Sinusoidal Excitations in Two Component Bose-Einstein Condensates in a Trap

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The non-linear coupled Gross-Pitaevskii equation governing the dynamics of the two component Bose-Einstein condensate (TBEC) is shown to admit sinusoidal, propagating wave solutions in quasi one dimensional geometry in a trap. The solutions exist for a wide parameter range, which illuminates the procedure for coherent control of these modes through temporal modulation of the parameters, like scattering length and oscillator frequency. The effects of time dependent coupling and the trap variation on the condensate profile are explicated. The TBEC has also been investigated in presence of an optical lattice potential, where the superfluid phase is found to exist under general conditions.

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Much theoretical work has already gone into studying the ground state solutions of the coupled Gross-Pitaevskii (GP) equations describing multi-component BECs [1, 2, 3, 4]. TBEC has been observed, where the two hyperfine levels of $^{87}$Rb act as the two components. In this case, a fortuitous coincidence in the triplet and singlet scattering lengths has led to the preservation of exoergic spin-exchange collisions, which lead to heating and resultant loss of atoms. A number of interesting features, like the preservation of the total density profile and coherence for a characteristically long time, in the face of the phase-diffusing couplings to the environment and the complex relative motions [1], point to the extremely interesting dynamics of the TBEC. TBEC has been produced in a system comprising of $^{41}$K and $^{87}$Rb, in which sympathetic cooling of Rb atoms was used to condense the $^{133}$Rb atoms [5]. It has also been observed in $^{7}$Li and $^{87}$Rb $^{133}$Cs systems [10].

The presence of nonlinearities in BECs [11], make them ideal candidates for observation of solitary waves, ubiquitous to non-linear media [12, 13, 14, 15]. In the TBEC, a number of investigations, primarily devoted to the study of localized solitons, have been carried out recently [16, 17, 18, 19, 20]. The coincidence of singlet-triplet coupling in $^{87}$Rb, leads to the well known Manakov system [21] in weak coupling quasi-one dimensional scenario [22, 23]. The rich dynamics of solitons in this integrable system has received considerable attention in the literature [24, 25, 26, 27]. The effect of spatial inhomogeneity, three-dimensional geometry, and dissipation on TBEC have been examined. However, the periodic solitary waves have not received much attention in the literature, particularly in the presence of the harmonic trap [28]. Periodic sinusoidal excitations are natural in linear systems. In nonlinear models periodic cnoidal waves can be present. It is worth mentioning that, in nonlinear resonant atomic media, cnoidal excitations have been experimentally generated [29, 30], where relaxation naturally led to the atomic level population necessary for the existence of these nonlinear periodic waves [31].

Here we analyze the solutions of a generic TBEC model in a quasi-one dimensional geometry for periodic solutions. Interestingly, we find exact sinusoidal wave solutions in this system in the presence of a harmonic trap, which do not occur in the single component case. The presence of two components leads to these waves, whose energy difference are controlled by the cross phase modulation (XPM). In presence of time dependent trap and scattering length, these waves can be compressed and accelerated. This leads to the possibility of their coherent control. We then consider this system in an optical lattice [32, 33, 34], where a superfluid phase is found to exist under general conditions.

In the case of two species condensate with a wave function $\psi_i(x,t)$ for the species $i$, the coupled quasi-1D GP equation in the presence of an external potential $V_i$, can be written as,

\[ i\hbar \psi_i = -\frac{\hbar^2}{2m} \psi_i'' + V_i(x,t)\psi_1 + [g_1|\psi_1|^2 + g_{12}|\psi_2|^2 - \nu_1]\psi_1 \quad (1a) \]

and \[ i\hbar \psi_2 = -\frac{\hbar^2}{2m} \psi_2'' + V_2(x,t)\psi_2 + [g_{21}|\psi_1|^2 + g_2|\psi_2|^2 - \nu_2]\psi_2. \quad (1b) \]

The strength of the intra-species interactions is $g_i$ and $\nu_j$ is the chemical potential. We assume the interspecies...
interaction to be same for both the components: $g_{12} = g_{21}$; $V_j$ is the trapping potential.

In the absence of any potential, the general traveling wave solutions of Eq. (1a) and (1b) have the following form:

$$\psi_1(x, t) = \sqrt{\sigma_{01}}[1 - (1 - \frac{m^2u^2}{h^2}) \sin^2(x - ut)]e^{i[k_1(x, t)]},$$  
(2a)

$$\psi_2(x, t) = \sqrt{\sigma_{02}}[1 - (1 - \frac{m^2u^2}{h^2}) \cos^2(x - ut)]e^{i[k_2(x, t)]},$$  
(2b)

where, $\sigma_{0j}$'s are the equilibrium densities of the atoms in the condensed phase. The phase velocity is given by,

$$v_j = \frac{\hbar}{m} (\lambda_j + \kappa_j) = u(1 - \frac{\sigma_{0j}}{\sigma_j}).$$  
(3)

For these solutions to exist, it is found that the interactions need to satisfy $g_{12}^2 = g_1g_2$ and the background densities are related by $g_1\sigma_{01} = g_{12}\sigma_{02}$. The difference between cross phase modulation and self phase modulation leads to a difference in chemical potentials:

$$\nu_1 - \nu_2 = (g_1 - g_{12})\sigma_{01}(1 + \frac{m^2u^2}{h^2})$$
$$= (g_{12} - g_2)\sigma_{02}(1 + \frac{m^2u^2}{h^2}).$$  
(4)

For the limiting case $u = 0$, the above solutions coincide with the solutions mentioned in Ref. [35] subjected to the zero external periodic potential.

Recently the effect of the longitudinal trap on the condensate and soliton profile has been investigated quite intensively [35]. In the general scenario, the scattering length, oscillator frequencies can be time dependent, in addition to the presence of a phenomenological loss term [36, 37, 38, 39]. Below we employ this method to the sinusoidal waves in the two component scenario. As will be seen later, this can be used for controlling the excitations. They may be compressed or accelerated, through suitable temporal modulations of various parameters. We consider self-similar solutions in the oscillator form:

$$\psi_j(x, t) = \sqrt{A(t)\sigma_j[A(t)(x - l(t))]}e^{i[k_j(x, t)] + \phi(x, t)].}$$  
(5)

Here, $\phi(x, t)$ is a density independent phase having the form $\phi(x, t) = a(t) + b(t)x - \frac{1}{2}c(t)x^2$ and $l(t) = \int_0^t v(t')dt'$.

The sinusoidal wave, in this case, is a propagating wave with the velocity $v(t)$ in the moving condensate. The consistency conditions lead to,

$$a(t) = a_0 - \frac{\hbar^2}{8m} \int_0^t A^2(t)dt'$$  
(6)

where $\mu = \nu_j + \lambda_j$. Here $\nu_j(t) = \mu_jA^2(t)$ ($j = 1, 2$) and $\lambda_j$'s are constant parameters controlling the energy of the excitations. The time dependent wave vector $b(t) = A(t)$ and $c(t)$ can be determined by the Ricatti type equation

$$\hbar \frac{\partial c(t)}{\partial t} - \frac{\hbar^2}{m}c^2(t) = M(t).$$  
(7)

From current conservation, amounting to solving the imaginary part of the coupled GP equations, one gets Eq. (9), with the consistency conditions:

$$l_t(t) + \frac{\hbar}{m}c(t)l(t) - \frac{\hbar}{m}A(t) = A(t)u$$  
(8a)

$$A(t) = \frac{\hbar A_0}{m} \exp(\int_0^t c(t')dt'),$$  
(8b)

$$g_j(t) = \kappa_jA(t) \quad \text{and} \quad g_{12}(t) = \kappa_{12}A(t).$$  
(8c)

The real part of the coupled GP equations reduces to,

$$\frac{\hbar^2}{4m}\sigma_1{\sigma_1''} - \frac{\hbar^2}{8m}\sigma_1' + \left(\frac{1}{2}mu^2 + \lambda_1\right)\sigma_1^2 - \kappa_1\sigma_1^2 - \kappa_{12}\sigma_2\sigma_1^2 - \frac{1}{2}mu^2\sigma_{01} = 0$$  
(9a)

and

$$\frac{\hbar^2}{4m}\sigma_2{\sigma_2''} - \frac{\hbar^2}{8m}\sigma_2' + \left(\frac{1}{2}mu^2 + \lambda_2\right)\sigma_2^2 - \kappa_2\sigma_2^2 - \kappa_{12}\sigma_1\sigma_2^2 - \frac{1}{2}mu^2\sigma_{02} = 0.$$  
(9b)
Consistency conditions further require \( \mu = \bar{\mu} = \mu_1 + \lambda_1 = \mu_2 + \lambda_2 \) and \( \lambda_1 - \lambda_2 = (\kappa_1 - \kappa_2)\sigma_{01}(1 + \frac{m^2u^2}{\hbar^2}) = (\kappa_1 - \kappa_2)\sigma_{02}(1 + \frac{m^2u^2}{\hbar^2}) \) with the constraint \( \kappa_1^2 = \kappa_1\kappa_2 \) and \( \kappa_1\sigma_{01} = \kappa_1\sigma_{02} \). The form of the densities have been found to retain their earlier forms:

\[
\psi_1(x, t) = \sqrt{A(t)\sigma_{01}[1 - (1 - \frac{m^2u^2}{\hbar^2})\sin^2[A(t)(x - l(t))]]}e^{i[\chi_1(x, t) + \phi(x, t)]} \tag{10a}
\]

and

\[
\psi_2(x, t) = \sqrt{A(t)\sigma_{02}[1 - (1 - \frac{m^2u^2}{\hbar^2})\cos^2[A(t)(x - l(t))]]}e^{i[\chi_2(x, t) + \phi(x, t)]}. \tag{10b}
\]

The non-trivial phases are now controlled by the trap:

\[
\chi_1 = \frac{mu}{\hbar}A(t)[x - l(t)] - \tan^{-1} \left[ \frac{mu}{\hbar} \tan[A(t)(x - l(t))] \right], \tag{11}
\]

with a corresponding expression for the second component. The superfluid current densities in presence of the trap takes the form

\[
j_1 = \frac{\hbar\sigma_{01}}{2m}((u + A(t) - c(t)x)(\frac{m^2u^2}{\hbar^2} - 1)\sin^2[A(t)(x - l(t))]) \tag{12}
\]

with a similar expression for the second component. The flow density gets modulated by the chirped phase and as expected it depends on the oscillator potential. Hence, by tuning the trap the current densities can be controlled suitably.

For illustration, we first consider a trap with \( M(t) = \alpha = \text{const.} \), and inter-species interactions \( \kappa_1 = 0.4 \) and \( \kappa_2 = 0.1 \). Mass of the \(^{87}\)Rb atom is \( m = 1.41 \times 10^{-25} \) kg. The equality of the SPM and XPM leads to the same background, along with the same chemical potentials for both the components. Fig. (10) shows the traveling wave, with a time dependent velocity in the presence of the trap. In presence of oscillator, the atoms can be accelerated and suitably controlled.

It needs to be mentioned that, unlike experimentally observed localized solitons, sinusoidal solutions have infinite extent, which should be excited in a finite sized trap. In a single component BEC, periodic solutions, existing in the finite condensate, have been experimentally seen as Faraday waves \(^{17}\), which manifest when the scattering length is time dependent in a periodic manner \(^{17}\). We expect similar behavior for the sinusoidal excitations in two component Bose-Einstein condensates, since these are exact solutions.

Recently, restricted sinusoidal solutions have been found for TBEC in an optical lattice \(^{10}\), where the form of the optical lattice potential is taken as, \( V(x) = V_0 \cos^2 x \), where \( V_0 \) is the amplitude of the optical lattice. The spatial co-ordinate and \( V_0 \) are scaled in the units of wavelength of incident laser light and recoil energy respectively. We find that under general conditions the following type of solutions exist:

\[
\psi_1(x, t) = \sqrt{A + B\cos^2(x)}e^{i\chi_1(x) + \omega_1t} \tag{13}
\]

and

\[
\psi_2(x, t) = \sqrt{C + D\cos^2(x)}e^{i\chi_2(x) + \omega_2t}, \tag{14}
\]

with \( \omega_j = \frac{1}{2} + \bar{\mu}_j \) and \( \chi_{2j} = \frac{2\omega_j}{\mu_j} \) \( (j = 1, 2) \). Here, \( c_j\)'s are the integration constants. Considering the scenario of independent chemical potentials for the two species, the consistency conditions yield:

\[
A = \frac{\mu_2g_{12} - \mu_1g_2 - 2V_2g_{12}}{g_{12} - g_{12}} = \frac{V_2g_{12} - V_1g_2}{g_{12} - g_{12}}, \tag{15a}
\]

\[
C = \frac{\mu_1g_{12} - \mu_2g_1 + 2V_2g_{12}}{g_{12} - g_{12}} = \frac{V_2g_{12} - V_1g_2}{g_{12} - g_{12}}, \tag{15b}
\]

with \( \mu_j = \nu_j + \bar{\mu}_j \). Dispersion only affects the super-current through the integration constants: \( I_1 = \frac{1}{2}AB + (\frac{1}{2} + \mu_1)A^2 - g_2A^2 - g_{12}(C + D)A^2 + V_1A^2 \), where, \( I_j = \frac{\mu_j}{2m} \). The condensate phase for the first component has the explicit form: \( \chi_1(z) = c_1\tan^{-1}[\frac{\sqrt{A + B}\tan(z)}{\sqrt{A} + B}] / \sqrt{A + B} \). Similar type of expression holds true for the second component. The difference between the solutions found here, as compared to the earlier one obtained in \(^{10}\), lies in the integration constants \( I_i \). These constants acquire an additional contribution from the dispersion term in the form of \( \frac{1}{2}AB \), not present in the restricted solutions found earlier. When both the components have identical chemical potentials \( (\mu_1 = \mu_2) \), the parameter values coincide with Ref. \(^{10}\).

In summary, the two component BEC is found to sustain sinusoidal excitations in a trap, which is not possible in the single component case. It is shown that appropriate changes in the trap and scattering length can be used to control the BEC profile. The superfluid velocity can also be changed by controlling the experimental parameters. We note that difference between the ground state energy of the two components can arise because of the XPM. The roles of both harmonic and optical trap together is an area worthy of future investigation. It may
provide additional parameters for controlling the dynamical phase transitions found in this system [41, 42, 43]. One can also study the Faraday patterns in this system with time dependent scattering length [44, 45]. The presence of the two components may affect the nature of these excitations.

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