On the equivalence of approximate Gottesman-Kitaev-Preskill codes

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(Dated: October 30, 2019)

The Gottesman-Kitaev-Preskill (GKP) quantum error correcting code attracts much attention in continuous variable (CV) quantum computation and CV quantum communication due to the simplicity of error correcting routines and the high tolerance against Gaussian errors. Since the GKP code state should be regarded as a limit of physically meaningful approximate ones, various approximations have been developed until today, but explicit relations among them are still unclear. In this paper, we rigorously prove the equivalence of these approximate GKP codes with an explicit correspondence of the parameters. We also propose a standard form of the approximate code states in the position representation, which enables us to derive closed-from expressions for the Wigner functions, the inner products, and the average photon numbers in terms of the theta functions. Our results serve as fundamental tools for further analyses of fault-tolerant quantum computation and channel coding using approximate GKP codes.

I. INTRODUCTION

Continuous variable (CV) systems [1–4] have attracted a growing interest in the field of quantum information science as a promising candidate for implementing quantum information processing. For reliable implementations of information processing tasks, one needs to construct an error correcting routine to fight against the inevitable noise in the real world. Intensive research has thus been made on the CV error correcting codes [5–22]. Among them, the Gottesman-Kitaev-Preskill (GKP) code [7] gathers much attention in terms of both fault-tolerant CV quantum computation [8, 23–30] and CV quantum communication [22, 31, 32] as it needs only Gaussian operations to implement the whole Clifford gates (or even the universal gate set using the distillation protocol with the GKP code state [33]), and it is highly robust against the random displacement errors and the loss errors [34].

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We have to regard an ideal GKP code state as a limit of its approximation. This is because the ideal GKP code state is unnormalizable, while physically meaningful states in quantum mechanics are normalizable. Various approximations of the GKP code states, which are considered to be roughly equivalent, appeared in the past literature [7, 8, 22, 30, 32, 35–38], each of which uses a convenient form of approximation in its respective context. However, exact relations of these approximations are unclear, and thus we lack the way to compare these results directly. Our aim here is to find rigorous relations among the different approximations of the GKP code states, and bridge the gap of the results in the past literature. Along the way, we also derive a closed expression of the Wigner functions, the normalization constants, and average photon numbers of these approximate code states.

This paper is organized as follows. In Section II, we define the notations used throughout this paper. In Section III, we review the formulation of the GKP code, and introduce its three approximations which have been conventionally used. In Section IV, which contains the main results of our paper, we explicitly give the position and momentum representations of these approximate code states. They allow us to derive the exact relations among these approximate code states as shown in Theorem I. Using the equivalence, we introduce a standard form of the approximate GKP code state. In Section V, we derive Wigner representations, inner products, and average photon numbers of the approximate code states using the standard form. Finally in Section VI, we give concluding remarks.

II. NOTATIONS

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II. NOTATIONS

We denote canonical operators by  \( \hat{q} \) and  \( \hat{p} \), which satisfy the commutation relation [\( \hat{q}, \hat{p} \)] = i, where we set  \( \hbar = 1 \). We denote annihilation and creation operators by  \( \hat{a} \) and  \( \hat{a}^\dagger \), respectively, which are associated with  \( \hat{q} \) and  \( \hat{p} \) as  \( \hat{q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2} \) and  \( \hat{p} = (\hat{a} - \hat{a}^\dagger)/\sqrt{2i} \). This leads to the commutation relation [\( \hat{a}, \hat{a}^\dagger \)] = 1. The Weyl-Heisenberg displacement operators are represented by  \( \hat{X}(r) := \exp(-ir\hat{p}) \) and  \( \hat{Z}(r) := \exp(irq) \), which displace a state by +r in position and momentum coordinates in the phase space, respectively. General Weyl-Heisenberg displacement operators are represented by  \( \hat{V}(r) := \exp(-ir_p r_q/2)\hat{Z}(r_p)\hat{X}(r_q) \), where  \( r = (r_p, r_q) \). The relation between  \( \hat{V}(r) \) and the conventional definition of the displacement operator  \( \hat{D}(\alpha) := \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \) is  \( \hat{V}(r) = \hat{D}(\sqrt{2r} + ir_p)/\sqrt{2} \). Squeezing operator  \( \hat{S}(\xi) \) (\( \xi \in \mathbb{R} \)) is defined as  \( \hat{S}(\xi) := \exp(i\xi (\hat{q}\hat{p} + \hat{p}\hat{q})/2) \), which satisfies  \( \hat{S}^\dagger(\xi)\hat{g}\hat{S}(\xi) = e^{\xi}\hat{g} \) and  \( \hat{S}^\dagger(\xi)\hat{p}\hat{S}(\xi) = e^{\xi}\hat{p} \). The number operator \( \hat{n} \) is defined as  \( \hat{n} := \hat{a}^\dagger\hat{a} \), and the Fourier operator  \( \hat{F} \) is defined as  \( \hat{F} := \exp(\pi i\hat{n}/2) \). Let  \( \hat{I} \) denote the identity
FIG. 1. The theta function in the form of (2) with respect to $x$ when $b = 1/2$ and $t = 1/15$ (blue solid line), and when $b = 1/2$ and $t = 1/3$ (yellow dashed line). The theta function in this form is a sequence of the same Gaussian functions with respect to $x$ which has peaks at $b, b \pm 1, b \pm 2, \ldots$, and the width of each Gaussian is determined by $t$ as shown in the figure. Note that (2) approaches the Dirac comb as $t \to 0$.

Throughout the paper, $|·\rangle$ denotes the logical states of (approximate) GKP codes. Other representations are specified by subscripts of ket vectors. For example, $|n\rangle_f$ denotes the Fock state, $|q\rangle_{\hat{q}}$ denotes the (generalized) eigenstate of the position operator $\hat{q}$, and $|p\rangle_{\hat{p}}$ is of the momentum operator $\hat{p}$. The latter two satisfy $\hat{q} \langle q | q' \rangle_{\hat{q}} = \delta(q - q'), \hat{p} \langle p | p' \rangle_{\hat{p}} = \delta(p - p'), \hat{q} \langle q | p \rangle_{\hat{p}} = \frac{1}{\sqrt{2\pi}} e^{iqp},$ and $\hat{F} |x\rangle_{\hat{q}} = |x\rangle_{\hat{p}},$ where $\delta(·)$ denotes the Dirac delta function.

We also line up functions that are used throughout the paper. For $z \in \mathbb{C}, \tau \in \mathbb{C},$ and $\text{Im}(\tau) > 0$, let $\vartheta(z, \tau) := \sum_{s \in \mathbb{Z}} \exp(\pi i \tau s^2 + 2\pi i z s)$ be the theta function (we follow the notation in [39]), and

$$\vartheta[a/b](z, \tau) := \sum_{s \in \mathbb{Z}} \exp[\pi i \tau s^2 + 2\pi i (z + b)(s + a)] = \exp[\pi i a^2 + 2\pi i a(z + b)] \vartheta(z + \tau a + b, \tau)$$

(1)

be the theta function with rational characteristics $(a, b)$ [39]. The theta functions which we mainly use are in the form

$$\vartheta[0/b](x, it),$$

(2)

where $x, t \in \mathbb{R}$, and $b \in \mathbb{Q}$. The theta function in this form is a sequence of the same Gaussian functions with respect to $x$ which has peaks at $b, b \pm 1, b \pm 2, \ldots$, and the width of each Gaussian is determined by $t$ as shown in Figure [1]. Note that (2) approaches the Dirac comb as $t \to 0$. Let $G_\sigma^2(x)$ be a probability density function of the normal distribution with variance $\sigma^2$, which is
defined as follows:

\[ G_{\sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right). \] (3)

For an operator \( \hat{A} \) acting on a Hilbert space, the Wigner function \( W_{\hat{A}}(q,p) \) of \( \hat{A} \) is given by

\[ W_{\hat{A}}(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, e^{2ipx} \langle \hat{A} | q - x \rangle_{\hat{A}}^\dagger \langle q + x \rangle_{\hat{A}}. \] (4)

Finally, let \( f * g(x) := \int dy f(y)g(x - y) \) denote the convolution of two functions \( f(x) \) and \( g(x) \).

### III. GOTTESMAN-KITAEV-PRESKILL CODE

Gottesman-Kitaev-Preskill (GKP) code [7] is an error correcting code which encodes \( d \)-dimensional logical Hilbert space in an oscillator mode. It has a lattice-like periodic structure when represented in the phase space; the Wigner function of the code states \( |j\rangle \) and \( |j+1\rangle \) have the same period but \( |j+1\rangle \) is shifted from \( |j\rangle \) by \( \frac{1}{d} \) of the period in position coordinate in the phase space. In the present paper, we treat the square lattice GKP code, while the generalization of our results to the case of hexagonal lattice GKP code is possible. The ideal (square lattice) GKP code states are defined as [7]

\[ |j^{(\text{ideal})}\rangle := \sqrt{\alpha d} \sum_{s \in \mathbb{Z}} |\alpha(ds + j)\rangle_{\hat{q}}, \] (5)

where \( d \) denotes the dimension of the logical Hilbert space, \( j \in \{0, \ldots, d - 1\} \), and the pre-factor \( \sqrt{\alpha d} \) is for later convenience. In position representation, it has a comb-like shape consisting of the Dirac delta functions (i.e., a Dirac comb) at intervals \( \alpha d \), and \( |j+1^{(\text{ideal})}\rangle \) is shifted from \( |j^{(\text{ideal})}\rangle \) by \( \alpha \). These states form the basis of the \( d \)-dimensional logical Hilbert space in an oscillator system, and therefore, we call them ideal logical basis states. In the momentum representation, the logical basis states are given by

\[
|j^{(\text{ideal})}\rangle = \int dy \sqrt{\alpha d} \sum_{s \in \mathbb{Z}} |y\rangle_{\hat{p}} \langle y|\alpha(ds + j)\rangle_{\hat{q}} \\
= \sqrt{\alpha d} \int dy \sum_{s \in \mathbb{Z}} e^{-i\alpha(ds+j)p} |y\rangle_{\hat{p}} \\
= \sqrt{2\pi\alpha d} \int dy \sum_{t \in \mathbb{Z}} \delta(\alpha dy - \frac{2\pi t}{\alpha d}) e^{-ijp} |y\rangle_{\hat{p}} \\
= \sqrt{\frac{2\pi}{\alpha d}} \sum_{t \in \mathbb{Z}} e^{-i\frac{2\pi j t}{d}} |2\pi t/\alpha d\rangle_{\hat{p}},
\] (6)

where we used the Poisson summation formula \( \sum_{s \in \mathbb{Z}} e^{-isx} = 2\pi \sum_{t \in \mathbb{Z}} \delta(x - 2\pi t) \).
In the rest of this section as well as Section IV A and IV B, we set
\[
\alpha = \sqrt{\frac{2\pi}{d}} =: \alpha_d,
\]
which symmetrizes the code space in position and momentum coordinates in the phase space [7]. This property of the code is meaningful even when the logical basis states are non-orthogonal, which is the case in approximate GKP codes. In this paper, we adopt the following definition for this property.

**Definition 1** (The code which is symmetric in position and momentum coordinates in the phase space). Let \(\{|j\rangle : j = 0, \ldots, d - 1\}\) be the logical qudit basis encoded in an oscillator mode. The code is symmetric in position and momentum coordinates if it satisfies

\[
\text{span}\{|j\rangle : j = 0, \ldots, d - 1\} = \text{span}\{\hat{F}|j\rangle : j = 0, \ldots, d - 1\}.
\]

(8)

Note that we can use \(\hat{F}^\dagger\) instead of \(\hat{F}\) in the definition. The symmetric code is beneficial if we aim at minimizing logical-level errors caused by physical-level phase-insensitive errors, that is, errors which occur symmetrically in position and momentum coordinates in the phase space.

The ideal GKP code can be regarded as a stabilizer code. The stabilizer generators are given by the two commuting displacement operators \(X_{st} := \hat{X}(\alpha_d d)\) and \(Z_{st} := \hat{Z}(2\pi/\alpha_d d) = \hat{Z}(\alpha_d d)\). Similarly, logical Pauli operators can be defined as \(X_L := \hat{X}(\alpha_d)\) and \(Z_L := \hat{Z}(2\pi/\alpha_d d) = \hat{Z}(\alpha_d)\), which satisfy \(Z_L X_L = \exp(2\pi i/d)X_L Z_L\) as expected. Using these stabilizer generators and logical Pauli operators, we have an alternative expression of the ideal GKP logical state as follows [7, 22]:

\[
|j^{\text{ideal}}\rangle = \frac{(d/2)^{\frac{d}{2}}}{\vartheta(0, i/d)} \sum_{(s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}} \hat{X}(\alpha_d (ds_1 + j)) \hat{Z}(\alpha_d s_2) |0\rangle_f
\]

\[
= \frac{(d/2)^{\frac{d}{2}}}{\vartheta(0, i/d)} X_L^j \left( \sum_{l=0}^{d-1} Z_L^l \right) \sum_{(s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}} X_{st}^{s_1} Z_{st}^{s_2} |0\rangle_f
\]

\[
=: X_L^j \left( \sum_{l=0}^{d-1} Z_L^l \right) P_{\text{GKP}} |0\rangle_f,
\]

(9)

where \(\vartheta(0, i/d)\) is the theta function, and the last line defines an operator \(P_{\text{GKP}}\), which is interpreted as the projection onto the code space ignoring the normalization. The consistency with (6) (and
hence with (5) can be confirmed as follows [22]:

\[
\frac{(d/2)^{\frac{1}{2}}}{\vartheta(0,i/d)} \sum_{(s_1,s_2) \in \mathbb{Z} \times \mathbb{Z}} \hat{X}(\alpha_d(ds_1 + j)) \hat{Z}(\alpha_d s_2) |0\rangle_f \\
= \frac{(d/2)^{\frac{1}{2}}}{\vartheta(0,i/d)} \sum_{(s_1,s_2) \in \mathbb{Z} \times \mathbb{Z}} \int dq \, dp \ e^{-ip\alpha_d(ds_1 + j) + ip\alpha_d s_2} |p\rangle \hat{p} \langle p| |q\rangle \hat{q} \langle q| |0\rangle_f \\
= \frac{1}{\sqrt{2\pi\alpha_d \vartheta(0,i/d)}} \sum_{(s_1,s_2) \in \mathbb{Z} \times \mathbb{Z}} \int dp \ \exp\left[-\frac{1}{2}(p - \alpha_d s_2)^2\right] \exp(-ip\alpha_d s_1) \exp(-ip\alpha_d j) |p\rangle \hat{p} \\
= \frac{1}{\sqrt{\alpha_d \vartheta(0,i/d)}} \sum_{(s_1,s_2) \in \mathbb{Z} \times \mathbb{Z}} \int dp \ \exp\left[-\frac{1}{2}(p - \alpha_d s_2)^2\right] \exp(-ip\alpha_d j) \exp\left(i\frac{2\pi j s_1}{d}\right) |p\rangle |2\pi s_1/\alpha_d\rangle \hat{p} \\
= \sqrt{\frac{(2\pi)^2/\alpha_d^2 d^2}{\vartheta(0,i/d)}} \sum_{(s_1',s_2') \in \mathbb{Z} \times \mathbb{Z}} \exp\left(-\frac{\alpha_d^2 d}{2} s_2^2\right) \exp\left(-i\frac{2\pi j s_1'}{d}\right) |(p =) 2\pi s_1'/\alpha_d\rangle \hat{p} \\
= |j^{(\text{ideal})}\rangle,
\]

where we used \( \hat{q} \langle q| |0\rangle_f = \pi^{-\frac{1}{4}} \exp(-q^2/2) \) in the second equality, used the Poisson summation formula \( \sum_{x \in \mathbb{Z}} e^{-ix x} = 2\pi \sum_{t \in \mathbb{Z}} \delta(x - 2\pi t) \) in the fourth equality, and defined \( s_2' := s_1' - s_2 \) in the fifth equality.

In the phase space, the Wigner function of the state \(|j^{(\text{ideal})}\rangle\) is given by [7]

\[
W_{j^{(\text{ideal})}}(q,p) = \frac{1}{2} \sum_{(t,t') \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi i t t'} \delta\left(p - \frac{\alpha_d t + \alpha_d t'}{2}\right) \delta\left(q - \frac{\alpha_d t + \alpha_d t'}{2} - \alpha_d j\right) \\
= \frac{1}{2} \sum_{(t,t') \in \mathbb{Z} \times \mathbb{Z}} \delta\left(p + \frac{\alpha_d t}{2}\right) \left[\delta\left(q - \alpha_d \left(t' + \frac{j}{d}\right)\right) + (-1)^t \delta\left(q - \alpha_d \left(t' + \frac{j}{d} + \frac{1}{2}\right)\right)\right] \\
\]

This shows that the Wigner function of the ideal logical basis states forms a square lattice consisting of the Dirac delta functions, which has half the period of the Dirac comb in the position and momentum representations. Since its sublattice formed of the odd periods starting from \((q,p) = (\alpha_d j,0)\) consists of the Dirac delta functions with negative signs, the comb at the odd periods in position coordinate cancel out when integrated along momentum coordinate, and vice versa.

As defined so far, the ideal GKP code states are nonnormalizable and thus unphysical. Therefore, the ideal GKP code should be regarded as a limiting case of physically meaningful approxi-
mate ones. Various approximations of the GKP code states are considered in the past literature [7, 8, 22, 32, 35–38]. Especially, the following three approximations are conventionally used.

• (Approximation 1)

$$|j^{(1)}_{\kappa, \Delta}\rangle := \frac{1}{\sqrt{N^{(1)}_{\kappa, \Delta, j}}} \sum_{s \in \mathbb{Z}} e^{-\frac{1}{2} \kappa^2 \alpha_j^2 (ds + j)^2} \tilde{X}(\alpha_d(ds + j)) \tilde{S}(-\ln \Delta) |0\rangle_f,$$

where $\kappa, \Delta > 0$, and $N^{(1)}_{\kappa, \Delta, j}$ is a normalization constant. This approximate code state approaches the ideal one in the limit of $\kappa, \Delta \to 0$. This approximation first appeared in the original paper [7]. The idea of this approximation is to replace the superposition of the position “eigenstates” with that of squeezed coherent states with a squeezing parameter $\ln(1/\Delta)$, which are weighted by a Gaussian envelope of the width $1/\kappa$. This gives us an insight about how to generate the GKP code state experimentally [40].

• (Approximation 2)

$$|j^{(2)}_{\gamma, \delta}\rangle := \frac{1}{\sqrt{N^{(2)}_{\gamma, \delta, j}}} \int \int \frac{dr_1 dr_2}{2 \pi \gamma \delta} e^{-\frac{r_1^2}{2 \gamma^2} - \frac{r_2^2}{2 \delta^2}} \hat{V}(r) |j^{(\text{ideal})}\rangle,$$

where $0 < \gamma \delta < 2$, and it approaches the ideal code state when $\gamma, \delta \to 0$. This approximation also appeared in the original paper to regard the approximation as an error, and treat $\frac{1}{2 \pi \gamma \delta} e^{-\frac{r_1^2}{2 \gamma^2} - \frac{r_2^2}{2 \delta^2}}$ as an error “wave function” [7]. They use the word “wave function” because the state given in (12) is not an ideal code state subject to the error caused by the random displacement channel, but a coherent superposition of randomly displaced ideal code states. The error “wave function” later turned out to have more profound meanings; it is actually a wave function in the “grid representation” [9, 38, 41–43], which is an analogous representation to the position representation, but with respect to the so-called “shifted grid states” instead of the position eigenstates. In Appendix A we make remarks on the “grid representation” in terms of the representation theory of the Heisenberg group.

• (Approximation 3)

$$|j^{(3)}_\beta\rangle := \frac{1}{\sqrt{N^{(3)}_{\beta, j}}} e^{-\beta(n + \frac{1}{2})} |j^{(\text{ideal})}\rangle,$$

where $\beta$ satisfies $\beta > 0$, and it approaches the ideal code state when $\beta \to 0$. Contrary to the former two approximations, Approximation 3, first appearing in [8], only deals with symmetric envelope in position and momentum coordinates. Since the approximation factor
$e^{-\beta(n+\frac{1}{2})}$ is diagonal in the fock basis, this approximation may be useful for computing the statistical properties of the quantity which is diagonal in the fock basis, as shown in [8]. On the other hand, though this approximate code state could conceptually be prepared by one’s feeding the ideal code states to the beamsplitter followed by post-selecting the vacuum click at the idler port [32], it provides few implications about their realistic experimental generation.

IV. THE EQUIVALENCE OF THE APPROXIMATIONS

A. Position and momentum representations

In order to analyze the relation among the three approximations, we derive the position and momentum representations, $\hat{q}\langle q\vert j \rangle$ and $\hat{p}\langle p\vert j \rangle$, of the approximate code states. Note that the position and momentum representations of Approximation 1 have already appeared in the past literature [7, 9, 23, 35, 37, 38, 40, 42, 44, 45], but we rewrite them for the completeness. For that, we first define the following functions.

**Definition 2.** Define $E_{\mu, \Gamma, a}(x) := \exp\left(-\frac{x^2}{2\mu}\right)\sum_{s \in \mathbb{Z}} \delta(x - (s + a)\Gamma)$,

\begin{equation}
E_{\mu, \Gamma, a}(x) := \exp\left(-\frac{x^2}{2\mu}\right)\sum_{s \in \mathbb{Z}} \delta(x - (s + a)\Gamma),
\end{equation}

\begin{equation}\tilde{E}_{\mu, \Gamma, a}(x) := \exp\left(-\frac{x^2}{2\mu}\right)\sum_{s \in \mathbb{Z}} e^{2\pi i s} \delta(x + s\Gamma).\end{equation}

The function $E_{\mu, \Gamma, a}(x)$ is a Dirac comb with its interval given by $\Gamma$, which is shifted by the rational $a$ of the interval from the origin and weighted by the Gaussian $\exp(-x^2/2\mu)$ of the width $\mu$. It can also be interpreted as a Fourier transform of the theta function in the form of $\frac{1}{\sqrt{2\pi}} \Theta^{[0]}_{\mathbb{Z}}(\frac{\Gamma}{2\pi} x, \frac{i\Gamma^2}{2\pi\mu})$ with respect to $x$, which can be confirmed by its definition (1). On the other hand, the function $\tilde{E}_{\mu, \Gamma, a}(x)$, a Dirac comb with the Gaussian weight which has a phase factor for each peak, is a Fourier transform of the theta function in the form of $\frac{1}{\sqrt{2\pi}} \Theta^{[0]}_{\mathbb{Z}}\left(-\frac{\Gamma}{2\pi} x, \frac{i\Gamma^2}{2\pi\mu}\right)$, which can also be confirmed by (1).

Now, under Definition 2, we show the following proposition.

**Proposition 1** (The position representation). Let $\kappa, \Delta, \beta > 0$ and $0 < \gamma\delta < 2$. Define $\lambda(\gamma, \delta) := 1 + \frac{\gamma^2 \delta^2}{\Gamma}$. Then, the position representations of the states (11), (12), and (13) are given as follows:
For each approximation, we gave the two expressions in which we replace the Dirac delta functions in the definition of the ideal GKP code state with the Gaussian functions in different orders. In the first representations (16), (18), and (20), each peak of the Dirac comb, which is weighted by a Gaussian as shown in the definition of \( E_{\mu, \Gamma, a} \), is convoluted with another Gaussian \( G_{\nu}(q) \). In the second representations (17), (19), and (21), the infinite sequence of Gaussian spikes as defined in \( \vartheta[a](q, it) \) is multiplied by another Gaussian function \( G_{\nu'}(q) \) which works as an overall envelope. The first representations are suited for understanding the physical structure of the approximation such as the interval of the neighboring Gaussian peaks. The second ones are convenient for the numerical calculations because the algorithms to calculate the theta function with arbitrary precision are well known [46].

**Sketch of the proof.** We derive (16), (18), and (20) with straightforward but cumbersome calculations, and then apply the following lemma to derive (17), (19), and (21).

**Lemma 1.** For \( \mu, \nu > 0, \Gamma \in \mathbb{R} \), and \( a \in \mathbb{Q} \), the following equality holds:

\[
E_{\mu, \Gamma, a} \ast G_{\nu}(q) = \sqrt{\frac{2\pi \mu}{\Gamma^2}} G_{\mu+\nu}(q) \vartheta[0] \left( -\frac{q}{(1+\nu/\mu)\Gamma}, \frac{2\pi i\nu}{(1+\nu/\mu)\Gamma^2} \right).
\] (22)
The full proof of Proposition 1 as well as the proof of Lemma 1 is in Appendix B.

Under Definition 2, the momentum representations of the approximate code states can also be given by the following corollary.

**Corollary 1** (The momentum representation). Let \( \kappa, \Delta, \beta > 0 \) and \( 0 < \gamma \delta < 2 \). Let \( \lambda(\gamma, \delta) := 1 + \frac{\gamma^2 \delta^2}{4} \). Then, the momentum representations of the states (11), (12), and (13) are given as follows:

- **(Approximation 1)**
  
  \[
  \hat{\rho} \langle p | j^{(1)}_{\kappa, \Delta} \rangle = \left( \frac{2 \sqrt{\pi \Delta^2}}{1 + \kappa^2 \Delta^2} \right)^{\frac{1}{2}} \tilde{E} \left( \frac{1}{\Delta^2 (1 + \kappa^2 \Delta^2)} \right)^{\frac{1}{2}} \cdot \alpha_d \cdot \frac{\lambda(\gamma, \delta)}{\gamma, \delta} \cdot \beta^{\frac{1}{2}} G \left( \frac{\kappa^2}{1 + \kappa^2 \Delta^2} \right) (p).
  \]

- **(Approximation 2)**
  
  \[
  \hat{\rho} \langle p | j^{(2)}_{\gamma, \delta} \rangle = \left( \frac{\alpha_d}{\lambda(\gamma, \delta)} \right)^{\frac{1}{2}} \tilde{E} \left( \frac{\lambda(\gamma, \delta)}{\Delta^2} \right)^{\frac{1}{2}} \cdot \alpha_d \cdot \left( 1 - \frac{\gamma^2 \delta^2}{2 \lambda(\gamma, \delta)} \right)^{\frac{1}{2}} \cdot \beta^{\frac{1}{2}} G \left( \frac{\gamma^2}{\lambda(\gamma, \delta)} \right) (p).
  \]

- **(Approximation 3)**
  
  \[
  \hat{\rho} \langle p | j^{(3)}_{\beta} \rangle = \left( \frac{\alpha_d}{\cosh \beta} \right)^{\frac{1}{2}} \tilde{E} \left( \frac{1}{\sinh \beta \cosh \beta} \right)^{\frac{1}{2}} \cdot \alpha_d \cdot \beta^{\frac{1}{2}} \cdot \beta^{\frac{1}{2}} G \left( \tanh \beta \right) (p).
  \]

**Proof.** We use the fact that the momentum representation of a state is a Fourier transform of its position representation, i.e., \( \hat{\rho} \langle p | j \rangle = \frac{1}{\sqrt{2\pi}} \int dq \, e^{-ipq} \langle q | j \rangle \). We can thus derive (23), (24), and (25) as Fourier transforms of (17), (19), and (21), respectively, exploiting the fact that the Fourier transform of the product of two functions is given by the convolution of the Fourier transforms of the respective functions, and the Fourier transform of \( \frac{1}{\sqrt{2\pi}} \hat{\Theta}^{[\beta]} \left( -\frac{\Gamma}{2\pi} x, \frac{\Gamma^2}{2\pi^2} \right) \) is \( \hat{E}_{\mu, \Gamma, a} \) while the Fourier transform of \( G_\nu \) is \( \sqrt{1/\nu} G_{1/\nu} \).

**B. Explicit relations of the three approximations**

The position and momentum representations of the three different approximate GKP code states lead to conditions for equivalence of these approximations. Since \( E_{\mu, \Gamma, a} * G_\nu (x) \) denotes the array of the Gaussian spikes \( G_\nu (x) \) at intervals \( \Gamma \), one can notice from (18) and (20) that the intervals of the Gaussian spikes of the approximate code states are narrower than that of the ideal one, \( \alpha_d d \), in the case of Approximation 2 and 3. Furthermore, from (24) and (25), the intervals of...
the Gaussian spikes of each of these approximate code states in the momentum representations get narrower in the same proportion as that of their respective position representations. With this observation, Approximation 3, which has symmetric envelope functions in position and momentum representations, (20) and (25), is expected to be a symmetric case \( (\gamma = \delta) \) of Approximation 2 in the sense of “symmetric” in Definition 1. This can be confirmed by the following.

**Corollary 2** (The symmetric code). Let \( \hat{F} \) be the Fourier operator defined in Section II. Then, the following relation holds for the logical basis states of the Approximation 3:

\[
\hat{F}^\dagger |j^{(3)}_\beta\rangle = \sum_{j' = 0}^{d-1} \sqrt{N^{(3)}_{\beta,j'}/N^{(3)}_{\beta,j}} |j^{(3)}_{\beta,j'}\rangle .
\]

(26)

The same relation holds for Approximation 2 iff \( \gamma = \delta \), i.e.,

\[
\hat{F}^\dagger |j^{(2)}_{\gamma,\gamma}\rangle = \sum_{j' = 0}^{d-1} \sqrt{N^{(2)}_{\gamma,j',\gamma}/N^{(2)}_{\gamma,j,\gamma}} |j^{(2)}_{\gamma,j',\gamma}\rangle .
\]

(27)

**Proof.** It can be observed by combining \( \hat{q} |x\rangle \hat{F}^\dagger = \hat{p} |x\rangle \) with (18), (20), (19), and (25).

In contrast with Approximation 2 and 3, the intervals of Gaussian spikes in the position representation (16) of Approximation 1 are the same as those in the position representation of the ideal code state, and the intervals in the momentum representation (23) of Approximation 1 are narrower than those in the momentum representation of the ideal code state; that is, Approximation 1 narrows the lattice spacing of the code space asymmetrically in position and momentum coordinates. This suggests that Approximation 1 may be related to Approximation 2 or 3 by a transformation that symmetrizes the deviation of the lattice spacing in position and momentum coordinates.

Let us confirm it by applying the squeezing operation \( \hat{S}(\ln \sqrt{1 + \kappa^2 \Delta^2}) \) for symmetrizing the intervals of the Gaussian spikes of the code state \( |j^{(1)}_{\kappa,\Delta}\rangle \) in position and momentum coordinates:

\[
\hat{q} |q\rangle \hat{S}(\ln \sqrt{1 + \kappa^2 \Delta^2}) |j^{(1)}_{\kappa,\Delta}\rangle = (1 + \kappa^2 \Delta^2)^{\frac{1}{4}} \hat{q} |\sqrt{1 + \kappa^2 \Delta^2} q |j^{(1)}_{\kappa,\Delta}\rangle
\]

\[
= \sqrt{m} E_{\kappa^2(1+\kappa^2 \Delta^2)}^{\frac{\alpha}{d^2} \sqrt{1+\kappa^2 \Delta^2}}, \hat{q} * G_{\Delta^2}^{(1+\kappa^2 \Delta^2)} (q)
\]

(28)

\[
\hat{p} |p\rangle \hat{S}(\ln \sqrt{1 + \kappa^2 \Delta^2}) |j^{(1)}_{\kappa,\Delta}\rangle = (1 + \kappa^2 \Delta^2)^{\frac{1}{4}} \hat{p} |\sqrt{1 + \kappa^2 \Delta^2} p |j^{(1)}_{\kappa,\Delta}\rangle
\]

\[
= \sqrt{\frac{m}{d}} \hat{E}_{\Delta^2}^{\frac{\alpha}{d^2} \sqrt{1+\kappa^2 \Delta^2}}, \hat{p} * G_{\kappa^2 \Delta^2} (p)
\]

(29)

where \( m = \frac{2}{\sqrt{N^{(1)}_{\kappa,\Delta,j}}} \sqrt{\frac{\pi \Delta^2}{1+\kappa^2 \Delta^2}} \). In order to derive (28) and (29), we used \( E_{\mu,\Gamma,a} * G_{\nu}(bx) = \frac{1}{b} E_{\mu,\Gamma,a} * G_{\nu}(x) \) and \( E_{\mu,\Gamma,a} * G_{\nu}(bx) = \frac{1}{b} E_{\mu,\Gamma,a} * G_{\nu}(x) \), which can be obtained from the definition of the
functions $E_{\mu,\Gamma,a}(x)$, $\tilde{E}_{\mu,\Gamma,a}(x)$, and $G_{\nu}(x)$. Comparing the position representation of the squeezed version of Approximation 1, (28), with the position representation of Approximation 2 (18) and 3 (20), we arrive at the following theorem.

**Theorem 1** (The equivalence of the approximate GKP code states). By choosing the parameters in Approximation 1 and 2 as

$$\kappa^2 = \frac{\gamma^2}{\lambda(\gamma, \delta)} = \tanh \beta,$$

$$\Delta^2 = \frac{\delta^2}{\lambda(\gamma, \delta)} \left(1 - \frac{\gamma^2\delta^2}{2\lambda(\gamma, \delta)}\right)^{-2} = \sinh \beta \cosh \beta,$$

$$\gamma^2 = \delta^2 = 2 \tanh \frac{\beta}{2}$$

where $\lambda(\gamma, \delta) := 1 + \frac{\gamma^2\delta^2}{4}$, we have

$$\hat{S} \left( \ln \sqrt{1 + \kappa^2 \Delta^2} \right) |j^{(1)}_{\kappa, \Delta}, \alpha\rangle = |j^{(2)}_{\gamma, \delta}, \alpha\rangle = |j^{(3)}_{\beta}, \alpha\rangle.$$

**Proof.** It directly follows from (18), (20), and (28).

Theorem 1 together with Corollary 2 shows that up to the moderate squeezing $\hat{S} \left( \ln \sqrt{1 + \kappa^2 \Delta^2} \right)$ for Approximation 1, which becomes even negligible in the limit of the good approximation, the symmetric code of Approximation 1, 2 and 3 are all equivalent, where “symmetric” is in the sense of Definition 1.

**Remark:** So far, we followed the convention to fix the lattice spacing parameter as $\alpha = \alpha_d$, and derived equivalence relations among symmetric approximate codes. We can extend the equivalence relations to asymmetric ones by regarding $\alpha$ as a free parameter in each approximation, and defining the states $|j^{(1)}_{\kappa, \Delta, \alpha}\rangle$, $|j^{(2)}_{\gamma, \delta, \alpha}\rangle$, and $|j^{(3)}_{\beta, \alpha}\rangle$ (See Appendix B). We can observe from (B2) and (B4) in Appendix B that $|j^{(1)}_{\kappa, \Delta, \alpha}\rangle = |j^{(2)}_{\gamma, \delta, \alpha'}\rangle$ with the following correspondence of the parameters:

$$\kappa^2 = \frac{\gamma^2}{\lambda(\gamma, \delta)} \left(1 - \frac{\gamma^2\delta^2}{2\lambda(\gamma, \delta)}\right)^{-2},$$

$$\alpha = \alpha' \left(1 - \frac{\gamma^2\delta^2}{2\lambda(\gamma, \delta)}\right),$$

$$\Delta^2 = \frac{\delta^2}{\lambda(\gamma, \delta)}.$$  

Compared to $|j^{(1)}_{\kappa, \Delta, \alpha}\rangle$ and $|j^{(2)}_{\gamma, \delta, \alpha}\rangle$, the third approximation $|j^{(3)}_{\beta, \alpha}\rangle$ has less parameters and thus cannot be made equivalent to $|j^{(1)}_{\kappa, \Delta, \alpha}\rangle$ and $|j^{(2)}_{\gamma, \delta, \alpha}\rangle$ in all the parameter region. However, if we apply the squeezing $\hat{S}(\ln \zeta)$ to $|j^{(3)}_{\beta, \alpha}\rangle$, then we have $|j^{(1)}_{\kappa, \Delta, \alpha}\rangle = |j^{(2)}_{\gamma, \delta, \alpha'}\rangle = \hat{S}(\ln \zeta) |j^{(3)}_{\beta, \alpha''}\rangle$ with the following
correspondence of the parameters in addition to (34), (35), and (36):

\[ \kappa^2 = \zeta^2 \sinh \beta \cosh \beta, \] (37)

\[ \alpha = \frac{\alpha''}{\zeta \cosh \beta}, \] (38)

\[ \Delta^2 = \tanh \beta \zeta^2. \] (39)

This can be confirmed from the fact that \[ \hat{q} \langle q | \hat{S} (\ln \zeta) | j \rangle = \sqrt{\zeta} \hat{q} \langle \zeta q | j \rangle, \] and \[ E_{\mu, \Gamma, a} \ast G_{\nu}(\zeta q) = \frac{1}{\zeta} E_{\mu, \frac{\zeta}{\zeta^2}, \zeta, a} \ast G_{\nu}(q). \]

C. The standard form

Now that we show the equivalence of Approximation 1, 2, and 3, we introduce a standard form of the approximate GKP code state, which we will use in the rest of the paper.

**Definition 3** (Standard form of the approximate GKP code states). Given three parameters \( \sigma_q^2 \), \( \sigma_p^2 \), and \( \Gamma \), the standard form of the approximate GKP code is defined as the code which is spanned by the logical qudit basis \( \{ | j \sigma_q^2, \sigma_p^2, \Gamma \rangle : j = 0, \ldots, d - 1 \} \) with its position representations given by

\[ \hat{q} \langle q | j \sigma_q^2, \sigma_p^2, \Gamma \rangle := \left( \frac{2\Gamma}{\sqrt{1 - 4\sigma_q^2\sigma_p^2 N_{\sigma_q^2, \sigma_p^2, \Gamma, j}}} \right)^{\frac{1}{2}} E_{\frac{1}{2\sigma_p^2}, \Gamma, \frac{1}{2}} \ast G_{2\sigma_q^2}(q), \] (40)

where \( N_{\sigma_q^2, \sigma_p^2, \Gamma, j} \) is a normalization constant, and \( 0 < \sigma_q^2, \sigma_p^2 < \frac{1}{2} \). For the symmetric code, the logical basis \( \{ | j \sigma \rangle : j = 0, \ldots, d - 1 \} \) is parametrized by only one parameter \( \sigma \) \( (0 < \sigma < \frac{1}{2}) \) as follows:

\[ \hat{q} \langle q | j \sigma \rangle := \left( \frac{2\alpha_d}{N_{\sigma^2, j}} \right)^{\frac{1}{2}} E_{\frac{1}{2\sigma^2} - 2\sigma^2, \alpha_d \sqrt{1 - 4\sigma^2}, \frac{1}{2}} \ast G_{2\sigma^2}(q). \] (41)

Note that \( | j \sigma \rangle \) is equal to \( | j \sigma, \sigma^2, \alpha_d \sqrt{1 - 4\sigma^2} \rangle \). The momentum representation of \( | j \sigma_q^2, \sigma_p^2, \Gamma \rangle \) is given by

\[ \hat{p} \langle p | j \sigma_q^2, \sigma_p^2, \Gamma \rangle = \left( \frac{4\pi}{\Gamma N_{\sigma_q^2, \sigma_p^2, \Gamma, j}} \right)^{\frac{1}{2}} \hat{E}_{\frac{1}{2\sigma_q^2} - 2\sigma^2, \frac{2\sigma^2}{\sqrt{1 - 4\sigma^2}}}, \frac{1}{2} \ast G_{2\sigma^2}(p), \] (42)

and thus, for the symmetric code, it is given by

\[ \hat{p} \langle p | j \sigma^2 \rangle = \left( \frac{2\alpha_d}{N_{\sigma^2, j}} \right)^{\frac{1}{2}} \hat{E}_{\frac{1}{2\sigma^2} - 2\sigma^2, \alpha_d \sqrt{1 - 4\sigma^2}, \frac{1}{2}} \ast G_{2\sigma^2}(p). \] (43)
The physical meanings of the parameters $\sigma_q^2$, $\sigma_p^2$ and $\Gamma$ of the state $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle$ (and hence $\sigma^2$ of the state $|j_{\sigma}\rangle$) will be clarified in the next section. Furthermore, an explicit form of the normalization constant $N_{\sigma_q^2,\sigma_p^2,\Gamma,j}$ (and hence $N_{\sigma^2,j}$) is given in Proposition 3 in the next section. The representation corresponding to (11), (12), and (13) for the state $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle$ can be obtained by simply substituting the corresponding parameters. For example, in the case of the representation corresponding to Approximation 1, we have from (11) and (16) that

$$|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle = \frac{1}{\sqrt{N(1)}} \sum_{s \in \mathbb{Z}} e^{-\sigma^2 q \frac{(s+j)^2}{2}} X((s+j/d)\Gamma) \hat{S}(-\ln \sqrt{2\sigma^2_q}) |0\rangle_f ,$$

where $N(1)$ is given by

$$N(1) = \sqrt{2\pi \sigma^2_q (1 - 4\sigma^2_q \sigma_p^2)} \frac{1}{\Gamma^2} N_{\sigma_q^2,\sigma_p^2,\Gamma,j} .$$

Likewise, the representations corresponding to Approximation 1 and 3 for the symmetric code $|j_{\sigma}\rangle$ are given by

$$|j_{\sigma}\rangle = \frac{1}{(\sigma^2/d)^{1/4}} \frac{1}{\sqrt{N_{\sigma^2,j}}} \sum_{s \in \mathbb{Z}} e^{-\sigma^2 \alpha^2 d(s+j)^2} X(\alpha_d (ds+j) \sqrt{1 - 4\sigma^4}) \hat{S}(-\ln \sqrt{2\sigma^2}) |0\rangle_f$$
$$= \frac{1}{(1/4 - \sigma^4)^{1/4}} \frac{1}{\sqrt{N_{\sigma^2,j}}} e^{-\text{arctanh}(2\sigma^2)(\hat{n} + \frac{1}{2})} |j\rangle_{\text{(ideal)}} .$$

V. EXPLICIT EXPRESSIONS OF THE WIGNER REPRESENTATIONS, INNER PRODUCTS, AND AVERAGE PHOTON NUMBERS

In this section, we derive the expressions of the Wigner representations, inner products, and average photon numbers for the standard form of the approximate code state $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle$ in Definition 3. Those for $|j_{\sigma}\rangle$ can also be given by substituting $\sigma_q^2 = \sigma_p^2 = \sigma^2$ and $\Gamma = \alpha_d d \sqrt{1 - 4\sigma^4}$.

A. Wigner representation

Here, we derive the Wigner representation of the operators $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle \langle j'_{\sigma_q^2,\sigma_p^2,\Gamma}|$. The Wigner representation of the approximate GKP code can be used for the analyses of quantum error correction as shown in [8, 24, 25, 29].

Proposition 2 (Wigner representation). For the approximate code states $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle$ and $|j'_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle$ in Definition 3, the Wigner representations $W_{j_{\sigma_q^2,\sigma_p^2,\Gamma}}(j'_{\sigma_q^2,\sigma_p^2,\Gamma} |(q,p)$ of the operators $|j_{\sigma_q^2,\sigma_p^2,\Gamma}\rangle \langle j'_{\sigma_q^2,\sigma_p^2,\Gamma}|$
is given as follows:

\[
W_{|j\sigma_q^2,\Gamma\rangle\langle j'\sigma_{q'}^2,\Gamma\rangle}(q,p) = \frac{1}{\sqrt{N_{\sigma_q^2,\sigma_{p}^2,\Gamma,j}N_{\sigma_{q'}^2,\sigma_{p'}^2,\Gamma,j'}}} \left[ \left( E_{\frac{1}{4\sigma_p^2} - \sigma_q^2,\Gamma,\frac{i + j'}{2\pi}} \ast G_{\sigma_q^2}(q) \right) \left( \tilde{E}_{\frac{1}{4\sigma_q^2} - \sigma_{p'}^2,\Gamma,\frac{i + j'}{2\pi}} \ast G_{\sigma_{p'}^2}(p) \right) + \left( E_{\frac{1}{4\sigma_p^2} - \sigma_q^2,\Gamma,\frac{i + j'}{2\pi} + \frac{1}{2}} \ast G_{\sigma_q^2}(q) \right) \left( \tilde{E}_{\frac{1}{4\sigma_q^2} - \sigma_{p'}^2,\Gamma,\frac{i + j'}{2\pi} + \frac{1}{2}} \ast G_{\sigma_{p'}^2}(p) \right) \right].
\]

(48)

The calculation for deriving the Wigner representation is similar to that for deriving the position and momentum representations, but is more complicated. The proof of Proposition 2 is in Appendix C.

The Wigner representation in Proposition 2 shows the physical meanings of \(\sigma_q^2\), \(\sigma_p^2\), and \(\Gamma\). The first term in the square bracket of (48) with \(j = j'\) denotes an infinite sequence of the Gaussian spikes each of which has variance \(\sigma_q^2\) in position coordinate and \(\sigma_p^2\) in momentum coordinate with the periods \(\Gamma\) and \(\pi(1 - 4\sigma_q^2\sigma_{p}^2)/\Gamma\), respectively, and has overall Gaussian envelopes with the variances \(\frac{1}{4\sigma_p^2} - \sigma_q^2\) and \(\frac{1}{4\sigma_q^2} - \sigma_{p'}^2\), respectively. The second term shows that the same structure is also at the places shifted by half periods in position coordinate, but with positive and negative signs alternately in the direction of momentum coordinate. The Gaussian spikes in the first and second terms with different signs interfere destructively when projected onto position or momentum coordinate, while constructively with the same signs. Since \(E_{\mu,\Gamma,\alpha}(x) \rightarrow \sum_{s \in \mathbb{Z}} \delta(x - (s + a)\Gamma)\) and \(\tilde{E}_{\mu,\Gamma,\alpha}(x) \rightarrow \sum_{s \in \mathbb{Z}} e^{2\pi i a s} \delta(x + s\Gamma)\) as \(\mu \rightarrow \infty\), and \(G_{\nu}(x) \rightarrow \delta(x)\) as \(\nu \rightarrow 0\), we can observe that (48) with \(\Gamma = \alpha dt\) approaches (10) as \(\sigma_q^2,\sigma_{p}^2 \rightarrow 0\), as expected. Note that the similar expression has already introduced in the literature [8] with a more intuitive explanation. Our contribution here is to derive the Wigner function corresponding to the approximate code states explicitly, which we will use the detailed analysis of average photon numbers.

B. Normalization constants and inner products of the approximate code states

Using the Wigner representation (48), we can provide an closed-form expression for \(N_{\sigma_q^2,\sigma_{p}^2,\Gamma}\), while the normalization constants were calculated numerically in previous works [8, 22, 42, 47]. Furthermore, since logical basis states of the approximate GKP codes are non-orthogonal, their inner products are non-zero in general, which we quantitatively analyze in the following.
Proposition 3 (Normalization constant and inner product). The normalization factor \(N_{\sigma_1^q, \sigma_2^p, \Gamma, j}\) of the approximate code state \(|j_{\sigma_1^q, \sigma_2^p, \Gamma, j}\rangle\) in Definition 3 is given in terms of the theta functions as follows:

\[
N_{\sigma_1^q, \sigma_2^p, \Gamma, j} = \vartheta \left[ \frac{j}{2} \right]_0 \left( 0, \frac{2i \Gamma^2 \sigma_2^p}{\pi(1 - 4 \sigma_1^q \sigma_2^p)} \right) \vartheta \left[ 0 \right]_0 \left( 0, \frac{2 \pi i \sigma_1^q (1 - 4 \sigma_1^q \sigma_2^p)}{\Gamma^2} \right) + \vartheta \left[ \frac{j}{2} + \frac{1}{2} \right]_0 \left( 0, \frac{2i \Gamma^2 \sigma_2^p}{\pi(1 - 4 \sigma_1^q \sigma_2^p)} \right) \vartheta \left[ \frac{j}{2} + \frac{1}{2} \right] \left( 0, \frac{2 \pi i \sigma_1^q (1 - 4 \sigma_1^q \sigma_2^p)}{\Gamma^2} \right). \tag{49}
\]

Furthermore, the inner product between \(|j_{\sigma_1^q, \sigma_2^p, \Gamma, j}\rangle\) and the approximate code state \(|j'_{\sigma_1^q, \sigma_2^p, \Gamma, j'}\rangle\) is given as follows:

\[
\langle j_{\sigma_1^q, \sigma_2^p, \Gamma, j} | j'_{\sigma_1^q, \sigma_2^p, \Gamma, j'} \rangle = \frac{1}{\sqrt{N_{\sigma_1^q, \sigma_2^p, \Gamma, j} N_{\sigma_1^q, \sigma_2^p, \Gamma, j'}}} \left\{ \vartheta \left[ \frac{j' + j}{2} \right]_0 \left( 0, \frac{2i \Gamma^2 \sigma_2^p}{\pi(1 - 4 \sigma_1^q \sigma_2^p)} \right) \vartheta \left[ 0 \right]_0 \left( 0, \frac{2 \pi i \sigma_1^q (1 - 4 \sigma_1^q \sigma_2^p)}{\Gamma^2} \right) + \vartheta \left[ \frac{j' + j}{2} + \frac{1}{2} \right]_0 \left( 0, \frac{2i \Gamma^2 \sigma_2^p}{\pi(1 - 4 \sigma_1^q \sigma_2^p)} \right) \vartheta \left[ \frac{j' + j}{2} + \frac{1}{2} \right] \left( 0, \frac{2 \pi i \sigma_1^q (1 - 4 \sigma_1^q \sigma_2^p)}{\Gamma^2} \right) \right\}. \tag{50}
\]

Since the theta function can be calculated with an arbitrary precision by a method in e.g. [46], the above results can be used for evaluating the code performance reliably.

Proof. We exploit the following facts:

\[
\langle j' | j \rangle = \text{Tr} \left[ | j \rangle \langle j' | \right] = \int dq dp \ W_{|j \rangle \langle j'|}(q, p), \tag{51}
\]

\[
\int dx \ f \ast g(x) = \int dx \ f(x) \int dx \ g(x), \tag{52}
\]

\[
\int dx \ E_{\mu, \Gamma, \alpha}(x) = \sum_{s \in \mathbb{Z}} \exp \left[ -(s + a)^2 \Gamma^2 / 2\mu \right] = \vartheta \left[ a \right]_0 \left( 0, \frac{i \Gamma^2}{2\pi \mu} \right), \tag{53}
\]

\[
\int dx \ \tilde{E}_{\mu', \Gamma', \alpha'}(x) = \sum_{s \in \mathbb{Z}} \exp \left[ -\Gamma'^2 s^2 / 2\mu' + 2\pi i a' s \right] = \vartheta \left[ a' \right]_0 \left( 0, \frac{i \Gamma'^2}{2\pi \mu'} \right), \tag{54}
\]

\[
\int dx \ G_{\nu}(x) = 1. \tag{55}
\]

Combining the above with the Wigner representation of \(W_{|j_{\sigma_1^q, \sigma_2^p, \Gamma, j}\rangle \langle j'_{\sigma_1^q, \sigma_2^p, \Gamma, j'}|}\) in (48), we obtain (49) and (50).

We investigate the asymptotic behavior of the inner product between two approximate code states. As shown in [48], the asymptotic behavior of the theta function in the form of \(\vartheta \left[ a \right]_0 (0, it)\) as \(t \to +0\) is given by

\[
\vartheta \left[ a \right]_0 (0, it) = \sum_{s=0}^{\infty} e^{-\pi t(s+a)^2} + \sum_{s=0}^{\infty} e^{-\pi t(s+1-a)^2} = \frac{1}{\sqrt{t}} + O \left( t^{\frac{1}{2}} \right). \tag{56}
\]
FIG. 2. The logarithms of the absolute values of the inner products $-\ln \left( |\langle 0_\sigma | 1_\sigma \rangle| \right)$ for the code state (41) in Definition 3 with $d = 2, 3,$ and 6. The horizontal axis, $-10 \log_{10}(2\sigma^2)$, is a squeezing level in decibels, which is a convention to express the degree of the squeezing. The vertical axis is in the log scale. One can observe that, in the region where the squeezing level is over 5 dB for $d = 3$ and 6, the minus of the logarithm of the inner products increases linearly with respect to the squeezing level in the log plot, that is, $-\ln |\langle 0_\sigma | 1_\sigma \rangle| \propto 1/\sigma^2$, as expected in the asymptotic behavior (59).

Furthermore, the asymptotic behavior of $\vartheta^{[0]}[a](0, it)$ as $t \to +0$ is given by

$$\vartheta^{[0]}[a](0, it) = \frac{1}{\sqrt{t}} \vartheta^{[a]}[0](0, i \frac{t}{\sqrt{t}}) = \frac{1}{\sqrt{t}} \sum_{s=-\infty}^{\infty} e^{-\frac{\pi}{8} (s+a)^2} \begin{cases} \frac{1}{\sqrt{t}} e^{-\frac{\pi}{8} a^2} & (|a| \ll \frac{1}{2}) \\ \frac{2}{\sqrt{t}} e^{-\frac{\pi}{8} a^2} & (|a| \simeq \frac{1}{2}) \end{cases}, \quad (57)$$

where we use (B9) in Appendix B in the first equality. Now we derive the asymptotic form of the normalization constant $N_{\sigma^2_q, \sigma^2_p, \Gamma, j}$ in (49) as $\sigma^2_q, \sigma^2_p \to +0$:

$$N_{\sigma^2_q, \sigma^2_p, \Gamma, j} \to \sqrt{\frac{\pi (1 - 4\sigma^2_q \sigma^2_p)}{2\Gamma^2 \sigma^2_p}} \frac{\Gamma^2}{2\pi \sigma^2_q (1 - 4\sigma^2_q \sigma^2_p)^2} \left[ 1 + 2 \exp \left( -\frac{\Gamma^2}{8\sigma^2_q (1 - 4\sigma^2_q \sigma^2_p)} \right) \right] \simeq \frac{1}{\sqrt{4\sigma^2_q \sigma^2_p}}. \quad (58)$$

In the same way, the asymptotic behavior of $\langle j'_{\sigma^2_q, \sigma^2_p, \Gamma} | j'_{\sigma^2_q, \sigma^2_p, \Gamma} \rangle$ for $\frac{j'-j}{2d} \ll \frac{1}{2}$ in (50) as $\sigma^2_q, \sigma^2_p \to +0$ is given by

$$\langle j'_{\sigma^2_q, \sigma^2_p, \Gamma} | j'_{\sigma^2_q, \sigma^2_p, \Gamma} \rangle \to \frac{1}{2\sigma_q \sigma_p} \exp \left( -\frac{(j'-j)^2 \Gamma^2}{8d^2 \sigma^2_q (1 - 4\sigma^2_q \sigma^2_p)} \right) + 2 \exp \left( -\frac{(d+j'-j)^2 \Gamma^2}{8d^2 \sigma^2_q (1 - 4\sigma^2_q \sigma^2_p)} \right) \frac{1}{2\sigma_q \sigma_p}$$

$$\simeq \exp \left( -\frac{(j'-j)^2 \Gamma^2}{8d^2 \sigma^2_q} \right). \quad (59)$$

The overlap between logical basis states thus decreases exponentially with respect to $1/\sigma^2_q$.

Using Proposition 3, we plot the inner products of the approximate code states against the degree of the approximation. Figure 2 shows the logarithms of the absolute values of the inner

[Diagram]

\[ -\ln |\langle 0_\sigma | 1_\sigma \rangle| \]

Squeezing level $-10 \log_{10}(2\sigma^2)$ [dB]

| d = 2 | d = 3 | d = 6 |
|------|------|------|
| 0.5  | 1    | 5    |
| 10   | 10   | 10   |

Using Proposition 3, we plot the inner products of the approximate code states against the degree of the approximation. Figure 2 shows the logarithms of the absolute values of the inner.
products $|\langle 0_{\sigma z}|1_{\sigma z} \rangle|$ of the approximate code states (41) in Definition 3 with $d = 2, 3,$ and $6,$ with respect to a squeezing level in decibels $-10 \log_{10}(2\sigma^2)$. One can observe that, in the region where the squeezing level is over 5 dB for $d = 3$ and 6, the minus of the logarithm of the inner products increases linearly with respect to the squeezing level in the log plot, that is, $-\ln |\langle 0_{\sigma z}|1_{\sigma z} \rangle| \propto 1/\sigma^2$, as expected in the asymptotic behavior [59]. In the case of $d = 2$, the inclination of the plot is larger than those in the case of $d = 3$ and 6, which may be caused by a constant factor in [59] when $\frac{\omega_j}{2\delta} < \frac{1}{2}$. Note that the squeezing levels of the code states when $d = 2$ in the recent experiments are 5.5-7.3 dB with the position and momentum degrees of freedom in trapped ion system [49], and 7.4-9.5 dB with the cavity mode of the superconducting system [50]. The required squeezing level for the fault-tolerant threshold of the universal quantum computation is considered to be 8-16 dB [25, 26, 29, 51], depending on experimental setups and noise models.

C. Average photon number

Using the Wigner representation (48) of the approximate code state $|j_{\sigma q^2,\sigma p^2,\Gamma_j}\rangle$, we can calculate the average photon number of the code state. In the following, we write $\langle \hat{n} \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}} := \langle j_{\sigma q^2,\sigma p^2,\Gamma_j} | \hat{A} | j_{\sigma q^2,\sigma p^2,\Gamma_j} \rangle$ for an operator $\hat{A}$.

**Proposition 4** (Average photon number). The average photon number $\langle \hat{n} \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}}$ of the approximate code state $|j_{\sigma q^2,\sigma p^2,\Gamma_j}\rangle$ in Definition 3 is given as follows:

$$\langle \hat{n} \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}} = \frac{\sigma_q^2 + \sigma_p^2 - 1}{2} - \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \ln \tilde{N}_{\sigma_q^2,\sigma_p^2,\Gamma_j}(x, y) \bigg|_{x = \frac{-2\sigma_q^2}{1 - 4\sigma_q^2\sigma_p^2}, y = \frac{-2\sigma_q^2}{1 - 4\sigma_q^2\sigma_p^2}}, \quad (60)$$

where $\ln \tilde{N}_{\sigma_q^2,\sigma_p^2,\Gamma_j}(x, y)$ is defined as

$$\ln \tilde{N}_{\sigma_q^2,\sigma_p^2,\Gamma_j}(x, y) := \theta \left[ \frac{1}{2} \right] \left( 0, \frac{i\Gamma^2 x}{\pi} \right) \theta \left[ 0 \right] \left( 0, \frac{\pi i(1 - 4\sigma_q^2\sigma_p^2)^2 y}{\Gamma^2} \right) + \theta \left[ \frac{1}{2} + \frac{1}{2} \right] \left( 0, \frac{i\Gamma^2 x}{\pi} \right) \theta \left[ 0 \right] \left( 0, \frac{\pi i(1 - 4\sigma_q^2\sigma_p^2)^2 y}{\Gamma^2} \right), \quad (61)$$

**Sketch of proof.** Using Wigner representation (48), we can derive the expectation values of the square of the position and momentum quadrature, $\langle \hat{q}^2 \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}}$ and $\langle \hat{p}^2 \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}}$. Then we can derive $\langle \hat{n} \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}}$ by exploiting the fact that $\langle \hat{q}^2 + \hat{p}^2 \rangle_{j_{\sigma q^2,\sigma p^2,\Gamma_j}} = (2\hat{n} + 1)_{j_{\sigma q^2,\sigma p^2,\Gamma_j}}$. The full proof is in Appendix D.
D. The relation between squeezing levels and average photon numbers

As an application of the results, we observe the relation between squeezing levels and average photon numbers of approximate code states. The “squeezing level” of the GKP code state is a quality measure of an approximate code state. It has a direct connection to the performance of the quantum error correction using GKP codes \[8, 24–26, 29, 51\]. On the other hand, the average photon number of the encoded state is relevant to the capacity of the CV quantum channel \[32, 52, 53\], which works as an effective dimension of the Hilbert space. Since it is found that the GKP code has high performance in the channel coding for Bosonic Gaussian channels \[22, 32\], the connections between these two notions are important for further analyses of the Gaussian channel coding.

“Squeezing level” of the (symmetric) GKP code state was first considered in \[8\] in order to characterize the variance \(\sigma^2\) of each convoluted Gaussian spike \(G_{\sigma^2}\) in the Wigner representation of the approximate code state, which directly affects the performance of the error correction with approximate GKP codes. On the analogy of the fact that squeezing level of a squeezed state is the logarithm of the ratio of the variances of the position quadrature \((\Delta q)^2\) of that state and the vacuum state, Ref. \[8\] defines the squeezing level of the symmetric GKP code state by 
\[-10 \log_{10}(2\sigma^2)\] for the variance \(\sigma^2\). In the case of asymmetric code state, there are two parameters 
\[-10 \log_{10}(2\sigma_q^2)\] and 
\[-10 \log_{10}(2\sigma_p^2)\], where \(\sigma_q^2\) and \(\sigma_p^2\) denote the variance of the Gaussian spike in position and momentum coordinates, respectively, in the Wigner representation of the standard form \(48\). Since the variance of the Gaussian spike of the Wigner representation of the code state in Approximation 1 is given by 
\(\approx \kappa^2\) when \(\kappa = \Delta\) and \(\kappa^2 \Delta^2 \ll 1\) as shown in \(48\), the “squeezing level” is often identified with 
\(-10 \log_{10}(\Delta^2) \approx -10 \log_{10}(\kappa^2)\) in \(11\). \[24, 29, 50, 51\]. Note that there also exists another definition of “effective squeezing parameter”, motivated by quantum metrology \(38, 43, 49\).

In this paper, we adopt the former definition as a “squeezing level” in order to observe the relation between the performance of the error correction and the average photon number of the approximate code states.

Previous literature estimates the average photon number of the encoded state as 
\(\approx \frac{1}{4\sigma^2} - \frac{1}{2}\) for the symmetric code for given squeezing level 
\(-10 \log_{10}(2\sigma^2) \gg 1\) \(7, 8, 22, 32, 36, 42\). This is because the variance of the envelope Gaussian in the Wigner representation of the approximate code states is roughly equal to \(\frac{1}{4\sigma^2}\), and the average photon number relates to the expectation values of the squares of the position and momentum quadratures by 
\(\langle q^2 + p^2 \rangle_{|j_{s2}\rangle} = \langle 2\hat{n} + 1 \rangle_{|j_{s2}\rangle}\).

It is also consistent with the expression of the average photon number given in \(60\) when the
FIG. 3. The average photon number of the code state $|j_{\sigma^2}\rangle$ in Definition 3 with $d = 2$. “Estimate” denotes a function $\frac{1}{4\sigma^2} - \frac{1}{2}$. These three are in good accordance when the squeezing level is over 10 dB, while they are not otherwise. Note again that the squeezing levels of the code states in the recent experiments are 5.5-7.3 dB in the trapped ion system [49], and 7.4-9.5 dB in the superconducting system [50].

We compute the average photon number of the code state $|j_{\sigma^2}\rangle$ defined in (41) in Definition 3 with $d = 2$, by using the formula (60). As mentioned above, the squeezing level of $|j_{\sigma^2}\rangle$ is given by $-10 \log_{10}(2\sigma^2)$. Figure 3 shows the average photon number of $|0_{\sigma^2}\rangle$ and $|1_{\sigma^2}\rangle$ with respect to the squeezing level $-10 \log_{10}(2\sigma^2)$. In Figure 3 we compare our result with a conventionally used estimate of the average photon number $\frac{1}{4\sigma^2} - \frac{1}{2}$. The figure reveals that, when the squeezing level is less than 10 dB, the conventionally used estimate of the average photon number deviates from the exact values. Note that 10 dB squeezing is considered to be near the threshold for fault-tolerant CV quantum computation [25, 26, 29, 51], which is a curious coincidence.
VI. CONCLUSION

In this paper, we explicitly showed conditions under which the conventional approximations of the GKP code, Approximations 1, 2, and 3, defined in (11), (12), and (13), are made equivalent. We observed that up to a slight squeezing for Approximation 1, Approximation 1, 2, and 3 are equivalent for the symmetric code, in which the logical basis states and their Fourier transforms span the same code space. Furthermore, we quantitatively showed that in all these approximations, the lattice spacing of the Gaussian spikes in the phase space appearing in the description of the approximate code states is narrower than that of the corresponding ideal GKP code state. Although this effect may be negligible in the limit of the large squeezing levels, it potentially affects the performance of the error correction since the error correction strategy explicitly depends on the lattice spacing of the code states. Quantitatively, in the case of approximate code state of $d = 2$ with 8 dB squeezing, the lattice spacing is about 1% narrower than that of the ideal one. It is thus needed to investigate error correction schemes taking the change in lattice spacing into account especially at a moderate squeezing level relevant to experimental realizations of GKP codes.

Exploiting the equivalence, we also gave the standard form of the approximate code states in terms of the position representation. Furthermore, we derived the explicit formulae of the Wigner functions, the normalization constants, inner products, and the average photon numbers of the logical basis states. We hope that these tools given in the present paper accelerate further theoretical developments of CV quantum information processing based on quantum error correction and channel coding with the GKP error correcting code.

ACKNOWLEDGMENTS

The authors thank K. Fukui, K. Maeda, Y. Kuramochi, and T. Sasaki for the helpful discussion. This work was supported by CREST (Japan Science and Technology Agency) JPMJCR1671 and Cross-ministerial Strategic Innovation Promotion Program (SIP) (Council for Science, Technology and Innovation (CSTI)).

Appendix A: Grid representation

Grid representation was considered in [9, 38, 41, 43], and we here review its basic notion. Let $(u, v) \in [0, 1) \times [0, 1)$, and $\hat{V}(u, v) := \hat{V}(2\pi v/\alpha_d, \alpha_d du)$. Then, $e^{-\pi i a} \hat{V}$ forms a Heisenberg
group $e^{-\pi it}\hat{\psi}(u,v)\cdot e^{-\pi i t'}\hat{\psi}(u',v') = e^{-\pi i(t+t'+uv' - uv)}\hat{\psi}(u+u',v+v')$. Define $|u,v\rangle_{\text{grid}}$ as

$$|u,v\rangle_{\text{grid}} := \hat{\psi}(u,v)|0^{(\text{ideal})}\rangle = e^{-\pi iuv}\hat{Z}(2\pi v/\alpha_d d)\hat{X}(\alpha_d du)|0^{(\text{ideal})}\rangle. \quad (A1)$$

In \cite{38 42 43}, $|u,v\rangle_{\text{grid}}$ with $d = 1$ is called the “shifted grid state”. The generalized “shifted grid state” $|u,v\rangle_{\text{grid}}$ with arbitrary $d$ satisfies an orthogonality and completeness relation in the following sense \cite{38 43}:

$$\langle u,v|v',u'\rangle_{\text{grid}} = \delta(u - u')\delta(v - v'),$$

$$\int_0^1 du \int_0^1 dv |u,v\rangle_{\text{grid}} \langle u,v| = \hat{I}. \quad (A2)$$

The “wave function” $\phi_f(u,v)$ of a state $|f\rangle$ with respect to the “shifted grid states”, i.e., the grid representation of $|f\rangle$, is defined as $\phi_f(u,v) := |u,v\rangle_{\text{grid}} \langle f|$, which satisfies

$$\int_0^1 du \int_0^1 dv \left| \phi_f(u,v) \right|^2 = 1. \quad (A2')$$

The “wave function” of the ideal GKP logical basis state $|j^{(\text{ideal})}\rangle$ can be regarded as a Dirac delta function centered at $(j/d,0)$, which does not satisfy (A2) and therefore, cannot be regarded as a physical state. However, the functions satisfying (A2) and localized at $(j/d,0)$ are well-defined approximate logical basis states.

Given the position representation $\psi_f(q) := q \langle q|f\rangle$ of a (pure) state $|f\rangle$, its grid representation $\phi_f(u,v)$ can be obtained as follows:

$$\phi_f(u,v) := |u,v\rangle_{\text{grid}} \langle f| = \int dq \langle u,v|q\rangle \hat{\psi}(q) \langle f| = \sqrt{\alpha_d d} \sum_{s \in \mathbb{Z}} e^{-2\pi i(s + \frac{u}{d})} \psi_f \left( \alpha_d d(u + s) \right). \quad (A3)$$

Using the last equality, we can expand the domain $[0,1) \times [0,1)$ of the “wave function” of the grid representation $\phi$ to $\mathbb{R}^2$. This redefined “wave function” $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfies (A2) and the following:

$$\phi(u + n_1, v + n_2) = e^{-\pi i(n_1 n_2 + un_2 - vn_1)} \phi(u,v), \quad \forall (n_1, n_2)^\top \in \mathbb{Z}^2, \quad (A4)$$

which can be confirmed from (A3). The functions $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ which satisfy (A2) and (A4) form a representation space of the Heisenberg group called $L^2(\mathbb{R}^2 \parallel \mathbb{Z}^2)$ \cite{54}, where the action of the group element $\text{Op}(\cdot)$ on $\phi$ is given by

$$\text{Op}(e^{-\pi it}\hat{\psi}(u,v))\phi_f(x,y) := |x,y\rangle_{\text{grid}} \langle f| e^{-\pi i t}\hat{\psi}(u,v) \langle f| = e^{-\pi i(t+xy-yu)}\phi_f(x-u, y-v). \quad (A5)$$
The formulation can easily be generalized to the $g$-mode case by considering the representation space $L^2(\mathbb{R}^2g / \mathbb{Z}^2g)$ [54].

Appendix B: The proof of Proposition [1] and Lemma [1]

First, we derive (16), (18), and (20) in Proposition [1]. In the main text, $\alpha$ is fixed to $\alpha_d$ for $|j^{(\text{ideal})}\rangle$ and all the approximations, but here, for later use, we perform the calculation for a general $\alpha$, that is, derive the position representation of $|j^{(1)}_{\kappa,\Delta,\alpha}\rangle$, $|j^{(2)}_{\gamma,\delta,\alpha}\rangle$, and $|j^{(3)}_{\beta,\alpha}\rangle$. We start with the derivation of (16). We have

\[
\hat{q} \langle q | j^{(1)}_{\kappa,\Delta,\alpha} \rangle = \frac{1}{\sqrt{N_{\kappa,\Delta,j}^{(1)}}} \sum_{s \in \mathbb{Z}} e^{-\frac{1}{4} \kappa^2 \alpha^2 (ds + j)^2} \hat{q} \langle q | \hat{X}(\alpha(ds + j)) \hat{S}( - \ln \Delta ) | 0 \rangle_f
\]

\[
= \sum_{s \in \mathbb{Z}} e^{-\frac{1}{4} \kappa^2 \alpha^2 (ds + j)^2} \frac{\sqrt{\Delta N_{\kappa,\Delta,j}^{(1)}}}{\sqrt{\Delta N_{\kappa,\Delta,j}^{(1)}}} \hat{q} \langle (q - \alpha(ds + j)) / \Delta | 0 \rangle_f
\]

\[
= \left( \sqrt{\pi \Delta^2} N_{\kappa,\Delta,j}^{(1)} \right)^{-\frac{1}{2}} \sum_{s \in \mathbb{Z}} \exp \left[ -\frac{\kappa^2 \alpha^2 d^2}{2} \left( s + \frac{j}{d} \right)^2 \right]
\]

\[
\times \exp \left[ -\frac{1}{2 \Delta^2} \left( q - \alpha d \left( s + \frac{j}{d} \right) \right)^2 \right]
\]

\[
= \left( \frac{2 \sqrt{\pi \Delta^2}}{\pi} \right)^{\frac{1}{2}} \left( \frac{\sqrt{\pi} \Delta}{N_{\kappa,\Delta,j}^{(1)}} \right)^{\frac{1}{2}} E_{\frac{1}{\pi \Delta^2}, \alpha d, \frac{1}{d}}^\ast \ast G_{\Delta^2}(q),
\]

where we used $X(a) | q \rangle_{\hat{q}} = | q + a \rangle_{\hat{q}}$ and $S(r) | q \rangle_{\hat{q}} = e^{-r^2/2} | e^{-r} q \rangle_{\hat{q}}$ in the second equality, and $\hat{q} \langle q | 0 \rangle_f = \pi^{-\frac{1}{4}} \exp(-q^2/2)$ in the third equality. Substituting $\alpha$ with $\alpha_d$ in (B2), we obtain (16).

The derivation of (18) goes in the similar way. We have

\[
\hat{q} \langle q | j^{(2)}_{\gamma,\delta,\alpha} \rangle = \frac{1}{\sqrt{N_{\gamma,\delta,j}^{(2)}}} \int d_1 \int d_2 e^{-\frac{r_1^2}{2\gamma^2} - \frac{r_2^2}{2\delta^2}} \langle q | \hat{V}(r) | j^{(\text{ideal})} \rangle
\]

\[
= \frac{1}{\sqrt{N_{\gamma,\delta,j}^{(2)}}} \int d_1 \int d_2 \exp \left( -\frac{r_1^2}{2\gamma^2} - \frac{r_2^2}{2\delta^2} - \frac{i r_1 r_2}{2} + i r_1 q \right) \hat{q} \langle q - r_2 | j^{(\text{ideal})} \rangle,
\]

where we used $\hat{V}(r) := \exp(-ir_{p}r_{q}/2) \hat{Z}(r_{p}) \hat{X}(r_{q})$, $\hat{Z}(r_{p}) | q \rangle_{\hat{q}} = e^{ir_{p}q}$, and $\hat{X}(r_{q}) | q \rangle_{\hat{q}} = | q + r_{q} \rangle$. 

Using \( \hat{q} \langle q - r_2 | j^{(\text{ideal})} \rangle = \sum_{s \in \mathbb{Z}} \delta (\alpha ds + j) - q + r_2 \), we have

\[
\text{(B3)} = \left( \frac{\alpha d}{N_{\gamma, \delta, j}^{(2)}} \right)^{\frac{1}{2}} \int \frac{dr_1 dr_2}{2\pi \gamma \delta} \exp \left( -\frac{r_1^2}{2\gamma^2} - \frac{r_2^2}{2\delta^2} + \frac{ir_1(2q - r_2)}{2} \right) \sum_{s \in \mathbb{Z}} \delta (r_2 - q + \alpha(ds + j))
\]

\[
= \frac{\sqrt{\alpha d}}{2\pi \gamma \delta N_{\gamma, \delta, j}^{(2)}} \sum_{s \in \mathbb{Z}} \int dr_1 \exp \left\{ -\frac{1}{2\gamma^2} \left[ r_1 - \frac{ir_1^2}{2}(q + \alpha(ds + j)) \right]^2 \right\}
\]

\[
\times \exp \left[ -\frac{\gamma^2}{8} (q + \alpha(ds + j))^2 - \frac{1}{2\delta^2} (q - \alpha(ds + j))^2 \right]
\]

\[
= \left( \frac{\alpha d}{2\pi \delta^2 N_{\gamma, \delta, j}^{(2)}} \right)^{\frac{1}{2}} \sum_{s \in \mathbb{Z}} \exp \left\{ -\frac{\lambda(\gamma, \delta) q^2}{2\delta^2} + \alpha d q \left( \frac{\lambda(\gamma, \delta) - \gamma^2 \delta^2}{2} \right) (s + j) \right\}
\]

\[
\times \exp \left[ -\frac{\alpha^2 d^2 \lambda(\gamma, \delta)}{2\delta^2} \left( s + \frac{j}{d} \right)^2 \right]
\]

\[
= \left( \frac{\alpha d}{\lambda(\gamma, \delta) N_{\gamma, \delta, j}^{(2)}} \right)^{\frac{1}{2}} E_{\frac{\lambda(\gamma, \delta)}{2\delta^2} (1 - \frac{\gamma^2 \delta^2}{2\lambda(\gamma, \delta)})^2, \alpha d (1 - \frac{\gamma^2 \delta^2}{2\lambda(\gamma, \delta)})} \psi_{n}(q),
\]

where we used Gaussian integral in the third equality, and used

\[
1 - \left( 1 - \frac{\gamma^2 \delta^2}{2\lambda(\gamma, \delta)} \right)^2 = \left( \frac{\lambda(\gamma, \delta)}{\lambda(\gamma, \delta)} \right)^2
\]

in the last equality. Substituting \( \alpha \) with \( \alpha_d \) in (B4) leads to (18).

The derivation of (20) needs a trick. We have

\[
\hat{q} \langle q | j^{(3)} \rangle = \frac{1}{\sqrt{N^{(3)}_{\beta, j}}} \hat{q} \langle q | \sum_{n \in \mathbb{N}} | n \rangle \langle n | e^{-\beta(n + \frac{1}{2})} | j^{(\text{ideal})} \rangle
\]

\[
= \left( \frac{\alpha d}{N^{(3)}_{\beta, j}} \right)^{\frac{1}{2}} \sum_{s \in \mathbb{Z}} \sum_{n \in \mathbb{N}} e^{-\beta(n + \frac{1}{2})} \psi_n(q) \psi_n^*(\alpha(ds + j)),
\]

where \( \psi_n(x) := (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) \) denotes the wave function of the fock state. Using the Mehler’s hermite polynomial formula [55]

\[
\sum_{n \in \mathbb{N}} \frac{(u/2)^n}{n!} H_n(x) H_n(y) \exp \left( -\frac{x^2 + y^2}{2} \right) = \frac{1}{\sqrt{1 - u^2}} \exp \left[ -\frac{1 + u^2(x^2 + y^2) - 4uxy}{2(1 - u^2)} \right],
\]

(B5)
we obtain

\[ \hat{q} \langle q | j_{\beta}^{(3)} \rangle = \left( \frac{e^{-\beta \alpha d}}{\pi(1 - e^{-2\beta}) N_{\beta,j}^{(3)}} \right)^{\frac{1}{2}} \sum_{s \in \mathbb{Z}} \exp \left[ -\frac{(1 + e^{-2\beta}) (q^2 + \alpha^2 (ds + j)^2)}{2(1 - e^{-2\beta})} - 4e^{-\beta \alpha (ds + j)q} \right] \]

\[ = \left( \frac{\alpha d}{2\pi \sinh \beta N_{\beta,j}^{(3)}} \right)^{\frac{1}{2}} \sum_{s \in \mathbb{Z}} \exp \left[ -\frac{\alpha^2 d^2}{2 \tanh \beta} \left( s + \frac{j}{d} \right)^2 + \frac{\alpha dq}{\sinh \beta} \left( s + \frac{j}{d} \right) - \frac{q^2}{2 \tanh \beta} \right] \]

\[ = \left( \frac{\alpha d}{2\pi \sinh \beta N_{\beta,j}^{(3)}} \right)^{\frac{1}{2}} \sum_{s \in \mathbb{Z}} \exp \left[ -\frac{1}{2 \tanh \beta} \left( q - \frac{\alpha d}{\cosh \beta} \left( s + \frac{j}{d} \right) \right)^2 \right] \times \exp \left[ -\frac{\alpha^2 d^2 \tanh \beta}{2} \left( s + \frac{j}{d} \right)^2 \right] \]

\[ = \left( \frac{\alpha d}{\cosh \beta N_{\beta,j}^{(3)}} \right)^{\frac{1}{2}} E_{\sinh \beta \cosh \beta} \frac{\alpha d}{\cosh \beta} G_{\sinh \beta}(q). \] (B6)

Substituting \( \alpha \) with \( \alpha_d \) in (B6) leads to (20).

Next, we prove Lemma 1 to derive (17), (19), and (21) from (16), (18), and (20), respectively.

From the definition of \( E_{\mu,\Gamma,a} \) in Definition 2 as well as the definition of \( G_{\nu} \) in (3), we have

\[ E_{\mu,\Gamma,a} \ast G_{\nu}(q) = \frac{1}{\sqrt{2\pi \nu}} \sum_{s \in \mathbb{Z}} \exp \left[ -\frac{(s + a)^2 \Gamma^2}{2\mu} - \frac{(q - (s + a)\Gamma)^2}{2\nu} \right] \]

\[ = \frac{1}{\sqrt{2\pi \nu}} \vartheta_{[0]}^{[a]} \left( \frac{\Gamma q}{2\pi \nu} \right) \exp \left( -\frac{q^2}{2\nu} \right). \] (B7)

The theta function has the following identity [39]

\[ \vartheta(z/\tau, -1/\tau) = (-i\tau)^{\frac{1}{2}} \exp(\pi iz^2/\tau) \vartheta(z, \tau), \] (B8)

which leads to

\[ \vartheta_{[a]}^{[0]}(z/\tau, -1/\tau) = (-i\tau)^{\frac{1}{2}} \exp(\pi iz^2/\tau) \vartheta_{[a]}^{[0]}(z, \tau). \] (B9)

Applying this to (B7), we have

\[ \sqrt{2\pi \nu} G_{\mu+\nu}(q) \vartheta_{[a]}^{[0]} \left( \frac{q}{(1 + \nu/\mu)\Gamma}, \frac{2\pi \nu}{(1 + \nu/\mu)\Gamma^2} \right) \exp \left( -\frac{q^2}{2\nu} + \frac{q^2}{2\nu(1 + \nu/\mu)} \right) \]

\[ \sqrt{2\pi \nu} G_{\mu+\nu}(q) \vartheta_{[a]}^{[0]} \left( \frac{q}{(1 + \nu/\mu)\Gamma}, \frac{2\pi \nu}{(1 + \nu/\mu)\Gamma^2} \right), \]

which proves Lemma 1. Then, as mentioned above, we obtain (17), (19), and (21) by applying Lemma 1 to (16), (18), and (20), respectively.
Appendix C: Proof of Proposition 2

We compute $W_{[j_{\sigma_q^p, \sigma_p^q, \Gamma}]}(j_{\sigma_q^p, \sigma_p^q, \Gamma}^{'}, q + x)$ as follows:

$$W_{[j_{\sigma_q^p, \sigma_p^q, \Gamma}]}(j_{\sigma_q^p, \sigma_p^q, \Gamma}^{'}, q + x) = \frac{1}{\pi} \int dx \ e^{2ipx} \langle q - x | j_{\sigma_q^p, \sigma_p^q, \Gamma} \rangle \langle j_{\sigma_q^p, \sigma_p^q, \Gamma}^{'}, q + x \rangle \times \int dx \ e^{2ipx} \sum_{s} \exp \left[ -\frac{(s + i \frac{j}{d})^2}{2} \Gamma^2 \sigma_p^2 \frac{1}{1 - 4\sigma_q^2 \sigma_p^2} \right] \langle q - x - \left(s + \frac{j}{d}\right) \Gamma \rangle^2 \right] \times \sum_{s'} \exp \left[ -\frac{(s' + i \frac{j'}{d})^2}{2} \Gamma^2 \sigma_p^2 \frac{1}{1 - 4\sigma_q^2 \sigma_p^2} \right] \langle q + x - \left(s' + \frac{j'}{d}\right) \Gamma \rangle^2 \right]

= \frac{(1 - 4\sigma_q^2 \sigma_p^2)^{-\frac{1}{2}} \Gamma}{2\pi \sqrt{2\pi \sigma_q^2 N_{\sigma_q^p, \sigma_p^q, \Gamma}, \sigma_q^p N_{\sigma_q^p, \sigma_p^q, \Gamma}} \times \left( \frac{1}{2} \frac{\sigma_p^2}{1 - 4\sigma_q^2 \sigma_p^2} + \frac{1}{2} \sigma_q^2 \right) \left( s + \frac{j}{d} + s' + \frac{j'}{d} \right)^2 + \left( s + \frac{j}{d} - s' - \frac{j'}{d} \right)^2 \right]

= \frac{(1 - 4\sigma_q^2 \sigma_p^2)^{-\frac{1}{2}} \Gamma}{2\pi \sqrt{2\pi \sigma_q^2 N_{\sigma_q^p, \sigma_p^q, \Gamma}, \sigma_q^p N_{\sigma_q^p, \sigma_p^q, \Gamma}} \times \left( \frac{1}{2} \frac{\sigma_p^2}{1 - 4\sigma_q^2 \sigma_p^2} + \frac{1}{2} \sigma_q^2 \right) \left( s + \frac{j}{d} + s' + \frac{j'}{d} \right)^2 + \left( s + \frac{j}{d} - s' - \frac{j'}{d} \right)^2 \right]

\text{where we used the standard form}\ [40] \text{in the second equality. At this stage, we will change the}
\text{variables for the summation from} s \text{ and} s' \text{ to} s + s' \text{ and} s - s'. \text{Since} s + s' \text{ and} s - s' \text{ has the same}
\text{parity, the summation splits into two parts: one with}\ s + s' = 2t, \ s - s' = 2t', (t, t' \in \mathbb{Z}) \text{and the}
other with $s + s' = 2t + 1$, $s - s' = 2t' + 1$. Thus, we have

\begin{equation}
\frac{(1 - 4\sigma_p^2\sigma_q^2)^{-\frac{1}{2}}\Gamma}{\pi \sqrt{2\pi\sigma_p^2\sigma_q^2}} \sum_{t,t'} \left( \exp \left\{ -\frac{1}{2\sigma_q^2} \left[ q - \Gamma \left( t + \frac{j + j'}{2d} \right) \right]^2 - 2\sigma_q^2p^2 - 2i\Gamma p \left( t' + \frac{j - j'}{2d} \right) \right\} \right.
\times \exp \left\{ -\frac{2\Gamma^2\sigma_p^2}{1 - 4\sigma_p^2\sigma_q^2} \left[ \left( t + \frac{j + j'}{2d} \right)^2 + \left( t' + \frac{j - j'}{2d} \right)^2 \right] \right\} \nonumber
+ \exp \left\{ -\frac{1}{2\sigma_q^2} \left[ q - \Gamma \left( t + \frac{j + j'}{2d} + \frac{1}{2} \right) \right]^2 - 2\sigma_q^2p^2 - 2i\Gamma p \left( t' + \frac{j - j'}{2d} + \frac{1}{2} \right) \right\} \nonumber
\times \exp \left\{ -\frac{2\Gamma^2\sigma_p^2}{1 - 4\sigma_p^2\sigma_q^2} \left[ \left( t + \frac{j + j'}{2d} + \frac{1}{2} \right)^2 + \left( t' + \frac{j - j'}{2d} + \frac{1}{2} \right)^2 \right] \right\} \nonumber
= \frac{(1 - 4\sigma_p^2\sigma_q^2)^{-\frac{1}{2}}\Gamma}{\pi \sqrt{N_{\sigma_q^2,\sigma_p^2,\Gamma,j}N_{\sigma_q^2,\sigma_p^2,\Gamma,j'}}} \left\{ \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} * G_{\sigma_q^2}(q) \right) e^{-\frac{\sigma^2}{2\sigma_q^2} \partial^2} \left[ \Gamma p, \frac{1}{\pi(1 - 4\sigma_q^2\sigma_p^2)} \right] \right. \nonumber
\left. + \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} + \frac{1}{2} * G_{\sigma_q^2}(q) \right) e^{-\frac{\sigma^2}{2\sigma_q^2} \partial^2} \left[ \Gamma p, \frac{1}{\pi(1 - 4\sigma_q^2\sigma_p^2)} \right] \right\} \nonumber
= \frac{1}{\sqrt{2\pi\sigma_p^2\sigma_q^2}} \sum_{t} \left\{ \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} * G_{\sigma_q^2}(q) \right) \exp \left( 2\pi it \frac{j - j'}{2d} \right) \right. \times \exp \left[ -\frac{1}{2\sigma_p^2} \left( p + \frac{\pi(1 - 4\sigma_q^2\sigma_p^2)t}{\Gamma} \right)^2 - \frac{2\pi^2(1 - 4\sigma_q^2\sigma_p^2)\sigma_q^2t^2}{\Gamma^2} \right] \nonumber
\left. + \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} + \frac{1}{2} * G_{\sigma_q^2}(q) \right) \exp \left[ 2\pi it \left( \frac{j - j'}{2d} + \frac{1}{2} \right) \right] \times \exp \left[ -\frac{1}{2\sigma_p^2} \left( p + \frac{\pi(1 - 4\sigma_q^2\sigma_p^2)t}{\Gamma} \right)^2 - \frac{2\pi^2(1 - 4\sigma_q^2\sigma_p^2)\sigma_q^2t^2}{\Gamma^2} \right] \right\} \nonumber
= \frac{1}{\sqrt{N_{\sigma_q^2,\sigma_p^2,\Gamma,j}N_{\sigma_q^2,\sigma_p^2,\Gamma,j'}}} \left\{ \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} * G_{\sigma_q^2}(q) \right) \left( E_{1/\sigma_p^2} - \sigma_q^2, \frac{(1 - 4\sigma_q^2\sigma_p^2)}{\Gamma \sigma_q^2}, \frac{j + j'}{2d} * G_{\sigma_q^2}(p) \right) \right. \nonumber
\left. + \left( E_{1/\sigma_p^2} - \sigma_q^2, \Gamma, \frac{j + j'}{2d} + \frac{1}{2} * G_{\sigma_q^2}(q) \right) \left( E_{1/\sigma_p^2} - \sigma_q^2, \frac{(1 - 4\sigma_q^2\sigma_p^2)}{\Gamma \sigma_q^2}, \frac{j + j'}{2d} + \frac{1}{2} * G_{\sigma_q^2}(p) \right) \right\},
\end{equation}
where we used \([B3]\) in the third equality.

**Appendix D: The proof of Proposition 4**

In order to derive the average photon number of the approximate code state \(|j_{\sigma_q^2, \sigma_p^2,r}\rangle\) in Definition 3, we first calculate the expectation values \(\langle q^2 \rangle_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}\) and \(\langle p^2 \rangle_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}\) of the square of the quadrature operators \(\hat{q}^2\) and \(\hat{p}^2\) with respect to \(|j_{\sigma_q^2, \sigma_p^2,r}\rangle\), using its Wigner representation \([48]\).

Then, one can obtain the average photon number \(\langle \hat{n} \rangle_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}\) of the state \(|j_{\sigma_q^2, \sigma_p^2,r}\rangle\) by exploiting the fact that \(\langle \hat{q}^2 + \hat{p}^2 \rangle_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle} = \langle 2\hat{n} + 1 \rangle_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}\). We frequently use \([52], [53], [54],\) and \([55]\) in the following calculation. Let \(\text{Pr}_q(q)\) and \(\text{Pr}_p(p)\) be the probability densities to obtain the values \(q\) and \(p\) in the \(q\)- and \(p\)-quadrature measurements, respectively. Then, they can be given by

\[
\text{Pr}_q(q) = \int dq \, W_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}(q,p) = \frac{1}{N_{\sigma_q^2, \sigma_p^2,\Gamma,j}} \left[ c_1 E_{\frac{1}{4\sigma_q^2} - \sigma_q^2, \Gamma, \frac{1}{2}} + c_2 E_{\frac{1}{4\sigma_q^2} - \sigma_q^2, \Gamma, \frac{1}{2}} \right] \ast G_{\sigma_q^2}(q), \tag{D1} \]

\[
\text{Pr}_p(p) = \int dp \, W_{|j_{\sigma_q^2, \sigma_p^2,r}\rangle}(q,p) = \frac{1}{N_{\sigma_q^2, \sigma_p^2,\Gamma,j}} \left[ c_3 \tilde{E}_{\frac{1}{4\sigma_q^2} - \sigma_p^2, \frac{1}{2} - (1 - 4\sigma_q^2 \sigma_p^2)} + c_4 \tilde{E}_{\frac{1}{4\sigma_q^2} - \sigma_p^2, \frac{1}{2} - (1 - 4\sigma_q^2 \sigma_p^2)} \right] \ast G_{\sigma_p^2}(p), \tag{D2} \]

where \(c_1, c_2, c_3, \) and \(c_4\) are defined as

\[
c_1 := |0\rangle_{|0\rangle} \left( 0, \frac{2\pi i \sigma_q^2 (1 - 4 \sigma_q^2 \sigma_p^2)}{\Gamma^2} \right)
\]

\[
c_2 := |0\rangle_{|0\rangle} \left( 0, \frac{2\pi i \sigma_q^2 (1 - 4 \sigma_q^2 \sigma_p^2)}{-\Gamma^2} \right)
\]

\[
c_3 := |\frac{1}{2}\rangle_{|0\rangle} \left( 0, \frac{2 i \Gamma^2 \sigma_p^2}{\pi (1 - 4 \sigma_q^2 \sigma_p^2)} \right)
\]

\[
c_4 := |\frac{1}{2} + \frac{1}{2}\rangle_{|0\rangle} \left( 0, \frac{2 i \Gamma^2 \sigma_p^2}{\pi (1 - 4 \sigma_q^2 \sigma_p^2)} \right)
\]

Note that the normalization constant \(N_{\sigma_q^2, \sigma_p^2,\Gamma,j}\) satisfies \(N_{\sigma_q^2, \sigma_p^2,\Gamma,j} = c_1 c_3 + c_2 c_4\) as shown in \([49]\).
Using \( \Pr_q(q) \), we calculate the expectation value of \( \bar{q}^2 \) as follows:

\[
\langle \bar{q}^2 \rangle_{\tilde{N}_q^2, \sigma_q^2, r} := \int dq \int dr \frac{q^2}{N_{\sigma_q^2, \sigma_q^2, r, j}} \left[ c_1 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} (r) + c_2 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} + \frac{1}{2} (r) \right] G_{\sigma_q^2}(q - r) \]

\[
= \int dq \int dr \frac{r^2}{N_{\sigma_q^2, \sigma_q^2, r, j}} \left[ c_1 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} (r) + c_2 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} + \frac{1}{2} (r) \right] G_{\sigma_q^2}(q - r) \]

\[
= \int dq' q^2 G_{\sigma_q^2}(q') + \int dr \frac{r^2}{N_{\sigma_q^2, \sigma_q^2, r, j}} \left[ c_1 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} (r) + c_2 E_{1, \frac{1}{4\sigma_q^2}} - \sigma_q^2, i, \frac{1}{4} + \frac{1}{2} (r) \right] \]

\[= \sigma_q^2 - \frac{1}{N_{\sigma_q^2, \sigma_q^2, r, j}} \frac{\partial}{\partial y} \left[ c_1 \theta \left[ \frac{y}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right) + c_2 \theta \left[ \frac{y + 1/2}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right) \right] \bigg|_{y = \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2}} \]

(D3)

where we used the fact that \( G_{\sigma_q^2}(x) \) has the zero mean in the fourth and the fifth equality. In the same way, for the expectation value of \( \bar{p}^2 \), we have

\[
\langle \bar{p}^2 \rangle_{\tilde{N}_q^2, \sigma_q^2, r} = \sigma_p^2 - \frac{1}{N_{\sigma_q^2, \sigma_q^2, r, j}} \frac{\partial}{\partial y} \left[ c_3 \theta \left[ \frac{y}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right) + c_4 \theta \left[ \frac{y + 1/2}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right) \right] \bigg|_{y = \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2}} .
\]

(D4)

Now we define \( \tilde{N}_{\sigma_q^2, \sigma_q^2, r, j}(x, y) \) as

\[
\tilde{N}_{\sigma_q^2, \sigma_q^2, r, j}(x, y) := \theta \left[ \frac{y}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right) + \theta \left[ \frac{y + 1/2}{\pi} \right] \left( 0, \frac{i \Gamma^2 x}{\pi} \right)
\]

where \( N_{\sigma_q^2, \sigma_q^2, r, j} = \tilde{N}_{\sigma_q^2, \sigma_q^2, r, j} \left( \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2}, \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2} \right) \). Then, the average photon number is given by

\[
\langle n \rangle_{\tilde{N}_q^2, \sigma_q^2, r} = \frac{\langle \bar{q}^2 + \bar{p}^2 \rangle_{\tilde{N}_q^2, \sigma_q^2, r} - 1}{2}
\]

\[
= \sigma_q^2 + \sigma_p^2 - 1 - \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \ln \tilde{N}_{\sigma_q^2, \sigma_q^2, r, j}(x, y) \bigg|_{x = \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2}, y = \frac{2\sigma_q^2}{1 - 4\sigma_q^2 \sigma_p^2}} .
\]

(D6)
which proves (60).
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