NewtonPlus: Approximate Relativity for Supernova Simulations

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Abstract

We propose an approximation to general relativity that captures the main gravitational effects of dynamical importance in supernovae. The conceptual link between this formalism and the Newtonian limit is such that it could likely be implemented in existing multidimensional Newtonian gravitational hydrodynamics codes employing a Poisson solver. As a test of the formalism’s utility, we display results for rapidly rotating (and therefore highly deformed) neutron stars.

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I. INTRODUCTION

The collapsed cores of massive stars are relativistic bodies. In addition to relativistic effects, convection and rapid rotation may play important roles in core collapse supernovae, making their accurate simulation a three-dimensional (3D) endeavor. Because accurate and stable numerical solutions to 3D relativistic problems are difficult to achieve, an approximation that captures the relativistic phenomena most relevant to supernovae is desirable. (More detailed discussion of these points, together with references, are given in the following paragraphs.)

While relativistic treatments would be necessary to follow the detailed processes leading to black hole formation, even collapse events leading to neutron stars ought to be treated relativistically, particularly at late times. Denoting the gravitational potential by $\Phi$, the Newtonian limit is obtained from the Einstein equations using the metric

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)d\mathbf{x}^2,$$

provided $\Phi \ll 1$ and velocities are much less than the speed of light. Taking $M$ and $R$ to be the mass and radius of the collapsed core, the gravitational potential is expected to be of order

$$\Phi \sim -\frac{M}{R} = -0.21 \left( \frac{M}{1.4 M_\odot} \right) \left( \frac{R}{10 \text{ km}} \right)^{-1},$$

while infall and/or ejecta velocities of order

$$v_{\text{ejecta}} \sim \sqrt{\frac{2M}{R}} = 0.45 \left( \frac{M}{1.4 M_\odot} \right)^{1/2} \left( \frac{R}{10 \text{ km}} \right)^{-1/2}$$

might be encountered. If some collapsed cores are born with rotation periods comparable to observed millisecond pulsars, equatorial velocities would be of order

$$v_{\text{equator}} = \Omega R = 0.21 \left( \frac{T_{\text{ms}}}{R} \right)^{-1} \left( \frac{R}{10 \text{ km}} \right).$$

Crudely taking the core as a cold degenerate gas of nucleons of mass $m_B$ and Fermi momentum $p_F$, nucleon velocities of order

$$v_{\text{nucleon}} \sim \frac{p_F}{m_B} = 0.69 \left( \frac{M}{1.4 M_\odot} \right)^{1/3} \left( \frac{R}{10 \text{ km}} \right)^{-1}$$

clearly indicate that pressure and internal energy density cannot be neglected as gravitational sources. The numerical coefficients in equations (2-5) may seem to overstate the case for relativity immediately after core bounce, when a less massive inner core has a larger radius. They are perhaps more relevant numbers for late times, when the final explosion energy \[1\] and remnant nature (neutron star vs. black hole) are determined. However, even at earlier times relativity cannot be ignored: detailed comparisons of Newtonian and relativistic simulations in spherical symmetry show more compact cores and higher neutrino luminosities and average energies in relativistic treatments \[2,3\]. Because of indications that
small (several percent) variations in, for example, neutrino luminosities and shock stagnation radius can make the difference between a successful (exploding) and a failed supernova, it is clear that even modest relativistic effects comprise an indispensable component of realism in supernova studies.

The multidimensional nature of supernovae must also be recognized as an important aspect of realism. Various observations, especially data from SN 1987A, point to the asphericity of supernova explosions (see e.g. for an overview, and for a recent polarimetry analysis of several supernovae). Convection—either deep in the core and driven by a lepton gradient or doubly-diffusive phenomena (“neutron fingers”), or in the more tenuous region between the neutrinosphere and the stalled shock, driven by an entropy gradient—has for some time been suggested as a means of boosting neutrino luminosities or increasing the efficiency of heating the material behind the stalled shock. The results of various simulations in up to two dimensions (2D) differ as to the significance of convective effects; clearly further study is necessary. Beyond convection, a pioneering 2D simulation studied jet production by magnetic fields in a supernova context, and simulations of explosions from collapsed stars with phenomenologically introduced jets have also been performed in 2D.

These 2D simulations have yielded interesting results, but ultimately consideration of the third dimension will be necessary. Based on an initial exploration of 3D effects, it has been reported that the sizes of convective cells in 3D simulations are about half as large as in 2D simulations. Moreover, rapid rotation can significantly affect the strength and spatial distribution of convection. Detailed studies of magnetic field generation, jet formation, and neutron star kicks also invite 3D treatments.

Including all the physics necessary for realism in a supernova simulation is a daunting task. The multidimensional simulations mentioned above, which had simplified neutrino transport, taxed the computational resources of their time; the same is true of recent simulations involving Boltzmann transport in spherical symmetry. Adding general relativity to the list of desired physics makes things all the more challenging. While numerical relativity has been successful in spherical and axisymmetric cases, “...in the general three-dimensional (3D) case which is needed for the simulation of realistic astrophysical systems, it has not been possible to obtain stable and accurate evolutions...,” and it is argued that the difficulties are more fundamental than insufficient resolution. While the difficulties with 3D numerical relativity are beginning to be overcome, the computations are “prohibitively slow” when the central gravitating body has a modest compactness ($M/R \lesssim 0.15$). Hence it will be some time before supernova simulations with sophisticated microphysics and transport can cover the full process of collapse, explosion, and wind formation over hundreds of milliseconds in a fully relativistic context.

In order to overcome the difficulties associated with 3D general relativistic simulations, and to save resources for 3D hydrodynamics and accurate neutrino transport, an approximate treatment of gravity that captures the phenomena of dynamical importance in supernovae would be desirable. The list of new gravitational phenomena introduced by relativity includes the nonlinearity of the gravitational field, the inclusion of all forms of energy and stress as sources, and gravitational waves. The first two of these effects are of dynamical importance in supernovae, while gravitational waves will probably not exert a strong back-reaction (unless bar or breakup instabilities of some sort become operative in the core).
Given the effects we wish to capture, what sort of approximation might we try? A common procedure in the literature has been to assume a spherical mass distribution in implementing gravity, which allows a relativistic treatment. However, for rapidly rotating progenitors this introduces errors of about 10% during collapse and bounce, with errors rising at later times as the nascent neutron star cools \[19\]. A multidimensional approach that captures relativistic phenomena could reduce these inaccuracies. A 3D post-Newtonian approach might be considered, but the post-Newtonian limit is known to be inadequate for quantitative determinations of neutron star properties. The case of spherical neutron stars constructed with a polytropic equation of state (EOS) with adiabatic index $\Gamma = 2$ (a common proxy for more realistic dense matter EOSs) is illustrative: when the polytropic constant is chosen to make the maximum mass in a fully relativistic calculation equal to $\sim 1.6 \, M_\odot$, the maximum mass determined by first-order post-Newtonian calculations is $\sim 2.5 \, M_\odot$—an error of close to 60% \[26\].

We conclude that an approach that both probes the nonlinear nature of gravity more deeply, and does so in multidimensions, would be desirable in the supernova context. A method that could be incorporated into existing Newtonian hydrodynamics codes would be even more useful. We describe such an approximation—which we call “NewtonPlus”—in §2, displaying a full set of Einstein equations in order to see where inconsistencies arise and how serious they are in the supernova environment. Hydrodynamics equations for the radial direction are presented in §3 in order to argue that the approach could be implemented in existing codes. In §4 we display results for rapidly rotating (and therefore highly deformed) stars, which show that this simple “NewtonPlus” approach to gravity is indeed a significant improvement over the Newtonian limit. Concluding remarks, including comments about certain approximate approaches to gravity employed in binary neutron star calculations, are given in §5. An appendix shows how the static limit of our formulation relates to the familiar equations of stellar structure in Schwarzschild coordinates.

**II. EINSTEIN EQUATIONS IN THE NEWTONPLUS APPROXIMATION**

As mentioned in §1, use of the metric of Eq. (1) in the Einstein equations yields the Newtonian limit, provided the gravitational potential $\Phi \ll 1$ and velocities (including microscopic velocities, large values of which lead to significant stresses) are much less than the speed of light. In order to capture the nonlinearity of gravity, the significance of stresses as gravitational sources, and relativistic fluid velocities, we propose the use of the following metric:

$$ds^2 = -e^{2\Phi+2\delta} dt^2 + e^{-2\Phi}(dr^2 + r^2 d\Omega^2),$$ (6)

where $d\Omega^2 \equiv d\theta^2 + r^2 \sin^2 \theta \, d\phi^2$. In comparison with Eq. (1), the “linearized” metric functions have been promoted to full exponentials, and a second metric function, $\delta$, has been added. Eq. (6) will then reduce to the Newtonian case if $\delta \to 0$; we shall see that this is in fact the case under conditions in which the Newtonian limit is valid. This provides a tight conceptual link with the Newtonian limit. Since it has two independent metric functions, this “NewtonPlus” metric also provides an exact solution in spherical symmetry. (For the static case, the Appendix shows the connection between our formulation and the familiar equations in Schwarzschild coordinates.)
A convenient formulation of the Einstein equations is the (3+1) formalism, in which spacetime is foliated into spacelike slices labeled by a time coordinate $t$. The metric can be expressed

$$ds'^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j,$$

where the latin indices run over the three spatial coordinates. As originally conceptualized, the four quantities $\alpha$ (the lapse function) and $\beta_i$ (the shift vector) could be chosen at will as a "gauge choice" or as "coordinate conditions," corresponding to the invariance of general relativity under coordinate transformations. Four constraint equations—the so-called Hamiltonian constraint and momentum constraints—would be solved to provide self-consistent data on the initial spacelike slice. The six degrees of freedom to evolve in time would then be the $\gamma_{ij}$, determined by equations second order in space and time. Alternatively, one could evolve equations first order in time for $\gamma_{ij}$ and $K_{ij}$; the latter is the extrinsic curvature tensor, which describes the embedding of the spacelike slices in spacetime. As an alternative to prescribing $\alpha$ and $\beta_i$, one can, for example, place conditions on $K_{ij}$; then $\alpha$ and $\beta_i$ become quantities for which solutions must be found.

It is apparent that the NewtonPlus metric of Eq. (6) contains only two of the six degrees of freedom that should be present. This means that examination of a complete set of Einstein equations should reveal inconsistencies; we here explore these and discuss their seriousness in the supernova environment.

We begin with the Hamiltonian constraint. This yields

$$\nabla^2 \Phi = 4\pi e^{-2\Phi} E + \frac{1}{2} (\partial \Phi)^2 - \frac{3}{2} e^{-4\Phi - 2\delta} (\partial_t \Phi)^2.$$

In this expression $\nabla^2$ is the usual 3D flat-space Laplacian. The energy density as viewed by an “Eulerian” observer (i.e., one whose 4-velocity is orthogonal to the spacelike slices, having covariant components $n_\mu = (-\alpha, 0, 0, 0)$, where $\alpha$ is the lapse function) is denoted by $E \equiv T^{\mu\nu} n_\mu n_\nu$, where $T^{\mu\nu}$ is the stress-energy tensor. For a perfect fluid, $E = \Gamma^2 (\rho + p) - p$, where $\Gamma = (1 - v^2)^{-1/2}$, $v$ is the magnitude of the physical fluid velocity as measured by an Eulerian observer, and $\rho$ and $p$ are respectively the total energy density and pressure in the fluid rest frame. We have employed the notation

$$\partial X \partial Y \equiv \partial_r X \partial_r Y + \frac{1}{r^2} \partial_\theta X \partial_\theta Y + \frac{1}{r^2 \sin^2 \theta} \partial_\phi X \partial_\phi Y.$$

As expected from the conceptual link between the Newtonian and NewtonPlus metrics, Eq. (8) identifies $\Phi$ as a glorified gravitational potential. In addition to the rest energy, the source includes internal and kinetic energies and pressure, all boosted by nonlinear contributions from $\Phi$ itself.

Next we turn to the momentum constraints. These can be summarized by

$$\nabla (e^{-\Phi - \delta} \partial_t \Phi) = -4\pi e^{-\Phi} s.$$

The three momentum constraint equations are obtained by reading this as a vector equation in orthonormal spherical coordinates. For a perfect fluid, $s = \Gamma^2 (\rho + p) v$, where $v$ is the physical velocity measured by an Eulerian observer. Equation (10) is the first challenge to the
consistency of the use of the NewtonPlus metric: Since the curl of any gradient vanishes, equation (10) will only have a solution if the fluid flow is such that \( \nabla \times (e^{-\Phi}s) = 0 \). This condition is satisfied in spherical symmetry, but it typically will not be true when convection is present. For the use of the NewtonPlus metric to be meaningful, there are two possibilities. One is that the transverse (or solenoidal) portion of \( e^{-\Phi}s \) is small compared with the longitudinal (or irrotational) part, that is, \( \nabla \times (e^{-\Phi}s) \ll \nabla \cdot (e^{-\Phi}s) \); this will obtain if, for example, radial flows dominate convective effects. In that case, \( e^{-\Phi}s \) is determined by the Poisson equation

\[
\nabla^2 (e^{-\Phi} - \delta \partial_t \Phi) = -4\pi \nabla \cdot (e^{-\Phi}s).
\]

(11)

The second possibility—the one we expect to follow in practice—is to neglect \( \partial_t \Phi \). This is reasonable in supernova environment: In contrast to the binary merger problem, one does not have entire stars moving at relativistic velocities. In the absence of bar or breakup instabilities, there is a quasispherical core, with asphericities due to rotation and convection. Aside from secular changes due to viscosity, rotation alone can be considered relatively stationary, not contributing greatly to explicit time variation of \( \Phi \). Core convection may occur, but probably will involve nonrelativistic velocities. There may be relativistic radial infall velocities, as seen at late times in failed explosions [3], and also relativistic outflow velocities in jets [15]; but these situations involve matter at low density in comparison with the core. Hence in the various phenomena occurring in the supernova environment there is reason to suppose that either the densities or velocities are such that \( s = \Gamma^2 (\rho + p)v \) is everywhere small enough for \( \partial_t \Phi \) to be neglected altogether. Heuristically, ignoring explicit time derivatives in determining \( \Phi \) based on this physical reasoning is consistent with the expectation that the back-reaction due to gravitational waves will not be dynamically important.

Next, we consider the evolution equations of \( \gamma_{ij} \):

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \gamma_{jk} D_i \beta^k + \gamma_{ik} D_j \beta^k,
\]

(12)

where \( D \) represents a covariant derivative with respect to the 3-metric \( \gamma_{ij} \). For the Newton-Plus metric, the extrinsic curvature turns out to be

\[
K_{ij} = e^{-\Phi} - \delta (\partial_t \Phi) \gamma_{ij}.
\]

(13)

Referring back to Eqs. (1) and (2), it is easy to see that the vanishing shift vector conspires with the explicit form of the extrinsic curvature tensor to make Eq. (12) an identity leading to no new information. (This is the case even if we choose not to neglect \( \partial_t \Phi \).)

Finally, we come to the evolution equations for \( K_{ij} \). In considering these equations we will take \( \partial_t \Phi = 0 \) in accordance with the physical reasoning discussed in connection with the momentum constraints. From Eq. (13), we see that our neglect of explicit time derivatives of \( \Phi \) implies a vanishing extrinsic curvature tensor. Nevertheless, the evolution equations for \( K_{ij} \) provide nontrivial conditions, and will lead us to an equation for \( \delta \). Various combinations of the Einstein equations could be employed to give an equation for \( \delta \), and we have explored some of these in numerical computations of stationary rotating stars. Because of numerical convergence issues discussed at the end of this section, our experience shows that convergence is achieved when the equation for \( \delta \) is derived from a combination of the individual evolution equations for \( K_{ij} \); the “lapse equation,” which is the trace of the evolution equations for \( K_{ij} \); and the Hamiltonian constraint.
The equations for δ obtained in this manner are as follows. Combining the evolution equation of \( K^r_r \) with the lapse equation and the Hamiltonian constraint yields

\[
\partial_r \partial_r \delta + \frac{1}{r} \partial_r \delta + \frac{1}{r^2} \tan \theta \partial_\theta \delta + \frac{1}{2r^2} \partial_\theta \partial_\theta \delta + \frac{1}{2r^2 \sin^2 \theta} \partial_\phi \partial_\phi \delta = 4\pi e^{-2\Phi} (S^\theta_\theta + S^\phi_\phi) - (\partial_r \Phi)^2 - 2\partial_r \Phi \partial_r \delta - (\partial_r \delta)^2 - \frac{1}{2r^2 \sin^2 \theta} (\partial_\phi \delta)^2 - \frac{1}{2r^2} (\partial_\theta \delta)^2,
\]

(14)

while combining the \( K^\theta_\theta \) evolution equation with the lapse equation and Hamiltonian constraint yields

\[
\partial_r \partial_r \delta + \frac{1}{r} \partial_r \delta + \frac{1}{r^2} \tan \theta \partial_\theta \delta + \frac{1}{r^2} \sin^2 \theta \partial_\theta \partial_\theta \delta = 8\pi e^{-2\Phi} S^\theta_\theta - (\partial_r \Phi)^2 + \frac{1}{r^2} (\partial_\theta \Phi)^2 - \frac{1}{r^2 \sin^2 \theta} (\partial_\phi \Phi)^2 + \frac{2}{r^2} \partial_\theta \Phi \partial_\theta \delta - (\partial_r \delta)^2 - \frac{1}{r^2 \sin^2 \theta} (\partial_\phi \delta)^2,
\]

(15)

and similar use of the \( K^\phi_\phi \) evolution equation gives

\[
\partial_r \partial_r \delta + \frac{1}{r} \partial_r \delta + \frac{1}{r^2} \partial_\theta \partial_\theta \delta = 8\pi e^{-2\Phi} S^\phi_\phi - (\partial_r \Phi)^2 - \frac{1}{r^2} (\partial_\theta \Phi)^2 + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi \Phi)^2 - 2\partial_r \Phi \partial_r \delta - \frac{2}{r^2 \sin^2 \theta} \partial_\phi \Phi \partial_\phi \delta - (\partial_r \delta)^2 - \frac{1}{r^2} (\partial_\theta \delta)^2.
\]

(16)

In these equations, \( S_{\mu\nu} \equiv T^{\mu\sigma} h_{\mu\rho} h_{\nu\sigma} \), where the spacelike projection tensor is defined by \( h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \). For a perfect fluid, \( S^\theta_\theta = (E + p)(v^\theta)^2 + p \), where \( v^\theta \) is the component of the fluid’s physical velocity in the \( \theta \) direction as measured by an Eulerian observer, and \( S^\phi_\phi \) is given by a similar expression with \( v^\theta \) replaced by \( v^\phi \). Given the fact that we expect inconsistencies in the Einstein equations due to the reduced number of degrees of freedom that we keep, it seems likely that Eqs. (14)-(16) for \( \delta \) are inconsistent, though it is not obvious (to us) how to prove this rigorously. However, we note that stresses (e.g., pressure) constitute the primary source terms in the equations determining \( \delta \), confirming the expectation expressed previously, that \( \delta \) should vanish as the Newtonian limit is approached. The observation that \( \delta \) will only be appreciable at the highest densities, where pressure begins to make a nontrivial contribution in comparison with energy density, suggests a reasonable path forward. In typical cases it is expected that the deepest portion of the core will be roughly spherical, even if rapid rotation causes an equatorial bulge of lower density material (e.g., Fig. 2 of Ref. [31]). If this is the case, neglecting the angular derivatives of \( \delta \) is justified, removing many of the apparent inconsistencies in the three equations for \( \delta \). The remaining discrepancies have to do with angular derivatives of \( \Phi \) and particular components of the stress tensor that appear in Eqs. (14)-(16). As it happens, if one adds (or averages) Eqs. (14)-(16), these remaining discrepancies disappear. Hence the equation we shall use to determine \( \delta \) is

\[
\partial_r \partial_r \delta + \frac{1}{r} \partial_r \delta = 4\pi e^{-2\Phi} (S^\theta_\theta + S^\phi_\phi) - [\partial_r (\Phi + \delta)]^2.
\]

(17)
The source on the right-hand side is to be angle-averaged in solving for $\delta$. Recalling that $S_{\theta}^\theta = S_{\phi}^\phi$ in spherical symmetry, Eq. (17) is the equation for $\delta$ obtained in the spherical case.

It should be straightforward to include the solution of $\Phi$ and, if desired, $\delta$, in existing multidimensional gravitational hydrodynamics codes. The solution of $\Phi$ would make use of the Poisson solver normally used to solve for the Newtonian gravitational potential, the only difference being that one would have to iterate on Eq. (8) (with the $\partial_t \Phi$ term dropped) to get a self-consistent $\Phi$. The simplest approximation would be to simply solve for $\Phi$ in this manner, and ignore $\delta$ altogether. The next level of approximation would involve solving Eq. (17) for $\delta$, but ignoring $\delta$ on the right hand side. Since $\delta$ is already something of a correction, $\delta$ appearing on the right hand side is essentially a “correction to the correction.” If desired, however, Eq. (17) could be solved as it stands, with iteration being required. In any case, solving Eq. (17), while requiring only a couple of angular and radial integrations over the computational domain, is a bit subtle. This is because the Green’s function of the operator on the left-hand side is a logarithm, which must be delicately handled in order to get a vanishing boundary condition at infinity [31]. It can be shown that $\delta$ is in fact the same function as that denoted $\zeta$ in the fully relativistic treatment of axisymmetric stars of Ref. [32], where the proper handling of this subtlety is described and an integral expression (Green’s function expansion) for $\zeta$ is given. The first term of this expansion can be adapted as the solution to Eq. (17).

**III. HYDRODYNAMICS IN THE NEWTONPLUS APPROXIMATION**

The utility of the NewtonPlus approximation will be greatly enhanced if existing multidimensional hydrodynamics codes can be adapted to its use. As a concrete case we consider the Virginia Hydrodynamics code (VH-1) [33], an implementation of the piecewise parabolic method [34]. This code treats a multidimensional problem with operator splitting: Successive sweeps through the spatial dimensions are taken, with each sweep involving a Lagrangian step forward in time followed by a remapping of the fluid variables to a fixed Eulerian grid. By way of example, we here present the equations for a spherically symmetric calculation in the NewtonPlus metric, which are analogous to those solved by VH-1 in its Lagrangian time steps in a similar Newtonian calculation. Similar equations can be derived for use in the angular sweeps.

The basic hydrodynamic equations can be expressed as conservation laws which follow from the vanishing divergences of the baryon flux vector and perfect fluid stress-energy tensor. It has been known for some time [35] that relativistic Eulerian hydrodynamics can be put into a conservative form amenable to methods developed for Newtonian hydrodynamics. In seeking to adapt VH-1 to an approximate relativistic treatment, we follow Appendix B of Ref. [32], which deals with spherically symmetric relativistic hydrodynamics in Eulerian coordinates. It was shown that these Eulerian equations can be cast in a form that, while not truly “Lagrangian” in the sense of being in a comoving frame, is nevertheless similar to the Newtonian Lagrangian equations.

Consider a vector of state variables, \( \mathbf{u} \equiv (D, S, E) \). \( \text{(18)} \)
The components of this vector are defined by

\[ D \equiv \Gamma \rho_B, \quad S \equiv \Gamma^2 (\rho + p)v, \quad E \equiv \Gamma^2 (\rho + p) - p, \]

where \( \Gamma \equiv (1 - v^2)^{1/2} \) is the boost factor between the fluid rest frame and the fixed “Eulerian” frame; \( \rho_B, \rho, \) and \( p \) are respectively the baryon rest mass density, total energy density, and pressure in the fluid rest frame; and \( v \) is the fluid physical radial velocity as measured by an Eulerian observer. This state vector is governed by the hydrodynamics equations in the metric of Eq. (6),

\[ \partial_t u + \frac{e^{3\Phi}}{r^2} \partial_r (r^2 e^{\delta - \Phi} f) = \sigma, \quad (22) \]

where the fluxes are

\[ f = (Dv, Sv + p, S), \quad (23) \]

and the sources are

\[ \sigma_D = 3D \partial_t \Phi, \quad (24) \]
\[ \sigma_S = -e^{\delta + 2\Phi} (E + p) \partial_r (\Phi + \delta) + e^{\delta + 2\Phi} p \left[ \frac{2}{r} + \partial_r (\delta - \Phi) \right] + 4S \partial_t \Phi, \quad (25) \]
\[ \sigma_E = -e^{\delta + 2\Phi} S \partial_r (\Phi + \delta) + (3p + Sv + 3E) \partial_t \Phi. \quad (26) \]

It is convenient to separate out the terms involving the advection of the state variables; to this end, Eq. (22) can be rewritten as

\[ \partial_t u + \frac{e^{3\Phi}}{r^2} \partial_r (r^2 e^{\delta - \Phi} u \nu) + \frac{e^{3\Phi}}{r^2} \partial_r (r^2 e^{\delta - \Phi} k) = \sigma, \quad (27) \]

where the vector

\[ k = (0, p, pv) \quad (28) \]

has been defined.

Next we consider how the conservation laws of Eq. (27) can be treated by a hydrodynamics code like VH-1, which performs Lagrangian hydrodynamic time steps followed by a “remap” to an Eulerian grid. We begin with baryon number conservation. The baryon flux vector is \( A^\mu = \rho_B u^\mu \), where \( u^\mu \) is the fluid 4-velocity. The number of baryons in a proper three-volume described by the 1-form \( \Delta \Sigma_\mu \) with edges \( \Delta r, \Delta \theta, \) and \( \Delta \phi \) is

\[ \text{constant} = A^\mu \Delta \Sigma_\mu = (e^{-3\Phi} r^2 \sin \theta \Delta r \Delta \theta \Delta \phi) \Gamma \rho_B. \quad (29) \]

In VH-1, the mass density is updated during a Lagrangian step as follows. From the time averaged fluid velocities at the zone edges obtained from the solution of the Riemann problem, the new positions of the zone edges are computed. From the new zone edge positions, the new zone volume is computed, and the new zone density is given by
Clearly, Eq. (29) can be used in precisely the same way. There is a factor of $\Gamma$ that must be dealt with, but presumably a relativistic Riemann solver [37] can be used if allowance for relativistic velocities is desired.

In VH-1 the updates of the velocity $v$ and specific internal energy $e$ make use of the (Newtonian) Lagrangian equations

$$\partial_t v + r^2 \partial_m p = g,$$

$$\partial_t e + \partial_m (r^2 pv) = v g.$$  

(31)  

(32)

Here the mass coordinate is defined by

$$m = \int_0^r \rho_B r^2 dr,$$

(33)

and $g$ is a force—gravity, for instance.

In the NewtonPlus approximation a similar set of equations can be obtained. We transform from the variables $t, r$ to $\bar{t}, m$, where

$$\bar{t} = t,$$

$$m = \int_0^r \Gamma \rho_B e^{-3\Phi} r^2 dr = \int_0^r De^{-3\Phi} r^2 dr.$$  

(34)  

(35)

Even though $\bar{t} = t$, we still write derivatives as $\partial_{\bar{t}}$ or $\partial_t$ to indicate whether $m$ or $r$ is being held constant respectively. We now use this change of coordinates to bring equation (27) into a useful form. The first component of equation (27) is

$$\partial_{\bar{t}} D + \frac{e^{3\Phi}}{r^2} \partial_r (r^2 e^{\delta - \Phi} Dv) = 3D \partial_t \Phi.$$  

(36)

Subtracting $u/D$ times Eq. (36) from Eq. (27) yields

$$D \partial_{\bar{t}} \left( \frac{u}{D} \right) + De^{\delta + 2\Phi} v \partial_r \left( \frac{u}{D} \right) + \frac{e^{3\Phi}}{r^2} \partial_r (r^2 e^{\delta - \Phi} k) = \sigma - 3u \partial_t \Phi.$$  

(37)

Equations (34-36) can be used to show that

$$\partial_t = \partial_{\bar{t}} - e^{\delta - \Phi} r^2 vD \partial_m,$$

$$\partial_r = r^2 e^{-3\Phi} D \partial_m,$$

(38)  

(39)

so that Eq. (37) can be expressed as

$$\partial_{\bar{t}} \left( \frac{u}{D} \right) + \partial_m (r^2 e^{\delta - \Phi} k) = \frac{\sigma}{D} - 3u \partial_t \Phi.$$  

(40)

The second and third components of this equation are

$$\partial_{\bar{t}} \left( \frac{S}{D} \right) + r^2 e^{\delta - \Phi} \partial_m p = g + \frac{S}{D} \partial_t \Phi,$$

$$\partial_{\bar{t}} \left( \frac{E}{D} \right) + \partial_m (r^2 e^{\delta - \Phi} pv) = v g + \frac{3p + Sv}{D} \partial_t \Phi,$$

(41)  

(42)
where the “gravitational force” \( g \) is defined by

\[
g = -e^{\delta+2\Phi} \frac{(E+p)}{D} \partial_r (\Phi + \delta) = -e^{\delta-\Phi} (E + p) \partial_m (\Phi + \delta).
\] (43)

In keeping with our practice of dropping explicit time derivatives of \( \Phi \) (at constant \( r \)), the last two terms of Eqs. (41) and (42) can be neglected. Furthermore, it is convenient to define a specific internal energy \( U \) by \( U \equiv E - D \), so that on the left hand side one can write \( \partial_t (E/D) = \partial_t ([U/D] + 1) = \partial_t (U/D) \).

The resulting equations,

\[
\partial_t \left( \frac{S}{D} \right) + r^2 e^{\delta-\Phi} \partial_m p = g,
\]

\[
\partial_t \left( \frac{U}{D} \right) + \partial_m (r^2 e^{\delta-\Phi} p v) = v g,
\]

are of the same form as Eqs. (31) and (32), allowing similar computational methods to be employed in their solution. It may seem surprising that the equations look so similar in the NewtonPlus and Newtonian formulations. As mentioned previously, it has been known for some time \([35]\) that relativistic conservation laws can be cast in a “conservative” form similar to the Eulerian Newtonian equations. Because we have taken care to define a mass coordinate based on the proper relativistic 3-volume element, and chosen a metric with a close connection to the one giving the Newtonian limit, we have also been able to find equations quite close to the Newtonian Lagrangian formulation.

\section*{IV. TESTING NEWTONPLUS GRAVITY WITH RAPIDLY ROTATING STARS}

In this section we present calculations of neutron stars undergoing rapid uniform rotation in order to assess the strengths and weaknesses of the NewtonPlus approximation to relativistic gravity. Our models were computed with a code described in Ref. [32], which was written to compute the structure of relativistic axisymmetric stars. We have modified the code to include the ability to perform computations in the Newtonian and various NewtonPlus limits: with vanishing metric function \( \delta \), with “linearized” \( \delta \) (i.e. ignoring \( \delta \) on the right hand side of Eq. (17)), and “full” \( \delta \) (solving Eq. (17) as written). All of the NewtonPlus limits solve a two-dimensional (and stationary) version of the nonlinear Poisson-type Eq. (8) for the glorified “gravitational potential” \( \Phi \). For the results presented here, the high-density portion of the equation of state (EOS) is taken from Ref. [38], and is based on a field-theoretic description of cold dense matter. We also performed calculations with a polytropic EOS of adiabatic index 2, and found qualitatively similar results.

Panel (a) of Fig. 1 shows mass vs. radius curves for spherical stars. The gravitational mass, a measure of total mass-energy, is defined for asymptotically flat spacetimes by (see e.g. Ref. [34])

\footnote{Computationally, this prevents finite differences in the internal energy from being swamped by finite differences of the (rest+internal) energy.}
\[ M = \int (T^i_i - T^t_t) \sqrt{-g} \, dx^1 \, dx^2 \, dx^3, \quad (46) \]

where here \( g \) is the determinant of the metric and \( x^i \) are spatial coordinates. For the NewtonPlus metric,

\[ M = \int e^{\delta-2\Phi} (E + S^i_i) r^2 \sin \theta \, dr \, d\theta \, d\phi. \quad (47) \]

From Eq. (46), it is evident that the proper equatorial radius is

\[ R = \exp[-\Phi(r = r_{\text{surface}}, \theta = \pi/2)] r_e, \quad (48) \]

where \( r_e \) is the coordinate radius of the equatorial surface. Panel (a) shows that while the Newtonian limit exhibits no maximum mass with this EOS, the NewtonPlus approximation does yield a maximum mass. Even with vanishing \( \delta \), the approximation captures this consequence of nonlinear gravity. The “linear \( \delta \)” approximation follows the exact relativistic curve until the most dense configurations are reached, where the approximation “overshoots” the true mass (i.e. masses are too large by a significant factor when \( \delta \) is neglected, and a bit too small when the “linear \( \delta \)” approximation is employed). Since pressure is the main source for \( \delta \) (see equation (17)), the large pressures associated with such high densities raise \( \delta \) to large enough values that it cannot be neglected on the right-hand side of equation (17). As expected, the “full \( \delta \)” approximation is indistinguishable from the relativistic results in spherical symmetry, where only two metric functions are needed to describe the spacetime exactly. The behavior of the “vanishing \( \delta \)” and “linear \( \delta \)” approximations in comparison with the exact result is given further explanation in the Appendix.

Panels (b)-(f) of Fig. 1 show various physical parameters of rapidly rotating configurations. In order to test the NewtonPlus approximation in a nonspherical setting, we ask the question: Given a definite number of baryons rotating at a given uniform angular velocity \( \Omega \), what do the various treatments of gravity do with those baryons? (The fact that baryon number is a conserved quantity makes this an obvious way to compare different treatments of gravity.) To answer this question we have computed, for each treatment of gravity, a constant baryon mass sequence beginning at zero rotation (marked by squares) and ending at the mass shedding limit (marked by stars). The value of baryon mass chosen, 1.8 \( M_\odot \), is close to the maximum baryon mass of 1.95 \( M_\odot \) for the equation of state we employed. The baryon mass is defined by (e.g., Ref. [31])

\[ \text{No turnover in the mass vs. radius curve appears in the Newtonian limit, up to the high-density boundary of the tabulated EOS. A configuration with central baryon mass density (total energy density) of } 3.07 \times 10^{15} \text{ g cm}^{-3} \text{ (4.65 } \times 10^{15} \text{ g cm}^{-3}) \text{ has a gravitational mass of } 15.6 \ M_\odot \text{ and radius 18.9 km in the Newtonian limit, while the relativistic configuration with this central density has a gravitational mass of } 1.68 \ M_\odot \text{ and a radius of 9.40 km. We remind the reader that the Chandrasekhar mass phenomenon is a property of stars built on a polytropic equation of state with adiabatic index equal to } 4/3, \text{ and that “realistic” nuclear equations of state will not generally exhibit this behavior. Instead, the upper mass limit of neutron stars derives from the the general relativistic instability indicated by the turning point in the mass vs. radius curve.} \]
\[ M_B = \int (-n_\mu A^\mu) \sqrt{h} \, dx^1 \, dx^2 \, dx^3, \]  
(49)  
\[ = \int e^{-3\Phi} \Gamma_{\rho B} r^2 \sin \theta \, dr \, d\theta \, d\phi, \]  
(50)  
where \( n_\mu \) is defined after Eq. (3), \( A^\mu \) is defined before Eq. (29), and \( h \) is the determinant of \( h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \). The second line (here and in the equations below) is specialized to the NewtonPlus metric. The quantities plotted as a function of the (uniform) stellar angular velocity in panels (b)-(f) are (b) gravitational mass; (c) equatorial radius; (d) angular momentum, given by (e.g., Ref. [31])  
\[ J = \int T^t_\phi \sqrt{-g} \, dx^1 \, dx^2 \, dx^2, \]  
(51)  
\[ = \int e^{-\delta-6\Phi} (E + p) \Omega r^4 \sin^3 \theta \, dr \, d\theta \, d\phi; \]  
(52)  
(e) the eccentricity, defined here as \( 1 - r_p/r_e \), where \( r_p \) and \( r_e \) are respectively the polar and equatorial coordinate radii; and (f) the linear equatorial velocity, given by (e.g. Ref. [31])  
\[ U = \frac{1}{\Gamma}(e_\phi)_\mu u^\mu \]  
(53)  
\[ = e^{-\delta-2\Phi} r \sin \theta \Omega, \]  
(54)  
where \( (e_\phi)\mu \) is the basis vector corresponding to the coordinate \( \phi \).

In panels (b)-(f), the efficacy of the NewtonPlus approximations can be judged by choosing a value of angular velocity and seeing how close the approximate quantities come to the fully relativistic value. While the “full \( \delta \)” approximation is indistinguishable from full relativity in the spherical case, the two curves representing these treatments deviate from one another with increasing angular velocity. In panels (b) and (c), the “overshoot” of the “linear \( \delta \)” approximation, discussed in connection with panel (a), is again visible at low angular velocities. At the highest angular velocities, the “linear \( \delta \)” approximation actually appears to give slightly better results than the “full \( \delta \)” calculations, but it must be remembered that this is a lingering result of the “overshoot” at low angular velocities. As expected, the angular velocities at mass shedding of the NewtonPlus approximations are closer to the relativistic values than the Newtonian case. The NewtonPlus treatments are quite successful at approximating the gravitational mass, radius, and eccentricity, while the success of the results for angular momentum and equatorial velocity is more modest.

The relative success of the NewtonPlus approximation in determining these observables can be understood by examining expressions for these quantities in the fully relativistic case. Axisymmetric, stationary spacetimes can be described by four metric functions; the metric can be expressed [31,32]

\[ g_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\nu} dt^2 + e^{-2\nu} G^2 r^2 \sin^2 \theta (d\phi - N^\phi dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2). \]  
(55)  
In comparison, the NewtonPlus metric is conformally flat (equivalent to requiring \( G = e^\zeta \) in Eq. (55)), and lacks the shift vector component \( N^\phi \). In the relativistic case the angular momentum and equatorial velocity are given by

\[ J = \int e^{2\zeta-6\nu} G^3 (E + p)(\Omega - N^\phi) r^4 \sin^3 \theta \, dr \, d\theta \, d\phi, \]  
(56)  
\[ U = e^{-2\nu} G r \sin \theta (\Omega - N^\phi). \]  
(57)
In comparison with Eqs. (52) and (54), an important difference is that $\Omega$ is replaced by $\Omega - N^\phi$. Since values of $N^\phi$ can be a significant fraction of $\Omega$ in extreme configurations (e.g., Refs. [31,39]), the neglect of this shift vector component in the NewtonPlus approximations probably accounts for most of the difference in $J$ and $U$ with the relativistic case.

The gravitational and baryon mass in the relativistic case are given by

\begin{align*}
M &= \int e^{2(\zeta - \nu)} G (E + S^i_i + 2e^{-\nu} N^\phi J_\phi) r^2 \sin \theta \, dr \, d\theta \, d\phi, \\
M_B &= \int e^{2\zeta - 3\nu} G \Gamma \rho_B r^2 \sin \theta \, dr \, d\theta \, d\phi,
\end{align*}

where $J_\phi = (E + p)e^{-\nu} G r \sin \theta U$, and the proper equatorial radius is

$$R = \left( e^{-\nu} G \right|_{r=r_{\text{surface}}, \theta=\pi/2} r_e. \quad (60)$$

In Eqs. (59) and (60), $N^\phi$ does not appear at all, and it plays a minor role in Eq. (58). This, together with the fact that the approximation of conformal flatness is known to work well for rapidly rotating stars [42], shows why the NewtonPlus approximation gives good results for the gravitational mass, baryon mass, and equatorial radius.

V. CONCLUSION

Accurate neutrino transport, 3D hydrodynamics, and relativity are all essential for realistic supernova simulations. Given the constraints of current hardware, these cannot all be treated simultaneously with the detail they deserve. We have therefore presented an approximation to full relativity (or set of related approximations) that captures important relativistic effects in the quasispherical supernova environment: nonlinearity creating a deeper potential well, and pressure and internal energy density being nontrivial sources of gravitation.

This “NewtonPlus” approach to gravity has a tight conceptual link with the Newtonian limit that yields certain advantageous features. The gravitational portion of multidimensional Newtonian calculations involves only the solution of the Poisson equation for the gravitational potential $\Phi$, and taking its gradient to find the gravitational force. The basic idea of our NewtonPlus approach is to promote the Newtonian metric functions—which are

\[ 3 \text{Since the lapse function and shift vector were quantities to be chosen at will in the (3+1) formalism as originally conceptualized, at first glance it may seem strange that the shift vector component } N^\phi \text{ should play a significant role in the determination of physical quantities. However, other coordinate choices have been made in arriving at Eq. (55), making the lapse and shift vector component quantities with physical content for which solutions must be found. Specifically, two of the four degrees of coordinate freedom have been used to choose the time coordinate } t \text{ to label the spacelike hypersurfaces which are invariant under time translations, and the azimuthal coordinate } \phi \text{ to measure the angle about the axis of symmetry. Once these two choices have been made, the metric components } g_{rr}, g_{\theta\theta}, g_{\phi r}, \text{ and } g_{\phi\theta} \text{ vanish (see Ref. [41], which draws on [40]). The remaining two degrees of coordinate freedom have been used to set } g_{r\theta} = 0 \text{ and } g_{\theta\theta} = r^2 g_{rr}. \]
linear in $\Phi$—to full exponentials. We also add a second metric function, $\delta$, whose main source is pressure; hence this metric function vanishes in the Newtonian limit. The Einstein equations yield a nonlinear Poisson-type equation for a (now glorified) “gravitational potential,” whose solution in 3D can be obtained in a manner similar to what would be done in the Newtonian limit. The inconsistencies in the Einstein equations arising from the reduced number of degrees of freedom turn out to be relegated to the subdominant metric function $\delta$; they can be removed by ignoring angular variations in $\delta$. This is expected to be successful in the supernova context because the region where $\delta$ makes the greatest difference is where pressure is significant in comparison with rest mass density. Normally, this is the deepest portion of the collapsed core, which is roughly spherical even when the outer layers bulge at the equator due to rapid rotation.\footnote{Ultra-strong magnetic fields \cite{43} or differential rotation (e.g., Ref. \cite{44}) can give rise to off-center density maxima, making the spherical correction for stresses via the metric function $\delta$ less useful. The NewtonPlus approximation with $\delta = 0$ could still be employed in such (probably exceptional) cases, however.}

This strategy—allowing the main contribution to the gravitational field to be multidimensional and nonlinear, while allowing a spherical correction for the contribution of stresses—reproduces fairly well many of the physical characteristics of rapidly rotating (stationary) relativistic stars. In future dynamic calculations, we intend to ignore explicit time derivatives of $\Phi$ that appear in the Einstein equations, and we have given specific arguments for the validity of this approximation in the supernova environment. (This can be tested in the future by comparing the results of spherically symmetric collapse simulations using NewtonPlus gravity with similar fully relativistic computations.) Importantly, the hydrodynamics equations in spherical symmetry obtained from the NewtonPlus metric are of the same form as those used in a popular Newtonian hydrodynamics algorithm, providing the expectation that existing Newtonian codes might be adapted fairly easily to the NewtonPlus approach.

Finally, we make a few comments on approximations to full relativity employed by Mathews and Wilson \cite{45} and by Shibata, Baumgarte, and Shapiro \cite{46} in binary neutron star merger calculations. While to our knowledge these approaches have not been implemented in supernova simulations, they conceivably could be; hence we offer a few comments by way of comparison. These approaches involve the use of a conformally flat metric, the imposition of a traceless extrinsic curvature tensor, and the neglect of explicit time derivatives of gravitational variables. Some arguments in favor of these approximations are given in Ref. \cite{47}.\footnote{Mathews and Wilson’s initial calculations produced the controversial result that the individual stars collapsed to black holes prior to merger, and it was suspected that this might be caused by their approximations rather than a physical effect. However, an error in their equations was pointed out in Ref. \cite{48}, and the discrepancies with previous expectations and the results of other groups’ calculations have been greatly mitigated in their most recent results \cite{45}.} These approximations are similar to ours in that both retain a reduced number of degrees of freedom (they keep four, while we keep two), both assume conformal flatness, and both ignore explicit time derivatives of gravitational variables. An important difference between Refs. \cite{45} and \cite{46} is that the latter neglect certain nonlinear gravitational terms that do not
arise in the limit of spherical symmetry. We would argue that an advantage of our approach is the conceptual link with the Newtonian limit. In the quasispherical supernova context this makes for only one dominant metric function—the glorified “gravitational potential”—while in the other approaches both the conformal factor and the lapse function are significant and must be solved for in multidimensions. They retain the entire shift vector; we have suggested dropping it altogether, though our work shows that this causes errors in the angular momentum and equatorial velocities. In the context of assumed conformal flatness, Wilson and Mathews also found that proper treatment of the shift vector was critical to accurate treatment of the angular momentum [43]. While angular momentum is perhaps less critical in supernovae than in binary mergers, it is known that the assumption of conformal flatness alone (while retaining an azimuthal shift vector component) yields accurate results for rapidly rotating stars [12]. This suggests a future extension to the work presented here that would improve results for angular momentum and equatorial velocity. The azimuthal shift vector component $N^\phi$ obeys an elliptic equation, whose solution can be written as a Green’s function expansion (see Ref. [32] for an explicit formula). Like the solution of Eq. (17) for $\delta$, obtaining the leading term of $N^\phi$ would involve only a pair of angular and radial integrations over the spacetime, but it would likely improve results for angular momentum and equatorial velocity significantly.

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APPENDIX:

This appendix shows the relationship between the NewtonPlus metric in the static limit and the (interior) Schwarzschild metric. We point out that the equation of hydrostatic equilibrium derived with the NewtonPlus metric is equivalent to the familiar Tolman-Oppenheimer-Volkov (TOV) equation, as expected if the NewtonPlus metric provides an exact solution in spherical symmetry. The connection to the Schwarzschild metric also allows greater insight into the behavior of the “vanishing $\delta$” and “linear $\delta$” approximations in the static limit.

The Schwarzschild line element is given by

$$ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{2m}{\tau}\right)^{-1}d\tau^2 + \tau^2d\Omega^2,$$

(A1)

where $\Phi$ and $m$ are functions of $\tau$. Comparing with a static version of Eq. (6) in which $\Phi$ and $\delta$ are only functions of $r$, we find that equivalence between the two metrics requires

$$\tau = e^{-\Phi}r,$$

(A2)

$$\Phi = \Phi + \delta,$$

(A3)

$$\left(1 - \frac{2m}{\tau}\right)^{-1} = \left(1 + r\frac{d\Phi}{dr}\right)^2 = \left(1 + r\frac{d\Phi}{dr}\right)^2.$$  

(A4)
We now describe how equations derived from the NewtonPlus metric are equivalent to the TOV equation,

\[
\frac{dp}{dr} = -\frac{\rho + p}{r} \left(1 - \frac{2m}{r}\right)^{-1} \left(\frac{m}{r} + 4\pi r^2 p\right). \tag{A5}
\]

The \(rr\) component of the Einstein equations in the static NewtonPlus metric is

\[
2 \frac{d\delta}{r \, dr} - 2 \frac{d\delta \, d\Phi}{dr \, dr} - \left(\frac{d\Phi}{dr}\right)^2 = 8\pi e^{-2\Phi} p. \tag{A6}
\]

Using Eqs. (A2), (A4), and (A6), it is a straightforward exercise to show that the TOV equation becomes

\[
\frac{dp}{dr} = -(\rho + p) \left(\frac{d\Phi}{dr} + \frac{d\delta}{dr}\right), \tag{A7}
\]

which is the static version of Eq. (31) (see also Eqs. (39) and (43)).

It is well known that \(m(\overline{r})\) evaluated at the stellar surface is equal to the gravitational mass. Using Eqs. (A4) and (A6), we find

\[
\frac{m}{\overline{r}} = \frac{r}{r} \frac{d\Phi}{dr} - \frac{r^2}{2} \left(\frac{d\Phi}{dr}\right)^2
= \frac{r}{r} \frac{d\Phi}{dr} - \frac{r^2}{2} \left[\left(2 \frac{d\delta}{r \, dr} - 2 \frac{d\delta \, d\Phi}{dr \, dr}\right) - 8\pi e^{-2\Phi} p\right]. \tag{A9}
\]

By evaluating these expressions at the stellar surface, we gain insight into the behavior of the “vanishing \(\delta\)” and “linear \(\delta\)” limits. In the exact solution, the second term in Eqs. (A8) and (A9) is negative. In the “vanishing \(\delta\)” approximation, this term becomes positive, explaining why the gravitational masses obtained in this limit are too large. Because the exact second term is negative, the quantity in parentheses in Eq. (A9) must be positive and greater in magnitude than the pressure term. Also, since \(\delta\) is subdominant, Eq. (A8) shows that \(d\delta/dr\) is positive. The “linear \(\delta\)” approximation involves neglecting terms containing \(\delta\) that are nonlinear in metric functions, which here involves neglecting \(2(d\delta/dr)(d\Phi/dr)\). This makes the quantity in parentheses in Eq. (A9) more positive, which makes the overall second term too negative. This explains the “overshoot” (gravitational masses too small) seen this limit.
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FIGURES

FIG. 1. Panel (a): Mass vs. radius curves for spherical configurations computed with various treatments of gravity. Panels (b)-(f): Various physical quantities characterizing uniformly rotating configurations, plotted as a function of angular velocity. Each curve represents a constant baryon mass sequence computed with the treatments of gravity labeled in panel (a).