Abstract

We explore some general consequences of a proper, full enforcement of the “twisted Poincaré” covariance of Chaichian et al [14], Wess [52], Koch et al [35], Oeckl [43] upon many-particle quantum mechanics and field quantization on a Moyal-Weyl noncommutative space(time). This entails the associated braided tensor product with an involutive braiding (or ⋆-tensor product in the parlance of Aschieri et al [3, 4]) prescription for any coordinates pair of \(x, y\) generating two different copies of the space(time); the associated nontrivial commutation relations between them imply that \(x - y\) is central and its Poincaré transformation properties remain undeformed. As a consequence, in QFT (even with space-time noncommutativity) one can reproduce notions (like space-like separation, time- and normal-ordering, Wightman or Green’s functions, etc), impose constraints (Wightman axioms), and construct free or interacting theories which essentially coincide with the undeformed ones, since the only observable quantities involve coordinate differences. In other words, one may thus well realize QM and QFT’s where the effect of space(time) noncommutativity amounts to a practically unobservable common noncommutative translation of all reference frames.
1 Introduction: Moyal-Weyl spaces, twisted Poincaré “group” and QFT

In the last decade a broad attention has been devoted to the construction of Quantum Field Theories (QFT) on the perhaps simplest examples of noncommutative spaces, the so-called Moyal-Weyl spaces. These are characterized by coordinates $\hat{x}^\mu$ fulfilling the commutation relations
\[
[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu},
\]
where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix. The $\theta^{\mu\nu} = 0$ limit is the undeformed algebra $A$ generated by commuting coordinates $x^\mu$. For the sake of definiteness we shall suppose (with the exception of Section 3) $\mu = 0,1,2,3$ and endow the space with the ordinary Minkowski metric $\eta_{\mu\nu}$, to obtain a deformation of the $3 + 1$-dimensional Minkowski spacetime. As $\theta^{\mu\nu}$ is not an isotropic tensor, relations (1) are not covariant (i.e. not form-invariant) under Lorentz transformations of the reference frame (although they are invariant under translations).

The unital algebra $\hat{A}$ generated by these $\hat{x}^\mu$ is isomorphic to the one $A_\theta$ which is obtained by endowing the vector space underlying $A$ (extended over the formal power series in $\theta^{\mu\nu}$) with a deformed product, the $\star$-product, which can be formally defined by
\[
a \star b := (F^{(1)} \triangleright a)(F^{(2)} \triangleright b).
\]
For typographical convenience we have denoted by $F \equiv F^{-1}$ the inverse of the so-called twist $F$. It (and therefore also the associated isomorphism $\phi : \hat{A} \to A_\theta$) is not uniquely determined, but what follows does not depend on the specific choice of $F$. The simplest is
\[
F \equiv F^{(1)} \otimes F^{(2)} : = \exp \left( -\frac{i}{2} \theta^{\mu\nu} P_\mu \otimes P_\nu \right).
\]
$P_\mu$ denote the generators of translations, and $\triangleright$ in general denotes the action of the Universal Enveloping algebra (UEA) $UP$ of the Poincaré Lie algebra $P$ (on $A$ this amounts to the action of the corresponding algebra of differential operators, e.g. $P_\mu$ can be identified with $i\partial_\mu := i\partial/\partial x^\mu$). In the second expression and in (2) we have used a Sweedler notation with suppressed summation index: $F^{(1)} \otimes F^{(2)}$ stands in fact for a (infinite) sum $\sum_i F^{(1)}_i \otimes F^{(2)}_i$. Relation (2) with the specific choice (3) of the twist gives in particular
\[
\hat{x}^\mu \hat{x}^\nu \xrightarrow{\phi} x^\mu \star x^\nu = x^\mu x^\nu + i\theta^{\mu\nu}/2.
\]
As a result, $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, i.e. again (1), as claimed. One advantage of working with $A_\theta$ instead of $\hat{A}$ is that integration over the original commutative space can be used also on the noncommutative one without loosing its properties (in particular Stoke’s theorem). In addition,
\[
\int d^4x \ a \star b = \int d^4x \ a b
\]
for any regular $a,b$ functions in the vector space underlying $A$ vanishing sufficiently fast at infinity. The definition (2-3) involves a power series in $\theta^{\mu\nu}$ and for the moment
should be regarded as formal: it can be applied to a much larger domain if \( \mathcal{F} \) is rather realized as the integral operator, as we shall explain in [13].

Different (obviously not Lorentz-covariant) approaches to quantization of field theory on Moyal-Weyl spaces have been proposed (see [19], [22], [20], [50] and references therein). New complications appear, like non-unitarity [29], violation of causality [17], UV-IR mixing of divergences [42] and subsequent non-renormalizability, alleged change of statistics, etc. Some of these problems, like non-unitarity [5], or the very occurrence of divergences [6], may be due simply to naive (and unjustified) applications of commutative QFT rules (path-integral methods, Feynman diagrams, etc) and could disappear adopting a sounder field-operator approach.

In Ref. [14], [52], [35] it has been recognized that the commutation relations of \( \hat{A} \sim A_\theta \) are in fact covariant under a deformed version of the Poincaré group, namely the triangular nonco commutative Hopf *-algebra \( H \) obtained from \( U\mathcal{P} \) by “twisting” [21] with \( \mathcal{F} \) (this result had been in fact anticipated in terms of corepresentations of the dual Hopf algebra in section 4.4.1 of [43]. For a general introduction to the twist see e.g. [18]). This means that (up to possible isomorphisms) the algebra structure and the counit \( \varepsilon \) of \( U\mathcal{P}, H \) (extended over the formal power series in \( \theta^{\mu\nu} \)) are the same, but the coproduct is changed through the similarity transformation

\[
\Delta(g) \equiv g^{(1)} \otimes g^{(2)} \quad \longrightarrow \quad \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv g^{(1)} \otimes g^{(2)}, \quad g \in H = U\mathcal{P} \quad (5)
\]

(at the rhs’s we have again used Sweedler notation with suppressed summation indices), and the antipode \( S \) accordingly. A straightforward computation gives

\[
\hat{\Delta}(P_\mu) = P_\rho \otimes 1 + 1 \otimes P_\mu = \Delta(P_\mu), \quad \hat{\Delta}(M_\omega) = M_\omega \otimes 1 + 1 \otimes M_\omega + P[\omega, \theta] \otimes P \neq \Delta(M_\omega),
\]

where we have set \( M_\omega := \omega^{\mu\nu} M_{\mu\nu} \) and used a row-by-column matrix product on the right. The left identity shows that the Hopf \( P \)-subalgebra remains undeformed and equivalent to the abelian translation group \( T \sim \mathbb{R}^3 \). Denoting by \( \triangleright, \langle \rangle \) the (say, left) actions of \( U\mathcal{P}, H \), they coincide on first degree polynomials in \( x^\nu, \hat{x}^\nu \),

\[
P_\mu \triangleright x^\rho = i \delta_\mu^\rho = P_\mu \langle \hat{x}^\rho, \quad M_\omega \triangleright x^\rho = 2i(x\omega)^\rho, \quad M_\omega \langle \hat{x}^\rho = 2i(x\omega)^\rho, \quad (6)
\]

and more generally on irreps (irreducible representations); as noted in [14], this yields the same classification of elementary particles as unitary irreps of \( \mathcal{P} \). But \( \triangleright, \langle \rangle \) differ on products of coordinates, and more generally on tensor products of representations, as \( \triangleright \) is extended by the rule \( g \triangleright (ab) = (g^{(1)} \triangleright a)(g^{(2)} \triangleright b) \) involving \( \Delta(g) \) (the rule reduces to the usual Leibniz rule for \( g = P_\mu, M^{\mu\nu} \)), whereas \( \langle \rangle \) is extended on products of elements in both \( \hat{A}, A_\theta \) by the rule

\[
g\langle \rangle \langle \hat{a}\hat{b} \rangle = (g^{(1)} \langle \rangle \hat{a})(g^{(2)} \langle \rangle \hat{b}) \quad \Leftrightarrow \quad g\langle \rangle \langle a \ast b \rangle = (g^{(1)} \langle \rangle a) \ast (g^{(2)} \langle \rangle b), \quad (7)
\]

which respects the commutation relations (1), making \( \hat{A}, A_\theta \) isomorphic \( H \)-module algebras; this deforms in particular the Leibniz rule of \( M^{\mu\nu} \) (but not of \( P_\mu \)).

How to implement this twisted Poincaré covariance in QFT is subject of debate and different proposals [15], [16], [51], [7], [8], [9], [12], [55], [68], [1], two main issues being whether one should: \( a) \) take the *-product of fields at different spacetime points; \( b) \) deform the
canonical commutation relations (CCR) of creation and annihilation operators \( a, a^\dagger \) for free fields.

The aim of this work is to point out that a proper enforcement of twisted Poincaré covariance answers affirmatively to a) and brings a radical simplification to the framework, in that all coordinate differences become \( \ast \)-central, i.e. central w.r.t. the \( \ast \)-product (section 2). We first explore (section 3) some consequences of the latter fact in \( n \)-particle Quantum Mechanics (QM): we find that twisted Galilei covariance is compatible with Bose or Fermi statistics and that the dynamics of an isolated system of \( n \)-particles is the same as its counterpart on commutative space. As for QFT, which we treat in field-operator approach, we sketch the general consequences of (slightly adapted) Wightman axioms in section 4, show in section 5 that the latter can be satisfied by free (for simplicity scalar) fields if we also suitably deform the CCR of the \( a, a^\dagger \)'s so that the \( \ast \)-commutator of the fields is equal to the undeformed counterpart, show in section 6 that then the time-ordered perturbative computation of Green functions of a scalar \( \varphi^n \) interacting theory gives the same results as the undeformed theory. In other words, we end up in this way with twisted Poincaré covariant QFT’s which are physically equivalent to their counterparts on commutative Minkowski space, with the obvious consequence that the above-mentioned complications will disappear. In Section 7 we draw some conclusions and briefly comment on the alternatives implying violation of the cluster property by the Wightman functions.

2 The action of the twisted Poincaré “group” on several spacetime variables

Dealing with \( n \)-point (Green’s, or Wightman’s, etc) functions in QFT requires \( n \) sets of noncommutative Minkowski spacetime coordinates \( \hat{x}^\mu_i, i = 1, ..., n \), of type (1). Similarly, dealing with \( n \)-particle QM requires \( n \) sets of noncommutative Euclidean space coordinates \( \hat{x}^\mu_i \), (one for each particle) of type (1).

Our starting, basic observation is that to consistently adopt the viewpoint of twisted Poincaré covariance one should require that also the larger algebra \( \hat{A}^n \) generated by them is a \( H \)-module algebra, meaning in particular that within the latter (7) still holds. This is also the philosophy adopted in Ref. [4]. To this end one cannot adopt as \( \hat{A}^n \) the tensor product algebra of \( n \) copies of \( \hat{A} \), or equivalently assume trivial commutation relations

\[
[\hat{x}^\mu_i, \hat{x}^\nu_j] = 0 \quad i \neq j,
\]

as done e.g. in [28] [9], because the latter are incompatible with (7) by the non-cocommutativity of \( \Delta \) (this can be checked e.g. by letting the Lorentz generators \( M^{\rho\sigma} \) act on both sides). In fact it is a basic property of quasitriangular Hopf algebra theory (see e.g. [11]) that one has to adopt as \( \hat{A}^n \) rather the deformation of the tensor product algebra, usually called braided tensor product algebra, dictated by the quasitriangular structure \( R \) of \( H \). Given two left \( H \)-module algebras \( \hat{M}, \hat{M}' \) the braided tensor product algebra \( \hat{M} \otimes \hat{M}' \) is still \( \hat{M} \otimes \hat{M}' \) as a vector space, but is characterized by the product

$$
(\hat{m} \otimes \hat{m}') \cdot (\hat{n} \otimes \hat{n}') = \hat{m}(\mathcal{R}(2)\hat{\sigma}\hat{n}) \otimes (\mathcal{R}(1)\hat{\sigma}\hat{m}')\hat{n}',
$$

(8)
where we have again used a Sweedler notation with suppressed summation index: \( \mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \) stands in fact for an (infinite) sum \( \sum_i \mathcal{R}_i^{(1)} \otimes \mathcal{R}_i^{(2)} \). In the present case \( \mathcal{R} = \mathcal{F}_2 \mathcal{F}^{-1} = (\mathcal{F})^{-2} \) is even triangular, i.e. \( \mathcal{R} \mathcal{R}^{-1} = 1 \otimes 1 \), implying that these rules are symmetric w.r.t. to the exchange of \( \hat{M}, \hat{M}' \), or equivalently the braiding coincides with the ordinary flip up to a similarity transformation. If \( \hat{M}, \hat{M}' \) are \( H \)-module algebras, deformations of two \( UP \)-module algebras \( M, M' \), so that the isomorphisms \( \hat{M} \sim M_\theta, \hat{M}' \sim M'_\theta \) hold, the braided tensor product \( \otimes_* \) is isomorphic to the \( \ast \)-tensor product \( \otimes_* \) of \( \mathbb{A} \), which is defined by setting for any \( m \in M_\theta, m' \in M'_\theta \)

\[
    m \otimes_* m' = (\mathcal{F}^{(1)} \triangleright m) \otimes (\mathcal{F}^{(2)} \triangleright m').
\]

That this is the ‘right’ deformation of the tensor product follows also from the observation that this is nothing but the extension of the \( \ast \)-product law \([2]\) to the whole tensor product algebra \( M \otimes M' \), in the sense

\[
    m \otimes_* m' = (m \otimes 1) \ast (1 \otimes m').
\]

If \( \hat{M}, \hat{M}' \) are unital \([5]\) reduces to the ordinary tensor algebra rule if either \( \hat{m}' = 1 \) or \( \hat{n} = 1 \), as \( \varepsilon(\mathcal{R}^{(1)})\mathcal{R}^{(2)} = \varepsilon(\mathcal{R}^{(2)})\mathcal{R}^{(1)} = 1 \). As for ordinary tensor product algebras, because of the trivial algebra isomorphisms \( 1 \otimes \hat{M}' \sim \hat{M}' \), \( \hat{M} \otimes 1 \sim \hat{M} \), one can simplify the notation by dropping the units, i.e. denote \( \hat{m} \otimes 1 \) and \( 1 \otimes \hat{m}' \) resp. by \( \hat{m}, \hat{m}' \), whereby the only novelty of \([5]\) remains concentrated in the nontrivial “cross” commutation relation

\[
    \hat{m}' \hat{n} = (\mathcal{R}^{(2)} \triangleright \hat{n}) (\mathcal{R}^{(1)} \triangleright \hat{m}').
\]

Similarly, we can simplify the notation denoting the sides of \([10]\) as \( m \ast m' \) and replacing the previous relation by \( m' \ast n = (\mathcal{R}^{(2)} \triangleright n) \ast (\mathcal{R}^{(1)} \triangleright m') \).

Choosing as \( \hat{M}, \hat{M}' \) two copies of the \( \ast \)-algebra of functions \( \hat{A} \) on the Moyal-Weyl noncommutative space, calling \( \hat{x}, \hat{y} \) the respective sets of coordinates, and noting that the action of the translation generators on the coordinates is given by

\[
    P_{\mu} \hat{x}^\nu = P_{\mu} \hat{y}^\nu = i \delta^\nu_{\mu},
\]

we find

\[
    \hat{x}^\mu \hat{y}^\nu = (\mathcal{R}^{(2)} \triangleright \hat{y}^\nu) (\mathcal{R}^{(1)} \triangleright \hat{x}^\mu) = \hat{y}^\nu \hat{x}^\mu + i \theta^{\mu\nu}.
\]

These are also automatically compatible with the \( \ast \)-structure (a straightforward check, beside a consequence of \( \mathcal{R} \ast \otimes \ast = \mathcal{R}_{21} = \mathcal{R}^{-1} \)), and with setting \( \hat{x} = \hat{y} \). More generally, applying the above rule iteratively, the braided tensor product of \( n \) copies of \( \hat{A} \) and the \( \ast \)-tensor product of \( n \) copies of \( \mathcal{A}_\theta \) will be isomorphic \( H \)-module \( \ast \)-algebras \( \hat{A}^n, \mathcal{A}_\theta^n \) respectively generated by real variables \( \hat{x}_i^\mu \) and \( x_i^\mu, i = 1, 2, ..., n, \) fulfilling the commutation relations

\[
    [\hat{x}_i^\mu, \hat{x}_j^\nu] = i \theta^{\mu\nu} \quad \iff \quad [x_i^\mu, x_j^\nu] = i \theta^{\mu\nu}.
\]

This formula summarizes all the commutation relations defining \( \hat{A}^n \sim \mathcal{A}_\theta^n \): for \( i = j \) these are the defining commutation relations of the \( i \)-th copy, for \( i \neq j \) these are consequences of the braided tensor (or \( \ast \)-tensor) product between the \( i \)-th and the
j=th copy. Summing up, the algebra $A^n_\theta$ is obtained by endowing the vector space underlying the n-fold tensor product $A^n$ of $A$ with a new product, the $\star$-product, related to the product in $A^n$ by formula (2) for any $a, b \in A^n$. This encodes both the usual $\star$-product within each copy of $A$, and the $\star$-tensor product of \cite{3, 4}. More explicitly, on analytic functions $a, b$ \cite{2} reads

$$a(x_i) \star b(x_j) = \exp \left( i \frac{1}{2} \partial_{x_i} \theta \partial_{x_j} \right) a(x_i) b(x_j),$$

and must be followed by the identification $x_i = x_j$ after the action of the bi-pseudodifferential operator $\exp [i \frac{1}{2} \partial_{x_i} \theta \partial_{x_j}]$ if $i = j$.

Strictly speaking, the definitions (2-3) or (12) make sense if we choose $a, b$ in a suitable subspace $A' \subset A$ ensuring that the involved power series in $\theta^\mu$ is termwise well-defined and converges. One such subspace can be looked for within the space of (analytic) functions that are the Fourier transforms $\hat{g}$ of functions $g$ with compact support. The determination of the largest possible $A'$ is a delicate issue, about which little is known (see \cite{23} and references therein). Anyway for field-theoretic purposes it would not be enough to work with $A'$, and it is much better to define the $\star$-product as the integral with a non-local kernel

$$a(x_i) \star b(x_j) = \int d^4 h \int d^4 k \ e^{i (h \cdot x_i + k \cdot x_j - \frac{\theta h}{2})} \hat{a}(h) \hat{b}(k),$$

where $\hat{\cdot}$ denotes the antiFourier transform. This is well-defined if $a, b \in L^1(\mathbb{R}^4) \cap \dot{L}^1(\mathbb{R}^4)$, can be defined even if $a, b$ are distributions, and is designed so as to have the series (2 ) as a formal power expansion; see \cite{23} for the conditions under which the latter is in fact an asymptotic expansion. More generally, one should adopt as proper definition of the action of $\mathcal{F}, \mathcal{F}$ and of derived operators like $(\Delta \otimes \text{id}) \mathcal{F}$ the corresponding non-local integral operators. They also fulfill the cocycle condition $\mathcal{F}_{12} (\Delta \otimes \text{id}) \mathcal{F} = \mathcal{F}_{23} (\text{id} \otimes \Delta) \mathcal{F}$, ensuring the associativity of the $\star$-product.

We now define an alternative set of real generators of $A^n_\theta$ (or, correspondingly, of $\hat{A}^n$):

$$\xi^\mu_i := x^\mu_{i+1} - x^\mu_i, \quad i = 1, ..., n-1, \quad X^\mu := \sum_{i=1}^n a_i x^\mu_i$$

where $a_i$ are real numbers such that $\sum_i a_i = 1$ (in particular one could choose $X^\mu = x^\mu_j$, for some special $j$). It is immediate to verify that:

1. All $\xi^\mu_i$ are invariant under translations, (whereas $X^\mu$ is not):

$$P_\mu \delta \xi^\nu_i = 0, \quad P_\mu \hat{\delta} X^\nu = i \delta^\nu_\mu.$$  \hspace{1cm} (15)

2. $X^\mu$ generate a copy $A_{\theta,X}$ of Moyal-Weyl noncommutative space, whereas the $\star$-product with $\xi^\mu_i$ (or any function thereof) reduces to the ordinary product

$$\xi^\mu_i \star b = \xi^\mu_i b = b \star \xi^\mu_i$$

$b \in A^n_\theta$, \hspace{1cm} (16)
implying that the \( \xi^\mu_i \) are \( \ast \)-central in \( A^n_\theta \) (i.e. \( \ast \)-commute with everything),

\[
[\xi^\mu_i \ast A^n_\theta] = 0. \tag{17}
\]

Thus the central \( \ast \)-subalgebra \( A^{n-1}_{\theta,\xi} \) generated by the \( \xi^\mu_i \) reduces to the ordinary tensor product algebra of \( n-1 \) copies of the undeformed \( A \) [because of the trivial action (15) of the \( P_\mu \) contained in the twist \( \mathcal{F} = \exp (\frac{i}{\hbar} \theta^{\mu\nu} P_\mu \otimes P_\nu) \) and in \( \mathcal{R} = \mathcal{F}^{-2} \) on the tensor factors], whereas \( A^n_\theta \) reduces to the tensor product algebra \( A^n_\theta = A^{n-1}_{\theta,\xi} \otimes A_{\theta,X} \). Moreover, the \( \xi^\mu_i \) have the same spectral decomposition on the whole \( \mathbb{R} \) as classical variables \( \xi^\mu_i \); in particular, 0 is in their spectrum.

3. \( A^{n-1}_{\theta,\xi} , A_{\theta,X} \) are actually \( H \)-module subalgebras, and

\[
g(\ast b) = g((a \ast b)) = (g(a) \ast b), \quad a \in A^{n-1}_{\theta,\xi}, \quad b \in A^n_\theta, \quad g \in H, \tag{18}
\]

implying in particular \( g \circ a = g \circ a \), i.e. on \( A^{n-1}_{\theta,\xi} \) the \( H \)-action is undeformed. In fact the Leibniz rule reduces to the undeformed one whenever a twist leg acts on \( a \), again because of the trivial action (15) of the \( P_\mu \)’s contained in \( \mathcal{F} \). The previous relation holds also without the two \( \ast \)-products, by (16).

Summing up, any coordinate difference like \( \xi^\mu_i \) can be treated as a classical, commutative variable. Any \( x^\mu_i \) is a combination of \( n-1 \) \( \ast \)-commutative variables \( \xi^\mu_i \) and 1 \( \ast \)-noncommutative one \( X^\mu \); or equivalently can be obtained from the zero 4-vector and \( n-1 \) \( \ast \)-commutative 4-vectors by the global “noncommutative translation” \( X \), e.g. if \( X := x_1 \) then

\[
x_i = \sum_{j=1}^{i-1} \xi_j + X.
\]

Of course, all the previous statements [with the exception of (16)] can be formulated in the isomorphic setting removing all \( \ast \)'s, putting a \( \ast \) over any coordinate and replacing \( A_{\theta,\xi} , A^n_\theta , A^{n-1}_{\theta,\xi} , A_{\theta,X} \) with the isomorphic objects \( \hat{A}_{\xi} , \hat{A}_n , \hat{A}^{n-1}_{\xi} , \hat{A}_X \) The result for \( \hat{X} \) is like the “quantum shift operator” of (15).

**Remark 1.** One immediate consequence is that on any irreducible representation \( \ast \)-multiplication by a spacetime coordinate difference \( x-y \) equals multiplication by \( x-y \), which is either a space-like, or a null, or a time-like 4-vector, in the usual sense.

**Remark 2.** Relation (18) holds also for an infinitesimal general coordinate transformation, i.e. if \( g \) is an element of the (deformed) U.E.A. \( U_{\mathfrak{e}} \) of the Lie algebra of general vector fields on the Moyal-Weyl NC space.

We recall that the differential calculus over \( \mathbb{R}^n \) remains unchanged under deformation of this space into a Moyal Weyl NC space. This is true also if we consider the differential calculus on the larger algebra \( A^n_\theta \) (or the isomorphic \( \hat{A}^n_\theta \)), and follows again from (2), (3) and the fact that \( P_\mu \) have trivial action on the derivatives. Explicitly,

\[
\partial x^\mu_i \ast x^\nu_j = \delta^\nu_j \delta^\mu_i + x^\nu_j \ast \partial x^\mu_i \quad \quad \left[ \partial x^\mu_i \ast \partial x^\nu_j \right] = 0 \tag{19}
\]

with self-explaining notation. Since the presence of the \( \ast \) product has no effect on the action of the derivatives on \( A^n_\theta \), in the sequel we shall drop it.
Given two sets $x, y$ of coordinates, integrating over some $x^\mu$ both sides of the identity
\[
g(y) \star f(x) = (\mathcal{R}^{(2)} \hat{\gamma} f(x)) \star (\mathcal{R}^{(1)} \hat{\gamma} g(y)) = \exp \left( -i \theta^{\mu\nu} \partial_{x^\mu} \partial_{y^\nu} \right) f(x) \star g(y)
\]
we see that any integration $\int dx^\mu$ commutes with $g(y) \star f$ if $f$ rapidly decreases at infinity; in fact, if we define the $\star$-product by the integral \[13\] we realize that
\[
\int dx^\mu \ g(y) \star f(x) = g(y) \star \int dx^\mu \ f(x)
\]
is true also for $f, g \in L^1(\mathbb{R}^4) \cap \hat{L}^1(\mathbb{R}^4)$ or even some distributions, as on commutative space [of course, since the rhs of \[20\] is independent of $x^\mu$, terms with $\theta^{\mu\nu} \partial_{y^\nu}$ will be ineffective and disappear, as if it were $\theta^{\mu\nu} = 0$ for all $\nu$. Therefore, for our purposes we can consider integration over any set of coordinates as an operation commuting with $\star$-products.

3 General consequences for many-particle QM

In configuration space the Hamiltonian of an isolated system of $n$ non-relativistic (for simplicity spinless) particles
\[
H = H_0 + \sum_{i<j} V_{ij}(|x_i - x_j|) \quad H_0 := -\sum_{i=1}^n \frac{\hbar^2}{2m_i} \nabla_{x_i}^2
\]

involves only derivatives and relative coordinates $\xi$. Denoting as $X$ the coordinates of the center of mass, as $M$ the total mass of the system, the kinetic part $H_0$ can be written as the sum of $-\hbar^2 \nabla_X^2 / 2M$ and a second order differential operator in the $\xi$-derivatives only. As a consequence, the dynamics of the center of mass is free. This means that an orthogonal basis of eigenfunctions of $H$ is $\{\exp(i k \cdot X) \psi_j(\xi)\}$, where $\psi_j$ are eigenfunctions of the rest Hamiltonian $H_\xi := H + \hbar^2 \nabla_X^2 / 2M$, depending on the $\xi$ and $\xi$-derivatives only.

Going to the noncommutative Euclidean space (the time remaining commutative) brings no change: the deformed Hamiltonian $H_\ast \equiv H \ast$ can be still split into a free part $-\hbar^2 \nabla_X^2 / 2M \ast$ for the center-of-mass degrees of freedom and an interacting part $H_\xi \ast$ depending only on the relative coordinates, and both parts act on the vector space underlying both $A^n$ and $\mathbb{A}^n$ (and therefore also on the subspace consisting of square-integrable wave-functions) exactly as their undeformed counterparts, implying that $\{\exp(i k \cdot X) \psi_j(\xi)\}$ is also an orthonormal basis of eigenfunctions of $H_\ast$ with the same eigenvalues. As a result, the deformed dynamics coincides with the undeformed one.

Assume now that the particles are identical. If the space is commutative, a wavefunction $\Psi(x_1, ..., x_n)$ completely (anti)symmetric under particles’ permutations can be decomposed as $\Psi = \sum_{I,J} \Psi_{I,J} \phi_I \chi_J$ in any tensor product basis $\{\phi_I(X) \chi_J(\xi)\}$, where $\chi_J(\xi)$ are completely (anti)symmetric $[\phi_I(X)$ are automatically completely symmetric]. These symmetries are preserved by the dynamical evolution, since this is ruled by the completely symmetric evolution operator $U(t-t_0) = \exp[-\frac{i}{\hbar} \hat{H}'(t-t_0)]$, where $H' = \sum_{i<j} V_{ij}(|x_i - x_j|)$. The operator $H'$ can be split into a free part $-\hbar^2 \nabla_X^2 / 2M \ast$ for the center-of-mass degrees of freedom and an interacting part $H_\xi \ast$ depending only on the relative coordinates, and both parts act on the vector space underlying both $A^n$ and $\mathbb{A}^n$ (and therefore also on the subspace consisting of square-integrable wave-functions) exactly as their undeformed counterparts, implying that $\{\exp(i k \cdot X) \psi_j(\xi)\}$ is also an orthonormal basis of eigenfunctions of $H_\ast$ with the same eigenvalues. As a result, the deformed dynamics coincides with the undeformed one.
H + \sum_i V_e(x_i) is the total Hamiltonian with V_\text{ij} \equiv V and V_e the external potential (if the system is not isolated). For the same reason this is true both in the Schrödinger and in the Heisenberg picture, which are related by the unitary transformation U(t-t_0), and also in the interaction picture, which is related to the Schrödinger by the completely symmetric evolution operator U_0(t-t_0) = \exp[-\frac{i}{\hbar}H_0(t-t_0)]. All the corresponding deformed statements remain true, as H'_i \equiv H'| and H_{0*} \equiv H_0* are also completely symmetric.

The action of the Galilei Lie algebra, and therefore also of its universal enveloping algebra U_\mathcal{G}, maps A_X \to A_X, A_\xi^{n-1} \to A_\xi^{n-1} preserving these complete (anti)symmetries, hence amounts to a change of the coefficients \Psi_{IJ}. Interpreting \Psi, \phi_I(X), \chi_J(\xi) as elements respectively of A_\theta_\theta, A_\theta_X, A_\theta_\xi, the same will be true of the action of the twisted Galilei U.E.A. H, as the latter maps A_\theta,X \to A_\theta,X, A_\theta_\xi \to A_\theta_\xi, by \mathbf{(18)}. Therefore there is no incompatibility between the standard complete (anti)symmetry conditions on a wave-function \Psi(x_1, ..., x_n) and the action of H. Consequently, the standard Bose, Fermi (and similarly anyon, in 2 space dimensions) statistics are compatible with twisted Galilei symmetry (in first quantization). This agrees with the general (and physically reassuring!) results of Ref. \cite{26}, where it was shown (by a unitary equivalence in a n-particle, abstract Hilbert space formalism) that covariance under a noncocommutative Hopf algebra obtained by twisting from a cocommutative one is compatible with usual statistics.

4 General consequences for QFT

In field-operator approach quantization of fields on Minkowski space obeys a set of general conditions, the Wightman axioms, which (as done e.g. in Ref. \cite{49}) can be divided into a subset (in the sequel labelled by QM) encoding the quantum mechanical interpretation of the theory, its symmetry under space-time translations and stability, and a subset (in the sequel labelled by R) encoding the relativistic properties (full Lorentz-covariance and locality). We now try to translate this into a field quantization procedure on a Moyal-Weyl noncommutative space keeping the QM conditions, “fully”

\footnote{We recall that \mathcal{G} is generated by H_0 (kinetic term in the Hamiltonian: generates time translations of a free system), m (mass: is central), P^a (momentum components: generate space translations), L^a (angular momentum components: generate rotations), K^a (generate boosts), with a = 1, 2, 3, where the only nontrivial commutation relations are

\begin{align}
[K^a, P^b] &= i\hbar \delta^{ab}, & [K^a, H_0] &= i\hbar P^a, \\
[L^a, L^b] &= i\epsilon^{abc} L^c, & [L^a, P^b] &= i\epsilon^{abc} P^c, & [L^a, K^b] &= i\epsilon^{abc} hK^c.
\end{align}

The generators are realized as the differential operators H_0 = -\hbar \nabla^2/2m, P^a = -i\hbar \partial^a, L^a = -i\hbar \epsilon^{abc} x^b \partial^c, K^a = mx^a + i\hbar \partial^a in the configuration space of each single particle. Hence the observable K^a + tP^a gives the mass times the space coordinate x^a of the particle. The coproducts are defined by the fact that these generators are primitive. The coproducts of m, H_0, P^a, L^a respectively give the addition laws for the total mass, the total kinetic energy, the total momentum and the total angular momentum of the system, whereas the coproduct of K^a + tP^a gives the total mass times the space coordinate X^a of the center of mass of the system.}
twisting Poincaré-covariance and being ready to weaken locality if necessary.

**QM1. (Hilbert space structure)** The states are described by vectors of a (separable) Hilbert space $\mathcal{H}$.

**QM2. (Energy-momentum spectral condition)** The group of space-time translations $T \sim \mathbb{R}^4$ is a symmetry of the theory and is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(a)$, $a \in \mathbb{R}^4$: the fields transform according to $[37]$ with unit $A, U(A), \Lambda(A)$. The spectrum of the generators $P_\mu$ is contained in the closed forward cone $V_+ = \{ p_\mu : p^2 \geq 0, p_0 \geq 0 \}$. There is a vacuum state $\Psi_0$, with the property of being the unique Poincaré invariant state (uniqueness of the vacuum).

**QM3. (Field operators)** The theory is formulated in terms of fields (in the Heisenberg representation) $\phi^\alpha(x)$, $\alpha = 1, \ldots, N$, that are operator (on $\mathcal{H}$) valued tempered distributions on Minkowski space, with $\Psi_0$ a cyclic vector for the fields, i.e. by applying polynomials of the (smeread) fields to $\Psi_0$ one gets a set $D_0$ dense in $\mathcal{H}$.

By taking vacuum expectation values (v.e.v.) of $\ast$-products of fields one can introduce different kinds of $n$-point functions, that will be (mere) distributions: Wightman functions

$$W^\alpha_1, \ldots, \alpha_n(x_1, \ldots, x_n) = (\Psi_0, \phi^{\alpha_1}(x_1) \ast \ldots \ast \phi^{\alpha_n}(x_n)\Psi_0), \quad (23)$$

where $\alpha_1, \ldots, \alpha_n$ enumerate possible different field species and/or $SL(2,C)$-tensor (spinor, vector,...) components, or (their linear combinations) Green’s functions

$$G^\alpha_1, \ldots, \alpha_n(x_1, \ldots, x_n) = (\Psi_0, T[\phi^{\alpha_1}(x_1) \ast \ldots \ast \phi^{\alpha_n}(x_n)]\Psi_0), \quad (24)$$

or retarded functions, etc. In the second definition there appears the time-ordering $T$, but there is in fact no ambiguity in defining $T$ as on commutative Minkowski space$[4]$.

$$T[\phi^{\alpha_1}(x_1) \ast \phi^{\alpha_2}(x_2) \ast \ldots \ast \phi^{\alpha_n}(x_n)] = \phi^{\alpha_1}(x_1) \ast \phi^{\alpha_2}(x_2) \ast \ldots \ast \phi^{\alpha_n}(x_n) \partial(x_1^0 - x_2^0) \ldots \partial(x_{n-1}^0 - x_n^0) + \phi^{\alpha_1}(x_1) \ast \phi^{\alpha_2}(x_2) \ast \ldots \ast \phi^{\alpha_n}(x_n) \partial(x_1^0 - x_2^0) \ast \ldots \ast \partial(x_{n-1}^0 - x_n^0), \quad (25)$$

as this definition involves multiplication by the $\ast$-central $\partial(x_i^0 - x_j^0)$ ($\partial$ denotes the Heavyside function). [The $\ast$’s preceding all $\partial$ can be and have been dropped, by $[16]$.

Arguing as in $[38]$ for ordinary QFT, exactly the same properties follow from QM1-3 (alone). Applying a pure translation, from QM2 we find that **Wightman and Green’s functions are translation invariant** and therefore may depend only on the commutative spacetime variables $\xi^\alpha$:

$$W^\alpha_1, \ldots, \alpha_n(x_1, \ldots, x_n) = W^\alpha_1, \ldots, \alpha_n(\xi_1, \ldots, \xi_n), \quad (26)$$

$$G^\alpha_1, \ldots, \alpha_n(x_1, \ldots, x_n) = G^\alpha_1, \ldots, \alpha_n(\xi_1, \ldots, \xi_n). \quad (27)$$

Moreover, from QM3, QM2, QM1 it respectively follows $^2$

---

$^2$In the standard approach $[37][11][2]$ this was found to be safe and unambiguous only in the case of spacetime commutativity ($\partial^{\alpha} = 0$), which gives commuting time variables $x_i^0$, so that time-ordering commutes with the $\ast$-product.
\( \textbf{W1.} \) \( W^{\alpha_1,...,\alpha_n}(x_1,...,x_n) \) are tempered distributions depending only on the \( \xi_i \).

\( \textbf{W2.} \) (Spectral condition) The support of the Fourier transform \( \tilde{W} \) of \( W \) is contained in the product of forward cones, i.e.

\[
\tilde{W}^{\alpha_1,...,\alpha_n}(q_1,...,q_{n-1}) = 0, \quad \text{if} \quad \exists j: \quad q_j \notin \mathbb{V}_+.
\]  

(28)

\( \textbf{W3.} \) (Hermiticity and Positivity) The transformation properties of Wightman functions under complex conjugation follow from

\[
(\Psi_0, \varphi^{\alpha_1}(x_1) \ast \cdots \ast \varphi^{\alpha_n}(x_n)\Psi_0) = (\Psi_0, \varphi^{\alpha_n\dagger}(x_1) \ast \cdots \ast \varphi^{\alpha_1\dagger}(x_n)\Psi_0).
\]

In particular, if all fields are Hermitean scalar then \( \overline{W(x_1,...,x_n)} = W(x_n,...,x_1) \). For any terminating sequence \( f = (f_0, f_1, ... f_N) \), \( f_j \in S(\mathbb{R}^4)^j \) one has

\[
\sum_{j,k} \int dx dy \, \bar{f}_j(x_j,...x_1) \, f_k(y_1,...,y_k) \, W(x_1,...x_j;y_1,...,y_k) \geq 0.
\]  

(29)

We now recall the ordinary relativistic conditions on QFT:

\( \textbf{R1.} \) (Lorentz Covariance) The universal covering group \( SL(2,\mathbb{C}) \) of the restricted Lorentz group is a symmetry of the theory and is represented on \( \mathcal{H} \) by (strongly continuous) unitary operators \( U(A) \). The fields transform covariantly under the inhomogeneous \( SL(2,\mathbb{C}) \) (i.e. generalized Poincaré) transformations \( U(a, A) = U(a) U(A) \):

\[
U(a, A) \varphi^\alpha(x) U(a, A)^{-1} = S_\alpha^\beta(A^{-1}) \varphi^\beta(\Lambda(A)x + a),
\]  

(30)

with \( S \) a finite dimensional representation of \( SL(2,\mathbb{C}) \) and \( \Lambda(A) \) the Lorentz transformation associated to \( A \in SL(2,\mathbb{C}) \).

\( \textbf{R2.} \) (Microcausality or locality) The fields either commute or anticommute at spacelike separated points

\[
[\varphi^\alpha(x), \varphi^\beta(y)]_\mp = 0, \quad \text{for} \quad (x-y)^2 < 0.
\]  

(31)

As a consequence of QM2,R1 in QFT on commutative Minkowski space one finds

\( \textbf{W4.} \) (Lorentz Covariance of Wightman functions)

\[
W^{\alpha_1,...,\alpha_n}(\Lambda(A)x_1,...,\Lambda(A)x_n) = S_\alpha^{\beta_1}(A)...S_\alpha^{\beta_n}(A) W^{\beta_1,...,\beta_n}(x_1,...,x_n).
\]  

(32)

Wightman functions of scalar fields are Lorentz invariant. (Similarly for Green functions).

In order to translate R1 into a corresponding condition \( R1_\star \) in the twisted Hopf algebra setting we could go either to the infinitesimal formulation (i.e. first to \( \mathcal{P} \), and

\[ \Psi_\underline{f} = f_0 \Psi_0 + \varphi(f_1) \Psi_0 + \varphi(f_2^{(1)}) \varphi(f_2^{(2)}) \Psi_0 + ..., \]

where \( \underline{f} = (f_0, f_1, ... f_N) \), \( f_j = \prod_{k=1}^j f_j^{(k)}(x_k) \).
then deform to $H$, or to the dual functions-on-the-group Hopf algebra. We do not attempt this here, because it would be rather technical (especially translating the strong continuity requirement), and moreover some subtlety might be hidden in the interplay of active (or system) and passive (or coordinates) twisted Poincaré transformations appearing at the two sides of (30). We content ourselves with requiring the deformed analog of $W_4$, which should follow from $R_1 \star$, however this will look like, namely that Wightman (and Green) functions transform under a twisted version of (32), in particular are invariant if all involved fields are scalar. On the other hand, as these functions should be built only in terms of the $\xi^\mu_i$ and of ordinary $SL(2,C)$ tensors, like $\partial x^\mu_i$, the isotropic tensor $\delta^\mu_\nu$, spinors, $\gamma$-matrices, etc., which are all annihilated by the action of $P_\mu$, the action of the twist “legs” $F^{(1)}, F^{(2)}$ should be trivial and the transformation properties under the Lorentz generators should remain undeformed: so these functions should admit exactly the same decomposition in Lorentz tensors as in the undeformed case (in particular should be invariant if all fields are scalar fields). Therefore, deferring the formulation of $R_1 \star$, to possible future works, here we shall require $W_4$ also in the deformed case as a temporary substitute of $R_1$.

As for $R_2$, it is natural to ask whether in the deformed theory one can adopt the twisted version

**R2.** (Microcausality or locality) The fields either $\star$-commute or $\star$-anticommute at spacelike separated points

\[ [\varphi^\alpha(x) \star \varphi^\beta(y)]_+ = 0, \quad \text{for } (x - y)^2 < 0. \] (33)

and whether there also viable alternatives. That the conditions QM1-3, W4, R2 are independent and compatible can be proved arguing along the lines [48]; in particular compatibility is proved by showing that they can be fulfilled by free fields (see next section). We thus find in particular that the noncommutativity structure of a Moyal-Weyl space is compatible with locality $R''$. Whether reasonable weakenings of $R_2$, exist is in fact an open question also in the ordinary theory. Phenomenology suggests that rhs(31) should at least rapidly decrease with increasing space-like distances, if it is not zero. On the other hand, the same results as in [54, 10] should apply, namely requiring that rhs(31) is zero only in some space-like separated open subsets (see [54], or Theorem 4-1 in [48]), or is a $c$-number decreasing faster than an exponential with space-like distances [10], are actually only apparent weakenings, in that they imply again R2.

As consequences of $R_2$, one again finds [48]

**W5.** (Local commutativity conditions) If $(x_j - x_{j+1})^2 < 0$ then

\[ W^{\alpha_1...\alpha_n}(x_1,...x_j,x_{j+1},...x_n) = \pm W^{\alpha_1...\alpha_j\alpha_{j+1}\alpha_j...\alpha_n}(x_1,...x_{j+1},x_j,...x_n); \] (34)

the sign is negative if $\varphi^{\alpha_j}, \varphi^{\alpha_{j+1}} \star$-anticommute, is positive otherwise.

**W6.** (Cluster property) For any spacelike vector $a$ and for $\lambda \to \infty$

\[ W^{\alpha_1...\alpha_n}(x_1,...x_j,x_{j+1}+\lambda a,...,x_n+\lambda a) \to W^{\alpha_1...\alpha_j}(x_1,...,x_j) W^{\alpha_{j+1}...\alpha_n}(x_{j+1},...,x_n), \] (35)

4As already noted, space-like separation is well-defined, so that the latter condition makes sense.
(convergence is in the distributional sense); this is true also with permuted coordinates.

In the proof of W6 the uniqueness of the invariant state $\Psi_0$ plays an essential role.

Summarizing, we end up with a QFT framework on these NC spaces with QM1-3, W4, R2, or alternatively with exactly the same constraints W1-6 on Wightman functions as in ordinary QFT on Minkowski space. Reasoning as described in [38, 32, 31], one should be able to prove the same, other well-known fundamental results in ordinary QFT:

1. That Wightman functions are boundary values

$$W^{\alpha_1,\ldots,\alpha_n}(\xi_1, \ldots, \xi_{n-1}) = \lim_{\eta_1, \ldots, \eta_{n-1} \to 0} W^{\alpha_1,\ldots,\alpha_n}(\zeta_1, \ldots, \zeta_{n-1})$$

of holomorphic functions $W(\zeta_1, \ldots, \zeta_{n-1})$ in the complex variables $\zeta_i = \xi_i - i\eta_i$, the domain of holomorphy being $\{\xi_1, \ldots, \xi_{n-1} \mid \eta_j \in V^+\}$.

2. The analogs of the Spin-Statistics and CPT theorems.

3. That the cluster property W6 implies (Haag-Ruelle theory) the existence of asymptotic (free) fields and, under the assumption of asymptotic completeness ($\mathcal{H} = \mathcal{H}^\text{in} = \mathcal{H}^\text{out}$), of a unitary $S$-matrix. This allows to derive [32] the LSZ formulation of QFT, and subsequently dispersion relations for scattering amplitudes, etc.

4. That the Wightman functions have an analytic continuation to the so-called Euclidean points, thus allowing to derive the existence and the general properties of Euclidean QFT with the analog of Schwinger functions.

We stress that these results should hold for all $\theta^{\mu\nu}$, and not only if $\theta^{0i} = 0$ as in the approach e.g. of [16, 17].

### 5 Free fields

As in ordinary QFT, things become much more definite for free fields. By [19], the kinetic differential operators (D’Alambertian, Dirac operator, etc) remain undeformed, therefore so will remain the free field equations and the consequent constraints on Wightman, Green’s functions and on the field commutation relations. For simplicity we stick to the case of a free Hermitean scalar field $\varphi_0(x)$ of mass $m$:

$$(-\Box_x + m^2)\varphi_0 = 0. \quad (36)$$

In momentum space this becomes $(p^2 - m^2)\tilde{\varphi}(p) = 0$, so the spectrum is contained in (the two sheets of) the hyperboloid $p^2 = m^2$. We can therefore Fourier decompose $\varphi_0(x)$ into a positive and a negative frequency part in a (twisted) Lorentz invariant way,

$$\varphi_0(x) = \varphi_0^+(x) + \varphi_0^-(x)$$

$$\varphi_0^+(x) := \int d\mu(p) e^{-ip \cdot x} a_p,$$

$$\varphi_0^-(x) := \int d\mu(p) a_p^\dagger e^{ip \cdot x} = (\varphi_0^+(x))^\dagger, \quad (37)$$
where \( dp(p) = \delta(p^2 - m^2) \vartheta(p^0) d^4 p = dp^0 \delta(p^0 - w_p) d^3 p / 2w_p \) is the invariant measure 
(\( w_p := \sqrt{p^2 + m^2} \)). From \([36]\) it immediately follows \((\Delta + m^2) W(\xi) = 0\) or equivalently 
\((p^2 - m^2) W(p) = 0\) in momentum space, whence the Fourier decomposition

\[
W(x - y) = \int dp(p) [w^+ (p)e^{-ip(x - y)} + w^-(p)e^{ip(x - y)}].
\]

On the other hand, using QM1-3 one finds first \( \varphi_+^+(x) \Psi_0 = 0 \), i.e. \( a^p \Psi_0 = 0 \), then

\[
W(x - y) = (\Psi_0, \varphi_0(x) \star \varphi_0(y) \Psi_0) = (\Psi_0, \varphi_+^+(x) \star \varphi_0^-(y) \Psi_0),
\]

showing that \( x \) (resp. \( y \)) is associated only to the positive (resp. negative) frequencies, i.e. \( w^- (p) \) has to vanish, and \( w^+(p) \) has to be positive. But by W4 \( w^+(p) \) has to be Lorentz invariant, i.e. constant, so we conclude that up to a positive factor \( W \) is given by

\[
W(x - y) = -i F^+(x - y), \quad F^+(\xi) := i \int \frac{dp(p)}{(2\pi)^3} e^{-ip \xi}
\]

and therefore coincides with the undeformed counterpart. Moreover,

\[
(\Psi_0, a^p a_q^\dagger \Psi_0) = 2\omega_p \delta^3(p - q)
\]

as in the undeformed case. The 2-point Green’s function is now immediately obtained as the time-ordered combination of \( W(x - y) \) and \( W(y - x) \):

\[
G(x, y) := (\Psi_0, T[\varphi_0(x) \star \varphi_0(y)] \Psi_0) = G(x - y),
\]

\[
G(\xi) := -i [\vartheta(\xi^0) F^+(\xi) + \vartheta(-\xi^0) F^+(\xi)] = \int \frac{dp(p)}{(2\pi)^3} e^{-ip \xi} \vartheta(\xi^0)
\]

and therefore coincides with (the undeformed) Feynman’s propagator. Note that \([38], [39], [40]\) are independent of \( R_2 \), or any other assumption about the field commutation relations, which we have not used in the proof.

On the other hand, if one postulates all the axioms of the preceding section (including \( R_2 \)) and reasons as in the proof of the Jost-Schroer theorem, Thm 4-13\(^5\) in [48], one proves up to a positive factor the free field commutation relation

\[
[\varphi_0(x) \star \varphi_0(y)] = i F(x - y), \quad F(\xi) := F^+(-\xi) - F^+(\xi),
\]

which coincides with the undeformed one. Incidentally, this can be proved also from just the free field equation and the assumption that the commutator is a (twisted, and therefore also untwisted) Poincaré invariant c-number (see e.g. [46], page 178-179). Applying \( \partial_{\rho} \) to \([41]\) and then setting \( y^0 = x^0 \) [as already noted, this is compatible with \([41]\)] one finds the canonical commutation relation

\[
[\varphi_0(x^0, x) \star \varphi_0(x^0, y)] = i \delta^3(x - y).
\]

As a consequence of \([42]\), also the \( n \)-point Wightman functions coincide with the undeformed ones, i.e. vanish if \( n \) is odd and are sum of products of two point functions

\(^5\)More precisely, as it is done after the proof that \([36]\) follows from \([38]\).
the last follows from the first and iteration of the previous result one finds
$x$ which holds also if $e$
Campbell-Hausdorff formula 6
$e$
In this and other proofs one has to use the following properties of $e$ exponentials. Recalling the Baker-
$x$ ⋆$e$
it is Lorentz invariant by (38) and factorized. The second, n onlocal term comes from
$θ$
although only for scalar fields and for
$[38]$
commutation relations of the form (41) are proposed in a 1+1 dimensional model
$θ$
which are non-local unless $θ' = θ$. As said, the authors of [9] adopt $θ' = θ$. In
$38$
commutation relations of the form (41) are proposed in a 1+1 dimensional model in order to close the chiral current algebra; in [11] (11) are proposed in any dimension, although only for scalar fields and for $θ^{αi} = 0$; whereas the authors of [8, 9] find non-local relations [see formula (3.23) in [9]] similar to (45), because they do not perform the ⋆-product between functions of different sets $x, y$ of coordinates.

Let us consider two typical contributions to the 4-point Wightman function:
$W(x_1, x_2, x_3, x_4) = W(x_1 - x_2)W(x_3 - x_4) + e^{i∂_2(θ' - θ')∂_3}W(x_1 - x_3)W(x_2 - x_4) + ...
$ The first term at the rhs comes from the v.e.v.'s of $ϕ_0(x_1)⋆ϕ_0(x_2)$ and $ϕ_0(x_3)⋆ϕ_0(x_4)$; it is Lorentz invariant by (38) and factorized. The second, nonlocal term comes from
6In this and other proofs one has to use the following properties of exponentials. Recalling the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B+C}$ (with $C := [A, B]/2$ commuting with $A, B$) one finds
$e^{ip \cdot x} e^{ip \cdot y} = e^{ip \cdot x + ip \cdot y - 5ρq} \Rightarrow e^{ip \cdot x} e^{ip \cdot y} = e^{ip \cdot y} e^{ip \cdot x} e^{-ip \cdot q} = e^{ip \cdot (x - y)}$; the last follows from the first and $pθq = 0$. These relations hold in particular for $y = x$. More generally, by iteration of the previous result one finds
$e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} \ldots e^{ip_n \cdot x_n} = \exp \left( i \sum_{j=1}^{n} p_j \cdot x_j - \frac{i}{2} \sum_{j < k} p_j θ p_k \right)$, (43)
which holds also if $x_j = x_k$ for some $j, k$. 
the v.e.v.’s of \( \varphi_0(x_1) \star \varphi_0(x_3) \) and of \( \varphi_0(x_2) \star \varphi_0(x_4) \), after using (45) to commute \( \varphi_0(x_2), \varphi_0(x_4) \). Only if \( \theta' = \theta \) it is Lorentz invariant and factorizes into \( W(x_1-x_3)W(x_2-x_4) \). As it depends only on \( x_1-x_3, x_2-x_4 \), it is invariant under the replacements \( (x_1, x_2, x_3, x_4) \to (x_1, x_2 + \lambda a, x_3, x_4 + \lambda a) \), even in the limit \( \lambda \to \infty \) By taking a space-like, we conclude that if \( \theta' \neq \theta \) \( W \) violates \( W4 \) and \( W6 \), as expected.

We present a second way to realize (41), which at first sight might appear “exotic”, but we are going to theoretically motivate elsewhere. It follows from assuming nontrivial transformation laws \( P_\mu \triangleright a_p = p_\mu a_p, P_\mu \triangleright a^\dagger_p = -p_\mu a^\dagger_p \) and extending the \( \star \)-product law (2) also to \( a^\dagger_p, a_p \). It amounts to choosing \( \theta' = -\theta \) in (41) [inserting for uniformity of notation a \( \star \)-symbol in each product, also in (37)] and to adding nontrivial commutation relations between the \( a^\dagger_p, a_p \) and functions, in particular exponentials, of the form

\[
\begin{align*}
    a^\dagger_p \star a_q &= e^{-ip\theta q} a^\dagger_q \star a_p, \\
    a^\dagger_p \star a_q &= e^{-ip\theta q} a_q \star a^\dagger_p, \\
    a^\dagger_p \star a_q &= e^{ip\theta q} a^\dagger_q \star a^\dagger_p + 2\omega p\delta^3(p-q), \\
    a^\dagger_p \star e^{iq\cdot x} &= e^{-ip\theta q} e^{iq\cdot x} \star a^\dagger_p, \\
    a^\dagger_p \star e^{iq\cdot x} &= e^{ip\theta q} e^{iq\cdot x} \star a^\dagger_p.
\end{align*}
\]

As a consequence, \([\varphi_0(x) \star f(y)] = 0\). In other words, the first three relations in (46) define an example of a general deformed Heisenberg algebra (25)

\[
\begin{align*}
    a^\dagger_s \star a^\dagger_r &= R^{rs}_{pq} a^\dagger_q \star a^\dagger_p, \\
    a^\dagger_p \star a^\dagger_q &= R^{pq}_{rs} a^\dagger_r \star a^\dagger_s, \\
    a^\dagger_p \star a^\dagger_q &= \delta^p_q + R^{pq}_{rs} a^\dagger_r \star a^\dagger_s
\end{align*}
\]

covariant under a triangular Hopf algebra \( H \). Here the \( R \)-matrix is the universal \( R \) in the infinite-dimensional representation of \( H \) spanned by the basis of vectors \( a^\dagger_p \), the (on-shell) 4-momenta \( p,q,r,s \) playing the role of (continuous) indices, and \( \delta^p_q = 2\omega p\delta^3(p-q) \) is Dirac’s delta (up to normalization). The first three relations in (41) also can be considered of the form (47) after a replacement \( \theta \to -\theta' \). Such \( a^\dagger_p, a^\dagger_p \) can be realized (25) as composite of operators \( c^r, c^{\dagger r} \), fulfilling the CCR (for the case at hand of the \( \theta \)-deformed Poincaré this has been done also in (2)), so that they act on the same (undeformed) Fock space. In doing so one finds that the action of \( P_\mu \) can be realized as a commutator with the operator \( P_\mu = \int d\mu(p)c^\dagger_p c^\dagger_p = \int d\mu(p)a^\dagger_pa^\dagger_p \).

As historical remarks we add that, up to normalization of \( R \), and with \( p,q,r,s \in \{1,\ldots,N\} \), relations (47) are also identical to the ones defining the older \( q \)-deformed Heisenberg algebras of (45, 53), based on a quasitriangular \( R \) in (only) the fundamental representation of \( H = U_qsu(N); \) allowing a different (possibly infinite-dimensional) representation has been considered in (30) for the \( U_qsu(2) \)-covariant quantization of fields on the \( q \)-deformed fuzzy sphere. Going further back in the past, (40,47) are reminiscent of the Zamolodchikov-Faddeev (50, 24) algebra, generated by deformed creation/annihilation operators of scattering states of some completely integrable 1+1-dimensional QFT; there again the indices are discrete, but the \( R \)-matrix depends on a
(continuous) spectral parameter, the rapidity of the particles. In \cite{36} the Zamolodchikov-Faddeev creation/annihilation operators have been realized as acting on the (undeformed) Fock space.

6 Theoretical developments. Perturbative expansion for interacting QFT

Normal ordering should be a $A_\theta^0$-bilinear map of the field algebra into itself, such that any normal ordered expression has a trivial v.e.v., in particular $:1: = 0$. Applying it to the sides of (44) we find that it is consistent to define

$$
:a^p a^q: = a^p a^q, \quad :a^q_p a^q: = a^q_p a^q, \quad :a^q_p a^q_\dagger: = a^q_p a^q_\dagger, \quad :a^q_p a^q_\dagger: = a^q_p a^q e^{-ip\theta q}.
$$

The phase in the last relation is to account for (44) and $:1: = 0$. More generally, it is consistent to define normal ordering on any monomial in $a^p, a^q_\dagger$, as a map which reorders all $a^p$ to the right of all $a^q_\dagger$ introducing a factor $e^{-ip\theta q}$ for each flip $a^p \leftrightarrow a^q_\dagger$. For $\theta = 0$ one finds the undeformed definition. Using $A_\theta^0$-bilinearity normal ordering is extended to fields.

We first consider the assumptions leading to (44), namely (44) or (46). One finds that exactly as in the undeformed case it maps each monomial $M$ in the fields (and their derivatives) into itself minus all lower degree monomials obtained by taking v.e.v.'s of pairs of fields appearing in $M$, e.g.

$$
:\varphi_0(x): = \varphi_0(x)
$$

$$
:\varphi_0(x) \ast \varphi_0(y): = \varphi_0(x) \ast \varphi_0(y) - (\Psi_0, \varphi_0(x) \ast \varphi_0(y) \Psi_0)
$$

$$
:\varphi_0(x) \ast \varphi_0(y) \ast \varphi_0(z): = \varphi_0(x) \ast \varphi_0(y) \ast \varphi_0(z) - (\Psi_0, \varphi_0(x) \ast \varphi_0(y) \Psi_0) \varphi_0(z) - (\Psi_0, \varphi_0(x) \ast \varphi_0(z) \Psi_0) \varphi_0(y) - \varphi_0(x) (\Psi_0, \varphi_0(y) \ast \varphi_0(z) \Psi_0)
$$

... 

By construction $(\Psi_0, :M: \Psi_0) = 0$. These are well-defined operators also in the limit of coinciding coordinates (e.g. $y \to x$). The above substractions amount to flipping step by step each $\varphi_0^+(x)$ to the right of each $\varphi_0^-(y)$. For instance on $\varphi_0^+(x)\varphi_0^-(y)$ ($\varepsilon, \eta \in \{-, +\}$) normal ordering acts as the identity unless $\varepsilon = +$ and $\eta = -$, whereas

$$
:\varphi_0^+(x) \ast \varphi_0^-(y): = \varphi_0^-(y) \ast \varphi_0^+(x).
$$

As a consequence we find that for any monomial $M'$ obtained from $M$ by permutation of the field factors $:M: = :M'$; for instance

$$
:\varphi_0(x) \ast \varphi_0(y): = \varphi_0(y) \ast \varphi_0(x):
$$

... 

\footnote{The authors of \cite{8} omit this phase. However, their conclusions about the S-matrix being undeformed remain valid, as the effect of this phase disappears when exploiting global energy-momentum conservation.}
Moreover, as \( \varphi_0(x) \) is Hermitean, any normal ordered monomial :\( \varphi_0(x_1)\cdots \varphi_0(x_n) : \) is (a fortiori for coinciding coordinates). Summing up, under these assumptions normal ordering \([29]\) and its properties are written in terms of the fields exactly as in the undeformed setting (apart from the occurrence of the \( \ast \)-product symbol). Since the same occurs with time-ordering \([25]\), another straightforward consequence is that the same Wick theorem will hold:

\[
T[\varphi_0(x) \ast \varphi_0(y)] = : \varphi_0(x) \ast \varphi_0(y) : + \left( \Psi_0, T[\varphi_0(x) \ast \varphi_0(y)] \Psi_0 \right) \\
T[\varphi_0(x) \ast \varphi_0(y) \ast \varphi_0(z)] = : \varphi_0(x) \ast \varphi_0(y) \ast \varphi_0(z) : + \left( \Psi_0, T[\varphi_0(x) \ast \varphi_0(y) \ast \varphi_0(z)] \Psi_0 \right) : \varphi_0(z) : + \left( \Psi_0, T[\varphi_0(y) \ast \varphi_0(z)] \Psi_0 \right) : \varphi_0(y) : + \left( \Psi_0, T[\varphi_0(z)] \Psi_0 \right) : \varphi_0(z) : \\
\vdots
\]

Let us apply now time-ordered perturbation theory to an interacting field. We use the Gell-Mann–Low formula (rigorously valid under the assumption of asymptotic completeness, \( \mathcal{H} = \mathcal{H}^m = \mathcal{H}^{out} \))

\[
G(x_1, \ldots, x_n) = \frac{\left( \Psi_0, T \left\{ \varphi_0(x_1) \ast \cdots \ast \varphi_0(x_n) \ast \exp \left[ -i \lambda \int d^4 y \, H_I(y^0) \right] \right\} \Psi_0 \right)}{\left( \Psi_0, T \exp \left[ -i \int d^4 y \, H_I(y^0) \right] \Psi_0 \right)}. \tag{51}
\]

Here \( \varphi_0 \) denotes the free “in” field, i.e. the incoming field in the interaction representation, and \( H_I(x^0) \) is the interaction Hamiltonian in the interaction representation. The derivation of \((51)\) is heuristic and goes as on commutative space. It involves unitary evolution operators of the form

\[
U(x^0, y^0) = \lim_{N \to \infty} \prod_{m=0}^{N-1} \exp \left[ -i \frac{\Delta}{\hbar} N H_I(y^0 + \Delta m / N) \right] = T \exp \left[ -i \frac{\hbar}{\Delta} \int_{y^0}^{x^0} dt H_I(t) \right],
\]

where \( \Delta = x^0 - y^0 > 0 \) and again \( T \) always uses \( \ast \)-central time coordinate differences as arguments of the Heavyside function. For the sake of definiteness we choose

\[
H_I(x^0) = \lambda \int d^3 x : \varphi_0^m(x) : \ast, \quad \varphi_0^m(x) \equiv \varphi_0(x) \ast \cdots \ast \varphi_0(x). \tag{52}
\]

This is a well-defined, Hermitean \([by \langle 51 \rangle ]\) operator, with zero v.e.v. Expanding the exponential the generic term of order \( O(\lambda^0) \) in the numerator of \((51)\) will be the v.e.v.

\[
\frac{\lambda^{\hbar}}{i \hbar} \int d^4 y_1 \ldots d^4 y_n \left( \Psi_0, T \left[ \varphi_0(x_1) \ast \cdots \ast \varphi_0(x_n) \ast : \varphi_0^m(y_1) : \ast \cdots \ast : \varphi_0^m(y_n) : \right] \Psi_0 \right), \tag{53}
\]

where we have also used the property \([20]\) that integration over any space-time variable commutes with taking \( \ast \)-products. We proceed to evaluate this expression as in the undeformed case: applying Wick theorem to the field monomial and the fact that all normal-ordered field monomials have trivial v.e.v. we end up with exactly the same sum of terms given by integrals over \( y \)-variables, as in the undeformed case, of products of
free propagators \((40)\) having coordinate differences as arguments. Each of these terms is represented by a Feynman diagram. So the result for the generic term \((53)\) will be the same as the undeformed one. On the other hand, the generic term of order \(O(\lambda^b)\) in the denominator of \((51)\) will be a special case of \((53)\), the one with \(n = 0\). Summing up, numerator and denominator of \((51)\), and therefore also the Green functions \((51)\) coincide with the undeformed ones (at least perturbatively). They can be computed by Feynman diagrams with the undeformed Feynman rules.

In other words, at least perturbatively, this QFT is completely equivalent to the undeformed counterpart, and therefore also pathologies like UV-IR mixing disappear. Thus, also for the interacting theory the \(a_p, a_p^\dagger\) and the spacetime noncommutativities somehow compensate each other.

We now sketch how perturbation theory changes if \(\theta' \neq \theta\), starting from normal ordered field monomials. Relations \((44)\) lead to a non-local (pseudodifferential) operator for each flip of a \(\varphi_0^+(x)\) to the right of a \(\varphi_0^-(y)\), e.g.

\[
\varphi_0^+(x) \ast \varphi_0^-(y) : = e^{i\partial_x (\theta - \theta')} \partial_y \varphi_0^-(y) \ast \varphi_0^+(x),
\]

whereas on \(\varphi_0^\varepsilon(x) \varphi_0^\eta(y)\) with \((\varepsilon, \eta) \neq (+, -)\) normal ordering still acts as the identity. As a consequence, property \((50)\) and Wick theorem are modified, so are the Feynman rules, and UV/IR mixing for nonplanar Feynman diagrams reappears. Just to get a feeling one can consider the \(\lambda \varphi^4\) theory without normal ordering and, as in \((42)\), one finds UV/IR mixing already in several contributions (of nonplanar tadpole diagram type) to the \(O(\lambda)\) correction to the propagator.

### 7 Final remarks and conclusions

There is still no convincing and generally accepted formulation of QFT on noncommutative spaces, even on the simplest one, the Moyal-Weyl space. One crucial aspect under debate is the form of its covariance under space(time) symmetry transformations. In this work we have argued that a Moyal-Weyl deformation of Minkowski space is compatible with the Wightman axioms (including locality) and time-ordered perturbation theory, provided one replaces products of fields by \(\ast\)-products (also at different spacetime points) and the Lorentz covariance axiom (R1) by the appropriate twisted version \(R1_\ast\) (which we have not formulated yet). Both for free and interacting fields the resulting QFT's appear physically equivalent to the undeformed counterparts on commutative Minkowski space, in that their Wightman and Green's functions coincide. This can be understood as a sort of compensation of the effects of the \(a_p, a_p^\dagger\) and of the spacetime noncommutativities, if these are matched to each other. (To keep the size of this work contained we have not developed other important aspects, like those mentioned at the end of section \(4\) which we hope to treat soon elsewhere. For the moment, regarding the question whether QFT on noncommutative spaces violate standard Bose or Fermi statistics, as claimed e.g. in \([7, 9, 13]\), we content with drawing the reader's attention on Ref.'s \([26, 27]\).)

The main positive aspect of this outcome is a way to avoid all the additional complications (non-unitarity, macroscopic violation of causality, UV-IR mixing and sub-
sequent non-renormalizability, change of statistics,...) appeared in other approaches and to end up with a theoretically and phenomenologically satisfactory QFT, the undeformed theory (to the extent the latter can be considered satisfactory). For free field this is achieved by matching the commutation relations of the deformed creation/annihilation operators to the spacetime noncommutativity (however, we have found even two different ways to realize such a matching).

The related, obvious disappointing aspect is that in this resulting QFT there appears neither new physics nor a more satisfactory formulation of the old one (e.g. by an intrinsic UV regularization), in that all effects of spacetime noncommutativity are confined in an “unobservable common noncommutative translation of all reference frames”. This may indicate that Moyal-Weyl deformations considered in the framework of twisted Poincaré covariance are too trivial for this scope.

As a general remark, we would like to emphasize that the cluster property W6 is an important test for QFT on noncommutative as well as on commutative spaces: its violation implies a macroscopic (and therefore contrasting with experiments) violation of causality. It is also an easy theoretical test to carry out on free fields. For the noncommutative space at hand, our two possible prescriptions for free fields fulfill the cluster property whereas other prescriptions proposed in the literature ($\theta' \neq \theta$, see the end of section 5) lead to its violation.

As already noted in the paper, our results have some overlap or links with those of other works. To this regard we add some further remarks. At the 21st Nishinomiya-Yukawa Memorial Symposium on Theoretical Physics (11-15 Nov. 2006) “Noncommutative Geometry and Quantum Spacetime in Physics”, after presenting our results, the author of Ref. [1] pointed out his work. We have realized that there he proposes a quantization procedure for scalar fields and $\theta^{\delta_{i}} = 0$ which finally coincides with the first of our two proposals and arrives at very similar conclusions, although the derivation is different and various steps of it appear to us not completely clear or justified.

In [40] field quantization on the $h$-deformed Lobachevsky plane was performed adopting a braided tensor product among coordinates of different spacetime points, as done here; by a proper treatment the authors found that the result for the 2-point function also coincides with the undeformed one. Finally, already in [44, 43] Oeckl used the relation between the deformed and undeformed covariance to determine a mapping between deformed and undeformed theories (in the Euclidean formulation of QFT); in the Moyal-Weyl case this mapping allows [43] to immediately compute the deformed Green functions in terms of the undeformed ones (however they do not coincide).

Acknowledgments

G. F. acknowledges financial support by the Alexander von Humboldt Stiftung for a stay at the “Arnold Sommerfeld Center for Theoretical Physics” of the University LMU of Munich, where the main part of this work has been done. We are grateful to P. Aschieri, H. Steinacker, S. Doplicher, G. Piacitelli, F. Lizzi, P. Vitale for useful and stimulating discussions.
References

[1] Y. Abe, “Noncommutative Quantization for Noncommutative Field Theory”, Int. J. Mod. Phys. A22 (2007), 1181-1200.
[2] L. Alvarez-Gaume, M.A. Vazquez-Mozo, “General Properties of Noncommutative Field Theories”, Nucl. Phys. B668 (2003), 293-321.
[3] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, “Noncommutative Geometry and Gravity”, Class. Quant. Grav. 22 (2005), 3511-3532.
[4] P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, “Noncommutative Geometry and Gravity”, Class. Quant. Grav. 23 (2006), 1883-1912.
[5] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, “On the unitarity problem in space/time noncommutative theories”, Phys. Lett. B533 (2002), 178-181.
[6] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, “Ultraviolet Finite Quantum Field Theory on Quantum Spacetime”, Commun. Math. Phys. 237 (2003), 221-241; “Field Theory on Noncommutative Spacetimes: Quasiplanar Wick Products”, Phys. Rev. D71 (2005), 025022.
[7] A. P. Balachandran, G. Mangano, A. Pinzul, S. Vaidya, “Spin and Statistics on the Groenewold-Moyal Plane: Pauli-Forbidden Levels and Transitions”, Int. J. Mod. Phys. A21 (2006), 3111-3126.
[8] A. P. Balachandran, A. Pinzul, B. A. Qureshi, “UV-IR Mixing in Non-Commutative Plane”, Phys. Lett. B634 (2006), 434-436.
[9] A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi, S. Vaidya, “Statistics and UV-IR Mixing with Twisted Poincaré Invariance”, hep-th/0608179.
[10] H.-J. Borchers, K. Pohlmeyer, “Eine scheinbare Abschwächung der Lokalitätsbedingung. II”, Commun. Math. Phys. 8 (1968), 269-281.
[11] H. Bozkaya, P. Fischer, H. Grosse, M. Pittschmann, V. Putz, M. Schweda, R. Wulkenhaar, “Space/time noncommutative field theories and causality”, Eur. Phys. J. C29 (2003), 133-141.
[12] J.-G. Bu, H.-C. Kim, Y. Lee, C. H. Vac, J. H. Yee, “Noncommutative Field Theory from twisted Fock space” Phys. Rev. D73 (2006), 125001.
[13] B. Chakraborty, S. Gangopadhyay, A. G. Hazra, F. G. Scholtz, “Twisted Galilean symmetry and the Pauli principle at low energies”, J. Phys. A39 (2006), 9557-9572.
[14] M. Chaichian, P. Kulish, K. Nishijima, A. Tureanu, “On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT”, Phys. Lett. B604 (2004) 98-102.
[15] M. Chaichian, K. Nishijima, A. Tureanu, “An Interpretation of Noncommutative Field Theory in Terms of a Quantum Shift”, Phys. Lett. B633 (2006), 129-133.
[16] M. Chaichian, P. Presnajder, A. Tureanu, “New concept of relativistic invariance in NC space-time: twisted Poincaré symmetry and its implications”, Phys. Rev. Lett. 94 (2005), 151602.
[17] M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A. Tureanu, Yu. S. Vernov, “Towards an Axiomatic Formulation of Noncommutative Quantum Field Theory”, hep-th/0402212.

[18] V. Chari and A. Pressley, “A guide to quantum groups”, Cambridge University Press, 1994.

[19] S. Doplicher, K. Fredenhagen, J. E. Roberts, “The quantum structure of spacetime at the Planck scale and quantum fields”, Commun. Math. Phys. 172 (1995), 187-220.

[20] M. R. Douglas, N. A. Nekrasov, “Noncommutative Field Theory”, Rev. Mod. Phys. 73 (2001), 977-1029.

[21] V. G. Drinfeld, “On constant quasiclassical solutions of the Yang-Baxter quantum equation”, Sov. Math. Dokl. 28 (1983), 667. See also: L. A. Takhtadjan, Lectures on quantum groups, in: Introduction to quantum group and integrable massive models of quantum field theory (Nankai, 1989), 69–197, Nankai Lectures Math. Phys., World Sci. Publishing, River Edge, NJ, 1990.

[22] T. Filk, “Divergences in a field theory on quantum space”, Phys. Lett. B376 (1996), 53-58.

[23] R. Estrada, J. M. Gracia-Bondía, J. C. Várilly, “On asymptotic expansions of twisted products”, J. Math. Phys. 30 (1989), 2789-2796.

[24] L. D. Faddeev “Quantum completely integral models of field theory”, Sov. Sci. Rev. C1 (1980), 107-155.

[25] G. Fiore, “Deforming Maps for Lie Group Covariant Creation & Annihilation Operators”, J. Math. Phys. 39 (1998), 3437-3452.

[26] G. Fiore and P. Schupp, “Identical Particles and Quantum Symmetries”, Nucl. Phys. B470 (1996), 211; “Statistics and Quantum Group Symmetries”, Banach Center Publications, vol 40, Inst. of Mathematics, Polish Academy of Sciences, Warszawa, P. Budzynski, W. Pusz, S. Zakrzeski (editors), hep-th/9605133.

[27] G. Fiore, “On Bose-Fermi Statistics, Quantum Group Symmetry, and Second Quantization”, proceedings of “XXI International Colloquium on Group Theoretical Methods in Physics” (Group21), 15 - 20 July 1996, Goslar, Germany, Ed. K.-H. Doebner e V. Dobrev. hep-th/9611144.

[28] D. H. T. Franco, “On the Borchers class of a non-commutative field”, J. Phys. A: Math. Gen. 38 (2005), 5799-5807.

[29] J. Gomis, T. Mehen, “Space-Time Noncommutative Field Theories And Unitarity”, Nucl.Phys. B591 (2000), 265-276

[30] H. Grosse, J. Madore, H. Steinacker, Field Theory on the q-deformed Fuzzy Sphere II: Quantization, J. Geom. Phys. 43 (2002), 205-240.

[31] R. Haag, Local Quantum Physics. Fields, Particles, Algebras, Springer, 1996.

[32] K. Hepp, “On the connection between the Wightman and the LSZ quantum field theory”, in Brandeis Summer Inst. Theor. Phys. 1965, Vol.I, Axiomatic Field theory, M. Chretien and S. Deser eds., Gordon and Breach 1966.
[33] C. Itzykson, J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill Inc., 1980.

[34] R. W. Jost, “Properties of Wightman functions”, in *Lectures on Field Theory and Many-Body Problem*, E. R. Caianiello (ed.), Academic Press, New York, 1961; *The General Theory of Quantized Fields* Lectures on Applied Mathematics, 4, AMS, Providence, 1965.

[35] F. Koch, E. Tsouchnika, “Construction of θ-Poincaré Algebras and their Invariants on Mθ”, Nucl.Phys. B717 (2005), 387-403.

[36] P. P. Kulish, “Representation of the Zamolodchikov-Faddeev algebra”, Jour. Sov. Math. 24 (1984), 208-215: English translation from the Russian, Zapiski Nauchn. Semin. LOMI, vol.109 (1981), 83-92.

[37] Y. Liao, K. Sibold, “Time-ordered Perturbation Theory on Noncommutative Spacetime: Basic Rules” Eur. Phys.J. C25 (2002), 469-477.

[38] F. Lizzi, S. Vaidya, P. Vitale, “Twisted Conformal Symmetry in Noncommutative Two-Dimensional Quantum Field Theory”, Phys. Rev. D73 (2006), 125020

[39] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cim.* 1, 1425 (1955); for an excellent survey see e.g. Sect. 18 b in [46].

[40] J. Madore, H. Steinacker, “Propagator on the h-deformed Lobachevsky plane”, J. Phys. A33 (2000), 327-342.

[41] S. Majid, *Foundations of Quantum Groups*, Cambridge Univ. Press (1995); and references therein.

[42] S. Minwalla, M. Van Raamsdonk, N. Seiberg, “Noncommutative Perturbative Dynamics”, JHEP 0002 (2000), 020.

[43] R. Oeckl, “Untwisting Noncommutative R^d and the Equivalence of Quantum Field Theory”, Nucl. Phys. B581 (2000), 559-574.

[44] R. Oeckl, “Braided Quantum Field Theory”, Commun. Math. Phys. 217 (2001), 451-473.

[45] W. Pusz, S. L. Woronowicz, “Twisted Second Quantization”, Reports on Mathematical Physics 27 (1989), 231.

[46] S. S. Schweber, “An introduction to relativistic quantum field theory” (1961), Row, Petersen & Co., New York.

[47] N. Seiberg, L. Susskind, N. Toumbas, *Space/Time Non-Commutativity and Causality*, JHEP 0006 (2000), 044.

[48] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin 1964.

[49] F. Strocchi, “Relativistic Quantum Mechanics and Field Theory” Found. Phys. 34 (2004) 501-527. [hep-th/0401143]

[50] R.J. Szabo, “Quantum Field Theory on Noncommutative Spaces”, Phys. Rept. 378 (2003), 207-299.

[51] A. Tureanu, “Twist and Spin-Statistics Relation in Noncommutative Quantum Field Theory”, Phys. Lett. B638 (2006), 296-301.
[52] J. Wess, “Deformed Coordinate Spaces; Derivatives”, Lecture given at BW2003 Workshop "Mathematical, Theoretical and Phenomenological Challenges Beyond Standard Model” 29 August-02 September, 2003 Vrnjacka Banja, Serbia. hep-th/0408080

[53] J. Wess and B. Zumino, Nucl. Phys. Proc. Suppl. 18B (1991), 302.

[54] A.S. Wightman, J. Indian Math. Soc., 24 625 (1960-61);
D. Ya. Petrina, Ukr. Mat. Zh. 13, 109 (1961);
V.S. Vladimirov, Sov. Math. Dokl. 1 (1960) 1039; Methods of the Theory of Functions of Several Complex Variables, Cambridge, Massachusetts, MIT Press, 1966.

[55] J. Zahn, “Remarks on twisted noncommutative quantum field theory”, Phys. Rev. D73 (2006), 105005.

[56] A. B. Zamolodchikov, A. B. Zamolodchikov, “Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Model”, Annals Phys. 120 (1979), 253-291.