Chiral Random Matrix Models:
Thermodynamics, Phase Transitions and Universality

Romuald Janik\textsuperscript{1}, Maciej A. Nowak\textsuperscript{1,2} and Ismail Zahed\textsuperscript{3}

\textsuperscript{1} Department of Physics, Jagiellonian University, 30-059 Krakow, Poland.
\textsuperscript{2} GSI, Planckstr. 1, D-64291 Darmstadt, Germany
\textsuperscript{3} Institut für Kernphysik, TH Darmstadt, D-64289 Darmstadt, Germany
\textsuperscript{4}Department of Physics, SUNY, Stony Brook, New York 11794, USA.

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For one flavour, we observe that standard chiral random matrix models are schematic variants of the Nambu-Jona-Lasinio (NJL) models whether in vacuum or matter. The ensuing thermodynamics is that of constituent quarks, with mean-field universality in general. For two and three flavours non-standard chiral random matrix models with \( U_A(1) \) breaking are suggested. For three flavours the transition is reminiscent of the isotropic-nematic transition in liquid crystals.

1. A large number of finite-temperature and density studies in the recent past have focused on the Nambu-Jona-Lasinio model (NJL) or extension thereof \[1]. These models capture the essentials of the spontaneous breaking of chiral symmetry in the vacuum, and have been used to model the properties of the low-lying pseudoscalar nonet. They bear much in common with more microscopically motivated thermodynamical descriptions of the QCD vacuum such as the instanton model \[2].

In these models, the low-lying spectrum is composed of constituent quarks and mesons. In the ground state, the quark-antiquark interaction is attractive in the singlet-isosinglet providing a simple mechanism for the spontaneous breaking of chiral symmetry, through quark condensation. With increasing density or temperature, the constituent quarks in matter overcome the "asymmetry" produced by the condensation of the quarks, leading to a chirally symmetric phase of screened quarks. The transition is mean-field in nature, and mostly entropy driven.

Since chiral symmetry breaking is encoded in the eigenvalue distribution at zero virtuality, it is clear that a chiral phase transition would affect quantitatively this distribution. Chiral random matrix models have proven to be a very efficient way to get at the eigenvalue distribution of QCD inspired Dirac spectra. In this letter, we investigate the relationship between chiral random matrix models and Nambu-Jona-Lasinio models both in vacuum and matter. In section 2, we make this relation quantitative. In section 3 we discuss the thermodynamics in the mean-field approximation. In section 4, we address some issues related to the Dirac spectrum in the mean-field approximation. In section 5, we discuss new and non-standard random matrix models with \( U_A(1) \) breaking as inspired by effective models. In section 6, we comment on how the issue of confinement may affect the thermodynamical arguments. Our conclusions are summarized in section 7.

2. To illustrate these points, consider first the case of one flavour, with the schematic Lagrangian in four Euclidean space as given by

\[
\mathcal{L}_4 = +\psi^\dagger (i\gamma \cdot \partial + im + i\mu \gamma_4)\psi + \frac{g^2}{2} \left( (\psi^\dagger \psi)^2 + (\psi^\dagger i\gamma_5 \psi)^2 \right)
\]  

(1)

or equivalently

\[
\mathcal{L}_4 = +\psi^\dagger_R (i\gamma \cdot \partial + i\mu \gamma_4)\psi_L + \psi^\dagger_L (i\gamma \cdot \partial + i\mu \gamma_4)\psi_R \\
+ \psi^\dagger_R i(P + m)\psi_R + \psi^\dagger_L i(P^\dagger + m)\psi_L + \frac{1}{2g^2} P P^\dagger
\]  

(2)

in the chiral basis \( \psi = (\psi_R, \psi_L) \). Here \( P, P^\dagger \) stand for independent auxiliary fields, \( g \) is a fixed coupling and \( \mu \) is a real chemical potential. Note that the Minkowski fields follow from the Euclidean fields through \((i\psi^\dagger, \psi) \rightarrow (\bar{\psi}, \psi)\). Equation (2) is defined on the strip \( \beta \times V_3 \) in Euclidean space, with \( P(\tau + \beta, \vec{x}) = P(\tau, \vec{x}) \), and \( \psi(\tau + \beta, \vec{x}) = -\psi(\tau, \vec{x}) \).

To establish the connection to chiral random matrix models \[3\], further simplifications are needed. The anti-periodicity of the quark fields yields

\[
\psi(\tau, \vec{x}) = \sum_{n=-\infty}^{+\infty} e^{-i\omega_n \tau} \psi^x_n
\]  

(3)

where \( \omega_n = (2n+1)\pi T \) are the Matsubara frequencies \( (T = 1/\beta) \), and \( x = 1, 2, ..., N \) label discrete points in space. Space is here a grid of dimension \( N \), where each point contains a quark of frequency \( n \). If we further assume that the auxiliary fields \( P, P^\dagger \) are constant in space and time, the action in (2) reduces dimensionally to a 0-dimensional one with infinitely many Matsubara modes. The corresponding partition function can be readily found in the form

\[
Z[T, \mu] = \int dP e^{-N \beta \Sigma P P^\dagger} \\
\times \prod_{n=-\infty}^{+\infty} \det^N_2 \beta \begin{pmatrix} i(m + P) & \omega_n + i\mu \\ \omega_n + i\mu & i(m + P^\dagger) \end{pmatrix}
\]  

(4)
following the rescaling \( q_n^x = \sqrt{T_0} \psi_n^x \) and \( \Sigma = V_3/2q^2 \), where \( q_n^x \) are now dimensionless Grassmann variables. The determinant in (3) is over \( 2 \times 2 \) matrices. The corresponding 0-dimensional Lagrangian is

\[
\mathcal{L}_0 = +q^\dagger((\Omega + i\mu)\gamma_4 + im)q + \frac{1}{N} \Omega q^\dagger q_L q^\dagger_R q_R
\]  

(5)

with \( \Omega = \omega_n 1_x \otimes 1_x \).

Consider now the new auxiliary matrix \( A^x_{n,m} \) with entries both in ordinary space \( x, y \) and frequency space \( n, m \). \( A \) is a doubly banded, complex matrix with dimensions \((N \times N) \otimes (\infty \times \infty)\). In contrast with \( P \), the matrix \( A \) bosonizes pairs of quarks of opposite chirality. For the lowest two Matsubara frequencies it is simply \((N \otimes N) \otimes (2 \times 2)\) matrix. In terms of \( A \), the analogue of (3) is

\[
\mathcal{L}_0 = +q^\dagger((\Omega + i\mu)\gamma_4 + im)q + N\Sigma \text{Tr}_{x,n}(AA^\dagger) + q^\dagger_R A q_L + q^\dagger_L A^\dagger q_R
\]  

(6)

The trace in (6) is over \( x \) and \( n \). The partition function associated to (6) is simply

\[
Z[T, \mu] = \int dA \ e^{-\beta N\Sigma \text{Tr}_{x,n}(AA^\dagger)} \det_{x,n} \beta Q
\]  

(7)

with the medium Dirac operator in a random background,

\[
Q = \begin{pmatrix}
im & \Omega + i\mu \\
\Omega + i\mu & -im
\end{pmatrix} + \begin{pmatrix}
0 & A^\dagger \\
A & 0
\end{pmatrix}
\]  

(8)

The determinant in (7) is over chirality (2), space (\( x \)), and frequency space (\( n \)). This is an example of a chiral random matrix model.

3. To discuss the thermodynamics of the random matrix model, it is best to use (4) in large \( N \) with \( n = N/V_3 \) fixed. The Gibbs free energy associated to (4) reads

\[
\Omega = -N \left( \omega + T \ln \prod_{\pm} (1 + e^{-(\omega + \mu)/T}) \right) + N\Sigma PP^\dagger
\]  

(9)

with \( \omega^2 = (P + m)(P^\dagger + m) \). The first term in (9) is just the Hartree contribution to the Gibbs free energy (Fig. 1a), while the second term corrects for the double counting of the interaction energy (Fig. 1b), that is

\[
N\Sigma PP^\dagger = -\frac{1}{N\Sigma} <q^\dagger_R q_L> - <q^\dagger_L q_R>
\]  

(10)

In this schematic model, the Dirac spectrum is simplified to two-levels in frequency space (\( \pm \omega \)) for each \( x = 1, 2, ..., N \). The quark fields do not carry spin. Recall that \( N_T = 1 \). There is no kinetic energy associated to the quarks in either (4) or (5). These degrees of freedom are also manifest in the pressure at high temperature,

\[
P = -\frac{\Omega}{V_3} = 2nT \ln 2 - n\Sigma PP^\dagger + \mathcal{O}(1/T)
\]  

(11)

where \( 2n \) is the number of quarks and antiquarks, and ln2 their respective entropy at \( T = \infty \) since their occupation number is 1/2. The \( 1/T \) term in the entropy cancels against the vacuum energy in the Hartree contribution. The first term in (11) is the thermal pressure of free constituent quarks, while the second term is the left out interaction energy.

![FIG. 1. (a) is the Hartree contribution to the Gibbs free energy, and (b) removes the overcounting.](a) (b)

In large \( N \), the extremum of the Gibbs free energy (4) yields a gap equation for \( P \star \)

\[
2P \Sigma = 1 - n - \pi
\]  

(12)

where \( n = (e^{(\omega + \mu)/T} + 1)^{-1} \) for particles and \( \pi = (e^{(\omega + \mu)/T} + 1)^{-1} \) for antiparticles. The constituent quark density is

\[
\rho_Q = i\langle \psi^\dagger \gamma_4 \psi \rangle = -\frac{1}{V_3} \frac{\partial \Omega}{\partial \mu} = n (n - \pi)
\]  

(13)

while the quark condensate is

\[
i\langle \psi^\dagger \psi \rangle = \frac{1}{V_3} \frac{\partial \Omega}{\partial m} = -2n \Sigma P \star
\]  

(14)

The relations (12-14) are just schematic versions of the usual gap, density and condensate relations in the NJL models.

For zero chemical potential, (12) admits solutions for \( 4\Sigma T \leq 1 \) in the massless case. For zero temperature, \( n = 1 \) for \( \mu \geq \omega \) and zero otherwise, and \( \pi = 0 \). For \( \mu < \omega \) the pressure of the constituent quarks is constant \( P = n/4\Sigma \), while that of the free quarks increases linearly with \( \mu \), \( P = n\mu \). This behaviour is not compatible with confinement at small \( \mu \) (see section 6). A first order transition to the free phase takes place at \( \mu_c = \omega/2 \), in which chiral symmetry is restored. Clearly the quark number density is zero in the constituent quark phase (Fermi level in the Dirac gap) and \( n \) in the free phase. Since \( \omega \) plays the role of the constituent quark mass, chiral symmetry is qualitatively restored for \( \mu_c \sim m_N/6 \), where \( m_N \) is the nucleon mass. The extra factor of \( 1/2 \) follows from the peculiar form of the pressure in the absence of kinetic energy.

If we were to retain only few Matsubara modes in the random matrix model then the thermodynamical relations are altered qualitatively at zero chemical potential and quantitatively otherwise. Indeed, at zero chemical...
potential the Helmholtz free energy with one Matsubara mode becomes

\[ F = -\frac{N}{\beta} \ln \beta (\omega^2 + \pi^2 T^2) + N\Sigma PP^\dagger \]  

(15)

in comparison with \[ \text{(9)} \]. Setting \( \beta = \Sigma P \) to ensure a finite \( T \to 0 \) limit, yields the gap equation \( \omega^2 + \pi^2 T^2 = \omega / P_0 \Sigma^2 \), for which the pressure is \( P = P_F = -n \Sigma P_0^2 \). For \( \beta = \Sigma \) a single Matsubara mode contributes zero to the Hartree part of the free energy (Fig. 1a). This is reminiscent of the high temperature cancelation mentioned above. We note that due to the choice \( \beta = \Sigma \) the pressure grows quadratically in this phase as opposed to linearly in \[ \text{(12)} \]. Since the pressure of a free Matsubara mode is \( P_F = 2n \ln (\Sigma \pi T) / \Sigma \), a phase change at \( \pi \Sigma T = 1 \) is expected, for which \( P = P_F = 0 \). This transition is characterized by \( i < \psi^\dagger \psi > = -2n \Sigma P_0 \), as an order parameter, and was first discussed in \[ \text{(4)} \].

In the case where no Matsubara mode is retained and \( \mu \) is finite, the results follow through the substitution \( \pi T \to i \mu \) above. In particular, the constituent quark mass \( P_0 = \sqrt{1 + \Sigma^2 \mu^2} / \Sigma \) (massless case) grows with \( \mu \). Hence \( i < \psi^\dagger \psi > = -2n \Sigma P_0 \) increases in strength, while \( \rho_G = -2n \Sigma \mu \) decreases. These behaviours are counter-intuitive. Indeed, in this phase the pressure \( P = -n \Sigma P_0^2 \) decreases with increasing \( \mu \). Such a system will tend to cavitate (collapse) unless a phase transition sets in. This is unlikely since the pressure of the free phase is complex

\[ P_F = 2n \frac{1}{\Sigma} \ln \mu \Sigma \pm n \frac{i \pi}{\Sigma} \]  

(16)

The free system is unstable. The present analysis suggests that the correct thermodynamical behaviour is only maintained if all Matsubara modes are retained in the random matrix model, with the zero temperature limit taken after the large (three) volume limit.

At zero chemical potential, the random matrix models we constructed depend on the number and mixing of the various Matsubara modes. At high temperature we expect only the lowest modes \( \omega_0 \) and \( \omega_{-1} \) to survive, making the various matrix models equivalent to each other at the critical point. Indeed, at high temperature and for \( T \approx T_c \) the gap equation \[ \text{(12)} \] simplifies to

\[ g(P_0 + m)^3 + \mu^2 (P_0 + m) - h = 0 \]  

(17)

with \( h = m, \mu^2 = (T - T_c) / T_c \) and \( g = (12 T_c^2)^{-1} \). Eq. \[ \text{(17)} \] has the generic form of a cubic equation, as expected from mean-field treatments of chiral phase transitions \[ \text{(3)} \]. Indeed, \[ \text{(17)} \] is just the gap equation generated from the effective potential \[ \text{(3)} \].

\[ \mathcal{L}_0(T, \Phi) = -h \Phi + \frac{1}{2} \mu^2 \Phi^2 - \frac{c}{N_F} \Phi^N + \frac{\mu}{4} \Phi^4 \]  

(18)

where in the present case \( c = 0 \). This observation implies that the chiral random matrix model \[ \text{(3)} \] much like the NJL model \[ \text{(3)} \] enjoys mean-field critical exponents \( (\alpha, \beta, \gamma, \delta, \nu, \eta) = (0, \frac{1}{2}, 1, 3, \frac{5}{2}, 0) \).

4. To discuss the spectral distribution of the Dirac operator in the medium, it is best to use \[ \text{(6)} \], instead of \[ \text{(4)} \]. For one Matsubara mode, the resolvent of \[ \text{(6)} \] is

\[ G(z) = \frac{1}{N} \left( \text{Tr}_N \left( \frac{1}{z - \Phi} \right) \right) \]  

(19)

where the averaging is carried using \[ \text{(1)} \] in the unquenched approximation. The spectral distribution associated to \[ \text{(19)} \] follows from its discontinuity through the real axis, \( \pi \nu(\lambda) = -\text{Im} G(\lambda + i0) \). In the large \( N \) limit and in the quenched approximation, the resolvent \[ \text{(19)} \] can be readily found to satisfy in general Pastur’s equation \[ \text{(6)} \]. For \( m = \mu = 0 \) and at high temperature (only one Matsubara mode retained), Pastur’s equation for \( G(z) \) is a cubic equation of the form

\[ G^3 - 2G^2 + (z^2 - \pi^2 T^2) + 1)G - z = 0 \]  

(20)

This equation was first discussed in \[ \text{(1)} \] in a general context, and used by others \[ \text{(1,4,0,1)} \]. The cubic equation \[ \text{(20)} \] is a direct manifestation of \[ \text{(18)} \] and hence mean-field universality \[ \text{(3)} \]. The spectral distribution at the critical point following from \[ \text{(20)} \] is consistent with a numerical analysis \[ \text{(6)} \] using large matrices.

For \( T = m = 0 \) and finite \( \mu \) the situation is more subtle. Naively, the resolvent in this case satisfies \[ \text{(20)} \] with the substitution \( \pi T \to i \mu \). The discontinuities of \( G(z) \) are valued in the \( z \)-plane, with end-points given by the zeroes of the discriminant

\[ 4\mu^2 z^4 + z^2(8\mu^4 - 20\mu^2 - 1) + 4(\mu + 1)^3 = 0 \]  

(21)

For \( \mu = 0 \) there are two real roots \( z = \pm \pm \). The result is a cut along the real axis between \(-2 + 2\). The discontinuity along the cut is Wigner’s sunrise distribution for the spectral density. With increasing \( \mu \), the two roots approach each other and tend to \( \pm \sqrt{27/8} \), followed by the emergence of two new roots from real infinity. These new roots are branch points of cuts that are rejected to infinity (unbound spectrum). For \( \mu^2 = 1/8 \), the branch points coalesce pair-wise on the real axis, and for \( \mu^2 > 1/8 \) they move off the real axis resulting into four endpoints that are symmetric about the real axis. The present large \( N \) analysis suggests that the spectral density jumps to zero at \( \mu_* = 1/\sqrt{8} \). Since the pressure \[ \text{(14)} \] of the free system is complex, this jump is simply spurious. Indeed,

1The other choice \( \beta = 2 \Sigma \) amounts for a global shift in the pressure, and would not affect quantitatively the discussion.

2The value of \( \mu_* \approx \sqrt{0.278} \) quoted in \[ \text{(6)} \] uses the real part of a complex pressure.
direct numerical analyses\cite{12,13} have revealed that the spectral distribution at finite $\mu$ is complex valued whatever $\mu > 0$. Technically, it is sufficient to note that the complex character of the Dirac operator at finite $\mu$ causes large fluctuations in the eigenvalue distributions, thereby upsetting the mean-field approximation for the resolvent in certain parts of the $z$-plane\cite{12,13}.

These effects are symptomatic of the fact that the bulk pressure shows signs of cavitation (decreasing $P$) and instability (complex $P_R$). A simple fix would be to include all Matsubara modes as we indicated above. Since chiral symmetry is only restored for $\mu_* \sim m_N/6$ in this case, and the thermodynamical system is stable, it would be interesting to investigate the properties of the resolvent and the Dirac spectrum.

Quenched lattice simulations\cite{14} suggest that the transition sets in at $\mu > 0$. This may mean at least two things: either that the lattice simulations do not generate the proper thermodynamical ensemble, or that quenched simulations do not follow from the conventional spectrum. The former can be checked by simulating the free massive quark ensemble at finite temperature and chemical potential, and probing the various limits. The latter has been suggested recently in\cite{12}, where it was argued that quenched lattice simulations follow from a new spectrum composed of quarks $q$ and their conjugates $\bar{Q}$ (copies of opposite baryon number). Specifically, the analog of (1) is now

$$L_0 = +q^\dagger \gamma_4 i\mu q - Q^\dagger \gamma_4 i\mu Q$$

$$+ \frac{1}{N\Sigma} (q_L^\dagger g_L q_R^\dagger g_R + Q_L^\dagger g_L Q_R^\dagger g_R)$$

$$+ \frac{1}{N\Sigma} (q_L^\dagger g_L Q_R^\dagger q_R + Q_L^\dagger g_L Q_R^\dagger q_R)$$

(22)

As a result, the pressure of the free phase is now real $P_F = 4n\Sigma(\Sigma\mu)/\Sigma$ and monotonously increasing with $\mu$. This pressure is physically meaningful only for $\Sigma\mu \geq 1$, since a free system with negative pressure will cavitate. Besides the chirally broken phase with $i < q^\dagger q \neq 0$ and an unphysical pressure (monotonously decreasing and negative), (22) allows for a chirally symmetric phase with a mixed condensate $< Q^\dagger q >$ and a physical pressure. In the large $N$ (Hartree) approximation, the mixed condensate satisfies the gap equation

$$ip_* = \frac{1}{N\Sigma} < Q_R^\dagger q_R >$$

$$= -\frac{1}{4\Sigma^2} \text{Tr} \left( \tau_1 \gamma_4 i\mu \gamma_3 + ip_* \tau_1 \right)$$

(23)

where $\tau$’s are Pauli matrices active on the doublet $\xi = (q, Q)$. Hence $p_*^2 + \mu^2 = 1/\Sigma^2$. The appearance of the mixed condensate, is tantamount to the spontaneous breaking of a vector symmetry, that is $< \xi^\dagger \tau_1 \xi > \neq 0$

\footnote{In gauge theories this is not in conflict with the Vafa-Witten theorem\cite{15} since $\mu$ is nonzero.}

The effect of the conjugate quarks is to change $\mu^2$ to $-\mu^2$ in the gap equation (thanks to $\tau_1$) thereby fixing the pressure at low $\mu$.

$$P = +\frac{2}{N\Sigma V_3} < Q_R^\dagger q_R >^2 = -2n\Sigma p_*^2$$

(24)

as expected from Fig. 1b, since Fig. 1a contributes to zero due to (23) for $\beta = \Sigma$. The quark number density in this case is $i < \xi^\dagger \gamma_q \gamma_3 \xi > /V_3 = 4n\Sigma\mu$. This behaviour is not compatible with confinement at small $\mu$ (see section 6). For $\Sigma\mu_* = 1$, the quark number density and the pressure are those of the free constituents, so a phase admixture is expected. Using mean-field arguments, the resolvent associated to (22) was discussed in\cite{12} in the quenched approximation, and overall consistency with the numerically generated Dirac spectrum was found.

5. For two and more flavours, we can use the above analogy with the NJL model to construct non-standard chiral random matrix models with explicit $U_4(1)$ breaking. Indeed,\cite{11} suggests the minimal generalization to two flavours

$$Z[T, \mu] = \int dP dP^\dagger e^{-N\Sigma(\text{det}_P + \text{det}_P^\dagger)}$$

$$\times \prod_{n=-\infty}^{+\infty} \text{det}_{2j/\Sigma} \left( \begin{array}{cc} i(m + P) & \omega_n + i\mu \\ \omega_n + i\mu & i(m + P) \end{array} \right)$$

(25)

Here $P$ and $P^\dagger$ are complex $2 \times 2$ valued auxiliary fields, that transform respectively as $V_L P^\dagger V^\dagger_R$ and $V_R P^\dagger V^\dagger_L$ under $U(2)_L \times U(2)_R$. In the massless case, (25) is $SU(2)_L \times SU(2)_R$ symmetric. We note that the $P$-integration can be traded to an integration over Grassmann variables, leading to a standard two-flavour NJL model with a determinantal interaction. The QCD motivation for this model can be found in\cite{11} and references therein. A rerun of the above arguments allows a rewriting of (25) into

$$Z[T, \mu] = \int dA_R dA_L e^{-\Sigma \text{Tr}_{f,x,n} (A_R^a A_R^a + A_L^a A_L^a)}$$

$$\times \text{det}_{f,x,n} \left( \begin{array}{cc} im + \tau^a A_R^a & \Omega + i\mu \\ \Omega + i\mu & im + \tau^a A_L^a \end{array} \right)$$

(26)

where $\tau = (1, i\vec{\tau})$ and $a = 0, 1, 2, 3$. For each frequency $\omega$, the entries in (26) are $(2, 2) \otimes (N, N)$ valued, with $\Omega$ deterministic and $A_{R,L}$ random and hermitean.

In the large $N$ limit, the parity even and isospin symmetric saddle point associated to (26) hence (25) belongs to the universality class described by the effective potential $(N_F = 2)$
\[ \mathcal{L}_0(T, \Phi) = -h\Phi - \frac{c}{N_F} \Phi^{N_F} + \frac{g}{4} \Phi^4 \]  

(27)

with \( \Phi \sim \text{diag} \ P_{R,L} \). Eq. (27) is only valid for temperatures \( T \sim T_c \sim 1/4\Sigma \), with \( c = 1 - 4\Sigma T \) and for a weak external field \( h = -m \). The quadratic term in (27), originates from the two \( 2 \times 2 \) determinants in (25) which are seen to break explicitly the \( U_A(1) \) symmetry.

For three flavours we can just pursue the analogy with one flavour (1) and two flavours (23) and write

\[
Z[T, \mu] = \int dP dP' e^{-\frac{N^3}{3} \sum_{L,F} (PP^t + \theta \Gamma_{L,F})} \times \prod_{n=-\infty}^{\infty} \det_{2f} \beta \left( i(m + P) \omega_n + i\mu \right) \left( i(m + P^t) \right)
\]

(28)

where \( P \) is a complex \( 3 \times 3 \) matrix. In the large \( N \) limit, the \( P \)-integration can be again traded for a Grassmann integration, leading to a three-flavour NJL model with a determinantal interaction. (28) is \( SU(3)_L \times SU(3)_R \) symmetric in the massless case. The QCD motivation for (28) can be found in [1,2] and references therein. In these models, we note that the additional parameter \( \theta > 0 \) plays the role of the instanton density.

If we were to expand the \( U_A(1) \) breaking determinantal part of the exponential in (28), rewrite each determinant using new auxiliary Grassmann variables \( (X_i^a, i = 1, \ldots, N_a, \text{ and } Y_j^b, j = 1, \ldots, N_b, a,b \text{ are flavor indices}), \) and bosonize pairs of opposite chirality as explained above (using \( A \)'s instead of \( P \)'s) then we can rewrite (28) in the form

\[
Z[T, \mu] = \sum_{N_+} \frac{1}{N_+!N_-!} \left( \frac{\theta \Gamma_{L,F}}{(i\beta)^3} \right)^{N_+ + N_-} \times \int dR \ e^{-\frac{N^3}{3} \sum_{L,F} (RR^t)} \det \beta(D + R)
\]

(29)

with the (rectangular) random matrix

\[
R = \begin{pmatrix}
0 & A & 0 & \Gamma_R^I

A^I & 0 & \Gamma_L^0 & 0

0 & \Gamma_L & 0 & \alpha

\Gamma_R & 0 & \alpha^+ & 0
\end{pmatrix}
\]

(30)

and the sparse and deterministic (square) matrix

\[
D = \begin{pmatrix}
im \mu & \Omega & \omega + i\mu & 0

\Omega + i\mu & im & 0 & 0

0 & 0 & 0 & 0

0 & 0 & 0 & 0
\end{pmatrix}
\]

(31)

The block matrices above correspond to the following bosonized pairs (the Matsubara indices connected to \( q \)’s are suppressed)

\[
A_{x,y}^{ab} = \frac{1}{N} \sum_{l} q_{x,l} q_{y,l}^b

\Gamma_{L,xx}^{ab} = X_i^a \delta_{l_x}^b,

\Gamma_{R,xx}^{ab} = Y_j^b \delta_{l_x}^a

\alpha_{ij}^{ab} = X_i^a Y_j^b
\]

(32)

Each of the four rows and columns in the above matrices (23) correspond to a vector of length : 1 : 3 \( \otimes \) N \( \otimes \) infinite, 2 : 3 \( \otimes \) N \( \otimes \) infinite, 3 : 3 \( \otimes \) N \( + \), 4 : 3 \( \otimes \) N \( - \), where infinite stands for the number of Matsubara modes. We observe that (24) with the substitution \( 3 \rightarrow 2 \), hold for two flavours as well, illustrating the model character of the present discussion.

The generic behaviour of these models occur at the critical points. For instance, (24) undergoes a first order transition. At the critical point (SU(3) \(_V \) symmetric phase) the universal potential for the parity even saddle point with \( \Phi \sim \text{diag} P \) follows from (18), with \( N_F = 3 \), \( h = -m \), \( \mu^2 = 4\Sigma T - 1 \), \( c = \theta g T > 0 \) and \( g = 1/12T^2 \). In the parameter range \( (4\Sigma T - 1) \geq 0 \) and \( (48\theta^2 T^4 - 4\Sigma T + 1) \geq 0 \), (18) allows for a first order transition (weak external field) with a critical temperature \( T_c \) fixed by the conditions \( L_0(T_c, \Phi) = L_0(T_c, 0) \) and similarly for its derivative \( L_0'(T_c, \Phi) = L_0'(T_c, 0) \). As \( T \rightarrow T_c \) the number of pairs \( \Phi(1) \) breaking determinan-
tion of the molecules) in liquid crystals [16,17]. Indeed, if we were to denote by \( Q \) the order parameter (mean orientation of the molecules) in liquid crystals then in the

mean-field approximation the free energy per molecule is easily found to be

\[
G = -T \ln 4\pi + (1 - 0.4T)Q^2
\]

\[
-\frac{8}{1057T^2} Q^2 + \frac{4}{175T^3} Q^4 + ...
\]

(33)

in analogy with (18). The nematic phase corresponds to molecules lined up along a preferential direction, but with no long-range order. This phase is different from the smectic phase which is characterized by various degrees of translational ordering. We note that in the presence of an external magnetic field (above quark masses) the first order transition becomes second order through a tricritical point.

As a final point in this section, we note that in the mean-field approximation, the equation of state \( \Phi[h] \) for \( T \sim T_c \), allows for a simple understanding of the distribution of the quark eigenvalues \( \nu(\lambda) \) near zero virtuality \( \lambda \sim 0 \) through the identification \( \pi \nu(\lambda) = \Re \Phi(h = i\lambda + 0) \), in the quenched approximation. For two flavours the transition is second order with \( \beta = 1/2 \), hence \( \Phi[h] = h^{1/3} \), so that \( \nu(\lambda) \sim \lambda^{1/3} \). For three flavours the transition is first order for weak field with \( \nu(0) \sim 2c/3\pi g \) for \( T < T_c \), and \( \nu(0) \sim 0 \) for \( T > T_c \).

\(^4\text{From universality and Widom scaling, a similar argument would in general suggest } \nu(\lambda) \sim \lambda^{3/(3+\gamma)} \text{ for the distribution of eigenvalues, still in the quenched approximation. Such behaviour, however, calls for a non-local equation of state as opposed to a local one in the Landau-Ginzburg analysis.} \)
6. The models we have discussed here are schematic version of the NJL model and its relatives. For one flavour, the spectrum is that of constant $\sigma$ and $\pi$ (fluctuations in $P$) as well as constituent quarks with masses $\omega_n = (P_n + m)$. In confining theories, constituent quarks are barred from the low temperature part of the spectrum. Indeed, in two-dimensional QCD it can be shown that the quarks in light-cone gauge satisfy the following dispersion relations

$$E = -k + \frac{g_\pi^2}{\sqrt{2}\pi\lambda}\text{sgn}(E - k)$$

where $g_\pi^2$ is the fixed QCD coupling in large $N_c$ and $\lambda \to 0$ is an infrared regulator. For spatially constant modes, the spectrum reduces to $E = \pm g_\pi^2/\sqrt{2}\pi\lambda$. The spectrum collapses to two levels that are rejected to infinity when the infrared regulator is removed. Thermodynamics with the spectrum yields heavy Boltzmann penalty factors $e^{-1/T/\lambda}$ or $e^{-\mu/\lambda}$. The contribution of the constant constituent quark modes drop from the thermodynamics at low temperature or chemical potential, for $\lambda \to 0$. The mass gap becomes infinite, and Fig. 1 would be identically zero. As a result, the contribution to the pressure at low temperature or chemical potential is hadronic. So the presently discussed models do not reflect properly on confining theories. They are, however, interesting to use around the critical point $T_\ast$, since the latter seems to be mainly driven by universality. In this regime, there may exist a window of coexistence between just about to be freed massless quarks and long-wavelength mesonic correlations, depending on the nature and character of the transition.

7. We have shown how standard chiral random matrix models can be understood in terms of two-level NJL models, whether in vacuum or matter. The thermodynamics of these models is that of constituent quarks, with mean-field universality at large chemical potential. Our analysis indicates that the large volume limit does not necessarily commute with the zero temperature limit in the presence of a finite chemical potential. This point may be relevant for lattice simulations of QCD (quenched and unquenched) at finite chemical potential. In this respect, it would be interesting to reassess the Dirac spectrum at finite chemical potential in the presence of all Matsubara modes.

We have suggested non-standard chiral random matrix models for two and three flavours, with explicit $U_A(1)$ breaking as inspired by a number of effective models to the QCD vacuum. In large $N$ the character of the transition follows the general lore of universality. In particular for three flavours, and for a specific choice of the external parameters, the transition is first order and analogous to the isotropic-nematic transition encountered in liquid crystals.

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