Representations of $SO(3)$ and angular polyspectra

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Abstract

We characterize the angular polyspectra, of arbitrary order, associated with isotropic fields defined on the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Our techniques rely heavily on group representation theory, and specifically on the properties of Wigner matrices and Clebsch-Gordan coefficients. The findings of the present paper constitute a basis upon which one can build formal procedures for the statistical analysis and the probabilistic modelization of the Cosmic Microwave Background radiation, which is currently a crucial topic of investigation in cosmology. We also outline an application to random data compression and “simulation” of Clebsch-Gordan coefficients.

Key Words. Group Representations; Isotropy; Polyspectra; Spherical Random Fields.

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1 Introduction

The connection between probability theory and group representation theory has led to a long tradition of fruitful interactions. A well-known reference is provided by [10]; see e.g. [2, Section 40-41], [1], [3], [4], [25], [26], and the references therein, for other relevant contributions. In this paper we shall focus in particular on the profound connection between the probabilistic notion of isotropy, i.e. invariance in law under the action of a group, and the representation theory of the group itself. One instance of this connection is well-known, i.e. the celebrated Peter-Weyl Theorem, which allows the construction of spectral representations for isotropic random fields on homogeneous spaces of general compact groups, see [24] for a general construction and [23], [22] for examples related, respectively, to the torus and the sphere. Our aim here is to use these representations in order to characterize random fields by means of a higher order spectral theory; in particular, one of our main goals will be to establish the link between the so-called polyspectra (or higher order spectra) and alternative (tensor product and direct sum) representations of the underlying isotropy group. In particular, we shall provide a general expression for higher order spectra of isotropic spherical random fields in terms of convolutions of Clebsch-Gordan or Wigner coefficients. The latter where introduced in Mathematics in the XIX century for the analysis of Algebraic Invariants; they have since then played a crucial role in the development of Quantum Physics in the XX century (see for instance [31] for a comprehensive reference); their role in Group Representation theory will be discussed below, while more details can be found for instance in [32].

Our analysis may have an intrinsic mathematical interest, but it is also strongly motivated by applications to Physics and Cosmology. Concerning the latter, the analysis of higher order spectra for isotropic spherical random fields is currently at the core of several research efforts which are related to the analysis of Cosmic Microwave Background (CMB) radiation data, see for instance [12] for a general introduction and [15], [17], [20], [21] for some references on the bi- and trispectrum. A general characterization of the theoretical properties of higher order angular power spectra can yield several insights into the statistical analysis of the massive datasets that are or will be made available by satellite experiments such as WMAP or Planck. For instance, the current understanding of the behaviour of the bispectrum for some simple physical models has already led to many applications ([1], [3], [5]), aiming at obtaining constraints on nonlinearity parameters of utmost physical significance; needless to say, a proper understanding of higher order spectra can lead to more efficient statistical procedures and better constraints, which may help to
solve some of the important scientific issues at stake in CMB analysis (primarily a proper understanding of the Big Bang inflationary dynamics, which is tightly linked with the CMB nonlinear structure, see [12, 9, 21, 41]).

The relevance of the current results need not be limited to cosmological applications. Indeed, the analysis of spherical random fields has currently led to remarkable developments in the Geophysical and Planetary Sciences, and even in Medical Imaging, see for instance (3, 27, 33). Moreover, we shall show below how the relationship which we establish leads very naturally to some numerical algorithms for the estimation of Clebsch-Gordan and Wigner coefficients. The latter represent probability amplitudes of quantum interactions and as such a rich literature in Mathematical Physics has been concerned with recipes for their numerical estimation: our procedure lends itself to easy implementation and can be simply extended to very general compact groups, although in this paper we focus solely on $SO(3)$.

The plan of this paper is as follows: in Section 2 we introduce our general probabilistic setting and provide some preliminary notation and background material. In Section 3 we present some background material on representation theory, while in Section 4 and Section 5 we obtain our main results, including the explicit characterization of polyspectra. These results are applied in Section 6 to derive explicit material on representation theory, while in Section 4 and Section 5 we obtain our main results, including the numerical estimation of Clebsch-Gordan coefficients. The latter represent probability amplitudes of quantum interactions and as such a rich literature in Mathematical Physics has been concerned with recipes for their numerical estimation: our procedure lends itself to easy implementation and can be simply extended to very general compact groups, although in this paper we focus solely on $SO(3)$.

In the subsequent sections, every random element is defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$.

## 2 General setting

In this paper, we focus on real-valued, centered, square-integrable and isotropic random fields on the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. A centered and square integrable random field $T$ on $S^2$ is just a collection of random variables of the type $T(x) : x \in S^2$ such that, for every $x \in S^2$, $ET(x) = 0$ and $ET^2(x) < \infty$. In the following, whenever we write that $T$ is a field on $S^2$, we will implicitly assume that $T$ is real-valued, centered and square-integrable. From now on, we shall distinguish between two notions of isotropy, which we name strong isotropy and weak isotropy of order $n (n \geq 2)$.

**Strong isotropy** – The field $T$ is said to be strongly isotropic if, for every $k \in \mathbb{N}$, every $x_1, ..., x_k \in S^2$ and every $g \in SO(3)$ (the group of rotations in $\mathbb{R}^3$) we have

$$\{T(x_1), ..., T(x_k)\} \overset{d}{=} \{T(gx_1), ..., T(gx_k)\},$$

(2.1)

where $\overset{d}{=}$ denotes equality in distribution.

**Weak isotropy** – The field $T$ is said to be $n$-weakly isotropic ($n \geq 2$) if $E|T(x)|^n < \infty$ for every $x \in S^2$, and if, for every $x_1, ..., x_n \in S^2$ and every $g \in SO(3)$,

$$E[T(x_1) \times \cdots \times T(x_n)] = E[T(gx_1) \times \cdots \times T(gx_n)].$$

The following statement, whose proof is elementary, indicates some relations between the two notions of isotropy described above.

**Proposition 1**

1. A strongly isotropic field with finite moments of some order $n \geq 2$ is also $n$-weakly isotropic.

2. Suppose that the field $T$ is $n$-weakly isotropic for every $n \geq 2$ (in particular, $E|T(x)|^n < \infty$ for every $n \geq 2$ and every $x \in S^2$) and that, for every $k \geq 1$ and every $(x_1, ..., x_k)$, the law of the vector $\{T(x_1), ..., T(x_k)\}$ is determined by its moments. Then, $T$ is also strongly isotropic.
Now suppose that $T$ is a strongly isotropic field, and denote by $dx$ the Lebesgue measure on $S^2$. Since the variance $ET(x)^2$ is finite and independent of $x$ (by isotropy), one deduces immediately that

$$E \left[ \int_{S^2} T(x)^2 \, dx \right] < \infty,$$
from which one infers that the random path $x \to T(x)$ is a.s. square integrable with respect to the Lebesgue measure. Then, it is a standard result that the following spectral representation holds:

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x), \quad \text{where} \quad a_{lm} \triangleq \int_{S^2} T(x) Y_{lm}(x) \, dx,$$

(2.2)

and where the complex-valued functions $\{Y_{lm} : l \geq 0, \ m = -l, \ldots, l\}$ are the so-called spherical harmonics, to be defined below. The spectral representation (2.2) must be understood in the $L^2(\Omega \times S^2)$ sense, i.e.

$$\lim_{L \to \infty} E \left[ T - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm} \right]^2_{L^2(S^2)} = 0,$$

where $L^2(S^2)$ is the complex Hilbert space of functions on $S^2$, which are square-integrable with respect to $dx$. If moreover the trajectories of $T(x)$ are a.s. continuous, then the representation (2.2) holds pointwise, i.e.

$$\lim_{L \to \infty} T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) = 0 \quad \text{for all} \ x \in S^2, \ a.s.-P,$$

see for instance [3] or [30]. The spherical harmonics $\{Y_{lm}\}_{m=-l,...,l}$ are the eigenfunctions of the Laplace-Beltrami operator on the sphere, denoted by $\Delta_{S^2}$, satisfying the relation $\Delta_{S^2} Y_{lm} = -l(l+1) Y_{lm}$. These functions can be represented by means of spherical coordinates $x = (\theta, \varphi)$ as follows:

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta) \exp(i m \varphi), \quad \text{for} \ m > 0,$$

$$Y_{lm}(\theta, \varphi) = (-1)^m Y_{l,-m}(\theta, \varphi), \quad \text{for} \ m < 0, \ 0 \leq \theta \leq \pi, \ 0 \leq \varphi < 2\pi,$$

where $P_l^m(\cos \theta)$ denotes the associated Legendre polynomial of degree $l, m$, i.e.

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l,$$

$$m = 0, 1, 2, ..., l, \ l = 0, 1, 2, 3, ....$$

The random spherical harmonics coefficients $\{a_{lm}\}$ appearing in (2.2) form a triangular array of zero-mean and square-integrable random variables, which are complex-valued for $m \neq 0$ and such that $E a_{lm} \overline{a_{lm'}} = \delta_l^0 \delta_{m}^{m'} C_l$, the bar denoting complex conjugation. Here, and for the rest of the paper, the symbol $\delta^*_k$ is equal to one if $a = b$ and zero otherwise. We also write $C_l = E |a_{lm}|^2, \ l \geq 0$, to indicate the angular power spectrum of $T$ (we stress that the quantity $C_l$ does not depend on $m$ – see e.g. [4] for a proof of this fact). Observe that, by definition of the spherical harmonics, $a_{lm} = (-1)^m \overline{a_{lm}}$. Note also that a convenient route to derive (2.2) is by means of an appropriate version of the stochastic Peter-Weyl theorem – see for instance [3] or [24], as well as Section 3.1 below.

Observe that the representation (2.2) still holds for fields $\{T(x)\}$ that are not necessarily isotropic, but such that the random path $x \to T(x)$ is $P$-a.s. square integrable with respect to the Lebesgue measure $dx$. Indeed, if the last property holds, then one has that, $P$-almost surely,

$$\lim_{L \to \infty} \int_{S^2} \left( T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \right)^2 \, dx = 0.$$

(2.3)
In this case, however, none of the previously stated properties on the array \( \{a_{lm}\} \) holds in general. By an argument similar to those displayed above, a sufficient condition to have that \( x \rightarrow T(x) \) is \( P \)-a.s. Lebesgue-square integrable is that \( \sup_{x \in S^2} ET(x)^2 < \infty \).

The next result, that we record for future reference, is proved in [1].

**Proposition 2** Let \( T \) be a centered, square-integrable and strongly isotropic random field. Let the coefficients \( \{a_{lm}\} \) be defined according to (2.3). Then, for every \( l,m \), one has that \( E|a_{lm}|^2 < \infty \). Moreover, for every \( l \geq 1 \), the coefficients \( \{a_{0l},...,a_{ll}\} \) are independent if, and only if, they are Gaussian. If the vector \( \{a_{0l},...,a_{ll}\} \) is Gaussian, one also has that \( \Re(a_{lm}) \) and \( \Im(a_{lm}) \) are independent and identically distributed for every fixed \( m = 1,...,l \) (\( \Re(z) \) and \( \Im(z) \) stand, respectively, for the real and imaginary parts of \( z \)).

The following result formalizes the fact that, in general, one cannot deduce strong isotropy from weak isotropy. The proof makes use of Proposition 1.

**Proposition 3** For every \( n \geq 2 \), there exists a \( n \)-weakly isotropic field \( T \) such that \( T \) is not strongly isotropic.

**Proof.** Fix \( l \geq 1 \), and consider a vector

\[
b_m, \quad m = -l,...,l,
\]

of centered complex-valued random variables such that: (i) \( b_0 \) is real, (ii) \( b_{-m} = (-1)^m b_m \), (iii) the vector \( \{b_0,...,b_l\} \) is not Gaussian and is composed of independent random variables, (iv) for every \( k = 1,...,n \), the (possibly mixed) moments of order \( k \) of the variables \( \{b_0,...,b_l\} \) coincide with those of a vector \( \{a_0,...,a_l\} \) of independent, centered and complex-valued Gaussian random variables with common variance \( C_l \) and such that \( a_0 \) is real and, for every \( m = 1,...,l \), the real and imaginary parts of \( a_m \) are independent and identically distributed (the existence of a vector such as \( \{b_0,...,b_l\} \) is easily proved). Now define the two fields

\[
T(x) = \sum_{m=-l}^{l} b_m Y_{lm}(x) \quad \text{and} \quad T^*(x) = \sum_{m=-l}^{l} a_m Y_{lm}(x).
\]

By Proposition 2, \( T^* \) is strongly isotropic, and also \( n \)-weakly isotropic by Proposition 1. By construction, one also has that \( T \) is \( n \)-weakly isotropic. However, \( T \) cannot be strongly isotropic, since this would violate Proposition 2 (indeed, if \( T \) was isotropic, one would have an example of an isotropic field whose harmonic coefficients \( \{b_0,...,b_l\} \) are independent and non-Gaussian). \( \blacksquare \)

In what follows, we use the symbol \( A \otimes B \) to indicate the Kronecker product between two matrices \( A \) and \( B \). Given \( n \geq 2 \), we denote by \( \Pi(n) \) the class of partitions of the set \( \{1,...,n\} \). Given an element \( \pi \in \Pi(n) \), we write \( \pi = \{b_1,...,b_k\} \) to indicate that the sets \( b_j \subseteq \{1,...,n\} \), \( j = 1,...,k \), are the blocks of \( \pi \). The blocks of a partition are always listed according to the lexicographic order, that is: the block \( b_1 \) always contains 1, the block \( b_2 \) contains the least element of \( \{1,...,n\} \) not contained in \( b_1 \), and so on. Also the elements within each block \( b_j \) are written in increasing order. For instance, if a partition \( \pi \) of \( \{1,...,5\} \) is composed of the blocks \( \{1,3\}, \{5,4\} \) and \( \{2\} \), we will write \( \pi \) in the form \( \pi = \{1,3\}, \{2\}, \{4,5\} \).

**Definition A. (A1)** Let the field \( T \) admit the representation (2.4), and suppose that, for some \( n \geq 2 \), one has that \( E|a_{lm}|^n < \infty \) for every \( l,m \). Then, \( T \) is said to have finite spectral moments of order \( n \).

(A2) Suppose that \( T \) has finite spectral moments of order \( n \geq 2 \), and, for \( l \geq 0 \), use the notation

\[
a_l = (a_{l-1},...,a_{0l},...,a_{ll}).
\]

The polyspectrum of order \( n-1 \), associated with \( T \), is given by the collection of vectors

\[
S_{l_1...l_n} = E[a_{l_1} \otimes a_{l_2} \otimes \cdots \otimes a_{l_n}],
\]

(2.5)
where \( 0 \leq l_1, l_2, \ldots, l_n \). Note that the vector \( S_{l_1 \ldots l_n} \) appearing in \((2.3)\) has dimension \((2l_1 + 1) \times \cdots \times (2l_n + 1)\).

\(\text{(A3)}\) Suppose that \( T \) has finite spectral moments of order \( n \geq 2 \). The (mixed) cumulant polyspectrum of order \( n \) is associated with \( T \), is given by the vectors

\[
S_{l_1 \ldots l_n} = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)!E[\otimes_{i \in b_1} a_{l_i}] \otimes \cdots \otimes E[\otimes_{i \in b_k} a_{l_i}],
\]

where \( 0 \leq l_1, l_2, \ldots, l_n \), and, for every block \( b_j = \{i_1, \ldots, i_p\} \), we use the notation

\[
E[\otimes_{i \in b_j} a_{l_i}] = E\left[a_{l_{i_1}} \otimes \cdots \otimes a_{l_{i_p}}\right]
\]

(recall that we always list the elements of \( b_j \) in such a way that \( i_1 \leq \cdots \leq i_p \)). Plainly, the vector \( S_{l_1 \ldots l_n}^C \) in \((2.4)\) has also dimension \((2l_1 + 1) \times \cdots \times (2l_n + 1)\).

Remark. Suppose that \( T \) has finite spectral moments of order \( n \geq 2 \). Then, by selecting frequencies \( l_1 = b_2 = \cdots = b_3 = l \geq 0 \), one obtains that

\[
S_{l_1 \ldots l_n}^C := S_{l_1 \ldots l}^{(n)}(n) = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)!E\left[(a_{l_1})^{\otimes |b_1|}\right] \otimes \cdots \otimes E\left[(a_{l_1})^{\otimes |b_k|}\right]
\]

where \( |b_j| \) stands for the size of the block \( b_j \), and we use the notation

\[
(a_{l_1})^{\otimes |b_j|} = \underbrace{a_{l_1} \otimes \cdots \otimes a_{l_1}}_{|b_j| \text{ times}}.
\]

3 Preliminary material

3.1 Representation Theory for \( SO(3) \)

We start by reviewing briefly some background material on the special group of rotations \( SO(3) \), i.e. the space of \( 3 \times 3 \) real matrices \( A \) such that \( A^T A = I_3 \) (the three-dimensional identity matrix) and \( \det(A) = 1 \).

We first recall that each element \( g \in SO(3) \) can be parametrized by the set \((\varphi, \vartheta, \psi)\) of the so-called Euler angles \((0 \leq \varphi < 2\pi, 0 \leq \vartheta \leq \pi, 0 \leq \psi < 2\pi)\); indeed each rotation in \( \mathbb{R}^3 \) can be realized sequentially as

\[
A = A(g) = R(\varphi, \vartheta, \psi) = R_z(\varphi)R_x(\vartheta)R_z(\varphi)
\]

where \( R_x(\varphi), R_x(\vartheta), R_z(\psi) \in SO(3) \) can be expressed by means of the following general definitions, valid for every angle \( \alpha \),

\[
R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}.
\]

The representation \((3.8)\) is unique except for \( \vartheta = 0 \) or \( \vartheta = \pi \), in which case only the sum \( \varphi + \psi \) is determined. In words, the rotation is realized by rotating first by \( \varphi \) around the axis \( z \), then rotating around the new \( x \) axis by \( \vartheta \), then rotating by \( \psi \) around the new \( z \) axis. It is clear that the first two rotations identify one point on the sphere, so the whole operation could be also interpreted as moving the North Pole to a new orientation in \( S^2 \) and then rotating by \( \psi \) the tangent plane at the new location.

In these coordinates, a complete set of irreducible matrix representations for \( SO(3) \) is provided by the Wigner’s \( D \) matrices \( D^l(\varphi, \vartheta, \psi) = \{D^l_{mn}(\varphi, \vartheta, \psi)\}_{m,n=0,1,\ldots,l} \) of dimensions \((2l + 1) \times (2l + 1)\) for \( l = 0, 1, 2, \ldots \); we refer to classical textbooks, such as [32], [2] or [10], for any unexplained definition or
result concerning group representation theory. An analytic expression for the elements of Wigner’s $D$ matrices is provided by

$$D^l_{mn}(\psi, \vartheta, \varphi) = e^{-im\psi} d^l_{mn}(\vartheta) e^{im\varphi}, \quad m, n = -(2l + 1), \ldots, 2l + 1$$

where the indices $m, n$ indicate, respectively, columns and rows, and

$$d^l_{mn}(\vartheta) = (-1)^{l-n}[ (l + m)!(l - m)!(l + n)!(l - n)! ]^{1/2} \sum_k (-1)^k \Bigg( \frac{\cos \frac{\vartheta}{2}}{k!} \Bigg)^{m+n+2k} \Bigg( \frac{\sin \frac{\vartheta}{2}}{2^{l-m-n-2k}} \Bigg)^{2l-m-n-2k} \sin^k \vartheta, \quad k \in \mathbb{N} \cup \{0\}$$

and the sum runs over all $k$ such that the factorials are non-negative; see [31, Chapter 4] for a huge collection of alternative expressions. Here we simply recall that the elements of $D^l(\psi, \vartheta, \varphi)$ are related to the spherical harmonics by the relationship

$$D^0_{lm}(\varphi, \vartheta, \psi) = (-1)^m \sqrt{\frac{4\pi}{2l + 1}} Y_{l-m}(\vartheta, \varphi) = \sqrt{\frac{4\pi}{2l + 1}} Y^*_{lm}(\vartheta, \varphi). \quad (3.9)$$

In other words, the spherical harmonics correspond (up to a constant) to the elements of the “central” column in the Wigner’s $D$ matrix. Such matrices operate irreducibly and equivalently on $(2l + 1)$ spaces (the so-called isotypical spaces), each of them spanned by a different column $n$ of the matrix representation itself. The elements of column $n$ correspond to the so-called spin $n$ spherical harmonics, which enjoy a great importance in particle physics and in harmonic expansions for tensor valued random fields. In this paper, we restrict our attention only to the usual $n = 0$ spherical harmonics, which correspond to usual scalar functions.

**Remark.** By exploiting relation (3.9), it is not difficult to show that the usual spectral representation for random fields on the sphere, as given in (2.2), is just the stochastic Peter-Weyl Theorem on the quotient space $S^2 = SO(3)/SO(2)$. Indeed, by the stochastic Peter-Weyl Theorem (see e.g. [24]) we obtain, for any square integrable, isotropic random field $\{T(g) : g \in SO(3)\}$

$$T(g) = T(\varphi, \vartheta, \psi) = \sum_l \sum_{m,n} a_{lmn} \sqrt{\frac{2l + 1}{8\pi^2}} D^l_{mn}(\varphi, \vartheta, \psi),$$

where $dg$ is the Haar (uniform) measure on $SO(3)$ with total mass $8\pi^2$. Now if we consider the restriction of $T(g)$ to $S^2 = SO(3)/SO(2)$, denoted by $T_{S^2}(\varphi, \vartheta)$, we deduce that

$$a_{lmn} = \int_{SO(3)} T_{S^2}(g) \sqrt{\frac{2l + 1}{8\pi^2}} D^l_{mn}(g) dg$$

$$= \int_{S^2} T_{S^2}(\varphi, \vartheta) \left\{ \int_0^{2\pi} e^{im\psi} d\psi \right\} \sqrt{\frac{2l + 1}{8\pi^2}} d^l_{mn}(\vartheta) e^{-im\varphi} \sin \vartheta d\vartheta d\varphi,$$

$$= \int_{S^2} T_{S^2}(\varphi, \vartheta) \delta^0_n(2\pi) \sqrt{\frac{2l + 1}{8\pi^2}} d^l_{mn}(\vartheta) e^{-im\varphi} \sin \vartheta d\vartheta d\varphi,$$

the second equality following from the fact that $T_{S^2}(g)$ is constant with respect to $\psi$. We can thus conclude that

$$a_{lmn} = \left\{ \begin{array}{ll} 0 & \text{for } n \neq 0, \\
\sqrt{2\pi} a_{lm} & \text{for } n = 0, \end{array} \right.$$
3.2 The Clebsch-Gordan matrices

It follows from standard representation theory that we can exploit the family \( \{D^l\}_{l=0,1,2,\ldots} \) to build alternative (reducible) representations, either by taking the tensor product family \( \{D_{l_1} \otimes D_{l_2}\}_{l_1,l_2} \), or by considering direct sums \( \{\oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l\}_{l_1,l_2} \). These representations have dimensions

\[
(2l_1 + 1)(2l_2 + 1),
\]

and are unitarily equivalent, whence there exists a unitary matrix \( C_{l_1,l_2} \) such that

\[
\{D_{l_1} \otimes D_{l_2}\} = C_{l_1,l_2} \left\{ \oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l \right\} C_{l_1,l_2}^* .
\]

(3.10)

The matrix \( C_{l_1,l_2} \) is a \( \{(2l_1 + 1)(2l_2 + 1) \times (2l_1 + 1)(2l_2 + 1)\} \) block matrix, whose blocks, of dimensions \( (2l_2 + 1) \times (2l_1 + 1) \), are customarily denoted by \( C_{l_1(m_1)l_2}^i(m_2) \), \( m_1 = -l_1,\ldots,l_1 \); the elements of such a block are indexed by \( m_2 \) (over rows) and \( m \) (over columns; note that \( m = -(2l_1 + 1),\ldots,2l_1 + 1 \)). More precisely,

\[
C_{l_1,l_2} = \left\{ C_{l_1(m_1)l_2}^i(m_2) \right\}_{m_1=-l_1,\ldots,l_1; i=|l_2-l_1|,\ldots,l_2+l_1} \quad (3.11)
\]

\[
C_{l_1(m_1)l_2}^l = \left\{ C_{l_1,m_1,l_2}^{i(m_2)} \right\}_{m_2=-l_2,\ldots,l_2; m=-l,\ldots,l} .
\]

(3.12)

Remark. The fact that the two matrices \( D_{l_1} \otimes D_{l_2} \) and \( \oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l \) have the same dimension follows from the elementary relation (valid for any integers \( l_1, l_2 \geq 0 \)):

\[
\sum_{l=|l_2-l_1|}^{l_2+l_1} (2l + 1) = (2l_1 + 1)(2l_2 + 1) .
\]

(3.13)

By induction, one also obtains that, for every \( n \geq 3 \),

\[
\sum_{l_1=|l_2-l_1|}^{l_2} \sum_{\lambda_1=|l_2-l_1|}^{\lambda_1+l_1} \cdots \sum_{\lambda_{n-1}=|l_n-l_{n-1}|}^{\lambda_{n-1}+l_{n-1}} (2\lambda_{n-1} + 1) = \prod_{j=1}^{n} (2l_j + 1) ,
\]

for any integers \( l_1, \ldots, l_n \geq 0 \) (relation (3.14) is needed in Section 4.2).

The Clebsch-Gordan coefficients for \( SO(3) \) are then defined as the collection \( \{C_{l_1,m_1,l_2}^{m_2}\} \) of the elements of the unitary matrices \( C_{l_1,l_2} \). These coefficients were introduced in Mathematics in the XIX century, as motivated by the analysis of invariants in Algebraic Geometry; in the 20th century, they have gained an enormous importance in the quantum theory of angular momentum, where \( C_{l_1,m_1,l_2}^{m_2} \) represents the probability amplitude that two particles with total angular momentum \( l_1, l_2 \) and momentum projection on the z-axis \( m_1 \) and \( m_2 \) are coupled to form a system with total angular momentum \( l \) and projection \( m \) (see e.g. [15]). Their use in the analysis of isotropic random fields is much more recent, see for instance [13] and the references therein.

Remark (More on the structure of the Clebsch-Gordan matrices). To ease the reading of the subsequent discussion, we provide an alternative way of building a Clebsch-Gordan matrix \( C_{l_1,l_2} \), starting from any enumeration of its entries. Fix integers \( l_1, l_2 \geq 0 \) such that \( l_1 \leq l_2 \) (this is just for notational convenience), and consider the Clebsch-Gordan coefficients \( \{C_{l_1,m_1,l_2}^{m_2}\} \) given in (3.11)–(3.12). According to the above discussion, we know that: (i) \( -l_i \leq m_i \leq l_i \), for \( i = 1, 2 \), (ii) \( l_2 - l_1 \leq l \leq l_1 + l_2 \), (iii) \( -m \leq l \leq m \), and (iv) the symbols \( (l_1, m_1, l_2, m_2) \) label rows, whereas the pairs \((l, m)\) are attached to columns. Now introduce the total order \( \prec_c \) on the “column pairs” \((l, m)\), by setting that \((l, m) \prec_c (l', m')\), whenever either \( l < l' \) or \( l = l' \) and \( m < m' \). Analogously, introduce a total order \( \prec_r \) over the “row symbols” \((l_1, m_1, l_2, m_2)\), by...
setting that \((l_1, m_1, l_2, m_2) \prec_r (l'_1, m'_1, l'_2, m'_2)\), if either \(m_1 < m'_1\), or \(m_1 = m'_1\) and \(m_2 < m'_2\) (recall that \(l_1\) and \(l_2\) are fixed). One can check that the set of column pairs (resp. row symbols) can now be written as a saturated chain\(^1\) with respect to \(\prec_c\) (resp. \(\prec_r\)) with a least element given by \((l_2 - l_1, -(l_2 - l_1))\) (resp. \((l_1, -l_1, -l_2, -l_2))\) and a maximal element given by \((l_1 + l_2 + l_1)\) (resp. \((l_1 + l_2, l_2, l_2))\). Then, (A) dispose the columns from west to east, increasingly according to \(\prec_c\), (B) dispose the rows from north to south, increasingly according to \(\prec_r\). For instance, by setting \(l_1 = 0\) and \(l_2 \geq 1\), one obtains that \(C_{l_1 l_2}\) is the \((2l_2 + 1) \times (2l_2 + 1)\) square matrix \(\{C_{l_2 m}^{l_1 m_1 m_2}\}\) with column indices \(m = -(2l_2 + 1), \ldots, (2l_2 + 1)\) and row indices \(m_2 = -(2l_2 + 1), \ldots, (2l_2 + 1)\) (from the subsequent discussion, one also deduces that, in general, \(C_{l_2 m}^{l_1 m_1 m_2} = \delta_{l_2}^{l_1} \delta_{m_1}^{m_2}\)). By selecting \(l_1 = l_2 = 1\), one sees that \(C_{11}\) is the \(9 \times 9\) matrix with elements \(C_{1m_1 m_2}^{1m_1 m_2}\) (for \(m_1, m_2 = -1, 0, 1; l = 0, 1, 2, m = -l, \ldots, l\)) arranged as follows:

\[
\begin{pmatrix}
C_{1,1,1,1}^{1,1,1,1} & C_{1,1,1,1}^{1,1,0,0} & C_{1,1,1,1}^{0,1,1,1} & C_{1,1,1,1}^{0,1,0,1} & C_{1,1,1,1}^{0,0,1,1} & C_{1,1,1,1}^{0,0,0,1} & C_{1,1,1,1}^{0,0,0,0} \\
C_{1,1,1,1}^{1,1,0,1} & C_{1,1,1,1}^{1,1,0,0} & C_{1,1,1,1}^{1,0,1,1} & C_{1,1,1,1}^{1,0,0,1} & C_{1,1,1,1}^{0,1,0,1} & C_{1,1,1,1}^{0,0,0,1} & C_{1,1,1,1}^{0,0,0,0} \\
C_{1,1,1,1}^{1,1,1,0} & C_{1,1,1,1}^{1,1,1,1} & C_{1,1,1,1}^{1,0,1,0} & C_{1,1,1,1}^{1,0,1,1} & C_{1,1,1,1}^{1,0,0,0} & C_{1,1,1,1}^{1,0,0,1} & C_{1,1,1,1}^{1,0,0,0} \\
C_{1,1,1,1}^{1,1,1,1} & C_{1,1,1,1}^{1,1,1,1} & C_{1,1,1,1}^{1,1,1,1} & C_{1,1,1,1}^{1,0,1,1} & C_{1,1,1,1}^{1,0,1,1} & C_{1,1,1,1}^{1,0,1,1} & C_{1,1,1,1}^{1,0,1,1} \\
\end{pmatrix}
\]

Explicit expressions for the Clebsch-Gordan coefficients of \(SO(3)\) are known, but they are in general hardly manageable. We have for instance (see \([53]\), expression 8.2.1.5)

\[
C_{l_1 m_1 m_2}^{l_3 m_3} = (-1)^{l_1 + l_2 - l_3 + m_2} \sqrt{2l_3 + 1} \left\{ \begin{array}{l}
(l_1 + l_2 - l_3)!(l_1 - l_2 + l_3)!(l_1 - l_2 + l_3)!
\end{array} \right. 1/2
\]

\[
\times \left[ \frac{(l_2 + m_3)!(l_3 - m_3)!}{(l_1 + 1)!} \right] 1/2
\]

\[
\times \sum_z \left[ \frac{(l_2 + l_3 + m_1 - z)!(l_1 - m_2 + z)!}{(l_2 - l_3)!} \right] 1/2,
\]

where the summation runs over all \(z\)'s such that the factorials are non-negative. This expression becomes much neater for \(m_2 = m_3 = 0\), where we have

\[
C_{l_1,0}^{l_3,0} = \begin{cases}
0 & \text{for } l_1 + l_2 + l_3 \text{ odd} \\
(-1)^{l_1 + l_2 + l_3} \sqrt{2l_3 + 1} \left( (l_1 + l_2 + l_3)! / 2 \right) & \text{for } l_1 + l_2 + l_3 \text{ even}
\end{cases}
\]

The coefficients, moreover, enjoy a nice set of symmetry and orthogonality properties, playing a crucial role in our results to follow. From unitary equivalence we have the two relations:

\[
\sum_{m_1, m_2} C_{l_1 m_1 m_2}^{l_3 m_3} C_{l_2 m_1 m_2}^{l_3 m_1'} C_{l_3 m_2}^{l_3 m_2'} = \delta_{l_1}^{l_3} \delta_{m_1}^{m_1'} \delta_{m_2}^{m_2'};
\]

\[
\sum_{l_1, m} C_{l_1 m_1 m_2}^{l_3 m} C_{l_2 m_1 m_2}^{l_3 m_1'} C_{l_3 m_2}^{l_3 m_2'} = \delta_{m_1}^{m_1'} \delta_{m_2}^{m_2'};
\]

\(^1\)Given a finite set \(A = \{a_j : j = 1, \ldots, N\}\) and an order \(\prec\) on \(A\), one says that \(A\) is a saturated chain with respect to \(\prec\) if there exists a permutation \(\pi\) of \(\{1, \ldots, N\}\) such that

\[
a_{\pi(1)} < a_{\pi(2)} < \cdots < a_{\pi(N - 1)} < a_{\pi(N)}.
\]

In this case, \(a_{\pi(1)}\) and \(a_{\pi(N)}\) are called, respectively, the least and the maximal elements of the chain (see \([23]\, p. 99)\)
in particular, (3.15) is a consequence of the orthogonality of row vectors, whereas (3.16) comes from the orthogonality of columns. Other properties are better expressed in terms of the Wigner’s coefficients, which are related to the Clebsch-Gordan coefficients by the identities (see [31], Chapter 8)

\[
\begin{pmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & -m_3
\end{pmatrix}
= (-1)^{l_1+l_2+l_3} \frac{1}{\sqrt{2l_1+1}} \begin{pmatrix}
  l_3{m_3} & l_{1-m_1}l_{2-m_2}
\end{pmatrix}
\]  

(3.17)

\[
C_{t_{1},m_{1},l_{2}m_{2}}^{l_{3}m_{3}} = (-1)^{l_1-l_2+m_3} \sqrt{2l_3+1} \begin{pmatrix}
  l_3 & l_2 & -l_1 \\
  m_1 & m_2 & -m_3
\end{pmatrix}
\]  

(3.18)

The Wigner’s 3j (and, consequently, the Clebsch-Gordan coefficients are real-valued, they are different from zero only if \(m_1 + m_2 + m_3 = 0\) and \(l_i \leq l_j + l_k\) for all \(i, j, k = 1, 2, 3\) (triangle conditions), and they satisfy the symmetry conditions

\[
\begin{pmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3
\end{pmatrix}
= (-1)^{l_1+l_2+l_3} \begin{pmatrix}
  l_1 & l_2 & l_3 \\
  -m_1 & -m_2 & -m_3
\end{pmatrix}
\]

(3.19)

\[
\begin{pmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3
\end{pmatrix}
= (-1)^{\text{sign}(\pi)} \begin{pmatrix}
  l_{\pi(1)} & l_{\pi(2)} & l_{\pi(3)} \\
  m_2 & m_3 & m_1
\end{pmatrix}
\]

where \(\pi\) is a permutation of \(\{1, 2, 3\}\), and \(\text{sign}(\pi)\) denotes the sign of \(\pi\). It follows also that for \(m_1 = m_2 = m_3 = 0\), the coefficients \(C_{t_{1},m_{1},l_{2}m_{2}}^{l_{3}m_{3}}\) are different from zero only when the sum \(l_1 + l_2 + l_3\) is even. Later in the paper, we shall also need the so-called Wigner’s 6j coefficients, which are defined by

\[
\begin{pmatrix}
  a & b & c \\
  c & d & e \\
  e & f & g
\end{pmatrix} := \sum_{\alpha,\beta,\gamma}(-1)^{e+f+e+\phi} \begin{pmatrix}
  a & b & c \\
  \alpha & \beta & \epsilon \\
  c & d & e
\end{pmatrix} \begin{pmatrix}
  d & e & f \\
  \gamma & \delta & -\epsilon \\
  \alpha & \delta & -\phi
\end{pmatrix} \begin{pmatrix}
  f & g & e \\
  e & f & g \\
  \gamma & \beta & \phi
\end{pmatrix}
\]

(3.20)

For future reference, we also recall some further standard properties of Kronecker (tensor) products and direct sums of matrices: we have

\[
\oplus_{i=1}^{n} (A_i B_i) = (\oplus_{i=1}^{n} A_i) (\oplus_{i=1}^{n} B_i)
\]

(3.21)

and, provided all matrix products are well-defined,

\[
(A B \otimes C) = (A \otimes I_n) (B \otimes C)
\]

(3.22)

Here, \(\oplus_{i=1}^{n} A_i\) is defined as the block diagonal matrix \(\text{diag}\{A_1, ..., A_n\}\) if \(A_i\) is a set of square matrices of order \(r_i \times r_i\), whereas it is defined as the stacked column vector of order \((\sum_{i=1}^{n} r_i) \times 1\) if the \(A_i\) are \(r_i \times 1\) column vectors.

### 4 Characterization of polyspectra

#### 4.1 Four general statements

The following result is well-known. As it is crucial in our arguments to follow and we failed to locate any explicit reference, we shall provide a short proof for the sake of completeness. Note that, in the sequel, we use the symbol \(a_l\) to indicate the \((2l+1)\)-dimensional complex-valued random vector defined in (2.4).
Lemma 4 Let \( T \) be a strongly isotropic field on \( S^2 \), and let the harmonic coefficients \( \{a_{lm}\} \) be defined according to \((2.3)\). Then, for every \( l \geq 0 \) and every \( g \in SO(3) \), we have

\[
D^l(g) a_l \overset{d}{=} a_l, \quad l = 0, 1, 2, \ldots .
\]  

(4.23)

The equality \((4.23)\) must be understood in the sense of finite-dimensional distributions for sequences of random vectors, that is, \((4.23)\) takes place if, and only if, for every \( k \geq 1 \) and every \( 0 \leq l_1 < l_2 < \cdots < l_k \),

\[
\{D^{l_1}(g)a_{l_1}, \ldots , D^{l_k}(g)a_{l_k}\} \overset{d}{=} \{a_{l_1}, \ldots , a_{l_k}\}.
\]  

(4.24)

Proof. We provide the proof of \((4.24)\) only when \( k = 1 \) and \( l_1 = l \geq 1 \). The general case is obtained analogously. By strong isotropy, we have that, for every \( l \geq 1 \), every \( g \in SO(3) \) and every \( x_1, \ldots , x_n \in S^2 \), the equality \((2.1)\) takes place. Now, \((2.1)\) can be rewritten as follows:

\[
\left\{ \sum_{l} \sum_{m} a_{lm} Y_{lm}(x_1), \ldots , \sum_{l} \sum_{m} a_{lm} Y_{lm}(x_n) \right\} \overset{d}{=} \left\{ \sum_{l} \sum_{m} a_{lm} Y_{lm}(gx_1), \ldots , \sum_{l} \sum_{m} a_{lm} Y_{lm}(gx_n) \right\}
\]

\[
= \left\{ \sum_{l} \sum_{m} a_{lm} \sum_{m'} D^{l'}_{m'm}(g) Y_{lm'}(x_1), \ldots , \sum_{l} \sum_{m} a_{lm} \sum_{m'} D^{l'}_{m'm}(g) Y_{lm'}(x_n) \right\}
\]

\[
= \left\{ \sum_{l} \sum_{m'} \tilde{a}_{lm'} Y_{lm'}(x_1), \ldots , \sum_{l} \sum_{m'} \tilde{a}_{lm'} Y_{lm'}(x_n) \right\},
\]  

(4.25)

where we write

\[
\tilde{a}_{lm'} \triangleq \sum_{m} a_{lm} D^{l'}_{m'm}(g),
\]  

(4.26)

and we have used

\[
\{Y_{lm}(gx_1), \ldots , Y_{lm}(gx_n)\} \equiv \left\{ \sum_{m'} D^{l'}_{m'm}(g) Y_{lm'}(x_1), \ldots , \sum_{m'} D^{l'}_{m'm}(g) Y_{lm'}(x_n) \right\},
\]  

(4.27)

which follows from the group representation property and the identity \((4.25)\). To conclude, just observe that \((4.27)\) implies that

\[
\tilde{a}_{lm'} = \int_{S^2} T(gx) Y_{lm'}(x) dx, \quad m' = -l, \ldots , l,
\]

yielding that, due to strong isotropy and with obvious notation, \(\tilde{a}_l \overset{d}{=} a_l\). The conclusion follows from the fact that, thanks to \((4.26)\),

\[
\tilde{a}_l = D^l(g) a_l.
\]  

The next theorem connects the invariance properties of the vectors \(\{a_l\}\) to the representations of \(SO(3)\). We need first to establish some notation. For every \(0 \leq l_1, l_2, \ldots , l_n\), we shall write

\[
\Delta_{l_1 \ldots l_n} \triangleq \int_{SO(3)} \{D^{l_1}(g) \otimes D^{l_2}(g) \otimes \cdots \otimes D^{l_n}(g)\} dg,
\]  

(4.28)

\[
\Delta_{l_1 \ldots l_n}(g) \triangleq D^{l_1}(g) \otimes D^{l_2}(g) \otimes \cdots \otimes D^{l_n}(g), \quad g \in SO(3),
\]  

(4.29)

and use the symbol \(S_{l_1 \ldots l_n}\) (whenever is well-defined), as given in formula \((2.5)\). We stress that \(\Delta_{l_1 \ldots l_n}\) and \(\Delta_{l_1 \ldots l_n}(g)\) are square matrices with \((2l_1 + 1) \times \cdots \times (2l_n + 1)\) rows and \(S_{l_1 \ldots l_n}\) is a column vector with \((2l_1 + 1) \times \cdots \times (2l_n + 1)\) elements. The following result applies to an arbitrary \(n \geq 2\): see \[13\] for some related results in the case \(n = 3, 4\).
Proposition 5 Let $T$ be a strongly isotropic field with moments of order $n \geq 2$. Then, for every $0 \leq l_1, l_2, \ldots, l_n$ and every fixed $g^* \in SO(3)$
\[
\Delta_{l_1 \ldots l_n} S_{l_1 \ldots l_n} = S_{l_1 \ldots l_n} \tag{4.30}
\]
\[
\Delta_{l_1 \ldots l_n} (g^*) S_{l_1 \ldots l_n} = S_{l_1 \ldots l_n}. \tag{4.31}
\]
On the other hand, fix $n \geq 2$ and assume that $T(x)$ is a not necessarily isotropic random field on the sphere s.t. $\sup_x (E|T(x)|^n) < \infty$. Then $T(\cdot)$ is $P$-almost surely Lebesgue square integrable and the $n$th order spectral moments of $T$ exist and are finite. If moreover $(4.30)$ holds for every $0 \leq l_1 \leq \cdots \leq l_n$, then one has that, for every $g \in SO(3)$,
\[
E \left[ D^{l_1}(g) a_{l_1} \otimes \cdots \otimes D^{l_n}(g) a_{l_n} \right] = E \left[ a_{l_1} \otimes \cdots \otimes a_{l_n} \right], \tag{4.32}
\]
and $T$ is $n$-weakly isotropic.

Proof. By strong isotropy and Lemma 4, one has
\[
E \left\{ D^{l_1}(g) a_{l_1} \otimes \cdots \otimes D^{l_n}(g) a_{l_n} \right\} = E \left\{ a_{l_1} \otimes \cdots \otimes a_{l_n} \right\} \quad \text{for all } g \in SO(3), \ l_1, \ldots, l_n \in \mathbb{N}^n.
\]
Now assume that $g$ is sampled randomly (and independently of the $\{a_i\}$) according to some probability measure, say $P_0$, on $SO(3)$. From the property $[3.22]$ of tensor products and trivial manipulations, we obtain (with obvious notation and independence)
\[
E \left\{ D^{l_1}(\cdot) a_{l_1} \otimes \cdots \otimes D^{l_n}(\cdot) a_{l_n} \right\} = E \left\{ \left[ D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot) \right] a_{l_1} \otimes \cdots \otimes a_{l_n} \right\} = E_0 \left\{ D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot) \right\} E \left\{ a_{l_1} \otimes \cdots \otimes a_{l_n} \right\}.
\]
Now, if one chooses $P_0$ to be equal to the Haar (uniform) measure on $SO(3)$, one has that
\[
E_0 \left\{ D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot) \right\} = \Delta_{l_1 \ldots l_n},
\]
thus giving $(4.31)$. On the other hand, if one chooses $P_0$ to be equal to the Dirac mass at some $g^* \in SO(3)$, one has that
\[
E_0 \left\{ D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot) \right\} = \Delta_{l_1 \ldots l_n}(g^*),
\]
which shows that $(4.31)$ is satisfied.

Now let $T$ satisfy the assumptions of the second part of the statement for some $n \geq 2$. We recall first that the representation $[2.3]$ continues to hold, in a pathwise sense. To see that the $n$th order joint moments of the harmonic coefficients $a_{lm}$ are finite it is enough to use Jensen’s inequality, along with a standard version of the Fubini theorem, to obtain that
\[
E |a_{lm}|^n = E \left[ \int_{S^2} T(x) \overline{Y_{lm}(x)} dx \right]^n \leq E \int_{S^2} |T(x)|^n |Y_{lm}(x)|^n dx \\
\leq \left\{ \sup_{x \in S^2} |Y_{lm}(x)|^n \right\} \left\{ \sup_{x \in S^2} E|T(x)|^n \right\} \\
\leq \left( \frac{2l+1}{4\pi} \right)^{n/2} \left\{ \sup_{x \in S^2} E|T(x)|^n \right\} < \infty.
\]
It is then straightforward that, if $S_{l_1 \ldots l_n}$ satisfies $(4.31)$, one also has that for any fixed $\overline{g} \in SO(3)$
\[
E \left\{ \left[ D^{l_1}(\overline{g}) \otimes \cdots \otimes D^{l_n}(\overline{g}) \right] a_{l_1} \otimes \cdots \otimes a_{l_n} \right\} \\
= \left[ D^{l_1}(\overline{g}) \otimes \cdots \otimes D^{l_n}(\overline{g}) \right] E \left\{ a_{l_1} \otimes \cdots \otimes a_{l_n} \right\} \\
= \left[ D^{l_1}(\overline{g}) \otimes \cdots \otimes D^{l_n}(\overline{g}) \right] \Delta_{l_1 \ldots l_n} S_{l_1 \ldots l_n} \\
= \left\{ \left[ D^{l_1}(\overline{g}) \otimes \cdots \otimes D^{l_n}(\overline{g}) \right] \int_{SO(3)} \{ D^{l_1}(g) \otimes \cdots \otimes D^{l_n}(g) \} dg \right\} S_{l_1 \ldots l_n} \\
= \left\{ \int_{SO(3)} \{ D^{l_1}(\overline{g}) \otimes D^{l_2}(\overline{g}) \otimes \cdots \otimes D^{l_n}(\overline{g}) \} dg \right\} S_{l_1 \ldots l_n} \\
= \Delta_{l_1 \ldots l_n} S_{l_1 \ldots l_n} = E \left\{ a_{l_1} \otimes \cdots \otimes a_{l_n} \right\},
\]
which proves the \(n\)-th spectral moment is invariant to rotations. The fact that \(T\) is \(n\)-weakly isotropic is a consequence of the spectral representation (2.2). \(\blacksquare\)

Note that relation (4.34) can be rephrased by saying that, for a strongly isotropic field, the joint moment vector \(E \{a_{t_1} \otimes a_{t_2} \otimes \cdots \otimes a_{t_n}\}\) must be an eigenvector of the matrix (4.28) for every \(n \geq 2\) and every \(0 \leq l_1 \leq \cdots \leq l_n\). A similar characterization holds for cumulants polyspectra. Recall the notation \(S_{l_1 \ldots l_n}^c\) introduced in (2.6).

**Proposition 6** Let \(T\) be a strongly isotropic field with moments of order \(n \geq 2\). Then, for every \(0 \leq l_1, l_2, \ldots, l_n\) and every fixed \(g^* \in SO(3), \)

\[
\Delta_{l_1 \cdots l_n} S_{l_1 \ldots l_n}^c = S_{l_1 \ldots l_n}^c \tag{4.33}
\]

\[
\Delta_{l_1 \cdots l_n} (g^*) S_{l_1 \ldots l_n}^c = S_{l_1 \ldots l_n}^c. \tag{4.34}
\]

On the other hand, fix \(n \geq 2\) and assume that \(T(x)\) is a not necessarily isotropic random field on the sphere s.t. \(\sup_2 (E |T(x)|^2) < \infty\). Then \(T(\cdot)\) is \(P\)-almost surely Lebesgue square integrable and the \(n\)th order spectral moments of \(T\) exist and are finite. If moreover (4.33) holds for every \(0 \leq l_1 \leq \cdots \leq l_n\), then one has that, for every \(g \in SO(3), \) relation (4.32) holds, and \(T\) is \(n\)-weakly isotropic.

**Proof.** For every \(x_1, \ldots, x_n \in S^2\), write \(\text{Cum} \{T(x_1), \ldots, T(x_n)\}\) the joint cumulant of the random variables \(T(x_1), \ldots, T(x_n)\). By using isotropy, one has that, for every \(g \in SO(3), \)

\[
\text{Cum} \{T(x_1), \ldots, T(x_n)\} = \text{Cum} \{T(gx_1), \ldots, T(gx_n)\}. \tag{4.35}
\]

Hence, by using the well-known multilinearity properties of cumulants, one deduces that (with obvious notation)

\[
\text{Cum} \{T(x_1), \ldots, T(x_n)\} = \sum_{l_1 m_1, \ldots, l_n m_n} \text{Cum} \{a_{l_1 m_1}, \ldots, a_{l_n m_n}\} Y_{l_1, m_1}(x_1) \cdots Y_{l_n, m_n}(x_n)
\]

\[
= \sum_{l_1 m_1, \ldots, l_n m_n} \text{Cum} \{a_{l_1 m_1}, \ldots, a_{l_n m_n}\} Y_{l_1, m_1}(gx_1) \cdots Y_{l_n, m_n}(gx_n), \tag{4.36}
\]

and relations (4.33)–(4.34) are deduced by rewriting (4.36) by means of the identity

\[
\{Y_{l_1, m_1}(gx_1), \ldots, Y_{l_n, m_n}(gx_n)\} \equiv \left\{ \sum_{m'} D_{m'm_1}^{l_1}(g) Y_{l_1 m'}(x_1), \ldots, \sum_{m'} D_{m'm_n}^{l_n}(g) Y_{l_n m'}(x_n) \right\}.
\]

The second part of the statement is proved by arguments analogous to the ones used in the proof of Proposition 5. \(\blacksquare\)

We now present an alternative (and more involved) characterization of the cumulant polyspectra associated with an isotropic field. Given \(n \geq 2\) and a partition \(\pi = \{b_1, \ldots, b_k\} \in \Pi(n)\), we build a permutation \(v^\pi = (v^\pi(1), \ldots, v^\pi(n)) \in S_n\) as follows: (i) write the partition

\[
\pi = \left\{ b_1, \ldots, b_k \right\} = \left\{ \left( i_1^1, \ldots, i_{|b_1|}^1 \right), \ldots, \left( i_1^k, \ldots, i_{|b_k|}^k \right) \right\} \tag{4.37}
\]

(where \(|b_j| \geq 1\) stands for the size of \(b_j\)) by means of the convention outlined in Section 2 (that is, order the blocks and the elements within each block according to the lexicographic order); (ii) define \(v^\pi = S_n\) by simply removing the brackets in (4.37), that is, set

\[
v^\pi = (v^\pi(1), \ldots, v^\pi(n)) = \left( i_1^1, \ldots, i_{|b_1|}, i_1^2, \ldots, i_{|b_2|}, \ldots, i_1^k, \ldots, i_{|b_k|} \right).
\]
For instance, if a partition $\pi$ of $\{1, \ldots, 6\}$ is composed of the blocks $\{1,3\}, \{6,4\}$ and $\{2,5\}$, one first writes $\pi$ in the form $\pi = \{1,3\}, \{2,5\}, \{4,6\}$, and then defines $v^{\pi} = (v^{\pi}(1), \ldots, v^{\pi}(6)) = (1,3,2,5,4,6)$. Given $n \geq 2$, $0 \leq l_1 \leq \cdots \leq l_n$, and $\pi \in \Pi(n)$, we define the matrix

$$\Delta_{l_1, \ldots, l_n}^{\pi} \triangleq \int_{SO(3)} \{D^{l_1,\pi(1)}(g) \otimes D^{l_2,\pi(2)}(g) \otimes \cdots \otimes D^{l_n,\pi(n)}(g)\} \, dg,$$

obtained from the matrix $\Delta_{l_1, \ldots, l_n}$ in (4.28), by permuting the indexes $l_i$ according to $v^{\pi}$. Plainly, if $v^{\pi}$ is equal to the identity permutation, then $\Delta_{l_1, \ldots, l_n}^{\pi} = \Delta_{l_1, \ldots, l_n}$. We also set, for every fixed $g \in SO(3)$,

$$\Delta_{l_1, \ldots, l_n}^{\pi} (g) \triangleq D^{l_1,\pi(1)}(g) \otimes D^{l_2,\pi(2)}(g) \otimes \cdots \otimes D^{l_n,\pi(n)}(g).$$

**Proposition 7** Let $T$ be a strongly isotropic field with finite moments of order $n \geq 2$. For $0 \leq l_1, l_2, \ldots, l_n$, define $S_{l_1, \ldots, l_n}^{\pi}$ according to (2.4). Then, for every $0 \leq l_1, l_2, \ldots, l_n$, and every $g \in SO(3)$

$$S_{l_1, \ldots, l_n}^{\pi} = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! \Delta_{l_1, \ldots, l_n}^{\pi} E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}],$$

(4.39)

and assume that $T(x)$ is a (not necessarily isotropic) random field on the sphere $s.t. \sup_{x} (E |T(x)|^n) < \infty$. Then, the $n$th order spectral moments and cumulants of $T$ exist and are finite. If moreover (4.40) holds for every $0 \leq l_1, l_2, \ldots, l_n$ and every $g \in SO(3)$, then one has that $T$ is $n$-weakly isotropic.

**Proof.** Fix $\pi = \{b_1, \ldots, b_k\} \in \Pi(n)$. By strong isotropy and Lemma [1], one has that, for a fixed $g^* \in SO(3)$, the quantity

$$E [\otimes_{i \in b_i} D^{l_i}(g^*) a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} D^{l_k}(g^*) a_{l_k}]$$

(4.40)

does not depend on $g^*$, so that

$$E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}]$$

$$= \Delta_{l_1, \ldots, l_n}^{\pi} (g^*) E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}]$$

(4.41)

$$= \int_{SO(3)} \Delta_{l_1, \ldots, l_n}^{\pi} (g) E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}] \, dg$$

$$= \Delta_{l_1, \ldots, l_n}^{\pi} E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}].$$

To prove the second part of the statement, suppose that $T(x)$ verifies $\sup_{x} (E |T(x)|^n) < \infty$, and that its associated harmonic coefficients verify (4.41). Then, for every fixed rotation $g^* \in SO(3)$,

$$\sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! E [\otimes_{i \in b_i} D^{l_i}(g^*) a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} D^{l_k}(g^*) a_{l_k}]$$

(4.42)

$$= \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! \times$$

$$\times [D^{l_1,\pi(1)}(g^*) \otimes \cdots \otimes D^{l_n,\pi(n)}(g^*)] E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}]$$

$$= \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! \times \Delta_{l_1, \ldots, l_n}^{\pi} (g^*) E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}]$$

$$= \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! E [\otimes_{i \in b_i} a_{l_i}] \otimes \cdots \otimes E [\otimes_{i \in b_k} a_{l_k}].$$
By the definition of cumulants, this last equality gives that
\[
E \left[ D^1(g^*)a_{l_1} \otimes \cdots \otimes D^{l_n}(g^*)a_{l_n} \right] = E \left[ a_{l_1} \otimes \cdots \otimes a_{l_n} \right].
\]
Since \(g^*\) is arbitrary, the \(n\)-weak isotropy follows from (2.2). \(\blacksquare\)

**Remark.** By combining (4.33) and (4.39) we obtain for instance that the \(n\)th cumulant polyspectrum of an isotropic field verifies the identity
\[
S^c_{l_1 \ldots l_n} = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! \Delta^\pi_{l_1 \ldots l_n} E \left[ \otimes_{i \in b_1} a_{l_i} \right] \otimes \cdots \otimes E \left[ \otimes_{i \in b_k} a_{l_i} \right].
\]

5 Angular polyspectra and the structure of \(\Delta_{l_1 \ldots l_n}\)

5.1 Spectra of strongly isotropic fields

Our aim in this section is to investigate more deeply the structure of the matrix \(\Delta_{l_1 \ldots l_n}\) appearing in (4.28), in order to derive an explicit characterization for the angular polyspectra. As a preliminary example, we deal with the case \(n = 2\).

**Proposition 8** For integers \(l_1, l_2 \geq 0\), one has that
\[
\Delta_{l_1 l_2} = \int_{SO(3)} \left\{ D^{l_1}(g) \otimes D^{l_2}(g) \right\} dg = \delta_{l_1 l_2} C^{00}_{l_1 l_2} \left( C^{00}_{l_1 l_2} \right)^*, \tag{5.41}
\]
that is: if \(l_1 \neq l_2\), then \(\Delta_{l_1 l_2}\) is a \((2l_1 + 1)(2l_2 + 1) \times (2l_1 + 1)(2l_2 + 1)\) zero matrix; if \(l_1 = l_2\), then \(\Delta_{l_1 l_2} = \Delta_{l_1 l_1}\) is given by \(C^{00}_{l_1 l_1} \left( C^{00}_{l_1 l_1} \right)^*\).

**Proof.** Using the equivalence of the two representations \(D^{l_1}(g) \otimes D^{l_2}(g)\) and \(\oplus_{\lambda=|l_2-l_1|}^{l_2+l_1} D^{l_1}(g)\), as well as the definition of the Clebsch-Gordan matrices, we obtain that
\[
\int_{SO(3)} \left\{ D^{l_1}(g) \otimes D^{l_2}(g) \right\} dg = C_{l_1 l_2} \left[ \int_{SO(3)} \left\{ \oplus_{\lambda=|l_2-l_1|}^{l_2+l_1} D^{l_1}(g) \right\} dg \right] C_{l_1 l_2}^* \tag{5.42}
\]
Now, if \(l_1 \neq l_2\), then the RHS of (5.42) is equal to the zero matrix since, as a consequence of the Peter-Weyl theorem and for \(\lambda \neq 0\), the entries of \(D^{l_1}(\cdot)\) are orthogonal to the constants. If \(l_1 = l_2\), then the integrated matrix on the RHS of (5.42) becomes \(\int_{SO(3)} \left\{ \oplus_{\lambda=0}^{2l_1} D^{l_1}(g) \right\} dg\), that is, a \((2l_1 + 1)^2 \times (2l_1 + 1)^2\) matrix which is zero everywhere, except for the entry in the top-left corner, which is equal to one (since \(\int_{SO(3)} dg = 1\)). The proof is concluded by checking that
\[
C_{l_1 l_1} \left[ \int_{SO(3)} \left\{ \oplus_{\lambda=0}^{2l_1} D^{l_1}(g) \right\} dg \right] C_{l_1 l_1}^* = C^{00}_{l_1 l_1} \left( C^{00}_{l_1 l_1} \right)^*. \]
\(\blacksquare\)

**Remark.** Recall that \(C^{00}_{l_1 l_2}\) is a *column* vector of dimension \((2l_1 + 1)(2l_2 + 1)\), corresponding to the first column of the matrix \(C_{l_1 l_2}\). Also, according e.g. to [31, formula 8.5.1.1], one has that
\[
C^{00}_{l_1 l_2} = \left\{ \begin{array}{c} (-1)^{m_2} \\ 2l_1 + 1 \end{array} \right\}_{l_1 \otimes \cdots \otimes l_2}^{m_1 = -l_1, \ldots, l_1 = -l_2, \ldots, l_2}.
\]

Proposition 8 provides a characterization of the spectrum of a strongly isotropic field.
Corollary 9 Let $T$ be a strongly isotropic field with second moments, and let the vectors of the harmonic coefficients $\{a_l\}$ be defined according to (2.23). Then, for any integers $l_1, l_2 \geq 0$, one has that

$$E \{a_{l_1} \otimes a_{l_2}\} = \left\{ \frac{(-1)^{m_1}}{2l_1 + 1} \delta_{l_1}^{l_2} \delta_{m_1}^{m_2} C_{l_1} \right\}$$

for some $C_{l_1} \geq 0$ depending uniquely on $l_1$.

Proof. According to (5.30), one has that

$$E \{a_{l_1} \otimes a_{l_2}\} = \delta_{l_1}^{l_2} C_{l_1} (C_{l_2})' E \{a_{l_1} \otimes a_{l_2}\},$$

implying that $E \{a_{l_1} \otimes a_{l_2}\}$ is (a) equal to the zero vector for $l_1 \neq l_2$, and (b) of the form $C_{l_1} \times C_{l_2}$, for some constant $C_{l_1}$, when $l_1 = l_2$. To see that $C_{l_1}$ cannot be negative, just observe that $a_{l_1,0}$ is real-valued for every $l_1 \geq 0$, so that (5.43) yields that

$$C_{l_1} = (2l_1 + 1) \times E \{a_{l_1,0}^2\}.$$  

In the subsequent two subsections, we shall obtain, for every $n \geq 3$, a characterization of $\Delta_{l_1,\ldots,l_n}$ and $E\{a_{l_1} \otimes \cdots \otimes a_{l_n}\}$, respectively analogous to (5.41) and (5.43).

5.2 The structure of $\Delta_{l_1,\ldots,l_n}$

We first need to establish some further notation.

Definition B. Fix $n \geq 3$. For integers $l_1,\ldots,l_n \geq 0$, we define $C_{l_1,\ldots,l_n}$ to be the unitary matrix, of dimension

$$\prod_{j=1}^{n} (2l_j + 1) \times \prod_{j=1}^{n} (2l_j + 1),$$

connecting the following two equivalent representations of $SO(3)$

$$D^{l_1}(\cdot) \otimes D^{l_2}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot)$$

and

$$\bigoplus_{l_1=0}^{l_2+l_1} \bigoplus_{l_2=0}^{l_1} \bigoplus_{l_3=0}^{l_2} \cdots \bigoplus_{l_n=0}^{l_{n-1}} \Delta_{l_{n-1}} (\cdot) \bigoplus_{l_{n-1}=0}^{l_n} \delta^{l_{n-1}}(\cdot).$$

(5.45)

Remarks. (1) Fix $l_1,\ldots,l_n \geq 0$, as well as $g \in SO(3)$. Then, the matrix

$$\bigoplus_{l_1=0}^{l_2+l_1} \bigoplus_{l_2=0}^{l_1} \bigoplus_{l_3=0}^{l_2} \cdots \bigoplus_{l_n=0}^{l_{n-1}} \Delta_{l_{n-1}} (g)$$

is a block-diagonal matrix, obtained as follows. (a) Consider vectors of integers $(\lambda_1,\ldots,\lambda_{n-1})$ satisfying the relations $|l_2 - l_1| \leq \lambda_1 \leq l_1 + l_2$, and $|l_{k+1} - l_{k-1}| \leq \lambda_k \leq l_{k+1} + l_{k-1}$, for $k = 2,\ldots,n - 1$. (b) Introduce a (total) order $\prec_0$ on the collection of these vectors by saying that

$$(\lambda_1,\ldots,\lambda_{n-1}) \prec_0 (\lambda_1',\ldots,\lambda_{n-1}'),$$

whenever either $\lambda_1 < \lambda_1'$, or there exists $k = 2,\ldots,n - 2$ such that $\lambda_j = \lambda_j'$ for every $j = 1,\ldots,k$, and $\lambda_{k+1} < \lambda_{k+1}'$. (c) Associate to each vector $(\lambda_1,\ldots,\lambda_{n-1})$ the matrix $D^{\lambda_{n-1}}(g)$. (d) Construct a block-diagonal matrix by disposing the matrices $D^{\lambda_{n-1}}(g)$ from the top-left corner to the bottom-right corner, in increasing order with respect to $\prec_0$. As an example, consider the case where $n = 3$ and $l_1 = l_2 = l_3 = 1$. Here, the vectors $(\lambda_1,\lambda_2)$ involved in the direct sum (5.43) are (in increasing order with respect to $\prec_0$)

$$(0,1), (1,0), (1,1), (1,2), (2,1), (2,2) \text{ and } (2,3),$$

and
and the matrix (5.46) is therefore given by

$$
\begin{pmatrix}
D^1(g) & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & D^1(g) & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & D^2(g) & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & D^1(g) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & D^3(g)
\end{pmatrix}
$$

(5.48)

where the dots indicate zero entries, and we have used the fact that $D^0(g) \equiv 1$.

(2) The fact that the representation (5.45) has dimension $\prod_{j=1}^{n} (2l_j + 1)$ is a direct consequence of formula (3.14).

(3) The fact that the two representations (5.44) and (5.45) are equivalent can be proved by iteration. Indeed, by standard representation theory, on has that (5.44) is equivalent to

$$
\oplus_{\lambda_1=|l_1-l_1|}^{l_2+l_1} \oplus_{\lambda_2=|l_2-l_1|}^{l_3+l_2} \cdots \oplus_{\lambda_n=|l_n-l_1|}^{l_n+l_{n-1}} D^{\lambda_1}(\cdot) \otimes D^{\lambda_2}(\cdot) \otimes \cdots \otimes D^{\lambda_n}(\cdot),
$$

which is in turn equivalent to

$$
\oplus_{\lambda_1=|l_1-l_1|}^{l_2+l_1} \oplus_{\lambda_2=|l_2-l_1|}^{l_3+l_2} \cdots \oplus_{\lambda_n=|l_n-l_1|}^{l_n+l_{n-1}} D^{\lambda_1}(\cdot) \otimes D^{\lambda_2}(\cdot) \otimes \cdots \otimes D^{\lambda_n}(\cdot).
$$

By iterating the same procedure until all tensor products have disappeared (that is, by successively replacing the tensor product $D^{\lambda_1}(\cdot) \otimes D^{\lambda_2}(\cdot)$ with $\oplus_{\lambda_1=|l_1-l_1|}^{l_2+l_1} D^{\lambda_1}(\cdot)$ for $k = 2, \ldots, n-1$), one obtains the desired conclusion.

For every $n \geq 3$ and every $l_1, \ldots, l_n \geq 0$, the elements of the matrix $C_{l_1, \ldots, l_n}$, introduced in Definition B, can be written in the form $C_{l_1, \ldots, l_n}^{\lambda_1, \lambda_n-1, \mu_{n-1}}$. The indices $(m_1, \ldots, m_n)$ are such that $-l_i \leq m_i \leq l_i$ ($i = 1, \ldots, n$) and label rows; on the other hand, the indices $(\lambda_1, \ldots, \lambda_{n-1}, \mu_{n-1})$ label columns, and verify the relations $|l_2-l_1| \leq \lambda_1 \leq l_1 + l_2$, $|l_{k+1} - l_k| \leq \lambda_k \leq l_{k+1} + l_k - 1$ ($k = 2, \ldots, n-1$) and $-\lambda_{n-1} \leq \mu_{n-1} \leq \lambda_{n-1}$. It is well known (see e.g. [31]) that the quantity $C_{l_1, \ldots, l_n}^{\lambda_1, \lambda_n-1, \mu_{n-1}}$ can be represented as a convolution of the Clebsch-Gordan coefficients introduced in Section 3.2, namely:

$$
C_{l_1, m_1, \ldots, l_n, m_n}^{\lambda_1, \lambda_{n-1}, \mu_{n-1}} = 
\begin{cases}
C_{l_1, m_1, \ldots, l_n, m_n}^{\lambda_1, \lambda_{n-1}, \mu_{n-1}} \\
\sum_{\mu_{n-2}} \left\{ \cdots \sum_{\mu_3} \left\{ \sum_{\mu_2} \left\{ \sum_{\mu_1} C_{l_1, m_1, l_2, m_2}^{\lambda_1, \mu_1} C_{l_2, m_2, l_3, m_3}^{\lambda_2, \mu_2} \cdots C_{l_{n-2}, m_{n-2}, l_{n-1}, m_{n-1}}^{\lambda_{n-2}, \mu_{n-2}} \right\} \right\} \right\}
\end{cases}
$$

Remark. Given an enumeration of the coefficients $C_{l_1, m_1, \ldots, l_n, m_n}^{\lambda_1, \lambda_{n-1}, \mu_{n-1}}$, the matrix $C_{l_1, \ldots, l_n}$ can be built (analogously to the case of the Clebsch-Gordan matrices of Section 3.2) by disposing rows (from top to bottom) and columns (from left to right) increasingly according to two separate total orders. The order $\prec_r$ on the symbols $(m_1, \ldots, m_n)$ is obtained by setting that $(m_1, \ldots, m_n) \prec_r (m'_1, \ldots, m'_n)$ whenever either $m_1 < m'_1$, or there exists $k = 2, \ldots, n-1$ such that $m_j = m'_j$ for every $j = 1, \ldots, k$, and $m_{k+1} < m'_{k+1}$. The order $\prec_c$ on the symbols $(\lambda_1, \ldots, \lambda_{n-1}, \mu_{n-1})$ is obtained by setting that $(\lambda_1, \ldots, \lambda_{n-1}, \mu_{n-1}) \prec_c (\lambda'_1, \ldots, \lambda'_{n-1}, \mu'_{n-1})$ whenever either $(\lambda_1, \ldots, \lambda_{n-1}) \prec_0 (\lambda'_1, \ldots, \lambda'_{n-1})$, as defined in (5.47), or $\lambda_i = \lambda'_i$ for every $i = 1, \ldots, n-1$ and $\mu_{n-1} < \mu'_{n-1}$.

One has also the following (useful) alternative representation of generalized Clebsch-Gordan matrices.
Proposition 10 For every \( n \geq 3 \) and every \( l_1, \ldots, l_n \geq 0 \), one can represent the matrix \( C_{l_1 \ldots l_n} \), as follows

\[
C_{l_1 \ldots l_n} = \left\{ C_{l_1l_2l_3 \ldots l_{n-1}} \otimes I_{2l_n+1} \right\} \left\{ \left( \oplus_{l_1=\lceil l_2-l_1 \rceil}^{l_1+\lambda_{n-3}} \oplus_{\lambda_{n-2}=\lceil n-l_2+\lambda_{n-3} \rceil}^{l_1+\lambda_{n-2}} C_{\lambda_{n-2}l_n} \right) \right\},
\]

where \( I_m \) indicates a \( m \times m \) identity matrix. Also, one has that

\[
C_{l_1 \ldots l_n} = \left( C_{l_1l_2} \otimes I_{2l_3+1} \otimes \cdots \otimes I_{2l_n+1} \right) \times \left( \left( \oplus_{l_1=\lceil l_2-l_1 \rceil}^{l_1+\lambda_{n-3}} \oplus_{\lambda_{n-2}=\lceil n-l_2+\lambda_{n-3} \rceil}^{l_1+\lambda_{n-2}} C_{\lambda_{n-2}l_n} \right) \right),
\]

where \( \times \) stands for the usual product between matrices.

Definition C. For every \( n \geq 3 \) and every \( l_1, \ldots, l_n \geq 0 \), we define \( E_{l_1 \ldots l_n} \) to be the \( \Pi^m \times \Pi^m \) square matrix

\[
E_{l_1 \ldots l_n} := \left( \oplus_{l_1=\lceil l_2-l_1 \rceil}^{l_1+\lambda_{n-3}} \oplus_{\lambda_{n-2}=\lceil n-l_2+\lambda_{n-3} \rceil}^{l_1+\lambda_{n-2}} \delta_{\lambda_{n-1}}^0 \right) I_{2l_n+1}.
\]

In other words, \( E_{l_1 \ldots l_n} \) is the diagonal matrix built from the matrix \((5.46)\), by replacing every block of the type \( D^{\lambda_{n-1}} (g) \), with \( \lambda_{n-1} > 0 \), with \( (2\lambda_{n-1} + 1) \times (2\lambda_{n-1} + 1) \) zero matrix, and by letting the \( 1 \times 1 \) blocks \( D^0 (g) = 1 \) unchanged. For instance, by setting \( n = 3 \) and \( l_1 = l_2 = l_3 = 1 \) (and by using \((5.48)\)) one obtains a \( 27 \times 27 \) matrix \( E_{111} \) whose entries are all zero, except for the fourth element (starting from the top-left corner) of the main diagonal.

The following result states that the matrix \( \Delta_{l_1 \ldots l_n} \) can be diagonalized in terms of \( C_{l_1 \ldots l_n} \) and \( E_{l_1 \ldots l_n} \).

Proposition 11 The matrix \( \Delta_{l_1 \ldots l_n} \) can be diagonalized as

\[
\Delta_{l_1 \ldots l_n} = C_{l_1 \ldots l_n} E_{l_1 \ldots l_n} C^{\star}_{l_1 \ldots l_n},
\]

where \( E_{l_1 \ldots l_n} \) is the matrix introduced in Definition C.

Proof. One has that

\[
\Delta_{l_1 \ldots l_n} = \int_{SO(3)} D^{l_1} (g) \otimes D^{l_2} (g) \otimes \cdots \otimes D^{l_n} (g) \, dg
\]

\[
= \int_{SO(3)} \left[ C_{l_1 \ldots l_n} \left( \oplus_{l_1=\lceil l_2-l_1 \rceil}^{l_1+\lambda_{n-3}} \oplus_{\lambda_{n-2}=\lceil n-l_2+\lambda_{n-3} \rceil}^{l_1+\lambda_{n-2}} D^{\lambda_{n-1}} (g)C^{\star}_{l_1 \ldots l_n} \right) \right] \, dg.
\]

By linearity and by the definition of the integral of a matrix-valued function, one has that the last line of \((5.51)\) equals

\[
C_{l_1 \ldots l_n} \left[ \ominus_{l_1=\lceil l_2-l_1 \rceil}^{l_1+\lambda_{n-3}} \ominus_{\lambda_{n-2}=\lceil n-l_2+\lambda_{n-3} \rceil}^{l_1+\lambda_{n-2}} \int_{SO(3)} D^{\lambda_{n-1}} (g) \, dg \right] \left[ C^{\star}_{l_1 \ldots l_n} \right].
\]

Now observe that, if \( \lambda_{n-1} > 0 \), then \( \int_{SO(3)} D^{\lambda_{n-1}} (g) \, dg \) equals a \( (2\lambda_{n-1} + 1) \times (2\lambda_{n-1} + 1) \) zero matrix, whereas \( \int_{SO(3)} D^0 (g) \, dg = \int_{SO(3)} 1 \, dg = 1 \). The conclusion is obtained by resorting to the definition of \( E_{l_1 \ldots l_n} \) given in \((5.48)\). \( \blacksquare \)
5.3 Existence and characterization of reduced polyspectra of arbitrary orders

Combining the previous Proposition with (5.3), we obtain the main result of this paper.

**Theorem 12** If a random field is strongly isotropic with finite moments of order \( n \geq 3 \), then for every \( l_1, \ldots, l_n \) there exists two arrays \( P_{1 \ldots l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}) \) and \( P^C_{1 \ldots l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}) \), with \( |l_2 - l_1| \leq \lambda_1 \leq l_2 + l_1 \), \( |l_3 - \lambda_1| \leq \lambda_2 \leq l_3 + \lambda_1 \), \( |l_{n-2} - \lambda_{n-4}| \leq \lambda_{n-3} \leq l_{n-2} + \lambda_{n-4} \), such that

\[
E_{a_1 m_1 \ldots a_n m_n} = (-1)^m \sum_{\lambda_1 = l_2 - l_1}^{l_2 + l_1} \ldots \sum_{\lambda_{n-3} = l_{n-2} - \lambda_{n-4}}^{l_{n-2} + \lambda_{n-4}} C^{\lambda_1 \ldots \lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} P_{l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}) \tag{5.52}
\]

\[
\text{Cum} \{a_1 m_1, \ldots, a_n m_n\} = (-1)^m \sum_{\lambda_1 = l_2 - l_1}^{l_2 + l_1} \ldots \sum_{\lambda_{n-3} = l_{n-2} - \lambda_{n-4}}^{l_{n-2} + \lambda_{n-4}} C^{\lambda_1 \ldots \lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} P^C_{l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}) \tag{5.53}
\]

\[
C^{\lambda_1 \ldots \lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} = \sum_{\mu_1} \ldots \sum_{\mu_{n-3}} C^{\lambda_1 \mu_1}_{l_1 m_1} C^{\lambda_2 \mu_2}_{l_1 m_1} \ldots C^{\lambda_{n-3} \mu_{n-3}}_{l_1 m_1} C^{\lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} \tag{5.54}
\]

**Remark.** For a fixed \( n \geq 2 \), the real-valued arrays \( \{P_{1 \ldots l_1 \ldots l_n}(\cdot) : l_1, \ldots, l_n \geq 0\} \) and \( \{P^C_{1 \ldots l_1 \ldots l_n}(\cdot) : l_1, \ldots, l_n \geq 0\} \) are, respectively, the reduced polyspectrum of order \( n - 1 \) and the reduced cumulant polyspectrum of order \( n - 1 \) associated with the underlying strongly isotropic random field.

**Proof of Theorem 12.** We shall prove only (5.52), since the proof of (5.53) is entirely analogous. By Proposition 3 and Proposition 1, if the random field is isotropic, then

\[
S_{l_1 \ldots l_n} = C_{l_1 \ldots l_n} E_{l_1 \ldots l_n} C^*_{l_1 \ldots l_n} S_{l_1 \ldots l_n} ;
\]

that is, because \( C_{l_1 \ldots l_n} \) is unitary

\[
C^*_{l_1 \ldots l_n} S_{l_1 \ldots l_n} = E_{l_1 \ldots l_n} C^*_{l_1 \ldots l_n} S_{l_1 \ldots l_n} .
\]

It follows that \( S_{l_1 \ldots l_n} \) is a solution if and only if the column vector \( C^*_{l_1 \ldots l_n} S_{l_1 \ldots l_n} \) has zeroes corresponding to the zeroes of \( E_{l_1 \ldots l_n} \), whereas the elements corresponding to unity can be arbitrary. In view of the orthonormality properties of \( C^*_{l_1 \ldots l_n} \), this condition is met if, and only if, \( S_{l_1 \ldots l_n} \) is a linear combination of the columns in the matrix \( C^*_{l_1 \ldots l_n} \) corresponding to non-zero elements of the diagonal \( E_{l_1 \ldots l_n} \). These linear combinations can be written explicitly as

\[
\sum_{l_2 - l_1}^{l_2 + l_1} \sum_{l_3 - l_1}^{l_3 + l_1} \ldots \sum_{l_{n-2} - l_1}^{l_{n-2} + l_1} C^{\lambda_1 \ldots \lambda_{n-2} m_1 \ldots m_n}_{l_1 m_1 \ldots l_n m_n} \tilde{P}_{l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}, \lambda_{n-2}) \delta_0^0 = \sum_{l_2 - l_1}^{l_2 + l_1} \sum_{l_3 - l_1}^{l_3 + l_1} \ldots \sum_{l_{n-2} - l_1}^{l_{n-2} + l_1} C^{\lambda_1 \mu_1}_{l_1 m_1} C^{\lambda_2 \mu_2}_{l_1 m_1} \ldots C^{\lambda_{n-3} \mu_{n-3}}_{l_1 m_1} C^{\lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} \delta_0^0 \times \tilde{P}_{l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}, \lambda_{n-2})
\]

\[
= \sum_{l_2 - l_1}^{l_2 + l_1} \sum_{l_3 - l_1}^{l_3 + l_1} \ldots \sum_{l_{n-2} - l_1}^{l_{n-2} + l_1} \sum_{l_{n-1} - l_1}^{l_{n-1} + l_1} C^{\lambda_1 \mu_1}_{l_1 m_1} C^{\lambda_2 \mu_2}_{l_1 m_1} \ldots C^{\lambda_{n-3} \mu_{n-3}}_{l_1 m_1} C^{\lambda_{n-3} m_1 \ldots m_{n-1}}_{l_1 m_1 \ldots l_n m_n} \delta_0^0 \times \tilde{P}_{l_1 \ldots l_n}(\lambda_1, \ldots, \lambda_{n-3}, \lambda_{n-2})
\]

Recalling again that

\[
C^{0m}_{l_1 m_1 l_2 m_2} = \frac{(-1)^{m_1}}{2l_1 + 1} \delta_{l_1}^{l_1} \delta_{m_1}^{m_1} \delta_{m_2}^{m_2} ,
\]

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where we have set $\text{Fix } C$ (see [31], 8.5.1.1), we obtain that

$$
= \sum_{\lambda_1=\ell_1-1}^{\ell_2-1} \sum_{\lambda_2=\ell_2-1}^{\ell_3-1} \cdots \sum_{\lambda_{n-1}=\ell_{n-1}-1}^{\ell_n-1} \left\{ \sum_{\mu_1=\mu_2=\cdots=\mu_{n-2}} C_{t_1, \mu_1} \mu_2 \cdots \frac{(-1)^{\mu_{n-2}}}{2^{\ell_{n-2}}+1} \delta_{\lambda_{n-2}} \mu_{n-2} \right\} \times \tilde{P}_{t_1, \ldots, t_n}(\lambda_1, \ldots, \lambda_{n-3}, \lambda_{n-2})
$$

$$
= \sum_{\lambda_1=\ell_1-1}^{\ell_2-1} \sum_{\lambda_2=\ell_2-1}^{\ell_3-1} \cdots \sum_{\lambda_{n-1}=\ell_{n-1}-1}^{\ell_n-1} \left\{ \sum_{\mu_1=\mu_2=\cdots=\mu_{n-2}} C_{t_1, \mu_1} \mu_2 \cdots C_{t_{n-3}, \mu_{n-3}} \mu_{n-3} \cdots P_{t_1, \ldots, t_n}(\lambda_1, \ldots, \lambda_{n-3}), \right\}
$$

where we have set

$$
P_{t_1, \ldots, t_n}(\lambda_1, \ldots, \lambda_{n-3}) := \frac{1}{2^{\ell_n}+1} \tilde{P}_{t_1, \ldots, t_n}(\lambda_1, \ldots, \lambda_{n-3}, t_n)
$$

All there is left to show is that the coefficients of this linear combination are necessarily real. To see this, it is sufficient to specialize the previous discussion to the case where $m_1 = m_2 = \ldots = m_n = 0$, and to observe that, in this case

$$
Ea_{t_1, \ldots, t_n} = \sum_{\lambda_1} \cdots \sum_{\lambda_{n-3}} C_{t_1, \mu_1, \ldots, \mu_{n-3}}^0 P_{t_1, \ldots, t_n}(\lambda_1, \ldots, \lambda_{n-3})
$$

is real by definition (note indeed that the columns of $C_{t_1, \ldots, t_n}$ are linearly independent).

Let us illustrate the previous results by some more examples.

**Examples.** For $n = 3$, Theorem 12 implies that, under isotropy

$$
Ea_{t_1, t_2, t_3} = (-1)^{m_3} C_{t_1, t_2, t_3} \tilde{P}_{t_1, t_2, t_3}.
$$

From this last relation, we can recover the so-called reduced bispectrum, noted $b_{t_1, t_2, t_3}$, defined for instance in [13], [20] and [21], which satisfies indeed the relationship

$$
P_{t_1, t_2, t_3} = b_{t_1, t_2, t_3} C_{t_1, t_2, t_3}^0 \frac{(2t_1+1)(2t_2+1)}{(2t_3+1)4\pi}.
$$

For $n = 4$ (i.e. the trispectrum, [16]) we obtain the expression

$$
Ea_{t_1, t_2, t_3, t_4} = (-1)^{m_4} \sum_{\lambda=|l_3-l_1|}^{l_2+l_4} C_{t_1, t_2, t_3, t_4}^{l_4-m_4} P_{t_1, t_2, t_3, t_4}(\lambda)
$$

$$
= \sum_{\lambda=|l_3-l_1|}^{l_2+l_4} \sum_{\mu=\lambda}^{l_1+l_2} \sum_{\mu=\lambda}^{l_1+l_2} C_{t_1, t_3, t_4}^{l_1-l_1} C_{t_3, t_4}^{l_1-l_1} P_{t_1, t_2, t_3, t_4}(\lambda).
$$

The next result gives a further probabilistic characterization of the reduced bispectrum.

**Proposition 13** Fix $n \geq 2$. A real-valued array $\{A_{t_1, \ldots, t_n}(\cdot) : l_1, \ldots, l_n \geq 0\}$ is the reduced polyspectrum of order $n-1$ (resp. the reduced cumulant polyspectrum of order $n-1$) of some strongly isotropic random
field if, and only if, there exists a sequence \( \{ X_l : l \geq 0 \} \) of zero-mean real-valued random variables such that
\[
\sum_{l \geq 0} (2l + 1) E [X_l^2] < +\infty
\]
and, for every \( l_1, \ldots, l_n \geq 0 \)
\[
E (X_{l_1} \cdots X_{l_n}) = \sum_{\lambda_1 + \cdots + \lambda_n = l_1 + \cdots + l_n} \cdots \cdots \sum_{\lambda_{n-3} + \cdots + \lambda_n = l_{n-3} + \cdots + l_n} C_{l_1,0,\cdots,0}^{\lambda_1,\cdots,\lambda_{n-3},0} A_{l_1,\cdots,l_n} (\lambda_1,\cdots,\lambda_{n-3}) \quad (5.55)
\]
(resp.
\[
\text{Cum} \{ X_{l_1}, \cdots, X_{l_n} \} = \sum_{\lambda_1 \cdots \lambda_{n-3} + \cdots + \lambda_n = l_{n-3} + \cdots + l_n} \cdots \cdots \sum_{\lambda_{n-3} \cdots \lambda_1 + \cdots + \lambda_n = l_{n-3} + \cdots + l_n} C_{l_1,0,\cdots,0}^{\lambda_1,\cdots,\lambda_{n-3},0} A_{l_1,\cdots,l_n} (\lambda_1,\cdots,\lambda_{n-3}). \quad (5.56)
\]

**Proof.** We shall only prove (5.55). For the necessity it is enough to take \( X_l = a_{l0} \), where \( a_{l0} \) is the harmonic coefficient of index \((l,0)\) associated with a strongly isotropic field with moments of all orders. For the sufficiency, we consider first the (anisotropic) random field \( Z(x) = \sum_{l \geq 0} X_l Y_{l0}(x) \).

Then, by taking \( T(x) = Z(gx) \), where \( g \) is sampled randomly with the uniform Haar measure on \( SO(3) \), one obtains a random field with the desired characteristics.

There are two very important issues that are left open by Theorem 12. As a first issue, it seems natural to look for characterizations of the reduced polyspectra \( P_{l_1,\ldots,l_n} \), at least under natural models of physical interest. As a second point, we note that the explicit expressions provided in Theorem 12 depend on the ordering \( l_1,\ldots,l_n \) we chose for the decomposition of \( \Delta_{l_1,\ldots,l_n} \). In the next two sections, we try to address these (and other) points.

### 6 Some Explicit Examples

In this section we provide explicit computations for the reduced polyspectra \( P_{l_1,\ldots,l_n} (n \geq 2) \), or \( P_{l_1,\ldots,l_n}^C \), for some models of physical interest. Of course, the Gaussian isotropic fields can be easily dealt with. Indeed, in this case one has that \( P_{l_1,\ldots,l_n}^C = 0 \) for all \( n \geq 3 \). In what follows, we shall therefore be concerned with polyspectra of Gaussian subordinated isotropic fields, that is, random fields that can be written as a deterministic and non-linear function of some collection of Gaussian isotropic fields. In general, this class of random fields allow for a clear-cut mathematical treatment, whilst covering a great array of empirically relevant circumstances.

#### 6.1 A simple physical model

The general Gaussian-subordinated model has the form
\[
T = \sum_{j=1}^q f_j H_j \left( T_G / \sqrt{E (T_G^2)} \right) = f_1 T_G + f_2 (T_G^2 / E (T_G^2) - 1) + \ldots, \quad (6.57)
\]
where \( f_j \) is a real constant, \( H_j(\cdot) \) denotes the \( j \)th Hermite polynomial (see e.g. [30]), and \( T_G \) is a Gaussian, zero-mean isotropic random field. Note that we have implicitly defined the sequence of Hermite polynomials in such a way that \( H_1(x) = x \), \( H_2(x) = x^2 - 1 \), \( H_3(x) = x^3 - 3x \), and so on. In this section,
when no further specification is needed, the spectral decomposition of the underlying Gaussian field $T_G$ is written

$$T_G = \sum_{lm} a_{lm} Y_{lm}.$$  

We shall sometimes use the following notation

$$T = \sum_{lm} \tilde{a}_{lm} Y_{lm} = \sum_{j=1}^q f_j a_{lm}(j) Y_{lm},$$  

$$a_{lm}(j) = \int_{S^2} H_j \left( T_G(x) / \sqrt{E(T_G^2)} \right) Y_{lm}(x) \, dx,$$  

$$\tilde{a}_{lm} = \sum_{j=1}^q a_{lm}(j).$$  

It follows that

$$T = T_G + f_{NL}(T_G^2 - ET_G^2),$$

where $f_{NL}$ is a nonlinearity parameter which depends upon physical constants in the associated “slow-roll” inflationary model (see e.g. [6] or [12]). The latter can be written down explicitly as

$$(f_{NL}^2 + f_{NL} E T_G^2) = \sum_{l=0}^{\infty} (2l+1) C_{nl}^0 C_{nl}^0,$$

where $f_{NL}$ is a nonlinearity parameter which depends upon physical constants in the associated “slow-roll” inflationary model (see e.g. [6] or [12]). Note that (6.61) has been written in the form (6.57), by setting $f_1 = 1$, $f_2 = f_{NL} \times E(T_G^2)$ and $f_j = 0$, for $j \geq 3$. The value of the constant $f_{NL} \times E(T_G^2)$ is expected to be very small, namely of the order $10^{-4}$ [5]. To simplify the discussion, we now assume that $E T_G^2 = 1$. In this case, by using (6.53)–(6.60), one has that

$$\tilde{a}_{lm} = a_{lm} + f_{NL} a_{Nl}(2),$$

$$a_{lm}(2) = \int_{S^2} T^2 Y_{lm} \, dx = \int_{S^2} \sum_{l_1 l_2 m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} Y_{l_1 m_1} Y_{l_2 m_2} T_{lm} \, dx$$

$$(l_1 l_2 m_1 m_2) = \sum_{l_1 l_2 m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} \sqrt{(2l_1 + 1)(2l_2 + 1)(2l + 1)} \, C_{l_1 m_1}^0 C_{l_2 m_2}^0 C_{l m}^0.$$

It follows that

$$\tilde{C}_l := E[\tilde{a}_{lm}]^2 = C_l + 2 f_{NL}^2 \sum_{l_1 l_2} C_{l_1} C_{l_2} \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \, (C_{l_1}^0) \, (C_{l_2}^0),$$

so that

$$Var(T) = \sum_l \frac{2l + 1}{4\pi} C_l = \sum_l \frac{2l + 1}{4\pi} C_l + 2 f_{NL}^2 \sum_{l_1 l_2} C_{l_1} C_{l_2} \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi} \, \sum_l (C_{l_1}^0) \, (C_{l_2}^0),$$

as expected, due to the orthogonality properties of Hermite polynomials. For the bispectrum, we obtain therefore

$$E\tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} = E \left\{ (a_{l_1 m_1} + f_2 a_{l_1 m_1}(2)) (a_{l_2 m_2} + f_2 a_{l_2 m_2}(2)) (a_{l_3 m_3} + f_2 a_{l_3 m_3}(2)) \right\}$$

$$= f_2 E a_{l_1 m_1}(2) a_{l_2 m_2} a_{l_3 m_3} + f_2 E a_{l_1 m_1} a_{l_2 m_2}(2) a_{l_3 m_3}$$

$$+ f_2 E a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}(2) + f_2^2 E a_{l_1 m_1}(2) a_{l_2 m_2}(2) a_{l_3 m_3}(2)$$

$$= (-1)^{l_1 + l_2 + l_3} C_{l_1 m_1 l_2 m_2}^0 C_{l_3 m_3}^0.$$
where
\[ P_{l_1 l_2 l_3} = 6f_2 \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{(2l_3 + 1)4\pi}} C_{l_1 l_2 l_3}^{000} \left\{ C_{l_1} C_{l_2} + C_{l_1} C_{l_3} + C_{l_2} C_{l_3} \right\} \] (6.62)

\[ + \frac{f_2^3}{2} \sum_{\ell_1, \ell_2, \ell_3} C_{\ell_1 \ell_2 \ell_3}^{000} C_{\ell_1 \ell_2 \ell_3}^{000} \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{(4\pi)^3} \times \] (6.63)
\[ \times \frac{8(-1)^{l_1}}{2^{l_3} + 1} \left\{ \ell_1 \ell_2 \ell_3 \right\} \left\{ C_{\ell_1} C_{\ell_2} C_{\ell_3} \right\} . \]

The lack of symmetry with respect to the \( l_3 \) term is only apparent and can be easily dispensed with by permuting the multipoles in \( C_{l_1 m_1 m_2} \) or using expression (5.18). Formula (6.62) is consistent with the cosmological literature, where (6.63) is considered a higher order term and hence neglected (see again (5.3)).

### 6.2 The Connection with Higher Order Moments
We now provide a simple result, connecting the reduced polyspectrum with the higher order moments of the associated spherical random field.

**Proposition 14** The following identity holds for every isotropic field with finite moments of order \( p \) and with a reduced polyspectrum \( \{ P_{l_1 \ldots l_p} (\cdot) : l_1, \ldots, l_p \geq 0 \} \): for every \( x \in S^2 \),

\[ ET(x)^p = \sum_{l_1 \ldots l_p} \sqrt{\frac{(2l_1 + 1) \cdots (2l_p + 1)}{(4\pi)^p}} \sum_{\lambda_1 \ldots \lambda_{p-3}} P_{l_1 \ldots l_p} (\lambda_1, \ldots, \lambda_{p-3}) C_{l_1 0 \ldots l_p - 3 0}^{\lambda_1 \ldots \lambda_{p-3} \lambda_p 0} . \]

**Proof.** We use the trivial fact that

\[ T(x) \triangleq T(0) = \sum_l a_{l0} Y_{l0}(0) = \sum_l a_{l0} \sqrt{\frac{2l + 1}{4\pi}} , \]

where 0 is the North Pole and we used the fact that, for \( m \neq 0 \), \( Y_{lm}(0) = 0 \) and \( Y_{l0}(0) = \sqrt{\frac{2l + 1}{4\pi}} \) (see e.g. [31], Chapter 5). Hence,

\[ ET^p = \sum_{l_1 \ldots l_p} \sqrt{\frac{(2l_1 + 1) \cdots (2l_p + 1)}{(4\pi)^p}} E \left\{ a_{l_1 0 \ldots a_{l_p 0} \right\} \]
\[ = \sum_{l_1 \ldots l_p} \sqrt{\frac{(2l_1 + 1) \cdots (2l_p + 1)}{(4\pi)^p}} \sum_{\lambda_1 \ldots \lambda_{p-3}} P_{l_1 \ldots l_p} (\lambda_1, \ldots, \lambda_{p-3}) C_{l_1 0 \ldots l_p - 3 0}^{\lambda_1 \ldots \lambda_{p-3} \lambda_p 0} . \]

**Example.** Take \( T = H_q(T_G) \), where \( H_q \) is the \( q \)th Hermite polynomial. Then \( ET^p = c_{pq} \{ ET^2 \}^{qp/2} \), where \( c_{pq} \in \mathbb{N} \) denotes the number of Gaussian diagrams without flat edges with \( p \) rows and \( q \) columns (see [31]). Therefore, one has the identity

\[ \sum_{l_1 \ldots l_p} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{(4\pi)^p}} \sum_{\lambda_1 \ldots \lambda_{p-3}} P_{l_1 \ldots l_p} (\lambda_1, \ldots, \lambda_{p-3}) C_{l_1 0 \ldots l_p - 3 0}^{\lambda_1 \ldots \lambda_{p-3} \lambda_p 0} \]
\[ = c_{pq} \left\{ \sum_l \frac{(2l + 1)}{4\pi} C_l \right\}^{pq/2} . \]
6.3 The $\chi^2_l$ polyspectrum

Previously in (6.64), we have implicitly derived the “$\chi^2_l$ bispectrum”, that is, the bispectrum associated with a field of the type $T = H_2(T_G)$, where $T_G$ is Gaussian, centered, isotropic and with unit variance. More precisely, with the notation (6.58)–(6.60), one deduces from (6.63) that

$$E \{ a_{\ell_1 m_1}(2) a_{\ell_2 m_2}(2) a_{\ell_3 m_3}(2) \} = \sum_{\ell_1 \ell_2 \ell_3} \sum_{\mu_1 \mu_2 \mu_3} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \mathcal{C}^{(l_3 m_3)}_{\ell_3} \times$$

$$\times \frac{(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)}{(2 \ell_1 + 1)4\pi (2 \ell_2 + 1)4\pi (2 \ell_3 + 1)4\pi} \times$$

$$\times \mathcal{E} \{ a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \}$$

$$= 8 (-1)^{l_1-m_3} \sum_{\ell_1 \ell_2 \ell_3} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \mathcal{C}^{(l_3 m_3)}_{\ell_3} \left( \frac{2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)}{(4\pi)^3} \right)$$

$$(6.64)$$

$$\times \mathcal{C}^{(l_4 m_4)}_{\ell_4}(2 \ell_1 + 1) \left\{ \ell_1 \ell_2 \ell_3 \ell_4 \right\} \left\{ C_{\ell_4} C_{\ell_3} C_{\ell_2} C_{\ell_1} \right\}$$

$$= 8 (-1)^{l_1-m_3} \sum_{\ell_1 \ell_2 \ell_3} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \mathcal{C}^{(l_3 m_3)}_{\ell_3} \left( \frac{2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)}{(4\pi)^3} \right)$$

$$(6.65)$$

see [11 p. 260 : p. 454]. We now wish to extend these results to polyspectra of order $p = 4, 5, 6$ for random fields of the type $T = H_2(T_G)$, where (as above) $T_G$ is Gaussian, centered, isotropic and with unit variance. As anticipated, here we focus on cumulants instead of moments. We have the following result.

**Proposition 15** The cumulant $\chi (a_{\ell_1 m_1}(2), ..., a_{\ell_p m_p}(2))$ (where $\ell_1, ..., \ell_p$ is Gaussian and isotropic, with angular power spectrum $\{ C_l : l \geq 0 \}$) is given by

$$\chi (a_{\ell_1 m_1}(2), ..., a_{\ell_p m_p}(2)) = (-1)^{p-3} \sum_{\ell_1, ..., \ell_p} \mathcal{C}^{(l_1 m_1)\ldots(l_{p-3} m_{p-3})} X \mathcal{P}^{(l_1 \ldots l_p)}(\ell_1, ..., \ell_{p-3})$$

where the reduced cumulant polyspectrum $\mathcal{P}^{(l_1 \ldots l_p)}(\cdot) : l_1, ..., l_p \geq 0 \}$ is given by

$$P^{(l_1 \ldots l_p)}(\lambda_1, ..., \lambda_{l_p}) = 48 \sqrt{\frac{(2 \ell_1 + 1)(2 \ell_2 + 1)}{(4\pi)^4(2 \ell_1 + 1)\ell_1 \ldots \ell_p} \sum_{\ell_1 \ldots \ell_p} C_{\ell_1} C_{\ell_2} C_{\ell_3} \ldots C_{\ell_{p-3}} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \ldots \mathcal{C}^{(l_{p-3} m_{p-3})}_{\ell_{p-3}} \times$$

$$(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)(2 \ell_{p-1} + 1) \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\} \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\}$$

for $p = 4$,

$$P^{(l_1 \ldots l_p)}(\lambda_1, \lambda_2) = 384 \sqrt{\frac{(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)}{(4\pi)^6(2 \ell_1 + 1)\ell_1 \ldots \ell_p} \sum_{\ell_1 \ldots \ell_5} C_{\ell_1} C_{\ell_2} C_{\ell_3} \ldots C_{\ell_{p-3}} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \ldots \mathcal{C}^{(l_{p-3} m_{p-3})}_{\ell_{p-3}} \times$$

$$(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)(2 \ell_{p-1} + 1) \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\}$$

$$\times \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\}$$

for $p = 5$,

and

$$P^{(l_1 \ldots l_p)}(\lambda_1, \lambda_2, \lambda_3) = 384 \sqrt{\frac{(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)(2 \ell_{p-1} + 1)}{(4\pi)^8(2 \ell_1 + 1)\ell_1 \ldots \ell_p} \sum_{\ell_1 \ldots \ell_5} C_{\ell_1} C_{\ell_2} C_{\ell_3} \ldots C_{\ell_{p-3}} \mathcal{C}^{(l_1 m_1)}_{\ell_1} \mathcal{C}^{(l_2 m_2)}_{\ell_2} \ldots \mathcal{C}^{(l_{p-3} m_{p-3})}_{\ell_{p-3}} \times$$

$$(2 \ell_1 + 1)(2 \ell_2 + 1)(2 \ell_3 + 1)(2 \ell_{p-1} + 1) \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\}$$

$$\times \left\{ l_1 l_2 l_3 \ldots l_p \lambda \right\}$$

for $p = 6$. 

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Proof. The result can be proved by means of the standard graphical techniques for convolutions of Clebsch-Gordan coefficients, as described in [31, Chapters 11 and 12]. Here, we only provide the complete proof for the case \( p = 6 \). Let \( \{a_{\ell m}\} \) be the random harmonic coefficients associated with the underlying Gaussian field \( T_G \). By definition, the field \( H_2(T_G) \) admits the expansion

\[
H_2(T_G) = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{lm}(2) Y_{lm},
\]

where

\[
a_{lm}(2) = \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \int_{S^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 m_2}(x) \overline{Y_{lm}}(x) dx
\]

\[
= \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \left( \frac{\ell_1}{m_1} \frac{\ell_2}{m_2} \frac{l}{-m} \right) \times (-1)^m \times \frac{\sqrt{(2\ell_1 + 1)(2\ell_2 + 1)(2l + 1)}}{4\pi}
\]

\[
= \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} C_{\ell_1 m_1 \ell_2 m_2}^{lm} C_{\ell_1 m_1 \ell_2 m_2}^{lm} \frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi (2l + 1)}.
\]

By using once again the multilinearity of cumulants, one obtains that

\[
\text{Cum} \{a_{\ell_1 m_1}(2), \ldots, a_{\ell_6 m_6}(2)\} = \sum_{\ell_1 m_1 \ldots \ell_6 m_6} \prod_{j=1}^{6} C_{\ell_j m_j}^{lm} \frac{(2\ell_j + 1)}{4\pi (2l + 1)}.
\]

For a given \( \text{lm} = (\ell_{11} m_{11}, \ell_{12} m_{12}, \ldots, \ell_{61} m_{61}, \ell_{62} m_{62}) \), the quantity \( \text{Cum} \{a_{\ell_{11} m_{11}} a_{\ell_{12} m_{12}}, \ldots, a_{\ell_{61} m_{61}} a_{\ell_{62} m_{62}}\} \) is computed as follows:

- **Build the** \( 6 \times 2 \) **matrix**

\[
\Lambda(\text{lm}) = \begin{bmatrix}
\ell_{11} m_{11} & \ell_{12} m_{12} \\
\ell_{21} m_{21} & \ell_{22} m_{22} \\
\ell_{31} m_{31} & \ell_{32} m_{32} \\
\ell_{41} m_{41} & \ell_{42} m_{42} \\
\ell_{51} m_{51} & \ell_{52} m_{52} \\
\ell_{61} m_{61} & \ell_{62} m_{62}
\end{bmatrix}
\]

- **Define the class** \( M(\Lambda(\text{lm})) \) **of connected, Gaussian non-flat diagrams over** \( \Lambda \), **that is, every** \( \gamma \in M(\Lambda(\text{lm})) \) **is a partition of the entries of** \( \Lambda(\text{lm}) \), **into pairs belonging to different rows; moreover, such a partition has to be connected, in the sense that** \( \gamma \) **cannot be divided into two separate diagrams. For instance, an element of** \( M(\Lambda(\text{lm})) \) **is**

\[
\gamma = \{\{\ell_{11} m_{11}, \ell_{21} m_{21}\} \{\ell_{22} m_{22}, \ell_{32} m_{32}\} \{\ell_{31} m_{31}, \ell_{41} m_{41}\} \{\ell_{42} m_{42}, \ell_{52} m_{52}\} \{\ell_{51} m_{51}, \ell_{61} m_{61}\} \{\ell_{62} m_{62}, \ell_{12} m_{12}\}\}
\]

- **For every** \( \gamma \in M(\Lambda(\text{lm})) \), **write**

\[
\delta(\gamma) = \prod_{\{\ell_{ab} m_{ab}, \ell_{cd} m_{cd}\} \in \gamma} \delta_{\ell_{cd} m_{ab}}^{\ell_{ab} m_{cd}} (-1)^{m_{cd}} C_{\ell_{ab}}
\]

(where \( \delta_{a}^{b} \) is the usual Kronecker symbol)
• Use the standard diagram formula (see again [31]), to obtain that

\[ \text{Cum} \{ a_{\ell_1 m_1}, a_{\ell_2 m_2}, \ldots, a_{\ell_6 m_6}, a_{\ell_6 m_2} \} = \sum_{\gamma \in M(\Lambda(\text{Im}))} \delta(\gamma). \]

It follows that

\[
\text{Cum} \{ a_{1 m_1} (2), \ldots, a_{m m_6} (2) \} = \sum_{\text{Im}} \sum_{\gamma \in M(\Lambda(\text{Im}))} \delta(\gamma) \prod_{j=1}^{6} \left\{ c_{\ell_j m_j}^{\ell_{j+1} m_{j+1}} c_{\ell_{j+1} m_{j+1}}^{\ell_{j+2} m_{j+2}} \left( \frac{(2\ell_{j+1} + 1)(2\ell_{j+2} + 1)}{4\pi(2\ell_j + 1)} \right) \right\},
\]

where the first sum runs over all vectors of the type \( \text{Im} = (\ell_{11} m_{11}, \ell_{12} m_{12}; \ldots; \ell_{61} m_{61}, \ell_{62} m_{62}) \). The proof now follows directly from graphical techniques. In particular, the previous term can be associated with an hexagon, having in each vertex an outward line corresponding to a “free” (i.e. not summed up) index \( l_j m_j \). An expression for convolutions of Clebsch-Gordan coefficients corresponding to such a configuration can be found in [31, p. 461], eq. 12.1.6.30. From this, standard combinatorial arguments and a convenient relabelling of the indexes, we obtain that

\[
P_{l_1 \ldots l_p}^{C;1}(\lambda_1, \lambda_2, \lambda_3) = 3840 \times \sum_{\ell_1, \ldots, \ell_6} (2\ell_1 + 1) \cdots (2\ell_6 + 1) C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6} C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6} C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6} C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6} C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6} C_{\ell_1 \ell_2 \cdots \ell_6}^{\ell_{p+1} \ell_{p+2} \cdots \ell_6}.
\]

Note that 3840 = \( 2^{p-1}(p-1)! = 2^55! \) is the number of automorphisms between graphs belonging to \( M(\Lambda(\text{Im})) \).

We recall that the Clebsch-Gordan coefficients \( \{ C_{\alpha \beta \gamma}^{\delta \epsilon \zeta} \} \) are identically zero unless \( a + b + c \) is even; it is hence easy to see that the previous polyspectra are non-zero only if the sum \( \{l_1 + \ldots + l_p\} \) is even as well.

From the previous Proposition, we can derive the corresponding expressions for the cumulant polyspectra for \( \chi^2_\nu \) random field.

**Definition B.** We say the random field \( T_{\chi^2_\nu} \) has a chi-square law with \( \nu \geq 1 \) degrees of freedom if there exist \( \nu \) independent and identically distributed Gaussian random fields \( T_i \) such that

\[ T_{\chi^2_\nu} \overset{\text{law}}{=} T_1^2 + \ldots + T_\nu^2. \]

It is trivial to show that \( T_{\chi^2_\nu} \) is mean-square continuous and isotropic if \( T_i \) is. We have the following

**Proposition 16** The cumulant polyspectra of \( T_{\chi^2_\nu} \) (for \( p \geq 2 \)) are given by

\[ P_{l_1 \ldots l_p}^{C;\nu}(\lambda_1, \ldots, \lambda_{\nu-3}) = \nu P_{l_1 \ldots l_p}^{C;1}(\lambda_1, \ldots, \lambda_{\nu-3}). \]

**Proof.** Note that the cumulant polyspectra of order \( p \geq 2 \) of \( T_{\chi^2_\nu} \) coincide with those of the centered field \( T_{\chi^2_\nu} - ET_{\chi^2_\nu} \) (due to the translation-invariance properties of cumulants). Then, the proof is an immediate consequence of Proposition 15 and the of the standard multilinearity properties of cumulants.
7 Further Issues and Applications

The purpose of this final Section is to introduce what we view as promising directions for further research, where the ideas of this paper may perhaps yield further insights. We shall delay to future work a more thorough investigation of the issues which are left open below.

7.1 Representations of the Symmetric Group

As a further link between representation theory and higher order angular power spectra, we mention the following. It is to be stressed that the decomposition of $\Delta_{l_1...l_n}$ that we achieved in the previous Proposition 4 is by no means unique. In particular, what we did was to choose a particular sequence of “couplings”, i.e. we partitioned tensor products of the Wigner’s matrices $D^l$ in a specific order before decomposing them into direct sums. Alternative partitions yield different eigenvectors and therefore, different expressions for the polyspectra/joint moments. Alternatively, we could maintain the same coupling scheme (for instance, “start always from the first pair on the left”, as we did earlier) but acting on $(l_1,...,l_n)$ by the symmetric group $S_n$. However, not all coupling schemes can be achieved by simply permuting the elements of $(l_1,l_2,...,l_n)$. This is the well-known problem of parentheses in Mathematical Physics (see for instance [3]).

We suggest here that one can establish a link between alternate expressions for the angular polyspectra and representations of the symmetric group. More precisely the alternate expressions that we find for the polyspectra $P_{l_1,...,l_n}(\lambda_1,...,\lambda_{n-3})$ of a strongly isotropic field (with $n$-moments) must be such that, for every permutation $\pi \in S_n$,

$$\sum_{\lambda_1} ... \sum_{\lambda_{n-3}} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1...\lambda_{n-3};l_n-m_n} P_{l_1...l_n}(\lambda_1,...,\lambda_{n-3}) = \sum_{\lambda_1} ... \sum_{\lambda_{n-3}} C_{\pi(l_1)m_1...\pi(l_{n-1})m_{n-1}}^{\lambda_1'...\lambda_{n-3}';l_n-m_n} P_{\pi(l_1)...\pi(l_n)}(\lambda_1',...,\lambda_{n-3}') .$$

Now let us multiply both sides by $C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n}$, where $(\lambda_1',...,\lambda_{n-3}')$ is fixed, and sum over $(m_1,...,m_n)$. In view of the unitary properties of Clebsch-Gordan coefficients we obtain for the left-hand side

$$\sum_{m_1...m_n} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n} \sum_{\lambda_1} ... \sum_{\lambda_{n-3}} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1...\lambda_{n-3};l_n-m_n} P_{l_1...l_n}(\lambda_1,...,\lambda_{n-3})$$

$$= \sum_{\lambda_1} ... \sum_{\lambda_{n-3}} \left\{ \sum_{m_1...m_n} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1...\lambda_{n-3};l_n-m_n} P_{l_1...l_n}(\lambda_1,...,\lambda_{n-3}) \right\}$$

$$= \sum_{\lambda_1} ... \sum_{\lambda_{n-3}} \left\{ \delta_{\lambda_1'}^{\lambda_1} ... \delta_{\lambda_{n-3}'}^{\lambda_{n-3}} P_{l_1...l_n}(\lambda_1, ..., \lambda_{n-3}) \right\} = P_{l_1...l_n}(\lambda_1',...,\lambda_{n-3}'); \quad (7.66)$$

on the right-hand side we get

$$\sum_{m_1...m_n} C_{l_1m_1...l_{n-1}m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n} \sum_{\lambda_1} ... \sum_{\lambda_{n-3}} C_{\pi(l_1)m_1...\pi(l_{n-1})m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n} P_{\pi(l_1)...\pi(l_n)}(\lambda_1',...,\lambda_{n-3}')$$

$$= \sum_{\lambda_1'} ... \sum_{\lambda_{n-3}'} \sum_{m_1...m_n} C_{\pi(l_1)m_1...\pi(l_{n-1})m_{n-1}}^{\lambda_1';...\lambda_{n-3}';l_n-m_n} C_{\pi(l_1)m_1...\pi(l_{n-1})m_{n-1}}^{\lambda_1...\lambda_{n-3};l_n-m_n} P_{\pi(l_1)...\pi(l_n)}(\lambda_1',...,\lambda_{n-3}'). \quad (7.67)$$

Similarly as in the previous section, the sum of products of Clebsch-Gordan coefficients on the right hand side can be expressed in terms of higher order Wigner’s coefficients. Since this section is just informal, for brevity’s sake we do not give explicit expressions (see e.g. [3] Chapter 10). The two expressions
(7.66) and (7.67) imply that, for every fixed \((l_1, ..., l_n)\) and every permutation \(\pi\), there exists a square matrix \(A((l_1, ..., l_n); \pi)\) such that

\[
P_{l_1, ..., l_n} = A \{ (l_1, ..., l_n); \pi \} P_{\pi(l_1), ..., \pi(l_n)},
\]

where \(P_{l_1, ..., l_n}\) is the vector with entries \(P_{l_1, ..., l_n}(\lambda_1, ..., \lambda_n)\). We conjecture that in this way one can build a representation of the symmetric group \(S_n\) on the vector space generated by admissible polyspectra \(P_{l_1, ..., l_n}\). If this is indeed the case, some important questions are left open: for instance, whether or not the representation is faithful (see [41]), and whether these ideas can lead to algorithms for the numerical simulation of representation matrices, along the lines of what we shall pursue in the next subsection.

7.2 Random data compression

In this subsection we shall show how we can exploit the previous results to develop a probabilistic algorithm to compress information on Clebsch-Gordan coefficients. Note first that

\[
\# \left\{ C_{l_1m_1l_2m_2}^{l_3m_3} : l_1, l_2, l_3 \leq L, \left| C_{l_1m_1l_2m_2}^{l_3m_3} \right| \neq 0 \right\} \approx O(L^6); 
\]

it is therefore clear how for most applications the storage of Clebsch-Gordan coefficients for future usage is simply unfeasible, whatever the supercomputing facilities (for instance, for CMB data analysis, \(L \approx 3 \times 10^3\) is currently required, so that the number of Clebsch-Gordan coefficients to be saved would exceed \(10^{20}\)). Let us consider again a chi-square field as defined before, i.e.

\[
T_{\chi^2}(x) = H_2(T_G(x)) = \sum_{lm} a_{lm}(2)Y_{lm}(x);
\]

we have proved earlier in (6.65) that

\[
Ea_{l_1m_1}(2)a_{l_2m_2}(2)a_{l_3m_3}(2) = (-1)^{m_1}C_{l_1m_1l_2m_2}^{l_3m_3}h_{l_1l_2l_3}
\]

where

\[
h_{l_1l_2l_3} := 8 \sum_{\ell_1\ell_2\ell_3} C_{\ell_10\ell_20\ell_30}^{l_10l_20l_30} \frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{(4\pi)^3} \frac{1}{2\ell_3+1} \left\{ \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ i_1 & i_2 & i_3 \end{array} \right\} \{ C_{i_1}C_{i_2}C_{i_3} \},
\]

which can be calculated analytically and stored, with storage dimension

\[
\# \left\{ h_{l_1l_2l_3} : l_1, l_2, l_3 \leq L, \left| C_{l_10l_20l_30}^{l_10l_20l_30} \right| \neq 0 \right\} \approx O(L^3).
\]

Let us assume we simulate \(B\) times \(T_{\chi^2}(x)\), which is trivially done by simply squaring a Gaussian field: the latter is obtained by sampling independent complex Gaussian variables with variance \(C_\ell\). We store the triangular arrays \(\{a_{lm}^{(i)}\}_{l_1=-\ell_1, ..., l_m=-\ell_m, ..., l_n=-\ell_n, ..., l_n=l_n}\), \(i = 1, ..., B\); here the dimension is of order \(B \times L^2\). We can then recover any value \(C_{l_1m_1l_2m_2}^{l_3m_3}\) by means of the Monte Carlo estimate

\[
\hat{C}_{l_1m_1l_2m_2}^{l_3m_3} = \frac{1}{B} \sum_{i=1}^{B} a_{l_1m_1}^{(i)}a_{l_2m_2}^{(i)}a_{l_3m_3}^{(i)}B,
\]

which requires \(B\) steps and \(B \times L^2 + L^3\) storage capacity, as opposed to \(L^6\) storage capacity by the direct method. We leave for further research a more thorough investigation on the convergence properties of this algorithm; we stress, however, that the procedure we advocate is completely general, i.e. it does not depend on peculiar features of the group \(SO(3)\) we are currently considering. We believe, hence, that similar ideas can be implemented for the numerical estimation of Clebsch-Gordan coefficients for other compact groups of interest for theoretical physicists. We leave this and the previous issues in this Section as topics for further research.
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