On $\kappa$-reducibility of pseudovarieties of the form $V \ast D$

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Abstract

This paper deals with the reducibility property of semidirect products of the form $V \ast D$ relatively to graph equation systems, where $D$ denotes the pseudovariety of definite semigroups. We show that, if the pseudovariety $V$ is reducible with respect to the canonical signature $\kappa$ consisting of the multiplication and the $(\omega - 1)$-power, then $V \ast D$ is also reducible with respect to $\kappa$.

Keywords. Pseudovariety, definite semigroup, semidirect product, implicit signature, graph equations, reducibility.

1 Introduction

A semigroup (resp. monoid) pseudovariety is a class of finite semigroups (resp. monoids) closed under taking subsemigroups (resp. submonoids), homomorphic images and finite direct products. It is said to be decidable if there is an algorithm to test membership of a finite semigroup (resp. monoid) in that pseudovariety. The semidirect product of pseudovarieties has been getting much attention, mainly due to the Krohn-Rhodes decomposition theorem [18]. In turn, the pseudovarieties of the form $V \ast D$, where $D$ is the pseudovariety of all finite semigroups whose idempotents are right zeros, are among the most studied semidirect products [23, 25, 8, 11, 12]. For a pseudovariety $V$ of monoids, $LV$ denotes the pseudovariety of all finite semigroups $S$ such that $eSe \in V$ for all idempotents $e$ of $S$. We know from [17, 23, 24, 25] that $V \ast D$ is contained in $LV$ and that $V \ast D = LV$ if and only if $V$ is local in the sense of Tilson [23]. In particular, the equalities $Sl \ast D = LSl$ and $G \ast D = LG$ hold for the pseudovarieties $Sl$ of semilattices and $G$ of groups.
It is known that the semidirect product operator does not preserve decidability of pseudovarieties [20, 11]. The notion of *tameness* was introduced by Almeida and Steinberg [7, 8] as a tool for proving decidability of semidirect products. The fundamental property for tameness is *reducibility*. This property was originally formulated in terms of graph equation systems and latter extended to any system of equations [2, 21]. It is parameterized by an *implicit signature* $\sigma$ (a set of implicit operations on semigroups containing the multiplication), and we speak of $\sigma$-reducibility. For short, given an equation system $\Sigma$ with rational constraints, a pseudovariety $V$ is $\sigma$-reducible relatively to $\Sigma$ when the existence of a solution of $\Sigma$ by implicit operations over $V$ implies the existence of a solution of $\Sigma$ by $\sigma$-words over $V$ and satisfying the same constraints. The pseudovariety $V$ is said to be $\sigma$-*reducible* if it is $\sigma$-reducible with respect to every finite graph equation system. The implicit signature which is most commonly encountered in the literature is the *canonical signature* $\kappa = \{ab, a^{\omega-1}\}$ consisting of the multiplication and the $(\omega - 1)$-power. For instance, the pseudovarieties $D$ [11], $G$ [10, 8], $J$ [1, 2] of all finite $J$-trivial semigroups, $LSl$ [16] and $R$ [6] of all finite $R$-trivial semigroups are $\kappa$-reducible.

In this paper, we study the $\kappa$-reducibility property of semidirect products of the form $V \ast D$. This research is essentially inspired by the papers [15, 16] (see also [13] where a stronger form of $\kappa$-reducibility was established for $LSl$). We prove that, if $V$ is $\kappa$-reducible then $V \ast D$ is $\kappa$-reducible. In particular, this gives a new and simpler proof (though with the same basic idea) of the $\kappa$-reducibility of $LSl$ and establishes the $\kappa$-reducibility of the pseudovarieties $LG$, $J \ast D$ and $R \ast D$. Combined with the recent proof that the $\kappa$-word problem for $LG$ is decidable [14], this shows that $LG$ is $\kappa$-tame, a problem proposed by Almeida a few years ago. This also extends part of our work in the paper [15], where we proved that under mild hypotheses on an implicit signature $\sigma$, if $V$ is $\sigma$-reducible relatively to pointlike systems of equations (i.e., systems of equations of the form $x_1 = \cdots = x_n$) then $V \ast D$ is pointlike $\sigma$-reducible as well. As in [14], we use results from [5], where various kinds of $\sigma$-reducibility of semidirect products with an order-computable pseudovariety were considered. More specifically, we know from [5] that a pseudovariety of the form $V \ast D_k$ is $\kappa$-reducible when $V$ is $\kappa$-reducible, where $D_k$ is the order-computable pseudovariety defined by the identity $yx_1 \cdots x_k = x_1 \cdots x_k$. As $V \ast D = \bigcup_k V \ast D_k$, we utilize this result as a way to achieve our property concerning the pseudovarieties $V \ast D$. The method used in this paper is similar to that of [15]. However, some significant changes, inspired by [16], had to be introduced in order to deal with the much more intricate graph equation systems.

## 2 Preliminaries

The reader is referred to the standard bibliography on finite semigroups, namely [1, 21], for general background and undefined terminology. For basic definitions and results about combinatorics on words, the reader may wish to consult [19].
2.1 Words and pseudowords

Throughout this paper, $A$ denotes a finite non-empty set called an alphabet. The free semigroup and the free monoid generated by $A$ are denoted respectively by $A^+$ and $A^*$. The empty word is represented by $1$ and the length of a word $w \in A^*$ is denoted by $|w|$. A word is called primitive if it cannot be written in the form $a^n$ with $n > 1$. Two words $u$ and $v$ are said to be conjugate if $u = w_1 w_2$ and $v = w_2 w_1$ for some words $w_1, w_2 \in A^*$. A Lyndon word is a primitive word which is minimal in its conjugacy class, for the lexicographic order on $A^+$.

A left-infinite word on $A$ is a sequence $w = (a_n)_n$ of letters of $A$ indexed by $\mathbb{Z}$, and the free monoid generated by $A$ is represented by $\mathbb{N}$ and the length of a word $w$ there is a unique continuous homomorphism $\varphi : A \rightarrow S$ extending $\varphi$. The elements of $\varphi_A V$ are called pseudowords (or implicit operations) over $V$. A pseudovariety $V$ is called order-computable when the subsemigroup $\varphi_A V$ of $\varphi_A V$ generated by $A$ is finite, in which case $\varphi_A V = \varphi_A V$, and effectively computable. Recall that, for the pseudovariety $\mathbb{S}$ of all finite semigroups, $\varphi_A \mathbb{S}$ is (identified with) the free semigroup $A^+$. The elements of $\varphi_A \mathbb{S} \setminus A^+$ will then be called infinite pseudowords.

A pseudoidentity is a formal equality $\pi = \rho$ of pseudowords $\pi, \rho \in \varphi_A \mathbb{S}$ over $\mathbb{S}$. We say that $V$ satisfies the pseudoidentity $\pi = \rho$, and write $V \models \pi = \rho$, if $\varphi \pi = \varphi \rho$ for every continuous homomorphism $\varphi : \varphi_A \mathbb{S} \rightarrow S$ into a semigroup $S \in V$, which is equivalent to saying that $p_V \pi = p_V \rho$ for the natural projection $p_V : \varphi_A \mathbb{S} \rightarrow \varphi_A V$.

2.2 Pseudoidentities over $V \ast D_k$

For a positive integer $k$, let $D_k$ be the pseudovariety of all finite semigroups satisfying the identity $yx_1 \cdots x_k = x_1 \cdots x_k$. Denote by $A^k$ the set of words over $A$ with length $k$ and by $A_k$ the set $\{w \in A^+ : |w| \leq k\}$ of non-empty words over $A$ with length at most $k$. We notice that $\varphi_A D_k$ may be identified with the semigroup whose support set is $A_k$ and whose multiplication is given by $u \cdot v = t_k(uv)$, where $t_k uv$ denotes the longest suffix of length at most $k$ of a given (finite or left-infinite) word $w$. Then, the $D_k$ are order-computable pseudovarieties such that $D = \bigcup_k D_k$. Moreover, it is well-known that $\varphi_A D$ is isomorphic to the semigroup $A^{-\infty}$.

For each pseudoword $\pi \in \varphi_A \mathbb{S}$, we denote by $t_k \pi$ the unique smallest word (of $A_k$) such that $D_k \models \pi = t_k \pi$. Simetrically, we denote by $i_k \pi$ the smallest word (of $A_k$) such
that $K_k \models \pi = i_k \pi$, where $K_k$ is the dual pseudovariety of $D_k$ defined by the identity $x_1 \cdots x_k y = x_1 \cdots x_k$. Let $\Phi_k$ be the function $A^+ \to (A^{k+1})^*$ that sends each word $w \in A^+$ to the sequence of factors of length $k + 1$ of $w$, in the order they occur in $w$. We still denote by $\Phi_k$ (see [2] and [1, Lemma 10.6.11]) its unique continuous extension $\overline{\Omega}_A S \to (\overline{\Omega}_A S)^1$. This function $\Phi_k$ is a $k$-superposition homomorphism, with the meaning that it verifies the conditions:

i) $\Phi_k w = 1$ for every $w \in A_k$;

ii) $\Phi_k(\pi \rho) = \Phi_k \pi \Phi_k((t_k \pi \rho) = \Phi_k(\pi(i_k \rho)) \Phi_k \rho$ for every $\pi, \rho \in \overline{\Omega}_A S$.

Throughout the paper, $V$ denotes a non-locally trivial pseudovariety of semigroups. For any pseudowords $\pi, \rho \in \overline{\Omega}_A S$, it is known from [1] Theorem 10.6.12 that

$V \ast D_k \models \pi = \rho \iff i_k \pi = i_k \rho, \ t_k \pi = t_k \rho$ and $V \models \Phi_k \pi = \Phi_k \rho$.  

(2.1)

### 2.3 Implicit signatures and $\sigma$-reducibility

By an implicit signature we mean a set $\sigma$ of pseudowords (over $S$) containing the multiplication. In particular, we represent by $\kappa$ the implicit signature $\{ab, a^{\omega-1}\}$, usually called the canonical signature. Every profinite semigroup has a natural structure of a $\sigma$-algebra, via the natural interpretation of pseudowords on profinite semigroups. The $\sigma$-subalgebra of $\overline{\Omega}_A S$ generated by $A$ is denoted by $\overline{\Omega}_A S$. It is freely generated by $A$ in the variety of $\sigma$-algebras generated by the pseudovariety $S$ and its elements are called $\sigma$-words (over $S$). To a (directed multi)graph $\Gamma = V(\Gamma) \uplus E(\Gamma)$, with vertex set $V(\Gamma)$, edge set $E(\Gamma)$, and edges $ae \xrightarrow{e} \omega e$, we associate the system $\Sigma_\Gamma$ of all equations of the form $(\alpha e) w = \omega e$, with $e \in E(\Gamma)$. Let $S$ be a finite $A$-generated semigroup, $\delta : \overline{\Omega}_A S \to S$ be the continuous homomorphism respecting the choice of generators and $\varphi : \Gamma \to S^1$ be an evaluation mapping such that $\varphi E(\Gamma) \subseteq S$. We say that a mapping $\eta : \Gamma \to (\overline{\Omega}_A S)^1$ is a $V$-solution of $\Sigma_\Gamma$ with respect to $(\varphi, \delta)$ when $\delta \eta = \varphi$ and $V \models \eta u = \eta v$ for all $(u = v) \in \Sigma_\Gamma$. Furthermore, if $\eta \Gamma \subseteq (\overline{\Omega}_A S)^1$ for an implicit signature $\sigma$, then $\eta$ is called a $(V, \sigma)$-solution. The pseudovariety $V$ is said to be $\sigma$-reducible relatively to the system $\Sigma_\Gamma$ if the existence of a $V$-solution of $\Sigma_\Gamma$ with respect to a pair $(\varphi, \delta)$ entails the existence of a $(V, \sigma)$-solution of $\Sigma_\Gamma$ with respect to the same pair $(\varphi, \delta)$. We say that $V$ is $\sigma$-reducible, if it is $\sigma$-reducible relatively to $\Sigma_\Gamma$ for all finite graphs $\Gamma$.

### 3 $\kappa$-reducibility of $V \ast D$

Let $V$ be a given $\kappa$-reducible non-locally trivial pseudovariety. The purpose of this paper is to prove the $\kappa$-reducibility of the pseudovariety $V \ast D$. So, we fix a finite graph $\Gamma$ and a finite $A$-generated semigroup $S$ and consider a $V \ast D$-solution $\eta : \Gamma \to (\overline{\Omega}_A S)^1$ of the system $\Sigma_\Gamma$ with respect to a pair $(\varphi, \delta)$, where $\varphi : \Gamma \to S^1$ is an evaluation mapping such that $\varphi E(\Gamma) \subseteq S$ and $\delta : \overline{\Omega}_A S \to S$ is a continuous homomorphism respecting the choice of generators. We have to construct a $(V \ast D, \kappa)$-solution $\eta' : \Gamma \to (\overline{\Omega}_A S)^1$ of $\Sigma_\Gamma$ with respect to the same pair $(\varphi, \delta)$.
3.1 Initial considerations

Suppose that $g \in \Gamma$ is such that $\eta g = u$ with $u \in A^*$. Since $\eta$ and $\eta'$ are supposed to be $V \ast D$-solutions of the system $\Sigma_\Gamma$ with respect to $(\varphi, \delta)$, we must have $\delta g = \varphi = \delta' g$ and so, in particular, $\delta \eta' g = \delta u$. As the homomorphism $\delta : \overline{\Omega}_A S \to S$ is arbitrarily fixed, it may happen that the equality $\delta \eta' g = \delta u$ holds only when $\eta' g = u$. In that case we would be obliged to define $\eta' g = u$. Since we want to describe an algorithm to define $\eta'$ that should work for any given graph and solution, we will then construct a solution $\eta'$ verifying the following condition:

$$\forall g \in \Gamma, \ (\eta g \in A^* \implies \eta' g = \eta g).$$  \(\mathcal{C}_1(\Gamma, \eta, \eta')\)

Suppose next that a vertex $v \in V(\Gamma)$ is such that $D \models \eta v = u^\omega$ with $u \in A^+$, that is, suppose that $p_D \eta v = u^\omega$. Because $\Gamma$ is an arbitrary graph, it could include, for instance, an edge $e$ such that $\alpha e = \omega e = v$ and the labeling $\eta$ could be such that $\eta e = u$. Since $D$ is a subpseudovariety of $V \ast D$, $\eta$ is a $D$-solution of $\Sigma_\Gamma$ with respect to $(\varphi, \delta)$. Hence, as by condition $\mathcal{C}_1(\Gamma, \eta, \eta')$ we want to preserve finite labels, it would follow in that case that $D \models (\eta' v) u = \eta' v$ and, thus, that $D \models \eta' v = u^\omega = \eta v$. This observation suggests that we should preserve the projection into $\overline{\Omega}_A D$ of labelings of vertices $v$ such that $p_D \eta v = u^\omega$ with $u \in A^+$. More generally, we will construct the $(V \ast D, \kappa)$-solution $\eta'$ in such a way that the following condition holds:

$$\forall v \in V(\Gamma), \ (p_D \eta v = u^\omega z \text{ with } u \in A^+ \text{ and } z \in A^* \implies p_D \eta' v = p_D \eta v).$$  \(\mathcal{C}_2(\Gamma, \eta, \eta')\)

Let $\ell_\eta = \max\{|u| : u \in A^* \text{ and } \eta g = u \text{ for some } g \in \Gamma\}$ be the maximum length of finite labels under $\eta$ of elements of $\Gamma$. To be able to make some reductions on the graph $\Gamma$ and solution $\eta$, described in Section 3.2, we want $\eta'$ to verify the extra condition below, where $L \geq \ell_\eta$ is a non-negative integer to be specified later, on Section 3.3

$$\forall v \in V(\Gamma), \ (\eta v = u \pi \text{ with } u \in A_L \implies \eta' v = u \pi' \text{ with } \delta \pi = \delta \pi').$$  \(\mathcal{C}_3(\Gamma, \eta, \eta')\)

3.2 Simplifications on the solution $\eta$

We begin this section by reducing to the case in which all vertices of $\Gamma$ are labeled by infinite pseudowords under $\eta$. Suppose first that there is an edge $v \xrightarrow{\varphi} w$ such that $\eta v = u_\varphi$ and $\eta e = u_\omega$ with $u_\varphi \in A^*$ and $u_\omega \in A^+$, so that $\eta w = u_\varphi u_\omega$. Drop the edge $e$ and consider the restrictions $\eta_1$ and $\varphi_1$, of $\eta$ and $\varphi$ respectively, to the graph $\Gamma_1 = \Gamma \setminus \{e\}$. Then $\eta_1$ is a $V \ast D$-solution of the system $\Sigma_{\Gamma_1}$ with respect to the pair $(\varphi_1, \delta)$. Assume that there is a $(V \ast D, \kappa)$-solution $\eta'_1$ of $\Sigma_{\Gamma_1}$ with respect to $(\varphi_1, \delta)$ verifying condition $\mathcal{C}_1(\Gamma_1, \eta_1, \eta'_1)$. Then $\eta'_1 v = u_\varphi$ and $\eta'_1 w = u_\varphi u_\omega$. Let $\eta'$ be the extension of $\eta'_1$ to $\Gamma$ obtained by letting $\eta' e = u_\omega$. Then $\eta'$ is a $(V \ast D, \kappa)$-solution of $\Sigma_\Gamma$ with respect to $(\varphi, \delta)$. By induction on the number of edges labeled by finite words under $\eta$ beginning in vertices also labeled by finite words under $\eta$, we may therefore assume that there are no such edges in $\Gamma$.

Now, we remove all vertices $v$ of $\Gamma$ labeled by finite words under $\eta$ such that $v$ is not the beginning of an edge, thus obtaining a graph $\Gamma_1$. As above, if $\eta'_1$ is a $(V \ast D, \kappa)$-solution of
\(\Sigma_{\Gamma_1}\), then we build a \((V \ast D, \kappa)\)-solution \(\eta'\) of \(\Sigma_{\Gamma}\) by letting \(\eta'\) coincide with \(\eta'_1\) on \(\Gamma_1\) and letting \(\eta'v = \eta v\) for each vertex \(v \in \Gamma \setminus \Gamma_1\). So, we may assume that all vertices of \(\Gamma\) labeled by finite words under \(\eta\) are the beginning of some edge.

Suppose next that \(v \xrightarrow{\delta} w\) is an edge such that \(\eta v = u\) and \(\eta e = \pi \in A^*\) and \(\pi \in \overline{\Omega_4}S \setminus A^+.\) Notice that, since it is an infinite pseudoword, \(\pi\) can be written as \(\pi = \pi_1\pi_2\) with both \(\pi_1\) and \(\pi_2\) being infinite pseudowords. Drop the edge \(e\) (and the vertex \(v\) in case \(e\) is the only edge beginning in \(v\)) and let \(v_1\) be a new vertex and \(v_1 \xrightarrow{\delta \eta'_1} w\) be a new edge thus obtaining a new graph \(\Gamma_1\). Let \(\eta_1\) and \(\varphi_1\) be the labelings of \(\Gamma_1\) defined as follows:

- \(\eta_1\) and \(\varphi_1\) coincide, respectively, with \(\eta\) and \(\varphi\) on \(\Gamma' = \Gamma_1 \cap \Gamma\);
- \(\eta_1v_1 = u\pi_1, \eta_1e_1 = \pi_2, \varphi_1v_1 = \delta\eta_1v_1\) and \(\varphi_1e_1 = \delta\eta_1e_1\).

Then \(\eta_1\) is a \((V \ast D)\)-solution of the system \(\Sigma_{\Gamma_1}\) with respect to the pair \((\varphi_1, \delta)\). Assume that there is a \((V \ast D, \kappa)\)-solution \(\eta'_1\) of \(\Sigma_{\Gamma_1}\) with respect to \((\varphi_1, \delta)\) verifying conditions \(\mathcal{C}_3(\Gamma_1, \eta_1, \eta'_1)\) and \(\mathcal{C}_3(\Gamma_1, \eta_1, \eta'_1)\). In particular, since \(L\) is chosen to be greater than \(\ell_\eta, \eta'_1v_1 = u\pi'_1 \text{ with } \delta\pi'_1 = \delta\pi'_1.\) Let \(\eta''\) be the extension of \(\eta'_1|_{\Gamma'}\) to \(\Gamma\) obtained by letting \(\eta'' e = \pi'_1(\eta'_1e_1)\) (and \(\eta''v = u\) in case \(v \notin \Gamma'\). As one can easily verify, \(\eta''\) is a \((V \ast D, \kappa)\)-solution of \(\Sigma_{\Gamma}\) with respect to \((\varphi, \delta)\). By induction on the number of edges beginning in vertices labeled by finite words under \(\eta\), we may therefore assume that all vertices of \(\Gamma\) are labeled by infinite pseudowords under \(\eta\).

Suppose at last that an edge \(e \in \Gamma\) is labeled under \(\eta\) by a finite word \(u = a_1 \cdots a_n\), where \(n > 1\) and \(a_i \in A\). Denote \(v_0 = \alpha e\) and \(v_n = \omega e\). In this case, we drop the edge \(e\) and, for each \(i \in \{1, \ldots, n - 1\},\) we add a new vertex \(v_i\) and a new edge \(v_{i-1} \xrightarrow{\delta \eta'_1} v_i\) to the graph \(\Gamma\). Let \(\Gamma_1\) be the graph thus obtained and let \(\eta_1\) and \(\varphi_1\) be the labelings of \(\Gamma_1\) defined as follows:

- \(\eta_1\) and \(\varphi_1\) coincide, respectively, with \(\eta\) and \(\varphi\) on \(\Gamma' = \Gamma_1 \setminus \{e\}\);
- for each \(i \in \{1, \ldots, n - 1\}, \eta_1v_i = (\eta v)a_1 \cdots a_i, \eta_1e_i = a_i, \varphi_1v_i = \delta\eta_1v_i\) and \(\varphi_1e_i = \delta\eta_1e_i\).

Hence, \(\eta_1\) is a \((V \ast D)\)-solution of the system \(\Sigma_{\Gamma_1}\) with respect to the pair \((\varphi_1, \delta)\). Suppose there exists a \((V \ast D, \kappa)\)-solution \(\eta'_1\) of \(\Sigma_{\Gamma_1}\) with respect to \((\varphi_1, \delta)\) verifying condition \(\mathcal{C}_3(\Gamma_1, \eta_1, \eta'_1)\). Let \(\eta''\) be the extension of \(\eta'_1|_{\Gamma'}\) to \(\Gamma\) obtained by letting \(\eta'' e = u\). Then \(\eta''\) is a \((V \ast D, \kappa)\)-solution of \(\Sigma_{\Gamma}\) with respect to \((\varphi, \delta)\). By induction on the number of edges labeled by finite words under \(\eta\), we may further assume that each edge of \(\Gamma\) labeled by a finite word under \(\eta\) is, in fact, labeled by a letter of the alphabet.

### 3.3 Borders of the solution \(\eta\)

The main objective of this section is to define a certain class of finite words, called borders of the solution \(\eta\). Since the equations (of \(\Sigma_{\Gamma}\)) we have to deal with are of the form \((\alpha e) e = \omega e,\) these borders will serve to signalize the transition from a vertex \(\alpha e\) to the edge \(e,\)

For each vertex \(v\) of \(\Gamma,\) denote by \(d_v \in A^{-N}\) the projection \(pD\eta v\) of \(\eta v\) into \(\overline{\Omega_4}D\) and let \(D_\eta = \{d_v \mid v \in V(\Gamma)\}.\) We say that two left-infinite words \(v_1, v_2 \in A^{-N}\) are confinal if they
have a common prefix \( y \in A^{-N} \), that is, if \( v_1 = yz_1 \) and \( v_2 = yz_2 \) for some words \( z_1, z_2 \in A^* \). As one easily verifies, the relation \( \propto \) defined, for each \( d_{v_1}, d_{v_2} \in D_\eta \), by

\[
d_{v_1} \propto d_{v_2} \quad \text{if and only if} \quad d_{v_1} \text{ and } d_{v_2} \text{ are confinal}
\]

is an equivalence on \( D_\eta \). For each \( \propto \)-class \( \Delta \), we fix a word \( y_\Delta \in A^{-N} \) and words \( z_v \in A^* \), for each vertex \( v \) with \( d_v \in \Delta \), such that \( d_v = y_\Delta z_v \).

Moreover, when \( d_v \) is ultimately periodic, we choose \( y_\Delta \) of the form \( u^{-\infty} \), with \( u \) a Lyndon word, and fix \( z_v \) not having \( u \) as a prefix. The word \( u \) and its length \( |u| \) will be said to be, respectively, a root and a period of the solution \( \eta \). Without loss of generality, we assume that \( \eta \) has at least one root (otherwise we could, easily, modify the graph and the solution in order to include one).

We fix a few of the integers that will be used in the construction of the \((V * D, \kappa)\)-solution \( \eta' \). They depend only on the mapping \( \eta \) and on the semigroup \( S \).

**Definition 3.1 (constants \( n_S, p_\eta, L, E \) and \( Q \))** We let:

- \( n_S \) be the exponent of \( S \) which, as one recalls, is the least integer such that \( s^{n_S} \) is idempotent for every element \( s \) of the finite \( A \)-generated semigroup \( S \);
- \( p_\eta = \text{lcm}\{|u| : u \in A^+ \text{ is a root of } \eta\} \);
- \( L = \max\{\ell_\eta, |z_v| : v \in V(\Gamma)\} \);
- \( E \) be an integer such that \( E \geq n_S p_\eta \) and, for each word \( w \in A^E \), there is a factor \( e \in A^+ \) of \( w \) for which \( \delta e \) is an idempotent of \( S \). Notice that, for each root \( u \) of \( \eta \), \( |u^{n_S}| \leq E \) and \( \delta(u^{n_S}) \) is an idempotent of \( S \);
- \( Q = L + E \).

For each positive integer \( m \), we denote by \( B_m \) the set

\[
B_m = \{ t_m y_\Delta \in A^m \mid \Delta \text{ is a } \propto \text{-class} \}.
\]

If \( y_\Delta = u^{-\infty} \) is a periodic left-infinite word, then the element \( y = t_m y_\Delta \) of \( B_m \) will be said to be periodic (with root \( u \) and period \( |u| \)). For words \( y_1, y_2 \in B_m \), we define the gap between \( y_1 \) and \( y_2 \) as the positive integer

\[
g(y_1, y_2) = \min\{|u| \in \mathbb{N} : u \in A^+ \text{ and, for some } v \in A^+, y_1 u = vy_2 \text{ or } y_2 u = vy_1\},
\]

and notice that \( g(y_1, y_2) = g(y_2, y_1) \leq m \).

**Proposition 3.2** Consider the constant \( Q \) introduced in Definition 3.1. There exists \( q_Q \in \mathbb{N} \) such that for all integers \( m \geq q_Q \) the following conditions hold:
(a) If \( y_1 \) and \( y_2 \) are distinct elements of \( B_m \), then \( g(y_1, y_2) > Q \);

(b) If \( y \) is a non-periodic element of \( B_m \), then \( g(y, y) > Q \).

**Proof.** Suppose that, for every \( q_Q \in \mathbb{N} \) there is an integer \( m \geq q_Q \) and elements \( y_{m,1} \) and \( y_{m,2} \) of \( B_m \) such that \( g(y_{m,1}, y_{m,2}) \leq Q \). Hence, there exist a strictly increasing sequence \( (m_i)_i \) of positive integers and an integer \( r \in \{1, \ldots, Q\} \) such that \( (g(y_{m_i,1}, y_{m_i,2}))_i \) is constant and equal to \( r \). Moreover, since the graph \( \Gamma \) is finite, we may assume that \( y_{m_i,1} = t_{m_i} y_{\Delta_1} \) and \( y_{m_i,2} = t_{m_i} y_{\Delta_2} \) for every \( i \) and some \( \alpha \)-classes \( \Delta_1 \) and \( \Delta_2 \). It then follows that \( y_{\Delta_1} u = y_{\Delta_2} \) or \( y_{\Delta_2} u = y_{\Delta_1} \) for some word \( u \in \Sigma^r \). Hence, \( y_{\Delta_1} \) and \( y_{\Delta_2} \) are confinal left-infinite words, whence \( \Delta_1 \) and \( \Delta_2 \) are the same \( \alpha \)-class \( \Delta \). Therefore, for every \( m, y_{m,1} \) and \( y_{m,2} \) have the same length and are suffixes of the word \( y_{\Delta} \) and, so, \( y_{m,1} \) and \( y_{m,2} \) are the same word. This proves already \( (a) \). Now, notice that \( y_{\Delta} u = y_{\Delta} \), meaning that \( y_{\Delta} \) is the periodic left-infinite word \( u^{-\infty} \). This shows \( (b) \) and completes the proof of the proposition.

We now fix two more integers.

**Definition 3.3 (constants \( M \) and \( k \))** We let:

- \( M \) be an integer such that \( M \) is a multiple of \( p_\eta \) and \( M \) is greater than or equal to the integer \( q_Q \) of Proposition \( \ref{prop:3.2} \) and notice that \( M > Q \);
- \( k = M + Q \).

The elements of the set \( B_M \) will be called the **borders of the solution** \( \eta \). We remark that the borders of \( \eta \) are finite words of length \( M \) such that, by Proposition \( \ref{prop:3.2} \) for any two distinct occurrences of borders \( y_1 \) and \( y_2 \) in a finite word, either these occurrences have a gap of size at least \( Q \) between them, or \( y_1 \) and \( y_2 \) are the same periodic border \( y \). In this case, \( y \) is a power of its root \( u \), since \( M \) is a multiple of the period \( |u| \), and \( g(y, y) \) is \( |u| \).

### 3.4 Getting a \( (V \ast D_k, \kappa) \)-solution

As \( V \ast D_k \) is a subpseudovariety of \( V \ast D \), \( \eta \) is a \( V \ast D_k \)-solution of \( \Sigma_\Gamma \) with respect to \( (\varphi, \delta) \). The given pseudovariety \( V \) was assumed to be \( \kappa \)-reducible. So, by \cite[Corollary 6.5]{[5]} \( V \ast D_k \) is \( \kappa \)-reducible too. Therefore, there is a \( (V \ast D_k, \kappa) \)-solution \( \eta'_k : \Gamma \rightarrow (\Omega_A S)^1 \) of \( \Sigma_\Gamma \) with respect to the same pair \( (\varphi, \delta) \). Moreover, as observed in \cite[Remark 3.4]{[6]} one can constrain the values \( \eta'_k g \) of each \( g \in \Gamma \) with respect to properties which can be tested in a finite semigroup. Since the prefixes and the suffixes of length at most \( k \) can be tested in the finite semigroup \( \Omega_A K_k \times \Omega_A D_k \), we may assume further that \( \eta'_k g \) and \( \eta g \) have the same prefixes and the same suffixes of length at most \( k \). We then denote

\[
i_g = i_{k, \eta'_k g} = i_k \eta g \quad \text{and} \quad t_g = t_{k, \eta'_k g} = t_k \eta g,
\]

for each \( g \in \Gamma \). Notice that, by the simplifications introduced in Section \( \ref{sec:3.2} \) if \( \eta g \) is a finite word, then \( g \) is an edge and \( \eta g \) is a letter \( a_g \) and so \( i_g = t_g = a_g \). Otherwise, \( i_g \) and \( t_g \) are
length \( k \) words. In particular, condition \( C_1(\Gamma, \eta, \eta'_k) \) holds. That is, \( \eta'_k e = \eta e \) for every edge \( e \) such that \( \eta e \) is a finite word. On the other hand, Lemma 2.3 (ii) of [12], which is stated only for edges, can be extended easily to vertices, so that \( \eta'_k g \) can be assumed to be an infinite pseudoword for every \( g \in \Gamma \) such that \( \eta g \) is infinite. Thus, in particular, \( \eta'_k v \) is an infinite pseudoword for all vertices \( v \).

Notice that, for each vertex \( v \), there exists a border \( y_v \) of \( \eta \) such that the finite word \( y_v z_v \) is a suffix of \( \eta v \). On the other hand, by Definitions 3.1 and 3.3 \( |z_v| \leq L < Q \) and \( k = M + Q \). So, as \( |y_v| = M \),

\[
\tau_v = x_v y_v z_v \quad \text{and} \quad \eta'_k v = \pi_v \tau_v \tag{3.1}
\]

for some infinite \( \kappa \)-word \( \pi_v \) and some word \( x_v \in A^+ \) with \( |x_v| = Q - |z_v| \).

3.5 Basic transformations

The objective of this section is to introduce the basic steps that will allow to transform the \((V * D_k, \kappa)\)-solution \( \eta'_k \) into a \((V * D, \kappa)\)-solution \( \eta' \). The process of construction of \( \eta' \) from \( \eta'_k \) is close to the one used in [15] to handle with systems of pointlike equations. Both procedures are supported by (basic) transformations of the form

\[
a_1 \cdots a_k \mapsto a_1 \cdots a_j (a_i \cdots a_j)^\omega a_{j+1} \cdots a_k,
\]

which replace words of length \( k \) by \( \kappa \)-words. Those procedures differ in the way the indices \( i \leq j \) are determined. In the pointlike case, the only condition that a basic transformation had to comply with was that \( j \) had to be minimum such that the value of the word \( a_1 \cdots a_k \) under \( \delta \) as well, but the equations \((\omega e) e = \omega e \) impose an extra restriction that is not required by pointlike equations. Indeed, we need \( \eta' \) to verify, in particular, \( \delta \eta' o e = \delta \eta_k o e (= \delta \eta o e) \) and \( \delta \eta' e = \delta \eta'_k e (= \delta \eta e) \). So, somewhat informally, for a word \( a_1 \cdots a_k \) that has an occurrence overlapping both the factors \( \eta'_k o e \) and \( \eta'_k e \) of the pseudoword \( (\eta'_k o e)(\eta'_k e) \), the introduction of the factor \( (a_i \cdots a_j)^\omega \) by the basic transformation should be done either in \( \eta'_k o e \) or in \( \eta'_k e \), and not in both simultaneously. The borders of the solution \( \eta \) were introduced to help us to deal with this extra restriction. Informally speaking, the borders will be used to detect the “passage” from the labeling under \( \eta'_k \) of a vertex \( o e \) to the labeling of the edge \( e \) and to avoid that the introduction of \( (a_i \cdots a_j)^\omega \) affect the labelings under \( \delta \) of \( \eta'_k o e \) or \( \eta'_k e \).

Consider an arbitrary word \( w = a_1 \cdots a_n \in A^+ \). An integer \( m \in \{ M, \ldots, n \} \) will be called a bound of \( w \) if the factor \( w_m = a_m' \cdots a_m \) of \( w \) is a border, where \( m' = m - M + 1 \). The bound \( m \) will be said to be periodic or non-periodic according to the border \( w_m \) is periodic or not. If \( w \) admits bounds, then there is a maximum one that we name the last bound of \( w \). In this case, if \( \ell \) is the last bound of \( w \), then the border \( w_{[\ell]} \) will be called the last border of \( w \). Notice that, by Proposition 3.2 and the choice of \( M \), if \( m_1 \) and \( m_2 \) are two bounds of \( w \) with \( m_1 < m_2 \), then either \( m_2 - m_1 > Q \) or \( w_{[m_1]} \) and \( w_{[m_2]} \) are the same periodic border.
Let \( w = a_1 \cdots a_k \in A^+ \) be a word of length \( k \). Notice that, since \( k = M + Q \), if \( w \) has a non-periodic last bound \( \ell \), then \( \ell \) is the unique bound of \( w \). We split the word \( w \) in two parts, \( 1_w \) (the left-hand of \( w \)) and \( r_w \) (the right-hand of \( w \)), by setting
\[
1_w = a_1 \cdots a_s \quad \text{and} \quad r_w = a_{s+1} \cdots a_k
\]
where \( s \) (the splitting point of \( w \)) is defined as follows: if \( w \) has a last bound \( \ell \) then \( s = \ell \); otherwise \( s = k \). In case \( w \) has a periodic last bound \( \ell \), the splitting point \( s \) will be said to be periodic. Then, \( s \) is not periodic in two situations: either \( w \) has a non-periodic last border or \( w \) has not a last border. The factorization
\[
w = 1_w r_w
\]
will be called the splitting factorization of \( w \). We have \( s \geq M > Q \geq E \). So, by definition of \( E \), there exist integers \( i \) and \( j \) such that \( s - E < i < j \leq s \) and the factor \( e = a_i \cdots a_j \) of \( 1_w \) verifies \( de = (de)^2 \). We begin by fixing the maximum such \( j \) and, for that \( j \), we fix next an integer \( i \) and a word \( e_w = a_i \cdots a_j \), called the essential factor of \( w \), as follows. Notice that, if the splitting point \( s \) is periodic and \( u \) is the root of the last border of \( w \), then \( \delta(u^ns) \) is idempotent and the left-hand of \( w \) is of the form \( 1_w = 1'_w u^ns \). Hence, in this case, \( j = s \) and we let \( e_w = u^w \), thus defining \( i \) as \( j - ns|u| + 1 \). Suppose now that the splitting point is not periodic. In this case we let \( i \) be the maximum integer such that \( \delta(a_i \cdots a_{j-1}) \) is idempotent. The word \( w \) can be factorized as \( w = 1'_w e_w 1'_w r_w \), where \( 1'_w = a_1 \cdots a_{i-1} \). We then denote by \( \hat{w} \) the following \( \kappa \)-word
\[
\hat{w} = 1'_w e_w e_w^1 1'_w r_w = a_1 \cdots a_j (a_i \cdots a_j)^w a_{j+1} \cdots a_k
\]
and notice that \( \delta \hat{w} = \delta w \). Moreover \( |e_w 1'_w| \leq E \) and so \( |1'_w| \geq M - E > Q - E = L \). It is also convenient to introduce two \( \kappa \)-words derived from \( \hat{w} \)
\[
\lambda_k w = a_1 \cdots a_j (a_i \cdots a_j)^w, \quad \varrho_k w = (a_i \cdots a_j)^w a_{j+1} \cdots a_k.
\]
This defines two mappings \( \lambda_k, \varrho_k : A^k \to \Omega^*_A \mathcal{S} \) that can be extended to \( \Omega_A \mathcal{S} \) as done in [15]. Although they are not formally the same mappings used in that paper, because of the different choice of the integers \( i \) and \( j \), we keep the same notation since the selection process of those integers is absolutely irrelevant for the purpose of the mappings. That is, with the above adjustment the mappings maintain the properties stated in [15].

The next lemma presents a property of the \( \sim \)-operation that is fundamental to our purposes.

**Lemma 3.4** For a word \( w = a_1 \cdots a_{k+1} \in A^+ \) of length \( k + 1 \), let \( w_1 = a_1 \cdots a_k \) and \( w_2 = a_2 \cdots a_{k+1} \) be the two factors of \( w \) of length \( k \). If \( \hat{w}_1 = a_1 \cdots a_{j_1} (a_i_1 \cdots a_{j_1})^w a_{j_1+1} \cdots a_k \) and \( \hat{w}_2 = a_2 \cdots a_{j_2} (a_{i_2} \cdots a_{j_2})^w a_{j_2+1} \cdots a_{k+1} \), then \( a_1 1_{w_2} = 1_{w_1} x \) for some word \( x \in A^* \). In particular \( j_1 \leq j_2 \).
Proof. Write \( w_2 = b_1 \cdots b_k \) with \( b_i = a_{i+1} \). Let \( s_1 \) and \( s_2 \) be the splitting points of \( w_1 \) and \( w_2 \) respectively, whence \( 1_{w_1} = a_1 \cdots a_{s_1} \) and \( 1_{w_2} = b_1 \cdots b_{s_2} = a_2 \cdots a_{s_2+1} \). To prove that there exists a word \( x \) such that \( a_1 1_{w_2} = 1_{w_1} x \), we have to show that \( s_1 \leq s_2 + 1 \). Under this hypothesis, we then deduce that \( a_{i_1} \cdots a_{j_1} \) is an occurrence of the essential factor \( e_{w_1} \) in \( 1_{w_2} \) which proves that \( j_1 \leq j_2 \).

Assume first that \( w_1 \) has a last bound \( \ell_1 \), in which case \( s_1 = \ell_1 \). By definition, \( \ell_1 \geq M \). If \( \ell_1 > M \), then the last border of \( w_1 \) occurs in \( w_2 \), one position to the left relatively to \( w_1 \). Hence \( \ell_1 - 1 \) is a bound of \( w_2 \) and, so, \( w_2 \) has a last bound \( \ell_2 \) such that \( \ell_2 \geq \ell_1 - 1 \). It follows in this case that \( s_2 = \ell_2 \) and \( s_1 \leq s_2 + 1 \). Suppose now that \( \ell_1 = M \). Since \( s_2 \geq M \) by definition, the condition \( s_1 \leq s_2 + 1 \) holds trivially in this case. Suppose now that \( w_1 \) has not a last bound. Then \( s_1 = k \). Moreover, either \( w_2 \) does not have a last bound or \( k \) is its last bound. In both circumstances \( s_2 = k \), whence \( s_1 = s_2 \leq s_2 + 1 \). This concludes the proof of the lemma.

In the conditions of the above lemma and as in \([15]\), we define \( \psi_k : (\overline{\Omega}_{A^{k+1}}S)^1 \to (\overline{\Omega}_A S)^1 \) as the only continuous monoid homomorphism which extends the mapping

\[
A^{k+1} \to \Omega_A^k S
a_1 \cdots a_{k+1} \mapsto (a_1 \cdots a_{j_1})^{\omega} a_{j_1+1} \cdots a_{j_2} (a_{j_2} \cdots a_{j_2})^{\omega}
\]

and let \( \theta_k = \psi_k \Phi_k \). The function \( \theta_k : \overline{\Omega}_A S \to (\overline{\Omega}_A S)^1 \) is a continuous \( k \)-superposition homomorphism since it is the composition of the continuous \( k \)-superposition homomorphism \( \Phi_k \) with the continuous homomorphism \( \psi_k \). We remark that a word \( w = a_1 \cdots a_n \) of length \( n > k \) has precisely \( r = n - k + 1 \) factors of length \( k \) and

\[
\theta_k(w) = \psi_k(a_1 \cdots a_{k+1}, a_2 \cdots a_{k+2}, \ldots, a_{r-1} \cdots a_n)
= \psi_k(a_1 \cdots a_{k+1}) \psi_k(a_2 \cdots a_{k+2}) \cdots \psi_k(a_{r-1} \cdots a_n)
= (e_1^\omega f_1^\omega f_2^\omega f_3^\omega \cdots e_r^\omega f_{r-1}^\omega f_r^\omega)
= e_1^\omega f_1^\omega f_2^\omega f_3^\omega \cdots e_r^\omega f_{r-1}^\omega f_r^\omega.
\]

where, for each \( p \in \{1, \ldots, r\} \), \( e_p \) is the essential factor \( e_{w_p} = a_{i_p} \cdots a_{j_p} \) of the word \( w_p = a_p \cdots a_{k+p-1} \) and \( f_p = a_{j_p+1} \cdots a_{j_{p+1}} \) \((p \neq r)\). Above, for each \( p \in \{2, \ldots, r-1\} \), we have replaced each expression \( e_p^\omega f_p^\omega \) with \( e_p^\omega \) since, indeed, these expressions represent the same \( \kappa \)-word. More generally, one can certainly replace an expression of the form \( x^\omega \cdot x^n \cdot x^\omega \) with \( x^\omega \cdot x^n \).

Using this reduction rule as long as possible, \( \theta_k(w) \) can be written as

\[
\theta_k(w) = e_n^\omega f_1^\omega e_n^\omega f_2^\omega \cdots e_q^\omega f_q^\omega.
\]

called the reduced form of \( \theta_k(w) \), where \( q \in \{1, \ldots, r\}, 1 = n_1 < n_2 < \cdots < n_q \leq r, f_p = f_{n_p} \cdots f_{n_{p+1}} \) \((p \in \{1, \ldots, q-1\})\) and \( f_q \) is \( f_{n_q} \cdots f_{r-1} \) if \( n_q \neq r \) and it is the empty word otherwise.
3.6 Definition of the \((V \ast D, \kappa)\)-solution \(\eta'\)

We are now in conditions to describe the procedure to transform the \((V \ast D_k, \kappa)\)-solution \(\eta'_k\) into the \((V \ast D, \kappa)\)-solution \(\eta'\). The mapping \(\eta' : \Gamma \rightarrow (\Omega^k_A(S))^1\) is defined, for each \(g \in \Gamma\), as

\[
\eta'g = (\tau_1g)(\tau_2g)(\tau_3g),
\]

where, for each \(i \in \{1, 2, 3\}\), \(\tau_i : \Gamma \rightarrow (\Omega^k_A(S))^1\) is a function defined as follows.

First of all, we let

\[
\tau_2 = \theta_k\eta'_k.
\]

That \(\tau_2\) is well-defined, that is, that \(\tau_2g\) is indeed a \(\kappa\)-word for every \(g \in \Gamma\), follows from the fact that \(\eta'_k g\) is a \(\kappa\)-word and \(\theta_k\) transforms \(\kappa\)-words into \(\kappa\)-words (see [15]). Next, for each vertex \(v\), consider the length \(k\) words \(i_v = i_k\eta'_k v = i_k\eta v\) and \(t_v = t_k\eta'_k v = t_k\eta v\). We let

\[
\tau_1v = \lambda_k i_v \quad \text{and} \quad \tau_3v = \varrho_k t_v,
\]

where the mappings \(\lambda_k\) and \(\varrho_k\) were defined in (3.2). Note that, by (3.1), \(\tau_v = x_{\eta\eta_k} z_v\). Moreover, the occurrence of \(y_v\), shown in this factorization is the last occurrence of a border in \(t_v\). Hence, the right-hand \(r_{\tau_v}\) of \(t_v\) is precisely \(z_v\). Therefore, one has

\[
\tau_1v = \lambda_k i_v = 1^l_{\lambda_k} e_{i_v} \quad \text{and} \quad \tau_3v = \varrho_k t_v = e^w_{\varrho_k} 1^l_{\varrho_k} z_v.
\]

Consider now an arbitrary edge \(e\). Suppose that \(\eta e\) is a finite word. Then, \(\eta e\) is a letter \(a_e\) and \(\eta'_k e\) is also \(a_e\) in this case. Then \(\tau_2 e = \theta_k a_e = 1\) because \(\theta_k\) is a \(k\)-superposition homomorphism. Since we want \(\eta' e\) to be \(a_e\), we then define, for instance,

\[
\tau_1 e = a_e \quad \text{and} \quad \tau_3 e = 1.
\]

Suppose at last that \(\eta e\) (and so also \(\eta'_k e\)) is an infinite pseudoword. We let

\[
\tau_3 e = \varrho_k t_e
\]

and notice that \(\tau_2 e = \tau_3 e\). Indeed, as \(\eta'_k\) is a \(V \ast D_k\)-solution of \(\Sigma\), it follows from (2.1) that \(t_e = t_k\eta'_k e = t_k\eta'_k \omega e = t_{\omega e}\). The definition of \(\tau_1 e\) is more elaborate. Let \(v\) be the vertex \(\omega e\) and consider the word \(t_v i_e = a_1 \cdots a_{2k}\). This word has \(r = k + 1\) factors of length \(k\). Suppose that \(\theta_k(t_v i_e)\) is \(e^w_1 f_1 e^w_2 f_2 \cdots e^w_{r-1} f_{r-1} e^w_r\) and consider its reduced form

\[
\theta_k(t_v i_e) = e^w_1 f_1 e^w_{n_2} f_2 \cdots e^w_{n_r} f_q.
\]

Notice that \(t_v i_e = \tilde{f}_0 \tilde{f}_1 \cdots \tilde{f}_q \tilde{f}_{q+1}\) for some words \(\tilde{f}_0, \tilde{f}_{q+1} \in A^*\). Hence, there is a (unique) index \(m \in \{1, \ldots, q\}\) such that \(t_v = \tilde{f}_0 \tilde{f}_1 \cdots \tilde{f}_{m-1} \tilde{f}_m\) and \(f_m = \tilde{f}_m \tilde{f}_m^m\) with \(\tilde{f}_m \in A^*\) and \(\tilde{f}_m^m \in A^+\). Then \(\theta_k(t_v i_e) = \beta_1 \beta_2\), where \(\beta_1 = e^w_1 f_1 e^w_{n_2} f_2 \cdots e^w_{n_m} f_m\) and \(\beta_2 = f_m e^w_{n_{m+1}} f_{m+1} \cdots e^w_{n_q} f_q\) and we let

\[
\tau_1 e = \beta_2 = f_m^m e^w_{n_{m+1}} f_{m+1} \cdots e^w_{n_q} f_q.
\]

Note that the word \(\beta'_2 = f_m^m f_{m+1} \cdots f_q\) is \(a_{k+1} \cdots a_{q}\), whence \(\beta'_{2} e^w_r = \lambda_k i_e\).

The next lemma is a key result that justifies the definition of the \(^\sim\) operation.
Lemma 3.5 Let \( e \) be an edge such that \( pe \) is infinite. Then, with the above notation, \( \beta_1 = \tau_3 \nu \) and so \( \theta_k(t_v i_e) = (\tau_3 \nu)(\tau_1 e) \). Moreover, \( \delta \tau_1 e = \delta \lambda_k i_e \).

Proof. We begin by recalling that \( t_v i_e = a_1 \cdots a_{2k} \) and

\[
\theta_k(t_v i_e) = e_1^\omega f_1 e_2^\omega f_2 \cdots e_r^\omega f_r e_{r-1}^\omega = e_1^\omega f_1 e_2^\omega f_2 \cdots e_r^\omega f_r q,
\]

where \( e_p \) is the essential factor \( e_{w_p} = a_{i_p} \cdots a_{j_p} \) of the word \( w_p = a_p \cdots a_{k+p-1} \) and \( f_p = a_{j_p+1} \cdots a_{j_p+1} \) for each \( p \). Note also that \( \lambda_k i_e = \beta_2^2 e_r^\omega \), \( e_r \) is a suffix of \( \beta_2^2 \) and \( \delta \epsilon_r \) is idempotent. So, to prove the equality \( \delta \tau_1 e = \delta \lambda_k i_e \) it suffices to show that \( \delta \tau_1 e = \delta \beta_2^2 \). We know from [3.1] that \( t_v = x_v y_v z_v \) with \( 1 \leq |x_v| \leq Q \). So, \( x_v = a_1 \cdots a_{h-1}, y_v = a_h \cdots a_{M+h-1} \) and \( z_v = a_{M+h} \cdots a_k \) for some \( h \in \{2, \ldots, Q+1\} \). There are two cases to verify.

Case 1. \( y_v \) is a non-periodic border. Consider the factor \( w_h = a_h \cdots a_{k+h-1} \) of \( t_v i_e \). By the choice of \( M \) and \( k \), the prefix \( y_v \) is the only occurrence of a border in \( w_h \). Hence, \( M \) is the last bound of \( w_h \) and, so, its splitting point. It follows that \( w_h = y_v \cdot z_v a_{k+1} \cdots a_{k+h-1} \) is the splitting factorization of \( w_h \). Therefore, as one can verify for an arbitrary \( p \in \{1, \ldots, h\} \), there is only one occurrence of a border in \( w_p \), precisely \( y_v \), and the splitting factorization of \( w_p \) is

\[
w_p = a_p \cdots a_{h-1} y_v \cdot z_v a_{k+1} \cdots a_{k+p-1},
\]

whence \( e_p = e_1 \) with \( j_p = j_1 \leq M + h - 1 \) and, so, \( f_p = 1 \) for \( p < h \). So, the prefix \( e_1^\omega f_1 e_2^\omega f_2 \cdots f_r e_r^\omega \) of \( \theta_k(t_v i_e) \) reduces to \( e_1^\omega \). Consider now the factor \( w_{h+1} = a_{h+1} \cdots a_{k+h} \).

Hence, either \( w_{h+1} \) does not have a last bound or \( k \) is its last bound. In both situations, the splitting point of \( w_{h+1} \) is \( k \) and its splitting factorization is \( w_{h+1} = w_{h+1}^1 \cdot 1 \). Therefore, one deduces from Lemma [3.1] that, for every \( p \in \{h+1, \ldots, r\} \), the occurrence \( a_{i_p} \cdots a_{j_p} \) of the essential factor \( e_{w_p} \) in \( w_p \) is, in fact, an occurrence in the suffix \( w' = a_{k+h-E} \cdots a_{2k} = a_{M+L} a_{L+h} \cdots a_{2k} \) of \( t_v i_e \). Since \( |x_v y_v| = M + h - 1 \) and \( |z_v| \leq L \), it follows that \( k = |x_v y_v z_v| < M + L + h \), whence \( w' \) is a suffix of \( i_e \) and so \( k < i_p < j_p \) for all \( p \in \{h+1, \ldots, r\} \). This means, in particular, that the \( \omega \)-power \( e_{h+1}^\omega \) is introduced at the suffix \( i_e \) of \( t_v i_e \). Hence \( \beta_1 = e_1^\omega f_1 e_2^\omega f_2 \cdots f_{r-1} e_r^\omega a_{h+1} \cdots a_k \) and its reduced form is \( e_1^\omega a_{h+1} \cdots a_k = \tau_1 \nu \), which proves that \( \beta_1 \) and \( \tau_1 \nu \) are the same \( \omega \)-word. Moreover, from \( k < i_p \), one deduces that the word \( e_p \) is a suffix of \( a_{k+1} \cdots a_{j_p} \), which proves that \( \delta \tau_1 e = \delta \beta_2^2 \).

Case 2. \( y_v \) is a periodic border. Let \( u \) be the root of \( y_v \). Then, since \( M \) was fixed as a multiple of \( |u| \), \( y_v = u^M u \) where \( M_u = M \). If the prefix \( y_v \) is the only occurrence of a border in \( w_h \), then one deduces the lemma as in Case 1 above. So, we assume that there is another occurrence of a border \( y \) in \( w_h \). Hence, by Proposition [3.22] and the choice of \( M \) and \( k \), \( y \) is precisely \( y_v \). Furthermore, since \( u \) is a Lyndon word and \( k = M + Q \) with \( Q < M \), \( w_h = y_v u^d w_h' \) for some positive integer \( d \) and some word \( w_h' \in A^* \) such that \( u \) is not a prefix of \( w_h' \). Notice that, since \( u \) is not a prefix of \( z_v \) by definition of this word, \( z_v \) is a proper prefix of \( u \). On the other hand \( w_h = u^d y_v w_h' \) and the occurrence of \( y_v \) shown in
this factorization is the last occurrence of $y_v$ in $w_h$. Thus, 
$$w_h = u^d y_v \cdot u'_h$$
is the splitting factorization of $w_h$. Therefore, $\bar{w}_h = u^d y_v (u^w)\omega u'_h$ and $e_h = u^w$. More generally, for any $p \in \{1, \ldots, h\}$, $y_v$ is a factor of $w_p$ and it is the only border that occurs in $w_p$. Hence, the splitting point of $w_p$ is periodic and $e_p = u^w$. Moreover, as one can verify, $j_1 = M + h - 1$ and the prefix $e_1 f_1 e_2 f_2 \cdots f_{h-1} e_h$ of $\theta_k(\tau_v \lambda_e)$ is $e_1^\omega (u(e_1^\omega)[u])d$ and so, analogously to Case 1, it reduces to $e_1^\omega u^d$. Since $z_v$ is a proper prefix of $u$ and $d \geq 1$, $k < j_h$. This allows already deduce that the reduced form of $\beta_1$ is $(u^w)^\omega z_v = \tau_3 v$, thus concluding the proof of the first part of the lemma. Now, there are two possible events. Either $m = q$ and $\beta_2 = \bar{f}_m = \beta_2'$, in which case $\delta \tau_1 e = \delta \beta_2'$ is trivially verified. Or $m \neq q$ and the $\omega$-power $e_{n_{m+1}}^\omega$ was not eliminated in the reduction process of $\theta_k(\tau_v \lambda_e)$. This means that the splitting point of the word $w_{n_{m+1}}$ is not determined by one of the occurrences of the border $y_v$ in the prefix $a_1 \cdots a_{k+h-1}$ of $\tau_v \lambda_e$. Then, as in Case 1 above, one deduces that $k < i_p$ for each $p \in \{n_{m+1}, \ldots, r\}$ and, so, that $\delta \tau_1 e = \delta \beta_2'$.

In both cases $\beta_1 = \tau_3 v$ and $\delta \tau_1 e = \delta \lambda_k \lambda_e$. Hence, the proof of the lemma is complete. \qed

Notice that, as shown in the proof of Lemma 3.5 above, if a vertex $v$ is such that $y_v$ is a periodic border with root $u$, then $\tau_3 v = (u^w)^\omega z_v$. So, the definition of the mapping $\tau_3$ on vertices assures condition $\mathcal{C}_2(\Gamma, \eta, \eta')$.

### 3.7 Proof that $\eta'$ is a $(\mathbf{V} \ast \mathbf{D}, \kappa)$-solution

This section will be dedicated to showing that $\eta'$ is a $(\mathbf{V} \ast \mathbf{D}, \kappa)$-solution of $\Sigma_{\Gamma}$ with respect to the pair $(\varphi, \delta)$ verifying conditions $\mathcal{C}_1(\Gamma, \eta, \eta')$ and $\mathcal{C}_3(\Gamma, \eta, \eta')$.

We begin by noticing that $\eta' g$ is a $\kappa$-word for every $g \in \Gamma$. Indeed, as observed above, each $\tau_2 g$ is a $\kappa$-word. That both $\tau_1 g$ and $\tau_3 g$ are $\kappa$-words too, is easily seen by their definitions. Let us now show the following properties.

**Proposition 3.6** Conditions $\delta \eta' = \varphi$, $\mathcal{C}_1(\Gamma, \eta, \eta')$ and $\mathcal{C}_3(\Gamma, \eta, \eta')$ hold.

**Proof.** As $\eta'_k$ is a $\mathbf{V} \ast \mathbf{D}_k$-solution of $\Sigma_{\Gamma}$ with respect to $(\varphi, \delta)$ and, so, the equality $\delta \eta'_k = \varphi$ holds, to deduce that $\delta \eta' = \varphi$ holds it suffices to establish the equality $\delta \eta' = \delta \eta'_k$. Consider first a vertex $v \in \Gamma$. Then $\tau_1 v = \lambda_k \hat{i}_v = 1^\varphi_i \lambda_v \lambda_e^\varphi_i$ and $\tau_3 v = \varphi_k t_v = e_1^\varphi_i 1^\varphi_i z_v$. In this case, the equality $\delta \eta'_k v = \delta \eta' v$ is a direct application of [13, Proposition 5.3], where the authors proved that

$$\delta \pi = \delta (\lambda_k \hat{i}_k \pi) (\theta_k \pi) (\varphi_k t_k \pi)$$

(3.3)

for every pseudoword $\pi$. Moreover, by definition of the $\hat{}$-operation, $|1^\varphi_i| > L$. Therefore, $\eta v$ and $\eta' v$ are of the form $\eta v = u \pi$ and $\eta' v = u \pi'$ with $u \in A^L$ and $\delta \pi = \delta \pi'$. So, condition $\mathcal{C}_3(\Gamma, \eta, \eta')$ holds.
Consider next an edge \( e \in \Gamma \). If \( \eta'_k e \) is a finite word \( a_e \), then \( \eta' e = (\tau_1 e)(\tau_2 e)(\tau_3 e) = a_e \cdot 1 = a_e = \eta'_k e \), whence \( \delta \eta' e = \delta \eta'_k e \) holds trivially. Moreover, since \( \eta'_k e = \eta e \) in this case and every vertex is labeled under \( \eta \) by an infinite pseudoword, it follows that condition \( \mathcal{E}'(\Gamma, \eta, \eta') \) holds. Suppose at last that \( \eta'_k e \) is infinite and let \( v = \alpha e \). Then \( \tau_3 e = \varrho \kappa \tau_3 e \). On the other hand, by Lemma 3.3, \( \delta \tau_3 e = \delta \lambda_k \iota e \). Hence, by \( \{3.3\} \) and since \( \delta \) is a homomorphism, \( \delta \eta' e = \delta((\tau_1 e)(\tau_2 e)(\tau_3 e)) = \delta((\lambda_k \iota e)(\theta_k \eta'_k e)(\varrho \kappa e)) = \delta \eta'_k e \). This ends the proof of the proposition. \( \blacksquare \)

Consider an arbitrary edge \( v \xrightarrow{\varphi} w \) of \( \Gamma \). To achieve the objectives of this section it remains to prove that \( V * D \) satisfies \( (\eta' v)(\eta' e) = \eta' w \). Since \( \eta'_k e \) is a \( V * D_k \)-solution of \( \Sigma \Gamma \), \( V * D_k \) satisfies \( (\eta'_k v)(\eta'_k e) = \eta'_k w \). Hence, by \( \{2.1\} \), \( \iota v = \iota_k ((\eta'_k v)(\eta'_k e)) = \iota_k (\eta'_k w) = \iota w \) and \( \tau_k ((\eta'_k v)(\eta'_k e)) = \tau_k (\eta'_k w) = \tau w \). Thus, \( \tau_1 v = \lambda_k \iota v = 1'_{v} e_{1v} e_{1v} e_{1w} = 1'_{iw} e_{1w} e_{1w} = \lambda_k \iota w = \tau_1 w \) and \( \tau_3 w = \varrho \kappa t_w = e_{1w} 1'_{t_w} z_w \). As shown in the proof of \( \{15\} \) Proposition 5.4, it then follows that \( V * D \) satisfies \( e_{1w} \lambda_k (\eta'_k v)(\eta'_k e)) e_{1w} = e_{1w} \lambda_k (\eta'_k w) e_{1w} \) and, so,

\[
V * D \models (\tau_1 v) \theta_k ((\eta'_k v)(\eta'_k e)) (\tau_3 w) = (\tau_1 w) \theta_k (\eta'_k w) (\tau_3 w) = \eta' w. \tag{3.4}
\]

On the other hand, from the fact that \( \theta_k \) is a \( k \)-superposition homomorphism one deduces

\[
\theta_k ((\eta'_k v)(\eta'_k e)) = \theta_k (\eta'_k v) \theta_k (\tau_1 v) \theta_k (\eta'_k e) = \theta_k (\eta'_k v) \theta_k (\tau_1 e) \theta_k (\eta'_k e). \tag{3.5}
\]

Suppose that \( \eta'_k e \) is an infinite pseudoword. In this case \( t_v = t_w \), whence \( \tau_3 e = \tau_3 w \). Moreover, by Lemma 3.3, \( \theta_k (\tau_1 e) = (\tau_3 v)(\tau_1 e) \). Therefore, by conditions \( \{2.1\} \) and \( \{3.3\} \), \( V * D \) satisfies \( (\eta' v)(\eta' e) = \eta' w \). Assume now that \( \eta'_k e \) is a finite word, whence \( \eta'_k e = a_e \in A \) and \( \eta' e = a_e \). Since \( \eta \) is a \( D \)-solution of \( \Sigma \Gamma \), \( D \models (\eta' v) a_e = \eta' w \) and, thus, \( d_v a_e = d_w \). Hence the left-infinite words \( d_v \) and \( d_w \) are confinal and, so, \( \infty \)-equivalent. Hence \( d_v = y_{\Delta} z_v \), \( d_w = y_{\Delta} z_w \) and \( y_v = y_w = t_v y_{\Delta} \), where \( \Delta \) is the \( \infty \)-class of \( d_v \) and \( d_w \). It follows that \( y_{\Delta} z_v a_e = y_{\Delta} z_w \) and \( t_k (\tau_1 v) a_e = t_w \). In this case, \( \theta_k ((\eta'_k v)(\eta'_k e)) = \theta_k (\eta'_k v) \theta_k (\tau_1 v) a_e \). On the other hand, \( \tau_v a_e = a_1 \cdots a_k a_{k+1} a_1 t_v w \) is a word of length \( k + 1 \) and, so, \( \theta_k (\tau_v a_e) = \psi_k (\tau_v a_e) \) is of the form

\[
\theta_k (\tau_v a_e) = e_{1w}^{\omega} f e_{2}^{\omega}.
\]

The splitting factorizations of \( \tau_v \) and \( \tau_w \) are, respectively, \( \tau_v = x_v y_v \cdot z_v \) and \( \tau_w = x w y w \cdot z w \). Since \( y_v = y_w \), it follows that \( e_1 = e_{tv} = e_{tw} = e_2 \).

Suppose that \( z_v a_e = z_w \). In this case it is clear that \( f = 1 \), so that \( \theta_k (\tau_v a_e) = e_{tv}^{\omega} \).

Since \( \theta_k (\eta'_k v) \) ends with \( e_{tv}^{\omega} \), it then follows that \( \theta_k ((\eta'_k v)(\eta'_k e)) = \theta_k \eta'_k v = \tau_2 v \). Therefore, \( (\tau_1 v) \theta_k ((\eta'_k v)(\eta'_k e)) (\tau_3 w) = (\tau_1 v) (\tau_2 v) (\tau_3 w) \).

On the other hand,

\[
\tau_3 w = \theta_k \tau w = e_{tv}^{\omega} \tau' \tau w z_w = e_{tv}^{\omega} 1''_{tw} z_w a_e = (\tau_3 v) a_e.
\]

So, by \( \{3.3\} \), one has that \( V * D \) satisfies \( (\eta' v) a_e = (\tau_1 v)(\tau_2 v) (\tau_3 w) a_e = (\tau_1 v)(\tau_2 v)(\tau_3 w) = \eta' w \).

Suppose now that \( z_v a_e \neq z_w \). In this case, one deduces from the equality \( y_{\Delta} z_v a_e = y_{\Delta} z_w \), that \( y_{\Delta} \) is a periodic left-infinite word. Let \( u \) be its root, so that \( y_{\Delta} = u^{\infty} \), \( e_{tv} = u^{e_{tv}} \) and \( 1_{tw}^{''} = 1_{tw}^{'} = 1 \). Since, by definition, \( u \) is a primitive word which is not a prefix of \( z_v \) nor a prefix of
We conclude that $z_w a_e = u$ and $z_w = 1$. In this case $f = u$, whence $\theta_1(t_v a_e) = e_w^u u$. Then, $\theta_1((\eta^w k_v) (\eta^w e)) = (\theta_1 k_v^w u) = (\tau_2 u)$. Therefore, $(\tau_1 u) \theta_1((\eta^w k_v) (\eta^w e)) (\tau_3 u) = (\tau_1 u) (\tau_2 u) u (\tau_3 u)$. Moreover,
\[
\theta_1((\eta^w k_v) (\eta^w e)) (\tau_3 u) = (\tau_1 u) (\tau_2 u) u (\tau_3 u).
\]
Therefore, using (3.4), one deduces as above that \( V * D \) satisfies $$(\eta^w k_v) (\eta^w e) = (\tau_3 u) a_e.$$ Therefore, using (3.4), one deduces as above that \( V * D \) satisfies $$(\eta^w k_v) (\eta^w e) = (\tau_3 u) a_e.$$ We have proved the main theorem of the paper.

**Theorem 3.7** If \( V \) is \( \kappa \)-reducible, then \( V * D \) is \( \kappa \)-reducible.

This result applies, for instance, to the pseudovarieties \( L, G, J \) and \( R \). Since the \( \kappa \)-word problem for the pseudovariety \( LG \) of local groups is already solved [14], we obtain the following corollary.

**Corollary 3.8** The pseudovariety \( LG \) is \( \kappa \)-tame.

**Final remarks.** In this paper we fixed our attention on the canonical signature \( \kappa \), while in [15] we dealt with a more generic class of signatures \( \sigma \) verifying certain undemanding conditions. Theorem 3.7 is still valid for such generic signatures \( \sigma \) but we preferred to treat only the instance of the signature \( \kappa \) to keep the proofs clearer and a little less technical.

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