POSITIVE SOLUTIONS FOR RESONANT \((p,q)\)-EQUATIONS WITH CONCAVE TERMS

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Abstract. We consider a parametric \((p,q)\)-equation with competing nonlinearities in the reaction. There is a parametric concave term and a resonant Carathéodory perturbation. The resonance is with respect to the principal eigenvalue and occurs from the right. So the energy functional of the problem is indefinite. Using variational tools and truncation and comparison techniques we show that for all small values of the parameter the problem has at least two positive smooth solutions.

1. Introduction. Let \(\Omega \subseteq \mathbb{R}^N\) be a bounded domain with a \(C^2\)-boundary \(\partial \Omega\). In this paper we study the following nonlinear nonhomogeneous parametric Robin problem

\[
\begin{cases}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{\tau-1} + f(z, u(z)) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial \Omega, \ u > 0, \ \lambda > 0, \ 1 < \tau < q < p < \infty.
\end{cases}
\]

\((P_\lambda)\)

For every \(1 < r < \infty\) we let \(\Delta_r\) denote the \(r\)-Laplace differential operator:

\[\Delta_r u = \text{div}(|Du|^{r-2}Du)\]

for all \(u \in W^{1,r}(\Omega)\).

In problem \((P_\lambda)\) we have the sum of a \(p\)-Laplacian and a \(q\)-Laplacian and this differential operator is nonhomogeneous. This is a source of difficulties in the analysis of the problem. Boundary value problems with a combination of differential operators of different nature (such as the \((p,q)\)-equation above), arise in the study of various physical phenomena such as reaction-diffusion equations (see Cherfils-Ilyasov [5]), particle physics (see Benci-Fortunato-Pisani [3]) and plasma physics (see Wilhelmsson [31]). The potential function \(\xi \in L^\infty(\Omega)\) and \(\xi(z) \geq 0\) for a.a. \(z \in \Omega\). In the right hand side (forcing term) we have the competing effects of two nonlinearities, one is a parametric concave term while the other is a Carathéodory perturbation \(f\). Namely, \(z \to f(z, x)\) is measurable and \(x \to f(z, x)\) is continuous. We assume that \(f(z, \cdot)\) exhibits \((p-1)\)-linear growth near \(+\infty\) and it can be resonant.
with respect to the principal eigenvalue of the operator $u \to -\Delta u + \xi(z)|u|^{p-2}u$ with Robin boundary condition. In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of $u$ defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to [|Du|^{p-2} + |Du|^{q-2}] \frac{\partial u}{\partial n},$$

with $n$ being the outward unit normal on $\partial \Omega$. The boundary coefficient $\beta$ is non-negative.

Problems with competition phenomena were studied primarily for Dirichlet problems driven by the Laplacian or the $p$-Laplacian and with the competition taking place between a concave $((p-1)$-sublinear) and a convex $((p-1)$-superlinear) terms. See Ambrosetti-Brezis-Cerami [2], García Azorero-Manfredi-Peral Alonso [12], Guo-Zhang [16], Hu-Papageorgiou [17], Marano-Papageorgiou [21], Papageorgiou-Winkert [29]. Concave-convex problems for nonlinear Neumann and Robin problems were investigated by Papageorgiou-Radulescu [24]. Recent works of Filippakis-Papageorgiou [29] and Marano–Mosconi-Papageorgiou [22] deal with resonant ($p,q$)-equations. A nice survey of the recent existence and multiplicity results for ($p,q$)-equations can be found in Marano-Mosconi [19]. Related problems and results can be found in [4, 6, 7, 12, 16, 30].

Here we look for positive solutions for ($p,q$)-equations in which the reaction exhibits the competing effects of a concave term and a resonant term. The resonance is with respect to the principal eigenvalue $\hat{\lambda}_1(p,\xi,\beta) > 0$ of the differential operator $u \to -\Delta_p u + \xi(z)|u|^{p-2}u$ with the Robin boundary condition $-\frac{\partial u}{\partial n_p} + \beta(z)|u|^{q-2}u = 0$ on $\partial \Omega$. The resonance occurs from the right of $\hat{\lambda}_1$ in the sense that

$$\hat{\lambda}_1 x^{p-1} - pF(z,x) \to -\infty$$

uniformly for a.a. $z \in \Omega$ as $x \to -\infty$, where $F(z,x) = \int_0^x f(z,s)ds$. This makes the energy functional $\varphi_\lambda$ of the problem $(P_\lambda)$ indefinite (unbounded from above and below) and so we can not use the direct method of the calculus of variations directly on $\varphi_\lambda$. We use instead variational methods based on the critical point theory together with suitable truncation and comparison techniques. We show that for all $\lambda > 0$ small, $(P_\lambda)$ has two positive smooth solutions.

2. Preliminaries - hypotheses. Let $X$ be a Banach space and $X^*$ its dual. $\langle \cdot, \cdot \rangle$ denotes the duality brackets for $(X^*,X)$. Given $\varphi \in C^1(X,\mathbb{R})$, we say that $\varphi$ satisfies the “Cerami condition” (the C-condition for short) if

“every sequence $\{x_n\} \subseteq X$ such that $\{\varphi(x_n)\}$ is bounded and $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in $X^*$, admits a strongly convergent subsequence”.

This is a compactness-type condition on $\varphi$ that compensates for the fact that the ambient space $X$ may not be locally compact. The C-condition is more general than the usual Palais-Smale condition (PS-condition for short). Nevertheless, the C-condition suffices to prove a deformation theorem and then from it we can derive the minimax theory of certain critical values of $\varphi$. One such minimax theorem which will be used shortly is the so-called mountain pass theorem. See, for example, Gasinski-Papageorgiou [13, p.648].

**Theorem 1.** If $X$ is a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$ with $\|x_1 - x_0\| > r,

$$\max\{\varphi(x_0),\varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r,$$
and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$, with

$$\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1 \},$$

then $c \geq \eta$, and $c$ is a critical value of $\varphi$.

In the study of problem $(P_\lambda)$ we will use the Sobolev space $W^{1,p}(\Omega)$, the space $C^1(\bar{\Omega})$ and the boundary Lebesgue spaces $L^r(\partial \Omega)$, $1 \leq r \leq \infty$. For $u \in W^{1,p}(\Omega)$, define

$$||u|| = \left[ ||u||_p^r + ||Du||_p^{r/p} \right]^{1/r}.$$

The Banach space $C^1(\bar{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{ u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega} \}.$$

This cone has nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+: u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial \Omega \cap u^{-1}(0)} < 0 \right\}.$$

Note that if $D_+ = \{ u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega} \}$, then $D_+$ is open in $C^1(\bar{\Omega})$ and $D_+ \subseteq \text{int } C_+$. In fact $D_+$ is the interior of $C_+$ when $C^1(\bar{\Omega})$ is equipped with the relative $C(\bar{\Omega})$-norm topology.

On $\partial \Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma$. Using $\sigma(\cdot)$ we can define in the usual way the boundary Lebesgue spaces $L^r(\partial \Omega)$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega)$, known as the “trace map” such that

$$\gamma_0(u) = u|_{\partial \Omega}$$

for all $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. So the trace map extends the notion of boundary values to all Sobolev functions. We know that

$$\text{im } \gamma_0 = W^{1,p}(\partial \Omega) \text{ and ker } \gamma_0 = W^{1,p}_0(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Also, we know that $\gamma_0$ is compact into $L^r(\partial \Omega)$ for all $r \in \left[ 1, \frac{(N-1)p}{N-p} \right]$ if $p < N$, and into $L^r(\partial \Omega)$ for all $1 \leq r < \infty$ if $N \leq p$. In the sequel for the sake of notational simplicity we drop the use of the trace map. All restrictions of Sobolev functions on $\partial \Omega$ are understood in the sense of traces.

For every $r \in (1, \infty)$ let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be the nonlinear map defined by

$$\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} dz$$

for all $u, h \in W^{1,r}(\Omega)$. The next proposition summarizes the main properties of this map (see Motreanu-Motreanu-Papageorgiou [22, p.40].

**Proposition 2.** The map $A_r$ is bounded, continuous, monotone (hence maximal monotone too), and of type $(S)_+$, namely we have $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, provided that $u_n \overset{w}{\rightharpoonup} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n \to \infty} (A(u_n), u_n - u) \leq 0$.

We introduce the hypotheses on the potential function $\xi$ and on the boundary coefficient $\beta$ as follows.

**H**($\xi$): $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$.

**H**($\beta$): $\beta \in C^{0,\alpha}(\partial \Omega)$ with $\alpha \in (0, 1)$ and $\beta(z) \geq 0$ for all $z \in \partial \Omega$.

**H**$_0$: $\xi \not= 0$ or $\beta \not= 0$. 


Remark. The above hypotheses include the case \( \beta \equiv 0 \), which corresponds to the Neumann problem.

From Mugnai-Papageorgiou [21, Lemma 4.11] we have

**Proposition 3.** If \( H(\xi) \) holds and \( \xi \neq 0 \), then there exists \( c_0 > 0 \) such that for \( u \in W^{1,p}(\Omega) \),

\[
\|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p dz \geq c_0 \|u\|^p.
\]

Also, from Gasinski-Papageorgiou [13, Proposition 2.4] we have

**Proposition 4.** If \( \beta \in L^\infty(\partial \Omega) \), \( \beta(z) \geq 0 \) and \( \beta \neq 0 \), then

\[
|u| = \left( \|Du\|_p^p + \int_{\partial \Omega} \beta(z)|u|^p d\sigma \right)^{1/p}
\]

is an equivalent norm on \( W^{1,p}(\Omega) \).

In the sequel we let \( \mu_p : W^{1,p}(\Omega) \to \mathbb{R} \) be the \( C^1 \)-functional defined by

\[
\mu_p(u) = \|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p dz + \int_{\partial \Omega} \beta(z)|u|^p d\sigma.
\]

Consider a Caratheodory function \( f_0 : \Omega \times \mathbb{R} \to \mathbb{R} \) such that

\[
|f_0(x,z)| \leq \alpha_0(z)(1 + |z|^{r-1}),
\]

with some \( \alpha_0 \in L^\infty(\Omega) \), \( 1 \leq r \leq p^* = \frac{Np}{N-p} \) if \( p < N \), and \( +\infty \) if \( N \leq p \) (the critical Sobolev exponent).

We set \( F_0(z,x) = \int_0^x f_0(z,s)ds \) and consider the \( C^1 \)-functional \( \varphi_0 : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
\varphi_0(u) = \frac{1}{p} \mu_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} F_0(z,u)dz.
\]

The next proposition is a particular case of a more general result of [24, Proposition 8]. The result is an outgrowth of the nonlinear regularity theory of Lieberman [18].

**Proposition 5.** If \( H(\xi), H(\beta) \) hold and \( u_0 \in W^{1,p}(\Omega) \) is a local \( C^1(\overline{\Omega}) \)-minimizer of \( \varphi_0 \), then \( u_0 \in C^{1,\eta}(\overline{\Omega}) \) with \( \eta \in (0,1) \) and \( u_0 \) is a local \( W^{1,p}(\Omega) \)-minimizer of \( \varphi_0 \).

As we already mentioned in the Introduction, our approach involves comparison techniques. So, we will need the following strong comparison principle, due to Papageorgiou-Radulescu-Repovs [27, Proposition 7].

**Proposition 6.** If \( H(\xi) \) holds, \( h_1, h_2 \in L^\infty(\Omega) \), \( 0 < c_2 \leq h_2(z) - h_1(z) \) for a.a. \( z \in \Omega \) and \( u, v \in C^{1,\eta}(\overline{\Omega}) \), \( 0 < \eta < 1 \) with \( 0 \leq v \leq u \) and

\[
-\Delta_p v - \Delta_q v + \xi(z)v^{p-1} = h_1(z), \quad -\Delta_p u - \Delta_q u + \xi(z)u^{p-1} = h_2(z),
\]

then \( u - v \in \text{int} C_+ \).

We consider the following nonlinear eigenvalue problem:

\[
\begin{cases}
-\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{p-2}u(z) & \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \frac{\partial u}{\partial n_p} = |Du|^{p-2} \frac{\partial u}{\partial n} \) for \( u \in W^{1,p}(\Omega) \).
We say that \( \tilde{\lambda} \) is an eigenvalue if problem (1) admits a nontrivial solution \( \tilde{u} \in W^{1,p}(\Omega) \), known as an eigenfunction corresponding to the eigenvalue \( \lambda \). The nonlinear regularity theory implies that every eigenfunction is in \( C^1(\overline{\Omega}) \). Let \( \tilde{\sigma} \) denote the set of all the eigenvalues. Problem (1) was studied by Mugnai-Papageorgiou [21] (Neumann problem) and Fragnelli-Mugnai-Papageorgiou [10], Papageorgiou-Radulescu [24] (Robin problem). The next proposition states some of their results.

**Proposition 7.** If \( H(\xi), H(\beta), \) and \( H_0 \) hold, then problem (1) admits a smallest eigenvalue \( \tilde{\lambda}_1 > 0 \) such that

(a) \( \tilde{\lambda}_1 \) is isolated, namely, \((\tilde{\lambda}_1, \tilde{\lambda}_1 + \epsilon) \cap \tilde{\sigma} = \emptyset \) for some \( \epsilon > 0 \);

(b) \( \tilde{\lambda}_1 \) is simple, namely, any pair of eigenfunctions is linearly dependent;

(c) \( \tilde{\lambda}_1 = \inf \left\{ \frac{\mu_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\} \);

(d) every eigenfunction corresponding to an eigenvalue other than \( \tilde{\lambda}_1 \) is nodal, i.e., sign changing.

The above properties imply that the elements of the principal eigenspace do not change sign. Let \( \tilde{u}_1 \) denote the positive \( L^p \)-normalized eigenfunction corresponding to \( \tilde{\lambda}_1 > 0 \). From the strong nonlinear maximum principle (see [13, Proposition 6.2.8, p.738] and Pucci-Serrin [30, pp.111, 120], we have that \( \tilde{u}_1 \in D_+ \). From (2) and Propositions 3 and 4 it is clear that under \( H(\xi), H(\beta) \) and \( H_0 \) we have \( \tilde{\lambda}_1 > 0 \). In (2) the infimum is realized in the corresponding one dimensional space.

We also have a weighted version of the eigenvalue problem (1). So let \( 0 \leq m \in L^\infty(\Omega) \) with \( m \neq 0 \), and consider

\[
\begin{cases}
-\Delta_p u(z) + \xi(z) |u(z)|^{p-2} u(z) = \tilde{\lambda} m(z) |u(z)|^{p-2} u(z) \text{ in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z) |u|^{p-2} u = 0 \text{ on } \partial \Omega.
\end{cases}
\]

Again we have a smallest eigenvalue \( \tilde{\lambda}_1(m) > 0 \), which has the same properties as \( \lambda_1 \). In this case the variational characterization of \( \tilde{\lambda}_1 \) takes the following form:

\[
\tilde{\lambda}_1(m) = \inf \left\{ \frac{\mu_p(u)}{\int_\Omega m |u|^p dz} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.
\]

As before the infimum is realized on the corresponding one dimensional eigenspace, whose elements have constant sign. Now let \( \tilde{u}_1(m) \) denote the positive \( L^p \)-normalized eigenfunction corresponding to \( \tilde{\lambda}_1(m) > 0 \). We have \( \tilde{u}_1(m) \in D_+ \). These properties lead to the following monotonicity property of the map \( m \to \tilde{\lambda}_1(m) \).

**Proposition 8.** If \( m_1, m_2 \in L^\infty(\Omega) \), \( 0 \leq m_1(z) \leq m_2(z) \) and the two inequalities are strict on sets of positive measure in \( \Omega \), then \( \tilde{\lambda}_1(m_2) < \tilde{\lambda}_1(m_1) \).

For \( x \in \mathbb{R} \) set \( x^\pm = \max\{\pm x, 0\} \) and for \( u \in W^{1,p}(\Omega) \) set \( u^\pm(\cdot) = u^\pm(\cdot) \). We know that

\[
u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.
\]

Given a measurable function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \), \( N_g \) denotes the Nemitskii operator corresponding to \( g \), defined by \( N_g(u)(z) = g(z, u(z)) \) for all \( u \in W^{1,p}(\Omega) \). If \( u, v \in W^{1,p}(\Omega) \) with \( v \leq u \), then

\[
[v, u] = \left\{ y \in W^{1,p}(\Omega) : u(z) \leq y(z) \leq u(z) \text{ for a.e. } z \in \Omega \right\}.
\]
Let \( \inf_{C^1(\Omega)}[v, u] \) denote the interior in \( C^1(\overline{\Omega}) \) of \([v, u] \cap C^1(\overline{\Omega}) \). Also, for given \( u \in W^{1,p}(\Omega) \), we set
\[
[u] = \{ y \in W^{1,p}(\Omega) : u(z) \leq y(z) \text{ for a.a. } z \in \Omega \}.
\]

Next we introduce the conditions on the perturbation term \( f \).

**H(f) :** \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is Caratheodory such that \( f(z, 0) = 0 \) for a.a. \( z \in \Omega \) and

(i): for any \( \varrho > 0 \) there is \( \alpha_\varrho \in L^\infty(\Omega) \) such that
\[
|f(z, x)| \leq \alpha_\varrho(z) \text{ for a.a. } z \in \Omega \text{ and all } 0 \leq x \leq \varrho;
\]

(ii): \( \lambda_1 \leq \liminf_{z \to +\infty} \frac{f(z, x)}{x^p} \leq \limsup_{z \to -\infty} \frac{f(z, x)}{x^p} \leq \lambda_0 \) uniformly a.e. on \( \Omega \);

(iii):
\[
\lim_{x \to +\infty} \frac{f(z, x) - pF(z, x)}{x^p} = -\infty \text{ uniformly a.e. on } \Omega;
\]

(iv): there exist \( w \in C^{1,\alpha}(\overline{\Omega}) \cap D_+ \) and \( \eta_0 > 0 \) such that in \( W^{1,p}(\Omega)^* \)
\[
A_p(w) + A_q(w) + \xi(z)w^{p-1} \geq 0,
\]
\[
\Delta_p w + \Delta_q w \in L^p(\Omega), \eta_0 w(z)^{r-1} + f(z, w(z)) \leq -c_w < 0;
\]

(v): if \( m_w = \min w > 0 \), then we can find \( \delta_0 \in (0, m_w) \) such that \( 0 < c_m \leq f(z, x) \) for all \( m \in (0, \delta_0) \), all \( x \in [m, \delta_0] \) and a.a. \( z \in \Omega \);

(vi): for any \( \varrho > 0 \) there is \( \xi_\varrho > 0 \) such that \( x \to f(z, x) + \xi_\varrho x^{p-1} \) is nondecreasing on \([0, \varrho]\).

**Remarks.** Since we are interested on positive solutions and all the above hypotheses concern the positive semiaxis, without loss of generality we may assume that
\[
f(z, x) = 0 \text{ for a.a. } z \in \Omega \text{ and all } x \leq 0. \tag{4}
\]

\( H(f)(ii) \) implies that at \(+\infty\) we can have resonance with respect to the principal eigenvalue \( \lambda_1 > 0 \). As we shall see in the process of the proof, \( H(f)(iii) \) implies that the resonance occurs from the right of \( \lambda_1 \) in the sense that
\[
pF(z, x) - \lambda_1 x^p \to +\infty
\]

uniformly for a.a. \( z \in \Omega \) as \( x \to +\infty \).

This makes the energy functional unbounded below. In \( H(f)(iv) \) the inequality in \( W^{1,p}(\Omega)^* \) means that
\[
\langle A_p(w), h \rangle + \langle A_q(w), h \rangle + \int_{\Omega} \xi(z)w^{p-1}h \, dz \leq 0,
\]
for all \( 0 \leq h \in W^{1,p}(\Omega) \). This assumption is satisfied if \( \xi = 0 \) and there are \( t > 0 \) and \( \eta_0 > 0 \) such that
\[
\eta_0 t^{r-1} + f(z, t) \leq -c_t < 0
\]
for a.a. \( z \in \Omega \).

\( H(f)(iv)(v) \) imply that \( f(z, \cdot) \) exhibits an oscillatory behavior near \( 0^+ \). \( H(f)(vi) \) is satisfied if \( f(z, \cdot) \) is differentiable and there is \( \xi_\varrho > 0 \) for any \( \varrho > 0 \) such that
\[
f'_x(z, x) \geq -\xi_\varrho x^{p-1} \text{ for a.a. } z \in \Omega \text{ and all } 0 \leq x \leq \varrho.
\]

Note that no asymptotic condition is assumed as \( x \to 0^+ \).
Example. The following function satisfies \( H(f) \). For the sake of simplicity we dropped the \( z \)-dependence.
\[
f(x) = \begin{cases} 
  x^{\tau-1} - 2x^\theta - 1 & \text{if } 0 \leq x \leq 1 \\
  \eta x^{p-1} + x^{\gamma-1} - (\eta + 2)x^{\delta-1} & \text{if } 1 < x 
\end{cases}
\]
with \( 1 < \tau < \theta < \infty, \eta \geq \lambda_1 \) and \( 1 < \delta \leq q < \gamma < p \).

3. An auxiliary Robin Problem. In this section we examine the following auxiliary nonlinear parametric Robin problem:
\[
\begin{aligned}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} &= \lambda(z)^{\tau-1} \text{ in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(z)u^{p-1} &= 0 \text{ on } \partial \Omega, u > 0, \lambda > 0. 
\end{aligned}
\]
(5)

Recall that \( \mu_p : W^{1,p}(\Omega) \to \mathbb{R} \) is the \( C^1 \)-functional on \( W^{1,p}(\Omega) \), defined by
\[
\mu_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma.
\]

On account of Propositions 3 and 4, together with \( H(\xi), H(\beta) \) and \( H_0 \), we have
\[
c_1 \|u\|^p \leq \mu_p(u)
\]
for some \( c_1 > 0 \) and all \( u \in W^{1,p}(\Omega) \).

Proposition 9. If \( H(\xi), H(\beta) \) and \( H_0 \) hold, then for any \( \lambda > 0 \) problem \( (Au_\lambda) \) has a unique positive solution, \( \tilde{u}_\lambda \in D_+ \). The map \( \lambda \to \tilde{u}_\lambda \) is strictly increasing from \( (0, \infty) \) into \( C^1(\overline{\Omega}) \), namely,
\[
\vartheta < \lambda \Rightarrow \tilde{u}_{\lambda} - \tilde{u}_\vartheta \in \text{int} C_+,
\]
and \( \|\tilde{u}_\lambda\|_{C^1(\overline{\Omega})} \to 0 \) as \( \lambda \to 0^+ \).

Proof. For \( \lambda > 0 \) let \( \psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) be the \( C^1 \)-functional defined by
\[
\psi_\lambda(u) = \frac{1}{p} \mu_p(u) + \frac{1}{q} \|Du\|_q^q - \frac{\lambda}{\tau} \|u^+\|^\tau.
\]

We have \( \psi_\lambda(u) \geq \frac{c_1}{p} \|u\|^p - \lambda c_2 \|u\|^\tau \) for some \( c_2 > 0 \) and all \( u \in W^{1,p}(\Omega) \). Thus, \( \psi_\lambda \) is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that \( \psi_\lambda \) is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass-Tonelli theorem we can find some \( \tilde{u}_\lambda \in W^{1,p}(\Omega) \) such that
\[
\psi_\lambda(\tilde{u}_\lambda) = \inf \{ \psi_\lambda(u) : u \in W^{1,p}(\Omega) \}.
\]
(7)

Since \( \tau < q < p \), for \( \tilde{h} \in (0, 1) \) small we have \( \psi_\lambda(\tilde{h}) < 0 \). Thus, \( \psi_\lambda(\tilde{u}_\lambda) < 0 = \psi_\lambda(0) \) and \( \tilde{u}_\lambda \neq 0 \). From (7) we have \( \psi'_\lambda(\tilde{u}_\lambda) = 0 \), hence for \( h \in W^{1,p}(\Omega) \),
\[
\begin{aligned}
\langle A_p(\tilde{u}_\lambda), h \rangle + \langle A_q(\tilde{u}_\lambda), h \rangle + \int_\Omega \xi(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h dz \\
+ \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h d\sigma &= \lambda \int_\Omega (\tilde{u}_\lambda^+)^{\tau-1} h dz.
\end{aligned}
\]
(8)

In (8) we choose \( h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega) \), then
\[
\mu_p(\tilde{u}_\lambda^-) + \|D\tilde{u}_\lambda^-\|_q^q = 0,
\]
thus $c_1 \|\bar{u}_\lambda\|^p \leq 0$ (see (6)), hence $0 \neq \bar{u}_\lambda \geq 0$. Then from (8) we have ([24])
\[
\begin{cases}
-\Delta_p \bar{u}_\lambda(z) - \Delta_q \bar{u}_\lambda(z) + \xi(z)\bar{u}_\lambda(z)^{p-1} = \lambda \bar{u}_\lambda(z)^{r-1} \text{ for a.e. } z \in \Omega, \\
\frac{\partial \bar{u}_\lambda}{\partial n_{pq}} + \beta(z)\bar{u}_\lambda^{r-1} = 0 \text{ on } \partial \Omega, \lambda > 0.
\end{cases}
\] (9)

From (9) and [25, Proposition 7] we have
\[
\bar{u}_\lambda \in L^\infty(\Omega).
\]

Then the nonlinear regularity theory of Lieberman [18] implies that
\[
\bar{u}_\lambda \in C_+ \setminus \{0\}.
\]

From (9) we have for a.a. $z \in \Omega$,
\[
\Delta_p \bar{u}_\lambda(z) + \Delta_q \bar{u}_\lambda(z) \leq \|\xi\|_{\infty} \bar{u}_\lambda(z)^{p-1},
\]
thus $\bar{u} \in D_+$ (see [30, pp. 111, 120]).

Next we show the uniqueness of this positive solution. To this end we consider the integral functional $k : L^1(\Omega) \to \mathbb{R} = \mathbb{R} \cup \{+\infty\}$, defined by
\[
k(u) = \frac{1}{p} \|Du^{1/q}\|_p^p + \frac{1}{q} \|Du^{1/q}\|_q^q + \frac{1}{p} \int_\Omega \xi(z)u^{p/q}dz + \frac{1}{p} \int_{\partial \Omega} \beta(z)u^{p/q}d\sigma
\]
if $u \geq 0$ and $u^{1/q} \in W^{1,q}(\Omega)$, and $k(u) = +\infty$ otherwise. From Diaz-Saa [8, Lemma 1] and $H(\xi)$ and $H(\beta)$, $k$ is convex and by Fatou’s Lemma it is also lower semicontinuous.

Suppose $\bar{v}_\lambda$ is another solution of $(Au_\lambda)$. For this solution too we have
\[
\bar{v}_\lambda \in D_+.
\]

Then for $h \in C^1(\overline{\Omega})$ and all $|t| < 1$ small we have
\[
\bar{v}_\lambda + th \in \text{dom } k \text{ and } \bar{u}_\lambda + th \in \text{dom } k,
\]
where dom $k = \{ u \in L^1(\Omega) : k(u) < +\infty \}$, which is the effective domain of $k$. We can easily show that $k$ is Gateaux differentiable at $\bar{u}_\lambda^k \in D_k$ and at $\bar{v}_\lambda^k \in D_+$ in the direction of $h$. Moreover, the Chain Rule and the nonlinear Green’s identity (see [13, p.211] imply that
\[
k'(\bar{u}_\lambda^k)(h) = \frac{1}{q} \int_\Omega \frac{-\Delta_p \bar{u}_\lambda - \Delta_q \bar{u}_\lambda + \xi(z)\bar{u}_\lambda^{p-1}}{\bar{u}_\lambda^{q-1}}hdz,
\]
and
\[
k'(\bar{v}_\lambda^k)(h) = \frac{1}{q} \int_\Omega \frac{-\Delta_p \bar{v}_\lambda - \Delta_q \bar{v}_\lambda + \xi(z)\bar{v}_\lambda^{p-1}}{\bar{v}_\lambda^{q-1}}hdz.
\]

The convexity of $k$ implies the monotonicity of $k'$. Hence we have
\[
0 \leq \frac{1}{\bar{u}_\lambda^{q-1}} - \frac{1}{\bar{v}_\lambda^{q-1}} \left( \frac{\bar{u}_\lambda^q - \bar{v}_\lambda^q}{\bar{u}_\lambda^q - \bar{v}_\lambda^q} \right) dz
\]
\[
\leq \frac{\bar{u}_\lambda^q - \bar{v}_\lambda^q}{\bar{u}_\lambda^q - \bar{v}_\lambda^q} dz
\]
\[
\leq 0,
\]

thus $\bar{u}_\lambda = \bar{v}_\lambda$. This proves the uniqueness of the positive solution of problem $(Au_\lambda)$.
Next we examine the monotonicity of the map \( \lambda \to \tilde{u}_\lambda \) from \((0, \infty)\) into \(C^1(\overline{\Omega})\). So, suppose that \(0 < \vartheta < \lambda\). Then

\[
- \Delta_p \tilde{u}_\lambda(z) - \Delta_q \tilde{u}_\lambda(z) + \xi(z) \tilde{u}_\lambda(z)^{p-1} = \lambda \tilde{u}_\lambda(z)^{r-1} > \vartheta \tilde{u}_\lambda(z)^{r-1}.
\]

(10)

We now introduce the Caratheodory function \(e_\vartheta(z, x)\) defined by

\[
e_\vartheta(z, x) = \begin{cases} 
\vartheta(x^+)^{r-1} & \text{if } x \leq \tilde{u}_\lambda(z) \\
\vartheta(\tilde{u}_\lambda(z))^{r-1} & \text{if } \tilde{u}_\lambda(z) < x.
\end{cases}
\]

(11)

Set \(E_\vartheta(z, x) = \int_0^x e_\vartheta(z, s)ds\) and consider the \(C^1\)-functional \(\tilde{\psi}_\vartheta : W^{1,p}(\Omega) \to \mathbb{R}\) defined by

\[
\tilde{\psi}_\vartheta(u) = \frac{1}{p} \mu_p(u) + \frac{1}{q} ||Du||_q^q - \int_{\Omega} E_\vartheta(z, u)dz.
\]

From (6) and (11), \(\tilde{\psi}_\vartheta\) is coercive and sequentially weakly lower semicontinuous. Hence we can find some \(\varpi_\vartheta \in W^{1,p}(\Omega)\) such that

\[
\tilde{\psi}_\vartheta(\varpi_\vartheta) = \inf \left\{ \tilde{\psi}_\vartheta(u) : u \in W^{1,p}(\Omega) \right\}.
\]

(12)

Also as before, for \(\hat{t} \in (0, 1)\) small, \(\hat{t} \leq \min \varpi_\vartheta \tilde{u}_\lambda\), we have

\[
\tilde{\psi}_\vartheta(\hat{t}) < 0,
\]

thus \(\tilde{\psi}_\vartheta(\varpi_\vartheta) < 0 = \tilde{\psi}_\vartheta(0)\) (see (12)). Therefore, \(\varpi_\vartheta \neq 0\).

From (12) we have \(\tilde{\psi}_\vartheta'(\varpi_\vartheta) = 0\), thus for all \(h \in W^{1,p}(\Omega)\)

\[
(A_p(\varpi_\vartheta), h) + (A_q(\varpi_\vartheta), h) + \int_{\Omega} \xi(z)|\varpi_\vartheta|^{p-2}\varpi_\vartheta hdz + \int_{\partial\Omega} \beta(z)|\varpi_\vartheta|^{p-2}\varpi_\vartheta hds = \int_{\Omega} e_\vartheta(z, \varpi_\vartheta)hdz.
\]

(13)

In (13) we first choose \(h = -\varpi_\vartheta^- \in W^{1,p}(\Omega)\) to get

\[
\mu_p(\varpi_\vartheta^-) + ||D\varpi_\vartheta^-||_q^q = 0
\]

(see (11)), thus \(c_1||\varpi_\vartheta^-||_p \leq 0\) (see (6)). Hence \(0 \leq \varpi_\vartheta^- \neq 0\).

Next in (13) we choose \(h = (\varpi_\vartheta - \tilde{u}_\lambda)^+ \in W^{1,p}(\Omega)\). Then

\[
(A_p(\varpi_\vartheta), (\varpi_\vartheta - \tilde{u}_\lambda)^+) + (A_q(\varpi_\vartheta), (\varpi_\vartheta - \tilde{u}_\lambda)^+) + \int_{\Omega} \xi(z)|\varpi_\vartheta|^{p-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+dz
\]

\[
+ \int_{\partial\Omega} \beta(z)|\varpi_\vartheta|^{p-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+d\sigma
\]

\[
= \int_{\Omega} \vartheta \tilde{u}_\lambda^{r-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+dz \text{ (see (11))}
\]

\[
\leq \int_{\Omega} \lambda \tilde{u}_\lambda^{r-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+dz \text{ (see (10))}
\]

\[
= (A_p(\tilde{u}_\lambda), (\varpi_\vartheta - \tilde{u}_\lambda)^+) + (A_q(\tilde{u}_\lambda), (\varpi_\vartheta - \tilde{u}_\lambda)^+)
\]

\[
+ \int_{\Omega} \xi(z)\tilde{u}_\lambda^{p-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+dz + \int_{\partial\Omega} \beta(z)\tilde{u}_\lambda^{p-1}(\varpi_\vartheta - \tilde{u}_\lambda)^+d\sigma,
\]

thus \(\varpi_\vartheta \leq \tilde{u}_\lambda\). Hence we have proved that \(\varpi_\vartheta \in [0, \tilde{u}_\lambda]\), \(\varpi_\vartheta \neq 0\), and \(\varpi_\vartheta\) is a positive solution of \((Au_\vartheta)\), therefore,

\[
\varpi_\vartheta = \tilde{u}_\lambda \in D_+ \text{ and so } u_\vartheta \leq \tilde{u}_\lambda.
\]

(14)
Then (10), (14) and Proposition 6 imply that
\[ u_\lambda - \bar{u}_\lambda \in \text{int } C_+, \]
thus \( \lambda \to \bar{u}_\lambda \) is strictly increasing from \((0, \infty)\) into \(C^1(\overline{\Omega})\). Now let \( \lambda > 0 \) and \( \bar{u}_\lambda \in D_+ \) be the unique solution of problem \((Au_\lambda)\). We have for all \( h \in W^{1,p}(\Omega) \)
\[ \langle A_p(\bar{u}_\lambda), h \rangle + \langle A_q(\bar{u}_\lambda), h \rangle + \int_\Omega \xi(z)\bar{u}_\lambda^{p-1}hdz + \int_{\partial\Omega} \beta(z)\bar{u}_\lambda^{p-1}hds = \lambda \int_\Omega \bar{u}_\lambda^{r-1}hdz. \]  
(15)

In (15) we choose \( h = \bar{u}_\lambda \in W^{1,p}(\Omega) \) to obtain
\[ \mu_p(\bar{u}_\lambda) + \|Du_\lambda\|_q^q = \lambda \|\bar{u}_\lambda\|_r^r, \]
thus \( c_1 \|\bar{u}_\lambda\|_p^p \leq \lambda c_3 \|\bar{u}_\lambda\|_r^r \) for some \( c_3 > 0 \) (see (6)), hence \( \|\bar{u}_\lambda\|^{r-p} \leq \lambda c_4 \) for some \( c_4 > 0 \).
So, for \( \eta > 0 \) we have
\[ \{\bar{u}_\lambda\}_{\lambda \in (0, \eta)} \subseteq W^{1,p}(\Omega) \text{ bounded and } \|\bar{u}_\lambda\| \to 0 \text{ as } \lambda \to 0^+. \]  
(16)

Then from (16) and [24] (Proposition 7) we can find \( c_5 > 0 \) such that \( \|\bar{u}_\lambda\|_\infty \leq c_5 \) for all \( 0 < \lambda \leq \eta \). From this bound and the nonlinear regularity theory of [18] (p. 320), there exist \( \alpha \in (0, 1) \) and \( c_6 > 0 \) such that
\[ \bar{u}_\lambda \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|\bar{u}_\lambda\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_6 \]
for all \( 0 < \lambda \leq \eta \). Hence the compact embedding of \( C^{1,\alpha}(\overline{\Omega}) \) into \( C^1(\overline{\Omega}) \) and (16) imply that
\[ \bar{u}_\lambda \to \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to 0^+. \]  
\[ \square \]

4. Multiple positive solutions. By Proposition 9 we can find \( \lambda_0 > 0 \) such that
\[ \bar{u}_\lambda(z) \in (0, \delta_0) \text{ for all } z \in \overline{\Omega} \text{ and all } 0 < \lambda \leq \lambda_0. \]  
(17)
Here \( \delta_0 \) is from \( H(f)(v) \). Let \( \eta_0 \) be from \( H(f)(iv) \) and \( \lambda^* = \min\{\lambda_0, \eta_0\} \).

**Proposition 10.** If \( H(\xi), H(\beta), H_0, H(f) \) hold and \( 0 < \lambda \leq \lambda^* \), then problem \((P_\lambda)\) admits a positive solution \( u_0 \in D_+ \) such that
\[ u_0 \in \text{int } C^1(\overline{\Omega})[\bar{u}_\lambda, w]. \]

**Proof.** From \( H(f)(v) \), \( \delta_0 < \min_{\overline{\Omega}} w \). This together with (17) indicate that we may define the following truncation of the reaction term:
\[ r_\lambda(z, x) = \begin{cases} \lambda \bar{u}_\lambda(z)^{r-1} + f(z, \bar{u}_\lambda(z)) & \text{if } x < \bar{u}_\lambda(z) \\ \lambda x^{r-1} + f(z, x) & \text{if } \bar{u}_\lambda(z) \leq x \leq w(z) \\ \lambda w(z)^{r-1} + f(z, w(z)) & \text{if } w(z) < x. \end{cases} \]  
(18)

Set \( R_\lambda(z, x) = \int_0^x r_\lambda(z, s)ds \) and consider the \( C^1\)-functional \( \tilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) defined by
\[ \tilde{\varphi}_\lambda(u) = \frac{1}{p} \mu_p(u) + \frac{1}{q} \|Du\|_q^q - \int_\Omega R_\lambda(z, u)dz. \]

From (6) and (18) we infer that \( \tilde{\varphi}_\lambda \) is coercive. It is also sequentially weakly lower semicontinuous. So we can find \( u_0 \in W^{1,p}(\Omega) \) such that
\[ \tilde{\varphi}_\lambda(u_0) = \inf\{\tilde{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\}. \]
Thus, $\hat{\varphi}'(u_0) = 0$. Therefore, for all $h \in W^{1,p}(\Omega)$ we have
\begin{equation}
\langle A_p(u_0), h \rangle + \langle A_q(u_0), h \rangle + \int_{\Omega} \xi(z)|u_0|^{p-2}u_0hd\sigma + \int_{\partial \Omega} \beta(z)|u_0|^{p-2}u_0hd\sigma = \int_{\Omega} r(\lambda, u_0)hd\sigma.
\end{equation}
(19)
In (19) we choose $h = (\bar{u}_\lambda - u_0)^+ \in W^{1,p}(\Omega)$, then

$$
\langle A_p(u_0), (\bar{u}_\lambda - u_0)^+ \rangle + \langle A_q(u_0), (\bar{u}_\lambda - u_0)^+ \rangle + \int_{\Omega} \xi(z)|u_0|^{p-2}u_0(\bar{u}_\lambda - u_0)^+dz
$$
\begin{align*}
&+ \int_{\partial \Omega} \beta(z)|u_0|^{p-2}u_0(\bar{u}_\lambda - u_0)^+d\sigma

&= \int_{\Omega} [\lambda \bar{u}_\lambda^{r-1} + f(z, \bar{u}_\lambda)](\bar{u}_\lambda - u_0)^+dz
\end{align*}
\begin{align*}
&\geq \int_{\Omega} \lambda \bar{u}_\lambda^{r-1}(\bar{u}_\lambda - u_0)^+dz

&= (A_p(\bar{u}_\lambda), (\bar{u}_\lambda - u_0)^+) + (A_q(\bar{u}_\lambda), (\bar{u}_\lambda - u_0)^+)

&+ \int_{\Omega} \xi(z)\bar{u}_\lambda^{p-1}(\bar{u}_\lambda - u_0)^+dz + \int_{\partial \Omega} \beta(z)\bar{u}_\lambda^{p-1}(\bar{u}_\lambda - u_0)^+d\sigma.
\end{align*}

Thus, $\bar{u}_\lambda \leq u_0$. Next in (19) we choose $h = (u_0 - w)^+ \in W^{1,p}(\Omega)$, we have

$$
\langle A_p(u_0), (u_0 - w)^+ \rangle + \langle A_q(u_0), (u_0 - w)^+ \rangle + \int_{\Omega} \xi(z)u_0^{p-1}(u_0 - w)^+dz
$$
\begin{align*}
&+ \int_{\partial \Omega} \beta(z)u_0^{p-1}(u_0 - w)^+d\sigma

&= \int_{\Omega} [\lambda w^{r-1} + f(z, w)](u_0 - w)^+dz
\end{align*}
\begin{align*}
&\leq (A_p(w), (u_0 - w)^+) + (A_q(w), (u_0 - w)^+) + \int_{\Omega} \xi(z)w^{p-1}(u_0 - w)^+dz

&+ \int_{\partial \Omega} \beta(z)w^{p-1}(u_0 - w)^+d\sigma.
\end{align*}

(recall that $\lambda \leq \lambda^* \leq \eta_0$ and see $H(f)(iv)$ and $H(\beta)$). Thus, $u_0 \leq w$. Therefore we have proved that
\begin{equation}
\begin{aligned}
&u_0 \in [\bar{u}_\lambda, w].
\end{aligned}
\end{equation}
(20)
From (18)-(20), $u_0$ is a positive solution of the problem $(P_\lambda)$ and the nonlinear regularity theory implies $u_0 \in D_+$.

Let $\varrho = \|w\|_\infty$ and $\hat{\xi}_\varrho > 0$ be from $H(f)(vi)$. We have
\begin{align}
-\Delta_p u_0(z) - \Delta_q u_0(z) + [\xi(z) + \hat{\xi}_\varrho]u_0(z)^{p-1}
&= \lambda u_0(z)^{r-1} + f(z, u_0(z)) + \hat{\xi}_\varrho u_0(z)^{p-1}
\geq \lambda \bar{u}_\lambda(z)^{r-1} + f(z, \bar{u}_\lambda(z)) + \hat{\xi}_\varrho \bar{u}_\lambda(z)^{p-1}

&\geq \lambda \bar{u}_\lambda(z)^{r-1} + \hat{\xi}_\varrho \bar{u}_\lambda(z)^{p-1}

&= -\Delta_p \bar{u}_\lambda(z) - \Delta_q \bar{u}_\lambda(z) + [\xi(z) + \hat{\xi}_\varrho]\bar{u}_\lambda(z)^{p-1}.
\end{align}
(21)

Let
\begin{equation}
\begin{aligned}
h_1(z) &= \lambda \bar{u}_\lambda(z)^{r-1} + f(z, \bar{u}_\lambda(z)) + \hat{\xi}_\varrho \bar{u}_\lambda(z)^{p-1},
\end{aligned}
\end{equation}
\[ h_2(z) = \lambda u_0(z)^{p-1} + f(z, u_0(z)) + \tilde{\xi}_0 u_0(z)^{p-1}, \]

and

\[ \tilde{h}(z) = \lambda \tilde{u}_\lambda(z)^{p-1} + f(z, \tilde{u}_\lambda(z)) + \tilde{\xi}_0 \tilde{u}_\lambda(z)^{p-1}. \]

Evidently, \( h_1, h_2, \tilde{h} \in L^\infty(\Omega) \). Note that \( \tilde{m}_\lambda = \min_{\Omega} \tilde{u}_\lambda > 0 \) implies

\[ 0 < c_{\tilde{m}_\lambda} \leq f(z, \tilde{u}_\lambda(z)), \]

(see (17) and \( H(f)(v) \)). Therefore, \( \tilde{h}(z) - h_1(z) \geq c_{\tilde{m}_\lambda} > 0 \) for a.a. \( z \in \Omega \). Also from (20) and \( H(f)(vi) \) we have \( \tilde{h}(z) \leq h_2(z) \) a.a. on \( \Omega \). Thus we finally can write

\[ 0 < c_{\tilde{m}_\lambda} \leq h_2(z) - h_1(z) \text{ for a.a. } z \in \Omega. \]

So, from (21) and Proposition 6 we conclude that

\[ u_0 - \tilde{u}_\lambda \in \text{int } C_+. \]

Similarly we have

\[ -\Delta_p u_0(z) - \Delta_q u_0(z) = [\xi(z) + \tilde{\xi}_0] u_0(z)^{p-1} \]

\[ = \lambda u_0(z)^{q-1} + f(z, u_0(z)) + \tilde{\xi}_0 u_0(z)^{p-1} \]

\[ \leq \lambda w(z)^{q-1} + f(z, w(z)) + \tilde{\xi}_0 w(z)^{p-1} \text{ (see (20) and } H(f)(vi)) \]

\[ \leq \eta w(z)^{q-1} + f(z, w(z)) + \tilde{\xi}_0 w(z)^{p-1} \text{ (since } \lambda \leq \lambda^* \leq \eta) \]

\[ \leq -c_w + \tilde{\xi}_0 w(z)^{p-1} \text{ (see } H(f)(iv)) \]

\[ < -\Delta_p w(z) - \Delta_q w(z) + [\xi(z) + \tilde{\xi}_0] w(z)^{p-1}. \]

From (22) and Proposition 6 we have \( w - u_0 \in \text{int } C_+ \). Thus we conclude that

\[ u_0 \in \text{int } C^1(\Omega) \tilde{u}_\lambda, w]. \]

Using \( u_0 \) and suitable variational, truncation and comparison arguments, we can generate a second positive solution for problem \((P_\lambda) \) when \( 0 < \lambda \leq \lambda^* \).

**Proposition 11.** If \( H(\xi), H(\beta), H_0 \) and \( H(f) \) hold and \( 0 < \lambda \leq \lambda^* \), then problem \((P_\lambda) \) has a second positive solution \( \tilde{u} \in \text{int } D_+ \) such that \( u_0 \neq \tilde{u} \) and \( \tilde{u} - \tilde{u}_\lambda \in \text{int } C_+ \).

**Proof.** Let \( \tilde{u}_\lambda \in D_+ \) be the unique positive solution of \((Au_\lambda)\) established in Proposition 9. Consider the following truncation of the reaction:

\[ k_\lambda(z, x) = \begin{cases} 
\lambda \tilde{u}_\lambda(z)^{p-1} + f(z, \tilde{u}_\lambda(z)) & \text{if } x \leq \tilde{u}_\lambda(z) \\
\lambda x^{p-1} + f(z, x) & \text{if } \tilde{u}_\lambda(z) < x.
\end{cases} \]  

This is Caratheodory. Set \( K_\lambda(z, s) = \int_0^s k_\lambda(z, s)ds \) and consider the \( C^1 \)-functional \( \varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[ \varphi_\lambda(u) = \frac{1}{p} \mu_p(u) + \frac{1}{q} \|Du\|_q^q - \int_\Omega K_\lambda(z, u)dz. \]

Also, let \( \tilde{\psi}_\lambda \in C^1(W^{1,p}(\Omega)) \) be as in the proof of Proposition 10. From (18) and (23) we have

\[ \varphi_\lambda[\tilde{u}_\lambda, w] = \tilde{\psi}_\lambda[\tilde{u}_\lambda, w]. \]  

From Proposition 10 and its proof we have that

\[ u_0 \in \text{int } C^1(\Omega) \tilde{u}_\lambda, w] \text{ and } u_0 \text{ is a minimizer of } \tilde{\psi}_\lambda. \]
From (24)-(25) we infer that \( u_0 \) is a local \( C^1(\Omega) \)-minimizer of \( \varphi_\lambda \), thus
\[
\text{(see Proposition 5).}
\]

**Claim 1:** \( K_{\varphi_\lambda} \subseteq \overline{\{u\}} \cap D_+ \).

Let \( u \in K_{\varphi_\lambda} \). Then for all \( h \in W^{1,p}(\Omega) \) we have
\[
\langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_{\Omega} \xi(z)|u|^{p-2}u\,hdz + \int_{\partial\Omega} \beta(z)|u|^{p-2}u\,hd\sigma = \int_{\Omega} k_\lambda(z,u)h\,dz.
\]

In (27) we choose \( h = (\bar{u}_\lambda - u)^+ \in W^{1,p}(\Omega) \). Then
\[
\langle A_p(u), (\bar{u}_\lambda - u)^+ \rangle + \langle A_q(u), (\bar{u}_\lambda - u)^+ \rangle + \int_{\Omega} \xi(z)|u|^{p-2}u(\bar{u}_\lambda - u)^+\,dz
\]
\[
+ \int_{\partial\Omega} \beta(z)|u|^{p-2}u(\bar{u}_\lambda - u)^+\,d\sigma
\]
\[
= \int_{\Omega} \lambda \bar{u}_\lambda^{-1} + f(z, \bar{u}_\lambda) (\bar{u}_\lambda - u)^+\,dz \text{ (see (23))}
\]
\[
\geq \int_{\Omega} \lambda \bar{u}_\lambda^{-1}(\bar{u}_\lambda - u)^+\,dz \text{ (see (17) and } H(f)(v))
\]
\[
= \langle A_p(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle + \langle A_q(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle + \int_{\Omega} \xi(z)\bar{u}_\lambda^{-1}(\bar{u}_\lambda - u)^+\,dz
\]
\[
+ \int_{\partial\Omega} \beta(z)\bar{u}_\lambda^{-1}(\bar{u}_\lambda - u)^+\,d\sigma,
\]

Thus \( \bar{u}_\lambda \preceq u \).

Then from (27) and (23) we have for all \( h \in W^{1,p}(\Omega) \)
\[
\langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_{\Omega} \xi(z)|u|^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-1}h\,d\sigma = \int_{\Omega} \lambda u^{-1} + f(z, u)h\,dz.
\]

Thus
\[
\begin{cases}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-1} + f(z, u(z)) \text{ a.e. in } \Omega, \\
-\partial_{n_p} u + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \ u > 0.
\end{cases}
\]

From (28) and [24, Proposition 7] we have \( u \in L^\infty(\Omega) \), thus the nonlinear regularity theory of [18] implies that \( u \in D_+ \) and hence
\[
K_{\varphi_\lambda} \subseteq \overline{\{u\}} \cap D_+.
\]

This proves Claim 1. Now we assume that
\[
K_{\varphi_\lambda} \text{ is finite.}
\]

Otherwise on account of Claim 1 problem \((P_\lambda)\) has an infinity of positive smooth solutions (see (23)) and so we are done. Then from (26) and (29) we can find \( \varrho \in (0,1) \) small such that
\[
\varphi_\lambda(u_0) < \inf \{\varphi_\lambda(u) : \|u - u_0\| = \varrho\} = m_\lambda,
\]
(see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29]). By \( H(f)(iii) \) for any \( \eta > 0 \) we can find \( M_1 > 0 \) such that
\[
f(z,x)x - pF(z,x) \leq -\eta x^q \text{ for a.a. } z \in \Omega \text{ and all } x \geq M_1.
\]
For a.a. $z \in \Omega$ and all $x \geq M_1$ we have (see (31))
\[
\frac{d}{dx} \frac{F(z,x)}{x^p} = \frac{f(z,x)x - px^{p-1}F(z,x)}{x^{2p}}
= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} 
\leq -\frac{\eta}{x^{p-q+1}},
\]
thus
\[
\frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \leq \frac{\eta}{p-q} \left[ \frac{1}{x^{p-q}} - \frac{1}{v^{p-q}} \right]
\] (32)
for $x \geq v \geq M_1$.

Integrating $H(f)(ii)$ we obtain
\[
\hat{\lambda}_1(p,\xi,\beta) \pi \leq \liminf_{x \to +\infty} \frac{F(z,x)}{x^p} \leq \limsup_{x \to +\infty} \frac{F(z,x)}{x^p} \leq \frac{\eta}{p}
\] (33)
uniformly for a.a. $z \in \Omega$. So, if in (32) we let $x \to +\infty$ and use (33), then
\[
\hat{\lambda}_1(p,\xi,\beta) \pi - \frac{F(z,v)}{v^p} \leq -\frac{\eta}{p-q} \frac{1}{v^{p-q}},
\]
thus
\[
\hat{\lambda}_1(p,\xi,\beta) v^p - pF(z,v) \leq \frac{np}{p-q} \leq -\eta
\]
for a.a. $z \in \Omega$ and all $v \geq M_1$. Since $\eta > 0$ is arbitrary we infer that
\[
\lim_{x \to +\infty} \frac{\hat{\lambda}_1(p,\xi,\beta) x^p - pF(z,x)}{x^q} = -\infty
\] (34)
uniformly for a.a. $z \in \Omega$. In what follows we set
\[
\hat{\lambda}_1 = \hat{\lambda}_1(p,\xi,\beta) \text{ and } \tilde{u}_1 = \tilde{u}_1(p,\xi,\beta) \in D_+.
\]

Claim 2: $\varphi_\lambda(t\tilde{u}_1) \to -\infty$ as $t \to +\infty$.

For some $c_\tau > 0$ (see (23)) we have
\[
p\varphi_\lambda(t\tilde{u}_1) = \mu_p(t\tilde{u}_1) + \frac{p}{q} \|D(t\tilde{u}_1)\|_q^q - \int_{\Omega} K_\lambda(z,t\tilde{u}_1)dz
\leq \mu_p(t\tilde{u}_1) + \frac{p}{q} \|D(t\tilde{u}_1)\|_q^q - \int_{\Omega} \left[ \frac{\lambda}{\tau}(t\tilde{u}_1)^\tau + pF(z,t\tilde{u}_1) \right]dz + c_\tau
\leq \frac{p}{q} \|D(t\tilde{u}_1)\|_q^q + \int_{\Omega} \left[ \hat{\lambda}_1(t\tilde{u}_1)^p - pF(z,t\tilde{u}_1) \right]dz - \frac{lp}{\tau} \|t\tilde{u}_1\|_\tau^\tau + c_\tau.
\]
Thus,
\[
p\varphi_\lambda(t\tilde{u}_1) \leq \frac{p}{q} \|D\tilde{u}_1\|_q^q + \int_{\Omega} \left[ \frac{\hat{\lambda}_1(t\tilde{u}_1)^p - pF(z,t\tilde{u}_1)}{(t\tilde{u}_1)^q} \right] \tilde{u}_1^q dz - \frac{lp}{\tau p^{\tau-1}} \|\tilde{u}_1\|_{\tau q} \tau + c_\tau \frac{p}{q},
\]
hence $\frac{p\varphi_\lambda(t\tilde{u}_1)}{t^q} \to +\infty$ as $t \to \infty$ (see (34)), and use Fatou’s Lemma to obtain
\[
\varphi_\lambda(t\tilde{u}_1) \to -\infty \text{ as } t \to +\infty.
\]
This proves Claim 2.

Claim 3: $\varphi_\lambda$ satisfies the C-condition.

Consider a sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ such that
\[
|\varphi_\lambda(u_n)| \leq M_2
\] (35)
for some $M_2 > 0$ and all $n \in \mathbb{N}$ and that
\[(1 + \|u_n\|)\varphi'_n(u_n) \to 0 \text{ in } W^{1,p}(\Omega)\] (36)
as $n \to +\infty$.

From (36) we have
\[\frac{\epsilon_n\|h\|}{1 + \|u_n\|} \geq \left| \langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_{\Omega} \xi(z)|u_n|^{p-2}u_n h dz \right.\]
\[+ \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h d\sigma - \int_{\Omega} k_\lambda(z, u_n) h dz \] (37)
for all $h \in W^{1,p}(\Omega)$ as $\epsilon_n \to 0^+$.

In (37) we choose $h = -u_n \in W^{1,p}(\Omega)$. Using (23) we obtain
\[\mu_p(u_n) + \|Du_n\|_q^q - \int_{\Omega} [\lambda u_n^{q-1} + f(z, u_n)](-u_n^+) dz \leq \epsilon_n \]
for all $n \in \mathbb{N}$. Thus, $c_1\|u_n^+\|^p \leq c_8\|u_n\|$ for some $c_8 > 0$ and all $n \in \mathbb{N}$. Therefore,
\[\{u_n^+\} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \] (38)

Suppose that $\|u_n^+\| \to \infty$. Set $y_n = \frac{u_n^+}{\|u_n^+\|}$. Then $\|y_n\| = 1$ and $y_n \geq 0$. We may assume that for some $y \in W^{1,p}(\Omega)$ with $y \geq 0$
\[y_n \rightharpoonup y \text{ in } W^{1,p}(\Omega), \ y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \] (39)

From (37), (38) we have for all $h \in W^{1,p}(\Omega)$, with $\epsilon_n' \to 0^+$,
\[\epsilon_n'^{\prime} \|h\| \geq \left| \langle A_p(y_n), h \rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q(y_n), h \rangle + \int_{\Omega} \xi(z)y_n^{p-1} h dz \right.\]
\[+ \int_{\partial\Omega} \beta(z)y_n^{p-1} h d\sigma - \int_{\Omega} k_\lambda(z, u_n) h dz \].

Thus for all $h \in W^{1,p}(\Omega)$ and with $\epsilon_n'' \to 0^+$,
\[\epsilon_n'' \|h\| \geq \langle A_p(y_n), h \rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q(y_n), h \rangle + \int_{\Omega} \xi(z)y_n^{p-1} h dz \]
\[+ \int_{\partial\Omega} \beta(z)y_n^{p-1} h d\sigma - \int_{\Omega} \left[ \frac{\lambda(u_n^+)^{q-1} + f(z, u_n^+)}{\|u_n^+\|^{p-1}} \right] |h| dz. \] (40)

Note that
\[\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\} \subseteq L^{p^*}(\Omega) \text{ is bounded (see } H(f)(i)(ii)). \text{ So for at least a subsequence we have} \]
\[\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \rightharpoonup \eta_0(z)y^{p-1} \text{ in } L^{p^*}(\Omega), \] (41)
with $\lambda_1 \leq \eta_0(z) \leq \tilde{\eta}$ (see [1], proof of Proposition 16). In (40) we choose $h = y_n - y \in W^{1,p}(\Omega)$, and use (39) and (41),
\[\lim_{n \to \infty} \left| \langle A_p(y_n), y_n - y \rangle + \langle A_q(y_n), y_n - y \rangle \right| = 0, \]
thus $\lim_{n \to \infty} \left[ \langle A_p(y_n), y_n - y \rangle + \langle A_q(y_n), y_n - y \rangle \right] \leq 0$ since $A_q$ is monotone. Hence (see (39)),
\[\lim_{n \to \infty} \sup_{n \to \infty} \{A_p(y_n), y_n - y \} \leq 0, \]
therefore (see Proposition 2) we obtain
\[y_n \to y \text{ in } W^{1,p}(\Omega), \text{ and hence } \|y\| = 1, y \geq 0. \] (42)
In (40), let $n \to \infty$ and use (41)-(42), for $h \in W^{1,p}(\Omega)$ we have
\[
\langle A_p(y), h \rangle + \int_{\Omega} \xi(z) y^{p-1} h dz + \int_{\partial \Omega} \beta(z) y^{p-1} h d\sigma = \int_{\Omega} \eta_0(z) y^{p-1} h dz.
\]
Thus, we obtain
\[
\begin{cases}
-\Delta_p y(z) + \xi(z) y(z)^{p-1} = \eta_0(z) y(z)^{p-1} & \text{for a.a. } z \in \Omega \\
\frac{\partial y}{\partial n_p} + \beta(z) y^{p-1} = 0 & \text{on } \partial \Omega, y \neq 0,
\end{cases}
\] (see [24]).

If $\eta_0 \neq \hat{\lambda}_1$ (see (41)), then by Proposition 6 we have
\[
\hat{\lambda}_1(\eta_0) < \hat{\lambda}_1(\hat{\lambda}_1) = 1.
\]
Hence, $y$ must be a nodal (see (43)), a contradiction (see (42)).

If $\eta_0(z) = \hat{\lambda}_1$ for a.a. $z \in \Omega$, then from (43) we have $y = \partial \hat{\mu}_1$ with $\hat{\theta} > 0$ (see (42)), thus $y(z) > 0$ for all $z \in \Omega$ and $u_n^+(z) \to \infty$ for all $z \in \Omega$. Therefore,
\[
\frac{f(z, u_n^+(z)) u_n^+(z) - pF(z, u_n^+)}{u_n(z)^q} \to -\infty
\]
(see $H(f)(iii)$). Hence by Fatou’s Lemma,
\[
\int_{\Omega} \frac{f(z, u_n^+(z)) u_n^+(z) - pF(z, u_n^+)}{\|u_n^+\|_q^q} dz \to -\infty. \tag{44}
\]

From (35) and (38) we have for some $M_3 > 0$ and all $n \in \mathbb{N}$ (see (23))
\[
\mu_p(u_n^+) + \frac{p}{q} \|Du_n^+\|_q^q - \int_{\Omega} \left[ \frac{\lambda p}{\tau} (u_n^+) \right]^\tau + pF(z, u_n^+) \right] dz \geq -M_3. \tag{45}
\]

Similarly from (37), (38) and using $h = u_n^+ \in W^{1,p}(\Omega)$, for some $M_4 > 0$ we have
\[
-\mu_p(u_n^+) - \|Du_n^+\|_q^q + \int_{\Omega} \left[ \lambda (u_n^+) \right]^\tau + f(z, u_n^+) u_n^+ \right] dz \geq -M_4. \tag{46}
\]

Adding (45) and (46), with $M_5 = M_3 + M_4$ and all $n \in \mathbb{N}$, we have
\[
\left( \frac{p}{q} - 1 \right) \|Du_n^+\|_q^q + \int_{\Omega} \left[ f(z, u_n^+) u_n^+ - pF(z, u_n^+) \right] dz \geq -M_5 + \left( \frac{\lambda p}{\tau} - 1 \right) \|u_n^+\|_\tau^\tau.
\]
Thus,
\[
\left( \frac{p}{q} - 1 \right) \|Dy_n\|_q^q + \int_{\Omega} \frac{f(z, u_n^+) u_n^+ - pF(z, u_n^+)}{\|u_n^+\|_q^q} dz \geq -M_5 + \left( \frac{\lambda p}{\tau} - 1 \right) \|u_n^+\|_\tau^\tau \|y_n\|_\tau^\tau. \tag{47}
\]

Since $\tau < q$, taking limit as $n \to \infty$ we reach a contradiction to (44). Therefore, \{u_n^+\} $\subseteq W^{1,p}(\Omega)$ is bounded hence \{u_n\} $\subseteq W^{1,p}(\Omega)$ is bounded (see (38)). So, we may assume that
\[
u_n^+ \to u \text{ in } W^{1,p}(\Omega), \ u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial \Omega). \tag{48}
\]

If in (37) we choose $h = u_n - u \in W^{1,p}(\Omega)$ and use (47), then
\[
\lim_{n \to \infty} \left[ \langle A_p(u_n), u_n - u \rangle + \langle A_q(u_n), u_n - u \rangle \right] = 0,
\]
thus $u_n \to u$ in $W^{1,p}(\Omega)$ (as before using Proposition 2). Therefore, $\varphi_\lambda$ satisfies the $C$-condition, which proves Claim 3.
Now (30) and Claims 2 and 3 permit the use of Theorem 1 (the mountain pass theorem). So we can find \( \tilde{u} \in W^{1,p}(\Omega) \) such that

\[
\tilde{u} \in K_{\varphi_2} \subseteq [\tilde{u}] \cap D_+ \text{ and } m_\lambda \leq \varphi_\lambda(\tilde{u}). \tag{49}
\]

From (23), (30) and (48) we see that \( \tilde{u} \in D_+ \) is a solution of problem \((P_\lambda)\), with \( \tilde{u} \neq u_0 \).

Moreover, as in the proof of Proposition 10 using Proposition 5 we can show that \( \tilde{u} - \tilde{u}_\lambda \in \text{int } C_+ \).

We can state the following multiplicity theorem for problem \((P_\lambda)\).

**Theorem 12.** If \( H(\xi), H(\beta), H_0 \) and \( H(f) \) hold, then there exists \( \lambda^* > 0 \) such that for all \( 0 < \lambda \leq \lambda^* \) problem \((P_\lambda)\) has at least two positive solutions: \( u_0, \tilde{u} \in D_+ \), with \( u_n \neq \tilde{u} \).

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