Concept of Time in Canonical Quantum Gravity and String Theory

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Abstract. The accommodation of gravity into the quantum framework will entail changes for the concepts of space and time at the most fundamental level. Here I shall review the concept of time, but include also some remarks on the concept of space. After a general introduction, I shall discuss time in canonical quantum gravity and its application to cosmology. The most important feature is the fundamental timelessness of the theory. I shall conclude with a brief discussion of space and time in string theory.

1. The Problem of Time
There exist many reasons why a final quantum theory of gravity is not yet at our disposal. First, there is the persistent lack of experiments which could serve as a trustful guide. Second, there are many mathematical problems. An third, there are conceptual problems which have no counterparts in other quantum theories. Among them is the problem of time, which is the subject of my contribution. In the following, I shall give a brief introduction to this problem and present some ideas about the role that time plays in some of the current approaches to quantum gravity – canonical quantum gravity and string theory. A more detailed exposition can be found, for example, in my monograph [1]; the reader can find there also references to the original literature and other sources.

The problem of time in quantum gravity is directly related with the background independence of general relativity. Consider, first, ordinary quantum mechanics, which is governed by the Schrödinger equation,

\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi. \]  

(1)

The parameter \( t \) in this equation is Newton’s absolute time which serves here as a background structure par excellence. Does the situation change with the advent of Einstein’s theory of special relativity? Not in this respect. Although space and time by themeself are no longer absolute, but are melted into a four-dimensional spacetime, this ensuing Minkowski spacetime is again an absolute structure. It exerts influence on all matter and fields, for example in the form of inertia, but cannot be acted upon by them. This is why Minkowski spacetime is a background structure. Standard quantum field theory, for example QED, is defined on this background. Moreover, the symmetry of this background, as given by the Poincaré group, is mandatory for the formulation of these theories.

The situation changes dramatically if general relativity is taken into account. Governed by the Einstein field equations,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \]  

(2)

spacetime (as specified by its metric \( g_{\mu\nu} \)) is dynamical and no longer a background structure. The general movement of matter and fields generates gravitational waves which modify spacetime. If the
gravitational field is to be quantized, one can no longer quantize it on a background. Everything must be quantized and, consequently, a quantum theory of gravity must contain new features; it can, in particular not contain any absolute time or spacetime.

In his closing speech at the Bern relativity meeting in 1955, Wolfgang Pauli seemed to have realized this problem clearly. He writes (in my translation, see [1] for the original reference):

It seems to me . . . that it [the main obstacle in quantizing gravity, C.K.] is not so much the linearity or non-linearity which forms the heart of the matter, but the very fact that here a more general group than the Lorentz group is present . . . .

With “more general group” he means the diffeomorphism group of general relativity, which is the counterpart of background independence in the mathematical formalism.

Approaches to quantum gravity can roughly be divided into two classes. The first is the oldest and the most conservative. It is the attempt to search directly for a quantum theory of the gravitational field, without embarking on a unification of all forces. Since one mostly deals with general relativity, I shall call this class quantum general relativity. Its members are covariant gravity (e.g. perturbation theory and the path-integral approach) and canonical gravity. Depending on the chosen canonical variables, the latter can be subdivided further into geometrodynamics, loop quantum gravity, and others. The second approach is the attempt to solve the problem of quantum gravity within a unified quantum framework of all interactions. The only known example is string theory. In the following, I shall first study canonical quantum gravity, address then its application to cosmology, and finally turn to string theory.

2. Canonical Quantum Gravity

One can arrive at the central equations of canonical quantum gravity by various means. Perhaps the most physical one is to proceed analogously to Erwin Schrödinger when he arrived at his wave equation. In 1926, he started from the Hamilton–Jacobi equation of classical mechanics, from which he guessed a wave equation that would yield this equation in the limit of “geometrical optics” (or WKB limit, as we would say today). In [3] he writes (given here in English translation)

We know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is in strict analogy with the failure of geometrical optics . . . that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wavelength. . . . Then it becomes a question of searching for an undulatory mechanics, and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.

Could one not proceed similarly for general relativity? Yes, this is possible. The first non-trivial step in this direction is, of course, to find the Hamilton–Jacobi version of the Einstein equations (2). This was achieved by Asher Peres in 1962 [4]. Instead of the ten Einstein field equations (2) one is now confronted with two functional equations for a functional $S[h_{ab}]$, where $h_{ab}$ denotes the three-dimensional metric on a spacelike slice through spacetime. Because of their functional nature, these equations are fully equivalent to Einstein’s equations. Restricting, for simplicity, to the vacuum case, they read (with $c = 1$)

\[ 16\pi G G_{abcd} \frac{\delta S}{\delta h_{ab}} \frac{\delta S}{\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} (^{(3)} R - 2\Lambda) = 0 , \]

\[ D_a \frac{\delta S}{\delta h_{ab}} = 0 . \]

The second equation guarantees that $S[h_{ab}]$ is invariant under coordinate transformations on the spatial section. In (3), $G_{abcd}$ is an ultralocal function of the three-metric, which plays the role of a metric on the configuration space of all three-metrics; it is called “DeWitt metric”. Furthermore, $h$ is the
determinant of the three-metric, \( (3)R \) the three-dimensional Ricci scalar, \( \Lambda \) the cosmological constant, and \( D_a \) denotes the three-dimensional covariant derivative. It is important to emphasize that these are purely spatial equations; there is, in particular, no analogue of the \( \partial S/\partial t \)-term in classical mechanics. The time \( t \) has disappeared! But where, then, does the classical spacetime come from? Spacetime can, in fact, be constructed by an equation analogously to \( p = \partial S/\partial q \) in classical mechanics, which allows the construction of a particle trajectory. In this sense, the classical spacetime can be interpreted as a generalized trajectory in which a single \( q \) corresponds to a whole three-geometry.

In fact, there hold interesting and important “interconnection theorems” between the Hamilton–Jacobi formulation and the ensuing spacetime picture. For example, the validity of (3) on each spatial slice guarantees the validity of all ten Einstein equations on the resulting spacetime.

So far for the classical part. Adopting the path originally followed by Schrödinger, we shall look for a wave functional, \( \Psi[h_{ab}] \), obeying a wave equation which in the WKB limit

\[
\Psi[h_{ab}] = C[h_{ab}] \exp \left( \frac{i}{\hbar} S[h_{ab}] \right), \tag{4}
\]

in which the metric-dependence of \( C[h_{ab}] \) is neglected compared to the one of \( S[h_{ab}] \), leads back to (3). In the vacuum case, this equation has the form

\[
\hat{H}_\perp \Psi \equiv \left( -2\kappa \hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - (2\kappa)^{-1} \sqrt{\hbar} (\nabla^3 R - 2\Lambda) \right) \Psi = 0, \tag{5}
\]

where \( \kappa = 8\pi G \), which is called the “Wheeler–DeWitt equation”. It generalizes the first equation in (3). The second equation is generalized by

\[
\hat{H}^x \Psi \equiv -2\nabla^x \frac{\hbar}{\sqrt{\Gamma}} \frac{\delta \Psi}{\delta h_{ab}} = 0 \tag{6}
\]

and is called the quantum diffeomorphism (momentum) constraint. Equations (5) and (6) are quantum constraints for the wave functional and constitute the central equations of canonical quantum gravity (quantum geometrodynamics). Provided the linear structure of quantum theory is universally valid, these equations should hold – independent of their exact validity at the most fundamental level – at least approximately on length scales bigger than the Planck length. One may thus expect to get at least some reliable results from this framework.

What about the problem of time in the light of canonical quantum gravity? The solution to this problem seems to be straightforward – external time has disappeared! Instead of the Schrödinger equation (1) one now possesses an equation of the form \( \hat{H} \psi = 0 \). Spacetime has disappeared, too, and only space remains: the argument of the wave functional is the three-dimensional metric (plus non-gravitational fields, if present).

A closer inspection of (5) reveals that it is of a local hyperbolic form. This follows from the structure of the DeWitt metric. Despite the absence of an external time, one can thus define an intrinsic timelike variable from the structure of the kinetic term in (5). This variable is constructed solely from three-dimensional quantities (the three-metric and perhaps part of the non-gravitational degrees of freedom).

Since the presence of an external time is mandatory for the interpretational structure of ordinary quantum theory – in particular, for the use of a Hilbert-space structure and the conservation of probability in time –, the question arises what is the fate of this structure in a timeless world. This is also circumscribed by the “Hilbert-space problem” and the “problem of observables” [1]. A final solution is not yet available.

The standard approach to canonical quantum gravity starts with a 3 + 1-decomposition of spacetime (assumed to be globally hyperbolic), that is, a decomposition of spacetime into a foliation of spacelike
hypersurfaces. Consider a timelike vector field $t^\nu$ that points from a point at a hypersurface with $t$ constant to a point at an infinitesimally neighbouring hypersurface with the same coordinate values,

$$X^\nu \equiv t^\nu = N n^\nu + N^\alpha X^\nu_{\alpha}, \quad \text{see Figure 1.}$$

It is decomposed into its normal component (specified by the lapse function $N$) and its tangential components (specified by the shift vector $N^\alpha$). The four-dimensional line element can accordingly be decomposed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$$

$$= (h_{ab}N^a N^b - N^2)dt^2 + 2h_{ab} N^a dx^b dt + h_{ab} dx^a dx^b .$$

The configuration variable turns out to be the three-metric $h_{ab}$ and the conjugated momentum is a linear function of the extrinsic curvature. Rewriting the Einstein–Hilbert action in terms of these variables, one arrives at a form in which it is the sum of constraints only (up to surface terms). Upon quantization, then, these constraints become the Wheeler–DeWitt equation (5) and the momentum constraints (6).

We can now follow the path that originally led us to (5) and (6) in the opposite direction. That is, we can perform a semiclassical approximation for the wave functional. We write

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]e^{i\frac{m_P^2}{2}S[h_{ab}]}|\psi[h_{ab}]\rangle,$$

where $m_P$ is the Planck mass. This ansatz is close to the WKB form (4), but includes also non-gravitational fields. The notation chosen here is such that the "ket" in the state refers to the non-gravitational degrees of freedom for which we assume that a standard Hilbert-space structure is available. That is, we could instead also write $\psi[h_{ab}, \phi]$, where $\phi$ stands for the non-gravitational fields.

The semiclassical approximation scheme [1] then yields in the highest orders the Hamilton–Jacobi equations (3) for the functional $S$ corrected by the expectation value of the Hamiltonian for the non-gravitational fields. In this sense the classical Einstein equations are recovered as approximate equations. The next order yields the following equations for $|\psi[h_{ab}]\rangle$:

$$\left(\hat{H}_m^m - \langle \psi|\hat{H}_m^m|\psi\rangle - iG_{abcd}\frac{\delta S}{\delta h_{ab}\delta h_{cd}}\right)|\psi[h_{ab}]\rangle = 0,$$

$$\left(\hat{H}_a^a - \langle \psi|\hat{H}_a^a|\psi\rangle - \frac{2}{\hbar}h_{ab}D_c\frac{\delta}{\delta h_{bc}}\right)|\psi[h_{ab}]\rangle = 0,$$

where the subscript ‘$m$’ refers to the non-gravitational parts of the Hamiltonian and momentum constraints (not considered in (5) and (6)). One now evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical
Einstein equations, \( h_{ab}(x, t) \), corresponding to a solution, \( S[h_{ab}] \), of the Hamilton–Jacobi equations; this solution is obtained from
\[
\hat{h}_{ab} = NG_{abcd} \frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}.
\]

Defining now, in this limit, a time parameter \( t \) by
\[
\frac{\partial}{\partial t} |\psi(t)\rangle = \int d^3x \hat{h}_{ab}(x, t) \frac{\delta}{\delta h_{ab}(x)} |\psi[h_{ab}]\rangle,
\]
one obtains a functional Schrödinger equation for the quantized non-gravitational fields in the chosen external classical gravitational field,
\[
i \hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}^m|\psi(t)\rangle,
\]
\[
\hat{H}^m = \int d^3x \left\{ N(x)\hat{H}^m_{+}(x) + N^\circ(x)\hat{H}^m_{-}(x) \right\}.
\]

Here, \( \hat{H}^m \) denotes the full Hamiltonian of the non-gravitational fields in the Schrödinger picture, which parametrically depends on the (generally non-static) metric coefficients of the curved space–time background.\(^1\) The emerging “WKB time” \( t \) controls the dynamics in this limit: time has been regained from timeless quantum gravity under well-defined special circumstances.

This recovery of time only works in a straightforward way if the ansatz (10) is used in which the dominating factor is of the complex form \( i \), where \( i = \sqrt{-1} \). If instead one uses, for example, a superposition containing the terms \( i \) and \( -i \), the question arises how (15) with its \( i \) emerges.

The answer is decoherence [5]. Consider an analogy from molecular physics: the emergence of chirality. Figure 2 shows the simplest possibility of a chiral molecule and the corresponding effective potential with respect to the transition from one configuration into the other.

\[ \text{Figure 2. Simplest possible structure of an optically active molecule and a schematic picture of the effective potential for the inversion coordinate as well as of the two lowest-lying eigenstates.} \]

Low-mass molecules (such as ammonia) indeed behave quantum mechanically, that is, they exhibit a superposition between the two configurations shown in Figure 2. For heavier mass this superposition is suppressed by the interaction with the environment, for example, by the scattering with light or air molecules. This is called decoherence. Since the Hamiltonian for such molecules is parity-invariant, the observation of one of the two definite molecular structures in Figure 2 constitutes a symmetry breaking – a chiral state has emerged.

In the case of the semiclassical approximation to quantum gravity, decoherence can lead to a similar symmetry breaking: although the full quantum state obeying the Wheeler–DeWitt equation is real, the actual observed semiclassical component is complex. Decoherence by interaction with small gravitational waves and tiny density fluctuations makes the \( \exp(iS_0) \) - and the \( \exp(-iS_0) \)-part of the

\(^1\) There may, of course, be situations in which part of the gravitational field (e.g. the gravitons) contribute to \( \hat{H}^m \) and part of the non-gravitational fields (e.g. macroscopic matter) contribute to \( S \).
wave function dynamically independent \([2, 5]\). This justifies in retrospect the use of the complex ansatz (10).

If the functional Schrödinger equation can be recovered from full quantum gravity in an appropriate limit, the question arises whether one can go beyond this limit and calculate quantum gravitational correction terms. This can be done at least at a formal level, that is, at the level where one treats the functional derivatives like partial derivatives.

The next order in the Born–Oppenheimer approximation then gives corrections to the Hamiltonian for the non-gravitational fields,

\[
\hat{H}^m \rightarrow \hat{H}^m + \frac{1}{m^2 \rho} \text{ (various terms) .} 
\] (16)

The detailed form of these terms can be found in [1]. Future investigations should deal with a concrete application of these terms in cosmology, for example, in the search for quantum gravitational effects in the anisotropy spectrum of the Cosmic Background Radiation.

A simple example is the calculation of the quantum gravitational correction to the trace anomaly in de Sitter space. For a conformally coupled scalar field, the trace of the energy–momentum tensor, although being zero classically, is non-vanishing in the quantum theory; this “anomalous trace” is proportional to \(h\). It corresponds to the following expectation value, \(\varepsilon\), of the Hamiltonian density,

\[
\varepsilon = \frac{\hbar H^4_{\text{dS}}}{1440 \pi^2 \rho^3} ,
\] (17)

where \(H_{\text{dS}}\) is the constant Hubble parameter of de Sitter space. It turns out that the ratio of the first quantum gravitational correction \(\delta \varepsilon\) to \(\varepsilon\), as calculated from the Born–Oppenheimer expansion, reads

\[
\frac{\delta \varepsilon}{\varepsilon} \approx -\frac{1}{2160 \pi} \left( \frac{t_P}{H_{\text{dS}}} \right)^2 ,
\] (18)

where \(t_P\) denotes the Planck time. Numerically, the ratio (18) is, of course, small. Using values motivated by inflationary cosmology, one can assume that \(H_{\text{dS}}\) lies between \(10^{13}\) and \(10^{15}\) GeV, leading for the ratio (18) to values between roughly \(10^{-16}\) and \(10^{-22}\). It is at present an open question whether these correction terms can become big enough to be observable.

At the end of this section I shall make some brief remarks on the situation in loop quantum gravity [6]. This is a particular case of the canonical theory and differs from geometrodynamics by the choice of variables. Since it contains the same timeless constraints, the situation is similar to the one discussed above. In addition, loop quantum gravity seems to give some new insights into space. At the kinematical level, it turns out that geometric operators can have a discrete spectrum of eigenvalues. Therefore, at least in an operational sense, three-dimensional space exhibits discrete features. One example is the quantization of area,\(^2\)

\[
\hat{A}(S) \Psi_S[A] = 8\pi \beta \ell_P^2 \sum_{P \in S \cap \mathcal{S}} \sqrt{j_P (j_P + 1)} \Psi_S[A] \] (19)

where \(\ell_P\) is the Planck length and \(\beta\) is a free parameter in this approach.

3. Quantum Cosmology
Quantum cosmology is the application of quantum theory to the Universe as a whole. Applying canonical quantum gravity to cosmology, one encounters intriguing consequences of the concept of time.

\(^2\) The sum is over all crossing points \(P\) of a spin network \(S\) with a surface \(S\), and \(j_P\) can assume integer and half-integer values.
As in classical cosmology, one starts with an ansatz for a highly symmetric metric. For a closed Friedmann–Lemaître universe with scale factor $a$, the metric can be chosen to be of the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega^2_3.$$  

As a representative for matter, we choose in addition a homogeneous massive scalar field $\phi$. The Wheeler–DeWitt equation then reads (with units $2G/3\pi = 1$)

$$\frac{1}{2} \left( \frac{\hbar^2}{a^2} \frac{\partial}{\partial a} \left( a \frac{\partial}{\partial a} \right) - \frac{\hbar^2}{a^3} \frac{\partial^2}{\partial \phi^2} - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2 \right) \psi(a, \phi) = 0.$$  

The configuration space here is only two-dimensional (consisting of $a$ and $\phi$); such drastically reduced configuration spaces are called “minisuperspaces”. The factor ordering (which is a general problem for the Wheeler–DeWitt equation) has been chosen in order to achieve covariance in minisuperspace.

The remaining part of this section is similar to the corresponding section in [2] and is therefore adapted from that source.

It is evident that equations such as (21) do not possess the mathematical problems of the full functional equation (5). One can thus focus attention on physical applications. One important application is the imposition of boundary conditions. Other important applications include the discussion of wave packets, the validity of the semiclassical approximation, the origin of classical behaviour and the arrow of time, and the possible quantum avoidance of classical singularities [1, 7].

Before picking out one particular model, I want to emphasize one important conceptual point which is relevant for the problem of time discussed above, see Figure 3.

**Figure 3.** The classical and the quantum theory of gravity exhibit drastically different notions of determinism [1].

Consider a two-dimensional minisuperspace model with the variables $a$ and $\phi$ as above. The figure on the left shows the classical trajectory in configuration space for a universe which is expanding and recollapsing. Classically, one can give initial conditions, for example, on the left end of the trajectory for small $a$ and then determine the whole trajectory. In this sense, the recollapsing part of the trajectory is the deterministic successor of the expanding part. One could, of course, also start from the right end of the trajectory because there is no distinguished direction; but the important point is that a trajectory exists. Not so in the quantum theory where both the trajectory and the time parameter $t$ are absent! If one wants to find a solution of the Wheeler–DeWitt equation which describes a wave packet following the classical trajectory, one has to specify two packets at the would-be ends of the classical trajectory, see the right figure. The reason is that (21) is a hyperbolic equation with respect to intrinsic time $a$, and the natural formulation of boundary conditions is to specify the wave function (and its derivative) at constant $a$. If one imposed only one of the two wave packets, the full solution would be a smeared-out wave function...
which does not resemble anything like a wave packet following the classical trajectory. In this sense, the “recollapsing” wave packet must be present “initially”. This is a direct consequence of the concept of time in quantum gravity.

Let me now turn to a specific example [8]: a cosmological model with a “big brake”. Classically, the model is characterized by an equation of state of the form \( p = A/\rho \), where \( A > 0 \) ("anti-Chaplygin gas"). This can be realized by a scalar field \( \phi \) with the following potential:

\[
V(\phi) = V_0 \left( \sinh \left( \sqrt{3\kappa^2} |\phi| \right) - \frac{1}{\sinh \left( \sqrt{3\kappa^2} |\phi| \right)} \right); \quad V_0 = \sqrt{A/4}.
\]  
\[
(22)
\]

This model universe develops a pressure singularity at the end of its evolution where it comes to an abrupt halt: \( \dot{a} \) remains finite there but \( \ddot{a}(t) \) tends to minus infinity; this is why it is called a “big brake”. Since this model does not describe an accelerating universe, it is as such in conflict with present observations. However, it can easily be generalized in order to accommodate such an acceleration, without modifying the following discussion. The total lifetime of this universe would be

\[
t_0 \approx 7 \times 10^2 \frac{1}{\sqrt{V_0 \left[ \frac{\text{eV}}{\text{cm}^3} \right]}} \text{s},
\]

which would be much bigger than the current age of our Universe for

\[
V_0 \ll 2.6 \times 10^{-30} \frac{\text{g}}{\text{cm}^3}.
\]

The classical trajectory in configuration space is shown in Figure 4. The big-brake singularity is at \( \phi = 0 \). In addition, there are the usual big-bang and big-crunch singularities at \( a = 0 \) and \( \phi \to \pm \infty \).

\[\text{Figure 4. Classical trajectory in configuration space [8].}\]

In the quantum theory, one encounters the following Wheeler–DeWitt equation:

\[
\frac{\hbar^2}{2} \left( \frac{\kappa^2}{6} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} \right) \Psi(\alpha, \phi) + V_0 e^{6\alpha} \left( \frac{\sinh \left( \sqrt{3\kappa^2} |\phi| \right)}{\sinh \left( \sqrt{3\kappa^2} |\phi| \right)} - \frac{1}{\sinh \left( \sqrt{3\kappa^2} |\phi| \right)} \right) \Psi(\alpha, \phi) = 0 ,
\]

\[
(23)
\]
where $\alpha = \ln a$. In order to study the behaviour near the region of the classical singularity, it is sufficient to study the limit of small $\phi$. One can then use the approximate equation

$$\frac{\hbar^2}{2} \left( \frac{\kappa^2}{\phi} \frac{\partial}{\partial \phi^2} - \frac{\partial^2}{\partial \phi^2} \right) \Psi (\alpha, \phi) - \frac{\tilde{V}_0}{\phi} \phi^{6\alpha} \Psi (\alpha, \phi) = 0,$$

where $\tilde{V}_0 = V_0/3\kappa^2$. A crucial input is now the choice of boundary conditions. Firstly, we have to impose the condition that the wave function go to zero for large $a$; this is because the classical evolution stops at finite $a$. Secondly, we demand normalizability with respect to $\phi$. The resulting solutions are then of the form

$$\Psi (\alpha, \phi) = \sum_{k=1}^{\infty} A(k) k^{-3/2} K_0 \left( \frac{1}{\sqrt{6\kappa}} \frac{V_0}{k} \right) \left( 2 \frac{V_0}{k} |\phi| \right) e^{-\frac{\sqrt{6\kappa}}{k} L_{k-1} \left( 2 \frac{V_0}{k} |\phi| \right)},$$

where $K_0$ is a Bessel function, $L_{k-1}$ denotes the Laguerre polynomials, and $V_0 \equiv \tilde{V}_0 \phi^{6\alpha}$. Inspection of this solution shows that it vanishes at $\phi = 0$, that is, at the classical big-brake singularity. Therefore, this singularity is avoided in the quantum theory. In fact, the normalization condition with respect to $\phi$ also guarantees that the big-bang singularity is absent. One is thus left with a singularity-free quantum universe.

A wave-packet solution following the classical solution of Figure 4 and approaching zero when $\phi \to 0$ (that is, when approaching the region of the classical big-brake singularity), is shown in Figure 5.

![Figure 5](image-url)

**Figure 5.** The wave packet for the big-brake model. The packet follows the classical trajectory but becomes zero at the classical singularity [8].

We have seen in the last section that a semiclassical time parameter $t$ exists only under special circumstances. These are no longer fulfilled in the regions of the classical singularities where the timelessness of the fundamental quantum world takes over.
4. String Theory

In contrast to the approaches discussed above, the ambition for string theory (or M-theory) is to construct a unified quantum theory of all interactions from the very beginning, see, for example, [9, 1] for an introduction. Quantum gravity then emerges in an appropriate limit where the various interactions can be conceptually separated from each other. Although string theory, too, does not yet exist in a final form, it possesses a couple of attractive features. Besides the unification of interactions, the inclusion of gravity is unavoidable; as Ed Witten has put it [10]: “String theory forces general relativity upon us.” Necessary ingredients are gauge invariance, supersymmetry, but also the presence of higher dimensions from which so far no trace has been seen. The perturbation theory is probably finite to all orders, although the sum of all terms diverges. And there are only three fundamental dimensionful constants, \( c, \hbar, \) and the string length, \( l_s, \) from which in principle all other scales can be derived.

What can be said about the concept of time (and space) in string theory [11]? A central ingredient of the formalism is the Euclidean path integral,

\[
Z = \int \mathcal{D}X \mathcal{D}h \, e^{-S/\hbar},
\]  

(26)

where the sum is over the embedding variables, \( X, \) describing the embedding of the string into a higher-dimensional auxiliary spacetime, and the metric on the two-dimensional worldsheet, shortly denoted by \( \hbar. \) Therefore, \( X \) and \( \hbar \) are the quantum degrees of freedom. In addition, however, the action depends on external ‘background’ fields, such as the metric of the auxiliary embedding spacetime, the dilaton field, and others. They are at this stage not considered as quantum variables. Since there should be no background fields also in string theory at the most fundamental level, this cannot yet be the final stage. Expression (26) is at the heart of the string perturbation theory.

Demanding the absence of quantum anomalies, it follows that the background fields obey Einstein’s equations up to order \( \hbar^2; \) these equations can be derived from an effective action. It also follows that the number of dimensions of the embedding spacetime is restricted to be 10 (or 11).

From an operational point of view, one can argue that there is a lower bound on the uncertainty, \( \Delta x, \) of a spacetime point. This is, of course, a consequence of the extended nature of the string. In heuristic gedanken experiments one can derive a generalized uncertainty relation of the form

\[
\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\hbar^2}{\hbar} \Delta p,
\]  

(27)

which expresses this fact.

Since string theory contains general relativity, the arguments leading to the timelessness of canonical quantum gravity apply here, too. Thus, for scales larger than the Planck length, the Wheeler–DeWitt equation should be approximately valid.

The “problem of time” is thus the same in string theory and canonical quantum gravity. New insights are obtained for the concept of space, although a final picture is not available. In this context, the following two approaches are relevant. First, motivated by M-theory one can devise matrix models which contain a finite number of degrees of freedom connected with a system of D0-branes for the description of M-theory; the fundamental scale is the 11-dimensional Planck scale. Second, the \( \text{AdS/CFT correspondence} \) plays a key role: non-perturbative string theory in a background spacetime which is asymptotically anti-de Sitter (AdS) is dual to a conformal field theory (CFT) defined in a flat spacetime of one less dimension. This is often considered as a mostly background-independent definition of string theory, since the background metric enters only through boundary conditions at infinity.

The AdS/CFT correspondence is also connected with the realization of the “holographic principle”: it turns out that laws including gravity in space dimension \( d = 3 \) are equivalent to laws excluding gravity in \( d = 2. \) In a sense, space has here vanished, too, and it has been claimed that gravity is only an “illusion” [12]. This should, however, not be interpreted literally, since the holographic principle only refers to the equivalence of two theories in an operational sense and does not postulate the absence of gravity in our world.
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