ON GENERALIZED $p$–VALENT NON-BAZILEVIĆ FUNCTIONS OF ORDER $\alpha + i\beta$

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Abstract. In this paper, we introduce a subclass $N_{p,\mu}^\alpha(\alpha, \beta, A, B)$ of $p$-valent non-Bazilević functions of order $\alpha + i\beta$. Some subordination relations and the inequality properties of $p$–valent functions are discussed. The results presented here generalize and improve some known results.

1. Introduction and preliminaries

Let $A_p$ denote the class of functions $f$ of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p, n \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and $p$–valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$, and we write:

$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z), \quad z \in U,$$

if there exists a Schwarz function $w(z)$, which is analytic in $U$ with

$$|w(0)| = 0 \text{ and } |w(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(w(z)), \quad z \in U.$$

Furthermore, if the function $g(z)$ is univalent in $U$, then we have the following equivalence, see Miller & Mocanu ([3], [4]),

$$f(z) \prec g(z) \ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

We define a subclass of $A_p$ as follows:

**Definition 1.1.** Let $N_{p,\mu}^\alpha(\alpha, \beta, A, B)$ denote the class of functions $f(z) \in A_p$ satisfying the inequality:

$$\left\{ (1 + \mu) \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu \left(\frac{z f'(z)}{pf(z)}\right) \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \right\} < \frac{1 + Az}{1 + Bz}, \quad (z \in U),$$

where $\mu \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $-1 \leq B \leq 1$, $A \neq B$, $p \in \mathbb{N}$ and $A \in \mathbb{R}$. All the powers in (1.3) are principal values.

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We say that the function $f(z)$ in this class is $p$-valent non-Bazilevič functions of type $\alpha + i\beta$.

**Definition 1.2.** Let $f \in N_{\mu}^n(\alpha, \beta, \rho)$ if and only if $f(z) \in \mathcal{A}_p$ and it satisfies

$$\Re \left\{ (1 + \mu) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} - \mu \left( \frac{zf'(z)}{pf(z)} \right)^{\alpha + i\beta} \right\} > \rho, \quad (z \in \mathbb{U}),$$

(1.4)

where $\mu \in \mathbb{C}, \alpha \geq 0, \beta \in \mathbb{R}, p \in \mathbb{N}$ and $0 \leq \rho < p$.

**Special Cases:**

(1) When $p = 1$, then $N_{\mu}^n(\alpha, \beta, A, B)$ is the class studied by AlAmoush and Darus [6].

(2) When $p = 1, \beta = 0$, then $N_{\mu}^n(\alpha, 0, A, B)$ is the class studied by Wang et al [1].

(3) When $p = 1, \beta = 0, \mu = -1, A = 1$ and $B = -1$, then $N_{\mu}^n(\alpha)$ is the class studied by Obradovic [10].

(4) If $p = 1, \beta = 0, \mu = B = -1$ and $A = 1 - 2\rho$, then the class $N_{\mu}^n(\alpha, 0, 1 - 2\rho, -1)$ reduces to the class of non-Bazilevič functions of order $\rho$ ($0 \leq \rho < 1$). The Fekete-Szegö problem of the class $N_{\mu}^n(\alpha, 0, 1 - 2\rho, -1)$ were considered by Tuneski and Darus [2].

We will need the following lemmas in the next section.

**Lemma 1.3.** Let the function $h(z)$ be analytic and convex in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $\Phi(z)$ given by

$$\Phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ...$$

is analytic in $\mathbb{U}$. If

$$\Phi(z) + \frac{1}{\gamma} z\Phi'(z) < h(z) \quad (z \in \mathbb{U}, \Re \gamma \geq 0, \gamma \neq 0),$$

(1.5)

then

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{n} z^{-\gamma/n} \int_0^z t^{(\gamma/n)-1} h(t) dt < h(z) \quad (z \in \mathbb{U}),$$

and $\Psi(z)$ is the best dominant for the differential subordination (1.3).

**Lemma 1.4.** Let $-1 \leq B_1 \leq B_2 < A_2 < A_1 \leq 1$, then

$$\frac{1 + A_2 z}{1 + B_2 z} < \frac{1 + A_1 z}{1 + B_1 z}.$$

**Lemma 1.5.** Let $\Phi(z)$ be analytic and convex in $\mathbb{U}$, $f(z) \in \mathcal{A}_p$, $g(z) \in \mathcal{A}_p$.

If $f(z) \prec \Phi(z)$, $g(z) \prec \Phi(z)$, $0 \leq \mu \leq 1$ then

$$\mu f(z) + (1 - \mu)g(z) \prec \Phi(z).$$

**Lemma 1.6.** Let $q(z)$ be a convex univalent function in $\mathbb{U}$ and let $\sigma \in \mathbb{C}, \eta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\sigma}{\eta} \right) \right\}.$$
If the function $\Phi(z)$ is analytic in $U$ and

$$\sigma \Phi(z) + \eta z \Phi'(z) \prec \sigma q(z) + \eta z q'(z),$$

then, $\Phi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.7. [12] Let $q(z)$ be a convex univalent in $U$ and $\eta \in \mathbb{C}$. Further, assume that $\text{Re}\{\eta\} > 0$. If $\Phi(z) \in H[q(0), 1] \cap Q$, and $\Phi(z) + \eta \Phi'(z) \in H$ is univalent in $U$. Then

$$q(z) + \eta z q'(z) \prec \Phi(z) + \eta z \Phi'(z),$$

signifies that $q(z) \prec \Phi(z)$ and $q(z)$ are the best subordinat.

We employ techniques similar to these used earlier by Yousef et al. [13], Amourah et al. ([14], [15]), AlAmoush and Darus [16] and Al-Hawary et al. [13].

In the present paper, we shall obtain results concerning the subordination relations and inequality properties of the class $N_{p,\mu}(\alpha, \beta, A, B)$. The results obtained generalize the related works of some authors.

2. Main Result

Theorem 2.1. Let $\mu \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $-1 \leq B \leq 1$, $A \neq B$, $\alpha + i\beta \neq 0$, $p \in \mathbb{N}$ and $A \in \mathbb{R}$. If $f(z) \in N_{p,\mu}(\alpha, \beta, A, B)$, Then

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + AZu}{1 + Bzu} e^{\frac{(\alpha+i\beta)}{\mu n}} - 1 du \prec \frac{1 + Az}{1 + Bz}. \quad (2.1)$$

Proof. Let

$$\Phi(z) = \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}. \quad (2.2)$$

Then $\Phi(z)$ is analytic in $U$ with $\Phi(0) = 1$. Taking logarithmic differentiation of (2.2) in both sides, we obtain

$$p(\alpha + i\beta) \frac{zf'(z)}{f(z)} - (\alpha + i\beta) \frac{zf(z)'}{f(z)} = \frac{z\Phi'(z)}{\Phi(z)}. \quad (2.3)$$

In the above equation, we have

$$1 - \frac{zf'(z)}{pf(z)} = \frac{1}{p(\alpha + i\beta)} \frac{z\Phi'(z)}{\Phi(z)}. \quad (2.4)$$

From this we can easily deduce that

$$\left\{ (1 + \mu) \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right) \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \right\}. \quad (2.3)$$

On a class of $p$–valent non-Bazilevič functions

$$\Phi(z) + \frac{\mu z \Phi'(z)}{p(\alpha + i\beta)} \prec \frac{1 + Az}{1 + Bz}, \quad (2.4)$$
Now, by Lemma 1.3 for $\gamma = \frac{p(\alpha + i\beta)}{\mu}$, we deduce that
\[
\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} < q(z) = \frac{p(\alpha + i\beta)}{\mu n} z^{-\frac{p(\alpha + i\beta)}{\mu n}} \int_0^z t^{-\frac{p(\alpha + i\beta)}{\mu n} - 1} \left(\frac{1 + At}{1 + Bt}\right) dt.
\]
Putting $t = zu \Rightarrow dt = zdu$. Then we have the above equation with
\[
= \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 1 + Az u^{\frac{p(\alpha + i\beta)}{\mu n} - 1} du < 1 + Az
\]and the proof is complete. \qed

**Corollary 2.2.** Let $\mu \in \mathbb{C}, \alpha \geq 0, \beta \in \mathbb{R}, \alpha + i\beta \neq 0, p \in \mathbb{N}$ and $\rho \neq 0$. If $f(z) \in A_p$ satisfies
\[
(1 + \mu) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} < \frac{1 + (1 - 2\rho)z}{1 - z}, \quad (z \in U),
\]
then
\[
\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} < \frac{p(\alpha + i\beta)(1 - \rho)}{\mu n} \int_0^1 1 + \frac{zu}{1 - z} u^{-\frac{p(\alpha + i\beta)}{\mu n} - 1} du, \quad (z \in U),
\]
or equivalent to
\[
\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} < \rho + \frac{p(\alpha + i\beta)(1 - \rho)}{\mu n} \int_0^1 1 + \frac{zu}{1 - z} u^{-\frac{p(\alpha + i\beta)}{\mu n} - 1} du, \quad (z \in U).
\]

**Corollary 2.3.** Let $\mu \in \mathbb{C}, \alpha \geq 0, \beta \in \mathbb{R}, \alpha + i\beta \neq 0, p \in \mathbb{N}$ and $\text{Re} \{\mu\} \geq 0$, then
\[
N^n_{p,\mu}(\alpha, \beta, A, B) \subset N^n_{p,0}(\alpha, \beta, A, B).
\]

**Theorem 2.4.** Let $0 \leq \mu_1 \leq \mu_2$, $\alpha \geq 0, \beta \in \mathbb{R}, p \in \mathbb{N}, \alpha + i\beta \neq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then
\[
N^n_{p,\mu_2}(\alpha, \beta, A_2, B_2) \subset N^n_{p,\mu_1}(\alpha, \beta, A_1, B_1). \tag{2.5}
\]

**Proof.** Suppose that $f(z) \in N^n_{p,\mu_2}(\alpha, \beta, A_2, B_2)$ we have $f(z) \in A_p$ and
\[
\left\{ (1 + \mu_2) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu_2 \left(\frac{zf'(z)}{pf(z)}\right) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \right\} < \frac{1 + A_2z}{1 + B_2z}, \quad (z \in U).
\]
Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore it follows from Lemma 1.4 that
\[
\left\{ (1 + \mu_2) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu_2 \left(\frac{zf'(z)}{pf(z)}\right) \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \right\} < \frac{1 + A_1z}{1 + B_1z}, \quad (z \in U), \tag{2.6}
\]
that is $f(z) \in N^n_{p,\mu_2}(\alpha, \beta, A_1, B_1)$. So Theorem 2.4 is proved when $\mu_1 = \mu_2 \geq 0$.

When $\mu_2 > \mu_1 \geq 0$, then we can see from Corollary 2.3 that $f(z) \in N^n_{p,0}(\alpha, \beta, A_1, B_1)$, then
\[
\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} < \frac{1 + A_1z}{1 + B_1z}. \tag{2.7}
\]
Proof.
Suppose that differential subordinations (2.6) and (2.7) that
\[ A \in \text{Theorem 2.6.} \]
Let \( \text{Corollary 2.5.} \)
Therefore, from the definition of the subordination, we have
\[ 1 + \mu_1 \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} - \mu_1 \left( \frac{zf'(z)}{pf(z)} \right) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \]
\[ = (1 - \frac{\mu_1}{\mu_2}) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} + \mu_1 \left\{ \frac{1 + \mu_1}{\mu_2} \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \right\}. \]
It is obvious that \( \frac{1 + A_1 z}{1 + B_1 z} \) is analytic and convex in \( U \). So we obtain from Lemma 1.5 and differential subordinations (2.6) and (2.7) that
\[ \left\{ (1 + \mu_1) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} - \mu_1 \left( \frac{zf'(z)}{pf(z)} \right) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \right\} 1 + A_1 z \]
\[ = \frac{1 + A_1 z}{1 + B_1 z}, \]
that is, \( f(z) \in N^n_{\mu_1}(\alpha, \beta, A_1, B_1) \). Thus we have
\[ N^n_{\mu_2}(\alpha, \beta, A_2, B_2) \subset N^n_{\mu_1}(\alpha, \beta, A_1, B_1). \]

\[ \square \]

Corollary 2.5. Let \( 0 \leq \mu_1 \leq \mu_2, 0 \leq \rho_1 \leq \rho_2, \alpha \geq 0, \beta \in \mathbb{R}, p \in \mathbb{N} \) and \( \alpha + i\beta \neq 0 \) then
\[ N^n_{\mu_2}(\alpha, \beta, \rho_2) \subset N^n_{\mu_1}(\alpha, \beta, \rho_1). \]

\[ \square \]

Theorem 2.6. Let \( \mu \in \mathbb{C}, \alpha \geq 0, \beta \in \mathbb{R}, \mu + i\beta \neq 0, p \in \mathbb{N}, -1 \leq B \leq 1, A \neq B \) and \( A \in \mathbb{R} \). If \( f(z) \in N^n_{\mu}(\alpha, \beta, A, B) \), then
\[ \inf_{z \in U} \text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{p(\alpha + i\beta)} - 1 du \right\} \]
\[ < \text{Re} \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} < \sup_{z \in U} \text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{p(\alpha + i\beta)} - 1 du \right\}. \]

Proof. Suppose that \( f(z) \in N^n_{\mu}(\alpha, \beta, A, B) \), then from Theorem 2.1 we know that
\[ \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} < \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{p(\alpha + i\beta)} - 1 du. \]
(2.8)
Therefore, from the definition of the subordination, we have
\[ \text{Re} \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} > \inf_{z \in U} \text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{p(\alpha + i\beta)} - 1 du \right\}, \]
\[ \text{Re} \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} < \sup_{z \in U} \text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{p(\alpha + i\beta)} - 1 du \right\}. \]

\[ \square \]
Corollary 2.7. Let \( \mu \in \mathbb{C} \), \( \alpha \geq 0 \), \( \beta \in \mathbb{R} \), \( \mu + i\beta \neq 0 \), \( p \in \mathbb{N} \) and \( \rho < 1 \). If \( f(z) \in N_{p,\mu}^n(\alpha, \beta, 1-2\rho, -1) \), then
\[
\rho + (1 - \rho) \inf_{z \in U} \Re \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + zu}{1 - zu} u^{\frac{p(\alpha + i\beta)}{\mu n} - 1} du \right\} < \Re \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \]
\[
< \rho + (1 - \rho) \sup_{z \in U} \Re \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + zu}{1 - zu} u^{\frac{p(\alpha + i\beta)}{\mu n} - 1} du \right\}. \tag{2.9}
\]

Corollary 2.8. Let \( \mu \in \mathbb{C} \), \( \alpha \geq 0 \), \( \beta \in \mathbb{R} \), \( \mu + i\beta \neq 0 \), \( p \in \mathbb{N} \) and \( \rho > 1 \). If \( f(z) \in N_{p,\mu}^n(\alpha, \beta, 1-2\rho, -1) \), then
\[
\rho + (1 - \rho) \sup_{z \in U} \Re \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + zu}{1 - zu} u^{\frac{p(\alpha + i\beta)}{\mu n} - 1} du \right\} < \Re \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \]
\[
< \rho + (1 - \rho) \inf_{z \in U} \Re \left\{ \frac{p(\alpha + i\beta)}{\mu n} \int_0^1 \frac{1 + zu}{1 - zu} u^{\frac{p(\alpha + i\beta)}{\mu n} - 1} du \right\}. \tag{2.10}
\]

Next, several new differential subordination results of non-Bazilevič class of order \( \alpha + i\beta \) are established.

3. Subordination for \( p \)-Valent Non-Bazilevič Class of Order \( \alpha + i\beta \)

By employing lemma 1.6, the following result is given.

Theorem 3.1. Let \( q \) be univalent in \( U \), \( \mu \in \mathbb{C}^* \), \( \alpha \geq 0 \), \( \beta \in \mathbb{R} \) and \( \alpha + i\beta \neq 0 \). Suppose that \( q \) satisfies
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, - \Re \left\{ \frac{p(\alpha + i\beta)}{\mu} \right\} \right\}. \tag{3.1}
\]

If \( f \in A_p \) satisfies the subordination
\[
\left\{ (1 + \mu) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} - \mu \left( \frac{zf'(z)}{pf(z)} \right) \left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \right\} \prec q(z) + \mu \frac{zq'(z)}{p(\alpha + i\beta)}, \tag{3.2}
\]
then
\[
\left( \frac{z^p}{f(z)} \right)^{\alpha + i\beta} \prec q(z) \tag{3.3}
\]
and \( q(z) \) is the best dominant.
Proof. Define the function $\Phi(z)$ by

$$\Phi(z) = \left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta}. \quad (3.4)$$

Having (3.4) differentiated logarithmically in connection with $z$, we have

$$\frac{z\Phi'(z)}{\Phi(z)} = p(\alpha + i\beta) \left( 1 - \frac{zf'(z)}{pf(z)} \right), \quad (3.5)$$

which, with respect to hypothesis (3.2) of Theorem 3.1, the following subordination is obtained:

$$\Phi(z) + \mu z \Phi'(z) \preceq q(z) + \mu \frac{zp'(z)}{p\alpha + i\beta}. \quad (3.6)$$

Theorem 3.1 assertion is now followed by the use of Lemma 1.6 with $\eta = \mu p(\alpha + i\beta)$ and $\sigma = 1$. \qed

Remark 3.2. For the choice $q(z) = \frac{1+z}{1+Bz}$ in Theorem 3.1, the following the corollary is obtained.

Corollary 3.3. Let $\mu \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, and

$$\text{Re} \left\{ \frac{1-Bz}{1+Bz} \right\} > \max \left\{ 0, -\text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu} \right\} \right\}, \ (z \in \mathbb{U}). \quad (3.7)$$

If $f \in \mathcal{A}_p$, and

$$\left\{ (1+\mu) \left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} - \mu \left( \frac{zf'(z)}{pf(z)} \right) \left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} \right\} \prec \frac{\mu(A-B)z}{p(\alpha + i\beta)(1+Bz)^2} + \frac{1+Az}{1+Bz}, \quad (3.8)$$

then

$$\left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} \prec \frac{1+Az}{1+Bz} \quad (3.9)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Remark 3.4. For the choice $q(z) = \frac{1+z}{1-z}$ in Theorem 3.1, the following the corollary is obtained.

Corollary 3.5. Let $\mu \in \mathbb{C}^*$, and

$$\text{Re} \left\{ \frac{1+z}{1-z} \right\} > \max \left\{ 0, -\text{Re} \left\{ \frac{p(\alpha + i\beta)}{\mu} \right\} \right\}, \ (z \in \mathbb{U}). \quad (3.10)$$

If $f \in \mathcal{A}_p$, and

$$\left\{ (1+\mu) \left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} - \mu \left( \frac{zf'(z)}{pf(z)} \right) \left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} \right\} \prec \frac{2\mu z}{p(\alpha + i\beta)(1-z)^2} + \frac{1+z}{1-z}, \quad (3.11)$$

then

$$\left( \frac{z^p}{f(z)} \right)^{\alpha+i\beta} \prec \frac{1+z}{1-z} \quad (3.12)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.
4. Superordination for $p$–Valent Non-Bazilevič Class of Order $\alpha + i\beta$

**Theorem 4.1.** Let $q$ be convex univalent in $U$, $\mu \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\alpha + i\beta \neq 0$. Suppose that $q$ satisfies

$$\text{Re} \{ \mu \} > 0$$

(4.1)

and \((\frac{z^p}{f(z)})^{\alpha + i\beta} \in H[q(0), 1] \cap Q\). Let

\[(1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \] 

(4.2)

be univalent in $U$, If

\[q(z) + \mu zq'(z) < (1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta},\] 

(4.3)

then

\[q(z) < \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \] 

(4.4)

and $q(z)$ is the best subordinant.

**Proof.** Define the function $\Phi(z)$ by

\[\Phi(z) = \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta}.\] 

(4.5)

Then based on a computation, it indicates that

\[\Phi(z) + \frac{\mu z\Phi'(z)}{p(\alpha + i\beta)} < (1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta}.\] 

(4.6)

Theorem 4.1 follows from Lemma 1.7.

□

**Remark 4.2.** Taking $q(z) = \frac{1 + Az}{1 + Bz}$ in Theorem 4.1 the following the corollary is obtained.

**Corollary 4.3.** Let $-1 \leq B < A \leq 1$. Let $q$ be convex univalent in $U$. Suppose that $q$ satisfies $\text{Re} \{ \mu \} > 0$, and \((\frac{z^p}{f(z)})^{\alpha + i\beta} \in H[q(0), 1] \cap Q\). Let

\[(1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \] 

(4.7)

be univalent in $U$, If

\[\frac{\mu(A - B)z}{p(\alpha + i\beta)(1 + Bz)^2} + \frac{1 + Az}{1 + Bz} < \left\{ (1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} - \mu \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \right\},\] 

(4.8)

then

\[\frac{1 + Az}{1 + Bz} < \left(\frac{z^p}{f(z)}\right)^{\alpha + i\beta} \] 

(4.9)

and $\frac{1 + Az}{1 + Bz}$ is the best subordinant.
5. Sandwich Results for p−Valent Non-Bazilevic Class of Order $\alpha + i\beta$

Combining the differential subordination and supordination results, the sandwich results are highlighted as follows.

**Theorem 5.1.** Let $q_1$ be convex univalent and let $q_2$ be univalent in $U$, $\mu \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\alpha + i\beta \neq 0$. Suppose $q_1$ satisfies (4.1) and $q_2$ satisfies (3.1). If $0 \neq (z^p f(z))^{\alpha + i\beta} \in H[q(0), 1] \cap Q$, $(1 + \mu)(z^p f(z))^{\alpha + i\beta} - \mu (z^p f(z))'(z^p f(z))^{\alpha + i\beta} \in H[q(0), 1] \cap Q$, and if $f \in A_p$ satisfies

$$q_1(z) + \mu \frac{zq_1'(z)}{p(\alpha + i\beta)} < (1 + \mu) \left( z^p \frac{f(z)}{f'(z)} \right)^{\alpha + i\beta} - \mu \left( z^p f'(z) \right) \left( z^p f(z) \right)^{\alpha + i\beta} < q_2(z) + \mu \frac{zq_2'(z)}{p(\alpha + i\beta)},$$

then

$$q_1(z) < \left( z^p \frac{f(z)}{f'(z)} \right)^{\alpha + i\beta} < q_2(z)$$

(5.1)

and $q_1(z)$ and $q_2(z)$ are respectively, the best subordinant and best dominant.

**Proof.** Simultaneously applying the techniques of the proof of Theorem 3.1 and Theorem 4.1. □

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**References**

[1] Z. Wang, C. Gao & M. Liao, *On certain generalized class of non-Bazilevic functions*, Acta Mathematica. Academiae Paedagogicae Nyıregyháziensis. New Series 21(2005), 147-154.

[2] N. Tuneski & M. Darus, *Fekete-Szegö functional for non-Bazilevic functions*, Acta Mathematica. Academiae Paedagogicae Nyıregyháziensis. New Series 18(2002), 63-65.

[3] S. S. Miller & P. T. Mocanu, *Differential subordinations and univalent functions*, The Michigan Mathematical Journal 28. 2(1981), 157-172.

[4] S. S. Miller & P. T. Mocanu, *Differential Subordination: Theory and Applications*, Ser, Monogr. Textbooks Pure Appl. Math 225(2000).

[5] J. L. Liu & K. I. Noor, *Some properties of Noor integral operator*, Journal of Natural Geometry 21. 1/2(2002), 81-90.

[6] A. G. Alamoush & M. Darus, *On Certain Class of Non-Bazilevič Functions of Order $\alpha + i\beta$ Defined by a Differential Subordination*, International Journal of Differential Equations 2014(2014).

[7] S. S. Miller & P. T. Mocanu, *Differential subordinations and univalent functions*, The Michigan Mathematical Journal 28. 2(1981), 157-172.

[8] L. Mingsheng, *On a Subclass of $p$−Valent Close-to-convex Functions of Order $\beta$ and Type $\alpha$*, Journal of Mathematical Study, 30. 1(1997), 102–104.

[9] L. Mingsheng, *On certain class of analytic functions defined by differential subordination*, Acta Mathematica Scientia 22. 3(2002), 388-392.

[10] M. Obrodovic, *A class of univalent functions*, Hokkaido Mathematical Journal 27. 2(1998), 329-335.

[11] T. N. Shanmugam, V. Ravichandran & S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions*, J. Austr. Math. Anal. Appl 3. 1(2006), 1-11.

[12] S. S. Miller & P. T. Mocanu, *Subordinants of differential superordinations*, Complex variables 48 .10(2003), 815-826.

[13] F. Yousef, A. A. Amourah, and M. Darus, *Differential sandwich theorems for $p$-valent functions associated with a certain generalized differential operator and integral operator*, Italian Journal of Pure and Applied Mathematics 36 (2016), 543-556.
[14] A. A. Amourah, F. Yousef, T. Al-Hawary and M. Darus, *A certain fractional derivative operator for p-valent functions and new class of analytic functions with negative coefficients*, Far East Journal of Mathematical Sciences, 99. (1)(2016), 75-87.

[15] A. A. Amourah, F. Yousef, T. Al-Hawary and M. Darus, *On a Class of p-Valent non-Bazilevič Functions of Order $\mu + i\beta$*, International Journal of Mathematical Analysis, 10. (15) (2016), 701-710.

[16] A. G. Alamoush and M. Darus, *Subordination results for a certain subclass of non-bazilevic analytic functions defined by linear operator*, Italian Journal of Pure and Applied Mathematics, 34 (2015), 375-388.

[17] T. Al-Hawary, A. A. Amourah, and M. Darus, *Differential sandwich theorems for p-valent functions associated with two generalized differential operator and integral operator*, International Information Institute (Tokyo). Information, 17. (8)(2014), 3559.

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