Arrival time for the fastest among $N$ switching stochastic particles

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Abstract. The first arrivals among $N$ Brownian particles is ubiquitous in the life sciences, as it often triggers cellular processes from the molecular level. We study here the case where stochastic particles, which represent proteins or molecules can switch between two states inside the non-negative real line. The switching process is modeled as a two-state Markov chain and particles can only escape in state 1. We estimate the fastest arrival time by solving asymptotically the Fokker–Planck equations for three different initial distributions: Dirac-delta, uniformly distributed and long-tail decay. The derived formulas reveal that the fastest particle avoids switching when the switching rates are much smaller than the diffusion time scale, but switches twice when the diffusion in state 2 is much faster than in state 1. The present results are compared to stochastic simulations revealing the range of validity of the derived formulas.

1 Introduction

The process of escaping for the fastest particle is often studied as an extreme statistical event [1–10] where Brownian particles have to find a narrow window, which represents a small fraction of the explored space. The probability distribution function for the arrival time and the mean time $\bar{\tau}_N$ for the fastest particles among $N$, depend on: the geometry of the domain, the size and shape of the target, the boundary conditions and several other parameters. Sometimes, explicitly computations can be derived [5,7,11–13]. Interestingly, the mean time of the fastest particle depends also on the initial distribution [6,12,14], leading to various decay with respect to $N$.

In this paper we study the extreme statistic properties for an ensemble of Brownian particles that can switch between two states characterized by two different diffusion coefficients $D_1$ and $D_2$ (Fig. 1A). This question has gained much attention in the recent years due to the possibility of deriving explicit formulas [15–18]. The case of $J$ possible switching states, modeled by a Markov chain have been studied before in [16,19] as a discrete version of diffusing diffusivity models. In both papers, explicit formulas are derived to compute first arrival distributions for the non-negative real line. Extreme statistics formulas for the mean first arrival time have been derived in [15] for two-channel Markov additive diffusion in a 3-dimensional spherical domain. They proved that the first arrival behaviour of a Markov additive process cannot be adequately captured by averaging quantities of the effective constants.

In this paper we compute the distribution and the mean arrival time (MFAT) for the fastest particle to reach a target, and clarify how they depend on the initial distribution in the non-negative real line. To study the variability of the MFAT with respect to the number of particles $N$, we consider four different initial distributions (Fig. 1B). The particles can only escape in state 1, as shown in Fig. 1C. The main computations rely on the short-time asymptotics of the survival probability for the Fokker–Planck equation based on the dominant exponentially small terms.

The manuscript is organized as follows: Sect. 2 presents the stochastic model and the associated mixed boundary value problem for two coupled backward Fokker–Planck equations. In Sect. 3, we compute the asymptotic solutions for two Dirac delta initial conditions and different diffusion coefficients. In Sect. 4, we study the case where particles are initially uniformly distributed in an interval. In Sect. 5, we present the distribution of the arrival times for a long-tail initial distribution. Finally, in Sect. 6, we discuss some possible applications of the present results to elementary signaling in cell biology. In the appendix we discuss cases where short time asymptotic is not enough to obtain relevant estimations.

2 Stochastic model

In the present model, we consider $N$ identical independently distributed Brownian particles that can switch at Poissonian random times between two states
Kolmogorov equation is given by \[19\]

The limit as \(\Delta t \to 0\) of Brownian trajectories starting at \(x, t\) is absorbed at \(x = 2\) and \(\lambda p_1(x, t)\) for escaping. The fastest trajectory should move in the shortest path and be in state 1 at the absorbing boundary for escape.

**Fig. 1** Schematic figure for switching Brownian dynamics. A. Example of two switching Brownian motions with two states in a line and an absorbing boundary. The particle can escape only in state 1 (blue). The fastest trajectory (green) should move in the shortest path and be in state 1 at the absorbing boundary for escape. B. Four initial distributions. C. Realization of Brownian trajectories starting at \(x = 2\) and absorbed at \(x = 0\).

\[
\begin{align*}
1 \overset{k_{ij}}{\to} 2,
\end{align*}
\]

with rates \(k_{12}\) and \(k_{21}\). The particles can escape only in state 1. The stochastic equation for the position \(x(t, i)\) in state \(i\) of the particle is given for \(i, j = 1, 2\) by

\[
x(t+\Delta t, i) = \begin{cases} x(t, i) + \sqrt{2D} \Delta w_i(t) & \text{w.p } 1-k_{ij}\Delta t + o(\Delta t) \\ x(t, j) & \text{w.p } k_{ij}\Delta t + o(\Delta t), i \neq j \end{cases}
\]

where \(w_i(t) (i = 1, 2)\) are independent standard Brownian motions, \(\Delta w_i(t) = w_i(t+\Delta t) - w_i(t)\), and \(k_{ij}\) are the transition rates from state \(i\) to \(j\). This stochastic model postulates that at each time \(\Delta t\) a particle can either move by Brownian motion in a single state only or switch to a different state (Fig. 1C). The transition probability density function \(p(x, i, t|y, s, j)\) of the trajectory \(x(t, i)\) with the initial condition \(x(s, j) = x\), is the limit as \(\Delta t \to 0\) of the integral equations

\[
p(x, i, t + \Delta t|y, j, s) = \frac{1-k_{ij}\Delta t}{\sqrt{2\pi D_{ij}\Delta t}} \int_{\Omega} p(z, i, t|y, j, s) \times \exp \left\{ -\frac{|x-z|^2}{2D_{ij}\Delta t} \right\} \, dz + k_{ji}\Delta t p(x, l|y, j, s) + o(\Delta t)
\]

for \(i, j, l = 1, 2, i \neq j\). (2)

In the present case, we shall use the notation \(k_{12} = \lambda, k_{21} = \mu\), and \(p(x, 1, t|y, 0) = p_1(x, t)\) and \(p(x, 2, t|y, 0) = p_2(x, t)\). In the limit \(\Delta t \to 0\), the backward system of Kolmogorov equation is given by [19]

\[
\frac{\partial p_i}{\partial t}(x, t) = D_{ij} \frac{\partial^2 p_i}{\partial x^2}(x, t) - \lambda p_i(x, t) + \mu p_2(x, t)
\]

Our goal is to find the probability density function and the MFAT for the fastest particle in the domain \([0, +\infty)\), with the boundary conditions

\[
p_1(0, t) = 0, \quad \frac{\partial p_2}{\partial x}(0, t) = 0.
\]

When the particle starts in state 1 at point \(y > 0\), the initial conditions are given by

\[
P_1(x, 0) = \delta(x - y) \quad P_2(x, 0) = 0.
\]

We impose the normalization condition

\[
\int_0^{\infty} (p_1(x, 0) + p_2(x, 0)) \, dx = 1.
\]

When the particle starts in state 2 at point \(y > 0\), the associated initial conditions are

\[
P_1(x, 0) = 0 \quad P_2(x, 0) = \delta(x - y).
\]

### 3 Explicit expressions for the mfat when the Brownian particles start in state 1

To solve the system (3-5), we use the Laplace transform in the time domain

\[
\mathcal{L}(p_i(x, t)) = \hat{p}_i(x, q) = \int_0^{\infty} p_i(x, t)e^{-qt} \, dt \text{ for } i = 1, 2,
\]
and we get the system of two ordinary differential equations

\[ D_1 \frac{\partial^2 \hat{p}_1}{\partial x^2}(x, q) - (\lambda + q) \hat{p}_1(x, q) + \mu \hat{p}_2(x, q) + \delta(x - y) = 0 \]
\[ D_2 \frac{\partial^2 \hat{p}_2}{\partial x^2}(x, q) - (\mu + q) \hat{p}_2(x, q) + \lambda \hat{p}_1(x, q) = 0. \]

(7)

### 3.1 Particles start in state 1 and \( D_2 = 0 \)

To find the Laplace transform of the survival probability we set \( D_2 = 0 \) in the Laplace transformed backward system given by (7) and we use the initial condition (5) to set that all particles start in state 1. We also add the boundary conditions given by (4) to have a well defined problem. Then, we get the differential equation

\[ \left[ \frac{\partial^2}{\partial x^2} - \frac{q(q + \theta)}{D_1(q + \mu)} \right] \hat{p}_1(x, q) = -\frac{\delta(x - y)}{D_1}, \]

(8)

where \( \theta = \lambda + \mu \). We find the general solution of this equation by following the well known theory on diffusion equation [20,21]. The solution is given by the formula

\[ \hat{p}_1(x, q) = \sqrt{\frac{q + \mu}{4D_1[q + \theta]}} \exp \left\{ -\sqrt{\frac{q(q + \theta)}{D_1(q + \mu)}} |x - y| \right\}. \]

(9)

To satisfy the boundary condition \( \hat{p}_1(0, q) = 0 \), we finally get

\[ \hat{p}_1(x, q) = \sqrt{\frac{q + \mu}{4D_1[q + \theta]}} \exp \left\{ -\sqrt{\frac{q(q + \theta)}{D_1(q + \mu)}} |x - y| \right\} - \exp \left\{ -\sqrt{\frac{q(q + \theta)}{D_1(q + \mu)}} |x + y| \right\}, \]

(10)

and

\[ \hat{p}_2(x, q) = \frac{\lambda}{\sqrt{4D_1[q + \theta](q + \mu)}} \exp \left\{ -\sqrt{\frac{q(q + \theta)}{D_1(q + \mu)}} |x - y| \right\} + \exp \left\{ -\sqrt{\frac{q(q + \theta)}{D_1(q + \mu)}} |x + y| \right\}. \]

(11)

Thus, the Laplace transform of the survival probability is given by

\[ \hat{S}(q) = \frac{1}{q} - \frac{\exp \left\{ -\sqrt{\frac{q}{D_1 \lambda}} \right\}}{q} - \frac{\exp \left\{ -\sqrt{\frac{q}{D_1 \mu}} \right\}}{q^2} + \lambda y \exp \left\{ -\sqrt{\frac{q}{D_1 \mu}} \right\} \]

\[ + O \left( \frac{\exp \left\{ -\sqrt{\frac{q}{D_1 \mu}} \right\}}{q^2} \right). \]

(13)

Since \( \hat{S}(q) \) has no branch points we can then apply the inverse Laplace transform, and we have for \( t \) small, the approximation

\[ S(t) \approx 1 - \exp \left\{ -\sqrt{\frac{y}{4D_1 t}} \right\} + \lambda \text{erfc} \left( \sqrt{\frac{y}{4D_1 t}} \right) \approx 1 \]

\[ -\sqrt{\frac{y}{4D_1 t}} \exp \left\{ -\frac{y^2}{4D_1 t} \right\}. \]

(14)

This leads for large \( N \), following [5], to the asymptotic formula for the mean first arrival time

\[ \tau^N = \mathbb{E} \left\{ \tau^1 \right\} \approx \int_0^\infty \exp \left\{ N \ln \left( 1 - \sqrt{\frac{4D_1 t}{y \sqrt{\pi} e^{-\frac{y^2}{4D_1 t}}}} \right) \right\} dt \]

\[ \approx \frac{y^2}{4D_1 \ln \left( \frac{N}{\sqrt{\pi}} \right)}. \]

(15)

The probability density function for the first arrival time, denoted by \( Pr \{ \tau^1 = t \} \), in the case of \( t \) small is
The formula (15) is the same as in the case when no switching is considered [5]. This result, give us the idea that when the diffusion coefficient in state 2 is zero, the fastest particle does not waste time switching to a state where it cannot moves.

When we consider large switching rates, the distribution for the first arrival time is given by the inverse of the long-time expansion of the Laplace transform.

$$Pr\{\tau^1 = t\} = -\frac{d}{dt} S^N(t) \approx \frac{d}{dt} \left[ \exp \left\{ -\sqrt{4D_1 t} N \mu^2 \frac{y^2 - y^2}{\sqrt{\pi}} \right\} \right]$$

$$= \frac{N \mu^2}{y \theta^2 / \sqrt{\pi}} \exp \left\{ -\frac{y^2}{4D_1 \mu t} \right\}$$

$$\times \exp \left\{ -\sqrt{4D_1 t} N \mu^2 \frac{y^2 - y^2}{\sqrt{\pi}} \right\} \left[ \frac{y^2 \theta}{4D_1 \mu t^2} + \frac{1}{2t} \right].$$

Eq. (17)

To evaluate the range of validity of the present formula, we decided to compare the asymptotic distributions with stochastic simulations made in a finite interval. The statistics for the escape time of the fastest particle on the finite interval [0, a] where the initial point satisfies 2y < a is similar to the ones in the non-negative real line when the number N is large [5]. Indeed the trajectory of the fastest particle is located near the shortest geodesic ending to the shortest distance to the absorbing boundary [22, 23]. This means that the pdf of the exit time in the non-negative real line is a good approximation of the pdf of the arrival time in a finite interval, result that allow us to do the simulations in a finite interval that here we consider as [0, 5].

We generated trajectories until they reach the origin and selected the fastest one. Figure 2 shows the statistics for the simulation of a switching process, starting in state 1 with diffusion coefficients $D_1 = 1$ and $D_2 = 0$. We use for the initial number of particles $N = [500, 1000, 2500, 5000, 10,000]$ with a time step $\Delta t = 0.0001$. The switching rates are $\lambda = 100$ and $\mu = [1000, 2500, 5000, 7500, 9000]$. We obtain a good agreement between the long-time expression for the distribution (Eq. 17) of the arrival time and the stochastic simulations (Fig. 2A). We further confirm the decay of the MFAT vs N. We use a shift $\alpha$ in formula (15) to correct for possible switching: indeed the fastest particle should not switch when it starts in state 1 and $D_1 \gg D_2$. In addition, we report here a large value for the mean number $M_{\text{swit}}$ of switching (Fig. 2C) for the fastest particles. The numerical simulations suggest that this number is independent of the switching rate $\mu$ when $\mu \gg \lambda$. To clarify this property, we compute the number of switching during time $t$ using the probability distribution function for a given switching from 1 to 2 that occurs during a time $\tau^{1-2}$ and then back from 2 to 1, during the time $\tau^{2-1}$. Since the underlying processes are Poissonian with rate $\lambda$ and $\mu$ respectively, the probability that a full cycle of switching time occurs at time $t$ is given by the convolution

$$Pr\{\tau^{\text{swit}} = t\} = Pr\{\tau^{1-2} + \tau^{2-1} = t\}$$

$$= \int_0^t \lambda \exp(-\lambda(t-s)) \mu \exp(-\mu s) ds$$

$$= \frac{\lambda \mu}{\mu - \lambda} \exp(-\lambda t) - \exp(-\mu t) = f(t).$$

Thus, the probability of $m$ switching occurs in a time $t$ is the convolution

$$Pr\{m \text{ switching at time } t\} = Pr\{\tau^{\text{swit}} = t\} + \cdots + t^{m \text{ swit}} = f(t) * \cdots * f(t).$$

When the rate $\lambda$ dominates, formula (19) is the classical Poisson distribution for having $m$ events during time $t$

$$Pr\{m \text{ switching at time } t\} = \frac{(\lambda t)^m}{m!} \exp(-\lambda t).$$

Thus, the mean number of switching at time $t$ is given by

$$E(M(t)) = \sum_{m=1}^{\infty} m Pr\{M(t) = m\} = \lambda t,$$

where $M(t)$ is the amount of switching occurring until time $t$. Thus, during the fastest arrival, the mean number of switching is given by

$$M_{\text{swit}} = E(M(\tau^N)) = \lambda \tau^N.$$

We can now use this asymptotic behavior to fit the curve of Fig. 2C (black line) with 1/log(N). Interestingly, the condition for having no switching is $\lambda \tau^N < 1$, leading to

$$N > \sqrt[\mu]{\exp(\lambda \tau_1)},$$

where $\tau_1$ is the mean time for a Brownian particle to escape at 0, while starting at position $y$. We observe a good agreement between the mean number of switching with the logarithmic law for $\lambda = 210$ (Fig. 2C ) with a factor 2.1, due probably to the approximations made in (20).

### 3.2 Particles start in state 1 and $D_2 \neq 0$

To compute the escape time for the fastest particle when $D_2 \neq 0$, we start from system (7). We shall now distinguish two cases, when $D_1 = D_2$, and when they are different but strictly positive.
3.2.1 Particles start in state 1 and the diffusion coefficients satisfy $D_1 = D_2 = D$

The equation coming from the backward Kolmogorov equations (3) and the initial condition (5) is given by

$$\begin{aligned}
\frac{\partial^4}{\partial x^4} - \left( \frac{\lambda + q}{D_1} + \frac{\mu + q}{D_2} \right) \frac{\partial^2}{\partial x^2} + \left( \frac{(\lambda + q)(\mu + q) - \lambda \mu}{D_1 D_2} \right) 
\end{aligned}$$

$$\times \hat{p}_2(x, q) = \frac{\lambda}{D_1 D_2} \delta_0. \quad (24)$$

Now, we use the boundary conditions (4), but we set $D_1 = D_2 = D$. In this case, the solutions are given by the expressions

$$\hat{p}_1(x, q) = \frac{\lambda}{2\theta \sqrt{D_1 q \sqrt{q}}} \left( e^{-\frac{\lambda}{\sqrt{D_1 q}} |x-y|} - e^{-\frac{\lambda}{\sqrt{D_1 q}} |x+y|} \right) + \frac{\mu}{2\theta \sqrt{D_2 q \sqrt{q}}} \left( e^{-\frac{\mu}{\sqrt{D_2 q}} |x-y|} - e^{-\frac{\mu}{\sqrt{D_2 q}} |x+y|} \right),$$

$$\hat{p}_2(x, q) = -\frac{\lambda}{2\theta \sqrt{D_1 q \sqrt{q}}} \left( e^{-\frac{\lambda}{\sqrt{D_1 q}} |x-y|} + e^{-\frac{\lambda}{\sqrt{D_1 q}} |x+y|} \right) + \frac{\mu}{2\theta \sqrt{D_2 q \sqrt{q}}} \left( e^{-\frac{\mu}{\sqrt{D_2 q}} |x-y|} + e^{-\frac{\mu}{\sqrt{D_2 q}} |x+y|} \right), \quad (25)$$

where $\theta = \lambda + \mu$. The Laplace transform of the survival probability is
\[ \dot{S}(q) = \int_{\Omega} (\dot{\rho}_1(x,q) + \dot{\rho}_2(x,q)) \, dx \]
\[ = \int_0^\infty \left( \frac{1}{\sqrt{4Dq}} e^{-\sqrt{\frac{D}{q}x-y}} + \frac{\lambda - \mu}{\theta \sqrt{4Dq}} e^{-\sqrt{\frac{D}{q}x+y}} \right) \, dx, \tag{26} \]
and using the integrals
\[ \int_0^\infty e^{-|x-y|^2/\sigma^2} \, dx = \frac{\sqrt{4\pi}}{\sqrt{4\pi}} \left( 2 - e^{-y/\sigma^2} \right) \] and
\[ \int_0^\infty e^{-|x+y|^2/\sigma^2} \, dx = \frac{\sqrt{4\pi}}{\sqrt{4\pi}} e^{-y/\sigma^2}, \]
we get
\[ \dot{S}(q) = \frac{1}{q} - \mu e^{-\sqrt{\frac{D}{q}y}} - \lambda e^{-\sqrt{\frac{D}{q}y}} \left( \frac{1}{q} + \frac{1}{\sqrt{4Dq}} \right) \]
\[ + O \left( \frac{e^{-\sqrt{\frac{D}{q}y}}}{q^2} \right). \tag{28} \]

The inverse Laplace transform gives the approximation
\[ S(t) \approx \mathcal{L}^{-1} \left( \frac{1}{q} \right) - \mathcal{L}^{-1} \left( \frac{e^{-\sqrt{\frac{D}{q}y}}}{q} \right) \approx 1 \]
\[ - \text{erfc} \left( \frac{y}{\sqrt{4Dt}} \right) \approx 1 - \frac{e^{-\pi^2 t}}{y \sqrt{\pi}}. \tag{29} \]

We obtain, thus, following [5]
\[ \bar{\tau}^N = \int_0^\infty [S(t)]^N \, dt \]
\[ \approx \int_0^\infty \exp \left\{ \ln \left( 1 - e^{-\pi^2 t} \frac{\sqrt{4D}}{y \sqrt{\pi}} \right)^N \right\} \, dt \]
\[ \approx \frac{y^2}{4D \ln \left( \frac{N}{\sqrt{\pi}} \right)}. \tag{30} \]

We conclude that the fastest arriving particle does not switch between states, but escape in state 1, avoiding to change state. We see this clearly from the inverse Laplace transform of the survival probability. The leading order term here came from the diffusion process, so the fastest particle does no switch.

### 3.2.2 Particles start in state 1 and the diffusion coefficients satisfy \( D_1 \neq D_2 \)

In Appendix A.1, we present the details of the computations when the diffusion coefficients differ for each state and both are not zero: we derived an equation of order 4 that we solve using the non-homogeneous linear equation theory. We use the solution to compute asymptotically the survival probability that we shall now use. It is given by
\[ \dot{S}(q) \approx \frac{1}{q} - \frac{e^{-\sqrt{\frac{D}{q}y}}}{q} - \frac{\lambda \mu D_2^3}{(D_2 - D_1)^2} \frac{e^{-y/\sqrt{\pi}}}{q^3}. \tag{31} \]

When \( D_1 \gg D_2 \), using the inverse Laplace transform, we obtain the approximation for \( t \) small
\[ S(t) \approx \mathcal{L}^{-1} \left( \frac{1}{q} \right) - \mathcal{L}^{-1} \left( \frac{e^{-\sqrt{\frac{D}{q}y}}}{q} \right) \]
\[ \approx 1 - \text{erfc} \left( \frac{y}{\sqrt{4D t}} \right) \]
\[ \approx 1 - \frac{e^{-\pi^2 t}}{y \sqrt{\pi}}. \tag{32} \]

Finally, the mean arrival time of the fastest particle is given by
\[ \bar{\tau}^N = \int_0^\infty [S(t)]^N \, dt \]
\[ \approx \int_0^\infty \exp \left\{ \ln \left( 1 - e^{-\pi^2 \frac{\sqrt{4D t}}{y \sqrt{\pi}}} \right)^N \right\} \, dt \]
\[ \approx \frac{y^2}{4D_1 \ln \left( \frac{N}{\sqrt{\pi}} \right)}. \tag{33} \]

This result is identical to the one when the fastest particle does not switch. Thus, the fastest particle starting in state 1, with \( D_1 \gg D_2 \) does not switch to the state 2 where it moves slowly, but it keeps moving in state 1 until it reaches the absorbing boundary.

We shall now consider the case where \( D_2 \gg D_1 \); the leading order term in the expansion of \( \dot{S}(q) \) for large \( q \) is \( \frac{\lambda \mu D_2^3}{(D_2 - D_1)^2} \frac{e^{-y/\sqrt{\pi}}}{q^3} \). Thus, the approximation of the survival probability for \( t \) small is given by
\[ S(t) \approx \mathcal{L}^{-1} \left( \frac{1}{q} \right) - \frac{\lambda \mu D_2^3}{(D_2 - D_1)^2} \mathcal{L}^{-1} \left( \frac{e^{-\sqrt{\frac{D}{q}y}}}{q^3} \right) \approx 1 \]
\[ - \frac{\lambda \mu D_2^3 e^{-\pi^2 \sqrt{\frac{4D \sqrt{t}}{\pi}}}}{2(D_2 - D_1)^2 y \sqrt{\pi}}. \tag{34} \]
leading to the mean first arrival formula

\[ \bar{\tau}^N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{\lambda \mu D_2 e^{-\frac{\pi^2}{8D_2 t^2}}}{2\sqrt{\pi}} \right) \right\} \, dt \]

\[ \approx \frac{y^2}{4D_2 \ln \left( \frac{N\lambda \mu D_2}{2\sqrt{\pi}} \frac{y^2}{4D_2} \right)}. \] (35)

Formulas (33, 35) reveal the role of the diffusion coefficients in the escape process. There are two different strategies for the fastest particles depending on the diffusion coefficients: Starting in state 1 (the only state where particles can escape at the absorbing boundary), with \( D_1 \gg D_2 \), the fastest particle will simply diffuse until it reaches the target (Fig. 3A). However, when starting in state 1 with \( D_2 \gg D_1 \), the fastest particle will switch to state 2 where it can diffuse faster, it will then diffuse to the target and then switch back to state 1 before escaping (Fig. 3B). Interestingly the exponent of the factor \( \frac{y^2}{4D} \) inside the logarithm of the asymptotic formula (35) confirms that the fastest particle switches twice before it escapes.

### 3.3 Particles start in state 2

When the particles start in state 2 at position \( y > 0 \) we use the initial condition (6). To determine the distribution of arrival time, we use Laplace transform in time to get the ordinary differential equation

\[ \frac{\partial^4}{\partial x^4} - \left( \frac{q + \lambda}{D_1} + \frac{q + \mu}{D_2} \right) \frac{\partial^2}{\partial x^2} + \left( \frac{(q + \lambda)(q + \mu)}{D_1 D_2} - \lambda \mu \right) \]

\[ \hat{p}_1(x, q) = \frac{\mu}{D_1 D_2} \delta_y. \] (36)

#### 3.3.1 Solution when the particles start in state 2 for \( D_1 = D_2 = D \)

Similarly to the steps presented in the previous subsections, we find from the theory of non homogeneous linear equations the solution for system (3) with the initial condition (6) and boundary conditions (4). This solution is given by

\[ \hat{p}_1(x, q) = \frac{\mu}{\theta \sqrt{4Dq}} \left( e^{-\frac{\pi^2}{4D} |x-y|} + e^{-\frac{\pi^2}{4D} |x+y|} \right) \]

\[ - \frac{\mu}{\theta \sqrt{4D \sqrt{q + \theta}}} \left( e^{-\frac{\pi^2}{4D} |x-y|} - e^{-\frac{\pi^2}{4D} |x+y|} \right). \] (37)

The Laplace transform of the survival probability is

\[ \hat{S}(q) = \int_0^\infty (\hat{p}_1(x, q) + \hat{p}_2(x, q)) \, dx \]

\[ = \int_0^\infty \left( \frac{1}{\sqrt{4Dq}} e^{-\frac{\sqrt{\pi}}{\sqrt{q}} |x-y|} + \frac{\lambda - \mu}{\theta \sqrt{4Dq}} e^{-\frac{\sqrt{\pi}}{\sqrt{q}} |x+y|} \right) \, dx, \]

\[ = \frac{1}{q} - \frac{\mu e^{-y \sqrt{q}}}{\lambda + \mu} q + \frac{\mu e^{-y \sqrt{q}}}{\lambda + \mu} q + \theta. \] (39)

An expansion for large \( q \) give us the asymptotic expression

\[ \hat{S}(q) \approx \frac{1}{q} - \frac{\mu e^{-y \sqrt{q}}}{\lambda + \mu} \left( \frac{1}{q} + \frac{y}{\sqrt{4Dq}} \right) \]

\[ + \frac{\mu \theta}{8Dq^3} e^{-y \sqrt{\frac{\pi q}{8D}}} \left( qy^2 + D \left( 8 + 5 \sqrt{\frac{q}{D}} y \right) \right). \] (40)

Applying then the inverse Laplace transform, we have

\[ S(t) \approx L^{-1} \left( \frac{1}{q} \right) - \mu L^{-1} \left( \frac{y e^{-y \sqrt{q}/\sqrt{4Dq}}}{2q^2 + y e^{-y \sqrt{q}/\sqrt{4Dq}}} \right) \approx 1 \]

\[ - \mu \text{erfc} \left( \frac{y}{\sqrt{4Dt}} \right) \approx 1 - \mu e^{-y \sqrt{\frac{\pi q}{8D}}} \sqrt{\frac{4Dt}{y \sqrt{\pi}}}. \] (41)

Considering the second order term in the expansion, we get

\[ S(t) \approx 1 - \mu t \left( 1 - \frac{\lambda + \mu}{2} t \right) e^{-\frac{\pi^2}{8D} \sqrt{4Dt}} \frac{y}{\sqrt{\pi}}. \] (42)

Thus, using the asymptotic computation for the MFAT [5], we obtain

\[ \bar{\tau}^N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \mu t e^{-\frac{\pi^2}{8D} \sqrt{4Dt}} \frac{y}{\sqrt{\pi}} \right) \right\} \, dt \]

\[ \approx \frac{y^2}{4D \ln \left( \frac{N}{\sqrt{\pi}} \mu \frac{y^2}{4D} \right)}. \] (43)
To conclude, this result shows that the fastest particle switches only once before it escapes.

3.3.2 Particles start in state 2 with $D_1 \neq D_2$

We derived in Appendix A.2 the general solution for equation (36) when the two diffusion coefficients are different and strictly positives with boundary conditions (4). We use this solution to compute the survival probability by expanding in large $q$, so that when $D_2 \gg D_1$ we obtain

$$\hat{S}(q) \approx \frac{1}{q} - \frac{D_2 \mu}{D_2 - D_1} \exp \left\{ - \frac{\sqrt{q}}{D_2} y \right\}. \quad (44)$$

This leads to the asymptotic formula for $N \gg 1$

$$\varpi_N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \mu t \frac{e^{-\frac{y^2}{4D_2 t}}}{y \sqrt{\pi}} \right) \right\} \, dt \approx \frac{y^2}{4D_2 \ln \left( \frac{N}{\sqrt{\pi} \mu \frac{y^2}{4D_2}} \right)}. \quad (45)$$

This result shows that the fastest particle diffuses in state 2 until it reaches the target and then switches back to state 1 before escape. This is the optimal strategy that the fastest particle follows, as shown in Fig. 3C.

When $D_1 \gg D_2$, we have the Laplace transform of the survival probability given by the asymptotic expansion in large $q$

$$\hat{S}(q) \approx \frac{1}{q} - \frac{D_1 \mu}{D_1 - D_2} \exp \left\{ - \frac{\sqrt{q}}{D_2} y \right\}, \quad (46)$$

which leads to the asymptotic formula for $N \gg 1$

$$\varpi_N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \mu t \frac{e^{-\frac{y^2}{4D_1 t}}}{y \sqrt{\pi}} \right) \right\} \, dt \approx \frac{y^2}{4D_1 \ln \left( \frac{N}{\sqrt{\pi} \mu \frac{y^2}{4D_1}} \right)}. \quad (47)$$

This formula reveals that in this case, it is fastest if the particle switches from the beginning to state 1 and then diffuses with the biggest coefficient to arrive at the absorbing boundary, as shown by Fig. 3D. To conclude this section, our results reveal how the diffusion coefficients play a role in the order of the 2 possible actions to be done by the fastest particle (switching and diffusion). The limit case, when $D_1$ and $D_2$ are closed, is presented in Appendix A.2 showing the same law of decay in the asymptotic formula.

4 Particles are initially uniformly distributed in an interval

We now consider the system (3) in the domain $\Omega = [0, +\infty)$ with the boundary condition (4), and a uniform initial distribution in the interval $[0, y_0]$. When particles start in state 1, the initial condition is given by

$$p_1(x, 0) = \frac{1}{y_0} \mathbb{1}_{\{x \in [0, y_0]\}} \quad p_2(x, 0) = 0, \quad (48)$$

Fig. 3 Strategies followed by the fastest particle depending on the initial state and the diffusion coefficients. A Diffusion in state 1 onto the target (Initial state = 1 and $D_1 \gg D_2$). B Switching from state 1 to 2. Diffusion in state 2 onto the target. Switching from state 2 to 1 (Initial state = 1 and $D_2 \gg D_1$). C Diffusion in state 2 onto the target. Switching from state 2 to 1 (Initial state = 2 and $D_2 \gg D_1$). D Switching from state 2 to 1. Diffusion in state 1 onto the target. (Initial state = 2 and $D_1 \gg D_2$)
and when the particles start in state 2 we have the initial condition given by

\[ p_1(x, 0) = 0 \]
\[ p_2(x, 0) = \frac{1}{y_0} \mathbb{1}_{x \in [0, y_0)}. \]  

(49)

We impose the normalization condition

\[ \int_0^\infty (p_1(x, 0) + p_2(x, 0)) \, dx = 1. \]  

(50)

### 4.1 Particles start in state 1

Applying the Laplace transform to the system (3) with the initial condition (48) we get an ordinary differential equation

\[
\begin{align*}
\frac{\partial^4}{\partial x^4} - \left( \frac{\lambda + q}{D_1} + \frac{\mu + q}{D_2} \right) \frac{\partial^2}{\partial x^2} + \left( \frac{(\lambda + q)(\mu + q) - \lambda \mu}{D_1 D_2} \right) \\
\times \dot{p}_2(x, q) = \frac{\lambda}{y_0 D_1 D_2}. 
\end{align*}
\]

(51)

This equation has same homogeneous part as in (24) but with an indicator function as a non-homogeneous term. The solution of this equation is obtained by the convolution between the non-homogeneous function and the solution obtained with the Dirac delta function. Then, the Laplace transform of the survival probability in this case is also the convolution between the Laplace transform of the survival probability for the Dirac delta initial condition and the uniform distribution. When \( D_1 = D_2 = D \), from the expansion in large \( q \) (28), we get the approximation for the Laplace transform of the survival probability

\[
\tilde{S}(q) \approx \frac{1}{y_0} \int_0^{y_0} \left( 1 - e^{-\frac{y}{\sqrt{D}}} \right) \, dy \approx \frac{1}{q} - \frac{\sqrt{D}}{y_0 q^2}. 
\]

(52)

The inverse Laplace transform leads to the short-time asymptotic,

\[ S(t) \approx 1 - \frac{\sqrt{4Dt}}{y_0 \sqrt{\pi}}, \]  

(53)

and

\[
\tilde{\tau}^N = \int_0^\infty [S(t)]^N \, dt \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{\sqrt{4Dt}}{y_0 \sqrt{\pi}} \right) \right\} \, dt \\
\approx \frac{y_0 \sqrt{\pi}}{2DN^2}. 
\]

(54)

When \( D_1 \neq D_2 \), we obtain the asymptotic expansion in large \( q \) for the Laplace transform of the survival probability

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{\sqrt{D_1} \mu D_1^3}{y_0 q^2} - \frac{\lambda \mu D_2^3}{(D_2 - D_1)^3 y_0 q^2}. 
\]  

(55)

Since the leading order terms are not given anymore by exponential terms, then the approximation for the survival probability is given by

\[ S(t) \approx 1 - \frac{\sqrt{4D_1 t}}{y_0 \sqrt{\pi}}, \]  

(56)

and the asymptotic formula for the MFAT

\[
\tilde{\tau}^N = \int_0^\infty [S(t)]^N \, dt \\
\approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{\sqrt{4D_1 t}}{y_0 \sqrt{\pi}} \right) \right\} \, dt \\
\approx \frac{y_0^2 \sqrt{\pi}}{2D_1 N^2}.
\]

(57)

The fact that in this case the fastest particle does not switch is given by the initial distribution of the particles. In this case, for \( N \) large is always possible to find a particle that at the beginning is very close to the target, and then, the fastest particle does not need to switch states to escape faster.

### 4.2 Particles start in state 2

We have derived in the Appendix A.3 the computations to find the Laplace transform of the survival probability for the system (3) with the initial condition (49). When \( D_1 = D_2 \), the expansion in large \( q \) for the Laplace transform of the survival probability is given by the formula (Appendix A.3.1)

\[
\hat{S}(q) \approx \frac{1}{y_0} \int_0^{y_0} \left( 1 - \frac{\mu e^{-\frac{y}{\sqrt{D}}} (1 + \frac{y}{\sqrt{ADq}})}{y_0 \sqrt{\pi}} \right) \, dy \\
\approx \frac{1}{q} - \frac{3 \mu \sqrt{D}}{2y_0 q^2}. 
\]

(58)

Thus, we have to leading order the formula

\[
\tilde{\tau}^N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{\mu \sqrt{4D_1 t}}{y_0 \sqrt{\pi}} \right) \right\} \, dt \\
\approx \left( \frac{\sqrt{\pi y_0}}{\mu \sqrt{4DN}} \right)^{\frac{2}{3}} \Gamma \left( \frac{5}{3} \right),
\]

(59)

where \( \Gamma(x) \) is the Gamma function.

When \( D_1 \neq D_2 \), we have the asymptotic expansion in large \( q \) for the survival probability given by

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_1^3}{y_0 (D_2 - D_1)^3 q^2} - \frac{\mu D_2^3}{y_0 (D_2 - D_1)^3 q^2}. 
\]

(60)
For an initial condition given by
\[ N \]
To leading order we obtain the asymptotic formula
\[ 1 - D \]

When \( D_1 \gg D_2 \) we have the approximation
\[ \hat{S}(q) \approx \frac{1}{q} - \frac{3\mu D_1^2}{2y_0q^2}, \] (61)
leading to the asymptotic formula
\[ \tilde{\pi}^N \approx \left( \frac{\sqrt{\pi} y_0}{\mu \sqrt{4D_1 N}} \right)^\frac{3}{2} \Gamma\left( \frac{5}{3} \right). \] (62)

When \( D_2 \gg D_1 \) we have the approximation
\[ \hat{S}(q) \approx \frac{1}{q} - \frac{3\mu D_2^2}{2y_0q^2}, \] (63)
leading to the asymptotic formula
\[ \tilde{\pi}^N \approx \left( \frac{\sqrt{\pi} y_0}{\mu \sqrt{4D_2 N}} \right)^\frac{3}{2} \Gamma\left( \frac{5}{3} \right). \] (64)

With this initial condition, the fastest particles switch and diffuse depending on the diffusion coefficients: when \( D_1 \gg D_2 \) the fastest particle switches first and diffuses to the target, but if \( D_2 \gg D_1 \) it diffuses first to the target and then switches back to the state 1 before escape.

### 4.4 Particles start in state 2 uniformly distributed in \([y_1, y_2]\) with \( y_1 > 0 \)

In the case where the particles follow the initial condition \( p_2(x, 0) = \frac{1}{y_2 - y_1} \mathbb{1}_{(x \in [y_1, y_2])} \) with \( y_1 > 0 \), which means that the particles start in state 2, uniformly distributed in \([y_1, y_2]\), we have from Appendix A.3.3 that the asymptotic expansion in large \( q \) for the Laplace transform of the survival probability given by
\[ \hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_2^3}{(y_2 - y_1)(D_2 - D_1)q^{\frac{5}{2}}} \times \left( \exp \left\{ -\sqrt{q D_1} y_1 \right\} - \exp \left\{ -\sqrt{q D_1} y_2 \right\} \right) \]
\[ + \frac{\mu D_1^3}{(y_2 - y_1)(D_2 - D_1)q^{\frac{5}{2}}} \times \left( \exp \left\{ -\sqrt{q D_2} y_1 \right\} - \exp \left\{ -\sqrt{q D_2} y_2 \right\} \right). \] (69)

When \( D_2 \gg D_1 \) we have the approximation
\[ \hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_2^3}{(y_2 - y_1)q^{\frac{5}{2}}} \exp \left\{ -\sqrt{q D_1} y_1 \right\}. \] (70)

This leads to the asymptotic formula
\[ \tilde{\pi}^N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{(\sqrt{4D_1 t})^3}{2\sqrt{\pi}(y_2 - y_1)y_2^2} e^{-\frac{y_2^2}{2\sqrt{\pi} t}} \right) \right\} dt \approx \frac{y_2^2}{4D_1 \ln \left( \frac{N y_1}{2\sqrt{\pi}(y_2 - y_1)} \right)}. \] (71)
This leads to the asymptotic formula

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_2^\frac{1}{2}}{(y_2 - y_1)q^\frac{3}{2}} \exp \left\{ -\sqrt{\frac{q}{D_2}} y_1 \right\}. 
\]  

(72)

This leads to the asymptotic formula

\[
\pi^N \approx \int_0^\infty \exp \left\{ \ln \left[ 1 - \frac{\mu \sqrt{4D_2 t} e^{-\frac{q}{D_2} y_1^2}}{3\sqrt{\pi} (y_2 - y_1)} \right] \right\} \, dt \approx \frac{y_1^2}{4D_2 \ln \left( \frac{N\mu}{\sqrt{\pi}} \left( \frac{y_1^2}{D_2} \right) \right)}.
\]  

(73)

In this case, since \( D_2 \gg D_1 \), the fastest particle diffuses to the target and then it switches to state 1 to escape.

5 Initial distribution with a long tail

5.1 Particles start in state 1

We shall study the MFAT when the initial distribution of particles is given by

\[
p_1(x,0) = \frac{2b^{\frac{1}{1+\alpha}}}{\Gamma(\frac{1+\alpha}{2})} x^\alpha \exp \{ -bx^2 \}
\]

\[
p_2(x,0) = 0.
\]  

(74)

For this new initial condition the solution is also the convolution between this initial distribution and the solution for the Dirac delta case starting in state 1. Then, the Laplace transform of the survival probability is given by the formula

\[
\hat{S}(q) = \frac{2b^{\frac{1}{1+\alpha}}}{\Gamma(\frac{1+\alpha}{2})} \int_0^\infty \left[ \frac{1}{q} + \frac{(\mu + q - D_2 w_+^2) e^{w_+^2}}{D_1 D_2 w_+^2 (w_+^2 - w_-^2)} \right] y^\alpha \exp \{ -by^2 \} \, dy
\]

\[= \frac{1}{q} - \frac{\Gamma(1+\alpha) (\mu + q - D_2 w_+^2)}{\Gamma(\frac{1+\alpha}{2}) 2^{1+\alpha} b} D_1 D_2 w_+(w_+^2 - w_-^2) \times U \left( 1 + \frac{\alpha}{2}, \frac{3}{2}, \frac{w_+^2}{4b} \right) + \frac{\Gamma(1+\alpha) (\mu + q - D_2 w_-^2)}{\Gamma(\frac{1+\alpha}{2}) 2^{1+\alpha} b} D_1 D_2 w_-(w_+^2 - w_-^2) \times U \left( 1 + \frac{\alpha}{2}, \frac{3}{2}, \frac{w_-^2}{4b} \right), \]

(75)

where, \( U(a, b, z) \) is the confluent hypergeometric function, \( w_+ \) and \( w_- \) are given by the formula (90). Using the large \( q \) expansion of the survival probability given by formula (75), we have

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{\Gamma(1+\alpha) 2b^{\frac{\alpha+1}{2}}}{\Gamma(\frac{1+\alpha}{2})} \left[ \frac{D_1^{\frac{1+\alpha}{2}}}{q^{\frac{3}{2}}} + \lambda \mu D_2^{\frac{1+\alpha}{2}} q^{\frac{3+\alpha}{2}} \right].
\]  

(76)

In both cases, when \( D_1 \gg D_2 \) or \( D_2 \gg D_1 \), we obtain the approximation

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{2b^{\frac{\alpha+1}{2}} \Gamma(1+\alpha) D_1^{\frac{1+\alpha}{2}}}{\Gamma(\frac{1+\alpha}{2}) q^{\frac{3+\alpha}{2}}}. \]

(77)

Thus the inverse Laplace transform leads to

\[
S(t) \approx 1 - \frac{2b^{\frac{\alpha+1}{2}} \Gamma(1+\alpha) D_1^{\frac{1+\alpha}{2}}}{\Gamma(\frac{1+\alpha}{2}) (\frac{3+\alpha}{2})} e^{-\frac{t^{\frac{3+\alpha}{2}}}{2\Gamma(1+\alpha)}}. \]

(78)

Following the steps described in the previous section, we obtain that the mean arrival time for the fastest particle is given by

\[
\pi^N = \int_0^\infty \left[ S(t) \right]^N \, dt
\]

\[\approx \int_0^\infty \exp \left\{ \ln \left[ 1 - \frac{2b^{\frac{\alpha+1}{2}} \Gamma(1+\alpha) D_1^{\frac{1+\alpha}{2}}}{\Gamma(\frac{1+\alpha}{2}) q^{\frac{3+\alpha}{2}}} \right] \right\} \, dt
\]

\[\approx \frac{\Gamma(\frac{1+\alpha}{2}) \Gamma(\frac{3+\alpha}{2})}{2\Gamma(1+\alpha)} \left[ \frac{a + 3}{b D_1} \right]^\alpha \frac{1}{N \pi^{\frac{\alpha+3}{2}}}. \]

(79)

This means that the fastest particle never switches to state 2, even when \( D_2 \gg D_1 \). This is due to the initial distribution of the particles. As in the case where the particles where initially distributed in [0, \( y_0 \)], with this initial distribution is always possible to find a particle very closed to the absorbing boundary.

5.2 Particles start in state 2

When the particles are initially distributed in state 2 according to

\[
p_1(x,0) = 0
\]

\[
p_2(x,0) = \frac{2b^{\frac{1}{1+\alpha}}}{\Gamma(\frac{1+\alpha}{2})} x^\alpha \exp \{ -bx^2 \}, \]

(80)

we have derived the computations to find the survival probability in Appendix A.4. When \( D_1 = D_2 = D \), an asymptotic expansion in large \( q \) give us the approximation

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{(bD)^{\frac{1+\alpha}{2}} \mu (3+\alpha) \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2}) q^{\frac{3+\alpha}{2}}}, \]

(81)
leading to the short-time asymptotic

\[ S(t) \approx 1 - \left( \frac{bD}{\Gamma(\frac{1+\alpha}{2})} \mu (3+\alpha) \Gamma(1+\alpha) t^{\frac{2+\alpha}{2}} \right), \tag{82} \]

and the mean first arrival time

\[ \tau_N \approx \int_0^\infty \exp \left\{ \ln \left\{ 1 - \left( \frac{bD}{\Gamma(\frac{1+\alpha}{2})} \mu (3+\alpha) \Gamma(1+\alpha) t^{\frac{2+\alpha}{2}} \right) N \right\} \right\} dt \]

\[ \approx \left[ \frac{\Gamma(\frac{1+\alpha}{2}) \Gamma(\frac{5+\alpha}{2})}{\mu (3+\alpha) \Gamma(1+\alpha) N} \right]^{1/2} \frac{\Gamma(\frac{3+\alpha}{2})}{(bD) \mu^{3/2}}. \tag{83} \]

This decay for the mean first arrival time of the fastest particle for large \( N \), shows a single switch before escapes. When \( D_1 \neq D_2 \), we can have the approximation for the survival probability in the large \( q \) expansion given by

\[ \hat{S}(q) \approx \frac{1}{q} \left( 1 - \frac{2b^{\frac{1+\alpha}{2}} \mu \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2}) (D_2 - D_1) q^{\frac{3+\alpha}{2}}} \right)D_{1}^{\frac{2+\alpha}{2}} - D_{2}^{\frac{2+\alpha}{2}}. \tag{84} \]

When \( D_1 \gg D_2 \), we have then the approximation

\[ \hat{S}(q) \approx \frac{1}{q} \cdot \left( 1 - \frac{2b^{\frac{1+\alpha}{2}} \mu \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2}) q^{\frac{3+\alpha}{2}}} \right). \tag{85} \]

This, leads to the asymptotic formula

\[ \tau_N \approx \int_0^\infty \exp \left\{ \ln \left\{ 1 - \left( \frac{2b^{\frac{1+\alpha}{2}} \mu \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2})} \right) \right\} N \right\} dt \]

\[ \approx \left[ \frac{\Gamma(\frac{1+\alpha}{2}) \Gamma(\frac{5+\alpha}{2})}{2b^{\frac{1+\alpha}{2}} D_1^{\frac{1+\alpha}{2}} \Gamma(1+\alpha)} \right]^{1/2} \frac{\Gamma(\frac{3+\alpha}{2})}{N^{3/2}}. \tag{86} \]

When \( D_2 \gg D_1 \), we have the approximation in large \( q \)

\[ \hat{S}(q) \approx \frac{1}{q} \cdot \left( 1 - \frac{2b^{\frac{1+\alpha}{2}} \mu \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2}) q^{\frac{3+\alpha}{2}}} \right)D_{2}^{\frac{2+\alpha}{2}}. \tag{87} \]

This, leads to

\[ \tau_N \approx \int_0^\infty \exp \left\{ \ln \left\{ 1 - \left( \frac{2b^{\frac{1+\alpha}{2}} \mu \Gamma(1+\alpha)}{\Gamma(\frac{1+\alpha}{2})} \right) \right\} N \right\} dt \]

\[ \approx \left[ \frac{\Gamma(\frac{1+\alpha}{2}) \Gamma(\frac{5+\alpha}{2})}{2b^{\frac{1+\alpha}{2}} D_1^{\frac{1+\alpha}{2}} \Gamma(1+\alpha)} \right]^{1/2} \frac{\Gamma(\frac{3+\alpha}{2})}{N^{3/2}}. \tag{88} \]

To conclude, in this case the fastest particle switches to state 1 only once, but depending if \( D_1 \gg D_2 \) it switches at the very beginning before diffuses, or in the opposite, if \( D_2 \gg D_1 \) it diffuses and then it switches at the very end once the particle is on the absorbing boundary.

6 Discussion and concluding remarks

In the present manuscript, we obtained several asymptotic formulas \([15, 33, 35, 45, 47, 43, 54, 59, 79, 83]\) for the mean time of the fastest Brownian particles arriving at the absorbing target. These particles can switch between two states and escape from the non-negative real line only in state 1. Resulting formulas are associated with different initial distributions and whether the particles start all in state 1 or 2. We found a decay in \( \frac{1}{\ln N} \) when the initial distribution does not intersect the target location. However, when the initial distribution \( \rho_0(x) \) intersects the absorbing boundary, we obtain an algebraic decay for \( N \), e.g. \( \frac{1}{N^\alpha} \).

When the initial distribution of particles is given a Dirac delta distribution in state 1, where escape is possible, the escape time strategy depends on the diffusion coefficients: when \( D_1 \gg D_2 \) the fastest particle does not switch before escaping. However, when \( D_2 \gg D_1 \) the fastest particle switches twice. When particles start in state 2, depending on the value for the largest diffusion coefficient, the fastest particles will either switch from the beginning and then diffuses or it can diffuse and then switch near the absorbing target. We remark that the number of switchings is given by the exponent of the factor \( \frac{2}{bD} \) inside the logarithm of formulas \([33, 35, 45, 47]\).

These extreme statistics formulas are relevant in the context of fast molecular signaling in cell biology. The activation is a MFAT and depends on the main parameters, involving the geometrical organisation of the domain, the initial distribution of the molecules as well as the initial number of them, and the specific dynamics of the particles (diffusion, switching and/or other stochastic dynamics). This approach can be used to compute the time of activation by diffusing molecules crossing a region where switching is possible. Indeed, the behaviour of the fastest particle described in this paper can be extended to the case of a finite interval \([5]\).

The analytical computations presented here provide a quantitative description of switching associated to transcription factors (TF) moving in the nucleus cell toward their promoter site. If a large number of switching states are considered, the computations and formulas would rapidly become impractical and unwieldy. However, we believe that the present work could be generalized for the problem in 2d or 3d, but in these domains the size of the target \( \varepsilon \) becomes a relevant parameter that needs to be integrated in the analytical formulas. For a narrow target, we predict a term of the type \( \frac{1}{\ln(\frac{1}{\varepsilon})} \) in the leading order term of the survival probability, as previously derived for the non-switching case \([5]\).

To explore the range of validity for formulas \([33, 35, 45, 47]\), when Brownian particles start at position \( y > 0 \) (Dirac-Dirac case), we performed several stochastic simulations for different values of \( \lambda, \mu, D_1, \) and \( D_2 \) corresponding to each scenario of Fig. 3. The results of these simulations for small values of \( \lambda \) and \( \mu \) are shown in Appendix A.5. In all cases, where at least one switching is necessary before escaping, our asymptotic for-
The response of the reaction network [18]. Cytoplasm or the nucleus can be used to characterize ous states of phosphorylation and location (inside the

Fig. 4 Application of the MFAT. A Example of Calcium ions entering through a ionic channel. These ions can switch states between free \( Ca^{2+} \) or bound to calmodulin \( CaM \), occurring diffusion. This binding interferes with the fast activation of the Ryanodine receptor [23]. B Example of a TF moving randomly inside a cell nucleus, alternating between 1d and 3d diffusion. The time for the first TF to activate an enhancer falls into the class of MFAT

...mula (15) for the MFAT as shown in Fig. 2B, for \( N \) not so large.

Key chemical reactions occurring in cells depend on the arrival of the first molecules to small targets [24]. This is the case for a large class of agonist molecules arriving to a gated channel [25] located on the cell membrane, triggering signaling cascades [19,26,27]. Upon arrival of the first molecules, channels open and thus it is not necessary to track the rest of the agonist population in the process of cellular activation [28,29]. This shows that the statistics of the fastest arrival time is a key event in revealing the time scale of sub-cellular processes triggered by single channel activation.

In the field of cellular signaling transduction, the time of the first signaling protein to reach a target in various states of phosphorylation and location (inside the cytoplasm or the nucleus) can be used to characterize the response of the reaction network [18].

Distinguishing between single- and multi-state dynamics at molecular level remains difficult [30], there are ubiquitous examples where particles such as molecules and TF have to switch between different states before reaching a small target site [29]. For example, when particles are injected slowly in a domain, an extended initial distribution can build up, leading to a long-tail distribution. This distribution could be approximated by a Gaussian or any other related distribution with even an algebraic decay, especially when the motion can be modeled as anomalous diffusion [31]. To cover this case, we studied here different initial distributions such as exponentials. For example, calcium ions enter in less than a few milliseconds inside a dendrite or neuronal synapses through few channels located on the membrane, as shown in Fig. 4A. When channels are closed again, the calcium concentration has already spread over the entire domain. These calcium ions can also change their state due to possible chemical reactions. One classical example is the interaction between calcium ions and calmodulin molecules \( CaM \) [32]. Calcium signaling in dendritic spines, is often mediated by the fastest escaping calcium ion to Ryanodine receptors. During this fast process, the fastest calcium ions do not bind to a \( CaM \) molecule, as it would lead to a much longer arrival time compared to the one observed experimentally [33]. This result can be explained by the present theory, showing that the fast particle does not switch to a state were it diffuses slower. Another example is the case of a TF, the motion of which can be described as Brownian inside the nucleus. TFs can alternate motion between a 1D sliding along the DNA and a 3D motion inside the nucleus (Fig. 4B), with diffusion coefficients \( D_1 \neq D_2 \) and switching at Poissonian rates \( \lambda \) and \( \mu \). Although a lot of the literature was dedicated to the case of a single TF [34], it is possible that the fastest \( TF \) arrives to the promoter site directly without switching, a scenario that should be further studied.

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**Appendix**

In this appendix, we summarized the main computations involving the Laplace transform of the survival probability taking into account the different initial conditions (spatial and state). We also added numerical simulations that emphasize a deviation with the asymptotic formulas for the Dirac-delta initial condition, under some specific values of the parameters.

### A.1 Computations of the survival probability when all particles start at position $y > 0$ in state 1 with $D_1 \neq D_2$

In the general case, when $D_1 \neq D_2$ and both are strictly positive, we can derive from system (7) the differential equation of order 4

\[
\frac{\partial^4 \varphi}{\partial x^4} - \left( \frac{\lambda + q}{D_1} + \frac{\mu + q}{D_2} \right) \frac{\partial^2 \varphi}{\partial x^2} + \left( \frac{\lambda + q}{D_1} - \frac{\lambda \mu}{D_1 D_2} \right) \varphi = f(x).
\]

\[
p_2(x, y) = \frac{\lambda}{D_1 D_2} \delta(y).
\]

We are going to solve these equation with the boundary conditions (4) and the initial condition given for $p_2(x, y)$ in (5). Using the set of smooth compact support $f(x)$ in $\mathbb{R}$ such that

\[
\frac{\partial^4 \varphi}{\partial x^4} - a \frac{\partial^2 \varphi}{\partial x^2}(x) + b \varphi(x) = f(x),
\]

we shall compute $\int_{\mathbb{R}} G(x) f(x) dx$, where $G(x)$ is the solution of the homogeneous equation and $a = \frac{\lambda + q}{D_1} + \frac{\mu + q}{D_2}$ and $b = \frac{\lambda + q - \lambda \mu}{D_1 D_2}$. Integrating by parts and using the fact that the derivatives of $G(x)$ until order 2 are continuous in $\mathbb{R}$, we get

\[
\int_{\mathbb{R}} G(x) f(x) dx = \int_{\mathbb{R}} G(x) \left( \frac{\partial^4 \varphi}{\partial x^4} - a \frac{\partial^2 \varphi}{\partial x^2}(x) + b \varphi(x) \right) dx
\]

\[
= \varphi(y) \left( \frac{\partial^3 G_+}{\partial x^3} (y) - \frac{\partial^3 G_-}{\partial x^3} (y) \right)
\]

\[
= \varphi(y) (2Au_1^3 + 2Bw_+^3 + 2Cw_3^3 + 2 Dw_{-3}).
\]

If we choose $A$, $B$, $C$ and $D$ such that $2Au_1^3 + 2Bw_+^3 + 2Cw_3^3 + 2Dw_{-3} = \frac{\lambda}{D_1 D_2}$, we get

\[
\int_{\mathbb{R}} G(x) f(x) dx = \langle G(x), \frac{\partial^4 \varphi}{\partial x^4} (x) - a \frac{\partial^2 \varphi}{\partial x^2} (x) + b \varphi(x) \rangle
\]

\[
= \langle \frac{\partial^4 G}{\partial x^4} (x) - a \frac{\partial^2 G}{\partial x^2} (x) + b G(x), \varphi(x) \rangle
\]

\[
= \frac{\lambda}{D_1 D_2} \varphi(y) = \frac{\lambda}{D_1 D_2} \delta(x - y), \varphi(x).
\]

Then,

\[
\frac{\partial^4 G}{\partial x^4} (x) - a \frac{\partial^2 G}{\partial x^2} (x) + b G(x) = \frac{\lambda}{D_1 D_2} \delta(x - y).
\]

Using the previous condition for $G(x)$ be a solution in the sense of Distributions

\[
2Au_1^3 + 2Bw_+^3 + 2Cw_3^3 + 2 Dw_{-3} = \frac{\lambda}{D_1 D_2},
\]

and the condition coming from the fact that all derivatives until order 2 are continuous

\[
2Au_1 + 2Bw_+ + 2Cw_3 + 2 Dw_{-3} = 0,
\]

we can write the solution under the following form

\[
G(x, q) = \left( \frac{Bw_++Cw_3+Dw_{-}}{w_1} \right) e^{w_1|x-y|} + B e^{w_1|x+y|}
\]

\[
+ \left( \frac{2D_1D_2w_+(w_3^2-w_1^2)}{w_3(w_3^2-w_2^2)} \right) e^{w_1|x+y|} - \frac{2Dw_{-}(w_2^2-w_3^2)}{2w_3(w_3^2-w_1^2)} e^{w_2|x+y|} + D e^{w_2|x-y|}.
\]

Since the solution is bounded, we obtain that $A = C = 0$, and this leads to

\[
B = \frac{\lambda}{2D_1 D_2 w_+(w_3^2-w_2^2)}, \quad D = \frac{\lambda}{2D_1 D_2 w_-(w_2^2-w_3^2)},
\]

and the solution is

\[
p_2(x, y) = \frac{\lambda}{2D_1 D_2 w_+(w_3^2-w_2^2)} e^{w_1|x-y|} + \frac{\lambda}{2D_1 D_2 w_-(w_2^2-w_3^2)} e^{w_2|x-y|}.
\]

Finally, the boundary conditions leads to

\[
p_2(x, y) = \frac{\lambda}{2D_1 D_2 w_+(w_3^2-w_2^2)} e^{w_1|x-y|} + \frac{\lambda}{2D_1 D_2 w_-(w_2^2-w_3^2)} e^{w_2|x+y|}.
\]

Using the relation between $p_1(x, q)$ and $p_2(x, q)$ given in (7), we find that

\[
p_1(x, q) = \frac{(\mu + q - D_2 w_2^2)}{2D_1 D_2 w_+(w_3^2-w_2^2)} e^{w_1|x-y|} + \frac{(\mu + q - D_2 w_2^2)}{2D_1 D_2 w_-(w_2^2-w_3^2)} e^{w_2|x-y|},
\]

and using the boundary conditions $p_1(0, q) = 0$, we get

\[
p_1(x, q) = \frac{(\mu + q - D_2 w_2^2)}{2D_1 D_2 w_+(w_3^2-w_2^2)} (e^{w_1|x-y|} - e^{w_1|x+y|})
\]

\[
+ \frac{(\mu + q - D_2 w_2^2)}{2D_1 D_2 w_-(w_2^2-w_3^2)} (e^{w_2|x-y|} - e^{w_2|x+y|}).
\]
where

$$
w_{\pm} = \sqrt{\frac{\lambda + \mu}{D_1} + \mu \pm \frac{\mu}{D_2} \pm q^2 \left( \frac{1}{D_1} - \frac{1}{D_2} \right)^2} + 2q \left( \frac{1}{D_1} - \frac{1}{D_2} \right) \left( \frac{\lambda}{D_1} - \frac{\mu}{D_2} \right) + \left( \frac{\lambda}{D_1} + \frac{\mu}{D_2} \right)^2 - \sqrt{2}.
$$

(90)

To compute the Laplace transform of the survival probability, we start with

$$
\hat{S}(q) = \int_0^{\infty} (\hat{p}_1(x, q) + \hat{p}_2(x, q)) dx
$$

$$
= \frac{(\mu + q - D_2 w_+^2)}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \int_0^{\infty} \left( e^{w_+ |x-y|} - e^{w_- |x+y|} \right) dx
$$

$$
+ \frac{(\mu + q - D_2 w_-^2)}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \int_0^{\infty} \left( e^{w_- |x-y|} - e^{w_+ |x+y|} \right) dx
$$

$$
+ \frac{\lambda}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \int_0^{\infty} \left( e^{w_+ |x-y|} + e^{w_- |x+y|} \right) dx
$$

$$
+ \frac{\lambda}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \int_0^{\infty} \left( e^{w_- |x-y|} + e^{w_+ |x+y|} \right) dx
$$

$$
= \frac{1}{q} + \frac{T_1(q) + T_2(q)}{q},
$$

Expanding $T_1(q)$ and $T_2(q)$ for $q$ large, we have the approximation for the survival probability

$$
\hat{S}(q) \approx \frac{1}{q} - \frac{e^{-\sqrt{\frac{T_1}{D_2}}} e^{-\sqrt{\frac{T_2}{D_1}}}}{T_1 D_2 D_2 \delta y},
$$

and this leads to a solution under the form

$$
p_1(x, q) = \frac{\mu}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} e^{w_+ |x-y|}
$$

$$
+ \frac{\mu}{2D_1 D_2 w_- (w_-^2 - w_+^2)} e^{w_- |x+y|},
$$

as in the above computations. Finally, the boundary conditions impose that

$$
p_1(x, q) = \frac{\mu}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \left( e^{w_+ |x-y|} - e^{w_+ |x+y|} \right)
$$

$$
+ \frac{\mu}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \left( e^{w_- |x-y|} - e^{w_- |x+y|} \right),
$$

Then, we obtain

$$
p_2(x, q) = \frac{(\lambda + q - D_1 w_+^2)}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} e^{w_+ |x-y|}
$$

$$
+ \frac{(\lambda + q - D_1 w_-^2)}{2D_1 D_2 w_- (w_-^2 - w_+^2)} e^{w_- |x+y|},
$$

and from the boundary condition, $\frac{\partial p_2}{\partial x}(0, q) = 0$, we get

$$
p_2(x, q) = \frac{(\lambda + q - D_1 w_+^2)}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \left( e^{w_+ |x-y|} + e^{w_+ |x+y|} \right)
$$

$$
+ \frac{(\lambda + q - D_1 w_-^2)}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \left( e^{w_- |x-y|} + e^{w_- |x+y|} \right).
$$

The Laplace transform of the survival probability leads to

$$
\hat{S}(q) = \int_0^{\infty} (\hat{p}_1(x, q) + \hat{p}_2(x, q)) dx
$$

$$
= \frac{\mu}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \int_0^{\infty} \left( e^{w_+ |x-y|} - e^{w_+ |x+y|} \right) dx
$$

$$
+ \frac{\mu}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \int_0^{\infty} \left( e^{w_- |x-y|} - e^{w_- |x+y|} \right) dx
$$

$$
+ \frac{\lambda + q - D_1 w_+^2}{2D_1 D_2 w_+ (w_+^2 - w_-^2)} \int_0^{\infty} \left( e^{w_+ |x-y|} + e^{w_+ |x+y|} \right) dx
$$

$$
+ \frac{\lambda + q - D_1 w_-^2}{2D_1 D_2 w_- (w_-^2 - w_+^2)} \int_0^{\infty} \left( e^{w_- |x-y|} + e^{w_- |x+y|} \right) dx.$$
$$\int_{0}^{\infty} (e^{w_{-}|z-y|} + e^{w_{-}|z+y|}) \, dx$$

$$= -\frac{\mu}{D_{1}D_{2}w_{+}^{2}(w_{+}^{2} - w_{-}^{2})} (1 - e^{w_{+}y})$$

$$- \frac{\mu}{D_{1}D_{2}w_{-}^{2}(w_{+}^{2} - w_{-}^{2})} (1 - e^{w_{-}y})$$

$$- \frac{\lambda + q - D_{1}w_{-}^{2}}{D_{1}D_{2}w_{-}^{2}(w_{+}^{2} - w_{-}^{2})} - \frac{\mu}{D_{1}D_{2}w_{+}^{2}(w_{+}^{2} - w_{-}^{2})}$$

$$= \frac{\mu}{D_{1}D_{2}w_{+}^{2}(w_{+}^{2} - w_{-}^{2})} (1 - e^{w_{+}y})$$

$$- \frac{\mu}{D_{1}D_{2}w_{-}^{2}(w_{+}^{2} - w_{-}^{2})} (1 - e^{w_{-}y})$$

$$- \frac{\lambda + q - D_{1}w_{-}^{2}}{D_{1}D_{2}w_{-}^{2}(w_{+}^{2} - w_{-}^{2})} - \frac{\mu}{D_{1}D_{2}w_{+}^{2}(w_{+}^{2} - w_{-}^{2})}$$

$$= \frac{1}{q} + T_{3}(q) - T_{4}(q),$$

where $w_{+}$ and $w_{-}$ are given by the formula (90), and

$$T_{3}(q) = \frac{\mu}{D_{1}D_{2}w_{+}^{2}(w_{+}^{2} - w_{-}^{2})} e^{w_{+}y},$$

$$T_{4}(q) = \frac{\mu}{D_{1}D_{2}w_{-}^{2}(w_{+}^{2} - w_{-}^{2})} e^{w_{-}y}.$$

Rewriting $\alpha = \frac{\lambda}{D_{1}} + \frac{\mu}{D_{2}}$, $\beta = \frac{1}{D_{1}} - \frac{1}{D_{2}}$, $\gamma = \frac{\lambda}{D_{1}} - \frac{\mu}{D_{2}}$ and $\eta = \frac{1}{D_{1}^{2}} + \frac{1}{D_{2}^{2}}$, and working in $T_{3}(q)$ and $T_{4}(q)$, we obtain

$$T_{3}(q) = \frac{2\mu}{D_{1}D_{2}} \exp \left\{ -\frac{\alpha + \eta q + \sqrt{\beta^{2}q^{2} + 2\gamma \beta q + \alpha^{2}}}{2} \right\}$$

$$- \ln \left( \alpha + \eta q + \sqrt{\beta^{2}q^{2} + 2\gamma \beta q + \alpha^{2}} \right),$$

$$T_{4}(q) = \frac{2\mu}{D_{1}D_{2}} \exp \left\{ -\frac{\alpha + \eta q - \sqrt{\beta^{2}q^{2} + 2\gamma \beta q + \alpha^{2}}}{2} \right\}$$

$$- \ln \left( \alpha + \eta q - \sqrt{\beta^{2}q^{2} + 2\gamma \beta q + \alpha^{2}} \right).$$

Expanding $T_{3}(q)$ and $T_{4}(q)$ for $q$ large, we obtain an expansion for the survival probability given by the expression

$$\hat{S}(q) = \frac{1}{q} - \frac{D_{2}\mu}{D_{2} - D_{1}} \exp \left\{ -\frac{\sqrt{q}}{\sqrt{\tau_{2}(1+\gamma)}} \right\}$$

$$+ \frac{D_{1}\mu}{D_{2} - D_{1}} \exp \left\{ -\frac{\sqrt{q}}{\sqrt{\tau_{1}(1+\gamma)}} \right\}$$

$$+ O \left( \frac{\exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\} + \exp \left\{ -\sqrt{\frac{q}{\tau_{2}(1+\gamma)}} \right\}}{q^{2}} \right).$$

The expression above contains two exponentially small terms. In the limit $D_{1}$ close to $D_{2}$ when $D_{2} > D_{1}$, we use the expansion $D_{2} = D_{1}(1 + \varepsilon)$ and studying the limit when $\varepsilon$ goes to zero, we have

$$\hat{S}_{\varepsilon}(q) \approx \frac{1}{q} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}$$

$$\times \left[ 1 + \frac{\exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\} - \exp \left\{ -\sqrt{\frac{q}{\tau_{2}(1+\gamma)}} \right\}}{\varepsilon \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}} \right].$$

The expansion in $\varepsilon$ and $q\varepsilon$ leads to

$$\hat{S}_{\varepsilon}(q) = \frac{1}{q} - \frac{\mu}{q^{2}} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}$$

$$+ \frac{1}{q^{2}} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\} + \frac{1}{q} \exp \left\{ -\sqrt{\frac{q}{\tau_{2}(1+\gamma)}} \right\}$$

$$+ O(\varepsilon) + O(\varepsilon q).$$

When $\varepsilon \to 0$, the survival probability $\hat{S}_{\varepsilon}(q)$ converges to $\hat{S}_{0}(q)$ corresponding to the solution for $D_{1} = D_{2}$, defined by Eq. (41). However, the convergence is not uniform in $t$ in the interval $[0, \infty]$, preventing us to use this expansion to estimate the MFAT for this case. Thus to leading order, using that

$$\lim_{D_{2} \to D_{1}} \Pr \{ t_{1} > t \} \approx 1 - \frac{e^{\frac{\mu^{2}}{\sqrt{\pi} D_{1} t}}}{y \sqrt{\pi}},$$

we obtain the asymptotic formula for $N \gg 1$

$$\tau_{N} \approx \int_{0}^{\infty} \exp \left\{ \left( \frac{1}{q} - \frac{\mu}{q^{2}} \right) \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\} \right\} \, dt$$

$$\approx \frac{N^{2} \varepsilon}{4D_{1} \ln \left( \frac{N^{2} \varepsilon}{\sqrt{\tau_{1}(1+\gamma)}} \right)} + A_{c},$$

where $A_{c} = A_{0} + \varepsilon A_{1} + \ldots$ and $A_{k}$ are constants. To conclude, to leading order in $\varepsilon$, the MFAT for the case when $D_{1} \neq D_{2}$ is similar to the case $D_{1} = D_{2}$.

When $D_{1} > D_{2}$, we study the limit case by making the expansion $D_{1} = D_{2}(1 + \varepsilon)$ and studying the limit when $\varepsilon$ goes to zero. In that case, we have

$$\hat{S}_{\varepsilon}(q) \approx \frac{1}{q} - \frac{\mu}{q^{2}} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}$$

$$+ \frac{1}{q^{2}} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}$$

$$+ \frac{1}{q} \exp \left\{ -\sqrt{\frac{q}{\tau_{2}(1+\gamma)}} \right\} - \exp \left\{ -\sqrt{\frac{q}{\tau_{2}(1+\gamma)}} \right\} \right\}.$$

The expansion in $\varepsilon$ and $q\varepsilon$ leads to

$$\hat{S}_{\varepsilon}(q) = \frac{1}{q} - \frac{\mu}{q^{2}} \exp \left\{ -\sqrt{\frac{q}{\tau_{1}(1+\gamma)}} \right\}$$

$$+ O(\varepsilon) + O(\varepsilon q).$$

When $\varepsilon \to 0$, the survival probability $\hat{S}_{\varepsilon}(q)$ converges to $\hat{S}_{0}(q)$ (equation (41)) corresponding to the solution for $D_{1} = D_{2}$. Working as we did before, we have the limit

$$\lim_{D_{1} \to D_{2}} \Pr \{ t_{1} > t \} \approx 1 - \frac{e^{\frac{\mu^{2}}{\sqrt{\pi} D_{2} t}}}{y \sqrt{\pi}}.$$
A.3 Particles start uniformly distributed in an interval

A.3.1 Particles start in state 2 with \( D_1 \neq D_2 \) uniformly distributed in \([0, y_0]\)

When the particles start in state 2 uniformly distributed, the Laplace transform applied to the survival probability is given by the convolution

\[
\hat{S}(q) = \frac{1}{y_0} \int_0^{y_0} \left[ 1 + \frac{\mu e^{w-y}}{D_1 D_2 w_+^2 (w_+^2 - w_-^2)} \right] dq
\]

\[
\approx \frac{1}{q} - \frac{\mu}{y_0 D_1 D_2 w_+^2} \left( w_+^2 - w_-^2 \right)
\]

\[
\approx \frac{1}{q} - \frac{\mu}{y_0 D_1 D_2 w_+^2} \left( w_+^2 - w_-^2 \right)
\]

\[
= \frac{1}{q} + T_7(q) + T_8(q),
\]

where \( T_7(q) = -\frac{\mu D_1^2}{y_0 (D_2 - D_1)^2 q^2} \) and \( T_8(q) = \frac{\mu D_1^2}{y_0 D_1 D_2 w_+^2 (w_+^2 - w_-^2)} \). Using the expansion in \( q \) large for \( T_7(q) \) and \( T_8(q) \), we obtain

\[
T_7(q) \approx \frac{\mu D_1^2}{y_0 (D_2 - D_1)^2 q^2}
\]

and

\[
T_8(q) \approx -\frac{\mu D_1^2}{y_0 (D_2 - D_1)^2 q^2}.
\]

In the limit \( D_2 = D_1 (1 + \varepsilon) \), when \( D_2 > D_1 \), an expansion in \( \varepsilon \) leads to

\[
\hat{S}_t(q) \approx \frac{1}{q} - \frac{\mu D_1^2}{y_0 q^2} \left( 1 + \varepsilon \right)^2 - D_1 \varepsilon^2
\]

\[
- \frac{\mu D_1^2}{y_0 q^2} \left( \sum_{n=1}^{\infty} \left( \frac{2}{n+1} \right) \varepsilon^n \right).
\]

When \( \varepsilon \to 0 \), the survival probability \( \hat{S}_t(q) \) converges to \( \hat{S}_0(q) \) corresponding to the solution when \( D_1 = D_2 \), given by equation (58). Thus to leading order, using that

\[
\lim_{D_2 \to D_1} \Pr \{ t_1 > t \} \approx 1 - \mu \frac{\sqrt{AD_2 t}}{y_0 \sqrt{\pi}},
\]

we obtain the asymptotic formula for \( N \gg 1 \)

\[
\tau_N \approx \int_0^{\infty} \exp \left\{ \ln \left( 1 - \frac{\mu \sqrt{AD_2 t}}{y_0 \sqrt{\pi}} \right) \right\} dt
\]

\[
\approx \Gamma \left( \frac{5}{3} \right) \left( \frac{y_0 \sqrt{\pi}}{\mu \sqrt{AD_2 t}} \right)^{\frac{2}{3}} \frac{1}{N^{\frac{4}{3}} + A_\varepsilon},
\]

where \( A_\varepsilon = A_0 + \varepsilon A_1 + \ldots \) and \( A_k \) are constants as before. When \( D_1 > D_2 \), we have then,

\[
\hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_1^2}{y_0 (D_1 - D_2)^2 q^2} + \frac{\mu D_1^2}{y_0 (D_1 - D_2)^2 q^2}.
\]

Making now, \( D_1 = D_2 (1 + \varepsilon) \) and expanding in \( \varepsilon \), we have

\[
\hat{S}_t(q) \approx \frac{1}{q} - \frac{\mu D_1^2}{y_0 q^2} \left( 1 + \varepsilon \right)^2 - D_1 \varepsilon
\]

\[
- \frac{\mu D_1^2}{y_0 q^2} \left( \sum_{n=1}^{\infty} \left( \frac{3}{n+1} \right) \varepsilon^n \right).
\]

When \( \varepsilon \to 0 \), the survival probability \( \hat{S}_t(q) \) converges to \( \hat{S}_0(q) \), corresponding to the solution when \( D_1 = D_2 \), given by the Eq. (58). For the same reasons as above, to leading order, using that

\[
\lim_{D_1 \to D_2} \Pr \{ t_1 > t \} \approx 1 - \mu \frac{\sqrt{AD_2 t}}{y_0 \sqrt{\pi}},
\]

we obtain the asymptotic formula for \( N \gg 1 \)

\[
\tau_N \approx \int_0^{\infty} \exp \left\{ \ln \left( 1 - \frac{\mu \sqrt{AD_2 t}}{y_0 \sqrt{\pi}} \right) \right\} dt
\]

\[
\approx \Gamma \left( \frac{5}{3} \right) \left( \frac{y_0 \sqrt{\pi}}{\mu \sqrt{AD_2 t}} \right)^{\frac{2}{3}} \frac{1}{N^{\frac{4}{3}} + A_\varepsilon},
\]

where \( A_\varepsilon = A_0 + \varepsilon A_1 + \ldots \) and \( A_k \) are constants. To conclude, to leading order, we obtain a formula for the MFAT when \( D_1 \) and \( D_2 \) are closed and the initial condition is uniformly distributed in \([0, y_0]\) with the same law \( \frac{3}{2} \) when they are different.

A.3.2 Particles start in state 1 uniformly distributed in \([y_1, y_2]\) with \( y_1 > 0 \)

Considering now the initial condition \( p_1(x, 0) = \frac{1}{y_2 - y_1} \) \( \mathbb{1}_{x \in [y_1, y_2]} \), when the particles start in state 1, we obtain the Laplace transform of the survival probability given by the convolution

\[
\hat{S}(q) = \frac{1}{q} + \frac{(q + \mu - D_2 w_+^2)}{(y_2 - y_1) D_1 D_2 w_+^2 (w_+^2 - w_-^2)}
\]

\[
- \frac{(q + \mu - D_2 w_+^2)}{(y_2 - y_1) D_1 D_2 w_+^2 (w_+^2 - w_-^2)}
\]

\[
= \frac{1}{q} + T_9(q) + T_{10}(q),
\]

where \( T_9(q) = \frac{(q + \mu - D_2 w_+^2)}{(y_2 - y_1) D_1 D_2 w_+^2 (w_+^2 - w_-^2)} \) and \( T_{10}(q) = \frac{(q + \mu - D_2 w_+^2)}{(y_2 - y_1) D_1 D_2 w_+^2 (w_+^2 - w_-^2)} \). Using the expansion in \( q \) large we get

\[
T_9(q) \approx \frac{D_1^2}{2^2 q^2} \left( \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_2 \right\} - \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_1 \right\} \right).
\]

\[
T_{10}(q) \approx \frac{\lambda \mu D_1^2}{2^2 (D_1 - D_2)^2 q^2} \left( \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_2 \right\} - \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_1 \right\} \right).
\]

This leads to

\[
\hat{S}_t(q) \approx \frac{1}{q} - \frac{D_1^2}{2^2 (y_2 - y_1) q^2}
\]

\[
\times \left( \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_2 \right\} - \exp \left\{ -\sqrt{\frac{2 \pi}{3}} y_1 \right\} \right)
\]

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\[ \frac{\lambda y \mu D_2}{2(y_2 - y_1)(D_1 - D_2)^2 q} \times \left( \exp \left\{-\frac{q}{D_1} y_2 \right\} - \exp \left\{-\frac{q}{D_2} y_1 \right\} \right). \]

A.3.3 Particles start in state 2 uniformly distributed in \([y_1, y_2] \]

In the case where the particles follows the initial condition
\[ \rho_2(x, 0) = \frac{1}{y_2 - y_1} \mathbf{1}_{\{x \in [y_1, y_2]\}}, \]
meaning that the particles start in state 2 uniformly distributed in \([y_1, y_2]\), the Laplace transform of the survival probability given by the convolution
\[ \hat{S}(q) = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \left( \frac{\mu e^{w-y}}{D_1 D_2 w_2^+ (w_2^+ - w^2)} - \frac{\mu e^{y-w}}{D_2 D_1 w_1^- (w_1^- - w^2)} \right) dy \]
\[ = \frac{1}{q} \left( \frac{\mu (e^{w-y_2} - e^{w-y_1})}{D_1 D_2 w_2^+ (w_2^+ - w^2)} - \frac{\mu (e^{y-w_2} - e^{y-w_1})}{D_2 D_1 w_1^- (w_1^- - w^2)} \right) \]
\[ = \frac{1}{q} T_{11}(q) + T_{12}(q), \]

where \(T_{11}(q) = \frac{\mu (e^{w-y_1} - e^{w-y_2})}{D_1 D_2 w_2^+ (w_2^+ - w^2)} \) and \(T_{12}(q) = \frac{\mu (e^{y-w_2} - e^{y-w_1})}{D_2 D_1 w_1^- (w_1^- - w^2)} \), and making the Taylor expansion in \(q \) large for \(T_{11}(q)\) and \(T_{12}(q)\) we get

\[ T_{11}(q) \approx -\frac{D_2^3 \mu}{(D_2 - D_1) q^2} \times \left( \exp \left\{-\frac{q}{D_1} y_2 \right\} - \exp \left\{-\frac{q}{D_2} y_1 \right\} \right), \]
\[ T_{12}(q) \approx \frac{D_1^3 \mu}{(D_1 - D_2) q^2} \times \left( \exp \left\{-\frac{q}{D_2} y_1 \right\} - \exp \left\{-\frac{q}{D_1} y_2 \right\} \right). \]

We can have the approximation for the Laplace transform of the survival probability given by

\[ \hat{S}(q) \approx \frac{1}{q} - \frac{\mu D_1^2}{y_2 - y_1} \times \left( \exp \left\{-\frac{q}{D_1} y_1 \right\} - \exp \left\{-\frac{q}{D_2} y_2 \right\} \right) \]
\[ + \frac{\mu D_2^2}{y_2 - y_1} \times \left( \exp \left\{-\frac{q}{D_2} y_1 \right\} - \exp \left\{-\frac{q}{D_1} y_2 \right\} \right), \]

and making \(y_2 = y_1 (1 + \epsilon)\), we have

\[ \hat{S}_\epsilon(q) = \frac{1}{q} - \frac{\mu D_1^2}{y_1 \epsilon(D_2 - D_1) q^2} \exp \left\{-\frac{q}{D_1} y_1 \right\} \]
\[ \times \left( 1 - \exp \left\{-\frac{q}{D_2} y_1 \epsilon \right\} \right) \]
\[ + \frac{\mu D_2^2}{y_1 \epsilon(D_2 - D_1) q^2} \exp \left\{-\frac{q}{D_2} y_1 \right\} \times \left( 1 - \exp \left\{-\frac{q}{D_1} y_1 \epsilon \right\} \right). \]

Using the expansion for the exponential function we have

\[ \hat{S}_\epsilon(q) \approx \frac{1}{q} - \frac{\mu D_1^2}{(D_2 - D_1) q^2} \times \sum_{n=1}^{\infty} \left( -\frac{q}{D_1} y_1 \epsilon \right)^n \]
\[ + \frac{\mu D_2^2}{(D_2 - D_1) q^2} \times \sum_{n=1}^{\infty} \left( -\frac{q}{D_2} y_1 \epsilon \right)^n. \]

When \(\epsilon \to 0\), the survival probability \(S_\epsilon(t)\) converges to \(S_0(t)\) corresponding to an initial condition for the Dirac delta function at position \(y_1\) for \(D_1 \neq D_2\). Thus to leading order, using that

\[ \lim_{\epsilon \to 0} \Pr \{t_1 > t\} \approx 1 - \frac{\mu t \sqrt{4D_1 t}}{\sqrt{\pi}} \left[ e^{-\frac{y_1^2}{4D_1 t}} \right], \]

we obtain the asymptotic formula for \(N \gg 1\)

\[ \tau_\epsilon^N \approx \int_0^\infty \exp \left\{ \ln \left( 1 - \frac{\mu t \sqrt{4D_1 t}}{\sqrt{\pi}} \left[ e^{-\frac{y_1^2}{4D_1 t}} \right] \right) \right\} dt \]
\[ \approx \frac{y_1^2}{4D_1 \ln \left( \frac{N \mu y_1^2}{4\pi} \right) + A_\epsilon}, \]

where \(A_\epsilon = A_0 + \epsilon A_1 + \ldots\), where \(A_k\) are constants. When \(D_1 > D_2\), we have the expansion for the Laplace transform of the survival probability

\[ \hat{S}(q) = \frac{1}{q} - \frac{\mu D_1^2}{y_2 - y_1} \times \left( \exp \left\{-\frac{q}{D_1} y_1 \right\} - \exp \left\{-\frac{q}{D_2} y_2 \right\} \right) \]
\[ + \frac{\mu D_2^2}{y_2 - y_1} \times \left( \exp \left\{-\frac{q}{D_2} y_1 \right\} - \exp \left\{-\frac{q}{D_1} y_2 \right\} \right), \]

and making \(y_2 = y_1 (1 + \epsilon)\), we have

\[ \hat{S}_\epsilon(q) = \frac{1}{q} - \frac{\mu D_1^2}{y_1 \epsilon(D_2 - D_1) q^2} \exp \left\{-\frac{q}{D_1} y_1 \right\} \]
\[ \times \left( 1 - \exp \left\{-\frac{q}{D_2} y_1 \epsilon \right\} \right) \]
\[ + \frac{\mu D_2^2}{y_1 \epsilon(D_2 - D_1) q^2} \exp \left\{-\frac{q}{D_2} y_1 \right\} \times \left( 1 - \exp \left\{-\frac{q}{D_1} y_1 \epsilon \right\} \right). \]
Then, using the Taylor expansion of the exponential functions we have
\[
\hat{S}_\varepsilon(q) = \frac{1}{q} - \frac{\mu D_1}{(D_1 - D_2)q^2} \left( -\sqrt{\frac{q}{D_1}} \varepsilon \right) + \frac{\mu D_1}{(D_1 - D_2)q^2} \left( -\sqrt{\frac{q}{D_1}} \varepsilon \right) n + \frac{\mu D_2}{(D_1 - D_2)q^2} \left( -\sqrt{\frac{q}{D_2}} \varepsilon \right) n.
\]

When \( \varepsilon \to 0 \), the survival probability \( S_\varepsilon(t) \) converges to \( S_0(t) \) corresponding to an initial condition for the Dirac-delta function at position \( y_1 \) for \( D_1 \neq D_2 \). Using the same reasoning as above, to leading order, using that
\[
\lim_{D_1 \to D_2} \text{Pr} \{ t_1 > t \} \approx 1 - \frac{\mu \sqrt{4D_2 t}}{\sqrt{\pi}} \left[ \frac{e^{-\frac{y_1^2}{4D_2 t}}}{y_1} \right],
\]
we obtain the asymptotic formula for \( N \gg 1 \)
\[
\Psi_N \approx \int_0^{\infty} \exp \left\{ n \left[ 1 - \frac{\mu \sqrt{4D_2 t}}{\sqrt{\pi}} \left[ \frac{e^{-\frac{y_1^2}{4D_2 t}}}{y_1} \right] \right] \right\} dt
\approx \frac{y_1^2}{4D_2 \ln \left( \frac{N \sqrt{\mu \frac{y_1^4}{4D_2}}}{a} \right) + A_e},
\]
where \( A_e = A_0 + \varepsilon A_1 + \ldots \), where \( A_k \) are constants.

**A.4 The initial distribution has a long tail and the particles start in state 2**

The Laplace transform of the survival probability when \( D_1 = D_2 = D \) and the initial condition is given by (80), is the convolution
\[
\hat{S}(q) = \frac{2b^{1+\alpha}}{\Gamma \left( \frac{1+\alpha}{2} \right)} \int_0^{\infty} \left[ \frac{1}{q} - \frac{\mu e^{-y \sqrt{\frac{\pi}{2}} \frac{q}{D}}} \theta q + \frac{\mu e^{-y \sqrt{\frac{\pi}{2}} \frac{q}{D}} \theta (q + \theta)} \gamma \right] y^\alpha \exp \left\{ -bx^2 \right\} \times \exp \left\{ -by^2 \right\}
\]
\[
= \frac{1}{q} - \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) \sqrt{D}} \left[ \frac{1}{\sqrt{q}} \Gamma \left( 1 + \alpha, \frac{3 q}{2 b D} \right) \right] - \frac{1}{\sqrt{\theta (q + \theta)}} \Gamma \left( 1 + \alpha, \frac{3 q}{2 b D} \right),
\]
where \( U(a, b, z) \) is the confluent hypergeometric function and \( \theta = \lambda + \mu \). When \( D_1 \neq D_2 \), the Laplace transform of the survival probability gives
\[
\hat{S}(q) = \frac{2b^{1+\alpha}}{\Gamma \left( \frac{1+\alpha}{2} \right)} \int_0^{\infty} \left[ \frac{1}{q} + \frac{\mu e^{-y \sqrt{\frac{\pi}{2}} \frac{q}{D_1 D_2 w_+^2 (w_2^2 - w_2^2)}}}{D_1 D_2 w_+^2 (w_2^2 - w_2^2)} \right] y^\alpha \exp \left\{ -bx^2 \right\}
\]
\[
= \frac{1}{q} \left[ 1 + \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_1 D_2 \left( w_2^2 - w_2^2 \right)} \times \frac{U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right) - \frac{w_2^2}{4 \theta}} \right] U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right),
\]
where \( U(a, b, z) \) is the confluent hypergeometric function.

We can approximate the survival probability for \( q \) large
\[
\hat{S}_\varepsilon(q) \approx \frac{1}{q} - \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_1 D_2 \left( w_2^2 - w_2^2 \right)} \times \frac{U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right) - \frac{w_2^2}{4 \theta}} \right] U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right),
\]
For \( D_2 = D_1 (1 + \varepsilon) \), we have
\[
\hat{S}_\varepsilon(q) \approx \frac{1}{q} - \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_1 \varepsilon q \frac{w_2^2}{4 \theta}} \left( 1 + \varepsilon \frac{3 w_2^2}{4 \theta} - 1 \right),
\]
and expanding the above expression in \( \varepsilon \), we have
\[
\hat{S}_\varepsilon(q) \approx \frac{1}{q} - \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_1 \varepsilon q \frac{w_2^2}{4 \theta}} \left( 1 + \varepsilon \frac{3 w_2^2}{4 \theta} - 1 \right),
\]
When \( \varepsilon \to 0 \), the survival probability \( S_\varepsilon(t) \) converges to \( S_0(t) \) corresponding to an initial condition with a long tail for \( D_1 = D_2 \). Using the same reasoning as before, to leading order, using that
\[
\lim_{D_2 \to D_1} \text{Pr} \{ t_1 > t \} \approx 1 - \frac{(bD_1)^{\frac{1+\alpha}{2}} \mu (3 + \alpha) \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) 2 \Gamma \left( \frac{2+\alpha}{2} \right) N^{\frac{\alpha}{2}} + A_e},
\]
we obtain the asymptotic formula for \( N \gg 1 \)
\[
\Psi_N \approx \int_0^{\infty} [S(t)]^N dt \approx \int_0^{\infty} \exp \left\{ n \left[ 1 - \frac{(bD_1)^{\frac{1+\alpha}{2}} \mu (3 + \alpha) \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) 2 \Gamma \left( \frac{2+\alpha}{2} \right) N^{\frac{\alpha}{2}} + A_e} \right] \right\} dt
\approx \frac{\Gamma \left( 1 + \alpha \right) \left( \frac{\varepsilon^2}{2} \right)}{\left( \frac{\varepsilon^2}{2} \right) \frac{\alpha + \varepsilon}{\left( \varepsilon^2 \right) \left( \varepsilon^2 \right) N^{\frac{\alpha}{2}} + A_e},
\]
where \( A_e = A_0 + \varepsilon A_1 + \ldots \), where \( A_k \) are constants.

To conclude, leading order in \( \varepsilon \), the MFAT when we have the initial distribution with a long tail and different diffusion coefficients starting in state 2 is similar to the case \( D_1 = D_2 \). When \( D_1 \geq D_2 \) we have the expression for the survival probability written as
\[
\hat{S}(q) \approx \frac{1}{q} \left[ 1 + \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_1 D_2 \left( w_2^2 - w_2^2 \right)} \times \frac{U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right) - \frac{w_2^2}{4 \theta}} \right] U \left( 1 + \frac{a}{2}, \frac{3 w_2^2}{4 \theta} \right),
\]
Making \( D_1 = D_2 (1 + \varepsilon) \), we have
\[
\hat{S}_\varepsilon(q) \approx \frac{1}{q} - \frac{2e^{-(1+\alpha)}b^{-\frac{\alpha}{2}} \mu \Gamma(1 + \alpha)}{\Gamma \left( \frac{1+\alpha}{2} \right) D_2 \varepsilon q \frac{w_2^2}{4 \theta}} \left( 1 + \varepsilon \frac{3 w_2^2}{4 \theta} - 1 \right),
\]
and expanding the above expression in $\varepsilon$, we have

\[
\hat{S}_\varepsilon(q) \approx \frac{1}{q} - \frac{\mu(3 + \alpha)\Gamma(1 + \alpha) b \frac{\varepsilon}{1 + \alpha} D_2^{\frac{1+\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}\right) q^{\frac{1+\alpha}{2}}} \\
- \frac{2\mu\Gamma(1 + \alpha) b \frac{\varepsilon}{1 + \alpha} D_2^{\frac{1+\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}\right) q^{\frac{1+\alpha}{2}}} \sum_{n=1}^{\infty} \left(\frac{1+\alpha}{n+1}\right)^n \varepsilon^n.
\]

When $\varepsilon \to 0$, the survival probability $S_\varepsilon(t)$ converges to $S_0(t)$ corresponding to an initial condition with a long tail for $D_1 = D_2$. Thus to leading order, using that

\[
\lim_{D_1 \to D_2} \text{Pr}\{t_1 > t\} \approx 1 - \frac{(bD_2)^{\frac{1+\alpha}{2}} \mu(3 + \alpha)\Gamma(1 + \alpha) t^{\frac{1+\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{3+\alpha}{2}\right)}
\]

we obtain for $N \gg 1$

\[
\tau^N = \int_0^\infty [S(t)]^N dt \\
\approx \int_0^\infty \exp\left\{ \ln\left(1 - \frac{(bD_2)^{\frac{1+\alpha}{2}} \mu(3 + \alpha)\Gamma(1 + \alpha) t^{\frac{1+\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{3+\alpha}{2}\right)}\right)^N \right\} dt \\
\approx \left[ \frac{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{3+\alpha}{2}\right)}{\mu(3 + \alpha)\Gamma(1 + \alpha)} \right]^{\frac{1}{N}} \Gamma\left(\frac{3+\alpha}{2}\right) \frac{1}{(bD_2)^{\frac{1+\alpha}{2}}} \frac{1}{N^{\frac{1+\alpha}{2}} + A_\varepsilon},
\]

where $A_\varepsilon = A_0 + \varepsilon A_1 + ..$, where $A_k$ are constants. To conclude, the MFAT for an initial distribution with a long tail and different diffusion coefficients starting in state 2 is similar to the case $D_1 = D_2$.

### A.5 Numerical simulations for the Dirac-delta case

We present here the results of stochastic simulations in 4 cases of parameters, and 3 of them present a deviation compared to the analytical formula. This result shows that the asymptotic formula should be valid only in a restricted region of the parameter space (Figs. 5, 6, 7, 8).
Fig. 5 Statistics of the fastest arrival time when particles start in state 1 with $D_1 \gg D_2$. A Distribution of arrival times $\bar{\tau}_1$: analytical short-time formula 16 vs stochastic simulations (blue histogram) for particles starting at position $y = 2$ for $N = 10,000$ with 1000 runs and $D_1 = 1$ and $D_2 = 0.01$ in the interval $[0,5]$. B MFAT vs $N$ for stochastic simulations (colored disks) and the asymptotic formula 33 (continuous line). C Mean number of switchings for the fastest particles until arrival.
Fig. 6 Statistics when particles start in state 1 with \( D_2 \gg D_1 \). A Distribution of the arrival time \( \bar{\tau}_1 \): analytical short-time formula for \( \Pr \{ \tau_1 = t \} \) using (34) vs stochastic simulations (blue histogram) for particles starting at position \( y = 2 \) for \( N = 10,000 \) with 1000 runs and \( D_1 = 0.1 \) and \( D_2 = 1 \) in the interval \([0, 5]\). B MFAT vs \( N \) for the stochastic simulations (colored disks) and the asymptotic formula 35 (continuous line). C Mean number of switchings for the fastest particles until its arrival to the target. D Distribution of switching instants (blue first switching and red second switching) for all particles that only switch twice (\( T_N = 498 \)) before escaping among all iterations (1000) for \( N = 10,000 \). E Arrival times vs the switching instants for the fastest particles that only switched twice (\( T_N = 498 \)) before escaping among all iterations (1000) for \( N = 10,000 \), with first (blue) and second (red) switching.
Fig. 7 Statistics when particles start in state 2 with \( D_1 \gg D_2 \). A Distribution of the arrival time \( \bar{\tau}_1 \); analytical short-time formula for \( \Pr \{ \tau_1 = t \} \) using (44) vs stochastic simulations (blue histogram) for particles starting at position \( y = 2 \) for \( N = 10,000 \) with 1000 runs and \( D_1 = 1 \) and \( D_2 = 0.1 \) in the interval [0, 5]. B MFAT vs \( N \) for the stochastic simulations (colored disks) and the asymptotic formula (continuous line). C Mean number of switchings for the fastest particles until its arrival to the target. D Distribution of the switching instants (blue histogram) for all the particles that only switch once \( (T_N = 1000) \) before escaping among all the iterations done (1000) for \( N = 10,000 \). E Arrival times vs the switching instants for the fastest particles that only switch once \( (T_N = 1000) \) before escaping with 1000 iterations for \( N = 10,000 \).
Fig. 8 Statistics when particles start in state 2 with $D_2 \gg D_1$. A Distribution of the arrival time $\bar{\tau}_1$; analytical short-time formula for $Pr \{\tau_1 = t\}$ using (46) vs stochastic simulations (blue histogram) for particles starting at position $y = 2$ for $N = 10,000$ with 1000 runs and $D_1 = 0.1$ and $D_2 = 1$ in the interval $[0, 5]$. B MFAT vs $N$ for the stochastic simulations (colored disks) and the asymptotic formula 47 (continuous line). C Mean number of switchings for the fastest particles until its arrival to the target. D Distribution of the switching instants (blue histogram) for all the particles that only switch once ($T_N = 552$) before escaping among all the iterations done (1000) for $N = 10000$. E Arrival times vs switching instants for the fastest particles that only switch once ($T_N = 552$) before escaping with 1000 iterations for $N = 10000$

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