CURVATURE COUPLING IN EINSTEIN-YANG-MILLS THEORY AND NON-MINIMAL SELF-DUALITY

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A self-consistent non-minimal non-Abelian Einstein-Yang-Mills model, containing three phenomenological coupling constants, is formulated. The ansatz of a vanishing Yang-Mills induction is considered as a particular case of the self-duality requirement for the gauge field. Such an ansatz is shown to allow obtaining an exact solution of the self-consistent set of equations when the space-time has a constant curvature. An example describing a pure magnetic gauge field in the de Sitter cosmological model is discussed in detail.

1 Introduction

The theory of non-minimal coupling of gravity with fields and media has numerous applications to cosmology and astrophysics. Non-minimal theory has been elaborated in detail for scalar and electromagnetic fields (see, e.g., the review [1] and references therein). A detailed theory of non-minimal coupling of gravity with gauge fields is still at its development stage. A version of the non-minimal Einstein-Yang-Mills (EYM) model was obtained by Müller-Hoissen in 1988 [2] from dimensional reduction of the Gauss-Bonnet action. This model contains one coupling parameter. We follow an alternative derivation of the non-minimal EYM theory, formulated as a non-Abelian generalization of non-minimal non-linear Einstein-Maxwell theory (see [3]) along the line proposed by Drummond and Hathrell for linear electrodynamics [4]. As a particular case, this theory gives the non-minimal, linear in curvature, EYM model which can be characterized as a three-parameter model since it contains three coupling constants $q_1$, $q_2$ and $q_3$. The problem of a curvature induced backreaction of the Yang-Mills field on the gravitational field seems to be important at least in two aspects. First, non-minimal coupling of the Yang-Mills field with gravity can modify the rate of the Universe evolution, providing the accelerated expansion analogously to the one in the non-minimal Einstein-Maxwell theory [5]. Second, a curvature coupling of gauge fields with gravity gives a new degree of freedom in modeling (regular) spherically symmetric objects [6].

In this note, we introduce a three-parameter self-consistent EYM model in which the EYM Lagrangian satisfies three special requirements: it is gauge-invariant, linear in space-time curvature, and quadratic in the Yang-Mills field strength tensor $F_{ik}$. Then we consider an exact solution of the obtained model for a specific case when the non-Abelian induction tensor $H_{ik}$ is proportional to the dual field stress tensor $F_{ik}^*$, i.e., when a generalized self-duality condition is satisfied. In this context, we consider an exact solution of the EYM model with vanishing induction of the gauge field when the space-time is characterized by a constant curvature and describe in detail the example of pure magnetic gauge field.

2 Non-minimal Einstein-Yang-Mills field equations

The three parameter non-minimal Einstein-Yang-Mills theory can be formulated in terms of the action functional

$$S_{NMEYM} = \int d^4x \sqrt{-g} \mathcal{L}, \quad \mathcal{L} = \frac{R + 2\Lambda}{\kappa} + \frac{1}{2} \left[ \frac{1}{2} g^{ikmn} + R^{ikmn} \right] F_{ik}^{(a)} F_{mn}^{(a)}.$$  \hspace{1cm} (1)

Here $\Lambda$ is the cosmological constant, $g = \det(g_{ik})$ is the determinant of the metric tensor $g_{ik}$, $R$ is the Ricci scalar, the constant $\kappa$ is equal to $8\pi\gamma$, where $\gamma$ is the gravitational constant, Latin indices without parentheses run from 0 to 3. The symbol $g^{ikmn}$ is a standard abbreviation for the tensor quadratic in the metric

$$g^{ikmn} = g^{im} g^{kn} - g^{in} g^{km},$$  \hspace{1cm} (2)

the tensor $R^{ikmn}$ is defined as follows:

$$R^{ikmn} = \frac{1}{2} q_1 R g^{ikmn} + \frac{1}{2} q_2 (R^{im} g^{kn} - R^{in} g^{km} + R^{km} g^{im} - R^{km} g^{in}) + q_3 R g^{ikmn},$$  \hspace{1cm} (3)

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where $R^{ik}$ and $R^{ikmn}$ are the Ricci and Riemann tensors, respectively, and $q_1$, $q_2$, $q_3$ are the phenomenological parameters describing the non-minimal coupling of the Yang-Mills and gravitational fields. Following [7], we consider the Yang-Mills field $F_{mn}$ taking values in the Lie algebra of the gauge group $SU(n)$:

$$F_{mn} = -i\mathcal{G}t^{(a)}F_{mn}^{(a)}, \quad A_m = -i\mathcal{G}t^{(a)}A_m^{(a)}.$$  

Here $t^{(a)}$ are the Hermitian traceless generators of the $SU(n)$ group, $A_m^{(a)}$ and $F_{mn}^{(a)}$ are the Yang-Mills field potential and strength, respectively, the group index $(a)$ runs from 1 to $n^2 - 1$, and the constant $\mathcal{G}$ is the strength of gauge coupling. Scalar products of the Yang-Mills fields, indicated by the bold letters, are defined in terms of the traces of the corresponding matrices (see [7]), a scalar product of the generators $t^{(a)}$ and $t^{(b)}$ is chosen to be equal to

$$\langle t^{(a)} , t^{(b)} \rangle \equiv 2\text{Tr} t^{(a)} t^{(b)} \equiv G^{(a)(b)}.$$  

The symmetric tensor $G^{(a)(b)}$ plays the role of a metric in the group space, and the generators can be chosen so that the metric is equal to the Kronecker delta. The Yang-Mills fields $F_{mn}^{(a)}$ are connected with the potentials of the gauge field $A_m^{(a)}$ by the well-known formulae (see, e.g., [7])

$$F_{mn}^{(a)} = \nabla_m A_n^{(a)} - \nabla_n A_m^{(a)} + \mathcal{G} f^{(a)}_{(b)(c)} A_m^{(b)} A_n^{(c)}.$$  

Here $\nabla_m$ is a covariant space-time derivative, the symbols $f^{(a)}_{(b)(c)}$ denote real structure constants of the gauge group $SU(n)$. The definition of the commutator in (6) is based on the relation

$$[t^{(a)} , t^{(b)}] = if^{(c)}_{(a)(b)} t^{(c)} ,$$

providing the formula

$$f^{(c)}_{(a)(b)} \equiv G^{(c)(d)} f^{(d)}_{(a)(b)} = -2i \text{Tr} [t^{(a)} , t^{(b)}] t^{(c)}.$$  

The structure constants $f^{(a)}_{(b)(c)}$ are supposed to be completely antisymmetric under exchange of any two indices [7]. The metric $G^{(a)(b)}$ and the structure constants $f^{(d)}_{(a)(c)}$ are supposed to be constant tensors in the standard and covariant manner. This means that

$$\partial_m G^{(a)(b)} = 0 , \quad \check{D}_m G^{(a)(b)} = 0 , \quad \partial_m f^{(a)}_{(b)(c)} = 0 , \quad \check{D}_m f^{(a)}_{(b)(c)} = 0 ,$$

where the following rule for the derivative of the tensors in the group space is used:

$$\check{D}_m Q^{(a) \ldots (d)} \equiv \nabla_m Q^{(a) \ldots (d)} + \mathcal{G} f^{(a)}_{(b)(c)} A_m^{(b)} Q^{(c) \ldots (d)} - \mathcal{G} f^{(c)}_{(b)(d)} A_m^{(b)} Q^{(a) \ldots (c)}.$$  

### 3 Non-minimal master equations

#### 3.1 Non-minimal extension of Yang-Mills equations

Variation of the action $S_{(NMEYM)}$ with respect to the Yang-Mills potential $A_m^{(a)}$ yields

$$\check{D}_k H^{ik} = 0 , \quad H^{ik} = \left[ \frac{1}{2} g^{ikmn} + \mathcal{R}^{ikmn} \right] F_{mn} .$$

The tensor $H^{ik}$ is a non-Abelian analogue of the induction tensor known in the electrodynamics [8, 9]. This analogy allows us to consider $\mathcal{R}^{ikmn}$ as a susceptibility tensor [3].

The Bianchi identity for the gauge field strength can be written as

$$\check{D}_k F^{*ik} = 0 ,$$

where the asterisk denotes the dual tensor

$$F^{*ik} = \frac{1}{2} \epsilon^{ikls} F_{ls} .$$

Here $\epsilon^{ikls}$ is the Levi-Civita tensor while $E^{ikls}$ is the completely antisymmetric symbol with $E^{0123} = -E_{0123} = 1$. 


3.2 Master equations for the gravitational field

Variation of the action $S_{(NMEYM)}$ with respect to the metric yields

$$R_{ik} - \frac{1}{2} R g_{ik} = \Lambda g_{ik} + \kappa T_{ik}^{(\text{eff})}. \quad (14)$$

The effective stress-energy tensor $T_{ik}^{(\text{eff})}$ can be divided into four parts:

$$T_{ik}^{(\text{eff})} = T_{ik}^{(YM)} + q_1 T_{ik}^{(I)} + q_2 T_{ik}^{(II)} + q_3 T_{ik}^{(III)}. \quad (15)$$

The first term

$$T_{ik}^{(YM)} = \frac{1}{4} g_{ik} F_{mn}^{(a)} F_{mn}^{(a)} - F_{in}^{(a)} F_{k(n)}^{(a)} \quad (16)$$

is the stress-energy tensor of the pure Yang-Mills field. The definitions of the other three tensors are related to the corresponding coupling constants $q_1$, $q_2$, $q_3$:

$$T_{ik}^{(I)} = R T_{ik}^{(YM)} - \frac{1}{2} R_{ik} F_{mn}^{(a)} F_{mn}^{(a)} + \frac{1}{2} \left[ \hat{D}_i \hat{D}_k - g_{ik} \hat{D}^l \hat{D}_l \right] F_{mn}^{(a)} F_{mn}^{(a)} , \quad (17)$$

$$T_{ik}^{(II)} = -\frac{1}{2} g_{ik} \left[ \hat{D}_m \hat{D}_n \left( F_{im}^{(a)} F_n^{(a)} \right) - R_{lm} F_{mn}^{(a)} F_{mn}^{(a)} \right] - F_{in}^{(a)} \left( R_{il} F_{kn}^{(a)} + R_{kl} F_{in}^{(a)} \right) - R_{mn} F_{im}^{(a)} F_{kn}^{(a)} - \frac{1}{2} \hat{D}_m \hat{D}_n \left( F_{im}^{(a)} F_{kn}^{(a)} \right) , \quad (18)$$

$$T_{ik}^{(III)} = \frac{1}{4} g_{ik} R_{mnls} F_{mn}^{(a)} F_{ls}^{(a)} - \frac{3}{4} F_{ls}^{(a)} \left( F_{in}^{(a)} R_{knls} + F_{kn}^{(a)} R_{inls} \right) - \frac{1}{2} \hat{D}_m \hat{D}_n \left( F_{i}^{(a)} F_{k}^{(a)} + F_{k}^{(a)} F_{i}^{(a)} \right) . \quad (19)$$

3.2.1 Trace of the effective stress-energy tensor

In contrast to the traceless stress-energy tensor of the Yang-Mills field $T_{ik}^{(YM)}$, the stress-energy tensors $T_{ik}^{(I)}$, $T_{ik}^{(II)}$, $T_{ik}^{(III)}$ have non-vanishing traces:

$$T^{(I)} = -\frac{1}{2} \left( R + 3 \hat{D}^l \hat{D}_l \right) F_{mn}^{(a)} F_{mn}^{(a)} , \quad (20)$$

$$T^{(II)} = -\hat{D}^m \hat{D}^k \left( F_{mn}^{(a)} F_{kn}^{(a)} \right) - \frac{1}{2} \hat{D}_k \hat{D}_i \left( R_{mn} F_{im}^{(a)} F_{kn}^{(a)} \right) , \quad (21)$$

$$T^{(III)} = -\frac{1}{2} R_{mnls} F_{mn}^{(a)} F_{ls}^{(a)} - \hat{D}_m \hat{D}_n \left[ F_{i}^{(a)} F_{k}^{(a)} \right] . \quad (22)$$

3.2.2 Bianchi identities

The right-hand side of the Einstein equations (14) must be divergence-free. This is valid automatically if $F_{ik}^{(a)}$ is a solution to the Yang-Mills equations (11) and (12). To check this fact directly, one has to use the Bianchi identities and the properties of the Riemann tensor:

$$\nabla_i R_{klmn} + \nabla_l R_{ikmn} + \nabla_k R_{limn} = 0, \quad R_{klmn} + R_{mkln} + R_{limkn} = 0, \quad (23)$$

as well as the commutation rules for covariant derivatives

$$(\nabla_i \nabla_k - \nabla_k \nabla_i) W^i = W^m R_{milk}^i . \quad (24)$$

Note that the susceptibility tensor $\mathcal{R}_{iklmn}$ has the same symmetry of indices as the Riemann tensor, moreover, the second identity in (23) will be valid if the Riemann tensor is replaced with $\mathcal{R}_{iklmn}$. 
4 Generalized self-duality and exact solutions in the EYM model

The well-known self-duality problem for the Yang-Mills field (see, e.g., [10]) can be generalized to the case when, in addition to the gauge field strength, one has a gauge field induction. When

$$H^{(a)}_{ik} = \lambda^{(a)} (b) F^{(b)}_{ik} ,$$

(25)

the Yang-Mills equations (11) are satisfied due to (12), when $$\hat{D}_m \lambda^{(a)}_{(b)} = 0$$. The latter requirement is valid for arbitrary $$A_m$$ if

$$\partial_m \lambda^{(a)}_{(b)} = 0 , \quad j^{(a)}_{(c)(d)} \lambda^{(d)}_{(b)} = j^{(d)}_{(c)(b)} \lambda^{(a)}_{(b)} .$$

(26)

For the SU(n) gauge group, the conditions (26) yield

$$\lambda^{(a)}_{(b)} = \lambda \delta^{(a)}_{(b)} , \quad \lambda \equiv \frac{1}{(n^2 - 1)} \lambda^{(a)}_{(a)} .$$

(27)

For arbitrary gauge groups, the matrix $$\lambda^{(a)}_{(b)}$$ is not necessarily diagonal (see, e.g., [11]).

Thus, for SU(n) symmetry, the Eq.(25) takes the form

$$\lambda F^{*}_{ik} = F_{ik} + \mathcal{R}_{ikmn} F^{mn} .$$

(28)

Second dualization of the relation (28) gives

$$F_{ik}(1 + \lambda^2) + \mathcal{R}_{ikmn} F^{mn} + \ast \mathcal{R}_{ikmn} \lambda F^{mn} = 0 .$$

(29)

In the minimal EYM theory with $$\mathcal{R}_{ikmn} = 0$$, the relation (29) requires that $$F_{ik}(1 + \lambda^2) = 0$$ and thus $$F_{ik} = 0$$ if $$\lambda$$ is a real constant. If $$\mathcal{R}_{ikmn} \neq 0$$, a non-trivial solution for $$F_{ik}$$ can exist.

The relation (28) generalizes the well-known self-duality condition in electrodynamics and Yang-Mills theory. We distinguish two different cases. In the first one, the relation (28) is satisfied for arbitrary $$F_{ik}$$ due to a special choice of the coupling constants $$q_1$$, $$q_2$$, $$q_3$$ and the symmetry of space-time. In the second case, the relation (28) is satisfied for a specific structure of $$F_{ik}$$. In this note we focus on the first case only.

4.1 Non-minimal EYM models with vanishing induction

The equation (28) is satisfied for arbitrary $$F_{ik}$$ if

$$\frac{\lambda}{2} c_{ikmn} = \frac{1}{2} g^{ikmn} + \mathcal{R}^{ikmn} .$$

(30)

A cyclic transposition of the last three indices in (30) yields $$\lambda = 0$$, providing the relation

$$\mathcal{R}_{ikmn} = - \frac{1}{2} g_{ikmn} .$$

(31)

Direct consequences of (31) are

$$\frac{3q_1 + q_2}{2} R_{gkn} + (q_2 + q_3) R_{kn} = \frac{3}{2} g_{kn} ,$$

(32)

$$R = - 12 K , \quad K = \frac{1}{2(6q_1 + 3q_2 + q_3)} .$$

(33)

When $$6q_1 + 3q_2 + q_3 \neq 0$$, one obtains

$$R = - 12 K , \quad K = \frac{1}{2(6q_1 + 3q_2 + q_3)} .$$

(34)

If $$q_2 + q_3 \neq 0$$, the Ricci tensor $$R_{kn}$$ can be extracted from (32):

$$R_{kn} = - 3 K g_{kn} .$$

(35)

Similarly, if $$q_3 \neq 0$$, the Riemann tensor can be obtained from (31):

$$R_{ikmn} = - K g_{ikmn} .$$

(36)
Eqs. (34)-(36) show that a self-dual non-minimal EYM model with a vanishing induction tensor requires the spacetime to be of constant curvature $K$. The condition $K = 0$ is incompatible with (30). Thus, varying the parameters $q_1$, $q_2$, $q_3$, one can formulate five different submodels. The first submodel is the general one with $q_3 \neq 0$, $q_2 + q_3 \neq 0$, $6q_1 + 3q_2 + q_3 \neq 0$. These conditions allow one to find $R$, $R_{ik}$ and $R_{ikm}$ unambiguously in the form (34)-(36). The second (first special) submodel with $q_3 = 0$, $q_2 \neq 0$, $2q_1 + q_2 \neq 0$, allows one to find the Ricci tensor in the form (35) and Ricci scalar according to (34), but the Riemann tensor itself cannot be extracted. The third (the second special) submodel with $q_2 + q_3 = 0$, $q_3 \neq 0$, $3q_1 + q_2 \neq 0$ gives the Ricci scalar according to (34) and the Riemann tensor in the form

$$R_{ikmn} = \frac{1}{2} (R_{im}g_{kn} + R_{kn}g_{im} - R_{in}g_{km} - R_{km}g_{in}) + \frac{1}{2 (3q_1 + q_2)} g_{ikmn},$$

the Ricci tensor being unresolvable. Note that the relation (37) is valid when the Weyl tensor vanishes. The last special model with $q_2 = q_3 = 0$, $q_1 \neq 0$ yields $R = -\frac{1}{q_1}$, but $R_{ikmn}$ and $R_{kn}$ cannot be identified.

### 4.2 Example of an exact solution

The self-duality discussed above guarantees the non-minimal Yang-Mills equations to be satisfied identically for an arbitrary potential $A_r$ if the relation (31) is valid. To solve the whole set of non-minimal EYM field equations, one needs to consider the remaining equations (14)-(19). In this note we consider a de Sitter-type model with $q_1 = q_2 = 0$ and the SU(2)-symmetric gauge field. This model is related to the non-minimal constant curvature $K = \frac{1}{2q_1}$ and is a non-Abelian generalization of Prasanna’s electrodynamical model [12]. For this model, the gravitational field equations reduce to

$$(3K - \Lambda) g_{ik} = -\frac{\kappa}{2K} \hat{D}^m \hat{D}^n \left( F_{in}^{(a)} F_{km(a)} \right) + \frac{\kappa}{2} g^{mn} F_{in}^{(a)} F_{km(a)}. \quad (38)$$

In the static representation, de Sitter space-time is characterized by the metric

$$ds^2 = (1 - Kr^2) dt^2 - \frac{dr^2}{(1 - Kr^2)} - r^2 [d\theta^2 + \sin^2 \theta d\varphi^2]. \quad (39)$$

Consider the so-called pure magnetic solution with the following ansatz (see, e.g., [6])

$$A_0 = A_r = 0, \quad A_\theta = -i (w(r) - 1) \ t_\varphi, \quad A_\varphi = i (w(r) - 1) \sin \theta \ t_\theta. \quad (40)$$

Here $t_r$, $t_\theta$ and $t_\varphi$ are the position-dependent generators of the SU(2) group:

$$t_r = \cos \varphi \sin \theta \ t_{(1)} + \sin \varphi \sin \theta \ t_{(2)} + \cos \theta \ t_{(3)},$$

$$t_\theta = \partial_\theta t_r, \quad t_\varphi = \frac{1}{\sin \theta} \partial_\varphi t_r, \quad (41)$$

which satisfy the relations

$$[t_r, t_\varphi] = i t_\varphi, \quad [t_\theta, t_\varphi] = i t_r, \quad [t_\varphi, t_r] = i t_\theta. \quad (42)$$

Non-vanishing components of the field strength tensor are

$$F_{r\theta} = -iw' \ t_\varphi, \quad F_{r\varphi} = iw' \sin \theta \ t_\theta, \quad F_{\theta\varphi} = -i (w^2 - 1) \sin \theta \ t_r. \quad (43)$$

The gravity field equations yield, first, the standard relation between cosmological constant and curvature, $\Lambda = 3K$, and, second, the only non-trivial differential equation for the function $w(r)$

$$\left( \frac{(1 - Kr^2)w'}{r} \right)' = \frac{(w^2 - 1)^2}{r^4}. \quad (44)$$

(The prime denotes a derivative with respect to $r$.) Thus, the whole set of differential equations in the non-minimal EYM model is reduced to a single second order differential equation. It has the constant solution $w(r) = \pm 1$, however, it describes a pure gauge ($F_{ik} = 0$). Another exact solution expressed in terms of elementary functions is

$$w(r) = \pm \sqrt{1 - Kr^2}. \quad (45)$$

This solution is regular at the origin for arbitrary $K$. When $K$ is positive, the metric (39) has a horizon at $r_h = \frac{1}{\sqrt{K}}$ and $w(r_h) = 0$. Other solutions of Eq.(44) can be found numerically.
5 Discussion

One of the results presented in the paper is a derivation of a new self-consistent non-minimal set of master equations for the coupled Yang-Mills and gravity fields from a gauge-invariant non-minimal Lagrangian. The obtained mathematical model contains three arbitrary parameters, and thus admits a wide choice of special submodels interesting for applications to non-minimal cosmology (isotropic and anisotropic) and non-minimal coloured spherically symmetric objects.

To present an exact solution of the formulated model, we have considered the ansatz of self-duality of the Yang-Mills field and focused attention on its particular case, a model with vanishing gauge field induction. This ansatz happens to be compatible with the non-minimal EYM model, when the space-time is characterized by a constant curvature. To show that this model contains at least one non-trivial solution, we have considered an example describing a pure magnetic non-Abelian gauge field in de Sitter space-time. We have shown that the whole set of equations reduces to one differential equation for one required function. This circumstance guarantees the consistency of the model. A new exact solution expressed in terms of elementary functions regular both at the origin and at the horizon is obtained.

Since the model contains three arbitrary parameters (coupling constants), there arises the problem of introduction of “new constants of Nature” with the dimensionality of area. We show that the coupling constants introduced phenomenologically can be interpreted in terms of the cosmological constant.

Acknowledgement
This work was supported by the Deutsche Forschungsgemeinschaft.

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