LATTICE POLYTOPES, HECKE OPERATORS, AND THE EHRHART POLYNOMIAL

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Abstract. Let $P$ be a simple lattice polytope. We define an action of the Hecke operators on $E(P)$, the Ehrhart polynomial of $P$, and describe their effect on the coefficients of $E(P)$. We also describe how the Brion-Vergne formula transforms under the Hecke operators for nonsingular lattice polytopes $P$.

1. INTRODUCTION

1.1. Let $L$ be a rank $n$ lattice, embedded in a real $n$-dimensional vector space $V$. Let $\mathcal{P}(L)$ be the set of $n$-dimensional convex polytopes in $V$ with vertices in $L$. For any $P \in \mathcal{P}(L)$, and for any nonnegative integer $t$, let $tP$ be $P$ scaled by the factor $t$. Then by a result of Ehrhart [8], the function $t \mapsto \#(tP \cap L)$ is a degree $n$ polynomial with rational coefficients, called the Ehrhart polynomial of $P$. Hence one can think of the Ehrhart polynomial as giving a map $E$ from $\mathcal{P}(L)$ to the polynomial ring $\mathbb{Q}[t]$.

Write $E(P) = \sum_{l=0}^{n} c_l t^l$. Formulas for the coefficients $c_l$, in various settings and with varying degrees of generality, have been given by several authors [3–5, 7, 11, 13, 15, 16]. Some coefficients are easy to understand, for example

$$c_0 = 1, \quad c_n = \text{Vol}(P), \quad \text{and} \quad c_{n-1} = \text{Vol}(\partial P)/2.$$  

Here $\text{Vol}(P)$ is taken with respect to the measure that gives a fundamental domain of $L$ volume 1; if a polytope has dimension less than $n$, we compute its volume with respect to the lattice obtained by intersecting its affine hull with $L$. For a general lattice polytope, expressions for the Ehrhart coefficients involve not only volumes but also subtle arithmetic information, namely higher-dimensional Dedekind sums as studied by Carlitz and Zagier [6, 18].

1.2. The Ehrhart polynomial depends not just on the combinatorial type of $P$, but rather on the pair $(P, L)$. Hence it is natural to consider how $E(P)$ changes as $L$ is varied. The theory of automorphic forms provides a powerful machine to accomplish this, namely the technique of Hecke operators.

Thus let $p$ be a prime, and let $k \leq n$ be a positive integer. Given a lattice polytope $P$ with Ehrhart polynomial $E(P)$, we define a new polynomial $T(p, k)E(P)$ as follows. Let $p^{-1}L$ be the canonical superlattice of $L$ of coindex $p^n$. We have $p^{-1}L/L \simeq \mathbb{F}_p^\infty$,

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and any lattice $M$ satisfying $p^{-1}L \supseteq M \supseteq L$ determines a subspace $\overline{M} \subset \mathbb{F}_p^n$. Let $\mathcal{L}_k$ be the set of such lattices with $\dim \overline{M} = k$. Then we define

$$T(p, k)E(P) = \sum_{M \in \mathcal{L}_k} E(P_M),$$

where $P_M \in \mathcal{P}(M)$ denotes the lattice polytope with vertices in $M$ canonically determined by $P$.

1.3. In this paper we consider the relationship between $T(p, k)E(P)$ and $E(P)$. To state our results, we require more notation. For any nonnegative integer $l \leq n$, choose and fix an $l$-dimensional subspace $U$ of $\mathbb{F}_p^n$, and define

$$\nu_{n,k,l}(p) = \sum_{W \subset \mathbb{F}_p^n} p^{\dim W \cap U}.$$

Note that this value is independent of the choice of $U$. Finally for any polynomial $f \in \mathbb{Q}[t]$ let $c_l(f)$ be the coefficient of $t^l$ in $f$. Then our first result can be stated as follows:

**Theorem 1.4.** We have

$$c_l(T(p, k)E(P))/c_l(E(P)) = \nu_{n,k,l}(p),$$

independently of $P$. The ratios $\nu$ satisfy

$$\nu_{n,k,l}(p)/\nu_{n,k,n-l}(p) = p^{k+l-n}.$$

Moreover, for each triple $(n, k, l)$, there is a polynomial with positive coefficients

$$\Phi_{n,k,l}(t) \in \mathbb{Z}[t],$$

independent of $p$, such that $\Phi_{n,k,l}(p) = \nu_{n,k,l}(p)$.

The sum (3) can be viewed as a sum of $p$-powers over a certain geometrically-defined stratification of the finite Grassmannian $\text{Gr}(k, n)(\mathbb{F}_p)$, and thus it is not surprising that for any given $p$ the quantity $\nu_{n,k,l}(p)$ can be expressed as an integral polynomial in $p$. However, the existence of $\Phi$, as well as the statement that it has positive coefficients, does not follow immediately from (3) since the number of terms in the sum grows with $p$ and since the strata are only locally closed.

As an example of Theorem 1.4, if $l = 0$, then $c_0(E(P)) = 1$ for any $P$. Hence the ratio on the left of (4) is the cardinality of $\text{Gr}(k, n)(\mathbb{F}_p)$ (cf. Lemma 2.5), which equals $\nu_{n,k,0}(p)$. For further examples, Table 1 shows the Hecke eigenvalues that arise for the Ehrhart coefficients of 4-dimensional polytopes.
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Table 1. Eigenvalues for \( n = 4 \).

| \( c_4 \) | \( T(p, 1) \) | \( T(p, 2) \) | \( T(p, 3) \) |
|----------|----------|----------|----------|
| \( p^4 + p^2 + p^2 + p \) | \( p^6 + p^9 + p^4 + p^4 \) | \( p^6 + p^9 + p^4 + p^4 \) |
| \( 2p^3 + p^2 + p \) | \( p^4 + 2p^4 + 2p^3 + p^2 \) | \( p^4 + 2p^4 + 2p^3 + p^2 \) |
| \( p^3 + 2p^2 + p \) | \( 2p^3 + 2p^3 + 2p^2 \) | \( p^4 + 4p^3 + p^2 \) |
| \( p^3 + p^2 + 2p \) | \( p^4 + 2p^3 + 2p^2 + p \) | \( 2p^3 + p^2 + p \) |
| \( p^3 + p^2 + p + 1 \) | \( p^4 + 2p^3 + 2p^2 + p + 1 \) | \( p^3 + p^2 + p + 1 \) |

1.5. A geometric interpretation of the eigenvalue \([3]\) is the following. Consider the map

\[ \text{Vol}_l: \mathcal{P}(L) \rightarrow \mathbb{Q} \]

taking \( P \) to the sum of the volumes of all faces of dimension \( l \). Then we can define an action of the Hecke operators on \( \text{Vol}_l \) as in \([2]\), and one can show that \( T(p, k) \text{Vol}_l = \nu_{a,k,l}(p) \text{Vol}_l \) (Proposition \( 2.8 \)). Hence Theorem 1.4 says that the \( l \)th coefficient of the Ehrhart polynomial transforms under the Hecke operators exactly as the volumes of the \( l \)-dimensional faces do. For another interpretation, in terms of counting the number of \( \mathbb{F}_p \)-points on certain varieties, see Remark \( 3.4 \).

1.6. Recall that an \( n \)-dimensional lattice polytope is called simple if every vertex meets exactly \( n \) edges, and is called nonsingular if for any vertex \( v \), the primitive lattice vectors parallel to the edges emanating from \( v \) form a \( \mathbb{Z} \)-basis of \( L \). Our next result concerns how the Hecke operators interact with certain formulas for the coefficients of the Ehrhart polynomial in the special case that \( P \) is simple.

Let \( \mathcal{F}(n-1) \) be the set of facets of \( P \), and let \( h = (h_F)_{F \in \mathcal{F}(n-1)} \) be a real multivariable indexed by the facets of \( P \). Let \( P(h) \) be the convex region obtained by parallel translation of the facets of \( P \) by the parameter \( h \), normalized by \( P(0) = P \) (\( \S 4.1 \)). For small \( h \) the region \( P(h) \) is bounded, and the volume \( \text{Vol} P(h) \) is a polynomial function of \( h \).

Let \( \Sigma \) be the normal fan to \( P \) (\( \S 2.2 \)). Then the polytope \( P \) determines a differential operator \( Td(\Sigma, \partial/\partial h) \), called the Todd operator (\( \S 4.7 \)). In the special case that \( P \) is nonsingular, this operator is defined as follows. Let \( Td(x) \) be the power series expansion of \( x/(1 - e^{-x}) \), i.e.

\[ Td(x) = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j, \]

where \( B_j \) are the Bernoulli numbers. For each \( h_F \) let \( Td(\partial/\partial h_F) \) be the differential operator obtained by formally replacing \( x \) with \( \partial/\partial h_F \) in \( Td(x) \). Then \( Td(\Sigma, \partial/\partial h) \)
is defined to be the product
\begin{equation}
Td(\Sigma, \partial/\partial h) = \prod_{F \in \mathcal{F}(n-1)} Td(\partial/\partial h_F).
\end{equation}

Note that product may be taken in any order, since the derivatives mutually commute. This is an infinite-degree differential operator, and we denote by
\[ Td_l(\Sigma, \partial/\partial h) \]
the homogeneous terms of degree \( l \). By Khovanskii-Pukhlikov [16] one has
\[ c_{n-l}(E(P)) = Td_l(\Sigma, \partial/\partial h) \left. \text{Vol } P(h) \right|_{h=0} \]
If the polytope \( P \) is simple and not nonsingular, then one must enlarge (6) with additional terms involving higher-dimensional Dedekind sums; the corresponding formula is due to Brion-Vergne [3].

1.7. Let \( f \) be a face of \( P \) of codimension \( \leq l \), and let \( \pi = (\pi(F))_{F \supset f} \) be a an ordered partition of \( l \) into positive parts indexed by the facets containing \( f \). The pair \((f, \pi)\) determines a differential operator
\[ \partial_f^\pi = \prod_{F \supset f} (\partial/\partial h_F)^{\pi(F)}, \]
and we can collect common terms in (6) to write
\begin{equation}
Td_l(\Sigma, \partial/\partial h) = \sum_{(f, \pi)} A(f, \pi) \partial_f^\pi.
\end{equation}

The coefficient \( A(f, \pi) \) is rational, and for simple \( P \) is essentially a rank \( l \) Dedekind sum. Our next result shows that if \( P \) is nonsingular, then these individual terms transform under the Hecke operators exactly as the coefficients of \( E(P) \) do:

**Theorem 1.8.** Let \( P \in \mathcal{P}(L) \) be a nonsingular lattice polytope. For any superlattice \( M \supset L \), let \( f_M \) be the face \( f \), thought of as a face of \( P_M \). Then for each degree \( l \) term \( A(f, \pi) \partial_f^\pi \in Td_l(\Sigma, \partial/\partial h) \) in the Brion-Vergne formula, we have
\begin{equation}
\sum_{M \in \mathcal{L}_k} A(f, \pi) \partial_f^\pi \left. \text{Vol } P_M(h) \right|_{h=0} = \nu_{n,k,n-l}(p) A(f, \pi) \partial_f^\pi \left. \text{Vol } P(h) \right|_{h=0}.
\end{equation}

Note that the Hecke images \( P_M \) in (8) are in general singular, even if \( P \) is nonsingular. Also, the proof of Theorem 1.8 is independent from that of Theorem 1.4, and hence provides another proof Theorem 1.4 for nonsingular lattice polytopes.
1.9. We comment briefly on the proofs of Theorems 1.4 and 1.8. The proof of Theorem 1.4 is a counting argument. The new lattice points appearing in $P$ in the sum (2) all lie in the superlattice $p^{-1}L$, and to compute $T(p, k)E(P)$ one keeps track of which lattice points appear in a given Hecke image. This gives an expression for $T(p, k)E(P)$ in terms of $E(P)(t)$, $E(P)(pt)$, and the cardinalities of some finite Grassmannians. An additional argument shows that this expression implies (1).

The proof of Theorem 1.8 is more complicated. At the heart of (3) are certain “distribution relations” of Dedekind sums, essentially coming from a distribution relation satisfied by the Hurwitz zeta function (§6.2). In the proof of Theorem 1.8, these relations appear in identities involving Dedekind sums and the cardinalities of strata in certain stratifications of finite Grassmannians.

Rather than proving these identities directly, we show that they occur in the computation of the constant term of $T(p, j)E(P')$ for lower-dimensional polytopes $P'$ and for $j \leq k$. Since these constant terms are always 1, by appropriately choosing $P'$ we show that our identities hold. Then we use induction to complete the argument.

1.10. Here is a fanciful interpretation of Theorem 1.4. The Ehrhart polynomial is clearly invariant under the action of $GL(L)$, the linear automorphisms of $V$ preserving $L$. One can think of $P(L)$ as being like the upper halfplane $\mathbb{H}$, and the equivalence class of $P \in P(L)$ as being a point on the modular curve $\text{SL}_2(\mathbb{Z})\backslash \mathbb{H}$. Then the $l$th coefficient $c_l$, thought of as a function $\text{GL}(L) \backslash P(L) \rightarrow \mathbb{Q}$, plays the role of a weight $l$ modular form, and Theorem 1.4 says that $c_l$ is a “weight $l$ Hecke eigenform of level 1.” Furthermore, the simple description of its Hecke eigenvalues indicates that $c_l$ should be thought of as being like an Eisenstein series. Indeed, the analogy between coefficients of $E$ and modular forms was our original motivation to consider this problem. These reflections lead to natural questions unanswered in this paper:

- What is the dimension of the space of eigenforms? Is it finite-dimensional?
- What are the analogues of level $N$ modular forms?
- Are there analogues of modular forms over number fields of higher degree, e.g. Hilbert modular forms?

1.11. The paper is organized as follows. Section 2 recalls background about lattice polytopes and their normal fans, and discusses the connection between Hecke operators and finite Grassmannians. Section 3 gives the proof of Theorem 1.4. Section 4 discusses the computation of the Ehrhart polynomial using the Todd operator, and Section 5 gives the proof of Theorem 1.8. Section 6 discusses explicit examples of Theorem 1.8 for three-dimensional polytopes, and relates the identities occurring in the proof of Theorem 1.8 to Dedekind sums and the Hurwitz zeta function. Finally, Section 7 addresses the problem of computing the average Ehrhart polynomial as one varies over a family of superlattices.
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2. Hecke operators and finite Grassmannians

2.1. Let $P$ be a simple lattice polytope in the vector space $V$ with vertices in the lattice $L$. For convenience we fix a nondegenerative bilinear form $\langle \ , \ \rangle$ and use it to identify $V$ with its dual. We also assume that $L$ is self-dual with respect to this form.

Let $\mathcal{F}$ be the set of faces of $P$, and for any $l$ let $\mathcal{F}(l)$ be the subset of faces of dimension $l$. Let $F \in \mathcal{F}(n-1)$ be a facet of $P$. Then $F$ is the intersection of $P$ with an affine hyperplane

$$H_F = \{ x \mid \langle x, u_F \rangle + \lambda_F = 0 \},$$

where the normal vector $u_F$ is taken to be a primitive vector in $L$, and points into the interior of $P$.

2.2. Let $f \in \mathcal{F}(n-l)$ be a face of codimension $l$, and let $H_f$ be the affine subspace spanned by $f$. Since $P$ is simple, there are exactly $l$ hyperplanes in $\{ H_F \mid F \in \mathcal{F}(n-1) \}$ whose intersection is $H_f$. Let $\sigma_f \subset V$ be the convex cone generated by the corresponding normal vectors $\{ u_F \}$. The cone $\sigma_f$ is called the normal cone to $f$.

The set $\Sigma$ of all normal cones $\{ \sigma_f \mid f \in \mathcal{F} \}$ forms an acute rational polyhedral fan in $V$. This means the following:

(1) Each $\sigma \in \Sigma$ contains no nontrivial linear subspace.
(2) If $\sigma'$ is a face of $\sigma \in \Sigma$, then $\sigma' \in \Sigma$.
(3) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of each.
(4) Given $\sigma \in \Sigma$, there exists a finite set $S \subset L$ such that any point in $\sigma$ can be written as $\sum \rho_s s$, where $s \in S$ and $\rho_s \geq 0$.

Moreover, $P$ simple implies $\Sigma$ is simplicial, which means that in (4) we can take $\#S = \dim \sigma$ for all $\sigma$. The fan $\Sigma$ is called the normal fan to $P$.

2.3. Let $\rho \in \Sigma$ be a 1-dimensional cone. Then $\rho$ contains a unique normal vector $u_F$, which we call the spanning point of $\rho$. For any cone $\sigma$, we denote by $\sigma(1)$ the set of spanning points of all 1-dimensional faces of $\sigma$, and write

$$\Sigma(1) = \bigcup_{\sigma \in \Sigma} \sigma(1).$$

There is bijection between $\Sigma(1)$ and $\mathcal{F}(n-1)$.

For any rational cone $\sigma$, let $U(\sigma)$ be the sublattice of $L$ generated by the spanning points of $\sigma$. Put $L(\sigma) = L \cap (U(\sigma) \otimes \mathbb{Q})$, and let $\text{Ind} \sigma = [L(\sigma) : U(\sigma)]$. If $\text{Ind} \sigma = 1$,
then \( \sigma \) is called \textit{unimodular}. Then \( P \) is nonsingular if and only if all its normal cones are unimodular.

2.4. Now we recall some basic facts about Hecke operators for the linear group \( \text{GL}_n \).

Let \( p \) be a prime, and let \( \overline{V} \) be the finite vector space \( \mathbb{F}_p^n \). For any rational subspace \( W \subset V \), let \( \overline{W} \) be the corresponding subspace of \( \overline{V} \). Fix a positive integer \( k \leq n \), and let \( \text{Gr}(k, n) \) be the Grassmannian of \( k \)-dimensional subspaces of an \( n \)-dimensional vector space.

**Lemma 2.5.** The set \( \mathcal{L}_k \) of superlattices \( p^{-1}L \supseteq M \supseteq L \) of coindex \( p^k \) is in bijection with the set \( \mathcal{T} \) of upper triangular matrices of the form

\[
\begin{pmatrix}
p^{e_1} & a_{ij} \\
\vdots & \ddots \\
p^{e_n}
\end{pmatrix},
\]

where

- \( e_i \in \{0, 1\} \), and exactly \( k \) of the \( e_i \) are equal to 0, and
- \( a_{ij} = 0 \) unless \( e_i = 0 \) and \( e_j = 1 \), in which case \( a_{ij} \) satisfies \( 0 \leq a_{ij} < p \).

Moreover, the map \( M \mapsto \overline{M} \) induces a bijection between \( \mathcal{L}_k \) and \( \text{Gr}(k, n)(\mathbb{F}_p) \).

**Proof.** It is well known that the set of \textit{sublattices} \( L \supseteq N \supseteq pL \) of index \( p^{n-k} \) is in bijection with \( \mathcal{T} \) [14, Prop. 7.2]. To realize this bijection, we take \( L = \mathbb{Z}^n \), and then any \( N \) is constructed as the sublattice generated by the rows of some \( A \in \mathcal{T} \). The sublattice \( N \) determines a subspace \( \overline{N} \subset \overline{V} \), which is the subspace generated by the \( k \) rows with diagonal entry 1. It is clear that we obtain all \( k \)-dimensional subspaces of \( \overline{V} \) in this way, for example by considering the decomposition of \( \text{Gr}(k, n)(\mathbb{F}_p) \) into Schubert cells [10, p. 147]. Finally, both statements of the lemma follow from the isomorphism \( p^{-1}L/L \simeq L/pL \) given by scaling by \( p \), and from the fact that a sublattice has coindex \( p^k \) if and only if it has index \( p^{n-k} \). \( \square \)

2.6. Let \( f \in \mathcal{F} \) be a face of \( P \), and let \( \sigma_f \) be the normal cone to \( f \). Let \( V_f \subset V \) be the linear subspace parallel to \( H_f \), and let \( C_f \) be the linear span of \( \sigma_f \). The subspace \( C_f \) contains the distinguished 1-dimensional subspaces \( \{ C_\rho \mid \rho \in \sigma_f(1) \} \).

**Proposition 2.7.** Let \( M \in \mathcal{L}_k \), and for any \( f \in \mathcal{F} \), let \( f_M \) be the corresponding face of \( P_M \). Then

1. \( \text{Vol} f_M = p^{\dim(M \cap \overline{V})} \text{Vol} f \), and
2. \( \text{Ind} \sigma_{f_M} = p^{\dim(M \cap \overline{C_f}) - r} \text{Ind} \sigma \),

where

\[ r = \# \{ \overline{C}_\rho \mid \rho \in \sigma_f(1) \text{ and } \overline{C}_\rho \subset \overline{M} \} . \]
Proof. Choose a \( \mathbb{Z} \)-basis \( B \) of \( L \) such that \( B \cap V_f \) is a \( \mathbb{Z} \)-basis for \( L \cap V_f \). By Lemma 2.5, with respect to \( B \) any \( M \in \mathcal{L}_k \) is spanned by the rows of \( p^{-1} A \) for some \( A \in \mathcal{T} \). Each row of \( A \) with diagonal entry 1 contributes a factor of \( p \) to \( \text{Vol} f_M / \text{Vol} f \), which proves (1).

For \( C_f \) we argue similarly. The only difference is that each row of \( A \) with diagonal entry 1 contributes a factor of \( p \) to \( \text{Ind} \sigma_{f_M} / \text{Ind} \sigma_f \), unless the diagonal entry is the only nonzero entry in the row. This situation corresponds to some subspace \( \overline{C}_\rho \) being contained in \( \overline{M} \), and (2) follows. \( \square \)

Proposition 2.7 allows us to give a geometric interpretation for the eigenvalue \( \nu(p) \).

**Proposition 2.8.** Fix nonnegative integers \( k, l \leq n \), and let \( p \) be a prime. Let \( \text{Vol}_l : \mathcal{P}(L) \to \mathbb{Q} \) be the function

\[
\text{Vol}_l(P) = \sum_{f \in \mathcal{F}(l)} \text{Vol}(f),
\]

and define

\[
T(p, k) \text{Vol}_l(P) = \sum_{M \in \mathcal{L}_k} \text{Vol}_l(P_M).
\]

Then \( T(p, k) \text{Vol}_l(P) = \nu_{n, k, l}(p) \text{Vol}_l(P) \).

**Proof.** Suppose \( f \in \mathcal{F}(l) \). According to Proposition 2.7, we have

\[
\sum_{M \in \mathcal{L}_k} \text{Vol} f_M = \sum_{M \in \mathcal{L}_k} p^{\dim(M \cap V_f)} \text{Vol} f.
\]

The right of (9) equals \( \nu_{n, k, l}(p) \text{Vol} f \), and the statement follows immediately. \( \square \)

3. **Proof of Theorem 1.4**

3.1. Throughout this section we allow \( P \) to be a general lattice polytope. Let \( U \subset \overline{V} \) be a fixed subspace of dimension \( l \) as in §1.3, and recall

\[
\nu_{n, k, l}(p) = \sum_{W \subset \mathbb{F}_p^n \atop \dim W = k} p^{\dim W \cap U}.
\]

Let \( G_{k, n} \) be the cardinality of number the finite Grassmannian \( \text{Gr}(k, n)(\mathbb{F}_p) \). It is well known that

\[
G_{k, n} = \frac{[n]_p!}{[k]_p! [n-k]_p!},
\]

where \( [n]_p = (p^n - 1)/(p - 1) \), and \( [n]_p! = \prod_{i=1}^n [i]_p \).
Lemma 3.2. Let \( E = E(t) \) be the Ehrhart polynomial of \( P \). Then
\[
T(p, k)E(t) = G_{k-1,n-1}E(pt) + (G_{n,k} - G_{k-1,n-1})E(t).
\]
In particular,
\[
c_t(T(p, k))E(t)/c_t(E) = G_{k,n} + (p^l - 1)G_{k-1,n-1}.
\]

Proof. We have
\[
\bigcup_{M \in \mathcal{L}_k} \ M = p^{-1}L,
\]
and since counting points in \( p^{-1}L \cap P \) is done by \( E(pt) \), we must count how often a point \( x \in p^{-1}L \) appears in the union (13). There are two separate cases, namely (i) \( x \in p^{-1}L \setminus L \), and (ii) \( x \in L \). The former contribute to \( E(pt) \), and the latter to \( E(t) \).

For (i), note that the point \( x \) determines a line \( \Lambda_x \in V \), and the number of \( k \)-dimensional subspaces containing \( \Lambda_x \) is \( G_{k-1,n-1} \). For (ii), each \( x \in L \) will appear in every Hecke image, which gives \( G_{k,n} \) in total. However, such points are also counted in the sublattices contributing to (i). When these contributions are subtracted, we obtain (11). This proves the first statement.

Finally, (12) follows easily from (11), since \( c_t(E(pt)) = p^l c_t(E(t)) \).

Lemma 3.3. We have
\[
\nu_{n,k,l}(p) = G_{k,n} + (p^l - 1)G_{k-1,n-1}.
\]
Moreover,
\[
\nu_{n,k,l}(p)/\nu_{n,n-k,n-l}(p) = p^{k+l-n}.
\]

Proof. We treat the case \( k \geq l \); the case \( k < l \) is similar.

For \( j = 0, \ldots, l \), let \( Y_j \) be the locally closed subvariety of \( \text{Gr}(k,n)(\mathbb{F}_p) \) defined by
\[
Y_j = \{ W \mid \dim W = k, \dim(W \cap U) = j \},
\]
and let \( y_j = \#Y_j \). Note that \( \sum_{j \geq 0} y_j = G_{k,n} \), and that \( \nu_{n,k,l}(p) = \sum_{j \geq 0} y_j p^j \). Since \( y_0 = G_{k,n} - \sum_{j \geq 1} y_j \), it follows that
\[
\nu_{n,k,l}(p) = G_{k,n} + \sum_{j \geq 1} y_j (p^j - 1).
\]

We prove the lemma by showing
\[
[l]_p G_{k-1,n-1} = \sum_{j \geq 1} [j]_p y_j,
\]
which is equivalent to (14) and (16) taken together. To do this, we explicitly describe \( Y_j \) recursively in terms of \( \{ Y_i \mid i > j \} \), and show that the right of (17) telescopes to the left of (17).
Consider first $Y_l$. Any point in $Y_l$ is given by choosing a $k$-dimensional subspace $W$ in $\mathbf{V}$ containing $U$. Such subspaces are in bijection with $(k-l)$-dimensional subspaces of $\mathbf{V}/U$, and thus $y_l = G_{k-l,n-l}$.

Next, any point in $Y_{l-1}$ is given by choosing an $(l-1)$-dimensional subspace $S$ of $U$, and then choosing a $k$-dimensional subspace $W$ of $\mathbf{V}$ with $W \cap U = S$. The subvariety of those $W$ with $W \cap U \supset S$ gives $G_{l-1,i} G_{k-(l-1),n-(l-1)}$ points; this is not $y_{l-1}$ since for each $S$ we have included those $W$ that contain $U$, instead of just meeting $U$ in a subspace of codimension 1. The correct value of $y_{l-1}$ is given by subtracting the contributions corresponding to points in $Y_l$, which gives

$$y_{l-1} = G_{l-1,i} (G_{k-(l-1),n-(l-1)} - G_{k-l,n-l}).$$

For the general $Y_j$ similar considerations apply. We summarize the results as follows. For $j = 1, \ldots, l$ let $U_j \subset \mathbb{F}^{n-j}$ be a fixed subspace of dimension $l - j$, and let $Z_j$ be the subvariety of the Grassmanian $\text{Gr}(k-j, n-j)(\mathbb{F}_p)$ of all $(k-j)$-dimensional subspaces $W$ such that $W \cap U_j = \{0\}$. Putting $z_j = \# Z_j$, we have

$$z_j = \begin{cases} G_{k-l,n-l} & j = l, \\ G_{k-j,n-j} - \sum_{i=1}^{l-j} G_{i,l-1} z_{i+j} & j < l. \end{cases}$$

Then

$$y_j = G_{j,l} z_j, \quad j = 1, \ldots, l,$$

and in particular

$$y_1 = G_{1,l} (G_{k-1,n-1} - G_{1,l-1} z_2 - G_{2,l-1} z_3 - \cdots - G_{l-1,l-1} z_l).$$

Finally, using (10) we see

$$[1]_p G_{1,l} G_{k-1,n-1} = [l]_p G_{k-1,n-1},$$

and

$$[j]_p G_{j,l} = [1]_p G_{1,l} G_{j-1,l-1}.$$  

Using (19) and (20) with (18) shows that the right of (17) telescopes to the left of (17), which proves (14). A simple computation obtains (15) from (14), and Lemma 3.3 is proved.

Lemmas 3.2 and 3.3 imply almost all of Theorem 1.4. Equations (12) and (14) imply (11), and the existence of the polynomial $\Phi_{n,k,l}$ from (5) is clear from (10) and (14). The only remaining statement is the positivity of the coefficients of $\Phi_{n,k,l}$. To see this, fix a complete flag in $\mathbf{V}$

$$\{0\} = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n = \mathbf{V},$$
where \( \dim U_j = j \). We define a polynomial \( \hat{\Phi}_{n,k} \in \mathbb{Z}[x_0, \ldots, x_n] \) by
\[
(21) \quad \hat{\Phi}_{n,k} = \sum_{W \subseteq V} \prod_{\dim W = k} x_j^{\dim W \cap U_j}.
\]
Clearly \( \Phi_{n,k,l}(p) \) is obtained from \( \hat{\Phi}_{n,k} \) by the substitutions \( x_l = p \) and \( x_j = 1 \) if \( j \neq l \). We claim \( \hat{\Phi}_{n,k} \) is a polynomial with positive coefficients. Indeed, the distinct monomials \( x^\alpha := \prod x_\alpha^j \) in (21) correspond to the different possibilities of intersections of \( W \) with the fixed flag, which correspond to the decomposition of \( \text{Gr}(n,k)(\mathbb{F}_p) \) into Schubert cells \( S_\alpha \) [10]. Thus we can rewrite (21) as
\[
\hat{\Phi}_{n,k} = \sum_\alpha \#S_\alpha(\mathbb{F}_p)x^\alpha.
\]
But each Schubert cell is isomorphic to an affine space, and hence the coefficients \( \#S_\alpha(\mathbb{F}_p) \) are pure \( p \)-powers. This completes the proof of Theorem 1.4.

**Remark 3.4.** We have the following additional geometric interpretation of the eigenvalue \( \nu_{n,k,l}(p) \). Let \( T \) be the total space of the rank \( n \) trivial bundle over \( G(k,n)(\mathbb{F}_p) \), and let \( T_l \subseteq T \) be the subbundle corresponding to a fixed \( l \)-dimensional subspace. Let \( B \) be the total space of the tautological bundle over \( G(k,n)(\mathbb{F}_p) \), i.e. for any \( x \in G(k,n)(\mathbb{F}_p) \) the fiber \( B_x \) over \( x \) is the \( k \)-dimensional subspace corresponding to \( x \). Then
\[
\nu_{n,k,l}(p) = \#(B \cap T_l).
\]

## 4. The Todd operator

4.1. In this section we describe the Todd operator \( \text{Td}(\Sigma, \partial/\partial h) \) and how it can be used to compute the Ehrhart polynomial of a simple lattice polytope \( P \). We closely follow [3].

Recall that \( \mathcal{F} \) is the set of faces of \( P \), and that each facet \( F \in \mathcal{F}(n-1) \) determines an affine hyperplane
\[
H_F = \{ x \mid \langle x, u_F \rangle + \lambda_F = 0 \},
\]
where the normal vector \( u_F \in L \) is a primitive vector pointing into the interior of \( P \).

Let \( h = (h_F)_{F \in \mathcal{F}(n-1)} \) be a real multivariable indexed by the facets of \( P \), and let \( P(h) \) be the convex region determined by the inequalities
\[
(22) \quad \{ \langle x, u_F \rangle + \lambda_F + h_F \geq 0 \mid F \in \mathcal{F}(n-1) \}.
\]
Note that \( P(0) = P \). Then \( P(h) \) is isomorphic to \( P \) for small \( h \), and thus for small \( h \) one can consider the volume \( \text{Vol} P(h) \). The following examples will play an important role in the proof of Theorem 1.8.
Example 4.2. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$, and let $e_0 = 0$. Let $P = \Delta_n$ be the convex hull of the vectors $\{e_0, \ldots, e_n\}$. Then $\Delta_n$ is the $n$-dimensional simplex. Let $h_i$ be the parameter attached to the facet obtained by deleting the vertex $e_i$. It is easy to check that

$$\text{Vol } \Delta_n(h) = \left(1 + \sum_{i=0}^{n} h_i\right)^n / n!.$$ 

Example 4.3. Let $P$ and $P'$ be two lattice polytopes, and let $h$ and $h'$ be multivari-ables indexed by their facets. Then

$$\text{Vol}(P \times P')(h, h') = \text{Vol } P(h) \text{ Vol } P'(h').$$

In particular, for the unit $n$-cube $P = (\Delta_1)^n$ we obtain

$$\text{Vol } P(h) = \prod_{i=1}^{n} (1 + h_i + h'_i).$$

4.4. Let $\Sigma$ be the normal fan to $P$. For any $\sigma \in \Sigma$, define

$$Q(\sigma) = \{ \sum_{s \in \sigma(1)} \rho_s s \mid 0 \leq \rho_s < 1 \}.$$ 

Note that $\text{Vol } Q(\sigma) = \text{Ind } \sigma$, and $Q(\sigma) \cap U(\sigma) = \{0\}$ if and only if $\sigma$ is unimodular. Put

$$\Gamma_\Sigma = \bigcup_{f \in \mathcal{F}} Q(\sigma_f) \cap L.$$ 

We have $\Gamma_\Sigma = \{0\}$ if and only if $P$ is nonsingular.

4.5. For each $F \in \mathcal{F}(n-1)$, let $\xi_F : V \to \mathbb{R}$ be the unique piecewise-linear continuous function defined by

- $\xi_F(s) = 1$ if $s \in \Sigma(1)$ is the spanning point corresponding to $F$,
- $\xi_F(s') = 0$ for all other $s' \in \Sigma(1)$, and
- $\xi_F$ is linear on all the cones of $\Sigma$.

Put $a_F(x) = \exp(2\pi i \xi_F(x))$ for all $x \in V$.

Suppose $g \in \Gamma_\Sigma \cap \sigma$. Then the pair $(g, \sigma)$ determines a tuple of roots of unity as follows. If $s_1, \ldots, s_l$ are the spanning points of $\sigma$, and $F_1, \ldots, F_l$ are the corresponding facets, then we can attach to $(g, \sigma)$ the tuple $(a_1(g), \ldots, a_l(g))$, where we have written $a_i$ for $a_{F_i}$.

4.6. Let $a$ be a complex number and $x$ a real variable. We define $T_d(a, \partial/\partial x)$ to be the differential operator given formally by the power series

$$\frac{\partial/\partial x}{1 - a \exp(-\partial/\partial x)} = \sum_{k=0}^{\infty} c(a, k) \left( \frac{\partial}{\partial x} \right)^k.$$
Note that \( c(1, k) = B_k/k! \), where \( B_k \) is the \( k \)th Bernoulli number.\(^1\) If \( a \neq 1 \), then \( c(a, k) \) is a rational function in \( a \) of degree \(-1\) closely related to the \( k \)th circle function of Euler (§6.2). Table 2 gives some examples of the \( c(a, k) \).

\[
\begin{array}{c|c}
 k & c(a, k) \\
\hline
 1 & -1/(a - 1) \\
 2 & -a/(a^2 - 2a + 1) \\
 3 & -(a^2 + a)/(2a^3 - 6a^2 + 6a - 2) \\
 4 & -(a^3 + 4a^2 + a)/(6a^4 - 24a^3 + 36a^2 - 24a + 6) \\
\end{array}
\]

Table 2. The coefficients \( c(a, k) \).

4.7. Now let \( h \) be a multivariable with components \( h_F \) indexed by the facets of \( P \). Let \( g \in \Gamma_{\Sigma} \), and define

\[
Td(g, \partial/\partial h) = \prod_{F \in \mathcal{F}(n-1)} Td(a_F(g), \partial/\partial h_F)
\]

and

\[
(23) \quad Td(\Sigma, \partial/\partial h) = \sum_{g \in \Gamma_{\Sigma}} Td(g, \partial/\partial h).
\]

We have the following theorem, proved by Khovanskii-Pukhlikov if \( P \) is nonsingular, and by Brion-Vergne for general simple lattice polytopes.

**Theorem 4.8.** [3, 16] Suppose \( P \) is a simple lattice polytope. Then the coefficients of the Ehrhart polynomial \( E_P(t) = \sum_{i=0}^{n} c_i t^i \) are given by

\[
c_{n-l} = Td_l(\Sigma, \partial/\partial h) \ Vol P(h) \big|_{h=0},
\]

where \( Td_l(\Sigma, \partial/\partial h) \) is the degree \( l \) part of \( Td(\Sigma, \partial/\partial h) \).

For the connection between coefficients of the Todd operator and higher-dimensional Dedekind sums, we refer to [1, §9].

5. **Proof of Theorem 4.8**

5.1. We recall some notation from §1.7. Let \( f \in \mathcal{F} \) be a face of codimension \( \leq l \), and let \( \pi = (\pi(F))_{F \supset f} \) be an ordered partition of \( l \) indexed by the facets containing \( f \). We expand (23) as a sum over pairs

\[
Td_l(\Sigma, \partial/\partial h) = \sum_{(f, \pi)} A(f, \pi) \partial^\pi_f,
\]

\(^1\)With our conventions the Bernoulli numbers are \( B_1 = 1/2, B_2 = 1/6, B_4 = -1/30, \ldots, \) and \( B_{2k-1} = 0 \) for \( k > 1 \). Note that for many authors \( B_1 = -1/2 \), cf. [6.2].
where
\[
\partial_f^\pi = \prod_{F \supset f} \left( \partial / \partial h_F \right)^{\pi(F)}
\]
and
\[
A(f, \pi) = \sum_{g \in \Gamma \cap \sigma_f} \prod_{F \supset f} c(a_F(g), \pi(F)).
\]

Note that if \(\sigma_f\) is unimodular, then
\[
A(f, \pi) = \prod_{F \supset f} \frac{B_{\pi(F)}}{\pi(F)!}.
\]

5.2. Now fix a total ordering on (unordered) partitions of \(l\) by using the lexicographic order. In other words, let \(\pi = \{\pi_1, \ldots, \pi_j\}\) and \(\pi' = \{\pi'_1, \ldots, \pi'_k\}\) be two partitions of \(l\) with parts arranged in nonincreasing order. Then we have \(\pi < \pi'\) if and only if there exists an index \(m\) with \(\pi_i = \pi'_i\) for \(i < m\) and \(\pi_i < \pi'_i\) for \(i \geq m\). For example, if \(l = 6\), then in increasing order (and in obvious notation) the partitions are
\[
1^6, 21^4, 2^21^2, 31^3, 321, 3^2, 41^2, 42, 51, 6.
\]

5.3. We say the pair \((f, \pi)\) is squarefree if \(\pi(F) = 1\) for all \(F \supset f\), and we write \(\pi = 1\). We begin with two lemmas. Lemma 5.4 gives a geometric interpretation of the squarefree terms, and Lemma 5.6 allows us to compute nonsquarefree terms using squarefree terms.

**Lemma 5.4.** Let \(P\) be simple. For any face \(f \in \mathcal{F}\), we have
\[
\partial_f^1 \Vol_P(h) \big|_{h=0} = \frac{\Vol f}{\Ind \sigma_f}.
\]
In particular, if \(P\) is nonsingular and \(f\) has codimension \(l\), then
\[
A(f, 1) \partial_f^1 \Vol_P(h) \big|_{h=0} = \frac{\Vol f}{2^l}.
\]

**Proof.** The first statement is Lemma 4.7 in [3]. The second statement follows from \([25]\) since the Bernoulli number \(B_1\) is \(1/2\), and \(\Ind \sigma_f = 1\) if \(P\) is nonsingular.

5.5. The following result is well known to experts, and is stated (for nonsingular \(P\)) in [16, Theorem, p. 795]. For the convenience of the reader we present a proof for \(P\) simple. For unexplained concepts from toric geometry, we refer to [9]. What we will need from Lemma 5.6 is \([25]\).

**Lemma 5.6.** [2] Let \(X\) be the projective toric variety associated to the simple lattice polytope \(P\). Then the rational Chow ring \(H^*(X, \mathbb{Q})\) is isomorphic to the quotient of
\[
\mathbb{Q} [\partial / \partial h_F \mid F \in \mathcal{F}(n-1)]
\]
by the ideal \(I\) of differential operators that annihilate the function \(\Vol P(h)\).
Proof. The rational Chow ring $H^*(X, \mathbb{Q})$ has generators the classes of the divisors $[D_F], F \in \mathcal{F}(n-1)$, and the following relations:

- square-free monomial relations $\prod_{F \in I} [D_F] = 0$ unless the facets in $I$ intersect transversally along a face of $P$, and
- linear relations $\sum_F \langle w, u_F \rangle [D_F] = 0$, where $w \in L$.

But the analogous relations hold for $\mathbb{Q}[\partial/\partial h_F | F \in \mathcal{F}(n-1)]$ applied to $\text{Vol}_P(h)$; for example, the linear relations express invariance of volume under translation. Thus, we obtain a surjective homomorphism of graded rings

$$H^*(X, \mathbb{Q}) \longrightarrow \mathbb{Q}[\partial/\partial h_F | F \in \mathcal{F}(n-1)]/I, \quad [D_F] \mapsto \partial/\partial h_F,$$

where the $\partial/\partial h_F$ have degree 2. To show its injectivity, it is enough (by Poincare duality) to show that all intersection numbers of the form $[D_{F_1}] \cdots [D_{F_n}]$ can be read off the images of the $D_F$. But this follows from the formula

$$(\sum_F (\lambda_F + h_F)[D_F])^n = \text{Vol}_P(h),$$

where the $\lambda_F$ come from the inequalities determining $P(h)$. Indeed, since the $h_F$ are independent variables, any monomial of degree $n$ in the $[D_F]$ can be expressed in terms of partial derivatives of $\text{Vol}_P(h)$. \hfill \square

5.7. Let $w \in L$. Then by Lemma 5.6 the differential operator

$$(26) \quad \sum_{F \in \mathcal{F}(n-1)} \langle w, u_F \rangle \partial/\partial h_F$$

annihilates $\text{Vol}_P(h)$. Hence if $\pi > 1$, by repeatedly applying (26) we can write

$$(27) \quad \partial_\pi^f \text{Vol}_P(h) = \varepsilon_f(\pi) \sum_{f'} \left( \prod_{w \in W(f')} \langle w, u_w \rangle \right) \partial_1^{f'} \text{Vol}_P(h),$$

where the quantities in (27) satisfy the following:

- The integer $\varepsilon_f(\pi) \in \{\pm 1\}$ depends only on the pair $(f, \pi)$;
- The sum ranges over a finite set of codimension $l$ faces $f'$, each of which is contained in $f$;
- For each $f'$, the set $W(f') \subset L \otimes \mathbb{Q}$ satisfies
  - $\langle w, u \rangle = 1$ for some $u \in \sigma_f(1)$,
  - $\langle w, v \rangle = 0$ for all $v \in \sigma_f(1) \setminus \{u\}$;
- The $\{u_w\} \subset \Sigma(1)$ are such that for each $f'$, we have
  $$\sigma_{f'}(1) = \sigma_f(1) \cup \{u_w\}_{w \in W(f')}$$
- The sets $W(f')$ are ordered and
  $$\langle w', u_w \rangle = 0 \quad \text{for all } w < w'.$$

We choose and fix an expression of the form (27) for each pair $(f, \pi)$.\hfill \square
5.8. We are now ready to prove Theorem 1.8. Our goal is to show
\begin{equation}
\sum_{M \in \mathcal{L}_k} A(f_M, \pi) \partial_{\pi}^f \text{Vol } P_M(h) = \nu_{n,k,n-l}(p) A(f, \pi) \text{Vol } P(h).
\end{equation}

Let \( \nu(p) = \nu_{n,k,n-l}(p) \). Applying (27) in (28) and using Lemma 5.4, we see that it suffices to verify
\begin{equation}
\sum_{M \in \mathcal{L}_k} A(f_M, \pi) \sum_{f'_M} \left( \prod_{w \in W(f'_M)} \langle w, u_w \rangle \right) \frac{\text{Vol } f'_M}{\text{Ind } \sigma_{f'_M}} = \nu(p) A(f, \pi) \sum_{f'} \left( \prod_{w \in W(f')} \langle w, u_w \rangle \right) \text{Vol } f'.
\end{equation}

Since the faces \( f' \) appearing in (27) are independent of the lattice \( M \), we can interchange the sum over \( \mathcal{L}_k \) and the sum over \( f'_M \), and focus on a single \( f' \). Furthermore, Lemma 2.5 implies that the sum over \( M \) in (29) is really a sum over \( \text{Gr}(k, n)(\mathbb{F}_p) \).

We construct a stratification \( \{X_{ij}\} \) of \( \text{Gr}(k, n)(\mathbb{F}_p) \) by defining
\begin{equation}
X_{ij} = \{W \subset V | \dim W = k, \dim W \cap V_f = i, \dim W \cap \overline{C}_f = j\},
\end{equation}
and the left of (29) becomes
\begin{equation}
\sum_{i,j} \sum_{M \in X_{ij}} A(f_M, \pi) \left( \prod_{w \in W(f'_M)} \langle w, u_w \rangle \right) \frac{\text{Vol } f'_M}{\text{Ind } \sigma_{f'_M}}.
\end{equation}

Now let \( S_j \subset \overline{C}_f \) be a fixed subspace of dimension \( j \), and put
\begin{equation}
m_{ij} = \# \{M \in X_{ij} | M \supset S_j\}.
\end{equation}
The number \( m_{ij} \) is independent of the choice of \( S_j \). If \( \overline{M} \in X_{ij} \), then
\begin{equation}
\text{Vol } f'_M = p^i \text{Vol } f',
\end{equation}
and equation (29) becomes
\begin{equation}
\sum_{i,j} p^i m_{ij} \sum_{S \subset \overline{C}_f} A(f_S, \pi) (\text{Ind } \sigma_{f_S})^{-1} \left( \prod_{w \in W(f'_S)} \langle w, u_w \rangle \right)
= \nu(p) A(f, \pi) \prod_{w \in W(f')} \langle w, u_w \rangle,
\end{equation}
where we have written
\begin{equation}
A(f_S, \pi) = \sum_{g \in W(\sigma_{f_S})} \prod_{F \supset f} c(a_F(g), \pi(F)).
\end{equation}
Note that it makes sense to replace the subscript \( M \) with \( S \) in (33) and (34), since \( \text{Ind } \sigma_{f_M} \) (respectively \( \Gamma \cap \sigma_{f_M} \)) depends only on \( S = M \cap \overline{C}_f \) (resp., \( \overline{C}_f \)). The notation
$W(f'_S)$ also makes sense, because all points in $W(f'_M)$ are multiples of points in $W(f')$ (in fact they differ at most by a factor of $p$), and which multiples we take depend only on $S$.

To verify (33), we show that for each $j$ the identity

\[(35) \sum_{S \subset C \text{ dim } S = j} A(f, \pi) \prod_{w \in W(f'_S)} \langle w, u_w \rangle \left( \Ind_{\sigma f'_S} \right)^{-1} = G_{j,l} A(f, \pi) \sum_{j'} \prod_{w \in W(f')} \langle w, u_{w'} \rangle \]

holds. This will complete the proof of the theorem, since

\[\sum_{i,j} p^i m_{ij} G_{j,l} = \nu(p).\]

We verify (35) by induction on the partition order; the main idea is to show that (35) appears in the computation of the constant term of $T(p, j) E(P)$ for some easily understood polytope $P$. Since we know how the constant terms transform under the Hecke operators, our identity is forced to hold. In particular, let

\[P = \prod_{F \supset \Delta} \Delta_{\pi(F)},\]

where in the product the facets $F$ are ordered so that $\pi$ has nonincreasing parts. Using Examples 4.2 and 4.3, we see that the highest order terms contributing to $E(P)$ and $T(p, j) E(P)$ are those of type $(f, \pi)$, where $f$ is a vertex. Now assume that all weight $l$ terms of type $(f, \pi')$, with $\pi' < \pi$ satisfy (35). Since the constant term of $T(p, j) E(P)$ equals $G_{j,l}$, and since each vertex of $P$ contributes equally to the constant term, this implies (35).

Hence to complete the proof, we must check (35) in the case $\pi = 1$. In this case we don’t need to apply (27), since the terms are already squarefree. Using (25), the identity to be proved is

\[(36) \sum_{S \subset C \text{ dim } S = j} (\Ind_{\sigma f'_S})^{-1} A(f, 1) = \frac{G_{j,l}}{2^l}.\]

To prove (36), we let $P = (\Delta_1)^l$ and consider the action of $T(p, j)$ on the constant term of its Ehrhart polynomial. By Example 4.3 we have

\[\text{Vol } P(h) = \prod_{i=1}^l (1 + h_i + h'_i).\]

We see from applying $\text{Td}_l$ to $\text{Vol } P(h)$ that only squarefree terms contribute to the constant term of $E(P)$, and that this contribution is the same for all vertices of $P$ (in fact it’s $2^{-l}$). Moreover, using the matrices given in Lemma 2.5 it’s easy to see
that only squarefree terms contribute to the constant term of $T(p, j)E(P)$, and that the contribution for any vertex $f$ is equal to

\[(37) \sum_{M \in \mathcal{L}} (\text{Ind } \sigma_M)^{-1} A(f_M, 1).\]

But under $T(p, j)$ the constant term of $E(P)$ is multiplied by $G_{j, l}$, and because the contribution of each vertex is the same, we have that (37) equals $G_{j, l}/2^l$. This completes the proof of (36), and the proof of Theorem 1.8.

**Remark 5.9.** We expect that Theorem 1.8 holds if $P$ is replaced by a general simple lattice polytope, although the argument presented here doesn’t prove this. In fact, Theorem 1.4 suggests that the analogous result for a general lattice polytope should hold, and indeed for the vector partition functions studied in [4].

**Remark 5.10.** The role of the polytopes $\prod_{F \supset J} \Delta_{\pi(F)}$ in the proof of Theorem 1.8 is very similar to the role of “basis sequences” in the theory of characteristic classes and genera, cf. [12, p. 79]. This is not a coincidence, since the machine behind the computation of $c_l$ in Theorem 1.8 is the Hirzebruch-Kawasaki-Riemann-Roch theorem.

### 6. Examples of distribution relations

6.1. In this final section, we give examples of the identities appearing in the proof of Theorem 1.8 and directly prove them by exhibiting their connection with special values of the Hurwitz zeta function.

6.2. Let $u$ be a real number, and let $k$ be a positive integer. Consider the special value of the (symmetrized) Hurwitz zeta function

$$\zeta(k, u) = \sum_{m \in \mathbb{Z}}' \frac{1}{(m + u)^k}.$$ 

Here the prime next to the summation means to omit the meaningless term that arises when $u \in \mathbb{Z}$. The series is absolutely convergent unless $k = 1$, in which case we define the value of $\zeta(1, u)$ to be the limit of the partial sums with $|m| < C$ as $C \to \infty$. Define the circle functions $\theta_k(u)$ by the series expansion

$$\frac{z}{\exp(z - 2\pi i u) - 1} = \sum_{k=0}^{\infty} \theta_k(u) \frac{z^k}{k!}.$$ 

If $u > 0$ and $k > 1$, then $\theta_k(0) = B_k$, the $k$th Bernoulli number as in (1.6). However, note that $c_1(0) = -B_1$. 
By a result of Euler, we have for all $u$

$$\zeta(k, u) = \begin{cases} 
-\frac{(2\pi i)^k}{k!} \theta_k(u) & k > 1, \\
-\frac{(2\pi i)^k}{k!} (\theta_k(u) + \frac{1}{2}) & k = 1.
\end{cases}$$

6.3. Now fix a positive integer $n$, and suppose $k > 1$. It is easy to see that

$$\sum_{j=0}^{n-1} \zeta(k, j/n) = n^k \zeta(k, 0).$$

Using (38), this becomes

$$\sum_{j=1}^{n-1} \theta_k(j/n) = (n^k - 1) B_k.$$

Comparing the definition of $c(a, k)$ from §4.6 yields

$$c(a, k) = \frac{(-1)^k}{k!} \theta_k(u), \quad a = \exp(-2\pi i u),$$

which in (39) gives

$$\sum_{j=1}^{n-1} c(\omega^j, k) = \frac{n^k - 1}{k!} B_k, \quad k > 1.$$

Here we have written $\omega = \exp(2\pi i/n)$ and used the fact that the sum on the left of (40) is real. In fact, (40) remains true if we take $k = 1$.

6.4. Let now $P$ be a 3-dimensional nonsingular lattice polytope; we investigate the computation of $T(p, 1)$ on $c_1$. We focus on the squarefree case, since no Dedekind sums arise in the nonsquarefree case.

So let $f$ be an edge of $P$. The key identity (28) becomes

$$\sum_{M \in \mathcal{L}_1} A(f_M, 1) \frac{\text{Vol} f_M}{\text{Ind} \sigma_{f_M}} = \frac{p^2 + 2p}{4} \text{Vol} f.$$

We break the coefficient $A = A(f_M, 1)$ into two parts

$$A = A_{ns} + A_a,$$

where $A_{ns}$ corresponds to $g = 0$ in (24), and $A_a$ corresponds to $g \neq 0$. The latter term appears only if $\text{Ind} \sigma_{f_M} \neq 1$. Note that $A_{ns} = \frac{1}{4}$.

To analyze the left of (41), we use Proposition 2.7. Figure 1 shows $\mathcal{V}$ with the two subspaces $\mathcal{V}_f$ and $\mathcal{C}_f$. The subspaces $\mathcal{C}_1$ and $\mathcal{C}_2$ are the 1-dimensional subspaces corresponding to the two facets containing $f$. For simplicity, we draw these subspaces, and the subspaces that follow, by drawing their images in $\mathbb{P}(\mathcal{V}) = \mathbb{P}^2(\mathbb{F}_p)$. By abuse
of notation, we denote a subspace of \( \overrightarrow{V} \) and the subspace it induces in \( \mathbb{P}(\overrightarrow{V}) \) by the same symbol.

\[
\begin{figure}
\centering
\begin{tikzpicture}
\draw (-1,0) -- (0,1) -- (1,0) -- cycle;
\filldraw[black] (0,1) circle (2pt);
\filldraw[black] (-1,0) circle (2pt);
\filldraw[black] (1,0) circle (2pt);
\draw[->] (0,1) -- (-1,0);
\draw[->] (0,1) -- (1,0);
\node at (0,1) {$\mathcal{C}_2$};
\node at (-1,0) {$\mathcal{C}_1$};
\node at (1,0) {$\mathcal{C}_f$};
\node at (0,-1) {$\overrightarrow{V}_f$};
\end{tikzpicture}
\caption{Subspaces in \( \overrightarrow{V} \) for an edge in a 3-dimensional polytope.}
\end{figure}
\]

Each \( M \in \mathcal{L}_1 \) corresponds to a point \( \overrightarrow{M} \in \mathbb{P}(\overrightarrow{V}) \). By Proposition 2.7 we have \( \text{Vol}_{\overrightarrow{f}} M = \text{Vol}_f \) unless \( \overrightarrow{M} = \overrightarrow{V}_f \), in which case \( \text{Vol}_{\overrightarrow{f}} M = p \text{Vol}_f \). Also \( A_s = 0 \) unless \( \overrightarrow{M} \) meets \( \mathcal{C}_f \setminus \{ \mathcal{C}_1 \cup \mathcal{C}_2 \} \). Hence there are \( p - 1 \) nonzero \( A_s \), and since \( c(a, 1) = 1/(1 - a) \) each nonzero \( A_s \) has the form

\[
A_s(\alpha, \beta) = \frac{1}{(1 - \omega^\alpha)(1 - \omega^\beta)}, \quad \omega = \exp(2\pi i/p),
\]

for some nonzero integers \( 1 \leq \alpha, \beta \leq p - 1 \). The value of \( A_s(\alpha, \beta) \) depends only on the point \( [\alpha : \beta] \in \mathbb{P}^1(\mathbb{F}_p) \). See Figure 2 for the four nonzero \( A_s(\alpha, \beta) \) when \( p = 5 \). The pairs \( (\alpha, \beta) \) are given below each lattice, and the four terms in \( A_s(\alpha, \beta) \) correspond to the four grey dots.

By (40), the contribution from the singular Hecke images is

\[
\sum_{[\alpha : \beta] \in \mathbb{P}^1(\mathbb{F}_p)} A_s(\alpha, \beta) = \sum_{i,j=1}^{p-1} \frac{1}{(1 - \omega^i)(1 - \omega^j)} = \frac{(p - 1)^2}{4}.
\]

With this in hand it is easy to complete the analysis of (41). We break \( \mathbb{P}(\overrightarrow{V}) \) into four disjoint subsets

\[
\mathbb{P}(\overrightarrow{V}) = S_1 \cup S_2 \cup S_3 \cup S_4,
\]

where

- \( S_1 = \overrightarrow{V}_f \),
- \( S_2 = \overrightarrow{C}_1 \cup \overrightarrow{C}_2 \),
- \( S_3 = \overrightarrow{C}_f \setminus S_2 \), and
- \( S_4 = \mathbb{P}(\overrightarrow{V}) \setminus \{ S_1 \cup S_2 \cup S_3 \} \).

The relevant contributions are given in Table 3 and one easily sees that (41) holds.
6.5. The computation of \( T(p, 2) \) on \( c_1 \) is similar. The only difference is that the sum over \( M \) corresponds to a sum over lines in \( \mathbb{P}(V) \), and that we obtain a nonzero \( A_s \) exactly when a line meets \( S_3 \) in a point. For example, in Figure 3 a nonzero \( A_s(\alpha, \beta) \) arises from the solid triangle. Hence each nonzero \( A_s(\alpha, \beta) \) occurs with multiplicity \( p \). Taking this into account, as well as which lines meet \( V_f \), yields

\[
\sum_{M \in \mathcal{L}_2} A(f_M, 1) \frac{\text{Vol } f_M}{\text{Ind } \sigma_{f_M}} = \frac{2p^2 + p}{4} \text{Vol } f.
\]

| \( S_i \) | \( \#S_i \) | \( \frac{\text{Vol } f_M}{\text{Vol } f} \) | \( \frac{\text{Ind } \sigma_f}{\text{Ind } \sigma_{f_M}} \) | \( \sum_{M \in S_i} A(f_M, 1) \) |
|-------|-------|----------------|----------------|----------------|
| \( S_1 \) | 1     | \( p \)       | 1              | \( \frac{1}{4} \) |
| \( S_2 \) | 2     | 1             | 1              | \( \frac{1}{2} \) |
| \( S_3 \) | \( p^2 - 1 \) | 1             | 1              | \( \frac{(p^2 - 1)}{4} \) |
| \( S_4 \) | \( p - 1 \)  | 1             | \( \frac{1}{p} \) | \( \frac{(p^2 - 1 + p - 1)}{4p} \) |

Table 3. Summary of \( T(p, 1) \) on \( c_1 \) for a 3-dimensional polytope.

Figure 2. Four superlattices giving a nonzero \( A_s(\alpha, \beta) \).

Figure 3. Computing \( T(p, 2) \) on \( c_1 \).
7. The regularized Ehrhart polynomial on average

7.1. Let \( P \) be a fixed \( n \)-dimensional lattice polytope respect to the lattice \( L \). We can define a “regularized” version \( \tilde{E}(P) \) of \( E(P) \) by

\[
\tilde{E}(P)(t) := E(P)(t) - \text{Vol}(P)t^n.
\]

Suppose \( \mathcal{M} \) is a finite set superlattices of \( L \) of finite coindex. We can define the average regularized Ehrhart polynomial of \( P \) with respect to the family \( \mathcal{M} \) by

\[
\tilde{E}_{\text{avg}}(P, \mathcal{M}) = \frac{1}{\# \mathcal{M}} \sum_{M \in \mathcal{M}} \tilde{E}(P_M).
\]

Our goal in this section is to show how Theorem 1.4 can be used to derive limiting formulas for \( \tilde{E}_{\text{avg}}(P, \mathcal{M}) \) as \( \mathcal{M} \) ranges over families of superlattices satisfying certain arithmetical conditions.

7.2. As a first example, fix a prime \( p \), and suppose \( \mathcal{M} = \mathcal{L}_1(p) \) consists of all superlattices of \( L \) of coindex \( p \). Then by definition

\[
\tilde{E}_{\text{avg}}(P, \mathcal{M}) = G_{1,n}^{-1} \sum_{l=0}^{n-1} T(p, 1)c_l t^l
\]

\[
= G_{1,n}^{-1} \sum_{l=0}^{n-1} \nu_{n,1,l}(p)c_l t^l.
\]

By Lemma 3.2 we have

\[
\nu_{n,1,l}(p) = G_{1,n} + p^l - 1 = p^n + \cdots + p^{l+1} + 2p^l + p^{l-1} + \cdots + p.
\]

This implies the following result:

**Proposition 7.3.**

\[
\lim_{p \to \infty} \tilde{E}_{\text{avg}}(P, \mathcal{L}_1(p)) = 2c_{n-1} t^{n-1} + c_{n-2} t^{n-2} + \cdots + c_1 t + 1.
\]

7.4. We can use the relations in the Hecke algebra to derive similar results for more general sets of superlattices. Let \( T_p(n, k) \) be the operator \( T(n, k) \) at the prime \( p \), and write \( T(N) \) for the operator that associates to any lattice \( L \) the set of superlattices of coindex \( N \). Suppose \( N \) has prime factorization \( \prod p_j^{e_j} \). Then, in the algebra \( \mathcal{H} \) generated by the \( T_p(n, k) \) as \( p \) ranges over all primes \( p \), we have [17, Theorem 3.21]

\[
T(N) = \prod T(p_j^{e_j}),
\]

and the operators \( T(p^e) \) satisfy the (formal) identity

\[
\sum_{e=0}^{\infty} T(p^e)X^e = \left( \sum_{i=0}^{n} (-1)^i p^{i(i-1)/2} T_p(n, k)X^i \right)^{-1}.
\]
As an example of this, suppose $\mathcal{M}(p^2)$ is the set of all superlattices of $L$ of coindex $p^2$. Note that $\mathcal{M}(p^2) \not= \mathcal{L}_2$, i.e. $T(p^2) \not= T_p(n, 2)$. In fact in $\mathcal{H}$ we have the relation

$$T(p^2) = T_p(n, 1)^2 - p T_p(n, 2).$$

One can easily show

$$\#\mathcal{M}(p^2) = G_{1,n}^2 - p G_{2,n} = G_{2,n+1},$$

and then from Lemma 3.2 we find the following:

**Proposition 7.5.**

$$\lim_{p \to \infty} \overline{E}_{\text{avg}}(P, \mathcal{M}(p^2)) = 3c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \cdots + c_1 t + 1.$$

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