Noncommutative Topological Quantum Field Theory, Noncommutative Floer Homology, Noncommutative Hodge Theory

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Abstract

We present some ideas for a possible Noncommutative Topological Quantum Field Theory (NCTQFT) and Noncommutative Floer Homology (NCFH). Our motivation is two-fold and it comes both from physics and mathematics: On the one hand we argue that NCTQFT is the correct mathematical framework for a quantum field theory of all known interactions in nature (including gravity). On the other hand we hope that a possible NCFH will apply to practically every 3-manifold (and not only to homology 3-spheres as ordinary Floer Homology currently does). The two motivations are closely related since, at least in the commutative case, Floer Homology Groups constitute the space of quantum observables of (3+1)-dim Topological Quantum Field Theory. Towards this goal we present some "Noncommutative" Versions of Hodge Theory for noncommutative differential forms and tangential cohomology for foliations.

Classification: theoretical physics, mathematical physics, geometric topology, differential geometry, quantum algebra

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1 Introduction and Motivation

This article describes some ideas which emerged during our most recent visit at the IHES a few years ago. Our motivation is twofold: it comes both from physics and from mathematics.

1.1 Physical Motivation: Why NCTQFT?

Why should one want a Noncommutative extension of Topological Quantum Field Theory (NCTQFT)?

We shall argue that NCTQFT should be an adequate framework for a unified quantum field theory incorporating all known interactions in nature, including gravity.

The cornerstone of quantum theory is the principle of particle-wave duality. Although neither gravitons nor gravitational waves have been experimentally observed until now, most physicists take the point of view that quantum gravity—which is currently an elusive theory—should exist; one of the main arguments in favour of its existence is mathematical consistency and it goes back to P.A.M. Dirac: let us consider Einstein’s classical field equations which describe gravity (we assume no cosmological constant and we set the speed of light $c = 1$):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

In the above equation, $G$ denotes Newton’s constant, $T_{\mu\nu}$ denotes the energy-momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor which is equal, by definition, to $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, where $g_{\mu\nu}$ is the Riemannian metric, $R_{\mu\nu}$ is the Ricci curvature tensor and $R$ is the scalar curvature. One can see clearly that the RHS of the above equation, namely the energy-momentum tensor, contains mass and energy coming from the other two interactions in nature; mass for instance (of ordinary matter), consists of fermions (quarks and leptons) and we know that these interactions (strong and electroweak) are quantized and hence the RHS of the equation contains quantized quantities. So for consistency of the equations, the LHS, which encodes geometry, should also be quantized.

[Aside 1: one may argue that the LHS may remain classical while the RHS may involve the average value of an operator; however such a theory]
will not be essentially different from classical general relativity and probably not qualified to be called quantum gravity, what we have in mind is the Ehrenfert Theorem from Quantum Mechanics. We think of the above field equations as describing, in the quantum level, an actual equality between operators.

There are a number of other reasons why physicists would like quantum gravity to exist like the elimination of spacetime singularities etc.

Now the famous and very well-known Holography Principle, which has attracted a lot of attention since 1993 when it was proposed originally by G. 't Hooft (see [8]), states that quantum gravity should be a topological quantum field theory as defined by Atiyah in [1]. There has been strong evidence from the GEO 600 experiment towards the validity of holography (see [3]). In fact quantum gravity should be a topological quantum field theory even without holography: given (for simplicity) a closed Riemannian 4-manifold and the Einstein-Hilbert action which contains the square root of the scalar curvature of the metric as Lagrangian density, in order to compute the partition function of the theory one would have to integrate over all metrics. It is clear that if one was able to perform this functional integral, the result should be a topological invariant of the underlying manifold simply because "there is nothing else left" apart from the topology of the Riemannian manifold. We take for brevity the 4-manifold to be closed, so Atiyah's axioms for a topological quantum field theory will reduce to obtaining numerical invariants and not elements of a vector space associated to the boundary (eg Floer Homology Groups of the boundary 3-manifold). But here there is an important question: the partition function of the Einstein-Hilbert action on a Riemannian manifold should be a topological invariant, but should it be a diffeomorphism or a homeomorphism (or even homotopy) invariant? We know from the stunning work of S.K. Donaldson in the '80s (see [12]) that the DIFF and the TOP categories in dimension 4 are two entirely different worlds (existence of "exotic" $\mathbb{R}^4$'s). So particularly for the case of 4-manifolds (which is our intuitive idea for spacetime, at least macroscopically) this question is crucial.

[Aside 2: We would like to make a remark here: in physics literature the term "topological" really means "metric independent" without further specification but for 4-dim geometry, this point is particularly important].

We do not have a definite answer on this but it is an issue which in most cases it is not addressed to in the physics literature; however we feel
that for quantum physics TOP should be more appropriate as the working
category since for example in quantum mechanics (solutions of Schrodinger
equation) one requires only continuity and not smoothness of solutions at
points connecting different regions.

But this is not enough; if we want a unifying theory of all interactions,
we must have other fields present apart from the metric (eg gauge fields for
electroweak and strong interactions and/or matter fields). We know from
the case of the Quantum Hall Effect, QHE for short and Bellissard’s work
(and others’) (see eg [2]) that the existence of external fields ”make things
noncommutative”. For the particular case of the QHE the presence of a
uniform magnetic field turns the Brilluim zone of a periodic crystal from a
2-torus to a noncommutative 2-torus (see [2]). Moreover as Connes et al.
have shown recently (see [4]), the full 4-dim standard model Lagrangian
of electroweak and strong interactions with Yukawa couplings and neutrino
mixing can be geometrically interpreted as the fundamental K-Homology
class of a noncommutative manifold arising as the discrete product of a spin
4-dim Riemannian manifold with a discrete space of metric dimension 0
and KO-dimension 6 mod 8. Further evidence for this phenomenon, namely
the appearance of noncommutative spaces when external fields are present,
comes from string theory: the Connes-Douglas-Schwarzc article ([9]) indi-
cates that when a constant 3-form $C$ (acting as a potential) of $D=11$
supergravity is turned on, M-theory admits additional compactifications on
noncommutative tori. Also in string theory, the Seiberg-Witten article (see
[10]) also discusses noncommutative effects on open strings arising from a
nonzero $B$-field. So we believe there is good motivation to try to see what
a possible noncommutative topological quantum field theory should look like
since from what we mentioned above, it is reasonable to expect that a uni-
fying quantum theory should have some noncommutativity arising from the
extra gauge or other fields present; it should also be a topological quantum
field theory since it should contain quantum gravity.

1.2 Mathematical motivation: Why NCFH?

We would like to deepen our understanding on 3-manifolds. Floer Homology
is a very useful device since it is the only known homology theory which is
only homeomorphism and not homotopy invariant. (This distinction lies at
the heart of manifold topology and it captures the essence of the Poincare
conjecture). Yet computations are particularly hard and the theory itself is very complicated; moreover the notorious reducible connections make things even worse and at the end Floer Homology Groups are defined only for homology 3-spheres. We would like to have a hopefully simpler theory which would apply to a larger class of 3-manifolds. We shall elaborate more on this in the next sections.

Let us start by recalling some well-known facts from 3-manifold topology: we fix a nice Lie group $G$, say $G = SU(2)$; if $M$ is a 3-manifold with fundamental group $\pi_1(M)$, then the set $R(M) := \text{Hom}(\pi_1(M), G)/\text{ad}(G)$ consisting of equivalence classes of representations of the fundamental group $\pi_1(M)$ of $M$ onto the Lie group $G$ modulo conjugation tends to be discrete. If $M$ is a homology 3-sphere, i.e. $H_1(M; \mathbb{Z}) = 0$, (this is a sufficient condition but not in any way necessary), then $R(M)$ has a finite number of elements and the trivial representation is isolated.

There is a well-known 1:1 correspondence between the elements of the set $R(M)$ and elements of the set $A(M) := \{\text{flat } G\text{-connections on } M\}/(\text{gauge equivalence})$

The bijection is nothing other than the holonomy of the flat connections.

Although $R(M)$ depends on the homotopy type of $M$, we can get topological invariants of $M$, i.e. invariants under homoeomorphisms, if we use the moduli space $A(M)$: depending on how we “decorate” the elements of $A(M)$, namely by giving different “labels” to the elements of $A(M)$, we can get the following topological invariants for the 3-manifold $M$:

1. The (semi-classical limit of the) Jones-Witten invariant.

Pick $G = O(n)$ and for each (gauge equivalence class of) flat $O(n)$-connection $a$ say on $M$, we have a flat $O(n)$-bundle $E$ over $M$ with flat $O(n)$-connection $a$ along with its exterior covariant derivative denoted $da$; now since $a$ is flat, $d^2_a = 0$ and hence we can form the twisted de Rham complex of $M$ by the flat connection $a$ denoted $(\Omega^*(M, E), da)$, where $\Omega^*(M, E)$ denotes smooth $E$-valued differential forms on $M$. If we equip $M$ with a Riemannian metric then we can define a Hodge star operator $*$ and thus we can also define the
adjoint operator \( d_a^* \) of \( d_a \) which is equal to

\[
d_a^* = (-1)^{kn+n+1} d_a^* 
\]

(acting on \( k \)-forms on an \( n \)-dim Riemannian manifold) and then finally one can define the twisted Laplace operator by the flat connection \( a \) to be:

\[
\Delta_a := d_a^* d_a + d_a d_a^*. 
\]

Then the Ray-Singer analytic torsion \( T(M, a) \) is a non-negative real number defined by the formula (see [13]):

\[
\log[T(M, a)] := \frac{1}{2} \sum_{i=0}^{3} (-1)^i i \zeta'_{\Delta_i,a}(0)
\]

where \( \Delta_{i,a} \) denotes the twisted Laplace operator acting on \( i \)-forms and

\[
\zeta'_{\Delta_i,a}(0) := -\frac{d}{ds} \zeta_{\Delta_i,a} |_{s=0} = log D(\Delta_{i,a})
\]

and where we call \( D(\Delta_{i,a}) \) the \( \zeta \)-function regularised determinant of the Laplace operator \( \Delta_{i,a} \) (this is a generalisation of the logarithm of the determinant of a self-adjoint operator).

The \( \zeta \)-function of the Laplace operator \( \zeta_{\Delta_i} \) is by definition (for \( s \in \mathbb{C} \)):

\[
\zeta_{\Delta_i}(s) := \sum_{\{\lambda_n \geq 0\}} \lambda_n^{-s} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-t\Delta_i}) dt
\]

for \( Re(s) \) large. Then \( \zeta_{\Delta_i} \) extends to a meromorphic function of \( s \) which is analytic at \( s = 0 \).

One can prove that the Ray-Singer analytic torsion is independent of the Riemannian metric if the twisted de Rham cohomology groups are trivial.

If \( M \) is a homology 3-sphere (or any other 3-manifold such that the set \( A(M) \) has finite cardinality), then if we sum-up the Ray-Singer analytic torsions of all the flat connections (since these are finite in number we know the sum will converge), what we shall get as a result is a topological invariant of the 3-manifold which is closely related to the “low energy limit” (or the semi-classical limit) of the Jones-Witten (or Reshetikin-Turaev) quantum invariants for 3-manifolds (see [11]). More precisely the low energy limit of the Jones-Witten quantum invariants for homology 3-spheres is a finite sum of combinations of the Ray-Singer torsions with the corresponding Chern-Simons numbers (ie the integral of the Chern-Simons 3-form over
the compact 3-manifold $M$) of the flat connections.

2. The Casson invariant.
Let $M$ be a homology 3-sphere and pick $G = SU(2)$. If we choose a Hegaard splitting on $M$, then assuming that $R(M)$ is regular (i.e., that the 1st twisted de Rham cohomology groups vanish for all flat connections), then each element of $R(M)$ acquires an orientation, namely a “label” +1 or -1. Let us denote by $c_-$ (resp $c_+$) the number of elements of $R(M)$ with orientation -1 (resp +1). Both $c_-$ and $c_+$ depend on the Hegaard splitting chosen but their difference $c := c_- - c_+$ does not (in fact it behaves like an index) and this integer $c$ is the Casson invariant of the 3-manifold $M$. Clearly $c$ is well defined since the cardinality of $R(M)$ is finite and hence both $c_-$ and $c_+$ are finite.

3. Floer Homology Groups.
Again $M$ is a homology 3-sphere (and hence both $R(M)$ and $A(M)$ have a finite number of elements); we pick $G = SU(2)$, we denote by $B(M)$ the space of all $SU(2)$-connections on $M$ modulo gauge transformations and we denote by $B^*(M)$ the irreducible ones (a connection is irreducible if its stabiliser equals the centre of $SU(2)$ where the stabiliser is the centraliser of the holonomy group of a connection). We want to do Morse Theory on the $\infty$-dim Banach manifold $B(M)$:
(i). We find a suitable “Morse function” $I : B^*(M) \to \mathbb{R}$: this is the integral over $M$ of the Chern-Simons 3-form

$$I(A) = \frac{1}{8\pi^2} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

with a finite number of critical points; these are precisely the elements of $A(M)$. This is true since the solutions of the Euler-Lagrange equations for the Chern-Simons action are the flat connection 1-forms.
(ii). Then each element of $A(M)$ acquires a “label” which is the Morse index of the critical point; in ordinary finite dim Morse theory this is equal to the number of negative eigenvalues of the Hessian. But the Hessian of the Chern-Simons function is unbounded below and we get $\infty$ as Morse index for every critical point. So naive imitation of ordinary finite dim Morse theory techniques do not work.

Floer in [7] observed the following crucial fact: if we pick a Riemannian metric on $M$, then considering the noncompact 4-manifold $\mathbb{R} \times M$ along with
its corresponding Riemannian metric, a continuous 1-parameter family of connections $A_t$ on $M$ corresponds to a unique connection $A$ on $\mathbb{R} \times M$; then, choosing the axial gauge (0th component of the connection vanishes), the gradient flow equation for the Chern-Simons function $I$ on $M$ corresponds to the instanton equation on the noncompact 4-manifold $\mathbb{R} \times M$:

$$\partial_t A_t = *F_{A_t} \iff F_{A} = 0.$$ 

Then consider the linearised instanton equation $d_A a = 0$, where $a$ is a small perturbation. This operator is not elliptic; we perturb it to $D_A = -d_A^* \oplus d_A^+$ to make it elliptic. Then the finite integer Morse index for each critical point comes as the relative (with respect to the trivial flat connection) Fredholm index of the perturbed elliptic operator $D_A$. In this way the moduli space $\mathcal{A}(M)$ acquires a $\mathbb{Z}/8$ grading and then we follow ideas from Morse theory:

(iii). We define the Floer-Morse complex using as generators the critical points and the “differential” is essentially defined by the flow lines of the critical points. Taking the cohomology in the usual way we get the Floer homology groups of $M$. The Euler characteristic of the Floer-Morse complex equals twice the Casson invariant (see [12]).

Remarks:

(a). The structure Lie group $SU(2)$ can be replaced by another group, say $U(2)$.

(b). We assumed that all critical points were not only non-degenerate (i.e. $H^1_A(M) = 0$, this denotes the first twisted de Rham cohomology group of $M$ by the flat connection $A$), but in fact acyclic (i.e. $H^0_A(M) = H^1_A(M) = 0$). If this is not the case, then the theory just becomes more complicated and one has to use weighted spaces.

(c). One needs a restriction of the form $b^+ > 1$ in order to be able to prove independence on the choice of the Riemannian metric (the Riemannian metric defines a Hodge star operator whose square equals 1, hence its eigenvalues are $\pm 1$; this gives a splitting of the space of 2-forms into positive and negative eigenspaces and $b^+$ simply denotes the positive part of the 2nd Betti number).

(d). Reducible connections create more severe problems; this is the main reason why people usually work with homology 3-spheres: apart from having a finite number of gauge equivalence classes of flat connections, they have a unique reducible connection which is the trivial flat connection which is moreover isolated. If one wants to take the reducible connections into account as well, then one has to use equivariant Floer homology. This is a lot
more complicated and less satisfactory as a theory since equivariant Floer Homology groups may be infinite dimensional and hence there is no Euler characteristic for the equivariant Morse-Floer complex; also there is no Casson invariant known in this case.

All the above depend crucially on the fact that $R(M)$ (or equivalently $A(M)$) has finite cardinality; the most convenient case that this is guaranteed is if $M$ is a homology 3-sphere. So the question is: what happens if $M$ is such that $R(M)$ does not have finite cardinality? Is there a chance to define the analogue of the Casson invariant say in this case or even more than that, a Floer homology?

We believe “yes” and this is precisely the point we are trying to develop here.

The key idea is the following: we want to replace $R(M)$ by another more stable and better behaving moduli space. To do that we use as our basis a recent result by David Gabai (see [17]): For practically any 3-manifold $M$ (closed, oriented and connected), the moduli space $N(M)$ of taut codim-1 foliations modulo coarse isotopy has finite cardinality.

More concretely: a codim-1 foliation $F$ on a given manifold $M$ is given by an integrable subbundle $F$ of the tangent bundle $TM$ of our 3-manifold $M$. A codim-1 foliation $F$ on $M$ is called topologically taut if there exists a circle $S^1$ which intersects transversely all leaves. A codim-1 foliation is called geometrically taut if there exists a Riemannian metric on $M$ for which all leaves are minimal surfaces (ie they have mean curvature zero). One can prove that a codim-1 foliation is geometrically taut if and only if it is topologically taut. Foliations in general are very flexible structures and the taut foliations are the most rigid ones. Let us call the quotient bundle $Q := TM/F$ the transverse bundle to our foliation.

Let $M$ be a Riemannian 3-manifold. Two codim-1 foliations on $M$ are called coarse isotopic if up to isotopy of each one of them their oriented tangent planes differ pointwise by angles less than $\pi$. Then Gabai proves the following (Theorem 6.15 in [17]): Given any closed, orientable, atoroidal 3-manifold $M$ with a triangulation, there exists a finite non-negative integer $n(M)$ such that any taut codim-1 foliation on $M$ is coarse isotopic to one of the $n(M)$ taut codim-1 foliations. The condition that $M$ should be atoroidal
may be relaxed as Gabai points out. It is clear that $n(M)$ is the cardinality of the Gabai moduli space $N(M)$.

The crucial fact is that although the definition of coarse isotopy depends on the Riemannian metric, the number $n(M)$ does not.

Let us emphasise here that although the Gabai moduli space is finite practically for any 3-manifold, it may turn out to be empty [for example, $S^3$ has no taut codim-1 foliations].

The key idea then is to try to mimic the constructions of the (commutative) topological invariants described above (Ray-Singer torsion, Casson, Floer homology groups) by replacing the moduli space of flat connections modulo gauge with taut codim-1 foliations modulo coarse isotopy. From the moment that foliations enter the scene, noncommutative geometry becomes relevant since it can supply a wealth of new mathematical tools. This means that in principle one could use noncommutative geometric tools to define new invariants for ordinary (commutative) manifolds. But this is not the end of the story: One might even also try to use noncommutative tools and the aforementioned strategies in order to define topological invariants for noncommutative spaces (noncommutative manifolds).

2 Available mathematical tools to study and classify foliations

As it is clear from our previous discussion, (commutative) topological invariants for 3-manifolds are constructed by giving various "labels" to gauge classes of connection 1-forms. Following the same strategy then, the next order of business is to find ways to "decorate" or "label" (coarse isotopy classes of) taut codim-1 foliations. What are the known topological invariants for foliations?

[Aside 3: It seems that the simplest commutative invariant is the Casson invariant, so the simplest idea would be to try to see if one can imitate the definition of the Casson invariant using the Gabai moduli space. Namely if one chooses a Heegaard splitting, can one define a Casson type of invariant by giving "orientations" to taut codim-1 foliations? We have no definite
answer to this question].

2.1 Foliation invariants

Coming back to the topological invariants for foliations, the first we encounter is the Godbillon-Vey invariant (see for example [19]): This is the integral over our compact 3-manifold $M$ of the Godbillon-Vey class which for codim-1 foliations on $M$ is a 3-dim real de Rham cohomology class defined as follows: Suppose that the foliation on $M$ is defined via a transversely oriented, codim-1, integrable subbundle $F$ of the tangent bundle $TM$ of our closed, oriented and connected 3-manifold $M$. Locally $F$ is defined by a non-singular 1-form say $\omega$ where $F$ consists precisely of the vector fields which vanish on $\omega$ (ie the fibre $F_x$ where $x \in M$ equals Ker $\omega_x$). The integrability condition of $F$ means that $\omega \wedge d\omega = 0$. This is equivalent to $d\omega = \theta \wedge \omega$ for another 1-form $\theta$. Then the Godbillon-Vey class is the 3-dim real de Rham cohomology class $[\theta \wedge d\theta] \in H^3(M; \mathbb{R})$. The problem however with the GV invariant is that it is only invariant under foliation cobordisms (see [19]) which is a more narrow equivalence relation than coarse isotopy, hence we may lose the finiteness of the Gabai moduli space (equivalently if we use the GV-invariant, we should restrict ourselves to only those 3-manifolds with a finite number of taut codim-1 foliations modulo foliation cobordisms).

A possibly useful second foliation invariant is the invariant for foliated manifolds that the author introduced some years ago (see [18]) using indeed noncommutative geometry tools, in particular Connes’ pairing between cyclic cohomology and K-Theory. The foliation has to be transversely oriented with a holonomy invariant transverse measure, these restrictions are quite mild. Connes’ approach to foliations as described in [2] is to complete the holonomy groupoid of a foliation to a $C^*$-algebra and then study its corresponding K-Theory and cyclic cohomology. The invariant in [18] is constructed by defining a canonical K-class in the K-Theory of the foliation $C^*$-algebra and then pair it with the transverse fundamental cyclic cocycle of the foliation. To give a flavour of what that means we describe it in the commutative case, ie when the foliation is a fibration, in particular a principal $G$-bundle (where $G$ is a nice Lie group): if we have a fibration seen as a foliation over a compact manifold (the foliated manifold is the total space of the fibre bundle), then this transverse fundamental cyclic cocycle is the fundamental homology class of the base manifold which is transverse to
the leaves=fibres; the $C^*$-algebra is Morita equivalent to the commutative algebra of functions on the base manifold. By the Serre-Swan theorem the K-Theory of this commutative algebra coincides with the Atiyah topological K-Theory of the base manifold and Connes’ pairing reduces to evaluating say Chern classes over the fundamental homology class of the base manifold (here we use the Chern-Weil theory to go from K-Theory to the de Rham cohomology). The key property of the canonical K-class constructed in [18] is that it takes into account the natural action of the holonomy groupoid onto the transverse bundle of the foliation.

[Aside 4: In some sense this class is similar to the canonical class in $G$-equivariant K-Theory, for $G$ some Lie group acting freely on a manifold, the situation is more complicated in the foliation case since instead of a Lie group we have the holonomy groupoid of the foliation acting naturally on the transverse bundle].

We also need the result that the $G$-equivariant K-Theory of the total space of the principal $G$-bundle is isomorphic to the topological K-Theory of the quotient by the group action (since this is a $G$-bundle, the quotient by the $G$-action is the base manifold). But this invariant has not yet been properly understood: obviously if it is to be used to define invariants for 3-manifolds using the Gabai moduli space it should be invariant under coarse isotopy or under a broader equivalence relation. For the moment this point is unclear.

The Heitsch-Lazarov analytic torsion in [21] is defined for foliated flat bundles and it does not seem to be of any use here since it is exactly the flat connections moduli space which we want to replace.

A third possibility for a new topological invariant for foliations which seems interesting, following what we know from the commutative case, is to try to define a Ray-Singer torsion for foliated manifolds and then try to see if this is invariant under coarse isotopy. In order to define the Ray-Singer analytic torsion one needs a flat connection. For the case of foliations, a flat connection always exists, it is our friend the 1-form $\theta$ appearing in the definition of the GV-class; this can indeed be seen in a natural way as a connection on the transverse bundle (for arbitrary codimension $q$, $\theta$ can be seen as a flat connection on the $q$th exterior power of the transverse bundle, this is always a line bundle). This 1-form is sometimes referred to as the (partial) flat Bott connection; it is flat (=closed since this is real valued ie Abelian), only when restricted to the leaf directions (which justifies the
term partial; this is harmless, it can be extended to a full connection by, for example, using a Riemannian metric).

2.2 (Co)homology theories for foliations

The next level of complication is the use of more refined tools to study foliations than numerical invariants and this is (co)homology theory. In general, there are three known cohomology theories which can be used to study foliations: the tangential cohomology, the Hochschild (and cyclic) cohomology of the corresponding foliation $C^*$-algebra and the so-called Haefliger cohomology (which has been used in the construction of the Heitsch-Lazarov analytic torsion).

For the definition of the corresponding foliation algebra and Hochschild and cyclic cohomology of the corresponding foliation algebra, one can see [2].

Perhaps the easiest way to describe tangential cohomology is to start thinking of foliations as generalisations of flat vector bundles. Following these lines, one way to manifest the integrability of a flat connection $\alpha$ say is to point out that its exterior covariant derivative $d_\alpha$ has square zero $d_\alpha^2 = 0$, ie it is a differential. Something similar happens for foliations if one considers the “tangential” (or “leafwise”) exterior derivative on the foliated manifold which is taking derivatives along the leaf directions only; the integrability condition means that the leafwise (or tangential) exterior derivative has square zero, hence it is a differential and this in turn enables one to define the “tangential Laplace operator” along with the so called tangential cohomology and it has corresponding tangential Chern classes (see [16]) by using a Riemannian metric and following the same strategy as one does for ordinary de Rham cohomology. In the above sense tangential cohomology can be seen somehow as a generalisation of the twisted de Rham cohomology by a flat connection. Under the light of this note the analytic torsion defined by Heitsch-Lazarov in [21] has some unsatisfactory properties for our purpose since it is a torsion for a foliated flat bundle (namely a flat bundle whose base space is in addition, foliated, and so the total space carries essentially 3 structures: the fibration, the foliation where the leaves are covering spaces of the base space–flatness–and another foliation which under the bundle projection projects leaves to leaves. One of the main points in this article is to develop a Hodge theory for tangential cohomology. This will be done in
However the most ambitious goal is to try to define a sort of Floer homology using the Gabai moduli space. In order to do that one needs to develop a Morse theory for foliated manifolds. One has at first to find a Morse function whose critical points will be the taut codim-1 foliations. Immitating perhaps naively the Floer homology case we have two natural candidates for a Morse function: tangential Chern-Simons forms and Chern-Simons forms for cyclic cohomology as developed by Quillen not very long ago in [20] (that’s a noncommutative geometry tool). The hope is that by using the Gabai moduli space one might have a chance to avoid the problems with reducible connections (ie the “bubbling phenomenon”, see [12]) when trying to define Floer homology groups for 3-manifolds which are not homology 3-spheres. There are some more versions of Floer homology available but they need some extra structure: a $\text{spin}^c$ structure for the Seiberg-Witten version (and use of the monopole equation instead of the instanton equation), or a symplectic structure (as in the original Floer attempt) or a complex structure (as in the Oszvath-Szabo approach where one uses complex holomorphic curves instead of instantons).

As we shall see, some kind of Morse theory is also needed in order to define Hodge theory for tangential cohomology. In ordinary Morse theory, given a compact smooth manifold, one considers a real valued function (called the Morse function) defined on the manifold and under favourable cases one can reconstruct the homology of the manifold by using the flows of the critical points of the Morse function. In a would-be Morse theory for foliated manifolds one would like to reconstruct the homology of the space of leaves using a suitable Morse function, but it is currently unclear which homology of the 3 above is more suitable. Moreover the critical points should correspond to taut foliations in order to use the Gabai moduli space. We think the above challenge is fascinating. Some progress towards a Morse theory for foliated manifolds has been accomplished and it will be presented again in section 4 below.

Let us sum up the situation: The basic idea is to try to construct topological invariants like the Ray-Singer torsion, the Casson invariant, Floer Homology groups etc (along with their quantum field theoretic analogues—correlation functions) by replacing flat bundles with taut codim-1 foliations. This will enable one to use noncommutative geometric tools. In order to
make some progress towards any of the aforementioned tasks, there is something which seems essential: noncommutative versions of Hodge theory.

3 Hodge theory for noncommutative differential forms

In this section, an analogue of the Hodge theorem will be proved for NC differential forms and as an immediate corollary a NC free bosonic propagator will be constructed.

Let us briefly recall that the "classical" Hodge Theorem states that on every smooth, compact, Riemannian manifold (also assumed oriented), each de Rham cohomology class has a unique harmonic representative (namely the Laplace operator vanishes). As a consequence, every closed form can be written as the sum of an exact form plus a harmonic form and moreover every form can be written as the sum of a harmonic form plus an exact form plus a coexact form.

We follow Quillen ([20]): Let $A$ be a complex, unital associative algebra (in our case at hand, this role will be played by the foliation $C^*$-algebra but the theorem can be proved in this slightly more general setting) and let

$$\Omega^n A := A \otimes_{\mathbb{C}} \bar{\mathbb{A}}^n,$$

for $n > 0$, where

$$\bar{\mathbb{A}} = A/\mathbb{C}$$

whereas

$$\Omega^n A = 0, n < 0$$

and

$$\Omega^0 A = A.$$

Hence we get an identification

$$a_0 da_1...da_n \leftrightarrow (a_0, a_1, ..., a_n).$$

Then we also define

$$\Omega A = \oplus_n \Omega^n A$$
which is the graded algebra (GA) of noncommutative differential forms over $A$, the multiplication being defined via

$$(a_0, a_1, \ldots, a_n)(a_{n+1}, a_{n+2}, \ldots, a_k) = \sum_{i=0}^{k} (-1)^{k-i}(a_0, \ldots, a_i a_{i+1}, \ldots, a_k)$$

for $k > n$. Moreover we define the differential $d : \Omega^n A \to \Omega^{n+1} A$ as follows:

$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$$

or in an equivalent notation

$$d(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n)$$

and hence

$$d\Omega^n A \simeq \bar{A} \otimes^{n+1}$$

for $n \geq 0$. Thus $(\Omega A, d)$ becomes a DGA.

On $\Omega A$, we can also define the Hochschild differential $b : \Omega^n A \to \Omega^{n-1} A$ given by

$$b(a_0, a_1, \ldots, a_n) = \sum_{j=0}^{n-1} (-1)^j (a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_n) + (-1)^n (a_n a_1, a_2, \ldots, a_{n-1}).$$

Thus one has that

$$b(\omega da) = (-1)^{|\omega|} (\omega a - a \omega) = (-1)^{|\omega|} [\omega, a]$$

and

$$b(a) = 0,$$

where $|\omega|$ denotes the degree of the differential form $\omega$.

One also has the Karoubi operator (see [14]) which is a degree zero operator on $\Omega A$ given by

$$k : \Omega^n A \to \Omega^n A$$

where

$$k(\omega da) = (-1)^{|\omega|} (da) \omega$$

(for negative degrees it is given by the identity).
Lemma 1. One has the following relation:

\[ bd + db = 1 - k. \]

Proof: One has

\[(bd+db)(\omega da) = b(d\omega da) + (-1)^{|\omega|} d[\omega, a] = (-1)^{|\omega|+1} [d\omega, a] + (-1)^{|\omega|} d[\omega, a] = [\omega, da] = \omega da - (-1)^{|\omega|} (da)\omega. \]

□

An immediate corollary of the above is that \( k \) commutes both with \( d \) and \( b \), namely

\[ bk = kb \]

and

\[ dk = kd. \]

The above also shows that \( k \) is homotopic to the identity with respect to either of the differentials \( b \) or \( d \).

One can formally see \( d \) and \( b \) as adjoint to each other, playing the roles of \( d \) and \( d^* \) respectively on the de Rham complex of a Riemannian manifold and call

\[ bd + db = 1 - k \]

the NC Laplacian. The natural thing to do next is to examine the spectrum of the NC Laplacian and focus on the zero eigenvalue. We have the following result:

Proposition 1. On \( \Omega A \) one has the harmonic decomposition

\[ \Omega A = Ker(1 - k)^2 \oplus Im(1 - k)^2, \]

where the generalised nullspace \( Ker(1 - k)^2 \) is analogous to the space of harmonic forms.

We can define the harmonic projection \( P \) to be the projection operator which is one on the first term of the harmonic decomposition and zero on the second; it is the spectral projection for \( k \) associated to the eigenvalue 1. Hence the harmonic decomposition can be written

\[ \Omega A = P\Omega A \oplus P^\perp \Omega A \]
where by definition

\[ P^\perp = 1 - P \]

is the spectral projection of \( k \) associated to the set of eigenvalues which are different from 1.

**Proof:** The proof is based on the following technical Lemma:

**Lemma 2.** The Karoubi operator \( k \) on \( \Omega^n A \) satisfies the polynomial relation

\[(k^n - 1)(k^{n+1} - 1) = 0.\]

**Proof of Lemma 2:** We have

\[ k(a_0 da_1 ... da_n) = (-1)^{n-1} da_n a_0 da_1 ... da_{n-1} = \]

\[ = (-1)^n a_n da_0 ... da_{n-1} + (-1)^{n-1} d(a_n a_0) da_1 ... da_{n-1} \]

which in the second equivalent notation reads

\[ k(a_0, a_1, ..., a_n) = (-1)^n (a_n, a_0, ..., a_{n-1}) + (-1)^{n-1} (1, a_n a_0, ..., a_{n-1}). \]

Moreover

\[ k(da_0 da_1 ... da_n) = (-1)^n da_n da_0 ... da_{n-1}. \]

In particular on \( \Omega^n A \) we have that \( k^{n+1} d = d. \)

Next we consider

\[ k^j(a_0 da_1 ... da_n) = (-1)^j (n-1) da_{n-j+1} ... da_n a_0 da_1 ... da_{n-j} \]

for \( 0 \leq j \leq n. \) Hence

\[ k^n(a_0 da_1 ... da_n) = da_1 ... da_n a_0 = a_0 da_1 ... da_n + [da_1 ... da_n, a_0] = \]

\[ = a_0 da_1 ... da_n + (-1)^n b(da_1 ... da_n da_0) \]

which yields

\[ k^n = 1 + bk^n d \]

on \( \Omega^n A. \) Then

\[ k^{n+1} = k + bk^{n+1} d = k + bd \]

and using the definition of the NC Laplacian we get

\[ k^{n+1} = 1 - db. \]
Thus from
\[ k^n = 1 + bk^n d \]
and
\[ k^{n+1} = 1 - db \]
we obtain that \( k \) on \( \Omega^n A \) satisfies the polynomial relation
\[ (k^n - 1)(k^{n+1} - 1) = 0. \]
\[ \square \]
This polynomial relation implies that \( k \) is invertible since the polynomial has constant term 1.

We return to the proof of Proposition 1: Since an operator satisfies a polynomial equation, it gives rise to a direct sum decomposition into generalised eigenspaces corresponding to the distinct roots of the polynomial.

The roots of
\[ (k^n - 1)(k^{n+1} - 1) \]
are the \( n \) different \( n \)-th roots of unity and the \( n + 1 \) different roots of unity of order dividing \( n + 1 \). Yet \( n \) and \( n + 1 \) are relatively prime which means that these two sets of roots have only \( k = 1 \) in common. Hence 1 is a double root and all other roots are simple.

Consequently \( \Omega^n A \) decomposes into the direct sum of the generalised eigenspace \( \text{Ker}(1 - k)^2 \) corresponding to the eigenvalue \( z = 1 \) and the ordinary eigenspaces \( \text{Ker}(k - z) \) for each root of unity \( z \neq 1 \) of order dividing \( n \) or \( n + 1 \).

Combining the above \( \forall n \) we obtain the following spectral decomposition with respect to \( k \)
\[ \Omega A = \text{Ker}(1 - k)^2 \oplus \bigoplus_{z \neq 1} \text{Ker}(k - z). \]
Lumping the eigenvalues \( z \neq 1 \) together we have
\[ \Omega A = \text{Ker}(1 - k)^2 \oplus \text{Im}(1 - k)^2 \]
which completes the proof.

\[ \square \]

Note however that the NC Laplacian

\[ 1 - k = [b, d], \]

contrary to the Riemannian manifold situation, is only nilpotent on the first factor (and invertible on the second). This defect can be cured by introducing the rescaled NC Laplacian

\[ L = [b, Nd] \]

where \( N \) is the numbering operator (this is a degree zero operator which acting on forms gives the scalar multiple of the form by its degree). The rescaled NC Laplacian then vanishes on \( P\Omega A \) (and is invertible on its complement).

On the complementary space \( P^\perp \Omega A \) the NC Laplacian is invertible and homotopic to zero with respect to either differential \( b \) or \( d \). Thus we can define the Green’s operator \( G \) for the NC Laplacian which is equal to its inverse on \( P^\perp \Omega A \), namely

\[ G = (1 - k)^{-1} \]

and \( G = 0 \) on \( P\Omega A \). This can be seen as the NC free bosonic propagator.

As in the classical Hodge theory, the complementary space \( P^\perp \Omega A \) to the ”harmonic forms” splits into subspaces of exact and coexact forms:

**Proposition 2.** One has

\[ P^\perp \Omega A = dP\Omega A \oplus bP\Omega A. \]

**Proof:** This is a formal consequence of the identity

\[ G(bd + db) = 1 \]

on \( P^\perp \Omega A \) and the fact that \( G \) commutes with both differentials \( b,d \) (as one can check via a direct computation). Thus

\[ (Gdb)d = G(bd + db)d = d \]
implies that $Gdb$ is a projection with image $dP^\perp \Omega A$. Similarly $Gbd$ is a projection with image $bP^\perp \Omega A$ and as these projections add to 1 we get the desired decomposition.

\[ \square \]

Let us close this section with some comments: There are 5 basic TQFT’s known up to now: The (2+1) Abelian Chern-Simons theory due to Albert Schwarz, its non-Abelian generalisation (this is the so-called Jones-Witten theory), the (3+1) Donaldson-Floer-Witten theory, its dual (the so-called Seiberg-Witten theory) and the Kontsevich-Gromov-Witten theory (topological $\sigma$ models) and their generalisations.

The simplest of all is the Abelian Chern Simons theory where the Lagrangian density is given by the Abelian Chern-Simons 3-form

$$S = \int_{N^3} A \wedge dA$$

and the partition function is given by the following product of zeta-function regularised determinants of Laplacians

$$Z(N^3) = (Det_\zeta \Delta_1)^{-1/4}(Det_\zeta \Delta_0)^{3/4}.$$  

The NC version of this should be obtained in a straightforward way for the case of a, say, noncommutative 3-sphere and using the NC Laplacian. We hope to be able to report on this elsewhere (see [20] for Chern-Simons forms in NCG and [5] and [6] for noncommutative 3-spheres).

### 4 Hodge Theory for Tangential Cohomology

Let $(M,F)$ be a smooth foliation on a closed $n$-manifold $M$ (and $F$ is an integrable subbundle of the tangent bundle $TM$ of $M$ where $\dim F = p$, $\text{codim} F = q$ with $p + q = n$), equipped with a holonomy invariant transverse measure $\Lambda$ (we need that in order to be able to perform the analogue of ”integration along the fibres” which we do for vector or principal $G$-bundles using the Haar measure which is invariant under the group action). We consider the tangential cohomology coming from the differential graded
complex \((d_F, \Omega^*(M, F))\), where \(d_F\) denotes the tangential exterior derivative (namely taking derivatives only along the tangential (leaf) directions) and \(\Omega^*(M, F)\) denotes forms on \(M\) with values on the bundle \(F\). Due to the integrability of \(F\), the tangential exterior derivative is also a differential, namely \(d^2_F = 0\), hence we can take the cohomology of the above complex.

We pick a Riemannian metric \(g\) on \(M\) (which, when restricted to every leaf gives a Riemannian metric on every leaf), we consider the adjoint operator \(d^*_F\) and we form the tangential Laplacian \(\Delta_F := d^*_F d_F + d_F d^*_F\). We denote by \(\beta_k\) the \(k\)-th tangential Betti number \((0 \leq k \leq p)\), where clearly \(\beta_k = \dim_{\Lambda} \text{Ker}(\Delta^k_F)\).

[Aside 5: We must make an important remark here: this is the Murray-von Neumann dimension defined by Connes using the invariant transverse measure, it is finite; the tangential cohomology groups may be infinite dimensional as linear spaces (see [16])].

It is well known that there exist real valued smooth functions on \(M\) having only Morse or birth-death singularities. We shall denote by \(h\) and \(v\) cummulatively the horizontal (or tangential) and vertical (or transverse) local coordinates respectively and by \(L_x\) the leaf through the point \(x \in M\). For any smooth real function \(\phi\) on \(M\) we denote by \(d_F \phi\) the differential of \(\phi\) in the leaf (horizontal or tangential) directions. A point \(a \in M\) for which the leaf differential vanishes will be called a tangential singularity for \(\phi\). For such a singularity the horizontal (or tangential) Hessian \(d^2_F \phi\) makes sense and in local coordinates \((h, v)\) one has

\[
d_F \phi(h, v) = \sum_{1 \leq i \leq p} \frac{\partial \phi}{\partial h^i}(h, v)
\]

and

\[
d^2_F \phi(h, v) = \left(\frac{\partial^2 \phi}{\partial h^i \partial h^j}(h, v)\right)_{ij}
\]

The index of a tangential singularity \(a\) on \(M\) is defined as the number of minus signs in the signature of the quadratic form \(d^2_F \phi(a)\).

**Definition 1:** A tangential singularity \(a\) on \(M\) of a smooth real function \(\phi\) on \(M\) as above is called a Morse singularity if \(d^2_F \phi(a)\) is non-singular.

We denote by \(T(\phi)\) (resp. \(M(\phi), M_i(\phi)\)) the set of all tangential singularities (resp. of Morse singularities, Morse singularities of index \(i\), where
$0 \leq i \leq p$) of the function $\phi$. The first complication emerges since in this case a good definition for a tangential (or horizontal) Morse function cannot be reduced to simply a smooth function on $M$ having only tangential Morse singularities. This is explained in the following Lemma:

**Lemma 3:** Let $(M, F)$ be as above. Assume that there exists a smooth function $\phi$ again as above with only tangential Morse singularities. Then the set of all tangential Morse singularities of $\phi$ is a closed $q$-dim submanifold transverse to the foliation.

**Proof:** We suppose that $\phi$ is a smooth function on $M$ such that for any leaf $L$ in the quotient space $M/F$ the restriction $\phi|_L$ of $\phi$ on the leaf $L$ has no degenerate critical points. Then the map

$$x \mapsto (x, d_F \phi(x))$$

from $M$ to $T^*F$ is transverse to the zero section of $T^*F$ since its differential is given on any foliation chart $\Omega = U \times T$ by

$$d(d_F \phi)(h, v)(X_h, X_v) = ((X_h, X_v), \phi_{hh}(h, v)X_h + \phi_{hv}(h, v)X_v)$$

where the subscripts $h$, $v$ denote partial derivative with respect to the corresponding coordinates and $det(\phi_{hh}(h, v)) \neq 0$ for a Morse singularity with coordinates $(h, v)$. This implies first that the set of all Morse singularities $M(\phi)$ is a closed submanifold of $M$ with $dim(M(\phi)) = codim(F)$ and second that $M(\phi)$ is transverse to the foliation $F$ because for any non-zero tangent vector $X = (X_h, X_v)$ of $M(\phi)$ at the point $(h, v)$, one has that

$$\phi_{hh}(h, v)X_h + \phi_{hv}(h, v)X_v = 0.$$ 

This means that the transverse component $X_v$ of $X \neq 0$ is non-zero which proves that $M(\phi)$ is transverse to the foliation $F$. This concludes the proof.

$\square$

It turns out that many interesting foliations have no closed transversals and hence any good notion of tangential Morse function should allow degenerate critical points in the leaf direction. (However taut foliations which are the ones appearing in the Gabai moduli space do have closed transversals...
by definition).

**Definition 2:** We call *almost Morse function* a smooth function $\phi$ as above with degenerate critical points which only occur at a $\Lambda$-negligible set of leaves (namely we allow degenerate critical points but not too many).

**Definition 3:** A *good* almost Morse function is an almost Morse function which is generically unfolded in the sense of Igusa Parametrised Morse Theory (see [23]) (roughly this means that it has only birth-death singularities, namely points where critical points cancel or create in pairs). More concretely, the last requirement means that there exist *normal forms* describing the function in a neighbourhood of a birth-death singularity. A birth-death singularity is a degenerate tangential singularity (i.e. tangential Hessian vanishes) for which the restriction of the map $x \mapsto (d_F(\phi)(x), \det[d^2_F(\phi)(x)])$ has rank $p$ at $x$.

### 4.1 Witten’s perturbation by a Morse function-Tangential version

Let $(M, F)$ be a foliation as above equipped with a holonomy invariant transverse measure $\Lambda$. We choose a smooth Riemannian metric on $M$ and denote by $\Delta^k_L$ ($0 \leq k \leq p$) the corresponding Laplace operator on the leaf $L$ acting on $k$-forms. We know that the bundle of Hilbert spaces is square integrable and thus has a well-defined Murray-von Neumann dimension

$$\beta_k = \text{dim}_\Lambda(\text{Ker}(\Delta^k_L)) < \infty$$

which does not depend on the choice of metric. Assume moreover that $\text{codim}(F) \leq \text{dim}(F)$. Let $\phi$ be a smooth real function on $M$ which is good almost Morse function and $\tau$ a positive real parameter. For each leaf $L$ and $0 \leq k \leq p$ we denote by $d^k_{\tau,L}$ the closure (in $L^2(L, \wedge^k F)$, the space of square integrable forms on the leaf $L$) of the operator which sends each smooth $k$-form $\omega$ on $L$ to the smooth $(k + 1)$-form $e^{-\tau\phi}d^k_L(e^{\tau\phi}\omega)$ again on $L$.

**Definition 4:** We shall call *Witten tangential Laplacian* the measurable filed $(\Delta^k_{\tau,L})_L$ which is defined in the obvious way, namely

$$\Delta^k_{\tau,L} = d^k_{\tau,L} \cdot (d^{k-1}_{\tau,L})^* + (d^k_{\tau,L})^*d^k_{\tau,L}.$$
Then we prove that $\Delta^k_\tau$ computes the $(L^2)$ tangential cohomology of $(M, F)$:

**Proposition 3:** The fields $(Ker(\Delta^k_\tau,L))_L$ and $(Ker(\Delta^k_L))_L$ of Hilbert spaces are measurably isomorphic and one has that

$$\beta_k = dim_\Lambda[Ker(\Delta^k_\tau,L)_L] < +\infty$$

for any positive real $\tau$ and $0 \leq k \leq p$.

**Proof:** The proposition can be proved following the steps below:

1. The operator $d^k_\tau = (d^k_{\tau,L})_L$ is a differential operator which is elliptic along the leaves of $F$. This can be proved using an argument similar to the one used by Connes in [2] to prove the transversal index theorem.

2. Observe that the adjoint of $d^k_{\tau,L}$ is the closure of the operator which sends each smooth $(k+1)$-form $\omega$ on $L$ to the smooth $k$-form $e^{\tau\phi}(d^k_L)^*(e^{-\tau\phi})\omega$ again on $L$, where

$$(d^k_L)^* = (-1)^{pk+1} * d^k_L *,$$

and where "*" denotes the Hodge star operator on the leaf $L$ defined via the Remanian metric.

3. We note that $\Delta^k_\tau = (\Delta^k_{\tau,L})_L$ is a field of measurable positive operators acting on the Hilbert space of square integrable $k$-forms on the leaf $L$. Moreover $\Delta^k_\tau$ is elliptic along the leaves.

4. For any leaf $L$ and $0 \leq k \leq p$, we denote by $T^k_L$ the bounded operator on $L^2(L, \wedge^k T^* F)$ defined by

$$T^k_L(\omega)(x) = e^{-\tau\phi(x)}\omega(x),$$

for $\omega \in L^2(L, \wedge^k T^* F)$.

It is clear that $T^k_L$ is invertible and defines an element of $L^\infty(M/F, \wedge^k T^* F)$. Next we set

$$U^k_{\tau,L} = Q^k_{\tau,L} T^k_L Q^k_L,$$
where $Q^k_{\tau,L}$ (resp. $Q^k_L$) denotes the orthogonal projection onto the sub-space $\text{Ker}(\Delta^k_{\tau,L})$ (resp. onto $\text{Ker}(\Delta^k_L)$). We thus define a measurable field $(U^k_{\tau,L})_L$ of endomorphisms of the random Hilbert space $(L^2(L, \wedge^k T^* F))_L$, such that $\text{Ker}(\Delta^k_{\tau,L})$ is a superset of $U^k_{\tau,L}(\text{Ker}(\Delta^k_L))$.

We want to show that $(U^k_{\tau,L})_L$ belongs to $L^\infty(M/F, \wedge^k T^* F)$ and defines an isomorphism of Hilbert spaces from $(\text{Ker}(\Delta^k_L))_L$ to $(\text{Ker}(\Delta^k_{\tau,L}))_L$. One then has (omitting the subscript $L$):

$$d^k_{\tau} = T^k_{\tau} + 1_{\tau} d^k_{\tau}(T^k_{\tau})^{-1}$$

and hence

$$T^k_{\tau}(\text{Kerd}^k) = \text{Kerd}^k_{\tau}(\text{equation 1})$$

$$T^k_{\tau+1}(\text{cl.Im}(d^k_{\tau})) = \text{cl.Im}(d^k_{\tau})(\text{equation 2}).$$

But it follows from Hodge theory that one has the following orthogonal decompositions:

$$\text{Ker}(d^k) = \text{Ker}(\Delta^k) \oplus \text{cl.Im}(d^k)$$

and

$$\text{Ker}(d^k_{\tau}) = \text{Ker}(\Delta^k_{\tau}) \oplus \text{cl.Im}(d^k_{\tau}),$$

and then from equations (1) and (2) it follows that $T^k_{\tau}$ is given in those decompositions by a $2 \times 2$ matrix with the upper left entry being $U^k_{\tau}$, the lower right entry being $B^k_{\tau}$, the upper right entry being 0 and the lower left entry being any element, namely

$$T^k_{\tau} = \begin{pmatrix} U^k_{\tau} & 0 \\ \ast & B^k_{\tau} \end{pmatrix}$$

and where the entry $B^k_{\tau,L} = T^k_{\tau,L}[\text{cl.Im}(d^k_{L})]$, (namely $T^k_{\tau,L}$ restricted to $\text{cl.Im}(d^k_{L})$, the closure of the Image of $d^k_{L}$), is invertible. We thus deduce that $U^k_{\tau}$ is an isomorphism from $(\text{Ker}(\Delta^k_{L}))_L$ onto $(\text{Ker}(\Delta^k_{\tau,L}))_L$ and hence

$$\beta_k = \dim_{\Lambda}[\text{Ker}(\Delta^k_{L})_L] = \dim_{\Lambda}[\text{Ker}(\Delta^k_{\tau,L})_L]$$

and this holds $\forall \tau > 0$. As $\beta_k < +\infty$, then an argument similar to Connes [2] completes the proof. □
Aside 6: Connes and Fack have proved that every measured foliation with \( q \leq p \) has at least one good tangential almost Morse function; their proof is based on an astounding theorem due to K. Igusa: it was a well-known fact that a generic smooth real valued function on a closed manifold has only nondegenerate critical points; however a generic 1-parameter family of real valued smooth functions has in addition birth-death points where critical points are created or canceled in pairs. A multi-parameter family has a zoo of complicated singularities; K. Igusa proved that more complicated singularities can be avoided: for any foliation on a closed manifold it is always possible to find a smooth real valued function such that singularities associated with the critical points of its restriction to every leaf are at most of degree 3! Clearly we think of a foliation as a more complicated parametrised family of manifolds than a fibre bundle: the family of manifolds (leaves—they correspond to the tangential directions) is parametrised by the space of leaves (corresponds to the transverse directions); in a fibre bundle we have a family of manifolds (fibre) parametrised by the base manifold.

Comments:

It is not true that any measured foliation with \( q \leq p \) has a tangential Morse function, namely the foliations with tangential Morse functions are rather special (they must have a closed transversal); taut foliations nevertheless, which is what we are mostly interested in, do have, by definition, a complete closed transversal).

If we denote by \( A(M, F) \), \( J(M, F) \) and \( R(M, F) \) the sets of tangential almost Morse functions, tangential generalised Morse functions and tangential generalised Morse functions which are generically unfolded respectively, then the good tangential almost Morse functions are those in the intersection of \( A(M, F) \) and \( R(M, F) \) (clearly the 3rd set is a subset of the second). The hard piece due to K. Igusa is to prove that for a closed \( M \) and an \( F \) with \( \text{codim} F \leq \dim F \), the set \( J(M, F) \) is nonempty.

For \( \phi \) a good tangential almost Morse function, we have that the critical manifold \( S_F(\phi) \) is a \( q \)-dim submanifold of \( M \) transverse to \( F \), the set of tangential Morse singularities of index \( i \) \( S^i_{1,F}(\phi) \) is also a \( q \)-submanifold of \( M \) transverse to \( F \) and open inside the critical manifold (but not closed in \( M \) in general) and the set of tangential birth-death singularities \( S^i_{2,F}(\phi) \) of index \( i \) of \( \phi \) is a closed \((q - 1)\)-submanifold of the critical manifold and it is both in the closure of \( S^i_{1,F}(\phi) \) and of \( S^{i+1}_{1,F}(\phi) \).
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References

[1] M. F. Atiyah: "Topological Quantum Field Theories", IHES Publ. Math. 68 (1989), 175-186.

M. F. Atiyah: "The Geometry and Physics of Knots", Accademia Nazionale dei Lincei, Cambridge University Press 1990.

[2] A. Connes: “Noncommutative Geometry”, Academic Press 1994.

[3] C.J. Hogan: Measurement of Quantum Fluctuations in Geometry, Phys. Rev. D, Vol 77 (2008).

[4] A.H. Chamseddine, A. Connes, M. Marcolli: "Gravity and the Standard Model with Neutrino Mixing", Adv. Theor. Math. Phys. 11 (2007), 991-1089.

[5] A. Connes, M. Dubois-Violette: "Moduli Space and Structure of Noncommutative 3-spheres", Lett. Math. Phys. 66 (2003), 91-121.

[6] A. Connes, M. Dubois-Violette: "Non commutative finite dimensional manifolds II. Moduli space and structure of non commutative 3-spheres", math/0511337.
[7] A. Floer: "An instanton invariant for 3-manifolds", Commun. Math. Phys. (1989).

[8] G. ’t Hooft: “Dimensional Reduction in Quantum Gravity”, Essay dedicated to Abdus Salam in “Salamfestschrift: a collection of talks”, editors: A. Ali, J. Ellis and S. Randjbar-Daemi, World Scientific 1993 and gr-qc/9310026.

[9] A. Connes, M.R. Douglas and A. Schwarz: “Noncommutative geometry and Matrix theory: compactification on tori” JHEP02(1998)003.

[10] N. Seiberg, E. Witten: ”String theory and Noncommutative Geometry”, JHEP09(1999)032.

[11] E. Witten: ”Quantum Field Theory and the Jones Polynomial”, Commun. Math. Phys. 121 (1989), 353-386.

E. Witten: ”Supersymmetry and Morse Theory”, J. Diff. Geom. 17 (1982).

[12] S.K. Donaldson: “Floer Homology Groups in Yang-Mills Theory”, Cambridge University Press 2002.

S.K. Donaldson and P.B. Kronheimer: “The geometry of 4-manifolds”, Oxford University Press 1991.

[13] D. B. Ray and I. M. Singer: ”R-torsion and the Laplacian on Riemannian manifolds”, Adv. Math. 7 (1971) 145-210.

[14] M. Karoubi: ”Homologie cyclique et K-Theorie”, Asterisque 149 (1987).

[15] W. Luck: “$L^2$-invariants: Theory and Applications to Geometry and K-Theory”, A Series of Modern Surveys in Mathematics Vol 44, Springer,
2002.

[16] C.C. Moore and C. Schochet: ”Global Analysis on Foliated Manifolds”, Springer (1988).

[17] D. Gabai: ”Essential Laminations and Kneser Normal Form”, Jour. Diff. Geom. Vol 53 No 3 (1999).

[18] I.P. Zois: “A new invariant for σ-models”, Commun. Math. Phys. Vol 209 No 3 (2000).

[19] A. Candel and L. Conlon: “Foliations I and II”, Graduate Studies in Mathematics Vol 23, AMS, Oxford University Press (2000) and (2003).

[20] D.G. Quillen: ”Chern-Simons Forms and Cyclic Cohomology”, in “The Interface of Mathematics and Particle Physics”, editors D.G. Quillen, G.B. Segal and S.T. Tsou, Oxford University Press 1990.

J. Cuntz and D.G. Quillen: ”Operators on noncommutative differential forms and cyclic homology” in “Geometry, Topology and Physics for R. Bott”, International Press 1995, edited by S-T Yau.

[21] J.L. Heitsch and C. Lazarov: “Riemann-Roch-Grothendieck and Torsion for Foliations” (unpublished).

[22] J.-M. Bismut and J. Lott: ”Flat Vector Bundles, Direct Images and Higher Real Analytic Torsion”, J. Am. Math. Soc. 8 (1995).

[23] K. Igusa: ”Higher singularities of smooth functions are unnecessary”, Annals of Math. 119, (1984), 1-58.

K. Igusa: ”Parametrised Morse Theory and its Applications”, Proc. Internat. Congr. Math. (Kyoto 1990), Math. Soc. Japan, Tokyo, (1991),
643-651.