An Ambiguous Statement Called ‘Tetrad Postulate’ and the Correct Field Equations Satisfied by the Tetrad Fields

Waldyr A. Rodrigues Jr. and Quintino A. G. de Souza
Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP, CP 6065
13083-970 Campinas, SP, Brazil
walrod@ime.unicamp.br, quintino@ime.unicamp.br

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The names *tetrad*, *tetrads*, *cotetrads*, have been used with many different meanings in the physical literature, not all of them, equivalent from the mathematical point of view. In this paper we introduce unambiguous definitions for each one of those terms, and show how the old miscellanea made many authors to introduce in their formalism an ambiguous statement called ‘tetrad postulate’, which has been source of many misunderstandings, as we show explicitly examining examples found in the literature. Since formulating Einstein’s field equations intrinsically in terms of cotetrad fields $\theta^a$, $a = 0, 1, 2, 3$ is an worth enterprise, we derive the equation of motion of each $\theta^a$ using modern mathematical tools (the Clifford bundle formalism and the theory of the square of the Dirac operator). Indeed, we identify (giving all details and theorems) from the square of the Dirac operator some noticeable mathematical objects, namely, the Ricci, Einstein, covariant D’Alembertian and the Hodge Laplacian operators, which permit to show that each $\theta^a$ satisfies a well defined wave equation. Also, we present for completeness a detailed derivation of the cotetrad wave equations from a variational principal. We compare the cotetrad wave equation satisfied by each $\theta^a$ with some others appearing in the literature, and which are unfortunately in error.
1 Introduction

In what follows we identify an ambiguous statement called ‘tetrad postulate’ (a better name, as we shall see would be ‘naive tetrad postulate’) that appears often in the Physics literature (see e.g., [5, 12, 24, 47, 49, 50], to quote only a few examples here). We identify the genesis of the wording ‘tetrad postulate’ as a result of a deficient identification of some mathematical objects of differential geometry. Note that we used the word ambiguous, not the word wrong. This is because, as we shall show, the equation dubbed ‘tetrad postulate’ can be rigorously interpreted as meaning that the components of a covariant derivative in the direction of a vector field $\partial_{\mu}$ of a certain tensor field $Q$ (Eq. (34)) are null (see Eq. (80)). This equation is not a postulate. Indeed, it is nothing more than the intrinsic expression of an obvious identity of differential geometry that we dubbed the freshman identity (Eq. (63)). However, if the freshman identity is used naively as if meaning a ‘tetrad postulate’ misunderstandings may arise, and in what follows we present some of them, by examining some examples that we found in the literature. We comment also on a result called ‘Evans Lemma’ of differential geometry and claimed in [12] to be as important as the Poincaré lemma. We show that ‘Evans Lemma’ as presented in [12] is a false statement, the proof offered by that author being invalid because in trying to use the naive tetrad postulate he did incorrect use of some fundamental concepts of differential geometry, as, e.g., his (wrong) Eq. (41E).

We explain all that in details in what follows. We observe also that in [12, 13, 14, 15, 16, 17] it is claimed that ‘Evans Lemma’ is the basic pillar of a (supposed) generally covariant unified field theory developed there. So, once we prove that ‘Evans Lemma’ is a wrong premise, all the theory developed in [12, 13, 14, 15, 16, 17] is automatically disproved.

Using modern mathematical tools (namely the theory of Clifford bundles and the theory of the square of the Dirac operator), we present two derivations (which includes all the necessary mathematical theorems) of the correct differential equations satisfied by the cotetrad fields $\theta^a = \theta^a_{\mu} dx^\mu$ on a Lorentzian manifold, modelling a gravitational field in General Relativity. The first derivation find the tetrad field equations directly from Einstein’s field equations, once we identify, playing with the square of the Dirac operator acting on sections of the Clifford bundle, the existence of some remarkable mathematical objects, namely, the Ricci, Einstein, covariant D’Alembertian, and Hodge Laplacian operators. The second derivation (presented here for completeness) is achieved

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1In order to not confuse the numeration of equations in [12] with the numeration of the equations in the present report we denote in what follows an equation numered Eq.(x) in [12] by Eq.(xE).

2The Dirac operator used in this paper acts on sections of a Clifford bundle. So, it is not to be confused with the (spin) Dirac operator that acts on section of a spin-Clifford bundle. Details can be found in [33]. In particular the square of the Dirac operator is different form the square of the spin-Dirac operator, first calculated by Lichnerowicz [31]. The difference of these squares will be presented in another publication.

3These equations already appeared in [13, 15], but the necessary theorems (proved in this report) needed to prove them have not been given there.
using a variational principle, after expressing the Einstein-Hilbert Lagrangian in terms of the tetrad fields. Our objective in presenting those derivations was the one of comparing the correct equations with the ones presented, e.g., in [13, 14, 15, 16, 17, 28] and which appears as Eq.(49E) in [12].

The functions $q^a_\mu$ appearing as components of the cotetrad fields $\theta^a$ in a coordinate basis can be used to define a tensor $Q = q^a_\mu e^\mu_a \otimes dx^\mu$ (see Eq.(29)). $Q$ satisfies trivially in any general Riemann-Cartan spacetime a second order differential equation which in intrinsic form is $g^{\nu\mu} \nabla_\nu \nabla_\mu Q = 0$. From that equation (numbered Eq.(113) below) we can, of course, write a wave equation for the each one of the functions $q^a_\mu$ in any Riemann-Cartan spacetime. It is this equation that author of [12] attempted to obtain and that he called 'Evans lemma'. However, as we already said, his final result is not correct. In what follows, we use the intrinsic Eq.(113) to write wave equations for the functions $q^a_\mu$ only in the particular case of a general Lorentzian spacetime. This restriction is done here for the following reason. Wave equations for the functions $q^a_\mu$ can also be derived from the correct equations satisfied by the $\theta^a$ in General Relativity (see Eq.(164) below). Then, by comparing both equations we obtain a constrain equation, involving these functions, the components of the Ricci tensor and the components of the energy-momentum tensor and its trace (Eq.(183)). The paper has three appendices. In Appendix A we give a very simple example of the many misunderstandings that the use of the naive 'tetrad postulate' may produce. We hope that this example may be understood even by readers with only a small knowledge of differential geometry. In appendix B we give the details of the calculations needed for deriving the equations for the tetrad fields in General Relativity from a variational principle. Those calculations require a working knowledge of the Clifford bundle formalism. Finally, in Appendix C we give a brief answer to a comment of the author of [12] who just posted a refutation to a preliminary version of this paper. We show that all his claims in his refutation are without valid foundation.

2 Recall of Some Basic Results

In what follows $M$ is a real differential manifold with dim $M = 4$ which will be made part of the definition of a spacetime (whose points are events) of General Relativity, or of a general Riemann-Cartan type theory. As usual we denote the tangent and cotangent spaces at $e \in M$ by $T_e M$ and $T^*_e M$. Elements of $T_e M$ are called vectors and elements of $T^*_e M$ are called covectors. The structures $TM = \cup_e T_e M$ and $T^* M = \cup_{e \in M} T^*_e M$ are vector bundles called respectively the tangent and cotangent bundles. Sections of $TM = \cup_{e \in M} T_e M$ are called vector fields and sections of $T^* M = \cup_{e \in M} T^*_e M$ are called covector fields (or 1-form fields). We denote moreover by $T^{r,s} M$ the bundle of r-covariant and s-contravariant tensor fields and by $\tau M = \bigoplus_{r,s=0}^\infty T^{r,s} M$, the tensor bun-

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4For the best of our knowldge, the Einstein-Hilbert Lagrangian write explicitly in terms of the tetrad fields appears in [57]. See also [48] and related material in [38, 55].

5If necessary these equations can be also written for a Riemman-Cartan spacetime.
dle of $M$. Also, $\bigwedge TM = \bigoplus_{i=0}^{4} \bigwedge^{i} TM$ and $\bigwedge T^{\ast} M = \bigoplus_{i=0}^{1} \bigwedge^{i} T^{\ast} M$, denote respectively the bundles of (nonhomogeneous) multivector fields and multiform fields.

**Remark 1** It is important to keep in mind, in order to appreciate some of the comments presented in the next section, that $T_{e} M$ and $T^{\ast}_{e} M$ are 4-dimensional vector spaces over the real field $\mathbb{R}$, i.e., $\dim T_{e} M = \dim T^{\ast}_{e} M = 4$. Also note the identifications $\bigwedge^{0} T_{e} M = \bigwedge^{0} T^{\ast}_{e} M = \mathbb{R}$, $\bigwedge^{1} T_{e} M = T_{e} M$ and $\bigwedge^{1} T^{\ast}_{e} M = T^{\ast}_{e} M$. Keep also in mind that $\dim \bigwedge^{i} T_{e} M = \dim \bigwedge^{i} T^{\ast}_{e} M = (4)^{i}$. More details on these structures will be given in Section 6, where they are to be used.

To proceed we suppose that $M$ is a connected, paracompact and noncompact manifold. We give the following standard definitions.

### 2.1 Spacetimes

**Definition 2** A Lorentzian manifold is a pair $(M, g)$, where $g \in \sec T^{2,0} M$ is a Lorentzian metric of signature $(1,3)$, i.e., for all $e \in M$, $T_{e} M \simeq T^{\ast}_{e} M \simeq \mathbb{R}^{4}$. For each $e \in M$ the pair $(\mathbb{R}^{4}, g_{e}) \equiv \mathbb{R}^{1,3} = (\mathbb{R}^{4}, \eta)$ is a Minkowski vector space.

**Remark 3** We shall always suppose that the tangent space at $e \in M$ is equipped with the metric $g_{e}$ and so, we eventually write by abuse of notation $T_{e} M \simeq T^{\ast}_{e} M \simeq \mathbb{R}^{1,3}$. Take into account also, that in general the tangent spaces at different points of the manifold $M$ cannot be identified, unless the manifold possess some additional appropriate structure.

**Definition 4** A spacetime $\mathfrak{M}$ is a pentuple $(M, g, \nabla, \tau_{g}, \uparrow)$ where $(M, g, \tau_{g}, \uparrow)$ is an oriented Lorentzian manifold (oriented by $\tau_{g}$) and time oriented by an appropriate equivalence relation (denoted $\uparrow$) for the timelike vectors at the tangent space $T_{e} M$, $\forall e \in M$. $\nabla$ is a linear connection for $M$ such that $\nabla g = 0$.

**Remark 5** In General Relativity, Lorentzian spacetimes are models of gravitational fields.

**Definition 6** Let $T$ and $R$ be respectively the torsion and curvature tensors of $\nabla$. If in addition to the requirements of the previous definitions, $T(\nabla) = 0$, then $\mathfrak{M}$ is said to be a Lorentzian spacetime. The particular Lorentzian spacetime where $M \simeq \mathbb{R}^{4}$ and such that $R(\nabla) = 0$ is called Minkowski spacetime and will

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6$\eta$ is a metric of Lorentzian signature $-2$ in $\mathbb{R}^{4}$.

7See [51] for details.

8More precisely, $\nabla$ is a covariant derivative operator associated to a connection $\omega$, which is a section of a principal bundle called the frame bundle of $M$. $\nabla$ acts on sections of the tensor bundle. We will need to specify with more details the precise nature of $\nabla$ in order to present in an intelligible way the ambiguities associated with the ‘tetrad postulate’. This will be done in Section 4.

9It is important to not confound Minkowski spacetime with $\mathbb{R}^{1,3}$; the Minkowski vector space.
be denoted by $\mathcal{M}$. When $T(\nabla)$ is possibly nonzero, $\mathcal{M}$ is said to be a Riemann-Cartan spacetime (RCST). A particular RCST such that $R(\nabla) = 0$ is called a teleparallel spacetime.

We will also denote by $F(M)$ the frame bundle of $M$ and by $P_{SO_{1,3}}(M)$ the principal bundle of oriented Lorentz tetrads. Those bundles will be used in Section 4 to give some additional details concerning the nature of the tangent, cotangent and tensor bundles, as associated vector bundles to $F(M)$ or $P_{SO_{1,3}}(M)$, which are necessary to clarify misunderstandings related to the naive ‘tetrad postulate’.

### 2.2 On the Nature of Tangent and Cotangent Fields I

Let $U \subset M$ be an open set and let $(U, \varphi)$ be a coordinate chart of the maximal atlas of $M$. We recall that $\varphi$ is a differentiable mapping from $U$ to an open set of $\mathbb{R}^4$. The coordinate functions of the chart are denoted by $x^\mu : U \to \mathbb{R}$, $\mu = 0, 1, 2, 3$.

Consider the subbundles $TU \subset TM$ and $T^*U \subset T^*M$. There are two types of vector fields (respectively covector fields) in $TU$ (respectively $T^*U$) which are such that at each point (event) $e \in U$ define interesting bases for $T_eU$ (respectively $T^*_eU$).

**Definition 7** coordinate basis for $TU$. A set $\{e_\mu\}$, $e_\mu \in \text{sec}TU$, $\mu = 0, 1, 2, 3$ is called a coordinate basis for $TU$ if there exists a coordinate chart $(U, \varphi)$ and coordinate functions $x^\mu : U \to \mathbb{R}$, $\mu = 0, 1, 2, 3$, such that for each (differentiable) function $f : M \to \mathbb{R}$ we have $(\varphi(e) \equiv x)$

$$e_\mu(f)|_e = \frac{\partial}{\partial x^\mu}(f \circ \varphi^{-1})\bigg|_x$$ \hspace{1cm} (1)

**Remark 8** Due to this equation mathematicians often write $e_\mu = \partial_\mu$ and sometimes even $e_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu$. Also by abuse of notation it is usual to see (in physics texts) $f \circ \varphi^{-1}$ written simply as $f$ or $f(x)$, and here we eventually use such sloppy notation, when no confusion arises.

**Definition 9** coordinate basis for $T^*U$. A set $\{\theta^\mu\}$, $\theta^\mu \in \text{sec}T^*U$, $\mu = 0, 1, 2, 3$ is called a coordinate basis for $T^*U$ if there exists a coordinate chart $(U, \varphi)$ and coordinate functions $x^\mu : U \to \mathbb{R}$, $\mu = 0, 1, 2, 3$, such that $\theta^\mu = dx^\mu$.

Recall that the basis $\{\theta^\mu\}$ is the dual basis of $\{\partial_\mu\}$ and we have $\theta^\mu(\partial_\nu) = \delta^\mu_\nu$.

Now, in general the coordinate basis $\{\partial_\mu\}$ is not orthonormal, this means that if the pullback of $g$ in $T^2,0 \varphi(U)$ is written as usual (with abuse of notation) as $g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$ then,

$$g(\partial_\mu, \partial_\nu)|_x = g(\partial_\nu, \partial_\mu)|_x = g_{\mu\nu}(x)$$ \hspace{1cm} (2)

\[\text{Also we say that } \{e_\mu\} \in \text{sec}F(U) \subset \text{sec}F(M), \text{i.e., is a section of the frame bundle.}\]
and in general the real functions $g_{\mu\nu} : \varphi(U) \to \mathbb{R}$ are not constant functions.

Also, if $g \in \sec T^0\varphi M$ is the metric of the cotangent bundle, we have (writing for the pullback of $g$ in $T^0\varphi(U)$, $g = g^{\mu\nu}(x)\partial_\mu \otimes \partial_\nu$)
\[
g(dx^\mu, dx^\nu)|_x = g^{\mu\nu}(x),
\]
and the real functions $g^{\mu\nu} : \varphi(U) \to \mathbb{R}$ satisfy
\[
g^{\mu\nu}(x)g_{\mu\nu}(x) = \eta^{\nu}, \quad \forall x \in \varphi(U).
\]

### 2.3 Tetrads and Cotetrads

**Definition 10** orthonormal basis for $TU$. A set $\{e_a\}, e_a \in \sec TU$, with $a = 0, 1, 2, 3$ is said to be an orthonormal basis for $TU$ if and only if for any $x \in \varphi(U),
\[
g(e_a, e_b)|_x = \eta_{ab}
\]
where the $4 \times 4$ matrix with entries $\eta_{ab}$ is the diagonal matrix $\text{diag}(1, -1, -1, -1)$. When no confusion arises we shall use the sloppy (but very much used) notation $\eta_{ab} = \text{diag}(1, -1, -1, -1)$.

**Definition 11** orthonormal basis for $T^*U$. A set $\{\theta^a\}, \theta^a \in \sec T^*U$, with $a = 0, 1, 2, 3$ is said to be an orthonormal basis for $T^*U$ if and only if for any $x \in \varphi(U),
\[
g(\theta^a, \theta^b)|_x = \eta^{ab} = \text{diag}(1, -1, -1, -1).
\]

Recall that the basis $\{\theta^a\}$ is the dual basis of the basis $\{e_a\}$, i.e., $\theta^a(e_b) = \delta^a_b$.

**Definition 12** The set $\{e_a\}$ considered as a section of the orthonormal frame bundle $P_\text{SO}(\mathbb{R})U \subset P_\text{SO}(\mathbb{R})M$ is called a tetrad basis for $TU$. The set $\{\theta^a\}$ is called a cotetrad basis for $T^*U$.

**Remark 13** We recall that a global (i.e., defined for all $e \in M$) tetrad (cotetrad) basis for $TM$ ($T^*M$) exists if and only if $M$ in Definition 4 is a spin manifold (see, e.g., [34, 35]). This result is the famous Geroch theorem [21].

**Remark 14** Besides that bases, it is also convenient to define reciprocal bases. So, the reciprocal basis of $\{\partial_\mu\} \in \sec F(U)$ is the basis of $\{\partial^\nu\} \in \sec F(U)$ such that $g(\partial_\mu, \partial^\nu) = \delta^\nu_\mu$. Also, the reciprocal basis of the basis $\{\theta^\mu = dx^\mu\}$ of $T^*U$, $\theta^a \in \sec T^*U$, $a = 0, 1, 2, 3$ is the basis $\{\theta^\mu\}$ of $T^*U$, $\theta^\mu \in \sec T^*U$, $\mu = 0, 1, 2, 3$ such that $g(\theta^\mu, \theta^\nu) = \delta^\nu_\mu$. Also $\{e^a\}, e^a \in \sec TU$, $a = 0, 1, 2, 3$ with $g(e^a, e_b) = \delta^a_b$ is called the reciprocal basis of the basis $\{e_a\}$. Finally, $\{\theta_a\}, \theta_a \in \sec T^*U$, $a = 0, 1, 2, 3$ with $g(\theta_a, \theta^b) = \delta^b_a$ is called the reciprocal basis of $\{\theta^a\}$.

Now, consider a vector field $V \in \sec TU$ and a covector field $C \in \sec T^*U$. We can express $V$ and $C$ in the coordinate basis $\{\partial_\mu\}, \{\partial^\nu\}$ and $\{\theta^\mu = dx^\mu\}, \{\theta_\mu\}$ by
\[
V = V^\mu \partial_\mu = V_b \partial^b, \quad C = C_\mu dx^\mu = C^a \theta_\mu
\]
and in the tetrad basis $\{e_a\}, \{e^a\}$ and $\{\theta^a\}, \{\theta_a\}$ by
\[
V = V^a e_a = V_a \theta^a, \quad C = C_a \theta^a = C^a \theta_a.
\]
3 Some Misconceptions and Misunderstandings Involving Tetrads

In this section we analyze some statements found in section 1 [12] which is said to be dedicated to give many distinct definitions of ‘tetrads’. Unfortunately that section is full of misconceptions and misunderstandings, which are the origin of many errors in papers signed by author of [12]. In order to appreciate that statement, let us recall some facts.

First, recall that each one of the tetrad fields (as defined in the previous Section, Definition 12), \( e_a \in \text{sec}TU \), \( a = 0, 1, 2, 3 \), as any vector field, can be expanded using Eq.(7) in the coordinate basis \( \{ \partial_\mu \} \), as

\[
e_a = q_\mu^a \partial_\mu. \tag{9}\]

Also, each one of the cotetrad fields \( \{ \theta^a \} \), \( \theta^a \in \text{sec}TU \), \( a = 0, 1, 2, 3 \), as any covector field, can be written as

\[
\theta^a = q^a_\mu dx^\mu. \tag{10}\]

Remark 15 The functions \( q_\mu^a, q^a_\mu : \varphi(U) \to \mathbb{R} \) are real functions and satisfy

\[
q_\mu^a q^b_\nu = \delta^b_a, \quad q^a_\mu q^a_\nu = \delta^\mu_\nu. \tag{11}\]

It is trivial to verify the formulas

\[
g_{\mu\nu} = q_\mu^a q^b_\nu \eta_{\alpha\beta}, \quad g^{\mu\nu} = q^a_\mu q^b_\nu \eta^{\alpha\beta},
\eta_{\alpha\beta} = q^a_\mu q^b_\nu g_{\mu\nu}, \quad \eta^{\alpha\beta} = q^a_\mu q^b_\nu g^{\mu\nu}. \tag{12}\]

Now to some comments.

(c1) In Eq.(9E) and Eq.(10E) it is written\(^{11}\)

\[
q^{\mu(A)}_{\nu} = q^a_\mu \wedge q^b_\nu, \tag{9E}\]
\[
q^{\mu B}_{\nu} = q^a_\mu q^b_\nu = q^a_\mu \otimes q^b_\nu. \tag{10E}\]

Of course, these unusual notations used to multiply scalar functions in the above equations, if they are to have any meaning at all, must be understood as a notation suggested from the result of correct mathematical operations. The problem is that in [12] they are not well specified and we have some ambiguity. Indeed, we have the possibilities:

\[
\theta^a \otimes \theta^b = q^a_\mu q^b_\nu \eta_{\alpha\beta} dx^\mu \otimes dx^\nu \tag{13}
= \theta^a \wedge \theta^b + \theta^a \otimes \theta^b, \tag{14}\]

\(^{11}\)In [12] instead of the symbol \( \wedge \) the symbol \( \wedge \) has been used for the exterior product.
This distinction is necessary here because the convention for the exterior product that we used in the second part of the paper is different from the one used in [12].
where the algebraists definitions \[4, 9\] of \(\theta^a \wedge \theta^b\) and \(\theta^a \otimes \theta^b\) are:

\[
\theta^a \wedge \theta^b = \frac{1}{2} \left( \theta^a \otimes \theta^b - \theta^b \otimes \theta^a \right)
\]
\[
= \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} - q^b_{\mu} q^a_{\nu} \right) dx^\mu \otimes dx^\nu \tag{15}
\]
\[
= q^a_{\mu} q^b_{\nu} dx^\mu \wedge dx^\nu = q^a_{\mu} q^b_{\nu} dx^\mu \otimes dx^\nu \tag{16}
\]
\[
= \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} - q^b_{\mu} q^a_{\nu} \right) dx^\mu \wedge dx^\nu \tag{17}
\]

\[
\theta^a \otimes \theta^b = \frac{1}{2} \left( \theta^a \otimes \theta^b + \theta^b \otimes \theta^a \right)
\]
\[
= \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} + q^b_{\mu} q^a_{\nu} \right) dx^\mu \otimes dx^\nu \tag{18}
\]
\[
= q^a_{\mu} q^b_{\nu} dx^\mu \wedge dx^\nu = q^a_{\mu} q^b_{\nu} dx^\mu \otimes dx^\nu \tag{19}
\]
\[
= \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} + q^b_{\mu} q^a_{\nu} \right) dx^\mu \wedge dx^\nu \tag{20}
\]

So, we have the following possibilities for identification of symbols:

(a) Use Eq.(17) and Eq.(20). This results in

\[
q^a_{\mu} \wedge q^b_{\nu} = \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} - q^b_{\mu} q^a_{\nu} \right) , \tag{21}
\]

\[
q^{ab}_{\mu\nu} = q^a_{\mu} \otimes q^b_{\nu} = \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} + q^b_{\mu} q^a_{\nu} \right) , \tag{22}
\]

\[
q^a_{\mu} \otimes q^b_{\nu} = q^a_{\mu} \otimes q^b_{\nu} + q^a_{\mu} \wedge q^b_{\nu} . \tag{23}
\]

(b) Use now Eq. (15) and Eq. (18). This results in the alternative possibility

\[
q^a_{\mu} \wedge q^b_{\nu} = \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} - q^a_{\mu} q^b_{\nu} \right) , \tag{24}
\]

\[
q^{ab}_{\mu\nu} = q^{ab}_{\mu\nu} = \frac{1}{2} \left( q^a_{\mu} q^b_{\nu} + q^b_{\mu} q^a_{\nu} \right) , \tag{25}
\]

\[
q^a_{\mu} \otimes q^b_{\nu} = q^a_{\mu} \otimes q^b_{\nu} + q^a_{\mu} \wedge q^b_{\nu} . \tag{26}
\]

To decide what the author of \[12\] had in mind, we need to look at line 19 in Table 1 of \[12\]. There, we learn that the definition of the exterior product (\(\wedge\)) used there is
d\(,\) given \(A, B \in \sec T^*M\),

\[
A \wedge B = A \otimes B - B \otimes A , \tag{27}
\]

12The definition given in Eq. (15) is used mainly (for very good reasons, that we refrain to discuss here) by algebraists \[4, 9\]. However, many physicists working in General Relativity use it, as, e.g. \[3\]. The definition given by Eq. (27) is eventually more popular among authors working on differential geometry, see, e.g. \[2\] and some authors working in General Relativity. In particular, this definition is also the one used in \[5\] (see his Eq.(1.79)), and also the one used, e.g., in \[22, 39\] both may be used, each one has its merits, but it is a good idea for a reader to first knows what the author means. We have discussed this issue in details in \[18\].
since line 19 of Table 1 in ME reads

\[(A \land B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu\]  \hspace{1cm} (28)

But, the author of [12] forgot to inform his readers that from the genuine notation given by Eq.(28) he starting using that \((A \land B)_{\mu\nu} := A_\mu \land B_\nu\). Without that explanation the symbols \(A_\mu \land B_\nu\) look as a product of scalars, and as we just showed that symbols can be interpreted in the alternative ways given above, which are different from the one eventually intended by author of [12]. Indeed, he should write

\[\left(\theta^a \land \theta^b\right)_{\mu\nu} = \left(\theta^a_\mu \land \theta^b_\nu\right) = q^a_\mu q^b_\nu - q^a_\nu q^b_\mu\]

and then advise his readers that he was going to represent \(\left(\theta^a \land \theta^b\right)_{\mu\nu}\) by the symbol \(q^a_\mu \land q^b_\nu\), i.e., \(\left(\theta^a \land \theta^b\right)_{\mu\nu} := q^a_\mu \land q^b_\nu\).

At first sight it may seem that we are being very pedantic. But if we insist in notational issues, it is because as we are going to see in the following sections, if the exact meaning of the symbols used are not precise, ambiguities may appear in calculations a bit more sophisticated than the ones above, resulting inevitably in nonsense.

(c2) Consider the statement following Eq.(22E) in page 437 of [12], namely:

"...The dimensionality of the tetrad matrix depends on the way it is defined: for example, using Eqs.(6E) (7E), (11E) or (12E), the tetrad is a 4 \times 4 matrix; using Eq.(13E), it is a 2 \times 2 complex matrix."

This is a very misleading statement, which is a source in [12, 13, 14, 15, 16, 17] of confusion. That statement has probably origin in some statements appearing [40, 43] of confusion. That statement has probably origin in some statements appearing [40, 43]. Indeed, suppose that we consider Clifford valued differential forms, i.e., objects that are sections of the bundle \(\mathcal{C}(T^\ast M) \otimes \bigwedge T^\ast M\), where \(\mathcal{C}(T^\ast M)\) is the Clifford bundle of nonhomogeneous multivector fields. We consider as usual that \(T^\ast M = \bigwedge^1 T^\ast M \hookrightarrow \mathcal{C}(T^\ast M)\) (details may be found in [40, 43]). Consider the object

\[Q = e_a \otimes \theta^a = e_\mu \otimes dx^\mu \in \text{sec} \bigwedge^1 T^\ast M \otimes \bigwedge^1 T^\ast M \hookrightarrow \mathcal{C}(T^\ast M) \otimes \bigwedge T^\ast M\]  \hspace{1cm} (29)

We define now the object \(S \in \text{sec} \mathcal{C}(T^\ast M) \otimes \bigwedge T^\ast M\) by

\[S := Qe_0 := e_\mu e_0 \otimes dx^\mu = q_\mu \otimes dx^\mu\]

(30)

As showed in details in [40], the objects

\[q_\mu = e_\mu e_0 \in \text{sec} \mathcal{C}^{(0)}(T^\ast M),\]

13Note that in section 8 and the following ones we work with \(\mathcal{C}(T^\ast M)\), the bundle of nonhomogeneous multiforms fields.
where $\mathcal{C}^{(0)}(TM)$ is the even subalgebra of $\mathcal{C}(TM)$. As it is well known, for each $e \in M$, $\mathcal{C}^{(0)}(T_eM) = \mathbb{R}_{3,0}$, a Clifford algebra also known as Pauli algebra, the reason being the fact that as a matrix algebra, $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of $2 \times 2$ complex matrices. Sachs thought that the $q_{\mu}$ would be quaternion fields, but indeed they are not. They are paravector fields. Important for our comments is the fact that the matrix representation of the $q_{\mu}$ are $2 \times 2$ complex matrices that as well known may be expanded in terms of the identity matrix and the Pauli matrices. Now, having in mind that we can write $q_{\nu} = e_{\nu}e_0 = q_{\nu}^a q_a$, we can understand that the real functions $q_{\nu}^a$ appears as components of complex functions in the matrix representations of the $q_{\nu}$. But this, of course, does not mean that the tetrads are complex matrices, as stated in [12]. We can define a covariant derivative $\nabla^c$ operator (see details in [33]) acting on sections of the Clifford bundle of multivectors $\mathcal{C}(TM)$. Then, we can define the covariant derivative of the paravector fields $q_{\nu}$ (or their matrix representations) in the direction of the coordinate vector field $e_{\mu} = \partial_{\mu}$. This would be written as $\nabla_{\partial_{\mu}} q_{\nu} = \nabla_{\partial_{\mu}} (q_{\nu}^a q_a) := (\nabla_{\mu} q_{\nu}^a) q_a$, thereby defining unambiguously the symbols $\nabla_{\mu} q_{\nu}^a$ as the components of the covariant derivatives of the paravector fields $q_{\nu}$ in the paravector field basis $\{q_a\}$.

Of course, it is possible to think of another matrix involving the real functions $q_{\nu}^a$. Indeed, forget for a while the bundle $\mathcal{C}(TM) \otimes \wedge T^* M$ and consider an object $P \in \sec T^{1,1} M$. Such object (sometimes called a vector valued 1-form) can be written in the ‘hybrid’ basis $\{e_a \otimes dx^\mu\}$ of $T^{1,1} U$ as

$$P = P^a_{\nu} e_a \otimes dx^\nu$$

and can, of course be represented by a $4 \times 4$ real matrix in the standard way. In particular, we can imagine a $Q \in \sec T^{1,1} M$ such that $Q^a_{\mu} = q^a_{\mu}$. This

$$Q = q^a_{\mu} e_a \otimes dx^\nu$$

can be, of course, be appropriately identified in an obvious way with the $Q$ defined in Eq. (29), this being the reason that we used the same symbol. As we shall show below we cannot identify the components of the covariant derivative of $Q$ in the direction of the vector field $\partial_{\mu}$, which we will denote by $\nabla_{\mu} q_{\nu}^a = (\nabla_{\partial_{\mu}} Q)^a_{\nu}$ with the components of the covariant derivative of the $Q^a_{\mu}$ in the direction of the vector field $\partial_{\mu}$, which will be denote by $\nabla_{\mu} q_{\nu}^a$, which is given by Eq. (34) below. It is also not licit to identify $\nabla_{\mu} q_{\nu}^a$ with $\nabla_{\mu} q_{\nu}^a$.

As we shall see, it is this wrong identification that leads to the ambiguous statement called ‘tetrad postulate’.

**Remark 16** Any how, before proceeding we have an observation concerning the symbols $q_{\mu}^{a \nu} \wedge q_{\nu}^b$ and $q_{\mu}^{a b}$. The idea of associating a linear combination of $q_{\mu}^{a b}$, as defined in Eq. (25) with a gravitational field and a multiple of $q_{\mu}^{a \nu} \wedge q_{\nu}^b$ as defined by Eq. (24) with an electromagnetic field already appeared in the old Sachs book [49] (see also Sachs recent book [50]). The only difference is that Sachs introduces the fields $q_{\mu}^a : \varphi(U) \to \mathbb{R}$ as coefficients of the matrix
representations of the paravector vector fields $q_{\mu}$ defined in Eq. (32) (which he incorrectly identified with quaternion fields). Unfortunately that idea does not work as proved in [40, 43], and much the same arguments can be given for the theory proposed in [12, 8, 13, 14, 15, 16, 17] and will not be repeated here.

For what follows we need to keep in mind that—as explained in the previous section—the functions $q_{a \mu}$, $q_{\mu a}$: $\varphi(U) \to \mathbb{R}$ are always real functions, and that set $\{q_{a \mu}\}$, can appear as components of very, distinct objects, e.g., for each fixed $a$, $\{q_{a \mu}\}$ can be interpreted as the components of a covector field (namely $\theta^a$) in the basis $\{dx^\mu\}$ or for fixed $\mu$, $\{q_{\mu a}\}$ as the components of the vector field $\partial_{\mu}$ in the basis $\{e_a\}$. Also, the set $\{q_{a \mu}\}$ for each fixed $a$ can be interpreted as the components of the vector field $e_a$ in the basis $\partial_{\mu}$. Also, for varying $a$ and $\mu$ the $\{q_{a \mu}\}$ can be thought as the components of the tensor $Q$ given by Eq. (34), etc. So, it is crucial to distinguish without ambiguity in what context the set of real functions $\{q_{a \mu}\}$ (or $\{q_{\mu a}\}$) is being used.

(c3) Consider the statement before Eq. (23E) of [12]:

"The tetrad is a vector-valued one-form, i.e., is a one-form $q_\mu$ with labels $a$. If $a$ takes values 1, 2 or 3 of a Cartesian representation of the tangent space, for example, the vector

$$q_\mu = q_{1 \mu}^i + q_{2 \mu}^j + q_{3 \mu}^k$$

(23E)

can be defined in this space. Each of the components $q_{1 \mu}^i, q_{2 \mu}^j$ or $q_{3 \mu}^k$ are scalar-valued one-forms of differential geometry [2], and each of the $q_{1 \mu}^i, q_{2 \mu}^j$, and $q_{3 \mu}^k$ is therefore a covariant four vector in the base manifold. The three scalar-valued one-forms are therefore the three components of the vector-valued one-forms $q_\mu^a$, the tetrad form."

Well, that sentence contains a sequence of misconceptions.

The first part of the statement namely ‘The tetrad is a vector-valued one-form, i.e., is a one-form $q_\mu$ with labels $a$’ only has meaning if the functions $q_{a \mu}$ are interpreted as the components of the tensor $Q$ defined by Eq. (34). So, the next part of the statement, namely Eq. (23E) is meaningless.

First, the tangent space to each $e \in M$, where $M$ is the manifold where the theory was supposed to be developed is a real 4-dimensional space. So, as we observed in Remark 1, $a$ must take the values 0, 1, 2, 3. More, as observed in Remark 3 the tangent spaces at different points of a general manifold $M$ in general cannot be identified, unless the manifold possess some additional appropriate structure, which is not the case in Evans paper. As such, the objects defined in Eq. (23E) have nothing to do with the concept of tangent vectors, as Evans would like for future use in some identifications that he used in [12] (and [8, 13, 14, 15, 16, 17] and also in some old papers that he signed alone or with the AIAS group and that were published in FPL and other journals) to justify some (wrong) calculations of his $B(3)$ theory. This means also that $q_\mu$ in Eq. (23E) cannot be identified with the basis vectors $\partial_{\mu}$. They

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14 A very detailed discussion of the many non sequitur results of those papers is given in [6]. A reply by Evans to that paper is to be found in Evans website: http://www.aias.us/pub/rebutal/finalrebutaldocument.pdf. A reply to Evans note can be found in: http://www.ime.unicamp.br/rel_pesq/2003/ps/rp28-03.pdf.
are simply mappings \( U \rightarrow F(U) \otimes \mathbb{R}^3 \), where \( F(U) \) is a subset of the set of (smooth) functions in \( U \). We emphasize again: The vectors in set \((i, j, k)\) as introduced by Evans are not tangent vector fields to the manifold \( M \), i.e., they are not sections of \( TU \). The set \((i, j, k)\) is simply a basis of the real three-dimensional vector space \( \mathbb{R}^3 \), which has been introduced by Evans without any clear mathematical motivation.

4 Some Results from the Theory of Connections

(i) In what follows we denote by \( F(M) \) the principal bundle of linear frames. The structural group of this bundle is \( \text{Gl}(4, \mathbb{R}) \), the general linear group on 4-dimensions.\(^{15}\)

(ii) The elements of \( F(M) \) are called frame fields (or simply frames). A frame \( \{e_\alpha\} \in \text{sec} F(M) \) can be identified with a basis of \( TM \), the tangent bundle.

(iii) We suppose that the manifold \( M \) is equipped with a Lorentz metric \( g \in \text{sec} T^{(2,0)}M \). We denote by \( P_{\text{SO}_{1,3}}(M) \) the bundle of orthonormal frames. Its structural group is \( \text{SO}_{1,3} \), the homogeneous orthochronous Lorentz group. \( P_{\text{SO}_{1,3}}(M) \) is said to be a reduction of \( F(M) \). A frame \( \{e_\alpha\} \in \text{sec} P_{\text{SO}_{1,3}}(M) \) is called an orthonormal frame.

(iv) A linear connection on \( F(M) \) is a 1-form with values in the Lie algebra \( \text{gl}(4, \mathbb{R}) \), which needs to satisfy a set of well specified properties, which we are not going to specify here, since they will be not necessary in what follows.

(v) It is a theorem of the theory of connections that each connection defined in \( P_{\text{SO}_{1,3}}(M) \) determines a connection in \( F(M) \) (a linear connection)\(^{15}\).

(vi) Given the pair \((M, g)\), a linear connection on \( F(M) \), which is determined by a connection on the bundle of orthonormal frames \( P_{\text{SO}_{1,3}}(M) \) is called metric compatible.

(vii) Any connection in a principal bundle determines a connection in each associated vector bundle to it.

4.1 On the Nature of Tangent and Cotangent Fields II

(viii) We are going to work exclusively with spacetime structures in this paper which have the pair \((M, g)\) as substructure. Under this condition we recall that the tangent and cotangent bundles \( TM \) and \( T^*M \) (already introduced in section 2) can also be written as the associated vector bundles

\[
TM = F(M) \times_{\rho^+(\text{Gl}(4, \mathbb{R}^4))} \mathbb{R}^4 = P_{\text{SO}_{1,3}}(M) \times_{\rho^+(\text{SO}_{1,3})} \mathbb{R}^4,
\]

\(15\) For details, the reader may consult as an introduction the books \([7, 22]\). A more advanced view of the subject can be acquired studying, e.g., \([29, 42]\).
and the cotangent bundle is

\[ T^*M = F(M) \times_{\rho^-(Gl(4,\mathbb{R}^4))} \mathbb{R}^4 = P_{SO_{1,3}^*}(M) \times_{\rho^-} (SO_{1,3}) \mathbb{R}^4. \] (36)

In the above equations, \( \rho^+(Gl(4,\mathbb{R}^4)) \) (\( \rho^+(SO_{1,3}^*) \)) refers to the \textit{standard} representations of the groups \( Gl(4,\mathbb{R}^4) \) (\( SO_{1,3}^* \)) and \( \rho^- (Gl(4,\mathbb{R}^4)) \) (\( \rho^- (SO_{1,3}^*) \)) refers to the \textit{dual} representations. Given these results the bundle of \((r, s)\) tensors is (in obvious notation)

\[ T^{(r,s)}(M) = \bigotimes_r F(M) \times \bigotimes_s \rho^+(Gl(4,\mathbb{R}^4)) \mathbb{R}^4 \]

\[ = \bigotimes_r P_{SO_{1,3}^*}(M) \times \bigotimes_s \rho^+(SO_{1,3}) \mathbb{R}^4 \] (37)

(ix) The tensor bundle is denoted here, as in Section 2 by \( \tau M = \bigoplus_{r,s=0}^{\infty} T^{(r,s)}(M) \).

(x) Any connection in a principal bundle determines a connection in each associated vector bundle to it. A connection on a vector bundle is also called a \textit{covariant derivative}.

### 4.2 \( \nabla^+, \nabla^- \) and \( \nabla \)

Let \( X, Y \in \sec TM \), any vector fields, \( \alpha \in \sec T^*M \) any covector (also called a 1-form field) and \( P \in \sec \tau M \) any general tensor. Then, we have the following three covariant derivatives operators, \( \nabla^+, \nabla^- \) and \( \nabla \), defined as follows:

\[ \nabla^+ : \sec TM \times \sec TM \to \sec TM, \]

\[ (X, Y) \mapsto \nabla^+_X Y, \] (38)

\[ \nabla^- : \sec TM \times \sec T^*M \to \sec TM, \]

\[ (X, \alpha) \mapsto \nabla^-_X \alpha, \] (39)

\[ \nabla : \sec TM \times \sec \tau M \to \sec \tau M, \]

\[ (X, P) \mapsto \nabla_X P, \] (40)

(xi) Each one of the covariant derivative operators introduced above satisfy the following properties: Given, differentiable functions \( f, g : M \to \mathbb{R} \), vector fields \( X, Y \in \sec TM \) and \( P, Q \in \sec \tau M \) we have

\[ \nabla_{fX + gY} P = f\nabla_X P + g\nabla_Y P, \]

\[ \nabla_X (P + Q) = \nabla_X P + \nabla_X Q, \]

\[ \nabla_X (fP) = f\nabla_X (P) + X(f)P, \]

\[ \nabla_X (P \otimes Q) = \nabla_X P \otimes Q + P \otimes \nabla_X Q. \] (41)
(xii) The absolute differential of $P \in \sec T^{(r,s)}(M)$ is given by the mapping

$$\nabla : \sec T^{(r,s)}(M) \to \sec T^{(r,s+1)}(M),$$

(42)

$$\nabla P(X_1, X_2, ..., X_s, \alpha_1, ..., \alpha_r) = \nabla X P(X_1, X_2, ..., X_s, \alpha_1, ..., \alpha_r),$$

(43)

$$X_1, ..., X_s \in \sec TM, \alpha_1, ..., \alpha_r \in \sec T^* M.$$  

(44)

(45)

(xiii) To continue we must give the relationship between $\nabla^+, \nabla^-$ and $\nabla$. So, let us suppose that a connection has been chosen according to what have been said in (vi) above.

Then, given the coordinate bases $\{\partial_\mu\}, \{\partial^\mu\}, \{\theta_\mu = dx^\mu\}, \{\theta^\alpha\}$ and the orthonormal bases $\{e_a\}, \{e^a\}, \{\theta_a\}, \{\theta^a\}$ defined in Section 2, we have the definitions of the connection coefficients associated to the respective covariant derivatives in the respective basis, e.g.,

$$\nabla^+_{\partial_\mu} \partial_\nu = \Gamma^\mu_{\sigma\nu} \partial_\sigma, \quad \nabla^-_{\partial_\mu} \partial^\nu = -\Gamma^\nu_{\sigma\alpha} \partial^\sigma,$$

(46)

$$\nabla^-_{e_a} e_b = \omega^a_{cb} e_c, \quad \nabla^+_{e_a} e_b = -\omega^b_{ac} e_c,$$

(47)

$$\nabla^-_{e_a} \theta_b = \omega^b_{ca} \theta_c = -\omega_{bac} \theta^c,$$

(48)

$$\omega_{abc} = \eta_{ad} \omega^d_{bc} = -\omega_{cba}, \quad \omega^a_{bc} = \eta^{ab} \omega_{kal} \eta^{cl}, \quad \omega_{bc} = -\omega^a_{eb}$$

$$\nabla^+_{\partial_\mu} e_b = \omega^b_{\mu c} e_c,$$

(49)

To understand how $\nabla$ works, consider its action, e.g., on the sections of $T^{(1,1)}M = TM \otimes T^* M$. For that case, if $X \in \sec TM, \alpha \in \sec T^* M$, we have that

$$\nabla = \nabla^+ \otimes \Id_{T^* M} + \Id_{TM} \otimes \nabla^-,$$

(50)

and

$$\nabla(X \otimes \alpha) = (\nabla^+ X) \otimes \alpha + X \otimes \nabla^- \alpha.$$  

(51)

The general case, of $\nabla$ acting on sections of $\tau M$ is an obvious generalization of the precedent one, and details are left to the reader.

(xiv) We said that a connection determined under the conditions given in (vi) above is metric compatible. This is given explicitly by the condition that for the metric tensor $g \in \sec (T^* M \otimes T^* M)$ we have

$$\nabla g = 0$$

(52)

Note that the metric compatibility condition $\nabla g = 0$, does not necessarily imply that the torsion tensor is null, $T = 0$. When $\nabla g = 0$ and $T = 0$, $\nabla$ is called the Levi-Civita connection, and it is unique. In that case, the connection coefficients (Cristoffel symbols) in a coordinate basis are symmetric. But, the connection coefficients in a tetrad basis can be written in a very useful way for computations as antisymmetric. See, e.g., Eq. 16.
(xvi) Also, we assume that we are studying a connection which is not teleparallel, i.e., there is no orthonormal basis for $TU \subset TM$ such that $\nabla_{e_a} e_b = 0$, for all $a, b = 0, 1, 2, 3$. So, in general, $\omega_{ab} \neq 0$ and

$$\nabla_{e_a} \theta^b = -\omega_{ac} \theta^c \neq 0.$$  

(xvii) For every vector field $V \in \text{sec} T U$ and a covector field $C \in \text{sec} T^* U$ we have

$$\nabla^+ \partial_\mu V = \nabla^+ \partial_\mu (V^\alpha \partial_\alpha), \quad \nabla^- \partial_\mu C = \nabla^- \partial_\mu (C_\alpha \theta^\alpha)$$

and using the properties introduced in (xi) above, $\nabla^+ \partial_\mu V$ can be written as:

$$\nabla^+ \partial_\mu V = \nabla^+ \partial_\mu (V^\alpha \partial_\alpha) = (\nabla^+ \partial_\mu V)^\alpha \partial_\alpha = (\partial_{\mu} V^\alpha) \partial_\alpha + V^\alpha \nabla^+ \partial_\mu \partial_\alpha = \left(\frac{\partial V^\alpha}{\partial x^\mu} + V^\rho \Gamma^\alpha_{\mu \rho}\right) \partial_\alpha = (\nabla^+ \partial_\mu V)^\alpha \partial_\alpha,$$

where it is to be kept in mind that

$$(\nabla^+ \partial_\mu V)^\alpha \equiv \nabla^+ \partial_\mu V^\alpha.$$ 

Also, we have

$$\nabla^- \partial_\mu C = \nabla^- \partial_\mu (C_\alpha \theta^\alpha) = (\nabla^- \partial_\mu C)_\alpha \theta^\alpha = \left(\frac{\partial C^\alpha}{\partial x^\mu} - C_\beta \Gamma^\alpha_{\mu \beta}\right) \theta^\alpha \equiv (\nabla^- \partial_\mu C^\alpha) \theta^\alpha$$

where it is to be kept in mind that

$$(\nabla^- \partial_\mu C^\alpha) \equiv \nabla^- \partial_\mu C^\alpha.$$ 

**Remark 17** Eqs. (55) and (57) define the symbols $\nabla^+ \partial_\mu V^\alpha$ and $\nabla^- \partial_\mu C^\alpha$. The symbols $\nabla^+ \partial_\mu V^\alpha : \varphi(U) \to \mathbb{R}$ are real functions, which are the components of the vector field $\nabla^+ \partial_\mu V$ in the basis $\{\partial_\alpha\}$. Also, $\nabla^- \partial_\mu C^\alpha : \varphi(U) \to \mathbb{R}$ are the components of the covector field $C$ in the basis $\{\theta^\alpha\}$.

**Remark 18** The standard practice of many Physics textbooks of representing, $\nabla^+ \partial_\mu V^\alpha$ and $\nabla^- \partial_\mu V^\alpha$ by $\nabla^\mu V^\alpha$ will be avoided here. This is no pedantism, as we are going to see. Moreover, we observe that the standard practice of calling $\nabla^+ \partial_\mu V^\alpha$ the covariant derivative of the "vector" field $V^\alpha$ generates a lot

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17 Recall that other authors prefer the notations $(\nabla^- \partial_\mu V^\alpha) = V^\alpha_\mu$ and $(\nabla^- \partial_\mu C^\alpha) \equiv C_\alpha^\mu$. What is important is always to have in mind the meaning of the symbols.
of confusion, for many people, confounds the symbol $\nabla^+_{\mu}$ (appearing in $\nabla^+_{\mu} V^\alpha$) with the real covariant derivative operator, which is $\nabla_{\mu}$ \cite{13}. Also, in many Physics textbooks the symbol $\nabla^+_{\mu}$ is sometimes also used as a sloppy notation for the symbol $\nabla_{\mu} \partial^\mu$, something that generates yet more confusion. The author of \cite{12,13,14,15,16,17}, e.g., has not escaped from that confusion, and generated more confusion yet.

Remark 19 In analyzing Eqs. \eqref{59} and \eqref{60} we see that in the process of taking the covariant derivative the action of the basis vector fields $\partial_\alpha$ on a vector field $V$ and on a covector field $C$ are

$$\partial_\mu V = \partial_\mu (V^\alpha \partial_\alpha) = \frac{\partial V^\alpha}{\partial x^\mu} \partial_\alpha, \quad (59)$$

$$\partial_\mu C = \partial_\mu (C_\alpha \theta^\alpha) = \frac{\partial C_\alpha}{\partial x^\mu} \theta^\alpha, \quad (60)$$

from where we infer the rule\cite{13} (to be used with care)

$$\partial_\mu (\partial_\nu) = 0,$$

$$\partial_\mu (\theta^\alpha) = 0. \quad (61)$$

Next we recall that our given connection has been assumed to be not teleparallel, a statement that implies also

$$\nabla_{\mu} \theta^a \neq 0, \quad a, b = 0, 1, 2, 3. \quad (62)$$

Take notice also that in general the $q^a_\mu$ cannot be all null (otherwise the $e_\mu = q^\mu_0 e_\mu$ would be null). Also in the more general case, $\partial_\mu q^b_\nu \neq 0$. Moreover, $\theta^a = q^\alpha_\mu \theta^a = q^\alpha_\mu d x^\alpha$, and in general $q^a_\alpha \neq 0$ and $\partial_\nu q^a_\alpha \neq 0$. It is now a well-known freshman exercise presented in many good textbooks to verify that the following identity holds:

$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^a_{\mu b} q^b_\nu = 0. \quad (63)$$

Indeed, from Eq.\eqref{62} we have,

$$\nabla_{\mu} \theta^a = \omega^a_{\mu c} \theta^c = q^\nu_\mu \nabla^-_{\nu} \theta^a = q^\nu_\mu \omega^a_{\mu \nu} \theta^c \neq 0. \quad (64)$$

Then, since in general $\nabla_{\mu} \theta^a \neq 0$ and $q^a_\mu \neq 0$, we must have in general, $\omega^a_{\mu \nu} \theta^c \neq 0$ and thus

$$\nabla^-_{\nu} \theta^a \neq 0. \quad (65)$$

\footnote{An explicit warning concerning this observation can be found at page 210 of \cite{39}.}

\footnote{These rules are crucial for the writing of the covariant derivative operator on the Clifford bundles $\mathcal{C}l(TM)$ and $\mathcal{C}l(T^*M)$. See Eq.\eqref{129}.}
Now, using Eq. (57) we can write

$$\nabla_{\partial_{\mu}} \theta^a = \nabla_{\partial_{\mu}} (q^a_\alpha dx^\alpha) = (\nabla_{\partial_{\mu}} \theta^a)_\alpha dx^\alpha = (\nabla_{\partial_{\mu}} q^a_\nu) dx^\nu = (\partial_{\mu} q^a_\nu - \Gamma^a_{\mu\beta} q^b_\beta) dx^\nu$$  \hspace{1cm} (66)

Then, from Eq. (65) and Eq. (66) it follows that (in general)

$$\nabla_{\partial_{\mu}} q^a_\nu \neq 0.$$  \hspace{1cm} (67)

Having proved that crucial result for our purposes, recall that (see Eq. (47))

$$\nabla_{\partial_{\nu}} \theta^a = -\omega^a_{\mu b} \theta^b = -q^b_\nu \omega^a_{\mu b} \theta^\mu.$$  \hspace{1cm} (68)

Then from Eq. (66) and Eq. (68) we get the proof of Eq. (63), i.e.,

$$\partial_{\mu} q^a_\nu - q^a_\beta \Gamma^\beta_{\mu\nu} = \partial_{\mu} q^a_\nu - \Gamma^a_{\mu b} q^b_\nu = -\omega^a_{\mu b} q^b_\nu \neq 0.$$  \hspace{1cm} (69)

## 5 Comments on the ‘Tetrad Postulate’

At page 438 of [12] the following equation (that the author, says to be known as the tetrad postulate)

$$D_\mu q^a_\nu = \partial_{\mu} q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^a_{\mu b} q^b_\nu = 0,$$ \hspace{1cm} (24E)

is said to be the basis for the ‘demonstration’ of Evans Lemma. In truth, that ‘demonstration’ needs also his Eq.(41E), which as we shall see is completely wrong. Of course, several other authors (see below) also call an equation like Eq.(24E) ‘tetrad postulate’

So, we need to investigate if Eq.(24E) has any meaning within the theory of covariant derivatives. This is absolutely necessary if someone is going to use that equation as a basis for applications, in particular, in applications to physical theories.

(a) To start, we immediately see that the statement contained in the first member of Eq.(24E) cannot be identified with the statement \(\nabla_{\partial_{\mu}} q^a_\nu = 0\). Indeed, to make such an identification is simply wrong, because we just showed that in general, \(\nabla_{\partial_{\mu}} q^a_\nu \neq 0\)

(b) The freshman identity (Eq.(63)) is simply a compatibility condition. It results from the condition introduced in (vi) in the previous section. There is nothing of mysterious in it. However, for reasons that we are going to explain below, such a compatibility condition generated a lot of misunderstandings. To see this we need to do some other (almost trivial) calculations.

(c) So, let us next calculate \(\nabla_{\partial_{\mu}} \theta^a\) in two different ways, as we did for \(\nabla_{\partial_{\mu}} q^a_\nu\). Recalling that

$$\partial_{\nu} = q^a_\nu e^a,$$  \hspace{1cm} (70)
we have:

\[
\nabla^+_{\partial_\mu} \partial_\nu = \nabla^+_{\partial_\mu} (q^a_\mu e_a) \\
= \partial_\mu (q^a_\nu) e_a + q^a_\mu (\nabla^+_{\partial_\nu} e_a) \\
= (\partial_\mu q^a_\nu + q^b_\nu \omega^a_{\mu b}) e_a \\
= (\nabla^+_{\partial_\mu} q^a_\nu) e_a
\] (71)

Now, writing

\[
\nabla^+_{\partial_\mu} \partial_\nu = \Gamma^\rho_{\mu\nu} \partial_\rho = \Gamma^\rho_{\mu\nu} q^a_\rho e_a,
\] (72)

and from Eqs. (71) and (72) we get again the freshman identity,

\[
\partial_\mu q^a_\nu + q^b_\nu \omega^a_{\mu b} - \Gamma^\rho_{\mu\nu} q^a_\rho = 0
\] (73)

Remark 20 It is very important before proceeding to keep in mind that \( \nabla^-_{\partial_\mu} q^a_\nu \) given by Eq. (66) and \( \nabla^+_{\partial_\mu} q^a_\nu \) given by Eq. (71) are different functions, i.e., in general,

\[
\nabla^+_{\partial_\mu} q^a_\nu \neq \nabla^-_{\partial_\mu} q^a_\nu
\] (74)

Remark 21 This shows that the statement contained in the first member of Eq. (24E) cannot be identified with the statement \( \nabla^+_{\partial_\mu} q^a_\nu = 0 \). Indeed, to make such an identification is simply wrong, since in general \( \nabla^+_{\partial_\mu} q^a_\nu \neq 0 \).

(d) As our last exercise in this section we now calculate \( \nabla_{\partial_\mu} P_\nu \), where \( P_\nu \in \text{sec}(TU \otimes T^*U) \). Objects of this kind are, as we already observed, often called vector valued differential forms. First taking into account the structures of the associated vector bundles recalled in Eq. (37), we expand \( P_\nu \) in the “hybrid” basis \( \{e_a \otimes dx^\nu\} \) of \( TU \otimes T^*U \), i.e., we write

\[
P_\nu = P^a_\nu e_a \otimes dx^\nu.
\] (75)

Then by definition, we have

\[
\nabla_{\partial_\mu} P_\nu = \nabla_{\partial_\mu} (P^a_\nu e_a \otimes dx^\nu) \\
= (\nabla_{\partial_\mu} P^a_\nu) e_a \otimes dx^\nu \\
= (\nabla_{\partial_\mu} P^a_\nu) e_a \otimes dx^\nu
\] (76)

A standard computation yields

\[
\nabla_{\partial_\mu} P^a_\nu = \partial_\mu P^a_\nu - \Gamma^\beta_{\mu\nu} P^a_\beta + \omega^a_{\mu b} P^b_\nu
\] (77)

and in general, \( \nabla_{\partial_\mu} P^a_\nu \neq 0 \).

We have the
Proposition 22 Let

\[ Q = q^a_b e_a \otimes dx^b \in \text{sec} \, T \otimes T^* U, \quad (79) \]

where the functions \( q^a_b \) are the ones appearing in Eqs. (10) and (70). Then

\[ \nabla Q = 0, \quad (80) \]

\[ \nabla^\mu a_{\nu} = (\nabla \partial^\mu Q)_{\nu} = \partial^\mu q^a_{\nu} - \Gamma^\beta_{\mu\nu} q^a_{\beta} + \omega^a_{\mu b} q^a_{\nu} = 0. \quad (81) \]

Proof. Since the functions \( q^a_{\nu} \) are the ones appearing in Eqs. (10) and (70) then satisfy the true ‘freshman’ identity (Eq.(63)). Then from that equation and Eq.(78) the proof follows. ■

Remark 23 The tensor \( Q \) given by Eq. (79) is for each \( e \in U \subset M \), \( Q|_e \) simply the identity (endomorphism) mapping \( T_e U \rightarrow T_e U \), as any reader can easily verify. We have more to say about \( Q \) in section 6.

5.1 Some Misunderstandings

We just calculated the covariant derivatives in the direction of the vector field \( \partial^\nu \) of the following tensor fields: the 1-forms \( \theta^a = q^a_0 dx^b \), the vector fields \( \partial^a = q^a_0 e_a \) and \( Q \), the identity tensor in \( TU \). The components of those objects in the basis above specified have been denoted respectively by \( \nabla^-_{\mu} q^a_{\nu}, \nabla^+_{\mu} q^a_{\nu} \) and \( \nabla_{\mu} q^a_{\nu} \) and we arrived at the conclusion that in general,

\[ \nabla^-_{\mu} q^a_{\nu} \neq 0, \]
\[ \nabla^+_{\mu} q^a_{\nu} \neq 0, \]
\[ \nabla^-_{\mu} q^a_{\nu} \neq \nabla^+_{\mu} q^a_{\nu}, \]
\[ \nabla_{\mu} q^a_{\nu} = 0. \]

With this in mind we can now identify from where the ambiguities referred in the introduction come from. Almost all physical authors use instead of the three distinct symbols \( \nabla^-_{\mu}, \nabla^+_{\mu} \) and \( \nabla_{\mu} \) which represent, as we emphasized above three different connections, the same symbol, say \( D_{\mu} \), for all of them. This clearly generated the absurd conclusions that we have \( D_{\mu} q^a_{\nu} = 0 \) and \( D_{\mu} q^a_{\nu} \neq 0. \)

The reader at this point may be thinking: What you explained until now is so trivial that nobody will make such an stupidity of confusing symbols. Are you sure, dear reader? Let us see.

5.1.1 Misunderstanding 1

As we just observed the majority of physical textbooks and physical articles, as, e.g., [5, 23, 24, 27, 30, 41, 47, 56] give first rules for the covariant derivative (denoted in general by \( \nabla \)) for the components of tensors in a given basis, and when introducing tetrads first state what they mean by using Eqs. (10) and
But then, immediately after that they say that the ‘covariant derivative’ of the tetrads $q^a_\nu$ must be calculated by

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu - \Gamma^a_{\mu\beta} q^\beta_\nu + \omega^a_{\mu b} q^b_\nu. \quad (82)$$

without specifying that $D_\mu q^a_\nu$ must means the $(\nabla_\mu Q)^a_\nu$.

After that they stated that we need a tetrad postulate\(^\text{20}\), which is introduced as the statement

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu - \Gamma^a_{\mu\beta} q^\beta_\nu + \omega^a_{\mu b} q^b_\nu = 0. \quad (83)$$

Of course, this statement has meaning only if the $q^a_\nu$ are the components of the tensor $Q$ (Eq.(79)), which is, as we already recalled the identity endomorphism on $TU$. However, the statement appears in many books and articles, e.g., in [5, 24] without the crucial information and this generates inconsistencies. Let us show two of them, using the physicists convention that all three operators $\nabla_\mu, \partial_\mu$, and $\nabla_\mu$ are to be represented by a unique symbol, that we choose as being same symbol $D_\mu$. In\(^\text{21}\) the authors correctly calculate $\nabla_\mu, \partial_\mu$ (called there $D_\mu, \partial_\mu$) and correctly obtained the freshman identity (their Eq.(5.32). After that they state in their comment 5.1.: ‘Eq.(5.32) is frequently written as the vanishing of a ‘total covariant derivative of the tetrad’. Then, they print the equivalent of Eq.(83). This clearly means that they did not grasp the meaning of the different symbols necessary to be used in an unambiguous presentation the theory of connections, and they are not alone. Statements of the same nature appears also, e.g., in [5, 23, 24, 27, 30, 41, 47, 56] and also in \([12, 13, 14, 15, 16, 17]\).

Specifically, we recall, e.g. that in [23], authors asserts that the true identity given by Eq.(63) is (unfortunately) written as $D_\mu q^a_\nu = 0$ as in Eq.(83) (or Eq.(24E)) and confused with the metric condition $D_\mu g^\alpha_\mu = 0$.

This old confusion of symbols, it seems, propagated also to papers and books on supersymmetry, superfields and supergravity, as it will be clear for any reader that has followed our discussion above and give a look, e.g., at pages 141-144 of [54]. That author defines two different covariant derivatives for the ‘tetrads’ $q^a_\nu$ without realizing that in truth he was calculating the covariant derivative of different objects, living in different vector bundles associated to $P_{SO_{1,3}}(M)$. The fact is that unfortunately many authors use mathematical objects in their papers without to know exactly their real mathematical nature. This generates many misunderstandings that propagate in the literature. For example, in [50], where equations for the gravitational field are derived from the Palatini method, which allows both the tetrads and the connections to vary independently in the variation of the action, there are two ‘tetrad postulates’. Both are expressions of the freshman identity. What this author and many others forget to say is that the postulate’ (here a better name would be, a constrain) is necessary to assure

\(^{20}\)Since in general no convincing explanation is given for Eq.(83), it should be better to call it the naive tetrad postulate.

\(^{21}\)Take care that some authors also use $D_\mu$ as meaning $D_\partial_\mu$. 

21
that a connection in $P_{SO_1^+(M)}$ determines a metric compatible connection in $F(M)$, as we note in (vi) of section 4.

We now show the ‘tetrad postulate’ generates even much more misunderstandings than those already described above.

5.1.2 Misunderstanding 2

Observe that for a covector field $C$ we have from Eq.(57) if the symbol $D_{\mu}$ (without any comment) is used in place of the correct symbol $\nabla_{\mu}$ that

$$D_{\mu}C = D_{\mu}(C_{\nu}) = \left(D_{\mu}C\right)_{\nu} \theta^\nu$$

$$\equiv (D_{\mu}C_{\alpha}) \theta^\alpha$$

$$= (\partial_{\mu}C_{\nu} - C_{\beta}\Gamma^\beta_{\mu\nu}) \theta^\nu$$

$$= D_{\mu}(C_{\nu}) \theta^\nu$$

$$\equiv (D_{\mu}C_{\alpha}) \theta^\alpha$$

$$= \left(\partial_{\mu}C_{\alpha} - C_{\beta}\omega^b_{\mu\alpha}\right) \theta^a$$

(84)

(85)

(86)

(87)

Now, since $C = C_{\nu}\theta^\nu = C_a\theta^a$, we have that $C_{\nu} = q^a_{\nu}C_a$ and we can write

$$D_{\mu}C_{\alpha} = \partial_{\mu}(q^a_{\nu}C_a) - C_{\beta}\Gamma^\beta_{\mu\nu}$$

$$= (\partial_{\mu}q^a_{\nu})C_a + q^a_{\nu}(\partial_{\mu}C_a) - C_{\beta}\Gamma^\beta_{\mu\nu}$$

$$= q^a_{\nu}(\partial_{\mu}C_a - \omega^b_{\mu\alpha}C_b) + C_a(\partial_{\mu}q^a_{\nu} - \Gamma^\beta_{\mu\nu}q^a_{\beta} + \omega^a_{\mu\beta}q^b_{\nu})$$

$$= q^a_{\nu}(D_{\mu}C_a),$$

(88)

where in going to the last line we used the ‘freshman identity’, i.e., Eq.(63).

Now, if someone confounds the meaning of the symbols $D_{\mu}C_{\alpha}$ with the covariant derivative of a vector field, taking into account that $C_{\alpha} = q^a_{\alpha}C_a$ he will use Eq.(88) to write the misleading equation

$$D_{\mu}(q^a_{\nu}C_a) = q^a_{\nu}(D_{\mu}C_a).$$

(89)

and someone must be tempted to think that the ‘tetrad postulate’, i.e., the statement that $D_{\mu}q^a_{\nu} = 0$ is necessary, for in that case he could apply the Leibniz rule to the first member of Eq.(89), i.e., he could write

$$D_{\mu}(q^a_{\nu}C_a) = (D_{\mu}q^a_{\nu})C_a + q^a_{\nu}(D_{\mu}C_a) = q^a_{\nu}(D_{\mu}C_a).$$

(90)

The fact is that:

(i) Whereas the symbols $D_{\mu}C_{\alpha}$ (meaning of course $\nabla_{\mu}C_{\alpha}$) are well defined, the symbol $D_{\mu}(q^a_{\nu}C_a)$ has not the meaning of being equal to $D_{\mu}C_{\alpha}$.

(ii) It is not licit to apply the Leibniz rule for the first member of Eq.(89), i.e., he could not write

$$D_{\mu}(q^a_{\nu}C_a) = (D_{\mu}q^a_{\nu})C_a + q^a_{\nu}(D_{\mu}C_a) = q^a_{\nu}(D_{\mu}C_a).$$

The reason is the label $a$ in each of the factors have different ontology. In $q^a_{\nu}$, it is the $\nu$ component of the tetrad $\theta^a$, i.e., $\theta^a = q^a_{\nu}dx^\nu$. In the second factor $a$
labels the components of the covector field $C$ in the tetrad basis, i.e., $C = C_a \theta^a$.

In that way the term $q^a C_a$ is not the contraction of a vector with a covector field and as such to apply the Leibniz rule to it, writing Eq.(90) is a nonsequitur. Some authors, like in [24] say that $D_\mu q_a^\nu C_a = q_a^\nu (D_\mu C_a)$ and say that this is a property of a spin connection. The fact is that $D$ must be understood in any case as the appropriate connection acting on a well specified vector bundle, as discussed in the previous section and satisfying the rules given in (vi) there.

5.1.3 Misunderstanding 3.

But what is a spin connection, an object said to be used, e.g., by [24]? Spin connection is the name that mathematicians give to a connection defined on the covering bundle $SE(M)$ (called here spinor bundle structure) of $P_{SO_{1,3}}(M)$, when $SE(M)$ exists, which necessitates that $(M,g)$ is a spin manifold. This imposes constraints in the topology of the manifold $M$. For the particular case of a manifold $M$ which is part of a spacetime structure, the constraint on the topology of $M$ is given by the famous Geroch [21] theorem, which says that $P_{SO_{1,3}}(M)$ must be trivial, i.e., has a global section. Thus, in that case, global tetrad fields (and of course, cotetrad fields) exist. Also, the wording spin connection can be used as meaning the covariant derivatives acting on appropriate spinor bundles. A spinor bundle $S(M)$ is an associated vector bundle to the principal bundle $SE(M)$. Sections of a given $S(M)$ are called spinor fields. The spin connection coefficients are related with the objects called $\omega^b_{\mu a}$ introduced, e.g., in Eq.(47) in a very natural way, but this will not be discussed here, because these results will be not needed for what follows. Interested readers, may, e.g., consult [33].

Now, let us present one more serious misunderstanding in the next subsection.

5.1.4 Misunderstanding 4

Of course, we can introduce in $M$ many different connections [7, 29, 42]. In particular, if $M$ is a spin manifold [33], which as we just explained above means that $M$ has a global tetrad $\{e_a\}$, $e_a \in \text{sec} T^*M$, $a = 0, 1, 2, 3$ and has also a global cotetrad field $\{\theta^a\}$, $\theta^a \in \text{sec} T^*M$, $a = 0, 1, 2, 3$ we can introduce a teleparallel connection—call it $D_-$—such that

$$D_\epsilon^a \theta^a = 0.$$  \hspace{1cm} (91)

From Eq.(91) we get immediately after multiplying by $q^b_\mu$ and summing in the index $b$ that

$$q^b_\mu D^-_{\epsilon_a} (q^a_\nu dx^\nu) = D^-_\theta (q^a_\nu dx^\nu) = (D^-_\mu q^a_\nu)dx^\nu = 0.$$  \hspace{1cm} (92)
Then, in this case we must have

\[ D^-_\mu q^a_\nu = \left( D^-_\mu \theta^a_\nu \right)_\nu = 0 \]  

(93)

The important point here is that for the teleparallel connection, as it is well-known the Riemann curvature tensor is null, but the torsion tensor is not null. Indeed, given vector fields \( X, Y \in \text{sec} TM \), the torsion operator is given by (see, e.g., [7])

\[ \tau : (X,Y) \rightarrow \tau(X,Y) = D^\pm_X Y - D^\pm_Y X - [X,Y]. \]  

(94)

First choose \( X = e_a, Y = e_b \), with \([e_a, e_b] = \epsilon_{ab} e_d \). Then since the \( \epsilon_{ab} \) are not all null, we have

\[ \tau(e_a, e_b) = T^d_{ab} e_d = \epsilon_{ab} e_d, \]  

(95)

and the components \( T^d_{ab} \) of the torsion tensor are not all null. Now, if we choose \( X = \partial_\mu \) and \( Y = \partial_\nu \), then since \([\partial_\mu, \partial_\nu] = 0\), we can write

\[ \tau(\partial_\mu, \partial_\nu) = T^a_{\mu\nu} e_a = D^\pm_\mu \partial_\nu e_a - D^\pm_\nu \partial_\mu e_a = (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}) \partial^\rho e_a, \]  

(96)

\[ = (D^\pm_\mu \partial_\nu)^a e_a - (D^\pm_\nu \partial_\mu)^a e_a = (D^\pm_\mu q^a_\nu)^a e_a - (D^\pm_\nu q^a_\mu)^a e_a, \]  

(97)

where

\[ D^\pm_\mu q^a_\nu = (D^\pm_\mu \theta^a_\nu)^a, D^\pm_\nu q^a_\mu = (D^\pm_\nu \theta^a_\mu)^a. \]  

(98)

and \( (D^\pm_\mu q^a_\nu) \) and \( (D^\pm_\nu q^a_\mu) \) are in general non null. Indeed

\[ T^a_{\mu\nu} = D^\pm_\mu q^a_\nu - D^\pm_\nu q^a_\mu, \]  

(99)

and the \( T^a_{\mu\nu} \neq 0 \), as just proved. Now, e.g., in [24], page 275, where the all the three distinct covariant derivatives \( \nabla^\pm_{\partial_\mu}, \nabla^\pm_{\partial_\nu} \) and \( \nabla_{\partial_\mu} \) introduced above are represented by the same symbol \( D_\mu \), we read: “The nonminimality of a nonminimal spin connection is conveniently measured by the so-called ‘torsion’ \( T^a_{\mu\nu} \), defined by

\[ T^a_{\mu\nu} = D_\mu q^a_\nu - D_\nu q^a_\mu. \]  

((12.1.7 gsw))

Now, application of Eq.(12.1.7 gsw) to calculate the components of torsion tensor, for the case of a teleparallel connection, instead of correct Eq.(99) may generate a big confusion if as in [24], authors adopt the tetrad postulate with the meaning given in Eq.(83). Indeed, observe that if the ‘tetrad postulate’ is adopted then, the torsion tensor results null for a teleparallel connection \( D_\mu \), and this is false, as we just showed. The use of Eq.(12.1.7 gsw) may generate confusion also in the case of a Levi-Civita connection a shown in Appendix A, if we compute compute the components of the torsion tensor for the case of the structure \((S^2, g, \nabla)\) using Eq.(83) and Eq.(12.1.7 gsw).
Remark 24 It is very important to have in mind that author of \cite{12} identified first the symbols $q^a_\nu$ as the components of $\theta^a$ in a coordinate basis $\{dx^e\}$ (line 55 in Table 1 of \cite{13} and as the components of the coordinate basis vectors $e_\nu = \partial_\nu$ in the tetrad basis $\theta^a$ (line 53 in Table 1 of \cite{12}). He never identified the $q^a_\nu$, explicitly as the components of the tensor $Q$ given by our Eq.\,(75), since such a tensor did not appear in his text. So, he can never claim that his ‘tetrad postulate’ has any meaning at all, but eventually he will do that after reading our paper. With the choices given above, he can tell you that he was just thinking about the tensor $Q$. So, in Section 7 we shall identify a crucial mathematical error in \cite{12} that invalidates completely his supposed unified theory.

6 What is $Q$?

In the theory of connections\cite{29, 46, 10} we introduce an affine connection as a connection $\omega$ on the principal fiber bundle $F(M)$, with canonical projection $\pi : F(M) \to M$. As already recalled $\hat{\omega}$ is a 1-form on $F(M)$ with values in the Lie algebra $\mathfrak{gl}(4, \mathbb{R})$ of the general linear group $\mathfrak{gl}(4, \mathbb{R})$. For each $p \in F(M)$ the tangent space $T_p F(M)$ has a canonical decomposition $T_p F(M) = H_p F(M) \oplus V_p F(M)$. Recall that each $p = (e,\{ e_a\}_{|_e})$, where $\{ e_a\}_{|_e}$ is a frame for $e \in M$. The derivative mapping $\pi_* : \pi^* F(M) \to TM$, i.e., $\pi_* : TF(M) \to TM$. To continue, we need to know that if $T_p F(M) \ni v = v_h + v_r$, $v_h \in H_p F(M)$, $v_r \in V_p F(M)$, then $\hat{\omega}(v_h) = 0$. Let $V$ be some vector space and consider objects $\phi \in \text{sec} \Lambda^r F(M) \otimes V$ called $r$-forms on $F(M)$ with values in $V$. The exterior covariant derivative $D^\omega$ of $\phi$ is defined by

$$D^\omega \phi(v_1, ..., v_r) = d\phi(v_{h1}, ..., v_{hr}),$$

where $d$ is the ordinary exterior derivative operator.

Take $V = \mathfrak{gl}(4, \mathbb{R})$ with basis $\{ g^i_j \}$, $i,j = 1,2,3,4$. Then $\hat{\omega} = \omega^i_j \otimes g^i_j$ and the curvature (2-form) of the connection is defined by

$$\hat{\Omega} = D^\omega \hat{\omega}.\quad (101)$$

Let $M_p : T_e(M) \to \mathbb{R}^4$ be a mapping that sends any vector $V \in T_e(M)$ into its components with respect to the basis $\{ e_a\}_{|_e}$. Then,

$$M_p(V) = (\theta^0(V), \theta^1(V), \theta^2(V), \theta^3(V)).\quad (102)$$

Now, take $V = \mathbb{R}^4$, with canonical basis $\{ E_a \}$ consider the object $\hat{\theta} \in \text{sec} \Lambda^1 T^* F(M) \otimes \mathbb{R}^4$ such that

$$\hat{\theta}(v) = M_p \pi_* (v), v \in T_p F(M)\quad (103)$$

is called the soldering form of the manifold. Unfortunately some authors also call the soldering form, by the name of tetrad, which only serves the purpose of increasing even more the confusion involving the issue under analysis.
The torsion of the connection $\hat{\omega}$ is defined as the 2-form
\[ \hat{\Theta} = D^\omega \hat{\theta}, \] (104)
We can show that
\[ \hat{\Theta} = d \hat{\theta} + [\hat{\omega}, \hat{\theta}], \] (105)
where $[\ , \ ]$ denotes the commutator in the Lie algebra $\mathfrak{a}^4$ of the affine group $\mathbb{A}^4 = GL(4, \mathbb{R}) \rtimes \mathbb{R}^4$. A basis of $\mathfrak{a}^4$ is taken as $\{ g^i_j, E_a \}$.

Let be $U \subset M$ and $\varsigma : U \to \varsigma(U) \subset F(M)$. We are interested in the pullbacks $\theta = \varsigma^* \hat{\theta}$ and $\omega = \varsigma^* \hat{\omega}$, once we give a local trivialization of the respective bundles. Now, $\omega$ has components $\omega^a_b \in \sec T^*U$ which are the connection 1-forms that we already introduced and used above. On the other hand we can show that [29], chart $\langle x^a \rangle$ covering $U$
\[ \theta = \varsigma^* \hat{\theta} = \theta^a \otimes E_a = q^a_d x^d \otimes E_a \in \bigwedge^1 T^* M \otimes \mathbb{R}^4. \] (106)

Now, $\theta = \varsigma^* \hat{\theta}$, i.e., it is the pullback of the soldering form under a local trivialization of the bundle $T^*F(M) \otimes \mathbb{R}^4$. $\theta$ is called by some physical authors "tetrad". We think that use of this name is an unfortunate one.

We recall that if we calculate the pullback of the torsion tensor $\hat{\Theta}$ we get the tensor $\Theta$ in the basis manifold $M$. Explicitly we have (taking into account Eq.(105) and the fact that the operator $d$ commutes with pullbacks) that
\[ \Theta = (d \theta^a + \omega^a_b \wedge \theta^b) \otimes E_a. \] (107)

The objects $T^a = d \theta^a + \omega^a_b \wedge \theta^b$, $T^a \in \bigwedge^2 T^* M$ are called the torsion 2-forms.

Now, given an orthonormal basis $\{ e_a \}$ for $TU$ any vector field $v = v^a e_a \in \sec TU$ we have
\[ \theta(v) = v^a E_a \] (108)
On the other hand recalling the definition of $Q = e_a \otimes \theta^a \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^* M$, we have
\[ Q(v) = v^a e_a \] (109)
and we see that $\theta$ is a kind of representation of $Q$. On the other hand the exterior covariant derivative, denoted $d^\omega$ of the vector valued 1-form $Q$ is (see, e.g., [22]) the torsion tensor of the connection in the basis manifold.
\[ T = d^\omega Q := e_a \otimes (d \theta^a + \omega^a_b \wedge \theta^b) \] (110)
and we see that $\Theta$ is a representation of $T$.

It is very important to keep in mind that for a general Riemann-Cartan manifold $T = d^\omega Q \neq 0$. However, if $\nabla$ is the covariant derivative operator acting on the sections of the tensor bundle, then as we showed above, we have always
\[ \nabla Q = (\nabla \partial_a Q) \otimes dx^a = 0 \] (111)

\[ ^{22}\text{The symbol } \rtimes \text{ means semi-direct product.} \]
7 Comments on the ‘Evans Lemma’

At page 440 of [12] no distinction is made between the connections \( \nabla_{\partial_{\mu}}, \nabla_{\partial^{\mu}} \) and \( \nabla_{\partial_{\mu}} \). As, e.g., in [3], all these three different connections are represented by \( D_{\mu} \) and the naive tetrad postulate is introduced by the equation \( D_{\mu} q^a_\mu = 0 \), which as just showed is misleading if its precise meaning is not well specified, which is just what happen in [12]. Then, author of [12] states that the ‘Evans Lemma’ is a direct consequence of the (naive) ‘tetrad postulate’.

We shall now show that author of [12] did a fatal flaw in the derivation of his ‘Evans lemma’. Indeed, from a true equation, namely, Eq.(40E) that should more correctly be written

\[ \nabla^{\mu}_{\partial_{\nu} V^\mu} = \left( \frac{\partial V^\mu}{\partial x^\mu} + V^\rho \Gamma^\mu_{\rho \nu} \right), \]  
(40E)

where the symbol \( \nabla^{\mu}_{\partial_{\nu}} \) has the precise meaning discussed above, Remark [17] he inferred Eq.(41E), i.e., he wrote 23

\[ \nabla^{\mu}_{\partial_{\nu}} \partial^\mu = \partial_{\mu} \partial^\nu + \Gamma^\mu_{\mu \lambda} \partial^\lambda \]  
(41E)

This equation has no mathematical meaning at all. Indeed, if the symbol \( \nabla^{\mu}_{\partial_{\nu}} \) is to have the precise mathematical meaning disclosed in Section 4, then it can only be applied (with care) to components of vector fields (as, e.g., in Eq. (40E)), and not to vector fields as it is the explicit case in Eq.(41E). If \( \nabla^{\mu}_{\partial_{\nu}} \) is to be understood as really having the meaning of \( \nabla^{\mu}_{\partial_{\nu}} \) then Eq. (41E) is incorrect, because the correct equation in that case is, as recalled in Eq.(??) must be:

\[ \nabla^{\mu}_{\partial_{\nu}} \partial^\mu = \Gamma^\mu_{\mu \lambda} \partial^\lambda. \]  

(112)

Now, it is from the completely wrong Eq.(41E), that author of (12) infers after a nonsense calculation (that we are not going to show here) that the tetrad functions \( q^a_\mu : \varphi(U) \to \mathbb{R} \) must satisfy his Eq.(49E), namely the ‘Evans Lemma’

\[ \Box q^a_\mu = R q^a_\mu. \]  

(49E)

where the symbol \( \Box \) in [12] is defined as meaning \( \Box = \partial_{\mu} \partial^\mu \) and called the D’Alembertian operator 24 and it said that \( R \) is the usual curvature scalar.

One more comment is in order. After arriving (illicitly) at Eq.(49E), author of [12] assumes the validity of Einstein’s gravitational equations 25 and write his ‘Evans field equations’, which he claims to give an unified theory of all fields...

23That the symbols \( \partial_{\mu} \) and \( \partial^\mu \) used by Evans are to be interpreted as meaning the basis vector fields \( \partial_{\mu} \) and \( \partial^\mu \), as its clear from Evans’ Eq.(25E), one of the equations with correct mathematical meaning in [12].

24Of course, in any case it is not, as well known, the covariant D’Alembertian operator on a general Riemann-Cartan spacetime. Indeed, the covariant D’Alembertian operator is given in Eq. (137a).

25Einstein equations, by the way, are empirical equations and have nothing to do with the foundations of differential geometry.
That equations are giving by Eq.(2E) and are
\[(\square + \kappa T)q^a = 0,\] (2E)
where \(\kappa\) is the gravitational constant and \(T\) is the trace of the energy-momentum tensor. We claim that Eq.(2E) is also \textit{wrong}. Since an equation somewhat similar to Eq.(2E) appears also\(^{26}\), e.g., in \cite{28} it is necessary to complete this paper by finding the correct equations satisfied by the functions \(\{q^a\}_a\), at least for the case of a Lorentzian spacetime. This will be done below, in two different ways. First we use the wave equation satisfied by the tensor \(Q\). Next we find the correct equations satisfied by the \textit{tetrad fields} \(\theta^a\) representing a gravitational field in General Relativity, something that was missing in the papers quoted above. Finally in Section 11 we describe the Lagrangian formalism for the tetrad fields and derive the field equations from a variational principle. In order to achieve that last goals we shall need to introduced some mathematics of the theory of Clifford bundles as developed, e.g., in \cite{33, 43}. See also \cite{44} for details of the Clifford calculus and some of the ‘tricks of the trade’.

\textbf{Remark 25} Before leaving this section, we remark that since we already showed that the identity tensor \(Q = q^a e_a \otimes dx^a \in \text{sec}TU \otimes T^*U\) (Eq.(79) is such that \(\nabla \partial_Q = 0\). It follows immediately that in any Riemann-Cartan spacetime
\[g^{\mu\nu} \nabla_{\partial_{\mu}} \nabla_{\partial_{\nu}} Q = 0\] (113)
This can be called a wave equation for \(Q = q^a e_a \otimes dx^a\), but it is indeed a very trivial result. It cannot have any fundamental significance. Indeed, all encoded differential geometry information is already encoded in the simply equation \(\nabla Q = 0\), which as we already know is an intrinsic writing of the freshman identity (Eq.(63)) derived above.

\section{Clifford Bundles \(\mathcal{C}\ell(T^*M)\) and \(\mathcal{C}\ell(TM)\)}

In this section, we restrict ourselves, for simplicity to the case where \((M, g, \nabla, \tau, \uparrow)\) refers to a \textit{Lorentzian} spacetime as introduced in Section 2\(^{27}\). This means that \(\nabla\) is the Levi-Civita connection of \(g\), i.e., \(\nabla g = 0\), and \(T(\nabla) = 0\), but in general \(R(\nabla) \neq 0\). Recall that \(R\) and \(T\) denote respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms \(\mathcal{C}\ell(T^*M)\) is the bundle of algebras \(\mathcal{C}\ell(T^*M) = \cup_{\tau \in M} \mathcal{C}\ell(T^*_\tau M)\), where \(\forall \tau \in M, \mathcal{C}\ell(T^*_\tau M) = R_{1,3}\), the so-called \textit{spacetime algebra} (see, e.g., \cite{44}). Locally as a linear space over the real field \(\mathbb{R}\), \(\mathcal{C}\ell(T^*_\tau M)\) is isomorphic to the Cartan algebra \(\bigwedge(T^*_\tau M)\) of

\(^{26}\)Reference \cite{25} has been criticized in \cite{38}.

\(^{27}\)The general case of a Riemann-Cartan spacetime will be discussed elsewhere.

\(^{28}\)Of course, in what follows the connection \(\nabla\) has the precise meaning presented in previous sections, but for simplicity of notation, we shall use only the symbol \(\nabla\), instead of the more precise symbols \(\nabla^+, \nabla^-, \nabla\).
the cotangent space and \( \bigwedge^k T^*_e M = \bigoplus_{k=0}^4 \bigwedge^k T^*_e M \), where \( \bigwedge^k T^*_e M \) is the \((k)\)-dimensional space of \(k\)-forms. The Cartan bundle \( \bigwedge T^*M = \cup_{e \in M} \bigwedge T^*_e M \) can then be thought of as “imbedded” in \( \mathcal{C}l(T^*M) \). In this way sections of \( \mathcal{C}l(T^*M) \) can be represented as a sum of nonhomogeneous differential forms.

Let \( \{e_a\} \in \text{sec} T^* M \), \((a = 0, 1, 2, 3)\) be an orthonormal basis \( g(e_a, e_b) = \eta_{ab} = \text{diag}(1, -1, -1, -1) \) and let \( \{\vartheta^a\} \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C}l(T^*M) \) be the dual basis. Moreover, we denote as in Section 2 by \( g \) the metric in the cotangent bundle.

An analogous construction can be done for the tangent space. The corresponding Clifford bundle is denoted \( \mathcal{C}l(TM) \) and their sections are called multivector fields. All formulas presented below for \( \mathcal{C}l(T^*M) \) have corresponding ones in \( \mathcal{C}l(TM) \).

Remark 26 Let \( V \) be a real \( n \)-dimensional vector space equipped with a non-degenerate metric \( G : V \times V \to \mathbb{R} \) of signature \((p, q)\), with \( n = p + q \). Let \( TV = \bigoplus_{r=0}^\infty T^r V \) be the tensor algebra. Let \( I \subset TV \) be the bilateral ideal generated by elements of the form \( a \otimes b + b \otimes a \), with \( a, b \in V \). Let \( J \subset T(V) \) be the bilateral ideals generated by elements of the form \( a \otimes b + b \otimes a - 2G(a, b) \). Then, we may define the exterior algebra of \( V \) (denoted \( \bigwedge V \)) by the quotient set \( TV/I \) and the Clifford algebra of the pair \( (V, G) \) (denoted \( \mathbb{R}_{p,q} \)) by \( \mathbb{R}_{p,q} = TV/J \).

With these definitions, the exterior product of \(a, b \in V \) is given by
\[
a \wedge b = \frac{1}{2} (a \otimes b - b \otimes a). \tag{114}
\]
and the Clifford product of \(a, b \in V \) (denoted by juxtaposition of symbols) satisfy the relation
\[
ab + ba = 2g(a, b), \tag{115}
\]
Moreover, we have
\[
ab = g(a, b) + a \wedge b \tag{116}
\]
There exists another way for defining the Clifford product and the exterior product. The algebraic structure of the alternative definition is of course, equivalent to the one given above. However, the components of \(p\)-forms in a given basis differ in the two cases. The interested reader may consult [18].

8.1 Clifford product, scalar contraction and exterior products

The fundamental Clifford product (in what follows to be denoted by juxtaposition of symbols) is generated by \( \vartheta^a \vartheta^b + \vartheta^b \vartheta^a = 2\eta^{ab} \) and if \( C \in \text{sec} \mathcal{C}l(T^*M) \) we have [43, 44]

\[\text{If the reader need more detail on the Clifford calculus of multivetors he may consult, e.g.,} [44] \text{and the list of references therein.}\]

29
\[
C = s + v_n \theta^n + \frac{1}{2!} b_{cd} \theta^c \theta^d + \frac{1}{3!} a_{abc} \theta^a \theta^b \theta^c + p \theta^5 ,
\]

where \( \theta^5 = \theta^0 \theta^1 \theta^2 \theta^3 \) is the volume element and \( s, v_n, b_{cd}, a_{abc}, p \in \sec \bigwedge^0 T^* M \subset \sec \mathcal{C}(T^* M) \).

Let \( A_r, \in \sec \bigwedge^r T^* M \mapsto \sec \mathcal{C}(T^* M), \) \( B_s \in \sec \bigwedge^s T^* M \mapsto \sec \mathcal{C}(T^* M) \).

For \( r = s = 1 \), we define the scalar product as follows:

For \( a, b \in \sec \bigwedge^1 T^* M \mapsto \sec \mathcal{C}(T^* M) \),

\[
a \cdot b = \frac{1}{2} (ab + ba) = g(a, b).
\]

We also define the exterior product (\( \forall r, s = 0, 1, 2, 3 \)) by

\[
A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}, \\
A_r \wedge B_s - (-1)^{rs} B_s \wedge A_r
\]

where \( \langle \cdot \rangle_k \) is the component in the subspace \( \bigwedge^k T^* M \) of the Clifford field. The exterior product is extended by linearity to all sections of \( \mathcal{C}(T^* M) \).

For \( A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r \), the scalar product is defined as

\[
A_r \cdot B_r = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) = \det \begin{bmatrix}
a_1 \cdot b_1 & \ldots & a_1 \cdot b_k \\
\ldots & \ldots & \ldots \\
a_k \cdot b_1 & \ldots & a_k \cdot b_k
\end{bmatrix}.
\]

We agree that if \( r = s = 0 \), the scalar product is simple the ordinary product in the real field.

Also, if \( r \neq s \), \( A_r \cdot B_s = 0 \).

For \( r \leq s \), \( A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s \) we define the left contraction by

\[
\mathcal{L} : (A_r, B_s) \mapsto A_r \mathcal{L} B_s = \sum_{i_1 < ... < i_r} e_{i_1 ... i_r}^{i_1 ... i_{r-s}} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r}) \sim b_{i_1+1} \wedge ... \wedge b_i
\]

where \( \sim \) denotes the reverse mapping (reversion)

\[
\sim : \sec \bigwedge^p T^* M \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1
\]

and extended by linearity to all sections of \( \mathcal{C}(T^* M) \). We agree that for \( \alpha, \beta \in \sec \bigwedge^0 T^* M \) the contraction is the ordinary (pointwise) product in the real field and that if \( \alpha \in \sec \bigwedge^0 T^* M, A_r \in \sec \bigwedge^r T^* M, B_s \in \sec \bigwedge^s T^* M \) then \( (\alpha A_r) \mathcal{L} B_s = A_r \mathcal{L} (\alpha B_s) \). Left contraction is extended by linearity to all pairs of elements of sections of \( \mathcal{C}(T^* M) \), i.e., for \( A, B \in \sec \mathcal{C}(T^* M) \)
\[
A_B = \sum_{r,s} \langle A \rangle^r \langle B \rangle^s, \ r \leq s.
\] (123)

It is also necessary to introduce in \( \mathcal{C}(T^*M) \) the operator of right contraction denoted by \( \cdot \). The definition is obtained from the one presenting the left contraction with the imposition that \( r \geq s \) and taking into account that now if \( A_r, B_s \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M \) then
\[
A_r \cdot B_s = (\alpha A_r) \cdot B_s.
\]
Finally, note that
\[
A_r = A_r \cdot B_r = \tilde{A_r} \cdot B_r = A_r \cdot B_r = A_r \cdot B_r = \tilde{A_r} \cdot B_r = A_r \cdot B_r = A_r \cdot B_r = \tilde{A_r} \cdot B_r
\] (124)

8.2 Some useful formulas

The main formulas used in the Clifford calculus in the main text can be obtained from the following ones, where \( a \in \sec \bigwedge^1 T^*M \) and \( A_r \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M \):
\[
aB_s = a_\cdot B_s + a \wedge B_s, B_s a = B_s \cdot a + B_s \wedge a,
\] (125)
\[
a_\cdot B_s = \frac{1}{2}(aB_s - (-)^s B_s a),
\]
\[
A_r_\cdot B_s = (-)^{r(s-1)} B_s \cdot A_r,
\]
\[
a \wedge B_s = \frac{1}{2}(aB_s + (-)^s B_s a),
\]
\[
A_r B_s = \langle A_r B_s \rangle_{r-s} + \langle A_r_\cdot B_s \rangle_{r-s-2} + \ldots + \langle A_r B_s \rangle_{r+s}
\]
\[
= \sum_{k=0}^{m} \langle A_r B_s \rangle_{r-s+2k}, \ m = \frac{1}{2}(s + r - |r - s|).
\] (126)

8.3 Hodge star operator

Let \( \star \) be the usual Hodge star operator \( \star : \bigwedge^k T^*M \rightarrow \bigwedge^{4-k} T^*M \). If \( B \in \sec \bigwedge^k T^*M, A \in \sec \bigwedge^{4-k} T^*M \) and \( \tau \in \sec \bigwedge^4 T^*M \) is the volume form, then \( \star B \) is defined by
\[
A \wedge \star B = (A \cdot B) \tau.
\]

Then we can show that if \( A_p \in \sec \bigwedge^p T^*M \rightarrow \sec \mathcal{C}(T^*M) \) we have
\[
\star A_p = \tilde{A_p} \theta^5.
\] (127)

This equation is enough to prove very easily the following identities (which are
used below):

\[ A_r \wedge \star B_s = B_s \wedge \star A_r; \quad r = s, \]
\[ A_r \star \star B_s = B_s \star \star A_r; \quad r + s = 4, \]
\[ A_r \wedge \star B_s = (-1)^{r(s-1)} \star (\bar{A}_r \wedge \bar{B}_s); \quad r \leq s, \]
\[ A_r \star \star B_s = (-1)^{r} \star (\bar{A}_r \wedge \bar{B}_s); \quad r + s \leq 4 \quad (128) \]

Let \( d \) and \( \delta \) be respectively the differential and Hodge codifferential operators acting on sections of \( \bigwedge T^* M \). If \( \omega_p \in \sec \bigwedge^p T^* M \leftrightarrow \sec C\ell(T^* M) \), then \( \delta \omega_p = (-1)^p \star^1 d \star \omega_p \), where \( \star^1 \star = \text{identity} \). When applied to a \( p \)-form we have

\[ \star^{-1} = (-1)^{p(4-p)+1} \star \].

### 8.4 Action of \( \nabla e_a \) on Sections of \( C\ell(TM) \) and \( C\ell(T^*M) \)

Let \( \nabla e_a \) be a metrical compatible covariant derivative operator acting on sections of the tensor bundle. It can be easily shown (see, e.g., [33]) that \( \nabla e_a \) is also a covariant derivative operator on the Clifford bundles \( C\ell(TM) \) and \( C\ell(T^*M) \).

Now, if \( A_p \in \sec \bigwedge^p T^* M \leftrightarrow \sec C\ell(M) \) we can show, very easily by explicitly performing the calculations \(^{30}\) that

\[ \nabla e_a A_p = e_a(A_p) + \frac{1}{2} [\omega_{e_a}, A_p], \quad (129) \]

where the \( \omega_{e_a} \in \sec \bigwedge^2 T^* M \leftrightarrow \sec C\ell(M) \) may be called Clifford connection 2-forms. They are given by:

\[ \omega_{e_a} = \frac{1}{2} \omega_{ab} \theta_b \theta_c = \frac{1}{2} \omega_{bac} \theta^c = \frac{1}{2} \omega_{a}^{bc} \theta_b \wedge \theta_c, \quad (130) \]

where we use the (simplified) notation

\[ \nabla e_a \theta_b = \omega^c_{ab} \theta_c, \quad \nabla e_a \theta^b = -\omega_{ac}^b \theta^c, \quad \omega_{a}^{bc} = -\omega^c_{ab} \]

### 8.5 Dirac Operator, Differential and Codifferential

**Definition 27** The Dirac (like) operator acting on sections of \( C\ell(T^*M) \) is the invariant first order differential operator

\[ \partial = \theta^a \nabla e_a. \quad (132) \]

We can show (see, e.g., [35]) that when \( \nabla e_a \) is the Levi-Civita covariant derivative operator (as assumed here), the following important result holds:

\[ \partial = \partial \wedge + \partial \star = d - \delta. \quad (133) \]

\(^{30}\)A derivation of this formula from the general theory of connections can be found in [43].
Definition 28 The square of the Dirac operator $\nabla^2$ is called Hodge Laplacian.

Some useful identities are:

\[
\begin{align*}
dd &= \delta\delta = 0, \\
n\n\n\delta\n\n\n\partial^2 &= \partial^2d; \\
\delta\partial^2 &= \partial^2\delta, \\
\delta\star &= (-1)^{p+1}\star d; \\
\star\delta &= (-1)^p\star d, \\
d\delta\star &= \star d\delta; \\
\star\delta &= \delta\star; \\
\partial^2 &= \partial^2\star.
\end{align*}
\] (134)

8.6 Covariant D’Alembertian, Ricci and Einstein Operators

In this section we study in details the Hodge Laplacian and its decomposition in the covariant D’Alembertian operator and the very important Ricci operator, which do not have analogous in the standard presentation of differential geometry in the Cartan and Hodge bundles, as given e.g., in [7].

Remembering that $\nabla = \theta^\alpha \nabla_{e^\alpha}$, where $\{e^\alpha\} \in F(M)$ is an arbitrary frame and $\{\theta^\alpha\}$ its dual frame on the manifold $M$ and $\nabla$ is the Levi-Civita connection of the metric $g$, such that

\[
\nabla_{e^\alpha} e^\beta = \gamma^\mu_{\alpha\beta} e^\mu, \\
\nabla_{e^\alpha} \theta^\beta = -\gamma^\mu_{\alpha\beta} \theta^\mu
\] (135)

we have:

\[
\nabla^2 = (\theta^\alpha \nabla_{e^\alpha})(\theta^\beta \nabla_{e^\beta}) = \theta^\alpha(\theta^\beta \nabla_{e^\alpha} \nabla_{e^\beta} + (\nabla_{e^\alpha} \theta^\beta) \nabla_{e^\beta}) \\
= g^{\alpha\beta}(\nabla_{e^\alpha} \nabla_{e^\beta} - \gamma^\rho_{\alpha\beta} \nabla_{e^\rho}) + \theta^\alpha \land \theta^\beta(\nabla_{e^\alpha} \nabla_{e^\beta} - \gamma^\rho_{\alpha\beta} \nabla_{e^\rho}). \] (136)

Next we introduce the operators:

\[
\begin{align*}
(a) \quad \Box &= \partial \cdot \partial = g^{\alpha\beta}(\nabla_{e^\alpha} \nabla_{e^\beta} - \gamma^\rho_{\alpha\beta} \nabla_{e^\rho}) \\
(b) \quad \partial \land \partial &= \theta^\alpha \land \theta^\beta(\nabla_{e^\alpha} \nabla_{e^\beta} - \gamma^\rho_{\alpha\beta} \nabla_{e^\rho}), \quad (137)
\end{align*}
\]

Definition 29 We call $\Box = \partial \cdot \partial$ the covariant D’Alembertian operator and $\partial \land \partial$ the Ricci operator.

The reason for the above names will become obvious through propositions 31 and 32.

Note that we can write:

\[
\nabla^2 = \partial \cdot \partial + \partial \land \partial \] (138)

or,

\[
\nabla^2 = (\partial \land + \partial \land)(\partial \land + \partial \land) \] (139)

\[
= \partial \cdot \partial \land + \partial \land \land \partial \land \\
= -(d\delta + \delta d). \] (140)

31 This means that it can be a cordiante basis or an orthonormal basis.
Before proceeding, let us calculate the commutator $[\theta_\alpha, \theta_\beta]$ and anticommutator $\{\theta_\alpha, \theta_\beta\}$. We have immediately

$$[\theta_\alpha, \theta_\beta] = c^p_{\alpha\beta} \theta_p,$$

(141)

where $c^p_{\alpha\beta}$ are the structure coefficients (see, e.g., [7]) of the basis $\{e_\alpha\}$, i.e., $[e_\alpha, e_\beta] = c^p_{\alpha\beta} e_p$.

Also,

$$\{\theta_\alpha, \theta_\beta\} = \nabla_{e_\alpha} \theta_\beta + \nabla_{e_\beta} \theta_\alpha,$$

(142)

Eq.(142) defines the coefficients $b^p_{\alpha\beta}$ which have a very interesting geometrical meaning as discussed in [53].

**Proposition 30** The covariant D’Alembertian $\partial \cdot \partial$ operator can be written as:

$$\partial \cdot \partial = \frac{1}{2} g^{\alpha\beta} \left[ \nabla_{e_\alpha} \nabla_{e_\beta} \omega + \nabla_{e_\beta} \nabla_{e_\alpha} - b^p_{\alpha\beta} \nabla_{e_p} \omega \right].$$

(143)

**Proof.** It is a simple computation left to the reader. ■

**Proposition 31** For every $r$-form field $\omega \in \text{sec} \bigwedge^r M$, $\omega = \frac{1}{r!} \omega_{\alpha_1...\alpha_r} \theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_r}$, we have:

$$(\partial \cdot \partial) \omega = \frac{1}{r!} g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \omega_{\alpha_1...\alpha_r} \theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_r},$$

(144)

where $\nabla_{\alpha} \nabla_{\beta} \omega_{\alpha_1...\alpha_r}$ is to be calculated with the standard rule for writing the covariant derivative of the components of a covector field.

**Proof.** We have $\nabla_{e_\beta} \omega = \frac{1}{r!} \nabla_{\beta} \omega_{\alpha_1...\alpha_r} \theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_r}$, with $\nabla_{\beta} \omega_{\alpha_1...\alpha_r} = (e_\beta(\omega_{\alpha_1...\alpha_r}) - \gamma^\sigma_{\beta\alpha_1} \omega_{\sigma \alpha_2 ... \alpha_r} - ... - \gamma^\sigma_{\beta \alpha_r} \omega_{\alpha_1 ... \alpha_{r-1} \sigma})$. Therefore,

$$\nabla_{e_\alpha} \nabla_{e_\beta} \omega = \frac{1}{r!} (e_\alpha(\nabla_{\beta} \omega_{\alpha_1...\alpha_r}) - \gamma^\sigma_{\alpha \alpha_1} \nabla_{\beta} \omega_{\sigma \alpha_2 ... \alpha_r} - ... - \gamma^\sigma_{\alpha \alpha_r} \nabla_{\beta} \omega_{\alpha_1 ... \alpha_{r-1} \sigma}) \theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_r}$$

and we conclude that:

$$\nabla_{e_\alpha} \nabla_{e_\beta} \omega - \gamma^\rho_{\alpha \beta} \nabla_{e_\rho} \omega = \frac{1}{r!} \nabla_{\alpha} \nabla_{\beta} \omega_{\alpha_1...\alpha_r} \theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_r}.$$

Finally, multiplying this equation by $g^{\alpha\beta}$ and using the Eq.(137a), we get the Eq.(144). ■

The Ricci operator $\partial \wedge \partial$ can be written as:

$$\partial \wedge \partial = \frac{1}{2} g^{\alpha\beta} \left[ \nabla_{e_\alpha} \nabla_{e_\beta} \omega - \nabla_{e_\beta} \nabla_{e_\alpha} - c^p_{\alpha\beta} \nabla_{e_p} \omega \right].$$

(145)
Proof. It is a trivial exercise, left to the reader. ■

Applying this operator to the 1-forms of the frame \( \{ \theta^\mu \} \), we get:

\[
(\partial \wedge \partial) \theta^\mu = -\frac{1}{2} R^\mu_{\alpha \beta}(\theta^\alpha \wedge \theta^\beta) \theta^\rho = -R^\mu_\rho \theta^\rho,
\]

where \( R^\mu_{\alpha \beta} \) are the components of the Riemann curvature tensor of the connection \( \nabla \). We can write using the first line in Eq. (125)

\[
R^\mu_\rho \theta^\rho = R^\mu_\rho \theta^\rho + R^\mu_\rho \wedge \theta^\rho.
\]

The second term in the r.h.s. of this equation is identically null because of the Bianchi identity satisfied by the Riemann curvature tensor, as can be easily verified. That result can be encoded in the equation:

\[
(\partial \wedge \partial) \theta^\mu = 0,
\]

The second term in the r.h.s. of this equation is identically null because of the Bianchi identity satisfied by the Riemann curvature tensor, as can be easily verified. That result can be encoded in the equation:

\[
(\partial \wedge \partial) \theta^\mu = 0,
\]

For the term \( R^\mu_\rho \theta^\rho \) we have (using Eq. (121) and the third line in Eq. (125)):

\[
R^\mu_\rho \theta^\rho = \frac{1}{2} R^\mu_{\alpha \beta}(\theta^\alpha \wedge \theta^\beta) \theta^\rho = \frac{1}{2} R^\mu_{\alpha \beta} (\theta^\alpha \wedge \theta^\beta - g^{\rho \beta} \theta^\rho) = -\frac{1}{2} R^\rho_{\alpha \beta} \theta^\rho = -R^\rho_{\alpha \beta} \theta^\rho,
\]

where \( R^\rho_{\alpha \beta} \) are the components of the Ricci tensor of the Levi-Civita connection \( \nabla \). The above results can be put in the form of the following

**Proposition 32**

\[
(\partial \wedge \partial) \theta^\mu = R^\mu,
\]

where \( R^\mu = R^\rho_{\alpha \beta} \theta^\rho \) are the Ricci 1-forms of the manifold.

The next proposition shows that the Ricci operator can be written in a purely algebraic way:

**Proposition 33** The Ricci operator \( \partial \wedge \partial \) satisfies the relation:

\[
\partial \wedge \partial = R^\sigma \wedge \theta^\sigma \wedge + R^{\rho \sigma} \wedge \theta^\rho \wedge \theta^\sigma \wedge,
\]

where \( R_{\rho \sigma} = g^{\rho \mu} R^\mu_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \) are the curvature 2-forms.

**Proof.** The Hodge Laplacian of an arbitrary r-form field \( \omega = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r} \) is given by: (e.g., [7]—recall that our definition differs by a sign from that given there) \( \partial^2 \omega = \frac{1}{r!} (\partial^2 \omega)_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r} \), with:

\[
(\partial^2 \omega)_{\alpha_1 \ldots \alpha_r} = g^{\alpha \beta} \nabla_\alpha \nabla_\beta \omega_{\alpha_1 \ldots \alpha_r} - \sum_p (-1)^p R^\alpha_{\beta \gamma} \omega_{\alpha_1 \ldots \hat{\alpha}_p \ldots \alpha_r} - 2 \sum_{p < q} (-1)^{p+q} R^\sigma_{\alpha \gamma} \sigma^\rho_{\alpha \rho \sigma \alpha_1 \ldots \hat{\alpha}_p \ldots \hat{\alpha}_q \ldots \alpha_r},
\]
where the notation $\bar{\alpha}$ means that the index $\alpha$ was excluded of the sequence.

The first term in the r.h.s. of this expression are the components of the covariant D'Alembertian of the field $\omega$. Then,

$$\mathcal{R}^\sigma \wedge \theta_\sigma \omega = -\frac{1}{r!} \left[ \sum_p (-1)^p R_{\alpha_p}^\sigma \omega_{\sigma_1 \ldots \sigma_p} \ldots \theta_1 \ldots \theta_r \right] \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$$

and also,

$$\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\rho \wedge \theta_\sigma \wedge \omega = -\frac{2}{r!} \left[ \sum_{p < q} (-1)^{p+q} R_{\alpha_q}^\rho \omega_{\rho \sigma_1 \ldots \sigma_p} \ldots \theta_1 \ldots \theta_r \right] \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}.$$ 

Hence, taking into account Eq. (138), we conclude that:

$$(\partial \wedge \partial) \omega = \mathcal{R}^\sigma \wedge \theta_\sigma \omega + \mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\rho \wedge \theta_\sigma \wedge \omega,$$

(153)

for every $r$-form field $\omega$. $\blacksquare$

Observe that applying the operator given by the second term in the r.h.s. of Eq. (151) to the dual of the 1-forms $\theta_\mu$, we get:

$$\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu = \mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu$$

$$= \mathcal{R}^\rho_\sigma \wedge \star (\theta_\rho \wedge \theta_\sigma \wedge \theta_\mu)$$

$$= \star (\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \theta_\mu),$$

(154)

where we have used the Eqs. (128). Then, recalling the definition of the curvature forms and using the Eq. (121), we conclude that:

$$\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu = 2 \star (\mathcal{R}^\rho_\sigma - \frac{1}{2} R \theta_\mu) = 2 \star \mathcal{G}^\mu,$$

(155)

where $R$ is the scalar curvature of the manifold and the $\mathcal{G}^\mu$ may be called the Einstein 1-form fields.

That observation motivates us to introduce the

**Definition 34** The Einstein operator of the manifold associated to the Levi-Civita connection $\nabla$ of $g$ is the mapping $\nabla : \text{secCl}(T^* M) \to \text{secCl}(T^* M)$ given by:

$$\nabla = \frac{1}{2} \star^{-1} (\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu) \star.$$ 

(156)

Obviously, we have:

$$\nabla \theta_\mu = \mathcal{G}^\mu = \mathcal{R}^\rho_\sigma - \frac{1}{2} R \theta_\mu.$$

(157)

In addition, it is easy to verify that $\star^{-1} (\partial \wedge \partial) \star = -\partial \wedge \partial$ and $\star^{-1} (\mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu) \star = \mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu$. Thus we can also write the Einstein operator as:

$$\nabla = -\frac{1}{2} (\partial \wedge \partial + \mathcal{R}^\rho_\sigma \wedge \theta_\rho \wedge \theta_\sigma \wedge \star \theta_\mu).$$

(158)

Another important result is given by the following proposition:
Proposition 35 Let $\omega_{\mu}^\rho$ be the Levi-Civita connection 1-forms fields in an arbitrary moving frame $\{\theta^\mu\} \in \text{sec}(F(M))$ on $(M, \nabla, g)$. Then:

\[
(a) \ (\partial \cdot \partial) \theta^\mu = - (\partial \cdot (\omega_{\rho}^\mu - \omega_{\sigma}^\rho \cdot \omega_{\sigma}^\mu)) \theta^\rho,
\]
\[
(b) \ (\partial \land \partial) \theta^\mu = -(\partial \land (\omega_{\rho}^\mu - \omega_{\sigma}^\rho \land \omega_{\mu}^\sigma)) \theta^\rho.
\]

that is,

\[
\partial^2 \theta^\mu = - (\partial \omega_{\rho}^\mu - \omega_{\rho}^\sigma \omega_{\sigma}^\mu) \theta^\rho.
\]

Proof. We have

\[
\partial \cdot \omega_{\rho}^\mu = \theta^\alpha \cdot \nabla_{ea} (\gamma_{\delta \rho}^\mu \theta^\beta)
= \theta^\alpha \cdot (e_{a} (\gamma_{\delta \rho}^\mu) \theta^\beta - \gamma_{\beta \rho}^\alpha \gamma_{\alpha \beta}^\mu \theta^\beta)
= \eta^{\alpha \beta} (e_{a} (\gamma_{\delta \rho}^\mu) - \gamma_{\beta \rho}^\alpha \gamma_{\alpha \beta}^\mu) \theta^\rho
\]

and $\omega_{\rho}^\sigma \cdot \omega_{\sigma}^\mu = (\gamma_{\delta \rho}^\mu \theta^\beta) \cdot (\gamma_{\delta \sigma}^\mu \theta^\alpha) = g^{\delta \alpha} \gamma_{\delta \sigma}^\mu \gamma_{\delta \rho}^\mu$. Then,

\[
- (\partial \cdot \omega_{\rho}^\mu - \omega_{\rho}^\sigma \omega_{\sigma}^\mu) \theta^\rho
= g^{\alpha \beta} (e_{a} (\gamma_{\delta \rho}^\mu) - \gamma_{\beta \rho}^\alpha \gamma_{\alpha \beta}^\mu) \theta^\rho
= -\frac{1}{2} g^{\alpha \beta} (e_{a} (\gamma_{\delta \rho}^\mu) + e_{\beta} (\gamma_{\delta \rho}^\mu) - \gamma_{\alpha \beta}^\mu \gamma_{\delta \rho}^\mu - b_{\alpha \beta}^\mu \gamma_{\delta \rho}^\mu) \theta^\rho
= (\partial \cdot \partial) \theta^\mu.
\]

Proof. Eq.(159b) is proved analogously.

Now, for an orthonormal coframe $\{\theta^a\}$ we have immediately, taken into account that $\nabla_{ea} \theta^b = -\omega_{ae}^b \theta^e$,

\[
\partial \cdot \partial = \eta^{ab} (\nabla_{ea} \nabla_{eb} - \omega_{eb}^c \nabla_{ea}),
\]
\[
\partial \land \partial = \theta^a \land \theta^b (\nabla_{ea} \nabla_{eb} - \omega_{eb}^c \nabla_{ea}).
\]

and\[32\]
\[
(\partial \land \partial) \theta^a = R^a,
\]

9 Equations for the Tetrad Fields $\theta^a$

Here we want to recall a not well known face of Einstein equations, i.e., we show how to write the field equations for the tetrad fields $\theta^a$ in such a way that the obtained equations are equivalent to Einstein field equations. This is done in order to compare the correct equations satisfied by those objects with equations proposed for those objects that appeared in [12] and also in other papers authored by Evans (some quoted in the reference list).

\[32\]In [25] there is an analogous equation, but there is a misprint of a factor of 2.
Proposition 36  Let \( \mathcal{M} = (M, g, \nabla, \tau_g, \uparrow) \) be a Lorentzian spacetime and also a spin manifold, and suppose that \( g \) satisfies the classical Einstein’s gravitational equation, which reads in standard notation (and in natural units)

\[
\text{Ricci} - \frac{1}{2} R g = \mathcal{T}.
\]  

(163)

Then, Eq. (163) is equivalent to Eq. (164) satisfied by the fields \( \theta^a \) \((a = 0, 1, 2, 3)\) of a cotetrad \( \{ \theta^a \} \) on \( \mathcal{M} \). Also, under the same conditions Eq. (164) is equivalent to Einstein’s equation.\(^{33}\):

\[
-(\partial \cdot \partial) \theta^a + \partial \wedge (\partial \cdot \theta^a) + \partial \partial (\partial \wedge \theta^a) = \mathcal{T}^a - \frac{1}{2} T \theta^a.
\]  

(164)

In Eq. (163) and Eq. (164), Ricci is the Ricci tensor, \( \mathcal{T} \) is the energy momentum tensor (with components \( T^a_{\ b} \)), \( R \) is the curvature scalar and \( \mathcal{T}^a = T^a_{\ b} \theta^b \in \text{sec} \bigwedge T^* M \hookrightarrow \text{sec} \text{Cl}(T^* M) \) are the energy momentum 1-form fields and \( T = T^a_{\ a} = -R = -R^a_{\ a} \).

Proof. We prove that Einstein’s equations are equivalent to Eq. (164). The proof that Eq. (164) is equivalent to Einstein’s equation is left to the reader. Einstein’s equation reads in components relative to a tetrad \( \{ e^a \} \in \text{sec} \text{P}_\text{SO}_{1,3}(M) \) and the cotetrad \( \{ \theta^a \} \), \( \theta^a \in \text{sec} \bigwedge T^* M \hookrightarrow \text{sec} \text{Cl}(T^* M) \) as:

\[
R^a_{\ b} - \frac{1}{2} \delta^a_{\ b} R = T^a_{\ b}
\]  

(165)

Multiplying the above equation by \( \theta^b \) and summing we get,

\[
\mathcal{R}^a - \frac{1}{2} \mathcal{R} \theta^a = \mathcal{T}^a
\]  

(166)

Next we use in Eq. (166) the Eq. (162), Eq. (138), Eq. (139), and that \( T = -R \) to write Eq. (166) as:

\[
-(\partial \cdot \partial v) \theta^a + \partial \wedge (\partial \cdot \theta^a) + \partial \partial (\partial \wedge \theta^a) = \mathcal{T}^a - \frac{1}{2} T \theta^a,
\]  

(167)

and the proposition is proved. \( \blacksquare \)

Note that in a coordinate chart \( \{ x^\mu \} \) of the maximal atlas of \( M \) covering \( U \subset M \), Eq. (166) can be written as

\[
\mathcal{R}^\mu - \frac{1}{2} R \theta^\mu = \mathcal{T}^\mu,
\]  

(168)

\(^{33}\)Of course, there are analogous equations for the \( e_a \), where in that case, the Dirac operator must be defined (in an obvious way) as acting on sections of the Clifford bundle \( \text{Cl}(T^* M) \) of non homogeneous multivector fields. See, e.g., [25], but take notice that the equations in [25] have an (equivocated) extra factor of 2.
with $R^\mu = R_\nu^\rho dx^\nu$ and $T^\mu = T_\nu^\rho dx^\nu$, $\theta^\mu = dx^\mu$. Eq. (168) looks like an equation appearing in some of Evans papers, but the meaning here is very different. From Eq. (168) we can show that an equation identical to Eq. (167) is also satisfied by the moving coordinate coframe $\{\theta^\mu = dx^\mu\}$. If we suppose moreover that the coordinate functions are harmonic, i.e., $\delta\theta^\mu = -\nabla\theta^\mu = 0$, Eq. (164) becomes

$$\Box \theta^\mu + \frac{1}{2} R\theta^\mu = -T^\mu,$$

(169)

We recall that in [12] it is wrongly derived that the equations for $\theta^a$, $a = 0, 1, 2, 3$ are the equations

$$\Box (\Box - R(x))\theta^a = 0.$$  

(49E)

**Remark 37** An equation looking similar to (49E), namely,

$$-\partial^2 \theta^a + \lambda(x)\theta^a = 0$$

(170)

has been proposed in [28] as vacuum field equations for a theory of the gravitational field not equivalent to General Relativity. Note that in Eq. (170) the wave equation is written with the Hodge Laplacian and moreover $\lambda(x) \neq R(x)$. Such a theory has been criticized in [38], who point some particularizations in the derivations of [28], but that paper is really interesting. See also [26]. We shall discuss this issue in another publication. However, even in [38], the wave equations for the tetrad fields in General Relativity are not given.

### 9.1 Correct Equations for the $q^a_\nu$ functions in a Lorentzian Manifold

First we obtain that equations for the functions $q^a_\nu$ in a Lorentzian manifold. This will be done using Eq. (113) for that situation. We have:

$$\nabla_{\partial_\nu} Q = \nabla_{\partial_\nu} (e_a \otimes \theta^a)$$

$$= \nabla^+_{\partial_\nu} e_a \otimes \theta^a + e_a \otimes \nabla^-_{\partial_\nu} \theta^a.$$  

(171)

Then,

$$g^{\mu\nu} \nabla_{\partial_\nu} \nabla_{\partial_\nu} Q$$

$$= \nabla^+_{\partial_\mu} \nabla^+_{\partial_\nu} e_a \otimes \theta^a + \nabla^+_{\partial_\nu} e_a \otimes \nabla^-_{\partial_\mu} \theta^a + \nabla^+_{\partial_\mu} e_a \otimes \nabla^-_{\partial_\nu} \theta^a + e_a \otimes \nabla^+_{\partial_\mu} \nabla^+_{\partial_\nu} \theta^a,$$

(172)

34 A somewhat similar equation with some (equivocated) extra factors of 2 appears in [25].

35 Here we wrote the equation in units where $\kappa = 1$. Note also that in [12] it is explicitly stated that the symbol $\Box$ means $\partial_\mu \partial^\mu$. It is not to be confused with the covariant D’Alembertian, which in our paper is represented by $\Box$.

36 We shall discuss this issue in another publication.
and

\[ g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} Q = g^{\mu\nu} \nabla^+_{\mu} \nabla^+_{\nu} e_a \otimes \theta^a + 2g^{\mu\nu} \nabla^+_\mu e_a \otimes \nabla^-_{\nu} \theta^a + e_a \otimes g^{\mu\nu} \nabla^-_{\mu} \nabla^-_{\nu} \theta^a \]  \tag{173} 

Now, write the difference of Eq.(173) and the quantity \( g^{\nu\mu} \nabla_{\nu} \nabla_{\mu} Q \). This gives

\[ g^{\mu\nu} \left[ \left( \nabla^+_{\mu} - \nabla^+_{\nu} \nabla^+_{\nu} \right) e_a \right] \otimes \theta^a + e_a \otimes g^{\mu\nu} \left[ \left( \nabla^-_{\mu} - \nabla^-_{\nu} \nabla^-_{\nu} \right) \theta^a \right] = 0 \]  \tag{174} 

Now, recalling the operator identity \( \Box \) (Eq.(175)),

\[ g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} = \Box + g^{\rho\nu} \Gamma^\rho_{\mu\nu} \nabla^-_{\rho} \]  \tag{175} 

and Eq.(144), we have

\[ g^{\mu\nu} \nabla^-_{\mu} \nabla^-_{\nu} \theta^a = \theta \cdot \theta \theta^a + g^{\mu\nu} \Gamma^\rho_{\mu\nu} \nabla^-_{\rho} \theta^a. \]  \tag{176} 

Also,

\[ g^{\mu\nu} \nabla^-_{\mu} \nabla^-_{\nu} \theta^a = g^{\mu\nu} \left( -\partial_{\nu} \omega^a_{\mu b} + \omega^a_{\mu c} \omega^c_{\nu b} \right) \theta^b. \]  \tag{177} 

Also,

\[ g^{\mu\nu} \left[ \left( \nabla^+_{\mu} - \nabla^+_{\nu} \nabla^+_{\nu} \right) e_a \right] = g^{\mu\nu} R^b_{\alpha \mu \nu} e_b = R^b_{\alpha} e_b \]  \tag{178} 

Using Eqs. (176), (177) and (178) in Eq.(174), we get

\[ g^{\alpha\beta} \nabla^-_{\alpha} \nabla^-_{\beta} \theta^b + R^b_{a \mu \nu} \theta^a - g^{\mu\nu} (\partial_{\nu} \omega^a_{\mu b} - \Gamma^\rho_{\mu\nu\rho a} - \omega^a_{\mu c} \omega^c_{\nu b}) \theta^a = 0 \]  \tag{179} 

So, this is the ‘wave equation’ satisfied by the functions \( q^a_{\mu b} \) in a Lorentzian manifold. It is to be compared with Eq.(2E) found in [12], which it has been used there to derive the false ‘Evans lemma’ used by the author of [12]. It is our opinion that as an wave equation Eq.(179) has no utility. However, since as it is well known, we can write the \( \omega^a_{\mu b} \) and \( g^{\mu\nu} \) in terms of the functions \( q^a_{\mu b} \) and their inverses \( q^b_{\nu} \). Doing that we can use Eq.(179) to write an explicit expression (in the tetrad basis) for the components \( R^b_{\alpha} \) of the Ricci tensor in terms of the functions \( q^a_{\mu b} \) and \( q^b_{\nu} \). However, at the moment we cannot see any advantage in writing such equation, for there are more efficient methods to obtain the components of the Ricci tensor.

\[ \text{The operator identity given by Eq.(175) is to be compared with the wrong Eq.(42E) and also with the equation in line 11 of table 1 in [12].} \]
9.2 Correct Equations for the $q_a^\nu$ functions in General Relativity

Having obtained the correct equations for the tetrad fields $\theta^a$ in General Relativity (Eq.(167)), we now derive the corresponding equation for the $q_a^\nu$ functions in a Lorentzian spacetime representing a gravitational field.

We first observe that

$$\partial \wedge (\partial \cdot \theta^a) + \partial \mu (\partial \wedge \theta^a) = \left[ -\partial_{\mu} \left( \omega^a_{\ d} q^\mu_b + q^\mu_k \partial_{\mu} \left( \eta^{kd} \omega^a_{\ bd} - \omega^a_{\ kb} \right) \right) \right] \theta^b$$

(180)

Next we define

$$K^b_a = - \left[ -\partial_{\mu} \left( \omega^b_{\ d} q^\mu_a + q^\mu_k \partial_{\mu} \left( \eta^{kd} \omega^b_{\ ad} - \omega^b_{\ ka} \right) \right) + T^b_a - \frac{1}{2} T \delta^b_a \right]$$

(181)

Using these results in Eq.(167) we get,

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta q^b_a + K^b_a q^\alpha_a = 0.$$  

(182)

Comparing that equation with Eq.(179) we get the constraint

$$R^b_a - g^{\mu\nu} (\partial_{\mu} \omega^a_{\ \nu} b - \Gamma^\mu_\mu \omega^b_\nu a - \omega^a_{\ \nu c} \omega^c_{\ \nu b}) + \partial_{\mu} \left( \omega^b_{\ d} q^\mu_a + q^\mu_k \partial_{\mu} \left( \eta^{kd} \omega^b_{\ ad} - \omega^b_{\ ka} \right) \right)$$

$$= \frac{1}{2} T \delta^b_a - T^b_a.$$

(183)

which is a compatibility equation that must hold if the tetrad field equations are to be equivalent to Einstein’s equations.

10 Correct Equation for the Electromagnetic Potential $A$

In [13, 14, 15] it is explicitly written several times that the ”electromagnetic potential" $A$ of the ”unified theory" (a 1-form with values in a vector space) satisfies the following wave equation, $(\Box = \partial_{\mu} \partial^\mu)$$\Box (\Box + T) A = 0.$

Now, this equation cannot be correct even for the usual $U(1)$ gauge potential of classical electrodynamics$^{38}$  

$A \in \sec \wedge^1 T^* M \subset \sec \mathcal{U}(T^* M)$. To show that let us first recall how to write electrodynamics in the Clifford bundle.

$^{38}$Which must be one of the gauge components of the gauge field.
10.1 Maxwell Equation

Maxwell equations when can be written in the Clifford bundle formalism of differential forms as single equation. Indeed, if \( F \in \text{sec} \bigwedge^2 T^*M \subset \text{sec} \mathcal{C}l(T^*M) \) is the electromagnetic field and \( J_e \in \text{sec} \bigwedge^1 T^*M \subset \text{sec} \mathcal{C}l(T^*M) \) is the electromagnetic current, we have Maxwell equation \(^{39}\)

\[
\partial F = J_e. \tag{184}
\]

Eq. (184) is equivalent to the pair of equations

\[
dF = 0, \tag{185}
\]

\[
\delta F = -J_e. \tag{186}
\]

Eq. (185) is called the homogeneous equation and Eq. (186) is called the non-homogeneous equation. Note that it can be written also as:

\[
d \ast F = - \ast J_e. \tag{187}
\]

Now, in vacuum Maxwell equation reads

\[
\partial F = 0, \tag{188}
\]
where \( F = \partial A = \partial \wedge A = dA \), if we work in the Lorenz gauge \( \partial \cdot A = \partial_j A = -\delta A = 0 \). Now, since we have according to Eq. (140) that \( \partial^2 = -(d \delta + \delta d) \), we get

\[
\partial^2 A = 0. \tag{189}
\]

Using Eq. (152) (or Eq. (140) coupled with Eq. (151)) and the coordinate basis introduced above we have,

\[
(\partial^2 A)_\alpha = g^{\mu\nu} \nabla_\mu \nabla_\nu A_\alpha + R^\nu_\mu A_\nu. \tag{190}
\]

Then, we see that Eq. (189) reads in components \(^{40}\)

\[
\nabla_\alpha \nabla^\alpha A_\mu = 0. \tag{191}
\]

Finally, we observe that in Einstein’s theory, \( R^\nu_\mu = 0 \) in vacuum, and so in vacuum regions we end with:

\[
\nabla_\alpha \nabla^\alpha A_\mu = 0. \tag{192}
\]

---

\(^{39}\)Then, there is no misprint in the title of this subsection.

\(^{40}\)Sometimes the symbol \( \Box \) is used to denote the operator \( D_\alpha D^\alpha \). Eq. (191) can be found, e.g., in Eddington’s book [11] on page 175.
11 Lagrangian Field Theory for the Tetrad Fields

We show here how the Einstein-Hilbert Lagrangian (modulo an exact differential) can be written in the suggestive form given by

$$ L_g = -\frac{1}{2} d\theta^a \wedge *d\theta_a + \frac{1}{2} \delta \theta^a \wedge *\delta \theta_a + \frac{1}{4} (d\theta^a \wedge \theta_a) \wedge * (d\theta^b \wedge \theta_b). \quad (193) $$

Here, $g = \eta_{ab} \theta^a \otimes \theta^b$ and

$$ \theta^a \theta^b + \theta^b \theta^a = 2 \eta^{ab}. \quad (194) $$

Now, the classical Einstein-Hilbert Lagrangian density in appropriate (geometrical) units is given by

$$ L_{EH} = \frac{1}{2} R_g = \frac{1}{2} R \theta^5, \quad (195) $$

where $R = \eta^{cd} R_{cd}$ is the scalar curvature. We observe that we can write $L_{EH}$ as

$$ L_{EH} = \frac{1}{2} R_{cd} \wedge * (\theta^c \wedge \theta^d) \quad (196) $$

$$ = \frac{1}{2} R_{cd} \wedge (\theta^d \wedge \theta^c) \theta^5 \quad (197) $$

Indeed, we have immediately that

$$ R_{cd} \wedge * (\theta^c \wedge \theta^d) = (\theta^c \wedge \theta^d) \wedge *R_{cd} = -(\theta^c) \wedge *(\theta^d, R_{cd}) $$

$$ = -* [\theta^c \wedge (\theta^d \wedge R_{cd})], \quad (198) $$

and since

$$ \theta^d, R_{cd} = \frac{1}{2} R_{cdab} (\theta^a \wedge \theta^b) = \frac{1}{2} R_{cdab} (\eta^{da} \eta^b - \eta^{db} \eta^a) $$

$$ = -R_{ca} \theta^b = -R_c. \quad (199) $$

it follows that $-\theta^c \wedge (\theta^d \wedge R_{cd}) = \theta^c \wedge R_c = R$.

Now, taking into account that $R_{cd} = d\omega_{cd} + \omega_{ca} \wedge \omega_{db}$, we can obtain the free Einstein’s field equations $*G_a = 0$ by varying the Einstein-Hilbert action $\int L_{EH}$ with respect to the fields $\theta^a$ and $\omega_{ca}$. Indeed, after a very long calculation (see Appendix B) which requires the notion of derivative of multivector functions and functionals [13, 14, 20, 34, 35, 36, 37] we get

$$ \delta L_{EH} = -\frac{1}{2} d [*(\theta^c \wedge \theta^d) \wedge \delta \omega_{cd}] + \delta \theta^a \wedge \left[ \frac{1}{2} *(\theta^c \wedge \theta^d \wedge \theta_a) \right] \wedge R_{cd}. \quad (200) $$

Now, taking into account that

$$ -\frac{1}{2} [*(\theta^c \wedge \theta^d \wedge \theta_a)] \wedge R_{cd} = *G_a = *(R_a - \frac{1}{2} R \theta_a). \quad (201) $$
Of course, in order to obtain Einstein’s equations in the presence of matter we have to vary the total Lagrangian density \( \mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_m \), where we explicitly suppose that \( \mathcal{L}_m(\theta^a, d\theta^a, \phi^A, d\phi^A) \), the matter Lagrangian of a set of fields \( \phi^A \) (which may be Clifford or spinor fields, the latter fields, also represented in each spin frame as a sum of non homogeneous differential forms, as explained in [33]) does not depend explicitly on the \( \omega_{cd} \). In that case, we have

\[
\delta \mathcal{L} = -\frac{1}{2} d \left[ \ast \left( \theta^c \wedge \theta^d \right) \right] + \delta \theta^a \wedge \left\{ \frac{1}{2} \ast \left( \theta^c \wedge \theta^d \wedge \theta^a \right) \right\} \wedge \mathcal{R}_{cd} + \ast \mathcal{T}_a,
\]

and the field equations results in

\[
\ast \mathcal{G}_a = \ast \mathcal{T}_a
\]

but this equation as we know, gives by use of the Ricci, Einstein, covariant D'Alembertian and the Hodge Laplacian, directly the equations for the tetrad fields.

**Remark 38** We observe that \( \mathcal{L}_g \) is the first order Lagrangian density (first introduced by Einstein) written in intrinsic form. Indeed, the dual of Eq.(198), i.e., \( \left[ \theta^c \lrcorner (\theta^d \lrcorner d\omega_{cd}) \right] \) is given by

\[
\left[ \theta^c \lrcorner (\theta^d \lrcorner d\omega_{cd}) \right] = \theta^c \lrcorner (\theta^d \lrcorner d\omega_{cd}) + \theta^a \lrcorner \theta^b \lrcorner (\omega_{ac} \wedge \omega^c_{db}).
\]

Writing \( \omega^c_b = \omega^c_{bc} \) we verify that

\[
\theta^a \lrcorner \theta^b \lrcorner (\omega_{ac} \wedge \omega^c_{db}) = \eta^{bk} (\omega^c_{bc} \omega^e_{db} - \omega^c_{ac} \omega^e_{kb}),
\]

and moreover,

\[
\ast \left[ \theta^c \lrcorner (\theta^d \lrcorner d\omega_{cd}) \right] = -d (\theta^a \wedge \ast d\theta^a).
\]

Now, since

\[
\omega_{cd} = \frac{1}{2} \left[ \theta^e \lrcorner d\theta^f - \theta^e \lrcorner d\theta^f + \theta^e \lrcorner (\theta^d \lrcorner d\theta^a) \theta^a \right],
\]

using Eq.(207) in Eq.(202) we get,

\[
\mathcal{L}_g = -\frac{1}{2} \partial_a \theta^b \lrcorner \partial_b \lrcorner \left\{ \frac{1}{2} \left[ \theta_a \lrcorner d\theta_c + \theta_c \lrcorner d\theta_a + \theta_a \lrcorner (\theta_c \lrcorner d\theta_d) \theta^k \right] \wedge \frac{1}{2} \left[ \theta_b \lrcorner d\theta^e + \theta^e \lrcorner d\theta_b + \theta^e \lrcorner (\theta_b \lrcorner d\theta_b) \theta^1 \right] \right\},
\]

which after some algebraic manipulations reduces to Eq.(193), i.e.,

\[
\mathcal{L}_g = -\frac{1}{2} \delta \theta^a \wedge \ast d\theta^a + \frac{1}{2} \delta \theta^a \wedge \ast \delta \theta^a + \frac{1}{4} (d\theta^a \wedge \theta_a) \wedge \ast (d\theta^b \wedge \theta_b).
\]

The Lagrangian density \( \mathcal{L}_g \) looks like the Lagrangians of gauge theories. The first term is of the Yang-Mills type. The second term, will be called the gauge fixing term, since as can be verified \( \delta \theta^a = 0 \) is equivalent to the harmonic gauge as we already observed above. The third term is the auto-interacting
term, responsible for the nonlinearity of Einstein’s equations. Lagrangians of this type have been discussed by some authors, see, e.g., [57], where no use of the Clifford bundle formalism is used. In [45] \( L_g \) has been used to give a theory of the gravitational field in Minkowski spacetime, by writing \( \star \) in terms of the Hodge dual associated to a constant Minkowski metric defined in the world manifold \( M \) which is supposed diffeomorphic to \( \mathbb{R}^4 \).

12 Conclusions

We discussed in details in this paper the genesis of an ambiguous statement called ‘tetrad postulate’, which should be more precisely called naive tetrad postulate. We show that if the naive ‘tetrad postulate’ is not used in a very special context—where it has a precise meaning as a correct mathematical statement—namely, that the Eq. \( \nabla Q = 0 \) is satisfied (an intrinsic expression of the obvious freshman identity given in Eq. \( 63 \)) it may produce some serious misunderstandings. We give explicit examples of such misunderstandings appearing in many books and articles.

We presented moreover a detailed derivation\(^{13}\) (including all the necessary mathematical theorems) of the correct differential equations satisfied by the \((co)\)tetrad fields \( \theta^a = q^a_i dx^i \) on a Lorentzian manifold, modelling a gravitational field in General Relativity. This has been done using modern mathematical tools, namely the theory of Clifford bundles and the theory of the square of the Dirac operator. The correct equations are to be compared with the ones given, e.g., in \([13, 14, 15, 17, 28]\) and which also appears as Eq. \( 49E \) in [12]. We derive also the tetrad equations in General Relativity from a variational principle.

The functions \( q^a_i \) appearing as components of the tetrads \( \theta^a \) in a coordinate basis, appear also as components of the tensor \( Q = q^a_i e_a \otimes dx^i \) (see Eq. \( 29 \)) that satisfies trivially in any general Riemann-Cartan spacetime a second order differential equation, namely Eq. \( 123 \). From that equation, we derived for the particular case of a general Lorentzian spacetime a ‘wave equation’ for the functions \( q^a_i \). Since a wave equation for the functions \( q^a_i \) can also be derived from the correct equations satisfied by the \( \theta^a \) in General Relativity, by comparing both equations we obtained a constraint equation (Eq. \( 183 \)). That equation couples the functions \( q^a_i \), the components of the Ricci tensor and the components of the energy-momentum tensor and its trace.

In a series of papers \([12, 13, 14, 15, 16, 17, 28]\) (to quote some of them) a ‘unified field theory’ is proposed. In \([12]\) it is claimed that such ‘unified theory’ follows from a so called ‘Evans Lemma’ of differential geometry. We proved that as presented in \([12]\) ‘Evans Lemma’ is a \textit{false} statement. Then it follows that the ‘unified field theory’ is wrong. Before closing, it is eventually worth to give additional pertinent comments concerning some other statements in \([12]\).

\(^{13}\) These equations already appeared in \([43, 45]\), but the necessary theorems (proved in this report) needed to prove them have not been given there.
At page 442 of [12], concerning his discovery of the ‘Evans Lemma’, i.e.,
the wrong Eq.(2E), the author said:

‘The Lemma is an identity of differential geometry, and so is comparable in
generality and power to the well-known Poincaré Lemma [14]. In other words,
new theorems of topology can be developed from the Evans Lemma in analogy
with topological theorems [2,14] from Poincaré Lemma.’

Well, a correct corollary (not lemma, please) of Eq.(63), which in intrinsic
form reads \( \nabla Q = 0 \), is simply our Eq.(113),

\[ g^{\nu\mu} \partial_\nu \partial_\mu Q = 0. \]

The author of [12] derived a wrong equation from the components \( q^\mu_\nu \) of \( Q \) and dubbed this
equation Evans lemma of differential geometry. So, we leave to the reader to
judge if such a triviality has the same status of the Poincaré lemma.

Note that we did not comment on many other errors in [12] and in particular
on Section 3 of that paper. But we emphasize that they are subtle confusions
there as some of the ones we have enough patience to describe above. Those
confusions are of the same caliber as the following on that we can find in [8] and
which according to our view is a very convincing proof of the sloppiness of [12
13 14 15 16 17] and other papers from that author and collaborators. Indeed,
e.g., in [8], Evans and his coauthor Clements try to identify Sachs supposed
‘electromagnetic’ field (which Sachs believes to follow from his ‘unified’ theory)
with a supposed existing longitudinal electromagnetic field predicted by Evans
‘theory’, the so-called \( B(3) \) mentioned several times in [12] and the other papers
we quoted. Well, on [8] we can read at the beginning of section 1.1:

“The antisymmetrized form of special relativity [1] has spacetime metric
given by the enlarged structure

\[ \eta^{\mu\nu} = \frac{1}{2} (\sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu), \quad (1.1.) \]

where \( \sigma^\mu \) are the Pauli matrices satisfying a Clifford algebra

\[ \{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}, \]

which are represented by

\[
\begin{align*}
\sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.2)
\end{align*}
\]

The * operator denotes quaternion conjugation, which translates to a spatial
parity transformation.”

Well, we comment as follows: the * is not really defined anywhere in [8]. If
it refers to a spatial parity operation, we infer that \( \sigma^{0*} = \sigma^0 \) and \( \sigma^{i*} = -\sigma^i \).
Also, \( \eta^{\mu\nu} \) is not defined, but Eq.(3.5) of [8] makes us to infer that \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \). In that case Eq.(1.1) above (with the first member under-
stood as \( \eta^{\mu\nu} \sigma^0 \)) is true but the equation \( \{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu} \) is false. Enough is
to see that \( \{\sigma^0, \sigma^1\} = 2\sigma^1 \neq 2\delta^{01} \).

\[42\] At the time of publication, a Ph.D. student at Oxford University.
\[43\] On this issue see [40, 43].
We left to the reader who fells expert enough on Mathematics matters to set the final judgment.

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Beauty is truth, truth beauty. That is all ye know on Earth, and all ye need to know.  
J. Keats

A Counterexample to the Naive ‘Tetrad Postulate’

(i) Consider the structure $(\mathcal{S}^2, g, \nabla)$, where the manifold $\mathcal{S}^2 = \{S^2 \setminus \text{north pole}\} \subset \mathbb{R}^3$ is a sphere of radius $R$ excluding the north pole, $g \in \text{sec} T^2_0 \mathcal{S}^2$ is a metric field for $\mathcal{S}^2$, the natural one that it inherits from euclidean space $\mathbb{R}^3$, and $\nabla$ is the Levi-Civita connection on $\mathcal{S}^2$, which may be understood as $\nabla^+, \nabla^-$ or $\nabla$ in each appropriate case.

(ii) Introduce the usual spherical coordinate functions $(x_1, x_2) = (\vartheta, \varphi)$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, which covers all the open set $U$ which is $\mathcal{S}^2$ with the exclusion of a semi-circle uniting the north and south poles.

(iii) Introduce first coordinate bases

\[ \{\partial_{\mu}\}, \{\theta_{\mu} = dx^\mu\} \quad (210) \]

for $TU$ and $T^*U$.

(iv) Then,

\[ g = R^2 dx^1 \otimes dx^1 + R^2 \sin^2 x^1 dx^2 \otimes dx^2 \quad (211) \]

(v) Introduce now the orthonormal bases $\{e_a\}, \{\theta^a\}$ for $TU$ and $T^*U$ with

\[ e_1 = \frac{1}{R} \partial_1, \quad e_2 = \frac{1}{R \sin x^1} \partial_2, \quad (212) \]

\[ \theta^1 = R dx^1, \quad \theta^2 = R \sin x^1 dx^2. \quad (213) \]

We immediately get that

\[ [e_i, e_j] = c_{ij}^k e_k, \quad c_{12}^2 = -c_{21}^2 = -\cot x^1 \]

(vi) Writing

\[ e_a = q^a_\mu \partial_{\mu}, \quad \theta^a = q^a_\mu dx^\mu, \quad (214) \]
we read from Eq. (212) and Eq. (213),
\[ q_1^1 = \frac{1}{R}, \quad q_1^2 = 0, \]  
\[ q_2^1 = 0, \quad q_2^2 = \frac{1}{R \sin x^1}, \]  
\[ q_1^1 = R, \quad q_1^2 = 0, \]  
\[ q_2^1 = 0, \quad q_2^2 = R \sin x^1. \]  

(vii) Christoffel symbols. Before proceeding we put for simplicity \( R = 1 \).
Then, the non zero Christoffel symbols are:
\[ \nabla^+_{\partial_\mu} \partial_\nu = \Gamma^\rho_{\mu \nu} \partial_\rho, \]
\[ \Gamma^1_{22} = \Gamma^\theta_{\varphi \varphi} = -\cos \theta \sin \vartheta, \quad \Gamma^2_{21} = \Gamma^{\varphi}_{\theta \varphi} = \Gamma^2_{12} = \Gamma^{\varphi \theta} = \cot \vartheta. \]  

(viii) Then we have, e.g.,
\[ \nabla^-_{\partial_2} \theta^2 = \cot x^1 \theta^1 = \cot \theta^1 \]
\[ \nabla^-_{\partial_2} \theta^2 = \cos x^1 \sin x^1 \theta^2 = \cos \theta \sin \theta^2 \]
\[ \nabla^-_{\partial_1} \theta^1 = -\cos x^1 \theta^2 = -\cos \theta \theta^2, \]
\[ \nabla^-_{\partial_1} \theta^1 = 0. \]

(ix) We also have, e.g.,
\[ \nabla^-_{\partial_2} \theta^2 = \nabla^-_{\partial_2} (q_1^2 d^\mu) = \nabla_{\partial_2} (q_1^2 d^\mu) \]
\[ = \nabla^-_{\partial_2} (\sin x^1 dx^2) = \sin x^1 \nabla^-_{\partial_2} dx^2 = -\cos x^1 dx^1 \]
\[ = (\nabla^-_{\partial_2} q_1^2) dx^\mu. \]

Then, the symbols \( \nabla^-_{\partial_2} q_1^2 \) and \( \nabla^-_{\partial_2} q_2^2 \) are according to Eq. (66)
\[ \nabla^-_{\partial_2} q_2^1 = -\cot x^1 \neq 0, \]
\[ \nabla^-_{\partial_2} q_2^2 = 0. \]  

This seems strange, but is correct, because of the definition of the symbols \( \nabla^-_{\partial_\mu} q^a_\nu \) (see Eq. (55) and Eq. (57)). Now, even if \( q_1^2 = 0 \), and \( q_2^2 = \sin x^1 \), we get,
\[ \nabla^-_{\partial_1} q_2^2 = \partial_{x^1} q_1^2 - \Gamma_{12}^1 q_1^2 - \Gamma_{12}^2 q_2^2 = -\Gamma_{21}^2 q_2^2 = \cos x^1 - \cos x^1 = 0, \]
\[ \nabla^-_{\partial_2} q_2^2 = \partial_{x^2} q_1^2 - \Gamma_{22}^1 q_1^2 - \Gamma_{22}^2 q_2^2 = \partial_{x^2} (\sin x^1) - (\sin x^1 \cos x^1)(0) - (0)(\sin x^1) = 0. \]  

For future reference we note also that
\[ \nabla^-_{\partial_1} q_1^1 = 0, \quad \nabla^-_{\partial_2} q_1^1 = 0, \quad \nabla^-_{\partial_2} q_2^1 = 0, \]
\[ \nabla^-_{\partial_1} q_1^2 = 0, \quad \nabla^-_{\partial_2} q_2^1 = \cos x^1 \sin x^1, \quad \nabla^-_{\partial_1} q_2^2 = -\cos x^1. \]
So, in definitive we exhibit a counterexample to the naive ‘tetrad postulate’ (when $\nabla_{\mu} q_{\nu}^a$ is written $\nabla_{\mu} q_{\nu}^a$ and interpreted by an equation Eq. (66)), because we just found, e.g., that $\nabla_{2} q_{1}^2 = -\cos x^1 \neq 0$.

Note that in our example, if it happened that all the symbols $\nabla_{\mu} q_{\nu}^a = 0$, it would result that $\nabla_{e_b} e_a = 0$, for $a, b = 1, 2$. In that case the Riemann curvature tensor of $\nabla$ would be null and the torsion tensor would be non null. But this would be a contradiction, because in that case $\nabla$ would not be the Levi-Civita connection as supposed.

Suppose now that we calculate the symbols $\nabla_{\mu} q_{\nu}^a$ for our problem

We get,

$$\nabla_{1} q_{2}^1 = 0, \quad \nabla_{1} q_{2}^2 = \cos \vartheta, \quad \nabla_{2} q_{1}^1 = 0, \quad \nabla_{2} q_{1}^2 = \cos \vartheta,$$

and the torsion tensor is zero, as it may be. However, if we forget about the necessary distinction of symbols and use the symbols $\nabla_{\mu} q_{\nu}^a$ to calculate the torsion tensor we would get the wrong result.

$$T_{12}^2 = \cos \vartheta, \quad T_{21}^2 = -\cos \vartheta.$$  

Of course, we can define for the manifold $\tilde{S}_2$ introduced above a metric compatible teleparallel connection $\tilde{\nabla}$ (the so-called navigator or Columbus connection [39]), by imposing that

$$\tilde{\nabla}_{e_a} e_b = 0, \quad a, b = 1, 2$$  

This corresponds to the following transport law. A vector is parallel transported along a curve $C$ if the angle between the vector and the latitude line intersecting the curve $C$ is kept constant. For that particular connection the statement $\tilde{\nabla}_{-1} q_{1}^a = 0$ is correct. However, we may verify that for that connection the $\tilde{\nabla}_{+\mu} q_{\nu}^a$ are not all null and the torsion is not null, for we have $T_{12}^2 = -T_{21}^2 = \cot \vartheta$ And of course, should we use naively the always true equation $\tilde{\nabla}_{\mu} q_{\nu}^a = 0$, and use $\tilde{\nabla}_{\mu} q_{\nu}^a$ instead $\tilde{\nabla}_{+\mu} q_{\nu}^a$ of to calculate the components of torsion tensor we would obtain that it would be null, a contradiction.

**B Variation of $L_{EH}$**

Given a Lagrangian density $L_\wedge(\phi) = L_\wedge(x, \phi, d\phi)$ for a homogenous field $\phi \in \text{sec} \wedge^n T^\ast M \hookrightarrow \text{sec} \mathcal{U}(M, g)$ the functional derivative (or Euler Lagrange functional) of $L_\wedge$ is the functional $\star \Sigma$, with $\star \Sigma(\phi) = \frac{\delta L_\wedge}{\delta \phi}(\phi) \in \text{sec} \wedge^{n-1} T^\ast M \hookrightarrow$
sec $\mathcal{C}(M, g)$ such that
\[ \delta L_\wedge(\phi) = \delta \phi \wedge \frac{\partial L_\wedge(\phi)}{\partial \phi} + \delta (d\phi) \wedge \frac{\partial L_\wedge(\phi)}{\partial d\phi} \]
\[ \delta \phi \wedge \frac{\partial L_\wedge(\phi)}{\partial \phi} + d(\delta \phi) \wedge \frac{\partial L_\wedge(\phi)}{\partial d\phi} \]
\[ \delta \phi \wedge \frac{\partial L_\wedge(\phi)}{\partial \phi} \]
\[ \delta \phi \wedge \star \Sigma(\phi) + d \left( \delta \phi \wedge \frac{\partial L_\wedge(\phi)}{\partial d\phi} \right), \quad (232a) \]
\[ \star \Sigma(\phi) = \frac{\partial L_\wedge(\phi)}{\partial \phi} - (-1)^r d \left( \frac{\partial L_\wedge(\phi)}{\partial d\phi} \right) \quad (233) \]

Definition 39 The terms $\frac{\partial L_\wedge}{\partial \phi}$ and $\frac{\partial L_\wedge}{\partial d\phi}$ are called in what follows algebraic derivatives of $L_\wedge$ and $L_\wedge^4$.

For our present problem, we are fortunate, since we only need to know the following rule [55, 57] (besides, of course a series of identities of the Clifford bundle formalism that we summarized in Section 8): Given two action functionals depending, say, only on $\phi$ such that $F(\phi) \in \sec \wedge^r T^* M$ and $K(\phi) \in \sec \wedge^q T^* M$,
\[ \frac{\partial}{\partial \phi} [F(\phi) \wedge K(\phi)] = \frac{\partial}{\partial \phi} F(\phi) \wedge K(\phi) + (-1)^r F(\phi) \wedge \frac{\partial}{\partial \phi} K(\phi). \quad (234) \]

With these preliminaries, we can find the algebraic derivatives $\frac{\partial L_\wedge}{\partial \theta^a}$ and $\frac{\partial L_\wedge}{\partial d\theta^a}$ of Einstein-Hilbert Lagrangian density $L_{EH}$, necessary to obtain its variation. We know that
\[ L_{EH} = -d(\theta^a \wedge \star d\theta_a) - \frac{1}{2} d\theta^a \wedge \star d\theta_a + \frac{1}{2} \delta \theta^a \wedge \star \delta \theta_a + \frac{1}{4} (d\theta^a \wedge \theta_a) \wedge \star (d\theta^b \wedge \theta_b) \quad (235) \]

Before proceeding we must take into account that for any $\phi \in \sec \wedge^r T^* M \mapsto \sec \mathcal{C}(M, g)$, it holds as can be easily verified
\[ [\delta, \star] \phi = \delta \star \phi - \star \delta \phi \]
\[ = \delta \theta^a \wedge (\theta_a \star \phi) - \star [\delta \theta^a \wedge (\theta_a \star \phi)] \quad (236) \]

Then since if $\phi = \theta_c$, any variation induced by a local Lorentz rotation or by an arbitrary diffeomorphism must be a constrained variation of the Lorentz type

\[ \text{44This terminology was originally introduced in [7]. The exterior product } \delta \phi \wedge \frac{\partial}{\partial \phi} \text{ is a particular instance of the } A \wedge \frac{\partial}{\partial \phi} \text{ directional derivatives introduced in the multiform calculus developed in [13, 19, 20, 34, 35, 36, 37] with } \delta \phi = A. \]
Then, writing \( \delta \theta_c = \chi_{cd} \theta^d \), \( \chi_{cd} = - \chi_{dc} \). If that is the case, we have that for any product of 1-forms \( \theta^a \wedge ... \wedge \theta^d \) since \( \star (\theta^a \wedge ... \wedge \theta^d) = (\theta^a \wedge ... \wedge \theta^d) \lrcorner (\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3) \) it holds that

\[
\delta \star (\theta^a \wedge ... \wedge \theta^d) = \delta \theta_c \wedge [\theta^c \lrcorner \star (\theta^a \wedge ... \wedge \theta^d)] = \delta \theta_c \wedge \star (\theta^a \wedge ... \wedge \theta^d \wedge \theta^c).
\]

The first term in Eq. (236), \( \mathcal{L}^{(1)} = -d(\theta^a \wedge \star d \theta_a) \) is an exact differential and so did not contribute for the variation of the action. The variation of the second term, \( \mathcal{L}^{(2)} = -\frac{1}{2}d \theta^a \wedge \star d \theta_a \) is calculated as follows. We have

\[
\theta^b \wedge \theta^c \wedge \star d \theta_a = d \theta_a \wedge \star (\theta^b \wedge \theta^c).
\]

Then, writing \( \theta^{bc} = \theta^b \wedge \theta^c \), we have

\[
\theta^{bc} \wedge \delta \star d \theta_a = \delta \theta_a \wedge \star \theta^{bc} + d \theta_a \wedge \delta \star \theta^{bc} - \delta \theta_a \wedge \star d \theta_a
= \delta \theta_a \wedge \star \theta^{bc} + d \theta_a \wedge \delta \theta^d \wedge (\theta_d \star \theta^{bc}) - \delta \theta_a \wedge (\theta_d \wedge \theta^{bc}) \wedge \star d \theta_a
= \delta \theta_a \wedge \star \theta^{bc} + \delta \theta^d \wedge [\theta_a \wedge (\theta_d \star \theta^{bc}) - (\theta_d \wedge \theta^{bc}) \wedge \star d \theta_a].
\]

Multiplying Eq. (239) by \( \frac{1}{2} \lbrack (d \theta_a)_{bc} = -\frac{1}{2} \theta a \lrcorner \theta b \rbrack \) we get

\[
d \theta^a \wedge \delta \star d \theta_a = \delta \theta_a \wedge (\theta d \star d \theta^a) - (\theta d \wedge d \theta^a) \wedge \star d \theta_a.
\]

Taking into account that \( \delta (d \theta^a \wedge \star d \theta_a) = \delta d \theta^a \wedge \star d \theta_a + d \theta^a \wedge \delta \star d \theta_a \) we have

\[
\delta \mathcal{L}^{(2)} = -\delta d \theta^d \wedge \star d \theta^d + \frac{1}{2} \delta d \theta^d \wedge [\theta d \wedge d \theta^a \wedge \star d \theta_a - d \theta_a \wedge (\theta d \wedge d \theta^a)].
\]

From Eq. (241) it follows that the algebraic derivatives of \( \mathcal{L}^{(2)} \) relative to \( \theta^d \) and \( d \theta^d \) are:

\[
\frac{\partial \mathcal{L}^{(2)}}{\partial \theta^d} = \frac{1}{2} \lbrack \theta d \wedge d \theta^a \wedge \star d \theta_a - d \theta_a \wedge (\theta d \wedge d \theta^a) \rbrack,
\]

\[
\frac{\partial \mathcal{L}^{(2)}}{\partial d \theta^d} = - \star d \theta_a \]

The variation of the third term in \( \mathcal{L}_{EH}, \mathcal{L}^{(3)} = \frac{1}{2} \delta \theta^a \wedge \star d \theta_a = -\frac{1}{2} \star d \theta^a \wedge d \theta_a \) is calculated as follows. First, we observe that \( \theta^{bcrs} \wedge \star d \theta^a = d \star \theta^a \wedge \theta^{bcrs} \).

Then

\[
\theta^{bcrs} \wedge \delta \star d \theta_a = \delta \theta_a \wedge \star \theta^{bcrs} + d \star \theta_a \wedge \delta \star \theta^{bcrs} - \delta \theta_a \wedge \star d \theta_a
= \delta \theta_a \wedge \star \theta^{bcrs} - \delta \theta_a \wedge \wedge (\theta d \wedge \theta^{bcrs}) \wedge \star d \theta_a.
\]

Multiplying Eq. (243) by the coefficients \( \frac{1}{2} \lbrack \theta d \wedge d \theta^a \rbrack \) we get

\[
d \theta^a \wedge \delta \star d \theta_a = \delta d \theta_a \wedge \star d \theta^a - \delta \theta_a \wedge (\theta d \wedge d \theta^a) \wedge \star d \theta_a.
\]
The first member of the right hand side of Eq. (244) gives
\[
\delta d \wedge \ast d \wedge \theta = \delta \theta \wedge \ast d \wedge \theta = \delta \theta \wedge \ast d \wedge \theta - \delta \theta \wedge \ast d \wedge \theta - \delta \theta \wedge \ast d \wedge \theta,
\]
and recalling that \( \delta (d \wedge \ast d \wedge \theta) = \delta d \wedge \ast d \wedge \theta + d \wedge \theta \wedge \delta d \wedge \ast d \wedge \theta \)
we get
\[
\delta \mathcal{L}^{(3)} = -\delta d \theta \wedge \ast d \wedge \theta + \delta d \theta \wedge \ast d \wedge \theta + \delta \theta \wedge \ast d \wedge \theta.
\]
Then,
\[
\frac{\partial \mathcal{L}^{(3)}}{\partial d \theta} = \frac{1}{2} d (\theta \wedge \ast d \wedge \theta) + \frac{1}{2} (\theta \wedge \ast d \wedge \theta) - \delta \theta \wedge \ast d \wedge \theta.
\]

The variation of the fourth term of \( \mathcal{L}_{EH} \), \( \mathcal{L}^{(4)} = \frac{1}{4} (d \theta \wedge \ast (d \theta \wedge \theta)) \) is done as follows. First, we observe that since \( \theta \wedge \ast (d \theta \wedge \theta) = d \theta \wedge \theta \wedge \ast (d \theta \wedge \theta) \)
we can write
\[
\theta \wedge \ast (d \theta \wedge \theta) = \delta d \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta).
\]
Multiplying Eq. (248) by \( \frac{1}{4} (d \theta \wedge \theta) \) we get
\[
d \theta \wedge \ast (d \theta \wedge \theta) = \delta d \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \ast (d \theta \wedge \theta).
\]
Then, since
\[
\delta (d \theta \wedge \theta \wedge \ast (d \theta \wedge \theta)) = \delta d \theta \wedge \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \theta \wedge \ast (d \theta \wedge \theta) + \delta \theta \wedge \theta \wedge \ast (d \theta \wedge \theta),
\]

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it follows that

\[
\delta L^{(4)} = \frac{1}{2} \delta \theta^d \wedge \theta_a \wedge (d\theta^a \wedge \theta_a) \\
+ \frac{1}{2} \delta \theta^d \wedge d\theta_a \wedge (d\theta^a \wedge \theta_a) \\
- \frac{1}{4} \delta \theta^d \wedge d\theta^a \wedge \theta_a \wedge [\theta_d \lrcorner (d\theta^e \wedge \theta_e)] \\
- \frac{1}{4} \delta \theta^d \wedge [\theta_d \lrcorner (d\theta^e \wedge \theta_e)] \wedge (d\theta^a \wedge \theta_a).
\] (251)

Then,

\[
\frac{\partial L^{(4)}}{\partial \theta} = \frac{1}{2} d\theta_d \wedge * (d\theta^a \wedge \theta_a) - \frac{1}{4} d\theta^a \wedge \theta_a \wedge [\theta_d \lrcorner * (d\theta^e \wedge \theta_e)] \\
- \frac{1}{4} [\theta_d \lrcorner (d\theta^e \wedge \theta_e)] \wedge * (d\theta^a \wedge \theta_a),
\] (252)

\[
\frac{\partial L^{(4)}}{\partial \theta} = \frac{1}{2} \theta_d \wedge * (d\theta^a \wedge \theta_a).
\] (253)

Finally, disregarding the contribution of the exact differential and collecting all terms in Eqs. (242), (247) and (253) we get:

\[
\frac{\partial L}{\partial \theta} = \frac{1}{2} \left[ \theta_d \wedge d\theta - (\theta_d \wedge \theta_a) \wedge (d\theta^a \wedge \theta_a) \right] \\
+ \frac{1}{2} d (\theta_d \wedge \theta_a) \wedge * d * \theta_a + \frac{1}{2} (\theta_d \wedge d \theta) \wedge * d * \theta_a \\
+ \frac{1}{2} d\theta_a \wedge * (d\theta^a \wedge \theta_a) - \frac{1}{4} d\theta^a \wedge \theta_a \wedge [\theta_d \lrcorner * (d\theta^e \wedge \theta_e)] \\
- \frac{1}{4} [\theta_d \lrcorner (d\theta^e \wedge \theta_e)] \wedge * (d\theta^a \wedge \theta_a),
\] (254)

and

\[
\frac{\partial L}{\partial \theta} = - * d\theta_d - (\theta_d \wedge \theta_a) \wedge * d * \theta_a + \frac{1}{2} \theta_d \wedge * (d\theta^a \wedge \theta_a).
\] (255)

Collecting all these terms we arrive at the Euler Lagrange equation,

\[
\frac{\partial L}{\partial \theta} + d \left( \frac{\partial L}{\partial \theta^a} \right) = * t^a + d * S^a = - (\star R^a - \frac{1}{2} R \star \theta^a)
\] (256)

where \(S^a\) are the superpotentials and \(* t^a\) the (pseudo) energy-momentum 1-forms of the gravitational field (see, e.g., [43, 55])

\[
* S^c = \frac{1}{2} \omega_{ab} \wedge * (\theta^a \wedge \theta^b \wedge \theta^c) \in \sec \bigwedge^2 T^* M \hookrightarrow C^\ell (T^* M),
\]

\[
* t^c = - \frac{1}{2} \omega_{ab} \wedge [\omega_a^c \wedge (\theta^a \wedge \theta^b \wedge \theta^d) + \omega_b^c \wedge (\theta^a \wedge \theta^d \wedge \theta^e)]
\] (257)

\[\in \sec \bigwedge^3 T^* M \hookrightarrow C^\ell (T^* M).
\]

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C A Note on a Reply to a Previous Version of this Paper

After taking notice of a preliminary version of our paper posted at the arXiv [math-ph/0411085] the author of [12] posted a paper entitled: Refutation of Rodrigues: The Correctness of Differential Geometry (called simply reply, in what follows). As anticipated he said that he interpreted correctly the tetrad postulate, since for him a tetrad is a vector valued 1-form, although he did not explain if by this statement he means the pullback \( \theta \) of the soldering form \( \theta \) (see Eq. (106)), or the \((1−1)\) tensor \( Q \) (see Eq. (34)). He then claims that his conclusions concerning the proof of his ‘Evans lemma’ are correct. However a look to his reply reveals, that he did not really grasp what is going on. Indeed, the way he introduces into the game the symbols \( q^a_\nu \) in equation (6) of the reply, is as a ‘matrix’ connecting the components of a vector field \( V \) from the coordinate basis \( \{ \partial_\mu \} \) to the orthonormal basis. He explicitly wrote: \( V^a = q^a_\mu V^\mu \). This immediately requires that for each \( \nu \), the \( q^a_\nu \) are the components of the coordinate vector field \( \partial_\nu \) in the orthonormal basis \( \{ e_a \} \). To obtain the covariant derivative of \( \partial_\nu \) in the direction of the vector field \( \partial_\mu \) we need (as discussed above) use the covariant derivative \( \nabla^+_{\partial_\mu} \) which acts on the sections of \( TM \). Once we do this, as showed in detail in Section 4, we arrive at the calculation of \( \nabla^+_{\partial_\mu} q^a_\nu \). And, in general \( \nabla^+_{\partial_\mu} q^a_\nu \neq 0 \). The only licit way of obtaining the ‘tetrad postulate’ (and in this case it is a proposition, as we showed in the main text) is by calculation of the covariant derivative of the tensor field \( Q \) in the direction of the vector field \( \partial_\mu \), i.e., \( \nabla^+_{\partial_\mu} Q \). This has not be done in [12] nor has it been done in the reply. Thus, we state here: the would be proof of the ‘tetrad postulate’ offered in the reply is unfortunately one more example of wishful thinking.

Besides that, the reply, the author of [12], did not address himself to the other strong criticisms we done to his work (and which already appeared in the preliminary version of this paper) as, e.g.,

(i) our statement and demonstration that his proof of the ‘Evans lemma’ is a nonsequitur, that his Eq.(41) is completely meaningless,

(ii) our statement that the tetrad differential equations of his paper are wrong,

(ii) our statement that the sequence of calculations done by him in paper written sometime ago with collaborator (at that time Ph.D. student at Oxford) and that we reproduced in the conclusions our paper, shows that (unfortunately) he effectively does not know what a Clifford algebra is and worse, does not know how to multiply \( 2 \times 2 \) matrices.

These statements are sad facts that cannot be hidden anymore, and so cannot be considered as ad Hominen attack, contrary to many of the arguments that author of [12] used in his reply against one of us.

Also, it must be registered here that instead of directing himself to the

\[ \text{Posted at } \text{http://www.aias.us/Comments/comments01062005b.html} \]
mathematical questions, the author of [12] preferred to suggest to his readers that we must succumb under the weight of authorities. Indeed, he said that we are contradicting authors like, e.g., Carroll, Greene, Wheeler and Witten. What the author of [12] forgot is that a name does not mean authority in science. In the formal sciences a valid argument must fulfil the rules of logic. What we did was simply to find serious ambiguities in a statement that some authors called ‘tetrad postulate’, and the bad use made of that statement in some papers.

So, whereas it is true that we criticize some writings of the above authors (and some others, quoted in the references), we express here our admiration and respect for all of them, and also to any honest researcher that has at least enough humility to recognize errors. We are sure that our comments have been fair, educated and constructive. Besides that we think that our clarification of the necessity to explicitly distinguish the different covariant derivatives acting on different associate vector associate to the principal bundles $F(M)$ (and $P_{SO(3)}(M)$) will be welcome.

And to end, we must say that we agree with at least one statement of the reply, namely: that differential geometry is correct. However, the use that author of [12] made of this notable theory in his many papers is not correct. Certainly, the reader that knows enough Mathematics and had enough patience to arrive here already knows that.

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