ANALYSIS OF THE RECONNECTION PROCESS IN NONTWIST CUBIC MAPS

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Abstract. The reconnection process in the dynamics of cubic nontwist maps, introduced in [3], is studied. The present paper extends the work presented in [8]. As in that work, in order to describe the route to reconnection of the involved Poincaré–Birkhoff chains or dimerised chains we investigate an approximate interpolating Hamiltonian of the map under study revealing again that the scenario of reconnection of cubic nontwist maps is different from that occurring in the dynamics of quadratic nontwist maps.

Key words: area preserving maps, nontwist maps, reconnection bifurcation.

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1. Introduction

Nontwist maps arise naturally in the study of Hamiltonian systems, because they are models for Poincaré maps associated to sections in an energy manifold of an iso–energetically degenerate two degree of freedom Hamiltonian system [2, 14, 15], in transport problems in plasma physics, accelerator physics and in other areas. Transport problems in plasma physics can be modelled by an area preserving map where the twist condition fails [7]. Applications of nontwist maps in accelerator physics can be found in [9]. In [4] are studied the quadratic nontwist standard-like maps both from theoretical and numerical point of view. Nontwist standard-like maps exhibit both time-reversal and spatial symmetry being observed the appearance of the meanders. Meanders are invariant circles that exhibit foldings in such a way that they are not graphs of functions. During the last decade numerical and theoretical studies of quadratic non–twist maps [3, 2, 4, 5] revealed a global bifurcation (called reconnection) of the invariant manifolds of two distinct regular hyperbolic periodic orbits having the same rotation number. At the threshold of reconnection the involved hyperbolic orbits are connected by a common arc of their invariant manifolds. The physical model of reconnection is met in Tokamaks [13] which are experimental machines for achievement of controlled thermonuclear fusion reactions.

For a rigorous analysis of local and global bifurcations occurring in a family of area preserving maps defined on an annulus $\mathbb{T} \times [a, b]$ ($\mathbb{T}$ denotes the circle identified with $[0, 2\pi]$) one derives an approximate interpolating Hamiltonian of the map under study [6].

The present work deals with reconnection in the cubic nontwist area preserving diffeomorphism of the annulus $\mathbb{T} \times \mathbb{R}$, $f : (x, y) \mapsto (x', y')$:

\begin{align*}
x' &= x + F(a, b; y') \mod 2\pi \\
y' &= y + k \sin x
\end{align*}

where the rotation number function $F$ is a cubic map depending on two parameters $a > 0, b \in \mathbb{R}, b \neq 0$, i.e. $F(a, b; y) = y - ay^2 + by^3$ and $k > 0$ is a perturbation parameter.

We recall that an area preserving diffeomorphism $g : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $g : (x, y) \mapsto (x', y')$ is a twist map if $\partial_y x' \neq 0$ ($\partial_y$ denotes the partial derivative with respect to $y$). Twist property is a basic assumption of KAM theorem, as well as for the Aubry–Mather theory [11]. The map (1) is a non–twist map because it...
violates the twist condition. Our purpose is to study its dynamics as well as the route to reconnection in the case when the shape parameter $a$ and the perturbation parameter $k$ are fixed and the other shape parameter $b$ varies on the real line or on an interval.

2. Properties of cubic nontwist map

First we recall some properties of the map under study [8]. The motion in the unperturbed map (1), i.e the map corresponding to $k = 0$:

$$\begin{align*}
x' &= x + y - ay^2 + by^3 \pmod{2\pi} \\
y' &= y
\end{align*}$$

occurs along the circle $y = \text{cst}$. The rotation number of an orbit starting at $(x, y)$ is:

$$\rho = \lim_{n \to \infty} \frac{X_n - X}{2n\pi} = \frac{F(a,b;y)}{(2\pi)}$$

where $(X_n, Y_n)$ is the orbit of the point $(X, Y) = (x, y)$ under the lift of the map (the map defined on $\mathbb{R}^2$ having the same expression, without modulo $2\pi$ for the first component). The map (2) violates the twist condition for the parameter values $(a, b)$ such that $a^2 - 3b \geq 0$, along the circles:

$$\begin{align*}
C_1 : y &= \frac{a + \sqrt{a^2 - 3b}}{3b} \\
C_2 : y &= \frac{a - \sqrt{a^2 - 3b}}{3b}
\end{align*}$$

These circles are called twistless or shearless circles. At the same time along the circle $C_1$ the rotation number has a global minimum,

$$F|_{C_1} = \frac{-2a^3 - 3a^2 \sqrt{a^2 - 3b} + 9ab + (a^2 + 6b)\sqrt{a^2 - 3b}}{54b^2} \frac{\sqrt{a^2 - 3b}}{54b^2}$$

while along $C_2$, a global maximum,

$$F|_{C_2} = \frac{-2a^3 + 3a^2 \sqrt{a^2 - 3b} + 9ab - (a^2 + 6b)\sqrt{a^2 - 3b}}{54b^2} \frac{\sqrt{a^2 - 3b}}{54b^2}$$

Let us denote by $\omega_m, \omega_M$:

$$\begin{align*}
\omega_m &= \frac{a + \sqrt{a^2 - 3b}}{3b} \quad \omega_M &= \frac{a - \sqrt{a^2 - 3b}}{3b}
\end{align*}$$

the points of minimum, respectively maximum, for the rotation number function $F$. For $y \in (-\infty, \omega_M)$ the unperturbed map has positive twist (the rotation number function is increasing), for $y \in (\omega_M, \omega_m)$ has a negative twist (the rotation number function is decreasing), while for $y \in (\omega_m, +\infty)$ has again a positive twist (Fig.1).

![Figure 1. The graph of the rotation number function $F(a,b;y)$ for $a = 2.5$ and $b = 1.26$](image)
The orbits lying on the circles \( y = y_0 \) with \( F(a, b; y_0)/2\pi \) a rational number in lowest terms, \( p/q \), are periodic orbits. If such a periodic orbit lies in a region of monotone twist property of the map, after a slight perturbation, it gives rise generically to at least two periodic orbits of the same rotation number, one elliptic and the second regular hyperbolic. Elliptic points are surrounded by invariant circles, and hyperbolic points are connected by heteroclinic connections. Such a pair of periodic orbits and the associated invariant sets form a Poincaré–Birkhoff chain. For \( y \in (\omega_M, \omega_m) \) can exist three circles \( y = \text{ct} \) of the unperturbed map, on which lie periodic orbits of the same rotation number \( p/q \). Our aim is to study the bifurcations of the periodic orbits or the invariant manifolds belonging to three distinct Poincaré–Birkhoff chains created after a slight perturbation, as the shape parameter \( b \in \mathbb{R} \), \( a^2 - 3b > 0 \) defining the rotation number function \( F \) varies.

3. RECONNECTION SCENARIO

In order to analyze the changes in the topology of invariant manifolds of the involved \( p/q \)-type hyperbolic periodic orbits consider the interpolating Hamiltonian associated to the map \( F \):

\[
H_{a,b,k}(x, y) = -y^2/2 + ay^3/3 - by^4/4 - k \cos x
\]

It defines the vector field

\[
X_H = \left(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}\right) = (y - ay^2 + by^3, k \sin x)
\]

which is reversible with respect to the involution \( R(x, y) = (-x, y) \), i.e. \( R \circ X_H = -X_H \circ R \). The fixed point set, \( \text{Fix}(R) \), consists in the lines \( x = 0 \) and \( x = \pi \), called symmetry lines. The equilibrium points \( X_H \) lying on the symmetry lines are called symmetric. The Hamiltonian system associated to the vector field \( X_H \) can display at most three chains: Poincaré–Birkhoff chains or dimerised chains. A dimerised chain is a structure formed by elliptic points surrounded by homoclinic circles to the corresponding hyperbolic points.

In order to describe the scenario of reconnection and the local bifurcations of the equilibrium points, we analyze the position on the symmetry lines of the equilibrium points, their stability type and bifurcations occurring as \( b \) varies and \( a, k \) are fixed in the parametric space \( (a, b, k) \), with \( a, k > 0 \) and \( b \in \mathbb{R} \). Therefore, if \( b < a^2/4 \), the Hamiltonian system has six equilibrium points (e stands for elliptic and h for hyperbolic):

\[
P_{1h}(0, 0); P_{2e}(\pi, 0); P_{3e}(0, \frac{a-\sqrt{a^2-4b}}{2b});
\]

\[
P_{4h}(\pi, \frac{a-\sqrt{a^2-4b}}{2b}); P_{5h}(0, \frac{a+\sqrt{a^2-4b}}{2b}); P_{6e}(\pi, \frac{a+\sqrt{a^2-4b}}{2b})
\]

If \( \frac{a^2}{4} < b < \frac{a^2}{3} \) (Fig. 2) the vector field \( X_H \) has only two equilibrium points: \( P_1(0, 0), P_2(\pi, 0) \), while for \( a^2 = 4b \) it has four equilibrium points: \( P_1(0, 0), P_2(\pi, 0), A(0, \frac{\sqrt{6}}{\sqrt{6}}), B(\pi, \frac{\sqrt{6}}{\sqrt{6}}) \). In the latter case, \( a^2 - 4b = 0 \), the two eigenvalues are zero and a bifurcation of equilibrium points occurs.

In the following we want to describe the local changes in the topology of the invariant manifolds of the Hamiltonian system, when \( a, k > 0 \) are fixed and \( b \neq 0 \) varies on the real line. We remark that to get connected any two neighboring chains when \( b \) varies, we need to consider both positive and negative values of the parameter \( b \). When the parameter \( a \) varies \( \mathbb{R} \), it is sufficient to consider the case \( a > 0 \). The systems whose phase portraits are illustrated in different figures correspond to \( a = 1.5 \) and \( k = 0.018 \). Denote by I, II and III the three chains containing the equilibrium points, more precisely, the chain I contains the points \( P_{1h}, P_{2e} \), the chain II contains the points \( P_{3e}, P_{4h} \) and III the points \( P_{5h}, P_{6e} \). For \( b \)
small enough \((b < -2\) for example), the all six equilibrium points are born, Fig. 3a). The chains I and II are two dimerised chains. Between these chains the trajectories of the Hamiltonian vector field are not graphs of real functions of \(x\), but they are meanders. Each two neighboring points on the same symmetry line have opposite stability type. Increasing further the parameter \(b\), the equilibrium points lying on the same symmetry line go away. At a critical value, called threshold of reconnection, the hyperbolic equilibrium points of these two chains get connected by common branches of their invariant manifolds, Fig. 3b). The common branches are \(W^u(P_{1h})\) and \(W^s(P_{4h})\) respectively.

\[
6b^2 + a^4 - 6a^2b + 48b^3k + 4ab\sqrt{a^2 - 4b} - a^3\sqrt{a^2 - 4b} = 0
\]

For the numerical values, the threshold of reconnection is \(b := b_{1\text{rec}} = -1.9538\).

Increasing \(b\) slightly from \(b_{1\text{rec}}\), the two dimerised chains become two Poincaré–Birkhoff chains, Fig. 4b), so the system displays three Poincaré–Birkhoff distinct chains, Fig. 4c) and the Poincaré–Birkhoff chains II and III approach each other. At the threshold of reconnection the hyperbolic points of these two chains get connected by common branches of invariant manifolds, Fig. 4d). The common branches are \(W^u(P_{4h})\) and \(W^s(P_{5h})\) respectively.

\[
k = \frac{1}{24}\frac{a}{b^3}(a^2 - 4b)^\frac{1}{2}\sqrt{a^2 - 4b}
\]

Remark that we can simultaneously get connected the all three chains I, II and III. Call this triple reconnection. Let us describe the triple reconnection scenario. Consider in this case \(b > 0\). For \(b\) slightly beyond 0, the chains I and II are two dimerised chains while III is a Poincaré–Birkhoff chain, Fig. 5a).
Increasing further the parameter $b$, the equilibrium points of the two dimerised chains lying on the same symmetry line go away while the equilibrium points of the Poincaré–Birkhoff chain approaches the points...
Figure 5. Reconnection scenario of the three chains I, II and III. The values of the parameter $b$ are: a) $b = 0.3$ b) $b = 0.5$ c) $b = 0.53$ d) $b = 0.5625$ e) $b = 0.7$

of the dimerised chain II. At a critical value, called triple threshold of reconnection, the hyperbolic points of the three chains get connected by common branches of their invariant manifolds Fig.3b). Imposing $H_{a,b,k}(P_{1h}) = H_{a,b,k}(P_{4h}) = H_{a,b,k}(P_{5h})$, we get from (8) and (9) the following reconnection curve:

$$a^4 - 6a^2b + 6b^2 + a(a^2 - 4b)\sqrt{a^2 - 4b} = 0$$

Solving numerically (10) for $a = 1.5$ one get $b := b_{3\text{rec}} = 0.5$ and from $k := k_{3\text{rec}} = 0.0625$. Consequently, to get connected the three chains I, II and III, in the case when $b$ varies, we have to keep the perturbation parameter $k$ at the constant value $k_{3\text{rec}}$. For $b > b_{3\text{rec}}$ the scenario is similar to the case $b > b_{2\text{rec}}$, Fig(3)-e) and Fig(4)-e)

4. Conclusions

In this paper we have extended the studies reported in [8] on reconnection scenario of a three-parameter cubic non-twist map depending on the parameters $a$, $b$ and $k$. Using an approximate interpolating Hamiltonian of the map we have described the reconnection process of any two neighboring chains in the case when the parameters $a, k$ are fixed and $b$ varies. By numerically computations we found the exact values of the thresholds of reconnection. At the end we presented the triple reconnection of the all three involved chains.
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References

[1] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
[2] D. del-Castillo-Negrete, J.M. Greene, P.J. Morrison, *Area preserving nontwist maps: periodic orbits and transition to chaos*, Physica D91 (1996) 1-23.
[3] J. E. Howard, J. Humpherys, *Nonmonotonic twist maps*, Physica D80 (1995) 256-276.
[4] E. Petrisor, *Reconnection scenarios and the threshold of reconnection in the dynamics of nontwist maps*, Chaos, Solitons and Fractals, 14 (2002) 117-127.
[5] E. Petrisor, *Nontwist area preserving maps with reversing symmetry group*, Int. J. Bif. Chaos, 11 (2001) 497-511.
[6] C. Simó, *Invariant curves of analytic perturbed nontwist area preserving maps*, Regular and Chaotic Dynamics, 3 (1998) 180-195.
[7] E.J. Doyle et al., *Modifications in turbulence and edge electric fields at the LH transition in the DIII-D tokamak*, Physics of Fluids B Vol.3(8), (1991), 2300-2307.
[8] Gh. Tigan, *On the scenario of reconnection in nontwist cubic maps*, Chaos, Soliton and Fractals (accepted), to appear.
[9] A. Gerasimov, F.M. Isailev, J.L. Tennyson, A.B. Temnykh, *Springer Lectures Notes in Physics*, Vol.247, (154), 1986.
[10] S.M Soskin, *Phys.Rev.*, E 50(1), (1994), R44.
[11] R. Egydio de Carvalho and A.M. Ozorio de Almeida, *Integrable approximation to the overlap of resonances*, Phys. Letters A, (1992), 162, 457-63.
[12] H.W. Chapel and T. Post, *The birth process of periodic orbits in nontwist maps*, Physica D, (1995), 80, 256-276.
[13] R.L. Viana, *Chaotic magnetic field lines in a Tokamak with resonant helical windings*, Chaos 11, (2000), 765-778.
[14] G. Voyatzis, S. Ichtiaroglou, *Degenerate bifurcations of resonant tori in Hamiltonian systems*, Int. J. Bif. Chaos 9, (1999), 849–863.
[15] A. Apte, A. Wurm, P.J. Morrison, *Renormalization and destruction of 1/γ^2 tori in the standard nontwist map*, Chaos 13 (2003).