Automorphisms of the Cube $n^d$

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Abstract. Consider a hypergraph $H^d_n$ where the vertices are points of the $d$-dimensional combinatorial cube $n^d$ and the edges are all sets of $n$ points such that they are in one line. We study the structure of the group of automorphisms of $H^d_n$, i.e., permutations of points of $n^d$ preserving the edges. In this paper we provide a complete characterization. Moreover, we consider the Colored Cube Isomorphism problem of deciding whether for two colorings of the vertices of $H^d_n$ there exists an automorphism of $H^d_n$ preserving the colors. We show that this problem is GI-complete.

1 Introduction

Combinatorial cube $n^d$ (or simply a cube $n^d$) is a set of points $[n]^d$, where $[n] = \{0, \ldots, n - 1\}$. A line $\ell$ of a cube $n^d$ is a set of $n$ points of $n^d$ which lie in a geometric line in the $d$-dimensional space where the cube $n^d$ is embedded. We denote the set of all lines of the cube $n^d$ by $\mathbb{L}(n^d)$. Thus, the hypergraph $H^d_n$ is defined as $(n^d, \mathbb{L}(n^d))$. We denote the group of all permutations on $n$ elements by $S_n$. A permutation $P \in S_n$ is an automorphism of the cube $n^d$ if $\ell = \{v_1, \ldots, v_n\} \in \mathbb{L}(n^d)$ implies $P(\ell) = \{P(v_1), \ldots, P(v_n)\} \in \mathbb{L}(n^d)$. Informally, an automorphism of the cube $n^d$ is a permutation of the cube points which preserves the lines. We denote the set of all automorphisms of $n^d$ by $T^d_n$. Note that all automorphisms of $n^d$ with a composition $\circ$ form a group $T^d_n = (T^d_n, \circ, Id)$.

Our main result is the characterization of the generators of the group $T^d_n$ and computing the order of $T^d_n$. Surprisingly, the structure of $T^d_n$ is richer than only the obvious rotations and symmetries. We use three groups of automorphisms for characterization of the group $T^d_n$ as follows. The first one is a group $R_d$ of rotations of the $d$-dimensional hypercube. Generators of $R_d$ are the rotations

$$R_{ij}([x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d]) = [x_1, \ldots, n - x_j - 1, \ldots, x_i, \ldots, x_d]$$

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Theorem 2. The problem Colored Cube Isomorphism is GI-complete even if both input colorings has a form \( n^d \rightarrow [2] \).

The paper is organized as follows. First we count the order of the group \( T_n^d \), whose structure is different from other automorphism groups. Next, for clarity reasons we characterize the generators for \( T_n^d \), and then we generalize the results for the general group \( T_n^d \). In Section \([5]\) we count the order of the group \( T_n^d \). In the last section we study the complexity of Colored Cube Isomorphism and show some idea of a prove of Theorem \([2]\).

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3 The parameter is the maximum number of vertices colored by the same color.
1.1 Motivation

A natural motivation for this problem comes from the game of Tic-Tac-Toe. It is usually played on a 2-dimensional square grid and each player puts his tokens (usually crosses for the first player and rings for the second) at the points on the grid. A player wins if he occupies a line with his token vertically, horizontally or diagonally (with the same length as the grid size) faster than his opponent. Tic-Tac-Toe is a member of a large class of games called strong positional games. For an extraordinary reference see Beck [5]. The size of a basic Tic-Tac-Toe board is $3 \times 3$ and it is easy to show by case analysis that the game ends as a draw if both players play optimally. However, the game can be generalized to larger grid and more dimensions. The $d$-dimensional Tic-Tac-Toe is played on the points of a $d$-dimensional combinatorial cube and it is often called the game $n^d$. With larger boards the case analysis becomes unbearable even using computer search and clever algorithms have to be devised.

The only (as far as we know) non-trivial solved 3-dimensional Tic-Tac-Toe is the game $4^3$, which is called Qubic. Qubic is a win for the first player, which was shown by Patashnik [11] in 1980. It was one of the first examples of computer-assisted proofs based on a brute-force algorithm, which utilized several clever techniques for pruning the game tree. Another remarkable approach for solving Qubic was made by Allis [1] in 1994, who introduced several new methods. However, one technique is common for both authors: the detection of isomorphisms of game configurations. As the game of Qubic is highly symmetric, this detection substantially reduces the size of the game tree.

For the game $n^d$, theoretical result are usually achieved for large $n$ or large $d$. For example, by the famous Hales and Jewett theorem [9], for any $n$ there is (an enormously large) $d$ such that the hypergraph $H_n^d$ is not 2-colorable, that means, the game $n^d$ cannot end in a draw. Using the standard Strategy Stealing argument, $n^d$ is thus a first player’s win. In two dimensions, each game $n^2$, $n > 2$, is a draw (see Beck [5]). Also, several other small $n^d$ are solved.

All automorphisms for Qubic were characterized by Rolland Silver [12] in 1967. As in the field of positional games the game $n^d$ is intensively studied and many open problems regarding $n^d$ are posed, the characterization of the automorphism group of $n^d$ is a natural task.

The need to characterize the automorphism group came from our real effort to devise an algorithm and computer program that would be able to solve the game $5^3$, which is the smallest unsolved Tic-Tac-Toe game. While our effort of solving $5^3$ is currently not yet successful, we were able to come up with the complete characterization of the automorphism group $n^d$, giving an algorithm for detection of isomorphic positions not only in the game $5^3$, but also in $n^d$ in general.

A game configuration can be viewed as a coloring $s$ of $n^d$ by crosses, rings and empty points, i.e., $s : n^d \to \{3\}$. Since we know the structure of the group $\mathbb{T}_n^d$, this characterization yields an algorithm for detecting isomorphic game positions by simply trying all combinations of the generators (the number of the combinations is given by the order of the group $\mathbb{T}_n^d$). A natural question arises: can one obtain
a faster algorithm? Note that the hypergraph $H_n^d$ has polynomially many edges in the number of vertices. Therefore, from a polynomial point of view it does not matter if there are hypergraphs $H_n^d$ with colorings or only colorings on the input. Due to Theorem 2, we conclude that deciding if two game configurations are isomorphic is as hard as deciding if two graphs are isomorphic.

Although our primary motivation came from the game of Tic-Tac-Toe, we believe our result has much broader interest as it presents an analogy of automorphism characterization results of hypercubes (see e.g. [7,10]).

## 2 Preliminaries

Beck [5] gives a different point of view on the lines of $n^d$. Let $s = (s_i^1, \ldots, s_i^n)$ be a sequence of $n$ distinctive points of a cube $n^d$. Let $s = [s_i^1, \ldots, s_i^n]$ for every $1 \leq i \leq n$. We say that $s$ is linear if for every $1 \leq j \leq d$ a sequence $\tilde{s}_j = (s_j^1, \ldots, s_j^n)$ is strictly increasing, strictly decreasing or constant and at least one sequence $\tilde{s}_j$ has to be nonconstant. A set of points $\{p^1, p^2, \ldots, p^n\} \subseteq n^d$ is a line if it can be ordered into a linear sequence $(q^1, q^2, \ldots, q^n)$. Beck [5] worked with ordered lines (the linear sequences in our case). However, for us it is more convenient to have unordered lines because some automorphisms will change the order of points in the line.

Let $\ell$ be a line and $q = (q^1, \ldots, q^n)$ be an ordering of $\ell$ into a linear sequence. Note that every line in $\mathbb{L}(n^d)$ has two such orderings. Another ordering of $\ell$ into a linear sequence is $(q^n, \ldots, q^1)$. We define a type of a sequence $\tilde{q}_j = (q_j^1, \ldots, q_j^n)$ as + if $\tilde{q}_j$ is strictly increasing, − if $\tilde{q}_j$ is strictly decreasing, c if $\tilde{q}_j$ is constant and $q^1_j = c$ for every $1 \leq i \leq n$. A type of $q$ is $\text{type}(q) = (\text{type}(\tilde{q}_1), \ldots, \text{type}(\tilde{q}_n))$.

Type of a line $\ell$ is a type of an ordering of $\ell$ into a linear sequence. Since every line has two such orderings, every line has also two types. However, the second type of $\ell$ can be obtained by switching + and − in the first type. For example, let $\ell = \{[0,0,3],[0,1,2],[0,2,1],[0,3,0]\} \subseteq \mathbb{L}(4^3)$ then $\text{type}(\ell) = \{(0,+,−),(0,−,+))\}$. However, for better readability we write only $\text{type}(\ell) = (0,+,−)$. We denote the $i$-th entry in $\text{type}(\ell)$ by $\text{type}(\ell)_i$.

Let us now define several terms we use in the rest of the paper. A dimension $\dim(\ell)$ of a line $\ell \in \mathbb{L}(n^d)$ is $\dim(\ell) = |\{i \in \{1,\ldots,d\}| \text{type}(\ell)_i \in \{+,-\}\}$. A degree $\deg(p)$ of a point $p \in n^d$ is a number of incident lines, formally $\deg(p) = |\{\ell \in \mathbb{L}(n^d) | p \in \ell\}|$. Two points $p_1, p_2 \in n^d$ are collinear, if there exists a line $\ell \in \mathbb{L}(n^d)$, such that $p_1 \in \ell$ and $p_2 \in \ell$. A point $p \in n^d$ is called a corner if $p$ has coordinates only 0 and $n-1$. A point $p = [x_1, \ldots, x_d] \in n^d$ is an outer point if there exists at least one $i \in \{1,\ldots,d\}$ such that $x_i \in \{0,n-1\}$. If a point $p \in n^d$ is not an outer point then $p$ is called an inner point.

A line $\ell \in \mathbb{L}(n^d)$ is called an edge if $\dim(\ell) = 1$ and $\ell$ contains two corners. Two corners are neighbors if they are connected by an edge. A line $\ell \in \mathbb{L}(n^d)$ with $\dim(\ell) = d$ is called main diagonal. We denote the set of all main diagonals by $\mathbb{L}_m(n^d)$. For better understanding the notions see Figure 1 with some examples in the cube $4^3$. 
A $k$-dimensional face $F$ of the cube $n^d$ is a maximal set of points of $n^d$, such that there exist two index sets $I, J \subseteq \{1, \ldots, d\}$, $I \cap J = \emptyset$, $|I| + |J| = d - k$ and for each point $[x_1, \ldots, x_d]$ in $F$ holds that $x_i = 0$ for each $i \in I$ and, $x_j = n - 1$ for each $j \in J$. For example, $\{[x, y, 0, n-1] \mid x, y \in [n]\}$ is a 2-dimensional face of the cube $n^4$.

A point $p \in n^d$ is fixed by an automorphism $T$ if $T(p) = p$. A set of points $\{p_1, \ldots, p_k\}$ is fixed by an automorphism $T$ if $\{p_1, \ldots, p_k\} = \{T(p_1), \ldots, T(p_k)\}$.

Note that if a set $S$ is fixed it does not necessarily mean every point of $S$ is fixed.

2.1 Order of $T_2^d$

The cube $2^d$ is different from other cubes because every two points are collinear. Thus, we have the following proposition.

**Proposition 1.** Order of the group $T_2^d$ is $(2^d)!$.

**Proof.** Every permutation of the points of the cube $2^d$ is an automorphism, as the graph $H^d_2$ is the complete graph on $2^d$ vertices. \qed

We further assume that $n > 2$.

3 Automorphisms of $n^3$

For better understanding of our technique, we first show the result for the 3-dimensional case of the group $T_3^n$. Here we state several general lemmas how an arbitrary automorphism maps main diagonals, edges and corners. The proofs are technical and are omitted from this conference paper.

**Lemma 1.** Let $F = \{[x, y, 0, \ldots, 0] \mid x, y \in [n] \}$ be a face of $n^d$, and let an automorphism $T \in T_3^n$ fixes all 4 corners of $F$, i.e., points $[0, 0, \ldots, 0]$, $[n-1, 0, \ldots, 0]$, $[0, n-1, 0, \ldots, 0]$ and $[n-1, n-1, 0, \ldots, 0]$. Then, if $T$ fixes a point $[i, 0, \ldots, 0]$, $i \in [n]$ it also fixes a point $[n-i-1, 0, \ldots, 0]$.

**Lemma 2.** Every automorphism $T \in T_3^n$ maps a main diagonal $m \in L_m(n^d)$ onto a main diagonal $m' \in L_m(n^d)$. 

Fig. 1. A cube $4^3$ with some examples of lines. An edge $e$ has a type $(+, 0, 0)$, a line $d$ has a dimension 2 and a type $(+, 3, -)$ and a main diagonal $m$ has a type $(+, -, +)$. 

| $x$: 0 1 2 3 | $z = 0$ | $z = 1$ | $z = 2$ | $z = 3$ |
|-------------|---------|---------|---------|---------|
| $y = 3$     |         |         |         |         |
| $y = 2$     |         |         |         |         |
| $y = 1$     |         |         |         |         |
| $y = 0$     |         |         |         |         |
| $e$         |         |         |         |         |
| $d$         |         |         |         |         |
| $m$         |         |         |         |         |
| $m$         |         |         |         |         |
Lemma 3. Let $T \in T_n^d$, $e$ be an edge and $p$ be a corner, such that $p \in e$. If the corner $p$ is fixed by $T$, then $T(e) = e'$ is an edge such that $p \in e'$.

Lemma 4. If an automorphism $T \in T_n^d$ fixes the corner $[0, \ldots, 0]$ and all its neighbors, then $T$ fixes all corners of the cube $n^d$.

We also use the following easy observations.

Observation 3 If an automorphism $T \in T_n^d$ fixes two collinear points $p, q \in n^d$, then $T$ also fixes a line $\ell \in \text{L}(n^d)$ such that $p, q \in \ell$.

Proof. For any two distinct points $p_1, p_2 \in n^d$ there is at most one line $\ell \in \text{L}(n^d)$ such that $p_1, p_2 \in \ell$. Therefore, if the points $p$ and $q$ are fixed then the line $\ell$ has to be fixed as well. □

Observation 4 If two lines $\ell_1, \ell_2 \in \text{L}(n^d)$ are fixed by $T \in T_n^d$ then their intersection, a point $p = \ell_1 \cap \ell_2$, is fixed by $T$.

Proof. For any two lines $\ell, \ell'$ there is at most one point in $\ell \cap \ell'$. Therefore, if the lines $\ell_1$ and $\ell_2$ are fixed then the point $p$ has to be fixed as well. □

3.1 Generators of $T_n^3$

In this section we characterize generators of the group $T_n^3$. We use two basic groups of automorphisms. The group of permutation automorphisms $F_n$. The group second group is the group of rotations $R$ of a 3-dimensional cube. The generators of $R$ are rotations

$$
R_x([x, y, z]) = [x, n - z - 1, y],$$
$$
R_y([x, y, z]) = [n - z - 1, y, x],$$
$$
R_z([x, y, z]) = [n - y - 1, x, z].
$$

Definition 1. Let $A_n^3$ be a group generated by elements of $R \cup F_n$.

We prove that $A_n^3 = T_n^3$. The idea of the proof, that resembles a similar proof of Silver [12], is composed of two steps:

1. For any automorphism $T \in T_n^3$ we find an automorphism $A \in A_n^3$ such that $T \circ A$ fixes every point in a certain set $S$.
2. If an automorphism $T' \in T_n^3$ fixes every point in $S$ then $T'$ is the identity.

Hence, for every $T \in T_n^3$ we find an inverse element $T'$ such that $T'$ is composed only by elements of $R \cup F_n$, therefore $T \in A_n^3$. The proof of the second part is very similar to the proof for a general cube $n^d$. Thus, it is proved only for the general cube in the next section.

Theorem 5. For every $T \in T_n^3$ there exists $A \in A_n^3$, such that $T \circ A$ fixes all corners and every point of the line $\ell = \{[i, 0, 0] | i \in [n]\}$. 

Proof. First we find an automorphism $A' \in \mathbb{A}_3$ such that $T \circ A'$ fixes all corners. We start with the point $p_0 = [0, 0, 0]$. A point $T(p_0)$ has to be on a main diagonal (by Lemma 2). Without loss of generality $T(p_0) = [i, i, n-i-1]$. We take $f_x \in \mathbb{F}_n$ such that $\pi(i) = 0$, $\pi(0) = i$, $\pi(n-i-1) = n-1$, $\pi(n-1) = n-i-1$, and $\pi(k) = k$ otherwise. Therefore, $T \circ f_x(p_0)$ is a corner. Then we take $R_1 \in \mathbb{R}$ such that the automorphism $T_1 = T \circ f_x \circ R_1$ fixes $p_0$.

By Lemma 3 the line $T_1(\ell)$ must be mapped onto an edge $e$ such that $p_0 \in e$. If the corner $p_1 = [n-1, 0, 0]$ is fixed by $T_1$, we take $T_2 = T_1$. Otherwise it can be mapped onto $[0, n-1, 0]$ or $[0, 0, n-1]$. We take a rotation $R_2([x, y, z]) = [y, z, x]$ or $[z, x, y])$. Thus, the automorphism $T_2 = T_1 \circ R_2$ fixes corners $p_1$ and $p_0$. Note that $R_2([0, 0, 0]) = [0, 0, 0]$.

If a corner $p_2 = [0, n-1, 0]$ is fixed by $T_2$ we take $T_3 = T_2$. Otherwise it can be mapped only onto $[0, 0, n-1]$. We take a permutation $F_\sigma$, where $\sigma(i) = n-i-1$. Hence, $T_3 = T_2 \circ R_3 \circ F_\sigma$ fixes the points $p_0, p_1, p_2$ as follows. For $p_0$,

$$T_2 \circ R_3 \circ F_\sigma ([0, 0, 0]) = R_3 \circ F_\sigma ([0, 0, 0]) = F_\sigma ([0, n-1, n-1, n-1]) = [0, 0, 0].$$

For $p_1$,

$$T_2 \circ R_3 \circ F_\sigma ([n-1, 0, 0]) = R_3 \circ F_\sigma ([n-1, 0, 0]) = F_\sigma ([n-1, n-1, n-1]) = [n-1, 0, 0].$$

For $p_2$,

$$T_2 \circ R_3 \circ F_\sigma ([0, n-1, 0]) = R_3 \circ F_\sigma ([0, n-1, 0]) = F_\sigma ([0, 0, n-1]) = [0, n-1, 0].$$

A corner $p_3 = [0, 0, n-1]$ is fixed by $T_3$ automatically, because it is neighbor of $p_0$ and all others neighbors are already fixed. All other corners are fixed due to Lemma 4. The automorphism $T_3 = T \circ A'$ for some $A' \in \mathbb{A}_n^3$ fixes all corners of the cube $n^3$.

Now we find an automorphism $A$ such that $T \circ A$ fixes all corners and all points on the line $\ell$. The line $\ell$ is fixed by $T_3$ due to Observation 5. Let $k = \lfloor \frac{n}{2} \rfloor - 1$. We construct the automorphism $A$ by induction over $i \in \{0, \ldots, k\}$. We show that in a step $i$ an automorphism $Y_i$ fixes all corners and every point in a set

$$Q_i = \{[j, 0, 0], [n-j-1, 0, 0] | 0 \leq j \leq i\}.$$

First, let $i = 0$ and $Y_0 = T_3$. The automorphism $Y_0$ fixes all corners and $Q_0$ contains only $[0, 0, 0]$ and $[n-1, 0, 0]$, which are also corners. Suppose that $i > 0$. By induction hypothesis, we have an automorphism $Y_{i-1}$ which fixes all corners and every point in the set $Q_{i-1}$. If $Y_{i-1}([i, 0, 0]) = [i, 0, 0]$ then $Y_i = Y_{i-1}$. Otherwise $Y_{i-1}([i, 0, 0]) = [j, 0, 0]$. Note that $i < j < n-i-1$ because points from $Q_{i-1}$ are already fixed. Let us consider $F^*_x \in \mathbb{F}_n$ where $\pi(j) = i$, $\pi(i) = j$, $\pi(n-j-1) = n-i-1$, $\pi(n-i-1) = n-j-1$, and $\pi(k) = k$ otherwise. The automorphism $Y_i = Y_{i-1} \circ F^*_x$ fixes the following points:

1. All corners, as the automorphism $Y_{i-1}$ fixes all corners by the induction hypothesis and $\pi(0) = 0$ and $\pi(n-1) = n-1$. 

2. Set \( Q_{i-1} \), as the automorphism \( Y_{i-1} \) fixes the set \( Q_{i-1} \) by the induction hypothesis and \( \pi(k) = k \) for all \( k < i \) and \( k > n - i - 1 \).

3. Point \([i,0,0]: Y_{i-1} \circ F_3^i([i,0,0]) = F_3^i([j,0,0]) = [i,0,0] \).

4. Point \([n-i-1,0,0]\) by Lemma [1]\

Note that if \( n \) is odd a point \([\frac{n-1}{2},0,\ldots,0]\) is fixed as well by an automorphism \( Y_k \). Thus, the automorphism \( Y_k = T \circ A \) for some \( A \in A_n^d \) fixes all points of the line \( \ell \) and all corners of the cube. \( \square \)

4 Generators of the Group \( T_n^d \)

In this section we characterize the generators of the general group \( T_n^d \). As we stated in Section [1] we use the groups \( \mathbb{R}_d, \mathbb{F}_n \) and \( \mathbb{X} \).

Definition 2. Let \( A_n^d \) be a group generated by elements of \( \mathbb{R}_d \cup \mathbb{F}_n \cup \mathbb{X} \).

We prove that \( A_n^d = T_n^d \) in the same two steps as we proved \( A_n^3 = T_n^3 \).

1. For any automorphism \( T \in T_n^d \) we find an automorphism \( A \in A_n^d \), such that \( T \circ A \) fixes all corners of the cube \( n^d \) and one edge.

2. If an automorphism \( T' \in T_n^d \) fixes all corners and one edge then \( T' \) is identity.

Theorem 6. For all \( T \in T_n^d \) there exists \( A \in A_n^d \) such that \( T \circ A \) fixes every corner of the cube \( n^d \) and every point of a line \( \ell = \{[i,0,\ldots,0]|i \in [n]\} \).

Proof (Sketch). First we construct an automorphism \( A' \in A_n^d \) such that \( T \circ A' \) fixes all corners. We start with the point \( p_0 = [0,\ldots,0] \). By Lemma [2] the point \( T(p_0) \) has to be on a main diagonal. We choose \( F \in \mathbb{F}_n \) such that \( T \circ F(p_0) \) is a corner. Then, we choose \( R \in \mathbb{R}_d \) such that \( T \circ F \circ R(p_0) = p_0 \).

By induction over \( i \) we can construct automorphisms \( Z_i \) to fix the points \( p_0 \) and \( p_i = [0,\ldots,n-1,\ldots,0] \) for all \( i \in \{0,\ldots,d-2\} \). We start with the automorphism \( Z_0 = T \circ F \circ R \) and in a step \( i \) we compose the automorphism \( Z_{i-1} \) with a suitable rotation in \( \mathbb{R}^d \). If \( Z_{d-2} \) fixes \( p_{d-1} \), then \( Z_{d-1} = Z_{d-2} \). Otherwise \( p_{d-1} \) is mapped onto \( p_d \) and then \( Z_{d-1} = Z_{d-2} \circ X \), where \( X \in \mathbb{X} \) and \( X \neq Id \). Thus, the automorphism \( Z_{d-1} \) fixes all points of \( P_{d-1} \) and the corner \( p_d \) is fixed automatically because there is no other possibility where the corner \( p_d \) can be mapped. The automorphism \( Z_{d-1} \) fixes the corner \( p_0 = [0,\ldots,0] \) and all its neighbors. Therefore by Lemma [3] the automorphism \( Z_{d-1} = T \circ A' \) for some \( A' \in A_n^d \) fixes all corners of the cube.

The automorphism fixing points on the line \( \ell \) is constructed in the same way as in the proof of Theorem [3] We find an automorphism \( Y \) fixing all corners and points on the line \( \ell \) by induction. We start with the automorphism \( Z_{d-1} \). In step \( i \) of the induction we compose the automorphism from the step \( i-1 \) and an automorphism \( F_i \in \mathbb{F}_n \) which fixes points \([i,0,\ldots,0]\) and \([n-i-1,0,\ldots,0]\). \( \square \)
It remains to prove that if an automorphism \( T \in T^d_n \) fixes all corners and all points in the line \( \ell = \{ [i,0,\ldots,0] | i \in [n] \} \) then \( T \) is the identity. We prove it in two parts. First, we prove that if \( d = 2 \) then the automorphism \( T \) is the identity. Then, we prove it for a general dimension by an induction argument.

**Theorem 7.** Let an automorphism \( T \in T^d_n \) fixes all corners of the cube and all points in the line \( \ell = \{ [i,0,\ldots,0] | i \in [n] \} \). Then, the automorphism \( T \) is the identity.

**Proof.** Let \( d_1, d_2 \in \mathbb{L}(n^2) \). Thus, \( \text{type}(d_1) = (+, +) \) and \( \text{type}(d_2) = (+, -) \). Since all corners are fixed, the diagonals \( d_1 \) and \( d_2 \) are fixed as well due to Observation 3. Let \( p \in d_1 \cup d_2 \) such that \( p \) is not a corner. The point \( p \) is collinear with the only one point \( q \in \ell \) such that \( q \) is not a corner. Therefore, every point on the diagonals \( d_1 \) and \( d_2 \) is fixed.

Now we prove that every line in \( \mathbb{L}(n^2) \) is fixed. Let \( \ell_1 \in \mathbb{L}(n^2) \) be a line of a dimension 1. Suppose \( n \) is even. The line \( \ell_1 \) intersects the diagonals \( d_1 \) and \( d_2 \) in distinct points, which are fixed. Therefore, the line \( \ell_1 \) is fixed as well by Observation 3.

Now suppose \( n \) is odd. If \( \ell_1 \) does not contain the face center \( c_1 = \left[ \frac{n-1}{2}, \frac{n-1}{2}, 0, \ldots, 0 \right] \) then \( \ell \) is fixed by the same argument as in the previous case. Thus, suppose \( c_1 \in \ell_1 \). There are two lines \( \ell_2, \ell_3 \) in \( \mathbb{L}(n^2) \) of dimension 1 which contains \( c_1 \). Their types are \( \text{type}(\ell_2) = (\frac{n-1}{2}, +) \) and \( \text{type}(\ell_3) = (+, \frac{n-1}{2}) \). The line \( \ell_2 \) also intersects the line \( \ell \). Therefore, the lines contains two fixed points \( c_1 \) and \( \left[ \frac{n-1}{2}, 0 \right] \) and thus the line \( \ell_2 \) is fixed. The line \( \ell_3 \) is fixed as well because every other line is fixed. For better understanding of all lines and points used in the proof see Figure 2 with example of the cube 5^2.

![Fig. 2. Points and lines used in the proof of Theorem 7.](image)

Every point in 5^2 is fixed due to Observation 4 because every point is in an intersection of at least two fixed lines.

**Theorem 8.** Let an automorphism \( T \in T^d_n \) fix all corners of the cube \( n^d \) and all points of an arbitrary edge \( e \). Then, the automorphism \( T \) is the identity.

**Proof.** We prove the theorem by induction over dimension \( d \) of the cube \( n^d \). The basic case for \( d = 2 \) is Theorem 7.
Therefore, we can suppose \( d > 2 \) and the theorem holds for all dimensions smaller then \( d \). Without loss of generality, \( e = \{ [i, 0, \ldots, 0] | i \in [n] \} \). We consider the face \( F = \{ [x_1, \ldots, x_{d-1}, 0] | x_1, \ldots, x_{d-1} \in [n] \} \). The face \( F \) has a dimension \( d - 1 \) and \( e \subset F \). Therefore, all points of \( F \) are fixed by the induction hypothesis. Then we take all faces \( G \) of dimension \( d - 1 \) such that \( F \cap G \neq \emptyset \). Corners \( c \in G \) are fixed. There is at least one edge \( f \) such that \( f \subseteq F \cap G \). Therefore the points of \( f \) are also fixed and the points \( p \in G \) are fixed by the induction hypothesis. By this argument we show that every outer point is fixed. Every line \( \ell \in L(n^d) \) is fixed due to Observation 3 because every line contains at least two outer points. Therefore by Observation 4, every point \( q \in n^d \) is fixed because every point is an intersection of at least two lines. □

5 Order of the Group \( T_{n^d}^d \)

In the previous section we characterized the generators of the group \( T_{n^d}^d \). Now we compute the order of \( T_{n^d}^d \). First, we state several technical lemmas whose proofs are omitted in this conference paper.

**Lemma 5.** Orders of the basic groups are as follows.

1. \( |R_d| = 2d! \), \( |R_{d-1}| = 2^{d-1}d! \), \( |R_2| = 4 \)
2. \( |F_n| = \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (2 \lfloor \frac{n}{2} \rfloor - 2i) \)
3. \( |X| = 2 \)

**Lemma 6.** The groups \( R_d \) and \( F_n \) commute, and the groups \( X \) and \( F_n \) commute.

**Lemma 7.** Let \( X \in X \) such that \( X \neq Id \). Then, for all \( R_1 \in R_d \) there exists \( R_2 \in R_d \) such that \( R_1 \circ X = X \circ R_2 \).

By Lemma 5 and Lemma 7 we can conclude that any automorphism \( A \in T_{n^d}^d \) can be written as \( A = R \circ F \circ X \) where \( R \in R_d \), \( F \in F_n \) and \( X \in X \). Thus, the product \( R_d F_n X = \{ R \circ F \circ X | R \in R_d, F \in F_n, X \in X \} \) is exactly the group \( T_{n^d}^d \). We state the well-known product formula for a group product.

**Lemma 8 (Product formula [4]).** Let \( S \) and \( T \) be subgroups of a finite group \( G \). Then, for an order of a product \( ST \) holds that

\[
|ST| = \frac{|S| \cdot |T|}{|S \cap T|}.
\]

Thus, for computing the order of \( T_{n^d}^d \) we need to compute the orders of intersections of the basic groups \( R_d, F_n \) and \( X \).

**Lemma 9.** If \( d \) is odd, then \( R_d \cap F_n = \{ Id \} \). If \( d \) is even, then \( R_d \cap F_n = \{ Id, F_\sigma \} \) where \( \sigma(i) = n - i - 1 \).
Lemma 10. The group $\mathbb{X}$ can be generated by elements of the groups $\mathbb{R}_d$ and $\mathbb{F}_n$ if and only if $d$ is odd.

Theorem 9. The order of the group $\mathbb{T}_n^d$ is $|\mathbb{R}_d| \cdot |\mathbb{F}_n|$.

Proof. If $d$ is odd $|\mathbb{R}_d \cap \mathbb{F}_n| = 1$ due to Lemma 8. Moreover, the group $\mathbb{X}$ is a subset of $\mathbb{R}_d \mathbb{F}_n$ due to Lemma 10. Therefore, the group $\mathbb{T}_n^d$ is exactly a product $\mathbb{R}_d \mathbb{F}_n$ and the theorem holds by Lemma 8.

Now suppose $d$ is even. By Lemma 9 $|\mathbb{R}_d \cap \mathbb{F}_n| = 2$. Thus by Lemma 8 $|\mathbb{R}_d \mathbb{F}_n| = |\mathbb{R}_d| \cdot |\mathbb{F}_n|/2$. The order of the intersection $|\mathbb{X} \cap \mathbb{R}_d \mathbb{F}_n|$ is 1 by Lemma 10. Hence, $|\mathbb{T}_n^d| = 2|\mathbb{R}_d \mathbb{F}_n| = |\mathbb{R}_d| \cdot |\mathbb{F}_n|$.

As a corollary of Theorem 9 we get the second part of Theorem 1.

Corollary 1. Let $k = \lfloor \frac{d}{2} \rfloor$. Then, $|\mathbb{T}_n^d| = 2^{d-1+k}d!k!$.

Proof. By Theorem 9 the order $|\mathbb{T}_n^d|$ is $2^{d-1}d! \prod_{i=0}^{k-1} (2k - 2i)$ for $k = \lfloor \frac{d}{2} \rfloor$. There are $k$ even numbers from 2 to $2k$ in the product $\prod_{i=0}^{k-1} (2k - 2i)$. Therefore, it can be rewritten as $2^k k!$.

Corollary 2. The groups $\mathbb{T}_n^d$ and $\mathbb{T}_n^{d+1}$ are isomorphic for $k \geq 2$.

Proof. The rotation group for generating $\mathbb{T}_n^d$ and $\mathbb{T}_n^{d+1}$ is the same. For every permutation $\pi \in S_{2k+1}$ with the symmetry property holds that $\pi(k) = k$. Therefore, the group $\mathbb{F}_{2k}$ is isomorphic to the group $\mathbb{F}_{2k+1}$. Whether $\mathbb{X}$ is generated by $\mathbb{F}_n$ and $\mathbb{R}_d$ depends only on the dimension.

6 The Complexity of Colored Cube Isomorphism

In this section we prove Theorem 2. As we stated before, CHI is in GI. Therefore, Colored Cube Isomorphism as a subproblem of CHI is in GI as well. It remains to prove the problem is GI-hard.

First, we describe how we reduce the input of Graph Isomorphism to the input of Colored Cube Isomorphism. Let $G = (V, E)$ be a graph. Without loss of generality $V = \{0, \ldots, n-1\}$. We construct the coloring $s^G : [k]^2 \to [2], k = 2n + 4$ as follows. The value of $s^G([i, j])$ is 1 if $[i, j] = [n, n]$ or $[i, j] = [n, n+1]$ or $i, j \leq n - 1$ and $\{i, j\} \in E$. The value of $s^G(p)$ for all other point $p$ is 0. We can view the coloring $s^G$ as a matrix $M^G$ such that $M^G_{i,j} = s^G([i, j])$. The submatrix of $M^G$ consisting of the first $n$ rows and $n$ columns is exactly the adjacency matrix of the graph $G$.

The idea of the reduction is as follows. If two colorings $s^{G_1}, s^{G_2}$ are isomorphic via a cube automorphism $A \in \mathbb{T}_n^d$ then $A$ can be composed only of permutation automorphisms in $\mathbb{F}_k$. Moreover, if $A = F_\pi$ for some permutation $\pi$ then the permutation $\pi$ maps the numbers in $[n]$ to the numbers in $[n]$ and describes the isomorphism between the graphs $G_1$ and $G_2$. 


Lemma 11. Let $G_1, G_2$ be graphs without vertices of degree 0. If colorings $s^{G_1}, s^{G_2}$ are isomorphic via a cube automorphism $A$ then $A = F_x \in F_k$. Moreover, $\pi(i) \leq n - 1$ if and only if $i \leq n - 1$.

Proof (Sketch). Let $A = R \circ X \circ F$ where $R \in \mathbb{R}_2, X \in \mathcal{X}, F \in F_k$ and $m_1, m_2$ be main diagonals of $[k]^2$ of type $(+, +)$ and $(+, -)$, respectively. Since $G_1$ and $G_2$ are simple graphs without loops, there is exactly one point with color 1 on the main diagonal $m_1$ (on the point $[n, n]$) and 0 points with color 1 on the main diagonal $m_2$ in both colorings $s^{G_1}$ and $s^{G_2}$. Therefore, $A$ has to fix $m_1$ and $m_2$. Every automorphism in $F_k$ and in $\mathcal{X}$ fixes the lines $m_1$ and $m_2$. Thus, the automorphism $R$ has to fix the main diagonals. There are two automorphisms in $\mathbb{R}_2$ such that they fix the main diagonals—the identity and $R'([i, j]) = [k - i - 1, k - j - 1]$. The identity and $R'$ are also in $F_k$ (see Lemma 9). Therefore, the automorphism $A$ can be composed as $X \circ F_x, X \in \mathcal{X}, F_x \in F_k$.

Let us suppose that $X \neq Id$. Note that $A([n, n]) = [\pi(n), \pi(n)]$ and $A([n, n + 1]) = [\pi(n + 1), \pi(n)]$. The automorphism $A$ has to fix the point $[n, n]$, thus $\pi(n) = n$. There is exactly one point with color 1 on a line $\ell$ of type $(+, n)$ in both colorings $s^{G_1}$ and $s^{G_2}$—on the point $[n, n]$. However, the automorphism $A$ mapped two points with 1 ([n, n] and [n, n + 1]) to the line $\ell$ which is a contradiction and $X$ has to be the identity.

Now we prove the last part of the lemma. We already know that $\pi(n) = n$ and $\pi(n + 1) = n + 1$.

For every $i \leq n - 1$ there is at least one point with color 1 on a line of type $(+, i)$ in both colorings $s^{G_1}, s^{G_2}$ because graphs $G_1$ and $G_2$ do not contain any vertex of degree 0. On the other hand, for every $i \geq n + 2$ there are only points with color 0 on a line of type $(+, i)$ in both colorings. Therefore, if $i \leq n - 1$ then $i$ has to be mapped on $j \leq n - 1$ by $\pi$. \hfill \Box

The proof of the following theorem follows from Lemma 11.

Theorem 10. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs without vertices of degree 0. Then, the graphs $G_1$ and $G_2$ are isomorphic if and only if the colorings $s^{G_1}$ and $s^{G_2}$ are isomorphic.

We may suppose that inputs graphs $G_1$ and $G_2$ have minimum degree at least 1 for the purpose of the polynomial reduction of GRAPH ISOMORPHISM to COLORED CUBE ISOMORPHISM. Thus, Theorem 2 follows from Theorem 10.

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A Corners, Main Diagonals and Edges in $n^d$ automorphisms

In this section we investigate how every automorphism $T \in T_n^d$ maps main diagonals, edges and corners. We restate and prove the technical lemmas from Section 3. First we prove Lemma 1.

**Lemma 1 (stated in Section 3)** Let $F = \{ [x, y, 0, \ldots, 0] | x, y \in [n] \}$ be a face of $n^d$, and let an automorphism $T \in T_n^d$ fixes all 4 corners of $F$, i.e., points $[0, \ldots, 0]$, $[n - 1, 0, \ldots, 0]$, $[0, n - 1, 0, \ldots, 0]$ and $[n - 1, n - 1, 0, \ldots, 0]$. Then, if $T$ fixes a point $[i, 0, \ldots, 0]$, $i \in [n]$ it also fixes a point $[n - i - 1, 0, \ldots, 0]$.

**Proof.** The automorphism $T$ fixes all 4 corners of $F$, therefore by Observation 3, it fixes both diagonals $d_1, d_2 \subset F$. Types of $d_1$ and $d_2$ are type($d_1$) = $(+, +, 0, \ldots, 0)$ and type($d_2$) = $(+, -, 0, \ldots, 0)$.

Suppose that $T$ fixes a point $p = [i, 0, \ldots, 0]$, where $i \in \{1, \ldots, n - 2\}$ (corners are already fixed). In three steps we show that the point $p_2 = [n - i - 1, 0, \ldots, 0]$ is fixed (note that for $i = n - i - 1$ the proof is trivial). Fixed points in a face $7 \times 7$ are depicted in Figure 3.

**Fig. 3.** How to fix points by diagonals in a front 2-dimensional face.

First we show that $p_1 = [i, i, 0, \ldots, 0]$ is fixed by $T$. A point $T(p_1)$ must be on $d_1$ and it must be collinear with $p$. There are 2 points collinear with $p$ on $d_1$: $[i, i, 0, \ldots, 0]$ and $[0, 0, \ldots, 0]$, but the second one is already fixed as a corner. Therefore, the point $p_1$ is fixed. A point $p_2 = [i, n - i - 1, 0, \ldots, 0] \in d_2$ is fixed by a similar argument.

Next we show that $T$ fixes a point $p_3 = [n - i - 1, i, 0, \ldots, 0]$. A point $T(p_3)$ must be on $d_2$ and it must be collinear with $p_1$. If $n$ is even there are two points on $d_2$ collinear with $p_1$: $p_2$ and $p_3$, but $p_2$ is fixed due to step 1. If $n$ is odd, there are 3 points collinear with $p_1$: $p_2, p_3$ and the face center $c_1 = \frac{n - 1}{2}, \frac{n - 1}{2}, 0, \ldots, 0$. However, the point $c_1$ is fixed due to Observation 4 because it is an intersection
of the lines $d_1$ and $d_2$. Therefore, the point $p_4 = [n - i - 1, n - i - 1, 0, \ldots, 0]$ is fixed by a similar argument.

Let $\ell_1$ be a line such that $type(\ell_1) = (n - i - 1, +, 0, \ldots, 0)$ and $\ell_2$ be a line such that $type(\ell_2) = (+, 0, \ldots, 0)$. Both lines $\ell_1$ and $\ell_2$ are fixed because $p_4, p_3 \in \ell_1$ and $\ell_2$ connects two fixed corners. Therefore, the point $p_5$, which is an intersection of $\ell_1$ and $\ell_2$, is fixed by Observation 3 as well. □

For proving the next lemma from Section 3, we use the following notions. Let $p = [x_1, \ldots, x_d]$. We call a set $B_j(p) = \{i \in \{1, \ldots, d\} | x_i = j \lor x_i = n - j - 1\}$ as $j$-block of $p$. Note that $j$-block and $(n - j - 1)$-block are the same set. We say a line $\ell$ such that $p \in \ell$ is active in the $j$-block $B_j(p)$ if there exists some $i \in B_j(p)$ such that $type(\ell)_i \in \{+,-\}$. For example $p = [0,0,1,3]$ be a point of the cube $4^4$ then $0$-block of $p$ is the set $\{1,2,4\}$ and a line $\ell \in L(4^4)$ of type $type(\ell) = (+,0,2,-)$ is active in $B_0(p)$. We consider only non-empty blocks. Thus, we say that the point $p$ from the example has only blocks $B_0(p)$ and $B_1(p) = \{3\}$. For $n$ odd, we denote $\gamma = \frac{n-1}{2}$ and the center of the cube $n^d$ is the point $c = [\gamma, \ldots, \gamma]$.

Lemma 12. Let $p$ be a point of $n^d$ and $\ell$ be a line such that $p \in \ell$. Then, there is exactly one $j \in [n]$ such that $\ell$ is active in $B_j(p)$.

Proof. It is clear that there is at least one $j \in [n]$ such that $\ell$ is active in $B_j(p)$. Suppose $\ell$ is active in $B_i(p)$ and $B_j(p)$, $i \neq j$. Therefore, $p$ has some coordinates equal to $i$ and some coordinates equal to $j$. Without loss of generality $p = [i, j, \ldots]$. Since $i \neq j$, $type(\ell)_i \neq (+, +, \ldots)$. Thus, $type(\ell) = (+, - , \ldots)$. However, it means that $j = n - i - 1$ and $B_i(p) = B_{n-i-1}(p)$.

Lemma 13. Let $p = [x_1, \ldots, x_d]$ be a point of $n^d$ and it has a block $B_j(p)$ for $j \neq \gamma$. Then, there is $2^k - 1$ lines active in $B_j(p)$ where $k = |B_j(p)|$.

Proof. Every $J \subseteq B_j(p), J \neq \emptyset$ defines an active line $\ell_J$ in $B_j(p)$, such that $p \in \ell_J$ and for $i \in \{1, \ldots, d\}$,

$$type(\ell_J)_i = \begin{cases} x_i, & i \notin J, \\ +, & i \in J \text{ and } x_i = j, \\ -, & i \in J \text{ and } x_i = n - j - 1. \end{cases}$$

For example,

$$p = [n-j-1, j, \ldots, j, x_{k+1}, \ldots, x_d] \text{ and } J = \{1, 2\}$$

the line $\ell_J$ has a type

$$type(\ell_J) = (-, +, j, \ldots, j, x_{k+1}, \ldots, x_d).$$

On the other hand, every line $\ell$ active in $B_j(p)$ defines a non-empty subset of $B_j(p)$ as coordinates where $\ell$ has non-constant coordinate sequences. Therefore, the number of lines active in $B_j(p)$ is the number of non-empty subsets of $B_j(p)$, which is $2^k - 1$. □
Lemma 14. Let \( p = [x_1, \ldots, x_d] \) be a point of \( n^d \) and it has block \( B_\gamma(p) \). Then, there is \( \frac{3^k-1}{2} \) lines active in \( B_\gamma(p) \) where \( k = |B_\gamma(p)| \).

Proof. The proof is very similar to the previous one. Every \( J \subseteq B_\gamma(p), J \neq \emptyset \) defines a line \( \ell_J \) such that \( p \in \ell_J \) and for \( i \in \{1, \ldots, d\} \)

\[
\text{type}(\ell_J)_i = \begin{cases} x_i & j \notin J, \\ + & i \in J. \end{cases}
\]

However, every \( K \subseteq J \) defines a line \( \ell'_JK \) such that \( p \in \ell'_JK \), and for \( j \in \{1, \ldots, d\} \),

\[
\text{type}(\ell'_JK)_j = \begin{cases} \text{type}(\ell_J)_j & j \notin K, \\ - & j \in K. \end{cases}
\]

For example, \( p = [\gamma, \ldots, \gamma, x_{k+1}, \ldots, x_d] \) and \( J = \{1, 2, 3\}, K = \{3\} \)
the line \( \ell'_JK \) has a type

\[
\text{type}(\ell'_JK) = (+, +, -, \gamma, \ldots, \gamma, x_{k+1}, \ldots, x_d).
\]

For fixed \( J, K \) and \( M = J \setminus K \) the lines \( \ell'_JK \) and \( \ell'_JM \) are the same. Again every line active in \( B_\gamma(p) \) defines two pairs of the set \( J, K \subseteq B_\gamma(p) \). Therefore, the numbers of lines active in \( B_\gamma(p) \) is a half of the number of pairs \( (J, K) \) such that \( J \) is a non-empty subset of \( B_\gamma(p) \) and \( K \) is a subset of \( J \). We have \( \sum_{m=1}^{k} \binom{k}{m} \) choices for the set \( J \). For fixed \( J \) of size \( m \), we have \( 2^m \) choices for \( K \subseteq J \). Therefore the number of these lines is

\[
\frac{1}{2} \sum_{m=1}^{k} \binom{k}{m} 2^m = \frac{3^k-1}{2}.\]

\[\square\]

Lemma 15. Let \( n \) be odd and \( \ell \in \mathbb{L}(n^d) \) such that the cube center \( c \) is in \( \ell \) and \( \dim(\ell) = k \). Let \( p \in \ell, p \neq c \). Then, \( \deg(p) = 2k - 1 + \frac{\sqrt{3^k+1}}{2} \).

Proof. Since \( p \in \ell \) and \( p \neq c \), the point \( p \) has to have exactly 2 blocks \( B_j(p) \) and \( B_\gamma(p) \). Note that \( |B_j(p)| = k \). Thus, the point \( p \) is incident with \( 2^k - 1 \) lines active in \( B_j(p) \) and with \( \frac{\sqrt{3^k+1}}{2} \) lines active in \( B_\gamma(p) \) (by Lemma [14] and Lemma [13]). By Lemma [12] the lines active in \( B_j(p) \) are disjoint from the lines active in \( B_\gamma(p) \) and there is no other lines incident with \( p \). \[\square\]

Now we are ready to prove Lemma [2]

Lemma 2 (stated in Section [3]). Every automorphism \( T \in \mathbb{T}_n \) maps a main diagonal \( m \in \mathbb{L}_m(n^d) \) onto a main diagonal \( m' \in \mathbb{L}_m(n^d) \).
Proof. Every point on a main diagonal has only one block. For $n$ even the proof is trivial. For every point $q \in n$ it holds that any of blocks of $q$ is not the $\gamma$-block. Therefore, every point $p \in m$ has degree $2^d - 1$ and any point which is not in any main diagonal has at least two blocks and thus the degree at most $2^{d-1}$ (by Lemma \[15\]). Every automorphism $T \in T^n$ has to preserve the point degree. Thus, a point $p \in m$ has to be mapped onto a point $p' \in m'$.

Now we prove the lemma for $n$ odd. The center of the cube $c$ is always mapped onto $c$ ($c$ is the only point of degree $2d - 1$). Therefore, the main diagonal $m \in L_m(n^d)$ has to be mapped onto a line $\ell \in L(n^d)$ such that $c \in \ell$. By Lemma \[15\], we know the degree of a non-central point $p \in \ell$ is $\deg(p) = 2k - 1 + \frac{3^{d-k}-1}{2}$.

The degree of a non-central point $q \neq c$ on a main diagonal $m \in L_m(n^d)$ is $\deg(q) = 2^d - 1$. We show that if $k \neq d$ then $2^k - 1 + \frac{3^{d-k}-1}{2} \neq 2^d - 1$. For contradiction let us suppose that $2^d - 2^k = \frac{3^{d-k}-1}{2}$ and $k < d$. We rewrite the formula into binary numbers:

$$
\begin{array}{c}
2^d & 1 & 0 & \ldots & 0 \\
-2^k & -1 & 0 & \ldots & 0 \\
\frac{3^{d-k}-1}{2} & 1 & \ldots & 0 & 0
\end{array}
= \beta > 0.
$$

It is easy to prove by induction that $4$ divides $3^{d-k} - 1$ if and only if $d - k$ is even. The number $\beta$ must be even so $d - k$ must be even as well. We use the well-known divisibility-by-3 test in the binary system for $\delta = 2\beta + 1$ (it should be equal to $3^{d-k} > 1$). The binary number is divisible by 3 if and only if the number $E$ of even order digits and the number $O$ of odd order digits are equal modulo 3. Note that

$$
\delta = 1\ldots10\ldots01.
$$

The number $d - k$ is even, thus the numbers of digits of the orders 1 to $d$ are equal, but $|E - O| = 1$ (because of the 1 at the order 0). Therefore $\delta$ is not divisible by 3, which is the contradiction.

Lemma 3 (stated in Section 3) Let $T \in T_n$, $e$ be an edge and $p$ be a corner, such that $p \in e$. If the corner $p$ is fixed by $T$, then $T(e) = e'$ is an edge such that $p \in e'$.

Proof. Without loss of generality the corner $p$ is $[0, \ldots, 0]$ and the type of $e$ is $type(e) = (+, 0, \ldots, 0)$. First we prove the lemma for odd $n$. Let $c_1$ be the center of $e$ i.e., the point $[\gamma, 0, \ldots, 0]$. Note that $c_1$ is collinear with the cube center $c$. Thus, the point $T(c_1)$ has to be also collinear with the cube center. Hence by Lemma \[15\] we can compute the degree of $c_1$ and $T(c_1)$. The function $2^k - 1 + \frac{3^{d-k}-1}{2}$ is increasing and it uniquely defines the sizes of blocks of the
point \( T(c_1) \). Thus, the point \( T(c_1) \) has a block \( B_1 \) of size 1 and a block \( B_0 \) of size \( d - 1 \). Therefore, the line \( c' \) is an edge.

We now complete the proof for \( n \) even. For a contradiction suppose that \( \dim(e') \geq 2 \). Without loss of generality the type of \( e' \) is \((+, \ldots, +, 0, \ldots, 0)\). Let \( p_1 = [1, 0, \ldots, 0] \) and \( p_2 = [2, 0, \ldots, 0] \). Since \( n \geq 4 \), the points \( p_1 \) and \( p_2 \) are not corners. Therefore, the point \( p_i \) (for \( i \in \{1, 2\} \)) has blocks \( B_i(p_i) = \{1\} \) and \( B_0(p_1) = \{2, \ldots, d\} \). Let \( L_1 \) be a set of lines incident with \( p_1 \) without the edge \( e \) and similarly \( L_2 \) be a set of lines incident with \( p_2 \) without \( e \). Note that lines in \( L_i \) (for \( i \in \{1, 2\} \)) can be active only in the block \( B_0(p_i) \). Let \( \ell_1 \in L_1 \) and \( \ell_2 \in L_2 \). For \( \ell_1 \) holds that \( \text{type}(\ell_1)_1 = 1 \) and for \( \ell_2 \) holds that \( \text{type}(\ell_2)_1 = 2 \). Therefore, the lines \( \ell_1 \) and \( \ell_2 \) cannot intersect.

Now take images of \( p_1 \) and \( p_2 \). Let \( q_1 = T(p_1) = [i, \ldots, i, 0, \ldots, 0] \) and \( q_2 = T(p_2) = [j, \ldots, j, 0, \ldots, 0] \). Since \( \dim(e') \geq 2 \), the point \( q_1 \) is incident with a line \( k_1 \) such that \( \text{type}(k_1) = (+, i, \ldots, i, 0, \ldots, 0) \). Similarly, the point \( q_2 \) is incident with a line \( k_2 \) such that \( \text{type}(k_2) = (j, +, \ldots, +, 0, \ldots, 0) \). The lines \( k_1 \) and \( k_2 \) have to be images of some lines in \( L_1 \) and \( L_2 \), respectively. However, the lines \( k_1 \) and \( k_2 \) intersect in a point \([j, i, \ldots, i, 0, \ldots, 0]\), which is the contradiction. \( \square \)

At the end we prove Lemma 4

**Lemma 4 (stated in Section 3)** If an automorphism \( T \in \mathcal{W}_n^d \) fixes the corner \([0, \ldots, 0]\) and all its neighbors, then \( T \) fixes all corners of the cube \( n^d \).

**Proof.** We prove the automorphism \( T \) fixes all corners \( p = [x_1, \ldots, x_d] \) by induction over

\[ k(p) = |\{i \in [n], x_i = n - 1\}|. \]

By the assumption, the automorphism \( T \) fixes corners \( p \) such that \( k(p) \in \{0, 1\} \). Without loss of generality, a corner \( q \) such that \( k(q) > 1 \) has coordinates

\[ q = [n - 1, \ldots, n - 1, 0, \ldots, 0]. \]

We take neighbors \( q_1, q_2 \) of the corner \( q \) as

\[ q_1 = [n - 1, \ldots, n - 1, 0, \ldots, 0], \]

\[ q_2 = [0, n - 1, \ldots, n - 1, 0, \ldots, 0]. \]

The corners \( q_1 \) and \( q_2 \) have two common neighbors: \( q \) and

\[ q_3 = [0, n - 1, \ldots, n - 1, 0, \ldots, 0]. \]

Corner \( q_1 \), \( q_2 \), and \( q_3 \) are fixed by the induction hypothesis. Therefore, corner \( q \) is also fixed as it must be the neighbor of \( q_1 \) and \( q_2 \). \( \square \)
B Relationships between Basic Groups

In this section we restate and prove lemmas from Section 5.

Lemma 5 (stated in Section 5) Orders of the basic groups are as follows.

1. $|R_d| = 2d|R_{d-1}| = 2^{d-1}d!$, $|R_2| = 4
2. If $n$ is even, then
   
   $$|F_n| = \prod_{i=0}^{n-1} (n - 2i).$$
   
   If $n$ is odd, then
   
   $$|F_n| = |F_{n-1}| = \prod_{i=0}^{n-2} ((n - 1) - 2i).$$
   
   The general formula is
   
   $$|F_n| = \prod_{i=0}^{\lfloor n/2 \rfloor - 1} (2\lfloor n/2 \rfloor - 2i).$$
   
3. $|X| = 2$

Proof. Size of the hypercube rotation group $R_d$ is well known \[8\]. The first equality can be deduced as follows. Hypercube $n^d$ has $2d$ faces of dimension $d - 1$ and each $(d - 1)$-dimensional face can be rotated to the front. Every $(d - 1)$-dimensional face of the cube $n^d$ has $|R_{d-1}|$ rotations. The second equality can be easily proved by induction.

Size of the group $F_n$ is the number of permutations $\pi \in S_n$ with the symmetry property. If $n$ is even, we have $n$ possibilities how to choose the image of the first element, we have $n - 2$ possibilities for the second element, and so on. If $n$ is odd, the element $n - \frac{n-1}{2}$ has to be mapped onto itself. Therefore, the order of $F_n$ where $n$ is odd is the same as the order of $F_{n-1}$.

Size of $X$ is clearly 2. □

Lemma 6 (stated in Section 5) The groups $R_d$ and $F_n$ commute, and the groups $X$ and $F_n$ commute.

Proof. It suffices to prove the lemma for rotations which generates $R_d$. Let $R_{ij} \in R_d$ and $F_\pi \in F_n$. Then,

$$R_{ij} \circ F_\pi ([x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d]) = F_\pi ([x_1, \ldots, n - x_j - 1, \ldots, x_i, \ldots, x_d])$$

$$= [\pi(x_1), \ldots, \pi(n - x_j - 1), \ldots, \pi(x_i), \ldots, \pi(x_d)]$$

$$= [\pi(x_1), \ldots, n - \pi(x_j) - 1, \ldots, \pi(x_i), \ldots, \pi(x_d)].$$
Similarly,

\[ F_n \circ R_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) \]
\[ = R_{ij}(\pi(x_1), \ldots, \pi(x_i), \ldots, \pi(x_j), \ldots, \pi(x_d)) \]
\[ = \pi(x_1), \ldots, n - \pi(x_j) - 1, \ldots, \pi(x_i), \ldots, \pi(x_d)] \]

The proof that \( X \) and \( F_n \) commute is analogous.

\[ \square \]

**Lemma 7 (stated in Section 5)** Let \( X \in \mathbb{K} \) such that \( X \neq \text{Id} \). Then for all \( R_1, R_2 \in \mathbb{R}_d \), such that \( R_1 \circ X = X \circ R_2 \).

**Proof.** It is sufficient to prove the lemma for the rotations \( R_{ij} \). If \( \{i, j\} \cap \{d - 1, d\} = \emptyset \) then the proof is obvious and \( R_{ij} \circ X = X \circ R_{ij} \). Let us suppose \( \{|i, j\} \cap \{d - 1, d\} \} = 1 \) and \( i = d - 1 \) (proofs of other cases are similar):

\[ R_{d-1,j} \circ X([x_1, \ldots, x_j, \ldots, x_{d-1}, x_d]) = [x_1, \ldots, x_{d-1}, \ldots, x_d, n - x_j - 1] \]
\[ = X \circ R_{dj}([x_1, \ldots, x_j, \ldots, x_{d-1}, x_d]) \]

Let us suppose the last possibility \( i = d - 1 \) and \( j = d \):

\[ R_{d-1,d} \circ X([c_1, \ldots, c_{d-1}, c_d]) = [c_1, \ldots, c_{d-1}, n - c_d - 1] \]
\[ = X \circ R_{d-1,d}([c_1, \ldots, c_{d-1}, c_d]) \]

\[ \square \]

Each rotation \( R \) permutes the coordinates of points and some coordinates are “flipped” to the opposite value, i.e., changed the value \( i \) to \( n - i - 1 \). Thus, every rotation \( R \in \mathbb{R}_d \) can be viewed as a permutation \( \pi_R \in \mathbb{S}_d \) on coordinates and characteristic vector \( \chi^R \in \{0, 1\}^d \) where the coordinates are flipped. Let \( \text{flip} : [n] \times \{0, 1\} \rightarrow [n] \) be a function such that

\[ \text{flip}(i, b) = \begin{cases} i & b = 0 \\ n - i - 1 & b = 1 \end{cases} \]

A rotation \( R \) can be written as follows

\[ R([x_1, \ldots, x_d]) = [\text{flip}(x_{\pi_R(1)}, \chi^R_1), \ldots, \text{flip}(x_{\pi_R(d)}, \chi^R_d)] \]

For example a rotation \( R(x, y, z) = (y, n - x - 1, z) \) can be represented by \( \pi^R = (2, 1, 3) \) and \( \chi^R = (0, 1, 0) \).

A flip value \( f(R) \) of a rotation \( R \) is a number of 1 in the vector \( \chi^R \). Note that for \( R = R_1 \circ R_2 \) holds that \( \pi^R = \pi^{R_1} \circ \pi^{R_2} \) and

\[ f(R_1) + f(R_2) \equiv f(R) \mod 2. \]

In the following lemmas we also use a permutation \( \sigma \in \mathbb{S}_n \) which is defined as \( \sigma(i) = n - i - 1 \). Note that we can represent automorphisms in \( \mathbb{K} \) and the automorphism \( F_\sigma \) by a permutation in \( \mathbb{S}_d \) and characteristic vector in \( \{0, 1\}^d \) in the same way as we can represent rotations.
After composing these rotations we get $\sigma$. Proof. Every rotation preserves an order of points on the line $\ell = \{[i, \ldots, i] | i \in [n]\}$. There are two permutation automorphisms, which preserve the order on the line $\ell$: identity and $F_\sigma$.

If $d$ is odd then $F_\sigma \not\in \mathbb{R}_d$. For a contradiction, let us suppose that $F_\sigma \in \mathbb{R}_d$. For every rotation $R_{ij}$, the permutation $\pi_{R_{ij}} \in \mathbb{S}_d$ is a transposition and $\pi_{F_\sigma}$ is the identity. Therefore, the automorphism $F_\sigma$ must be composed of even number of $R_{ij}$. Every $R_{ij}$ has the flip value exactly one. Hence, the rotation $F_\sigma$ must have even flip value, which is a contradiction as $d$ is odd and $F_\sigma$ has an odd flip value.

If $d$ is even, then $F_\sigma \in \mathbb{R}_d$. We composed $F_\sigma$ as follows: for each pair $\{i, j\}$ such that $i \in \{1, \ldots, d\}$ is odd and $j = i + 1$, we use the rotation

$$R_{ij} \circ R_{ij} = R_{ij}^2([x_1, \ldots, x_i, x_{i+1}, \ldots x_d]) = [x_1, \ldots, x_i, x_{i+1}, \ldots x_d].$$

When we compose all these rotations we get the automorphism $F_\sigma$. We know that there is no other automorphism in $\mathbb{R}_d \cap \mathbb{F}_n$. □

**Lemma 10 (stated in Section 5)** The group $\mathbb{X}$ can be generated by elements of the groups $\mathbb{R}_d$ and $\mathbb{F}_n$ if and only if $d$ is odd.

**Proof.** Let $X \in \mathbb{X}$. The case $X = \text{Id}$ is trivial. Further suppose that

$$X([x_1, \ldots, x_{d-1}, x_d]) = [x_1, \ldots, x_d, x_{d-1}].$$

If $d$ is odd we use the following rotations.

1. $R_{d,d-1}([x_1, \ldots, x_{d-1}, x_d]) = [x_1, \ldots, x_d, n-x_{d-1}-1]$.
2. Note that $d-1$ is even. We consider pairs $\{i, j\}$ such that $i \in \{1, \ldots, d-1\}$ is odd and $j = i + 1$. For each such pair $\{i, j\}$ we use the rotation $R_{ij}^2$ (like in the proof of Lemma 9).

After composing these rotations we get

$$\overline{R([x_1, \ldots, x_{d-1}, x_d])} = [n-x_1-1, \ldots, n-x_{d-1}-1, n-x_{d-1}-1].$$

We use $F_\sigma$ and get the automorphism $X = \overline{R} \circ F_\sigma$.

If $d$ is even, we use a similar argument as in the proof of Lemma 9. Every automorphism generated by elements from $\mathbb{R}_d$ and $\mathbb{F}_n$ can be written as $F \circ R$ where $R \in \mathbb{R}_d$, $F \in \mathbb{F}_n$. For a contradiction, let us suppose that $X = F \circ R$. First we prove that we can use only permutation automorphism $F_\lambda$ such that $\lambda(i) \in \{i, n-i-1\}$ for every $i \in [n]$. Suppose that $X = F_\rho \circ R$ and there exists $i$ such that $\rho(i) = j, j \not\in \{i, n-i-1\}$. Then, $X([i, \ldots, i]) = [i, \ldots, i]$ and

$$F_\rho \circ R([i, \ldots, i]) = R([j, \ldots, j])$$

$$= [\text{flip}(j, \lambda^R_{1}), \ldots, \text{flip}(j, \lambda^R_{d})].$$
However, the value \( \text{flip}(j, \chi_R^k) \) is \( j \) or \( n - j - 1 \) and both values are not equal to \( i \).

Further, we prove that \( \lambda \) is equal to \( \sigma \) or the identity. For a contradiction, let us suppose that there exists \( i, j \) such that \( \lambda(i) = i \) and \( \lambda(j) = n - j - 1 \). We consider two points \( p_1 = [i, \ldots, i, j] \) and \( p_2 = [j, \ldots, j, i] \). Thus,

\[
X(p_1) = [i, \ldots, i, j, i]
\]

\[
F_{\lambda} \circ R(p_1) = R([i, \ldots, i, n - j - 1])
\]

We can conclude that \( \pi_R(d) = d - 1 \) and \( \chi_R = (0, \ldots, 0, 1) \). However, \( X(p_2) = [j, \ldots, j, i, j] \) and the point

\[
F_{\lambda} \circ R(p_2) = R([n - j - 1, \ldots, n - j - 1, i])
\]

has a value \( n - i - 1 \) on the \((d - 1)\)-th coordinate, which is the contradiction.

Suppose that \( X = F \circ R \) such that \( F \) is \( F_{\sigma} \) or the identity. Thus, the automorphism \( F \) has an even flip value (0 or \( d \)). Permutation of coordinates \( \pi_X \in \mathbb{S}_d \) is a transposition and permutation \( \pi_F \) is the identity. Therefore, the rotation \( R \) must be composed of odd number of \( R_{ij} \) rotations and \( R \) has odd flip value. Thus, the automorphism \( F \circ R \) has odd flip value, which is the contradiction as the flip value of \( X \) is 0. \( \square \)

C Polynomial Reduction

First we state an example of a reduction for a path \( P \) on 3 vertices. Then we prove Theorem 10. Let a matrix \( M \) be defined as

\[
M_{i,j} = s_P([i, j])
\]

Then

\[
M^G = \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Theorem 10 (stated in Section 6) Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs without vertices of degree 0. Then, the graphs \( G_1 \) and \( G_2 \) are isomorphic if and only if the colorings \( s^{G_1} \) and \( s^{G_2} \) are automorphic.

Proof. Let sets \( V_1, V_2 \) be \([n]\). By Lemma 11 we know that \( s^{G_1} \) and \( s^{G_2} \) are isomorphic via a cube automorphism \( F_{\pi} \in \mathbb{F}_k \). We define an isomorphism \( f : V_1 \rightarrow V_2 \) as \( f(i) = \pi(i) \). By Lemma 11 the function \( f \) is a well defined bijection \([n] \rightarrow [n]\). It remains to prove that \( f \) is a graph isomorphism:

\[\{i, j\} \in E_1 \Leftrightarrow s^{G_1}([i, j]) = 1 \Leftrightarrow s^{G_2}([\pi(i), \pi(j)]) = 1 \Leftrightarrow \{f(i), f(j)\} \in E_2.\]
Now we prove the other implication. Let \( f : V(G_1) \to V(G_2) \) be an isomorphism between \( G_1 \) and \( G_2 \). We construct a permutation \( \pi : [k] \to [k] \) as follows:

\[
\pi(i) = \begin{cases} 
  i & n \leq i \leq n + 1 \\
  f(i) & i \leq n - 1 
\end{cases}
\]

We define values of \( \pi(i) \) for \( i \geq n + 2 \) in such a way the symmetric property would hold for the permutation \( \pi \).

We prove that \( F_\pi \in \mathbb{F}_k \) is an isomorphism between \( s^{G_1} \) and \( s^{G_2} \). Let us suppose that \( s^{G_1}([i, j]) = 1 \). If \( [i, j] = [n, n] \) or \( [i, j] = [n, n+1] \) then \( s^{G_2}(F_\pi([i, j])) = 1 \) as well. Otherwise, \( i, j \leq n - 1 \) because there is not any other point colored by 1. Thus,

\[
s^{G_1}([i, j]) = 1 \iff \{i, j\} \in E_1 \iff \{f(i), f(j)\} \in E_2 \iff s^{G_2}([\pi(i), \pi(j)]) = 1.
\]

Hence, we proved that \( s^{G_1}([i, j]) = 1 \) if and only if \( s^{G_2}(F_\pi[i, j]) = 1 \). \( \square \)