A chaotic 'turnstile' for atoms in periodic potentials

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Using a new type of chaotic ratchet generated by pulsed standing waves of light, we propose a mechanism which would allow packets of atoms travelling through a pulsed optical lattice in one direction to pass almost undisturbed, while strongly heating atoms drifting through in the opposite direction. An analytical formula for the diffusive energy growth is derived and shown to give good agreement with numerical calculations.

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There is much ongoing interest in ratchets- in other words the use of periodic (but spatially and/or temporally asymmetric) systems which can be used to generate a current when there is no net force: see [1] for a comprehensive review. Much of the work has focussed on Brownian ratchets [2] which play an important role in biophysical systems such as molecular motors. Corresponding quantum ratchets have also been investigated [3].

There has been comparatively little work on Hamiltonian ratchets, which are 'clean' ratchets without extrinsic noise or dissipation. Mixed phase-space mechanisms (involving tori/stable islands) for ratchet transport have been investigated [4, 5, 6]. In [4, 5] general properties of ratchets and a mechanism involving desymmetrization of ballistic flights was proposed. In [6] a sum rule connecting the current in regular islands with that in adjoining chaotic manifolds was derived.

Until recently it was not thought that a fully chaotic system could admit a ratchet effect. But in [7] a fully chaotic ratchet mechanism was proposed for atoms in a double-well periodic lattice, pulsed with unequal periods. The ratchet depends on the generic properties of the chaotic diffusion so the initial state requires no specific preparation: in all cases it was found that in general there is a classical timescale $t_r$, during which particles moving with negative momenta (relative to the initial value) absorb energy at a different rate from those moving in the opposite direction. In the corresponding quantum ratchet, for a significant effect to develop, it was found in [7] that it is simply necessary that the quantum break-time $t^*$ be of similar magnitude to $t_r$.

Atoms in pulsed optical lattices have become a paradigm in the study of quantum chaos; experiments on sodium and cesium atoms [8] have provided a convincing demonstration of dynamical localization (DL) [9, 10], the quantum suppression of chaotic diffusion. The dynamics for the usual experiment is given by the kicked-particle Hamiltonian: $H = \frac{p^2}{2} - K \cos x \sum \delta(t - nT)$ where $K$ is the kick strength. The classical dynamics is obtained by iterating the well-known 'Standard Map' $x_{i+1} = x_i + p_i T; p_{i+1} = p_i + K \sin x_{i+1}$.

We can take $T = 1$, without loss of generality, in the Standard Map, but for the ratchet, we use a repeating cycle of unequal kicks. The ratchet Hamiltonian is given by $H = \frac{p^2}{2} + V(x) \sum \delta(t - nT_j)$ where for the $n$-th cycle, the period of the $j$-th kick is $T_j$. For example, in [7] the kick spacings cycle between $T_1 = 1 + b, T_2 = 1, T_3 = 1 - b$ where $|b| < 1$.

In [7] an asymmetric potential of the form $V(x) = K(\sin x + a \sin(2x + \Phi))$ was investigated, but in [10] the spatial symmetry was broken by a 'rocking' linear term: $V(x) = -(K \cos x + A x)$. In [7] the kick spacing alternated between $T_1 = 1 + b$ and $T = 1 - b$.

The rocking ratchet [7] was analysed in the regime where regular tori are still present: it was found that although it no longer has the $2\pi$ periodicity in $p$ of the Standard Map, there is a long range periodicity in $p$ which is of order $2\pi/b$. Also, since pairs of corresponding tori are located asymmetrically about $p = 0$, the resulting confinement of classical trajectories can yield transport.

Here we investigate this system in the chaotic regime. One aim of this work is to demonstrate that the physics is similar to that found in [7] and hence that the new chaotic ratchet mechanism is quite generic: both ratchets depend on the ratchet timescale $t_r \sim \frac{1}{D}$ in the classical case, where $D$ is the diffusion rate, and the break-time $t^* \sim D/\hbar^2$ in the quantum case. The ratchet currents were found in [7] to originate predominantly from correlations of the form $\langle V'(x_i)V'(x_{i+j}) \rangle$. We show below that the resulting analytical formulae accurately predict phenomena such as current reversals, without any detailed consideration of the structure of phase-space.

Further, while [7] considered only the ratchet current which arises for a system with zero initial current, i.e. $p(t = 0) = 0$, here we present results for the case where there is an initial drift $p(t = 0) = p_0$. We show that this can form the basis of a device to manipulate traffic of cold atoms moving along some channel in a trap. This is not so far-fetched, since the trapping of atoms in devices such as atom chips is now a reality. The proposed device represents a sort of 'turnstile' which would selectively heat only atoms moving in one direction, while imparting little or no energy to atoms moving in the opposite direction, since we can tune the parameters of the pulsed lattice so as to generate correlations which almost cancel the energy diffusion in one direction, but enhance it in the other. Since the rocking ratchet produces much simpler analytical expressions than the double well ratchet, it provides a better didactic
example of the underlying physics.

For the Standard Map, at the lowest level of approximation, the momenta at consecutive kicks are uncorrelated and evolve in time as a 'random-walk’. The average momentum of a large ensemble of particles is unchanged. The average energy grows linearly: if the momenta are uncorrelated the average kinetic energy grows by \( K^2/4 \) at each consecutive kick. In the absence of phase-space barriers the energy is unbounded and this diffusive increase in energy continues indefinitely. It is characterized by a diffusion rate \( D_0 \), i.e. \( <p^2> = D_0 t \) so for uncorrelated momenta \( D_0 = K^2/2 \). However, this results neglects correlations between sequences of consecutive kicks; if included, they result in well-known corrections to the diffusion constant in the form of Bessel functions hence \( D_0 = \frac{K^2}{2}(1 - 2(J_1(K))^2 - 2J_2(K)) \ldots \) \[\text{[8]}\]. These corrections have even been measured experimentally with cold cesium atoms in pulsed optical lattices \[\text{[4]}\].

Here, the \( C(2) = -K^2 J_2(K) \) term is of particular interest. It corresponds to a two-kick correlation, given by evaluating the phase-space average of \( 2 < V'(x_{i-1})V'(x_{i+1}) \) hence:

\[
C(2) = 2K^2 < \sin x_{i-1} \sin x_{i+1} > . \quad (0.1)
\]

It is easy to show, by direct substitution from the map that one obtains separate integrals over the \( x \) and \( p \). For chaotic diffusion it is usual to neglect odd integrals, for example \( < \sin p \cos p > \simeq 0 \), while even integrals yield a positive average, eg \( < \sin^2 p > \simeq \cos^2 p > \simeq 1/2 \), since after a few kicks the ensemble has a substantial spread in \( p \). In effect the only terms which contribute to \( C(2) \) in the standard map are even in \( p \) and do not differentiate between \( \pm p \). The integral over \( x \) yields \( J_2(K) \), hence the form of \( C(2) \).

We consider now the map of \[\text{[8]}\], with a rocking linear potential, which consists of a repeating cycle of two kicks. Around the \( i-th \) kick we have:

\[
\begin{align*}
x_i &= x_{i-1} + p_{i-1}(1 + b) \\
p_i &= p_{i-1} + K \sin x_{i} + A \\
x_{i+1} &= x_{i} + p_{i}(1 - b) \\
p_{i+1} &= p_{i} + K \sin x_{i+1} - A
\end{align*}
\]

Following the usual procedure, we now obtain a modified form for the 2-kick average, but find that the momentum averages now include terms of the form \(< \sin^2 p, \sin 2pb >\) and \(< \cos^2 p, \sin 2pb >\). At small \( t \), \( pb \) is small while \( \sin^2 p \) oscillates rapidly, hence we can approximate these by \( 1/2 < \sin 2pb > \sim p_{av}(t)b \) and \( 1/2 < \cos 2pb > \sim 1/2 \) where \( p_{av} \) is the average momentum relative to the initial momentum. We note that such terms are the origin of the ratchet effect proposed in \[\text{[8]}\]. These terms are odd in \( p \) and hence unlike the standard map case- depend on whether we average our momenta from \( 0 \rightarrow \infty \) or from \( 0 \rightarrow -\infty \). As in \[\text{[8]}\] below we consider the case of particles with positive momenta separately from those with negative momenta, since the chaotic ratchet depends on differential energy diffusion rates for particles moving in opposite directions.

We consider first the case where \( p(t = 0) = 0 \) (or we take a cloud of particles with a narrow gaussian distribution peaked about \( p = 0 \)), but later consider the case where the initial current is non-zero. We take \( p_{av} \sim \pm \sqrt{D_0 t} \) for particles with positive or negative momenta respectively; hence \( |p_{av}| \) estimates the width of the momentum distributions about the origin.

With the approximations \(< \cos 2pb > \simeq 1 \) and \(< \sin 2pb > \simeq 2b \) it is easy to show:

\[
C^\pm(2) \simeq -K^2 J_2(K) [\cos A(1 \pm b) + 2p_{av}(t)b \sin A(1 \pm b)] \quad (0.2)
\]

The \( \pm \) correspond to the correlations between the two possible sequences of two kicks in the map, and we must average these to obtain \( C(2) = 1/2[C(2)_- + C(2)_+] \). However, since \( b \) is usually a very small parameter, here we can take:

\[
C(2) \simeq -K^2 J_2(K) [\cos A + 2p_{av}(t)b \sin A] \quad (0.3)
\]

or considering separately the energy diffusion for positive and negative momenta:

\[
C(2, \pm) \simeq -K^2 J_2(K) [\cos A \pm 2\sqrt{D_0 tb} \sin A] \quad (0.4)
\]

We see immediately from this expression that only the terms in \( \sin A \) differentiate between positive and negative momenta and hence allow a ratchet effect. This implies that for \( A = n\pi \), where \( n = 0, 1, 2, \ldots \) is an integer, there is no transport, while for \( A = (2n+1)\pi/2 \), there is maximal transport.

![FIG. 1: Figure compares the behavior of the classical current as a function of kick strength \( K \) for \( A \simeq \pi/2 \), \( b = 0.03 \) and \( A \simeq \pi \). It shows that the latter case is not transporting (ie \( <P > \simeq 0 \)), but that for \( A = \pi/2 \) the current direction follows the oscillations of a Bessel function \(-J_2(K)\).](image-url)
initially sharply peaked about \( p = 0 \) (ie a gaussian distribution of width \( \sigma = 1 \)). For \( A \sim \pi \) there is essentially no current, while for \( A \sim \pi/2 \), the current is appreciable and current reversals follow exactly the oscillations of \( -J_2(K) \), as expected from the formula.

In [8], the origin of the classical current was also found to be due to the fact that for \( p_{av} > 0 \) we have a \( C(2) \) correction of the opposite sign to the correction for \( p_{av} < 0 \), though in the case of the double-well ratchet the analytical form of the correction is much more complicated than Eq. (0.4).

In Fig.2 we test Eq.0.3 against a numerical calculation of the energies as a function of time for \( K = 3 \) and \( K = 10 \) and \( A = \pi/2 \). For an ensemble of trajectories initially peaked about \( p = 0 \), we expect that the average energy, \( E_+ \), of particles with positive momenta, and the average energy, \( E_- \), for those with negative momenta, are given by:

\[
2E_{\pm} \sim D_0 t \pm \frac{4}{3} b K^2 J_2(K) \sqrt{D_0 t}^{3/2} \tag{0.5}
\]

in the regime where \( pb \) is small. For large \( pb \), \(< \sin 2pb \geq < \cos 2pb \geq 0 \) hence \( C(2) = 0 \); as shown in [8], beyond a time scale \( t_r \) that it takes the momenta to increase to \( |p_{av}| \sim 1 \) both the positive \( p \) and negative \( p \) part of the ensemble of trajectories diffuse at the same rate \( 2E = \langle p^2 \rangle = D_0 t \), where for this system \( D_0 \approx K^2/2(1 - 2J_1(K))^2 \). Hence \( t_r \sim 1/(D_0 b^2) \).

The graph shows that for short times, the equation gives a good estimate to the correction to the energy.(\(< \ p^2 \ > - D_0 t \)). It also shows that after \( t \sim t_r \) the correction vanishes, and \(< \ p^2 \ > - D_0 t \) becomes a constant.

It is very interesting to consider also the case where at \( t = 0 \) we already have a drift current; in other words we start our trajectories with non-zero \( p = p_0 \). It is easy to show that the corresponding correction to the energy diffusion, \( C(2, p_0, p_{av}^\pm) \), now takes the form:

\[
\sim -K^2 J_2(K)\{ \cos A(\cos 2p_0 b + 2p_{av}^\pm(t) b \sin 2p_0 b) \\
+ \sin A(\sin 20 b + 2p_{av}^\pm(t) b \cos 2p_0 b) \}
\]

\( p_{av}^\pm(t) \) now represents the average momenta relative to \( p_0 \) at time \( t \) for particles with momenta greater or less than \( p_0 \) respectively. We see that the average energy absorption rate of the cloud of atoms depends on \( p_0 \) as well the effective widths of the distribution \( p_{av} \sim \pm \sqrt{D_0 t} \) about \( p_0 \). In other words, initially, if all particles are at \( p = p_0 \), we have a local diffusion rate correction \( -K^2 J_2(K) \cos (A - 2p_0 b) \); but as the cloud spreads in phase space one has to consider the average over the width of the cloud.

When \( pb \) is large, the \( \cos 2pb \) and \( \sin 2pb \) terms average to zero, hence \( C(2, p_0) \rightarrow 0 \) asymptotically for \( t > t_r \) regardless of the value of \( p_0 \). After this, the average energy growth for the particles with positive and negative momenta (relative to \( p_0 \)) is equal to \( D_0 t \). Regardless of \( p_0 \), the timescale for these correlations to be ‘averaged-out’ is given by the ratchet time of \( \pi/b \), \( t_r \sim (D_0 b^2)^{-1} \), the time required for the distribution to broaden to a substantial width in \( pb \). In [8] it was found that although the classical map does not have the \( 2\pi \) periodicity of the standard map, it has a new long-ranged periodicity in \( p \), with periodicity \( \sim 2\pi/b \). From the formula, we see that the periodicity of the correction is half of this: \( C(2, p_0 = 0) = C(2, p_0 = \pi/b) \). We note that since the \( C(2) \) correction vanishes when the distribution size becomes of order one cell in \( p \), the long range periodicity in this respect is not significant to the magnitude of current accumulated: by the time the expanding distribution samples the periodicity boundary, \( < p > \) is almost constant: we have tested this numerically.

We now consider how one might exploit Eq.0.4 to construct a cold atom ‘turnstile’. For simplicity we take the case \( A = \pi/2 \). Consider the case of a packet of cold atoms drifting through the lattice with a constant drift, which may be positive or negative, \( p_0 b = \pm \frac{\pi}{4} \), we see that for short times (where the equation is valid) its energy grows linearly, so \( < p^2 > \approx D t \approx [D_0 - (\pm K^2 J_2(K))]t \approx K^2 [1 - 2J_1(K)^2] - (\pm 2J_2(K))]t \). In other words for positive drift, the energy absorption is reduced relative to the basic diffusion term, while for negative drift it is accelerated. For \( K \approx 2 \) we can cancel the diffusion term almost completely so for \( p_0 b = \pi/4 \) the packet absorbs very little energy.

Fig.3a shows the effect on two classical ensembles drifting through the ratchet with speeds of \( p_0 = \pm 27 \) (for an effective turnstile it is not essential to have exactly \( p_0 = \pm \pi/(4b) \)) and \( b = 0.03, K = 1.7 \). Here \( D_0 \approx 0.8 \) so for positive \( p \), \( D \sim 0 \), while for negative \( p D \sim 1.6 \). The figure shows the initial distributions as well as the average momentum distribution after \( \sim 125 \) kicks. We see that the effect of the pulsed lattice on the two components is drastically different. While the cloud moving right is only slightly perturbed, the cloud moving left has been
FIG. 3: Shows the effect of a pulsed lattice on a system with finite initial current. Figure 3a demonstrates the 'turnstile' effect on particles moving through the lattice with initial momentum $p_0 = \pm 27 \approx \pi/(4b)$. The initial momentum distributions are shown, as well as the effect of $\pm 125$ kicks on the final distributions. We see that while particles moving right have absorbed relatively little energy, the particles moving left have substantially increased average kinetic energy. In Fig 3b we show that Eq.1.4 accurately describes the energy absorption at short times. The upper figure shows energy growth for $\cos A = \pi$ and $p_0b = \pm \pi/4$. Energy absorption is asymmetric about $p_0$ and is greater than $D_{\text{ot}}$ if $p_0, p_{\text{av}}$ are of different sign, but smaller otherwise. The lower figure corresponds closely to Fig3a. Energy absorption rates are symmetric about $p_0$ but are $\approx 15$ times faster for negative momenta relative to positive momenta.

heated to much larger average kinetic energies. Since for this case $C(2, p_0, p_{\text{av}}^+) now has no dependence on $p_{\text{av}}$ for short times, the particles absorb energy symmetrically about $p_0 = \pm \pi/(4b)$.

The corresponding quantum case was investigated in [8]; for the equivalent quantum turnstile to show similar behaviour we require only that the quantum break-time $t^*$ should be of the same order or longer than the duration of the classical turnstile (ie $\sim 100$ kicks for the above example, though we can adjust this by varying $b$). For $t < t^*$, the quantum behaviour follows closely the classical behaviour. For $t > t^*$, the quantum wavepacket localizes and absorbs no more energy, thus 'freezing-in' the ratchet asymmetry. For the turnstile, if the lattice region is of finite extent, the limiting time could be the time the atom spends within the optical lattice if this is less than $t^*$.

In Fig.3b we further verify Eq.1.3 by plotting the actual energy growth for the case $A = \pi$ (upper graph) and $A = \pi/2$ (lower graph). For the upper graph, only the $\cos A$ terms contribute; since we take $p_0 = \pm \pi/4$, we can see from both the formula $(C(2, p_0, p_{\text{av}}^+)) \approx -2bp_{\text{av}} \sin 2p_0b$) and the numerics that for $p_0, p_{\text{av}}$ of the same sign, trajectories absorb energy slower that $D_{\text{ot}}$ while for $p_0, p_{\text{av}}$ of different sign the converse is true. The lower figure corresponds closely to the 'turnstile' shown in Fig.3a since we have $\sin A = \pi/2$. The figure shows that for short times, the energy of particles initially with $p_0 = \pm \pi/4$ grows linearly. However the negative momentum particles absorb energy $\approx 15$ times faster than the particles with a positive drift.

In conclusion, we have shown the chaotic ratchet effect found in [8] is generic in character and applies to another class of ratchets. We show that by means of a simple analytical formula the effect can be maximised and use this to propose a mechanism for manipulating cold atoms. In particular we show that we can have a pulsed optical lattice that after $\sim 100$ kicks has imparted considerable energy to particles moving in one direction, while particles moving in the opposite direction absorb little energy. Since the mechanism is based on fully chaotic dynamics rather than a regime with stable islands, no position is special so no preparation of the initial state is required.

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