Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect

Ted Jacobson

Department of Physics, University of Maryland
College Park, MD 20742-4111
jacobson@physics.umd.edu

Abstract

These notes introduce the subject of quantum field theory in curved spacetime and some of its applications and the questions they raise. Topics include particle creation in time-dependent metrics, quantum origin of primordial perturbations, Hawking effect, the trans-Planckian question, and Hawking radiation on a lattice.

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1 Introduction

Quantum gravity remains an outstanding problem of fundamental physics. The bottom line is we don’t even know the nature of the system that should be quantized. The spacetime metric may well be just a collective description of some more basic stuff. The fact\(^1\) that the semi-classical Einstein equation can be derived by demanding that the first law of thermodynamics hold for local causal horizons, assuming the proportionality of entropy and area, leads one to suspect that the metric is only meaningful in the thermodynamic limit of something else. This led me at first to suggest that the metric shouldn’t be quantized at all. However I think this is wrong. Condensed matter physics abounds with examples of collective modes that become meaningless at short length scales, and which are nevertheless accurately treated as quantum fields within the appropriate domain. (Consider for example the sound field in a Bose-Einstein condensate of atoms, which loses meaning at scales below the so-called “healing length”, which is still several orders of magnitude longer than the atomic size of the fundamental constituents.) Similarly, there exists a perfectly good perturbative approach to quantum gravity in the framework of low energy effective field theory\(^2\). However, this is not regarded as a solution to the problem of quantum gravity, since the most pressing questions are non-perturbative in nature: the nature and fate of spacetime singularities, the fate of Cauchy horizons, the nature of the microstates counted by black hole entropy, and the possible unification of gravity with other interactions.

At a shallower level, the perturbative approach of effective field theory is nevertheless relevant both for its indications about the deeper questions and for its application to physics phenomena in their own right. It leads in particular to the subject of quantum field theory in curved spacetime backgrounds, and the “back-reaction” of the quantum fields on such backgrounds. Some of the most prominent of these applications are the Hawking radiation by black holes, primordial density perturbations, and early universe phase transitions. It also fits into the larger category of quantum field theory (qft) in inhomogeneous and/or time-dependent backgrounds of other fields or matter media, and is also intimately tied to non-inertial effects in flat space qft such as the Unruh effect. The pertubative approach is also known as “semi-classical quantum gravity”, which refers to the setting where there is a well-defined classical background geometry about which the quantum fluctuations are occuring.

The present notes are an introduction to some of the essentials and phenomena of quantum field theory in curved spacetime. Familiarity with quantum mechanics and general relativity are assumed. Where computational steps are omitted I expect that
the reader can fill these in as an exercise.

Given the importance of the subject, it is curious that there are not very many books dedicated to it. The standard reference by Birrell and Davies\cite{3} was published twenty years ago, and another monograph by Grib, Mamaev, and Mostapanenko\cite{4}, half of which addresses strong background field effects in flat spacetime, was published two years earlier originally in Russian and then in English ten years ago. Two books with a somewhat more limited scope focusing on fundamentals with a mathematically rigorous point of view are those by Fulling\cite{5} and Wald\cite{6}. This year DeWitt\cite{7} published a comprehensive two volume treatise with a much wider scope but including much material on quantum fields in curved spacetime. A number of review articles (see e.g. \cite{8, 9, 10, 11, 12, 13}) and many shorter introductory lecture notes (see e.g. \cite{14, 15, 16}) are also available. For more information on topics not explicitly referenced in the text of these notes the above references should be consulted.

In these notes the units are chosen with $c = 1$ but $\hbar$ and $G$ are kept explicit. The spacetime signature is $(-+++)$.

2 Planck length and black hole thermodynamics

Thanks to a scale separation it is useful to distinguish quantum field theory in a curved background spacetime (qftcs) from true quantum gravity (qg). Before launching into the qftcs formalism, it seems worthwhile to have a quick look at some of the interesting issues that qftcs is concerned with.

2.1 Planck length

It is usually presumed that the length scale of quantum gravity is the Planck length $L_P = (\hbar G/c^3)^{1/2} \approx 10^{-33}$ cm. The corresponding energy scale is $10^{19}$ GeV. Recent “braneworld scenarios”, in which our 4d world is a hypersurface in a higher dimensional spacetime, put the scale of quantum gravity much lower, at around a TeV, corresponding to $L_{\text{TeV}} = 10^{16}L_P \approx 10^{-17}$ cm. In either case, there is plenty of room for applicability of qftcs. (On the other hand, we are much closer to seeing true qg effects in TeV scale qg. For example we might see black hole creation and evaporation in cosmic rays or accelerators.)

Here I will assume Planck scale qg, and look at some dimensional analysis to give a feel for the phenomena. First, how should we think of the Planck scale? The Hilbert-
Einstein action is \( S_{HE} = (\hbar/16\pi L_P^2) \int d^4x |g|^{1/2} R \). For a spacetime region with radius of curvature \( L \) and 4-volume \( L^4 \) the action is \( \sim \hbar (L/L_P)^2 \). This suggests that quantum curvature fluctuations with radius less than the Planck length \( L < L_P \) are unsuppressed.

Another way to view the significance of the Planck length is as the minimum localization length \( \Delta x \), in the sense that if \( \Delta x < L_P \) a black hole swallows the \( \Delta x \). To see this, note that the uncertainty relation \( \Delta x \Delta p \geq \hbar/2 \) implies \( \Delta p \gtrsim \hbar/\Delta x \) which implies \( \Delta E \gtrsim \hbar c/\Delta x \). Associated with this uncertain energy is a Schwarzschild radius \( R_s(\Delta x) = 2G \Delta M/c^2 = 2G \Delta E/c^4 \), hence quantum mechanics and gravity imply \( R_s(\Delta x) \gtrsim L_P^2/\Delta x \). The uncertain \( R_s(\Delta x) \) is less than \( \Delta x \) only if \( \Delta x \gtrsim L_P \).

\subsection{2.2 Hawking effect}

Before Hawking, spherically symmetric, static black holes were assumed to be completely inert. In fact, it seems more natural that they can decay, since there is no conservation law preventing that. The decay is quantum mechanical, and thermal: Hawking found that a black hole radiates at a temperature proportional to \( \hbar \), \( T_H = (\hbar/2\pi)\kappa \), where \( \kappa \) is the surface gravity. The fact that the radiation is thermal is even natural, for what else could it be? The very nature of the horizon is a causal barrier to information, and yet the Hawking radiation emerges from just outside the horizon. Hence there can be no information in the Hawking radiation, save for the mass of the black hole which is visible on the outside, so it must be a maximum entropy state, i.e. a thermal state with a temperature determined by the black hole mass.

For a Schwarzschild black hole \( \kappa = 1/4GM = 1/2R_s \), so the Hawking temperature is inversely proportional to the mass. This implies a thermal wavelength \( \lambda_H = 8\pi^2 R_s \), a purely geometrical relationship which indicates two things. First, although the emission process involves quantum mechanics, its “kinematics” is somehow classical. Second, as long as the Schwarzschild radius is much longer than the Planck length, it should be possible to understand the Hawking effect using only qfts, i.e. semi-classical qg. (Actually one must also require that the back-reaction is small, in the sense that the change in the Hawking temperature due to the emission of a single Hawking quantum is small. This fails to hold for for a very nearly extremal black hole\cite{17}.)

A Planck mass (\( \sim 10^{-5} \text{ gm} \)) black hole—if it could be treated semi-classically—would have a Schwarzschild radius of order the Planck length (\( \sim 10^{-33} \text{ cm} \)) and a Hawking temperature of order the Planck energy (\( \sim 10^{19} \text{ GeV} \)). From this the Hawking temperatures for other black holes can be found by scaling. A solar mass black hole has a Schwarzschild radius \( R_s \sim 3 \text{ km} \) hence a Hawking temperature \( \sim 10^{-38} \text{ times} \)
smaller than the Planck energy, i.e. $10^{-19}$ GeV. Evaluated more carefully it works out to $T_H \sim 10^{-7}$ K. For a mini black hole of mass $M = 10^{15}$ gm one has $R_s \sim 10^{-13}$ cm and $T_H \sim 10^{11}$ K $\sim 10$ MeV.

The “back reaction”, i.e. the response of the spacetime metric to the Hawking process, should be well approximated by the semi-classical Einstein equation $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$ provided it is a small effect. To assess the size, we can compare the stress tensor to the background curvature near the horizon (not to $G_{\mu\nu}$, since that vanishes in the background). The background Riemann tensor components are $\sim 1/R_s^2$ (in suitable freely falling reference frame), while for example the energy density is $\sim T_H^4/\hbar^3 \sim \hbar/R_s^4$. Hence $G\langle T_{\mu\nu} \rangle \sim \hbar G/R_s^4 \sim (L_P/R_s)^2 R_s^{-2}$, which is much less than the background curvature provided $R_s \gg L_P$, i.e. provided the black hole is large compared to the Planck length.

Although a tiny effect for astrophysical black holes, the Hawking process has a profound implication: black holes can “evaporate”. How long does one take to evaporate? It emits roughly one Hawking quantum per light crossing time, hence $dM/dt \sim T_H/R_s \sim \hbar/R_s^2 \sim \hbar/G^2 M^2$. In Planck units $\hbar = c = G = 1$, we have $dM/dt \sim M^{-2}$. Integration yields the lifetime $\sim M^3 \sim (R_s/L_P)^2 R_s$. For the $10^{15}$ gm, $10^{-13}$ cm black hole mentioned earlier we have $(10^{20})^2 10^{-13}$ cm $= 10^{27}$ cm, which is the age of the universe. Hence a black hole of that mass in the early universe would be explosively ending its life now. These are called “primordial black holes” (pbh’s). None have been knowingly observed so far, nor their detritus, which puts limits on the present density of pbh’s (see for example the references in [18]). However, it has been suggested [18] that they might nevertheless be the source of the highest energy ($\sim 3 \times 10^{20}$ eV) cosmic rays. Note that even if pbh’s were copiously produced in the early universe, their initial number density could easily have been inflated away if they formed before inflation.

2.3 Black hole entropy

How about the black hole entropy? The thermodynamic relation $dM = T_H dS = (\hbar/8\pi GM)dS$ (for a black hole with no angular momentum or charge) implies $S_{BH} = 4\pi GM^2/\hbar = A_H/4L_P^2$, where $A_H = 4\pi R_s^2$ is the black hole horizon area. What is this huge entropy? What microstates does it count?

Whatever the microstates may be, $S_{BH}$ is the lower bound for the entropy emitted to the outside world as the black hole evaporates, i.e. the minimal missing information brought about by the presence of the black hole. By sending energy into a black hole one could make it last forever, emitting an arbitrarily large amount of entropy, so it
only makes sense to talk about the minimal entropy. This lower bound is attained when
the black hole evaporates reversibly into a thermal bath at a temperature infinitesimally
below $T_H$. (Evaporation into vacuum is irreversible, so the total entropy of the outside
increases even more \cite{19}, $S_{\text{emitted}} \sim (4/3)S_{BH}$. ) This amounts to a huge entropy increase.
The only way out of this conclusion would be if the semi-classical analysis breaks down
for some reason...which is suspected by many, not including me.

The so-called “information paradox” refers to the loss of information associated with
this entropy increase when a black hole evaporates, as well as to the loss of any other
information that falls into the black hole. I consider it no paradox at all, but many
think it is a problem to be avoided at all costs. I think this viewpoint results from
missing the strongly non-perturbative role of quantum gravity in evolving the spacetime
into and beyond the classically singular region, whether by generating baby universes or
otherwise (see section \ref{6.3.12} for more discussion of this point). Unfortunately it appears
unlikely that semi-classical qg can “prove” that information is or is not lost, though
people have tried hard. For it not to be lost there would have to be subtle effects not
captured by semi-classical qg, yet in a regime where semi-classical description “should”
be the whole story at leading order.

3 Harmonic oscillator

One can study a lot of interesting issues with free fields in curved spacetime, and I will
restrict entirely to that case except for a brief discussion. Free fields in curved spacetime
are similar to collections of harmonic oscillators with time-dependent frequencies. I will
therefore begin by developing the properties of a one-dimensional quantum harmonic
oscillator in a formalism parallel to that used in quantum field theory.

The action for a particle of mass $m$ moving in a time dependent potential $V(x,t)$ in
one dimension takes the form

$$S = \int dt \ L \quad L = \frac{1}{2}m\dot{x}^2 - V(x,t)$$

(3.1)

from which follows the equation of motion $m\ddot{x} = -\partial_x V(x,t)$. Canonical quantization
proceeds by (i) defining the momentum conjugate to $x$, $p = \partial L/\partial \dot{x} = m\dot{x}$, (ii) replacing
$x$ and $p$ by operators $\hat{x}$ and $\hat{p}$, and (iii) imposing the canonical commutation relations
$[\hat{x}, \hat{p}] = i\hbar$. The operators are represented as hermitian linear operators on a Hilbert
space, the hermiticity ensuring that their spectrum is real as befits a quantity whose
classical correspondent is a real number. In the Schrödinger picture the state is time-dependent, and the operators are time-independent. In the position representation for example the momentum operator is given by \( \hat{p} = -i\hbar \partial_x \). In the Heisenberg picture the state is time-independent while the operators are time-dependent. The commutation relation then should hold at each time, but this is still really only one commutation relation since the the equation of motion implies that if it holds at one initial time it will hold at all times. In terms of the position and velocity, the commutation relation(s) in the Heisenberg picture take the form

\[
[x(t), \dot{x}(t)] = i\hbar/m. \tag{3.2}
\]

Here and from here on the hats distinguishing numbers from operators are dropped.

Specializing now to a harmonic oscillator potential \( V(x, t) = \frac{1}{2}m\omega^2(t)x^2 \) the equation of motion takes the form

\[
\ddot{x} + \omega^2(t)x = 0. \tag{3.3}
\]

Consider now any operator solution \( x(t) \) to this equation. Since the equation is second order the solution is determined by the two hermitian operators \( x(0) \) and \( \dot{x}(0) \), and since the equation is linear the solution is linear in these operators. It is convenient to trade the pair \( x(0) \) and \( \dot{x}(0) \) for a single time-independent non-hermitian operator \( a \), in terms of which the solution is written as

\[
x(t) = f(t)a + \bar{f}(t)a^\dagger, \tag{3.4}
\]

where \( f(t) \) is a complex function satisfying the classical equation of motion,

\[
\ddot{f} + \omega^2(t)f = 0, \tag{3.5}
\]

\( \bar{f} \) is the complex conjugate of \( f \), and \( a^\dagger \) is the hermitian conjugate of \( a \). The commutation relations (3.2) take the form

\[
\langle f, f \rangle [a, a^\dagger] = 1, \tag{3.6}
\]

where the bracket notation is defined by

\[
\langle f, g \rangle = (im/\hbar)\left( \bar{f}\partial_t g - (\partial_t\bar{f})g \right). \tag{3.7}
\]

In a more abstract algebraic approach, one does not require at this stage a representation but rather requires that the quantum variables are elements of an algebra equipped with a star operation satisfying certain axioms. For quantum mechanics the algebraic approach is no different from the concrete representation approach, however for quantum fields the more general algebraic approach turns out to be necessary to have sufficient generality. In these lectures I will ignore this distinction. For an introduction to the algebraic approach in the context of quantum fields in curved spacetime see [6]. For a comprehensive treatment of the algebraic approach to quantum field theory see [20].
If the functions $f$ and $g$ are solutions to the harmonic oscillator equation (3.5), then the bracket (3.7) is independent of the time $t$ at which the right hand side is evaluated, which is consistent with the assumed time independence of $a$.

Let us now assume that the solution $f$ is chosen so that the real number $\langle f, f \rangle$ is positive. Then by rescaling $f$ we can arrange to have

$$\langle f, f \rangle = 1. \quad (3.8)$$

In this case the commutation relation (3.6) becomes

$$[a, a^\dagger] = 1, \quad (3.9)$$

the standard relation for the harmonic oscillator raising and lowering operators. Using the bracket with the operator $x$ we can pluck out the raising and lowering operators from the position operator,

$$a = \langle f, x \rangle, \quad a^\dagger = -\langle \bar{f}, x \rangle. \quad (3.10)$$

Since both $f$ and $x$ satisfy the equation of motion, the brackets in (3.10) are time independent as they must be.

A Hilbert space representation of the operators can be built by introducing a state $|0\rangle$ defined to be normalized and satisfying $a|0\rangle = 0$. For each $n$, the state $|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle$ is a normalized eigenstate of the number operator $N = a^\dagger a$ with eigenvalue $n$. The span of all these states defines a Hilbert space of “excitations” above the state $|0\rangle$.

So far the solution $f(t)$ is arbitrary, except for the normalization condition (3.8). A change in $f(t)$ could be accompanied by a change in $a$ that keeps the solution $x(t)$ unchanged. In the special case of a constant frequency $\omega(t) = \omega$ however, the energy is conserved, and a special choice of $f(t)$ is selected if we require that the state $|0\rangle$ be the ground state of the Hamiltonian. Let us see how this comes about.

For a general $f$ we have

$$H = \frac{1}{2}m \dot{x}^2 + \frac{1}{2}m \omega^2 x^2$$

$$= \frac{1}{2}m \left[ (\dot{f}^2 + \omega^2 f^2)aa + (\dot{f}^2 + \omega^2 f^2)^*a^\dagger a^\dagger + (|\dot{f}|^2 + \omega^2 |f|^2)(aa^\dagger + a^\dagger a) \right]. \quad (3.12)$$

Thus

$$H|0\rangle = \frac{1}{2}m(f^2 + \omega^2 f^2)^*a^\dagger a^\dagger|0\rangle + (|\dot{f}|^2 + \omega^2 |f|^2)|0\rangle, \quad (3.13)$$
where the commutation relation (3.9) was used in the last term. If $|0\rangle$ is to be an eigenstate of $H$, the first term must vanish, which requires

$$\dot{f} = \pm i\omega f.$$  

(3.14)

For such an $f$ the norm is

$$\langle f, f \rangle = \mp \frac{2m\omega}{\hbar} |f|^2,$$  

(3.15)

hence the positivity of the normalization condition (3.8) selects from (3.14) the minus sign. This yields what is called the normalized positive frequency solution to the equation of motion, defined by

$$f(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} e^{-i\omega t}$$  

(3.16)

up to an arbitrary constant phase factor.

With $f$ given by (3.16) the Hamiltonian (3.12) becomes

$$H = \frac{1}{2}\hbar\omega (aa^{\dagger} + a^{\dagger}a)$$  

(3.17)

$$= \hbar\omega (N + \frac{1}{2}),$$  

(3.18)

where the commutation relation (3.9) was used in the last step. The spectrum of the number operator is the non-negative integers, hence the minimum energy state is the one with $N = 0$, and “zero-point energy” $\hbar\omega/2$. This is just the state $|0\rangle$ annihilated by $a$ as defined above. If any function other than (3.16) is chosen to expand the position operator as in (3.4), the state annihilated by $a$ is not the ground state of the oscillator.

Note that although the mean value of the position is zero in the ground state, the mean of its square is

$$\langle 0 | x^2 | 0 \rangle = \hbar/2m\omega.$$  

(3.19)

This characterizes the “zero-point fluctuations” of the position in the ground state.

4 Quantum scalar field in curved spacetime

Much of interest can be done with a scalar field, so it suffices for an introduction. The basic concepts and methods extend straightforwardly to tensor and spinor fields.

To being with let’s take a spacetime of arbitrary dimension $D$, with a metric $g_{\mu\nu}$ of signature $(+-\cdots-)$. The action for the scalar field $\varphi$ is

$$S = \int d^Dx \sqrt{|g|^{\frac{1}{2}}} \left( g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - (m^2 + \xi R)\varphi^2 \right),$$  

(4.1)
for which the equation of motion is
\[(\Box + m^2 + \xi R) \varphi = 0, \quad \Box = |g|^{-1/2} \partial_\mu |g|^{1/2} g^{\mu\nu} \partial_\nu.\] (4.2)

(With \(\hbar\) explicit, the mass \(m\) should be replaced by \(m/\hbar\), however we’ll leave \(\hbar\) implicit here.) The case where the coupling \(\xi\) to the Ricci scalar \(R\) vanishes is referred to as “minimal coupling”, and that equation is called the Klein-Gordon (KG) equation. If also the mass \(m\) vanishes it is called the “massless, minimally coupled scalar”. Another special case of interest is “conformal coupling” with \(m = 0\) and \(\xi = (D - 2)/4(D - 1)\).

4.1 Conformal coupling

Let me pause briefly to explain the meaning of conformal coupling since it comes up often in discussions of quantum fields in curved spacetime, primarily either because Robertson-Walker metrics are conformally flat or because all two-dimensional metrics are conformally flat. Consider making a position dependent conformal transformation of the metric:
\[\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu},\] (4.3)

which induces the changes
\[\tilde{g}^{\mu\nu} = \Omega^{-2}(x)g^{\mu\nu}, \quad |\tilde{g}|^{1/2} = \Omega^D(x)|g|^{1/2}, \quad |\tilde{g}|^{1/2}\tilde{g}^{\mu\nu} = \Omega^{D-2}(x)|g|^{1/2}g^{\mu\nu},\] (4.4)
\[\tilde{R} = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} = \Omega^{-2}\left( R - 2(D - 1)\Box \ln \Omega - (D - 1)(D - 2)g^{\alpha\beta}(\ln \Omega)_{,\alpha}(\ln \Omega)_{,\beta}\right).\] (4.5)

In \(D = 2\) dimensions, the action is simply invariant in the massless, minimally coupled case without any change of the scalar field: \(S[\varphi, g] = S[\tilde{\varphi}, \tilde{g}]\). In any other dimension, the kinetic term is invariant under a conformal transformation with a constant \(\Omega\) if we accompany the metric change (4.3) with a change of the scalar field, \(\tilde{\varphi} = \Omega^{(2-D)/2}\varphi\). (This scaling relation corresponds to the fact that the scalar field has dimension \([\text{length}]^{(2-D)/2}\) since the action must be dimensionless after factoring out an overall \(\hbar\).) For a non-constant \(\Omega\) the derivatives in the kinetic term ruin the invariance in general. However it can be shown that the action is invariant (up to a boundary term) if the coupling constant \(\xi\) is chosen to have the special value given in the previous paragraph, i.e. \(S[\varphi, g] = S[\tilde{\varphi}, \tilde{g}]\). In \(D = 4\) dimensions that value is \(\xi = 1/6\).
4.2 Canonical quantization

To canonically quantize we first pass to the Hamiltonian description. Separating out a

time coordinate \( x^0, x^\mu = (x^0, x^1) \), we can write the action as

\[
S = \int dx^0 L, \quad L = \int d^{D-1}x \mathcal{L}. \tag{4.6}
\]

The canonical momentum at a time \( x^0 \) is given by

\[
\pi(x) = \frac{\delta L}{\delta (\partial_0 \varphi(x))} = \frac{|g|^{1/2} g^{\mu 0} \partial_\mu \varphi(x)}{\pm \frac{1}{2} n^\mu \partial_\mu \varphi(x)}. \tag{4.7}
\]

Here \( x \) labels a point on a surface of constant \( x^0 \), the \( x^0 \) argument of \( \varphi \) is suppressed,

\( n^\mu \) is the unit normal to the surface, and \( h \) is the determinant of the induced spatial

metric \( h_{ij} \). To quantize, the field \( \varphi \) and its conjugate momentum \( \pi \) are now promoted
to hermitian operators\(^2\) and required to satisfy the canonical commutation relation,

\[
[\varphi(x), \pi(y)] = i\hbar \delta^{D-1}(x, y) \tag{4.8}
\]

It is worth noting that, being a variational derivative, the conjugate momentum is a
density of weight one, and hence the Dirac delta function on the right hand side of

(4.8) is a density of weight one in the second argument. It is defined by the property

\[
\int d^{D-1}y \delta^{D-1}(x, y) f(y) = f(x) \text{ for any scalar function } f, \text{ without the use of a metric}

volume element. In analogy with the bracket (3.7) defined for the case of the harmo

nic oscillator, one can form a conserved bracket from two complex solutions to the scalar wave equation

(4.2),

\[
\langle f, g \rangle = \int \Sigma d\Sigma_\mu j^\mu, \quad j^\mu(f, g) = (i/\hbar)|g|^{1/2} g^{\mu \nu} \left( T \partial_\nu g - (\partial_\nu T) g \right). \tag{4.9}
\]

This bracket is sometimes called the \textit{Klein-Gordon inner product}, and \( \langle f, f \rangle \) the \textit{Klein-

Gordon norm} of \( f \). The current density \( j^\mu(f, g) \) is divergenceless (\( \partial_\mu j^\mu = 0 \)) when the

functions \( f \) and \( g \) satisfy the KG equation (4.2), hence the value of the integral in (4.9) is

independent of the spacelike surface \( \Sigma \) over which it is evaluated, provided the functions

vanish at spatial infinity. The KG inner product satisfies the relations

\[
\overline{\langle f, g \rangle} = -\langle \overline{f}, \overline{g} \rangle = \langle g, f \rangle, \quad \langle f, \overline{f} \rangle = 0 \tag{4.10}
\]

Note that it is not positive definite.

\(^2\)See footnote 1.
4.3 Hilbert space

At this point it is common to expand the field operator in modes and to associate annihilation and creation operators with modes, in close analogy with the harmonic oscillator (3.4), however instead I will begin with individual wave packet solutions. My reason is that in some situations there is no particularly natural set of modes, and none is needed to make physical predictions from the theory. (An illustration of this statement will be given in our treatment of the Hawking effect.) A mode decomposition is a basis in the space of solutions, and has no fundamental status.

In analogy with the harmonic oscillator case (3.10), we define the annihilation operator associated with a complex classical solution \( f \) by the bracket of \( f \) with the field operator \( \varphi \):

\[
a(f) = \langle f, \varphi \rangle
\]  

(4.11)

Since both \( f \) and \( \varphi \) satisfy the wave equation, \( a(f) \) is well-defined, independent of the surface on which the bracket integral is evaluated. It follows from the above definition and the hermiticity of \( \varphi \) that the hermitian conjugate of \( a(f) \) is given by

\[
a^\dagger(f) = -a(\bar{f}).
\]  

(4.12)

The canonical commutation relation (4.8) together with the definition of the momentum (4.7) imply that

\[
[a(f), a^\dagger(g)] = \langle f, g \rangle.
\]  

(4.13)

The converse is also true, in the sense that if (4.13) holds for all solutions \( f \) and \( g \), then the canonical commutation relation holds. Using (4.12), we immediately obtain the similar relations

\[
[a(f), a(g)] = -\langle f, \bar{g} \rangle, \quad [a^\dagger(f), a^\dagger(g)] = -\langle \bar{f}, g \rangle
\]  

(4.14)

If \( f \) is a positive norm solution with unit norm \( \langle f, f \rangle = 1 \), then \( a(f) \) and \( a^\dagger(f) \) satisfy the usual commutation relation for the raising and lowering operators for a harmonic oscillator, \([a(f), a^\dagger(f)] = 1\). Suppose now that \(|\Psi\rangle \) is a normalized quantum state satisfying \( a(f)|\Psi\rangle = 0 \). Of course this condition does not specify the state, but rather only one aspect of the state. Nevertheless, for each \( n \), the state \(|n, \Psi\rangle = (1/\sqrt{n!})(a^\dagger(f))^n|\Psi\rangle \) is a normalized eigenstate of the number operator \( N(f) = a^\dagger(f)a(f) \) with eigenvalue \( n \). The span of all these states defines a Fock space of \( f \)-wavepacket “\( n \)-particle excitations” above the state \(|\Psi\rangle \).

If we want to construct the full Hilbert space of the field theory, how can we proceed? We should find a decomposition of the space of complex solutions to the wave equation...
\(S\) into a direct sum of a positive norm subspace \(S_p\) and its complex conjugate \(\overline{S_p}\), such that all brackets between solutions from the two subspaces vanish. That is, we must find a direct sum decomposition
\[
S = S_p \oplus \overline{S_p}
\] (4.15)
such that
\[
\langle f, f \rangle > 0 \quad \forall f \in S_p \tag{4.16}
\]
\[
\langle f, g \rangle = 0 \quad \forall f, g \in S_p. \tag{4.17}
\]
The first condition implies that each \(f\) in \(S_p\) can be scaled to define its own harmonic oscillator sub-algebra as in the previous paragraph. The second condition implies, according to (4.14), that the annihilators and creators for \(f\) and \(g\) in the subspace \(S_p\) commute amongst themselves: \([a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]\).

Given such a decomposition a total Hilbert space for the field theory can be defined as the space of finite norm sums of possibly infinitely many states of the form \(a^\dagger(f_1) \cdots a^\dagger(f_n)|0\rangle\), where \(|0\rangle\) is a state such that \(a(f)|0\rangle = 0\) for all \(f\) in \(S_p\), and all \(f_1, \ldots, f_n\) are in \(S_p\). The state \(|0\rangle\) is called a Fock vacuum. It depends on the decomposition (4.15), and in general is not the ground state (which is not even well defined unless the background metric is globally static). The representation of the field operator on this Fock space is hermitian and satisfies the canonical commutation relations.

4.4 Flat spacetime

Now let’s apply the above generalities to the case of a massive scalar field in flat spacetime. In this setting a natural decomposition of the space of solutions is defined by positive and negative frequency with respect to a Minkowski time translation, and the corresponding Fock vacuum is the ground state. I summarize briefly since this is standard flat spacetime quantum field theory.

Because of the infinite volume of space, plane wave solutions are of course not normalizable. To keep the physics straight and the language simple it is helpful to introduce periodic boundary conditions, so that space becomes a large three-dimensional torus with circumferences \(L\) and volume \(V = L^3\). The allowed wave vectors are then \(k = (2\pi/L)n\), where the components of the vector \(n\) are integers. In the end we can always take the limit \(L \to \infty\) to obtain results for local quantities that are insensitive to this formal compactification.
A complete set of solutions ("modes") to the classical wave equation (4.2) with the flat space d'Alembertian and $R = 0$ is given by

$$f_k(t, x) = \sqrt{\frac{\hbar}{2V\omega(k)}} e^{-i\omega(k)t} e^{ik\cdot x}$$

(4.18)

where

$$\omega(k) = \sqrt{k^2 + m^2},$$

(4.19)

together with the solutions obtained by replacing the positive frequency $\omega(k)$ by its negative, $-\omega(k)$. The brackets between these solutions satisfy

$$\langle f_k, f_l \rangle = \delta_{k,l}$$

(4.20)

$$\langle \overline{f_k}, f_l \rangle = -\delta_{k,l}$$

(4.21)

$$\langle f_k, \overline{f_l} \rangle = 0, \quad (4.22)$$

so they provide an orthogonal decomposition of the solution space into positive norm solutions and their conjugates as in (4.17), with $S_p$ the space spanned by the positive frequency modes $f_k$. As described in the previous subsection this provides a Fock space representation.

If we define the annihilation operator associated to $f_k$ by

$$a_k = \langle f_k, \varphi \rangle,$$

(4.23)

then the field operator has the expansion

$$\varphi = \sum_k \left( f_k a_k + \overline{f_k} a_k^\dagger \right).$$

(4.24)

Since the individual solutions $f_k$ have positive frequency, and the Hamiltonian is a sum over the contributions from each $k$ value, our previous discussion of the single oscillator shows that the vacuum state defined by

$$a_k |0\rangle = 0$$

(4.25)

for all $k$ is in fact the ground state of the Hamiltonian. The states

$$a_k^\dagger |0\rangle$$

(4.26)

have momentum $\hbar k$ and energy $\hbar \omega(k)$, and are interpreted as single particle states. States of the form $a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$ are interpreted as $n$-particle states.
Note that although the field Fourier component \( \varphi_k = f_k a_k + \bar{f}_{-k} a_{-k} \) has zero mean in the vacuum state, like the harmonic oscillator position it undergoes “zero-point fluctuations” characterized by
\[
\langle 0 | \varphi_k^\dagger \varphi_k | 0 \rangle = |f_{-k}|^2 = \frac{\hbar}{2V\omega(k)},
\]
which is entirely analogous to the oscillator result (3.19).

4.5 Curved spacetime, “particles”, and stress tensor

In a general curved spacetime setting there is no analog of the preferred Minkowski vacuum and definition of particle states. However, it is clear that we can import these notions locally in an approximate sense if the wavevector and frequency are high enough compared to the inverse radius of curvature. Slightly more precisely, we can expand the metric in Riemann normal coordinates about any point \( x_0 \):
\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\nu\alpha\beta}(x_0)(x - x_0)^\alpha(x - x_0)^\beta + O((x - x_0)^3).
\]
(4.28)

If \( k^2 \) and \( \omega^2(k) \) are much larger than any component of the Riemann tensor \( R_{\mu\nu\alpha\beta}(x_0) \) in this coordinate system then it is clear that the flat space interpretation of the corresponding part of Fock space will hold to a good approximation, and in particular a particle detector will respond to the Fock states as it would in flat spacetime. This notion is useful at high enough wave vectors locally in any spacetime, and for essentially all wave vectors asymptotically in spacetimes that are asymptotically flat in the past or the future or both.

More generally, however, the notion of a “particle” is ambiguous in curved spacetime, and one should use field observables to characterize states. One such observable determines how a “particle detector” coupled to the field would respond were it following some particular worldline in spacetime and the coupling were adiabatically turned on and off at prescribed times. For example the transition probability of a point monopole detector is determined in lowest order perturbation theory by the two-point function \( \langle \Psi | \varphi(x)\varphi(x')| \Psi \rangle \) evaluated along the worldline\([21][11]\) of the detector. (For a careful discussion of the regularization required in the case of a point detector see \([22]\).) Alternatively this quantity—along with the higher order correlation functions—is itself a probe of the state of the field.

Another example of a field observable is the expectation value of the stress energy tensor, which is the source term in the semi-classical Einstein equation
\[
G_{\mu\nu} = 8\pi G \langle \Psi | T_{\mu\nu}(x) | \Psi \rangle.
\]
(4.29)
This quantity is infinite because it contains the product of field operators at the same point. Physically, the infinity is due to the fluctuations of the infinitely many ultraviolet field modes. For example, the leading order divergence of the energy density can be attributed to the zero-point energy of the field fluctuations, but there are subleading divergences as well. We have no time here to properly go into this subject, but rather settle for a few brief comments.

One way to make sense of the expectation value is via the difference between its values in two different states. This difference is well-defined (with a suitable regulator) and finite for any two states sharing the same singular short distance behavior. The result depends of course upon the comparison state. Since the divergence is associated with the very short wavelength modes, it might seem that to uniquely define the expectation value at a point \( x \) it should suffice to just subtract the infinities for a state defined as the vacuum in the local flat spacetime approximation at \( x \). This subtraction is ambiguous however, even after making it as local as possible and ensuring local conservation of energy (\( \nabla^\mu \langle T_{\mu \nu} \rangle = 0 \)). It defines the expectation value only up to a tensor \( H_{\mu \nu} \) constructed locally from the background metric with four or fewer derivatives and satisfying the identity \( \nabla^\mu H_{\mu \nu} = 0 \). The general such tensor is the variation with respect to the metric of the invariant functional

\[
\int d^D x \sqrt{|g|} \left( c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu \nu} R_{\mu \nu} + c_4 R_{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} \right).
\]  

(4.30)

(In four spacetime dimensions the last term can be rewritten as a combination of the first two and a total divergence.) Thus \( H_{\mu \nu} \) is a combination of \( g_{\mu \nu} \), the Einstein tensor \( G_{\mu \nu} \), and curvature squared terms. In effect, the ambiguity \(- H_{\mu \nu} \) is added to the metric side of the semi-classical field equation, where it renormalizes the cosmological constant and Newton’s constant, and introduces curvature squared terms.

A different approach is to define the expectation value of the stress tensor via the metric variation of the renormalized effective action, which possesses ambiguities of the same form as (4.30). Hence the two approaches agree.

### 4.6 Remarks

#### 4.6.1 Continuum normalization of modes

Instead of the “box normalization” used above we could “normalize” the solutions \( f_k \) with the factor \( V^{-1/2} \) replaced by \( (2\pi)^{-3/2} \). Then the Kronecker \( \delta \)'s in (4.20, 4.21) would be replaced by Dirac \( \delta \)-functions and the discrete sum over momenta in (4.24)
would be an integral over \( k \). In this case the annihilation and creation operators would satisfy \([a_k, a^\dagger_l] = \delta^3(k, l)\).

### 4.6.2 Massless minimally coupled zero mode

The massless minimally coupled case \( m = 0, \xi = 0 \) has a peculiar feature. The spatially constant function \( f(t, x) = c_0 + c_1 t \) is a solution to the wave equation that is not included among the positive frequency solutions \( f_k \) or their conjugates. This “zero mode” must be quantized as well, but it behaves like a free particle rather than like a harmonic oscillator. In particular, the state of lowest energy would have vanishing conjugate field momentum and hence would be described by a Schrödinger wave function \( \psi(\varphi_0) \) that is totally delocalized in the field amplitude \( \varphi_0 \). Such a wave function would be non-normalizable as a quantum state, just as any momentum eigenstate of a non-relativistic particle is non-normalizable. Any normalized state would be described by a Schrödinger wavepacket that would spread in \( \varphi_0 \) like a free particle spreads in position space, and would have an expectation value \( \langle \varphi_0 \rangle \) growing linearly in time. This suggests that no time-independent state exists. That is indeed true in the case where the spatial directions are compactified on a torus for example, so that the modes are discrete and the zero mode carries as much weight as any other mode. In non-compact space one must look more closely. It turns out that in 1+1 dimensions the zero mode continues to preclude a time independent state (see e.g. [23], although the connection to the behavior of the zero mode is not made there), however in higher dimensions it does not, presumably because of the extra factors of \( k \) in the measure \( k^{D-2}dk \). A version of the same issue arises in deSitter space, where no deSitter invariant state exists for the massless, minimally coupled field [24]. That is true in higher dimensions as well, however, which may be related to the fact that the spatial sections of deSitter space are compact.

### 5 Particle creation

We turn now to the subject of particle creation in curved spacetime (which would more appropriately be called “field excitation”, but we use the standard term). The main applications are to the case of expanding cosmological spacetimes and to the Hawking effect for black holes. To begin with I will discuss the analogous effect for a single harmonic oscillator, which already contains the essential elements of the more complicated cases. The results will then be carried over to the cosmological setting. The following section then takes up the subject of the Hawking effect.
5.1 Parametric excitation of a harmonic oscillator

A quantum field in a time-dependent background spacetime can be modeled in a simple way by a harmonic oscillator whose frequency $\omega(t)$ is a given function of time. The equation of motion is then (3),

$$\ddot{x} + \omega^2(t) x = 0.$$  \hfill (5.1)

We consider the situation where the frequency is asymptotically constant, approaching $\omega_{\text{in}}$ in the past and $\omega_{\text{out}}$ in the future. The question to be answered is this: if the oscillator starts out in the ground state $|0_{\text{in}}\rangle$ appropriate to $\omega_{\text{in}}$ as $t \to -\infty$, what is the state as $t \to +\infty$? More precisely, in the Heisenberg picture the state does not evolve, so what we are really asking is how is the state $|0_{\text{in}}\rangle$ expressed as a Fock state in the out-Hilbert space appropriate to $\omega_{\text{out}}$? It is evidently not the same as the ground state $|0_{\text{out}}\rangle$ of the Hamiltonian in the asymptotic future.

To answer this question we need only relate the annihilation and creation operators associated with the in and out normalized positive frequency modes $f_{\text{in}}(t)$, which are solutions to (5.1) with the asymptotic behavior

$$f_{\text{out}} (t) \xrightarrow{t \to \pm \infty} \sqrt{\frac{\hbar}{2m\omega_{\text{in}}}} \exp(-i\omega_{\text{in}} t).$$  \hfill (5.2)

Since the equation of motion is second order in time derivatives it admits a two-parameter family of solutions, hence there must exist complex constants $\alpha$ and $\beta$ such that

$$f_{\text{out}} = \alpha f_{\text{in}} + \beta \bar{f}_{\text{in}}.$$  \hfill (5.3)

The normalization condition $\langle f_{\text{out}}, f_{\text{out}} \rangle = 1$ implies that

$$|\alpha|^2 - |\beta|^2 = 1.$$  \hfill (5.4)

The out annihilation operator is given (see (3.10)) by

$$a_{\text{out}} = \langle f_{\text{out}}, x \rangle = \langle \alpha f_{\text{in}} + \beta \bar{f}_{\text{in}}, x \rangle = \alpha a_{\text{in}} - \bar{\beta} a_{\text{in}}^\dagger.$$  \hfill (5.5)
the Bogoliubov coefficients. The mean value of the out number operator $N_{\text{out}} = a_{\text{out}}^\dagger a_{\text{out}}$ is nonzero in the state $|0_{\text{in}}\rangle$:

$$\langle 0_{\text{in}}|N_{\text{out}}|0_{\text{in}}\rangle = |\beta|^2.$$  

(5.8)

In this sense the time dependence of $\omega(t)$ excites the oscillator, and $|\beta|^2$ characterizes the excitation number.

To get a feel for the Bogoliubov coefficient $\beta$ let us consider two extreme cases, adiabatic and sudden.

### 5.1.1 Adiabatic transitions and ground state

The adiabatic case corresponds to a situation in which the frequency is changing very slowly compared to the period of oscillation,

$$\frac{\dot{\omega}}{\omega} \ll 1.$$  

(5.9)

In this case there is almost no excitation, so $|\beta| \ll 1$. Generically in the adiabatic case the Bogoliubov coefficient is exponentially small, $\beta \sim \exp(-\omega_0 T)$, where $\omega_0$ is a typical frequency and $T$ characterizes the time scale for the variations in the frequency.

In the Schrödinger picture, one can say that during an adiabatic change of $\omega(t)$ the state continually adjusts to remain close to the instantaneous adiabatic ground state. This state at time $t_0$ is the one annihilated by the lowering operator $a_{\text{f}_{t_0}}$ defined by the solution $f_{t_0}(t)$ satisfying the initial conditions at $t_0$ corresponding to the “instantaneous positive frequency solution”,

$$f_{t_0}(t_0) = \sqrt{\frac{\hbar}{2m\omega(t_0)}},$$

(5.10)

$$\dot{f}_{t_0}(t_0) = -i\omega(t_0)f_{t_0}(t_0).$$

(5.11)

The instantaneous adiabatic ground state is sometimes called the “lowest order adiabatic ground state at $t_0$”. One can also consider higher order adiabatic ground states as follows (see e.g. [3][5]). A function of the WKB form $(\hbar/2mW(t))^{1/2}\exp(-i \int^t W(t') dt')$ is a normalized solution to (5.11) provided $W(t)$ satisfies a certain second order differential equation. That equation can be solved iteratively, yielding an expansion $W(t) = \omega(t) + \cdots$, where the subsequent terms involve time derivatives of $\omega(t)$. The lowest order adiabatic ground state at $t_0$ is defined using the solution whose initial conditions (5.11) match the lowest order truncation of the expansion for $W(t)$. A higher order adiabatic ground state is similarly defined using a higher order truncation.
5.1.2 Sudden transitions

The opposite extreme is the sudden one, in which $\omega$ changes instantaneously from $\omega_{in}$ to $\omega_{out}$ at some time $t_0$. We can then find the Bogoliubov coefficients using (5.3) and its first derivative at $t_0$. For $t_0 = 0$ the result is

$$\alpha = \frac{1}{2} \left( \sqrt{\frac{\omega_{in}}{\omega_{out}}} + \sqrt{\frac{\omega_{out}}{\omega_{in}}} \right)$$  \hspace{1cm} (5.12)

$$\beta = \frac{1}{2} \left( \sqrt{\frac{\omega_{in}}{\omega_{out}}} - \sqrt{\frac{\omega_{out}}{\omega_{in}}} \right).$$  \hspace{1cm} (5.13)

(For $t_0 \neq 0$ there are extra phase factors in these solutions.) Interestingly, the amount of excitation is precisely the same if the roles of $\omega_{in}$ and $\omega_{out}$ are interchanged. For an example, consider the case where $\omega_{out} = 4\omega_{in}$, for which $\alpha = 5/4$ and $\beta = -3/4$. In this case the expectation value (5.8) of the out number operator is $9/16$, so there is about “half an excitation”.

5.1.3 Relation between in and out ground states & the squeeze operator

The expectation value of $N_{out}$ is only one number characterizing the relation between $|O_{in}\rangle$ and the out states. We shall now determine the complete relation

$$|O_{in}\rangle = \sum_n c_n |n\rangle_{out}$$ \hspace{1cm} (5.14)

where the states $|n\rangle_{out}$ are eigenstates of $N_{out}$ and the $c_n$ are constants.

One can find $a_{in}$ in terms of $a_{out}$ and $a_{out}^\dagger$ by combining (5.3) with its complex conjugate to solve for $f_{in}$ in terms of $f_{out}$ and $\bar{f}_{out}$. In analogy with (5.7) one then finds

$$a_{in} = \alpha a_{out} + \beta a_{out}^\dagger.$$  \hspace{1cm} (5.15)

Thus the defining condition $a_{in}|0_{in}\rangle = 0$ implies

$$a_{out}|0_{in}\rangle = -\frac{\bar{\beta}}{\alpha} a_{out}^\dagger|0_{in}\rangle.$$  \hspace{1cm} (5.16)

A transparent way to solve this is to note that the commutation relation $[a_{out}, a_{out}^\dagger] = 1$ suggests the formal analogy $a_{out} = \partial/\partial a_{out}^\dagger$. This casts (5.16) as a first order ordinary differential equation, with solution

$$|0_{in}\rangle = \mathcal{N} \exp \left[ -\left( \frac{\bar{\beta}}{2\alpha} \right) a_{out}^\dagger a_{out} \right] |0_{out}\rangle$$ \hspace{1cm} (5.17)

$$= \mathcal{N} \sum_n \frac{\sqrt{2n!}}{n!} \left( -\frac{\bar{\beta}}{2\alpha} \right)^n |2n\rangle_{out}.$$  \hspace{1cm} (5.18)

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Note that the state $|0_{\text{out}}\rangle$ on which the exponential operator acts is annihilated by $a_{\text{out}}^\dagger$, so there is no extra term from $a_{\text{out}}^\dagger$-dependence.

The state $|0_{\text{in}}\rangle$ thus contains only even numbered excitations when expressed in terms of the out number eigenstates. The normalization constant $\mathcal{N}$ is given by

$$|\mathcal{N}|^{-2} = \sum_n \frac{2^n!}{(n!)^2} \left| \frac{\bar{\beta}}{2\alpha} \right|^{2n}.$$  \hfill (5.19)

For large $n$ the summand approaches $|\bar{\beta}/\alpha|^{2n} = [|\beta|^2/(|\beta|^2 + 1)]^n$, where the relation (5.4) was used in the last step. The sum therefore converges, and can be evaluated\(^3\) to yield

$$|\mathcal{N}| = \left(1 - |\beta/\alpha|^2\right)^{1/4} = |\alpha|^{-1/2}. \hfill (5.20)$$

An alternate way of describing $|0_{\text{in}}\rangle$ in the out Hilbert space is via the squeeze operator

$$S = \exp \left[ \frac{z}{2} a^\dagger a - \frac{\bar{z}}{2} aa \right]. \hfill (5.21)$$

Since its exponent is anti-hermitian, $S$ is unitary. Conjugating $a$ by $S$ yields

$$S^\dagger a S = \cosh |z| a + \sinh |z| \frac{z}{|z|} a^\dagger. \hfill (5.22)$$

This has the form of the Bogoliubov transformation (5.15) with $\alpha = \cosh |z|$ and $\beta = \sinh |z|(z/|z|)$. With $a_{\text{out}}$ in place of $a$ in $S$ this gives

$$a_{\text{in}} = S^\dagger a_{\text{out}} S. \hfill (5.23)$$

The condition $a_{\text{in}} |0_{\text{in}}\rangle = 0$ thus implies $a_{\text{out}} S |0_{\text{in}}\rangle = 0$, so evidently

$$|0_{\text{in}}\rangle = S^\dagger |0_{\text{out}}\rangle \hfill (5.24)$$

up to a constant phase factor. That is, the $\text{in}$ and $\text{out}$ ground states are related by the action of the squeeze operator $S$. Since $S$ is unitary, the right hand side of (5.24) is manifestly normalized.

\(^3\)Alex Maloney pointed out that it can be evaluated by expressing the binomial coefficient $2n!/(n!)^2$ as the contour integral $\oint (dz/2\pi i z)(z + 1/z)^{2n}$ and interchanging the order of the sum with the integral.
5.2 Cosmological particle creation

We now apply the ideas just developed to a free scalar quantum field satisfying the KG equation (4.2) in a homogeneous isotropic spacetime. The case when the spatial sections are flat is slightly simpler, and it is already quite applicable, hence we restrict to that case here.

The spatially flat Robertson-Walker (RW) line element takes the form:

\[ ds^2 = dt^2 - a^2(t)dx^i dx^i = a^2(\eta)(d\eta^2 - dx^i dx^i) \] (5.25)

It is conformally flat, as are all RW metrics. The coordinate \( \eta = \int dt/a(t) \) is called the conformal time, to distinguish it from the proper time \( t \) of the isotropic observers. The d’Alembertian \( \Box \) for this metric is given by

\[ \Box = \partial_t^2 + (3 \dot{a}/a) \partial_t - a^{-2} \partial_{x^i}^2, \] (5.26)

where the dot stands for \( \partial/\partial t \). The spatial translation symmetry allows the spatial dependence to be separated from the time dependence. A field

\[ u_k(x, t) = \zeta_k(t) e^{i k \cdot x} \] (5.27)

satisfies the field equation (4.2) provided \( \zeta_k(t) \) satisfies an equation similar to that of a damped harmonic oscillator but with time-dependent damping coefficient \( 3 \dot{a}/a \) and time-dependent frequency \( a^{-2}k^2 + m^2 + \xi R \). It is worth emphasizing that in spite of the “damping”, the field equation is Hamiltonian, and the Klein-Gordon norm (4.9) of any solution is conserved in the evolution.

The field equation can be put into the form of an undamped oscillator with time-dependent frequency by using the conformal time \( \eta \) instead of \( t \) and factoring out an appropriate power of the conformal factor \( a^2(\eta) \):

\[ \zeta_k = a^{-1} \chi_k. \] (5.28)

The function \( u_k \) satisfies the field equation if and only if

\[ \chi_k'' + \omega^2(\eta) \chi_k = 0, \] (5.29)

where the prime stands for \( d/d\eta \) and

\[ \omega^2(\eta) = k^2 + m^2 a^2 - (1 - 6 \xi)(a''/a). \] (5.30)
(In the special case of conformal coupling $m = 0$ and $\xi = 1/6$, this becomes the time-independent harmonic oscillator, so that case is just like flat spacetime. All effects of the curvature are then incorporated by the prefactor $a(\eta)^{-1}$ in (5.28).

The $u_k$ are orthogonal in the Klein-Gordon inner product (4.9), and they are normalized\(^4\) provided the $\chi_k$ have unit norm:

$$\langle u_k, u_l \rangle = \delta_{k,l} \iff (iV/\hbar) (\bar{\chi}_k \chi'_k - \bar{\chi}'_k \chi_k) = 1.$$  \hspace{1cm} (5.31)

Note that the relevant norm for the $\chi_k$ differs from that for the harmonic oscillator (3.7) only by the replacement $m \rightarrow V$, where $V$ is the $x^i$-coordinate volume of the constant $t$ surfaces. We also have $\langle u_k, \bar{u}_l \rangle = 0$ for all $k, l$, hence these modes provide an orthogonal positive/negative norm decomposition of the space of complex solutions. As discussed in section 4.3, this yields a corresponding Fock space representation for the field operators. The field operator can be expanded in terms of the corresponding annihilation and creation operators:

$$\varphi(x, t) = \sum_k \left( u_k(x, t) a_k + \bar{u}_k(x, t) a_k^\dagger \right).$$  \hspace{1cm} (5.32)

Consider now the special case where there is no time dependence in the past and future, $a(\eta) \rightarrow$ constant. The in and out “vacua” are the ground states of the Hamiltonian at early and late times, and are the states annihilated by the $a_k$ associated with the $u_k^{in,out}$ constructed with early and late time positive frequency modes $\chi_k^{in,out}$, as explained in section 4.4:

$$\chi_k^{out}(\eta) \rightarrow \infty \iff \sqrt{\hbar \over 2V(\omega_{in,out})} \exp(-i\omega_{in,out} \eta).$$  \hspace{1cm} (5.33)

The Bogoliubov transformation now takes the form:

$$u_k^{out} = \sum_{k'} \left( \alpha_{kk'} u_k^{in} + \beta_{kk'} \bar{u}_k^{in} \right).$$  \hspace{1cm} (5.34)

Matching the coefficients of $\exp(ik \cdot x)$, we see that

$$\alpha_{kk'} = \alpha_{k} \delta_{k,k'}, \quad \beta_{kk'} = \beta_{k} \delta_{k,-k'},$$  \hspace{1cm} (5.35)

i.e. the Bogoliubov coefficients mix only modes of wave vectors $k$ and $-k$, and they depend only upon the magnitude of the wavevector on account of rotational symmetry.

\(^4\)The two inverse factors of $a$ coming from (5.28) are cancelled by the $a^3$ in the volume element and the $a^{-1}$ in the relation $\partial/\partial t = a^{-1} \partial/\partial \eta$.
(eqn. 5.29) for $\chi_k$ does not depend on the direction of $k$). The normalization condition on $u_{k}^{\text{out}}$ implies

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (5.36)$$

As in the harmonic oscillator example (5.8), if the state is the in-vacuum, then the expected excitation level of the $k$ out-mode, i.e. the average number of particles in that mode, is given by

$$\langle 0_{\text{in}} | N_{k}^{\text{out}} | 0_{\text{in}} \rangle = |\beta_k|^2. \quad (5.37)$$

To convert this statement into one about particle density, we sum over $k$ and divide by the physical spatial volume $V_{\text{phys}} = a^3 V$, which yields the number density of particles. Alternatively one can work with the continuum normalized modes. The relation between the discrete and continuous sums is

$$\frac{1}{V_{\text{phys}}} \sum_k \longleftrightarrow \frac{1}{(2\pi a)^3} \int d^3k. \quad (5.38)$$

the number density of out-particles is thus

$$n_{\text{out}} = \frac{1}{(2\pi a)^3} \int d^3k |\beta_k|^2. \quad (5.39)$$

The mean particle number characterizes only certain aspects of the state. As in the oscillator example, a full description of the in-vacuum in the out-Fock space is obtained from the Bogoliubov relation between the corresponding annihilation and creation operators. From (5.34) we can solve for $u_{k}^{\text{in}}$ and thence find

$$a_{k}^{\text{in}} = \alpha_k a_{k}^{\text{out}} + \bar{\beta}_k \alpha_{-k}^{\text{out}}, \quad (5.40)$$

whence

$$a_{k}^{\text{out}} |0_{\text{in}}\rangle = -\frac{\bar{\beta}_k}{\alpha_k} a_{-k}^{\text{out}} a_{-k}^{\text{out}} |0_{\text{in}}\rangle, \quad (5.41)$$

which can be solved to find

$$|0_{\text{in}}\rangle = \left( \prod_{k'} N_{k}' \right) \exp \left[ -\sum_k \left( \frac{\bar{\beta}_k}{2\alpha_k} a_{k}^{\text{out}} a_{-k}^{\text{out}} \right) \right] |0_{\text{out}}\rangle \quad (5.42)$$

where the $N_{k}'$ are normalization factors. This solution is similar to the corresponding expression (5.18) for the harmonic oscillator and it can be found by a similar method. Although the operators $a_{k}^{\text{out}}$ and $a_{-k}^{\text{out}}$ are distinct, each product appears twice in the sum, once for $k$ and once for $-k$, hence the factor of 2 in the denominator of the exponent is required. Using the two-mode analog of the squeeze operator (5.21) the state (5.42) can also be written in a manifestly normalized fashion analogous to (5.24). It is sometimes called a squeezed vacuum.
5.3 Remarks

5.3.1 Momentum correlations in the squeezed vacuum

The state (5.42) can be expressed as a sum of terms each of which has equal numbers of $k$ and $-k$ excitations. These degrees of freedom are thus entangled in the state, in such a way as to ensure zero total momentum. This is required by translation invariance of the states $|0_{in}\rangle$ and $|0_{out}\rangle$, since momentum is the generator of space translations.

5.3.2 Normalization of the squeezed vacuum

The norm sum for the part of the state (5.42) involving $k$ and $-k$ is a standard geometric series, which evaluates to $|\alpha_k|^2$ using (5.36). Hence to normalize the state one can set $N_k = |\alpha_k|^{-1/2}$ for all $k$, including $k = 0$ as in (5.20). The overall normalization factor is a product of infinitely many numbers less than unity. Unless those numbers converge rapidly enough to unity the state is not normalizable. The condition for normalizability is easily seen to be $\sum_k |\beta_k|^2 < \infty$, i.e. according to (5.37) the average total number of excitations must be finite. If it is not, the state $|0_{in}\rangle$ does not lie in the Fock space built on the the state $|0_{out}\rangle$. Note that, although formally unitary, the squeeze operator does not act unitarily on the out Fock space if the corresponding state is not in fact normalizable.

5.3.3 Energy density

If the scale factor $a$ changes by over a time interval $\Delta \tau$, then for a massless field dimensional analysis indicates that the in vacuum has a resulting energy density $\rho \sim \hbar(\Delta \tau)^{-4}$ after the change. To see how the formalism produces this, according to (5.39) we have $\rho = (2\pi a)^{-3} \int d^3k |\beta_k|^2 (\hbar \omega / a)$. The Bogoliubov coefficient $\beta_k$ is of order unity around $k_c/a \sim 1/\Delta \tau$ and decays exponentially above that. The integral is dominated by the upper limit and hence yields the above mentioned result.

5.3.4 Adiabatic vacuum

Modes with frequency much larger than $a'/a$ see the change of the scale factor as adiabatic, hence they remain relatively unexcited. The state that corresponds to the instantaneously defined ground state, in analogy with (5.11) for the single harmonic oscillator, is called the adiabatic vacuum at a given time.
5.4 de Sitter space

The special case of de Sitter space is of interest for various reasons. The first is just its high degree of symmetry, which makes it a convenient arena for the study of qft in curved space. It is the maximally symmetric Lorentzian space with (constant) positive curvature. Maximal symmetry refers to the number of Killing fields, which is the same as for flat spacetime. The Euclidean version of de Sitter space is just the sphere.

de Sitter (dS) space has hypersurface-orthogonal timelike Killing fields, hence is locally static, which further simplifies matters, but not to the point of triviality. The reason is that all such Killing fields have Killing horizons, null surfaces to which they are tangent, and beyond which they are spacelike. Hence dS space serves as a highly symmetric analog of a black hole spacetime. In particular, a symmetric variant of the Hawking effect takes place in de Sitter space, as first noticed by Gibbons and Hawking\[25\]. See \[26\] for a recent review.

Inflationary cosmology provides another important use of deSitter space, since during the period of exponential expansion the spacetime metric is well described by dS space. In this application the dS line element is usually written using spatially flat RW coordinates:

\[ ds^2 = dt^2 - e^{2Ht}dx^i dx^i. \]

These coordinates cover only half of the global dS space, and they do not make the existence of a time translation symmetry manifest. This takes the conformal form (5.25) with \( \eta = -H^{-1}\exp(-Ht) \) and \( a(\eta) = -1/H\eta \). The range of \( t \) is \((-\infty, \infty)\) while that of \( \eta \) is \((-\infty, 0)\).

The flat patch of de Sitter space is asymptotically static with respect to conformal time \( \eta \) in the past, since \( a'/a = -1/\eta \rightarrow 0 \) as \( \eta \rightarrow -\infty \). Therefore in the asymptotic past the adiabatic vacuum (with respect to positive \( \eta \)-frequency) defines a natural initial state. This is the initial state used in cosmology. In fact it happens to define a deSitter invariant state, also known as the Euclidean vacuum or the Bunch-Davies vacuum.

5.4.1 Primordial perturbations from zero point fluctuations

Observations of the Cosmic Microwave Background radiation support the notion that the origin of primordial perturbations lies in the quantum fluctuations of scalar and tensor metric modes (see \[27\] for a recent review and \[28\] for a classic reference.) The scalar modes arise from (and indeed are entirely determined by, since the metric has no
independent scalar degree of freedom) coupling to matter.\(^5\) Let’s briefly discuss how this works for a massless minimally coupled scalar field, which is just how these perturbations are described. First I’ll describe the scenario in words, then add a few equations.

Consider a field mode with a frequency high compared to the expansion rate \(\dot{a}/a\) during the early universe. To be specific let us assume this rate to be a constant \(H\), i.e. de Sitter inflation. Such a mode was presumably in its ground state, as the prior expansion would have redshifted away any initial excitation. As the universe expanded the frequency redshifted until it became comparable to the expansion rate, at which point the oscillations ceased and the field amplitude approached a time-independent value. Just before it stopped oscillating the field had quantum zero point fluctuations of its amplitude, which were then preserved during the further expansion. Since the amplitude was frozen when the mode had the fixed proper wavenumber \(H\), it is the same for all modes apart from the proper volume factor in the mode normalization which varies with the cosmological time of freezeout. Finally after inflation ended the expansion rate dropped faster than the wavenumber, hence eventually the mode could begin oscillating again when its wavelength became shorter than the Hubble length \(H^{-1}\). This provided the seeds for density perturbations that would then grow by gravitational interactions. On account of the particular wavevector dependence of the amplitude of the frozen spatial fluctuations, the spectrum of these perturbations turns out to be scale-invariant (when appropriately defined).

More explicitly, the field equation for a massless minimally coupled field is given by (5.29), with

\[
\omega^2(\eta) = k^2 - \frac{a''}{a} = k^2 - 2H^2a^2,
\]

(5.44)

where the last equality holds in de Sitter space. In terms of proper frequency \(\omega_p = \omega/a\) and proper wavenumber \(k_p = k/a\) we have \(\omega_p^2 = k_p^2 - 2H^2\). The first term is the usual flat space one that produces oscillations, while the second term tends to oppose the oscillations. For proper wavenumbers much higher than \(H\) the second term is negligible.

The field oscillates while the proper wavenumber redshifts exponentially. Eventually the two terms cancel, and the mode stops oscillating. This happens when

\[
k_p = \sqrt{2}H.
\]

(5.45)

As the wavenumber continues to redshift into the region \(k_p \ll H\), to a good approximation the amplitude satisfies the equation \(\chi''_k - (a''/a)\chi_k = 0\), which has a growing and a

\(^5\)The Cosmic Microwave Background observations supporting this account of primordial perturbations thus amount to quantum gravity observations of a limited kind.
decaying solution\(^6\). The growing solution is \( \chi_k \propto a \), which implies that the field mode \( u_k \) \((5.27,5.28)\) is constant in time. (This conclusion is also evident directly from the fact that the last term of the wave operator \((5.26)\) vanishes as \( a \) grows.)

The squared amplitude of the fluctuations when frozen is, according to \((4.27)\),

\[
\langle 0 | \varphi^\dagger_k \varphi_k | 0 \rangle = |u_{-k}|^2 \sim \frac{\hbar}{k_p V_p} = \frac{\hbar H^2}{Vk^3},
\]

where the proper values are used for consistent matching to the previous flat space result. This gives rise to the scale invariant spectrum of density perturbations.

### 6 Black hole evaporation

The vacuum of a quantum field is unstable to particle emission in the presence of a black hole event horizon. This instability is called the Hawking effect. Unlike the cosmological particle creation discussed in the previous section, this effect is not the result of time dependence of the metric exciting the field oscillators. Rather it is more like pair creation in an external electric field\(^{[13]}\). (For more discussion of the role of time dependence see section \(6.3.13\).) A general introduction to the Hawking effect was given in section \(2.2\). The present section is devoted to a derivation and discussion related topics.

The historical roots of the Hawking effect lie in the classical Penrose process for extracting energy from a rotating black hole. We first review that process and indicate how it led to Hawking’s discovery. Then we turn to the qft analysis.

#### 6.1 Historical sketch

The Kerr metric for a rotating black hole is stationary, but the asymptotic time translation Killing vector \( \chi \) becomes spacelike outside the event horizon. The region where it is spacelike is called the ergoregion. The conserved Killing energy for a particle with four-momentum \( p \) is \( E = \chi \cdot p \). Physical particles have future pointing timelike 4-momenta, hence \( E \) is positive provided \( \chi \) is also future timelike. Where \( \chi \) is spacelike however, some physical 4-momenta have negative Killing energy.

In the Penrose process, a particle of energy \( E_0 > 0 \) is sent into the ergoregion of a rotating black hole where it breaks up into two pieces with Killing energies \( E_1 \) and \( E_2 \), so that \( E_0 = E_1 + E_2 \). If \( E_2 \) is arranged to be negative, then \( E_1 > E_0 \), that is, more energy comes out than entered. The black hole absorbs the negative energy \( E_2 \) and thus

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\(^6\)The general solution is \( c_1 a + c_2 a \int^\eta d\eta/a^2 \), which is \( c_3 \eta^{-1} + c_4 \eta^2 \) in de Sitter space.
loses mass. It also loses angular momentum, hence the process in effect extracts the rotational energy of the black hole.

The Penrose process is maximally efficient and reversible if the horizon area is unchanged. That condition is achievable in the limit that the absorbed particle enters the black hole on a trajectory tangent to one of the null generators of the horizon. This role of the horizon area in governing efficiency of energy extraction exhibits the analogy between area and entropy. Together with Bekenstein’s information theoretic arguments, it gave birth to the subject of black hole thermodynamics.

When a field scatters from a rotating black hole a version of the Penrose process called superradiant scattering can occur. The analogy with stimulated emission suggests that quantum fields should exhibit spontaneous emission from a rotating black hole. To calculate this emission rate from an “eternal” spinning black hole one must specify a condition on the state of the quantum field that determines what emerges from the past horizon. In order to avoid this unphysical specification Hawking considered instead a black hole that forms from collapse, which has no past horizon. In this case only initial conditions before the collapse need be specified.

Much to his surprise, Hawking found that for any initial state, even a non-rotating black hole will spontaneously emit radiation. This Hawking effect is a pair creation process in which one member of the pair lies in the ergoregion inside the horizon and has negative energy, while the other member lies outside and escapes to infinity with positive energy. The Hawking radiation emerges in a steady flux with a thermal spectrum at the temperature \( T_H = \frac{\hbar \kappa}{2\pi} \). The surface gravity \( \kappa \) had already been seen to play the role of temperature in the classical first law of black hole mechanics, which thus rather remarkably presaged the quantum Hawking effect.

6.2 The Hawking effect

Two different notions of “frequency” are relevant to this discussion. One is the “Killing frequency”, which refers to time dependence with respect to the time-translation symmetry of the background black hole spacetime. In the asymptotically flat region at infinity the Killing frequency agrees with the usual frequency defined by the Minkowski observers at rest with respect to the black hole. The other notion is “free-fall frequency” defined by an observer falling across the event horizon. Since the Killing flow is tangent to the horizon, Killing frequency there is very different from free-fall frequency, and that distinction lies at the heart of the Hawking effect.
6.2.1 Average number of outgoing particles

The question to be answered is this: if a black hole forms from collapse with the quantum field in any ‘regular’ state $|\Psi\rangle$, then at late times, long after the collapse, what will be the average particle number and other observables for an outgoing positive Killing frequency wavepacket $P$ of a quantum field far from the black hole? We address this question here for the case of a noninteracting scalar field and a static (non-rotating) black hole. At the end we make some brief remarks about generalizations. Figure 1 depicts the various ingredients in the following discussion.

![Spacetime diagram of black hole formed by collapsing matter. The outgoing wavepacket $P$ splits into the transmitted part $T$ and reflected part $R$ when propagated backwards in time. The two surfaces $\Sigma_{f,i}$ are employed for evaluating the Klein-Gordon inner products between the wavepacket and the field operator. Although $P$, and hence $R$ and $T$, have purely positive Killing frequency, the free-fall observer crossing $T$ just outside the horizon sees both positive and negative frequency components with respect to his proper time.]

To begin with we evaluate the expectation value $\langle \Psi | N(P) | \Psi \rangle$ of the number operator.
\(N(P) = a^\dagger(P)a(P)\) in the quantum state \(|\Psi\rangle\) of the field. This does not fully characterize the state, but it will lead directly to considerations that do.

The annihilation operator (4.11) corresponding to a normalized wavepacket \(P\) is given by

\[a(P) = \langle P, \varphi \rangle_{\Sigma_f}, \tag{6.1}\]

where the Klein-Gordon inner product (4.9) is evaluated on the “final” spacelike slice \(\Sigma_f\). To evaluate the expectation value of \(N(P)\) we use the field equation satisfied by \(\varphi\) to relate \(N(P)\) to an observable on an earlier slice \(\Sigma_i\) on which we know enough about the quantum state. Specifically, we assume there are no incoming excitations long after the black hole forms, and we assume that the state looks like the vacuum at very short distances (or high frequencies) as seen by observers falling across the event horizon. Hawking originally propagated the field through the time-dependent collapsing part of the metric and back out all the way to spatial infinity, where he assumed \(|\Psi\rangle\) to be the incoming vacuum at very high frequencies. As pointed out by Unruh \cite{21} (see also \cite{29, 30}) the result can be obtained without propagating all the way back, but rather stopping on a spacelike surface \(\Sigma_i\) far to the past of \(\Sigma_f\) but still after the formation of the black hole. This is important since propagation back out to infinity invokes arbitrarily high frequency modes whose behavior may not be given by the standard relativistic free field theory.

If \(\Sigma_i\) lies far enough to the past of \(\Sigma_f\), the wavepacket \(P\) propagated backwards by the Klein-Gordon equation breaks up into two distinct parts,

\[P = R + T. \tag{6.2}\]

(See Fig. 11.) \(R\) is the “reflected” part that scatters from the black hole and returns to large radii, while \(T\) is the “transmitted” part that approaches the horizon. \(R\) has support only at large radii, and \(T\) has support only in a very small region just outside the event horizon where it oscillates very rapidly due to the backwards gravitational blueshift. Since both the wavepacket \(P\) and the field operator \(\varphi\) satisfy the Klein-Gordon equation, the Klein-Gordon inner product in (6.1) can be evaluated on \(\Sigma_i\) instead of \(\Sigma_f\) without changing \(a(P)\). This yields a corresponding decomposition for the annihilation operator,

\[a(P) = a(R) + a(T). \tag{6.3}\]

Thus we have

\[\langle \Psi | N(P) | \Psi \rangle = \langle \Psi | (a^\dagger(R) + a^\dagger(T))(a(R) + a(T)) | \Psi \rangle. \tag{6.4}\]
Now it follows from the stationarity of the black hole metric that the Killing frequencies in a solution of the KG equation are conserved. Hence the wavepackets $R$ and $T$ both have the same, purely positive Killing, frequency components as $P$. As $R$ lies far from the black hole in the nearly flat region, this means that it has purely positive asymptotic Minkowski frequencies, hence the operator $a(R)$ is a *bona fide* annihilation operator—or rather $\langle R, R \rangle^{1/2}$ times the annihilation operator—for incoming excitations. Assuming that long after the black hole forms there are no such incoming excitations, we have $a(R)\ket{\Psi} = 0$. Equation (6.4) then becomes

$$\langle \Psi | N(P) | \Psi \rangle = \langle \Psi | a^\dagger(T)a(T) | \Psi \rangle.$$  (6.5)

If $a(T)\ket{\Psi} = 0$ as well, then no $P$-particles are emitted at all. The state with this property is called the “Boulware vacuum”. It is the state with no positive Killing frequency excitations anywhere, including at the horizon.

The Boulware vacuum does not follow from collapse however. The reason is that the wavepacket $T$ does not have purely positive frequency with respect to the time of a free fall observer crossing the horizon, and it is this latter frequency that matches to the local Minkowski frequency in a neighborhood of the horizon small compared with the radius of curvature of the spacetime.

More precisely, consider a free-fall observer (i.e. a timelike geodesic $x(\tau)$) with proper time $\tau$ who falls across the horizon at $\tau = 0$ at a point where the slice $\Sigma_i$ meets the horizon (see Fig. 1). For this observer the wavepacket $T$ has time dependence $T(\tau)$ (i.e. $T(x(\tau))$) that vanishes for $\tau > 0$ since the wavepacket has no support behind the horizon. Such a function cannot possibly have purely positive frequency components. To see why, recall that if a function vanishes on a continuous arc in a domain of analyticity, then it vanishes everywhere in that domain (since its power series vanishes identically on the arc and hence by analytic continuation everywhere). Any positive frequency function

$$h(\tau) = \int_0^\infty d\omega e^{-i\omega\tau} \tilde{h}(\omega)$$  (6.6)

is analytic in the lower half $\tau$ plane, since the addition of a negative imaginary part to $\tau$ leaves the integral convergent. The positive real $\tau$ axis is the limit of an arc in the lower half plane, hence if $h(\tau)$ were to vanish for $\tau > 0$ it would necessarily vanish also for $\tau < 0$. (Conversely, a function that is analytic on the lower half-plane and does not blow up exponentially as $|\tau| \to \infty$ must contain only positive frequency components, since $\exp(-i\omega\tau)$ does blow up exponentially as $|\tau| \to \infty$ when $\omega$ is negative.)

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The wavepacket $T$ can be decomposed into its positive and negative frequency parts with respect to the free fall time $\tau$,

$$T = T^+ + T^-,$$  \hspace{1cm} (6.7)

which yields the corresponding decomposition of the annihilation operator

$$a(T) = a(T^+) + a(T^-)$$  \hspace{1cm} (6.8)

$$= a(T^+) - a^\dagger(T^-).$$  \hspace{1cm} (6.9)

Since $T^-$ has negative KG norm, Eqn. (4.12) has been used in the last line to trade $a(T^-)$ for the bona fide creation operator $a^\dagger(T^-)$. The $\tau$-dependence of $T$ consists of very rapid oscillations for $\tau < 0$, so the wavepackets $T^+$ and $\overline{T^-}$ have very high energy in the free-fall frame.

A free fall observer crossing the horizon long after the black hole forms would presumably see the ground state of the field at short distances, that is, such an observer would see no very high positive free-fall frequency excitations. The reason is that the collapse process occurs on the much longer time scale of the Schwarzschild radius $r_s$, so the modes with frequency much higher than $1/r_s$ should remain in their ground state. We therefore assume that the wavepackets $T^+$ and $\overline{T^-}$ are in their ground states,

$$a(T^+)\Psi = 0, \quad a(\overline{T^-})\Psi = 0.$$  \hspace{1cm} (6.10)

For further discussion of this assumption see section 7.

Using (6.9) the number expectation value (6.5) can be evaluated as

$$\langle \Psi | N(P) | \Psi \rangle = \langle \Psi | a(\overline{T^-})a^\dagger(\overline{T^-}) | \Psi \rangle$$  \hspace{1cm} (6.11)

$$= \langle \Psi | [a(\overline{T^-}), a^\dagger(\overline{T^-})] | \Psi \rangle$$  \hspace{1cm} (6.12)

$$= \langle \overline{T^-}, \overline{T^-} \rangle_{\Sigma_i}$$  \hspace{1cm} (6.13)

$$= -\langle T^-, T^- \rangle_{\Sigma_i},$$  \hspace{1cm} (6.14)

where (6.10) is used in the first and second lines, (4.12) is used in the third line, and (4.10) is used in the last step. The problem has thus been reduced to the computation of the Klein-Gordon norm of the negative frequency part of the transmitted wavepacket $T$. This requires that we be more explicit about the form of the wavepacket.

### 6.2.2 Norm of the negative frequency part & thermal flux

For definiteness we consider a spherically symmetric vacuum black hole in 3+1 dimensions, that is a Schwarzschild black hole. The Schwarzschild line element can variously
be expressed as
\[ds^2 = (1 - \frac{r_s}{r})dt^2 - (1 - \frac{r_s}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2)\] (6.15)
\[= (1 - \frac{r_s}{r})(dt^2 - dr^2) - r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2)\] (6.16)
\[= (1 - \frac{r_s}{r})du \, dv - r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2).\] (6.17)

The first form is in “Schwarzschild coordinates” and \(r_s = 2GM\) is the Schwarzschild radius. The second form uses the “tortoise coordinate” \(r^*\), defined by \(dr^* = dr/(1 - r_s/r)\) or \(r^* = r + r_s \ln(r/r_s - 1)\), which goes to \(-\infty\) at the horizon. The third form uses the retarded and advanced time coordinates \(u = t - r^*_s\) and \(v = t + r^*_s\), which are also called outgoing and ingoing null coordinates respectively.

A scalar field satisfying the Klein-Gordon equation \((\Box + m^2)\varphi = 0\) can be decomposed into spherical harmonics
\[\varphi(t, r, \theta, \phi) = \sum_{lm} \frac{\varphi_{lm}(t, r)}{r} Y_{lm}(\theta, \phi),\] (6.18)
where \(\varphi_{lm}(t, r)\) satisfies the 1+1 dimensional equation
\[(\partial_t^2 - \partial_{r^*}^2 + V_{lm})\varphi_{lm} = 0\] (6.19)
with the effective potential
\[V_{lm}(r) = \left(1 - \frac{r_s}{r}\right)\left(\frac{r_s}{r^3} + \frac{l(l+1)}{r^2} + m^2\right).\] (6.20)

As \(r \to \infty\) the potential goes to \(m^2\). As \(r \to r_s\), the factor \((r - r_s)\) approaches zero exponentially as \(\exp(r_s/r)\) with respect to \(r_s\). Near the horizon \(\varphi_{lm}(t, r_s)\) therefore satisfies the massless wave equation, hence has the general form \(f(u) + g(v)\).

Since the wavepacket \(P = \sum_{lm} P_{lm}(t, r)Y_{lm}(\theta, \phi)\) is purely outgoing with support only at large radii at late times, near the horizon \(P_{lm}(t, r)\) must be only a function of the ‘retarded time’ \(u = t - r^*_s\). That is, there can be no ingoing component. Since the metric is static, i.e. invariant with respect to \(t\)-translations, we can decompose any solution into components with a fixed \(t\)-frequency \(\omega\). A positive frequency outgoing mode at infinity has \(t\)-dependence \(\exp(-i\omega t)\), hence its form near the horizon must be \(\exp(-i\omega u)\).

Consider now a late time outgoing positive frequency wavepacket \(P\) that is narrowly peaked in Killing frequency \(\omega\). Propagating backwards in time, \(T\) is the part of the
wavepacket that is squeezed up against the horizon, and its $Y_{lm}$ component has the form
\[ T_{lm} \sim \exp(-i\omega u) \] for all $l, m$. The coordinate $u$ diverges as the horizon is approached.

It is related to the proper time $\tau$ of a free-fall observer crossing the horizon at $\tau = 0$ via $\tau \simeq -\tau_0 \exp(-\kappa u)$, where $\kappa = 1/2r_s$ is the surface gravity of the black hole and the constant $\tau_0$ depends on the velocity of the free-fall observer.\(^7\)

Hence the $\tau$-dependence of the wavepacket along the free-fall worldline is
\[ T \sim \exp\left(i\omega \kappa \ln(-\tau)\right) \] (6.21)
for $\tau < 0$, and it vanishes for $\tau > 0$.

To find the positive frequency part we use a method introduced by Unruh, which exploits the fact that a function analytic and bounded as $|\tau| \to \infty$ in the lower half complex $\tau$ plane has purely positive frequency (see the discussion after Eqn. (6.6)). The positive frequency extension of $T(\tau)$ from $\tau < 0$ to $\tau > 0$ is thus obtained by analytic continuation of $\ln(-\tau)$ in the lower half complex $\tau$-plane. This continuation is given by $\ln \tau + i\pi$, provided the branch cut of $\ln \tau$ is taken in the upper half-plane. The positive frequency extension of $T(\tau)$ to $\tau > 0$ is therefore obtained by replacing $\ln(-\tau)$ with $\ln \tau + i\pi$ in (6.21), which yields
\[ T(\tau) = T(-\tau) \exp(-\pi \omega/\kappa) \] for $\tau > 0$. Similarly, the negative frequency extension of $\ln(-\tau)$ is given by $\ln \tau - i\pi$, provided the branch cut of $\ln \tau$ is taken instead in the lower half-plane. The negative frequency extension of $T(\tau)$ to $\tau > 0$ is therefore $T(-\tau) \exp(+\pi \omega/\kappa)$.

Knowing these two extensions, we proceed as follows.

Define a new wavepacket $\tilde{T}$, with support only inside the horizon, by “flipping” the wavepacket $T(u)$ across the horizon (see Fig. 2). That is, $\tilde{T}$ vanishes outside the horizon and inside is constant on the outgoing null lines, with $\tilde{T}(\tau) = T(-\tau)$ for $\tau > 0$. The above argument shows that the wavepackets
\[ T^+ = c_+ (T + e^{-\pi \omega/\kappa} \tilde{T}) \] (6.22)
\[ T^- = c_- (T + e^{+\pi \omega/\kappa} \tilde{T}) \] (6.23)
have positive and negative free-fall frequency respectively. The two constants $c_\pm$ can be chosen so that $T^+ + T^-$ agrees with $T$ outside the horizon and vanishes (as does $T$) inside the horizon. This yields $c_- = (1 - e^{2\pi \omega/\kappa})^{-1}$ and $c_+ / c_- = -e^{2\pi \omega/\kappa}$.

Now $\langle T, \tilde{T} \rangle = 0$ (since the two wavepackets do not overlap) and $\langle \tilde{T}, \tilde{T} \rangle = -\langle T, T \rangle$ (since the flipped wavepacket has the reverse $\tau$-dependence), so using (6.23) one finds
\[ \langle T^-, T^- \rangle = \frac{\langle T, T \rangle}{1 - e^{2\pi \omega/\kappa}}. \] (6.24)

\(^7\)This can be obtained by noting from the third form of the line element (6.17) that along a timelike line $(1 - r_s/r)\ddot{u} = 1$, where the dots represent the proper time derivative. As the horizon is crossed $\dot{v}$ is finite, hence $\dot{u} \sim (r - r_s)^{-1} \sim e^{-r_s/r_s} = e^{(u-v)/2r_s} \sim e^{\kappa u}$. 

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Inserting this in the expression (6.14) for the number operator expectation value yields Hawking’s result,

$$\langle \Psi | N(P) | \Psi \rangle = \frac{\langle T, T \rangle}{e^{2\pi \omega / \kappa} - 1}. \quad (6.25)$$

The number expectation value (6.25) corresponds to the result for a thermal state at the Hawking temperature $T_H = \hbar \kappa / 2 \pi$, multiplied by the so-called “greybody factor”

$$\Gamma = \langle T, T \rangle. \quad (6.26)$$

This factor is the probability for an excitation described by the the wavepacket $P$ to pile up just outside the event horizon when propagated backwards in time, rather than being scattered back out to infinity. Equivalently, $\Gamma$ is the probability for the excitation with wavepacket $P$ to fall across the horizon when sent in forwards in time. This means that the black hole would be in detailed balance with a thermal bath at the Hawking temperature. Yet another interpretation of $\Gamma$ is the probability for an excitation originating close to the horizon with normalized wavepacket $T / \langle T, T \rangle^{1/2}$ to escape to infinity rather than scattering back and falling into the black hole.
6.2.3 The quantum state

The local free-fall vacuum condition (6.10) can be used to find the quantum state of the near horizon modes. We can do that here just “pair by pair”, since the field is noninteracting so the state is the tensor product of the states for each outside/inside pair.

Using (6.23) the vacuum conditions become

\[ a(T^+) |\Psi\rangle \propto \left[ a(T) - e^{-\pi \omega / \kappa} a(\tilde{T}^*) \right] |\Psi\rangle = 0 \]  
\[ a(\tilde{T}^-) |\Psi\rangle \propto \left[ -a^\dagger(T) + e^{+\pi \omega / \kappa} a(\tilde{T}^*) \right] |\Psi\rangle = 0 \]

(here we use * instead of “bar” for complex conjugation of \( \tilde{T} \) for typographical reasons). Note that since \( T \) and \( \tilde{T} \) have negative norm (as explained before (6.24)), we have replaced the corresponding annihilation operators by minus the creation operators of their conjugates. These equations define what is called the Unruh vacuum \( |\mathcal{U}\rangle \) for these wavepacket modes.

Now let \( |B\rangle \) denote the quantum state of the \( T \) and \( \tilde{T}^* \) modes such that

\[ a(T) |B\rangle = 0 \quad (6.29) \]
\[ a(\tilde{T}^*) |B\rangle = 0 \quad (6.30) \]

This state is called the Boulware vacuum for these modes. In analogy with (5.16) and (5.17) the vacuum conditions (6.27, 6.28) imply that the Unruh and Boulware vacua are related by

\[ |\mathcal{U}\rangle \propto \exp \left[ e^{-\pi \omega / \kappa} a^\dagger(\hat{T})a^\dagger(\hat{\tilde{T}}^*) \right] |B\rangle \]  

(6.31)

where the hats denote the corresponding normalized wavepackets.

The Unruh vacuum is thus a two-mode squeezed state, analogous to the one (5.42) found at each wavevector \( k \) in the case of cosmological pair creation. There each pair had zero total momentum since the background was space translation invariant. In the present case, each pair has zero total Killing energy, since the background is time translation invariant. The mode \( \tilde{T} \) has the same positive Killing frequency as \( T \) has (because the Killing flow is symmetric under the flipping across the horizon operation that defines \( \tilde{T} \)), however its conjugate has negative Killing frequency, and therefore negative Killing energy.

The Unruh vacuum is a pure state, but because of its entangled structure it becomes mixed when restricted to the exterior. To find that mixed state we expand the exponential in a series. Denoting by \( |n_{R,L}\rangle \) the level-\( n \) excitations of the modes \( T \) and \( \tilde{T}^* \)
respectively, we have

\[ |U\rangle \propto \sum_n e^{-n\pi\omega/\kappa} |n_L\rangle |n_R\rangle. \quad (6.32) \]

The reduced density matrix is thus

\[ Tr_L |U\rangle \langle U| \propto \sum_n e^{-2n\pi\omega/\kappa} |n_R\rangle \langle n_R|, \quad (6.33) \]

a thermal canonical ensemble at the Hawking temperature.

The essence of the Hawking effect is the correlated structure of the local vacuum state at short distances near the horizon and its thermal character outside the horizon. Given this, the outgoing flux at infinity is just a consequence of the propagation of a fraction \( \Gamma \) (6.26) of each outgoing wavepacket from the horizon to infinity.

6.3 Remarks

In this subsection we make a large number of brief remarks about related topics that we have no time or space to go into deeply. Where no references are given see the sources listed at the end of the Introduction.

6.3.1 Local temperature

The Hawking temperature refers to the Killing energy, or, since the Schwarzschild Killing vector is normalized at infinity, to the energy defined by a static observer at infinity. A static observer at finite radius will perceive the thermal state to have the blueshifted temperature \( T_{loc} = T_H / |\xi| \), where \( |\xi| \) is the local norm of the Schwarzschild time translation Killing vector. At infinity this is just the Hawking temperature, whereas it diverges as the horizon is approached. This divergence is due to the infinite acceleration of the static observer at the horizon and it occurs even for an accelerated observer in the Minkowski vacuum of flat spacetime (see section 6.3.4 below). A freely falling observer sees nothing divergent.

6.3.2 Equilibrium state: Hartle-Hawking vacuum

A black hole will be in equilibrium with an incoming thermal flux at the Hawking temperature. The state that includes this incoming flux is called the Hartle-Hawking vacuum.
6.3.3 Stimulated emission

Suppose that the field is not in the free-fall vacuum at the horizon (6.10), but rather that there are $n$ excitations in the mode $T^+$, so that $a^\dagger(T^+)a(T^+)|\Psi\rangle = n\langle T^+, T^+ | \Psi \rangle$. Then instead of (6.14) the expectation value of the number operator will be

$$\langle \Psi | N(P) | \Psi \rangle = n \langle T, T \rangle + (n + 1) \langle T^-, T^- \rangle.$$  \hfill (6.34)

That is, if $n$ quanta are present to begin with in the $T^+$ mode, the observer at infinity will observe in the $P$ mode $n + 1$ times the usual number of Hawking quanta, in addition to $n$ times the greybody factor (6.26). To produce a state in which the $T^+$ mode is occupied in standard physics one would have to send in particles of enormous energy just before the black hole formed\[32\]. As explained in section 7.4.2, however, trans-Planckian considerations could in principle allow stimulated emission at times long after the black hole formed. (This has nothing to do with the standard late time stimulated emission of the super-radiant modes of a rotating black hole\[33, 34\].)

6.3.4 Unruh effect

The argument given above for the structure of the vacuum near a black hole horizon applies equally well to the Minkowski vacuum near an acceleration horizon in flat spacetime, where it is known as the Unruh effect. From a logical point of view it might be better to introduce the Unruh effect first, and then export it to the neighborhood of a black hole horizon to infer the Hawking effect. However, I chose here to go in the other direction.

In the Unruh effect the boost Killing field $\xi_B = x \partial_t + t \partial_x$ (which generates hyperbolic rotations) plays the role of the Schwarzschild time translation, and the corresponding “temperature” is $\hbar/2\pi$. The Minkowski vacuum is the analog of the Hartle-Hawking equilibrium state, rather than the Unruh evaporating state. A uniformly accelerated observer following a hyperbolic orbit of the Killing field will perceive the Minkowski vacuum as a thermal state with temperature $\hbar/2\pi |\xi_B|$. The norm $|\xi_B|$ is just $(x^2 - t^2)^{1/2}$, which is also the inverse of the acceleration of the orbit, hence the local temperature is the \textit{Unruh temperature} $T_U = \hbar a/2\pi$. As $(x^2 - t^2)^{1/2} \to \infty$ this temperature is redshifted to zero, so a Killing observer at infinity sees only the zero temperature vacuum. As the acceleration horizon $x = \pm t$ is approached a Killing observer sees a diverging temperature. The same temperature divergence is seen by a static observer approaching the horizon of a black hole in the Unruh or Hartle-Hawking states (cf. section 6.3.1 above).
6.3.5 Rotating black hole

A small portion of the event horizon of a rotating black hole is indistinguishable from that of a Schwarzschild black hole, so the Hawking effect carries over to that case as well. The frequency $\omega$ in (6.25) should be replaced by the frequency with respect to the horizon generating Killing field $\partial_t + \Omega_H \partial_\phi$, where $\partial_t$ and $\partial_\phi$ are the asymptotic time translation and rotation Killing vectors, and $\Omega_H$ is the angular velocity of the horizon. Thus $\omega$ is replaced by $\omega - m\Omega_H$, where $m$ denotes the angular momentum. In effect there is a chemical potential $m\Omega_H$. See e.g. [35] for a discussion of the quantization of the “super-radiant” modes with $\omega - m\Omega_H < 0$.

6.3.6 de Sitter space

The reasoning in the black hole case applies mutatis mutandis to de Sitter spacetime, where an observer is surrounded by a horizon that is locally indistinguishable from a black hole horizon. This leads to the temperature of deSitter spacetime [25, 26].

6.3.7 Higher spin fields

The Hawking effect occurs also for higher spin fields, the only difference being (1) the greybody factors are different, and (2) for half-integer spin fields the Fermi distribution rather than the Bose distribution arises for the Hawking emission.

6.3.8 Interacting fields

Our discussion here exploited the free field equation of motion, but the Hawking effect occurs for interacting fields as well. The essence, as in the free field case, is the Unruh effect, the interacting version of which can easily be established with the help of a Euclidean functional integral representation of the Minkowski vacuum [36]. (This result was found a decade earlier, at about the same time as the original Unruh effect, via a theorem [37] in the context of axiomatic quantum field theory, although the interpretation in terms of the thermal observations of uniformly accelerated observers was not noted until later [38].) The direct analog for the Hawking effect involves a Euclidean functional integral expression for the interacting Hartle-Hawking equilibrium state (for an introduction see [39, 40] and references therein).

More directly, in an asymptotically free theory one can presumably use the free field analysis to discover the structure of the vacuum near the horizon as in the free field case. The propagation of the field from that point on will involve the interactions. If
the Hawking temperature is much higher than the scale $\Lambda$ of asymptotic freedom then free particles will stream away from the black hole and subsequently be “dressed” by the interactions and fragment into asymptotic states\[41, 42\]. If the Hawking temperature is much lower than $\Lambda$ then it is not so clear (to me at least) how to determine what is emitted.

### 6.3.9 Stress-energy tensor

The Unruh and Hartle-Hawking states are “regular” on the horizon, i.e. they look like the Minkowski vacuum at short distances. (Recall that it is precisely the local vacuum property \[6.10\] or \[6.27, 6.28\] that determines the thermal state of the outgoing modes at infinity.) Hence the mean value of the stress energy tensor is finite. The Boulware state $|B\rangle$ referred to above is obtained by removing the $T$, $\bar{T}^*$ pair excitations. This produces a state with a negative mean energy density that diverges as the horizon is approached. In fact, if even just one Hawking quantum is removed from the Unruh state $|U\rangle$ to obtain the Boulware state for that mode, a negative energy density divergence will be produced at the horizon. This can be viewed as the result of the infinite blueshift of the negative energy “hole”. That is quite odd in the context of flat space, since the Minkowski vacuum should be the lowest energy state, hence any other state should have higher energy. The explanation is that there is a positive energy density divergence on the horizon that more than compensates the negative energy off the horizon\[43\].

### 6.3.10 Back-reaction

As previously noted the Unruh state corresponds to an entangled state of positive and negative Killing energy excitations. As the positive energy excitations escape to infinity, there must be a corresponding negative energy flux into the black hole. Studies of the mean value of the stress tensor confirm this. Turning on the gravitational dynamics, this would lead to a mass loss for the black hole via the Einstein equation. The backreaction driven by the mean value is called the “semi-classical” evolution. There are quantum fluctuations about this mean evolution on a time and length scale of the Schwarzschild radius, unless a large number of matter fields is invoked to justify a large $N$ limit that suppresses the quantum fluctuations.

### 6.3.11 Statistical entropy

The entangled structure \[6.32\] of the Unruh state leads to a mixed state \[6.33\] when observations are restricted to the region outside the horizon. The “entanglement entropy”
$-Tr \rho \ln \rho$ of this state is the same as the thermal entropy of the canonical ensemble (6.33). Summing over all modes this entropy diverges due to the infinite density of modes. To characterize the divergence one can use the thermodynamic entropy density $s \propto T^3$ of a bath of radiation at temperature $T$. The local temperature measured by a static observer (cf. section 6.3.1) is given by

$$T_{\text{loc}} = T_H/|\xi| \simeq T_H/\kappa \ell = 1/2\pi \ell,$$  \hspace{1cm} (6.35)

where $\ell$ is the proper distance to the horizon on a surface of constant $t$, and the relation $\kappa = (d|\xi|/d\ell)_H$ has been used. Thus the entropy diverges like

$$S = \int s \, dv \sim \int T_{\text{loc}}^3 \, d\ell \, d^2A \sim A/\ell_c^2,$$ \hspace{1cm} (6.36)

where $\ell_c$ is a cutoff length above the horizon.$^8$

What is the meaning of this entropy? It seems clear on the one hand that it must be included in the black hole entropy, but on the other hand it must somehow be meaningfully cut off. It is natural to try to understand the scaling of the black hole entropy with area in this way, but this is only the contribution from one quantum field, and there is also the classical contribution from the gravitational field itself that emerges from the partition function for quantum gravity.[47] The apparent dependence on the number of fields is perhaps removed by the corresponding renormalization of Newton’s constant[48]. (The last reference in [48] is a review.) However this is regularization dependent and hence difficult to interpret physically, and moreover at least in dimensional regularization vector fields and some non-minimally coupled scalars contribute negatively to renormalizing $G$, which does not seem to match the entanglement entropy. There has been much work in this area but it remains to be fully understood.

### 6.3.12 Information loss

Two types of potential information loss occur in black hole physics. First, when something falls into a black hole any information it carries is apparently lost to the outside world. Second, when a black hole radiates Hawking quanta, each radiated quantum is entangled with a partner lying inside the horizon, as Eqn. (6.32) shows. As long as the

$^8$Sorkin[44] introduced the notion of black hole entanglement entropy, and with collaborators[45] computed it in the presence of a regulator. A mode-by-mode version of the thermal entropy calculation was first done by ’t Hooft[46], who called it the “brick wall model” because of a Dirichlet boundary condition applied at $\ell_c$. 

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black hole does not evaporate completely, all information remains available on a space-
lke surface that crosses the horizon and enters the black hole. If on the other hand the
black hole evaporates completely, then no single spacelike surface stretching to infinity
and filling “all space” can capture all information, according to the semi-classical anal-
ysis of the Hawking effect. Rather a disconnected surface behind the horizon must be
included. The information on this disconnected surface flows into the strong curvature
region at the singularity, where its fate is not yet understood.

This situation has generated much discussion. Some researchers (myself included)
see the information loss to the outside world not as a sign of the breakdown of quantum
mechanics but just as a consequence of the mutability of spatial topology in quantum
gravity. When a black hole is about to evaporate completely it looks very small to the
world outside the horizon. However this outside smallness has absolutely nothing to do
with the size of the region inside available for storing information. Let us look into this
a bit more.

Consider for example the spacelike singularity at \( r = 0 \) inside the Schwarzschild black
hole. The metric in Eddington-Finkelstein coordinates is

\[
ds^2 = (1 - r_s/r)dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( v = t + r_s \) is the advanced time coordinate defined below (6.17). Inside the
horizon, where \( r < r_s \), a line of constant \( r, \theta, \phi \) is spacelike, and has a proper length
\( L(r, \Delta v) = (r_s/r - 1)^{1/2} \Delta v \). This goes to infinity as the singularity is approached, for
any interval of advanced time \( \Delta v \). Hence there is no dearth of space inside. On the
other hand, the transverse angular dimensions go to zero size, and also we do not know
how to describe the spacetime too near to the singularity. Therefore let’s stop at the
radius where the curvature \( \sim r_s/r^3 \) is equal to the Planck curvature, i.e at \( r \sim r_s^{1/3} \) in
Planck units. Then then proper length goes as \( L(r_s^{1/3}, \Delta v) \sim r_s^{1/3} \Delta v \). For a solar mass
black hole we have \( r_s \sim 1 \text{ km} \sim 10^{38} \) in Planck units, so \( r_s^{1/3} \sim 10^{13} \). That means that
for an external advanced time of \( \Delta v = 1 \text{ second} \), the proper length inside is one million
light years. After a day or so, the length is the size of the universe, and so on over the
Hawking lifetime \( r_s^3 \sim 10^{114} \).

What happens to the future of this spacelike \( r = r_s^{1/3} \) cylinder inside the black hole
is governed by quantum gravity. We don’t know what form the evolution takes. It is
conceivable that time just stops running, like a frozen engine, producing a boundary of
spacetime. It seems much more likely however that spacetime persists beyond, either
into “quantum foam” or into a plump baby universe (see e.g. [49] and references therein)
or universes [50]. In any of these scenarios, the tiny outside size of a black hole at the
end stage of Hawking evaporation is no indication of the information carrying capacity of the interior.

Others believe that the validity of quantum mechanics requires that all information winds up available on the exterior slice after the evaporation. Results from string theory are often invoked to support this viewpoint. (For a critique of these arguments see [51].) If true, it would require some breakdown of the semi-classical description where none seems to be otherwise called for. The role of ultra high frequencies in the Hawking effect is often brought up in this context, however they are irrelevant since the derivation of the correlated structure of the vacuum and the Hawking effect does not need to access those frequencies. For more on this see the discussion of the trans-Planckian question in section 7.

6.3.13 Role of the black hole collapse

The Hawking flux continues in a steady state long after the collapse that forms a black hole, which suggests that the collapse phase has nothing to do with the Hawking effect. Indeed the derivation given above makes use of only the free-fall vacuum conditions near the horizon, and the collapse phase plays no role. On the other hand if, as Hawking originally did, we follow the positive Killing frequency wavepacket $T$ all the way backwards in time, it would go through the collapse phase and back out to infinity where Killing frequency and free-fall frequency coincide. Were it not for the time dependence of the background during the collapse, there would be no change of the Killing frequency, so there would be no negative frequency part of the ingoing wavepacket and there would be no particle creation. As discussed in the following section this propagation through the collapse phase invokes ultra high frequency field modes, and therefore may not even be physically relevant. However it seems clear is that whatever physics delivers the outgoing vacuum near the horizon, it must involve some violation of the time translation symmetry of the classical black hole background, even if it does not involve the collapse phase. In the lattice model of section 7.3 the violation arises from microscopic time dependence of the lattice spacing. In quantum gravity it may come just from the quantum gravitational fluctuations (see section 7.4.3).

7 The trans-Planckian question

A deep question arises on account of the infinite redshift at the black hole horizon: do the outgoing modes that carry the Hawking radiation really emerge from a reservoir
of modes with frequency arbitrarily far beyond the Planck frequency just outside the horizon, or is there another possibility? Does the existence or properties of the Hawking effect depend on the existence of such a trans-Planckian reservoir?

The reasoning leading to the expression (6.32) for the Unruh state in terms of positive Killing frequency modes is largely shielded from this question. The essential input is the free-fall vacuum conditions (6.10), which can be applied at any length or time scale much shorter than the Schwarzschild radius and inverse surface gravity (which are roughly the same unless the black hole is near extremally rotating or charged). There is no need to appeal to Planckian or trans-Planckian frequencies.

From this perspective it is clear that, as far as the derivation of the Hawking effect is concerned, the only question is whether or not the free-fall vacuum in fact arises at short distances near the horizon from the initial conditions before collapse. As mentioned earlier, the modes with frequencies much higher than the inverse of the collapse time scale would be expected to remain unexcited. Nevertheless, in standard relativistic field theory these modes arise from trans-Planckian modes.

Consider an outgoing wavepacket near the horizon and peaked around frequency $\omega_1^{ff}$ as measured by a free-fall observer crossing the horizon at the advanced time $v_1$. At an earlier time $v_2 = v_1 - \Delta v$, the wavepacket would be blueshifted and squeezed closer to the horizon, with exponentially higher free-fall frequency,

$$\frac{\omega_2^{ff}}{\omega_1^{ff}} \sim e^{\kappa \Delta v} = e^{\Delta v/2R_s}.$$ (7.1)

For a solar mass black hole and $\Delta v = 2$ seconds, the ratio is $\exp(10^5)$.

To predict the state of the positive free-fall frequency modes $T^+$ and $\overline{T}^-$ from the initial state thus seems to require trans-Planckian physics. This is a breakdown of the usual separation of scales invoked in the application of effective field theory and it leaves some room for doubt\[52, 30, 53\] about the existence of the Hawking effect.

While the physical arguments for the Hawking effect do seem quite plausible, the trans-Planckian question is nevertheless pressing. After all, there are reasons to suspect that the trans-Planckian modes do not even exist. They imply an infinite contribution to black hole entanglement entropy from quantum fields, and they produce other divergences in quantum field theory that are not desirable in a fundamental theory.

The trans-Planckian question is really two-fold:

1. Is the Hawking effect universal, i.e. insensitive to short distance physics, or at least can it be reliably derived in a quantum gravity theory with acceptable short distance behavior?
2. If there is no trans-Planckian reservoir, from where do the outgoing black hole modes arise?

### 7.1 String theory viewpoint

String theory has made impressive progress towards answering the first question, at least for some special black holes. In particular\[54, 55\], some near-extremal black holes become well-understood D-brane configurations in the weak coupling limit, and supersymmetry links the weak coupling to strong coupling results. Thus the Hawking effect can be reliably analyzed in a full quantum gravity theory. By a rather remarkable and unexpected correspondence the computations yield agreement with the semi-classical predictions, at least in the long wavelength limit. Moreover, the D-brane entropy is understood in terms of the counting of microstates, and agrees with the corresponding black hole entropy at strong coupling, just as the supersymmetry reasoning says it should. From yet another angle, the AdS/CFT duality in string theory offers other support\[56\]. There the Hawking effect and black hole entropy are interpreted in terms of a thermal state of the CFT (a conformally invariant super-Yang-Mills theory). However, neither of these approaches from string theory has so far been exploited to address the origin of the outgoing modes, since a local spacetime picture of the black hole horizon is lacking. This seems to be a question worth pursuing.

### 7.2 Condensed matter analogy

Condensed matter physics provides an analogy for effective field theory with a fundamental cutoff, hence it can be used to explore the consequences of a missing trans-Planckian reservoir. (For a review of these ideas see \[57\], and for a very brief summary see \[58\].) The first such black hole analog was Unruh’s sonic black hole, which consists of a fluid with an inhomogeneous flow exceeding the speed of sound at a sonic horizon. A molecular fluid does not support wavelengths shorter than the intermolecular spacing, hence the sonic horizon has no “trans-molecular” reservoir of outgoing modes. Unruh found that nevertheless outgoing modes are produced, by a process of “mode conversion” from ingoing to outgoing modes. This phenomenon comes about already because of the alteration of the dispersion relation for the sound waves. It has been studied in various field theoretic models, however none are fully satisfactory since the unphysical short distance behavior of the field is always eventually called into play. The model that most closely mirrors the fluid analogy is a falling lattice\[59\] which has sensible short distance physics.
The mode conversion on the lattice involves what is known as a Bloch oscillation in the condensed matter context. Here I will briefly explain how it works.

7.3 Hawking effect on a falling lattice

The model is 2d field theory on a lattice of points falling freely into a black hole. We begin with the line element in Gaussian normal coordinates,

\[ ds^2 = dt^2 - a^2(t, z) \, dz^2. \]  (7.2)

A line of constant \( z \) is an infalling geodesic, at rest at infinity, and the “local scale factor” \( a(t, z) \) satisfies

\[
    a(t, z \to \infty) = 1 \quad (7.3) \\
    a(0, z) = 1 \quad (7.4) \\
    a(t, z_H(t)) \sim \kappa t \quad \text{for } \kappa t \gtrsim 1, \quad (7.5)
\]

expressing the facts that the metric is asymptotically flat, the coordinate \( z \) measures proper distance on the \( t = 0 \) time slice, and at the horizon \( z_H(t) \) the time scale for variations of the local scale factor is the surface gravity \( \kappa \). The specific form of \( a(t, z) \) is not required for the present discussion. (For details see [59].) A scalar field on this spacetime is governed by the action

\[
    S = \frac{1}{2} \int d^2x \, \sqrt{-g} \, g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \quad (7.6)
\]

\[
    = \frac{1}{2} \int dt \, dz \left[ a(z, t)(\partial_t \varphi)^2 - \frac{1}{a(t, z)} (\partial_z \varphi)^2 \right]. \quad (7.7)
\]

Now we discretize the \( z \) coordinate with spacing \( \delta \):

\[
    z \to z_m = m\delta, \quad \partial_z \varphi \to D \varphi = \frac{\varphi_{m+1} - \varphi_m}{\delta}, \quad (7.8)
\]

where \( \varphi_m = \varphi(z_m) \). The proper distance between the lattice points \( z_m \) and \( z_{m+1} \) on a constant \( t \) slice is approximately \( a(t, z_m)\delta \). At \( t = 0 \) this is just \( \delta \) everywhere, that is the points start out equidistant. However, since they are on free fall trajectories at different distances from the horizon, they spread out as time goes on. In particular the lattice spacing at the horizon grows with time like \( \sim \kappa t \).

The discretized action is

\[
    S_{\text{lattice}} = \frac{1}{2} \int dt \sum_m \left[ a_m(t)(\partial_t \varphi_m)^2 - \frac{2(D \varphi_m(t))^2}{a_{m+1}(t) + a_m(t)} \right]. \quad (7.9)
\]
The discrete field equations produce the dispersion relation

$$\omega^{\text{ff}}(k) = \pm \frac{2}{a(z,t)\delta} \sin(k\delta/2)$$ (7.10)

for a mode of the form $\exp(-i\omega^{\text{ff}}t + ikz)$, provided $\partial_t a \ll \omega^{\text{ff}}$ and $\partial_z a \ll k$. (See Fig. 3.) For small wave numbers the lattice dispersion agrees with the continuum, however it is periodic in translation of $k$ by $2\pi/\delta$. At the wavenumber $k = \pi/\delta$ there is a maximum frequency $2/a\delta$ and vanishing group velocity $d\omega/dk$. Beyond that wavenumber the group velocity reverses, and $k$ is equivalent to $k - 2\pi/\delta$ which lies in the Brillouin zone $|k| \leq \pi/\delta$.

On the lattice the trans-Planckian redshift cannot take place, because of the lattice cutoff. Hence the outgoing modes—provided they exist—must come from ingoing modes. Both WKB “eikonal” trajectory and numerical evolution of the discrete wave equation confirm that indeed this occurs. The behavior of a typical wavepacket throughout the process of bouncing off the horizon is illustrated in Fig. 4. The real part of the wavepacket is plotted vs. the static coordinate at several different times. Following backwards in time, the packet starts to squeeze up against the horizon and then a trailing dip freezes and develops oscillations that grow until they balloon out, forming into a compact high frequency wavepacket that propagates neatly away from the horizon backwards in time.

The mode conversion can be understood as follows, following a wavepacket peaked around a long wavelength $\lambda \gg \delta$ backwards in time. The wavepacket blueshifts as it approaches the horizon, eventually enough for the lattice structure and therefore the curvature of the dispersion relation to be felt. At this point its group velocity begins
to drop. In the WKB calculation the wavepacket motion reverses direction at a turning point outside the horizon. This occurs before its group velocity in the falling lattice frame is negative, so it is falling in at that stage because its outward velocity is not great enough to overcome the infalling of the lattice.

As the wavepacket continues backwards in time now away from the black hole its wavevector continues to grow until it goes past the edge of the Brillouin zone, thus becoming an ingoing mode also in the lattice frame. This reversal of group velocity is precisely what happens in a “Bloch oscillation” when a quantum particle in a periodic potential is accelerated. As the turnaround is occurring, another equally important effect is that the time-dependence of the underlying lattice is felt. Thus, unlike in the continuum limit of this stationary background, the Killing frequency of the wavepacket is no longer conserved.

Following the wavepacket all the way backwards in time out to the asymptotic region, it winds up with a short wavelength of order $\delta$ and a large frequency of order $1/\delta$. This frequency shift is absolutely critical to the existence of the outgoing modes, since an ingoing low frequency mode would simply fall across the horizon. Only the “exotic” modes with sufficiently high frequency will undergo the mode conversion process.

The Hawking flux is determined by the negative frequency part of the ingoing
wavepacket. The eikonal approximation just described does not capture the negative frequency mixing which occurs during the turnaround at the horizon. Just as in the continuum, the wavepacket squeezed against the horizon has both positive and negative free-fall frequency components. As these components propagate backwards in time away from the horizon, their frequency slowly shifts, but their relative amplitude remains fixed. Hence the norm of the negative frequency part of the incoming wave packet turns out to be just what the continuum Hawking effect indicates, with small lattice corrections. Put differently, the infalling vacuum is adiabatically modified by the underlying microscopic time dependence of the lattice in such a way that the Unruh conditions \( (6.10) \) on the state of the outgoing modes at the horizon are satisfied.

### 7.4 Remarks

#### 7.4.1 Finite Entanglement entropy

Since the lattice has a short distance cutoff the entanglement entropy between the modes just inside and outside of the horizon is finite at any time. As illustrated in Figure 5, any given entangled pair of vacuum modes began in the past outside the horizon, propagated towards the horizon where it was “split”, and then separated, with one half falling in and the other converted into an outgoing mode. As time goes on, new pairs continually propagate in and maintain a constant entanglement entropy.

![Figure 5: The ancestors of a Hawking quantum and its negative energy partner. In standard relativistic field theory the ancestors are trans-Planckian and pass through the collapsing matter at the moment of horizon formation. On the lattice the ancestors are Planckian and propagate in towards the black hole at late times.](image-url)
7.4.2 Stimulated emission of Hawking radiation at late times

As discussed in section 6.3.3, if the Unruh vacuum conditions (6.10) do not hold at the horizon then stimulated emission of Hawking radiation will occur. In the falling lattice model, these horizon modes arise from ingoing modes long after the black hole formed as shown in Figure 5. Thus it is possible to stimulate the emission of Hawking radiation by sending in radiation at late times, in contrast to the usual continuum case of a static black hole. This seems a generic feature of theories with a cutoff, for which the outgoing modes must arise from modes that are ingoing after the collapse. (Something like it should happen also in string theory if, as many suppose, the trans-Planckian reservoir at the horizon is also eliminated there.) Note however that the linear model described here is surely a gross oversimplification. Turning on the gravitational interactions between the modes and the background, one is led to a picture in which the modes “dissipate” when propagated backwards in time to the Planckian regime. Hence what really produces the outgoing Hawking quantum must be a complicated collective mode of the interacting vacuum that “anti-dissipates” as it approaches the horizon and turns around. Calculations exploring this in quantum gravity were carried out in [60].

7.4.3 Lattice time dependence and geometry fluctuations

The microscopic time dependence of the lattice, i.e. the slow spreading of the lattice points, plays a critical role in transforming an ingoing mode with high Killing frequency to an outgoing mode with low Killing frequency, and in allowing for mixing of positive and negative frequencies despite the stationarity of the continuum black hole background. This suggests the conjecture that in quantum gravity the underlying quantum fluctuations of the geometry that do not share the stationarity of the black hole background metric might play this role. A step towards understanding this might be provided by a two or three dimensional version of the lattice model, in which the density of lattice points remains fixed but their microscopic positions fluctuate. This is precisely what happens with an inhomogeneous flow of a real molecular fluid. The quantum gravity analysis of [60] lends some support to this conjecture, though it is not clear whether the Lorentz violation seen there is just an artifact of a non-covariant cutoff or a feature introduced by the global geometry of the black hole spacetime.
7.5 Trans-Planckian question in cosmology

The modes producing the inflationary perturbation spectrum (cf. section 5.4.1) redshift exponentially from their trans-Planckian origins. It has been suggested that this might leave a visible imprint on the perturbation spectrum, via a modified high frequency dispersion relation and/or a modified initial quantum state for the field modes. As illustrated by the Hawking process on the lattice however, as long as the redshifting is adiabatic on the timescale of the modes, they would remain in their ground states. If the Hubble rate $H$ during inflation were much less than the Planck mass $M_P$ (or whatever scale the modified dispersion sets in) the modes could be treated in the standard relativistic manner by the time the perturbation spectrum is determined. At most one might expect an effect of order $H/M_P$, and some analyses suggest the effect will be even smaller. This all depends on what state the modes are in. They cannot be too far from the vacuum, since otherwise running them backwards in time they would develop an exponentially growing energy density which would be incompatible with the inflationary dynamics. Hence it seems the best one can say at present is that there may be room for noticeable deviations from the inflationary predictions, if $H/M_P$ is large enough. (See for example [61] and references therein for discussions of these issues.)

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