Fermionic vacuum polarization around a cosmic string in compactified AdS spacetime

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Received September 20, 2021
Accepted November 27, 2021
Published January 4, 2022
Abstract. We investigate topological effects of a cosmic string and compactification of a spatial dimension on the vacuum expectation value (VEV) of the energy-momentum tensor for a fermionic field in (4+1)-dimensional locally AdS spacetime. The contribution induced by the compactification is explicitly extracted by using the Abel-Plana summation formula. The mean energy-momentum tensor is diagonal and the vacuum stresses along the direction perpendicular to the AdS boundary and along the cosmic string are equal to the energy density. All the components are even periodic functions of the magnetic fluxes inside the string core and enclosed by compact dimension, with the period equal to the flux quantum. The vacuum energy density can be either positive or negative, depending on the values of the parameters and the distance from the string. The topological contributions in the VEV of the energy-momentum tensor vanish on the AdS boundary. Near the string the effects of compactification and gravitational field are weak and the leading term in the asymptotic expansion coincides with the corresponding VEV in (4+1)-dimensional Minkowski spacetime. At large distances, the decay of the cosmic string induced contribution in the vacuum energy-momentum tensor, as a function of the proper distance from the string, follows a power law. For a cosmic string in the Minkowski bulk and for massive fields the corresponding fall off is exponential. Within the framework of the AdS/CFT correspondence, the geometry for conformal field theory on the AdS boundary corresponds to the standard cosmic string in (3+1)-dimensional Minkowski spacetime compactified along its axis.

Keywords: Cosmic strings, domain walls, monopoles, cosmology with extra dimensions, cosmological applications of theories with extra dimensions, particle physics - cosmology connection

ArXiv ePrint: 2109.04505
1 Introduction

The formation of topological defects is among the interesting implications of symmetry breaking phase transitions \cite{1, 2}. This phenomenon plays an important role in cosmology, in astrophysics and in a wide range of condensed-matter systems. Within the framework of unified gauge theories of particle interactions a sequence of phase transitions, accompanied by the formation of a variety of topologically stable structures, is predicted in the early Universe. Depending on the type of symmetry that is broken, the corresponding defects can be cosmic strings, domain walls, monopoles and textures. The investigation of physical effects induced by those defects is an important issue to understand the early Universe. The corresponding observable consequences provide a link between high-energy particle physics and cosmology. Among other topological defects, the astrophysical and cosmological implications of cosmic strings are most thoroughly studied. The early interest was motivated by the possibility for cosmic strings to seed perturbations into the energy density of the early Universe that can serve as sources for the formation of the large-scale structures \cite{3}. The spectrum of primordial density perturbations is encoded in statistical characteristics of the cosmic microwave background (CMB) radiation and recent precision measurements of the CMB temperature anisotropy have excluded the cosmic strings as the main source for perturbations. However, the cosmic strings may have effects on the CMB properties that include the creation of small non-Gaussianities and influence on the tensor modes. Among the other interesting effects sourced by cosmic strings we mention here the gravitational lensing, the radiation of gravitational waves, and the generation of gamma ray bursts and high-energy cosmic rays. As it has been discussed recently in \cite{4}, cosmic string loops may seed black holes with a continuous range of masses. This mechanism is especially interesting in the case of intermediate mass black holes. An alternative mechanism for the formation of cosmic string-type linear structures has been suggested in brane inflation model within the framework of fundamental string theory (for reviews see \cite{5}--\cite{8}).

The tension of cosmic string is proportional to the square of the symmetry breaking energy scale and the corresponding energy density is localized inside the core with the radius
determined by the Compton wavelength of the fields originating the cosmic string (complex scalar and gauge fields in Abelian Higgs theory). Away from the core the geometry around a straight cosmic string is well approximated by conical spacetime with planar angle deficit determined by the string tension. Despite the fact that this geometry is flat, the corresponding nontrivial topology gives rise to a number of interesting physical effects. In particular, the vacuum fluctuations of quantum fields are modified and, as a consequence, the vacuum expectation values (VEVs) of local physical characteristics receive topological contributions. Among those characteristics the VEV of the energy-momentum tensor is of special interest. In addition to describing the distribution of the energy density and vacuum stresses around the cosmic string, it appears as a source of gravity in the semiclassical Einstein equations and determines the backreaction of quantum effects on the gravitational field. The VEV of the energy-momentum tensors for the electromagnetic field and for massless scalar and fermionic fields in the idealized geometry of a straight cosmic string with zero thickness core has been investigated in [9]–[13]. Massive scalar and fermionic fields have been discussed in [14]–[24]. The presence of boundaries induces additional contributions in the VEV of the energy-momentum tensor (the Casimir effect). The combined effects of topology and of various types of boundaries in the geometry of a cosmic string have been discussed in [25]–[33]. The compactification of cosmic string along its axis gives rise to topological Casimir contributions in the vacuum energy-momentum tensor [34]–[36]. For cosmic strings in curved backgrounds additional vacuum polarization is induced by gravitational fields. Quantum effects near cosmic strings in the Schwarzschild geometry have been studied in [37]–[39]. The VEVs of the energy momentum tensor for scalar, fermionic and electromagnetic fields around cosmic strings in de Sitter and anti-de Sitter (AdS) spacetimes were investigated in [40]–[46]. It has been shown that the gravitational field essentially changes the behavior of the vacuum densities at distances from the string larger than the curvature radius of the background spacetime.

Continuing our previous study of the combined effects of cosmic string, background gravitational field and compactification on the local properties of the vacuum state, in the present paper we investigate the vacuum energy-momentum tensor for a massive fermionic field in background of (4+1)-dimensional locally AdS spacetime with a compactified dimension and in the presence of a cosmic string. The VEV of the current density and the fermion condensate in the same geometry have been considered recently in [47, 48]. The effects induced by a brane parallel to the AdS boundary were discussed in [49]. Our choice of the AdS spacetime as a background geometry is motivated by its crucial role in braneworld models with extra dimensions and in AdS/CFT correspondence (for reviews see [50–52]). It is also important that the AdS spacetime is maximally symmetric and closed analytic expressions can be obtained for physical characteristics of the vacuum state. The corresponding results may shed light on the influence of gravitational fields on quantum matter in more complicated geometries.

The organization of the paper is as follows. The problem setup is presented in the next section. The VEV of the energy-momentum tensor in the geometry of a straight cosmic string in background of AdS spacetime is studied in section 3. The contribution in the vacuum energy-momentum tensor induced by compactification of a spatial dimension is considered in section 4. The main results of the paper are summarized in section 5. In appendix A, an alternative representation for the topological contribution in the VEV of the energy-momentum tensor is provided.
2 Problem setup

In the presence of an external gauge field with the vector potential $A_\mu$ and on a spacetime background described by the metric tensor $g_{\mu\nu}$, the quantum fermionic field $\psi(x)$ obeys the equation

$$(i\gamma^\mu \mathcal{D}_\mu - sm) \psi(x) = 0, \quad \mathcal{D}_\mu = \partial_\mu + \Gamma_\mu + ieA_\mu, \tag{2.1}$$

where the curved spacetime Dirac matrices $\gamma^\mu$ form the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\Gamma_\mu$ is the spin connection. The matrices $\gamma^\mu$ are related to the corresponding flat spacetime Dirac matrices $\gamma^{(a)}$ through the vielbein fields $e^\mu_{\ (a)}$: $\gamma^\mu = e^\mu_{\ (a)}\gamma^{(a)}$. The spin connection is expressed in terms of those fields as $\Gamma_\mu = \gamma^{(a)}\gamma^{(b)}e^\mu_{\ (a)}\nabla_\mu e^{(b)\nu}/4$, where $\nabla_\mu e^{(b)\nu}$ stands for the covariant derivative of the vector field with the covariant components determined by the index $\nu$. The parameter $s$ takes the values $s = +1$ and $s = -1$ and corresponds to the two irreducible representations of the Clifford algebra in odd-dimensional spacetimes (this point will be discussed in detail below).

The background geometry under consideration in the present paper is described by the $(4+1)$-dimensional line element

$$ds^2 = e^{-2y/a} \left( dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \right) - dy^2, \tag{2.2}$$

with the spacetime coordinates $x^\mu = (t, r, \phi, y, z)$ varying in the ranges $-\infty < t, y < +\infty$, $0 \leq r < \infty$, and $0 \leq \phi \leq \phi_0$. Introducing a new coordinate $w = ae^{y/a}$, $0 \leq w < \infty$, the line element is presented in the conformally-flat form (for $r > 0$) with the metric tensor $g_{\mu\nu} = (a/w)^2 \text{diag}(1, -1, -r^2, -1, -1)$. For the $z$-coordinate we will consider two cases. In the first one $-\infty < z < +\infty$ and for the special value $\phi_0 = 2\pi$, the line element (2.2) corresponds to the slice of the AdS spacetime described in Poincaré coordinates. For the corresponding cosmological constant one has $\Lambda = -6/a^2$ and the Ricci tensor is expressed as $R_{\mu\nu} = -4g_{\mu\nu}/a^2$. For $\phi_0 < 2\pi$ the line element (2.2) generalizes the idealized geometry of a cosmic string for $(4+1)$-dimensional background AdS spacetime. Although the surface $r = 0$ corresponds to a two-dimensional spatial surface with the induced line element $dy^2 + e^{-2y/a}dz^2$, in the discussion below we will use the term cosmic string. The $(3+1)$-dimensional geometry of a cosmic string on the AdS bulk (see [53–55]) corresponds to the hypersurface $z = \text{const}$. The geometry of another hypersurface with $y = \text{const}$ corresponds to the standard geometry of a cosmic string in $(3+1)$-dimensional Minkowski spacetime. In particular, this also applies to the background geometry for the conformal field theory within the framework of the AdS/CFT correspondence. For $r > 0$ the local geometry corresponding to (2.2) coincides with that for the AdS spacetime. The second case corresponds to the $z$-direction compactified on a circle and it will be discussed in section 4.

We are interested in the effects of the topology change, induced by the cosmic string and by compactification, on the VEV of the energy-momentum tensor $\langle 0|T_{\mu\nu}|0 \rangle = \langle T_{\mu\nu} \rangle$ for the fermionic field $\psi(x)$. For the gauge field, a simple configuration $A_\mu = (0, 0, A_2, 0, A_4)$ will be considered, where the covariant components $A_2$ and $A_4$ are constants. The component $A_2$ is expressed in terms of the magnetic flux $\Phi_s$ running along the string’s core as $A_2 = -q\Phi_s/(2\pi)$. In the problem with uncompactified $z$-direction the component $A_4$ will not appear in the expressions for the VEVs. That component is physically relevant for the geometry with compact $z$-coordinate.
The VEV of the energy-momentum tensor can be evaluated by using the mode-sum formula
\[ \langle T_{\mu\nu} \rangle = -\frac{i}{4} \sum_{\sigma} \sum_{\chi=\pm} \chi \left[ \overline{\psi}_{\sigma}^{(\chi)} \gamma_{(\mu} \mathcal{D}_{\nu)} \psi_{\sigma}^{(\chi)} - (\mathcal{D}_{(\mu} \overline{\psi}_{\sigma}^{(\chi)}) \gamma_{\nu)} \psi_{\sigma}^{(\chi)} \right], \]  
(2.3)
where \( \{ \psi_{\sigma}^{(+)} , \psi_{\sigma}^{(-)} \} \) is the complete set of the positive and negative energy fermionic modes, specified by the set of quantum numbers \( \sigma \). In (2.3), the brackets in the index expression mean the symmetrization over the enclosed indices, the Dirac adjoint is defined as \( \overline{\psi}^{(\chi)} = \psi^{(\chi)\dagger} \gamma^{(0)} \) and \( \mathcal{D}_{\mu} \psi_{\sigma}^{(\chi)} = \partial_{\mu} \psi_{\sigma}^{(\chi)} - ieA_{\mu} \psi_{\sigma}^{(\chi)} - \psi_{\sigma}^{(\chi)} J_{\mu} \) for \( \chi = +, - \). The expression in the right-hand side of (2.3) is divergent and a regularization is required. For example, we can introduce a cutoff function or employ the point-splitting procedure. In the discussion below we will extract from the VEV the topological part. For \( r > 0 \) it is finite and does not depend on the specific regularization procedure.

In what follows it will be convenient to work in the coordinate system \( x^\mu = (t, r, \phi, w, z) \) with the conformal coordinate \( w \). We will take the vielbein fields \( e^\mu_{(b)} = \delta^\mu_b w/a \) and the flat spacetime Dirac matrices
\[ \gamma^{(0)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^{(4)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]
\[ \gamma^{(l)} = -i \text{ diag}(\sigma^l, -\sigma^l), \quad l = 1, 2, 3. \]
(2.4)
Here, \( 2 \times 2 \) Pauli matrices in the cylindrical coordinates \( (r, \phi, w) \) are given by
\[ \sigma^l = \begin{pmatrix} i & 0 \\ 0 & e^{iq\phi} \end{pmatrix}^{l-1}, \quad l = 1, 2, \]
(2.5)
with the notation \( q = 2\pi/\phi_0 \). The geometries with uncompact and compact \( z \)-directions will be considered separately.

3 Vacuum energy-momentum tensor in the uncompactified geometry

In this section we consider the VEV of the energy-momentum tensor in the geometry with uncompactified \( z \)-direction, \(-\infty < z < +\infty \). The corresponding fermionic modes are specified by the set of quantum numbers \( \sigma = (\lambda, p, k, j, \eta) \) with the variation ranges \( 0 \leq \lambda, p < \infty, -\infty < k < +\infty, j = \pm 1/2, \pm 3/2, \ldots, \eta = \pm 1 \). The mode functions are expressed as (for the modes with compact \( z \)-dimension see [47])
\[ \psi_{\sigma}^{(\pm)}(x) = C_{\sigma}^{(\pm)} u_{\sigma}^{5/2} \begin{pmatrix} J_{\beta_j}(\lambda r)J_{\nu_1}(pw)e^{-iq\phi/2} \\ \mp s\epsilon_{j,\lambda} b^\eta J_{\beta_j+\epsilon_j}(\lambda r)J_{\nu_2}(pw)e^{iq\phi/2} \\ i\sigma_\lambda J_{\beta_j}(\lambda r)J_{\nu_1}(pw)e^{-iq\phi/2} \\ \mp i\epsilon_{j,\lambda} b_\eta J_{\beta_j}(\lambda r)J_{\nu_1}(pw)e^{iq\phi/2} \end{pmatrix} e^{iq\phi + ikz \pm iEt}, \]
(3.1)
where \( \nu_1 = m a + (-1)^j \sqrt{s}/2 \), \( J_{\nu}(x) \) is the Bessel function [56, 57] and the energy is given by \( E = \sqrt{\lambda^2 + p^2 + k^2} \). The order of the Bessel function is defined by
\[ \beta_j = q[j + \alpha] - \epsilon_j/2, \]  
(3.2)
where \( \epsilon_j = \text{sgn}(j + \alpha) \) with \( \alpha = eA_2/q = -\Phi_s/\Phi_0 \). Here \( \Phi_s \) is the magnetic flux running along the string’s core and \( \Phi_0 \) is the flux quantum. The coefficients \( \kappa_\eta \) and \( b^{(\pm)}_\eta \) are given by the expressions

\[
\kappa_\eta = \frac{1}{p} \left( -k_z + \eta \sqrt{p^2 + k_z^2} \right),
\]
\[
b^{(\pm)}_\eta = \frac{1}{\lambda} \left( E \mp \eta \sqrt{p^2 + k_z^2} \right).
\]

(3.3)

For the normalization coefficient one gets

\[
|C^{(\pm)}_\sigma|^2 = \frac{(4\pi)^{-2} \eta a^{-4} q p^2 \lambda^2}{E \kappa_\eta b^{(\pm)}_\eta \sqrt{p^2 + k_z^2}}.
\]

(3.4)

Having the normalized mode functions (3.1), the VEV of the energy-momentum tensor is evaluated by using the formula (2.3), where the summation over \( \sigma \) in is understood as

\[
\sum_\sigma = \sum_j \int_0^\infty d\lambda \int_0^\infty dp \int_{-\infty}^{\infty} dk_z \sum_{\eta=\pm 1},
\]

(3.5)

with \( \sum_j = \sum_{j=\pm 1/2, \pm 3/2,...} \). We can show that the VEVs for \( s = +1 \) and \( s = -1 \) coincide and in what follows the formulas will be presented for \( s = +1 \). The VEVs of the diagonal components are presented as (no summation over \( \mu \))

\[
\langle T^\mu_{\mu} \rangle_{\text{cs}}^{\text{AdS}} = -\frac{qw^6}{4\pi^2 v^6} \sum_j \int_0^\infty d\lambda \int_0^\infty dp \int_{-\infty}^{\infty} dk_z E^{2\delta_{j\mu} - 1} (k_z^2)^{\delta_{j\mu}} R^{(\mu)}_{\beta_j} (\lambda r) W^{(\mu)}_{\nu}(pw),
\]

(3.6)

where \( \nu = ma - 1/2, \)

\[
R^{(0)}_{\beta_j} (\lambda r) = R^{(3)}_{\beta_j} (\lambda r) = -R^{(4)}_{\beta_j} (\lambda r) = J^2_{\beta_j} (\lambda r) + J^2_{\beta_j + \epsilon_j} (\lambda r),
\]

\[
R^{(1)}_{\beta_j} (\lambda r) = \epsilon_j \lambda^2 [J^1_{\beta_j} (\lambda r)J_{\beta_j + \epsilon_j} (\lambda r) - J^1_{\beta_j} (\lambda r)J_{\beta_j + \epsilon_j} (\lambda r)],
\]

\[
R^{(2)}_{\beta_j} (\lambda r) = -\frac{\lambda}{r} (2\beta_j + \epsilon_j)J^1_{\beta_j} (\lambda r)J_{\beta_j + \epsilon_j} (\lambda r),
\]

(3.7)

and

\[
W^{(\mu)}_{\nu}(pw) = J^2_{\nu}(pw) + J^2_{\nu+1}(pw), \quad \mu = 0, 1, 2, 4,
\]

\[
W^{(3)}_{\nu}(pw) = p^2 [J^1_{\nu}(pw)J_{\nu+1}(pw) - J_{\nu}(pw)J'_{\nu+1}(pw)].
\]

(3.8)

Note that we have the relation

\[
R^{(1)}_{\beta_j} (\lambda r) = \lambda^2 R^{(0)}_{\beta_j} (\lambda r) + R^{(2)}_{\beta_j} (\lambda r).
\]

(3.9)

The off-diagonal components of the vacuum energy-momentum tensor vanish.

For the extraction of the topological part in (3.6) it is convenient to use the integral representation

\[
E^{2\delta_{j\mu} - 1} = \frac{2}{\sqrt{\pi}} \int_0^\infty d\tau (\partial_{-\tau^2})^{\delta_{j\mu}} e^{-(\lambda^2 + p^2 + k_z^2)\tau^2}.
\]

(3.10)
After evaluating the integral over $k_z$, the $p$-integral is evaluated by using the formula
\[
\int_0^{\infty} dp e^{-p^2/2} W_{\nu}(pw)(pw) = \frac{(-1)^{\delta_{\nu w}} e^{-x}}{(2\pi^2)^{1+\delta_{\nu w}}} [I_{\nu}(x) + I_{\nu+1}(x)]_{x=w^2/2\tau^2},
\] (3.11)
where $I_{\nu}(x)$ is the modified Bessel function [56]. The integral over $\lambda$ involving the functions $R^{(\mu)}_{\beta_j}(\lambda r)$ with $\mu \neq 2$ has the structure similar to (3.11) and the integral for $R^{(2)}_{\beta_j}(\lambda r)$ is evaluated by using the relation (3.9). The components with $\mu = 0, 1, 3, 4$ are presented in the form (no summation over $\mu$)
\[
\langle T_{\mu \mu}^{\text{AdS}} \rangle_{\text{cs}} = \frac{qa^{-5}}{8\pi^2} \int_0^{\infty} dx x^2 e^{-x(1+\rho^2)} [I_{\nu}(x) + I_{\nu+1}(x)] \mathcal{J}(q, \alpha_0, x\rho^2),
\] (3.12)
with $\rho = r/w$ and
\[
\mathcal{J}(q, \alpha, y) = \sum_j \left[ I_{\beta_j}(y) + I_{\beta_j+\epsilon_j}(y) \right].
\] (3.13)
It can be seen that the remaining component is related to the energy density by the formula
\[
\langle T_2^2 \rangle_{\text{AdS}}^{\text{AdS}} = (1 + r\partial_r) \langle T_0^0 \rangle_{\text{AdS}}^{\text{AdS}}.
\] (3.14)
The proper distance from the string is given by
\[
r_p = ar/w
\] (3.15)
and $\rho$ in (3.12) is the proper distance measured in units of the curvature radius $a$. Note that the parameter $\alpha$ enters in the expression for the VEV $\langle T_{\mu \mu}^{\text{AdS}} \rangle_{\text{cs}}$ in the form $j + \alpha$. Consequently, if we present it as $\alpha = \alpha_0 + n_0$, with $|\alpha_0| < 1/2$, then, after the redefinition $j + n_0 \to j$, we see that the VEV does not depend on the integer part $n_0$. Hence, in the discussion below we can take $\alpha = \alpha_0$ without loss of generality.

For the further transformation of the VEV we use the representation [58]
\[
\mathcal{J}(q, \alpha_0, y) = \frac{2}{q} e^y + \frac{4}{q} \sum_{k=1}^{[q/2]} (-1)^k c_k \cos(2\pi k\alpha_0) e^{y\cos(2\pi k/q)}
+ \frac{4}{\pi} \int_0^{\infty} du \frac{H(q, \alpha_0, u) e^{-u \cos(2\alpha_0)}}{\cosh(2qu) - \cos(q\pi)},
\] (3.16)
where $[q/2]$ is the integer part of $q/2$ and the prime on the summation sign means that for even values of $q$ the term with $k = q/2$ should be halved. In (3.16), the notations $c_k = \cos(\pi k/q)$ and
\[
H(q, \alpha_0, u) = \sinh u \sum_{\chi = +, -} \cos[\pi q(1/2 + \chi\alpha_0)] \sinh[(1 - \chi\alpha_0)qu]
\] (3.17)
have been introduced. In the special case $\alpha_0 = 0$ (the magnetic flux along the string is a multiple of the flux quantum) the expression (3.17) is simplified to
\[
H(q, 0, u) = 2 \cos(\pi q/2) \sinh(qu) \sinh u,
\] (3.18)
and in (3.16) the integral term vanishes for odd values of $q$. 

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Substituting the representation (3.16) in (3.12), the VEV is decomposed as (no summation over $\mu$)

$$\langle T^\mu_\mu \rangle_{\text{AdS}} = \langle T^\mu_\mu \rangle_{\text{AdS}} + \langle T^\mu_\mu \rangle_{\text{cs}}. \quad (3.19)$$

The first contribution in the right-hand side comes from the first term in the right-hand side of (3.16) and is given by the expression

$$\langle T^\mu_\mu \rangle_{\text{AdS}} = \frac{a^{-5}}{4\pi^2} \int_0^\infty dx \ x^2 e^{-x} [I_\nu(x) + I_{\nu+1}(x)], \quad (3.20)$$

for all values of $\mu$. The part $\langle T^\mu_\mu \rangle_{\text{cs}}$ in (3.19) comes from the second and third terms in the right-hand side of (3.16) and for the components with $\mu = 0, 1, 3, 4$ it is presented as

$$\langle T^\mu_\mu \rangle_{\text{cs}} = \frac{a^{-5}}{\sqrt{2} \pi^{5/2}} \left[ \sum_{k=1}^{[q/2]} (-1)^k c_k \cos(2\pi k\alpha_0) F_{ma}(u_k) + \frac{q}{\pi} \int_0^\infty dx \ H(q,\alpha_0, x) F_{ma}(u_x) \right], \quad (3.21)$$

with the notations

$$u_k = 1 + 2\rho^2 s_k^2, \quad s_k = \sin(\pi k/q),$$

$$u_x = 1 + 2\rho^2 \cosh^2 x, \quad (3.22)$$

and

$$F_{ma}(u) = \sqrt{\frac{\pi}{2}} \int_0^\infty dx \ x^2 e^{-ux} [I_\nu(x) + I_{\nu+1}(x)]. \quad (3.23)$$

The component $\langle T^2_2 \rangle_{\text{cs}}$ is obtained from

$$\langle T^2_2 \rangle_{\text{cs}} = (1 + r \partial_r) \langle T^0_0 \rangle_{\text{cs}}. \quad (3.24)$$

The VEV $\langle T^\mu_\mu \rangle_{\text{cs}}$ depends on the coordinates $r$ and $w$ through the ratio $r/w$. This property is a consequence of the maximal symmetry of the AdS spacetime. Note that the VEV $\langle T^\mu_\mu \rangle_{\text{cs}}$ is an even function of the parameter $\alpha_0$. In terms of the magnetic flux along the string core, $\langle T^\mu_\mu \rangle_{\text{cs}}$ is an even periodic function of the magnetic flux with the period of the flux quantum.

We have shown that the mean energy-momentum tensor is diagonal. This property is not a consequence of the problem symmetry. In general, we could have a nonzero component $\langle T^4_4 \rangle_{\text{cs}}$ allowed by the symmetry. As it has been discussed in [44], this type of nonzero off-diagonal component is present for a massive scalar field around a cosmic string in $(D + 1)$-dimensional AdS spacetime. Note that for a scalar field, in general, we could have a nonzero component $\langle T^\mu_\mu \rangle_{\text{cs}}$ with $\mu = 0, 1, 3, 4$ and the property that the VEV depends on $r$ and $w$ through the ratio $r/w$. The continuity equation

$$\nabla \mu \langle T^\nu_\nu \rangle_{\text{cs}} = 0 \quad \text{and} \quad \text{trace relation} \quad \langle T^\mu_\mu \rangle_{\text{cs}} = sm \langle \bar{\psi} \psi \rangle_{\text{cs}},$$

with the fermion condensate $\langle \bar{\psi} \psi \rangle_{\text{cs}}$ from [48]. In particular, for a massless field the tensor $\langle T^\mu_\nu \rangle_{\text{cs}}$ is traceless. The trace anomaly is contained in the part $\langle T^\mu_\mu \rangle_{\text{AdS}}$. The continuity equation is reduced to the relations

$$\langle T^2_2 \rangle_{\text{cs}} = \partial_r \langle T^1_1 \rangle_{\text{cs}}, \quad \langle T^3_3 \rangle_{\text{cs}} = (1 - w \partial_w) \langle T^3_3 \rangle_{\text{cs}}. \quad (3.25)$$

These relations directly follow from (3.24) by taking into account that $\langle T^1_1 \rangle_{\text{cs}} = \langle T^3_3 \rangle_{\text{cs}} = \langle T^0_0 \rangle_{\text{cs}}$ and the property that the VEV depends on $r$ and $w$ through the ratio $r/w$. 

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The special case with $q = 1$ and $\alpha_0 = 0$ corresponds to the absence of the cosmic string and magnetic flux and the corresponding geometry is reduced to the AdS spacetime. In this special case one has $\langle T_\mu^\mu \rangle_{cs} = 0$ and, hence, the part $\langle T_\mu^\mu \rangle_{AdS}$ is the VEV in AdS spacetime. For it one has $\langle T_\mu^\mu \rangle_{AdS} = \text{const} \cdot g_{\mu\nu}$ and this property is a consequence of the maximal symmetry of AdS spacetime. For $r > 0$ the contribution $\langle T_\mu^\mu \rangle_{cs}$ is finite and the renormalization is required for the pure AdS part $\langle T_\mu^\mu \rangle_{AdS}$ only. This feature is a consequence of the fact that in the region $r > 0$ the local geometrical characteristics are the same as those in AdS spacetime and, hence, the divergences are the same as well. The AdS part $\langle T_\mu^\mu \rangle_{AdS}$ is widely considered in the literature and here we are interested in the topological effects induced by the cosmic string and magnetic flux.

The integral in (3.23) is expressed in terms of the associated Legendre function $Q_{\nu}^\mu(z)$ [56, 57] and one finds

$$F_{ma}(u) = \frac{Q^{5/2}_{\nu-1/2}(u) + Q^{7/2}_{\nu-1/2}(u)}{e^{5\pi/2(u^2 - 1)^{5/4}}}. \quad (3.26)$$

For the evaluation of the derivative $\partial_u F_{ma}(u)$ in (3.24) one can use the relation

$$\partial_u F_{ma}(u) = -\frac{Q^{7/2}_{\nu-1/2}(u) + Q^{5/2}_{\nu-1/2}(u)}{e^{7\pi u/2(u^2 - 1)^{7/4}}}. \quad (3.27)$$

For a massless field $\nu = -1/2$ and the modified Bessel functions in (3.23) are expressed in terms of elementary functions. In this special case one gets

$$F_0(u) = \frac{3\sqrt{\pi}}{4(u - 1)^{5/2}}. \quad (3.28)$$

For the discussion of the behavior of the VEV in the asymptotic regions of the parameters we will need the asymptotics of the function $F_{ma}(u)$ for large values of the argument and for $u \to 1$. By using the asymptotic expression of the function $Q^{5/2}_{\nu-1/2}(u)$ for $u \to 1^+$ (see, for example, [57]) we can see that

$$F_{ma}(u) \approx \frac{3\sqrt{\pi}}{4(u - 1)^{5/2}}. \quad (3.29)$$

Note that the leading term given by (3.29) coincides with (3.28). For large argument the leading term in the corresponding asymptotic expansion is given by

$$F_{ma}(u) \approx \frac{\sqrt{\pi}(2ma + 3)(2ma + 1)}{2ma + 2\sqrt{\pi}ma + 5/2}. \quad (3.30)$$

This result is obtained by using the asymptotic for the associated Legendre function [57] or directly from the integral representation (3.23).

Let us consider some special and limiting cases of the general formula (3.21). For a massless field, by using (3.28), one finds (no summation over $\mu$)

$$\langle T_\mu^\mu \rangle_{cs} = \frac{3h_5(q, \alpha_0)}{32\pi^2 r_p^5}, \quad (3.31)$$
for \( \mu = 0, 1, 3, 4 \) and \( \langle T_{2}^{0} \rangle_{cs} = -4 \langle T_{0}^{0} \rangle_{cs} \). Here and in what follows the notation

\[
h_{n}(q, \alpha_{0}) = \sum_{k=1}^{(q/2)} (-1)^{k} \frac{c_{k}}{s_{k}^{q}} \cos(2\pi k \alpha_{0}) + \frac{q}{\pi} \int_{0}^{\infty} dx \frac{H(q, \alpha_{0}, x) \cosh^{-n} x}{\cosh(2qx) - \cos(qx)},
\]

(3.32)
is introduced. In the special case \( q = 1 \), corresponding to the absence of planar angle deficit, the expression (3.32) is simplified to

\[
h_{n}(1, \alpha_{0}) = \frac{\sin(\pi \alpha_{0})}{\pi} \int_{0}^{\infty} dx \frac{\sin(2\alpha_{0}x)}{\cosh^{n+1} x} \sinh x.
\]

(3.33)

As seen from (2.2), the limit \( a \to \infty \) for fixed \( y \) corresponds to the geometry of a cosmic string in background of (4+1)-dimensional Minkowski spacetime. In this limit \( ma \gg 1 \) and the order of the associated Legendre function in (3.26) is large. For the \( w \)-coordinate one has \( w \approx a + y \) and in (3.21) \( u_{k}, u_{\nu} \to 1^{+} \). Consequently, we need the asymptotic of the function \( Q_{\nu-1/2}^{5/2}(u) \) for \( \nu \gg 1 \) and \( u - 1 \ll 1 \). By using that asymptotic from [57], we can see that

\[
\mathcal{F}_{ma}(u) \approx 2m^{5}a^{5}f_{5/2}(ma \sqrt{2(u - 1)}),
\]

(3.34)

where \( f_{\mu}(x) = K_{\mu}(x)/x^{\mu} \) and \( K_{\mu}(x) \) is the modified Bessel function. As a result of the limiting transition we get the VEV of the energy-momentum tensor induced by a cosmic string in (4+1)-dimensional Minkowski spacetime (no summation over \( \mu = 0, 1, 3, 4 \)):

\[
\langle T_{\mu}^{\nu} \rangle^{(M)}_{cs} = \frac{\sqrt{2}m^{5}}{\pi^{5/2}} \left[ \sum_{k=1}^{(q/2)} (-1)^{k} c_{k} \cos(2\pi k \alpha_{0}) f_{5/2}(2mr s_{k}) \right. \\
\left. + \frac{q}{\pi} \int_{0}^{\infty} dx \frac{H(q, \alpha_{0}, x) f_{5/2}(2mr \cosh x)}{\cosh(2qx) - \cos(qx)} \right].
\]

(3.35)

The component \( \langle T_{2}^{2} \rangle^{(M)}_{cs} \) is found from the relation similar to (3.24). For the function \( f_{5/2}(x) \) in this expression one has

\[
f_{5/2}(x) = \sqrt{\frac{x^{2}}{2}} \frac{x^{2} + 3x + 3}{x^{5}} e^{-x}.
\]

(3.36)

In the case of a massless field the formula (3.35) is reduced to

\[
\langle T_{\mu}^{\mu} \rangle^{(M)}_{cs} = \frac{3h_{5}(q, \alpha_{0})}{32\pi^{2}r^{5}}.
\]

(3.37)

This result is also seen from (3.31) by taking into account that for fixed \( y \) one has \( \lim_{a \to \infty} w/a = 1 \). At small distances from the string, \( mr \ll 1 \), the leading term in the expansion of (3.35) does not depend on the mass and coincides with (3.37). The effect of the mass is essential at distances \( mr \gg 1 \). In that region the VEVs are suppressed by the factor \( e^{-2mr \sin(\pi q)} \) for \( q \geq 2 \) and \( e^{-2mr} \) for \( 1 \leq q < 2 \).

The special case \( q = 1 \) corresponds to a magnetic flux in background of AdS spacetime. The VEV of the energy-momentum tensor induced by the magnetic flux we will denote as \( \langle T_{\mu}^{\mu} \rangle_{mt} = \langle T_{\mu}^{\mu} \rangle_{cs}|_{q=1} \). For this case

\[
H(1, \alpha_{0}, x) = \sin(\pi \alpha_{0}) \sinh(2\alpha_{0}x) \sinh(2x),
\]

(3.38)
and the general formula (3.21) is simplified to (no summation over $\mu$)

$$\langle T_{\mu}^{\mu} \rangle_{mf} = \frac{\sin(\pi \alpha_0)}{\sqrt{2\pi} a^5} \int_0^\infty dx \sinh(2\alpha_0 x) \tan(x) \mathcal{F}_{ma}(1 + 2\rho^2 \cosh^2 x),$$

(3.39)

for $\mu = 0, 1, 3, 4$, and $\langle T_\mu^\mu \rangle_{mf} = \partial_\mu (\langle r T_\mu^0 \rangle_{mf})$.

Now let us consider the behavior of the string induced VEV in the asymptotic regions of the radial coordinate $r$. In the region where the proper distance from the string is much smaller than the curvature radius, $r_p \ll a$, one has $\rho \ll 1$. For the arguments of the function $\mathcal{F}_{ma}(u)$ in (3.21) we have $u_k, u_x \to 1+$. By using the asymptotic (3.29), we conclude that near the string the effect of the mass is weak and to the leading order the VEV coincides with that for a massless field. In that region the influence of the background gravitational field on the string induced effects is weak and the VEV $\langle T_{\mu}^{\mu} \rangle_{cs}$ behaves as $(w/r)^5$. By taking into account that the VEV depends on the coordinates $\rho, w$ through the ratio $r/w$, from here we obtain the asymptotic near the horizon for fixed value of $r$: near the horizon, $w \gg r$, the VEV behaves like $w^5$. In the opposite limit of large distances from the string one has $r_p \ll a$ and $\rho \gg 1$. By taking into account (3.30), the leading term in the asymptotic expansion is given as (no summation over $\mu$)

$$\langle T_{\mu}^{\mu} \rangle_{cs} \approx \frac{(ma + 3/2)(ma + 1/2)}{2^{2ma+3} \pi^2 a^5 (r/w)^{2ma+5}} b_{2ma+5}(q, \alpha_0),$$

(3.40)

for $\mu = 0, 1, 3, 4$ and $\langle T_\mu^\mu \rangle_{cs} \approx -2 (ma + 2) \langle T_0^0 \rangle_{cs}$. For a massless field this result coincides with the exact formula (3.31). For a fixed $r$ and $w \ll r$, the expression (3.40) describes the behavior of the VEV near the AdS boundary: the VEV tends to zero as $w^{2ma+5}$. As seen from (3.40), for a massive field on the AdS bulk the decay of the string induced VEV at large distance from the string follows a power law. This is in contrast to the case of the Minkowski bulk where, as it has been mentioned above, the fall off is exponential.

In figure 1 the energy density $\langle T_0^0 \rangle_{cs}$ (in units of $1/a^5$) is plotted versus the proper distance from the string (in units of the curvature scale $a$), $r_p/a = r/w$. In the numerical evaluation we have taken $ma = 1$ and $\alpha_0 = 0.3$. The corresponding values of the parameter $q$ are written near the curves. In accordance with the asymptotic analysis given above, near the string the energy density behaves as $(w/r)^5$ and at large distances the VEV is suppressed by the factor $(w/r)^{2ma+5}$.

The dependence of the energy density induced by the string on the parameter $\alpha_0$ and on the mass (in units of $1/a$) is displayed in figure 2 for $r/w = 1.5$. The left panel is plotted for $ma = 0.5$ and for the right panel $\alpha_0 = 0.3$. For both panels, the numbers near the curves present the corresponding values of $q$. As seen, for fixed values of the other parameters, the absolute value of the energy density increases with increasing planar angle deficit.

It is well known that in odd-dimensional spacetimes there are two inequivalent irreducible representations of the Clifford algebra. Let us denote the corresponding sets of Dirac gamma matrices as $\gamma^{(b)}_{(s)}$ with $s = +1$ and $s = -1$ for separate representations. In the geometry under consideration with (4+1)-dimensional spacetime we will take $\gamma^{(b)}_{(+1)} = \gamma^{(b)}$, where the matrices $\gamma^{(b)}$ are given by (2.4), and for the second representation $\gamma^{(b)}_{(-1)} = (-1)^{\delta_{\mu_4,4}} \gamma^{(b)}$. The Lagrangian density for the field $\psi_{(s)}$ realizing the representation for a given $s$ reads $L_{(s)} = \bar{\psi}_{(s)} (i\gamma^{(b)}_{(s)} D_{\mu} - m) \psi_{(s)}$, where in the definition (2.1) of the covariant derivative the corresponding spin connection $\Gamma^{(s)}_{\mu}$ is taken. Note that $\Gamma_{\mu}^{(+1)} = \Gamma_{\mu}$. The operator of the
Figure 1. The dependence of the mean energy density, induced by the cosmic string, on the radial coordinate for $ma = 1$, $\alpha_0 = 0.3$. The numbers near the curves are the values of the parameter $q$.

Figure 2. The energy density $\langle T_\mu^\nu \rangle_{cs}$ as a function of the parameter $\alpha_0$ (left panel) and of the mass (right panel) for $r/w = 1.5$. On the left panel $ma = 0.5$ and on the right one $\alpha_0 = 0.3$. The numbers near the curves are the values of $q$.

The energy-momentum tensor has the form

$$T_{\mu\nu}^{(s)} = \frac{i}{2} \left[ \bar{\psi}(s) (\gamma_\mu D_\nu - \gamma_\nu D_\mu) \psi(s) - (D_\mu \bar{\psi}(s) \gamma_\nu \gamma(s) \psi(s) \bar{\psi}(s) \right].$$

(3.41)

Instead of the field $\psi(-1)$, we can introduce a new field $\psi' = -\gamma^{(4)}_\mu \psi_{(-1)}$. By taking into account that $\gamma^{(4)}_\mu \gamma^{(-1)}_\mu \gamma^{(4)} = \gamma^{\mu} \Gamma_\mu$, the corresponding Lagrangian density is rewritten in the form $L_{(-1)} = \bar{\psi}' (i\gamma^{\mu} D_\mu + m) \psi'$ with $D_\mu$ from (2.1). The equation of motion for the field $\psi'$ is given by (2.1) with $s = -1$. The expression of the energy-momentum tensor $T_{\mu\nu}^{(-1)}$, written in terms of the new field $\psi'$, coincides with (3.41) for $s = +1$ replacing $\psi_{(+1)}$ by $\psi'$. As seen, the parameter $s$ introduced in (2.1) is interpreted in terms of the parameter that distinguishes inequivalent representations. Hence, the VEV of the energy-momentum tensor $T_{\mu\nu}^{(-1)}$ for the field $\psi(-1)$ coincides with the VEV $\langle T_{\mu\nu} \rangle$ investigated above for the value of the parameter $s = -1$ in the Dirac equation (2.1). From here we conclude that the VEVs of the energy-momentum tensors coincide for fields realizing two irreducible representations of
Figure 3. A part of the spatial surface corresponding to the string core \( r = 0 \) with compactified \( z \)-coordinate, embedded in a 3-dimensional Euclidean space. The surface is plotted for \( \zeta = 1 \).

the Clifford algebra. We recall that the fermionic condensates for the fields \( \psi^{(+1)} \) and \( \psi^{(-1)} \) have opposite signs [48].

4 Vacuum energy-momentum tensor induced by compactification

4.1 General formula

In this section we consider the VEV of the energy-momentum tensor in background of the spacetime with the line element (2.2) assuming that the \( z \)-direction is compactified to a circle \( S^1 \) with the length \( L \) and, hence, one has \( 0 \leq z \leq L \). Introducing a new angular coordinate \( \varphi, 0 \leq \varphi \leq 2\pi \), in accordance with \( z = L\varphi/(2\pi) \), the line element on the 3-dimensional hypersurface corresponding to the string core \( r = 0 \) is presented in the form

\[
 ds^2_{\text{core}} = e^{-2y/a} \left( dt^2 - a^2 \zeta^2 d\varphi^2 \right) - dy^2, \tag{4.1}
\]

where \( \zeta = L/(2\pi a) \). The spatial line element is expressed as \( ds^2_{\text{core}} = dy^2 + a^2 \zeta^2 e^{-2y/a} d\varphi^2 \). A part of the 2-dimensional spatial surface corresponding to this line element, determined by \( y_0 \leq y < +\infty \), with \( y_0 = a \ln \zeta \), can be embedded in a 3-dimensional Euclidean space with the coordinates \((X,Y,Z)\) by using the relations

\[
 X = a\zeta e^{-y/a} \cos \varphi, \quad Y = a\zeta e^{-y/a} \sin \varphi, \quad Z = \int_{y_0}^{y/a} du \sqrt{1 - \zeta^2 e^{-2u}}. \tag{4.2}
\]

In figure 3 we have plotted the surface presenting the string core with this embedding for \( \zeta = 1 \). Note that the length of the compact dimension measured by an observer having a fixed \( y \)-coordinate is given by \( Le^{-y/a} \) and it exponentially decreases with increasing \( y \).

For the further discussion we need to specify the periodicity condition on the field operator along the compact \( z \)-direction. Here, the quasi-periodic condition

\[
 \psi(t, r, \phi, w, z + L) = e^{2\pi i \beta} \psi(t, r, \phi, w, z), \tag{4.3}
\]
will be imposed, where $\beta$ is an arbitrary constant. This leads to the quantization of the eigenvalues for the momentum $k_z$:

$$k_z = \tilde{k}_l = 2\pi \frac{l + \tilde{\beta}}{L}, \quad l = 0, \pm 1, \pm 2, \ldots, \quad (4.4)$$

where $\tilde{\beta} = \beta - \Phi_c/\Phi_0$ and $\Phi_c = -LA_4$. The component $A_4$ can be formally interpreted in terms of the magnetic flux $\Phi_c$ enclosed by the compact $z$-dimension.

The fermionic modes in the problem at hand are obtained from (3.1) taking $k_z = \tilde{k}_l$ and adding an additional factor $2\pi/L$ in the expression (3.4) for $|C_\nu^{(\pm)}|^2$ (see also [47]). For the energy one has $E = \sqrt{\lambda^2 + p^2 + \tilde{k}_l^2}$. The VEV of the energy-momentum tensor is evaluated by using the formula (2.3), where now the collective summation is understood as (3.5) with the replacement $\int_{-\infty}^{\infty} dk_z \rightarrow \sum_{l=-\infty}^{\infty}$. As before, the off-diagonal components vanish. For the VEV of the diagonal components we find the representation (no summation over $\mu$)

$$\langle T_\mu^\mu \rangle = -\frac{qw^6}{2\pi a^5 L} \sum_j \int_0^\infty d\lambda \int_0^\infty dp \sum_{l=-\infty}^{\infty} (\lambda^2 + p^2 + \tilde{k}_l^2)^{\delta_{0u} - 1/2} (\tilde{k}_l^2)^j \delta_{\mu,\lambda} R_x(\mu)(\lambda r) W_x(pw). \quad (4.5)$$

The VEVs (4.5) coincide for $s = +1$ and $s = -1$ and the consideration below will be continued for $s = +1$. Here we are interested in the effects of compactification and the corresponding contribution in the VEV can be explicitly extracted by using the Abel-Plana formula [59]

$$\sum_{l=\infty}^{\infty} g(\tilde{k}_l) = \frac{L}{\pi} \int_0^\infty dx \frac{g(x) + iL}{2\pi} \int_0^\infty dx \sum_{n=\pm 1} \frac{g(ix) - g(-ix)}{e^{Lx} + 2\pi i n \beta} - 1. \quad (4.6)$$

The part in the VEV $\langle T_\mu^\mu \rangle$ coming from the first term in the right-hand side (4.6) gives the corresponding quantity in the uncompactified geometry and we get the representation (no summation over $\mu$)

$$\langle T_\mu^\mu \rangle = \langle T_\mu^\mu \rangle_{\text{AdS}} + \langle T_\mu^\mu \rangle_c. \quad (4.7)$$

The contribution in the VEV induced by compactification is expressed as

$$\langle T_\mu^\mu \rangle_c = -\frac{(-1)^{\delta_{0u} + \delta_{\mu,\lambda}} q w^6}{2\pi a^5} \sum_j \int_0^\infty d\lambda \lambda R_x(\mu)(\lambda r) \int_0^\infty dp W_x(pw) \times \int_{\sqrt{\lambda^2 + p^2}}^\infty dx x^{2\delta_{\mu,\lambda}} \sum_{n=\pm 1} \frac{(x^2 - \lambda^2 - p^2)^{\delta_{0u} - 1/2}}{e^{Lx} + 2\pi i n \beta} - 1. \quad (4.8)$$

The compactification does not change the local geometry and for $r > 0$ the renormalization in (4.7) is only needed for the part $\langle T_\mu^\mu \rangle_{\text{AdS}}$. Consequently, the implicit regularization, assumed in the discussion above, can be safely removed in the topological part $\langle T_\mu^\mu \rangle_c$.

By making use of the relation $(e^v - 1)^{-1} = \sum_{l=1}^{\infty} e^{-lv}$ in (4.8), with $u = Lx + 2\pi i n \tilde{\beta}$, the integral over $x$ is evaluated in terms of the modified Bessel function $K_\nu(lL/\sqrt{\lambda^2 + p^2})$ with $\nu = 0, 1$. As the next step, we use the integral representation

$$K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty dt \frac{e^{-t - z^2/4t}}{t^{\nu+1}}. \quad (4.9)$$
The integrals over \( \lambda \) and \( p \) are evaluated in the way similar to that we have discussed in the previous section for the part \( \langle T^k_{\mu \nu} \rangle_{\text{AdS}} \). Introducing the function (3.13), the following representation is obtained for the energy density

\[
\langle T^0_0 \rangle_c = \frac{q}{4\pi^2 a^3} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \int_0^\infty dx \, x^2 e^{-[1+\rho^2+L^2/(2w^2)]x} [I_\nu(x) + I_{\nu+1}(x)] J(q, \alpha_0, \rho^2 x). \tag{4.10}
\]

The remaining components are related to the energy density by the formulas

\[
\langle T^1_1 \rangle_c = \langle T^3_3 \rangle_c = \langle T^0_0 \rangle_c, \quad \langle T^2_2 \rangle_c = (1 + r \delta_p) \langle T^0_0 \rangle_c, \quad \langle T^4_4 \rangle_c = \partial_L (L \langle T^0_0 \rangle_c). \tag{4.11}
\]

For the evaluation of the derivative \( \partial_u F_{ma}(u) \) in the integrands for \( \langle T^2_2 \rangle_c \) and \( \langle T^4_4 \rangle_c \) we can use the relation (3.27).

Substituting the representation (3.16) for function (3.13) in (4.10), the integral over \( x \) is expressed in terms of the function (3.23) and for the energy density we get

\[
\langle T^0_0 \rangle_c = \frac{\sqrt{2}}{\pi^{1/2} a^3} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \left[ \sum_{k=0}^{[q/2]} (-1)^k c_k \cos(2\pi k \alpha_0) F_{ma}(u_{lk}) + \frac{q}{\pi} \int_0^\infty du \frac{H(q, \alpha_0, u) F_{ma}(u_{lw})}{\cosh(2qu) - \cos(q\pi)} \right], \tag{4.12}
\]

with the notations

\[
u_{lk} = 1 + 2\rho^2 s_k^2 + \frac{\nu^2 L^2}{2w^2}, \quad u_{lw} = 1 + 2\rho^2 \cosh^2 u + \frac{\nu^2 L^2}{2w^2}. \tag{4.13}
\]

The asterisk near the sign of summation over \( k \) in (4.12) indicates that the term \( k = 0 \) must be divided by 2. As a consequence of the maximal symmetry of AdS spacetime the contribution \( \langle T^{\mu}_{\mu} \rangle_c \) depends on \( r, L, w \) through the combinations \( r/w \) and \( L/w \). By using \( \langle T^1_1 \rangle_c = \langle T^3_3 \rangle_c = \langle T^0_0 \rangle_c \), the covariant continuity equation \( \nabla_{\mu} \langle T^\mu_{\nu} \rangle_c = 0 \) is reduced to the following two relations \( (\langle T^2_2 \rangle_c = \partial_L (L \langle T^0_0 \rangle_c) \) and \( \langle T^2_2 \rangle_c + \langle T^4_4 \rangle_c = (2 - w \partial_u) \langle T^0_0 \rangle_c \). By taking into account that the VEV \( \langle T^{\mu}_{\mu} \rangle_c \) is a function of the ratios \( r/w \) and \( L/w \), it can be seen that these two relations follow from the last two relations in (4.11). In addition, we have the trace relation \( \langle T^{\mu}_{\mu} \rangle_c = sm \langle \bar{\psi} \psi \rangle_c \).

We have considered two contributions in the VEV of the energy-momentum tensor having a topological nature. The first one is generated by cosmic string in the uncompactified geometry and the second one is induced by the compactification. The combined topological part is expressed as \( \langle T^{\mu}_{\mu} \rangle_t = \langle T^{\mu}_{\mu} \rangle_{cs} + \langle T^{\mu}_{\mu} \rangle_{AdS} \). Alternatively, we can write \( \langle T^{\mu}_{\mu} \rangle_t = \langle T^{\mu}_{\mu} \rangle_c - \langle T^{\mu}_{\mu} \rangle_{AdS} \). By using the expressions for the separate parts, the topological contributions in the components with \( \mu = 0, 1, 3 \) are expressed as

\[
\langle T^{\mu}_{\mu} \rangle_t = \langle T^{\mu}_{\mu}\rangle^{(0)}_c + \frac{\sqrt{2}}{\pi^{1/2} a^3} \sum_{l=0}^{\infty} \cos(2\pi l \beta) \left[ \sum_{k=1}^{[q/2]} (-1)^k c_k \cos(2\pi k \alpha_0) F_{ma}(u_{lk}) + \frac{q}{\pi} \int_0^\infty du \frac{H(q, \alpha_0, u) F_{ma}(u_{lw})}{\cosh(2qu) - \cos(q\pi)} \right], \tag{4.14}
\]
and the remaining components are determined from the relations \( \langle T^2_{\mu \nu} \rangle_t = (1 + r \partial_r) \langle T^0_{\mu \nu} \rangle_t \) and \( \langle T^4_{\mu \nu} \rangle_t = \partial_t (L \langle T^0_{\mu \nu} \rangle_t) \). The prime in \( \sum_{l=0}^{\infty} \) indicates that the term with \( l = 0 \) is taken with the coefficient \( 1/2 \). The first term in the right-hand side of (4.14) is defined as (no summation over \( \mu \))

\[
\langle T^\mu_\mu \rangle^{(0)}_c = \frac{a^{-5}}{\sqrt{2\pi}^{5/2}} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \mathcal{F}_{ma} \left( 1 + \frac{l^2 L^2}{2w^2} \right), \tag{4.15}
\]

for \( \mu \neq 4 \) and \( \langle T^4_\mu \rangle^{(0)}_c = \partial_t [L \langle T^0_\mu \rangle^{(0)}_c] \). For \( q = 1 \) and \( \alpha_0 = 0 \) the second term in the right-hand side of (4.14) vanishes and, hence, the part (4.15) presents the VEV in the geometry where the cosmic string is absent ((4+1)-dimensional AdS spacetime with compactified \( z \)-direction). Consequently, the last term in (4.14) is induced by the planar angle deficit and by the magnetic flux along the string’s core. An alternative representation for the topological part, given by (A.3), is provided in appendix A.

### 4.2 Special cases

For a massless field one has the expression (3.28), and (4.12) is reduced to

\[
\langle T^0_\mu \rangle^{(0)}_c = \frac{6w^5}{\pi^2 a^5} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \left[ \sum_{k=0}^{[q/2]} (-1)^{k} c_k \cos(2\pi K) \sum_{l=0}^{\infty} \frac{1}{4r^2 s_k^2 + l^2 L^2} \right] + \frac{q}{\pi} \int_0^\infty du \frac{H(q, \alpha_0, u)}{\cosh(2qu) - \cos(q\pi)} \frac{1}{4r^2 \cosh^2(u + l^2 L^2)} \right] \tag{4.16}
\]

This result can be written as \( \langle T^0_\mu \rangle^{(0)}_c = (w/a)^5 \langle T^0_\mu \rangle^{(M)}_c \), where \( \langle T^0_\mu \rangle^{(M)}_c \) is the corresponding VEV in the Minkowski bulk. In the Minkowskian limit, \( a \rightarrow \infty \), and for a massive field, by using (3.34), we get

\[
\langle T^0_\mu \rangle^{(M)}_c = \frac{2^{3/2} m^0}{\pi^{5/2}} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \left[ \sum_{k=0}^{[q/2]} (-1)^{k} c_k \cos(2\pi K) f_{5/2}(m \sqrt{4r^2 s_k^2 + l^2 L^2}) \right] + \frac{q}{\pi} \int_0^\infty du \frac{f_{5/2}(m \sqrt{4r^2 \cosh^2(u + l^2 L^2)})}{\cosh(2qu) - \cos(q\pi)} H(q, \alpha_0, u) \right], \tag{4.17}
\]

where the function \( f_{5/2}(x) \) is given by (3.36). For large values of \( L \) the dominant contribution in (4.17) comes from the \( l = 1 \) term and the VEV is exponentially suppressed by the factor \( e^{-mL} \).

The case with \( q = 1 \) and \( \alpha_0 = 0 \) corresponds to (4+1)-dimensional locally AdS spacetime with a compactified \( z \)-coordinate in the absence of cosmic string. In this special case the nonzero contribution to the compactification part \( \langle T^\mu_\mu \rangle_c \) comes from the \( k = 0 \) term in (4.12). That term coincides with (4.15). The corresponding expression is further simplified for a massless field:

\[
\langle T^\mu_\mu \rangle^{(0)}_c \big|_{m=0} = \frac{3\pi^{-2}}{(aL/w)^5} \sum_{l=1}^{\infty} \frac{\cos(2\pi l \beta)}{l^5}. \tag{4.18}
\]

for \( \mu \neq 4 \) and \( \langle T^4_\mu \rangle^{(0)}_c = -4 \langle T^0_\mu \rangle^{(0)}_c \). This expression is conformally related (with the conformal factor \( (w/a)^3 \)) to the corresponding result in (4+1)-dimensional Minkowski spacetime.
with a single compact dimension of the length \( L \). For small values of the proper length of the compact dimension, compared with the curvature radius, one has \( L/w \ll 1 \) and by using the asymptotic expression (3.29) we see that to the leading order \( \langle T_{\mu}^\mu \rangle_{c}^{(0)} \approx \langle T_{\mu}^\mu \rangle_{c}^{(0)} \big|_{m=0} \). In this region the effect of the mass is weak. In the opposite limit of large values of the length of the compact dimension, \( L/w \gg 1 \), we use the approximation (3.30). In the leading order this gives

\[
\langle T_{\mu}^\mu \rangle_{c}^{(0)} \approx \frac{(2ma + 3)(2ma + 1)}{\pi^2 a^5 (L/w)^{2ma+5}} \sum_{l=1}^{\infty} \frac{\cos(2\pi l\beta)}{l^{2ma+5}},
\]

(4.19)

for \( \mu \neq 4 \) and \( \langle T_{4}^{4} \rangle_{c}^{(0)} \approx -2(ma + 2) \langle T_{0}^{0} \rangle_{c}^{(0)} \). For a massless field this result is exact.

Another special case corresponds to a magnetic flux in the absence of planar angle deficit. In this case \( q = 1 \) and the general formula (4.12) is simplified to

\[
\langle T_{0}^{0} \rangle_{c} = \langle T_{0}^{0} \rangle_{c}^{(0)} + \frac{\sqrt{2}\sin(\pi\alpha_0)}{\pi^2/2a^5} \sum_{l=1}^{\infty} \cos(2\pi l\beta) \int_{0}^{\infty} du \sinh(2\alpha_0 u) \tanh(u) F_{ma}(uL),
\]

(4.20)

with \( \langle T_{0}^{0} \rangle_{c}^{(0)} \) defined in (4.15).

### 4.3 Asymptotics and numerical results

Now let us investigate the asymptotic behavior of the compactification contribution at small and large distances from the string. For \( 2|\alpha_0| < 1 - 1/q \) we can directly put \( r = 0 \) and one gets

\[
\langle T_{\mu}^\mu \rangle_{c} \big|_{r=0} = [1 + 2h_0(q, \alpha_0)] \langle T_{\mu}^\mu \rangle_{c}^{(0)},
\]

(4.21)

for \( \mu \neq 4 \) and \( \langle T_{4}^{4} \rangle_{c} = [1 + 2h_0(q, \alpha_0)] \partial_\mu(L \langle T_{0}^{0} \rangle_{c}^{(0)}) \) at \( r = 0 \). In this range of the parameter \( \alpha_0 \) the part \( \langle T_{\mu}^\mu \rangle_{c} \) is finite on the string. For \( 2|\alpha_0| > 1 - 1/q \) the main contribution in (4.12) comes from the integral term. In the integral the contribution from large values of \( u \) dominates. Introducing a new integration variable \( x = \rho^2 e^{2u}/2 \), to the leading order we get

\[
\langle T_{0}^{0} \rangle_{c} \approx q \frac{2^{[\alpha_0]-1/2}q^{-1} \cos[\pi q(1/2 - |\alpha_0|)]}{\pi^7/2a^5(r/w)^{1+(2|\alpha_0|-1)q}} \sum_{l=1}^{\infty} \cos(2\pi l\beta) \times \int_{0}^{\infty} dx x^{(\alpha_0)-1/2}q^{-1/2} F_{ma} \left( 1 + x + \frac{l^2 L^2}{2w^2} \right).
\]

(4.22)

In this case the compactification contributions in the VEVs diverge on the string as \( 1/r^{1+(2|\alpha_0|-1)q} \).

At large distances from the string, corresponding to \( r/w \gg 1 \), to the leading order we get \( \langle T_{\mu}^\mu \rangle_{c} \approx \langle T_{\mu}^\mu \rangle_{c}^{(0)} \). The effects induced by the planar angle deficit and by the magnetic flux appear in the next order. In the corresponding expressions we use the asymptotic formula (3.30). In the expressions (4.13) we can ignore 1 and, then, the series over \( l \) is estimated as

\[
\sum_{l=1}^{\infty} \frac{\cos(2\pi l\beta)}{(b_0 + l^2 x^2)^{ma+5/2}} \approx -\frac{1}{2\tilde{b}^{2ma+5}},
\]

(4.23)

valid for \( x \ll 1 \). In this way we get

\[
\langle T_{0}^{q} \rangle_{c} \approx \langle T_{0}^{q} \rangle_{c}^{(0)} - \frac{(2ma + 3)(2ma + 1)}{\pi^2 a^5 (2r/w)^{2ma+5}} h_{2ma+5}(q, \alpha_0).
\]

(4.24)
Note that the correction induced by the planar angle deficit and by the magnetic flux does not depend on the compactification radius. From here it follows that \( \langle T_{\mu}^\nu \rangle_c - \langle T_{\mu}^\nu \rangle_c^{(0)} \approx \langle T_{0}^0 \rangle_c - \langle T_{0}^0 \rangle_c^{(0)} \). The correction to the component \( \langle T_{\mu}^\nu \rangle_c \) is obtained from (2.2) omitting VEV induced by cosmic string in (3+1)-dimensional AdS spacetime (the corresponding line

The second term in the right-hand side of (4.26), multiplied by \( q/\pi \), we see that the fall-off of the difference \( \langle T_{\mu}^\nu \rangle_t - \langle T_{\mu}^\nu \rangle_c^{(0)} \) at large distances from the string is stronger than the decay of separate terms \( \langle T_{\mu}^\nu \rangle_c \) and \( \langle T_{\mu}^\nu \rangle_c - \langle T_{\mu}^\nu \rangle_c^{(0)} \) in the topological part (4.14) of the energy-momentum tensor.

Now we consider the asymptotics with respect to the length of the compact dimension \( L \). For large values, \( L \gg r, w \), the function \( F_{ma}(u) \) in the expression (4.12) for the compactification contribution is approximated by (3.30). In the range \( 2|\alpha_0| < 1 - 1/q \) of the parameter \( \alpha_0 \), in (4.12) we can use the approximations \( u_{tk} \approx u_{tu} \approx (L/w)^2/2 \) and the leading term reads

\[
\langle T_{\mu}^\nu \rangle_c \approx [1 + 2h_0(q, \alpha_0)] \langle T_{\mu}^\nu \rangle_c^{(0)},
\]

for \( \mu = 0, 1, 2, 3 \), and \( \langle T_{4}^{4} \rangle_c \approx -2(ma + 2) \langle T_{0}^{0} \rangle_c^{(0)} \). The asymptotic for \( \langle T_{\mu}^\nu \rangle_c \) in (4.25) is given by (4.19) and, to the leading order, the effects of compactification do not depend on \( r \). For the region \( 2|\alpha_0| > 1 - 1/q \), the contribution of the integral term in (4.12) dominates. The main contribution to the integral comes from the integration range with \( u \sim \ln(L/r) \). Replacing the integrand by its large \( u \) asymptotic, one can show that \( \langle T_{\mu}^\nu \rangle_c \propto (w/L)^{2ma+5}(L/r)^{2|\alpha_0|q+1-q} \). Hence, for \( 2|\alpha_0| > 1 - 1/q \) the leading term in the compactification contribution depends on the radial coordinate and its decay is weaker compared to the case \( 2|\alpha_0| < 1 - 1/q \). As seen, the fall off of the contribution \( \langle T_{\mu}^\nu \rangle_c \) for large values of \( L \) follows power law as a function of the length of the compact dimension. Recall that for a massive field the corresponding decay in the Minkowski bulk is exponential, like \( e^{-\mu L} \).

For the investigation of the asymptotic in the region \( L \ll w \), we use an alternative representation (A.3) of the topological contribution. The leading term of the part \( \langle T_{0}^{0} \rangle_c^{(0)} \) is given by the right-hand side of (4.18) and it behaves as \( \langle T_{0}^{0} \rangle_c^{(0)} \sim (w/L)^5 \). The asymptotic behavior of the difference \( \langle T_{\mu}^\nu \rangle_t - \langle T_{\mu}^\nu \rangle_c^{(0)} \) for small \( L \) crucially depends on the value of \( \beta \). We have a periodicity with respect to that parameter, with the period 1, and without loss of generality it will be assumed that \( |\beta| < 1/2 \). For \( \beta = 0 \) the last contribution in (A.3) is dominated by the \( l = 0 \) term and, after evaluation of the integral over \( x \), for \( \mu = 0, 1, 3 \) we get

\[
\langle T_{\mu}^\nu \rangle_t \approx \langle T_{\mu}^\nu \rangle_c^{(0)} + \frac{\pi^2w}{\alpha L} \sum_{k=1}^{[g/2]} (-1)^k c_k \cos(2\pi k\alpha_0) F_{ma}^{(2)}(u_k) + \frac{q}{\pi} \int_0^\infty \frac{H(q, \alpha_0, x) F_{ma}^{(2)}(u_x)}{\cosh(2qx) - \cos(q\pi)} dx,
\]

where

\[
F_{ma}^{(2)}(u) = \frac{Q_{ma}^2(1-u) + Q_{ma}^2(u)}{u^2 - 1}.
\]

The second term in the right-hand side of (4.26), multiplied by \( aL/w \), coincides with the VEV induced by cosmic string in (3+1)-dimensional AdS spacetime (the corresponding line element is obtained from (2.2) omitting \( -dz^2 \)). In the special case \( \alpha_0 = 0 \) the latter coincides with the result given in [45]. For \( \beta = 0 \) one has \( \langle T_{0}^{0} \rangle_c^{(0)} \approx 3\pi^2\zeta(5)(aL/w)^5 \) and this term is dominant in (4.26).

For \( L \ll w \) and \( 0 < |\beta| < 1/2 \), assuming that \( |\beta| \) is not too small, the integral over \( x \) in (A.3) is dominated by the contribution from the region with \( x \sim w^2\pi|\beta - l/(Lr)| \gg 1 \). By
using the asymptotic of the modified Bessel function for large arguments [56], we can see that
the leading contribution to the last term in (A.3) coincides with the corresponding result for
a massless field. The latter is given by the last term in (A.4). If in addition \( L \ll r \), the
dominant contribution comes from the \( l = 0 \) term. Replacing the modified Bessel function
by its large argument asymptotic, we can see that the difference \( \langle T_{\mu}^{\nu} \rangle_1 - \langle T_{\mu}^{\nu} \rangle_0 \) behaves as
\( (rL/w^2)^{-5/2} \exp[-4\pi|\beta| \sin(\pi/q)r/L] \) for \( q \geq 2 \) and like \( (rL/w^2)^{-5/2}e^{-4\pi|\beta| r/L} \) for \( 1 \leq q < 2 \).
Hence, we conclude that for small \( L \) the topological contribution \( \langle T_{\mu}^{\nu} \rangle_1 \) is dominated by the
part \( \langle T_{0}^{0} \rangle_0 \) with the leading behavior given by the right-hand side of (4.18).

It remains to consider the behavior of the VEV near the boundary and horizon of the
AdS spacetime. Near the boundary we have \( w \ll L, r \) and the arguments of the function
\( \mathcal{F}_{ma}(u) \) in (4.12) are large. By using the asymptotic (3.30), it can be seen that the compact-
ification contribution \( \langle T_{\mu}^{\nu} \rangle_c \) vanishes on the AdS boundary like \( w^{2ma+5} \). Near the horizon
the ratios \( w/L \) and \( w/r \) are large and we use the asymptotic expression (3.29). The effect of
the mass is weak and the leading term coincides with the VEV for a massless field. Hence,
neart the horizon the compactification induced part behaves as \( w^5 \).

Figure 4 presents the dependence of the compactification part \( \langle T_{0}^{0} \rangle_c \) (in units of \( 1/a^5 \))
on the radial coordinate (left panel) and on the mass of the field (right panel). The left panel is plotted for \( q = 1, 2, 5, 3 \) (the numbers near the curves) and \( a_0 \rightarrow 0.3, \beta = 0.25, ma = 1, \frac{L}{w} = 1 \). On the right panel, the numbers near the curves correspond to the values of the ratio \( L/w \) (proper length of the compact dimension in units of the curvature scale \( a \) ) and the graphs are plotted for \( a_0 = 0.3, \beta = 0.25, q = 2.5, r/w = 1 \). For the graph on the left panel corresponding to \( q = 1 \) has \( 2|a_0| > 1 - 1/q \) and, in accordance with the asymptotic analysis given above, the VEV \( \langle T_{0}^{0} \rangle_c \) diverges on the string as \( 1/(r/w)^{1+(2|a_0|-1)q} \).

For \( q = 2.5 \), and \( q = 3 \) the compactification contribution is finite for \( r = 0 \). The corresponding
limiting values are given by (4.21) and they are equal to \(-6.7 \times 10^{-3}/a^5 \) and \(-2.4 \times 10^{-3}/a^5 \)
for \( q = 2.5, 3 \), respectively. For \( r/w \gg 1 \) one has \( \langle T_{0}^{0} \rangle_c - \langle T_{0}^{0} \rangle_0 \propto (w/r)^{2ma+5} \) (see (4.24)) and \( \langle T_{0}^{0} \rangle_c \) tends to \( \langle T_{0}^{0} \rangle_0 \). The latter does not depend on the parameters \( q \) and \( a_0 \) and in
that region the effects of the cosmic string are weak.

In figure 5 we plot the VEV \( \langle T_{0}^{0} \rangle_c \) versus the parameters \( a_0 \) (left panel) and \( \beta \) (right panel).
The graphs on the left panel we have taken \( \beta = 0.25, ma = 0.5, L/w = 1, r/w = 0.5 \), and the numbers near the curves represent the values of \( q \). On the right panel \( a_0 = 0.2, q = 2, ma = 0.5, r/w = 0.5 \), and the numbers near the curves correspond to the values of \( L/w \). The compactification contribution \( \langle T_{\mu}^{\nu} \rangle_c \) and its derivative with respect to the
parameter \( \beta \) are continuous at points \( \beta = \pm 1/2, \pm 3/2, \ldots \). As a function of the parameter \( \alpha \), the VEV \( \langle T_{\mu}^{\nu} \rangle_c \) is continuous at the points \( \alpha = \pm 1/2, \pm 3/2, \ldots \), but the derivative \( \partial_{\alpha} \langle T_{\mu}^{\nu} \rangle_c \) is discontinuous at those points.

The dependence of the vacuum energy density on the proper length of the compact
dimensions (in units of the curvature radius \( a \) ) is depicted in figure 6 for the values of the parameters \( q = 2.5, ma = 0.5, a_0 = 0.3, r/w = 1 \). The numbers near the curves correspond
to the values of \( \beta \). On the left panel we have plotted the contribution \( \langle T_{0}^{0} \rangle_0 \) that corresponds
to the geometry in the absence of the cosmic string (locally AdS spacetime with compactified
\( z \)-dimension). The right panel presents the quantity \( \langle T_{0}^{0} \rangle_1 - \langle T_{0}^{0} \rangle_0 \). It describes the effects
induced in the topological part by the cosmic string and magnetic flux along its axis. For
small values of \( L/w \) the effects of the mass are subdominant and the VEV on the left panel
behaves as \( \langle T_{0}^{0} \rangle_0 \propto (L/w)^{-5} \). In the opposite limit \( L/w \gg 1 \) one has \( \langle T_{0}^{0} \rangle_0 \propto (w/L)^{2ma+5} \).
As it has been explained by the asymptotic analysis above, the behavior of the difference $(T_{0}^{(0)}_{t} - T_{0}^{(0)}_{c})$ for small values of $L/w$ is essentially different for $\tilde{\beta} = 0$ and $0 < |\tilde{\beta}| < 1/2$. For $\tilde{\beta} = 0$ that difference behaves like $w/L$, whereas for $0 < |\tilde{\beta}| < 1/2$ it is exponentially suppressed (under the condition $L \ll r$). For large values of the ratio $L/w$ the effects of compactification are weak and $(T_{0}^{(0)}_{t} - T_{0}^{(0)}_{c})$ tends to $(T_{0}^{(0)}_{c})_{\text{cs}}$.

5 Conclusion

We have studied the influence of two sources of nontrivial topology on the VEV of the energy-momentum tensor for a massive fermionic field propagating in background of (4+1)-dimensional locally AdS spacetime. The effects are induced by cosmic string and by the compactification of a spatial dimension. An additional polarization of the vacuum is induced
Figure 6. The topological part in the vacuum energy density in AdS spacetime with a compact dimension, \(\langle T_0^0 \rangle_{\text{cs}}(0)\), (left panel) and the difference \(\langle T_0^0 \rangle_t - \langle T_0^0 \rangle_{\text{cs}}(0)\) (right panel) versus the proper length of the compact dimension (for the values of the parameters see the text).

by the background gravitational field. In odd number of spacetime dimensions one has two inequivalent irreducible representations for the Clifford algebra. We have shown that if the masses of the fields realizing different representations are the same then the corresponding VEVs of the energy-momentum tensor coincide. Note that the fermion condensates have opposite signs for those fields.

We have divided the investigation in two stages. The first corresponds to the geometry described by the line element (2.2) with \(-\infty < z < +\infty\). The mean energy-momentum tensor is decomposed into two contributions (see (3.19)). The first one is a purely gravitational contribution and corresponds to the VEV in AdS spacetime. The latter geometry is maximally symmetric and the part \(\langle T_{\mu\nu} \rangle_{\text{AdS}}\) is proportional to the metric tensor. The second contribution, denoted here as \(\langle T_{\mu\nu} \rangle_{\text{cs}}\), is induced by the cosmic string. For \(r > 0\) the local geometry is not changed by the presence of the string and the renormalization is needed only for the part \(\langle T_{\mu\nu} \rangle_{\text{AdS}}\). The vacuum energy-momentum tensor is diagonal. The components \(\langle T_{\mu\mu} \rangle_{\text{cs}}\) with \(\mu = 0, 1, 3, 4\), given by (3.21), coincide and the component \(\langle T_2^2 \rangle_{\text{cs}}\) is obtained from the relation (3.24). Note that, unlike in the fermionic case, for a massive scalar field the vacuum energy-momentum tensor will have a nonzero off-diagonal component. As a consequence of the maximal symmetry of the AdS spacetime, the string induced VEVs depend on the coordinates \(r\) and \(z\) in the form of the ratio \(r/z\). The latter presents the proper distance from the string measured in units of the curvature radius \(a\). At small distances from the string the effects of mass and of the background gravitational field are weak and the leading term in the asymptotic expansion is given by (3.31) for the components \(\mu = 0, 1, 3, 4\) and \(\langle T_2^2 \rangle_{\text{cs}} \approx -4\langle T_0^0 \rangle_{\text{cs}}\). In that region the total VEV of the energy-momentum tensor is dominated by the string induced contribution and behaves as \((w/r)^5\). At large distances the influence of the gravitational field is essential and the VEV \(\langle T_\mu^\mu \rangle_{\text{cs}}\) decays like \((w/r)^{2ma+5}\).

As a limiting case, we have obtained the VEV of the energy-momentum tensor for a fermionic field, induced by a cosmic string in background of (4+1)-dimensional Minkowski spacetime. It is expressed as (3.35) with an exponential decay at large distances from the cosmic string. The latter is in contrast to the power law fall off for the AdS bulk. Another special case corresponds to the absence of planar angle deficit. In this case the VEV is entirely induced by magnetic flux and is expressed as (3.39).
The compactification of the coordinate \( z \) on a circle with the length \( L \) induces additional topological contributions to the characteristics of the vacuum. The VEV of the corresponding energy density is given by the formula (4.12) and the other components are obtained by using the relations (4.11). An alternative representation of the topological part is derived in appendix A. Note that in the compactified geometry the components \( \langle T^0_0 \rangle_c \) and \( \langle T^4_4 \rangle_c \) differ. For a massless field, we have the conformal relation \( \langle T^\mu_\nu \rangle_c = (w/a)^5 \langle T^\mu_\nu \rangle_c^{(M)} \) with the corresponding VEV for \((4+1)\)-dimensional Minkowski bulk having spatial topology \( R^3 \times S^1 \).

In the case of a massive field the VEVs in the Minkowski spacetime are obtained taking the limit \( a \to \infty \), with fixed value of the coordinate \( y \), and the corresponding energy density is expressed as (4.17). For a magnetic flux in the absence of planar angle deficit the contribution in the energy density induced by compactification is given by (4.20). The compactification contribution in the VEV of the energy-momentum tensor is finite on the cosmic string in the range of parameters \( 2|\alpha_0| < 1 - 1/q \) and diverges like \( 1/r^{1+(2|\alpha_0|-1)/q} \) for \( 2|\alpha_0| > 1 - 1/q \).

In both cases, near the string the total VEV is dominated by the part \( \langle T^\mu_\mu \rangle_{cs} \). At large distances from the cosmic string the asymptotic of the compactification part in the VEV of the energy density is described by (4.24). The last term, with the power law decay \((w/r)^{2ma+5}\), corresponds to the correction induced by the planar angle deficit and by the magnetic flux and it does not depend on the compactification radius. For large values of the compactification length and for \( 2|\alpha_0| < 1 - 1/q \) the leading term in the asymptotic expansion for \( \langle T^\mu_\mu \rangle_c \) is given by (4.25) and the effects of compactification do not depend on \( r \). In the region \( 2|\alpha_0| > 1 - 1/q \) the leading term in the compactification contribution depends on the radial coordinate and its decay is weaker. For small \( L \) the leading term in the asymptotic expansion of the topological contribution \( \langle T^\mu_\mu \rangle_{t} \) does not depend on the planar angle deficit and on the magnetic flux and behaves as \((w/L)^5\). For small \( L \) the behavior of the difference \( \langle T^\mu_\mu \rangle_{t} - \langle T^\mu_\mu \rangle_c^{(0)} \) crucially depends on the parameter \( \tilde{\beta} \). For \( \tilde{\beta} = 0 \) it decays like \( w/L \) (see (4.26)), whereas for \( 0 < |\tilde{\beta}| < 1/2 \) one has an exponential fall off as a function of \( r/L \). The compactification contribution \( \langle T^\mu_\mu \rangle_c \) vanishes on the AdS boundary like \( w^{2ma+5} \) and behaves near the horizon as \( w^5 \).

Acknowledgments

A.A.S. was supported by the grant No. 20RF-059 of the Committee of Science of the Ministry of Education, Science, Culture and Sport RA. E.R.B.M. is partially supported by CNPQ under Grant no. 301.783/2019-3. W.O.S. thanks Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) for financial support.

A Alternative representation of the topological part

An alternative representation for the topological contribution (4.14) is obtained by using the relation (see also [48])

\[
\sum_{l=0}^{\infty} \cos(2\pi l \tilde{\beta}) e^{-\frac{L^2}{2L^2 x^2}} = \frac{w}{L^2} \sum_{l=0}^{\infty} e^{-\frac{2\pi^2 w^2}{L^2 x^2} (\tilde{\beta} - \lambda)^2},
\]

(A.1)

and the representation (3.23) for the function \( \mathcal{F}_{ma}(u) \). This relation is a direct consequence of the Poisson resummation formula. By taking into account the representation (3.23) and (A.1)
we get
\[\sum_{l=0}^{\infty} \cos(2\pi l \beta) \mathcal{F}_{ma} \left( b + \frac{L^2}{2w^2} \right) = \frac{\pi w}{2L} \int_0^\infty dx x^3/2 e^{-bx} [I_\nu(x) + I_{\nu+1}(x)] \sum_{l=-\infty}^{\infty} e^{-2x\pi^2 (\beta-l)^2}. \] (A.2)

Using this result in (4.14), for the components with \( \mu = 0, 1, 3 \) one finds
\[\langle T_{\mu}^{(0)} \rangle_t = \frac{\pi^{-3/2} w}{\sqrt{2a^6 L}} \int_0^\infty dx x^{3/2} e^{-x} [I_\nu(x) + I_{\nu+1}(x)] \sum_{l=-\infty}^{\infty} e^{-2x\pi^2 (\beta-l)^2} \]
\[\times \sum_{k=1}^{[q/2]} (-1)^k c_k \cos(2\pi k \alpha_0) e^{-2x\pi^2 s^2_{k,0}} + \frac{q}{\pi} \int_0^\infty du \frac{H(q,\alpha_0, u) e^{-2x\pi^2 \cosh^2 u}}{\cosh(2qu) - \cos(q\pi)}. \] (A.3)

The other components are obtained from the relations \( \langle T_2^{(0)} \rangle_t = (1 + r\partial_r) \langle T_0^{(0)} \rangle_t \) and \( \langle T_4^{(4)} \rangle_t = \partial_L (\langle T_4^{(0)} \rangle_t) \).

The representation (A.3) of the topological contribution is further simplified for a massless fermionic field. By taking into account that \( \nu = -1/2 \) and \( I_\nu(x) + I_{\nu+1}(x) = e^{x/\sqrt{2\pi}} \), after evaluating the integral over \( x \), we get
\[\langle T_0^{(0)} \rangle_t = \frac{2 (w/a)^5}{L^3 \pi^2} \sum_{l=-\infty}^{\infty} \langle \beta-l \rangle \left[ \sum_{k=1}^{[q/2]} (-1)^k c_k \cos(2\pi k \alpha_0) K_2 \left( 4\pi |\beta-l| s_k r/L \right) \right. \]
\[+ \left. \frac{q}{\pi} \int_0^\infty du \frac{H(q,\alpha_0, u) \cosh^{-2} u}{\cosh(2qu) - \cos(q\pi)} K_2 \left( 4\pi (r/L) |\beta-l| \cosh u \right) \right]. \] (A.4)

Note that convergence of the series in this representation is stronger than in (4.16).

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