Nonparametric Pattern-Mixture Models for Inference with Missing Data

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Abstract

Pattern-mixture models provide a transparent approach for handling missing data, where the full-data distribution is factorized in a way that explicitly shows the parts that can be estimated from observed data alone, and the parts that require identifying restrictions. We introduce a nonparametric estimator of the full-data distribution based on the pattern-mixture model factorization. Our approach uses the empirical observed-data distribution and augments it with a nonparametric estimator of the missing-data distributions under a given identifying restriction. Our results apply to a large class of donor-based identifying restrictions that encompasses commonly used ones and can handle both monotone and nonmonotone missingness. We propose a Monte Carlo procedure to derive point estimates of functionals of interest, and the bootstrap to construct confidence intervals.

Keywords: Bootstrap; Missingness mechanism; Nonignorable nonresponse; Nonparametric identification; Nonparametric inference.

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1 Introduction

Statistical inference with missing data requires handling the joint distribution of the study variables and their missingness indicators (Rubin, 1976). Two natural ways of factorizing this distribution arise. The most popular approach is the selection model factorization, where we work with the product of the marginal distribution of the study variables and the missingness mechanism. The popularity of this approach is at least partly due to the fact that analysts are able to use the same, typically parametric, models that they would use with complete data, and that they do not have to handle the missingness mechanism under the assumption of ignorability (e.g., Little & Rubin, 2002, Chapter 6.2). Nevertheless, selection models are difficult to work with under nonignorable missing data, as ensuring that they are identifiable is difficult in general (e.g., Sadinle & Reiter, 2019) and it often heavily relies on parametric assumptions (e.g., Daniels & Hogan, 2008, p. 107).

Pattern-mixture models offer an alternative, where the full-data distribution is factorized as the distribution of the observed data times the conditional distribution of the missing data given the observed data (Little, 1993). This factorization shows that the full-data distribution is not identifiable, and it clearly separates what can and cannot be recovered from observed data alone. Specifically, the observed-data distribution can be directly estimated from the observed data, but the missing-data distribution is only obtainable after imposing identifying restrictions that tie the full-data distribution to the observed-data distribution. Despite this transparent separation of what one can and cannot recover from data alone, implementations of pattern-mixture models have predominantly been done under parametric assumptions for the observed-data distribution (e.g., Little, 1993, 1994, 1995; Hogan & Laird, 1997; Daniels & Hogan, 2000; Fitzmaurice, 2003; Kenward et al., 2003; Roy &
Daniels, 2008; Wang & Daniels, 2011), although more flexible Bayesian nonparametric modeling approaches have recently been developed (Linero & Daniels, 2015; Linero, 2017; Linero & Daniels, 2018). Existing approaches also mainly focus on handling monotone missingness in longitudinal studies.

In this article we propose a methodology for nonparametric inference under pattern-mixture models. We show how inferences can be derived under a very general class of identifying restrictions that can handle both monotone and nonmonotone missingness. Our proposed procedure guarantees that inferences only depend on the observed data and a given identifying restriction.

Main Contributions.

1. We introduce the concept of donor-based identification in Section 2, as a way of unifying different identification restrictions commonly used to handle monotone missingness. The application of pattern-mixture models to nonmonotone nonresponse has been prevented partly due to the lack of identifying restrictions applicable to that more general scenario. Here we also extend donor-based identification to handle nonmonotone missingness.

2. We propose nonparametric estimators of the full-data distribution, where the part corresponding to the observed-data distribution is estimated using its empirical version, and the missing-data distributions are estimated under identifying restrictions using a surrogate estimate of the observed-data distribution that relies on conditional kernel-density estimators (Section 3.1).

3. We show how to estimate statistical functionals of interest (Section 3.2), where the point estimator is obtained numerically using a Monte Carlo approach (Section 3.2.1), which we prove to be the valid (Theorem 4).
4. We introduce a bootstrap approach for constructing confidence intervals for statistical functionals of interest (Section 4) and prove its validity (Theorem 6).

5. Finally, we use Efron’s bootstrap diagram to make explicit the estimand that our method is inferring (Section 5.4). This diagram helps clarify the validity of our procedures even under misspecified identifying assumptions.

Outline. We give a general introduction to pattern-mixture models, and present some generalizations of commonly used identifying restrictions in Section 2. We introduce our nonparametric estimator along with a Monte Carlo approach for parameter estimation in Section 3. In Section 4, we explain how one can use the bootstrap approach to construct valid confidence intervals. Theoretical results, including convergence rates, asymptotic normality, validity of Monte Carlo methods, and the validity of bootstrap methods, are given in Section 5. We provide a data analysis in Section 6. Finally, we give a short discussion about future directions in Section 7.

2 Pattern-Mixture Models

2.1 Setup

We consider $d$ study variables $X = (X_1, \ldots, X_d)$, taking values on a sample space $\mathcal{X} = \otimes_{j=1}^{d} \mathcal{X}_j$, and its vector of response indicators $R = (R_1, \ldots, R_d)$, taking values on $\mathcal{R} \subseteq \{0, 1\}^d$, with $R_j = 0$ when variable $j$ is missing and $R_j = 1$ when it is observed. An element $r = (r_1, \ldots, r_d) \in \{0, 1\}^d$ is called a response pattern, which we will often represent as the string $r_1 \ldots r_d$. Given $r \in \mathcal{R}$, we define $\bar{r} = 1_d - r$ to be the missingness pattern, where $1_d = (1, 1, \cdots, 1)$ is a vector of ones of length $d$. For a response pattern $r$, we define $X_r = (X_j : r_j = 0)$ to be the missing variables
and $X_r = (X_j : r_j = 1)$ to be the observed variables, which have sample spaces $X_r$ and $X_r^c$, respectively. For example, if $X = (X_1, X_2, X_3)$ and $r = 101$, that is, the realizations of random variables $X_1$ and $X_3$ are observed, then $X_{101} = (X_1, X_3)$ and $X_{101}^c = X_{010} = X_2$.

We refer to the true joint distribution $F$ of $X$ and $R$ as the full-data distribution, and denote the collection of full-data distributions by $\mathcal{F}$. We denote the distribution of $X$ given $R = r$ by $F_r$. We write $f(x \mid r)$ for the density of $F_r$ with respect to an appropriate dominating measure $\mu$, and $f(x, r)$ for the density of $F$ with respect to the product of $\mu$ and the counting measure on $\{0, 1\}^d$. $F$ cannot be estimated from observed data alone, since when $R = r$ we only get to see the realization of $X_r$. The observed-data distribution $G$ involves the response indicators and the corresponding observed study variables, with density derived as $g(x_r, r) = \int_{X_r} f(x, r) \mu(dx_r)$. We denote the collection of observed-data distributions by $\mathcal{G}$. Throughout the document we more generally use $g$ to denote density functions that can be directly derived from $G$, and $f$ for density functions that more generally depend on $F$ and not exclusively on $G$. Namely, quantities denoted using $g$ can be derived from the observed-data distribution alone, whereas quantities denoted using $f$ will depend on identifying assumptions. For example, we write

$$f(x, r) = g(r)f(x \mid r) = g(x_r, r)f(x_r \mid x_r, r),$$

where $g(r) = \int_{X_r} g(x_r, r) \mu(dx_r)$, and $f(x_r \mid x_r, r)$ is the density of the distribution of $X_r \mid X_r = x_r, R = r$, which represents the non-identifiable parts of the full-data distribution, referred to as the extrapolation or missing-data distribution (Daniels & Hogan, 2008, p. 166).

Little (1993) refers to models based on the factorization given by (1) as pattern-mixture models, since the marginal distribution of $X$ is obtained as a mixture of the
distributions of $X$ given each response pattern, that is,

$$f(x) = \sum_r g(r) f(x | r) = \sum_r g(x_r, r) f(x_r | x_r, r).$$

### 2.2 Identifying assumptions and modeling

The pattern-mixture factorization explicitly shows the parts of the full-data distribution that cannot be identified from data, represented by the extrapolation densities $f(x_r | x_r, r)$. The role of an identifying restriction or assumption $A$ is to map an observed-data distribution $G$ into a full-data distribution $F_A \equiv A(G) \in \mathcal{F}$. Under the pattern-mixture approach, based on (1), identifying assumptions amount to indicating how to construct extrapolation distributions based on the observed-data distribution. In terms of densities, we write $f_A \equiv A(g)$ to indicate that we obtain $f_A(x_r | x_r, r)$, and therefore $f_A(x, r)$ in (1), under restriction $A$.

Parametric implementations of pattern-mixture models work under a restricted family of observed-data distributions $\mathcal{M} \subset \mathcal{G}$; for example, Little (1993) used multivariate Gaussian distributions for each $F_r$. The combination of modeling of the observed data and specification of an identifying restriction leads to three types of parameters that we need to distinguish, here depicted in Figure 1. The observed-data distribution $G$ is obtained from a true full-data distribution $F$ under a true missingness mechanism that implicitly defines a mapping $A_0$ such that $F = A_0(G)$, and correspondingly, $\theta_0 \equiv \theta(F) \equiv \theta(A_0(G))$ is the true value of a population parameter of interest. As we have argued, $A_0$ cannot be learned from $G$ alone, and therefore we must work under an assumption $A$ on the missing-data mechanism that conceptually maps $G$ into $A(G)$ in the space of full-data distributions. The population parameter of interest under assumption $A$ is then $\theta_A \equiv \theta(A(G))$. Furthermore, if one uses a parametric approach restricting the observed-data distribution to a class $\mathcal{M} \subset \mathcal{G}$, and if $G$ is not
Figure 1: Population parameters of interest. Solid arrow: modeling; dashed arrow: mapping implied by missing-data restriction; dotted arrow: derivation of parameter of interest.

contained in $\mathcal{M}$, all we can hope to recover is the distribution $G_{\mathcal{M}}$ in $\mathcal{M}$ that diverges the least from $G$, for example in a Kullback-Leibler sense (e.g., van der Vaart, 1998, p. 55). From that distribution, we could then obtain a full-data distribution $A(G_{\mathcal{M}})$ under a missing-data restriction $A$, thereby leading to the population parameter $\bar{\theta}_A \equiv \theta(A(G_{\mathcal{M}}))$. If the model $\mathcal{M}$ contains the true $G$, then the right and center branches of Figure 1 are the same. If the identifying assumption $A$ is correct, then the left and center branches of Figure 1 are the same. If model $\mathcal{M}$ and identifying restriction $A$ are both correct then the three branches coincide. However, we can never recover $A_0$ from observed data alone, so the gap between the left and middle branches of Figure 1 completely depends on assumption $A$ and cannot be reduced based on observed-data. It is however possible to asymptotically derive inferences under $G$, by modeling it in a nonparametric way. Our goal can then be seeing as closing the gap between the middle and right branches of Figure 1.

We now present a unified view and a generalization of commonly used identifying assumptions for pattern-mixture models.
2.3 Identifying assumptions under monotone missingness

In the context of a longitudinal study, where the study variables \( X_1, \ldots, X_d \) correspond to measurements over time, it is common to obtain monotone missingness, that is, when missingness in \( X_t \) implies missingness in every \( X_{t'} \) for \( t' > t \), due to dropout of study participants (e.g., Daniels & Hogan, 2008, p. 88). In such situations, the response pattern \( R \) is uniquely determined by the dropout time \( T \), that is, \( T \equiv |R| \), the number of non-zero entries of \( R \). Given a response pattern \( r \) such that \( |r| = t \), we denote \( x_r = (x_1, \ldots, x_t) \equiv x_{\leq t} \); likewise \( x_{<t} \equiv (x_1, \ldots, x_{t-1}) \) and \( x_{>t} \equiv (x_{t+1}, \ldots, x_d) \). Then, the extrapolation density can be written as

\[
    f(x_r \mid x_r, r) = f(x_{>t} \mid x_{\leq t}, T = t) = \prod_{s=t+1}^{d} f(x_s \mid x_{<s}, T = t). \tag{2}
\]

Therefore, under monotone missingness, the specification of identifying restrictions for pattern-mixture models amounts to indicating a way of deriving \( f(x_s \mid x_{<s}, T = t) \) as a function of the observed-data distribution, for each \( t = 0, \ldots, d - 1 \) and \( s = t + 1, \ldots, d \).

We now present a unifying framework that encompasses common approaches to identify \( f(x_s \mid x_{<s}, T = t) \), for \( s > t \), which assumes that this conditional distribution is the same as the analogous distribution among observations with later dropout times, called donors.

**Definition 1 (Donor-based identification for monotone missingness).** For each \( t \) and \( s \) with \( t < s \), let \( A_{ts} \subseteq \{s, s + 1, \ldots, d\} \) denote a set of dropout times called donor times. A donor-based identification restriction \( A \) sets

\[
    f_A(x_s \mid x_{<s}, T = t) \equiv g(x_s \mid x_{<s}, T \in A_{ts}). \tag{3}
\]

The observations with dropout times in \( A_{ts} \) can be thought of as distribution-donors, because they are such that their values of \( X_1, \ldots, X_s \) are observed, and therefore we
can use them to obtain the right-hand side of (3). The notation $A_{ts}$ indicates that this set is associated with an identifying restriction $A$ which can use different sets of donor times for each $t$ and $s$.

**Remark 1 (Donor-based selection-model restriction).** For selection models, Definition 1 can be equivalently formulated in terms of restrictions of the missingness mechanism, namely, assuming (3) is equivalent to assuming that $f(T = t \mid x \leq s)/f(T \in A_{ts} \mid x \leq s)$ is constant as a function of $x$.

Donor-based restrictions encompass different popular identifying restrictions, which can be specified via different subsets $A_{ts}$.

**Example 1 (Complete-case).** Little (1993) proposed the complete-case (CC) restriction, under which identification is tied to the completely observed cases only, that is, $cc_{ts} = \{d\}$ for all $s > t$, leading to $f_{cc}(x_s \mid x < s, T = t) \equiv g(x_s \mid x < s, T = d)$, or, in terms of the missingness mechanism, it assumes that $f(T = t \mid x \leq s)/f(T = d \mid x \leq s)$ is constant as a function of $x$, for all $s$ and $t$ with $s > t$.

**Example 2 (Available-case).** Molenberghs et al. (1998) proposed the available-case (AC) restriction, where we use all available cases to identify the missing-data distribution, that is, $ac_{ts} = \{s, s + 1, \cdots, d\}$ for all $s > t$, and $f_{ac}(x_s \mid x < s, T = t) \equiv g(x_s \mid x < s, T \geq s)$. Equivalently, this assumes that $f(T = t \mid x \leq s)/f(T \geq s \mid x \leq s)$ is constant as a function of $x$, for all $s$ and $t$ with $s > t$. Under monotone missingness, this assumption is equivalent to the missing at random assumption of Rubin (1976).

**Example 3 (Neighboring-case).** Thïjs et al. (2002) introduced the neighboring-case (NC) restriction, which uses the nearest case available for identification, namely, $nc_{ts} = \{s\}$ for all $s > t$, and $f_{nc}(x_s \mid x < s, T = t) \equiv g(x_s \mid x < s, T = s)$. This corresponds to assuming that $f(T = t \mid x \leq s)/f(T = s \mid x \leq s)$ does not change with $x$, for all $s > t$. Under monotone missingness, this is equivalent to the itemwise conditionally independent nonresponse assumption of Sadinle & Reiter (2017).
Example 4 (k-nearest-case). Other reasonable restrictions are possible under our general formulation in (3). For example, one could think of a k-nearest-case (kNC) subclass of restrictions, where $kNC_{ts} = \{s, \ldots, \min\{s + k, d\}\}$, and $f_{kNC}(x_s | x_{<s}, T = t) \equiv g(x_s | x_{<s}, s \leq T \leq s + k)$. This is equivalent to assuming that $f(T = t | x_{<s})/f(s \leq T \leq s + k | x_{<s})$ does not depend on $x_s$, for all $s$ and $t$ with $s > t$.

Note that the sets $A_{ts}$ in the particular cases presented above are the same for all $t$, although they change with $s$. However, in the formulation given by Definition 1 these sets could change over both $t$ and $s$.

Remark 2 (More general restrictions). We also point out that it is possible to devise even more general strategies for identification compared to our donor-based approach; for example, for each $t$ and $s$, $t < s$, we could choose different sets $A_{ts}^{(u)}$, and then identify $f_A(x_s | x_{<s}, T = t) = \sum_u \omega_{ts}^{(u)} g(x_s | x_{<s}, T \in A_{ts}^{(u)})$, for some set of weights $\{\omega_{ts}^{(u)}\}_u$ that adds up to 1. This is similar to a general strategy presented by Thijs et al. (2002), but such approaches in general lack the interpretability in terms of the missingness mechanism given by Remark 1.

2.4 Identifying assumptions under nonmonotone missingness

Most existing identifying restrictions for pattern-mixture models have been developed to handle monotone missingness. Among the restrictions mentioned in the previous section, the CC restriction is the only one that is readily applicable for nonmonotone nonresponse. Under that restriction, the distribution of the missing variables $X_r$ given the observed data $(X_r, R = r)$ is the same as the distribution of $X_r$ given $(X_r, R = 1_d)$, coming from fully-observed responses, regardless of the response pattern $r$. We obtain $f_{CC}(x_r | x_r, r) \equiv g(x_r | x_r, 1_d)$, for all $r \in \{0, 1\}^d$. Other donor-based restrictions, which are intuitively reasonable under monotone missingness, do not
currently have a clear analog in the nonmonotone case.

We note that donor-based restrictions for monotone missingness build on the factorization of the extrapolation density in (2), which naturally follows the longitudinal order of the study variables, which in turn defines the monotonicity in the response patterns. With nonmonotone missingness, to identify a full-data distribution, we need to identify \( f(x_r \mid x_r, r) \) for each response pattern \( r \in \{0,1\}^d \).

Each \( f(x_r \mid x_r, r) \) can be factorized as a product of sequential conditional densities analogously to (2), but the order in which the variables appear in the factorization do not have to follow the indexing of the study variables, especially if that indexing is arbitrary. For example, with \( d = 4 \) variables and response pattern \( r = 0101 \), we need to identify 

\[
\begin{align*}
  f(x_r \mid x_r, r) &= f(x_3 \mid x_2, x_4, R = 0101) f(x_1 \mid x_2, x_3, x_4, R = 0101) \\
  &= f(x_3 \mid x_2, x_4, R = 0101) f(x_1 \mid x_2, x_3, x_4, R = 0101).
\end{align*}
\]

Denote the index set of the study variables by \([d] = (1, \ldots, d)\), and let \([d]_r = (j : r_j = 1)\) and \([d]_\bar{r} = (j : r_j = 0)\) be the indices of the observed and missing variables according to \( r \), respectively. Consider a permutation \( \pi^{(r)} \) of \([d]\) such that its first \( |r| \) entries, \( \pi^{(r)}_{\leq |r|} \), contain the indices of the entries of \( r \) that equal 1, \( [d]_r \). For example, with \( r = 0101 \), \( \pi^{(r)} \) is a permutation of \([4]\) such that its first \( |r| = 2 \) entries equal 2 and 4, that is, either \( \pi^{(r)} = (2, 4, 1, 3) \) or \( \pi^{(r)} = (2, 4, 3, 1) \). Denote the \( j \)th entry of \( \pi^{(r)} \) by \( \pi^{(r)}_j \). The variables \( X \) can be reordered using the permutation \( \pi^{(r)} \) as 

\[
X_{\pi^{(r)}} = (X_{\pi^{(r)}_j} : j \in [d]),
\]

and we can use this to base the factorization

\[
f(x_r \mid x_r, r) = \prod_{j=|r|+1}^{d} f(x_{\pi^{(r)}_j} \mid x_{\pi^{(r)}_{<j}}, r),
\]

where \( x_{\pi^{(r)}_{<j}} \) denotes a value of the \( X \) variables with indices corresponding to the first \( j - 1 \) entries of \( \pi^{(r)} \). For example, with \( r = 0101 \) and \( \pi^{(r)} = (2, 4, 3, 1) \),...
\( X_{\pi^{(r)}} = (X_2, X_4, X_3, X_1) \), and (4) corresponds to \( f(x_1, x_3 \mid x_2, x_4, R = 0101) = f(x_3 \mid x_2, x_4, R = 0101)f(x_1 \mid x_2, x_3, x_4, R = 0101). \) Note that with the notation used above, the observed variables according to \( r \) are \( X_r = X_{\pi^{(r)}_{\leq |r|}}. \)

Based on the factorization in (4), we need to identify \( f(x_{\pi_j^{(r)}} \mid x_{\pi_{<j}^{(r)}}, r) \) for each \( j > |r| \). To do this, we need to use the response patterns where all of the variables in \( X_{\pi_{<j}^{(r)}} \) are observed. Denote \( 1[\pi_{\leq j}^{(r)}] \) the response pattern with ones in the entries \( \pi_{\leq j}^{(r)} \). We write \( r \leq r' \) if \( r' \) indicates at least the same observed variables as \( r \). Denote \( \mathcal{D}(\pi_{\leq j}^{(r)}) \equiv \{ r' : 1[\pi_{\leq j}^{(r)}] \preceq r' \} \) as the set of response patterns where all of the variables in \( X_{\pi_{<j}^{(r)}} \) are observed, and therefore \( \mathcal{D}(\pi_{\leq j}^{(r)}) \) represents the set of potential donors for identification of \( f(x_{\pi_j^{(r)}} \mid x_{\pi_{<j}^{(r)}}, r) \). For example, with \( r = 0100 \) and \( \pi^{(r)} = (2, 4, 3, 1) \), we have \( \pi_{\leq 2}^{(r)} = (2, 4), \pi_{\leq 3}^{(r)} = (2, 4, 3) \) and \( \pi_{\leq 4}^{(r)} = (2, 4, 3, 1) \); \( 1[\pi_{\leq 2}^{(r)}] = 0101, 1[\pi_{\leq 3}^{(r)}] = 0111 \) and \( 1[\pi_{\leq 4}^{(r)}] = 1111 \); and finally \( \mathcal{D}(\pi_{\leq 2}^{(r)}) = \{0101, 1101, 0111, 1111\}, \mathcal{D}(\pi_{\leq 3}^{(r)}) = \{0111, 1111\} \) and \( \mathcal{D}(\pi_{\leq 4}^{(r)}) = \{1111\} \). Donor-based identification strategies use subsets of the possible sets of donors that can be used for identification.

**Definition 2 (Donor-based identification for nonmonotone missingness).** Given a response pattern \( r \in \{0, 1\}^d \), let \( \pi^{(r)} \) be a permutation of \([d]\) such that its first \(|r|\) entries contain the indices \([d]_r\), that is, \( \pi_{\leq |r|}^{(r)} = [d]_r \). Let \( \mathcal{D}(\pi_{\leq j}^{(r)}) \equiv \{ r' : 1[\pi_{\leq j}^{(r)}] \preceq r' \} \) be the set of response patterns where the \( X \) variables with indices \( \pi_{\leq j}^{(r)} \) are observed. Let \( A(r, \pi_{\leq j}^{(r)}) \subseteq \mathcal{D}(\pi_{\leq j}^{(r)}) \) denote a set of response patterns called donor patterns, for \( j = |r| + 1, \ldots, d \). A donor-based identification restriction \( A \) sets

\[
f_A(x_{\pi_j^{(r)}} \mid x_{\pi_{<j}^{(r)}}, R = r) \equiv g(x_{\pi_j^{(r)}} \mid x_{\pi_{<j}^{(r)}}, R \in A(r, \pi_{\leq j}^{(r)})).
\]

(5)

As in the monotone case, the observations with response patterns in \( A(r, \pi_{\leq j}^{(r)}) \) can be thought of as donor cases, because they are such that their values of \( X_{\pi_{\leq j}^{(r)}} \) are observed, and therefore we can use them to obtain the right-hand side of (5). Likewise, as in the monotone case, donor-based restrictions can also be redefined in terms of
We notice that, naturally, different factorizations based on different permutations \( \pi \) will lead to different identified full-data distributions.

Remark 3 (Nonmonotone donor-based selection-model restriction). Definition 2 can be equivalently formulated for selection models in terms of restrictions of the missingness mechanism, namely, (5) is equivalent to assuming that \( f(R = r \mid x_{\pi^{(r)}_{\leq j}}) / f(R \in A(r, \pi^{(r)}_{\leq j}) \mid x_{\pi^{(r)}_{\leq j}}) \) is constant as a function of \( x_{\pi^{(r)}_{\leq j}} \).

We now explain how the CC restriction fits into this general formulation and present some examples that generalize the AC, NC and kNC restrictions to nonmonotone nonresponse.

Example 5 (Complete-case). Regardless of the values of \( r, \pi^{(r)} \), and \( j \), the set of possible donors \( D(\pi^{(r)}_{\leq j}) \equiv \{ r' : 1[\pi^{(r)}_{\leq j}] \leq r' \} \) always contains the pattern 1\( d \). Taking all sets of donor patterns to contain 1\( d \) only, that is, \( CC(r, \pi^{(r)}_{\leq j}) = \{ 1_d \} \), leads to the CC restriction, \( f_{CC}(x_r \mid x_r, r) \equiv g(x_r \mid x_r, 1_d) \). Note that this holds regardless of the pattern \( r \) and is invariant to the permutation \( \pi^{(r)} \).

Example 6 (Available-case). We can also use the full sets of available donor patterns for identification, that is, \( AC(r, \pi^{(r)}_{\leq j}) = D(\pi^{(r)}_{\leq j}) \), leading to
\[
 f_{AC}(x_{\pi^{(r)}_{j}} \mid x_{\pi^{(r)}_{\leq j}}, R = r) \equiv g(x_{\pi^{(r)}_{j}} \mid x_{\pi^{(r)}_{\leq j}}, R \geq 1[\pi^{(r)}_{\leq j}]),
\]
for all \( r, \pi^{(r)} \) and \( j > |r| \). For example, with \( r = 0100 \) and \( \pi^{(r)} = (2, 4, 3, 1) \), we obtain \( f_{AC}(x_4 \mid x_2, R = 0100) \equiv g(x_4 \mid x_2, R \geq 0101) \), \( f_{AC}(x_3 \mid x_2, x_4, R = 0100) \equiv g(x_3 \mid x_2, x_4, R \geq 0111) \), and \( f_{AC}(x_1 \mid x_2, x_3, x_4, R = 0100) \equiv g(x_1 \mid x_2, x_3, x_4, R = 1111) \).

We notice that, naturally, different factorizations based on different permutations \( \pi^{(r)} \) will lead to different identified full-data distributions.

Example 7 (Neighboring-case). The pattern \( 1[\pi^{(r)}_{\leq j}] \) is the closest pattern to \( r \) under which all the variables \( X_{\pi^{(r)}_{\leq j}} \) are available, and therefore it represents observations with the minimal variables needed to identify \( f(x_{\pi^{(r)}_{j}} \mid x_{\pi^{(r)}_{\leq j}}, R = r) \). Therefore, it is reasonable to take \( NC(r, \pi^{(r)}_{\leq j}) = \{ 1[\pi^{(r)}_{\leq j}] \} \) for all \( r, \pi^{(r)} \) and \( j > |r| \). For example, with \( r = 0100 \) and \( \pi^{(r)} = (2, 4, 3, 1) \), we obtain \( f_{NC}(x_4 \mid x_2, R = 0100) \equiv
\[ g(x_4 \mid x_2, R = 0101), \ f_{\text{NC}}(x_3 \mid x_2, x_4, R = 0100) \equiv g(x_3 \mid x_2, x_4, R = 0111), \text{ and} \]
\[ f_{\text{NC}}(x_1 \mid x_2, x_3, x_4, R = 0100) \equiv g(x_1 \mid x_2, x_3, x_4, R = 1111). \] Here again, the final identified full-data distribution depends on the chosen factorization based on permutation \( \pi^{(r)} \).

**Example 8 (k-nearest-case).** We can define the donor patterns within \( k \) distance of \( 1[\pi_{\leq j}^{(r)}] \) as \( D_k(\pi_{\leq j}^{(r)}) \equiv \{ r' : 1[\pi_{\leq j}^{(r)}] \preceq r', H(1[\pi_{\leq j}^{(r)}], r') \leq k \} \), where \( H(r, r') \) is the Hamming distance between two response patterns, which counts the number of entries where \( r \) and \( r' \) disagree. Then, the \( k\text{NC} \) restriction takes \( k\text{NC}(r, \pi_{\leq j}^{(r)}) = D_k(\pi_{\leq j}^{(r)}) \).

**Remark 4 (More general restrictions).** Along the lines of Remark 2, here we also point out that it is possible to devise even more general identification strategies compared to our donor-based approach. In Definition 2 we tie identification to a set of donors, but a more general approach would be to take a mixture of \( g(x_{\pi_{\leq j}^{(r)}} \mid x_{\pi_{\leq j}^{(r)}}, R \in A^{(u)}(r, \pi_{\leq j}^{(r)})) \) across \( u \), as in Remark 2, for fixed permutations \( \{\pi^{(r)}\}_r \). Yet, an even more general approach is to identify multiple versions of \( f(x_{\bar{r}} \mid x_r, r) \) based on different factorizations induced by multiple sets of permutations \( \{\pi^{(r)}\}_r \), and then take a weighted average of them. Again, these approaches lack the interpretability in terms of the missingness mechanism given by Remark 3.

### 2.5 Nonparametric identification and sensitivity analysis

The previous sections indicate that, with pattern-mixture models, we start with an observed-data density \( g(x_r, r) \), and complement it with extrapolation densities \( f_A(x_{\bar{r}} \mid x_r, r) \) obtained from an identifying restriction \( A \), leading to a full-data distribution with density \( f_A(x, r) \). This construction is such that \( \int_{X_r} f_A(x, r) \mu(dx_r) = g(x_r, r) \), that is, the observed-data distribution implied by the full-data distribution under \( A \) is the same as the true observed-data distribution, meaning that the identifying
restrictions only impose constraints on what cannot be recovered from observed data. This implies that the full-data distribution under \( A \) is observationally equivalent to the true full-data distribution, and it means that assumption \( A \) cannot be tested based on the observed data (e.g., Sadinle & Reiter, 2019). This desirable characteristic is known as nonparametric identification, nonparametric saturation or just-identification (Robins, 1997; Vansteelandt et al., 2006; Daniels & Hogan, 2008; Hoonhout & Ridder, 2018).

The choice of the appropriate identifying restriction is no trivial matter, as it cannot be justified based on the observed data. Regardless of whether one feels confident about an identifying restriction, it is desirable to perform sensitivity analyses, where inferences are obtained under different identifying restrictions (Scharfstein et al., 2018). The advantage of the construction that we have presented is that the same observed-data distribution can be used to obtain different full-data distributions under different identifying restrictions. Since all of such full-data distributions will be observationally equivalent, discrepancies in inferences will be entirely due to the different identifying restrictions, making pattern-mixture models ideal for conducting sensitivity analyses.

3 Nonparametric Estimation

As mentioned before, the observed-data distribution \( G \), with density \( g(x_r, r) \), is directly estimable from the observed data. The observed data \( S_n = (X_i, R_i)_{i=1}^n \) corresponds to a random sample from \( G \). The extrapolation densities, represented by \( f(x_r | x_r, r) \), have to be recovered under an identifying assumption \( A \), which provides a recipe for writing \( f_A \) in terms of \( g \), and we write \( f_A = A(g) \). Definitions 1 and 2
provide explicit forms of $f_A = A(g)$ under donor-based identification. Under $A$, the full-data density is given by

$$f_A(x, r) = g(x_r, r) f_A(x_r | x_r, r),$$

and the associated full-data distribution is given by

$$F_A(B, r) = \int_B F_A(B_r | x_r, r) G(dx_r, r),$$

(6)

where $B = B_r \otimes B_r = \otimes_{j=1}^d B_j$, for measurable subsets $B_j$ of $X_j$. In this expression, $G(B_r, r) = \int_{B_r} g(x_r, r) \mu(dx_r)$ and $F_A(B_r | x_r, r) = \int_{B_r} f_A(x_r | x_r, r) \mu(dx_r)$.

### 3.1 Estimating the Full-Data Distribution

To construct a nonparametric estimator of $F_A$, we propose to directly replace $G$ in (6) by the empirical observed-data distribution $\hat{G}$,

$$\hat{G}(B_r, r) = \frac{1}{n} \sum_{i=1}^n I(X_{i,r} \in B_r, R_i = r).$$

(7)

Handling $F_A(B_r | x_r, r)$ is more challenging, because identifying restrictions are typically expressed in terms of density functions, and so $f_A(x_r | x_r, r)$ will usually have an explicit expression in terms of $g$, as seen in Section 2. This means that we cannot typically obtain an estimate of $F_A(B_r | x_r, r)$ directly as a function of the empirical observed-data distribution $\hat{G}$. We then propose to obtain a nonparametric estimate of $F_A(B_r | x_r, r)$ via a surrogate estimate of $g$, that is, an estimate of the observed-data density obtained for the sole purpose of being used in the estimation of $F_A(B_r | x_r, r)$. In particular, we take the surrogate as the kernel density estimator

$$\hat{g}_h(x_r, r) = \frac{1}{n} \sum_{i=1}^n I(R_i = r) \prod_{j=1}^d K_j(x_j; X_{ij}, h_j)^{r_j}.$$  

(8)
In this expression, \( K_j \) represents a density with respect to an appropriate dominating measure, which depends on the nature of variable \( X_j \), a location \( X_{ij} \) and a smoothing parameter \( h_j \). For continuous variables, we take \( K_j \) to be a Gaussian density centered at \( X_{ij} \) with variance \( h_j^2 \), and so when all \( d \) variables are continuous, expression (8) closely corresponds to a traditional kernel density estimator (Rosenblatt, 1956; Parzen, 1962; Silverman, 1986). For an unordered categorical variable with \( C_j \) categories, we take \( K_j(x_j; X_{ij}, h_j) = I(x_j = X_{ij})h_j + I(x_j \neq X_{ij})(1-h_j)/(C_j-1), \) for \( C_j^{-1} \leq h_j \leq 1 \), which corresponds to the kernel method proposed by Aitchison & Aitken (1976). Different types of variables can be handled in this fashion as long as appropriate kernels are used (Titterington, 1980; Wang & van Ryzin, 1981; Kokonendji et al., 2009; Chen & Tang, 2011; Li & Racine, 2003).

We then obtain \( \hat{f}_{A,h}(x_r | x_r, r) \) using \( \hat{g}_h(x_r, r) \) and the mapping induced by assumption \( A \), that is, \( \hat{f}_{A,h} = A(\hat{g}_h) \). Then, the extrapolation distributions are obtained as

\[
\hat{F}_{A,h}(B_r | x_r, r) = \int_{B_r} \hat{f}_{A,h}(x_r | x_r, r) \mu(dx_r). \tag{9}
\]

Finally, our estimator of the full-data distribution is obtained by plugging into equation (6):

\[
\hat{F}_{A,h}(B, r) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_{A,h}(B_r | X_{i,r}, r) I(X_{i,r} \in B_r, R_i = r). \tag{10}
\]

To illustrate this estimator we now provide details for monotone nonresponse, and provide the construction of \( \hat{F}_{A,h}(B, r) \) under the CC restriction in Appendix A.

**Example 9 (Monotone missingness).** As a concrete example, we present our estimation approach for the case of monotone missingness. The nonmonotone case can be handled similarly under the identification approach presented in Section 2.4, albeit with a more intricate notation. The observed sample in this case is represented by \( S_n = (X_{i, \leq T_i}, T_i)_{i=1}^{n} \), where \( X_{i, \leq T_i} = (X_{i,1}, \ldots, X_{i,T_i}) \). Following the notation in
Section 2.3, we have that

\[
\hat{f}_{A,h}(x_r \mid x_r, r) = \hat{f}_{A,h}(x_{\leq t} \mid x_{\leq t}, T = t) = \prod_{s=t+1}^{d} \hat{f}_{A,h}(x_s \mid x_{<s}, T = t),
\]

(11)

where

\[
\hat{f}_{A,h}(x_s \mid x_{<s}, T = t) = \hat{g}_h(x_s \mid x_{<s}, T \in A_{ts}),
\]

(12)

and

\[
\hat{g}_h(x_s \mid x_{<s}, T \in A_{ts}) = \sum_{i=1}^{n} W_i(x_{<s}; A_{ts}) K_s(x_s; X_{is}, h_s),
\]

(13)

with

\[
W_i(x_{<s}; A_{ts}) \propto I(T_i \in A_{ts}) \prod_{j=1}^{s-1} K_j(x_j; X_{ij}, h_j).
\]

(14)

Remark 5 (Finite-sample vs asymptotic nonparametric saturation). For an estimator of the full-data distribution \(\hat{F}\), we say that it is finite-sample nonparametrically saturated if for any \(r\) and any Borel measurable set \(B_r \subseteq X_r\), \(\hat{F}(B_r \otimes X_r, r) = \hat{G}(B_r, r)\) for every sample size \(n\). If \(\hat{F}(B_r \otimes X_r, r) = \hat{G}(B_r, r)\) only when \(n \to \infty\), \(\hat{F}\) is called asymptotically nonparametrically saturated. It is clear that the estimator in equation (10) satisfies \(\hat{F}_{A,h}(B_r \otimes X_r, r) = \hat{G}(B_r, r)\) for all \(n\), which means that \(\hat{F}_{A,h}\) is finite-sample nonparametrically saturated. In Remark 7, we give an example of an estimator which is only asymptotically nonparametrically saturated.

3.2 Estimation of Functionals

Typically, we are interested in estimating a statistical functional \(\theta \equiv \theta(F)\). We can construct an estimator by evaluating \(\theta(\cdot)\) on (10), that is,

\[
\hat{\theta}_{A,h} \equiv \theta(\hat{F}_{A,h}).
\]

(15)
We will show that under some conditions this estimator is consistent and has asymptotic normality (Corollary 5). While \( \hat{\theta}_{A,h} \) may have a closed-form under some simple cases (see an example in Appendix A), evaluating \( \theta(\cdot) \) on \( \hat{F}_{A,h} \) is not easy in general. For example, computing moments of \( X \) under the assumptions presented before involves computing integrals of ratios of kernel density estimates. Here we propose a Monte Carlo approach to approximate \( \hat{\theta}_{A,h} \), in particular under the large class of donor-based assumptions presented in Sections 2.3 and 2.4.

3.2.1 Monte Carlo Approximation

Under our estimate \( \hat{F}_{A,h} \), the conditional distribution of \( (X_r \mid X_r = x_r, R = r) \) has a density \( \hat{f}_{A,h}(x_r \mid x_r, r) \), whereas the distribution of \( (X_R, R) \) is discrete given by the empirical observed-data distribution. We propose to approximate \( \Theta(\hat{F}_{A,h}) \) via Monte Carlo. In particular, we draw \( X^{(v)}_{i,R_i} \) from \( \hat{F}_{A,h}(\cdot \mid X_{i,R_i}, R_i) \), for \( v = 1, \ldots, V \), for each observed sample point \( (X_{i,R_i}, R_i) \). The result can be organized into \( nV \) completed sample points \( S_{n,V} = \{(X^{(v)}_{i,R_i}, R_i)_{i=1}^n\}_{v=1}^V \), where \( X^{(v)}_{i,R_i} = (X_{i,R_i}, X^{(v)}_{i,R_i}) \). We can then define a Monte Carlo approximation of \( \hat{F}_{A,h} \) as

\[
\hat{F}_{A,h}^{MC}(B, r) = \frac{1}{nV} \sum_{i=1}^n I(X_{i,r} \in B, R_i = r) \sum_{v=1}^V I(X^{(v)}_{i,R_i} \in B_r),
\]

and we approximate \( \Theta(\hat{F}_{A,h}) \) by \( \Theta(\hat{F}_{A,h}^{MC}) \). Computing the latter is simple, as it entails evaluating the sample version of the functional \( \Theta(\cdot) \) on \( S_{n,V} \). For instance, if the parameter of interest is a correlation coefficient, we can use the sample correlation coefficients computed on \( S_{n,V} \).

We note that this Monte Carlo procedure is analogous to the one used in the multiple imputation approach of Rubin (1987), in the sense that \( S_{n,V} \) corresponds to the observed data \( S_n \) being completed \( V \) times. Unlike in Rubin’s multiple imputation,
Algorithm 1 Sampling for Monte Carlo Approximation under Monotone Missingness

for \( v = 1, \ldots, V; i = 1, \ldots, n \) do

\( t \leftarrow T_i, X^{(v)}_{\leq t} \leftarrow X_{i,\leq t} \)

for \( s = t + 1, \ldots, d \) do

Draw \( \ell \in \{1, \ldots, n\} \) with probability \( W_\ell(X_{i,\leq t}; A_{ts}) \)

Draw \( X^{(v)}_s \) from the distribution with density \( K_s(\cdot; X_{\ell,s}, h_s) \)

\( X^{(v)}_{\leq s} \leftarrow (X^{(v)}_{\leq s}, X^{(v)}_s) \)

end for

\( X^{(v)}_{i,>t} \leftarrow X^{(v)}_{>t} \)

end for

return \{\( (X^{(v)}_{i,\leq T_i}, X^{(v)}_{i,>T_i})_{i=1}^n \)\}_{v=1}^V

our goal here is to simply approximate \( \theta(\hat{F}_{A,h}) \), as we assess the variability of \( \theta(\hat{F}_{A,h}) \) via the bootstrap, as explained in Section 4.

Example 10 (Monotone missingness). Continuing with Example 9, we now need to draw \( X^{(v)}_{i,>T_i} \) from \( \hat{F}_{A,h}(\cdot \mid X_{i,\leq T_i}, T_i) \), for \( v = 1, \ldots, V \), and for each observed sample point \( (X_{i,\leq T_i}, T_i) \). From (11) we can see that a draw \( X^{(v)}_{i,>T_i} \) can be obtained by sequentially sampling \( X^{(v)}_{i,s} \), \( s = T_i + 1, \ldots, d \), from a distribution with density \( \hat{g}_h(x_s \mid X^{(v)}_{i,<s}, T \in A_{Ts}) \), where \( X^{(v)}_{i,<s} = (X^{(v)}_{i,<s-1}, X^{(v)}_{i,s-1}) \) and \( X^{(v)}_{i,<s+1} = X_{i,\leq T_i} \).

Sampling from \( \hat{g}_h(x_s \mid X^{(v)}_{i,<s}, T \in A_{Ts}) \) can be accomplished using the mixture representation given by (13), by sampling an index \( \ell \) with probability \( W_\ell(X^{(v)}_{i,<s}; A_{Ts}) \) as in (14), and then drawing \( X^{(v)}_{i,s} \) from a distribution with density \( K_s(x_s; X_{\ell,s}, h_s) \). We summarize this procedure in Algorithm 1. In the Online Supplementary Material we provide R code implementing Algorithm 1 for handling monotone missingness under donor-based identification.

Remark 6 (Alternative Monte Carlo approximation). For each completed sam-
ple $S_n^v = \{(X_i^{(v)}, R_i) : i = 1, \cdots, n\}$, let $\hat{F}_{A,h}^{(v)}$ be its corresponding empirical distribution. Then the average of the corresponding estimates

$$\tilde{\theta}_V = \frac{1}{V} \sum_{v=1}^{V} \theta\left(\hat{F}_{A,h}^{(v)}\right)$$

provides another Monte Carlo approximation to $\theta(\hat{F}_{A,h})$. This approach might be appealing if computing $\theta(\cdot)$ on $\hat{F}_{A,h}^{\text{MC}}$ is too computationally intensive. In such case, each $\theta\left(\hat{F}_{A,h}^{(v)}\right)$ can be computed in parallel, and the results can be easily combined to obtain a final approximation $\tilde{\theta}_V$ of $\theta(\hat{F}_{A,h})$. Nevertheless, we expect any potential bias of $\theta(\hat{F}_{A,h}^{\text{MC}})$ to be smaller than the bias of $\tilde{\theta}_V$, and therefore $\theta(\hat{F}_{A,h}^{\text{MC}})$ should be the preferred approximation of $\theta(\hat{F}_{A,h})$.

### 3.2.2 Parameters Defined by Estimating Equations

If the parameter of interest $\theta$ is determined by an estimating function $E(\xi(X, \theta)) = 0$ for some function $\xi$, then our method can be viewed as a generalization of the work of Wang & Chen (2009), who consider the problem of estimating parameters defined by estimating functions when a continuous response variable is subject to missingness but covariates are fully observed. Wang & Chen (2009) proposed a Monte Carlo method to impute the missing data, and apply the empirical likelihood approach to find parameter of interest. When only one continuous variable is subject to missingness and we set the bandwidth $h_s = 0$ in Algorithm 1, our method will be the same as the Monte Carlo method of Wang & Chen (2009).

Note that Wang & Chen (2009) also discussed how to use the bootstrap for constructing a confidence interval for the parameter of interest. We will also analyze the same idea in Section 4 with a more general setting in terms of both the missingness assumption and the parameter of interest, i.e., we do not restrict ourselves to
estimators from empirical likelihood methods.

### 3.3 Estimating the Full-Data Density

If the goal is to recover the joint density function of the study variables, one possible estimator would be the one that uses the surrogate estimate of $g$, that is, $\hat{f}_{A,h}(x) = \sum_r \hat{f}_{A,h}(x,r)$, where

$$\hat{f}_{A,h}(x,r) = \hat{g}_h(x,r)\hat{f}_{A,h}(x_r | x_r, r).$$

(17)

Note that the distribution generated by $\hat{f}_{A,h}$ is not the same as our proposed $\hat{F}_{A,h}$, because the latter does not involve any smoothing of the observed-data distribution, whereas the surrogate estimate $\hat{g}_h(x_r, r)$ does involve smoothing.

**Remark 7.** As is mentioned before, the estimator $\hat{F}_{A,h}$ is finite-sample nonparametrically saturated. However, the distribution induced by the density estimator $\hat{f}_{A,h}(x,r)$ is not because of the effect of smoothing in the construction of the surrogate estimator $\hat{g}_h(x_r, r)$. The distribution induced by $\hat{f}_{A,h}(x,r)$ is nevertheless asymptotically nonparametrically saturated under a good choice of smoothing bandwidths. For example for continuous variables, a good choice of smoothing bandwidth $h = h_n \to 0$ leads $\hat{g}_h$ to be a consistent estimator of the true observed-data density function $g$.

### 4 Bootstrap Confidence Intervals

In what follows, we discuss how to use the bootstrap approach to obtain inferences on functions of the full-data distribution. In particular, we use the empirical bootstrap (Efron, 1979; Efron & Tibshirani, 1994). Let $\theta = \theta(F)$ be the parameter of interest and $\hat{\theta}_{A,h} = \theta(\hat{F}_{A,h})$ be our estimate. Let $S^*_n = \{(X^*_{1,R_1}, R^*_1), \cdots, (X^*_{n,R_n}, R^*_n)\}$ be a
bootstrap sample obtained by sampling with replacement from \( S_n \), and let \( \hat{\theta}^*_A,h = \theta(\hat{F}^*_{A,h}) \) be the corresponding estimator, where \( \hat{F}^*_{A,h} \) is constructed using equation (10) with the bootstrap sample. After repeating the bootstrap procedure \( B \) times we obtain \( \hat{\theta}^{(1)}_{A,h}, \ldots, \hat{\theta}^{(B)}_{A,h} \), \( B \) bootstrap estimates of the parameter of interest. We use the upper and lower \( \alpha/2 \) quantiles of these \( B \) numbers as the confidence interval of \( \theta \). Specifically, we propose to use the interval

\[
\tilde{C}_{n,\alpha} = [\ell_\alpha, u_\alpha] = \left[ \hat{\Gamma}^{-1}(\alpha/2), \hat{\Gamma}^{-1}(1 - \alpha/2) \right],
\]

where

\[
\hat{\Gamma}(t) = \frac{1}{B} \sum_{b=1}^{B} I(\hat{\theta}^{(b)}_{A,h} \leq t),
\]

as the confidence interval of \( \theta \). In Theorem 6, we prove that \( \tilde{C}_{n,\alpha} \) is asymptotically valid under appropriate assumptions.

Note that the above method is called the bootstrap percentile approach (Efron & Tibshirani, 1994; Hall, 2013). There are other possible approaches for constructing a confidence interval, such as using the bootstrap variance estimator to construct a normal confidence interval, or bootstrapping the \( t \)-distribution. See Efron & Tibshirani (1994) and Hall (2013) for more details on other approaches.

When our estimator is constructed using the Monte Carlo approximation described in Section 3.2.1, for each bootstrap sample \( S^*_n \), we apply Algorithm 1 to obtain a completed sample \( S^*_{n,V} \). Using \( S^*_{n,V} \) and its corresponding estimate of the full-data distribution \( \hat{F}^\text{MC*}_{A,h} \), we then obtain our bootstrap estimate \( \theta(\hat{F}^\text{MC*}_{A,h}) \). Repeating this procedure for \( B \) bootstrap samples leads to \( \theta(\hat{F}^\text{MC*}_{A,h}(1)), \ldots, \theta(\hat{F}^\text{MC*}_{A,h}(B)) \), from which we obtain the quantiles to construct the confidence interval \( \tilde{C}_{n,\alpha} \).

**Remark 8.** One can also use the bootstrap to infer the density function. This is often done by bootstrapping the \( L_\infty \) error of the density estimate. Specifically, we use
the quantile of \{\sup_x |\hat{f}_{A,h}^{(b)}(x,r) - \hat{f}_{A,h}(x,r)|\}_{b=1}^{B} as the width of the confidence band. It can be shown that this leads to a valid confidence band for \( f_{A,h}(x,r) \) for each \( r \). For more details, we refer to Chernozhukov et al. (2014) and Chen (2017).

5 Theory

In the theoretical analysis, we will show that the estimator \( \hat{F}_{A,h} \) is a consistent estimator of \( F \) under appropriate assumptions. To simplify the analysis, we assume that all variables in \( X \) are continuous and we will focus on the case where the sets \( B_j = (-\infty, x_j] \) for some \( x_j \) so that we can simply write \( F(B, r) = F(x, r) \) for \( B = \otimes_{j=1}^{d} B_j = \otimes_{j=1}^{d} (-\infty, x_j] \). Our results, with the exception of the ones involving estimating densities, can be generalized to the case with categorical variables very easily. Also, to simplify the problem, we assume that the smoothing bandwidth of each variable is the same, i.e., \( h_s = h \) for all \( s = 1, \cdots, d \), and all kernel functions are the same, i.e., \( K_s = K \) for all \( s = 1, \cdots, d \). We will prove theory for the monotone missingness case, but the general case in Section 2.4 can be derived in a similar way, albeit with a more complicated notation. With monotone missingness, it is easier to use the notation \( T = |R|, t = |r| \in \{0, 1, \cdots, d\} \) to represent the missingness pattern, so the joint distribution function is written as \( F(x, t) \) and its estimator is \( \hat{F}_{A,h}(x, t) \).

We note that \( F(x, t) \) will denote the joint distribution function of study variables \( X \) with \( T = t \), namely,

\[
F(x, t) = F(x \mid t)g(t) = \int_{-\infty}^{x} f(x', t)\mu(dx').
\]

(18)

An important note is that in the definition of \( F(x, t) \) we are integrating over the sample space of \( X \) until \( x \), but not over the sample space of \( T \), that is, \( F(\infty, t) \) gives us the probability of \( T = t \), \( g(t) \). Before moving forward, we define two asymptotic
functions related to $\hat{F}_{A,h}(x,t)$ and $\hat{f}_{A,h}(x,t)$ as

$$
\bar{F}_{A,h}(x,t) = \int_{-\infty}^{x \leq t} \bar{F}_{A,h}(x > t \mid x' \leq t,t) G(dx' \leq t,t),
$$

$$
\bar{f}_{A,h}(x,t) = g(t)\bar{g}_h(x < t \mid t)\bar{f}_{A,h}(x > t \mid x \leq t,t),
$$

where

$$
\bar{F}_{A,h}(x > t \mid x \leq t,t) = \int_{-\infty}^{x > t} \bar{f}_{A,h}(x' > t \mid x \leq t,t) \mu(dx' > t)
$$

is the extrapolation distribution, and $\bar{f}_{A,h}(x > t \mid x \leq t,t) = A(\bar{g}_h)$ is the extrapolation density constructed by the smoothed density function $\bar{g}_h(x < t \mid t) = \mathbb{E}(\bar{g}_h(x < t \mid t))$ and $\bar{g}_h(x < t) = \bar{g}_h(x < t \mid t)g(t)$. Essentially, $\bar{f}_{A,h}(x,t)$ is constructed using the expected value of the kernel density estimator (KDE). However, $\bar{f}_{A,h}(x,t) \neq \mathbb{E}(\hat{f}_{A,h}(x,t))$ although we do have $\bar{f}_{A,h}(x,t) = \mathbb{E}(\hat{f}_{A,h}(x,t)) + o(1)$ when $n \to \infty$ and $h \to 0$. Later we will see that $\bar{f}_{A,h}(x,t)$ plays a central role in estimation theory and asymptotic normality.

Note that the density function corresponding to $\bar{F}_{A,h}(x,t)$ is not $\bar{f}_{A,h}(x,t)$, since the marginal distribution function $\bar{F}_{A,h}(x \leq t,t) = G(x \leq t,t)$, which does not have $\bar{g}_h(x \leq t,t)$ as a density. If we are thinking about the conditional distribution, the density corresponding to $\bar{F}_{A,h}(x > t \mid x \leq t,t)$ is $\bar{f}_{A,h}(x > t \mid x \leq t,t)$, and therefore $\bar{F}_{A,h}(x,t)$ and $\bar{f}_{A,h}(x,t)$ agree only on the extrapolation densities.
5.1 Estimation Theory

Let $\text{UBC}_2$ denote the collection of all functions with uniformly bounded second derivatives. Moreover, we use the simplified notation

$$
K_h (x; X_i) \equiv \prod_{j=1}^d K_h (x_j; X_{ij}) \equiv \prod_{j=1}^d K_j (x_j; X_{ij}, h),
$$

$$
K_h (x_{\leq t}; X_i \leq t) \equiv \prod_{j=1}^t K_h (x_j; X_{ij}) \equiv \prod_{j=1}^t K_j (x_j; X_{ij}, h).
$$

For univariate variables $w, z$, the kernel function $K_h (w; z)$ can be written as

$$
K_h (w; z) = \frac{1}{\bar{h}} K \left( \frac{w - z}{h} \right),
$$

where $K (z)$ is the conventional kernel function. For instance, the Gaussian kernel has the form $K (z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

Assumptions.

(A1) The true full-data distribution function $F(x, t)$ has a density function $f_0(x, t)$ satisfying

1. $\inf_{x \in X} f_0(x, t) > 0$ for each $t = 1, \cdots, d$.

2. $f_0(x, t) \in \text{UBC}_2$ for each $t = 1, \cdots, d$.

(A2) The statistical functional $\theta$ is Hadamard differentiable.

(K1) $K(z)$ has at least second-order bounded derivative and

$$
\int z^2 K(z) \mu(dz) < \infty, \quad \int K^2(z) \mu(dz) < \infty.
$$

(K2) Let $\mathcal{K} = \{ z \mapsto K \left( z - \frac{w}{h} \right) : w \in \mathbb{R}, \bar{h} > h > 0 \}$, for some fixed constant $\bar{h}$. We assume that $\mathcal{K}$ is a VC-type class. Namely, there exists constants $A,v$ and a
constant envelope $b_0$ such that

$$\sup_Q N(\mathcal{K}, \mathcal{L}^2(Q), b_0 \epsilon) \leq \left( \frac{A}{\epsilon} \right)^v,$$

(22)

where $N(T, d_T, \epsilon)$ is the $\epsilon$-covering number for a semi-metric set $T$ with metric $d_T$, and $\mathcal{L}^2(Q)$ is the $L_2$ norm with respect to the probability measure $Q$.

Assumption (A1) is to ensure that the probability density function (PDF) of the true full-data distribution is bounded away from 0 on the support $\mathcal{X}$. Note that this implies that the corresponding observed-data PDF $g_0$ has a density value uniformly bounded away from 0. We need bounded second derivatives so that the bias of the KDE will be of the order $O(h^2)$. Assumption (A2) ensures the smoothness of the parameter of interest. Most common statistical functions, such as quantiles and moments, satisfy this condition.

Assumption (K1) is a common assumption for the kernel function so that we can obtain the conventional rate of the bias and variance (Scott, 2015; Wasserman, 2006). Assumption (K2) is an assumption to guarantee uniform convergence of a KDE (Giné & Guillou, 2002; Einmahl & Mason, 2005). Most common kernel functions, such as the Gaussian kernel or a compact support kernel, satisfy this condition (Giné & Guillou, 2002). Note that under assumption (K2), the requirements of the smoothing bandwidth $\frac{\log n}{nh} \to 0$ and $h \to 0$ amount to the uniform convergence of the KDE, see, e.g., Rinaldo & Wasserman (2010); Genovese et al. (2014); Chen (2016).

For an identifying assumption $A$, we define

$$F_A(x, t) = \int_{-\infty}^x f_A(x', t) \mu(dx'),$$

$$f_A(x, t) = g(t)g(x_{\leq t} \mid t)f_A(x_{>t} \mid x_{\leq t}, t).$$

Note the function $f_A$ is the corresponding PDF of $F_A = A(G)$. If the identifying assumption is correct, we have $f_A(x_{>t} \mid x_{\leq t}, t) = f(x_{>t} \mid x_{\leq t}, t)$. Namely, the
extrapolation density based on the identifying assumption is the same as the true extrapolation density. This further implies \( f_A(x) = f(x) \) and \( F_A(x) = F(x) \). Because the extrapolation density \( f_A(x > t \mid x \leq t, t) \) plays a crucial role in our analysis, we first derive perturbation theory for \( f_A(x > t \mid x \leq t, t) \).

**Theorem 1 (Perturbation theory of pattern mixture model).** Assume (A1) and consider a donor-based identifying restriction with an identifying set \( A_{ts} \). Let \( \tilde{g}_n \) be a sequence of observed-data density functions that is close to \( g \) in the sense that

\[
\Delta g(x \leq s \mid T \in A_{ts}) \equiv \tilde{g}_n(x \leq s \mid T \in A_{ts}) - g(x \leq s \mid T \in A_{ts}) \to 0,
\]

\[
\Delta g(x < s \mid T \in A_{ts}) \equiv \tilde{g}_n(x < s \mid T \in A_{ts}) - g(x < s \mid T \in A_{ts}) \to 0,
\]

\[
\Delta g(t) \equiv \tilde{g}_n(t) - g(t) \to 0
\]

uniformly for all \( x \in \mathcal{X} \) and \( t = 1, \ldots, d \). Define

\[
\Delta f_A(x > t \mid x \leq t, t) \equiv \tilde{f}_A(x > t \mid x \leq t, t) - f_A(x > t \mid x \leq t, t).
\]

Then

\[
\frac{\Delta f_A(x > t \mid x \leq t, t)}{f_A(x > t \mid x \leq t, t)} = \sum_{s=t+1}^{d} \left\{ \frac{\Delta g(x \leq s \mid T \in A_{ts})}{g(x \leq s \mid T \in A_{ts})} + \frac{\Delta g(x < s \mid T \in A_{ts})}{g(x < s \mid T \in A_{ts})} \right\} + \tilde{W}_n(x, t),
\]

where \( \sup_x \left| \frac{\tilde{W}_n(x, t)}{\Delta f_A(x > t \mid x \leq t, t)} \right| \to 0. \)

Theorem 1 shows the linear perturbation theory for \( f_A(x > t \mid x \leq t, t) \): when the observed-data distribution is slightly perturbed, the corresponding extrapolation distribution is also slightly perturbed, and the perturbation in the extrapolation distribution is of a linear order to the perturbation in the observed-data distribution. Using this theorem, we can derive the asymptotic theory for \( \hat{F}_{A,h} \).

**Theorem 2.** Assume (A1), (K1-2), \( h \to 0 \) and \( \frac{\log n}{nh^d} \to 0. \) Then for a pattern-mixture
model with a donor-based identifying restriction $A$,

$$
\hat{F}_{A,h}(x,t) - F_{A,h}(x,t) = O_P \left( \sqrt{\frac{1}{n}} \right) \\
\hat{F}_{A,h}(x,t) - F_A(x,t) = O(h^2)
$$

for each $t = 1, \cdots, d$. Moreover, under the same assumptions,

$$
\hat{f}_{A,h}(x,t) - \bar{f}_{A,h}(x,t) = O_P \left( \sqrt{\frac{1}{nh^d}} \right) \\
\bar{f}_{A,h}(x,t) - f_A(x,t) = O(h^2)
$$

for each $t = 1, \cdots, d$. Thus, if the identifying assumption $A$ is correct, then we immediately have

$$
\hat{F}_{A,h}(x,t)-F(x,t) = O(h^2) + O_P \left( \sqrt{\frac{1}{n}} \right), \\
\hat{f}_{A,h}(x,t)-f(x,t) = O(h^2) + O_P \left( \sqrt{\frac{1}{nh^d}} \right)
$$

for each $t = 1, \cdots, d$.

The first part of Theorem 2 describes the convergence rate of $\hat{F}_{A,h}$. Unlike the case of estimating the density function, estimating the cumulative density function (CDF) has a $\sqrt{n}$ convergence rate even if it is built by integrating a KDE. This result is known in the literature, see, e.g., Reiss (1981); Liu & Yang (2008).

The estimation rate in Theorem 2 implicitly uses a decomposition of the uncertainty:

$$
\hat{F}_{A,h}(x,t) - F(x,t) = \underbrace{\hat{F}_{A,h}(x,t) - \bar{F}_{A,h}(x,t)}_{\text{Stochastic error}} + \underbrace{\bar{F}_{A,h}(x,t) - F_A(x,t)}_{\text{Smoothing/model bias}} + \underbrace{F_A(x,t) - F(x,t)}_{\text{Restriction bias}}.
$$

(24)

The stochastic error comes from the sampling variability. Confidence intervals are often designed to capture this type of uncertainty (later we will see this more explicitly).
The smoothing/model bias comes from the bias of our model. When using a KDE, it is of the order $O(h^2)$. We can reduce its effect by choosing a small smoothing bandwidth (undersmoothing). The restriction bias comes from misspecification of the identifying restriction.

With the convergence rate of $\hat{F}_{A,h}$, the rate for estimating $\theta_{A,h} = \theta(F_{A,h})$ is

$$\hat{\theta}_{A,h} - \theta_{A,h} = O_P\left(\sqrt{\frac{1}{n}}\right),$$

when $\theta$ is a smooth functional (such as being Hadamard differentiable, e.g., van der Vaart 1998).

In addition to the convergence rate, the estimated distribution function $\hat{F}_{A,h}$ also has a beautiful asymptotic behavior as illustrated in the following theorem.

**Theorem 3.** Assume (A1), (K1-2), $h \to 0$ and $\frac{\log n}{nh^2} \to 0$. Then for each $t = 1, \ldots, d$, 

$$\sqrt{n}\left(\hat{F}_{A,h}(\cdot, t) - \bar{F}_{A,h}(\cdot, t)\right)$$

converges weakly to a Brownian bridge in $L_\infty$ norm.

Theorem 3 shows that the difference between the estimated CDF $\hat{F}_{A,h}$ and its population counterpart $\bar{F}_{A,h}$ forms a Brownian bridge. This implies several powerful results. For instance, for any given point $x$ and pattern $t$, $\hat{F}_{A,h}(x, t) - \bar{F}_{A,h}(x, t)$ has asymptotic normality. For any Hadamard differentiable functional $\theta(\cdot)$, $\hat{\theta}_{A,h} - \bar{\theta}_{A,h}$ also has asymptotic normality (see Corollary 5).

We now provide a high-level idea of the proof of Theorem 3. Recall that the CDF estimator can be written as an integral of the PDF estimator. Because of Theorem 1, the difference $\hat{F}_{A,h} - \bar{F}_{A,h}$ can be rewritten as the difference between an integrated KDE and a CDF. Then, we apply the uniform central limit theorem of a smoothed empirical process (the main theorem of van der Vaart (1994) or Theorem 2 of Giné & Nickl (2008)), which implies that the difference between an integrated KDE and its expectation (a CDF) converges uniformly to a Brownian bridge and the result
5.2 Theory of Monte Carlo Estimate

Now we show that the Monte Carlo approach of Algorithm 1 indeed generates points from the estimated full-data distribution.

**Theorem 4.** The observations drawn using Algorithm 1 are such that

\[ X_{i>T_i}^{(1)}, \ldots, X_{i>T_i}^{(V)} \overset{\text{iid}}{\sim} \hat{F}_{A,h}(x_{>T_i} \mid X_{i \leq T_i}, T_i). \]

Here is an intuitive explanation of why Theorem 4 works. Equations (12) and (13) imply that given \( T = t \) and any \( s > t \), the extrapolation density

\[ \hat{f}_{A,h}(x_s \mid x_{<s}, T = t) = \hat{g}_h(x_s \mid x_{<s}, T \in A_{ts}) = \sum_{i=1}^{n} W_i(x_{<s}; A_{ts}) K_s(x_s; X_{is}, h_s), \]

which can be viewed as a density mixture such that with a probability of \( W_i(x_{<s}; A_{ts}) \), we will sample from the density \( K_s(x_s; X_{is}, h_s) \). Essentially, Algorithm 1 is following this feature of a density mixture for each \( s = t + 1, t + 2, \ldots, d \), so the imputed observations are iid from the desired distribution.

5.3 Bootstrap Theory

We now discuss the validity of bootstrap confidence bands and intervals. We first introduce a corollary showing that the estimator \( \hat{\theta}_{A,h} \) is asymptotically normal.

**Corollary 5.** Assume (A1-2), (K1-2), and \( h \to 0, \frac{\log n}{nh^d} \to 0 \). Then

\[ \sqrt{n}(\hat{\theta}_{A,h} - \bar{\theta}_{A,h}) \xrightarrow{D} N(0, \sigma^2), \]

for a constant \( \sigma^2 > 0 \).
This corollary follows from the functional delta method and Theorem 3, so we omit the proof. The crucial assumption here is that the statistical functional $\theta(\cdot)$ is Hadamard differentiable. This assumption holds for many common statistical functionals such as the mean, median, variance, correlation between two variables, etc. The formal definitions and more details are given in Appendix B.

With Corollary 5, we derive the validity of the bootstrap confidence interval as follows.

**Theorem 6.** Under the assumptions (A1-2) and $h \to 0$, $\frac{\log n}{nh^2} \to 0$,

$$\sqrt{n}(\hat{\theta}^*_{A,h} - \bar{\theta}_{A,h}) \xrightarrow{D} N(0,\sigma^2)$$

and

$$\sup_q \left| P\left(\sqrt{n}(\hat{\theta}^*_{A,h} - \bar{\theta}_{A,h}) < q | S_n\right) - P\left(\sqrt{n}(\hat{\theta}_{A,h} - \bar{\theta}_{A,h}) < q \right) \right| \xrightarrow{P} 0.$$

Thus,

$$P(\bar{\theta}_{A,h} \in \hat{C}_{n,\alpha}) = 1 - \alpha + o(1).$$

With the convergence towards a Brownian bridge (Theorem 3), this result follows from the Theorem of the bootstrap for the delta method; see, e.g., Theorem 23.9 of van der Vaart (1998) and Theorem 3.9.11 of van der Vaart & Wellner (1996). Thus, we omit the proof.

Theorem 6 shows that the bootstrap method can recover the uncertainty of $\hat{\theta}_{A,h}$, which further leads to the validity of a bootstrap confidence interval. An advantage of using the bootstrap is that there is no need to calculate $\sigma^2$. If $\sigma^2$ has a closed form and can be estimated, we can use the estimated $\sigma^2$ to construct a normal confidence interval or use the bootstrap $t$-distribution (Hall, 2013) to construct another bootstrap confidence interval. Even when we do not know $\sigma^2$, we can use the sample variance of the bootstrap estimates to estimate this quantity and use it to construct a confidence interval.
5.4 Bootstrap Diagram

Figure 2: Extended Efron’s bootstrap diagram. The following concepts are represented by the arrow types in parenthesis: sampling (solid gray), modeling/density estimation (dashed black), extrapolating (dotted black), reconstructing CDF (solid black), and derivation of parameter of interest (dashed dotted black).

In Theorem 6, we see that the bootstrap provides a valid confidence interval of $\bar{\theta}_{A,h} = \theta(\bar{F}_{A,h})$, the parameter of interest corresponding to $\bar{F}_{A,h}$ in (19). We now explain why this is the underlying population quantity using the concept of bootstrap diagram from Efron (1994). Figure 2 shows an expanded version of Efron’s bootstrap diagram to explicitly illustrate the effect of modeling (or smoothing) and the role of identifying assumptions. Initially we shall think of $\hat{G}$ as the empirical distribution based on the observed sample $S$, and the observed sample is from the true distribution

33
function $G$. During our construction of the estimator $\hat{F}_{A,h}$, we first apply smoothing to $\hat{G}$ to obtain the KDE $\hat{g}_h$ and then combine these densities using the identifying assumption $A$ to obtain an estimate of the extrapolation distribution $\hat{F}_{A,h}(\cdot | \cdot)$ defined in equation (9). Thus, we can view the kernel smoothing as a mapping from $\hat{G}$ to $\hat{G}_h$, where $\hat{G}_h$ denotes the CDF generated by $\hat{g}_h$. The estimator of the joint CDF $\hat{F}_{A,h}$ in equation (10) is constructed using a mapping with two inputs: the extrapolation distribution $\hat{F}_{A,h}(\cdot | \cdot)$ and the empirical distribution function $\hat{G}$. Therefore, in the bootstrap diagram we use two arrows ($\hat{F}_{A,h}(\cdot | \cdot) \rightarrow \hat{F}_{A,h}$ and $\hat{G} \rightarrow \hat{F}_{A,h}$) to denote this. The estimate of the parameter of interest $\hat{\theta}_{A,h} = \theta(\hat{F}_{A,h})$ is simply a plug-in estimate so it can be viewed as a mapping from $\hat{F}_{A,h}$ to $\hat{\theta}_{A,h}$.

In the bootstrap process, we are generating observations from the empirical distribution function $\hat{G}$, so $\hat{G}$ now serves the role of $G$. Therefore, the bootstrap sample $S^\ast$ generates another empirical distribution function $\hat{G}^\ast$, which also leads to $\hat{g}_h^\ast$, the bootstrap estimates of the density functions of the observed-data distribution. Just like the case of original estimate, $\hat{G}_h^\ast$ is the CDF corresponds to $\hat{g}_h^\ast$ and the identifying assumption maps it into the extrapolation distribution $\hat{F}_{A,h}^\ast(\cdot | \cdot)$. The extrapolation distribution $\hat{F}_{A,h}^\ast(\cdot | \cdot)$, together with the bootstrap empirical distribution $\hat{G}^\ast$, leads to the CDF estimator $\hat{F}_{A,h}^\ast$, and the bootstrap estimate of the parameter of interest $\hat{\theta}_{A,h}^\ast = \theta(\hat{F}_{A,h}^\ast)$.

So far, we have explained the middle and the right-hand branches of the bootstrap diagram in Figure 2. We will now explain the left-hand branch, which will also clarify why the population quantity that the confidence interval covers is $\bar{\theta}_{A,h} = \theta(\bar{F}_{A,h})$. The key step is to notice that there is a mapping between $\hat{G}$ to $\hat{G}_h$ and $\hat{G}^\ast$ to $\hat{G}_h^\ast$ that represents the effect of smoothing/modeling. If we start with $G$, we also need to define a quantity $G_h$ that is a quantity based on smoothing $G$. Thus, it is not hard to see that $G_h$ denotes the CDF corresponding to the smoothed densities $g_h = E(\hat{g}_h)$. Then
the identifying assumption \( A \) maps \( G_h \) to an extrapolation distribution \( \bar{F}_{A,h}(\cdot | \cdot) \) defined in equation (20). The quantity \( \bar{F}_{A,h} \) in equation (19) is constructed using the extrapolation distribution \( \bar{F}_{A,h}(\cdot | \cdot) \) and the observed-data distribution \( G \) so again we have two arrows from \( \bar{F}_{A,h}(\cdot | \cdot) \) and \( G \) to \( \bar{F}_{A,h} \). With the full-data distribution \( \bar{F}_{A,h} \), we can view the corresponding parameter of interest, \( \bar{\theta}_{A,h} = \theta(\bar{F}_{A,h}) \), as a mapping from \( \bar{F}_{A,h} \) to \( \bar{\theta}_{A,h} \), which is represented by the dotted dashed arrow at the bottom of Figure 2. It is now easy to see why \( \bar{\theta}_{A,h} \) is the parameter of interest that the bootstrap confidence interval is covering – the bootstrap difference \( \hat{\theta}^*_A - \bar{\theta}_{A,h} \) is mimicking the difference \( \hat{\theta}_{A,h} - \bar{\theta}_{A,h} \).

Although Figure 2 displays the bootstrap diagram when we are using the KDE in the modeling stage, this diagram can be generalized to an arbitrary density estimator or modeling method. If we use a model \( M \) that maps \( \hat{G} \) into a density estimator \( \hat{g}_M \) with a CDF estimator \( \hat{G}_M \), the diagram remains the same except that we replace every element with a subscript \( h \) by a subscript \( M \). For example, one could model the observed-data density functions using Gaussians as in Little (1993), in which case each \( \hat{g}_M(x_r | r) \) will correspond to a fitted Gaussian PDF, and \( \hat{g}_M(r) \) could simply correspond to the empirical frequency of pattern \( r \). With these components, we can construct the extrapolation distribution \( \hat{F}_{A,M}(\cdot | \cdot) \), and with these elements a joint full-data distribution can be constructed in a similar manner as equation (10).

6 Data Analysis

To illustrate the usage of nonparametric pattern-mixture models, we now present an application to the analysis of data coming from a clinical trial on schizophrenia. These data had been previously analyzed under parametric or semiparametric longitudinal
models (e.g., Diggle et al., 2007). The purpose of the trial was to evaluate the effectiveness of four different doses of a new treatment (risperidone, an antipsychotic medication, with 2, 6, 10 or 16 mg/day) compared with placebo and with a standard of care (20 mg/day of haloperidol, a standard antipsychotic), in patients with chronic schizophrenia (Marder & Meibach, 1994). The Positive and Negative Syndrome Scale for Schizophrenia (PANSS) score was measured on patients one week before, the day of, and on weeks 1, 2, 4, 6, and 8 after randomization. In the left panel of Figure 3 we summarize the frequency of dropout times for each arm of the trial. Here for simplicity the new treatment is taken as the arm of 6 mg/day of risperidone, since this was found to be the most effective dose, and so we omit the results for the other risperidone arms. The center and right panels of Figure 3 present the observed means of the PANSS score among those who were last seen in week 4 and those who completed the study.

In this context we are interested in estimating average treatment effects (ATEs) over time. Let $X_j$ denote the PANSS score at time $j$. Denote $\mu_j^N = E(X_j | \text{New})$, where
\( \mu^S_j = E(X_j \mid \text{Standard}) \), and \( \mu^P_j = E(X_j \mid \text{Placebo}) \). The ATEs that we are interested in are \( \mu^N_j - \mu^P_j \), new treatment vs placebo; \( \mu^S_j - \mu^P_j \), standard treatment vs placebo; and \( \mu^N_j - \mu^S_j \), new vs standard treatment. Since the true ATEs are not accessible to us, not even with infinite samples, we focus on estimating ATEs under a given identifying restriction \( A \), and denote the corresponding means as \( \mu^N_{j,A} \), \( \mu^S_{j,A} \), \( \mu^P_{j,A} \) to emphasize the dependence on the assumption.

Our approach estimates the full-data distribution under an assumption \( A \), approximates it via a Monte Carlo procedure, which is then used to evaluate functionals of interest. To implement our methodology, we used Gaussian kernels in the construction of the surrogate estimate (8), using Silverman’s rule (Silverman, 1986) to compute the bandwidths with the observed PANSS scores for each week. We then implemented our Monte Carlo approximation using Algorithm 1, taking \( V = 100 \) Monte Carlo samples. We used the AC, 3NC and NC assumptions, as explained in Section 2.3. We note that the AC restriction is equivalent to the missing at random (MAR) assumption under monotone missingness, and in this case also equivalent to the 5NC restriction. This approach thus provides us with a way of performing sensitivity analysis to the commonly used MAR assumption.

We compute confidence intervals repeating our estimating procedure over 1000 bootstrap samples, as described in Section 4. In Figure 4 we present the point-wise bootstrap 95% confidence intervals and point estimates of the ATEs. Often the main interest is in estimating the ATE at the last time point. Furthermore, one could argue that here the most interesting ATE is \( \mu^N_{8,A} - \mu^S_{8,A} \), as it compares the new treatment with the standard of care at week 8. From Figure 4 we can see that all the confidence intervals for this ATE fall below zero, under each of the three missing data assumptions considered here. This gives us compelling evidence for declaring superiority of the new treatment over the standard of care, as this conclusion seems
Figure 4: Estimated ATEs from a clinical trial on schizophrenia under three different missing-data identifying assumptions. Dashed lines indicate results for the ATEs that compare the new treatment vs the placebo, $\mu_{j,A}^N - \mu_{j,A}^P$; dotted lines compare the standard treatment vs placebo, $\mu_{j,A}^S - \mu_{j,A}^P$; and solid lines compare new vs standard treatment $\mu_{j,A}^N - \mu_{j,A}^S$; for times $j = 1, 2, 4, 6, 8$, and assumptions $A = AC, 3NC, NC$. Vertical lines represent point-wise 95% bootstrap confidence intervals.

to be insensitive to the missing data assumption.

7 Discussion

We studied how to conduct nonparametric inference with missing data under the pattern-mixture model framework. We introduced the concept of donor-based identification under monotone missingness and generalized it to nonmonotone missing data. We proposed an estimator of the full-data distribution based on a surrogate estimator of the observed-data distribution based on kernel smoothing, derived the corresponding convergence rates and proved asymptotic normality. To numerically compute the estimator of functionals of interest, we proposed a Monte Carlo method that allows us to easily sample from the estimated full-data distribution. We also
introduced a bootstrap method for constructing confidence intervals, and we revisited Efron’s bootstrap diagram to explain why and how the bootstrap method works. We presented the underlying theory supporting our methodology, and numerical analyses to illustrate the applicability of our approach in practice.

Our current results were developed under donor-based identification, which cover important identifying restrictions used in pattern-mixture models, but it is of interest to study extensions of our methods to handle other identifying restrictions that analysts might want to specify. Two challenges arise when thinking of working with totally generic restrictions. First, it is not clear how to design a Monte Carlo method for sampling from the estimated full-data distribution under arbitrary restrictions. Second, perturbation theory similar to the one in Theorem 1 needs to be derived, although once one has it, convergence rates, asymptotic normality, and the validity of the bootstrap can all be derived in a similar manner.

Another open problem is how to reduce the computational cost for the bootstrap method when the estimator is approximated via Monte Carlo. Our current approach requires using Monte Carlo (Algorithm 1) for every bootstrap sample. When the sample size is large, this procedure may not be computationally appealing, although one could parallelize it. An interesting observation is that some functionals under some restrictions might be obtainable in closed form, such as the mean under the CC restriction, as shown in Appendix A, thereby avoiding the need for the Monte Carlo approximation.

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A Examples: CC restriction

We provide a detailed analysis under the CC restriction. Consider the case with \( d \) variables. Let \( 1_d \) be a vector of ones of length \( d \). The CC restriction assumes that the extrapolation density has the following form:

\[
f_{cc}(x_r | x_r, r) = g(x_r | x_r, 1_d) = \frac{g(x, 1_d)}{g(x_r, 1_d)} = \frac{g(x, 1_d)}{\int g(x, 1_d) \mu(dx_r)}.
\]

To further simplify the notation, we assume all variables are continuous and all kernel functions and smoothing bandwidths are the same. We use the notation

\[
K_h (x_r; X_{i,r}) \equiv \prod_{j \in r} K_h (x_j; X_{ij}) \equiv \prod_{j=1}^{d} K_j(x_j; X_{ij}, h)^{r_j},
\]

where \( j \in r \) stands for the indices of the elements of \( r \) that are 1 and \( |r| = \sum_{j=1}^{d} r_j \) is the number of 1’s in \( r \).
CDF estimation. The estimator of the extrapolation density is
\[ \hat{f}_{cc,h}(x_r | x_{\bar{r}}, r) = \frac{\hat{g}_h(x, 1_d)}{\int \hat{g}_h(x, 1_d) \mu(dx_r)}, \tag{25} \]
where
\[ \hat{g}_h(x, 1_d) = \frac{1}{n} \sum_{i=1}^{n} K_h(x; X_i) I(R_i = 1_d). \]
Namely, \( \hat{g}_h(x, 1_d) \) is the estimated PDF of all study variables using the observations without any missing entries. Let \( \hat{F}_{cc,h}(x_r | x_{\bar{r}}, r) = \int_{-\infty}^{x_r} \hat{f}_{cc,h}(x'_{\bar{r}} | x_{r}, r) \mu(dx'_r) \) be the estimator of \( F_{cc}(x_r | x_{\bar{r}}, r) \) using the KDE. The CDF of the full-data distribution is estimated by
\[ \hat{F}_{cc,h}(x, r) = \int_{-\infty}^{x_r} \hat{F}_{cc,h}(x_{\bar{r}} | x_r, r) \hat{G}(dx'_r, r) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_{cc,h}(x_{\bar{r}} | X_{i,r}, r) I(R_i = r, X_{i,r} \leq x_r). \tag{26} \]

PDF estimation. If the goal is to estimate the PDF of the full-data distribution, we use
\[ \hat{f}_{cc,h}(x, r) = \hat{g}_h(x, r) \hat{f}_{cc,h}(x_{\bar{r}} | x_r, r) = \frac{\hat{g}_h(x, r)}{\hat{g}_h(x, 1_d)} \hat{g}_h(x, 1_d), \tag{27} \]
where
\[ \hat{g}_h(x, r) = \frac{1}{n} \sum_{i=1}^{n} K_h(x_r; X_{i,r}) I(R_i = r). \tag{28} \]

Estimating the mean parameter. As a special case, we analyze the estimator for the mean parameter. To simplify the problem, we consider the case with two study variables \( X_1 \) and \( X_2 \) so the possible missing patterns are \( \mathcal{R} = \{(00), (10), (01), (11)\} \). The goal is to estimate the marginal mean of \( X_2 \), i.e., \( \mu_2 = \mathbb{E}(X_2) \), under the CC
restriction. Using equation (26),

\[ \hat{\mu}_2 = \int x_2 \hat{F}_{cc,h}(dx_1 dx_2) \]

\[ = \sum_r \int \int x_2 \hat{f}_{cc,h}(x_1 | x_r, r) \hat{G}(dx_r) \]

\[ = \sum_r \int \int x_2 \hat{f}_{cc,h}(x_1 | x_r, r) \mu(dx_r) \hat{G}(dx_r, r) \]

\[ = \sum_r \int \int x_2 \hat{f}_{cc,h}(x_1 | x_r, r) \mu(dx_r) \hat{G}(dx_r | r) \hat{g}(r) \]

\[ = \sum_r \hat{g}(r) \hat{\mu}_{2,r}, \]

where \( \hat{\mu}_{2,r} = \int \int x_2 \hat{f}_{cc,h}(x_1 | x_r, r) \mu(dx_r | r) \) and \( \hat{g}(r) = n_r / n \), with \( n_r = \sum_{i=1}^n I(R_i = r) \).

When \( r = (1, 1) \), we have

\[ \hat{\mu}_{2,11} = \int \int x_2 \hat{G}(dx_1 dx_2 | 11) = \int x_2 \hat{G}(dx_2 | 11) = n_{11}^{-1} \sum_{i=1}^n X_{i,2} I(R_i = 11). \]

When \( r = (0, 0) \), we have

\[ \hat{\mu}_{2,00} = \int \int x_2 \hat{f}_{cc,h}(x_1, x_2 | 00) \mu(dx_1 dx_2) \]

\[ = \int \int x_2 \hat{g}_h(x_1, x_2 | 11) \mu(dx_1 dx_2) \]

\[ = n_{11}^{-1} \sum_{i=1}^n \int x_2 K_h(x_2; X_{i,2}) \mu(dx_2) I(R_i = 11) \]

\[ = n_{11}^{-1} \sum_{i=1}^n X_{i,2} I(R_i = 11). \]
When $r = (0, 1)$, we have
\[
\hat{\mu}_{2,01} = \int \int x_2 \hat{f}_{cc,h}(x_1 \mid x_2, 01) \mu(dx_1) \hat{G}(dx_2 \mid 01) \\
= \int x_2 \hat{G}(dx_2 \mid 01) \\
= n_{01}^{-1} \sum_{i=1}^{n} X_{i,2} I(R_i = 01).
\]

The case for $r = (1, 0)$ is more involved:
\[
\hat{\mu}_{2,10} = \int \int x_2 \hat{f}_{cc,h}(x_2 \mid x_1, 10) \mu(dx_2) \hat{G}(dx_1 \mid 10) \\
= \frac{1}{n_{10}} \sum_{i=1}^{n} \int x_2 \hat{f}_{cc,h}(x_2 \mid X_{i,1}, 10) \mu(dx_2) I(R_i = 10).
\]

Recall that the CC restriction implies $\hat{f}_{cc,h}(x_2 \mid x_1, 10) = \frac{\hat{g}_h(x_1, x_2, 11)}{\hat{g}_h(x_1, 11)}$. Thus,
\[
\int x_2 \hat{f}_{cc,h}(x_2 \mid x_1, 10) \mu(dx_2) = \frac{\int x_2 \hat{g}_h(x_1, x_2, 11) \mu(dx_2)}{\hat{g}_h(x_1, 11)} \\
= \frac{\sum_{j=1}^{n} X_{j,2} K_h(x_1; X_{j,1}) I(R_j = 11)}{\sum_{k=1}^{n} K_h(x_1; X_{k,1}) I(R_k = 11)} \\
= \sum_{j=1}^{n} X_{j,2} W_j(x_1),
\]

where $W_j(x_1) = \frac{K_h(x_1; X_{j,1}) I(R_j=11)}{\sum_{k=1}^{n} K_h(x_1; X_{k,1}) I(R_k=11)}$ are such that $\sum_{i=1}^{n} W_i(x_1) = 1$. With this, we can then obtain the estimator
\[
\hat{\mu}_{2,10} = \frac{1}{n_{10}} \sum_{i,j=1}^{n} X_{j,2} W_j(X_{i,1}) I(R_i = 10) \\
= \frac{1}{n_{10}} \sum_{i=1}^{n} \sum_{j=1}^{n} X_{j,2} \frac{K_h(X_{i,1}; X_{j,1}) I(R_j = 11)}{\sum_{k=1}^{n} K_h(X_{i,1}; X_{k,1}) I(R_k = 11)} I(R_i = 10) \\
= \frac{1}{n_{10}} \sum_{j=1}^{n} X_{j,2} \sum_{i=1}^{n} \frac{K_h(X_{i,1}; X_{j,1}) I(R_i = 10)}{\sum_{k=1}^{n} K_h(X_{i,1}; X_{k,1}) I(R_k = 11)} I(R_j = 11).
\]
We note that \( \hat{\mu}_{2,10} \) is essentially a weighted average of the fully observed data, so we may rewrite it as

\[
\hat{\mu}_{2,10} = \sum_{j=1}^{n} \alpha_j X_{j,2} I(R_j = 11),
\]

where \( \alpha_j = \frac{1}{n_{10}} \sum_{i=1}^{n} \frac{K_h(X_i; X_{j,1}) I(R_i = 10)}{K_h(X_i; X_{k,1}) I(R_k = 11)} \). Therefore, combining all estimators \( \hat{\mu}_{2,r} \) together, we conclude that

\[
\hat{\mu}_2 = \sum_r g(r) \hat{\mu}_{2,r} = \sum_{i=1}^{n} X_{i,2} \omega_i, \tag{29}
\]

where

\[
\omega_i = \begin{cases} 
\frac{1+n_{00}+n_{10}}{n}, & \text{if } R_i = 11, \\
\frac{1}{n}, & \text{if } R_i = 01, \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( \omega_i \geq 0 \) and \( \sum_{i=1}^{n} \omega_i = 1 \) so the estimator is essentially a weighted estimator, where the weight is determined by the identifying assumption.

### B Hadamard Differentiation

The Hadamard differentiation is one type of functional derivative that is key for a statistical functional to have asymptotic normality. Here we briefly describe the definition of Hadamard differentiation.

Let \((\mathbb{D}_1, \| \cdot \|_{\mathbb{D}_1})\) and \((\mathbb{D}_2, \| \cdot \|_{\mathbb{D}_2})\) be two normed spaces and let \( \Psi : \mathbb{D}_1 \mapsto \mathbb{D}_2 \) be a mapping. \( \Psi \) is called Hadamard differentiable at \( \omega \in \mathbb{D}_1 \) with a differentiation \( \dot{\Psi}_\omega : \mathbb{D}_1 \mapsto \mathbb{D}_2 \) if for any \( \eta_t \to \eta \),

\[
\lim_{t \to 0} \left\| \frac{\Psi(\omega + t \cdot \eta_t) - \Psi(\omega) - \dot{\Psi}_\omega(\eta)}{t} \right\|_{\mathbb{D}_2} = 0.
\]
The Hadamard differentiation is commonly used to derive the bootstrap theory. Roughly speaking, if a statistical functional $\theta : \mathcal{F} \mapsto \mathbb{R}$ is Hadamard differentiable at $F_0 \in \mathcal{F}$ and $\sqrt{n}(\hat{F}_n - F_0) \overset{D}{\to} \mathcal{B}$ for some random process $\mathcal{B}$, then
\begin{equation}
\sqrt{n} \left( \theta(\hat{F}_n) - \theta(F_0) \right) \overset{D}{\to} \dot{\theta}_{F_0}(\mathcal{B}).
\end{equation}
Equation (30) is known as the functional delta method (van der Vaart, 1998). Many common population quantities can be expressed as a Hadamard-differentiable statistical functional; for instance, population mean, population quantiles, population median, correlation between two variables, regression coefficients.

C Proofs

For any function $\eta(x)$ we write $\eta(x)dx \equiv \eta(x)\mu(dx)$ to simplify the notation for integration.

**Proof.** [Proof of Theorem 1]

Because
\begin{equation*}
f_A(x > t \mid x \leq t, t) = \prod_{s=t+1}^{d} f_A(x_s \mid x < s, t) = \prod_{s=t+1}^{d} g(x_s \mid x < s, T \in A_{ts}).
\end{equation*}
we have
\begin{equation*}
\Delta f_A(x > t \mid x \leq t, t) = \prod_{s=t+1}^{d} \tilde{g}(x_s \mid x < s, T \in A_{ts}) - \prod_{s=t+1}^{d} g(x_s \mid x < s, T \in A_{ts}).
\end{equation*}
Using the fact that
\begin{equation*}
\tilde{g}(x_s \mid x < s, T \in A_{ts}) = g(x_s \mid x < s, T \in A_{ts}) + \Delta g(x_s \mid x < s, T \in A_{ts}),
\end{equation*}
where $\Delta g(x_s \mid x_{<s}, T \in A_{ts})$ is a small quantity (due to equation (23)), the above equality becomes

$$
\Delta f_A(x_{>t} \mid x_{\leq t}, t) = \prod_{s=t+1}^{d} \tilde{g}(x_s \mid x_{<s}, T \in A_{ts}) - \prod_{s=t+1}^{d} g(x_s \mid x_{<s}, T \in A_{ts})
$$

$$
= \prod_{s=t+1}^{d} \left[ g(x_s \mid x_{<s}, T \in A_{ts}) + \Delta g(x_s \mid x_{<s}, T \in A_{ts}) \right] - \prod_{s=t+1}^{d} g(x_s \mid x_{<s}, T \in A_{ts})
$$

$$
= \left\{ \sum_{s=t+1}^{d} \frac{\Delta g(x_s \mid x_{<s}, T \in A_{ts})}{g(x_s \mid x_{<s}, T \in A_{ts})} \right\} \prod_{\tau=t+1}^{d} g(x_{\tau} \mid x_{<\tau}, t \in A_{\tau}) + O(\Delta_1^2),
$$

where $O(\Delta_1^2)$ involves products of two small terms ($\Delta$ of some functions) so it is negligible. Note that the identifying restriction implies that $\prod_{\tau=t+1}^{d} g(x_{\tau} \mid x_{<\tau}, t \in A_{\tau}) = f_A(x_{>t} \mid x_{\leq t}, t)$, so we can rewrite the above as

$$
\frac{\Delta f_A(x_{>t} \mid x_{\leq t}, t)}{f_A(x_{>t} \mid x_{\leq t}, t)} = \sum_{s=t+1}^{d} \frac{\Delta g(x_s \mid x_{<s}, T \in A_{ts})}{g(x_s \mid x_{<s}, T \in A_{ts})} + O(\Delta_1^2). 
$$

(31)

Using the fact that

$$
g(x_s \mid x_{<s}, T \in A_{ts}) = \frac{g(x_s \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})},
$$

we conclude that

$$
\Delta g(x_s \mid x_{<s}, T \in A_{ts}) = \frac{\tilde{g}(x_s \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} - \frac{g(x_{<s} \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} - \frac{g(x_{<s} \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} g(x_{<s} \mid T \in A_{ts})
$$

$$
= \frac{\Delta g(x_s \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} - \frac{\Delta g(x_{<s} \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} g(x_{<s} \mid T \in A_{ts}) + O(\Delta_2^2),
$$

where $O(\Delta_2^2)$ is again a quantity involving the product of two or more small terms so it is negligible. Diving both sides by $g(x_s \mid x_{<s}, T \in A_{ts})$, we conclude

$$
\frac{\Delta g(x_s \mid x_{<s}, T \in A_{ts})}{g(x_s \mid x_{<s}, T \in A_{ts})} = \frac{\Delta g(x_{<s} \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} - \frac{\Delta g(x_{<s} \mid T \in A_{ts})}{g(x_{<s} \mid T \in A_{ts})} g(x_{<s} \mid T \in A_{ts}) + O(\Delta_2^2).
$$

51
Putting this back to equation (31), we conclude

$$
\frac{\Delta f_A(x > t | x \leq t, t)}{f_A(x > t | x \leq t, t)} = \sum_{s=t+1}^{d} \left\{ \frac{\Delta g(x \leq s | T \in A_{ts})}{g(x \leq s | T \in A_{ts})} - \frac{\Delta g(x < s | T \in A_{ts})}{g(x < s | T \in A_{ts})} \right\} + O(\Delta_1^2 + \Delta_2^2),
$$

which is the desired result by identifying \( \tilde{W}_n = O(\Delta_1^2 + \Delta_2^2) \). Note that since \( O(\Delta_1^2 + \Delta_2^2) \) involves product of two small quantities, it is easy to see that \( \frac{O(\Delta_1^2 + \Delta_2^2)}{\Delta f_A(x > t | x \leq t, t)} \to 0 \) uniformly for all \( x \). □

**Proof.** [Proof of Theorem 2]

**Part 1: convergence of CDF estimator.**

Recall that

$$
\bar{F}_{A,h}(x > t | x \leq t, t) = \int_{-\infty}^{x > t} \bar{f}_{A,h}(x' > t | x \leq t, t)dx'.
$$

We define

$$
\hat{F}^\dagger_{A,h}(x, t) = \sum_t \frac{1}{n} \sum_{i=1}^{n} \bar{F}_{A,h}(x > t | X_{i,r}, t) I(T_i = t, X_{i,\leq t} \leq x \leq t)
$$

for each \( t \). We will use this quantity to decompose the uncertainty of \( \hat{F}_{A,h}(x) \). Specifically, We bound the rate of \( \hat{F}_{A,h} - \bar{F}_{A,h} \) using the decomposition

$$
\hat{F}_{A,h} - \bar{F}_{A,h} = (I) + (II)
$$

and control the rates of (I) and (II). After deriving this rate, we will then analyze the bias \( F_{A,h} - F_A \).

Because \( \hat{F}^\dagger_{A,h} \) is the summation of IID random elements, the rate of (II) is easier to
derive so we focus on it first. Because of the IID assumption,

\[
E \left\{ \frac{1}{n} \sum_{i=1}^{n} \tilde{F}_{A,h}(x > t | X_{i \leq t}, t) I(T_i = t, X_{i \leq t} \leq x_{\leq t}) \right\}
\]

\[
= E \left\{ \tilde{F}_{A,h}(x > t | X_{1 \leq t}, t) I(T_1 = t, X_{1 \leq t} \leq x_{\leq t}) \right\}
\]

\[
= \int_{-\infty}^{x_{\leq t}} \int_{-\infty}^{x_{\geq t}} \tilde{f}_{A,h}(x' > t | x'_{\leq t}, t) g(x'_{\leq t}, t) dx'_{\leq t} dx'_{\geq t}
\]

\[
= \int_{-\infty}^{x'_{\leq t}=x_{\leq t}} \int_{x'_{\geq t}=x_{\geq t}} \tilde{f}_{A,h}(x' > t | x'_{\leq t}, t) G(dx'_{\leq t}, t) dx'_{\geq t}
\]

\[
= \tilde{F}_{A,h}(x, t)
\]

for each \( t = 1, \cdots, d \). Therefore,

\[
E(\tilde{F}_{A,h}^\dagger) = \tilde{F}_{A,h}
\]

so we only need to focus on the variance. The variance is very easy to derive because the variance of each individual

\[
\text{Var}(\tilde{F}_{A,h}(x > t | X_{1 \leq t}, t) I(T_1 = t, X_{1 \leq t} \leq x_{\leq t})) \leq 1
\]

because both \( F_A \) and indicator function are uniformly bounded by 1. So \( \text{Var}(\tilde{F}_{A,h}^\dagger) = O(1/n) \), which implies

\[
(II) = O_P \left( \frac{1}{\sqrt{n}} \right).
\]

To bound (I), note that

\[
\tilde{F}_{A,h}(x, t) - \tilde{F}_{A,h}^\dagger(x, t) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \tilde{F}_{A,h}(x > t | X_{i \leq t}, t) - \tilde{F}_{A,h}(x > t | X_{i \leq t}, t) \right\}
\]

\[
\times I(T_i = t, X_{i \leq t} \leq x_{\leq t}).
\]

(32)

So the key is to bound the difference

\[
\tilde{F}_{A,h}(x > t | X_{i \leq t}, t) - \tilde{F}_{A,h}(x > t | X_{i \leq t}, t)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{x_{\geq t}} \left\{ \tilde{f}_{A,h}(x' > t | X_{i \leq t}, t) - \tilde{f}_{A,h}(x' > t | X_{i \leq t}, t) \right\} dx'_{\geq t},
\]
which boils down to the difference between the density estimator.

Using Theorem 1,
\[
\hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) - \hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) = \hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) \times \sum_{s=t+1}^{d} \left\{ \frac{\hat{g}_h(x_{<s} | T \in A_{ts}) - \hat{g}_h(x_{<s} | T \in A_{ts})}{\hat{g}_h(x_{<s} | T \in A_{ts})} \right\} + \tilde{W}_n(x),
\]

(33)

where \(\tilde{W}_n(x)\) is a negligible term. Recall that
\[
\hat{g}_h(x_{<s} | T \in A_{ts}) = \frac{1}{n_{ts}} \sum_{i=1}^{n} K_h(x_{<s}; X_{i,s}) I(T_i \in A_{ts})
\]
\[
\hat{g}_h(x_{<s} | T \in A_{ts}) = \frac{1}{n_{ts}} \sum_{i=1}^{n} K_h(x_{<s}; X_{i,s}) I(T_i \in A_{ts})
\]
are both KDEs and \(\hat{g}_h(x_{<s} | T \in A_{ts}) = \mathbb{E}(\hat{g}_h(x_{<s} | T \in A_{ts}))\) and \(\hat{g}_h(x_{<s} | T \in A_{ts}) = \mathbb{E}(\hat{g}_h(x_{<s} | T \in A_{ts}))\) and \(n_{ts} = \sum_{i=1}^{n} I(T_i \in A_{ts}).\)

Let
\[
\mathcal{E}_{1,s}(x,t) = \hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) \frac{\hat{g}_h(x_{<s} | T \in A_{ts}) - \hat{g}_h(x_{<s} | T \in A_{ts})}{\hat{g}_h(x_{<s} | T \in A_{ts})},
\]
\[
\mathcal{E}_{2,s}(x,t) = \hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) \frac{\hat{g}_h(x_{<s} | T \in A_{ts}) - \hat{g}_h(x_{<s} | T \in A_{ts})}{\hat{g}_h(x_{<s} | T \in A_{ts})},
\]

(34)

Equation (33) can be written as
\[
\hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) - \hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) = \sum_{s=t+1}^{d} \left\{ \mathcal{E}_{1,s}(x,t) - \mathcal{E}_{2,s}(x,t) \right\} + \tilde{W}_n(x,t).
\]

Because the difference \(F_{A,h}(x_{>t} | X_{i,s}, t) - \tilde{F}_{A,h}(x_{>t} | X_{i,s}, t)\) comes from integrating
\[
\hat{F}_{A,h}(x_{>t} | X_{i,s}, t) - \hat{F}_{A,h}(x_{>t} | X_{i,s}, t),
\]
ignoring the negligible terms leads to
\[
\hat{F}_{A,h}(x_{>t} | X_{i,s}, t) - \hat{F}_{A,h}(x_{>t} | X_{i,s}, t) = \sum_{s=t+1}^{d} \int_{-\infty}^{x_{>t}} \mathcal{E}_{1,s}(x_{>t}, t) dx_{>t} - \int_{-\infty}^{x_{>t}} \mathcal{E}_{2,s}(x_{>t}, t) dx_{>t},
\]

54
So we only need to bound each quantity \( \int_{-\infty}^{x>_{t}} \mathcal{E}_{j,s}(x',t)dx'_{>t} \) for \( j = 1,2 \) and \( s \in \{t+1, \ldots, d\} \).

Because \( \mathcal{E}_{j,s}(x',t) \) is essentially a KDE minus its expectation and rescale by a function, the quantity \( \int_{-\infty}^{x>_{t}} \mathcal{E}_{j,s}(x',t)dx'_{>t} \) is just the corresponding convergence rate of using this as a CDF estimator. Using the convergence rate of the CDF estimator from the KDE (Reiss, 1981; Liu & Yang, 2008) along with our assumption on the smoothing bandwidth \( \frac{\log n}{nh^{d}} \to 0 \), we conclude that

\[
\mathbb{E}\left( \int_{-\infty}^{x>_{t}} \mathcal{E}_{j,s}(x',t)dx'_{>t} \right) = o\left( \frac{1}{n} \right),
\]

\[
\text{Var}\left( \int_{-\infty}^{x>_{t}} \mathcal{E}_{j,s}(x',t)dx'_{>t} \right) = O\left( \frac{1}{n} \right),
\]

for \( s \in \{t+1, \ldots, d\} \) and \( j = 1,2 \) and uniformly for all \( x \leq t \). Therefore,

\[
\mathbb{E}\left( \hat{F}_{A,h}(x_{>t} | x \leq_{t}, t) \right) - \bar{F}_{A,h}(x_{>t} | x \leq_{t}, t) = o\left( \frac{1}{n} \right),
\]

\[
\text{Var}\left( \hat{F}_{A,h}(x_{>t} | x \leq_{t}, t) \right) = O\left( \frac{1}{n} \right).
\]

Note that the construction of \( \mathcal{E}_{j,s} \) does not involve data points with \( T_{i} = t \), so the convergence rate will not change if we are conditioning on \( \{(X_{i},T_{i}) : T_{i} = t\} \). Thus, using the law of total expectation and variance and applying the above results into equation (32), we conclude that

\[
(I) = \hat{F}_{A,h}(x,t) - \bar{F}_{A,h}(x,t) = O_{P}\left( \frac{1}{\sqrt{n}} \right).
\]

To analyze the bias \( \bar{F}_{A,h} - F_{A} \), note that the difference between \( \bar{F}_{A,h}(x) - F_{A}(x) \) is due to the quantity

\[
\bar{F}_{A,h}(x_{>t} | x \leq_{t}, t) - F_{A}(x_{>t} | x \leq_{t}, t) = \int_{-\infty}^{x>_{t}} \left\{ \bar{f}_{A,h}(x_{>t} | x \leq_{t}, t) - f_{A}(x_{>t} | x \leq_{t}, t) \right\} dx'_{>t}.
\]
Theorem 1 shows that the difference $\bar{f}_{A,h}(x > t \mid x \leq t, t) - f_A(x > t \mid x \leq t, t)$ is due to the difference $f_h(x \leq s \mid T \in A_{ts}) - f(x_s \mid T \in A_{ts})$, which is the bias rate of the KDE $O(h^2)$ under the regular smoothness condition (assumption (A1) and (K1)). Therefore, we conclude that

$$\bar{F}_{A,h}(x,t) - F_A(x,t) = O(h^2).$$

**Part 2: convergence of PDF estimator.**

Recall that

$$\bar{f}_{A,h}(x,t) = \hat{g}(t)\bar{g}_h(x \leq t \mid t)\bar{f}_{A,h}(x > t \mid x \leq t, t),$$

$$\bar{f}_{A,g}(x,t) = g(t)\bar{g}_h(x \leq t \mid t)\bar{f}_{A,h}(x > t \mid x \leq t, t),$$

$$f_A(x,t) = g(t)g(x \leq t \mid t)f_A(x > t \mid x \leq t, t).$$

We first derive the bound on $\hat{f}_{A,h}(x,t) - \bar{f}_{A,h}(x,t)$.

It is clear that the difference comes from $\hat{g}(t) - g(t)$ and $\hat{g}_h(x \leq t \mid t) - g_h(x \leq t \mid t)$ and $\bar{f}_{A,h}(x > t \mid x \leq t, t) - \bar{f}_{A,h}(x > t \mid x \leq t, t)$. The first quantity is just the population proportion versus the empirical ratio so the rate is $\hat{g}(t) - g(t) = O_P\left(\sqrt{1/n}\right)$. The second quantity is the classical result about the KDE versus its expectation so the convergence rate is

$$\hat{g}_h(x \leq t \mid t) - g_h(x \leq t \mid t) = O_P\left(\sqrt{1/nh^l}\right),$$

see, e.g. Wasserman (2006) and Scott (2015). The last quantity can be reduced to the
problem of KDE versus its expectation using Theorem 1 and thus, we conclude
\[
\hat{f}_{A,h}(x_{>t} | x_{\leq t}, t) - \tilde{f}_{A,h}(x_{>t} | x_{\leq t}, t) = \sum_{s=t+1}^{d} O_P(\hat{g}_h(x_{\leq s} | T \in A_{ts}) - \tilde{g}_h(x_{\leq s} | T \in A_{ts})) \\
+ O_P(\tilde{g}_h(x_{<s} | T \in A_{ts}) - \bar{g}_h(x_{<s} | T \in A_{ts})) \\
= \sum_{s=t+1}^{d} O_P \left( \sqrt{\frac{1}{nh^d}} \right) \\
= O_P \left( \sqrt{\frac{1}{nh^d}} \right).
\]

Therefore, after combing all the difference, the dominant term is \(O_P \left( \sqrt{\frac{1}{nh^d}} \right)\) so we conclude
\[
\hat{f}_{A,h}(x, t) - \tilde{f}_{A,h}(x, t) = O_P \left( \sqrt{\frac{1}{nh^d}} \right).
\]

The analysis on \(\tilde{f}_{A,h}(x, t) - f_{A}(x, t)\) is just the bias analysis of the KDE. Using the bias of the KDE and the smoothness condition (A1), \(\tilde{g}_h(x_{\leq t} | t) - g(x_{\leq t} | t) = O(h^2)\). Moreover, by applying Theorem 1, the the difference \(\tilde{f}_{A,h}(x_{>t} | x_{\leq t}, t) - \bar{f}_{A,h}(x_{>t} | x_{\leq t}, t)\) is the difference between KDEs and their expectations, which is the bias of the KDE. So we conclude \(\tilde{f}_{A,h}(x_{>t} | x_{\leq t}, t) - f_{A}(x_{>t} | x_{\leq t}, t) = O(h^2)\) which completes the proof. \(\square\)
Proof. [Proof of Theorem 3] First note that the difference can be written as

\[
\hat{F}_{A,h}(x,t) - \bar{F}_{A,h}(x,t) = \left\{ \frac{1}{n} \sum_{i=1}^{n} \hat{F}_{A,h}(x_{>t} \mid X_{i\leq t}, t) \times \right.
\]

\[
I(T_i = t, X_{i\leq t} \leq x_{\leq t}) - \mathbb{E}\left( F_{A,h}(x_{>t} \mid X_{1\leq t}, t) \right) \}
\]

\[
= \left\{ \int_{x'_{\leq t} = -\infty}^{x'_{\leq t} = x_{\leq t}} \hat{F}_{A,h}(x_{>t} \mid x'_{\leq t}, t)\hat{G}(dx'_{\leq t} \mid t) \right.
\]

\[
- \int_{x'_{\leq t} = -\infty}^{x'_{\leq t} = x_{\leq t}} \bar{F}_{A,h}(x_{>t} \mid x'_{\leq t}, t)G(dx'_{\leq t} \mid t) \}
\]

\[
= \mathcal{E}_{n,t}'(x).
\]

Thus, we will focus on a given \( t \) and show that \( \sqrt{n}\mathcal{E}_{n,t}' \) converges to a Brownian bridge.

To simplify the problem, we define

\[
\Delta_{1,n}(x_{\leq t}, t) = \hat{G}(x_{\leq t}, t) - G(x_{\leq t}, t),
\]

\[
\Delta_{1,n}(dx_{\leq t}, t) = \hat{G}(dx_{\leq t}, t) - G(dx_{\leq t}, t),
\]

\[
\Delta_{2,n}(x_{>t} \mid x_{\leq t}, t) = \hat{F}_{A,h}(x_{>t} \mid x_{\leq t}, t) - \bar{F}_{A,h}(x_{>t} \mid x_{\leq t}, t),
\]

\[
\Delta_{2,n}(dx_{>t} \mid x_{\leq t}, t) = \hat{F}_{A,h}(dx_{>t} \mid x_{\leq t}, t) - \bar{F}_{A,h}(dx_{>t} \mid x_{\leq t}, t).
\]

The quantity \( \Delta_{1,n}(x_{\leq t}, t) \) is just the difference between an empirical distribution function and the corresponding CDF so the DKW inequality (Dvoretzky et al., 1956) implies

\[
\sup_{x_{\leq t}, t} |\Delta_{1,n}(x_{\leq t}, t)| = O_P \left( \frac{1}{\sqrt{n}} \right). \tag{35}
\]

Also, because \( \Delta_{2,n}(x_{>t} \mid x_{\leq t}, t) \) is related to the smoothed CDF estimator versus its expectation, it is known that

\[
\sup_{x_{>t}, x_{\leq t}, t} |\Delta_{2,n}(x_{>t} \mid x_{\leq t}, t)| = O_P \left( \sqrt{\frac{\log n}{n}} \right); \tag{36}
\]

58
see, e.g., Reiss (1981) and Liu & Yang (2008).

Using \( \Delta_{1,n} \) and \( \Delta_{2,n} \), we can rewrite \( \sqrt{n} \mathcal{E}_{n,t} \) as

\[
\sqrt{n} \mathcal{E}_{n,t} = \sqrt{n} \int_{x_{t}^{*}=-\infty}^{x_{t}^{*}=x_{t}} \hat{F}_{A,h}(x_{t}^{*} | x_{t}^{*} \leq t) \hat{G}(dx_{t}^{*}, t) - \int_{x_{t}^{*}=-\infty}^{x_{t}^{*}=x_{t}} \hat{F}_{A,h}(x_{t}^{*} | x_{t}^{*} \leq t) G(dx_{t}^{*}, t) \\
= \int_{x_{t}^{*}=-\infty}^{x_{t}^{*}=x_{t}} \hat{F}_{A,h}(x_{t}^{*} | x_{t}^{*} \leq t) \sqrt{n} \Delta_{1,n}(dx_{t}^{*}, t) \\
+ \int_{x_{t}^{*}=-\infty}^{x_{t}^{*}=x_{t}} \sqrt{n} \Delta_{2,n}(x_{t}^{*} | x_{t}^{*} \leq t) G(dx_{t}^{*}, t) \\
+ \sqrt{n} \int_{x_{t}^{*}=-\infty}^{x_{t}^{*}=x_{t}} \Delta_{2,n}(x_{t}^{*} | x_{t}^{*} \leq t) \Delta_{1,n}(dx_{t}^{*}, t).
\]

To show the convergence toward a Brownian bridge, we separately analyze each term. Due to equation (35) and (36), (III) = \( o_P(1) \) so we only need to focus on (I) and (II).

**Analysis of (I):**

Observe that the quantity

\[
\sqrt{n} \Delta_{1,n}(x_{t}^{*}, t) = \sqrt{n} \left( \hat{G}(x_{t}^{*}, t) - G(x_{t}^{*}, t) \right) = \mathcal{G}_t(x_{t}^{*})
\]

defines an empirical process. Thus, (I) can be written as

\[
(I) = \int \hat{F}_{A,h}(x_{t}^{*} | x_{t}^{*} \leq t) I(x_{t}^{*} \leq x_{t}) \mathcal{G}_t(dx_{t}^{*}) = \mathcal{G}_t(f_{x_{t}^{*}, x_{t}}),
\]

where the last equality is the common expression in the empirical process theory (van der Vaart & Wellner, 1996; van der Vaart, 1998) and \( f_{x_{t}^{*}, x_{t}} : \mathbb{R}^t \mapsto \mathbb{R} \) is the function \( f_{x_{t}^{*}, x_{t}}(y) = \hat{F}_{A,h}(x_{t}^{*} | y, t) I(y \leq x_{t}) \) with two indices \( x_{t}^{*}, x_{t} \).
show that (I) converges to a Brownian bridge, we need to show that the class of functions

\[ \mathcal{F}_{0,h} = \{ f_{x>t,x\leq}(y) = \bar{F}_{A,h}(x_t \mid y, t) I(y \leq x_t) : (x_t, x_{\leq}) \in \mathcal{X} \} \]

has some nice property for every \( h \leq 1 \). Let \( \mathcal{X}_{>t} \) be the support of \( x_{>t} \) and \( \mathcal{X}_{\leq t} \) be the support of \( x_{\leq t} \). Due to assumption (A1), the derivative of \( \bar{F}_{A,h}(x_t \mid y, t) \) with respect to \( x_{>t} \) is uniformly bounded so the class

\[ \mathcal{F}_{1,h} = \{ f_{x>t}(y) = \bar{F}_{A,h}(x_t \mid y, t) : x_{>t} \in \mathcal{X}_{>t} \} \]

has an \( \epsilon \)-bracketing number of the order \( O(1/\epsilon^{d-t}) \) (see example 19.7 of van der Vaart 1998) so its uniform entropy integral converges (loosely speaking, \( \int_0^\delta \log (\epsilon\text{-bracketing number}) \, d\epsilon \) converges to 0 when \( \delta \to 0 \)) for every \( h \leq 1 \). Moreover,

\[ \mathcal{F}_{2} = \{ f_{x\leq}(y) = I(y \leq x_{\leq}) : x_{\leq} \in \mathcal{X}_{\leq} \} \]

has an \( \epsilon \)-bracketing number of the order \( O(1/\epsilon^{2t}) \) (see example 19.7 of van der Vaart 1998; the power is \( 2t \) because we have \( t \) variables) so again its uniform entropy integral converges. Thus, the class \( \mathcal{F}_{0,h} = \{ f_1 \cdot f_2 : f_1 \in \mathcal{F}_{1,h}, f_2 \in \mathcal{F}_{2} \} \) has a covering number shrinking at a polynomial rate of \( \epsilon \) so the uniform entropy integral converges for every \( h \leq 1 \). Therefore, according to Theorem 19.28 in van der Vaart (1998), (I) converges to a Brownian bridge.

**Analysis of (II):** Recalled that from Theorem 1,

\[
\hat{f}_{A,h}(x_{>t} \mid x_{\leq}, t) - \bar{f}_{A,h}(x_{>t} \mid x_{\leq}, t) \\
= \hat{f}_{A,h}(x_{>t} \mid x_{\leq}, t) \times \sum_{s=t+1}^{d} \left\{ \bar{g}_h(x_{<s} \mid T \in A_{ts}) - \bar{g}_h(x_{<s} \mid T \in A_{ts}) \right\} + \bar{W}_n(x, t) \\
= \sum_{s=t+1}^{d} \{ \mathcal{E}_{1,s}(x, t) - \mathcal{E}_{2,s}(x, t) \} + \bar{W}_n(x, t)
\]
where $\tilde{W}_n(x, t)$ is a negligible term and $E_{1,s}(x, t), E_{2,s}(x, t)$ are defined in equation (34). Because $\tilde{W}_n(x, t)$ is negligible, we ignore it in our analysis. After ignoring it, we obtain

$$
\Delta_{2,n}(x > t \mid x \leq t, t) = F_{A,h}(x > t \mid x, t) - \bar{F}_{A,h}(x > t \mid x, t)
$$

$$
= \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \left\{ f_{A,h}(x'_{r,t} \mid x \leq t, t) - \bar{f}_{A,h}(x'_{r,t} \mid x \leq t, t) \right\} dx'_{r,t}
$$

$$
= \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \sum_{s = t+1}^{d} \left\{ E_{1,s}(x', t) - E_{2,s}(x', t) \right\} dx'_{r,t}
$$

$$
= \sum_{s = t+1}^{d} \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \left\{ E_{1,s}(x', t) - E_{2,s}(x', t) \right\} dx'_{r,t}
$$

Using the fact that $G(dx'_{r,t}, t) = g(x'_{r,t}, t)dx'_{r,t}$ and the above expression, the quantity (II) can be written as

$$
(II) = \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \sqrt{n} \Delta_{2,n}(x \leq t \mid x', t) G(dx'_{r,t}, t)
$$

$$
= \sqrt{n} \sum_{s = t+1}^{d} \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \left\{ E_{1,s}(x', t) - E_{2,s}(x', t) \right\} dx'_{r,t} G(dx'_{r,t}, t)
$$

$$
= \sum_{s = t+1}^{d} \Omega_{1,s}(x, t) - \Omega_{2,s}(x, t).
$$

Since the result is an additive form of each quantity involving $j$ and $s$, we will show that

$$
\Omega_{j,s}(x, t) = \sqrt{n} \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \int_{x'_{r,t} = -\infty}^{x'_{r,t} = x > t} \mathcal{E}_{j,s}(x', t) dx'_{r,t} G(dx'_{r,t}, t)
$$

converges to a Brownian bridge for each $j = 1, 2$ and $s \in \{t, t + 1, \cdots, d\}$.

For simplicity, we focus on $j = 1$ case. The case of $j = 2$ can be derived similarly.
Using the fact that \( G(dx_{<t}^t, t) = g(x_{<t}^t, t)dx_{<t}^t \), we can rewrite \( \Omega_{j,s}(x) \) as

\[
\Omega_{1,s}(x,t) = \sqrt{n} \int_{x'=-\infty}^{x'=x} E_{1,s}(x', t)g(x_{<t}^t, t)dx_{<t}^t
\]

\[
= \sqrt{n} \int_{x'=-\infty}^{x'=x} \frac{\tilde{f}_{A,h}(x'_{>t}^t | x'_{<t}^t, t)g(x_{<t}^t, t)}{g(x_{<s}^t | T \in A_{ts})} \times (\tilde{g}_h(x'_{<s}^t | T \in A_{ts}) - \tilde{g}_h(x'_{<s}^t | T \in A_{ts})) dx'
\]

\[
= \sqrt{n} \int_{x'=s}^{x'=x} \omega_{0,h}(x', s, t) (\tilde{g}_h(x'_{<s}^t | T \in A_{ts}) - \tilde{g}_h(x'_{<s}^t | T \in A_{ts})) dx'
\]

\[
= \sqrt{n} \int_{x'_{<s}=-\infty}^{x'_{<s}=x_{<s}} \omega_{1,h}(x_{>s}, x'_{<s}, s, t) (\tilde{g}_h(x'_{<s}^t | T \in A_{ts}) - \tilde{g}_h(x'_{<s}^t | T \in A_{ts})) dx'_{<s},
\]

where

\[
\omega_{0,h}(x', s, t) = \frac{\tilde{f}_{A,h}(x'_{>t}^t | x'_{<t}^t, t)g(x_{<t}^t, t)}{g(x_{<s}^t | T \in A_{ts})}
\]

\[
\omega_{1,h}(x_{>s}, x'_{<s}, s, t) = \int_{x'_{>s}=-\infty}^{x'_{>s}=x_{>s}} \omega_{0,h}(x', s, t) dx'_{>s}.
\]

For a function \( f_z : \mathbb{R}^s \mapsto \mathbb{R} \) with index \( z \in \mathbb{T} \), we define a smoothed empirical process \( \tilde{G}_s \) such that

\[
\tilde{G}_s(f_z) = \sqrt{n} \int f_z(x_{<s}^t) (\tilde{g}_h(x_{<s}^t | T \in A_{ts}) - \tilde{g}_h(x_{<s}^t | T \in A_{ts})) dx_{<s}.
\]

Let \( F = \{f_z : z \in \mathbb{T}\} \). By Theorem 2 and Section 4 of Giné & Nickl (2008), the smoothed empirical process defined on \( F \) converges to a Brownian bridge if the function space \( F \) is Donsker (there are assumptions on the kernel functions and smoothing bandwidth; assumption (K1-2) satisfy the assumptions). In our case, the function space is

\[
\mathcal{G}_{t,s,h} = \{g_x(y) : x \in \mathcal{X}\}
\]

\[
g_x(y) = \omega_{1,h}(x_{>s}, y, s, t)I(y \leq x_{<s}).
\]

Namely,

\[
\Omega_{1,s}(x) = \tilde{G}_r(g_x).
\]
So we only need to show that $G_{t,s,h}$ is a Donsker class for every $h \leq 1$.

To show that $G_{t,s,h}$ is Donsker, first note that any function $g_x \in G_{t,s,h}$ can be written as

$$g_x(y) = \psi_x(y) \cdot q_x(y),$$

where $\psi_x(y) = I(y \leq x \leq s)$ and $q_x(y) = \omega_{1,h}(x > s, y, s, t)$ so

$$G_{t,s,h} = \mathcal{H} \times Q_{s,t,h},$$

with $\mathcal{H} = \{\psi_x(y) : x \in \mathcal{X}\}$ is a collection of indicator functions and $Q_{s,t,h} = \{q_x(y) : x \in \mathcal{X}\}$ is a collection of smooth functions with a bounded derivative (due to assumption (A1)– note that $\omega_{1,h}$ are product of densities with integral so it has a bounded derivative). Thus, both $\mathcal{H}$ and $Q_{s,t,h}$ for every $h \leq 1$ are Donsker so $G_{t,s,h} = \mathcal{H} \times Q_{s,t,h}$ is also Donsker for every $h \leq 1$. Thus, we have shown that $\Omega_{1,s}(x)$ converges to a Brownian bridge for each $s$. The same argument works for $\Omega_{2,s}(x)$ so

$$(II) = \sum_{s=T_i+1}^{d} \Omega_{1,s}(x) - \Omega_{2,s}(x)$$

converges to a Brownian bridge.

Putting the analysis of (I) and (II) together and use the fact that (III) is negligible, we conclude that $\sqrt{n}(\hat{F}_{A,h}(x,t) - \bar{F}_{A,h}(x,t))$ converges to a Brownian bridge for each $t$. □

**Proof.**[Proof of Theorem 4] Because each $X_{i,T_i}^{(v)}$ is generated independently and identically from each other, we only need to show that they are from the distribution function $\hat{F}_{A,h}(x_{>T_i} | X_{i,T_i}, t = T_i)$.

Recall that from Algorithm 1, each $X_{i,T_i}^{(v)}$ is created by the ordering $X_{i,T_i+1}^{(v)}, X_{i,T_i+2}^{(v)}, \cdots, X_{i,d}^{(v)}$. Thus, we prove this by induction. To start with, we consider the case $s = T_i + 1$ and we show that $X_{i,T_i+1}^{(v)}$ is from the distribution function

$$\hat{F}_{A,h}(x_{T_i+1} | X_{i,T_i+1}, t = T_i)$$

63
or its PDF $\hat{f}_{A,h}(x_{T_i+1} \mid X_{i,\leq T_i}, t = T_i)$.

By Algorithm 1, $X_{i,T_i+1}^{(v)}$ is obtained by first sampling an index $\ell \in \{1, \cdots, n\}$ such that $P(\ell = N \mid \text{data}) = W_N(X_{<T_i+1}^{(v)}; A_{t,T_i+1})$ and then drawing from the PDF $K_{T_i+1}(\cdot; X_{\ell,T_i+1}, h_{T_i+1})$. This is essentially sampling from a mixture distribution so the PDF of $X_{i,T_i+1}^{(v)}$ is

$$
\sum_{\ell=1}^n W_\ell(X_{<T_i+1}^{(v)}; A_{t,T_i+1}) K_{T_i+1}(\cdot; X_{\ell,T_i+1}, h_{T_i+1}),
$$

which by equation (12) and (13) is $\hat{f}_{A,h}(\cdot \mid X_{i,\leq T_i+1}, t = T_i)$. Thus, we have proved the case of $T_i + 1$.

Now assume that it is true for all $s = T_i + 1, \cdots, \tau$. Namely, $X_{i,s}^{(v)}$ has a PDF $\hat{f}_{A,h}(x_s \mid X_{i,<s}, t = T_i)$ so the joint PDF is $\prod_{s=T_i+1}^\tau \hat{f}_{A,h}(x_s \mid X_{i,<s}, t = T_i)$. We will show that it implies that $s = \tau + 1$ is also true. By the same derivation as the case of $s = T_i + 1$, one can easily see that conditioned on $X_{i,T_i+1}^{(v)}, \cdots, X_{i,\tau}$, $X_{i,\tau+1}^{(v)}$ is from a mixture distribution whose PDF is

$$
\sum_{\ell=1}^n W_\ell(X_{<\tau+1}^{(v)}; A_{t,\tau+1}) K_{\tau+1}(\cdot; X_{\ell,\tau+1}, h_{\tau+1}) = \hat{f}_{A,h}(\cdot \mid X_{i,\leq \tau}, t = T_i).
$$

Thus, the joint PDF of $X_{i,T_i+1}^{(v)}, \cdots, X_{i,\tau+1}$ is the product

$$
\prod_{s=T_i+1}^{\tau+1} \hat{f}_{A,h}(x_s \mid X_{i,<s}, t = T_i)
$$

so the result holds for $s = \tau + 1$ case. By induction, it holds for every $s = T_i + 1, \cdots, d$, which proves that $X_{i,>T_i}^{(v)}$ has a PDF

$$
\prod_{s=T_i+1}^d \hat{f}_{A,h}(x_s \mid X_{i,<s}, t = T_i) = \hat{f}_{A,h}(x_{>T_i} \mid X_{i,\leq T_i}, t = T_i),
$$

which is the desired result. □

Proof.[Proof of Theorem 6] With the convergence toward a Brownian bridge (Theorem 3), this result follows from the Theorem of the bootstrap for delta method; see,
e.g., Theorem 23.9 of van der Vaart (1998) and Theorem 3.9.11 of van der Vaart & Wellner (1996). □