THE EIGENVALUES OF THE HESSIAN MATRICES
OF THE GENERATING FUNCTIONS FOR TREES
WITH $k$ COMPONENTS

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Abstract. Let us consider a truncated matroid $M^r_\Gamma$ of rank $r$ of a graphic matroid of a graph $\Gamma$. The basis for $M^r_\Gamma$ is the set of the forests with $r$ edges in $\Gamma$. We consider this basis generating function and compute its Hessian. In this paper, we show that the Hessian of the basis generating function of the truncated matroid of the graphic matroid of the complete or complete bipartite graph does not vanish by calculating the eigenvalues of the Hessian matrix. Moreover, we show that the Hessian matrix of the basis generating function of the truncated matroid of the graphic matroid of the complete or complete bipartite graph has exactly one positive eigenvalue. As an application, we show the strong Lefschetz property for the Artinian Gorenstein algebra associated to the truncated matroid.

1. Introduction

Various applications of the strong Lefschetz property to other areas, e.g. combinatorics, representation theory and so on (see [1] for details), has been found in the last two decades. Recently, algebras associated to matroids are studied, e.g. [2], [5], [8], [9], [11].

In [2], Huh and Wang defined a chow ring $A(M)$ associated to a loop-less matroid $M$ on a set $E = \{0, 1, \ldots, n\}$. They show that the ring $A(M)$ has the strong Lefschetz property in the narrow sense (see Definition 3.4). Moreover an element $L$ in $A^1(M)$ such that $L$ is strictly submodular function on the family of the subsets of $E$ is a strong Lefschetz element. They also defined algebra $B^*(M)$ associated to a matroid on a set $E = \{1, 2, \ldots, n\}$ which is a subring of $A(M)$. They show that $B^*(M)$ has the “injective” Lefschetz property in the case where $M$ is representable. Moreover the element $L$ in $B^1(M)$ such that all the coefficients of all variables are one is a strong Lefschetz element.

In [5], Maeno and Numata defined algebras $Q/J_M$ and $A_M$ for a matroid $M$ to give an algebraic proof of the Sperner property for the lattice $L(M)$ consisting of flats of $M$. The algebra $Q/J_M$ is isomorphic to the vector space with basis the set of flats of $M$ as vector spaces.

This work was partly supported by the Sasakawa Scientific Research Grant from The Japan Science Society.
The algebra $A_M$ is defined to be the quotient algebra of the ring of the differential polynomials by the annihilator of $F_M$ (the algebra $A_M$ is isomorphic to $B^*(M)$). They show that $Q/J_M$ has the strong Lefschetz property in the narrow sense if and only if the lattice $\mathcal{L}(M)$ is modular, and that $Q/J_M = A_M$ if and only if $\mathcal{L}(M)$ is modular. They conjectured that the algebra $A_M$ has the strong Lefschetz property for an arbitrary matroid $M$ in an extended abstract [4] of the paper [5].

In general, a graded Artinian Gorenstein algebra has a representation $A = \mathbb{K}[x_1, x_2, \ldots, x_N]/\text{Ann}(\Phi)$, where $\Phi$ is a homogeneous polynomial (see Section 3 for details). For a graded Artinian Gorenstein algebra, there is a criterion for the strong Lefschetz property. This uses the Hessian matrices (see Theorem 3.6). Roughly, a graded Artinian Gorenstein algebra has the strong Lefschetz property if and only if the Hessians (the determinant of the Hessian matrices) do not vanish. Hence the Hessian matrices and Hessians are important.

In this paper, for the generating function for forests with $k$ components, we consider its Hessian matrix and Hessian. In [11], for the generating function for forests with one components (it is called the Kirchhoff polynomial of the complete graph), its Hessian matrix and Hessian are computed. The Hessian matrix of the generating function for forests with one components has exactly one positive eigenvalue and its Hessian does not vanish. More general, in [9] and [8], the Hessian matrix has exactly one positive eigenvalue and its Hessian does not vanish for the generating function for any simple graphic matroid and any simple matroid, respectively. Our main theorem is that for the generating function for forests with $k$ components, its Hessian matrix and its Hessian are in the same situation. That is the Hessian matrix has exactly one positive eigenvalue and its Hessian does not vanish. We gives another proof of the theorem in the case of the truncated matroids of graphic matroids of the complete and complete bipartite graphs by directly calculation of the eigenvalues of the Hessian matrix in [8]. See also Remark 3.11.

This paper is organized as follows: In Section 2, we consider the generating function for the forests. Our main result is that the Hessian of some generating functions for forests does not vanish (Theorems 2.4 and 2.13). In Section 3, we consider the strong Lefschetz property of an algebra associated to a matroid. We see a definition of a matroid and its example, and conclude that our main result gives applications to algebras associated to truncated matroids of graphic matroids of the complete and complete bipartite graphs.

2. MAIN RESULT

In this section, we show that the Hessian of some generating functions for forests does not vanish (Theorems 2.4 and 2.13).
A forest is a graph without cycles. Note that a forest is a simple graph. For a finite set $V$, define
\[
\binom{V}{2} = \{ \{x, y\} \mid x, y \in V, x \neq y\}.
\]

For a graph $\Gamma$, $V(\Gamma)$ and $E(\Gamma)$ are the set of vertices and edges of $\Gamma$, respectively. For a graph $\Gamma$ and an edge $e$, $\Gamma \cup e$ stands for a graph such that the vertex set is $V(\Gamma) \cup e$, and the edge set is $E(\Gamma) \cup \{e\}$. For graphs $\Gamma$ and $\Gamma'$ where $V(\Gamma) \cap V(\Gamma') = \emptyset$, $\Gamma \cup \Gamma'$ stands for a graph such that the vertex set is $V(\Gamma) \cup V(\Gamma')$, and the edge set is $E(\Gamma) \cup E(\Gamma')$.

2.1. The generating function for the forests in the complete graph. For a finite set $V$, we define $\mathcal{F}_V^k$ to be the collection of the forests with the vertex set $V$ and $k$ components. We define the generating function $\Phi_{V,k}$ for $\mathcal{F}_V^k$ by
\[
\Phi_{V,k} = \sum_{F \in \mathcal{F}_V^k} \prod_{e \in E(F)} x_e.
\]

**Remark 2.1.** Let $K_V$ be the complete graph with the vertex set $V$. The set $\mathcal{F}_V^k$ can be regarded as the set of subgraphs of $K_V$ with $k$ components. A forest with $r$ edges in the complete graph $K_n = K_{\{1, 2, \ldots, n\}}$ is a forest consisting of $n - r$ components. For $k = 1$, an element in $\mathcal{F}_V^1$ is called a spanning tree in $K_V$, and the generating function $\Phi_{V,1}$ is called the Kirchhoff polynomial of $K_V$.

**Example 2.2.** Consider $V = \{1, 2, 3, 4\}$. Then, the generating functions are as follows:

\[
\Phi_{V,3} = x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34},
\]
\[
\Phi_{V,2} = x_{12}x_{13} + x_{12}x_{14} + x_{12}x_{23} + x_{12}x_{24} + x_{12}x_{34} + x_{13}x_{14} + x_{13}x_{23} + x_{13}x_{24} + x_{13}x_{34} + x_{14}x_{23} + x_{14}x_{24} + x_{14}x_{34} + x_{23}x_{24} + x_{23}x_{34} + x_{24}x_{34},
\]
\[
\Phi_{V,1} = x_{12}x_{13}x_{23} + x_{12}x_{13}x_{14} + x_{12}x_{14}x_{23} + x_{12}x_{13}x_{24} + x_{12}x_{23}x_{24} + x_{12}x_{23}x_{34} + x_{12}x_{14}x_{34} + x_{12}x_{14}x_{24} + x_{12}x_{13}x_{14} + x_{12}x_{23}x_{34} + x_{12}x_{24}x_{34} + x_{13}x_{23}x_{24} + x_{13}x_{23}x_{34} + x_{13}x_{24}x_{34} + x_{14}x_{23}x_{34} + x_{14}x_{24}x_{34}.
\]

By definition, the generating function $\Phi_{V,k}$ is a homogeneous polynomial of degree $\#V - k$. Moreover, the generating function $\Phi_{V,k}$ is a sum of square-free monomials.

For the generating function $\Phi_{V,k}$, consider the matrix
\[
H_{\Phi_{V,k}} = \left( \frac{\partial}{\partial x_e} \frac{\partial}{\partial x_{e'}} \Phi_{V,k} \right)_{e, e' \in \binom{V}{2}}.
\]
The matrix $H_{\Phi_{V,k}}$ is called the Hessian matrix of $\Phi_{V,k}$, and the determinant $\det H_{\Phi_{V,k}}$ is called the Hessian of $\Phi_{V,k}$. We define $\tilde{H}_{\Phi_{V,k}}$ to be the special value of $H_{\Phi_{V,k}}$ at $x_e = 1$ for all $e$.

Example 2.3. Let $V = \{1, 2, 3, 4\}$. In the case of $\Phi_{V,2}$, we have

$$\tilde{H}_{\Phi_{V,2}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$  

The eigenvalues of $\tilde{H}_{\Phi_{V,2}}$ are 5, $-1$, $-1$, $-1$, $-1$, $-1$.

In the case of $\Phi_{V,1}$, we have

$$\tilde{H}_{\Phi_{V,1}} = \begin{pmatrix} 0 & 3 & 4 & 3 & 3 & 3 \\ 3 & 0 & 3 & 4 & 3 & 3 \\ 4 & 3 & 0 & 3 & 3 & 3 \\ 3 & 4 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 3 & 0 & 4 \\ 3 & 3 & 3 & 3 & 4 & 0 \end{pmatrix}.$$  

The eigenvalues of $\tilde{H}_{\Phi_{V,1}}$ are 16, $-2$, $-2$, $-4$, $-4$, $-4$.

Note that the matrices $H_{\Phi_{V,1}}$ and $\tilde{H}_{\Phi_{V,1}}$ are always the zero matrices for $V = \{1, 2, \ldots, n\}$.

Theorem 2.4 (Main theorem). Consider a set $V = \{1, 2, \ldots, n\}$. Let $n \geq 3$ and $0 < k < n - 2$. The determinant $\det \tilde{H}_{\Phi_{V,k}}$ does not vanish. Moreover, the matrix $\tilde{H}_{\Phi_{V,k}}$ has exactly one positive eigenvalue.

Now, we prove Theorem 2.4. The first step is to compute the eigenvalues of $\tilde{H}_{\Phi_{V,k}}$ (Proposition 2.7). The second step is to show that each eigenvalue of $\tilde{H}_{\Phi_{V,k}}$ is non-zero (Proposition 2.9).

The eigenvalues of $\tilde{H}_{\Phi_{V,k}}$ are known for $k = 1$ [11, Proposition 3.2]. The proof of [11, Proposition 3.2] can be generalized as follows.

Lemma 2.5. Let $E$ be the set of the edges of the complete graph $K_V$ and $#V = n$. Let $H = (h_{e,e'})_{e,e' \in E}$ be the matrix defined by

$$h_{e,e'} = \begin{cases} \alpha, & e = e', \\ \beta, & #e \cap e' = 1, \\ \gamma, & #e \cap e' = 0. \end{cases}$$
The eigenvalues of $H$ are
\[
\alpha + (2n - 4)\beta + \frac{(n - 2)(n - 3)}{2} \gamma,
\]
\[
\alpha - 2\beta + \gamma,
\]
\[
\alpha + (n - 4)\beta - (n - 3)\gamma.
\]

The dimensions of the eigenspaces of $H$ associate with $\alpha + (2n - 4)\beta + \frac{(n - 2)(n - 3)}{2} \gamma, \alpha - 2\beta + \gamma$ and $\alpha + (n - 4)\beta - (n - 3)\gamma$ are
\[
1, \quad \binom{n}{2} - n, \quad n - 1,
\]
respectively.

Fix $V = \{1, 2, \ldots, n\}$ and $n \geq 4$. Define
\[
P = \{ F \in \mathcal{F}_V^k \mid \{1, 2\}, \{2, 3\} \in E(F)\},
\]
\[
R = \{ F \in \mathcal{F}_V^k \mid \{1, 2\}, \{3, 4\} \in E(F)\},
\]
and
\[
p = \#P, \quad q = \#Q.
\]

Lemma 2.6. We set $\tilde{H}_{\Phi_{V,k}} = (h_{e,e'})$. Then, we have
\[
h_{e,e'} = \begin{cases} 
0, & e = e' \\
p, & \#e \cap e' = 1 \\
q, & \#e \cap e' = 0.
\end{cases}
\]

Proof. Since $\Phi_{V,k}$ is a sum of square-free monomials, each diagonal component of $H_{\Phi_{V,k}}$ is zero.

Let $V = \{1, 2, \ldots, n\}$. For any $\sigma \in S_n$, consider a map $\sigma : V \to V$ such that $i \mapsto \sigma(i)$. This map induces an isomorphism between the complete graphs with $n$ vertices. For $e, e' \in \binom{V}{2}$ such that $\#e \cap e' = 1$, the number of the forests with $k$ components which contain the edges $e$ and $e'$ is $p$, since there is an isomorphism $\sigma$ such that $\sigma \{1, 2\} = e$ and $\sigma \{2, 3\} = e'$.

Similarly, we can prove the case where $\#e \cap e' = 0$. \hfill \qed

Proposition 2.7. For $0 < k < n - 2$, the eigenvalues of $\tilde{H}_{\Phi_{V,k}}$ are
\[
(2n - 4)p + \frac{(n - 2)(n - 3)}{2} q, \quad -2p + q, \quad (n - 4)p - (n - 3)q.
\]

The dimensions of the eigenspaces of $\tilde{H}_{\Phi_{V,k}}$ associate with $(2n - 4)p + \frac{(n - 2)(n - 3)}{2} q, -2p + q$ and $(n - 4)p - (n - 3)q$ are
\[
1, \quad \binom{n}{2} - n, \quad n - 1,
\]
respectively.
Next, we show that each eigenvalue of $\tilde{H}_{\Phi_{V,k}}$ is non-zero. Let $W$ be a subset of $V$ such that $\{1,2,3,4\} \subset W$. For $W$, define
\[
\mathcal{T}_W = \{ T \in \mathcal{F}_W \mid \{1,2\}, \{2,3\} \in E(T) \},
\]
\[
\mathcal{T}_W'' = \{ T \in \mathcal{F}_W \mid \{1,2\}, \{3,4\} \in E(T) \}.
\]
Due to [7], we have
\[
\text{Then, we have } f = \mathcal{T}_W = 3W^{W-4}, \quad \mathcal{T}_W'' = 4W^{W-4}.
\]
For $W$, define
\[
\mathcal{F}_W' = \{ T \cap T' \in \mathcal{F}_W^2 \mid \{1,2\}, \{2,3\} \in E(T), 4 \in V(T') \},
\]
\[
\mathcal{F}_W'' = \{ T \cap T' \in \mathcal{F}_W^2 \mid \{1,2\} \in E(T), \{3,4\} \in E(T') \}.
\]
Lemma 2.8. For $\{1,2,3,4\} \subset W \subset \{1,2,\ldots,n\}$, we have
\[
\#\mathcal{F}_W' = \#\mathcal{F}_W''.
\]
Proof. We construct a bijection between $\mathcal{F}_W'$ and $\mathcal{F}_W''$.

We define a map $f$ from $\mathcal{F}_W'$ to $\mathcal{F}_W''$ in the following manner: Let $T = T_a \cup T_b \in \mathcal{F}_W'$, where $T_a$ contains the edges $\{1,2\}$ and $\{2,3\}$, and $T_b$ contains the vertices 4. Since $T_a$ is a tree, if we remove the edge $\{2,3\}$ from $T_a$, then the tree $T_a$ is decomposed into two trees. One of them contains the edges $\{1,2\}$, and we set $T_a^{1,2}$ for this tree. The other one contains the vertex 3, and we set $T_a^{3}$ for this tree. Then we have a decomposition $T_a = T_a^{1,2} \cup \{2,3\} \cup T_a^{3}$. For $T$, define
\[
f(T) = T_a^{1,2} \cup (\{2,3\} \cup T_a^{3} \cup T_b).
\]
Then, we have $f(T) \in \mathcal{F}_W''$. Hence, the map $f$ is well-defined.

We define a map $g$ from $\mathcal{F}_W''$ to $\mathcal{F}_W'$ in the following manner: Let $T = T_c \cup T_d \in \mathcal{F}_W''$, where $T_c$ contains the edge $\{1,2\}$, and $T_d$ contains the edge $\{3,4\}$. Since $T_d$ is a tree, if we remove the edge $\{3,4\}$ from $T_d$, then the tree $T_d$ is decomposed into two trees. One of them contains the vertex 3, and we set $T_d^{3}$ for this tree. The other one contains the vertex 4, and we set $T_d^{4}$ for this tree. Then we have a decomposition $T_d = T_d^{3} \cup \{3,4\} \cup T_d^{4}$. For $T$, define
\[
g(T) = \left(\{2,3\} \cup T_d^{3} \cup T_c\right) \cup T_d^{4}.
\]
Then, we have $g(T) \in \mathcal{F}_W'$. Hence, the map $g$ is well-defined.

The maps $f$ and $g$ are inverses of each other.

We are ready to show that each eigenvalue of $\tilde{H}_{\Phi_{V,k}}$ is non-zero.

Proposition 2.9. Let $\#V = n(n \geq 3)$ and $0 < k < n - 2$. The matrix $\tilde{H}_{\Phi_{V,k}}$ does not have the zero-eigenvalues. Moreover we have the
following:

\[(2n - 4)p + \frac{(n - 2)(n - 3)}{2}q > 0,
\]

\[-2p + q < 0,
\]

\[(n - 4)p - (n - 3)q < 0.
\]

\textbf{Proof.} Since \(p\) and \(q\) are the number of some forests, we have \(p, q > 0\).

Therefore \((2n - 4)p + \frac{(n - 2)(n - 3)}{2}q\), the eigenvalue of \(\tilde{H}_{\Phi_{V,k}}\), is positive.

Let us show that the other eigenvalues of \(\tilde{H}_{\Phi_{V,k}}\) are negative. Let

\[P = \left( \bigsqcup_{W} (T'_W \times F_{W,c}^{(k-1)}) \sqcup \bigsqcup_{W} (F'_W \times F_{W,c}^{(k-2)}) \right),\]

\[Q = \left( \bigsqcup_{W} (T''_W \times F_{W,c}^{(k-1)}) \sqcup \bigsqcup_{W} (F''_W \times F_{W,c}^{(k-2)}) \right),\]

where the sums run over \(\{1, 2, 3, 4\} \subset W \subset \{1, 2, \ldots, n\}\). Then we have \(#P = p\) and \(#Q = q\). For \(W\), define \(f_W = \#F'_W\) and \(t_W = \#W\#W^{W-4}\).

By (1) and Lemma 2.8, we have

\[\#T'_W = 3t_W, \quad \#T''_W = 4t_W, \quad \#F'_W = \#F''_W = f_W.\]

Then

\[p = \sum_{W} 3t_W \#F_{W,c}^{(k-1)} + \sum_{W} f_W \#F_{W,c}^{(k-2)} = 3 \sum_{W} t_W \#F_{W,c}^{(k-1)} + \sum_{W} f_W \#F_{W,c}^{(k-2)},\]

and

\[q = \sum_{W} 4t_W \#F_{W,c}^{(k-1)} + \sum_{W} f_W \#F_{W,c}^{(k-2)} = 4 \sum_{W} t_W \#F_{W,c}^{(k-1)} + \sum_{W} f_W \#F_{W,c}^{(k-2)}.\]

If we set \(t = \sum_{W} t_W \#F_{W,c}^{(k-1)}\) and \(f = \sum_{W} f_W \#F_{W,c}^{(k-2)}\), then we have

\[p = 3t + f,\]

\[q = 4t + f.\]

Note that \(t, f > 0\). Hence

\[-2p + q = -2(3t + f) + (4t + f) = -2t - f < 0,\]

\[(n - 4)p - (n - 3)q = (n - 4)(3t + f) - (n - 3)(4t + f) = -nt - f < 0.\]
By Propositions 2.7 and 2.9, we have Theorem 2.4.

2.2. The generating function for the forests in the complete bipartite graph. Let $X$ and $Y$ be finite sets and $X \cap Y = \emptyset$. Let $V = X \sqcup Y$. For $X$ and $Y$, define

$$F^k_{X,Y} = \left\{ F \in F^k_V \mid \text{if } e \in E(F), \text{ then } e \notin \binom{X}{2} \text{ and } e \notin \binom{Y}{2} \right\},$$

where $F^k_V$ is in 2.1. We define the generating function $\Phi_{X,Y,k}$ for $F^k_{X,Y}$ by

$$\Phi_{X,Y,k} = \sum_{F \in F^k_{X,Y}} \prod_{e \in E(F)} x_e.$$ 

Remark 2.10. Let $K_{X,Y}$ be the complete bipartite graph with the vertex sets $X$ and $Y$. The set $F^k_{X,Y}$ can be regarded as the set of forests of $K_{X,Y}$ with $k$ components. A forest with $r$ edges in the complete bipartite graph $K_{m,n} = K\{1,2,\ldots, m\},\{\bar{1},\ldots,\bar{n}\}$ is a forest consisting of $m + n - r$ components. For $k = 1$, an element in $F^1_{X,Y}$ is called a spanning tree in $K_{X,Y}$, and the generating function $\Phi_{X,Y,1}$ is called the Kirchhoff polynomial of $K_{X,Y}$.

2.11. Let $X = \{1, 2\}$ and $Y = \{\bar{1}, \bar{2}\}$. Then, the generating functions are as follows:

$$\Phi_{X,Y,3} = x_{1\bar{1}} + x_{1\bar{2}} + x_{2\bar{1}} + x_{2\bar{2}},$$
$$\Phi_{X,Y,2} = x_{1\bar{1}}x_{1\bar{2}} + x_{1\bar{1}}x_{2\bar{1}} + x_{1\bar{1}}x_{2\bar{2}} + x_{1\bar{2}}x_{2\bar{1}} + x_{1\bar{2}}x_{2\bar{2}} + x_{2\bar{1}}x_{2\bar{2}},$$
$$\Phi_{X,Y,1} = x_{1\bar{1}}x_{1\bar{2}}x_{2\bar{1}} + x_{1\bar{1}}x_{1\bar{2}}x_{2\bar{2}} + x_{1\bar{1}}x_{2\bar{1}}x_{2\bar{2}} + x_{1\bar{2}}x_{2\bar{1}}x_{2\bar{2}}.$$ 

By definition, the generating function $\Phi_{X,Y,k}$ is a homogeneous polynomial of degree $\#X + \#Y - k$. Moreover, the generating function $\Phi_{X,Y,k}$ is a sum of square-free monomials.

Let us consider the Hessian matrix $H_{\Phi_{X,Y,k}}$ and Hessian det $H_{\Phi_{X,Y,k}}$ of the generating function $\Phi_{X,Y,k}$. We define $\tilde{H}_{\Phi_{X,Y,k}}$ to be the special value of $H_{\Phi_{X,Y,k}}$ at $x_e = 1$ for all $e$.

2.12. Let $X = \{1, 2\}$ and $Y = \{\bar{1}, \bar{2}\}$. In the case of $\Phi_{X,Y,2}$, we have

$$\tilde{H}_{\Phi_{X,Y,2}} = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{pmatrix}.$$ 

The eigenvalues of $\tilde{H}_{\Phi_{X,Y,2}}$ are $3, -1, -1, -1$. 
In the case of $\Phi_{X,Y,1}$, we have
\[
\tilde{H}_{\Phi_{X,Y,1}} = \begin{pmatrix}
0 & 2 & 2 & 2 \\
2 & 0 & 2 & 2 \\
2 & 2 & 0 & 2 \\
2 & 2 & 2 & 0
\end{pmatrix}.
\]
The eigenvalues of $\tilde{H}_{\Phi_{X,Y,1}}$ are $6$, $-2$, $-2$, $-2$.

Note that the matrices $H_{\Phi_{X,Y,1}}$ and $\tilde{H}_{\Phi_{X,Y,1}}$ are always the zero matrices for any $X$ and $Y$.

**Theorem 2.13** (Main theorem). Consider sets $X$ and $Y$ such that $X \cap Y = \emptyset$, $\#X \geq 2$ and $\#Y \geq 2$. For $0 < k < \#X + \#Y - 2$, the determinant $\det \tilde{H}_{\Phi_{X,Y,k}}$ does not vanish. Moreover, the matrix $\tilde{H}_{\Phi_{X,Y,k}}$ has exactly one positive eigenvalue.

Now, we prove Theorem 2.13. The first step is to compute the eigenvalues of $\tilde{H}_{\Phi_{X,Y,k}}$ (Proposition 2.16). The second step is to show that each eigenvalue of $\tilde{H}_{\Phi_{X,Y,k}}$ is non-zero (Proposition 2.24).

The eigenvalues of $\tilde{H}_{\Phi_{X,Y,k}}$ are known for $k = 1$ [11, Proposition 3.15]. The proof of [11, Proposition 3.15] can be generalized as follows.

**Lemma 2.14.** Let $E$ be the set of the edges of the complete bipartite graph $K_{X,Y}$, $\#X = m$ and $\#Y = n$. Let $H = (h_{e,e'})_{e,e' \in E}$ be the matrix defined by
\[
h_{e,e'} = \begin{cases}
\alpha, & e = e', \\
\beta, & e \cap e' \in X, \\
\gamma, & e \cap e' \in Y, \\
\delta, & e \cap e' = \emptyset.
\end{cases}
\]
The eigenvalues of $H$ are
\[
\alpha + (n - 1)\beta + (m - 1)\gamma + (m - 1)(n - 1)\delta, \\
\alpha + (n - 1)\beta - \gamma - (n - 1)\delta, \\
\alpha - \beta + (m - 1)\gamma - (m - 1)\delta, \\
\alpha - \beta - \gamma + \delta.
\]
The dimensions of the eigenspaces of $H$ associate with $\alpha + (n - 1)\beta + (m - 1)\gamma + (m - 1)(n - 1)\delta$, $\alpha + (n - 1)\beta - \gamma - (n - 1)\delta$, $\alpha - \beta + (m - 1)\gamma - (m - 1)\delta$ and $\alpha - \beta - \gamma + \delta$ are
\[
1, \quad m - 1, \quad n - 1, \quad (m - 1)(n - 1),
\]
respectively.

Fix
\[
X = \{ 1, 2, \ldots, m \}, \\
Y = \{ \tilde{1}, \tilde{2}, \ldots, \tilde{n} \},
\]
and \(m, n \geq 2\). By definition, we have \(\#X = m\) and \(\#Y = n\). Define
\[
P = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{1, 2\} \in E(F) \},
\]
\[
Q = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{\bar{1}, 2\} \in E(F) \},
\]
\[
R = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{2, \bar{2}\} \in E(F) \},
\]
and
\[
p = \#P, \quad q = \#Q, \quad r = \#R.
\]

**Lemma 2.15.** We set \(\tilde{H}_{\Phi_{X,Y,k}} = (h_{e,e'})\). Then, we have
\[
h_{e,e'} = \begin{cases} 
0, & e = e', \\
p, & e \cap e' \in X, \\
q, & e \cap e' \in Y, \\
r, & e \cap e' = \emptyset.
\end{cases}
\]

**Proof.** Since \(\Phi_{X,Y,k}\) is a sum of square-free monomials, each diagonal component of \(\tilde{H}_{\Phi_{X,Y,k}}\) is zero.

For any \((\sigma, \tau) \in S_m \times S_n\), consider a map on the vertex set \(X \sqcup Y\) such that
\[
X \ni i \mapsto \sigma(i), \quad Y \ni j \mapsto \tau(j).
\]
The map induces an automorphism of \(K_{X,Y}\). Similarly to Lemma 2.6, we can prove Lemma 2.15. \(\square\)

As a corollary of Lemma 2.14, we obtain the following.

**Proposition 2.16.** For \(0 < k < m + n - 2\), the eigenvalues of \(\tilde{H}_{\Phi_{X,Y,k}}\) are
\[
(n - 1)p + (m - 1)q + (m - 1)(n - 1)r,
\]
\[
(n - 1)p - q - (n - 1)r,
\]
\[
-p + (m - 1)q - (m - 1)r,
\]
\[
-p - q + r.
\]
The dimensions of the eigenspaces of \(\tilde{H}_{\Phi_{X,Y,k}}\) associate with \((n - 1)p + (m - 1)q + (m - 1)(n - 1)r, (n - 1)p - q - (n - 1)r, -p + (m - 1)q - (m - 1)r\) and \(-p - q + r\) are
\[
1, \quad m - 1, \quad n - 1, \quad (m - 1)(n - 1),
\]
respectively.

Next, we show that each eigenvalue of \(\tilde{H}_{\Phi_{X,Y,k}}\) is non-zero. Define
\[
Z = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{1, 2\}, \{2, \bar{2}\} \in E(F) \},
\]
\[
P' = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{\bar{1}, 2\} \in E(F), \{2, \bar{2}\} \notin E(F) \},
\]
\[
R' = \{ F \in \mathcal{F}_{X,Y}^k \mid \{1, \bar{1}\}, \{2, \bar{2}\} \in E(F), \{1, \bar{2}\} \notin E(F) \}.
\]
Then we have a decomposition
\[ P = Z \cup P', \quad R = Z \cup R', \]
respectively. Let \( F \in P' \). Consider the component \( T \) of \( F \) with the vertex 1. Note that the vertices \( 1 \) and \( 2 \) are in \( T \). If we remove the edges \( \{1, 1\} \) and \( \{1, 2\} \) from \( T \), then the tree \( T \) is decomposed into three trees. One of them contains the vertex 1, denoted by \( F_1 \). One of them contains the vertex \( 1 \), denoted by \( \overline{F}_1 \). One of them contains the vertex \( 2 \), denoted by \( \overline{F}_2 \). Let
\[
\begin{align*}
P_1 &= \{ F \in P' \mid 2 \in F_1 \}, \\
P_2 &= \{ F \in P' \mid 2 \in \overline{F}_1 \}, \\
P_3 &= \{ F \in P' \mid 2 \not\in F_1, 2 \not\in \overline{F}_1, 2 \not\in \overline{F}_2 \}, \\
P_4 &= \{ F \in P' \mid 2 \in \overline{F}_2 \}.
\end{align*}
\]
Then we have a decomposition
\[ P = Z \cup P_1 \cup P_2 \cup P_3 \cup P_4. \]
Let \( F \in R' \). Consider the component \( T \) of \( F \) with the vertex 1. Note that \( 1 \in T \). If we remove the edge \( \{1, 1\} \) from \( T \), then the tree \( T \) is decomposed into two trees. One of them contains the vertex 1, denoted by \( F_1 \). The other one contains the vertex \( 1 \), denoted by \( \overline{F}_1 \). Let
\[
\begin{align*}
R_1 &= \{ F \in R' \mid 2 \in F_1, 2 \not\in F_1, 2 \not\in \overline{F}_1 \}, \\
R_2 &= \{ F \in R' \mid 2 \in \overline{F}_1, 2 \not\in F_1, 2 \not\in \overline{F}_1 \}, \\
R_3 &= \{ F \in R' \mid 2 \not\in F_1, 2 \not\in \overline{F}_1, 2 \not\in \overline{F}_1 \}, \\
R_4 &= \{ F \in R' \mid 2 \not\in F_1, 2 \not\in \overline{F}_1, 2 \in \overline{F}_1 \}, \\
R_5 &= \{ F \in R' \mid 2 \not\in F_1, 2 \not\in \overline{F}_1, 2 \in \overline{F}_1 \}.
\end{align*}
\]
Then we have a decomposition
\[ R = Z \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5. \]

**Lemma 2.17.** We have \( \#P_1 = \#R_1 \), \( \#P_2 = \#R_2 \) and \( \#P_3 = \#R_3 \).

**Proof.** For \( 1 \leq i \leq 3 \), we define each map \( f_i \) from \( P_i \) to \( R_i \) in the following manner: Let \( F \in P_i \). Define \( f_i(F) \) to be the forest removing the edge \( \{1, 2\} \) from \( F \) and adding the edge \( \{1, 2\} \). Then \( f_i(F) \in R_i \). We define each map \( g_i \) from \( R_i \) to \( P_i \) in the following manner: Let \( F \in R_i \). Define \( g_i(F) \) to be the forest removing the edge \( \{1, 2\} \) from \( F \) and adding the edge \( \{1, 2\} \). Then \( g_i(F) \in P_i \). The maps \( f_i \) and \( g_i \) are inverses of each other. \( \square \)

**Lemma 2.18.** We have \( \#P_4 = \#R_4 \).
Proof. We define a map \( h \) from \( P_4 \) to \( R_4 \) in the following manner: Let \( F \in P_4 \). Let \( F' \) be the forest such that transpose the vertices 1 and 2 of \( F \). Note that the vertices 1 and \( \bar{2} \) in \( F' \) are not connected by an edge since the vertices 2 and \( \bar{2} \) in \( F \) are not connected by an edge. Define \( h(F) \) to be the forest removing the edge \( \{ \bar{1}, 2 \} \) from \( F' \) and adding the edge \( \{ 1, \bar{1} \} \). Then \( h(F) \in R_4 \). We define a map \( h' \) from \( R_4 \) to \( P_4 \) in the following manner: Let \( F \in R_4 \). Let \( F' \) be the forest such that transpose the vertices 1 and 2 of \( F \). Note that the vertices 2 and \( \bar{2} \) in \( F' \) are not connected by an edge since the vertices 1 and \( \bar{2} \) in \( F \) are not connected by an edge. Define \( h'(F) \) to be the forest removing the edge \( \{ \bar{1}, 2 \} \) from \( F' \) and adding the edge \( \{ 1, \bar{1} \} \). Then \( h'(F) \in P_4 \). The maps \( h \) and \( h' \) are inverses of each other. \( \square \)

We obtain from Lemmas 2.17 and 2.18 the following.

**Lemma 2.19.** We have \( \#P < \#R \).

Define

\[
Z' = \{ F \in F_{X,Y}^k \mid \{ 1, \bar{1} \}, \{ 1, 2 \}, \{ 2, \bar{2} \} \in E(F) \},
\]

\[
Q' = \{ F \in F_{X,Y}^k \mid \{ 1, \bar{1} \}, \{ \bar{1}, 2 \} \in E(F), \{ 2, \bar{2} \} \not\in E(F) \},
\]

\[
R'' = \{ F \in F_{X,Y}^k \mid \{ 1, \bar{1} \}, \{ 2, \bar{2} \} \in E(F), \{ \bar{1}, 2 \} \not\in E(F) \}.
\]

Note that \( Z' = Q \cap R, Q' = Q \setminus Z' \) and \( R'' = R \setminus Z' \). Then, the sets \( Q \) and \( R \) are decomposed into

\[
Q = Z' \sqcup Q', \quad R = Z' \sqcup R'',
\]

respectively. Let \( F \in Q' \). Consider the component \( T \) of \( F \) with the vertex 1. Note that the vertices \( \bar{1} \) and 2 are in \( T \). If we remove the edges \( \{ 1, \bar{1} \} \) and \( \{ \bar{1}, 2 \} \) from \( T \), then the tree \( T \) is decomposed into three trees. One of them contains the vertex 1, denoted by \( F_1 \). One of them contains the vertex \( \bar{1} \), denoted by \( F_1 \). One of them contains the vertex 2, denoted by \( F_2 \). Let

\[
Q_1 = \{ F \in Q' \mid \bar{2} \in F_1 \},
\]

\[
Q_2 = \{ F \in Q' \mid \bar{2} \in F_1 \},
\]

\[
Q_3 = \{ F \in Q' \mid \bar{2} \not\in F_1, \bar{2} \not\in F_2 \},
\]

\[
Q_4 = \{ F \in Q' \mid \bar{2} \not\in F_2 \}.
\]

Then we have a decomposition

\[
Q = Z' \sqcup Q_1 \sqcup Q_2 \sqcup Q_3 \sqcup Q_4.
\]

Let \( R'_i = R_i \cap R'' \) for \( 1 \leq i \leq 5 \). Then we have a decomposition

\[
R = Z' \sqcup R'_1 \sqcup R'_2 \sqcup R'_3 \sqcup R'_4 \sqcup R'_5.
\]

Similarly to Lemma 2.19, we obtain the following.

**Lemma 2.20.** We have \( \#Q < \#R \).
Similarly to Lemma 2.17, the number of elements of $Q_2$ and $R'_2$ are the same. We, however, consider a relation between $Q_2$ and $R_5$ in the following.

**Lemma 2.21.** We have $\#Q_2 = \#R_5$.

**Proof.** We define a map $f'$ from $Q_2$ to $R_5$ in the following manner: Let $F \in Q_2$. Define $f'(F)$ to be the forest removing the edge $\{1, 2\}$ from $F$ and adding the edge $\{2, 2\}$. Then $f'(F) \in R_5$. We define a map $g'$ from $R_5$ to $Q_2$ in the following manner: Let $F \in R_5$. Define $g'(F)$ to be the forest removing the edge $\{2, 2\}$ from $F$ and adding the edge $\{1, 2\}$. Then $g'(F) \in Q_2$. The maps $f'$ and $g'$ are inverses of each other. □

Combining Lemmas 2.17, 2.18 and 2.21, we obtain the following.

**Lemma 2.22.** We have $\#R < \#P + \#Q$.

Summarize Lemmas 2.19, 2.20 and 2.22, and we obtain the following.

**Lemma 2.23.** We have

$$p - r < 0, \quad q - r < 0, \quad -p - q + r < 0.$$  

We are ready to show that each eigenvalue of $\tilde{H}_{\Phi_{X,Y,k}}$ is non-zero.

**Proposition 2.24.** Let $0 < k < m + n - 2$. The matrix $\tilde{H}_{\Phi_{X,Y,k}}$ does not have the zero-eigenvalues. Moreover we have the following:

$$(n - 1)p + (m - 1)q + (m - 1)(n - 1)r > 0,$$

$$(n - 1)p - q - (n - 1)r < 0,$$

$$-p + (m - 1)q - (m - 1)r < 0,$$

$$-p - q + r < 0.$$  

**Proof.** Since $p, q$ and $r$ are the number of some forests, we have $p, q, r > 0$. Therefore $(n - 1)p + (m - 1)q + (m - 1)(n - 1)r$, the eigenvalue of $\tilde{H}_{\Phi_{X,Y,k}}$, is positive.

Let us show that the other eigenvalues of $\tilde{H}_{\Phi_{X,Y,k}}$ are negative. For the other eigenvalues, we have

$$(n - 1)p - q - (n - 1)r = (p - r)n + (-p - q + r),$$

$$-p + (m - 1)q - (m - 1)r = (q - r)m + (-p - q + r).$$

It follows from Lemma 2.23 that the other eigenvalues are negative. □

By Propositions 2.16 and 2.24, we have Theorem 2.13.
3. Application

In this section, we consider the strong Lefschetz property of a graded Artinian Gorenstein algebra associated to a matroid, which is defined by Maeno and Numata in [5]. They showed that the strong Lefschetz property for the algebra associated to the uniform matroid [3]. Here, we discuss the Lefschetz property of the algebra associated to the truncated matroids of the graphic matroids of the complete and complete bipartite graphs.

First of all, we recall definitions of matroids and the strong Lefschetz property.

A matroid $M$ is an ordered pair $(E, \mathcal{B})$ consisting of a finite set $E$ and a collection $\mathcal{B}$ of subsets of $E$ satisfying the following properties:

- $\mathcal{B} \neq \emptyset$.
- If $B_1$ and $B_2$ are in $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element $y \in B_2 \setminus B_1$ such that $\{y\} \cup (B_1 \setminus \{x\}) \in \mathcal{B}$.

In this case, we call each $B \in \mathcal{B}$ a basis of $M$ and $E$ the ground set of $M$.

**Proposition 3.1.** Let $M$ be a matroid with the basis set $\mathcal{B}$. If $B$ and $B'$ are basis of $M$, then the number of elements of them are the same. In other words, if $B, B' \in \mathcal{B}$, then $\#B = \#B'$.

We say that a matroid $M$ has rank $r$ if the number of elements of a basis of $M$ is $r$. The rank of $M$ is denoted by $\text{rank} M$.

**Example 3.2.** We see some examples of matroids.

(a) For any finite graph $\Gamma = (V, E)$ with the vertex set $V$ and the edge set $E$, we call a subgraph $T \subseteq \Gamma$ a spanning tree in $\Gamma$ if $T$ does not contain any cycles and $T$ passes through all vertices of $\Gamma$. Let $\mathcal{B}_T$ be the set of all spanning trees in $\Gamma$. Then $M(\Gamma) = (E, \mathcal{B}_T)$ is a matroid. In this case, $\text{rank} M_\Gamma = \#V - 1$. These matroids are called graphic matroids.

(d) Let $M = (E, \mathcal{B})$ be a matroid and

$$\mathcal{B}_r = \left\{ B' \in \binom{E}{r} \mid \text{there exists } B \in \mathcal{B} \text{ such that } B' \subset B \right\}.$$ 

Then $M = (E, \mathcal{B}_r)$ is a matroid. In this case, $\text{rank} M = r$. These matroids are called truncated matroids of $M$.

Let $M$ be a matroid with the ground set $E$ and $\mathcal{B}$ the set of basis for $M$. For $M$, define

$$\Phi_M = \sum_{B \in \mathcal{B}} \prod_{b \in B} x_b.$$ 

We call $\Phi_M$ the basis generating function of $M$. By Proposition 3.1, for a matroid $M = (E, \mathcal{B})$ of rank $r$, its basis generating function $\Phi_M$
is a homogeneous polynomial of degree $r$ in $|E|$ variables with positive coefficients.

**Remark 3.3.** Let $M'_{K_n}$ be the truncated matroid of rank $r$ of the graphic matroid $M_{K_n}$ of the complete graph $K_n$. Its bases are the forests with $r$ edges. Hence, its basis generating function is $\Phi_{n-r}$ in Section 2.

**Definition 3.4.** Let $A = \bigoplus_{k=0}^{s} A_k$, $A_s \neq 0$, be a graded Artinian algebra. We say that $A$ has the strong Lefschetz property if there exists an element $L \in A_1$ such that the multiplication map $\times L^{s-2k} : A_k \to A_{s-k}$ is bijective for all $k \leq \frac{s}{2}$. We call $L \in A_1$ with this property a strong Lefschetz element.

Let $K$ be a field of characteristic zero. For a homogeneous polynomial $\Phi \in K[x_1, x_2, \ldots, x_N]$, we define $\text{Ann}(\Phi)$ by

$$\text{Ann}(\Phi) = \left\{ P \in K[x_1, \ldots, x_N] \mid P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) \Phi = 0 \right\}.$$ 

Then $\text{Ann}(\Phi)$ is a homogeneous ideal of $K[x_1, \ldots, x_N]$. We consider $A = K[x_1, \ldots, x_N]/\text{Ann}(\Phi)$. Since $\text{Ann}(\Phi)$ is homogeneous, the algebra $A$ is graded. Furthermore $A$ is an Artinian Gorenstein algebra. Conversely, a graded Artinian Gorenstein algebra $A$ has the presentation

$$A = K[x_1, \ldots, x_N]/\text{Ann}(\Phi)$$

for some homogeneous polynomial $\Phi \in K[x_1, x_2, \ldots, x_N]$. We decompose $A$ into the homogeneous spaces $A_k$. Then $A_k$ is a vector space over $K$ for all $k$. Let $\Lambda_k$ be the basis for $A_k$. We define the matrix $H^{(k)}_\Phi$ by

$$H^{(k)}_\Phi = \left( e_i \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) e_j \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) \Phi \right)_{e_i, e_j \in \Lambda_k}.$$ 

The determinant of $H^{(k)}_\Phi$ is called the $k$th Hessian of $\Phi$ with respect to the basis $\Lambda_k$.

**Remark 3.5.** Since $A_0 \cong K$ in this case, we can take the basis $\{1\}$ for $A_1$. Hence the 0th Hessian of $\Phi$ with respect to the basis $\{1\}$ is $\Phi$.

There is a criterion for the strong Lefschetz property for a graded Artinian Gorenstein algebra.

**Theorem 3.6** (Watanabe [10], Maeno–Watanabe [6]). Consider the graded Artinian Gorenstein algebra $A$ with the following presentation and decomposition: $A = K[x_1, x_2, \ldots, x_N]/\text{Ann}(\Phi) = \bigoplus_{k=0}^{s} A_k$. Let $L = a_1 x_1 + a_2 x_2 + \cdots + a_N x_N$. The multiplication map $\times L^{s-2k} : A_k \to A_{s-k}$ is bijective if and only if

$$\det H^{(k)}_\Phi(a_1, a_2, \ldots, a_N) \neq 0.$$
For a matroid $M$ with the ground set $E$, the algebra $A_M$ is defined by
\[ \mathbb{K}[x_e | e \in E] / \text{Ann}(\Phi_M). \]

Theorem 3.7 follows from Theorems 3.6 and 2.4.

**Theorem 3.7.** Let $E$ be the edge set of the complete graph $K_n$. In this case, the ground set of $M_{rK_n}$ is $E$. Let $N = \#E$, and identify $E$ with \{1, 2, \ldots, N\}. Consider the algebra $A_{M_{rK_n}} = \bigoplus_{k=0}^{r} A_k$ for $2 < r < n$. Let $L = x_1 + \cdots + x_N$. The multiplication map $\times L^{r-2}$ from $A_1$ to $A_{r-1}$ is bijective.

**Corollary 3.8.** The algebra $A_{M_{rK_n}}$ has the strong Lefschetz property for $n \leq 5$ and $2 < r < n$. The element $x_1 + \cdots + x_N$ is a strong Lefschetz element.

Theorem 3.9 follows from Theorems 3.6 and 2.13.

**Theorem 3.9.** Let $E$ be the edge set of the complete bipartite graph $K_{m,n}$. In this case, the ground set of $M_{rK_{m,n}}$ is $E$. Let $N = \#E$, and identify $E$ with \{1, 2, \ldots, N\}. Consider the algebra $A_{M_{rK_{m,n}}} = \bigoplus_{k=0}^{r} A_k$ for $2 < r < n$. Let $L = x_1 + \cdots + x_N$. The multiplication map $\times L^{r-2}$ from $A_1$ to $A_{r-1}$ is bijective.

**Corollary 3.10.** The algebra $A_{M_{rK_{m,n}}}$ has the strong Lefschetz property for $n \leq 5$ and $2 < r < n$. The element $x_1 + \cdots + x_N$ is a strong Lefschetz element.

Finally, we refer a recent work [8].

**Remark 3.11.** It is shown that the Hessian of the generating function for simple matroid does not vanish in [8]. Moreover, its Hessian matrix has exactly one positive eigenvalue. In [8], the authors show that the strong Lefschetz property and the Hodge–Riemann relation are equivalent, and all variables are form of a basis for the algebra associated to simple matroid. They also show that the strong Lefschetz property of the algebra associated to any matroid by simplifying matroids.

Our results in this paper are similar to the main result in special case in [8]. But this paper calculates the eigenvalues of the Hessian matrix concretely, and gives another proof of a part of the result in [8].

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