INEQUALITIES FOR THE \((q, k)\)-DEFORMED GAMMA FUNCTION EMANATING FROM CERTAIN PROBLEMS OF TRAFFIC FLOW

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Abstract. In this paper, the authors establish some double inequalities concerning the \((q, k)\)-deformed Gamma function. These inequalities emanate from certain problems of traffic flow. The procedure makes use of the integral representation of the \((q, k)\)-deformed Gamma function.

1. Introduction

The celebrated classical Euler’s Gamma function, \(\Gamma(x)\) is usually defined for \(x > 0\) by

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} dt
= \lim_{n \to \infty} \left[ \frac{n!n^x}{x(x+1)\ldots(x+n)} \right].
\]

The \(k\)-deformed Gamma Function, \(\Gamma_k(x)\) (also known as the \(k\)-analogue of the Gamma function or simply the \(k\)-Gamma function) is defined by (see [3])

\[
\Gamma_k(x) = \int_0^\infty e^{-\frac{x}{k}} t^{x-1} dt, \quad k > 0, \quad x > 0.
\]

It satisfies the following properties (see [3]).

\[
\Gamma_k(x + k) = x\Gamma_k(x),
\Gamma_k(k) = 1.
\]
The Jackson’s $q$-integral from 0 to $a$ and from 0 to $\infty$ are defined as follows
\[
\int_0^a f(t) \, dq_t = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n,
\]
\[
\int_0^{\infty} f(t) \, dq_t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n
\]
provided that the sums converge absolutely.

In a generic interval $[a, b]$, the Jackson’s $q$-integral takes the following form:
\[
\int_a^b f(t) \, dq_t = \int_0^b f(t) \, dq_t - \int_0^a f(t) \, dq_t.
\]

For more information on this special integral, reference is made to [7].

The $q$-deformed Gamma function is also defined for $q \in (0, 1)$ and $x > 0$ by
\[
\Gamma_q(x) = \int_0^{[\frac{1}{1-q}]} t^{x-1} E^{-qt}_q \, dq_t
\]
\[
= \int_0^{[\frac{1}{1-q}]} t^{x-1} E^{-qt}_q \, dq_t
\]
where $[x]_q = \frac{1-q^x}{1-q}$, and $E^t_q = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{t^n}{[n]_q^n} = (-1+q) t$ is a $q$-analogue of the classical exponential function. See also [1], [2], [5], [6] and the references therein. For $a \in C$, the set of complex numbers, we have the following notations:
\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1-aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty} (1-aq^i) \quad \text{and} \quad [n]_q! = \frac{(aq)_n}{(1-q)^n}.
\]

Just as the $k$-deformed Gamma function, the $q$-deformed Gamma function also satisfies the following properties:
\[
\Gamma_q(x + 1) = [x]_q \Gamma_q(x),
\]
\[
\Gamma_q(1) = 1.
\]

Similarly, the $(q, k)$-deformed Gamma function, $\Gamma_{q,k}(t)$ was defined by Díaz and Teruel [4] for $x > 0$, $q \in (0, 1)$ and $k > 0$ as (See also [9])
\[
\Gamma_{q,k}(x) = \int_0^{\left[\frac{k}{1-q}\right]} t^{x-1} E^{-qt}_{q,k} \, dq_t.
\]
It also exhibits the following properties:

\[(1) \quad \Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x), \]
\[\Gamma_{q,k}(k) = 1.\]

In 1978, Lew, Frauenthal and Keyfitz [8] established the double inequalities:

\[(2) \quad 2 \Gamma \left( n + \frac{1}{2} \right) \leq \Gamma \left( \frac{1}{2} \right) \Gamma(n+1) \leq 2^n \Gamma \left( n + \frac{1}{2} \right), \quad n \in \mathbb{N}\]

when studying certain problems of traffic flow.

In 2006, Sándor [12] by using the following inequalities due Wendel [13]:

\[(3) \quad \left( \frac{x}{x+s} \right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s \Gamma(x)} \leq 1\]

for \(x > 0\) and \(s \in (0,1)\), extended and refined the inequalities (2) as follows:

\[(4) \quad \sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma \left( x + \frac{1}{2} \right)} \leq \sqrt{\frac{x+1}{2}}\]

for \(x > 0\).

In a recent paper [10], the authors established the \(q\)-analogue of (4) as follows:

\[(5) \quad \sqrt{[x]_q} \leq \frac{\Gamma_q(x+1)}{\Gamma_q \left( x + \frac{1}{2} \right)} \leq \sqrt{\left[ x + \frac{1}{2} \right]_q}\]

for \(x \in (0,1)\) and \(q \in (0,1)\).

The purpose of this short paper is to prove the \((q,k)\)-analogues of the inequalities (4) by utilizing similar techniques as in [12] and [10]. Consequently, we deduce the \(k\)-analogue of (4) and as applications, we recover the inequalities (4) and (5). We present our results in the following section.

2. Main Results

We begin with the following Lemma which is crucial to the theme of the paper.
Lemma 2.1. Suppose that \( s \in (0, 1), q \in (0, 1) \) and \( k > 0 \). Then for any \( x > 0 \), the following inequalities are valid.

\[
\left( \frac{[x]_q}{[x + sk]_q} \right)^{1-s} \leq \frac{\Gamma_q(x + sk)}{[x]_q^s \Gamma_q(x)} \leq 1
\]

Proof. In order to prove this Lemma, we need Hölder’s inequality for the Jackson’s \( q \)-integral stated as follows.

\[
\int_0^\infty |f(t)g(t)| \, dq_t \leq \left[ \int_0^\infty |f(t)|^\alpha \, dq_t \right]^\frac{1}{\alpha} \left[ \int_0^\infty |g(t)|^\beta \, dq_t \right]^\frac{1}{\beta}
\]

where \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \) and \( \alpha > 1 \).

Let \( \alpha = \frac{1}{1-s} \), \( \beta = \frac{1}{s} \), \( f(t) = t^{(1-s)(x-1)}E_{q,k}^{-\frac{(1-s)k}{[s]_q}} \), \( g(t) = t^s(x+k-1)E_{q,k}^{-\frac{sk}{[s]_q}} \)
and \( A = \left( \frac{[k]_q}{[1-q]} \right)^{\frac{1}{s}} \).

Then by the Hölder’s inequality we obtain

\[
\Gamma_q(x + sk) = \int_0^A t^{x+sk-1}E_{q,k}^{-\frac{sk}{[s]_q}} \, dq_t
\]

\[
\leq \left[ \int_0^A \left( t^{(1-s)(x-1)}E_{q,k}^{-\frac{(1-s)k}{[s]_q}} \right)^{\frac{1}{1-s}} \, dq_t \right]^{1-s} \times
\]

\[
\left[ \int_0^A \left( t^s(x+k-1)E_{q,k}^{-\frac{sk}{[s]_q}} \right)^{\frac{1}{s}} \, dq_t \right]^s
\]

\[
= \left[ \int_0^A t^{x-1}E_{q,k}^{-\frac{sk}{[s]_q}} \, dq_t \right]^{1-s} \left[ \int_0^A t^{x+k-1}E_{q,k}^{-\frac{sk}{[s]_q}} \, dq_t \right]^s
\]

Thus,

\[
\Gamma_q(x + sk) \leq [\Gamma_q(x)]^{1-s} [\Gamma_q(x + k)]^s.
\]

Substituting equation (1) into inequality (7) yields:

\[
\Gamma_q(x + sk) \leq [x]_q^s \Gamma_q(x).
\]

Replacing \( s \) by \( 1 - s \) in equation (8) gives

\[
\Gamma_q(x + k - sk) \leq [x]_q^{1-s} \Gamma_q(x).
\]
By substituting $x$ by $x + sk$, we obtain
\[ \Gamma_{q,k}(x + k) \leq [x + sk]_q^{1-s} \Gamma_{q,k}(x + sk). \]

Now combining equations (8) and (10) gives
\[ \frac{\Gamma_{q,k}(x + k)}{[x + sk]_q^{1-s}} \leq \Gamma_{q,k}(x + sk) \leq [x]_q^s \Gamma_{q,k}(x) \]
which can be rearranged as:
\[ \frac{[x]_q}{[x + sk]_q^{1-s}} \Gamma_{q,k}(x) \leq \Gamma_{q,k}(x + sk) \leq [x]_q^s \Gamma_{q,k}(x) \]
Finally, equation (11) may be written as
\[ \left( \frac{[x]_q}{[x + sk]_q^{1-s}} \right)^{1-s} \leq \frac{\Gamma_{q,k}(x + sk)}{[x]_q^s \Gamma_{q,k}(x)} \leq 1 \]
ending the proof of Lemma 2.1.

\[ \square \]

**Theorem 2.2.** Assume that $q \in (0,1)$ and $k > 0$. Then the double inequality
\[ \frac{[x]_q}{[x + \frac{1}{2}]_q^{1-s}} \leq \frac{\Gamma_{q,k}(x + k)}{\Gamma_{q,k}(x + \frac{1}{2})} \leq [x + \frac{1}{2}]_q^{1-s} \]
is valid for each $x > 0$.

**Proof.** By inverting and setting $s = \frac{1}{2k}$ in the $(q,k)$-analogue of Wendel’s inequalities (6), we obtain:
\[ [x]_q^{-\frac{1}{2k}} \leq \frac{\Gamma_{q,k}(x)}{\Gamma_{q,k}(x + \frac{1}{2})} \leq [x + \frac{1}{2}]_q^{1-\frac{1}{2k}} \]
Utilizing the functional equation (1), inequalities (13) can be rearranged as follows.
\[ [x]_q^{-\frac{1}{2k}} \leq \frac{\Gamma_{q,k}(x + k)}{\Gamma_{q,k}(x + \frac{1}{2})} \leq [x + \frac{1}{2}]_q^{1-\frac{1}{2k}} \]
That completes the proof. \[ \square \]

**Remark 2.3.** Inequalities (6) implies that
\[ \lim_{x \to \infty} \frac{\Gamma_{q,k}(x + sk)}{[x]_q^s \Gamma_{q,k}(x)} = 1. \]
Remark 2.4. Let \( \lambda \) and \( \mu \) be real numbers. Then, as a consequence of (14), we obtain

\[
\lim_{x \to \infty} [x]_q^{\mu - \lambda} \frac{\Gamma_{q,k}(x + \lambda k)}{\Gamma_{q,k}(x + \mu k)} = 1
\]

since

\[
[x]_q^{\mu - \lambda} \frac{\Gamma_{q,k}(x + \lambda k)}{\Gamma_{q,k}(x + \mu k)} = \frac{\Gamma_{q,k}(x + \lambda k)}{\Gamma_{q,k}(x + \mu k)} \frac{[x]_q^\mu \Gamma_{q,k}(x)}{[x]_q^\mu \Gamma_{q,k}(x)}.
\]

We note that the limits (14) and (15) are the \((q, k)\)-deformations of the classical Wendel’s asymptotic relation given by (see [11], [13])

\[
\lim_{x \to \infty} \frac{\Gamma(x + s)}{x^s \Gamma(x)} = 1.
\]

Corollary 2.5. Assume that \( k > 0 \). Then the double inequality

\[
x^{1 - \frac{1}{2k}} \leq \frac{\Gamma_k(x + k)}{\Gamma_k(x + \frac{1}{2})} \leq \left(x + \frac{1}{2}\right)^{1 - \frac{1}{2k}}
\]

is valid for each \( x > 0 \).

Proof. This follows from Theorem 2.2 by letting \( q \to 1^- \).

As applications, we make the following remarks:

Remark 2.6. If in (12) we allow \( q \to 1^- \) as \( k \to 1 \) then, we recover the inequalities (4) as presented in [12].

Remark 2.7. If in (12) we allow \( k \to 1 \) then, we recover the inequalities (5) as presented in [10].

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