Nearly Localised States in Weakly Disordered Conductors.  
II. Beyond Diffusion Approximation.

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We use optimal fluctuation method for a new ballistic σ-model to study the long time dispersion of conductance $G(t)$ of a mesoscopic sample. In the long time limit the conductance of a $d$-dimensional sample decays as $\exp(-\Lambda n^d t)$. At shorter times the new results match those in our previous paper [1]. It is found that at very long times the diffraction effects are important and the ballistic treatment is not valid. We also suggest a physical picture of trapping.

I. INTRODUCTION

This article is a continuation of our paper [1] where we suggested an optimal fluctuation method to study exponentially rare fluctuations in weakly disordered conductors. The quantity of interest was expressed using the super-matrix nonlinear σ-model [2], and the resulting functional integral was evaluated by the saddle point method. In this way we obtained an intermediate asymptote of conductance time dispersion in one- and two-dimensional metals. It was, however, pointed out that this method has only a limited scope of validity, e.g., it fails to describe satisfactorily three-dimensional conductors. The failure occurs when the super-matrix corresponding to the optimal fluctuation changes rapidly over the mean free path $l$. Then the diffusion approximation breaks down and the use of the nonlinear σ-model [2] can no longer be justified.

In this article we treat the problem using the recently proposed ballistic σ-model [3]: the generalisation of the standard one that correctly accounts for large gradients. To be specific, we consider the long time asymptote of conductance dispersion. If a voltage $V(t)$ is applied to a sample, the current through it is given by the Ohm law

$$I(t) = \int_{-\infty}^{t} G(t - t') V(t') dt',$$  

and we are interested in the behaviour of the conductance $G(t)$ in the long time limit.

We first discuss this problem under conditions when the diffusion approximation is valid. The super-matrix theory is defined by the the partition function (see [2] and [3] for review):

$$Z = \int DQ e^{-F}, \quad F = \frac{\pi \nu}{8} \text{str} \int dr \{ D(\nabla Q)^2 + 2i\omega\Lambda Q \},$$  

where the functional integral over super-matrices $Q$ is subjected to the constraint

$$Q^2 = 1.$$  

The expression for the averaged conductance $G(t)$ looks as follows

$$G(t) = G_0 e^{-t/\tau} + \int \frac{d\omega}{2\pi} e^{-i\omega t} \int_{Q^2 = 1} DQ P(Q) e^{-F}$$  

In Eqs. (2), (3) $\nu$ is the density of states, $D$ is diffusion coefficient and $\tau$ is the mean free time. The strategy suggested in Ref [3] consists in studying the condition of the extremum of the free energy $F$ in Eq. (4)

$$2D\nabla(Q \nabla Q) + i\omega[\Lambda, Q] = 0$$

together with the condition at the boundary $\Gamma$ between the mesoscopic sample and a bulk electrode

$$Q|_{\Gamma} = \Lambda,$$

and a self-consistency condition

$$\frac{4t\Delta}{\pi \hbar} = -\int \frac{dr}{V} \text{str}\{\Lambda Q\}; \quad \Delta = \frac{1}{\nu V},$$

that arises after integrating out $\omega$ in Eq. (4). After solving equation (3) with boundary conditions (5), expressing frequency $\omega$ through time $t$, using self-consistency condition (6) and substituting the solution $Q(r)$ to the free energy (4), we obtain the conductance $G(t)$ with exponential accuracy.

Long time retardation in electric response $G(t)$ is caused by relatively improbable quasi-localised states that are weakly coupled to the bulk electrodes and have life time of the order of $t$. Since the mean square of the wave function $|\Psi(r)|^2$ is connected to the super-matrix $Q(r)$ via

$$|\Psi(r)|^2 \sim -\text{str}\{\Lambda Q\},$$

the self-consistency condition (3) can be regarded as a relation between the life-time of a quasi-stationary state and its wave function. The super-matrix $Q$ is fixed at the boundary of the sample (see Eq. (3)), so, to satisfy the self-consistency condition (6) $\text{str}\{\Lambda Q\}$ must grow towards the middle of the sample. The gradients of the $Q$-matrix increase simultaneously with the delay time $t$.

Since theory (3) correctly accounts only for the lowest
term in $\nabla Q$, it cannot be used for sufficiently long times $t$.

The importance of high gradients for long time asymptotes and tails of distribution functions was first announced by Altschuler, Kravtsov and Lerner (AKL) (6), who found the growth of the corresponding invariant charges under renormalization group flow. The optimal fluctuation method has been used recently by Falko and Efetov (6) and by Mirlin (7), who studied the influence of the nearly localised states on the distribution of wave function amplitudes (7) and local density of states (7). These authors found that the gradients of $Q$ become large near the centre of the sample and, therefore, the $\sigma$-model description breaks down.

To treat consistently this problem, along with the others where ballistic motion of electrons is essential, we suggested a new version of the nonlinear $\sigma$-model (8). This theory operates with a super-matrix distribution function $g_n(r)$, where $n$ is the unit vector of the electron momentum direction ($p = n p_F$). It effectively accounts for the infinite series in $i\nabla Q$ of which only the leading term is kept in the action (9), and, therefore, correctly describes fluctuations of the super-matrix $Q$ with wave vectors $q \sim 1/l$. The theory is still restricted, however, to the region of validity of semi-classical approximation. The condition of extremum for the new action is related to the kinetic equation in the same manner as Eq. (9) is related to the diffusion equation. We solve the kinetic equation for one- two- and three-dimensional geometries and obtain very long time asymptotes of conductance.

The results are summarised in Table I. The exponential decay law for times $t_D \ll t \ll h/\Delta$ can be obtained analysing the time dependence of weak localisation corrections. The “ballistic” part of the long time asymptote at $t > t_b$ was first studied by AKL who used the frequency representation and expanded the conductance $G(\omega)$ in powers of $\omega$. They estimated the growth rate of coefficients in this expansion and pointed out the importance of high gradients. Unfortunately, the Fourier transformation to $t$-representation in two-dimensions was carried out with insufficient precision (see (6) for discussion) and the factor $g$ was lost under the sign of $\ln^2$. The AKL approach also failed to predict the existence of intermediate asymptote at $h \Delta^{-1} \ll t \ll h/\Delta$ which was discovered by the authors (6) using the optimal fluctuations method for the diffusive $\sigma$-model. We found the region of validity for this intermediate asymptote and obtained some estimates for even longer times. These estimates were recently improved by Mirlin (9) who imposed somewhat arbitrary effective boundary conditions on the super-matrix $Q$ at the point where the diffusion approximation breaks down. The value of the action turned out to be rather insensitive to the exact form of these boundary conditions which enabled Mirlin to rederive the AKL result in 2d and obtain $\exp(-A \ln^2 t)$ asymptote in 3d, although without the value of the coefficient $A$ in the exponent.

The full ballistic treatment presented in this article gives the coefficient in the exponent for 3d; confirms the AKL result in 2d; and discovers a new regime in the case of a thick wire. We have also found a restriction on the validity of ballistic treatment. It turns out, that at ultralong delay times the super-matrix distribution function depends strongly on the direction $n$ of the momentum developing sharp features with characteristic width $\delta \phi$. At times $t \sim t_Q$ this width becomes comparable with the diffraction angle $\delta \phi \sim \lambda/a$, where $a$ is a typical size. At longer times the diffraction effects become important and the ballistic treatment is no longer valid. The values of times $t_Q$ are presented in Table I. Apart from derivation of these results from the first principles the paper describes the optimal fluctuation of random potential in a strictly one-dimensional wire that traps an electron for time $t$. We believe that the same mechanism is responsible for long time delays in higher dimensions.

The material is organised as follows. In section II we present physical motivations behind the ballistic $\sigma$-model, find its condition of extremum; and show how the ballistic description transforms into the diffusive one when the gradients are small. We also discuss the geometry of the super-matrix distribution function $g_n(r)$ and introduce its convenient parametrisation. The optimal fluctuation method is described in section III. In section IV the solution of kinetic saddle-point equation is found and the long time asymptote of the conductance is evaluated in two and three dimensions. The one dimensional case is discussed in section V. Physical picture of trapping is presented in section VI. Finally, in section VII we discuss the results and the limits of their validity.

II. EFFECTIVE ACTION FOR QUANTUM BALLISTICS

A. Outline of Derivation

In this section we present a generalised non-linear super-matrix $\sigma$-model (6), which is valid in the ballistic regime.

If the quantum effects are neglected, the ballistic regime is described by the Boltzmann kinetic equation for the distribution function $f_n(r)$ of coordinate $r$ and momentum $p = n p_F$. The quantum description operates with density matrix $\hat{g}_n(r)$. To enable averaging over disorder, $\hat{g}_n(r)$ should be a super-matrix (6), analogously to the matrix $\hat{Q}$ in the standard $\sigma$-model (6). The quantum generalisation of the kinetic equation has the form

$$2 v n \frac{\partial g_n(r)}{\partial r} = \left[\left(\frac{i \omega A - \langle g(r) \rangle}{\tau}\right), g_n(r)\right]$$

with the additional constraint

$$g_n^2 = 1,$$
equation (13) and serves as an extremum condition for the ballistic action we are constructing. To generate the first derivative in Eq. (13), the action has to have a Wess-Zumino type term

$$W\{g_n\} = \int \int_0^1 du \, \text{str} \left( \tilde{g}_n(r, u) \left[ \frac{\partial \tilde{g}_n}{\partial u} - n \frac{\partial \tilde{g}_n}{\partial r} \right] \right) dr$$  \hspace{1cm} (11)

where an arbitrary smooth interpolation can be chosen as $\tilde{g}_n(r, u)$, and the angular brackets denote averaging over directions of $n$. The functional derivative $\delta W/\delta g_n(r)$ is taken with the constraint (14) which guarantees that $g_n \delta g_n + \delta g_n g_n = 0$ and an arbitrary variation $\delta g_n$ has the form $\delta g_n = [g_n, a_n]$. As a result of variation

$$\delta W = 4 \int dr \, \text{str} \left( n \frac{\partial g_n}{\partial r} a_n \right)$$  \hspace{1cm} (13)

the first derivative appears. There is another way of writing the functional $W$ which employs the decomposition $g_n = U \Lambda U^{-1}$:

$$W\{g_n\} = 4 \int dr \, \text{str} \left( U \Lambda^{-1} 2 \frac{\partial U}{\partial r} \right).$$  \hspace{1cm} (14)

This representation can be verified by comparing the variation of $W$ with Eq. (13). The quantum ballistic partition function $Z$ can be presented as an integral over distribution functions $g_n(r)$ with effective action $F$:

$$Z = \int Dg_n(r) e^{-F},$$  \hspace{1cm} (15a)

$$F = \frac{\pi \nu}{4} \int dr \, \text{str} \left\{ \omega \Lambda \langle g(r) \rangle - \frac{1}{2\tau} \langle g(r) \rangle^2 \right\} - \frac{\pi \nu \nu F}{8} W\{g_n\}.$$  \hspace{1cm} (15b)

The details of the derivation can be found in Ref [3].

Field theory [4] enables us to study any chaotic problem, for which the semi-classical approach is valid, irrespective of validity of the diffusion approximation. If space gradients are small ($|\nabla g| \ll g$), the standard treatment recovers (see [3]) the Q-matrix theory [5]. To show this we expand the matrix $g_n$ into a sum over angular harmonics keeping only the zeroth and first terms:

$$g_n = Q(r) \left( 1 - \frac{Q J^2}{2} (n^2) \right) + J(r) \cdot n$$  \hspace{1cm} (16)

The constraint $g_n^2 = 1$ now reads

$$Q^2 = 1, \quad Q J + J Q = 0.$$  \hspace{1cm} (17)

Substituting Eq. (16) into Eqs. (15) and using conditions (7), we obtain the partition function in the form

$$Z = \int DQ \int D\mathbf{J} e^{-F_{Q, J}},$$

$$F_{Q, J} = \frac{\pi \nu}{4} \int dr \, \text{str} \left\{ \omega \Lambda Q + \frac{J^2}{2\tau} - \frac{\nu F}{3} \langle \nabla Q \rangle Q J \right\}$$  \hspace{1cm} (18)

The Gaussian integral over $J$ in Eq. (18) is dominated by the vicinity of

$$J = l(\nabla Q) Q$$  \hspace{1cm} (19)

and leads finally to Eq. (5).

### B. Symmetries of $g$-matrices

Both $Q$ and $\Lambda$ are $8 \times 8$ matrices which act on the 8-component super-vectors $\Psi$ with the basis [2]:

$$\Psi^T = (\chi_1, \chi_2^*, S_1, S_1^*, \chi_2, \chi_2^*, S_2, S_2^*),$$  \hspace{1cm} (20)

where $\chi$ are the fermionic and $S$ are bosonic variables; indices 1, 2 correspond to the retarded and advanced Green functions with energies $E_{\pm} = \pm \omega/2$. For a super-matrix $M$ the super-trace is defined as follows:

$$\text{str} M = M_{11} + M_{22} - M_{33} - M_{44} + M_{55} + M_{66} - M_{77} - M_{88}$$

In this basis the super-matrix $\Lambda$ has the form

$$\Lambda_{ij} = \lambda_i \delta_{ij}, \quad \lambda_i = \begin{cases} 1, & i = 1 \ldots 4 \\ -1, & i = 5 \ldots 8 \end{cases}$$

For our purposes it is more convenient to use another basis where each variable is stands next to its charge conjugate partner and fermionic variables are separated from bosonic:

$$\Psi^T = (\chi_1, \chi_2^*, \chi_2, \chi_1^*, S_1, S_2^*, S_2, S_1^*).$$  \hspace{1cm} (21)

In this new basis, which will be used everywhere in this paper, the matrix $\Lambda$ has the form:

$$\Lambda = \begin{pmatrix} \sigma_3 \otimes \tau_3 & 0 \\ 0 & \sigma_3 \otimes \tau_3 \end{pmatrix},$$  \hspace{1cm} (22)

where the blocks in Eq. (22) correspond to the fermionic and bosonic sectors. The matrices $\tau$ act inside the $2 \times 2$ blocks, while matrices the $\sigma$ mix these blocks inside the $4 \times 4$ sectors.

We first discuss the symmetry of the Q-matrix from the standard $\sigma$-model [2] focusing on the properties of the boson-boson ($B$) and fermion-fermion ($F$) sectors only. We consider only the unitary ensemble when an additional condition

$$[Q^{B,F}, 1 \otimes \tau_i] = 0$$  \hspace{1cm} (23)

is fulfilled. Together with constraint (3), requirement (23) permits to parametrise the matrices $Q^{B,F}$ with four...
complex 3-vectors $l^{B,F}, m^{B,F}$ subject to the constraint $l^2 = m^2 = 1$:

$$Q^{B,F} = \frac{1 + \tau_3}{2} \otimes l\sigma + \frac{1 - \tau_3}{2} \otimes m\sigma. \quad (24)$$

Parametrisation (24) is still too general because the matrix $Q$ has additional symmetries: charge neutrality

$$\tilde{Q} \equiv CQ^T C^T = Q, \quad (25a)$$

with the charge conjugation matrix

$$C = \begin{pmatrix} i\sigma_2 \otimes \tau_1 & 0 \\ 0 & -\sigma_2 \otimes \tau_2 \end{pmatrix},$$

and pseudo-hermiticity

$$Q^\dagger \equiv KQ^* K = Q, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_3 \otimes \tau_3 \end{pmatrix}. \quad (25b)$$

Requirements (25a) are satisfied when

$$l^B = -m^B, \quad l^F = -m^F, \quad (26a)$$

and

$$l^B = \tilde{\mu}(l^B)^*, \quad l^F = (l^F)^*, \quad \tilde{\mu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (26b)$$

Thus, the fermionic sector is parametrised by a real vector $l^F$ subjected to constraint $(l^F)^2 = 1$, i.e. a sphere $S^2$

$$S^2 = \{l^F, \quad l_1^2 + l_2^2 + l_3^2 = 1, \quad \text{Im } l = 0 \}. \quad (27)$$

while the bosonic sector is represented by the vector $l^B$ with two imaginary components $l_1^B$ and $l_2^B$ and one real $l_3^B$. Due to constraint $l^B = 1$ the bosonic sector is represented by a hyperboloid $H^2_3$:

$$H^2_3 = \{l^B, \quad -|l_1|^2 - |l_2|^2 + l_3^2 = 1, \quad l = \tilde{\mu}l^* \}. \quad (28)$$

The matrix $g_n$ of the ballistic $\sigma$-model (15) obeys the constraint $g^2 = 1$ and, therefore, can be parametrised with Eq. (24). The condition of charge neutrality $g^B_n = 1$ must be replaced by the generalised version

$$\tilde{g}_n \equiv Cg_n^\dagger C^\dagger = g_{-n} \quad (29)$$

because the charge conjugation changes the sign of the $W$-term in action (15).

The pseudo-hermitian transformation $g_n \rightarrow Kg_n^* K$ not only changes the sign of the $W$-term but also replaces $\omega$ by $-\omega^*$. In the long time asymptote calculation the frequency $\omega$ is purely imaginary and the generalisation of Eq. (25b) reads:

$$Kg_n^* K = g_{-n} \quad (30)$$

Symmetries (29) and (30) impose a new restriction on the vectors $l$ and $m$ (compare with Eqs. (26)):

$$l^B_+= -m^B_-, \quad l^F_+= -m^F_-, \quad l^B_-= \tilde{\mu}(l^B)^*_+, \quad l^F_-= (l^F)^*_+. \quad (31)$$

Since conditions (29) are imposed on the components of two vectors $l$ and $m$, they are less restrictive than Eqs. (25). Both $g^B_n$ and $g^F_n$ can be parametrised with a complex unit vector $l_n = \xi_n + i\eta_n$:

$$g^B_n = \frac{1 + \tau_3}{2} \otimes \sigma l_n + \frac{1 - \tau_3}{2} \otimes \sigma l_{-n}. \quad (32)$$

with the constraint

$$l^2 = \xi^2 - \eta^2 + 2i(\xi\eta) = 1. \quad (33)$$

The geometric meaning of Eq. (33) can be described as follows. The vector $\xi = \nu\xi$ is characterised by its absolute value $\xi$ and a unit vector $\nu$, which corresponds to a point on a sphere $S^2$. Due to the condition $0 = \xi\eta = \xi\nu\eta$, the vector $\eta$ belongs to the plane tangential to the sphere $S^2$ at the point $\nu$. The condition $\xi^2 - \eta^2 = 1$ means that at every point of the sphere there is an upper part of the two sheet hyperboloid $H^2_3$ with the axis along the radius of the sphere. In other words, the vector $l$ subject to constraint (33), belongs to the fibre bundle with the base $S^2$ and the fibre $H^2_3$. The $Q$-matrices in this geometric picture are represented by the sub-manifold of this fibre bundle: $Q^F$ belongs to the base and $Q^B$ lies on the fibre over the North Pole of $S^2$ (see Eq. (24)). The A-matrix corresponds to the bottom of $H^2_3$ in the fibre over the North Pole.

### III. STEEPEST DESCENT PROCEDURE FOR QUANTUM BALLISTICS

The long time asymptote of the conductance $G(t)$ is found following the procedure outlined in the introduction. The generalised version of Eq. (4) has the form

$$G(t) = G_0 e^{-t/\tau} + \int \frac{d\omega}{2\pi} e^{-i\omega t} \int_{s^2=1}^\mathcal{D} g_n P \{ g_n \} e^{-F}, \quad (34)$$

where $F$ is the ballistic action (15), and the explicit form of the functional $P$ is irrelevant within exponential accuracy. The functional integral over $g_n$ is evaluated using the steepest descent method, which consists in:

1. finding a solution $g_n (r)$ of the saddle point equation (4);
2. expressing the frequency $\omega$ through the time $t$ using the self-consistency condition

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\[ \frac{4t\Delta}{\pi\hbar} = -\int \frac{d\mathbf{r}}{V} \text{str}\{\Lambda(g)\}, \]  

which arises as a result of integration over \( \omega \) in Eq. (34):

3. substituting \( g_n(r) \) in Eq. (34) and obtaining the result with exponential accuracy.

We do not specify boundary conditions for Eq. (1) bearing in mind that far from the centre of the sample the space gradients become small and \( g_n(r) \) approaches a solution of diffusion equation (46). It was shown in Ref. [1] that for the long times \( t \gg \hbar/\Delta \) the solutions of diffusion equation (46) can be written in the form:

\[ Q = U \left( \begin{array}{cc} \sigma_3 \otimes \tau_3 & 0 \\ 0 & (\sigma_3 \cosh \theta + i\sigma_2 \sinh \theta) \otimes \tau_3 \end{array} \right) U^{-1}, \]  

(36)

where the real angle \( \theta \) obeys the equation

\[ D\nabla^2 \theta + i\omega \sinh \theta = 0; \quad \theta|_{\text{lead}} = 0 \]  

(37)

and the constant super-matrices \( U \) commute with \( \Lambda \). They do not enter the action and thus will be omitted. As can be seen from Eq. (36), only the bosonic block \( Q^B \) has a non-trivial dependence on the coordinates. The same holds true in the ballistic region, and from now we consider only the bosonic sector of the theory.

Combining the diffusion asymptote (46) with Eqs. (16, 19, 32), we find that the vector \( l_n \) that parametrises \( g_n \) approaches the limit

\[ l_1 = -ln \nabla \theta, \quad l_2 = i \sinh \theta, \quad l_3 = \cosh \theta \]  

(38)

far from the centre of the sample. One can see from Eq. (38) that

\[ \text{Im} \ l_1 = \text{Re} \ l_2 = \text{Im} \ l_3 = 0 \]  

(39)

It turns out that the condition (33) remains valid even in the ballistic regime. Therefore, for the solution of kinetic equation (1) the complex vector \( l_n(r) \) is restricted to the real three dimensional subspace (29). The additional constraint \( l^2 = 1 \) thus acquires the form

\[ l_1^2 - l_2^2 + l_3^2 = 1 \]  

(40)

and defines a one sheet hyperboloid. Convenient coordinates on the subspace (29) are associated with the cone \( l_1^2 - l_2^2 + l_3^2 = 0 \) and described in appendix A.

Due to the symmetry (34), \( l_1 \) is an odd and \( l_2, l_3 \) are even functions of \( n \). Averaging Eq. (1) over directions of \( n \) and using the parametrisation (32), we arrive at the continuity condition

\[ 2v_F \text{div} \langle n l_1 \rangle + \omega \langle l_2 \rangle = 0 \]  

(41)

which reduces to the conservation law for the current:

\[ J \equiv \langle n l_1 \rangle, \quad \text{div} J = 0, \]  

(42)

if the frequency is small and \( \gamma \equiv i\omega \tau \rightarrow 0 \).

**IV. SOLUTION OF KINETIC EQUATION AND LONG TIME ASYMPTOTE OF CONDUCTANCE**

In Ref. [1] we considered the solution of the diffusion equation (37) either for a disordered wire (1D), or for a disc (2D), or for a droplet (3D) of radius \( R \) (see Fig 1). In analysing the kinetic equation (1) we stick to the same geometry. In this section we find the solutions of Eq. (1) in space dimensions 2 and 3, postponing the discussion of one-dimensional case to the next section.

Long times correspond to small frequencies \( \omega (\gamma \ll 1) \). This means that the terms containing \( \gamma \) in the kinetic equation can be neglected everywhere, except the central part of the sample. It can be seen from Eq. (34) that the total flux of the current \( J \) is conserved in the outer area. Since the current \( J \) is directed along the radius, flux conservation means that in two- and three-dimensional samples the current density decays towards the outer boundary together with the space gradients of all relevant quantities, and the solution of the kinetic equation approaches the diffusion asymptote. The situation is, however, different in the one-dimensional case when the current and gradients do not decay and the diffusion regime is reached only outside the sample.

**A. Qualitative description**

In the space dimensions 2 and 3 the qualitative behaviour of the solutions is the same for both the diffusive and kinetic equation. We illustrate it with the diffusion Equation (37) which can be regarded as describing a chemical reaction where \( \theta \) is the concentration of a reacting agent. The term \( D\nabla^2 \theta \) describes the propagation of the agent in a porous medium and the rate of the agent reproduction depends on its concentration as \( \gamma \sinh \theta \). Since \( \gamma \ll 1 \), generation takes place only in the central part of the sample, where the value of \( \theta \) is high. We refer to this central zone as the “zone of reaction”. In the outer part of the sample the agent diffuses freely (the “run-out zone”), i.e. \( \theta \) is a solution of the Laplace equation \( \nabla^2 \theta = 0 \). An azimuthally symmetric solution \( \theta(r) \) obeying the boundary condition \( \theta(R) = 0 \) decays like

\[ \theta = C \ln \frac{R}{r} \]  

(2D)

\[ \theta = C \left( \frac{1}{r} - \frac{1}{R} \right) \]  

(3D)

as \( r \rightarrow R \).

The separation into two zones is also valid for solutions of the kinetic equation. It turns out that in the
run-out zone the space gradients are small and the diffusion asymptote \( \bar{3} \) with \( \theta \) given by \( \bar{3} \) is reached.

In the centre of the sample \( I_n \) does not depend on \( n \), because the solution is azimuthally symmetric. Taking into account that \( l_1 \) is an odd function of \( n \) and using the constraint \( \bar{1} \), we can define the value of \( I_n \) at \( r = 0 \) with a single parameter \( \theta_0 \):

\[
l_1(0) = 0, \quad l_2(0) = i \sin \theta_0, \quad l_3(0) = \cosh \theta_0. \tag{44}
\]

In the next subsection we find the exact solution of the kinetic equation for \( \theta_0 = \ln(1/\gamma) \). Although it does not match the asymptote \( \bar{3} \), \( \bar{1} \) at large \( r \), it plays an important role. Any solution of the kinetic equation that starts at \( \theta_0 < \ln(1/\gamma) \) eventually approaches the diffusion asymptote, while the one that starts at \( \theta_0 > \ln(1/\gamma) \) does not. Therefore this exact solution is a separatrix.

Now we are ready to describe the shape of the solution that obeys the boundary conditions \( \bar{1} \) and reaches the diffusion asymptote at large \( r \). If \( \gamma \ll 1 \), it starts at \( \theta_0 \), being only slightly smaller than \( \ln(1/\gamma) \), and, therefore, runs close to the separatrix up to the large radius \( r_+ > 1 \). For even larger \( r \) the deviation becomes significant and our solution crosses over to the diffusion asymptote \( \bar{3} \) with \( \theta \) given by \( \bar{3} \). The cross-over region is of the order of the mean free path \( l \) and is much smaller than both the reaction and run-out zones. We match the separatrix in the reaction zone with the diffusion asymptote in the run-out zone keeping the mean value \( l \) continuous at the point \( r_+ \) (see Fig. \( \bar{1} \)), thus finding both the position of the cross-over \( r_+ \) and the coefficient \( C \) in Eq. \( \bar{1} \).

B. Solutions

An azimuthally symmetric solution of the kinetic equation depends only on the radius \( r \) and the angle \( \phi \) between the radius and the vector \( n \). Therefore, the space derivative in the kinetic equation has the form

\[
n \nabla = \cos \phi \frac{\partial}{\partial r} - \sin \phi \frac{\partial}{\partial \phi} = \left( \frac{\partial}{\partial s} \right)_\rho, \tag{45}
\]

where the impact parameter \( \rho = r \sin \phi \) and the distance along a straight line trajectory \( s = r \cos \phi \) are introduced (see Fig. \( \bar{1} \)).

If the parameterisation \( \bar{1} \) for the matrix \( g_n \) is used together with the conic basis for the vector \( I_n \) (see appendix \( \bar{1} \)):

\[
l_1 = k_1, \quad l_3 - il_2 = k_+ + \gamma, \quad l_3 + il_2 = k_- + \gamma, \tag{46}
\]

the kinetic equation is simplified to the form (in this section we measure all distances in the units of the mean free path \( l \)):

\[
\begin{align*}
\frac{\partial}{\partial s} k_+ &= -\langle k_+ \rangle k_1, \tag{47a} \\
\frac{\partial}{\partial s} k_- &= \langle k_- \rangle k_1, \tag{47b} \\
k_+^2 + (k_+ + \gamma)(k_- + \gamma) &= 1. \tag{47c}
\end{align*}
\]

The condition at the origin \( \bar{1} \) has the form:

\[
k_1(0) = 0, \quad k_+(0) = e^{\theta_0} - \gamma, \quad k_-(0) = e^{-\theta_0} - \gamma. \tag{48}
\]

The separatrix solution starts at \( \theta_0 = \ln(1/\gamma) \) and therefore, \( k_-(0) = 0 \). Equation \( \bar{1} \) for \( k_+ \) now gives \( k_+ = 0 \) for all \( r \). Taking into account the strong inequality \( \gamma \ll 1 \), we simplify the system \( \bar{1} \):

\[
\begin{align*}
\frac{\partial k_+}{\partial s} &= -(k_+) k_1, \tag{49a} \\
k_+^2 + \gamma k_+ &= 1 \tag{49b}
\end{align*}
\]

and obtain a closed integro-differential equation for \( k_+ \):

\[
\frac{\partial k_+}{\partial s} = -(k_+) \sqrt{1 - \gamma k_+} \tag{50}
\]

The solution of this equation is given in appendix \( \bar{1} \) at distances \( r \geq 1 \) it has the form:

\[
\begin{align*}
\gamma k_+ &= 1 - \frac{a^2}{4} \ln^2 \cot \frac{\phi}{2} + \frac{ab}{\gamma r} \frac{(\frac{\phi}{2} - \langle |\phi| \rangle \ln \cot \frac{\phi}{2})}{\sin \phi} \tag{51a} \\
a &= 2 \left( \ln^2 \cot \frac{\phi}{2} \right)^{-1/2} \left\{ \frac{1}{\sqrt{3}} \right\} \tag{51b} \\
b &= 4 \left( \ln^2 \frac{\phi}{2} \right)^{-1} \left\{ \frac{4 \ln^2 r}{\sin \phi} \right\} \approx .95 \tag{51c}
\end{align*}
\]

The divergence of \( k_+ \) at \( \phi = 0 \) is cutoff at the the angles \( \phi \sim 1/r^2 \). Since all integrals with \( k_+ \) converge, the exact form of this cut-off is relevant only for the criterium of validity of the ballistic treatment (see section VII). The first term in \( \bar{1} \) has zero average and does not depend on \( r \), while the mean value \( \langle k_+ \rangle = a/(\gamma r) \) decays when \( r \to \infty \). Using Eq. \( \bar{1} \), we find the expressions for the component \( k_1 \) and the current \( J = \langle \cos \phi k_1 \rangle \):

\[
\begin{align*}
k_1 &= -\frac{a}{2} \ln |\cot \frac{\phi}{2}| \tag{52a} \\
J &= \frac{a}{2} \left\langle \cos \phi \ln |\cot \frac{\phi}{2}| \right\rangle = \frac{1}{\sqrt{3}} \tag{52b}
\end{align*}
\]

Note, that the the current \( J \) don’t depend on the coordinates.

We are looking at a solution of Eq. \( \bar{1} \) that transfers to the diffusion regime at a certain radius \( r_+ \gg 1 \). The transition region has a width of the order of unity, and, therefore, both the mean value \( \langle k_+ \rangle \) and the current \( J \) change negligibly across the cross-over region. Combining Eqs. \( \bar{1} \), \( \bar{1} \) and \( \bar{1} \) we obtain in the diffusion region:

\[
\langle k_+ \rangle = \frac{R}{r}^C, \quad J = \frac{Cl}{2r^2} \tag{53a} \\
\langle k_+ \rangle = \exp \frac{C}{r}, \quad J = \frac{Cl}{r^2} \tag{53b}
\]
Using continuity of \( J \) and \( \langle k_+ \rangle \) at the point \( r_\ast \) we find the values of the parameters \( C, r_\ast \):

\[
\begin{align*}
    r_\ast &= \frac{2\pi}{3} \ln \frac{1}{\gamma}, \quad C = \frac{\ln \frac{1}{\gamma}}{3\sqrt{3}} \quad (2D), \\
    r_\ast &= \frac{2\pi}{3} \ln \frac{1}{\gamma}, \quad C = \frac{\ln \frac{1}{\gamma}}{3\sqrt{3}} \quad (3D)
\end{align*}
\]

The sketch of the solution is given on the Fig. 2.

C. Long-time Asymptote of Conductance

We begin with the self-consistency condition (35). In the conic coordinates (40) it looks as follows:

\[
\frac{\Delta t}{\pi R} = \int dr \frac{\langle k_+ - k_- \rangle + 2\gamma}{V} \approx \int dr \frac{\langle k_+ \rangle}{\sqrt{2}},
\]

where it is taken into account that for solution (51) the inequality \( k_- \ll k_+ \) holds. The last integral can be calculated using the continuity condition (41), where \( -i\Delta_2 = (k_+ - k_-)/2 \) can be replaced by \( k_+/2 \). Integrating Eq. (41), we express the integral (53) through the total current \( J \) through the outer boundary of the sample:

\[
\int_{r=R} J(r) dS = \frac{\gamma}{21} \int dr \langle k_+ \rangle.
\]

Although the integral (53) contains comparable contributions from both the reaction and run-out zones, with the help of Eq. (66) it can be expressed through the current in the run-out zone only. Using the asymptote (53) we finally arrive at the expression for \( \gamma \):

\[
\begin{align*}
    \gamma &= \frac{\pi}{2} p_F l^2 \ln \frac{t}{p_F l^2 \tau}, & (2D), \\
    \gamma &= \frac{2\pi}{\sqrt{3}} (p_F l)^2 \ln \frac{t}{(p_F l)^2 \tau}, & (3D)
\end{align*}
\]

The long time asymptote of the conductance \( G(t) \) is determined by the value of the action \( F \) from Eq. (51) for solution (51) - (52). This action is a sum of two contributions: from the run-out zone \( (F_b) \) and from the reaction zone \( (F_e) \). The action is dominated by \( F_b \):

\[
F_b = \frac{\pi \nu D}{2} \int_{r_*}^R dr \left( \frac{d\theta}{dr} \right)^2
\]

\[
= \frac{\pi^2 \nu DC^2}{2} \left\{ \ln \frac{R/r_*}{2/r_*} \right\} \approx \frac{\ln(R/r_*)}{2/r_*} \quad (2D)
\]

\[
= \frac{\pi^2 \nu DC^2}{2} \ln \frac{R/l}{2/l} \quad (3D)
\]

Substituting the values of the parameters (54) we obtain:

\[
F_b = \frac{\pi g \ln^2 t/(\gamma g)}{2 \ln R/l} \quad (2D)
\]

\[
F_b = \frac{\pi}{9\sqrt{3}} (p_F l)^2 \ln^3 \frac{t}{\tau g} \quad (3D)
\]

where \( g = 2\pi \hbar vD \) is the dimensionless conductance. The contribution from the reaction zone \( F_r \) is calculated, using ballistic action (15c), in appendix (3):

\[
F_b \sim g \ln(t/g) \quad (2D)
\]

\[
F_b \sim (p_F l)^2 \ln^2 \frac{t}{(p_F l)^2 \tau} \quad (3D)
\]

Comparing Eqs. (59), (60) and (61) we see that the action is dominated by the contribution from the diffusion zone. In two-dimensions it comes from the whole diffusion zone \( r_* \ll r \ll R \) because the integral in Eq. (58) diverges logarithmically. In three-dimensions only the region near \( r = r_* \) of width of the order of \( r_* \) is important. Since \( r_* \ll l \), the contribution from the crossover region, which has the width of the order of \( l \), can be neglected. The long time conductance asymptote is given by

\[
G(t) = \exp \left( -F_D \right)
\]

with the action \( F_D \) defined Eqs. (54) and (60).

V. LONG-TIME ASYMPTOTE FOR CONDUCTANCE IN DISORDERED WIRE

In this section we consider a 1D thick wire of length \( L \) and cross-section \( w \) (\( w \ll l \ll L \)) with specular boundary conditions and assume that the distribution function \( g_n \) is uniform across the wire. For not very long times the diffusion equation (37) is valid and has the solution

\[
\theta = \theta_0 - \frac{x}{\xi}, \quad \xi = L/\log t\Delta.
\]

The space gradient \( \nabla \theta = 1/\xi \) does not depend on the coordinate \( x \) and is smaller than \( 1/l \) as long as the time is shorter than \( \hbar \Delta^{-1} \exp(L/l) \). For longer times the diffusion regime breaks down simultaneously in the whole wire. The separation into reaction and run-out zone is still valid, and the term in \( \omega \) in the kinetic equation is important in the reaction zone near the centre of the wire and can be neglected elsewhere. Unlike the 2D and 3D cases however, the space gradients in the run-out zone do not decay towards the outer ends of the wire. We now present a solution of the kinetic equation in the run-out zone which is valid for arbitrary gradients.

The distribution function \( g_n \) depends on the coordinate \( x \) along the wire and the angle \( \phi \) between the axis of the wire and the direction \( n \) of the momentum (see Fig. 6). In these coordinates the kinetic equation (8) takes the form:

\[
2l \cos \phi \frac{\partial g}{\partial x} = \left[ (\gamma \Lambda - \langle g \rangle), g \right] \quad (64)
\]

In the run-out zone the term in \( \gamma \) is negligible and, using the conic coordinates, from appendix (3) as \( \gamma \to 0 \), we rewrite Eq. (64):
Finally, the constants \( \theta \) where the function \( \theta \) produces to
\[
\cos \phi \frac{\partial}{\partial x} k_\pm = \mp (k_\pm) k_1, \quad \text{(65a)}
\]
\[
k_1^2 + k_+ k_- = 1. \quad \text{(65b)}
\]
Dividing both sides of Eq. (65a) by \( \cos \phi \) and averaging over \( \phi \), we find a closed equation for \( \langle k_\pm \rangle \) with the solutions:
\[
\langle k_\pm \rangle = q \exp(\mp \theta), \quad \text{(66)}
\]
where \( q \) is a constant and \( \theta \) obeys the equation:
\[
\frac{d \theta}{dx} = \left( \frac{k_1}{\cos \phi} \right). \quad \text{(67)}
\]
In one-dimension the current conservation law (42) reduces to \( J \equiv \langle k_1 \cos \phi \rangle = \text{const} \) and suggests that \( k_1 \) does not depend on \( x \). Therefore, Eq. (51) gives
\[
\frac{d \theta}{dx} = \text{const} = \theta' \quad \text{(68)}
\]
and, substituting this back into Eq. (55), we obtain the solution in the factorised form:
\[
k_\pm = \beta(\phi) \langle k_\pm \rangle, \quad k_1 = \theta' \beta(\phi) \cos \phi, \quad \text{(69)}
\]
where the function \( \beta(\phi) \) is determined from Eq. (55)
\[
\beta = [q^2 + (\theta')^2 \cos^2 \phi]^{-1/2}. \quad \text{(70)}
\]
Finally, the constants \( \theta' \) and \( q \) are related by the condition
\[
1 = \langle \beta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{\sqrt{q^2 + (\theta')^2 \cos^2 \phi}} \quad \text{(71)}
\]
Using the asymptotic values of the elliptic integral in Eq. (71), we find
\[
q = \begin{cases} 
1 - \frac{\theta'^2}{4}, & \theta' \ll 1, \\
4\theta' \exp\left( - \frac{\pi \theta'}{2} \right), & \theta' \gg 1
\end{cases} \quad \text{(72)}
\]
and obtain expressions for the current \( J = \langle k_1 \cos \phi \rangle \):
\[
J = \frac{q'}{4} \left\{ 1 - 4\pi|\theta'| \exp\left( - \pi|\theta'| \right) \right\} \quad \theta' \ll 1, \quad \theta' \gg 1 \quad \text{(73)}
\]

The above solution is not valid in the reaction zone in the vicinity of \( x = 0 \), whose contribution to both the action and self-consistency condition is negligible. The boundary condition for the ballistic problem is determined by the requirement that the distribution function matches the one in the bulk electrodes where \( g_n = \Lambda \).

For small gradients (\( \theta' \ll 1 \)) the angular dependence of the distribution function in the wire is weak and the above condition is fulfilled provided \( \theta(\pm L/2) = 0 \). On the other hand, in the ballistic regime (\( \theta' \gg 1 \)) the distribution function in the wire strongly depends on the angle \( \phi \) (see Eq. (70)). Thus, there is a cross-over region near the outer ends of the wire where the solution of the kinetic equation deviates from Eqs. (66), (69) and (70).

We consider the wires which are long enough to neglect the change of \( \theta \) in the cross-over region. Therefore, the boundary condition \( \theta(\pm L/2) = 0 \) is still valid in the ballistic regime, and the function \( \theta(x) \) can be presented in the form (62) with \( \xi = l/\theta' \) and \( \theta_0 = \theta' l/(2l) \).

### A. Action and self-consistency condition

The self-consistency condition (52) gives
\[
\frac{t \Delta L}{\pi \hbar l} = \frac{1}{|\theta'| l} \exp\left( - \frac{|\theta'| L}{2l} \right) \quad \text{(74)}
\]
and should be compared with the continuity equation (73) which in the 1D case reads:
\[
J = \frac{\gamma}{2|\theta'| l} \exp\left( - \frac{|\theta'| L}{2l} \right). \quad \text{(75)}
\]

Using Eq. (73) for the current and Eqs. (74), and (75) we express \( \gamma \) and \( \theta' \) through \( t \):
\[
|\theta'| = \frac{2l}{L} \left( \ln \frac{t \Delta L}{\pi \hbar l} + \ln \frac{\Delta L}{\pi \hbar l} \right), \quad \text{(76a)}
\]
\[
\gamma = \frac{2\pi \hbar l}{m \Delta L} J = \frac{\gamma'}{l} \left\{ 2g \ln \frac{\Delta L}{\hbar l} \quad \frac{\Delta L}{\hbar l} \ll \exp\left( \frac{\gamma'}{l} \right) \right\}, \quad \text{(76b)}
\]
where \( g = 2\pi \hbar v D w / L \) is dimensionless conductance and \( N = w \nu F / \hbar \) is the number of transverse channels, which is equal to ballistic conductance. The calculation of the W-term in action (15b) for the solution (68) is performed in appendix D and gives:
\[
W \{ g_n \} = -8Lw \theta' \frac{J}{l} \quad \text{(77)}
\]

Evaluating the other terms in the action we get:
\[
F = \frac{\pi}{2\Delta r} \left( 2l J' + q^2 - 1 \right). \quad \text{(78)}
\]

Thus, the long time asymptote of the conductance is given by:
\[
G(t) \sim \exp\left\{ -g \ln\left( \frac{1}{2\pi \hbar k} \right) \right\} \quad 1 \ll \frac{\Delta L}{2\pi \hbar k} \ll e^{L/l}, \quad \text{(79)}
\]
\[
G(t) \sim \left( \frac{1}{2\pi \hbar k} \right)^{2N} e^{L/l} \quad \frac{1}{2\pi \hbar k} \ll N e^{L/l}
\]

The last inequality in Eq. (79) ensures the validity of semiclassical approximation (see section VII).
VI. PHYSICAL PICTURE OF TRAPPING

In this section we consider the time dispersion of the conductance $G(t)$ in a purely one-dimensional wire. This problem was solved by Altshuler and Prigodin \[12\] who obtained the long time asymptote in the form

$$G(t) \sim \exp\left(-\frac{t}{L} \ln^2 t \Delta\right),$$  \hspace{1cm} (80)

which can be treated as a limiting case of a multi-channel formula for a thick wire (see table \[3\]), assuming that in a one-channel case $g$ is given by $l/L$. Formula (80) can be understood as a probability of an optimal potential fluctuation that traps an electron of Fermi energy $E_f$ for time $t$. In a weak potential $U(x) \ll E_f$ the wave function can be presented in the form

$$\Psi(x) = \phi_+(x)e^{ip_F x} + \phi_-(x)e^{-ip_F x}$$  \hspace{1cm} (81)

with the amplitudes $\phi_{\pm}(x)$ changing slowly: $\nabla \phi_{\pm} \ll p_F \phi$. Let us consider a quasi-stationary state obeying the open boundary conditions

$$\phi_+(0) = \phi_-(L) = 0,$$  \hspace{1cm} (82)

which correspond to the outward flow of current through the ends of the wire. The life time of such a state is inversely proportional to the outward current

$$t = \frac{\int dx |\Psi|^2}{v_F (|\phi_-(0)|^2 + |\phi_+(L)|^2)}.$$  \hspace{1cm} (83)

The maximum delay time is achieved when the currents through both ends are equal ($\phi_-(L/2) = \phi_+(L/2)$). Fixing normalisation by

$$|\phi_-(0)| = |\phi_+(L)| = 1$$  \hspace{1cm} (84)

we reduce Eq. (83) to the from

$$\frac{t \Delta}{\pi \hbar} = \frac{\int dx |\Psi|^2}{L}$$  \hspace{1cm} (85)

which resembles the self-consistency condition \[5\] obtained for arbitrary dimensions.

To achieve life times $t \gg \hbar/\Delta$ the wave function must grow towards the middle of the wire; assuming the growth is exponential, $\Psi \sim \exp[(L/2 - |x|)/\xi]$, we obtain for the localisation length of the quasi-stationary state

$$\xi = \frac{L}{\ln t \Delta}.$$  \hspace{1cm} (86)

A typical random potential $\bar{U}$ causes one-dimensional wave functions to be localised with $\xi \sim l$. The shorter localisation length $\xi \ll l$ corresponds to life times longer than $\hbar \Delta^{-1} \exp(L/l)$ and can be achieved in the potential

$$U(x) = \bar{U} + U_0 \cos(2p_F x).$$  \hspace{1cm} (87)

with the additional $2p_F$-Fourier component having the amplitude

$$U_0 = \frac{2\hbar v_F}{\xi}.$$  \hspace{1cm} (88)

The probability of this potential realization in given by the Gaussian distribution:

$$\exp\left(-\frac{\pi v_F}{2} \int U(x)^2 dx\right) \sim \exp\left(-\frac{t}{L} \ln^2 t \Delta\right).$$  \hspace{1cm} (89)

and coincides with (80) with a correct numerical factor in the exponent. We therefore conclude that the states with long life times are locked by the Bragg reflection and can be found with the probability proportional to that of potential fluctuation with the Bragg mirror of appropriate strength.

We believe that the same mechanism is responsible for nearly localised states in multi-channel wires and the samples of higher dimensional. In multi-channal cases, however, adding a single $2p_F$-Fourier harmonics cannot localise the wave function, because the random part $\bar{U}$ mixes different directions of the momentum. To localise the wave function in a two- or three-dimensional sample, the potential fluctuation should be effective for all directions of the momentum; so we expect it to have the form

$$U(x) = \int d\Omega_n U_n(r) \cos(2p_F nr)$$  \hspace{1cm} (90)

with the amplitude $U_n(r)$ slowly depending on $n$ and $r$.

VII. DISCUSSION

This paper continues our study of the conductance $G(t)$ asymptote at long times $t$ that began in Ref. \[3\]. We use the steepest descent approach which enables us to obtain $g(t)$ for different ranges of time. The purpose of this section is dual: we want to analyse the restrictions of our treatment and compare our results with those in the literature.

Let us first discuss the conductance $G(t)$ of a thick wire made from a two-dimensional strip of length $L \gg l$ and width $w < l \ll L$. The solution $g(x, \phi)$ of the kinetic equation has typical gradients (see Eq. (71)):

$$\frac{\partial g}{\partial x} \sim \frac{\theta'}{l} \sim \frac{\ln(t \Delta/\hbar)}{L}$$  \hspace{1cm} (91)

At long times the distribution function changes rapidly with the angle $\phi$ acquiring a sharp maximum at $\phi = \pi/2$ with the width $\delta \phi \sim \exp(-\pi |\theta'|/4)$ (see Eq. (70)-(73)). The semi-classical approximation remains valid if this width is much larger than the diffraction angle $\delta \phi \sim p_F w$. This imposes an upper limit on the gradients:

$$\theta' \ll \ln(p_F w/\hbar)$$  \hspace{1cm} (92)
Thus, the ballistic asymptote is valid only for times smaller than
\[ t_Q \sim \Delta^{-1} p_F e^{t/\lambda} \]  
(93)
Since the width of the wire is limited by the condition \[ w < t \] the gradient \( \theta' \) at times \( t \sim t_Q \) is still much smaller than \( p_F \). Thus, an intermediate interval of gradients arises
\[ \ln(l p_F) < \theta' < l p_F \]  
(94)
which corresponds to the interval of delay times:
\[ \frac{L}{l} \ln \frac{l}{\lambda} < \ln(t \Delta) < \frac{L}{\lambda} \]  
(95)
We assume that the time dispersion of the conductance in this region can be recovered by some kind of a ballistic treatment with diffraction proper accounted for.

A similar phenomenon restricts the range of applicability of ballistic asymptotes in two- and three-dimensional samples. Analogously to the one-dimensional case, the distribution function \( g(\phi, r) \) in the reaction zone has a sharp feature at \( \phi = 0 \) with the width \( \delta \phi = l/r \). Diffraction can be neglected when \( \delta \phi \gg \lambda/l \). Therefore, the reaction zone radius \( r_\ast \) must obey the constraint:
\[ r_\ast \ll l^2/\lambda. \]  
(96)
Substituting the radius of reaction zone from Eq. (54), we obtain the upper time limit \( t \) for which the semi-classical approach is still valid:
\[ t \ll t_Q \sim \hbar \Delta^{-1} \exp \left( \frac{p_F t}{\hbar} \right) \left\{ \begin{array}{ll} l/R & (2D) \\ (l/R)^3 & (3D) \end{array} \right. \]  
(97)
Similarly to the one-dimensional case the gradients of \( g \) at the time \( t_Q \) are still much smaller than \( p_F \), being only of the order of \( 1/l \). The limiting times \( t_Q \) for a different geometries are summarised in Table I.

As it has been mentioned in the introduction, the whole field was pioneered by AKL. They added high powers of frequency and gradients of \( Q \) to the diffusive \( \sigma \)-model and analysed the renormalization flow of the corresponding coupling constants in two dimensions. This gave the growth rate of the coefficients \( C_n \) in the expansion
\[ G(\omega) = \sum_{n=0}^{\infty} C_n \omega^n. \]  
(98)
The Fourier transform of Eq. (18) with the AKL asymptote for \( C_n \) leads to the result presented in Table I. AKL also put forward a general conjecture that the logarithmically normal asymptote was valid for all dimensions.

As one can see from Table I, there is a variety of different regimes in the time dispersion of the conductance. The AKL procedure predicts one of them and fails to describe the others. It is also difficult, using this procedure, to find out the criteria when this or that results are valid.

Another attack on the problem has been recently carried out by Mirlin [4]. He used our optimal fluctuation method for the diffusive \( \sigma \)-model in dimensions two and three and supplied Eq. (3) by somewhat arbitrary the conditions near the origin. In this way he obtained the asymptotes of conductance for 2D and 3D samples. The reason for his success lies in the fact established in this paper that the action for the dimensions \( d \geq 2 \) is dominated by the contribution from the run-out zone. The solution of the ballistic problem confirms the result of Ref. [3] for two-dimensional conductance. In the three-dimensional case we obtain the numerical coefficient \( A \) in the expression \( G(t) \sim \exp(-A \ln^3 t) \). Although the form of the density-matrix fluctuation suggested by Mirlin deviates strongly in the reaction zone from the correct one, the method of his work could be useful for qualitative estimates.

It should be noted, however, that the ballistic treatment is needed to find the upper limit \( t_Q \) on the delay time interval where these results are valid. We regard the occurrence of the the new range of times \( t \geq t_Q \) where the diffraction effects are important we regard as one of the most interesting results of this paper.

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APPENDIX A: CONIC COORDINATES

Throughout this paper we use a convenient parametrisation of the bosonic sector of the distribution function \( g_n \). To obtain it, we introduce a basis in the space of traceless \( 2 \times 2 \) real matrices:
\[ \hat{e}_1 = \sigma_1, \quad \hat{e}_\pm = \frac{\sigma_3 \pm i \sigma_2}{2}. \]  
(A1)
The new unit vectors have the following properties:
\[ \hat{e}_1^2 = 1, \quad \hat{e}_\pm^2 = 0, \quad [\hat{e}_\pm, \hat{e}_1] = \pm 2 \hat{e}_\pm, \]  
(A2a)
\[ [\hat{e}_+, \hat{e}_-] = -\hat{e}_1, \quad [\hat{e}_+, \hat{e}_-] = 1. \]  
(A2b)
Now, instead of parametrisation \( \{12\} \) for the \( g_n^B \), we use:
\[ g_n^B = \frac{1 + \tau_3}{2} \otimes (l_1(n)\hat{e}_1 + l_+(n)\hat{e}_+ + l_-(n)\hat{e}_-) + \frac{1 - \tau_3}{2} \otimes (l_1(-n)\hat{e}_1 + l_+(n)\hat{e}_+ + l_-(n)\hat{e}_-), \]  
(A3)
where, due to requirement \( \{3\} \), all components \( l_1 \) and \( l_\pm \) are real. The kinetic equation can be further simplified by introducing the functions
k_1 = l_1; \quad k_+ = l_+ - \gamma; \quad k_- = l_- - \gamma \quad (A4)

The constraint \( g^2 = 1 \) now can be written as:

\[
k_1^2 + (k_+ + \gamma)(k_- + \gamma) = 1, \quad (A5)
\]

and the kinetic equation (5) in coordinates (15) has the form:

\[
\begin{align*}
\frac{\partial}{\partial s} k_+ &= -(k_+)k_1, \\
\frac{\partial}{\partial s} k_- &= (k_-)k_1, \\
-\lambda \frac{\partial}{\partial s} k_1 &= -k_+(k_-) + (k_+)k_- + \gamma(k_+ - k_-). \quad (A6c)
\end{align*}
\]

It is more convenient to use the constraint (A5) instead of the last equation in this set (see Eq. (47) in the text).

APPENDIX B: INTEGRO-DIFFERENTIAL EQUATION FOR SEPARATRIX

Solving Eq. (5) with respect to \( k_+ \) and considering its average value \( \langle k_+ \rangle \) as a function of radius \( r \), we obtain:

\[
1 - \gamma k_+ = \frac{\gamma^2}{4} \left( \int_0^s \langle k_+ \rangle ds' \right)^2. \quad (B1)
\]

By taking the average of both the right- and left-hand sides, this expression is reduced to an integral equation for \( \lambda(r) = \gamma \langle k_+ \rangle \):

\[
1 - \lambda(r) = \frac{1}{4} \left\{ \left( \int_0^r \cos \phi \, d\lambda \left( \sqrt{r^2 \sin^2 \phi + s^2} \right) \right)^2 \right\}_\phi, \quad (B2)
\]

where

\[
\begin{align*}
\langle \phi \rangle &= \int_0^{2\pi} \frac{d\phi}{2\pi}, \\
\langle \phi \rangle &= \int_0^{\sin \phi} \frac{d\phi}{\sin \phi}. \quad (2D) \quad (3D)
\end{align*}
\]

The solution of Eqs. (B2) has the following asymptotes:

\[
\lambda(0) = 1; \quad \lambda(r) = \frac{a}{r} + \frac{b}{r^2}, \quad r \gg 1, \quad (B3)
\]

where the constants \( a \) and \( b \) are given by Eqs. (51) in the text.

If \( r \sin \phi \gg 1 \), then the asymptote of (B3) can be substituted into kernel of Eq. (B1), and give the result

\[
\gamma k_+ = 1 - \frac{\gamma^2}{4} \ln^2 \left( \frac{\phi}{2} \right) + \frac{ab}{2r} \ln \left( \frac{\phi}{2} + \frac{1}{2} \right) \cot \left( \frac{\phi}{2} \right) \frac{1}{|\sin \phi|} \quad (B4)
\]

Expression (B4) is valid only if \( r |\sin \phi| \gg 1 \) and is free of singularities in this region.

APPENDIX C: CALCULATION OF BALLISTIC ACTION IN THE REACTION ZONE

The contribution from the reaction zone to the action is given by Eq. (51) and contains two terms:

\[
F_1 = -\frac{\pi \nu \nu}{8\tau} W, \quad F_2 = -\frac{\pi \nu}{8\tau} \int \langle g \rangle^2; \quad (C1)
\]

the term with \( \omega \) dissapears after integration over \( \omega \) in Eq. (B4).

To calculate the \( W \) term we use Eq. (14) and present the solution of kinetic equation in the form \( g_n = U^2 \). Using parametrisation (32) we find for the bosonic block of \( g \)-matrix:

\[
g^B = \begin{pmatrix} U(\phi) & 0 \\
0 & -U(\pi - \phi) \end{pmatrix}, \quad (C2)
\]

where \( 2 \times 2 \) matrix \( U \) is given by:

\[
U = \exp \left( \frac{-k_1}{\gamma} \frac{1}{\tilde{e}_+} \right) \exp \left( \frac{\ln \gamma}{2} \sigma_1 \right). \quad (C3)
\]

Using

\[
\frac{\partial}{\partial r} U = \frac{1}{\gamma} \frac{\partial k_1}{\partial r} \tilde{e}_+ U
\]

we obtain from Eq. (14):

\[
W = 8 \int dr \langle n \frac{\partial k_1}{\partial r} \rangle = 8 \int dSJ, \quad (C4)
\]

where the integral in the last expression is taken over the boundary of the reaction zone. This gives

\[
F_1 \sim \begin{pmatrix} \frac{g \ln(t/\tau)}{\ln(R/l)} \\
(pF)^2 \ln^2 \left( \frac{t}{(pF)^2} \right) \end{pmatrix} \quad (2D) \quad (3D) \quad (C5)
\]

which is one power of the logarithm smaller than the contribution from the run-out zone.

The calculation of the term \( F_2 \) is straight-forward:

\[
F_2 = \frac{\pi \nu}{8\tau} \int dr \frac{\nu^2}{r^2} \sim \begin{pmatrix} g \ln \left( \frac{\ln(t/\tau)}{\ln(R/l)} \right) \\
(pF)^2 \ln \left( \frac{t}{(pF)^2} \right) \end{pmatrix} \quad (2D) \quad (3D) \quad (C6)
\]
APPENDIX D: CALCULATION OF $W$ TERM FOR ONE-DIMENSIONAL SOLUTION.

To employ the expression \[14\] for the $W$-term we should find a decomposition $g_n = UAU^{-1}$. Combining Eqs. \[22\] and \[63\] with the parametrisation \[12\] and formulae from appendix \[8\], we write the bosonic block of $g_n$ in the form of \[24\], where the $2 \times 2$ matrix $U$ is given by:

$$U = \exp\left(\frac{\theta}{2} \sigma_x\right) \exp\left(-i \frac{\chi}{2} \sigma_y\right),$$ \hspace{1cm} (D1)

where $\chi = \arcsin (\theta' \beta(\phi) \cos \phi)$ does not depend on $x$. The straight-forward application of Eq. \[14\] gives:

$$W = -8Lw \frac{\theta|J|}{l} \hspace{1cm} (D2)$$

where the minus sign comes from the super-trace of the bosonic block. The result of averaging over $n$ is expressed through the current $J = \langle \cos \phi k_1 \rangle$.

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[9] This equation has the same form as the Eilenberger equation, first introduced in the theory of superconductivity (see G. Eilenberger Z. Phys. 214, 195, (1968); K.D.Ussadell Phys.Rev.Lett. 25, 507 (1970), A.I.Larkin, Yu.N.Ovchinnikov JETP 46, 155 (1977), Schmid A., in Non-equilibrium Superconductivity, Phonons and Kapitza Boundaries, edited by K.E.Grey, p.423 (1981) Plenum Press, NY.). The difference is that our distribution function $g_n(t)$ is a super-matrix with both commuting and anti-commuting elements.
[10] The value of the $W$-term does not depend on the interpolating function; for detailed discussion see Fradkin E. Field Theory for Condensed Matter Systems, Addison-Wesley 1991;[11] The symmetry of the solutions of the kinetic equation \[1\] differs from the symmetry of $g$-matrices in the functional integral \[15\], i.e. the integration contour should be deformed to reach the saddle point. Analogous situation occurs in the diffusive $\sigma$-model. In the text we describe the symmetries of the solutions of the kinetic equation.

APPENDIX D: CALCULATION OF $W$ TERM FOR ONE-DIMENSIONAL SOLUTION.

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where the minus sign comes from the super-trace of the bosonic block. The result of averaging over $n$ is expressed through the current $J = \langle \cos \phi k_1 \rangle$.

| $t_D \ll t \ll \hbar/\Delta$ | $h/\Delta \ll t \ll t_b$ | $t_b \Delta$ | $t_b \ll t \ll t_Q$ | $t_Q \Delta$ |
|-----------------|-----------------|-------------|---------------|---------------|
| 1D<sup>a</sup>  | $t/t_D$         | $g \ln^2(\Delta t)$ | $\exp(L/l)$ | $2N \frac{\ln t \Delta}{\pi g} \frac{\ln^2(t/\tau)}{\ln^2(R/l)}$ | $N \exp(L/l)$ |
| 2D<sup>b</sup>  | $\pi t/t_D$     | $4g \ln(\Delta t)$ | $(L/l)^2$     | $\frac{\pi}{2\sqrt{3}} (pF/l)^2 \ln^3 \left(\frac{t}{\tau(pF/l)}\right)$ | $\exp(pF/l/h) \frac{\hbar}{R}$ |
| 3D<sup>c</sup>  | $\pi t/t_D$     | none         | 1            | $1$           | $\exp(pF/l/h) \left(\frac{t}{R}\right)^3$ |

TABLE I. Long time asymptotes of the conductance $G(t)$. The cells contain formulae for $-\ln G$ at different time intervals for one- two- and three- dimensional samples. The diffusion time $t_D$ is defined as $L^2/D$ (1d) and $R^2/D$ (2d, 3d); the diffusion coefficient $D$ is given by $\frac{1}{2} v_F^2 \tau$ (1d, 2d) and $\frac{1}{4} v_F^2 \tau$ (3d); $\tau$ is the mean free time and $\Delta = (\nu V)^{-1}$ is the mean level spacing.

<sup>a</sup>strip of width $w$ and length $L$, $N = w p_F/(\pi \hbar)$ is the number of ballistic channels

<sup>b</sup>disk of radius $R$

<sup>c</sup>ball of radius $R$
FIG. 1. The geometry of the 1d sample (a) and 2d and 3d samples (b)

FIG. 2. Sketch of kinetic equation solutions in two- and three-dimensional samples. (dotted line is the separatrix solution)
FIG. 1. The geometry of the 1d sample (a) and 2d and 3d samples (b).

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