Mass of Neutron Star in SdS space-time

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Abstract In this work we present a modified TOV equation which incorporates the cosmological constant with regard to the recent astronomical observations that the Universe is in a phase of accelerated expansion. Using this modified TOV equation we considered the structure of a neutron star in SdS space-time and calculated maximum mass limit for neutron stars.

Keywords TOV equation · Neutron Star · Maximum mass · cosmological constant

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1 Introduction

Neutron stars can be included in the class of compact stars which are stars with high mass and small radius and their density will be very high. So we need to take into account effects from general relativity, like the curvature of space–time in the study of neutron stars. Baade and Zwicky were the first to propose the idea of neutron stars (Baade and Zwicky, 1934), pointing out that they have very high density and small radius. They also made a very important suggestion that neutron stars would be formed in supernova explosions. We know that there are two forces acting on a star, one of them is gravitation and the second one arises from the pressure. The opposing force of gravitation will be thermal pressure for ordinary stars and degeneracy pressure for white dwarfs and neutron stars. Thus we can obtain two coupled differential equations for mass and pressure which tell their change with the radius of the
star. Tolman-Oppenheimer-Volkoff (TOV) equation (Tolman, 1939; Volkoff and Oppenheimer, 1939) derived from the Einstein’s field equation gives the pressure gradient of neutron stars. Numerical model of a neutron star with a relativistic mean field theory was given by Walecka (Walecka and Chin, 1974; Walecka, 1975).

Our Universe can be represented by SdS metric, since the recent astronomical observations show that the expansion of our universe is accelerating (Guzzo et al., 2008; Perlmutter et al., 1999; Riess et al., 1998; Weinberg, 1972). Thus Schwarzschild-de Sitter metric described with a positive cosmological constant and having two horizons is considered to be a model of this universe. In such a case, the use of TOV equation, derived from the Einstein’s field equation in the case of accelerating universe, will be more apt for the theoretical constructions of stellar models. Thus we will get a modified TOV equation for pressure, for calculating the maximum mass of a neutron star. Anisotropic models for compact self gravitating objects have been studied extensively (Corchero, 2001; Dev and Gleiser, 2003; Herrera and Santos, 1997; Ivanov, 2002; Mak and Harko, 2003). Our aim is to establish a modified TOV equation which is the prime structure equation for compact stars.

In §2 we will show the derivation of the new equation to replace the TOV equation, starting with the standard metric for a spherically symmetric star and the consecutive use of Einstein’s field equation in the case of accelerating universe. Then we will show the numerical calculation of maximum mass of Neutron stars by plotting the mass-radius relation.

2 Derivation of the modified TOV equation

We have the standard metric (Weinberg, 1972) for a spherically symmetric star,

\[ ds^2 = -U(r)dt^2 + V(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]  

where,

\[ V(r) = \frac{1}{U(r)}. \]

Ricci tensor is given by,

\[ R_{\mu k} = \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\lambda} + \Gamma^\eta_{\mu k} \Gamma^\lambda_{\eta -} - \Gamma^\eta_{\mu k} \Gamma^\lambda_{\eta f}. \]

Substituting for the Christoffel symbols, we will get the equations as,

\[ R_{rr} = \frac{U''}{2U} - \frac{U'}{4U} \left( \frac{V''}{V} + \frac{U''}{U} \right) - \frac{1}{r} \frac{V'}{V}, \]

\[ R_{\theta \theta} = -1 + \frac{r}{2V} \left( \frac{V'}{V} + \frac{U'}{U} \right) + \frac{1}{V}, \]

\[ R_{\phi \phi} \approx R_{\theta \theta}. \]
\[ R_{tt} = -\frac{U''}{2V} + \frac{U'}{4V} \left( \frac{V'}{V} + \frac{U'}{U} \right) - \frac{U'}{rV}. \]  

The result \( R_{\phi\phi} \approx R_{\theta\theta} \) is the consequence of the rotational invariance of the metric.

Einstein’s field equation in the case of accelerating universe (Weinberg, 1972) is given by,

\[ G_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \]  

where

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \]  

Substituting Eq. (9) in Eq. (8), we get

\[ R_{\mu\nu} = -8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right). \]  

From hydrodynamics, we have the relation for energy-momentum tensor in the case of models with cosmological constant,

\[ \tilde{T}_{\mu\nu} = \tilde{p} g_{\mu\nu} + (\tilde{\epsilon} + \tilde{\rho}) u_\mu u_\nu, \]

where \( \tilde{p} \)-the modified pressure, \( \tilde{\epsilon} \)-the modified total energy density and \( u_\mu \)-the velocity four vector. Here

\[ \tilde{p} = p - \frac{\Lambda}{8\pi G}, \quad \tilde{\epsilon} = \epsilon + \frac{\Lambda}{8\pi G}, \]

Also the velocity vector is defined so that,

\[ g^{\mu\nu} u_\mu u_\nu = -1. \]

Assuming fluid at rest, we have

\[ u_r = u_\theta = u_\phi = 0, \]

\[ u_t = -\left( -g^t_t \right)^{-\frac{1}{2}} = -\sqrt{U(r)}. \]

Our assumptions of time independence and spherical symmetry imply that \( p \) and \( \epsilon \) are functions only of the radical co-ordinate \( r \).

Using Eq. (10), we will get

\[ R_{rr} = -\frac{U''}{2U} + \frac{U'}{4U} \left( \frac{V'}{V} + \frac{U'}{U} \right) - \frac{1}{r} \frac{V'}{V} \]

\[ = -4\pi G(\epsilon - p)V - \Lambda V, \]  

\[ R_{\theta\theta} = -1 + \frac{r}{2V} \left( \frac{V'}{V} + \frac{U'}{U} \right) + \frac{1}{V} \]

\[ = -4\pi G(\epsilon - p)r^2 - \Lambda r^2, \]
Here single prime denotes first derivative and double prime denotes second derivative.

We have the equation for hydrostatic equilibrium,

\[
\frac{U'}{U} = -2p' \rho + \epsilon. \tag{19}
\]

Let us derive the equation for \( V(r) \) alone. Using Eq. (16), Eq. (17) and Eq. (18), we will get

\[
-\frac{V'}{rV^2} - \frac{1}{r^2} + \frac{1}{Vr^2} = -8\pi G \epsilon - \Lambda, \tag{20}
\]

which can be written as

\[
\left( \frac{r}{V} \right)' = 1 - 8\pi G \epsilon r^2 - Ar^2. \tag{21}
\]

Integrating,

\[
V(r) = \left[ 1 - \frac{2GM(r)}{r} - \frac{A}{3} r^2 \right]^{-1}, \tag{22}
\]

where

\[
M(r) = \int_0^r 4\pi r^2 \epsilon(r') dr'. \tag{23}
\]

Eq. (22) tells that outside the star, the space-time looks like SdS.

Now, we can use Eq. (19), Eq. (22) and Eq. (23) to eliminate \( U(r) \) and \( V(r) \) from Eq. (17), which becomes

\[
- r^2 p' = GM(r) \epsilon(r) \left[ 1 + \frac{p(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4\pi r^3 p(r)}{M(r)} + \frac{A r^3}{GM(r)} \right] \left[ 1 - \frac{2GM(r)}{r} - \frac{A r^2}{4} \right]. \tag{24}
\]

Thus the modified TOV equation is,

\[
\frac{dp}{dr} = -\frac{GM(r) \epsilon(r)}{r^2} \left[ 1 + \frac{p(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4\pi r^3 p(r)}{M(r)} + \frac{A r^3}{GM(r)} \right] \left[ 1 - \frac{2GM(r)}{r} - \frac{A r^2}{4} \right]. \tag{25}
\]

This equation is different from TOV equation and this becomes TOV equation for \( \Lambda = 0 \).

Now we can use this new equation to find the maximum mass of neutron stars. In this work, neutron stars are considered to be composed of a Fermi gas of degenerate neutrons. The mass-radius and density-radius relations are plotted and maximum mass limit is checked.
We have the standard structure equations for a spherically symmetric star which describe how the mass and the pressure of a star change with radius:

\[
\frac{dm}{dr} = \frac{4\pi r^2 \epsilon (r)}{c^2}, \tag{26}
\]

\[
\frac{dp}{dr} = -\frac{G\epsilon (r)m(r)}{c^2 r^2}, \tag{27}
\]

where \(G\) is Newton’s Gravitational constant, \(\epsilon\) is the mass density, \(m\) is the mass up to radius \(r\) and \(c\) is the velocity of light.

The modified TOV equation is similar to the differential equation for the pressure, Eq. (27), but here has three correction factors. All three correction factors are larger than one, i.e. they strengthen the term from Newtonian gravity. Here one has to solve (Chandrasekhar, 1931) the coupled differential Equations (26) and (27). We must notice that while \(\frac{dm}{dr}\) is positive, \(\frac{dp}{dr}\) must always be negative. Starting with certain positive values for \(m\) and \(p\) in a small central region of the star the mass will increase while the pressure decreases eventually reaching zero. We can set \(m(r = 0) = 0\) and , in addition we specify the central pressure \(p(r = 0) = p_0\) in order to solve for Equations (26) and (25). The behaviour of \(m\) and \(p\) as a function of radius will become important for numerical calculations.

Both the coupled differential equations depend on energy density. So we need a relation between pressure \(p\) and energy density \(\epsilon\) which is known as the Equation of State (EoS) (Shapiro and Teukolsky, 1983) for the neutron stars. For mass calculation, we will use the well known EoS:

\[
p = K \epsilon^\gamma, \tag{28}
\]

where \(K\) is a constant and \(\gamma\) is the polytropic exponent. \(\gamma = \frac{4}{3}\) in general relativistic case (Silbar and Reddy, 2004).

We will give the value of cosmological constant (Carmeli and Kuzmenko, 2001; Carroll, 2000) as \(\Lambda \approx 10^{-46} \text{km}^{-2}\).

3 Result and Discussion

The mass-radius and density-radius relations obtained is shown in Fig. 1 and Fig. 2. For calculational purpose, we put \(G=1\) and \(c=1\) and then pick the units of all dimensionful quantities to be powers of kilometers.

Converting the results back from natural units, we will get the maximum mass of neutron star \(M=0.74 M_\odot\) and radius \(R=6.98\text{km}\). Comparing with the result that obtained when calculated using the normal TOV equation, we found that the obtained values are lesser. So we can conclude that there is a slight influence of cosmological constant in the stellar structure. The influence of cosmological constant will be higher if the value is higher than the recently accepted one. Future advancements in dark energy
research may strengthen the importance of cosmological constant. The modified TOV equation could also be used in the calculations of surface tension of compact stars (Bagchi, et al., 2005; Dey, et al., 1998).

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