Image inpainting using directional wavelet packets

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Abstract

The paper presents a new algorithm for the image inpainting problem. The algorithm is using a recently designed versatile library of quasi-analytic complex-valued wavelet packets (qWPs) which originate from polynomial splines of arbitrary orders. Tensor products of 1D qWPs provide a diversity of 2D qWPs oriented in multiple directions. For example, a set of the fourth-level qWPs comprises 62 different directions. The properties of the presented qWPs such as refined frequency resolution, directionality of waveforms with unlimited number of orientations, (anti-)symmetry of waveforms and windowed oscillating structure of waveforms with a variety of frequencies, make them efficient in image processing applications, in particular, in dealing with the inpainting problem addressed in the paper. The obtained results for this problem are quite competitive with the best state-of-the-art algorithms. The inpainting is implemented by an iterative scheme, which, in essence, is the Split Bregman Iteration (SBI) procedure supplied with an adaptive variable soft thresholding based on the Bivariate Shrinkage algorithm. In the inpainting experiments, performance comparison between the qWP-based methods and the state-of-the-art algorithms is presented.

1 Introduction

Multimedia images as well as biomedical, seismic, and hyper-spectral images, to name a few, comprise smooth regions, edges oriented in various directions and texture, which can have an oscillating structure. One of the main goals of the image processing is to reconstruct the image from a degraded available data. The degradation can result from missing pixels, noise, blurring, for example, and different combinations of these factors. Another goal is the extraction of a limited number of characteristic features from images for the pattern recognition and machine learning. Achievement of the above goals relies on the fact that practically all images to be processed have a sparse representation in a proper transform domain. The sparse representation of an image means that the image can be approximated by a linear combination of a relatively small number of 2D “basic” elements, while retaining the above mentioned components of the image. The dictionary of such elements should comprise waveforms that

\begin{enumerate}
  \item \textbf{Are oriented in multiple directions} (for capturing edges),
  \item \textbf{Have oscillating structure with multiple frequencies} (for retaining texture patterns), and
  \item \textbf{Have vanishing moments, at least locally} (for sparse representation of smooth regions).
\end{enumerate}

In addition, such properties of the waveforms as

\begin{enumerate}
  \item \textbf{Refined frequency separation},
  \item \textbf{Good localization in the spatial domain}
\end{enumerate}

are desirable. Last but not least is \textbf{Fast implementation of the corresponding transforms}.
In the recent years, a number of elements’ dictionaries that meet some of the above requirements were described such as pseudo-polar processing [4, 5], contourlets [11], curvelets [8, 7] and shearlets [20, 12]. These dictionaries were used in various image processing applications. For example, one of successful applications of the shearlet transforms to the image processing is the methodology based on the Digital Affine Shear transforms (DAS-1) [24]. However, while successfully capturing edges in images, these dictionaries did not demonstrate a satisfactory texture restoration due to the lack of oscillatory waveforms in the dictionaries libraries.

Another approach to design directional dictionaries consists of the tensor product of complex wavelets ([19, 22]), wavelet frames and wavelet packets (WPs) ([17, 6, 15, 14, 16], to name a few). The tight tensor-product complex wavelet frames (TP\_CTF\_n) with different number of directions, are designed in [15, 16, 14] and some of them, in particular cptTP\_CTF\_6, TP\_CTF\_6 and TP\_CTF\_6↓, demonstrate impressive performance for image denoising and inpainting. The waveforms in these frames are oriented in 14 directions and, due to the 2-layer structure of their spectra possess some oscillatory properties.

Some of the disadvantages in the above 2D TP\_CTF\_6 and TP\_CTF\_6↓ frames such as, for example, limited and fixed number of directions (14 directions at each decomposition level) are overcome in [9] (algorithm Digital Affine Shear Filter Transform with 2-Layer Structure (DAS-2)) by the incorporation of the two-layer structure, which is inherent in the TP\_CTF\_6 frames, into directional filter banks introduced in [24]. This improves the performance of the DAS-2 compared to the TP\_CTF\_6 on texture-rich images such as “Barbara”, which is not the case on smoother images like “Lena”.

Recently we designed a family of dictionaries that maximally meet the requirements 1–6 (preprint [1]) and utilize them in image processing applications. Building blocks for such a design are orthonormal WPs \(\{\psi\}\) originated from discretized polynomial splines of multiple orders (see Chapter 4 in [2]).

The design scheme is outlined in Sections 2 and 3. Briefly, it consists of the following steps: 1. Application of the Hilbert transform (HT) to the set \(\{\psi\}\) of orthonormal WPs producing the set \(\{\vartheta = H(\psi)\}\). 2. A slight correction of the set \(\{\vartheta\}\) provides an orthonormal set \(\{\varphi\}\) of the so-called complimentary WPs (cWPs), which are anti-symmetric and whose magnitude spectra coincides with the magnitude spectra of respective WPs from the set \(\{\psi\}\). 3. Two sets of complex quasi-analytic WPs (qWPs) \(\{\Psi^+ \overset{\text{def}}{=} \psi + i \varphi\}\) and \(\{\Psi^- \overset{\text{def}}{=} \psi - i \varphi\}\) whose spectra are localized in the positive and negative half-bands of the frequency domain, respectively, are defined. 4. Two sets of 2D complex qWPs by the tensor product of the qWPs \(\{\Psi^\pm\}\) as: \(\{\Psi^+ \overset{\text{def}}{=} \Psi^+ \otimes \Psi^+\}\) and \(\{\Psi^- \overset{\text{def}}{=} \Psi^- \otimes \Psi^-\}\) are defined. 5. The dictionaries we use are the real parts of these qWPs: \(\{\vartheta^\pm \overset{\text{def}}{=} \Re(\Psi^\pm)\}\).

The DFT spectra of dictionaries elements \(\{\vartheta^+\}\) and \(\{\vartheta^-\}\) form various tiling of the pairs of quadrants \(Q_0 \cup Q_3\) and \(Q_1 \cup Q_2\) (see Eq. (1.1)), respectively, by squares of different sizes depending on the decomposition level. The waveforms’ shapes are close to windowed cosines with multiple frequencies oriented in multiple directions \(2(2^m+1) - 1\) directions at level \(m\). Combinations of waveforms from the sets \(\{\vartheta^+\}\) and \(\{\vartheta^-\}\) provide a variety of frames in the space of 2D signals. The transforms are executed in a fast way using FFT.

The designed dictionaries \(\{\vartheta^\pm \overset{\text{def}}{=} \Re(\Psi^\pm)\}\) are applied to image inpainting to restore images from corrupted data where many pixels are missing and additive noise is present. The paper.
introduces two qWP-based iterative methods: Method1 (M1) and Method2 (M2) to deal with the inpainting problem. The performances of M1 and M2 are compared with the performance of Algorithm I introduced in [23] that uses the SET-4 filter banks. Algorithm I consists of iterative thresholding of the transform coefficients with decreasing localized thresholds. The threshold levels are determined by using the Bivariate Shrinkage algorithm (BSA) ([10]). The computational scheme in M1 is close to Algorithm I with the difference that the qWP transforms are used instead of the SET-4 filter banks. Due to multiple orientations and the oscillatory structure of the waveforms, M1 outperforms in PSNR and Structural Similarity Index (SSIM) all the SET-4 algorithms applied to “Barbara” and “Fingerprint” images. Although on “Lena” and “Boat” images, at least one of the SET-4 algorithms produces the highest PSNR values, in many experiments with “Lena” and almost all experiments with “Boat”, the SSIM outputs from M1 exceed those from SET-4.

M2 couples M1 with split Bregman iteration (SBI) scheme that uses the so-called analysis-based approach ([13, 18]). In essence, it is the SBI algorithm supplied with decreasing localized thresholds determined by BSA. Practically, in all the experiments, M2 outperforms M1. In many cases, the outperformance gap is substantial.

The paper is organised as follows. In order for the paper to be self-contained, Sections 2 and 3 briefly outline the design and implementation of the qWP transforms in one and two dimensions, respectively. Section 4 describes the qWP-based approaches for image inpainting. Section 5 presents the experimental results from the restoration of images degraded by missing up to 80% of its pixels while additive noise is present. The performances of the qWP-based methods are compared with the performances of the algorithms that use the SET-4 filter banks. Section 5 discusses the results.

Notations and abbreviations: N = 2^j, ω = e^{2πi/N} and Π[N] is a space of real-valued N-periodic signals. Π[N,N] is the space of two-dimensional N-periodic arrays in both vertical and horizontal directions. The sequence δ[k] ∈ Π[N] is the N-periodic Kronecker delta.

DFT is Discrete Fourier transform and FFT is Fast Fourier transform. ·* is complex conjugate.

PR is perfect reconstruction. HT is Hilbert transform, H(x) is the discrete periodic HT of a signal x. DTS is discrete-time spline. DTSWP, cWP and qWP is discrete-time-spline-based wavelet packets ϕ_{m,l}^{p}, complimentary wavelet packets ψ_{m,j}^{p} and quasi-analytic wavelet packets Ψ_{±m,l}^{p}, respectively, in 1D case, and wavelet packets ϕ_{m,j}^{p}, complimentary wavelet packets ψ_{m,j,l}^{p} and quasi-analytic wavelet packets Ψ_{±m,l,j}^{p}, respectively, in 2D case. WPT is wavelet packet transform.

Peak Signal-to-Noise ratio (PSNR) in decibels (dB) is \( \text{PSNR} \overset{\text{def}}{=} 10 \log_{10} \left( \frac{255^2}{\sum_{k=1}^{N} (x_k - \tilde{x}_k)^2} \right) \) dB.

SSIM is the Structural Similarity Index. BSA stands for Bivariate Shrinkage algorithm and p-filter is periodic filter. SBI stands for split Bregman iteration. M1 and M2 stand for Method1 and Method2, respectively. SET-4 is the set of the filter banks DAS-2, DAS-1, TP-CTF₆ and TP-CTF₆.\(^{\text{1}}\)

Notation \( l_0 \overset{\text{def}}{=} 0, \quad l_m \overset{\text{def}}{=} 2^m - 1 \). Quadrants of the frequency domain:

\[
Q_0 \overset{\text{def}}{=} [0, N/2 - 1] \times [0, N/2 - 1], \quad Q_1 \overset{\text{def}}{=} [0, N/2 - 1] \times [-N/2, -1], \quad Q_2 \overset{\text{def}}{=} [-N/2, -1] \times [0, N/2 - 1], \quad Q_3 \overset{\text{def}}{=} [-N/2, -1] \times [-N/2, -1].
\]

\(^{\text{1}}\) SET-4 is the set of the filter banks DAS-2, DAS-1, TP-CTF₆ and TP-CTF₆.
2 Quasi-analytic WPs

In this section and Section 3, we briefly outline the design and properties of quasi-analytic WPs (qWPs). For details see [1].

2.1 Orthonormal WPs originated from polynomial splines

In this section, we list the main properties of periodic discrete-time wavelet packets originated from polynomial splines and corresponding transforms. For details see Chapter 4 in [2].

The centered N-periodic polynomial B-spline \( B^p(t) \) of order \( p \) is an \( N \)-periodization of the function \( b^p(t) = \frac{1}{(p-1)!} \sum_{k=0}^{p} (-1)^k \frac{t^k}{k!} \left( t + \frac{p}{2} - k \right)^{p-1} \) where \( x_+ \) is \( \max \{ x, 0 \} \). The B-spline \( B^p(t) \) is supported on the interval \((-p/2, p/2)\) up to periodization. It is strictly positive inside this interval and symmetric about zero, where it has its single maximum and has \( p-2 \) continuous derivatives.

The functions \( S^p(t) = \sum_{l=0}^{N-1} q[l] B^p[t-l] \) are referred to as the order-\( p \) periodic splines. The following two sequences will be used in the further presentation:

\[
\begin{align*}
    u^p[n] & \overset{\text{def}}{=} \sum_{k=-N/2}^{N/2-1} \omega^{-kn} b^p(k), &
    v^p[n] & \overset{\text{def}}{=} \omega^{-n/2} \sum_{k=-N/2}^{N/2-1} \omega^{-kn} b^p \left( k + \frac{1}{2} \right). 
\end{align*}
\]

Denote by \( b^p_d(t) = b^p(t)/2 \), which is the two-times dilation of the B-spline \( b^p(t) \).

**Definition 2.1** The span-two discrete-time B-spline \( b^p_{[1]} \) of order \( p \) is defined as an \( N \)-periodization of the sampled B-spline \( b^p_d(t) \). It is an \( N \)-periodic signal from \( \Pi[N] \):

\[
b^p_{[1]}[k] \overset{\text{def}}{=} b^p_d(k), \quad k = -N/2, ..., N/2 - 1 \text{(mod } N\text{)}, \quad \hat{b}^p_{[1]}[n] = \frac{u^p[2n] + v^p[2n]}{2}.
\]

Linear combinations of two-sample shifts of the B-splines \( s^p_{[1]}[k] = \sum_{l=0}^{N/2-1} q[l] b^p_{[1]}[k - 2l] \), \( \hat{s}^p_{[1]}[n] = \hat{q}[n] \hat{b}^p_{[1]}[n] \), are referred to as periodic discrete-time splines (DTSs) of span 2. The \( N/2 \)-dimensional space of the DTSs is denoted by \( pS^0_{[1]} \subset \Pi[N] \) and its orthogonal complement in the signal space \( \Pi[N] \) is denoted by \( pS^1_{[1]} \).

Define the DTS \( \psi^p_{[1],0} \) and the signal \( \psi^p_{[1],1} \) by their DFTs:

\[
\begin{align*}
    \hat{\psi}^p_{[1],0}[n] &= \beta[n] \overset{\text{def}}{=} 2 \frac{u^p[2n] + v^p[2n]}{\sqrt{u^p[2n]^2 + v^p[2n]^2}}, \\
    \hat{\psi}^p_{[1],1}[n] &= \alpha[n] \overset{\text{def}}{=} 2 \omega^n \frac{u^p[2n] - v^p[2n]}{\sqrt{u^p[2n]^2 + v^p[2n]^2}}.
\end{align*}
\]

Two-sample shifts of the discrete-time spline \( \psi^p_{[1],0}[k] \) and of the signal \( \psi^p_{[1],1}[k] \) form orthonormal bases of the subspaces \( pS^0_{[1]} \) and \( pS^1_{[1]} \), respectively.

The signals \( \psi^p_{[1],0}[k] \) and \( \psi^p_{[1],1}[k] \) are real-valued and symmetric about \( k_0 = 0 \) and \( k_1 = -1 \), respectively. They are referred to as the discrete-time-spline wavelet packets (DTSWPs) of order \( p \) from the first decomposition level.

The one-level DTSWP transform of a signal \( x \in \Pi[N] \) and its inverse are implemented by application of the analysis \( \mathbf{H} = \{ \hat{h}^0_{[1]}, \hat{h}^1_{[1]} \} \) and synthesis \( \mathbf{H} = \{ h^0_{[1]}, h^1_{[1]} \} \) p-filter banks to the signal \( x \) and the transform coefficients, respectively. The p-filters \( h^\lambda_{[1]} = \hat{h}^\lambda_{[1]}, \lambda = 0, 1, \) and
their impulse responses are \( h_k^{[1]}[n] = w_{[1],\lambda}[k] \) and the frequency responses are \( \hat{h}_k^{[1]}[n] = \beta[n] \) and \( \hat{h}_k^{[1]}[n] = \alpha[n] \).

The transforms can be represented in a matrix form:

\[
\begin{pmatrix}
\hat{y}_k^{[1]}[n] \\
\hat{y}_k^{[1]}[\bar{n}]
\end{pmatrix} = \frac{1}{2} \hat{M}[-n] \cdot \begin{pmatrix}
x[n] \\
\bar{x}[\bar{n}]
\end{pmatrix}, \quad \begin{pmatrix}
x[n] \\
\bar{x}[\bar{n}]
\end{pmatrix} = M[n] \cdot \begin{pmatrix}
\hat{y}_k^{[1]}[n] \\
\hat{y}_k^{[1]}[\bar{n}]
\end{pmatrix},
\]

where \( \bar{n} = n + N/2 \) and \( \hat{M}[n] \) and \( M[n] \) are the modulation matrices of the analysis and synthesis p-filter banks:

\[
M[n] = \sqrt{2} \begin{pmatrix}
\beta[n + \frac{N}{2}] & \alpha[n + \frac{N}{2}]
\end{pmatrix} = \hat{M}[n]^T.
\]

The sequences \( \beta[n] \) and \( \alpha[n] \) are defined in Eq. \( 2.2 \).

**Extension of transforms to deeper decomposition levels** The WP transforms to deeper decomposition levels are implemented iteratively, while the transform coefficients \( \{y_p^{[m+1]} \} \) are derived by filtering the coefficients \( \{y_\lambda^{[m]} \} \) with the p-filters \( h^{[m+1]}_{\lambda} \), where \( \lambda = 0, \ldots, 2^m - 1, \mu = 0, 1 \) and \( \rho = \begin{cases} 2\lambda + \mu, & \text{if } \lambda \text{ is even} \\ 2\lambda + (1 - \mu), & \text{if } \lambda \text{ is odd} \end{cases} \). The frequency responses of the p-filters \( h^{[m+1]}_{\lambda} \), are \( \hat{h}_{[m+1]}^{[1]}[n] = \beta[2^m n] \) and \( \hat{h}_{[m+1]}^{[1]}[n] = \alpha[2^m n] \).

The transform coefficients are \( y_\lambda^{[m]}[l] = \langle x, w_{[m],\lambda}[\cdot, -2^m l] \rangle \), where the signals \( w_{[m],\lambda} \) are normalized, orthogonal to each other in the space \( \Pi[N] \), and their \( 2^m l \)-sample shifts are mutually orthogonal. They are referred to as level-\( m \) DTSWPs of order \( p \). The set \( \{w_{[m],\lambda}[\cdot, -2^m l] \} \), \( \lambda = 0, \ldots, 2^m - 1, l = 0, \ldots N/2^m - 1 \), constitutes an orthonormal basis of the space \( \Pi[N] \) and generates its split into \( 2^m \) orthogonal subspaces. The next-level wavelet packets \( w_{[m+1],\rho} \) are derived by filtering the wavelet packets \( w_{[m],\lambda} \) with the p-filters \( h^{[m+1]}_{\lambda} \) such that

\[
\psi_{[m+1],\rho}[n] = \sum_{k=0}^{N/2^m-1} h_{[m+1]}^{[1]}[k] \psi_{[m],\lambda}[n - 2^m k].
\]

The transforms are executed in the spectral domain using the Fast Fourier transform (FFT) by the application of critically sampled two-channel filter banks with the modulation matrices \( \hat{M}[2^m n] \) and \( M[2^m n] \) and to the DFTs of the transform coefficients.

### 2.2 2D WPTs

A standard way to extend one-dimensional (1D) WPTs to multiple dimensions is the tensor-product extension. The 2D one-level WPT of a signal \( x = \{x[k, n] \} \), \( k, n = 0, \ldots, N - 1 \), which belongs to \( \Pi[N, N] \), consists of the application of 1D WPT to columns of \( x \), which is followed by the application of the transform to rows of the coefficients array. As a result of the 2D WPT of signals from \( \Pi[N, N] \), the space becomes split into four mutually orthogonal subspaces \( \Pi[N, N] = \bigoplus_{j,l=0}^{1} \mathcal{P} \).

The 2D DTSWPs are \( \psi_{[1],j,l}[n, m] \triangleq \psi_{[1],j}[n] \psi_{[1],l}[m] \), \( j, l = 0, 1 \). They are normalized and orthogonal to each other in the space \( \Pi[N, N] \). It means that
analytic periodic signals

A signal \( x \in \Pi[N] \) is represented by its inverse DFT which can be written as follows:

\[
x[k] = \frac{\hat{x}[0]}{N} + \frac{2}{N} \sum_{n=1}^{N/2-1} \hat{x}[n] \omega^{kn} + \left( \hat{x}[n] \omega^{kn} \right)^*.
\]

Define the real-valued signal \( h \in \Pi[N] \) and two complex-valued signals \( x_+ \) and \( x_- \) such that

\[
h[k] \overset{\text{def}}{=} \frac{2}{N} \sum_{n=1}^{N/2-1} \hat{x}[n] \omega^{kn} - \hat{x}[n]^* \omega^{-kn},
\]

\[
x_{\pm}[k] \overset{\text{def}}{=} \frac{\hat{x}[0] + (-1)^k \hat{x}[N/2]}{2} \pm \frac{2}{N} \sum_{n=1}^{N/2-1} \hat{x}[n] \omega^{kn} \pm \hat{x}[n] \omega^{kn}.
\]

The spectrum of \( x_+ \) comprises only non-negative frequencies and vice versa for \( x_- \). We have \( x = \Re(x_+) \) and \( \Im(x_+) = \pm h \). The signals \( x_\pm \) are referred to as periodic analytic signals.

The signal \( h \) is a discrete periodic version of the Hilbert transform (HT) of a discrete-time periodic signal \( x \), that is \( h = H(x) \) (see [21], for example). Note that the DFT spectrum \( \hat{h}[n] \) lacks the values \( \hat{h}[0] \) and \( \hat{h}[N/2] \).

2.3 Analytic and quasi-analytic WPs

2.3.1 Analytic periodic signals

The analytic DTSWPs and their DFT spectra are derived from the corresponding WPs \( \{ \psi^{p}_{[m],l} \} \), \( m = 1, \ldots, M \), \( l = 0, \ldots, 2^m - 1 \), in line with the scheme in Section 2.3.1. Recall that for all \( l \neq 0 \), the DFT \( \psi^{p}_{[m],0}[0] = 0 \) and for all \( l \neq 2^m - 1 \), the DFT \( \psi^{p}_{[m],N/2} = 0 \).

Denote by \( \theta_{[m],l} = H(\psi^{p}_{[m],l}) \) the HT of the wavelet packet \( \psi^{p}_{[m],l} \). Then, the corresponding analytic DTSWPs are \( \psi^{p}_{\pm,[m],l} = \psi^{p}_{[m],l} \pm i\theta_{[m],l} \).

2Especially it is true for WPs derived from higher-order DTSs.
For a given decomposition level \( m \) by the addition of the two values \( \phi \) and their magnitude spectra. Addition of \( \hat{\phi} \) results in an antisymmetry distortion.

The sets of complex-valued WPs, which we refer to as the quasi-analytic differrent orthonormal bases for the space \( \Pi \frac{N}{m} \) are defined by \( \Psi \) wavelet packets (qWP). Similarly to the WPs \( \phi \) waveforms. For \( \phi \) \( \leq m \), such that
\[
\varphi_{m,l}^p[n] = \tilde{\varphi}_{m,l}^p[n] + \varphi_{m,l}^0[0] + \varphi_{m,l}^p[N/2].
\]

For all \( l \neq 0, 2^m - 1 \), the signals \( \varphi_{m,l}^p \) coincide with \( \theta_{m,l}^p = H(\varphi_{m,l}^p) \).

**Proposition 2.2**

- The magnitude spectra \( |\varphi_{m,l}^p[n]| \) coincide with the magnitude spectra of the respective WPs \( \psi_{m,l}^p \).

- For any \( m = 1, \ldots, M \), and \( l = 1, \ldots, 2^m - 2 \), the signals \( \varphi_{m,l}^p \) are antisymmetric oscillating waveforms. For \( l = l_0, l_m \), the shapes of the signals are near antisymmetric.

- The orthonormality properties that are similar to the properties of WPs \( \psi_{m,l}^p \) hold for the signals \( \varphi_{m,l}^p \) such that
\[
\langle \varphi_{m,l}^p[-p 2^m], \varphi_{m,l}^p[-s 2^m] \rangle = \delta[l, p] \delta[l, s].
\]

Figure 2.1 displays the signals \( \psi_{3,l}^0 \) and \( \varphi_{3,l}^0 \), \( l = 0, \ldots, 7 \), from the third decomposition level and their magnitude spectra. Addition of \( \psi_{3,l}^0[0] \) and \( \varphi_{3,l}^0[N/2] \) to the spectra of \( \varphi_{3,l}^0 \), \( l = 0, 7 \) results in an antisymmetry distortion.

We call the signals \( \psi_{m,l}^{p^\dagger} \), \( m = 1, \ldots, M \), \( l = 0, \ldots, 2^m - 1 \), the **complementary wavelet packets** (cWP). Similarly to the WPs \( \psi_{m,l}^p \), different combinations of the cWPs can provide different orthonormal bases for the space \( \Pi \frac{N}{m} \).

**Quasi-analytic WPs:** The sets of complex-valued WPs, which we refer to as the quasi-analytic wavelet packets (qWP), are defined by \( \Psi_{\pm m,l}^p = \psi_{m,l}^p \pm i\varphi_{m,l}^p \), \( m = 1, \ldots, M \), \( l = 0, \ldots, 2^m - 1 \), where \( \varphi_{m,l}^p \) are the cWPs from Eq. (2.5). The qWPs \( \Psi_{\pm m,l}^p \) differ from the analytic WPs \( \psi_{m,l}^p \) by the addition of the two values \( \pm i\varphi_{m,l}^p[0] \) and \( \pm i\varphi_{m,l}^p[N/2] \) into their DFT spectra, respectively. For a given decomposition level \( m \), these values are zero for all \( l \) except for \( l_0 = 0 \) and \( l_m = 2^m - 1 \).
It means that for all \( l \) except for \( l_0 \) and \( l_m \), the qWPs \( \Psi^p_{\pm[m+1],\rho} \) are analytic. The DFTs of qWPs are

\[
\hat{\Psi}^p_{[m],l}[n] = \begin{cases} 
(1 + i)\hat{\psi}^p_{[m],l}[n], & \text{if } n = 0, N/2; \\
2\hat{\psi}^p_{[m],l}[n], & \text{if } 0 < n < N/2; \\
0, & \text{if } N/2 < n < N,
\end{cases}
\hat{\psi}^p_{[m],l}[n] = \begin{cases} 
(1 - i)\hat{\psi}^p_{[m],l}[n], & \text{if } n = 0, N/2; \\
0, & \text{if } 0 < n < N/2; \\
2\hat{\psi}^p_{[m],l}[n], & \text{if } N/2 < n < N.
\end{cases}
\]

### 2.3.3 Design of cWPs and qWPs

The DFTs of the first-level DTSWPs are \( \hat{\psi}^p_{[1],0}[n] = \beta[n], \) \( \hat{\psi}^p_{[1],1}[n] = \omega^n \beta[n + N/2] = \alpha[n], \) where the sequences \( \beta[n] \) and \( \alpha[n] \) are defined in Eq. (2.2). The DFTs of the first-level cWPs are

\[
\hat{\varphi}^p_{[1],0}[n] = \begin{cases} 
-i \beta[n], & \text{if } 0 < n < N/2; \\
i \beta[n], & \text{if } N/2 < n < N; \\
\sqrt{2}, & \text{if } n = 0; \\
0, & \text{if } n = N/2,
\end{cases}
\hat{\varphi}^p_{[1],1}[n] = \begin{cases} 
-i \alpha[n], & \text{if } 0 < n < N/2; \\
i \alpha[n], & \text{if } N/2 < n < N; \\
0, & \text{if } n = 0; \\
-\sqrt{2}, & \text{if } n = N/2.
\end{cases}
\]

The cWPs and qWPs from the second and further decomposition levels are derived iteratively using the same p-filter banks as the DTFWPs.

**Proposition 2.3** ([1]) Assume that for a DTSWP \( \psi^p_{[m+1],\rho} \) the relation in Eq. (2.4) holds. Then, for the cWP \( \varphi^p_{[m+1],\rho} \) and qWP \( \Psi^p_{\pm[m+1],\rho} \) we have

\[
\varphi^p_{[m+1],\rho}[n] = \sum_{k=0}^{N/2-1} h^\mu_{[m+1]}[k] \varphi^p_{[m],\lambda}[n - 2^m k] \iff \varphi^p_{[m],\lambda}[\nu] = \hat{h}^\mu_{[1]}[2^m \nu] m \varphi^p_{[m],\lambda}[\nu],
\]

\[
\Psi^p_{\pm[m+1],\rho}[n] = \sum_{k=0}^{N/2-1} h^\mu_{[m+1]}[k] \Psi^p_{\pm[m],\lambda}[n - 2^m k] \iff \Psi^p_{\pm[m],\lambda}[\nu] = \hat{h}^\mu_{[1]}[2^m \nu] m \Psi^p_{\pm[m],\lambda}[\nu],
\]

\[
\hat{h}^0_{[1]}[\nu] = \hat{\psi}^p_{[1],0}[\nu] = \beta[\nu], \quad \hat{h}^1_{[1]}[\nu] = \hat{\psi}^p_{[1],1}[\nu] = \alpha[\nu].
\]

### 2.3.4 Implementation of cWP and qWP transforms

Implementation of transforms with DTSWPs \( \psi^p_{[m],\lambda} \) was discussed in Section 2.1. In this section, we extend the transform scheme to the transforms with cWPs \( \varphi^p_{[m],\lambda} \) and qWPs \( \Psi^p_{\pm[m],\lambda} \). The coefficients for the transform \( x \rightarrow \{ c^0_{[1]} \cup c^1_{[1]} \} \) with the cWPs \( \varphi^p_{[1],\mu}, \mu = 0, 1, \) are \( c^p_{[1],[l]} = \langle x, \varphi^p_{[1],\mu}[-2l] \rangle .\)

Denote \( \beta[n] = \begin{cases} \beta[0], & \text{if } n = 0; \\
-i\beta[n], & \text{otherwise},
\end{cases} \) \( \alpha[n] = \begin{cases} \alpha[N/2], & \text{if } n = N/2; \\
-i\alpha[n], & \text{otherwise}.
\end{cases} \) The sequences \( \beta[n] \) and \( \alpha[n] \) are defined in Eq. (2.2). The direct and inverse transforms are implemented using the analysis \( \mathbf{M}^c[n] \) and the synthesis \( \mathbf{M}^c[n] \) modulation matrices, respectively:

\[
\begin{bmatrix} \hat{c}^0_{[1]}[n] \\ \hat{c}^1_{[1]}[n] \end{bmatrix} = \frac{1}{2} \mathbf{M}^c[-n] \cdot \begin{bmatrix} \hat{x}[n] \\ \hat{x}[\bar{n}] \end{bmatrix}, \quad \begin{bmatrix} \hat{x}[n] \\ \hat{x}[\bar{n}] \end{bmatrix} = \mathbf{M}^c[n] \cdot \begin{bmatrix} \hat{c}^0_{[1]}[n] \\ \hat{c}^1_{[1]}[n] \end{bmatrix},
\]

where \( \bar{n} = n + N/2 \) and the modulation matrices are \( \mathbf{M}^c[n] = \begin{bmatrix} \beta[n] & -\beta[n + N/2] \\ \alpha[n] & -\alpha[n + N/2] \end{bmatrix} \), \( \mathbf{M}^c[n] = \mathbf{M}^c[n]^T, \ n = 0, \ldots N/2 - 1. \)
The coefficients for the transform \( \mathbf{x} \rightarrow \{ \mathbf{z}^{0}_{\pm1} \cup \mathbf{z}^{1}_{\pm1} \} \) with the qWPs \( \Psi^{p}_{\pm[1],\mu} \) \( \mu = 0, 1 \), are
\[
z^{j}_{\pm[1],[l]} = \langle \mathbf{x}, \Psi^{p}_{\pm[1],[l]} \rangle \].

The direct and inverse transforms are implemented using the analysis \( \mathbf{M}^{q}_{\pm}[n] \) and the synthesis \( \mathbf{M}^{c}[n] \) modulation matrices, respectively:
\[
\mathbf{M}^{q}_{\pm}[n] \overset{\text{def}}{=} \mathbf{M}[n] \pm i \mathbf{M}^{c}[n], \quad \mathbf{M}^{c}_{\pm}[n] \overset{\text{def}}{=} \mathbf{M}[n] \pm i \mathbf{M}^{c}[n].
\]

Application of the matrices \( \mathbf{M}^{q}_{\pm}[n] \) to the vector \((\hat{x}[n], \hat{x}[n])^{T}, \) where \( n = N/2 \), produces the vectors
\[
\left( z^{0}_{\pm[1],[n]} z^{1}_{\pm[1],[n]} \right) = \frac{1}{2} (\mathbf{M}^{q}_{\pm}[n])^{*} \cdot \left( \hat{y}^{0}_{[1],[1]} \hat{y}^{1}_{[1],[1]} \right) \mp i \left( z^{0}_{\pm[1],[1]} z^{1}_{\pm[1],[1]} \right).
\]

**Proposition 2.3** \( \square \) *Successive application of the filter banks \( \mathbf{H}^{q}_{\pm} \) and \( \mathbf{H}^{c}_{\pm} \) defined by the analysis and synthesis modulation matrices \( \mathbf{M}^{q}_{\pm}[n] \) and \( \mathbf{M}^{c}_{\pm}[n] \), respectively, to a signal \( \mathbf{x} \in \Pi[N] \) produces the analytic signals \( \mathbf{x}_{\pm} \) associated with \( \mathbf{x} : \mathbf{H}^{q}_{\pm} \mathbf{H}^{c}_{\pm} \mathbf{x} = 2\mathbf{x}_{\pm} \iff \mathbf{x} = 2\Re(\mathbf{H}^{q}_{\pm} \mathbf{H}^{c}_{\pm} \mathbf{x}).
\]

Proposition 2.3 implies that the cWPs \( \varphi^{p}_{[m+1],\rho} \) and qWPs \( \Psi^{p}_{\pm[1],\rho} \), \( m = 1, ..., M \), starting from the second decomposition level are derived from the WPs \( \varphi^{p}_{[m],\lambda} \) and \( \Psi_{\pm[1],\lambda} \) using the same p-filters as used for the transfer from \( \Psi^{p}_{[m],\lambda} \) to \( \Psi^{p}_{[m+1],\rho} \). A similar fact takes place for the transform from level \( m \) to level \( m + 1 \). To be specific, the transforms are implemented in an iterative way:
\[
\left( z^{\rho0}_{\pm[1],[m],[m+1]} z^{\rho1}_{\pm[1],[m],[m+1]} \right) = \frac{1}{2} \mathbf{M}[-2^{m}n] \cdot \left( z^{\lambda}_{\pm[1],[m],[m]} z^{\lambda}_{\pm[1],[m],[m]} \right), \quad \left( z^{\lambda}_{\pm[1],[m],[m]} z^{\lambda}_{\pm[1],[m],[m]} \right) = \mathbf{M}[2^{m}n] \cdot \left( z^{\rho0}_{\pm[1],[m],[m+1]} z^{\rho1}_{\pm[1],[m],[m+1]} \right),
\]
where \( \rho0 = \begin{cases} 2\lambda, & \text{if } \lambda \text{ is even}; \\ 2\lambda + 1, & \text{if } \lambda \text{ is odd}, \end{cases} \) and vice versa for \( \rho1, n = N/2^{m+1} \) and \( m = 1, ..., M \).

The modulation matrices \( \mathbf{M}[n] \) and \( \mathbf{M}[n] \) are given in Eq. \( \text{(2.3)} \).

By the application of the inverse DFT to the arrays \( \{ z^{\rho}_{\pm[1],[m],[m+1]} \} \), we get the arrays
\[
\{ z^{\rho}_{\pm[1],[m],[k]} = y^{\rho}_{[m+1],[k]} \pm i c^{\rho}_{[m+1],[k]} \}
\]
of the transform coefficients with the qWPs \( \Psi^{p}_{\pm[1],\rho} \). The transforms are executed in the spectral domain using the FFT by the application of critically sampled two-channel filter banks to the half-band spectral components \((x[n], x[n+1/2])^{T}) \) of a signal.

### 3 Two-dimensional complex wavelet packets

#### 3.1 Design of 2D directional WPs

The 2D wavelet packets defined by the tensor products of 1D DTSWPs \( \psi^{p}_{[m],\lambda,d}[k,n] = \psi^{p}_{[m],\lambda}[k] \psi^{p}_{[m],d}[n] \) and cWPs \( \varphi^{p}_{[m],\lambda,d}[k,n] = \varphi^{p}_{[m],\lambda}[k] \varphi^{p}_{[m],d}[n] \), possess many valuable properties but they lack the directionality, which is needed in many applications that process 2D data. However, real-valued 2D WPs oriented in multiple directions are derived from tensor products of complex qWPs \( \Psi^{p}_{\pm[1],\rho} \). The complex 2D qWPs are defined as follows:
\[
\Psi^{p}_{+\pm[1],d}[k,n] = \Psi^{p}_{+\pm[1],d}[k] \Psi^{p}_{-\pm[1],d}[n], \quad \Psi^{p}_{-\pm[1],d}[k,n] = \Psi^{p}_{+\pm[1],d}[k] \Psi^{p}_{-\pm[1],d}[n], \quad \Psi^{p}_{+\pm[1],d}[k,n] = \Psi^{p}_{+\pm[1],d}[k] \Psi^{p}_{-\pm[1],d}[n], \quad \Psi^{p}_{-\pm[1],d}[k,n] = \Psi^{p}_{+\pm[1],d}[k] \Psi^{p}_{-\pm[1],d}[n],
\]
Figure 3.1: WPs \( \varphi_{+2,j,l} \) from the second decomposition level and their magnitude spectra

Figure 3.2: WPs \( \varphi_{-2,j,l} \) from the second decomposition level and their magnitude spectra

where \( m = 1, ..., M \), \( j, l = 0, ..., 2^m - 1 \), and \( k, n = -N/2, ..., N/2 - 1 \). The real parts of these 2D qWPs are

\[
\begin{align*}
\varphi_{+m,j,l}^p[k,n] & \overset{\text{def}}{=} \Re\{\Psi_{+m,j,l}^p[k,n]\} = \psi_{+m,j,l}^p[k,n] - \varphi_{+m,j,l}^p[k,n], \\
\varphi_{-m,j,l}^p[k,n] & \overset{\text{def}}{=} \Re\{\Psi_{-m,j,l}^p[k,n]\} = \psi_{-m,j,l}^p[k,n] + \varphi_{-m,j,l}^p[k,n],
\end{align*}
\]

The DFT spectra of the 2D qWPs \( \Psi_{+m,j,l}^p \), \( j, l = 0, ..., 2^m - 1 \), are the tensor products of the one-sided spectra of the qWPs: \( \hat{\Psi}_{+m,j,l}^p[p,q] = \hat{\Psi}_{+m,j}^p[p] \hat{\Psi}_{+m,l}^p[q] \) and, as such, they tile the quadrant \( Q_0 \) of the frequency domain, while the spectra of \( \Psi_{-m,j,l}^p \), \( j, l = 0, ..., 2^m - 1 \), tile the quadrant \( Q_1 \) (see Eq. (1.1)). Consequently, the spectra of the real-valued 2D WPs \( \varphi_{+m,j,l}^p \) and \( \varphi_{-m,j,l}^p \) tile the pairs of quadrants \( Q_+ \overset{\text{def}}{=} Q_0 \cup Q_2 \) and \( Q_- \overset{\text{def}}{=} Q_1 \cup Q_3 \), respectively by relatively small squares. Such localizations and shapes of the spectra determine the directionality and shapes of the waveforms \( \varphi_{+m,j,l}^p \). They are, approximately, windowed cosines with multiple frequencies, which are oriented, at the level \( m \), in \( 2(2^m+1) \) directions.

Figures 3.1 and 3.2 display WPs \( \varphi_{+2,j,l}^p \) and \( \varphi_{-2,j,l}^p \), \( j, l = 0, 1, 2, 3, \) respectively, from the second decomposition level and their magnitude spectra.

### 3.2 Implementation of 2D qWP transforms

**One-level transform:** The spectra of the real-valued 2D WPs \( \{ \varphi_{+m,j,l}^p \}, j, l = 0, ..., 2^m - 1, \) and \( \{ \varphi_{-m,j,l}^p \} \) fill the pairs of quadrant \( Q_+ \overset{\text{def}}{=} Q_0 \cup Q_2 \) and \( Q_- \overset{\text{def}}{=} Q_1 \cup Q_3 \) (see Eq. (1.1)).
respectively (Figs. 3.1 and 3.2).

By this reason, none linear combination of the WPs \( \{ \vartheta_p^{[m],[j,l]} \} \) and their shifts can serve as a basis for the signal space \( \Pi[N,N] \). The same is true for WPs \( \{ \vartheta_{-p}^{[m],[j,l]} \} \). However, combinations of the WPs \( \{ \vartheta_p^{[m],[j,l]} \} \) provide frames of the space \( \Pi[N,N] \). The one-level 2D qWP transforms of a signal \( X = \{ X[k,n] \} \in \Pi[N,N] \) are implemented by a tensor-product scheme.

Denote by \( \tilde{T}_h \) and \( \tilde{T}_v \) the 1D transforms applicable to rows and columns of signals, respectively, with the analysis modulation matrices \( \tilde{M}_q^\pm \) which are defined in Eq. (2.6). We have

\[
\tilde{T}_h \pm X = (\zeta_0, \zeta_1), \quad \tilde{T}_v \zeta_j = Z_j^{[1]}, \quad \tilde{T}_v \zeta_j = Z_j^{-[1]}, \quad j = 0, 1.
\]

Denote by \( T_v \) the 1D inverse transform with the synthesis modulation matrix \( M_q^+ \) applicable to columns of the coefficient arrays and by \( T_h \) the 1D inverse transforms applicable to rows of the coefficient arrays with the synthesis modulation matrices \( M_q^\pm \). Subsequent application of these transforms to the coefficient arrays produces the 2D analytic signals:

\[
X_+ \overset{\text{def}}{=} T_h^+ \cdot T_v^+ \cdot Z_j^{[1]} \quad \text{and} \quad X_- \overset{\text{def}}{=} T_h^- \cdot T_v^+ \cdot Z_j^{-[1]}.
\]

Then, the signal \( X \in \Pi[N,N] \) is restored by \( X = \Re(2X_+ + X_-)/8 \).

Figure 3.3 illustrates the image restoration the “Barbara” image by the 2D signals \( \Re(2X_+ + X_-) \). The signal \( \Re(X_+) \) captures edges oriented to north-east, while \( \Re(X_-) \) captures edges oriented to north-west. The signal \( \tilde{X} = \Re(2X_+ + X_-)/8 \) perfectly restores the image achieving PSNR=313.8596 dB.

**Multi-level 2D transforms:** The 1D qWP transforms of a signal \( x \in \Pi[N] \) to the second and further decomposition levels and the respective inverse transforms are implemented by the successive application of the filter banks that are determined by their analysis \( \tilde{M}[2^m,n] \) and synthesis \( M[2^m,n] \) modulation matrices to the coefficients arrays. The same scheme is applicable to the tensor-product 2D qWP transforms. The coefficient arrays \( Z_j^{[1]} \) and \( Z_j^{-[1]} \), \( j = 0, 1 \) are processed separately using the same modulation matrices \( \tilde{M}[2^m,n] \) and \( M[2^m,n] \). The processing can be implemented in a parallel mode in the frequency domain using FFT.

### 4 Image inpainting methodology

This section presents two qWP-based methods: Method1 (M1) and Method2 (M2) that are applied to image inpainting. Image inpainting means restoration of an image that was degraded by having many missing pixels and, possibly, being corrupted by Gaussian noise. Both methods rely on the redundancy of the qWP transforms and on the interdependency between the transform coefficients in the horizontal (between neighboring coefficients) and vertical (parent-child) directions. This interdependency is utilized via the Bivariate Shrinkage algorithm (BSA) ([10]).
Further on, we refer to the set of the filter banks DAS-2, DAS-1, TP-CTF$_6$ and TP-CTF$_6^\downarrow$ as SET-4.

4.1 Method 1

This method is a slight modification of the inpainting Algorithm I introduced in [23]. Algorithm I uses the TP-CTF$_6$ filter bank that produces excellent image inpainting results that outperform (in a PSNR sense) most state-of-the-art methods (see [23] for methods comparison). Algorithm I in [9] is implemented using the SET-4 filter banks. On the cartoon-type images such as “Lena” and “Boat”, TP-CTF$_6$ achieves better performance than the rest of the members in SET-4. However, images with texture such as “Fingerprint” and, especially, “Barbara” are inpainted better by the filter bank DAS-2 with increased directionality compared to TP-CTF$_6$.

Algorithm I consists of iterative thresholding of the transform coefficients with decreasing localized thresholds. After setting a few parameters, the threshold levels are determined automatically by using BSA. In M1, the qWP transforms are used instead of applying SET-4 filter banks. The image is restored separately from the transform coefficients from the third and fourth decomposition levels and results are combined.

4.2 Method 2

The split Bregman iteration (SBI) scheme [13] is widely used for image restoration (see for example [18], Chapter 18 in [3]). M2 to be described couples M1 with the SBI scheme. In essence, it is the SBI algorithm supplied with decreasing localized thresholds determined by BSA. The problem to be solved is the estimation $\tilde{X}$ of an image $X$ from the available data array $\tilde{X} = P_\Theta(X + \varepsilon)$, where $\varepsilon = \{e_{k,n}\}$ is the i.i.d. Gaussian error array and $P_\Theta$ denotes the projection operator on the available set of pixels. This operator can be expressed via the application of a mask $\Theta$ to an image $X$. The mask $\Theta$ is a matrix of the same size as $X$, which consists of ones and zeros. Ones indicate the places of available pixels.

The solution scheme is based on the assumption that the original image $X$ can be sparsely represented in the qWP transform domain.

Description of M2:

Thresholding parameters $\lambda$: Assume that the image to be restored comprises $M$ pixels while $L$ pixels are missing. Denote by $\rho = L/M$ the percentage of missing pixels. Similarly to Algorithm I in [23], M2 uses decreasing thresholding values that are determined by the sequences $\Lambda_1[i]$ and $\Lambda_2[i]$ described in Eq. (4.2). Assume that the noise STD=$\sigma$ is known. Let $\lambda_{max} \overset{\text{def}}{=} 512$,

$$\lambda_{min} \overset{\text{def}}{=} \max \left\{ 1, \sigma \left( 1 - \frac{\rho^2}{2} \right) \right\}, \quad \lambda_{mid} \overset{\text{def}}{=} \min \{ \max \{ 2 \lambda_{min} + 10 \}, 20 \}.$$  \hspace{1cm} (4.1)

Let $r_1 \overset{\text{def}}{=} \lambda_{mid}/\lambda_{max} < 1$, $r_2 \overset{\text{def}}{=} \lambda_{min}/\lambda_{mid} < 1$. The sequences $\Lambda_1[i]$ and $\Lambda_2[i]$ are defined by

$$\Lambda_1[i] = \sqrt{2} r_1^{i-1} \lambda_{mid}, \quad i = 1, ..., R_1, \quad \Lambda_2[i] = \sqrt{2} r_2^{i-1} \lambda_{min}, \quad i = 1, ..., R_2$$  \hspace{1cm} (4.2)

where $R_1$ and $R_2$ are free parameters. In our experiments, we use $R_1 = 5$, $R_2 = 8$. 

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Bivariate shrinkage operator: Assume that \(c_m[k, n]\) is a transform coefficient from a decomposition level \(m\). The averaged variance for the coefficient \(c_m[k, n]\) is determined by the equation
\[
\tilde{\sigma}_W[k, n]^2 = \frac{1}{W^2} \sum_{\kappa, \nu = -W/2}^{W/2-1} c_m[k + \kappa, n + \nu]^2,
\]
where \(W\) is an even integer.

The marginal variance for the coefficient \(c[k, n]\) is estimated:
\[
\tilde{\sigma}_W[k, n]^2 = (\tilde{\sigma}_W[k, n]^2 - \lambda^2)_+,
\]
where \(\lambda\) is a parameter taken from either the sequence \(\Lambda_1\) or \(\Lambda_2\) by the rule explained below.

Then, the BSA operator \(B^\lambda_W\) is defined by
\[
B^\lambda_W c_m[k, n] \overset{\text{def}}{=} \begin{cases} 
  c_m[k, n] - \lambda_W(c) \frac{c_m[k, n]}{|c_m[k, n]|}, & \text{if } |c_m[k, n]| > \lambda_W(c); \\
  0, & \text{otherwise},
\end{cases}
\]
where \(c_m[k, n]\) is the transform coefficient from the coarser decomposition level \(m + 1\) which is located at approximately the same spatial position as the coefficient \(c_m[k, n]\) and, in our case, it is related to the waveform with approximately the same directionality as the waveform related to \(c_m[k, n]\). If \(C_m = \{c_m[k, n]\}\) is the set of all transform coefficients from the decomposition level \(m\), then \(B^\lambda_W C_m\) is the application of the operator \(B^\lambda_W\) to each coefficient \(c_m[k, n] \in C_m\).

Implementation of image inpainting by M2: Denote by \(\hat{F}^+\) and \(\hat{F}^-\) the operators of the transform of an image \(X\) with the complex qWPs \(\Psi^{2r}_{++}[m,j,l}\) and \(\Psi^{2r}_{+-}[m,j,l}\), respectively, to the coefficients of decomposition level \(m\) such that \(\hat{F}^\pm X = C^\pm_m\). Denote by \(F^+_m\) and \(F^-_m\) the operators of the images \(X^+_m\) and \(X^-_m\), respectively. Denote by \(F_m\) the operator of the full reconstruction of the image from the coefficient array \(C_m = C^+_m \cup C^-_m\), that means \(F_m C_m = X_m = \Re(X^+_m + X^-_m)/8\). The solution is obtained by a weighted average of reconstruction results from either two or three sets of the transform coefficients \(C_m\), where either \(m = 3, 4\) or \(m = 2, 3, 4\).

- In order to eliminate boundary effects, the degraded image \(\hat{X}\) of size \(N \times N\) is symmetrically extended to the image \(Y\) of size \(N_1 \times N_1\), where \(N_1 = N + 2T\). Typically, \(T = N/8\). Respectively, the mask \(\Theta\) is extended.

- The free parameters’ values are set in the following way: 1. Order \(p\) of the DTS that generates qWPs (in most cases \(p = 3, 4\) or 5). 2. Integers \(R_1\) and \(R_2\) and tolerance parameters \(tol_1\) and \(tol_2\) (typically, \(tol_1 = 0.05\) and \(tol_2 = 0.01\)). 3. Iterations limits \(L_1\), \(L_2\) and \(L_3\); 4. Windows spans \(\{W_m\}\) for the averaged variances’ \(\tilde{\sigma}_W[k, n]\) calculation and balance factors \(\{\beta_m\}\), where \(m = (2,)3, 4\). 5. Regularization parameter \(\mu\).

- Initialization: Let \(v^0 = 0, b^0_+ = 0, b^0_- = 0, b^0 = b^0_+ \cup b^0_-, d^0_+ = 0, d^0_- = 0, d^0 = d^0_+ \cup d^0_-, \) where \(m = (2,)3, 4\). The thresholding parameter is \(\lambda = \Lambda_1[1]\). Then, the iterations start:

  1. Apply the inverse qWP transforms to the arrays \(D^k_m = d^k_m - b^k_m\), where \(m = (2,)3, 4,\)

\[
\left\{V^k_m \overset{\text{def}}{=} F_m D^k_m, \quad \sum_m \beta_m V^k_m \overset{\text{def}}{=} \sum_m \beta_m V^k_m \right\},
\]

\[
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\]
2. The next iteration $v^{k+1}$ is derived from Eq. (4.4) that is solved by the conjugate gradient algorithm:
\[
v^{k+1} := (P_\Theta + \mu I) v = Y + \mu V^k.
\] (4.4)

3. Apply the forward qWP transforms to the array $v^{k+1}$: $F_m^\pm v^{k+1} = C_m^\pm$, and denote $\bar{C}_m^\pm = C_m^\pm + b_{x_m}^\pm$, where $m = (2,3,4,5$).

4. Select the thresholding parameter $\lambda/(stop the iterations)$ by the Select/stop rule (below).

5. Update the arrays $d_{x_m}^k$ by application of the BSA operators $B_{x_m}^{1/2}$, $m = (2,3,4,5$, to the arrays $\bar{C}_m^\pm$: $d_{x_m}^{k+1} = B_{x_m}^{1/2} \bar{C}_m^\pm$, $d_{x_m}^{k+1} = d_{x_m}^{k+1} \bigcup d_{x_m}^{k+1}$, $m = (2,3,4$.

6. Update the arrays $b_{x_m}^k$ by $b_{x_m}^{k+1} = b_{x_m}^k + \left(\bar{C}_m^\pm - d_{x_m}^{k+1}\right)$, $b_{x_m}^{k+1} = b_{x_m}^{k+1} \bigcup b_{x_m}^{k+1}$, $m = (2,3,4$.

7. Go to item 1.

Select/stop rule: The rule is, to some extent, similar to the rule in Algorithm I in [23]. Iterations limits are $L_1$, $L_2$ and $L_3$. Initial thresholding parameter is $\lambda = \Lambda_1[1]$ and $\nu = 1$. Denote the number of iterations with a certain index $\nu$ by $K_\nu$.

- The $l_2$ norm of the difference $\Delta^{k+1} = \|v^{k+1} - v^k\|_2$ is computed,
- If $\nu < R_1$ and \( \{ K_\nu > L_1 \text{ or } \Delta^{k+1} < \text{tol}\} \), then $\nu = \nu + 1$ and $\lambda = \Lambda_1[\nu]$.
- If $R_1 \leq \nu < R_1 + R_2$ and \( \{ K_\nu > L_2 \text{ or } \Delta^{k+1} < \text{tol}_2\} \), then $\nu = \nu + 1$ and $\lambda = \Lambda_2[\nu - R_1]$.
- If $\nu = R_1 + R_2$ and \( \{ K_\nu > L_3 \text{ or } \Delta^{k+1} < \text{tol}_2\} \), then STOP iterations.
- The array $v^{k+1}$ is shrunk to the original size $N \times N$ as $v^{k+1} \rightarrow \bar{X}$, and the array $\bar{X}$ is taken as a solution to the inpainting problem.

5 Experimental results

For the experiments, we used a standard set of benchmark images: “Lena”, “Boat”, “Barbara”, and “Fingerprint”. The “clean” images of size 512 × 512 pixels are displayed in Fig. 5.1. The performances of M1 and M2 are compared with the performances of the state-of-the-art inpainting algorithms that use the SET-4 filter banks. The inpainting results by the SET-4 algorithms of the four images from Fig. 5.1 which are degraded by application of four masks displayed in Fig. 5.1 and additive Gaussian noise of various intensities (noise STD $\sigma = 0, 5, 10, 30, 50$ dB), are presented in [9]. The Matlab codes, which produce the results in [9], are available at the web site [http://staffweb1.cityu.edu.hk/xzhuang7/softs/index.html#bdTPCTF]. For each triple image-mask-$\sigma$, the inpainting results from the application of our M1 and M2 are compared with the best result produced by the application of SET-4 algorithms. The results are compared according to PSNR and SSIM values and by visual perception.

5.1 Experiments with “Barbara” and “Fingerprint” images

These images have complicated structure which comprises smooth regions, multiple edges (in “Barbara” ) and oscillatory patterns (in both images). The restoration results in PSNR and SSIM measurements are given in Table 5.1.
| σ  | M1       | M2       | Best from SET-4 | M1       | M2       | Best from SET-4 |
|----|----------|----------|-----------------|----------|----------|-----------------|
|    | mask1    | mask1    | mask2           | mask2    | mask3    | mask4           |
| 0  | 38.13/0.958 | 38.33/0.965 | 36.68/0.958 4  | 31.93/0.978 | 32.64/0.980  | 31.72/0.976 4  |
| 5  | 34.93/0.8256 | 34.94/0.831 | 34.05/0.8138(1) | 30.69/0.963 | 30.86/0.964  | 30.36/0.960 1  |
| 10 | 32.57/0.748 | 32.56/0.753 | 32.02/0.739 1  | 28.96/0.939 | 29.09/0.940  | 28.77/0.935 1  |
| 30 | 28.09/0.601 | 28.01/0.603 | 27.74/0.592 1  | 25.15/0.843 | 25.22/0.848  | 25.04/0.839(1) |
| 50 | 25.85/0.501 | 25.77/0.504 | 25.51/0.489 1  | 23.26/0.767 | 23.42/0.778  | 23.21/0.766 1  |
|    | 0        | 34.26/0.922 | 34.70/0.932   | 33.66/0.923 1  | 28.76/0.949 | 28.94/0.950  | 28.31/0.942 4  |
| 5  | 32.78/0.80  | 32.75 /0.806 | 32.24/0.787 1  | 28.02/0.938 | 28.31/0.936  | 27.68/0.926 1  |
| 10 | 31.13/0.730 | 31.11 /0.734 | 30.71/0.718 1  | 27.01/0.908 | 27.08/0.911  | 26.76/0.903 1  |
| 30 | 27.22/0.579 | 27.2/0.585  | 26.95/0.570 1  | 24.16/0.812 | 24.35/0.822  | 24.15/0.809 1  |
| 50 | 25.21/0.487 | 25.13/0.487 | 24.96/0.471 1  | 22.58/0.739 | 22.77/0.753  | 22.65/0.738 1  |
|    | 0        | 37.30/0.901 | 37.45/0.913   | 35.72/0.9051 1  | 34.53/0.979 | 34.54/0.979  | 34.19/0.977 4  |
| 5  | 34.24/0.789 | 34.21/ 0.793 | 33.29/0.778 1  | 31.72/0.958 | 31.84/0.960  | 31.54/0.957 4  |
| 10 | 31.82/0.721 | 31.85/0.723 | 31.12/0.707 1  | 29.14/0.926 | 29.37/0.932  | 29.09/0.927 4  |
| 30 | 27.1/0.571  | 27.17/0.571 | 26.77/0.554 1  | 24.51/0.809 | 24.73/0.825  | 24.43/0.806 4  |
| 50 | 24.84/0.460 | 24.9/0.467  | 24.60/0.441 1  | 22.48/0.721 | 22.69/0.748  | 22.53/0.722 1  |
|    | 0        | 30.17/0.7609 | 30.34/0.779 | 29.12/0.753 1  | 27.03/0.899 | 27.02/0.9   | 26.77/0.894 4  |
| 5  | 29.28/0.687 | 29.44/0.696 | 28.41/0.673 1  | 26.27/0.878 | 26.29/0.881  | 25.87/0.872 4  |
| 10 | 28.01/0.626 | 28.19/0.633 | 27.25/0.609 1  | 25.11/0.843 | 25.11/0.845  | 24.60/0.834 4  |
| 30 | 24.59/0.452 | 24.45/0.461 | 24.23/0.434(1) | 22.09/0.712 | 22.15/0.731  | 22.11/0.715 1  |
| 50 | 22.67/0.347 | 22.57/0.363 | 22.35/0.325 1  | 20.32/0.613 | 20.62/0.646  | 20.60/0.614 1  |

Table 5.1: PSNR/SSIM values for the restoration of “Barbara” (columns 2–4) and “Fingerprint” (columns 5–7) images by M1, M2 and the best algorithm from SET-4. Top – mask1; Second from top – mask2; Second from bottom – mask3; Bottom –mask4. Boldface highlights the best results. σ – noise STD. Numbers in brackets indicate which is the best algorithm from SET-4 that produces the best result: (1) means DAS-2, (2)–DAS-1, (3)–TP-CTF<sub>6</sub>, (4)–TP-CTF<sub>↓6</sub>.
Figure 5.1: Left: Clean images: “Lena”, “Boat”, “Barbara” and “Fingerprint”. Right: Four types of masks: Mask1 and Mask2 – text and curves; Mask3 and Mask4 – 50% and 80% of random pixels missing, respectively.

Figures 5.2 and 5.3 illustrate the restoration of “Barbara”, which was degraded by the application of mask2 and strong additive Gaussian noise with $\sigma = 50$ dB, by M2 and DAS-2. Both algorithms successfully inpainted the mask and suppressed the noise but, in doing so, DAS-2 loses thin texture details and adds some artifacts into the image. Figure 5.3 demonstrates it.

Figures 5.4 and 5.5 illustrate the restoration of the “Fingerprint” image, which was severely degraded by the application of mask4 (80% of the pixels are missing) and strong additive Gaussian noise with $\sigma = 50$ dB, by M2 and DAS-2. In this case, the PSNR for restoration by DAS-2 is almost the same as for the restoration by M2, which is not the case for the SSIM. Both algorithms successfully inpainted the mask and suppressed the noise but DAS-2 over-smoothed some parts of the image. It is seen in Fig. 5.5.

5.2 Experiments with “Lena” and “Boat” images

These images are characterized by a number of edges separating smooth areas and some fine details such as hair and the hat decoration in “Lena” and ground in “Boat”. The restoration results in PSNR and SSIM measurements are given in Table 5.2.

Figures 5.6 and 5.7 display the restoration results of the “Lena” image, which was severely degraded by the application of mask3 (50% of pixels are missing), and strong Gaussian noise with $\sigma = 50$ dB, by M2 and TP-CTF6. Although the PSNR of the restoration by TP-CTF6 is higher than the one from M2. On the other hand, the SSIM values are close to each other while fine details, which were lost by TP-CTF6, are retained by M2.

Figures 5.8 and 5.9 display the results of the “Boat” restoration, which was degraded by the application of mask3 (50% of pixels are missing) and strong additive Gaussian noise with $\sigma = 50$ dB is added, by M2 and TP-CTF6. Although the PSNR for restoration by TP-CTF6 is higher than
Figure 5.2: Restoration of “Barbara” image. Left: Top left: clean image. Top right: image degraded by application of mask2 and Gaussian noise with $\sigma = 50$ dB. Bottom: Restoration by M2 (left), PSNR=25.13 dB and by DAS-2 (right), PSNR=24.96 dB. Right: SSIM maps for image restored by DAS-2 (top), SSIM=0.4709 and by M2 (bottom), SSIM=0.4868.

Figure 5.3: Fragments of “Barbara” images displayed in Fig. 5.2 restored by M2 (left) and DAS-2 (right).
Figure 5.4: Restoration of “Fingerprint” image. Left: Top: clean image (left) and image degraded by the application of mask4 and Gaussian noise with $\sigma = 50$ dB (right). Bottom: Restoration by $\textbf{M2}$ (left), PSNR=20.62 dB, and by DAS-2 (right), PSNR=20.61 dB. Right: SSIM maps for image restored by DAS-2 (top), SSIM=0.6138, and by $\textbf{M2}$ (bottom), SSIM=0.6456.

Figure 5.5: Fragments of “Fingerprint” images displayed in Fig. 5.4 restored by $\textbf{M2}$ (left) and DAS-2 (right).
| σ  | M1     | M2     | Best from SET-4 | M1     | M2     | Best from SET-4 |
|----|--------|--------|-----------------|--------|--------|-----------------|
|    |        |        | mask1           |        |        |                 |
| 0  | 37.50/0.9371 | 37.84/0.9488 | **38.02/0.9487 (3)** | 34.78/0.936 | 34.95/0.94072 | **34.96/0.94071 (3)** |
| 5  | 34.94/0.702 | 34.95/0.724 | 35.19/0.672 (3) | 32.68/0.778 | 32.68/0.792 | **32.81/0.752 (3)** |
| 10 | 33.18/0.605 | 33.22/0.614 | **33.42/0.586 (3)** | 30.79/0.659 | 30.75/0.674 | **31.04/0.630 (3)** |
| 30 | 29.34/0.449 | 29.43/0.454 | **29.81/0.449 (3)** | 27.03/0.454 | 27.02/0.463 | **27.41/0.433 (3)** |
| 50 | 27.31/0.370 | 27.40/0.377 | **27.85/0.377 (3)** | 25.22/0.348 | 25.15/0.360 | **25.6/0.334 (2)** |
|    |        |        | mask2           |        |        |                 |
| 0  | 33.72/0.884 | 33.96/0.9005 | **34.3/0.9001 (3)** | 30.23/0.874 | 30.57/0.880 | **30.80/0.881 (3)** |
| 5  | 32.48/0.661 | 32.43/0.692 | **32.97/0.647 (3)** | 29.30/0.721 | 29.29/0.741 | **29.79/0.718 (1)** |
| 10 | 31.39/0.582 | 31.37/0.593 | **31.78/0.567 (3)** | 28.18/0.611 | 28.25/0.630 | **28.85/0.59 (2)** |
| 30 | 28.39/0.432 | 28.51/0.440 | **28.89/0.434 (3)** | 25.78/0.417 | 25.83/0.430 | **26.34/0.407 (2)** |
| 50 | 26.68/0.356 | 26.65/0.364 | **27.22/0.362 (3)** | 24.33/0.319 | 24.37/0.334 | **24.84/0.312 (2)** |
|    |        |        | mask3           |        |        |                 |
| 0  | 37.70/0.813 | 37.93/0.850 | **38.0/0.852 (3)** | 34.39/0.8518 | 34.43/0.862 | **34.42/0.858 (3)** |
| 5  | 35.14/0.647 | 35.17/0.660 | **35.4/0.632 (3)** | 32.26/0.700 | 32.11/0.724 | **32.50/0.684 (3)** |
| 10 | 33.03/0.568 | 33.04/0.577 | **33.40/0.558 (3)** | 30.3/0.598 | 30.19/0.617 | **30.65/0.586 (3)** |
| 30 | 28.67/0.418 | 28.81/0.425 | **29.18/0.421 (3)** | 26.33/0.405 | 26.45/0.427 | **26.66/0.380 (3)** |
| 50 | 26.51/0.341 | 26.65/0.349 | **27.06/0.346 (3)** | 24.43/0.299 | 24.49/0.335 | **24.75/0.278 (3)** |
|    |        |        | mask4           |        |        |                 |
| 0  | 32.09/0.6311 | 32.12/0.67 | **32.33/0.674 (3)** | 28.39/0.629 | 28.25/0.644 | **28.58/0.643 (3)** |
| 5  | 31.18/0.552 | 31.15/0.553 | **31.44/0.551 (3)** | 27.7/0.544 | 27.71/0.558 | **27.98/0.546 (3)** |
| 10 | 29.86/0.484 | 29.84/0.4874 | **30.25/0.487(3)** | 26.77/0.465 | 26.73/0.480 | **27.08/0.465 (3)** |
| 30 | 26.40/0.3406 | 26.52/0.349 | **26.95/0.347 (3)** | 24.01/0.278 | 24.0/0.308 | **24.46/0.277 (3)** |
| 50 | 24.25/0.260 | 24.47/0.2753 | **25.15/0.2746 (3)** | 22.50/0.196 | 22.22/0.233 | **22.96/0.193 (3)** |

Table 5.2: PSNR/SSIM values for the restoration of “Lena” (columns 2–4) and “Boat” (columns 5–7) images by M1, M2 and the best from SET-4 algorithms. Top – mask1; Second from top – mask2; Second from bottom – mask3; Bottom –mask4. Boldface highlights the best results. σ – noise STD. Numbers in brackets indicate which algorithm in SET-4 produces the best result: (1) means DAS-2, (2)–DAS-1, (3)–TP-CTF₆, (4)–TP-CTF⁶↓.
that by \textbf{M2}, the \textbf{M2}-SSIM value is much higher than that produced by TP-\text{CTF}_6. The image restored by TP-\text{CTF}_6 looks better over-smoothed areas compared to restoration by \textbf{M2}. Some edges and almost the entire texture were lost by the application of TP-\text{CTF}_6 but were retained by \textbf{M2}.

6 Discussion

The paper presents two methods for image inpainting: \textbf{M1} and \textbf{M2}. These methods are based on the directional quasi-analytic WPs (qWPs) originated from polynomial splines of arbitrary order that are designed in [1]. The computational scheme in \textbf{M1} is a slight modification of the inpainting by \textbf{Algorithm I} introduced in [23]. \textbf{M2} is a couples \textbf{M1} with the Split Bregman Iterations (SBI) scheme. In essence, it is the SBI algorithm supplied with decreasing localized thresholds that is determined by BSA.

The successful application of \textbf{M1} and \textbf{M2} to image inpainting stems from the exclusive properties of qWPs such as:

- The qWP transforms provide a variety of 2D waveforms oriented in multiple directions. For example, fourth-level qWPs are oriented in 62 different directions.

- The waveforms are close to directional cosines with a variety of frequencies modulated by spatially localized low-frequency 2D signals.
The waveforms may have any number of local vanishing moments (see [1]).

The DFT spectra of the waveforms produce a refined tiling of the frequency domain.

Fast implementation of the transforms by using FFT enables us to use the transforms with increased redundancy.

In multiple experiments, the performances of $\mathbf{M}_1$ and $\mathbf{M}_2$ are compared with the performances of the-state-of-the-art inpainting algorithms that use the SET-4 filter banks. The inpainting results for 4 images by the SET-4 algorithms, which are degraded by the application of 4 masks (Fig. 5.1) and additive Gaussian noise of various intensities (noise STD $\sigma = 0, 5, 10, 30, 50$ dB), are presented in [9]. For each triple image-mask-$\sigma$, the inpainting results by $\mathbf{M}_1$ and $\mathbf{M}_2$ vs. SET-4 algorithms are compared through PSNR, SSIM values and visual perception. In all the experiments, our methods succeeded in inpainting even in severely degraded images and in suppressing noise. In most cases, thin structures were recovered. On images with complicated structures such as “Barbara” and “Fingerprint”, $\mathbf{M}_1$ and especially $\mathbf{M}_2$ significantly outperform the SET-4 algorithms in all three parameters - PSNR, SSIM and visual perception. As for “Lena” and “Boat” images, while in most experiments SET-4 algorithms produce higher PSNR values compared to $\mathbf{M}_1$ and $\mathbf{M}_2$, the SSIM values are higher for $\mathbf{M}_1$ and especially for $\mathbf{M}_2$. This fact indicates that our $\mathbf{M}_1$ and $\mathbf{M}_2$ are superior over the SET-4 algorithms for image structure recovery. Visual observations confirm that claim, which is illustrated in Figs. 5.6–5.9.

Generally, we claim that our $\mathbf{M}_1$ and especially $\mathbf{M}_2$, which are based on directional quasi-analytic wavelet packets originated from polynomial splines, are powerful tools for images restoration that are severely degraded by loss of the majority of the pixels and have strong additive noise.
Figure 5.8: Restoration of “Boat” image. Left: Top: clean image (left) and image degraded by the application of mask3 and Gaussian noise with $\sigma = 50$ dB (right). Bottom: Restoration by M2 (left), PSNR=24.49 dB, and by TP-CTF$_6$ (right), PSNR=24.74 dB. Right: SSIM maps for image restored by TP-CTF$_6$ (top), SSIM=0.2777, and by M2 (bottom), SSIM=0.3347.

Figure 5.9: Fragments of images displayed in Fig. 5.8. Left: restored by M2. Right: restored by TP-CTF$_6$. 

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Figure 6.1: Restoration of the “Mandrill” image. Top left: clean image. Top right: image degraded by the application of mask4 with additive Gaussian noise with $\sigma = 50$ dB. Bottom left: M2 restoration, PSNR=19.37 dB, SSIM=0.2185. Bottom right: DAS-2 restoration, PSNR=19.81 dB, SSIM=0.1414

These methods succeed in capturing fine details in the images in cases where other state-of-the-art algorithms fail. This fact is illustrated in Fig. 6.1 which displays the restoration results of the ‘Mandrill” image from the input where 80% of pixels are missing and additive Gaussian noise with $\sigma = 50$ dB is present. The output from DAS-2 algorithm (which was the best for this image in SET-4) has PSNR=19.81 dB compared to 19.37 dB produced by M2. On the other hand, the SSIM from the M2 restoration is 0.2185 compared to 0.1414 produced by DAS-2. Figure 6.1 is a good illustration to the fact that SSIM is a much more informative characteristics than PSNR.

Summarizing, by having such a versatile and flexible tool at hand, we are in a position to address several data processing problems such as image denoising, deblurring, superresolution, segmentation and classification, target detection (here the directionality is of utmost importance). The 3D directional wavelet packets, whose design is underway, may be beneficial for seismic and hyper-spectral processing.

We did not compare the qWP-based methods performances with the performance of the schemes based on deep learning (DL). However, we believe that the designed directional qWPs can boost the quality of DL-based inpainting (and others) methods by serving as a powerful tool for characteristic features extraction from images. This is a subject of our current research work.

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