POSITIVE OPERATOR-VALUED MEASURES AND DENSELY-DEFINED OPERATOR-VALUED FRAMES

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Abstract. In the signal-processing literature, a frame is a mechanism for performing analysis and reconstruction in a Hilbert space. By contrast, in quantum theory, a positive operator-valued measure (POVM) decomposes a Hilbert-space vector for the purpose of computing measurement probabilities. Frames and their most common generalizations can be seen to give rise to POVMs, but does every reasonable POVM arise from a type of frame? In this paper we answer this question using a Radon-Nikodym-type result.

1. Introduction

Originally studied in quantum mechanics, positive operator-valued measures (POVMs) have recently been suggested in the signal-processing literature as a natural general framework for the analysis and reconstruction of square-integrable functions [18, 14, 22, 17]. Examples include frames [10, 7, 6], g-frames [23, 16, 15], continuous frames [2], and the overarching continuous g-frames [1], all of which we will call simply operator-valued frames (OVFs). The precise relationship between OVF s and POVMs is captured by a Radon-Nikodym-type theorem [22, Theorem 3.3.2]. But this theorem applies only to POVMs with sigma-finite total variation. The purpose of this paper is to show that the above relationship extends to non-sigma-finite POVMs and densely-defined OVF s.

Even the sigma-finite case has been attempted by many authors. In finite dimensions, for example, Chiribella et al. recently solved the problem using barycentric decompositions [5]. In infinite dimensions, extra assumptions have traditionally been used, such as the finiteness of the POVM’s trace [11, VI.8.10]. One of the main complications in this case is that the space of bounded linear operators on an infinite-dimensional Hilbert space does not have the so-called Radon-Nikodym property [3]. Another reason the problem is challenging is that it

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requires proving the conclusion of Naimark’s Theorem [19] [20]. The non-sigma-finite case is at least as hard, and as far as we know no one has yet attempted it.

The paper is organized as follows. In Section 2, we make the relationship we wish to establish precise. Then, in Section 3 we prove a Radon-Nikodym-type theorem from which the desired relationship follows.

2. The Question

Let \( \mathcal{H} \) be a separable complex Hilbert space throughout, whose inner product is conjugate-linear in its second variable. A \( g \)-frame for \( \mathcal{H} \) is a sequence of bounded operators \( T_1, T_2, \ldots \) mapping \( \mathcal{H} \) into the sequence of separable Hilbert spaces \( \mathcal{K}_1, \mathcal{K}_2, \ldots \) such that the operator \( T : x \in \mathcal{H} \mapsto \{ T_i x \}_{i \geq 1} \) maps into \( \bigoplus_{i \geq 1} \mathcal{K}_i \) and is bounded above and below [23]. A related concept is the concept of a frame for \( \mathcal{H} \). This is usually defined as follows.

**Definition 1.** A sequence \( x_1, x_2, \ldots \) in \( \mathcal{H} \) is said to be a frame for \( \mathcal{H} \) if the operator \( T : x \in \mathcal{H} \mapsto \{ \langle x, x_i \rangle \}_{i \geq 1} \) maps into \( \ell^2(\mathbb{N}) \) and is bounded above and below, or equivalently, if there are \( A, B > 0 \) such that

\[
A \| x \|^2_{\mathcal{H}} \leq \sum_{i \geq 1} |\langle x, x_i \rangle|^2 \leq B \| x \|^2_{\mathcal{H}}
\]

for all \( x \in \mathcal{H} \). The numbers \( A \) and \( B \) are called the frame bounds.

Using the self-duality of Hilbert spaces, we can easily see that every frame \( \{ x_i \}_{i \geq 1} \) for \( \mathcal{H} \) can be uniquely expressed as the \( g \)-frame \( \{ T_i \}_{i \geq 1} \) for \( \mathcal{H} \) with \( T_i = \langle \cdot, x_i \rangle \), so that the concept of a \( g \)-frame subsumes the concept of a frame. For examples of frames of interest, see [10] [7] [8] [6]. For examples of \( g \)-frames of interest that are not frames, see Examples 2.3.7 and 2.3.8 in [22].

The importance of frames and \( g \)-frames is that they provide a framework for analysis, synthesis, and reconstruction of a function. The operator \( T \) above is usually referred to as the analysis operator. Reconstruction can be performed if we can calculate the result of applying \( S^{-1}T^* \) to \( Tx \), where \( T^* \) is the adjoint of \( T \) and \( S = T^*T \) is known as the frame operator. The intermediate step of applying \( T^* \) to \( Tx \) is called synthesis. In practice, if one knows only \( T \) and \( Tx \), one way to recover \( x \) without directly inverting \( S \) is by way of the so-called frame algorithm:

**Proposition 2.** [13] Let \( \{ x_i \}_{i \geq 1} \) be a frame for \( \mathcal{H} \) with frame bounds \( A, B > 0 \). Then every \( x \in \mathcal{H} \) can be reconstructed from \( T \) and \( Tx \) alone using the iteration

\[
x^{(n)} = x^{(n-1)} + \frac{2}{A+B} S \left( x - x^{(n-1)} \right),
\]

for \( n \geq 1 \) with \( x^{(0)} = 0 \). Further, \( x^{(n)} \rightarrow x \) according to

\[
\| x - x^{(n)} \| \leq \left( \frac{B-A}{B+A} \right)^n \| x \|.
\]
This algorithm applies equally well to frames and g-frames.

Another paradigm for analysis and synthesis is that of continuous frames [2], which we now describe.

**Definition 3.** Let \((\Omega, \Sigma, \mu)\) be a measure space and let \(\{x_t\}_{t \in \Omega} \subset \mathcal{H}\) be such that for all \(x \in \mathcal{H}\), the map \(t \in \Omega \mapsto \langle x, x_t \rangle\) is measurable. Then \((\mu, \{x_t\}_{t \in \Omega})\) is a continuous frame if \(T : x \in \mathcal{H} \mapsto \{\langle x, x_t \rangle\}_{t \in \Omega}\) maps into \(L^2(\mu)\) and is bounded above and below. This boundedness condition can also be expressed by saying there are constants \(A, B > 0\) such that

\[
A \|x\|^2_\mathcal{H} \leq \int_{\Omega} |\langle x, x_t \rangle|^2 \, d\mu(t) \leq B \|x\|^2_\mathcal{H}
\]

for all \(x \in \mathcal{H}\).

Examples of interest occur in wavelet and Gabor analysis [7, 8].

Encompassing both continuous frames and g-frames is the overarching concept of a continuous g-frame [1]. We will generalize this concept slightly and call the result an operator-valued frame. (We note that our terminology differs from the term “operator-valued frame” in [16], which refers to what we call here g-frames.)

**Definition 4.** Let \((\Omega, \Sigma, \mu)\) be a measure space. Let \(\mathcal{M}\) be a dense linear subspace of \(\mathcal{H}\). Let \(\{\mathcal{K}(t)\}_{t \in \Omega}\) be a \(\Sigma\)-measurable family of Hilbert spaces and \(T(t) : \mathcal{M} \rightarrow \mathcal{K}(t)\) for every \(t \in \Omega\) be a family of linear maps such that for every \(x \in \mathcal{M}\) both \(\{T(t)x\}_{t \in \Omega} \in \int_{\Omega} \mathcal{K}(t) \, d\mu(t)\) and the map \(T : x \in \mathcal{M} \mapsto \{T(t)x\}_{t \in \Omega}\) is bounded above and below. Then we say \((\mu, \mathcal{M}, \{T(t)\}_{t \in \Omega})\) is an operator-valued frame. This boundedness condition can be expressed by saying that there are \(A, B > 0\) such that

\[
A \|x\|^2_\mathcal{H} \leq \int_{\Omega} \langle T(t)^*T(t)x, x \rangle \, d\mu(t) \leq B \|x\|^2_\mathcal{H},
\]

for every \(x \in \mathcal{M}\).

Examples of operator-valued frames that are neither continuous frames nor g-frames arise from the Plancherel theorem for non-compact non-commutative groups, for which examples \(A = B\). (See [12, Equation 7.46] for more information.)

If \((\mu, \mathcal{M}, \{T(t)\}_{t \in \Omega})\) is an operator-valued frame for \(\mathcal{H}\), then the frame operator

\[
S = \int_{\Omega} T(t)^*T(t) \, d\mu(t)
\]

may be defined as before. Here, \(S\) is interpreted as an operator that satisfies

\[
\langle Sx, y \rangle = \int_{\Omega} \langle T(t)^*T(t)x, y \rangle \, d\mu(t)
\]

\(^1\text{See [10] or [22] for the definition of this and the direct integral} \int_{\Omega} K(t) \, d\mu(t) \text{of these spaces.}\)
for all $x, y \in \mathcal{M}$. By the boundedness of $T$ and the density of $\mathcal{M}$, such an operator exists and is unique. As a result, both the frame algorithm and the procedure of applying $S^{-1}T^*$ to $Tx$ for $x \in \mathcal{M}$ recover $x$, as before.

A related concept is that of a positive operator-valued measure (POVM). POVMs represent measurements that occur in open quantum systems and generalize the concept of a projection-valued measure or von Neumann measure. Formally, if $(\Omega, \Sigma)$ is a measurable space, a POVM is a map from $\Sigma$ to $\mathcal{L}^+(\mathcal{H})$, the positive semi-definite bounded operators on $\mathcal{H}$, which is $\sigma$-additive in the weak operator topology. That is, it is a tuple $(\Omega, \Sigma, \mathcal{M})$, where $\mathcal{M} : \Sigma \rightarrow \mathcal{L}^+(\mathcal{H})$ is a map such that

- $M(\emptyset) = 0$ and
- if $E_1, E_2, \ldots$ are pairwise disjoint members of $\Sigma$ and $x, y \in \mathcal{H}$, then

$$\langle M(\bigcup_{i=1}^{\infty} E_i) x, y \rangle = \sum_{i=1}^{\infty} \langle M(E_i) x, y \rangle.$$

If, in addition, $M(\Omega)$ is invertible, we say $M$ is a framed POVM. We will say that an OVF $(\mu, \mathcal{M}, \{T(t)\}_{t \in \Omega})$ gives rise to the POVM $M$ if

$$\langle M(E) x, y \rangle = \int_E \langle T(t)^* T(t)x, y \rangle \, d\mu(t)$$

for all $x, y \in \mathcal{M}$ and all $E \in \Sigma$.

It is easily seen that every g-frame and every continuous frame give rise to a framed POVM. With a little more work, it can be shown that every OVF gives rise to a framed POVM. The question of this paper is the converse question: “Does every framed POVM arise from an OVF?”

3. Main Result

For this section, recall that a closed operator on $\mathcal{H}$ is a map $A : D(A) \rightarrow \mathcal{H}$ with a closed graph, where $D(A)$ is a dense linear subspace of $\mathcal{H}$. In other words, it is a map for which if $x_n \in D(A) \rightarrow x \in \mathcal{H}$ and $Ax_n$ converges to $y \in \mathcal{H}$, then $x \in D(A)$ and $y = Ax$.

The answer to the question of the last section hinges on the following Radon-Nikodym-type theorem.

**Theorem.** Suppose $(\Omega, \Sigma, \mathcal{M})$ is a POVM. Suppose that $\mathcal{M}$ is a dense linear manifold in $\mathcal{H}$ and that for each $x \in \mathcal{M}$, the total variation of the vector measure $\mu_x : E \in \Sigma \mapsto M(E)x$ is sigma-finite. Then there is a sigma-finite measure $(\Omega, \Sigma, \mu)$ and a positive closed operator-valued function $Q(t) : \mathcal{M} \rightarrow \mathcal{H}$, defined for $\mu$-a.e. $t \in \Omega$, such that

$$M(E)x = \int_E Q(t)x \, d\mu(t),$$

weakly, for all $E \in \Sigma$ and all $x \in \mathcal{M}$. Further, if $(Q_1, \mu_1)$ and $(Q_2, \mu_2)$ are operator-valued functions defined on $\mathcal{M}$ and sigma-finite measures satisfying...
we have the following operator equality for \((\mu_1 + \mu_2)\)-a.e. \(t\):

\[
Q_1(t) \frac{d\mu_1}{d(\mu_1 + \mu_2)}(t) = Q_2(t) \frac{d\mu_2}{d(\mu_1 + \mu_2)}(t).
\]

**Proof.** Let \(\mu_{x,y} : \Sigma \to \mathbb{C}\) for \(x, y \in \mathcal{H}\) be the complex measure defined by \(\mu_{x,y}(E) = \langle M(E)x, y \rangle\). Let \(\mu\) be any sigma-finite measure dominating each \(\mu_{x,y}\). For example, if \(\{x_j\}\) is a countable dense subset of the unit ball, then we may define

\[
\mu(E) = \sum_{j \geq 1} \frac{1}{2^j} \mu_{x_j, x_j}(E).
\]

Observe that

\[
|\mu_{x,y}(E)| = |\langle M(E)x, y \rangle| \\
\leq ||M(E)x|| \cdot ||y||
\]

so that

\[
|\mu_{x,y}(E)| (E) \leq |\mu_x| (E) \cdot ||y||
\]

for all \(x \in \mathcal{M}\) and all \(y \in \mathcal{H}\). We use the term *null set* to mean “\(\mu\)-null set.”

For each \(x \in \mathcal{M}\), fix a Radon-Nikodym derivative \(g_x\) of \(|\mu_x|\) with respect to \(\mu\). The fact that \(|\mu_x|\) is sigma-finite implies that we may assume \(g_x\) is finite everywhere. For each \(x, y \in \mathcal{M}\), fix a Radon-Nikodym derivative \(g_{x,y}\) of \(\mu_{x,y}\) with respect to \(|\mu_x|\). By (3), we may assume \(g_{x,y}\) does not exceed \(||y||\) in absolute value. Similarly, we may choose \(g_{x,x}(t)\) to be non-negative for all \(t \in \Omega\). Let \(f_{x,y} = g_{x,y}g_x\) for \(x, y \in \mathcal{M}\). Then, since both \(f_{x,y}\) and \(f_{y,x}\) are valid Radon-Nikodym derivatives of \(\mu_{x,y}\) with respect to \(\mu\) for \(x, y \in \mathcal{M}\), so is the following:

\[
q(t; x, y) := \begin{cases} 
  f_{x,y}(t), & \text{if } |f_{x,y}(t)| \leq |f_{y,x}(t)| \\
  f_{y,x}(t), & \text{else}.
\end{cases}
\]

Let \(\mathbb{F} = \mathbb{Q} + i\mathbb{Q}\) and let \(\mathcal{M}_0\) be the finite \(\mathbb{F}\)-linear span of a countable dense subset of \(\mathcal{M}\). Assume \(x, y \in \mathcal{M}_0\) and \(t \in \Omega\). By our choice of \(q\), we have \(|q(t; x, y)| \leq g_y(t) ||x||\). Further, \(g_y(t)\) is finite, so \(q(t; \cdot, y)\) is bounded. Let us restrict \(t\) to a null-complemented set \(\Omega_y \subset \Sigma\) such that \(q(t; \cdot, y)\) is \(\mathbb{F}\)-linear. The functional \(q(t; \cdot, y)\) extends to a unique \(\mathbb{C}\)-linear continuous functional on all of \(\mathcal{H}\). Thus, by the Riesz representation theorem, for all \(t \in \Omega' := \cap_{y \in \mathcal{M}_0} \Omega_y\) there is a vector \(z(t; y) \in \mathcal{H}\) such that

\[
(4) \quad q(t; x, y) = \langle x, z(t; y) \rangle
\]

for all \(x, y \in \mathcal{M}_0\), and all \(t \in \Omega'\).

Let \(x \in \mathcal{M}_0\). There is a measurable null-complemented set \(\Omega'_x \subset \Omega'\) such that \(y \in \mathcal{M}_0 \mapsto q(t; x, y)\) is \(\mathbb{F}\)-conjugate-linear for all \(t \in \Omega'_x\). Letting \(\Omega'' = \cap_{x \in \mathcal{M}_0} \Omega'_x\), we may assume this map is \(\mathbb{F}\)-conjugate-linear for all \(x \in \mathcal{M}_0\) and all \(t \in \Omega''\). Thus, we have

\[
\langle x, z(t; ay + by') \rangle = \langle x, az(t; y) + bz(t; y') \rangle
\]
for all \( x,y,y' \in \mathcal{M}_0 \), all \( a,b \in \mathbb{F} \), and all \( t \in \Omega'' \). Letting \( x \) range over \( \mathcal{M}_0 \) this gives \( \mathbb{F} \)-linearity of the map \( y \in \mathcal{M}_0 \mapsto z(t;y) \) for all \( t \in \Omega'' \). Thus, there is an \( \mathbb{F} \)-linear map \( Q_0(t) : \mathcal{M}_0 \to \mathcal{H} \) such that \( z(t;y) = Q_0(t)y \) for all \( y \in \mathcal{M}_0 \). Combining this with \( \text{(4)} \) we get

\[ q(t;x,y) = \langle x,Q_0(t)y \rangle \]

for all \( x,y \in \mathcal{M}_0 \), and \( t \in \Omega'' \).

Let \( y \in \mathcal{M}_0 \) and \( E \in \Sigma \). We have now shown that

\[ \mu_{x,y}(E) = \int_E \langle x,Q_0(t)y \rangle \, d\mu(t) \]

for all \( x \in \mathcal{M}_0 \). It follows that

\[ |\mu_{x,y}(E)| = \int_E |\langle x,Q_0(t)y \rangle| \, d\mu(t) \]

for all \( x \in \mathcal{M}_0 \). We now wish to show that \( \text{(5)} \) extends to all \( x \in \mathcal{H} \). In other words, we wish to show that the functional \( \phi : \mathcal{H} \to \mathbb{C} \) defined by

\[ \phi(x) = \int_E \langle x,Q_0(t)y \rangle \, d\mu(t) \]

is well-defined and satisfies \( \phi(x) = \mu_{x,y}(E) \) for all \( x \in \mathcal{H} \). For this, suppose \( x \in \mathcal{H} \) and \( x_n \) is a sequence in \( \mathcal{M}_0 \) converging to \( x \). To show \( \phi \) is well-defined, we first note that by Fatou’s lemma we have

\[
\left( \int_E |\langle x,Q_0(t)y \rangle| \, d\mu(t) \right)^2 \\
= \left( \int_E \liminf_n |\langle x_n,Q_0(t)y \rangle| \, d\mu(t) \right)^2 \\
\leq \liminf_n \left( \int_E |\langle x_n,Q_0(t)y \rangle| \, d\mu(t) \right)^2
\]

By Cauchy-Schwarz, this is bounded by

\[
\leq \liminf_n \int_E \langle x_n,Q_0(t)x_n \rangle \, d\mu(t) \int_E \langle y,Q_0(t)y \rangle \, d\mu(t).
\]

Further, by \( \text{(5)} \), this is equal to

\[ \leq \liminf_n \langle M(E)x_n,x_n \rangle \langle M(E)y,y \rangle \\
= \langle M(E)x,x \rangle \langle M(E)y,y \rangle \]

This means that \( \phi \) is well-defined. Futher, \( \text{(7)} \) means that \( \phi \) is continuous. Since \( x \in \mathcal{H} \mapsto \mu_{x,y}(E) \) is also continuous and restricts to \( \phi \) on \( \mathcal{M}_0 \), we have \( \phi(x) = \mu_{x,y}(E) \), as desired.

We will now show that there is a closed, positive operator-valued function \( Q(t) \) with domain \( \mathcal{M} \) such that

\[ \mu_{x,y}(E) = \int_E \langle Q(t)x,y \rangle \, d\mu(t), \]

for all \( x,y \in \mathcal{M}_0 \), and \( t \in \Omega'' \).
for all $x \in \mathcal{M}$, all $y \in \mathcal{M}_0$, and all $E \in \Sigma$. For this, let $x \in \mathcal{M}$, $y \in \mathcal{M}_0$, and $E \in \Sigma$. By the conclusion of the last paragraph and the definition of total-variation measure, we have:

$$\int_E |\langle x, Q_0(t) y \rangle| \, d\mu(t) = |\mu_{x,y}(E)|.$$

Using (3), the right hand side is bounded by $|\mu_x|(E) \|y\|$. But since $E$ was arbitrary, this means that

$$|\langle x, Q_0(t) y \rangle| \leq g_x(t) \|y\|,$$

for every $t$ in a null-complemented set $\Omega'''$. Fix $t \in \Omega'''$. By finiteness of $g_x(t)$, the above display equation means that if $y_n$ is a sequence in $\mathcal{M}_0$ converging to $y$ we have

$$\langle x, Q_0(t) y_n \rangle - \langle x, Q_0(t) y \rangle \to 0.$$

Thus, by the Riesz representation theorem, there is a densely defined operator $Q(t)$ with domain $\mathcal{M}$ such that

$$\langle x, Q_0(t) y \rangle = \langle Q(t)x, y \rangle.$$

Further, $Q(t)$ is closed by [21, 5.1.5] and positive since $Q_0(t)$ is positive.

Let $x \in \mathcal{M}$ and $E \in \Sigma$. We will now argue that (3) extends to all $y \in \mathcal{H}$. For this we define $\psi : \mathcal{H} \to \mathbb{C}$ by

$$\psi(y) = \int_E \langle Q(t)x, y \rangle \, d\mu(t).$$

It suffices to show that $\psi$ is well-defined and agrees with $\mu_{x,y}(E)$ for all $y \in \mathcal{H}$. But this follows from the extension argument just applied to $\phi : \mathcal{M}_0 \to \mathbb{C}$ in the paragraph before last. This concludes the proof of (1).

For (2), suppose that

$$\int_E \langle Q_1(t)x, y \rangle \, d\mu_1(t) = \int_E \langle Q_2(t)x, y \rangle \, d\mu_2(t),$$

for all $E \in \Sigma$ and all $x \in \mathcal{M}$ and $y \in \mathcal{H}$. Then for $(\mu_1 + \mu_2)$-a.e. $t$ and all $x \in \mathcal{M}$ and $y \in \mathcal{H}$, we have

$$\langle Q_1(t)x, y \rangle \frac{d\mu_1}{d(\mu_1 + \mu_2)}(t) = \langle Q_2(t)x, y \rangle \frac{d\mu_2}{d(\mu_1 + \mu_2)}(t).$$

But this means

$$Q_1(t)\frac{d\mu_1}{d(\mu_1 + \mu_2)}(t) = Q_2(t)\frac{d\mu_2}{d(\mu_1 + \mu_2)}(t),$$

as desired.

The above yields the sigma-finite case ([22, Theorem 3.3.2]) as an immediate corollary. Here, we say that $E \mapsto M(E)$ is sigma-finite if its total variation with respect to the operator norm is sigma-finite.

**Corollary.** Suppose $(\Omega, \Sigma, M)$ is a sigma-finite POVM. Then $\mu_x$ is sigma-finite for all $x \in \mathcal{H}$ and $Q(t)$ is bounded.
Proof. We may replace $\mu$ in the previous proof by the total variation measure of $M$ since it dominates $\mu_{x,y}$ for all $x, y \in \mathcal{H}$. Further, $\mu(E) \|x\|$ dominates the total variation measure of $E \mapsto M(E)x$, so the latter is sigma-finite for all $x \in \mathcal{H}$. By the definition of the total-variation measure of $\mu_{x,y}$ and $M$, we have
\[
\int_E \|\langle Q(t)x, y \rangle\| d\mu(t) \leq \mu(E) \|x\| \|y\|
\]
for all $E \in \Sigma$ and all $x, y \in \mathcal{H}$. Thus, the Radon-Nikodym theorem tells us that for $\mu$-a.e. $t$ and all $x, y \in \mathcal{H}$,
\[
|\langle Q(t)x, y \rangle| \leq \|x\| \|y\|,
\]
which means that $Q(t)$ is bounded for $\mu$-a.e. $t$. □

It follows immediately from the Theorem that any framed POVM $M$ satisfying the given sigma-finiteness condition arises from an OVF, as desired. Indeed, letting $\mu$, $Q(t)$, and $M$ be as in the Theorem, such an OVF is $(\mu, M, \{T(t)\}_{t \in \Omega})$, where $T(t) = Q(t)^{1/2}$. Further, the uniqueness condition of the Theorem shows that if $(\mu_1, M, \{T_1(t)\}_{t \in \Omega})$ and $(\mu_2, M, \{T_2(t)\}_{t \in \Omega})$ are any two OVFs giving rise to $M$, then they are essentially the same in the sense that
\[
T_1(t)^*T_1(t) \frac{d\mu_1}{d(\mu_1 + \mu_2)}(t) = T_2(t)^*T_2(t) \frac{d\mu_2}{d(\mu_1 + \mu_2)}(t)
\]
for $(\mu_1 + \mu_2)$-a.e. $t$. This concludes our argument.

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