Dimension of Restricted Classes of Interval Orders

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Received: 3 January 2022 / Revised: 7 July 2022 / Accepted: 18 July 2022
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Abstract
Rabinovitch showed in 1978 that the interval orders having a representation consisting of only closed unit intervals have order dimension at most 3. This article shows that the same dimension bound applies to two other classes of posets: those having a representation consisting of unit intervals (but with a mixture of open and closed intervals allowed) and those having a representation consisting of closed intervals with lengths in \([0, 1]\).

Keywords
Interval order · Dimension · Semiorder · Unit OC interval order

Mathematics Subject Classification
06A07

1 Introduction and Background
Recent years have seen a number of efforts to characterize the classes of interval orders between the unit interval orders (or semiorders) in which each interval has the same length and the full class of interval orders in which intervals of any nonnegative length are allowed. See, for instance, \([3, 4, 11, 12, 20, 22, 23]\). To date, this work has focused on forbidden subposet characterizations and recognition algorithms. Similar...
questions involving interval graphs have also been studied. See, for example, [5, 7, 15–17]. This paper begins to address how these classes interact with the order dimension of these posets. For example, Rabinovitch showed in 1978 in [19] that the dimension of a unit interval order is at most three. On the other hand, a series of papers have shown that for every positive integer \( d \), there exists an interval order of dimension at least \( d \) and developed a thorough understanding of the rate of growth of dimension in terms of typical poset parameters. (See Sect. 3 for references.) However, the growth rate of the maximum dimension of an interval order having a representation using at most \( r \) distinct interval lengths has not been carefully studied. For the classes of interval orders studied in this paper, we show that the dimension is at most 3, which raises the compelling question of finding the “smallest” (in any of several reasonable senses) interval order of dimension 4.

1.1 Special Classes of Interval Orders

Following Trotter’s survey article [25], we consider posets to be reflexive. An interval order is a poset \( P \) for which each element \( x \) of the ground set of \( P \) can be assigned an interval \([\ell(x), r(x)]\) so that for distinct \( x \) and \( y \), \( x < y \) in \( P \) if and only if \( r(x) < \ell(y) \). That is, \( x < y \) in \( P \) if and only if the interval assigned to \( x \) lies completely to the left of the interval assigned to \( y \). It follows that two elements are incomparable in \( P \) if and only if their intervals intersect. Such an assignment of a collection of intervals is called an interval representation or just representation of \( P \). Interval orders were shown by Fishburn in [6] to be characterized by being the class of posets that exclude \( 2 + 2 \), the disjoint union of two chains on two elements. General background on interval orders can be found in the monograph of Golumbic and Trenk [9] or Trotter’s survey article [25].

A semiorder or unit interval order is an interval order having a representation in which all intervals have the same (typically unit) length. Scott and Suppes showed in [21] that unit interval orders can be characterized as the posets that exclude \( 2 + 2 \) and \( 1 + 3 \), where \( 1 + 3 \) is the disjoint union of an isolated point and a chain on three elements. While interval representations are typically assumed to consist of closed, bounded intervals, for finite interval orders, it is equivalent to restrict to open, bounded intervals. Working from the perspective of graph theory, Rautenbach and Szwarcfiter showed in [20] that allowing both open and closed intervals (all bounded) in a representation does not expand the class of posets beyond the interval orders. In addition, they give a forbidden graph characterization of the class of interval graphs having a representation in which all intervals have unit length with both open and closed intervals allowed. Indeed, the following result follows immediately from the construction provided in part (i) of the proof of [20, Proposition 1].

Lemma 1 If \( P \) is a poset that has an interval representation consisting of any combination of open, closed, or half-open intervals, then \( P \) can be represented by a collection of closed intervals.

In [22], Shuchat et al. considered the analogous class of interval orders having a representation in which all intervals have unit length with both open and closed
intervals allowed. They called these posets unit OC interval orders and gave both a forbidden subposet characterization and a polynomial time recognition algorithm for the class. It is straightforward to see that $1 + 3$ is a unit OC interval order, so the class of unit OC interval orders is strictly larger than the class of unit interval orders. In [4], Boyadzhiyska et al. studied the interval orders having a representation in which each (closed) interval has length 0 or 1. The authors use digraph methods to provide a forbidden subposet characterization of this class of posets. These two classes of posets are our objects of study in this paper. Because we are able to prove our results using only the representations of these posets, we do not reproduce the forbidden subposet characterizations from [4, 22] here.

In [1], Bogart et al. first studied the dimension of interval orders and showed that there are interval orders of arbitrarily large dimension. They did so by using the canonical interval order on $n$ endpoints, which consists of all intervals having endpoints in the set $\{1, 2, \ldots, n\}$. Subsequently, Rabinovitch showed in [19] that if $P$ is a unit interval order, then its dimension is at most 3. The independent characterization of the 3-irreducible posets (with respect to dimension) by Kelly in [13] and Trotter and Moore in [27] subsequently made it easy to show that the dimension of a unit interval order that is not a total order is 3 if and only if it contains one of three posets on seven points shown in Fig. 1.

In this paper, we show that if $P$ is a unit OC interval order or has a representation consisting of intervals of lengths 0 and 1, then the dimension of $P$ is at most 3. While Rabinovitch’s argument for unit interval orders appears to rely on the lack of $1 + 3$, we are able to give our proof by using the representation for unit OC interval orders. The proof for interval orders that are $\{0, 1\}$-representable is more intricate and relies on other techniques in dimension theory. The next subsection provides an overview of the background in dimension theory that is required to read the remainder of the paper. We conclude with some open questions regarding the dimension of interval orders.

1.2 Definitions

The down set of $x$ in a poset $P$ is $D(x) = \{z \in P : z < x \text{ in } P\}$, and we write $D[x] = D(x) \cup \{x\}$. The up sets $U(x)$ and $U[x]$ are defined dually. We say that a poset

![FX2](image1) ![H0](image2) ![G0](image3)

Fig. 1 The three subposets that can force a unit interval order to have dimension 3
If \( P \) is a poset and \( A, B \subseteq P \), we write \( A < B \) to mean that for all \( a \in A \) and all \( b \in B \), \( a < b \). (And similarly for \( A > B \).) If \( P \) is an interval order, this means that in any interval representation of \( P \), every interval assigned to a point of \( A \) lies completely to the left of every interval assigned to a point of \( B \).

Given a poset \( P \), let \( \text{inc}(P) = \{(x,y) \in P \times P : x \text{ is incomparable to } y \text{ in } P\} \). We call an ordered pair \( (x,y) \in \text{inc}(P) \) an incomparable pair. A linear extension \( L \) of \( P \) is a total order on the ground set of \( P \) such that if \( x < y \) in \( P \), then \( x < y \) in \( L \). A realizer of \( P \) is a set \( \mathcal{R} \) of linear extensions of \( P \) such that \( x < y \) in \( P \) if and only if \( x < y \) in \( L \) for all \( L \in \mathcal{R} \). Equivalently, \( \mathcal{R} \) is a realizer for \( P \) if and only if for each incomparable pair \( (x,y) \) of \( P \), there exist \( L, L' \in \mathcal{R} \) such that \( x < y \) in \( L \) and \( y < x \) in \( L' \). In this case, we say that \( L \) reverses the incomparable pair \( (y,x) \) and \( L' \) reverses the incomparable pair \( (x,y) \). Since \( (x,y) \) being an incomparable pair means that \( (y,x) \) is also an incomparable pair, it is sufficient to say that \( \mathcal{R} \) is a realizer provided that each incomparable pair is reversed by a linear extension in \( \mathcal{R} \).

The dimension of a poset \( P \) is the least positive integer \( t \) such that there exists a realizer of \( P \) having cardinality \( t \). Note that for posets of dimension at least 2, it is always possible to assume that \( P \) has no duplicated holdings without changing the dimension, since elements with the same up set and same down set can be placed consecutively in one linear extension of a realizer and consecutively in the reverse order in another to ensure that all incomparable pairs are reversed.

A strict alternating cycle of length \( k \) in a poset \( P \) is a sequence \( \{(x_i, y_i) : 1 \leq i \leq k\} \) such that \( x_i \leq y_{i+1} \) cyclically for \( i = 1, 2, \ldots, k \) and \( x_i \) and \( y_j \) are incomparable when \( j \neq i + 1 \) (cyclically).

In [28], Trotter and Moore showed that there is a linear extension reversing all incomparable pairs in a set \( S \subseteq P \times P \) if and only if \( S \) does not contain a strict alternating cycle. Figure 2 shows a strict alternating cycle of length 5. The only comparabilities amongst the 10 points are those illustrated, but note that since our posets are reflexive, it is possible for the comparabilities to be equality. For instance, \( x_3 = y_4 \) is a possibility.

When \( P \) is an interval order, a strict alternating cycle takes a particularly restrictive form, since in an interval order, a strict alternating cycle \( S \) of length at least 2 may only contain one strict comparability or else \( S \) witnesses that \( P \) contains \( 2 + 2 \). We note for its usefulness later that a strict alternating cycle of length 2 in an interval order that contains a strict comparability must therefore be \( \{(x_1, y_1), (x_2, x_1)\} \) with \( x_2 < y_1 \) and \( x_1 \) incomparable to \( y_1 \) and \( x_2 \).

We now establish a useful consequence of the preceding observation. This originally appears in Rabinovitch’s thesis [18], but can also be found in Trotter’s monograph [24, p. 196].

**Lemma 2** If \( P \) is an interval order and \( A \) and \( B \) are disjoint subsets of \( P \), then there exists a linear extension \( L \) of \( P \) with \( a > b \) in \( L \) for all \( a \in A \) and \( b \in B \) for which \( a \) and \( b \) are incomparable in \( P \).
Proof Let $S = (A \times B) \cap \text{inc}(Q)$. Because $A$ and $B$ are disjoint and the only strict alternating cycles $C$ in the interval order $P$ have an element that appears as both the first coordinate and a second coordinate of distinct pairs in $C$, we know that $S$ contains no strict alternating cycles. Thus, there is a linear extension $L$ of $P$ reversing all the incomparable pairs in $S$. Such a linear extension has the desired property.

2 Dimension of Unit OC Interval Orders and Length $\{0,1\}$-Representable Interval Orders

Our first main result concerns the dimension of unit OC interval orders. Since unit OC interval orders are not necessarily unit interval orders, in the proof we build a related unit interval order $Q$. We do this in order to take advantage of the ideas behind Rabinovitch’s proof that unit interval orders have dimension at most 3.

Theorem 1 If $P$ is a unit OC interval order, then $\dim(P) \leq 3$.

Proof Let $P$ be a unit OC interval order and fix a representation $\mathcal{I}$ of $P$ in which every interval has unit length. Let $Q$ be the unit interval order formed from $P$ by making all of the intervals in $\mathcal{I}$ open. (This may result in more than one point of $Q$ being represented by the same interval. However, this is not a problem, as it merely introduces points with duplicated holdings.) Thus, in $Q$ there may be pairs of points $x,y$ such that $x$ and $y$ are comparable in $Q$ and incomparable in $P$. However, all comparabilities in $P$ remain comparabilities in $Q$. This means that any linear extension of $Q$ is also a linear extension of $P$. We now partition $Q$ into antichains $A_1,A_2,\ldots,A_t$ by successively removing the minimal elements of $Q$. Since incomparabilities in $Q$ are also incomparabilities in $P$, each $A_i$ is also an antichain in $P$.

We claim that the only incomparabilities in $P$ are within an $A_i$ or between consecutive antichains. First we show that for any $a \in A_i$ and $b \in A_{i+2}$, $a < b$ in $P$. To prove this, assume for a contradiction that $a$ and $b$ are incomparable in $P$. Hence, $\ell(b) \leq r(a)$. Since $b \in A_{i+2}$, there exists $x \in A_{i+1}$ with $x < b$ in $Q$. Similarly, there exists $y \in A_i$ with $y < x$ in $Q$. Since $y < x < b$ in $Q$ and all intervals are open,
If any of the inequalities above is strict, then \( r(y) < \ell(a) \) and \( y < a \) in \( Q \), which contradicts that \( a \) and \( y \) are both in the antichain \( A_i \). Thus, equality must hold throughout and \( \ell(x) = \ell(a) \), meaning that \( x \) and \( a \) have the same interval in our representation of \( Q \). However, this would mean that \( x \) and \( a \) should be in the same antichain of our partition instead of having \( x \in A_{i+1} \) and \( a \in A_i \). Now for \( j > i + 2 \), \( c \in A_j \) requires that \( c \) is greater than some element of \( A_{i+2} \), and thus \( c \) is greater than all elements of \( A_i \). Therefore, the only incomparabilities in \( P \) are within an \( A_i \) or between consecutive antichains.

We now form a witness \( R = \{L_1, L_2, L_3\} \) to demonstrate \( \text{dim}(P) \leq 3 \). The total orders \( L_1 \) and \( L_2 \) will be linear extensions of \( P \), while \( L_3 \) will be a linear extension of both \( P \) and \( Q \). Let \( A = \bigcup_i A_{2i} \) and \( B = \bigcup_i A_{2i+1} \). As \( P \) is an interval order by Lemma 1 and \( A \) and \( B \) are disjoint, we may apply Lemma 2. In particular, we define \( L_1 \) as the linear extension of \( P \) formed by applying Lemma 2 to place \( a > b \) for all \( a \in A \) and \( b \in B \) that are incomparable in \( P \). We similarly define \( L_2 \) to place \( b > a \) for all \( b \in B \) and \( a \in A \) that are incomparable in \( P \). Thus, we have that if \( x \) and \( y \) are incomparable in \( P \) with \( x \in A \) and \( y \in B \), then \( x > y \) in \( L_1 \) and \( y > x \) in \( L_2 \).

For \( L_3 \), we first order the elements by the antichains to which they belong \( A_1, A_2, \ldots, A_i \), and when ordering the elements of \( A_i \), we place them in the dual order to their order in \( L_1 \). Now \( R \) is a realizer of \( P \) because we know all incomparabilities are either within an antichain \( A_i \), in which case the incomparable pairs appear in opposite orders in \( L_1 \) and \( L_3 \), or between consecutive antichains, in which case the incomparable pairs appear in opposite orders in \( L_1 \) and \( L_2 \). \( \square \)

Notice that the argument that proves Theorem 1 works equally well if we allow posets \( P \) having representations including half-open intervals \( [a, a + 1] \) and \( [b, b + 1] \), since the construction of \( Q \) still preserves all comparabilities of \( P \) and Lemma 1 ensures that \( P \) is an interval order so that Lemma 2 can be applied. When unit half-open intervals are allowed, the resulting class of posets is larger than the unit OC interval orders. For example, the poset in Fig. 3 is not a unit OC interval order. There is a \( 1 \) \( + \) \( 3 \) on \( y \) and \( \{x_1, x_2, x_3\} \). This forces \( y \) and \( x_2 \) to be represented by intervals with the same endpoints, \( y \) to be represented by a closed interval, \( x_2 \) to be represented by an open interval, and \( x_3 \) to be represented by an interval that is closed on the left. Similarly, there is a \( 1 \) \( + \) \( 3 \) on \( y \) and \( \{x_1, x_2, z\} \), so \( z \) must be represented by an interval having the same endpoints as the interval of \( x_3 \). Furthermore, the interval of \( z \) must be closed on the left. We now try to add an interval representing \( x_4 \). It must be greater than \( x_3 \) but incomparable with \( z \). Since the intervals representing \( x_3 \) and \( z \) have the same endpoints, the only way to make this happen is for \( x_3 \) to be represented by an interval that is open on the right and \( z \) to be represented by an interval that is closed on the right. Thus \( x_3 \) must be represented by a half-open interval. This leads to the unit mixed interval representation shown at the right in Fig. 3. Joos [12] and Shuchat et al. [23] study this larger class from the perspective of graph theory and independently give a forbidden graph characterization of these unit mixed interval graphs.
Theorem 2  If $P$ is an interval order having a representation with each interval of length 0 or 1, then $\dim(P) \leq 3$.

Proof  Without loss of generality, we may assume that $P$ has no duplicated holdings, as duplicated holdings do not change dimension when posets have dimension 2 or more. Thus every representation of $P$ has all distinct intervals. Let $\mathcal{I} = \{[\ell(x), r(x)) : x \in P\}$ be a representation of $P$ in which each interval has length 0 or 1. Let $Q$ be the subposet of $P$ consisting of the points represented by unit intervals in $\mathcal{I}$. Partition $Q$ into antichains $A_1, A_2, \ldots, A_t$ by successively removing the minimal elements. For $i$ with $1 \leq i \leq t$, let $p_i = \min\{r(x) : x \in A_i\}$. That is, $p_i$ is the smallest right endpoint of an interval in $A_i$. We now partition the length 0 intervals into $t + 1$ sets based on their location relative to the $p_i$. For $i$ with $1 \leq i \leq t - 1$, let $D_i$ be the length 0 intervals $[c, c]$ with $c \in (p_i, p_{i+1})$. Let $D_0$ be the length 0 intervals $[c, c]$ with $c \leq p_1$, and let $D_t$ be the length 0 intervals $[c, c]$ with $c > p_t$.

Note that a length 1 interval belongs to the antichain $A_i$ if and only if it contains $p_i$. Since each interval in $A_{i+1}$ is greater than an interval in $A_i$, all intervals in $A_{i+1}$ lie to the right of $p_i$. Therefore, $A_{i+2} > D_i$ for $i = 0, \ldots, t - 2$. Furthermore, $D_i > A_{i-1}$ for $i = 2, \ldots, t$ since all intervals in $A_{i-1}$ must end before $p_i$. If $x \in A_i$ and $y \in A_{i+2}$, then there is $z \in A_{i+1}$ satisfying $z < y$. Hence,

$$\ell(y) > r(z) \geq p_{i+1} > r(x).$$

This guarantees that $A_j > A_i$ if $j \geq i + 2$.

Hence, the incomparable pairs we must reverse involve pairs of points

- in $A_i$ and $A_{i+1}$,
- in $D_i$ and $A_i$,
- in $D_i$ and $A_{i+1}$, and
- both in $A_i$.

We will now define two disjoint sets of incomparable pairs and show that neither contains a strict alternating cycle. Let

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**Fig. 3**  A poset that is not a unit OC interval order and a unit mixed interval representation of it.
\[ S_1 = \{(x, y) \in \text{inc}(P) : x \in A_{2i-1}, y \in A_{2i} \text{ or } x \in D_{2i-1}, y \in A_{2i} \text{ or } x \in A_{2i-1}, y \in D_{2i-1}\} \]

\[ S_2 = \{(x, y) \in \text{inc}(P) : x \in A_{2i}, y \in A_{2i+1} \text{ or } x \in D_{2i}, y \in A_{2i+1} \text{ or } x \in A_{2i}, y \in D_{2i}\} \]

Suppose that \( S_j \) contains a strict alternating cycle \( C \). By our earlier argument about strict alternating cycles in interval orders, \( C = \{(d, y), (x, d)\} \) is the entire strict alternating cycle with \( d \) represented by an interval of length 0. (If \( d \) had length 1, then \( d \) would be in both \( A_i \) and \( A_{i+1} \) for some \( i \), which is impossible.) By our definition of \( S_1 \) and \( S_2 \), this forces \( x \in A_k \) and \( y \in A_{k+1} \) for some \( k \). However, the definition of a strict alternating cycle implies that \( d \) is incomparable with both \( x \) and \( y \) and that \( x < y \). This is a contradiction since a length 0 interval cannot intersect two disjoint intervals.

Since \( S_j \) does not contain a strict alternating cycle, there is a linear extension \( L_j \) of \( P \) that reverses all the incomparable pairs of \( S_j \) for \( j = 1, 2 \). Form a third linear extension of \( P \) with \( D_0 < A_1 < D_1 < A_2 < \cdots < A_i < D_i \). Each \( D_i \) is a chain, so the ordering of those points is fixed. For each \( A_i \), order the points in the dual order to how they appear in \( L_1 \).

We now verify that \( \{L_1, L_2, L_3\} \) realizes \( P \) and therefore \( \dim(P) \leq 3 \). In particular, a pair \((a_i, a_{i+1}) \in \text{inc}(P) \cap (A_i \times A_{i+1})\) is reversed in \( L_1 \) if \( i \) is odd and in \( L_2 \) if \( i \) is even. The pair \((a_{i+1}, a_i)\) is reversed in \( L_3 \). Similarly, the pairs \((a_i, d_i) \in \text{inc}(P) \cap (A_i \times D_i)\) and \((d_{i+1}, a_{i+1}) \in \text{inc}(P) \cap (D_i \times A_{i+1})\) are reversed in \( L_1 \) if \( i \) is odd and in \( L_2 \) if \( i \) is even, while \((d_i, a_i)\) and \((a_{i+1}, d_i)\) are reversed in \( L_3 \). Finally all pairs \((a_i, a'_i) \in \text{inc}(P) \cap A_i^2\) are either reversed in \( L_1 \) or in \( L_3 \), where the elements of \( A_i \) appear in the dual order as in \( L_1 \).

Notice that the proof of Theorem 2 works equally well if all intervals have length 0 and \( r \) for some fixed positive real number \( r \), since the subposet consisting of the length \( r \) intervals is a unit interval order. While it seems possible to relax the requirement of the unit intervals being closed in Theorem 2, the modifications to the proof appear to be more intricate than those required for Theorem 1.

### 3 Conclusion

We conclude briefly with some related open questions inspired by this work.

1. Here we address the case of interval orders that can be represented using intervals of only length 0 and some positive length. What is the bound on the dimension of interval orders having a representation consisting only of intervals of lengths \( r \) and \( s \) with \( r, s > 0 \)?

2. More generally, what is the growth rate of the function \( f \) such that if \( P \) is an interval order having a representation using at most \( r \) different interval lengths, then \( \dim(P) \leq f(r) \)?

It is easy to see that the answer to the first question is at most 8 by partitioning the poset into two semiorders, \( R \) and \( S \), consisting of the length \( r \) and length \( s \) intervals.
The incomparabilities in \(R\) and \(S\) can each be reversed using 3 linear extensions, and two additional linear extensions suffice to reverse incomparabilities between \(R\) and \(S\). This idea can be generalized [26] to show that \(f(r) \leq 3r + \binom{r}{2}\). However, based on the work of Füredi et al. in [8] showing that the dimension of the canonical interval order with \(n\) different interval lengths is asymptotically

\[
\text{lg}(\text{lg}(n)) + \left(\frac{1}{2} + o(1)\right) \text{lg}(\text{lg}(\text{lg}(n))),
\]

we expect that this bound on \(f(r)\) is extremely loose and that \(f(r)\) in fact grows extremely slowly, i.e., \(O(\text{lg} \text{lg}(r))\). It is worth noting that using improved estimates for the dimension of the shift graph [10, 14] the error term on the estimate of the dimension of the canonical interval order is actually at most 5 [26].

In [2], Bosek et al. use marking functions to provide an alternative proof of Rabinovitch’s theorem that the dimension of a unit interval order is at most 3. These methods may provide alternative (but likely no shorter) proofs of the theorems in this paper, but care would have to be taken because the marking function arguments require representations with distinct endpoints, which cannot be assured when allowing both open and closed unit intervals. (For instance, \(1 + 3\) can only be represented as such an order by allowing repetition of endpoints.)

It would also be interesting to find the smallest \(n\) such that there exists an interval order on \(n\) points having dimension 4 and to determine the minimum number of interval lengths required to force the dimension to 4. To the best of the authors’ knowledge, all arguments showing the existence of a 4-dimensional interval order require a large poset with a relatively large number of interval lengths. An answer to this problem may be related to the first question above, were it possible to show that the best possible bound is 4 by giving a straightforward construction of a four-dimensional interval order that can be represented using intervals of two lengths.

Acknowledgements The authors would like to thank an anonymous referee for valuable suggestions that improved the paper.

Funding This work was supported by a grant from the Simons Foundation (#426725, Ann Trenk).

Availability of Data and Material Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest Not applicable.
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