A numerical characterization of reduction for arbitrary modules

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Abstract

Let $(R, \mathfrak{m})$ be a $d$-dimensional Noetherian local ring and $E$ a finitely generated $R$-submodule of a free module $R^p$. In this work we introduce a multiplicity sequence $c_k(E)$, $k = 0, \ldots, d+p-1$ for $E$ that generalize the Buchsbaum-Rim multiplicity defined when $E$ has finite colength in $R^p$ as well as the Achilles-Manaresi multiplicity sequence that applies when $E \subseteq R$ is an ideal. Our main result is that the new multiplicity sequence can indeed be used to detect integral dependence of modules. Our proof is self-contained and implies known numerical criteria for integral dependence of ideals and modules.

1 Introduction

Let $(R, \mathfrak{m})$ be a local Noetherian ring, $N$ a finitely generated $d$-dimensional $R$-module, and $I \subseteq J$ be two ideals in $A$. Recall that $I$ is a reduction of $(J, N)$ if $IJ^n N = J^{n+1} N$ for sufficiently large $n$. If $I \subseteq J$ are $\mathfrak{m}$-primary and

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I is a reduction of \((J, N)\) then it is well known and easy to prove that the Hilbert-Samuel multiplicities \(e(J, N)\) and \(e(I, N)\) are equal. D. Rees proved his famous result, which nowadays has his name, that the converse also holds:

**Theorem 1.1.** (Rees’s Theorem, [R]) Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring, \(N\) a finitely generated \(d\)-dimensional \(R\)-module and \(I \subseteq J\) \(\mathfrak{m}\)-primary ideals of \(R\). Then, the following conditions are equivalent:

(i) \(I\) is a reduction of \((J, N)\);

(ii) \(e(J, N) = e(I, N)\).

Now assume that \(I \subseteq J\) are arbitrary ideals with the same radicals. If \(I\) is a reduction of \(J\) then we have always \(e(J_p, R_p) = e(I_p, R_p)\) for all minimal primes of \(J\). However, the converse is not true, in general. Under additional assumption E. Böger [B] was able to prove a converse as follows: let \(J \subseteq I \subseteq \sqrt{I}\) be ideals in a quasi-unmixed local ring \(R\) such that \(s(I) = \text{ht}(I)\), where \(s(I)\) denotes the analytic spread of \(I\). Then \(I\) is a reduction of \(J\) if and only if \(e(J_p, R_p) = e(I_p, R_p)\) for all minimal primes of \(J\).

Using the \(j\)-multiplicity defined by R. Achilles and M. Manaresi [AM1] (a generalization of the classical Hilbert-Samuel multiplicity), H. Flenner and M. Manaresi [FM] gave numerical characterization of reduction ideals which generalize Böger’s theorem to arbitrary ideals: let \(I \subseteq J\) be ideals in a quasi-unmixed local ring \(R\) and \(N\) a finitely generated \(d\)-dimensional \(R\)-module. Then \(I\) is a reduction of \((J, N)\) if and only if \(j(J_p, N_p) = j(I_p, N_p)\) for all \(p \in \text{Spec}(R)\).

There is another generalization of the classical Hilbert-Samuel multiplicity for arbitrary ideals due to R. Achilles and M. Manaresi [AM2]. They introduced, for each ideal \(I\) of a \(d\)-dimensional local ring \((R, \mathfrak{m})\) and \(N\) a finitely generated \(d\)-dimensional \(R\)-module, a sequence of multiplicities \(c_0(I, N), \ldots, c_d(I, N)\) which generalize Böger’s theorem to arbitrary ideals: let \(I \subseteq J\) be ideals in a quasi-unmixed local ring \(R\) and \(N\) a finitely generated \(d\)-dimensional \(R\)-module. Then \(I\) is a reduction of \((J, N)\) if and only if \(j(J_p, N_p) = j(I_p, N_p)\) for all \(p \in \text{Spec}(R)\).

Using the above multiplicity sequence defined by R. Achilles and M. Manaresi [AM2], the authors gave the following numerical characterization of reduction of ideals which generalize Rees’s theorem for arbitrary ideals [CP2, Theorem 5.5]:

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Theorem 1.2. Let \((R, \mathfrak{m})\) be a quasi-unmixed \(d\)-dimensional local ring and \(N\) a finitely generated \(d\)-dimensional \(R\)-module. Let \(I \subseteq J\) be proper arbitrary ideals of \(R\) such that \(ht_N(I) > 0\). Then the following conditions are equivalent:

(i) \(I\) is a reduction of \((J, N)\);

(ii) \(c_k(I, N) = c_k(J, N)\) for all \(k = 0, \ldots, d\).

On the other hand, the Buchsbaum-Rim multiplicity \(e_{BR}(E)\) is a generalization of the Samuel multiplicity and is defined for submodules of free modules \(E \subseteq R^p\) such that \(R^p/E\) has finite length. These were first described by D. A. Buchsbaum and D. S. Rim in [BR]. The Buchsbaum-Rim multiplicity has been generalized, in the finite colength case, by D. Kirby [K], D. Kirby and D. Rees [KR], D. Katz [K], S. Kleiman and A. Thorup [KT1] and A. Simis, B. Ulrich and W. Vasconcelos [SUV]. For an extensive history of Buchsbaum-Rim multiplicity we refer to [KT1]. Using the Buchsbaum-Rim multiplicity S. Kleiman and A. Thorup [KT1], D. Katz [K], and A. Simis, B. Ulrich and W. Vasconcelos [SUV] proved the following generalization of the Rees’s theorem for modules:

Theorem 1.3. Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring, \(E \subseteq F\) finitely generated \(R\)-submodule of a free module \(R^p\) such that \(R^p/E\) has finite length. Then, the following conditions are equivalent:

(i) \(E\) is a reduction of \(F\);

(ii) \(e_{BR}(E) = e_{BR}(F)\).

In the last fifteen years the Buchsbaum-Rim multiplicity of a submodule of a free module has played an important role in the theory of equisingularity of families of complete intersections with isolated singularities (ICIS). The Buchsbaum-Rim multiplicity has been used in the context to control the \(A_f\), \(W_f\), \(A\) and \(W\) conditions of equisingularity, which are analogous to the Whitney conditions (cf. [Ga1], [GaK], [GaM], [KT2] and the reference therein). The usefulness of the Buchsbaum-Rim multiplicity is restricted to families of ICIS, because it is only for these singularities that the submodules associated to the equisingularity conditions have finite colength and only for these types is the Buchsbaum-Rim multiplicity well defined. In order to generalize those works for families of arbitrary complete intersection singularities
(ACIS) it is strictly necessary to generalize first the notion of Buchsbaum-Rim multiplicities for submodules $M$ of a free module $F$ of arbitrary colength. For this new notion of multiplicity to be useful in equisingularity theory it must characterize the integral closure of arbitrary modules, that is, it must generalize Theorem 1.3.

There have been some generalizations of the Buchsbaum-Rim multiplicity for arbitrary submodules $E$ of a free module $R^p$ which we now describe. T. Gaffney in [Ga2] introduced a sequence of multiplicities $e_i(E)$, $0 \leq i \leq d = \dim R$ in the analytic context. This sequence satisfies a Rees type theorem: Suppose that $E \subseteq F \subseteq R^p$ are $R := \mathcal{O}_{X,x}$-modules where $X^{d}$ is a complex analytic space which as a reduced space is equidimensional, and which is generically reduced. Suppose that $e_i(E, x) = e_i(F, x)$, $0 \leq i \leq d$. Then $E$ is a reduction of $F$. Also, if $E$ is of finite colength in $R^p$, then $e_d(E)$ is the standard Buchsbaum-Rim multiplicity of $E$, and the others $e_i$’s are zero. Unfortunately, for ideals of non-finite colength, Gaffney’s multiplicity sequence does not coincide with the Achilles-Manaresi multiplicity sequence and also the codimension condition of $E$ in $R^p$ is built into the definition of the multiplicity which uses a codimension filtration ascending from the integral closure of the module.

On the other hand, the authors in [CP1] extended the notion of the Buchsbaum-Rim multiplicity of a submodule of a free module to the case where the submodule no longer has finite colength. For a submodule $E$ of $R^p$ they introduced a sequence $e^k_{BR}(E)$, $k = 0, \ldots, d + p - 1$ which in the ideal case coincides with the multiplicity sequence $c_0(I, R), \ldots, c_d(I, R)$ defined for an arbitrary ideal $I$ of $R$ by R. Achilles and M. Manaresi [AM2]. They also proved that if $E = I_1 \oplus \cdots \oplus I_p \subseteq R^p$ has finite colength then $e^0_{BR}(E) = p ![e_{BR}(E)]$ and $e^k_{BR}(E) = 0$ for $k = 1, \ldots, d - 1$. Nevertheless, no relation with reduction of modules and their multiplicity sequence was shown in their work.

There is also a particularly beautiful generalization of Flenner-Manaresi theorem for arbitrary submodules of a free modules due to B. Ulrich and J. Validashti (see [UV]), they introduced a multiplicity $j(E)$ for a submodule of the free module $R^p$ that generalizes the Buchsbaum-Rim multiplicity defined when $E$ has finite colength in $R^p$ as well as the $j$-multiplicity of Achilles-Manaresi that applies when $E \subseteq R$ is an ideal. Their result is as follows:

**Theorem 1.4.** Let $(R, m)$ be a universally catenary ring, $E \subseteq F$ finitely generated $R$-submodule of a free module $R^p$, and $N$ a finitely generated lo-
cally equidimensional Noetherian \( R \)-module. Assume that \( E_p = F_p \) for every minimal prime \( p \) of \( R \). Then, the following are equivalent:

(i) \( E \) is a reduction of \( F \);

(ii) \( j(E_q) = j(F_q) \) for every \( q \in \text{Spec}(R) \).

The above theorem characterize reduction of arbitrary modules by using numerical data in all localizations of the modules which is hard to verify algebraically and doesn’t seems (at least for the authors) to be useful in equisingularity theory. In this work we introduce a multiplicity sequence \( c_k(E, N) \) with \( k = 0, \ldots, d + p - 1 \) for the pair \( (E, N) \) that generalize the Buchsbaum-Rim multiplicity defined when \( E \) has finite colength in \( R^p \) as well as the Achilles-Manaresi multiplicity sequence that applies when \( E \subseteq R \) is an ideal. Our main result is that the new multiplicity sequence can indeed be used to detect integral dependence of modules:

**Theorem 1.5.** Let \( (R, m) \) be a Noetherian local ring, \( E \subseteq F \subseteq R^p \) be \( R \)-modules and write \( I := \mathcal{R}_1(E) \) for the corresponding ideal of \( A := \text{Sym}(R^p) \). Let \( N \) be a \( d \)-dimensional finitely generated \( R \)-module and set \( M := A \otimes_R N \). Assume that \( \text{ht}_M(I) > 0 \). Consider the following statements:

(i) \( E \) is a reduction of \( (F, N) \);

(ii) \( c_k(E, N) = c_k(F, N) \) for all \( k = 0, \ldots, d + p - 1 \).

Then, (i) implies (ii) and if \( N \) is quasi-unmixed the converse also holds.

We strongly believe that this multiplicity sequence, apart of being important in commutative algebra, it will be very useful for studying equisingularity conditions for families of ACIS. In particular we expect that it will characterize the \( A_f \) condition, answering positively Gaffney and Kleiman’s conjecture stated in [GaK, p. 546]. For this generalization of all equisingularity conditions to be carried out it is also necessary to develop a geometric theory for this multiplicity sequence, a theory involving blowups and intersection numbers as in the work of J. P. Henry and M. Merle [HM] and S. Kleiman and A. Thorup [KTI]. The authors hope to present this geometric approach elsewhere.

The paper is organized as follows. In section 2, we recall the basic results of Hilbert functions of bigraded algebras and we define the \( c^D \)-multiplicity
sequence associated to a graded module. The important result of this sec-
tion is the additivity formula for this multiplicity sequence. In sections 3 
and 4, we define two multiplicity sequences associated to ideals generated 
by linear forms, which we call $c^*$-multiplicity sequence and $c^\sharp$-multiplicity 
sequence. The important results of this sections are the additivity formula 
for those multiplicity sequences which immediately implies that they remains 
constant when passing to a reduction (Theorem 3.4 and Theorem 4.6). They 
are related by a third multiplicity sequence, which is denoted by $b^D$, which 
also satisfies the additivity formula (see Lemma 4.2). The $c^*$, $b^D$ and $c^\sharp$-
multiplicity sequences serve different purposes: the first two are more readily 
seen to be additive on short exact sequences of graded modules (Theorem 3.2 
and the proof of Proposition 4.3) and they were introduced in this work with 
the only purpose of proving the additivity property of the $c^\sharp$-multiplicity se-
quencies (Proposition 4.3). The last multiplicity sequence on the other hand 
is more suited for proving that conversely, the constancy of the multiplicity 
sequence implies integral dependence (Theorem 5.3). In section 5, we re-
call the notion of intertwining algebras and modules and state the reduction 
criterium we use here which was proved for algebras by A. Simis, B. Ulrich 
and W. Vasconcelos [SUV]. The important result of this section is Theorem 
5.3 which contain as special case the multiplicity sequence of an arbitrary 
module, defined in section 6, which in turn generalize the Buchsbaum-Rim 
multiplicity defined only for finite colength modules as well as the Achilles-
Manaresi multiplicity sequence defined for arbitrary ideals. The main result 
of section 6 is Theorem 6.3 which is an immediate consequence of Theorem 
5.3. Our approach is partly inspired by [CP2], [UV] and [AM2].

2 Multiplicity sequence

In this section we recall some well-known facts on Hilbert functions and 
Hilbert polynomials of bigraded modules, which will be essential for defining 
the Buchsbaum-Rim multiplicity sequence associated to a pair $(I, M)$. 

Let $R = \bigoplus_{i,j=0}^\infty R_{i,j}$ be a bigraded ring and let $T = \bigoplus_{i,j=0}^\infty T_{i,j}$ be a bigraded 
$R$-module. Assume that $R_{0,0}$ is an Artinian ring and that $R$ is finitely generated 
as an $R_{0,0}$-algebra by elements of $R_{1,0}$ and $R_{0,1}$ (i.e., $R$ is a standard 
bigraded algebra) The Hilbert function of $T$ is defined to be 

$$h_T(i, j) = \ell_{R_{0,0}}(T_{i,j}).$$
For $i, j$ sufficiently large, the function $h_T(i, j)$ becomes a polynomial $P_T(i, j)$. If $D$ denotes the dimension of the module $T$, we can write this polynomial in the form

$$P_T(i, j) = \sum_{k, l \geq 0} a_{k,l}(T) \binom{i + k}{k} \binom{j + l}{l}$$

with $a_{k,l}(T) \in \mathbb{Z}$ and $a_{k,l}(T) \geq 0$ if $k + l = D - 2$ [W, Theorem 7, p. 757 and Theorem 11, p. 759].

We also consider the sum transform of $h_T$ with respect to the first variable defined by

$$h^{(1,0)}_T(i, j) = \sum_{u=0}^i h_T(u, j).$$

From this description it is clear that, for $i, j$ sufficiently large, $h^{(1,0)}_T$ becomes a polynomial with rational coefficients of degree at most $D - 1$. As usual, we can write this polynomial in terms of binomial coefficients

$$P^{(1,0)}_T(i, j) = \sum_{k, l \geq 0} a^{(1,0)}_{k,l}(T) \binom{i + k}{k} \binom{j + l}{l}$$

with $a^{(1,0)}_{k,l}(T)$ integers and $a^{(1,0)}_{k,D-k-1}(T) \geq 0$.

Since

$$h_T(i, j) = h^{(1,0)}_T(i, j) - h^{(1,0)}_T(i - 1, j)$$

we get $a^{(1,0)}_{k+1,0}(T) = a_{k,0}(T)$ for $k, l \geq 0, k + l \leq D - 2$.

**Definition 2.1.** For the coefficients of the terms of highest degree in $P^{(1,0)}_T$ we introduce the symbols

$$c_k(T) := a^{(1,0)}_{k,D-k-1}(T), \quad k = 0, \ldots, D - 1$$

which are called the **multiplicity sequence** of $T$.

We define next the $c^{D}$-multiplicity sequence associated to a module. Let $(R, m)$ be a local ring, $S = \oplus_{j \in \mathbb{N}} S_j$ a standard graded $R$-algebra, $N = \oplus_{j \in \mathbb{N}} N_j$
a finitely generated graded $S$-module, and

$$T := G_{m}(N) = \bigoplus_{i,j \in \mathbb{N}} \frac{m^{i}N_{j}}{m^{i+1}N_{j}}$$

the bigraded $F$-module with

$$F := G_{m}(S) = \bigoplus_{i,j \in \mathbb{N}} \frac{m^{i}S_{j}}{m^{i+1}S_{j}}.$$  

Notice that $F_{0,0} = R/m$ is a field.

**Definition 2.2.** Consider an integer $D$ such that $D \geq \dim N$. For all $k = 0, \ldots, D - 1$, we set

$$c_{k}^{D}(N) = \begin{cases} 0 & \text{if } \dim N < D \\ c_{k}(T) & \text{if } \dim N = D \end{cases}$$

which is called the $c^{D}$-multiplicity sequence of $N$. Moreover, we set $c_{k}(N) := c_{k}^{\dim N}(N)$.

First we show that this $c^{D}$-multiplicity sequence behaves well with respect to short exact sequences.

**Proposition 2.3.** ([CP2, Proposition 2.3]) Let $(R, m)$ be a local ring, $S = \bigoplus_{j \in \mathbb{N}} S_{j}$ a standard graded $R$-algebra, and $0 \rightarrow N_{0} \rightarrow N_{1} \rightarrow N_{2} \rightarrow 0$ an exact sequence of finitely generated graded $S$-modules. Then for $D \geq d := \dim N_{1}$

$$c_{k}^{D}(N_{1}) = c_{k}^{D}(N_{0}) + c_{k}^{D}(N_{2})$$

for all $k = 0, \ldots, D - 1$.

**Proof.** Let $M_{s} := R(m, N_{s})^+ := \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} m^{i}(N_{s})_{j}$ be the extended Rees module associated to $N_{s}$, $s = 0, 1, 2$. For any bigraded module $T$ and for $i, j \gg 0$, we define the polynomial $h_{T}^{2}(i, j)$ of degree $D - 2$ as the Hilbert polynomial of $h_{T}(i, j)$ adding coefficient zero to the terms of degree between $\dim(T) - 2$ and $D - 2$.

Let $u$ be an indeterminate, which we consider with degree one. Set $M'_{0} := \ker(M_{1} \rightarrow M_{2}) = \bigoplus_{i \in \mathbb{Z}, j \in \mathbb{N}} (N_{0})_{j} \cap m^{i}(N_{1})_{j}$. We consider the natural diagram
which gives an exact sequence of cokernels

\[ 0 \rightarrow G' := \frac{M_0'}{u^{-1}M_0'} \rightarrow G_m(N_1) \rightarrow G_m(N_2) \rightarrow 0. \tag{1} \]

Denote the cokernel of the natural injection \( M_0 \hookrightarrow M_0' \) by \( L \). Using the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M_0(1,0) & \rightarrow & M_0'(1,0) & \rightarrow & L(1,0) & \rightarrow & 0 \\
0 & \rightarrow & M_0' & \rightarrow & M_0 & \rightarrow & L & \rightarrow & 0
\end{array}
\]

the snake-lemma yields an exact sequence

\[ 0 \rightarrow V \rightarrow G_m(N_0) \rightarrow G' \rightarrow W \rightarrow 0. \tag{2} \]

where \( V \) and \( W \) are the kernel and cokernel of \( u^{-1} : L(1,0) \rightarrow L \) respectively, i.e., we have the exact sequence

\[ 0 \rightarrow V \rightarrow L(1,0) \rightarrow L \rightarrow W \rightarrow 0. \tag{3} \]

For \( n \leq 1 \) the coefficient modules of \( u^n \) in \( \mathcal{R}(m, N_0)^+ \) and in \( M_0' \) coincide, hence the action of \( u^{-1} \) on \( L \) is nilpotent. Therefore the dimension of \( L \) is at most that of \( G' \), which is bounded by \( D \). Thus all modules occurring in the exact sequence (3) have dimension at most \( D \).

Now (1), (2) and (3) are exact sequences of finitely generated modules of dimension at most \( D \). We denote by \( h_{N_s}(i,j) \) the Hilbert-Samuel function of \( G_m(N_s) \).

From (1) and (2) we have

\[ h_{N_0}^{D(1,0)}(i,j) + h_{N_2}^{D(1,0)}(i,j) - h_{N_1'}^{D(1,0)}(i,j) = h_V^{D(1,0)}(i,j) - h_W^{D(1,0)}(i,j). \tag{4} \]

Because of (3) we have
\[ h^{D(1,0)}_V(i, j) - h^{D(1,0)}_W(i, j) = h^{D(1,0)}_L(i + 1, j) - h^{D(1,0)}_L(i, j) = h^D_L(i, j) \]  

Hence by (4) and (5):

\[ h^{D(1,0)}_{N_0}(i, j) + h^{D(1,0)}_{N_2}(i, j) - h^{D(1,0)}_{N_1}(i, j) = h^D_L(i, j) \]

is a polynomial of degree at most \( D - 2 \), which concludes the proof. \( \square \)

### 3 \( c^* \)-multiplicity sequence

In this section we introduce the \( c^* \)-multiplicity sequence. The main idea here is to consider a suitable grading on the extended Rees module as in the work of B. Ulrich and J. Validashti [UV].

Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(A\) a standard graded Noetherian \(R\)-algebra, \(I\) an ideal of \(A\) generated by elements of degree one and \(M\) a finitely generated graded \(A\)-module.

Let \(t\) be a variable. Consider the extended Rees ring of \(I\)

\[ \mathcal{R}(I, A)^+ := \bigoplus_{i \in \mathbb{Z}} I^i t^i \subseteq A[t, t^{-1}], \]

and the extended Rees module

\[ \mathcal{R}(I, M)^+ := \bigoplus_{i \in \mathbb{Z}} I^i Mt^i \subseteq M \otimes_R R[t, t^{-1}], \]

where we set \(I^i = R\) for \(i \leq 0\). Notice that \(\mathcal{R}(I, M)^+\) is a module over \(\mathcal{R}(I, A)^+\) which gives rise to the associated graded module of \(M\) with respect to \(I\),

\[ G_I(M) := \frac{\mathcal{R}(I, M)^+}{t^{-1}\mathcal{R}(I, M)^+} = \bigoplus_{i \in \mathbb{N}} \frac{I^i M}{I^{i+1} M} t^i, \]

which is a module over the associated graded ring \(G_I(A)\) of the same dimension as \(M\).

Assigning degree zero to the variable \(t\), the Laurent polynomial ring \(A[t, t^{-1}]\) becomes a standard graded Noetherian \(R[t, t^{-1}]\)-algebra, and \(M[t, t^{-1}] := M \otimes_R R[t, t^{-1}]\) a finitely generated graded module over this algebra. The extended Rees ring \(\mathcal{R}(I, A)^+\) is a homogeneous \(R[t^{-1}]\)-subalgebra
of $A[t, t^{-1}]$, and hence a standard graded Noetherian $R[t^{-1}]$-algebra. Furthermore $\mathcal{R}(I, M)^+$ is a homogeneous $\mathcal{R}(I, A)^+$-submodule of $M[t, t^{-1}]$, thus a finitely generated graded module over $\mathcal{R}(I, A)^+$. With respect to this grading, $G_I(A) := \mathcal{R}(I, A)^+/t^{-1}\mathcal{R}(I, A)^+$ becomes a standard graded Noetherian $R$-algebra and $G_I(M) := \mathcal{R}(I, M)^+/(t^{-1}\mathcal{R}(I, M)^+$ a finitely generated graded module over this algebra. Notice that

$$[G_I(M)]_n = \oplus_{i \in \mathbb{N}} [I^i M/I^{i+1} M]_n.$$  

The grading so defined on the extended Rees module and the associated graded module is called internal grading-for it is induced by the grading on the module $M$ (see [UV]).

**Definition 3.1.** Let $D$ be any integer with $D \geq \dim M$. We define the $c^*$-multiplicity sequence of $M$ with respect to $I$, as

$$c^*_{k, D}(I, M) := c^D_k(G_I(M)), \quad k = 0, \ldots, D - 1,$$

where $G_I(M)$ is graded by the internal grading. In the case where $D = \dim M$ we simply write $c^*_k(I, M)$ instead of $c^*_{k, \dim M}(I, M), \quad k = 0, \ldots, \dim M - 1$.

To be more explicit, consider the standard bigraded $R$-algebra

$$S^* := G_m(G_I(A)) = \oplus_{s, n=0}^{\infty} S^*_{s, n} \text{ with }$$

$$S^*_{s, n} = \oplus_{i=0}^{\infty} \left[ \frac{m^s I^i A + I^{i+1} A}{m^{s+1} I^i A + I^{i+1} A} \right]_n,$$

where $G_I(A)$ is graded by the internal grading, and the finitely generated bigraded module over this algebra

$$T^* = G_m(G_I(M)) = \oplus_{s, n=0}^{\infty} T^*_{s, n}$$

with

$$T^*_{s, n} = \oplus_{i=0}^{\infty} \left[ \frac{m^s I^i M + I^{i+1} M}{m^{s+1} I^i M + I^{i+1} M} \right]_n,$$

where $G_I(M)$ is graded by the internal grading.

Observe that $S^*_{0, 0} = R/m$ is a field and $T^*$ has dimension $\dim M$. We denote the Hilbert-Samuel function $\ell_{s, 0}(T^*_{s, n})$ of $T^* = G_m(G_I(M))$ by $h^*_0(I,M)(s, n)$ and its first Hilbert sum by $h_0^{*(1, 0)}(I,M)(s, n)$. Thus
For $s, n \gg 0$, we define the polynomial $h_{(I, M)}^*(s, n)$ of degree $D - 2$ as the Hilbert polynomial of $h_{(I, M)}^*(s, n)$ adding coefficient zero to the terms of degree between $\dim M - 2$ and $D - 2$.

Thus, if $s, n \gg 0$, the sequence $c_{k, D}^*(I, M)$, $k = 0, \ldots, D - 1$ are the numerators of the leading coefficients of the polynomial $h_{(I, M)}^*(1, 0)(s, n)$.

We will need the fact that the $c^*$-multiplicity sequence is additive on short exact sequences:

**Theorem 3.2.** (Additivity) Let $(R, \mathfrak{m})$ be a local ring, $A$ a standard graded Noetherian $R$-algebra, and $I$ an ideal of $A$ generated by linear forms. If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of finitely generated graded $A$-modules and $D$ an integer with $D \geq d := \dim M_1$. Then, for $s, n \gg 0$,

$$h_{(I, M_0)}^*(s, n) + h_{(I, M_2)}^*(s, n) - h_{(I, M_1)}^*(s, n)$$

is a polynomial of degree at most $D - 2$. In particular we have that

$$c_{k, D}^*(I, M_1) = c_{k, D}^*(I, M_0) + c_{k, D}^*(I, M_2)$$

for all $k = 0, \ldots, D - 1$.

**Proof.** Let $N_j := \mathcal{R}(I, M_j)^+$ be the extended Rees module associated to $M_j$, graded by the internal grading. Set

$$N'_0 := \ker(N_1 \to N_2) = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i=0}^\infty [M_0 \cap I^i M_1]_n.$$
This gives an exact sequence of cokernels

\[0 \rightarrow G' := \frac{N'_0}{t^{-1}_0 N'_0} \rightarrow G_I(M_1) \rightarrow G_I(M_2) \rightarrow 0.\]

Notice that

\[G' = \bigoplus_{n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{M_0 \cap I^i M_1}{M_0 \cap I^{i+1} M_1} \right]_n.\]

Let \(u\) be another indeterminate which we consider with degree one. Set

\[K_j := \mathcal{R}(\text{m}G_I(A), G_I(M_j))^+ = \bigoplus_{s \in \mathbb{Z}} \bigoplus_{n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{\text{m}^s I^i M_j + I^{i+1} M_j}{I^{i+1} M_j} \right]_n.\]

Notice that \(\frac{K_j}{u^{-1} K_j} = G_m(G_I(M_j)).\) Set

\[K'_0 := \ker(K_1 \rightarrow K_2) = \bigoplus_{s \in \mathbb{Z}} \bigoplus_{n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{M_0 \cap (\text{m}^s I^i M_1 + I^{i+1} M_1)}{M_0 \cap I^{i+1} M_1} \right]_n.\]

We consider the natural diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K'_0(1,0) & \rightarrow & K_1(1,0) & \rightarrow & K_2(1,0) & \rightarrow & 0 \\
& | & \downarrow u^{-1} & | & \downarrow u^{-1} & | & \downarrow u^{-1} & & \\
0 & \rightarrow & K'_0 & \rightarrow & K_1 & \rightarrow & K_2 & \rightarrow & 0 \\
\end{array}
\]

which gives an exact sequence of cokernels

\[0 \rightarrow G'' := \frac{K'_0}{u^{-1} K'_0} \rightarrow G_m(G_I(M_1)) \rightarrow G_m(G_I(M_2)) \rightarrow 0. \quad (6)\]

Notice that

\[G'' = \bigoplus_{s,n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{M_0 \cap (\text{m}^s I^i M_1 + I^{i+1} M_1)}{M_0 \cap (\text{m}^{s+1} I^i M_1 + I^{i+1} M_1)} \right]_n.\]

Let \(Q := \text{Img}(K_0 \rightarrow K'_0)\) and let

\[P := \ker(K_0 \rightarrow K'_0) = \bigoplus_{s \in \mathbb{Z}} \bigoplus_{n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{I^{i+1} M_0 + \text{m}^s I^i M_0 \cap I^{i+1} M_1}{I^{i+1} M_0} \right]_n.\]
We consider the natural diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & P(1,0) & \longrightarrow & K_0(1,0) & \longrightarrow & Q(1,0) & \longrightarrow & 0 \\
& & \downarrow u^{-1} & & \downarrow u^{-1} & & \downarrow u^{-1} & & \\
0 & \longrightarrow & P & \longrightarrow & K_0 & \longrightarrow & Q & \longrightarrow & 0
\end{array}
\]

which gives an exact sequence of cokernels

\[
0 \longrightarrow P' := \frac{P}{u^{-1}P} \longrightarrow G_{m}(G_{1}(M_0)) \longrightarrow Q' := \frac{Q}{u^{-1}Q} \longrightarrow 0. \tag{7}
\]

Notice that

\[
P' = \bigoplus_{s,n \in \mathbb{N}} \bigoplus_{i=0}^{\infty} \left[ \frac{I^{i+1}M_0 + m^sI^iM_0 \cap I^{i+1}M_1}{I^{i+1}M_0 + m^{s+1}I^iM_0 \cap I^{i+1}M_1} \right]_n.
\]

We consider the natural diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Q(1,0) & \longrightarrow & K_0'(1,0) & \longrightarrow & L(1,0) & \longrightarrow & 0 \\
& & \downarrow u^{-1} & & \downarrow u^{-1} & & \downarrow u^{-1} & & \\
0 & \longrightarrow & Q & \longrightarrow & K_0' & \longrightarrow & L & \longrightarrow & 0
\end{array}
\]

where \(L := \text{coker}(Q \to K_0')\). The snake-lemma yields an exact sequence

\[
0 \longrightarrow W \longrightarrow Q' \longrightarrow G'' \longrightarrow V \longrightarrow 0 \tag{8}
\]

where \(W := \ker(L(1,0) \to L)\) and \(V := \text{coker}(L(1,0) \to L)\). We also have the exact sequence

\[
0 \longrightarrow W \longrightarrow L(1,0) \longrightarrow L \longrightarrow V \longrightarrow 0 \tag{9}
\]

For \(n \leq 1\) the coefficient modules of \(u^n\) in \(K_0 = R(m, G_{1}(M_0))^+\) and in \(K'_0\) coincide, hence the action of \(u^{-1}\) on \(L\) is nilpotent. Therefore the dimension of \(L\) is at most that of \(G''\), which is bounded by \(D\). Thus all modules occurring in the exact sequence \((\ref{9})\) have dimension at most \(D\).

For any bigraded algebra \(E = \bigoplus_{s,n \in \mathbb{N}} E_{s,n}\) consider the Hilbert-Samuel functions \(h_E(s, n) := \ell(E_{s,n})\) and its Hilbert sums

\[
h_E^{(1,0)}(s, n) := \sum_{v=0}^{s} h_E(v, n).
\]
For $s, n \gg 0$, we define the polynomial $h_E^D(s, n)$ of degree $D - 2$ as the Hilbert polynomial of $h_E(s, n)$ adding coefficient zero to the terms of degree between $\dim E - 2$ and $D - 2$.

Using the additivity of the length function in (6), (7) and (8) leads to

$$
h^D(1, 0)(s, n) = h^D(1, 0)(s, n) - h^D(1, 0)(s, n) + h^D(1, 0)(s, n) - h^D(1, 0)(s, n)
$$

(10)

Because of (9) we have that

$$
h^D(1, 0)(s, n) - h^D(1, 0)(s, n) = h^D(1, 0)(s + 1, n) - h^D(1, 0)(s, n) = h^D_L(s, n),
$$

which, for $s, n \gg 0$, is a polynomial of degree at most $D - 2$ because $L$ has dimension at most $D$.

Because of (10), for concluding the result we will prove next that $h^D_P(s, n)$ is a polynomial of degree at most $D - 2$, for $s, n \gg 0$, or equivalently that $h^D_P(s, n)$ is a polynomial of degree at most $d - 2$, for $s, n \gg 0$. Notice that

$$
h^D_P(s, n) = \sum_{i=0}^{\infty} \ell \left[ \frac{I^i M_0 \cap I^{i+1} M_1}{I^{i+1} M_0 + m^{s+1} I^i M_0 \cap I^{i+1} M_1} \right]_n.
$$

Set

$$S^i_{s, n} := \left[ \frac{I^i M_0 \cap I^{i+1} M_1}{I^{i+1} M_0 + m^{s+1} I^i M_0 \cap I^{i+1} M_1} \right]_n,$$

$$C^i_{s, n} := \left[ \frac{I^i M_0}{m^{s+1} I^i M_0 + I^i M_0 \cap I^{i+1} M_1} \right]_n,$$

and

$$F^i_{s, n} := \left[ \frac{m^s I^i M_0 + I^i M_0 \cap I^{i+1} M_1}{m^{s+1} I^i M_0 + I^{i+1} M_0} \right]_n.$$

Consider the exact sequences

$$0 \longrightarrow S^i_{s, n} \longrightarrow \left[ \frac{I^i M_0}{m^{s+1} I^i M_0 + I^{i+1} M_0} \right]_n \longrightarrow C^i_{s, n} \longrightarrow 0
$$

(11)

and

$$0 \longrightarrow F^i_{s, n} \longrightarrow \left[ \frac{I^i M_0}{m^{s+1} I^i M_0 + I^{i+1} M_0} \right]_n \longrightarrow C^i_{s-1, n} \longrightarrow 0
$$
which yields
\[ h^\text{(1,0)}_{p'}(s, n) + \sum_{i=0}^{\infty} \ell(C^i_{s,n}) - \sum_{i=0}^{\infty} \ell(C^i_{s-1,n}) = \sum_{i=0}^{\infty} \ell(F^i_{s,n}). \quad (12) \]

Notice that, by equality (11), for \( s, n \gg 0 \), \( \sum_{i=0}^{\infty} \ell(C^i_{s,n}) \) is a polynomial of degree at most \( d - 1 \). Hence, for \( s, n \gg 0 \), \( \sum_{i=0}^{\infty} \ell(C^i_{s,n}) - \sum_{i=0}^{\infty} \ell(C^i_{s-1,n}) \) is a polynomial of degree at most \( d - 2 \). On the other hand, if \( h_F(s, n) := \sum_{i=0}^{\infty} \ell(F^i_{s,n}) \) then, it is clear that \( h(1,0)(s, n) = h^\text{(1,0)}_{(I,M)}(s, n) \). Therefore, for \( s, n \gg 0 \), \( h_F(s, n) \) is a polynomial of degree at most \( d - 2 \), for \( s, n \gg 0 \), as we claimed.

In order to be able to formulate the main result in an efficient way, we need a generalization of the notion of height of an ideal to the case of modules (see [FM]). We call the number
\[ \text{ht}_M(I) := \min \{ \dim M_p \mid p \in \text{supp} \, M \cap V(I) \} \]
the \( M \)-height of \( I \).

**Lemma 3.3.** Let \((R, m)\) be a local ring, \( A \) a standard graded Noetherian \( R \)-algebra and \( M \) a finitely generated graded \( A \)-module of dimension \( D \). Let \( I \subseteq J \) be ideals of \( A \) generated by linear forms such that \( \text{ht}_M(I) > 0 \). Then, for all \( k = 0, \ldots, D - 1 \) we have that

(i) \( c^*_k(J, I^tM) = c^*_k(I, M) \) for all \( t \in \mathbb{N} \);

(ii) \( c^*_k(I, J^tM) = c^*_k(I, M) \) for all \( t \in \mathbb{N} \);

(iii) \( c^*_k(I, I^tM) = c^*_k(I, M) \) and \( c^*_k(J, J^tM) = c^*_k(J, M) \) for all \( t \in \mathbb{N} \).

**Proof.** In order to prove (i) consider the exact sequence of graded \( A \)-modules
\[ 0 \to I^tM \to M \to \frac{M}{I^tM} \to 0. \]

From Theorem 3.2 we have that
\[ c^*_k(J, M) = c^*_k,J^tM(J, I^tM) + c^*_k,J^tM \left( J, \frac{M}{I^tM} \right). \]
Since \( \text{ht}_M(I) > 0 \) we have that \( \dim(M/I^tM) < D \) and \( \dim(I^tM) = D \). Thus \( c^*_k, D(J, I^tM) = c^*_k(J, I^tM) \) and \( c^*_k, D \left( J, \frac{M}{I^tM} \right) = 0 \), which proves (i).

The proof of (ii) and (iii) follows analogously. \( \square \)

**Theorem 3.4.** Let \((R, m)\) be a local ring, \(A\) a standard graded Noetherian \(R\)-algebra and \(M\) a finitely generated graded \(A\)-module of dimension \(D\). Let \(I \subseteq J\) be ideals of \(A\) generated by linear forms such that \( \text{ht}_M(I) > 0 \). If \(I\) is a reduction of \((J, M)\) then \(c^*_k(I, M) = c^*_k(J, M)\) for all \(k = 0, \ldots, D - 1\).

**Proof.** Since \(I\) is a reduction of \((J, M)\) we have that \(I(J^rM) = J^{r+1}M = J(J^rM)\) for all \(r \gg 0\). Hence we have that \(G_m(G_I(J^rM)) = G_m(G_J(J^rM))\) and thus
\[
c^*_k(I, J^rM) = c^*_k(J, J^rM) \quad \text{for all} \quad k = 0, \ldots, D - 1.
\]
Therefore the result follows by items (ii) and (iii) of Lemma 3.3. \( \square \)

### 4 \(c^\sharp\)-multiplicity sequence

To prove a converse of Theorem 3.4 we introduce another multiplicity sequence, the \(c^\sharp\)-multiplicity sequence, that is more suited for this purpose. The definition is inspired by [UV] and [AM2].

In addition to the assumptions of the above section suppose that \(M\) is generated in degree zero. Again consider \(G_I(M)\) as graded by the internal grading.

**Definition 4.1.** Let \(D\) be any integer with \(D \geq \dim M\). We define the **\(c^\sharp\)-multiplicity sequence** of \(M\) with respect to \(I\), as
\[
c^\sharp_k, D(I, M) := c^D_k(A_1G_I(M)), \quad k = 0, \ldots, D - 1.
\]

where \(G_I(M)\) is graded by the internal grading. In the case where \(D = \dim M\) we simply write \(c^\sharp_k(I, M)\) instead of \(c^\sharp_k, \dim M(I, M), \quad k = 0, \ldots, \dim M - 1\).

To be more explicit, consider the standard bigraded \(R\)-algebra \(S^\sharp := G_m(A_1G_I(A)) = \oplus_{s,n=0}^\infty S^\sharp_{s,n}\) with
\[
S^\sharp_{s,n} = \oplus_{i=0}^{\infty} \left[ \frac{m^sI^iA_1 + I^{i+1}}{m^{s+1}I^iA_1 + I^{i+1}} \right]_n = \oplus_{i=0}^{n-1} \left[ \frac{m^sI^iA_1 + I^{i+1}}{m^{s+1}I^iA_1 + I^{i+1}} \right]_n,
\]
where $G_I(A)$ is graded by the internal grading, and the finitely generated bigraded module over this algebra

$$T^\sharp = G_m(A_1G_I(M)) = \bigoplus_{s,n=0}^\infty T^\sharp_{s,n}$$

with

$$T^\sharp_{s,n} = \bigoplus_{i=0}^\infty \left[ \frac{m^s I^i A_1 M + I^{i+1} M}{m^{s+1} I^i A_1 M + I^{i+1} M} \right]_n = \bigoplus_{i=0}^{n-1} \left[ \frac{m^s I^i M + I^{i+1} M}{m^{s+1} I^i M + I^{i+1} M} \right]_n,$$

where $G_I(M)$ is graded by the internal grading.

We denote the Hilbert-Samuel function $\ell_R(T^\sharp_{s,n})$ of $T^\sharp = G_m(A_1G_I(M))$ by $h^\sharp_{(I,M)}(s,n)$ and its first Hilbert sum by $h^\sharp_{(I,M)}^{(1,0)}(s,n)$.

Thus

$$h^\sharp_{(I,M)}(s,n) = \sum_{i=0}^{n-1} \ell_R \left[ \frac{m^s I^i M + I^{i+1} M}{m^{s+1} I^i M + I^{i+1} M} \right]_n$$

and

$$h^\sharp_{(I,M)}^{(1,0)}(s,n) = \sum_{i=0}^{n-1} \ell_R \left[ \frac{I^i M}{m^{s+1} I^i M + I^{i+1} M} \right]_n.$$

For $s,n \gg 0$, we define the polynomial $h^\sharp_{(I,M)}^{D}(s,n)$ of degree $D - 2$ as the Hilbert polynomial of $h^\sharp_{(I,M)}(s,n)$ adding coefficient zero to the terms of degree between $\dim(A_1G_I(M)) - 2$ and $D - 2$.

Thus, if $s,n \gg 0$, the sequence $c^*_{k,D}(I,M)$, $k = 0, \ldots, D - 1$ are the numerators of the leading coefficients of the polynomial $h^\sharp_{(I,M)}^{D(1,0)}(s,n)$.

It will be useful to clarify the relationship between the two multiplicity sequences $c^*$ and $c^\sharp$.

**Lemma 4.2.** We use the same notation of Definition 4.1. Denote the graded $G_I(A)$-module $G_I(M)/A_1G_I(M)$ by $B(I,M)$. Set $b^D_{k}(I,M) := c^D_{k}(B(I,M))$, $k = 0, \ldots, D - 1$. Then we have that

$$c^*_{k,D}(I,M) = c^\sharp_{k,D}(I,M) + b^D_{k}(I,M)$$

**Proof.** Consider the exact sequence of $G_I(A)$-modules

$$0 \longrightarrow A_1G_I(M) \longrightarrow G_I(M) \longrightarrow B(I,M) \longrightarrow 0.$$
By the additivity of the $c^D$-multiplicity sequence, Proposition 2.3, we have
\[ c^D_k(G_I(M)) = c^D_k(A_1G_I(M)) + c^D_k(B(I, M)). \]
Recall that $c^D_k(G_I(M)) = c^*_k, D(I, M)$, $c^D_k(A_1G_I(M)) = c^*_k, D(I, M)$ and $c^D_k(B(I, M)) = b^D_k(I, M)$. Hence the result follows.

We will need the fact that the $c^\sharp$-multiplicity sequence is additive on short exact sequences:

**Proposition 4.3.** (Additivity) Let $(R, \mathfrak{m})$ be a local ring, $A$ a standard graded Noetherian $R$-algebra, and $I$ an ideal of $A$ generated by linear forms. If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of finitely generated graded $A$-modules and $D$ an integer with $D \geq \text{dim } M_1$. Then,
\[ c^\sharp_k, D(I, M_1) = c^\sharp_k, D(I, M_0) + c^\sharp_k, D(I, M_2) \]
for all $k = 0, \ldots, D - 1$.

**Proof.** By Lemma 4.2 and Theorem 3.2 it is enough to show that the $b^D_k$-sequence is additive, that is
\[ b^D_k(I, M_1) = b^D_k(I, M_0) + b^D_k(I, M_2) \]
for all $k = 0, \ldots, D - 1$.

Notice that $B(I, M_j) = \bigoplus_{n \in \mathbb{N}} [I^nM_j]_n$, $j = 0, 1, 2$. Set
\[ G' := \text{ker}(B(I, M_1) \to B(I, M_2)) = \bigoplus_{n \in \mathbb{N}} [M_0 \cap I^nM_1]_n. \]
We have the exact sequence
\[ 0 \to G' \to B(I, M_1) \to B(I, M_2) \to 0. \tag{13} \]
Set
\[ L := \text{coker}(B(I, M_0) \to G') = \bigoplus_{n \in \mathbb{N}} \left[ \frac{M_0 \cap I^nM_1}{I^nM_0} \right]_n. \]
We have the exact sequence
\[ 0 \to B(I, M_0) \to G' \to L \to 0. \tag{14} \]
Now (13) and (14) are exact sequences of finitely generated graded modules of dimension at most $D$. Hence we may compute the $c^D$-multiplicity.
sequence of graded modules along these sequences. Using the additivity of this multiplicity sequence as stated in Proposition 2.3 we deduce that

\[ b^D_k(I, M_0) = b^D_k(I, M_1) + b^D_k(I, M_2) + c^D_k(L). \]

To obtain that \( c^D_k(L) = 0 \) we show that \( L \) has dimension less than \( D \). In fact, by Artin-Rees we have that

\[ M_0 \cap I^nM_1 = I^n - c(M_0 \cap I^nM_2) \subseteq I^n \cap M_0. \]

Hence \( \dim(L) \leq \dim(P) \) where \( P := \bigoplus_{n \in \mathbb{N}} \left[ \frac{I^n - cM_0}{I^nM_0} \right]_n \). Clearly \( P \) has dimension less than the dimension of \( G_1(M_0) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{i=0}^\infty \left[ \frac{I^nM_0}{I^{n+1}M_0} \right]_n \) which is at most \( D \).

**Lemma 4.4.** Let \((R, m)\) be a local ring, \( A \) a standard graded Noetherian \( R \)-algebra and \( M \) a finitely generated graded \( A \)-module of dimension \( D \). Let \( I \subseteq J \) be ideals of \( A \) generated by linear forms such that \( \text{ht}_M(I) > 0 \). Then, for all \( k = 0, \ldots, D - 1 \) we have that

1. \( c^\ast_k(J, I^tM) = c^\ast_k(J, M) \) for all \( t \in \mathbb{N} \);
2. \( c^\ast_k(I, J^tM) = c^\ast_k(I, M) \) for all \( t \in \mathbb{N} \);
3. \( c^\ast_k(I, I^tM) = c^\ast_k(I, M) \) and \( c^\ast_k(I, J^tM) = c^\ast_k(J, M) \) for all \( t \in \mathbb{N} \).

**Proof.** In order to prove (i) consider the exact sequence of graded \( A \)-modules

\[ 0 \rightarrow I^tM \rightarrow M \rightarrow \frac{M}{I^tM} \rightarrow 0. \]

From Proposition 4.3 we have that

\[ c^\ast_k(J, M) = c^\ast_{k, D}(J, I^tM) + c^\ast_{k, D} \left( J, \frac{M}{I^tM} \right). \]

Since \( \text{ht}_M(I) > 0 \) we have that \( \dim(M/I^tM) < D \) and \( \dim(I^tM) = D \). Thus \( c^\ast_{k, D}(J, I^tM) = c^\ast_k(J, I^tM) \) and \( c^\ast_{k, D} \left( J, \frac{M}{I^tM} \right) = 0 \), which proves (i).

The proof of (ii) and (iii) follows analogously.

**Remark 4.5.** Let \( D = \dim M \). If \( \text{ht}_M(I) > 0 \) we have from the above Lemma that, for all \( k = 0, \ldots, D - 1 \)
(i) \( c^\sharp_k(I, J^r M) = c^\sharp_k(I, I^r M) = c^\sharp_k(I, M) \) for all \( r \in \mathbb{N} \) and

(ii) \( c^\sharp_k(J, I^s M) = c^\sharp_k(J, J^s M) = c^\sharp_k(J, M) \) for all \( s \in \mathbb{N} \).

That is, if

\[
V(1,0,0,0)(i, j, r, n) := \ell \left( \frac{I^j J^r M}{m^{i+1} I^j J^r M + I^{j+1} J^r M} \right)_n,
\]

\[
W(1,0,0,0)(i, j, s, n) := \ell \left( \frac{I^s J^j M}{m^{i+1} I^s J^j M + I^s J^{j+1} M} \right)_n,
\]

\[
H_{(I,M)}^{(1,0,0)}(i, j, n) := \ell \left( \frac{I^j M}{m^{i+1} I^j M + I^{j+1} M} \right)_n,
\]

and

\[
H_{(J,M)}^{(1,0,0)}(i, j, n) := \ell \left( \frac{J^j M}{m^{i+1} J^j M + J^{j+1} M} \right)_n,
\]

then for all \( i, j, s, r, n \gg 0 \) they become polynomials which satisfy the following relations

\[
V(1,0,0,0)(i, j, r, n) = H_{(I,M)}^{(1,0,0)}(i, j + r, n),
\]

\[
W(1,0,0,0)(i, j, s, n) = H_{(J,M)}^{(1,0,0)}(i, j + s, n),
\]

\[
h^\sharp_{(I,M)}^{(1,0)}(i, n) = H_{(I,M)}^{(1,1,0)}(i, n - 1, n),
\]

and

\[
h^\sharp_{(J,M)}^{(1,0)}(i, n) = H_{(J,M)}^{(1,1,0)}(i, n - 1, n)
\]

where \( F(i, j, r, n) \) denotes, from now on, the leading homogeneous part of the polynomial function \( F(i, j, r, n) \). The equalities (15) and (16) follow by (i) and (ii) respectively and the equalities (17) and (18) follow by the definition of \( h^\sharp_{(I,M)}^{(1,0)}(i, n) \) and \( h^\sharp_{(J,M)}^{(1,0)}(i, n) \) respectively. By equations (17) and (18) we have that, for all \( i, j, n \gg 0 \), \( H_{(J,M)}^{(1,0,0)}(i, j, n) \) and \( H_{(J,M)}^{(1,0,0)}(i, j, n) \) become polynomials of degree at most \( D - 2 \).

**Theorem 4.6.** Let \((R, m)\) be a local ring, \( A \) a standard graded Noetherian \( R \)-algebra and \( M \) a finitely generated graded \( A \)-module of dimension \( D \). Let \( I \subseteq J \) ideals of \( A \) generated by linear forms such that \( \text{ht}_M(I) > 0 \). If \( I \) is a reduction of \((J, M)\) then \( c^\sharp_k(I, M) = c^\sharp_k(J, M) \) for all \( k = 0, \ldots, D - 1 \).
Proof. Since \( I \) is a reduction of \((J, M)\) we have that \( I(J^r M) = J^{r+1} M = J(J^r M) \) for all \( r \gg 0 \). Hence we have that \( G_m(A_1 G_f(J^r M)) = G_m(A_1 G_J(J^r M)) \) and thus
\[
c_k(I, J^r M) = c_k(J, J^r M) \quad \text{for all } k = 0, ..., D - 1.
\]
Therefore the result follows by items (ii) and (iii) of Lemma 4.4

\[\square\]

5 Intertwining algebra and module

In this section we recall the notions of intertwining algebras and intertwining modules introduced in [SUV] in the context of graded algebras which provide a strong criterium for reductions of algebras. This algebras has been exploited on several occasions by D. Kirby and D. Rees [KR], S. Kleiman and A. Thorup [KT1] and D. Katz [K]. Their presentation can be immediately extended to the version for modules we present here.

Let \((\mathcal{R}, m)\) be a Noetherian local ring, \( A \) a standard graded Noetherian \( \mathcal{R} \)-algebra, \( I \subseteq J \) ideals of \( A \) generated by linear forms and \( M \) a finitely generated graded \( A \)-module. Set
\[
\mathcal{A} := \mathcal{R}(I, A) = \bigoplus_{i \in \mathbb{N}} I^i t^i \subseteq A \otimes_{\mathcal{R}} \mathcal{R}[t];
\]
\[
\mathcal{B} := \mathcal{R}(J, A) = \bigoplus_{i \in \mathbb{N}} I^i t^i \subseteq A \otimes_{\mathcal{R}} \mathcal{R}[t],
\]
\[
\mathcal{R}(I, M) := \bigoplus_{i \in \mathbb{N}} I^i Mt^i \subseteq M \otimes_{\mathcal{R}} \mathcal{R}[t],
\]
and
\[
\mathcal{R}(J, M) := \bigoplus_{i \in \mathbb{N}} J^i Mt^i \subseteq M \otimes_{\mathcal{R}} \mathcal{R}[t].
\]

Assigning degree zero to the variable \( t \), the polynomial ring \( A[t] := A \otimes_{\mathcal{R}} \mathcal{R}[t] \) becomes a standard graded Noetherian \( \mathcal{R}[t] \)-algebra, and \( M[t] := M \otimes_{\mathcal{R}} \mathcal{R}[t] \) a finitely generated module over this algebra. The Rees algebras \( \mathcal{A} \) and \( \mathcal{B} \) are homogeneous \( \mathcal{R}[t] \)-subalgebras of \( A[t] \), and hence standard graded Noetherian \( \mathcal{R}[t] \)-algebras. Furthermore \( \mathcal{R}(I, M) \) and \( \mathcal{R}(J, M) \) are homogeneous \( \mathcal{A} \) and \( \mathcal{B} \) submodules of \( M[t] \) respectively, thus they are finitely generated graded modules over \( \mathcal{A} \) and \( \mathcal{B} \) respectively. The grading so
defined on this Rees algebras and modules are also called internal grading. Notice that with respect to this grading we have

$$[\mathcal{R}(I, M)]_n = \bigoplus_{i \in \mathbb{N}} [I^i M]_n,$$

and so on.

Let $u$ be a new variable which we also consider of degree zero and set

$$C := \mathcal{R}(I, J) := \mathcal{B} \left[ \bigoplus_{i \in \mathbb{N}} AI^i u^i \right] \subseteq A \otimes_R R[u, t]$$

Notice that $A[u, t] := A \otimes_R R[u, t]$ becomes a standard graded Noetherian $R[u, t]$-algebra and $C$ a homogeneous $R[u, t]$-subalgebra of $A[u, t]$, and hence a standard graded Noetherian $R[u, t]$-algebra. Furthermore with respect to this grading, $C$ becomes a standard graded Noetherian $R[u, t]$-algebra. This grading on $C$ is also called internal grading. Notice that

$$C_n = \bigoplus_{i, j \in \mathbb{N}} [AI^i J^j]_n.$$

Set

$$T_{J/I}(M) := \frac{C(\mathcal{R}(J, M))}{C(\mathcal{R}(I, M))},$$

where all the graded algebras and graded modules involved are considered with the internal grading. This grading on $T_{J/I}(M)$ is also called internal grading. With this grading $T_{J/I}(M)$ becomes a finitely generated Noetherian graded $C$-module which is called the intertwining module of $I$ and $J$ with respect to $M$. Notice that

$$[T_{J/I}(M)]_n = \bigoplus_{s, r \in \mathbb{N}} \left[ \frac{I^{s-1} J^{r+1} M}{I^s J^r M} \right]_n.$$

We say that $I$ is a reduction of $(J, M)$ if $IJ^n M = J^{n+1} M$ for at least one positive integer $n$.

The following Theorem has been proved by A. Simis, B. Ulrich and W.Vasconcelos [SUV] in the context of graded algebras (see also [V, Theorem 1.153, p. 85]). Their proof can be immediately extended to the version for modules we present here.
Theorem 5.1. Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(A\) a standard graded Noetherian \(R\)-algebra and \(M\) a finitely generated quasi-unmixed graded \(A\)-module. Let \(I \subseteq J\) be ideals of \(A\) generated by linear forms such that \(\text{ht}_M(I) > 0\). Then the following are equivalent:

(i) \(I\) is a reduction of \((J, M)\);

(ii) \(\dim T_{J/I}(M) \leq \dim(\mathcal{R}(J, M)) - 1 = \dim M\).

Remark 5.2. The implication \((i) \Rightarrow (ii)\) does not need the quasi-unmixedness hypotheses for \(M\) (see for example, [V, Proposition 1.149]) this requirement is needed only for the converse.

Theorem 5.3. Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(A\) a standard graded Noetherian \(R\)-algebra, \(M\) a \(D\)-dimensional graded \(A\)-module generated by finitely many homogeneous elements of degree zero and \(I \subseteq J\) ideals of \(A\) generated by linear forms such that \(\text{ht}_M(I) > 0\). Consider the following statements:

(i) \(I\) is a reduction of \((J, M)\);

(ii) \(c^k(I, M) = c^k(J, M)\) for all \(k = 0, ..., D - 1\).

Then, \((i)\) implies \((ii)\) and if \(M\) is quasi-unmixed the converse also holds.

Proof. The implication \((i) \Rightarrow (ii)\) has been proved in Theorem 4.6.

Conversely assume that \(c^k(I, M) = c^k(J, M)\) for all \(k = 0, ..., D - 1\). Notice that, by Theorem 5.1, it is enough to prove that \(\dim_{C(I, J)}(T_{J/I}(M)) \leq \dim(\mathcal{R}_J(M)) - 1 = D\). We will compute \(\dim_{C(I, J)}(T_{J/I}(M))\) or, equivalently, \(\dim (G_m(T_{J/I}(M)))\). Notice that

\[
G := G_m(T_{J/I}(M)) = \bigoplus_{i,n \in \mathbb{N}} G_{i,n}
\]

where

\[
G_{i,n} := \bigoplus_{j,r \in \mathbb{N}} \left\lbrack \frac{m^i I^{j+1} J^{r+1} M + I^j J^r M}{m^{i+1} I^{j-1} J^{r+1} M + I^j J^r M} \right\rbrack_n.
\]

Let \(h_G(i, n) = \ell(G_{i,n})\) and let \(h_G^{(1,0)}(i, n) = \sum_{u=0}^i h_G(u, n)\) be its Hilbert-sum. Notice that

\[
h_G^{(1,0)}(i, n) = \sum_{j+r \leq n} \ell \left( \left[ \frac{I^{j-1} J^{r+1} M}{m^{i+1} I^{j-1} J^{r+1} M + I^j J^r M} \right]_n \right).
\]
For concluding the proof it is sufficient to show that, for \( i, n \gg 0 \), \( h^{(1,0)}_G(i, n) \) is a polynomial of degree at most \( D - 1 \). Set

\[
F_{i,n}^{j,r} := \left[ \frac{I^{j-1} J^{r+1} M}{m^{i+1} I^{j-1} J^{r+1} M + I^j J^r M} \right]_n,
\]

\[
P_{i,n}^{j,r} := \left[ \frac{m^{i+1} I^{j-1} J^{r+1} M \cap I^j J^r M + I^{j+1} J^r M}{m^{i+1} I^j J^r M + I^{j+1} J^r M} \right]_n
\]

and

\[
Q_{i,n}^{j,r} := \left[ \frac{I^j J^r M}{m^{i+1} I^j J^r M + I^{j+1} J^r M} \right]_n.
\]

Notice that

\[
h^{(1,0)}_G(i, n) = \sum_{j+r \leq n} \ell \left( \left[ F_{i,n}^{j,r} \right]_n \right).
\]

Therefore to conclude the proof it is sufficient to show that, for \( i, j, r, n \gg 0 \), \( \ell \left( \left[ F_{i,n}^{j,r} \right]_n \right) \) becomes a polynomial of degree at most \( D - 3 \).

Consider the exact sequence

\[
0 \rightarrow P_{i,n}^{j,r} \rightarrow Q_{i,n}^{j,r} \rightarrow Q_{i,n}^{j-1,r+1} \rightarrow F_{i,n}^{j,r} \rightarrow 0
\]

which yields

\[
\ell \left( \left[ F_{i,n}^{j,r} \right]_n \right) = \ell(Q_{i,n}^{j-1,r+1}) - \ell(Q_{i,n}^{j,r}) + \ell(P_{i,n}^{j,r}) \tag{19}
\]

Notice that, by equality (15) of Remark 4.5, \( \ell(Q_{i,n}^{j,r}) = H^{(1,0)}_{(i,M)}(i,j+r,n) \) for all \( i, j, r, n \gg 0 \). Hence, for all \( i, j, r, n \gg 0 \), \( \ell(Q_{i,n}^{j-1,r+1}) - \ell(Q_{i,n}^{j,r}) \) is a polynomial of degree at most \( D - 3 \). Therefore, by equality (19), it remains to prove that, for \( i, j, r, n \gg 0 \), \( \ell(P_{i,n}^{j,r}) \) is polynomial of degree at most \( D - 3 \).

We observe that, by Artin-Rees,

\[
P_{i,n}^{j,r} \subseteq N_{i,n}^{j,r} := \left[ \frac{m^{i+1-c} I^j J^r M + I^{j+1} J^r M}{m^{i+1} I^j J^r M + I^{j+1} J^r M} \right]_n.
\]

Thus, it is enough to prove that, for \( i, j, r, n \gg 0 \), \( \ell(N_{i,n}^{j,r}) \) is a polynomial of degree at most \( D - 3 \).

Consider now the exact sequences
\[ 0 \rightarrow \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \rightarrow N_{i,n}^{j,r} \rightarrow \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \rightarrow 0, \tag{20} \]

\[ 0 \rightarrow \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \rightarrow Q_{i-1,n}^{j,r} \rightarrow \left[ \frac{I^J M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \rightarrow 0 \tag{21} \]

and

\[ 0 \rightarrow \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \rightarrow Q_{i,n}^{j,r} \rightarrow Q_{i-1,n}^{j,r} \rightarrow 0 \tag{22} \]

which yields

\[ \ell(N_{i,n}^{j,r}) = \ell\left( \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \right) + \ell\left( \left[ \frac{m^{i+1-c}I^J M + I^{j+1}M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \right) \]

\[ = \ell(Q_{i,n}^{j,r}) - \ell\left( \left[ \frac{I^J M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \right) \]

where the first equality follows by (20) and the last equality follows by (21) and (22).

Therefore, for \( i, j, r, n \gg 0 \),

\[ \ell(N_{i,n}^{j,r}) = \ell(Q_{i,n}^{j,r}) - \ell\left( \left[ \frac{I^J M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \right) \tag{23} \]

Notice that, by equality (15) of Remark 4.5, \( \ell(Q_{i,n}^{j,r}) = H_{(i,M)}^{(1,0,0)}(i, j + r, n) \) for all \( i, j, r, n \gg 0 \) and by equality (16) of Remark 4.5,

\[ \ell\left( \left[ \frac{I^J M}{m^{i+1-c}I^J M + I^{j+1}M} \right]_{n} \right) = H_{(j,M)}^{(1,0,0)}(i, j + r, n) \]

Hence, for \( i, j, r, n \gg 0 \),

\[ \ell(N_{i,n}^{j,r}) = H_{(i,M)}^{(1,0,0)}(i, j + r, n) - H_{(j,M)}^{(1,0,0)}(i, j + r, n) \]

26
Notice that the leading coefficients of the Hilbert-Samuel polynomials of $H_{(I,M)}^{(1,0,0)}(i, j + r, n)$ and $H_{(J,M)}^{(1,0,0)}(i, j + r, n)$ are described as specific sums of $c_k^*(I, M)$ and $c_k^*(J, M)$, with $k = 0, ..., D - 1$, respectively, which by assumption must coincide. Hence, for $i, j, r, n \gg 0$, $\ell(N_{i,n})$ is a polynomial of degree at most $D - 3$ as we claimed.

6 Multiplicity sequence for arbitrary modules

We are now ready to introduce the main object of this paper, the multiplicity sequence of a module. Here the ideal of the previous section will be replaced by a module $E$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E$ a submodule of the free $R$-module $R^p$, and $N$ a finitely generated $R$-module of dimension $d$. The symmetric algebra $A := \text{Sym}(R^p) = \bigoplus S_n(R^p)$ of $R^p$ is a polynomial ring $R[T_1, \ldots, T_p]$. If $h = (h_1, \ldots, h_p) \in R^p$, then we define the element $w(h) = h_1T_1 + \ldots + h_pT_p \in A$. We denote by $\mathcal{R}(E) := \oplus \mathcal{R}_n(E)$ the subalgebra of $A$ generated in degree one by $\{w(h) : h \in E\}$ and call it the Rees algebra of $E$. Then $\mathcal{R}(E)$ has dimension $d + p$. Consider the $A$-ideal $I$ generated by $\mathcal{R}_1(E)$ and the $A$-module $M := A \otimes_R N$. Notice that $I$ is an $A$-ideal generated by linear forms and $M$ is a finitely generated graded $A$-module of dimension $d + p$ that is generated in degree zero.

**Definition 6.1.** We define the multiplicity sequence associated to the module $E$ with respect to $N$ by

$$c_k(E, N) := c_k^*(I, M), \quad k = 0, \ldots, d + p - 1.$$ 

To be more explicit,

$$\left[ \frac{I^iM}{m^{s+1}I^iM + I^{i+1}M} \right]_n = \frac{\mathcal{R}_i(E)S_{n-i}(R^p)N}{m^{s+1}\mathcal{R}_i(E)S_{n-i}(R^p)N + \mathcal{R}_{i+1}(E)S_{n-i-1}(R^p)N}$$

for $0 \leq i \leq n - 1$. Thus the Hilbert function of $T^s = G_m(A_1G_I(M))$ is

$$h_{(I,M)}^{(1,0)}(s, n) = \sum_{i=0}^{n-1} \ell_R \left[ \frac{\mathcal{R}_i(E)S_{n-i}(R^p)N}{m^{s+1}\mathcal{R}_i(E)S_{n-i}(R^p)N + \mathcal{R}_{i+1}(E)S_{n-i-1}(R^p)N} \right],$$

27
which, for \( s, n \gg 0 \), becomes a polynomial of degree at most \( d + p - 1 \) whose leading coefficients are \( c_k(E, N), k = 0, \ldots, d + p - 1 \). If \( N = R \) we simply write \( c_k(E) \) instead of \( c_k(E, N) \) for \( k = 0, \ldots, d + p - 1 \).

**Remark 6.2.** If \( E \) has finite colength in \( R^p \) then, for \( s \gg 0 \), we have that \( m^{s+1} \mathcal{R}_i(E) S_{n-i}(R^p) \subseteq \mathcal{R}_{i+1}(E) S_{n-i-1}(R^p) \). Hence, in this context

\[
\ell_R \left[ \frac{S_n(R^p)}{\mathcal{R}_{n}(E)} \right] = \ell_R \left[ \frac{S_{n-1}(R^p)}{\mathcal{R}_{n-1}(E)} \right]
\]

Thus in this case \( c_0(E) = e_{BR}(E) \) and \( c_k(E) = 0 \) for all \( k = 1, \ldots, d + p - 1 \). In case that \( E \) is an ideal \( J \) of \( R \) then the Buchsbaum-Rim multiplicity sequence \( c_k(E, N) \) coincides with the Achilles-Manaresi multiplicity sequence \( c_k(J, N) \) for all \( k = 0, \ldots, d \).

Theorem 5.3 immediately gives the following result:

**Theorem 6.3.** Let \( (R, m) \) be a Noetherian local ring, \( E \subseteq F \subseteq R^p \) be \( R \)-modules and write \( I := \mathcal{R}_1(E) \) for the corresponding ideal of \( A := \text{Sym}(R^p) \). Let \( N \) be a \( d \)-dimensional finitely generated \( R \)-module and set \( M := A \otimes_R N \). Assume that \( \text{ht}_M(I) > 0 \). Consider the following statements:

(i) \( E \) is a reduction of \( (F, N) \);

(ii) \( c_k(E, N) = c_k(F, N) \) for all \( k = 0, \ldots, d + p - 1 \).

Then, (i) implies (ii) and if \( N \) is quasi-unmixed the converse also holds.

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