Three-state Neural Network: from Mutual Information to the Hamiltonian

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Abstract
The mutual information, $I$, of the three-state neural network can be obtained exactly for the mean-field architecture, as a function of three macroscopic parameters: the overlap, the neural activity and the activity-overlap, i.e., the overlap restricted to the active neurons. We perform an expansion of $I$ on the overlap and the activity-overlap, around their values for neurons almost independent on the patterns. From this expansion we obtain an expression for a Hamiltonian which optimizes the retrieval properties of this system. This Hamiltonian has the form of a disordered Blume-Emery-Griffiths model. The dynamics corresponding to this Hamiltonian is found. As a special characteristic of such network, we see that information can survive even if no overlap is present. Hence the basin of attraction of the patterns and the retrieval capacity is much larger than for the Hopfield network. The extreme diluted version is analyzed, the curves of information are plotted and the phase diagrams are built.

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1 Introduction
The collective properties of neural networks, such as the storage capacity and the overlap with the memorized patterns, have been a subject of intensive research in the last decade[9]. However, more precise measures of their performance as an associative memory, as the information capacity and the basins of attraction of their retrieval states, have received comparatively less attention[11, 12]. For some models as the sparse-code networks[13, 14], or the three-state networks[15, 16], where the patterns are not uniformly distributed, an information-theoretical approach[17, 18] seems crucial.

Calculations of the Shannon mutual information $(I)$ for the sparse-code network were made[13, 14]. For low storage of patterns, a few time steps are needed to retrieve them[13, 14]. However, for large storage, only imperfect retrieval is possible. The closer to saturation, the larger the time steps required to dynamical retrieval. So, first-time retrieval is not enough and it is interesting to study the information capacity of recurrent networks. To improve $I$ for this recurrent network, a scheme was proposed[19]. This Self-Control Neural Network (SCNN) is an adaptive scheme induced by the dynamics itself instead of imposing any external constraint on the activity of the neurons. Such procedure successfully increases both $I$ and the basins of attraction of the patterns. Similar mechanisms can improve $I$ for three-state low-activity networks[20], with diluted and fully-connected architectures.

Here we propose a new method, based on direct use of the $I$ calculated in the mean-field approximation, to obtain a Hamiltonian which maximizes $I$ within a large range of values for the activity of the network.

A three-state neural network is defined by the use of a set of $\mu = 1, \ldots, p$ ternary patterns, $\{\xi^\mu_i\} \in \{0, \pm 1\}$, $i = 1, \ldots, N$, which are independent random variables given by the probability distribution

$$p(\xi^\mu_i) = a\delta(|\xi^\mu_i|^2 - 1) + (1 - a)\delta(\xi^\mu_i),$$

where $a$ is the activity of the patterns ($\xi^\mu_i = 0$ are the inactive states). A low-activity three-state neural network corresponds to the case where the distribution is not uniform, i.e., $a < 2/3$. In the limit $a = 1$ the binary Hopfield model is reproduced.

The information enclosed in a simple unit $\xi^\mu_i$ is given by the entropy of its probability,

$$H_{\mu i} = -a \ln(a/2) - (1 - a) \ln(1 - a).$$

One can define as sparse a code whose fraction of active neurons is very small and tends to zero in the thermodynamic limit[11]. Sparse-code binary patterns can have a large load rate $\alpha \sim [a \ln(a)]^{-1}$, where $a$ is the ratio between the number of patterns $p$ and number of connection per neuron $N$.

However, the information per unit for the sparse code is $H_{\mu i} \sim a \ln(a) \ll 1$, and it is not clear if the total information per connection, $i = \sum_{\mu} H_{\mu i} / \langle J_{ij} \rangle = \alpha H_{\mu i}$ of such network is larger than the uniform (non-sparse) one.

Although ternary patterns with low-activity have not been studied in the same proportion, they present a similar behavior[13]. Besides of the fact that ternary patterns are a step towards an analog neural model, they have the

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advantage that they can be generated with a bias but keeping their symmetric distribution (both ±1 states are considered active). An important question related to the three-state model is the measurement of the retrieval quality in the cases where this is imperfect. The overlap alone is not anymore a good measure because it accounts only for the active states. For the homogeneous ternary patterns, the Hamming distance can be considered a good measure, since it takes equally into account all errors in retrieving, the active and the inactive one. For the low-activity case, however, also the Hamming distance is not a good parameter of the retrieval quality because the errors in retrieving the active states are much more relevant (they contain much more information) than the errors in retrieving the inactive states.

To solve this problem, we use the conditional probability of neuron states given the pattern states \( p(\sigma_i = \pm 1 | \xi_t) \) to obtain the mutual information \( I \) of the attractor neural network (ANN). This quantity measures directly the amount of dependence between the random variables, neurons and patterns. To accomplish that, we must use a new variable, we call activity-overlap, which is the overlap between the active states of the ternary neurons and the active states of the ternary patterns, taken with the absolute value.

This \( I \) is thus a function of three parameters: the overlap \( m \), the neural-activity \( q \), and the activity-overlap \( n \). We then expand the \( I \) around the values of the parameters when the neurons are independent on the patterns. This expansion gives us an expression that can be interpreted as a Hamiltonian, a function only of the neuron states and the synaptic couplings. This Hamiltonian is similar to the Blume-Emery-Griffiths [5, 6] spin-1 model (BEG), but with random interactions. The BEG model, originally proposed to study \( H_{13} - H_{44} \) mixtures, was later used to describe several systems, like memory alloys, fluid mixtures, micro-emulsions, etc., and displays a variety of new thermodynamic phases.

Some disordered BEG models have been recently studied [7, 8], where either the exchange-interactions or the crystal-field are random variables. However, from our knowledge, no random biquadratic-interactions model has been treated up to this date.

We describe our model in Section II. In Section III we describe the \( I \) measures used to evaluate the performance of the ANN, and derive the BEG Hamiltonian from \( I \). After solving the thermodynamics for this model in Section IV, we present some results for the dynamics and the phase diagrams in the section V, comparing the results with previous works. We conclude in the last Section with some comments about possible improvements of the network.

## 2 The model

As well as the pattern states, the neuron states at time \( t \) are three-state variables, defined as

\[
\sigma_{it} \in \{0, \pm 1\}, \quad i = 1, \ldots, N. \tag{3}
\]

They are updated according to a stochastic dynamics which depends on the previous states \( \{\sigma_{i,t-1}\} \) and on synaptic interactions between different neurons. The specific form of the synapses will be obtained later, by construction. We will see they are of the Hebbian type, that is, the learning is local (the synapses depend only on the two neurons interacting). Moreover, the updating rule will be also obtained by construction, no supposition being done here except that the patterns have the same three-state symmetry as the neuron states.

The three-state patterns \( \xi_{t}^\mu \in \{0, \pm 1\}, \mu = 1, \ldots, p \), are independent identically distributed random variables (IIDRV) chosen according to the probability distribution in Eq. (1). There is no bias (\( \langle \xi_{t}^\mu \rangle = 0 \)) neither correlation between patterns (\( \langle \xi_{t}^\mu \xi_{t}^\nu \rangle = 0 \)), and \( a = \langle |\xi_{t}^\mu|^2 \rangle \) is the activity of the patterns.

The mean-field networks have the property of being site-independent, that means, the correlations between different sites are negligible in the thermodynamic limit, \( N \to \infty \). This implies that every macroscopic quantity satisfies the conditions of the law of large numbers (LLN), so they can be defined as an average on the probability distribution of a state in a single site. If \( f \) is the thermodynamic limit of the variable \( f_N \), we have

\[
F_N \equiv \frac{1}{N} \sum_{i} f(\sigma_{i}, \xi_{t}) \to F \equiv f(\sigma, \xi) >_{\sigma, \xi}. \tag{4}
\]

where the brackets represent averages over the distribution of a single, typical state \( \sigma, \xi \) (we can drop the index \( t \)).

An example of this is the special case of the overlap, but this property is valid also for every function \( F_N \), since it comes from a property of the probability distribution of the states itself,

\[
p(\sigma_{i}, \xi_{t}) = \prod_{i} p(\sigma_{i}, \xi_{i}). \tag{5}
\]

The task of retrieval is successful if the distance between the state of the neuron \( \{\sigma_{it}\} \) and the pattern \( \{\xi_{it}\} \), defined as

\[
d_{it}^\mu = \frac{1}{N} \sum_{i} |\xi_{it}^\mu - \sigma_{it}|^2 = a - 2am_{it}^\mu + q_{Nt}, \tag{6}
\]

becomes small after some time \( t \). This is the so-called Hamming distance (an Euclidean quadratic measure for discrete sets). The overlap of the \( j \)th pattern with the neuron-state is defined as:

\[
m_{Nt}^\mu = \frac{1}{aN} \sum_{i} \xi_{it}^\mu \sigma_{it}, \tag{7}
\]

while the neural activity is

\[
q_{Nt} = \frac{1}{N} \sum_{i} |\sigma_{it}|^2. \tag{8}
\]

The \( m_{it}^\mu \) are called the retrieval overlaps, and they are normalized parameters within the interval \([-1, 1]\), which attain the extreme values \( m_{Nt}^\mu = \pm 1 \) whenever \( \sigma_j = \pm \xi_{jt}^\mu \), as we see from Eq. (1).

Another parameter is need to define completely the macroscopic state of the ANN. This is

\[
n_{Nt}^\mu = \frac{1}{aN} \sum_{i} |\sigma_{it}|^2 |\xi_{it}^\mu|^2. \tag{9}
\]

We call this quantity the activity-overlap [9], as long as \( n_{Nt}^\mu \) represents the overlap between the sites, where the neurons are active, \( |\sigma_i| = 1 \), and the sites where the patterns are active, \( |\xi_{it}^\mu| = 1 \). For the dynamics used in most work found in the literature [10, 11, 12], where the synapses used are of the Hopfield form \( J_{ij} = \sum_{\mu} \xi_{it}^\mu \xi_{jt}^\mu \), this parameter \( n_{Nt}^\mu \) does not seem to play any role in the evolution of the network, independent of the architecture considered (diluted, layered or fully-connected for instance). However \( n_{Nt}^\mu \) is necessary to define the mutual information of the network, as well as this is necessary in computing the network’s performance [13, 14].
For this Hopfield three-state network a self-control (SC) mechanism was recently introduced with the following threshold dynamics \[ \theta_t = c(a) \Delta t, \] where \( c(a) = \sqrt{-2 \ln(a)} \) is a function only of the pattern activity, while the variance of the cross-talk noise (due to the \( p - 1 \) non-retrieved patterns) has the simple form \( \Delta t = \sqrt{\alpha q_N t} \) for the diluted architecture \[ 3.1 \] Here we take the alternative approach of starting from the mutual information for the model, which we describe in the next Section.

### 3 Mutual Information

#### 3.1 Mean-Field-Theory

Compared to the binary neural network (NN), where the natural parameter is the overlap \( m^a \), to describe the statistical macro-dynamics of the three-state NN, there should be two additional parameters. Although the only variables appearing in the usual Hopfield dynamics are the overlap \( m_N^a \) and the neural activity \( q_N \), the activity-overlap \( n^a_N \) is an independent parameter which complete the macroscopic description. For a long-range system, as the one we are considering, it is enough to observe the distribution of a single typical neuron in order to know the global distribution.

The conditional probability of having a neuron in a state \( \sigma_{it} = \sigma \) in a time \( t \), given that in the same site the pattern being retrieved is \( \xi_t = \xi \), is:

\[
p(\sigma|\xi) = (s_\xi + m(\sigma)\delta(\sigma^2 - 1) + (1 - s_\xi)\delta(\sigma),
\]

\[
s_\xi \equiv s + \frac{n - q}{1 - a} \xi^2, \quad s \equiv \frac{q - na}{1 - a}.
\]

One can verify that this probability satisfies the averages:

\[
m = \langle \langle \sigma \sigma_{it} \rangle \rangle_\xi; \quad q = \langle \langle \sigma^2 \sigma_{it} \rangle \rangle_\xi; \quad n = \langle \langle \sigma^2 \rangle \rangle_\xi.
\]

These are the thermodynamic limits \( (N \to \infty) \) of Eqs.\[ 1.5 \] and \[ 1.6 \], for a given time \( t \) and pattern \( \mu \). Due to the symmetry of the patterns, we have also \( \langle \langle \sigma \sigma_{it} \rangle \rangle_\xi = \langle \langle \sigma^2 \sigma_{it} \rangle \rangle_\xi = \langle \langle \sigma \sigma_t \rangle \rangle_\xi \equiv 0 \). The averages are over the pattern distribution, Eq.\[ 1.4 \], and over the conditional distribution, Eq.\[ 1.4 \].

\[
\langle \ldots \rangle_{\sigma_{it}} \equiv \sum_{\xi} \sum_{\sigma} p(\xi) \sum_{\sigma} p(\sigma|\xi) \ldots = \langle \ldots \rangle_{\sigma_{it}}.
\]

Together with the distribution of the patterns, the conditional probability leads also to the probability

\[
p(\sigma) \equiv \sum_{\xi} p(\xi)p(\sigma|\xi) = q\delta(\sigma^2 - 1) + (1 - q)\delta(\sigma).
\]

With the above expressions we can calculate the Mutual Information \( I[\sigma; \xi] \), a theoretical information quantity used to measure the average amount of information that can be received by the user by observing the symbol (or the signal) at the output of a channel. We can regard all the dynamical process, or rather each time step of it, as a channel, and write the \( I \) as:

\[
I[\sigma; \xi] = S[\sigma] - \langle S[\sigma|\xi]\rangle_\xi;
\]

\[
S[\sigma] = -\sum_{\sigma} p(\sigma) \ln[p(\sigma)],
\]

\[
S[\sigma|\xi] = -\sum_{\sigma} p(\sigma|\xi) \ln[p(\sigma|\xi)]. \tag{14}
\]

\( S[\sigma] \) and \( S[\sigma|\xi] \) are the entropy of the output and the conditional entropy of the output, respectively. The quantity \( S[\sigma|\xi] \) is also called the equivocation term of the \( I[\sigma; \xi] \).

For an ANN of homogeneous distributed patterns the \( I \) is not a necessary measure, because the Hamming distance is enough to quantify the quality of the retrieval. The latter distinguishes well between a situation where most of the wrong neurons were turned off (\( d = 1 \)) and another situation where they were flipped (\( d = 4 \)). However, for the low-activity ANN, \( I \) can be a very useful measure, since the Hamming distance is not so good in distinguishing the cases where neurons were turned off from the cases where they were turned on (\( d = 1 \)). This distinction is critical in the sparse coded three-state \( NN \), where \( a \ll 1 \), because the inactive states have less information than the active ones.

For instance, let be an ANN with pattern activity \( a \) and denote the active (inactive) sites as \( A (\bar{A}) \), such that \( \xi_A = \pm 1 \) (\( \xi = 0 \)). Now suppose that all neurons where turned off, \( \sigma_i = 0 \), then \( m = 0 \), and \( q = 0 \), so the Hamming distance is \( d = a \) and there is no information transmitted, \( I = 0 \). If instead of turning off the A neurons among the inactive \( \sigma_{\bar{A}} \), one had turned on \( A \) neurons among the inactive \( \sigma_{\bar{A}} \), one get \( m = 1 \), and \( q = 2a \). So the Hamming distance is still \( d = a \), but now there is some transmitted information. It is intuitive that the first kind of errors have erased all the meaningful bits, while the second situation have not affected essentially the code, and obviously have much less important errors.

The expressions for the entropies defined above are:

\[
S[\sigma] = -q \ln \frac{q}{2} - (1 - q) \ln(1 - q),
\]

\[
\langle S[\sigma|\xi]\rangle_\xi = aS_a + (1 - a)S_{\bar{A}},
\]

\[
S_a = - \frac{n + m + m}{2} \ln \frac{n + m + m}{2} - \frac{n - m}{2} \ln \frac{n - m}{2} - (1 - n) \ln(1 - n),
\]

\[
S_{\bar{A}} = - n \ln \frac{n}{2} - (1 - s) \ln(1 - s). \tag{15}
\]

Applying to the second case cited above, the entropy of the output is \( S[\sigma] = -2a \ln a - (1 - 2a) \ln(1 - 2a) \), while the equivocation is \( <S[\sigma]\rangle_A = -a \ln a - 2 \ln(1 - a) - (1 - 2a) \ln(1 - 2a) - 2 \ln a - 2 \ln(1 - a) \), so that the mutual information is \( I = -a \ln(2a) - (1 - a) \ln(1 - a) = S[\xi] - 2a \ln(2a) \) which is not so smaller than the entropy of the original patterns, \( S[\xi], \) Eq.\[ 1.4 \]. It is easy to understand why we must use \( I \) for the sparse code case, instead of the Hamming distance.

#### 3.2 Derivation of the Hamiltonian

We search for a Hamiltonian which is symmetric in any permutations of the patterns \( \xi^t \), since they are not known during the retrieval process. This imposes that the retrieval of any pattern \( \xi^t \) is week, i.e., \( \sigma \) is almost independent of it. Then obviously the overlap \( m^a \to 0 \) vanishes. An expansion of \( I \) with \( a = 1 = q \) around \( m^a \to 0 \) yields the Hopfield Hamiltonian. If afterwards some particular overlap becomes eventually large, this should be a consequence of the network evolution.

However, for general \( a, q \), this is not the only quantity which vanishes in this limit. The variable \( \sigma^a \) is also almost independent of \( (\xi^t)^2 \), so that \( m^a \to q \). Hence, the parameter
\[ I^\mu = \frac{\eta^\mu}{1-a} < \sigma^2 \eta^\mu >, \quad \eta^\mu = \frac{(\xi^\mu)^2 - a}{a(1-a)}, \quad (16) \]

also vanishes when the states of the neurons and the patterns are independent.

We use this fact to look at the information close to the non-retrieval regime. An expansion of the expression for the \( I^\mu \) around \( m^\mu = 0, \eta^\mu = 0 \) gives

\[ I^\mu \approx \frac{1}{2} \sigma^2 (m^\mu)^2 + \frac{1}{2} a (1-a) (\eta^\mu)^2. \quad (17) \]

Since this expression gives the information for a single pattern \( \mu \), \( I(m^\mu, \eta^\mu) \equiv I^\mu \), it should be summed \( I^\mu \rightarrow I \) to give the total information of the network. It is natural to associate this quantity with the opposite of the Hamiltonian, because the maximum of the information gives the minimal energy.

We suppose, as a further simplification of the model, that the neural activity is of the same order of the pattern activity, \( q \sim a \). With this assumption, \( I \) from Eq.(17) depends on the same way on \( m^\mu \) and \( \eta^\mu \). Substituting these expressions for these parameters, given by the definitions (7), (8) and (9) (i.e., Eqs. (11) before the thermodynamic limit), we obtain the following expression for the \( I \):

\[ \mathcal{H} = -I_{\mu N} \equiv \mathcal{H}_1 + \mathcal{H}_2, \quad (18) \]

where

\[ \mathcal{H}_1 = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j \quad (19) \]

and

\[ \mathcal{H}_2 = -\frac{1}{2} \sum_{i,j} K_{ij} \sigma_i^2 \sigma_j^2 \quad (20) \]

are the quadratic and the biquadratic terms, respectively. The above expression for the Hamiltonian, obtained from the mutual information close to the non-retrieval regime, has the same form as of the BEG model \[21\]. We call our model the BEG Neural Network (BEGNN).

The interactions are randomly distributed, given by

\[ J_{ij} = \frac{1}{a^2 N} \sum_{\mu=1}^p \xi^\mu_{ij}, \quad (21) \]

and

\[ K_{ij} = \frac{1}{N} \sum_{\mu=1}^p \eta^\mu_{ij}. \quad (22) \]

The first term of the Hamiltonian is the usual Hopfield model with the Hebbian rule given by Eq.(21). The second term, arising from the term depending on \( I^\mu \) in Eq.(17), related to the activity-overlap, is also Hebbian-like, but is associated, as will be seen latter, with the quadrupolar order of the system.

Note that the Hamiltonian formulation of the problem is only possible in the case of fully-connected neural network, where the interaction matrix is symmetric. In the next Section we will present the dynamical formulation of the problem, which can be applied to the cases of asymmetric couplings \[23\].

As is well known, the phase diagram of the usual BEG model is very rich, showing different phases, depending on the sign and the strength of the biquadratic coupling constant. Without any disorder and for very negative biquadratic coupling constant, a quadrupolar phase, related to the quadrupolar moment \( < \sigma^2 > \) also appear, apart of the usual disordered and ferromagnetic phases \[24\]-\[28\]. However, our variables \( \xi^\mu \) are quenched, so we have a disordered system. BEG models with disordered quadratic coupling have been recently studied \[24\]-\[28\], showing some new phases (spin-glass, quadrupolar spin-glass phases, etc), but, from our knowledge, no disordered biquadratic BEG model has been studied up to this date.

\section{4 Asymptotic Macro-dynamics}

For the derivation of the asymptotic macro-dynamics we will use a naive mean-field (MF) approach using the Hamiltonian Eqs.(14)-(20). Since the Hamiltonian is quadratic in the overlaps, we can linearize it, using Gaussian transformation, to obtain the partition function:

\[ Z = Tr (\sigma) e^{-\beta \mathcal{H}} = \int \prod_{\mu} [D\Phi(\sqrt{\beta N m^\mu}) D\Phi(\sqrt{\beta N l^\mu})] \prod_{\mu} \prod_{i=\pm 1,0} e^{\xi_i^\mu}, \quad (23) \]

where \( D\Phi(z) \equiv dz e^{-z^2/2} \sqrt{2\pi} \), and \( \beta = 1/T \). The effective Hamiltonian is

\[ \mathcal{H}_i = h_i \sigma_i + \theta_i \sigma_i^2, \quad (24) \]

where the local fields are

\[ h_i = \frac{1}{a} \sum_{\mu} \xi^\mu m^\mu, \]

\[ \theta_i = \sum_{\mu} \eta^\mu l^\mu. \quad (25) \]

After taking the trace over the spin variables, we apply a saddle point integration and use Eq.(4) for the thermodynamic limit, to get the free energy in terms of the parameters \( m, l \) and \( q \):

\[ f = -\frac{T}{N} \ln Z = \frac{1}{2} (\bar{m}^2 + \bar{l}^2) - T < \ln \bar{Z} > \bar{\xi}. \quad (26) \]

where the effective partition function is:

\[ \bar{Z} = 1 + 2e^{\beta h} \cosh(\beta H). \quad (27) \]

The fields \( h, \theta \) are defined in Eq.(25), but the indices \( i \) can be dropped out. The saddle-point equations \( \partial f / \partial m^\mu = 0 \) and \( \partial f / \partial l^\mu = 0 \), leads to the following expressions for the stationary states:

\[ m^\mu = < \frac{1}{a} \xi^\mu > \bar{\xi}, \quad (28) \]

\[ l^\mu = < \eta^\mu \sigma^2 > \bar{\xi}, \]

where the angular brackets mean the average over the patterns, and the thermal averages of the states are:
\[ \sigma = F_\beta(h, \theta) = \frac{2e^{\beta \theta} \sinh(3h)}{Z}, \]
\[ \sigma_t = G_\beta(h, \theta) = \frac{2e^{\beta \theta} \cosh(3h)}{Z}. \]  
\hfill (29)

For zero-temperature, the behavior of the averages are:

\[ F_\infty(h, \theta) = \text{sign}(\theta) \Theta(|h| + \theta), \]
\[ G_\infty(h, \theta) = \Theta(|h| + \theta), \]  
\hfill (30)

where \( \Theta(\ldots) \) is the step function.

This result, obtained from the naive MF theory, can be easily understood if we write the Hamiltonian in Eqs. (19, 20) in the form:

\[ H = \frac{1}{2} \sum_i \tilde{H}_i = \frac{1}{2} \sum_i \left( h_i \sigma_i + \theta_i \sigma_i^2 \right); \]
\[ h_i = \sum_j J_{ij} \sigma_j, \]
\[ \theta_i = \sum_j K_{ij} \sigma_j^2. \]  
\hfill (31)

So, the deterministic parallel dynamics, which leads to the minimization of the Hamiltonian, is

\[ \sigma_{i,t+1} = \text{sign}(h_i^t) \Theta(|h_i^t| + \theta_i^t), \]  
\hfill (32)

where the local fields \( h_i^t, \theta_i^t \) (associated to the variables \( \sigma_{j,t}, \sigma_{j,t}^2 \) respectively), are given in the time step \( t \). Such dynamics has the same form as the zero-temperature function in Eq. (24).

Alternatively to the thermodynamic approach, in the noise case, we can also start from the stochastic parallel dynamics [22, 33]

\[ p(\sigma_{t+1} | \{ \sigma_t \}) = \exp[\beta \tilde{H}_{t+1}]/\tilde{Z}, \]  
\hfill (33)

where \( \tilde{H}_{t+1} \) is given by Eq. (23) (in the time step \( t \)), and \( \tilde{Z} \) by Eq. (24). Differently from the dynamics for the (Q=3)-Ising model [22, 33], here the field \( \theta = \theta(\{ \sigma_j^2 \} \) in the effective Hamiltonian is a function of the states in the previous time steps. The resulting noise-averaged states coincide with Eq. (24) in the stationary regime.

Because we are mainly interested on the retrieval properties of our network, we take an initial configuration whose retrieval overlaps are only macroscopic of order \( O(1) \) for a given pattern, let say the first one. We singled out the term \( \mu = 1 \) in the local fields of Eq. (25) in order to study the retrieval of the first pattern.

Supposing an initial configuration \( \{ \sigma_{t=0} \} \) as a collection of IDRV with zero-mean and variance \( q_{t=0} \), the fields \( h_{t=0} \) and \( \theta_{t=0} \) in the zeroth time step are given by:

\[ h_{t=0} = \frac{1}{a} \xi_{t=0} + \omega_{t=0}; \]
\[ \omega_{t=0} \equiv \sum_{\nu \geq 2} \frac{1}{a^{\nu-2}} \xi^{\nu} \eta_{t=0}; \]
\[ \theta_{t=0} = \eta^{t=0} + \Omega_{t=0}; \]
\[ \Omega_{t=0} \equiv \sum_{\nu \geq 2} \frac{1}{a^{\nu-2}} \xi^{\nu} \eta_{t=0}. \]  
\hfill (34)

where the indices \( \mu = 1 \) where dropped, and the rest of the patterns is regarded as some additive noise. According to the Central Limit Theorem (CLT), they are independent Gaussian distributed [12, 32], with zero mean and variance

\[ \text{Var}[\omega_{t=0}] = \frac{1}{a^2} \sigma q_{t=0} \equiv \Delta^2 \]
\[ \text{Var}[\Omega_{t=0}] = \frac{\Delta^2}{(1-a)^2}. \]  
\hfill (35)

Although the dynamics for the parameters \( m_t, n_t \) and \( q_t \) in the first time step is a function of the initial step, the expression for the noises in the next steps evolves with time in more complicated way then Eqs. (15). In the extremely diluted synaptic case [12], however, the first time step describes the dynamics for every time step \( t \). From now on we will adopt this limiting case.

Thus, in the asymptotic limit \( N \to \infty \), the expression for the overlap \( m_t = \lim_{N \to \infty} m_{N_t} \) becomes, after averaging over the pattern \( \xi \):

\[ m_{t+1} = \langle \xi^2 \rangle_\alpha > \xi > \int D\Phi(y) \text{D}\Phi(z) F_\beta(m_t + y \Delta_t; \frac{l_t}{a} + \frac{\Delta_t}{1-a}), \]  
\hfill (36)

where the averages over the weights on the brackets should be done with the Gaussian distributions, Eq. (35).

The neural activity is the thermodynamic limit of Eq. (9), which reads

\[ q_t = \langle \xi^a > \xi > \text{anh} + (1-a) s_t, \]
\[ s_{t+1} \equiv \int D\Phi(y) \int D\Phi(z) G_\beta(y \Delta_t; -\frac{l_t}{a} + \frac{\Delta_t}{1-a}). \]  
\hfill (37)

Here \( s \) is the variable defined in Eq. (10) and the activity-overlap is given by

\[ n_{t+1} = \langle \xi^2 > \xi > \xi > \int D\Phi(y) \text{D}\Phi(z) G_\beta(m_t + y \Delta_t; \frac{a}{l_t} + \frac{\Delta_t}{1-a}). \]  
\hfill (38)

The equation for \( l_t \) is obtained using the definition in Eq. (11), \( l_t = (n_t - q_t)/(1-a) \). It is worth to note that the definitions of the parameters \( m, q, n \) in Eqs. (13) are the same as that in Eqs. (8), (23), since the average over the conditional probability \( p(\sigma | \xi) \) is equivalent to the average over the noise due the \( p - 1 \) remaining patterns \( \{ \xi \} \). Eqs. (35-38) describe the macro-dynamics of the diluted BEGNN by adapting self-consistently the threshold during the time-evolution of the system. With these equations we can calculate the mutual information from Eqs. (11-12).

### 5 Phase diagram

In this Section we present some explicit results for the BEGNN model. We first calculated the stable fixed-points of the Eqs. (8-18) for the asymptotic \( N \to \infty \) network, and obtained the curves for the order parameters \( m, q, n \) in Eqs. (13) for two values of the activity \( a \) (Fig.1). For small load \( a < 0.2 \), the overlap remains close to \( m \sim 1 \) and the neural activity is \( q \sim a \). When more patterns are stored in the network, \( a \) increases almost linearly, up to an optimal value,
\(t_{opt}(\alpha_{opt})\), after which \(i\) decreases to zero in \(\alpha_{max}\). The comparison is done with the self-control neural network (SCNN) model\[3, 4\]. It is seen that for small activities (\(\alpha = 0.3\)), the BEGNN model gives worst results compared with the SCNN model, with a smaller value for \(i\), while for \(\alpha = 0.6\) (close to the uniform distribution of patterns, \(\alpha = 2/3\)), the BEGNN performs better, with an optimal value of the information \(i \sim 0.15\), although it is attained for a smaller value of load, \(\alpha \sim 0.2\). The reason for this behavior is that the third order parameter (related to the activity-overlap), is \(l \sim 1\) for the BEGNN (SCNN) and \(l < 1\) for the SCNN (BEGNN) with \(\alpha = 0.6\) (\(\alpha = 0.3\)).

The behavior of the order parameters and the \(i\) with load, for the zero-temperature case is presented for three different values of the activities (Fig.2). The initial conditions used were with \(m_0 = 10^{-6}\), \(l_0 = 1\), \(q_0 = a\), such that there is almost no initial overlap. In this case there is always a sharp fall on the information for \(\alpha\) not so larger than \(\alpha_{opt}\). We see different behaviors depending on the activities.

The corresponding dynamical phase diagram is drawn in Fig.3. Four possible phases are present: the retrieval phase \((m = 0, l \neq 0, q \sim a)\) and \((m \neq 0, l < 0.5m, q \sim a)\), the quadrupolar phase \(Q\) \((m = 0, l \neq 0, q \sim a)\) and the zero phase \(Z\) \((m = 0, l = 0, q \sim a)\). The last phase \(Z\) is called because there is no information transmitted, is analogous to the Self-Sustained \((S)\) activity phase of the \((Q = 3)\)-Ising ANN\[3, 4\], since the parameter related to the spin-glass order is \(q \neq 0\). We have not found any paramagnetic \((P)\) phase, with all \((m = 0, l = 0, q = 0)\) for the BEGNN. Note that the quadrupolar phase is a quite new phase, compared to the other NN models and is a special one for the BEGNN-model\[21\]. This phase is also present in the original BEG-model\[20\], as well as in all its generalizations including disorder\[22\]. It is seen in Fig.2, for \(\alpha = 0.9\), where the overlap goes to \(m = 0\) at \(\alpha \sim 0.13\) (much before \(l\), which goes to zero at \(\alpha \sim 0.3\)); this phase corresponds to non-zero information, although there is no retrieval overlap. The phase \(R\) appears for \(\alpha = 0.5\), where both \(m\) and \(l\) are large and so is \(i\). On the other hand, the phase \(M\) is observed for \(\alpha = 0.1\) where the parameter \(l\) is much smaller than \(m\). The phase transitions from \(R\) or \(M\) to \(Z\) are usually sharp.

The behavior of the order parameters and information with the temperature \(T\) for fixed activity \(\alpha = 0.5\) is shown on Fig.4. We observe an increase of \(l\) with the temperature, showing an optimal value for \(T \sim 0.2\). Such an improvement of a feeble signal with noise, similar to the stochastic resonance phenomena, appears also in other physical systems\[4\]. A further increase in temperature leads to decreasing the information of the model. We note that this behavior doesn’t hold for \(\alpha \geq 2/3\), nevertheless there is still an increase of the storage capacity \(\alpha_{max}\).

The last result is in agreement with other investigations of dynamical activity of real and model neurons, where the observed stochastic resonance disappears by increasing the amplitude of the external stimulus.

A cut of the phase diagram in the plane \(T \times \alpha\) for a fixed value of the activity \(\alpha = 0.7\) is shown in Fig.5. The dashed line, which corresponds to the optimal case, \(t_{opt}(\alpha)\), is within either the phase \(R\) or \(Q\). It is also interesting to observe that there are two separate \(Q\)-phase islands, for either small temperature \(T\) and large load \(\alpha\) or large temperature \(T\) and small load \(\alpha\). The phase transitions become smoother with the temperature.

Finally on Fig.6 we present the evolution of the information and of the order parameters with the time \(l\), for a given temperature \(T = 0.2\) and activity \(\alpha = 0.7\), for two values of the load parameter \(\alpha\). As can be seen from this figure, for \(\alpha = 0.4\), which is close to the transition R-M, the change to the behavior of the order parameters needs more time steps than for \(\alpha = 0.2\). This is not strange due to the critical slowing down near the transition. However, an interesting new fact appears here: the parameters \(l_t\) and \(q_t\) have a fast felt down to a much smaller value, after which the network stays a long while with an almost zero overlap, and finally the BEGNN is able to retrieve quite well the pattern. For instance, for \(\alpha = 0.2\), \(l\) falls to \(l \sim 0.6\) and \(m\) stays near \(m \sim 0\) during the first \(l \sim 20\) time steps, then they jump up to \(l \sim 0.8\), \(m \sim 0.9\), which means the memory pattern was (partially) attained. This result, caused by the instability of the Z-phase in this region, makes the BEGNN capacity much larger than that of the usual Hopfield model, in all its versions so far as we know.

The behavior of the continuous phase transitions can be analytically studied within the mean-field approximation by expanding Eqs.\[3, 4\] for small values of the order parameters. A standard calculation, for example, for the transition line \(QZ\) \((m = 0, l << 1)\) leads to the following expression:

\[
l^{QZ} = \beta T^{QZ} l + \left\langle e^{\beta \Omega} \sinh \beta \omega \frac{1 - 2e^{\beta \Omega} \sinh \beta \omega}{1 + 2e^{\beta \Omega} \sinh \beta \omega} \right\rangle \frac{1 - 2a}{a^2 + \hat{a}^2}\]

where the transition temperature between the phases \(Q\) and \(Z\) is:

\[
T_c^{QZ} = \frac{1}{a \hat{a}} \left\langle \sinh \beta \omega \frac{2e^{\beta \Omega} \sinh \beta \omega}{1 + 2e^{\beta \Omega} \sinh \beta \omega} \right\rangle \Omega, \omega \]

with

\[
q_c^{QZ} = \frac{1}{a \hat{a}} \left\langle \sinh \beta \omega \frac{2e^{\beta \Omega} \sinh \beta \omega}{1 + 2e^{\beta \Omega} \sinh \beta \omega} \right\rangle \Omega, \omega \]

Expanding the above expressions for small value of the load rate and large temperatures, \(\beta, \Omega < 1\), and calculating the averages over the noise up to the leading terms, one obtains the following equation for the transition line:

\[
T_c = \frac{2}{\Omega} - \frac{1 + \hat{a}^2}{2a \hat{a}}(42)
\]

The last expression for \(T_c\) is in qualitative agreement with the previous results shown on Fig.5.

Regarding the equation for the order parameter \(l\), one can verify that in leading order:

\[
l^{QZ} = \beta T^{QZ} l - \frac{1}{2} \left(1 - 2a\right) \frac{1}{a^2 + \hat{a}^2} \Omega^2 + O(l^3)\]

By use of Eq.\[4\], it is seen that the quadratic term of the above expansion changes sign when the activity \(a = 0.5\), thus defining a tricritical line between the transition of second order \((\alpha > 0.5)\) and of first order \((\alpha < 0.5)\). Note that similar tricritical behavior has been described also in the other versions of the BEG model\[12, 33\]. Similar analysis can be also performed for the other continuous transition between the different phases.

6 Conclusions

In this paper we proposed a BEG-like Hamiltonian for a ternary neural network, which couplings arise from an expansion of its mean-field mutual information, \(\Omega\), resulting in a system evolving with a self-consistently adapting threshold. The stationary and dynamical equations for this model were obtained as functions of three order parameters, the overlap \(m\), the neural activity \(q\), and the activity-overlap \(n\). Their solutions were explicitly calculated as functions of the variables: the pattern activity \(a\), the load \(\alpha\) and the temperature \(T\).
When the activity is near $a = 2/3$, corresponding to the uniform ternary patterns, the BEGNN improves the information, compared with a previous SCNN model \cite{11,12}. Improvement of the information content by increasing the noise, effect similar to the stochastic resonance, is also observed for activities $a < 2/3$.

There are four possible phases for the BEGNN, which were displayed in phase diagrams $a \times a$ and $T \times a$. In particular, a quadrupolar phase, $Q$, with $m = 0,1 \sim 1$, holds whenever the activity is large enough. This phase, known in the BEG literature, but new in an ANN context, carries out some nonzero information about the patterns even without any overlap $m$.

As the main result we obtained that, while the phase $Z$ is not stable in a large range of the variables, the basin of attraction of the retrieval phase is increased with respect to the usual ternary neural network models. States with initial conditions having very small overlap flow to final states with large overlap.

We believe that the BEGNN has a quite large range of applications for real systems. We also think that this way to obtain an Hamiltonian starting from a mean-field calculation of $I$, which yields an almost optimal retrieval dynamics, can be generalized to other spin systems, as the Ising with $Q > 3$ or the Potts models, for instance. Such method, based on the maximization of the entropy, can be an universal approach to information systems.

Then, we expect that the same improvement should happen for analogue neurons and for networks of binary synapses. It would be also interesting to investigate the case of local field for multi-neuron synapses, which comes up from higher order terms in the expansion of the mutual information, such that a better use of a network with fixed size is expected.

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Figure 1: The information $i = I\alpha$ and the order parameters $m, l, q$ against $\alpha$ with activities $a = 0.3$ (left) and $a = 0.6$ (right). The temperature $T = 0$ and the initial conditions are $m_0 = 1, l_0 = 1, q_0 = a$. The continuous line is for the BEGNN while the dashed line is for the SCNN.

Figure 2: The information $i$ and the order parameters $m$ (solid line), $l$ (thin line) and $q$ (dashed line) against $\alpha$ with activities $a = 0.1$ (left), $a = 0.5$ (center) and $a = 0.9$ (right); The temperature $T = 0$ and the initial conditions are $m_0 = 10^{-6}, l_0 = 1, q_0 = a$.

Figure 3: The dynamical phase diagram $\alpha \times a$, for $T = 0$ with initial conditions $m_0 = 10^{-6}, l_0 = 1, q_0 = a$. The different phases are explained in the text.
Figure 4: The information $i$ and the order parameters $m$ (solid line), $l$ (thin line) and $q$ (dashed line) against $\alpha$ with temperatures $T = 0.0$ (left), $T = 0.2$ (center) and $T = 0.4$ (right); The activity $a = 0.5$ and the initial conditions are $m_0 = 10^{-6}$, $l_0 = 1$, $q_0 = a$.

Figure 5: The dynamical phase diagram $\alpha \times T$, for $a = 0.7$ with initial conditions $m_0 = 10^{-6}$, $l_0 = 1$, $q_0 = a$. The dashed line corresponds to the optimal information.
Figure 6: The information $i$ and the order parameters $m, l, q$, against the time $t$ for temperature $T = 0.2$ and activity $a = 0.7$, with $\alpha = 0.4$ (left) and $\alpha = 0.2$ (right). The initial conditions are $m_0 = 10^{-6}$, $l_0 = 1$, $q_0 = a$. 