Time variability of viscosity parameter in differentially rotating discs

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Abstract

We propose a mechanism to produce fluctuations in the viscosity parameter ($\alpha$) in differentially rotating discs. We carried out a nonlinear analysis of a general accretion flow, where any perturbation on the background $\alpha$ was treated as a passive/slave variable in the sense of dynamical system theory. We demonstrate a complete physical picture of growth, saturation and final degradation of the perturbation as a result of the nonlinear nature of coupled system of equations. The strong dependence of this fluctuation on the radial location in the accretion disc and the base angular momentum distribution is demonstrated. The growth of fluctuations is shown to have a time scale comparable to the radial drift time and hence the physical significance is discussed. The fluctuation is found to be a power law in time in the growing phase and we briefly discuss its statistical significance.

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1 Introduction

The presence of accretion discs around compact objects like the neutron star and the black hole in both galactic and extragalactic X-ray sources is now a well established phenomenon. Some radio objects such as active galactic nuclei have accretion discs around supermassive black holes. Apart from several details such as environment, size, strength of magnetic field, cooling mechanism etc., all the global models of accretion systems share a common hydrodynamic structure. The central idea being, the turbulent shear stress causes dissipation of angular momentum and energy of the rotating fluid particles such that accretion can take place. The origin of turbulence and hence turbulent viscosity was an issue for the founders of the field of ‘accretion powered astrophysical systems’ and still remains to be a major issue. The closure model proposed by Shakura & Sunyaev (1973) remains the only working model for turbulent shear stress in astrophysical accretion discs. In this model, the physics behind the turbulent shear stress is parametrised by a dimensionless number $\alpha$. Thus the $\alpha$ viscosity continues to be the central idea in any model for hydrodynamic transport in accretion systems.

The spirit of the $\alpha$ viscosity is as follows: any eddy velocity which is greater than the local sound speed will dissipate quickly and cannot be the cause of eddy viscosity. Hence the turbulent stress must be less than the local isotropic pressure. Thus the shearing stress is taken to be proportional to the local isotropic pressure where the proportionality factor is called $\alpha$, where $0 < \alpha < 1$. When $\alpha \sim 1$ the flow is called a high viscosity flow, whereas when $\alpha \lesssim 10^{-2}$, the flow is called a low viscosity flow. With this model, the spectrum of cool Keplerian discs could be explained (Pringle & Rees, 1972, Novikov & Thorne, 1973, Shakura & Sunyaev, 1973). The idea of a sub-Keplerian disc was proposed to explain the nonthermal tail of the spectra from X-ray sources (Liang & Thompson, 1980, Paczyński & Bisnovatyi-Kogan, 1981, Muchotrzeb & Paczyński, 1982). In the case of a sub-Keplerian accretion disc the turbulent energy dissipated locally, is partly advected radially and partly emitted as radiation via nonthermal processes. In this model also, $\alpha$ closure remained unchanged although additional ram pressure was added to the total pressure (Chakrabarti & Titarchuk, 1995, Mandal & Chakrabarti, 2005, Rajesh & Mukhopadhyay, 2010). Since $\alpha$ is the ratio of two flow variables, namely, the turbulent shear stress and isotropic pressure, $\alpha$ should also be considered as a flow (continuum) variable. Since there is no known equation for the evolution of $\alpha$, it is treated as a disc parameter, and its value is fixed globally.

Apart from the fundamental problem to explain the origin of turbulent viscosity, there are other phenomena which await complete understanding, such as rapid X-ray variabilities in black hole accretion discs, aperiodic X-ray fluctuations and quasi periodic oscillations (QPOs) in accretions discs. Global mode oscillations and waves are invoked to explain some
of these phenomena (Mukhopadhyay, 2009). As the $\alpha$ viscosity is the source of energy dissipation in accretion discs, it is logical to attribute some of the time variabilities of the spectra to the temporal variation of $\alpha$. For example, in order to explain the $1/f$ (flicker) noise in X-ray sources, Lyubaraskii (1997) considered the local temporal fluctuations of $\alpha$ at outer radii. This would cause a change in mass accretion rate at inner radii where most of the X-rays are emitted. In order to have such an effect, a time varying component of $\alpha$ was assumed. The fluctuation was assumed to grow enough in a local accretion time scale. Thus the fluctuations in $\alpha$ resulted in a variable mass accretion rate, which would lead to variations in X-ray luminosity. Although the ultimate aim of our work in the present paper is similar to that of Lyubaraskii (1997), i.e., to study a mechanism to produce variabilities in the observed luminosities (say, X-ray) from accretion discs, our approach is somewhat different, and may be stated as follows: considering a steady state accretion in an annular region of an accretion disc with self-similar base flow profiles, we wish to study how the background $\alpha$ changes in response to any perturbation on the radial velocity field? Such perturbations of the radial velocity field (i.e., the mass accretion rate) may, in general, be of internal or external origin. These kind of studies have direct implications on the observed variabilities in X-ray luminosities from accretions discs.

A stable accretion system tries to maintain the steady mass flow across all radii. Any cause, internal or external which disturbs this steady state, will be quickly nullified by viscous dissipation. In § (2.1) we model the steady state flow variables in a local annular region as a power law in radial coordinate. The global flow domain can be thought of as a collection of such annular regions. In § (2.2) the evolution equations for perturbations in the mean density and the radial velocity, causing perturbation in $\alpha$ are discussed. We use the standard $\alpha$ model for viscous stress. In § (2.3) we reduce the perturbation equations to a set of nonlinear dynamical systems of equations, by specializing to the case when the Lagrangian derivatives (defined with respect to the radial velocity field) of the perturbations in the flow variables vanish. In § (3) we demonstrate that the growth of the viscosity parameter is always followed by saturation and degradation, and the fluctuation asymptotically goes to zero. The behaviour of the fluctuation is strongly dependent on the radial location, base angular momentum distribution and the mean viscosity of the flow. We demonstrate that the growth of the fluctuation in viscosity parameter always scales as the local accretion time, and that it shows a power law growth phase in time, in the astrophysically relevant time scale. We conclude in § (4).
2 Model Equation Describing The System And The Solution Procedure

Let us consider a cylindrical coordinate system, with spacetime coordinates denoted by \((r, \phi, z, t)\), whose origin is at the centre of the compact object. The angular velocity vector, \(\Omega\), is pointed along \(z\) (vertical) direction, and the midplane of the accretion disc is at \(z = 0\). We begin by considering a vertically integrated, axisymmetric, steady-state accretion flow, in which, we focus on an annular region of the accretion disc. We consider axisymmetric perturbations on base radial velocity field and mean density in this annular region to study the response of such perturbations on the evolution of the viscosity parameter. Thus all the base flow variables in this study are functions only of the radial coordinate \((r)\), whereas the perturbations depend on both, the radial coordinate \((r)\) and time \((t)\).

2.1 Base flow

For a general accretion flow, we consider a small annular region specified by the mean velocity field, where \(V\) and \(\Omega\) are the magnitudes of the radial and angular velocity fields, respectively. Let us specify the unperturbed axisymmetric, steady-state accretion (base) flow where the radial velocity and the angular velocity are power laws in radial coordinate, i.e., \(V = V_0 r^{-j}\) and \(\Omega = \Omega_0 r^{-q}\). The explicit radial dependence of other fluid variables in an unperturbed state can be obtained by solving the conservation equations for mass, radial momentum and angular momentum, given as:

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \Sigma V) = 0 \quad (1)
\]

\[
\Sigma \left( V \frac{\partial V}{\partial r} - r \Omega^2 \right) = \frac{k \Sigma}{r^2} - \frac{\partial P}{\partial r} \quad (2)
\]

\[
\frac{\Sigma V}{r^2} \frac{\partial}{\partial r} (r^2 \Omega) = -\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 W^{\phi r}) \quad (3)
\]

where \(\Sigma\) and \(P\) are vertically integrated density and pressure, respectively. The quantity \(k = -GM\), where \(G\) is the universal gravitational constant and \(M\) is the mass of the central object. We solve the above set of equations along with the equation of state, \(P = \Sigma T\),
where $T$ is the effective temperature of the flow. We impose the boundary condition that all the physical quantities go to zero as $r \to \infty$. For the turbulent stress, we use the $\alpha$ viscosity model, i.e., $W_{r}^{\phi r} = \alpha P/r$, where $\alpha$ is the Shakura-Sunyaev viscosity parameter. We can write the solution to the above set of equations as,

$$V(r) = V_0 r^{-j}; \quad \Omega(r) = \Omega_0 r^{-q}; \quad \Sigma(r) = \Sigma_0 r^{j-1}$$

(4)

$$T(r) = \left(\frac{k}{j-2}\right) r^{-1} - \left(\frac{jV_0^2}{j+1}\right) r^{-2j} + \left(\frac{\Omega_0^2}{j-2q+1}\right) r^{2(1-q)}$$

(5)

where $V_0 < 0$, as the radial flow is directed towards the central object, and $\Sigma_0 = \dot{M}/(4\pi V_0)$, where $\dot{M}$ is a negative quantity called the mass accretion rate. Since both, the radial velocity and density, decrease with increasing values of $r$, we get from Eq. (4) that $0 < j < 1$. The value of $q$ indicates the angular momentum distribution of the base flow; $q = 1$, $q = 3/2$ and $q = 2$ describe, respectively, the flat rotation, Keplerian rotation and constant angular momentum disc profiles. From the angular momentum balance equation, we get

$$\alpha(r) T(r) = -V_0 \Omega_0 r^{1-q-j} + \frac{c}{\Sigma_0} r^{-j-1}$$

(6)

As $\alpha T$ is a physical quantity, it approaches zero as $r \to \infty$, according to the boundary condition that we have chosen. In Eq. (6), since the term containing the integration constant $c$ goes to zero as $r \to \infty$, $c$ is nonzero in general. The physics behind $c$ comes from the actual physical mechanism which produces the $\alpha$ viscosity. The origin of $\alpha$ is beyond the scope of the present analysis, therefore, we can only choose $\alpha$ at a particular radius which automatically fixes the value of $c$ from Eq. (6).

### 2.2 Perturbation

Any closure model in hydrodynamics is based on the assumption that the stress tensor can be written as a functional of space and time through the mean flow variables, such as mean density, velocity etc. In the case of an accretion disc it is this unknown dependence which is characterized by $\alpha$. Thus the perturbation in mean hydrodynamic quantities should cause an effective change in $\alpha$ or vice versa due to their dynamical coupling. The evolution of $\alpha$ cannot be traced rigorously since we do not yet have any equation describing $\alpha$. In the following
perturbation analysis, we perturb the mean density and radial velocity fields, which cause an effective change in the mass accretion rate, thus effectively changing the angular momentum flux. In the present work, we explore the scenario in which the stress tensor fluctuates instantaneously, whereas the angular momentum distribution remains unchanged. Assuming that $\sigma, v$ and $\beta$ denote perturbations on the base quantities $\Sigma, V$ and $\alpha$ respectively, such that, $\Sigma \rightarrow \Sigma + \sigma$, $V \rightarrow V + v$ and $\alpha \rightarrow \alpha + \beta$, we can write the following equations for the perturbations:

$$
\frac{\partial \sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \Sigma v + r \sigma V + r \sigma v \right] = 0 \quad (7)
$$

$$(\Sigma + \sigma) \frac{\partial v}{\partial t} + \sigma V \frac{\partial V}{\partial r} + (\Sigma + \sigma) \left[ V \frac{\partial v}{\partial r} + v \frac{\partial V}{\partial r} + v \frac{\partial v}{\partial r} \right] - \sigma r \Omega^2 = \frac{k \sigma}{r^2} - \frac{\partial}{\partial r} (\sigma T) \quad (8)
$$

$$
\left[ \frac{\sigma V}{r^2} + \frac{(\Sigma + \sigma)v}{r^2} \right] \frac{\partial}{\partial r} (r^2 \Omega) = -\frac{1}{r^3} \frac{\partial}{\partial r} \left[ r^2 (\alpha \sigma + \beta \Sigma + \beta \sigma) T \right] \quad (9)
$$

We note that the perturbations are functions of $r$ and $t$, as discussed above, i.e., $\sigma = \sigma(r, t)$, $v = v(r, t)$ and $\beta = \beta(r, t)$.

### 2.3 Dynamical model

Any general accretion disc is expected to possess all sorts of perturbations, the exact form of which could be completely unknown and hence arbitrary. We consider the perturbation such that the observer, moving along with a fluid particle radially, sees no change in the perturbation, i.e., the perturbation is frozen to the radially advecting fluid particle. Thus the Lagrangian or advective time derivative of the perturbation is zero. We define the advective time derivative with respect to base radial flow, which is denoted as,

$$
\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V \frac{\partial}{\partial r} \quad (10)
$$

Let $r_0$ be an arbitrary length scale. We scale the radial coordinate by $r_0$ and denote the scaled radial coordinate as $\tilde{r} = r/r_0$. We scale the other variables as,
\[ \tilde{V} = \frac{V}{V_0 r_0^{-j}} = \tilde{r}^{-j} \; \tilde{\Sigma} = \frac{\Sigma}{\Sigma_0 r_0^{-j-1}} = \tilde{r}^{j-1} \; \tilde{\Omega} = \frac{\Omega}{\Omega_0 r_0^{-q}} = \tilde{r}^{-q} \]  

(11)

We scale the time and the temperature as,

\[ \tilde{t} = \frac{t}{(\Omega_0 r_0^{-q})^{-1}} = (\Omega_0 r_0^{-q}) t \; \tilde{T} = \frac{T}{(\Omega_0 r_0^{1-q})^{2}} \]  

(12)

Where \( r_0 \Omega(r_0) = r_0 \Omega_0 r_0^{-q} = \Omega_0 r_0^{1-q} \) is the azimuthal velocity at distance \( r_0 \). We scale the perturbed quantities \( \sigma \) and \( v \) as following:

\[ \tilde{\sigma} = \frac{\sigma}{\Sigma_0 r_0^{-j-1}} \; \tilde{v} = \frac{v}{V_0 r_0^{-j}} \]  

(13)

Thus we scale Eq. (5) and write the following expression for temperature in scaled units:

\[ \tilde{T}(\tilde{r}) = \left( \frac{\epsilon_1}{j - 2} \right) \tilde{r}^{-1} + \left( \frac{1}{j - 2q + 1} \right) \tilde{r}^{2(j-1-q)} - \left( \frac{\epsilon_2 j}{j + 1} \right) \tilde{r}^{-2j} \]  

(14)

where,

\[ \epsilon_1 = \frac{k r_0^{-1}}{(\Omega_0 r_0^{1-q})^2} \; \epsilon_2 = \frac{(V_0 r_0^{-j})^2}{(\Omega_0 r_0^{1-q})^2} \]  

(15)

We note that \( \epsilon_1 \) is the ratio of the local gravitational energy to the angular kinetic energy which is a negative number; \(|\epsilon_1| \simeq 1\) for a Keplerian accretion disc and \(|\epsilon_1| > 1\) for a sub-Keplerian disc. From Eq. (15), we see that \( \epsilon_2 \) is the ratio of the local radial kinetic energy to the angular kinetic energy; \( \epsilon_2 << 1 \) for a Keplerian accretion disc and \( \epsilon_2 > 1 \) for inner sub-Keplerian disc very close to the central object. Without loss of generality we can choose \( \tilde{r} = 1 \). The information of radial location in the disc is implicitly contained in the parameters \( j, q, \epsilon_1 \) and \( \epsilon_2 \), and thus one can sample different radial locations by suitably varying these parameters. Using Eq. (10) and the scaling relations in Eqs. (7)-(9) we find that the set of dynamical equations for perturbations then reduces to
\[
\frac{\partial \tilde{\sigma}}{\partial t} = F_1(\tilde{\sigma}, \tilde{v}; j, q, \epsilon_1, \epsilon_2) \tag{16}
\]

\[
\frac{\partial \tilde{v}}{\partial t} = F_2(\tilde{\sigma}, \tilde{v}; j, q, \epsilon_1, \epsilon_2) \tag{17}
\]

\[
\frac{\partial \beta}{\partial t} = F_3(\tilde{\sigma}, \tilde{v}, \beta; j, q, \epsilon_1, \epsilon_2) \tag{18}
\]

Details of the functions \( F_1, F_2, F_3 \) are given in the Appendix B.

### 3 Global behaviour of dynamical system

Our aim is to solve the system of coupled nonlinear dynamical equations, (16)–(18), as an initial value problem. The density (\( \sigma \)) and radial velocity (\( v \)) perturbations are nonlinearly coupled to each other, whereas the dynamical evolution of \( \beta \) is governed by its nonlinear coupling to \( \sigma \) and \( v \), as may be seen from Eq. (18). The quantity \( \beta \) may be thought of as a passive/slave mode in the sense of dynamical system theory (Manneville, 1990). We wish to study a mechanism which can produce temporal variations in the viscosity parameter, \( \alpha \). Therefore \( \beta \), signifying any departure from a chosen base value of \( \alpha \), is universally assumed to be zero at an initial time. The initial density perturbation is also chosen to be zero throughout. In this setup, an initial fluctuation of the radial velocity is assumed and we analyze the evolution of \( \sigma \), \( v \) and \( \beta \) as a function of time, for different values of free parameters, \( j, q, \epsilon_1 \) and \( \epsilon_2 \). In Fig. (1) a typical plot of \( \beta \) as a function of time is shown for the Keplerian angular velocity profile for different values of \( \epsilon_2 \), keeping everything else same. When the system is pumped with a random external velocity fluctuation the density and radial velocity fields change. Since \( \beta \) is dynamically coupled to the density and velocity fluctuations, it gets excited and shows growth in the present example. We can see from Fig. (1) that \( \beta \) grows with time, reaches a saturation and finally degrades. The major difficulty with standard linear perturbation analysis is that they could either show growth or decay. A complete physical picture of growth, saturation and final degradation of the perturbation is the result of the nonlinear nature of the system of equations, which is a non-trivial result.

When the system is pumped with a random initial radial velocity fluctuation, the density and radial velocity perturbation evolve according to the rule governed by Eqns. (16)–(17). We observe that for the physically acceptable range of free parameters \( j, q, \epsilon_1 \) and \( \epsilon_2 \) (the
Figure 1: The temporal variation of $\beta$ is plotted corresponding to the Keplerian disc (i.e., $q = 1.5$) for $\alpha \simeq 10^{-2}$, $\epsilon_1 = -1.1$ and $j = 0.9$. The solid bold line corresponds to $\epsilon_2 = 0.01$, the dashed-dotted line corresponds to $\epsilon_2 = 0.005$ and the dotted line is for $\epsilon_2 = 0.0025$. The horizontal axis denotes the time measured in units of the rotational time-scale (see Eq. (12)).
allowed range of \( j, q, \epsilon_1 \) and \( \epsilon_2 \) will be discussed in the next subsection) all initial values form orbits attracted towards origin \((\sigma = 0, v = 0)\) in the \( \sigma - v \) plane. In other words, for the physically acceptable range of free parameters, we find that \( \sigma = 0 \) and \( v = 0 \) is a universal attractor. This observation along with Eq. (18) means that asymptotically \( \beta \) is always zero.

The physical reason behind the time variability of \( \beta \) may be explained as follows: as the viscous stress dissipates the angular momentum the fluid particle gains more radial drift velocity and moves to the inner orbit. The radial drift velocity towards the central object is larger for a high viscosity flow (larger \( \alpha \)) than a low viscosity flow (smaller \( \alpha \)); see, for example, Rajesh & Mukhopadhyay (2010). When a positive radial velocity component is added to the fluid particle, the particle moves with less radial drift velocity towards the central object. The tendency of the fluid particle to remain at a larger orbit is thus overcome by increasing viscous dissipation and hence \( \beta \) shows increasing tendency. Finally when the density/radial velocity perturbations damp, the \( \beta \) asymptotically becomes zero. The reverse is true when a negative initial radial velocity fluctuation is added to the system. The spatial dependence of the evolution of the perturbation is contained in the parameters of the system such as \( q, j, \epsilon_1, \epsilon_2 \). In Fig. (1), we demonstrate the evolution of \( \beta \) as a response to a positive radial velocity perturbation in a Keplerian disc for different values of \( \epsilon_2 \), while keeping rest of the parameters the same. We note that different values of \( \epsilon_2 \), while keeping other parameters the same, represent different radial distances in the accretion disc, with larger values corresponding to the smaller orbits. Therefore the bold, the dashed-dotted and the dotted curves in Fig. (1) correspond to the innermost, the intermediate and the outermost radial location from the central object. Thus we find that, for the same initial trigger, the stronger fluctuation in viscosity is seen for relatively inner orbit. The reason for this nature could be understood as follows: the fluid particle stays in the inner orbit for less time as compared to the outer orbit. So the fluid particle in the inner orbit gets relatively less time to neutralise the external energy input. To dissipate this energy the viscosity has to increase much larger in relatively smaller times.

### 3.1 Solution of dynamical system

We have a whole range of parameters in this problem, whose physical meanings could be briefly stated as follows: (i) \( q \) determines the distribution of base angular momentum; (ii) \( \epsilon_1 \), which is the ratio of gravitational energy to the angular kinetic energy of the fluid particle at a particular radius, signifies the competition between gravity and angular momentum, with value unity corresponding to strict balance of gravity and angular momentum as in the case of the Keplerian motion; (iii) \( \epsilon_2 \), which is the ratio of radial kinetic energy to the angular kinetic energy, represents the strength of advection, and is expected to be larger in the inner
parts of the accretions discs where the flow could be advection dominated depending on the efficiency of the cooling mechanism and strength of turbulent viscosity; and (iv) $j$ represents the base distribution of density and radial velocity fields. We can find the Bernoulli number ($B$) of the flow by expressing the radial velocity equation in integral form. We can also define the Mach number ($M$) as the ratio of radial speed to the local sound speed. In the scaled units at $\tilde{r} = 1$, $B$ and $M$ have the following form:

$$B = \frac{\epsilon_2}{2} + \frac{1}{2(q - 1)} + \epsilon_1 + \tilde{T} + (j - 1) \left[ \frac{\epsilon_1}{(2 - j)} + \frac{1}{(j - 2q + 1)(2 - 2q)} + \frac{\epsilon_2}{2(j + 1)} \right]$$

$$M = \sqrt{\frac{\epsilon_2}{\tilde{T}}}$$

3.1.1 The parameter search

As our system of equations (Eqns. (16)-(18)) depends on many parameters discussed above, we need to carefully choose the values of these parameters in order to get physically acceptable solutions. Many of these parameters are restricted and should be chosen such that they satisfy the conditions, $T > 0$ and $B < 0$. We note from Eq. (4) that $0 < j < 1$, and also restrict ourselves to cases when $1.5 \leq q < 2$. We are not considering the effect of shocks which may be significant at much of the inner part of the accretion disc. Hence throughout the analysis we consider subsonic flow, i.e.; $M < 1$. We search for some suitable values of these parameters in the following two regimes:

(i) We consider outer regions of an accretion disc such that the value of $q$ is close to the Keplerian value ($q \simeq 1.5$). Thus we explore the cases when $1.5 \leq q < 1.7$. For such a disc, we consider both, the low viscosity ($\alpha \lesssim 10^{-2}$) and high viscosity ($\alpha \sim 1$) flows. Having chosen these parameters, we now study our set of dynamical equations as a function of remaining parameters, where we always ensure that our choices are consistent with conditions, $T > 0$, $B < 0$ and $M < 1$.

(ii) Next, we focus on regions of the accretions disc which have dissipated much of their angular momentum, like, for example, inner parts of the disc, where the value of $q$ would be closer to the case of constant angular momentum disc, for which $q = 2$. Thus we explore the cases when $1.7 \leq q < 2$. Again, we study low viscosity and high viscosity discs in a self-consistent manner as discussed above.
In the standard Keplerian disc proposed by Shakura & Sunyaev (1973) there is an exact balance between the gravitational force and the centrifugal reaction experienced by the rotating fluid particle. Therefore for a flow with \( q \) more close to the Keplerian value the magnitude of \( \epsilon_1 \) should be more close to unity. In this case the radial drift velocity is too small compared to the corresponding rotational velocity. So \( \epsilon_2 \) must be much smaller than unity. On the other hand, if the accretion system has much less angular momentum at larger radii or the accretion disc is highly viscous at larger radii, the flow may have even less angular momentum at inner radii. In such cases if the fluid moves radially without change in angular momentum, the angular momentum distribution is represented by \( q = 2 \). Thus the flow with \( q \) more close to 2 will have less angular kinetic energy. Because of this the radial drift velocity will be larger than the the corresponding rotational velocity, thus \( \epsilon_2 \) will be larger than unity. In this case, the gravitational energy will be much larger than the angular kinetic energy and hence the magnitude of \( \epsilon_1 \) will be much larger than unity. Apart from the above reasoning, range of \( \epsilon_1, \epsilon_2 \) and \( j \) must be consistent with physically acceptable values of \( T, B \) and \( M \).

3.1.2 Numerical results

By taking into account all the points presented previously, we numerically solve the dynamical set of equations, given in Eqns. (16)-(18). Here we only show the temporal variations of the quantity \( \beta \) and plot, in Fig. (2), its time-evolution for variety of parameters. The four different panels in Fig. (2) correspond to four classes of parameters, as described below:

(i) \textit{Keplerain/Nearly Keplerian, low viscosity case (Fig. (2a))} : The solid, the dashed-dotted and the dotted curves in Fig. (2a) correspond to \( q = 1.5, q = 1.6 \) and \( q = 1.7 \), respectively. For all the three curves, we chose \( \alpha \sim 10^{-2}, \epsilon_1 = -1.1, \epsilon_2 = 0.01 \) and \( j = 0.9 \).

(ii) \textit{Keplerain/Nearly Keplerian, high viscosity case (Fig. (2b))} : The solid, the dashed-dotted and the dotted curves in Fig. (2b) correspond to \( q = 1.5, q = 1.6 \) and \( q = 1.7 \), respectively. For all the three curves, we chose \( \alpha \sim 0.99, \epsilon_1 = -1.1, \epsilon_2 = 0.01 \) and \( j = 0.9 \).

(iii) \textit{Sub-Keplerian, low viscosity case (Fig. (2c))} : The solid, the dashed-dotted and the dotted curves in Fig. (2c) correspond to \( q = 1.7, q = 1.8 \) and \( q = 1.9 \), respectively. For all the three curves, we chose \( \alpha \sim 10^{-2}, \epsilon_1 = -3, \epsilon_2 = 1.125 \) and \( j = 0.9 \).

(iv) \textit{Sub-Keplerian, high viscosity case (Fig. (2d))} : The solid, the dashed-dotted and the dotted curves in Fig. (2d) correspond to \( q = 1.7, q = 1.8 \) and \( q = 1.9 \), respectively. For all the three curves, we chose \( \alpha \sim 0.99, \epsilon_1 = -3, \epsilon_2 = 1.125 \) and \( j = 0.9 \).
Figure 2: The temporal evolution of the perturbation of the viscosity parameter is shown. The solid, the dashed-dotted and the dotted curves in panels (a) and (b) correspond to $q = 1.5$, $q = 1.6$ and $q = 1.7$, respectively, whereas they correspond to $q = 1.7$, $q = 1.8$ and $q = 1.9$, respectively in panels (c) and (d). We set $\alpha \sim 10^{-2}$ for (a) and (c), whereas $\alpha \sim 0.99$ for (b) and (d). For panels (a) and (b) $\epsilon_1 = -1.1$, $\epsilon_2 = 0.01$, and for (c) and (d) $\epsilon_1 = -3$, $\epsilon_2 = 1.125$. We set $j = 0.9$ and $\nu(t = 0) = 0.5$, throughout. The horizontal axes in all the panels denote the time measured in units of the rotational time-scale (see Eq. (12)).
The initial value of the radial velocity perturbation was identical for all the cases described above, which was chosen to be 0.5. Thus we have all the four regimes of parameters from high angular momentum, low viscosity flow to low angular momentum, high viscosity flow. In all these regimes, we notice that the flow with lower $q$ value has a stronger response to the same initial trigger, with all other parameters being the same. This is because for the flow with lower $q$, the angular momentum of the base flow is larger and to nullify the effect of an external trigger, relatively more angular momentum has to be dissipated, and hence the viscous stress has to increase by a larger amount, as characterized by $\beta$.

Generally, radiatively inefficient sub-Keplerain components of accretion discs are thought to be highly viscous, and in such situations the huge radial velocity advects the fluid much before the radiative mechanisms could cool the system. In Fig. (2d), the extreme case of flow with low angular momentum and high viscosity is shown. Such a disc has already dissipated much of its angular momentum and is rapidly heading towards an inherent unstable situation. The only way to stabilise the system is by channelising the available external energy input to reduce the viscous dissipation and radial velocity. Thus the fluctuation in viscosity, $\beta$ shows a decreasing tendency in this extreme case.

We note that there are two time scales involved in the problem, one, the radial drift time of the particle, denoted by $\tau_D$, and the other, the saturation time of the perturbation, denoted by $\tau_S$. In our scaling convention, $\tau_D = 1/\sqrt{\epsilon_2}$ and $\tau_S$ is defined as the time that it takes for the perturbation ($\beta$) to reach its extremum value. The mechanism proposed here to produce a variation in the viscosity parameter will be appreciable only if the saturation time is smaller than or of the order of radial drift time. This is because if the saturation time is much larger than radial drift time the fluid particle will be drifted to the inner orbit much before the perturbation grows appreciably and hence the mechanism is inefficient to have any practical relevance. We see from the above mentioned cases that under proper choice of parameters, the condition $\tau_S \lesssim \tau_D$ is always met. Thus we conclude that this mechanism is physically efficient to produce an appreciable variability in viscosity.

### 3.2 Power Law Growth

A simple analytical model for the statistical theory (random saturation) of a nonlinear growth process was first introduced in astrophysics by Fermi (1949) in the context of cosmic rays. The most common growth processes in nature are either exponential growth or power law growth. Processes where a discrete number of basic units interact multiplicatively to form an avalanche (such as nuclear chain reactions) are well represented by an exponential growth law. A power law growth commonly occurs when the process scales with expansion in spatial domain (a brilliant account of such models can be found in Aschwanden (2011)). In our case,
Figure 3: Plot for the growing phase of $\beta$ for the Keplerian base flow ($q = 1.5$) with $\alpha \sim 10^{-2}$, $j = 0.9$ and $\epsilon_1 = -1.1$ with $\epsilon_2 = 10^{-3}$ (thick line), $10^{-2}$ (dot dashed line), $10^{-5}$ (thin line).

The power emitted by the system is due to the process of viscous dissipation. If we neglect the density fluctuation, which is always very small compared to the mean density, the power emitted (after subtracting the background power) by the process is

$$P(t) = \left( \Sigma r \frac{\partial \Omega}{\partial r} \right) \beta(t) \quad (19)$$

The power emitted by the system is directly proportional to $\beta$, where in scaled units at $\tilde{r} = 1$, the proportionality constant depends on radial location through $q, j, \epsilon_1$ and $\epsilon_2$.

As noted earlier the time scale of growth of the viscosity fluctuation is always less than or of the order of the radial drift time (advection time). Hence if we model the viscosity fluctuation as a nonlinear ‘process’ of extracting energy, the relevant time of the process is
the saturation time $\tau_S$, defined as the time when $\beta$ is extremum. In Fig. (3), $\log(\beta)$ as a function of $\log(\tilde{t})$ has been plotted up to the saturation time, $\tau_S$. The curve is a straight line until it approaches the saturation time $\tau_S$. From now onwards we represent $\tilde{t}$ by $t$ for simplicity. Hence for the time interval $0 \leq t \leq \tau_S$ we can approximately write

$$\beta = \beta_0 t^\gamma$$

(20)

where $\gamma$ is a positive number. The value of $\beta_0$ and $\gamma$ can be found by fitting the curve numerically. The analytical expression given above deviates from the curves shown in Fig. (3) only when $t$ approaches the saturation time. This is because the power law growth cannot capture the saturation of the perturbation. Nevertheless the power law is the best analytical choice and the physics is well represented until $t \leq \tau_S$. The power law nature of the growing phase of the process is an indicator of its strong dependence on spatial location. Our perturbation analysis is done at a particular radial location of the general flow domain of an accretion disc, and the strong dependence of the perturbation on radial location is evident through the restrictive range of values for the free parameters. Hence

$$P(t) = P_0 t^\gamma \quad \text{for} \quad (0 \leq t \leq \tau_S)$$

where

$$P_0 = q \left( \frac{\epsilon_1}{j-2} + \frac{1}{j-2q+1} - \frac{j\epsilon_2}{j+1} \right) \beta_0$$

(21)

Therefore the total energy emitted by the system in the time interval $0 < t < \tau$ is

$$E(\tau) = \int_0^\tau P(t) \, dt = P_0 \int_0^\tau t^\gamma \, dt = \left( \frac{P_0}{\gamma + 1} \right) \tau^{\gamma+1}$$

(22)

The real probabilistic quantity in this theory is the time $\tau$ up to which the ‘process’ continues. The probability distribution for $\tau$ is beyond the scope of the standard hydrodynamic theory. If the dynamical equation for the probability distribution is thought of as a diffusion equation in $\tau$ then the probability distribution has the form (see e.g. Aschwanden (2011)),

$$N(\tau) \, d\tau \propto e^{-a\tau} \, d\tau$$

(23)

Thus the probability distribution becomes
\[ N(E) \, dE \propto E^{-(\frac{\gamma+1}{\gamma+1})} e^{-n E^{\frac{1}{\gamma+1}}} \, dE, \quad \text{where} \quad n = a \left( \frac{\gamma + 1}{P_0} \right)^{1/(\gamma+1)} \] (24)

where \( N \) is defined as the probability that the nonlinear process extracts an amount of energy \( E \) from the background shear and \( a \sim 1/\tau_S \).

We note that there is a difference between the ‘nonlinear process’ discussed in Aschwanden (2011) and the one we analysed in this work. In the analytical models of Aschwanden (2011) it is assumed that the growing phase of the ‘process’ could either be an exponential or a power law function of time, and the process remains nonlinear up to the saturation time. Whereas, we numerically find that the growing phase of our process is strictly a power law in time. In Aschwanden (2011), final dissipation is treated as a linear process, whereas, in our real physical system another time scale enters which is the radial drift time, \( \tau_D \). It would be physically irrelevant to consider any growth of the perturbation beyond \( \tau_D \), as the fluid particle itself would have moved to inner orbits due to radial advection at times of order \( \tau_D \). Since \( \tau_S \) is always comparable to or less than \( \tau_D \), we need to focus only on the growing phase. The real probabilistic quantity is \( \tau \), which signifies how long the ‘process’ continues in the growth phase. It is this probability which we assumed to be as an exponential function. We also note that the saturation time of the growth process studied in this work shows strong dependence on the radial location through the parameters \( q, j, \epsilon_1 \) and \( \epsilon_2 \).

4 Conclusion

In the above analysis we have proposed a mechanism which causes fluctuations in the \( \alpha \) viscosity parameter for a general angular momentum distribution, Keplerian or non-Keplerian (\( 1.5 \leq q < 2 \)). We began by considering a vertically integrated, axisymmetric, steady-state accretion flow (base flow), in which, we focussed on an annular region of the accretion disc. The global flow domain can be thought of as a collection of such annular regions. We modelled the steady state flow variables in the annular region as the power law in the radial coordinate. We assumed the closure model proposed by Shakura & Sunyaev (1973) and treated the viscosity parameter \( \alpha \) as a continuum variable. Our aim was to study how the background \( \alpha \) could change in response to any perturbation on the radial velocity and density fields. Ignoring any instantaneous perturbation of the base angular momentum, we perturbed the density and radial velocity fields, which enabled us to obtain an expression for the evolution of the perturbation as a function of the viscosity parameter. We reduce the perturbation equations to a set of nonlinear dynamical systems of equations, by specializing...
to the case when the Lagrangian derivatives (defined with respect to the radial velocity field) of the perturbations in the flow variables vanish. This special choice enabled us to reduce the coupled nonlinear perturbation equations to a set of first-order, coupled, nonlinear, autonomous, dynamical equations in time. It may be worthwhile to remark that such analysis would provide enough motivation to isolate this class of perturbations in future numerical experiments whose effects are interesting as demonstrated in the present work. We found that the base \( \alpha \) could change due to its nonlinear dynamical coupling to other fluid variables and perturbations of them. Thus we treated the perturbation of \( \alpha \) (denoted by \( \beta \)) as a passive/slave variable in the sense of dynamical system theory.

Such studies have direct implications on the observed variability in X-ray luminosities from accretion discs. Lyubaraskii (1997) has shown that local fluctuations of the \( \alpha \) parameter could cause the mass accretion rate at inner radii to vary and hence cause a temporal variation in energy dissipation. We have shown in the present work that the fluctuations in \( \alpha \) could come about due to perturbations on the mass accretion rate at some radius, and they could behave in a complicated manner as a function of radial distance from the central object. Our main conclusions may be stated as follows:

(i) We demonstrate that the viscosity parameter in the accretion disc can change in appreciably short and astrophysically relevant time scale.

(ii) We find that the saturation time \( (\tau_S) \) of the perturbation is smaller or of the order of the radial advection time scale \( (\tau_D) \). This is an important point because if the saturation time is larger than radial drift time, the fluid particle will be drifted to the inner orbit much before the perturbation grows appreciably, and hence the mechanism would be inefficient to have any practical relevance. Thus we conclude that this mechanism proposed here is physically efficient to produce appreciable variability in viscosity.

(iii) The perturbation \( (\beta) \) shows growth, saturation and eventual degradation, which is a non-trivial result due to nonlinear nature of the system of equations. This offers a complete physical picture of nonlinear evolution of the perturbation.

(iv) We have demonstrated that the variability in \( \alpha \) in the growing phase is an exact power law in time.

(v) We have also demonstrated that not only the time scale of the fluctuation, but also the amplitude of the fluctuation strongly depends on radial location. The time scale of the fluctuation is found to be independent of the initial trigger.

A stable accretion system tries to maintain the steady mass flow across all radii. Any cause, internal or external which disturbs this steady state, gets quickly nullified by sufficient
variation of viscous dissipation. We can think of the local fluctuation in $\alpha$ and the associated change in mass accretion rate as a chain of coupled processes across radii $r_1 > r_2 > ...r_i$. A zero initial trigger in active modes (density perturbation and radial velocity perturbation) could cause no change in $\alpha$. Let us consider an annular region at radius $r_1$ where the mass accretion rate is slightly changed/perturbed due to some process happening at outer radii, say, due to the fluctuation in $\alpha$ proposed by Lyubaraskii (1997). This will seed the initial trigger in radial velocity perturbation at radius $r_1$. Thus the system will evolve according to the set of equations described in § (2.3). This induces local fluctuation in $\alpha$ at $r_1$ which would cause change in the mass accretion rate in the accretion time scale at $r_1$, $\tau_D (r_1)$. This will excite a fluctuation in $\alpha$ at the next inner orbit at $r_2$ in a time scale $\tau_D (r_2)$. This will cause a larger change in mass accretion rate compared to the change in mass accretion rate at $r_1$ (this is because $\beta$ is independent of local scaling and the real time involved in the fluctuation decreases as radius decreases). This chain of process continues until the radius $r_i << r_1$, where most of the energy is released. The variation in the mass accretion rate at the inner orbit is therefore a cumulative effect of $\alpha$ fluctuations at different outer radii and hence a small variation of $\alpha$ at the outer radius could cause a huge effect at the inner orbit. The amplitude of the fluctuation in $\alpha$ depends on the radial location through $j, q, \epsilon_1$ and $\epsilon_2$. But these parameters at each radius should be found from a global solution of conservation laws. Thus each accreting system found observationally, should be explained by the branch of a global solution which can account for the observed time variability in luminosity (Appendix A).

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Appendix A

The accreting matter is a mixture of ions and electrons. The mass content of the fluid is mainly due to ions and the radiative process (cooling mechanism) is effectively due to electrons. In the most general case the ion and electron temperatures are different. Let $T_i$ be the ion temperature and $T_e$ be the electron temperature and $T$ be the effective temperature. Then the equation of state is \( P = \Sigma T = \frac{\Sigma}{f c^2} \left( \frac{K_B T_i}{\mu_i m_i} + \frac{K_B T_e}{\mu_e m_e} \right) \) \( (25) \)

\( T \) in the units of light speed squared and $T_i, T_e$ measured in the units of Kelvin. $\mu_i, \mu_e$ are the effective molecular weights of ion and electron. $m_i$ is the mass of proton and $m_e$ is the mass of electron. $f$ is the ratio of gas pressure to the total pressure. $K_B$ is the Boltzmann’s constant and $c$ is the speed of light in vacuum.)
For a general advective flow the turbulent energy dissipated ($Q_{vis}$) is partly advected radially ($Q_{adv}$) and partly radiated away ($Q_{rad}$) by electrons. $Q_{vis} = r \frac{\partial \Omega}{\partial r} W^\phi r$ and $Q_{adv} = Q_{adv}(\Sigma, V, T)$. $Q_{rad}$ depends on the radiative process we choose. In the case of a sub-Keplerian disc $Q_{rad}$ is mainly bremsstrahlung, synchrotron processes and inverse Comptonization due to soft synchrotron photons (Narayan & Yi, 1995, Rajesh & Mukhopadhyay, 2010). In the case of a Keplerian disc $Q_{adv}$ is neglected and it is assumed that the local energy dissipated is radiated thermally. In both cases $Q_{rad} = Q_{rad}(\Sigma, T_e)$. Thus in general we can write

$$Q_{vis}(\alpha, \Sigma, \Omega, T) = Q_{adv}(\Sigma, V, T) + Q_{rad}(\Sigma, T_e) \quad (26)$$

Solving the above equations (25) and (26) simultaneously we can find the ion temperature ($T_i$), electron temperature ($T_e$) and the corresponding radiation output. In the case of perturbed system discussed in the main text the fluctuation in radiation output due to $\alpha$ fluctuations can be computed.

### Appendix B

\[ F_1 = \frac{B_2 f_1 - B_1 f_2}{A_1 B_2 - B_1 A_2} \]
\[ F_2 = \frac{A_1 f_2 - A_2 f_1}{A_1 B_2 - B_1 A_2} \]
\[ F_3 = \frac{B_1 A_3 f_2 - A_3 B_2 f_1 + (A_1 B_2 - A_2 B_1) f_3}{A_1 B_2 C_3 - B_1 A_2 C_3} \quad (27) \]

\[ A_1 = \tilde{v} ; \quad B_1 = (1 + \bar{\sigma}) ; \quad C_1 = 0 \]
\[ A_2 = \tilde{T} ; \quad B_2 = \epsilon_2 (1 + \bar{\sigma}) \tilde{v} ; \quad C_2 = 0 \]
\[ A_3 = \tilde{T}(\alpha + \beta) ; \quad B_3 = 0 ; \quad C_3 = \tilde{T}(1 + \bar{\sigma}) \quad (28) \]
\begin{align*}
  f_1 &= \epsilon_2^{1/2} [(1-j)\tilde{\sigma} + j\tilde{v} + \tilde{\sigma}\tilde{v}] \\
  f_2 &= -\epsilon_2^{3/2} j\tilde{\sigma} - \epsilon_2^{1/2} \tilde{\sigma} - \epsilon_2^{3/2} j\tilde{v}(1 + \tilde{\sigma}) + \epsilon_2^{1/2} \tilde{\sigma} \left[ \frac{2j^2 \epsilon_2}{j+1} + \frac{2(1-q)}{j-2q+1} - \frac{\epsilon_1}{j-2} \right] - \epsilon_2^{1/2} \epsilon_1 \tilde{\sigma} \\
  f_3 &= -(3-q-j)\epsilon_2 \tilde{\sigma} + \epsilon \tilde{\sigma} - \frac{j(1-j)}{1+j} \epsilon^{1/2} \beta + \frac{(j-2q+3)}{(j-2q+1)} \epsilon^{1/2} \beta + \frac{j}{j-2} \epsilon_1 \epsilon_2^{1/2} \beta \\
  &\quad - \frac{2j(1-j)}{1+j} \epsilon_2^{3/2} \beta \tilde{\sigma} + \frac{2(2-q)}{(j-2q+1)} \epsilon_2^{1/2} \beta \tilde{\sigma} + \frac{\epsilon_1 \epsilon_2^{1/2}}{j-2} \beta \tilde{\sigma} + (2-q)\epsilon_2 [\tilde{\sigma} + (1+\tilde{\sigma})\tilde{v}] \quad (29)
\end{align*}