WEAK COUPLING LIMIT FOR DIRECTED POLYMERS ON TUBE

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Abstract. In this paper, we study a model of directed polymers in random environment, where the environment region is a dynamical tube with its width growing polynomially along with time. It can be viewed as an interpolation of the pinning model and the classic directed polymer model. We prove weak coupling limits for the directed polymers on tube in all dimensions, as the inverse temperature vanishes in a suitable rate. Phase transitions among disorder irrelevance, marginal relevance and disorder relevance are observed.

1. INTRODUCTION

1.1. The directed polymer model. We study a model of directed polymers in random environment in this paper. This model investigates the behavior of polymer chains (chemical compounds consisting of repeating units) when they stretch in some solvent with charges or impurities (called environment). It can be imagined that the configurations of polymer chains are influenced by the physical interaction between polymers and environment. Whether this interaction essentially changes the behavior of polymer chains plays a crucial role in the study of the directed polymer model. When the behavior of the model is essentially changed, the system undergoes a phase transition, which is an important and interesting phenomenon in statistical physics.

The directed polymer model was first introduced by Huse and Henley in [18] to study the domain walls in Ising systems. The first mathematical study on this model was later done by Imbrie and Spencer in [19]. Thereafter, the directed polymer model had become increasingly popular in mathematical physics society and attracted both mathematicians and physicists. During the last thirty years, a large amount of study has been carried out by many authors, e.g. [6, 11, 12, 13, 21, 24]. We refer to [10] for a comprehensive introduction of the directed polymer model.

In most of the previous work, the interaction between polymers and environment is assumed to take place on some fixed subset of the whole space for all time (usually the subset is the whole space itself or the origin, see Subsection 1.2 below for more details). We consider a variant of the directed polymer model, where we allow the interaction region to change along with time. The intuition can be explained as follows: at the beginning, the charges or the impurities are highly concentrated, and then they spread out in solvent as time goes by.

Let us introduce our model in details. Let $S := (S_n)_{n \geq 0}$ be a simple symmetric random walk on $\mathbb{Z}^d$, representing the polymer chain. The law and expectation of $S$, with $S_0 = x$, are denoted by $P_x$ and $E_x$, respectively. We omit the subscript if $x$ is the origin. The environment (also called disorder) is simulated by a family of i.i.d. random variables $\omega := (\omega_z)_{z \in \mathbb{N} \times \mathbb{Z}^d}$ with $\mathbb{N}$ for time and $\mathbb{Z}^d$ for space. The law and expectation of $\omega$ is denoted by $P$ and $E$, respectively. We assume that $\omega$ has finite exponential moment and denote its logarithmic moment generating function by

$$\lambda(\beta) := \log E[\exp(\beta \omega_z)] < \infty, \quad \forall \beta \in \mathbb{R}.$$  

Without loss of generality and for computational simplicity, we further assume that

$$E[\omega_z] = 0, \quad E[\omega_z^2] = 1.$$  

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Our directed polymer model, up to time $N$, is then defined via a Gibbs transform
\begin{equation}
\frac{d\mathbf{P}_{N,\beta}^\omega}{d\mathbf{P}}(S) := \frac{1}{Z_{N,\beta}^\omega} \exp \left( \sum_{n=1}^{N} (\beta \omega_n, S_n - \lambda(\beta)) \mathbb{1}_{\{(n,S_n) \in \Omega_N\}} \right),
\end{equation}
where $\beta$ denotes the inverse temperature, $\Omega_N$ is a subset of $\mathbb{N} \times \mathbb{Z}^d$ denoting the dynamical environment region, $\mathbf{P}_{N,\beta}^\omega$ is called polymer measure, and
\begin{equation}
Z_{N,\beta}^\omega = \mathbb{E} \left[ \exp \left( \sum_{n=1}^{N} (\beta \omega_n, S_n - \lambda(\beta)) \mathbb{1}_{\{(n,S_n) \in \Omega_N\}} \right) \right]
\end{equation}
is called partition function, which makes $\mathbf{P}_{N,\beta}^\omega$ a (random) probability measure. Although this model is described by the probability measure $\mathbf{P}_{N,\beta}^\omega$, we usually turn to study the partition function $Z_{N,\beta}^\omega$, since it has already carried rich enough information of the system and is more treatable.

1.2. Motivation of the paper. We first review two celebrated polymer models, which will naturally induce the problem studied in this paper.

- **The classic $(1+d)$-dimensional directed polymer model:** The most standard and well-known definition of the directed polymer model is \cite{13} with $\Omega_N \equiv \mathbb{N} \times \mathbb{Z}^d$. That is, the disorder is full of the space and the indicator in (1.3) can thus be omitted.

- **The pinning model:** $\Omega_N \equiv \mathbb{N} \times \{0\}$, namely the disorder only lies on a defect line (or a membrane). Since the distribution of $S$ is disturbed only when it returns to the origin, an alternative (and also more general) way to define the pinning model is by:
\begin{equation}
\frac{d\mathbf{P}_{N,\beta}^{\omega,h}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,\beta}^{\omega,h}} \exp \left( \sum_{n=1}^{N} (\beta \omega_n - h) \mathbb{1}_{\{\tau \in \tau\}} \right),
\end{equation}
where $h$ is an external field, and $\tau = \{0, \tau_1, \tau_2, \ldots\}$ is a renewal process with
\begin{equation}
K(n) := \mathbb{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}}, \quad \alpha \geq 0, \quad \sum_{n=1}^{\infty} K(n) \leq 1.
\end{equation}
Here $L(\cdot)$ is a slowly varying function (see \cite{5} for reference), i.e., $L(x) > 0$ on $(0, \infty)$ and $\lim_{x \to \infty} L(bx)/L(x) = 1$ for any $b > 0$. In the scope of this paper, we set $h = \lambda(\beta)$.

**Remark 1.1.** Clearly, the times a simple random walk on $\mathbb{Z}^d$ hitting 0 give rise to a renewal process and hence define an associated pinning model. Indeed if $d = 1$, then in (1.6) $K(2n) \sim cn^{-3/2}$ \cite{13}, which was dealt with subtly in \cite{14}. If $d = 2$, then $K(2n) \sim c/(n \log^2 n)$ \cite{20}, and the study was first carried out in \cite{2}. For these two cases, the random walks are recurrent and thus $\sum_{n=1}^{\infty} K(n) = 1$. But if $d \geq 3$, then $K(2n) \sim cn^{-d/2}$ \cite{16} and $\sum_{n=1}^{\infty} K(n) < 1$, since the walk is transient.

We have mentioned in the previous subsection that whether the environment influences the behavior of polymer chains, at least at high temperature, is a crucial topic in the study of the polymer model. It can be seen that this influence is depicted by the perturbation from disorders to the distributions of the pure model, namely the underlying random walks. If disorders do not change the behavior of the model for small enough $\beta > 0$, then we say the model is disorders irrelevant, while if disorders change the model essentially for arbitrarily small $\beta > 0$, then we say the model is disorder relevant.

Harris criterion \cite{17} asserted that the classic $(1+d)$-dimensional directed polymer model is disorder relevant for $d=1$ and disorder irrelevant for $d=3$. For the pinning model, it is disorder relevant for $\alpha \in (0, \frac{1}{2})$ and disorder irrelevant for $\alpha > \frac{1}{2}$ (cf. \cite{10, 15}). The $(1+2)$-dimensional directed polymer and the pinning model for 1-d simple random walk are critical, where the Harris criterion is inclusive and the situation is more sophisticated. Recently, it has been shown in \cite{14, 18} that the disorders are still relevant for the two critical models, whose property, as found in \cite{8}, is dramatically different from that of disorder relevant models. They belong to the so-called marginal relevant regime.
Based on the same simple random walk $S$ on $\mathbb{Z}^d$, if the region $\Omega_N$ is widened from the time-space line $\mathbb{N} \times \{0\}$ to $\mathbb{N} \times \mathbb{Z}^d$, then the model (1.3) varies from the pinning model to the classic directed polymer model. In particular, when $d = 1$, it is expected to observe a phase transition for the model from marginal relevance to disorder relevance. A natural question then arises: how does the behavior of the model change as the dynamical environment region enlarges gradually?

1.3. Directed polymers on tube and the main result. Motivated by the question above, we define a new variant of the classic directed polymer model, called directed polymer on tube, by

$$
\frac{dP_{N,\beta}^{\omega}}{dP}(S) := \frac{1}{Z_{N,\beta}} \exp \left( \sum_{n=1}^{N} (\beta \omega_n S_n - \lambda(\beta)) \mathbb{1}_{\{|S_n| \leq R N^n\}} \right),
$$

where $|\cdot|$ denotes the Euclidean distance, and $R \geq 0$ and $a \in [0, 1]$ are parameters for the environment region that can be tuned. It is an interpolation of the classic directed polymer and the pinning model, where the former refers to $a = 1$ and $R \geq 1$ and the latter refers to $R = 0$. The regularity assumption that the width of the environment region grows polynomially fast is for feasibility. To our best knowledge, a similar setting that the environment region widens gradually has only been considered in [9], where the authors studied the first passage percolation.

Generally speaking, the model with fixed $\beta > 0$ is difficult to treat. During the last ten years, an alternative approach has been developed, that is, studying the weak coupling limits of the partition functions $Z_{N,\beta}^{\omega}$ by sending $\beta_N \downarrow 0$ in a suitable rate as $N \to \infty$. Roughly speaking, if we suitably tune down $\beta_N$, such that it is perfectly balanced with the increasing energy gain, then the environment will be barely felt by the system. However, if $\beta_N$ is too small, then the polymer will not feel disorders and perform like the underlying process, while if $\beta_N$ is too large, then the contribution to the partition function will concentrate on paths picking extremely large disorders, making the polymer measure asymptotically singular with respect to the underlying measure. Notice that this idea is only valid for disorder (marginal) relevant models, resulting in some non-trivial limit $\mathcal{Z}$ for $Z_{N,\beta_N}^{\omega}$, while for disorder irrelevant model, $Z_{N,\beta_N}^{\omega}$ trivially converges to 1 in probability.

The first work studying the weak coupling limit was due to [11], where the authors considered the classic $(1 + 1)$-dimensional directed polymer model. They proved that for $\beta_N = \hat{\beta} N^{-1/4}$ with any $\hat{\beta} > 0$, $Z_{N,\beta_N}^{\omega}$ converges in distribution to a non-trivial limit $\mathcal{W}_{\beta}$. Later in [8], the authors proved that for a general class of marginal relevant polymer models, all their partition functions converge in distribution to the same weak limit $\mathcal{Z}_{\hat{\beta}}$ for $\beta_N = \hat{\beta} / L(N)$ with $\hat{\beta} < \hat{\beta}_c$ and $L(\cdot)$ some slowly varying function going to infinity as $N \to \infty$, where $\beta_c$ is an explicit critical value, while for $\beta_N \geq \hat{\beta}_c / L(N)$, the weak limit is 0.

The above facts also illustrate an essential difference between disorder relevance and marginal relevance. In the disorder relevant regime, the suitable $\beta_N$ decays polynomially fast and the value of $\hat{\beta}$ is not crucial, i.e., there is no phase transition in $\hat{\beta}$. By contrast, in the marginal relevant regime, the suitable $\beta_N$ decays as a slowly varying function and a critical $\hat{\beta}_c$ splits the marginal relevant regime into two sub-regimes: for $\hat{\beta} < \hat{\beta}_c$, temperature and energy are perfectly balanced, while for $\hat{\beta} \geq \hat{\beta}_c$, the disorders are too strong, i.e., there is a phase transition in $\hat{\beta}$ and the property of the system depends on finer information.

In this paper, we consider the weak coupling limits for directed polymer on tube in all dimensions $d \geq 1$ and parameters $R \geq 0$ and $a \in [0, 1]$. Note that the case $a \in (0, 1]$ but $R = 0$ is the same as $a = 0$ and $R < 1$. We determine the weak limit of the partition function in all cases in the following theorem, where we write $Z_N = Z_{N,\beta_N}^{\omega}$ for simplicity. From the theorem we can also observe two phase transitions taking place at $d = 1, a = \frac{1}{2}$ and $d = 2, a = 0$.

**Theorem 1.1.** For directed polymers on tube (1.7), the following weak coupling limits hold.
• **Disorder relevant regime:** For $d = 1$, $a \in [\frac{1}{2}, 1]$ and any $R > 0$, let $\beta_N = \hat{\beta} N^{-\frac{1}{3}}$ for some $\hat{\beta} > 0$. We have

\[
Z_N \xrightarrow{d} Z_{\hat{\beta}} := 1 + \left\{ \begin{array}{ll}
\sum_{k=1}^{\infty} \frac{\hat{\beta}^k}{k!} \int_{[0,1] \times \mathbb{R}^d} \psi_k((t_1, x_1), \ldots, (t_k, x_k)) \prod_{j=1}^{k} W(dt_j, dx_j), & \text{for } a \in \left(\frac{1}{2}, 1\right), \\
\sum_{k=1}^{\infty} \frac{\hat{\beta}^k}{k!} \int_{[-R, 0]} \psi_k((t_1, x_1), \ldots, (t_k, x_k)) \prod_{j=1}^{k} W(dt_j, dx_j), & \text{for } a = \frac{1}{2},
\end{array} \right.
\]

where $\xrightarrow{d}$ denotes the weak convergence, $\psi_k((t_1, x_1), \ldots, (t_k, x_k)) = \prod_{j=1}^{k} \sqrt{2} g_{t_j-t_{j-1}}(x_j-x_{j-1})$ with $g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ the Gaussian density, and $W(t, x)$ is the time-space white noise. Furthermore, $\mathbb{E}[(Z_N)^2] \xrightarrow{N \to \infty} \mathbb{E}[(Z_{\hat{\beta}})^2]$.

• **Marginal relevant regime:** If

(i) $d = 1$, $a = 0$, $R \geq 0$ and $\beta_N = \hat{\beta} \sqrt{\frac{\pi}{2(1-R) \log N}}$; or

(ii) $d = 1$, $a \in (0, \frac{1}{2})$, $R > 0$, and $\beta_N = \hat{\beta} \sqrt{\frac{2(1-2a) \log N}{(1-R) \log N}}$; or

(iii) $d = 2$, $a \in (0, 1]$, $R > 0$, and $\beta_N = \hat{\beta} \sqrt{\frac{2a \log N}{(2a \log N) \log N}}$, then

\[
Z_N \xrightarrow{d} Z_{\hat{\beta}} := \begin{cases} 
\exp \left( \sigma^2 \beta W_1 - \frac{\sigma^2}{2} \right), & \text{if } \hat{\beta} \in (0, 1), \\
0, & \text{if } \hat{\beta} \geq 1,
\end{cases}
\]

where $W_1$ is standard normal and $\sigma^2 = \log \frac{1}{1-\beta^2}$. Furthermore, $\mathbb{E}[(Z_N)^2] \xrightarrow{N \to \infty} \mathbb{E}[(Z_{\hat{\beta}})^2]$.

• **Disorder irrelevant regime:** If

(i) $d = 2$, $a = 0$, $R \geq 0$;

(ii) $d \geq 3$, $a \in [0, 1]$, $R \geq 0$,

then $Z_N$ converges to 1 in $\mathbb{P}$-probability for any $\beta_N \downarrow 0$.

**Remark 1.2.** Note that for (i) in the marginal relevant regime, we have that $\mathbb{P}(|S_n| \leq R) = \sum_{|x| \leq \frac{R}{\sqrt{m}}} \mathbb{P}(S_n = x)$, which exactly falls in the scope of [3], since for $|2m|$, $|2m+1| \leq R$, $\mathbb{P}(S_{2k} = 2m) \sim \mathbb{P}(S_{2k} = 0)$ and $\mathbb{P}(S_{2k+1} = 2m+1) \sim \mathbb{P}(S_{2k+1} = 1)$ as $k \to \infty$.

1.4. **Discussion.** A challenge open problem for the classic $(1 + 1)$-dimensional directed polymer model is that for any fixed $\beta > 0$,

\[
\mathbb{E}_{\kappa, \beta}^N |S_N| \approx N^{\frac{2}{3}} \quad \text{and} \quad \frac{\log Z_{N, \beta}^N - c_N}{N^{\frac{2}{3}}} \xrightarrow{d} \text{Tracy-Widom law}.
\]

If we consider the directed polymer on tube with fixed $\beta$, then it is reasonable that the polymers still have transversal fluctuations of order $N^{2/3}$ when $a > 2/3$. When $\beta \in (\frac{1}{2}, \frac{1}{3})$, since the extra entropy cost outside the environment tube with width of order $N^a$ cannot be balanced by the energy gain due to the lack of disorders, the transversal fluctuations of polymers $S_N$ is expected to be $N^a$.

Theorem 1.4 suggests that when $a < 1/2$, the model is comparable to the pinning model associated to 1-d simple random walk. Such pinning model with $\beta > 0$ falls in the so-called delocalization phase (see [4]), in which regime the polymer only interacts with the environment for a negligible fraction of time and it thus behaves close to a simple random walk. This suggests the typical fluctuations of $S_N$ is of order $\sqrt{N}$. Therefore, we conjecture that for the directed polymer on tube (1.7),

\[
\mathbb{E}_{\kappa, \beta}^N |S_N| \approx N^{\xi},
\]
where the transversal exponent $\xi = (1/2 \lor a) \land 2/3$. But we have no ideas for the variance of $\log Z_{N,\beta}^\omega$.

The rest of the paper is devoted to proofs.

2. Proof of Theorem 1.1

The general strategy of proofs for weak coupling limits has been elaborated in [1] [7] [8]. We are thus able to provide, based on existing results, a short proof of Theorem 1.1 in Subsection 2.2 without going through the whole picture. Nevertheless, to illustrate the key ideas behind the proof, we will first sketch some heuristics in Subsection 2.1.

2.1. Heuristics. We start with a standard polynomial chaos expansion for the partition function $Z_N$. For every $N$, we define

$$\xi_{n,x}^{(N)} := \xi_{n,x} := e^{\beta_N \omega_{n,x} - \lambda(\beta_N) - 1}/\beta_N.$$  

(2.1)

It is easy to check that $E[\xi_{n,x}] = 0$ and $\text{Var}(\xi_{n,x}) = 1 + o(1)$ as $\beta_N \downarrow 0$. Using $e^{x^3} = 1 + (e^x - 1)1_A$ in the expression (1.4) of $Z_N$, we can rewrite

$$Z_N = E \left[ \prod_{n=1}^{N} \left( 1 + \beta_N \xi_{n,S_n} 1_{|S_n| \leq RN^a} \right) \right]$$

$$= 1 + \sum_{k=1}^{N} \beta_N^k Z_{N,k}$$

$$=: 1 + \sum_{k=1}^{N} \beta_N^k Z_{N,k},$$

(2.2)

where $B(r)$ is the ball centered at the origin with radius $r$, and $p_n(x) = P(S_n = x)$. We also use the convention that $n_0 = 0$ and $x_0$ is the origin.

Since for $k \neq l$, there must be some $\xi_{n,x}$ appears exactly once in the product $Z_{N,k}$ and $Z_{N,l}$, by independence we have $E[Z_{N,k} Z_{N,l}] = 0$ and thus $(Z_{N,k})_{1 \leq k \leq N}$ are orthogonal in $L_2$ space. Therefore to establish the weak convergence of $Z_N$, it suffices to approximate $\beta_N^k Z_{N,k}$ in $L_2$ given that $\sum_{k=K}^{\infty} \beta_N^k Z_{N,k}$ vanishes uniformly in $N$ as $K \to \infty$. For any $k$, a second moment calculation reveals that $\text{Var}(Z_{N,k}) \lesssim \text{Var}(Z_{N,1})^k$. Hence, to achieve a non-trivial limit, it is natural to choose $\beta_N$ of order $\text{Var}(Z_{N,1})^{-1/2}$ in view of (2.2).

To calculate the variance of $Z_{N,1}$, we can decompose

$$\text{Var}(Z_{N,1}) = \sum_{1 \leq n \leq N} \sum_{|x| \leq RN^a} p_n^2(x) = \left( \sum_{n \leq N^{3/2 - 2a}} \sum_{|x| \leq RN^a} + \sum_{N^{1/2 - 2a} < n \leq N} \sum_{|x| \leq RN^a} \right) p_n^2(x) := \sigma_1^2 + \sigma_2^2,$$

where $\sigma_1^2$ (resp. $\sigma_2^2$) is contributed from the short (resp. long) time interval. A simple comparison shows that only when $d = 1, a \in (0, 1/2)$, the variance is dominated by $\sigma_2^2$, which suggests that in that case the model should behave similarly to the pinning model associated to the 1-d simple random walk, as the environment region is thin (sub-diffusive) for large time scale. On the other hand, when $d = 1, a \in (1/2, 1)$ or $d \geq 2, a \in (0, 1)$, the term $\sigma_1^2$ dominates, and it suggests that the model should be comparable to the classic directed polymer model. By [1] the classic $(1 + 1)$-dimensional directed polymer model is disorder relevant, while the $(1 + 2)$-dimensional directed polymer and the pinning model associated to the 1-d simple random walk are marginal relevant by [8]. For $d \geq 3$, since $\text{Var}(Z_{N,1})$ is finite, the second moment of $Z_{N,\beta}^\omega$ is uniformly bounded for small enough $\beta > 0$, and hence the model is disorder irrelevant.

In the disorder relevant and marginal relevant regimes, the limiting object turns out to be the limit of a Wiener chaos expansion. Indeed, by a version of the Lindeberg principle [8, Theorem 2.6] and [8, Theorem 4.2], $\xi_{n,x}$ in (2.2) can be replaced by independent standard normal random variables $W_{n,x}$ without affecting the limiting distribution. On the diffusive scale, the local limit theorem allows us
to approximate the weight function $p_n(x)$ by the Gaussian kernel. Therefore, $Z_{N,k}$ can be written asymptotically as a multiple integral with respect to white noise with Gaussian weight over diffusively rescaled space and time. With $\beta_N$ on the right scale as described above, $Z_N$ can be seen to converge to $[\mathbb{E}]$ or $[\mathbb{L}]$, once the covariance structure of $Z_{N,k}$ is determined.

2.2. **Proof.** Following the heuristics above, the first crucial step for the proof is to estimate $\mathbb{V}ar(Z_{N,1})$, which is given by

$$\mathbb{V}ar(Z_{N,1}) = \sum_{1 \leq n \leq N} \sum_{x \in B(RN^a)} P(S_n = x)^2 \mathbb{E}[\xi_{n,x}^2] = \sum_{1 \leq n \leq N} P(S_n = S_n', |S_n| \leq RN^a) (1 + o(1)),$$

where in the first equality we used $\mathbb{E}[\xi_{n,x} \xi_{n',x'}] = 0$ for $(n, x) \neq (n', x')$, and in the second equality $S'$ is an independent copy of $S$. The following lemma determines its asymptotics.

**Lemma 2.1.** Let $a \in [0, 1]$ and let $R \geq 0$. Denote the expected intersection times within the ball $B(RN^a)$ of two independent simple random walks by

$$I_N := I_N^{a,R} := \sum_{n=1}^{N} P(S_n = S_n', |S_n| \leq RN^a)$$

Then as $N \to \infty$, the following asymptotics hold:

$$I_N \sim \begin{cases} \frac{2}{\sqrt{\pi}} \sqrt{N}, & \text{for } d = 1, a \in \left(\frac{1}{2}, 1\right], R > 0, \\
C_{1,R} \sqrt{N}, & \text{for } d = 1, a = \frac{1}{2}, R > 0, \\
\frac{2(1-2a)}{\pi} RN^a \log N, & \text{for } d = 1, a \in (0, \frac{1}{2}), R > 0, \\
\frac{2|t|+1}{\pi} \log N, & \text{for } d = 1, a = 0, R \geq 0, \\
\frac{1+2a}{\pi} \log N, & \text{for } d = 2, a \in (0, 1], R > 0 \\
C_{d,a,R}, & \text{for } d = 2, a = 0, R \geq 0 \text{ or } d \geq 3, a \in [0, 1], R \geq 0, 
\end{cases}$$

where $C_{1,R} = \frac{1}{\sqrt{\pi}} \int_{|x| \leq R} e^{-|x|^2} dx.$

**Proof.** Note that

$$P(S_n = S_n', |S_n| \leq RN^a) = \sum_{|x| \leq RN^a \wedge n} P(S_n = x)^2.$$

Recall that the case $a > 0, R = 0$ is the same as $a = 0, R < 1$, and the case $d = 1, a = 0$ has actually been proved in Remark 1.2. For $d = 2, a = 0$, (2.3) $\leq C_R n^{-2}$ and for $d \geq 3$, (2.3) $\leq P(S_{2n} = 0) \sim c_d n^{-d/2}$ by the local limit theorem, where both are summable and the last case for $I_N$ follows. We then focus on $d = 1, 2$ and $a \in (0, 1]$.

If $x$ can be visited at time $n$, then we write $x \leftrightarrow n$. By a finer local limit theorem [22 Theorem 1.2.1],

$$I_N = \sum_{n=1}^{N} \sum_{x \leq RN^a} \left(2 \left(\frac{d}{2 \pi n}\right)^{\frac{d}{2}} e^{-\frac{dx^2}{2n}} + E(n, x)\right)^2.$$
where $E(n, x) = O(n^{-1} - d/2 \land |x|^{-2} n^{-d/2})$. Note that $I_N$ is the sum of

$$J_N^{(1)} := 4 \left( \frac{d}{2\pi} \right) \frac{d}{n^2} \sum_{n=1}^{N} \sum_{|x| \leq R N^a} e^{-\frac{d|x|^2}{n}},$$

$$J_N^{(2)} := 4 \left( \frac{d}{2\pi} \right) \frac{d}{n^2} \sum_{n=1}^{N} \sum_{|x| \leq R N^a} e^{-\frac{d|x|^2}{n}} E(n, x),$$

$$J_N^{(3)} := \sum_{n=1}^{N} \sum_{|x| \leq R N^a} E(n, x)^2.$$

**Term $J_N^{(1)}$.** Note that $S$ has period 2. By a Riemann sum approximation, we have

$$J_N^{(1)} \sim 2 \left( \frac{d}{2\pi} \right) \frac{d}{n^2} \int_{1}^{\epsilon a \sqrt{N}} e^{-\frac{d|x|^2}{n}} \, dx \, dn = \frac{2d^2}{(2\pi)^d} \int_{1}^{\epsilon a \sqrt{N}} \int_{|x| \leq R N^a} e^{-\frac{d|x|^2}{n}} \, dx \, dn.$$

We need to consider (i) $a > \frac{1}{2}$, (ii) $a = \frac{1}{2}$, and (iii) $a < \frac{1}{2}$ separately.

**Case $a > \frac{1}{2}$.** The inner integral converges to $\int_{\mathbb{R}^d} \exp(-x^2) \, dx$ for any $R > 0$, and hence

$$J_N^{(1)} \sim \begin{cases} \frac{2}{\sqrt{\pi}} \sqrt{N}, & \text{for } d = 1, \\ \frac{1}{\sqrt{\pi}} \log N, & \text{for } d = 2. \end{cases}$$

**Case $a = \frac{1}{2}$.** We split the domain $[1, N]$ of $n$ into two parts: (1) $[1, \epsilon N]$ and (2) $[\epsilon N, N]$, where $\epsilon$ is small. For the first part, there exists $\delta_\epsilon \xrightarrow{\epsilon \to 0} 0$, such that

$$\int_{1}^{\epsilon N} \frac{1}{\sqrt{n}} \int_{|x| \leq R \sqrt{\frac{dx}{N}}} e^{-\frac{d|x|^2}{n}} \, dx \, dn \sim \begin{cases} 2(1 - \delta_\epsilon) \sqrt{\epsilon a N}, & \text{for } d = 1, \\ \pi(1 - \delta_\epsilon) \log N, & \text{for } d = 2. \end{cases}$$

For the second part,

$$\int_{\epsilon N}^{N} \frac{1}{\sqrt{n}} \int_{|x| \leq R \sqrt{\frac{dx}{n}}} e^{-\frac{d|x|^2}{n}} \, dx \, dn \sim \begin{cases} \int_{1}^{\epsilon N} \int_{|x| \leq R \sqrt{\frac{dx}{n}}} e^{-\frac{d|x|^2}{n}} \, dx \, dn, & \text{for } d = 1, \\ \frac{1}{\sqrt{\pi}} \log N, & \text{for } d = 2. \end{cases}$$

**Case $a < \frac{1}{2}$.** For $d = 1$, we split the range of $[1, N]$ into two parts: (1) $[1, KN^{2a}]$ and (ii) $[KN^{2a}, N]$, where $K$ is large. For the first part,

$$\int_{1}^{KN^{2a}} \frac{1}{\sqrt{n}} \int_{|x| \leq R \sqrt{\frac{dx}{n}}} e^{-\frac{d|x|^2}{n}} \, dx \, dn \leq C_{K,R} N^a.$$

For the second part, note that $e^x \xrightarrow{x \to 0} 1$. Let $N \to \infty$ and then let $K \to \infty$. We obtain

$$\int_{KN^{2a}}^{N} \frac{1}{\sqrt{n}} \int_{|x| \leq R N^a / \sqrt{n}} e^{-\frac{d|x|^2}{n}} \, dx \, dn \sim 2RN^a \int_{KN^{2a}}^{N} \frac{1}{n} \, dn \sim 2(1 - 2a) R N^a \log N.$$

Hence $J_N^{(3)} \sim \frac{2(1 - 2a)}{\pi} R N^a \log N$ for $d = 1, a \in (0, \frac{1}{2})$. 


For $d = 2$, we split the range of $[1, N]$ into three parts: (1) $[1, \epsilon N^{2\alpha}]$, (2) $[\epsilon N^{2\alpha}, K N^{2\alpha}]$ and (3) $[K N^{2\alpha}, N]$, where $K$ is large and $\epsilon$ is small. For the first part, there exists $\delta_\epsilon \xrightarrow{\epsilon \to 0} 0$, such that

$$
\int_1^{\epsilon N^{2\alpha}} \frac{1}{n} \int_{|x| \leq R N^{\alpha}} e^{-|x|^2} \, dx \, dn \sim 2a(1 - \delta_\epsilon) \pi \log N.
$$

For the second part,

$$
\int_{\epsilon N^{2\alpha}}^{KN^{2\alpha}} \frac{1}{n} \int_{|x| \leq R N^{\alpha}} e^{-|x|^2} \, dx \, dn \xrightarrow{\epsilon \to 0} \int_K^1 \frac{1}{t} \int_{|x| \leq R \sqrt{\frac{2}{t}}} e^{-|x|^2} \, dx \, dt,
$$

which is finite for any fixed $\epsilon, K$. For the third part, note that $e^x \xrightarrow{x \to 0} 1$. Let $N \to \infty$ and then let $K \to \infty$. We obtain

$$
\int_{KN^{2\alpha}}^N \frac{1}{n} \int_{|x| \leq R N^{\alpha}} e^{-|x|^2} \, dx \, dn \sim C_d R^2 N^{2\alpha} \int_{KN^{2\alpha}}^N \frac{1}{n^2} \, dn \sim \frac{C_d R^2}{K}.
$$

Therefore, $J_N^{(1)} \sim \frac{2a}{\pi} \log N$ for $d = 2, a \in (0, \frac{1}{2})$.

**Term $J_N^{(2)}$.** Note that $E(n, x) = o(n^{d-2})$, so $J_N^{(2)}$ is negligible compared to $J_N^{(1)}$.

**Term $J_N^{(3)}$.** Recall that $E(n, x) = O(n^{-1-d/2} \wedge |x|^{-2} n^{-d/2})$. We have

$$
J_N^{(3)} \leq \sum_{n=1}^N \sum_{|x| \leq n} n^{-(d+2)} \leq \sum_{n=1}^\infty n^{-2} = C < \infty.
$$

Finally, we combine all computations above together with (2.5) to yield the lemma for $d = 1, 2$. □

We are now able to complete the proof of Theorem 1.1. We will separately treat the cases of disorder irrelevance, disorder relevance, and marginal relevance.

**Disorder irrelevant regime.** We prove the result by a second moment estimate. Recall $Z_{N,k}$ in (2.2) and the definition of $\beta_N$. Using $\mathbb{E} \text{Var}(Z_{N,k}) \leq \mathbb{E} \text{Var}(Z_{N,1})^k$, by Lemma 2.1 (also see (4.3) in [3]),

$$
\mathbb{E}|Z_N - 1|^2 = \sum_{k=1}^N \beta_N^{2k} \mathbb{E} \text{Var}(Z_{N,k}) \leq \sum_{k=1}^\infty (C \beta_N)^{2k} = \frac{C \beta_N^2}{1 - C \beta_N^2} \to 0, \quad \text{as } \beta_N \downarrow 0.
$$

**Disorder relevant regime.** For disorder relevant systems, [7] Theorem 2.3 states a general convergence criterion. We will prove this case by verifying the conditions of [7] Theorem 2.3. Let us introduce a space-time lattice $T_N$ and its associated rescaled lattice $\hat{T}_N$ by

$$
T_N := \{(n, x) \in N \times Z : n \leftrightarrow x\}, \quad \hat{T}_N := \{(t, y) \in [0, 1] \times \mathbb{R} : (Nt, \sqrt{N}y) \in T_N\}.
$$

For $z = (t, y)$, let $\zeta_N := (\zeta_{N,z})_{z \in \hat{T}_N}$ be i.i.d. with $\zeta_{N,(t,y)} = \xi_{n,x}$, where $(Nt, \sqrt{N}y) = (n, x)$. Then

$$
\mathbb{E}[\zeta_{N,z}] = 0 \quad \text{and} \quad \mathbb{E}[\zeta_{N,z}] = 1 + o(1) \quad \text{as } N \to \infty.
$$

Let $\Psi_N$ be the formal multi-linear polynomial

$$
\Psi_N(\zeta_N) = 1 + \sum_{k=1}^N \sum_{z_1, \ldots, z_k \in \hat{T}_N^{\otimes k}} \psi_N(z_1, \ldots, z_k) \prod_{j=1}^k \xi_{N,z_j},
$$

where $\psi_N$ is a symmetric function of $(z_1, \ldots, z_k) \in \hat{T}_N^{\otimes k}$, which vanishes if $z_i = z_j$ for any $i \neq j$, and for distinct $z_1 = (t_1, y_1), \ldots, z_k = (t_k, y_k)$ with $0 < t_1 < \ldots < t_k \leq 1$,

$$
\psi_N((t_1, y_1), \ldots, (t_k, y_k)) = \prod_{j=1}^k N^{-1/4} p_{(t_j-t_{j-1})N}((x_j-x_{j-1})\sqrt{N}) 1_{|x_j| \leq RN^{\alpha-1/2}}.
$$
Let $C_N$ be the tessellation indexed by $\mathbb{T}_N$ with $C_N(z) = (t - \frac{1}{2N}, t + \frac{1}{2N}) \times (y - \frac{1}{\sqrt{N}}, y + \frac{1}{\sqrt{N}})$ and $v_N := |C_N(z)| = 2N^{-3/2}$. We extend the domain of $\psi$ to $([0, 1] \times \mathbb{R})^{\otimes k}$ by defining $\psi_N(\tilde{z}) := \psi_N(z)$ for $\tilde{z} \in C(z)$.

We then check the three conditions of [7] Theorem 2.3. To check condition (i), we simply note that $\mu_N = \mathbb{E}[\xi_{n,x}] = 0$ and $\sigma_N^2 = \mathbb{Var}(\xi_{n,x}) = 1 + o(1)$ as $N \to \infty$. For condition (ii), first the local limit theorem shows that the following pointwise convergence holds,

$$v_N^{-1/2} \psi_N(z_1, \ldots, z_k) = 2^{-k/2} N^{3k/4} N^{-k/4} \prod_{j=1}^{k} \lim_{N \to \infty} \psi(z_1, \ldots, z_k) = \prod_{j=1}^{k} \sqrt{2} g_{t_j-t_{j-1}} (x_j - x_{j-1}).$$

Then, to show the convergence in $L_2$, by the dominated convergence theorem it suffices to show that $v_N^{-1/2} \psi_N(z_1, \ldots, z_k)$ are uniformly integrable and $\|\psi(z_1, \ldots, z_k)\|_2$ is finite. By a local large deviation for the simple random walk (cf. [23] Theorem 3), we have that $p_t(x) \leq Ct^{-1/2} \exp(-cx^2/t)$. By integrating out $x$ for $\exp(-x^2/t)$, we obtain the integral $\int_{0<t_1<\ldots<t_k} \prod_{j=1}^{k} (t_j - t_{j-1})^{-1/2} dt_1 \cdots dt_k$, which decays like $k^{-c} \log(k)$. Hence condition (ii) has been verified. It remains to check condition (iii), which follows immediately from the super-exponential decays for $\|v_N^{-1/2} \psi_N(z_1, \ldots, z_k)\|_2$. Then the proof is completed by directly applying [7] Theorem 2.3.

• **Marginal relevant regime.** We prove the results in this regime by showing that $Z_N$ is arbitrarily close to the partition function of pinning or directed polymer model whose weak coupling limit is already known. Recall that $\beta_N = \hat{\beta}/\sqrt{N}$. We will first treat the case $\hat{\beta} \in (0, 1)$ and then $\hat{\beta} \geq 1$.

**Case $\hat{\beta} \in (0, 1)$**. We perform a second moment estimate similar to (2.6).

$$\sum_{k=K}^{\infty} \beta_N^{2k} \mathbb{Var}(Z_{N,k}) \leq \sum_{k=K}^{\infty} \beta_N^{2k} \stackrel{K \to \infty}{\converges} 0.$$

Therefore, we only need to tackle $Z_{N,k}$ for finitely many $k = 1, \ldots, K$. It suffices to show that as $N \to \infty$, $Z_{N,k}$ is close enough in $L_2$ to its *constrained free* version, that is

$$\hat{Z}_{N,k} := \left\{ \begin{array}{ll} \sum_{1 \leq n_1 < \ldots < n_k \leq N} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(\gamma) \xi_{n_j,z}, & \text{for } d = 1, a \in (0, \frac{1}{2}), \\ \sum_{(x_1, \ldots, x_k) \in \mathbb{Z}^d} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(x_j - x_{j-1}) \xi_{n_j,x_j}, & \text{for } d = 2, a \in (0, 1], \end{array} \right.$$  

(2.11)

where $\gamma$ is 0 if $n_j-n_{j-1}$ is even and 1 otherwise. Since $\hat{Z}_{N,k}$ is the $k$-th term of the polynomial chaos expansion for some partition function that satisfies [8] Hypothesis 2.4, the convergence of $Z_N$ then follows by [8] Theorem 2.8.

The details for (a) $d = 1, a \in (0, \frac{1}{2})$, (b) $d = 2, a \in \left[\frac{1}{2}, 1\right]$ and (c) $d = 2, a \in (0, \frac{1}{2})$ are slightly different and hence we will separate the proofs for each case. The first case is the most lengthy one, where we give full details. Many computations can be reused in the other two cases.

**Case $d = 1, a \in (0, \frac{1}{2})$**. As we mentioned above, our strategy is to approximate $\hat{Z}_{N,k}$ by $Z_{N,k}$. Note that the transition kernel $p_n(\gamma)$ satisfies [8] Hypothesis 2.4 for pinning model.

Since $a \in (0, \frac{1}{2})$, we can find some $\epsilon > 0$ such that $2a + \epsilon < 1$. And then we write

$$Z_{N,k} = Z_{N,k}^c + Z_{N,k}^c,$$

(2.12)
where

\begin{equation}
Z_{N,k}^{<} = \sum_{1 \leq n_1 < \cdots < n_k \leq N, j < j_1 \leq N^{2a+\varepsilon}} \prod_{j=1}^{k} \rho_{n_j-n_{j-1}}(x_j - x_{j-1}) \xi_{n_j,x_j},
\end{equation}

\begin{equation}
Z_{N,k}^{>} = \sum_{1 \leq n_1 < \cdots < n_k \leq N, j < j_1 \geq N^{2a+\varepsilon}} \prod_{j=1}^{k} \rho_{n_j-n_{j-1}}(x_j - x_{j-1}) \xi_{n_j,x_j}.
\end{equation}

Since for any time sequence \((n_1, \cdots, n_k)\) in (2.13) and \((m_1, \cdots, m_k)\) in (2.14), there must be some \(j\) such that \(n_j - n_{j-1} < m_j - m_{j-1}\) and hence \(Z_{N,k}^{<}\) and \(Z_{N,k}^{>}\) are independent. Therefore, we only need to compare their second moment. We have that

\begin{equation}
\var{Z_{N,k}^{<}} = \sum_{1 \leq n_1 < \cdots < n_k \leq N, j < j_1 < N^{2a+\varepsilon}} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(x_j - x_{j-1})(1 + o(1)),
\end{equation}

\begin{equation}
\var{Z_{N,k}^{>}} = \sum_{1 \leq n_1 < \cdots < n_k \leq N, j < j_1 \geq N^{2a+\varepsilon}} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(x_j - x_{j-1})(1 + o(1)).
\end{equation}

For (2.15), we have that

\begin{equation}
\var{Z_{N,k}^{<}} \leq \sum_{1 \leq n_1 < \cdots < n_k \leq N, j < j_1 < N^{2a+\varepsilon}} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(x_j - x_{j-1}) \leq \sum_{j=1}^{k} \binom{N}{j} \left( \sum_{n=1}^{N} \Pr(S_n = S'_n, |S_n| \leq 2RN^n) \right)^{k-j} \left( \sum_{n=1}^{N^{2a+\varepsilon}} \Pr(S_n = S'_n, |S_n| \leq 2RN^n) \right)^{j} \leq C_{K,a,R} (N^a \log N)^k,
\end{equation}

where in the last inequality, we follow the same lines in proving Lemma 2.1 to get that for \(d = 1, a = \frac{1}{2}\),

\begin{equation}
\sum_{n=1}^{N^{2a+\varepsilon}} \Pr(S_n = S'_n, |S_n| \leq RN^n) \sim 2eRN^n \log N \text{ as } N \to \infty \text{ and } C_{K,a,R} \text{ is uniform for } k = 1, \cdots, K \text{ and independent of } N.
\end{equation}

For (2.16), we have that

\begin{equation}
\var{Z_{N,k}^{>}} \geq \sum_{N^{2a+\varepsilon} \leq n_1 < \cdots < n_k \leq N/k} \prod_{j=1}^{k} p_{n_j-n_{j-1}}(x_j - x_{j-1}) = \left( \sum_{n=N^{2a+\varepsilon}}^{N/k} \Pr(S_n = S'_n, |S_n| \leq RN^n/k) \right)^{N \to \infty} \left( \frac{2(1 - 2a - \epsilon)R}{k\pi} \right)^k N^a \log N \right)^k,
\end{equation}

where the last asymptotic behavior is uniform for \(k = 1, \cdots, K\) (check the proof of Lemma 2.1). Hence, the contribution from \(Z_{N,k}^{<}\) can be made arbitrarily small by choosing small enough \(\epsilon\), so we only need to deal with \(Z_{N,k}^{>}\) in the following.

We show that \(\tilde{Z}_{N,k}\) can be approximated by \(Z_{N,k}^{>}\). We start by restricting the space on \(\gamma\) (recall (2.11)). Since \(\forall j, |x_j - x_{j-1}| \leq 2RN^n\), and \(n_j - n_{j-1} \geq N^{2a+\varepsilon}\), by local limit theorem, we have that \(p_{n_j-n_{j-1}}(x_j - x_{j-1}) \sim (1 + o_N(1))p_{n_j-n_{j-1}}(\gamma)\), uniformly \(\forall j, x_j - x_{j-1} \leftrightarrow n_j - n_{j-1}\). Then, note that
\( N^a \) is sub-diffusive, so intuitively we can “glue” all disorders at any time. To be precise, we introduce 
\( \zeta_n(N) := \sum_{x} |x| \leq R N^a \frac{\zeta_{n,x}}{\sqrt{2RN^2}} \) and note that \( \mathbb{E} \zeta_n = 0 \), \( \forall n(\zeta_n) \underset{N \to \infty}{\to} 1 \). Now we can write

\begin{equation}
Z_{N,k}^2 = \sqrt{2RN^a} \sum_{1 \leq n_1 < \cdots < n_k \leq N} \prod_{j=1}^k (1 + o_N(1))p_{n_j - n_{j-1}}(\gamma)\zeta_{n_j}.
\end{equation}

Finally, we show \( \|\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2\|_2 \overset{N \to \infty}{\to} 0 \) and then the result follows, where \( \bar{\beta} N^k = (\bar{\beta}/\sqrt{\log N})^k \).

We adapt the idea in [8, Lemma 6.1] here,

\begin{equation}
\mathbb{E}[(\beta N Z_{N}^2 - \beta N \tilde{Z}_{N,k}^2) - e^{(2k) N \tilde{Z}_{N,k}^2}] - 2\mathbb{E}[(\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2) - 2\beta N (\tilde{Z}_{N,k}^2) Z_{N,k}^2].
\end{equation}

Reasoning the independence as (2.12), the last term above is 0. It remains to show that the difference for the first two terms goes to 0. By a similar computation as (2.17), we find that the contribution to the partition function from \( n_j - n_{j-1} < N^{2a+c} \) is negligible and then the proof is completed.

(b) \( d = 2, a \in [\frac{1}{2}, 1] \). Similar to (2.20), we have

\[\mathbb{E}[(<\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2)] - 2\mathbb{E}[(\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2) - 2\beta N (\tilde{Z}_{N,k}^2) Z_{N,k}^2].\]

The last term is again 0, and the first term goes to 1 by a similar computation as (2.17) and (2.18) (Check the proof of Lemma 2.1 that the constant \( k \) does not matter the asymptotics for \( d = 2 \)). We only need to show that the second term also goes to 1, but the computation is the same as the first term and therefore the result follows, where \( \tau_k = N / \beta N \).

(c) \( d = 2, a \in (0, \frac{1}{2}] \). In this case, we show that \( \|\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2\|_2 \overset{N \to \infty}{\to} 0 \). We write

\[\mathbb{E}[(\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2)] - 2\mathbb{E}[(\beta N Z_{N,k}^2 - \beta N \tilde{Z}_{N,k}^2) - 2\beta N (\tilde{Z}_{N,k}^2) Z_{N,k}^2].\]

Reasoning as the case \( d = 2, a \in [\frac{1}{2}, 1] \), the first two terms both go to 1. However, the last term is not 0 since \( 2a < 1 \) and thus the summands in \( \tilde{Z}_{N,k}^2 \) is no longer a subset of those in \( \tilde{Z}_{N,k}^2 \). By independence of \( \xi_{n,x} \), we have that

\[\mathbb{E}[(\tilde{Z}_{N,k}^2 - \tilde{Z}_{N,k}^2)] = \mathbb{E}[(\tilde{Z}_{N,k}^2 - \tilde{Z}_{N,k}^2)] = \mathbb{E}[(\tilde{Z}_{N,k}^2 - \tilde{Z}_{N,k}^2)],\]

where

\[\mathbb{E}[(\tilde{Z}_{N,k}^2 - \tilde{Z}_{N,k}^2) = \sum_{1 \leq n_1 < \cdots < n_{N+1} \leq N} \prod_{j=1}^k p_{n_j - n_{j-1}}(x_j - x_{j-1}).\]

Again, by argument in the proof of Lemma 2.1, \( \mathbb{E}[(\tilde{Z}_{N,k}^2)] \sim (2a \log N / \pi)^k \), we show that the last term also converges to 0, and then we conclude the last case.

(ii) Case \( \beta \geq 1 \). The proof is identical to that in [8]. We sketch it for completeness.

It is enough to show that any \( \theta \in (0, 1) \), \( \mathbb{E}[(Z_{N,\beta})^\theta] \) converges to 0. We have that

\[\frac{d}{d\beta} \mathbb{E}[(Z_{N,\beta})^\theta] = \theta \sum_{n=1}^N \mathbb{E}\left[1_{\{|S| \leq RN^a\}} \mathbb{E}_S[(\omega_{n,S} - \lambda(\beta))(Z_{N,\beta})^\theta-1]\right],\]

where \( \mathbb{E}_S \) is a probability measure with \( \mathbb{E}_S[X] = \mathbb{E}[\exp(\sum_{n=1}^N (\beta \omega_{n,S} - \lambda(\beta)))X] \). Then by the FKG inequality, and note that \( \mathbb{E}_S[\omega_{n,S} - \lambda(\beta)] = 0 \), we get that

\[\frac{d}{d\beta} \mathbb{E}[(Z_{N,\beta})^\theta] \leq \theta \sum_{n=1}^N \mathbb{E}\left[1_{\{|S| \leq RN^a\}} \mathbb{E}_S[(\omega_{n,S} - \lambda(\beta))] \mathbb{E}_S[(Z_{N,\beta})^\theta-1]\right] = 0.\]

Thus, \( \theta \in (0, 1) \), \( \mathbb{E}[(Z_{N,\beta})^\theta] \) is non-increasing in \( \beta \). Then for any \( \beta' \geq \beta \),

\[\limsup_{N \to \infty} \mathbb{E}[(Z_{N,\beta'})^\theta] \leq \limsup_{N \to \infty} \mathbb{E}[(Z_{N,\beta})^\theta] = \mathbb{E}[(Z_{\beta})^\theta] = (1 - \beta^2)^{2(1-\theta)/2},\]

where the equality is due to the convergence result for \( \beta < 1 \). Let \( \beta \uparrow 1 \) and the proof is completed.
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