Counting rainbow triangles in edge-colored graphs

Xueliang Li*, Bo Ning†, Yongtang Shi‡, Shenggui Zhang§

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Abstract

Let \( G \) be an edge-colored graph on \( n \) vertices. The minimum color degree of \( G \), denoted by \( \delta^c(G) \), is defined as the minimum number of colors assigned to the edges incident to a vertex in \( G \). In 2013, H. Li proved that an edge-colored graph \( G \) on \( n \) vertices contains a rainbow triangle if \( \delta^c(G) \geq \frac{n+1}{2} \). In this paper, we obtain several estimates on the number of rainbow triangles through one given vertex in \( G \). As consequences, we prove counting results for rainbow triangles in edge-colored graphs. One main theorem states that the number of rainbow triangles in \( G \) is at least \( \frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n \), which is best possible by considering the rainbow \( k \)-partite Turán graph, where its order is divisible by \( k \). This means that there are \( \Omega(n^2) \) rainbow triangles in \( G \) if \( \delta^c(G) \geq \frac{n+1}{2} \), and \( \Omega(n^3) \) rainbow triangles in \( G \) if \( \delta^c(G) \geq cn \) when \( c > \frac{1}{2} \). Both results are tight in sense of the order of the magnitude. We also prove a counting version of a previous theorem on rainbow triangles under a color neighborhood union condition due to Broersma et al., and an asymptotically tight color degree condition forcing a colored friendship subgraph \( F_k \) (i.e., \( k \) rainbow triangles sharing a common vertex).

1 Introduction

Throughout this paper, we only consider finite undirected simple graphs. Let \( G \) be a graph. By an edge-coloring of \( G \), we mean a function \( C : E \to \mathbb{N} \), where \( \mathbb{N} \) is the

*Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China. Email: lxl@nankai.edu.cn.
†Corresponding author. College of Cyber Science, Nankai University, Tianjin 300350, China. Email: bo.ning@nankai.edu.cn.
‡Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China. Email: shi@nankai.edu.cn.
§School of Mathematics and Statistics, Northwestern Polytechnical University, Xi’an, Shaanxi 710129, China; Xi’an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi’an, Shaanxi 710129, China. Email: sgzhang@nwpu.edu.cn.
set of non-negative integers. If \( G \) has such an edge-coloring, we call \( G \) an edge-colored graph and denote it by \((G, C)\). For a vertex \( v \in V(G) \), the color neighborhood \( CN_G(v) \) is defined as the set \( \{C(e) : e \text{ is incident with } v\} \), and the color degree of \( v \) is denoted by \( d^c_G(v) := |CN_G(v)| \). We denote by \( \delta^c(G) := \min\{d^c_G(v) : v \in V(G)\} \), and by \( c(G) \) the number of colors appearing on \( E(G) \). Let \( \sigma_2(G) = \min\{d^c(x) + d^c(y) : xy \in E(G)\} \). For a vertex \( v \in V(G) \), the monochromatic degree of \( v \) (in \( G \)), denoted by \( d^c_G^{\text{mon}}(v) \), is defined as the maximum number of edges incident to \( v \) colored with a same color. A subgraph \( H \) of \( G \) is called properly-colored if every two incident edges are assigned with different colors, and is called rainbow if all of its edges have distinct colors. When there is no possibility of confusion, we will drop the subscript \( G \). For example, we use \( \delta^c \) instead of \( \delta^c(G) \). For notation and terminology not defined here, we refer to Bondy and Murty [3].

Rainbow and properly-colored subgraph problems have received much attention from graph theorists, see [1, 4, 5, 7, 15]. For surveys, see [6, 18]. In 2013, H. Li [21] proved a Theorem 1 motivated much attention on rainbow subgraphs. Czygrinow, Molla, Nagle, and Oursler [7] recently proved that the same condition in Theorem 1 ensures a rainbow \( \ell \)-cycle \( C_\ell \) whenever \( n > 432\ell \), which is sharp for a fixed odd integer \( \ell \geq 3 \) when \( n \) is sufficiently large. The authors in [20] proposed a new type condition, i.e., every edge-colored graph \((G, C)\) on \( n \) vertices satisfying \( e(G) + c(G) \geq \frac{n(n+1)}{2} \) contains a rainbow triangle, where \( e(G) \) is the number of edges in \( G \) and \( c(G) \) is the number of all colors appearing on \( E(G) \). This motivated further studies on rainbow cliques [24] and properly-colored \( C_4 \)'s [25].

The original purpose of this article is to study the supersaturation problem of rainbow triangles in edge-colored graphs. This problem is obviously motivated by the study of supersaturation problem of triangles in graphs. It studies the following function: for triangle \( C_3 \) and for integers \( n, t \geq 1 \),

\[
 h_{C_3}(n, t) = \min\{t(G) : |V(G)| = n, |E(G)| = ex(n, C_3) + t\},
\]

where \( t(G) \) is the number of \( C_3 \) in \( G \) and \( ex(n, C_3) \) is the Turán function of \( C_3 \). Improving Mantel’s theorem, Rademacher (unpublished, see [9]) proved that \( h_{C_3}(n, 1) \geq \lfloor \frac{n}{2} \rfloor \). Erdős [10, 11] proved that \( h_{C_3}(n, k) \geq k|\frac{n}{2}| \) where \( k \leq cn \) for some constant \( c \). In fact, Erdős
conjectured that $h_{C_3}(n, k) \geq k\lfloor \frac{n}{2} \rfloor$ for all $k < \lfloor \frac{n}{2} \rfloor$, which was finally resolved by Lovász and Simonovits \[23\].

One can ask for a rainbow analog of the above Erdős’ conjecture. In this direction, answering an open problem in \[16\], Ehard and Mohr \[8\] proved there are at least $k$ rainbow triangles in an edge-colored graph $(G, C)$ such that $e(G) + c(G) \geq \binom{n+1}{2} + k - 1$. If we consider $e(G) + c(G)$ as a variant of Turán function in edge-colored graphs, then the theorem above tells us that the supersaturation phenomenon of rainbow triangles under this type of condition is quite different from the original one. On the other hand, the problem of finding a counting version of Theorem \[1\] is still open.

We denote by $G_n^*$ the family of edge-colored graphs on $n$ vertices with the minimum color degree at least $\frac{n+2}{2}$, by $rt(G)$ the number of rainbow triangles in an edge-colored graph $G$, and by $rt(G; v)$ be the number of rainbow triangles through a vertex $v$ in $G$. Denote by

$$f(n) := \min\{rt(G) : G \in G_n^*\}.$$  

Proving a special case of a conjecture which states that every edge-colored graph on $n \geq 20$ vertices contains two disjoint rainbow triangles if the minimum color degree is at least $\frac{n+2}{2}$, Hu, Li, and Yang developed a key lemma \[17\] (Lemma 1), from which one can easily obtain $f(n) = \Omega(n)$. One may dare to guess that $f(n) \geq \Omega(n^2)$. Our first humble contribution confirms this.

**Remark 1.** Throughout this paper, we repeatedly assume that an edge-colored graph $(G, C)$ satisfying $\delta_c(G) \geq \frac{n+1}{2}$ and subject to this, $e(G)$ is minimal. Here the word “minimal” means that deleting any edge in $G$ will decrease the color degree of some vertex of $G$. That is, $G$ contains no monochromatic $C_3$ or $P_4$. Furthermore, we can see that a spanning subgraph of $G$ with a same color should be a star forest.

**Theorem 3.** Let $(G, C)$ be an edge-colored graph on $n$ vertices. Suppose that $\delta_c(G) \geq \frac{n+1}{2}$, and subject to this, $e(G)$ is minimal. Then

$$rt(G) \geq \frac{e(G)(\sigma_2(G) - n)}{3} + \frac{1}{6} \sum_{v \in V(G)} (n - d(v) - 1)(d(v) - d_c(v)).$$

As consequences of Theorem \[3\] we obtain two counting versions of Theorem \[1\]

**Theorem 4.** Let $(G, C)$ be an edge-colored graph on $n$ vertices. Then

$$rt(G) \geq \frac{1}{6}\delta_c(G)(2\delta_c(G) - n)n.$$  

In particular, if $\delta_c(G) > cn$ for $c > \frac{1}{2}$, then

$$rt(G) \geq \frac{c(2c - 1)}{6}n^3.$$
One may wonder the tightness of Theorem 4. The following example shows that Theorem 4 is the best possible.

**Example 1.** Let $G$ be a rainbow $k$-partite Turán graph on $n$ vertices where $k|n$ and $k \geq 3$. Then there are exactly \( \left( \frac{k}{3} \right)^3 \binom{n}{k} \) rainbow triangles. By Theorem 4 there are at least \( \left( \frac{k}{3} \right)^3 \binom{n}{k} \) rainbow triangles.

Setting $\delta^r(G) = \frac{n+1}{2}$ in Theorem 4 we obtain the right hand of the following.

**Theorem 5.** For even $n \geq 4$, we have $\frac{n^3}{4} \geq f(n) \geq \frac{n^3 + 2n}{6}$; for odd $n \geq 3$, we have $\frac{n^3}{8} \geq f(n) \geq \frac{n^3 + n}{12}$.

For Theorem 4 the leftmost of each inequality (for $f(n)$) of Theorem 5 was shown by the following two examples. From Theorem 5, we infer $f(n) = \Theta(n^2)$.

**Example 2.** Let $(G, C)$ be a rainbow graph of order $n$ where $n$ is divisible by 4. Let $V(G) = X_1 \cup X_2$, $|X_1| = |X_2| = \frac{n}{2}$, and each of $G[X_1]$ and $G[X_2]$ consists of a perfect matching of size $\frac{n}{2}$. In addition, $G - E(X_1) - E(X_2)$ is balanced and complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n}{4}$ rainbow triangles. So does each edge in $G[X_2]$. Therefore, there are exactly $\frac{n^2}{4}$ rainbow triangles in $G$.

**Example 3.** Let $(G, C)$ be a rainbow graph of order $n$ where $n \equiv 1 \pmod{4}$. Let $V(G) = X_1 \cup X_2$, $|X_1| = \frac{n+1}{2}$ and $|X_2| = \frac{n-1}{2}$, and $G[X_1]$ consists of a perfect matching of size $\frac{n+1}{4}$. In addition, $G - E(X_1)$ is complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n-1}{2}$ rainbow triangles and so does each edge in $G[X_1]$. Therefore, there are exactly $\frac{n^2}{8}$ rainbow triangles in $G$.

In 2005, Broersma, X. Li, Woeginger, and Zhang proved an edge-colored graph $(G, C)$ on $n \geq 4$ vertices contains a rainbow $C_3$ or a rainbow $C_4$ if $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices $u, v \in V(G)$. Define $G$ to be a rainbow $K_{\frac{n+1}{2}, \frac{n-1}{2}}$ where $n$ is even. Then $|CN(u) \cup CN(v)| = n - 1$ for each pair of vertices $u$ and $v$, and $G$ contains no rainbow triangles. Thus, one need slightly enhance the color degree condition when finding rainbow triangles. Broersma et al.’s theorem was generalized by Fujita, Ning, Xu and Zhang to the one forcing rainbow triangles under the same condition.

In this paper, we extend both theorems mentioned to a counting version as follows.

**Theorem 6.** Let $(G, C)$ be an edge-colored graph of order $n \geq 4$ such that $|CN(u) \cup CN(v)| \geq n$ for every pair of vertices $u, v \in V(G)$. Then $G$ contains $\frac{n^2 - 2n}{24}$ rainbow $C_3$'s.

We also prove some better estimate on the number of rainbow triangles through vertices with high monochromatic degree.
Theorem 7. Let \((G, C)\) be an edge-colored graph on \(n\) vertices with \(\delta^c(G)\) and furthermore, \(e(G)\) is minimal. Let \(V(G) = \{v_1, v_2, \ldots, v_n\}\) such that \(d_{G}^{mon}(v_1) \geq d_{G}^{mon}(v_2) \geq \cdots \geq d_{G}^{mon}(v_n)\). Then for each \(1 \leq k \leq \delta^c(G) - 1\),

\[
\sum_{i=1}^{k} rt(G; v_i) \geq \frac{1}{2} \left( \sum_{i=1}^{k} d_{G}^{mon}(v_i) + k(\delta^c(G) - 1) \right) \left( \sigma_2^c(G) - n \right) + \frac{\Delta_k(G)}{2}.
\]

where \(\Delta_k(G) = \left( \delta^c(G) \sum_{i=1}^{k} d_{G}^{mon}(v_i) - k \sum_{i=1}^{\delta^c(G)} d_{G}^{mon}(v_i) \right)\).

The above theorem has the following simple but useful corollary.

Theorem 8. Let \((G, C)\) be an edge-colored graph on \(n\) vertices with \(\delta^c(G) \geq \frac{n+1}{2}\). Let \(V(G) = \{v_1, v_2, \ldots, v_n\}\) such that \(d_{G}^{mon}(v_1) \geq d_{G}^{mon}(v_2) \geq \cdots \geq d_{G}^{mon}(v_n)\). Then for each \(1 \leq k \leq \delta^c(G) - 1\),

\[
\sum_{i=1}^{k} rt(G; v_i) \geq \frac{k\delta^c(G)}{2}.
\]

The friendship graph \(F_k\) is a graph consisting of \(k\) triangles sharing a common vertex. Finally, we obtain some color degree condition for the existence of some rainbow triangles sharing one common vertex, i.e., the underlying graph is a friendship subgraph. This extends Theorem 1 in another way.

Theorem 9. Let \(k \geq 2\) and \(n \geq 50k^2\). Let \((G, C)\) be an edge-colored graph on \(n\) vertices. If \(\delta^c(G) \geq \frac{n}{2} + k - 1\) then \(G\) contains \(k\) rainbow triangles sharing one common vertex.

This paper is organised as follows. In Section 2, we prove one estimate on the number of rainbow triangles through one given vertex. As consequences, we prove Theorems 3 and 6 (together with some corollaries). In Section 3, we prove another estimate on the number of rainbow triangles through vertices with high monochromatic degree, and prove Theorem 7. In Section 4 we prove a theorem slightly stronger than Theorem 9. We conclude this paper with some open problems in the last section.

2 Counting rainbow triangles

The main purpose of this section is to prove Theorem 8. We first prove the following lemma, whose proof is partly inspired by [17, Lemma 1].
Lemma 1. Let \((G, C)\) be an edge-colored graph on \(n\) vertices with \(\delta^c(G) \geq \frac{n+1}{2}\) and furthermore, \(e(G)\) is minimal. Then for each \(v \in V(G)\),
\[
rt(G; v) \geq \frac{1}{2} \left( (n - d(v) - 1)(d(v) - d^c(v)) + \sum_{1 \leq j \leq d^c(v)} \sum_{a \in N_j(v)} (d_j(v) - d_j(a) + \sum_{a \in N_v} (d^c(v) + d^c(a) - n)) \right).
\]

Proof. Since \(G\) is edge-minimal, there is no monochromatic path of length 3 and no monochromatic triangle in \(G\).

Let \(N_v := N_G(v)\). Without loss of generality, assume that \(CN_G(v) = \{1, 2, \ldots, s\}\), where \(s = d^c(v)\). Let \(N_j(v) := \{u : C(au) = j, v \in N_v\}\) and \(d_j(v) := |N_j(v)|\), where \(1 \leq j \leq s\). Furthermore, assume that \(d_1(v) \geq d_2(v) \geq \cdots \geq d_s(v)\). So \(d^\text{mon}(v) = d_1\).

For the vertex \(v \in V(G)\), define a digraph \(D_v\) on \(N_v\) as follows: \(\overrightarrow{ab} \in A(D_v)\) if and only if \(ab \in E(G)\) and \(C(ab) \neq C(va)\), i.e., \(vab\) is a rainbow path of length 2. Therefore, for any two vertices \(x, y \in N_j(v)\) (if \(|N_j(v)| \geq 2\)), there is either a 2-cycle \(xyx\) or no arc between \(x\) and \(y\) (since otherwise, there is a monochromatic \(C_3\), a contradiction).

For \(a \in N_v\), let \(S_a \subseteq N_v \setminus \{a\}\) be maximal such that \(C(au), C(au')\) and \(C(av)\) are pairwise different for any two distinct vertices \(u, u' \in N_v\). According to the definition of \(D_v\), every edge \(au, u \in S_a\), corresponds to an out-arc from \(a\) in \(D_v\). Notice that
\[
d^+_D(v) \geq |S_a| \geq CN_{G[N_v \cup \{v\}]}(a) \geq d^c(a) - 1 - |V(G) \setminus (N_v \cup \{v\})|.
\]
Thus, we have
\[
d^+_D(v) \geq d^c(a) + d_G(v) - n.
\]
Therefore,
\[
\sum_{a \in N_v} d^+_D(v) \geq d_G(v)(d_G(v) - n) + \sum_{a \in N_v} d^c(a)
\]
\[
= d_G(v) \left( \sum_{j=1}^{s} (d_j - 1) \right) + \sum_{a \in N_v} (d^c(a) + d^c(v) - n).
\]

(1)

Next, consider \(\sum_{a \in N_v} d^-_D(v)\). Note that for any vertices \(a, b \in N_j(v)\) such that \(ab \in E(G)\) \((d_j(v) \geq 2)\), if \(\overrightarrow{ba} \in A(D_v)\) then \(c(va) = c(ab)\), and so \(ba\) is not contained in a 2-cycle. Hence for every \(j \in [1, s]\) and every \(a \in N_j(v)\), there are at most \(d_j(a) - 1\) in-arcs which are not contained in 2-cycles, such that \(a\) is a common sink. Observe that for any such vertex \(a \in N_v\), if \(a \in N_j\) then \(d_j = 1\); otherwise there is a monochromatic \(P_3\) in \(G\), a contradiction.
For \(1 \leq j \leq s\), let \(n_j\) be the number of 2-cycles in \(D_v[N_j]\). Let \(n_0\) be the number of all 2-cycles \(xyx\) in \(D_v\) such that \(C(xv) \neq C(yv)\). That is, \(n_0 = \text{rt}(G; v)\).

Thus, we have

\[
\sum_{a \in N_v} d_{D_v}^-(a) \leq \sum_{1 \leq j \leq d^c(v)} \sum_{d_j(v) = 1} (d_j(a) - 1) + 2n_0 + 2 \sum_{j=1}^s n_j. \tag{2}
\]

Since

\[2n_j \leq d_j(v)(d_j(v) - 1),\]

From (2), we can obtain that

\[
\sum_{a \in N_v} d_{D_v}^+(a) \leq \sum_{1 \leq j \leq d^c(v)} \sum_{d_j(v) = 1} (d_j(a) - 1) + 2n_0 + \sum_{j=1}^s d_j(v)(d_j(v) - 1). \tag{3}
\]

As

\[\sum_{a \in N_v} d_{D_v}^+(a) = \sum_{a \in N_v} d_{D_v}^-(a),\]

combining (1) and (3), we have

\[
2n_0 \geq \sum_{a \in N_v} (d^c(v) + d^c(a) - n) + d(v) \left( \sum_{j=1}^s (d_j - 1) \right) - 2 \sum_{j=1}^s d_j(v)(d_j(v) - 1)
- \sum_{1 \leq j \leq s, d_j(v) = 1} \sum_{a \in N_j(v)} (d_j(a) - 1) + \sum_{j=1}^s d_j(v)(d_j(v) - 1). \tag{4}
\]

Set

\[A = d(v) \left( \sum_{j=1}^s (d_j(v) - 1) \right) - 2 \sum_{j=1}^s d_j(v)(d_j(v) - 1),\]

and

\[B = - \sum_{1 \leq j \leq s, d_j(v) = 1} \sum_{a \in N_j(v)} (d_j(a) - 1) + \sum_{j=1}^s d_j(v)(d_j(v) - 1).\]

Then (4) is equivalent to the following

\[2n_0 \geq \sum_{a \in N_v} (d^c(v) + d^c(a) - n) + A + B. \tag{5}\]

By simple algebra,

\[A = \sum_{j=1}^s (d(v) - 2d_j(v))(d_j(v) - 1) = \sum_{j=1, d_j(v) \geq 2}^s (d(v) - 2d_j(v))(d_j(v) - 1).\]
As
\[ d_1(v) \leq d(v) - d^c(v) + 1 \]
and
\[ d^c(v) \geq \frac{n + 1}{2}, \]
we have
\[ d(v) - 2d_j(v) \geq d(v) - 2d_1(v) \geq d(v) - 2(d(v) - d^c(v) + 1) = 2d^c(v) - d(v) - 2 \geq n - d(v) - 1, \]
and so
\[ A \geq \sum_{j=1}^{s} (n - d(v) - 1)(d_j(v) - 1) = (n - d(v) - 1)(d(v) - d^c(v)). \quad (6) \]

Furthermore, we obtain
\[
\begin{align*}
B = & - \sum_{1 \leq j \leq s, d_j(v) = 1} \sum_{a \in N_j(v)} (d_j(a) - d_j(v)) + \sum_{1 \leq j \leq s, d_j(v) \geq 2} \sum_{a \in N_j(v)} (d_j(v) - d_j(a)) \\
= & \sum_{1 \leq j \leq d^c(v)} \sum_{a \in N_j(v)} (d_j(v) - d_j(a)), \quad (7)
\end{align*}
\]

where \( d_j(a) = 1 \) when \( d_j(v) \geq 2 \), since \( G \) contains no monochromatic \( P_4 \).

Now, together with (5), (6), and (7), we infer
\[
2n_0 \geq \left( \sum_{a \in N_v} (d^c(v) + d^c(a) - n) \right) + (n - d(v) - 1)(d(v) - d^c(v)) \\
+ \sum_{1 \leq j \leq d^c(v)} \sum_{a \in N_j(v)} (d_j(v) - d_j(a)).
\]

This proves Lemma 1. \( \square \)

By simple technique of counting in two ways, we have the following.

Proposition 10. \( \square \) Let \( (G, C) \) be an edge-colored graph with vertex set \( V(G) \) and \( \delta^c(G) \geq \frac{n+1}{2} \), and furthermore, \( e(G) \) is minimal. Let \( I_v = \{C(uv) : uv \in E(G)\} \). For \( k \in I_v \), \( N_k(v) := \{u \in N_v : C(uv) = k\} \) and \( d_k(v) := |N_k(v)| \). Then
\[
\sum_{v \in V(G)} \sum_{k \in I_v} \sum_{a \in N_k(v)} (d_k(v) - d_k(a)) = 0. \quad (8)
\]

Note that in the proof of Lemma 1, without loss of generality, we assume that \( I_v = \{1, 2, \ldots, d^c(v)\} \) for simply. In fact, for distinct vertices \( u, v \in V(G) \), \( I_u \) may be not equal to \( I_v \), and may be not a subset of \([1, C(G)]\).
Proof. By definition of $N_k(v)$, we can see
\[
\sum_{k \in I_v} \sum_{a \in N_k(v)} (d_k(v) - d_k(a)) = \sum_{a \in N_G(v)} d_{C(va)}(v) - d_{C(va)}(a).
\]
By counting in two ways, we have
\[
\sum_{v \in V(G)} \sum_{a \in N_G(v)} (d_{C(va)}(v) - d_{C(va)}(a)) = \sum_{xy \in E(G)} (d_{C(xy)}(x) - d_{C(xy)}(y)) + (d_{C(xy)}(y) - d_{C(xy)}(x)) = 0.
\]
This proves Proposition 10. \qed

Now we can obtain one main result in this paper, which is stronger than Theorem 3.\footnote{Note: This should be cited or referred to as Theorem 3.}

**Theorem 11.** Let $(G, C)$ be an edge-colored graph with vertex set $V(G)$. Let $n = |V(G)|$. If $\delta^c \geq \frac{n+1}{2}$, and furthermore, $e(G)$ is minimal, then we have,
\[
rt(G) \geq \frac{1}{6} \sum_{v \in V(G)} \left( (n - d(v) - 1)(d(v) - d^c(v)) + \sum_{a \in N_G(v)} (d^c(v) + d^c(a) - n) \right).
\]

Proof. The theorem follows from Lemma 1, Proposition 10 and the fact that $3rt(G) = \sum_{v \in V(G)} rt(G; v)$. \qed

Finally, we present the proof of Theorem 6.\footnote{This should be cited or referred to as Theorem 6.}

**Proof of Theorem 6.** If $\delta^c \geq \frac{n+1}{2}$, then by Theorem 3, $G$ contains $\frac{n^2+n}{12}$ rainbow $C_3$’s. Thus, $\delta^c \leq \frac{n}{2}$. Choose $v \in V(G)$ such that $d^c_G(v) = \delta^c \leq \frac{n}{2}$. Set $G' = G - v$.

First we furthermore suppose that $d_G^c(v) \leq \frac{n-1}{2}$. For a vertex $u$ adjacent to $v$, $|CN(u) \cup CN(v)| \geq n$. It follows that
\[
d_G^c(u) + d_G^c(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n + 1.
\]
It follows that $d^c_G(u) \geq \frac{n+3}{2}$. For a vertex $u$ non-adjacent to $v$, we also have
\[
d_G^c(u) + d_G^c(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n.
\]
Thus, $d^c_G(u) \geq \frac{n+1}{2}$. It follows that $d_{G'}(u) \geq \frac{n+1}{2} > \frac{|G'|+1}{2}$. Then by Theorem 3, we have
\[
rt(G') \geq \frac{1}{6} \cdot \frac{n+1}{2} - \left(2 \cdot \frac{n+1}{2} - (n-1)\right) n \geq \frac{n^2 + n}{6}.
\]
So $d^c(v) = \frac{n}{2}$, i.e., $\delta^c = \frac{n}{2}$. In this case, for an edge $uv \in E(G)$,
\[
d^c(u) + d^c(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n + 1.
\]
By setting \( k = \delta^c - 1 \) in Theorem \( \square \) we have
\[
rt(G) \geq \frac{1}{3} \sum_{i=1}^{\delta^c-1} rt(G; v_i) \geq \frac{n^2 - 2n}{24},
\]
where \( V(G) = \{v_1, v_2, \ldots, v_n\} \) such that \( d^{\text{mon}}(v_1) \geq d^{\text{mon}}(v_2) \geq \cdots \geq d^{\text{mon}}(v_n) \). This proves the theorem.

### 3 Rainbow triangles through a specified vertex

The main purpose of this section is to obtain better estimate of the number of rainbow triangles through a specified vertex when the monochromatic degree of this vertex is large. Before the proof, we need introduce some additional notations.

For a vertex \( v \in V(G) \), let \( X_v \) be the maximal subset of \( N_G(v) \) such that \( c(va) = c(vb) \) for any two distinct vertices \( a, b \in X_v \). Then \( |X_v| = d^{\text{mon}}(v) \). Let \( Y_v \subseteq (N_G(v) \setminus X_v) \) such that \( c(va) \neq c(vb) \) for any two vertices \( a, b \in Y_v \). Thus, we have \( |Y_v| \leq d^c(v) - 1 \). In the following, set
\[
f(v) := \min\{d^c(u) + |Y_v| + 1 : u \in X_v \cup Y_v\}.
\]

We first prove the following lemma, whose proof is a variant of Lemma \( \square \).

**Lemma 2.** Let \((G, C)\) be an edge-colored graph on \( n \) vertices with \( \delta^c(G) \), and subject to this, \( e(G) \) is minimal. Then for each \( v \in V(G) \),
\[
rt(G; v) \geq \frac{1}{2} \left( (d_G^{\text{mon}}(v) + |Y_v|)(f(v) - n) + (|Y_v|d_G^{\text{mon}}(v) - \sum_{a \in Y_v} d_G^{\text{mon}}(a)) \right).
\]

**Proof.** For a vertex \( v \in V(G) \), define a digraph \( D_v \) on \( X_v \cup Y_v \) as follows: \( \overrightarrow{ab} \in A(D_v) \) if and only if \( ab \in E(G) \) and \( c(ab) \neq c(va) \). Let \( n_1 \) be the number of 2-cycles in \( D_v[X] \). Let \( n_2 \) be the number of other 2-cycles in \( D_v \). Apparently, \( rt(v) \geq n_2 \).

For \( a \in X_v \cup Y_v \), let \( S \subseteq (X_v \cup Y_v) \setminus \{a\} \) be maximal such that \( c(au), c(au'), c(av) \) are pairwise different for two distinct vertices \( u, u' \in X_v \cup Y_v \). According to the definition of \( D_v \), every edge \( au, u \in S \) gives an out-arc of \( a \) in \( D_v \). Hence, we have
\[
d^{+}_{D_v}(a) \geq d^c(a) - 1 - |V(G) \setminus (X_v \cup Y_v \cup \{v\})| \\
\geq f(v) + d^{\text{mon}}(v) - n - 1.
\]

Therefore,
\[
\sum_{a \in X_v \cup Y_v} d^{+}_{D_v}(a) \geq (d^{\text{mon}}(v) + |Y_v|)(f(v) + d^{\text{mon}}(v) - n - 1).
\]
Next, consider $\sum_{a \in X_v \cup Y_v} d^-_D(a)$. By reasoning the proof of Lemma 1 and a similar analysis, we obtain
\[
\sum_{a \in X_v \cup Y_v} d^-_D(a) \leq \sum_{a \in Y_v} d^{\text{mon}}(a) - |Y_v| + 2(n_1 + n_2).
\]
(11)
Since
\[
\sum_{a \in X_v \cup Y_v} d^+_D(a) = \sum_{a \in X_v \cup Y_v} d^-_D(a)
\]
and
\[
2n_1 \leq d^{\text{mon}}(v)(d^{\text{mon}}(v) - 1),
\]
combining (10) and (11), we have
\[
2n_2 \geq d^{\text{mon}}(v)(f(v) - n) + |Y_v|(f(v) + d^{\text{mon}}(v) - n) - \sum_{a \in Y_v} d^{\text{mon}}(a)
\]
\[
= (d^{\text{mon}}(v) + |Y_v|)(f(v) - n) + (|Y_v|d^{\text{mon}}(v) - \sum_{a \in Y_v} d^{\text{mon}}(a)).
\]
Hence,
\[
rt(G; v) \geq n_2 \geq \frac{1}{2} \left( (d^{\text{mon}}(v) + |Y_v|)(f(v) - n) + (|Y_v|d^{\text{mon}}(v) - \sum_{a \in Y_v} d^{\text{mon}}(a)) \right).
\]
The proof is complete. 

Specially, set $|Y_v| = d^c(v) - 1$. Then
\[
f(v) := \min\{d^c(v) + d^c(u) : u \in X_v \cup Y_v\} \geq \sigma^2_2(G),
\]
and Lemma 2 has the following form:

**Lemma 3.** Let $(G, C)$ be an edge-colored graph on $n$ vertices with $\delta^c(G)$ and furthermore, $e(G)$ is minimal. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d^{\text{mon}}(v_1) \geq d^{\text{mon}}(v_2) \geq \cdots \geq d^{\text{mon}}(v_n)$. Then for each $1 \leq i \leq \delta^c(G)$,
\[
rt(G; v_i) \geq \frac{1}{2} \left( (d^{\text{mon}}_G(v_i) + d^c_G(v_i) - 1)(\sigma^2_2(G) - n) + (|Y_{v_i}|d^{\text{mon}}_G(v_i) - \sum_{a \in Y_{v_i}} d^{\text{mon}}_G(a)) \right).
\]

**Proposition 12.** Let $(G, C)$ be an edge-colored graph on $n$ vertices with $\delta^c(G)$ and furthermore, $e(G)$ is minimal. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d^{\text{mon}}_G(v_1) \geq d^{\text{mon}}_G(v_2) \geq \cdots \geq d^{\text{mon}}_G(v_n)$. Let $Y_{v_i}$ be defined as in Lemma 2 with $|Y_{v_i}| = \delta^c(G) - 1$. Then for each $1 \leq k \leq \delta^c(G) - 1$,
\[
\sum_{i=1}^{k} (|Y_{v_i}|d^{\text{mon}}_G(v_i) - \sum_{a \in Y_{v_i}} d^{\text{mon}}_G(a)) \geq \left( \delta^c \sum_{i=1}^{k} d^{\text{mon}}_G(v_i) - k \sum_{i=1}^{\delta^c} d^{\text{mon}}_G(v_i) \right) \geq 0.
\]
Proof. Note that for \( v_i \in V(G) \), \( v_i \notin Y_i := Y_i \). Hence,

\[
\sum_{a \in Y_i} d_{\text{mon}}(a) \leq \sum_{j=1}^{i-1} d_{\text{mon}}(v_j) + \sum_{j=i+1}^{\delta_c} d_{\text{mon}}(v_j).
\]

Thus,

\[
\sum_{i=1}^{k} \sum_{a \in Y_i} d_{\text{mon}}(a) \leq \sum_{i=1}^{k} \left( \sum_{j=1}^{i-1} d_{\text{mon}}(v_j) + \sum_{j=i+1}^{\delta_c} d_{\text{mon}}(v_j) \right) = \sum_{i=1}^{k} \delta_c d_{\text{mon}}(v_i) - \sum_{i=1}^{k} d_{\text{mon}}(v_i).
\]

It follows that

\[
\sum_{i=1}^{k} \left( d_{\text{mon}}(v_i)(\delta_c - 1) - \sum_{a \in Y_i} d_{\text{mon}}(a) \right) \geq \left( \delta_c \sum_{i=1}^{k} d_{\text{mon}}(v_i) - k \sum_{i=1}^{\delta_c} d_{\text{mon}}(v_i) \right).
\]

The proof of Proposition 12 is complete. \( \square \)

Now we can obtain the following theorem.

**Theorem 13.** Let \((G, C)\) be an edge-colored graph on \( n \) vertices with \( \delta_c(G) \) and furthermore, \( e(G) \) is minimal. Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) such that \( d_{\text{mon}}^G(v_1) \geq d_{\text{mon}}^G(v_2) \geq \ldots \geq d_{\text{mon}}^G(v_n) \). Then for each \( 1 \leq k \leq \delta_c(G) - 1 \),

\[
\sum_{i=1}^{k} \text{rt}(G; v_i) \geq \frac{1}{2} \left( \sum_{i=1}^{k} d_{\text{mon}}^G(v_i) + k(\delta_c(G) - 1) \right) (\sigma_2(G) - n) + \frac{\Delta_k(G)}{2}.
\]

where

\[
\Delta_k(G) = \left( \delta_c(G) \sum_{i=1}^{k} d_{\text{mon}}^G(v_i) - k \sum_{i=1}^{\delta_c(G)} d_{\text{mon}}^G(v_i) \right).
\]

Proof. This theorem directly follows from Lemma 3 and Proposition 12. \( \square \)

### 4 Edge-colored friendship subgraphs

In this section, we shall prove a result slightly stronger than Theorem 9. For a graph \( G \), we denote by \( \Delta_{\text{mon}}(G) := \max\{d_{\text{mon}}^G(v) : v \in V(G)\} \).

**Theorem 14.** Let \( k, n \) be positive integers, and \( G \) be an edge-colored graph on \( n \) vertices with \( n \geq 50k^2 \) where \( k \geq 2 \), and \( \delta^c(G) \geq \frac{n}{2} + k - 1 \). Let \( v \in V(G) \) such that \( d_{\text{mon}}^G(v) = \Delta_{\text{mon}}(G) \). Then \( G \) contains \( k \) rainbow triangles sharing only the vertex \( v \) as the center (i.e., the underly graph is \( F_k \) with \( v \) as its center).
The following result on Turán number of friendship graphs is well known.

**Theorem 15** ([12]). For every $k \geq 1$ and every $n \geq 50k^2$, if a graph $G$ of order $n$ satisfies $e(G) > \text{ex}(n, F_k)$, then $G$ contains a copy of a $k$-friendship graph, where $\text{ex}(n, F_k) = \left\lceil \frac{n^2}{4} \right\rceil + k^2 - k$ if $k$ is odd; and $\text{ex}(n, F_k) = \left\lceil \frac{n^2}{4} \right\rceil + k^2 - \frac{3k}{2}$ if $k$ is even.

The matching number of a graph $G$, denoted by $\alpha'(G)$, is defined to be the maximum number of pairwise disjoint edges in $G$. Our proof of Theorem 14 uses a famous result on Turán number of a matching with given number of edges due to Erdős and Gallai [13].

**Theorem 16** ([13]). Let $G$ be a graph on $n$ vertices. If $\alpha'(G) \leq k$ then $e(G) \leq \max\{\binom{2k+1}{2}, \binom{n}{2} - \binom{n-k}{2}\}$.

We also need a special case of Lemma 4.

**Lemma 4.** Let $(G, C)$ be an edge-colored graph on $n$ vertices with $\delta^c(G)$ such that $e(G)$ is minimal. Then for a vertex $v \in V(G)$ with $d^\text{mon}_G(v) = \Delta^\text{mon}(G)$, we have

$$rt(G; v) \geq \frac{1}{2} (\Delta^\text{mon}(G) + d^c_G(v) - 1)(\delta^c(G) + d^c_G(v) - n)).$$

**Proof.** Putting the vertex $v$ as one with $d^\text{mon}(v) = \Delta^\text{mon}(G)$, and $Y_v \subset N(v) \setminus X_v$, such that for each $u, u' \in Y_v$, we have $C(uv) \neq C(u'v)$ and $|Y_v| = \delta^c(v) - 1$ in Lemma 2, from the fact

$$|Y_v|d^\text{mon}(v) - \sum_{a \in Y_v} d^\text{mon}(a) \geq 0,$$

we obtain the lemma. \(\square\)

**Proof of Theorem 14.** Without loss of generality, assume that $G$ is edge-minimal subject to the condition $\delta^c \geq \frac{n}{2} + k - 1$. We prove the theorem by contradiction. Choose $v \in V(G)$ such that $d^\text{mon}(v) = \Delta^\text{mon}(G)$.

If $\Delta^\text{mon}(G) = 1$, then $G$ is properly-colored. Note that $e(G) \geq \delta^c n \geq \frac{n^2}{4} + \frac{kn}{2} - \frac{n}{2}$, and $\text{ex}(n, F_k) \leq \left\lceil \frac{n^2}{4} \right\rceil + k^2 - \frac{3k}{2}$ when $n \geq 50k^2$ by Theorem 15. When $n \geq 50k^2$, we have

$$\frac{n^2}{4} + \frac{kn}{2} - \frac{n}{2} > \left\lceil \frac{n^2}{4} \right\rceil + k^2 - \frac{3k}{2},$$

(recall $k \geq 2$), and so $G$ contains a properly-colored $F_k$, and hence $k$ rainbow triangles sharing one common vertex. Next we assume that $\Delta^\text{mon}(G) \geq 2$.

By Lemma 4,

$$rt(G; v) \geq \frac{1}{2} ((d^\text{mon}_G(v) + d^c(v) - 1)(\delta^c + d^c(v) - n)) \geq (k - 1)(d^\text{mon}_G(v) + d^c(v) - 1).$$

(12)
Consider the graph $G' = G[X_v \cup Y_v]$. Then $|G'| = d^{mon}(v) + d^c(v) - 1 \geq \frac{n}{2} + k$. Notice that each edge in $G'$ corresponds to a rainbow triangle through the vertex $v$. From \cite{12}, we have that

$$e(G') \geq (k - 1)(d^{mon}(v) + d^c(v) - 1) \geq \frac{(k - 1)(n + 2k)}{2}. \quad (13)$$

Since $G$ contains no $k$ rainbow triangles sharing one common vertex, $G'$ contains no matching of size $k$. That is, $\alpha'(G') \leq k - 1$. So by Theorem \cite{16}

$$e(G') \leq \max \left\{ \binom{2k - 1}{2}, \binom{k - 1}{2} + (k - 1)(|G'| - k + 1) \right\}. \quad (14)$$

By simple algebra, we have $(\frac{2k - 1}{2}) < \frac{(k - 1)(n + 2k)}{2}$ when $n \geq 2k - 3$. Furthermore,

$$(k - 1)(d^{mon}(v) + d^c(v) - 1) - \binom{k - 1}{2} - (k - 1)(|G'| - k + 1)$$

$$\geq -\binom{k - 1}{2} + (k - 1)^2 > 0$$

Thus, (13) contradicts (14) since $n \geq 2k - 3$. The proof is complete. \hfill \blacksquare

5 Concluding remarks

In this paper, we present a tight color degree condition (up to a constant) for $k$ rainbow triangles sharing one common vertex (when $k$ is a fixed integer), and highly suspect the tight one is $\frac{n + 1}{2}$ for $n = \Omega(k^2)$ (by considering Theorem \cite{15}).

Erdős et al. \cite{12} conjectured Theorem \cite{15} holds for $n \geq 4k$. If the answer to this conjecture is positive, then Theorem \cite{9} can be improved to all graphs with order $n \geq 4k$. On the other hand, maybe an answer to the following is positive.

**Problem 1.** Let $n, k$ be two positive integers. Let $(G, C)$ be an edge-colored graph on $n$ vertices with $d^c(G) \geq \frac{n + 1}{2}$. Does there exist a constant $c$, such that if $n \geq ck$ then $G$ contains a properly-colored $F_k$?

Recall that $f(n) := \min \{rt(G) : G \in \mathcal{G}_n^* \}$ (see Section 1). We conclude this paper with the following more feasible problem.

**Problem 2.** Determine the value of $\lim_{n \to \infty} \frac{f(n)}{n^2}$.

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