STABILITY OF A NATURAL SHEAF OVER THE CARTESIAN SQUARE 
OF THE HILBERT SCHEME OF POINTS ON A $K3$ SURFACE

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ABSTRACT. Let $S$ be a $K3$ surface and $S^{[n]}$ the Hilbert scheme of length $n$ subschemes of $S$. Over the cartesian square $S^{[n]} \times S^{[n]}$ there exists a natural reflexive rank $2n - 2$ coherent sheaf $E$, which is locally free away from the diagonal. The fiber of $E$ over the point $(I_{Z_1}, I_{Z_2})$, corresponding to ideal sheaves of distinct subschemes $Z_1 \neq Z_2$, is $\text{Ext}^1_S(I_{Z_1}, I_{Z_2})$. We prove that $E$ is slope stable if the rank of the Picard group of $S$ is $\leq 19$. The Chern classes of $\text{End}(E)$ are known to be monodromy invariant. Consequently, the sheaf $\text{End}(E)$ is polystable-hyperholomorphic.

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References

1. Introduction

Let $S$ be a $K3$ surface, not necessarily projective, and $n$ an integer $\geq 2$. Denote by $S^{[n]}$ the Hilbert scheme, or Douady space, of length $n$ zero-dimensional subschemes of $S$. Let $U$ be the ideal sheaf of the universal subscheme of $S \times S^{[n]}$. Let $\pi_{ij}$ be the projection from $S^{[n]} \times S \times S^{[n]}$ onto the product of the $i$-th and $j$-th factors. The relative extension sheaf

\begin{equation}
E := \text{Ext}^1_{\pi_{13}}(\pi_{12}^*U, \pi_{23}^*U)
\end{equation}

is a rank $2n - 2$ reflexive sheaf over $S^{[n]} \times S^{[n]}$. The sheaf $E$ is infinitesimally rigid, i.e., $\text{Ext}^1(E, E)$ vanishes, by [MM2 Prop. 4.1]. We prove in this note the following statement.

**Theorem 1.1.**

1. When $\text{Pic}(S)$ is trivial the sheaf $E$ is slope-stable with respect to every Kähler class.
2. If $0 < \text{rank}(\text{Pic}(S)) \leq 19$, then $E$ is $\omega \boxplus \omega$-slope-stable with respect to some Kähler class $\omega$ on $S^{[n]}$.

The Kähler class $\omega \boxplus \omega$ above is the sum of the pullbacks of $\omega$ via the two projections. The theorem is proven in Section 4. The class $c_2(\text{End}(E))$ remains of Hodge type $(2, 2)$ on $X \times X$. 

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for all Kähler manifolds $X$ deformation equivalent to $S^{[n]}$, by [M] Lemma 3.2 and Prop. 3.4. Such a manifold $X$ is called of $K3^{[n]}$-type. The slope stability result of the above theorem implies that $\mathcal{E}nd(E)$ deforms to a reflexive sheaf $A$ of Azumaya algebras over $X \times X$, for all manifolds $X$ of $K3^{[n]}$-type, by a theorem of Verbitsky [Ve1, Theorem 3.19] (see also [M, Cor. 6.11 and Prop. 6.16]). Slope stability was proven earlier in [M, Theorem 7.4] for the analogue of the sheaf $E$ over the cartesian square $M \times M$ of a moduli space $M$ of stable coherent sheaves of rank $2n - 2$ on a projective $K3$ surface $S$. There $M$ was chosen so that the sheaf $E$ is twisted by a Brauer class of maximal order equal to the rank 2 of $\mathcal{E}nd(E)$, with respect to every Kähler class [M, Prop. 6.5].

Theorem 1.1 plays an important role in two joint works with S. Mehrotra [MM1, MM2]. There we need the stability of $E$ in Theorem 1.1 (rather than over a moduli space $M$ of higher rank sheaves), since only when the moduli space is a Hilbert scheme we could prove that the integral transform using the universal sheaf from the derived category of the $K3$ surface to that of the moduli space is a $\mathbb{P}^n$-functor [MM2, Theorem 1.1] (see also [Ad]). Theorem 1.1 is used in the construction of a 21-dimensional moduli space $\mathcal{M}$ of isomorphism classes of triples $(X, \eta, A)$, where $X$ is a manifold of $K3^{[n]}$-type, $\eta : H^2(X, \mathbb{Z}) \to \Lambda_n$ is an isometry with a fixed lattice $\Lambda_n$, and $A$ is a slope stable reflexive sheaf of Azumaya algebras over $X \times X$ [MM1]. We prove a Torelli theorem for the pairs $(X, A)$ deformation equivalent to $(S^{[n]}, \mathcal{E}nd(E))$, stating that if $(X', A')$ is another such pair, and $A$ and $A'$ are both $\mathcal{E}nd$-slope-stable, with respect to the same Kähler class $\omega$ on $X$, then $A' \cong \mathcal{E}nd_A(A \cap \mathcal{E}nd_A) \cap \mathcal{E}nd_A(A \cap \mathcal{E}nd_A)$ (the latter is the same sheaf as $A$, but with the dual multiplication) [MM1, Theorem 1.11]. In [MM2] we associate to a triple $(X, \eta, A)$ in $\mathcal{M}$ a pre-triangulated $K3$-category, resulting in a global generalized (non-commutative and gerby) deformation of the derived categories of coherent sheaves on $K3$ surfaces (which are associated to the triples $(S^{[n]}, \eta, \mathcal{E}nd(E))$ by the construction).

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2. Density

Let $\Lambda$ be the $K3$ lattice, namely the unique even unimodular lattice of rank 22 and signature $(3, 19)$. A marked $K3$ surface $(S, \eta)$ consists of a $K3$ surface $S$ and an isometry $\eta : H^2(S, \mathbb{Z}) \to \Lambda$. Choose one of the two connected components of the moduli space of isomorphism classes of marked $K3$ surfaces, not necessarily projective, and denote it by $\mathcal{M}_{K3}$. Let $\Omega_{K3}$ be the corresponding period domain, and let $P : \mathcal{M}_{K3} \to \Omega_{K3}$ be the period map [LP]. The map $P$ is a local homeomorphism.

The signed isometry group $O^+(\Lambda)$ acts on $\mathcal{M}_{K3}$, by $g(S, \eta) = (S, g\eta)$, and on $\Omega_{K3}$, and the period map is $O^+(\Lambda)$-equivariant. Over $\mathcal{M}_{K3}$ we have a universal family of $K3$ surfaces $S \to \mathcal{M}_{K3}$, since the automorphism group of a $K3$ surface acts faithfully on its degree 2 cohomology. We get over $\mathcal{M}_{K3}$ a universal Douady space $S^{[n]} \to \mathcal{M}_{K3}$, as well as the fiber square of the latter $S^{[n]} \times_{\mathcal{M}_{K3}} S^{[n]} \to \mathcal{M}_{K3}$. Let $\mathcal{U}$ be the ideal sheaf of the universal subscheme of $S \times_{\mathcal{M}_{K3}} S^{[n]}$ and let $\pi_{ij}$ be the projection from $S^{[n]} \times_{\mathcal{M}_{K3}} S \times_{\mathcal{M}_{K3}} S^{[n]}$ onto the fiber product of the $i$-th and $j$-th factors. The universal version $\mathcal{E}$ over $S^{[n]} \times_{\mathcal{M}_{K3}} S^{[n]}$ of the rank $2n - 2$ sheaf $E$ is the relative extension sheaf

$$E = \mathcal{E}xt^1_{\pi_{13}^*\mathcal{U}, \pi_{23}^*\mathcal{U}}.$$  

(2.1)
The universal family $\mathcal{S} \to \mathfrak{M}_{K3}$ is $O^+(\Lambda)$-equivariant, by the universal property of the universal family. Hence, so are the universal Douady space and the universal subscheme. The sheaves $\mathcal{U}$ and $\mathcal{E}$ are thus $O^+(\Lambda)$-equivariant as well. The following is a consequence of a density theorem of Verbitsky.

**Lemma 2.1.** Let $W \subset \mathfrak{M}_{K3}$ be a non-empty open $O^+(\Lambda)$-invariant subset. Then $W$ contains every marked pair $(S, \eta)$ such that the rank of the Picard group of $S$ is $\leq 19$.

**Proof.** The image $P(W)$ is an open $O^+(\Lambda)$-invariant subset of $\Omega_{K3}$. The stabilizer of $P(S, \eta)$ in $O^+(\Lambda)$ acts transitively on the fiber of $P$ over $P(S, \eta)$, by the Global Torelli Theorem [BR] (see also [LP] Lemma 10.4). Hence, $W = P^{-1}(P(W))$. If the rank of $\text{Pic}(S)$ is less than or equal to 19, then the $O^+(\Lambda)$-orbit of $P(S, \eta)$ is dense in $\Omega_{K3}$, by [Ve2], so it intersects $P(W)$ and is thus contained in $P(W)$. Hence, the $O^+(\Lambda)$-orbit of $(S, \eta)$ is contained in $W$.

### 3. A canonical subsheaf

A coherent sheaf $F$ on a complex manifold of dimension $d$ is said to be pure of codimension $c$ if the support of every subsheaf of $F$ has codimension $c$.

**Lemma 3.1.** [H Cor. 1.5] Let $0 \to F' \to F \to Q \to 0$ be an exact sequence of coherent sheaves on a complex manifold.

1. Assume that $F$ is reflexive. Then $F'$ is reflexive, if and only if either the torsion subsheaf of $Q$ is pure of codimension 1 or $Q$ is torsion free.
2. If $F$ is torsion free and $F'$ is reflexive, then either $Q$ is torsion free, or the torsion subsheaf of $Q$ is pure of codimension 1.

**Proof.** Part [H] is proven in [H Cor. 1.5]. Part (2): The torsion subsheaf of $F''/F'$ either vanishes, or is pure of codimension 1, by Part [H], and the composition $Q \cong F/F' \to F''/F'$ is injective.

Let $\iota : Z \to S \times S^{[n]}$ be the inclusion of the universal subscheme and let $I_Z$ be its ideal sheaf. Let $q_i$ be the projection from $S \times S^{[n]}$ to the $i$-th factor, $i = 1, 2$. We get the split short exact sequence of locally free sheaves

$$0 \to \mathcal{O}_{S^{[n]}} \xrightarrow{\iota} q_2^* \mathcal{O}_Z \to A_0 \to 0.$$  

We have $c_1(A_0) = c_1(q_2^* \mathcal{O}_Z) = -\delta$, where $2\delta \in H^2(S^{[n]}, \mathbb{Z})$ is the class of the divisor of non-reduced subschemes [ELG] Sec. 5]. Let $\mathcal{I} \subset S^{[n]} \times S^{[n]}$ be the incidence subvariety consisting of pairs $(Z_1, Z_2)$ of length $n$ subschemes, which are not disjoint, $Z_1 \cap Z_2 \neq \emptyset$. Let $p_i$ be the projection from $S^{[n]} \times S^{[n]}$ to the $i$-th factor, $i = 1, 2$.

**Proposition 3.2.** The sheaf $E$, given in [1.7], fits in the left exact sequence

$$0 \to p_2^* A_0 \xrightarrow{b} E \xrightarrow{j} p_1^* A_0^*$$

and the co-kernel of $j$ is supported, set theoretically, on $\mathcal{I}$.

**Proof.** Set $\mathcal{O} := \mathcal{O}_{S^{[n]} \times S \times S^{[n]}}$. Apply the functor $R\text{Hom}(\pi_{12}^* I_Z, \bullet)$ to the short exact sequence

$$0 \to \pi_{23}^* I_Z \to \mathcal{O} \to \pi_{23}^* \mathcal{O}_Z \to 0$$

to get the exact triangle

$$R\text{Hom}(\pi_{12}^* I_Z, \pi_{23}^* I_Z) \to R\text{Hom}(\pi_{12}^* I_Z, \mathcal{O}) \to R\text{Hom}(\pi_{12}^* I_Z, \pi_{23}^* \mathcal{O}_Z).$$
Note the isomorphism $\mathcal{H}om(\pi^*_1 I_Z, \mathcal{O}) \cong \mathcal{O}$. Applying the functor $R\pi_{13,*}$ and taking sheaf cohomology of the resulting exact triangle we get the long exact sequence

$$
0 \to \mathcal{O}_{S[n_xS[n]} \to \pi_{13,*}\mathcal{H}om(\pi^*_1 I_Z, \pi^*_2 I_* O_Z) \to E \to \mathcal{E}xt^1_{\pi_{13}}(\pi^*_1 I_Z, \mathcal{O}) \to \ldots
$$

Away from $\mathcal{I}$ the natural homomorphism

$$
\pi_{13,*}\mathcal{H}om(\mathcal{O}, \pi^*_2 I_* O_Z) \to \pi_{13,*}\mathcal{H}om(\pi^*_1 I_Z, \pi^*_2 I_* O_Z)
$$
is an isomorphism, and the left hand sheaf is naturally isomorphic to $p^*_2 q^*_2, I_* O_Z$. Composing the above displayed homomorphism with $\tilde{h}$ we get the injective homomorphism $\tilde{h}$ in Equation (3.2). The exactness of the sequence (3.3) implies that the image of $\tilde{h}$ is saturated, away from $\mathcal{I}$. The image of $\tilde{h}$ must be a saturated subsheaf of $E$ everywhere, by Lemma 3.1 (1), since the image is locally free and the codimension of $\mathcal{I}$ is 2.

The relative extension sheaves $\mathcal{E}xt^1_{\pi_{13,*}}(\mathcal{O}, \mathcal{O})$ and $\mathcal{E}xt^2_{\pi_{13,*}}(\pi^*_1 I_Z, \mathcal{O})$ both vanish. Hence, we get the short exact sequence

$$
0 \to \mathcal{E}xt^1_{\pi_{13}}(\pi^*_1 I_Z, \mathcal{O}) \to \mathcal{E}xt^2_{\pi_{13}}(\pi^*_2 I_* O_Z, \mathcal{O}) \to \mathcal{E}xt^2_{\pi_{13}}(\pi^*_1 I_Z, \mathcal{O}) \to 0.
$$

Grothendieck-Verdier Duality, combined with the triviality of the relative canonical line bundle $\omega_{\pi_{13}}$, identifies the pullback $p^*_1(u^*)$ of the dual of the homomorphism $u$ in Equation (3.1) with the of the homomorphism $\tilde{u}^*$ above. Hence, $\mathcal{E}xt^1_{\pi_{13}}(\pi^*_1 I_Z, \mathcal{O})$ is isomorphic to the kernel $p^*_1 A^0$ of $p^*_1(u^*)$. Using the latter isomorphism we obtain the homomorphism $j$ in Equation (3.2) from the homomorphism $j$ in the long exact sequence (3.3). The co-kernel of $j$ is equal to the kernel of $k$ in (3.3) from the sheaf $\mathcal{E}xt^1_{\pi_{13}}(\pi^*_1 I_Z, \pi^*_2 I_* O_Z)$ to $\mathcal{E}xt^2_{\pi_{13}}(\pi^*_1 I_Z, \pi^*_2 I_* O_Z)$. The former is isomorphic to $\mathcal{E}xt^2_{\pi_{13}}(\pi^*_2 I_* O_Z, \pi^*_2 I_* O_Z)$ and is thus supported set theoretically on $\mathcal{I}$ and the latter is supported on the diagonal. Hence, the co-kernel of $j$ is supported on $\mathcal{I}$.

4. Blow-up of the diagonal and restriction to the exceptional divisor

We recall one more crucial property of the relative extension sheaf $E$ given in (1.1). Let $\beta : Y \to S[n] \times S[n]$ be the blow-up of $S[n] \times S[n]$ along the diagonal $\Delta$. Denote by $\tilde{\Delta}$ the exceptional divisor in $Y$. Note that $\tilde{\Delta}$ is naturally isomorphic to $\mathbb{P}(T^*\Delta)$. Set

$$
V := \beta^* E(\tilde{\Delta})/\text{torsion}.
$$

Let $\pi : \tilde{\Delta} \to \Delta$ be the restriction of $\beta$. Let $\ell \subset \pi^* T^* \Delta$ be the tautological line sub-bundle. The restriction of $\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})$ to $\tilde{\Delta}$ is isomorphic to $\ell$. Let $\ell^\perp$ be the sub-bundle of $\pi^* T^* \Delta$ symplectic-orthogonal to $\ell$. Note that the symplectic structure on $T^* \Delta$ induces one on $\ell^\perp/\ell$.

**Lemma 4.1.** The vector space $\text{End}(\ell^\perp/\ell)$ is one-dimensional.

**Proof.** The restriction of $\ell^\perp/\ell$ to each fiber of $\pi$ is simple, by [MM1, Lemma 7.3 (2)]. Hence, the sheaf $\pi_* \mathcal{E}nd(\ell^\perp/\ell)$ is the trivial line-bundle over $S[n]$. The statement follows from the isomorphism $H^0(\tilde{\Delta}, \mathcal{E}nd(\ell^\perp/\ell)) = H^0(S[n], \pi_* \mathcal{E}nd(\ell^\perp/\ell))$. \hfill $\square$

**Proposition 4.2.** $V$ is locally free. The restriction of $V$ to $\tilde{\Delta}$ is isomorphic to $\ell^\perp/\ell$.

**Proof.** When the $K3$ surface $S$ is projective, the statement is precisely [M, Prop. 4.1 parts (3) and (6)]. The proof provided there uses a global complex over $S[n] \times S[n]$ of locally free sheaves, representing the object $R\pi_{13}(R\mathcal{H}om(\pi^*_1 U, \pi^*_2 U))$. However, the argument provided there is
local and goes through when such a complex of locally free sheaves is given only in a complex analytic neighborhood of a point of the diagonal in $S[n] \times S[n]$. Hence, the statement that $V$ is locally free holds without the assumption that $S$ is projective. It remains to be proved that the restriction of $V$ to $	ilde{\Delta}$ is isomorphic to $\ell^\perp/\ell$ even when $S$ is non-projective.

Let $\beta : \mathcal{Y} \to S[n] \times \mathcal{M}_{K3}$ be the blow-up of the diagonal and $D \subset \mathcal{Y}$ the exceptional divisor. The sheaf $\mathcal{V} := (\beta^*\mathcal{E})(D)/\text{torsion}$ is locally free, by the above argument. Let $\phi : D \to \mathcal{M}_{K3}$ and $\psi : S[n] \to \mathcal{M}_{K3}$ be the natural morphisms. $\mathcal{D}$ is naturally isomorphic to $\mathbb{P}(T_{\phi})$. Let $\mathcal{L}$ be the tautological subbundle of $\phi^*\mathcal{T}_{\phi}$ over $\mathcal{D}$. The sheaf $\psi_*\Omega^2_\psi$ is a line-bundle over $\mathcal{M}_{K3}$. We have a natural injective homomorphism $\psi^*\psi_*\Omega^2_\psi \to \Omega^2_\psi$. We get a well defined symplectic-orthogonal subbundle $\mathcal{L}^\perp \subset \phi^*\mathcal{T}_{\phi}$ as well as the quotient $\mathcal{L}^\perp/\mathcal{L}$.

The fiber of the sheaf $R^{4n-1}\phi_*\left(\left((\mathcal{L}^\perp/\mathcal{L})^* \otimes \mathcal{V}_{1}\right) \otimes \omega_{\phi}\right)$ at every marked pair $(S, \eta)$ maps isomorphically onto the vector space $\text{Ext}^{4n-1}(\ell^\perp/\ell, \mathcal{V}_{1}\otimes \omega_{\tilde{\Delta}})$, by the Base-Change Theorem. The latter vector space is isomorphic to $\text{Hom}(\mathcal{V}_{1}\otimes \ell^\perp/\ell^*, \mathcal{L}^\perp)$ and is thus one-dimensional whenever $S$ is projective, by [M Prop. 4.1 parts (3) and (6)] and Lemma 4.1. The locus of projective $K3$ surfaces is dense in $\mathcal{M}_{K3}$. Hence, $\text{Hom}(\mathcal{V}_{1}\otimes \ell^\perp/\ell)$ is one-dimensional over a non-empty Zariski open subset of $\mathcal{M}_{K3}$, by the semi-continuity theorem.

Considering the sheaf $R^{4n-1}\phi_*\left(\mathcal{V}_{1}\otimes \mathcal{L}^\perp/\mathcal{L} \otimes \omega_{\phi}\right)$ we conclude similarly that $\text{Hom}(\ell^\perp/\ell, \mathcal{V}_{1}\otimes \omega_{\tilde{\Delta}})$ is one-dimensional over a non-empty Zariski open subset of $\mathcal{M}_{K3}$. Hence, the sheaves

$$L_1 := \phi_\ast \text{Hom}(\mathcal{L}^\perp/\mathcal{L}, \mathcal{V}_{1})$$

and

$$L_2 := \phi_\ast \text{Hom}(\mathcal{V}_{1}, \mathcal{L}^\perp/\mathcal{L})$$

are both locally free of rank 1 over a non-empty Zariski open subset $U'$ of $\mathcal{M}_{K3}$. The fiber of $L_i$, $i = 1, 2$, is spanned by an isomorphism over every projective marked $K3$ surface. Hence, the composition maps $L_1 \otimes L_2 \to \phi_\ast \text{End}(\mathcal{L}^\perp/\mathcal{L})$ and $L_1 \otimes L_2 \to \phi_\ast \text{End}(\mathcal{V}_{1})$ are isomorphisms over a non-empty Zariski open subset $U$ of $U'$. We conclude that $\mathcal{V}_{1}\otimes \omega_{\tilde{\Delta}}$ is isomorphic to $\ell^\perp/\ell$ for every marked pair $(S, \eta)$ in $U$. The set $U$ is $O^*(\Lambda)$-invariant and thus contains every marked pair $(S, \eta)$ with Picard rank $\leq 19$, by Lemma 2.1. Such is the Picard rank of every non-projective $K3$ surface.

Proposition 4.3. When Pic$(S)$ is trivial the vector bundle $\ell^\perp/\ell$ has a unique non trivial subsheaf $C$ of rank less than $2n - 2$. The rank of $C$ is $n - 1$.

Proposition 4.3 will be proven in Section 5.

Proof of Theorem 4.1 We prove first that every non-trivial proper subsheaf of $V$ has rank $n - 1$. Let $F$ be a non-trivial subsheaf of $V$ of lower rank. We may assume that $F$ is saturated in $V$. Then $F$ is a subsheaf of $V$ away from the locus where $V/F$ is not locally free. That locus has codimension at least 2, since $V/F$ is torsion free. Thus, $F$ restricts to a subsheaf of the restriction of $V$ to $\tilde{\Delta}$ of the same rank as $F$. The restriction of $V$ to $\tilde{\Delta}$ is isomorphic to $\ell^\perp/\ell$, by Proposition 4.2. We conclude that rank$(F) = n - 1$, by Proposition 4.3.

Let $F$ be a non-trivial proper saturated subsheaf of $E$. Then $F$ is reflexive, being saturated in the reflexive sheaf $E$, and rank$(F) = n - 1$. In particular, the image $F_0$ of the homomorphism $h$ in the sequence (3.2) does not have any non-trivial subsheaf of lower rank. Furthermore, either $F = F_0$, or $F \cap F_0 = 0$.

Assume that $F \cap F_0 = 0$. Composing the inclusion $F \to E$ with the homomorphism $j$ in the sequence (3.2) we get the injective homomorphism $g : F \to p_1^*A^*_0$. The sheaf $F$ is reflexive, $p_1^*A^*_0$
is locally free, and the rank of both is \( n - 1 \). Hence, every irreducible component of the support of the co-kernel of \( g \) must be of codimension 1, by Lemma 5.1. The co-kernel of \( g \) surjects onto the co-kernel of the homomorphism \( j \), and the latter is supported, set theoretically, on the codimension two subvariety \( \mathcal{I} \), by Proposition 3.2. Hence, \( \mathcal{I} \) is contained in some effective divisor of \( S^{[n]} \times S^{[n]} \). But such a divisor does not exists, since the only effective divisors on \( S^{[n]} \) are multiples of the divisor of non-reduced subschemes. A contradiction. Hence, \( F = F_0 \).

We have the equalities \( c_1(F_0) = -p_2^*c_1(A_0) \), \( c_1(E/F_0) = p_2^*c_1(A_0^*), \) and \( c_1(A_0) = -\delta \), where \( 2\delta \) is an effective divisor, as noted in Section 3. Hence, the unique non-trivial subsheaf \( F_0 \) of \( E \) does not slope-destabilize \( E \) with respect to any Kähler class on \( S^{[n]} \times S^{[n]} \).

(2) Let \( W \subset \mathcal{M}_{K3} \) be the subset consisting of pairs \((S, \eta)\) for which the sheaf \( \mathcal{E} \), given in (2.1), restricts to \( S^{[n]} \times S^{[n]} \) as an \( \omega \oplus \omega \)-slope-stable sheaf with respect to some Kähler class \( \omega \) on \( S^{[n]} \). \( W \) is an open subset, since every Kähler class extends to a section of Kähler classes for the universal Douady space over an open subset of \( \mathcal{M}_{K3} \) (see [Vo1, Th. 9.3.3]) and since slope-stability is an open condition. \( W \) is clearly \( O^+(\Lambda) \)-invariant. Hence, \( W \) contains every marked pair \((S, \eta)\) in \( \mathcal{M}_{K3} \), with rank\( (\text{Pic}(S)) \leq 19 \), by Lemma 2.1.

5. Proof of Proposition 4.3

Let \( S \) be a K3 surface with a trivial Picard group.

Lemma 5.1.  
1. \( H^0(S^n, \text{Sym}^k(T^*S^n)) = 0 \), for all \( n > 0 \) and \( k > 0 \).
2. \( \mathbb{P}(T^*S^n) \) does not contain any effective divisors.

Proof. 1. The vector bundles \( \text{Sym}^d(T^*S) \) are stable, as the holonomy of \( T^*S \) is \( Sp(2) \cong SL(2) \) and the \( d \)-th symmetric power of the standard rank 2 representation of \( SL(2) \) is irreducible, for all \( d \geq 0 \). The spaces \( H^0(S, \text{Sym}^d(T^*S)) \) thus vanish for \( d > 0 \), since \( c_1(\text{Sym}^d(T^*S)) = 0 \). Now \( H^0(S^n, \text{Sym}^k(T^*S^n)) \) is the direct sum, over all ordered partitions \( k = d_1 + \cdots + d_n \), of the tensor product \( \otimes_{i=1}^n H^0(S, \text{Sym}^{d_i}(T^*S)) \). The latter tensor product vanishes if \( k > 0 \), since at least one \( d_i \) is positive.

2. Any line bundle is a tensor power of the tautological line bundle \( \mathcal{O}_{\mathbb{P}(T^*S^n)}(1) \). The statement follows immediately from part 1.

Lemma 5.2. Let \( X \) be a compact complex manifold, which does not have any effective divisors, and \( L \) a line bundle on \( X \). Let \( V \) be a finite dimensional vector space. Then every saturated subsheaf of \( V \otimes_C L \) is of the form \( W \otimes_C L \), for some subspace \( W \) of \( V \).

Proof. Let \( F \) be a saturated subsheaf of \( V \otimes_C L \) and \( \iota : F \to V \otimes_C L \) the inclusion homomorphism. Choose a generic quotient space \( Q \) of \( V \), such that the composite homomorphism \( h \), given by \( F \to V \otimes_C L \to Q \otimes_C L \), is injective. The sheaf \( F \) is reflexive, being a saturated subsheaf of a locally free sheaf. Thus, the cokernel of \( h \) either vanishes, or it is supported on a codimension 1 subscheme, by Lemma 3.1. The latter case is excluded by the assumption that \( X \) does not have any effective divisors. Hence \( h \) is an isomorphism. Let \( \iota_0 : Q \to V \) be the composition \( Q \cong \text{Hom}(L,F) \to \text{Hom}(L,V \otimes_C L) \cong V \). Then \( F \) is the image of \( \iota_0 \otimes 1 : Q \otimes_C L \to V \otimes_C L \).

Lemma 5.3. Let \( X, L, \) and \( V \) be as in Lemma 5.2. Let \( G \) be a finite group of automorphisms of \( X \). Assume that \( L \) is endowed with a \( G \)-equivariant structure, \( V \) is an irreducible \( G \)-representation, and endow \( V \otimes_C L \) with the associated \( G \)-equivariant structure. Then \( V \otimes_C L \) does not have any non-trivial saturated \( G \)-equivariant subsheaf of lower rank.
Proof. Let $F \subset V \otimes_{\mathbb{C}} L$ be a saturated subsheaf. Then $F = W \otimes_{\mathbb{C}} L$, for some subspace $W$ of $V$, by Lemma 5.2. $G$-equivariance of $F$ implies that $\text{Hom}(L, F)$ is a $G$-subrepresentation of the irreducible representation $V$. Hence, $W = 0$ or $W = V$. \hfill \Box

Proof of Proposition 4.3. Let $S^n$ be the $n$-th Cartesian product, $S^{(n)}$ the $n$-th symmetric product, and $q : S^n \to S^{(n)}$ the quotient morphism. Denote by $U \subset S^{(n)}$ the complement of the diagonal subscheme and set $\tilde{U} := q^{-1}(U)$. Denote by $q : \tilde{U} \to U$ the covering map. We identify $U$ also as an open subset of the Hilbert scheme $S^{[n]}$.

Let $p_n : \mathbb{P}(T^* S^n) \to S^n$ be the projection and denote by $\tilde{\ell}$ the tautological line sub-bundle of $p_n^* T^* S^n$. Denote by $\mathfrak{S}_n$ the symmetric group on $n$ letters. Let $\sigma_n$ be a $\mathfrak{S}_n$-invariant symplectic structure on $S^n$. Note that $\sigma_n$ is unique, up to a scalar factor. Denote by $\ell_{\tilde{\ell}}$ the sub-bundle of $p_n^* T^* S^n$ symplectic-orthogonal to $\tilde{\ell}$ with respect to $\sigma_n$. Both $\ell$ and $\ell_{\tilde{\ell}}$ are $\mathfrak{S}_n$-invariant sub-bundles of $p_n^* T^* S^n$. Hence, $\ell_{\tilde{\ell}} / \ell$ is endowed with the structure of an $\mathfrak{S}_n$-equivariant vector bundle over $\mathbb{P}(T^* S^n)$.

Let $\pi_k : S^n \to S$ be the projection on the $k$-th factor. Let $\tilde{\pi}_k : \mathbb{P}(T^* S^n) \to S$ be the composition $\pi_k \circ p_n$. Let $\hat{\ell}_i$ be the projection of $\tilde{\ell}$ to $\tilde{\pi}_i^* T^* S$. The projection $\ell_{\hat{\ell}_i}$ is an isomorphism. Let $\tilde{C}$ be the quotient $(\oplus_{i=1}^n \hat{\ell}_i) / \ell$. Then $C$ is an $\mathfrak{S}_n$-invariant subsheaf of $\tilde{\ell}_{\hat{\ell}} / \ell$ of rank $n - 1$. It thus corresponds to a saturated subsheaf $C$ of $\ell_{\tilde{\ell}} / \ell$ of rank $n - 1$. Set $\tilde{Q} := [\ell_{\tilde{\ell}} / \ell] / \tilde{C}$. We get the short exact sequence

$$0 \to \tilde{C} \to \ell_{\tilde{\ell}} / \ell \to \tilde{Q} \to 0$$

of $\mathfrak{S}_n$-equivariant coherent sheaves over $\mathbb{P}(T^* S^n)$. $\tilde{C}$ is isomorphic, as a $\mathfrak{S}_n$-equivariant sheaf, to $\ell \otimes_{\mathbb{C}} W$, where $W$ is the reflection representation of $\mathfrak{S}_n$. If the torsion subsheaf of $\tilde{Q}$ is non-zero, then its support has codimension $\geq 2$. But $\tilde{C}$ is locally free and hence reflexive. Consequently, the sheaf $Q$ is torsion free, by Lemma 3.1 (1). The dual sheaf $\tilde{Q}^*$ is isomorphic to $\tilde{C}$, since $\tilde{C}$ is a Lagrangian subsheaf with respect to the $\mathfrak{S}_n$-invariant symplectic structure on $\ell_{\tilde{\ell}} / \ell$. We conclude that neither $\tilde{C}$ nor $\tilde{Q}$ admit any non-trivial saturated $\mathfrak{S}_n$-equivariant subsheaf of lower rank, by Lemmas 5.1 and 5.3.

Assume that $F$ is a non-trivial saturated subsheaf of $\ell_{\tilde{\ell}} / \ell$ of rank $< 2n - 2$. Then $F$ restricts to a subsheaf of the restriction of $\ell_{\tilde{\ell}} / \ell$ to $\mathbb{P}(T^* U)$. Now $q^* \mathbb{P}(T^* U)$ is isomorphic to $\mathbb{P}(T^* \tilde{U})$, and $q^* F$ extends to a non-trivial saturated $\mathfrak{S}_n$-invariant subsheaf $\tilde{F}$ of $\tilde{\ell}_{\hat{\ell}} / \ell$ of rank $\leq 2n - 3$. We may assume that $\tilde{F}$ has rank $\leq n - 1$, possibly after replacing $\tilde{F}$ with its symplectic-orthogonal subsheaf.

If $\tilde{F}$ is not contained in $\tilde{C}$, then the natural homomorphism $h : \tilde{F} \to \tilde{Q}$ is $\mathfrak{S}_n$-equivariant and it does not vanish. Its image is an equivariant subsheaf of $\tilde{Q}$ and it thus must have rank $n - 1$. The support of the quotient $\tilde{Q} / h(\tilde{F})$ has codimension $\geq 2$, since $\mathbb{P}(T^* S^n)$ does not contain any effective divisor, by Lemma 5.1. The sheaf $\tilde{F}$ is reflexive, being a saturated subsheaf of a locally free sheaf. The quotient $\tilde{Q} / h(\tilde{F})$ must thus vanish, by Lemma 3.1 (2), since $\tilde{Q}$ is torsion free. Hence, $h$ is an isomorphism and the short exact sequence (5.1) splits. But the bundle $\ell_{\tilde{\ell}} / \ell$ restricts to a stable bundle with trivial determinant over every $\mathbb{P}^{2n-1}$ fiber of $p_n$ [MM1, Lemma 7.4]. A contradiction. Hence, $\tilde{F}$ is contained in $\tilde{C}$. Consequently, $\tilde{F} = \tilde{C}$, since the latter does not have any $\mathfrak{S}_n$-equivariant subsheaf of lower rank. This completes the proof of Proposition 4.3. \hfill \Box
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