A Unifying Framework for the $\nu$-Tamari Lattice and Principal Order Ideals in Young’s Lattice

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Abstract

We present a unifying framework in which both the $\nu$-Tamari lattice, introduced by Préville-Ratelle and Viennot, and principal order ideals in Young’s lattice indexed by lattice paths $\nu$, are realized as the dual graphs of two combinatorially striking triangulations of a family of flow polytopes which we call the $\nu$-caracol flow polytopes. The first triangulation gives a new geometric realization of the $\nu$-Tamari complex introduced by Ceballos et al. We use the second triangulation to show that the $h^*$-vector of the $\nu$-caracol flow polytope is given by the $\nu$-Narayana numbers, extending a result of Mészáros when $\nu$ is a staircase lattice path. Our work generalizes and unifies results on the dual structure of two subdivisions of a polytope studied by Pitman and Stanley.

Keywords Flow polytope · Triangulation · $\nu$-Dyck path · $\nu$-Tamari lattice · Young’s lattice

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1 Introduction

Flow polytopes are a family of beautiful mathematical objects. They appear in optimization theory as the feasible sets in maximum flow problems and they also appear in other areas of mathematics including representation theory and algebraic combinatorics. In the following, \( G = (V, E) \) denotes a connected directed graph with vertex set \( V = \{1, 2, \ldots, n+1\} \) and edge multiset \( E \) with \( m \) edges, with \( n, m \in \mathbb{N} \). We assume that any edge \((i, j) \in E\) is directed from \( i \) to \( j \) whenever \( i < j \) and hence \( G \) is acyclic. At each vertex \( i \in V \) we assign a net flow \( a_i \in \mathbb{Z} \) satisfying the balance condition \( \sum_{i=1}^{n+1} a_i = 0 \), and hence \( a_{n+1} = - \sum_{i=1}^{n} a_i \). For \( a = (a_1, \ldots, a_n, - \sum_{i=1}^{n} a_i) \in \mathbb{Z}^{n+1} \), an \( a \)-flow on \( G \) is a tuple \((x_e)_{e \in E} \in \mathbb{R}_0^m \) such that

\[
\sum_{e \in \text{out}(j)} x_e - \sum_{e \in \text{in}(j)} x_e = a_j
\]

where \( \text{in}(j) \) and \( \text{out}(j) \) respectively denote the set of incoming and outgoing edges at \( j \), for \( j = 1, \ldots, n \). In what follows, by a graph \( G \) we mean a connected directed acyclic graph whose sets \( \text{out}(1), \text{in}(n+1) \), and \( \text{in}(j) \) and \( \text{out}(j) \) for \( j = 2, \ldots, n \), are nonempty. The flow polytope of \( G \) with net flow \( a \) is the set \( F_G(a) \) of \( a \)-flows on \( G \). In this article we only consider flow polytopes with unitary flow \( a = e_1 - e_{n+1} \), where \( e_i \) for \( i = 1, \ldots, n+1 \) denotes the standard basis in \( \mathbb{R}^{n+1} \), and we will abbreviate the flow polytope of \( G \) with unitary flow as \( F_G \). In this case, the only integral points of \( F_G \) are its vertices, which correspond to the unitary flows along maximal directed paths of \( G \) from vertex 1 to \( n+1 \). Such maximal paths are called routes (see Fig. 7).

A \( d \)-simplex is the convex hull of \( d + 1 \) points in general position in \( \mathbb{R}^k \) with \( k \geq d \). A (lattice) triangulation of a \( d \)-polytope \( P \) is a collection \( \mathcal{T} \) of \( d \)-simplices each of whose vertices are in \( P \cap \mathbb{Z}^d \), such that the union of the simplices in \( \mathcal{T} \) is \( P \), and any pair of simplices intersect in a (possibly empty) common face. If in a triangulation \( \mathcal{T} \) we consider, in addition to the collection of \( d \)-simplices, their sets of lower dimensional faces we get the structure of a simplicial complex. In this context we will be referring to the collection of \( d \)-simplices as the top-dimensional faces (or facets) of the triangulation viewed as a simplicial complex. The normalized volume of a \( d \)-polytope is \( d! \) times its Euclidean volume. Since the Euclidean volume of a unimodular \( d \)-simplex in \( \mathbb{R}^d \) is \( \frac{1}{d!} \), then by enumerating the top-dimensional simplices in a unimodular triangulation, one can compute the normalized volume of the polytope.

Baldoni and Vergne [1] gave a set of formulas to determine the normalized volume of \( F_G(a) \), these are known as the Lidskii formulas. Mészáros and Morales [2] described a triangulation approach due to Postnikov and Stanley, providing an alternative proof of the Lidskii formulas. In [3], Benedetti et al. introduced two families of combinatorial objects, known as gravity diagrams and unified diagrams, as combinatorial interpretations for the Lidskii formulas. These families, which are based on familiar objects like parking functions, are combinatorial aids that are useful for computing the normalized volume of \( F_G(a) \). Mészáros et al. [4], showed that a family of Postnikov–Stanley triangulations for \( F_G \) are equivalent to the triangulations introduced by Danilov et al. in [5], which depend on the notion of a framing on a graph (see
Sect. 3). Different framings of a graph give different regular unimodular triangulations of the associated flow polytope.

The combinatorial structure of a triangulation $T$ of a polytope is encoded in its dual graph. This is a graph on the set of top-dimensional simplices in $T$ with edges between simplices sharing a common facet. In this article we introduce the family of $\nu$-caracol graphs $\text{car}(\nu)$ (see Definition 2.1) which are indexed by lattice paths $\nu$ in $\mathbb{Z}^2$.

In [2, Corollary 6.17] Mészáros and Morales considered a family of graphs similar to $\text{car}(\nu)$ and computed the volume of their flow polytopes, which they denoted as $\mathcal{F}_{\Pi_1^\nu}(\nu)$. After a simple transformation one can check that the flow polytopes $\mathcal{F}_{\text{car}(\nu)}$ and $\mathcal{F}_{\Pi_1^\nu}(\nu)$ are integrally equivalent (see the definition after Remark 2.2). In particular, these two polytopes have the same normalized volume.

Using the combinatorial interpretation of the Lidskii formula developed in [3] we give a new proof of the following theorem which can be derived from the closely-related result first proved in [2].

**Theorem 1.1** ([2, Corollary 6.17]) The normalized volume of the flow polytope $\mathcal{F}_{\text{car}(\nu)}$ is given by the number of $\nu$-Dyck paths, that is, the $\nu$-Catalan number $\text{Cat}(\nu)$.

In Sects. 4 and 5 we discuss two particular framings on $\text{car}(\nu)$ which we call the length and the planar framings. The triangulations arising from these framings have connections to two lattices on $\nu$-Catalan objects (see Sect. 2.1) that appear recurrently in the literature:

1. The $\nu$-Tamari lattice $\text{Tam}(\nu)$ introduced by Préville-Ratelle and Viennot [6].
2. The principal order ideal $I(\nu)$ determined by $\nu$ in Young’s lattice $Y$.

Thus we find that the collection of framed triangulations on $\mathcal{F}_{\text{car}(\nu)}$ provides a unifying framework for studying these two ubiquitous lattice structures. The family of $\nu$-Catalan objects is a generalization of the classical Catalan and rational Catalan families of objects that have been extensively studied in the recent literature (see for example [6–9]).

Our main results are the following:

**Theorem 1.2** The length-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is a regular unimodular triangulation whose dual graph is the Hasse diagram of the $\nu$-Tamari lattice $\text{Tam}(\nu)$.

**Theorem 1.3** The planar-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is a regular unimodular triangulation whose dual graph is the Hasse diagram of the principal order ideal $I(\nu)$ in Young’s lattice $Y$.

**Theorem 1.4** The $h^*$-polynomial of $\mathcal{F}_{\text{car}(\nu)}$ is the $\nu$-Narayana polynomial.

Theorems 1.2 and 1.3 provide two alternative proofs of Theorem 1.1.

To describe the combinatorial structure of the length-framed and planar-framed triangulations we use three different $\nu$-Catalan families of objects, as each highlights the combinatorics in crucial and distinct ways. These are the $(I, J)$-trees introduced by Ceballos et al. [10], the $\nu$-Dyck paths introduced by Préville-Ratelle and Viennot [6], and the $\nu$-trees which were also introduced by Ceballos et al. [11] (see Sects. 4.1.2, 4.1.1, and 4.1.3 respectively). The role that they play in the combinatorics of the
trianulations is summarized in Table 1 below. The interested reader can visit [10, 11] for the correspondences between \((I, \overline{J})\)-trees, \(\nu\)-trees and \(\nu\)-Dyck paths and [12] for their generalizations to \((I, \overline{J})\)-forests, \(\nu\)-Schröder trees and \(\nu\)-Schröder paths.

We point out that the study of length-framed and planar-framed triangulations of flow polytopes on the \(\nu\)-caracol graphs can be extended systematically to all graphs. On the \(\nu\)-caracol graphs, the length framing can be viewed as ordering both the incoming and outgoing sets of edges at each inner vertex according to decreasing length, while the planar framing can be viewed as ordering the set of incoming edges at each inner vertex with respect to decreasing length while the set of outgoing edges is ordered with respect to increasing length. Particularly for graphs which are symmetric with respect to the vertical axis (such as the \(\nu\)-caracol graph for \(\nu = (1^n)\)), then our viewpoint suggests that these two framings are in a sense dual to each other, so perhaps it is not surprising that both framings lead to combinatorially interesting triangulations of the flow polytope. In a forthcoming article we study aspects of the secondary polytope of triangulations for general \(\nu\).

Ceballos et al. [10, Section 1.4] compiled a list of polytopes and related structures which possess an ‘associahedral’ triangulation or subdivision, meaning that the dual graph of the triangulation is the Hasse diagram of the classical Tamari lattice. One can ask whether all of these polytopes also possess a ‘root’ triangulation whose dual graph is a lattice of filters of the type \(A\) root poset \(I(1, \ldots, 1)\).

Indeed, in the classical case when \(\nu = (1, 1, \ldots, 1) =: (1^n)\), the simultaneous appearance of the Tamari lattice \(\text{Tam}(1^n)\) and \(I(1^n)\) in the study of polytopes has been observed before in the work of Pitman and Stanley [13] on subdivisions of a family of polytopes \(\text{PS}_n(x)\) (see Remark 2.3 and Sect. 5.4). Combined with the work of Mészáros and Morales [2, Section 7], our results imply that the two triangulations of \(\mathcal{F}_{\text{car}(1^n)}\) in Theorems 1.2 and 1.3 induce two subdivisions of \(\text{PS}_n(x)\) whose dual graphs are the Hasse diagrams of \(\text{Tam}(1^n)\) and \(I(1^n)\) respectively. While Pitman and Stanley obtained the two combinatorial structures as two subdivisions of \(\text{PS}_n(x)\) via distinct methods, we are able to obtain the two combinatorial structures as two subdivisions in a uniform manner, since they both correspond to framed triangulations of \(\mathcal{F}_{\text{car}(1^n)}\).
The graph \( \text{car}(\nu) \) is a planar graph. Consequently, the results of Mészáros et al. in [4] on flow polytopes of planar graphs imply that the polytope \( \mathcal{F}_{\text{car}(\nu)} \) is integrally equivalent to an order polytope \( O(Q_{\nu}) \). Under this integral equivalence the planar-framed triangulation of \( \mathcal{F}_{\text{car}(\nu)} \) corresponds to the canonical triangulation of \( O(Q_{\nu}) \) (see Fig. 11).

This article is organized as follows. In Sect. 2 we introduce the \( \nu \)-caracol graph \( \text{car}(\nu) \) and its associated flow polytope \( \mathcal{F}_{\text{car}(\nu)} \), proving in Theorem 1.1 that its normalized volume is given by the \( \nu \)-Catalan number \( \text{Cat}(\nu) \). In Sect. 3 we describe the theory of framed triangulations as presented in [4]. In Sect. 4 we define the length framing of \( \text{car}(\nu) \), prove Theorem 1.2, and as consequence we conclude that the associated triangulation is a geometric realization of the \( \nu \)-Tamari complex. In Sect. 5 we define the planar framing of \( \text{car}(\nu) \) and prove Theorem 1.3. We explain how our unifying framework relates to results of Pitman and Stanley in [13], and also describe the relationship to order polytopes. As an application, in Sect. 6 we use the dual graph of the planar-framed triangulation of \( \mathcal{F}_{\text{car}(\nu)} \) to obtain the \( h^* \)-polynomial, which proves Theorem 1.4. This result also gives a new proof that the \( h \)-vector of the \( \nu \)-Tamari complex is given by the \( \nu \)-Narayana numbers.

2 The Family of \( \nu \)-Caracol Flow Polytopes

In [3], the second and fourth authors studied the flow polytope of the caracol graph, whose normalized volume is the number of Dyck paths from \((0, 0)\) to \((n, n)\), a Catalan number. We now extend this construction.

2.1 \( \nu \)-Dyck Paths and \( \nu \)-Catalan Numbers

Let \( a, b \) be nonnegative integers, and let \( \nu \) be a lattice path from \((0, 0)\) to \((b, a)\), consisting of a sequence of \( a \) north steps \( N = (0, 1) \) and \( b \) east steps \( E = (1, 0) \). A \( \nu \)-Dyck path is a lattice path from \((0, 0)\) to \((b, a)\) that lies weakly above \( \nu \).

An alternative description of a \( \nu \)-Dyck path that will prove to be useful in this work is one in terms of compatibility of points as follows. Let \( \mathcal{L}_\nu \) denote the set of lattice points in the plane which lie weakly above \( \nu \) inside the rectangle defined by \((0, 0)\) and \((b, a)\). Two lattice points \((x_1, y_1)\) and \((x_2, y_2)\) in \( \mathcal{L}_\nu \) with \( x_1 < x_2 \) are said to be path-incompatible if \( y_1 > y_2 \). Otherwise, any other pair of lattice points are said to be path-compatible. Maximal sets of path-compatible lattice points in \( \mathcal{L}_\nu \) determine a unique \( \nu \)-Dyck path.

When \( a \) and \( b \) are coprime positive integers and \( \nu \) is the lattice path that borders the squares which intersect the line \( y = \frac{a}{b} x \), this is the special case of the rational \( (a, b) \)-Dyck path studied by Armstrong, Loehr and Warrington in [7] who showed that the number of rational \( (a, b) \)-Dyck paths is the \( (a, b) \)-Catalan number \( \text{Cat}(a, b) = \frac{1}{a+b} \binom{a+b}{a} \). See Fig. 5 for an example of a \((3, 5)\)-rational Dyck path. When \((a, b) = (n, n+1)\), this is the case of the classical Catalan number \( \text{Cat}(n) = \frac{1}{2n+1} \binom{2n+1}{n} = \frac{1}{n+1} \binom{2n}{n} \). For general \( \nu \), the number \( \text{Cat}(\nu) \) of \( \nu \)-Dyck paths is calculated by a determinantal formula which can be derived by an application of
the Gessel–Viennot Lemma [14]:

\[
\text{Cat}(v) = \det \left( \left( \begin{array}{c}
1 + \sum_{k=1}^{a-j} v_k \\
1 + j - i
\end{array} \right) \right)_{1 \leq i, j \leq a-1},
\]

but a closed-form positive formula that does not involve cancellations due to signs is not known. For more on \(v\)-Dyck paths, see for example Ceballos and González D’León [9], or Préville-Ratelle and Viennot [6].

### 2.2 The \(v\)-Caracol Graph

**Definition 2.1** Let \(a, b\) be nonnegative integers, and let \(v\) be a lattice path from \((0, 0)\) to \((b, a)\) where \(v = NE^{v_1}NE^{v_2} \cdots NE^{v_a}\). The \(v\)-caracol graph \(\text{car}(v)\) is the graph on the vertex set \([a + 3]\), together with \(v_i\) copies of the edge \((1, i + 2)\) for \(i = 1, \ldots, a\), the edges \((i, a + 3)\) for \(i = 2, \ldots, a + 1\), and the edges \((i, i + 1)\) for \(i = 1, \ldots, a + 2\).

Note that in this construction, the graph \(\text{car}(v)\) has \(n + 1 := a + 3\) vertices, and the in-degree \(\text{in}_i\) of the vertex \(i\) in \(\text{car}(v)\) is \(\text{in}_2 = 1, \text{in}_i = v_{i-2} + 1\) for \(i = 3, \ldots, n\) and \(\text{in}_{n+1} = n - 1\). The number of edges \(m\) of \(\text{car}(v)\) is computed by summing the in-degrees of its vertices, so that

\[
m = \sum_{i=2}^{n+1} \text{in}_i = 1 + \sum_{i=1}^{a} (v_i + 1) + (a + 1) = 2a + b + 2.
\]

The (intrinsic) dimension of a flow polytope is given by \(\dim F_G = |E(G)| - |V(G)| + 1\), so we can conclude from this that \(\dim F_{\text{car}(v)} = m - n = a + b\).

The flow polytope on the graph \(\text{car}(v)\) in the special case when \(v = (1^n)\) has previously been studied by Mészáros [15] and by Benedetti et al. [3].

**Remark 2.2** The careful reader will notice that in Definition 2.1 of the graph \(\text{car}(v)\) we chose to use a lattice path \(v\) that begins with an \(N\) step. This choice was made for convenience of the presentation, and is not a true restriction. From the results in Sect. 3 one can verify that the combinatorial structure of a framed triangulation of the flow polytope \(F_{\text{car}(v)}\) is not affected by changing the number of \(N\) steps at the beginning of \(v\) (or by changing the number of \(E\) steps at the end of \(v\)). Hence without loss of generality and unless otherwise specified, all lattice paths \(v\) begin with at least one \(N\) step.
Two integral polytopes \( \mathcal{P} \subseteq \mathbb{R}^m \) and \( \mathcal{Q} \subseteq \mathbb{R}^n \) are integrally equivalent if there exists an affine transformation \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n \) whose restriction to \( \mathcal{P} \) is a bijection \( \varphi : \mathcal{P} \rightarrow \mathcal{Q} \) that preserves the lattice. That is, \( \varphi \) is a bijection between \( \mathbb{Z}^m \cap \text{aff}(\mathcal{P}) \) and \( \mathbb{Z}^n \cap \text{aff}(\mathcal{Q}) \). Integrally equivalent polytopes have the same Ehrhart polynomial, and hence the same volume.

**Remark 2.3** Mészáros and Morales [2, Corollary 6.17] have previously considered a closely-related variant of the flow polytope \( \mathcal{F}_{\text{car}(v)} \), denoted as \( \mathcal{F}_{\Pi_1^+(v)} \) in their work. The underlying graph \( \Pi_1^+(v) \) can be obtained from \( \text{car}(v) \) by deleting the edge \((2, n+1)\) and contracting the edge \((1, 2)\), and a simple transformation reveals that the flow polytopes \( \mathcal{F}_{\text{car}(v)} \) and \( \mathcal{F}_{\Pi_1^+(v)} \) are integrally equivalent.

They observed that the normalized volume of \( \mathcal{F}_{\Pi_1^+(v)} \) is the number of lattice points in the Pitman–Stanley polytope \( \text{PS}_a(v) = \{y \in \mathbb{R}_{\geq 0}^a \mid \sum_{i=1}^k y_i \leq \sum_{i=1}^k v_i\} \), which is equal to the number of \( v \)-Dyck paths. In the next section, we obtain a proof of this result by giving a combinatorial interpretation to the vector partitions enumerated by the Kostant partition function in the Lidskii volume formula. This method was first considered in [3] and further developed in [16].

### 2.3 The Volume of the \( v \)-Caracol Flow Polytope

We begin by defining the Kostant partition function of a graph, and the special case of the Lidskii volume formula which we will use.

For \( i = 1, \ldots, n \), we call \( \alpha_i = e_i - e_{i+1} \) the \( i \)-th simple root. For each edge \( e = (i, j) \) of a graph \( G \), let \( \alpha_e = e_i - e_j = \alpha_i + \cdots + \alpha_{j-1} \), and \( \Phi_G^+ = \{ \alpha_e \mid e \in E(G) \} \) will be called the multiset of positive roots associated to \( G \).

A vector partition of the vector \( v \) with respect to \( \Phi_G^+ \) is a decomposition of \( v \) into a non-negative linear combination of the positive roots associated to \( G \). The Kostant partition function of \( G \) evaluated at \( v \), denoted by \( K_G(v) \), is the number of vector partitions of \( v \) with respect to \( \Phi_G^+ \). Integral \( v \)-flows on \( G \) are equivalent to vector partitions of \( v \), so the number of integral \( v \)-flows on \( G \), and hence the number of lattice points in \( \mathcal{F}_G(v) \), is \( K_G(v) \).

Let \( G \) be a graph on the vertex set \( \{1, \ldots, n+1\} \). For \( i = 2, \ldots, n+1 \), let \( u_i = \text{in}_i - 1 \) be one less than the in-degree of the vertex \( i \).

**Proposition 2.4** (Baldoni and Vergne [1, Theorem 38]) Let \( G \) be a graph with \( n+1 \) vertices and \( m \) edges. The normalized volume of the flow polytope \( \mathcal{F}_G \) with unitary net flow \( a = e_1 - e_{n+1} \) is given by

\[
\text{vol} \mathcal{F}_G = K_G(v_{in}),
\]

where \( v_{in} = (0, u_2, \ldots, u_n, -(m-n-u_{n+1})) \).

For flow polytopes of \( v \)-caracol graphs with unitary net flow, the Kostant partition function \( K_{\text{car}(v)}(v_{in}) \) has a simple combinatorial interpretation which we now describe. This generalizes the construction for the case \( v = NE_1k \cdots NE_1k \) considered in [16, Section 2.4].
Definition 2.5 Let $v = N E^{v_1} \cdots N E^{v_a}$ be a lattice path from $(0, 0)$ to $(b, a)$. An in-degree gravity diagram for the flow polytope $\mathcal{F}_{\text{car}(v)}$ consists of a collection of dots and line segments with the following properties:

(i) The dots are arranged in columns indexed by the simple roots $\alpha_3, \ldots, \alpha_{a+2}$, with $v_1 + \cdots + v_{j-2}$ dots in the column indexed by $\alpha_j$, and all dots are drawn justified upwards.

(ii) Horizontal line segments may be drawn between dots in consecutive columns so that each dot is incident to at most one line segment. A trivial line segment is a singleton dot. All non-trivial line segments must contain a dot in the column indexed by $\alpha_{a+2}$ (that is, all line segments are justified to the right). Longer line segments appear above shorter line segments.

We denote the set of all in-degree gravity diagrams by $G_{\text{car}(v)}(v_{\text{in}})$. See Fig. 2 for an example of an in-degree gravity diagram.

The proof of the following Lemma is analogous to the one in [3, Theorem 3.1] for out-degree gravity diagrams. See also [16].

Lemma 2.6 There is a bijection between the set of vector partitions of $v_{\text{in}}$ with respect to $\Phi_1^{+}$ and the set of in-degree gravity diagrams for the flow polytope $\mathcal{F}_{\text{car}(v)}$. Consequently, $K_{\text{car}(v)}(v_{\text{in}}) = |G_{\text{car}(v)}(v_{\text{in}})|$.

Example 2.7 Let $v = N E^2 N E N E^3 N E$. A vector partition of $v_{\text{in}} = 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 6\alpha_6 + 7\alpha_7$ with respect to the positive roots in $\Phi_1^{+}$ is

$$v_{\text{in}} = \alpha_{(3,8)} + \alpha_{(5,8)} + 2\alpha_{(6,8)} + \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + 3\alpha_7.$$ 

This vector partition is represented by the gravity diagram on the left of Fig. 2.

We are ready to make a connection from in-degree gravity diagrams to $v$-Dyck paths.

Lemma 2.8 There is a bijection between the set $G_{\text{car}(v)}(v_{\text{in}})$ of in-degree gravity diagrams for the flow polytope $\mathcal{F}_{\text{car}(v)}$ and the set $\mathcal{D}_v$ of $v$-Dyck paths.

Proof First recall that a $v$-Dyck path is a lattice path in the rectangular grid from $(0, 0)$ to $(b, a)$ that lies weakly above the path $v$. Also recall that in an in-degree gravity diagram for $\text{car}(v)$, the column indexed by $\alpha_k$ has $v_1 + \cdots + v_k$ dots, for $k = 3, \ldots, a+2$. This is precisely the number of squares in the row between the lines $x = 0, y = k - 1, y = k$, and above $v$.

Therefore, given an in-degree gravity diagram $\Gamma \in G_{\text{car}(v)}(v_{\text{in}})$, we may rotate it 90 degrees counterclockwise and embed the array of dots into the squares of $\mathbb{Z}^2$ so that the dots in the column indexed by $\alpha_{a+2}$ lie in the row of squares just above the line $y = a$, and the dots in the first row of $\Gamma$ lie in the column of squares just right of the line $x = 0$. By the previous observation, we see that the dots of $\Gamma$ occupy every square in $\mathbb{Z}^2$ between the lines $x = 0, x = b$ and $y = a + 1$, and which lie above the path $v$. See Fig. 2 for an illustration.
Fig. 2 A gravity diagram (left) representing a vector partition of $v_{in}$ associated to car($v$) for $v = NE^2 NENNE^3 NE$. The bijection $\Xi$ of Theorem 1.1 sends the gravity diagram to the $v$-Dyck path via a 90 degree rotation (right).

Line segments of the rotated embedded gravity diagram $\Gamma$ are now vertical, and they extend down from just above the top row of the rectangular grid. The lengths of these vertical line segments are weakly decreasing from left to right, so the line segments of $\Gamma$ define a unique $v$-Dyck path that separates the dots in $\Gamma$ which are incident to a line segment in $\Gamma$, from the dots which are not incident to any (proper) line segment in $\Gamma$. This construction defines a map $\Xi : G_{car}(v)(v_{in}) \rightarrow D_v$.

Conversely, any $v$-Dyck path defines an in-degree gravity diagram $\Gamma$ for $F_{car}(v)$, where every dot of $\Gamma$ that occupies a square that is above the $v$-Dyck path is incident to a line segment of $\Gamma$, and every dot of $\Gamma$ that occupies a square that is below the $v$-Dyck path is not incident to any (proper) line segment of $\Gamma$. Therefore, $\Xi$ is a bijection. □

Theorem 1.1 ([2, Corollary 6.17]) The normalized volume of the flow polytope $F_{car}(v)$ is given by the number of $v$-Dyck paths, that is, the $v$-Catalan number $Cat(v)$.

Proof Combining Proposition 2.4 and Lemmas 2.6 and 2.8, the normalized volume of $F_{car}(v)$ is

$$vol F_{car}(v) = K_{car}(v)(v_{in}) = |G_{car}(v)(v_{in})| = |D_v| = Cat(v).$$

In Sects. 4 and 5, we construct two regular unimodular triangulations for the flow polytope $F_{car}(v)$ with combinatorially interesting dual graph structures, giving two more proofs that the normalized volume of $F_{car}(v)$ is the number of $v$-Dyck paths.

3 Framed Triangulations of a Flow Polytope

We now describe the family of triangulations defined by Danilov, et al. [5], interpreted as special cases of the Postnikov–Stanley triangulations described by Mészáros et al. in [4].

Any non-source and non-sink vertex in the directed graph $G$ is an inner vertex. A framing at the inner vertex $i$ is a pair of linear orders ($\prec_{in(i)}$, $\prec_{out(i)}$) on the incoming and outgoing edges at $i$. A framed graph, denoted $(G, \prec)$, is a graph with a framing at every inner vertex. In Sects. 4 and 5, we will consider two specific framings of the caracol graphs car($v$), which lead to combinatorially interesting triangulations of $F_{car}(v)$. An example of these two framings is given in Fig. 3.
For an inner vertex $i$ of a graph $G$, let $\text{In}(i)$ and $\text{Out}(i)$ respectively denote the set of maximal paths ending at $i$ and the set of maximal paths beginning at $i$. For a route $R$ containing an inner vertex $i$, let $Ri$ (respectively $iR$) denote the maximal subpath of $R$ ending (respectively beginning) at $i$. Define linear orders $\prec_{\text{In}(i)}$ and $\prec_{\text{Out}(i)}$ on $\text{In}(i)$ and $\text{Out}(i)$ as follows. Given paths $R$, $Q \in \text{In}(i)$, let $j \leq i$ be the smallest vertex after which $Ri$ and $Qi$ coincide. Let $e_R$ be the edge of $R$ entering $j$ and let $e_Q$ be the edge of $Q$ entering $j$. Then $R \prec_{\text{In}(i)} Q$ if and only if $e_R \prec_{\text{In}(j)} e_Q$. Similarly for $R$, $Q \in \text{Out}(i)$, let $j \geq i$ be the largest vertex before which $iR$ and $iQ$ coincide. Then $R \prec_{\text{Out}(i)} Q$ if and only if $e_R \prec_{\text{Out}(j)} e_Q$.

Two routes $R$ and $Q$ containing an inner vertex $i$ are coherent at $i$ if $Ri$ and $Qi$ are in the same relative order as $iR$ and $iQ$. Routes $R$ and $Q$ are coherent if they are coherent at each common inner vertex. A set of mutually coherent routes is a clique. For a maximal clique $C$, let $\Delta C$ denote the convex hull of the vertices of $\mathcal{F}_G$ corresponding to the unitary flows along the routes in $C$.

**Proposition 3.1** (Danilov et al. [5]) Let $(G, \prec)$ be a framed graph. Then

\[
\{\Delta C \mid C \text{ is a maximal clique of } (G, \prec)\}
\]

is the set of the top-dimensional simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

### 4 The Length-Framed Triangulation and the $\nu$-Tamari Lattice

The goal of this section is to show that the flow polytope $\mathcal{F}_{\text{car}(\nu)}$ has a regular unimodular triangulation whose dual graph structure is given by the Hasse diagram of the $\nu$-Tamari lattice. The triangulation in question arises as a DKK triangulation with the length framing. We show that this length-framed triangulation is combinatorially equivalent to the $\nu$-Tamari complex introduced by Ceballos, Padrol, and Sarmiento in [10].

#### 4.1 The $\nu$-Tamari Lattice

The $\nu$-Tamari lattice $\text{Tam}(\nu)$ was introduced by Préville-Ratelle and Viennot [6, Theorem 1] as a partial order on the set of $\nu$-Dyck paths. Using an alternative description with $(I, J)$-trees, Ceballos, Padrol and Sarmiento [10] realized the $\nu$-Tamari lattice as the one-skeleton of a polyhedral complex known as the $\nu$-associahedron, which
generalizes the classical associahedron. In [11], they gave further descriptions of the \(v\)-Tamari lattice using \(v\)-trees and \(v\)-bracket vectors, proving a special case of Rubey’s lattice conjecture. In [12] the first and fourth authors generalize \(v\)-Dyck paths and \(v\)-trees to \(v\)-Schröder paths and \(v\)-Schröder trees in their study of the face poset of the \(v\)-associahedron.

In this article we use three descriptions of Tam(\(v\)), as each provides a useful viewpoint. The \((I, J)\)-tree description shows that the length-framed triangulation is combinatorially the \((I, J)\)-Tamari complex, which is constructed using \((I, J)\)-trees. The \(v\)-Dyck path perspective enables us to determine when the length-framed and planar-framed triangulations coincide (see Proposition 5.5). Finally, the \(v\)-tree description captures the combinatorial structure of the triangulation succinctly, with \(v\)-trees playing an analogous role to \(v\)-Dyck paths in the planar-framed triangulation (compare Figs. 8 and 9). See Fig. 4 for an example of each of these characterizations of Tam(\(v\)), and see Fig. 5 for an additional example of a \(v\)-Dyck path, an \((I, J)\)-tree, and a \(v\)-tree.

### 4.1.1 The \(v\)-Tamari Lattice as the Rotation Lattice on \(v\)-Dyck Paths

We first give the description of Tam(\(v\)) in terms of \(v\)-Dyck paths. A valley of a lattice path is a point \(p\) at the end of an east step that is immediately followed by a north step. Let \(\mu\) be a \(v\)-Dyck path. For any lattice point \(p\) on \(\mu\), let \(\text{horiz}_v(p)\) denote the horizontal distance at \(p\), which is defined to be the maximum number of east steps that can be added to the right of \(p\) without crossing \(v\). For example, \(\text{horiz}_v(p)\) of the lattice points on the \(v\)-Dyck path in Fig. 5 are 0, 1, 3, 2, 1, 3, 2, 1, 0 as it is traversed from (0, 0) to (5, 3). The set of \(v\)-Dyck paths can then be endowed with the structure of a poset with the covering relation \(\prec_v\) defined as follows.

Let \(p\) be a valley of \(\mu\). Notice that the next point \(p'\) following \(p\) in \(\mu\) satisfies \(\text{horiz}_v(p') \geq \text{horiz}_v(p)\). Since the end point \(e\) of \(\mu\) has \(\text{horiz}_v(e) = 0\) and horizontal distances decrease by one after each east step in \(\mu\), there is a point \(q\) in \(\mu\) after \(p\) with \(\text{horiz}_v(q) = \text{horiz}_v(p)\). Let \(q\) be the first such lattice point in \(\mu\) after \(p\), and let \(\mu_{[p,q]}\) denote the subpath of \(\mu\) between \(p\) and \(q\). Define a rotation on \(\mu\) at \(p\) by switching the east step preceding \(p\) with the subpath \(\mu_{[p,q]}\). If \(\mu'\) is the lattice path obtained by rotating \(\mu\) at \(p\), then \(\mu \prec_v \mu'\) is a covering relation in Tam(\(v\)). Let \(\prec_v\) denote the transitive closure of the relation \(\prec_v\).
Definition 4.1 (Préville-Ratelle and Viennot [6]) The $v$-Tamari lattice $\text{Tam}(v)$ is the lattice on the set of $v$-Dyck paths induced by the relation $<_v$.

The leftmost lattice in Fig. 4 gives an example of the $v$-Tamari lattice indexed by $v$-Dyck paths.

4.1.2 The $v$-Tamari Lattice as the Flip Lattice on $(I, \overline{J})$-Trees

Next, we consider the description of $\text{Tam}(v)$ in terms of $(I, \overline{J})$-trees as introduced by Ceballos et al. [10].

Definition 4.2 Let $r \in \mathbb{Z}$ and let $I \sqcup \overline{J}$ be a bipartition of $[r]$ such that $1 \in I$ and $r \in \overline{J}$. An $(I, \overline{J})$-tree is a maximal subgraph of the complete bipartite graph $K_{|I|,|\overline{J}|}$ that is

(i) increasing: each arc $(i, \overline{j})$ satisfies $i < \overline{j}$; and
(ii) non-crossing: the graph does not contain arcs $(i, \overline{j})$ and $(i', \overline{j'})$ with $i < i' < j < \overline{j}$.

To a pair $(I, \overline{J})$ we can associate a unique lattice path $v$ as follows. Assign to the $i$-th element of $I$ the label $E_{i-1}$ and to the $j$-th element of $\overline{J}$ the label $N_{j-1}$. Reading the labels of the nodes $k = 2, \ldots, r - 1$ in increasing order yields a lattice path $v$ from $(0, 0)$ to $(|I| - 1, |\overline{J}| - 1)$. See Fig. 5 for an example. Conversely, a lattice path $v$ determines a unique pair $(I, \overline{J})$, and hence a unique set of $(I, \overline{J})$-trees. Let $\mathcal{T}_v$ denote the set of $(I, \overline{J})$-trees determined by $v$.

Given two $(I, \overline{J})$-trees $T$ and $T'$ in $\mathcal{T}_v$, we say that $T'$ is an increasing flip of $T$ if $T'$ is obtained from $T$ by replacing an arc $(i, j)$ with an arc $(i', j')$, where $i < i'$ and $j < j'$.

Proposition 4.3 ([10, Theorem 3.4]) The increasing flip lattice on the set of $(I, \overline{J})$-trees in $\mathcal{T}_v$ is isomorphic to $\text{Tam}(v)$.

The lattice in the center of Fig. 4 gives a $v$-Tamari lattice with vertices indexed by $(I, \overline{J})$-trees. The following corollary is then immediate.

Corollary 4.4 The Hasse diagram of the $v$-Tamari lattice is the graph whose vertices are the $(I, \overline{J})$-trees determined by $v$, with edges between $(I, \overline{J})$-trees that differ by exactly one arc.
4.1.3 The $\nu$-Tamari Lattice as the Rotation Lattice on $\nu$-Trees

Given a lattice path $\nu$ from $(0, 0)$ to $(b, a)$, recall that $\mathcal{L}_\nu$ denotes the set of lattice points in the plane which lie weakly above $\nu$ inside the rectangle defined by $(0, 0)$ and $(b, a)$. Let $\mathcal{A}_\nu$ denote the set of possible arcs in the $(I, \overline{J})$-trees in $\mathcal{T}_\nu$. In [10, Remark 2.2], Ceballos et al. constructed a bijection $\rho : \mathcal{A}_\nu \rightarrow \mathcal{L}_\nu$, which we describe below.

Recall that the lattice path $\nu$ determines the pair $(I, \overline{J})$, where the elements in the sets $I$ and $\overline{J}$ respectively correspond to the $E$ and $N$ steps in the path $\nu E \nu N$, as in Fig. 5. Describing $\rho$ in terms of the $N$ and $E$ steps is easier than using the elements of $I$ and $\overline{J}$, so we add indices to the $N$ and $E$ steps in order to distinguish between them. In $\nu = E \nu N$, index the $E$ steps left to right by $0, 1, \ldots, b$, and index the $N$ steps left to right by $0, 1, \ldots, a$ (see Fig. 5). Then arcs in the $(I, \overline{J})$-trees in $\mathcal{T}_\nu$ can be expressed as pairs of the form $(E_x, N_y)$. Then the bijection $\rho$ is given by $\rho(E_x, N_y) = (x, y)$.

The collection of points in $\mathcal{L}_\nu$ corresponding to the arcs of an $(I, \overline{J})$-tree $T$ is called the grid representation of $T$. These grid representations were studied in detail under the name of $\nu$-trees (see [11, Remark 3.7]). The word ‘tree’ is justified by the fact that each point except $(0, a)$ in a grid representation has either one point above it in the same column or one point to its left in the same row, but not both [11, Lemma 2.2]. Thus we can connect each point to the point above it or to its left, forming a rooted binary tree with a root at $(0, a)$. Figure 5 (right) gives an example of a $\nu$-tree, which is the grid representation of the $(I, \overline{J})$-tree in the center. The non-crossing condition for arcs in an $(I, \overline{J})$-tree can be translated to $\nu$-trees and a $\nu$-tree can then be defined without reference to an $(I, \overline{J})$-tree as follows.

**Definition 4.5** ([11, Definition 2.1]) Two lattice points $p$ and $q$ in $\mathcal{L}_\nu$ are said to be tree-incompatible if $p$ is southwest or northeast of $q$, and the smallest rectangle containing $p$ and $q$ contains only lattice points of $\mathcal{L}_\nu$. The points $p$ and $q$ are tree-compatible if they are not tree-incompatible. A $\nu$-tree is a maximal set of tree-compatible points in $\mathcal{L}_\nu$.

**Remark 4.6** The concept of tree-compatibility is called $\nu$-compatibility in [11]. We call it tree-compatibility here to contrast with the concept of path-compatibility (Sect. 2.1).

If a $\nu$-tree has points $p$, $q$ and $r$ such that $r$ is the southwest corner of the rectangle determined by $p$ and $q$ (with $p$ northwest of $q$ or vice versa), then replacing $r$ with the lattice point at the point at the northeast corner of the rectangle is called a (right) rotation. For example, in Fig. 5, the only possible rotation in the $\nu$-tree replaces the lattice point $(1, 2)$ with $(2, 3)$. Rotations in a $\nu$-tree are a direct translation of increasing flips for $(I, \overline{J})$-trees. Define a partial order $\prec_\nu$ on the set of $\nu$-trees given by a covering relation $T \prec_\nu T'$ if and only if $T'$ is formed from $T$ by a rotation. This partial order is the rotation lattice of $\nu$-trees [11, Theorem 2.8]. The rightmost lattice in Fig. 4 shows the $\nu$-Tamari lattice indexed with $\nu$-trees.

**Proposition 4.7** ([11, Theorem 3.3]) The rotation lattice on the set of $\nu$-trees is isomorphic to $\text{Tam}(\nu)$. 

\[\text{Springer}\]
4.2 The Length- Framed Triangulation

In this subsection we study the length-framed triangulation of $F_{\text{car}(\nu)}$ and show its connection with $\text{Tam}(\nu)$. Before defining the length framing of $\text{car}(\nu)$, or any framing for that matter, we need to be able to distinguish between multiedges. To that end, we assign a labeling to the edges of $\text{car}(\nu)$ that are incident to the source and sink as follows.

First fix a planar embedding of $\text{car}(\nu)$ so that the path $1, \ldots, n+1$ lies on the $x$-axis. The set of edges incident to the source vertex $1$ are labeled $0, 1, \ldots, b$ in order from the shortest to the highest longest edge, and the set of edges incident to the terminal vertex $n+1$ are likewise labeled $0, 1, \ldots, a$ in order from the longest to the highest shortest edge. See Fig. 6 for an illustration. This edge labeling also provides a succinct way of denoting the routes in $\text{car}(\nu)$. Recall from Sect. 1 that the vertices of $F_{\text{car}(\nu)}$ are determined by routes (unitary flows) in $\text{car}(\nu)$. These are completely characterized by the initial edge from the source vertex 1, and the terminal edge to the sink vertex $n+1$. We will therefore denote by $R_{x,y}$ the route in $\text{car}(\nu)$ with initial edge labeled $x$ and terminal edge labeled $y$.

**Definition 4.8** Let $G$ be a graph on the vertex set $\{1, \ldots, n+1\}$. Define the length of a directed edge $(i, j)$ to be $j - i$. Given an inner vertex $i \in [2, n]$ of $G$, the length framing for $G$ at $i$ is the pair of linear orders $(\prec_{\text{in}(i)}, \prec_{\text{out}(i)})$ where longer edges precede shorter edges and multiedges with smaller labels precede ones with larger labels. Figure 3 gives an example of the length framing at vertex 6 of $\text{car}(\nu)$ with $\nu = NEN E^2 NENNE^3 NE$.

We are now ready to describe a key bijection between the set of routes $\mathcal{R}_\nu$ in the $\nu$-caracol graph $\text{car}(\nu)$ and the set $\mathcal{A}_\nu$ of possible arcs in the $(I, J)$-trees in $\mathcal{T}_\nu$. Recall from Sect. 4.1.3 that arcs in $\mathcal{A}_\nu$ can be expressed as pairs of the form $(E_x, N_y)$. Define the map $\varphi : \mathcal{R}_\nu \rightarrow \mathcal{A}_\nu$ by $\varphi(R_{x,y}) = (E_x, N_y)$. To see this is well-defined, suppose the initial and terminal edges of $R_{x,y}$ are $(1, k)$ and $(\ell, n+1)$ respectively. Then $k \leq \ell$, which implies $E_x$ appears before $N_y$ in $\nu$, so $(E_x, N_y)$ is a valid arc in $\mathcal{A}_\nu$. Figure 7 shows an example of this correspondence between routes and arcs in a particular $(I, J)$-tree.

**Lemma 4.9** The map $\varphi : \mathcal{R}_\nu \rightarrow \mathcal{A}_\nu$ is a bijection.

**Proof** Define the inverse map $\varphi^{-1} : \mathcal{A}_\nu \rightarrow \mathcal{R}_\nu$ by $\varphi^{-1}((E_x, N_y)) = R_{x,y}$. To see this is well-defined, suppose the edge in $\text{car}(\nu)$ that is incident to the vertex 1 having the label $x$ is $(1, k)$, and the edge that is incident to the vertex $n+1$ having the label $y$ is
Two routes \( R_1 \) and \( R_2 \) in the framed graph \((\text{car}(\nu), \prec)\) are coherent if and only if \( \varphi(R_1) \) and \( \varphi(R_2) \) are non-crossing arcs in \( \mathcal{A}_\nu \).

**Proof** Let \( R_{x,y} \) and \( R_{x',y'} \) be two routes in \( \text{car}(\nu) \) and \( \varphi(R_{x,y}) = (x, y) \), \( \varphi(R_{x',y'}) = (x', y') \). If \( x = x' \), then the routes are coherent and the corresponding arcs \((E_x, N_y)\) and \((E_{x'}, N_{y'})\) are non-crossing. Otherwise assume \( x < x' \). If the arcs \((E_x, N_y)\) and \((E_{x'}, N_{y'})\) cross then \( y < y' \) necessarily and \( E_x, E_{x'}, N_y, N_{y'} \) appear in that order in \( \nu = EnE^2N \). Denote the terminal edge of the route \( R_{x,y} \) by \((\ell, n+1)\). Then \( x < x' \) and \( y < y' \) imply the routes \( R_{x,y} \) and \( R_{x',y'} \) are incoherent at the vertex \( \ell \). Conversely, let \((1, k)\) and \((1, k')\) respectively denote the initial edges of \( R_{x,y} \) and \( R_{x',y'} \) so that \( k \leq k' \), and suppose these routes are incoherent. They must be incoherent at a maximal vertex \( \ell \) for which \((\ell, n+1)\) is the terminal edge of \( R_{x,y} \), and \( \ell \leq \ell' \) where \((\ell', n+1)\) is the terminal edge of \( R_{x',y'} \). Moreover, since \( R_{x,y} \) and \( R_{x',y'} \) coincide at \( \ell \), then \( k' \leq \ell \), and hence \( E_x, E_{x'}, N_y, N_{y'} \) appear in that order in \( \nu \) and the arcs \((E_x, N_y)\) and \((E_{x'}, N_{y'})\) cross.

**Theorem 1.2** The length-framed triangulation of \( \mathcal{F}_{\text{car}(\nu)} \) is a regular unimodular triangulation whose dual graph is the Hasse diagram of the \( \nu \)-Tamari lattice \( \text{Tam}(\nu) \).

**Proof** By Lemma 4.10, the bijection \( \varphi \) in Lemma 4.9 extends to a bijection \( \Phi \) from the set of maximal cliques of routes in the length-framed \( \text{car}(\nu) \) to the set \( \mathcal{T}_\nu \) of \((I, \mathcal{T})\)-trees determined by \( \nu \). Two top-dimensional simplices in a DKK triangulation of a flow polytope are adjacent if and only if they differ by a single vertex, that is, if the corresponding maximal cliques differ by a single route. Under the bijection \( \Phi \), two top-dimensional simplices are adjacent if and only if their corresponding \((I, \mathcal{T})\)-trees differ by a single arc, which by Corollary 4.4 is precisely the description of the Hasse diagram of the \( \nu \)-Tamari lattice.

**Example 4.11** Let \( \nu = NNE^2NE^2 \). One example of the bijection \( \Phi \) between cliques of routes of \((\text{car}(\nu), \prec)\) and \((I, \mathcal{T})\)-trees is illustrated in Fig. 7. The dual graph of the length-framed triangulation of \( \mathcal{F}_{\text{car}(\nu)} \) is shown in Fig. 4.
In [10] Ceballos, Padrol, and Sarmiento introduced the \((I, \overline{J})\)-Tamari complex \(A_{I, \overline{J}}\) as the flag simplicial complex on \(\{(i, j) \in I \times \overline{J} \mid i < j\}\) whose minimal non-faces are the pairs \(\{(i, j), (i', j')\}\) with \(i < i' < j < j'\), that is, the complex on collections of non-crossing arcs of \((I, \overline{J})\)-trees. The following result is then a corollary of Theorem 1.2.

**Corollary 4.12** Let \(v\) be the lattice path from \((0, 0)\) to \((b, a)\) associated to the pair \((I, \overline{J})\). The length-framed triangulation of \(\mathcal{F}_{\text{car}(v)}\) is a geometric realization of the \((I, \overline{J})\)-Tamari complex of dimension \(a + b\) in \(\mathbb{R}^{2a+b+2}\).

**Remark 4.13** A simple projection of the coordinates along the edges of the form \((i, i+1)\) produces a lower dimensional geometric realization of the \((I, \overline{J})\)-Tamari complex in \(\mathbb{R}^{a+b}\). This geometric realization is integrally equivalent to the first of three realizations given in [10, Theorem 1.1].

### 4.3 A Second Description in Terms of \(v\)-Trees

We conclude this section by indexing top-dimensional simplices in the length-framed triangulation of \(\mathcal{F}_{\text{car}(v)}\) by \(v\)-trees, which in terms of the lattice path \(v\) is an analogous counterpart to the \(v\)-Dyck path description of the planar-framed triangulation in Sect. 5. The vertices (routes) of \(\mathcal{F}_{\text{car}(v)}\) are encoded by the lattice points in \(\mathcal{L}_v\), with the lattice points in each \(v\)-tree corresponding to a top-dimensional simplex in the length-framed triangulation. Furthermore, it then follows that two top-dimensional simplices are adjacent in the length-framed triangulation if their corresponding \(v\)-trees differ by a rotation.

**Lemma 4.14** Let \(v\) be a lattice path from \((0, 0)\) to \((b, a)\). There exists a bijection \(\theta : \mathcal{R}_v \rightarrow \mathcal{L}_v\) between the set \(\mathcal{R}_v\) of routes in \(\text{car}(v)\) and the set \(\mathcal{L}_v\) of lattice points in the rectangle defined by \((0, 0)\) and \((b, a)\) that lie weakly above \(v\).

**Proof** Recall that routes in \(\text{car}(v)\) are completely characterized by their initial and terminal edges, and a route is denoted by \(R_{x,y}\) if these edges are respectively labeled \(x\) and \(y\) (see Fig. 6). Define the map \(\theta : \mathcal{R}_v \rightarrow \mathcal{L}_v\) by \(\theta(R_{x,y}) = (x, y)\). By construction, \(0 \leq x \leq b\) and \(0 \leq y \leq a\) so \((x, y)\) is a lattice point within the rectangle defined by \((0, 0)\) and \((b, a)\). Moreover, \(R_{x,y}\) is a route means that \(E_x\) appears before \(N_y\) in \(\overline{v} = EvN\), so \((x, y)\) lies weakly above the lattice path \(v\), and \(\theta\) is well-defined.

It is clear that \(\theta\) is one-to-one. To see that \(\theta\) is onto, consider \((x, y)\) \(\in \mathcal{L}_v\). In \(\text{car}(v)\), the edge \((y + 2, n + 1)\) is labeled \(y\), and \((x, y) \in \mathcal{L}_v\) means that there are at least \(x\) \(E_x\’s\) appearing before \(N_y\) in \(\overline{v}\). If the edge \((1, k)\) is labeled \(x\) in \(\text{car}(v)\), then \((1, k), (k, k+1), \ldots, (y+1, y+2), (y+2, n+1)\) is the route \(R_{x,y}\) that maps to \((x, y)\). Therefore, \(\theta\) is a bijection. \(\square\)

**Remark 4.15** In summary, \(\theta = \phi \circ \rho\), and we have the following bijections between routes in \(\text{car}(v)\), possible arcs in \((I, \overline{J})\)-trees in \(\mathcal{T}_v\), and lattice points lying weakly above \(v\).
The bijection $\theta$ leads to a characterization of the routes that appear in every top-dimensional simplex of the length-framed triangulation of $F_{\text{car}(v)}$.

**Corollary 4.16** The set of $v$-trees are in bijection with the set of top-dimensional simplices in the length-framed triangulation of $F_{\text{car}(v)}$. Under this bijection, the lattice points of a given $v$-tree correspond with the routes that appear in the associated top-dimensional simplex.

**Proof** Recall from Lemma 4.10 that two routes in $R_v$ are coherent if and only if their corresponding arcs in $A_v$ are non-crossing. Since the non-crossing condition for arcs in $(I, J)$-trees translates to the tree-compatibility condition of $v$-trees, we have that routes in $R_v$ are coherent if and only if their corresponding lattice points via $\theta$ are tree-compatible. If we associate each lattice point in $L_v$ with the corresponding route in $\text{car}(v)$ via the bijection $\theta$ in Lemma 4.14, then $\theta$ extends to a bijection $\Theta_1$ between maximal cliques of routes with respect to the length framing and lattice points in a $v$-tree. Two adjacent top-dimensional simplices in the length-framed triangulation of $F_{\text{car}(v)}$ differ by a single vertex, and the corresponding $v$-trees differ by a single lattice point via a rotation. $\square$

**Remark 4.17** A $v$-tree always contains the root $(0, a)$, the valley points of the lattice path $v$, along with each initial point of any initial $N$ steps of $v$, and each terminal point of any terminal $E$ steps of $v$. Hence under the bijection $\theta$, these points correspond to the routes which are coherent with all other routes in the length-framing, and thus appear in every top-dimensional simplex of the length-framed triangulation.

In the example in Fig. 8, the routes which appear in every top-dimensional simplex of the length-framed triangulation of $F_{\text{car}(v)}$ are labeled 1, 3, 5, 7, 8 and 9.

## 5 The Planar-Framed Triangulation and Young’s Lattice

The goal of this section is to show that the flow polytope $F_{\text{car}(v)}$ has a regular unimodular triangulation whose dual graph structure is given by the Hasse diagram of
a principal order ideal \( I(v) \) in Young’s lattice. The triangulation in question arises as a DKK triangulation with the planar framing. A consequence of this is that we can construct a family of posets \( Q_v \) so that the dual graph structure of the canonical triangulation of the order polytope \( \mathcal{O}(Q_v) \) is also \( I(v) \).

### 5.1 Principal Order Ideals in Young’s Lattice

Recall that Young’s lattice \( Y \) is the poset on integer partitions \( \lambda \) with covering relations \( \lambda \lessdot \lambda' \) if \( \lambda \) is obtained from \( \lambda' \) by removing one corner box of \( \lambda' \). Note that a lattice path \( \nu \) in the rectangular grid defined by \((0, 0)\) to \((b, a)\) defines a partition \( \lambda(\nu) \) by letting \( \lambda_k = b - \sum_{i=a-k+1}^{a} v_i \) for \( k = 1, \ldots, a \). The Young diagram for \( \lambda(\nu) \) may be visualized as the region within the rectangle from \((0, 0)\) to \((b, a)\) which lies NW of \( \nu \). For example in Fig. 1, \( \nu = NE^2NENNE^3NE \) defines the partition \( \lambda(\nu) = (6, 3, 3, 2) \).

An order ideal of a poset \( P \) is a subset \( I \subseteq P \) with the property that if \( x \in I \) and \( y \leq x \), then \( y \in I \). An ideal is said to be principal if it has a single maximal element \( x \in P \), and such an ideal will be denoted by \( I(x) \).

If \( \mu \) is a \( \nu \)-Dyck path, then it lies weakly above the path \( \nu \) and so \( \mu \) can be identified with a partition \( \lambda(\mu) \) that is contained in \( \lambda(\nu) \). Thus there is a one-to-one correspondence between the set of \( \nu \)-Dyck paths with the set of elements in the order ideal \( I(\nu) := I(\lambda(\nu)) \) in \( Y \). Under this correspondence, in terms of \( \nu \)-Dyck paths, a path \( \pi \) covers a path \( \mu \) if and only if \( \pi \) can be obtained from \( \mu \) by replacing a consecutive \( NE \) pair by a \( EN \) pair. See Fig. 10 (right) for an example of \( I(\nu) \) with \( \nu = NEN^2NE^2 \).

### 5.2 The Planar-Framed Triangulation

**Definition 5.1** Let \( G \) be a planar graph that affords a planar embedding in the plane such that if vertex \( i \) is at the coordinates \((x_i, y_i)\), then \( x_i < x_j \) for all \( i < j \). This leads to natural orderings \((<_{\text{in}(i)}, <_{\text{out}(i)})\) at every inner vertex \( i \) of \( G \) as follows: with respect to the planar embedding of \( G \), the incoming edges at the vertex \( i \) are ordered in increasing order from the top to the bottom, and the same for the outgoing edges from the vertex \( i \). This is the planar framing for \( G \).

It is clear that the graphs \( \text{car}(\nu) \) have a planar embedding with the properties of Definition 5.1 if it is embedded so that the path 1, \ldots, \( n+1 \) lies on the x-axis. Figure 3 gives an example of the planar framing at vertex 6 of \( \text{car}(\nu) \) with \( \nu = NE^2NENNE^3NE \).

In Lemma 4.14, we showed that \( \theta : \mathcal{R}_v \rightarrow \mathcal{L}_v \) is a bijection between routes in \( \text{car}(\nu) \) and lattice points lying weakly above \( \nu \). We already saw that \( \theta \) extends to a bijection \( \Theta_1 \) in which a maximal clique of routes in the length framing of \( \text{car}(\nu) \) corresponds to the collection of lattice points in a \( \nu \)-tree. As we will see in Theorem 1.3, the bijection \( \theta \) extends to a second bijection \( \Theta_2 \) in which a maximal clique of routes in the planar framing of \( \text{car}(\nu) \) corresponds to the collection of lattice points in a \( \nu \)-Dyck path.

Recall from Sect. 2.1 that Maximal sets of path-compatible lattice points in \( \mathcal{L}_v \) (lying above \( \nu \)) determine a unique \( \nu \)-Dyck path.
A result of Mészáros et al. [4, Lemma 6.5] states that two routes in a planar triangulation
is denoted by $\theta_1$ that routes in car $(\nu)$ are coherent if and only if
Lemma 5.2 Let $\prec_p$ denote the planar framing, and let $\theta$ be the bijection in Lemma 4.14. Two routes $R_1$ and $R_2$ in the framed graph $(\text{car}(\nu), \prec_p)$ are coherent if and only if $\theta(R_1) \sqsubset \theta(R_2)$ are path-compatible lattice points.

Proof A result of Mészáros et al. [4, Lemma 6.5] states that two routes in a planar framing of a graph $G$ are coherent if and only if they are non-crossing in $G$. Recall that routes in $\text{car}(\nu)$ are completely characterized by their initial and terminal edges, and a route is denoted by $R_{x,y}$ if these edges are respectively labeled $x$ and $y$ (see Fig. 6). Let $R_{x,y}$ and $R_{x',y'}$ be two routes in $\text{car}(\nu)$ such that $\theta(R_{x,y}) = (x, y)$ and $\theta(R_{x',y'}) = (x', y')$.

Suppose $R_{x,y}$ and $R_{x',y'}$ are coherent with $y \preceq y'$. Then the fact that $R_1$ and $R_2$ are non-crossing in $\text{car}(\nu)$ implies that $x \preceq x'$, and hence $(x, y)$ and $(x', y')$ are path-compatible. On the other hand suppose $R_{x,y}$ and $R_{x',y'}$ are not coherent so that they cross at a vertex $c$. Without loss of generality, we may assume that $y < y'$ (for otherwise, if $y = y'$ then the routes are coherent). The routes cross, so $x' \preceq x$, and hence $(x, y)$ and $(x', y')$ are not path-compatible.

Theorem 1.3 The planar-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is a regular unimodular triangulation whose dual graph is the Hasse diagram of the principal order ideal $I(\nu)$ in Young’s lattice $Y$.

Proof By Lemma 5.2, the bijection $\theta$ in Lemma 4.14 extends to a bijection $\Theta_2$ from maximal cliques of routes in the planar-framed $\text{car}(\nu)$ to maximal sets of path-compatible lattice points lying above $\nu$, which are $\nu$-Dyck paths. Two top-dimensional simplices in a DKK triangulation of a flow polytope are adjacent if and only if they differ by a single vertex. Under the bijection $\Theta_2$, two top-dimensional simplices are adjacent if and only if their corresponding $\nu$-Dyck paths $\pi_1$ and $\pi_2$ differ by a single lattice point. Let $(x_1, y_1) \in \pi_1$ and $(x_2, y_2) \in \pi_2$ be the lattice points which are not contained in both paths. Assume without loss of generality that $x_1 < x_2$. Since these lattice points are not path-compatible, we must have $y_1 > y_2$. Thus $(x_1, y_1)$ is in the top left corner of the single square determined by $(x_1, y_1)$ and $(x_2, y_2)$, while $(x_2, y_2)$ is in the bottom left. In other words, $\pi_1$ and $\pi_2$ differ by a transposition of a consecutive $NE$ pair, which is precisely the description of the covering relation in the principal order ideal $I(\nu)$.

Example 5.3 Let $\nu = NENE^2NE^2$. One example of the bijection $\Theta_2$ between cliques of routes of $(\text{car}(\nu), \prec_p)$ and $\nu$-Dyck paths is shown in Fig. 9. The dual graph of the planar-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is shown in Fig. 10 on the right.
Remark 5.4 A $\nu$-Dyck path always contains the lattice points of any initial $N$ steps of $\nu$ and any terminal $E$ steps of $\nu$. Hence, under this bijection, these points correspond to the routes which are coherent with all other routes, and thus appear in every top-dimensional simplex of the planar-framed triangulation.

For example in Fig. 9, the routes which appear in every top-dimensional simplex of the planar-framed triangulation of $F_{\text{car}}(\nu)$ are labeled 1, 2, 7, 8 and 9.

5.3 Comparing the Length-Framed and Planar-Framed Triangulations

A special case when the dual graph of the length-framed and planar-framed triangulations of $F_{\text{car}}(\nu)$ are the same is given by the following proposition.

Proposition 5.5 When $\nu = E^aN^b$, so that the set of $\nu$-Dyck paths is the set of all lattice paths from $(0,0)$ to $(b,a)$, the length-framed triangulation and the planar-framed triangulation of $F_{\text{car}}(\nu)$ have the same dual graph.

Proof We use the $\nu$-Dyck path description (see Sect. 4.1) of the $\nu$-Tamari lattice $\text{Tam}(\nu)$ in this proof. Let $\mu$ be a $\nu$-Dyck path. For any valley point $p$ of $\mu$, the next lattice point $q$ in $\mu$ with $\text{horiz}_\nu(p) = \text{horiz}_\nu(q)$ is the next lattice point after $p$. This is because the horizontal distance of any of the lattice points in a run of consecutive $N$ steps is the same when $\nu = E^aN^b$. Performing a rotation on $\mu$ at the valley point $p$ to obtain the $\nu$-Dyck path $\mu'$ is then the same as exchanging the $EN$ pair centered at $p$ with an $NE$ pair of steps in $\mu$. Thus $\mu \prec_\nu \mu'$ is a covering relation in the lattice $\text{Tam}(\nu)$ if and only if it is a covering relation in the dual order ideal $I(\nu)^*$. Therefore, $\text{Tam}(\nu) = I(\nu)^*$. Lastly, since the partition $\lambda(\nu) = (b^a)$ is rectangular, then $I(\nu)$ is self-dual. Therefore, $\text{Tam}(\nu)$ and $I(\nu)$ are isomorphic. □
5.4 A Connection with the Pitman–Stanley Polytope

As mentioned in the Introduction, for \((x_1, \ldots, x_n) \in \mathbb{R}_+^n\), the lattices \(\text{Tam}(1^n)\) and \(I(1^n)\) appear as duals of polytopal subdivisions of the Pitman–Stanley polytope

\[
\text{PS}_n(x_1, \ldots, x_n) = \{(y_1, \ldots, y_n) \in \mathbb{R}_+^n \mid y_1 + \cdots + y_i \leq x_1 + \cdots + x_i \text{ for each } i = 1, \ldots, n\}.
\]

We briefly explain how we can recover these results of Pitman and Stanley in a uniform way by combining Theorems 1.2 and 1.3 and the work of Mészáros and Morales [2, Section 7].

First, the Pitman–Stanley polytope \(\text{PS}_n(x_1, \ldots, x_n)\) is integrally equivalent to the flow polytope \(\mathcal{F}_{\Pi_n}(x_1, \ldots, x_n, -\sum_{i=1}^n x_i)\), where \(\Pi_n\) is the graph on the vertex set \([n+1]\), with the (multi)edges \((i, i+1)\) and \((i, n+1)\) for \(i = 1, \ldots, n\). In turn, this flow polytope can be expressed as a Minkowski sum

\[
\mathcal{F}_{\Pi_n}(x_1, \ldots, x_n, -\sum_{i=1}^n x_i) = x_1 P_1 + \cdots + x_n P_n,
\]

where \(P_i = \mathcal{F}_{\Pi_n}(e_i - e_{i+1})\). On the other hand (see [2, Proposition 7.2]), the Cayley embedding \(\mathcal{C}(P_1, \ldots, P_n)\) of \(P_1, \ldots, P_n\) is the flow polytope \(\mathcal{F}_{\text{car}(1^n)}\) up to an integral equivalence. In the case \(\sum_{i=1}^n x_i = 1\), the Cayley trick states that there is a correspondence between polytopal subdivisions of \(\mathcal{C}(P_1, \ldots, P_n)\) intersected by \((x_1, \ldots, x_n) \times \mathbb{R}^m\), and mixed subdivisions of \(x_1 P_1 + \cdots + x_n P_n\). Thus, we may conclude that triangulations of \(\mathcal{F}_{\text{car}(1^n)}\) are in bijection with fine mixed subdivisions of \(\text{PS}_n(x_1, \ldots, x_n)\).

5.5 A Connection with Order Polytopes

In this subsection, \(G\) is a planar graph on the vertex set \([n+1]\) with a planar embedding satisfying the properties outlined in Definition 5.1. We further assume that the in-degree and out-degree of each vertex \(i\) for \(i = 2, \ldots, n\) is at least one. A result of Mészáros et al. [4, Theorem 3.11] states that for such a graph \(G\), the flow polytope \(\mathcal{F}_G\) is integrally equivalent to the order polytope \(\mathcal{O}(P_G)\), where \(P_G\) is the poset that is induced by the bounded faces of the planar embedding of \(G\). We discuss some of the implications of this connection, and refer the reader to [4] for the background details.

Given a lattice path \(\nu = NE^{v_1} \cdots NE^{v_a}\) from \((0, 0)\) to \((b, a)\), we label the sequence of steps in the path by

\[
N_1, E_{1,1}, E_{1,2}, \ldots, E_{1,v_1}, N_2, E_{2,1}, \ldots, E_{2,v_2}, \ldots, N_a, E_{a,1}, \ldots, E_{a,v_a}.
\]

Define the poset \(Q_{\nu}\) on the above labeled elements, with the following cover relations: \(N_1 \prec N_2 \prec \cdots \prec N_a, E_{1,1} \prec E_{1,2} \prec E_{1,v_1} \prec E_{2,1} \prec \cdots \prec E_{a,v_a}\), and \(N_i \prec E_{i,1}\) for each \(i = 1, \ldots, a\) where \(E_{i,1}\) exists. Then \(\mathcal{F}_{\text{car}(\nu)}\) is integrally equivalent to \(\mathcal{O}(Q_{\nu})\). See the right side of Fig. 11 for an example.
Fig. 11 Let \( v = NE^2NENNE^3NE \). The truncated dual graph \( P_{\text{car}(v)} \) (left) is the graph whose vertices are the bounded faces of the embedded \( \text{car}(v) \). The Hasse diagram of the poset \( Q_v \) (right) is induced by the truncated dual.

It was first observed by Postnikov (also see Mészáros et al. [4, Theorem 1.3]) that the canonical triangulation of \( \mathcal{O}(P_G) \) is the same as the planar-framed DKK triangulation of \( \mathcal{F}_G \) up to an integral equivalence. Combined with Theorems 1.2 and 1.3, we have the following two corollaries.

**Corollary 5.6** The canonical triangulation of the order polytope \( \mathcal{O}(Q_v) \) has dual graph which is the Hasse diagram of the principal order ideal \( I(v) \) in Young’s lattice.

**Corollary 5.7** The order polytope \( \mathcal{O}(Q_v) \) has a regular unimodular triangulation whose dual graph is the \( v \)-Tamari lattice \( \text{Tam}(v) \).

Since the volume of \( \mathcal{O}(Q_v) \) is given by the number of linear extensions of \( Q_v \) (see [17]), we also have the following result.

**Corollary 5.8** The number of linear extensions of the poset \( Q_v \) is \( \text{Cat}(v) \).

Having obtained results for order polytopes via methods for flow polytopes, we now end this section by obtaining the face structure of \( \mathcal{F}_{\text{car}(v)} \) via methods for order polytopes. Stanley [17, Section 1] gave a full description of the faces of an order polytope \( \mathcal{O}(P) \) via partitions of the poset \( \hat{P} := P \cup \{\hat{0}, \hat{1}\} \) into connected blocks satisfying a compatibility criterion. Translating these results to our setting, we can describe the face lattice of \( \mathcal{F}_{\text{car}(v)} \) using valid subwords of \( \hat{v} := NvE \).

**Definition 5.9** Let \( \hat{v} := NvE \), with its letters indexed by their position in the word. A subword \( \sigma \) of \( \hat{v} \) is valid if it satisfies the following conditions.

1. If \( \sigma \) contains an \( E \) step and an \( N \) step of \( \hat{v} \), then it contains a peak of \( \hat{v} \).
2. If \( \sigma \) contains \( E_i \) and \( E_j \) with \( i < j \), then \( \sigma \) contains all \( E_k \) with \( i < k < j \).
3. If \( \sigma \) contains \( N_i \) and \( N_j \) with \( i < j \), then \( \sigma \) contains all \( N_k \) with \( i < k < j \).
4. If \( \sigma \) contains \( N_i \) and \( E_j \) with \( i < j \), then \( \sigma \) contains all steps \( X_k \) of \( \hat{v} \) with \( i < k < j \).

In particular, \( \hat{v} \) itself is a valid subword, as are each of the steps in \( \hat{v} \). The word \( \hat{v} \) can now be partitioned into valid subwords. For example, if \( \hat{v} = N_1N_2E_3E_4N_5E_6N_7N_8E_9 \), then \( \hat{v} \) can be partitioned into the valid subwords \( \sigma_1 = N_1N_2E_3N_5 \), \( \sigma_2 = E_4 \), and \( \sigma_3 = E_6N_7N_8E_9 \). The following is a direct consequence of [17, Theorem 1.2] when translated to our setting (with connectedness corresponding with conditions 1, 2, and 3 in Definition 5.9, and compatibility corresponding with condition 4). We omit the details of the proof.
Proposition 5.10 The face lattice of \( \mathcal{F}_{\text{car}(\nu)} \) is the poset of partitions of \( \hat{\nu} \) into valid subwords, ordered by reverse inclusion of the partitions. The face lattice is ranked by the number of valid subwords in the partition.

The empty face of \( \mathcal{F}_{\text{car}(\nu)} \) corresponds to \( \hat{\nu} \) itself and the top dimensional face is the partition into \( a + b + 2 \) subwords, each consisting of a single step of \( \hat{\nu} \). We have the following corollary.

Corollary 5.11 Let \( \nu = NE^{\nu_1} \cdots NE^{\nu_a} \) be a lattice path from \( (0, 0) \) to \( (b, a) \). The number of facets of \( \mathcal{F}_{\text{car}(\nu)} \) is \( a + b + \text{peak}(\hat{\nu}) \), where \( \text{peak}(\hat{\nu}) \) denotes the number of consecutive \( \text{NE} \) pairs in \( \hat{\nu} \).

Proof The facets of \( \mathcal{F}_{\text{car}(\nu)} \) correspond with the partitions of \( \hat{\nu} \) into \( a + b + 1 \) valid subwords. In such a partition, exactly one subword contains two letters. From the conditions of a valid subword, we see that such a subword is either a peak, or consists of two \( E \) steps, or two \( N \) steps. As there are exactly \( a \) valid subwords with two \( E \) steps and \( b \) valid subwords with two \( N \) steps, the result follows. \( \square \)

6 The \( h^* \)-Vector of the \( \nu \)-Caracol Flow Polytope

The Ehrhart series of a polytope \( P \subseteq \mathbb{R}^n \) is defined as \( Ehr_P(z) = 1 + \sum_{t \geq 1} |tP \cap \mathbb{Z}^n| z^t \) where \( tP \) is the dilation of \( P \) by a factor of \( t \). In the case when \( P \) is a lattice polytope of dimension \( d \) it is known since Ehrhart [18] that

\[
Ehr_P(z) = \frac{h^*(z)}{(1 + z)^{d+1}}
\]

where \( h^*(z) \) is a polynomial of degree \( d \). The polynomial \( h^*(z) \) is known as the \( h^* \)-polynomial of \( P \) and its sequence of coefficients is known as the \( h^* \)-vector of \( P \). Stanley’s nonnegativity theorem [19] also says that \( h^*(z) \) has non-negative coefficients. The \( h^* \)-vector of a lattice polytope coincides with the \( h^* \)-vector of any of its unimodular triangulations [20, Theorem 10.3], so we will compute the \( h^* \)-vector of the planar-framed triangulation of \( \mathcal{F}_{\text{car}(\nu)} \). This extends a result of Mészáros [15, Theorem 4.4] for the classical case when \( \nu = (1^n) \).

We begin by recalling some relevant definitions from [21]. Given a simplicial complex, a shelling is an ordering \( F_1, \ldots, F_s \) of its facets such that for every \( i < j \) there is some \( k < j \) such that the intersection \( F_i \cap F_j \subseteq F_k \cap F_j \), and \( F_k \cap F_j \) is a facet of \( F_j \). A simplicial complex is said to be shellable if it admits a shelling. The \( h \)-vectors of shellable simplicial complexes have non-negative entries which can be computed combinatorially from the shelling order as follows. For a fixed shelling order \( F_1, \ldots, F_s \) define the restriction \( R_j \) of the facet \( F_j \) as the set \( R_j := \{ v \in F_j : v \) is a vertex in \( F_j \) and \( F_j \setminus v \subseteq F_i \) for some \( 1 \leq i < j \} \). Then the \( i \)-th entry of the \( h \)-vector is given by \( h_i = |\{ j : |R_j| = i , 1 \leq j \leq s \}| \).
Lemma 6.1 Let $C$ be the planar-framed triangulation of $\mathcal{F}_{\text{car}}(v)$ interpreted as a simplicial complex. Any linear extension of $I(v)$ gives a shelling order of $C$.

Proof By Theorem 1.3 we can give the dual graph of $C$ the structure of $I(v)$, identifying each facet in $C$ with the associated $v$-Dyck path in $I(v)$. For a linear extension $L$ of $I(v)$, we can order the facets $F_1, \ldots, F_s$ of $C$ according to $L$. Let $\pi_i$ and $\pi_j$ be two $v$-Dyck paths in $L$, with $i < j$. Let $\pi_{s_1}$ be the minimal $v$-Dyck path that covers both $\pi_i$ and $\pi_j$, i.e. $\pi_{s_1} = \pi_i \cup \pi_j$ in $I(v)$. Now $\pi_{s_1}$ contains the lattice points in $\pi_i \cap \pi_j$, and so $F_i \cap F_j \subseteq F_{s_1}$. It is clear that there exists a sequence of $v$-Dyck paths $\pi_{s_1}, \pi_{s_2}, \ldots, \pi_j$ such that each path contains the lattice points $\pi_i \cap \pi_j$, and each path is formed from the previous path by replacing a consecutive $NE$ pair with $EN$. Given such a sequence of paths, let $\pi_k$ be the second to last path in the sequence. Replacing a consecutive $NE$ pair with $EN$ in $\pi_k$ yield $\pi_j$. Now $k < j$, and $F_i \cap F_j$ is contained in every facet $F_{s_\ell}$ for $1 \leq \ell \leq k$. In particular, $F_i \cap F_j \subseteq F_k \cap F_j$. Furthermore, $\pi_k$ and $\pi_j$ differ by a single lattice point, so $F_k \cap F_j$ is a facet of $F_j$.

Let $v$ be a lattice path from $(0, 0)$ to $(b, a)$. For $i = 0, \ldots, a$, the $v$-Narayana number $Nar_v(i)$ is the number of $v$-Dyck paths with $i$ valleys (recall that a valley is a consecutive $EN$ pair). The $v$-Narayana polynomial is $N_v(x) = \sum_{i \geq 0} Nar_v(i)x^i$. For more on these definitions, see [12] or [10], for example.

Theorem 1.4 The $h^*$-polynomial of $\mathcal{F}_{\text{car}}(v)$ is the $v$-Narayana polynomial.

Proof As previously mentioned, it suffices to find the $h$-vector of the planar-framed triangulation of $\mathcal{F}_{\text{car}}(v)$, which can be computed from a shelling order of the planar-framed triangulation of $\mathcal{F}_{\text{car}}(v)$. We fix a linear extension of $I(v)$, which by Lemma 6.1 a gives a shelling order $F_1, \ldots, F_s$. For a facet $F_i$, $|R_i|$ is the number of facets incident to $F_i$ appearing before $F_i$ in the shelling order. Since the shelling order is given by a linear extension of $I(v)$, $|R_i|$ is the number of elements covered by $F_i$ in $I(v)$. By the cover relation in $I(v)$, the number of elements covered by $F_i$ is the number of valleys in the corresponding $v$-Dyck path. The $i$-th entry of the $h$-vector can now be computed as follows

$$h_i = |\{j : |R_j| = i, 1 \leq j \leq s\}|$$
$$= |\{\text{paths in } I(v) \text{ that cover exactly } i \text{ paths}\}|$$
$$= |\{v \text{-Dyck paths with exactly } i \text{ valleys}\}|$$
$$= Nar_v(i).$$

$\square$

Example 6.2 Let $v = NE\bar{N}E^2NE^2$. The dual graph of the planar-framed triangulation of $\mathcal{F}_{\text{car}}(v)$ is then the Hasse diagram of $I(v)$ as shown on the right in Fig. 10. The number of $v$-Dyck paths with 0, 1, and 2 valleys are respectively 1, 4, and 2. Thus the $v$-Narayana polynomial is $Nar_v(x) = 1 + 4x + 2x^2$.

A different proof of Theorem 1.4 can be obtained by computing the $h$-vector of the length-framed triangulation of $\mathcal{F}_{\text{car}}(v)$, which by Corollary 4.12 is combinatorially
equivalent to the \((I, J)\)-Tamari complex with the pair \((I, J)\) associated to \(\nu\), which we also call the \((I, J)\)-Tamari complex. In \cite[Lemma 4.5]{Reference10} a shelling order on facets of this complex was used to show that the \(h\)-vector of the \((I, J)\)-Tamari complex is given by the \(\nu\)-Narayana numbers. Since any lattice unimodular triangulation can be used to calculate the \(h^*\)-vector of \(\mathcal{F}_{\text{car}(\nu)}\), Theorem 1.4 provides a new proof that the \(h\)-vector of the \((I, J)\)-Tamari complex is given by the \(\nu\)-Narayana numbers.

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**References**

1. Baldoni, W., Vergne, M.: Kostant partitions functions and flow polytopes. Transform. Groups 13(3), 447–469 (2008)
2. Mészáros, K., Morales, A.H.: Volumes and Ehrhart polynomials of flow polytopes. Math. Z. 293(4), 1369–1401 (2019)
3. Benedetti, C., et al.: A combinatorial model for computing volumes of flow polytopes. Trans. Am. Math. Soc. 372(5), 3369–3404 (2019)
4. Mészáros, K., Morales, A.H., Striker, J.: On flow polytopes, order polytopes and certain faces of the alternating sign matrix polytope. Discrete Comput. Geom. 62(1), 128–163 (2019)
5. Danilov, V.I., Karzanov, A.V., Koshevoy, G.A.: Coherent fans in the space of flows in framed graphs. In: 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). Discrete Mathematics and Theoretical Computer Science. pp. 481–490 (2012)
6. Préville-Ratelle, L., Viennot, X.: The enumeration of generalized Tamari intervals. Trans. Am. Math. Soc. 369(7), 5219–5239 (2017)
7. Armstrong, D., Loehr, N.A., Warrington, G.S.: Rational parking functions and Catalan numbers. Ann. Comb. 20(1), 21–58 (2016)
8. Armstrong, D., Rhoades, B., Williams, N.: Rational associahedra and noncrossing partitions. Electron. J. Combin. 20(3), 51 (2013)
9. Ceballos, C., González D’León, R.S.: Signature Catalan combinatorics. J. Comb. 10(4), 725–773 (2019)
10. Ceballos, C., Padrol, A., Sarmiento, C.: Geometry of \(\nu\)-Tamari lattices in types A and B. Trans. Am. Math. Soc. 371(4), 2575–2622 (2019)
11. Ceballos, C., Padrol, A., Sarmiento, C.: The \(\nu\)-Tamari lattice via \(\nu\)-trees, \(\nu\)-bracket vectors, and subword complexes. Electron. J. Comb. 27, 18–62 (2020)
12. von Bell, M., Yip, M.: Schröder combinatorics and \(\nu\)-associahedra. Eur. J. Combinatorics 98, 103415 (2021)
13. Pitman, J., Stanley, R.P.: A polytope related to empirical distributions, plane trees, parking functions, and the Associahedron. Discrete Comput. Geom. 27(4), 603–634 (2002)
14. Gessel, I., Viennot, G.: Binomial determinants, paths, and hook length formulae. Adv. Math. 58(3), 300–321 (1985)
15. Mészáros, K.: Pipe dream complexes and triangulations of root polytopes belong together. SIAM J. Discrete Math. 30(1), 100–111 (2016)
16. Yip, M.: A Fuss-Catalan variation of the caracol flow polytope. arXiv: 1910.10060
17. Stanley, R.P.: Two poset polytopes. Discrete Comput. Geom. 1(1), 9–23 (1986)
18. Ehrhart, E.: Sur les polyèdres rationnels homothétiques à n dimensions. C. R. Acad. Sci. Paris 254, 616–618 (1962)
19. Stanley, R.P.: Decompositions of rational convex polytopes. Ann. Discrete Math. 6(6), 333–342 (1980)
20. Beck, M., Robins, S.: Computing the Continuous Discretely. Undergraduate Texts in Mathematics, Springer, New York (2007)
21. Ziegler, G.M.: Lectures on Polytopes, vol. 152. Springer Science & Business Media, New York (2007)

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