On the Solutions of Einstein Equations with Massive Point Source

P.P. Fiziev

Abstract

We show that Einstein equations are compatible with the presence of massive point particles and find corresponding two parameter family of solutions. They are defined by the bare mechanical mass $M_0 > 0$ and the Keplerian mass $M < M_0$ of the point source of gravity. The global analytical properties of these solutions in the complex plane define a unique preferable radial variable of the problem.

1 Introduction

A clear physical motivation for consideration of massive point particle sources of gravitational field in GR, both electrically neutral and charged ones, can be found as early as in 1962-63 Feynman lectures on gravity [1]. In spite of this fact the problem is still open.

At present the vast majority of relativists do not accept the consideration of point particles in general relativity (GR) as incompatible with Einstein equations (EE) idealization. There are different reasons: some doubts about consistence of the theory of mathematical distributions (like Dirac $\delta$-function $\delta(r)$) with the nonlinear character of EE; the clear understanding of the drastic change of geometry of the Riemannian space-time $M^{(1,3)}\{g_{\mu\nu}\}$ in a vicinity of a point with infinite concentration of energy in it, etc.

On the other hand it is obvious that in Nature very distant objects like stars look like "points" of finite mass and finite luminosity. This fact has a proper mathematical description in Newton theory of gravity in the language of mathematical distributions. A formal mathematical problem is to find a corresponding idealized treatment of such objects in GR, as well, but up to recently no reasonable approach was known.

Here we show that correct mathematical solutions of EE with $\delta(r)$ term in the rhs do exist. Such solutions describe a two parameter family of analytical space-times $M^{(1,3)}\{g_{\mu\nu}\}$ with specific strong singularity at the place of the massive point source with bare mechanical mass $M_0 > 0$ and Keplerian mass $M < M_0$.

The price, one has to pay for this enlargement of standard GR framework, is:

1) To consider metric coefficients $g_{\mu\nu}(x)$ as functions of class $C^0$ of the coordinates $x$. Some of them are to have definite finite jumps in their first derivatives at the place of the point source, necessary to reproduce the $\delta(r)$ term in the lhs of EE from the Einstein tensor $G^\nu_\mu \sim \delta(r)$.

*Department of Theoretical Physics, University of Sofia,Boulevard 5 James Bourchier, Sofia 1164, Bulgaria E-mail: fiziev@phys.uni-sofia.bg
2) To accept the unusual geometry of space-time around the matter point in GR which actually appeared at first in the original Schwarzschild article [2] and has been discussed by Brillouin [3] as early as in 1923. This geometry is essentially different from the geometry around space-time points with finite energy density in them. The global properties of the space-time are different from the ones of popular Hilbert-Droste-Weyl form of the Schwarzschild solution [4].

At present the original Schwarzschild geometry and other similar geometries of space-time are widely ignored in GR. In addition, the historical facts unfortunately are not reproduced in completely correct way in the most of the modern literature on this subject. The important physical consequences, which can be derived using the new solutions of EE, discussed in the present article, can be considered after these solutions will be properly studied.

2 Formulation of the Mathematical Problem

In its proper frame the single point particle with bare mechanical mass \(M_0\), placed at the origin of the standard spherical coordinate system in the 3D Riemannian space \(M(3)\{g_{ij}\} \subset M(1,3)\{g_{\mu\nu}\}\) yields the familiar static metric \[ds^2 = g_{tt}(r) dt^2 + g_{rr}(r) dr^2 - \rho(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2)\]

with three unknown functions \(g_{tt}(r) \geq 0, g_{rr}(r) \leq 0, \rho(r)\) of the radial variable \(r \geq 0\). The variable \(r\) is not defined by the \(SO(3)\) symmetry of the problem, nor by its global-time translation invariance with respect to the group \(T_t(1)\). The only clear thing is that the value \(r = 0\) must correspond to the center of spherical symmetry, \(C\), where the massive matter point is placed.

In contrast, the quantity \(\rho\) has a clear geometrical meaning: it defines the area \(A_\rho = 4\pi \rho^2\) of the sphere, centered at the point \(C\). Physically \(\rho\) defines the luminosity of the objects, which is reciprocal to \(A_\rho\). Therefore we shall refer to \(\rho\) as "a luminosity variable".

At first glance the function \(\rho(r)\) may be chosen in quite arbitrary way, thus fixing the remaining (radial) gauge freedom of the problem – the only one, which is not fixed by symmetry reasons. In this article we shall show that choosing a definite class of functions \(\rho(r)\) one can solve correctly the Einstein equations:

\[G^\mu_\nu = \kappa T^\mu_\nu\quad (2.1)\]

with stress-energy tensor \(T^\mu_\nu \sim M_0 \delta(r)\), which describes a massive point source with bare mass \(M_0\) and mechanical action \(A_{M_0} = -M_0 \int ds\).

It may seems strange that to solve the problem, one needs to fix the class of coordinates by choosing the radial gauge. As we shall see, the proper choice of coordinates is dictated naturally by the boundary conditions. In the problem at hand these conditions are masked in Dirac \(\delta\)-function, which describes in a formal mathematical way the properties of source of gravity and its boundary.

The shortest way to derive the field equation of our specific problem is to perform a restriction on the orbits of the group \(SO(3) \times T_t(1)\) of the total action \(A_{tot} = A_{GR} + A_{M_0}\), where \(A_{GR} = -\frac{1}{16\pi G_n} \int d^4x \sqrt{|g|} R\) is the Hilbert-Einstein action. Neglecting the inessential
surface terms and using $r$ as an independent variable, we arrive at one dimensional problem
with Lagrangian

$$\mathcal{L} = \frac{1}{2G_N} \left( \frac{2\rho \rho'}{\sqrt{-g_{rr}}} + (\rho')^2 \sqrt{g_{tt}} + \sqrt{g_{tt}} \sqrt{-g_{rr}} \right) - M_0 \sqrt{g_{tt}} \delta(r).$$

Here $G_N$ is the Newton gravitational constant, $\delta(r)$ is the 1D Dirac $\delta$-function \[6\]. (We are using units $c=1$.) The corresponding Euler-Lagrange equations read:

$$\left( \frac{2\rho \rho'}{\sqrt{-g_{rr}}} \right)' - \frac{\rho^2}{\sqrt{-g_{rr}}} - \sqrt{-g_{rr}} + 2G_N M_0 \delta(r) = 0,$$

$$\left( \frac{\rho \sqrt{g_{tt}}}{\sqrt{-g_{rr}}} \right)' - \frac{\rho' (\sqrt{g_{tt}})'}{\sqrt{-g_{rr}}} = 0,$$

$$\frac{2\rho \rho'}{\sqrt{-g_{rr}}} (\sqrt{g_{tt}})' + (\rho')^2 \sqrt{g_{tt}} - \sqrt{g_{tt}} \sqrt{-g_{rr}} \overset{w}{=} 0 \quad (2.2)$$

where the symbol $\overset{w}{=} \equiv$ denotes a weak equality in the sense of theory of constrained dynamical systems. (As a result of the rho-gauge freedom the field variable $\sqrt{-g_{rr}}$ is not a true dynamical variable but rather plays the role of a Lagrange multiplier, which is needed in a description of constrained dynamics.) Of course, the same equations (2.2) can be derived directly from EE (2.1).

Outside the point source of the gravitational field (i.e. for $r > 0$, where $\delta(r) \equiv 0$) one obtains the standard solution of this system \[7\]:

$$g_{tt}(r) = 1 - \frac{\rho_G}{\rho(r)}, \quad g_{tt}(r) g_{rr}(r) = - (\rho(r)')^2,$$  \[2.3\]

where $\rho_G = 2G_N M$ is a new integration constant – the Schwarzschild radius, $M$ is the gravitational (Keplerian) mass of the source, and $\rho(r)$ is an arbitrary $C^2$ function of $r$.

As usual, this solution describes the gravitational field outside the point source (and outside any body with spherically symmetric mass distribution). According to the Birkhoff theorem \[5\] it is unique (up to the choice of the radial variable $r$). Nevertheless, being a pure vacuum solution of EE, it is not able to describe the mass distribution $M_0 \delta(r)$ in the very point source, as well as the mass distribution in any other matter source.

To see this in a more transparent way, let us consider the Hilbert gauge (HG) of the radial variable: $r \equiv \rho$ \[4\], in which the field equations (2.2) can be rewritten in the form:

$$(\sqrt{-g_{\rho\rho}} - \frac{1}{\sqrt{-g_{\rho\rho}}}) \frac{d}{d \ln \rho} \ln \left( \rho \left( 1 - \frac{1}{g_{\rho\rho}} \right) \right) = 2\sigma_0 G_N M_0 \delta(\rho - \rho_0),$$

$$\frac{d^2 \ln g_{tt}}{(d \ln \rho)^2} + \frac{1}{2} \left( \frac{d \ln g_{tt}}{d \ln \rho} \right)^2 + \left( 1 + \frac{1}{2} \frac{d \ln g_{tt}}{d \ln \rho} \right) \frac{d \ln g_{\rho\rho}}{d \ln \rho} = 0,$$

$$\frac{d \ln g_{tt}}{d \ln \rho} + g_{\rho\rho} + 1 \overset{w}{=} 0 \quad (2.4)$$

Here we have utilized the properties of Dirac $\delta$-function and the assumption that the metric coefficients are usual $C^2$ functions of $r$ (not distributions). Then the only remnant
of the function $\rho(r)$ in the system Eq. (2.4) are the numbers $\rho_0 = \rho(0)$ and $\sigma_0 = \text{sign}(\rho'(0))$, which enter only the first equation, related to the source of gravity.

One usually ignores the general case of an arbitrary value $\rho_0 \neq 0$ accepting the value $\rho_0 = 0$, which seems to be natural in HG. Indeed, if we consider the luminosity variable as a measure of the real geometrical distance to the point source of gravity in the 3D space (which is not the case), we have to accept the value $\rho_0 = 0$ for the position of the point source. Otherwise the $\delta$-function term in the Eq. (2.4) will describe a shell with radius $\rho_0 \neq 0$, instead of a point source. Such conclusion is based on the above interpretation of the variable $\rho$.

Actually the point source has to be described using the function $\delta(r)$ and is placed at the point $r = 0$ by definition. There is no reason to change this original position of the source, or the interpretation of the variables in the problem at hand. To what value of the luminosity variable $\rho_0 = \rho(0)$ corresponds the real position of the point source is not known a priori. This depends strongly on the choice of the rho-gauge. One can not exclude such a nonstandard behavior of a physically or mathematically reasonable rho-gauge function $\rho(r)$ which leads to some value $\rho_0 \neq 0$. Physically this means that instead to infinity, the luminosity of the point source will go to a finite value, when the distance to the source goes to zero (see, for example, [2]). This interesting possibility appears in curved space-times due to their unusual geometrical properties and is not supported by our Euclidean experience. If one accepts the value $\rho_0 = 0$, one has to recognize that the HG singularity at $\rho = 0$ will be space-like, not time-like, because for $\rho \in [0, \rho_G]$ the variables $\rho$ and $t$ are changing their roles. This will be a quite unusual and non-physical property for physical source of a physical field of any kind.

The solution of the subsystem formed by the last two equations of (2.4) is well known:

$$g_{tt}(\rho) = 1 - \rho G/\rho, \quad g_{\rho\rho}(\rho) = -1/g_{tt}(\rho)$$

(2.5)

– the Hilbert form of the Schwarzschild solution. Note that in this subsystem one of the equations is a field equation, but the other one is a constraint. However, these functions do not solve the first of the Eq. (2.4) for any value of $\rho_0$, if $M_0 \neq 0$. Indeed, for these functions the left hand side of the first field equation equals identically zero and does not have a $\delta$-function-type singularity, in contrast to the right hand side. Hence, the first field equation remains unsolved by the functions (2.5). Thus we see that the assumption that $g_{tt}(r)$, $g_{rr}(r)$ and $\rho(r)$ are usual $C^2$ smooth functions, instead of distributions, yields a contradiction, if $M_0 \neq 0$. Indeed, in this way one is not able to describe correctly the gravitational field of a massive point source of gravity in GR.

This mathematical result is the real basis of the widespread opinion, according to which it’s impossible to describe a massive point in GR. Actually, the right conclusion is that the class of $C^2$ (and even the class of $C^1$) metrics is not proper for this purpose. To reach such goal, the first derivative with respect to the variable $r$ of at least one of the metric coefficients $g_{\mu\nu}$ must have a strictly definite jump. It is needed to reproduce the Dirac $\delta$-function in the energy-momentum tensor $T_\mu^\nu$ of the massive point particle via the Einstein tensor $G_\mu^\nu$. Otherwise the EE (2.1) can not be fulfilled.

The above consideration illustrates one more important juncture. It is obvious that physical results of any theory must not depend on the choice of the variables and, in particular, these results must be invariant under changes of coordinates. This requirement is a basic principle in GR. It is fulfilled for any already fixed mathematical problem.
Nevertheless, the change of the interpretation of the variables may change the formulation of the mathematical problem and thus, the physical results, because we are using the variables according to their meaning. For example, if we are considering the luminosity variable $\rho$ as a radial variable of the problem, it seems natural to put the point source at the point $\rho = 0$. In general, we may obtain a physically different model, if we are considering another variable $r$ as a radial one. In this case we shall place the source at a different geometrical point $r = 0$, which now seems to be the natural position for the center $C$. The relation between these two geometrical "points" and between the corresponding physical models strongly depends on the choice of the function $\rho(r)$, i.e. on the radial gauge. Thus, applying the same physical requirements in different "natural" variables, we arrive at different physical theories, because we are solving EE under different boundary conditions, coded in corresponding Dirac $\delta$-functions. One has to find a theoretical or an experimental reasons to resolve this essential ambiguity.

3 Field Variables and Radial Gauges, Suitable for a Correct Treatment of the Point Source of Gravity

An obstacle for the description of the gravitational field of a point source at the initial stage of development of GR was the absence of an adequate mathematical formalism. Even after the development of the correct theory of mathematical distributions [6] there still exist an opinion that this theory is inapplicable to GR because of the nonlinear character of Einstein equations (2.1), see for example [8]. In a recent article [9] the authors have considered singular lines and surfaces, using mathematical distributions. They have stressed, that "there is apparently no viable treatment of point particles as concentrated sources in GR".

Here we propose a novel approach to this problem, based on a specific choice of the field variables in the metric:

$$ds^2 = e^{2\varphi_1} dt^2 - e^{-2\varphi_1+2\varphi_2} dr^2 - \rho^2 e^{-2\varphi_1+2\varphi_2} (d\theta^2 + \sin^2 \theta d\phi^2)$$  (3.1)

where $\varphi_1(r)$, $\varphi_2(r)$ and $\bar{\varphi}(r)$ are unknown functions of the variable $r$ and $\bar{\rho}$ is a constant – the unit for luminosity variable $\rho = \bar{\rho} e^{-\varphi_1+\varphi_2}$. The corresponding form of the Lagrangian:

$$\mathcal{L} = \frac{1}{2G_N} \left( e^\varphi \left( -(-\bar{\rho}\varphi_1')^2 + (\bar{\rho}\varphi_2')^2 \right) + e^{-\varphi} e^{2\varphi_2} \right) - M_0 e^{\varphi_1} \delta(r)$$  (3.2)

shows that the field variables $\varphi_1(r)$, $\varphi_2(r)$ and $\bar{\varphi}(r)$ diagonalize the kinetic-like part in $\mathcal{L}$. Hence, they play the role of a normal fields’ coordinates in the problem at hand. Now the field equations read:

$$\bar{\Delta}_r \varphi_1(r) = \frac{G_N M_0}{\rho^2} e^{\varphi_1(r)-\bar{\varphi}(r)} \delta(r), \quad \bar{\Delta}_r \varphi_2(r) = \frac{1}{\rho^2} e^{2(\varphi_2(r)-\bar{\varphi}(r))}$$  (3.3)

where $\bar{\Delta}_r = e^{-\varphi} \frac{d}{dr} \left( e^\varphi \frac{d}{dr} \right)$ is related to the radial part of the 3D-Laplacean. The variation of the total action with respect to the auxiliary variable $\bar{\varphi}$ gives the constraint:

$$e^\varphi \left( -(-\bar{\rho}\varphi_1')^2 + (\bar{\rho}\varphi_2')^2 \right) - e^{-\varphi} e^{2\varphi_2} \equiv 0.$$  (3.4)
The advantage of the above normal fields’ coordinates is that when expressed in them the field equations (3.3) are linear with respect to the derivatives of the unknown functions \( \varphi_{1,2}(r) \). This circumstance legitimates the correct application of the mathematical theory of distributions and makes our normal coordinates privileged field variables.

The choice of the function \( \varphi(r) \) fixes the rho-gauge in the normal coordinates. We have to choose this function in a way that makes the first of the equations (3.3) mathematically meaningful. Note that this inhomogeneous equation is quasi-linear and has a correct mathematical meaning if, and only if, the condition \( |\varphi(0) - \varphi(0)| < \infty \) is satisfied.

Let’s consider once again the domain \( r > 0 \). In this domain the first of the equations (3.3) gives \( \varphi_1(r) = C_1 \int e^{-\varphi(r)} dr + C_2 \) with arbitrary constants \( C_{1,2} \). Suppose that the function \( \varphi(r) \) has an asymptotic \( \exp(-\varphi(r)) \sim kr^n \) in the limit \( r \to +0 \) (with some arbitrary constants \( k \) and \( n \)). Then one easily obtains \( \varphi_1(r) - \varphi(r) \sim C_1 kr^{n+1}/(n+1) + n \ln r + n k + C_2 - \text{if } n \neq -1 \), and \( \varphi_1(r) - \varphi(r) \sim (C_1 k - 1) \ln r + n k + C_2 - \text{for } n = -1 \). Now we see that one can satisfy the condition \( \lim_{r \to 0} |\varphi_1(r) - \varphi(r)| = \text{constant} < \infty \) for arbitrary values of the constants \( C_{1,2} \) if, and only if \( n = 0 \). This means that we must have \( \varphi(r) \sim k = \text{const} \neq \pm \infty \) for \( r \to 0 \). We call such gauges regular gauges for the problem at hand. Then \( \varphi_1(0) = \text{const} \neq \pm \infty \). Obviously, the simplest choice of a regular gauge is \( \varphi(r) \equiv 0 \). Further on we shall use this basic regular gauge (BRG). Other regular gauges defer from it by a regular rho-gauge transformation which describes a diffeomorphism of the fixed by the BRG gauge manifold \( M(3) \{g_{mn}(r)\} \). In terms of the metric components the BRG fixing condition reads \( \rho^4 g_{tt} + \bar{\rho}^4 g_{rr} = 0 \).

Under this gauge the field equations (3.3) acquire a simple quasi-linear form:

\[
\varphi_1''(r) = \frac{G_N M_0}{\bar{\rho}^2} e^{\varphi_1(0)} \delta(r), \quad \varphi_2''(r) = \frac{1}{\bar{\rho}^2} e^{2\varphi_2(r)}
\]

and constraint (3.4) is:

\[
-(\bar{\rho} \varphi_1')^2 + (\bar{\rho} \varphi_2')^2 - e^{2\varphi_2} w = 0.
\]

Hence, the BRG gauge \( \varphi(r) \equiv 0 \) has the unique property to split the system of field equations (3.3) and the constraint (3.4) into three independent relations (3.5), (3.6).

\section{Solution of Einstein Equations with Massive Point Source}

\subsection{Solution in the Basic Regular Gauge}

The new form of Einstein field equations (3.5) of our problem has a simple general solution in terms of mathematical distributions (called sometimes ”generalized functions”[6]):

\[
\varphi_1(r) = \frac{G_N M_0}{\bar{\rho}^2} e^{\varphi_1(0)} (\Theta(r) - \Theta(0)) r + \varphi_1'(0) r + \varphi_1(0),
\]

\[
\varphi_2(r) = -\ln \left( \frac{1}{\sqrt{2\varepsilon_2}} \sinh \left( \sqrt{2\varepsilon_2} \frac{r_c - r}{\bar{\rho}} \right) \right).
\]

The first expression in Eq.(3.1) represents a distribution \( \varphi_1(r) \). In it \( \Theta(r) \) is the Heaviside step function. Here we use the additional assumption \( \Theta(0) := 1 \). It gives a specific
regularization of the products, degrees and functions of the distribution $\Theta(r)$ and makes them definite.

The second expression $\varphi_2(r)$ in Eq. (4.1) is a usual function of the variable $r$. The symbol $r_\infty$ is used as an abbreviation for the constant expression $r_\infty = \text{sign} (\varphi_2'(0)) \bar{\rho} \sinh \left( \sqrt{2 \bar{\varepsilon}_2} e^{-\varphi_2(0)} \right) / \sqrt{2 \bar{\varepsilon}_2}$. The constants

$$\varepsilon_1 = -\frac{1}{2} \bar{\rho}^2 \varphi'_1(r)^2 + \frac{G N M_0}{\bar{\rho}^2} \varphi'_1(0) e^{\varphi_1(0)} (\Theta(r) - \Theta(0)), \quad \varepsilon_2 = \frac{1}{2} \left( \bar{\rho}^2 \varphi'_2(r)^2 - e^{2\varphi_2(r)} \right)$$

(4.2)

are the values of the corresponding first integrals (4.2) of the differential equations (3.5) for a given solution (4.1).

Then for the regular solutions (4.1) the condition (3.6) reads:

$$
\varepsilon_1 + \varepsilon_2 + \frac{G N M_0}{\bar{\rho}^2} e^{\varphi_1(0)} (\Theta(r) - \Theta(0)) \equiv 0.
$$

(4.3)

An unexpected property of this relation is that it cannot be satisfied for any value of the variable $r \in (-\infty, \infty)$, because $\varepsilon_{1,2}$ are constants. The constraint (4.3) can be satisfied either on the interval $r \in [0, \infty)$, or on the interval $r \in (-\infty, 0)$. If, from physical reasons we chose it to be valid at only one point $r^* \in [0, \infty)$, this relation will be satisfied on the whole interval $r \in [0, \infty)$ and this interval will be the physically admissible real domain of the radial variable. Thus one can see that our approach gives a unique possibility to derive the admissible real domain of the variable $r$ from the dynamical constraint (3.6), i.e., this dynamical constraint yields a geometrical constraint on the values of the radial variable. As a result, in the physical domain the values of the first integrals (4.2) are related by the standard equation

$$
\varepsilon_{\text{tot}} = \varepsilon_1 + \varepsilon_2 \equiv 0,
$$

(4.4)

which reflects the fact that our variation problem is invariant under local reparameterization of the independent variable $r$. At the end, as a direct consequence of relation (4.4) one obtains the inequality $\varepsilon_2 = -\varepsilon_1 > 0$, because in the real physical domain $r \in [0, \infty)$ we have $\varepsilon_1 = -\frac{1}{2} \bar{\rho}^2 \varphi'_1(r)^2 = \text{const} < 0$.

For the function $\rho_{\text{BRG}}(r) \geq 0$, which corresponds to the BRG, we obtain

$$
\rho_{\text{BRG}}(r) = \rho_G \left( 1 - \exp \left( \frac{4 (r - r_\infty)}{\rho_G} \right) \right)^{-1}.
$$

(4.5)

Now we can fix the arbitrary integration constants $\varphi_1(0), \varphi'_1(0), r_\infty$ and $\varepsilon_2$:

1) Imposing a number of additional requirements: i) to have an asymptotically flat space-time, taking into account that in the present variable $r$ we have $\rho(r) \to \infty$ when $r \to r_\infty$; ii) to have a correct Keplerian mass $M$ for $\rho \to \infty$; iii) to have a consistence with the relation $g_{\mu \nu} g_{\mu \nu} + \rho^2 = 0$;

2) Making use of suitable choice of the units for the luminosity variable in the form $\bar{\rho} = G N M = \rho_G / 2$;

3) Introducing a gravitational mass defect for the point particle in the following way:

Representing the bare mechanical mass $M_0$ of the point source in the form $M_0 = \int_0^{r_\infty} M_0 \delta(r) \, dr = 4 \pi \int_0^{r_\infty} \sqrt{-g_{\mu \nu}(r)} \rho^2(r) \mu(r) \, dr$, one obtains for the mass distribution of the point particle the expression $\mu(r) = M_0 \delta(r) / \left( 4 \pi \sqrt{-g_{\mu \nu}(r)} \rho^2(r) \right) = M_0 \delta_g(r)$, where
\( \delta_g(r) := \delta(r) / \left( 4\pi \sqrt{-g_{rr}(r)} \rho^2(r) \right) \) is the 1D *invariant* Dirac delta function. The Keplerian gravitational mass \( M \) can be calculated using the Tolman formula \[5\]:

\[
M = 4\pi \int_0^{r_\infty} \rho'(r) \rho^2(r) \mu(r) dr = M_0 \sqrt{g_{tt}(0)}. \tag{4.6}
\]

Here we use the relation \( \rho' = \sqrt{-g_{tt} g_{rr}} \). As a result we reach the relations: \( g_{tt}(0) = e^{2\varphi_1(0)} = \exp\left(-2 \frac{\rho}{G_N M}\right) \leq 1 \) and \( r_\infty = G_N M \ln \left( \frac{M_0}{M} \right) \geq 0 \). (Note that due to our convention \( \Theta(0) := 1 \) the component \( g_{tt}(r) \) is a continuous function in the interval \( r \in [0, \infty) \) and \( g_{tt}(0) = g_{tt}(+0) \) is a well defined quantity.)

The ratio \( \varrho = \frac{M}{M_0} = \sqrt{g_{tt}(0)} \in [0, 1] \) describes the gravitational mass defect of the point particle as a second physical parameter in the problem. The Keplerian mass \( M \) and the ratio \( \varrho \) define completely the solutions \[4.1\].

Then for the initial constants of the problem one obtains:

\[
\begin{align*}
\varphi_1(0) &= \ln \varrho, \quad \varphi_2(0) = -\ln \frac{1 - \varrho^2}{2\varrho}, \\
\varphi'_1(0) &= \frac{1}{G_N M}, \quad \varphi'_2(0) = \frac{1 + \varrho^2}{G_N M} \frac{1 - \varrho^2}{1 - \varrho^2}.
\end{align*}
\tag{4.7}
\]

Thus we arrive at the following form of the solutions \[4.1\]:

\[
\begin{align*}
\varphi_1(r) &= \frac{r \Theta(r)}{G_N M} - \ln(1/\varrho), \\
\varphi_2(r) &= -\ln \left( \frac{1}{2} \left( \frac{1 - \varrho^2}{G_N M} e^{-r/G_N M} - \varrho e^{r/G_N M} \right) \right).
\end{align*}
\tag{4.8}
\]

and the rho-gauge fixing function

\[
\rho_{BRG}(r) = \rho_G \left( 1 - \varrho^2 \exp \left( \frac{4r}{\rho_G} \right) \right)^{-1}. \tag{4.9}
\]

An unexpected feature of this *two parametric* variety of solutions for the gravitational field of a point particle is that each solution must be considered only in the domain \( r \in [0, G_N M \ln (1/\varrho)] \), if we wish to have a monotonic increase of the luminosity variable in the interval \( [\rho_0, \infty) \).

It is easy to check that away from the source (i.e., for \( r > 0 \)) the solutions \[4.8\] coincide with the HG solution and acquire the well known standard form, when represented using the variable \( \rho \). This means that these solutions strictly respect a generalized Birkhoff theorem. Its generalization requires only a justification of the physical domain of variable \( \rho \). In a remarkable accord with Dirac’s intuition \[10\] the minimal value of the luminosity variable for the solutions \[4.8\] is

\[
\rho_0 = 2G_N M / (1 - \varrho^2) \geq \rho_G. \tag{4.10}
\]

This changes the Gauss theorem and leads to a variety of different important consequences. One of them is that we must apply the Birkhoff theorem only in the interval \( \rho \in [\rho_0, \infty) \iff r \in [0, r_\infty) \). As a result, in this domain all local GR effects like gravitational redshift, perihelion shift, deflection of light rays, time-delay of signals, etc., will have their standard *exact* values in gravitational field of the solutions \[4.8\].
4.2 Regular Mapping of the interval \( r \in [0, r_\infty) \) to the whole interval \( r \in [0, \infty) \)

It does not seem to be convenient to work with the unusual radial variable \( r \in [0, r_\infty) \). One can easily overcome this problem using the regular radial gauge transformation

\[
r \rightarrow r_\infty \frac{r/\tilde{r}}{r/\tilde{r} + 1}
\]  

(4.11)

with an arbitrary scale \( \tilde{r} \) of the new radial variable \( r \) (Note that in the present article we are using the same notation \( r \) for different radial variables.) This linear fractional diffeomorphism does not change the number and the character of the singular points of the solutions in the whole compactified complex plane \( \tilde{C}_r \) of the variable \( r \). The transformation (4.11) simply places the point \( r = r_\infty \) at infinity: \( r = \infty \), at the same time preserving the initial place of the origin \( r = 0 \). Now the new variable \( r \) varies in the standard interval \( r \in [0, \infty) \) and the regular solutions (4.8) acquire the final form

\[
\varphi_1(r) = -\ln(1/\rho) \left( 1 - \frac{r/\tilde{r}}{r/\tilde{r} + 1} \Theta \left( \frac{r/\tilde{r}}{r/\tilde{r} + 1} \right) \right),
\]

\[
\varphi_2(r) = -\ln \left( \frac{1}{2} \left( (1/\rho)^{r/\tilde{r} + 1} - \rho^{r/\tilde{r} + 1} \right) \right),
\]

\[
\bar{\varphi}(r) = 2 \ln(r/\tilde{r} + 1) + \ln(\tilde{r}/r_\infty).
\]

(4.12)

The final form of the rho-gauge fixing function reads:

\[
\rho_{PRG}(r) = \rho_G \left( 1 - \rho_\tilde{r}/r_\infty \right)^{-1}.
\]

(4.13)

The last expression shows that the mathematically admissible interval of values of the ratio \( \rho \) is the open interval \((0, 1)\). This is so, because for \( \rho = 0 \) and for \( \rho = 1 \) we would have impermissible trivial gauge-fixing functions \( \rho_{PRG}(r) \equiv 1 \) and \( \rho_{PRG}(r) \equiv 0 \), respectively.

The expressions [1.12] and [1.13] still depend on the choice of unit for the new variable \( r \). We have to fix the arbitrary scale of this variable in the form \( \tilde{r} = \rho_G/\ln(1/\rho^2) = G_N M/\ln \left( \frac{M_0}{M} \right) \) to ensure validity of the standard asymptotic expansion: \( g_{tt} \sim 1 - \rho_G/r + \mathcal{O} ((\rho_G/r)^2) \) when our last radial variable \( r \) goes to infinity. Then the final form of the 4D interval, defined by the new regular solutions outside the source (i.e., for \( r > 0 \)) is:

\[
ds^2 = e^{2\tilde{\varphi}_G} \left( dt^2 - \frac{dr^2}{N_G(r)^4} \right) - \rho_{PRG}(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

(4.14)

Here we are using a modified (Newton-like) gravitational potential:

\[
\varphi_G(r; M, M_0) := -\frac{G_N M}{r + G_N M/\ln \left( \frac{M_0}{M} \right)}
\]

(4.15)

a coefficient \( N_G(r) = (2\varphi_G)^{-1} (e^{2\tilde{\varphi}_G} - 1) \), and a luminosity variable

\[
\rho_{PRG}(r) = 2G_N M/ \left( 1 - e^{2\tilde{\varphi}_G} \right) = \frac{r + G_N M/\ln \left( \frac{M_0}{M} \right)}{N_G(r)}.
\]

(4.16)
These basic formulas describe in a more usual way our regular solutions of Einstein equations for $r \in (0, \infty)$. Therefore we call this radial gauge physical radial gauge (PRG).

The Eqs. (4.14)-(4.16) show immediately that in the limit $\varrho \to 0$ our solutions tend to the Pugachev-Gun’ko-Menzel one \[11\], and $\varphi_G(0; M, M_0) = \ln \varrho \to -\infty$. In the limit: $\varrho \to 1$ we obtain for any value of the ratio $r/\rho_g$: $g_{tt}(r/\rho_g, \varrho) \to 1$, $g_{rr}(r/\rho_g, \varrho) \to -1$, and $\rho(r/\rho_g, \varrho) \to \infty$. Because of the last result the 4D geometry does not have a meaningful limit when $\varrho \to 1$. In this case $\varphi_G(r; M, M_0) \to 0$ at all 3D space points. Physically this means that solutions with point source without mass defect are not admissible in GR.

5 Total energy of a point source and its gravitational field

In the problem at hand we have an extreme example of an "island universe". In it a privileged reference system and a well defined global time exist. It is well known that under these conditions the energy of the gravitational field can be defined unambiguously \[5\]. Moreover, we can calculate the total energy of the aggregate of a mechanical particle and its gravitational field in a canonical way. Indeed, the canonical procedure produces a total Hamilton density $\mathcal{H}_{tot} = \Sigma_{a=1,2,\mu=t,r} \pi_a^{\mu} \varphi_{a,\mu} - L_{tot} = \frac{1}{2 \mathcal{N}} \left( -\bar{\rho}^2 \varphi_1'{}^2 + \bar{\rho}^2 \varphi_2'{}^2 - e^2 \varphi_2 \right) + M_0 e^{\varphi_1} \delta(r)$. Using the constraint (4.4) and the first of the relations (4.7), one immediately obtains for the total energy of the GR universe with one point particle in it:

$$E_{tot} = \int_0^{\infty} \mathcal{H}_{tot} dr = M = \varrho M_0 < M_0.$$ \hfill (5.1)

This result completely agrees with the strong equivalence principle of GR. The energy of the gravitational field, created by a point particle is the negative quantity: $E_{GR} = E_{tot} - E_0 = M - M_0 = -M_0(1 - \varrho) < 0$.

The above consideration gives a clear physical explanation of the gravitational mass defect for a point particle.

6 Invariants of the Riemann Tensor

To reach a coordinate independent description of the geometry of space-time manifold with metric \[3.1\] we calculate the invariants of the Riemann curvature tensor. Starting from the general consideration in \[12\] we find in BRG ($\bar{\varphi} \equiv 0$) the following four independent invariants:

$$I_1 := e^{2(\varphi_1^2 - \varphi_2)} \varphi_1''$$
$$I_2 := e^{2(\varphi_1^2 - \varphi_2)} \left( \varphi_2'' - e^2 \varphi_2 / \bar{\rho}^2 \right),$$
$$I_3 := e^{2(\varphi_1^2 - \varphi_2)} \left( -\varphi_1^2 - \varphi_2^2 - e^2 \varphi_2 / \bar{\rho}^2 \right)/2,$$
$$I_4 := e^{2(\varphi_1^2 - \varphi_2^2)} \varphi_1' \left( \varphi_1' - \varphi_2' \right)/2.$$

These are linear with respect to the second derivatives of the functions $\varphi_{1,2}''$, which is of critical importance when we have to work with distributions $\varphi_{1,2}''$. For the
regular solutions (1.8) one obtains:

\[ I_1 = \frac{1}{8 \rho_G^2} \frac{(1-\delta^2)^4}{\rho^2} \delta \left( \frac{r}{\rho_G} \right) = -\frac{1}{2} R(r), \quad I_2 = 0, \]
\[ I_3 = \frac{\Theta(r/\rho_G)-1}{8 \rho_G^2 \rho^2} (1-\delta^2 e^{4r/\rho_G})^4 = \frac{\rho_G^2 \Theta(r/\rho_G)-1}{8 \rho^2}, \]
\[ I_4 = \frac{\Theta(r/\rho_G)}{4 \rho_G^2} (1-\delta^2 e^{4r/\rho_G})^3 = \frac{\rho_G \Theta(r/\rho_G)}{4 \rho^3}. \]  

(6.2)

The invariants \( I_1, \ldots, I_4 \) of the Riemann tensor are well defined distributions. This confirms the general expectations, described in the articles [13], where one can find a correct mathematical treatment of distribution-valued curvature tensors in GR.

As we see, the manifold \( \mathbb{M}^{(1,3)} \{ g_{\mu\nu}(x) \} \) for our regular solutions has a definite geometrical singularity at the point, where the physical source – the massive point particle is placed. The fractional-linear transformation (4.11) does not change the character of this singularity and does not add new ones.

7 Global Analytical Properties of the Solutions in the Complex Plane of the Radial Variable

To simplify the notations we introduce dimensionless variables: \( \zeta = r/G_N M \) – instead of the radial variable \( r \) in BRG, \( z = (r/G_N M) \ln(M_0/M) \) - instead of the radial variable \( r \) in PRG, and \( w = \rho/\rho_G \) – instead of the luminosity variable \( \rho \). In this Section we shall consider these variables as complex valued ones in the corresponding complex plains \( \mathbb{C}_\zeta, \mathbb{C}_z, \mathbb{C}_w \), and study the basic analytical properties of the corresponding functions. Then, up to inessential constant factors, the independent curvature invariants (6.2) read:

\[ I_1 \sim \delta(\zeta), \quad I_3 \sim (\Theta(\zeta)-1) (1-\delta^2 e^{2\zeta})^4, \quad I_4 \sim \Theta(\zeta) (1-\delta^2 e^{2\zeta})^3 \]  
in BRG, or \[ I_1 \sim \delta(z), \quad I_3 \sim \left( \Theta \left( \frac{z}{z+1} \right) -1 \right) (1-\delta^2 e^{2z})^4, \quad I_4 \sim \Theta \left( \frac{z}{z+1} \right) (1-\delta^2 e^{2z})^3 \]  
in PRG. (7.2)

As we see, the BRG-invariants have a unique analytical property: their only singularity in \( \mathbb{C}_\zeta \) is the center \( C: \zeta = 0 \). On the compactified complex plain \( \mathbb{C}_\zeta \) these invariants have an additional essentially singular point \( \zeta = \infty \), where the jump of the luminosity variable \( w(\zeta) = (1-\delta^2 e^{2\zeta})^{-1} \) on the real axes equals 1. As seen from Eq. (7.2), the transition to PRG, defined by the fractional linear mapping (4.11), which now reads

\[ \zeta(z) = \frac{\ln 1/\theta}{z+1} = \frac{\ln 1/\theta - \zeta}{z+1} \]  

simply translates this essentially singular point at the position \( z = -1 \). From analytical point of view the existence of such essentially singular point is the basic difference between the descriptions of massive point source in GR and in Newton gravity.

It is obvious that transition to any other radial variable \( z \), defined by nonlinear transformation \( z = z(\zeta) \), more general than a fractional linear one: \( z(\zeta) = \frac{\theta+\zeta}{\theta+\zeta} \), will create
new singular points in the plane $\tilde{C}_z$, or will change the character of the existing ones, thus changing the mathematical and the physical properties of the solution in the $\tilde{C}_z$ plane.

Hence, from analytical point of view the PRG is a unique radial gauge which defines a preferable radial variable $r$ in the problem at hand, satisfying simultaneously the following two conditions:

i) In the physical domain it varies in the natural interval $r \in [0, \infty)$ and has no other singularities in this interval than the very point source.

ii) The only singular points of the solution in the compactified complex plain $\tilde{C}_r$ are: the place of the source at $r = 0$ and the unavoidable in GR essentially singular point at $r = -1$, which is placed in the nonphysical domain.

This important result solves the longstanding problem of the choice of radial variable $r$ for point source in GR on a clear theoretical basis.

The only independent curvature invariant for the HG form of Schwarzschild solution (2.5) is $\left(I_4\right)_{HG} \sim w^{-3} = \left(1 - \rho^2 z^2\right)^{-3}$. The comparison with Eqs. (7.2) makes it clear that from geometrical point of view the HG solution of EE (2.5) is a vacuum one and essentially differs from the massive point particle solutions.

Now we are ready to describe the singular character of the coordinate transition from the Hilbert form of Schwarzschild solution to the regular one (4.12), considered outside the source. Eq. (4.13) shows that in this domain the change of the coordinates is described, in both directions, by the functions:

$$w(z) = \left(1 - \rho^2 z^2\right)^{-1} \Rightarrow z(w) = \frac{\ln(1/\rho^2)}{\ln w - \ln(w - 1)} - 1, \quad \rho \in (0, 1). \quad (7.4)$$

The function $w(z)$ is regular at the place of the point source $z = 0$; it has a simple pole at $z = \infty$ and an essentially singular point at $z = -1$. At the same time the inverse function $z(w)$ has a logarithmic branch points both at the HG center of symmetry $w = 0$ and the event horizon $w = 1$. Thus we see how one produces the HG singularities at $\rho = 0$ and at $\rho = \rho_H$, starting from a regular solution. The derivative

$$dz/dw = \frac{\ln(1/\rho^2)}{w(w - 1)(\ln w - \ln(w - 1))^2}$$

approaches infinity at these two points, hence the singular character of the change of the variables in the whole complex domain. The restriction of the change of the radial variables on the corresponding physical interval outside the source: $z \in (0, \infty) \Rightarrow w \in (1/(1 - \rho^2), \infty)$, is a regular one.

8 Conclusion

In the present article we have studied a new, two parameter class of solutions of Einstein equations. These static spherically symmetric solutions describe the gravitational field of massive point particle with bare mass $M_0 > 0$ and Keplerian mass $M (0 < M < M_0)$. The difference between these masses, or their ratio $\rho = M/M_0 \in (0, 1)$, define the gravitational mass defect for the point particle. Such mass defect was not considered and studied until now, because for the standard Hilbert form (2.5) of the Schwarzschild solution "the bare rest-mass density is never even introduced" [14] correctly.
The new solutions form a two parameter family of metrics on singular manifolds $\mathbb{M}^{(1,3)}\{g_{\mu\nu}\}$, described in details in the present article.

We have shown the principal role of the massive point source of gravity. It presents a natural cutting factor for the physical values of the luminosity variable $\rho \in [\rho_0, \infty)$, where $\rho_0 > \rho_G$ (4.10). This happens because the infinite mass density of the matter point changes drastically the geometry of the space-time around it.

Similar geometry of space-time with $\rho_0 \equiv \rho_G$ was discovered at first in the original article by Schwarzschild $\mathbf{[2]}$. According to Eq. (4.10), such limiting value of the luminosity variable corresponds to zero value $\varrho = 0$ of mass defect ratio. For finite value of $M$ this is possible only if $M_0 = \infty$. In this sense our work is a proper extension of the Schwarzschild one to the physically and mathematically admissible values of the mass defect ratio $\varrho \in (0, 1)$.

In full accord with Dirac’s suggestion $\mathbf{[10]}$ our cutting of the domain of luminosity variable places the event horizon in the nonphysical domain of the variables. This effect is well known from the solutions of Einstein equations with massive matter sources of finite dimension.

The mathematical and the physical properties of the new solutions are essentially different in comparison with the well known other spherically symmetric static solutions to the Einstein equations. All they have different type of singularities at the center of the symmetry, which is surrounded by empty space. The previously known solutions were often erroneously considered as a solutions for describing of single point mass in GR.

It is clear that our solutions in generalized functions define in mathematical sense the fundamental solutions of Einstein equations, which are complete analogous to the fundamental solutions of Poisson equation in Newton theory of gravity. Thus the problem, formulated by Feynman in $\mathbf{[1]}$ is solved.

Further study of the new solutions and their physical applications will be given in separate articles.

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