Phase space analysis of quintessence fields trapped in a Randall–Sundrum braneworld: anisotropic Bianchi I brane with a positive dark radiation term

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Received 26 January 2012, in final form 12 July 2012
Published 14 August 2012
Online at stacks.iop.org/CQG/29/175006

Abstract
In this paper we investigate, from the dynamical systems perspective, the evolution of a scalar field with an arbitrary potential trapped in a Randall–Sundrum’s braneworld of type 2. We consider a homogeneous but anisotropic Bianchi I (BI) brane filled with a perfect fluid. We also consider the effect of the projection of the five-dimensional (5D) Weyl tensor onto the three-brane in the form of a positive dark radiation term. Using the center manifold theory, we obtain sufficient conditions for the asymptotic stability of the de Sitter solution with standard 4D behavior. We also prove that there are no late-time de Sitter attractors with 5D modifications since they are always saddle-like. This fact correlates with a transient primordial inflation. We present here sufficient conditions on the potential for the stability of the scalar field–matter scaling solution, the scalar-field-dominated solution and the scalar field–dark radiation scaling solution. We illustrate our analytical findings using a simple \( f \)-deviser as a toy model. All these results are generalizations of our previous results obtained for FRW branes.

PACS numbers: 04.20.−q, 04.20.Cv, 04.20.Jb, 04.50.Kd, 11.25.−w, 11.25.Wx, 95.36.+x, 98.80.−k, 98.80.Bp, 98.80.Cq, 98.80.Jk

1. Introduction
Randall–Sundrum braneworlds were first proposed in \cite{1, 2}. In these references it was proved that for non-factorizable geometries in five dimensions, there exists a single massless bound
state confined in a domain wall or three-brane. This bound state is the zero mode of the Kaluza–Klein dimensional reduction and corresponds to the four-dimensional graviton. The Randall–Sundrum brane of type 2 model, as an alternative mechanism to the Kaluza–Klein compactifications [1], has been intensively studied in the last few years because of its appreciable cosmological impact in the inflationary scenario [3–5], among other reasons. The setup of the model starts with the particles of the standard model confined in a four-dimensional hypersurface with positive tension embedded in a five-dimensional (5D) bulk with a negative cosmological constant. It is well known that the cosmological field equations on the brane are essentially different from the standard four-dimensional cosmology [6–8].

Friedmann–Robertson branes with a scalar field trapped on it have been investigated widely in the literature [9–13]. In [14], it was shown that the potential $V \propto \text{cosech}^2(A\phi)$ leads to a scaling solution in an RS2 scenario. The dynamics of a scalar field with a constant and exponential potentials was investigated by [15]. These results were extended to a wider class of self-interaction potential in [16] using a method proposed in [17], supporting the idea that this scenario modifies gravity only at very high energy/short scales (UV modifications only) having an appreciable impact on the primordial inflation but does not affect the late-time dynamics of the Universe unless the energy density of the matter trapped in the brane increases at late times [18]. In [19], we investigated a scalar field with an arbitrary potential trapped in a RS2 braneworld. There we presented sufficient conditions for the asymptotic stability of the de Sitter solution and for the stability of scaling solutions as well as for the stability of the scalar-field-dominated solution extending the results in [20] to the higher dimensional framework. We prove the non-existence of late-time attractors with 5D modifications—a fact that correlates with a transient primordial inflation. Finally, as an example, we studied a scalar field with the exponential potential

$$V(\phi) = V_0 e^{-\chi \phi} + \Lambda$$

confined in the brane. We proved that for $\chi < 0$, the de Sitter solution is asymptotically stable. However, for $\chi > 0$, we have proved that the de Sitter solution is unstable (of saddle type). The exponential potential (1) has been widely investigated in the literature. It was studied for quintessence models in [21] where it is considered a negative cosmological constant $\Lambda$. In our case, we assume $\Lambda \geq 0$ to avoid dealing with negative values of $\gamma$. However, we can apply our procedure by permitting negative values for $\gamma$ for the case $\Lambda < 0$. The dark energy models with the exponential potential and negative cosmological constant were baptized as quinstant cosmologies. They were investigated in [22] using an alternative compactification scheme. The asymptotic properties of a cosmological model with a scalar field with the exponential potential have been investigated in the context of the general relativity by the authors of [17, 20], and in the context of the RS2 braneworlds by [15, 23]. In both cases, the pure exponential potential ($\Lambda = 0$) was studied. Potentials of exponential orders at infinity were studied in the context of scalar–tensor theories and conformal $F(R)$ theories by the authors of [24–26].

Here, we move a step further by considering the natural generalization of FRW, i.e. Bianchi I (BI) cosmologies. The results in [19] concerning the exponential potential are applied here too.

BI models are the minimal extension of the FRW metric to the anisotropic framework. Homogeneous but anisotropic geometries are well known [27, 28]. BI, Bianchi III and Kantowski–Sachs can be very good representations for the homogeneous but anisotropic universe. They were investigated in the framework of $f(R)$ cosmology from both numerical and analytical viewpoints also incorporating the matter content (see [29] and references therein). The evolution of cosmological braneworld models was investigated, for instance, in [30–35].
In [32], a systematic analysis of FRW, BI and Bianchi V metrics in these scenarios is presented. The changes in the structure of the phase space with respect to the general-relativistic case are discussed there. In [36], the dynamics of a BI brane in the presence of an inflationary scalar field is studied and it is shown that the high-energy effects from extra dimensional gravity remove the anisotropic behavior near the initial singularity which is found in general relativity. However, if anisotropic stresses are included, then the model’s behavior in the vicinity of the initial singularity changes. Barrow and Hervik [37] studied a BI braneworld model with a pure magnetic field and a perfect fluid with a linear barotropic $\gamma$-law equation of state. It was shown that if $\gamma \leq \frac{4}{3}$ the past attractor corresponds to a critical point with non trivial magnetic field, which is also a local source, bringing the anisotropic behavior back in the initial singularity.

In this paper we are interested in investigating the evolution of the scalar field with an arbitrary potential trapped in a Randall–Sundrum braneworld of type 2. We consider a homogeneous but anisotropic BI brane filled with a mixture of a scalar field with the arbitrary potential and a perfect fluid with EoS parameter $\omega = \gamma - 1$, with $1 \leq \gamma \leq 2$. For the treatment of the potential, we use the method developed in [17]. We extend previous results in [32, 23, 15, 16, 19] by considering scalar fields with arbitrary potentials in an anisotropic background and the effect of the projection of the 5D Weyl tensor onto the three-brane in the form of a positive dark radiation term. In order to illustrate our analytical results, we consider a toy model using the simple $f$-deviser given by $f(s) = s + \alpha$. The cosmological implications of such a model are also discussed.

2. Bianchi cosmology in the brane

The setup is as follows. Using the Gauss–Codacci equations, relating the four- and five-dimensional spacetimes, we obtain the modified Einstein equations on the brane [39, 40]:

$$
G_{ab} = -\Lambda_4 g_{ab} + \kappa^2 T_{ab} + \kappa^4 S_{ab} - \mathcal{E}_{ab},
$$

where $g_{ab}$ is the four-dimensional metric on the brane and $G_{ab}$ is the Einstein tensor, $\kappa$ is the four-dimensional gravitational constant and $\Lambda_4$ is the cosmological constant induced in the brane. $S_{ab}$ are quadratic corrections in the matter variables. Finally, $\mathcal{E}_{ab}$ are corrections coming from the extra dimension. More precisely, $\mathcal{E}_{ab}$ are the components of the electric part of the Weyl tensor of the bulk (see [39] and the review [41]). Following [42, 31, 23] from the energy–momentum tensor conservation equations ($\nabla_a T_{ab} = 0$) and equations (2) we obtain a constraint on $S_{ab}$ and $\mathcal{E}_{ab}$:

$$
\nabla_a (\mathcal{E}_{ab} - \kappa^2 S_{ab}) = 0.
$$

In general, we can decompose $\mathcal{E}_{ab}$ with respect to a chosen 4-velocity field $u^a[42]$ as

$$
\mathcal{E}_{ab} = -\left(\frac{\kappa^4 S_{ab}}{\kappa^4}ight) \left[ \mathcal{U} (u_a u_b + \frac{1}{3} h_{ab}) + \mathcal{P}_{ab} + 2u_a Q_b \right],
$$

where

$$
\mathcal{P}_{(ab)} = \mathcal{P}_a u_b, \quad \mathcal{P}_a = 0, \quad \mathcal{P}_{ab} u^b = 0, \quad Q_a u^a = 0.
$$

Here the scalar component $\mathcal{U}$ is referred as the dark radiation energy density due to the fact that it has the same form as the energy–momentum tensor of a radiation perfect fluid. $Q_a$ is a

If $\gamma > \frac{4}{3}$, then the previous fixed point is not the only past attractor possibility. It was argued, in both cases, that chaotic behavior is not possible [37, 38].

In this section, we follow [31, 23] in order to give a brief outline of the considerations that lead to the effective Einstein equations for BI models ((9)-(13)).
spatial vector that corresponds to an effective nonlocal energy flux on the brane and \( \mathcal{P}_{ab} \) is a spatially symmetric trace-free tensor which is an effective nonlocal anisotropic stress. Another important point is that the nine independent components in the trace-free \( E_{ab} \) are reduced to five degrees of freedom by equation (3) \cite{41, 43}, i.e. the constraint equation (3) provides evolution equations for \( U \) and \( Q_a \), but not for \( \mathcal{P}_{ab} \).

Taking into account the effective Einstein equations (2), the consequence of having a BI model\(^8\) on the brane is \cite{36}

\[
Q_a = 0,
\]

but we do not obtain any restriction on \( \mathcal{P}_{ab} \) \cite{31, 23}. Since there is no way of fixing the dynamics of this tensor, we will study the particular case in which

\[
\mathcal{P}_{ab} = 0;
\]

this condition, together with (6) and (3), implies

\[
D_a U = 0 \iff U = U(t).
\]

Using the above conditions over \( Q_a \) and \( \mathcal{P}_{ab} \) (6)–(7), setting the effective cosmological constant in the brane to zero, i.e. \( \Lambda_b = 0 \)\(^9\), the effective Einstein equations (2) for BI models (which have zero 3-curvature, i.e. \( R^3 = 0 \)) become

\[
\begin{align*}
H^2 &= \frac{1}{3} \kappa^2 \rho_T \left( 1 + \frac{\rho_T}{2\Lambda} \right) + \frac{1}{3} \sigma^2 + \frac{2U}{\Lambda} \quad \text{(9)} \\
\dot{H} &= -\frac{1}{2} \left( 1 + \frac{\rho_T}{\Lambda} \right) (\dot{\phi}^2 + \gamma \rho_m) - \frac{4U}{\Lambda} - \sigma^2 \\
\dot{\sigma} &= -3H \sigma \quad \text{(11)} \\
\dot{\rho}_m + 3H (\rho_m + p_m) &= 0 \quad \text{(12)} \\
\dot{\phi} - 3H \phi + \partial_\phi V &= 0, \quad \text{(13)}
\end{align*}
\]

where \( \sigma \) is the shear term.

It is also convenient to relate \( p \) and \( \rho \) by

\[
p = (\gamma - 1) \rho. \quad \text{(14)}
\]

The dark radiation term in (9) and (10) evolves as \( U(t) = \frac{C}{a(t)^2} \) \cite{42}, where \( C \) is a constant parameter. An important issue comes from the sign that \( C \) can take. From the point of view of the brane (brane base formalism), \( C \) is just an integration constant and can take any sign. However it was shown, from the bulk base formalism, that for all possible homogeneous and isotropic solutions on the brane, the bulk spacetime is Schwarzschild–AdS being possible to identify \( C \) with the mass of a bulk black hole (\( \mu = \frac{3C}{2\lambda} \)), and use this to constrain to be positive \cite{45, 43, 46}. For anisotropic models such constraints do not exist, i.e. \( C \), and therefore \( U \), can take any sign \cite{31, 47, 23}. In this first approach for investigating arbitrary potentials, we consider the case \( C > 0 \implies U > 0 \). The case of \( U < 0 \) is also worthy of note and deserves a careful analysis. For investigating these classes of models, i.e. with \( U < 0 \), it is required to consider a newer set of variables than that in this paper (more precisely, normalizing with \( D = \sqrt{H^2 - \frac{2U}{\Lambda}} \) instead of \( H \) and re-emplacing the analogue with the variable \( \gamma \) by the new one \( Q = \frac{\dot{Q}}{H^2} \))\(^10\). Since the resulting system has its own subtleties and also has a rich cosmological behavior, these results would enlarge this paper, so we prefer to address this point in a forthcoming paper \cite{48}.

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\(^8\) The metric in the brane is given by \( ds^2 = -dt^2 + \sum a_i(t) dx_i^2 \).

\(^9\) The induced cosmological constant in the brane can be set to \( \Lambda_b = 0 \) by fine tuning the negative cosmological constant of the AdS with the positive brane tension \( \lambda > 0 \) \cite{44, 41}.

\(^10\) This allows for the construction of a phase space compact in all the variables, but \( s \) defined in equation (16), that runs, in principle, over all the real values.
3. Dynamical variables and dynamical system

3.1. Dynamical systems analysis for Bianchi I brane with a positive dark radiation term

In this section, we investigate the cosmological model (9)–(13) from the dynamical system’s perspective. First, we need to recast the cosmological equations (9)–(13) as an autonomous system of first-order ordinary differential equations [49, 50, 25, 26]. For this purpose, we introduce the normalized variables

\[
\begin{align*}
 x &= \frac{\phi}{\sqrt{6H}} \\
 y &= \frac{V}{3H^2} \\
 \Omega_m &= \frac{\rho_m}{3H^2} \\
 \Omega_\sigma &= \frac{\sigma}{\sqrt{3H}} \\
 \Omega_U &= \frac{2\lambda}{\lambda H^2},
\end{align*}
\]

the additional dynamical (non-compact) variable, \(s\), given by

\[
s = -\partial_\phi \ln V(\phi),
\]

which is a function of the scalar field and the new time variable \(\tau = \ln a(t)\).

For the scalar potential treatment, we proceed following the reference [17, 19]. Let the scalar function be defined as

\[
f = \Gamma - 1, \quad \Gamma = \frac{V'}{V^2}.\]

Since \(\Gamma\) is a function of the scalar field \(\Gamma(\phi)\) (see definition (17)), so is the variable \(s = S(\phi)\). Assuming that the inverse of \(S\) exists, we have \(\phi = S^{-1}(s)\). Thus, one can obtain the relation \(\Gamma = \Gamma(S^{-1}(s))\) and finally the scalar field potential can be parameterized by a function \(f(s)\).

Using the Friedmann equation (10), we obtain the following relation between the variables (15):\n
\[
x^2 + y + \Omega_m + \Omega_\sigma + \Omega_U^2 = 1.\]

Restriction (18) allows us to forget about one of the dynamical variables, e.g., \(\Omega_U\), for obtaining a reduced dynamical system. From the condition \(0 \leq \Omega_U \leq 1\), we have the following inequality:

\[
0 \leq x^2 + y + \Omega_m + \Omega_\sigma + \Omega_U^2 \leq 1.
\]

Using variables (15), the field equations (9)–(13) and the new time variable, we obtain the following autonomous system of ordinary differential equations (ODE):

\[
\begin{align*}
x' &= \frac{\sqrt{3}}{2} y z + x^3 + \left( \Omega_m^2 - 2y + \frac{3y^2}{2} \right) \Omega_m + \frac{3(y - 2)\Omega_m - 2y\Omega_\sigma}{x^2 + y + \Omega_m} + \frac{4\Omega_\lambda - 1}{3H^2} x \\
y' &= y \left( 2x^2 + \sqrt{6}xy + 2\Omega_m^2 - 4y + 3y\Omega_m - 4\Omega_m^2 + \frac{2(4x^2 - 2y + (3y - 2)\Omega_m)\Omega_\sigma}{x^2 + y + \Omega_m} + 4 \right)
\end{align*}
\]

\[
\begin{align*}
\Omega_m' &= \Omega_m(2x^2 - 4y - 3y^4 + (3y - 4)\Omega_m + 4) \\
&\quad + \frac{\Omega_m(8x^2 - 4y + (6y - 4)\Omega_m)\Omega_\sigma}{x^2 + y + \Omega_m} + 2\Omega_m\Omega_\sigma^2
\end{align*}
\]

\[
\begin{align*}
\Omega_\sigma' &= \frac{(8x^2 - 4y + (6y - 4)\Omega_m)\Omega_\sigma^2}{x^2 + y + \Omega_m} + \frac{2\left( x^2 - 2y - 3y^2 + \frac{3((y - 2)x^2 + y\gamma)}{x^2 + y + \Omega_m} + 2 \right)\Omega_\lambda}{\Omega_\sigma} \\
&\quad + \frac{(2\Omega_\sigma^2 + (3y - 4)\Omega_m)\Omega_\lambda}{\Omega_\sigma}
\end{align*}
\]
\[ \Omega'_\sigma = \Omega \sigma^3 + \left( x^2 - 2y + \left( \frac{3y}{2} - 2 \right) \Omega_m - 1 \right) \Omega \sigma + \left( \frac{4x^2 - 2y + (3y - 2)\Omega_m}{x^2 + y + \Omega_m} \right) \Omega \Omega \sigma \] (24)
\[ s' = -\sqrt{6s^2}xf(s). \] (25)

Using the Friedmann equation (18), we obtain the following useful relationship:
\[ \frac{\rho_T}{\lambda} = \frac{\Omega_b}{\Omega_m + x^2 + y}. \] (26)

From (26) it follows that the region \( \Omega_m + x^2 + y = 0 \) corresponds to cosmological solutions, where \( \rho_T \gg \lambda \) (corresponding to the formal limit \( \lambda \to 0 \)). Therefore, they are associated with high-energy regions, i.e. with cosmological solutions in a neighborhood of the initial singularity\(^{11}\). Due to its classic nature, our model is not appropriate to describe the dynamics near the initial singularity, where quantum effects appear. However, from the mathematical viewpoint, this region \( (\Omega_m + x^2 + y = 0) \) is reached asymptotically. In fact, as some numerical integrations corroborate, there exists an open set of orbits in the phase interior that tends to the boundary \( \Omega_m + x^2 + y = 0 \) as \( \tau \to -\infty \). Therefore, for mathematical motivations it is common to attach the boundary \( \Omega_m + x^2 + y = 0 \) to the phase space\(^{12}\). On the other hand, the points with \( (\Omega_b = 0) \) are associated with the standard 4D behavior \( (\rho_T \ll \lambda \text{ or } \lambda \to \infty) \) and correspond to the low-energy regime.

From definition (15) and from restriction (18), and taking into account the previous statements, it is enough to investigate the flow of (20)--(25) defined in the phase space
\[ \Psi = \{(x, y, \Omega_m, \Omega_b, \Omega_s) : 0 \leq x^2 + y + \Omega_m + \Omega_b + \Omega_s^2 \leq 1, -1 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq \Omega_m \leq 1, 0 \leq \Omega_b \leq 1, -1 \leq \Omega_s \leq 1 \} \times \{s \in \mathbb{R}\}. \] (27)

### 3.2. Phase space analysis

The critical points of the system (20)--(25) are summarized in table 1, where \( s^* \) corresponds with \( f(s^*) = 0 \). All the points shown in table 1, but \( P^x_8, P^\lambda_9 \) for \( x = 0 \), and \( P_{10} \) when \( y = 0 \) always satisfy the condition \( \Omega_m + x^2 + y \neq 0 \). The points where this condition does not hold will be excluded from our analysis, since they represent solutions near the initial singularity. As we commented in the last section, our model is not applicable near this singularity.

It will be helpful to have the important observational parameters in terms of the state variables. The dimensionless scalar field energy density parameter \( \Omega_\phi = \frac{\rho_\phi}{\rho} \) and deceleration parameter \( q = -1 + \frac{H}{H_0} \). They read
\[ \Omega_\phi = x^2 + y, \quad \omega_\phi = \frac{x^2 - y}{x^2 + y}, \] (28)
\[ q = -1 + 3x^2 + \frac{3y}{2} \Omega_m + \frac{3(2x^2 + y \Omega_m)}{x^2 + y + \Omega_m}. \] (29)

In table 1, the existence conditions for the critical point of the system (20)--(25) are displayed, whereas in table 2 the values of the cosmological parameters \( \omega_\phi, \Omega_\phi \) and \( q \) are shown.

\(^{11}\) See [51, 24] for a classical treatment of cosmological solutions near the initial singularity in FRW cosmologies.

\(^{12}\) We refer the reader to the end of section 3.2 for a discussion of how to deal with the singular points at the hypersurface \( \Omega_m + x^2 + y = 0 \).
Table 1. Existence conditions for the critical point of the system (20)–(25). We use the notations $s_c$ for an arbitrary real $s$-value and $s^*$ for an $s$-value such that $f(s^*) = 0$.

| $P_i$ | $x$ | $y$ | $\Omega_m$ | $\Omega_\lambda$ | $\Omega_\sigma$ | $s$ | Existence |
|-------|-----|-----|------------|----------------|----------------|-----|-----------|
| $P_1$ | 0   | 0   | 1          | 0              | 0              | $s_c$ | $s_c \in \mathbb{R}$ |
| $P_2^\pm$ | $\pm 1$ | 0 | 0 | 0 | 0 | 0 | Always |
| $P_3^\pm$ | $\pm 1$ | 0 | 0 | 0 | 0 | $s^*$ | Always |
| $P_4^\pm$ | $\sqrt[3]{\gamma}$ | $\frac{1}{3\gamma} - \frac{2}{3\gamma}$ | 1 | 0 | 0 | $s^*$ | $s^2 \geq 3\gamma$ |
| $P_5^\pm$ | $\frac{2}{3\gamma}$ | 1 | 0 | 0 | 0 | $s^*$ | $s^2 \leq 6$ |
| $P_6^\pm$ | $\sqrt[3]{\gamma}$ | 0 | 0 | 0 | 0 | $s^*$ | $s^2 \geq 2$ |
| $P_7^\pm$ | $x \in [-1, 1]$ | 0 | 0 | 0 | 0 | $s^*$ | $x \neq 0$ |
| $P_8^\pm$ | $x \in [-1, 1]$ | 0 | 0 | 0 | 0 | $s^*$ | $x \neq 0$ |
| $P_{10}$ | 0 | 1 - $\Omega_\lambda$ | 0 | $\Omega_\sigma \in [0, 1]$ | 0 | 0 | $\Omega_\sigma \in [0, 1]$ |
| $P_{11}$ | 0 | 0 | 1 | 0 | $s_c \in \mathbb{R}$ | Always |

Table 2. Some basic observables $\omega_\phi$, $\Omega_\phi$ and $q$ for the critical point of the system (20)–(25). We use the notations $s_c$ for an arbitrary real $s$-value and $s^*$ for an $s$-value such that $f(s^*) = 0$.

| $P_i$ | $\omega_\phi$ | $\Omega_\phi$ | $q$ |
|-------|---------------|---------------|-----|
| $P_1$ | Undefined | 0 | $\frac{2}{3\gamma} - 1$ |
| $P_2^{\pm}$ | 1 | 1 | 2 |
| $P_3^{\pm}$ | 1 | 1 | 2 |
| $P_4$ | $\gamma - 1$ | $\frac{2}{3\gamma}$ | $\frac{2}{\gamma}$ - 1 |
| $P_5^{\pm}$ | $\frac{1}{3\gamma}$ - 1 | 1 | $\frac{2}{\gamma}$ - 1 |
| $P_6^{\pm}$ | 1 | $\frac{1}{3\gamma}$ | $\frac{2}{\gamma}$ - 1 |
| $P_7^{\pm}$ | $\frac{1}{x^2}$ | $\frac{1}{x^2}$ | $\frac{2}{\gamma}$ - 1 |
| $P_8^{\pm}$ | 1 | $x$ | $\frac{3}{x^2} - 1$ |
| $P_9^{\pm}$ | 1 | $x^2$ | $\frac{3}{x^2} - 1$ |
| $P_{10}$ | $-1$ | $1 - \Omega_\lambda$ | $-1$ |
| $P_{11}$ | Undefined | 0 | Undefined |

Now, let us comment on the stability and physical interpretation of the critical points/curves.

The point $P_1$ represents a matter-dominated solution ($\Omega_m = 1$). Although it is non-hyperbolic, it behaves like a saddle point in the phase space of the RS model, since they both have nonempty stable and unstable manifolds (see table 3).

The critical points $P_2^{\pm}$ are solutions dominated by the kinetic energy of the scalar field and they represent solutions with a standard behavior ($\Omega_\lambda = 0$). These critical points are non-hyperbolic. However, they behave as saddle-like points in the phase space because of the instability in the eigendirection associated with two positive eigenvalues and the stability of the eigendirection associated with a negative eigenvalue.

The critical points $P_3^{\pm}$ are the solutions dominated by the kinetic energy of the scalar field. They are non-hyperbolic; however, they behave as saddle points since they have both nonempty stable and unstable manifolds (see table 3). Thus, they represent transient states in the evolution of the universe.

The critical point $P_4$ is non-hyperbolic for $x = 2, 4/3, s^* = \pm \sqrt[3]{3\gamma}$ and $f'(s^*) = 0$ corresponding to scalar field–matter scaling solutions ($\Omega_\phi \sim \Omega_m$).
\[
P_3 \text{ is a stable node in the cases } 0 < \gamma < \frac{2}{3}, s^* < -\sqrt{3\gamma}, f'(s^*) < 0 \text{ or } \frac{2}{9} < \gamma < \frac{4}{3}, -\frac{2\sqrt{\gamma}}{\sqrt[3]{9\gamma^2}} \leq s^* < -\sqrt{3\gamma}, f'(s^*) < 0 \text{, or } 0 < \gamma \leq \frac{2}{9}, s^* > \sqrt{3\gamma}, f'(s^*) > 0, \text{ or } 0 < \gamma < \frac{2}{9}, \sqrt{3\gamma} < s^* \leq \frac{2\sqrt{\gamma}}{\sqrt[3]{9\gamma^2}}, f'(s^*) > 0.
\]

It is a spiral stable point for either \(\frac{2}{3} < \gamma < \frac{4}{3}, s^* < -\sqrt{3\gamma}, f'(s^*) < 0 \text{ or } \frac{2}{9} < \gamma < \frac{4}{3}, s^* > \sqrt{3\gamma}, f'(s^*) < 0 \text{ or } 0 < \gamma < \frac{2}{9}, s^* > \sqrt{3\gamma}, f'(s^*) > 0 \text{. Otherwise, it is a saddle point.}

The critical point \(P_6\) represents a scalar-field-dominated solution (\(\Omega_\phi = 1\)) that is not hyperbolic for \(s^* \in (0, \pm \sqrt{6}, \pm \sqrt{15\gamma}, 2)\) or \(f'(s^*) = 0\).

In the hyperbolic case, \(P_3\) is a stable node for either \(0 < \gamma \leq \frac{2}{3}, -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0 \) or \(\frac{4}{3} < \gamma < 2, s^* < -2, f'(s^*) < 0 \text{ or } 0 < \gamma < \frac{4}{3}, 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0 \text{ or } \frac{4}{3} < \gamma < 2, 0 < s^* < 2, f'(s^*) > 0 \); otherwise, it is a saddle point.

The critical points \(P_{6}^\pm\) are non-hyperbolic and correspond to scalar field–anisotropic scaling solutions (\(\Omega_\phi \sim \Omega_\sigma\)). For \(\frac{4}{3} < \gamma \), the anisotropic term in the Friedmann equation (10) dominates the cosmological dynamic (\(\Omega_\sigma = \pm 1\)). However, they behave as saddle-like points since they have both nonempty stable and unstable manifolds (see table 3).

The critical point \(P_3\) is non-hyperbolic for \(\gamma = 4/3, s^* = \pm 2\) and \(f'(s^*) = 0\) and corresponds to scalar field–dark radiation scaling solutions (\(\Omega_\phi \sim \Omega_\sigma\)).

It is a stable node for either \(\frac{4}{3} < \gamma \leq 2, -\frac{4}{\sqrt{15}} \leq s^* < -2, f'(s^*) < 0 \) or \(\frac{4}{3} < \gamma \leq 2, s^* < -\frac{4}{\sqrt{15}}, f'(s^*) > 0 \).

It is a spiral point for either \(\frac{4}{3} < \gamma \leq 2, s^* < -\frac{4}{\sqrt{15}}, f'(s^*) < 0 \) or \(\frac{4}{3} < \gamma \leq 2, s^* > 2, f'(s^*) > 0 \); it is a saddle point otherwise.

The circles of critical points \(P_{6}^\pm\) and \(P_3\) are non-hyperbolic and correspond to scalar field–anisotropic scaling solutions (\(\Omega_\phi \sim \Omega_\sigma\)). Both solutions represent transient states in the
evolution of the universe. When $x \to 0$, the anisotropic term in the Friedmann equation (10) dominates the cosmological dynamics ($\Omega_\sigma = \pm 1$). In this limit, the corresponding cosmological solutions are in the vicinity of the initial singularity.

The line of critical points $P_{10}$ is non-hyperbolic. It represents the solutions with 5D corrections for $\Omega_\lambda \neq 0$, whereas for $\Omega_\lambda = 0$ it represents a standard 4D cosmological solution. From the relationship between $y$ and $\Omega_\lambda$, follows that this solution is dominated by the potential energy of the scalar field $\rho_T = V(\phi)$, that is, it is de Sitter-like solution ($\omega_\phi = -1$). In this case, the Friedmann equation can be expressed as

$$3H^2 = V \left( 1 + \frac{V}{2\lambda} \right).$$

(30)

In the early universe, where $\lambda \ll V$, the expansion rate of the universe for the RS model differs from the general relativity predictions:

$$\frac{H_{RS}}{H_{GR}} = \sqrt{\frac{V}{2\lambda}}.$$  

(31)

Due to the importance of de Sitter solutions in the cosmological context, in section 4 we explicitly calculate their center manifolds. Due to the physical differences between solutions with standard 4D and non-standard 5D behaviors, we consider the cases $\Omega_\lambda = 0$ and $\Omega_\lambda \neq 0$ in separate subsections.

Summarizing, the possible late-time stable solutions are as follows.

- The scalar field–matter scaling solution ($P_4$) provided $0 < \gamma < \frac{4}{3}, s^* < -\sqrt{3\gamma}, f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}, s^* > \sqrt{3\gamma}, f'(s^*) > 0$.
- The scalar-field-dominated solution ($P_5$) provided $0 < \gamma < \frac{4}{3}, -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}, 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0$ or $\frac{4}{3} < \gamma < 2, 0 < s^* < 2, f'(s^*) > 0$.
- The scalar field–dark radiation scaling solution ($P_7$) provided $\frac{4}{3} < \gamma < \frac{1}{2}, s^* < -2, f'(s^*) < 0$ or $\frac{4}{3} < \gamma < 2, s^* > 2, f'(s^*) > 0$.
- The de Sitter solution $P_{10}$ with $\Omega_\lambda = 0$ provided $f(0)$ is a real (finite) positive number, i.e. $f(0) > 0$.

It is worth noting that there are fractional terms with $x^2 + y + \Omega_m$ in the denominator, in equations (20)–(25). To finish this section we want to discuss the case when the denominator gets closer to zero and the effects this may have on the system.

It is easy to see that the only singular solution located on the hypersurface $x^2 + y + \Omega_m = 0$ is the fixed point $P_{11}$. In order to investigate the dynamics of (20)–(25) near $P_{11}$ we introduce the local coordinates:

$$\{x, y, \Omega_\lambda - 1, \Omega_\sigma, \Omega_m, s - s_c \} = \epsilon [\tilde{x}, \tilde{y}, \tilde{\Omega}_\lambda, \tilde{\Omega}_\sigma, \tilde{\Omega}_m, \tilde{s}] + O(\epsilon)^2,$$

(32)

where $\epsilon$ is a constant satisfying $\epsilon \ll 1$:

$$\tilde{x} = \sqrt{\frac{3}{2}} x \tilde{y} + \frac{3y r}{1 + r} - 3,$$

$$\tilde{y} = \frac{6y \tilde{y} r}{1 + r},$$

$$\tilde{\Omega}_\lambda' = \frac{3y r (\tilde{y} + \tilde{\Omega}_m + 2\tilde{\Omega}_\sigma)}{1 + r} - 4 (\tilde{y} + \tilde{\Omega}_m + \tilde{\Omega}_\lambda),$$

$$\tilde{\Omega}_\sigma' = 3\tilde{\Omega}_\sigma \left( \frac{y r}{1 + r} - 1 \right).$$
In this section, we will study the stability of the center manifold of the de Sitter solutions for \(4.\) Dynamics of the center manifold of the de Sitter solutions

\[
\dot{\widetilde{\Omega}}_m = \frac{3y r (\widetilde{\Omega}_m - \widetilde{y})}{1 + r},
\]

\[
\dot{\widetilde{y}} = -\sqrt{6x_0^2} \widetilde{\chi} f(x_c),
\]

where

\[
r = \frac{\widetilde{\Omega}_m}{\widetilde{y}}.
\]

From the above equations, the equation \(r' = -3y r\) is deduced. Since \(y > 0\), it follows that \(r\) goes to zero (resp. to infinity) at an exponential rate as \(\tau \to \infty\) (resp. \(\tau \to -\infty\)). Thus, in the limit \(\tau \to -\infty\), \(r/(1 + r) \to 1\), and the system (34) has the asymptotic structure:

\[
\dot{\widetilde{x}} = 3(\gamma - 1)\widetilde{x} + \sqrt{3} \widetilde{y},
\]

\[
\dot{\widetilde{y}} = 6y \widetilde{y},
\]

\[
\dot{\widetilde{\Omega}}_x = 2(3y - 2)\widetilde{\Omega}_x + (3y - 4)(\widetilde{y} + \widetilde{\Omega}_m),
\]

\[
\dot{\widetilde{\Omega}}_\sigma = 3(\gamma - 1)\widetilde{\Omega}_\sigma,
\]

\[
\dot{\widetilde{\Omega}}'_m = 3y (\widetilde{\Omega}_m - \widetilde{y}),
\]

\[
\dot{\widetilde{y}} = -\sqrt{6x_0^2} \widetilde{\chi} f(x_c).
\]

The system (35) admits the exact solution passing by \((\widetilde{x}_0, \widetilde{y}_0, \widetilde{\Omega}_x, \widetilde{\Omega}_\sigma, \widetilde{\Omega}_m, \widetilde{y}_0)\) at \(\tau = 0\) given by

\[
\widetilde{x}(\tau) = \frac{e^{3(y - 1)\tau}(\sqrt{6x_0} \widetilde{y}_0 (e^{3(y + 1)\tau} - 1) + 6\widetilde{y}_0 (y + 1))}{6(y + 1)},
\]

\[
\widetilde{y}(\tau) = \widetilde{y}_0 e^{6y\tau},
\]

\[
\dot{\widetilde{\Omega}}_x(\tau) = e^{2(3y - 2)\tau}(\widetilde{y}_0 + \widetilde{\Omega}_m + \widetilde{\Omega}_m) - (\widetilde{y}_0 + \widetilde{\Omega}_m) e^{3y\tau},
\]

\[
\dot{\widetilde{\Omega}}_\sigma(\tau) = e^{3y\tau}(\widetilde{y}_0 - e^{3y\tau}) + \widetilde{y}_0 + \widetilde{\Omega}_m),
\]

\[
\dot{\widetilde{\Omega}}'_m(\tau) = 6\widetilde{y}_0 (y + 1)(e^{3(y - 1)\tau} - 1),
\]

\[
\dot{\widetilde{y}}(\tau) = -\sqrt{6x_0^2} \widetilde{\chi} f(x_c)(\sqrt{6x_0} \alpha(\tau, y) + 2\sqrt{6x_0} y(y + 1)(e^{3(y - 1)\tau} - 1)),
\]

\[
\widetilde{\Omega}_\sigma(\tau) = \sqrt{6x_0} e^{3(y - 1)\tau},
\]

where \(\alpha(\tau, y) = (y - 1)e^{6y\tau} - 2y e^{3(y - 1)\tau} + y + 1\). By using the above linearization technique, we have obtained the eigenvalues of the linearization around the fixed points \(P_{11}\), given by (35), displayed in the last row of table 3. They are calculated under the hypothesis \(y \ll \Omega_m\) as \(y, \Omega_m \to 0\) (which is consistent with the limit \(r \to \infty\) as \(\tau \to -\infty\)). From the analysis of the system (35) it follows that the singular point \(P_{11}\) represents a big-bang singularity. It is non-hyperbolic with a 5D unstable manifold for \(1 < \gamma < 2\). In this case, it is a local source as numerical simulations suggest. For \(0 < \gamma < 1\), it behaves as a saddle point. Since the limit \(\Omega_\lambda \to 1\) corresponds to the domain where the brane corrections are important, and this regime is associated with the past dynamics, we conclude that the linear approximation (34) is not valid as \(\tau \to +\infty\).

4. Dynamics of the center manifold of the de Sitter solutions

In this section, we will study the stability of the center manifold of the de Sitter solutions for \(\Omega_\lambda = 0\) and for \(0 < \Omega_\lambda < 1\). For this purpose, we can use the center manifold theory (see the reviews [25, 26]).
4.1. Case $\Omega_\lambda = 0$

The solution $P_{10}$ with $\Omega_\lambda = 0$ can be a candidate to be a late-time de Sitter attractor without 5D corrections. To analyze their stability we carry out a detailed stability study using the center manifold theory.

Let us assume that $f(0) \in \mathbb{R} \setminus \{0\}$. To prepare the system for center manifold calculations we introduce the new variables as follows:

\[
\begin{align*}
    u_1 &= s, \\
    u_2 &= \Omega_\lambda, \\
    v_1 &= -1 + y + \Omega_m + \Omega_\lambda, \\
    v_2 &= \Omega_\sigma, \\
    v_3 &= x - s \sqrt{6}, \\
    v_4 &= \Omega_1 m.
\end{align*}
\]

Then, we Taylor expand the system $u'_1, u'_2, v'_1, v'_2, v'_3, v'_4$ in a neighborhood of the origin with an error of order $O(4)$.

According to the center manifold theorem, the local center manifold of the origin the vector field is given by the following graph:

\[
W^c_{\text{loc}}(0) = \{ (u_1, u_2, v_1, v_2, v_3, v_4) : v_i = f_i(x_1, x_2), \\
i = 1, \ldots, 4, f(0) = 0, Df(0) = 0, u_1^2 + u_2^2 < \delta \},
\]

with $\delta > 0$ being an small enough real value.

Deriving each one of the functions in (38) with respect to $\tau$ and substituting the vector field $u'_1, u'_2, v'_1, v'_2, v'_3, v'_4$ one can obtain a system of quasi-linear partial differential equations that the functions $f_i$ must satisfy.

Solving this system using the Taylor series up to an error term $O(4)$ we obtain

\[
\begin{align*}
    v_1 &= \frac{u_1^2 u_2}{3} - \frac{u_1^3}{6}, \\
    v_2 &= 0, \\
    v_3 &= \frac{u_1^2 f(0)}{3 \sqrt{6}} - \frac{u_1 u_2}{\sqrt{6}}, \\
    v_4 &= 0.
\end{align*}
\]

Thus, the dynamics on the center manifold is given by

\[
\begin{align*}
    u'_1 &= -f(0)u_1^3 + O(4), \\
    u'_2 &= -u_1^2 u_2 + O(4),
\end{align*}
\]

which is the same given by (36) and (37) in [19] with the identifications $u_1 \equiv x_1, u_2 \equiv x_2$.

Using the same arguments as in [19] we obtain that the origin is asymptotic stable under initial conditions in the vicinity of the origin whenever $f(0) > 0$.

From the asymptotic stability of the origin of (40) and (41) for the above choice of sign for $f(0)$, it follows that the center manifold of $P_{10}$ (for $\Omega_\lambda = 0$) is locally asymptotic stable, and hence the solution $P_{10}$ of the system (20)–(25).

Therefore, $P_{10}$ with $\Omega_\lambda = 0$ and $f(0) > 0$ corresponds to a late-time de Sitter attractor. This result for RS-2 brane cosmology is in perfect agreement with the standard four-dimensional TGR framework.

4.2. Case $\Omega_\lambda \neq 0$

In this section, we investigate the stability of the curve of critical points $P_{10}$ for $0 < \Omega_\lambda < 1$. This solution corresponds to a de Sitter solution with 5D corrections.

According to the RS model, this solution cannot behave like a late-time attractor since 5D corrections are typically of the high-energy regime (early universe) and not of the low-energy regime (late universe). If we can prove that this solution is of saddle type, this behavior can
be correlated with a transient inflationary stage for the universe. In order to verify our claim we appeal to the center manifold theory.

Let us consider an arbitrary critical point with coordinates \((x = 0, y = 1 - u_c, \Omega_\lambda = u_c, \Omega_\sigma = 0, \Omega_m = 0, s = 0)\) located at \(P_{10}\).

In order to prepare the system (20)–(25) for the application of the center manifold theorem we introduce the coordinate change

\[
\begin{align*}
  u_1 &= s, u_2 = -u_c (y + 2\Omega_m) - (u_c - 1) \Omega_\lambda, \\
  v_1 &= u_c (y + \Omega_m + \Omega_\lambda - 1), \quad v_2 = \Omega_\sigma, \\
  v_3 &= \frac{s(u_c - 1)}{\sqrt{6}} + x, \quad v_4 = u_c \Omega_m.
\end{align*}
\]

(42)

Then, we Taylor expand the system \(u'_1, u'_2, v'_1, v'_2, v'_3, v'_4\) in a neighborhood of the origin with an error of order \(O(4)\).

According to the center manifold theorem, the local center manifold of the origin for the resulting vector field is given by the following graph:

\[
\begin{align*}
  W^c_{\text{loc}}(0) = \{ (u_1, u_2, v_1, v_2, v_3, v_4) : v_i &= f_i(x_1, x_2), \\
  i &= 1 \ldots 4, f(0) = 0, Df(0) = 0, u_1^2 + u_2^2 < \delta \},
\end{align*}
\]

(43)

with \(\delta > 0\) being an small enough real value.

Deriving each function in (43) with respect to \(\tau\) and substituting the vector field \(u'_1, u'_2, v'_1, v'_2, v'_3, v'_4\) one can obtain a system of quasi-linear partial differential equations that the functions \(f_i\) must satisfy.

Solving this system using the Taylor series up to an error term \(O(4)\) we obtain

\[
\begin{align*}
  v_1 &= -\frac{1}{6} u_1^2 (u_c - 1) u_c (2u_2 + u_c - 1), \quad v_2 = 0, \\
  v_3 &= \frac{u_1 (u_1^2 (u_c - 1)^2 f(0) - 3u_2)}{3\sqrt{6}}, \quad v_4 = 0.
\end{align*}
\]

(44)

Thus, the dynamics on the center manifold is given by

\[
\begin{align*}
  u'_1 &= u_1^3 (u_c - 1) f(0) + O(4), \\
  u'_2 &= -u_1^2 \left( (u_c (3u_c - 4) + 1) u_2 + (u_c - 1)^2 u_c \right) + O(4),
\end{align*}
\]

(45)

(46)

which is the same as (41) and (42) in [19] under the variables rescaling

\[
\begin{align*}
  v_1 &\to \frac{(1 - u_c)}{\sqrt{6}} u_1, \quad u_2 \to -u_2.
\end{align*}
\]

Thus, using the same arguments as in [19], the origin of coordinates is locally asymptotically unstable (of saddle type) irrespective of the sign of \(f(0)\). Henceforth, the center manifold of \(P_{10}\) is locally asymptotic unstable (saddle type) for \(f(0) \neq 0\).

The physical interpretation of this result is that there are not late-time attractors with 5D modifications. These types of corrections are characteristics of the early universe. In this sense, the solution \(P_{10}\) with \(0 < \Omega_\lambda < 1\) is associated with the primordial inflation.

As in [19] for \(P_{10}\) with \(0 \leq \Omega_\lambda < 1\), we assume that \(f(0) \in \mathbb{R} \setminus \{0\}\). Otherwise, it is required to include higher order terms in the Taylor expansion, increasing the numerical complexity.
5. A toy model

The objective of this section is to illustrate our analytical results for the following toy model. Let us consider a simple $f$-deviser given by $f(s) = s + \alpha$. This function corresponds to the potential given in the implicit form
\[
\phi(s) = \frac{1}{s\alpha} + \ln\left(\frac{s}{s + \alpha}\right)^{\frac{1}{\gamma}},
\]
\[
V(s) = V_0\left(\frac{s}{s + \alpha}\right)^{\frac{1}{\gamma}},
\]
or alternatively
\[
V(\phi) = V_0\left(-\frac{1}{W\left(-e^{-\phi\alpha^{\alpha^2-1}}\right)}^{\frac{1}{\gamma}}, \right)
\]
where $W(z)$ is the special function ‘ProductLog’ that gives the principal solution for $w$ in $z = we^w$. $V(\phi)$ is defined for $\phi \geq 0$. For $\alpha < 0$, $V$ is a monotonic decreasing function taking values in the range $[0, V_0]$; for $\alpha > 0$, $V$ is a monotonic increasing function taking values in the range $[V_0, \infty)$.

The function $f$ satisfies $s^* = -\alpha$, $f'(s) = 1$, $f(0) = \alpha$ and $s^* f'(s^*) = s^* = -\alpha$. The sufficient conditions for the existence of late-time attractors are fulfilled easily. They read (see figure 1) as follows.

- The scalar field–matter scaling solution ($P_4$) is a late-time attractor provided $0 < \gamma < \frac{4}{3}$, $\alpha < -\sqrt{3\gamma}$.
- The scalar-field-dominated solution ($P_5$) is a late-time attractor provided $0 < \gamma \leq \frac{2}{3}$, $-\sqrt{3\gamma} < \alpha < 0$ or $\frac{4}{3} < \gamma \leq 2$, $-2 < \alpha < 0$.
- The scalar field–dark radiation scaling solution ($P_7$) is a late-time attractor provided $\frac{4}{3} < \gamma \leq 2$, $\alpha < -2$.
- The de Sitter solution $P_{10}$ with $\Omega_M = 0$ is the late-time attractor provided $\alpha > 0$.

In figures 2 and 3 some numerical integrations for the system (20)–(25) for the input function $f(s) = s + \alpha$ for $\alpha = -2$ are presented. In this case the local attractor is the scalar...
Figure 2. (a) Some orbits in the invariant set $\Omega_1 = \Omega_\sigma = 0, s = 2$ of the system (20)–(25) for the input function $f(s) = s + \alpha$ for $\alpha = -2$. (b) Projection in the planes $x$ and $y$. This numerical elaboration shows that $P_4$ is the local attractor and $P_3^\pm$ are early-time attractors for this invariant set (actually they are saddles in the 6D phase space.)
Figure 3. (a) Projection in the plane $y - \Omega_1$ for the system (20)-(25) restricted to the invariant set $\Omega_m = \Omega_n = 0, s = 2$ for the input function $f(s) = s + \alpha$ for $\alpha = -2$. The instability of the line $y + \Omega_1 = 1$ and the stability of $P_4$ have been shown there. (b) Some orbits projected in the plane $\Omega_\sigma$ and $\Omega_\lambda$. This figure illustrates that the universe evolves from a solution with non-standard 5D behavior to an isotropic solution ($P_1$).

local attractor and $P_2^\pm$ are saddles. The local source in this invariant set is $P_{11}$. In figure 6(c), some orbits projected in the plane $y - \Omega_\lambda$ for $\Omega_m = 0$ suggesting that the line $y + \Omega_1 = 1$ is a local attractor in the invariant set $\Omega_m = 0$ are drawn. However, they are saddles for the full dynamics.

6. Results and discussion

The main results of this investigation can be summarized as follows. The singular point $P_{11}$ represents a big-bang singularity. It is non-hyperbolic with a 5D unstable manifold for
Figure 4. (a) Some orbits in the invariant set $\Omega_1 = \Omega_m = 0$, $s = 1$ of the system (20)–(25) for the input function $f(s) = s + \alpha$ for $\alpha = -1$. (b) Projection in the plane $x$, $y$, $\Omega_1 = 0$. This numerical elaboration shows that $P_5$ is the local attractor and $P_{\pm 3}$ are early-time attractors for this invariant set (actually they are saddles in the 6D phase space). $P_4$ and $P_7$ do not exist.

$1 < \gamma < 2$. In this case, it is a local source, as numerical simulations in figures 3, 5 and 6(c) suggest. For $0 < \gamma < 1$, it behaves as a saddle point.

In the general case, $f(0) \in \mathbb{R}$, the solution $P_{10}$ with $\Omega_1 = 0$ is the late-time de Sitter attractor without 5D corrections for $f(0) > 0$. To analyze their stability we have used the center manifold theory. Using this technique we have obtained an analogous center manifold for the origin (up to a variable rescaling) to the one in (41) and (42) in [19]. This allows us to prove that $P_{10}$ with $\Omega_1 \in (0, 1)$ are not late-time attractors with 5D modifications since they are always saddle-like. This fact correlates with a transient primordial inflation. These results extend our results in [19] to the Bianchi model class.

The critical points $P_{\pm 2}$ are the solutions dominated by the kinetic energy of the scalar field, representing solutions with an standard behavior ($\Omega_\phi = 0$); although non-hyperbolic, they behave as saddle-like points in the phase space. Also, there are the critical points $P_{\pm 6}^k$ which are the solutions dominated by the kinetic energy of the scalar field. The critical points $P_{6}^k$ corresponding to scalar field–anisotropic scaling solutions ($\Omega_\phi \sim \Omega_\sigma$) are non-hyperbolic; however, they behave as saddle-like points since they have both nonempty stable and unstable
The possible late-time stable solutions are as follows.

- The scalar field–matter scaling solution ($P_4$) provided $0 < \gamma < \frac{4}{3}$, $s^* < -\sqrt{3\gamma}$, $f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}$, $s^* > \sqrt{3\gamma}$, $f'(s^*) > 0$.

- The scalar-field-dominated solution ($P_5$) provided $0 < \gamma < \frac{4}{3}$, $-\sqrt{3\gamma} < s^* < 0$, $f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}$, $0 < s^* < \sqrt{3\gamma}$, $f'(s^*) > 0$ or $\frac{4}{3} < \gamma \leq 2$, $0 < s^* < 2$, $f'(s^*) > 0$.

- The scalar field–dark radiation scaling solution ($P_7$) provided $\frac{4}{3} < \gamma \leq 2$, $s^* < -2$, $f'(s^*) < 0$ or $\frac{4}{3} < \gamma \leq 2$, $s^* > 2$, $f'(s^*) > 0$.

- The de Sitter solution $P_{10}$ with $\Omega_\lambda = 0$ provided $f(0)$ is a real (finite) positive number, i.e. $f(0) > 0$.

The main difference with respect to our analysis in [19] is that there are possible scalar field–dark radiation scaling late-time solutions ($P_7$) for a wide region in the parameter space (see figure 1 for a numerical elaboration).

Our results are quite general and apply to the scalar field potentials presented in table I in [19]. In the particular case of a scalar field with the potential $V = V_0 e^{-\chi \phi} + \Lambda$ it can be proved in the same way as in [19], that for $\chi < 0$ the de Sitter solution is asymptotically stable. However, for $\chi > 0$, the de Sitter solution is unstable (of saddle type). We omit the calculation here because it is straightforward and the result is consistent with our previous results in [19]. Instead of considering classical potentials discussed in the literature we have considered a toy model for illustrating our analytical findings. In this case, we have investigated a scalar field with potential $V(\phi) = V_0 \left( -\frac{1}{w \left( e^{\phi} - 1 \right)^2} \right)^\frac{1}{2}$, where $W(z)$ is the special function ‘ProductLog’ that gives the principal solution for $w$ in $z = we^w$. This potential corresponds to the $f$-deviser $f(s) = s + \alpha$.

In this paper, we are mainly interested in the late-time dynamics. For the above particular example, the late-time dynamics can be summarized as follows.
Figure 6. (a) Some orbits in the invariant set $\Omega_m = \Omega = 0, x = 0$ of the system (20)–(25) for the input function $f(s) = s + \alpha$ for $\alpha = 0.1$. (b) Projection in the planes $x$ and $y$. This numerical elaboration shows that $P_{10}, \Omega_\lambda = 0$ is the local attractor and $P_{\pm 2}$ are saddles. The local source in this invariant set is $P_{11}$. (c) Projection of some orbits in the plane $y - \Omega_\lambda$ for $\Omega_m = 0$ suggesting that the line $y + \Omega_\lambda = 1$ is a local attractor in the invariant set $\Omega_m = 0$. However they are saddles for the full dynamics.
• The scalar field–matter scaling solution \( P_9 \) is a late-time attractor provided \( 0 < \gamma < \frac{4}{3}, \alpha < -\sqrt{3} \gamma \).
• The scalar-field-dominated solution \( P_{11} \) is a late-time attractor provided \( 0 < \gamma \leq \frac{4}{3}, \alpha < 0 \) or \( \frac{4}{3} < \gamma \leq 2, -2 < \alpha < 0 \).
• The scalar field–dark radiation scaling solution \( P_7 \) is a late-time attractor provided \( \frac{4}{3} < \gamma \leq \frac{2}{3}, \alpha < -\frac{2}{3} \).
• The de Sitter solution \( P_{10} \) with \( \Omega_{\Lambda} = 0 \) is the late-time attractor provided \( \alpha > 0 \).

7. Conclusions

In this paper, we have investigated the phase space of the Randall–Sundrum braneworld models with a self-interacting scalar field trapped in a Bianchi I brane with an arbitrary potential. From our numerical experiments we claim that \( P_{11} \) is associated with the big-bang singularity type. The numerical investigations performed in this paper suggest that it is in general the past attractor in the phase space of the Randall–Sundrum cosmological models. Using the center manifold theory we have obtained sufficient conditions for the asymptotic stability of the de Sitter solution with standard 4D behavior. We have proved, using the center manifold theory, that there are no late-time de Sitter attractors with 5D modifications since they are always saddle-like. This fact correlates with a transient primordial inflation. We have obtained sufficient conditions on the potential for the stability of the scalar field–matter scaling solution, the scalar-field-dominated solution and the scalar field–dark radiation scaling solution. We illustrate our analytical findings using a simple \( f \)-deviser as a toy model. All these results are generalizations of our previous results obtained for FRW branes in [19].

Acknowledgments

This work was partially supported by PROMEP, DAIP and by CONACyT, Mexico, under grant 167335; by MECESUP FSM0806, from Ministerio de Educación de Chile; and by the National Basic Science Program (PNCB) and Territorial CITMA Project (no. 1115), Cuba. DE, CRF and GL wish to thank the MES of Cuba for partial financial support of this investigation. YL is grateful to the Departamento de Física and the CA de Gravitación y Física Matemática for their kind hospitality and their joint support for a postdoctoral fellowship.

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