

\[ \eta_{2m-1}^s = \eta_{2m}^s = \pi_1, \]

\[ (1 \leq m \leq M), \quad \epsilon \to 0, \quad N = 2M + P, \]

\[ e^{A_{js}} = 0, \quad (2M < j < s \leq N), \] (25)

where the trajectories of the lump wave determined by the parameters \( K_{2m-1}, K_{2m}, L_{2m-1} \) and \( L_{2m} \), and the central coordinates of the lump wave before and after the interaction with the P-resonance Y-type soliton are

\[ x_{\text{before}} = \left( \frac{K_{2m-1}L_{2m} + K_{2m}L_{2m-1}}{L_{2m-1}L_{2m}}c_3 \right) t \]

\[ + \sum_{s=2M+1}^{N} h_{\text{before}}(\lambda_s) \Phi_s, \]

\[ y_{\text{before}} = \left( \frac{c_3 K_{2m-1} K_{2m} - c_2 L_{2m-1} L_{2m}}{L_{2m-1} L_{2m}} \right) t \]

\[ + \sum_{s=2M+1}^{N} h_{\text{before}}(\lambda_s) \Theta_s, \] (26)
Fig. 10 (Color online) An interaction between a 2-resonance Y-type soliton and a breather wave obtained from N-soliton solution by setting the restrictive conditions (23) and \( k_1 = \frac{1}{2}, l_1 = \frac{1}{2}, k_2 = \frac{3}{2}, l_2 = -\frac{3}{2}, i, l_3 = \frac{1}{2}, k_3 = -\frac{1}{2} + \frac{1}{2} i, l_3 = \frac{1}{2}, k_4 = -\frac{1}{2} - \frac{1}{2} i, l_4 = \frac{1}{2}, c_1 = c_2 = c_3 = 1, n_1 = n_2 = n_3 = n_4 = 0. \)

\[
\begin{align*}
\text{x}_{\text{after}} &= \left( \frac{K_{2m-1}L_{2m} + K_{2m}L_{2m-1}}{L_{2m-1}L_{2m}} \right) c_3 t \\
&+ \sum_{s=2M+1}^{N} h_{\text{after}}(\lambda_s) \Phi_s, \\
\text{y}_{\text{after}} &= -\left( \frac{c_3 K_{2m-1}K_{2m-1}K_{2m-1}L_{2m}}{L_{2m-1}L_{2m}} \right) t \\
&+ \sum_{s=2M+1}^{N} h_{\text{after}}(\lambda_s) \Theta_s,
\end{align*}
\]

with

\[
\begin{align*}
\lambda_s &= \frac{c_3 K_{2m-1}K_{2m-1}L_{2m}}{L_{2m-1}L_{2m}} \\
&- \frac{c_3 k_{2m-1}k_{2m-1}L_{2m} + c_3 k_{2m-1}L_{2m-1} + c_3 k_{2m}L_{2m-1} + c_3 k_{2m}L_{2m}}{L_{2m-1}L_{2m}}, \\
\Phi_s &= -\frac{c_3 k_{2m-1}k_{2m-1}L_{2m} + c_3 k_{2m-1}L_{2m-1} + c_3 k_{2m}L_{2m-1} + c_3 k_{2m}L_{2m}}{L_{2m-1}L_{2m}}, \\
\Theta_s &= -\frac{c_3 k_{2m-1}k_{2m-1}L_{2m} + c_3 k_{2m-1}L_{2m-1} + c_3 k_{2m}L_{2m-1} + c_3 k_{2m}L_{2m}}{L_{2m-1}L_{2m}}, \\
h_{\text{before}}(x) &= \begin{cases} 1, & x < 0, \\ 0, & x \geq 0, \end{cases} \quad h_{\text{after}}(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}
\end{align*}
\]

The height of the lump wave remains constant, which have the fixed values

\[
h_{\text{lump}} = \frac{2c_3(-K_{2m}L_{2m-1} + K_{2m-1}L_{2m})^2}{3c_1L_{2m}L_{2m-1}(K_{2m-1}L_{2m} + K_{2m}L_{2m-1})},
\]

before and after the interaction.

It can be seen from formula (30) that for the model (8) with different parameters \( c_1, (i = 1, 2, 3), \) the ratio of \( c_3 \) to \( c_1 \) has an important impact on the amplitude of the lump wave. When the other parameters are fixed, the greater the ratio, the higher the amplitude of the lump wave.

If we take \( M = 1, P = 2 \) in formula (25), we gain a hybrid wave consisting of a 2-resonance Y-type soliton and single lump wave. Under this condition, the auxiliary function \( f \) can be expanded as

\[
f = \theta_1 \theta_2 + B_{12} + (\theta_1 \theta_2 + B_{23} \theta_1)
\]
Fig. 11 (Color online) 3D plots of hybrid solution \( u \) between a 2-resonance \( Y \)-type soliton and a single lump wave determined by formula (32) with \( K_1 = -\frac{1}{2} + \frac{1}{2}i \), \( K_2 = -\frac{1}{2} - \frac{1}{2}i \), \( L_1 = L_2 = 1 \), \( k_3 = \frac{1}{2} \), \( k_4 = \frac{1}{2} \), \( k_5 = \frac{5}{2} \), \( l_3 = \frac{1}{2} \), \( l_4 = \frac{10}{236} = \frac{\sqrt{37}}{236} \), \( c_1 = c_2 = c_3 = 1 \), \( \eta_0^1 = \eta_0^4 = 0 \), \( a \ t = -12 \), \( b \ t = -8 \), \( c \ t = -4 \), \( d \ t = 0 \), \( e \ t = 10 \), \( f \ t = 20 \).

Fig. 12 (Color online) Overhead view of the hybrid solution \( u \) between a 2-resonance \( Y \)-type soliton and a single lump wave determined by formula (32) with the same parameters presented in Fig. 11. \( a \ t = -12 \), \( b \ t = -8 \), \( e \ t = -4 \), \( d \ t = 0 \), \( e \ t = 10 \), \( f \ t = 20 \).
+ $B_{13} \theta_2 + B_{13} B_{23} + B_{12}) e^{\theta_3}$
+ $(\theta_1 \theta_2 + B_{24} \theta_1)$
+ $B_{14} \theta_2 + B_{14} B_{24} + B_{12}) e^{\theta_4},$ \quad (31)

where $\theta_j = K_j x + L_j y - \left(\frac{cK_j^2}{L_j} + c_2 L_j\right) t$. By choosing the parameters as $K_2 = -\frac{1}{2} + \frac{i}{2}, K_2 = -\frac{1}{2}, L_1 = L_2 = 1, k_3 = \frac{1}{2}, l_3 = \frac{1}{2}, k_4 = \frac{1}{5}, l_4 = \frac{49}{236} - \frac{1}{236}, c_1 = c_2 = c_3 = 1, \eta_3^0 = \eta_4^0 = 0$, the function $f$ can be explicitly written as

$$f = \left(\begin{array}{c}
-20412455871305 t^2 \\
+ (328351905624 - 24494947605566 x \\
+ 3265991934085 y) t \\
- 8164982348522 x^2 + (1901642544456 \\
+ 16329964697044 x + 314658217764 y - 50068062594444 \\
- 16329964697044 y^2, x^2 + y^2)
\end{array}\right)$$

Substituting Eq. (32) into the variable transformation (10), we obtain the interaction solution between a fusion-type resonance $Y$-type soliton and 1-order lump wave. The single lump wave travels black line $y = -\frac{1}{2} x$ [black line in Fig. 12] before the interaction. Then, the trajectory of the lump wave changes into line $y = -\frac{1}{2} x + \frac{294}{236}$ [red line in Fig. 12] after the interaction. It is notable that the center of the lump wave is located at the bifurcation point of the resonance soliton as $t = 0$. By observing Figs. 11 and 12, we can conclude that lump wave propagates with invariant amplitude, velocity and shape after the collision with the resonance $Y$-type soliton. It is notable the trajectory of the lump wave is shifted, which is valuable to predict the trend of the lump waves in the nonlinear water wave model. Based on the results of Proposition 2, we can further investigate more complex hybrid solutions of the resonance $Y$-type solitons and high-order lump waves.

5 Quasi-periodic waves and asymptotic properties

In this section, we shall use the Riemann-theta function to construct the quasi-periodic wave solutions and analyze the dynamical behaviors of the quasi-periodic waves. The multi-dimensional Riemann-theta function of genus can be written as

$$\vartheta(\xi) = \sum_{n \in \mathbb{Z}^N} e^{-\pi (\tau n, n) + 2 \pi i (\xi, n)}, \quad (33)$$

where the integer value vector $n = (n_1, n_2, \ldots, n_N)^T$, and the complex phase variables $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in C^N$. The inner product of two vectors is defined as

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N. \quad (34)$$

The periodic $N \times N$ matrix $\tau$ is positive and real-valued symmetric, in which the entries $\tau_{ij}$ can be considered as free parameters of the theta function (33). For an arbitrary vector $\xi \in C^N$, the series (33) converges to a real-valued function.

In order to find the Riemann-theta function quasi-periodic wave solutions, we consider the solution of Eq. (8) in the form

$$u = u_0 + 2 \varepsilon^2 \ln \vartheta(\xi), \quad (35)$$

where $u_0$ is a free constant and $\xi_j = \alpha_j x + \rho_j y + \delta_j t + \sigma_j, j = 1, \ldots, N$. Substituting (35) into (8) and integrating with respect to $x$, then we obtain the following bilinear form

$$(D_x D_y + c_1 D_x^3 D_y + 3 u_0 c_1 D_x D_y)$$
where \( c \) is an integral constant.

### 5.1 One-periodic wave solution and its asymptotic properties

**Theorem 1** If \( \vartheta (\xi) \) is a Riemann-theta function determined by formula (33) with \( N = 1 \) and \( \xi = ax + \rho y + \delta t + \sigma \), the generalized \((2+1)\)-dimensional nonlinear wave equation (8) admits one-periodic wave solution

\[
u = u_0 + 2\vartheta_x^2 \ln \vartheta (\xi),
\]

where

\[
\vartheta (\xi) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x - \pi n^2 \tau},
\]

\[
d = \frac{b_{12}a_{12} - b_{11}a_{12}}{a_{11}a_{12} - a_{11}a_{12}}, \quad c = \frac{b_{21}a_{21} - b_{11}a_{21}}{a_{11}a_{12} - a_{11}a_{12}},
\]

in which

\[
a_{11} = \sum_{n=-\infty}^{\infty} -16\pi^2 n^2 \rho \lambda 2n^2,
\]

\[
a_{12} = \sum_{n=-\infty}^{\infty} \rho 2n^2,
\]

\[
a_{21} = \sum_{n=-\infty}^{\infty} -4\pi^2 (2n - 1)^2 \rho \lambda 2n^2 - 2n + 1,
\]

\[
a_{22} = \sum_{n=-\infty}^{\infty} \rho 2n^2 - 2n + 1,
\]

\[
b_1 = \sum_{n=-\infty}^{\infty} \left[ -256c_1 \pi^4 n^4 \alpha^3 \rho 
+ 16\pi^2 n^2 \left( 3u_0 c_1 \alpha \rho + c_2 \rho^2 + c_3 \alpha^2 \right) \right] \lambda 2n^2,
\]

\[
b_2 = \sum_{n=-\infty}^{\infty} \left[ -16c_1 \pi^4 (2n - 1)^4 \alpha^3 \rho 
+ 4\pi^2 (2n - 1)^2 \left( 3u_0 c_1 \alpha \rho + c_2 \rho^2 + c_3 \alpha^2 \right) \right] \lambda 2n^2 - 2n + 1,
\]

\[
\lambda = e^{-\pi \tau}.
\]

The other parameters \( \alpha, \rho, u_0, \sigma, \) and \( \tau \) are free.

**Proof** Based on the theory of the quasi-periodic wave solutions in [39], in order to make the theta function (38) satisfies the bilinear form (36), the parameters of the one-periodic wave solution (37) should satisfy the following constraint equations

\[
\tilde{G}(0) = \sum_{n=-\infty}^{\infty} \left[ -16\pi^2 n^2 \rho \delta + 256c_1 \pi^4 n^4 \alpha^3 \rho 
- 16\pi^2 n^2 \left( 3u_0 c_1 \alpha \rho + c_2 \rho^2 + c_3 \alpha^2 \right) \right] e^{-2\pi n^2 \tau} = 0,
\]

\[
\tilde{G}(1) = \sum_{n=-\infty}^{\infty} \left[ -4\pi^2 (2n - 1)^2 \rho \delta 
+ 16c_1 \pi^4 (2n - 1)^4 \alpha^3 \rho 
- 4\pi^2 (2n - 1)^2 \left( 3u_0 c_1 \alpha \rho + c_2 \rho^2 + c_3 \alpha^2 \right) \right] e^{-\pi \tau (2n^2 - 2n + 1)} = 0.
\]

To write above constraint equations into a linear system, we introduce the notations as formula (39). Then, liner system about parameters \( \delta \) and \( c \) can be obtained

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
\delta \\
c
\end{pmatrix}
= \begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}.
\]

By solving the linear equations, the theorem of the one-periodic wave can be established.

Figure 13 shows the structure of the space for the one-periodic wave solution (37). As we can see, one-periodic wave is a parallel superposition of single solitons. The one-period wave has strict periodicity in each direction of the coordinate axis.

It is interesting that one can consider the asymptotic properties of the one-periodic wave solution. The relation between the one-periodic wave solution and one-soliton solution can be established as follows. Based on the form of the \( N \)-soliton solutions (12), the one-soliton solution can be given as

\[
u = 2 \left[ \ln (1 + e^0) \right]_{xx},
\]

\[
\eta = kx + ly
\]

\[
- \left( c_1 k^3 + c_2 l + \frac{c_3 k^2}{l} \right) t + \eta^0.
\]

For the case \( u_0 = 0 \), we write the coefficient matrix and the vector \((b_1, b_2)^T\) into power series of \( \lambda \) as

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
0 \ 1 \\
0 \ 0
\end{pmatrix}
+ \begin{pmatrix}
0 \ 0 \\
-8\pi^2 \rho \ 2
\end{pmatrix} \lambda
+ \begin{pmatrix}
-32\pi^2 \rho^2 \ 2 \\
0 \ 0
\end{pmatrix} \lambda^2 + o(\lambda^2).
\]
Fig. 13 (Color online) A one-periodic wave (37) with \( \alpha = 2, \rho = 1, u_0 = \tau = c_1 = c_2 = c_3 = 1 \). a Perspective view of the wave \( u \) when \( t = 0 \). b Overhead view of the wave. c The wave along the \( x \)-axis. d The wave along the \( y \)-axis.

\[
\begin{align*}
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -32c_1\pi^2\alpha^3 + 8c_2\rho^2 + 8c_3\pi^2\alpha^3 \end{pmatrix} + \begin{pmatrix} 0 \\ -512c_1\pi^4\alpha^3 + 32c_2\rho^2 + 32c_3\pi^2\alpha^3 \end{pmatrix} \lambda^2 + o(\lambda^2).
\end{align*}
\]

Substituting (43) into the liner system (41), we obtain the solution \((\delta, c)^T\) with the form of series

\[
\begin{align*}
\begin{pmatrix} \delta \\ c \end{pmatrix} &= \begin{pmatrix} 4c_1\pi^2\alpha^3 - c_2\rho - \frac{c_3\alpha^2}{\rho} \\ 0 \end{pmatrix} + \begin{pmatrix} -96c_1\pi^4\alpha^3 \\ -384c_1\pi^4\alpha^3 \end{pmatrix} \lambda^2 + o(\lambda^2).
\end{align*}
\]

Thus, it concludes that

\[
\begin{align*}
\delta &= 4c_1\pi^2\alpha^3 - c_2\rho - \frac{c_3\alpha^2}{\rho} + o(\lambda), \\
c &= o(\lambda) \rightarrow 0,
\end{align*}
\]

when \( \lambda \rightarrow 0 \), which indicates that

\[
\begin{align*}
2\pi i\delta &\rightarrow 8c_1\pi^3i\alpha^3 - 2\pi ic_2\rho - \frac{2\pi ic_3\alpha^2}{\rho} \\
\end{align*}
\]

as \( \lambda \rightarrow 0 \). From this fact, we know that the phase variable \( 2\pi i\xi \) tends to \( \eta - \eta^0 \) under the assumption

\[
\begin{align*}
u_0 = 0, \quad \alpha = \frac{k}{2\pi i}, \\
\rho = \frac{l}{2\pi i}.
\end{align*}
\]

In addition, if the parameters \( \sigma \) and \( \eta^0 \) satisfy the relation

\[
\sigma = \frac{\eta^0 + \pi \tau}{2\pi i},
\]

the theta function

\[
\begin{align*}
\vartheta(\xi) &= 1 + \left( e^{2\pi i\xi} + e^{-2\pi i\xi} \right) \lambda^2 + \left( e^{4\pi i\xi} + e^{-4\pi i\xi} \right) \lambda^4 + \cdots,
\end{align*}
\]

can be reduced into

\[
\begin{align*}
\vartheta(\xi) &= 1 + e^{\xi} + \left( e^{-\xi} + e^{2\xi} \right) \lambda^2 + \left( e^{-2\xi} + e^{3\xi} \right) \lambda^6 + \cdots, \\
&\rightarrow 1 + e^{\xi}, \quad \text{as} \quad \lambda \rightarrow 0,
\end{align*}
\]

where \( \xi = 2\pi i\xi - \pi \tau \). According to formulas (46) and (48), one concludes that

\[
\xi \rightarrow \eta, \quad \vartheta(\xi) \rightarrow 1 + e^{\eta}, \quad \text{as}
\]

\( \lambda \rightarrow 0 \).
\( \lambda \to 0. \) (51)

Therefore, the one-periodic wave solution (37) tends to the one-soliton solution (42) as \( \lambda \to 0. \)

5.2 Two-periodic wave solution and its asymptotic properties

**Theorem 2** If \( \vartheta (\xi) \) is a Riemann-theta function determined by formula (33) with \( N = 2 \) and \( \xi_j = \alpha_j x + \rho_j y + \delta_j t + \sigma_j, \ j = 1, 2, \) the generalized (2 + 1)-dimensional nonlinear wave equation (8) admits two-periodic wave solution

\[
\begin{align*}
u &= u_0 + 2a_1^2 \ln \vartheta (\xi), \\
\vartheta (\xi) &= \sum_{n \in \mathbb{Z}^N} e^{-\pi (\tau n, n) + 2\pi i (\xi, n)},
\end{align*}
\]

where

\[
\vartheta (\xi) = \sum_{n \in \mathbb{Z}^N} e^{-\pi (\tau n, n) + 2\pi i (\xi, n)},
\]

with \( n = (n_1, n_2)^T \in \mathbb{Z}^2, \xi = (\xi_1, \xi_2)^T \) and \( \tau \) is a positive definite and real-valued symmetric 2 \( \times \) 2 matrix, which has the form

\[
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}.
\]

The vector of the parameters \((\delta_1, \delta_2, u_0, c)^T\) is determined by a linear system \(H(\delta_1, \delta_2, u_0, c)^T = b, \) where

\[
H = \begin{pmatrix} a_{js} \end{pmatrix}_{4 \times 4},
\]

\[
b = (b_1, b_2, b_3, b_4)^T,
\]

\[
a_{j1} = -4c_1^2 \pi^2 \sum_{n_1, n_2 \in \mathbb{Z}^2} \left( 2n - s_j, \rho \right) \lambda_j(n),
\]

\[
a_{j2} = -4c_2^2 \pi^2 \sum_{n_1, n_2 \in \mathbb{Z}^2} \left( 2n - s_j, \rho \right) \lambda_j(n),
\]

\[
a_{j3} = -12c_1u_0 \pi^2 \sum_{n_1, n_2 \in \mathbb{Z}^2} \left( 2n - s_j, \alpha \right) \lambda_j(n),
\]

\[
a_{j4} = \sum_{n_1, n_2 \in \mathbb{Z}^2} \lambda_j(n),
\]

\[
b_j = -16c_1^2 \pi^4 \sum_{n_1, n_2 \in \mathbb{Z}^2} \left( 2n - s_j, \alpha \right)^3 \lambda_j(n),
\]

\[
= 0, \quad j = 1, 2, 3, 4.
\]

(55)

where \( \alpha = (\alpha_1, \alpha_2)^T, \rho = (\rho_1, \rho_2)^T \) and \( \delta = (\delta_1, \delta_2)^T. \) The system \((55)\) transforms into a linear system

\[
H(\delta_1, \delta_2, u_0, c)^T = b,
\]

with the aid of introducing the notations presented in formulas (59).

Figures 14, 15, and 16 are devoted to show the dynamical behaviors of the two-periodic wave. The two-periodic is a direct generalization of one-periodic wave, i.e., it has two phase variables \( \xi_1 \) and \( \xi_2. \) The velocities of the two-periodic wave along the \( x \)-axis and \( y \)-axis are \( \frac{\delta x_1 - \delta x_2}{\alpha_1 \rho_2 - \alpha_2 \rho_1} \) and \( \frac{\delta y_2 - \delta \rho_1}{\alpha_1 \rho_2 - \alpha_2 \rho_1}, \) respectively. Figure 14 reflects that the two-periodic has no strict...
Fig. 14 (Color online) A two-periodic wave (52) with \( \alpha_1 = 6, \alpha_2 = -2, \rho_1 = 2, \rho_2 = 3, \tau_{11} = 1, \tau_{12} = 0.5, \tau_{22} = 1.2, c_1 = c_2 = c_3 = 1 \). a Perspective view of the wave \( u + 1.43 \times 10^{12} \) when \( t = 0 \). b Overhead view of the wave. c The wave along the x-axis. d The wave along the y-axis.

Fig. 15 (Color online) A degenerate two-periodic wave (52) with \( \alpha_1 = 6, \alpha_2 = 3, \rho_1 = 2, \rho_2 = 1, \tau_{11} = 1, \tau_{12} = 0.5, \tau_{22} = 3, c_1 = c_2 = c_3 = 1 \). a Perspective view of the wave \( u + 3.36 \times 10^{11} \) when \( t = 0 \). b Overhead view of the wave. c The wave along the x-axis. d The wave along the y-axis.
Fig. 16 (Color online) A symmetric two-periodic wave (52) with $\alpha_1 = 4, \alpha_2 = -4, \rho_1 = 2, \rho_2 = 2, \tau_{11} = 2, \tau_{12} = 0.5, \tau_{22} = 2, c_1 = c_2 = c_3 = 1$. a Perspective view of the wave $u = 7.40 \times 10^9$ when $t = 0$. b Overhead view of the wave. c The wave along the $x$-axis. d The wave along the $y$-axis.

Periodicity in both $x$ and $y$ directions. In general, two-periodic waves are arranged periodically by independent lump waves. If the parameters $\frac{a_2}{a_1} = \frac{\rho_2}{\rho_1} = a$, where $a$ is a constant, and $\vartheta(\xi_1, \xi_2) \sim \vartheta(\xi_1, a \xi_1)$, the two-periodic wave degenerates to the one-periodic wave. This phenomenon can be seen in Fig. 15. By choosing $\alpha_1 = -\alpha_2, \rho_1 = \rho_2$ and $\tau_{11} = \tau_{22}$ in theta function (33), the two-periodic wave is symmetric about the $x$- and $y$-axis, which is shown in Fig. 16. The quasi-periodic waves can be further classified into two classifications, (i) singly periodic, i.e., periodic in any one dimension (Figs. 13, 15) and (ii) doubly periodic, i.e., periodic in both dimensions (Figs. 14, 16).

In what follows, we shall investigate the relation between two-periodic wave solution (52) and two-soliton solution

$$u = 2 \left[ \ln \left( 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}} \right) \right]_{xx},$$

$$\eta_j = k_j x + l_j y$$

$$- \left( c_1 k_j^3 + c_2 l_j + \frac{c_3 k_j^2}{l_j} \right) t$$

$$+ \eta_j^0, j = 1, 2,$$

$$e^{A_{12}} = \frac{3c_1 k_1 l_2 l_2 (k_1 - k_2) (l_1 - l_2) - c_3 (k_1 l_2 - k_2 l_1)^2}{3c_1 k_1 l_2 l_2 (k_1 + k_2) (l_1 + l_2) - c_3 (k_1 l_2 - k_2 l_1)^2}. \quad (57)$$

First of all, we expand the two-dimensional theta function $\vartheta(\xi_1, \xi_2)$ into the following form

$$\vartheta(\xi_1, \xi_2) = 1 + \left( e^{2\pi i \xi_1} + e^{-2\pi i \xi_1} \right) e^{-\pi \tau_{11}}$$

$$+ \left( e^{2\pi i \xi_2} + e^{-2\pi i \xi_2} \right) e^{-\pi \tau_{22}}$$

$$+ \left( e^{2\pi i (\xi_1 + \xi_2)} + e^{-2\pi i (\xi_1 + \xi_2)} \right)$$

$$e^{-\pi (\tau_{11} + 2\tau_{12} + \tau_{22})} + \ldots. \quad (58)$$

The coefficient matrix $H$ and the column vectors $b$ can be expanded as the following series

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -8\pi^2 \rho_1 & 0 & -24\pi \tau_{11} \rho_1 \rho_2 \end{pmatrix} \lambda_1$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8\pi^2 \rho_2 & -24\pi \tau_{11} \rho_2 \rho_1 \end{pmatrix} \lambda_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_1 \lambda_2$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2^2$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8\pi^2 (\rho_1 - \rho_2) - 8\pi^2 (\rho_2 - \rho_1) - 24\pi \rho_1 \rho_2 \end{pmatrix} \lambda_1 \lambda_2$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8\pi^2 (\rho_1 - \rho_2) - 8\pi^2 (\rho_2 - \rho_1) - 24\pi \rho_1 \rho_2 \end{pmatrix} \lambda_1 \lambda_2$$

$$+ \lambda_1 \lambda_2 \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right). \quad (59)$$
\[ b = \begin{pmatrix}
-32 c_1^2 \rho_0 c_1^2 + 8 c_1^2 \rho_0 c_1^2 + 8 c_1^2 \rho_1^2 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \lambda_1 + \\
\begin{pmatrix}
-32 \rho_0 c_1^2 \rho_2 + 8 \rho_0 c_1^2 \rho_2 + 8 \rho_1^2 \rho_2 \\
0 \\
0
\end{pmatrix} \lambda_2 + \\
\begin{pmatrix}
-32 c_1^2 \rho_1 c_1^2 + 8 c_1^2 \rho_1 c_1^2 + 8 c_1^2 \rho_1^2 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \lambda_1^2 + \\
\begin{pmatrix}
-32 (c_1 \rho_0 - c_2 \rho_2) (\rho_0 - \rho_2) c_1^2 - 8 (c_1 \rho_0 - c_2 \rho_2) c_1^2 + 8 (c_1 \rho_0 - c_2 \rho_2) \rho_1^2 \\
+ \frac{4}{3} c_1 \rho_0 \rho_2 - \frac{4}{3} \rho_0 \rho_2 - \frac{4}{3} c_1 \rho_0 - \rho_2 \rho_2^2 - \frac{4}{3} c_1 \rho_0 + \rho_0 \rho_2
\end{pmatrix} \lambda_1 \lambda_2 + \\
\lambda \left( \sin \left( \frac{\pi}{2} \right) \right), \quad i + j \geq 2. \tag{60}
\]

Similarly, the vector of the solution \((\delta_1, \delta_2, u_0, c)^T\) is expanded as
\[
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
u_0 \\
c
\end{pmatrix} = \begin{pmatrix}
\delta_1^0 \\
\delta_2^0 \\
u_0^0 \\
c^0
\end{pmatrix} + \begin{pmatrix}
\delta_1^1 \\
\delta_2^1 \\
u_0^1 \\
c^1
\end{pmatrix} \lambda_1 + \\
\begin{pmatrix}
\delta_1^2 \\
\delta_2^2 \\
u_0^2 \\
c^2
\end{pmatrix} \lambda_2 + \begin{pmatrix}
\delta_1^{11} \\
\delta_2^{11} \\
u_0^{11} \\
c^{11}
\end{pmatrix} \lambda_1^2 + \\
\begin{pmatrix}
\delta_1^{22} \\
\delta_2^{22} \\
u_0^{22} \\
c^{22}
\end{pmatrix} \lambda_2^2 + \begin{pmatrix}
\delta_1^{12} \\
\delta_2^{12} \\
u_0^{12} \\
c^{12}
\end{pmatrix} \lambda_1 \lambda_2 + \\
\lambda \left( \sin \left( \frac{\pi}{2} \right) \right), \quad i + j \geq 2. \tag{61}
\]

Substituting formulas (59)–(61) into the linear system (56) and comparing the same order of \(\lambda_1\) and \(\lambda_2\), we obtain
\[
\begin{align*}
\rho_2^0 \delta_2^0 + 3 c_1 \alpha_2 \rho_1 u_0^0 &= 0, \\
\rho_1^0 \delta_1^0 + 3 c_1 \alpha_1 \rho_1 u_0^0 &= 4 c_1^2 \alpha_3 \rho_0 - c_2 \rho_1^2 - c_3 \alpha_1^2, \\
\rho_2^0 \delta_2^0 + 3 c_1 \alpha_2 \rho_2 u_0^0 &= 4 c_1^2 \alpha_3 \rho_2 - c_2 \rho_2^2 - c_3 \alpha_2^2, \\
\rho_1^0 \delta_1^0 + 3 c_1 \alpha_1 \rho_1 u_0^0 &= 0,
\end{align*}
\]
\[
\begin{align*}
\rho_2^0 \delta_2^0 + 3 c_1 \alpha_2 \rho_2 u_0^0 &= 0, \\
\rho_1^0 \delta_1^0 + 3 c_1 \alpha_1 \rho_1 u_0^0 &= -32 c_1^2 \rho_1^2 - 96 c_1^2 \alpha_1 \rho_1 u_0^0 \\
&= -512 \alpha_1^4 \rho_1 + 32 \alpha_1^2 \rho_1^2 + 32 c_3 \alpha_1^2 \\
\rho_2^0 \delta_2^0 + 3 c_1 \alpha_2 \rho_2 u_0^0 &= -512 \alpha_1^4 \alpha_2 \rho_2 + 32 \alpha_1^2 \rho_2^2 + 32 c_3 \alpha_1^2. \tag{62}
\end{align*}
\]

If we choose \(u_0^0 = 0\), then we can obtain
\[
\begin{align*}
u_0 &= \alpha (\lambda_1, \lambda_2) \to 0, \\
\lambda &= \alpha (\lambda_1, \lambda_2) \to 0, \\
\delta_1 &= 4 c_1^2 \alpha_3^2 - c_2 \rho_1^2 \\
&- c_3 \alpha_1^2 + \alpha (\lambda_1, \lambda_2) \\
&\to 4 c_1^2 \alpha_3^2 - c_2 \rho_1^2 - c_3 \alpha_1^2, \\
\delta_2 &= 4 c_1^2 \alpha_3^2 - c_2 \rho_2^2 \\
&- c_3 \alpha_2^2 + \alpha (\lambda_1, \lambda_2) \\
&\to 4 c_1^2 \alpha_3^2 - c_2 \rho_2^2 - c_3 \alpha_2^2.
\end{align*}
\]

Then, we assume that
\[
\begin{align*}
u_0 &= 0, \quad \alpha_j = \frac{k_j}{2 \pi i}, \\
\rho_j &= \frac{l_j}{2 \pi i}, \quad \sigma_j = \frac{\eta_j + \pi \tau_{jj}}{2 \pi i}, \\
\tau_{12} &= \frac{\eta_{12}}{2 \pi}, \tag{64}
\end{align*}
\]

the two-dimensional theta function is reduced to
\[
\begin{align*}
\tilde{\theta} \left( \xi_1, \xi_2 \right) &= 1 + e^{\tilde{\xi}_1} + e^{\tilde{\xi}_2} \\
&+ e^{\tilde{\xi}_1 + \tilde{\xi}_2 - 2 \pi \tau_{12}} \\
&+ \tilde{\lambda}_1 e^{-\tilde{\xi}_1} + \tilde{\lambda}_2 e^{-\tilde{\xi}_2} \\
&+ \tilde{\lambda}_1 \tilde{\lambda}_2 e^{-\tilde{\xi}_1 - \tilde{\xi}_2 - 2 \pi \tau_{12}} + \cdots \to 1 \\
&+ e^{\tilde{\xi}_1} + e^{\tilde{\xi}_2} + e^{\tilde{\xi}_1 + \tilde{\xi}_2 - 2 \pi \tau_{12}}, \tag{65}
\end{align*}
\]

where
\[
\begin{align*}
\tilde{\xi}_j &= 2 \pi i \xi_j - \pi \tau_{jj} \\
&= k_j x + l_j y \\
&- \left( c_1 k_j^3 + c_2 l_j + \frac{c_3 k_j^2}{l_j} \right) t \\
&+ \eta_j, \quad j = 1, 2. \tag{66}
\end{align*}
\]

From above analysis, we can conclude that the two-periodic wave solution (52) tends to the two-soliton solution (57) under the condition \(\lambda_1, \lambda_2 \to 0\).
6 Conclusions

This paper mainly focuses on investigating the localized wave solutions of the 2DNW equation, which is useful in describing dynamical characteristics of the shallow water wave and the small-amplitude surface wave with nonlinearity and weak perturbation in fluid mechanics. Based on the bilinear method, the $N$-soliton solutions are obtained. By considering more generalized constraints of the parameters, the $N$-soliton solutions are reduced to the resonance $Y$-type solitons. We study three different interactions between the fusion-type and fission-type resonance $Y$-type solitons through numerical simulation. The new kind of two-opening resonance $Y$-type solitons that can be regarded as resonance $X$-type solitons are presented.

In order to study more complex structure of the solutions, we investigate the hybrid solutions consisting of resonance $Y$-type solitons, breather waves and high-order lump waves. The trajectories of the multiple lump waves before and after the interaction with the resonance $Y$-type solitons are explicitly given. Furthermore, on the basis of the bilinear form of the 2DNW equation, the multi-dimensional Riemann-theta function is employed to construct the quasi-periodic wave solutions. A limiting procedure is presented to analyze the asymptotic behaviors of the one-periodic and two-periodic wave solutions. It is shown that the quasi-periodic wave solutions tend to the corresponding soliton solutions under a limit of the small amplitude. The dynamical characteristics of the quasi-periodic waves are analyzed by means of comparing the different parameters in Riemann-theta function. The results may be helpful to explain some new nonlinear phenomena in the fields of solitons, fluid mechanics, shallow water waves and optical solitons.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.
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