Analytical design of controller orienting higher-order closed-loop systems to minimum deadbeat response

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Abstract. This paper aims at designing controllers or compensators analytically so that designed higher-order closed-loop systems are oriented to having minimum deadbeat response. In accordance with the standardization transfer function of higher-order closed-loop system which has the minimum deadbeat response, a characteristic polynomial equation is configurated. This characteristic polynomial equation includes the unknown parameters of designed series controller/compensator. Solving this characteristic polynomial equation, the designed controller/compensator parameters can be determined. If the closed-loop system cannot meet the desired specifications, a pre-filter is cascaded ahead of the just designed closed-loop system. All the parameters of this pre-filter can be derived directly from the transfer function numerator polynomial of just designed closed-loop system. Application examples indicate the designed higher-order closed-loop systems certainly have the minimum deadbeat responses. This method provides another approach to analytically design the controllers or compensators for higher-order closed-loop systems.

1 Introduction

For the different higher-order closed-loop systems (HOCLS, their orders are usually equal to or higher than the third), it is not easy to analytically design the series controllers or compensators, because there are many controller or compensator parameters waiting to be determined. Meanwhile, these parameters are still associated with many coefficients of controlled plant as well as with many desired specifications of closed-loop system [1, 2]. The usual designing method is to preliminarily select one of some typical controllers such as PID or phase-lag-lead (PLL) compensators, then to test whether the desired closed-loop specifications are met [3, 4]. Hence the desired closed-loop specifications often cannot be met at once, the controller or compensator parameters have to be modified and tuned many times and the attained closed-loop specifications have to be tested again and again. As a result, the system designing procedure become complicated and unanalytical, and even so the attained closed-loop specifications are not necessarily satisfactory [5, 6].

In this paper, an analytical method of designing controller or compensator for HOCLS is formulated, in which the designed closed-loop systems are oriented to having the minimum deadbeat responses (MDR) [7]. Section 2 introduces the MDR and elaborates the designing procedure of controller or compensator. Section 3 verifies the designing procedure by two applying examples. Section 4 concludes the full text.
2 Analytical design method oriented higher-order closed-loop systems to minimum deadbeat response

The minimum deadbeat response means that a closed-loop system has the output response, whose oscillation amplitudes and wave numbers are both the minimum, when this closed-loop system is given step input. Therefore, it is evaluated quantitatively by the multiplication of natural oscillation frequency $\omega_n$ and settling time $t_s$ of the closed-loop system. Table 1 shows the standardization transfer functions and the optimal multiplication $\omega_n t_s$ of different order HOCLS which have the MDR [8].

| Order | Closed-loop transfer function | $\omega_n t_s$ |
|-------|------------------------------|---------------|
| 2     | $\frac{\omega_n^2}{s^2 + 1.82\omega_n s + \omega_n^2}$ | 4.8           |
| 3     | $\frac{\omega_n^3}{s^3 + 1.9\omega_n^2 s^2 + 2.2\omega_n^2 s + \omega_n}$ | 4.0           |
| 4     | $\frac{\omega_n^4}{s^4 + 2.2\omega_n^3 s^3 + 3.5\omega_n^3 s^2 + 2.8\omega_n^3 s + \omega_n}$ | 4.8           |
| 5     | $\frac{\omega_n^5}{s^5 + 2.7\omega_n^4 s^4 + 4.9\omega_n^4 s^3 + 5.4\omega_n^4 s^2 + 3.4\omega_n^4 s + \omega_n}$ | 5.4           |

Table 1: The standardization transfer functions and the optimal multiplication $\omega_n t_s$ of different order closed-loop systems which have the minimum deadbeat response

Fig. 1 shows the basic control configuration of closed-loop system, in which $G_0(s)$ represents the transfer function of controlled plant and $G_c(s)$ represents the transfer function of controller or compensator.

![Fig. 1 Basic control configuration of closed-loop system](image)

2.1 Analytical design of PID controller

PID controllers are used widely in the control systems in different industrial field. It consists of the proportional, the differential and the integral parts. The transfer function of PID controllers can be represented as:

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + \tau s\right)$$

(1)

where $K_p$, $T_i$ and $\tau$ are the proportional, the differential and the integral coefficients, respectively. They are waiting-determined parameters.

Supposing that transfer function model of controlled plant, $G_o(s)$, is as follows:

$$G_o(s) = \frac{K}{(as + 1)(bs + 1)}$$

(2)

where $K$, $a$ and $b$ are plant structure parameters.

Deriving the closed-loop transfer function, it yields:

$$\Phi(s) = \frac{KK_p(T_s s^2 + T_s + 1)}{abT_s^3 + (a + b + KK_p \tau)T_s^2 + (T_i + KK_p T_i)s + KK_p}$$

(3)

Then comparing the characteristic polynomial (CP) in Eq. 3 with the CP of the 3rd order
standardization closed-loop transfer function in Table 1, letting their different order coefficients to be equal correspondingly, there constructs an algebraic equations:

\[
\begin{align*}
abT_i &= 1 \\
(a + b + KK_i T_i)T_i &= 1.90\omega_n \\
T_i + KK_i T_i &= 2.20\omega_n^2 \\
KK_i &= \omega_n^2
\end{align*}
\]

(4)

Since \( K, a \) and \( b \) are known structure parameters, the natural oscillation frequency \( \omega_n \), and all the controller parameters \( K_p, T_i \) and \( r \), can be solved from Eq. 4.

2.2 Analytical design of PLL compensator

PLL compensator is another often used controller form in industrial field and it is usually designed in frequency domain. Due to different parameter selections of phase-lag and phase-lead loops, it has many structure types. Here giving one type whose transfer function is represented as:

\[ G_i(s) = \frac{K_g}{s + z_i} \]

(5)

where \( K_g \) is the controller gain, \( z_1 \) and \( z_2 \) are negative values of zero points; \( p_1 \) is negative value of non-zero pole.

Assuming the transfer function of an undamped controlled plant, \( G_o(s) \), is as follows:

\[ G_o(s) = \frac{K_0}{ms^2 + K_i} \]

(6)

where \( K_0, K_i \) and \( m \) are plant structure parameters.

Thus, the closed-loop transfer function can be derived:

\[ \Phi(s) = \frac{K_0 K_g [s^2 + (z_1 + z_2) s + z_1 z_2]}{ms^4 + mp_1 s^3 + (K_1 + K_i K_g) s^2 + (K_1 p_i + K_g (z_1 + z_2)) s + K_i K_g z_1 z_2} \]

(7)

Then comparing the CP in Eq. 7 with the CP of the 4th order standardization closed-loop transfer function in Table 1, letting their different order coefficients to be equal correspondingly, there can also construct an algebraic equations:

\[
\begin{align*}
p_1 &= 2.1\omega_n \\
(K_1 + K_i K_g) / m &= 2.4\omega_n^2 \\
[K_i p_i + K_g (z_1 + z_2)] / m &= 2.7\omega_n^3 \\
K_g K_i z_1 z_2 / m &= \omega_n^4
\end{align*}
\]

(8)

Since \( K_0, K_i \) and \( m \) are the known plant structure parameters, if assigning the natural oscillation frequency \( \omega_n \) to be a positive number, all the controller parameters \( K_g, z_1, z_2, \) and \( p_1 \) can be solved from Eq. 8.

2.3 Analytical design of pre-filter

As observed from Eq. 3 and Eq. 7, comparing with the standardization closed-loop transfer functions in Table 1, designed closed-loop transfer functions are attached a numerator polynomial. Although the performances of HOCLS is dependent on the denominator polynomial, i.e. the characteristic polynomial, the numerator polynomial might slightly effect the MDR because of the added zero points. In order to cancel out the zero point effects, there is placed a pre-filter ahead of the designed closed-loop system. Fig. 2 shows this control configuration of closed-loop system with pre-filter.
denoted by $G_p(s)$.

$$R(s) \xrightarrow{G_p(s)} G_c(s) \xrightarrow{G_0(s)} C(s)$$

**Fig. 2** Control configuration of closed-loop system with pre-filter

If the numerator polynomial (except for the gain) of designed closed-loop transfer function is denoted by $N(s)$, then the transfer function of pre-filter, $G_p(s)$, can be easily designed as:

$$G_p(s) = \frac{1}{N(s)} \quad (9)$$

### 3 Designing examples

#### 3.1 Designing example of PID Controller

Giving a second order controlled plant, and its transfer function model is as follows:

$$G_c(s) = \frac{1}{(10s + 1)(0.5s + 1)} \quad (10)$$

substituting the structure parameters $K$, $a$ and $b$ into Eq. 4 and solving this equations, there can be solved the natural oscillation frequency $\omega_n$ and all the controller parameters $K_p$, $T_i$ and $\tau$. Thereby the designed PID controller transfer function can be derived to be:

$$G_c(s) = 1327.37 \times \left(1 + \frac{1}{0.2s} + 0.0707s\right) \quad (11)$$

then the closed-loop transfer function can be solved to be:

$$\Phi(s) = \frac{1327.37(0.014s^2 + 0.2s + 1)}{s^3 + 20.88s^2 + 265.7s + 1327.37} \quad (12)$$

and the unit step response of designed closed-loop system is shown in Fig. 3:

**Fig. 3** The unit step response of closed-loop system

There can be read from Fig. 3, the settling time $t_s = 0.51$ second ($\pm5\%$ error band) and the maximum overshoot is $31\%$. There is only one time oscillation and it is a desired MDR. The maximum overshoot is higher somehow. So a pre-filter is designed according to Eq. 9 as:

$$G_p(s) = \frac{1}{0.014s^2 + 0.2s + 1} \quad (13)$$
and the transfer function of closed-loop system with pre-filter turns into:

\[
\Phi'_t(s) = \Phi(t)G_p(s)
\]

\[
= \frac{1327.37}{s^3 + 20.88s^2 + 265.7s + 1327.37}
\]

The unit step response of designed closed-loop system with pre-filter is shown in Fig. 4. There can be read from Fig. 4, the maximum overshoot is reduced strikingly from 31% to 1.7% and the settling time \( t_s \) is also decreased from 0.51 second to 0.34 second. This illustrates that the closed-loop specifications is improved by the pre-filter.

![Unit step response of closed-loop system with pre-filter](image)

**Fig. 4** The unit step response of closed-loop system with pre-filter

### 3.2 Designing example of PLL compensator

Giving an undamped second order controlled plant, and its transfer function model is as follows:

\[
G_o(s) = \frac{4}{5s^2 + 1}
\]

Substituting the structure parameters \( K_0, K_1 \) and \( m \) into Eq. 8 and assigning the natural oscillation frequency \( \omega_n \) to be 1, the PLL compensator parameters \( K_p, z_1, z_2, \) and \( p_1 \) can be solved. Thereby the designed PLL compensator transfer function can be derived:

\[
G_c(s) = \frac{2.75(s + 0.518 + j0.4262)(s + 0.518 - j0.4262)}{s(s + 2.1)}
\]

Then the closed-loop transfer function can be solved to be:

\[
\Phi_t(s) = \frac{4.95(2.22s^2 + 2.3s + 1)}{5s^4 + 10.5s^3 + 12s^2 + 13.496s + 4.95}
\]

and the unit step response of designed closed-loop system is shown in Fig. 5:

![Unit step response of closed-loop system](image)
There can be read from Fig. 5, the settling time $t_s = 35.38$ second ($\pm 5\%$ error band) and the maximum overshoot is 72.5%. Due to the effects of added zero points in closed-loop system, obviously, the oscillation amplitude is too high. Thus, a pre-filter is designed according to Eq. 9 as:

$$G_p(s) = \frac{1}{(2.2s^2 + 2.3s + 1)}$$

and the transfer function of closed-loop system with this pre-filter is derived as:

$$\Phi'(s) = \Phi(s) G_p(s)$$

$$= \frac{4.95}{5s^4 + 10.5s^3 + 12s^2 + 13.496s + 4.95}$$

The unit step response of designed closed-loop system with pre-filter is shown in Fig.6. There can be read from Fig. 6, comparing with the unit step response shown in Fig. 5, the maximum overshoot is reduced strikingly from 72.5% to 14.2% and the settling time $t_s$ is also decreased from 35.38 second to 21.23 second. It illustrates that the closed-loop system is improved and the closed-loop specifications can be met by the pre-filter.

**Fig. 6** The unit step response of closed-loop system with pre-filter

**4 Conclusions**

Analytical designing method of controllers or compensators for higher-order closed-loop systems is formulated in this paper. Based on the standardization transfer function of higher-order closed-loop system which has the minimum deadbeat response, a characteristic polynomial equation, in which the unknown parameters of designed series controller or compensator are included, is configured. Solving this characteristic polynomial equation, the designed controller or compensator parameters can be determined. Furthermore, if the closed-loop systems with just designed controllers or
compensators cannot meet the desired specifications, a pre-filter is cascaded ahead of the just designed closed-loop system. All the parameters of the pre-filter can be derived directly from the numerator polynomial of transfer function of just designed closed-loop system. By this method, the controllers or compensators, and the pre-filters can be analytically designed to apply to higher-order closed-loop systems, such that the designed higher-order closed-loop systems have the minimum deadbeat responses.

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