Noncommutative Geometry and Gauge Theory on Discrete Groups

(revised version)

Andrzej Sitarz

Department of Field Theory
Institute of Physics, Jagiellonian University
Reymonta 4, PL-30-059 Kraków
Poland

Abstract:
We build and investigate a pure gauge theory on arbitrary discrete groups. A systematic approach to the construction of the differential calculus is presented. We discuss the metric properties of the models and introduce the action functionals for unitary gauge theories. A detailed analysis of two simple models based on $\mathbb{Z}_2$ and $\mathbb{Z}_3$ follows. Finally we study the method of combining the discrete and continuous geometry.

1e-mail: ufsitarz@plkrcy11.bitnet
1 INTRODUCTION AND NOTATION

The noncommutative geometry provides us with a far more general framework for physical theories than the usual approaches. Its basic idea is to substitute an abstract, associative and not necessarily commutative algebra for the algebra of functions on a smooth manifold $^1$–$^5$. This allows us to use nontrivial algebras as a geometrical setup for the field-theoretical purposes, in particular for the gauge theories, which are of special interest both from mathematical and physical points of view.

The construction of noncommutative gauge theories has led to a remarkable result, which is the description of the Higgs field in terms of a gauge potential. This suggests some possible nontrivial geometry behind the structure of the Standard Model. Several examples of this kind, with various choices of fundamental objects of the theory, have been investigated in such context $^6$–$^{13}$. The ”discrete geometry” models, which take as the algebra the set of functions on a discrete space, seem to be one of the most promising interpretations $^4,^5$ and suggest that such a geometry may play an important role in physics. Recently, some more analysis has been carried out for two- and three-point spaces $^{15,16}$ in the context of grand unification and general relativity.

We propose to develop here a systematic approach towards the construction of a pure gauge theory on arbitrary discrete groups. The choice of a group as our base space is crucial for our analysis and allows us to make use of the correspondence with the gauge theory on Lie groups. The formalism of finite derivations and invariant forms, which we use in our approach, is equivalent to the one used in various works $^2,^4,^{12,13}$ for the two-point space, it also extends considerably our earlier studies $^{14}$.

The paper is organized as follows: in the first section we construct the tools of the differential calculus, then we outline the general formalism of gauge theories in this case and some problems of the construction of actions. The discussion of two examples follows. Finally we discuss other possibilities originating from the symmetry principles and we present the method of combining the discrete and continuous geometry.
2 DIFFERENTIAL CALCULUS

Let $G$ be a finite group and $\mathcal{A}$ be the algebra of complex valued functions on $G$. We will denote the group multiplication by $\odot$ and the size of the group by $N_G$. The right and left multiplications on $G$ induce natural automorphisms of $\mathcal{A}$, $R_g$ and $L_g$, respectively,

$$(R_g f)(g) = f(g \odot h),$$  \hspace{1cm} (1)

with a similar definition for $L_g$.

Now we will construct the extension of $\mathcal{A}$ into a graded differential algebra. We shall follow the standard procedure of introducing the differential calculus on manifolds, in particular on Lie groups. Therefore we use almost the same terminology, although the definitions of certain objects may differ.

First let us identify the vector fields over $\mathcal{A}$ with linear operators on $\mathcal{A}$, which have their kernel equal to the space of constant functions. They form a subalgebra of $GL(N_G, \mathbb{C})$, with an additional structure of a finite dimensional left module over $\mathcal{A}$. Now, we can define the vector space $\mathcal{F}$ of left invariant vector fields as satisfying the following identity:

$$\partial \in \mathcal{F} \iff \forall f \in \mathcal{A} \quad L_h \partial(f) = \partial(L_h f).$$  \hspace{1cm} (2)

Before we discuss the algebraic structure of $\mathcal{F}$ let us observe that this vector space is $N_G - 1$ dimensional and it generates the module of vector fields. This means that for a given basis of $\mathcal{F}$, $\partial_i$, $i = 1 \ldots N_G$, every vector field can be expressed as a linear combination $f_i \partial_i$, with the coefficients $f_i$ from the algebra $\mathcal{A}$.

$\mathcal{F}$ forms an algebra itself and we find the relations of generators to be of the second order,

$$\partial_i \partial_j = \sum_k C^k_{ij} \partial_k,$$  \hspace{1cm} (3)

where $C^k_{ij}$ are the structure constants. Because of the associativity of the algebra they must obey the following set of relations,

$$\sum_l C^l_{ij} C^m_{lk} = \sum_l C^m_{il} C^l_{jk}.$$  \hspace{1cm} (4)

Now we choose a specific basis of $\mathcal{F}$ and calculate the relations (3) in this particular case. It is convenient for our purposes to introduce the basis of $\mathcal{F}$
labeled by the elements of $G' = G \setminus \{e\}$, where $e$ is the neutral element of $G$. Further on, if not stated otherwise, it should be assumed that all indices take values in $G'$.

$$\partial_g f = f - R_g f, \quad g \in G', \ f \in A,$$  \hfill (5)

The structure relations \((3)\) become in the chosen basis quite simple,

$$\partial_g \partial_h = \partial_g + \partial_h - \partial_{(h \circ g),} \quad g, h \in G'.$$  \hfill (6)

As a next step let us introduce the Haar integral, which is a complex valued linear functional on $A$ that remains invariant under the action of $R_g$,

$$\int f = \frac{1}{N_G} \sum_{g \in G} f(g),$$  \hfill (7)

where we normalized it, so that $\int 1 = 1$.

Although the elements of $F$ do not satisfy the Leibniz rule, they are inverse to the integration. Indeed, we notice that for every $f \in A$ and every $v \in F$ the integral \((4)\) of $v(f)$ vanishes. For this reason we can consider them as corresponding to the derivations on the algebra $A$.

We define now the space of one-forms $\Omega^1$ as a left module over $A$, which is dual to the space of vector fields. It could be also considered as a right module with an appropriate definition of the right action of $A$. This, however, is a straightforward consequence of the differential structure and we will discuss it later.

Now we can introduce the notion of left invariant forms, which, when acting on the elements of $F$, give constant functions. Having chosen the basis of $F$ we automatically have the dual basis of $F^*$ consisting of forms $\chi^g$, $g \in G'$, which satisfy

$$\chi^g(\partial_h) = \delta_h^g.$$  \hfill (8)

To build a graded differential algebra we need to construct n-forms and their products for an arbitrary positive integer $n$. Of course, we identify zero-forms with the algebra $A$ itself and their product with the product in the algebra. The definition for higher forms is natural, we take $\Omega^n$ to be the tensor product of $n$ copies of $\Omega^1$,

$$\Omega^n = \underbrace{\Omega^1 \otimes \cdots \otimes \Omega^1}_{\text{n times}}.$$  \hfill (9)
and the product of forms to be the tensor product over $\mathcal{A}$. However, let us remember that we use here the tensor product of modules with different right and left actions of $\mathcal{A}$.

To complete the construction of the differential algebra we need to define the external derivative $d$ and this is the subject of the following lemma:

**Lemma**  There exists exactly one linear operator $d : \Omega^n \to \Omega^{n+1}$, which is nilpotent, $d^2 = 0$, satisfies the graded Leibniz rule and for every $f \in \mathcal{A}$ and every vector field $v$ $df(v) = v(f)$, provided that the right and left action of $\mathcal{A}$ on $\mathcal{F}^*$ are related as follows,

$$\chi^g f = (R_g f) \chi^g, \quad g \in G', \ f \in \mathcal{A},$$

(10)

and that the following structure relations hold,

$$d\chi^g = - \sum_{h,k} C_{hk}^g \chi^h \otimes \chi^k, \quad g \in G'.$$

(11)

Before we prove the lemma, let us observe that due to the properties of the tensor product the condition (10) could be extended to the space of all one-forms. Therefore it gives to $\Omega^1$ the structure of a right module, mentioned earlier. The next requirement (11) is equivalent to the Maurer-Cartan structure relations for Lie groups.

**Proof:** Since we want $d$ to satisfy the graded Leibniz rule, it is sufficient to define the action of $d$ on $\mathcal{A}$ and on $\mathcal{F}^*$ because all other forms can be represented as tensor products of them. The action of $d$ on $\mathcal{A}$ is defined by the requirement stated in the lemma, from which we get that

$$df = \sum_g (\partial_g f) \chi^g,$$

(12)

The Leibniz rule applied to the product of any two elements $a, b \in \mathcal{A}$, gives the following identity:

$$\sum_g (ab - R_g(a) R_g(b)) \chi^g = \sum_g (a - R_g(a)) \chi^g b + a (b - R_g(b)) \chi^g,$$

(13)

which is satisfied only if (10) holds. The Maurer-Cartan relations arise from the requirement that $d^2$ acting on an arbitrary $a \in \mathcal{A}$ must vanish. Indeed, we calculate,

$$d^2 a = d \left( \sum_h (\partial_h a) \chi^h \right) =$$

(14)
\[ \sum_{h,k} C_{hk}^g (\partial_g a) \chi^h \otimes \chi^k + \sum_h (\partial_h a) d\chi^h, \]

and this expression vanishes only if \((11)\) is true. This ends the proof.

In our construction we have obtained the differential algebra over the algebra of complex functions on a discrete group, which may be a starting point for the analysis of this structure. One may attempt, for instance, to calculate its cohomology. Let us notice that although the basic algebra was commutative, in the end we obtained a noncommutative, infinite-dimensional algebra, which may be an interesting subject of further studies in the program of non-commutative geometry. However, in this paper we shall rather proceed towards the construction of gauge theories on the basis of introduced formalism.

Let us end this section by constructing the involution on our differential algebra, which agrees with the complex conjugation on \(\mathcal{A}\) and (graded) commutes with \(d\), i.e. \(d(\omega^*) = (-1)^{\deg \omega} (d\omega)^*\). Again, it is sufficient to calculate it for one-forms,

\[ (\chi^g)^* = -\chi^{g^{-1}}. \quad (15) \]

So far, we restricted ourselves in our approach to the complex-valued functions. Similarly we can consider a straightforward extension of the model if we take functions valued in any involutive algebra, for instance, the matrix valued functions. The quotient subalgebras of the obtained algebra may also be considered, the necessary formalism and the examples will be given in the last section.

3 GAUGE THEORY

3.1 General Formalism

In this section we shall construct the gauge theory on finite groups using the differential calculus we have just introduced. First let us explain some basic ideas. The starting point is the differential algebra \(\tilde{\Omega}^*\) with its subalgebra of zero-forms \(\tilde{A} \subset \tilde{\Omega}^*\). We take the group of gauge transformations to be any proper group \(\mathcal{H} \subset \tilde{A}\), which generates \(\tilde{A}\). In particular, we will often take \(\mathcal{H}\) to be the group of unitary elements of \(\tilde{A}\),

\[ \mathcal{H} = \mathcal{U}(\tilde{A}) = \{a \in \tilde{A} : a a^* = a^* a = 1\}. \]
Of course, the external derivative $d$ is not covariant with respect to the gauge transformations. Therefore we have to introduce the covariant derivative $d + \Phi$, where $\Phi$ is a one-form. The requirement that $d + \Phi$ is gauge covariant under gauge transformations,

$$d + \Phi \rightarrow H^{-1}(d + \Phi)H, \quad H \in \mathcal{H},$$

results in the following transformation rule of $\Phi$,

$$\Phi \rightarrow H^{-1}\Phi H + H^{-1}dH.$$  \hspace{1cm} (17)

$\Phi$ is the gauge potential, which we will also call connection. If the gauge group is unitary, we require also that the covariant derivative is hermitian,

$$(d + \Phi)(a^*b) = a^*(d + \Phi)b + (b^*(d + \Phi)a)^*, \quad a, b \in \tilde{A},$$  \hspace{1cm} (18)

which results in the condition that the connection is anti-selfadjoint, $\Phi = -\Phi^*$. Finally, we have the curvature two-form, $F = d\Phi + \Phi\Phi$, which, of course, is gauge covariant.

In order to proceed with the construction and analysis of the Yang-Mills theory we have to introduce a metric. We shall briefly mention here a general theory and concentrate our efforts in the next section on the analysis of the model under study.

Let us define the metric $\eta$ as a form on the left module of one-forms, valued in the algebra $\tilde{A}$ and bilinear over the algebra $\tilde{A}$,

$$\eta : \tilde{\Omega}^1 \times \tilde{\Omega}^1 \rightarrow \tilde{A},$$

$$\eta(au, ub) = a\eta(v, u)b, \quad a, b \in \tilde{A}, u, v \in \tilde{\Omega}^1.$$  \hspace{1cm} (19)

Note that $\eta$ can no longer be symmetric. One can consider, however, the $\mathbb{C}$-valued bilinear functional on $\tilde{\Omega}^1$, which is the composition of the metric $\eta$ and the integral on $\tilde{A}$. The latter can be any $\mathbb{C}$-linear functional on $\tilde{A}$, which is symmetric, gauge invariant and real-valued on self-adjoint elements of $\tilde{A}$, if the algebra is involutive. Then the new functional can be made symmetric provided that we impose some restrictions on $\eta$. We shall discuss it in details for our particular model. If we additionally require that $\eta(\omega, \omega^*)$ is self-adjoint for an involutive $\tilde{A}$, we immediately notice that the composition may be used to construct a semi-norm on $\Omega^1$. 

6
For the purpose of this paper and the studies of discrete geometry the above definition of the metric is sufficient, although in the case of more complicated non-commutative algebras a more detailed analysis would be necessary. However, this shall be the task of future investigations, here we restrict the detailed study to the main subject of discrete geometry. Let us only say that in general case one needs to extend the metric $g$ to the vector spaces of forms of higher order as well as to introduce the already mentioned integral.

3.2 Gauge Transformations, Connection and Curvature on Discrete Groups

Let us take the algebra $\tilde{A}$ to be the tensor product of the algebra $A$ of complex valued functions on $G$, which we introduced in the previous section, by a certain algebra $A$, which could be the algebra of complex $n \times n$ matrices $M_n$, for instance. In such case, the differential algebra $\tilde{Ω}^*$ is clearly the tensor product of $Ω^*$ by $A$. The group of gauge transformations, as defined above, can be identified with the group of functions on $G$ taking values in a group $H < A$. Similarly, gauge potentials are interpreted as $A$ valued one-forms. We will denote the involution on $A$ by $\dagger$.

Due to this simplified structure it is sufficient to construct the metric only for the differential algebra $Ω^*$, as described in the previous section, since it could be extended immediately for the whole algebra. A natural interpretation of this property is that the metric shall depend only on the base space of the gauge theory, characterized by $A$, and not on the target space, characterized by $A$. We take the integral to be the combination of the Harr integral, as defined in (7) and the trace operation on the algebra $A$, which we assume to exist.

Before we introduce the metric, let us work out the gauge transformation rules (17) for the connection and the curvature in the convenient basis we chose (8). If we write $\Phi = \sum_g \Phi_g \chi^g$, the transformation of $\Phi_g$ under gauge transformation $H \in \mathcal{H}$ is

$$\Phi_g \rightarrow H^{-1} \Phi_g (R_g H) + H^{-1} \partial_g H$$  \hspace{1cm} (20)

The condition $\Phi = -\Phi^*$ enforces the following relation of its coefficients,

$$\Phi_g^\dagger = R_g (\Phi_{g^{-1}})$$  \hspace{1cm} (21)
If we introduce a new field $\Psi = 1 - \Phi$, $\Psi_g = 1 - \Phi_g$, we can see that (20) is equivalent to

$$\Psi_g \rightarrow H^{-1}\Psi_g(R_g H).$$

(22)

The introduction of $\Psi$ is convenient for the calculations as it simplifies the formulas. We will discuss the physical meaning of this step later. It is instructive to compare formulas for the coefficients of the curvature,

$$F = \sum_{g,h} F_{gh} \chi^g \otimes \chi^h,$$

using both $\Phi$ and $\Psi$. In the first case, from the definition of $F$, the rules of differential calculus ([10, 11]) and the exact form of the structure constants in this basis ([3]), we obtain,

$$F_{gh} = \Phi(h \circ g) - \Phi_g - R_g(\Phi_h) + \Phi_g R_g(\Phi_h),$$

(23)

whereas the same formula written with $\Psi$ is much simpler,

$$F_{gh} = \Psi_g R_g(\Psi_h) - \Psi_{(h \circ g)}.$$

(24)

The transformation rule for $F_{gh}$ follows from the gauge covariance of $F$. However, since the algebra is noncommutative the coefficients are no longer gauge covariant:

$$F_{gh} \rightarrow H^{-1}F_{gh}(R_{(h \circ g)} H).$$

(25)

### 3.3 The Yang-Mills Action

In our investigation of gauge theories in the setup of discrete geometry we have come to the point when we need to introduce the action. Therefore we shall now discuss the problem of the metric. Suppose, we have a metric $\eta$ defined on the space of one-forms $\Omega^1$. By $\eta^{gh} \in \mathcal{A}$ we denote its values on the elements of the basis, $\eta^{gh} = \eta(\chi^g, \chi^h)$. Clearly, this metric is not symmetric, nevertheless, if we require that after the integration we should recover the symmetry,

$$\int \eta(u, v) = \int \eta(v, u),$$

(26)

we obtain from the definition of the Haar integral ([7]) and the relations ([14]) that this is possible only if $\eta^{gh}$ are $\mathbb{C}$-numbers such that $\eta^{gh} = \eta^{hg} \sim \delta^{gh}$. 8
Now let us consider $A$-valued one-forms. Since we are dealing with the gauge theory let us postulate the most natural requirement in such case, which is that the metric is gauge covariant, i.e.,

$$\int \text{Tr} \eta(u, v) = \int \text{Tr} \eta(H^{-1}uH, H^{-1}vH), \quad H \in \mathcal{H}, \ u, v \in \tilde{A}.$$  \hspace{1cm} (27)

The bilinearity of $\eta$, as defined previously, gives us immediately,

$$\eta(H^{-1}uH, H^{-1}vH) = H^{-1} \eta(uH, H^{-1}v)H,$$

so after taking the trace we need only to check the last part of the equality,

$$\eta(uH, H^{-1}v) = \sum_{g, h} u_R^g(H) \eta^{gh} R_{h^{-1}}(H^{-1}) v.$$ \hspace{1cm} (28)

The right-hand side of the last equality is gauge invariant provided that $\eta^{gh} \sim \delta^{gh^{-1}}$, which is again the condition obtained earlier by requiring the symmetry of the integrated metric. In addition, if we want $\eta(u, u^*)$ to be self-adjoint, we must fix $\eta^{gh}$ to be real numbers.

Now, we can tackle the analogous problem of the metric structure on the space of two-forms. This would allow us to construct the Yang-Mills action. Following the arguments above, for any two forms $u, v \in \Omega^2$, which have a unique representation as $u = \sum_{g, h} u_{gh} \chi^g \otimes \chi^h$ and $v = \sum_{g, h} \chi^g \otimes \chi^h v_{gh}$, let us construct the bilinear form,

$$\theta(u, v) = \sum_{g, h, g', h'} u_{gh} \theta^{ghg'h'} v_{g'h'},$$ \hspace{1cm} (29)

where each $\theta^{ghg'h'}$ is in the beginning an arbitrary element of $A$. The bilinearity is again mixed, i.e. from the left for the first entry and from the right for the other one. The elements $\theta^{ghg'h'}$ are the evaluation of the metric on the basis of two-forms, $\theta(\chi^g \otimes \chi^h, \chi^{g'} \otimes \chi^{h'})$.

Again, we require that after integration the metric must be symmetric and that it remains gauge invariant. This leads to the restriction that $\theta^{ghg'h'}$ has to be a $\mathbb{C}$-number, which vanishes unless $h' \circ g' = g^{-1} \circ h^{-1}$ and additionally, $\theta^{ghg'h'} = \theta^{g'h'gh}$. Since we want to construct $\theta$ from the metric $\eta$ we
obtain, after taking into account the conditions above, the following general expression,
\[
\theta_{ghg'\,h'} = \alpha \eta_{gh} \eta_{g'h'} + \beta \eta_{gh'} \eta_{g'h},
\] (30)
where \(\alpha, \beta\) are arbitrary constants.

In the case of a commutative group \(G\) we have also an additional term possible,
\[
\theta_{c\,gh} = \theta_{ghg'\,h'} + \gamma \eta_{gh} \eta_{gh'}.\] (31)

We want the Yang-Mills action to be constructed in the same way as in the case of the gauge theories on manifolds. Therefore, we postulate that for an involutive algebra and the structure group \(H \subset U(A)\), it has the following form:
\[
S_{YM} = \int \text{Tr} \theta(F, F^*),\] (32)
where \(\int\) is the Haar integral on \(A\) and \(\text{Tr}\) is the trace on \(A\). First of all, from the previous considerations we immediately notice that (32) is indeed gauge invariant. This formula, together with corresponding expressions for the metric \(\theta\) applied in the situation of calculus on manifolds yields the standard answer. For noninvolutive algebras or other structure groups one has to modify the expression for the ‘squared norm’ of \(F\), which we used in the formula (32).

Finally let us calculate the action (32) using the functions \(F_{gh}\). Let us denote by \(\eta^{gh}\) the value of \(\eta(\chi^g, (\chi^h)^*)\). From the definition of the metric (29,30) and from the involution rules on our algebra (15) we get,
\[
S_{YM} = \int \sum_{g,h,g',h'} \left( \alpha \eta_{gh} \eta_{g'h'} \text{Tr} F_{gh} F_{h'g'}^\dagger + \beta \eta_{gh'} \eta_{g'h} \text{Tr} F_{gh} F_{h'g'}^\dagger \right). \] (33)

Of course, for the commutative group \(G\) we get also another term coming from (31). Therefore, due to the arbitrary choice of the constants \(\alpha, \beta\), we can single out at least two possible independent actions, which are of the second order in \(F\). If we take into account the form of the metric,
\[\eta^{gh} = E_g \delta^{gh^{-1}},\] (34)

we obtain the following expressions:
\[
S_1 = \int \sum_{g,h} E_g E_{h^{-1}} \text{Tr} F_{gh} F_{gh}^\dagger, \] (35)
\[
S_2 = \int \sum_{g,h} E_g E_{h^{-1}} \text{Tr} F_{gh} F_{gh}^\dagger, \] (36)
and again, for commutative $G$, we additionally get,

$$S_c = \int \sum_{g,h} E_g E_h \text{Tr} \ F_{gh} F_{gh}^\dagger. \quad (37)$$

All these actions are of course gauge invariant and independent of our choice of the basis, the latter due to their construction, which involves the contraction of tensors (32). The properties of $\eta$ guarantee also that they are all real.

Let us point out that the constructed actions or rather each possible linear combination of them may pretend to be the Yang-Mills action of our theory. This ambiguity is the result of the fact that our differential algebra is noncommutative, furthermore it is interesting that it also depends on the group structure of the base space. Another significant feature of the theory is that the space of possible metrics on $\Omega^\ast$ is much smaller than one would expect.

Finally, let us point out that one could build more invariant quantities, which might be used in the construction of the general action of our gauge theory. For instance, let us notice that the following quantity,

$$S_m = \int \text{Tr} \sum_{g,h} \eta^{gh} F_{gh}, \quad (38)$$

is gauge invariant and independent of our choice of the basis. If we rewrite it using the shifted connection $\Psi$, use the exact form of $\eta^{gh}$ (34) and take into account (21) we obtain

$$S_m = \int \text{Tr} \sum_g E_g \Psi_g \Psi_g^\dagger, \quad (39)$$

which is of course real. So, this term is as good as all introduced earlier and therefore it also has to be taken into account.

In the above analysis we used the metric $\eta$ to construct the actions. Let us now discuss the issue of other possible candidates, which can replace the metric. For instance, we can introduce another bilinear functional on $\Omega^1$, which is this time bilinear with respect to the left action of $\mathcal{A}$ on its both entries, $\rho(au, bv) = ab \rho(u, v)$. This holds only for commutative $\mathcal{A}$ and can be extended for $\mathcal{H}$ valued forms after taking the trace. Additionally, $\rho$ can be made symmetric and gauge invariant if $\rho^{gh} = \rho(\chi^{g}, \chi^{h}) \sim \delta^{gh}$. The only
missing property is that $\rho(a, a^*)$ may not be self-adjoint and therefore the actions may appear not to be real valued.

We constructed the theory in a purely geometric way. Thus, although we chose our specific basis, which proved to be very convenient for the calculations, our results are independent of this choice. We shall not discuss this symmetry here, let us only mention that if we neglect it we end up with more candidates for gauge invariant objects.

4 EXAMPLES

In this section we will study two simple examples of the unitary gauge theory on the two- and three-point spaces, with the structure of abelian groups. We will construct the action functionals and discuss briefly the solutions and their geometry.

4.1 Gauge Theory on $Z_2$

Let us denote the group elements of $Z_2$ by $+$ and $-$. We take the group $H$ to be $U(N)$ and the algebra $A$ to be the algebra of complex matrices $M_n$. Since the structure group is unitary, the connection must be antihermitian, therefore we obtain the following relation,

$$\Phi_-(+) = \Phi_+^\dagger(-),$$

and the same applies to $\Psi_- = 1 - \Phi_-$. Thus, effectively we have got only one degree of freedom, which is an arbitrary complex matrix. We take it as $\hat{\Psi} = \Psi_-(+)$ and for convenience we drop here the subscript index.

Let us observe that for $n \geq 1$ all $\Omega^n$ are one-dimensional. Consequently, the curvature two-form $F = F_- \chi^- \otimes \chi^-$ is completely determined by one coefficient function $F_-$, which using Eqs. (24) and (40) can be calculated to be,

$$F_-(+) = \hat{\Psi}\hat{\Psi}^\dagger - 1,$$

$$F_-(--) = \hat{\Psi}^\dagger\hat{\Psi} - 1.$$  

Of course the metric is trivial in this case, for simplicity we take $\eta^{--} = 1$. Now one can easily see that all possibilities for the Yang-Mills action are
reduced to the following,

\[ S_{YM} = \text{Tr} (\hat{\Psi}^\dagger \hat{\Psi} - 1)^2, \quad (43) \]

where we have already done the Haar integration.

This has exactly the form of the potential of the Higgs model and was first obtained in Connes’ consideration of the \( \mathbb{C}^2 \) algebra \(^1\). Here, however, we can modify this expression slightly by adding the term linear in \( F \) \( (38) \), which is proportional to \( \text{Tr} (\hat{\Psi}^\dagger \hat{\Psi} - 1) \). In this case we get the total action equivalent to \( (43) \) with the field \( \Psi \) rescaled.

Let us now make some comments on the moduli space of the theory and the extremal points of the action functionals. The space of flat connections modulo gauge transformations is trivial. Indeed, the vanishing of \( F \) is equivalent to the unitarity of \( \hat{\Psi} \) and from its transformation rule \( (22) \) we see that arbitrary \( \hat{\Psi} \) can be obtained from the trivial flat connection, \( \Psi = 1 \), by choosing the appropriate gauge transformation. The Yang-Mills action has one absolute minimum, which is reached for the flat connections.

4.2 Gauge Theory on \( \mathbb{Z}_3 \)

Let us denote the elements of the group by \( 0, 1, -1 \) and the group action by \( + \). The space of one-forms is two-dimensional, spanned by the basis of invariant forms \( \chi^+, \chi^- \), (for the indices, \(+\) stands for \(+1\) and \(-\) for \( -1 \)).

We choose the metric \( \eta \) to be in its simplest form, so that \( \eta^{+\cdots} = \eta^{-\cdots} = 1 \) and the other two components vanish according to our requirements \( (34) \).

The algebra of derivations \( \partial_+, \partial_- \) obeys the following relations,

\[
\begin{align*}
\partial_- \partial_- &= 2\partial_\perp - \partial_+ , \\
\partial_+ \partial_+ &= 2\partial_+ - \partial_- , \\
\partial_- \partial_+ &= \partial_\perp \partial_\perp = \partial_+ + \partial_- ,
\end{align*}
\]

which determine the structure constants and the rules of the differential calculus in this case \( (11) \).

Now, let us again construct the \( U(N) \) gauge theory. The gauge potential one-form \( \Phi \) could be expressed as \( \Phi_+ \chi_+ + \Phi_- \chi_- \). The condition that \( \Phi \) is antihermietian \( (21) \) takes the form,

\[ \Phi_+(g) = \Phi_-^\dagger (g + 1), \quad g \in \mathbb{Z}_3. \quad (47) \]
From this relation we see that the connection is completely determined by either of its coefficients. Let us define \( \Psi = 1 - \Phi_+ \), and use it in further analysis. Its gauge transformation is as follows,

\[
\Psi(x) \rightarrow H^\dagger(g)\Psi(g)H(g + 1), \quad H(g) \in U(N), \quad g \in \mathbb{Z}_3. \tag{48}
\]

Now we can express the curvature in terms of the function \( \Psi \). Since \( \Phi_+ = 1 - \Psi \) and \( \Phi_- = R_-(1 - \Psi^\dagger) \) we obtain the coefficients \( F_{gh} \),

\[
F_{++} = \Psi(R_+\Psi) - R_-\Psi^\dagger, \tag{49}
\]

\[
F_{+-} = \Psi(\Psi^\dagger) - 1, \tag{50}
\]

\[
F_{-+} = (R_-\Psi^\dagger)(R_-\Psi) - 1, \tag{51}
\]

\[
F_{--} = (R_-\Psi^\dagger)(R_+\Psi^\dagger) - \Psi. \tag{52}
\]

One can easily notice that \( F_{--} = R_+F_{++}^\dagger \) and both \( F_{+-} \) and \( F_{-+} \) are hermitian.

Before we discuss the action functionals let us find the moduli space of flat connections in this case. If \( F \) vanishes, from (50) we obtain that the function \( \Psi \) must be unitary, whereas \( F_{++} = 0 \) gives us from (49) and from the previous result the following identity,

\[
(R_-\Psi)\Psi(R_+\Psi) = 1. \tag{53}
\]

Using the transformation rule (48) and the condition above we can again show that all flat connections are gauge equivalent.

Finally, let us present the actions. The action linear in \( F \) is,

\[
S_m = 2 \int \text{Tr} \left( \Psi\Psi^\dagger - 1 \right), \tag{54}
\]

and we are left with three possible terms for the Yang-Mills quartic action,

\[
S_1 = 2 \int \text{Tr} \left( (\Psi\Psi^\dagger - 1)R_-(\Psi^\dagger\Psi - 1) + (\Psi\Psi^\dagger - 1)(\Psi^\dagger\Psi - 1) \right), \tag{55}
\]

\[
S_2 = \int 2\text{Tr} \left( (\Psi\Psi^\dagger - 1)^2 + R_+(\Psi\Psi^\dagger)(\Psi^\dagger\Psi) \right.
+ \Psi^\dagger\Psi - R_-(\Psi)\Psi R_+(\Psi) - R_+(\Psi^\dagger)\Psi^\dagger R_-(\Psi^\dagger) \right), \tag{56}
\]

\[
S_c = 2 \int \text{Tr} \left( (\Psi\Psi^\dagger - 1)R_-(\Psi^\dagger\Psi - 1) + R_+(\Psi\Psi^\dagger)(\Psi^\dagger\Psi) \right.
+ \Psi^\dagger\Psi - R_-(\Psi)\Psi R_+(\Psi) - R_+(\Psi^\dagger)\Psi^\dagger R_-(\Psi^\dagger) \right) \tag{57}
\]
Each linear combination of them may pretend to be the global action of the theory. Let us observe the remarkable property that there exists a combination, which is a third order polynomial in $\Psi$,

$$S_3 = 2S_1 + 2S_m - S_1 - S_m = 2\int \text{Tr} R_-(\Psi)\Psi R_+ (\Psi) + c.c. \quad (58)$$

Such decomposition of quartic actions into the third-order invariants is not a general feature of the theory and is characteristic only for the model under study.

The problem of the extremal points of the presented actions is more complicated than in the previous example and in some cases the action might not even have an absolute minimum. The analysis becomes much simpler if we restrict ourselves to the consideration of smaller algebras, which we shall discuss in the next section.

## 5 SYMMETRIES AND SUBALGEBRAS

This section is devoted to a brief discussion of possible restrictions and modifications of the theory, which arise from employing symmetries of the considered algebra. Suppose we take a proper subalgebra of $\tilde{A}$, we shall denote it by $\tilde{\mathcal{B}} \subset \tilde{A}$. Then we can find a graded differential subalgebra of $\tilde{\Omega}$ in such a way that the zero-forms are the elements of $\tilde{\mathcal{B}}$ and all the rules of differential calculus remain unchanged. For instance, let us consider the subalgebra of $\mathcal{A}$, which contains all $\mathbb{C}$-valued constant functions on the group $G$ and denote it by $\mathcal{A}_0$. If we take as $\Omega^n$ all differential forms that have their coefficients in $\mathcal{A}_0$, we obtain the required subalgebra of $\Omega^n$. We can express that construction in a more formal way. Indeed, if we use the group of automorphisms of $\mathcal{A}$, $R_g, g \in G$, which can be easily extended to the whole of $\Omega^n$, we see that $\Omega^n_0$ remains invariant under the action of this group. Therefore, this construction can be generalized for any algebra and any group of its automorphisms. A trivial example from the differential calculus on Lie groups is set by the restriction to the algebra of left-invariant forms. In the particular case of the $SU(2)$ gauge theory constructed in such model was discussed in details in several papers $^6$–$^10$, although it was not formulated precisely in the language we introduced here. More sophisticated model, with the coset symmetry condition, was analysed using this formalism $^{17}$. 

15
Let us turn our attention back to the introduced examples. Having the differential grading algebra \( \Omega^* \) we can proceed with the construction of gauge theories following the steps we have already made for \( \Omega^* \).

The gauge transformations are now global, i.e. they are also constant, when considered as functions on \( G \). The same applies to the coefficients of the connection and the curvature. Let us see what is the effect of this for the constructed theories on \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \).

In the first case, from (40) we get that \( \Psi \) must be hermitian. This implies that the minima of the Yang-Mills action (43), which again correspond to the flat connections, are separated. The moduli space is equal to the space of equivalence classes of unitary and hermitian matrices. Notice that for \( U(1) \) we obtain \( \mathbb{Z}_2 \), which is the base space of our theory \( G \).

The same applies to the model with \( \mathbb{Z}_3 \). This time, the moduli space of flat connections is the space of equivalence classes of matrices that satisfy the relation \( \Psi^3 = 1 \). Again, for \( U(1) \) this space can be identified with \( \mathbb{Z}_3 \). The arbitrariness in the choice of action is reduced slightly when compared to the model with larger algebra, as in this case we have \( S_2 = S_m \).

We shall end here the discussion of possibilities arising from employing the symmetry principles encoded in the automorphisms of the algebras. Our aim was only to show this option and present briefly the implications for the given models and we leave a more detailed investigation of this topic for future studies.

6 PRODUCTS OF DISCRETE AND CONTINUOUS GEOMETRY

Suppose we have two graded differential algebras, \( \Omega_1, \Omega_2 \), with the external derivative operators \( d_1 \) and \( d_2 \) respectively. Then we can construct the tensor product of them, which can be made again a differential algebra provided that we take,

\[
d(\omega_1 \otimes \omega_2) = (d_1 \omega_1) \otimes \omega_2 + (-1)^{\deg \omega_2} \omega_1 \otimes (d_2 \omega_2),
\]

(59)

Let us take as the first component the algebra of differential forms on a manifold and the differential algebra over a discrete group as the other one. Their tensor product is the basis for the description of models combining both continuous and discrete geometry. Remember that we have to distinguish the
tensor product of these algebras, \( \tilde{\otimes} \) from the product in \( \Omega_2 \), which we denote again by \( \otimes \) and the product within \( \Omega_1 \), denoted as usually by \( \wedge \).

We shall calculate here, as a very illustrative example, the outcome of a unitary gauge theory on two base spaces. The first will be the product of the Euclidean space \( M \) and the two-point space and the second will arise from the same euclidean geometry and from the algebra of constant functions on the three-point space \( \mathbb{Z}_3 \).

In the first example, we may assume simply that the algebra of zero-forms consists of complex functions, with their arguments from \( M \times \mathbb{Z}_2 \). The gauge group is then \( U(1) \). We know the action of the external derivative on each separate algebra as well as the rule (59). Thus, the gauge connection \( \Phi \) is a one-form, comprising discrete and continuous geometry differential forms:

\[
\sum A_\mu dx^\mu + \Phi \chi^- .
\]

Both \( A_\mu \) and \( \Phi \) are again functions on the product space, \( A_\mu = A_\mu(x, g) \) and \( \Phi = \Phi(x, g) = R_-(\Phi(x, g))^* \). Let us calculate the curvature, which in addition to the 'continuous and discrete' terms has the mixed ones as well. We use the terminology from the section 2.1, with the shifted connection \( \Psi \). Notice that because \( \partial_\mu \) and \( \partial_- \) commute, the product of forms dual to them anticommutes: \( dx^\mu \tilde{\otimes} \chi^- = -\chi^- \tilde{\otimes} dx^\mu \). After some calculations we finally get,

\[
F = \sum_{\mu, \nu} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) dx^\mu \wedge dx^\nu + (\Psi \Psi^* - 1) \chi^\nu \otimes \chi^- \\
+ \left( \frac{\partial \Psi}{\partial x^\mu} - A_\mu \Psi + R_-(A_\mu) \Psi \right) dx^\mu \tilde{\otimes} \chi^- .
\]

(61)

If we take the metric to be the tensor product of metrics on each algebra and integrate over \( M \) with respect to the standard measure, we obtain the following Yang-Mills action,

\[
S_{\text{YM}} = \int_M F_{\mu\nu}(x, +) F^{\mu\nu}(x, +) + F_{\mu\nu}(x, -) F^{\mu\nu}(x, -) \\
+ (\Psi(x, +) \Psi^*(x, +) - 1)^2 \\
+ (D_\mu \Psi(x, +)) (D^\mu \Psi(x, +))^* ,
\]

(62)

where we have used properties of the metric and integration on the two-point space and the relation \( A_\mu = -A_\mu^* \). \( D_\mu \) is the abbreviation, which we use for
the following operator:

\[ D_\mu \Psi = \frac{\partial}{\partial x^\mu} - A_\mu(x, +) + A_\mu(x, -), \quad (63) \]

The action (62) describes a scalar \( U(1) \times \bar{U}(1) \) Higgs model, with a scalar complex field \( \Psi(x, +) \) and two gauge fields \( A_\mu(x, +) \) and \( A_\mu(x, -) \). The interaction term for the scalar field, which has the form of the quartic potential, arises in this way naturally. Let us point out the importance of the fact that the physical meaning is given not to the gauge potential \( \Phi \) but to the shifted connection \( \Psi \). Only by using \( \Psi \) we get the coupling between discrete and continuous parts in the form of the covariant derivative (63).

Similarly as above we will now briefly discuss the unitary gauge theory on the product of continuous geometry on \( M \) by the geometry set by the invariant subalgebra \( \Omega^0 \) over \( \mathbb{Z}_3 \). Here we shall not restrict ourselves to the \( U(1) \) case, as nonabelian case would be more interesting. So we have the gauge connection in the product algebra,

\[ A_\mu(x)dx^\mu + \Phi(x)\chi^+ + \Phi(x)^\dagger\chi^-, \quad (64) \]

and the curvature two-form \( F \),

\[
F = \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu] \right) dx^\mu \wedge dx^\nu + \sum_{g,h=+, -,} F_{gh} \chi^g \otimes \chi^h \\
+ \left( \frac{\partial \Psi}{\partial x^\mu} + [A_\mu, \Psi] \right) dx^\mu \otimes \chi^+ - \left( \frac{\partial \Psi^\dagger}{\partial x^\mu} + [A_\mu, \Psi^\dagger] \right) dx^\mu \otimes \chi^- , \quad (65) \]

where again \( \Psi = 1 - \Phi \) and one has to insert the exact form of \( F_{gh} \) from (49-52). We observe that due to the underlying \( \mathbb{Z}_3 \) space we have some freedom in the choice of our action. It is clear that it must include the following terms,

\[ \text{Tr} F_{\mu\nu}^\dagger F^{\mu\nu} + \text{Tr} (D_\mu \Psi)(D_\mu \Psi)^\dagger \quad (66) \]

where \( D_\mu \) is the covariant derivative \( D_\mu \Psi = \frac{\partial \Psi}{\partial x^\mu} + [A_\mu, \Psi] \). The ambiguity arises when we consider the self-interaction term for \( \Psi \). We can choose any of the possible actions (55-57), which could give the same quartic potential as in the previous example but we can get the \( \Psi^3 \) type action from (58) as well.

Finally let us notice that if the gauge group is abelian we have no coupling between the \( A \) and \( \Psi \) fields.
7 CONCLUSIONS

We presented in this paper a systematic approach to the problem of construction of the gauge theories on discrete spaces. This involved the introduction of the differential calculus, which we have carried out for spaces that possess the structure of a finite group. This structure was a key point of our analysis, so we established that in order to construct theories on discrete spaces one has to specify their group properties as well. It would be interesting, for instance, to investigate theories, which have the same number of points in the base space but which differ in their group structures.

We constructed gauge theories only for the unitary gauge groups, indicating that the formalism could be extended to arbitrary groups. In fact, they do not have to be continuous and one may as well use discrete groups for this purpose. Another spectacular property of this theory is the fact that we can take as a starting point the algebra, which is not necessarily the algebra of \( \mathbb{C} \)-valued functions or functions valued in any algebra but its proper subalgebra. For instance, if have an algebra \( A \), and its subalgebras, say \( A_0, A_1, \ldots \), we can construct the proper subalgebra of \( \tilde{A} \) as the set of functions such that \( f(x), x \in G \) takes values in \( A_j \) for some index \( j \). Now, following the same steps as we presented in this paper, we can construct the differential calculus and the gauge theories. It appears that if we consider the two-point space, as in the first example, with the algebra of functions taking values in \( \mathbb{C} \) at one point and at the other in \( \mathbb{H} \), which is the algebra of quaternions, its product with the continuous geometry of the Minkowski space gives us the precise description of the pure gauge part of the electroweak interactions. Of course, in this approach fermions stay out of the picture.

The discussion of the Yang-Mills action has led us to the investigation of the metric properties of the model. This seems to be another interesting point arising from our considerations and in the future investigations we shall attempt a thorough discussion of this topic. Consequently, one may try to investigate the general relativity of discrete geometries alone or of the product of them by the continuous manifolds, which may have a deep physical meaning if the geometry of the world involves discrete part, as suggested by the Standard Model.

Finally, let us notice that the restriction to the finite groups may be relaxed as well and one can analyze similar models for infinite discrete groups like \( \mathbb{Z}_n \), for example.
The program of noncommutative geometry has given us the possibility of considering a far more general class of models than the one arising from the analysis on manifolds. Since quantum groups and discrete geometry are the two most interesting and promising examples, their study seems to be important and we believe that their analysis, in particular the investigation of gauge theories in this framework, will help to a better understanding of the subject.
References

[1] A.Connes, *Non-commutative differential geometry, de Rham homology and non-commutative algebra*, Publ. Math. IHES 62 (1986), 44-144

[2] A.Connes, J.Lott, *Particle models and non-commutative geometry*, Nucl. Phys. (Proc. Suppl.) B18, (1990) 29-47

[3] A.Connes, *Essays on physics and non-commutative geometry*, in: ”The Interface of Mathematics and Particle Physics”, eds D.Quillen, G.Segal and S.Tsou, Oxford University Press (1990)

[4] A.Connes, *Geometrie non Commutative*, Intereditions, Paris (1990)

[5] A.Connes, *Lectures at the Les Houches Summer School 1992*,

[6] B.S.Balkrishna,F.Gürsey,K.C.Wali, *Noncommutative geometry and Higgs Mechanism in the Standard Model*, Phys.Lett. B254, (1991), 430

[7] M.Dubois-Violette,R.Kerner,J.Madore, *Gauge bosons in noncommutative geometry*, Class.Quant.Grav. 6, (1989) 1709-1724

[8] J.Madore, Int.J.Mod.Phys. A6 (1991), 1287-1300

[9] M.Dubois-Violette,J.Madore,R.Kerner, Phys.Lett. B217, (1989) 485

[10] M.Dubois-Violette,J.Madore,R.Kerner, *Non-commutative differential geometry and new models of gauge theory*, J.Math.Phys. 31(2), (1990) 323

[11] R.Coquereaux, *Non-commutative geometry and theoretical physics, Journal of Geometry and Physics*, 1990

[12] R.Coquereaux,G. Esposito-Farèse, G.Vaillant, *Higgs field as Yang-Mills field and discrete symmetries*, Nucl. Phys. B353, (1991) 689-706

[13] R.Coquereaux, G. Esposito-Farèse, F.Scheck, *The theory of electroweak interactions described by SU(2|1) Algebraic Superconnections*, preprint CPT-90/P.E. 2464, 1990

[14] A.Sitarz preprint TPJU-7/92,
[15] A.H.Chamseddine, G.Felder, J.Fröhlich, *Grand Unification in Non-Commutative Geometry*, preprint ZU-TH-30/1992 and ETH/TH/92-41,

[16] A.H.Chamseddine, G.Felder, J.Fröhlich, *Gravity in Non-Commutative Geometry*, preprint ETH/TH/92-41,

[17] A.Sitarz $SU(1|1)$ *Coset Gauge Theory* Acta Phys. Pol. **23**, (1992), 791

[18] A.Sitarz, *in preparation*