Computing Vertex-Weighted Multi-Level Steiner Trees

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Abstract

In the classical vertex-weighted Steiner tree problem (VST), one is given an undirected graph $G = (V, E)$ with nonnegative vertex weights, and a set $T \subseteq V$ of terminals. The objective is to compute a minimum-weight tree that spans $T$. The VST problem is NP-hard and it is NP-hard to approximate VST to within factor $(1-\varepsilon)\ln |T|$, but nearly-best approximation algorithms exist including the $2\ln |T|$-approximation algorithm of [Klein & Ravi, 1995].

Steiner tree problems and their variants have many applications, from combinatorial optimization and network routing to geometry and visualization. In some applications, the terminals may have different levels, priorities, or rate-of-service requirements. For problems of this type, we study a natural generalization of the VST problem to multiple levels, referred to as the vertex-weighted, multi-level Steiner tree (V-MLST) problem: given a vertex-weighted graph $G = (V, E)$ and $\ell \geq 2$ nested terminal sets $T_\ell \subset T_{\ell-1} \subset \cdots \subset T_1 \subseteq V$, compute a nested set of trees $G_\ell \subseteq G_{\ell-1} \subseteq \cdots \subseteq G_1$ where each tree $G_i$ spans its corresponding terminal set $T_i$, with minimum total weight.

We introduce a simple heuristic with approximation ratio $O(\ell \ln |T_1|)$, which runs in a top-down manner using a single-level VST subroutine. We then show that the V-MLST problem can be approximated to within $2 \ln |T_1|$ of the optimum with a greedy algorithm that connects “level-respecting trees” at each iteration with a minimum cost-to-connectivity ratio. This result is counterintuitive as it suggests that the seemingly harder multi-level version is not indeed harder than the single-level VST problem to approximate. The key tool in the analysis of our greedy approximation algorithm is a new “tailed spider decomposition.” We also provide an integer linear programming (ILP) formulation for the V-MLST problem and show that the V-MLST problem is a special case of the directed Steiner tree problem.

Keywords: Approximation algorithms; vertex-weighted Steiner tree; multi-level graph representation; spider decomposition
1 Introduction

Let \( G = (V, E) \) be an undirected, connected graph, and let \( T \subseteq V \) be a set of vertices, called terminals. A Steiner tree is a subtree of \( G \) that contains all vertices in \( T \). In the classical Steiner tree (ST) problem, each edge of the graph \( G \) has a positive weight, and the objective is to find a minimum-weight edge set \( E' \subseteq E \) that spans the terminals. ST is one of Karp’s initial NP-hard problems \[17\]. The ST problem is APX-hard \[3\] and cannot be approximated within a factor of 96/95 unless \( P = NP \) \[10\]. However, the ST problem admits a simple 2-approximation \[13\], by computing a minimum spanning tree on the metric closure of \( G \). The LP-based approximation algorithm of Byrka et al. \[5\] guarantees a ratio of \( \ln(4) + \varepsilon < 1.39 \). More details about the ST problem and its variants can be found in a survey \[30\], an online compendium \[16\], and a textbook \[27\]. More general edge-weighted graph design problems are studied in \[14\].

In the vertex-weighted (or node-weighted) Steiner tree problem, which we refer to as VST, it is the vertices of the graph that have positive weights (rather than the edges). Specifically, given a graph \( G = (V, E) \) with vertex weights and a set \( T \subseteq V \) of terminals, the VST problem is to compute a Steiner tree spanning \( T \) such that the sum of vertex weights is minimized.

Finding a minimum-weight VST is provably harder than ST, and cannot be approximated within a factor of \((1 - \varepsilon) \ln |T|\) unless \( P = NP \) \[12\], via a simple reduction from the set cover problem \[19\]. The VST problem generalizes the edge-weighted Steiner tree problem; an instance of ST can be reduced to an instance of VST by replacing each edge \( uv \) with weight \( w(u, v) \) with two edges \( ux \) and \( xv \), where \( w(u) = w(v) = 0 \) and \( w(x) = w(u, v) \) \[13\]. The VST problem occurs naturally in applications, including network design problems, where it is important to consider vertex costs (e.g., in telecommunication networks where installing routers/facilities is associated with different costs). There are nearly-best-possible approximation algorithms for the VST problem, such as the greedy \( 2 \ln |T| \)-approximation algorithm by Klein and Ravi \[19\].

In many applications, the terminals may have different levels, priorities, or rate of service requirements \[1, 2, 22, 8, 11, 21\]. For example, when connecting cities with a network, larger cities might require higher-grade facilities than smaller ones. Such multi-level graph problems have practical applications in network design and and graph visualization. The multi-level Steiner tree (MLST) problem is one such generalization of the edge-weighted ST problem \[11\]: given a graph \( G = (V, E) \) with positive edge weights, and \( \ell \geq 2 \) nested terminal sets \( T_\ell \subseteq T_{\ell-1} \subseteq \ldots \subseteq T_1 \subseteq V \), compute a sequence of trees \( \mathcal{G} = (G_1, \ldots, G_\ell) \) such that the spanning trees are nested (i.e., \( E_\ell \subseteq E_{\ell-1} \subseteq \ldots \subseteq E_1 \)), \( G_i \) spans its terminals \( T_i \), and the sum of the edge weights across all levels is minimized. In this paper, we extend the definition of the MLST problem to the vertex-weighted setting.

As all terminals must be connected in any solution of the VST problem, it is conventional to assume that terminals have weight zero. Indeed, one can construct an equivalent instance whose terminals have weight zero: for each terminal \( t \in T \), add edge \( tt' \), set \( w(t') = 0 \), and set \( t' \) to be a terminal instead of \( t \) \[15\]. With this in mind, we define a multi-level generalization of the VST problem, as follows:

**Definition 1.1. (Vertex-Weighted Multi-Level Steiner Tree (V-MLST) Problem)** Given an undirected, connected graph \( G = (V, E) \) with vertex weights \( w : V \rightarrow \mathbb{R}^+ \), and \( \ell \geq 2 \) nested terminal sets \( T_\ell \subseteq T_{\ell-1} \subseteq \ldots \subseteq T_1 \subseteq V \), the V-MLST problem is to compute a sequence of \( \ell \) nested trees \( \mathcal{G} = (G_1, \ldots, G_\ell) \) such that the sum of the weights of the Steiner vertices (non-terminals) across all levels, \( W = \sum_{i=1}^{\ell} w(G_i) \), is minimized, where

\[
 w(G_i) = \sum_{v \in V_i \setminus T_i} w(v). \tag{1.1}
\]

Figure \[1\] shows an instance of the V-MLST problem and an optimal solution. As in the VST problem, we wish to only include the weights of non-terminals. In the above definition of cost (Eq. 1.1), this is done by including \( w(v) \) only if \( v \) is present as a non-terminal.
An equivalent definition of the V-MLST problem is that each vertex \( v \) has a rate of service (demand) \( R : V \mapsto \{0, 1, 2, \ldots, \ell\} \), where \( R(v) = i \) if \( v \in T_i \) and \( v \notin T_{i+1} \), and \( R(v) = 0 \) if \( v \) is a non-terminal. The problem is equivalent to computing a single tree with edge rates such that for all \( u, v \in V \) with rate at least 1, the path from \( u \) to \( v \) uses edges of rates \( \min(R(u), R(v)) \) or higher. Given a solution, if \( y_v \) denotes the highest level that \( v \) is a vertex in, then the weight of a V-MLST is equivalently defined as

\[
W = \sum_{v \in V} (y_v - R(v))w(v)
\]

In Figure 1, for example, the vertex \( v \) with weight 3 has \( y_v = 3 \) and \( R(v) = 2 \), so the weight incurred by including \( v \) is \((3 - 2) \times 3 = 3\). The weight of the optimal solution is \((3 - 0) \cdot 1 + (3 - 2) \cdot 3 = 6\).

If we had defined \( W \) as the sum of all vertex weights on all levels, \( W = \sum_{i=1}^{\ell} \sum_{v \in V_i} w(v) = \sum_{v \in V} y_v w(v) \), the optimal solution is the same, as the weight of any solution under the two definitions of cost differ by a constant, namely \( \sum_{i=1}^{\ell} \sum_{v \in T_i} w(v) \). The definition of the weight of a solution (Eq. 1.1) is more natural in the context of approximation algorithms, as it does not include “required” weights of terminals.

### 1.1 Related Work

Klein and Ravi [19] give a greedy \( 2 \ln |T| \)-approximation to the VST problem. Their analysis relies on a spider decomposition. Guha and Khuller [15] improve the approximation ratio to \( 1.5 \ln |T| \) via minimum-weight 3+ branch spiders. Variants of the VST problem have also been studied, including bi-criteria constraints [23], buy-at-bulk [9], multi-commodity [20], \( k \)-connectivity [25], and degree constraints [28].

Several results are known on multi-level or rate-of-service Steiner tree problems, where edges are weighted. Balakrishnan et al. [2] gave a \( 4/3 \rho \)-approximation algorithm for the 2-level multi-level network design (MLND) problem with proportional edge costs, where \( \rho \) is the best known approximation ratio for the edge-weighted Steiner tree problem, currently \( \rho = \ln 4 + \varepsilon < 1.39 \) by Byrka et al. [4]. In the QoS Multicast Tree problem [8], one is given a graph, a source vertex \( s \), and a level between 1 and \( k \) for each terminal (1 for highest priority). The task is to find a minimum-cost Steiner tree that connects all terminals to \( s \). The level of an edge \( e \) in this tree is the minimum over the levels of the terminals that are connected to \( s \) via \( e \). Charikar et al. [8] describe a simple \( 4 \rho \)-approximation for the rate model, with proportional edge costs, which is improved to \( \varepsilon \rho \) through randomized doubling. Karpinski et al. [18] used an iterative contraction scheme to propose a \( 2.454 \rho \)-approximation. Ahmed et al. [11] further improved the approximation ratio for MLST problem by using composition of partial top-down and bottom-up methods and taking the solution with minimum cost. They show that this approach yields a roughly \( 2.351 \rho \)-approximation for \( \ell \leq 100 \).

The VST problem is a special case of the directed Steiner tree problem (or Steiner arborescence), which is the following: given a directed edge-weighted graph \( G = (V, E) \), set of terminals \( T \subseteq V \), and a root vertex \( r \in V \), find a minimum-weight directed tree rooted at \( r \), such that there is a directed path from \( r \) to each \( v \in T \). Segev [29] gives a simple approximation-preserving reduction from VST to DST, where each edge \( uv \in E \) is replaced with two directed edges \((u, v)\) and \((v, u)\). The weight of edge \((u, v)\) equals...
the weight of its incoming vertex, \( w(v) \). The root \( r \) is any terminal in \( T \). The \( \ell \)-level V-MLST problem is also a special case of DST; the reduction is shown in Section [5]. The DST problem with \( k \) terminals can be approximated with ratio \( i(i-1)k^{1/i} \) in time \( O(n^i k^{2i}) \) for fixed \( i \geq 1 \) [7] using a recursive greedy approach, which implies a polynomial time \( O(k^\varepsilon) \)-approximation for fixed \( \varepsilon > 0 \). By setting \( i = \log k \), DST can be \( O(\log^2 k) \)-approximated in quasi-polynomial time. As the reduced instance of DST has \( k = |T_i| \) terminals, this immediately implies a quasi-polynomial \( O(\log^2 |T_1|) \)-approximation to the V-MLST problem.

A somewhat related problem is the online node-weighted Steiner tree problem, in which the terminals \( T \) arrive in an online manner. At any stage, a subgraph must connect all terminals that have arrived thus far. Naor et al. [24] describe a randomized \( O(\log |V| \log^2 |T|) \)-approximation algorithm to the online problem. Note that the approximation ratio of the online algorithm is worse than that of the DST formulation.

1.2 Our Contributions. While edge-weighted, multi-level ST problems have been studied under various names, including Multi-Level Network Design (MLND) [2], Multi-Tier Tree (MTT) [22], Quality-of-Service (QoS) Multicast Tree [8], and Priority-Steiner Tree [11], to the best of our knowledge, the vertex-weighted analogue has not been studied yet.

We give a vertex-weighted analogue of the top-down and bottom-up approaches given in [1]. We show that, under the definition of the weight of a V-MLST solution (Eq. (1.1)), the top-down approach (Section 2.1) is an \( \ell \rho \)-approximation, where \( \rho = O(\ln |T_1|) \) (e.g., \( \rho = 2 \ln |T_1| \) using the Klein-Ravi greedy heuristic [19]). However, the bottom-up approach can perform arbitrarily badly.

Our main result is a \( (2 \ln |T_1|) \)-approximation algorithm for the V-MLST problem (Section 3) that relies on a generalization of the method by Klein and Ravi [19]. This result is surprising, as it suggests the seemingly harder multi-level problem can be approximated equally well as the single level problem.

Our greedy approximation algorithm GREEDYVMLST relies on growing a set of level-respecting trees and carefully merging a subset of these trees so that the newly-formed tree is also level-respecting. The main tool of our analysis is based upon the existence of a “tailed spider decomposition.” Similar graph decompositions are often used in graph design problems [19, 15, 6, 20, 23, 28, 26], but to the best of our knowledge, the tailed spider decomposition is novel, and we expect further applications in other multi-level graph design problems.

Finally, we provide an integer linear programming (ILP) formulation for the V-MLST problem and show that the V-MLST problem is a special case of directed Steiner tree problem (or Steiner arborescence).

Notation. Throughout, we refer to an undirected edge connecting vertices \( u \) and \( v \) as \( uv \) and to a directed edge from \( u \) to \( v \) as \( (u, v) \). Let \( R(v) \) denote the rate (demand) of \( v \), i.e., the highest terminal set that \( v \) appears in; in other words, \( R(v) = i \) if \( v \in T_i \) and \( v \notin T_{i+1} \), and \( R(v) = 0 \) if \( v \) is a non-terminal.

For \( 1 \leq i \leq \ell \), denote by \( \text{MIN}_i \) the weight of a minimum VST over terminals \( T_i \), regardless of other levels, with the convention that terminals have zero weight. For example, in Figure [1] \( \text{MIN}_1 = 0 \), \( \text{MIN}_2 = 1 \), and \( \text{MIN}_3 = 4 \). Let \( G^* \) denote an optimal solution to the V-MLST problem; note that \( G^* \) may be specified via a nested sequence of trees, or a single tree \( G^* \) with edge rates. Denote by OPT the weight of \( G^* \); for example, \( \text{OPT} = 6 \) in Figure [1].

2 Top-Down and Bottom-Up Approaches for V-MLST

Consider the following naïve approach to the V-MLST problem: for each level \( i \), compute a VST over \( T_i \) independently of other levels in order to compute \( \text{MIN}_i \). The collection of these Steiner trees is not necessarily a feasible V-MLST solution because the edge sets may not be nested. The top-down and bottom-up approaches can be used to ensure a feasible solution is returned.
2.1 Top-down Approach. Assume an oracle can compute the optimal (single-level) VST for an input graph \( G \) over terminals \( T \) with vertex weights \( w: E' = VST(G, w, T) \). The top-down approach is as follows: compute an optimal VST \( E_\ell = VST(G, w, T_\ell) \) over the topmost terminal set \( T_\ell \) with original vertex weights, then determine a VST over \( T_{\ell-1} \) by contracting all vertices spanned by \( G_\ell \) into a single vertex (equivalently, set \( w(v) = 0 \) for each vertex \( v \) spanned by \( G_\ell \)). We repeat this process until we have computed a solution, as shown in Algorithm 1.

**Algorithm 1** Top-down Approach

1. **procedure** TopDownVMLST\((G, w, T_1, \ldots, T_\ell)\)
2. Find top-level Steiner tree: \( E_\ell = VST(G, w, T_\ell) \)
3. for \( i = \ell - 1, \ldots, 1 \) do
4. Set \( w(v) = 0 \) for each \( v \) spanned by \( E_{i+1} \)
5. Find lower-level trees using the updated weights: \( E_i = VST(G, w, T_i) \)
6. end for
7. **end procedure**

Lemma 2.1. Denote by \( \text{OPT}_i \) the cost of the non-terminals on level \( i \) from the optimal V-MLST solution. Define \( \text{TOP}_i \) similarly as the cost of the level \( i \) tree from TopDownVMLST assuming a VST oracle. Then, the total cost \( \text{OPT} = \sum_{i=1}^\ell \text{OPT}_i \) and \( \text{TOP} = \sum_{i=1}^\ell \text{TOP}_i \) satisfy (1) \( \text{OPT} \geq \sum_{i=1}^\ell \text{MIN}_i \) and (2) \( \text{TOP} \leq \ell \text{MIN}_\ell + (\ell - 1)\text{MIN}_{\ell-1} + \ldots + \text{MIN}_1 \).

Proof. The first inequality is trivially true, as \( \text{OPT}_i \geq \text{MIN}_i \). For each \( i = \ell, \ldots, 1 \), when the top-down heuristic (Algorithm 1) computes a solution on level \( i \), the vertices spanned by \( G_i \) are inherited to levels \( G_{i-1}, \ldots, G_1 \). That is, when computing \( E_\ell = VST(G, w, T_\ell) \) (line 2), the total weight incurred is \( \text{TOP}_\ell = \text{MIN}_\ell \). In order to compute \( E_{\ell-1} = VST(G, w, T_{\ell-1}) \), a candidate solution is to take the union of the two trees \( E_\ell \) along with the vertex-weighted ST of cost \( \text{MIN}_{\ell-1} \), and remove edges from the latter, so that the result is a tree spanning \( T_{\ell-1} \). Thus, the weight of the tree on level \( i = \ell - 1 \) is at most \( \text{TOP}_{\ell-1} \leq \text{MIN}_\ell + \text{MIN}_{\ell-1} \). We can repeat this argument for levels \( \ell - 2, \ldots, 1 \) to show that \( \text{TOP}_i \leq \ell \text{MIN}_\ell + \ell \text{MIN}_{\ell-1} + \ldots + \text{MIN}_1 \). Summing \( \text{TOP}_i \) across all levels completes the proof.

Theorem 2.1. The top-down heuristic with a VST oracle is an \( \ell \)-approximation to the V-MLST problem.
Further, using a logarithmic \( O(\ln |T|) \)-approximation to VST (e.g., a \( 2 \ln |T| \)-approximation \[19\]) in place of \( VST(G, w, T_i) \) in Algorithm 1 leads to a polynomial-time \( O(\ell \cdot \ln |T_1|) \)-approximation.

Proof. Using Lemma 2.1, we have

\[
\frac{\text{TOP}}{\text{OPT}} \leq \frac{\ell \cdot \text{MIN}_\ell + (\ell - 1)\text{MIN}_{\ell-1} + \ldots + \text{MIN}_1}{\ell \text{MIN}_\ell + \ell \text{MIN}_{\ell-1} + \ldots + \text{MIN}_1} \leq \ell.
\]

Equality is attained when \( \text{MIN}_1 = \text{MIN}_2 = \ldots = \text{MIN}_{\ell-1} = 0 \), and the approximation ratio of \( \ell \) is tight, as shown by example in Figure 2 left.

If we use a logarithmic \( (1 + \alpha) \ln |T| \)-approximation to VST in place of \( VST(G, w, T_i) \) in Algorithm 1, Lemma 2.1 yields

\[
\text{TOP} \leq (1 + \alpha)\ell \ln |T_\ell|\text{MIN}_\ell + (1 + \alpha)(\ell - 1)\ln |T_{\ell-1}|\text{MIN}_{\ell-1} + \ldots + (1 + \alpha)\ln |T_1|\text{MIN}_1 \leq (1 + \alpha)\ln |T_1|((\ell \text{MIN}_\ell + \ldots + (\ell - 1)\text{MIN}_{\ell-1} + \ldots + \text{MIN}_1).)
\]

Hence \( \frac{\text{TOP}}{\text{OPT}} \leq (1 + \alpha)\ell \ln |T_1| \).
One important note is that when OPT = 0 (i.e., there is a solution that does not include any non-terminal), the solution returned by the top-down approach also has cost zero. This follows as TOP_\ell = 0 trivially, and TOP_i = 0 since a candidate solution for TOP_i is to take the union of edges found on level i + 1 with the edges in the optimal Steiner tree on level i.

2.2 Bottom-up Approach. A “bottom-up” approach computes a VST E_1 = VST(G, w, T_1) over T_1, which induces the remaining \ell - 1 trees. We can locally improve the solution by pruning edges that do not connect two terminals, for each level. For the edge-weighted, multi-level ST problem, the bottom-up approach yields an \ell−approximation assuming a ST oracle, even without pruning edges (1). However, this approach can perform arbitrarily badly for the V-MLST problem using the definition of cost (Eq. 1.1), as illustrated in Figure 2, right. In particular, since T_1 = V, any of the four spanning trees is a candidate solution for G_1.

3 The GREEDYVMLST Algorithm

In this section we describe our main result: a (2 ln |T|)−approximation algorithm for the V-MLST problem, called GREEDYVMLST. Our algorithm is a generalization of the (2 ln |T|)−approximation algorithm for the (single-level) VST problem by Klein and Ravi (19). We begin with a brief review of KLEIN-RAVI. The algorithm maintains a node-disjoint forest F of trees; initially, each terminal is a singleton tree. At each iteration, a vertex as well as a subset S \subseteq F consisting of two or more trees (|S| \geq 2) is connected to form a single tree, with the objective of minimizing the average node-to-tree distance. Specifically, for all vertices v and subsets S \subseteq F, the algorithm considers the ratio \frac{w(v) + \sum_{T \in S} d(v, T)}{|S|} where d(v, T) denotes the shortest distance from v to any vertex in the tree T (excluding endpoint costs). For any given vertex v, the optimal subset S and its corresponding ratio (quotient cost) can be found in polynomial time, as the only subsets S that need to be considered are those consisting of the 2, 3, …, |F| nearest trees from v. The algorithm terminates when the forest becomes a single tree, i.e., |F| = 1.

\begin{algorithm}
\caption{Klein-Ravi Greedy Algorithm for VST [19].}
\begin{algorithmic}[1]
\Procedure{KLEIN-RAVI}{G, w, T}
\State Initialize the set F of trees so that each terminal is a singleton tree.
\State Set the cost of terminals to zero, i.e., let w(v) = 0 if v \in T.
\While{|F| > 1}
\State Find the best center v_c, and subset S \subseteq F that minimizes \frac{w(v_c) + \sum_{T \in S} d(w, T)}{|S|}.
\State Update set F: replace trees in S with tree T_{new} that connects center v_c and trees in S.
\State Update cost functions: for v \in V(T_{new}), set w(v) = 0.
\EndWhile
\EndProcedure
\end{algorithmic}
\end{algorithm}
3.1 Setup. Our generalization of the KLEIN-RAVI algorithm to the V-MLST problem relies on growing a set of rooted level-respecting trees, $F$, defined below. The notion is similar to that of the rate-of-service Steiner tree or QoS Multicast Tree ([31], [8]), except that we need to grow and maintain a valid set of such trees during the algorithm. The trees in $F$ are not necessarily vertex-disjoint as the same vertex may appear in different trees on different levels; hence, we avoid the term “forest.” Recall that $R(v)$ denotes $v$’s rate (demand), or the largest $i$ such that $v \in T_i$, which is given as input. In this section, we will simply refer to any vertex $v \in T_1$ as a “terminal.”

For each tree $T \in F$, each vertex $v \in T$ is augmented with a label $\text{Lev}_T(v)$, specific to that tree:

**Definition 3.1.** Given a tree $T \in F$, let $\text{Lev}_T(v) : V(T) \to \{1, \ldots, \ell\}$ be a labeling function associated with the tree $T$, that maps each vertex $v$ to the level that $v$ appears on in $T$, at the current iteration of the algorithm.

We specifically require $\text{Lev}_T(v) \geq R(v)$ for all $v$ and for all trees $T$ at all iterations of the algorithm.

**Definition 3.2. (Level-Respecting Path)** Let $T$ be a tree, along with its labeling function $\text{Lev}_T$. A path in $T$ is level-respecting if the labels $\text{Lev}_T(\cdot)$ along the path are non-increasing.

**Definition 3.3. (Level-Respecting Tree)** Let $T$ be a tree, along with its labeling function $\text{Lev}_T$. We say that $T$ is a level-respecting tree if there exists a root vertex $r$ such that for all $v \in T$, the path from $r$ to $v$ is level-respecting. The level of a tree equals $\text{Lev}_T(r)$ and is denoted $\text{Lev}(T)$.

Figure 3 gives an example of a level-respecting tree. The V-MLST problem is equivalent to finding a minimum-cost level-respecting tree $T$ satisfying $\text{Lev}_T(v) \geq R(v)$ for all $v \in V$; given a level-respecting tree $T$, a solution to the V-MLST problem can be obtained by setting $V_i = \{v \in T \mid \text{Lev}_T(v) \geq i\}$ and $E_i = \{uv \in T \mid \text{Lev}_T(u) \geq i \text{ and } \text{Lev}_T(v) \geq i\}$.

In this greedy algorithm, we maintain a set $F$ of level-respecting trees. Initially, each terminal $v \in T_1$ is its own tree whose root is itself (so that $|F| = |T_1|$). Each singleton tree $T \in F$ is initialized with the simple labeling function $\text{Lev}_T$ that maps its own vertex $v$ to $R(v)$. On each iteration, a subset of the current set $F$ of level-respecting trees is greedily chosen and connected appropriately to form a new level-respecting tree, while updating $F$ and the level labels appropriately. In particular, the roots of all level-respecting trees are necessarily terminals. The size of $|F|$ is strictly decreasing at each iteration; once $|F| = 1$, the algorithm terminates with a single level-respecting tree that is a candidate solution to the V-MLST problem.

To decide which trees to connect, we will define cost functions $w_1, w_2, \ldots, w_\ell$ with the interpretation that $w_i(v)$ denotes the cost of “promoting” vertex $v$ from its current level to level $i$. Initially, $w_i(v) = \max(0, i - R(v))w(v)$ for all $1 \leq i \leq \ell$ and for all $v \in V$. The computation of $w_i(v)$ updates when
3.2 Iteration Step. An iteration in this algorithm consists of properly choosing a root tree $T_r \in F$ with root $v_r$, a center $v_c$ (the center may or may not be $v_r$, or may be in $T_r$), an integer $k \leq \text{Lev}(T_r)$, and a subset of level-respecting trees $S \subset F$ of size at least $|S| \geq 1$, such that $\text{Lev}(T) \leq k$ for all $T \in S$. By properly connecting $T_r$ to $v_c$ via a level $k$ path, connecting $v_c$ to each tree $T \in S$ via a level $\text{Lev}(T)$ path, we create a new level-respecting tree $T_{\text{new}}$ whose root is $v_r$. The labels $\text{Lev}_{T_{\text{new}}}$ are then set appropriately; see Fig. 4.

On each iteration, we wish to minimize the cost-to-connectivity ratio $\gamma$, defined as follows:

$$\gamma = \frac{d_k(T_r, v_c) + w_k(v_c) + \sum_{T \in S} d_{\text{Lev}(T)}(v_c, T)}{1 + |S|}.$$

Without loss of generality, $d_{\text{Lev}(T)}(v_c, T)$ is defined as the cost of the shortest path between $v_c$ and the root of $T$, not including endpoints, by “promoting” all vertices on the path to level $\text{Lev}(T)$. As $T$ is level-respecting, any intermediate path in $T$ from its root to any other vertex $v$ of the same level uses edges at level $\text{Lev}(T)$, so $v$ and any intermediate vertices incur zero cost in $w_{\text{Lev}(T)}$. Similarly, $d_k(T_r, v_c)$ can be defined as the cost of the shortest path between the root $v_r$ and $v_c$ via a level $k$ path. Additionally, $\gamma$ considers the sum of all path costs from $v_c$ to each $T \in S$, even though there may be repeat vertices. Figure 5 gives a more concrete example of the execution of GREEDYVMLST on an example input.

**Lemma 3.1.** The optimal choice of $T_r, v_c, k$, and $S$ that minimizes $\gamma$ can be found in polynomial time.

**Proof.** For a fixed center $v_c$ and integer $k$, sort all trees in $F$ with level $\text{Lev}(T) \leq k$ by their “distance” to $v_c$, namely $d_{\text{Lev}(T)}(v_c, T)$. The best choice of the subset $S$ can be found through checking only subsets with the nearest $2, 3, \ldots, |F| - 1$ trees. Therefore, we can find the subset that minimizes $\frac{w_k(v_c) + \sum_{T \in S} d_{\text{Lev}(T)}(v_c, T)}{|S|}$ in polynomial time. If $k > \max_{T \in S} \text{Lev}(T)$, then we can improve $\gamma$ by setting $k = \max_{T \in S} \text{Lev}(T)$.

Lastly, find the nearest tree $T' \in F$ to $v_c$ in $F$ with level strictly larger than $k$. If such a tree exists, and if choosing $T'$ as the root tree $T_r$, lowers the $\gamma$ ratio, then accept this tree as $T_r$. Otherwise, use one of the trees in $S$ at level $k$ as the root tree, remove it from $S$, and set it as the root tree $T_r$. Therefore, for a fixed center vertex $v_c$ and level $k$, we can find the optimal choice of $T_r$ and $S$. As such the optimal choice of $T_r, v_c, k,$ and $S$ that minimizes $\gamma$ can be found in polynomial time.

Compared to the KLEIN-RAVI algorithm, an iteration in the GREEDYVMLST algorithm requires determining two new elements: a root tree $T_r$ and an integer $k$ representing the level that $v_c$ is promoted to. However, the above proof indicates that a GREEDYVMLST iteration is only $\ell$ times more expensive than KLEIN-RAVI.

Once a root tree $T_r$, center $v_c$, integer $k$, and subset $S \subset F$ have been found, the labels for each vertex in $T_{\text{new}}$ should be set. Specifically, if $v \in T_{\text{new}}$ and $v$ is on the path from $v_r$ to $v_c$, then set $\text{Lev}_{T_{\text{new}}}(v) \leftarrow k$. Otherwise, $v$ is on the path from $v_c$ to some tree $T \in S$, and should be updated so that $\text{Lev}_{T_{\text{new}}}(v) \leftarrow \text{Lev}(T)$. In the case that $v$ is on the path from $v_c$ to multiple trees in $S$, set $\text{Lev}_{T_{\text{new}}}(v)$ to be the maximum of the levels of the trees that $v$ connects.
Figure 5: Illustration of \texttt{GREEDYVMLST}. (a): Input graph with $\ell = 2$, and eight vertices with vertex weights and demands $L(v)$ shown in red. (b): Initial set of singleton trees $F$ with $|F| = 4$. (c): Choosing $T_r$ and $v_c$ as shown, $k = 1$, and $S$ of size 2 minimizes $\gamma$. The value of $\gamma$ is $\gamma = \frac{1 \cdot 1 + 3 \cdot 0 + 0}{1 + 2} = 4$. Note that $w_1(v_c) = 3$ and $w_2(v_c) = 6$ initially; after the first iteration, we have $w_1(v_c) = 0$ and $w_2(v_c) = 3$. (d): Choosing $T_r$ and $v_c$ as shown, $k = 2$, and $S$ of size 1 minimizes $\gamma$. In this case, $d_2(v_r, v_c) = 2 \cdot 7 + 1 \cdot 8 = 22$, $w_2(v_c) = 3$, and $d_2(v_c, T) = 1$ where $T$ is the single tree in $|S|$. The value of $\gamma$ is $\gamma = \frac{22 + 3 + 1}{1 + 1} = 13$. Since $|F| = 1$, \texttt{GREEDYVMLST} terminates after two iterations.

### 3.3 The Algorithm

The greedy algorithm for the V-MLST problem, which we call \texttt{GREEDYVMLST}, is summarized in Algorithm 3.

**Algorithm 3 Multi-Level Greedy Approximation Algorithm**

1. **procedure** \texttt{GREEDYVMLST}(G, w, T_1, \ldots, T_\ell)
2. Initialize $F$ so that each terminal $v \in T_1$ is a singleton tree
3. Initialize $\text{Lev}_T(v) = R(v)$ for each singleton tree $T \in F$
4. For each $i = 1, \ldots, \ell$, initialize $w_i(v) = \max(0, i - R(v))w(v)$
5. **while** $|F| > 1$ **do**
6. Find root tree $T_r \in F$, center $v_c$, integer $k$, and subset $S \subset F$ that minimizes $\gamma$
7. Connect $T_r$ to $v_c$ via a level $k$ path, and $v_c$ to each $T \in S$ via a level $\text{Lev}(T)$ path to construct a new level-respecting tree $T_{\text{new}}$. Update $F$.
8. Update weight functions: for $v \in V(T_{\text{new}})$, set $w_j(v) = 0$ for $j \leq \text{Lev}_{T_{\text{new}}}(v)$, and $w_j(v) = w_j(v) - w_i(v)$ for $j > \text{Lev}_{T_{\text{new}}}(v)$.
9. **end while**
10. **end procedure**

### 3.4 Analysis

The analysis of \texttt{KLEIN-RAVI} \cite{19} involves a spider decomposition in the optimal VST solution.

**Definition 3.4.** (\texttt{SPIDER} \cite{19}) A **spider** is a tree where at most one vertex has degree greater than two.
spider is identified by its center, a vertex from which all paths to the leaves of the spider are vertex-disjoint. A nontrivial spider is a spider with at least two leaves.

A foot of a spider is a leaf; if the spider has at least three leaves, then its center is unique, and is also counted as a foot. Klein and Ravi show that given a connected graph $G = (V, E)$ and a subset of the vertices $M \subseteq V$, a set of vertex-disjoint nontrivial spiders can be found such that the union of the feet of the spiders contains $M$ [19]. We generalize the notion of a spider to the multi-level setting, which we refer to as a tailed spider.

**Definition 3.5. (Tailed Spider)** A tailed spider is a level-respecting tree containing at most one vertex whose degree is greater than two, identified by a center $v_c$ and a root $v_r$. The root $v_r$ is either the center, or a leaf of the tree, and there exists a level-respecting path from $v_r$ to each vertex in the spider. From the center $v_c$, there is a vertex-disjoint, level-respecting path to each leaf of the tree, possibly excluding the root $v_r$.

The resulting tree in Figure 5 is a tailed spider whose root is distinct from its center.

**Definition 3.6. ($M$-Optimized Level-Respecting Tree)** Let $T$ be a level-respecting tree with root $v_r$, and let $M$ be a subset of the vertices in $T$ such that $v_r \in M$. We say that $T$ is $M$-optimized if all leaves of $T$ belong to $M$, and for any vertex $v \notin M$, $\text{Lev}_T(v) = i$ if there exists a vertex $u \in M$ in the subtree rooted at $v$ for which $\text{Lev}_T(u) = i$.

The above definition indicates that an $M$-optimized, level-respecting tree does not unnecessarily use higher levels to reach $M$ from the root. Figure 5 right, gives an example. An optimal V-MLST $G^*$ is $T_1$-optimized due to the following simple argument: if there is a vertex $v \notin T_1$, for which $\text{Lev}_{G^*}(v) > \text{Lev}_{G_1}(w)$ for all $w \in T_1$ in the subtree rooted at $v$, then demoting $\text{Lev}_{G^*}(v)$ to equal $\max_w \text{Lev}_{G_1}(w)$ over all $w$ in the subtree rooted at $v$ leads to a solution with lower cost, contradicting the optimality of $G^*$.

**Lemma 3.2.** Given any level-respecting tree $T$ and a subset $M$ of its vertices, we can efficiently construct an $M$-optimized subtree of $T$.

**Proof.** To $M$-optimize a tree $T$, we can first trim any leaves of the tree $T$ until all leaves belong to $M$. Assign $\max_{v \in M} \text{Lev}_T(v)$ to any vertex $u \notin M$ for which $\text{Lev}_T(u) > \max_{v \in M} \text{Lev}_T(v)$. Finally, we can pick a vertex in $M$ with highest $\text{Lev}_T$ label, declare it the root, and run a depth-first search (DFS) to adjust any level labels of vertices not in $M$.

**Theorem 3.1.** Given a nonempty subset of the vertices $M$ with $|M| \geq 2$, any $M$-optimized level-respecting tree $T$ can be decomposed into vertex-disjoint tailed spiders such that the leaves and roots belong to $M$ (a center may or may not belong to $M$), and the leaves, roots or centers cover the set $M$.

The proof gives a way to construct such a decomposition, which is important.

**Proof.** We use induction on $|M|$. For the base case $|M| = 2$, the decomposition consists of a single tailed spider, namely the path from the root $v_r$ to the other vertex in $M$, where $v_r$ is also the root of its tailed spider.

For $|M| \geq 3$, set one of the vertices in $M$ with highest $\text{Lev}_T$ label as the root $v_r$ of $T$. Find the furthest vertex $v$ from $v_r$ (by number of edges) in $T$ with the property that the subtree $T'$ rooted at $v$ contains at least two vertices in $M$. If $v = v_r$, then $T$ is already a tailed spider, as no other vertex in $V - \{v\}$ can have degree greater than 2; if there existed $w \neq v$ with degree greater than 2, then the subtree rooted at $w$ has at
least two leaves (which belong to \(M\)), contradicting the choice of \(v\). In this case, set \(v_r\) to be the root and center of its tailed spider.

Thus, we assume \(v \neq v_r\). The vertex \(v\) and its subtree forms a tailed spider with center \(v\). If \(v \in M\), then set \(v\) to be the root of its tailed spider. If \(v \notin M\), then there is a vertex \(w \in M\) in the subtree rooted at \(v\) with label \(\text{Lev}_T(v)\); set \(w\) to be the root of its tailed spider. Remove this subtree from the original tree \(T\), as well as the edge from \(v\) to its parent, to produce a smaller tree \(T'\).

Let \(M' \subset M\) be the set of vertices in \(M\) that remain in \(T'\) upon removing the subtree rooted at \(v\). If \(|M'| = 0\), then the subtree rooted at \(v\) is the only tailed spider, giving a valid decomposition. If \(|M'| = 1\), then \(M' = \{v_r\}\), so connecting \(v_r\) to \(v\) produces a single tailed spider with root \(v_r\) and center \(v\) as \(\text{Lev}_T(v_r) \geq \text{Lev}_T(v)\). Otherwise, if \(|M'| \geq 2\), we may prune the path \(v_r \to v\) so that \(T'\) is an \(M'\)-optimized level-respecting tree. By the inductive hypothesis, \(T'\) can be decomposed into vertex-disjoint tailed spiders over \(M'\).

\[\text{Cost}(T) = \sum_{v \in T}(\text{Lev}_T(v) - R(v))w(v)\]

**Corollary 3.1.** Let \(T\) be an \(M\)-optimized level-respecting tree. Consider a tailed spider decomposition containing \(s\) spiders, generated using the method in the proof of Theorem [3.7]. Let \(S_j\) denote the set of vertices in \(M\) reachable from the \(j\)th spider’s center, not including the root. Then \(\sum_{j=1}^s (1 + |S_j|) = |M|\).

**Proof.** This statement follows as every vertex in \(v \in M\) is either a root of its tailed spider, or is reachable from its spider’s center. In particular, the path from the tailed spider’s center \(v\) to any leaf \(w \in M\) does not encounter any other vertices in \(M\), as this would contradict the choice of \(v\) when computing such a decomposition.

In the next lemma, we consider an arbitrary tailed spider \(T_j\) with root \(v_r\), center \(v_c\), and leaf vertices \(S\) not including the root. The meaning of \(j\) is irrelevant for now, but will be used to index into the \(j\)th spider in a tailed spider decomposition. Note that \(T_j\) has a level labeling function \(\text{Lev}_{T_j}\). Define by \(\text{Cost}(T_j)\) the cost (weight) of this spider, given by \(\text{Cost}(T_j) = \sum_{v \in T_j}(\text{Lev}_{T_j}(v) - R(v))w(v)\). On the first iteration, a candidate choice for GREEDYVMLST is to select the vertices characteristic of \(T_j\), namely \(v_r, v_c, k = \text{Lev}_{T_j}(v_c)\), and set \(S\). Recall on the first iteration, each leaf is its own singleton tree. Let \(C'_j = d_k(v_r, v_c) + w_k(v_c) + \sum_{T \in S} d_{\text{Lev}(T)}(v_c, T)\) denote the cost computed by the GREEDYVMLST algorithm for this choice of root \(v_r\), center \(v_c\), \(k\), and \(S\). Note that \(C'\) equals the numerator in the cost-to-connectivity ratio \(\gamma\).

**Lemma 3.3.** Let \(T_j\) be a tailed spider, with center \(v_c\), root \(v_r\), and leaves \(S\) not including the root. Define \(C'_j\) and \(\text{Cost}(T_j)\) as above. Then \(C'_j \leq \text{Cost}(T_j)\).

**Proof.** For a tailed spider \(T_j\), the paths from the root \(v_r\) to the center \(v_c\), and from \(v_c\) to each leaf are vertex-disjoint by definition. Consider a path \(P\) from \(v_c\) to a leaf \(T \in S\), and its incurred cost within the spider \(T_j\) (not including \(v_c\) or \(s\)), which equals \(\text{Cost}(P) = (\text{Lev}_{T_j}(v) - R(v))w(v)\). As \(\gamma\) considers the minimum weight path from \(v_c\) to \(T\) over cost function \(w_k\), we have \(d_{\text{Lev}(s)}(v_c, s) \leq \text{Cost}(P)\). Summing over all vertex-disjoint paths from \(v_c\) to each leaf in \(S\), and also noting that \(d_k(v_r, v_c)\) is not more than the distance from \(v_r\) to \(v_c\) in the spider \(T_j\), the result follows. Equality occurs if \(T_j\) is \(M\)-optimized where \(M = \{v_r\} \cup S\), and if \(d_k(v_r, v_c)\) chooses the path from \(v_r\) to \(v_c\) in the tailed spider.
Theorem 3.2. The multi-level greedy algorithm GREEDYVMLST is a \(2\ln |T_1|\)-approximation to V-MLST.

Proof. Let the optimal V-MLST solution \(G^*\) be represented by a tree \(G^* = (V^*, E^*)\), along with its labeling function \(\text{Lev}_{G^*} : V^* \rightarrow \{1, \ldots, \ell\}\) that assigns each vertex the highest level that it belongs to. In this way, the level \(i\) tree \(G^*_i\) is the subgraph of \(G^*\) induced by the vertex set \(V^*_i = \{v \in V^* \mid \text{Lev}_{G^*}(v) \geq i\}\). Furthermore, the weight of each vertex \(v\) in \(G^*\) is \((\text{Lev}_{G^*}(v) - R(v))w(v)\) so that the weight of \(G^*\) is \(\text{OPT} = \sum_{v \in V^*}(\text{Lev}_{G^*}(v) - R(v))w(v)\).

Let \(\mathcal{F}_i\) denote the set of trees at the beginning of iteration \(i\). Let \(M_i\) denote the set of roots of trees in \(\mathcal{F}_i\) at the beginning of iteration \(i\). That is, \(M_1 = T_1\) and \(|\mathcal{F}_1| = |T_1|\) as each terminal is initially a root. Let \(h_i\) denote the number of trees in \(\mathcal{F}_i\) that are merged on iteration \(i\), so that \(h_i \geq 2\). Let \(C^*_i\) denote the cost computed on iteration \(i\). Define \(\text{Cost}(T_j)\) to be the cost of the \(j\)th spider; this equals \(\text{Cost}(T_j) = \sum_v(\text{Lev}_{T_j}(v) - R(v))w(v)\) over all vertices \(v\) in \(T_j\). Optimize this spider to obtain a tailed spider with root \(v_{r_j}\), center \(v_{c_j}\), and leaves \(S_j\).

On iteration \(i = 1\), a candidate choice for GREEDYVMLST is to select root \(v_{r_j}\), center \(v_{c_j}\), integer \(k_j\), and a subset \(S \subset \mathcal{F}\) where \(S\) consists of all singleton trees corresponding to the leaves of the \(j\)th spider not including \(v_{r_j}\). Let \(C'_j\) denote the cost computed by GREEDYVMLST for this choice of root, center, \(k = k_j\), and leaves. By Lemma 3.3, \(C'_j \leq \text{Cost}(T_j)\). If this candidate choice had been chosen, the number of terminals merged is \(|S| - 1\). Hence, we have

\[
\frac{C^*_1}{h_1} \leq \frac{C_1}{h_1} \leq \frac{C'_j}{1 + |S_j|} \leq \frac{\text{Cost}(T_j)}{1 + |S_j|}.
\]

We use the simple algebraic fact that for non-negative numbers \(a, x_1, \ldots, x_s, y_1, \ldots, y_s\), if \(a \leq \frac{x_i}{y_i}\) for every \(i\), then \(a \leq \frac{\sum_{j=1}^s x_j}{\sum_{j=1}^s y_j}\). Applying this fact, we have

\[
\frac{C^*_1}{h_1} \leq \frac{\sum_{j=1}^s \text{Cost}(T_j)}{\sum_{j=1}^s (1 + |S_j|)}.
\]

The numerator, \(\sum_{j=1}^s \text{Cost}(T_j)\), is at most \(\text{OPT}\), as the vertices in a tailed spider decomposition of \(G^*\) are a subset (not necessarily a proper subset) of the vertices in \(G^*\). The denominator, \(\sum_{j=1}^s (1 + |S_j|)\), equals \(|T_1|\) by Corollary 3.1. The lemma follows.

After the first iteration, at least \(h_1\) singleton trees are replaced with a tree \(T_{new}\) whose root is a terminal \(v_r\). We have \(|\mathcal{F}_{i+1}| \leq |\mathcal{F}_i| - (h_i - 1)\). Lemma 3.4 generalizes as follows:

Lemma 3.5. For each iteration \(i \geq 1\), we have \(\frac{C^*_i}{h_i} \leq \frac{\text{OPT}}{|\mathcal{F}_{i+1}|} = \frac{\text{OPT}}{|\mathcal{F}_i| - (h_i - 1)}\).
Proof. Here, we consider a tailed spider decomposition of the optimal tree $G^*$ over $M_i$, the set of tree roots at the beginning of the $i$th iteration. Otherwise, the proof is identical to that of Lemma 3.4.

It is worth noting that on iteration $i$, the proof by Klein and Ravi [19, §4] uses a spider decomposition over a set of “supernodes”, where each supernode corresponds to a subtree computed thus far. Here, we use a tailed spider decomposition over the set $M_i$ of roots.

Lemma 3.5 rearranges to $h_i \geq C_i^* \cdot \text{OPT} |F_i|$. Division by zero may occur if $\text{OPT} = 0$, or if there is a solution that does not span any non-terminal. In this case, GREEDYVMLST necessarily returns a solution with zero cost, as $\gamma = 0$ on every iteration. Hence, we assume $\text{OPT} > 0$. As $h_i \geq 2$, we equivalently have $\frac{1}{2} h_i \leq h_i - 1$.

$$|F_{i+1}| \leq |F_i| - (h_i - 1) \leq |F_i| - \frac{1}{2} h_i \leq |F_i| \left(1 - \frac{1}{2} \cdot \frac{C_i^*}{\text{OPT}} \right)$$

The remainder of the proof is otherwise similar to that by Klein and Ravi [19, §4], by unraveling the inequalities and taking the logarithm of both sides. Suppose the total number of iterations equals $I$, so that $|F_I| \geq 2$ and $|F_{I+1}| = 1$.

$$|F_{I+1}| \leq |F_1| \prod_{i=1}^{I} \left(1 - \frac{1}{2} \cdot \frac{C_i^*}{\text{OPT}} \right)$$

$$\ln |F_{I+1}| \leq \ln |F_1| + \sum_{i=1}^{I} \ln \left(1 - \frac{1}{2} \cdot \frac{C_i^*}{\text{OPT}} \right) \leq \ln |F_1| - \sum_{i=1}^{I} \frac{C_i^*}{2 \cdot \text{OPT}}$$

using the fact that $\ln(1 - x) \leq -x$ for $x \in [0, 1)$. This inequality rearranges to

$$\sum_{i=1}^{I} C_i^* \leq 2 \cdot \text{OPT}(\ln |F_1| - \ln |F_{I+1}|) \leq 2 \cdot \text{OPT}(\ln |T_1| - \ln 1) = 2 \ln |T_1| \cdot \text{OPT}.$$ 

completing the proof.

It is worth noting that the proof of Theorem 3.2 differs from that by Klein and Ravi in how a (tailed) spider decomposition in $G^*$ is considered at each iteration. Once the inequality relating $h_i$ with $C_i^*$, OPT, and $|F_i|$ (Lemma 3.5) is established, the remainder of the analysis is similar to that of Klein and Ravi.

Note that $|F_i|$ is strictly decreasing on each iteration, so the number of iterations is upper bounded by $|T_1|$. Each iteration can be carried out in polynomial time (Lemma 3.1), thus, GREEDYVMLST runs in polynomial time.

**GREEDYVMLST Summary.** Several key techniques allow for the generalization of the Klein-Ravi VST approximation algorithm [19] to the V-MLST problem. First, GREEDYVMLST maintains a set of level-respecting trees and merges them greedily until a single level-respecting tree is left. Merging multiple level-respecting trees is non-trivial, as we must ensure that the resulting tree is also level-respecting. Simply connecting one vertex, the center, to a subset of level-respecting trees, does not necessarily work. Our approach is to connect a root tree, to a center, to a subset $S$ of level-respecting trees. This leads to an increase
in computational complexity of a factor of $\ell$. Second, the analysis of the KLEIN-RAVI algorithm [19] relies on the existence of a spider decomposition in the optimal VST solution. For our analysis, we introduce level-respecting “tailed spiders,” characterized by a root, center, and terminal leaves. Third, instead of contracting subtrees computed on iteration $i$ into “supernodes”, in the GREEDYVMLST algorithm, we only keep the root vertex.

4 Integer Linear Programming Formulation

ILP formulations for the (single level) VST are known, e.g., Segev [29]. An ILP formulation for the MLST problem using $O(|E|)$ variables and $O(|E| + |V|)$ constraints is given in [1]. With few modifications, this formulation extends nicely to the V-MLST problem.

Assume the input graph is directed, which can be done by replacing each undirected edge $uv$ with two edges $(u, v), (v, u)$. Let $x_{uv} = 1$ if edge $(u, v)$ is present anywhere in the solution (i.e., on the bottom level), and 0 otherwise. Let $y_{uv}$ denote the rate of directed edge $(u, v)$, i.e., the highest level that $(u, v)$ is an edge in, and 0 if $(u, v)$ is not present. Let $f_{uv}$ denote the (integer) flow from $u$ to $v$. Let $y_v$ denote the highest level that $v$ is a vertex in, given a solution. Recall that the cost of a solution is defined by $\sum_{v \in V} w(v)(y_v - R(v))$.

Let $N(v)$ denote the neighborhood of $v$, i.e., $N(v) = \{ u \mid (u, v) \in E \}$. Let $s \in T_\ell$ be a source vertex. A flow-based ILP formulation for the V-MLST problem is as follows:

Minimize $\sum_{v \in V} w(v)(y_v - R(v))$ subject to

\[
\sum_{v \in V} w(v)(y_v - R(v)) = \sum_{(v, w) \in E} f_{vw} - \sum_{(u, v) \in E} f_{uv} = \begin{cases} |T_1| - 1 & v = s \\ -1 & v \in T_1 - \{ s \} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V \tag{4.2}
\]

\[
0 \leq f_{uv} \leq (|T_1| - 1)x_{uv} \quad \forall uv \in E \tag{4.3}
\]

\[
\sum_{u \in N(v)} x_{uv} \leq 1 \quad \forall v \in V \tag{4.4}
\]

\[
x_{uv} \leq y_{uv} \leq \ell x_{uv} \quad \forall uv \in E \tag{4.5}
\]

\[
\sum_{u \in N(v) - \{ w \}} x_{uv} \geq x_{vw} \quad \forall uv \in E, v \neq s \tag{4.6}
\]

\[
\sum_{u \in N(v) - \{ w \}} y_{uv} \geq y_{vw} \quad \forall uv \in E, v \neq s \tag{4.7}
\]

\[
\sum_{u \in N(v)} y_{uv} \geq R(v) \quad \forall v \in T_1 - \{ s \} \tag{4.8}
\]

\[
0 \leq x_{uv} \leq 1 \quad \forall uv \in E \tag{4.9}
\]

\[
y_v \geq y_{uv} \quad \forall v \neq s; u \in N(v) \tag{4.10}
\]

\[
y_s = \ell \tag{4.11}
\]

\[
\sum_{u \in N(s)} x_{us} = 0 \tag{4.12}
\]

Constraints (4.3) through (4.10) are exactly the same as in the MLST problem. (4.3) enforces that the source vertex sends out $|T_1| - 1$ units of flow to the remaining terminals in $T_1$, and that each other terminal receives net one unit of flow. Constraint (4.4) enforces that the flow along an edge $(u, v)$ is no more than $|T_1| - 1$ if $(u, v)$ is present in the solution. Constraint (4.5) enforces that each vertex has at most one incoming edge in the solution.

Similarly to [1], (4.6) enforces $1 \leq y_{uv} \leq \ell$ if $x_{uv} = 1$, and $y_{uv} = 0$ if $x_{uv} = 0$. (4.7) says that if $(v, w)$ is an edge in the solution, and $v$ is not the root, then $v$ has at least one incoming edge. Combined with constraint (4.5), $v$ necessarily has exactly one incoming edge, and it is not the edge $(w, v)$.

Constraint (4.8) enforces that if edge $(v, w)$ appears on levels $1, \ldots, i$ (i.e. $y_{uw} = i$), and $v$ is not the
root, then the sum over all neighbors $u$ of $v$ (other than $w$) of $y_{uw}$ is at least $i$. As $v$ must have exactly one incoming edge, constraint (4.8) combined with (4.5) and (4.6) ensure that if $(v, w)$ is an edge, then $v$ has exactly one incoming edge $(u, v)$, whose level $y_{uv}$ is greater than or equal to $y_{uw}$.

Constraint (4.9) enforces that for each terminal $v \in T_1 - \{s\}$, the sum $\sum_{u \in N(v)} y_{uw}$ is at least the rate of $v$, $R(v)$. Combined with (4.5) and (4.6), every non-root terminal $v$ has exactly one incoming edge $(u, v)$ such that $y_{uv} \geq R(v)$. Constraint (4.10) enforces that the $x_{uv}$ indicator variables are either 0 or 1, which implies $0 \leq y_{uv} \leq \ell$ in constraint (4.6).

The following constraints are specific to the V-MLST problem. Constraint (4.11) enforces that if $v$ is present in the solution, then its level $y_v$ is at least the level of its incoming edge. Combined with (4.11), this implies $y_v \geq R(v)$ for all non-root terminals $v$. The root $s$ is necessarily on level $\ell$ (4.12), i.e., $y_s = \ell$. Constraint (4.13) enforces that the root $s$ has no incoming edge thus avoiding cycles (which may occur as edges incur no cost).

**Lemma 4.1.** In the optimal solution to the above ILP, the graph $G_1 = (V, E_1)$ with $E_1 = \{uv \in E \mid x_{uv} = 1 \text{ or } x_{vu} = 1\}$ is a Steiner tree spanning $T_1$.

**Proof.** We show that i) $G_1$ contains no cycle, ii) there exists a path from $s$ to each terminal $v \in T_1$, and iii) $G_1$ is connected.

Assume otherwise $G_1$ contains a cycle $C$. As in [11], the edges of the cycle are oriented in the same direction, and such a cycle cannot have “incoming” edges (edges $(u, v)$ where $u \notin C$ and $v \in C$).

Because edges have weight zero in the V-MLST problem, unlike the MLST problem, such a cycle may contain the source $s$, in which removing its preceding edge does not lower the objective cost. Hence, adding the constraint that $s$ has no incoming edge removes this possibility. Otherwise, if $C$ does not contain $s$ but contains a different terminal $v \in T_1$, then the flow constraint on $v$ cannot possibly be satisfied. If $C$ contains no terminal, then as before, removing $C$ produces a solution with lower cost.

To show ii), consider an arbitrary terminal $v \in T_1$, $v \neq s$. By constraint (4.5), $v$ has at most one incoming edge; however, as $R(v) \geq 1$, constraint (4.9) implies that $v$ has exactly one incoming edge $(u, v)$, whose edge rate is at least $R(v)$. If $u = s$, we are done. Otherwise, continue the process until reaching the source $s$, noting that cycles cannot occur. iii) follows from i) and ii) assuming positive vertex weights; ii) implies that all terminals are in the same connected component as $s$, and if there had existed a different connected component in $G_1$, removing it and setting flows to zero lowers the objective cost.

**Lemma 4.2.** In the optimal solution to the above ILP, the graph $G_i = (V, E_i)$ with $E_i = \{uv \in E \mid y_{uv} \geq i \text{ or } y_{vu} \geq i\}$ is a Steiner tree spanning $T_i$.

**Proof.** The graph $G_i$ is necessarily a subgraph of $G_1$, as $y_{uw} \geq i$ or $y_{vu} \geq i$ implies $x_{uv} = 1$ or $x_{vu} = 1$ by (4.6), so $G_i$ is acyclic. Consider a terminal $v \in T_i$ where $R(v) \geq i$. By (4.9), it has exactly one incoming edge $(u, v)$ whose rate $y_{uv}$ is at least $R(v)$. If $u = s$, we are done. Otherwise, $u$ has exactly one incoming edge whose rate is at least $y_{uv}$ by (4.10); continuing this process, we will eventually reach the root $s$ via a path.

As such, Lemmas 4.1 and 4.2 imply that the optimal solution to the ILP is a feasible solution to the V-MLST problem.

**Theorem 4.1.** The optimal solution to the above ILP with objective value $\text{OPT}_{ILP}$, yields the optimal solution to the V-MLST problem.

**Proof.** The optimal solution to the V-MLST problem is a feasible solution to the above ILP, as the $x_{uv}$, $y_{uv}$, $y_v$, and flow variables $f_{uv}$ can be set accordingly with the optimal solution. Then $\text{OPT}_{ILP} \leq \text{OPT}$. By
Lemmas [4,1] and [4,2] the optimal solution to the ILP with objective value \( \text{OPT}_{ILP} \) yields a feasible solution to the V-MLST problem, so \( \text{OPT} \leq \text{OPT}_{ILP} \). Then \( \text{OPT}_{ILP} = \text{OPT} \), and the above ILP can be used to solve the V-MLST problem exactly.

This ILP formulation adds \( O(|E|) \) constraints when compared to the MLST problem, so the number of variables and constraints is still \( O(|V|) \) and \( O(|E| + |V|) \) respectively.

5 V-MLST as a Directed Steiner Tree (DST) Problem

The DST problem is as follows:

**Definition 5.1. (Directed Steiner Tree (DST) Problem)** Given a directed graph \( G = (V, E) \) with \( |V| = n \), non-negative edge weights \( w : E \to \mathbb{R}_{\geq 0} \), a set \( T \subseteq V \) of terminals, and a root vertex \( r \in V \), compute a minimum-weight directed tree rooted at \( r \) containing a directed path from \( r \) to each terminal in \( T \).

The reduction from V-MLST to DST generalizes the reduction by Segev [29] and is as follows: combine \( \ell \) “copies” of the input graph \( G \), denoted \( C_1, \ldots, C_\ell \), into a single graph containing \( \ell|V| \) vertices. Recall that the levels may be specified via vertex rates or demands \( R : V \to \{0, 1, \ldots, \ell\} \). For each copy \( C_i \), replace each edge with two intra-level edges, as in the single-level case. For all \( u, v \in V \), the directed edge \((u, v)\) on copy \( C_i \) will have weight equal to the weight incurred by including the incoming vertex, \( \max(0, i - R(v))w(v) \). For each vertex \( v \in V \), add inter-level edges of weight zero connecting \( v \) on copy \( C_{i+1} \) to \( v \) on copy \( C_i \), for all \( 1 \leq i \leq \ell - 1 \). Set the root to be any vertex in \( T_\ell \), appearing in \( C_\ell \).

To complete the reduction, set the terminals in the transformed graph as follows: for each vertex \( v \in G \), the vertex corresponding to \( v \) in copy \( C_{R(v)} \) becomes a terminal. No other terminals corresponding to \( v \) are added. Thus, the number of terminals in the directed instance is \( |T_1| \). Figure 7 shows an example reduction.

The resulting graph has \( \ell|V| \) vertices, \( 2\ell \cdot |E| + (\ell - 1) \cdot |V| \) directed edges and \( |T_1| \) terminals, and hence the reduction is polynomial in the size of the input.

Given an optimal solution \( G^* = (G^*_1, \ldots, G^*_\ell) \) to the V-MLST problem, one can construct a solution to the equivalent DST instance with the same cost, by taking advantage of inter-level edges. Conversely, given an optimal solution to the equivalent DST instance, one can construct a solution to the V-MLST problem with the same cost, by setting \( E^*_i \) to be the set of edges appearing on \( C_i \) (with directions removed), \( E^*_i \to E^*_{i-1} \) to be the set of intra-level edges appearing on either \( C_\ell \) or \( C_{\ell-1} \) (again, with edge directions removed), and so on. This produces a solution \( G^* \) to the V-MLST problem with the same weight.

6 Conclusions and Future Work

We presented a generalization of the VST problem to multiple levels and showed that the resulting V-MLST problem can be approximated within a factor of \( 2 \ln |T_1| \), matching the results for the single level problem. Our algorithm relies on the notions of tailed-spider decompositions and level-respecting trees, which could be of use in other multi-level graph design problems.

The definition of the weight or cost of a solution (Eq. 1.1) is somewhat restrictive in the sense that the cost of using a Steiner vertex \( v \) on a higher level (or rate) increases linearly with the number of levels. We make this assumption primarily to simplify the presentation of the algorithms. We can use a more general definition of weights, where each vertex \( v \) is associated with a sequence of \( \ell \) weights \( w_1(v) \leq w_2(v) \leq \ldots \leq w_\ell(v) \), where \( w_i(v) \) denotes the weight of including \( v \) on level \( i \). The only needed modifications for GreedyVMLST are in having the initial weights \( w_i(v) \) as input to the algorithm.

Our \( 2 \ln |T_1| \)–approximation results for the VMLST problem is rather surprising, as it suggests the seemingly harder multi-level problem can be approximated equally well as the single level problem. It will
be interesting to investigate whether for other graph sparsification problems, the multi-level variant can also be approximated equally well as their single-level counterpart. For example, for the (edge-weighted) MLST problem of [1], existing approximation results (see, [1, 8, 2, 18]) are up to a factor larger than 2 away from the single-level ST problem.

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