The Geometrical Structure of Phase Space of the Controlled Hamiltonian System with Symmetry

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Dedicated to My Advisor—Professor Hesheng Hu
on the Occasion of Her 90th Birthday
February 5, 2018

Abstract: In this paper, from the viewpoint of completeness of Marsden-Weinstein reduction, we illustrate how to give the definitions of a controlled Hamiltonian (CH) system and a reducible controlled Hamiltonian system with symmetry; and how to describe the dynamics of a CH system and the controlled Hamiltonian equivalence; as well as how to give the regular point reduction and the regular orbit reduction for a CH system with symmetry, by analyzing carefully the geometrical and topological structures of the phase space and the reduced phase space of the corresponding Hamiltonian system. We also introduce briefly some recent developments in the study of reduction theory for the CH systems with symmetries and applications.

Keywords: cotangent bundle, Marsden-Weinstein reduction, RCH system, CH-equivalence, Poisson reduction.

AMS Classification: 70H33, 53D20, 70Q05.

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1 Introduction

The reduction theory for a mechanical system with symmetry is an important subject and it is widely studied in the theory of mathematics and mechanics, as well as applications. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. Over forty years ago, the regular symplectic reduction for the Hamiltonian system with symmetry and coadjoint equivariant momentum map was set up by famous professors Jerrold E. Marsden and Alan Weinstein, which is called Marsden-Weinstein reduction, and great developments have been obtained around the work in the theoretical study and applications of mathematics, mechanics and physics; see Abraham and Marsden [1], Abraham et al. [2], Arnold [3], Libermann and Marle [9], Marsden [11], Marsden et al. [12], Marsden and Ratiu [16], Marsden et al. [17], Marsden and Weinstein [19] and Ortega and Ratiu [21].

Recently, in Marsden et al. [18], the authors found that the symplectic reduced space of a cotangent bundle may not be a cotangent bundle, and hence the set of Hamiltonian systems with symmetries on the cotangent bundle is not complete under the Marsden-Weinstein reduction. This is a serious problem. If we define directly a controlled Hamiltonian system with symmetry on a cotangent bundle, then it is possible that the Marsden-Weinstein reduced system may not have definition. The study of completeness of Hamiltonian reductions for a Hamiltonian system with symmetry is related to the geometrical and topological structures of Lie group, configuration manifold and its cotangent bundle, as well as the action way of Lie group on the configuration manifold and its cotangent bundle. In order to define the CH system and set up the various perfect reduction theory for the CH systems, we have to give the precise analysis of geometrical and topological structures of the phase spaces and reduced phase spaces of various CH systems.

A controlled Hamiltonian (CH) system is a Hamiltonian system with external force and control. In general, a CH system under the action of external force and control is not Hamiltonian, however, it is a dynamical system closely related to a Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of Hamiltonian systems. Thus, we emphasize explicitly the impact of external force and control in the study of CH systems. For example, in order to describe the feedback control law to modify the structures of CH system and the reduced CH system, we introduce the notions of CH-equivalence, RpCH-equivalence and RoCH-equivalence and so on. In this paper, at first, from the viewpoint of completeness of Marsden-Weinstein reduction, we shall illustrate how to give the definitions of a CH system and a reducible CH system; Next, we illustrate how to describe the dynamics of a CH system and the controlled Hamiltonian equivalence; The third, we give the regular point reduction and the regular orbit reduction for a CH system with symmetry, by analyzing carefully the geometrical and topological structures of the phase space and the reduced phase space of the corresponding Hamiltonian system. Finally, we also introduce briefly some recent developments.
in the study of reduction theory for the CH systems with symmetries and applications. These research works not only gave a variety of reduction methods for the CH systems, but also showed a variety of relationships of the controlled Hamiltonian equivalences of these systems.

2 Marsden-Weinstein Reduction on a Cotangent Bundle

It is well-known that, in mechanics, the phase space of a Hamiltonian system is very often the cotangent bundle \( T^*Q \) of a configuration manifold \( Q \), and the reduction theory on the cotangent bundle of a configuration manifold is a very important special case of general symplectic reduction theory. In the following we first give the Marsden-Weinstein reduction for a Hamiltonian system with symmetry on the cotangent bundle of a smooth configuration manifold with canonical symplectic structure, see Abraham and Marsden [1] and Marsden and Weinstein [19].

Let \( Q \) be a smooth manifold and \( TQ \) the tangent bundle, \( T^*Q \) the cotangent bundle with a canonical symplectic form \( \omega_0 \). Assume that \( \Phi : G \times Q \to Q \) is a left smooth action of a Lie group \( G \) on the manifold \( Q \). The cotangent lift is the action of \( G \) on \( T^*Q \), \( \Phi^*: G \times T^*Q \to T^*Q \) given by \( g \cdot \alpha_q = (T\Phi_g^{-1})^* \cdot \alpha_q, \forall \alpha_q \in T^*_qQ, q \in Q \). The cotangent lift of any proper (resp. free) \( G \)-action is proper (resp. free). Assume that the cotangent lift action is symplectic with respect to the canonical symplectic form \( \omega_0 \), and has an \( \text{Ad}^* \)-equivariant momentum map \( J : T^*Q \to \mathfrak{g}^* \), given by \( < J(\alpha_q), \xi > = \alpha_q(\xi\omega_0(q)) \), where \( \xi \in \mathfrak{g}, \xi\omega_0(q) \) is the value of the infinitesimal generator \( \xi\omega_0 \) of the \( G \)-action at \( q \in Q \), \( \langle, \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \) is the duality pairing between the dual \( \mathfrak{g}^* \) and \( \mathfrak{g} \). Assume that \( \mu \in \mathfrak{g}^* \) is a regular value of the momentum map \( J \), and \( G_\mu = \{ g \in G \mid \text{Ad}_g^* \mu = \mu \} \) is the isotropy subgroup of the coadjoint \( G \)-action at the point \( \mu \). From the Marsden-Weinstein reduction, we know that the reduced space \(( (T^*Q)_\mu, \omega_\mu ) \) is a symplectic manifold.

In the following we give further a precise analysis of the geometrical structure of the symplectic reduced space of \( T^*Q \). From Marsden and Perlmutter [14] and Marsden et al. [12], we know that the classification of symplectic reduced space of the cotangent bundle \( T^*Q \) as follows. (1) If \( \mu = 0 \), the symplectic reduced space of cotangent bundle \( T^*Q \) at \( \mu = 0 \) is given by \( ((T^*Q)_\mu, \omega_\mu ) = (T^*(Q/G), \omega_0) \), where \( \omega_0 \) is the canonical symplectic form of the reduced cotangent bundle \( T^*(Q/G) \). Thus, the symplectic reduced space \(( (T^*Q)_\mu, \omega_\mu ) \) at \( \mu = 0 \) is a symplectic vector bundle; (2) If \( \mu \neq 0 \), and \( G \) is Abelian, then \( G_\mu = G \), in this case the regular point symplectic reduced space \(( (T^*Q)_\mu, \omega_\mu ) \) is symplectically diffeomorphic to symplectic vector bundle \( (T^*(Q/G), \omega_0 - B_\mu) \), where \( B_\mu \) is a magnetic term; (3) If \( \mu \neq 0 \), and \( G \) is not Abelian and \( G_\mu \neq G \), in this case the regular point symplectic reduced space \(( (T^*Q)_\mu, \omega_\mu ) \) is symplectically diffeomorphic to a symplectic fiber bundle over \( T^*(Q/G_\mu) \) with fiber to be the coadjoint orbit \( \mathcal{O}_\mu \), see the cotangent bundle reduction theorem—bundle version, also see Marsden and Perlmutter [14] and Marsden et al. [12].

Thus, from the above discussion, we know that the symplectic reduced space on a cotangent bundle may not be a cotangent bundle. Therefore, the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle \( T^*Q \) may not be a Hamiltonian system on a cotangent bundle, that is, the set of Hamiltonian systems with symmetries on the cotangent bundle is not complete under the Marsden-Weinstein reduction.
The Definition of Controlled Hamiltonian System

In 2005, we hope to study the Hamiltonian system with control from geometrical viewpoint. I and my students read the two papers of professor Marsden and his students on a seminar of Nankai University, see Chang et al. [5,6]. We found that there are some serious wrong of rigor for the definitions of controlled Hamiltonian and controlled Lagrangian systems in the two papers.

There are the following three aspects of these wrong for CH system and the reduced CH system:

(1) The authors didn’t consider the phase spaces of CH system and the reduced CH system, that is, all of CH systems and the reduced CH systems given in [5,6], have not the spaces on which these systems are defined, see Definition 3.1 in [5] and Definition 3.1, 3.3 in [6]. Thus, it is impossible to give the actions of a Lie group on the phase spaces of CH systems and their momentum maps, also impossible to determine precisely the reduced phase spaces of CH systems.

(2) The authors didn’t consider the change of geometrical structures of the phase spaces of the CH systems. In fact, it is not that all of CH systems in [6] have same phase space $T^*Q$, same action of Lie group $G$, and same reduced phase space $T^*Q/G$. Different geometrical structures determine different CH systems and their phase spaces.

(3) The authors didn’t consider the momentum map of the CH system with symmetry, and hence cannot determine precisely the geometrical structures of phase spaces of the reduced CH systems. For example, we consider the cotangent bundle $T^*Q$ of a smooth manifold $Q$ with a free and proper action of Lie group $G$, and the Poisson tensor $B$ on $T^*Q$ is determined by the canonical symplectic form $\omega_0$ on $T^*Q$. Assume that there is an $Ad^*$-equivariant momentum map $J : T^*Q \to g^*$ for the symplectic, free and proper cotangent lifted $G$-action, where $g^*$ is the dual of Lie algebra $g$ of $G$. For $\mu \in g^*$, a regular value of $J$, from Abraham and Marsden [1], we know that the regular point reduced space $J^{-1}(\mu)/G_\mu$ and regular orbit reduced space $J^{-1}(O_\mu)/G$ at $\mu$ are different, and they are not yet the space $T^*Q/G$. The two reduced spaces are determined by the momentum map $J$, where $G_\mu$ is the isotropy subgroup of the coadjoint $G$-action at the point $\mu$, and $O_\mu$ is the orbit of the coadjoint $G$-action through the point $\mu$. Thus, in the two cases, it is impossible to determine the reduced CH systems by using the method given in [6]. Moreover, it is impossible to give precisely the relations of the reduced controlled Hamiltonian equivalences, if don’t consider the different Lie group actions and momentum maps.

To sum up the above statement, we think that there are a lot of wrong of rigor for the definitions of CH system and the reduced CH system, as well as CH-equivalence and the reduced CH-equivalence in Chang et al. [5,6], and we want to correct their work. In Marsden et al. [18], we corrected and renewed carefully these wrong definitions.

In order to deal with the above problems, and give the proper definition of CH system, and determine uniformly the reduced CH systems, our idea in Marsden et al. [18], is that we first define a CH system on $T^*Q$ by using the symplectic form, and such system is called a regular controlled Hamiltonian (RCH) system, and then regard a Hamiltonian system on $T^*Q$ as a spacial case of a RCH system without external force and control. Thus, the set of Hamiltonian systems on $T^*Q$ is a subset of the set of RCH systems on $T^*Q$. On the other hand, since the symplectic reduced space on a cotangent bundle is not complete under the Marsden-Weinstein
reduction, and the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle. So, we can not define directly a RCH system on the cotangent bundle $T^*Q$. However, from the classification of symplectic reduced space of the cotangent bundle $T^*Q$, we know that the regular point symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle over $T^*(Q/G_\mu)$ with fiber to be the coadjoint orbit $O_\mu$. Moreover, from the regular reduction diagram, see Ortega and Ratiu [21], we know that the regular orbit reduced space $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$, and hence is also symplectically diffeomorphic to a symplectic fiber bundle. In consequence, if we may define a RCH system on a symplectic fiber bundle, then it is possible to describe uniformly the RCH system on $T^*Q$ and its regular reduced RCH systems on the associated reduced spaces, and we can study regular reduction theory of the RCH systems with symplectic structures and symmetries, as an extension of the regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. This is why the authors in Marsden et al. [18] set up the regular reduction theory of the RCH system on a symplectic fiber bundle, by using momentum map and the associated reduced symplectic form and from the viewpoint of completeness of regular symplectic reduction.

Let $(E, M, N, \pi, G)$ be a fiber bundle and $(E, \omega_E)$ be a symplectic fiber bundle. If for any function $H : E \to \mathbb{R}$, we have a Hamiltonian vector field $X_H$ by $i_{X_H} \omega_E = dH$, then $(E, \omega_E, H)$ is a Hamiltonian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle $E$ as follows.

**Definition 3.1 (RCH System)** A RCH system on $E$ is a 5-tuple $(E, \omega_E, H, F, W)$, where $(E, \omega_E, H)$ is a Hamiltonian system, and the function $H : E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F : E \to E$ is called the (external) force map, and a fiber sub-manifold $W$ of $E$ is called the control subset.

Sometimes, $W$ also denotes the set of fiber-preserving maps from $E$ to $W$. When a feedback control law $u : E \to W$ is chosen, the 5-tuple $(E, \omega_E, H, F, u)$ denotes a closed-loop dynamic system. In particular, when $Q$ is a smooth manifold, and $T^*Q$ its cotangent bundle with a symplectic form $\omega$ (not necessarily canonical symplectic form), then $(T^*Q, \omega)$ is a symplectic vector bundle. If we take that $E = T^*Q$, from above definition we can obtain a RCH system on the cotangent bundle $T^*Q$, that is, 5-tuple $(T^*Q, \omega, H, F, W)$. Here for convenience, we assume that all controls appearing in this paper are the admissible controls.

The main contributions in Marsden et al. [18] are given as follows. (1) In order to describe uniformly RCH systems defined on a cotangent bundle and on the regular reduced spaces, we define a kind of RCH systems on a symplectic fiber bundle by using its symplectic form; (2) We give regular point and regular orbit reducible RCH systems by using momentum maps and the associated reduced symplectic forms, and prove regular point and regular orbit reduction theorems for the RCH systems, which explain the relationships between RpCH-equivalence, RoCH-equivalence for the reducible RCH systems with symmetries and RCH-equivalence for the associated reduced RCH systems; (3) We prove that rigid body with external force torque, rigid body with internal rotors and heavy top with internal rotors are all RCH systems, and as a pair of the regular point reduced RCH systems, rigid body with internal rotors (or external force torque) and heavy top with internal rotors are RCH-equivalent; (4) We describe the RCH system from the viewpoint of port Hamiltonian system with a symplectic structure, and state
the relationship between RCH-equivalence of RCH systems and equivalence of port Hamiltonian systems.

4 The Dynamics of Controlled Hamiltonian System

In order to describe the dynamics of a RCH system, we have to give a good expression of the dynamical vector field of the RCH system, by using the notations of vertical lifted maps of a vector along a fiber, see Marsden et al. [18].

At first, for the notations of vertical lifts along fiber, we need to consider three case: (1) $\pi : E \to M$ is a fiber bundle; (2) $\pi : E \to M$ is a vector bundle; (3) $\pi : E \to M$, $E = T^*Q$, $M = Q$, is a cotangent bundle, which is a special vector bundle. For the case (2) and (3), we can use the standard definition of the vertical lift operator given in Marsden and Ratiu [16]. But for the case (1), the operator cannot be used. This question is found by one of referees who give us that in a review report of our manuscript. In order to deal with uniformly the three cases, we have to give a new definition of vertical lifted maps of a vector along a fiber, and make it to be not conflict with that given in Marsden and Ratiu [16], and it is not and cannot be extension of the definition of Marsden and Ratiu.

It is worthy of noting that there are two aspects in our new definition. First, for two different points, $a_x$, $b_x$ in the fiber $E_x$, how define the moving vertical part of a vector in one point $b_x$ to another point $a_x$; Second, for a fiber-preserving map $F : E \to E$, we know that $a_x$ and $F_x(a_x)$ are the two points in $E_x$, how define the moving vertical part of a tangent vector in image point $F_x(a_x)$ to $a_x$. The eventual goal is to give a good expression of the dynamical vector field of RCH system by using the notation of vertical lift map of a vector along a fiber. Our definitions are reasonable and clear, and should be stated explicitly as follows.

For a smooth manifold $E$, its tangent bundle $TE$ is a vector bundle, and for the fiber bundle $\pi : E \to M$, we consider the tangent mapping $T\pi : TE \to TM$ and its kernel $ker(T\pi) = \{ \rho \in TE | T\pi(\rho) = 0 \}$, which is a vector subbundle of $TE$. Denote $VE := ker(T\pi)$, which is called a vertical bundle of $E$. Assume that there is a metric on $E$, and we take a Levi-Civita connection $\mathcal{A}$ on $TE$, and denote by $HE := ker(\mathcal{A})$, which is called a horizontal bundle of $E$, such that $TE = HE \oplus VE$. For any $x \in M$, $a_x, b_x \in E_x$, any tangent vector $\rho(b_x) \in T_{b_x}E$ can be split into horizontal and vertical parts, that is, $\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)$, where $\rho^h(b_x) \in H_{b_x}E$ and $\rho^v(b_x) \in V_{b_x}E$. Let $\gamma$ be a geodesic in $E_x$ connecting $a_x$ and $b_x$, and denote by $\rho^\gamma_v(a_x)$ a tangent vector at $a_x$, which is a parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic $\gamma$ from $b_x$ to $a_x$. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $T\pi(\rho^\gamma_v(a_x)) = 0$, and hence $\rho^\gamma_v(a_x) \in V_{a_x}E$. Now, for $a_x, b_x \in E_x$ and tangent vector $\rho(b_x) \in T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$vlift : TE_x \times E_x \to TE_x; \quad vlift(\rho(b_x), a_x) = \rho^\gamma_v(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of the geodesic $\gamma$.

If $F : E \to E$ is a fiber-preserving map, for any $x \in M$, we have that $F_x : E_x \to E_x$ and $TF_x : TE_x \to TE_x$, then for any $a_x \in E_x$ and $\rho \in TE_x$, the vertical lift of $\rho$ under the action of
$F$ along a fiber is defined by
\[
(vlift(F_x)\rho)(a_x) = vlift((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)_{\gamma}^{u}(a_x),
\]
where $\gamma$ is a geodesic in $E_x$ connecting $F_x(a_x)$ and $a_x$.

In particular, when $\pi : E \to M$ is a vector bundle, for any $x \in M$, the fiber $E_x$ is a vector space. In this case, we can choose the geodesic $\gamma$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho_x^\gamma(a_x) = \rho_x^\beta(b_x)$. Moreover, when $E = T^*Q$, by using the local trivialization of $TT^*Q$, we have that $TT^*Q \cong TQ \times T^*Q$.

Note that $\pi : T^*Q \to Q$, and $T\pi : TT^*Q \to TQ$, then in this case, for any $\alpha_x$, $\beta_x \in T^*_xQ$, $x \in Q$, we know that $(0, \beta_x) \in V^*_xT^*_xQ$, and hence we can get that
\[
vlift((0, \beta_x)(\beta_x), \alpha_x) = (0, \beta_x)(\alpha_x),
\]
which is consistent with the definition of vertical lift operator along a fiber given in Marsden and Ratiu [16].

For a given RCH System $(T^*Q, \omega, H, F, W)$, the dynamical vector field $X_H$ of the associated Hamiltonian system $(T^*Q, \omega, H)$ satisfies $i_{X_H}\omega = dH$. If considering the external force $F : T^*Q \to T^*Q$, which is a fiber-preserving map, by using the above notations of vertical lift maps of a vector along a fiber, the change of $X_H$ under the action of $F$ is that
\[
vlift(F)X_H(\alpha_x) = vlift((TFX_H)(F(\alpha_x)), \alpha_x) = (TFX_H)_{\gamma}^{u}(\alpha_x),
\]
where $\alpha_x \in T^*_xQ$, $x \in Q$ and the geodesic $\gamma$ is a straight line in $T^*_xQ$ connecting $F_x(\alpha_x)$ and $\alpha_x$.

In the same way, when a feedback control law $u : T^*Q \to W$, which is a fiber-preserving map, is chosen, the change of $X_H$ under the action of $u$ is that
\[
vlift(u)X_H(\alpha_x) = vlift((TuX_H)(F(\alpha_x)), \alpha_x) = (TuX_H)_{\gamma}^{u}(\alpha_x).
\]
In consequence, we can give an expression of the dynamical vector field of the RCH system as follows.

**Theorem 4.1** The dynamical vector field of a RCH system $(T^*Q, \omega, H, F, W)$ with a control law $u$ is the synthetic of Hamiltonian vector field $X_H$ and its changes under the actions of the external force $F$ and control $u$, that is,
\[
X_{(T^*Q, \omega, H, F, u)}(\alpha_x) = X_H(\alpha_x) + vlift(F)X_H(\alpha_x) + vlift(u)X_H(\alpha_x),
\]
for any $\alpha_x \in T^*_xQ$, $x \in Q$. For convenience, it is simply written as
\[
X_{(T^*Q, \omega, H, F, u)} = X_H + vlift(F) + vlift(u). \tag{4.1}
\]
We also denote that $vlift(W) = \bigcup\{vlift(u)X_H \mid u \in W\}$. It is worthy of noting that in order to deduce and calculate easily, we always use the simple expressions of the dynamical vector field $X_{(T^*Q, \omega, H, F, u)}$ and the $R_P$-reduced vector field $X_{((T^*Q)_{\mu, \mu}, H_{\mu}, F_{\mu}, u_{\mu})}$ and the $R_O$-reduced vector field $X_{((T^*Q)_{\omega, \mu}, H_{\omega}, F_{\omega}, u_{\omega})}$.

From the expression (4.1) of the dynamical vector field of a RCH system, we know that under the actions of the external force $F$ and control $u$, in general, the dynamical vector field is not Hamiltonian, and hence the RCH system is not yet a Hamiltonian system. However, it is a dynamical system closed relative to a Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of Hamiltonian system.
5 Controlled Hamiltonian Equivalence

It is worthy of noting that, when a RCH system is given, the force map \( F \) is determined, but the feedback control law \( u : T^*Q \to W \) could be chosen. In order to emphasize explicitly the impact of external force and control in the study of the RCH systems, by using the above expression of the dynamical vector field of the RCH system, we can describe the feedback control law how to modify the structure of a RCH system, and the controlled Hamiltonian matching conditions and RCH-equivalence are induced as follows.

**Definition 5.1 (RCH-equivalence)** Suppose that we have two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i)\), \(i = 1, 2\), we say them to be RCH-equivalent, or simply, \((T^*Q_1, \omega_1, H_1, F_1, W_1) \overset{RCH}{\sim} (T^*Q_2, \omega_2, H_2, F_2, W_2)\), if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that the following controlled Hamiltonian matching conditions hold:

**RCH-1:** The cotangent lifted map of \( \varphi \), that is, \( \varphi^* = T^\ast \varphi : T^*Q_2 \to T^*Q_1 \) is symplectic, and \( W_1 = \varphi^*(W_2) \).

**RCH-2:** \( \text{Im}[X_{H_1} + \text{vlift}(F_1) - T^\ast \varphi^*(X_{H_2}) - \text{vlift}(\varphi^*F_2\varphi_*)] \subset \text{vlift}(W_1) \), where the map \( \varphi_* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2 \), and \( T^\ast \varphi^* : TT^*Q_2 \to TT^*Q_1 \), and \( \text{Im} \) means the pointwise image of the map in brackets.

It is worthy of noting that (i) we define directly the RCH system \((T^*Q, \omega, H, F, W)\) with a space \( T^*Q \); (ii) we use a diffeomorphism \( \varphi : Q_1 \to Q_2 \) to describe that two different RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), are RCH-equivalent. These are very important for a rigorous definition of CH-equivalence, by comparing the definitions in Chang et al. \[5, 6\]. On the other hand, our RCH system is defined by using the symplectic structure on the cotangent bundle of a configuration manifold, we have to keep with the symplectic structure when we define the RCH-equivalence, that is, the induced equivalent map \( \varphi^* \) is symplectic on the cotangent bundle. Moreover, the following theorem explains the significance of the above RCH-equivalence relation, its proof is given in Marsden et al. \[18\].

**Theorem 5.2** Suppose that two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), are RCH-equivalent, then there exist two control laws \( u_i : T^*Q_i \to W_i, i = 1, 2 \), such that the two closed-loop systems produce the same equations of motion, that is, \( X_{(T^*Q_1, \omega_1, H_1, F_1, u_1)} \cdot \varphi^* = T^\ast (\varphi^*)X_{(T^*Q_2, \omega_2, H_2, F_2, u_2)} \), where the map \( T(\varphi^*) : TT^*Q_2 \to TT^*Q_1 \) is the tangent map of \( \varphi^* \). Moreover, the explicit relation between the two control laws \( u_i, i = 1, 2 \) is given by

\[
\text{vlift}(u_1) - \text{vlift}(\varphi^*u_2\varphi_*) = -X_{H_1} - \text{vlift}(F_1) + T^\ast (X_{H_2}) + \text{vlift}(\varphi^*F_2\varphi_*) \quad (5.1)
\]

6 Regular Reducible Controlled Hamiltonian System

We know that when the external force and control of a RCH system \((T^*Q, \omega, H, F, W)\) are both zeros, in this case the RCH system is just a Hamiltonian system \((T^*Q, \omega, H)\). Thus, we can regard a Hamiltonian system on \( T^*Q \) as a spacial case of a RCH system without external force and control. In consequence, the set of Hamiltonian systems with symmetries on \( T^*Q \) is a subset of the set of RCH systems with symmetries on \( T^*Q \). If we first admit the regular symplectic reduction of a Hamiltonian system with symmetry, then we may study the regular reduction of a RCH system with symmetry, as an extension of regular symplectic reduction of a Hamiltonian system under regular controlled Hamiltonian equivalence conditions. In order to do these, in this section we first give the regular point reducible RCH system and the regular orbit reducible RCH system, by using the Marsden-Weinstein reduction and regular orbit reduction
for a Hamiltonian system, respectively.

At first, we consider the regular point reducible RCH system. Let \( Q \) be a smooth manifold and \( T^*Q \) its cotangent bundle with the symplectic form \( \omega \). Let \( \Phi : G \times Q \to Q \) be a smooth left action of a Lie group \( G \) on \( Q \), which is free and proper. Then the cotangent lifted left action \( \Phi^* : G \times T^*Q \to T^*Q \) is also free and proper. Assume that the action is symplectic and admits an \( \text{Ad}^* \)-equivariant momentum map \( \mathbf{J} : T^*Q \to g^* \), where \( g \) is the Lie algebra of \( G \) and \( g^* \) is the dual of \( g \). Let \( \mu \in g^* \) be a regular value of \( \mathbf{J} \) and denote by \( G_{\mu} \) the isotropy subgroup of the coadjoint \( G \)-action at the point \( \mu \in g^* \), which is defined by \( G_{\mu} = \{ g \in G | \text{Ad}_g^* \mu = \mu \} \). Since \( G_{\mu} \subset G \) acts freely and properly on \( Q \) and on \( T^*Q \), then \( Q_{\mu} = Q/G_{\mu} \) is a smooth manifold and that the canonical projection \( \rho_{\mu} : Q \to Q_{\mu} \) is a surjective submersion. It follows that \( G_{\mu} \) acts also freely and properly on \( J^{-1}(\mu) \), so that the space \( (T^*Q)_{\mu} = J^{-1}(\mu)/G_{\mu} \) is a symplectic manifold with the symplectic form \( \omega_{\mu} \) uniquely characterized by the relation

\[
\pi_{\mu}^* \omega_{\mu} = i_{\mu}^* \omega.
\]

(6.1)

The map \( i_{\mu} : J^{-1}(\mu) \to T^*Q \) is the inclusion and \( \pi_{\mu} : J^{-1}(\mu) \to (T^*Q)_{\mu} \) is the projection. The pair \( ((T^*Q)_{\mu}, \omega_{\mu}) \) is called Marsden-Weinstein reduced space of \( (T^*Q, \omega) \) at \( \mu \).

Assume that \( H : T^*Q \to \mathbb{R} \) is a \( G \)-invariant Hamiltonian, the flow \( F_t \) of the Hamiltonian vector field \( X_H \) leaves the connected components of \( J^{-1}(\mu) \) invariant and commutes with the \( G \)-action, so it induces a flow \( f^\mu_t \) on \( (T^*Q)_{\mu} \), defined by \( f^\mu_t \cdot \pi_{\mu} = \pi_{\mu} \cdot F_t \cdot i_{\mu} \), and the vector field \( X_{h_{\mu}} \) generated by the flow \( f^\mu_t \) on \( (T^*Q)_{\mu} \) is Hamiltonian with the associated regular point reduced Hamiltonian function \( h_{\mu} : (T^*Q)_{\mu} \to \mathbb{R} \) defined by \( h_{\mu} \cdot \pi_{\mu} = H \cdot i_{\mu} \), and the Hamiltonian vector fields \( X_H \) and \( X_{h_{\mu}} \) are \( \pi_{\mu} \)-related. On the other hand, from Marsden et al. [18], we know that the regular point reduced space \( ((T^*Q)_{\mu}, \omega_{\mu}) \) is symplectically diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a kind of the regular point reducible RCH system as follows.

**Definition 6.1 (Regular Point Reducible RCH System)** A 6-tuple \( (T^*Q, G, \omega, H, F, W) \), where the Hamiltonian \( H : T^*Q \to \mathbb{R} \), the fiber-preserving map \( F : T^*Q \to T^*Q \) and the fiber submanifold \( W \) of \( T^*Q \) are all \( G \)-invariant, is called a regular point reducible RCH system, if there exists a point \( \mu \in g^* \), which is a regular value of the momentum map \( \mathbf{J} \), such that the regular point reduced system, that is, the 5-tuple \( ((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, W_{\mu}) \), where \( (T^*Q)_{\mu} = J^{-1}(\mu)/G_{\mu} \), is a RCH system, which is simply written as \( R_P \)-reduced RCH system. Where \( ((T^*Q)_{\mu}, \omega_{\mu}) \) is the \( R_P \)-reduced space, the function \( h_{\mu} : (T^*Q)_{\mu} \to \mathbb{R} \) is called the reduced Hamiltonian, the fiber-preserving map \( f_{\mu} : (T^*Q)_{\mu} \to (T^*Q)_{\mu} \) is called the reduced (external) force map, \( W_{\mu} \) is a fiber submanifold of \( (T^*Q)_{\mu} \) and is called the reduced control subset.

It is worthy of noting that for the regular point reducible RCH system \( (T^*Q, G, \omega, H, F, W) \), the \( G \)-invariant external force map \( F : T^*Q \to T^*Q \) has to satisfy the conditions \( F(J^{-1}(\mu)) \subset J^{-1}(\mu) \), and \( f_{\mu} \cdot \pi_{\mu} = \pi_{\mu} \cdot F \cdot i_{\mu} \), such that we can define the reduced external force map \( f_{\mu} : (T^*Q)_{\mu} \to (T^*Q)_{\mu} \). The condition \( W \cap J^{-1}(\mu) \neq \emptyset \) in above definition makes that the \( G \)-invariant control subset \( W \cap J^{-1}(\mu) \) can be reduced and the reduced control subset is \( W_{\mu} = \pi_{\mu}(W \cap J^{-1}(\mu)) \). If the control subset cannot be reduced, we cannot get the \( R_P \)-reduced RCH system. The study of RCH system which is not regular point reducible is beyond the limits in this paper, it may be a topic in future study.
Next, we consider the regular orbit reducible RCH system. For the cotangent lifted left action $\Phi^T : G \times T^*Q \to T^*Q$, which is symplectic, free and proper, assume that the action admits an $\text{Ad}^*$-equivariant momentum map $J : T^*Q \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of the momentum map $J$ and $O_\mu = G \cdot \mu \subset \mathfrak{g}^*$ be the $G$-orbit of the coadjoint $G$-action through the point $\mu$. Since $G$ acts freely, properly and symplectically on $T^*Q$, then the quotient space $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{O_\mu}$ uniquely characterized by the relation

$$i_{\partial_{\mu}}^* \omega = \pi_{O_\mu}^* \omega_{O_\mu} + J_{\partial_{\mu}}^* \omega_{O_\mu},$$

(6.2)

where $J_{O_\mu}$ is the restriction of the momentum map $J$ to $J^{-1}(O_\mu)$, that is, $J_{O_\mu} = J \circ i_{O_\mu}$ and $\omega_{O_\mu}^+$ is the $+$-symplectic structure on the orbit $O_\mu$ given by

$$\omega_{O_\mu}^+(\nu)(\xi, \eta) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in O_\mu, \xi, \eta \in \mathfrak{g}.$$  

(6.3)

The maps $i_{O_\mu} : J^{-1}(O_\mu) \to T^*Q$ and $\pi_{O_\mu} : J^{-1}(O_\mu) \to (T^*Q)_{O_\mu}$ are natural injection and the projection, respectively. The pair $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is called the symplectic orbit reduced space of $(T^*Q, \omega)$ at $\mu$.

Assume that $H : T^*Q \to \mathbb{R}$ is a $G$-invariant Hamiltonian, the flow $F_t$ of the Hamiltonian vector field $X_H$ leaves the connected components of $J^{-1}(O_\mu)$ invariant and commutes with the $G$-action, so it induces a flow $f_t^{O_\mu}$ on $(T^*Q)_{O_\mu}$, defined by $f_t^{O_\mu} \circ \pi_{O_\mu} = \pi_{O_\mu} \circ f_t$ and $F_t \circ i_{O_\mu}$, and the vector field $X_{\eta_{O_\mu}}$ generated by the flow $f_t^{O_\mu}$ on $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is Hamiltonian with the associated regular orbit reduced Hamiltonian function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ defined by $h_{O_\mu} \circ \pi_{O_\mu} = H \circ i_{O_\mu}$, and the Hamiltonian vector fields $X_H$ and $X_{\eta_{O_\mu}}$ are $\pi_{O_\mu}$-related. In general case, we maybe thought that the structure of the symplectic orbit reduced space $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is more complex than that of the symplectic point reduced space $((T^*Q)_{\mu}, \omega_{\mu})$, but, from the regular reduction diagram, see Ortega and Ratiu [21], we know that the regular orbit reduced space $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_{\mu}, \omega_{\mu})$, and hence is also symplectically diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a kind of the regular orbit reducible RCH systems as follows.

**Definition 6.2 (Regular Orbit Reducible RCH System)** A 6-tuple $(T^*Q, G, \omega, H, F, W)$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold $W$ of $T^*Q$ are all $G$-invariant, is called a regular orbit reducible RCH system, if there exists an orbit $O_\mu$, $\mu \in \mathfrak{g}^*$, where $\mu$ is a regular value of the momentum map $J$, such that the regular orbit reduced system, that is, the 5-tuple $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, W_{O_\mu})$, where $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$, $\pi_{O_\mu}^* \omega_{O_\mu} = i_{\partial_{\mu}}^* \omega - J_{\partial_{\mu}}^* \omega_{O_\mu}$, $h_{O_\mu} \circ \pi_{O_\mu} = H \circ i_{O_\mu}$, $F(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)$, $f_{O_\mu} \circ \pi_{O_\mu} = \pi_{O_\mu} \circ F \circ i_{O_\mu}$, and $W \cap J^{-1}(O_\mu) \neq \emptyset$, $W_{O_\mu} = \pi_{O_\mu}(W \cap J^{-1}(O_\mu))$, is a RCH system, which is simply written as $R_O$-reduced RCH system. Where $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is the $R_O$-reduced space, the function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ is called the reduced Hamiltonian, the fiber-preserving map $f_{O_\mu} : (T^*Q)_{O_\mu} \to (T^*Q)_{O_\mu}$ is called the reduced (external) force map, $W_{O_\mu}$ is a fiber submanifold of $(T^*Q)_{O_\mu}$, and is called the reduced control subset.

It is worthy of noting that for the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$, the $G$-invariant external force map $F : T^*Q \to T^*Q$ has to satisfy the conditions $F(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)$, and $f_{O_\mu} \circ \pi_{O_\mu} = \pi_{O_\mu} \circ F \circ i_{O_\mu}$, such that we can define the reduced external force map $f_{O_\mu} : (T^*Q)_{O_\mu} \to (T^*Q)_{O_\mu}$. The condition $W \cap J^{-1}(O_\mu) \neq \emptyset$ in above definition makes that the $G$-invariant control subset $W \cap J^{-1}(O_\mu)$ can be reduced and the reduced control subset is
$W_{O_u} = \pi_{O_u}(W \cap J^{-1}(O_\mu))$. If the control subset cannot be reduced, we cannot get the $R_O$-reduced RCH system. The study of RCH system which is not regular orbit reducible is beyond the limits in this paper, it may be a topic in future study.

## 7 Regular Point Reduction of the RCH System

In the following we consider the RCH system with symmetry and momentum map, and give the RpCH-equivalence for the regular point reducible RCH system, and prove the regular point reduction theorem. Denote $X_{(T^*Q,G,\omega,H,F,u)}$ the dynamical vector field of the regular point reducible RCH system $(T^*Q,G,\omega,H,F,W)$ with a control law $u$, then

$$X_{(T^*Q,G,\omega,H,F,u)} = X_H + \text{vlift}(F) + \text{vlift}(u). \quad (7.1)$$

Moreover, by using the above expression, for the regular point reducible RCH system we can also introduce the regular point reduced controlled Hamiltonian equivalence (RpCH-equivalence) as follows.

**Definition 7.1 (RpCH-equivalence)** Suppose that we have two regular point reducible RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, we say them to be RpCH-equivalent, or simply,

$(T^*Q_1,G_1,\omega_1,H_1,F_1,W_1) \sim_{\text{RpCH}} (T^*Q_2,G_2,\omega_2,H_2,F_2,W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following controlled Hamiltonian matching conditions hold:

**RpCH-1:** The cotangent lifted map $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic.

**RpCH-2:** For $\mu_i \in g_i^*$, the regular reducible points of RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, the map $\varphi^* \circ i_{\mu_2} : J_2^{-1}(\mu_2) \to J_1^{-1}(\mu_1)$ is $(G_{2\mu_2},G_{1\mu_1})$-equivariant and $W_1 \cap J_1^{-1}(\mu_1) = \varphi^*(W_2 \cap J_2^{-1}(\mu_2))$, where $\mu = (\mu_1,\mu_2)$, and denote by $i_{\mu_1}^{-1}(S)$ the pre-image of a subset $S \subset T^*Q_1$ for the map $i_{\mu_1} : J_1^{-1}(\mu_1) \to T^*Q_1$.

**RpCH-3:** $\text{Im}[X_{H_1} + \text{vlift}(F_1) - T\varphi^*(X_{H_2}) - \text{vlift}(\varphi^*F_2 \varphi_*)] \subset \text{vlift}(W_1)$.

It is worthy of noting that for the regular point reducible RCH system, the induced equivalence map $\varphi^*$ not only keeps the symplectic structure, but also keeps the equivariance of $G$-action at the regular point. If a feedback control law $u_\mu : (T^*Q)_\mu \to W_\mu$ is chosen, the $R_P$-reduced RCH system $(T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)$ is a closed-loop regular dynamic system with a control law $u_\mu$. Assume that its vector field $X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)}$ can be expressed by

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} = X_{h_\mu} + \text{vlift}(f_\mu) + \text{vlift}(u_\mu), \quad (7.2)$$

where $X_{h_\mu}$ is the dynamical vector field of the regular point reduced Hamiltonian $h_\mu$, $\text{vlift}(f_\mu) = \text{vlift}(f_\mu)e^{-i_{\mu_2}h_\mu}$, $\text{vlift}(u_\mu) = \text{vlift}(u_\mu)e^{-i_{\mu_2}h_\mu}$, and satisfies the condition

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} \cdot \pi_{\mu} = T\pi_{\mu} \cdot X_{((T^*Q,G,\omega,H,F,u))} \cdot i_{\mu}. \quad (7.3)$$

Then we can obtain the following regular point reduction theorem for the RCH system, which explains the relationship between the RpCH-equivalence for the regular point reducible RCH system with symmetry and the RCH-equivalence for the associated $R_P$-reduced RCH system, its proof is given in Marsden et al. [18].

**Theorem 7.2** Two regular point reducible RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, are RpCH-equivalent if and only if the associated $R_P$-reduced RCH systems $((T^*Q_i)_{\mu_i},\omega_{i\mu_i},h_{i\mu_i},f_{i\mu_i},W_{i\mu_i})$, $i = 1, 2$, are RCH-equivalent.

This theorem can be regarded as an extension of the regular point reduction theorem of a Hamiltonian system under regular controlled Hamiltonian equivalence conditions.
8 Regular Orbit Reduction of the RCH System

The orbit reduction of a Hamiltonian system is an alternative approach to symplectic reduction given by Marle [10] and Kazhdan, Kostant and Sternberg [8], which is different from the Marsden-Weinstein reduction. We note that the regular reduced symplectic spaces \(((T^*Q)_{\sigma_\mu}, \omega_{\sigma_\mu})\) and \(((T^*Q)_{\mu}, \omega_{\mu})\), of the regular orbit reduced Hamiltonian system and the regular point reduced Hamiltonian system, are different, and the symplectic forms on the reduced spaces, given by (6.2) for the regular orbit reduced Hamiltonian system and given by (6.1) for the regular point reduced Hamiltonian system, are also different. Thus, the assumption conditions for the regular orbit reduction case are not same as that for the regular point reduction case.

In the following we consider the RCH system with symmetry and momentum map, and give the RoCH-equivalence for the regular orbit reducible RCH system, and prove the regular orbit reduction theorem. Denote \(X_{(T^*Q,G,\omega,H,F,W)}\) the dynamical vector field of the regular orbit reducible RCH system \((T^*Q,G,\omega,H,F,W)\) with a control law \(u\), then

\[
X_{(T^*Q,G,\omega,H,F,u)} = X_H + \text{vlift}(F) + \text{vlift}(u). \tag{8.1}
\]

Moreover, by using the above expression, for the regular orbit reducible RCH system we can also introduce the regular orbit reduced controlled Hamiltonian equivalence (RoCH-equivalence) as follows.

**Definition 8.1 (RoCH-equivalence)** Suppose that we have two regular orbit reducible RCH systems \((T^*Q_i,G_i,\omega_i,H_i,F_i,W_i), i = 1, 2\), we say them to be RoCH-equivalent, or simply, \((T^*Q_1,G_1,\omega_1,H_1,F_1,W_1) \sim_{\text{RoCH}} (T^*Q_2,G_2,\omega_2,H_2,F_2,W_2)\), if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that the following Hamiltonian matching conditions hold:

**RoHM-1**: The cotangent lift map \(\varphi^* : T^*Q_2 \to T^*Q_1\) is symplectic.

**RoHM-2**: For \(\sigma_\mu \in g^*_\mu\), the regular reducible orbits of RCH systems \((T^*Q_i,G_i,\omega_i,H_i,F_i,W_i), i = 1, 2\), the map \(\varphi^*_\mu \sigma_\mu = i_{\mu_1}^{-1} \cdot \varphi \cdot i_{\mu_2} : J_2^{-1}(\sigma_{\mu_2}) \to J_1^{-1}(\sigma_{\mu_1})\) is \((G_2,G_1)\)-equivariant, \(W_1 \cap J_1^{-1}(\sigma_{\mu_1}) = \varphi^*_\mu_2(W_2 \cap J_2^{-1}(\sigma_{\mu_2}))\), and \(J^*_\mu_2 W_2 \cap J^*_\mu_2 = (\varphi^*_\mu_2)^* \cdot J_{1,\mu_1} \cdot \omega_{1,\mu_1}\), where \(\mu = (\mu_1,\mu_2)\), and denote by \(i_{\mu_1}^{-1}(S)\) the pre-image of a subset \(S \subset T^*Q_1\) for the map \(i_{\mu_1} : J_1^{-1}(\sigma_{\mu_1}) \to T^*Q_1\).

**RoHM-3**: \(\text{Im}[X_{H_1} + \text{vlift}(F_1) - T\varphi^*(X_{H_2}) - \text{vlift}(\varphi^*F_2\varphi) \subset \text{vlift}(W_1)]\).

It is worthy of noting that for the regular orbit reducible RCH system, the induced equivalent map \(\varphi^*\) not only keeps the symplectic structure and the restriction of the \((+)-\)symplectic structure on the regular orbit to \(J^{-1}(\sigma_\mu)\), but also keeps the equivariance of \(G\)-action on the regular orbit. If a feedback control law \(u_{\sigma_\mu} : (T^*Q)_{\sigma_\mu} \to W_{\sigma_\mu}\) is chosen, the RoO-reduced RCH system \(((T^*Q)_{\sigma_\mu},\omega_{\sigma_\mu},h_{\sigma_\mu},f_{\sigma_\mu},u_{\sigma_\mu})\) is a closed-loop regular dynamic system with a control law \(u_{\sigma_\mu}\). Assume that its vector field \(X_{((T^*Q)_{\sigma_\mu},\omega_{\sigma_\mu},h_{\sigma_\mu},f_{\sigma_\mu},u_{\sigma_\mu})}\) can be expressed by

\[
X_{((T^*Q)_{\sigma_\mu},\omega_{\sigma_\mu},h_{\sigma_\mu},f_{\sigma_\mu},u_{\sigma_\mu})} = X_{h_{\sigma_\mu}} + \text{vlift}(f_{\sigma_\mu}) + \text{vlift}(u_{\sigma_\mu}), \tag{8.2}
\]

where \(X_{h_{\sigma_\mu}}\) is the dynamical vector field of the regular orbit reduced Hamiltonian \(h_{\sigma_\mu}\), \(\text{vlift}(f_{\sigma_\mu}) = \text{vlift}(f_{\sigma_\mu})X_{h_{\sigma_\mu}} + \text{vlift}(u_{\sigma_\mu}) = \text{vlift}(u_{\sigma_\mu})X_{h_{\sigma_\mu}}\), and satisfies the condition

\[
X_{((T^*Q)_{\sigma_\mu},\omega_{\sigma_\mu},h_{\sigma_\mu},f_{\sigma_\mu},u_{\sigma_\mu})} \cdot \pi_{\sigma_\mu} = T\pi_{\sigma_\mu} \cdot X_{(T^*Q,G,\omega,H,F,u)} \cdot i_{\sigma_\mu}. \tag{8.3}
\]

Then we can obtain the following regular orbit reduction theorem for the RCH system, which explains the relationship between the RoCH-equivalence for regular orbit reducible RCH system with symmetry and the RCH-equivalence for the associated RoO-reduced RCH system, its proof is given in Marsden et al. [18].
Theorem 8.2 If two regular orbit reducible RCH systems \((T^*Q_i, G_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), are RoCH-equivalent, then their associated RoO-reduced RCH systems \(((T^*Q)_{O_{\mu_i}}, \omega_{O_{\mu_i}}, h_{O_{\mu_i}}, f_{O_{\mu_i}}, W_{O_{\mu_i}}), i = 1, 2\), must be RCH-equivalent. Conversely, if RoO-reduced RCH systems \(((T^*Q)_{O_{\mu_i}}, \omega_{O_{\mu_i}}, h_{O_{\mu_i}}, f_{O_{\mu_i}}, W_{O_{\mu_i}}), i = 1, 2\), are RCH-equivalent, and the induced map \(\varphi_{O_{\mu_i}} \circ J_2^{-1}(O_{\mu_2}) \to J_1^{-1}(O_{\mu_1})\), such that \(J_2^*\omega_2 \cdot J_1^*\omega_1 = (\varphi_{O_{\mu}})^* \cdot J_1^*\omega_{O_{\mu_1}}\), then the regular orbit reducible RCH systems \((T^*Q_i, G_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), are RoCH-equivalent.

This theorem can be regarded as an extension of the regular orbit reduction theorem of a Hamiltonian system under regular controlled Hamiltonian equivalence conditions.

Remark 8.3 If \((T^*Q, \omega)\) is a connected symplectic manifold, and \(J : T^*Q \to \mathfrak{g}^*\) is a non-equivariant momentum map with a non-equivariance group one-cocycle \(\sigma : G \to \mathfrak{g}^*,\) which is defined by \(\sigma(g) := J(g \cdot z) - \text{Ad}^*_{g^{-1}} J(z), \) where \(g \in G\) and \(z \in T^*Q.\) Then we know that \(\sigma\) produces a new affine action \(\Theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*\) defined by \(\Theta(g, \mu) := \text{Ad}^*_{g^{-1}} \mu + \sigma(g),\) where \(\mu \in \mathfrak{g}^*,\) with respect to which the given momentum map \(J\) is equivariant. Assume that \(G\) acts freely and properly on \(T^*Q,\) and \(G_{\mu}\) denotes the isotropy subgroup of \(\mu \in \mathfrak{g}^*\) relative to this affine action \(\Theta,\) and \(O_{\mu} = G \cdot \mu \subset \mathfrak{g}^*\) denotes the \(G\)-orbit of the point \(\mu\) with respect to the action \(\Theta,\) and \(\mu\) is a regular value of \(J.\) Then the quotient space \((T^*Q)_{O_{\mu}} = J^{-1}(\mu)/G_{\mu}\) is a symplectic manifold with the symplectic form \(\omega_{O_{\mu}}\) uniquely characterized by \((6.1),\) and the quotient space \((T^*Q)_{O_{\mu}} = J^{-1}(O_{\mu})/G\) is also a symplectic manifold with the symplectic form \(\omega_{O_{\mu}}\) uniquely characterized by \((6.2),\) see Ortega and Ratiu [21]. Moreover, in this case, for the given regular point or regular orbit reducible RCH system \((T^*Q, G, \omega, H, F, W),\) we can also prove the regular point reduction theorem or regular orbit reduction theorem, by using the above similar ways.

9 Some Developments

In this section, shall give some generalizations of the above results.

(1) Optimal Reductions of a CH System

It is a natural problem what and how we could do, if we define a controlled Hamiltonian system on the cotangent bundle \(T^*Q\) by using a Poisson structure, and if symplectic reduction procedure given by Marsden et al. [18] does not work or is not efficient enough. In Wang and Zhang [28], we study the optimal reduction theory of a CH system with Poisson structure and symmetry, by using the optimal momentum map and the reduced Poisson tensor (resp. the reduced symplectic form). We prove the optimal point reduction, optimal orbit reduction, and regular Poisson reduction theorems for the CH system, and explain the relationships between OpCH-equivalence, OoCH-equivalence, RPR-CH-equivalence for the optimal reducible CH systems with symmetries and the CH-equivalence for the associated optimal reduced CH systems. This paper is published in Jour. Geom. Phys., 62(5)(2012), 953-975.

The late professor Jerry Marsden had joined this research. When the paper was about to finish, he left us. We are extremely sad. H.Wang and Z.X.Zhang would like to acknowledge his understanding, support and help in more than two years of cooperation.

(2) Singular Reductions of a RCH System
It is worthy of noting that when Lie group \( G \) acts only properly on \( Q \), does not act freely, then the reduced space \((T^*Q)_\mu = J^{-1}(\mu)/G_\mu \) (resp. \((T^*Q)_{\Omega \mu} = J^{-1}(\Omega_\mu)/G \)) is not necessarily smooth manifold, but just quotient topological space, and it may be a symplectic Whitney stratified space. In this case, we study the singular reduction theory of the RCH system with symmetry, by using the momentum map and the singular reduced symplectic form in the stratified phase space. We prove the singular point reduction and singular orbit reduction theorems for the RCH system, and explain the relationships between SpCH-equivalence, SoCH-equivalence for the singular reducible RCH systems with symmetries and the RCH-equivalence for the associated singular reduced CH systems. Since before professor Marsden passed away, we have begun looking for a couple of practical examples which are RCH-equivalent for singular reduced RCH systems. But, eight years flew away, we have not found them. It is not easy to do for the complex structure of stratified phase space of a singular reduced RCH system.

(3) Poisson Reduction by Controllability Distribution for a CH System

It is worthy of noting that when there is no momentum map of Lie group action for our considered system, then the reduction procedures given in Marsden et al. \[18\] and Wang and Zhang \[28\] can not work. One must look for a new way. On the other hand, motivated by the work of Poisson reductions by distribution for Poisson manifolds, see Marsden and Ratiu \[15\], we note that the phase space \( T^*Q \) of the CH system is a Poisson manifold, and its control subset \( W \subset T^*Q \) is a fiber submanifold. If we assume that \( D \subset TT^*Q|_W \) is a controllability distribution of the CH system, then we can study naturally the Poisson reduction by controllability distribution for the CH system. For a symmetric CH system, and its control subset \( W \subset T^*Q \) is a \( G \)-invariant fiber submanifold, if we assume that \( D \subset TT^*Q|_W \) is a \( G \)-invariant controllability distribution of the symmetric CH system, then we can give Poisson reducible conditions by controllability distribution for this CH system, and prove the Poisson reducible property for the CH system and it is kept invariant under the CH-equivalence. We also study the relationship between Poisson reduction by \( G \)-invariant controllability distribution, for the regular (resp. singular) Poisson reducible CH system, and Poisson reduction by the reduced controllability distribution, for the associated reduced CH system. In addition, we can also develop the singular Poisson reduction and SPR-CH-equivalence for a CH system with symmetry, and prove the singular Poisson reduction theorem. See Ratiu and Wang \[22\] for more details.

(4) Regular Reduction of a CMH System with Symmetry of the Heisenberg Group

We consider the regular point reduction of a controlled magnetic Hamiltonian (CMH) system \((T^*Q, \mathcal{H}, \omega_Q, H, F, C)\) with symmetry of the Heisenberg group \( \mathcal{H} \). Here the configuration space \( Q = \mathcal{H} \times V, \mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R} \), and \( V \) is a \( k \)-dimensional vector space, and the cotangent bundle \( T^*Q \) with the magnetic symplectic form \( \omega_Q = \Omega_0 - \pi_Q^*\bar{B} \), where \( \Omega_0 \) is the usual canonical symplectic form on \( T^*Q \), and \( \bar{B} = \pi_1^*B \) is the closed two-form on \( Q \), \( B \) is a closed two-form on \( \mathcal{H} \) and the projection \( \pi_1 : Q = \mathcal{H} \times V \rightarrow \mathcal{H} \) induces the map \( \pi_1^* : T^*\mathcal{H} \rightarrow T^*Q \).

We note that there is a magnetic term on the cotangent bundle of the Heisenberg group \( \mathcal{H} \), which is related to a curvature two-form of a mechanical connection determined by the reduction of center action of the Heisenberg group \( \mathcal{H} \), see Marsden et al. \[13\], such that we can define a controlled magnetic Hamiltonian system with symmetry of the Heisenberg group \( \mathcal{H} \), and study the regular point reduction of this system. A CMH system is also a RCH system, but its symplectic structure is given by a magnetic symplectic form. Thus, the set of the CMH systems
is a subset of the RCH systems, and it is not complete under the regular point reduction. It is worthy of noting that it is different from the regular point reduction of a RCH system defined on a cotangent bundle with the canonical structure, it reveals the deeper relationship of the intrinsic geometrical structures of the RCH systems. See Wang \[24\] for more details.

10 Applications

In this section, shall give two applications for the regular reduction of a RCH system.

10.1 Rigid Body and Heavy Top with Internal Rotors

Before, as applications examples of theoretical results, Prof. Jerrold E. Marsden gave us two couples of examples in October 2008, that is, (i) the rigid body with external force torque and that with internal rotors; (ii) the rigid body with internal rotors and the heavy top with internal rotors. He told us that these systems are symmetric, and have $G$-invariant control, may consider reduction and the RCH-equivalence, but need to calculate them in detail. Thus, for these examples, we have done three things: (i) to do reduction by calculation in detail; (ii) to state the reduced systems to be RCH systems; (iii) to state every couple of the reduced systems to be RCH-equivalent. Because the configuration space of rigid body (resp. heavy top) is the Lie group $G = \text{SO}(3)$ (resp. $G = \text{SE}(3)$), and the configuration space of rigid body with internal rotors (resp. heavy top with internal rotors) is the generalized case $G \times V$ of Lie group, where $V$ is a vector space. In consequence, we first discuss the regular point reducible RCH systems on a Lie group and on its generalization, then apply these results to the concrete examples. In addition, when we calculated these examples in detail, we also found the third couple of RCH-equivalent systems, that is, the rigid body with internal rotors (or external force torque) and the heavy top, where the heavy top is regarded as a regular point reducible RCH system without the external force and control. See Marsden et al. \[18\] for more details. We can also give some other examples, but it is not easy to give a couple of RCH-equivalent examples, just like as Prof. J.E.Marsden had done.

When we consider the reduction of a RCH system on a Lie group $G$, the dual of Lie algebra $\mathfrak{g}^*$ is a Poisson manifold with respect to its Lie-Poisson bracket, and the coadjoint orbit $O_\mu$ through $\mu \in \mathfrak{g}^*$ is the symplectic leaf of $\mathfrak{g}^*$. We give the reduced RCH system on $O_\mu$ by using the reduced symplectic form $\omega_{O_\mu}$, which is the more detail symplectic reduction on $O_\mu$ than the Poisson reduction on $\mathfrak{g}^*$. Moreover, when we consider the reductions of rigid body and heavy top with internal rotors, the configuration space $Q = G \times V$, where $G$ is a Lie group and $V$ is a vector space. In this case our reduced phase space is $O_\mu \times T^*V \cong O_\mu \times V \times V^*$, and its reduced symplectic form is the synthetic of the symplectic form $\omega_{O_\mu}$ and the canonical symplectic form $\omega_V$ on $T^*V$. But, in this case we consider the space $O_\mu \times V \times V^*$ is the symplectic leaf of $\mathfrak{g}^* \times V \times V^*$ with whose Poisson structure is the synthetic of Lie-Poisson bracket on $\mathfrak{g}^*$ and the Poisson bracket induced from $\omega_V$ on $T^*V$. Thus, we can deal with uniformly the symplectic reduction of the rigid body, heavy top, as well as them with internal rotors, such that we can state that all these systems are the regular point reducible RCH systems and can give their RCH-equivalences. It is worthy of noting that the method of calculation for the reductions of rigid body with internal rotors and heavy top with internal rotors is important and efficient, and it is generalized and used to the more practical controlled Hamiltonian systems, see Wang \[26,27\] for more details.
On the other hand, we can also get the equations of rigid body and heavy top as well as them with internal rotors (or the external force torques), by using Lagrangian and Euler-Lagrangian equation, just like as that have been done in some references, see Marsden [11] and Bloch et al. [4]. But, it cannot state uniformly that these systems are the regular point reduced RCH systems, and hence it cannot state every couple of them to be RCH-equivalent. Thus, it is very important for a rigorous theoretical work to offer uniformly a composition of the research results from a global view point.

10.2 Hamilton-Jacobi Theorems

It is well-known that Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which states that the integral of Lagrangian of a system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let \( Q \) be a smooth manifold and \( TQ \) the tangent bundle, \( T^*Q \) the cotangent bundle with the canonical symplectic form \( \omega \), and the projection \( \pi_Q : T^*Q \to Q \) induces the map \( T\pi_Q : TT^*Q \to TQ \).

Theorem 10.1 Assume that the triple \((T^*Q, \omega, H)\) is a Hamiltonian system with Hamiltonian vector field \( X_H \), and \( W : Q \to \mathbb{R} \) is a given function. Then the following two assertions are equivalent:

(i) For every curve \( \sigma : \mathbb{R} \to Q \) satisfying \( \dot{\sigma}(t) = T\pi_Q(X_H(\partial W(\sigma(t)))) \), \( \forall t \in \mathbb{R} \), then \( \partial W \cdot \sigma \) is an integral curve of the Hamiltonian vector field \( X_H \).

(ii) \( W \) satisfies the Hamilton-Jacobi equation \( H(q^i, \frac{\partial W}{\partial q^i}) = E \), where \( E \) is a constant.

From Marsden et al. [18] we know that, since the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle \( T^*Q \) may not be a Hamiltonian system on a cotangent bundle, then we cannot give the Hamilton-Jacobi theorem for the Marsden-Weinstein reduced system just like same as the above theorem. We have to look for a new way. Recently, in Wang [23], some of formulations of Hamilton-Jacobi equations for Hamiltonian system and regular reduced Hamiltonian systems are given. It is worthy of noting that, at first, an important lemma is proved, and it is a modification for the corresponding result of Abraham and Marsden [1], such that we can prove two types of geometric Hamilton-Jacobi theorem for a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and dynamical vector field; Then these results are generalized to the regular reducible Hamiltonian system with symmetry and momentum map, by using the reduced symplectic form and the reduced dynamical vector field. The Hamilton-Jacobi theorems are proved and two types of Hamilton-Jacobi equations, for the regular point reduced Hamiltonian system and the regular orbit reduced Hamiltonian system, are obtained. As an application of the theoretical results, the regular point reducible Hamiltonian system on a Lie group is considered, and two types of Lie-Poisson Hamilton-Jacobi equation for the regular point reduced system are given. In particular, the Type I and Type II of Lie-Poisson Hamilton-Jacobi equations for the regular point reduced rigid body and heavy top systems are shown, respectively.

Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the (regular) controlled Hamiltonian system and its a variety of the reduced systems, as well as to the nonholonomic Hamiltonian system and the nonholonomic reducible Hamiltonian system, see Wang [25] and de León and
Wang [7].

Finally, we also note that there have been a lot of beautiful results of reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. So, it is an important topic to study the application of symmetric reduction and Hamilton-Jacobi theory for the controlled Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. These are our goals in future research.

**Acknowledgments:** This is a survey to introduce briefly some recent developments of symmetric reductions of the regular controlled Hamiltonian systems with symmetries and applications. The author would like to thank the cultivation and training of her advisor—Professor Hesheng Hu and to dedicate the article to her on the occasion of her 90th birthday.

Especially grateful to Professor Jerrold E. Marsden, Professor Tudor S. Ratiu, Professor Manuel de León, Professor Arjan Van der Schaft and Professor Juan-Pablo Ortega and MS. Wendy McKay for their understanding, guiding, support and help in the study of geometric mechanics.

Professor Jerrold E. Marsden is worthy to be respected. He knew and admitted their wrong and joined us to correct these wrong. Professor Marsden is a model of excellent scientists. It is an important task for us to correct and develop well the research work of Professor Marsden, such that we never feel sorry for his great cause.

Marsden-Weinstein reduction is just like an elegant cobblestone on the long river bank of history of mankind civilization, ones share its beauty and happiness.

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