Computation of Lyapunov Exponents in General Relativity

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Abstract

Lyapunov exponents (LEs) are key indicators of chaos in dynamical systems. In general relativity the classical definition of LE meets difficulty because it is not coordinate invariant and spacetime coordinates lose their physical meaning as in Newtonian dynamics. We propose a new definition of relativistic LE and give its algorithm in any coordinate system, which represents the observed changing law of the space separation between two neighboring particles (an “observer” and a “neighbor”), and is truly coordinate invariant in a curved spacetime.

Key words: chaotic dynamics, relativity and gravitation

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Chaos is a popular phenomenon in dynamical systems. One of its main features is the exponential sensitivity on small variations of initial conditions. The exhibition of chaos in the motion of Pluto makes it particularly attractive for scientists to investigate the dynamical behavior of the solar system[1].

In Newtonian mechanics, Lyapunov exponents (LEs), as a key index for mea-

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suring chaos in a dynamical system, can be calculated numerically with ei-
ther the variational method[2] or the two-particle method[3], and sometimes
a mixture of the two[4]. The former is more rigorous but one has to derive the
variational equations of the dynamical equations of the system and integrate
them numerically with the dynamical equations together. The latter is less
cumbersome especially when one wants to compute the maximum LE only,
which is the main index for chaos. We will discuss the extension of the two
particle method to relativistic models in this letter.

Let \( \mathbf{q}(t) \) and \( \dot{\mathbf{q}}(t) \) be the position and velocity vector of a dynamical system.
As a set of initial conditions is selected randomly and the correspond-
ing trajectories are restricted in a compact region in the phase space, the classical
definition of the maximum LE is

\[
\lambda_N = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)},
\]

where the separation \( d \) between two neighboring trajectories is required to be
sufficiently small so that the deviation vector \((\Delta \mathbf{q}, \Delta \dot{\mathbf{q}})\) can be regarded as a
good approximation of a tangent vector, and the distance \( d(t) \) at time \( t \) is of
the form

\[
d(t) = \sqrt{\Delta \mathbf{q}(t) \cdot \Delta \mathbf{q}(t) + \Delta \dot{\mathbf{q}}(t) \cdot \Delta \dot{\mathbf{q}}(t)}. \tag{2}
\]

It is the Euclidian distance in the phase space. \( \mathbf{q} \) and \( \dot{\mathbf{q}} \) are in different di-
-men-ion, so one has to carefully choose their units to assure that both terms
in the expression of \( d(t) \) are important. We suggest computing the LE in the
configuration space rather than in the phase space, that is,

\[
d'(t) = \sqrt{\Delta \mathbf{q}(t) \cdot \Delta \mathbf{q}(t)} \tag{3}
\]
\[ \lambda'_{N} = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d'(t)}{d'(0)}, \]  

(4)

We argue that both \( \lambda_{N} \) and \( \lambda'_{N} \) are effective in detecting chaotic behavior of orbits since \( \Delta \dot{q}(t) = \frac{d}{dt}(\Delta q(t)) \) and they should have the same Lyapunov exponents.

In general relativity the above definition and algorithm are questionable. First, there is no unified time for all the reference systems. Secondly, the separation of time and space of the 4-dimensional spacetime is different for different observers. Furthermore, time and space coordinates usually play book-keeping only for events and are not necessary with a physical meaning. Consequently, one would get different values of \( \lambda_{N} \) and \( \lambda'_{N} \) in different coordinate systems.

Up to now the references for exploring chaos in relativistic models have been mainly interested in studying the dynamical behavior of black hole systems[5-8] and the mixmaster cosmology[9-13]. Most of them adopt the classical definition of LE. Here we will abandon the variational method in the case of general relativity for it is rather cumbersome to derive variational equations. Actually there exist general expressions for geodesic deviation equations[5] but one has to derive the complicated curvature tensor. Furthermore, in many cases a particle does not follow a geodesic. For instance, a spinning particle in Schwarzschild spacetime doesn’t move along a geodesic[5]. Therefore, we will concentrate on constructing a revised edition of the two-particle method. The classical algorithm of LE lacks invariance in general relativity, which depends on a coordinate gauge and even could bring spurious chaos in some coordinate systems. The chaotic behavior in the mixmaster cosmology[9-13] has been debated for decade or so. Using different time parameterization [14,15], one would get distinct values of \( \lambda_{N} \). Especially, in a logarithmic time, chaos
becomes hidden[12]. Chaos, as an intrinsic nature of a given system, should not be affected by the choice of a coordinate gauge, and a chaos indicator, LE, should be defined as invariant under spacetime transformations. It means that the LE in general relativity should be expressed as a physical or so called proper quantity but not a coordinate quantity[16-19].

Now we consider a particle, called “observer”, moving along an orbit (not necessary to be a geodesic) in the spacetime with a metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. In this letter Greek letters run from 0 to 3 and Latin letters from 1 to 3. The observer can determine if his motion would be chaotic by observing whether the proper distances from his neighbors are increasing exponentially or not with his proper time. Both the distances and time are observables and should not depend on the choice of a coordinate system. Then we can apply the theory of observation in general relativity[16] to explore the dynamical behavior of the observer.

Fig. 1 illustrates the trajectories of the observer and its neighboring particle called “neighbor”. The initial condition of the neighbor is arbitrarily chosen as long as its 4-dimensional distance from the observer is small enough to assure that the neighbor is approximately in the tangent space of the observer. In an arbitrary spacetime coordinate system $x^\alpha$, one can derive the equations of motion of the observer and its neighbor. Here we have to notice that the independent variable of the equations has to be the coordinate time $t$ rather than their proper times $\tau$ because the two particles have different proper times but one unique independent variable has to be adopted when integrating numerically their equations of motion together. At the coordinate time $t$ the observer arrives at the point $O$ with the coordinate $x^\alpha$ and 4-velocity $U^\alpha$, and the corresponding proper time of the observer is $\tau$. At the same coordinate
time $t$, its neighbor reaches the point $P$ with the coordinate $y^\alpha$ along another orbit. A displacement vector $\delta x^\alpha = y^\alpha - x^\alpha$ from $O$ to $P$ should be projected into the local space of the observer. The space projection operator of the observer is constructed as $h^{\alpha\beta} = g^{\alpha\beta} + c^{-2}U^\alpha U^\beta$ (c represents the velocity of light)\[16\]. $\overrightarrow{OP}$ represents the projected vector $\delta x^\alpha_{\perp} = h^{\alpha\beta}_{\perp} \delta x^\beta$, and its length $\|\overrightarrow{OP}\| \perp$ is

$$\Delta L(\tau) = \sqrt{g_{\alpha\beta} \delta x^\alpha_{\perp} \delta x^\beta_{\perp}} = \sqrt{h_{\alpha\beta} \delta x^\alpha \delta x^\beta}.$$  \tag{5}$$

Here $g_{\alpha\beta}$ is calculated at $x^\alpha$. $\Delta L(\tau)$ is the proper distance to the neighbor observed by the observer at his time $\tau$ and it is a scalar. Hence the maximum LE in general relativity is defined as

$$\lambda_R = \lim_{\tau \to \infty} \frac{1}{\tau} \ln \frac{\Delta L(\tau)}{\Delta L(0)}.$$  \tag{6}$$

Let $\Sigma_\tau$ be the 3-dimensional subspace of the tangent space of the observer at $O$, which is orthogonal to the 4-velocity $U^\alpha$. It is the point $P'$ but not $P$ in $\Sigma_\tau$ as long as $P'$ is close enough to the observer $O$. This tells us that $\lambda_R$ is coordinate invariant. The next is an argument for this point. Let us carry out a time transformation $t \to \eta$. For convenience, we express the projection operation as $\overrightarrow{OP} \perp = H\overrightarrow{OP}$, where $H$ is the space projection operator and $H\overrightarrow{U}_O = 0$, where a subscript is added for $\overrightarrow{U}$ to describe the 4-velocity. Assume that the observer is located at the point $O$ at $\eta$ that corresponds to the old coordinate time $t$, then the neighbor at $\eta$ would situate at the point $Q$ but not necessary at $P$. Certainly we have to keep both $P$ and $Q$ inside an $\epsilon$ neighborhood of $O$ (see Fig.1) and $\epsilon$ is considered as a very small quantity. In this case we have $\overrightarrow{U}_P = \overrightarrow{U}_O + O(\epsilon)$. Furthermore, we have $\overrightarrow{OQ} = \overrightarrow{OP} - \overrightarrow{QP}$ and $\overrightarrow{PQ} = \Delta \tau_P \overrightarrow{U}_P + O(\Delta \tau_P^2) = k\epsilon \overrightarrow{U}_P + O(\epsilon^2)$, where $k$ is a constant and
the proper time interval of the neighbor between $P$ and $Q$, $\Delta \tau_P$, is in the magnitude of $\epsilon$. Hence, we get the relation $\overrightarrow{HOQ} = \overrightarrow{HOP} + O(\epsilon^2)$. This shows that $\lambda_R$ is invariant with time transformations.

As far as the numerical implementation of the computation of $\lambda_R$ is concerned, the following notes are worth noticing. (1) The observer and the neighbor have different proper times, so a coordinate time $t$ should be adopted as the independent variable. The equations of the motion of the particles should be transformed to use $t$ as the independent variable. The variables to be computed step by step are the space coordinates and velocities (with respect to $t$) of the observer and its neighbor, and the proper time $\tau$ and $d\tau/dt$ of the observer. In total there are 14 variables. (2) As is well known, renormalization after a certain time interval is essential in this procedure to keep the distance be-
tween the observer and its neighbor small enough. The renormalization must be proceeded in the phase space though our LE is calculated in the configuration space. (3) One important difference during renormalization between the relativistic and classical cases must be noticed. Let \( \Sigma_t \) be the local 3-space in which all the points have the same coordinate time \( t \). It is evident that the renormalization should be done in \( \Sigma_t \) otherwise the computation will commit an error. In Fig.1 \( P \) is pulled back to \( M \) and \( \overrightarrow{OM} = (\Delta L(0)/\Delta L(\tau))\overrightarrow{OP} \). The next integration step for the neighbor will start from the point \( M \). It is obvious that \( \overrightarrow{OP} \) is located in \( \Sigma_t \) because both points \( O \) and \( P \) are in \( \Sigma_t \). On the other hand \( \overrightarrow{OP}' \) is in the 3-space \( \Sigma_\tau \) and a pull back from \( P' \) to \( S \) as renormalization is not correct. Our practice has proven this argument.

In order to check the validity of our scheme for calculating the relativistic LE, we are going to reexamine the core-shell system studied by Vieira & Letelier[6]. In Schwarzschild coordinates \( (t, r, \theta, \phi) \), the 4-metric for this system is of the form

\[
ds^2 = -(1 - \frac{2}{r})e^{A}dt^2 + e^{B-A}[(1 - \frac{2}{r})^{-1}dr^2 + r^2d\theta^2] + e^{-A}r^2\sin^2\theta d\phi^2,
\]

where \( A \) and \( B \) are functions of \( r \) and \( \theta \) only. Here we adopt nondimensional variables and take \( c = 1 \). The metric does not explicitly depend on \( t \) and \( \phi \) and has an energy constant \( E \) and an angular momentum constant \( L \), so test particles in free fall are actually in a system with two degrees of freedom with an integral \( U^\alpha U_\alpha = -1 \). When \( A = B \equiv 0 \), Eq.(7) represents the Schwarzschild spacetime in which test particles in free fall move in regular orbits due to integrability of the system. For simplicity, we discuss the dynamical behavior of geodesics in a black hole plus a dipolar shell, and let
\begin{align*}
A &= 2\sigma\mu v, \\
B &= \gamma_0 + 4\sigma v - \sigma^2[\mu^2(1 - v^2) + v^2], \tag{8}
\end{align*}

where \(\mu = r - 1\) and \(v = \cos \theta\). The Newtonian limit of the model resembles the Stark problem\cite{20,21}, which is fully integrable. However, as far as the relativistic model is concerned, Vieira & Letelier\cite{6} and Saa & Venegroles\cite{7} have demonstrated strong chaos in Weyl coordinates using the Poincaré surface of section, and Guéron & Letelier\cite{8} estimated its maximum LE \(\lambda_N = (3.2 \pm 0.4) \times 10^{-4}\) with the classical definition of LE (see Eq.(1)) in prolate spheroidal coordinates.

![Graph](image)

**Fig. 2.** The maximum Newtonian Lyapunov exponent, \(\lambda_N\), with the time variable \(\eta\). It becomes one order of magnitude smaller after the time transformation (see Eq.(9)).

We choose parameters as \(E = 0.975, L = 3.8, \sigma = 2.5 \times 10^{-4}\), and \(\gamma_0 = \sigma^2\),
Fig. 3. The maximum relativistic Lyapunov exponent, $\lambda_R$, with the proper time $\tau$ of the observer. It remains invariant after the time transformation (see Eq.(9)). $\tau$ reaches about $9.168 \times 10^5$ when $\eta$ runs $10^7$.

and the initial conditions of the observer as $r = 32$, $\theta = \frac{\pi}{2}$, $\phi = 0$, $\dot{r} = 0$, and $\dot{\theta}$ from $U^\alpha U_\alpha = -1$. As to its neighbor, an initial separation $\Delta r = -10^{-8}$ is adopted, regarded as the best choice[3], and the others remain the same as the observer’s except $\dot{\theta}$. The values of $E$ and $L$ are carefully chosen to assure that the trajectories of the observer and its neighbors are bounded in a compact region. We integrate the geodesic equations using two integrators for comparison, Runge-Kutta-Fehlberg 7(8) and the 12th-order Adams-Cowell method, with a coordinate time step 0.01. When $t$ reaches $10^6$, we find $\lambda_N = \lambda'_N = (2.2 \pm 0.2) \times 10^{-4}$, which is close to the result of [8] as its integration time amounts to $10^5$. Meanwhile, we get $\lambda_R = (2.8 \pm 0.2) \times 10^{-4}$ by Eq.(6).

One may notice that $\lambda_N$ and $\lambda_R$ are in the same magnitude. This is because
the Euclidian distance \(d'(t)\) and the invariant proper distance \(\Delta L\) differ not very much and so do the coordinate time \(t\) and the proper time \(\tau\). In fact, \(\tau\) runs about \(9.174 \times 10^5\) when \(t\) passes through \(10^6\). A stronger gravitation by decreasing the angular momentum \(L\) will increase the difference between \(\lambda_N\) and \(\lambda_R\), but the smallest stable circular orbit in Schwarzschild spacetime is \(r = 6\).

To verify the invariance of \(\lambda_R\), we do a time transformation

\[
t \to \eta = 10t + r^2/2.
\]

(9)

We obtain \(\lambda_N = (2.6 \pm 0.2) \times 10^{-5}\) in the coordinate time \(\eta\), while \(\lambda_R\) retains the original value (see Fig.2 and Fig.3).

As a further experiment, we slightly change the model to put \(B/2 \equiv A = 2\sigma \mu \nu\). This system is still chaotic in Schwarzschild coordinates \((t, r, \theta, \phi)\). To remove the singularity of this metric at the horizon, we go to Lemaître coordinates \((T, R, \theta, \phi)\)[22] as

\[
T = \kappa(t + 2\sqrt{2r} + 2 \ln \frac{\sqrt{r} - \sqrt{2}}{\sqrt{r} + \sqrt{2}}),
\]

\[R = \frac{2\kappa r}{3} \sqrt{\frac{T}{2}} + T,\]

(10)

where the positive constant \(\kappa\) may be chosen arbitrarily. Our numerical tests display that the larger \(\kappa\) is, the smaller \(\lambda_N\) becomes. Particularly, the Lemaître coordinates hide chaos for sufficient large \(\kappa\) if \(\lambda_N\) is adopted as a chaotic index. However, our \(\lambda_R\) does not vary with the parameter \(\kappa\).

We would like to emphasize again that the computation of \(\lambda_R\) can be applied, whether the particles move along geodesics or not. The theory of observa-
tion in general relativity is not relevant to the 4-acceleration of the observer. For example, this method can be used to identify chaos in compact binary systems[23,24].

In this letter we concentrate on calculating an invariant maximum LE in general relativity, but this discussion can be easily extended to find all the relativistic LEs numerically with the technique proposed by Benettin et al.[25] (also see [26]). They suggest choosing an initial set of orthonormal tangent vectors as a base of the tangent space of the observer, then compute the evolution of the volume determined by these vectors. To avoid two vectors getting close to each other under evolution they use Gram-Schmidt procedure to orthonormalize the base after each time step. The only change in a relativistic model is in computing the evolution of a tangent vector, which can be realized by the technique in this letter. As to the inner product in the orthonormalizing procedure and the norm computation, one has to use the Riemanian inner product in place of the Euclidean one.

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