Examination of Hartert’s calculation of $G$ and recalculation using revised assumptions

**Definition of shear modulus**

The shear modulus $G$ of a material is defined by the shear stress divided by the shear strain:

$$ G = \frac{F/A}{\Delta x/\ell} = \frac{F\ell}{A\Delta x} $$

where $F$ is the force applied, $\ell$ is the height, $A$ is the area and $\Delta x$ is the distance the top surface moves in response to the force applied as shown in the following schematic diagram of the clot.

**Figure:** Sheared blood clot represented in a flat configuration.

Note that given a laminar shear of a blood clot with shear modulus $G$, the tangential force can be calculated as

$$ F = \frac{GA\Delta x}{\ell} = GA\left(\frac{\Delta x}{\ell}\right). $$
Examination of Hartert’s calculation (sheared thin cylindrical clot)

In the original paper by Hartert and Schaeder the geometry of the blood clot between the vessel and the plunger was described by a thin cylinder. The relative rotation of the vessel and plunger induced a torque on the wire and a shear of the blood clot as described below.

The outer vessel is rotated \( \theta' \) degrees causing a shearing of the cylindrical blood clot and a rotation of \( \theta \) degrees of the plunger and torsional wire. Experimentally, Hartert found \( 2\theta \approx \theta' \) as shown below.

For generality we write \( a\theta = \theta' \) for some \( a > 1 \).

From the torque equation \( T = \tau\theta = rF \) and the definition of shear modulus,

\[
\tau\theta = rF = r \frac{G A \Delta x}{\ell} = r \frac{G 2\pi r h (r\theta' - r\theta)}{\ell}.
\]

we may rearrange the expression to derive the shear modulus \( G \):

\[
G = \frac{\tau\theta\ell}{2\pi r^3 h (\theta' - \theta)} = \frac{\tau\ell}{2\pi r^3 h (a - 1)}.
\]
The following values are taken from the paper of Hartert:

- $\ell = 0.1 \ [cm]$
- $r = 0.3 \ [cm]$
- $h = 0.75 \ [cm]$
- $\alpha = 2 \ (\text{derived from observation})$
- $\tau = 6377 \ [dyn \ cm]$

Therefore the shear modulus can be calculated as follows:

$$G = \frac{(6377)(0.1)}{2(3.14159)(0.3)^3(0.75)(1)} = 5012 \ [dyn \ cm^{-2}].$$

This is consistent with the value of “approximately 5000 dyn/cm$^2$” reported by Hartert & Schaeder.
Recalculation of G using revised assumptions (extended geometry)

In the experimental set-up of Hartert the volume of blood clot formation between the vessel and the plunger is only partially represented by a thin cylindrical clot. The total volume of clotting substance used in Hartert’s experiment was reported as 0.36 cm$^3$. In the geometry above, less than half the blood clot volume is accounted for, since

$$V = \pi((\ell + r)^2 - r^2)h = \pi h(\ell^2 + 2\ell r) = 0.165 \text{ [cm}^3].$$

The experimental set-up representing the full volume of the blood clot is shown below with parts A, B and C. In this context, Hartert & Schaeder considered only volume B in their calculation and hence the only torque accounted for is the tangential force $F_b$ on the curved plunger wall of radius $r_b$.

The two other volumes of blood clot labelled A and C also contribute to the measured torque. In volume A the blood is sheared and a tangential force $F_a$ is applied to the curved plunger wall of radius $r_a$ along with a tangential force on the horizontal surface where the plunger widens from radius $r_a$ to $r_b$; the volume C is attached to the base and side of the vessel and to the base of the plunger and contributes to the measured torque via tangential forces across the whole surface of the base of the plunger denoted by $F_c$.

$$\tau = \tau_a F_a + \tau_b F_b + \tau_c,$$

where the term $\tau_c$ has been left general for later ease of computation.

When multiple forces apply their effect is additive

$$T = T_a + T_b + T_c.$$
Thus, we are required to calculate the three main forces on the plunger from the blood clot. In order to make the calculation of shear modulus analytically tractable we make some simplifying assumptions as follows.

**Simplifying assumption 1**

We assume that the horizontal surface at the shoulder of the plunger doesn’t contribute to resulting torque (equivalently that the blood clot does not attach to the horizontal surface of the plunger) so that the clot in A is sheared in the same manner as in the volume $B$ in the paper of Hartert. This assumption will mean that we slightly underestimate the resulting torque and hence overestimate the shear modulus of the blood clot. For volume $C$ we assume that the disc is twisted linearly; the actual movement may differ slightly due to the attachment of the clot to the curved wall of the vessel.

**Simplifying assumption 2**

Technical drawings of the device indicate that the base of the cup forms a slight upside-down cone (i.e. the center point is slightly lower than the edge). For simplification, in our calculations the base of the cup has been treated as a flat, horizontal surface.

From the definition of shear modulus, and with the same assumption about how curvature is incorporated into the calculation of $\Delta x$ as in the first section, we have:

$$F_a = \frac{G2\pi r_a^2 h_a \theta (\alpha - 1)}{\ell_a}.$$  

The shearing force from the thin cylinder has already been derived in the first section and is represented as:

$$F_b = \frac{G2\pi r_b^2 h_b \theta (\alpha - 1)}{\ell_b}.$$  

To calculate the torque from the disc we may consider it as the sum of concentric sheared rectangles:

$$T_c = \int_{u=0}^{u=r_b} \int_{u=0}^{u=r_b} u \cdot \frac{G2\pi u^2 \theta (\alpha - 1)}{h_c} du.$$  

Therefore

$$\tau \theta = r_a \cdot \frac{G2\pi r_a^2 h_a \theta (\alpha - 1)}{\ell_a} + r_b \cdot \frac{G2\pi r_b^2 h_b \theta (\alpha - 1)}{\ell_b} + \frac{G2\pi \theta (\alpha - 1) r_b^4}{4 h_c}.$$  

And hence under our assumptions:

$$G = \frac{\tau}{2\pi (\alpha - 1) \left( \frac{r_a^3 h_a}{\ell_a} + \frac{r_b^3 h_b}{\ell_b} + \frac{r_b^4}{4 h_c} \right)^{-1}}.$$
From the paper of Hartert & Schaeder we know:

- $\ell_b = 0.1 \text{ [cm]}$
- $r_b = 0.3 \text{ [cm]}$
- $h_b = 0.75 \text{ [cm]}$
- $\alpha = 2$ (derived from observation)
- $\tau = 6377 \text{ [dyn cm]}$

Other variables were obtained from measurements of the device as follows:

- $\ell_a = 0.25 \text{ [cm]}$
- $r_a = 0.15 \text{ [cm]}$

By inspecting Figure 1 in the paper of Hartert & Schaeder, $h_c$ was estimated as 0.1 cm. Given that the total reported volume was 0.36 cm$^3$, the value of $h_a$ could be calculated by considering the volumes of parts A, B and C:

$$\begin{align*}
\text{Total volume (V)} &= V_a + V_b + V_c = \pi (l_a^2 + 2l_ar_a)h_a + \pi (l_b^2 + 2l_br_b)h_b + \pi (l_b + r_b)^2h_c \\
h_a &= \frac{V - \pi (l_b^2 + 2l_br_b)h_b - \pi (l_b + r_b)^2h_c}{\pi (l_a^2 + 2l_ar_a)} = 0.3352 \text{ cm}
\end{align*}$$

Based on these assumptions the shear modulus can be calculated as:

$$\begin{align*}
G &= \frac{\tau}{2\pi(\alpha - 1)} \left\{ \frac{r_a^3h_a}{\ell_a} + \frac{r_b^3h_b}{\ell_b} + \frac{r_b^4}{4h_c} \right\}^{-1} \\
&= \frac{(6377)^2}{2\pi} \left\{ \frac{(0.15)^3(0.34)}{(0.25)} + \frac{(0.3)^3(0.75)}{(0.1)} + \frac{(0.3)^4}{4(0.1)} \right\}^{-1} = 4466 \text{ dyn cm}^{-2}
\end{align*}$$

Hartert considers only 46% of the blood clot volume and makes where not otherwise stated assumptions of frictionless contact surfaces. We have shown that, by considering the full clot volume and making no-slip assumptions on most of the contact surfaces, our estimate for the shear modulus of clotted blood is substantially lower.

Reference

Hartert H, Schaeder JA. The physical and biological constants of thrombelastography. Bioheology. 1962;1:31-9.

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