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NONEXISTENCE OF LEVI FLAT HYPERSURFACES WITH
POSITIVE NORMAL BUNDLE IN COMPACT KÄHLER
MANIFOLDS OF DIMENSION $\geq 3$

SÉVERINE BIARD AND ANDREI IORDAN

In memory of Gennadi M. Henkin

Abstract. Let $X$ be a compact connected Kähler manifold of dimension $\geq 3$ and $L$ a $C^\infty$ Levi flat hypersurface in $X$. Then the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This represents an answer to a conjecture of Marco Brunella.

1. Introduction

A classical theorem of Poincaré-Bendixson [28], [29], [5] states that every leaf of a foliation of the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}^n$, $n \geq 2$, does not contain any compact leaf and its singular set $\text{Sing } \mathcal{F}$ is not empty, a major problem in foliation theory is the following: can $\mathcal{F}$ contain a leaf $\mathcal{F}$ such that $\mathcal{F} \cap \text{Sing } \mathcal{F} = \emptyset$? If this is the case, then there exists a nonempty compact set $K$ called exceptional minimal, invariant by $\mathcal{F}$ and minimal for the inclusion such that $K \cap \text{Sing } \mathcal{F} = \emptyset$. The problem of the existence of an exceptional minimal in $\mathbb{CP}^n$, $n \geq 2$ is implicit in [12].

In [13] D. Cerveau proved a dichotomy under the hypothesis of the existence of a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}^n$ which admits an exceptional minimal $\mathfrak{M}$: $\mathfrak{M}$ is a real analytic Levi flat hypersurface in $\mathbb{CP}^n$ (i.e. $T(\mathfrak{M}) \cap JT(\mathfrak{M})$ is integrable, where $J$ is the complex structure of $\mathbb{CP}^n$), or there exists $p \in \mathfrak{M}$ such that the leaf through $p$ has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi flat hypersurface in $\mathbb{CP}^n$, $n \geq 2$.

The conjecture was proved for $n \geq 3$ by A. Lins Neto [22] for real analytic Levi flat hypersurfaces and by Y.-T. Siu [31] for $C^{1,2}$ smooth Levi flat hypersurfaces. The methods of proofs for the real analytic case are very different from the smooth case.

A real hypersurface of class $C^2$ in a complex manifold is Levi flat if its Levi form vanishes or equivalently, it admits a foliation by complex hypersurfaces. We say that a (non-necessarily smooth) real hypersurface $L$ in a complex manifold $X$ is Levi flat if $X \setminus L$ is pseudoconvex. An example of (non-smooth) Levi flat hypersurface in
\( \mathbb{C}P_2 \) is \( L = \{ [z_0, z_1, z_2] : |z_1| = |z_2| \} \), where \([z_0, z_1, z_2]\) are homogeneous coordinates in \( \mathbb{C}P_2 \) (see [19]).

In [21] Iordan and Mattthey proved the nonexistence of Lipschitz Levi flat hypersurfaces in \( \mathbb{C}P_n, n \geq 3 \), which are of Sobolev class \( W^s, s > 9/2 \). A principal element of the proof is that the Fubini-Study metric induces a metric of positive curvature on any quotient of the tangent space.

Nonexistence questions for the Levi flat hypersurfaces in compact Kähler manifolds were first discussed by T. Ohsawa in [24], who proved the nonexistence of real-analytic Levi flat hypersurfaces with Stein complement in compact Kähler manifolds of dimension \( \geq 3 \).

In [9], M. Brunella proved that the normal bundle to the Levi foliation of a closed real analytic Levi flat hypersurface in a compact Kähler manifold of dimension \( n \geq 3 \) does not admit any Hermitian metric with leafwise positive curvature. The real analytic hypothesis may be relaxed to the assumption of \( C^{2, \alpha}, 0 < \alpha < 1 \), such that the Levi foliation extends to a holomorphic foliation in a neighborhood of the hypersurface.

The main step in his proof is to show that the existence of a Hermitian metric with leafwise positive curvature on the normal bundle to the Levi foliation of a compact Levi flat hypersurface \( L \) in a Hermitian manifold \( X \), implies that \( X \setminus L \) is strongly pseudoconvex, i.e. there exists on \( X \setminus L \) an exhaustion function which is strongly plurisubharmonic outside a compact set. This was generalized in [10] for invariant compact subsets of a holomorphic foliation of codimension one. Of course, if \( X \) is the complex projective space, then every proper pseudoconvex domain in \( X \) is Stein [34].

Brunella stated also the following conjecture [9]: Let \( X \) be a compact connected Kähler manifold of dimension \( n \geq 3 \) and \( L \) a \( C^\infty \) compact Levi flat hypersurface in \( X \). Then the normal bundle to the Levi foliation does not admit any Hermitian metric with leafwise positive curvature.

The assumption \( n \geq 3 \) is necessary in this conjecture (see Example 4.2 of [9]).

In [11] Brunella and Perrone proved that every leaf of a holomorphic foliation \( F \) of codimension one of a projective manifold \( X \) of dimension at least 3 and such that \( Pic(X) = \mathbb{Z} \) accumulates on the singular set of the foliation. In this case the normal bundle to the foliation is ample.

In [25], T. Ohsawa considered a \( C^\infty \) Levi flat compact hypersurface \( L \) in a compact Kähler manifold \( X \) such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank \( \geq k \) on the holomorphic tangent space to the leaves and proved that \( X \setminus L \) admits an exhaustion plurisubharmonic function of logarithmic growth which is strictly \((n-k)\)-convex. Then, if \( \dim X \geq 3 \), he proved that there are no Levi flat real analytic hypersurfaces such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank \( \geq 2 \) on the holomorphic tangent space to \( L \). Some possibilities for generalization in the smooth case are also indicated.

In this paper we solve the above mentioned conjecture of Brunella for compact connected Kähler manifolds of dimension \( n \geq 3 \). The principal ingredient of the proof is a refinement of the proof of Brunella [9] of the strong pseudoconvexity of \( X \setminus L \) : we show that there exist a neighborhood \( U \) of \( L \) and a function \( v \) on \( U \) vanishing on \( L \), such that \(-i\partial \bar{\partial} \ln v \geq c \omega \) on \( U \setminus L \), where \( c > 0 \) and \( \omega \) is the \((1,1)\)-form associated to the Kähler metric. Then we use the \( L^2 \) estimates [2], [1], [20],
and the normal bundle to the Levi foliation admits a \( C \)-structure that the Levi foliation extends to a holomorphic foliation in a neighborhood of \( C \) in a compact Hermitian manifold of class \( L \). This leads in dimensions \( \geq 3 \) to the solution of Brunella’s conjecture.

2. Preliminaries

Let \( X \) be a complex \( n \)-dimensional manifold, \( \omega \) a Kähler metric on \( X \), \( \Omega \) a domain in \( X \) and \( \sigma \) a positive function on \( \Omega \). For \( \alpha \in \mathbb{R} \) denote

\[
L^2_{(p,q)}(\Omega, \sigma^\alpha, \omega) = \left\{ f \in L^2_{(p,q),\text{loc}}(\Omega) : \int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega < \infty \right\}
\]

endowed with the norm

\[
N_{\alpha,\omega,\sigma}(f) = \left( \int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega \right)^{1/2}.
\]

Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{CP}_n \) and \( \delta_\Omega \) the geodesic distance to the boundary for the Fubini-Study metric \( \omega_{FS} \). By using the \( L^2 \) estimates for the \( \overline{\partial} \)-operator of Hörmander with the weight \( e^{-\varphi} \), \( \varphi = -\alpha \log \delta_\Omega \) which is strongly plurisubharmonic by a theorem of Takeuchi [34], Henkin and Iordan proved in [19] the existence and regularity of the \( \overline{\partial} \) equation for \( \overline{\partial} \)-closed forms in \( L^2_{(p,q)}(\Omega, \delta_\Omega^\alpha, \omega_{FS}) \) verifying the moment condition. This gives the regularity of the \( \overline{\partial} \)-operator in pseudoconcave domains with Lipschitz boundary [19] and, by using a method of Siu [31], [32], the nonexistence of smooth Levi flat hypersurfaces in \( \mathbb{CP}_n \), \( n \geq 3 \) follows (see [21]). These techniques will be used in the 4th and the 5th paragraph.

We will use also the following theorem of regularity of \( \overline{\partial} \) equation of Brinkshulte [7]:

**Theorem 1.** Let \( \Omega \) be a relatively compact domain with Lipschitz boundary in a Kähler manifold \((X, \omega)\) and set \( \delta_\Omega \) the geodesic distance to the boundary of \( \Omega \). Let \( f \in L^2_{(p,q)}(\Omega, \delta_\Omega^{-k}, \omega) \cap C^0(\Omega) \) and \( u \in L^2_{(p,q)}(\Omega, \delta_\Omega^{-k}, \omega) \) such that \( \overline{\partial} u = f \) and \( \overline{\partial}_{-k} u = 0 \), where \( \overline{\partial}_{-k} \) is the Hilbert space adjoint of the unbounded operator \( \overline{\partial}_{-k} : L^2_{(p,q)}(\Omega, \delta_\Omega^{-k}, \omega) \to L^2_{(p,q)}(\Omega, \delta_\Omega^{-k}, \omega) \).

Then for \( k \) big enough \( u \in C^{s(k)}(\Omega) \) where \( s(k) \sim \sqrt{k} \).

3. Strong pseudoconvexity of the complement of a Levi flat hypersurface

Let \( L \) be a smooth Levi flat hypersurface in a Hermitian manifold \( X \). As was mentioned in [9] and in [25], by taking a double covering, we can assume that \( L \) is orientable and the complement of \( L \) has two connected components in a neighborhood of \( L \). This will be always supposed in the sequel and for an open neighborhood \( U \) of \( L \) we will denote by \( U^+ \) and \( U^- \) the two connected components of \( U \setminus L \). We will denote by \( \delta_L \) the signed geodesic distance to \( L \).

In [9] Brunella proved that the complement of a closed Levi flat hypersurface in a compact Hermitian manifold of class \( C^{2,\alpha} \), \( 0 < \alpha < 1 \), having the property that the Levi foliation extends to a holomorphic foliation in a neighborhood of \( L \) and the normal bundle to the Levi foliation admits a \( C^2 \) Hermitian metric with leafwise positive curvature is strongly pseudoconvex, i.e. there exists an exhaustion.
function which is strongly plurisubharmonic outside a compact set. The following proposition strengthens this result:

**Proposition 1.** Let \( L \) be a compact \( C^3 \) Levi flat hypersurface in a Hermitian manifold \( X \) of dimension \( n \geq 2 \), such that the normal bundle \( N^{1,0}_L \) to the Levi foliation admits a \( C^2 \) Hermitian metric with leafwise positive curvature. Then there exist a neighborhood \( U \) of \( L \), \( c > 0 \) and a non-negative function \( v \in C^2(U) \), vanishing on \( L \) and positive on \( U \setminus L \) such that \(-i\partial\bar{\partial}\ln v \geq \omega \) on \( U \setminus L \), where \( \omega \) is the \((1,1)\)-form associated to the metric. Moreover, there exists a nonvanishing continuous function \( g \) in a neighborhood of \( L \) such that \( v = g^2 \).

**Proof.** Let \( z_0 \in L \). There exist holomorphic coordinates \( z = (z_1, \cdots, z_{n-1}, z_n) = (z', z_n) \) in a neighborhood of \( z_0 \) such that the local parametric equations for \( L \) are of the form

\[ z_j = w_j, \quad j = 1, \ldots, n-1, \quad z_n = \varphi(w', t) \]

where \( \varphi \) is of class \( C^3 \) (see [4]) on a neighborhood of the origin in \( \mathbb{C}^{n-1} \times \mathbb{R} \), holomorphic in \( w' \) and \( \frac{\partial \varphi}{\partial t}(z_0) \in \mathbb{R}^n \). We consider a \( C^3 \) extension \( \psi = (\psi_1, \ldots, \psi_n) \) of \( \varphi \) on a neighborhood of the origin in \( \mathbb{C}^{n-1} \times \mathbb{C} \), \( \psi(w', t + is) = (w', \varphi(w', t) + is) \).

Then \( \psi \) is a \( C^3 \) diffeomorphism in a neighborhood of \( z_0 \) and holomorphic in \( w' \). It follows that

\[ L = \{(z', z_n) : \rho(z', z_n) = 0\} \]

where \( \rho = \text{Im} ((\psi^{-1})'_n) \). We denote \( f = (\psi^{-1})'_n (z', z_n) \). Since \( \partial_h f = 0 \) on \( L \), where \( \partial_h \) is the tangential Cauchy-Riemann operator on \( L \), there exists an extension \( \tilde{f} \) of class \( C^3 \) in a neighborhood of \( z_0 \) such that \( \partial \tilde{f} \) vanishes to order greater than 2 on \( L \), i.e. \( D^l \partial \tilde{f} = 0 \) for \( |l| \leq 2 \) on \( L \).

So there exists an open finite covering \( \{U_j\}_{j \in J} \) by holomorphic charts of \( L \) such that \( \tilde{U}_j \setminus L = U_j^+ \cup U_j^- \) such that \( U_j = L \cap \tilde{U}_j = \{z \in \tilde{U}_j : \text{Im} \tilde{f}_j = 0\} \), where \( \partial \tilde{f}_j \) vanishes to order greater than 2 on \( L \) and the Levi foliation is given on \( U_j \) by \( \{z \in U_j : \tilde{f}_j(z) = c_j\} \), \( c_j \in \mathbb{C} \). Thus \( D^l \partial \tilde{f}_j = 0 \) is a nonvanishing section of \( N^{1,0}_L \) on \( U_j \) and by shrinking \( \tilde{U}_j \), we may consider that \( D^l \partial \tilde{f}_j \neq 0 \) on \( \tilde{U}_j \).

We may suppose that \( N^{1,0}_L \) is represented by a cocycle \( \{g_{jk}\} \) of class \( C^2 \) subordinated to the covering \( \{U_j\}_{j \in J} \) and there exist closed \((1,0)\)-forms \( \alpha_j \) of class \( C^2 \) on \( U_j \) holomorphic along the leaves such that \( T^{1,0}(U_j) = \ker \alpha_j \) for every \( j \in J \) and \( \alpha_j = g_{jk} \alpha_k \) on \( U_j \cap U_k \). So \( \{\alpha_j\}_{j \in J} \) defines a global form \( \alpha \) on \( L \) with values in \( N^{1,0}_L \) such that locally on \( U_j \) we have \( \alpha(z) = \alpha_j(z) \otimes \alpha^*_j(z) \) where \( \alpha^*_j \) is the dual frame of \( \alpha_j \). In particular we have \( \alpha^*_k = g_{jk} \alpha^*_j \).

Let \( h \) be a \( C^2 \) Hermitian metric with positive leafwise curvature \( \Theta_h \left( N^{1,0}_L \right) \) on \( N^{1,0}_L \), \( h \) is defined on each \( U_j \) by a \( C^2 \) function \( h_j = |\alpha_j|^2 \) such that \( h_k = |g_{jk}|^2 h_j \) on \( U_j \cap U_k \).

Since \( \alpha_j = \eta_j d \tilde{f}_j \) on \( U_j \) for every \( j \), where \( \eta_j \) are nowhere vanishing functions of class \( C^2 \) on \( U_j \) holomorphic along the leaves and

\[ \frac{1}{\eta_k} \left( d \tilde{f}_k \right)^* = \frac{1}{\eta_j} g_{jk} \left( d \tilde{f}_j \right)^* \]
on $U_j \cap U_k$, it follows that
\[
|g_{jk}(z)|^2 = \left| \frac{\eta_j(z)}{\eta_k(z)} \right|^2 \left| \frac{(d\tilde{f}_k)}{(d\tilde{f}_j)} \right|^2 = \frac{h_k(z)}{h_j(z)}, \quad z \in U_j \cap U_k.
\]
So
\[
h_j|\eta_j|^2 \left( \text{Im} \tilde{f}_j \right)^2 - h_k|\eta_k|^2 \left( \text{Im} \tilde{f}_k \right)^2
\]
vanishes to order greater than 2 on $U_j \cap U_k$ and \( \left( h_j|\eta_j|^2 \right) \left( \text{Im} \tilde{f}_j \right)^2 \) defines a jet of order 2 on $L$. By Whitney extension theorem there exists a $C^2$ function $v$ on $X$ such that $v - h_j|\eta_j|^2 \left( \text{Im} \tilde{f}_j \right)^2$ vanishes to order 2 on $U_j$ for every $j \in J$. Let $\tilde{\eta}_j, \tilde{h}_j$ be $C^2$ extensions of $\eta_j, h_j$ on $\tilde{U}_j$ and set $\tilde{\alpha}_j = \tilde{\eta}_j d\tilde{f}_j, \tilde{v} = \tilde{h}_j |\tilde{\eta}_j|^2 \left( \text{Im} \tilde{f}_j \right)^2$.

For $z \in \tilde{U}_j$ denote $E_z' = \left\{ V' \in T_{z,0}^1(X) : \langle \partial \text{Im} \tilde{f}_j, V' \rangle = 0 \right\}$ and $E_z''$ the orthogonal of $E_z'$ in $T_{z,0}^1(X)$. Then for every $V \in T_{z,0}^1(X)$ there exists $V' \in E_z', V'' \in E_z''$ such that $V = V' + V''$. The curvature form $\Theta \left( \mathcal{N}_{\tilde{\lambda}}^{1,0} \right)$ is represented by $-i\partial\bar{\partial} \ln \left( h_j|\alpha_j|^2 \right)$ on $U_j$, so by shrinking $\tilde{U}_j$ we may suppose that there exists $\beta > 0$ such that $\left( -i\partial\bar{\partial} \ln \left( h_j|\alpha_j|^2 \right) \right) (V', V') \geq \beta \langle V', V' \rangle$ for every $z \in \tilde{U}_j$ and $V \in T_{z,0}^1(X)$.

On $\tilde{U}_j \setminus L$ we have
\[
-i\partial\bar{\partial} \ln \tilde{v} = -i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \left( \text{Im} \tilde{f}_j \right)^2
\]
\[
= -i\partial\bar{\partial} \ln \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 + i\partial\bar{\partial} \ln \left| d\tilde{f}_j \right|^2 - i\partial\bar{\partial} \ln \left( \text{Im} \tilde{f}_j \right)^2.
\]
Let $z \in \tilde{U}_j$ and $V \in T_{z,0}^1(X)$. Then $V = V' + V''$, $V' \in E_z'$ and $V'' \in E_z''$ and
\[
-i\partial\bar{\partial} \ln \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 (V, V') = \left( -i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \right) (V', V')
\]
\[
+ 2 \text{Re} \left( -i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \right) (V', V'')
\]
\[
+ \left( -i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \right) (V'', V'')
\]
There exists a constant $C > 0$ depending on the eigenvalues of $-i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right)$ with respect to $\omega$ such that for every $\varepsilon > 0$
\[
2 \left| \text{Re} \left( -i\partial\bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \right) (V', V'') \right| \leq C \left( \varepsilon \omega (V', V') + \frac{1}{\varepsilon} \omega (V'', V'') \right),
\]
so
\[
-i\partial\bar{\partial} \ln \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 (V, V') \geq \beta \omega (V', V') - C \left( \varepsilon \omega (V', V') - \frac{1}{\varepsilon} \omega (V'', V'') \right)
\]
(3.2)
\[
\left\| -i\partial\bar{\partial} \ln \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right\|_\omega (V'', V'')
\]
Since \( \overline{\partial} \tilde{f}_j \) vanishes to order greater than 2 on \( L \), for every \( \gamma > 0 \) there exists a neighborhood of \( L \) such that

\[
(3.3) \quad \left| i \partial \overline{\partial} \ln \left| d \tilde{f}_j \right| (V, \overline{V}) \right|^2 \leq \gamma \omega (V, \overline{V})
\]

and

\[
(3.4) \quad \left| i \partial \overline{\partial} \Im \tilde{f}_j (V, \overline{V}) \right| \leq \gamma \left( \Im \tilde{f}_j \right) \omega (V, \overline{V}).
\]

Let \( z \in \tilde{U}_j \setminus L \). By (3.4) it follows that

\[
-i \partial \overline{\partial} \left( \Im \tilde{f}_j \right)^2 (V, \overline{V}) = \left( -2 i \partial \overline{\partial} \Im \tilde{f}_j \frac{\partial \Im \tilde{f}_j + \overline{\partial} \Im \tilde{f}_j}{(\Im \tilde{f}_j)^2} \right) (V, \overline{V})
\]

\[
(3.5) \quad \geq -2 \gamma \omega (V, \overline{V}) + 2 i \partial \overline{\partial} \Im \tilde{f}_j \frac{\partial \Im \tilde{f}_j + \overline{\partial} \Im \tilde{f}_j}{(\Im \tilde{f}_j)^2} (V''', \overline{V'''})
\]

By using (3.2), (3.3) and (3.5), from (3.1) we obtain

\[
(-i \partial \overline{\partial} \ln \tilde{v}) (V, \overline{V}) \geq (\beta - C \varepsilon) \omega (V', \overline{V'})
\]

\[
+ \left( \frac{2}{(\Im \tilde{f}_j)^2} \inf_{U_j} \left\| \partial \Im \tilde{f}_j \right\|^2 - C \varepsilon - \left\| -i \partial \overline{\partial} \ln \tilde{h} \left| \alpha_j \right|^2 \right\| \omega (V'', \overline{V''}) - 2 \gamma \omega (V, \overline{V}) \right.
\]

By choosing \( 0 < C \varepsilon < \beta \) and by shrinking \( \tilde{U}_j \) such that \( \frac{2}{(\Im \tilde{f}_j)^2} \) is big enough and \( \gamma \) small enough, we obtain that there exists \( c > 0 \) such that \( -i \partial \overline{\partial} \ln \tilde{v} \geq c \omega \) on \( \tilde{U}_j \setminus L \). Finally, since \( v - \tilde{v} \) vanishes to order greater than 2 on \( L \), it follows that there exists a neighborhood \( U' \) of \( L \) such that \( - \ln v \) is strongly plurisubharmonic on \( U' \setminus L \). We can now take \( U = \{ z \in U' : v (z) < \mu \} \) for \( \mu > 0 \) small enough.

\( L \) is a \( C^3 \) manifold, so the signed distance function \( \delta_L \) is a defining function of class \( C^3 \) for \( L \). Since \( v \) is of class \( C^2 \) on \( U \) and vanishes to order greater than 2 on \( L \), we have \( v = g \delta^2_L \) with \( g \) continuous in a neighborhood of \( L \).

Suppose that there exists \( x \in L \) such that \( g (x) = 0 \). Then \( v = o \left( \delta^2_L \right) \) in a neighborhood of \( x \). But there exists \( j \) such that \( x \in U_j \) and \( v = h_j |y_j|^2 \left( \Im \tilde{f}_j \right)^2 + o \left( \delta^2_L \right) \). Since \( \Im \tilde{f}_j = 0 \) and \( d \Im \tilde{f}_j \neq 0 \) on \( L \) it follows that \( |\nabla^2 \psi| (x) \neq 0 \). This contradiction shows that \( g (x) \neq 0 \) on \( L \).

4. Weighted estimates for the \( \overline{\partial} \)-equation

Remark 1. Under the hypothesis and conclusions of Proposition 1, we consider a positive extension \( \tilde{v} \) of the restriction of \( v \) on a neighborhood of \( L \) to \( X \setminus L \). Let \( s > 0 \) such that \( \{ v < e^{-s} \} \subset U \) and let \( \varphi \) be a smooth function on \( \mathbb{R} \) such that \( \varphi = 0 \) on \( | - \infty, s] \) and \( \varphi \) is strictly convex increasing on \( |s, \infty| \). Then \( \psi = \varphi (- \ln \tilde{v}) \) is a
plurisubharmonic exhaustion function of $X\setminus L$, which is strongly plurisubharmonic outside a compact subset of $X\setminus L$.

In the sequel, $L$ will be a compact $C^\infty$ Levi flat hypersurface in a compact Kähler manifold $X$ of dimension $n \geq 2$, verifying the hypothesis and the conclusions of Proposition 1. We denote $X^\pm$ the connected components of $\{z \in X : v > 0\}$ endowed with a complete Kähler metric $\tilde{\omega}$ which will be defined later and we set

$$D_{(p,q)}(X^\pm) = \left\{ f \in C^\infty_{(p,q)}(X^\pm) : \text{supp } f \subset X^\pm \right\}$$

and

$$H_{(p,q)}(X^\pm, \tilde{\omega}, \tilde{\omega}) = \ker \tilde{\partial} \cap \ker \tilde{\partial}^* \subset L^2_{(p,q)}(X^\pm, \tilde{\omega}, \tilde{\omega})$$

where $\tilde{\partial}^*$ is the Hilbert space adjoint of the operator $\tilde{\partial} : L^2_{(p,q)}(X^\pm, \tilde{\omega}, \tilde{\omega}) \to L^2_{(p,q+1)}(X^\pm, \tilde{\omega}, \tilde{\omega})$.

**Proposition 2.** For every $\alpha > 0$, there exists a complete Kähler metric $\tilde{\omega}$ on $X\setminus L$, $\omega \leq \tilde{\omega} \leq \frac{C}{\alpha} \omega$, $C > 0$, such that the range $R^p_{(n,q)}(X^\pm)$ of the operator $\tilde{\partial} : L^2_{(n,q-1)}(X^\pm, \tilde{\omega}, \tilde{\omega}) \to L^2_{(n,q)}(X^\pm, \tilde{\omega}, \tilde{\omega})$ is closed for $1 \leq q \leq n$.

**Proof.** The proof is based on methods of [16] (see also [14]).

Denote by $\omega$ the Kähler metric of $X$. Since $i \partial \bar{\partial} (-\ln v) \geq c \omega$ on $U\setminus L$, $c > 0$, by a method developed in [27] it follows that there exists a neighborhood $V$ of $L$ and $\eta > 0$ such that $-v^\eta$ is strongly plurisubharmonic on $V\setminus L$. Then for $0 < \beta < \eta$, we have the Donnelly-Fefferman estimate [17]

$$i \bar{\partial} (-\ln v) \wedge \partial (-\ln v) \leq i r \partial \bar{\partial} (-\ln v).$$

on $V\setminus L$, with $0 < r = \beta/\eta < 1$. This is equivalent to say that the norm of $\partial (-\ln v)$ measured in the metric $i \partial \bar{\partial} (-\ln v)$ is smaller than $r$ on $V\setminus L$ (see also [6] and [19]).

Let $\alpha > 0$. We consider the trivial line bundle $E$ on $X\setminus L$ endowed with the Hermitian metric $h_{\alpha} = e^{\alpha \ln v}$. Set

$$\tilde{\omega} = i \partial \bar{\partial} (E) + K \omega = i \alpha \partial \bar{\partial} (-\ln v) + K \omega$$

with $K$ a positive constant. Since $-\ln v$ is an exhaustion function on $X\setminus L$, it follows by (4.1) that for $K$ big enough $\tilde{\omega}$ is a complete Kähler metric on $X\setminus L$ such that $\omega \leq \tilde{\omega} \leq \frac{c}{\alpha} \omega$, $C > 0$.

Denote $\lambda_j$ (respectively $\tilde{\lambda}_j$) the eigenvalues of $i \Theta (E)$ with respect to $\omega$ (respectively $\tilde{\omega}$), $1 \leq j \leq n$, in increasing order. By Proposition 1, there exists $c > 0$ such that $i \Theta (E) = i \alpha \partial \bar{\partial} (-\ln v) \geq c \omega$ on $\{\psi > b\}$ for $b$ big enough. So, as in [16] (1.3) we have

$$1 \geq \tilde{\lambda}_j = \frac{\lambda_j}{\alpha c + K} \geq \frac{\alpha c}{\alpha c + K} > 0, \ 1 \leq j \leq n$$

on $\{\psi > b\}$. By Bochner-Kodaira-Nakano inequality (see for ex. [14]) we have

$$N_{\alpha, \tilde{\omega}, \tilde{\omega}} (\tilde{\partial} u)^2 + N_{\alpha, \tilde{\omega}, \tilde{\omega}} (\tilde{\partial}^* u)^2 \geq \int_{X^\pm} \left( \left[ i \Theta (E), \Lambda_{\tilde{\omega}} \right] u, u \right)_{\alpha, \tilde{\omega}, \tilde{\omega}} dV_{\tilde{\omega}}$$

for every $u \in D_{(n,q)}(X\setminus L)$, where $N_{\alpha, \tilde{\omega}, \tilde{\omega}} = \int_{X^\pm} |u|^2_{\tilde{\omega}} \tilde{\omega} dV_{\tilde{\omega}}$.

Let $\chi$ be a smooth function on $X$ such that $0 \leq \chi \leq 1$, $\chi = 0$ on a neighborhood of $\{\psi < b\}$ and $\chi = 1$ on a neighborhood $\{\psi > b'\}$ of $L$, $b' > b$. By (4.3) and (4.2),
for every \( u \in D_{(n,q)} (X\backslash L) \) we have

\[
N_{\alpha,\tilde{\omega}} \left( \overline{\partial} (\chi u) \right)^2 + N_{\alpha,\tilde{\omega}} \left( \overline{\partial}_\alpha^* (\chi u) \right)^2 \geq \int_{X^\pm} \langle [i\Theta (E), \Lambda_{\tilde{\omega}}] \rangle \chi u, \chi u \rangle_{\alpha,\tilde{\omega},\tilde{\beta}} dV_{\tilde{\omega}}
\]

\[
\geq \int_{\{ \psi < b' \}} \langle [i\Theta (E), \Lambda_{\tilde{\omega}}] \rangle \chi u, \chi u \rangle_{\alpha,\tilde{\omega},\tilde{\beta}} dV_{\tilde{\omega}}
\]

\[
\geq \int_{\{ \psi > b' \}} (\lambda_1 + \cdots + \lambda_n) |\chi u|_{\tilde{\omega}}^2 \overline{\partial}^* dV_{\tilde{\omega}}
\]

so there exists \( C, c' > 0 \) such that

\[
2 N_{\alpha,\tilde{\omega}} \left( \overline{\partial}u \right)^2 + 2 N_{\alpha,\tilde{\omega}} \left( \overline{\partial}_\alpha^* u \right)^2 + C \int_{\text{supp}(\chi')} |u|_{\tilde{\omega}}^2 \overline{\partial}^* dV_{\tilde{\omega}}
\]

\[
\geq c' \int_{X\backslash L} |u|_{\tilde{\omega}}^2 \overline{\partial}^* dV_{\tilde{\omega}} - c' \int_{\{ \psi < b' \}} |u|_{\tilde{\omega}}^2 \overline{\partial}^* dV_{\tilde{\omega}}.
\]

Finally it follows that there exists a compact subset \( F = \text{supp} (\chi') \cup \{ \psi \leq b' \} \) of \( X^\pm \) such that for every \( u \in D_{(n,q)} (X\backslash L) \)

\[
(4.4) \quad c' N_{\alpha,\tilde{\omega}}(u)^2 \leq 2 N_{\alpha,\tilde{\omega}} \left( \overline{\partial}u \right)^2 + 2 N_{\alpha,\tilde{\omega}} \left( \overline{\partial}_\alpha^* u \right)^2 + (C + c') \int_F |u|_{\tilde{\omega}}^2 \overline{\partial}^* dV_{\tilde{\omega}}.
\]

Since \( \tilde{\omega} \) is a complete metric on \( X\backslash L \), (4.4) is valid for every \( u \in (\text{Dom} \overline{\partial}) \cap \left( \text{Dom} \overline{\partial}_\alpha^* \right) \). The conclusion of Proposition 2 is now a consequence of Proposition 1.2 of [23]. \( \Box \)

**Corollary 1.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have \( \mathcal{H}_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) = \{0\} \).

**Proof.** As \((X^\pm, \tilde{\omega})\) is a connected weakly 1-complete Kähler manifold and the bundle \( E \) defined in the proof of Theorem 2 is a semi-positive line bundle on \( X^\pm \) which is positive outside a compact subset of \( X^\pm \), the Corollary 1 is a consequence of [33], Corollary of the Main Theorem (see also [3], [30] and [26], Corollary 2.10). \( \Box \)

By taking in account Corollary 1, a classical application of Proposition 2 (see for example [18]) is the following:

**Corollary 2.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have:

1. There exists the \( \overline{\partial} \)-Neumann operator \( \mathcal{N}_{(n,q)}^\alpha : L^2_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) \rightarrow L^2_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) \) such that for every \( f \in L^2_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) \) we have the orthogonal decomposition

\[
f = \overline{\partial}_\alpha^* \mathcal{N}_{(n,q)}^\alpha f + \overline{\partial}_\alpha \mathcal{N}_{(n,q)} \mathcal{N}_{(n,q)}^\alpha f \quad \text{and} \quad \overline{\partial} \mathcal{N}_{(n,q)}^\alpha = \mathcal{N}_{(n,q+1)} \overline{\partial}, \quad \overline{\partial}^* \mathcal{N}_{(n,q)}^\alpha = \mathcal{N}_{(n,q-1)} \overline{\partial}_\alpha^*.
\]

2. For every \( \overline{\partial} \)-closed form \( f \in L^2_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) \), \( \overline{\partial} \left( \overline{\partial}_\alpha^* \mathcal{N}_{(n,q)}^\alpha f \right) = f \).

**Lemma 1.** Let \( f \in C^\infty_{(0,q)} (X) \), \( 1 \leq q \leq n - 1 \), be a \( \overline{\partial} \)-closed form such that \( f \) vanishes to infinite order on \( L \). Let \( \psi_1, \psi_2 \in \text{Dom} \overline{\partial} \subset L^2_{(n,q)} (X^\pm, \overline{\partial}^\alpha, \tilde{\omega}) \) such that \( \overline{\partial} \psi_1 = \overline{\partial} \psi_2 \). Then

\[
\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = 0.
\]
Proposition 3. Let $f \in C^\infty_{\overline{0},q}(X)$, $1 \leq q \leq n-1$, be a $\overline{\partial}$-exact form such that $f$ vanishes to infinite order on $L$. Then for every $\alpha > 0$, there exists $u \in L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})$ such that $\overline{\partial}u = f$ and $N_{-\alpha,\overline{\omega},\overline{\nu}}(u) \leq C_n N_{-\alpha,\overline{\omega},\overline{\nu}}(f)$, with $C_{\alpha} > 0$ independent of $f$.

Proof. Step 1. Definition by duality of $u \in L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})$, $1 \leq q \leq n-1$. The proof of this point is inspired from [19], Proposition 5.3. By Proposition 2, $R^\alpha_{(n,q)}(X^\pm)$ is closed for every $\alpha > 0$ and by Corollary 2 we can find a bounded operator $T^\alpha_{(n,q)} = \overline{\partial}^\alpha R^\alpha_{(n,q)} : R^\alpha_{(n,q)}(X^\pm) \to L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})$, such that $\overline{\partial}T^\alpha_{(n,q)} \varphi = \varphi$ for every $\varphi \in R^\alpha_{(n,q)}(X^\pm)$, $1 \leq q \leq n$.

Define now the continuous linear form $\Phi_f$ on $R^\alpha_{(n,n-q+1)}(X^\pm)$, $1 \leq q \leq n$, by

$$\Phi_f(\varphi) = \int_{X^\pm} f \wedge T^\alpha_{(n,n-q+1)} \varphi, \varphi \in R^\alpha_{(n,n-q+1)}(X^\pm).$$

By the Hahn-Banach theorem, we extend $\Phi_f$ as a linear form $\tilde{\Phi}_f$ on $L^2_{(n,n-q+1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})$ such that $\|\tilde{\Phi}_f\| = \|\Phi_f\|$. Since $\left(L^2_{(n,n-q+1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})\right)^\prime = L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{\alpha}, \overline{\omega})$

by the pairing

$$\langle \beta_1, \beta_2 \rangle = \int_{X^\pm} \beta_1 \wedge \beta_2, \beta_1 \in L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega}), \beta_2 \in L^2_{(n,n-q+1)}(X^\pm, \overline{\partial}^{\alpha}, \overline{\omega}),$$

there exists $u \in L^2_{(0,q-1)}(X^\pm, \overline{\partial}^{-\alpha}, \overline{\omega})$ such that

$$\tilde{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \varphi$$

for every $\varphi \in R^\alpha_{(n,n-q+1)}(X^\pm)$.

Step 2. We prove that $\overline{\partial}(-1)^q u = f$, $1 \leq q \leq n-1$.

Let $\varphi = \overline{\partial} \psi \in C^\infty_{(n,n+1)}(X^\pm)$ with $\psi \in D_{(n,n-q)}(X^\pm)$. Set $g_n = \overline{\partial}^n N^\alpha_{(n,n-q+1)} \overline{\partial} \psi \in L^2_{(n,n-q)}(X^\pm, \overline{\partial}^{\alpha}, \overline{\omega})$. By Corollary 2, $\overline{\partial}g_n = \varphi$ and by Lemma 1

$$\int_{X^\pm} f \wedge g_n = \int_{X^\pm} f \wedge \psi.$$  

But by step 1 we have

$$\tilde{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \overline{\partial} \psi = \Phi_f(\varphi) = \int_{X^\pm} f \wedge g_n$$

and by (4.5) and (4.6) it follows that

$$\int_{X^\pm} f \wedge \psi = \int_{X^\pm} u \wedge \overline{\partial} \psi$$

for every $\psi \in D_{(n,n-q)}(X^\pm)$. Therefore $\overline{\partial}(-1)^q u = f$ and the Proposition is proved. $\blacksquare$
Remark 2. Since $\omega \leq \overline{\omega} \leq \frac{C}{\sqrt{n}} \omega$, by Lemma VIII.6.3 of [14] it follows that:

a) Let $f$ be a smooth $(n, q)$-form on $X$ such that $f$ vanishes to order $k$ on $L$. Then $f \in L^2_{(n, q)}(X^\pm, \overline{\nu}^{-k}, \overline{\omega})$

Indeed

$$\int_{X^\pm} |f|^2 \overline{\nu}^{-k} dV_\omega \leq \int_{X^\pm} |f|^2 \overline{\nu}^{-k} dV_\omega < \infty$$

b) Let $f \in L^2_{(n, q)}(X^\pm, \overline{\nu}^{-k}, \overline{\omega})$, $k > 2$. Then $f \in L^2_{(n, q)}(X^\pm, \overline{\nu}^{-k+2}, \omega)$.

Indeed

$$\int_{X^\pm} |f|^2 \overline{\nu}^{-k+2} dV_\omega \leq C \int_{X^\pm} |f|^2 \overline{\nu}^{-k} dV_\omega < \infty, \ C > 0.$$

5. Nonexistence of Levi flat hypersurfaces

Proposition 4. Let $L$ be a compact $C^\infty$ Levi flat hypersurface in a Kähler manifold $X$ of dimension $n \geq 3$ such that the normal bundle $N_{L,0}^{1,0}$ to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature. Let $u \in C_{(0,q)}^\infty(L)$, $1 \leq q \leq n-2$, such that $\overline{\partial} u = 0$. Then for every $k \in \mathbb{N}^*$ there exist a $\overline{\partial}$-closed extension $U_k \in C_k^{(0,q)}(X)$ of $u$.

Proof. By Proposition 1 there exist a neighborhood $U$ of $L$, $c > 0$ and a non-negative function $v \in C^2(U)$ on $L$ such that $v = g \delta^2_L$ and $-i\partial \overline{\partial} \ln v \geq c \omega$ on $U \setminus L$. Let $u \in C_{(0,q)}^\infty(X)$ be an extension of $u$ such that $\overline{\partial} u$ vanishes to infinite order on $L$. Since $\overline{\partial} u \in L^2_{(0,q+1)}(X^\pm, \delta^{2k}_L, \omega)$, $q+1 \leq n-1$ and $L^2_{(0,q)}(X^\pm, \delta^{2k}_L, \omega) = L^2_{(0,q)}(X^\pm, \overline{\nu}^{-k}, \omega)$ for every $k \in \mathbb{N}$, by Remark 2 a) and Proposition 3 it follows that for every $k \in \mathbb{N}^*$ there exist a Hermitian complete metric $\overline{\omega}$ on $X \setminus L$, $\omega \leq \overline{\omega} \leq \frac{C}{\sqrt{n}} \omega$ and $h^\pm \in L^2_{(0,q)}(X^\pm, \delta^{2k}_L, \overline{\omega})$ such that $\overline{\partial} h^\pm = \overline{\partial} u$ on $X^\pm$. By Remark 2 b) we have $h^\pm \in L^2_{(0,q)}(X^\pm, \delta^{2k+4}_L, \omega)$. So by using Theorem 1, for $k$ big enough we can choose $h^\pm \in C^{(0,k)}_{(0,q)}(X^\pm)$, $s(k) \sim \sqrt{k}$. This means that for $k$ big enough, the form $h$ defined as $h^\pm$ on $X^\pm$ is of class $C^k$ on $X$ and vanishes on $L$. So $U_k = \overline{u} - h^\pm$ is a $C^k$-smooth $\overline{\partial}$-closed form on $X$ which is an extension of $u$. ♦

Theorem 2. Let $X$ be a compact connected Kähler manifold of dimension $n \geq 3$ and $L$ a $C^\infty$ compact Levi flat hypersurface. Then the normal bundle to the Levi foliation does not admit any Hermitian metric of class $C^2$ with leafwise positive curvature.

Proof. Suppose that the normal bundle $\mathcal{N}$ to the Levi foliation admits a Hermitian metric of class $C^2$ with leafwise positive curvature. Since $\mathcal{N}$ is topologically trivial, its curvature form $\Theta^\mathcal{N}$ for the Kähler metric of $X$ is $d$-exact. So there exists a 1-form $u$ of class $C^\infty$ on $L$ such that $du = \Theta^\mathcal{N}$; we may suppose that $u$ is real and $u = u^{0,1} + u^{0,\overline{1}}$, where $u^{0,1}$ is the $(0, 1)$ component of $u$. Since $\Theta^\mathcal{N}$ is a $(1, 1)$-form, it follows that $\overline{\partial} u^{0,1} = 0$, where $\overline{\partial}$ is the tangential Cauchy-Riemann operator. By Proposition 4 there exists a $C^k$-extension $U^{0,1}$ of $u^{0,1}$ to $X$, $k \geq 2$, such that $\overline{\partial} U^{0,1} = 0$.

By Hodge symmetry and Dolbeault isomorphism $H^{0,1}(X, C) \cong H^{1,0}(X, \overline{\omega}) \cong H^0(X, \Omega^1_X)$, where $\Omega^1_X$ is the sheaf of holomorphic 1-forms on $X$. So there exists $\eta \in H^0(X, \Omega^1_X)$ and $\Phi \in C^k(X)$ such that $U^{0,1} = \eta + \overline{\partial}\Phi$. It follows that $\Theta^\mathcal{N}$ =
Remark 3. A first version of this paper was announced on arXiv in 2014, but there was a gap in the proofs of §4, which is now corrected. Recently, Brinkschulte proved a generalization of Theorem 2 for compact Levi flat hypersurfaces in complex manifolds (see Theorem 1.1. of [8]). She uses crucially the Proposition 4.1 of [8], whose statement and proof are the same as Proposition 1 of this paper and which are unchanged from 2014 in our preprint arXiv:1406.5712. However she refers only to Proposition 1.1 of [25], where the lower positive bound for the eigenvalues of the strongly plurisubharmonic function is not mentioned.

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References

[1] A. Andreotti and E. Vesentini, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, Publ. Math. IHES 24-25 (1965), 81–150.
[2] A. Andreotti and E. Vesentini, Sopra un teorema di Kodaira, Ann. Scuola Norm. Sup. Pisa 15 (1961), no. 4, 283–309.
[3] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. 36 (1957), 235–249.
[4] D. E. Barrett and J. E. Fornaess, On the smoothness of Levi-foliations, Publ. Mat. 2 (1988), 171–177.
[5] I. Bendixson, Sur les courbes définies par une équation différentielle, Acta Mathematica 24 (1901), 1–88.
[6] B. Berndtsson and Ph. Charpentier, A Sobolev mapping property of the Bergman kernel, Math. Z. 235 (2000), 1–10.
[7] J. Brinkschulte, The \( \overline{\partial} \)-problem with support conditions on some weakly pseudoconvex domains, Ark. Mat. 42 (2004), 259–282.
[8] M. Brunella, On the dynamics of codimension one holomorphic foliations with ample normal bundle, Indiana Univ. Math. J. 57 (2008), 3101–3113.
[9] M. Brunella and C. Perrone, Exceptional singularities of codimension one holomorphic foliations, Publ. Mat. 55 (2011), 295–312.
[10] C. Camacho, A. Lins Neto and P. Sad, Minimal sets of foliations in complex projective space, Publ. Math. de l’I.H.E.S. 68 (1988), 187–203.
[11] D. Cerveau, Minimales des feuilletages algébriques de \( \mathbb{C}^n \), Ann. Inst. Fourier 43 (1993), 1535–1543.
[12] J.-P. Demailly, Complex Analytic Geometry and Differential Geometry, http://www-fourier.ujf-grenoble.fr/~demailly/books.html.
[13] H. Donnelly and C. Fefferman, \( L^2 \)-cohomology and index theorem for the Bergman metric, Ann. of Math. 118 (1983), no. 2, 593–618.
[14] G. B. Folland and J. J. Kohn, The Neumann problem for the Cauchy-Riemann complex, no. 75, Princeton Univ. Press, Princeton, N. J., 1972.
[15] G. M. Henkin and A. Iordan, Regularity of \( \overline{\partial} \) on pseudoconcave compact and applications, Asian J. Math. 4 (2000), no. 4, 855–884, and Erratum to : Regularity of \( \overline{\partial} \) on pseudoconcave compact and applications by G. M. Henkin and A. Iordan, Asian J. Math., 4, 855-884, 2000.
[20] L. Hörmander, \textit{L}^2 \textit{estimates and existence theorems for the } \overline{\partial} \textit{operator}, Acta Math. 113 (1965), 89–152.

[21] A. Iordan and F. Matthey, \textit{Régularité de l’opérateur } \overline{\partial} \textit{et théorème de Siu sur la nonexistence d’hypersurfaces Levi-plates dans l’espace projectif complexe } \mathbb{CP}^n, n \geq 3, C. R. Acad. Sc. Paris 346 (2008), 395–400.

[22] A. Lins Neto, \textit{A note on projective Levi flats and minimal sets of algebraic foliations}, Ann. Inst. Fourier 49 (1999), 1369–1385.

[23] T. Ohsawa, \textit{Isomorphism theorems for cohomology groups of weakly 1-complete manifolds}, Publ. RIMS, Kyoto Univ. 18 (1982), 191–232.

[24] \textit{On the complement of Levi flats in Kähler manifolds of dimension } \geq 3, \textit{Nagoya Math. J.} 185 (2007), 161–169.

[25] \textit{Nonexistence of certain Levi flat hypersurfaces in Kähler manifolds from the viewpoint of positive normal bundles}, Publ. RIMS Kyoto Univ. 49 (2013), 229–239.

[26] \textit{L}^2 \textit{approaches in Several Complex Variables}, Springer, 2015.

[27] T. Ohsawa and N. Sibony, \textit{Bounded P.S.H. functions and pseudoconvexity in a Kähler manifold}, Nagoya Math. J. 149 (1998), 1–8.

[28] \textit{Mémoire sur les courbes définies par une équation différentielle}, Journal de Math. Pures et Appl. 7 (1881), 375–422.

[29] \textit{Mémoire sur les courbes définies par une équation différentielle}, Journal de Math. Pures et Appl. 8 (1882), 251–296.

[30] O. Riemenschneider, \textit{Characterizing Moishezon spaces by almost positive coherent analytic sheaves}, Math. Z. 123 (1971), 263–284.

[31] Y.-T. Siu, \textit{Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension } \geq 3, \textit{Ann. of Math.} 151 (2000), 1217–1243.

[32] \textit{\overline{\partial}} \textit{-regularity for weakly pseudoconvex domains in compact Hermitian symmetric spaces with respect to invariant metrics}, Ann. of Math. 156 (2002), 595–621.

[33] K. Takegoshi, \textit{A generalization of vanishing theorems for weakly 1-complete manifolds}, Publ. RIMS Kyoto Univ 17 (1981), 311–330.

[34] A. Takeuchi, \textit{Domaines pseudoconvexes sur les variétés kählériennes}, J. Math. Kyoto Univ. 6 (1967), 323–357.