Atom canonicity, and omitting types in temporal and topological cylindric algebras

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Abstract

We study what we call topological cylindric algebras and tense cylindric algebras defined for every ordinal $\alpha$. The former are cylindric algebras of dimension $\alpha$ expanded with $S4$ modalities indexed by $\alpha$. The semantics of representable topological algebras is induced by the interior operation relative to a topology defined on their bases. Tense cylindric algebras are cylindric algebras expanded by the modalities $F$ (future) and $P$ (past) algebraising predicate temporal logic.

We show for both tense and topological cylindric algebras of finite dimension $n > 2$ that infinitely many varieties containing and including the variety of representable algebras of dimension $n$ are not atom canonical. We show that any class containing the class of completely representable algebras having a weak neat embedding property is not elementary. From these two results we draw the same conclusion on omitting types for finite variable fragments of predicate topologic and temporal logic. We show that the usual version of the omitting types theorem restricted to such fragments when the number of variables is $> 2$ fails dramatically even if we considerably broaden the class of models permitted to omit a single non principal type in countable atomic theories, namely, the non-principal type consisting of co atoms.

1 Introduction

1.1 General

We alert the reader to the fact that the introduction is quite long since we discuss extensively many concepts related to the subject matter of the paper, in the process emphasizing similarities and highlighting differences. The topic is rich, the paper is interdisciplinary between temporal logic, topological logic and algebraic logic. Besides some history is surveyed, dating back to the seminal work of Tarski and McKinsey up to the present time.
Universal logic addresses different logical systems simultaneously in essentially three ways. Either abstracting common features, or building new bridges between them, or constructing new logics.

We do the first two in what follows, though the third can also be implemented by combining temporal and topological logic getting a natural multidimensional modal logic of space and time. Such logics do exist in the literature; an example is dynamic topological logic that studies the modal aspects of dynamical systems.

However, we defer this unification to another paper. We apply cylindric algebra theory to topological and minimal temporal predicate logic, also known as tense predicate logic, with finitely many variables, so that the bridges are built via algebraic logic, and their common abstraction is cylindric algebra theory when we expand cylindric algebras by modalities like the interior box operator stimulated by a topology on the base of a cylindric algebra that happens to be representable, or the temporal $F$ (future) and $P$ (past).

We will also have occasion to make detours into the functionally maximal temporal logic with Still and Until.

Another unifying framework here is that we repeat an extremely rewarding deed of Henkin’s performed ages ago (in the sixties of the 20th century), when he decided to study cylindric algebras of finite dimension that can be seen in retrospect as the modal algebras of finite variable fragments of first order logic. Here the dimension is the same as the number of available variables.

The repercussions of such a view turned out absolutely astounding and it has initiated a lot of quite deep research till this day, involving algebraic and modal logicians from Andreka to Venema; this paper included.

We immitate Henkin, but we add a topological and temporal dimension. We study finite variable fragments of topological, tense and temporal predicate logic using the well developed machinery of algebraic logic. We reap the harvest of our algebraic results by deducing the metalogic one of omitting types.

We address the algebraic notions of atom-canonicity an important persistense property in modal logic, and complete representations an important notion for cylindric-like algebras, both notions proved closely related to the algebraic modal property of atom-canonicity and the meta-logical notion of omitting types [34].

We do such tasks for the modal algebras of both temporal and topological logics of finite dimension $\geq 2$, culminating in a very strong negative result on omitting types when semantics are restricted to clique guarded semantics, proving the result in the abstract.

The interconnection between the three notions mentioned above manifests itself blatantly in this paper, where we draw a deep conclusion concerning the failure of the Orey-Henkin omitting types. This is proved using quite
sophisticated machinery from algebraic logic, more specifically from cylindric algebra theory, namely, rainbow constructions, which were first used in the context of relation algebras.

Using this powerful technique, we construct certain representable algebras, having countably many atoms of finite dimension \( n > 2 \). In the first case, we deal with an algebra with no complete representations. In the second case we deal with an algebra whose Dedekind-MacNeille completion is outside infinitely many varieties containing properly the variety of representable algebras. These two constructions are worked out for both temporal and topological cylindric algebras of dimension \( n \).

Topological logic with \( n \) many variables can be viewed as a \( 2n \) dimensional propositional multimodal logic or a Kripke complete logic that is the \( n \) product of decidable bi-modal logics whose frames are of the form \((U, U \times U, R)\); with \( R \) is the accessibility relation of an \( S4 \) modality, in short \( S5 \times S4 \).

On the other hand, modal algebras of temporal logic of dimension \( m \), that are representable in a concrete sense to be specified below, can be seen as product of cylindric set algebras of the same dimension \( n \) indexed by a time flow \( T = (T, <) \), with navigating functions between the different worlds that are cartesian squares.

So, metaphorically cylindric set algebras of dimension \( n < \omega \) are snapshots of temporal semantics at one moment of time in the time flow.

1.2 Universal logic

Universal logic is the field of logic that is concerned with giving an account of what features are common to all logical structures. If slogans are to be taken seriously, then universal logic is to logic what universal algebra is to algebra. The term “universal logic” was introduced in the 1990s by Swiss Logician Jean Yves Beziau but the field has arguably existed for many decades. Some of the works of Alfred Tarski in the early twentieth century, on metamathematics and in algebraic logic, for example, can be regarded undoubtedly, in retrospect, as fundamental contributions to universal logic.

Indeed, there is a whole well established branch of algebraic logic, that attempts to deal with the universal notion of a logic. Pioneers in this branch include Andréka and Németi [8] and Blok and Pigozzi [13]. The approach of Andréka and Németi though is more general, since, unlike the approach in [13] which is purely syntactical, it allows semantical notions stimulated via so-called “meaning functions” [10], to be defined below. Other universal approaches to many cylindric-like algebras were implemented in [36] in the context of the very general notion of what is known in the literature as systems of varieties definable by a Monk’s schema [32, 19], and in the context of category theory in [40].
One aim of universal logic is to determine the domain of validity of such and such metatheorem (e.g. the completeness theorem, the Craig interpolation theorem, or the Orey-Henkin omitting types theorem of first order logic) and to give general formulations of metatheorems in broader, or even entirely other contexts. This is also done in algebraic logic, by dealing with modifications and variants of first order logic resulting in a natural way during the process of algebraisation, witness for example the omitting types theorem proved in [34].

This kind of investigation is extremely potent for applications and helps to make the distinction between what is really essential to a particular logic and what is not.

During the 20th century, numerous logics have been created, to mention only a few: intuitionistic logic, modal logic, topological logic, topological dynamic logic, spatial logic, dynamic logic, tense logic, temporal logic, many-valued logic, fuzzy logic, relevant logic, para-consistent logic, non monotonic logic, etc.

The rapid development of computer science, since the fifties of the 20th century initiated by work of giants like Godel, Church and Turing, ultimately brought to the front scene other logics as well, like logics of programs and lambda calculas (the last can be traced back to the work of Church).

After a while it became noticable that certain patterns of concepts kept being repeated albeit in different logics. But then the time was ripe to make in retrospect an inevitable abstraction like as the case with the field of abstract model theory (Lindstrom’s theorem is an example here).

1.3 Abstract algebraic logic and institutions

Other abstract approaches to logic, besides categorial logic, are topoi theory and abstract algebraic logic which focuses on the algebraic essence of numeras logics, like first order logic, modal logics, logics with infinitary predicates, a new addition introduced here, namely, finite variable fargments of tense, temporal and topological predicate logics.

Abstract algebraic logic provides a unifying algebraic framework viz the process of algebraisation, categorial logic via institutions, and topoi theory, and last but not least universal logic.

Institution theory is a major model theoretic trend of universal logic that formalizes within category theory the intuitive notion of a logical system, including syntax, semantics, and the satisfaction relation between them. It owes its birth to so-called computing science as a response to the explosion of new logics and it developed to an important foundational theory. Though computing science can be blamed for its poor intellectual value, but institutions is a very significant exception. In fact what is now labelled as computing science is hardly a science in the way mathematics or theoretical physics are. Comput-
ing science can be viewed as a playground where several actors, most notably mathematics, logic, engineering and philosophy, play.

Sometimes an interesting play which has not only brought significant changes and development to the actors but also revolutionized our way of thinking.

Institution theory, universal logic and abstract algebraic logic are no exceptions. All three branches have travelled well in model theory and logic. The way we think of logic and model theory will never be the same as before. The development of model and proof theory in the very abstract setting of arbitrary institutions, is free of commitment to a particular logical system. When we live without concrete models, sentences, satisfaction, and so on, we reep the harvest of another level of abstraction and generality.

A deeper understanding of model theoretic phenomena not hindered by the largely irrelevant details of a particular logical system, but instead guided by structurally by clear cut causality, necessarily follows.

The latter aspect is based upon the fact that concepts come naturally as presumed features that a “logic” might exhibit or not and are defined at the most appropriate level of abstraction; hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down regardless of their depth.

The continuous interplay between the specific and the general in institution theory brings a large array of new results for particular non-conventional approaches, unifies several known results, produces new results in well-studied conventional areas, reveals previously unknown causality relations, and dismantles some which are usually assumed as natural. Access to highly non-trivial results is also considerably facilitated.

The similarity of institutions to the notion of full fledged algebriazable logics in the sense of Andréka and Németi [10] is striking.

They both assume nothing about the specific nature of a logical system, models and sentences are arbitrary objects; the only assumption being that there is a satisfiability relation between models and texts or sentences, telling whether a sentence hold in a given model; this is implemented in the Andréka and Németi approach using meaning functions to be dealt with below. In institutions such connections are formulated in the extremely abstract level of category theory.

A crucial unifying feature is that notions of models, sentences and the satisfiability relation are defined abstractly; the definition is broad, in both cases, but narrow enough to obtain significant results at such a level of generality. Such thereby obtained ‘meta-theorems’ can even illuminate various aspects of their concrete instances.

A vital difference though is in the algebraic approach the signature, determining the logical connectives is fixed in advance.

In institutions signatures vary, and they are synchronized by two functors;
a syntactical functor navigating between sentences in different signatures, and
another, a semantical one, taking models in a given signature, to models in
a possibly different signature in such a way that semantics, expressed via a
satisfiability relation defined between sentences and models, is preserved.

1.4 Topological and minimal Temporal logic, tense logic

Motivated by questions like: which spatial structures may be characterized
by means of modal logic, what is the logic of space, how to encode in modal
logic different geometric relations, topological logic provides a framework for
studying the confluence of the topological semantics for S4 modalities, based
on topological spaces rather than Kripke frames, with the S4 modality induced
by the interior operator.

Topological logic was introduced by Makowsky, Ziegler and Sgro [42]. Such
logics have a classical semantics with a topological flavour, addressing spatial
logics and their study was approached using algebraic logic by Georgescu [17],
the task that we further pursue in this paper. Topological logics are apt for
dealing with logic and space; the overall point is to take a common mathematical
model of space (like a topological space) and then to fashion logical tools
to work with it.

One of the things which blatantly strikes one when studying elementary
topology is that notions like open, closed, dense are intuitively very transpar-
ent, and their basic properties are absolutely straightforward to prove. How-
ever, topology uses second order notions as it reasons with sets and subsets
of ‘points’. This might suggest that like second order logic, topology ought to
be computationally very complex. This apparent dichotomy between the two
paradigms vanishes when one realizes that a large portion of topology can be
formulated as a simple modal logic, namely, S4! This is for sure an asset for
modal logics tend to be much easier to handle than first order logic let alone
second order.

The project of relating topology to modal logic begins with work of Alfred
Tarski and J.C.C McKinsey [43]. Strictly speaking Tarski and McKinsey did
not work with modal logic, but rather with its algebraic counterpart, namely,
Boolean algebras with operators which is the approach we adopt here; the op-
erators they studied where the closure operator induced on what they called the
algebra of topology, certainly a very ambitious title, giving the impression that
the paper aspired to completely algebraise topology. This paper was published
around the time when there was an interest in algebraising metamathemat-
ics as well, culminating in defining cylindric and polyadic algebras obtained
independently by Tarski in Berkely and Halmos in Stanford, respectively.

Ever since there has been extensive research to highlight the similarities and
differences between such algebraisations of predicate logic and it is commonly
accepted now that they belong to different paradigms [34], though they are essentially the same when they are restricted algebraise first order logic with equality, giving the so-called infinite dimensional locally finite algebras, to be further elaborated upon in a while.

In retrospect McKinsey and Tarski showed, that the Stone representation theorem for Boolean algebras extend to algebras with operators to give topological semantics for classical propositional modal logic, in which the ‘necessity’ operation is modeled by taking the interior (dual operation) of an arbitrary subset of topological space. Although the topological completeness of $\mathbf{S4}$ has been well known for quite a long time, it was until recently considered as some exotic curiosity, but certainly having mathematical value. It was in the 1990-ies that the work of McKinsey and Tarski, came to the front scene of modal logic (particularly spatial modal logic), drawing serious attention of many researchers and inspiring a lot of work stimulated basically by questions concerning the ‘modal logic of space’; how to encode in modal logic different geometric relations? A point of contact here between topological spaces, geometry, and cylindric algebra theory is the notion of dimension.

From the modern point of view one introduces a basic modal language with a set $\mathbb{At}$ of atomic propositions, the logical Boolean connectives $\land$, $\neg$ and a modality $I$ to be interpreted as the interior operation. Let $X$ be a topological space. The modal language $L_0$ is interpreted on such a space $X$ together with an interpretation map $i : \mathbb{At} \to \wp(X)$. For atomic $p \in \mathbb{At}$, $i(p)$ says which points satisfy $p$. We do not require that $i(p)$ is open. $(X, i)$ is said to be a topological model. Then $i$ extends to all $L_0$ formulas by interpreting negation as complement relative to $X$, conjunction as intersection and $I$ as the interior operator. In symbols we have:

\[
i(\neg \phi) = X \sim i(\phi), \\
i(\phi \land \psi) = i(\phi) \cap i(\psi), \\
i(I(\phi)) = \text{int} i(\phi).
\]

The main idea here is that the basic properties of the Boolean operations on sets as well as the salient topological operations like interior and its dual the closure, correspond to to schemes of sentences. For example, the fact that the interior operator is idempotent is expressed by

\[i((II\phi) \leftrightarrow (I\phi)) = X.\]

The natural question to ask about this language and its semantics is: Can we characterize in an enlightening way the sentences $\phi$ with the property that for all topological models $(X, i)$, $i(\phi) = X$; these are the topologically valid sentences. They are true at all points in all spaces under whatever interpretation. More succinctly, do we have a nice completeness theorem?
Tarski and McKinsey proved that the topologically valid sentences are exactly those provable in the modal logic $S4$. $S4$ has a seemingly different semantics using standard Kripke frames. Now $X$ is viewed as the set of possible worlds. In $S4$, $I$ is read as all points which the current point relates to. To get a sound interpretation of $S4$ we should require that the current point is related to itself. Without this requirement we get only a transitive frame; if the order is further a linear order, then this represents what is known in temporal logic as a 'flow of time' getting a transitive model. (Strictly speaking, in a linear flow of time we have two transitive linear relations $<$ and $>$ that are converses, reflecting past and future; it is a bi-modal logic).

The reflexivity condition, together with transitivity leads naturally to preordered model.

A pre-ordered model is defined to be a triple $(X, \leq, i)$ where $(X, \leq)$ is a pre-order and $i : At \to \wp(X)$ where

$$i(I(\phi)) = \{x : \{y : x \leq y\} \subseteq i(\phi)\}.$$ 

Temporally world $x' \in X$ is a successor of world $x \in X$ if $x \leq x'$, $x$ and $x'$ are equivalent worlds if further $x \leq x'$ and $x' \leq x$. We have a completely analogous result here; $\phi$ is valid in pre-ordered models if $\phi$ is provable in $S4$.

One can prove the equivalence of the two systems using only topologies on finite sets. Let $(X, \leq)$ be a pre-order. Consider the Alexandrov topology on $X$, the open sets are the sets closed upwards in the order. This gives a topology, call it $O_{\leq}$. A correspondence between topological models and pre-ordered models can thereby be obtained, and as it happens we have for any pre-ordered model $(X, \leq, i)$, all $x \in X$, and all $\phi \in \mathcal{L}_0$

$$x \models \phi \text{ in } (X, \leq, i) \iff x \models \phi \text{ in } (X, O_{\leq}, i).$$

Using this result together with the fact that sentences satisfiable in $S4$ have finite topological models, thus they are Alexandrov topologies, one can show that the semantics of both systems each is interpretable in the other; they are equivalent. We can summarize the above discussion in the following neat theorem, that we can and will attribute to McKinsey, Tarski and Kripke; this historically is not very accurate. For a topological space $X$ and $\phi$ an $S4$ formula we write $X \models \phi$, if $\phi$ is valid topologically in $X$ (in either of the senses above). For example, $w \models \Box \psi$ iff for all $w' \leq w$, then $w' \models \psi$, where $\leq$ is the relation $x \leq y$ iff $y \in \text{cl}\{x\}$.

**Theorem 1.1.** (McKinsey-Tarski-Kripke) Suppose that $X$ is a dense in itself metric space (every point is a limit point) and $\phi$ is a modal $S4$ formula. Then the following are equivalent

$$(1) \quad \phi \in S4.$$
(2) $\models \phi$.

(3) $X \models \phi$.

(4) $\mathbb{R} \models \phi$.

(5) $Y \models \phi$ for every finite topological space $Y$.

(6) $Y \models \phi$ for every Alexandrov space $Y$.

One can say that finite topological space or their natural extension to Alexandrov topological spaces reflect faithfully the $S_4$ semantics, and that arbitrary topological spaces generalize $S_4$ frames. On the other hand, every topological space gives rise to a normal modal logic. Indeed $S_4$ is the modal logic of $\mathbb{R}$, or any metric that is separable and dense in itself space, or all topological spaces, as indicated above. Also a recent result is that it is also the modal logic of the Cantor set, which is known to be Baire isomorphic to $\mathbb{R}$.

But, on the other hand, modal logic is too weak to detect interesting properties of $\mathbb{R}$, for example it cannot distinguish between $[0, 1]$ and $\mathbb{R}$ despite their topological dissimilarities, the most striking one being compactness; $[0, 1]$ is compact, but $\mathbb{R}$ is not.

The huge field of Temporal logic, of which dynamic topological logic is an example is broadly used to cover approaches to the representation of temporal information with a logica (modal) framework.

When asked to think of time in an abstract way, many people will probably figure our a picture of a line. The mathematics of this picture is given by a set of time points, together with an irreflexive transitive order. To represent time on a frame we take a relational structure of the form $\mathfrak{T} = (T, <)$ where $T$ is the set of worlds namely the ‘moments’ and $<$ is an irreflexive transitive relation called the precedence relation. If $(s, t) \in <$, we say that $s$ is earlier than $t$. Essentially temporal logic extends classical propositional logic with a set of temporal operators that navigate between worlds using the accessibility relation $\prec$.

To capture such structures modally, in the syntax, one introduces two modal operators $G, H$ with intended meanings for $G, H$ are are follows:

$G$ is it will always be the case that and $H$ is it has always been the case that.

The spice of temporal logic, however, lies in a basic idea, namely, to use new, non classical connectives to relate the truth of formulated in possibly distinct moments.

The operators denoted by $F$ (short for future) and $P$ (short for past) are such, defined by $F\phi$ is $\neg G \neg \phi$ and $P\phi$ is $\neg H \neg \phi$, where $\phi$ is a formula in the modal language. Here $F\phi$ is to be read as ‘at some time in the future $\phi$ will
be the case, and $P\phi$ is read as ‘at some time in the past $\phi$ holds’. This can be easily seen by reading their duals $G$ and $H$ simply as ‘henceforth’ and ‘hitherto’, respectively.

Formally semantics are captured by Kripke frames of the form $((T, <))$ where $<$ is an irreflexive transitive order. A model is a triple $\mathfrak{M} = (T, <, i)$, where $i$ is a map from the propositional variables to the set of worlds (moments), and just truth relation defined inductively at moment $t$ by:

$$\mathfrak{M}, t \models q \iff \phi(t)(q) = 1$$

The Booleans are the usual and the temporal operators:

$$M, t \models G\phi \iff (\forall s)(t < s, \mathfrak{M}, s \models \phi).$$

$$M, t \models H\phi \iff (\forall s)(s < t, \mathfrak{M}, s \models \phi).$$

Now some valid formulas in the intended interpretation are $p \rightarrow HFp$: what is has always been going to be, $p \rightarrow GPp$: what is will always have been, $G(p \rightarrow q) \rightarrow Gp \rightarrow Gq$: whatever will always follow from what always will be.

And these in fact, together with $H(p \rightarrow q) \rightarrow Hp \rightarrow Hq)$, constitute a complete set of axioms for the specially significant Minimal Tense Logic $K_t$ which has the the following four axioms:

1. $p \rightarrow HFp,$
2. $p \rightarrow GPp,$
3. $H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq),$
4. $G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq).$

together with two rules of necessitation, or temporal inference, namely:

From $p$ derive $Hp$ and from $p$ derive $Gp$ and of course the rules of ordinary propositional logic.

The theorems of $K_t$ express, essentially, those properties of the tense operators which do not depend on any specific assumptions about the temporal order. Of obvious interest is tensed predicate logic, where the tense operators are added to classical first order logic. This enables us to express distinctions concerning the logic of both time and existence, the latter reflected by the existential quantifier. For example, the statement a minister will be president can be interpreted as ‘Someone who is now a minister will be a president at some future time’ Formally:

$$\exists x(minister(x) \land F(president(x)).$$
If we replace a minister by philosopher and president by king then another
away to express *A philosopher will be a king* is

\[ \exists x F(\text{minister}(x)) \land \text{king}(x). \]

In words: there now exists someone who will at some future time be both a
philosopher and a king.

\[ F(\exists x (\text{minister}(x) \land F\text{King}(x)) \]
works for both; it reads ‘there will exist someone who is a minister (philosopher) and later will be a president (king)’.

We shall study finite variable fragments of both predicate and topological
logic

### 1.5 Atom canonicity and omitting types

Assume that we have a class of Boolean algebras with operators for which
we have a semantical notion of representability (like Boolean set algebras or
cylindric set algebras). A weakly representable atom structure is an atom
structure such that at least one atomic algebra based on it is representable. It
is strongly representable if all atomic algebras having this atom structure are
representable. The former is equivalent to that the term algebra, that is, the
algebra generated by the atoms, in the complex algebra is representable, while
the latter is equivalent to that the complex algebra is representable.

Could an atom structure be possibly weakly representable but *not* strongly
representable? Ian Hodkinson [28], showed that this can indeed happen for
both cylindric algebras of finite dimension \( \geq 3 \), and relation algebras, in the
context of showing that the class of representable algebras, in both cases, is not
closed under \( \mathcal{D} \) completions. In fact, he showed that this can be witnessed on an
atomic algebras, so that the variety of representable relation algebras algebras
and cylindric algebras of finite dimension \( > 2 \) are not atom-canonical. (The
complex algebra of an atom structure is the completion of the term algebra.)
This construction is somewhat complicated using a rainbow atom structure.
It has the striking consequence that there are two atomic algebras sharing the
same atom structure, one is representable the other is not.

This construction was simplified and streamlined, by many authors, in-
cluding the author, but Hodkinson’s construction, as we indicate below, has
the supreme advantage that it has a huge potential to prove analogous the-
orems on Dedekind-MacNeille completions, and atom-canonicity for several
varieties of topological algebras including properly the variety of representable
cylindric-like algebras, whose members have a neat embedding property, such
as polyadic algebras with and without equality and Pinter’s substitution al-
gebras. In fact, in such cases atomic *representable countable algebras* will be
constructed so that their Dedekind-MacNeille completions are outside such
varieties.
Let $K_n$ be the class of either topological or tense cylindric algebras of dimension $n$, $RK_n$ the class of representable $K_n$s, and $\mathcal{R}_{ca}$ stands for forming cylindric reducts. We show, that for $n > 2$ finite, there is $\mathfrak{A} \in RK_n$ with countably many atoms such that $\mathcal{R}_{ca}\mathfrak{C}_m\mathcal{A}_\mathfrak{A} \notin S\mathfrak{M}_n CA_{n+4}$ inferring that the varieties $S\mathfrak{M}_n K_{n+k}$, for $n > 2$ finite for any $k \geq 4$, not closed under $d$ completions. Such results, as illustrated below will have non-trivial (to say the least) repercussions on omitting types for finite variable fragments.

Lately, it has become fashionable in algebraic logic to study representations of abstract algebras that has a complete representation [34] for an extensive overview. A representation of $\mathfrak{A}$ is roughly an injective homomorphism from $f : \mathfrak{A} \rightarrow \wp(V)$ where $V$ is a set of $n$-ary sequences; $n$ is the dimension of $\mathfrak{A}$, and the operations on $\wp(V)$ are concrete and set theoretically defined, like the Boolean intersection and cylindrifiers or projections. A complete representation is one that preserves arbitrary disjuncts carrying them to set theoretic unions. If $f : \mathfrak{A} \rightarrow \wp(V)$ is such a representation, then $\mathfrak{A}$ is necessarily atomic and $\bigcup_{x \in A_t} f(x) = V$.

Let us focus on cylindric algebras for some time to come. It is known that there are countable atomic $\mathcal{R}_{CA_n}$s when $n > 2$, that have no complete representations; in fact, the class of completely representable $CA_n$s when $n > 2$, is not even elementary [26, corollary 3.7.1].

Such a phenomena is also closely related to the algebraic notion of atom-canonicity, as indicated, which is an important persistence property in modal logic and to the metalogical property of omitting types in finite variable fragments of first order logic [34, theorems 3.1.1-2, p.211, theorems 3.2.8, 9, 10]. Recall that a variety $V$ of Boolean algebras with operators is atom-canonical, if whenever $\mathfrak{A} \in V$, and $\mathfrak{A}$ is atomic, then the complex algebra of its atom structure, $\mathcal{C}_m\mathcal{A}_\mathfrak{A}$ for short, is also in $V$.

If $\mathfrak{A}$ is a weakly representable but not strongly representable, then $\mathcal{C}_m\mathfrak{A}$ is not representable; this gives that $\mathcal{R}_{CA_n}$ for $n > 2$ $n$ finite, is not atom-canonical. Also $\mathcal{C}_m\mathfrak{A}$ is the $d$ completion of $\mathfrak{A}$, and so obviously $\mathcal{R}_{CA_n}$ is not closed under $d$ completions.

On the other hand, $\mathfrak{A}$ cannot be completely representable for, it can be shown without much ado, that a complete representation of $\mathfrak{A}$ induces a representation of $\mathcal{C}_m\mathfrak{A}$ [26, definition 3.5.1, and p.74].

Finally, if $\mathfrak{A}$ is countable, atomic and has no complete representation then the set of co-atoms (a co-atom is the complement of an atom), viewed in the corresponding Tarski-Lindenbaum algebra, $\mathfrak{F}_T$, as a set of formulas, is a non principal-type that cannot be omitted in any model of $T$; here $T$ is consistent if $|A| > 1$. This last connection was first established by the author leading up to [12] and more, see e.g [20].

The reference [34] contains an extensive discussion of such notions.
2 Guarded and locally guarded fragments of topological logic

We also focus on cylindric algebras of dimension $m$ in this part. We explain why non atom-canonicity of the varieties $S^\mathfrak{c}N^r_mCA_{n+k}$, $k \geq 3$, non-elementarily of any class containing the class of completely representable algebras and contained in $S^\mathfrak{c}N^r_nCA_{n+3}$; these are algebras having $n+3$ local relativized representations and failure of omitting types in $n+3$ flat frames, all three proved here go together.

The uniform cause of this phenomena, is that in a two player game over a rainbow atom structure, $\forall$ can win an Ehrenfeucht–Fraïssé forth game using and re-using $n+3$ pebbles, preventing $\forall$ from establishing the negation of the above negative properties.

This is too abrupt so let us start from the beginning of the story ultimately reaching the inter-connections of such results navigating between algebra and logic. Indeed results in algebraic logic are most attractive when they have impact on logic, particularly first order logic. But here the feedback between algebraic logic and logic works both ways.

We prove that infinitely many varieties containing and including the variety of representable algebras are not atom canonical; such varieties possess $n$ relativized representations.

Here $n$ measures the degree of the $n$ squareness or $n$ flatness of the model. This is a locally guarded relativization; $n$ is the measure of how much we have to zoom in by a ‘movable window’ to this $n$ relativized representation to mistake it for a real genuine one which is both infinitely flat and square.

When $n < \omega$, $n$ flatness is stronger than $n$ squareness by a threatening ‘Church Rosser condition’, but at the limit of ordinary representations this huge discrepancy disappears.

In $n$ squareness witnesses for cylindrifiers can only be found in $n$ Gaifman graphs on $m$ cliques $m < n$, where an $m$ clique is an $m$ version of usual cliques in graph theory. In $n$ flatness it is further required that cylindrifiers also commute on the $n$ square (the Church Rosser condition).

Another way of viewing an algebra $\mathfrak{A}$ possessing such representations is that such representations are induced by $n$ hypergraphs, whose $m$ cliques are labelled by the top element of the algebra. If $\mathfrak{A}$ is countable then an $\omega$ relativized (whether flat or square) representation is just a classical (Tarskian) representation.

Actually the $n$ flat relativization of an algebra $\mathfrak{A}$ also measures the degrees of freedom that $\mathfrak{A}$ has.

These are hidden but are coded in the Gaifman graph which can be looked upon as an $n$ hypergraph whose hyperedges are $n$ ary assignments from the base of the representation to formulas in a relational first order signature having an
$m$ relation symbols for each element $a \in \mathfrak{A}$ and using $n$ variables.

Algebraically, $n$ measures the dimension for which there is an algebra having this dimension in which $\mathfrak{A}$ neatly embeds into, it is how much we truncate $\omega$, it is the distance between $S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n = \mathcal{RCA}_m$ and $S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n$, which is infinite in the sense that for any finite $n > m$ the variety $S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n$ is not finitely axiomatizable over $\mathcal{RCA}_m$. This can be proved using Monk-like algebras constructed by Hirsch and Hodkinson \[22\].

This is a syntactical measure, and in this case we have the two following weak $n$ completeness theorems that can be seen as an $n$ approximations of Henkin’s algebraic completeness theorem implemented via the neat embedding theorem ($S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n = \mathcal{RCA}_m$):

$\mathfrak{A} \in S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n$ iff it has an $n$ flat representation and $\mathfrak{A} \in S\forall\mathfrak{m}_{\omega}C\mathfrak{A}_n$ if it has a complete $m$ flat representation.

On the other hand in case of $n$ squareness the algebra also has $n$ degrees of freedom, but such degrees of freedom are ‘looser’ so to speak, for the algebra neatly embeds into another $n$ dimensional algebra $\mathfrak{D}$, but such a ‘dilation’ $\mathfrak{D}$ is not a classical $C\mathfrak{A}_n$ for it may fail commutativity of cylindrifiers, though not entirely in the sense that a weakened version of commutativity of cylindrifiers, or a confluence property, holds here.

In fact, such dilations, which happen to be representable in a relativized sense, are globally guarded fragments of $n$ variable first order logic. However, in the case of $n$ flatness, such dilations are $C\mathfrak{A}_n$s; so that they are not guarded.

This discrepancy in the formed dilations blatantly manifests itself in a very important property. The Church Rosser condition in the $n$ flat case of commutativity of cylindrifiers, when $n \geq m + 3$ make this locally guarded fragment strongly undecidable, not finitely axiomatizable in $k$th order logic for any finite $k$, and there are finite algebras that have infinite $n$ flat representations but does not have finite ones, and the equational theory of algebras having only infinite $n$ flat representations is not recursively enumerable because the equational theory of those algebras having finite $n$ flat representations is recursively enumerable and the problem of telling whether a finite algebra has an $n$ flat representation is undecidable.

In $n$ squareness it is still undecidable and not finitely axiomatizable, but every finite algebra that has an $n$ square representation has a finite one.

The reason for such a discrepancy is that the equational theory of the $n$ dimensional dilations is undecidable while the universal (hence equational) theory of the dilations in case of $n$ squareness is decidable. The first is just the equational theory of $C\mathfrak{A}_n$ the second is the equation theory of $D_n$, the class of algebras introduced by Andréka et al in \[11\].

We also show in that infinitely many classes containing and including the completely representable algebras and characterized by having complete $n$ relativized whether square or flat representations are not elementary. Here com-
plete (relativized) representation refers to the fact that such representations preserve arbitrary (possibly infinitary meets) carrying them to set theoretic intersection. A necessary (but not sufficient) condition for an algebra to possess such complete relativized representations is atomicity, and so such complete relativized representations are also atomic in the following sense.

Every sequence in the top element of the representing algebra is in the range of an atom, equivalently the pre-image of such sequence, always an ultrafilter, turns out to be a principal one.

Such representations when \( n = \omega \), and \( \mathfrak{A} \) is countable, then a complete \( \omega \) representation of \( \mathfrak{A} \) is just a complete representation.

Analogous remarks of properties mentioned above for \( n \) flat representations works here by replacing representation by complete representation and neat embeddings by complete neat embeddings.

In both cases of an existence of \( n \) square or \( n \) flat representations of an algebra \( \mathfrak{A} \) such semantics can be viewed as both locally guarded and clique guarded semantics.

But here guards, semantically, exist in the \( n \) dilation, the \( n \)-ary assignments that satisfy formulas in \( \mathcal{L}(\mathfrak{A})^n \) are restricted to the the \( n \) Gaifman hypergraph

\[
\mathcal{C}(M) = \{ s \in M : \text{rng}(s) \text{ is an } m \text{ clique} \}.
\]

Here \( M = \bigcup_{s \in V} \text{rng}(s) \) where \( V \) is the set of permissable assignments. Every \( m \) variable formula \( \phi \) can be effectively translated to one in the packed fragment of \( m \) variable first order logic, call it \( \text{packed}(\phi) \).

Here we have a very interesting connection (*)

\[
M, \mathcal{C}(M), s \models \phi \iff M, s \models \text{packed}(\phi),
\]

which says that locally guarded fragments are a subfragment of the packed fragment which is an extension of the loosely GF.

Let us twist to topological logic \( \mathfrak{T}_{\mathcal{L}} \). The analogous algebraic results discussed above for \( \mathcal{L}_m(\mathfrak{CA}_m) \) hold here as well. Here \( \mathcal{L}_m \) denotes first order logic restricted to the first \( m \) variables.

Quite surprisingly perhaps at first glance we will draw the same conclusion from both results, namely, that a version of the famous Orey-Henkin omitting types theorem (\( \text{OTT} \)) fails even if count in \( m + k \), \( k \geq 3 \) square, \( a \text{ priori} \) square models as potential candidates for omitting single non principle types; and these can be chosen to consist of co-atoms, so that what we actually have, is that the \( \mathfrak{T}_{\mathcal{L}} \) theory for which \( \text{OTT} \) fails is an atomic one, that does not have an \( m + k \) flat atomic model, so we are showing that a famous theorem of Vaught which holds for first order logic - countable atomic theory have countable models- fails for \( \mathfrak{T}_{\mathcal{L}} \) when \( m > 2 \).
The topologizing of such multimodal logics are not finitely axiomatizable, nor decidable but their modal algebras have the finite algebra finite base property which means that if a finite algebra has an \( n \) square representation, then it has a finite one.

But \( n \) flat representations in the topological context as well is far more intricate. In fact, for \( n = 3 \) it is undecidable to tell whether a finite frame is has an \( 3 + m \) flat representation when \( m \geq 3 \).

In the case of global guarding negative properties of finite variable fragments of first order logic having at least three variables, in topological predicate logic starting from two variables, like robust undecidability simply disappear, while one retains a lot of positive properties concerning definability properties, like interpolation, Beth definability, completeness and \( OTT \). So in guarding we throw away some of the worlds but the accessibility relations are kept as they are but now restricted to the remaining worlds.

3 Preliminaries

Through \( \alpha, \beta \) denote arbitrary ordinals. We formulate the basic definitions in full generality, by throughout the paper ordinals considered will be finite or countable.

3.1 Topological cylindric algebras

Instead of taking ordinary set algebras, as in the case of cylindric algebras, with units of the form \( \alpha U \), one may require that the base \( U \) is endowed with some topology. This enriches the algebraic structure. For given such an algebra, for each \( k < \alpha \), one defines an interior operator on \( \phi(\alpha U) \) by

\[
I_k(X) = \{ s \in \alpha U; s_k \in \text{int}\{a \in U : s^k_a \in X\}, X \subseteq \alpha U. \}
\]

Here \( s^k_a \) is the sequence that agrees with \( s \) except possibly at \( k \) where its value is \( a \). This gives a topological cylindric set algebra of dimension \( \alpha \).

Now such algebras lend itself to an abstract formulation aiming to capture the concrete set algebras; or rather the variety generated by them.

This consists of expanding the signature of cylindric algebras by unary operators, or modalities, one for each \( k < \alpha \), satisfying certain identities.

We start with the standard definition of cylindric algebras [18, Definition 1.1.1]:

**Definition 3.1.** Let \( \alpha \) be an ordinal. A **cylindric algebra of dimension** \( \alpha \), a \( \text{CA}_\alpha \) for short, is defined to be an algebra

\[
\mathcal{C} = \langle C, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i, j < \alpha}
\]

obeying the following axioms for every \( x, y \in C, i, j, k < \alpha \).
1. The equations defining Boolean algebras,
2. \( c_i 0 = 0 \),
3. \( x \leq c_i x \),
4. \( c_i(x \cdot c_j y) = c_i x \cdot c_i y \),
5. \( c_i c_j x = c_j c_i x \),
6. \( d_{ii} = 1 \),
7. if \( k \neq i,j \) then \( d_{ij} = c_k (d_{ik} \cdot d_{jk}) \),
8. If \( i \neq j \), then \( c_i (d_{ij} \cdot x) \cdot c_i (d_{ij} \cdot -x) = 0 \).

For a cylindric algebra \( \mathfrak{A} \), we set \( q_i x = -c_i - x \) and \( s^i_j(x) = c_i(d_{ij} \cdot x) \). Now we want to abstract equationally the prominent features of the concrete interior operators defined on cylindric set and weak set algebras. We expand the signature of \( \mathcal{CA}_\alpha \) by a unary operation \( I_i \) for each \( i \in \alpha \). In what follows \( \oplus \) denotes the operation of symmetric difference, that is, \( a \oplus b = (\neg a + b) \cdot (\neg b + a) \). For \( \mathfrak{A} \in \mathcal{CA}_\alpha \) and \( p \in \mathfrak{A} \), \( \Delta p \), the dimension set of \( p \), is defined to be the set \( \{ i \in \alpha : c_i p \neq p \} \). In polyadic terminology \( \Delta p \) is called the support of \( p \), and if \( i \in \Delta p \), then \( i \) is said to support \( p \) [17].

**Definition 3.2.** A **topological cylindric algebra of dimension** \( \alpha \), \( \alpha \) an ordinal, is an algebra of the form \( (\mathfrak{A}, I_i)_{i<\alpha} \) where \( \mathfrak{A} \in \mathcal{CA}_\alpha \) and for each \( i < \alpha \), \( I_i \) is a unary operation on \( A \) called an **interior operator** satisfying the following equations for all \( p, q \in A \) and \( i, j \in \alpha \):

1. \( q_i(p \oplus q) \leq q_i(I_i p \oplus I_i q) \),
2. \( I_i p \leq p \),
3. \( I_i p \cdot I_i p = I_i (p \cdot q) \),
4. \( p \leq I_i I_i p \),
5. \( I_i 1 = 1 \),
6. \( c_k I_i p = I_i p, k \neq i, k \notin \Delta p \),
7. \( s^i_j I_i p = I_j s^i_j p, j \notin \Delta p \).
The class of all such topological cylindric algebras are denoted by $\text{TCA}_\alpha$.

For $\mathcal{B} = (\mathfrak{A}, I_i)_{i<\alpha} \in \text{TCA}_\alpha$ we write $\text{Nr}_\alpha \mathcal{B}$ for $\mathfrak{A}$. Notice too that every $\text{CA}_\alpha$ can be extended to a $\text{TCA}_\alpha$, by defining for all $i < \alpha$, $I_i$ to be the identity function.

Topological algebras in the form we defined are not Boolean algebras with operators because the interior operators do not distribute over the Boolean join.

We also need the notion of compressing dimensions and, dually, dilating them; expressed by the notion of neat reducts.

**Definition 3.3.**

1. Let $\alpha < \beta$ be ordinals and $\mathcal{B} \in \text{TCA}_\beta$. Then $\text{Nr}_\alpha \mathcal{B}$ is the algebra with universe $\text{Nr}_\alpha \mathfrak{A} = \{ a \in \mathfrak{A} : \Delta a \subseteq \alpha \}$ and operations obtained by discarding the operations of $\mathcal{B}$ indexed by ordinals in $\beta \sim \alpha$. $\text{Nr}_\alpha \mathcal{B}$ is called the neat $\alpha$ reduct of $\mathcal{B}$. If $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathcal{B}$, with $\mathcal{B} \in \text{TCA}_\beta$, then we say that $\mathcal{B}$ is a $\beta$ dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$.

2. An injective homomorphism $f : \mathfrak{A} \to \text{Nr}_\alpha \mathcal{B}$ is called a neat embedding; if such an $f$ exists, then we say that $\mathfrak{A}$ neatly embeds into its dilation $\mathcal{B}$. In particular, if $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathcal{B}$, then $\mathfrak{A}$ neatly embeds into $\mathcal{B}$ via the inclusion map.

Note that the algebra $\text{Nr}_\alpha \mathcal{B}$ is well defined; it is closed under the cylindric operations; this is well known and indeed easy to show, and it also closed under all the interior operators $I_i$ for $i < \alpha$, for if $x \in \text{Nr}_\alpha \mathcal{B}$, and $k \in \beta \sim \alpha$, then by axiom (6) of definition [3.2] $k \notin \alpha \supseteq \Delta x \cup \{i\} \supseteq \Delta(I_i(x))$, hence $c_k(I_i(x)) = I_i(x)$.

### 3.2 Cylindric tense algebras

We start with semantics for tense cylindric algebras.

A time flow is a pair $(T, <)$ where $T$ is a non empty (set of moments) and $<$ is an irreflexive transitive relation. If $s < t$ we say that $<$ is earlier than $\mathfrak{A}$.

**Definition 3.4.** A tense system based on a time flow $(T, <)$ is a tuple $\mathcal{K} = (X_t, V_{st}, <, >, Q_1, Q_2, 0)_{s,t \in T}$ such that

(i) $T$ is a non-empty set, the set of moments

(ii) $<$ and $>$ are two binary relations on $T$,

(iii) $0 \in T$ and $Q_1, Q_2 \subseteq T$,

(iv) $X_t \neq \emptyset$ for all $t \in T$,

(v) if $t < s$ or $s > t$ then $V_{ts} : X_t \to X_s$ is a function such that
(1) \( V_{st} = Id \),
(2) if \( V_{st}, V_{tr} \) are defined then \( V_{tr} = V_{st} \circ V_{tr} \),
(3) If \( t < s \) and \( s > t \) then \( V_{ts} \) is a bijection and \( V_{ts} = V_{st}^{-1} \).

Let \( \alpha \) be an ordinal \( > 0 \). For \( s,t \in T \) and \( x \in ^{\alpha}X_t \), \( V_{ts}(x) \) denotes the set \( \{V_{st}(x_i) : i < \alpha \} \) The one defines an algebra
\[
\mathfrak{F}_\alpha = \{(f_w : w \in W) ; f_w : ^{\alpha}D_w \to \mathfrak{O}\}.
\]
The operations are defined as follows: If \( x, y \in Y_w \) and \( j \in \alpha \) then we write \( x \equiv_y j \) if \( x(i) = y(i) \) for all \( i \neq j \). We write \((f_w)\) instead of \((f_w : w \in W)\). In \( \mathfrak{F}_\alpha \) we consider the following operations:
\[
(f_w) \lor (g_w) = (f_w \lor g_w)
\]
\[
(f_w) \land (g_w) = (f_w \land g_w).
\]
For any \((f_w)\) and \((g_w) \in \mathfrak{F} \), define
\[
\neg(f_w) = (-f_w).
\]
For any \( i, j \in ^{\alpha}\alpha \), we define
\[
s^i_j : \mathfrak{F} \to \mathfrak{F}
\]
by
\[
s^i_j(f_w) = (g_w)
\]
where
\[
g_w(x) = f_w(x \circ [j, i]) \text{ for any } w \in W \text{ and } x \in ^{\alpha}D_w.
\]
For any \( j \in \alpha \) and \((f_w) \in \mathfrak{F} \) define
\[
c_j(f_w) = (g_w)
\]
where for \( x \in ^{\alpha}D_w \)
\[
g_w(x) = \bigvee \{f_w(y) : y \in ^{\alpha}D_w, y \equiv_j x \}.
\]

\[ G(f_t : t \in T) =: (g_t : t \in T) \] such that for \( x \in Y_t \), \( g_t(x) = 1 \) iff \( t \in Q_1 \) and for all \( s > t \), \( f_s(T_{ts}(x)) = 1 \). Expressed otherwise for any \( x \in ^{\alpha}D_t \)
\[
g_t(x) = \bigwedge \{f_s(x) : t < s \}.
\]
and likewise \( H(f_t : t \in T) = (g_t : t \in T) \) where
\[
g_t(x) = \bigwedge \{f_s(x) : s < t \}.
\]
\[
d_{ij} = \{(f_t : t \in T) : f_t(i) = f_t(j) \forall t \in T\}.
\]
The class of abstract \( \text{TeCA}_\alpha \) is obtained by a process of abstraction. It consists of the cylindric axioms together with the following equations for the modalities \( G \) and \( H \):
(T1) \( G(x \cdot y) = G(x) \cdot G(y) \).
(T2) \( H(x \cdot y) = H(x) \cdot G(y) \).
(T3) \( Gx \leq GGx \)
(T4) \( x \leq GPx \) and \( x \leq HF \).

and interaction axioms, or rather non-interaction axioms, for each \( i < \alpha \)
\[ c_i G(x) = G(x) \& c_i H(x) = H(x). \]

It is tedious, but basically routine to verify that \( F_k \) endowed with the above operations satisfies the above equations. Let \( C = F_k \) be as above then it has a cylindric reduct which is isomorphic to \( \prod_{t \in T} \wp(\alpha X_t) \).

We can also study maximal temporal logic with Still and Until, using the axiomatization in [V].

3.3 Temporal cylindric algebras

A time flow is a pair \((T, <)\) where \( T \) is a non-empty (set of moments) and \( < \) is an irreflexive transitive relation. If \( s < t \) we say that \( < \) is earlier than \( \mathfrak{A} \).

**Definition 3.5.** A tense system based on a time flow \((T, <)\) is a tuple \( \mathfrak{R} = (X_t, V_{st}, <, >, Q_1, Q_2, 0)_{s,t \in T} \) such that

(i) \( T \) is a non-empty set, the set of moments
(ii) \( < \) and \( > \) are two binary relations on \( T \),
(iii) \( 0 \in T \) and \( Q_1, Q_2 \subseteq T \),
(iv) \( X_t \neq \emptyset \) for all \( t \in T \),
(v) if \( t < s \) or \( s > t \) then \( V_{ts} : X_t \to X_s \) is a function such that

\[ (1) \ V_{tt} = Id, \]
\[ (2) \text{ if } V_{st}, V_{tr} \text{ are defined then } V_{tr} = V_{st} \circ V_{tr}, \]
\[ (3) \text{ If } t < s \text{ and } s > t \text{ then } V_{ts} \text{ is a bijection and } V_{ts} = V_{st}^{-1}. \]

Let \( \alpha \) be an ordinal \( > 0 \). For \( s, t \in T \) and \( x \in ^\alpha X_t, V_{ts}(x) \) denotes the set \( (V_{st}(x_i) : i < \alpha) \) The one defines an algebra

\[ \mathfrak{F}_R = \{ (f_w : w \in W) ; f_w : ^\alpha D_w \to \wp \}. \]

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The operations are defined as follows: \( U[(f_t : t \in T), (h_t : t \in T)] =: (g_t : t \in T) \) such that for \( x \in \alpha D_t, g_t(x) = 1 \) iff \( t \in Q_1 \) and

\[
(\exists s)[s > t, f_s(T_{ts}(x)) = 1 \& (\forall u)(t < u < s, h_u(T_{us}(x)) = 1)].
\]

\( S[(f_t : t \in T), (h_t : t \in T)] =: (g_t : t \in T) \) such that for all \( x \in \alpha D_t, g_t(x) = 1 \) iff \( t \in Q_2 \) and

\[
(\exists s)[s < t, f_s(T_{ts}(x)) = 1 \& (\forall u)(s < u < t, h_u(T_{us}(x)) = 1)].
\]

It is tedious, but basically routine to verify that \( \mathcal{F}_R \) endowed with the above operations is a temporal algebra.

The completeness theorem is as before, so forming the family of dilations inductively. We have \( U[(f_t : t \in T), (h_t : t \in T)] =: (g_t : t \in T) \) such that for \( x \in \alpha D_t, g_t(x) = 1 \) iff \( t \in Q_1 \) and the following double infinitary join meet holds:

\[
g_t(x) = \bigvee_{s > t} [f_s(x) = 1 \& \bigwedge_{u < s} h_u(x) = 1],
\]

and its mirror for \( S \) is:

\[
g_t(x) = \bigvee_{s < t} [f_s(x) = 1 \& \bigwedge_{u > s} h_u(x) = 1],
\]

Venema proved that the following axioms are complete for temporal logic with since and until over \((\mathbb{N}, <)\). We pick up this axiomatization for the propositional part and prove a completeness theorem for predicate temporal logic.

**Definition 3.6.**

1. \( G(p \rightarrow q) \rightarrow U(p, r) \rightarrow U(q, r) \)
2. \( G(p \rightarrow q) \rightarrow (U(r, p) \rightarrow U(r, q)) \)
3. \( p \land U(q, r) \rightarrow U(a \land S(p, r), r) \)
4. \( U(p, q) \land \neg U(p, r) \rightarrow U(q \land \neg r, q) \)
5. \( U(p, q) \rightarrow U(p, rq \land U(p, q)) \)
6. \( U(p, q) \land U(r, s) \rightarrow U(p \land r, q \land s) \lor U(p \land s, q \land s) \lor U(q \land r, q \land s) \)
7. mirror images
8. \( F\top \rightarrow U(T, \bot) \land (PT \rightarrow S(T, \bot)) \)
9. \( H \bot \lor PH \bot \)
10. \( Fp \rightarrow U(p, \neg \phi) \)
Let \( \alpha \) be an infinite ordinal. Let \( L \) be the signature consisting of the Boolean operations, cylindrifiers, diagonal elements, with indices from \( \alpha \) and the two binary modalities \( S \) and \( U \). The above axioms can be translated to equations in \( L \). Now we take the CA\(_\alpha\) axioms together with such equations, call the finite resulting schema of equations \( \Sigma \).

### 3.4 Discrete topologizing and static temporalizing

We state very simple fact that allows us to recursively associate with every CA of any dimension a TCA, a TeCA and a TemCA of the same dimension, such that the last three algebras are representable if and only if the original CA is. This mechanical procedure will be the main technique we use to obtain negative results for both TCAs, TeCAs and TemCAs by bouncing them back to their cylindric counterpart.

**Definition 3.7.** Let \( \mathfrak{A} \in \text{CA}_\alpha \).

1. \( \mathfrak{A}^{\text{top}} \in \text{TCA}_\alpha \) is a topologizing of \( \mathfrak{A} \) if \( \mathfrak{R}_{\text{ca}} \mathfrak{A}^{\text{top}} = \mathfrak{A} \). The discrete topologizing of \( \mathfrak{A} \) is the TCA\(_\alpha\) obtained from \( \mathfrak{A} \) by expanding \( \mathfrak{A} \) with \( \alpha \) many identity operators.

2. \( \mathfrak{A}^{\text{tense}} \) is a tense expansion of \( \mathfrak{A} \), if \( \mathfrak{R}_{\text{ca}} \mathfrak{A}^{\text{tense}} = \mathfrak{A} \). The static temporal expansion of \( \mathfrak{A} \) is the TeCA\(_\alpha\) obtained from \( \mathfrak{A} \) by defining \( G = H = \text{Id} \), taking the time \( T \) to consist of only one moment, that is, \( T = \{ t \} \) say, and the flow \( < \) is taken to be the empty set.

3. In the same way we can form maximal temporal extensions of a cylindric algebra, by defining \( S(a, b) = U(a, b) = \text{Id} \) and defining the flow like in the previous item, call the resulting algebra \( \mathfrak{A}^{\text{temp}} \). In this case, \( \mathfrak{A}^{\text{tense}} = \mathfrak{A}^{\text{temp}} \) so that the minimal and maximal resulting temporal logic coincide.

**Theorem 3.8.** The discrete topologizing of \( \mathfrak{A} \in \text{CA}_\alpha \) is unique up to isomorphism. Furthermore, if \( \mathfrak{A}^{\text{top}} \) is the discrete topologizing of \( \mathfrak{A} \), then \( \mathfrak{A} \) is representable if and only if \( \mathfrak{A}^{\text{top}} \) is representable. A completely analogous statement holds for \( \mathfrak{A} \in \text{CA}_\alpha \) in connection to \( \mathfrak{A}^{\text{tense}} \) and \( \mathfrak{A}^{\text{temp}} \).

**Proof.** The first part is trivial. The second part is also very easy. If \( \mathfrak{A}^{\text{top}} \) is representable then obviously \( \mathfrak{A} = \mathfrak{R}_{\text{ca}} \mathfrak{A}^{\text{top}} \) is representable. For the last part if \( \mathfrak{A} \) is representable with base \( U \) then \( \mathfrak{A}^{\text{top}} \) have the same universe of \( \mathfrak{A} \), hence it is representable by endowing \( U \) with the discrete topology, which induces the identity interior operators.

**Theorem 3.9.** Let \( 1 < m < n \)
(1) Assume that $A \in \mathcal{Nr}_m\mathcal{C}A_n$, then $A^{\text{top}} \in \mathcal{Nr}_m\mathcal{T}\mathcal{C}A_n$.

(2) Assume that $A \in S\mathcal{Nr}_m\mathcal{C}A_n$, then $A^{\text{top}} \in S\mathcal{Nr}_m\mathcal{CA}_n$.

(3) Same if we apply the operation $S_c$ of forming complete subalgebras in $\mathcal{Nr}_m\mathcal{CA}_n$.

(4) Completely analogous statements hold for $\mathcal{TeCA}_m$s and $\mathcal{TemCAs}$ by static temporalizing.

Proof. We prove only the first same. The rest of the proof is the same. Assume that $A = \mathcal{Nr}_m\mathcal{B}$, $\mathcal{B} \in \mathcal{CA}_n$, then $A^{\text{top}} = \mathcal{Nr}_m\mathcal{B}^{\text{top}}$ and we are done. \qed

Form this easy lemma one can infer quite deep results proved for cylindric algebras. We mention two.

- For any pair of ordinal $1 < m < n$ the class $\mathcal{Nr}_m\mathcal{T}\mathcal{C}A_n$ not closed under elementary subalgebras \cite{35}.

- For any pair of ordinals $2 < m < n$ (infinite included), for any $r \in \omega$ and for any finite $k \geq 1$, there is a $\mathcal{B}^r \in S\mathcal{Nr}_m\mathcal{T}\mathcal{C}A_{m+k}$ such that $\mathcal{Nr}_m\mathcal{A} \notin S\mathcal{Nr}_m\mathcal{C}A_{m+k+1}$ and $\Pi_{r \in \omega}\mathcal{B}^r/U \in \mathcal{RT}\mathcal{C}A_m$ for any non principal ultrafilter on $\omega$.

In particular, for finite for $m$ and any finite $k \geq 1$, we can infer that the variety $S\mathcal{Nr}_m\mathcal{T}\mathcal{C}A_{m+k+1}$ is not finitely axiomatizable over the variety $S\mathcal{Nr}_m\mathcal{T}\mathcal{CA}_{m+k}$.

- The same result holds for infinite dimensions by replacing finite axiomatizability by finite schema axiomatizability \cite{22} \cite{27}.

The process of discrete topologizing , and for that matter static temporal expansions, work best in recovering negative results proved for cylindric algebras to the topological or temporal expansion by bouncing it back to the cylindric case. We will witness such a phenomena quite frequently.

4 Complete representability, atom canonicity and neat embeddings

4.1 Use of rainbows

In the proof of our main results mentioned in the abstract (concerning atom canonicity and complete representations) we use advanced sophisticated machinery of cylindric algebra theory, like so called rainbow constructions invented by Hirsch and Hodkinson \cite{21} \cite{22} \cite{28} \cite{26}, obtaining new results for
cylindric-like algebras, strengthening results proved for cylindric algebras in \cite{12, 21, 28}, and then lifting them to the topological and tense context, by discrete topologizing and static temporal expansions.

Rainbow algebras are only superficially similar to what is known in the literature of Monk-like algebras. Monk-like algebras are efficient in proving that certain algebras may not be representable and this type of results is proved by an application of the combinatorial Ramey’s theorem. The idea of Mon-like algebras is not too hard. Such algebras are finite, hence atomic and the atoms are coloured in such a way to forbid monochromatic triangles (triangles all of its three sides are labelled by the same colour) If the atoms are more than the colours then a representation will force a monochromatic triangle which a impossible.

On a very basic level in rainbow algebras, or more accurately on the atom structures of rainbow algebras, deterministic games are played between two players Elloise $\exists$ and Abelard $\forall$ one of them has to win, there are no draws.

We have almost all rainbow colours, red, green, white, black, if one considers that black and white are colours etc. These games lift very simple forth Ehrenfeucht–Fraïssé game played on two coloured relational structures (usually complete irreflexive graphs) $A$ is the ‘greens’ and $B$ is the ‘reds’ to the cylindric rainbow algebras.

It is a forth Ehrenfeucht–Fraïssé pebble game such that winning strategys for either player in the private Ehrenfeucht–Fraïssé game are preserved in the lifted rainbow algebra $\mathcal{C}A_{A,B}$, but the number of used pebbles and rounds increases.

Because we can control the number of pebbles in play, rainbow algebras prove very delicate results via quite sophisticated constructions from the cylindric algebra point of view, but such constructions which tend to seem complicated enough, actually use simple games Ehrenfeucht–Fraïssé games that serves the task at hand by appropriately choosing the structures $A$ and $B$.

Reducing complicated constructions, proving sophisticated subtle statements solving really hard problems, to a manageable simple case, namely, a forth Ehrenfeucht–Fraïssé pebble game, is precisely the ingenuity of such constructions.

This technique proved to be a nut cracker in addressing difficult problems, and it especially proved highly efficient in contexts when it is not obvious how to use Monk-like algebras, in both the relation algebra and cylindric algebra cases like for instance proving that $\mathsf{RA}_{n+1}$ is not finitely axiomatizable over $\mathsf{RA}_n$, when $n \geq$ where $\mathsf{RA}_m(m > 2)$ is the variety of relation algebras that embed into algebras having a relational basis in the sense of Maddux \cite{22}.

We will introduce a cylindric analogue of $\mathsf{RA}_n$, further on to be topologized, these are cylindric algebras of dimension $m$, that also has an $n$ dimensional basis that can be characterized in many ways that are equivalent.
First by simple games played on finite graphs with a set of nodes $n$, in which $\forall$ is offered only a ‘cylindrifier move’ there are no amalgamation moves, second by certain $n$ dimensional basis that are obtained from the cylindric basis of Maddux by discarding the amalgamation condition, third by a weak neat embedding property; such algebras embed into $n$ dimensional algebras and they have localized $n$ square representations to be elaborated upon in a short while.

It is always the case, like just indicated, that the number of pebbles used in the Ehrenfeucht–Fraïssé private game, appears on the algebra level, so for example if we want to show that a rainbow algebra is not in $S\forall\forall_m CA_n$ then we use $n$ pebbles.

It will turn out that the following three results:

• non atom canonicity proved by constructing a rainbow atom structure whose term algebra is representable but its Dedekind-MacNeille completion does not neatly embed into an $n + 3$ dimensional algebra,

• the non-existence of $n + 3$ flat models

• and the failure of OTT even if we allow clique guarded $n + 3$ flat semantic, is witnessed by a winning strategy for $\forall$ in a certain essentially forth Ehrenfeucht–Fraïssé translated to the coloured graphs involved in the rainbow construction, where he can use and re-use $n + 3$ pebbles, though in the case of atom canonicity $\forall$ can win only in finitely many rounds because the game is played on a finite algebra.

In this case actually $\forall$ can win without having to reuse pebbles, and this makes the situation worse. Even $n + 3$ square models are not enough for omitting types.

In the other case $\forall$ can win only in an $\omega$ rounded game. He cannot win the game truncated to finitely many rounds, because his strategy is forcing $\forall$ a decreasing sequence in the red $\mathbb{N}$; the double indices of the red colours come from $\mathbb{N}$. The strategy of $\forall$ is bombarding $\exists$ with ‘cones’ having the same base and green tints. Here the suffixes of the greens come from the green $\mathbb{Z}$.

By the rules of the game $\exists$ will have to choose a red colour to label the edges between apexes of cones having the same base. This cannot be achieved in finitely many rounds.

For such a construction we can infer that algebras having $n$ flat complete representations when $n \geq m + 3$ is not elementary. The $n$ square case does not follow here, though it can be proved using a rainbow construction, and this particular instance Monk-like algebras can do the job just as well.

Coloured graphs are complete graphs whose edges are labelled by the rainbow colours and some of its $n - 1$ hyperedges are also coloured. Shades of
yellow are reserved for that. Such coloured graphs can be also seen as models for an $L_{\omega_1,\omega}$ theory formulated in the rainbow signature, which consists of binary relation, green, red, white, etc and $n-1$-ary relation representing the shades of yellow.

When the greens are finite (as is the case with our first encounter with rainbows), the rainbow theory is a first order theory. If an edge in a coloured graph is coloured by a green, then this is interpreted model-theoretically, that this edge holds in the green binary relation in the corresponding model.

The atoms of the rainbow algebra are very roughly finite coloured graphs. Cones are special coloured graphs. The strategy for $\forall$ is always bombarding $\exists$ with cones whose tints are green and because $\exists$ can never play greens according to the rules of the game, so when she is to label an edge between apexes of cones having the same base she has to choose a red. It is always the case that $\forall$ wins on a ‘red clique.’

4.2 Atom canonicity

We will show using the so called blow up and blur construction, a very indicative name suggested in [12], that for any finite $n > 2$, any $K \in \{Sc, CA, PA, PEA\}$, and any $k \geq 3$, $SGn_n K_{n+k}$ is not atom canonical. Here $Sc$ denotes Pinter’s substitution algebras, $PA$ denotes polyadic algebras and $PEA$ denotes polyadic algebras with equality.

We will blow up and blur a finite rainbow algebra.

We give the general idea for cylindric algebras, though the idea is much more universal as we will see. Assume that $RCA_n \subseteq K$, and $K$ is closed under forming subalgebras. Start with a finite algebra $C$ outside $K$. Blow up and blur $C$, by splitting each atom to infinitely many, getting a new atom structure $At$. In this process a (finite) set of blurs are used.

They do not blur the complex algebra, in the sense that $C$ is there on this global level. The algebra $CmAt$ will not be in $K$ because $C \notin K$ and $C$ embeds into $CmAt$. Here the completeness of the complex algebra will play a major role, because every element of $C$, is mapped, roughly, to the join of its splitted copies which exist in $CmAt$ because it is complete.

These precarious joins prohibiting membership in $K$ do not exist in the term algebra, only finite-cofinite joins do, so that the blurs blur $C$ on this level; $C$ does not embed in $TmAt$.

In fact, the the term algebra will not only be in $K$, but actually it will be in the possibly smaller $RCA_n$. This is where the blurs play their other role. Basically non-principal ultrafilters, the blurs are used as colours to represent $TmAt$.

In the process of representation we cannot use only principal ultrafilters, because $TmAt$ cannot be completely representable, that is, it cannot have a
representation that preserves all (possibly infinitary) meets carrying them to set theoretic intersections, for otherwise this would give that \( \mathbb{C}m\mathbb{A}t \) is representable.

But the blurs will actually provide a complete representation of the canonical extension of \( \mathbb{I}m\mathbb{A}t \), in symbols \( \mathbb{I}m\mathbb{A}t^+ \); the algebra whose underlying set consists of all ultrafilters of \( \mathbb{I}m\mathbb{A}t \). The atoms of \( \mathbb{I}m\mathbb{A}t \) are coded in the principal ones, and the remaining non-principal ultrafilters, or the blurs, will be finite, used as colours to completely represent \( \mathbb{I}m\mathbb{A}t^+ \), in the process representing \( \mathbb{I}m\mathbb{A}t \).

We start off with a conditional theorem: giving a concrete instance of a blow up and blur construction for relation algebras due to Hirsch and Hodkinson. The proof is terse highlighting only the main ideas.

**Theorem 4.1.** Let \( m \geq 3 \). Assume that for any simple atomic relation algebra with atom structure \( S \), there is a cylindric atom structure \( H \), constructed effectively from \( S \), such that:

1. If \( \mathbb{I}mS \in \mathbb{R}RA \), then \( \mathbb{I}mH \in \mathbb{R}CA_{m} \),
2. If \( S \) is finite, then \( H \) is finite,
3. \( \mathbb{C}mS \) is embeddable in \( \mathbb{R}a \) reduct of \( \mathbb{C}mH \).

Then for all \( k \geq 3 \), \( S\mathbb{N}r_{m}CA_{m+k} \) is not closed under completions,

**Proof.** Let \( S \) be a relation atom structure such that \( \mathbb{I}mS \) is representable while \( \mathbb{C}mS \notin \mathbb{R}A_{6} \). Such an atom structure exists [22, Lemmas 17.34-35-36-37].

We give a brief sketch at how such algebras are constructed by allowing complete irreflexive graphs having an arbitrary finite set nodes, slightly generalizing the proof in op.cit, though the proof idea is essentially the same.

Another change is that we refer to non-principal ultrafilters (intentionally) by **blurs** to emphasize the connection with the blow up and blur construction in [12] as well as with the blow up and blur construction outlined above, to be encountered in full detail in a little while, witness theorem 4.6.

In all cases a finite algebra is blown up and blurred to give a representable algebra (the term algebra on the blown up and blurred finite atom structure) whose d completion does not have a neat embedding property.

We use the notation of the above cited lemmas in [26] without warning, and our proof will be very brief just stressing the main ideas. \( G_{r}^{n} \) denotes the usual atomic \( r \) rounded game played on atomic networks having \( n \) nodes of an atomic relation algebra, where \( n, r \leq \omega \), and \( K_{r} \) (\( r \in \omega \)) denotes the complete irreflexive graph with \( r \) nodes.

Let \( R \) be the rainbow algebra \( \mathbb{A}_{K_{m},K_{n}} \), \( m > n > 2 \). Let \( T \) be the term algebra obtained by splitting the reds. Then \( T \) has exactly two blurs \( \delta \) and \( \rho \).
If \(X \subseteq D\) of its copies. Let \(S\) subsets of \(D\), not in the latter class for \(R\). Then \(Tm\) the hypothesis of the theorem. Then \(EF^*_k(A, B)\) where \(A\) and \(B\) are relational structures. This game has \(r\) rounds and \(k\) pebbles. The rainbow theorem \([22, Theorem 16.5]\) says that \(\exists\) has a winning strategy in the game \(G^2_{1+r}(A_{A, B})\) if and only if she has a winning strategy in \(EF^*_r(A, B)\).

Using this theorem it is obvious that \(\exists\) has a winning strategy over \(AtR\) in \(m + 2\) rounds, hence \(R \notin RA_{m+2}\), hence is not in \(S\text{RaCA}_{m+2}\). \(CmAtT\) is also not in the latter class for \(R\) embeds into it, by mapping every red to the join of its copies. Let \(D = \{r^n_l : n < \omega, l \in n\}\), and \(R = \{r^n_{lm}, l, m \in n, l \neq m\}\).

If \(X \subseteq R\), then \(X \subseteq T\) if and only if \(X\) is finite or cofinite in \(R\) and same for subsets of \(D\) \([22]\) lemma 17.35]. Let \(\delta = \{X \subseteq T : X \cap D\) is cofinite in \(D\}\), and \(\rho = \{X \subseteq T : X \cap R\) is cofinite in \(R\}\). Then these are the non principal ultrafilters, they are the blurs and they are enough to (to be used as colours), together with the principal ones, to represent \(T\) as follows \([26]\) bottom of p. 533]. Let \(\Delta\) be the graph \(n \times \omega \cup m \times \{\omega\}\). Let \(B\) be the full rainbow algebras over \(At\mathcal{A}_{K_m, \Delta}\) by deleting all red atoms \(r_{ij}\) where \(i, j\) are in different connected components of \(\Delta\).

Obviously \(\exists\) has a winning strategy in \(EF^\omega_{2\omega}(K_m, K_m)\), and so it has a winning strategy in \(G^\omega_{\omega}(A_{K_m, K_m})\). But \(At\mathcal{A}_{K_m, K_m} \subseteq At\mathcal{B} \subseteq At\mathcal{K}_{m, \Delta}\), and so \(B\) is representable.

One then defines a bounded morphism from \(At\mathcal{B}\) to the the canonical extension of \(T\), which we denote by \(T^+,\) consisting of all ultrafilters of \(T\). The blurs are images of elements from \(K_m \times \{\omega\}\), by mapping the red with equal double index, to \(\delta\), for distinct indices to \(\rho\). The first copy is reserved to define the rest of the red atoms the obvious way. (The underlying idea is that this graph codes the principal ultrafilters in the first component, and the non principal ones in the second.) The other atoms are the same in both structures. Let \(S = CmAtT\), then \(CmS \notin S\text{RaCA}_{m+2}\) \([22]\) lemma 17.36].

Note here that the \(d\) completion of \(T\) is not representable while its canonical extension is completely representable, via the representation defined above. However, \(T\) itself is not completely representable, for a complete representation of \(T\) induces a representation of its \(d\) completion, namely, \(CmAt\mathcal{A}\).

Now let \(H\) be the \(CA_m\) atom structure obtained from \(AtT\) provided by the hypothesis of the theorem. Then \(\mathcal{A}\) \(\subseteq RCA_m\). We claim that \(CmH \notin S\text{Mt}_mCA_{m+k}, k \geq 3\). For assume not, i.e. assume that \(CmH \in S\text{Mt}_mCA_{m+k}, k \geq 3\). We have \(CmS\) is embeddable in \(\mathcal{A}\) \(CmH\). But then the latter is in \(S\text{RaCA}_0\) and so is \(CmS\), which is not the case.

Hodkinson constructs atom structures of cylindric and polyadic algebras of any pre-assigned finite dimension \(\geq 2\) from atom structures of relation algebras \([29]\). One could well be tempted to use such a construction with the above proof
to obtain an analogous result for cylindric and polyadic algebras. However, we emphasize that the next result cannot be obtained by lifting the relation algebra case \[22\], lemmas 17.32, 17.34, 17.35, 17.36] to cylindric algebras using Hodkinson’s construction in \[29\] as it stands. It is true that Hodkinson constructs from every atomic relation algebra an atomic cylindric algebra of dimension \(n\), for any \(n \geq 3\), but the relation algebras does not embed into the \(Ra\) reduct of the constructed cylindric algebra when \(n \geq 6\). If it did, then the \(Ra\) result would lift as indeed is the case with \(n = 3\).

Now we are faced with two options. Either modify Hodkinson’s construction, implying that the embeddability of the given relation algebra in the \(Ra\) reduct of the constructed cylindric algebras, or avoid completely the route via relation algebras. We tend to think that it is impossible to adapt Hodkinson’s construction the way needed, because if \(A\) is a non representable relation algebra, and \(CmAt(\mathfrak{A})\) embeds into the \(Ra\) reduct of a cylindric algebra of every dimension \(\geq 2\), then \(\mathfrak{A}\) will be representable, which is a contradiction.

Therefore we choose the second option. We instead start from scratch. We blow up and blur a finite rainbow cylindric algebra.

In \[26\] the rainbow cylindric algebra of dimension \(n\) on a graph \(\Gamma\) is denoted by \(R(\Gamma)\). We consider \(R(\Gamma)\) to be in \(PEA_n\) by expanding it with the polyadic operations defined the obvious way (see below). In what follows we consider \(\Gamma\) to be the indices of the reds, and for a complete irreflexive graph \(G\), by \(PEA_{G,\Gamma}\) we mean the rainbow cylindric algebra \(R(\Gamma)\) of dimension \(n\), where \(G = \{g_i : 1 \leq i < n - 1\} \cup \{g_0^i : i \in G\}\).

More generally, we consider a rainbow polyadic algebra based on relational structures \(A, B\), to be the rainbow algebra with signature the binary colours (binary relation symbols) \(\{r_{ij} : i, j \in B\} \cup \{w_i : i < n - 1\} \cup \{g_i : 1 \leq i < n - 1\} \cup \{g_0^i : i \in A\}\) and \(n - 1\) shades of yellow (\(n - 1\) ary relation symbols) \(\{y_S : S \subseteq A, \text{ or } S = A\}\).

We look at models of the rainbow theorem as coloured graphs \[21\]. This class is denoted by \(CRG_{A,B}\) or simply \(CRG\) when \(A\) and \(B\) are clear from context.

A coloured graph is a graph such that each of its edges is labelled by one of the first three colours mentioned above, namely, greens, whites or reds, and some \(n - 1\) hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

**Definition 4.2.** Let \(i \in A\), and let \(M\) be a coloured graph consisting of \(n\) nodes \(x_0, \ldots, x_{n-2}, z\). We call \(M\) an \(i\) - cone if \(M(x_0, z) = g_0^i\) and for every \(1 \leq j \leq n - 2\), \(M(x_j, z) = g_j\), and no other edge of \(M\) is coloured green. \((x_0, \ldots, x_{n-2})\) is called the center of the cone, \(z\) the apex of the cone and \(i\) the tint of the cone.

**Definition 4.3.** The class of coloured graphs \(CRG\) are
• $M$ is a complete graph.
• $M$ contains no triangles (called forbidden triples) of the following types:

\[(g, g', g^*)\] any $1 \leq i < n - 1$ (1)

\[(g^j, g^k, w_0)\] any $j, k \in A$ (2)

\[(r_{ij}, r_{j'k'}, r_{i*k'})\] any $j, j', k', i^*, k^* \in B$ (3)

unless $i = i^*$, $j = j'$ and $k' = k^*$ (4)

and no other triple of atoms is forbidden.

• If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge $(a_i, a_j)$ $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n - 1)$ tuples are coloured shades of yellow.

• If $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an $i$ cone with apex $\delta$, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade $y_S$ then $i \in S$.

One then can define a polyadic equality atom structure of dimension $n$ from the class CRG. It is a rainbow atom structure. Rainbow atom structures are what Hirsch and Hodkinson call atom structures built from a class of models [26]. Our models are, according to the original more traditional view [21] coloured graphs. So let CRG be the class of coloured graphs as defined above.

Let

$$\text{At} = \{a : n \to M, M \in \text{CRG} : a \text{ is surjective}\}.$$

We write $M_a$ for the element of At for which $a : n \to M$ is a surjection. Let $a, b \in \text{At}$ define the following equivalence relation: $a \sim b$ if and only if

• $a(i) = a(j) \iff b(i) = b(j)$,
• $M_a(a(i), a(j)) = M_b(b(i), b(j))$ whenever defined,
• $M_a(a(k_0), \ldots, a(k_{n-2})) = M_b(b(k_0), \ldots, b(k_{n-2}))$ whenever defined.

Let $\text{At}$ be the set of equivalences classes. Then define

$$[a] \in E_{ij} \text{ iff } a(i) = a(j).$$

$$[a]T_i[b] \text{ iff } a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}.$$

Define accessibility relations corresponding to the polyadic (transpositions) operations as follows:

$$[a]S_{ij}[b] \text{ iff } a \circ [i, j] = b.$$

This, as easily checked, defines a $\text{PEA}_n$ atom structure. The complex algebra of this atom structure is denoted by $\text{PEA}_{A,B}$ where $A$ is the greens and $B$ is the reds.
One can define a TCA\(_n\) atom structure by discrete topologizing setting for all \(i < n\), \([a]I\_n[b]\) iff \(a = b\), where \(I\_n\) is the accessibility relation corresponding to \(I\). Similar remark hold for tense and temporal algebras. For example in the former case one defines \([a]G[b]\) iff \(a = b\) and same for \(H\). The time consists of one moment and the flow is the empty set.

Now consider the following two games on coloured graphs, each with \(\omega\) rounds, and limited number of pebbles \(m > n\). They are translations of \(\omega\) atomic games played on atomic networks of a rainbow algebra using a limited number of nodes \(m\). Both games offer \(\forall\) only one move, namely, a cylindrifier move.

From the graph game perspective both games \([21]\) p.27-29 build a nested sequence \(M_0 \subseteq M_1 \subseteq \ldots\) of coloured graphs.

First game \(G^m\). \(\forall\) picks a graph \(M_0 \in \text{CRG}\) with \(M_0 \subseteq m\) and \(\exists\) makes no response to this move. In a subsequent round, let the last graph built be \(M_i\). \(\forall\) picks

- a graph \(\Phi \in \text{CRG}\) with \(|\Phi| = n\),
- a single node \(k \in \Phi\),
- a coloured graph embedding \(\theta : \Phi \setminus \{k\} \to M_i\). Let \(F = \phi \setminus \{k\}\). Then \(F\) is called a face. \(\exists\) must respond by amalgamating \(M_i\) and \(\Phi\) with the embedding \(\theta\). In other words she has to define a graph \(M_{i+1} \in C\) and embeddings \(\lambda : M_i \to M_{i+1}\) \(\mu : \phi \to M_{i+1}\), such that \(\lambda \circ \theta = \mu \upharpoonright F\).

\(F^m\) is like \(G^m\), but \(\forall\) is allowed to resuse nodes.

\(F^m\) has an equivalent formulation on atomic networks of atomic algebras.

Let \(\delta\) be a map. Then \(\delta[i \to d]\) is defined as follows. \(\delta[i \to d](x) = \delta(x)\) if \(x \neq i\) and \(\delta[i \to d](i) = d\). We write \(\delta^i\) for \(\delta[i \to \delta_j]\).

**Definition 4.4.** Let \(2 < n < \omega\). Let \(\mathfrak{C}\) be an atomic PEA\(_n\). An atomic network over \(\mathfrak{C}\) is a map

\[N : n\Delta \to \text{At}\mathfrak{C},\]

where \(\Delta\) is a non-empty set called a set of nodes, such that the following hold for each \(i, j < n\), \(\delta \in n\Delta\) and \(d \in \Delta\):

- \(N(\delta_j^i) \leq d_{ij}\),
- \(N(\delta[i \to d]) \leq c_i N(\delta)\),
- \(N(\bar{x} \circ [i, j]) = s_{i, j} N(\bar{x})\) for all \(i, j < n\).

**Definition 4.5.** Let \(2 \leq n < \omega\). For any Sc\(_n\) atom structure \(\alpha\) and \(n < m \leq \omega\), we define a two-player game \(F^m(\alpha)\), each with \(\omega\) rounds.
Let $m \leq \omega$. In a play of $F^m(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \ldots$ where $\text{nodes}(N_i)$ is a finite subset of $m = \{ j : j < m \}$, for each $i$.

In the initial round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$ must play a finite network $N_0$ with $\text{nodes}(N_0) \subseteq m$, such that $N_0(\bar{d}) = a$ for some $\bar{d} \in ^m\text{nodes}(N_0)$.

In a subsequent round of a play of $F^m(\alpha)$, $\forall$ can pick a previously played network $N$ an index $l < n$, a face $F = \langle f_0, \ldots, f_{n-2} \rangle \in ^{n-2}\text{nodes}(N)$, $k \in m \sim \{ f_0, \ldots, f_{n-2} \}$, and an atom $b \in \alpha$ such that

$$b \leq c_i(N(f_0, \ldots, f_i, x, \ldots, f_{n-2})).$$

The choice of $x$ here is arbitrary, as the second part of the definition of an atomic network together with the fact that $c_i(c_i x) = c_i x$ ensures that the right hand side does not depend on $x$.

This move is called a cylindrifier move and is denoted

$$(N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l)$$

or simply by $(N, F, k, b, l)$. In order to make a legal response, $\exists$ must play a network $M \supseteq N$ such that $M(f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2})) = b$ and $\text{nodes}(M) = \text{nodes}(N) \cup \{ k \}$.

$\exists$ wins $F^m(\alpha)$ if she responds with a legal move in each of the $\omega$ rounds. If she fails to make a legal response in any round then $\forall$ wins.

We start by proving the following algebraic result. Recall that $TCA_n$, $TeCA_n$ and $TemCA_n$ stand for the classes of topological cylindric, dense cylindric, and temporal cylindric algebras of dimension $n$, respectively. For $K$ any of the above, $RK$ stands for the class of representable algebras in $K$ of dimension $n$.

**Theorem 4.6.** For any finite $n > 2$, for any $K$ between $S\Nr TCA_{n+3}$ and $TRCA_n$ is not atom-canonical, hence is not closed under minimal completions, and is not Sahlqvist axiomatizable. In more detail, there exists an atomic countable completely additive algebra $\mathfrak{A} \in RTCA_n$ such its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure is not in $S\Nr TCA_{n+3}$. A completely analogous result holds for $TeCA_n$ and $TemCA_n$.

**Proof.** The proof uses a rainbow algebra [26].

We blow up and blur in the sense of [12] a finite rainbow cylindric algebra namely $R(\Gamma)$ where $\Gamma$ is the complete irreflexive graph $n + 1$, and the greens are $G = \{ g_i : 1 \leq i < n - 1 \} \cup \{ g_0 : 1 \leq i \leq n + 1 \}$, we denote this finite algebra endowed by the topologization induced $n$ identity interior operators by $TCA_{n+1,n}$. 

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Let $\mathbf{At}$ be the rainbow atom structure similar to that in \cite{28} except that we have $n+1$ greens and only $n$ indices for reds, so that the rainbow signature now consists of $g_i : 1 \leq i < n - 1$, $g_0^i : 1 \leq i \leq n + 1$, $w_i : i < n - 1$, $r_{kl}^t : k < l \in n$, $t \in \omega$, binary relations and $y_S$, $S \subseteq n + 1$, $n - 1$ ary relations.

We also have a shade of red $\rho$; the latter is a binary relation but is outside the rainbow signature, though it is used to label coloured graphs during a certain game devised to prove representability of the term algebra \cite{28}, and in fact $\exists$ can win the $\omega$ rounded game and build the $n$ homogeneous model $M$ by using $\rho$ whenever she is forced a red, as will be shown in a while.

So $\mathbf{At}$ is obtained from the rainbow atom structure of the algebra $\mathfrak{A}$ defined in \cite{28} section 4.2 starting p. 25) truncating the greens to be finite (exactly $n+1$ greens). In \cite{28} it shown that the complex algebra $\mathfrak{CmAt}\mathfrak{A}$ is not representable; the result obtained now, because the greens are finite but still out fit the red, is sharper; it will imply that $\mathfrak{CmAt} \notin S\mathfrak{Mr}_n \mathfrak{TCA}_{n+3}$.

The logics $L^n$, $L^n_\infty$ are taken in the rainbow signature (without $\rho$).

Now $\mathfrak{mAt} \in \mathfrak{TRCA}_n$; this can be proved exactly like in \cite{28}. Strictly speaking the cylindric reduct of $\mathfrak{MmAt}$ can be proved representable like in \cite{28}; giving, as usual, the base of the representation the discrete topology we get representability of the interior operators as well. The colours used for coloured graphs involved in building the finite atom structure of the algebra $\mathfrak{TCA}_{n+1,n}$, which is the rainbow algebra constructed on the irreflexive complete graphs $n + 1$, the greens, and $n$, the reds, are:

- **greens**: $g_i$ ($1 \leq i \leq n - 2$), $g_0^i$, $1 \leq i \leq n + 1$,
- **whites**: $w_i : i \leq n - 2$,
- **reds**: $r_{ij} \ i < j \in n$,
- **shades of yellow**: $y_S : S \subseteq n + 2$.

with forbidden triples

\[
(g, g', g^*), (g_i, g_0, w_i), \quad \text{any } 1 \leq i \leq n - 2
\]
\[
(g_j^k, g_0^k, w_0), \quad \text{any } 1 \leq j, k \leq n + 1
\]
\[
(r_{ij}, r_{j'k'}, r_{i*}, r_{i*}), \quad i, j, i', j', k', i^*, j^* \in n,
\]

unless $i = i^*$, $j = j'$ and $k' = k^*$.

and no other triple is forbidden.

Coloured graphs using such colours and the finite atom structure of $\mathfrak{TCA}_{n+1,n}$, build up of (quotients of ) surjections from $n$ into coloured graphs are defined the usual way \cite{21}.

A coloured graph is red if at least one of its edges is labelled red. For brevity write $r$ for $r_{jk}(j < k < n)$. If $\Gamma$ is a coloured graph using the colours in
AtTCA_{n+1,n}, and \(a : n \rightarrow \Gamma\) is in AtTCA_{n+1,n}, then \(a' : n \rightarrow \Gamma'\) with \(\Gamma' \in \text{CGR}\) is a copy of \(a : n \rightarrow \Gamma\) if \(|\Gamma'| = |\Gamma'|\), all non red edges and \(n - 1\) tuples have the same colour (whenever defined) and for all \(i < j < n\), for every red \(r\), if \((a(i), a(j)) \in r\), then there exits \(l \in \omega\) such that \((a'(i), a'(j)) \in r'\). Here we implicitly require that for distinct \(i, j, k < n\), if \((a(i), a(j)) \in r\), then \((a(k)) \in r'\), \((a(i), a(k)) \in r''\), and \((a'(i), a'(j)) \in r'\), \((a'(i), a'(k)) \in [r''|z]\) and \((a'(i), a'(k)) \in [r''|l]\), then \(l_1 = l_2 = l_3 = l\), say, so that \((r', [r']|r''|l])\) is a consistent triangle in \(\Gamma'\). If \(a' : n \rightarrow \Gamma'\) and \(\Gamma'\) is a red graph using the colours of the rainbow signature of At, whose reds are \(\{r_{kj} : k < j < n, l \in \omega\}\), then there is a unique red \(a : n \rightarrow \Gamma\) a red graph using the red colours in the rainbow signature of TCA_{n+1,n}, namely, \(\{r_{kj} : k < j < n\}\) such that \(a'\) is a copy of \(a\). We denote \(a\) by \(o(a')\), \(o\) short for original; \(a\) is the original of its copy \(a'\).

For \(i < n\), let \(T_i\) be the accessibility relation corresponding to the \(i\)th cylindrifier in At. Let \(T_i^s\), be that corresponding to the \(i\)th cylindrifier in TCA_{n+1,n}. Then if \(c : n \rightarrow \Gamma\) and \(d : n \rightarrow \Gamma'\) are surjective maps \(\Gamma, \Gamma'\) are coloured graphs for TCA_{n+1,n}, that are not red, then for any \(i < n\), we have

\[
T_i \iff \{[c], [d]\} \in T_i^s.
\]

If \(\Gamma\) is red using the colours for the rainbow signature of At (without \(\rho\)) and \(a' : n \rightarrow \Gamma\), then for any \(b : n \rightarrow \Gamma'\) where \(\Gamma'\) is not red and any \(i < n\), we have

\[
\{[a'], [b]\} \in T_i \iff \{[o(a')], [b]\} \in T_i^s.
\]

Extending the notation, for \(a : n \rightarrow \Gamma\) a graph that is not red in At, set \(o(a) = a\). Then for any \(a : n \rightarrow \Gamma\), \(b : n \rightarrow \Gamma'\), where \(\Gamma, \Gamma'\) are coloured graphs at least one of which is not red in At and any \(i < n\), we have

\[
[a]T_i[b] \iff \{o(a)\}|T_i^s[o(b)].
\]

Now we deal with the last case, when the two graphs involved are red. Now assume that \(a' : n \rightarrow \Gamma\) is as above, that is \(\Gamma \in \text{CGR}\) is red, \(b : n \rightarrow \Gamma'\) and \(\Gamma'\) is red too, using the colours in the rainbow signature At.

Say that two maps \(a : n \rightarrow \Gamma\), \(b : n \rightarrow \Gamma'\), with \(\Gamma\) and \(\Gamma'\) in CGR having the same size are \(r\) related if all non red edges and \(n - 1\) tuples have the same colours (whenever defined), and for all every red \(r\), whenever \(i < j < n, l \in \omega\), and \((a(i), a(j)) \in r\), then there exists \(k \in \omega\) such that \((b(i), b(j)) \in r\). Let \(i < n\). Assume that \((o(a')], [o(b)]) \in T_i^s\). Then there exists \(c : n \rightarrow \Gamma\) that is \(r\) related to \(a'\) such that \([c]T_i[b]\). Conversely, if \([c]T_i[b]\), then \([o(c)]T_i|o(b)]\).

Hence, by complete additivity of cylindrifiers, the map \(\Theta : \text{At}(\text{TCA}_{n+1,n}) \rightarrow \text{CMAt}\) defined via

\[
\Theta([a]) = \begin{cases} 
{[a'] : a' \text{ copy of } a} & \text{if } a \text{ is red,} \\
{[a]} & \text{otherwise.}
\end{cases}
\]

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induces an embedding from $\text{TCA}_{n+1,n}$ to $\mathfrak{CmAt}$, which we denote also by $\Theta$.

We first check preservation of diagonal elements. If $a'$ is a copy of $a$, $i, j < n$, and $a(i) = a(j)$, then $a'(i) = a'(j)$.

We next check cylindrifiers. We show that for all $i < n$ and $[a] \in \text{At}(\text{TCA}_{n+1,n})$ we have:

$$\Theta(c_i[a]) = \bigcup \{\Theta([b]) : [b] \in \text{At}\text{TCA}_{n+1,n}, [b] \leq c_i[a]\} = c_i\Theta([a]).$$

Let $i < n$. If $[b] \in \text{At}\text{TCA}_{n+1,n}$, $[b] \leq c_i[a]$, and $b' : n \to \Gamma, \Gamma \in \text{CGR}$, is a copy of $b$, then there exists $a' : n \to \Gamma', \Gamma' \in \text{CGR}$, a copy of $a$ such that $b' \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}$. Thus $\Theta([b]) \leq c_i\Theta([a])$.

Conversely, if $d : n \to \Gamma, \Gamma \in \text{CGR}$ and $[d] \in c_i\Theta([a])$, then there exist $a'$ a copy of $a$ such that $d \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}$. Hence $o(d) \upharpoonright n \setminus \{i\} = a \upharpoonright n \setminus \{i\}$, and so $[d] \in \Theta(c_i[a])$, and we are done.

But now we can show that $\forall$ can win a certain game on $\text{At}(\text{TCA}_{n+1,n})$ in only $n + 2$ rounds as follows. Viewed as an Ehrenfeucht–Fraïssé forth game pebble game, with finitely many rounds and pairs of pebbles, played on the two complete irreflexive graphs $n + 1$ and $n$, in each round $0, 1 \ldots n$, $\forall$ places a new pebble on an element of $n + 1$. The edge relation in $n$ is irreflexive so to avoid losing $\exists$ must respond by placing the other pebble of the pair on an unused element of $n$. After $n$ rounds there will be no such element, and she loses in the next round. This game, denoted by $F^{n+3}$ is the usual graph game in [21] except that the nodes are limited to $n + 3$ and $\forall$ can re-use nodes. So it is an atomic game with a limited number of pebbles allowing $\forall$ to re-use them. But in fact $\forall$ will win without needing to re-use pebbles.

We show that $\forall$ can win the graph game on $\text{At}(\text{TCA}_{n+1,n})$ in $n + 2$ rounds using $n + 3$ nodes.

$\forall$ forces a win on a red clique using his excess of greens by bombarding $\exists$ with $\alpha$ cones having the same base $(1 \leq \alpha \leq n + 2)$.

In his zeroth move, $\forall$ plays a graph $\Gamma$ with nodes $0, 1, \ldots, n-1$ and such that $\Gamma(i, j) = w_0(i < j < n - 1), \Gamma(i, n - 1) = g_0(i = 1, \ldots, n - 2), \Gamma(0, n - 1) = g_0^0$, and $\Gamma(0, 1, \ldots, n - 2) = y_{n+2}$. This is a 0-cone with base $\{0, \ldots, n - 2\}$. In the following moves, $\forall$ repeatedly chooses the face $(0, 1, \ldots, n - 2)$ and demands a node $\alpha$ with $\Phi(i, \alpha) = g_i, (i = 1, \ldots, n - 2)$ and $\Phi(0, \alpha) = g_0^\alpha$, in the graph notation – i.e., an $\alpha$-cone, without loss $n - 1 < \alpha \leq n + 1$, on the same base. $\exists$ among other things, has to colour all the edges connecting new nodes $\alpha, \beta$ created by $\forall$ as apexes of cones based on the face $(0, 1, \ldots, n - 2)$, that is $\alpha, \beta \geq n - 2$. By the rules of the game the only permissible colours would be red. Using this, $\forall$ can force a win in $n + 2$ rounds, using $n + 3$ nodes without needing to re-use them, thus forcing $\exists$ to deliver an inconsistent triple of reds.

Let $\mathcal{B} = \text{TCA}_{n+1,n}$. Then $\mathcal{B}$ is outside $S\mathfrak{Tr}_n\text{TCA}_{n+3}$ for if it was then because it is finite it would be in $S_c\mathfrak{Tr}_n\text{TCA}_{n+3}$ because $\mathcal{B}$ is the same as its
canonical extension $\mathfrak{D}$, say, and $\mathfrak{D} \in S_n\text{Mr}_n\text{TCA}_{n+3}$. But then $\exists$ would have won [20, Theorem 33]. The last theorem is formulated for relation algebras but it can be easily modified to the cylindric case.

Hence $\mathsf{CmAt} \notin S_n\text{Mr}_n\text{TCA}_{n+3}$, because $\mathsf{Rd}_ca\mathfrak{B}$ is embeddable in its CA reduct and $S_n\text{Mr}_n\text{CA}_{n+3}$ is a variety; in particular, it is closed under forming subalgebras. It now readily follows that $\mathsf{CmAt} \notin S_n\text{Mr}_n\text{TCA}_{n+3}$.

Notice that $\mathsf{Rd}_y\mathfrak{A}$ is not completely representable, because if it were then $\mathfrak{A}$, generated by elements whose dimension sets $< n$, as a $\text{TCA}_n$ would be completely representable and this induces a representation of its Dedekind-MacNeille completion, namely, $\mathsf{CmAt}\mathfrak{A}$.

We have shown that for any finite $n > 2$, any class $\mathcal{K}$ between the varieties $S_n\text{Mr}_n\text{TCA}_{n+3}$ and $\text{TRCA}_n$ is not atom-canonical hence not Sahqvist axiomatizable, and we are done.

The same proof works for $\text{TeCA}_n$ and $\text{TemCA}_n$ by statically temporalizing the algebras dealt with.

Let us prove the tense case, the temporal case is the same. For $\mathfrak{A} \in \text{CA}_n$ let $\mathfrak{A}^{te}$ denotes its static temporalization.

Now let $\mathfrak{C}$ be the CA reduct of the topological complex algebra constructed, then $\mathfrak{C}^{te} \notin S_n\text{Mr}_n\text{TeCA}_{n+3}$ for if it was then $\mathfrak{C}^{te} \subseteq \text{Mr}_n\mathfrak{D}$ for some $\mathfrak{D} \in \text{TeCA}_{n+3}$, hence $\mathfrak{C} \subseteq \mathsf{Rd}_ca\text{Mr}_n\mathfrak{D} = \text{Mr}_n\mathsf{Rd}_ca\mathfrak{D}$ which is impossible.

Statically temporalizing the term algebra $\mathsf{Rd}_ca\mathfrak{A}$, too we have that $\mathfrak{A}^{te}$ is representable (as a $\text{TeCA}_n$), and we still have that $\mathfrak{C}^{te}$ is the minimal completion of $\mathfrak{A}^{te}$, via the same map that embeds $\mathfrak{A}$ into $\mathfrak{C}$. By the neat embedding theorem for tense cylindric algebras we are done.

Now we use another rainbow construction. Coloured graphs and rainbow algebras are defined like above. The algebra constructed now is very similar to $\text{CA}_\omega$, but is not identical; for in coloured graphs we add a new triple of forbidden colours involving two greens and one red synchronized by an order preserving function. In particular, we consider the underlying set of $\omega$ endowed with two orders, the usual order and its converse. To prove the analogous result for relation algebra Robin Hirsch [20] uses a rainbow-like algebra based on the ordered structures $\mathbb{Z}$, $\mathbb{N}$ (with natural order). We could have used these structures, but we chose to replace $\mathbb{Z}$ with the natural number endowed with the converse to $\leq$, to make the analogy tighter with the rainbow algebra used in [21] proving a weaker result.

**Definition 4.7.** Let $\mathfrak{A} \in \text{TeCA}_n$, and assume that $\mathfrak{A} \subseteq \prod_{i \in T} \mathfrak{A}_i$ such that $\mathsf{Rd}_ca\mathfrak{A}_i \in C_n$ Then $\mathfrak{A}$ is completely representable if for each $t \in T$, we have $\mathsf{Rd}_ca\mathfrak{A}_t$ is completely representable.

The following is easy to prove by using known results on complete representability for cylindric algebras [21]. One such result is that complete representability of a cylindric algebra implies that is atomic, the converse however
is not true and \( f : \mathcal{A} \to \wp(X) \) is a complete representation if and only if is a complete one.

We start with the following easy but very useful lemma:

**Theorem 4.8.** Assume that \( n < m \). Let \( \mathcal{A}^{\text{top}} \in \text{TCA}_n \) be obtained from \( \mathcal{A} \in \text{CA}_n \) by topologization. If \( \mathcal{A}^{\text{top}} \notin S\text{Nr}_n \text{TCA}_m \), then \( \mathcal{A} \notin S\text{Nr}_n \text{CA}_m \). A completely analogous result holds for \( \text{TeCA}_n \).

**Proof.** We prove the contrapositive. Assume that \( \mathcal{A} \in S\text{Nr}_n \text{CA}_m \). Then \( \mathcal{A} \subseteq \text{Nr}_n \mathcal{B} \) where \( \mathcal{B} \in \text{CA}_m \). By re-topologizing \( \mathcal{A} \) and topologizing \( \mathcal{B} \) we get the required. \( \square \)

We approach the notion of complete representations for \( \text{TCA}_n \) when \( n \) is finite. Rainbows \([22, 26]\) will offer solace here. Throughout this subsection \( n \) will be finite and \( > 1 \). We identify notationally set algebras with their universes.

Let \( \mathcal{A} \in \text{TCA}_n \) and \( f : \mathcal{A} \to \wp(V) \) be a representation of \( \mathcal{A} \), where \( V \) is a generalized space of dimension \( n \). If \( s \in V \) we let \( f^{-1}(s) = \{ a \in \mathcal{A} : s \in f(a) \} \).

An atomic representation \( f : \mathcal{A} \to \wp(V) \) is a representation such that for each \( s \in V \), the ultrafilter \( f^{-1}(s) \) is principal.

A complete representation of \( \mathcal{A} \) is a representation \( f \) satisfying

\[
 f(\prod X) = \bigcap f[X]
\]

whenever \( X \subseteq \mathcal{A} \) and \( \prod X \) is defined.

**Theorem 4.9.** Let \( \mathcal{A} \in \text{TCA}_n \). Let \( f : \mathcal{A} \to \wp(V) \) be a representation of \( \mathcal{A} \). Then \( f \) is a complete representation iff \( f \) is an atomic one. Furthermore, if \( \mathcal{A} \) is completely representable, then \( \mathcal{A} \) is atomic and \( \mathcal{A} \in S\text{c} \text{Nr}_n \text{TCA}_\omega \).

**Proof.** Witness \([35], \text{Theorems 5.3.4, 5.3.6}, [26], \text{Theorem 3.1.1}\) for the first three parts. It remains to show that if \( \mathcal{A} \) is completely representable, then \( \mathcal{A} \in S\text{c} \text{Nr}_n \text{TCA}_\omega \). Assume that \( M \) is the base of a complete representation of \( \mathcal{A} \), whose unit is a generalized space, that is, \( 1^M = \bigcup_{i \in I} \wp U_i \), where \( U_i \cap U_j = \emptyset \) for distinct \( i \) and \( j \) in \( I \) where \( I \) is an index set \( I \). Let \( t : \mathcal{A} \to \wp(1^M) \) be the complete representation. For each \( i \in I \), \( U_i \) carries the subspace topology of \( M \). For \( i \in I \), let \( E_i = \wp U_i \), pick \( f_i \in \wp U_i \), let \( W_i = \{ f \in \wp U_i : |\{ k \in \omega : f(k) \neq f_i(k) \}| < \omega \} \), and let \( \mathcal{E}_i \) be the \( \text{TCA}_n \) with universe \( \wp(W_i) \), with the \( \text{CA} \) operations defined the usual way on weak set algebras, and the interior operator is induced by the topology on \( U_i \). Then \( \mathcal{E}_i \) is atomic; indeed the atoms are the singletons.

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Let \( x \in \mathfrak{M}_n \), that is \( c_jx = x \) for all \( n \leq j < \omega \). Now if \( f \in x \) and \( g \in W_i \) satisfy \( g(k) = f(k) \) for all \( k < n \), then \( g \in x \). Hence \( \mathfrak{M}_n \) is atomic; its atoms are \( \{g \in W_i : \{g(0), \ldots, g(n-1)\} \subseteq U_i\} \). Define \( h_i : \mathfrak{A} \rightarrow \mathfrak{M}_n \) by

\[
  h_i(a) = \{f \in W_i : \exists a \in \text{At}\mathfrak{A} : (f(0) \ldots f(n-1)) \in t(a)\}.
\]

Let \( \mathfrak{C} = \prod_i \mathfrak{C}_i \). Let \( \pi : \mathfrak{C} \rightarrow \mathfrak{C}_i \) be the \( i \)th projection map. Now clearly \( \mathfrak{C} \) is atomic, because it is a product of atomic algebras, and its atoms are \( \{\pi_i(\beta) : \beta \in \text{At}(\mathfrak{C}_i)\} \). Now \( \mathfrak{A} \) embeds into \( \mathfrak{M}_n \) via \( I : a \mapsto (\pi_i(a) : i \in I) \), and we may assume that the map is surjective.

If \( a \in \mathfrak{M}_n \), then for each \( i \), we have \( \pi_i(x) \in \mathfrak{M}_n \), and if \( x \) is non zero, then \( \pi_i(x) \neq 0 \). By atomicity of \( \mathfrak{C}_i \), there is a tuple \( \bar{m} \) such that \( \{g \in W_i : g(k) = [\bar{m}]_k\} \subseteq \pi_i(x) \). Hence there is an atom \( a \) of \( \mathfrak{A} \), such that \( \bar{m} \in t(a) \), so \( x \wedge I(a) \neq 0 \), and so the embedding is complete and we are done. Note that in this argument no cardinality condition is required. (The reverse inclusion does not hold in general for uncountable algebras, as will be shown in theorem \[4.22 \] though it holds for atomic algebras with countably many atoms as shown in our next theorem). Hence \( \mathfrak{A} \in S_c \mathfrak{M}_n \mathcal{TCA}_\omega \).

\[ \Box \]

Conversely, the following theorem can be obtained by topologizing \[35 \], Theorem 5.3.6] and applying the omitting types theorem proved in Part 1.

**Theorem 4.10.** If \( \mathfrak{A} \) is countable and atomic and \( \mathfrak{A} \in S_c \mathfrak{M}_n \mathcal{TCA}_\omega \), then \( \mathfrak{A} \) is completely representable.

Now we will prove that the notion of complete representability, like in the case of cylindric algebras of dimension \( > 2 \), is not first order definable. In fact, we prove much more, namely, that any class \( \mathcal{K} \) containing the class of completely representable algebras of finite dimension \( n > 2 \) and contained in \( S_c \mathfrak{M}_n \mathcal{TCA}_{n+3} \) is not elementary. This is meaningful since any completely representable algebra of dimension \( n \) is in \( S_c \mathfrak{M}_n \mathcal{CA}_\omega \) by theorem \[4.22 \] and obviously the latter class contained in \( S_c \mathfrak{M}_n \mathcal{CA}_{n+3} \).

Indeed this result is much stronger because for finite \( n > 2 \) and any \( k \leq \omega \), we have \( S_c \mathfrak{M}_n \mathcal{TCA}_{n+k+1} \subseteq S \mathfrak{M}_n \mathcal{TCA}_{n+k} \) for all \( k \in \omega \) where \( \subseteq \) denotes proper inclusion.

Also the strictness of such inclusions can be witnessed by ‘topologizing’ the finite algebras constructed in \[27 \], that is, expanding them with identity functions as interior operators; recall that this expansion preserves representability which can be induced by imposing the discrete topology on the base of the representing algebra of the original \( \mathcal{CA} \).

Now we use another rainbow construction. Coloured graphs and rainbow algebras are defined like above. The algebra constructed now is very similar to \( \mathcal{CA}_{\omega,\omega} \) but is not identical; for in coloured graphs we add a new triple of
forbidden colours involving two greens and one red synchronized by an order preserving function. In particular, we consider the underlying set of $\omega$ endowed with two orders, the usual order and its converse. To prove the analogous result for relation algebra Robin Hirsch [20] uses a rainbow-like algebra based on the ordered structures $\mathbb{Z}, \mathbb{N}$ (with natural order). We could have used these structures, but we chose to replace $\mathbb{Z}$ with the natural number endowed with the converse to $\leq$, to make the analogy tighter with the rainbow algebra used in [21] proving a weaker result.

**Definition 4.11.** Let $\mathfrak{A} \in \text{TeCA}_n$, and assume that $\mathfrak{A} \subseteq \prod_{t \in T} \mathfrak{A}_t$ such that $\mathfrak{R}_{ca}\mathfrak{A}_t \in \mathfrak{C}_n$. Then $\mathfrak{A}$ is completely representable if for each $t \in T$, we have $\mathfrak{R}_{ca}\mathfrak{A}_t$ is completely representable.

The following is easy to prove by using known results on complete representability for cylindric algebras [21]. One such result is that complete representability of a cylindric algebra implies that is atomic, the converse however is not true and $f : \mathfrak{A} \rightarrow \wp(X)$ is a complete representation if and only if is a complete one.

**Theorem 4.12.** (1) If $\mathfrak{A} \in \text{TeCA}_n$ is completely representable, $\mathfrak{A}$ based on $T$, and $\mathfrak{R}_{ca}\mathfrak{A} \cong \prod_{t \in T} \mathfrak{A}_t$, then every component $\mathfrak{A}_t$ is atomic ($t \in T$). Hence $\mathfrak{A}$ itself is atomic, since it is a product of atomic algebras.

(2) Furthermore, if $\pi_t : \mathfrak{A} \rightarrow \mathfrak{A}_t$ is the projection map, then $\mathfrak{A}$ is completely representable if and only if for all $X \subseteq \mathfrak{A}$ for all non-zero $a \in \mathfrak{A}$, whenever $\sum X = 1$, then there exists a homomorphism $f : \mathfrak{A} \rightarrow \wp_{\mathfrak{A}}$, for some tense system $K$ such that $\bigcup_{x \in X} f(\pi_t(x)) = 1^{\mathfrak{A}_t}$ and $f(a) \neq 0$.

(3) if $\mathfrak{A}$ has countably many atoms then $\mathfrak{A}$ is completely representable if and only if $\mathfrak{A} \in S_{\mathfrak{CN}}^{\mathfrak{C}_{\mathfrak{CN}}}{\text{TeCA}}_\omega$.

**Proof.** We prove only the last item. The idea is simple. Lifting complete representations of the component to one for their product and vice versa by noting that each component is a complete subalgebra of the product. Assume that $\mathfrak{R}_{ca}\mathfrak{A} = \prod_{t \in T} \mathfrak{A}_t$ and every $\mathfrak{A}_t$ is completely representable hence is in $S_{\mathfrak{CN}}^{\mathfrak{C}_{\mathfrak{CN}}}{\text{TeCA}}_\omega$, but the latter class as easily checked is closed under products.

The converse follows from that every $\mathfrak{A}_t$ is a complete subalgebra of $\mathfrak{A}$, the latter is in $S_{\mathfrak{CN}}^{\mathfrak{C}_{\mathfrak{CN}}}{\text{TeCA}}_\omega$, hence $\mathfrak{A}_t \in S_{\mathfrak{CN}}^{\mathfrak{C}_{\mathfrak{CN}}}{\text{TeCA}}_\omega$. But this means that every $\mathfrak{A}_t$ is completely representable with the homomorphism $f_t : \mathfrak{A} \rightarrow \wp(\mathfrak{A}_t)$ say then so is $\mathfrak{A}$, via $f : \mathfrak{A} \rightarrow \prod_{t \in T} \wp(\mathfrak{A}_t) = \wp(\bigcup \mathfrak{A}_t)$ where $\bigcup$ here denotes disjoint union. □
Part 2

In the next theorem we show that atomicity does not imply complete representability. In fact we show that for finite \( n > 2 \) that the class of completely representable \( \text{TCA}_n \)'s is not elementary, \textit{a fortiori} it is not atomic because atomicity is a first order definable notion. We use a rainbow construction and in fact we prove a result stronger than non elementarity of the class of completely representable algebras; we prove it for both \( \text{TCA}_n \) and \( \text{TeCA}_n \).

**Theorem 4.13.**

1. Let \( 3 \leq n < \omega \). Then there exists an atomic \( \mathcal{C} \in \text{TCA}_n \) with countably many atoms such that \( \mathcal{C} \notin S_c \text{Nr}_n \text{TCA}_{n+3} \), and there exists a countable \( \mathcal{B} \in S_c \text{Nr}_n \text{TCA}_\omega \) such that \( \mathcal{C} \equiv \mathcal{B} \) (hence \( \mathcal{B} \) is also atomic). Hence for any class \( L \), such that \( S_c \text{Nr}_n \text{TCA}_\omega \subseteq L \subseteq S_c \text{Nr}_n \text{TCA}_{n+3} \), \( L \) is not elementary, and the class of completely representable \( K \) algebras of dimension \( n \) is not elementary.

2. The same holds for \( \text{TeCA}_n \). For every time flow \((T, <)\) there is an atomic \( R\text{TeCA}_n \) based on \( T \) that is not completely representable, in fact not in \( S_c \text{Nr}_n \text{TeCA}_{n+3} \) but is elementary equivalent to a completely representable tense cylindric algebra of dimension \( n \).

**Proof.**

(1) Let \( N^{-1} \) denote \( N \) with reverse order, let \( f : N \to N^{-1} \) be the identity map, and denote \( f(a) \) by \( -a \), so that for \( n, m \in N \), we have \( n < m \) iff \( -m < -n \). We assume that 0 belongs to \( N \) and we denote the domain of \( N^{-1} \) (which is \( N \)) by \( N^{-1} \). We alter slightly the standard rainbow construction. The colours we use are the same colours used in rainbow constructions:

- greens: \( g_i \ (1 \leq i \leq n - 2), \ g_i^0, \ i \in N^{-1} \),
- whites: \( w_i : i \leq n - 2 \),
- reds: \( r_{ij} \ (i, j \in N) \),
- shades of yellow: \( y_S : S \subseteq \omega N^{-1} \) or \( S = N^{-1} \).

We define a subclass of coloured graphs \( M \) such that

1. \( M \) is a complete graph.

(2) \( M \) contains no triangles (called forbidden triples) defined exactly as in \[21\] together with the additional forbidden triple \( (g_0^0, g_0^1, r_{kl}) \), in more detail.
and no other triple of atoms is forbidden.

(3) The last two items concerning shades of yellow are as before.

But the forbidden triple \((g_{0}, g_{0}', r_{kl})\) is not present in standard rainbow constructions, adopted example in [21] and in a more general form in [26]. Therefore, we cannot use the usual rainbow argument adopted in [21]; we have to be selective for the choice of the indices of reds if we are labelling the apexes of two cones having green tints; not any red that works for usual rainbows will do.

One then can define (what we continue to call) a rainbow atom structure of dimension \(n\) from the class \(G\). Let

\[
\text{At} = \{a : n \to M, M \in G : a \text{ is surjective }\}.
\]

We write \(M_{a}\) for the element of \(\text{At}\) for which \(a : n \to M\) is a surjection. Let \(a, b \in \text{At}\) define the following equivalence relation: \(a \sim b\) if and only if

\[
\begin{align*}
&\bullet a(i) = a(j) \iff b(i) = b(j), \\
&\bullet M_{a}(a(i), a(j)) = M_{b}(b(i), b(j)) \text{ whenever defined}, \\
&\bullet M_{a}(a(k_{0}), \ldots, a(k_{n-2})) = M_{b}(b(k_{0}), \ldots, b(k_{n-2})) \text{ whenever defined}.
\end{align*}
\]

Let \(\text{At}\) be the set of equivalences classes. Then define

\[
[a] \in E_{ij} \text{ iff } a(i) = a(j).
\]

\[
[a]T_{i}[b] \text{ iff } a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}.
\]

And topologizing the atom structure we set

\[
[a]I_{i}[b] \text{ iff } a = b,
\]

that adds nothing to the cylindric structure inducing the interior topology. This, as easily checked, defines a \(\text{TCA}_{n}\) atom structure. Let \(\mathcal{C}\) be
the complex algebra. Let $k > 0$ be given. We show that $\exists$ has a winning strategy in the usual graph game in $k$ rounds (now there is no restriction here on the size of the graphs) on $\mathcal{A}t\mathcal{C}$. We recall the ‘usual atomic’ $k$ rounded game $G_k$ played on coloured graphs. $\forall$ picks a graph $M_0 \in \mathfrak{G}$ with $M_0 \subseteq n$ and $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $M_i$. $\forall$ picks

- a graph $\Phi \in \mathfrak{G}$ with $|\Phi| = n$,
- a single node $m \in \Phi$,
- a coloured graph embedding $\theta : \Phi \setminus \{m\} \to M_i$. Let $F = \phi \setminus \{m\}$. Then $F$ is called a face. The $\exists$ has to define a graph $M_{i+1} \in C$ and embeddings $\lambda : M_i \to M_{i+1} \mu : \phi \to M_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

Consider the following restricted version of $G_\omega$; only $m > n$ nodes are available. We denote this $\omega$ rounded game using $m$ nodes by $G^m$. In more detail, in a play of $G^m$ $\forall$ picks a graph $M_0 \in \mathfrak{G}$ with $M_0 \subseteq m$ and $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $M_i$. $\forall$ picks

- a graph $\Phi \in CRG$ with $|\Phi| = n$,
- a single node $k \in \Phi$,
- a coloured graph embedding $\theta : \Phi \setminus \{k\} \to M_i$. $\exists$ must respond like the in the usual atomic game by amalgamating $M_i$ and $\Phi$ with the embedding $\theta$.

$F^m$ is like $G^m$, but $\forall$ is allowed to resuse nodes.

Inspite of the restriction of adding a forbidden triple relating two greens and a red (that makes it harder for $\exists$ to win), we will show that $\exists$ will always succeed to choose a suitable red in the finite rounded atomic games $G^m$. On the other hand, this bonus for $\forall$ will enable him to win the game $F^{n+3}$ by forcing $\exists$ to play a decreasing sequence in $\mathbb{N}$. Using and re-using $n + 3$ nodes will suffice for this purpose. We define $\exists$’s strategy for choosing labels for edges and $n - 1$ tuples in response to $\forall$’s moves. Assume that we are at round $r + 1$. Our arguments are similar to the arguments in [20, Lemmas, 41-43].

Let $M_0, M_1, \ldots, M_r$, $r < k$ be the coloured graphs at the start of a play of $G^k(\alpha)$ just before round $r + 1$. Assume inductively that $\exists$ computes a partial function $\rho_s : \mathbb{N}^{-1} \to \mathbb{N}$, for $s \leq r$, that will help her choose the suffixes of the chosen red in the critical case. In our previous rainbow construction we had the additional shade of red $\rho$ that did the job. Now we do not have it, so we proceed differently. Inductively for $s \leq r$ we assume:
(1) If $M_s(x, y)$ is green then $(x, y)$ belongs $\forall$ in $M_s$ (meaning he coloured it),

(2) $\rho_0 \subseteq \ldots \rho_r \subseteq \ldots$,

(3) $\text{dom}(\rho_s) = \{i \in N^−1 : \exists t \leq s, x, x_0, x_1, \ldots, x_{n−2} \in \text{nodes}(M_t)$

where the $x_i$'s form the base of a cone, $x$ is its apex and $i$ its tint $\}$.

The domain consists of the tints of cones created at an earlier stage,

(4) $\rho_s$ is order preserving: if $i < j$ then $\rho_s(i) < \rho_s(j)$. The range of $\rho_s$

is widely spaced: if $i < j \in \text{dom} \rho_s$ then $\rho_s(i) - \rho_s(j) \geq 3^{m−r}$, where

$m − r$ is the number of rounds remaining in the game,

(5) For $u, v, x_0 \in \text{nodes}(M_s)$, if $M_s(u, v) = r_{\mu, \delta}, M_s(x_0, u) = g_0^i$, $M_s(x_0, v) = g_0^j$,

where $i, j$ are tints of two cones, with base $F$ such that $x_0$ is the

first element in $F$ under the induced linear order, then $\rho_s(i) = \mu$

and $\rho_s(j) = \delta$.

(6) $M_s$ is a a coloured graph,

(7) If the base of a cone $\Delta \subseteq M_s$ with tint $i$ is coloured $y_s$, then $i \in S$.

To start with if $\forall$ plays $a$ in the initial round then $\text{nodes}(M_0) = \{0, 1, \ldots, n−1\}$, the hyperedge labelling is defined by $M_0(0, 1, \ldots, n) = a$.

In response to a cylindrifier move for some $s \leq r$, involving a $p$ cone, $p \in N^−1$, $\exists$ must extend $\rho_r$ to $\rho_{r+1}$ so that $p \in \text{dom}(\rho_{r+1})$ and the gap between elements of its range is at least $3^{m−r−1}$. Properties (3) and (4) are easily maintained in round $r + 1$. Inductively, $\rho_r$ is order preserving and the gap between its elements is at least $3^{m−r}$, so this can be maintained in a further round. If $\forall$ chooses a green colour, or green colour whose suffix already belong to $\rho_r$, there would be fewer elements to add to the domain of $\rho_{r+1}$, which makes it easy for $\exists$ to define $\rho_{r+1}$.

Now assume that at round $r + 1$, the current coloured graph is $M_r$ and that $\forall$ chose the graph $\Phi$, $|\Phi| = n$ with distinct nodes $F \cup \{\delta\}$, $\delta \notin M_r$, and $F \subseteq M_r$ has size $n−1$. We can view $\exists$ move as building a coloured graph $M^*$ extending $M_r$ whose nodes are those of $M_r$ together with the new node $\delta$ and whose edges are edges of $M_r$ together with edges from $\delta$

to every node of $F$.

Now $\exists$ must extend $M^*$ to a complete graph $M^+$ on the same nodes and complete the colouring giving a graph $M_{r+1} = M^+$ in $\mathfrak{G}$ (the latter is the class of coloured graphs). In particular, she has to define $M^+(\beta, \delta)$ for all nodes $\beta \in M_r \sim F$, such that all of the above properties are maintained.

(1) If $\beta$ and $\delta$ are both apexes of two cones on $F$.

Assume that the tint of the cone determined by $\beta$ is $a \in N^−1$, and the two cones induce the same linear ordering on $F$. Recall that
we have $\beta \notin F$, but it is in $M_r$, while $\delta$ is not in $M_r$, and that $|F| = n - 1$. By the rules of the game $\exists$ has no choice but to pick a red colour. $\exists$ uses her auxiliary function $\rho_{r+1}$ to determine the suffixes, she lets $\mu = \rho_{r+1}(p)$, $b = \rho_{r+1}(q)$ where $p$ and $q$ are the tints of the two cones based on $F$, whose apexes are $\beta$ and $\delta$. Notice that $\mu, b \in \mathbb{N}$; then she sets $N_s(\beta, \delta) = t_{\mu,b}$ maintaining property (5), and so $\delta \in \text{dom}(\rho_{r+1})$ maintaining property (4). We check consistency to maintain property (6).

Consider a triangle of nodes $(\beta, y, \delta)$ in the graph $M_{r+1} = M^+$. The only possible potential problem is that the edges $M^+(y, \beta)$ and $M^+(y, \delta)$ are coloured green with distinct superscripts $p, q$ but this does not contradict forbidden triangles of the form involving $(g_0^p, g_0^q, \nu_{kl})$, because $\rho_{r+1}$ is constructed to be order preserving. Now assume that $M_r(\beta, y)$ and $M_{r+1}(y, \delta)$ are both red (some $y \in \text{nodes}(M_r)$). Then $\exists$ chose the red label $N_{r+1}(y, \delta)$, for $\delta$ is a new node. We can assume that $y$ is the apex of a $t$ cone with base $F$ in $M_r$. If not then $N_{r+1}(y, \delta)$ would be coloured $w$ by $\exists$ and there will be no problem. All properties will be maintained. Now $y, \beta \in M$, so by by property (5) we have $M_{r+1}(\beta, y) = t_{\rho+1(p),\rho+1(q)}$. But $\delta \notin M$, so by her strategy, we have $M_{r+1}(y, \delta) = t_{\rho+1(t),\rho+1(q)}$. But $M_{r+1}(\beta, \delta) = t_{\rho+1(p),\rho+1(q)}$, and we are done. This is consistent triple, and so have shown that forbidden triples of reds are avoided.

(2) If there is no $f \in F$ such that $M^*(\beta, f), M^*(\delta, f)$ are coloured $g_0^t$, $g_0^u$ for some $t, u$ respectively, then $\exists$ defines $M^+(\beta, \delta)$ to be $w_0$.

(3) If this is not the case, and for some $0 < i < n - 1$ there is no $f \in F$ such that $M^*(\beta, f), M^*(\delta, f)$ are both coloured $g_i$, she chooses $w_i$ for $M^+(\beta, \delta)$.

It is clear that these choices in the last two items avoid all forbidden triangles (involving greens and whites).

She has not chosen green maintaining property (1). Now we turn to colouring of $n - 1$ tuples, to make sure that $M^+$ is a coloured graph maintaining property (7).

Let $\Phi$ be the graph chosen by $\forall$ it has set of node $F \cup \{\delta\}$. For each tuple $\bar{a} = a_0, \ldots, a_{n-2} \in M^{n-1} = a \notin M^{n-1} \cup \Phi^{n-1}$, with no edge $(a_i, a_j)$ coloured green (we already have all edges coloured), then $\exists$ colours $\bar{a}$ by $y_s$, where

$$S = \{i \in A : \text{there is an } i \text{ cone in } M^* \text{ with base } \bar{a}\}.$$
Recall that $M$ is the current coloured graph, $M^* = M \cup \{\delta\}$ is built by $\forall$ s move and $M^+$ is the complete labelled graph by $\exists$ whose nodes are labelled by $\exists$ in response to $\forall$ s moves. We need to show that $M^+$ is labelled according to the rules of the game, namely, that it is in $\mathcal{G}$. It can be checked $(n - 1)$ tuples are labelled correctly, by yellow colours using the same argument in [28, p.16] [21, p.844] and [26].

We show that $\forall$ has a winning strategy in $F^{n+3}$, the argument used is the CA analogue of [20, Theorem 33, Lemma 41]. The difference is that in the relation algebra case, the game is played on atomic networks, but now it is translated to playing on coloured graphs, [21, lemma 30].

In the initial round $\forall$ plays a graph $\Gamma$ with nodes $0, 1, \ldots n - 1$ such that $\Gamma(i, j) = w_0$ for $i < j < n - 1$ and $\Gamma(i, n - 1) = g_i$ ($i = 1, \ldots, n - 2$), $\Gamma(0, n - 1) = g_0$ and $\Gamma(0, 1, \ldots, n - 2) = y_B$.

In the following move $\forall$ chooses the face $(0, \ldots n - 2)$ and demands a node $n$ with $\Gamma_2(i, n) = g_i$ ($i = 1, \ldots, n - 2$), and $\Gamma_2(0, n) = g_0^{-1}$. $\exists$ must choose a label for the edge $(n + 1, n)$ of $\Gamma_2$. It must be a red atom $r_{mn}$. Since $-1 < 0$ we have $m < n$. In the next move $\forall$ plays the face $(0, \ldots n - 2)$ and demands a node $n + 1$, with $\Gamma_3(i, n) = g_i$ ($i = 1, \ldots, n - 2$), such that $\Gamma_3(0, n + 2) = g_0^{-2}$. Then $\Gamma_3(n + 1, n)$ and $\Gamma_3(n + 1, n - 1)$ both being red, the indices must match. $\Gamma_3(n + 1, n) = r_{ln}$ and $\Gamma_3(n + 1, n - 1) = r_{lm}$ with $l < m$. In the next round $\forall$ plays $(0, 1, \ldots, n - 2)$ and reuses the node $2$ such that $\Gamma_4(0, 2) = g_0^{-3}$. This time we have $\Gamma_4(n, n - 1) = r_{jl}$ for some $j < l \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$.

Now that $\forall$ has a winning strategy in $F^{n+3}$, it follows by an easy adaptation of lemma [20, Lemma 33], baring in mind that games on coloured graphs are equivalent to games on atomic networks, that $\mathfrak{d}_{sc} \subseteq S, \mathfrak{m}_{sc} \subseteq \mathfrak{s}_{sc}^{n+3},$ but it is elementary equivalent to a countable completely representable algebra. Indeed, using ultrapowers and an elementary chain argument, we obtain $\mathfrak{b}$ such $\mathfrak{c} \equiv \mathfrak{b}$ [20, lemma 44], and $\exists$ has a winning strategy in $G_\omega$ (the usual $\omega$ rounded atomic game) on $\mathfrak{b}$ so by [26, Theorem 3.3.3], $\mathfrak{b}$ is completely representable.

In more detail we have $\exists$ has a winning strategy $\sigma_k$ in $G_k$. We can assume that $\sigma_k$ is deterministic. Let $\mathfrak{d}$ be a non-principal ultrapower of $\mathfrak{c}$. One can show that $\exists$ has a winning strategy $\sigma$ in $G(\mathfrak{d})$ — essentially she uses $\sigma_k$ in the $k'$th component of the ultraproduct so that at each round of $G(\mathfrak{d})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost.

We can also assume that $\mathfrak{c}$ is countable. If not then replace it by its subalgebra generated by the countably many atoms (the term algebra);
winning strategies that depended only on the atom structure persist for both players.

Now one can use an elementary chain argument to construct a chain of countable elementary subalgebras $\mathcal{C} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \ldots \mathcal{D}$ inductively in this manner. One defines $\mathcal{A}_{i+1}$ to be a countable elementary subalgebra of $\mathcal{D}$ containing $\mathcal{A}_i$ and all elements of $\mathcal{D}$ that $\sigma$ selects in a play of $\mathcal{G}(\text{At}\mathcal{D})$ in which $\forall$ only chooses elements from $\mathcal{A}_i$. Now let $\mathcal{B} = \bigcup_{i<\omega} \mathcal{A}_i$. This is a countable elementary subalgebra of $\mathcal{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $\mathcal{G}(\text{At}\mathcal{B})$, so by [26, Theorem 3.3.3], noting that $\mathcal{B}$ is countable, then $\mathcal{B}$ is completely representable; furthermore $\mathcal{B} \equiv \mathcal{C}$.

The same proof works for $\text{TecA}_\alpha$ for the last part using a static expansion of $\mathcal{C}$ and a static expansion of $\mathcal{B}$ expansion.

(4) As for the last part, let $(T,<)$ be a given flow of time. Choose representable atomic cylindric algebras $\mathcal{E}_t, \mathcal{B}_t$ for each $t \in T$ such that $\mathcal{B}_t \equiv \mathcal{E}_t$, $\mathcal{E}_t \notin S\text{Nr}_n \text{CA}_{n+3}$ while $\mathcal{B}_t$ is completely representable.

We know by theorem 4.13 that such algebras exist. In fact, we can take $\mathcal{B}_t = \mathcal{E}_t$ and $\mathcal{E}_t = \mathcal{C}$ for each moment $t \in T$, where $\mathcal{B}$ and $\mathcal{C}$ are algebras used in theorem 4.13. By the Feferman-Vaught theorem we have $\mathcal{B}^+ = \prod_{t \in T} \mathcal{B}_t \equiv \prod_{t \in T} \mathcal{E}_t = \mathcal{E}^+$. Now define $<$ in $T$ using $G$ and $H$ both defined as the identity map, getting the required flow of time, the tense system based on it, and finally the tense algebra constructed from such a tense system. The elementary equivalence remain to hold, we need to check that $\prod B_t$ is completely representable, this is not hard.

Now assume for contradiction that $\mathcal{E}^+ = \prod_{t \in T} \mathcal{E}_t \in S_c \text{Nr}_n \text{TeCA}_{n+3}$. Recall that $\mathcal{E}_t = \mathcal{C}$ for any $t \in T$. But it is clear that the projection map from $\mathcal{E} \to \mathcal{E}^+$ is a complete one, hence $\mathcal{E}^e \in S_c \text{Nr}_n \text{TeCA}_{n+3}$. But that $\mathcal{E} = \prod_{t \in \alpha} \mathcal{E}^e \in S_c \text{Nr}_n \text{CA}_{n+3}$ which is a contradiction.

Consider the following two statements for finite $n > 2$:

(1) Any class $K$ such that $S_c \text{Nr}_n \text{TCA}_\omega \subseteq K \subseteq S_c \text{TCA}_{n+3}$ is not elementary.

(2) Any $K$ such that $\text{Nr}_n \text{CA}_\omega \subseteq K \subseteq S_c \text{Nr}_n \text{CA}_{n+3}$ is not elementary.

The first statement is proved in our previous theorem. The second statement is stronger since $\text{Nr}_n \text{CA}_\omega \subseteq S_c \text{Nr}_n \text{CA}_\omega$ and the inclusion is strict. This inclusion can be distilled from a construction in [30, 40] where an infinite set algebra is used and which we now recall. Later, witness example 4.18 we will give an example of a finite algebra that witnesses the strictness of this inclusion for finite $n > 2$. Our next example though works for any $\alpha > 1$, but algebras
used are infinite. Of course this is to be expected for infinite dimensions but such algebras are also infinite for any finite dimension $> 1$.

**Example 4.14.** Here we give an example that works for any signature is between $\mathcal{Sc}$ and $\mathcal{QEA}$, where $\mathcal{Sc}$ denotes Pinter's substitution algebras and $\mathcal{QEA}$ denotes quasipolyadic algebras. (Here we are using the notation $\mathcal{QEA}$ instead of $\mathcal{PEA}$ because we are allowing infinite dimensions).

$FT_{\alpha}$ denotes the set of all finite transformations on $\alpha$. Let $\alpha$ be an ordinal $> 1$; could be infinite. Let $\mathfrak{F}$ is field of characteristic 0.

$V = \{s \in ^{\alpha}\mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$

$\mathfrak{C} = (\varphi(V), \cup, \cap, \sim, \emptyset, V, c_i, d_{i,j}, s_{\tau})_{i,j \in \alpha, \tau \in FT_{\alpha}}.$

Then clearly $\varphi(V) \in \mathfrak{N}_{\alpha}\mathcal{QPEA}_{\alpha+\omega}$. Indeed let $W = ^{\alpha+\omega}\mathfrak{F}(0)$. Then $\psi : \varphi(V) \rightarrow \mathfrak{N}_{\alpha}\varphi(W)$ defined via

$$X \mapsto \{s \in W : s \upharpoonright \alpha \in X\}$$

is an isomorphism from $\varphi(V)$ to $\mathfrak{N}_{\alpha}\varphi(W)$. We shall construct an algebra $\mathfrak{A}$, $\mathfrak{A} \notin \mathfrak{N}_{\alpha}\mathcal{QPEA}_{\alpha+1}$. Let $y$ denote the following $\alpha$-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

Let $y_s$ be the singleton containing $s$, i.e. $y_s = \{s\}$. Define $\mathfrak{A} \in \mathcal{QPEA}_{\alpha}$ as follows:

$$\mathfrak{A} = \mathfrak{S}g^{\mathfrak{F}}\{y, y_s : s \in y\}.$$

Now clearly $\mathfrak{A}$ and $\varphi(V)$ share the same atom structure, namely, the singletons. Then we claim that $\mathfrak{A} \notin \mathfrak{N}_{\alpha}\mathcal{QPEA}_{\beta}$ for any $\beta > \alpha$. The first order sentence that codes the idea of the proof says that $\mathfrak{A}$ is neither an elementary nor complete subalgebra of $\varphi(V)$. We use $\land$ and $\rightarrow$ in the meta language with their usual meaning. Let $\text{At}(x)$ be the first order formula asserting that $x$ is an atom. Let

$$\tau(x, y) = c_1(c_0 x \cdot s_0^0 c_1 y) \cdot c_1 x \cdot c_0 y.$$

Let

$$\text{Rc}(x) := c_0 x \cap c_1 x = x,$$

$$\phi := \forall x(x \neq 0 \rightarrow \exists y(\text{At}(y) \land y \leq x)) \land \forall x(\text{At}(x) \rightarrow \text{Rc}(x)),$$

$$\alpha(x, y) := \text{At}(x) \land x \leq y,$$

and $\psi(y_0, y_1)$ be the following first order formula

$$\forall z(\forall x(\alpha(x, y_0) \rightarrow x \leq z) \rightarrow y_0 \leq z) \land \forall x(\text{At}(x) \rightarrow \text{At}(c_0 x \cap y_0) \land \text{At}(c_1 x \cap y_0))$$
\[ \forall x_1 \forall x_2 (\alpha(x_1, y_0) \land \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq y_1) \land \forall z (\forall x_1 \forall x_2 (\alpha(x_1, y_0) \land \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq z) \rightarrow y_1 \leq z) \].

Then

\[ \forall \alpha \text{ QEA}_\beta \models \phi \rightarrow \forall y_0 \exists y_1 \psi(y_0, y_1). \]

But this formula does not hold in \( \mathfrak{A} \). We have \( \mathfrak{A} \models \phi \) and not \( \mathfrak{A} \models \forall y_0 \exists y_1 \psi(y_0, y_1). \)

In words: we have a set \( X = \{ y_s : s \in V \} \) of atoms such that \( \sum^\mathfrak{a} X = y \), and \( \mathfrak{A} \) models \( \phi \) in the sense that below any non-zero element there is a rectangular atom, namely a singleton.

Let \( Y = \{ \tau(y_r, y_s), r, s \in V \} \), then \( Y \subseteq \mathfrak{A} \), but it has no supremum in \( \mathfrak{A} \), but it does have one in any full neat reduct \( \mathfrak{B} \) containing \( \mathfrak{A} \), and this is \( \tau_\mathfrak{B}(y, y) \), where \( \tau_\mathfrak{A}(x, y) = c_\alpha (s_1^x c_\alpha x \cdot s_0^y c_\alpha y) \).

In \( \varphi(V) \) this last is \( w = \{ s \in \alpha \mathfrak{B}(0) : s_0 + 2 = s_1 + 2 \sum_{i=1} s_i \} \), and \( w \notin \mathfrak{A} \). The proof of this can be easily distilled from [39] main theorem. For \( y_0 = y \), there is no \( y_1 \in \mathfrak{A} \) satisfying \( \psi(y_0, y_1) \). Actually the above proof proves more. It proves that there is a \( \mathfrak{C} \in \mathfrak{N} \text{ QEA}_\beta \) for every \( \beta > \alpha \) (equivalently \( \mathfrak{C} \in \mathfrak{N} \text{ QEA}_{\omega} \)), and \( \mathfrak{A} \subseteq \mathfrak{C} \), such that \( \mathfrak{N}^\text{Sc}_\mathfrak{a} \mathfrak{A} \notin \mathfrak{N}^\text{Sc}_\mathfrak{a} \mathfrak{C} \). See [35] theorems 5.1.4-5.1.5 for an entirely different example. Now topologizing \( \mathfrak{N}^\text{Sc}_\mathfrak{a} \mathfrak{C} \) we get the required.

Indeed we have \( \mathfrak{N}^\text{Sc}_\mathfrak{a} \mathfrak{A} \in \mathfrak{N}^\text{Sc}_\mathfrak{a} \mathfrak{C} \), hence there exists \( \mathfrak{B}' \in \mathfrak{C} \) such that \( \mathfrak{A} = \mathfrak{N} \mathfrak{B}' \). Let \( \mathfrak{B} \) be the subalgebra of \( \mathfrak{B}' \) generated by \( \mathfrak{A} \). Then \( \mathfrak{B} \) is locally finite, hence representable with base \( U \) say; give \( U \) the discrete topology, then \( \mathfrak{N} \text{ Sc}_\mathfrak{a} \mathfrak{A}^{\text{top}} = \mathfrak{N} \text{ Sc}_\mathfrak{a} \mathfrak{B}^{\text{top}} \). Now it is obvious that \( \mathfrak{A}^{\text{top}} \notin \mathfrak{N} \text{ Sc}_\mathfrak{a} \mathfrak{TCA}_{n+1} \), for if it were then its \( \text{Sc} \) reduct would be in \( \mathfrak{N} \text{ Sc}_\mathfrak{a} \mathfrak{TCA}_{n+1} \) and we know is not the case.

Now give a more sophisticated example. We prove that the strictness of the inclusion can be witnessed by a finite algebra:

### 4.3 Neat games

Let \( \delta \) be a map. Then \( \delta[i \rightarrow d] \) is defined as follows. \( \delta[i \rightarrow d](x) = \delta(x) \) if \( x \neq i \) and \( \delta[i \rightarrow d](i) = d \). We write \( \delta^j \) for \( \delta[i \rightarrow \delta_j] \).

We recall the definition of polyadic equality networks.

**Definition 4.15.** Let \( 2 < n < \omega \). Let \( \mathfrak{C} \) be an atomic \( \text{PEA}_n \). An atomic network over \( \mathfrak{C} \) is a map

\[ N : {}^n \Delta \rightarrow \text{At} \mathfrak{C}, \]

where \( \Delta \) is a non-empty set called a set of nodes, such that the following hold for each \( i, j < n \), \( \delta \in {}^n \Delta \) and \( d \in \Delta \):

- \( N(\delta_j^j) \leq d_{ij} \)
- \( N(\delta[i \rightarrow d]) \leq c_i N(\delta) \)
\[ N(\bar{x} \circ [i,j]) = s_{[i,j]}N(\bar{x}) \text{ for all } i, j < n. \]

**Theorem 4.16.** There is an \( \omega \) rounded game \( J \) such that if \( \exists \) can win \( J \) on a \( \text{PEA}_n \) atom structure \( \alpha \) with \( 2 < n < \omega \), then there exists a locally finite \( \mathcal{D} \in \text{QEA}_\omega \) such that \( \alpha \cong \text{At}N\text{Tr}_n\mathcal{D} \). Furthermore, \( \mathcal{D} \) can be chosen to be complete, in which case we have \( \mathcal{C} \alpha = \text{NT}_n\mathcal{D} \).

**Proof.** We define a stronger game \( J \). This game is played on \( \lambda \) neat hypernetworks. We need some preparing to do. For an atomic network and for \( x, y \in \text{nodes}(N) \), we set \( x \sim y \) if there exists \( \bar{z} \) such that \( N(x,y,\bar{z}) \leq d_{01} \). Define the equivalence relation \( \sim \) over the set of all finite sequences over \( \text{nodes}(N) \) is defined by \( \bar{x} \sim \bar{y} \iff |\bar{x}| = |\bar{y}| \text{ and } x_i \sim y_i \text{ for all } i < |\bar{x}|.(\text{It can be checked that this indeed an equivalence relation.})

A hypernetwork \( N = (N^a,N^h) \) over an atomic polyadic equality algebra \( \mathcal{C} \) consists of a network \( N^a \) together with a labelling function for hyperlabels \( N^h: <^\omega\text{nodes}(N) \to \Lambda \) (some arbitrary set of hyperlabels \( \Lambda \) ) such that for \( \bar{x}, \bar{y} \in <^\omega\text{nodes}(N) \)

\[
\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y}).
\]

If \( |\bar{x}| = k \in \mathbb{N} \) and \( N^h(\bar{x}) = \lambda \) then we say that \( \lambda \) is a \( k \)-ary hyperlabel. \( (\bar{x}) \) is referred to a a \( k \)-ary hyperedge, or simply a hyperedge. (Note that we have atomic hyperedges and hyperedges)

When there is no risk of ambiguity we may drop the superscripts \( a,h \). There are short hyperedges and long hyperedges (to be defined in a while). The short hyperedges are constantly labelled. The idea (that will be revealed during the proof), is that the atoms in the neat reduct are no smaller than the atoms in the dilation. (When \( \mathfrak{A} = \text{NT}_n\mathfrak{B} \), it is common to call \( \mathfrak{B} \) a dilation of \( \mathfrak{A} \).)

There is a one to one correspondence between networks and coloured graphs [26, second half of p. 76]. If \( \Gamma \) is a coloured graph, then by \( N_\Gamma \) we mean the corresponding network.

- A hyperedge \( \bar{x} \in <^\omega\text{nodes}(\Gamma) \) of length \( m \) is short, if there are \( y_0, \ldots, y_{n-1} \in \text{nodes}(N) \), such that

\[
N_\Gamma(x_i,y_0,\bar{z}) \leq d_{01} \text{ or } N_\Gamma(x_i,y_1,\bar{z}) \ldots
\]

or \( N(x_i,y_{n-1},\bar{z}) \leq d_{01} \)

for all \( i < |x| \), for some (equivalently for all) \( \bar{z} \).

Otherwise, it is called long.

- A hypergraph \( (\Gamma,l) \) is called \( \lambda \) neat if \( N_\Gamma(\bar{x}) = \lambda \) for all short hyperedges.
This game is similar to the games devised by Robin Hirsch in [20] definition 28], played on relation algebras. However, lifting it to cylindric algebras is not straightforward at all, for in this new context the moves involve hyperedges of length $n$ (the dimension), rather than edges. We have to be careful here for we have two types of hyperedges. Hyperedges having length $n$ that are labelled by atoms of the algebra, and there are other hyperedges labelled from another set (of labels). The latter hyperedges can get arbitrarily long.

In the $\omega$ rounded game $J$, $\forall$ has three moves. $J_k$ will denote the game $J$ truncated to $k$ rounds.

The first is the normal cylindrifier move. There is no polyadic move for the polyadic information is coded in the networks. The next two are amalgamation moves. But now the games are not played on coloured graphs, they are played on coloured hypergraphs, consisting of two parts, the graph part that can be viewed as an $L_{\omega_1,\omega}$ model for the rainbow signature, and the part dealing with hyperedges with a labelling function. The game is played on $\lambda$ neat hypernetworks, translated to $\lambda$ neat hypergraphs, where $\lambda$ is a label for the short hyperedges.

For networks $M,N$ and any set $S$, recall that we write $M \equiv^S N$ if $N|_S = M|_S$, and we write $M \equiv_S N$ if the symmetric difference

$$\Delta(\text{nodes}(M), \text{nodes}(N)) \subseteq S$$

and $M \equiv^{(\text{nodes}(M) \cup \text{nodes}(N)) \setminus S} N$. We write $M \equiv_k N$ for $M \equiv_{\{k\}} N$.

Let $N$ be a network and let $\theta$ be any function. The network $N\theta$ is a complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),$$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N))$.

Concerning $\forall$ s moves:

$\forall$ can play a cylindrifier move, like before but now played on $\lambda_0$ neat hypernetworks.

$\forall$ can play a transformation move by picking a previously played hypernetwork $N$ and a partial, finite surjection $\theta : \omega \to \text{nodes}(N)$, this move is denoted $(N,\theta)$. $\exists$ must respond with $N\theta$.

Finally, $\forall$ can play an amalgamation move by picking previously played hypernetworks $M,N$ such that $M \equiv^{\text{nodes}(M) \cap \text{nodes}(N)} N$ and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M,N)$. Here there is no restriction on the number of overlapping nodes, so in principal the game is harder for $\exists$ to win, if there was (Below we will encounter such restricted amalgamation moves).

To make a legal response, $\exists$ must play a $\lambda_0$-neat hypernetwork $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$.
Now assume that $\exists$ has a winning strategy in $J$ on the $\text{PEA}_\alpha$ atom structure $\alpha$.

Fix some $a \in \alpha$. Using $\exists$'s winning strategy in the game of neat hypernetworks, one defines a nested sequence $N_0 \subseteq N_1 \ldots$ of neat hypernetworks where $N_0$ is $\exists$'s response to the initial $\forall$-move $a$, such that

1. If $N_r$ is in the sequence and and $b \leq c_i N_r(f_0, \ldots, x, \ldots, f_{n-2})$, then there is $s \geq r$ and $d \in \text{nodes}(N_s)$ such that $N_s(f_0, f_{l-1}, d, f_{l+1}, \ldots, f_{n-2}) = b$.

2. If $N_r$ is in the sequence and $\theta$ is any partial isomorphism of $N_r$ then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $N_s$ extending $\theta$ such that $\text{rng}(\theta^+) \supseteq \text{nodes}(N_r)$.

Now let $N_a$ be the limit of this sequence, that is $N_a = \bigcup N_i$, the labelling of $n - 1$ tuples of nodes by atoms, and the hyperedges by hyperlabels done in the obvious way. This limit is well-defined since the hypernetworks are nested. We shall show that $N_a$ is the base of a weak set algebra having unit $V = \omega N_a^{(p)}$, for some fixed sequence $p \in \omega N_a$.

Let $\theta$ be any finite partial isomorphism of $N_a$ and let $X$ be any finite subset of $\text{nodes}(N_a)$. Since $\theta, X$ are finite, there is $i < \omega$ such that $\text{nodes}(N_i) \supseteq X \cup \text{dom}(\theta)$. There is a bijection $\theta^+ \supseteq \theta$ onto $\text{nodes}(N_i)$ and $\geq i$ such that $N_j \supseteq N_i, N_i \theta^+$. Then $\theta^+$ is a partial isomorphism of $N_j$ and $\text{rng}(\theta^+) = \text{nodes}(N_i) \supseteq X$. Hence, if $\theta$ is any finite partial isomorphism of $N_a$ and $X$ is any finite subset of $\text{nodes}(N_a)$ then

$$\exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_a \text{ where } \text{rng}(\theta^+) \supseteq X$$  \hspace{1cm} (9)

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of $\text{nodes}(N_a)$ within its domain. Let $L$ be the signature with one $n$-ary relation symbol ($b$) for each $b \in \alpha$, and one $k$-ary predicate symbol ($\lambda$) for each $k$-ary hyperlabel $\lambda$. We work in usual first order logic.

Here we have a sequence of variables of order type $\omega$, and two 'sorts' of formulas, the $n$ relation symbols using $n$ variables; roughly these that are built up out of the first $n$ variables will determine the atoms of neat reduct, the $k$-ary predicate symbols will determine the atoms of algebras of higher dimensions as $k$ gets larger; the atoms in the neat reduct will be no smaller than the atoms in the dilations.

This process will be interpreted in an infinite weak set algebra with base $N_a$, whose elements are those assignments satisfying such formulas.

For fixed $f_a \in \text{"nodes}(N_a)$, let $U_a = \{ f \in \text{"nodes}(N_a) : \{ i < \omega : g(i) \neq f_a(i) \} \text{ is finite} \}$. Notice that $U_a$ is weak unit (a set of sequences agreeing cofinitely with a fixed one.)
We can make \( U_a \) into the universe an \( L \) relativized structure \( N_a \); here relativized means that we are only taking those assignments agreeing cofinitely with \( f_a \), we are not taking the standard square model. However, satisfiability for \( L \) formulas at assignments \( f \in U_a \) is defined the usual Tarskian way, except that we use the modal notation, with restricted assignments on the left:

For \( b \in \alpha \), \( l_0, \ldots, l_{n-1}, i_0, \ldots, i_{k-1} < \omega \), \( k \)-ary hyperlabels \( \lambda \), and all \( L \)-formulas \( \phi, \psi \), let

\[
\begin{align*}
N_a, f \models b(x_{l_0}, \ldots, x_{l_{n-1}}) & \iff N_a(f(l_0), \ldots, f(l_{n-1})) = b \\
N_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) & \iff N_a(f(i_0), \ldots, f(i_{k-1})) = \lambda \\
N_a, f \models \neg \phi & \iff N_a, f \not\models \phi, \\
N_a, f \models (\phi \lor \psi) & \iff N_a, f \models \phi \text{ or } N_a, f \models \psi \\
N_a, f \models \exists x_i \phi & \iff N_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(N_a).
\end{align*}
\]

For any \( L \)-formula \( \phi \), write \( \phi_{N_a} \) for \( \{ f \in \text{nodes}(N_a) : N_a, f \models \phi \} \). Let \( D_a = \{ \phi_{N_a} : \phi \text{ is an } L \text{-formula} \} \) and

\[ D_a = (D_a, \cup, \sim, d_{ij}, c_i, s_{[i,j]})_{i,j<\omega} \]

where \( d_{ij} = (x_i = x_j)^{N_a}, c_i(\phi_{N_a}) = (\exists x_i \phi)^{N_a} \) and \( s_{[i,j]} \) swaps the variables \( x_i \) and \( x_j \). Observe that \( T_{N_a} = U_a, (\phi \lor \psi)^{N_a} = \phi^{N_a} \cup \psi^{N_a} \), etc then \( D_a \in \text{RQE}_\omega \). (Weak set algebras are representable). In fact \( D_a \in Lf_\omega \cap \mathcal{W}_\omega \), for each atom \( a \in \alpha \).

Let \( \mathcal{D} = \prod_{a \in \alpha} D_a \). (This is not necessarily locally finite). Then \( \mathcal{D} \in \text{RQE}_\omega \) will be shown to be is the desired generalized weak set algebra, that is the desired dilation. Note that unit of \( \mathcal{D} \) is the disjoint union of the weak spaces. So \( \mathfrak{M}_n \mathcal{D} \) is atomic and \( \alpha \cong \text{At}\mathfrak{M}_n \mathcal{D} \) — the isomorphism is \( b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{D_a} : a \in A) \). To make \( \mathcal{D} \) locally finite, one can assume that \( \mathcal{D} \) is generated by \( \mathfrak{M}_n \mathcal{D} \).

Now we can work in \( L_{\infty, \omega} \) so that \( \mathcal{D} \) is complete by changing the defining clause for infinitary disjunctions to

\[ N_a, f \models (\bigvee_{i \in I} \phi_i) \iff (\exists i \in I)(N_a, f \models \phi_i). \]

By working in \( L_{\infty, \omega} \), we assume that arbitrary joins hence meets exist, so \( \mathcal{D}_a \) is complete, hence so is \( \mathcal{D} \). But \( \mathfrak{C}_\alpha \subseteq \mathfrak{M}_n \mathcal{D} \) is dense and complete, so \( \mathfrak{C}_\alpha = \mathfrak{M}_n \mathcal{C} \).

\( \square \)

**Remark 4.17.** The game \( J \) is stronger than the usual atomic game, since it allows \( \forall \) more moves. But it is also an atomic game played on *atomic hypernetworks*; which carry more information than networks. But all the same,
because it an atomic game, it will turn out that it captures only the atom 
structures of neat reducts. This is substantially weaker than capturing the 
neat reduct itself.

However, one can define another stronger neat game, denoted by $H$, that captures a neat reduct.

Part of $H$ is an atomic game, as far as $\exists$'s response to $\forall$'s moves, playing $\lambda$ neat hypernetworks, is concerned. In the hypernetwork part of $H$, like in the weaker neat game $J$, $\forall$ can deliver only atoms for $n$ hyperedges and, like $J$, labels for $\lambda$ neat hyperedges.

But there is also 'another part' of the game. On the board there are networks with no consistency conditions and $\exists$ is allowed to label the $n$ hyperedges of these ($n$ is the dimension) by arbitrary elements of the algebra. These do not exist in the game $J$.

The main play of the stronger game $H(\mathcal{B})$, $\mathcal{B} \in \mathbb{CA}_n$ is a play of the game $J(\mathcal{B})$. Recall that $J(\mathcal{B})$ was played on $\lambda$ neat hypernetworks. The base of the main board at a certain point will be the the neat $\lambda$ hypernetwork, call its network part $X$ and we write $X(\bar{x})$ for the atom that labels the edge $\bar{x}$ on the main board. But now $\forall$ can make other moves too, which makes it harder for $\exists$ to win and so a winning strategy for $\exists$ in this new $\omega$ rounded game will give a stronger result.

An $n$ network is a finite complete graph with nodes including $0, \ldots, n-1$ with all edges labelled by arbitrary elements of $\mathcal{B}$. No consistency properties are assumed.

$\forall$ can play an arbitrary $n$ network $N$, $\exists$ must replace $N(0, \ldots, n-1)$, by some element $a \in \mathcal{B}$. The idea, is that the constraints represented by $N$ correspond to an element of the $\text{RCA}_\omega$ being constructed on $X$, generated by $\mathcal{B}$.

The final move is that $\forall$ can pick a previously played $n$ network $N$ and pick any tuple $\bar{x}$ on the main board whose atomic label is below $N(0, \ldots, n-1)$.

$\exists$ must respond by extending the main board from $X$ to $X'$ such that there is an embedding $\theta$ of $N$ into $X'$ such that $\theta(0) = x_0 \ldots, \theta(n-1) = x_{n-1}$ and for all $i_0, \ldots i_{n-1} \in N$, we have $X(\theta(i_0) \ldots, \theta(i_{n-1})) \leq N(i_0, \ldots, i_{n-1})$. This ensures that in the limit, the constraints in $N$ really define the element $a$.

If $\exists$ has a winning strategy in the atomic game $J_k$ ($J$ truncated to $k$ rounds) for all $k$ on an atomic algebra $\mathfrak{A}$ with countably many atoms, then this algebra will have an elementary equivalent algebra $\mathcal{B}$ such that $\text{At}\mathfrak{A} \in \text{At}\mathcal{N}_n\mathbb{CA}_\omega$. It can be shown that $\exists$ has a winning strategy in $J_k$ for any finite $k$ on $\mathfrak{A} \in \mathbb{CA}_{n-1,n}$.

In [20] similar games are devised for relation algebras and Robin Hirsch deduces from the fact that $\exists$ can win the $k$ rounded game (analogous to $J$) on a certain relation algebra $\mathfrak{A}$, then $\mathfrak{A}$ is elementary equivalent to $\mathcal{B} \in \text{RaCA}_\omega$. This is a mistake. All we can deduce from $\exists$'s winning strategy is that $\mathcal{B} \in \mathbb{CA}_{n-1,n}$. 

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$S_c\mathcal{R}a\mathcal{CA}_\omega$.

But if $\exists$ has a winning strategy in $H(\mathcal{B})$, for $\mathcal{B} \in \mathcal{CA}_n$, then the extra move ensures that that every $n$ dimensional element generated by $\mathcal{B}$ in the $\mathcal{D} \in \mathcal{RCA}_\omega$ constructed in the play is an element of $\mathcal{B}$, so that $\mathcal{B}$ exhausts all $n$ dimensional elements of $\mathcal{D}$, hence $\mathcal{B} \cong \mathcal{N}_n\mathcal{D}$, and so $\mathcal{B} \in \mathcal{N}_n\mathcal{CA}_\omega$.

Hence if $\exists$ has a winning strategy in $H_k(\mathcal{A})$, $\mathcal{A} \in \mathcal{CA}_n$ atomic with countably many atoms, for all finite $k$, then $\mathcal{A}$ will have an elementary equivalent algebra $\mathcal{B}$, such that $\mathcal{B} \in \mathcal{N}_n\mathcal{CA}_\omega$.

This is much stronger, for like the case of representable algebras, and unlike the case of completely representable algebra even in the relativized sense, we may well have algebras $\mathcal{A}, \mathcal{B} \in \mathcal{CA}_n (n > 1)$, and even more in $\mathcal{RCA}_n$ such that $\text{At} \mathcal{A} = \text{At} \mathcal{B}$, $\mathcal{A} \in \mathcal{N}_n\mathcal{CA}_\omega$ but $\mathcal{B} \notin \mathcal{N}_n\mathcal{CA}_{n+1}$, a fortiori $\mathcal{B} \notin \mathcal{N}_n\mathcal{CA}_\omega$. Unfortunately, $\exists$ does not have a winning strategy on $\mathcal{CA}_{n-1,N}$ for every finite rounded game of $H$. So an open question is whether there is an atomic algebra $\mathcal{A} \in \mathcal{CA}_n$ with countably many atoms such that $\exists$ can win $H_k$ for every finite $k$, while $\exists$ can win $F^m$ for some $m > n$.

If there is such an algebra, then any $K$ between $\mathcal{N}_n\mathcal{CA}_\omega$ and $S_{c,\mathcal{N}_n}\mathcal{CA}_m$ is not elementary. This is stronger than the result proved in theorem 4.13. Indeed from our present proof of the statement in 4.13 we cannot replace $S_{c,\mathcal{N}_n}\mathcal{CA}_\omega$ by the strictly smaller $\mathcal{N}_n\mathcal{CA}_\omega$.

Example 4.14 alerts us to the fact that it is possible to separate classes that are ‘controlled by atom structure of their members’ but there those that cannot like the class of representable algebras of finite dimension $n > 2$, and more generally, the varieties $S\mathcal{N}_n\mathcal{TCA}_{n+k}$ for $k \geq 3$ theorem 4.16. So we can distinguish between two ‘types’ of classes of algebras, those that are gripped by their atom structures, call such a class $K$, meaning that if $\mathcal{A} \in K \subseteq TCA_\alpha$ and $\mathcal{B} \in \mathcal{TCA}_\alpha$ such that $\text{At} \mathcal{A} = \text{At} \mathcal{B}$, iff $\mathcal{B} \in K$.

An example here is the classes $S_{c,\mathcal{N}_n}\mathcal{CA}_m$ for $1 < m < n$; in particular this is true of the class of completely representable algebras of dimension $n$. This notion will be elaborated upon below.

Another example is the class $\mathcal{N}_\alpha\mathcal{CA}_\beta$ for any pair of ordinals $1 < \alpha < \beta$.

If $\mathcal{A}, \mathcal{B} \in K \subseteq \mathcal{TCA}_n$ then of course $\text{At} \mathcal{A}$ and $\text{At} \mathcal{B}$ is in $\text{At} K$. The converse as we have already seen shows that the converse is false. There are classes of algebras that are not gripped by their atom structures.

Example 4.14 also told us that $\mathcal{N}_\alpha\mathcal{TCA}_{\alpha+\omega}$ is properly contained in $S_{c,\mathcal{N}_\alpha}\mathcal{TCA}_{\alpha+\omega}$ for any ordinal $\alpha > 1$.

The next example shows that when $\alpha = n > 2$ is finite, then the strictness of such inclusion can be actually witnessed by a finite algebra.

**Example 4.18.** Take the finite cylindric algebra $\mathfrak{A}$ consisting of three dimensional matrices (as defined by Monk) over any integral non-permutational relation algebra $\mathfrak{R}$, discretely topologized. Such relation algebras exist \[20\].
The algebra $\mathcal{A}$ is finite, hence completely representable, hence $\mathcal{A} \in S_c \mathfrak{N}_3 \text{TCA}_\omega$.

Suppose for contradiction that $\mathcal{A} \in \mathfrak{N}_3 \text{TCA}_\omega$, so that $\text{At}\mathcal{A} \in \text{At}\mathfrak{N}_n \mathcal{A}_\omega$. Then we claim that $\mathcal{A}$ has a $3$-homogeneous complete representation, which is impossible, because $R$ does not have a homogeneous representation.

It can be shown, using arguments similar to [20, theorem 33], that $\exists$ has a winning strategy in an $\omega$ rounded game $K$ but played on atomic networks where $\forall$ is offered a cylindrifier move together with an amalgamation move except that in amalgamation moves on networks there is an additional restriction. The networks he chooses can overlap only on at most 3 nodes. $\exists$ uses her winning strategy to define a sequence of networks $N_0 \subseteq \ldots N_r$ such that this sequence respects the cylindrifier move in the sense that if $N_r(\bar{x}) \leq c_i a$ for $\bar{x} \in \text{nodes} (N_r)$, then there exists $N_s \supseteq N_r$ and a node $k \in \omega \sim N_r$ such that $N_s(\bar{y}) = a$; and also respects the partial isomorphism move, in the sense that if if $\bar{x}, \bar{y} \in \text{nodes} (N_r)$ such that $N_r(\bar{x}) = N_r(\bar{y})$, then there is a finite surjective map extending $\{ (x_i, y_i) : i < n \}$ mapping onto $\text{nodes} (N)$ such that $\text{dom} (\theta) \cap \text{nodes} (N_r) = \bar{y}$, and we seek an extension $N_s \supseteq N_r, N_r \theta$ (some $s \geq r$).

Then if $\tau$ is a partial isomorphism of $N_a$ and $X$ is any finite subset of $\text{nodes} (N_a)$ then there is a partial isomorphism $\theta \supseteq \tau, \text{rng}(\theta) \supseteq X$.

Define the representation $\mathcal{N}$ of $\mathcal{A}$ with domain $\bigcup_{a \in A} \text{nodes} (N_a)$, by
\[
S^\mathcal{N} = \{ \bar{x} : \exists a \in A, \exists s \in S, N_a(\bar{x}) = s \},
\]
for any subset $S$ of $\text{At}\mathcal{A}$. Then this representation of $\mathcal{A}$, is obviously complete, and by the definition of the game is $n$ homogeneous.

For higher dimension one uses the result in [29] by lifting the used relation algebra to a representable $\text{TCA}_n$ for any $n > 2$ preserving $n$-homogeneity.

**Theorem 4.19.** Let $n$ be finite $> 2$. Then the following hold:

(1) $\text{RTeCA}_n$ is not finitely axiomatizable by any universal formulas containing only finitely many variables.

(2) It is undecidable to tell whether a finite algebra is representable. In particular, the equational theory of $\text{RTeCA}_n$ is undecidable, the set of models having infinite bases is not recursively enumerable, and $\text{RTeCA}_n$ is not finitely axiomatizable in $m$th order logic for any finite $m$.

**Proof.** (1) Assume that $\mathcal{A} \in \text{RTeCA}_n$. Then $\mathcal{R} \mathcal{A}_n \mathcal{A} \subseteq \prod_{t \in T} \varphi^n (U_t) \in \mathcal{G}_n$. Conversely, $\mathcal{C}_n \subseteq \mathcal{R} \mathcal{A}_n \text{RTeCA}_n$ (by fixing a moment in time). Hence $\text{RCA}_n = \text{SP} \mathcal{R} \mathcal{A}_n \text{RTeCA}_n$, so $\text{RTeCA}_n$ is a finite expansion of $\text{RCA}_n$ by only two unary modalities, namely $P$ (past) and $F$ (future) hence the required follows from [1] Theorem 5.
(2) We show that from every simple atomic $\mathfrak{A} \in \mathbb{CA}_n$ we can construct recursively a $\mathbb{TeCA}_n$, such that the former is representable if and only if the latter is. This suffices by the main result in [29]. Expand $\mathfrak{A}$ recursively and temporally to a static $\mathbb{TeCA}_n$ as done before. Take $G = H = Id$ and $T = \{t\}$ with $\leq \emptyset$; then $<$ is obviously irreflexive and vacuously transitive and $G$ and $H$ distribute over the Boolean meet.

\[\square\]

### 4.4 Neat embeddings in connection to complete and strong representations

Here we approach the notion of complete representations and strong representations using neat embeddings. But before embarking on such connections we make an observation.

The algebras $\mathcal{E}(m, n)$ for $2 < m < n$, to be defined next witness the strictness of this inclusion are based on relations algebra similar to the algebras in [26] but such algebras are infinite since there are an infinite set of hypernetworks with hyperlabels from $\omega$.

Now the algebras consists of the set of the set of $m$ basic matrices ($m$ is the dimension) on this finite relation algebras hence, as stated, they are finite.

**Example 4.20.** Let $3 \leq m < n < \omega$. Then there are finite algebras $\mathcal{E}(m, n) \in \text{PEA}_m$ such that

I. $\mathcal{E}(m, n) \in \mathcal{N}_m \text{PEA}_n$,

II. $\mathcal{N}_n \mathcal{E}(m, n) \notin \mathcal{S} \mathcal{N}_m \mathcal{S}_{n+1}$.

In particular, for any class $K$ between $\mathcal{S}c$ and $\text{PEA}$, for any finite $m > 2$ and any finite $k \geq 1$, we have $\mathcal{S}_c \mathcal{N}_m K_{m+k+1} \subset \mathcal{S}_c \mathcal{N}_n K_{m+k}$ and the strictness of the inclusion is witnessed on finite algebras. We use a simpler version of algebras construction in [27], and they also have affinity to algebras constructed in [22, section 15.2]. So we will be sketchy highlighting the idea of proof.

Define a function $\kappa : \omega \times \omega \rightarrow \omega$ by $\kappa(x, 0) = 0$ (all $x < \omega$) and $\kappa(x, y+1) = 1 + x \times \kappa(x, y)$ (all $x, y < \omega$). For $n, r < \omega$ let

$$\psi(n) = \kappa((n - 1) \times 5, (n - 1) \times 5) + 1.$$  

All of this is simply to ensure that $\psi(n)$ is sufficiently big compared to $n$ for the proof of non-embeddability to work.

For any $n < \omega$, let

$$\text{Bin}(n) = \{Id\} \cup \{a^k(i, j) : i < n - 1, j \in 5, k < \psi(n)\}$$

where $Id, a^k(i, j)$ are distinct objects indexed by $k, i, j$. 

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Let $F(m, n)$ be the set of all functions $f : m \times m \to \text{Bin}(n)$ such that $f$ is symmetric ($f(x, y) = f(y, x)$ for all $x, y < m$) and for all $x, y, z < m$ we have $f(x, x) = \text{Id}$, $f(x, y) = f(y, x)$, and $(f(x, y), f(y, z), f(x, z)) \notin \text{Forb}$, where $\text{Forb}$ (the forbidden triples) is the following set of triples

$$
\{(\text{Id}, b, c) : b \neq c \in \text{Bin}(n)\} \cup \{(a^k(i, j), a^{k'}(i, j'), a^{k''}(i, j')) : k, k', k'' < \psi(n, r), i < n - 1, j' \leq j < 5\}.
$$

Here $\text{Bin}(n)$ is an atom structure of a finite relation and $\text{Forb}$ specifies its operations by specifying forbidden triples.

Now any such $f \in F(m, n)$ is a basic matrix on this atom structure in the sense of [26, definition 12.35], a term due to Maddux, and the whole lot of them will be a cylindric symmetric basis (closed under substitutions), a term also due to Maddux, constituting the atom structure of algebras we want. Now accessibility relations corresponding to substitutions, cylindrifiers are defined, as expected on matrices, as follows. For any $f, g \in F(m, n)$ and $x, y < m$ we write $f \equiv_{xy} g$ if for all $w, z \in m \setminus \{x, y\}$ we have $f(w, z) = g(w, z)$. We may write $f \equiv_x g$ instead of $f \equiv_{xx} g$. For $\tau : m \to m$ we write $(f \tau)$ for the function defined by

$$(f \tau)(x, y) = f(\tau(x), \tau(y)).$$

Clearly $(f \tau) \in F(m, n)$. Accordingly, the universe of $\mathcal{C}(m, n)$ is the power set of $F(m, n)$ and the operators (lifting from the atom structure) are

- the Boolean operators $+, -$ are union and set complement,
- the diagonal $d_{xy} = \{f \in F(m, n) : f(x, y) = \text{Id}\}$,
- the cylindrifier $c_x(X) = \{f \in F(m, n) : \exists g \in X \ f \equiv_x g\}$ and
- the polyadic $s_\tau(X) = \{f \in F(m, n) : f \tau \in X\}$,

for $x, y < m$, $X \subseteq F(m, n)$ and $\tau : m \to m$.

It is straightforward to see that $3 \leq m$, $2 \leq n$ the algebra $\mathcal{C}(m, n)$ satisfies all of the axioms defining $\text{PEA}_m$ except, perhaps, the commutativity of cylindrifiers $c_x c_y(X) = c_y c_x(X)$, which it satisfies because $F(m, n)$ is a symmetric cylindric basis, so that overlapping matrices amalgamate. Furthermore, if $3 \leq m \leq m'$ then $\mathcal{C}(m, n) \cong \mathfrak{M}_m \mathcal{C}(m', n)$ via

$$X \mapsto \{f \in F(m', n) : f |_{m \times m} \in X\}.$$

Now we prove [1.20][11], which is the heart and soul of the proof, and it is quite similar to its $\text{CA}$ analogue 4.69-475 in [26], and the proof in [27]. In the latter two proofs there is a third parameter $r$ which we fix to be 5 here so that
our proof is simpler. We will refer to the proof in [26] when the proofs overlap, or are very similar. Assume for contradiction that \( \mathcal{N}_\mathcal{N}_\mathcal{C}(m, n) \subseteq \mathcal{N}_\mathcal{m}_\mathcal{C} \) for some \( \mathcal{C} \in \mathcal{S}_\mathcal{n}_\mathcal{+}_1 \), some finite \( m, n \). Then it can be shown inductively that there must be a large set \( S \) of distinct elements of \( \mathcal{C} \), satisfying certain inductive assumptions, which we outline next. For each \( s \in S \) and \( i, j < n + 2 \) there is an element \( \alpha(s, i, j) \in \text{Bin}(n) \) obtained from \( s \) by cylindrifying all dimensions in \( (n + 1) \setminus \{i, j\} \), then using substitutions to replace \( i, j \) by 0, 1. Then one shows that \( (\alpha(s, i, j), \alpha(s, j, k), \alpha(s, i, k)) \notin \text{Forb} \).

The induction hypothesis say, most importantly, that \( c_n(s) \) is constant, for \( s \in S \), and for \( l < n \) there are fixed \( i < n - 1, j < 5 \) such that for all \( s \in S \) we have \( \alpha(s, l, n) \leq a(i, j) \). This defines, like in the proof of theorem 15.8 in [26] p.471, two functions \( I : n \to (n - 1) \), \( J : n \to 5 \) such that \( \alpha(s, l, n) \leq a(I(l), J(l)) \) for all \( s \in S \). The rank \( \text{rk}(I, J) \) of \((I, J)\) (as defined in definition 15.9 in [26]) is the sum (over \( i < n - 1 \)) of the maximum \( j \) with \( I(l) = i \), \( J(l) = j \) (some \( l < n \)) or \(-1\) if there is no such \( j \).

Next it is proved that there is a set \( S' \) with index functions \((I', J')\), still relatively large (large in terms of the number of times we need to repeat the induction step) where the same induction hypotheses hold but where \( \text{rk}(I', J') > \text{rk}(I, J) \). (See [26], where for \( t < n \times 5, S' \) was denoted by \( S_t \) and proof of property (6) in the induction hypothesis on p.474 of [26].)

By repeating this enough times (more than \( n \times 5 \)) we obtain a non-empty set \( T \) with index functions of rank strictly greater than \((n - 1) \times 4\), an impossibility. (See [26], where for \( t < n \times 5, S' \) was denoted by \( S_t \).)

We sketch the induction step. Since \( I \) cannot be injective there must be distinct \( l_1, l_2 < n \) such that \( I(l_1) = I(l_2) \) and \( J(l_1) \leq J(l_2) \). We may use \( l_1 \) as a "spare dimension" (changing the index functions on \( l \) will not reduce the rank). Since \( c_n(s) \) is constant, we may fix \( s_0 \in S \) and choose a new element \( s' \) below \( c_l s_0 \cdot s_i c_l s \), with certain properties. Let \( S^* = \{s' : s \in S \setminus \{s_0\}\} \). We wish to re-establish the induction hypotheses for \( S^* \), and many of these are simple to check. Although suitable functions \( I', J' \) may not exist on the whole of \( S \), but \( S \) remains large enough to enable selecting a subset \( S' \) of \( S^* \), still large in terms of the number of remaining times the induction step must be applied. The required functions \( I', J' \) now exist (for all but one value of \( l < n \) the values \( I'(l), J'(l) \) are determined by \( I, J \), for one value of \( l \) there are at most \( 5 \times (n - 1) \) possible values, hence on a large subset the choices agree).

Next it can be shown that \( J'(l) \geq J(l) \) for all \( l < n \). Since

\[ (\alpha(s, i, j), \alpha(s, j, k), \alpha(s, i, k)) \notin \text{Forb} \]

and by the definition of \( \text{Forb} \) either \( \text{rng}(I') \) properly extends \( \text{rng}(I) \) or there is \( l < n \) such that \( J'(l) > J(l) \), hence \( \text{rk}(I', J') > \text{rk}(I, J) \), and we are done.

Lifting a well known notion from atom structures [26], we set:
Definition 4.21. An atomic algebra \( \mathfrak{A} \in \text{TCA}_n \) is strongly representable if it is completely additive its Dedekind-MacNeille completion, namely \( \text{EmAt}\mathfrak{A} \) is representable.

It is not hard to see that a completely representable algebra is strongly representable. However, the converse is false, as we show in our next theorem. Also not every representable atomic algebra is strongly representable. The algebra \( \text{EmAt} \) constructed on the rainbow atom structure \( \mathfrak{At} \) in theorem 4.6 is such.

Let \( \text{CRTA}_n \) denote the class of completely representable \( \mathbf{K} \) algebras of dimension \( n \). We have proved above that \( S_c\text{Mt}_n\text{TCA}_\omega \) and \( \text{CRTCA}_n \) coincide on countable atomic algebras. The result can slightly generalized to allow algebras with countably many atoms, that may not be countable.

In our next theorem we show that this does not generalize any further as far as cardinalities are concerned.

Theorem 4.22. For any \( n > 2 \) we have, \( \text{Mt}_n\mathbf{K}_\omega \not\subseteq \text{CRTCA}_n \), while for \( n > 1 \), \( \text{CRTCA}_n \not\subseteq \text{UpUrMt}_n\text{TCA}_\omega \). In particular, there are completely representable, hence strongly representable algebras that are not in \( \text{Mt}_n\text{TCA}_\omega \). Furthermore, such algebras can be countable. However, for any \( n \in \omega \), if \( \mathfrak{A} \in \text{Mt}_n\text{TCA}_\omega \) is atomic, then \( \text{Rd}_{\text{ca}}\mathfrak{A} \) is strongly representable.

Proof. (1) We first show that \( \text{Mt}_n\text{TCA}_\omega \not\subseteq \text{CRTCA}_n \). This cannot be witnessed on countable algebras, so our constructed neat reduct that is not completely representable, must be uncountable.

In \([20]\) a sketch of constructing an uncountable relation algebra \( R \in \text{RaCA}_\omega \) (having an \( \omega \) dimensional cylindric basis) with no complete representation is given. It has a precursor in \([33]\) which is the special case of this example when \( \kappa = \omega \) but the idea in all three proofs are very similar using a variant of the rainbow relation algebra \( R_{\kappa,\omega} \) where \( \kappa \) is an uncountable cardinal \( \geq 2^{\aleph_0} \). We do not know whether it works for the least uncountable cardinal, namely, \( \aleph_1 \)

Assume that \( R = \text{Ra}\mathfrak{B} \) and \( \mathfrak{B} \in \text{PEA}_\omega \), then \( \text{Rd}_{\kappa}\text{Mt}_n\mathfrak{B} \) is as required, for a complete representation of it, induces easily a complete representation of \( \text{RaCA}_\omega \). The latter example shows that \( \text{UpUrMt}_n\mathbf{K}_\omega \not\subseteq \text{CRA}_n \).

Now give the details of the construction in \([20\text{ remark 31}]\). We prove more, namely, there exists an uncountable neat reduct, that is, an algebra in \( \text{Mt}_n\text{QEA}_\omega \), such that its \( \text{Df} \) reduct, obtained by discarding all operations except for cylindrifiers, is not completely representable.

Using the terminology of rainbow constructions, we allow the greens to be of cardinality \( 2^\kappa \) for any infinite cardinal \( \kappa \), and the reds to be of cardinality \( \kappa \). Here a winning strategy for \( \forall \) witnesses that the algebra
has no complete representation. But this is not enough because we want our algebra to be in \( \text{RaCA}_\omega \); we will show that it will be.

We specify the atoms and forbidden triples. The atoms are \( \text{ld}, \ g_0^i : i < 2^\kappa \) and \( r_j : 1 \leq j < \kappa \), all symmetric. The forbidden triples of atoms are all permutations of \( (\text{ld}, x, y) \) for \( x \neq y \), \( (r_j, r_j, r_j) \) for \( 1 \leq j < \kappa \) and \( (g_0^i, g_0^j, g_0^{i*}) \) for \( i, i', i^* < 2^\kappa \). In other words, we forbid all the monochromatic triangles.

Write \( g_0 \) for \( \{g_0^i : i < 2^\kappa \} \) and \( r_+ \) for \( \{r_j : 1 \leq j < \kappa \} \). Call this atom structure \( \alpha \).

Let \( \mathfrak{A} \) be the term algebra on this atom structure; the subalgebra of \( \mathfrak{C}\mathfrak{m}\alpha \) generated by the atoms. \( \mathfrak{A} \) is a dense subalgebra of the complex algebra \( \mathfrak{C}\mathfrak{m}\alpha \). We claim that \( \mathfrak{A} \), as a relation algebra, has no complete representation.

Indeed, suppose \( \mathfrak{A} \) has a complete representation \( M \). Let \( x, y \) be points in the representation with \( M \models r_1(x, y) \). For each \( i < 2^\kappa \), there is a point \( z_i \in M \) such that \( M \models g_0^i(x, z_i) \land r_1(z_i, y) \).

Let \( Z = \{z_i : i < 2^\kappa \} \). Within \( Z \) there can be no edges labeled by \( r_0 \) so each edge is labelled by one of the \( \kappa \) atoms in \( r_+ \). The Erdos-Rado theorem forces the existence of three points \( z^1, z^2, z^3 \in Z \) such that \( M \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z^1) \), for some single \( j < \kappa \). This contradicts the definition of composition in \( \mathfrak{A} \) (since we avoided monochromatic triangles).

Let \( S \) be the set of all atomic \( \mathfrak{A} \)-networks \( N \) with nodes \( \omega \) such that \( \{r_i : 1 \leq i < \kappa : r_i \text{ is the label of an edge in } N \} \) is finite. Then it is straightforward to show \( S \) is an amalgamation class, that is for all \( M, N \in S \) if \( M \equiv_{ij} N \) then there is \( L \in S \) with \( M \equiv_{i} L \equiv_{j} N \). Hence the complex cylindric algebra \( \mathfrak{C}a(S) \in \text{QEA}_\omega \).

Now let \( X \) be the set of finite \( \mathfrak{A} \)-networks \( N \) with nodes \( \subseteq \omega \) such that

1. each edge of \( N \) is either (a) an atom of \( \mathfrak{A} \) or (b) a cofinite subset of \( r_+ = \{r_j : 1 \leq j < \kappa \} \) or (c) a cofinite subset of \( g_0 = \{g_0^i : i < 2^\kappa \} \) and

2. \( N \) is ‘triangle-closed’, i.e. for all \( l, m, n \in \text{nodes}(N) \) we have \( N(l, n) \leq N(l, m) \land N(m, n) \). That means if an edge \( (l, m) \) is labeled by \( \text{ld} \) then \( N(l, n) = N(mn) \) and if \( N(l, m) \), \( N(m, n) \leq g_0 \) then \( N(l, n) \cdot g_0 = 0 \) and if \( N(l, m) = N(m, n) = r_j \) (some \( 1 \leq j < \omega \)) then \( N(l, n) \cdot r_j = 0 \).

For \( N \in X \) let \( N' \in \mathfrak{C}a(S) \) be defined by

\[
\{L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N)\}
\]
For \( i \in \omega \), let \( N \upharpoonright_{-i} \) be the subgraph of \( N \) obtained by deleting the node \( i \). Then if \( N \in X \), \( i < \omega \) then \( c_iN' = (N \upharpoonright_{-i})' \). The inclusion \( c_iN' \subseteq (N \upharpoonright_{-i})' \) is clear.

Conversely, let \( L \in (N \upharpoonright_{-i})' \). We seek \( M \equiv_i L \) with \( M \in N' \). This will prove that \( L \in c_iN' \), as required. Since \( L \in S \) the set \( X = \{ r_i \notin L \} \) is infinite. Let \( X \) be the disjoint union of two infinite sets \( Y \cup Y' \), say. To define the \( \omega \)-network \( M \) we must define the labels of all edges involving the node \( i \) (other labels are given by \( M \equiv_i L \)). We define these labels by enumerating the edges and labeling them one at a time. So let \( j \neq i < \omega \).

Suppose \( j \in \text{nodes}(N) \). We must choose \( M(i, j) \leq N(i, j) \). If \( N(i, j) \) is an atom then of course \( M(i, j) = N(i, j) \). Since \( N \) is finite, this defines only finitely many labels of \( M \). If \( N(i, j) \) is a cofinite subset of \( a_0 \) then we let \( M(i, j) \) be an arbitrary atom in \( N(i, j) \). And if \( N(i, j) \) is a cofinite subset of \( r_k \), then let \( M(i, j) \) be an element of \( N(i, j) \cap Y \) which has not been used as the label of any edge of \( M \) which has already been chosen (possible, since at each stage only finitely many have been chosen so far).

If \( j \notin \text{nodes}(N) \) then we can let \( M(i, j) = r_k \in Y \) some \( 1 \leq k < \kappa \) such that no edge of \( M \) has already been labeled by \( r_k \). It is not hard to check that each triangle of \( M \) is consistent (we have avoided all monochromatic triangles) and clearly \( M \in N' \) and \( M \equiv_i L \). The labeling avoided all but finitely many elements of \( Y' \), so \( M \in S \). So \( (N \upharpoonright_{-i})' \subseteq c_iN' \).

Now let \( X' = \{ N' : N \in X \} \subseteq \mathcal{C}a(S) \). Then the subalgebra of \( \mathcal{C}a(S) \) generated by \( X' \) is obtained from \( X' \) by closing under finite unions. Clearly all these finite unions are generated by \( X' \). We must show that the set of finite unions of \( X' \) is closed under all cylindric operations. Closure under unions is given. For \( N' \in X \) we have \( -N' = \bigcup_{m,n \in \text{nodes}(N)} N'_mn \) where \( N'_mn \in \mathcal{C}a(S) \). The diagonal \( d_{ij} \in \mathcal{C}a(S) \) is equal to \( N' \), where \( N \) is a network with nodes \( \{ i, j \} \) and labeling \( N(i, j) = 1' \).

Closure under cylindrification is given. Let \( \mathcal{C} \) be the subalgebra of \( \mathcal{C}a(S) \) generated by \( X' \). Then \( \mathfrak{A} = \mathcal{R}a(\mathcal{C}) \). Each element of \( \mathfrak{A} \) is a union of a finite number of atoms and possibly a co-finite subset of \( a_0 \) and possibly a co-finite subset of \( a_+ \). Clearly \( \mathfrak{A} \subseteq \mathcal{R}a(\mathcal{C}) \). Conversely, each element \( z \in \mathcal{R}a(\mathcal{C}) \) is a finite union \( \bigcup_{N \in F} N' \), for some finite subset \( F \) of \( X \), satisfying \( c_i z = z \), for \( i > 1 \). Let \( i_0, \ldots, i_k \) be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in \( F \). Then, \( c_{i_0} \ldots c_{i_k} z = \bigcup_{N \in F} c_{i_0} \ldots c_{i_k} N' = \bigcup_{N \in F} (N \upharpoonright_{\{0, 1\}})' \in \mathfrak{A} \). So \( \mathfrak{A} \subseteq \mathfrak{A} \). \( \mathfrak{A} \) is relation algebra reduct of \( \mathcal{C} \in \mathcal{C}A_\omega \) but has no complete representation; so \( \mathcal{C}^\text{top} \in \mathcal{T}CA_\omega \).

Let \( n > 2 \). Let \( \mathfrak{B} = \mathfrak{N}_n \mathfrak{A}^\text{top} \). Then \( \mathfrak{B} \in \mathfrak{N}_n \mathcal{T}CA_\omega \), is atomic, but has
no complete representation; in fact because it is binary generated its \( \text{Df} \) reduct is not completely representable.

(2) Now we show that \( \text{CRK}_n \nsubseteq \text{UpUr}\mathcal{N}_n\mathcal{C}_\omega \). The \( \text{K} \) reduct of the algebra \( \mathcal{A} \) in example 4.14 is such; it is completely representable, hence strongly representable, but it is not in \( \text{UpUr}\mathcal{N}_n\mathcal{K}_{n+1} \), a fortiori it is not in \( \text{UpUr}\mathcal{N}_n\mathcal{K}_\omega \). If the field \( \mathfrak{F} \) is countable, then \( \mathcal{A} \) is countable.

(3) Now for the last part, namely, that neat reducts are strongly representable. Let \( \mathcal{A} \in \mathcal{N}_n\mathcal{K}_\omega \) be atomic. Then \( \exists \) has a winning strategy in \( F^\omega \) as in theorems 4.6 and 4.13, hence it has a winning strategy in \( G \) (the usual \( \omega \) rounded atomic game) and so it has a winning strategy for \( G_k \) for all finite \( k \) \((G\text{ truncated to } k \text{ rounds.})\) Thus \( \mathcal{A} \models \sigma_k \) which is the \( k \) th Lyndon sentence coding that \( \exists \) has a winning strategy in \( G_k \), called the \( k \) the Lyndon condition. Since \( \mathcal{A} \) satisfies the \( k \)th Lyndon conditions for each \( k \), then any algebra on its atom structure is representable, so that \( \text{CmAt}\mathcal{A} \) is representable, hence it is strongly representable, and we are done.

\[ \square \]

Recall that \( S_c \) is the operation of forming complete subalgebras. We write \( \mathcal{A} \subseteq_c \mathcal{B} \) if \( \mathcal{A} \) is a complete subalgebra of \( \mathcal{B} \).

**Theorem 4.23.** For \( n > 1 \) the class \( \mathcal{N}_n\text{TCA}_\omega \) is not closed under \( S_c \). For every \( n \in \omega \) the class \( S_c\mathcal{N}_n\text{TCA}_\omega \) is closed under \( S_c \) but for \( n > 2 \) it is not elementary and is not closed under forming subalgebras, hence is not pseudo-universal. For \( n > 2 \), the class \( \text{UpUr}S_c\mathcal{N}_n\mathcal{K}_\omega \) is not finitely axiomatizable.

**Proof.** For \( n > 1 \) \( \mathcal{N}_n\text{TCA}_\omega \) is not closed under \( S_c \) follows, from example SL since the the algebra \( \mathcal{A} \) constructed therein is countable and completely representable but is not in \( \mathcal{N}_n\text{TCA}_{n+1} \), a fortiori it is not in \( \mathcal{N}_n\text{TCA}_\omega \). \( S_c\mathcal{N}_n\text{TCA}_\omega \) is not elementary follows from theorem 4.13

We now show that \( S_c\mathcal{N}_n\text{TCA}_\omega \) is not closed under forming subalgebras, hence it is not pseudo-universal. That it is closed under \( S_c \) follows directly from the definition. Consider rainbow algebra used in theorem 4.13, call it \( \mathcal{A} \). Because \( \mathcal{A} \) has countably many atoms, \( \text{TmA} \subseteq \mathcal{A} \subseteq \text{CmAt}\mathcal{A} \), and all three are completely representable or all three not completely representable sharing the same atom structure \( \text{At}\mathcal{A} \), we can assume without loss that \( \mathcal{A} \) is countable. Now \( \mathcal{A} \) is not completely representable hence it is not \( S_c\mathcal{N}_n\text{TCA}_\omega \).

On the other hand, \( \mathcal{A} \) is strongly representable (from the argument used in the second item of theorem 4.22 since \( \exists \) can win the finite rounded atomic game with \( k \) rounds for every \( k \), hence it satisfies the Lyndon conditions), so its canonical extension is representable, indeed completely representable. On the other hand, the canonical extension of \( \mathcal{A} \) is in \( S_c\mathcal{N}_n\text{TCA}_\omega \) and \( \mathcal{A} \)
embeds into its canonical extension, hence \( S_{t} \mathcal{Nr}_{n} \text{TCA} \omega \) is not closed under forming subalgebras. The first of these statements follow from the fact that if \( \mathcal{D} \subseteq \mathcal{Nr}_{n} \mathcal{B} \), then \( \mathcal{D}^{+} \subseteq c \mathcal{Nr}_{n} \mathcal{B}^{+} \).

To prove non-finite axiomatizability of the elementary closure of \( S_{t} \mathcal{Nr}_{n} \text{TCA} \omega \) we give two entirely different proofs.

(i) First we use the flexible construction in [12].

Now let \( l \in \omega, l \geq 2 \), and let \( \mu \) be a non-zero cardinal. Let \( I \) be a finite set, \( |I| \geq 3l \). Let \( J = \{(X,n) : X \subseteq I, |X| = l, n \ll \mu\} \). Here \( I \) is the atoms of \( \mathcal{M} \). \( J \) is the set of blurs, consult [12, definition 3.1] for the definition of blurs. Pending on \( l \) and \( \mu \), let us call these atom structures \( F(I,l,\mu) \). If \( \mu \gg \omega \), then \( J \) would be infinite, and \( \mathcal{Uf} \), the set of non principal ultrafilters corresponding to the blurs, will be a proper subset of the ultrafilters. It is not difficult to show that if \( l \geq \omega \) (and we relax the condition that \( I \) be finite), then \( \mathcal{CmF}(I,l,\mu) \) is completely representable, and if \( l \ll \omega \), then \( \mathcal{CmF}(I,l,\mu) \) is not representable.

Let \( \mathcal{D} \) be a non-trivial ultraproduct of the atom structures \( F(I,i,1), i \in \omega \). Then \( \mathcal{CmD} \) is completely representable. Thus \( \mathcal{TmF}(I,i,1) \) are \( \text{RRA} \)'s without a complete representation while their ultraproduct has a complete representation.

Also the sequence of complex algebras \( \mathcal{CmF}(I,i,1), i \in \omega \) consists of algebras that are non-representable with a completely representable ultraproduct.

Then because such algebras posses symmetric \( n \) dimensional cylindric basis (closed under substitutions), the result lifts easily to discretely topologized \( \text{CA} \)s

(ii) Alternatively, one can prove the cylindric case directly as follows. Take \( \mathcal{G}_{i} \) to be the disjoint union of cliques of size \( (n(n-1)/2) + i \), or let \( \mathcal{G}_{i} \) be the graph with nodes \( \mathbb{N} \) and edge relation \( (i,j) \in E \) if \( 0 < |i-j| < n(n-1)/2 + i \). Let \( \alpha_{i} \) be the corresponding atom structure, as defined in [38] and \( \mathcal{A}_{i} \) be the term polyadic equality algebra based on \( \alpha_{i} \). Then \( \mathcal{CmA}_{i} \) is not representable because, as proved in [38], \( \alpha \) is weakly but not strongly representable. In fact, the diagonal free reduct of \( \mathcal{CmA}_{i} \) is not representable.

But \( \Pi_{i \in \omega} \mathcal{CmA}_{i}/F = \mathcal{Cm}(\Pi_{i \in \omega} \mathcal{A}_{i}/F) \), so the one graph is based on the disjoint union of the cliques which is arbitrarily large, and the second on graphs which have arbitrary large chromatic number, hence both, by discrete topologizing are completely representable. 

\( \square \)
We prove our theorem on failure of omitting types for $\mathfrak{T}_n$, the first order topological logic with equality restricted to the first $n$ variables, viewed as a multi-modal logic. The corresponding class of modal algebras are $\mathcal{PEA}_n$. $n$ remains to be finite.

**Definition 4.24.** let $L$ be a signature and $M$ a structure for $L$ endowed with an Alexandrov topology. Let $\leq$ be the corresponding pre-order on $M$ and let $1 < m < n$ be finite.

(1) The *Gaifman hypergraph* $\mathcal{G}(M)$ of $M$ is the graph $(\text{dom}(M), E)$ where $E$ is the $m$ hyperedge relation such that for $a_1, \ldots, a_m \in M$, $E(a_1, \ldots, a_m)$ holds if there are $n$ and an $n$ ary relation symbol $R$ and $a_1, \ldots, a_m \in M$, such that $M \models R(a_1, \ldots, a_n)$ and $a_0, \ldots, a_{m-1} \in \{a_0, \ldots, a_n\}$.

2. An $m$ clique in $\mathcal{C}(M)$ is a set $C \subseteq M$ such $E(a_1 \ldots a_{m-1})$ for distinct $a_0, \ldots, a_{m-1} \in C$.

(2) The clique guarded semantics $M \models \phi(\bar{a})$ where $\phi$ an $L$ formula, and $\bar{a} \in M$ and $\text{rng}\bar{a}$ is an $m$ clique are defined by:

- For atomic $\phi$, $M \models \phi(\bar{a})$ iff $M \models \bar{a}$.
- The semantics of the Boolean connectives are defined the usual way.
- For $s \in \mathcal{C}(M)$, $M, s \models \exists x_i \phi$ iff there is a $t \in \mathcal{C}(M)$, $t \equiv_i s$ such that $M, t \models \phi$ and $\text{rng}(t)$ is in $m$ clique in $\mathcal{C}(M)$.
- For $s \in \mathcal{C}(M)$, $M \models \boxdot_i \phi(s)$ iff there exists $t \in \mathcal{C}(M)$ such that $t \equiv_i s$, $t_i \leq s_i$, $M, t \models \phi$ and $\text{rng}(t)$ is an $m$ clique in $\mathcal{C}(M)$.

**Definition 4.25.** Let $A \in \mathcal{TCA}_n$. Assume that $2 < n < k$. Let $\mathcal{L}(A)^k$ be the first order signature consisting of an $n$ ary relation symbol for each $a \in A$, using $k$ variables. Let $\mathcal{G}(M)$ be the $k$ Gaifman graph of an $\mathcal{L}_n$ structure $M$ carrying a topology.

(1) A topological space $M$ is a relativized representation of $A$ if there exists an injective homomorphism $f : A \to \wp(V)$ where $V \subseteq \mathcal{C}(M)$ and $\bigcup_{s \in V} \text{rng}(s) = M$.

(2) For $\bar{s} \in V$ and $a \in A$, we write $M \models a(\bar{s})$ iff $\bar{s} \in f(a)$. Then $M$ is said to be $k$ *square*, if whenever $\bar{s} \in \mathcal{G}(M) = \{s \in k M : \text{rng}(s)$ is an $n$ clique$\}$, $a \in A$, $i < n$, and injection $t : n \to k$, if $M \models c_i a(s_{t(0)} \ldots, s_{t(n-1)})$, then there is a $t \in \mathcal{G}(M)$ with $\bar{t} \equiv_i \bar{s}$, and $M \models a(t_{t(0)} \ldots, t_{t(n-1)})$.

(3) $M$ is said to be $k$ *flat* if it is $k$ square and for all $\phi \in \mathcal{L}(A)^k$, for all $\bar{s} \in \mathcal{G}(M)$, for all distinct $i, j < k$, we have

$$M \models [\exists x_i \exists x_j \phi \iff \exists x_j \exists x_i \phi](\bar{s}).$$
Here the subscript $G$ refers to the Giafman clique guarded semantics defined above, which we henceforth refer to as Giafman semantics.

$K_{n,\text{square}}$ denotes the class of square frames of dimension $n$, $K_{n,k,\text{square}}$ and $K_{n,k,\text{flat}}$ denote the class of Kripke frames of dimension $n$ whose modal algebras have $k$ square, $k$ flat representations, respectively.

That is

\[
K_{n,\text{square}} = \text{StrRTCA}_n,
K_{n,k,\text{square}} = \{ \mathcal{F} : \exists m \mathcal{F} \text{ has a } k \text{ square representation} \},
K_{n,k,\text{flat}} = \{ \mathcal{F} : \exists m \mathcal{F} \text{ has a } k \text{ flat representation} \}.
\]

We have the following deep results that can obtained from their CA analogues by discrete topologizing and same holds for tense and temporal cylindric algebras by static temporalizing. We formulate the plethora of negative results, with very few exception like undecidability of the equation theory of any class between RTCA$_2$ and TCA$_2$, a significant (negative) deviation from the theory of CA$_2$s, but admittedly utterly unsurprising due to the presence of two $S4$ modalities induced by the topology, witness theorem 4.26. However, in interesting and for that matter sharp contrast, like CA$_2$, the equational theories of TeCA$_2$ and TemCA$_2$ are decidable, theorem 4.26 and the corresponding 2 dimensional modal logic has the finite model property, and they (the corresponding modal algebras) have the finite algebra finite base property, any finite algebra has a finite representation.

If $\mathcal{L}_m$ is the multi modal topological logic corresponding to cubic frames, that is, frames whose domains are of the form $mU$, with $m$ still $> 2$. It is not to hard to distill from the literature the following:

- $\mathcal{L}_m$ is undecidable (this can be proved from non atomicity of free algebras or by coding the word problem for finitely presented semigroups), for $m = 2$ it is also undecidable; which is interesting (but expected) deviation from CA$_2$, because roughly the existence of the two $\S4$ modalities induced by the topology on the base of a representation, is stronger than the undecidable product logic $S4 \times S4$,

- $\mathcal{L}_m$ is not finitely axiomatizable,

- it is undecidable to tell whether a finite frame is a frame for $\mathcal{L}_m$, this implies that its modal algebras cannot be finitely axiomatizable in $k$th order first order logic for any finite $k > 0$,

- $\mathcal{L}_m$ lacks Craig interpolation and Beth definability; this will happen too for $m = 2$,

- the class of such frames cannot be Sahlqvist axiomatizable,
• even more it cannot be axiomatized by any set of first order sentences,

• though canonical, any axiomatization would necessarily contain infinitely many non canonical sentences (canonical sentences are sentences whose algebraic equivalents are preserved in canonical extensions).

• $\Sigma L_m$ has Godel’s incompleteness theorem.

The algebraic equivalences of the last theorem are (in the same order): $m$ is finite $> 2$ and $RTCA_m$ denotes the variety of representable algebras.

• The equational theory of $RTCA_m$ is undecidable [19, Theorem 5.1.66]; this holds even for $m = 2$, witness theorem 4.26 below,

• $RTCA_m$ is not finitely axiomatizable. This follows easily by topologizing Monk’s classical result proved for CAs, [19]. There are sharper results proved by Biro, Hirsch, Hodkinson, Maddux and others that can also be topologized, like for example making Monk’s algebras binary generated, thus it automorphisms group becomes rigid, this is far from being trivial.

Another such sharp result is that $SNrt_mTCA_{m+k}$ is not finitely axiomatizable over $SNrt_mTCA_{m+k}$ for any finite $k \geq 1$, and this lifts to infinite dimensions replacing finitely axiomatizable by axiomatizable by a Monk’s schema as defined in [19] under the name of systems of varieties definable by a finite schema [26, 27].

We have already dealt with Andréea’s complexity results indicating how they can all lift to the topological addition, witness theorem 4.19 above,

• It is undecidable to tell whether a finite algebra is representable [24, 22] witness theorem 4.19 above,

• $RTCA_m$ does not have the amalgamation property, this also works for $m = 2$ [2],

• $RTCA_m$ is not atom-canonical, hence not closed under Dedekind-MacNeille completions, theorem 4.6. Actually theorem 4.6 proves this result for $SNrt_mTCA_{m+k}$ with $k \geq 3$, which mean that for any $k \geq 3$ (infinite included giving $Str(TRCA_m)$ ) the class of frames $K_{m,m+k,flat} = \{ \mathcal{F} : \mathcal{C}e m \mathcal{F} in SNrt_mTCA_{m+k} \}$, and $K_{m,m+k,square}$ are not Sahlqvist axiomatizable. When $k = \omega$; it is furthermore not first order axiomatizable at all, as indicated in the next item so that the standard second order translation from modal formulas to second order formulas contain genuine second order formulas.

When $k$ is finite the question is open. It is not known whether there exists a sequence of weakly representable cylindric atom structures (that
can be easily topologized) whose ultrapoduct is outside \( S\text{Hit}_m T\text{CA}_{m+k} \) for some \( 3 \leq k < \omega \).

For \( k = \omega \) this is proved by constructing a sequence of good Monk-like algebras based on graphs with infinite chromatic number converging to a bad Monk-like algebra based on a graph that is 2 colourable. Witness \cite{25} for the terminology good and bad Monk algebras. Roughly a bad Monk’s algebra is one that is based on a graph that has a finite colouring and a good one is based on a graph that does not have finite chromatic number, in other words its chromatic number is infinite. Finite colorings forbid representations of Monk algebras based on them. Conversely, Monk-like algebras based on graphs with infinite chromatic number are representable \cite{26}.

Roughly this is a reverse process to Monk’s original construction back in 1969. Monk constructed a sequence of bad Monk’s algebras converging to a good one which is more believable, and intuitive. It boils down to constructing graphs having larger and larger finite chromatic number converging to one with infinite chromatic number. Indeed, for the reverse process, it took Hirsch and Hodkinson the use of Erdos’ probabilistic graphs to obtain their amazing result.

- The class of strongly representable atom structures, namely the class \( \text{Str}(\text{RTCA}_m) = \{ \mathfrak{A} : \mathfrak{C}m\mathfrak{A} \in \text{RTCA}_m \} \) is not elementary \cite{25}.

- Though canonical, i.e closed under canonical extensions, any axiomatization of \( \text{RTCA}_m \) must contain infinitely many non-canonical sentences (sentences that are not preserved in canonical extensions). Quoting Hodkinson: \( \text{RTCA}_m \) is only barely canonical \cite{14}.

- The free finitely generated algebras of any \( K \) between \( \text{RTCA}_m \) and \( \text{TCA}_m \) are not atomic since they are stronger than that of \( \text{TCA}_3 \), hence one can use the same construction of Németi’s by stimulating quasi-projections and coding \( ZFC \) in the equational theory of \( \text{TCA}_m \) \cite{9}.

- We use a deep (unpublished) result of Németi’s. Németi shows that there are three \( \text{CA}_3 \) terms \( \tau(x) \), \( \sigma(x) \) and \( \delta(x) \) such that for \( m \geq 3 \), we have \( \text{RCA}_m \models \sigma(\tau(x)) = x \) and \( \text{RCA}_m \models \delta(\tau(x)) = 1 \) but not \( \text{Cs}_m \models \delta(x) = 1 \). Then for every \( m \geq 3 \) we have (a) \( \text{TRCA}_m \models \sigma(\tau(x)) = x \) and (b) \( \text{TRCA}_m \models \delta(\tau(x)) = 1 \) but not \( \text{TCs}_m \models \delta(x) = 1 \). The first two validities follow from the fact that these terms do not contain modalities, and the last is obtained by giving the base of the the set algebra falsifying the equation the discrete topology; the resulting algebra falsifies the same equation since it also contains no modalities.
Let $0 < \beta$, and $n \geq 3$ and let $\{g_i : i < \beta\}$ be an arbitrary generator set of $\mathfrak{F}_{\beta}^{\text{TRCA}_m}$. Then $\{\tau(g_0)\} \cup \{g_i : 0 < i < \beta\}$ generates $\mathfrak{F}_{\beta}^{\text{TRCA}_m}$ by (a) but not freely by (b). Let $K$ be the class of all finite algebras in $\text{TRCA}_m$. Then $\text{HSP}K \neq \text{TRCA}_m$. In particular, there are equations valid in all finite $\text{TRCA}_m$’s but they are not valid in all $\text{TRCA}_m$’s.

**Theorem 4.26.**

(1) $\text{TRCA}_1$ is finitely axiomatizable, has fmp and is decidable.

(2) $\text{TRCA}_2$ and $\text{TalRCA}_2$ are finitely axiomatizable but they do not have the finite base property and their equational theory is undecidable. Furthermore, $S54^m$ does not have abstract fmp.

(3) However, the equational theory of any class between $\text{TeCA}_2$ and $\text{TemCA}_2$, and their concrete versions, namely, the representable algebras.

(4) For $K$ as in the previous item the free algebras are atomic, and $K = \text{HSP}\{\mathfrak{A} \in K : |\mathfrak{A}| < \omega\}$.

**Proof.**

(1) Follows from the fact that the bi-modal logic of Kripke frames of the form $(U, U \times U, R)$ where $R$ is a pre-order is decidable and has fmp

(2) Let $\Sigma$ be a finite set of equations that axiomatize $\text{RCA}_2$, together with the $S4$ axioms. Then if $\mathfrak{A} \models \Sigma$, then $\text{Rca}_A \in \text{RCA}_2$. A itself can be proved representable as follows:

We have $\mathfrak{B} \subseteq \mathfrak{N}_{\mathfrak{R}_2} \mathfrak{C}$ where $\mathfrak{C} \in \mathfrak{L}_{\omega}$. Let $\bar{h} : \mathfrak{C} \to \mathfrak{D}$ where $\mathfrak{D} \in \mathfrak{C}_{\omega \times \omega}^{\text{reg}} \cap \mathfrak{L}_{\omega}$ be such that $\bar{h} \models m = h$. Let $G$ be the corresponding Henkin ultrafilter in $\mathfrak{D}$, so that $\bar{h} = h_G$. For $p \in \mathfrak{A}$ and $i < m$, let $O_{p,i} = \{k \in \omega : s^k_i I(i) \mathfrak{A} p \in G\}$ and let $\mathcal{B} = \{O_{p,i} : i \in \omega, p \in A\}$. Then $\mathcal{B}$ is the base for a topology on $\omega$ and the concrete interior operations are defined for each $i < m$ via $J_i : \wp(\wp(\omega)) \to \wp(\wp(\omega))$

$$x \in J_i X \iff \exists U \in \mathcal{B}(x_i \in U \subseteq \{u \in \omega : x_i \in U\}),$$

where $X \subseteq \{x_i \}$. Then $h^* : \mathfrak{A} \to \wp(\wp(U, J_i))$ defined by $h^*(x) = h(x)$ for $x \in B$ is an isomorphism, such that $h^*(a) \neq 0$, which contradicts that $\Gamma \models \phi$, because $a = \neg \phi/\Gamma$.

There is formula using only the boxes such that if $\phi$ is satsifiable in a frame $\mathfrak{F} = (U, R_1) \times (U, R_2)$ where $R_1, R_2$ are symmetric and transitive that is $\mathfrak{F} \models \phi$ iff $\mathfrak{F}$ is an $\infty$ chessboard board. For such a frame $\mathfrak{F}$, let $\mathfrak{F}^+ = (U, U \times U, R_1) \times (U, U \times U, R_2)$, then because the constructed formula does not contain the two modalities $c_0$ and $c_1$, we also have $\mathfrak{F}^+ \models \phi$ iff $\mathfrak{F}$ is an $\infty$ chessboard board. Let $e$ be the equation corresponding to $\phi$ in the
sublanguage of that of TCA\textsubscript{2} obtained by dropping cylindrifiers. Such a formula forces infinite frames, and so the required follows for \textit{fmp} because \(\mathcal{C}m\mathfrak{F}^+\) cannot be represented on finite sets, since \(\mathcal{C}m\mathfrak{F}^+ \models \varepsilon\). Hence such varieties do not have \textit{fmp}.

To prove undecidability a sufficiently complex problem for Turing machines or tilings is reduced to the satisfiability problem for the multimodal logic at hand.

We use the result proved in [15], namely, that \(S4 \times S4\) is undecidable. There is a formula \(\psi_M\), without using the modalities \(c_0\) and \(c_1\) such that \(\psi_M\) is satisfiable in \(\log\{\mathfrak{F} : \mathcal{C}m\mathfrak{F} \in \text{TRCA}_n\}\) iff \(M\) stops having started for an all blank tape [15, p. 1014].

Another way is to show that there are three modal formulas using \(\Box_0, \Box_1\), denoted by \(\phi_\infty, \phi_{\text{grid}},\) and \(\phi_\Theta\) in [15, p. 1006-13] where \(\Theta\) is a finite set of tile types, \textit{encoding tilings; that is encoding the }\(\mathbb{N} \times \mathbb{N}\) \textit{grid, in the sense that }\Theta\textit{ tiles }\mathbb{N} \times \mathbb{N}, \text{iif their conjunction is satisfiable in any }\infty\textit{ chessboard. More explicitely }\Theta\textit{ tiles }\mathbb{N} \times \mathbb{N}, \text{iif for every frame }\mathfrak{F} = (U \times U, T_0, T_1, \Box_0, \Box_1, \delta), \text{there exists }s = (s_0, s_1) \in U \times U \text{ such that }\mathfrak{N} \mathfrak{F} = (U \times U, \Box_0, \Box_1), s \models \phi_\infty \land \phi_{\text{grid}} \land \phi_\Theta\text{ and furthermore }\mathfrak{N} \mathfrak{F}\text{ is an }\infty\textit{ chessboard.}

(3) Follows from the fact that the first order correspondence of the modal formulas of modal logic with \(U\) and \(S\) lands in the loosely guarded fragment. Hence we have a disjoint union of decidable first order theories that of \(\{\mathfrak{F} : \mathcal{C}m\mathfrak{F} \in \text{RCA}_2\}\) together with first order correspondants of the modal formulas axiomatizing TeCA\textsubscript{2} which is decidable. Since both have \textit{fmp} so does TeCA\textsubscript{2}.

(4) The last required is exactly like the CA\textsubscript{2} case [13], since every finite algebra has a finite representation.

\(\square\)

### 4.5 Further negative results

Now we prove that the omitting types theorem fails in a very strong sense for \(TL_m\), when \(m \geq 2\). We have \(S\mathcal{C}m\mathcal{K}_{n,\text{square}} = \text{RTCA}_n\). It can also be proved that \(S\mathcal{C}m\mathcal{K}_{n,k,\text{flat}} = S\mathfrak{N}r_n\text{CA}_k\); we will prove \(\subseteq\) below, which is known to be strictly contained in \(\text{TRCA}_n\) for any \(n, k\) such that \(2 < n < k < \omega\).

Given \(v : \omega \to \text{TM}(X), \mathfrak{A} \in \text{CA}_m\) and any map \(s : \omega \to \mathfrak{A}\), then by \(\bar{s}\) we denote the unique extension of \(s, \bar{s} : \text{TM}(X) \to \mathfrak{A}\), such that \(\bar{s} \circ i = s\). Here we are looking at \(\text{TM}(X)\) as the absolutely free algebra on \(X; i\) is the inclusion map from \(X\) into \(\text{TM}(X)\).
Definition 4.27.  (1) Let $K$ be a class of frames, and $A \in \text{TCA}_m$ be countable and $\mathfrak{F} \in K$. Then a model $\mathfrak{M} = (\mathfrak{F}, v)$ is a representation of $A$, if there exists a bijection $s : X \to A$ ($X$ is an infinite countable set of variables) and an injective homomorphism $f : A \to \text{Cm} \mathfrak{F}$ such that $f \circ s = v$. In this case, we say that $f$ is a representing function.

(2) If $A$ is atomic, then $f$ is called a complete representation of $A$ if furthermore $\bigcup_{x \in \text{At} A} f(x) = F$.

We show that when we broaden considerably the class of allowed models, permitting $m + 3$ flat ones, there still might not be countable models omitting a single non-principal type. Furthermore, this single-non principal type can be chosen to be the set of co-atoms in an atomic theory. In the next theorem by $T \models \phi$ we mean that $\phi$ is valid in any (classical) topological model of $T$. That is for any topological model $\mathfrak{M}$ of $T$, any $s \in m M$, $\mathfrak{M} \models \phi[s]$; using the modal notation $\mathfrak{M}, s \models \phi$.

Definition 4.28.  (1) Let $T$ be an $\mathfrak{L}_m$ theory. A set $\Gamma$ of formulas is isolated if there is a formula $\psi$ consistent with $T$ such that $T \models \psi \to \alpha$ for all $\alpha \in \Gamma$. Otherwise, $\Gamma$ is non-principal.

(2) Let $T$ be an $\mathfrak{L}_m$ theory. A frame $\mathfrak{M} = (\mathfrak{F}, v)$ is a model of $T$ if $v : \omega \to F$, is such that for all $\phi \in T$, and $s \in F$, $\mathfrak{M}, s \models_v \phi$.

(3) Let $L$ be a class of frames. $\mathfrak{L}_m$ has OTT with respect to $L$, if for every countable theory $T$, for any non-principal type $\Gamma$, there is a model $\mathfrak{M} = (\mathfrak{F}, v) \in L$ of $T$ that omits $\Gamma$, in the sense that for any $s \in F$, there exists $\phi \in \Gamma$ such that not $\mathfrak{M}, s \models_v \phi$.

The proof of the following lemma can be easily distilled from its relation algebra analogue. The statement of the lemma can be can seen as a weak soundness theorem truncating syntactically the well know neat embedding theorem of Henkin to $n$ if $n < \omega$ and classical represenations to ‘$n$ localized ones.’ Otherwise if $n \geq \omega$ it is the easy implication in the usual neat embedding theorem, for countable algebras having $\omega$ square or flat representations are representable in the classical sense. The converse also holds (using step by step constructions) so we have a weak completeness theorem, too, approximating the hard implication in the neat embedding theorem, but we do not need that much.

Lemma 4.29. Assume that $2 < m < n$

(1) If $A$ has $n$ flat representation, then $A \in S_{m} \text{TCA}_m$.

(2) If $A$ has a complete $n$ flat representation, then $A \in S_{c} \text{Nr}_m \text{TCA}_n$. 

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We are now ready to formulate and prove the metalogical result that can be inferred from both 4.6 and 4.13, thus we give two proof.

The proofs also depend on the last lemma.

**Theorem 4.30.** For any \( k \geq 3 \), \( \mathcal{TL}_m \) does not have \( \text{OTT} \) with respect \( K_{m,m+k,\text{flat}} \).

**Proof.** We give two proofs depending on our previous two rainbow constructions; one proving non atom canonicty of \( S_{\text{Nr}^m_n \text{TCA}_m^{n+3}} \) and the other proving non-elementarity of any class \( K \) containing the completely representable algebras and contained in \( S_{c\text{Nr}^m_n \text{TCA}_m^{n+3}} \).

**First proof:** Let \( A = \text{Term} \) be the term algebra as in 4.6. Then \( A \) is countable, simple and atomic. Let \( \Gamma' \) be the set of co-atoms. Then \( A = \mathcal{F}_{mT} \) for some countable consistent \( L_m \), complete theory \( T \). Then \( \Gamma = \{ \phi : \phi_T \in \Gamma' \} \) is not principal. If it can be omitted in an \( m + 3 \) flat model \( M \), then this gives an \( m + 3 \) complete flat representation of \( A = \mathcal{F}_{mT} \), which gives an \( m + 3 \) flat representation of \( \text{Term} \). By lemma 4.29 we get that \( \text{Term} \in S_{c\text{Nr}^m_n \text{TCA}_m^{n+3}} \) which contradicts theorem 4.6.

**Second proof:** Let \( A \) be the term algebra of the rainbow atom structure of that of \( \text{TCA}_{Z,N} \) constructed in theorem 4.13. Then \( A \not\in S_{c\text{Nr}^m_n \text{TCA}_m^{n+3}} \). Let \( T \) be such that \( A \cong \mathcal{F}_{mT} \) and let \( \Gamma \) be the set of co-atoms. Then \( \Gamma \) cannot be omitted in an \( m + 3 \) flat \( M \), for else, this gives a complete \( m + 3 \) flat representation of \( A \), which means by lemma 4.29 that \( A \in S_{c\text{Nr}^m_n \text{TCA}_m^{n+3}} \) and this contradicts theorem 4.13.

The same result on failure of \( \text{OTT} \) holds for the multi dimensional modal logics corresponding to both \( \text{TeCA}_m \) and \( \text{TemCA}_m \) for finite \( m > 2 \).

By noting that \( \forall \) can win the game \( G^{n+3}_\omega \) in theorem 4.6, which is the usual \( \omega \) rounded atomic game restricted to \( n + 3 \) nodes (pebbles), though \( n + 3 \) round suffice for him to force \( \exists \) an inconsistent red, this excludes ever \( n + 3 \) square models omitting types.

This class is substantially larger; for example the first order theory of its Kripke frames can be coded in the packed fragment, which is not the case with flat model, basically due to the additional Church Rosser condition of commutativity of cylindrifiers.

In fact, it can be shown that for \( m = 3 \) and \( k \geq 3 \), it is undecidable to tell whether a finite frame \( F \) has its modal algebra \( S_{\text{Nr}^3_n \text{TCA}_3^{3+k}} \), that is whether \( F \) is a frame for \( \mathcal{L}_3 \) with a proof system, more specifically a Hilbert style axiomatization, using \( 3 + k \) variables; corresponding to the equational axiomatization of \( \text{TCA} \) where formulas are built up from 3 variable restricted atomic formulas, that is, variables occur in the order \( x_0, x_1, x_2 \), i.e. in their natural order, applying usual connectives of first order logic and \( \Box_i \), \( i < 3 \).

This follows from the fact that \( S_{\text{RaCA}_5} \) has the same property and this property lifts to \( \text{TCA}_3 \), by discretely topologizing a well known construction.
of Monk’s associating to every simple atomic $\text{CA}_3$ a simple atomic relation algebra such that the relation algebra embeds into the relation algebra reduct of the constructed cylindric algebras $\text{CA}_3$.

There are finite algebras that have infinite $n$ flat representations, for any $m < n$ but not finite ones. This is not the case with square models.

Any finite algebra that has an $n > m$ square model has a finite one because the packed fragment of finite variable first order logic has the finite model property. However, the equational theory of its modal algebras in undecidable.

4.6 Various notions of representability

Here we restrict ourselves to Alexandrov topologies stimulated by pre-order on the base of Kripke frames and we deal with diamonds that are dual to boxes (the interior operators).

The definition of atomic networks on atomoc algebras are obtained from those for CAs by adding a further consistency condition, namely, $N(\delta_d^i) \leq \Diamond_i N(\delta)$, when $d \leq \delta(i)$.

Now we introduce quite an interesting class that lies strictly between $\text{SRTCA}_n$ and the class of atomic completely additive representable algebras in $\text{RTCA}_n$ which we call weakly representable algebras and denote by $\text{TWCA}_n$.

The introduction of this new elementary class is motivated by its relation algebra analogue proposed by Goldblatt and further investigated by Hirsch and Hodkinson. For a cylindric algebra atom structure $\mathfrak{F}$ the first order algebra over $\mathfrak{F}$ is the subalgebra of $C_m\mathfrak{F}$ consisting of all sets of atoms that are first order definable with parameters from $S$. $\text{TFOCA}_n$ denotes the class of atomic such algebras of dimension $n$.

This class is strictly bigger than $\text{SRTCA}_n$. Indeed, let $\mathfrak{A}$ be any the rainbow term algebra, constructed in the proof of theorem 4.6; recall that such an algebra is obtained by blowing up and blurring the finite rainbow algebra $T\text{CA}_{n+1, n}$ proving that $\mathfrak{S}\triangledown\triangledown T\text{CA}_{n+3}$ is not atom canonical.

This algebra was defined using first order formulas in the rainbow signature (the latter is first order since we had only finitely many greens). Though the usual semantics was perturbed, first order logic did not see the relativization, only infinitary formulas saw it, and that’s why the complex algebra could not be represented. Other simpler examples, that can be topologized, are given in [38, 28].

These examples all show that $\text{SRTCA}_n$ is properly contained in $\text{TFOCA}_n$. Another way to view this is to notice that $\text{TFOCA}_n$ is elementary, by definition, that $\text{STRCA}_n \subseteq \text{TFOCA}_n$, but but as mentioned above one can obtain by discretely topologizing Monk-like algebras, denoted by $\mathfrak{M}(\Gamma)$ in [26] based on Erdos’ graphs that $\text{STRCA}_n$ is not elementary as proved by Hirsh and Hodkinson for CAs.
Fix finite \( n > 2 \). Let \( \text{CRTCA}_n \) denote the non elementary class of completely representable algebras, \( \text{TLCA}_n \) denotes the class satifying the *diamond Lyndon conditions*. Here, however, the \( m \)th Lyndon condition is a first order sentence that codes that \( \exists \) has a winning strategy in the \( m \) rounded atomic game defined like the \( \text{CA} \) case, but here we have the extra diamond move, namely,

\[
\forall \text{ chooses } i < n, a \in \text{At}A, \text{ a tuple } \bar{x} \text{ and a previously played atomic network } N \text{ such that } N(\bar{x}) \leq \Diamond_i a. \text{ Now } \exists \text{ has to refine } N \text{ by a network } M \text{ and } \bar{y} \in M \text{ possible xtending the pre-order } \leq \text{ to } \leq_i \text{ with } \bar{x} \equiv_i \bar{y} \text{ and } x_i \leq_i y_i \text{ such that } M(\bar{y}) = a. \]

The Lyndon conditions characterizes representability for finite algebras but if an infinite algebra satisfies Lyndon conditions then all we can conclude is that it is elementary equivalent to a completely representable algebra.

Using this observation it can be easily shown that the elementary closure of \( \text{CRTCA}_n, \text{ELCRTRA}_n \) for short, coincides with \( \text{TLCA}_n \). Recall that the former class is not elementary by theorem 4.13 while \( \text{TLCA}_n \) is elementary by definition.

\( \text{TWRCA}_n \) is the class of weakly representable algebras defined simply to be the class consisting of atomic algebras in \( \text{RTCA}_n \). This class is obviously elementary because atomicity is a first order definable property. and \( \text{RTCA}_n \) is a variety.

It can be proved using Monk-like algebras that the elementary closure of all such classes is not finitely axiomatizable \( [4,22] \) and an that \( \text{El}(\text{CRTCA}_n) = \text{TLCA}_n \).

Let \( \mathcal{M}(\Gamma) \) be the the discretely topologized Monk like algebras constructed in \( [26] \) based on \( \Gamma \).

For an undirected graph \( \Gamma \) let \( \chi(\Gamma) \) denotes its chromatic number which is the size of the smallest finite set \( C \) such that there exist a \( C \) colouring of \( \Gamma \) and \( \infty \) otherwise. \( C \) is a colouring of \( \Gamma \) if there exists \( f : C \to \Gamma \) such that whenever \( (v, w) \) is an edge then \( f(v) \neq f(w) \).

Recall that \( \mathcal{M}(\Gamma) \) is good if \( \chi(\Gamma) = \infty \) this is equivalent to its representability, otherwise, that is when \( \chi(\Gamma) < \infty \), it is bad. This means that \( \Gamma \) has a finite colouring prohibiting a representation for \( \mathcal{M}(\Gamma) \).

To show that \( \text{SRTCA}_n \) is not finitely axiomatizable one constructs a sequence of algebras that are weakly but not strongly representable with an ultraproduct that is in \( \text{El}(\text{SRTCA}_n) \); which is a plausible task and not very hard to accomplish. Roughly one constructs Monk-like algebras based on graphs with arbitrary large chromatic number converging to one with infinite chromatic number (via an ultraproduct construction).

Monk’s original algebras can be seen this way. Indeed it not hard to show that the limit of Monk’s original finite algebras \( [19] \) is strongly representable; by observing first that it is atomic because it is an ultraproduct of atomic algebras. In fact, the limit is completely representable. Now we have the strict inclusions, that can be proved the topological coiunterpat of the arguments.
in [26]:

\[
\text{CRTCA}_n \subseteq \text{TLCA}_n \subseteq \text{SRCTA}_n \subseteq \text{TFOCA}_n \subseteq \text{WRTCA}_n.
\]

Strictly speaking, [26] did not deal with \text{TFOCA}_n, but the relation algebra analogue of such a class is investigated in [22], and the thereby obtained results lifts to the CA case without much ado.

Last inclusion follows from the following RA to CA adaptation of an example of Hirsch and Hodkinson which we use to show that \text{TFOCA}_n \subset \text{WRCA}_n and it will be used for another purpose as well. We note that the strictness of the inclusion is not so obvious because they are both elementary.

**Example 4.31.** Take an \( \omega \) copy of the \( n \) element graph with nodes \{1, 2, \ldots, n\} and edges \( 1 \rightarrow 2 \rightarrow \ldots \rightarrow n \). Then of course \( \chi(\Gamma) < \infty \). Now \( \Gamma \) has an \( n \) first order definable colouring. Since \( \mathfrak{M}(\Gamma) \) as defined above and in [26, top of p. 78] is not representable, then the algebra of first order definable sets, call it \( \mathfrak{A} \), is also not representable because \( \Gamma \) is first order interpretable in \( \rho(\Gamma) \), the atom structure constructed from \( \Gamma \) as defined in [26]. However, it can be shown that the term algebra is representable. (This is not so easy to prove).

Now \( \text{At}\mathfrak{ImA} = \text{At}\mathfrak{A} \) and \( \mathfrak{A} \) is not representable, least strongly representable. But since \( \text{SRCA}_n \subseteq \text{FOCA}_n \), then \( \text{At}(\text{SRCA}_n) \) grips \( \text{TFOCA}_n \) but it does not grip \( \text{WRTCA}_n \), for \( \mathfrak{A} \) is not strongly representable.

For a class of algebras \( K \) having a Boolean reduct \( K \cap \text{At} \) denotes the class of atomic algebras in \( K \).

**Theorem 4.32.** Let \( n > 2 \) be finite. Then we have the following inclusions (note that \( \text{At} \) commutes with \( \text{UpUr} \)):

\[
\mathfrak{M}_n \text{TCA}_\omega \cap \text{At} \subseteq \text{El}\mathfrak{M}_n \text{TCA}_\omega \cap \text{At} \\
\subseteq \text{ElS}_c\mathfrak{M}_n \text{TCA}_\omega \cap \text{At} = \text{ElCRTCA}_n = \text{TCA}_n \subseteq \text{SRCTA}_n \\
\subseteq \text{UpSRTCA}_n = \text{URSTRCA}_n = \text{ElSTRRCAC}_n \subseteq \text{TFOCA}_n \\
\subseteq \text{S\text{At}\text{TCA}}_\omega \cap \text{At} = \text{WRTCA}_n = \text{RACA}_n \cap \text{At}.
\]

**Proof.** The majority of inclusions, and indeed their strictness, can be distilled without much difficulty from our previous work. It is known that \( \text{UpSTRCA}_n = \text{URSTRCA}_n \) [23 [22].

The first inclusion is witnessed by a slight modification of the algebra \( \mathfrak{B} \) used in the proof of [35, Theorem 5.1.4], showing that for any pair of ordinals \( 1 < n < m \cap \omega \), the class \( \mathfrak{M}_n \text{CA}_m \) is not elementary.

In the constructed model \( \mathfrak{M} \) [35, lemma 5.1.3] on which (using the notation in op.cit), the two algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) are based, one requires (the stronger) that the interpretation of the 3 ary relations symbols in the signature in \( \mathfrak{M} \).
are disjoint not only distinct as above. Atomicity of $\mathcal{B}$ follows immediately, since its Boolean reduct is now a product of atomic algebras. For $n = 3$ these are denoted by $\mathfrak{A}_u$ except for one countable component $\mathfrak{B}_I$, $u \in \mathfrak{N} \sim \{\text{Id}\}$, cf. [35] p.113-114. Second inclusion follows from example 4.14. Third follows from theorem 4.6, and fourth inclusion follows from theorem 4.13. Last one follows from example 4.31.

We characterize the class of strongly representable atom structures via neat embeddings, modulo an inverse of Erdos’ theorem.

For an atomic algebra $\mathfrak{A}$, by an atomic subalgebra we mean a subalgebra of $\mathfrak{A}$ containing all its atoms, equivalently a superalgebra of $\mathfrak{mAt}\mathfrak{A}$. We write $S_{at}$ to denote this operation applied to an algebra or to a class of algebras. $\mathfrak{M}(\Gamma)$ denotes the Monk algebra based on the graph $\Gamma$ as in [26]. Notice that although we have $UpUrS_{at}\mathfrak{C}A_n \subseteq T\mathfrak{FOCA}_n$, the latter is not closed under forming atomic subalgebras, since forming subalgebras does not preserve first order sentence. The next characterization therefore seems to be plausible. However it is formulaed only for cylindric algebras. The notion is obtained by removing the ’ $T$ ’ to get the corresponding class of cylindric algebra, for example $SRCA_n = R_{\omega}^{\text{c}}S_{at}\mathfrak{C}A_n$, because every CA can be expanded with the interior identity operations. The other inclusion is problematic for the algebras in $TRCA_n$ that are not completely additive.

**Theorem 4.33.** Assume that for every atomic representable algebra that is not strongly representable, there exists a graph $\Gamma$ with finite chromatic number such that $\mathfrak{A} \subseteq \mathfrak{M}(\Gamma)$ and $At\mathfrak{A} = \rho(\mathfrak{M}(\Gamma))$. Assume also that for every graph $\Gamma$ with $\chi(\Gamma) < \infty$, there exists $\Gamma_i$ with $i \in \omega$, such that $\prod_{i \in F} \Gamma_i = \Gamma$, for some non principal ultrafilter $F$. Then $S_{at}UpS_{at}\mathfrak{C}A_n = WRCA_n = S\mathfrak{mAt}_n\mathfrak{C}A_\omega \cap At$.

**Proof.** Assume that $\mathfrak{A}$ is atomic, representable but not strongly representable. Let $\Gamma$ be a graph with $\chi(\Gamma) < \infty$ such that $\mathfrak{A} \subseteq \mathfrak{M}(\Gamma)$ and $At\mathfrak{A} = \rho(\mathfrak{M}(\Gamma))$. Let $\Gamma_i$ be a sequence of graphs each with infinite chromatic number converging to $\Gamma$, that is, their ultraproduct is $\Gamma$. Let $\mathfrak{A}_i = \mathfrak{M}(\Gamma_i)$. Then $\mathfrak{A}_i \in SRCA_n$, and we have:

$$\Pi_{i \in \omega} \mathfrak{M}(\Gamma_i) = \mathfrak{M}(\Pi_{i \in \omega} \Gamma_i) = \mathfrak{M}(\Gamma).$$

And so $\mathfrak{A} \subseteq_{at} \prod_{i \in \omega} \mathfrak{A}_i$, and we are done. $\square$

### 4.7 Neat embeddings

For the well known definitions of pseudo universal and pseudo elementary classes, the reader is referred to [26, definition 9.5, definition 9.6]. In fact all our results below hold for any class whose signature is between $\mathfrak{Sc}$ and $\mathfrak{PEA}$. (Here $Df$ is not counted in because the notion of neat reducts for this class is trivial [19, Theorem 5.1.31]).
One can topologize rainbow atom structures by defining for bijections \( f, g : n \to \Gamma, \Gamma \) a coloured graph, and for \( i < n \) \( [f]In_{i}[g] \) iff \( f = g \).

**Theorem 4.34.**

1. For \( n > 2 \), the inclusions \( \mathfrak{N}_n \text{TCA}_\omega \subseteq S_c \mathfrak{N}_n \text{TCA}_\omega \subseteq S\mathfrak{N}_n \text{TCA}_\omega \) are proper. The first strict inclusion can be witnessed by a finite algebra for \( n = 3 \), while the second cannot by witnessed by a finite algebra for any \( n \). In fact, \( m > n > 1 \), the inclusion \( \mathfrak{N}_n \text{TCA}_m \subseteq S_c \mathfrak{N}_n \text{TCA}_m \) is proper and for \( n > 2 \) and \( m \geq n + 3 \) the inclusion \( S_c \mathfrak{N}_n \text{TCA}_m \subseteq S\mathfrak{N}_n \text{TCA}_m \) is also proper.

2. For any pair of ordinals \( 1 < \alpha < \beta \) the class \( \mathfrak{N}_\alpha \text{TCA}_\beta \) is not elementary. In fact, there exists an uncountable atomic algebra \( \mathfrak{A} \in \mathfrak{N}_\alpha \text{QEA}_{\alpha+\omega} \), hence \( \mathfrak{A} \in \mathfrak{N}_\alpha \text{QEA}_\beta \) for every \( \beta > \alpha \), and \( \mathfrak{B} \subseteq \mathfrak{A} \), such that \( \mathfrak{B} \) is completely representable, \( \mathfrak{A} \equiv \mathfrak{B} \), so that \( \mathfrak{B} \) is also atomic, but \( \mathfrak{N}_\alpha \mathfrak{B} \not\equiv \mathfrak{N}_\alpha \text{Sc}_{\alpha+1} \). For finite dimensions, we have \( \text{At} \mathfrak{A} \equiv \infty \text{At} \mathfrak{B} \).

3. For finite \( n \), the elementary theory of \( \mathfrak{N}_n \text{TCA}_\omega \) is recursively enumerable.

4. For \( n > 1 \), the class \( S_c \mathfrak{N}_n \text{TCA}_\omega \) is not elementary (hence not pseudo-universal) but it is pseudo-elementary, and the elementary theory of \( S_c \mathfrak{N}_n \text{TCA}_\omega \) is recursively enumerable.

5. For \( n > 1 \), the class \( S_c \mathfrak{N}_n \text{TCA}_\omega \) is closed under forming strong subalgebras but is not closed under forming subalgebras.

6. For \( n > 2 \), the class \( \text{UpUr}S_c \mathfrak{N}_n \text{TCA}_\omega \) is not finitely axiomatizable. For any \( m \geq n + 2 \), \( S\mathfrak{N}_n \text{TCA}_m \) is not finitely axiomatizable and for \( p \geq 2 \), \( \alpha > 2 \) (infinite included) and TCA the variety \( S\mathfrak{N}_\alpha \text{TCA}_{\alpha+2} \) cannot be finitely axiomatized by a universal set of formulas containing only finitely many variables.

7. For \( n > 2 \), both \( \text{UpUr} \mathfrak{N}_n \text{TCA}_\omega \) and \( \text{UpUr}S_c \mathfrak{N}_n \text{TCA}_\omega \) are properly contained in \( \text{RK}_n \); by closing under forming subalgebras both resulting classes coincide with \( \text{RK}_n \). Furthermore, the former is properly contained in the latter.

**Proof.**

1. The inclusions are obvious.

The strictness of the inclusion follows from item example 4.14 since \( \mathfrak{A} \) is dense in \( \varphi(V) \), hence it is in \( S_c \mathfrak{N}_n \text{TCA}_\omega \), but it is not in \( \mathfrak{N}_n \text{TCA}_{n+1} \), a fortiori it is not in \( \mathfrak{N}_n \text{TCA}_\omega \) for the latter is clearly contained in the former.

For the strictness of the second inclusion, let \( \mathfrak{A} \) be the rainbow algebra \( \text{PEA}_{Z,N} \). Then \( \text{PEA}_{Z,N} \) is representable, in fact, it satisfies the Lyndon
conditions, hence is also strongly representable, but it is *not* completely representable, in fact its $Df$ reduct is not completely representable, and its $Sc$ reduct is not in $S_c\mathfrak{Nr}_n Sc_w$, for had it been in this class, then it would be completely representable, inducing a complete representation of $A$.

We have proved that there is a representable algebra, namely, $PEA_{Z,N}$ such that $\mathfrak{R}_{Df}PEA_{Z,N} \notin S_c\mathfrak{Nr}_n Sc_w$. The required now follows by using the neat embedding theorem that says that $RTCA_n = S\mathfrak{Nr}_n TCA_\omega$ for any TCA as specified above [18], indeed we have $\mathfrak{R}_{Df}PEA_{Z,N}$ is not in $S_c\mathfrak{Nr}_n TCA_\omega$ but it is (strongly) representable, that is, it is in $S\mathfrak{Nr}_n TCA_\omega$. On the other hand, its canonical extension is completely representable a result of Monk [19, Corollary 2.7.24].

For the second part concerning finite 3 dimensional algebras witnessing the strictness of inclusions by above example.

The strictness of the last inclusion cannot be witnessed by a finite representable algebra for any finite dimension $n > 2$, because any such algebra, is atomic (of course) and completely representable, hence it will be necessarily in $S\mathfrak{Nr}_n TCA_\omega$.

Concerning the second strictness of inclusions (concerning neat embeddability in finitely many extra dimensions), the first is witnessed by $A$ constructed in [14] or the $B$ constructed in the next item, while the second follows by noting $S_c\mathfrak{Nr}_n TCA_\omega$ is not elementary while $S\mathfrak{Nr}_n TCA_m$ is a variety theorem [4.13].

(2) We consider the case when $A$ and $B$ are cylindric algebras as constructed in [35, Theorem 5.1.3] but discretly topologized. We show that $A \equiv_{\infty} B$. A finite atom structure, namely, an $n$ dimensional cartesian square, with accessibility relations corresponding to the concrete interpretations of cylindrifiers and diagonal elements, is fixed in advance.

Then its atoms are split twice. Once, each atom is split into uncountably many, and once each into uncountably many except for one atom which is only split into *countably* many atoms. These atoms are called big atoms, which mean that they are cylindrically equivalent to their original. This is a general theme in splitting arguments. The first splitting gives an algebra $A$ that is a full neat reduct of an algebra in arbitrary extra dimensions; the second gives an algebra $B$ that is not a full neat reduct of an algebra in just one extra dimensions, hence in any higher dimensions. Both algebras are representable, and elementary equivalent because first order logic cannot see this cardinality twist. We will show in a minute that their atom structures are $L_{\infty,\omega}$ equivalent.
We, henceforth, work with dimension 3. The proof for higher finite dimensions is the same. However, we make a slight perturbation to the construction in *op.cit*; we require that the interpretation of the uncountably ternary relation symbols in the signature of \( M \) on which the set algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) are based are *disjoint*, not only distinct [35, Theorems 5.3.1, 5.3.2].

The Boolean reduct of \( \mathfrak{A} \) can be viewed as a finite direct product of disjoint Boolean relativizations of \( \mathfrak{A} \), denoted in [35, theorem 5.3.2] by \( \mathfrak{A}_u; \mathfrak{A}_u \) is the finite-cofinite algebra on a set having the same cardinality as the signature; it is relativized to \( 1_u \) as defined in *opcit*, \( u \in 3^3 \).

Each component will be atomic by our further restriction on \( M \), so that \( \mathfrak{A} \) itself, a product of atomic algebras is also atomic. The language of Boolean algebras can now be expanded so that \( \mathfrak{A} \) is interpretable in an expanded structure \( \mathfrak{P} \), based on the same atomic Boolean product. Now \( \mathfrak{B} \) can be viewed as obtained from \( \mathfrak{P} \), by replacing one of the components of the product with an elementary *countable* Boolean subalgebra, and then giving it the same interpretation. By the Feferman Vaught theorem (which says that replacing in a product one of its components by an elementary equivalent one, the resulting product remains elementary equivalent to the original product) we have \( \mathfrak{B} \equiv \mathfrak{A} \). In particular, \( \mathfrak{B} \) is also atomic.

First order logic will not see this cardinality twist, but a suitably chosen term not term definable in the language of \( \text{CA}_3 \), namely, the substitution operator, \( \_3s(0,1) \) will witnessing that the twisted algebra \( \mathfrak{B} \) is not a neat reduct.

The Boolean structure of both algebras are in fact very simple. For a set \( X \), let \( \text{Cof}(X) \) denote the finite co-finite Boolean algebra on \( X \), that is \( \text{Cof}(X) \) has universe \( \{ a \in \wp(X) : |a| < \omega, \text{or } |X \sim a| < \omega \} \).

Let \( J \) be any set having the same cardinality as the signature of \( M \) so that \( J \) can simply be \( M \). Then \( \mathfrak{A} \cong \prod_{u \in 3^3} \mathfrak{A}_u; \mathfrak{A}_u = \text{Cof}(J) \) and \( \mathfrak{B} = \prod_{u \in 3^3} \mathfrak{B}_u \), where \( \mathfrak{B}_{1d} = \text{Cof}(N) \) (\( N \) can in fact be any set such that \( |N| < |J| \) but for definiteness let it be the least infinite cardinal) and otherwise \( \mathfrak{B}_u = \mathfrak{A}_u \).

We show that \( \exists \) has a winning strategy in an Ehrenfeucht–Fraïssé-game over \((\mathfrak{A}, \mathfrak{B})\). At any stage of the game, if \( \forall \) places a pebble on one of \( \mathfrak{A} \) or \( \mathfrak{B} \), \( \exists \) must place a matching pebble on the other algebra. Let \( \bar{a} = \langle a_0, a_1, \ldots, a_{n-1} \rangle \) be the position of the pebbles played so far (by either player) on \( \mathfrak{A} \) and let \( \bar{b} = \langle b_0, \ldots, b_{n-1} \rangle \) be the the position of the pebbles played on \( \mathfrak{B} \). \( \exists \) maintains the following properties throughout the game.
For any atom $x$ (of either algebra) with $x.1_{id} = 0$ then $x \in a_i \iff x \in b_i$.

$a$ induces a finite partition of $1_{id}$ in $A$ of $2^n$ (possibly empty) parts $p_i : i < 2^n$ and $b$ induces a partition of $1_{id}$ in $B$ of parts $q_i : i < 2^n$. $p_i$ is finite iff $q_i$ is finite and, in this case, $|p_i| = |q_i|$.

It is easy to see that $\exists$ can maintain these properties in every round. Therefore she can win the game. Therefore $A \equiv_\infty B$.

(3) We show that $\mathfrak{Nr}_n TCA_\omega$ is pseudo-elementary. This is similar to the proof of [20, theorem 21] using a three sorted first order theory, from which we can infer the elementary theory $\mathfrak{Nr}_n CA_\omega$ is recursively enumerable for any finite $n$.

To show that $\mathfrak{Nr}_n TCA_\omega$ is pseudo-elementary, we use a three sorted defining theory, with one sort for a topological cylindric algebra of dimension $n$ ($c$), the second sort for the Boolean reduct of a cylindric algebra ($b$) and the third sort for a set of dimensions ($\delta$); the argument is analogous to that of Hirsch used for relation algebra reducts [20, theorem 21]. We use superscripts $n, b, \delta$ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the cylindric algebra of dimension $n$, the Boolean part of the cylindric algebra or the dimension set, respectively. We do it for CA$s$. The other cases can be dealt with in exactly the same way.

The signature includes dimension sort constants $i^\delta$ for each $i < \omega$ to represent the dimensions. The defining theory for $\mathfrak{Nr}_n TCA_\omega$ includes sentences stipulating that the constants $i^\delta$ for $i < \omega$ are distinct and that the last two sorts define a cylindric algebra of dimension $\omega$. For example the sentence

$$\forall x^\delta, y^\delta, z^\delta (d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta), d^b(z^\delta, y^\delta)))$$

represents the cylindric algebra axiom $d_{ij} = c_k(d_{ik}, d_{kj})$ for all $i, j, k < \omega$. We have have a function $I^b$ from sort $c$ to sort $b$ and sentences requiring that $I^b$ be injective and to respect the $n$ dimensional cylindric operations as follows: for all $x^r$:

$$I^b(d_{ij}) = d^b(i^\delta, j^\delta),$$
$$I^b(c_i x^r) = c_i^b(I^b(x^r)),$$
$$I^b(\diamond_i x^r) = \diamond_i^b(I^b(x^r)).$$

Finally we require that $I^b$ maps onto the set of $n$ dimensional elements

$$\forall y^b((\forall z^\delta (z^\delta \neq 0^\delta, \ldots, (n-1)^\delta \rightarrow c^b(z^\delta, y^\delta) = y^b)) \leftrightarrow \exists x^r(y^b = I^b(x^r))).$$
In all cases, it is clear that any algebra of the right type is the first sort of a model of this theory. Conversely, a model for this theory will consist of an $n$ dimensional cylindric algebra type (sort c), and a cylindric algebra whose dimension is the cardinality of the $\delta$-sorted elements, which is at least $|m|$. Thus the three sorted theory defines the class of neat reduct, furthermore, it is clearly recursive.

Finally, if $TCA$ be a pseudo elementary class, that is $\mathbf{K} = \{ M^a | L : M \models U \}$ of $L$ structures, and $L, L^a, U$ are recursive. Then there a set of first order recursive theory $T$ in $L$, so that for any $\mathfrak{A}$ an $L$ structure, we have $\mathfrak{A} \models T$ iff there is a $\mathfrak{B} \in \mathbf{K}$ with $\mathfrak{A} \equiv \mathfrak{B}$. In other words, $T$ axiomatizes the closure of $\mathbf{K}$ under elementary equivalence, see [22, theorem 9.37] for unexplained notation and proof.

(4) That $S_c\mathfrak{M}_n TCA_\omega$ is not elementary follows from theorem 4.13. It is pseudo-elementary because of the following reasoning. For brevity, let $L = S_c\mathfrak{M}_n TCA_\omega \cap \text{At}$, and let $\text{CRK}_n$ denote the class of completely representable algebras. Let $T$ be the first order theory that axiomatizes $\text{UpUrCRK}_n = \text{LCR}_n$. It suffices to show, since $\text{CRK}_n$ is pseudo-elementary [22], that $T$ axiomatizes $\text{UpUrL}$ as well. First, note that $\text{CRA}_n \subseteq L$. Next assume that $\mathfrak{A} \in \text{UpUrL}$, then $\mathfrak{A}$ has a countable elementary (necessarily atomic) subalgebra in $L$ which is completely representable by the above argument, and we are done.

(5) We show that $S_c\mathfrak{M}_n TCA_\omega$ is not closed under forming subalgebras, hence it is not pseudo-universal. That it is closed under $S_c$ follows directly from the definition. Consider the $\text{Sc}$ reduct of either $\text{PEA}_{\omega,\omega}$ or $\text{PEA}_{\omega,\omega}$ or $\text{PEA}_{TCA_{\omega,\omega},TCA}$ of the previous item. Fix one of them, call it $\mathfrak{A}$. Because $\mathfrak{A}$ has countably many atoms, $\text{Im}\mathfrak{A} \subseteq \mathfrak{A} \subseteq \text{EmAt}\mathfrak{A}$, and all three are completely representable or all three not completely representable sharing the same atom structure $\text{At}\mathfrak{A}$, we can assume without loss that $\mathfrak{A}$ is countable. Now $\mathfrak{A}$ is not completely representable, hence by the above 'omitting types argument' in item (1), its $\text{Sc}$ reduct is not in $S_c\mathfrak{M}_n \text{Sc}_\omega$.

On the other hand, $\mathfrak{A}$ is strongly representable, so its canonical extension is representable, indeed completely representable, hence as claimed this class is not closed under forming subalgebras, because the $\text{Sc}$ reduct of the canonical extension of $\mathfrak{A}$ is in $S_c\mathfrak{M}_n \text{Sc}_\omega$, $\text{Re}_n \mathfrak{A}$ is not in $S_c\mathfrak{M}_n \text{Sc}_\omega$ and $\mathfrak{A}$ embeds into its canonical extension. The first of these statements follow from the fact that if $\mathfrak{D} \subseteq \mathfrak{M}_n \mathfrak{B}$, then $\mathfrak{D}^+ \subseteq \mathfrak{M}_n \mathfrak{B}^+$.

(6) Witness theorem 4.22. The rest is known.

(7) The algebra $\mathfrak{A}$ in item (1) in theorem 4.14 witnesses the strictness of
the stated last inclusion. The strictness of second inclusion from the fact that the class in question coincides with the class of algebras satisfying the Lyndon conditions, and we have proved in that this class this is properly contained in even the class of (strongly) representable algebras.

Indeed, let $\Gamma$ be any graph with infinite chromatic number, and large enough finite girth. Let $\rho_k$ be the $k$ the Lyndon condition for $D_f$s. Let $m$ be also large enough so that any 3 colouring of the edges of a complete graph of size $m$ must contain a monochromatic triangle; this $m$ exists by Ramsey’s theorem. Then $\mathcal{M}(\Gamma)$, the complex algebra constructed on $\Gamma$, as defined in [22, definition sec 6.3, p.78] will be representable as a polyadic equality algebra but it will fail $\rho_k$ for all $k \geq m$. The idea is that $\forall$ can win in the $m$ rounded atomic game coded by $\sigma_m$, by forcing a forbidden monochromatic triangle.

The last required from the fact that the resulting classes coincide with $S\mathcal{N}r_\omega TCA_\omega$ which, in turn, coincides with the class of representable algebras by the neat embedding theorem of Henkin.

\[ \square \]

4.8 Neat embeddings for infinite dimensional algebras

**Theorem 4.35.** Let $K$ be any class between $S\mathcal{C}$ and $PEA$. Let $\alpha$ be an infinite ordinal. Then the following hold.

(1) Assume that for any $r \in \omega$ and $3 \leq m \leq n < \omega$, there is an algebra $\mathcal{C}(m,n,r) \in \mathcal{N}r_m PEA_n$, with $R_d sc \mathcal{C}(m,n,r) \notin S\mathcal{N}r_m Sc_{n+1}$ and $\Pi_{r/\mathcal{C}}(m,n,r) \in RPEA_m$. Furthermore, assume that if $3 \leq m < n$, $k \geq 1$ is finite, and $r \in \omega$, there exists $x_n \in \mathcal{C}(n,n+k,r)$ such that $\mathcal{C}(m,m+k,r) \equiv \mathcal{R}_x \mathcal{C}(n,n+k,r)$ and $c_i x_n \cdot c_j x_n = x_n$ for all $i, j < m$. Then for any any $r \in \omega$, for any finite $k \geq 1$, for any $l \geq k + 1$ (possibly infinite), there exist $\mathcal{B}^r \in \mathcal{N}r_\alpha QEA_{\alpha+k}$, $R_d sc \mathcal{B}^r \notin S\mathcal{N}r_\alpha Sc_{\alpha+k+1}$ such $\Pi_{r \in \omega} \mathcal{B}^r \in \mathcal{N}r_\alpha QEA_{\alpha+l}$. In particular, the same result holds for topological cylindric algebras.

(2) Let $k \geq 1$. Assume that for each finite $m \geq 3$, there exists $\mathcal{C}(m) \in TCA_m$ such that $R_d sc \mathcal{C}(m) \notin S\mathcal{N}r_m Sc_{m+k}$ and $\mathcal{A}(m) \in \mathcal{N}r_m TCA_\omega$ such that $\mathcal{A}(m) \equiv \mathcal{C}(m)$. Furthermore, assume that for $3 \leq m < n < \omega$, $\mathcal{C}(m) \subseteq c R_d m \mathcal{C}(n)$. Then any class $\mathcal{L}$ between $\mathcal{N}r_\alpha K_{\omega+\omega}$ and $S\mathcal{C} \mathcal{N}r_\alpha K_{\alpha+k}$ is not elementary.

(3) Assume that for each finite $m \geq 3$, there exists $\mathcal{C}(m) \in \mathcal{N}r_k TCA_\omega$ whose $D_f$ reduct is not completely representable such that for $3 \leq m < n < \omega$, there exists $x_m \in \mathcal{C}(n)$ such that $c_j x_m c_j x_m = x_k$ for all $j < m$ and
\[ \mathcal{C}(k) \cong \mathcal{R}_d \mathcal{A}_k \mathcal{C}(m). \] Then there exists an algebra \( \mathcal{A} \) in and \( \mathcal{N}_{\alpha \text{TCA}_{\alpha+\omega}} \)
whose SC reduct is not completely representable.

**Proof.**

1. Fix such \( r \). Let \( I = \{ \Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega \} \). For each \( \Gamma \in I \), let \( M_{\Gamma} = \{ \Delta \in I : \Gamma \subseteq \Delta \} \), and let \( F \) be an ultrafilter on \( I \) such that \( \forall \Gamma \in I, M_{\Gamma} \in F \). For each \( \Gamma \in I \), let \( \rho_{\Gamma} \) be a one to one function from \( |\Gamma| \) onto \( \Gamma \).

Let \( \mathcal{C}_{\Gamma} \) be an algebra similar to \( \text{QEA}_\alpha \) such that

\[ \mathcal{R}_d \mathcal{C}_{\Gamma} = \mathcal{C}(|\Gamma|, |\Gamma| + k, r). \]

Let

\[ \mathcal{B}^r = \Pi_{\Gamma \in F} \mathcal{C}_{\Gamma}. \]

We will prove that

1. \( \mathcal{B}^r \in \text{S} \mathcal{N}_{\alpha \text{QEA}_{\alpha+k}} \) and
2. \( \mathcal{R}_d \mathcal{B}^r \not\in \text{S} \mathcal{N}_{\alpha \text{SC}_{\alpha+k+1}} \).

For the first part, for each \( \Gamma \in I \) we know that \( \mathcal{C}(|\Gamma| + k, |\Gamma| + k, r) \in \text{K}_{|\Gamma| + k} \) and \( \mathcal{N}_{|\Gamma|} \mathcal{C}(|\Gamma| + k, |\Gamma| + k, r) \cong \mathcal{C}(|\Gamma|, |\Gamma| + k, r) \). Let \( \sigma_{\Gamma} \) be a one to one function \( (|\Gamma| + k) \to (\alpha + k) \) such that \( \rho_{\Gamma} \subseteq \sigma_{\Gamma} \) and \( \sigma_{\Gamma}(|\Gamma| + i) = \alpha + i \) for every \( i < k \). Let \( \mathcal{A}_{\Gamma} \) be an algebra similar to \( \text{CA}_{\alpha+k} \) such that \( \mathcal{R}_d \sigma_{\Gamma} \mathcal{A}_{\Gamma} = \mathcal{C}(|\Gamma| + k, |\Gamma| + k, r) \). We claim that \( \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma} \in \text{QEA}_{\alpha+k} \).

For this it suffices to prove that each of the defining axioms for \( \text{QEA}_{\alpha+k} \) hold for \( \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma} \). Let \( \sigma = \tau \) be one of the defining equations for \( \text{QEA}_{\alpha+k} \), and we assume to simplify notation that the number of dimension variables is one. Let \( i \in \alpha + k \), we must prove that \( \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma} \models \sigma(i) = \tau(i) \).

If \( i \in \text{rng}(\rho_{\Gamma}) \), say \( i = \rho_{\Gamma}(i_0) \), then \( \mathcal{R}_d \sigma_{\Gamma} \mathcal{A}_{\Gamma} \models \sigma(i_0) = \tau(i_0) \), since \( \mathcal{R}_d \sigma_{\Gamma} \mathcal{A}_{\Gamma} \in \text{QEA}_{|\Gamma|+k} \), so \( \mathcal{A}_{\Gamma} \models \sigma(i) = \tau(i) \). Hence \( \{ \Gamma \in I : \mathcal{A}_{\Gamma} \models \sigma(i) = \tau(i) \} \supseteq \{ \Gamma \in I : i \in \text{rng}(\rho_{\Gamma}) \} \in F \), hence \( \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma} \models \sigma(i) = \tau(i) \). Thus, as claimed, we have \( \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma} \in \text{QEA}_{\alpha+k} \).

We prove that \( \mathcal{B}^r \subseteq \mathcal{N}_{\alpha \Pi_{\Gamma \in F} \mathcal{A}_{\Gamma}} \). Recall that \( \mathcal{B}^r = \Pi_{\Gamma \in F} \mathcal{C}_{\Gamma}^r \) and note that \( \mathcal{C}_{\Gamma}^r \subseteq \mathcal{A}_{\Gamma} \) (the universe of \( \mathcal{C}_{\Gamma}^r \) is \( C(|\Gamma|, |\Gamma| + k, r) \), the universe of \( \mathcal{A}_{\Gamma} \) is \( C(|\Gamma| + k, |\Gamma| + k, r) \)). So, for each \( \Gamma \in I \),

\[
\mathcal{R}_d \sigma_{\Gamma} \mathcal{C}_{\Gamma}^r = \mathcal{C}(|\Gamma|, |\Gamma| + k, r) \\
\cong \mathcal{N}_{\Gamma} \mathcal{C}(|\Gamma| + k, |\Gamma| + k, r) \\
= \mathcal{N}_{\Gamma} \mathcal{R}_d \sigma_{\Gamma} \mathcal{A}_{\Gamma} \\
= \mathcal{R}_d \sigma_{\Gamma} \mathcal{N}_{\Gamma} \mathcal{A}_{\Gamma} \\
= \mathcal{R}_d \sigma_{\Gamma} \mathcal{N}_{\Gamma} \mathcal{A}_{\Gamma}
\]

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Thus (using a standard Los argument) we have: \( \Pi_{\Gamma/F} C_r^\Gamma \cong \Pi_{\Gamma/F} \mathfrak{N}_\Gamma \mathfrak{A}_\Gamma = \mathfrak{N}_\alpha \Pi_{\Gamma/F} \mathfrak{A}_\Gamma \), proving (1).

The above isomorphism \( \cong \) follows from the following reasoning. Let \( \mathcal{B}_\Gamma = \mathfrak{N}_\Gamma \mathfrak{A}_\Gamma \). Then universe of the \( \Pi_{\Gamma/F} C_r^\Gamma \) is identical to that of \( \Pi_{\Gamma/F} \mathfrak{N}_\alpha \mathfrak{C}_r^\Gamma \), which is identical to the universe of \( \Pi_{\Gamma/F} \mathcal{B}_\Gamma \). Each operator \( o \) of \( \mathfrak{QEA}_\alpha \) is the same for both ultraproducts because \( \{ \Gamma \in I : \dim(o) \subseteq \text{rng}(\rho_\Gamma) \} \in F \).

Now we prove (2). For this assume, seeking a contradiction, that \( \mathfrak{N}_\lambda \mathfrak{B}^\Gamma \subseteq \mathfrak{N}_\alpha \mathfrak{C} \), where \( \mathfrak{C} \in \mathfrak{Sc}_{\alpha+k+1} \). Let \( 3 \leq m < \omega \) and \( \lambda : m + k + 1 \to \alpha + k + 1 \) be the function defined by \( \lambda(i) = i \) for \( i < m \) and \( \lambda(m + i) = \alpha + i \) for \( i < k + 1 \). Then \( \mathfrak{N}_\lambda(\mathfrak{C}) \in \mathfrak{Sc}_{m+k+1} \) and \( \mathfrak{N}_\lambda \mathfrak{B}^\Gamma \subseteq \mathfrak{N}_\alpha \mathfrak{N}_\lambda(\mathfrak{C}) \).

For each \( \Gamma \in I \), let \( I|\Gamma \) be an isomorphism 
\[ \mathfrak{C}(m, m + k, r) \cong \mathfrak{N}_{x|\Gamma} \mathfrak{N}_m \mathfrak{C}(|\Gamma|, |\Gamma| + k, r). \]

Let \( x = (x|\Gamma) \) be an isomorphism from \( \mathfrak{C}(m, m + k, r) \) into \( \mathfrak{N}_{x|\Gamma} \mathfrak{N}_m \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \). Then by [18] theorem 2.6.38 we have \( \mathfrak{N}_{x|\Gamma} \mathfrak{N}_m \mathfrak{B}^\Gamma \subseteq \mathfrak{N}_\alpha \mathfrak{N}_m \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \). It follows that \( \mathfrak{N}_\lambda \mathfrak{C}(m, m + k, r) \subseteq \mathfrak{N}_\alpha \mathfrak{N}_m \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \) which is a contradiction and we are done.

Now we prove the third part of the theorem, putting the superscript \( r \) to use. Let \( k \) be as before; \( k \) is finite and \( \beta > 0 \) and let \( l \) be as in the hypothesis of the theorem, that is, \( l \geq k + 1 \), and we can assume without loss that \( l \leq \omega \). Recall that \( \mathfrak{B}^\Gamma = \Pi_{\Gamma/F} C_r^\Gamma \), where \( C_r^\Gamma \) has the signature of \( \mathfrak{QEA}_\alpha \) and \( \mathfrak{N}_\lambda \mathfrak{C}_r^\Gamma = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \). We know (this is the main novelty here) from item (2) that \( \Pi_{r/U} \mathfrak{N}_\lambda \mathfrak{C}_r^\Gamma = \Pi_{r/U} \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \subseteq \mathfrak{N}_{r|\Gamma} \mathfrak{A}_\Gamma \), for some \( \mathfrak{A} \in \mathfrak{QEA}_{|\Gamma|+\omega} \).

Let \( \lambda_\Gamma : |\Gamma| + k + 1 \to \alpha + k + 1 \) extend \( \rho_\Gamma : |\Gamma| \to \Gamma \) to a \( \alpha \) and satisfy 
\[ \lambda_\Gamma(|\Gamma| + i) = \alpha + i \]
for \( i < k + 1 \). Let \( \mathfrak{F}_\Gamma \) be a \( \mathfrak{QEA}_{\alpha+l} \) type algebra such that \( \mathfrak{N}_\lambda \mathfrak{F}_\Gamma = \mathfrak{N}_\lambda \mathfrak{A}_\Gamma \). As before, \( \Pi_{\Gamma/F} \mathfrak{F}_\Gamma \in \mathfrak{QEA}_{\alpha+l} \). And 
\[ \Pi_{r/U} \mathfrak{B}^\Gamma = \Pi_{r/U} \Pi_{\Gamma/F} C_r^\Gamma \]
\[ \cong \Pi_{\Gamma/F} \Pi_{r/U} C_r^\Gamma \]
\[ \subseteq \Pi_{\Gamma/F} \mathfrak{N}_{r|\Gamma} \mathfrak{A}_\Gamma \]
\[ = \Pi_{\Gamma/F} \mathfrak{N}_{r|\Gamma} \mathfrak{N}_\lambda \mathfrak{F}_\Gamma \]
\[ \subseteq \mathfrak{N}_\alpha \Pi_{\Gamma/F} \mathfrak{F}_\Gamma, \]

But \( \mathcal{B} = \Pi_{r/U} \mathfrak{B}^\Gamma \in \mathfrak{SMr}_\alpha \mathfrak{QEA}_{\alpha+l} \) because \( \mathfrak{F} = \Pi_{\Gamma/F} \mathfrak{F}_\Gamma \in \mathfrak{QEA}_{\alpha+l} \) and \( \mathcal{B} \subseteq \mathfrak{N}_\alpha \mathfrak{F} \).
(2) For \(k \geq 1\). For finite \(m > 2\), let \(\mathcal{C}(m), \mathfrak{A}(m)\) be the atomic algebras in \(\text{TCA}_m\) such that \(\mathfrak{R}_{sc}\mathcal{C}(m) \notin \mathcal{S}_c\mathcal{N}_m\mathcal{S}_{c_m+k}, \mathfrak{A}(m) \in \mathcal{N}_k\text{TCA}_m\) and \(\mathcal{C}(m) \equiv \mathfrak{A}(m)\) as in the hypothesis. For \(m < n\) we also have \(\mathcal{C}(m) \subseteq c \mathfrak{R}_m\mathcal{C}(n)\).

Let \(\alpha\) be an infinite ordinal. Then we claim that there exists \(\mathcal{B} \in \text{TCA}_\alpha\) such that \(\mathfrak{R}_{sc}\mathcal{B} \notin \mathcal{S}_c\mathcal{N}_\alpha\mathcal{S}_{c_\alpha+k}\) and \(\mathfrak{A} \in \mathcal{N}_\alpha\text{TCA}_{\alpha+\omega}\), such that \(\mathfrak{A} \equiv \mathcal{B}\).

Now we use the same lifting argument as before, we only fix finite \(m > 2\).

We do not have the parameter \(r\). Let \(I = \{\Gamma : m \leq \Gamma \leq \alpha, |\Gamma| < \omega\}\). For each \(\Gamma \in I\), let \(M_\Gamma = \{\Delta \in I : \Delta \subseteq \Delta\}\), and let \(F\) be an ultrafilter on \(I\) such that \(\forall \Gamma \in I, M_\Gamma \subseteq F\). For each \(\Gamma \in I\), let \(\rho_\Gamma\) be a one to one function from \(|\Gamma|\) onto \(\Gamma\). Let \(\mathcal{C}_\Gamma\) be an algebra similar to \(\text{TCA}_\alpha\), such that \(\mathfrak{R}_{sc}\mathcal{C}_\Gamma = \mathcal{C}(|\Gamma|)\). In particular, \(\mathcal{C}_\Gamma\) has an atomic Boolean reduct. Let \(\mathcal{B} = \prod_{\Gamma/F \in I} \mathcal{C}_\Gamma\).

We claim that \(\mathfrak{R}_{sc}\mathcal{B} \notin \mathcal{S}_c\mathcal{N}_\alpha\mathcal{S}_{c_\alpha+k}\). For assume, seeking a contradiction, that \(\mathfrak{R}_{sc}\mathcal{B} \in \mathcal{S}_c\mathcal{N}_\alpha\mathcal{S}_{c_\alpha+k}\). Let \(3 \leq m < \omega\) and \(\lambda : m + k \to \alpha + k\) be the function defined by \(\lambda(i) = i\) for \(i < m\) and \(\lambda(m + i) = \alpha + i\) for \(i < k\). Then \(\mathfrak{R}_{sc}\mathcal{C} \in \mathcal{P}_{\mathcal{E}_{m+k}}\) and \(\mathfrak{R}_m\mathcal{B} \subseteq c \mathfrak{R}_m\mathfrak{R}_{sc}\mathcal{C}\).

For each \(\Gamma \in I\), \(|\Gamma| \geq m\), we have

\[\mathcal{C}(m) \subseteq c \mathfrak{R}_m\mathcal{C}(|\Gamma|)\].

Let \(I_{|\Gamma|}\) be an injective complete homomorphism, witnessing this complete embedding. Let \(x = (x_{|\Gamma|} : \Gamma)/F\) and let \(\iota(b) = (I_{|\Gamma|}b : \Gamma)/F\) for \(b \in \mathcal{C}(m)\). Then \(\iota\) is an injective homomorphism that embeds \(\mathcal{C}(m)\) into \(\mathfrak{R}_m\mathcal{B}r\), and this embedding is complete. Now \(\mathfrak{R}_m\mathcal{B}r \in \mathcal{S}_c\mathcal{N}_m\mathcal{S}_{c_m+k}\), hence \(\mathfrak{R}_{sc}\mathcal{C}(m) \in \mathcal{S}_c\mathcal{N}_m\mathcal{S}_{c_m+k}\) which is a contradiction and we are done.

We now show that there exists \(\mathfrak{A} \in \mathcal{N}_\alpha\text{TCA}_{\alpha+\omega}\) such that \(\mathfrak{A} \equiv \mathcal{B}\). We use the \(\mathfrak{A}(k)\)s. For each \(\Gamma \in I\) we can assume that \(\mathfrak{N}_{I_{|\Gamma|}}\mathfrak{A}(|\Gamma| + k) \equiv \mathfrak{A}(|\Gamma|)\); if not then replace \(\mathfrak{A}(|\Gamma| + k)\) by an algebra \(\mathcal{D}(|\Gamma| + k)\) such that \(\mathfrak{N}_{I_{|\Gamma|}}\mathcal{D}(|\Gamma| + k) \equiv \mathfrak{A}(|\Gamma|)\). Such a \(\mathcal{D}\) obviously exists.

Let \(\sigma_\Gamma\) be an injective map \((|\Gamma| + \omega) \to (\alpha + \omega)\) such that \(\rho_\Gamma \subseteq \sigma_\Gamma\) and \(\sigma_\Gamma(|\Gamma| + i) = \alpha + i\) for every \(i < \omega\). Let \(\mathfrak{A}_\Gamma\) be an algebra similar to a \(\text{QEA}_{\alpha+\omega}\) such that \(\mathfrak{R}_{sc}\mathfrak{A}_\Gamma = \mathfrak{A}(|\Gamma| + k)\), and \(\mathfrak{N}_\alpha\mathfrak{A}_\Gamma \equiv \mathcal{C}_\Gamma\). Then \(\Pi_{I_{|\Gamma|}}\mathfrak{A}_\Gamma \in \mathcal{C}_\alpha_{\alpha+\omega}\).

We now prove that \(\mathfrak{A} = \mathfrak{N}_\alpha\Pi_{I_{|\Gamma|}}\mathfrak{A}_\Gamma\). Using the fact that neat reducts
commute with forming ultraproducts, for each $\Gamma \in I$, we have

$$R^\rho_{\Gamma} A_\Gamma = A(|\Gamma|)$$

$$\cong A_{|\Gamma|} A(|\Gamma| + k)$$

$$= A_{|\Gamma|} R^\sigma_{\Gamma} A_\Gamma$$

$$= R^\rho_{\Gamma} A_{|\Gamma|} A_\Gamma$$

$$= R^\rho_{\Gamma} A_{|\Gamma|}$$

We deduce that

$$A = \Pi_{\Gamma/F} C_\Gamma \cong \Pi_{\Gamma/F} A_{|\Gamma|} = A_{|\Gamma|} \Pi_{\Gamma/F} A_\Gamma \in A_{|\Gamma|} \text{QEA}_{\alpha+\omega}.$$  

Finally, we have $A \equiv B$, because $A = A_{|\Gamma|} \Pi_{\Gamma/F} A_\Gamma = \Pi_{\Gamma/F} A_{|\Gamma|} A_\Gamma$. $B = \Pi_{\Gamma/F} C_\Gamma$, and we chose $A_{|\Gamma|} A_\Gamma$ to be elementary equivalent to $C_\Gamma$ for each finite subset $\Gamma$ of $\alpha$.

(3) Let $I$ and $C(\Gamma)$ be defined as above for every $\Gamma$ finite subset of $\alpha$. For each $\Gamma \in I$, let $\rho_{\Gamma}$ be a one to one function from $|\Gamma|$ onto $\Gamma$. Let $C_\Gamma$ be an algebra similar to $\text{QEA}_\alpha$ such that $R^\rho_{\Gamma} C_\Gamma = C(|\Gamma|)$. In particular, $C_\Gamma$ has an atomic Boolean reduct. Let $B = \Pi_{\Gamma/F} C_\Gamma$. Then $B$ is atomic, because it is an ultraproduct of atomic algebras. We claim that $R_{sc} B$ is not completely representable.

Assume for contradiction that $R_{sc} B$ is completely representable, with complete representation $f$. Let $3 \leq m < \omega$. Then of course $R_{sc} R_{md} B$ is completely representable with the same $f$; notice that both $R_{sc} B$ and $R_{md} R_{sc} B$ have the same universe as $B$.

For each $\Gamma \in I$, let $I_{|\Gamma|$ be an isomorphism $C(m) \cong R_{|\Gamma|} R_{md} C(|\Gamma|)$. Let $x = (x_{|\Gamma|} : \Gamma)/F$ and let $\iota(b) = (I_{|\Gamma|} b : \Gamma)/F$ for $b \in C(m)$. Then $\iota$ is an isomorphism from $C(m)$ into $R_{md} R_{sc} B'$. Then by [13, theorem 2.6.38] we have $R_{sc} R_{md} B \in \text{PEA}_m$ and it is atomic. Indeed, if $a \leq x$ is non-zero, then there is an atom $c \in R_{md} B$ below $a$, so that $c \leq a \leq x$, hence $c \in R_{md} R_{sc} B$ is an atom. Atomicity is not enough to guarantee complete representability because the class of completely representable algebras is not axiomatizable even for $\text{Dfs}$, but as it happens we do have that $R_{sc} R_{md} R_{sc} B$ is a completely representable $\text{Sc}_m$, as we proceed to show. First of all, we know that $R_{md} R_{df} B$ is completely representable, so let $f : R_{md} R_{sc} B \rightarrow \varphi(V)$ be a complete representation; that is a representation that preserves joins. Here $\text{rng}(f)$ is a generalized set algebra; that is $V$, a generalized space, is of the form $\bigcup_{i \in I} \alpha U_i$, where for distinct $i$ and $j$, $U_i \cap U_j = \emptyset$.

Define $g : R_{md} R_{sc} B \rightarrow R_{f(x)} \varphi(V)$ by $g(b) = f(b)$ for $b \leq x$. We have $f(x) \subseteq V$, and $s^l_k x \cdot s^l_k x = x$ for all $l, k < m$, so this equation

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holds also for $f(x)$, that is, we have $s_k^i f(x) \cap s_k^j f(x) = f(x)$. By \cite{19} theorem 3.1.31, we have $f(x)$ is a generalized space, too, and indeed $\mathfrak{M}_f(x) \varphi(V) \cong \varphi(f(x))$.

Assume that $X \subseteq \mathfrak{M}_x \mathfrak{B} = \mathfrak{M}_x \mathfrak{R}_m \mathfrak{R}_sc \mathfrak{B}$ is such that $\sum X = x$, then

$$g(\sum X) = f(\sum X) = \bigcup_{y \in X} f(y) = f(x) = 1^\psi(f(x)).$$

Hence $g$ is a complete representation of $\mathfrak{M}_x \mathfrak{R}_m \mathfrak{R}_sc \mathfrak{B}$. The latter is isomorphic to $\mathfrak{R}_sc C(m)$, hence $\mathfrak{R}_sc C(m)$ is completely representable, too. This is a contradiction, and we are done.

Finally, $\mathfrak{B} \in \mathfrak{N}_\alpha TCA_{\alpha+\omega}$ exactly as above.

The second part follows from the second item in theorem \ref{4.14}.

Let us see how close we are to the hypothesis addressing finite dimensional algebras.

(1) For the first item for CA and PEA we have $\mathcal{C}(m, n, r) = \mathcal{C}(H^{n+1}_m(\mathfrak{A}(n, r), \omega))$, consisting of all $n+1$-wide $m$-dimensional wide $\omega$ hypernetworks \cite{22} definition 12.21] on $\mathfrak{A}(n, r)$, is in CA$_m$ and it can be easily expanded to a PEA$_m$.

Furthermore, for any $r \in \omega$ and $3 \leq m \leq n < \omega$, we have $\mathcal{C}(m, n, r) \in \mathfrak{N}_m PEA_n$. $\mathfrak{R}_{ca} C(m, n, r) \notin S\mathfrak{N}_m CA_{n+1}$ and $\Pi_\nu \mathcal{C}(m, n, r) \in RPEA_m$.

Lastly, let $3 \leq m < n$. Take

$$x_n = \{f : \leq n+k+1 \rightarrow \text{At}\mathfrak{A}(n+k,r) \cup \omega : m \leq j < n \rightarrow \exists i < m, f(i,j) = Id\}.$$

Then $x_n \in C(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$.

Furthermore

$$I_n : \mathcal{C}(m, m+k, r) \cong \mathfrak{M}_x \mathfrak{R}_m \mathfrak{C}(n, n+k, r),$$

via the map, defined for $S \subseteq H^{m+k+1}_m(\mathfrak{A}(m+k,r)\omega)$, by

$$I_n(S) = \{f : \leq n+k+1 \rightarrow \text{At}\mathfrak{A}(n+k,r) \cup \omega : m \leq j < n \rightarrow \exists i < m, f(i,j) = Id\}.$$

We have shown that we have all conditions in the hypothesis of the first item of theorem \ref{4.35} for cylindric and polyadic equality algebras.

However for Sc and QA the problem remains open. The algebras constructed in \cite{27} and recalled in theorem \ref{4.20} for another purpose, do not have representable ultraproducts, so the diagonal free cases in this stronger form is not yet confirmed.
(2) By theorem 4.13 there exist for every finite \( m > 2 \), polyadic equality atomic algebras \( \mathcal{C}(m) = \mathrm{PEA}_{\mathbb{N}^{-1}, \mathbb{N}} \) and \( \mathcal{B}(m) \) such that \( \mathfrak{R}_{\mathfrak{s}} \mathcal{C}(m) \notin \mathfrak{S}_c \mathfrak{R}_{\mathfrak{n}} \mathfrak{S}_{\mathfrak{c}m+3} \), \( \mathcal{C}(m) \equiv \mathcal{B}(m) \), \( \mathcal{B} \) is a countable (atomic) completely representable \( \mathrm{TCA}_m \) except that we could only succeed to show that \( \mathfrak{A}_\mathfrak{t} \mathcal{B}(m) \in \mathfrak{A}_\mathfrak{t} \mathfrak{R}_\mathfrak{m} \mathrm{TCA}_\omega \); we do not know whether we can remove \( \mathfrak{A}_\mathfrak{t} \) from both sides of the equation; that is, we do not know whether \( \mathcal{B}(m) \) can be chosen to be in \( \mathfrak{R}_\mathfrak{m} \mathrm{QEA}_\omega \). A conditional theorem depending on a game \( H \), stronger than the game \( J \) used was given to ensure the latter condition, but it can be shown without too much difficulty that \( \forall \) can win this finite rounded game on \( \mathfrak{A}_\mathfrak{t} \mathrm{TCA}_{\mathbb{N}^{-1}, \mathbb{N}} \) as long as the number of rounds are \( > m \). Also we not know whether \( \mathcal{C}(m) \) embeds completely into \( \mathcal{C}(n) \) for \( n > m \). However, we tend to think that the situation is not hopeless and that such algebras can be found, but further research is needed.

(3) For the third part for \( k \geq 3 \), let \( \mathcal{C}(k) \in \mathfrak{R}_\mathfrak{k} \mathrm{TCA}_\omega \) be the atomic uncountable \( \mathrm{PEA}_k \) such that its \( \mathrm{Df} \) reduct is not completely representable constructed in the first item theorem 4.22. Recall that \( \mathcal{C}(k) \) was obtained as a \( k \) neat reduct of an \( \omega \) dimensional algebra whose atom structure is an \( \omega \) basis over a relation algebra, denoted by \( \mathfrak{R} \), in the proof of the first item in theorem 4.22. Hence an atom in \( \mathcal{C}(k) \) is of the form \( \{N\} \), where \( N : \omega \times \omega \to \mathfrak{A}_\mathfrak{t} \mathfrak{R} \) is an \( \omega \) dimensional network. For \( 3 \leq m < n \), let

\[
x_n = \{N \in C(n) : m \leq j < n \to \exists i < m, N(i, j) = \mathrm{Id}\}.
\]

Then \( x_n \in C(n) \) and \( c_i x_n \cdot c_j x_n = x_n \) for distinct \( i, j < m \). Furthermore,

\[
I : \mathcal{C}(m) \cong \mathfrak{R}_{x_n} \mathfrak{R}_{m} \mathcal{C}(n)
\]

via

\[
I(S) = \{N \in C(n) : N \upharpoonright \omega \times \omega \in S, \forall j(m \leq j < n \to \exists i < m, N(i, j) = \mathrm{Id}\}.
\]

This is similar to the definition in the first item above, for in the former case we have a hyperbasis and now we have an amalgamation class.

**Corollary 4.36.** Let \( \alpha \geq \omega \). Then the following hold:

(1) For any \( k \geq 0 \), and any \( l \geq k + 2 \) and , then the variety \( S\mathfrak{R}_\alpha \mathrm{TCA}_{\alpha+l} \) cannot be axiomatized by a finite schema over the variety \( S\mathfrak{R}_\alpha \mathrm{K}_{\alpha+k} \)

(2) There are algebras in \( \mathfrak{R}_\alpha \mathrm{TCA}_{\alpha+\omega} \) that are not completely representable.

**Proof.** The second part follows immediately from the first item of theorem 4.22 and the third of the previous theorem 4.35.
For the second part let us first recall from [19] what we mean by finite schema. If \( \rho : \omega \to \alpha \) is an injection, then \( \rho \) extends recursively to a function \( \rho^+ \) from \( \text{CA}_\omega \) terms to \( \text{CA}_\alpha \) terms. On variables \( \rho^+(v_k) = v_k \), and for compound terms like \( c_k \tau \), where \( \tau \) is a \( \text{CA}_\omega \) term, and \( k < \omega \), \( \rho^+(c_k \tau) = c_{\rho(k)} \rho^+(\tau) \). For an equation \( e \) of the form \( \sigma = \tau \) in the language of \( \text{CA}_\omega \), \( \rho^+(e) \) is the equation \( \rho^+(\tau) = \rho^+(\sigma) \) in the language of \( \text{CA}_\alpha \). This last equation, namely, \( \rho^+(e) \) is called an \( \alpha \) instance of \( e \) obtained by applying the injection \( \rho \).

Let \( k \geq 1 \) and \( l \geq k + 1 \). Assume for contradiction that \( \text{SNr}_\alpha \text{TCA}_{\alpha+l} \) is axiomatizable by a finite schema over \( \text{SNr}_\alpha \text{TCA}_{\alpha+k} \). We can assume that there is only one equation, such that all its \( \alpha \) instances, axiomatize \( \text{SNr}_\alpha \text{TCA}_{\alpha+l} \) over \( \text{SNr}_\alpha \text{TCA}_{\alpha+k} \). So let \( \sigma \) be such an equation in the signature of \( \text{TCA}_\omega \) and let \( E \) be its \( \alpha \) instances; so that for any \( \mathfrak{A} \in \text{SNr}_\alpha \text{TCA}_{\alpha+k} \) we have \( \mathfrak{A} \models \mathcal{L} \text{TCA}_{\alpha+l} \) iff \( \mathfrak{A} \models E \). Then for all \( r \in \omega \), there is an instance of \( \sigma \), \( \sigma_r \), say, such that \( \mathcal{B}^r \) does not model \( \sigma_r \). \( \sigma_r \) is obtained from \( \sigma \) by some injective map \( \mu_r : \omega \to \alpha \).

For \( r \in \omega \), let \( v_r \in a^\alpha \), be an injection such that \( \mu_r(i) = v_r(i) \) for \( i \in \text{ind}(\sigma_r) \), and let \( \mathfrak{A}_r = \mathcal{A}_B^v \mathcal{B}^r \). Now \( \Pi_{r/v} \mathfrak{A}_r \models \sigma \). But then

\[
\{ r \in \omega : \mathfrak{A}_r \models \sigma \} = \{ r \in \omega : \mathcal{B}^r \models \sigma_r \} \subseteq U,
\]

contradicting that \( \mathcal{B}^r \) does not model \( \sigma_r \) for all \( r \in \omega \). \( \square \)

### 4.9 Gripping atom structures

Example 4.14 prompts the following definition. \( \mathfrak{A}, \mathcal{B} \in \mathcal{K} \subseteq \text{TCA}_n \) are atomic, then of course \( \text{At}\mathfrak{A} \) abd \( \text{At}\mathcal{B} \) is in \( \text{At}\mathcal{K} \). The next definition addresses the converse of this. There are classes of algebras \( \mathcal{K} \) such that there are two atomic algebras having the same atom structure, one in \( \mathcal{K} \) and the other is not, examplea are classes \( \text{TRCA}_n \) for finite \( n > 2 \) theorem 4.16 and the class \( \text{Mtr}_n \text{CA}_m \) for all \( m > n \), examples 4.11 and 4.18.

Conversely, there are classes of algebras, like the class of completely representable algebras in any finite dimension, and also \( \text{S}_{\omega} \text{Mtr}_n \text{CA}_m \) for any \( m > n > 2 \) finite, that do not have this property. This is not marked by first order definability, for the class of neat reducts and the last two classes not elementary, while \( \text{TRCA}_n \) is a variety for any \( n \).

**Definition 4.37.**  
(1) A class \( \mathcal{K} \) is gripped by its atom structures, if whenever \( \mathfrak{A} \in \mathcal{K} \cap \text{At} \), and \( \mathcal{B} \) is atomic such that \( \text{At}\mathcal{B} = \text{At}\mathfrak{A} \), then \( \mathcal{B} \in \mathcal{K} \).

(2) A class \( \mathcal{K} \) is strongly gripped by its atom structures, if whenever \( \mathfrak{A} \in \mathcal{K} \cap \text{At} \), and \( \mathcal{B} \) is atomic such that \( \text{At}\mathcal{B} \equiv \text{At}\mathfrak{A} \), then \( \mathcal{B} \in \mathcal{K} \).

(3) A class \( \mathcal{K} \) of atom structures is infinitary gripped by its atom structures if whenever \( \mathfrak{A} \in \mathcal{K} \cap \text{At} \) and \( \mathcal{B} \) is atomic, such that \( \text{At}\mathcal{B} \equiv_{\omega} \text{At}\mathfrak{A} \), then \( \mathcal{B} \in \mathcal{K} \).
An atomic game is strongly gripping for $K$ if whenever $\mathfrak{A}$ is atomic, and $\exists$ has a winning strategy for all finite rounded games on $\mathsf{At}\mathfrak{A}$, then $\mathfrak{A} \in K$.

An atomic game is gripping if whenever $\exists$ has a winning strategy in the $\omega$ rounded game on $\mathsf{At}\mathfrak{A}$, then $\mathfrak{A} \in K$.

Notice that infinitary gripped implies strongly gripped implies gripped (by its atom structures). For the sake of brevity, we write only (strongly) gripped, without referring to atom structures.

In the next theorem, all items except the first applies to all algebras considered.

**Theorem 4.38.**

1. The class of neat reducts for any dimension $> 1$ is not gripped, hence is neither strongly gripped nor infinitary gripped.

2. The class of completely representable algebras is gripped but not strongly gripped.

3. The class of algebras satisfying the Lyndon conditions is strongly gripped.

4. The class of representable algebras is not gripped.

5. The Lyndon usual atomic game is gripping but not strongly gripping for completely representable algebras, it is strongly gripping for $\mathsf{LCA}_m$, when $m > 2$.

6. There is a game, that is not atomic, that is strongly gripping for $\mathfrak{Nt}_n\mathsf{CA}_\omega$, call it $H$. In particular, if there exists an algebra $\mathfrak{A}$ and $k \geq 1$ such $\forall$ has a winning strategy in $F^{n+k}$ and $\exists$ has a winning strategy in $H_m$, the game $H$ truncated to $m$ rounds, for every finite $m$, then for any class $K$ such that $\mathfrak{Nt}_n\mathsf{CA}_\omega \subseteq K \subseteq S_c\mathfrak{Nt}_n\mathsf{CA}_{n+k}$, $K$ will not be elementary.

**Proof.**

1. Example 4.14

2. The algebra $\mathsf{TCA}_{Z,N}$ or $\mathsf{TCA}_{N-1,N}$ used in the proof of theorem 4.13 and its various reducts down to $S_c$s, shows that the class of completely representable algebras is not strongly gripped.

The former rainbow algebra based on the greens $Z$ and reds $N$ would also work just as well proving theorem 4.13, $\exists$ wins the $k$ rounded game $G_k$ is the same and the decreasing sequence in $N$ is forced on $\exists$ by $\forall$ 's moves using and reusing only $n + 3$ pebbles by exploiting the negative mirror image of $N$, namely, the set $\{-a : a \in N\} \subseteq Z$, so the game alternates between positive reds played by $\forall$ and negative greens played...
by $\exists$ synchronized by order preserving partial function inevitably forcing $\forall$ to play a decreasing sequence in $\mathbb{N}$. But this cannot go on for ever, hence $\forall$ wins the $\omega$ rounded game $F^{n+3}$ by pebbling successively $\mathbb{N}$ and its mirror image.

(3) This is straightforward from the definition of Lyndon conditions [22].

(4) Any weakly representable atom structure that is not strongly representable detects this, see e.g. the main results in [28] theorem 1.1, corolary 1.2, corolary 1.3], [38], [12] theorems 1.1, 1.2] and theorems 4.6.

(5) From item (2) and the rest follows directly from the definition.

(6) We allow more moves for $\forall$ in the neat game defined in remark 4.17 way above.

But nevertheless concerning the last item, we can say more. Indeed, more generally the following. Assume that If $\mathcal{L}$ is a class of algebras, $G$ be a game and $\mathfrak{A} \in \mathcal{L}$, then $\exists$ has a winning strategy in $G$. Assume also that $\mathfrak{N}_{tr_n}CA_\omega \subseteq \mathcal{L}$. If there is an atomic algebra $\mathfrak{A}$ such that $\exists$ can win all finite rounded games of $J$ and $\forall$ has a winning strategy in $G$ then any class between $\mathfrak{N}_{tr_n}CA_\omega$ and $\mathcal{L}$ is not elementary.

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