The Quantum Dissipative Villain Model

G. Falci\(^{(1)}\), G. Giaquinta\(^{(1)}\) and U. Weiss\(^{(2)}\)

\(^{(1)}\)Istituto di Fisica, Università di Catania, 95125 Catania (I)
& Istituto Nazionale per la Fisica della Materia – E-mail: gfalci@ing.unict.it
\(^{(2)}\)II. Institut für Theoretische Physik, Universität Stuttgart, 70550 Stuttgart (D)

Abstract

We introduce the Quantum Dissipative Villain (QDV) model as a prototype model to study tunneling in dissipative quantum mechanics. Dissipation is provided by a coupled linear environment. In the QDV model, the discrete character of a tunneling degree of freedom coupled to an environment is explicit, leading to a rich dual structure. We derive general exact mappings of the QDV model on several dual discrete representations, including pairs of self-dual models, for general linear environments and arbitrary temperatures. Self-duality allows to write exact equations for each correlation function of each representation. Analogies with the theory of classical network transformations are also presented. Finally we discuss the fundamental character of the QDV model. For instance, the standard Caldeira-Leggett model, which describes, e.g., mesoscopic Josephson junctions in a circuit and many other physical systems, is a special QDV model. The self-dual structure of the QDV model allows then the exact generalization of the Schmid approximate self-duality to general linear environments and arbitrary temperatures.

1 Introduction

Quantum dissipative systems can be described in terms of system plus environment hamiltonians where a special quantum variable \(\varphi\) interacts with an environment of harmonic oscillators. The reduced dynamics of \(\varphi\) obtained by integrating out the reservoir’s degrees of freedom is studied. An effective euclidean action is found which, for state-independent dissipation, reads

\[
S[\varphi] = \frac{1}{2} \int_{0}^{\beta} d\tau d\tau' \varphi(\tau) A(\tau - \tau') \varphi(\tau') + \int_{0}^{\beta} d\tau \mathcal{V}(\varphi) ,
\]  

(1)
where $V(\varphi)$ is the potential. The kernel, whose Fourier transform (FT) is $A(\omega) = m\omega^2 - \frac{1}{2}\alpha(\omega)$, with $\omega = 2\pi n/\beta$, contains the kinetic “mass” term and the damping term $\alpha(\omega)$ subsuming the spectral properties of the bath coupling $\Gamma$.

If $V(\varphi)$ describes a tunneling problem then the variable $\varphi$ has an underlying discrete character even if it is defined to be continuous. For instance, the low energy properties of the double-well potential $V(\varphi) = V(\varphi^2 - a^2)^2$ are approximately accounted for by a two-state system. However, it is impossible to find the exact relation for a generic environment. We now introduce a model which displays the discrete character of the continuous variable $\varphi$ by writing

$$\Lambda^{-1}\sum_\tau V(\varphi_\tau) - \ln\left[ \sum_{\{m_\tau\}} \exp\left\{ 1/(2\Lambda) \sum_\tau V(\varphi_\tau - 2\pi m_\tau)^2 \right\} \right] - J_\tau \varphi_\tau,$$

where we added a source $J_\tau$. Here we are considering a version of eq.(1) on a lattice with $N$ sites and spacing $\beta/N = 1/\Lambda$. We name it Quantum Dissipative Villain (QDV) model because eq.(2) is the Villain approximation of the potential $V(\varphi) = -V \cos \varphi - J_\varphi$ in the Caldeira-Leggett (CL) model.

## 2 Dual representations of the QDVM

### m-representation: We integrate $\varphi$ in the partition function of the QDV model, which becomes $Z^{QDV}[J] = Z^V[J] \cdot Z^{(m)}[J^{(m)}]$. Here $Z^V$ is the partition function of the damped harmonic oscillator (DHO), and

$$Z^{(m)}[J^{(m)}] = \sum_{\{m_\tau\}} e^{-\frac{1}{2}\sum_{\tau,\tau'} m_\tau (\frac{\Lambda}{m})^2 \left[ \delta_{\tau,\tau'} - \Lambda^2 G_V^{\tau,\tau'} \right] m_{\tau'} + \sum_\tau m_\tau J^{(m)}_\tau} / \Lambda$$

is a 1-D surface roughening model with interacting heights $m_\tau$. The source is $J^{(m)}_\omega = 2\pi V \Lambda G^V_\omega J_\omega$, and $G^V_\omega$ is the Green’s function of the discretized DHO, given by $G^V_\omega = \Theta(\Lambda - |\omega|) [m\Lambda \omega^2 - \Lambda \alpha(\omega)/2 + \Lambda V]^{-1}$, for large enough $\Lambda$.

### e-representation: We introduce $e_\tau := m_{\tau+1} - m_\tau$ and rewrite $Z^{(m)}$ as

$$Z^{(e)}[J^{(e)}] = \sum_{\{e_\tau\}} e^{-\frac{1}{2}\sum_{\tau,\tau'} e_\tau \Delta_{\tau,\tau'} e_{\tau'} + \sum_\tau J^{(e)}_\tau e_\tau},$$

$$\Delta_\omega = (2\pi V \Lambda / \omega)^2 \left[ 1/V - \Lambda G^V_\omega \right]; \quad J^{(e)}_\omega = -2\pi i V \Lambda / \omega G^V_\omega J_\omega,$$

obtaining a gas of interacting charges $e_\tau \in [0, \infty[$.

### n-representation: Another charge representation can be obtained starting
from the QDV model, performing a Poisson transformations (which changes \( m \to n \)) and then integrating out \( \varphi \). We obtain
\[
\mathcal{Z}^{QDV}[\mathcal{J}] = \mathcal{Z}^0[\mathcal{J}] = \mathcal{Z}^0[(\mathcal{J}^{(n)})^e] \quad \text{where} \quad \mathcal{Z}^0 \text{ describes a Brownian particle and} \quad \mathcal{Z}^{(n)} \text{ has the same structure of} \quad \mathcal{Z}^{(e)} \text{ eq.}\,(3), \quad \text{being a gas of} \quad n_\tau \in ]-\infty, \infty[ \quad \text{charges with interaction and source given by}
\]
\[
\mathcal{D}_{\omega} = \Lambda/V + \Lambda^2 \varphi^{0}_\omega ; \quad \mathcal{J}^{(n)}_{\omega} = i \Lambda \varphi^{0}_\omega \mathcal{J}^{(e)}_{\omega} .
\]

3 Exact Self-duality

The \( \mathcal{Z}^{(e)} \) and \( \mathcal{Z}^{(n)} \) represent the same model, with modified interaction and sources. This means that the QDV model has an exact self-dual structure. A simple reformulation of this self-dual mapping is obtained if we introduce the functions
\[
\zeta^{0}_\omega = |\omega|/(2\pi \Lambda \varphi^{0}_\omega) \quad \text{and} \quad \zeta(\omega) = \zeta^{0}_\omega + |\omega|/(2\pi V) .
\]

Then we rewrite \( \Lambda^{-1} \mathcal{D}_{\omega} = 2\pi/|\omega| \cdot \zeta(\omega) \) and \( \Lambda^{-1} \Delta_{\omega} = 2\pi/|\omega| \cdot [\zeta(\omega)]^{-1} \). The transformations of the interaction and of the source are finally given by
\[
\zeta(\omega) \rightarrow 1/\zeta(\omega) , \quad \text{and} \quad \mathcal{J}^{(n)}_{\omega} = -\omega/|\omega| \cdot \zeta(\omega) \mathcal{J}^{(e)}_{\omega} .
\]

We can also write exact relations between correlation functions of the representations of the QDV model. For instance, the FT of the correlation function \( \langle \varphi_\tau \varphi_0 \rangle \) of the QDV model is related to \( \langle n_\tau n_0 \rangle_{\omega} = |\omega|/(2\pi \Lambda) C_\omega[\zeta] \) by
\[
|\omega|/(2\pi \Lambda) \cdot \langle \varphi \varphi \rangle_{\omega} = \zeta^{0}(\omega) \left\{ 1 - \zeta^{0}(\omega) \cdot C_\omega[\zeta(\omega)] \right\} .
\]

Using self-duality, the relation between the e-e and n-n correlation functions becomes an exact equation for \( C \)
\[
\zeta(\omega) \cdot C_\omega[\zeta(\omega)] + \zeta^{-1}(\omega) \cdot C_\omega[\zeta^{-1}(\omega)] = 1 .
\]

We have not yet specified the environment, i.e. the function \( \zeta(\omega) \). Both \( \zeta(\omega) \) and \( 1/\zeta(\omega) \) have to be strictly positive for \( \omega \neq 0 \) since otherwise the integrations involved in the transformations cannot be performed. Moreover the calculation of dynamic correlation functions involves the analytic continuation \( |\omega| \to p \to i\Omega + 0^+ \), so it is desirable that \( \zeta(p) \) is analytic in \( \Re p > 0 \). Thus we require that \( \zeta(p) \) has the properties of the impedance of a linear passive bipole. The analogy with network theory involves also duality. Namely, eqs.\,(11) for \( \Re p > 0 \) can be reparaphrased by associating to each charge representation a circuit with a non linear quantum component \( \mathcal{X} \), the interaction \( \zeta(p) \) corresponding to the impedance seen by \( \mathcal{X} \) and the current bias being \( \mathcal{J}^{(n)}_{\omega} \) (see fig.1). Then the quantum self-dual transformation for the charge models, eqs.\,(3,4,5), correspond to transforming the linear elements and the source of the circuit using the known classical dual and Norton transformations, while keeping unchanged the non-linear quantum component \( \mathcal{X} \).
Figure 1: The $n$-representation corresponds to circuit (a), $\zeta(p)$ being the impedance seen by $X$; the circuit (b) is obtained by a dual plus Norton transformation and corresponds to the $e$-representation. For the CL model we have $2\pi m \leftrightarrow R_Q C$, $\alpha(p) \leftrightarrow -p R_Q / [\pi Z(p)]$, $2\pi V \leftrightarrow R_Q / L$, where $R_Q = h/(4e^2)$. Then $R_Q \zeta(p) = [Z^{-1}(p) + pC]^{-1} + pL$.

4 Further developments

The above results are significant in view of the fundamental character of the QDV model. Indeed, the well known CL model can be obtained exactly from the QDV model in the continuum limit if $V \rightarrow \Lambda/[2\ln(2\Lambda/V)]$. In other words this choice makes exact the Villain approximation. In this case the low frequency limit of eq.(6) reproduces both the approximate Schmid self-duality and the $\sigma \leftrightarrow -\sigma$ correspondence. The exact self-dual structure of the CL model was recently found for a special environment. Here we find that it holds true for arbitrary temperatures and general environments.

When the CL model describes a mesoscopic Josephson junction in a circuit the analogy with network theory (see fig.1) becomes more stringent. The lowest order in the Coulomb-gas representation of the $n$-model is the standard theory of the “influence of the environment” and calculations can be performed numerically for any external impedance. The same can be done for the lowest order in the $e$-representation, which corresponds to a single-instanton contribution. Duality network relations for a purely resistive environment, justified by the results of Schmid have been used for mesoscopic junctions since a long time. They are here substantiated and generalized.

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