I. INTRODUCTION

Quantum mechanics introduced the intrinsic discreteness of some physical quantities represented by polynomials of continuous quantities. This discrepancy was resolved by the non-commutativity of the canonical observables, the canonical commutation relation (CCR), found by Heisenberg. Another basic feature of quantum mechanics is that every measurement introduces an unavoidable and uncontrollable disturbance on the measured object. Heisenberg’s uncertainty relation interprets the physical content of the non-commutativity as the limitation to our ability of observation by quantifying the amount of unavoidable disturbance caused by measurement.

According to the celebrated paper by Heisenberg [1] in 1927, Heisenberg’s uncertainty relation can be formulated as follows: For every measurement of the position $Q$ of a mass with root-mean-square error $\epsilon(Q)$, the root-mean-square disturbance $\eta(P)$ of the momentum $P$ of the mass caused by the interaction of this measurement always satisfies the relation

$$\epsilon(Q)\eta(P) \geq \frac{\hbar}{2},$$

where $\hbar$ is Planck’s constant divided by $2\pi$. Here, we use the lower bound $\hbar/2$ for the consistency with the modern treatment.

Heisenberg [1] not only explained the physical intuition underlying the above relation by discussing the famous $\gamma$ ray microscope thought experiment, but also claimed that this relation is a straightforward mathematical consequence of the CCR, $QP - PQ = i\hbar$. Heisenberg’s argument runs as follows. He assumed that the mass state just after the measurement of position $Q$ with root-mean-square error $\epsilon(Q)$ is represented by a Gaussian wave function $\psi$ with the spread $Q_1 = \epsilon(Q)$. Then, by Fourier transform of $\psi$, he showed that the momentum spread $P_1$ in this state satisfies the relation

$$Q_1 P_1 \geq \frac{\hbar}{2}.$$  \hspace{1cm} (2)

He identified the momentum disturbance $\eta(P)$ with the momentum spread $P_1$ in the state just after the measurement, and concluded Eq. (1) (see [2] for the detailed discussion).

The mathematical part of his proof leading to Eq. (2) was refined by introducing the notion of standard deviation shortly afterward by Kennard [3]. He explicitly defined the spreads $Q_1$ and $P_1$ to be the standard deviations of position and momentum, $\sigma(Q)$ and $\sigma(P)$. Using Fourier analysis, he proved

$$\sigma(Q)\sigma(P) \geq \frac{\hbar}{2}$$  \hspace{1cm} (3)

in an arbitrary wave function $\psi$.

Kennard’s relation above was later generalized to arbitrary pair of observables by Robertson [4] as follows. For any pair of observables $A$ and $B$, their standard deviations, $\sigma(A)$ and $\sigma(B)$, satisfy the relation

$$\sigma(A)\sigma(B) \geq \frac{1}{2} |\langle \psi, [A, B]\psi \rangle|$$  \hspace{1cm} (4)

in any state $\psi$ with $\sigma(A), \sigma(B) < \infty$. In the above, $[A, B]$ stands for the commutator $AB - BA$, and the standard deviation is defined as $\sigma(A) = (\langle \psi, A^2 \psi \rangle - \langle \psi, A\psi \rangle^2)^{1/2}$, where $\langle \cdot, \cdot, \cdot \rangle$ denotes the inner product; in this paper, every state vector is assumed normalized and the domain of the commutator $[A, B]$ is considered extended appropriately.
Robertson proved the above relation using a simple application of the Schwarz inequality without using Fourier analysis. Thus, it was made clear that Heisenberg’s relation Eq. (2) is a straightforward mathematical consequence of the CCR. However, Heisenberg’s argument that leads to Eq. (1) from Eq. (2) has not been justified for more than 75 years since then.

In fact, Heisenberg himself appears to have changed his position from 1927 to 1929. Around this time, it was already known that an EPR type thought experiment violates Eq. (1). In this case, by the effect of entanglement between two masses, 1 and 2, the position of mass 1 at a time $t$ can be indirectly measured very precisely by measuring the entangled mass 2 without disturbing any observables of mass 1, and hence if the momentum of mass 1 is measured directly just after the position measurement, the momentum at the time $t$ can also be measured very precisely. Heisenberg’s response to this criticism appears in many ways [13, 14, 15, 6, 7, 8, 9, 10, 11, 12], whereas the universal validity of Eq. (1) has been also criticized in many examples including the $\gamma$ ray microscope. Such a view has been accepted for many years as to what are general states and what are general observables. However, the question was left unanswered for long time as to what are general measurements.

Towards this problem, Davies and Lewis (DL) [30] introduced the mathematical formulation of the notion of “instrument” as normalized positive map valued measures, to which we shall refer as DL instruments, and showed that this notion quite generally describes the statistical properties of a general measurement, so that for any sequence of measurements the joint probability distribution of those measurements are determined by their corresponding DL instruments.

In this paper, we start with presenting the above characterization of measurements in more accessible form. In Section II, we propose the two axioms for general measuring apparatuses, the mixing law (of joint output probability) and the extendability axiom, which characterize the statistical description of general measuring apparatuses. In Section III, we introduce mathematical models of measuring processes, called indirect measurement models, and pose the realizability postulate. Under the above three axioms, we show that (i) every apparatus corresponds to a unique CP instrument that describes the statistical properties of that apparatus, (ii) conversely, every CP instrument corresponds to at least one apparatus, (iii) the correspondence is a one-to-one correspon-
dence up to statistical equivalence of apparatuses, and (iv) any apparatus has a statistically equivalent apparatus which is described by an indirect measurement model. Thus, we establish the notion of “instrument” as the function of a measuring apparatus by the mathematical notion “CP instrument” that represents the statistical equivalence class of a measuring apparatus. In the above sections, we are also devoted to explain how the notion of CP instruments integrates such notions as effects, operations, probability operator valued measures (POVMs), and trace-preserving completely positive maps, widely accepted in the field of quantum information [34]. It should be also pointed out that since in an indirect measurement model, the measuring interaction is described purely quantum mechanically, the above results provide a useful approach to explore statistical properties of general quantum measurements using quantum mechanical laws.

In Section IV, a mathematical notion of the distance between POVMs and observables is introduced and the basic properties are explored. Then, we formulate the notion of measurement noise and obtain the basic properties. In particular, we clarify the meaning of noise in the indirect measurement model and show that this notion is equivalent to the distance of the POVM of the apparatus from the observable to be measured, and hence the noise is independent of particular models but depend only on the POVM of the apparatus. In Section V, we formulate the notion of disturbance caused by a measurement and we obtain the basic properties. Disturbance is rather straightforward notion for indirect measurement models, while it is not clear whether it is model independent. We show that the disturbance in a given observable is determined only by the trace-preserving completely positive map that describes the nonselective operation of the apparatus. In Section VI, under the formulation provided as above, Heisenberg’s noise-disturbance uncertainty relation is generalized to a relation that holds for any measuring apparatuses, from which conditions are obtained for measuring apparatuses to satisfy Heisenberg’s relation. In particular, every apparatus with the noise and the disturbance statistically independent from the measured object is proven to satisfy Heisenberg’s relation. Under this formulation, various uncertainty relations are also derived for apparatuses with independent noise, independent disturbance, unbiased noise, and unbiased disturbance as well as noiseless apparatuses and nondisturbing apparatuses. In Section VII, we examine von Neumann’s model of position measurement to show that this model typically satisfies Heisenberg’s relation. Then, we examine the position measurement model that was introduced in Ref. [19] and show that this model violates Heisenberg’s relation uniformly. The above model was shown in Ref. [19] to realize Yuen’s contractive state measurement [16] and to break the standard quantum limit for monitoring free-mass position claimed by Braginsky and collaborators [10, 11, 12] as a consequence of Heisenberg’s relation. An experimental proposal was given in Ref. [35] for realizing the above model in an equivalent linear optical setting. In Section VIII, based on the above model we show that Heisenberg’s relation can be violated even by approximately repeatable position measurements. Some discussions in the final section conclude the present paper.

II. STATISTICS OF GENERAL QUANTUM MEASUREMENTS

A. Postulates for quantum mechanics

Throughout this paper, we assume the following postulates introduced by von Neumann [6] for non-relativistic quantum mechanics without any superselection rules.

Postulate I. (Representations of states and observables) Any quantum system $S$ is associated with a unique separable Hilbert space $\mathcal{H}_S$, called the state space of $S$. Any state of $S$ is represented in one-to-one correspondence by a positive operator $\rho$ with unit trace, called a density operator on $\mathcal{H}_S$. Under a fixed unit system, any observable of $S$ is represented in one-to-one correspondence by a self-adjoint operator $A$ (densely defined) on $\mathcal{H}_S$.

Postulate II. (Schrödinger equation) If system $S$ is isolated in a time interval $(t, t')$, there is a unitary operator $U$, called the time evolution operator, such that if $S$ is in state $\rho$ at time $t$ then $S$ is in state $\rho' = U\rho U^\dagger$ at time $t'$.

Postulate III. (Born statistical formula) Any observable $A$ can be precisely measured in any state $\rho$ in such a way that $A$ takes the value in a Borel set $\Delta$ with probability $\text{Tr}[E^A(\Delta)\rho]$, where $E^A(\Delta)$ is the spectral projection of $A$ corresponding to Borel set $\Delta$.

Postulate IV. (Composition rule) The state space of the composite system $S + S'$ of two systems $S$ and $S'$ is the tensor product $\mathcal{H}_S \otimes \mathcal{H}_S'$ of their state spaces. An observable $A$ in $S$ and an observable $B$ in $S'$ are identified with the observables $A \otimes I$ and $I \otimes B$, respectively, in the system $S + S'$.

For any unit vector $\psi$, the state $\rho = |\psi\rangle\langle\psi|$ is called a vector state represented by $\psi$. In this case, $\psi$ is called a state vector representing the state $\rho$.

Let $p_1, \ldots, p_n$ be a sequence of density operators and let $p_1, \ldots, p_n$ be a probability distribution on $\{1, \ldots, n\}$, i.e., $p_j \geq 0$ for all $j$ and $\sum_j p_j = 1$. We say that a system $S$ is a random sample from the ensemble $\{\{p_j\}, \{p_j\}\}$, iff system $S$ is in state $\rho_j$ with probability $p_j$. In this case, an observable $A$ takes, in a precise measurement, the value in a Borel set $\Delta$ with probability

$$P(\Delta) = \sum_j p_j \text{Tr}[E^A(\Delta)\rho_j].$$

(6)

Let $\rho = \sum_j p_j \rho_j$. By linearity of trace, the density operator $\rho$ satisfies

$$P(\Delta) = \text{Tr}[E^A(\Delta)\rho].$$

(7)
In this case, we have of the mixture of states that the system \( S \) is in the state \( p\rho_1 + (1-p)\rho_2 \), if it is in state \( \rho_1 \) with probability \( p \) and in state \( \rho_2 \) with probability \( 1-p \).

The notion of precise measurements of observables is determined solely by Postulate III (Born statistical formula) without assuming any further conditions on the state change caused by measurement such as the projection postulate stating that the measurement projects the state onto the eigenspace corresponding to the eigenvalue to be obtained.

Postulate III (Born statistical formula) does not assume that the observable has a certain unknown value in the state just before the measurement that is reproduced by a precise measurement, but only requires that the precise measurement statistically reproduces the postulated probability.

A Hilbert space is separable if and only if its dimension is at most countable infinite. Throughout this paper, only separable Hilbert spaces are considered and simply called Hilbert spaces.

Throughout this paper, the word “Borel set” can be safely replaced by the word “interval” only with some modifications on mathematical technicality. Readers not familiar with measure theory are recommended to read with such replacements.

The relation between the present formulation based on spectral projections due to von Neumann [6] and Dirac’s formulation [36] is as follows. If the observable \( A \) has the Dirac type spectral decomposition

\[
A = \sum_{\mu} \sum_{\nu} \mu |\mu,\nu\rangle \langle \mu,\nu | + \int_{\mathbb{R}} \sum_{\nu} |\lambda,\nu\rangle \langle \lambda,\nu | \, d\lambda,
\]

where \( \mu \) varies over the discrete eigenvalues, \( \lambda \) varies over the continuous eigenvalues, and \( \nu \) is the degeneracy parameter, then we have

\[
E^A(\Delta) = \sum_{\mu \in \Delta} \sum_{\nu} |\mu,\nu\rangle \langle \mu,\nu | + \int_{\Delta} \sum_{\nu} |\lambda,\nu\rangle \langle \lambda,\nu | \, d\lambda.
\]

In this case, we have

\[
\text{Tr}[E^A(\Delta)\rho] = \sum_{\mu \in \Delta} \sum_{\nu} \langle \mu | p|\mu,\nu\rangle \langle \mu,\nu | + \int_{\Delta} \sum_{\nu} \langle \lambda | \rho|\lambda,\nu\rangle \langle \lambda,\nu | \, d\lambda.
\]

We do not allow unnormalizable states such as the one described by Dirac’s delta function, since they by no means define the normalized probability distribution of the output of every measurement consistent with the probability theory axiomatized by Kolmogorov [37].

### B. Output probability distributions

Every measuring apparatus has a macroscopic output variable that takes the output of each instance of measurement. The output variable is a random variable, in the sense of classical probability theory [37], the probability distribution of which depends only on the input state, the state of the system to be measured at the instant just before the measurement.

Let \( S \) be a quantum system, to be referred to the object, with state space \( \mathcal{H} \). Let \( A(x) \) be a measuring apparatus with output variable \( x \) to measure the object \( S \). We assume that \( x \) takes values in the real line \( \mathbb{R} \). For any Borel set \( \Delta \) in \( \mathbb{R} \), we shall denote by “\( x \in \Delta \)” the probabilistic event that the output \( x \) takes a value in \( \Delta \). The event “\( x \in \Delta \)” is called the outcome of measurement. The probability distribution of \( x \) on input state \( \rho \) is denoted by \( \text{Pr}\{x \in \Delta|\rho\} \), where \( \Delta \) varies over all Borel subsets of the real line, and called the output probability distribution of \( A(x) \). We shall write \( \text{Pr}\{x \in \Delta|\rho\} = \text{Pr}\{x \in \Delta|\psi\} \), if \( \rho = |\psi\rangle\langle \psi | \).

In this paper, any probability distribution is required to satisfy the positivity, the countable additivity, and the normalization condition [37], so that the output probability distribution satisfies the following conditions. \( x \in \Delta \|ho\}\}}_x \}

\[
\text{Pr}\{x \in \Delta|\rho\} = \sum_j \text{Pr}\{x \in \Delta_j|\rho\} \quad (8)
\]

for any disjoint sequence of Borel sets \( \Delta_1, \Delta_2, \ldots \) with \( \Delta = \bigcup_j \Delta_j \).

(iii) (Normalization condition) \( \text{Pr}\{x \in \mathbb{R}|\rho\} = 1 \).

In addition to the above, it is natural to require that the output probability distribution should satisfy the following postulate.

**Mixing law of output probability:** For any apparatus \( A(x) \), the function \( \rho \mapsto \text{Pr}\{x \in \Delta|\rho\} \) is an affine function of density operators \( \rho \) for every Borel set \( \Delta \), i.e.,

\[
\text{Pr}\{x \in \Delta|\rho_1 + (1-p)\rho_2\} = p \text{Pr}\{x \in \Delta|\rho_1\} + (1-p) \text{Pr}\{x \in \Delta|\rho_2\}, \quad (9)
\]

where \( \rho_1 \) and \( \rho_2 \) are density operators and \( 0 < p < 1 \).

The above postulate is justified as follows. If the system \( S \) is a random sample from the ensemble \( \{\rho_1, \rho_2\}, \{p, 1-p\} \) \}, then the event “\( x \in \Delta \)” occurs with probability \( p \text{Pr}\{x \in \Delta|\rho_1\} + (1-p) \text{Pr}\{x \in \Delta|\rho_2\} \). On the other hand, from Theorem II.1 in this case the system \( S \) is in the state \( p\rho_1 + (1-p)\rho_2 \), so that the above equality should hold.

### C. Probability operator valued measures

In order to characterize the output probability distributions, we need a mathematical definition: A mapping
\( \Pi : \Delta \to \Pi(\Delta) \) of the collection \( \mathcal{B}(\mathcal{R}) \) of Borel subsets in \( \mathcal{R} \) into the space \( \mathcal{L}(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \) is called a \textit{probability operator valued measure (POVM)}, if the following conditions are satisfied:

(i) (Positivity) \( \Pi(\Delta) \geq 0 \) for all \( \Delta \in \mathcal{B}(\mathcal{R}) \).

(ii) (Countable additivity) For any disjoint sequence \( \Delta_1, \Delta_2, \ldots \) of Borel sets with \( \Delta = \bigcup \Delta_j \), we have

\[
\Pi(\Delta) = \sum_j \Pi(\Delta_j),
\]

where the summation is convergent in the weak operator topology, i.e., we have \( \langle \psi | \Pi(\Delta) | \psi \rangle = \sum_j \langle \psi | \Pi(\Delta_j) | \psi \rangle \) for every state vector \( \psi \).

(iii) (Normalization condition) \( \Pi(\mathcal{R}) = I \), where \( I \) is the identity operator on \( \mathcal{H} \).

By the above decomposition, for every trace class operator \( \rho \), we can decompose as

\[
\rho = \sum_{j=1}^{4} \alpha_j \rho_j,
\]

where \( \rho_1, \ldots, \rho_4 \) are density operators and \( \alpha_1, \ldots, \alpha_4 \) are complex numbers; one of the decompositions can be easily found from the spectral decomposition. By the above decomposition, for every trace class operator \( \rho \) and every Borel set \( \Delta \), we can define a complex number \( \Pi(\Delta, \rho) \) by

\[
\Pi(\Delta, \rho) = \sum_{j=1}^{4} \alpha_j \text{Pr}\{x \in \Delta | \rho_j\}.
\]

Then, by linearity of \( \Pi(\Delta, \rho) \) in \( \rho \), we have

\[
\text{Tr}[\Pi(\Delta, \rho)] = \sum_{n,m} \Pi(\Delta, |m\rangle \langle n|)(m|\rho)n = \Pi(\Delta) \sum_{n,m} \langle m|\rho|n\rangle |m\rangle \langle n| = \Pi(\Delta, \rho) = \text{Pr}\{x \in \Delta | \rho\}.
\]

Thus, \( \Pi(\Delta) \) is a unique operator satisfying Eq. (11). Now, conditions (i)–(iii) for \( \Pi \) follow easily from Eq. (11), and hence \( \Pi \) is a POVM. This completes the proof.

The POVM \( \Pi \) defined by Eq. (11) is called the \textit{POVM of apparatus} \( \mathbf{A}(x) \). The operator \( \Pi(\Delta) \) is called the \textit{effect of apparatus} \( \mathbf{A}(x) \) associated with the outcome \( x \in \Delta \). For the general notion of effects, we refer to Kraus [40]. For applications of POVMs to quantum measurement, quantum estimation, and quantum information, we refer to the reader to Helstrom [41], Davies [42], Holevo [43], Peres [44], and Nielsen-Chuang [34].

Let \( \mathbf{A} \) be an observable of system \( \mathbf{S} \). Postulate III (Born statistical formula) naturally leads to the following definition. We say that apparatus \( \mathbf{A}(x) \) satisfies the \textit{Born statistical formula (BSF)} for observable \( \mathbf{A} \) on input state \( \rho \), if we have

\[
\text{Pr}\{x \in \Delta | \rho\} = \text{Tr}[E^\mathbf{A}(\Delta, \rho)]
\]

for every Borel set \( \Delta \). The mapping \( E^\mathbf{A} \) that maps every Borel set \( \Delta \) to the spectral projection \( E^\mathbf{A}(\Delta) \) of \( \mathbf{A} \) corresponding to \( \Delta \) is called the \textit{spectral measure of \( \mathbf{A} \)}. From Postulate III (Born statistical formula), apparatus \( \mathbf{A}(x) \) precisely measures an observable \( \mathbf{A} \) if and only if \( \mathbf{A}(x) \) satisfies the BSF for observable \( \mathbf{A} \) on every input state, and moreover for every observable \( \mathbf{A} \) of \( \mathbf{S} \) there is at least one apparatus that precisely measures \( \mathbf{A} \). From Eqs. (11) and (16), \textit{apparatus} \( \mathbf{A}(x) \) \textit{precisely measures observable} \( \mathbf{A} \) if and only if the POVM \( \Pi \) of \( \mathbf{A}(x) \) \textit{is the spectral measure} \( E^\mathbf{A} \), i.e.,

\[
\Pi = E^\mathbf{A}.
\]

**D. Quantum state reductions**

We have shown that every apparatus is associated with a POVM which determines the output probability distribution. However, POVMs of apparatuses do not determine the joint probability distributions of outputs from successive measurements using several apparatuses. In the following, we introduce the notion of quantum state reduction to determine such joint probability distributions.

Depending on the input state \( \rho \) and the outcome \( x \in \Delta \), let \( \rho(x \in \Delta) \) be the state just after the measurement
conditional upon the outcome \( x \in \Delta \). We assume that for any Borel set \( \Delta \) with \( \Pr\{x \in \Delta|\rho\} > 0 \) the state \( \rho_{(x \in \Delta)} \) is uniquely determined. If \( \Pr\{x \in \Delta|\rho\} = 0 \), the state \( \rho_{(x \in \Delta)} \) is taken to be indefinite and the notation \( \rho_{(x \in \Delta)} \) denotes an arbitrary state. The state \( \rho_{(x \in \Delta)} \) is called the output state given the outcome \( x \in \Delta \) on input state \( \rho \).

The state change from the input state to the output state is generally called the quantum state reduction; while the transformation from the input state to the output probability distribution, namely the state of the macroscopic meter, is called the objectification or the objective state reduction. Those two different notions have been mixed up for long time [46].

Two apparatuses are called statistically equivalent, if they have the same objective state reduction and quantum state reduction, or they have the same output probabilities and the same output states for any outcomes and any input states.

E. Mixing law

For notational convention, we distinguish apparatuses by their output variables. For instance, symbols \( A(x) \), \( A(y) \), and \( A(z) \) denote three apparatuses with output variables \( x \), \( y \), and \( z \), respectively.

The operational meaning of the state \( \rho_{(x \in \Delta)} \) is given as follows. Suppose that a measurement using the apparatus \( A(x) \) on input state \( \rho \) is immediately followed by a measurement using another apparatus \( A(y) \). Then, the joint probability distribution \( \Pr\{x \in \Delta, y \in \Delta'||\rho\} \) of the output variables \( x \) and \( y \) is given by

\[
\Pr\{x \in \Delta, y \in \Delta'||\rho\} = \Pr\{y \in \Delta'||\rho_{(x \in \Delta)}\} \Pr\{x \in \Delta|\rho\}, \tag{18}
\]

since the event \( x \in \Delta \) occurs with probability \( \Pr\{x \in \Delta|\rho\} \) and then the event \( y \in \Delta' \) occurs with probability \( \Pr\{y \in \Delta'||\rho_{(x \in \Delta)}\} \). We shall call the above joint probability distribution the joint output probability distribution of \( A(x) \) and \( A(y) \).

Thus, the joint probability distribution of outputs of successive measurements depends only on the input state of the first measurement and should satisfy the following postulate.

**Mixing law (of joint output probability):** For any apparatuses \( A(x) \) and \( A(y) \), the function \( \rho \mapsto \Pr\{x \in \Delta, y \in \Delta'||\rho\} \) is an affine function of density operators \( \rho \) for every pair of Borel sets \( \Delta, \Delta' \), i.e.,

\[
\Pr\{x \in \Delta, y \in \Delta'||\rho_1 + (1-p)\rho_2\} = p\Pr\{x \in \Delta, y \in \Delta'||\rho_1\} + (1-p)\Pr\{x \in \Delta, y \in \Delta'||\rho_2\}, \tag{19}
\]

where \( \rho_1 \) and \( \rho_2 \) are density operators and \( 0 < p < 1 \).

This requirement is justified as follows. The successive applications of two apparatuses \( A(x) \) and \( A(y) \) to a single system \( S \) can be considered as an application of one apparatus \( A(x, y) \) with two output variables \( x \) and \( y \). Thus, the above postulate follows from the mixing law of output probability (generalized to apparatuses with two output variables).

By substituting \( \Delta' = R \) in Eq. (18) and using the normalization condition \( \Pr\{y \in R|\rho_{(x \in \Delta)}\} = 1 \), we have

\[
\Pr\{x \in \Delta, y \in R|\rho\} = \Pr\{x \in \Delta|\rho\} \tag{20}
\]

for any \( \Delta \) and \( \rho \). Thus, we conclude that the mixing law of joint output probability implies the mixing law of output probability. From now on, the mixing law of joint output probability will be simply referred to as the mixing law.

Consider the case where \( \Delta = R \). The symbol \( \rho \) in \( \Pr\{x \in R, y \in \Delta'||\rho\} \) refers to the state just before \( A(x) \) measurement, while the symbol \( \rho \) in \( \Pr\{y \in \Delta'||\rho\} \) refers to the state just before \( A(y) \) measurement. Thus, the above two probabilities are not generally identical. According to Eq. (18), we have

\[
\Pr\{x \in R, y \in \Delta'||\rho\} = \Pr\{y \in \Delta'||\rho_{(x \in R)}\}. \tag{21}
\]

The above relation characterizes the state \( \rho_{(x \in R)} \).

If \( \Pr\{x \in \Delta|\rho\} = 1 \), by the additivity of probability, we have

\[
\Pr\{x \in \Delta, y \in \Delta'||\rho\} = \Pr\{x \in R, y \in \Delta'||\rho\}, \tag{22}
\]

and hence Eq. (18) leads to

\[
\Pr\{y \in \Delta'||\rho_{(x \in \Delta)}\} = \Pr\{y \in \Delta'||\rho_{(x \in R)}\}. \tag{23}
\]

Since apparatus \( A(y) \) is arbitrary, we have \( \rho_{(x \in \Delta)} = \rho_{(x \in R)} \). Thus, the condition \( x \in \Delta \) makes no selection. In this case, the state change \( \rho \mapsto \rho_{(x \in \Delta)} \) is called the nonselective state change.

From Eq. (18), the conditional probability distribution of \( y \) given \( x \in \Delta \) is determined as

\[
\Pr\{y \in \Delta'|x \in \Delta|\rho\} = \Pr\{y \in \Delta'||\rho_{(x \in \Delta)}\}. \tag{24}
\]

provided that \( \Pr\{x \in \Delta|\rho\} > 0 \). In particular, we have

\[
\Pr\{y \in \Delta'|x \in \Delta|\rho\} = \text{Tr}[B^R(\Delta')\rho_{(x \in \Delta)}], \tag{25}
\]

if \( A(y) \) precisely measures an observable \( B \). The above relation uniquely determines the output state \( \rho_{(x \in \Delta)} \).

F. Operational distributions

In 1970, Davies and Lewis [30] introduced the following mathematical notion for unified description of statistical properties of measurements. A mapping \( I : \Delta \mapsto I(\Delta) \) of \( B(R) \) into the space \( L(\tau c(H)) \) of bounded linear transformations on the space \( \tau c(H) \) of trace-class operators on \( H \) is called an DL instrument, iff the following conditions are satisfied.

(i) (Positivity) \( I(\Delta) \) is a positive linear transformation of \( \tau c(H) \) for every \( \Delta \in B(R) \).
(ii) (Countable additivity) For any disjoint sequence $\Delta_1, \Delta_2, \ldots$ of Borel sets with $\Delta = \bigcup_j \Delta_j$, we have

$$I(\Delta) = \sum_j I(\Delta_j),$$

where the summation is convergent in the strong operator topology of $L(\tau_c(H))$, i.e., $\lim_{n \to \infty} \|I(\Delta)\rho - \sum_{j=1}^n I(\Delta_j)\rho\|_{\tau_c} = 0$ for any $\rho \in \tau_c(H)$, where $\|X\|_{\tau_c} = \text{Tr}\sqrt{X^TX}$ for any $X \in \tau_c(H)$.

(iii) (Normalization condition) $I(R)$ is trace-preserving, i.e.,

$$\text{Tr}[I(R)] = \text{Tr}\rho$$

for any $\rho \in \tau_c(H)$.

For mathematical properties of DL instruments we refer to Davies [42]. One of important consequence from the mixing law is the following unified characterization of output probability distribution and quantum state reduction given in Ref. [47, 48].

**Theorem II.3** The mixing law is equivalent to the following requirement:

For any apparatus $A(x)$ there exists a unique DL instrument $I$ satisfying

$$I(\Delta)\rho = \text{Pr}\{x \in \Delta\|\rho\} \rho(x \in \Delta)$$

for any Borel set $\Delta$ and density operator $\rho$.

A sketch of the proof runs as follows. Let $A(x)$ be an apparatus. For any state $\rho$ and Borel set $\Delta$, we define an operator $I(\Delta, \rho)$ by

$$I(\Delta, \rho) = \text{Pr}\{x \in \Delta\|\rho\} \rho(x \in \Delta).$$

If $\text{Pr}\{x \in \Delta\|\rho\} = 0$, then $I(\Delta, \rho) = 0$, so that $I(\Delta, \rho)$ is determined definitely, despite that $\rho(x \in \Delta)$ is indefinite in this case. Then, for any apparatus $A(y)$ to precisely measure an observable $B$, we have

$$\text{Pr}\{x \in \Delta, y \in \Delta'\|\rho\} = \text{Tr}[E^B(\Delta')\rho(x \in \Delta)] \text{Pr}\{x \in \Delta\|\rho\}$$

$$= \text{Tr}[E^B(\Delta')I(\Delta, \rho)].$$

Thus, by the mixing law, we have

$$I(\Delta, p\rho_1 + (1-p)\rho_2) = pI(\Delta, \rho_1) + (1-p)I(\Delta, \rho_2),$$

where $\rho_1$ and $\rho_2$ are density operators and $0 < p < 1$. Thus, the definition of $I(\Delta, \rho)$ can be extended to all trace class operators $\rho$ by the relation

$$I(\Delta, \rho) = \sum_{j=1}^4 \alpha_j I(\Delta, \rho_j),$$

for any density operators $\rho_1, \ldots, \rho_4$ and complex numbers $\alpha_1, \ldots, \alpha_4$ such that $\rho = \sum_{j=1}^4 \alpha_j \rho_j$. Since every density operator has at least one such decomposition, and since Eq. (29) ensures the uniqueness of extension, the operator $I(\Delta, \rho)$ is well-defined for all Borel sets $\Delta$ and all trace class operators $\rho$. Then, we can see that the mapping that maps $\rho$ to $I(\Delta, \rho)$ is a bounded linear transformation of $\tau_c(H)$ for every Borel set $\Delta$. We denote this mapping by $I(\Delta)$. Then, we define $I$ as the mapping that maps $\Delta$ to $I(\Delta)$. Now, we have only to show three properties (i)–(iii) above; however, these are easy consequences from the positivity, countable additivity, and normalization condition of the probability distribution $\text{Pr}\{x \in \Delta\|\rho\}$. For the detail, see Refs. [47, 48, 49].

The mapping $I(\Delta)$ given above is called the operation of apparatus $A(x)$ associated with the outcome $x \in \Delta$. The mapping $I$ is called the operational distribution of apparatus $A(x)$. Then, the output probability and the output state can be expressed by

$$\text{Pr}\{x \in \Delta\|\rho\} = \text{Tr}[I(\Delta)\rho],$$

$$\rho(x \in \Delta) = \frac{I(\Delta)\rho}{\text{Tr}[I(\Delta)\rho]},$$

where the second relation assumes $\text{Pr}\{x \in \Delta\|\rho\} > 0$. Thus, if $I_x$ and $I_y$ are the operational distributions of $A(x)$ and $A(y)$, respectively, then the joint output probability distribution can be expressed by

$$\text{Pr}\{x \in \Delta, y \in \Delta'\|\rho\} = \text{Tr}[I_y(\Delta')I_x(\Delta)\rho]$$

for any state $\rho$ and any Borel sets $\Delta_1, \Delta_2$.

Both the output probability distribution and the output states are determined by the operational distribution. Thus, two apparatuses are statistically equivalent if and only if they have the same operational distribution.

Let us consider three apparatuses $A(x), A(y), A(z)$ with operational distributions $I_x, I_y, I_z$, respectively, and suppose that in a state $\rho$, these three apparatuses are applied to the system $S$ successively in this order. Then, the joint probability distribution of three outputs $x, y, z$ are given by

$$\text{Pr}\{x \in \Delta, y \in \Delta', z \in \Delta''\|\rho\} = \text{Pr}\{y \in \Delta', z \in \Delta''\|\rho(x \in \Delta)\} \text{Pr}\{x \in \Delta\|\rho\}$$

$$= \text{Tr}[I_z(\Delta'')I_y(\Delta')I_x(\Delta)\rho].$$

Thus, by mathematical induction, we obtain the relation

$$\text{Pr}\{x_1 \in \Delta_1, \ldots, x_n \in \Delta_n\|\rho\} = \text{Tr}[I_n(\Delta_n)\ldots I_1(\Delta_1)\rho]$$

for the joint probability distribution of the output variables $x_1, \ldots, x_n$ of the successive measurements on the initial input state $\rho$ using apparatuses $A(x_1), \ldots, A(x_n)$ in this order with operational distributions $I_1, \ldots, I_n$, respectively. Thus, joint probability distribution of the output variables in any successive measurements are determined by the operational distributions of apparatuses,
so that statistically equivalent apparatuses are mutually exchangeable without affecting the joint probability of their outcomes.

In this subsection, under the mixing law, we have shown that statistical properties of every apparatus are described by a DL instrument and that two apparatuses are statistically equivalent if and only if they corresponds to the same DL instrument.

G. Duality

For any bounded linear transformation $T$ on $\tau c(\mathcal{H})$, the dual of $T$ is defined to be the bounded linear transformation $T^*$ on $\mathcal{L}(\mathcal{H})$ satisfying

$$\text{Tr}[A(T\rho)] = \text{Tr}[(T^*A)\rho]$$

for any $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$. The dual $\mathcal{I}(\Delta)^*$ of the operation $\mathcal{I}(\Delta)$ is called the dual operation associated with $x \in \Delta$; by Eq. (35) it is defined by the relation

$$\text{Tr} [A\mathcal{I}(\Delta)\rho] = \text{Tr} [\{\mathcal{I}(\Delta)^*A\}\rho]$$

for any $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$.

The operator $\mathcal{I}(\Delta)^*I$ obtained by applying the dual operation $\mathcal{I}(\Delta)^*$ to the identity operator is called the effect of operation $\mathcal{I}(\Delta)$. By Eq. (31) and Eq. (35) we have

$$\Pr \{x \in \Delta | \rho\} = \text{Tr} [\{\mathcal{I}(\Delta)^*I\}\rho].$$

(37)

Since $\rho$ is arbitrary, comparing with Eq. (11), we have

$$\Pi(\Delta) = \mathcal{I}(\Delta)^*I$$

(38)

for any Borel set $\Delta$. Thus, the POVM of $A(x)$ is determined by the effects of the operational distribution $\mathcal{I}$.

Let $\mathcal{I}_x$ and $\Omega_y$ be the operational distributions of $A(x)$ and $A(y)$, respectively, and let $\Pi_x$ be the POVM of $A(y)$. Then, we have

$$\text{Tr} [\mathcal{I}_y(\Delta')\mathcal{I}_x(\Delta)\rho] = \text{Tr} [\{\mathcal{I}_y(\Delta')^*I\}\mathcal{I}_x(\Delta)\rho] = \text{Tr} [\{\Pi_y(\Delta')\mathcal{I}_x(\Delta)\rho]\} = \text{Tr} [\{\mathcal{I}(\Delta)^*\Pi_y(\Delta')\rho\}] = \{\mathcal{I}(\Delta)^*\Pi_y(\Delta')\rho\}. \quad (39)$$

Thus, the joint output probability distribution can be expressed by

$$\Pr \{x, y \in \Delta | \rho\} = \text{Tr} [\{\mathcal{I}(\Delta)^*\Pi_y(\Delta')\rho\}].$$

(40)

for any $\Delta, \Delta' \in \mathcal{B} (\mathcal{R})$.

Given the operational distribution $\mathcal{I}$ of an apparatus $A(x)$, the operation $T = \mathcal{I}(\mathcal{R})$ is called the nonselective operation of apparatus $A(x)$ and $T^* = \mathcal{I}(\mathcal{R})^*$ is called the nonselective dual operation of apparatus $A(x)$. The nonselective operation $T$ is trace-preserving, i.e.,

$$\text{Tr}(T\rho) = \text{Tr}\rho$$

for any trace-class operator $\rho$, while the nonselective dual operation $T^*$ is unit-preserving, i.e.,

$$T^* I = I. \quad (42)$$

H. Individual quantum state reductions

It is natural to assume that the output variable $x$ can be read out with arbitrary precision. It follows that each instance of measurement has the output value $x = x$. Let $\rho_{x=x}$ be the state of the system $S$ at the time just after the measurement on input state $\rho$ provided that the measurement yields the output value $x = x$. The individual quantum state reduction caused by the apparatus $A(x)$ is the state change $\rho \rightarrow \rho_{x=x}$ for any real number $x$. The state $\rho_{x=x}$ is called the output state given the output $x = x$ on input state $\rho$.

For distinction, we shall call the previously defined quantum state reduction $\rho \rightarrow \rho_{x=x}$ as the collective quantum state reduction.

If $\Pr \{x \in \{x\} | \rho\} > 0$, the state $\rho_{x=x}$ is determined by the relation

$$\rho_{x=x} = \rho_{x=x}. \quad (43)$$

However, the above relation determines no $\rho_{x=x}$ if the output probability is continuously distributed. In order to determine states $\rho_{x=x}$, the following mathematical notion was introduced in Ref. [50]. A family $\{\rho_{x=x}\} x \in \mathcal{R}$ of states is called a family of posterior states for a DL instrument $\mathcal{I}$ and a prior state $\rho$, if it satisfies the following conditions.

(i) The function $x \rightarrow \rho_{x=x}$ is Borel measurable.

(ii) For any Borel set $\Delta$, we have

$$\mathcal{I}(\Delta)\rho = \int_{\Delta} \rho_{x=x} \cdot \text{Tr}[d\mathcal{I}(x)\rho]. \quad (44)$$

It was shown in Ref. [50] that for any DL instrument $\mathcal{I}$ and prior state $\rho$, there exists a family of posterior states uniquely, where two families are taken identical, if they differ only on a set $\Delta$ such that $\text{Tr}[\mathcal{I}(\Delta)\rho] = 0$.

We define the individual quantum state reduction to be the correspondence from the input state $\rho$ to the family $\{\rho_{x=x}\} x \in \mathcal{R}$ of posterior states for the operational distribution $\mathcal{I}$ of $A(x)$ and prior state $\rho$.

According to the above definition, the individual quantum state reduction and the collective quantum state reduction are related by

$$\rho_{x=x} = \frac{1}{\Pr \{x \in \Delta | \rho\}} \int_{\Delta} \rho_{x=x} \cdot \Pr \{x \in dx | \rho\}. \quad (45)$$

Thus, the individual quantum state reduction and the collective quantum state reduction are equivalent under Eq. (45).

The operational meaning of the individual quantum state reduction is given as follows. Suppose that a measurement using the apparatus $A(x)$ on input state $\rho$ is immediately followed by a measurement using another apparatus $A(y)$. Then, the joint probability distribution $\Pr \{x \in \Delta, y \in \Delta' | \rho\}$ of the output variables $x$ and $y$ is given by Eq. (18). The conditional probability distribution of $y$ given $x \in \Delta$ is defined in probability theory.
by

$$\Pr\{y \in \Delta'|x \in \Delta\} = \frac{\Pr\{x \in \Delta, y \in \Delta' | \rho\}}{\Pr\{x \in \Delta | \rho\}}. \quad (46)$$

However, this definition does not cover the conditional probability distribution of $y$ given $x = x$, since it may happen that $\Pr\{x \in \{x\} | \rho\} = 0$ for every $x$. To avoid this difficulty, in probability theory the conditional probability distribution of $y$ given $x = x$ is defined as the function $x \mapsto \Pr\{y \in \Delta' | x = x | \rho\}$ satisfying

$$\Pr\{x \in \Delta, y \in \Delta' | \rho\} = \int_{\Delta} \Pr\{y \in \Delta' | x = x | \rho\} \Pr\{x \in dx | \rho\}. \quad (47)$$

From Eqs. (45) and (46), we have the following characterization of the individual quantum state reduction,

$$\Pr\{y \in \Delta' | x = x | \rho\} = \Pr\{y \in \Delta' | \rho_{\{x=x\}}\}. \quad (48)$$

Thus, the individual quantum state reduction is determined by the conditional probability distribution of the output of any succeeding measurement conditional upon the individual output.

I. Extendability postulate

In the previous discussions, under a sole hypothesis, the mixing law, we have shown that statistical properties of every apparatus are described by a DL instrument and that two apparatuses are statistically equivalent if and only if they correspond to the same DL instrument. Consequently, the set of statistical equivalence classes of apparatuses are considered to be a subset of the set of DL instruments. In the above sense, “apparatus” denotes a physical system for measurement and “DL instrument” is intended to denote the function of an apparatus or to mathematically denote the statistical equivalence class of an apparatus; of course, we consider that two apparatuses have the same function if and only if they are statistically equivalent. However, up to this point, some DL instruments represent statistical equivalence classes of apparatuses, but some of them may not. In this subsection, we shall eliminate physically irrelevant DL instruments by another physically plausible requirement.

We deal with any apparatus $A(x)$ as a mathematical description of a physical system which has a macroscopic variable $x$ to measure a quantum system $S$. However, even if we sufficiently specify the physical entity of the measuring apparatus described by $A(x)$, an ambiguity still remains as to what is the system to be measured. For example, let $S'$ be another quantum system which is remote from both $S$ and $A(x)$. Then, we always make the composite system $S + S'$. Since we identify an observable $A$ of $S$ with the observable $A \otimes I$ of $S + S'$, the physical apparatus measuring an observable $A$ of the system $S$ is also considered as the one measuring the observable $A \otimes I$ of the extended system $S + S'$. Thus, every real apparatus has the property that if it is described to measure a system $S$, then it is also described to measure the trivially extended system $S + S'$. The above consideration naturally leads to the following postulate.

**Extendability postulate:** For any apparatus $A(x)$ measuring a system $S$ and any quantum system $S'$ not interacting with $A(x)$ nor $S$, there exists an apparatus $A(x')$ measuring system $S + S'$ with the following statistical properties:

$$\Pr\{x' \in \Delta | \rho \otimes \rho'\} = \Pr\{x \in \Delta | \rho\}, \quad (49)$$

$$\rho \otimes \rho'(x' \in \Delta) = \rho(x \in \Delta) \otimes \rho'. \quad (50)$$

for any Borel set $\Delta$, any state $\rho$ of $S$, and any state $\rho'$ of $S'$.

In order to obtain a mathematical condition characterizing the models satisfying the above requirement, we need the following mathematical notions. Let $C^n$ be the Hilbert space of $n$-dimensional vectors. Since every linear operator on a finite dimensional space is bounded and of finite trace, we have $\tau_c(C^n) = L(C^n)$. Then, the space of trace class operators on the tensor product Hilbert space $H \otimes C^n$ is decomposed as $\tau_c(H \otimes C^n) = \tau_c(H) \otimes L(C^n)$. Thus, any linear transformation $T$ on $\tau_c(H)$ can be extended naturally to the linear transformation $T \otimes id_n$ on $\tau_c(H \otimes C^n)$ by

$$T \otimes id_n(\sum \rho_j \otimes \rho'_j) = \sum_j T(\rho_j) \otimes \rho'_j \quad (51)$$

for any $\rho_j \in \tau_c(H)$ and $\rho'_j \in L(C^n)$. Then, $T$ is called completely positive (CP), if $T \otimes id_n$ maps positive operators in $\tau_c(H \otimes C^n)$ to positive operators in $\tau_c(H \otimes C^n)$ for any positive integer $n$. A DL instrument $I$ is called a completely positive (CP) instrument, if $I(\Delta)$ is CP for every $\Delta$. Completely positive maps on $C^*$-algebras were introduced by Stinespring [51]. CP operations were introduced by Kraus [52], and CP instruments were introduced in Ref. [32]. For the general theory of CP maps we refer to Takesaki [53].

**Theorem II.4** Under the mixing law, the extendability postulate implies the following requirement:

The operational distribution of every apparatus should be a CP instrument.

The proof runs as follows. Let $A(x)$, $S$, and $S'$ be those given in the extendability postulate. The state space of $S$ is denoted by $H$ and the state space of $S'$ is supposed to be $C^n$. It has been proven that under the mixing law, every apparatus has its own operational distribution. Let $I$ be the operational distribution of $A(x)$. By the extendability postulate, there is an apparatus $A(x')$ satisfying Eqs. (49) and (50). The mixing law ensures the existence of the operational distribution $I'$ of the apparatus $A(x')$. Then, we have

$$I'(\Delta)(\rho \otimes \rho') = \Pr\{x' \in \Delta | \rho \otimes \rho'\}(\rho \otimes \rho')(x' \in \Delta)$$

$$= \Pr\{x \in \Delta | \rho\} \rho(x \in \Delta) \otimes \rho'$$

$$= [I(\Delta)\rho] \otimes \rho'. \quad (52)$$
It follows that the operation $I'(\Delta)$ of the extended apparatus $A(x')$ associated with $\Delta$ is represented by $I'(\Delta) = I(\Delta) \otimes id_{n}$. Then, by the positivity of operation $I'(\Delta)$, $I(\Delta) \otimes id_{n}$ should be a positive linear transformation on $\tau \epsilon (H \otimes C^{n})$. Since $n$ is arbitrary, we conclude that $I(\Delta)$ is completely positive, so that the proof is completed.

The transpose operation of matrices in a fix basis is a typical example of a positive linear map which is not CP [34]. Let $T$ be a transpose operation on $\tau \epsilon (H \otimes C^{n})$ for $H = C^{m}$, and let $\mu$ be any probability measure on $R$. Then, the relation

$$I(\Delta) \rho = \mu(\Delta) T(\rho) \tag{53}$$

for any Borel set $\Delta$ and any operator $\rho$ defines a DL instrument. However, since $T$ is not CP, the operation $I(\Delta)$ is not CP, so the $I$ is not a CP instrument. The extendability postulate implies that there is no apparatus corresponding to the above DL instrument.

As a consequence of the whole argument of this section, we have reached the following conclusion:

For any apparatus $A(x)$ there exists a unique CP instrument $I$ such that the output probability distribution and the quantum state reduction are described by

$$\Pr \{x \epsilon \Delta|\rho \} = \text{Tr}[I(\Delta)\rho], \tag{54}$$

$$\rho \mapsto \rho_{x \epsilon \Delta} = \frac{I(\Delta)\rho}{\text{Tr}[I(\Delta)\rho]} \tag{55}$$

for any Borel set $\Delta$ and any input state $\rho$, where the second equality assumes $\text{Tr}[I(\Delta)\rho] > 0$.

We have posed two plausible requirements, the mixing law and the extendability postulate, as a set of necessary conditions for every apparatus to satisfy. Under these conditions, we have shown that every apparatus corresponds uniquely to a CP instrument, called the operational distribution, that determines the output probability distributions and the quantum state reduction. Thus, the problem of determining all the possible quantum measurements is reduced to the problem as to which CP instrument corresponds to an apparatus. This problem will be discussed in the next section and it will be shown that every CP instrument corresponds to at least one apparatus. Thus, the statistical equivalence classes of all the possible measuring apparatuses are described in one-to-one correspondence by the CP instruments.

III. MEASURING PROCESSES

A. Indirect measurement models

The disturbance on the object caused by a measurement can be attributed to an interaction, called the measuring interaction, between the object and the apparatus. In this section, we shall consider indirect measurement models in which the measuring interactions are subject to the equations of motions in quantum mechanics [47, 54] and show that even though the indirect measurement models are only a subclass of all the possible quantum measurements, every measurement is statistically equivalent to one of indirect measurement models.

Let $A(x)$ be a measuring apparatus with macroscopic output variable $x$ to measure the object $S$. The measuring interaction turns on at time $t$, the time of measurement, and turns off at time $t + \Delta t$ between object $S$ and apparatus $A(x)$. We assume that the object and the apparatus do not interact each other before $t$ nor after $t + \Delta t$ and that the composite system $S + A(x)$ is isolated in the time interval $(t, t + \Delta t)$. The probe $P$ is defined to be the minimal part of apparatus $A(x)$ such that the composite system $S + P$ is isolated in the time interval $(t, t + \Delta t)$. By minimality, we naturally assume that probe $P$ is a quantum system represented by a Hilbert space $K$. Denote by $U$ the unitary operator on $H \otimes K$ representing the time evolution of $S + P$ for the time interval $(t, t + \Delta t)$.

At the time of measurement the object is supposed to be in an arbitrary input state $\rho$ and the probe is supposed to be prepared in a fixed state $\sigma$. Thus, the composite system $S + P$ is in the state $\rho \otimes \sigma$ at time $t$ and in the state $U(\rho \otimes \sigma)U^{\dagger}$ at time $t + \Delta t$. Just after the measuring interaction, the object is separated from the apparatus, and the probe is subjected to a local interaction with the subsequent stages of the apparatus. The last process is assumed to measure an observable $M$, called the probe observable, of the probe, and the output is represented by the value of the output variable $x$. The above measurement of $M$ is assumed to be local, in the sense that the measuring apparatus for the measurement of $M$ interacts with the probe $P$ but does not interact with the system $S$ [54].

The measuring process of the apparatus $A(x)$ described above is thus modeled by the state space $K$ of the probe $P$, the initial state $\sigma$ of $P$, the time evolution operator $U$ of $S + P$, and the probe observable $M$.

In order to develop the theory of measuring processes described above, we define an indirect measurement model to be any quadruple $(K, \sigma, U, M)$ consisting of a Hilbert space $K$, a density operator $\sigma$ on $K$, a unitary operator $U$ on $H \otimes K$, and an observable $M$ on $K$. An apparatus $A(x)$ is said to be described by an indirect measurement model $(K, \sigma, U, M)$, if the measuring process of $A(x)$ admits the above description with the state space $K$ of the probe, the initial state $\sigma$ of the probe, the time evolution operator $U$ of the object plus probe during the measuring interaction, and the probe observable $M$. Two indirect measurement models $(K, \sigma, U, M)$ and $(K', \sigma', U', M')$ are said to be unitarily equivalent, if there is a unitary transformation $U$ from $K$ onto $K'$ such that $\sigma = \sigma U' U^{\dagger}$, $U = (I \otimes U') (I \otimes U)$, and $M = U' M' U$. We shall not distinguish two unitarily equivalent models, since they may describe the same physical system. An indirect measurement model $(K, \sigma, U, M)$ is called pure, if $\sigma$ is a pure state; we shall write $(K, \sigma, U, M) = (K, \xi, U, M)$, if $\rho = |\xi\rangle \langle \xi|$. In an indirect measurement model, the role of the mea-
suring interaction is well characterized as a transducer, and the subsequent stages as an amplifier. In the Stern-Gerlach measurement of the z-component of spin, the object system models the spin-degree of freedom of the particle, the probe models the orbital-degrees of freedom of the particle, and the probe observable corresponds to the z-component of the linear momentum of the particle. Moreover, the amplification process models the free orbital-motion plus the interaction with a detector, and the output variable corresponds to the z-coordinate of the position of the detector that captures the particle.

**B. Output probability distributions**

Let $A(x)$ be an apparatus with indirect measurement model $(K, \sigma, U, M)$. Since the outcome of this measurement is obtained by the measurement of the probe observable $M$ at time $t + \Delta t$, by the BSF for observable $M$ on input state $U(\rho \otimes \sigma)U^\dagger$ the output probability distribution of $A(x)$ is determined by

$$\Pr\{x \in \Delta||\rho\} = \text{Tr}[[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger].$$

(56)

By linearity of operators and the trace, it is easy to check that the output probability distribution of $A(x)$ satisfies the mixing law of output probability. Thus, by Theorem II.2 there exists the POVM $\Pi$ of $A(x)$. To determine $\Pi$, using the partial trace operation $\text{Tr}_K$ over $K$ we rewrite Eq. (56) as

$$\Pr\{x \in \Delta||\rho\} = \text{Tr}\{\text{Tr}_K[U^\dagger[I \otimes E^M(\Delta)]U(I \otimes \sigma)]\rho\}. \quad \text{ (57)}$$

Since $\rho$ is arbitrary, comparing Eqs. (11) and (57), POVM of $A(x)$ is determined as

$$\Pi(\Delta) = \text{Tr}_K[U^\dagger[I \otimes E^M(\Delta)]U(I \otimes \sigma)]$$

(58)

for any Borel set $\Delta$.

**C. Conditional expectation**

Now we shall introduce a convenient mathematical notion to deal with such formulas as Eq. (58).

Let $H$ and $K$ be two Hilbert spaces and let $\sigma$ be a density operator on $K$. For any $C \in \mathcal{L}(H \otimes K)$, we define the operator $\mathcal{E}_\sigma(C) \in \mathcal{L}(H)$ by the relation

$$\mathcal{E}_\sigma(C) = \text{Tr}_K[C(I \otimes \sigma)].$$

(59)

The operator $\mathcal{E}_\sigma(C) \in \mathcal{L}(H)$ is called the conditional expectation of $C$ in $\sigma$.

The conditional expectations have the following properties easily obtained from the properties of partial trace operation.

(i) For $C = \sum_i A_i \otimes B_i \in \mathcal{L}(H \otimes K)$,

$$\mathcal{E}(C) = \sum_i \text{Tr}[B_i \sigma]A_i.$$  

(60)

(ii) For any $C \in \mathcal{L}(H \otimes K)$ and $\rho \in \tau(H)$,

$$\text{Tr}[\mathcal{E}_\sigma(C)\rho] = \text{Tr}[C(\rho \otimes \sigma)].$$

(61)

(iii) For any $A, D \in \mathcal{L}(H)$ and $B, E \in \mathcal{L}(K)$,

$$\mathcal{E}_\sigma[(A \otimes B)C(D \otimes E)] = A \mathcal{E}_\sigma[(I \otimes B)C(I \otimes E)]D$$

(62)

(iv) The transformation $C \mapsto \mathcal{E}_\sigma(C)$ is a linear transformation from $\mathcal{L}(H \otimes K)$ to $\mathcal{L}(H)$.

(v) If $C \geq 0$, then $\mathcal{E}_\sigma(C) \geq 0$.

Mathematically, Eq. (61) shows that the transformation $C \mapsto \mathcal{E}_\sigma(C)$ is the dual of the trace-preserving CP map $\rho \mapsto \rho \otimes \sigma$ from $\tau(H)$ to $\tau(H \otimes K)$. Thus, $C \mapsto \mathcal{E}(C)$ is a unit-preserving CP map from $\mathcal{L}(H \otimes K)$ to $\mathcal{L}(H)$.

If $\sigma$ is a vector state such that $\sigma = |\xi\rangle\langle\xi|$, we shall write $\mathcal{E}_\xi(C) = \mathcal{E}_\sigma(C)$ and call it the conditional expectation of $C$ in $\xi$. In this case, we have

$$\langle\psi|\mathcal{E}_\xi(C)|\psi\rangle = \langle\psi \otimes \xi|C|\psi \otimes \xi\rangle$$

(63)

for any $\psi \in H$. Thus, we shall also write

$$\mathcal{E}_\xi(C) = |\xi\rangle\langle\xi|C\langle\xi|.$$  

(64)

From Eq. (58), the POVM of apparatus $A(x)$ with indirect measurement model $(K, \sigma, U, M)$ is the conditional expectation of the spectral measure of the observable $M(t + \Delta t)$ in the state $\sigma$, i.e.,

$$\Pi(\Delta) = \mathcal{E}_\sigma[E^{M(t+\Delta t)}(\Delta)] = \mathcal{E}_\sigma[U^\dagger[I \otimes E^M(\Delta)]U].$$

(65)

(66)

**D. Quantum state reductions**

Since the composite system $S + P$ is in the state $U(\rho \otimes \sigma)U^\dagger$ at time $t + \Delta t$, it is standard that the observable $C$ at time $t + \Delta t$ is obtained by tracing out the probe part of that state, and, in fact, this rule is justified by Postulate IV (Composition rule) in Subsection II A. Thus, the nonselective state change is determined by

$$\rho \mapsto \rho' = \text{Tr}_K[U(\rho \otimes \sigma)U^\dagger].$$

(67)

In order to determine the quantum state reduction, suppose that at time $t + \Delta t$ the observer would locally measure an arbitrary observable $B$ of the same object $S$. Let $A(y)$ be an apparatus with output variable $y$ to make a precise measurement of $B$. Since both the $M$ measurement on $P$ and the $B$ measurement on $S$ at time $t + \Delta t$ are local, the joint probability distribution of their outputs satisfies the joint probability formula for the simultaneous measurement of $I \otimes M$ and $B \otimes I$ in the state $U(\rho \otimes \sigma)U^\dagger$ [54].

It follows that the joint output probability distribution of $A(x)$ and $A(y)$ is given by

$$\Pr\{x \in \Delta, y \in \Delta' ||\rho\} = \text{Tr}[[E^B(\Delta') \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger].$$

(68)
Thus, using the partial trace $\text{Tr}_K$ we have

$$
\Pr\{x \in \Delta, y \in \Delta' | \rho\} = \text{Tr}[E_B(\Delta')\text{Tr}_K\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger]\}. 
$$

(69)

On the other hand, from Eq. (18) the same joint output probability distribution can be represented by

$$
\Pr\{x \in \Delta, y \in \Delta' | \rho\} = \text{Tr}[E_B(\Delta')\rho_{(x \in \Delta)}\text{Pr}\{x \in \Delta | \rho\}]
= \text{Tr}[E_B(\Delta')\text{Pr}\{x \in \Delta | \rho\}\rho_{(x \in \Delta)}].
$$

(70)

Since $B$ and $\Delta'$ are chosen arbitrarily, comparing Eqs. (69) and (70), we have

$$
\Pr\{x \in \Delta | \rho\}\rho_{(x \in \Delta)} = \text{Tr}_K\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}
$$

(71)

From Eq. (56), the state $\rho_{(x \in \Delta)}$ is uniquely determined as

$$
\rho_{(x \in \Delta)} = \frac{\text{Tr}_K\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}}{\text{Tr}\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}}
$$

(72)

for any Borel set $\Delta$ with $\Pr\{x \in \Delta | \rho\} > 0$.

The above formula was obtained in Ref. [32]. It should be noted that Eq. (72) does not assume such an illegitimate use of the projection postulate as assuming that the composite system $S + P$ with the outcome $x \in \Delta$ is in the state

$$
\rho_{(x \in \Delta)}^{S+P} = \frac{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger [I \otimes E^M(\Delta)]}{\text{Tr}\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}}
$$

(73)

just after the measurement. It is true that the state $\rho_{(x \in \Delta)}^{S+P}$ leads to the same conclusion by defining $\rho_{(x \in \Delta)} = \text{Tr}_K[\rho_{(x \in \Delta)}^{S+P}]$, but such an assumption is by no means correct, since for any partition $\Delta = \Delta' U \Delta''$ the state $\rho_{(x \in \Delta)}^{S+P}$ should be a mixture of $\rho_{(x \in \Delta')}^{S+P}$ and $\rho_{(x \in \Delta'')}$ but this is not the case for Eq. (73). It is a significant merit of our derivation of Eq. (72) to make no assumptions on the state of the composite system $S + P$ after the measurement.

E. Operational distributions

In the previous subsection, we have confined our attention to the case where the measurement using $A(x)$ is followed by a precise measurement of an observable. Now, we generally suppose that at time $t + \Delta t$ the observer would locally measure the same system $S$ by an arbitrary apparatus $A(y)$. We shall show that the joint output probability distribution of $A(x)$ and $A(y)$ satisfies the mixing law. Let $\Pi_x$ be the POVM of $A(y)$. Under the condition that the measurement of $A(x)$ leads to the outcome $x \in \Delta$, the state at time $t + \Delta t$ is $\rho_{(x \in \Delta)}$.

It follows from Eq. (71) that the joint output probability distribution is given by

$$
\Pr\{x \in \Delta, y \in \Delta' | \rho\} = \Pr\{y \in \Delta' | \rho_{(x \in \Delta)}\} \Pr\{x \in \Delta | \rho\}
= \text{Tr}[\Pi_y(\Delta')\Pr\{x \in \Delta | \rho\}\rho_{(x \in \Delta)}]
= \text{Tr}[\Pi_y(\Delta') \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger].
$$

(74)

By linearity of operators and the trace, it is easy to check that the joint output probability distribution of $A(x)$ and $A(y)$ satisfies the mixing law. Thus, by Theorem II.3 there exists the operational distribution $I$ of $A(x)$.

By Eq. (71) the operational distribution $I$ is determined by

$$
I(\Delta)\rho = \text{Tr}_K\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}
$$

(75)

for any Borel set $\Delta$ and any state $\rho$. From the above relation, it is easy to see that $I$ satisfies the complete positivity; as an alternative characterization, it is well-known that a linear transformation $T$ on $\tau c(H)$ is completely positive if and only if

$$
\sum_{ij} (\xi_i | T(\rho_i | \rho_j)) | \xi_j | \geq 0
$$

(76)

for any finite sequences $\xi_1, \ldots, \xi_n \in H$ and $\rho_1, \ldots, \rho_n \in \tau c(H)$ [53]. Thus, we conclude that the operational distribution of any apparatus with indirect measurement model $(K, \sigma, U, M)$ is a CP instrument.

Let $I$ be the operational distribution of an apparatus $A(x)$ with indirect measurement model $(K, \sigma, U, M)$. From Eq. (75), the nonselective operation $T = I(R)$ is represented as

$$
T \rho = \text{Tr}_K[U(\rho \otimes \sigma)U^\dagger].
$$

(77)

For any bounded operator $A \in \mathcal{L}(H)$, trace-class operator $\rho \in \tau c(H)$, and Borel set $\Delta \in \mathcal{B}(R)$, we have

$$
\text{Tr}[A I(\Delta)\rho] = \text{Tr}[A \text{Tr}_K\{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}]
= \text{Tr}\{[A \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\}
= \text{Tr}\{U^\dagger[A \otimes E^M(\Delta)]U(\rho \otimes \sigma)\}
= \text{Tr}(\text{Tr}_K\{U^\dagger[A \otimes E^M(\Delta)]U(I \otimes \sigma)\}\rho).
$$

(78)

Thus, from Eqs. (36) and (78) we have

$$
\text{Tr}\{[I(\Delta)^* A | \rho]\} = \text{Tr}(\text{Tr}_K\{U^\dagger[A \otimes E^M(\Delta)]U(I \otimes \sigma)\}\rho).
$$

(79)

Since $\rho$ is arbitrary, we have

$$
I(\Delta)^* A = \mathcal{E}_\sigma \{U^\dagger[A \otimes E^M(\Delta)]U\}
$$

(80)

for any bounded operator $A \in \mathcal{L}(H)$. In particular, the POVM $\Pi$ of $A(x)$ satisfies

$$
\Pi(\Delta) = \mathcal{E}_\sigma \{U^\dagger[I \otimes E^M(\Delta)]U\}
$$

(81)
for any Borel set $\Delta$ and the nonselective dual operation $T^*$ satisfies

$$T^* A = \mathcal{E}_s \{ U^\dagger [A \otimes I] U \}$$  \hfill (82)

for any bounded operator $A$.

### F. Canonical Measurements

In this section, we shall consider a model which has been considered to describe a typical measuring process for an arbitrary observable [8, 55]. Let $A(x)$ be an apparatus to measure an observable $A$ of the object $S$ described by a Hilbert space $\mathcal{H}$. The measuring process of $A(x)$ is described by an indirect measurement model $(K, \sigma, U, M)$ as follows. The probe is modeled by a mass of one degree of freedom with position $\hat{q}$ and momentum $\hat{p}$, so that the Hilbert space $K$ is the $L^2$ space of wave functions on $\mathbb{R}$, i.e., $K = L^2(\mathbb{R})$. The measuring interaction is turned on during the time interval $(t, t + \Delta t)$ that couples $A$ and $\hat{p}$, so that the total Hamiltonian in the time interval $(t, t + \Delta t)$ is given by

$$H_{S+P} = H_S \otimes I + I \otimes H_P + K A \otimes \hat{p},$$  \hfill (83)

where $H_S$ and $H_P$ are free Hamiltonians of the object and the probe, respectively, and $K$ is the coupling constant. We suppose that the coupling is so strong that we can neglect the free evolutions and the duration $\Delta t$ is so small as to satisfy $K \Delta t = 1$. Thus, the time evolution of the composite system $S + P$ during the measuring interaction is given by

$$U = \exp \left( \frac{-i}{\hbar} A \otimes \hat{p} \right).$$  \hfill (84)

After the measuring interaction, the apparatus makes a precise measurement of the position $\hat{q}$ of the probe to output the measurement result. Let $\xi$ be the state vector of the probe at the time of the measurement. Then, the above apparatus is modeled by the indirect measurement model $M(A, \xi)$ defined by

$$M(A, \xi) = (L^2(\mathbb{R}), \xi, \exp(-iA \otimes \hat{p}/\hbar), \hat{q}).$$

We shall call this model the canonical model with observable $A$ and probe state $\xi$. In what follows, we shall denote its operational distribution by $I_\xi$ and its POVM by $\Pi_\xi$, respectively.

The Schrödinger equation for the wave function $\Psi_{t+\tau} \in \mathcal{H} \otimes L^2(\mathbb{R})$ in the time interval $t < t + \tau < t + \Delta t$ becomes

$$\frac{\partial \Psi_{t+\tau}(q)}{\partial \tau} = -KA \frac{\partial \Psi_{t+\tau}(q)}{\partial q}. $$  \hfill (85)

Now assume the initial condition

$$\Psi_t(q) = \xi(q) \psi,$$  \hfill (86)

where $\psi \in \mathcal{H}$ is a state vector of the measured system, and the vector valued function $\xi(q) \psi$ represents the tensor product $\psi \otimes \xi$ in $\mathcal{H} \otimes L^2(\mathbb{R})$. The solution of the Schrödinger equation is given by

$$\Psi_{t+\tau}(q) = \xi(qI - \tau K A) \psi,$$  \hfill (87)

where $I$ is the identity operator on $\mathcal{H}$. For $\tau = \Delta t$, we have

$$\Psi_{t+\Delta t}(q) = \xi(qI - A) \psi.$$  \hfill (88)

In order to determine the operational distribution of this measurement, we first obtain the following useful general result.

**Theorem III.1** For any unitary operator $U$ on $\mathcal{H} \otimes L^2(\mathbb{R})$, the indirect measurement model $(L^2(\mathbb{R}), \xi, U, \hat{q})$ has the operational distribution $I_\xi$ determined by

$$I(\Delta) |\psi\rangle \langle \psi| = \int_{\Delta} |U(\psi \otimes \xi)(q)|^2 \, dq$$  \hfill (89)

for any input state $\psi$ and Borel set $\Delta$.

A formal proof using the Dirac notation runs as follows. Let $\psi \in \mathcal{H}$ and $\Delta \in \mathcal{B}(\mathbb{R})$. Then, we have

$$I(\Delta) |\psi\rangle \langle \psi| = \text{Tr}_K \{ |I \otimes E^\dagger(\Delta)| [U(\psi \otimes \xi)(q)|U(\psi \otimes \xi)(q)] \}$$

$$= \text{Tr}_K \{ |I \otimes \int_{\Delta} |q \rangle \langle q| dq|U(\psi \otimes \xi)(q)|U(\psi \otimes \xi)(q)\} \}$$

$$= \int_{\Delta} \text{Tr}_K \{ |I \otimes |q \rangle \langle q| U(\psi \otimes \xi)(q)|U(\psi \otimes \xi)(q)\} \} dq$$

$$= \int_{\Delta} \langle U(\psi \otimes \xi)(q)|U(\psi \otimes \xi)(q) \rangle dq.$$  \hfill (90)

Thus, we obtain Eq. (86).

The statistics of the canonical model $M(A, \xi)$ is determined by the operational distribution $I_\xi$. From Eq. (88),

$$|U(\psi \otimes \xi)(q)| \langle U(\psi \otimes \xi)(q)$$

$$= |\xi(qI - A) \psi \rangle \langle \xi(qI - A) \psi|$$

$$= \xi(qI - A) \psi \langle \psi| \xi(qI - A) \dagger$$

for any $\psi \in \mathcal{H}$, and hence from Eq. (89),

$$I_\xi(\Delta) |\psi\rangle \langle \psi| = \int_{\Delta} \xi(qI - A) |\psi\rangle \langle \psi| \xi(qI - A) \dagger \} \} dq.$$

(90)

By linearity and continuity, we obtain

$$I_\xi(\Delta) \rho = \int_{\Delta} \xi(qI - A) \rho \xi(qI - A) \dagger dq.$$  \hfill (91)

Then, the dual operational distribution is given by

$$I_\xi(\Delta) X = \int_{\Delta} \xi(qI - A) \dagger X \xi(qI - A) \, dq,$$  \hfill (92)
where $X$ is an arbitrary bounded operator. The associated POM $\Pi_\xi$ is given by

$$\Pi_\xi(\Delta) = \int_\Delta |\xi(qI - A)|^2 dq. \quad (93)$$

It follows that the output probability distribution is given by

$$\Pr\{q \in \Delta||\rho\} = \int_\Delta \Tr[|\xi(qI - A)|^2 \rho] dq \quad (94)$$

for any input state $\rho$. From Eq. (91), the output state given the output $q = q$ is obtained as

$$\rho_{(q=q)} = \frac{\xi(qI - A)\rho\xi(qI - A)^\dagger}{\Tr[|\xi(qI - A)|^2 \rho]} \quad (95)$$

If the input state is a vector state $\rho = |\psi\rangle\langle\psi|$, we also have

$$\Pr\{q \in \Delta||\psi\} = \int_\Delta \|\xi(qI - A)\psi\|^2 dq \quad (96)$$

and the output state is also the vector state $\rho_{(q=q)} = |\psi_{(q=q)}\rangle\langle\psi_{(q=q)}|$ such that

$$\psi_{(q=q)} = \frac{\xi(qI - A)\psi}{\|\xi(qI - A)\psi\|} \quad (97)$$

By the function calculus of self-adjoint operator $A$, we have

$$\int_\Delta |\xi(qI - A)|^2 dq = \int_R dE^A(\lambda) \int_R \chi_\Delta(q)|\xi(q - \lambda)|^2 dq, \quad (98)$$

where $\chi_\Delta$ is the characteristic function of the Borel set $\Delta$, i.e., $\chi_\Delta(x) = 1$ if $x \in \Delta$ and $\chi_\Delta(x) = 0$ if $x \notin \Delta$. Let $f(q) = |\xi(-q)|^2$. From Eq. (93), we have

$$\Pi_\xi(\Delta) = \int_R (f * \chi_\Delta)(\lambda) dE^A(\lambda), \quad (99)$$

where $f * \chi_\Delta$ is the convolution, i.e.,

$$(f * \chi_\Delta)(\lambda) = \int_R f(q)\chi_\Delta(\lambda - q) dq. \quad (100)$$

Thus, if the initial state $\xi$ of the probe goes to the position eigenstate $|q = 0\rangle$, the initial position density function $|\xi(q)|^2$ and $f(q)$ approaches to the Dirac delta function, so that the effect $\Pi_\xi(\Delta)$ approaches to the spectral projection $E^A(\Delta)$ of the observable $A$. Similarly, if $\xi$ goes to the position eigenstate, the output state $\rho_{(q=q)}$ goes to the eigenstate of the observable $A$ corresponding to the output of measurement. Thus, the model $\mathcal{M}(A, \xi)$ describes an approximately precise measurement of $A$ that leaves the object in an approximate eigenstate of $A$ corresponding to the output. For the notion of approximate eigenvectors, see Halmos [45]. For the detailed discussion on the statistical properties of the model $\mathcal{M}(A, \xi)$, we refer to Ref. [55].

G. Realizability postulate

In the preceding section, we have considered the requirements that every measuring apparatus should satisfy. However, no postulates were posed as to what measuring apparatus exists, except for Postulate II (Schrödinger equation) requiring that for any observable there is at least one apparatus to make a precise measurement of that observable.

Here, we introduce a postulate that allows to construct another measuring apparatus from the apparatus allowed by Postulate II (Schrödinger equation).

Realizability postulate. For any indirect measurement model $(K, \sigma, U, M)$, there is an apparatus $\mathcal{A}(x)$ described by $(K, \sigma, U, M)$.

From the above postulate and the statistics of an apparatus with indirect measurement model, we conclude that for any indirect measurement model $(K, \sigma, U, M)$, there is an apparatus with the operational distribution $I$ such that

$$I(\Delta)\rho = \Tr_K \{[I \otimes E^M(\Delta)]U(\rho \otimes \sigma)U^\dagger\} \quad (101)$$

for any input state $\rho$.

The above postulate is justified by the assumption that our quantum systems obey no superselection rules. The argument runs as follows. By our assumption, every observable $A$ admits a precise measurement, so that we can assume that there is at least one plausible model of a measuring apparatus for the measurement of $A$. Although this is related to a long standing controversy on the measurement problem, for the simplicity of the coupling, the model $\mathcal{M}(\xi, A)$ has been considered be the first one to be plausible [6, 8]. Now, we shall argue that the realizability of any other indirect measurement model is, in principle, as feasible as the realizability of $\mathcal{M}(\xi, A)$. From the negation of any nontrivial superselection rules, any self-adjoint operator corresponds to an observable and any density operator corresponds to a state, so that we can prepare $P$ in $\sigma$ and measure $M$ within a given experimental error limit. Thus, we have only to show that the unitary operator $U$ is realizable. Since any unitary operator can be represented by an exponential of some observable, we can find an observable $A$ and a parameter $\theta$ such that $U = e^{-i\theta A}$. Then, in order to realize the model $(K, \sigma, U, M)$, we can follow the following steps:

(i) Prepare the probe $P$ in the state $\sigma$ at time $t$.

(ii) Prepare the model $\mathcal{M}(A, \xi)$ in the state $\xi$ near the momentum eigenstate $|\theta\rangle = |\theta\rangle$ at time $t$.

(iii) Couple the composite system $S + P$ to the model $\mathcal{M}(A, \xi)$.

(iv) At the time $t + \Delta t$, the coupling with the model $\mathcal{M}(A, \xi)$ is turned off, and the observer measures the probe observable $M$.

Now, let $P$ be the input state to the model $(K, \sigma, U, M)$. If the state $\xi$ were such that $\xi = |\theta\rangle$, the coupling between the composite system $S + P$ and the model $\mathcal{M}(A, \xi)$ changes the state of the $S + P$ from $\rho \otimes \sigma$ to $U(\rho \otimes \sigma)U^\dagger$. 

by the relation
\[ \exp(-iA \otimes \hat{p})(\rho \otimes \sigma \otimes |\theta\rangle\langle\theta|) \exp(-iA \otimes \hat{p})^\dagger = \exp(-i\theta A)(\rho \otimes \sigma) \exp(-i\theta A)^\dagger \otimes |\theta\rangle\langle\theta|. \]

Thus, the above procedure realizes the model \((K, \sigma, U, M)\) within the given error limit, if the state preparation is sufficiently near to the eigenstate \(|\hat{p} = \theta\rangle\rangle. Therefore, any indirect measurement model can be realized, in principle, by a physical system under a given unit system and in a given experimental error limit. This supports the realizability postulate.

In the preceding subsection, we concluded that the operational distribution of any apparatus with indirect measurement model \((K, \sigma, U, M)\) is a CP instrument. The converse of this assertion was proven by Ref. [32, 33] as follows.

**Theorem III.2 (Realization Theorem)** For any CP instrument \(I\) there exists a pure indirect measurement model \((K, \xi, U, M)\) satisfying
\[
\begin{align*}
I(\Delta)\rho &= \text{Tr}_K\{I \otimes E^M(\Delta)[U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger]\} \\
I(\Delta)^* X &= \xi_x \{U^\dagger [X \otimes E^M(\Delta)]U\}
\end{align*}
\]

for any state \(\rho\) and observable \(X\).

The above theorem has the following two significant corollaries.

**Theorem III.3** For any POVM \(\Pi\) there exists a pure indirect measurement model \((K, \xi, U, M)\) satisfying
\[ \Pi(\Delta) = \xi_x \{U^\dagger[I \otimes E^M(\Delta)]U\} \]

for any Borel set \(\Delta\).

Proof runs as follows. Note that given POVM \(\Pi\) and any fixed state \(\rho_0\), the relation
\[ I(\Delta)\rho = \text{Tr}[\Pi(\Delta)\rho]\rho_0, \]

defines a CP instrument \(I(\Delta)\) with
\[ I(\Delta)^* X = \Pi(\Delta)\rho_0(X). \]

Then, by the relation \(\Pi(\Delta) = I(\Delta)^* I\), the assertion follows immediately from Theorem III.2.

**Theorem III.4** For any trace-preserving CP map \(T\) on \(\tau_c(H)\) there exists a pure indirect measurement model \((K, \xi, U, E)\) with projection \(E\) satisfying
\[
\begin{align*}
T\rho &= \text{Tr}_K\{U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger\}, \\
T^* X &= \xi_x \{U^\dagger(X \otimes I)U\}
\end{align*}
\]

for any Borel set \(\Delta\).

This representation was also given by Kraus [40] independently. Proof runs as follows. Note that given trace-preserving CP map \(T\) and any fixed fixed probability measure \(\mu\), the relation
\[ I(\Delta)\rho = \mu(\Delta)T(\rho). \]
defines a CP instrument \(I(\Delta)\) with
\[ I(\Delta)^* X = \mu(\Delta)T^*(X). \]

Then, by the relation \(T = I(R)\), the assertion follows immediately from Theorem III.2.

We summarize the results.

**Theorem III.5** The operational distribution of any apparatus with indirect measurement model is a CP instrument, and conversely every CP instrument is obtained in this way with a pure indirect measurement model.

From the realization theorem, every CP instrument \(I\) has a pure indirect measurement model \((K, \xi, U, M)\). In this case, from Eqs. (36) and (78) we have
\[
\langle \psi | I(\Delta)^* A | \psi \rangle = \text{Tr}\{[I(\Delta)^* A]|\psi\rangle\langle\psi|\}
\]
\[ = \text{Tr}\{U^\dagger[A \otimes E^M(\Delta)]U(|\psi\rangle\langle\psi| \otimes |\xi\rangle\langle\xi|)\}
\]
\[ = \langle \psi \otimes \xi | U^\dagger[A \otimes E^M(\Delta)]U|\psi \otimes \xi\rangle. \]

Since \(\psi \in \mathcal{H}\) is arbitrary, we have
\[ I(\Delta)^* A = \langle \xi | U^\dagger[A \otimes E^M(\Delta)]U|\xi\rangle. \]

Let \(V\) be the linear transformation from \(\mathcal{H}\) to \(\mathcal{K}\) defined by
\[ V\psi = U(\psi \otimes \xi) \]
for all \(\psi \in \mathcal{H}\). Then, we have
\[ V^\dagger V = I. \]

Now, we have the following useful representation applied to every CP instruments:
\[ I(\Delta)^* A = V^\dagger[A \otimes E^M(\Delta)]V \]

for any \(A \in \mathcal{L}(\mathcal{H})\) and \(\Delta \in \mathcal{B}(\mathcal{H})\).

Under the realizability postulate, any CP instrument has a corresponding apparatus. From the three postulates discussed above, we conclude that the set of statistical equivalence classes of apparatuses is in one-to-one correspondence with the set of CP instruments.

We have also another useful conclusion: Any apparatus is statistically equivalent to an apparatus with indirect measurement model. Thus, when we discuss statistical properties of all the possible measurements, we can assume without any loss of generality that the apparatus under consideration has an indirect measurement model.
IV. NOISE IN MEASUREMENTS

A. Measurements of observables

Let $A(x)$ be a measuring apparatus with indirect measurement model $(X, \sigma, U, M)$. Let $A$ be an observable of the object $S$. As defined previously, $A(x)$ precisely measures $A$ if and only if $A(x)$ satisfies the BSF for observable $A$ on every input state.

In order to clarify the meaning of the above definition, let us examine the case where observable $A$ has a complete orthonormal basis of eigenvectors. Then, we can write

$$A = \sum_{n,\nu} a_n|a_n,\nu\rangle\langle a_n,\nu|,$$  \hspace{1cm} (115)

where $a_n$ varies over all eigenvalues and $\nu$ is the degeneracy parameter. From Eq. (16), it is obvious that if $A(x)$ precisely measures $A$, then $A(x)$ outputs $a_n$ with probability one on input state $|a_n,\nu\rangle$. In what follows, we shall show that the converse is also true. Suppose that $A(x)$ outputs $a_n$ with probability one on input state $|a_n,\nu\rangle$ for all $n$ and $\nu$. Then we have

$$\text{Pr}\{x = a_n|a_m,\nu\} = \delta_{m,n},$$  \hspace{1cm} (116)

so that the POVM $\Pi$ of $A(x)$ satisfies

$$\langle a_m,\nu|\Pi\{a_n\}|a_m,\nu\rangle = \delta_{m,n}.$$  \hspace{1cm} (117)

Consequently,

$$\Pi\{a_n\}|a_m,\nu\rangle = \delta_{m,n}|a_m,\nu\rangle.$$  \hspace{1cm} (118)

Hence, we have

$$\Pi\{a_n\} = \sum_{\nu} |a_n,\nu\rangle\langle a_n,\nu|$$  \hspace{1cm} (119)

for all $n$. Thus, we conclude

$$\Pi(\Delta) = E^A(\Delta)$$  \hspace{1cm} (120)

for all Borel set $\Delta$, so that apparatus $A(x)$ precisely measures observable $A$. Thus, apparatus $A(x)$ precisely measures observable $A$ if and only if it outputs the value of $A$ whenever the object has a definite value of $A$ just before the measurement.

B. Noise in direct measurements

Measurement noise should be defined to be the difference between the true value of the quantity to be measured and the output from the measuring apparatus. This is meaningful in classical mechanics, but there is a difficulty in quantum mechanics, since we cannot always expect that the definite true value exists. However, this does not mean that we cannot define the average amount of noise of the measurement in a given state. In fact, if we can identify the noise with a physical quantity, we can describe the statistical properties of the noise even in quantum mechanics. We shall call this physical quantity the noise operator.

In this subsection, we consider a case where the noise operator can be determined easily. We suppose that in order to measure an observable $A$ in a given state, the observer actually make a precise measurement of another observable $X$, the meter observable, in the same state. In this case, it is natural to define the noise operator to be the observable

$$N(A) = X - A.$$  \hspace{1cm} (121)

Accordingly, the root-mean-square (rms) noise of this measurement in the input state $\psi$ should be defined to be

$$\epsilon(A) = \langle \psi|N(A)^2|\psi\rangle^{1/2} = \langle \psi|(X - A)^2|\psi\rangle^{1/2}.$$  \hspace{1cm} (122)

The above formula is easily rewritten as

$$\epsilon(A) = \|X|\psi\rangle - A|\psi\rangle\|,$$  \hspace{1cm} (123)

and hence the rms noise $\epsilon(A)$ has properties of distance between two vectors $A|\psi\rangle$ and $X|\psi\rangle$.

If the observable $A$ has a definite value $a$ in the state $\psi$, i.e., $A|\psi\rangle = a|\psi\rangle$, we have

$$\epsilon(A) = \langle \psi|(X - a)^2|\psi\rangle^{1/2}$$  \hspace{1cm} (124)

$$= \left( \int_R \langle x - a|^2|\psi|dE^X(x)|\psi\rangle \right)^{1/2},$$  \hspace{1cm} (125)

and hence $\epsilon(A)$ is the root-mean-square of the difference between the output $x$ and the true value $a$.

Let $\langle A \rangle$, $\langle X \rangle$, $\sigma(A)$, and $\sigma(X)$ be the means and the standard deviations of observables $A$ and $X$, respectively, in state $\psi$. Then, we have $\sigma(A) = \|A|\psi\rangle - \langle A|\psi\rangle\|$ and so on. From the triangular inequality for the distance between vectors, we have

$$\|X|\psi\rangle - \langle X|\psi\rangle\| \leq \|X|\psi\rangle - A|\psi\rangle\| + \|A|\psi\rangle - \langle A|\psi\rangle\|$$

$$+ \|A|\psi\rangle - \langle X|\psi\rangle\|$$  \hspace{1cm} (126)

Thus, the geometric inequality Eq. (126) implies the statistical inequality

$$\sigma(X) \leq \epsilon(A) + \sigma(A) + \|\langle X \rangle - \langle A \rangle\|.$$  \hspace{1cm} (127)

From an analogous inequalities for vectors, we have

$$\sigma(A) \leq \epsilon(A) + \sigma(X) + \|\langle X \rangle - \langle A \rangle\|,$$  \hspace{1cm} (128)

$$\epsilon(A) \leq \sigma(A) + \sigma(X) + \|\langle X \rangle - \langle A \rangle\|.$$  \hspace{1cm} (129)

From the above, we have

$$\|\sigma(X) - \sigma(A)\| \leq \epsilon(A) + \|\langle X \rangle - \langle A \rangle\|,$$  \hspace{1cm} (130)

and hence the increase and decrease of the standard deviation of the output from the standard deviation of the
measured observable in the input state is bounded from above by the rms noise plus the bias, the difference of their means.

If $A$ has a definite value and the output is unbiased, i.e., $A|\psi\rangle = \langle X|\psi\rangle$, from Eqs. (127) and (129) we have

$$\epsilon(A) = \sigma(X).$$

(131)

Thus, the rms noise in this case is identical with the fluctuation of the meter observable.

If the output is constant, i.e., $X = x_0I$, from Eq. (129) we have

$$\epsilon(A) \leq \sigma(A) + |x_0 - \langle A\rangle|. \quad (132)$$

This inequality already shows that Heisenberg’s noise-disturbance uncertainty relation does not cover all the possible ways of measuring the same observable $A$. In fact, suppose that in order to measure the position observable $Q$ in a state $\psi$, the observer actually make a precise measurement of a constant observable $X = x_0I$. Then, this measurement can be done without disturbing any observables, in particular, the momentum $P$. However, the rms noise of this measurement is bounded by the finite number $\sigma(Q) + |x_0 - \langle Q\rangle|$ for any state $\psi$ with $||Q|\psi|| < \infty$. Thus, for this measurement the product of the root-mean-square noise and the root-mean-square disturbance vanishes uniformly over all states $\psi$ in the domain of the operator $Q$.

C. Noise in indirect measurements

Let $A(x)$ be an apparatus with indirect measurement model $(K, \sigma, U, M)$. We suppose that the apparatus $A(x)$ is used for measuring an observable $A$ in the state $\rho$ at time $t$. In the Heisenberg picture with the original state $\rho \otimes \sigma$, we write $A^{\text{in}} = A \otimes I$, $M^{\text{in}} = I \otimes M$, $A^{\text{out}} = U^\dagger(A \otimes I)U$, and $M^{\text{out}} = U^\dagger(I \otimes M)U$. In this subsection, for any observable $C$ of $S + P$, the mean value and the standard deviation of $C$ in state $\rho \otimes \sigma$ is denoted by $\langle C \rangle$ and $\sigma(C)$, respectively, i.e.,

$$\langle C \rangle = \text{Tr}[C(\rho \otimes \sigma)], \quad (133)$$

$$\sigma(C) = \text{Tr}[(C - \langle C \rangle)^2(\rho \otimes \sigma)]^{1/2}. \quad (134)$$

The above definition can be rewritten as

$$\sigma(C) = \|C - \langle C \rangle\|_{\text{HS}}, \quad (135)$$

where $\|\cdot\|_{\text{HS}}$ is the Hilbert-Schmidt norm defined by

$$\|X\|_{\text{HS}} = \sqrt{\text{Tr}X^\dagger X} \quad (136)$$

for any Hilbert-Schmidt class operator $X$, i.e., $\text{Tr}X^\dagger X < \infty$. Then, a simple application of the Schwarz inequality for the inner product $\langle X, Y \rangle = \text{Tr}X^\dagger Y$ on Hilbert-Schmidt class operators, we have

$$\sigma(C)\sigma(D) \geq \frac{1}{2} |\text{Tr}[(C - \langle C \rangle)(D - \langle D \rangle)\rho \otimes \sigma]| \geq \frac{1}{2} |\text{Tr}[C, D]_{\rho \otimes \sigma]| \quad (137)$$

for any observables $C, D$ with $\sigma(C), \sigma(D) < \infty$. We shall refer to the last inequality as the Heisenberg-Robertson uncertainty relation for standard deviations or Heisenberg-Robertson relation, for short.

In order to quantify the noise, we introduce the noise operator $N(A)$ of $A(x)$ for measuring $A$. According to the measuring process described in Section III, this measurement can be described as follows: in order to measure the observable $A^{\text{in}}$ in the state $\rho \otimes \sigma$ the observer actually make a precise measurement of the observable $M^{\text{out}}$ in the same state. It follows that we can apply the definition of the noise operator given in the preceding section. Thus, we define the noise operator $N(A)$ of $A(x)$ for measuring $A$ by

$$N(A) = M^{\text{out}} - A^{\text{in}} = U^\dagger(I \otimes M)U - A \otimes I. \quad (138)$$

(139)

The root-mean-square (rms) noise $\epsilon(A, \rho)$, or denoted by $\epsilon(A)$ for short, of $A(x)$ for measuring $A$ on input state $\rho$ is, then, defined by

$$\epsilon(A, \rho) = \langle N(A)^2 \rangle^{1/2}. \quad (140)$$

Using the Hilbert-Schmidt norm, the above definition can be rewritten as

$$\epsilon(A, \rho) = \|M^{\text{out}} - A^{\text{in}}\|_{\text{HS}}. \quad (141)$$

We shall write $\epsilon(A, \rho) = \epsilon(A, \psi)$, if $\rho = |\psi\rangle\langle \psi|$. In order to clarify the meaning of the above definition, suppose that the probe preparation is a pure state $\sigma = |\xi\rangle\langle \xi|$ and let us consider the observable $A$ in Eq. (115). Suppose that the input state is $|a_n, \nu\rangle$. Then, we have

$$N(A)|a_n, \nu\rangle = (M^{\text{out}} - a_n)|a_n, \nu\rangle \langle a_n, \nu| \quad (142)$$

and

$$\epsilon(A, |a_n, \nu\rangle) = \langle (M^{\text{out}} - a_n)^2 \rangle^{1/2}. \quad (143)$$

Thus, $\epsilon(A, |a_n, \nu\rangle)$ stands for the root-mean-square difference between the experimental output $M^{\text{out}}$ and the true value $a_n$ of observable $A$. If $\epsilon(A, |a_n, \nu\rangle) = 0$, we have

$$M^{\text{out}}|a_n, \nu\rangle = a_n|a_n, \nu\rangle \langle a_n, \nu|, \quad (144)$$

so that $A(x)$ outputs $a_n$ with probability one. Thus, we have shown that if $\epsilon(A, \psi) = 0$ for any eigenstates $\psi$ of a purely discrete observable $A$, then $A(x)$ precisely measures $A$.

For a general observable $A$, if the observable $A$ has a definite value $a$ in the state $\rho$, i.e., $A\rho = a\rho$, we have

$$\epsilon(A) = \left(\int_{\mathbb{R}}(x - a)^2\langle dE^{M^{\text{out}}}(x) \rangle\right)^{1/2}, \quad (145)$$

and hence $\epsilon(A)$ is the root-mean-square of the difference between the output $x$ and the true value $a$.
From the similar argument leading to Eqs. (127)–(129), we have
\[ \sigma(M^{\text{out}}) \leq \epsilon(A) + \sigma(A^{\text{in}}) + |\langle M^{\text{out}} \rangle - \langle A^{\text{in}} \rangle|, \]
\[ \sigma(A^{\text{in}}) \leq \epsilon(A) + \sigma(M^{\text{out}}) + |\langle M^{\text{out}} \rangle - \langle A^{\text{in}} \rangle|, \]
\[ \epsilon(A) \leq \sigma(A^{\text{in}}) + \sigma(M^{\text{out}}) + |\langle M^{\text{out}} \rangle - \langle A^{\text{in}} \rangle|. \]
From the above, we also have
\[ |\sigma(M^{\text{out}}) - \sigma(A^{\text{in}})| \leq \epsilon(A) + |\langle M^{\text{out}} \rangle - \langle A^{\text{in}} \rangle|. \]

D. Distance of POVMs from observables

In the preceding subsection, we have defined the root-mean-square noise of measurement using the associated indirect measurement model. Thus, this amount of noise apparently depends on the model; for example, two different models with different boundaries between the apparatus and the observer describing the physically identical apparatus might have different amounts of noise. In the next subsection, we shall show that this is only apparently the case. The root-mean-square noise depends only on the POVM of the apparatus and hence statistically equivalent apparatuses have the same amount of noise. In this subsection, we shall generically introduce a notion of distance between a POVM and an observable, which will play an important role in the study of quantum noise and disturbance in measurements.

Let \( \Pi \) be a POVM on a Hilbert space \( \mathcal{H} \). Let \( f(x) \) be a real Borel function on \( \mathbb{R} \). Denote by \( \int f(x)d\Pi(x) \), or \( \int f d\Pi \) for short, the symmetric operator defined by
\[ \langle \xi | \int f(x)d\Pi(x) | \eta \rangle = \int f(x) d\langle \xi | \Pi(x) | \eta \rangle \]
for any \( \xi, \eta \in \text{dom}(\int f(x)d\Pi(x)) \), where the domain is defined by
\[ \text{dom} \left( \int f(x)d\Pi(x) \right) = \left\{ \xi \in \mathcal{H} \mid \int f(x)^2 d\langle \xi | \Pi(x) | \xi \rangle < \infty \right\}. \]
The first and the second moment operators of \( \Pi \), denoted by \( O(\Pi) \) and \( O^{(2)}(\Pi) \), are defined by
\[ O(\Pi) = \int x d\Pi(x), \]
\[ O^{(2)}(\Pi) = \int x^2 d\Pi(x). \]

By the Naimark theorem [56], there is a Hilbert space \( \mathcal{W} \), an isometry \( V : \mathcal{H} \to \mathcal{W} \), and a self-adjoint operator \( C \) such that
\[ \Pi(\Delta) = V^\dagger E^C(\Delta)V \]
for every Borel set \( \Delta \). We shall call any triple \( (W, V, C) \) satisfying Eq. (154) a Naimark extension of \( \Pi \). By integrating both sides of Eq. (154), we have
\[ O(\Pi) = V^\dagger CV, \]
\[ O^{(2)}(\Pi) = V^\dagger CV^2 V. \]
Since \( V^\dagger CV^2 V \geq V^\dagger CVV^\dagger CV \), we have
\[ O^{(2)}(\Pi) \geq O(\Pi)^2. \]
Let \( A \) and \( \rho \) be an observable and a density operator on \( \mathcal{H} \). We define the distance \( d_\rho(\Pi, A) \) of POVM \( \Pi \) from observable \( A \) in \( \rho \) by
\[ d_\rho(\Pi, A) = \text{Tr}[(O^{(2)}(\Pi) - O(\Pi)^2 + (O(\Pi) - A)^2)\rho]^{1/2} \]
\[ = \text{Tr}[O^{(2)}(\Pi) - O(\Pi)A - AO(\Pi) + A^2\rho]^{1/2}. \]

We shall abbreviate \( d_{\psi}(|\psi\rangle \langle \psi|) \) as \( d_\psi \) for a vector state \( \psi \).

In the case where \( \Pi \) is the spectral measure of an observable \( X \), i.e., \( \Pi = E^X \), we have
\[ O(E^X) = \int x dE^X(x) = X, \]
\[ O^{(2)}(E^X) = \int x^2 d\Pi(x) = X^2. \]
Consequently, we have
\[ d_\rho(E^X, A) = \text{Tr}[(X - A)^2\rho]^{1/2} \]
\[ = \|X\sqrt{\rho} - A\sqrt{\rho}\|_{\text{HS}}. \]
Thus, the distance \( d_\rho \) generalizes the distance of two observables given by \( \|X\sqrt{\rho} - A\sqrt{\rho}\|_{\text{HS}} \).

Now, we have the following properties of the distance \( d_\rho \).

**Theorem IV.1** Let \( A \) and \( \rho \) be an observable and a density operator on \( \mathcal{H} \). For any Naimark extension \( (W, V, C) \) of a POVM \( \Pi \) on \( \mathcal{H} \), we have
\[ d_\rho(\Pi, A) = \|CV^\dagger - VA\sqrt{\rho}\|_{\text{HS}}. \]

**Proof.** The assertion follows from Eqs. (155), (156), (159), and the relations
\[ \|CV^\dagger - VA\sqrt{\rho}\|_{\text{HS}}^{2} = \text{Tr}[(CV^\dagger - VA\sqrt{\rho})^2(CV^\dagger - VA\sqrt{\rho})] 
= \text{Tr}[(V^\dagger C^2V - V^\dagger CVA - AV^\dagger CV + A^2\rho)] \].

**QED**

**Theorem IV.2** A POVM \( \Pi \) on \( \mathcal{H} \) is a spectral measure of an observable \( A \) on \( \mathcal{H} \), i.e., \( \Pi = E^A \) if and only if \( d_\psi(\Pi, A) = 0 \) for any state vector \( \psi \in \mathcal{H} \).
Proof. From Eq. (162), if \( \Pi = E^A \), we have \( d_\rho(\Pi, A) = 0 \) for any \( \rho \). Conversely, suppose that \( d_\rho(\Pi, A) = 0 \) for all state vector \( \psi \in \mathcal{H} \). Let \((W, V, C)\) be a Naimark extension of \( \Pi \). From Theorem IV.1, we have
\[
CV|\psi\rangle\langle\psi| = VA|\psi\rangle\langle\psi|
\]
(164)
for all \( \psi \in \mathcal{H} \). Thus, we have
\[
CV = VA,
\]
(165)
and hence \( CVV^\dagger = VAV^\dagger \). By taking the adjoint of the both sides, we have \( CVV^\dagger = VV^\dagger C \). Since \( VV^\dagger \) is a projection, it follows that all the spectral projections \( E^C(\Delta) \) commutes with \( VV^\dagger \). Since \( V \) is isometry, i.e., \( VV^\dagger = I \), we have
\[
VV^\dagger E^C(\Delta)VV^\dagger E^C(\Delta)V = VV^\dagger E^C(\Delta)V.
\]
(166)
Thus, \( \Pi(\Delta) = VV^\dagger E^C(\Delta)V \) is projection valued. From Eq. (165), we have also
\[
A = V^\dagger CV = \int \lambda d\Pi(\lambda).
\]
(167)
By the uniqueness of the spectral decomposition, we conclude that \( \Pi \) is the spectral measure of \( A \), i.e., \( \Pi = E^A \). QED

Corollary IV.3 For any POVM \( \Pi \) on \( \mathcal{H} \) and any observable \( A \) on \( \mathcal{H} \), the following conditions are equivalent.
(i) \( \Pi = E^A \).
(ii) \( d_\rho(\Pi, A) = 0 \) for any state \( \rho \).
(iii) \( d_\rho(\Pi, A) = 0 \) for a faithful state \( \rho \).
(iv) \( d_{|n\rangle}(\Pi, A) = 0 \) for any \( |n\rangle \) in an orthonormal basis \( \{|n\rangle\} \).
(v) \( d_\rho(\Pi, A) = 0 \) for any state vector \( \psi \in \mathcal{H} \).

Proof. The implication (i) \( \Rightarrow \) (ii) is an immediate consequence of Eq. (162), and the implication (ii) \( \Rightarrow \) (iii) is obvious, since a faithful state exists on any separable Hilbert space. To show the implication (iii) \( \Rightarrow \) (iv), assume that \( d_\rho(\Pi, A) = 0 \) for a faithful state \( \rho \). Let \((W, V, C)\) be a Naimark extension of \( \Pi \). From Theorem IV.1, we have
\[
(CV - VA)\rho = 0.
\]
(168)
By linearity, it follows easily that for any state vector \( \psi \in \mathcal{H} \), we have
\[
(CV - VA)|\psi\rangle\langle\psi| = 0.
\]
(169)
Let \( |1\rangle, |2\rangle, \ldots \) be an orthonormal basis consisting of eigenvectors of \( \rho \). Then, we have \( \rho|n\rangle = p_n|n\rangle \) with \( p_n > 0 \) for all \( n \). Thus, applying the both sides of Eq. (168) to the vector \( p_n^{-1}|n\rangle \), we have
\[
(CV - VA)|n\rangle = 0.
\]
(169)
By Theorem IV.1, we have \( d_{|n\rangle}(\Pi, A) = 0 \) for all \( |n\rangle \), and (iii) \( \Rightarrow \) (iv) has been shown. To show the implication (iv) \( \Rightarrow \) (v), assume that \( d_{|n\rangle}(\Pi, A) = 0 \) for an orthonormal basis \( \{|1\rangle, |2\rangle, \ldots \} \). From Theorem IV.1, we have
\[
(CV - VA)|n\rangle = 0.
\]
(170)
Thus, we conclude \( d_\rho(\Pi, A) = 0 \) for any state \( \psi \), so that (iv) \( \Rightarrow \) (v) has shown. Since the implication (v) \( \Rightarrow \) (i) has been proven in the proof of Theorem IV.2, this completes the proof. QED

Theorem IV.4 Let \( C \) be an observable on Hilbert space \( \mathcal{H} \otimes \mathcal{K} \) and let \( \sigma \) be a density operator on \( \mathcal{K} \). If \( \Pi_C \) is a POVM defined by
\[
\Pi_C(\Delta) = \mathcal{E}_C[E^C(\Delta)]
\]
(172)
for any \( \Delta \in \mathcal{B}(\mathbb{R}) \), then we have
\[
d_\rho(\Pi_C, A) = \|C\sqrt{\rho \otimes \sigma} - A \otimes I\sqrt{\rho \otimes \sigma}\|_{\text{HS}}.
\]
(173)
Proof. By integrating the both sides of Eq. (172), we have
\[
O(\Pi_C) = \text{Tr}_K[C(I \otimes \sigma)],
\]
(174)
\[
O(\Pi_C) = \text{Tr}_K[C^2(I \otimes \sigma)].
\]
(175)
Thus, by the properties of the partial trace, we have
\[
\text{Tr}[O(\Pi_C)A\rho] = \text{Tr}[C(A \otimes I)(\rho \otimes \sigma)],
\]
(176)
\[
\text{Tr}[A^2O(\Pi_C)\rho] = \text{Tr}[(A \otimes I)C(\rho \otimes \sigma)],
\]
(177)
\[
\text{Tr}[O(\Pi_C)^2\rho] = \text{Tr}[C^2(\rho \otimes \sigma)].
\]
(178)
Thus, we have
\[
\|C\sqrt{\rho \otimes \sigma} - A \otimes I\sqrt{\rho \otimes \sigma}\|_{\text{HS}}^2 = \text{Tr}[C^2(\rho \otimes \sigma)] - \text{Tr}[C(A \otimes I)(\rho \otimes \sigma)] - \text{Tr}[(A \otimes I)C(\rho \otimes \sigma)] - \text{Tr}[(A^2 \otimes I)(\rho \otimes \sigma)] + \text{Tr}[O(\Pi_C)^2 - O(\Pi_C)A - AO(\Pi_C) + A^2]\rho].
\]
Thus, Eq. (173) follows from Eq. (159). QED

E. Model independent definition of noise

The following theorem shows that the root-mean-square noise of an apparatus is determined only by its POVM, and hence statistically equivalent apparatuses have the same amount of noise.

Theorem IV.5 Let \( A(x) \) be an apparatus with indirect measurement model \( (K, \sigma, U, M) \). Then, the rms noise \( \epsilon(A, \rho) \) is determined by the POVM \( \Pi \) of \( A(x) \) as
\[
\epsilon(A, \rho) = d_\rho(\Pi, A).
\]
(179)
Proof. The assertion follows from Eq. (141) and Theorem IV.4 for \( C = U^\dagger(I \otimes M)U \). QED
We define the root-mean-square (rms) noise \( \epsilon(A, \rho) \) of apparatus \( A(x) \) for measuring observable \( A \) in state \( \rho \) to be the distance \( d_{\rho}(\Pi, A) \) of the POVM \( \Pi \) of \( A(x) \) from observable \( A \) in state \( \rho \). As above, this definition is consistent with the definition for apparatuses with indirect measurement models.

The following theorem asserts that apparatuses precisely measuring \( A \) and apparatuses with numerically zero rms noise for \( A \) are equivalent notions.

**Theorem IV.6** An apparatus \( A(x) \) precisely measures an observable \( A \) if and only if \( \epsilon(A, \rho) = 0 \) on any input state \( \rho \).

**Proof.** Let \( \Pi \) be the POVM of an apparatus \( A(x) \). Then, \( A(x) \) precisely measures \( A \) if and only if \( \Pi = E^A \). Thus, the assertion follows immediately from Theorem IV.2. QED

Let \( \langle x \rangle \) and \( \sigma(x) \) be the mean and the standard deviation of the output variable \( x \) of the apparatus \( A(x) \) in state \( \rho \). Then, we have

\[
\langle x \rangle = \int R x \Pr\{x \in dx | \rho\}, \quad (180)
\]

\[
\sigma(x) = \left( \int R (x - \langle x \rangle)^2 \Pr\{x \in dx | \rho\} \right)^{1/2}. \quad (181)
\]

From Eqs. (11), (152), and (153), we have

\[
\langle x \rangle = \text{Tr}[O(\Pi)\rho], \quad (182)
\]

\[
\sigma(x) = \left( \text{Tr}[O^2(\Pi)\rho] - \text{Tr}[O(\Pi)^2] \right)^{1/2}. \quad (183)
\]

From Eqs. (146)–(148), we have

\[
\sigma(x) \leq \epsilon(A) + \sigma(A) + |\langle x \rangle - \langle A \rangle|, \quad (184)
\]

\[
\sigma(A) \leq \epsilon(A) + \sigma(x) + |\langle x \rangle - \langle A \rangle|, \quad (185)
\]

\[
\epsilon(A) \leq \sigma(A) + \sigma(x) + |\langle x \rangle - \langle A \rangle|. \quad (186)
\]

In particular, we have

\[
|\sigma(x) - \sigma(A)| \leq \epsilon(A) + |\langle x \rangle - \langle A \rangle|. \quad (187)
\]

In this subsection, we have shown that the rms noise of an apparatus is defined independent of a particular model to describe the measuring process of the apparatus. This suggests that the rms noise can be statistically estimated from the experimental data. In fact, this can be done as follows. Let \( \Pi \) be a POVM and let \( A \) be an observable. By the relation

\[
O(\Pi)A + AO(\Pi) = (A + I)O(\Pi)(A + I) - AO(\Pi)A - O(\Pi), \quad (188)
\]

we have

\[
d_{\rho}(\Pi, A)^2 = \langle \psi | A^2 | \psi \rangle + \langle \psi | O^2(\Pi) | \psi \rangle + \langle \psi | O(\Pi) | A \psi \rangle + \langle A + I | \psi \rangle | O(\Pi) | (A + I) \psi \rangle - \langle (A + I) | \psi \rangle | O(\Pi) | (A + I) \psi \rangle. \quad (189)
\]

In the above, \( \langle \psi | A^2 | \psi \rangle \) is the theoretical mean value of \( A^2 \) in state \( \psi \), \( \langle \psi | O^2(\Pi) | \psi \rangle \) is the mean of the squared output \( x^2 \) in state \( \psi \), and the other terms are the means of the output \( x \) in the respective input states. Thus, the error \( \epsilon(A, \rho) \) can be statistically estimated, in principle, from experimental data of the measurements in states \( \psi \), \( A\psi/\|A\psi\| \), and \( (A + I)\psi/\|(A + I)\psi\| \).

### F. Relations to other approaches

In Refs. [19, 20, 23] the notion of rms noise was previously introduced for a restricted class of measurements. In what follows, we shall show that those definitions are equivalent to the general definition introduced above.

Let \( A \) be an observable of \( S \). A POVM \( \Pi \) of \( S \) is said to be compatible with \( A \), or \( A \) compatible for short, if it satisfies the relation

\[
[\Pi(\Delta_1), E^A(\Delta_2)] = 0 \quad (190)
\]

for all \( \Delta_1, \Delta_2 \in \mathcal{B}(R) \).

Let \( \rho \) be a state. For an \( A \)-compatible POM \( \Pi \), the joint probability distribution of \( \Pi \) and \( A \) in state \( \rho \) is defined by

\[
\mu_\rho^{(\Pi, A)}(\Delta_1 \times \Delta_2) = \text{Tr}[\Pi(\Delta_1)E^A(\Delta_2)\rho] \quad (191)
\]

for any \( \Delta_1, \Delta_2 \in \mathcal{B}(R) \). By Eq. (190) it is easy to see that Eq. (191) defines a unique Borel measure on \( R^2 \). As a notational convention, we shall write

\[
\int_{R^2} f(x, y) d\mu_\rho^{(\Pi, A)}(x, y)
\]

\[
= \int_{R^2} f(x, y) \text{Tr}[d\Pi(x)dE^A(y)\rho] \quad (192)
\]

for a Borel function \( f(x, y) \) on \( R^2 \). If \( f(x)g(y) \) is a \( \mu_\rho^{(\Pi, A)}(x, y) \)-integrable function on \( R^2 \), then we have

\[
\int_{R^2} f(x)g(y)\text{Tr}[d\Pi(x)dE^A(y)\rho]
\]

\[
= \text{Tr} \left[ \left( \int_{R^2} f(x) d\Pi \right) g(A)\rho \right]. \quad (193)
\]

Now, let us assume that the POVM \( \Pi \) of an apparatus \( A(x) \) is compatible with an observable \( A \). Then, we have

\[
\int_{R^2} (x - y)^2 \text{Tr}[d\Pi(x)dE^A(y)\rho]
\]

\[
= \int_{R^2} x^2 \text{Tr}[d\Pi(x)dE^A(y)\rho]
\]

\[
- 2 \int_{R^2} xy \text{Tr}[d\Pi(x)dE^A(y)\rho]
\]

\[
+ \int_{R^2} y^2 \text{Tr}[d\Pi(x)dE^A(y)\rho]
\]

\[
= \text{Tr}[O^2(\Pi)\rho] - 2\text{Tr}[O(\Pi)A\rho] + \text{Tr}[A^2\rho]
\]

\[
= \text{Tr}[(O^2(\Pi) - O(\Pi)^2 + (X - A)^2)\rho]
\]

\[
= d_\rho(\Pi, A).
\]
Thus, by Theorem IV.5, we have

$$\epsilon(A, \rho)^2 = \iint_{\mathbb{R}^2} (x - y)^2 \text{Tr}[d\Pi(x)dE^A(y)\rho].$$ (194)

The above relation shows that the rms noise \(\epsilon(A, \rho)\) represents the root-mean-square deviation of the output \(x\) of the measurement using \(A(x)\) from the output \(y\) of an precise \(A\) measurement using another apparatus \(A(y)\), when these two were made simultaneously in the state \(\rho\). In Ref. [23], the rms noise of an apparatus with \(A\)-compatible POVM was introduced by Eq. (194).

Let us consider the case where the object \(S\) is a one-dimensional mass and the observable to be measured is the position \(\hat{x}\) of the mass. Suppose that the POVM \(\Pi\) of apparatus \(A(x)\) to measure \(\hat{x}\) is compatible with \(\hat{x}\), i.e.,

$$\Pi(\Delta), E^x(\Delta_2) = 0$$ (195)

for all Borel sets \(\Delta_1, \Delta_2\). Under this condition, there is a kernel function \(G(a, x)\) called the resolution kernel, which may be a distribution or a generalized function, such that

$$\Pi(\Delta) = \int_{\Delta} da \int_{\mathbb{R}} G(a, x) dE^x(x)$$ (196)

or

$$d\Pi(a) = da \int_{\mathbb{R}} G(a, x)|x\rangle \langle x| dx$$ (197)

in the Dirac notation. Even if the apparatus \(A(x)\) measures position \(\hat{x}\) approximately, the output probability distribution \(\Pr\{x \in da|\psi\}\) on input state represented by a wave function \(\psi(x)\) is expected to be related to the position distribution \(|\psi(x)|^2\) — from Eq. (197), this relation is expressed in the following form

$$\Pr\{x \in da|\psi\} = da \int_{\mathbb{R}} dx G(a, x)|\psi(x)|^2.$$ (198)

Note that \(G(a, x)\) is independent of a particular wave function \(\psi(x)\). Obviously, \(A(x)\) precisely measures \(\hat{x}\), i.e.,

$$\Pr\{x \in da|\psi\} = |\psi(a)|^2 da$$ (199)

for all \(\psi\), if and only if \(G(a, x) = \delta(x - a)\). Roughly speaking, \(G(a, x)\) is the conditional probability density of the output \(x = a\), given that the mass is in the position \(\hat{x} = x\) at the time of measurement; hence the rms noise \(\epsilon(\hat{x}, |x\rangle)\) of the apparatus \(A(x)\) on input state \(|x\rangle\) should satisfy

$$\epsilon(\hat{x}, |x\rangle)^2 = \int_{\mathbb{R}} da (a - x)^2 G(a, x).$$ (200)

Since our definition of the rms noise excludes the case where the input state is an unnormalizable state like \(|x\rangle\), Eq. (200) cannot be justified. However, if the input mass state is a normalized wave function \(\psi(x)\), the rms noise \(\epsilon(\hat{x}, \psi)\) should satisfy

$$\epsilon(\hat{x}, \psi)^2 = \int_{\mathbb{R}} \epsilon(\hat{x}, |x\rangle)^2 |\psi(x)|^2 dx$$ (201)

or equivalently

$$\epsilon(\hat{x}, \psi)^2 = \int_{\mathbb{R}^2} da (a - x)^2 G(a, x)|\psi(x)|^2 dx.$$ (202)

The following computations show that Eq. (202) is actually derived from our general definition. For \(\rho = |\psi\rangle\langle\psi|\), we have

$$\text{Tr}[\Pi(\Delta_1)E^x(\Delta_2)|\psi\rangle\langle\psi|]$$

$$= \langle\psi|\Pi(\Delta_1)E^x(\Delta_2)|\psi\rangle$$

$$= \int_{\Delta_1} da \int_{\mathbb{R}} G(a, x)|\psi| dE^x(x)E^x(\Delta_2)|\psi\rangle$$

$$= \int_{\Delta_1} da \int_{\Delta_2} G(a, x)|\psi| dE^x(x)|\psi\rangle.$$

Thus, by properties of Lebesgue integral, we have

$$\int_{\mathbb{R}^2} (x - a)^2 \text{Tr}[d\Pi(a)dE^x(x)|\psi\rangle\langle\psi|]$$

$$= \int_{\mathbb{R}^2} (x - a)^2 da G(a, x)|\psi(x)|^2 dx.$$ (203)

Therefore, from Eq. (194) we conclude that Eq. (202) actually holds.

V. DISTURBANCE IN MEASUREMENT

A. Nondisturbing measurements

Let \(A(x)\) be an apparatus with indirect measurement model \((K, \sigma, U, M)\). We should generally say that apparatus \(A(x)\) does not disturb an observable \(B\) of \(S\), if the nonselective state change does not perturb the dynamical evolution of the probability distribution of \(B\), i.e.,

$$\text{Tr}\{[E^B(\Delta) \otimes I]U(\rho \otimes \sigma)U^\dagger\}$$

$$= \text{Tr}[E^B(\Delta)e^{-iH\Delta t/h}\rho e^{iH\Delta t/h}]$$ (204)

for any Borel set \(\Delta\) and any input state \(\rho\), where \(H\) is the Hamiltonian of the system \(S\). In this paper, we assume that the apparatus carries out instantaneous measurements in the sense that the time duration \(\Delta t\) is very small and the coupling between \(S\) and \(P\) is very large so that the free evolution of \(S\) in the time interval \((t, t + \Delta t)\) can be neglected. In this case, we say that apparatus \(A(x)\) does not change the probability distribution of an observable \(B\) of \(S\) on input state \(\rho\), if

$$\text{Tr}\{[E^B(\Delta) \otimes I]U(\rho \otimes \sigma)U^\dagger\} = \text{Tr}[E^B(\Delta)|\psi\rangle\langle\psi|].$$ (205)
or in the Heisenberg picture,
\[
\langle E^{B^\text{out}}(\Delta) \rangle = \langle E^{B^\text{in}}(\Delta) \rangle \tag{206}
\]
for every Borel set \(\Delta\), where we write \(B^\text{in} = B \otimes I\) and \(B^\text{out} = U^\dagger(B \otimes I)U\). We say that apparatus \(A(x)\) does not disturb observable \(B\), or \(A(x)\) is called \(B\)-nondisturbing, if apparatus \(A(x)\) does not disturb the probability distribution of observable \(B\) on any input state \(\rho\) [54].

The next theorem shows that nondisturbing measurements are characterized by nonselective operations, so that it is independent of the particular choice of the indirect measurement model associated with the apparatus.

**Theorem V.1** An apparatus \(A(x)\) with indirect measurement model \((K, \sigma, U, M)\) does not disturb an observable \(B\) if and only if we have
\[
T^*E^B(\Delta) = E^B(\Delta) \tag{207}
\]
for any Borel set \(\Delta\), where \(T\) is the nonselective operation of \(A(x)\).

**Proof.** By the property of the partial trace, we have
\[
\text{Tr} \{ [E^B(\Delta) \otimes I]U(\rho \otimes \sigma)U^\dagger \} = \text{Tr} \{ \text{Tr}_K \{ U^\dagger [E^B(\Delta) \otimes I]U(\rho \otimes \sigma) \} \rho \}. \tag{208}
\]
Thus, Eq. (205) is equivalent to
\[
\text{Tr}(\text{Tr}_K \{ U^\dagger [E^B(\Delta) \otimes I]U(\rho \otimes \sigma) \} \rho) = \text{Tr}[E^B(\Delta)\rho]. \tag{209}
\]
Since \(\rho\) is arbitrary, \(A(x)\) does not disturb \(B\) if and only if
\[
\text{Tr}_K \{ U^\dagger [E^B(\Delta) \otimes I]U(\rho \otimes \sigma) \} = E^B(\Delta) \tag{210}
\]
for any Borel set \(\Delta\). Thus, by Eq. (82) we conclude that \(A(x)\) does not disturb \(B\) if and only if Eq. (207) holds for any Borel set \(\Delta\). QED

**B. Joint measurements with nondisturbing apparatuses**

The relation between simultaneous measurements and nondisturbing measurements were investigated in Ref. [54] and it was proven that any apparatus \(A(x)\) precisely measuring an observable \(A\) does not disturb observable \(B\) if and only if successive precise measurements of observables \(A\) and \(B\), using \(A(x)\) for the \(A\) measurement, satisfies the joint probability formula for simultaneous measurements in the first input state. Here, we shall generalize the above result for apparatuses which do not necessarily make a precise measurement.

**Theorem V.2** Suppose that an apparatus \(A(y)\) precisely measures an observable \(B\) immediately after a measurement using an apparatus \(A(x)\) with POVM \(\Pi_x\). Then, apparatus \(A(x)\) does not disturb observable \(B\) if and only if their joint output probability distribution satisfies
\[
\Pr \{ x \in \Delta, y \in \Delta' | \rho \} = \text{Tr}[\Pi_x(\Delta)E^B(\Delta')\rho] \tag{211}
\]
for any input state \(\rho\) and any Borel sets \(\Delta\) and \(\Delta'\). In this case, \(\Pi\) is necessarily compatible with \(B\).

**Proof.** By the realization theorem, we can assume without any loss of generality that the apparatus \(A(y)\) has a pure indirect measurement model \((K, \sigma, U, M)\).

Since the apparatus \(A(y)\) precisely measures \(B\), the POVM \(\Pi_y\) of \(A(y)\) is such that \(\Pi_y = E^B\). Thus, from Eq. (40), we have
\[
\Pr \{ x \in \Delta, y \in \Delta' | \rho \} = \text{Tr}[\mathcal{I}_x(\Delta)^* [E^B(\Delta')] \rho]. \tag{212}
\]
By Eq. (114) we have
\[
\mathcal{I}_x(\Delta)^*E^B(\Delta') = V^\dagger [E^B(\Delta') \otimes E^M(\Delta)] V \tag{213}
\]
for any \(\Delta, \Delta' \in \mathcal{B}(R)\), where \(V\) is such that \(V\psi = U(\psi \otimes \xi)\) for all \(\psi \in H\). Suppose that apparatus \(A(x)\) does not disturb observable \(B\). Then, we have
\[
\mathcal{I}_x(\Delta)^*E^B(\Delta') = E^B(\Delta'), \tag{214}
\]
and hence
\[
E^B(\Delta') = V^\dagger [E^B(\Delta') \otimes I] V. \tag{215}
\]
Thus, we have
\[
|VE^B(\Delta') - [E^B(\Delta') \otimes I] V|^2 = 0. \tag{216}
\]
Consequently,
\[
VE^B(\Delta') = [E^B(\Delta') \otimes I] V. \tag{217}
\]
By Eq. (213), we have
\[
\mathcal{I}_x(\Delta)^*E^B(\Delta') = \text{Tr} \{ \mathcal{I}_x(\Delta)^*[E^B(\Delta')] \rho \} = \text{Tr}[E^B(\Delta') \rho]. \tag{218}
\]
Therefore, Eq. (211) follows. Conversely, suppose that Eq. (211) holds for any input state \(\rho\) and any Borel sets \(\Delta\) and \(\Delta'\). Let \(\Delta = R\). We have
\[
\text{Tr} \{ \mathcal{I}_x(\Delta)^*[E^B(\Delta')] \rho \} = \text{Tr}[E^B(\Delta') \rho] \tag{219}
\]
for any state \(\rho\). Thus, we conclude
\[
\mathcal{I}_x(\Delta)^*[E^B(\Delta')] = E^B(\Delta') \tag{220}
\]
for any Borel set \(\Delta'\), and the assertion follows from Eq. (207). QED
C. Disturbance in indirect measurement models

In order to quantify the disturbance, we introduce the disturbance operator \( D(B) \) of apparatus \( A(x) \) for observable \( B \) defined by

\[
D(B) = B^{\text{out}} - B^{\text{in}} = U^\dagger(B \otimes I)U - B \otimes I. \tag{221}
\]

The root-mean-square (rms) disturbance \( \eta(B, \rho) \) of observable \( B \) by apparatus \( A(x) \) on input state \( \psi \) is, then, defined by

\[
\eta(B, \rho) = \langle D(B) \rangle^{1/2}. \tag{223}
\]

We shall write \( \eta(B, \rho) = \eta(B, \psi) \) if \( \rho = |\psi\rangle\langle\psi| \). The above definition can be rewritten as

\[
\eta(B, \rho) = \|B^{\text{out}} - B^{\text{in}}\|_{HS}. \tag{224}
\]

From Eq. (222) we have

\[
D(B) = U^\dagger(B \otimes I,U). \tag{225}
\]

Thus, we have

\[
\eta(B, \rho) = \|B \otimes I,U\|^{1/2}, \tag{226}
\]

and \( \eta(B, \rho) = 0 \) if and only if \( [B \otimes I,U] \rho \otimes \sigma = 0 \).

D. Model independent definition of disturbance

In the preceding subsection, we have defined the rms disturbance of apparatus using the associated indirect measurement model. In what follows, we shall show that the rms disturbance is determined by the nonselective operation of the apparatus and hence depends only on the statistical equivalence class of the apparatus.

The following theorem shows that the rms disturbance of an apparatus determined only by its nonselective operation.

**Theorem V.3** Let \( A(x) \) be an apparatus with indirect measurement model \( (K, \sigma, U,M) \). Then, the rms disturbance \( \eta(B, \rho) \) is determined by the nonselective operation \( T \) as

\[
\eta(B, \rho) = d_\rho(T^*E_B^B, B), \tag{227}
\]

where \( T^*E_B^B \) stands for the POVM defined by

\[
(T^*E_B^B)(\Delta) = T^*[E_B^B(\Delta)] \tag{228}
\]

for any \( \Delta \in \mathcal{B}(\mathcal{R}) \).

**Proof.** Let \( \Pi \) be the POVM defined by

\[
\Pi(\Delta) = E_\sigma\{U^\dagger[E_B^B(\Delta) \otimes I]U\} \tag{229}
\]

for any \( \Delta \in \mathcal{B}(\mathcal{R}) \). Then, by Theorem IV.4, we have

\[
d_\rho(\Pi, B) = \|U^\dagger(B \otimes I)U \sqrt{\rho \otimes \sigma} - B \otimes I \sqrt{\rho \otimes \sigma}\|_{HS}. \tag{230}
\]

and hence by Eq. (224), we have

\[
d_\rho(\Pi, B) = \eta(B, \rho) \tag{231}
\]

On the other hand, by Eq. (82) we have

\[
\Pi(\Delta) = T^*E_B^B(\Delta) \tag{232}
\]

for any \( \Delta \in \mathcal{B}(\mathcal{R}) \). Thus, the assertion follows from Eq. (231) and Eq. (232). QED

We generally define the root-mean-square (rms) disturbance \( \eta(B, \rho) \) of an observable \( B \) by any apparatus \( A(x) \) in state \( \rho \) to be the distance \( d_\rho(T^*E_B^B, B) \). As above, this definition is consistent with the definition for apparatuses with indirect measurement models.

One of the fundamental properties of the rms disturbance is that non-disturbing apparatuses and apparatuses with zero disturbances are equivalent notions, as ensured by the following theorem.

**Theorem V.4** The apparatus \( A(x) \) does not disturb observable \( B \) if and only if \( \eta(B, \rho) = 0 \) for any state \( \rho \).

**Proof.** From Theorem V.3, \( \eta(B, \rho) = 0 \) if and only if \( d_\rho(T^*E_B^B, B) = 0 \). Thus, from Theorem IV.2, \( \eta(B, \rho) = 0 \) for all \( \rho \) if and only if \( T^*E_B^B = E_B^B \). By Theorem V.1, the last condition holds if and only if \( A(x) \) does not disturb \( B \). The proof is completed. QED

VI. NEW FORMULATION OF UNCERTAINTY PRINCIPLE

A. Universally valid uncertainty relation

Under the general definitions of rms noise and rms disturbance introduced in the preceding sections, we can rigorously investigate the validity of Heisenberg’s noise-disturbance uncertainty relation. For this purpose, let \( A(x) \) be an apparatus with indirect measurement model \( (K, \sigma, U,M) \). Let \( A \) and \( B \) be two observables of the object. Recall that the noise operator \( N(A) \) and the disturbance operator \( D(A) \) satisfy

\[
M^{\text{out}} = A^{\text{in}} + N(A), \tag{233}
\]

\[
B^{\text{out}} = B^{\text{in}} + D(B). \tag{234}
\]

Since \( M \) and \( B \) are observables in different systems, we have \( [M^{\text{out}}, B^{\text{out}}] = 0 \), and hence we obtain the following commutation relation for the noise operator and the disturbance operator,

\[
[N(A), D(B)] + [N(A), B^{\text{in}}] + [A^{\text{in}}, D(B)] = -[A^{\text{in}}, B^{\text{in}}]. \tag{235}
\]
Taking the moduli of means in the original state $\rho \otimes \sigma$ of the both sides and applying the triangular inequality, we have
\[
|\langle [N(A), D(B)] \rangle| + |\langle [N(A), B^{\text{in}}] \rangle + |\langle A^{\text{in}}, D(B) \rangle| \geq |\text{Tr}([A, B] \rho)|. \tag{236}
\]
Since the variance is not greater than the mean square, we have
\[
\epsilon(A, \rho) \geq \sigma(N(A), \rho \otimes \sigma), \tag{237}
\]
\[
\eta(B, \rho) \geq \sigma(D(B), \rho \otimes \sigma), \tag{238}
\]
and hence by the Heisenberg-Robertson relation, we have
\[
\epsilon(A, \rho)\eta(B, \rho) \geq \frac{1}{2}|\langle [N(A), D(B)] \rangle|. \tag{239}
\]
Thus, we obtain the universally valid noise-disturbance uncertainty relation for the pair $(A, B)$,
\[
\epsilon(A, \rho)\eta(B, \rho) + \frac{1}{2}|\langle [N(A), B^{\text{in}}] \rangle + |\langle A^{\text{in}}, D(B) \rangle| \geq \frac{1}{2}|\text{Tr}([A, B] \rho)|. \tag{240}
\]

The above relation immediately gives rigorous conditions on what apparatus satisfies Heisenberg’s noise-disturbance uncertainty relation. Some conditions are listed in the following.

**Theorem VI.1** Let $A$ and $B$ be a pair of observables. An apparatus $A(x)$ with indirect measurement model $(K, \sigma, U, M)$ satisfies Heisenberg’s noise-disturbance uncertainty relation, i.e.,
\[
\epsilon(A, \rho)\eta(B, \rho) \geq \frac{1}{2}|\text{Tr}([A, B] \rho)|
\]
for any state $\rho$ for which all the relevant terms are finite, if one of the following conditions holds:
(i) The noise operator commutes with $B^{\text{in}}$ and the disturbance operator commutes with $A^{\text{in}}$, i.e.,
\[
[N(A), B^{\text{in}}] = 0, \tag{241}
\]
\[
[D(B), A^{\text{in}}] = 0. \tag{242}
\]
(ii) The noise operator and the disturbance operator belong to the probe system, i.e., there are two observables $N$ and $D$ on $K$ such that
\[
N(A) = I \otimes N, \tag{243}
\]
\[
D(B) = I \otimes D. \tag{244}
\]

**B. Model-Independent formulation**

The above characterizations are easily obtained, but depend on the model. In order to obtain intrinsic characterizations of apparatuses satisfying Heisenberg’s relation, we reformulate the universally valid relation in terms of model independent notions.

Let $A(x)$ be an apparatus with POVM $\Pi$ and nonselective operation $T$. We now introduce the mean noise operator $n(A)$ for observable $A$ and the mean disturbance operator $d(B)$ for observable $B$ defined by
\[
n(A) = O(\Pi) - A, \tag{245}
\]
\[
d(B) = T^*(B) - B \tag{246}
\]
The meaning of the above operators will be clarified in the following argument.

By the realization theorem, there is an indirect measurement model $(K, \sigma, U, M)$ such that
\[
\Pi(\Delta) = \mathcal{E}_\sigma[M^{\text{out}}(\Delta)], \tag{247}
\]
\[
T^*(X) = \mathcal{E}_\sigma[U^\dagger(X \otimes I)U] \tag{248}
\]
for any Borel set $\Delta$ and any observable $X$ on $\mathcal{H}$. Then, we also have
\[
O(\Pi) = \mathcal{E}_\sigma(M^{\text{out}}), \tag{249}
\]
\[
T^*(B) = \mathcal{E}_\sigma(B^{\text{out}}). \tag{250}
\]
Thus,
\[
\mathcal{E}_\sigma[N(A)] = \mathcal{E}_\sigma[M^{\text{out}} - A^{\text{in}}] = O(\Pi) - A, \tag{251}
\]
and
\[
\mathcal{E}_\sigma[D(B)] = \mathcal{E}_\sigma[B^{\text{out}} - B^{\text{in}}] = T^*(B) - B. \tag{252}
\]
Thus, we have
\[
n(A) = \mathcal{E}_\sigma[N(A)], \tag{253}
\]
\[
d(B) = \mathcal{E}_\sigma[D(B)]. \tag{254}
\]

Note that for any observable $C$ on $\mathcal{H} \otimes K$ and any observable $X$ on $\mathcal{H}$, we have
\[
\mathcal{E}_\sigma\{[C, X \otimes I]\} = [\mathcal{E}_\sigma(C), X]. \tag{255}
\]
By the relations,
\[
\text{Tr}\{[N(A), B^{\text{in}}] \rho \otimes \sigma\} = \text{Tr}\{\mathcal{E}_\sigma([N(A), B^{\text{in}}]) \rho\} = \text{Tr}\{[\mathcal{E}_\sigma(N(A)), B] \rho\},
\]
we have
\[
\langle [N(A), B^{\text{in}}]\rangle = \text{Tr}\{[n(A), B] \rho\}. \tag{256}
\]
Similarly, we also have
\[
\langle [A^{\text{in}}, D(B)]\rangle = \text{Tr}\{[A, d(B)] \rho\}. \tag{257}
\]
Therefore, by substituting Eqs. (256) and (257), we obtain the model-independent universally valid noise-disturbance uncertainty relation as follows.
Theorem VI.2 Let $A$ and $B$ be a pair of observables. Every apparatus $\mathbf{A}(x)$ satisfies the relation
\[\epsilon(A, \rho)\eta(B, \rho) + \frac{1}{2}\text{Tr}[[n(A), B]\rho] + \text{Tr}[[A, d(B)]\rho] \geq \frac{1}{2}\text{Tr}([A, B]\rho)]\]
(258)
for any state $\rho$ for which all the relevant terms are finite, where $\Pi$ is the POVM of $\mathbf{A}(x)$ and $T$ is the nonselective operation of $\mathbf{A}(x)$.

Before stating the conditions for Heisenberg’s relation, we introduce some terminology. Let $A$ and $B$ be observables of the system $\mathbf{S}$ to be measured. We say that an apparatus $\mathbf{A}(x)$ makes an unbiased measurement of $A$, if the mean output is equal to the mean of the observable $A$ in the input state, i.e.,
\[\langle x \rangle = \langle A^{\text{in}} \rangle \]
(259)
for any input state $\rho$. From Eq. (182), this is the case if and only if the first moment operator of $\Pi$ is equal to $A$, i.e.,
\[O(\Pi) = A. \]
(260)
We say that an apparatus $\mathbf{A}(x)$ makes an unbiased disturbance of $B$, if $\mathbf{A}(x)$ does not change the mean of $B$, i.e.,
\[\langle B^{\text{in}} \rangle = \langle B^{\text{out}} \rangle \]
(261)
for any input state $\rho$. Since the state just after the measurement is $T(\rho)$, we have
\[\langle B^{\text{out}} \rangle = \text{Tr}[T^*(B)\rho], \]
(262)
by the relation $\text{Tr}[BT(\rho)] = \text{Tr}[T^*(B)\rho]$. The above relation is also obtained from indirect measurement models. In fact, if $\mathbf{A}(x)$ has an indirect measurement model $(K, \sigma, U, M)$, then, from Eq. (250) we have
\\[
\langle B^{\text{out}} \rangle &= \text{Tr}[B^{\text{out}}(\sigma \otimes \rho)] \\
&= \text{Tr}[\mathcal{E}_\sigma(B^{\text{out}})\rho] \\
&= \text{Tr}[T^*(B)\rho].
\]
Since $\langle B^{\text{in}} \rangle = \text{Tr}[B\rho]$ and $\rho$ is arbitrary, we conclude that apparatus $\mathbf{A}(x)$ makes an unbiased disturbance of $B$, if and only if
\[T^*(B) = B. \]
(263)
We say that $\mathbf{A}(x)$ has statistically independent noise for $A$, if the mean noise $\langle x \rangle - \langle A^{\text{in}} \rangle$ does not depend on the input state $\rho$, or equivalently, if the mean noise operator $n(A)$ is a constant operator, i.e., $n(A) = rI$ for some $r \in \mathbb{R}$. We say that $\mathbf{A}(x)$ has statistically independent disturbance for $B$, if the mean disturbance $\langle B^{\text{out}} \rangle - \langle B^{\text{in}} \rangle$ does not depend on the input state $\rho$, or equivalently, if the mean disturbance operator $d(B)$ is a constant operator, i.e., $d(B) = rI$ for some $r \in \mathbb{R}$.

The model-independent universally valid noise-disturbance uncertainty relation leads to rigorous conditions on what apparatus satisfies Heisenberg’s noise-disturbance uncertainty relation, as follows.

Theorem VI.3 Let $A$ and $B$ be a pair of observables. An apparatus $\mathbf{A}(x)$ satisfies Heisenberg’s noise-disturbance uncertainty relation, i.e.,
\[\epsilon(A, \rho)\eta(B, \rho) \geq \frac{1}{2}\text{Tr}([A, B]\rho)]\]
for any state $\rho$ for which all the relevant terms are finite, if one of the following conditions holds:
(i) The mean noise operator commutes with $B$ and the mean disturbance operator commutes with $A$, i.e.,
\[n(A), B] = 0, \quad [d(B), A] = 0. \]
(264)
(265)
(ii) The apparatus $\mathbf{A}(x)$ has both statistically independent noise for $A$ and statistically independent disturbance for $B$.
(iii) The apparatus $\mathbf{A}(x)$ makes both unbiased measurement of $A$ and unbiased disturbance of $B$.

C. Generalized noise-disturbance uncertainty relation

In order to obtain the trade-off among the rms noise $\epsilon(A, \rho)$, the disturbance $\eta(B, \rho)$, and the pre-measurement uncertainties $\sigma(A, \rho)$ and $\sigma(B, \rho)$, we apply the Heisenberg-Robertson relation to all terms in the left-hand-side of the universally valid noise-disturbance uncertainty relation. Then, we now obtain the generalized noise-disturbance uncertainty relation as follows.

Theorem VI.4 For any apparatus $\mathbf{A}(x)$ and observables $A$ and $B$, we have the relation
\[\epsilon(A, \rho)\eta(B, \rho) + \epsilon(A, \rho)\sigma(B, \rho) + \sigma(A, \rho)\eta(B, \rho) \geq \frac{1}{2}\text{Tr}([A, B]\rho)]\]
(266)
for any state $\rho$ for which all the relevant terms are finite.

Under the finite energy constraint, i.e., $\sigma(Q), \sigma(P) < \infty$, the above relation excludes the possibility of having both $\epsilon(Q) = 0$ and $\eta(P) = 0$. However, $\epsilon(Q) = 0$ is possible with $\sigma(Q)\eta(P) \geq \hbar/2$, and also $\eta(P) = 0$ is possible with $\epsilon(Q)\sigma(P) \geq \hbar/2$. In particular, even the case where $\epsilon(Q) = 0$ and $\eta(P) < \epsilon$ with arbitrarily small $\epsilon$ is possible for some input state with $\sigma(Q) > \hbar/2\epsilon$, and also the case where $\eta(P) = 0$ and $\epsilon(Q) < \epsilon$ is possible for some input state with $\sigma(P) > \hbar/2\epsilon$. Such extreme cases occur in compensation for large uncertainties in the
input state, while in the minimum uncertainty state with \( \sigma(Q) = \sigma(P) = (\hbar/2)^{1/2} \), we have

\[
e(Q)\eta(P) + \sqrt{\frac{\hbar}{2} [e(Q) + \eta(P)]} \geq \frac{\hbar}{2}.
\] (267)

Even in this case, it is allowed to have \( e(Q)\eta(P) = 0 \) with \( e(Q) = 0 \) and \( \eta(P) \geq (\hbar/2)^{1/2} \) or with \( \eta(P) = 0 \) and \( e(Q) \geq (\hbar/2)^{1/2} \).

For the general case, we have the following trade-off relations for precise \( A \) measurements or \( B \)-non-disturbing measurements.

**Theorem VI.5** For any apparatus \( A(x) \) and observables \( A \) and \( B \), if \( A(x) \) does not disturb \( B \), we have

\[
e(A, \rho)\sigma(B, \rho) \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\] (268)

for any state \( \rho \) for which all the relevant terms are finite.

**Theorem VI.6** For any apparatus \( A(x) \) and observables \( A \) and \( B \), if \( A(x) \) precisely measures \( A \), we have

\[
\sigma(A, \rho)\eta(B, \rho) \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\] (269)

for any state \( \rho \) for which all the relevant terms are finite.

For physical significance of the generalized noise-disturbance uncertainty relation, we refer the reader to Ref. [2, 35]. In the next section, we shall give an indirect measurement model that satisfies inequalities in Theorems VI.4 and VI.6 but does not satisfy Heisenberg’s relation in Theorem VI.3 for position measurement noise and momentum disturbance.

**D. Uncertainty relations for measurements with statistically independent noise**

Let \( A(x) \) be an arbitrary apparatus and let \( A, B \) be a pair of observable of the measured object. Denote by \( Z, T, \) and \( \Pi \) be its operational distribution, nonselective operation, and POVM respectively. Recall that the standard deviation of the output \( x \) on input state \( \rho \) is given by

\[
\sigma(x, \rho) = \langle (x - \langle x \rangle)^2 \rangle^{1/2} = \langle \text{Tr}(O^2(\Pi)\rho) - \text{Tr}(O(\Pi)\rho)^2 \rangle^{1/2}.
\]

From Eqs. (184)–(186), if \( A(x) \) makes an unbiased measurement of \( A \), i.e., \( \langle x \rangle = \langle A \rangle \), we have

\[
|\sigma(A, \rho) - \epsilon(A, \rho)| \leq \sigma(x, \rho) \leq \epsilon(A, \rho) + \sigma(A, \rho).
\] (270)

In what follows, we shall show that if \( A(x) \) has statistically independent noise or makes an unbiased measurement of \( A \), the standard deviation \( \sigma(x, \rho) \) obeys a reciprocal trade-off with the disturbance on any observable \( B \).

Let \( (K, \sigma, U, M) \) be an indirect measurement model statistically equivalent to \( A(x) \). Now, we shall return to the input-output relations, Eqs. (233) and (234), from which we have

\[
\begin{align*}
[M^\text{out}, B^\text{out}] &= [M^\text{out}, B^\text{in} + D(B)] \\
&= [M^\text{out}, B^\text{in}] + [M^\text{out}, D(B)] \\
&= [A^\text{in}, B^\text{in}] + [N(A), B^\text{in}] + [M^\text{out}, D(B)].
\end{align*}
\]

By the relation \([M^\text{out}, B^\text{out}] = 0\), we have

\[
\begin{align*}
[N(A), B^\text{in}] + [M^\text{out}, D(B)] &= -[A^\text{in}, B^\text{in}].
\end{align*}
\] (271)

Taking the moduli of the both sides in the original state \( \rho \otimes \sigma \) and applying the triangular inequality as before, we have

\[
|\langle [N(A), B^\text{in}] \rangle| + |\langle [M^\text{out}, D(B)] \rangle| \geq |\text{Tr}([A, B]\rho)|.
\]

By the Heisenberg-Robertson relation and the relation \( \sigma(M^\text{out}) = \sigma(x, \rho) \), we have

\[
\sigma(x, \rho)\eta(B, \rho) \geq \frac{1}{2} |\langle [M^\text{out}, D(B)] \rangle|.
\] (272)

From Eqs. (256) and (272), we have

**Theorem VI.7** Let \( A \) and \( B \) be a pair of observables. Every apparatus \( A(x) \) satisfies the relation

\[
\sigma(x, \rho)\eta(B, \rho) + \frac{1}{2} |\text{Tr}([n(A), B]\rho)| \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\] (273)

for any state \( \rho \) for which all the relevant terms are finite, where \( n(A) \) is the mean noise operator for \( A \).

From the above, we have the following reciprocal uncertainty relation for measurements with statistically independent noise and unbiased measurements.

**Theorem VI.8** Let \( A \) and \( B \) be a pair of observables. An apparatus \( A(x) \) satisfies the relation

\[
\sigma(x, \rho)\eta(B, \rho) \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\] (274)

for any state \( \rho \) for which all the relevant terms are finite, if one of the following conditions holds:

(i) The mean noise operator commutes with \( B \), i.e., \( [n(A), B] = 0 \).

(ii) The apparatus \( A(x) \) has an statistically independent noise for \( A \).

(iii) The apparatus \( A(x) \) makes an unbiased measurement of \( A \).
E. Uncertainty relations for measurements with statistically independent disturbance

Let \( A(x) \) be an arbitrary apparatus and let \( A, B \) be a pair of observables of the measured object. Denote by \( \mathcal{L}, \mathcal{T}, \) and \( \Pi \) by its operational distribution, nonselective operation, and POVM respectively. For any input state \( \rho \), the standard deviation \( \sigma(B, \rho) \) is called the pre-measurement uncertainty of \( B \) and the standard deviation \( \sigma(B, T \rho) \) of \( B \) in the state \( T \rho \) is called the post-measurement uncertainty of \( B \). By the definition of the root-mean-square disturbance \( \eta(B, \rho) \), they satisfy the relation

\[
|\sigma(B, \rho) - \eta(B, \rho)| \leq \sigma(B, T \rho) \leq \eta(B, \rho) + \sigma(B, \rho).
\] (275)

If the measurement does not disturb an observable \( B \), the rms noise \( \epsilon(A, \rho) \) is constrained by Eq. (268) so that

\[
\epsilon(A, \rho) \geq \frac{|\langle [A, B] \rangle|}{2\sigma(B, \rho)}.
\] (276)

In what follows, we consider the more general case where the statistically independent disturbance or unbiased disturbance is allowed and we shall show that the rms noise \( \epsilon(A) \) obeys another reciprocal trade-off that is obtained by replacing the pre-measurement uncertainty \( \sigma(B, \rho) \) by the post-measurement uncertainty \( \sigma(B, T \rho) \).

Let \( (\mathcal{K}, \sigma, U, M) \) be an indirect measurement model statistically equivalent to the apparatus \( A(x) \). From the input-output relations, Eqs. (233) and (234), we have

\[
[M_{\text{out}}^{\text{out}}, B_{\text{out}}^{\text{out}}] = [A_{\text{in}}^{\text{in}} + N(A), B_{\text{out}}^{\text{out}}] = [A_{\text{in}}^{\text{in}}, B_{\text{in}}^{\text{in}}] + [N(A), B_{\text{out}}^{\text{out}}] = [A_{\text{in}}^{\text{in}}, B_{\text{in}}^{\text{in}}] + [A_{\text{in}}^{\text{in}}, D(B)] + [N(A), B_{\text{out}}^{\text{out}}].
\]

By the relation \([M_{\text{out}}^{\text{out}}, B_{\text{out}}^{\text{out}}] = 0\), we have

\[
[N(A), B_{\text{out}}^{\text{out}}] + [A_{\text{in}}^{\text{in}}, D(B)] = -[A_{\text{in}}^{\text{in}}, B_{\text{in}}^{\text{in}}].
\] (277)

Taking the moduli of the both sides in the original state \( \rho \otimes \sigma \) and applying the triangular inequality as before, we have

\[
|\langle [N(A), B_{\text{out}}^{\text{out}}] \rangle| + |\langle [A_{\text{in}}^{\text{in}}, D(B)] \rangle| \geq |\text{Tr}([A, B] \rho)|.
\]

By the Heisenberg-Robertson relation and the relation \( \sigma(B_{\text{out}}^{\text{out}}) = \sigma(B, T \rho) \), we have

\[
\epsilon(A, \rho)\sigma(B, T \rho) \geq \frac{1}{2}|\langle [N(A), B_{\text{out}}^{\text{out}}] \rangle|.
\] (278)

From Eqs. (277) and (278), we have

**Theorem VI.9** Let \( A \) and \( B \) be a pair of observables. Every apparatus \( A(x) \) satisfies the relation

\[
\epsilon(A, \rho)\sigma(B, T \rho) + \frac{1}{2}|\text{Tr}([A, d(B)] \rho)| \geq \frac{1}{2}|\text{Tr}([A, B] \rho)|
\] (279)

for any state \( \rho \) for which all the relevant terms are finite, where \( n(A) \) is the mean noise operator for \( A \).

From the above, we have the following reciprocal uncertainty relation for measurements with statistically independent disturbance or unbiased disturbance.

**Theorem VI.10** Let \( A \) and \( B \) be a pair of observables. An apparatus \( A(x) \) satisfies the relation

\[
\epsilon(A, \rho)\sigma(B, T \rho) \geq \frac{1}{2}|\text{Tr}([A, B] \rho)|
\] (280)

for any state \( \rho \) for which all the relevant terms are finite, if one of the following conditions holds:

(i) The mean disturbance operator commutes with \( B \), i.e., \([A, d(B)] = 0\).

(ii) The apparatus \( A(x) \) has an statistically independent disturbance for \( B \).

(iii) The apparatus \( A(x) \) makes an unbiased disturbance of \( B \).

VII. THE MODEL BREAKING HEISENBERG’S RELATION

A. Von Neumann’s model

For comparison with the model to be presented later, we shall start with a canonical position measurement proposed by von Neumann [6], which turns out to typically satisfy Heisenberg’s noise-disturbance uncertainty relation.

Let us consider the case where the object \( S \) is a one-dimensional mass with position \( \hat{x} \), momentum \( \hat{\rho}_x \) \( (|\hat{x}, \hat{\rho}_x \rangle = i\hbar) \), and Hamiltonian \( H_S \) on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \). Under general definitions given in the previous sections, we can rigorously formulate Heisenberg’s noise-disturbance uncertainty relation as

\[
\epsilon(\hat{x})\eta(\hat{\rho}_x) \geq \frac{\hbar}{2}.
\] (281)

Let \( A(q) \) be the apparatus measuring the system \( S \) described as follows. The probe \( P \) of \( A(q) \) is supposed to be a one-dimensional system with canonical observables \( \hat{q} \) and \( \hat{\rho}_p \) \( (|\hat{q}, \hat{\rho}_p \rangle = i\hbar) \), and Hamiltonian \( H_P \) on the Hilbert space \( \mathcal{K} = L^2(\mathbb{R}) \). The probe observable is designed to be the coordinate \( \hat{q} \) of \( P \). The probe is also designed to be prepared in a state with a normalized wave function \( \xi(q) \) just before measurement. Mathematically, we assume that the wave function is rapidly decreasing, i.e., \( \xi(q) \in \mathcal{S}(\mathbb{R}) \), so that we have \( \sigma(\hat{q}), \sigma(\hat{\rho}) < \infty \) in the state \( \xi \). The object-probe interaction \( H \) is turned on from time \( t \) to \( t + \Delta t \). The total Hamiltonian for the object plus probe is taken to be

\[
H_{S+P} = H_S + H_P + KH,
\] (282)
where $K$ is the coupling constant. We assume that the coupling is so strong, i.e., $K \gg 1$, that the free Hamiltonians can be neglected and that the duration $\Delta t$ of the coupling is chosen so that $K \Delta t = 1$.

Von Neumann [6] introduced the measuring interaction

$$H = \hat{x} \hat{p}$$

(283)

for an approximate position measurement (see also Refs. [11, 55, 57]). Then, the unitary operator of the time evolution of $S + P$ from $t$ to $t + \Delta t$ is given by

$$U = \exp \left( -\frac{i}{\hbar} \hat{x} \hat{p} \right).$$

(284)

This measurement is, therefore, described by the indirect measurement model

$$\mathcal{M}(\hat{x}, \xi) = \left( L^2(\mathbb{R}), \xi, \exp \left( -\frac{i}{\hbar} \hat{x} \hat{p} \right), \hat{q} \right),$$

(285)

which has been generally introduced in Subsection III F. From general results in Subsection III F, the model $\mathcal{M}(\hat{x}, \xi)$ has the operational measure

$$\mathcal{I}_\xi(\Delta) \rho = \int_\Delta \xi(qI - \hat{x}) \rho \xi(qI - \hat{x})^\dagger dq,$$

(286)

the dual operational measure

$$\mathcal{I}_\xi(\Delta)^* X = \int_\Delta \xi(qI - \hat{x})^\dagger \rho \xi(qI - \hat{x}) dq,$$

(287)

the POVM

$$\Pi_\xi(\Delta) = \int_\Delta |\xi(qI - \hat{x})|^2 dq,$$

(288)

the output probability distribution

$$\Pr\{q \in \Delta || \rho\} = \int_\Delta \text{Tr}[|\xi(qI - \hat{x})|^2 \rho] dq,$$

(289)

and the output state

$$\rho_{(q=q)} = \frac{\xi(qI - \hat{x}) \rho \xi(qI - \hat{x})^\dagger}{\text{Tr}[|\xi(qI - \hat{x})|^2 \rho]}.$$

(290)

If the input state is a vector state $\rho = |\psi\rangle \langle \psi|$, we also have the output probability distribution

$$\Pr\{q \in \Delta || \psi\} = \int_\Delta dq \int_\mathbb{R} |\xi(q-x)|^2 |\psi(x)|^2 dx$$

(291)

and the output state

$$\psi_{(q=q)}(x) = \frac{\xi(q-x) \psi(x)}{\left( \int_\mathbb{R} |\xi(q-x)|^2 |\psi(x)|^2 dx \right)^{1/2}}$$

(292)

with $\rho_{(q=q)} = |\psi_{(q=q)}\rangle \langle \psi_{(q=q)}|$.

Solving the Heisenberg equations of motion

$$dX(t + \tau)/d\tau = -i/\hbar [X(t + \tau), KH]$$

(293)

for $t < t + \tau < t + \Delta t$, where $X(t + \tau)$ is any Heisenberg observable of $S + P$, we obtain

$$\dot{x}(t + \tau) = \dot{x}(t),$$

(294)

$$\dot{q}(t + \tau) = K \tau \dot{x}(t) + \dot{q}(t),$$

(295)

$$\dot{p}_x(t + \tau) = \dot{p}_x(t) - K \tau \ddot{p}(t),$$

(296)

$$\dot{p}(t + \tau) = \dot{p}(t).$$

(297)

For $\tau = \Delta t = 1/K$, we have

$$\dot{x}(t + \Delta t) = \dot{x}(t),$$

(298)

$$\dot{q}(t + \Delta t) = \dot{x}(t) + \dot{q}(t),$$

(299)

$$\dot{p}_x(t + \Delta t) = \dot{p}_x(t) - \ddot{p}(t),$$

(300)

$$\dot{p}(t + \Delta t) = \ddot{p}(t).$$

(301)

It follows that the noise operator, the disturbance operator, the mean noise operator, and the mean disturbance operator are given by

$$N(\bar{x}) = \bar{q}(t + \Delta t) - \dot{x}(t) = \bar{q}(t),$$

(302)

$$D(\bar{p}_x) = \bar{p}_x(t + \Delta t) - \bar{p}_x(t) = -\bar{p}(t),$$

(303)

$$n(\bar{x}) = \langle \xi | \bar{q} | \xi \rangle I,$$

(304)

$$d(\bar{p}_x) = -\langle \xi | \bar{p} | \xi \rangle I.$$ (305)

Thus, this measurement has statistically independent position-measurement noise and statistically independent momentum disturbance, so that this measurement satisfies Heisenberg’s noise-disturbance uncertainty relation. In fact, the mean-square position-measurement noise and the mean-square momentum disturbance are given by

$$\epsilon(\bar{x})^2 = \langle \bar{q}(t)^2 \rangle \geq \sigma(\bar{q})^2,$$

(306)

$$\eta(\bar{p}_x)^2 = \langle \bar{p}(t)^2 \rangle \geq \sigma(\bar{p})^2.$$ (307)

Therefore, we conclude that the von Neumann model obeys Heisenberg’s noise-disturbance uncertainty relation,

$$\epsilon(\bar{x}) \eta(\bar{p}_x) \geq \frac{\hbar}{2},$$

(308)

as a consequence of the Heisenberg-Kennard relation

$$\sigma(\bar{q}) \sigma(\bar{p}) \geq \frac{\hbar}{2},$$

(309)

applied to the probe state just before measurement.

This model represents a basic feature of the γ ray microscope on the point that the trade-off between the rms noise and the disturbance arises from the fundamental physical limitation on preparing the probe. It might be expected that such a basic feature is shared by every model in a reasonable class of position measurements. However, the next model suggests that it is not the case.
B. Time independent Hamiltonian model

In what follows, we modify the measuring interaction of the von Neumann model to construct a model that violates Heisenberg’s noise-disturbance uncertainty relation. In this new model, the object, the probe, the probe preparation, and the probe observable to be actually measured are the same systems, the same state, and the same observable as the von Neumann model. Instead of Eq. (283), the measuring interaction is now taken to be [19]

\[ H = \frac{\pi}{3\hbar^2} (2\hat{x}\hat{p} - 2\hat{p}_x\hat{q} + \hat{x}\hat{p}_x - \hat{q}\hat{p}) \]  

(310)
The measuring interaction \( H \) is turned on from time \( t \) to \( t + \Delta t \). The total Hamiltonian for the object plus probe is

\[ H_{S+P} = H_S + H_P + KH. \]  

(311)
The coupling constant \( K \) and the time duration \( \Delta t \) are chosen as before so that \( K \gg 1 \) and \( K\Delta t = 1 \). Then, the time evolution operator \( U \) for the time interval \( (t, t + \Delta t) \) is given by

\[ U = \exp \left[ -\frac{i\pi}{3\hbar^2} (2\hat{x}\hat{p} - 2\hat{p}_x\hat{q} + \hat{x}\hat{p}_x - \hat{q}\hat{p}) \right]. \]  

(312)

This measurement is, therefore, described by the indirect measurement model

\[ \left( L^2(\mathbb{R}), \xi, \exp \left[ -\frac{i\pi}{3\hbar^2} (2\hat{x}\hat{p} - 2\hat{p}_x\hat{q} + \hat{x}\hat{p}_x - \hat{q}\hat{p}) \right], \hat{q} \right). \]

We shall call this model the \((1, -2, 2)\) model, whereas the von Neumann model will be called the \((0, 0, 1)\) model; for general \((\alpha, \beta, \gamma)\) model we refer to Ref. [58].

For the time interval \( t < t + \tau < t + \Delta t \), the wave function \( \Psi_{t+\tau}(x, q) \) of the composite system \( S + P \) satisfies the Schrödinger equation

\[ i\hbar \frac{\partial \Psi_{t+\tau}(x, q)}{\partial \tau} = KH \Psi_{t+\tau}(x, q). \]  

(313)
The solution is

\[ \Psi_{t+\tau}(x, q) = \Psi_t \left( \frac{2}{\sqrt{3}} \left\{ x \sin \left( \frac{1 - K\tau}{3} \right) \frac{\pi}{3} + q \sin \left( \frac{K\tau\pi}{3} \right) \right\}, \right. \]

\[ \left. \frac{2}{\sqrt{3}} \left\{ -x \sin \left( \frac{K\tau\pi}{3} \right) + q \sin \left( \frac{1 + K\tau\pi}{3} \right) \right\} \right) \]  

(314)

For \( \tau = \Delta t = 1/K \), we have

\[ \Psi_{t+\Delta t}(x, q) = \Psi_t(q, q-x). \]  

(315)

Now, suppose that at time \( t \), just before the coupling is turned on, the object wave function is \( \psi(x) \) with \( \sigma(x), \sigma(\hat{p}_x) < \infty \) in the state \( \psi(x) \). Since the the probe is prepared in the wave function \( \xi(q) \), the total wave function is

\[ \Psi_t(x, q) = \psi(x)\xi(q). \]  

(316)
At time \( t + \Delta t \), the end of the interaction, the total wave function becomes

\[ \Psi_{t+\Delta t}(x, q) = \psi(q)\xi(q-x). \]  

(317)

Compare with Eq. (283); as simple as the von Neumann model, but the statistics is much different.

In the above state, the probe observable \( \hat{q} \) is measured to obtain the outcome. Thus the output probability distribution of this measurement is given by

\[ \text{Pr}\{q \in \Delta|\psi\} = \int_{\Delta} dq \int_{\mathbb{R}} |\Psi_{t+\Delta t}(x, q)|^2 dx. \]

(318)
The output probability distribution has the probability density function \( |\psi(q)|^2 \), which coincides with the Born probability density of the object \( x \) just before the measurement and shows that this measurement is precise position measurement.

The object wave function \( \psi_{\{q=q\}}(x) \) just after this measurement given the output \( q = q \) is obtained (up to normalization) by

\[ \psi_{\{q=q\}}(x) \propto \frac{\Psi_{t+\Delta t}(x, q)}{|\Psi_{t+\Delta t}(x, q)|^{1/2}} \]

\[ \propto \frac{\psi(q)\xi(q-x)}{|\psi(q)|^{1/2}} = C \xi(q-x), \]

where \( C \) \( (|C| = 1) \) is a constant phase factor depending only on the output \( q = q \). The above relation can be also derived from a general result in Section III. Let \( f(x) \) be the wave function in \( \mathcal{H} \) defined by

\[ f(x) = \xi(-x) \]  

(319)
for all \( x \in \mathbb{R} \). Then, we have

\[ \xi(q-x) = f(x-q) \]

\[ = \left[ \exp(-iq\hat{p}_x/\hbar) f \right](x). \]  

(320)
Thus, from

\[ U(\psi \otimes \xi)(x, q) = \psi(q)\xi(q-x), \]  

(321)
we have

\[ U(\psi \otimes \xi)(q) = \psi(q)\exp(-iq\hat{p}_x/\hbar) f \]  

(322)
From Theorem III.1, the operational distribution $I$ satisfies

$$I(\Delta)|\psi\rangle\langle\psi| = \int_{\Delta} \exp(-i\hat{p}_x/\hbar)|\psi(q)f\rangle\langle\psi(q)\exp(-i\hat{p}_x/\hbar)f|dq$$

$$= \int_{\Delta} \exp(-i\hat{p}_x/\hbar)|f\rangle\langle f|\exp(i\hat{p}_x/\hbar)|\psi(q)\rangle^2dq$$

$$= \int_{\Delta} \exp(-i\hat{p}_x/\hbar)|f\rangle\langle f|\exp(i\hat{p}_x/\hbar)\times \text{Tr}[dE^\theta(q)|\psi\rangle\langle\psi|]$$

(323)

It follows that the output state given $q = q$ is

$$|\psi(q=\xi)\rangle = \exp(-i\hat{p}_x/\hbar)|f\rangle(\exp(xq_0)),$$

and hence we have

$$\psi_0(x) = [\exp(-i\hat{p}_x/\hbar)|f\rangle = \xi(q-x),$$

(325)

up to constant phase factor.

By linearity and continuity, from Eq. (323) the operational distribution of this model is given by

$$I(\Delta) = \int_{\Delta} \exp(-i\hat{p}_x/\hbar)|f\rangle\langle f| \exp(i\hat{p}_x/\hbar)\times \text{Tr}[dE^\theta(q)|\psi\rangle\langle\psi|]$$

(326)

Solving the Heisenberg equations of motion for $t < t + \tau < t + \Delta t$, we obtain

$$\dot{x}(t + \tau) = \frac{2}{\sqrt{3}}(1 + K\tau)\pi - \frac{2}{\sqrt{3}}(1 - K\tau)\pi,$$

$$\dot{\hat{p}}(t + \tau) = \frac{2}{\sqrt{3}}(1 + K\tau)\pi - \frac{2}{\sqrt{3}}(1 - K\tau)\pi,$$

$$\hat{p}_x(t + \tau) = \frac{2}{\sqrt{3}}(1 + K\tau)\pi - \frac{2}{\sqrt{3}}(1 - K\tau)\pi,$$

$$\hat{p}(t + \tau) = \frac{2}{\sqrt{3}}(1 + K\tau)\pi - \frac{2}{\sqrt{3}}(1 - K\tau)\pi.$$

For $\tau = \Delta t = 1/K$, we have

$$\dot{x}(t + \Delta t) = \hat{x}(t) = \hat{\hat{p}}(t),$$

$$\dot{\hat{p}}(t + \Delta t) = \hat{x}(t),$$

$$\dot{\hat{p}}_x(t + \Delta t) = \hat{\hat{p}}(t),$$

(327)

(328)

(329)

(330)

It follows that the noise operator, the disturbance operator, the mean noise operator, and the mean disturbance operator are given by

$$N(\hat{x}) = \hat{x}(t + \Delta t) - \hat{x}(t) = 0,$$

$$D(\hat{p}_x) = \hat{p}_x(t + \Delta t) - \hat{p}_x(t) = 0,$$

$$n(\hat{x}) = 0,$$

$$d(\hat{p}_x) = -\langle\hat{p}_x\hat{\xi}\rangle - \hat{p}_x,$$

(331)

(332)

(333)

(334)

(335)

Thus, the position-measurement noise and the momentum disturbance are given by

$$\epsilon(\hat{x}) = 0,$$

$$\eta(\hat{p}_x)^2 = (\langle\hat{p}_x(t)\rangle)^2 = \sigma(\hat{p}_x)^2 + \sigma(\hat{p}_x)^2 + [\langle\hat{p}_x(t)\rangle + \langle\hat{p}_x(t)\rangle]^2.$$$$

(336)

(337)

Consequently, we have

$$\epsilon(\hat{x})\eta(\hat{p}_x) = 0.$$

(338)

Therefore, our model obviously violates Heisenberg's noise-disturbance uncertainty relation.

If $\langle\hat{p}_x(t)^2\rangle \rightarrow 0$ and $\langle\hat{p}_x(t)^2\rangle \rightarrow 0$ (i.e., $\psi$ and $\xi$ tend to the momentum eigenstate with zero momentum) then we have even $\eta(\hat{p}_x(t)) \rightarrow 0$ with $\epsilon(\hat{x}) = 0$. Thus, we can precisely measure position without effectively disturbing momentum in a near momentum eigenstate.

Taking advantage of the above model, we can refute the argument that the uncertainty principle generally leads to a general sensitivity limit, called the standard quantum limit, for monitoring free-mass position [16, 19].

**C. Time dependent Hamiltonian model**

The interaction of the preceding model Eq. (310), the $(1, -2, 2)$ model, includes the term $\hat{x}\hat{p}_x - \hat{p}_x\hat{q}$, which cannot be implemented by a simple coupling. Thus, it seems that this model is far more different than the von Neumann model. In this section, we shall show, however, that if we use time dependent interaction, the $(1, -2, 2)$ model can be implemented as feasibly as the von Neumann model.

Now, we shall consider the following model description, which will turn out statistically equivalent to the model discussed in the preceding subsection. In this model, the object, the probe, the probe preparation, and the probe observable are the same as the previous models. The object-probe interaction is turned on from time $t$ to $t + \Delta t$. For the time interval $t < t + \tau < t + \Delta t$, the time dependent total Hamiltonian $H_{S+P}(t + \tau)$ of $S + P$ is taken to be

$$H_{S+P}(t + \tau) = H_S \otimes I + I \otimes H_P - K_1(\tau)\hat{p}_x \otimes \hat{q} + K_2(\tau)\hat{x} \otimes \hat{p},$$

(339)

where the strengths of couplings, $K_1(\tau)$ and $K_2(\tau)$, satisfy

$$K_1(\tau) = 0 \text{ if } \tau \not\in (t, t + \Delta t),$$

(340)

$$K_2(\tau) = 0 \text{ if } \tau \not\in (t + \Delta t, t + \Delta t),$$

(341)

$$\int_t^{t + \Delta t} K_1(\tau)d\tau = 1, \int_{t + \Delta t}^{t + \Delta t} K_2(\tau)d\tau = 1.$$
We assume that \( \Delta t \) is so small that the system Hamiltonians \( H_S \) and \( H_P \) can be neglected from \( t \) to \( t + \Delta t \). Solving the Schrödinger equation, just as von Neumann model, the time evolution of \( S + P \) during the coupling is described by the unitary evolution operators

\[
U(t + \frac{\Delta t}{2}, t) = \exp \left( \frac{i}{\hbar} \vec{p} \cdot \vec{q} \right),
\]

\[
U(t + \Delta t, t + \frac{\Delta t}{2}) = \exp \left( \frac{i}{\hbar} \vec{x} \cdot \vec{p} \right).
\]

Then, in the position basis we have

\[
\langle x, y | U(t + \frac{\Delta t}{2}, t) | x', y' \rangle = \langle x + y, y' | x', y' \rangle,
\]

\[
\langle x, y | U(t + \Delta t, t + \frac{\Delta t}{2}) | x', y' \rangle = \langle x, y - x | x', y' \rangle,
\]

and hence

\[
\langle x, y | U(t + \Delta t, t + \frac{\Delta t}{2}) U(t + \frac{\Delta t}{2}, t) | x', y' \rangle = \langle y, y - x | x', y' \rangle.
\]

Thus, by Eq. (321), we conclude that the unitary evolution operator

\[
U = U(t + \Delta t, t + \frac{\Delta t}{2}) U(t + \frac{\Delta t}{2}, t)
\]

is the same as the unitary operator of the \((1, -2, 2)\) model. Thus, the above model is identical with the \((1, -2, 2)\) model. In particular, we have obtained the relation

\[
\exp \left[ \frac{-i}{3 \sqrt{3} \hbar} \left( 2 \vec{x} \cdot \vec{p} - 2 \vec{p} \cdot \vec{q} + \vec{x} \cdot \vec{q} \right) \right] = \exp \left( \frac{-i}{\hbar} \vec{x} \cdot \vec{p} \right) \exp \left( \frac{i}{\hbar} \vec{p} \cdot \vec{q} \right).
\]

Thus, we can avoid to implement the term \( \vec{x} \cdot \vec{p} \) and \( \vec{p} \cdot \vec{q} \), and only von Neumann type interactions \( \vec{x} \cdot \vec{p} \) and \( \vec{p} \cdot \vec{q} \) are suffice to implement the \((1, -2, 2)\) model.

VIII. REPEATABILITY AND UNCERTAINTY PRINCIPLE

A. Repeatability hypothesis and the projection postulate

In formulating the canonical description of the measurement of an observable, von Neumann required not only that the output probability distribution satisfies the Born statistical formula but also that the quantum state reduction satisfies the following hypothesis abstracted from the result of the Compton-Simons experiment [6].

Repeatability hypothesis. If an observable is measured twice in succession in a system, then we get the same value each time.

In what follows, we consider the rigorous formulation of this requirement for general measuring apparatuses. Let \( \mathbf{A}(x) \) be an apparatus with output variable \( x \). In order to formalize the repeatability hypothesis, we need to consider repeated measurements using the identical apparatuses on the same system. Since the same apparatus cannot be used twice in succession, we assume that immediately after the measurement using \( \mathbf{A}(x) \), another statistically equivalent apparatus \( \mathbf{A}(y) \) with output variable \( y \) is used for the succeeding measurement. Then, the repeatability hypothesis states that if \( x = x \) then \( y = x \) for any \( x \). This condition is well-formulated by the concept of conditional probability as follows. The apparatus \( \mathbf{A}(x) \) satisfies the repeatability hypothesis if and only if the conditional probability distribution of \( y \) given \( x = x \) satisfies

\[
\Pr \{ y \in \Delta | x = x | \rho \} = \chi_\Delta (x)
\]

for all \( x, \Delta \) and \( \rho \). Let \( \rho_{\{x=x\}} \) be the output state given \( x = x \) for input state \( \rho \). Then, from Eq. (48), the apparatus \( \mathbf{A}(x) \) satisfies the repeatability hypothesis if and only if we have

\[
\Pr \{ y \in \Delta | \rho_{\{x=x\}} \} = \chi_\Delta (x)
\]

for any Borel set \( \Delta \).

Now, we shall consider the case where apparatus \( \mathbf{A}(x) \) precisely measures an observable \( A \). From Eq. (349), in this case \( \mathbf{A}(x) \) satisfies the repeatability hypothesis if and only if we have

\[
\text{Tr} [ E^A(\Delta) \rho_{\{x=x\}} ] = \chi_\Delta (x)
\]

for any Borel set \( \Delta \). The last equality is equivalent to the condition

\[
E^A(\Delta) \rho_{\{x=x\}} E^A(\Delta) = \chi_\Delta (x) \rho_{\{x=x\}}.
\]

Suppose that \( A \) has purely discrete nondegenerate spectrum \( a_1, a_2, \ldots \) with corresponding orthonormal basis \( \phi_1, \phi_2, \ldots \) of eigenvectors. Then the repeatability hypothesis holds if and only if

\[
\rho_{\{x=a_n\}} = | \phi_n \rangle \langle \phi_n |
\]

for all \( n = 1, 2, \ldots \). In this case, the operational distribution \( \mathcal{I} \) of \( \mathbf{A}(x) \) is determined uniquely by

\[
\mathcal{I}(\Delta) \rho = \sum_{a_n \in \Delta} | \phi_n \rangle \langle \phi_n | \rho \langle \phi_n | \langle \phi_n |
\]

for any \( \rho \in \tau c(\mathcal{H}) \) and \( \Delta \in \mathcal{B}(\mathbb{R}) \). Thus for any observable with purely discrete nondegenerate spectrum the repeatability hypothesis determines an apparatus uniquely up to statistical equivalence.

If \( A \) has, however, purely discrete but degenerate spectrum then the repeatability hypothesis no longer determines the state after the measurement. In fact, in this case \( \rho_{\{x=a_n\}} \) can be one of any eigenstates \( | \phi \rangle \langle \phi | \) with
$A \phi = a_n \phi$ or even mixtures of them. In order to determine the state after the measurement in this case, Lüders [59] proposed the following requirement.

**Projection Postulate.** For any input state $\rho$ for a precise measurement of a purely discrete observable $A$, the output state $\rho_{(x=a)}$ is given by

$$
\rho_{(x=a)} = \frac{E^A(a) \rho E^A(a)}{\text{Tr}[E^A(a) \rho]} \quad (354)
$$

for any $a \in \mathbb{R}$ with $\Pr\{x = a \mid \rho\} > 0$.

According to the projection postulate, if the input state is a vector state $\psi$, i.e., $\rho = |\psi \rangle \langle \psi|$, then the output state $\rho_{(x=a)}$ is represented by the projection $E^A(\{a\}) \psi$ of $\psi$ on the eigenspace corresponding to the output $a$, i.e.,

$$
\rho_{(x=a)} = \frac{|E^A(\{a\}) \psi \rangle \langle E^A(\{a\}) \psi|}{||E^A(\{a\}) \psi||^2}. \quad (355)
$$

It is obvious that the projection postulate implies the repeatability hypothesis. The projection postulate yields the following operational distribution

$$
\mathcal{I}(\Delta) \rho = \sum_{a \in \Delta} E^A(a) \rho E^A(a) \quad (356)
$$

for all $\rho \in \tau(\mathcal{H})$.

**B. Discreteness of repeatable instruments**

Now we shall consider the general case where $A$ may have a continuous spectrum or even a vector $(x)$ makes no precise measurement of an observable. Let us assume that a measurement using $A(x)$ is immediately followed by a measurement using another statistically equivalent apparatus $A(y)$. Let $\mathcal{I}$ be the common operational distribution of those apparatuses. It follows from Eq. (47) and (348) that the repeatability hypothesis holds if and only if

$$
\Pr\{x \in \Delta, y \in \Delta' \mid \rho\} = \Pr\{x \in \Delta \cap \Delta' \mid \rho\} \quad (357)
$$

where $\Delta, \Delta' \in \mathcal{B}(\mathbb{R})$. Thus, from Eqs. (31) and (33) we conclude that apparatus $A(x)$ satisfies the repeatability hypothesis if and only if the operational distribution $\mathcal{I}$ satisfies

$$
\text{Tr}[\mathcal{I}(\Delta') \mathcal{I}(\Delta) \rho] = \text{Tr}[\mathcal{I}(\Delta \cap \Delta') \rho] \quad (358)
$$

or equivalently

$$
\mathcal{I}(\Delta')^* \mathcal{I}(\Delta)^* I = \mathcal{I}(\Delta \cap \Delta')^* I \quad (359)
$$

for any input state $\rho$ and $\Delta, \Delta' \in \mathcal{B}(\mathbb{R})$. The above conditions are also restated as $A(x)$ satisfies the repeatability hypothesis if and only if the operational distribution $\mathcal{I}$ and the POVM of $A(x)$ satisfies

$$
\mathcal{I}(\Delta')^* \Pi(\Delta') = \Pi(\Delta \cap \Delta'). \quad (360)
$$

Motivated by the above argument, any DL instrument satisfying Eq. (358) for all $\Delta, \Delta' \in \mathcal{B}(\mathbb{R})$ is said to be repeatable; note that Davies and Lewis [30] called originally such DL instruments as “weakly repeatable”.

Contrary to the fact that there can be many repeatable DL instruments corresponding to the same purely discrete observables, the following theorem, conjectured in Ref. [30] and proved in Ref. [50, Theorem 5.1] shows that there are no repeatable DL instruments corresponding to any observables with continuous spectrum.

**Theorem VIII.1** Every repeatable DL instrument is discrete in the sense that there is a countable subset $\Lambda_0$ of $\mathbb{R}$ such that $\mathcal{I}(\mathbb{R} \setminus \Lambda_0) = 0$.

It is concluded, therefore, that in order to model repeatable measurements of continuous observables it is necessary to describe them approximately with arbitrary closeness or to extend the formulation of quantum mechanics to include the limit of those approximate models [60]. In Ref. [55] it was shown that we have still satisfactory models of approximately repeatable measurement of continuous observables within arbitrarily small error limit in the standard formulation of quantum mechanics.

**C. Approximate repeatability**

Whereas von Neumann considers only precise measurements of observables and introduced the repeatability hypothesis for canonical description of state changes caused by measurements, the von Neumann model does not satisfy the preciseness nor the repeatability. One of the characteristic features of our $(1, -2, 2)$ model is that it precisely measures position, but our model does not satisfy the repeatability hypothesis either. Thus, it is tempting to understand that the $(1, -2, 2)$ model circumvents Heisenberg’s noise-disturbance uncertainty relation by paying the price of failing the repeatability. In what follows we shall show that such a view cannot be supported.

In the first place, as discussed in Subsection VIII A, the repeatability hypothesis can be satisfied only by measurements of purely discrete observables. Thus, no precise position measurements satisfy the repeatability hypothesis.

Secondly, if we consider the approximate repeatability, our model satisfies any stringent requirement of approximate repeatability. In order to show this, we need the measure of approximate repeatability introduce by Ref. [55].

Let $\mathcal{I}$ be a DL instrument. We define the root-mean-square repetition error of $\mathcal{I}$ on input state $\rho$, denoted by $R(\mathcal{I}, \rho)$, as follows.

$$
R(\mathcal{I}, \rho) = \left( \int_{\mathbb{R}^2} (x - y)^2 \text{Tr}[d\mathcal{I}(x) d\mathcal{I}(y) \rho] \right)^{1/2}. \quad (361)
$$
We shall write $R(I, \psi) = R(I, |\psi\rangle\langle\psi|)$. Since 
$\text{Tr}[I(\Delta)I(\Delta')\rho]$ represents the joint probability distribution of the outputs of the repeated measurements of statistically equivalent apparatuses with operational distribution $I$, the interpretation of the above error is obvious. Then we have the following.

**Theorem VIII.2**   
A DL instrument $I$ is repeatable if and only if $I$ satisfies  
$$R(I, \rho) = 0$$  
for any density operator $\rho$.

For the proof, we refer to Ref. [55].

A DL instrument $I$ is said to be $\varepsilon$-repeatable if $I$ satisfies $R(I, \rho) \leq \sqrt{2}\varepsilon$ for any density operator $\rho$. Now, it is natural to say that an apparatus or an indirect measurement model is said to be $\varepsilon$-repeatable if the corresponding operational measure is $\varepsilon$-repeatable.

Suppose that we measure the position of mass $\hat{x}$ in succession using two apparatuses described by the identical indirect measurement models with operational distribution $I$. Suppose that the first apparatus with probe $\hat{q}$ interacts with $\hat{x}$ in $(t, t + \Delta t)$ and the second apparatus with probe $\hat{q}'$ interacts with $\hat{x}$ in $(t + \Delta t, t + 2\Delta t)$. Then, the root-mean-square repetition error $R$ of the above apparatus is the root-mean-square difference between the first output $\hat{q}(t + \Delta t)$ and the second output $\hat{q}(t + 2\Delta t)$, i.e.,  
$$R^2 = \langle \psi \otimes \xi \otimes \xi | [\hat{q}'(t + 2\Delta t) - \hat{q}(t + \Delta t)]^2 | \psi \otimes \xi \otimes \xi \rangle. \tag{362}$$

If the apparatuses are described by the von Neumann model, we have  
$$\hat{q}'(t + 2\Delta t) = \hat{x}(t + \Delta t) + \hat{q}'(t + \Delta t)$$
$$= \hat{x}(t) + \hat{q}'(t + \Delta t)$$
$$\hat{q}(t + \Delta t) = \hat{x}(t) + \hat{q}(t).$$

Thus, we have  
$$[\hat{q}'(t + 2\Delta t) - \hat{q}(t + \Delta t)]^2$$
$$= \hat{q}'(t + \Delta t)^2 - 2\hat{q}'(t + \Delta t)\hat{q}(t) + \hat{q}(t)^2.$$  
Since $\hat{q}'(t + \Delta t)$ and $\hat{q}(t)$ are statistically independent and identically distributed we have  
$$\langle \psi \otimes \xi \otimes \xi | [\hat{q}'(t + 2\Delta t) - \hat{q}(t + \Delta t)]^2 | \psi \otimes \xi \otimes \xi \rangle$$
$$= 2\langle \xi | \hat{q}'(t) | \xi \rangle \langle \xi | \hat{q}(t) | \xi \rangle - \langle \xi | \hat{q}(t) | \xi \rangle^2$$
$$= 2\sigma(\hat{q}(t))^2.$$  
Thus, we have  
$$R = \sqrt{2}\sigma(\hat{q}(t)). \tag{363}$$

If the apparatuses are described by the $(1, -2, 2)$ model, we have  
$$\hat{q}'(t + 2\Delta t) = \hat{x}(t + \Delta t) = \hat{x}(t) - \hat{q}(t), \tag{364}$$
$$\hat{q}(t + \Delta t) = \hat{x}(t), \tag{365}$$
and hence  
$$\hat{q}'(t + 2\Delta t) - \hat{q}(t + \Delta t) = -\hat{q}(t). \tag{366}$$

Thus, we have  
$$R = \langle \hat{q}'(t)^2 \rangle^{1/2}. \tag{367}$$

Thus, for the probe preparation $\xi$ such that $\langle \hat{q}(t) \rangle = 0$, the von Neumann model has  
$$\epsilon(\hat{x}) = \sigma(\hat{q}(t)) \tag{368}$$
$$R = \sqrt{2}\sigma(\hat{q}(t)), \tag{369}$$
and the $(1, -2, 2)$ model has  
$$\epsilon(\hat{x}) = 0 \tag{370}$$
$$R = \sigma(\hat{q}(t)). \tag{371}$$

Thus, for the identical preparation of the probe, the $(1, -2, 2)$ model is concluded to be a $\sigma(\hat{q}(t))/\sqrt{2}$-repeatable precise position measurement, whereas the von Neumann model is a $\sigma(\hat{q}(t))$-repeatable $\sigma(\hat{q}(t))$-precise position measurement.

Therefore, we conclude for any small $\varepsilon > 0$ we have an $\varepsilon$-repeatable precise position measurement that violates Heisenberg’s noise-disturbance uncertainty relation (1). This suggests that how stringent conditions on preciseness and repeatability might be posed for a class of position measurements, we can find in that class at least one position measurement that violates Heisenberg’s noise-disturbance uncertainty relation.

**IX. CONCLUDING REMARKS**

In Ref. [35], we have obtained the universally valid noise-disturbance uncertainty relation Eq. (240) and the generalized noise-disturbance uncertainty relation Eq. (266), and also derived Theorems VI.1, VI.5, and VI.6 in the model dependent formulation. However, the following problems have been remained open concerning the foundations of the model dependent approach. (I) Can every measuring apparatus be described by an indirect measurement model? (II) Are the root-mean-square noise and disturbance uniquely determined independent of the model?

Indirect measurement models, originally introduced by von Neumann [6] and generally formalized in Ref. [32, 33], are powerful tool to study measuring processes, since the interaction between the measured object and the apparatus is described purely by quantum mechanics. This merit is strongly contrasted with a conventional view that the measuring interaction involves the macroscopic part of the apparatus. Although some measuring apparatuses, especially in the attempts for quantum nondemolition measurements [26], allow indirect measurement model descriptions, it is still difficult to convince any schools.
of measurement theory of the affirmative answer to question (I) above. However, the present paper has shown that in order to establish uncertainty relations for noise and disturbance the use of indirect measurement models is justified regardless of the answer to question (I).

The strategy taken in the present paper is as follows. We have started with listing up properties that obviously every measuring apparatus obeys, and then proven that every apparatus satisfying those properties is statistically equivalent to an apparatus described by an indirect measurement model. In the next step, we have proven that the root-mean-square noise and disturbance are determined by the POVM and the nonselective operation, respectively, of the apparatus, so that question (II) above has been answered affirmatively. This means that if two apparatuses are statistically equivalent, they have the same root-mean-square noise and disturbance. Thus, if a formula for root-mean-square noise and disturbance is proven for one apparatus with an indirect measurement model, every apparatus statistically equivalent to that apparatus obeys the same formula. In this way, we have justified the assertion of Ref. [35] that those formulas obtained for apparatuses with indirect measurement model are universally true for every apparatus irrespective of the model that describes the apparatus.

As properties that obviously every measuring apparatus obeys, we have proposed the following axioms for general measuring apparatuses.

(i) Mixing law: If two apparatuses are applied to a single system in succession, the joint probability distribution of outputs from those two apparatuses depends affinely on the input state.

(ii) Extendability axiom: Every apparatus measuring one system can be trivially extended to an apparatus measuring a larger system including the original system without changing the statistics.

(iii) Realizability postulate: Every indirect measurement model corresponds to an apparatus whose measuring process is described by that model.

From axioms (i) and (ii), we have demonstrated that statistical properties of any apparatus is described by a normalized completely positive map valued measure, called a CP instrument. Then, it has been shown that two apparatus are statistically equivalent if and only if they corresponds to the same CP instrument. Thus, the set of the statistical equivalence classes of all apparatuses is considered to be a subset of the set of all CP instruments. From the realization theorem of CP instruments (Theorem III.2) and axiom (iii), we have further concluded that the statistical equivalence class of apparatuses are in one-to-one correspondence with the CP instruments. Thus, we can conclude that every apparatus is statistically equivalent to at least one apparatus which is described by an indirect measurement model, in which the measuring interaction is simply described by a quantum mechanical interaction between two quantum mechanical systems, the object and the probe.

There have been many attempts to define the root-mean-square noise for some special classes of measurements. In Section IV we have shown that all those convincing attempts are equivalent to our notion of the distance of a POVM from an observable, based on which we define the root-mean-square noise of an arbitrary measurement in the model independent formulation. The empirical adequacy of our definition can be supported by the following reasons. (i) Our definition satisfies the requirement that if the measured observable has a definite value in the input state, the root-mean-square noise be the root-mean-square of the difference between the true value and the measured value (Eq. (145)). (ii) Our definition satisfies the requirement that measurements with uniformly zero root-mean-square noise coincide with precise measurements (Theorem IV.6). (iii) The difference between the standard deviations of the measured observable and of the measured value is bounded from above by the root-mean-square noise plus the bias, namely, the difference of their means (Eq. (187)). (iv) The root-mean-square noise in any input state can be statistically estimated from the experimental data (Eq. (189)). (v) The root-mean-square noise defined through the noise operator has a clear geometric interpretation (Eq. (123)). (vi) Even if another observer describes the same apparatus by a different indirect measurement model and identify the noise operator in a different way, the root-mean-square noises for both observers are equal (Eq. (140)).

In Ref. [2], we have discussed two distinct types of measurements in which Heisenberg’s noise-disturbance uncertainty relation is violated for position measurement noise and momentum disturbance uniformly for any input state. These cases are generalized in Theorem VI.5 for type I violation and Theorem VI.6 for type II violation. These relations clearly reveals possibilities of measurements beyond Heisenberg’s relation such as Yuen’s contractive state measurement [16] and clarifies the new constraints for measurements beyond Heisenberg’s relation. An experimental realization of a measurement with type II violation for optical quadrature measurement is proposed in Ref. [35]. This measuring interaction is equivalent to the (1, 2, 2) model, discussed in Section VII, which realizes Yuen’s contractive state measurement as demonstrated in Ref. [19], so that the realization of this measurement with required accuracy will open a way to the new technology for supersensitive sensors.

Acknowledgments

This work was supported by the Strategic Information and Communications R&D Promotion Scheme of the MPHPT of Japan, by the CREST project of the JST, and by the Grant-in-Aid for Scientific Research of the JSPS.
