FILLING MINIMALITY OF FINSLERIAN 2-DISCs

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Abstract. We prove that every Riemannian metric on the 2-disc such that all its geodesics are minimal, is a minimal filling of its boundary (within the class of fillings homeomorphic to the disc). This improves an earlier result of the author by removing the assumption that the boundary is convex. More generally, we prove this result for Finsler metrics with area defined as the two-dimensional Holmes–Thompson volume. This implies a generalization of Pu’s isosystolic inequality to Finsler metrics, both for Holmes–Thompson and Busemann definitions of Finsler area.

1. Introduction

For a Riemannian metric $g$ on a compact manifold $M$ with boundary, let $d_g$ denote the corresponding distance function on $M \times M$. The boundary distance function of $g$, denoted by $bd_g$, is the restriction of $d_g$ to $\partial M \times \partial M$. That is, $bd_g(x, y)$ is the length of a $g$-shortest path in $M$ between boundary points $x$ and $y$.

It is natural to ask what kind of information about $g$ can be recovered if one knows the boundary distance function (or an approximation of it). In some cases $bd_g$ determines $g$ uniquely up to an isometry but in general this is not the case. Attaching a large “bubble” with a narrow neck has very little effect (in the $C^0$ sense) on boundary distances, and changing the metric within the bubble has no effect at all. Thus a metric with a given boundary distance function can be arbitrarily large in terms of Riemannian volume.

However it cannot be arbitrarily small. The boundary distance function (or a lower bound for it) imposes a positive lower bound on the volume of the metric. Metrics realizing this lower bound are called minimal fillings, see below. Although minimal fillings are in a sense similar to minimal surfaces (cf. [11]), they do not have similar existence and regularity properties. Nevertheless there are many examples of smooth minimal fillings (including all metrics sufficiently close to a flat one, cf. [3]).

It is plausible that every smooth Riemannian metric with minimal geodesics (see below for a precise definition) is a minimal filling. In [10], this conjecture was proved in dimension 2 for discs with convex boundaries. In this paper we remove the convex boundary assumption and generalize the result to Finsler metrics. (The Finslerian case appears in [10] as well but the proof there is too sketchy.) The Finslerian result implies that Pu’s isosystolic inequality [14] holds for Finsler metrics. Even in the Riemannian case the resulting proof of Pu’s inequality differs from the original one, in particular, it does not use the uniformization theorem.

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1.1. **Riemannian minimal fillings.** Let $D$ denote the two-dimensional disc $D^2$ and $S = \partial D \cong S^1$. For a nonnegative function $d : S \times S \to \mathbb{R}$, define the *filling volume* (or *filling area*) of $(S, d)$, denoted by $\text{FillVol}_D(S, d)$, by

$$\text{FillVol}_D(S, d) = \inf_{g : \text{bd} g \geq d} \text{area}(D, g)$$

where the infimum is taken over all Riemannian metrics $g$ on $D$ such that the boundary distance function $\text{bd} g$ is bounded below by $d$. Here $\text{area}(D, g)$ denotes the two-dimensional Riemannian volume of $D$ with respect to $g$.

The notion of filling volume was introduced by Gromov [8]. The above definition differs from Gromov’s in one essential detail (indicated by the subscript $D$): we restrict ourselves to metrics on the disc while in Gromov’s definition one takes the infimum over Riemannian manifolds of varying topology (namely all orientable manifolds) whose boundaries are identified with $S$. In higher dimensions restricting the topology type does not change the filling volume [8, App. 2, Prop. A′], but in dimension 2 the two definitions are probably not equivalent.

A metric $g_0$ (or a space $(D, g_0)$) is said to be a *minimal filling* if it realizes the infimum in (1.1) for some $d$, or, equivalently, for $d = \text{bd} g_0$. Substituting the definitions yields the following reformulation: $g_0$ is a minimal filling if and only if for every Riemannian metric $g$ on $D$ satisfying

$$d_g(x, y) \geq d_{g_0}(x, y) \quad \text{for all } x, y \in \partial D,$$

one has

$$\text{area}(D, g) \geq \text{area}(D, g_0).$$

Classic examples of two-dimensional minimal fillings are the standard hemisphere (this follows from Pu’s inequality, see below) and regions in the Euclidean and hyperbolic planes, cf. [8]. In fact, Euclidean and hyperbolic regions are minimal in a stronger sense of Gromov’s definition while the hemisphere is known to be minimal only within the class of fillings homeomorphic to the disc.

We say that $g$ is a *metric with minimal geodesics* if every $g$-geodesic in the interior of $D$ is a shortest path between its endpoints. By continuity, minimality of all geodesics in the interior implies minimality of geodesics with endpoints on the boundary but otherwise contained in the interior. We do not consider geodesics that have points on $\partial D$ other than endpoints.

One of the goals of this paper is to prove the following theorem.

**Theorem 1.** *Every Riemannian metric with minimal geodesics on $D$ is a minimal filling in the above sense.*

This theorem was proved in [10] under an additional assumption that the boundary of the disc is convex with respect to the metric. The proof in this paper is essentially the same as in [10] modulo technical details allowing the proof to work in the case of a non-convex boundary. The key idea to estimate areas using cyclic order of gradients of distance functions is borrowed from [2], and it was used earlier by Croke and Kleiner [6].
1.2. Finslerian case. Theorem 1 is a partial case of Theorem 2 which asserts the same fact for Finsler metrics (including non-reversible ones). We do not use heavy machinery of Finsler geometry, all necessary definitions and facts are included here and in section 2. Details and proofs can be found e.g. in [15].

A Finsler metric on a smooth manifold \( M \) is a continuous function \( \varphi : T M \rightarrow \mathbb{R} \) satisfying the following conditions:

1. \( \varphi(tv) = t\varphi(v) \) for all \( v \in T M \) and \( t \geq 0 \);
2. \( \varphi \) is positive on \( T M \setminus 0 \);
3. \( \varphi \) is smooth on \( T M \setminus 0 \) (for our purposes, \( C^2 \) smoothness is sufficient);
4. \( \varphi \) is strictly convex in the following sense: for every \( x \in M \), the function \( \varphi^2|_{T_x M} \) has positive definite second derivatives on \( T_x M \setminus \{0\} \).

A Finsler metric \( \varphi \) is reversible (or symmetric) if \( \varphi(-v) = \varphi(v) \) for all \( v \in T M \).

A Finsler metric \( \varphi \) on \( M \) can be thought of as a family of (non-symmetric) norms \( \varphi_x := \varphi|_{T_x M}, x \in M \), on the fibers of \( T M \). Riemannian metrics are partial case of (reversible) Finsler metrics, they are characterized by the property that all norms \( \varphi_x \) are Euclidean. For a Finsler metric \( \varphi \), one naturally defines geodesics, lengths and a (non-symmetric) distance function \( d_\varphi : M \times M \rightarrow \mathbb{R}_+ \), cf. section 2. We define the boundary distance function and metrics with minimal geodesics in the same way as in the Riemannian case.

The definition of area is a more delicate subject. There are several non-equivalent definitions of area and volume in Finsler geometry, cf. [16] or [1] for a survey. The most widely used definition is Busemann’s [4] where Finsler volume is defined so that the volume of a unit ball in every \( n \)-dimensional normed space is the same as that of the standard Euclidean ball in \( \mathbb{R}^n \). The Busemann volume of a reversible Finsler metric equals the Hausdorff measure of the corresponding metric space. We denote the Busemann volume of an \( n \)-dimensional Finsler manifold \((M, \varphi)\) by \( \text{vol}_b^n(M, \varphi) \).

In this paper we mainly use another definition, namely the Holmes–Thompson volume [9]. By definition, the Holmes–Thompson volume \( \text{vol}_t^n(M, \varphi) \) of an \( n \)-dimensional Finsler manifold \((M, \varphi)\) equals the canonical (symplectic) volume of the bundle of unit balls in \( T^* M \), normalized by a suitable constant (namely divided by the Euclidean volume of the unit ball in \( \mathbb{R}^n \)). See section 2 for more details.

This choice of volume definition is enforced by the following fact: Finsler metrics (including metrics with minimal geodesics) admit non-isometric deformations preserving the boundary distances. These deformations preserve the Holmes–Thompson volume but not the Busemann (or any other) volume.

The main result of this paper is the following theorem.

**Theorem 2.** Let \( \varphi_0 \) be a Finsler metric with minimal geodesics on \( D \), and let \( \varphi \) be a Finsler metric on \( D \) such that

\[
d_\varphi(x, y) \geq d_{\varphi_0}(x, y) \quad \text{for all } x, y \in \partial D.
\]

Then

\[
\text{vol}_t^n(D, \varphi) \geq \text{vol}_t^n(D, \varphi_0)
\]

with equality if and only if \( \varphi \) is a metric with minimal geodesics whose boundary distance function equals that of \( \varphi_0 \).
The proof of Theorem 2 is contained in sections 2, 3 and 5. Section 2 contains preliminaries and some technical facts about Finsler metrics, in section 3 we obtain a lower bound for $\text{vol}^h_2(D, \varphi)$ in terms of a “cyclic map” (Proposition 3.7), and in section 5 we construct a special cyclic map such that the resulting lower bound equals $\text{vol}^h_2(D, \varphi_0)$. The equality case of the theorem follows from the inequality, this is explained in the end of section 5. Section 4 is a digression where we prove Pu’s inequality for Finsler metrics, see below.

1.3. Pu’s inequality. For a metric $\varphi$ on a manifold $M$, denote by $\text{sys}_1(M, \varphi)$ the one-dimensional homotopy systole, that is the length of the shortest noncontractible loop in $(M, \varphi)$. Certain types of manifolds (cf. [8] for details) admit an inequality of the form

$$\text{sys}_1(M, g) \leq C(M) \cdot \text{vol}_n(M, g)^{1/n}$$

for any Riemannian metric $g$ on $M$, where $n$ is the dimension and $C(M)$ depends only on the topology of $M$. The optimal value of the constant $C(M)$ is known only in a few cases, one of which is Pu’s theorem for $M = \mathbb{RP}^2$.

**Theorem (P. Pu [14]).** For every Riemannian metric $g$ on $\mathbb{RP}^2$, one has

$$\text{area}(\mathbb{RP}^2, g) \geq \frac{2}{\pi} \text{sys}_1(\mathbb{RP}^2, g)^2$$

with equality if and only if $g$ has constant curvature (or, equivalently, is isometric to a rescaling of a standard “round” metric).

It is easy to see that Pu’s inequality is equivalent to the fact that the hemisphere $S^2_+\mathbb{+}$ with its standard Riemannian metric is a minimal filling. To reduce Pu’s inequality to the filling minimality of the hemisphere, rescale the metric so that $\text{sys}_1(\mathbb{RP}^2, g) = \pi$ and cut $\mathbb{RP}^2$ along a $g$-shortest noncontractible loop. The resulting space is a disc with a Riemannian metric such that the length of the boundary is $2\pi$ and the distance between every pair of opposite points of the boundary is $\pi$. Then the triangle inequality implies that the distance between any two boundary points is realized by an arc of the boundary. This means that (1.2) is satisfied if $g_0$ is the metric of the standard hemisphere (whose boundary circle is identified with the boundary of our disc in a length-preserving way).

Since the geodesics in the hemisphere are minimal, Theorem 1 implies that the hemisphere is a minimal filling. Therefore the area of our metric on the disc (and hence of the original metric on $\mathbb{RP}^2$) is at least that of the hemisphere, that is, $2\pi$. Thus Pu’s inequality follows from Theorem 1.

The above argument applies without changes to Finsler metrics (except that the triangle inequality part requires symmetry of the metric). Thus Theorem 2 implies the following Finslerian generalization of Pu’s inequality.

**Theorem 3.** Let $\varphi$ be a reversible Finsler metric on $\mathbb{RP}^2$. Then

$$\text{vol}^h_2(\mathbb{RP}^2, \varphi) \geq \frac{2}{\pi} \text{sys}_1(\mathbb{RP}^2, \varphi)^2.$$
It is well-known (cf. e.g. [7] or [1]) that the Busemann volume of any Finsler metric is no less than its Holmes-Thompson volume, and they are equal if and only if the metric is Riemannian. This fact and Theorem 3 immediately imply the following Pu’s inequality for the Busemann area.

**Theorem 4.** Let $\varphi$ be a reversible Finsler metric on $\mathbb{RP}^2$. Then

$$\text{vol}_b^1(\mathbb{RP}^2, \varphi) \geq \frac{2}{\pi} \text{sys}_{\pi_1}(M, \varphi)^2$$

with equality if and only if $\varphi$ is a Riemannian metric of constant curvature.

1.4. **Remarks and open questions.**

1. It remains unclear whether Riemannian metrics with minimal geodesics on the disc are minimal fillings in a stronger sense, that is, within the class of Riemannian metrics on surfaces of arbitrary genus. This is not known even in the case of the hemisphere although the question dates back to Gromov’s paper [8]. The proof in this paper does not work for surfaces of higher genus since it uses the Jordan Curve Theorem in several places (most importantly, in Lemma 2.3 and Lemma 5.4).

2. One of the motivating reasons to study minimal fillings is their relation to boundary rigidity problems. A metric $g$ is said to be boundary (distance) rigid if its boundary distance function determines the metric uniquely up to an isometry. It is conjectured (cf. e.g. [12] and [5]) that all metrics with strongly minimal geodesics (that is, minimal and having no conjugate points up to and including endpoints at the boundary) are boundary rigid. In dimension 2 this conjecture was proved (for metrics with convex boundaries) by Pestov and Uhlmann [13]. A promising approach to boundary rigidity is studying the case of equality in the filling inequality (1.3), cf. [3] for a successful application of this approach. Unfortunately the proof of Theorem 1 in this paper does not suggest a way to study the equality case because of the Finslerian nature of the proof and non-rigidity of Finsler metrics.

3. One easily sees that the assumption about $\varphi_0$ in Theorem 2 is the weakest possible: a metric with a non-minimal geodesic cannot be a minimal filling in the Finsler category since it can be altered so as to reduce the volume while preserving the boundary distances (similarly to the proof of the equality case in section 5). This argument does not work in the Riemannian category, and this raises the following question: are there Riemannian minimal fillings with non-minimal geodesics? Probably the simplest example to study is the product metric on $S^1 \times [0, 1]$.

2. **FINSLER METRICS, GEODESICS AND DIRECTIONS**

Let $\varphi$ be a Finsler metric on $D$. We omit dependence on $\varphi$ in most terms and notations. For $x \in D$, we denote by $B_x$ and $U_x$ the unit ball and the unit sphere of the norm $\varphi_x = \varphi|_{T_x D}$, that is,

$$B_x = \{v \in T_x D : \varphi(v) \leq 1\},$$

$$U_x = \{v \in T_x D : \varphi(v) = 1\}.$$

Note that $B_x$ is a convex set whose boundary $U_x$ is a smooth strictly convex curve.

The length $L_\varphi(\gamma)$ of a piecewise smooth curve $\gamma : [a, b] \to D$ is defined by

$$L_\varphi(\gamma) = \int_a^b \varphi(\gamma(t)) \, dt$$
where the velocity vector $\dot{\gamma}(t)$ is regarded as an element of $T_{\gamma(t)}D$. The distance function $d_{\varphi} : D \times D \to \mathbb{R}_+$ is defined by $d_{\varphi}(x, y) = \inf_{\gamma} L_{\varphi}(\gamma)$ where the infimum is taken over all piecewise smooth curves $\gamma$ starting at $x$ and ending at $y$. Note that $d_{\varphi}$ is not symmetric (unless $\varphi$ is reversible) but it has other standard properties of a distance, in particular the triangle inequality

\begin{equation}
 d_{\varphi}(x, y) + d_{\varphi}(y, z) \geq d_{\varphi}(x, z).
\end{equation}

Once the distance is defined, the length functional extends to all continuous curves in a usual way, and a standard compactness argument shows that every pair of points $x, y \in D$ can be connected by a shortest path, i.e. is a curve from $x$ to $y$ whose length equals $d_{\varphi}(x, y)$. Note that a pointwise limit of shortest paths is a shortest path due to lower semi-continuity of length.

A geodesic is a curve which is contained in the interior of $D$ except possibly the endpoints and is a critical point of the energy functional $\gamma \mapsto \int \varphi^2(\gamma(t)) \, dt$. Finsler geodesics have standard properties such as existence and uniqueness of a geodesic with a given initial velocity and the fact that every shortest path in the interior of $D$ is a geodesic (and hence smooth). The only notable difference from Riemannian geodesics is that reversing direction may turn a geodesic into a non-geodesic. Unless otherwise stated, all geodesics and shortest paths are assumed parameterized by arc length.

As usual, $T^*D$ denotes the co-tangent bundle of $D$; an element of $T^*_x D \subset T^*D$ is a linear function on $T_x D$. The dual metric $\varphi^* : T^*D \to \mathbb{R}_+$ is defined by

$$
\varphi^*(u) = \sup\{u(v) : v \in U_x\} \quad \text{for } u \in T^*_x D, \ x \in M.
$$

The definition implies that $\varphi^*|_{T^*_x D}$ is a (possibly non-symmetric) norm on $T^*_x D$. We denote by $B^*_x$ and $U^*_x$ the unit ball and the unit sphere of this norm, that is,

$$
B^*_x = \{u \in T^*_x D : \varphi^*(u) \leq 1\},
$$

$$
U^*_x = \{u \in T^*_x D : \varphi^*(u) = 1\}.
$$

Note that $B_x$ and $B^*_x$ are polar to each other.

Recall that the co-tangent bundle $T^*D$ carries a canonical (four-dimensional) volume form. By definition, the Holmes–Thompson area $\text{vol}^h_2(D, \varphi)$ equals the canonical volume of the set $B^*D = \bigcup_{x \in D} B^*_x$, divided by $\pi$. In coordinates $(x_1, x_2)$ on $D$, this can be written as

\begin{equation}
\text{vol}^h_2(D, \varphi) = \frac{1}{\pi} \int_D |B^*_x| \, dx_1 dx_2
\end{equation}

where $|B^*_x|$ is the coordinate Lebesgue measure of the set $B^*_x \subset T^*_x D \approx \mathbb{R}^2$. In the sequel we always use this coordinate formula rather than any of the invariant expressions for the Holmes–Thompson area.

The Legendre transform associated with $\varphi$ is a norm-preserving positively homogeneous map $L = L_{\varphi} : TD \to T^*D$ that can be defined as follows: for every $x \in M$ and $v \in U_x$, $L(v)$ is the unique co-vector $u \in U^*_x$ such that $u(v) = 1$. Strict convexity of $\varphi$ implies that $L$ is a diffeomorphism between $U_x$ and $U^*_x$.

**Definition 2.1.** Let $x \in D \setminus \partial D$ and $y \in D$. Consider a shortest path $\gamma : [0, T] \to D$ from $x$ to $y$. Since $x \notin \partial D$, $\gamma$ is a geodesic near $x$ and hence is differentiable at 0. We refer to the
initial velocity vector $\dot{\gamma}(0)$ of $\gamma$ as a direction to $y$ at $x$ and denote it by $\vec{xy}$. If all shortest paths from $x$ to $y$ have the same initial velocity, we say that $\vec{xy}$ is uniquely defined.

Similarly, if $\gamma : [0, T] \to D$ is a shortest path from $y$ to $x$, we refer to the vector $\dot{\gamma}(T)$ as the direction from $y$ at $x$ and denote it by $\vec{yx}$; and we say that $\vec{xy}$ is uniquely defined if this vector is the same for all such paths.

**Lemma 2.2.** Suppose that $\varphi$ is a metric with minimal geodesics. Then for every pair of distinct points $x \in D \setminus \partial D$ and $y \in D$, the directions $\vec{xy}$ and $\vec{yx}$ are uniquely defined.

**Proof.** We prove uniqueness of $\vec{xy}$, the case of $\vec{yx}$ is similar. Suppose the contrary, then there exist shortest paths $\gamma_1, \gamma_2 : [0, T] \to D$ connecting $x$ to $y$ such that $\dot{\gamma}_1(0) \neq \dot{\gamma}_2(0)$. Let $z$ be the nearest to $x$ common point of $\gamma_1$ and $\gamma_2$. Then, by Jordan Curve Theorem, the intervals of $\gamma_1$ and $\gamma_2$ between $x$ and $z$ bound a region $\Omega \subset D$ containing no points of $\partial D$.

Let $\gamma : [0, T_1] \to D$ be a geodesic starting at $x$ with initial velocity $\dot{\gamma}(0)$ pointing into $\Omega$ and extended forward until it reaches the boundary of $\Omega$ (due to minimality of geodesics, $\gamma$ cannot have infinite length and hence hits the boundary eventually). Denote $p = \gamma(T_1)$ and assume for definiteness that $p$ lies on $\gamma_1$. Since $\varphi$ is a metric with minimal geodesics, $\gamma$ is a shortest path.

Let $s$ denote the interval of $\gamma_1$ from $x$ to $p$. Then $s$ and $\gamma$ are shortest paths, hence $L_\varphi(s) = L_\varphi(\gamma)$. Let $\gamma^- : [-\varepsilon, 0] \to D$ be a geodesic extending $\gamma$ backwards (that is, $\dot{\gamma}^-(0) = x$ and $\dot{\gamma}^-(0) = \dot{\gamma}(0)$). Then $\gamma^- \cup \gamma$ is a geodesic and hence a shortest path. Therefore the curve $\gamma^- \cup s$ is a shortest path because it has the same length. But this curve is not smooth at $0$ since $\dot{s}(0) = \dot{\gamma}_1(0) \neq \dot{\gamma}(0)$, a contradiction. \qed

Fix an orientation of $D$. This orientation induces orientations and hence cyclic orders on $\partial D$ and the circles $U_x$ and $U_x^*$ for all $x \in D$. Note that the Legendre transform $\mathcal{L} : U_x \to U_x^*$ is an orientation-preserving diffeomorphism.

**Lemma 2.3.** Let $x \in D \setminus \partial D$ and $p_1, p_2, p_3 \in \partial D$. Suppose that directions $\vec{xp}_1$, $\vec{xp}_2$ and $\vec{xp}_3$ are uniquely defined and distinct. Then the cyclic ordering of the vectors $\vec{xp}_1$, $\vec{xp}_2$ and $\vec{xp}_3$ in $U_x$ is the same as the cyclic ordering of the points $p_1$, $p_2$ and $p_3$ in $\partial D$.

The same is true for $\vec{xp}_1$, $\vec{xp}_2$ and $\vec{xp}_3$, provided that they are uniquely defined and distinct.

**Proof.** We will prove the first statement, the second one is similar. For every $i = 1, 2, 3$, let $\gamma_i : [0, T_i] \to D$ be a shortest path from $x$ to $p_i$ to $x$. Then $\dot{\gamma}_i(0) = \vec{xp}_i$.

We claim that the curves $\gamma_1$, $\gamma_2$ and $\gamma_3$ have no common points except $x$. Indeed, suppose that $\gamma_1$ and $\gamma_2$ have a common point $q \neq x$. Let $s_1$ and $s_2$ denote the intervals of $\gamma_1$ and $\gamma_2$ between $x$ and $q$. Since $s_1$ and $s_2$ are intervals of shortest paths, they are shortest paths themselves. In particular, $L_\varphi(s_1) = L_\varphi(s_2) = d_\varphi(x, q)$. Consider a new curve $\gamma$ composed from $s_2$ and the segment of $\gamma_1$ between $q$ and $p_1$. Since $L_\varphi(s_1) = L_\varphi(s_2)$, we have $L_\varphi(\gamma) = L_\varphi(\gamma_1)$, hence $\gamma$ is another shortest path from $x$ to $p_1$. However the initial velocity of $\gamma$ equals $\dot{\gamma}(0) = \vec{xp}_2 \neq \vec{xp}_1$, contrary to the assumption that $\vec{xp}_1$ is uniquely defined. The claim follows.

Since the curves $\gamma_1$, $\gamma_2$ and $\gamma_3$ have no common points except $x$ and our space $D$ is a topological 2-disc, the cyclic ordering of the points $p_1$, $p_2$ and $p_3$ on $\partial D$ is the same as that of the intersections $\gamma_1$, $\gamma_2$ and $\gamma_3$ with a small circle centered at $x$, and the latter is the same as the cyclic ordering of the initial velocity vectors $\dot{\gamma}_1(0)$, $\dot{\gamma}_2(0)$ and $\dot{\gamma}_3(0)$. \qed
Definition 2.4. A function $f: D \to \mathbb{R}$ is said to be forward 1-Lipschitz (with respect to $\varphi$) if $f(y) - f(x) \leq d_\varphi(x, y)$ for all $x, y \in D$.

Example 2.5. For any $p \in D$, the functions $x \mapsto d_\varphi(p, x)$ and $x \mapsto -d_\varphi(x, p)$ are forward 1-Lipschitz. The requirement of Definition 2.4 follows from the triangle inequality (2.1).

Obviously every forward 1-Lipschitz function is Lipschitz in any local coordinates. Hence, by Rademacher’s theorem, such a function is differentiable almost everywhere. We denote the derivative of $f$ at $x \in D$ by $d_x f$ and the map $x \mapsto d_x f \in T_x^* D$ by $df$. We regard $df$ as a differential 1-form on $D$ with Borel measurable coefficients (defined a.e.).

Note that $\varphi^*(d_x f) \leq 1$ if $f$ is a forward 1-Lipschitz function differentiable at $x$.

Lemma 2.6. Let $f: D \to \mathbb{R}$ be a forward 1-Lipschitz function and $x \in D \setminus \partial D$. Suppose that $f$ is differentiable at $x$. Then

1. If a point $y \in D$ satisfies
   \[ f(y) = f(x) + d_\varphi(x, y), \]
   then $\overrightarrow{xy}$ is uniquely defined and $d_x f = \mathcal{L}(\overrightarrow{xy}) \in U_x^*$.

2. If a point $y \in D$ satisfies
   \[ f(x) = f(y) + d_\varphi(y, x), \]
   then $\overrightarrow{yx}$ is uniquely defined and $d_x f = \mathcal{L}(\overrightarrow{yx}) \in U_x^*$.

Proof. Let $y \in D$ satisfy (2.3) and let $v$ be a direction to $y$ at $x$. Then $v = \dot{\gamma}(0)$ where $\gamma: [0, T] \to D$ is a shortest path from $x$ to $y$ parameterized by arc length. Since $f$ is forward 1-Lipschitz, the function $t \mapsto f(\gamma(t))$ is 1-Lipschitz on $[0, T]$. On the other hand,
   \[ f(\gamma(T)) - f(\gamma(0)) = f(y) - f(x) = d_\varphi(x, y) = T, \]
   therefore $f(\gamma(t)) = f(x) + t$ for all $t \in [0, T]$. Hence
   \[ d_x f(v) = \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) = 1. \]
   Since $\varphi^*(d_x f) \leq 1$ and $\varphi(v) = 1$, this identity implies that $\varphi^*(d_x f) = 1$ and $d_x f = \mathcal{L}(v)$. This determines $v$ uniquely (namely $v = \mathcal{L}^{-1}(d_x f)$) and the first assertion of the lemma follows.

The second assertion follows by a similar argument applied to a shortest path from $y$ to $x$ and its derivative at the endpoint. \qed

Corollary 2.7. Fix a point $p \in D$ and consider a function $f: D \to \mathbb{R}$ given by $f(x) = d_\varphi(p, x)$. Then, for every $x \in D \setminus \partial D$ where $f$ is differentiable, the direction $\overrightarrow{xp}$ is uniquely defined and $d_x f = \mathcal{L}(\overrightarrow{xp}) \in U_x^*$.

Proof. The function $f$ is forward 1-Lipschitz, cf. Example 2.5. For every $x \in D$, the point $y = p$ satisfies (2.4) from Lemma 2.6 hence the result. \qed
3. Cyclic maps

**Definition 3.1.** A Lipschitz map $f : D \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$ is said to be cyclic (with respect to $\varphi$) if the derivatives of its coordinate functions $f_i$ satisfy the following:

1. If $x \in D \setminus \partial D$ and $f_i$ is differentiable at $x$, then $d_x f_i \in U^*_x$.
2. For every $x \in D \setminus \partial D$ and $i, j, k \in \{1, \ldots, n\}$ such that $i < j < k$ and the derivatives $d_x f_i, d_x f_j, d_x f_k$ are well-defined and distinct, the cyclic ordering of the triple $(d_x f_i, d_x f_j, d_x f_k)$ in $U^*_x$ is positive.

Note that the second requirement depends only on the cyclic order of coordinate functions. Thus if a map $(f_1, f_2, \ldots, f_n)$ is cyclic, then so is $(f_2, f_3, \ldots, f_n, f_1)$.

**Example 3.2.** Let $p_1, p_2, \ldots, p_n$ be a cyclically ordered collection of points in $\partial D$. Define $f_i(x) = d_\varphi(p_i, x)$ for all $x \in D$, $i \leq n$. Then the map $f = (f_1, f_2, \ldots, f_n)$ is cyclic.

**Proof.** If $f_i$ is differentiable at $x \in D \setminus \partial D$, then $d_x f_i = \mathcal{L}(\frac{x}{x_P}) \in B^*_x$ by Corollary 2.7. Since $\mathcal{L}$ is an orientation-preserving diffeomorphism between $U_x$ and $U^*_x$, the cyclic ordering of any triple $(d_x f_i, d_x f_j, d_x f_k)$ in $U^*_x$ is the same as that of the triple $(\frac{x_P}{x_P}, \frac{x_P}{x_P}, \frac{x_P}{x_P})$ in $U_x$. By Lemma 2.3, the latter is the same as the cyclic ordering of $(p_i, p_j, p_k)$ in $\partial D$, provided that these (co-)vectors are well-defined and distinct. And the cyclic ordering of $(p_i, p_j, p_k)$ is positive if $i < j < k$. □

**Definition 3.3.** For a Lipschitz map $f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$, define

$$I(f) = \frac{1}{2\pi} \int_D \sum_{i=1}^n df_i \wedge df_{i+1}. \quad (3.1)$$

Here and in the sequel all indices like $i + 1$ are taken modulo $n$.

**Lemma 3.4.** $I(f)$ is determined by the restriction of $f$ to the boundary. That is, if $f$ and $\overline{f}$ are Lipschitz maps from $D$ to $\mathbb{R}^n$ and $f|_{\partial D} = \overline{f}|_{\partial D}$, then $I(f) = I(\overline{f})$.

**Proof.** The (Borel measurable) 2-form $\sum df_i \wedge df_{i+1}$ on $D$ is induced by $f$ from a 2-form $\omega = \sum dx_i \wedge dx_{i+1}$ on $\mathbb{R}^n$. Hence the integral in (3.1) is the integral of $\omega$ over a Lipschitz singular chain defined by $f$. This integral is determined by $f|_{\partial D}$ since $\omega$ is closed. □

**Remark 3.5.** Since the 2-form $\omega$ in the above proof is the exterior derivative of a 1-form $\sum x_i dx_{i+1}$, one can explicitly rewrite $I(f)$ using Stokes’ formula:

$$I(f) = \frac{1}{2\pi} \int_{\partial D} \sum_{i=1}^n f_i \cdot df_{i+1}.$$  

Each term $\int_D df_i \wedge df_{i+1} = \int_{\partial D} f_i \cdot df_{i+1}$ is the oriented area (with multiplicities) encircled by the loop $F_i|_{\partial D}$ in the plane where $F_i$ is a map from $D$ to $\mathbb{R}^2$ defined by $F_i(x) = (f_i(x), f_{i+1}(x))$.

Fix a coordinate system $(x_1, x_2)$ in $D$. This induces coordinates in the co-tangent bundle $T^*D$, that is, every fiber $T^*_x D$ is identified with $\mathbb{R}^2$. For a measurable set $E \subset T^*_x D$ we denote by $|E|$ its Lebesgue measure w.r.t. these coordinates.

For a set $A \subset T^*_x D$, let $\text{conv}(A)$ denote its convex hull (that is, the least convex set containing $A$). If $A$ is finite, then $\text{conv}(A)$ is either a convex polygon in the plane $T^*_x D \approx \mathbb{R}^2$ or a line segment or a single point.
Lemma 3.6. For a cyclic map \( f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n \) one has
\[
I(f) = \frac{1}{\pi} \int_D |\text{conv}\{d_x f_1, \ldots, d_x f_n\}| \, dx.
\]
where \( dx = dx_1 dx_2 \) denotes the coordinate integration.

Proof. Let \( S_i \) be a Borel measurable function on \( D \) defined by the relation
\[
df_i \wedge df_{i+1} = 2S_i \, dx_1 \wedge dx_2.
\]
Then
\[
I(f) = \frac{1}{\pi} \int_D \sum_{i=1}^n S_i(x) \, dx,
\]
and it suffices to prove that
\[
(3.2) \quad \sum_{i=1}^n S_i(x) = |\text{conv}\{d_x f_1, \ldots, d_x f_n\}|
\]
for a.e. \( x \in D \). We will show that (3.2) holds for every \( x \in D \setminus \partial D \) where \( f \) is differentiable. Fix such a point \( x \) and denote \( w_i = d_x f_i \) for all \( i = 1, \ldots, n \). Observe that \( S_i(x) = \frac{1}{2} \frac{w_i \wedge w_{i+1}}{dx_1 \wedge dx_2} \)
equals the oriented area of the Euclidean triangle \( \Delta_i := \triangle_{0w_i w_{i+1}} \) in the plane \( T_x^* D \sim \mathbb{R}^2 \).

Hence the left-hand side of (3.2) equals the sum of oriented areas of these triangles.

Since \( f \) is cyclic, we have \( \phi_x^*(w_i) = 1 \) (cf. Definition 3.1(1)). Hence the points \( w_1, \ldots, w_n \)
belong to the convex curve \( U_x^* = \partial B_x^* \). The second requirement of Definition 3.1 implies that \( w_i, w_j \) and \( w_k \) are positively ordered whenever they are distinct and \( i < j < k \).

If all the points \( w_i \) coincide, then all terms in (3.2) are zero. Otherwise we may assume that \( w_n \neq w_1 \) and furthermore that \( w_{i+1} \neq w_i \) for all \( i \). Indeed, if \( w_{i+1} = w_i \), we can remove \( w_{i+1} \) from the list; this clearly does not change the right- and left-hand side of (3.2).

If the points \( w_1, \ldots, w_n \) are distinct, then the second requirement of Definition 3.1 implies that they are positively cyclically ordered. That is, they are vertices of a convex polygon inscribed in \( U_x^* \), enumerated according to their cyclic order. Observe that this polygon and the convex hull in the right-hand side of (3.2) are the same set. Now (3.2) follows from the fact that the area of this polygon equals the sum of oriented areas of the triangles \( \Delta_i = \triangle_{0w_i w_{i+1}} \).

It remains to consider the case when some of the points \( w_i \) coincide. Recall that \( w_{i+1} \neq w_i \)
for all \( i \). We may assume that \( w_1 = w_k \) for some \( k \), \( 2 < k < n \). Then for every \( i > k \), the point \( w_i \) coincides with either \( w_1 \) or \( w_2 \), otherwise the triples \( (w_1, w_2, w_i) \) and \( (w_2, w_k, w_i) = (w_2, w_1, w_i) \) have opposite cyclic orderings, contrary to the definition of a cyclic map. In particular, \( w_{k+1} = w_2 \) since \( w_{k+1} \neq w_k = w_1 \). Then a similar argument shows that for all \( i \) between 1 and \( k \) the point \( w_i \) coincides with either \( w_k = w_1 \) or \( w_{k+1} = w_2 \).

Thus every point \( w_i \) coincides with either \( w_1 \) or \( w_2 \). Since \( w_{i+1} \neq w_i \) for all \( i \), the sequence \( w_1, \ldots, w_n \) consists of alternating \( w_1 \) and \( w_2 \) (in particular, \( n \) is even). Then \( \text{conv}\{w_1, \ldots, w_n\} \)
is the line segment \([w_1, w_2]\) and hence the right-hand side of (3.2) is zero. The left-hand side is the sum of oriented areas of the triangles \( \Delta_i \) where \( \Delta_i = \triangle_{0w_1 w_2} \) if \( i \) is odd and \( \Delta_i = \triangle_{0w_2 w_1} \) if \( i \) is even. These two types of triangles have the same area but opposite orientations, hence the total sum of their oriented areas is zero. \( \square \)
Proposition 3.7. \( I(f) \leq \text{vol}^{ht}_2(D, \varphi) \) for every cyclic map \( f : D \rightarrow \mathbb{R}^n \).

Proof. Let \( x \in D \setminus \partial D \) be a point where \( f \) is differentiable. Then \( \varphi^*(d_x f_i) = 1 \) for all \( i = 1, \ldots, n \) (cf. Definition 3.1(1)), hence \( d_x f_i \in B^*_x \). Since \( B^*_x \) is convex, it follows that
\[
\text{conv}\{d_x f_1, \ldots, d_x f_n\} \subset B_x,
\]
hence
\[
|\text{conv}\{d_x f_1, \ldots, d_x f_n\}| \leq |B_x|.
\]
Integrating over \( D \) yields
\[
I(f) = \frac{1}{\pi} \int_D |\text{conv}\{d_x f_1, \ldots, d_x f_n\}| \, dx \leq \frac{1}{\pi} \int_D |B_x| \, dx = \text{vol}^{ht}_2(D, \varphi)
\]
Here the first equality follows from Lemma 3.6 and the second one from (2.2). \( \square \)

The next proposition is used in section 3 but not in section 4.

Proposition 3.8. If \( \varphi \) is a metric with minimal geodesics, then for every \( \varepsilon > 0 \) there exist a cyclically ordered set of points \( p_1, \ldots, p_n \in \partial D \) such that the map \( f = (f_1, \ldots, f_n) : D \rightarrow \mathbb{R}^n \) where \( f_i(x) = d_{p_i}(p_i, x) \) (cf. Example 3.2) satisfies
\[
I(f) > \text{vol}^{ht}_2(D, \varphi) - \varepsilon.
\]

Proof. Let \( Q = \{q_i\}_{i=1}^\infty \) be a countable dense subset of \( \partial D \). For each \( n \), define a map \( f = f^{(n)} \) using points \( p_1, \ldots, p_n \) obtained from the set \( \{q_1, \ldots, q_n\} \) by enumeration according to the cyclic order. We are going to prove that \( I(f^{(n)}) \rightarrow \text{vol}^{ht}_2(D, \varphi) \) as \( n \rightarrow \infty \); then for a sufficiently large \( n \) the map \( f = f^{(n)} \) satisfies (3.3).

Since \( f^{(n)} \) is cyclic (cf. Example 3.2), Lemma 3.6 implies that
\[
I(f^{(n)}) = \frac{1}{\pi} \int_D |\text{conv}\{d_x g_1, \ldots, d_x g_n\}| \, dx,
\]
where \( g_i(x) = d_{p_i}(q_i, x) \). As \( n \rightarrow \infty \), the convex hull in the right-hand side monotonically converges to the convex hull of the set \( \{d_{p_i}\}_{i=1}^\infty \). Hence, by Levy’s theorem,
\[
\lim_{n \rightarrow \infty} I(f^{(n)}) = \frac{1}{\pi} \int_D |\text{conv}(\{d_{x}g_i\}_{i=1}^\infty)| \, dx
\]
Taking into account (2.2), it suffices to prove that
\[
(3.4) \quad |\text{conv}(\{d_x g_i\}_{i=1}^\infty)| = |B^*_x|
\]
for a.e. \( x \in D \). We will show that (3.4) holds for every \( x \in D \setminus \partial D \) where all functions \( g_i \) are differentiable. Fix such a point \( x \). It suffices to prove that the set \( \{d_x g_i\}_{i=1}^\infty \) is dense in \( U^*_x \).

By Lemma 2.2, for every \( q \in \partial D \) the vector \( \bar{x}q \) is uniquely defined and hence depend continuously on \( q \). Define a map \( \xi : \partial D \rightarrow U_x \) by \( \xi(q) = \bar{x}q \). This map is surjective. Indeed, for a vector \( v \in U_x \) consider a geodesic \( \gamma \) with initial data \( \gamma(0) = x \) and \( \dot{\gamma}(0) = v \) extended backwards up to the boundary. By minimality of geodesics it indeed reaches the boundary at some point \( q = \gamma(-T) \), and then \( v = \dot{\gamma}(q) \). Since \( \xi \) is surjective and \( Q \) is dense in \( \partial D \), the set \( \xi(Q) = \{\bar{x}q_i\}_{i=1}^\infty \) is dense in \( U_x \). By Corollary 2.7 we have \( d_x g_i = \mathcal{L}(\bar{x}q_i) \), hence the set \( \{d_x g_i\}_{i=1}^\infty \) is dense in \( U^*_x \). Therefore its convex hull contains the interior of \( B^*_x \). This implies (3.4), and the proposition follows. \( \square \)
4. Proof of Pu’s inequality

The goal of this section is to give a direct proof of Theorem 3. As explained in the introduction, it suffices to prove the following: if \( \varphi \) is a reversible Finsler metric on \( D \) such that \( L_\varphi(\partial D) = 2\pi \) and for every \( x, y \in \partial D \) the distance \( d_\varphi(x, y) \) is realized by an arc of \( \partial D \), then \( \text{vol}_2(D, \varphi) \geq 2\pi \).

Let \( \varphi \) be such a metric. Fix a large positive integer \( n \) and let \( p_1, p_2, \ldots, p_n \in \partial D \) be a cyclically ordered collection of points dividing \( \partial D \) into \( n \) arcs of length \( 2\pi/n \). Define a cyclic map \( f : D \to \mathbb{R}^n \) as in Example 3, namely \( f = (f_1, \ldots, f_n) \) where \( f_i(x) = d_\varphi(p_i, x) \).

Our plan is to compute \( I(f) \) and then use Proposition 3.7 to estimate the area of the metric. Recall that \( I(f) \) can be recovered from the restriction of \( f \) to the boundary (cf. Lemma 3). This restriction does not depend on \( \varphi \) since \( d_{\varphi|_{\partial D \times \partial D}} \) is just the intrinsic distance of the boundary (which is isometric to the standard circle of length \( 2\pi \)).

To find \( I(f) \), fix an \( i \leq n \) and consider a map \( F_i = (f_i, f_{i+1}) : D \to \mathbb{R}^2 \). Then the term

\[
I_i(f) := \int_D df_i \wedge df_{i+1}
\]

equals the oriented area encircled by the planar curve \( F_i(\partial D) \), cf. Remark 3. Let \( s : [0, 2\pi] \to \partial D \) be a positively oriented arc-length parameterization of \( \partial D \) such that \( s(0) = s(2\pi) = p_i \). Computing the distances along the circle yields that

\[
F_i(s(t)) = \begin{cases}
(t, 2\pi \cdot t), & t \in [0, \frac{2\pi}{n}], \\
(t, t - \frac{2\pi}{n}), & t \in [\frac{2\pi}{n}, \pi], \\
(2\pi - t, t - \frac{2\pi}{n}), & t \in [\pi, \pi + \frac{2\pi}{n}], \\
(2\pi - t, 2\pi + \frac{2\pi}{n} - t), & t \in [\pi + \frac{2\pi}{n}, 2\pi].
\end{cases}
\]

This means that the curve \( F_i(\partial D) \) bounds the planar rectangle with vertices \((0, \frac{2\pi}{n}), (\frac{2\pi}{n}, 0), (\pi, \pi - \frac{2\pi}{n}), (\pi - \frac{2\pi}{n}, \pi)\) whose area equals \( 2 \cdot \frac{2\pi}{n} \cdot (\pi - \frac{2\pi}{n}) \). Thus

\[
I_i(f) = 2 \cdot \frac{2\pi}{n} \cdot (\pi - \frac{2\pi}{n})
\]

for all \( i \), hence

\[
I(f) = \frac{1}{2\pi} \sum_{i=1}^{n} I_i(f) = \frac{n}{2\pi} \cdot I_1(f) = 2\pi(1 - \frac{2}{n}).
\]

By Proposition 3.7

\[
\text{vol}_2(D, \varphi) \geq I(f) = 2\pi(1 - \frac{2}{n}).
\]

Since \( n \) is arbitrarily large, this inequality implies that \( \text{vol}_2(D, \varphi) \geq 2\pi \), and Theorem 3 follows.

5. Proof of Theorem 2

Let \( \varphi \) and \( \varphi_0 \) be as in the theorem, that is, \( \varphi_0 \) is a metric with minimal geodesics and \( d_\varphi(x, y) \geq d_{\varphi_0}(x, y) \) for all \( x, y \in \partial D \).

For every \( p \in \partial D \), define a function \( f_p : D \to \mathbb{R} \) by

\[
f_p(x) = \sup_{q \in \partial D} \{ d_{\varphi_0}(p, q) - d_{\varphi}(x, q) \}.
\]

(5.1)
Observe that $f_p$ is forward 1-Lipschitz (with respect to $\varphi$). Indeed, for every $q \in \partial D$ the function $x \mapsto -d_\varphi(x, q)$ is forward 1-Lipschitz (cf. Example 2.5), hence so is the function $x \mapsto d_\varphi(p, q) - d_\varphi(x, q)$, and the supremum of a family of forward 1-Lipschitz functions is forward 1-Lipschitz as well.

**Lemma 5.1.** If $x \in \partial D$, then $f_p(x) = d_\varphi(p, x)$.

**Proof.** Let $x \in \partial D$. Then $d_\varphi(x, q) \geq d_\varphi(p, q)$ for every $q \in \partial D$ by the assumption of the theorem. Hence

$$d_\varphi(p, q) - d_\varphi(x, q) \leq d_\varphi(p, q) - d_\varphi(p, x) \leq d_\varphi(p, x)$$

by the triangle inequality. Taking the supremum over $q \in \partial D$ yields that

$$f_p(x) \leq d_\varphi(p, x).$$

The inverse inequality follows by substituting $q = x$ in (5.1):

$$f_p(x) = \sup_{q \in \partial D} \{d_\varphi(p, q) - d_\varphi(x, q)\} \geq d_\varphi(p, x) - d_\varphi(x, x) = d_\varphi(p, x).$$

Thus $f_p(x) = d_\varphi(p, x)$.

**Proposition 5.2.** Let $p_1, \ldots, p_n \in \partial D$ be a cyclically ordered collection of points. Then a map $f : D \to \mathbb{R}$ defined by $f = (f_{p_1}, \ldots, f_{p_n})$ is cyclic (with respect to $\varphi$).

**Proof.** Fix a point $x \in D \setminus \partial D$. We say that a point $q_0 \in \partial D$ is a point of maximum for $p \in \partial D$ if the supremum in (5.1) is attained at $q = q_0$, that is,

$$f_p(x) = d_\varphi(p, q_0) - d_\varphi(x, q_0).$$

By compactness, a point of maximum exists for every $p \in \partial D$.

**Lemma 5.3.** If $f_p$ is differentiable at $x$ and $q_0$ is a point of maximum for $p$, then the direction $\overrightarrow{xq_0}$ is uniquely defined and $d_x f_p = \mathcal{L}(\overrightarrow{xq_0}) \in U_x^*$. 

**Proof.** By Lemma 5.1 and (5.2) we have $f_p(q_0) = d_\varphi(p, q_0) = f_p(x) + d_\varphi(x, q_0)$. Since $f_p$ is forward 1-Lipschitz, this identity and Lemma 2.6(1) imply the desired assertion.

For every $i = 1, \ldots, n$, let $q_i \in \partial D$ be a point of maximum for $p_i$. Then by Lemma 5.3 we have $d_x f_{p_i} \in U_{x}^*$, hence the first requirement of Definition 3.1 is satisfied.

Since the second requirement deals with triples of coordinate functions, it suffices to verify it for $n = 3$. Suppose that the derivatives $d_x f_i$, $i = 1, 2, 3$, are well-defined and distinct. Then by Lemma 5.3 the directions $\overrightarrow{xq_i}$ are uniquely defined and $d_x f_{p_i} = \mathcal{L}(\overrightarrow{xq_i})$. Hence the vectors $\overrightarrow{xq_i}$ are distinct and their cyclic ordering in $U_x$ is the same as that of the co-vectors $d_x f_i$ in $U_x^*$. By Lemma 2.3 the former is the same as the cyclic ordering of points $q_i$ in $\partial D$ (note that the points $q_i$ are distinct since the directions $\overrightarrow{xq_i}$ are distinct). Thus it suffices to prove that the cyclic ordering of the points $q_1, q_2, q_3$ in $\partial D$ is the same as that of points $p_1, p_2, p_3$.

**Lemma 5.4.** 1. $q_i \neq p_i$ for every $i \in \{1, 2, 3\}$.

2. If $i, j \in \{1, 2, 3\}$ and $i \neq j$ then the set $\{p_i, q_j\}$ does not separate points $p_j$ and $q_i$ in $\partial D$, that is, $p_j$ and $q_i$ belong to the same connected component of $\partial D \setminus \{p_i, q_j\}$.
Proof. Suppose the contrary. Without loss of generality we may assume that \( p_1 = q_1 \) or \( \{p_1, q_2\} \) separates \( \{p_2, q_1\} \) in \( \partial D \). In either case, any curve connecting \( p_1 \) to \( q_2 \) intersects (possibly at an endpoint) any curve connecting \( p_2 \) to \( q_1 \). Let \( z \) be a common point of a \( \varphi_0 \)-shortest path from \( p_1 \) to \( q_2 \) and a \( \varphi_0 \)-shortest path from \( p_2 \) to \( q_1 \). Then
\[
d_{\varphi_0}(p_1, q_2) + d_{\varphi_0}(p_2, q_1) = d_{\varphi_0}(p_1, z) + d_{\varphi_0}(z, q_2) + d_{\varphi_0}(p_2, z) + d_{\varphi_0}(z, q_1) \\
\geq d_{\varphi_0}(p_1, q_1) + d_{\varphi_0}(p_2, q_2)
\]
by the triangle inequality. Therefore
\[
f_{p_1}(x) + f_{p_2}(x) \geq (d_{\varphi_0}(p_1, q_2) - d_\varphi(x, q_2)) + (d_{\varphi_0}(p_2, q_1) - d_\varphi(x, q_1)) \\
\geq d_{\varphi_0}(p_1, q_1) - d_\varphi(x, q_1) + d_{\varphi_0}(p_2, q_2) - d_\varphi(x, q_2) \\
= f_{p_1}(x) + f_{p_2}(x).
\]
Here the first inequality follows from the definition \((5.1)\) of \( f_{p_1} \) and \( f_{p_2} \) and the last identity from the fact that \( q_1 \) and \( q_2 \) are points of maximum for \( p_1 \) and \( p_2 \), resp. Since the left- and right-hand side are the same, the intermediate inequalities turn to equalities. In particular,
\[
f_{p_1}(x) = d_{\varphi_0}(p_1, q_2) - d_\varphi(x, q_2). \quad \text{This means that} \quad q_2 \quad \text{is a point of maximum for} \quad p_1, \quad \text{hence} \quad d_xf_{p_1} = \mathcal{L}(\overline{xq_2}) \neq \mathcal{L}(\overline{xq_1}), \quad \text{a contradiction.} \quad \square
\]

Now the desired coincidence of cyclic orderings follows from the following combinatorial lemma.

**Lemma 5.5.** Let \( \{p_i\}_{i=1}^3 \) and \( \{q_i\}_{i=1}^3 \) be two triples of distinct points in \( \partial D \) such that the assertion of Lemma 5.4 holds. Then the cyclic ordering of these two triples is the same.

*Proof.* This fact was proved in [10] by examination of possible configurations of six points on a circle. Here we give a more algebraic-style proof.

For distinct points \( a, b, c \in \partial D \), we define \([abc] = 1\) if the cyclic ordering of the triple \((a, b, c)\) is positive and \([abc] = -1\) otherwise. This notation satisfies a trivial identity
\[
[abc]^2 = 1,
\]
is skew-symmetric:
\[
[abc] = [bca] = [cab] = -[bac] = -[acb] = -[cba]
\]
and satisfies the standard cyclic order identity \([abc][bcd][cda][dab] = 1\) for any four distinct point \( a, b, c, d \in \partial D \). Taking into account \((5.4)\), this identity can be rewritten as
\[
[abc] = [abd][acd][bcd],
\]
and
\[
[abc][acd][abd] = [bcd].
\]
The fact that \( \{p_i, q_j\} \) does not separate \( p_j \) from \( q_i \) means that
\[
[p_ip_jq_j] = [p_iq_iq_j]
\]
promised that the four points are distinct.

Observe that a small perturbation of the configuration does not break the assumptions of the lemma and does not change the cyclic ordering of the triples \( \{p_i\} \) and \( \{q_i\} \). By means
of such a perturbation we can change the configuration so that all six points \( p_1, p_2, p_3, q_1, q_2, q_3 \) are distinct. Then

\[
[p_1 p_2 p_3] = [p_1 p_2 p_3; p_1 p_3 p_3; p_2 p_3 q_3]
\]

by (5.5)

\[
= [p_1 p_2 q_3; p_1 q_3 p_3; p_2 q_3 q_3]
\]

by (5.7)

\[
= [p_1 q_2 q_3; p_1 q_3 q_3; p_2 q_3 q_3]
\]

by (5.5)

\[
= [p_1 q_2 q_3; p_1 q_3 q_3; p_2 q_2 q_3]
\]

by (5.4)

\[
= [p_1 q_2 q_3; p_1 q_3 q_3; p_1 q_2 q_3] = [p_1 q_2 q_3; p_1 q_3 q_3; p_1 q_2 q_3]^2
\]

by (5.3) and (5.7)

\[
= [q_1 q_2 q_3]
\]

by (5.6),

and the lemma follows. \( \square \)

Lemma 5.4 and Lemma 5.5 imply that the second requirement of Definition 3.1 is satisfied for the map \((f_{p_1}, f_{p_2}, f_{p_3})\) from \(D\) to \(\mathbb{R}^3\). Applying this to all triples \((p_i, p_j, p_k)\) where \(1 \leq i < j < k \leq n\) yields that this requirement is satisfied for \(f = (f_{p_1}, \ldots, f_{p_n})\), and Proposition 5.2 follows. \( \square \)

**Proof of the inequality.** Fix an \(\varepsilon > 0\). By Proposition 3.8 applied to \(\varphi_0\), there exists a positively cyclically ordered set of points \(p_1, \ldots, p_n \in \partial D\) such that the map

\[
f^0 = (f_1^0, \ldots, f_n^0) : D \to \mathbb{R}^n
\]

where

\[
f_i^0(x) = d_{\varphi_0}(p_i, x) \quad x \in D, i = 1, \ldots, n,
\]

satisfies

\[
I(f^0) > \text{vol}^2_h(D, \varphi_0) - \varepsilon.
\]

Define \(f = (f_{p_1}, \ldots, f_{p_n}) : D \to \mathbb{R}^n\) where the functions \(f_{p_i}\) are defined by (5.1). Then Lemma 5.1 implies that \(f|_{\partial D} = f^0|_{\partial D}\), hence \(I(f) = I(f^0)\) by Lemma 3.4. On the other hand, Proposition 5.2 implies that \(f\) is cyclic (with respect to \(\varphi\)) and hence \(\text{vol}^2_h(D, \varphi) \geq I(f)\) by Proposition 3.7. Thus

\[
\text{vol}^2_h(D, \varphi) \geq I(f) = I(f^0) > \text{vol}^2_h(D, \varphi_0) - \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, it follows that \(\text{vol}^2_h(D, \varphi) \geq \text{vol}^2_h(D, \varphi_0)\). Thus we have proved the inequality part of Theorem 2.

**The equality case.** Suppose that \(\text{vol}^2_h(D, \varphi) = \text{vol}^2_h(D, \varphi_0)\). First we prove that \(\varphi\) is a metric with minimal geodesics. Arguing by contradiction, suppose that a geodesic \(\gamma : [0, T] \to D \setminus \partial D\) of \(\varphi\) is not minimal, that is, \(L_\varphi(\gamma) > d_\varphi(\gamma(0), \gamma(T))\). Then a similar inequality holds for every geodesic of length \(T\) with initial velocity sufficiently close to \(v := \dot{\gamma}(0)\). Such geodesics cannot be parts of shortest paths, hence there is a neighborhood of \(v\) in \(TD\) avoided by velocity vectors of shortest paths. A small perturbation of the metric in this neighborhood does not change the boundary distances, and one can choose a perturbation so that the resulting metric \(\tilde{\varphi}\) is such that \(\tilde{\varphi} \leq \varphi\) everywhere and \(\tilde{\varphi}(v) < \varphi(v)\). Then \(\text{vol}^2_h(D, \tilde{\varphi}) < \text{vol}^2_h(D, \varphi) = \text{vol}^2_h(D, \varphi_0)\), contrary to the inequality part of the theorem (applied to \(\tilde{\varphi}\) in place of \(\varphi\)). Thus \(\varphi\) is a metric with minimal geodesics.
Now suppose that the boundary distance functions of $\varphi_0$ and $\varphi$ differ. It is easy to see that then there are points $p, q \in \partial D$ such that $d_{\varphi}(p, q) > d_{\varphi_0}(p, q)$ and a shortest path from $p$ to $q$ is a geodesic which hits the boundary transversally. Then one gets a contradiction similarly to the above argument, namely slightly shrinking the metric in a neighborhood of a velocity vector of this geodesic.

Thus if $\text{vol}^b_2(D, \varphi) = \text{vol}^b_2(D, \varphi_0)$, then $\varphi$ is a metric with minimal geodesics and has the same boundary distance function as $\varphi_0$. Conversely, these two properties imply that the areas are equal. To see this, just interchange $\varphi$ and $\varphi_0$ in the inequality part of the theorem.

This finishes the proof of Theorem 2.

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