A memory-induced diffusive-superdiffusive transition: ensemble and time-averaged observables

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The ensemble properties and time-averaged observables of a memory-induced diffusive-superdiffusive transition are studied. The model consists in a random walker whose transitions in a given direction depend on a weighted linear combination of the number of both right and left previous transitions. The diffusion process is nonstationary and its probability develops the phenomenon of aging. Depending on the characteristic memory parameters, the ensemble behavior may be normal, superdiffusive, or ballistic. In contrast, the time-averaged mean squared displacement is equal to that of a normal undriven random walk, which renders the process non-ergodic. In addition, and similarly to Levy walks [Godec and Metzler, Phys. Rev. Lett. 110, 020603 (2013)], for trajectories of finite duration the time-averaged displacement apparently becomes random with properties that depend on the measurement time and also on the memory properties. These features are related to the non-stationary power-law decay of the transition probabilities to their stationary values. Time-averaged response to a bias is also calculated. In contrast with Levy walks [Froemberg and Barkai, Phys. Rev. E 87, 030104(R) (2013)], the response always vanishes asymptotically.

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I. INTRODUCTION

Anomalous superdiffusive processes describe a wide variety of systems arising in different disciplines such as physics and biology. Levy walks is one of the simpler models that lead to this feature [1][8]. It is a generalization of the classical Drude model where a particle moves, in successive random directions, with constant velocity during random periods of time. Depending of the mean sojourn times a transition between diffusive, superdiffusive and ballistic behaviors is obtained [1][2].

Similarly to other anomalous diffusive processes [7][22], the ergodic properties of Levy walks were recently studied [23][24]. While ensemble moments are defined in a usual way, time-averaged moments, as in single-particle tracking techniques [21], are defined by a temporal moving average performed with only one single trajectory of a given temporal length (see for example Refs. [7][8]),

$$\delta_\kappa(t, \Delta) \equiv \frac{\int_0^{t-\Delta} dt'[X(t'+\Delta) - X(t')]^\kappa}{t-\Delta}. \tag{1}$$

Here, $X(t)$ is the walker trajectory, $\Delta$ is called the lag (or delay) time and $\kappa = 1, 2$. For normal diffusive processes (independent random increments with a characteristic time scale), ensemble- and time-averaged moments coincide, $\lim_{t \to \infty} \delta_\kappa(t, \Delta) = \langle [X(\Delta) - X(0)]^\kappa \rangle$, situation that defines ergodicity. \langle \cdots \rangle denotes ensemble average. The initial condition $X(0)$ appears due to the traslational invariance of the definition (1). The so called weak ergodicity breaking is set by the condition $\lim_{t \to \infty} \delta_\kappa(t, \Delta) \neq \langle [X(\Delta) - X(0)]^\kappa \rangle$ even for long $\Delta$.

For Levy walks, the behavior of the time-averaged mean square displacement [$\kappa = 2$ in Eq. (1)] strongly deviates from that of subdiffusive continuous-time random walks [7][8], where, even at infinite measurement times $t$, they are intrinsically random objects. For Levy walks in the superdiffusive regime this randomness is absent. Ensemble and time-averaging only differ by a constant [23][24], effect called ultraweak ergodicity breaking [23]. Nevertheless, when considering trajectories made over a finite measurement time ($t < \infty$) an apparent randomness emerges both in the scaling exponents as well as in the amplitude of the time-averaged mean square displacement. This feature can be related to trajectories where the walker persists along a great fraction or even during the entire trajectory with the same velocity. On the other hand, in the ballistic regime an intrinsic randomness similar to that of subdiffusive processes arises when considering a shifted time-averaged moment [24]. Furthermore, time-averaged response to a bias [26][27] and a corresponding generalized Einstein relation were also studied [24][25].

The previous results were obtained from a renewal description [1] of the stochastic dynamics. Nevertheless, alternative underlying dynamics may also lead to superdiffusion. For example, similar analysis were performed by considering a deterministic diffusion model [27] and also correlated random walks [28]. In addition, globally correlated dynamics, where the walker dynamics depend on the whole previous history of transitions [29][39], also may lead to superdiffusion. Given that the ensemble properties may be similar to those of Levy walks it become of interest to study the ergodic properties of these strongly correlated dynamics. Added to its theoretical interest, given an experimental situation, one may obtains specific criteria for discriminating between different
possible underlying nonequilibrium stochastic dynamics.

In Ref. [40] we introduced a globally correlated diffusive dynamics that leads to (ensemble) ballistic behaviors and also characterized its time-averaged moments. Interestingly, the memory effects also lead to weak ergodicity breaking. Asymptotically the time averaged moments becomes intrinsically random. The first and second moments [Eq. (1)] grow respectively linearly and quadratically with the lag time $\Delta$. Nevertheless, the characteristic parameters of these dependences change realization to realization. A fluctuation-dissipation Einstein-like relation between the first and (a centered) second time-averaged moments for driven and undriven dynamics respectively was also established. These features are similar to that found in Levy walks in the ballistic regime [24]. Hence, it is natural to investigate if similar results can be obtained in a sub-ballistic regime and to explore up to which point previous results based on renewal memoryless dynamics are intrinsic to superdiffusive process and which are intrinsic properties of the model.

The main goal of this paper is to introduce an alternative description of superdiffusion based on global memory effects and to study its ensemble and time-averaged observables. The model interpolates between two previous known dynamics: the elephant model [29] and the urn-like model of Ref. [40]. The transition probability of the walker depends on a weighted linear combination of the number of both right and left previous transitions. Hence, jumps can be correlated or anticorrelated with the previous history. Depending on the memory parameters, the ensemble behavior suffers a transition between diffusion, superdiffusion and ballistic behaviors. The nonstationary character of the process is explicitly shown through its correlation. In addition, the probability evolution develops the phenomenon of aging [11–13].

We show that in contrast to Levy walks, for infinite measurement times the time-averaged moments strongly differ from their ensemble behavior. In fact, they are equal to that of an undriven diffusion process. Hence, ergodicity is broken, while an ultraweak ergodicity breaking effect only appears in the diffusive regime. On the other hand, averages performed with finite-time trajectories develop similar properties to that of Levy walks, that is, they become apparently random. This feature here is related to the non-stationary power-law decay of the transition probabilities to their stationary values. In contrast with previous results [24, 40], for the studied model we also show that time-averaged response to a bias dye out in the asymptotic regime.

The paper is outlined as follows. In Sec. II the global correlated dynamics is introduced. A detailed characterization of its realizations is performed. In Sec. III the ensemble properties are presented (statistical moments, correlation, and probability evolution). In Sec. IV the time-averaged observables are studied. Sec. V is devoted to the Conclusions. Analytical calculations that support the main results are presented in the Appendixes.

II. RANDOM WALK DYNAMICS

The model consists in a one-dimensional walker that at successive times perform random jumps. As in Refs. [29, 40], both the time and position coordinates are discrete. Hence, in each discrete time step $(t \rightarrow t + \delta t)$ the walker perform a jump of length $\delta x$ to the right or to the left. For simplicity, time and position are measured in units of $\delta x$ and $\delta t$ respectively. The stochastic position $X_t$ at time $t$ reads

$$X_t - X_0 = x_t = \sum_{\nu=1}^{t} \sigma_{\nu}. \tag{2}$$

Here, $X_0$ is the initial position, while $x_t$ gives the departure with respect to it. $\sigma_{\nu} = \pm 1$ is a random variable assigned to each step. The stochastic dynamics of the variables $\{\sigma_{\nu}\}_{\nu=1}$ as follows. At $t = 1$ (first jump or transition) the two possible values are chosen with probability $P(\sigma_1 = \pm 1) = q_\pm$, where the weights satisfy $q_+ + q_- = 1$. The next values are determined by a conditional probability $\mathcal{T}(\sigma_1, \ldots, \sigma_{\nu+1} | \sigma_1)$ (the notation is such that $\mathcal{T}(A|B)$ gives the probability of $B$ given $A$). This object depends on the whole previous trajectory: $\sigma_1, \ldots, \sigma_{\nu}$.

The present model relies in the selection

$$\mathcal{T}(\sigma_1, \ldots, \sigma_{\nu+1} | \sigma_1 = \pm 1) = \frac{\lambda q_{\pm} + \mu x_{\nu} + (1 - \mu)t_\mp}{t + \lambda}. \tag{3}$$

This transition probability depends on two free parameters, $\lambda$ and $\mu$. They satisfy the condition $\lambda \geq 0$ and $0 \leq \mu \leq 1$ respectively. Furthermore, $t_+$ and $t_-$ are the number of times that the walker jumped (up to time $t$) to the right and to the left respectively, $t = t_+ + t_-$. Depending on the memory parameters $\lambda$ and $\mu$, the present model recovers two previous studied dynamics. For $\mu = 1$, the urn-like dynamics of Ref. [40] is recovered, while for $\lambda = 0$ the elephant model arises [29] (see Refs. [30, 31] where this model is written in terms of the number of transitions $t_\pm$).

The parameter $\lambda$ allows to control the degree or “intensity” of the memory effects. In fact, in the limit $\lambda \rightarrow \infty$ a memoryless dynamics is recovered. On the other hand, the role of the parameter $\mu$ is to weight the two contributions $t_\pm$. For $\mu \geq 1/2$, the next jump probability is correlated (anticorrelated) with the previous trajectory. This feature can be lighted by using that $t_\pm = t_+ - t_-$, which implies

$$t_\pm = t \pm \frac{x_t}{2}. \tag{4}$$

Hence, Eq. (3) can be rewritten as

$$\mathcal{T}(\sigma_1, \ldots, \sigma_{\nu+1} | \sigma_1 = \pm 1) = \frac{\lambda q_{\pm} + (t \pm \alpha x_t)/2}{t + \lambda}, \tag{5}$$

where for shortening the expression we defined the parameter $\alpha = 2\mu - 1$. The previous equation say us that
the previous randomness is absent. These results were demonstrated in Ref. [44] by analyzing weak ergodicity breaking in globally correlated finite systems, which in contrast to diffusive ones are endowed with a stationary state [45].

From the previous limiting behaviors, it becomes of interest to determine the stationary transition probabilities for the present model. Denoting $T^{\pm}_{t} = T(\sigma_{1}, \cdots, \sigma_{t}|\sigma_{t+1} = \pm 1)$, these quantities are $T^{\pm}_{\infty} = \lim_{t \to \infty} T^{\pm}_{t}$, which from Eq. (2) can be written as

$$T^{\pm}_{\infty} = \lim_{t \to \infty} \frac{\lambda q_{\pm} + \mu t_{\pm} + (1 - \mu) t_{\mp}}{t + \lambda}, \quad (7a)$$

$$= \mu \lim_{t \to \infty} t_{\pm} + (1 - \mu) \lim_{t \to \infty} \frac{t_{\mp}}{t}. \quad (7b)$$

In this expression, $\lim_{t \to \infty} t_{\pm}/t$ are the asymptotic fraction of right-left transitions. Consistently, these values must to coincide with the asymptotic transition probabilities, that is, $T^{\pm}_{\infty} = \lim_{t \to \infty} t_{\pm}/t$. Hence, the previous equation leads to

$$T^{\pm}_{\infty} = T^{\pm}_{\infty} = \frac{1}{2}, \quad 0 \leq \mu < 1, \quad (8)$$

while for $\mu = 1$ none condition for $T^{\pm}_{\infty}$ is obtained. In fact, in this case $T^{\pm}_{\infty} = f_{\pm}$, where $f_{\pm}$ are Beta random variables whose probability density is $P(f_{\pm}) = N^{-1} f_{\pm}^{\lambda_{\pm} - 1} f_{\nicefrac{1}{2} - \lambda_{\pm} - 1}$, with $N = \Gamma(\lambda_{+})\Gamma(\lambda_{-})/\Gamma(\lambda)$ [46, 48], where $\lambda_{\pm} \equiv \lambda q_{\pm}$. Notice that Eq. (8) is equivalent to the probability transitions of a memoryless unbiased discrete diffusion process. This result is independent of the parameter $\lambda$ and the weights $q_{\pm}$.

In order to check the result (8) in Fig. 1, we plot the time dependence of the transition probabilities for different values of $\mu$, jointly with the corresponding realizations of the centered walker displacement $x_{t}$ [Eq. (2)]. For $\mu = 0.2$ and $\mu = 0.6$ the transition probabilities converges in a fast way to $1/2$. Consistently, the realization of $x_{t}$ looks like a standard diffusion process. On the other hand, for $\mu = 0.8$ the same asymptotic values ($1/2$) are attained. Nevertheless, the convergence is much slower. In fact, in a small time scale $T^{\pm}_{t}$ seem to attain stationary random values, property characteristic of the case $\mu = 1$ (see Fig. 1(d) in Ref. [44]). Due to this feature, the gap between $T^{+}_{t}$ and $T^{-}_{t}$ drives the walker in one single direction, property clearly seen in the trajectory of $x_{t}$.

**B. Power-law convergence to the stationary transition probabilities**

The plots of Fig. 1 are consistent with the asymptotic values defined by Eq. (8). On the other hand, the rate at which these values are attained strongly depend of $\mu$. In order to characterize this property we introduce the difference $\delta T_{t}$ between the transition probabilities

$$\delta T_{t} = T^{+}_{t} - T^{-}_{t} = \frac{\lambda \delta q_{+} + \alpha(t_{+} - t_{-})}{t + \lambda}, \quad (9)$$

**A. Stationary transition probabilities**

For $\mu = 1$, it is known that in the asymptotic regime ($t \gg \lambda$) the transition probability [Eq. (3)] becomes a random variable characterized by a Beta probability density [46]. On the other hand, for $\lambda = 0$ (elephant model) the previous randomness is absent. These results were
result that follows straightforwardly from Eq. (3), where \( \delta q \equiv (q_+ - q_-) \), \( \delta T_{t=0} = \delta t \), and as before \( \alpha = (2\mu - 1) \). Notice that \( \delta T_t \) can be read as the instantaneous drift felt by the walker.

In Figs. 2(a) and 2(b) we plot \( \sqrt{\delta T_t^2} \). We find that, after a initial transient, independently of the parameter values of the model, \( \sqrt{\delta T_t^2} \approx c/t^{\beta} \), where \( c \) and \( \beta \) change in each realization. As shown by the figures, this behavior is valid over many decades of time. For \( \mu < 1/2 \) (not shown) the signal \( \sqrt{\delta T_t^2} \) become more noisy [see Fig. 1(a)] but a power-law decay behavior is also present.

In order to characterize the previous decay behaviors in Fig. 2(c) and 2(d) we plot \( \langle \delta T_t \rangle \) and \( \langle \delta T_t^2 \rangle \) for the same value of \( \mu \). \( \langle \cdots \rangle \) denotes average over an ensemble of realizations. For both quantities we find that asymptotically a power-law fitting always apply

\[
\langle \delta T_t \rangle \simeq \frac{c_1}{t^{1/2}}, \quad \langle \delta T_t^2 \rangle \simeq \frac{c_2}{t^{\beta_2}}.
\]

The time scale where this fitting start to be valid strongly depend on \( \mu \). Nevertheless, when achieved, we found that the scaling exponents \( \beta_1 \) and \( \beta_2 \) only depend on the parameter \( \mu \).

C. Memory-induced transition

A (memory-induced) transition is found when analyzing the dependences of the scaling exponents \( \beta_1 \) and \( \beta_2 \) with the parameter \( \mu \). They can be determinate in a numerical way [see Figs. 2(c) and 2(d)], results shown in

\[
\beta_1 = 2(1 - \mu),
\]

while for the second moment as \( (\mu \neq 1/2, \mu \neq 1) \)

\[
\beta_2 = \begin{cases} 
4(1 - \mu) & \text{if } 3/4 \leq \mu < 1 \\
1 & \text{if } 0 \leq \mu \leq 3/4
\end{cases}.
\]

While \( \beta_1 \) presents a monotonous linear behavior, the dependence of \( \beta_2 \) with \( \mu \) suffer a transition at \( \mu = 3/4 \). This is an intrinsic property of the correlation mechanism defined by the transition probability \( [3] \), which in turn is independent of the parameters \( \lambda \) and \( q_\pm \).

For \( \beta_2 \) two values of \( \mu \) are not described by Eq. (12).

First, for \( \mu = 1 \), \( \delta T_t \) converges to \( \lim_{t \to \infty} \delta T_t = f_+ - f_- \), where \( f_\pm \) are Beta random variables that realization to realization satisfy \( f_\pm \neq 1/2 \) [10]. Therefore, in this case the exponent \( \beta_2 \) [Eq. (12)] loses its meaning.

For \( \mu = 1/2 \), from Eq. (3) [with \( \alpha = 0 \)] it follows the deterministic behavior \( \delta T_t = \lambda \beta q/(t + \lambda) \), leading to \( \beta_1 = 1 \) and \( \beta_2 = 2 \). Hence, this value of \( \beta_2 \) is not covered by the fitting [12]. Numerically, we checked that this is the only exception. Consistently, we found that around this point \( (\mu \approx 1/2) \) the \( t^{\beta_2} \) power-law decay of \( \langle \delta T_t^2 \rangle \) occurs at higher times. These properties and results are supported by analytical calculations presented in next sections.

D. Relation between average transition fluctuations and walker ensemble properties

The memory-induced transition defined from the asymptotic decay of \( \langle \delta T_t^2 \rangle \) is analytically demonstrated in the next section by studying the ensemble properties of the random walker trajectories. In fact, each trajectory of \( \delta T_t \) [Eq. (9)], given that \( x_t = t_++ - t_- \), can be written as \( \delta T_t = (\lambda \beta q + \alpha x_t)/(t + \lambda) \). Hence,

\[
\langle \delta T_t \rangle = \frac{\lambda \beta q + \alpha \langle x_t \rangle}{t + \lambda},
\]

Fig. 3. The scaling exponents can be fit as

\[
\beta_1 = 2(1 - \mu),
\]

while for the second moment as \( (\mu \neq 1/2, \mu \neq 1) \)

\[
\beta_2 = \begin{cases} 
4(1 - \mu) & \text{if } 3/4 \leq \mu < 1 \\
1 & \text{if } 0 \leq \mu \leq 3/4
\end{cases}.
\]

While \( \beta_1 \) presents a monotonous linear behavior, the dependence of \( \beta_2 \) with \( \mu \) suffer a transition at \( \mu = 3/4 \). This is an intrinsic property of the correlation mechanism defined by the transition probability \( [3] \), which in turn is independent of the parameters \( \lambda \) and \( q_\pm \).

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while the second moment becomes

\[
\langle \delta T_t^2 \rangle = \langle \delta T_t \rangle^2 + \frac{\alpha^2 \langle (x_t^2) -(x_t)^2 \rangle}{(t + \lambda)^2}. \tag{14}
\]

Furthermore, defining \( \widetilde{\delta T}_t = \delta T_t - \langle \delta T_t \rangle \), it follows that

\[
\langle \delta T_{t+\tau} \delta T_t \rangle = \frac{\alpha^2 \langle (x_{t+\tau}) - (x_t) \rangle \langle (x_{t+\tau}) \rangle}{(t + \tau + \lambda)(t + \lambda)}. \tag{15}
\]

The previous relations demonstrate that the statistical properties of \( \delta T_t \) and those of \( x_t \) can be put in one-to-one correspondence. This relation is also valid for the variable \( \sigma_t \), which can be read as the “walker velocity.” Given that \( T_t^\pm \) gives the probability for \( \sigma_{t+1} = \pm 1 \), it follows

\[
\langle \sigma_{t+1} \rangle = \langle \delta T_t \rangle, \quad \langle \sigma_{t+1}^2 \rangle = 1. \tag{16}
\]

On the other hand, defining the centered velocity \( \bar{\sigma}_t \equiv \sigma_t - \langle \sigma_t \rangle \), its correlation reads

\[
\langle \bar{\sigma}_{t+\tau+1} \bar{\sigma}_{t+1} \rangle = \langle \delta T_{t+\tau} \delta T_t \rangle. \tag{17}
\]

Therefore, the properties of \( \sigma_t \) can also be determined from the ensemble behavior of \( x_t \).

### III. ENSEMBLE PROPERTIES

In this section we study the ensemble properties (moments and correlation) of the random walk defined by Eq. (6). They not only determine the moments of the transition probability difference \( \delta T_t \), but also set the behavior of the time-averaged observables (next section). The probability of finding the walker at a given time is also obtained.

The walker statistical moments can be obtained in an exact analytical way by introducing the double characteristic function

\[
Q(k_1, t; k_2, \tau) \equiv \langle \exp[i(k_1 x_t + k_2 x_{t+\tau})] \rangle. \tag{18}
\]

In Appendix B we obtain an explicit recurrence relation for this object. As usual, recursive relations for the moments follows by differentiation with respect to \( k_1 \) and \( k_2 \). Below, we also provide the corresponding exact solutions. Numerical simulations support the following analytical results.

#### A. First moment

For the first moment, it follows the recursive relation

\[
\langle x_{t+1} \rangle = \langle x_t \rangle \left[ 1 + \frac{\alpha}{t + \lambda} \right] + \frac{\lambda}{t + \lambda} \delta q, \tag{19}
\]

\( \alpha = 2\mu - 1 \). Notice that for \( \lambda \neq 0 \), the factor \( \delta q = q_+ - q_- \) can be read as an external bias that drift the average dynamics.

![Graph](image-url)

**FIG. 4**: Ensemble moments of the random walker. (a) First moment \( \langle x_t \rangle \). (b) Second moment \( \langle x^2_t \rangle \). The full lines correspond to the exact expressions (20) and (21) respectively. In (a) we take \( q_+ = 1, q_- = 0 \), while in (b) \( q_+ = q_- = 1/2 \). In all cases \( \lambda = 2 \). The value of \( \mu \) is indicated in each curve. Numerical results (circles) were obtained from an average over 5 \times 10^3 realizations.

The solution of the previous equation is

\[
\langle x_t \rangle = \delta q \frac{\Gamma(\lambda + 1)}{\alpha \Gamma(\alpha + \lambda)} \frac{\Gamma(\alpha + \lambda + t)}{\Gamma(\lambda + t)} - \lambda. \tag{20}
\]

where the Gamma function. For \( \mu = 1/2 \), that is \( \alpha = 0 \), it follows \( \langle x_t \rangle = \delta q \lambda \psi(\lambda + t) - \psi(\lambda) \), where the digamma function is defined as \( \psi(z) = (d/dz) \ln[\Gamma(z)] \). At \( \mu = 1 \), \( \langle x_t \rangle = \delta q t \).

In the long-time limit, \( t \gg \lambda \), by using the approximation \( \Gamma(z + \nu)/\Gamma(z) \approx z^\nu \) valid for \( z \to \infty \), from Eq. (20) we get the asymptotic behaviors

\[
\langle x_t \rangle \approx \begin{cases} 
\frac{\delta q \lambda \ln(t)}{(2\mu-1) \Gamma(2\mu-1+\lambda)} & \mu > 1/2, \\
\delta q \lambda / (1-2\mu) & \mu = 1/2, \\
\frac{\delta q \lambda}{(1-2\mu)} & \mu < 1/2.
\end{cases} \tag{21}
\]

By taking into account these asymptotic behaviors, from Eqs. (13) and (20) it is possible to confirm the fitting for \( \beta_1 \) given by Eq. (11).

For \( \delta q = q_+ - q_- \neq 0 \) the first moment grows indefinitely with time when \( \mu \geq 1/2 \) and saturates to a constant value when \( \mu < 1/2 \). In order to check these properties, in Fig. 4(a) we plot \( \langle x_t \rangle \) obtained from an ensemble of stochastic realizations such as those shown in Fig. 1. Theoretical and numerical results are indistinguishable in the scale of the plots.

The different behaviors shown in Fig. 4(a) are a consequence of the power-law decay of \( \delta T_t \) to its stationary value, \( \lim_{t \to \infty} \delta T_t = 0 \) [see Eq. (5)]. In fact, from the dependence of the exponent \( \beta_1 \) with parameter \( \mu \) [Eq. (11)] and the relation between \( \langle \delta T_t \rangle \) and \( \langle x_t \rangle \) [Eq. (13)], which can be rewritten as \( \alpha \langle x_t \rangle = \langle \delta T_t \rangle (t + \lambda) - \delta q t \), it follows that the first moment grows in time only for \( \mu \geq 1/2 \).
B. Second moment

For the second moment, we get the recursive relation

$$\langle x_{t+1}^2 \rangle = \langle x_t^2 \rangle \left[ 1 + \frac{2\alpha}{t + \lambda} \right] + 1 + 2\delta q \frac{\lambda}{t + \lambda} \langle x_t \rangle, \quad (22)$$

whose solution is given by

$$\langle x_t^2 \rangle = \frac{1}{2\alpha - 1} \left[ \Gamma(\lambda + 1) \Gamma(2\alpha + \lambda + t) \Gamma(2\alpha + \lambda + 3) - (n + \lambda) \right] + \varphi(t) \quad (23)$$

where $2\alpha - 1 = 4\mu - 3$. The bracket term gives the solution when $\delta q = 0$, while $\varphi(t)$ takes into account the contributions proportional to $\delta q \neq 0$,

$$\varphi(t) \equiv \delta q^2 \lambda^2 \left\{ 1 + \frac{\Gamma(\lambda + 1)}{4\alpha - 1} \left[ \frac{\Gamma(\lambda + 2\alpha + t)}{\Gamma(\lambda + 2\alpha + 3)} - 2\Gamma(\lambda + \alpha + t) \right] \right\}. \quad (24)$$

In the long time limit, $t \gg \lambda$, by using the approximation $\Gamma(z + \mu)/\Gamma(z) \simeq z^{\mu}$ valid for $z \rightarrow \infty$, in the case $\delta q = 0$ we get the asymptotic behaviors

$$\langle x_t^2 \rangle \simeq \begin{cases} 
\frac{1}{4\mu - 1} \Gamma(4\mu - 2\lambda + 1) \lambda^{4\mu - 2}, & \mu > 3/4, \\
\frac{t}{4\mu}, & \mu < 3/4.
\end{cases} \quad (25)$$

By introducing these behaviors in Eq. (14) it is possible to recover analytically the fitting for $\beta_2$, Eq. (14). In fact, corrections proportional to $\langle x_t \rangle$ and $\varphi(t)$, Eqs. (21) and (24) respectively, gives higher order (inverse) power-law corrections that can be disregarded in the asymptotic regime.

We notice that $\langle x_t^2 \rangle$ [Eq. (25)] develops a transition between a normal diffusive behavior ($\mu < 3/4$) to a superdiffusive one ($\mu > 3/4$). These features are related to the exponent $\beta_2$ of the power-law decay of $\langle \delta T^2 \rangle$, Eqs. (10) and (14). In Fig. 4(b) we plot $\langle x_t^2 \rangle$ for different values of $\mu$. Numerical simulations confirm the theoretical predictions. On the other hand, at $\mu = 1$, a ballistic behavior is obtained asymptotically, $\langle x_t^2 \rangle = t(t + \lambda)/(1 + \lambda)$, result derived in Ref. [10].

C. Correlation

The correlation of the random walker is defined as

$$C_{t,\tau}^x \equiv \langle x_t x_{t+\tau} \rangle, \quad (26)$$

with initial condition $C_{t,0}^x = \langle x_t^2 \rangle$. From the double characteristic function [Appendix B], it is possible to obtain the recursive relation

$$C_{t,\tau+1}^x = C_{t,\tau}^x \left[ 1 + \frac{\alpha}{t + \tau + \lambda} \right] + \frac{\lambda \delta q}{t + \tau + \lambda} \langle x_t \rangle. \quad (27)$$

\[FIG. 5: Normalized correlation $\langle x_t x_{t+\tau} \rangle / \langle x_t^2 \rangle$ as a function of the difference time $\tau$. The full lines correspond to the exact result Eq. (28). In all cases we take $t = 10$, $q_+ = q_- = 1/2$, and $\lambda = 2$. The value of $\mu$ is indicated in each curve. Numerical results (circles) were obtained from an average over $2 \times 10^4$ realizations.\]

Its solution is

$$C_{t,\tau}^x = \left[ \langle x_t^2 \rangle + \frac{\lambda \delta q}{\alpha} \langle x_t \rangle \right] \Phi(t, \tau) - \frac{\lambda \delta q}{\alpha} \langle x_t \rangle, \quad (28)$$

where $\langle x_t \rangle$ and $\langle x_t^2 \rangle$ follow from Eqs. (21) and (28) respectively. The auxiliary function $\Phi(t, \tau)$ reads

$$\Phi(t, \tau) \equiv \frac{\Gamma(\lambda + t + \tau)}{\Gamma(\alpha + \lambda + t + \tau)} \frac{\Gamma(\lambda + \alpha + t + \tau)}{\Gamma(\alpha + \lambda + t + \tau)}. \quad (29)$$

For $\mu = 1/2$, that is $\alpha = 0$, the solution of the recursive relation (27) reads $C_{t,\tau}^x = \langle x_t^2 \rangle + \langle x_t \rangle \lambda \delta q [\psi(\lambda + t + \tau) - \psi(t + \tau)]$, where the digamma function is defined as $\psi(z) = (d/dz) \log \Gamma(z)$. In the limit $\mu \rightarrow 1$, $C_{t,\tau}^x$ is given by Eq. (28) with $\Phi(t, \tau) = [1 + \tau/(t + \lambda)]$.

In the asymptotic regime, by using the approximation $\Gamma(z + \mu)/\Gamma(z) \simeq z^{\mu}$, valid for $z \rightarrow \infty$, it follows $\Phi(t, \tau) \simeq [1 + \tau/(t + \lambda)]^\alpha$, which leads to

$$C_{t,\tau}^x \simeq \left[ \langle x_t^2 \rangle + \frac{\lambda \delta q}{\alpha} \langle x_t \rangle \right] \left( 1 + \frac{\tau}{t} \right)^\alpha - \frac{\lambda \delta q}{\alpha} \langle x_t \rangle. \quad (30)$$

Thus, in the asymptotic regime the correlation depends on the quotient $\tau/t$, showing the strong non-stationarity property of the diffusion process. A similar result is also valid for Lévy walks [24]. On the other hand, for $\mu \gtrless 1/2$ (equivalently $\alpha \gtrless 0$) $C_{t,\tau}^x$ increases (decreases) with $\tau$. This result is consistent with the correlation-anticorrelation mechanism introduced by $\mu$. In order to check these results, in Fig. (5) we plot the normalized correlation $\langle x_t x_{t+\tau} \rangle / \langle x_t^2 \rangle$ as a function of the interval $\tau$. Numerical simulations and analytical results are indistinguishable in the scale of the plots.

D. Joint-probability evolution

By Fourier inversion, $k_1 \rightarrow y$, $k_2 \rightarrow x$, the double characteristic function (18) also allows us to obtain the
joint probability $P(y, t; x, \tau)$ of observing the walker at position $y$ at time $t$ and at position $x$ at time $t + \tau$. From Eq. (15) we get

$$P(y, t; x, \tau + 1) = W_{i+}^\tau(x - 1)P(y, t; x - 1, \tau)$$

$$+ W_{i-}^\tau(x + 1)P(y, t; x + 1, \tau),$$

where the transition probabilities are

$$W_{i, \tau}^\pm(x) = \frac{1}{2} \left[ 1 \pm \frac{1}{t + \tau + \lambda} (ax + \lambda \delta q) \right].$$

Hence, the dynamics as a function of the interval $\tau$ develops aging [41–43], that is, here the transition probabilities $W_{i, \tau}^\pm(x)$ depend on the starting time $t$ with a power-law dependence. This property is closely related with the asymptotic power-law behavior of the walker correlational correlations, Eq. (30). These features are absent in the memoryless regime, $\lim_{\tau \to \infty} W_{i, \tau}^\pm(x) = q_{\pm}$.

In a continuous limit, where both the jump length $\delta x$ and the time interval $\delta t$ between consecutive transitions become small, the previous master equation leads to the Focker-Planck equation

$$\frac{\partial}{\partial \tau} P(y, t; x, \tau) = D \frac{\partial^2}{\partial x^2} P(y, t; x, \tau)$$

$$- \frac{\alpha}{t + \tau + t_\lambda} \frac{\partial}{\partial x} [xP(y, t; x, \tau)]$$

$$- \frac{t_\lambda}{t + \tau + t_\lambda} \frac{\partial}{\partial x} P(y, t; x, \tau),$$

where the parameters are $D = (1/2)\delta x^2/\delta t$, $V \equiv (q_+ - q_-)(\delta x/\delta t)$, and $t_\lambda \equiv \lambda \delta t$. Interestingly, this equation has the form of a diffusion process in a time- and age-(power-law) dependent inverted parabolic potential superimposed with a time-dependent linear drift. Notice that around $\mu = 1/2$ the potential is inverted, property related to the correlation-anticorrelation mechanism introduced by the parameter $\mu$.

IV. TIME-AVERAGED OBSERVABLES

Here we study the ergodic properties of the walker defined by Eq. (8). Given the discrete nature of the dynamics, the definition of the time-averaged moments is given by

$$\delta_\kappa(t, \Delta) = \frac{\sum_{t'=0}^{t-\Delta} [x(t') + \Delta) - x(t')]^\kappa}{t - \Delta}.$$  (33)

Here, we have used the translational invariance of Eq. (11), which allow to write the definition in terms of $x(t)$ [Eq. (8)]. The definitions of ergodicity and ergodicity breaking are those quoted in the Introduction.

A. Infinite-time trajectories

The walker ensemble properties are mainly determined by the decay behavior of the transition probability. In contrast, for infinite-time trajectories, $\lim_{t \to \infty} \delta_\kappa(t, \Delta)$, the time-averaged moments are settled by the asymptotic behavior of the transition probabilities. In fact, taking higher times $t$ in Eq. (33), the relevant walker transitions are those governed by the asymptotic value $T_\infty = 1/2$ [Eq. (5)]. Consequently, we expect that along a single trajectory the time averaged moments of $x(t)$ converge to those of an undriven normal diffusion process. Hence, it follows

$$\lim_{t \to \infty} \delta_1(t, \Delta) = 0, \quad 0 \leq \mu < 1,$$

while the time-averaged mean square displacement reads

$$\lim_{t \to \infty} \delta_2(t, \Delta) = \Delta, \quad 0 \leq \mu < 1.$$  (35)

For normal diffusion, these expressions follows straightforwardly from the definition (33) and by considering independent random walker increments.

The previous results imply that the process is not ergodic. In fact, $\langle x_\Delta \rangle \neq \lim_{t \to \infty} \delta_1(t, \Delta)$, and $\langle x_\Delta^2 \rangle \neq \lim_{t \to \infty} \delta_2(t, \Delta)$. These inequalities remain valid even for $\Delta \gg 1$ [see Eqs. (21) and (26) respectively] and are valid for any value of $\lambda$, $\mu$, and $q_{\pm}$.

Interestingly, while the bias induced by $\delta q = q_+ - q_-$ drives the ensemble behavior [see Eq. (21)], the time-averaged response $\lim_{t \to \infty} \delta_1(t, \Delta)$ vanishes (dye out) asymptotically [Eq. (34)]. In consequence, it is not possible to ask about an Einstein fluctuation dissipation relation formulated with time-averaged observables (infinite-time trajectories). This unusual property relies on both the power-law decay of the transitions probabilities and their stationary on-half values.

While for the second moment time and ensemble averages are always different, when $\mu < 3/4$ [diffusive-like regime, Eq. (25)] they differ only in terms of a constant, $\lim_{t \to \infty} \delta_2(t, \Delta) \Delta (3 - 4\mu) \sim (x_\Delta^2)$. In contrast, for Levy walks this ultraweak ergodicity breaking is valid in the superdiffusive regime.

In order to check these results, in Fig. 6 we plot a set of realizations corresponding to $\delta_2(t, \Delta)$ for different values of $\mu$. Around the origin all realizations approach the limit defined by Eq. (34). We checked that by increasing the measurement time $t$, the departure with respect to the linear behavior consistently occurs at larger delay times $\Delta$.

B. Randomness in finite-time trajectories

The previous results are valid for any (finite) value of $\lambda$ and $0 \leq \mu < 1$. When $\mu = 1$, the model reduces to the urn-like dynamics of Ref. (40). Thus,

$$\lim_{t \to \infty} \delta_1(t, \Delta) = (f_+ - f_-)\Delta,$$

while the second time-averaged moment reads

$$\lim_{t \to \infty} \delta_2(t, \Delta) = (f_+ - f_-)^2 \Delta^2 + [1 - (f_+ - f_-)^2] \Delta.$$  (37)
In (b) (25 trajectories) \( \lim_{\Delta \to \infty} \delta(t, \Delta) = \Delta \). The gray lines (light blue lines) correspond to the infinite trajectory limit, \( \lim_{t \to \infty} \delta(t, \Delta) = \Delta \). In (a) (25 trajectories) we take \( \mu = 0.2 \). In (b) (25 trajectories) \( \mu = 0.8 \). In (c) a few of the previous trajectories are shown in the time scale posterior to the linear regime. In (d) (5 trajectories) \( \mu = 0.9 \). In all cases, we take \( \lambda = 2 \), \( q_+ = q_- = 1/2 \), and \( t = 4 \times 10^4 \).

where \( f_\pm \) are Beta random variables, with \( f_+ + f_- = 1 \) (see Eqs. (32) and (33) in Ref. [40] where these results were derived). The transition between these scaling and those defined by Eqs. (34) and (35) can be described by analyzing the behavior of the time-averaged moments obtained with finite-time trajectories.

In the plots of Fig. 6 we observe that, even when \( \Delta \ll t \), beyond the linear regime the scaling of \( \delta_2(t, \Delta) \) can be subdiffusive or superdiffusive. Furthermore, the amplitude of the scaling can also be random [see Fig. 6(c)]. Here, these features are present for all values of \( \mu \). Hence, we associate these effects to the random behavior of the transition probabilities (see Fig. 1 and 2). In fact, independently of the values of the memory parameters \( \mu \) and \( \lambda \), they decay to their stationary values following a power-law behavior with parameters that are intrinsically random [see Eqs. (1) and (10)]. For \( \mu \approx 1 \), all realizations becomes superdiffusive [see Fig. 6(d), \( \mu = 0.9 \)], a consistent property necessary for approaching the scaling defined by Eq. (37).

**C. Ensemble average of finite-time trajectories**

Now we study how the finiteness of single trajectories affects the corresponding average over an ensemble of trajectories. From Eq. (33) the first moment reads

\[
\langle \delta_1(t, \Delta) \rangle = \frac{1}{t-\Delta} \sum_{t'=0}^{t-\Delta} (x_{t'+\Delta}) - \langle x_{t'} \rangle, \tag{38}
\]

while the second one can be written as

\[
\langle \delta_2(t, \Delta) \rangle = \frac{1}{t-\Delta} \sum_{t'=0}^{t-\Delta} (x_{t'}^2) + (x_{t'}^2) - 2(x_{t'+\Delta} x_{t'}). \tag{39}
\]

Given the exact analytical expressions for \( \langle x_t \rangle \) [Eq. (20)], \( \langle x_t^2 \rangle \) [Eq. (23)], and the correlation \( \langle x_{t'+\Delta} x_{t'} \rangle \) [Eq. (25)], we can also evaluate these objects in an exact way. Nevertheless, they cannot be expressed in terms of general simple expressions. Only for special values one get simpler ones. For example, for \( \mu = 1 \) Eq. (38) becomes

\[
\langle \delta_1(t, \Delta) \rangle = \delta q \Delta \left[ 1 + \frac{1}{t-\Delta} \right], \quad \mu = 1. \tag{40}
\]

Taking \( \delta q = q_+ - q_- = 0 \), the mean square displacement [Eq. (39)] reads

\[
\langle \delta_2(t, \Delta) \rangle = \frac{\Delta (\Delta + \lambda)}{1 + \lambda} \left[ 1 + \frac{1}{t-\Delta} \right], \quad \mu = 1. \tag{41}
\]

In the limit \( t \to \infty \), these expressions correspond to the average over realizations of Eqs. (36) and (37) (see Ref. [40]). On the other hand, the finite-time effects are given by the contributions proportional to \( 1/(t-\Delta) \).

For \( \mu < 1 \), given that not simple analytical expression can be obtained, in Appendix C we introduce a set of approximations that allow to obtaining the asymptotic behavior \( (t \gg \Delta) \) of the exact expressions Eqs. (35) and (39). We get

\[
\langle \delta_1(t, \Delta) \rangle \sim \delta q c_0 \left\{ \frac{2 \mu \Delta}{\Gamma(1-\mu)} - \frac{1}{t} [(\Delta + \lambda)^{2\mu} - \lambda^{2\mu}] \right\}, \tag{42}
\]

where \( c_0 = \Gamma(\lambda+1)/[2\mu \Gamma(\alpha+\lambda)] \). Consistently with Eq. (34), \( \langle \delta_1(t, \Delta) \rangle \) vanishes when \( t \to \infty \). This regime is approached following a power-law behavior. In fact, for \( \mu < 1/2 \), \( \langle \delta_1(t, \Delta) \rangle \sim \Delta^{2\mu}/t \), while for \( \mu > 1/2 \), \( \langle \delta_1(t, \Delta) \rangle \sim \Delta/(t\Delta^{1-\mu}) \).

Taking \( \delta q = q_+ - q_- = 0 \), for the mean squared displacement we obtain

\[
\langle \delta_2(t, \Delta) \rangle \sim \Delta + \frac{\Delta^2}{t} \left[ a + b \ln \left( \frac{\Delta}{t} \right) \right] + c \frac{\Delta^2}{t^{3(1-\mu)}} + d \frac{\Delta^{4\mu-1}}{t}, \tag{43}
\]

where \( a, b, c, \) and \( d \) are constants given in Appendix C. The factor proportional to \( d \) only contributes for \( \mu > 1/2 \).

In Eq. (43), the deviations with respect to the linear behavior (35) change around \( \mu = 3/4 \). For \( \mu < 3/4 \), the dominant terms are those proportional to \( a \) and \( b \), while for \( \mu > 3/4 \) are those proportional to \( c \) and \( d \). In fact,
the quadratic contribution $\Delta^2$ dominates at $\mu = 1$, which approximate the exact behavior \(^{(11)}\).

In contrast to Levy walks (see Eq. (18) in Ref. \([24]\)), by comparing the asymptotic behaviors of $\langle \delta_1(t, \Delta) \rangle$ [Eq. (42)] and $\langle \delta_2(t, \Delta) \rangle$ [Eq. (43)] we conclude that, for finite-time trajectories, it is not possible to establishing a simple relation between both objects (Einstein-like relation).

In order to check the previous results, in Fig. 7(a) we plot $\langle \delta_2(t, \Delta) \rangle$ for different times and the same $\mu$. An increasing convergence to the linear regime, $\langle \delta_2(t, \Delta) \rangle \simeq \Delta$, is observed for increasing $t$. In Fig. 7(b) we plot $\langle \delta_2(t, \Delta) \rangle$ for different values of $\mu$. For $\mu < 1/2$ it is a concave function of $\Delta$ while convex for $\mu > 1/2$. In all plots, the numerical results, the exact result \([39]\) and the approximation \([43]\) are indistinguishable in the scale of the graphs.

\section{V. Summary and Conclusions}

The studied model consists of a diffusive walker whose successive jumps depend on the whole previous history of transitions [Eq. (3)]. The second moment develops a randomness that appears in both the scaling exponents and their amplitudes (Fig. 6). The vanishing of the first time-averaged moment implies that the dynamics is (asymptotically) insensitive to the bias introduced by the characteristic parameters. On the other hand, in the diffusive regime an ultraweak ergodicity breaking phenomenon occurs, that is, for the second moment ensemble and time-averages only differ by a constant.

For finite-time trajectories, the time-averaged moments develop a randomness that appears in both the scaling exponents and their amplitudes (Fig. 6). This effect is induced by the intrinsic randomness of the power-law decay of the transition probabilities (Fig. 1). Departure between ensemble averages performed with finite-time trajectories (Fig. 7) and the corresponding infinite-time limit are also governed by power-law behaviors [Eqs. (42) and (43)]. None simple relation can be established between the mean asymptotic behaviors of the first two time-averaged moments (driven and undriven cases).

The studied model recovers many features that also arise in Levy walks. While their time-averaged properties are not equivalent, the present results demonstrate that many properties of anomalous diffusive processes can also be recovered with simple globally correlated dynamics.

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\section{Appendix A: Probability of sojourn times}

Here, we obtain the probability of the sojourn times, that is, the time intervals during which the walker moves in the same direction. Equivalently, they correspond to the time during which the “velocity” is the same. Given that the walker at time $t$ performed $t_\text{right}$ right-left transitions, the probability of performing $k$ successive jumps to the right, from the definition \([3]\), is given by

$$P_t(k) = \prod_{i=0}^{k-1} \frac{\lambda q_+ + \mu(t_+ + i) + (1 - \mu)t_-}{t + i + \lambda} \times \frac{\lambda q_- + \mu t_- + (1 - \mu)(t_+ + k)}{t + k + \lambda}.$$  \hspace{1cm} (A1)

The last term takes into account the beginning of a sojourn time with transitions in the opposite direction. By using the property of the Gamma function: $\Gamma(n + z)/\Gamma(z) = z(1 + z)(2 + z) \cdots (n - 1 + z)$, the pre-
Various equation becomes
\[
P_t(k) = \frac{\Gamma(t + \lambda)}{\Gamma(t + \lambda + k + 1)} [\tau_\pm + (1 - \mu)k]\mu^k \frac{\Gamma(k + t\tau_{+}/\mu)}{\Gamma(\tau_{+}/\mu)},
\]
where for shortening the expression we introduced the parameters
\[
\tau_{\pm} \equiv \lambda q_{\pm} + \mu t_{\pm} + (1 - \mu)t_{+} = \lambda q_{\pm} + \frac{1}{2}(t \pm \alpha x_t), \quad (A3)
\]
\[
\alpha = (2\mu - 1).
\]
The last equality straightforwardly follows from the relation \[(4)\]. In this way, \(P_t(k)\) depends on which time and which position the sojourn interval begins. The structure of this dependence is simpler in the asymptotic regime \(t \gg \lambda\). By using the Gamma function property \(\Gamma(z + v)/\Gamma(z) \simeq z^v\) valid for \(z \to \infty\), Eq. \[(A2)\] becomes
\[
P_t(k) \simeq \frac{1}{(t + \lambda)^{k+1}} [\tau_\pm + (1 - \mu)k]^{t\tau_{+}}. \quad (A4)
\]
Approximating \(\tau_{\pm} \simeq tw_{\pm}\) for \(t \gg \lambda\), where
\[
w_{\pm} \equiv \frac{1}{2}[1 \pm \alpha x_t/t],
\]
\((w_+ + w_- = 1)\) we obtain the final expression
\[
P_t(k) \simeq w_+^k w_-.
\]
Through the corrections to this expression are of order \((1/t)\). On the other hand, we notice that \(w_+\) are the asymptotic transition probabilities defined in Eq. \[(B)\].

For \(\mu \geq 1/2\), for increasing (decreasing) \(x_t\) the probability \(P_t(k)\) increase (decrease). That is, if the particle attains larger (smaller) values of \(x_t\) the possibility of larger sojourn times in the same direction increase (decrease). This dependence of \(P_t(k)\) with \(x_t\) is confirmed by its values in the boundary \(x_t = \pm t\), and \(x_t = 0\),
\[
P_t(k) \simeq \begin{cases} 
(1 - \mu)\mu^k, & x_t = +t, \\
\left(\frac{1}{2}\right)^{k+1}, & x_t = 0, \\
\mu(1 - \mu)^k, & x_t = -t,
\end{cases}
\]
which in turn also clarify the role of the parameter \(\mu\) in the walker realizations. Interestingly, in spite of the previous feature the average sojourn time is finite \(\langle k \rangle \equiv \sum_{k=0}^{\infty} k P_t(k) \simeq w_+/(1 - w_+)\), as well as the second moment, \(\langle k^2 \rangle \equiv \sum_{k=0}^{\infty} k^2 P_t(k) \simeq w_+ (1 + w_+)/(1 - w_+)^2\).

Straightforwardly, the same results apply for the probability of sojourn times in the opposite direction.

**Appendix B: Double characteristic function**

In this Appendix we obtain an exact recursive relation for the double characteristic function
\[
Q(k_1, t; k_2, \tau + 1) = \langle \exp[i(k_1 x_t + k_2 x_{t+\tau})] \rangle.
\]
It is obtained as follows. At time \(\tau + 1\), it can be written as
\[
Q(k_1, t; k_2, \tau + 1) = \left\langle \exp[i(k_1 x_t + k_2 x_{t+\tau})] \right\rangle,
\]
where the change of notation \(Q(k_1, t; k_2, \tau) \to Q_t, \tau\) was introduced for shortening the expression. Furthermore, the random function \(U^\tau\) is defined as
\[
U^\tau_{\pm} = \mu t_{\pm} + (1 - \mu)t_{+}, \quad (B4)
\]
which due to the relation \(t_{\pm} = (t \pm x_t)/2\) \[(B1)\] can be rewritten as \(U^\tau_{\pm} = (t \pm \alpha x_t)/2\). Using that
\[
\frac{\partial Q_t, \tau}{\partial k_2} = i[k x_t, x\exp[i(k_1 x_t + k_2 x_{t+\tau})]],
\]
jointly with the normalization condition \(q_+ + q_- = 1\), after some algebra, from Eq. \[(B3)\] it follows the closed recursive relation
\[
Q_t, \tau+1 = \cos(k_2)Q_t, \tau + i\lambda q_+ \frac{\sin(k_2)}{t + \tau + \lambda} Q_t, \tau + \alpha \frac{\sin(k_2)}{t + \tau + \lambda} \frac{\partial Q_t, \tau}{\partial k_2},
\]
where \(\alpha = (2\mu - 1),\) and \(\delta q = (q_+ - q_-)\). By differentiation with respect to \(k_1\) and \(k_2\) the recursive relations presented in Sec. III follows straightforwardly.

**Appendix C: Approximation for the ensemble time-averaged moments**

Here we show the procedure to obtain the asymptotic behavior of the time-averaged moments \(\langle \delta_n(t, \Delta) \rangle\). Their exact expressions, Eqs. \[(B8)\] and \[(B9)\], have the following structure
\[
F(t, \Delta) = \frac{1}{t - \Delta} \sum_{t'=0}^{t-\Delta} f(t', \Delta).
\]
The goal is to approximate \(F(t, \Delta)\) at large time scales, \(t \gg \Delta\), given that we have an exact expression for \(f(t', \Delta)\)
written in terms of the ensemble moments $\langle x_t \rangle$, $\langle x_t^2 \rangle$, and the correlation $\langle x_{t+\Delta} x_t \rangle$.

Defining the variable $\varepsilon \equiv \Delta/t$, the scaled time $\tau \equiv t'/t$, and $d\tau \equiv 1/t$, the previous general expression can be rewritten as

$$ F(t, \varepsilon) = \frac{1}{1 - \varepsilon} \sum_{\tau=0}^{1-\varepsilon} f(\tau, \varepsilon) d\tau. \quad (C2) $$

In this expression $d\tau = 1/t \ll 1$, and consistently $\tau = t'/t$ can be considered as a real continuous variable. Therefore, we can approximate the sum by an integral,

$$ F(t, \Delta) \approx \frac{1}{1 - \varepsilon} \int_0^{1-\varepsilon} f(\tau, \varepsilon) d\tau. \quad (C3) $$

In addition, given that only the asymptotic regime is of interest, before performing this integral $f(\tau, \varepsilon)$ can be approximated by its asymptotic behavior. Posteriorly, the integral can be expanded in the parameter $\varepsilon$. This procedure leads to the approximations given in Eqs. (42) and (43).

The parameters of Eq. (43) are

$$ a = \frac{(1 - \mu)(1 - 2\mu)}{4\mu - 3} \left( 3 - 2H[2(1 - \mu)] \right), \quad (C4) $$

where $H[x] = \gamma + \psi(x - 1)$, where $\gamma$ is the Euler constant and $\psi(x)$ is the digamma function,

$$ b = \frac{2}{4\mu - 3} \left[ (1 - \mu)(2\mu - 1) \right], \quad (C5) $$

and

$$ c = \frac{(1 - 2\mu)^2}{(4\mu - 3)^2} \frac{\Gamma(1 + \lambda)}{\Gamma(4\mu - 2 + \lambda)}, \quad (C6) $$

while

$$ d = \frac{-1}{(4\mu - 3)^2} \frac{\Gamma(1 + \lambda)}{\Gamma(4\mu - 2 + \lambda)} \times \left[ \frac{1}{4\mu - 1} + \frac{2(1 - 4\mu)}{\sqrt{\pi}} \Gamma(1 - 2\mu) \Gamma(2\mu) \right]. \quad (C7) $$
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