The Subelliptic Heat Kernel of the Octonionic Anti-De Sitter Fibration

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Received July 31, 2020, in final form January 29, 2021; Published online February 10, 2021
https://doi.org/10.3842/SIGMA.2021.014

Abstract. In this note, we study the sub-Laplacian of the 15-dimensional octonionic anti-de Sitter space which is obtained by lifting with respect to the anti-de Sitter fibration the Laplacian of the octonionic hyperbolic space $\mathbb{O}H^1$. We also obtain two integral representations for the corresponding subelliptic heat kernel.

Key words: sub-Laplacian; 15-dimensional octonionic anti-de Sitter space; the anti-de Sitter fibration

2020 Mathematics Subject Classification: 58J35; 53C17

1 Introduction and results

In this note we study the sub-Laplacian and the corresponding sub-Riemannian heat kernel of the octonionic anti-de Sitter fibration

$$S^7 \hookrightarrow \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1.$$ 

This paper follows the previous works [2, 3, 10] which respectively concerned:

1. The complex anti-de Sitter fibrations:

$$S^1 \hookrightarrow \text{AdS}^{2n+1}(\mathbb{C}) \rightarrow \mathbb{C}H^n.$$ 

2. The quaternionic anti-de Sitter fibrations:

$$S^3 \hookrightarrow \text{AdS}^{4n+3}(\mathbb{H}) \rightarrow \mathbb{H}H^n.$$ 

The 15-dimensional anti-de Sitter fibration is the last model space that remained to be studied of a sub-Riemannian manifold arising from a $H$-type semi-Riemannian submersion over a rank-one symmetric space, see the Table 3 in [4].

Similarly to the complex and quaternionic case, the sub-Laplacian is defined as the lift on $\text{AdS}^{15}(\mathbb{O})$ of the Laplace–Beltrami operator of the octonionic hyperbolic space $\mathbb{O}H^1$. However, in the complex and quaternionic case the Lie group structure of the fiber played an important role that we can not use here, since the fiber $S^7$ is not a group. Instead, we make use of some algebraic properties of $S^7$ that were already pointed out and used by the authors in [1] for the study of the octonionic Hopf fibration:

$$S^7 \hookrightarrow S^{15} \rightarrow \mathbb{O}P^1.$$ 

Let us briefly describe our main results. Due to the cylindrical symmetries of the fibration, the heat kernel of the sub-Laplacian only depends on two variables: the variable $r$ which is the
Riemannian distance on $O H^1$ (the starting point is specified with inhomogeneous coordinate in Section 3) and the variable $\eta$ which is the Riemannian distance starting at a pole on the fiber $S^7$. We prove in Proposition 3.1 that in these coordinates, the radial part of the sub-Laplacian $\tilde{L}$ writes

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right).$$

As a consequence of this expression for the sub-Laplacian, we are able to derive two equivalent formulas for the heat kernel. The first formula, see Proposition 4.1, reads as follows: for $r \geq 0$, $\eta \in [0, \pi)$, $t > 0$

$$p_t(r, \eta) = \int_0^\infty s_t(\eta, i u) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,$$

where $s_t$ is the heat kernel of the Jacobi operator

$$\tilde{\triangle}_{S^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}$$

with respect to the measure $\sin^6 \eta \, d\eta$, and where $q_{t,15}$ is the Riemannian heat kernel on the 15-dimensional real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). The second formula, see Proposition 4.2, writes as follows:

$$p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi,$$

where $q_{t,9}$ is Riemannian heat kernel on the 9-dimensional hyperbolic space $\mathbb{H}^{9}$ and $G_t(\eta, \varphi, u)$ is given in (4.3).

Similarly to [2, 3, 10], it might be expected that explicit integral representations of the heat kernel might be used to study small-time asymptotics, inside and outside of the cut-locus. Integral representations of heat kernels can also be used to obtain sharp heat kernel estimates, see [7]. Those applications of the heat kernel representations we obtain will possibly be addressed in a future research project.

2 The octonionic anti-de Sitter fibration

Let

$$\mathbb{O} = \left\{ x = \sum_{j=0}^7 x_j e_j, x_j \in \mathbb{R} \right\},$$

be the division algebra of octonions (see [9] for explicit representations of this algebra). We recall that the multiplication rules are given by

$$e_i e_j = e_j \quad \text{if } i = 0,$$

$$e_i e_j = e_i \quad \text{if } j = 0,$$

$$e_i e_j = -\delta_{ij} e_0 + \epsilon_{ijk} e_k \quad \text{otherwise},$$

where $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ijk}$ is the completely antisymmetric tensor with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$ (also see [1]). The octonionic norm is defined for $x \in \mathbb{O}$ by

$$||x||^2 = \sum_{j=0}^7 x_j^2.$$
The octonionic anti-de Sitter space $\text{AdS}^{15}(\mathbb{O})$ is the quadric defined as the pseudo-hyperbolic space by:

$$\text{AdS}^{15}(\mathbb{O}) = \{(x, y) \in \mathbb{O}^2, \| (x, y) \|^2_{\mathbb{O}} = -1 \},$$

where

$$\| (x, y) \|^2_{\mathbb{O}} := \| x \|^2 - \| y \|^2.$$

In real coordinates we have $x = \sum_{j=0}^7 x_j e_j$, $y = \sum_{j=0}^7 y_j e_j$, and the pseudo-norm can be written as

$$x_0^2 + \cdots + x_7^2 - y_0^2 - \cdots - y_7^2.$$

As such, $\text{AdS}^{15}(\mathbb{O})$ is embedded in the flat 16-dimensional space $\mathbb{R}^{8,8}$ endowed with the Lorentzian real signature $(8,8)$ metric

$$ds^2 = dx_0^2 + \cdots + dx_7^2 - dy_0^2 - \cdots - dy_7^2.$$  

Consequently, $\text{AdS}^{15}(\mathbb{O})$ is naturally endowed with a pseudo-Riemannian structure of signature $(8,7)$.

Let $\mathbb{O}H^1$ denote the octonionic hyperbolic space. The map $\pi: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$, given by $(x, y) \mapsto [x : y] = y^{-1}x$ is a pseudo-Riemannian submersion with totally geodesic fibers isometric to the seven-dimensional sphere $\mathbb{S}^7$. Notice that, as a topological manifold, $\mathbb{O}H^1$ can therefore be identified with the unit open ball in $\mathbb{O}$. The pseudo-Riemannian submersion $\pi$ yields the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1.$$

For further information on semi-Riemannian submersions over rank-one symmetric spaces, we refer to [6].

3 Cylindrical coordinates and radial part of the sub-Laplacian

The sub-Laplacian $L$ on $\text{AdS}^{15}(\mathbb{O})$ we are interested in is the horizontal Laplacian of the Riemannian submersion $\pi: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$, i.e., the horizontal lift of the Laplace–Beltrami operator of $\mathbb{O}H^1$. It can be written as

$$L = \Box_{\text{AdS}^{15}(\mathbb{O})} + \triangle_{\mathbb{S}^7}, \quad (3.1)$$

where $\Box_{\text{AdS}^{15}(\mathbb{O})}$ is the d’Alembertian, i.e., the Laplace–Beltrami operator of the pseudo-Riemannian metric and $\triangle_{\mathbb{S}^7}$ is the vertical Laplacian. Since the fibers of $\pi$ are totally geodesic and isometric to $\mathbb{S}^7 \subset \text{AdS}^{15}(\mathbb{O})$, we note that $\Box_{\text{AdS}^{15}(\mathbb{O})}$ and $\triangle_{\mathbb{S}^7}$ are commuting operators, and we can identify

$$\triangle_{\mathbb{S}^7} = \triangle_{\mathbb{S}^7}. \quad (3.2)$$

The sub-Laplacian $L$ is associated with a canonical sub-Riemannian structure on $\text{AdS}^{15}(\mathbb{O})$ which is of $H$-type, see [4].

To study $L$, we introduce a set of coordinates that reflect the cylindrical symmetries of the octonionic unit sphere which provides an explicit local trivialization of the octonionic anti-de Sitter fibration. Consider the coordinates $w \in \mathbb{O}H^1$, where $w$ is the inhomogeneous coordinate on $\mathbb{O}H^1$ given by $w = y^{-1}x$, with $x, y \in \text{AdS}^{15}(\mathbb{O})$. Consider the north pole $p \in \mathbb{S}^7$ and take
Let us denote $\exp_p$ the Riemannian exponential map at $p$ on $S^7$. Then the cylindrical coordinates we work with are given by

$$(w, \theta_1, \ldots, \theta_7) \mapsto \left(\frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i\right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i\right)}{\sqrt{1 - \rho^2}}\right) \in \text{AdS}^{15}(\mathbb{O}),$$

where $\rho = \|w\|$ and $\|\theta\| = \sqrt{\theta_1^2 + \cdots + \theta_7^2} < \pi$.

A function $f$ on $\text{AdS}^{15}(\mathbb{O})$ is called radial cylindrical if it only depends on the two coordinates $(\rho, \eta) \in [0, 1) \times [0, \pi]$ where $\eta = \sqrt{\sum_{i=1}^7 \theta_i^2}$. More precisely $f$ is radial cylindrical if there exists a function $g$ so that

$$f \left(\frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i\right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i\right)}{\sqrt{1 - \rho^2}}\right) = g(\rho, \eta).$$

We denote by $\mathcal{D}$ the space of smooth and compactly supported functions on $[0, 1) \times [0, \pi)$. Then the radial part of $L$ is defined as the operator $\tilde{L}$ such that for any $f \in \mathcal{D}$, we have

$$L(f \circ \psi) = (\tilde{L} f) \circ \psi. \quad (3.3)$$

We now compute $\tilde{L}$ in cylindrical coordinates.

**Proposition 3.1.** The radial part of the sub-Laplacian on $\text{AdS}^{15}(\mathbb{O})$ is given in the coordinates $(r, \eta)$ by the operator

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}\right),$$

where $r = \tanh^{-1} \rho$ is the Riemannian distance on $\mathbb{O}H^1$ from the origin.

**Proof.** Note that the radial part of the Laplace–Beltrami operator on the octonionic hyperbolic space $\mathbb{O}H^1$ is

$$\tilde{\Delta}_{\mathbb{O}H^1} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r},$$

and the radial part of the Laplace–Beltrami operator on $S^7$ is

$$\tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}. \quad (3.4)$$

Since the octonionic anti-de Sitter fibration defines a totally geodesic submersion with base space $\mathbb{O}H^1$ and fiber $S^7$, the semi-Riemannian metric on $\text{AdS}^{15}(\mathbb{O})$ is locally given by a warped product between the Riemannian metric of $\mathbb{O}H^1$ and the Riemannian metric on $S^7$. Hence the radial part of the d’Alembertian becomes

$$\tilde{\Box}_{\text{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + g(r) \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}\right), \quad (3.5)$$

for some smooth function $g$ to be computed.

On the other hand, from the isometric embedding $\text{AdS}^{15}(\mathbb{O}) \subset \mathbb{O} \times \mathbb{O}$, the d’Alembertian on $\text{AdS}^{15}(\mathbb{O})$ is a restriction of the d’Alembertian on $\mathbb{O} \times \mathbb{O} \simeq \mathbb{R}^{8,8}$ in the sense that for a smooth $f: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{R}$

$$\Box_{\text{AdS}^{15}(\mathbb{O})} f = \Box_{\mathbb{O} \times \mathbb{O}} f |_{\text{AdS}^{15}(\mathbb{O})},$$
where $\square_{\mathbb{O} \times \mathbb{O}} = \sum_{i=0}^{7} \left( \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2} \right)$ and for $x, y \in \mathbb{O}$ such that $\|y\|^2 - \|x\|^2 > 0$, $f^*(x, y) = f\left(\frac{x}{\sqrt{\|y\|^2 - \|x\|^2}}, \frac{y}{\sqrt{\|y\|^2 - \|x\|^2}}\right)$. For the specific choice of the function $f(x, y) = y_1$, one easily computes that $\square_{\mathbb{O} \times \mathbb{O}} f^*_\text{AdS}^{15}(\mathbb{O})(x, y) = 15y_1$, thus
\[
\square_{\text{AdS}^{15}(\mathbb{O})} f(x, y) = 15y_1.
\]

For the point with coordinates
\[
\left( \exp_p \left( \sum_{i=1}^{7} \theta_i Y_i \right) w, \exp_p \left( \sum_{i=1}^{7} \theta_i Y_i \right) \right) \in \text{AdS}^{15}(\mathbb{O})
\]
one has
\[
y_1 = \frac{\cos \eta}{\sqrt{1 - \rho^2}} = \cosh r \cos \eta.
\]
We therefore deduce that
\[
\square_{\text{AdS}^{15}(\mathbb{O})}(\cosh r \cos \eta) = 15 \cosh r \cos \eta.
\]
Using the formula (3.5), after a straightforward computation, this yields $g(r) = -\frac{1}{\cosh^2 r}$ and therefore
\[
\square_{\text{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - \frac{1}{\cosh^2 r} \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right)
\]
\[= \Delta_{\text{O}H^1} - \frac{1}{\cosh^2 r} \tilde{\Delta}_{S^7}.
\]
Finally, to conclude, one notes that the sub-Laplacian $L$ is given by the difference between the Laplace–Beltrami operator of $\text{AdS}^{15}(\mathbb{O})$ and the vertical Laplacian. Therefore by (3.1) and (3.2),
\[
\tilde{L} = \square_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right).
\]

**Remark 3.2.** As a consequence of the previous result, we can check that the Riemannian measure of $\text{AdS}^{15}(\mathbb{O})$ in the coordinates $(r, \eta)$, which is the symmetric and invariant measure for $\tilde{L}$, is given by
\[
d\mu = \pi^7 \frac{9}{80} \sinh^7 r \cos^7 r \sin^6 \eta \, dr \, d\eta.
\]
(See also Remark 2 in [1], which corresponds to the case of the octonionic Hopf fibration.)

### 4 Integral representations of the subelliptic heat kernel

In this section, we give two integral representations of the subelliptic heat kernel associated with $\tilde{L}$. We denote by $p_t(r, \eta)$ the heat kernel of $\tilde{L}$ issued from the point $r = \eta = 0$ with respect to the measure (3.6). We remark that studying the subelliptic heat kernel associated with $\tilde{L}$ is enough to study the heat kernel of $L$, because due to (3.3) the heat kernel $h_t(w, \theta)$ of $L$ issued from the point with cylindric coordinates $w = 0, \theta = 0$ is then given by
\[
h_t(w, \theta) = p_t\left( \tanh^{-1} \|w\|, \|\theta\| \right).
\]
4.1 First integral representation

We denote by \( s_t \) the heat kernel of the operator

\[
\tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}
\]

with respect to the reference measure \( \sin^6 \eta \, d\eta \). The operator \( \tilde{\Delta}_{S^7} \) belongs to the family of Jacobi diffusion operators which have been extensively studied in the literature, see for instance the appendix in [5] and the references therein. In particular, the spectrum of \( \tilde{\Delta}_{S^7} \) is given by

\[
\text{Sp}(-\tilde{\Delta}_{S^7}) = \{ m(m + 6), \, m \in \mathbb{N} \},
\]

and the eigenfunction corresponding to the eigenvalue \( m(m + 6) \) is \( P^5_{m/2,5/2}(\cos \eta) \) where \( P^5_{m/2,5/2}(\cdot) \) is the Jacobi polynomial

\[
P^5_{m/2,5/2}(x) = \frac{(-1)^m}{2^m m! (1 - x^2)^{5/2}} \, dx^m (1 - x^2)^{5/2 + m}.
\]

As a consequence, one has the following spectral decomposition for the heat kernel:

\[
s_t(\eta, u) = \frac{1}{\pi} \sum_{m=0}^{+\infty} \frac{2^{4m+7} m!(m+5)!((m+3)!)^2}{(2m+6)!(2m+5)!} e^{-m(m+6)t} P^5_{m/2,5/2}(\cos \eta) P^5_{m/2,5/2}(\cos u).
\]

**Proposition 4.1.** For \( r \geq 0, \, \eta \in [0, \pi], \) and \( t > 0 \) we have

\[
p_t(r, \eta) = \int_0^\infty s_t(\eta, u) q_{r,15}(\cosh r \cosh u) \sinh^6 u \, du,
\]

where

\[
q_{r,15}(\cosh s) := \frac{e^{-49t}}{(2\pi)^7 \sqrt{4\pi t}} \left( -\frac{1}{\sinh s \, ds} \right)^7 e^{-s^2/4t}
\]

is the Riemannian heat kernel on the 15-dimensional real hyperbolic space \( \mathbb{H}^{15} \).

**Proof.** Since \( \pi: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1 \) is a (semi-Riemannian) totally geodesic submersion, the operators \( \tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} \) and \( \tilde{\Delta}_{S^7} \) commute. Thus

\[
e^{t\tilde{L}} = e^{t(\tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_{S^7})} = e^{t\tilde{\Delta}_{S^7}} e^{t\tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})}}.
\]

We deduce that the heat kernel of \( \tilde{L} \) can be written as

\[
p_t(r, \eta) = \int_0^\pi s_t(\eta, u) p_t^{\text{AdS}^{15}(\mathbb{O})}(r, u) \sinh^6 u \, du,
\]

where \( s_t \) is the heat kernel of (3.4) with respect to the measure \( \sin^6 \eta \, d\eta, \, \eta \in [0, \pi] \), and \( p_t^{\text{AdS}^{15}(\mathbb{O})}(r, u) \) the heat kernel at \( (0,0) \) of \( \tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} \) with respect to the measure in (3.6), i.e.,

\[
d\mu(r, u) = \frac{\pi^7}{90} \sinh^7 r \cosh^7 r \sinh^6 u \, dr \, du, \quad r \in [0, \infty), \quad u \in [0, \pi].
\]

In order to write (4.2) more precisely, let us consider the analytic change of variables \( \tau: (r, \eta) \to (r, i\eta) \) that will be applied on functions of the type \( f(r, \eta) = h(r) e^{-i\lambda \eta} \), with \( h \) smooth and
compact supported on \([0, \infty)\) and \(\lambda > 0\). Then as we saw in the proof of Proposition 3.1 one can see that
\[
\tilde{\triangle}_{AdS^{15}(\mathcal{O})} (f \circ \tau) = (\tilde{\triangle}_{\mathbb{H}^{15}} f) \circ \tau,
\]
where
\[
\tilde{\triangle}_{\mathbb{H}^{15}} = \tilde{\triangle}_{\mathbb{O}^{11}} + \frac{1}{\cosh^2 r} \tilde{\triangle}_p, \quad \tilde{\triangle}_p = \frac{\partial^2}{\partial \eta^2} + 6 \coth \eta \frac{\partial}{\partial \eta}.
\]
Then, one deduces
\[
e^{t \tilde{L}}(f \circ \tau) = e^{t \tilde{\triangle}_{\mathbb{O}^{11}}} e^{t \tilde{\triangle}_{AdS^{15}(\mathcal{O})}} (f \circ \tau) = e^{t \tilde{\triangle}_{\mathbb{O}^{11}} ( (e^{t \tilde{\triangle}_{\mathbb{H}^{15}} f) \circ \tau) = (e^{-t \tilde{\triangle}_p} e^{t \tilde{\triangle}_{\mathbb{H}^{15}} f) \circ \tau}.
\]
Now, since for every \(f(r, \eta) = h(r) e^{-i \lambda \eta}\),
\[
(e^{t \tilde{\triangle}_{AdS^{15}(\mathcal{O})}} f)(0, 0) = (e^{t \tilde{\triangle}_{\mathbb{H}^{15}} f)(0, 0),
\]
once deduces that for a function \(h\) depending only on \(u\),
\[
\int_0^\pi h(u)p_t^{\tilde{\triangle}_{AdS^{15}(\mathcal{O})}}(r, u) \sin^6 u \, du = \int_0^\infty h(-iu) q_{t, 15}(\cosh r \cosh u) \sin^6 u \, du.
\]
Therefore, coming back to (4.2), one infers that using the analytic extension of \(s_t\) one must have
\[
\int_0^\pi s_t(\eta, u)p_t^{\tilde{\triangle}_{AdS^{15}(\mathcal{O})}}(r, u) \sin^6 u \, du = \int_0^\infty s_t(\eta, -iu) q_{t, 15}(\cosh r \cosh u) \sin^6 u \, du,
\]
where \(q_{t, 15}\) is the Riemannian heat kernel on the real hyperbolic space \(\mathbb{H}^{15}\) given in (4.1).

4.2 Second integral representation

**Proposition 4.2.** For \(r \geq 0, \eta \in [0, \pi]\), and \(t > 0\) we have
\[
p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t, 9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi.
\]
where \(q_{t, 9}\) is the 9-dimensional Riemannian heat kernel on the hyperbolic space \(\mathbb{H}^9\):
\[
q_{t, 9}(\cosh s) := \frac{e^{-16t}}{(2\pi)^4 \sqrt{4\pi t}} \left( \frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/4t},
\]
and
\[
G_t(\eta, \varphi, u) = \frac{15}{8} \sum_{m \geq 0} e^{-(m(m+6)+33)t}(\cos \eta + i \sin \eta \cos \varphi)^m \cosh((m+3)u).
\]

**Proof.** The strategy of the following method appeals to some results proved in [8]. Firstly, we decompose the subelliptic heat kernel in the \(\eta\) variable with respect to the basis of normalized eigenfunctions of \(\tilde{\triangle}_S^\tau = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}\). Accordingly,
\[
p_t(r, \eta) = \sum_{m \geq 0} f_m(t, r) h_m(\eta),
\]
where for each \(m\), \(h_m\) is given by
\[
h_m(\eta) = \frac{15}{16} \int_0^\pi (\cos \eta + i \sin \eta \cos \varphi)^m \sin^5 \varphi \, d\varphi.
\]
and \( f_m(t, \cdot) \) solves the following heat equation
\[
\frac{\partial}{\partial t} f_m(t, r) = \left( \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - m(m + 6) \tanh^2 r \right) f_m(t, r)
\]
\[
= \left( \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + m(m + 6) \cosh^2 r - m(m + 6) \right) f_m(t, r).
\]
We consider then the operator
\[
L_m := \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \frac{m(m + 6)}{\cosh^2 r} + 49,
\]
which was studied in [8, p. 229]. From [8, Theorem 2], with \( \alpha = 3 + \frac{m}{2}, \beta = -\frac{m}{2} \), we deduce that the solution to the wave Cauchy problem associated with the subelliptic Laplacian is given \( f \in C_0^\infty(\mathbb{O}H^1) \) by
\[
\cos \left( s \sqrt{-L_m} \right)(f)(w) = \frac{-\sinh s}{(2\pi)^4} \left( \frac{1}{\sinh s} \right) \frac{d}{ds} \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - ||y||^2)^8},
\]
where
\[
K_m(s, w, y) = \frac{(1 - (w, y)^2)^{3+m/2}}{(1 - (w, y)^2)^{m/2}} \frac{1}{\cosh^3(d(w, y)) \sqrt{\cosh^2(s) - \cosh^2(d(w, y))}} \times \binom{3}{-m-3} \frac{1}{2} \frac{\cosh(d(w, y)) - \cosh(s)}{2 \cosh(d(w, y))},
\]
where \( \binom{3}{-m-3} \) is the Gauss hypergeometric function and \( dy \) stands for the Lebesgue measure in \( \mathbb{R}^8 \). Using the spectral formula
\[
e^{tL} = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)} \cos \left( s \sqrt{-L} \right) ds,
\]
which holds for any non positive self-adjoint operator, we deduce that the solution to the heat Cauchy problem associated with \( L_m \):
\[
e^{tL_m}(f)(w) = \frac{e^{-m(m+6)t-7t^2}}{\sqrt{4\pi t}(2\pi)^4} \int_{\mathbb{R}} ds (-\sinh s) e^{-s^2/(4t)}
\]
\[
\times \left( \frac{1}{\sinh s} \right) \frac{d}{ds} \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - ||y||^2)^8}.
\]
Performing integration by parts 4-times,
\[
\int_{\mathbb{R}} ds (-\sinh s) \left( \frac{1}{\sinh s} \right) \frac{d}{ds} \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - ||y||^2)^8}
\]
\[
= \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^8} \int_{\mathbb{R}} ds (-\sinh s) K_m(s, w, y) \left( \frac{1}{\sinh s} \right) \frac{d}{ds} e^{-s^2/4t}
\]
\[
= 2 \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^8} \int_{d(w,y)}^\infty d(cosh(s)) K_m(s, w, y) \left( \frac{1}{\sinh s} \right) \frac{d}{ds} e^{-s^2/4t}.
\]
Thus we get
\[
e^{tL_m}(f)(0) = 2e^{-(m(m+6)+33)t} \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^8} \int_{d(0,y)}^\infty d(cosh(s)) K_m(s, 0, y) q_{t,9}(cosh s).
As a result, the subelliptic heat kernel of \( L_m \) reads

\[
\frac{dy}{(1-||y||^2)^8} \int_{d(0,y)}^{\infty} d(cosh \ s)K_m(s,0,y)q_{t,9}(cosh s)
\]

\[
= dr \sinh^7 r \cosh^7 r \int_{r}^{\infty} d(cosh \ s)K_m(s,0,y)q_{t,9}(cosh s).
\]

By changing the variable \( \cosh s = \cosh r \cosh u \) for \( u \geq 0 \), the last expression becomes

\[
dr \sinh^7 r \cosh^7 r \int_{0}^{\infty} 2F_1 \left( m + 3, -m - 3, -\frac{1}{2}; \frac{1 - \cosh u}{2} \right) q_{t,9}(cosh r \cosh u) \, du.
\]

Therefore \( p_t(r,\eta) \) has the integral representation

\[
2 \sum_{m \geq 0} e^{-\frac{(m(m+6)+33)t}{2}} h_m(\eta) \int_{0}^{\infty} 2F_1 \left( m + 3, -m - 3, -\frac{1}{2}; \frac{1 - \cosh u}{2} \right) q_{t,9}(cosh r \cosh u) \, du.
\]

Now, notice that \( 2F_1 \left( m + 3, -m - 3, -\frac{1}{2}; \frac{1 - \cosh u}{2} \right) \) is simply the Chebyshev polynomial of the first kind

\[
T_{m+3}(x) = 2F_1 \left( m + 3, -m - 3, -\frac{1}{2}; \frac{1 - \cosh u}{2} \right),
\]

for all \( x \in \mathbb{C} \). Therefore, one has

\[
2F_1 \left( m + 3, -m - 3, -\frac{1}{2}; \frac{1 - \cosh u}{2} \right) = T_{m+3}(\cosh u) = \cosh((m + 3)u),
\]

and the proof is over.

\[\blacksquare\]

Acknowledgements

F.B. is partially funded by the NSF grant DMS-1901315.

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