Hierarchical societies exhibit diverse swarming transitions

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Social hierarchy is central to decision-making such as the coordinated movement of many swarming species. Here we propose a hierarchical swarm model for collective motion in the spirit of the Vicsek model of self-propelled particles. We show that, as the hierarchy becomes more important, the swarming transition changes dramatically from the weak first-order transition observed for egalitarian populations, to a stronger first-order transition for intermediately strong hierarchies, and finally to a second-order phase transition for despotic societies. Associated to this we observe that the spatial structure of the swarm, as measured by the correlation between the density and velocity fields, depends very strongly on the hierarchy. A vectorial network model is developed that correctly explains all these features. Our results imply that diverse type of swarming transitions are possible, depending on the impact of hierarchy of the species under study.

Introduction.— Collective motion is one of most spectacular and fascinating emergent behaviors in nature, as exhibited in insects, bird flocks, fish shoals, and herds of ungulates, among others [1]. While detailed case studies are preferred in general by biologists [2–4], physicists usually seek for minimal models with the hope that there are universal features behind seemingly diverse observations, and simple models are sufficient to capture the fundamental laws [5, 6]. A prototype model of the second kind, which considerably advanced our understanding of collective motion, is the Vicsek model [7] inspired by statistical physics, and its many variants [8]. A major concern here is the nature of swarming transitions between the ordered and disordered states of movement. The second-order phase transition (PT) claimed in the original work was later challenged by Châte et al [9, 10], by showing that the observed continuous nature is actually due to finite-size effects, and a first-order transition should be expected in the thermodynamical limit with only local interactions – in line with even later theoretical studies [11–13]. Note that, analogous to the identical particle assumption in statistical mechanics, individuals in these models are supposed to be indistinguishable and thus equally important in decision-making for the movement coordination.

While the collective movements for some species can indeed be described as an equally shared consensus [14, 15], many more are based on partially shared or even unshared consensus decision-making [16, 17], where only a tiny fraction of individuals (or even a single one) lead the group movement. This is particularly true when hierarchical social structures are present. For example, recent experiments with high-resolution GPS data have revealed well-defined hierarchical structures among homing pigeons [18] and migratory white storks [19], where a small number of leaders direct the group flight. Highly asymmetrical dominance is also revealed in mammals such as African elephants [20], gray wolves [21], and primates [22, 23], as well as in fish schools [24] and honeybee swarms [25] etc. However, a generic model that interpolates from egalitarian to despotic swarms is still lacking. Revealing how the hierarchy impacts on the collective motion remains a crucial challenge for understanding hierarchical societies in general.

In this letter, we fill this gap by introducing a hierarchical swarm model, called the hierarchical Vicsek model (HVM), and investigate the impact of the hierarchy on the nature of order-disorder transitions. By tuning the swarm from egalitarian to despotism, we show numerically that the swarming transition changes non-monotonically from a weakly first-order to a continuous (2nd order) transition, with strongly first-order PTs occurring at intermediate levels of hierarchical impact.

Detailed analysis of the microscopic structures of swarms shows that the altered nature is due to hierarchy-induced changes in correlations between the density and orientation fields. A more quantitative account from a network perspective is provided and shows that a bimodal distribution of the neighborhood sizes naturally explains the enhanced discontinuity for the intermediate degree of hierarchical impact.

The hierarchical Vicsek model.— As in the standard Vicsek model (SVM) [7], N pointwise particles labeled as 1, 2,..., N are randomly placed on a two-dimensional domain with size L × L with periodic boundary conditions. They move synchronously at discrete time steps by a fixed distance v0Δt. Each particle i is endowed with an angle θi that determines the direction of the movement during the next time step, and its update is determined by the orientations of its neighbors (defined as particles within a unit circle centered around particle i, including itself). In the SVM, the influence of

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we have thus full influence on particle \(j \neq i\). Instead of Eq. (1) we could have implemented in our model more complicated hierarchical structures, by replacing simply the two order parameters defined as \(\phi\) and its temporal average \(\langle \phi \rangle\). To monitor jumps of \(\phi\), a useful quantity is the so-called Binder cumulant \(G(\eta, L) = 1 - \langle \phi^4 \rangle / (3 \langle \phi^2 \rangle^2)\), which is expected to fall to negative values in first-order phase transitions with phase coexistence. In the ordered phase we expect roughly Gaussian distributions of \(\phi\) and thus \(G \approx 2/3\), while in the disordered phase \(G \approx 1/3\) in two dimensions.

Here the key ingredient is the competition between the tendency towards local alignment and the noise \(\xi(t)\) (that might come from external perturbations and/or from uncertainties in individual’s perception), chosen from a uniform distribution within the interval \([-1/2, 1/2]\), and \(\eta\) is the noise amplitude.

In the limit \(\alpha = 1\) we recover the standard Vicsek model, which is egalitarian by nature. Notice also that we could have implemented in our model more complicated hierarchical structures, by replacing simply the two values \(\alpha, 1\) by any function \(\alpha(i, j)\).
Results and analysis.— Varying the hierarchical coefficient $\alpha$, we observe a rich spectrum of phase transitions (PTs) as a function of noise intensity $\eta$, see Fig. 1a. For the egalitarian case where $\alpha = 1$, a first-order phase transition is seen but the gap between the two branches is small, as is also the decrease in the Binder cumulant $G$ (Fig. 1b). Therefore it is weak, as also found in extensive previous studies [9, 10]. As $\alpha$ is decreased, the gap becomes larger and the minimum of $G$ becomes deeper, suggesting an increasingly stronger PT. Around $\alpha \approx 1/36$, this enhancement becomes maximal. Decreasing $\alpha$ further towards zero, the gap shrinks again and the minimum of $G$ becomes again shallower. Finally, for $\alpha \approx 0$ the curve of $\langle \phi \rangle$ against $\eta$ becomes continuous and the minimum of $G$ is at $G \approx 1/3$. Thus, the nature of the transition is completely converted into second order, when the particles have almost no influence on their higher-ranking neighbors. The impact of swarm density $\rho$ on the nature of PT at the interface of continuous-discontinuous transitions is shown in Supplemental Material [26].

The observation of an enhanced first-order PT is strengthened by the study of probability distribution functions (PDFs) of $\phi$. The PDFs shown in Fig. 1c are clearly bimodal in the bistable region. The peak at smaller values of $\phi$ corresponds to disordered motion, the other to the ordered phase. As expected, the ordered phase shrinks when the noise is increased, while the disordered phase expands. Another hallmark of first-order PTs, also clearly seen in Fig. 1c, is the presence of the hysteresis. Fig. 1d provides further evidence by the finite size effect analysis, showing more and more sharp bimodal peaks as the system becomes larger.

To understand how the hierarchy affects the features of the swarming transition, we first look at how the spatial distribution is influenced. A known feature of the SVM is that there exist localized, high-density, traveling bands corresponding to the ordered, symmetry-breaking phase. They are metastable on long time scales, i.e. they dissolve and reappear from time to time. On a much shorter time scale, particles enter and leave the bands. When they are not in bands, they perform random-walk-like movements. As seen in Fig. 2a, this is even more pronounced for the HVM at $\alpha = 1/36$, where the transition is most abrupt. To see the impact of the hierarchy, we divide the population into five subgroups based on their labels, and compare their density profiles $\rho_\perp$ along the longitudinal direction with respect to the mean velocity (Fig. 2b). While a kink-like profile is observed for all groups, it is most pronounced for the low-rank group and least pronounced for the group with highest rank (a detailed neighborhood analysis is presented in the Supplemental Material [26]). Intuitively, this is easily understood. Individuals on top of the hierarchy are least sensitive to their neighbors. Thus they feel the weakest collective force, and they have the least tendency to be trapped in the bands. For individuals at the bottom of the hierarchy, the opposite is true.

In particular, this explains immediately why the PT becomes continuous in the limit $\alpha \to 0$ by examining the stability of the band structures. In this limit, high-ranking individuals completely ignore nearly all other individuals and can only be influenced by neighbors of even higher rank. Assume now that there exists a band. The top ranking particle is blind to it, and will therefore soon leave it. But then the second-ranking particle becomes top-ranking in the band, and will also leave it. As the departure process repeats, the band will finally dissolve. Thus, there is simply no band structure for $\alpha \to 0$, nor can there be other ordered structures and no bistability in this limit. A continuous PT is expected instead.

To gain more insight, let us divide the domain into $L^2$ cells of size $1 \times 1$. We denote by $N_j$ the set of particles in cell $j$, and define the local order parameter as

$$\phi_{\text{local}}(j) = \frac{1}{N_j v_0} \sum_{i \in N_j} v_i,$$  \hspace{1cm} (5)

where $N_j = |N_j|$ is the number of particles located in cell $j$ (for cells with $N_j = 0$, we define $\phi_{\text{local}}(j) = 0$). Additional evidence for an enhanced discontinuity is provided.
by plotting the correlation between the global order parameter $\phi(t)$ and the spatially averaged local order parameter

$$
\phi_{\text{local,av}}(t) = \frac{1}{L^2} \sum_{j=1}^{L^2} \phi_{\text{local}}(j).
$$

In Fig. 3 (a) we show these correlations in the bistable region, when band structure is most pronounced. We see that the correlation is nearly zero for the original SVM, while it is rather strong in the case with hierarchy. Notice that the global order parameter can also be considered an extreme case of $\phi_{\text{local}}(j)$, where the cell size goes to the system size $L$. Thus Fig. 3 (a) suggests that the local order parameters in bands and outside of them are very different in the hierarchical model, while they would be more or less the same in the SVM (times series of $\phi_{\text{local,av}}(t)$ are provided in Supplemental Material [26]).

A consensus regarding the mechanism for band dynamics is that it is due to the intimate coupling between the particle density and orientation fields that in a cascaded manner leads to the band emergence and disappearance. Consider a moving patch with a slightly higher density than its surroundings, since its orientation field is better aligned, it is more likely to attract particles come across, which in turn makes the patch more ordered and is again even more likely to attract particles. In such a way, a band is formed out of the homogenous state near the transition points. Band dissolution occurs in just opposite cascade, that the loss of some particles in a band lowers the order of the local orientation field, which potentially leads to further reduction of band density as more particles leave. The feedback between density and orientation fields then dissolves the bands in the end. Fig. 3b shows that there is indeed a positive correlation between the local density $\rho_{\text{local}}$ (defined as the particle number in a cell) and the effective local order parameter $\phi_{\text{local}} = \phi_0$. Notice that, $\phi_0$ is the background value of $\phi_{\text{local}}$ that comes from finite particle effect — a smaller number of particles always produce finite $\phi_{\text{local}}$ even if their headings are completely uncorrelated (see the inset). Therefore $\phi_{\text{local}} - \phi_0$ measures the degree of order that purely comes from the ordering process. By comparison, the case with hierarchical impact hold a significant improvement in the orientation field given the same density field. More importantly, there is a considerably

FIG. 3. (Color online) Microscopic properties in the bistable regions (a)-(c) and in the bandless ordered region (d). (a) Correlation between the global and local order parameters in bistable region in both non-hierarchical and hierarchical swarms. (b) Effective local order parameter $\phi_{\text{local}} - \phi_0$ versus local density $\rho_{\text{local}}$ (mean ± standard deviation). The correlation can be fitted by $\phi_{\text{local}} - \phi_0 \sim \gamma * \lg \rho_{\text{local}}$, where $\gamma = 0.29$ for $\alpha = 1$, and $0.34$ for $\alpha = 1/36$. Inset shows the finite particle effect for the local order parameter $\phi_0$: the smaller number of particles in a given cell, the higher value of $\phi_0$. $\phi_0$ is computed for different densities $\rho_{\text{local}}$ by averaging an assemble of randomly orientated particles in a cell. (c) PDF of local order parameter $P(\phi_{\text{local}})$ in bands and outside of them are very different in the hierarchical model, while they would be more or less the same in the SVM (times series of $\phi_{\text{local,av}}(t)$ are provided in Supplemental Material [26]).

FIG. 4. (Color online) Vectorial network model (VNM). (a) Time series of order parameter $\phi$ in swarm simulations and the corresponding average degree $K_{\text{ave}}$ (defined as the average number of particles in each particle’s neighborhood) in the bistable state at $\eta = 2.12$; PDF of $K_{\text{ave}}$ is shown in the inset in (b). (b) The order parameter $\phi$ versus $\alpha$ for two $K$ in VNM as function of the hierarchy coefficient $\alpha$. (c) The discontinuity gap $\Delta \phi = \phi(K_{\text{high}}) - \phi(K_{\text{low}})$ reaches the maximal value around $\alpha = 0.03$. (d) The discontinuity gap $\Delta \phi$ versus the noise amplitude $\eta$ with different $\alpha$ and degree $K$ as in Fig. 1a ($\alpha = 1$ : $K_{\text{low}} = 12, K_{\text{high}} = 13; \alpha = 1/9$ : $K_{\text{low}} = 14, K_{\text{high}} = 16; \alpha = 1/100 : K_{\text{low}} = 25, K_{\text{high}} = 34$). Time average have been computed over $1 \times 10^5$ time step. Parameters: $N = 32468$, the noise $\eta = 0.66$ for (a)-(c).
high density region $\rho_{\text{local}} > 25$ that only appears in hierarchical swarms, and these highly dense patches usually are most crucial for nucleation processes. Actually, for identical $\phi$ for both cases, there is always a crossover in their profiles of $\phi_{\text{local}}$ that the spatial segregation is stronger for hierarchical swarms because its distribution of $\phi_{\text{local}}$ near full order is significantly higher (Fig. 3c). This means that the nucleated structures like the band are comparably much denser and the discontinuity is expected to be stronger once band is formed in hierarchical swarms. These features are found to be generic even in disordered or handleless ordered phase, see Supplemental Material [26].

Giant number fluctuations [10, 27, 28] in ordered phase are nevertheless present and even slightly enhanced compared to the SVM: the number variance $\Delta n$ of particles in sampling square boxes of linear size $l$ behaves as $\Delta n \sim (n)^{2/3}$, where $\langle n \rangle = \rho l^2$ is the average number of particles expected in the box (Fig. 3d). The reason for this enhanced fluctuations is rooted in the enhanced spatial segregation in hierarchical swarm as seen in Fig. 3c (see also Fig. S7 in SM) that a stronger impact of hierarchy leads to more heterogeneous spatial distribution and therefore gives rise to more anomalous density fluctuations.

**Vectorial network model.**— We next turn to a network perspective [29–31] to understand why and how the hierarchy influence the degree of first-order PT. Consider the swarming particles as network nodes, and a link is established when two particles are in each other’s neighborhood, the moving heading $\theta$ is incorporated in node states as a unit vector $e^{i\theta}$ (a complex number). In such a way, the collective motion can be described as networked dynamics. In principle, a dynamic network framework [32, 33] seems more reasonable as the nodes’ neighbors are always time-varying. However, here we resort to static regular random networks to approximate our swarming system, which means each individual is in contact with some fixed neighbors and we find this already captures the key features resulted from the hierarchical impact. The evolution of each node is as follows

$$\theta_j(t+1) = \Theta(\alpha \sum_{k \in \Omega_j, k < j} e^{i\theta_k} + \sum_{k \in \Omega_j, k \leq j} e^{i\theta_k}) + \eta \xi_j(t), \quad (7)$$

where $\Omega_j$ is neighborhood of node $j$ plus itself, and $K_j = |\Omega_j| - 1$ is node degree that is assumed to be uniform in our approximation.

From network point of view, the key structural difference in the bistable state is that there are two characteristic $K_{\text{ave}} = \langle K \rangle$, one for band and the other for homogeneous disordered states, as shown in Fig. 4a and inset in Fig. 4b. When we incorporate this bimodal degree distribution and other swarm parameters (like the population size $N$ and noise strength $\eta$) into our vectorial network model, and compute the order parameter $\phi$ as a function of the hierarchy coefficient $\alpha$, we obtain the values for two states (Fig. 4b). Accordingly, the difference $\Delta \phi$ of the two order parameters is supposed to be the gap of first-order PT (see also the illustration in $K - \eta$ phase diagram in the Supplemental Material [26]). Surprisingly, there is indeed an optimal hierarchy coefficient $\alpha$ that $\Delta \phi$ reaches maximal, and this optimal $\alpha \approx 0.03$, very close to $1/36$ as we obtained in swarm studies (Fig. 4c). We also compute other values of $\alpha$, the peaks corresponding to the maximal gap is highest for $1/36$, followed by $1/100, 1/9, 1$, correctly reproducing the variation trend of weak-strong first-order PT in HVM (Fig. 4d).

**Two group model.**— To show the generality of the revealed impact of hierarchy on the swarm transitions, we study a further simplified HVM, where we remove the fine layered hierarchy in HVM and only divide the population into two subgroups: one with high ranking with fraction $h$, the other of low ranking with the fraction $1 - h$. The alignment is determined by

$$\langle \theta_i(t) \rangle = \Theta(\alpha \sum_{d_{ij} < r, j > M} \mathbf{v}_j(t) + \sum_{d_{ij} < r, j \leq M} \mathbf{v}_j(t)), \quad (8)$$

where $M = Nh$ is the size of high-ranking group. Fig. 5 shows a similar dependence of transition nature on the degree of hierarchy in the case of $h = 0.1$. A weak 1st PT — strong 1st PT — 2nd PT scenario is also seen when the hierarchy coefficient $\alpha$ increases, though the enhancement in discontinuity is less significant than the case of HVM and the optimal level of hierarchy is also different, now around 1/7.

**Conclusions.**— To summarize, we have introduced, motivated by wide observations in many animal species, a simple model for hierarchical swarms in which the impact of hierarchy on the decision-making of where to go is our focus. The model exhibits a rich zoo of swarming transitions from weak to strong first-order PTs and to continuous transitions, depending on the impact of hierarchy. A two-level hierarchy swarm model verifies the robustness of these findings. Microscopically, this scenario is attributed to the altered correlation between the density and the orientation fields. We also developed a vectorial network model, it successfully reproduces the intermedi-
ate level of hierarchy impact that generates the strongest discontinuity. On the theoretic side, our results points out a wide possibility for swarming transitions given the ubiquitous hierarchy in many species; On the experimental side, we expect specific case studies with diverse hierarchical impacts that confirm our conclusion and reveal other complexities may induced from hierarchy impact.

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