Application of the Beta Probability Density Function for Representing Infiltration of Water-Repellent Soil

Changjiang Ren¹, Yong Zhao²*, Xinyu Zhao¹, Xianghui Lu¹ and Guohua He²

¹ Department of Hydraulic and Ecological Engineering, Nanchang Institute of Technology, Nanchang 330099, China
² State Key Laboratory of Simulation and Regulation of Water Cycle in River Basin, China Institute of Water Resources and Hydropower Research, Beijing 100038, China
*Email: Zhaoyong@iwhr.com

Abstract. Laboratory experiments were conducted to investigate the infiltration of water-repellent soil. The most suitable mathematical model to represent infiltration of water-repellent soil was investigated and water infiltration into this column was measured for both hydrophilic and water-repellent soils of the Guishui River Basin. The results showed a monotonous increase and decrease in cumulative infiltration and infiltration rate over time with hydrophilic soil, respectively. In contrast, the results for water-repellent soil showed: (1) The existence of a turning point in cumulative infiltration above which the infiltration rate increased sharply, with higher initial soil moisture content resulting in an earlier and larger increase in the infiltration rate; (2) A larger stable infiltration rate after the peak compared to that before the peak, with the overall infiltration rate presenting a single peak curve if the beginning of rapid infiltration is ignored. The applicability of the Kostiakov, segmented Kostiakov, Gaussian, segmented Gaussian, Beta and segmented Beta function infiltration models were analysed for the two soil types. The numerical analysis indicated that: (3) The Kostiakov and Beta function models showed better applicability for both hydrophilic and water-repellent soils; (4) For water-repellent soil, the Gaussian function infiltration model not only represented the monotonous decrease in infiltration rate, but also demonstrated a steady infiltration rate during the initial stage and a gradual increase and decrease in infiltration rate pre- and post-peak, respectively. Similarly, the Beta function model was able to represent the monotonous decrease in infiltration rate, and the segmented Beta function model represented the U-shaped change in rapid infiltration before the threshold as well as the gradual increase and right-skewed distribution curve of infiltration pre- and post-turning point in water-repellent soil. The Beta function model achieved the highest simulation accuracy and showed the widest applicability.

1. Introduction

Soil water repellency (SWR) is a measure of the hydrophobicity soil, i.e., the degree to which surface soil particles resist moisture [1], typically because of soil particles being coated by hydrophobic substances, predominantly of plant origin. Soil hydrophobicity results in the alteration of soil water dynamics, thereby lowering the soil matric potential and consequently also lowering water infiltration. SWR is a measure of impedance of water infiltration into soil, which limits the rate and capacity of soil water absorption [2]. Soil hydrophobicity is a phenomenon that has been observed globally, from the...
tropics to sub-Arctic regions, and over a wide range of soil types [3]. Soil hydrophobicity has significant impacts on ecological and hydrological processes, including reductions in water infiltration [4], the promotion of surface runoff [5] and associated soil erosion [6], increased wind erosion, accelerated fertilizer loss [7] by influencing preferential flow [8] and an increase in soil carbon content [9], which may result in reductions in crop yield [2].

Observed patterns of infiltration rate (IR) have been different for hydrophobic and hydrophilic soils. SWR of soil appears to be highly variable over time and is dependent on soil water content. Past studies have shown a general increase in SWR with decreasing water content, thereby demonstrating the complexity of the influence of SWR on water infiltration, which may result in hysteresis in soil water retention [8] and distinct preferential flow paths [8]. The process of water infiltration can be fairly complex due to the dynamic characteristic of SWR, which can result in unstable water flow within the soil matrix [10]. The effect of SWR on hydrological process remains poorly understood, with the majority of researchers assuming that SWR reduces soil IR [11]. However, it has been shown that the hydrophobicity of soil does not result in a monotonous decrease in IR, but rather that IR increases with time with continual infiltration of water into soils [12]. The effect of soil hydrophobicity is often clear within an infiltration process. For example, Vogelmann et al. [13] showed that IR first increased with an atypical infiltration curve with a double-slope shape as the repellency effect of soil decreased. Lichner et al. [14] proposed instead a “hockey stick” infiltration curve that represented the relationship between cumulative infiltration (CI) and the square root of time in hydrophobic and wettable soil states. Pierson et al. [15] found that the increase in runoff in soil post-fire is due to the destruction of ground cover and canopy cover by fire rather than the effect of fire on soil hydrophobicity, and that the severity of these influences can be considerably affected by short-term variations in soil hydrophobicity. Doerr et al. [16] demonstrated that the IR of water-repellent soil during an entire rainfall event first showed a decrease followed by an increase and finally a decrease to the minimum value, after which IR stabilized. Zhao et al. [17] showed a gradual increase in IR of water-repellent soil during the primary stage following a single-peak curve. However, the phenomenon of increasing IR has to date been neglected and the infiltration process has been assumed to decrease monotonously for ease of calculations. Although not entirely appropriate, traditional infiltration models and segmented function models continue to be used to represent the double-slope infiltration process [18]. Ren et al. [12] concluded that the use of the segmented Kostiakov function (SKF) to calculate the IR of water-repellent soil results in a maximum and minimum IR occurring simultaneously at the turning point. Hence, traditional water infiltration models are not appropriate for describing the gradual decrease in IR after the initiating in infiltration.

Therefore, considering the shortcomings of traditional water infiltration models, the present study aimed to: (1) quantify the applicability of the water infiltration models, such as the Kostiakov function (KF), segmented Kostiakov function (SKF), Gaussian function (GF), segmented Gaussian function (SGF), beta function (BF) and segmented beta function (SBF) for representing IR in hydrophilic and water-repellent soils; (2) construct a mathematical model to represent the water infiltration processes of both hydrophilic and water-repellent soils; (3) analyze the impact of soil hydrophobicity on water infiltration; (4) identify the differences and relationships between the BF model and classical water infiltration models.

2. Materials and Methods

2.1. Soil Sample Collection
The present study selected two soil samples (figure 1) of differing textures. The first soil sample representing hydrophilic soil had a low humus content and was collected from silt loam soil at a depth > 50 cm in an undisturbed flat area. The second soil sample representing water-repellent soil had a high humus content and was collected from forest topsoil in the Guishui River Basin China (115°44′–
116°21′E and 40°19′–40°38′N). Soil particle diameters (SPDs) of the samples were measured using a Master-Sizer 2000 laser sizer whereas organic matter contents (OMCs) were determined using the method proposed by Walkley and Black [19]. Soil pH was determined by the electrometric method [20] and soil bulk density (SBD) was determined using the oven drying method [21]. Soil saturated hydraulic conductivity (Ks) was determined using an UMS KSAT device. After drying of the soil samples in a controlled atmosphere of 25 °C, the average water drop penetration times (WDPTs) of seven replicates tested with distilled water were used to assess SWR [22]. Table 1 summarizes the soil physical and chemical indices (PCI) of the two soil samples.

### Table 1. Soil particle diameter (SPD), physical and chemical indices [17]

| SPD (mm)       | 0–0.005 | 0.005–0.01 | 0.01–0.05 | 0.05–0.1 | 0.1–2  |
|---------------|---------|------------|-----------|----------|--------|
| Hydrophilic soil | 17.47%  | 8.95%      | 45.54%    | 20.54%   | 7.50%  |
| Water-repellent soil | 3.54%  | 3.26%      | 28.24%    | 35.00%   | 29.96% |

| PCI | OMC (mg kg⁻¹) | pH | SBD (kg m⁻³) | Ks (cm h⁻¹) | WDPT (s) |
|-----|---------------|----|--------------|-------------|----------|
| Hydrophilic soil | 9584 | 7.834 | 1367 | 3.084 | 0.803 |
| Water-repellent soil | 51982 | 8.473 | 874 | 7.752 | 69.482 |

*Abbreviations: OMC- organic matter content*

### 2.2. Experimental Apparatus

Both soil samples were passed through a 2 mm sieve before the start of the experiment to remove impurities. The hydrophilic and water-repellent soil samples were then packed into three poly(methyl methacrylate) (PMMA) boxes with a diameter and height of 20 cm and 100 cm, respectively. The measured SBDs of the hydrophilic and water-repellent soils were 1,367 kg m⁻³ and 874 kg cm⁻³, respectively, similar to measured field values. The initial soil water contents (ISWCs) of the hydrophilic
soil in the three PMMA boxes were 4%, 8% and 12%, respectively, whereas they were 4.7%, 6.2% and 9.6%, respectively for the hydrophilic soil.

2.3. Experimental Test Device

An automatic water infiltration meter (figure 2) was devised in the present study to measure infiltrating water. The test device consisted of a scaffold with four movable universal wheels, a Mariotte bottle used for waterhead adjustment (#2 in figure 2), a PMMA box (#13 in figure 2), an adjustable card trough (#06 in figure 2) for adjustment of the water head differences between the surface of the soil (#14 in figure 2) and the air outlet of a thick tube placed within a Mariotte bottle with the air intake of tube (#01 in figure 2) connected to the outside air, a spoke type weighing transducer (#05 in figure 2), a data collector and a signal amplifier (#07 in figure 2). In the present study, the time step of the data collector was 2 s and the water head was 1 cm.

2.4. Methodology

2.4.1. KF/SKF Infiltration Model. As the simple form of the Kostiakov[23] infiltration model, this model is widely applied to represent infiltration of hydrophilic soil. The IR of the Kostiakov infiltration model[24] is expressed as:

\[ i(t) = A \rho t^{\rho - 1} \]  \hspace{1cm} (1)

The IR of the SKF model can be expressed as follows:

\[ i(t) = \begin{cases} A_f \rho_f t^{\rho_f - 1} & 0 < t < t_m \\ A_b \rho_b t^{\rho_b - 1} & t \geq t_m \end{cases} \] \hspace{1cm} (2)

In equation (1) and equation (2), \( t \) is infiltration time (min), \( i(t) \) is the IR at time \( t \)(cm min\(^{-1}\)), \( A \) and \( \rho \) are experiential coefficients without physical significance, \( A_f \) and \( \rho_f \) are parameters applicable pre-turning point , \( A_b \) and \( \rho_b \) are parameters applicable post-turning point.

2.4.2. GF/SGF Infiltration Model. The Gaussian distribution was first proposed by Gauss [25]. In the present study, the GF and SGF infiltration models were used to calculate IR. The forms of the GF and SGF [12, 17]are given by:

\[ i(t) = k \cdot \exp \left( -\left( \frac{t-t_p}{\delta} \right)^2 \right) + \psi \]  \hspace{1cm} (3)
SGF: 
\[
i(t) = \begin{cases} 
    k_f \cdot \exp\left(-\frac{(t-t_p)^2}{\delta_f^2}\right) + \psi_f, & 0 < t < t_m \\
    k_b \cdot \exp\left(-\frac{(t-t_p)^2}{\delta_p^2}\right) + \psi_b, & t \geq t_m
\end{cases}
\]  
(4)

In equation (3) and equation (4), \( k \) is a coefficient, \( t_p \) is the time at the turning point, \( \delta \) is the standard deviation, \( \psi \) is a constant, \( k_f, t_p, \delta_f \) and \( \psi_f \) are parameters of the SGF pre-turning point and \( k_b, t_b, \delta_b \) and \( \psi_b \) are the parameters of the SGF post-turning point.

2.4.3. BF/SBF infiltration model. The beta distribution is a set of continuous probability distributions defined on the interval [0,1] containing two positive real parameters \( p, q \) [26]. Figure 3 shows the shape properties of the probability density function (PDF).

Under \( 0 < a < 1 \) and \( 0 < b < 1 \), the PDF distribution follows an asymmetric U-shaped curve. For any given \( \beta < \alpha \), kurtosis is an increasing function of \( \alpha \). Similarly, for a fixed \( a < b \), kurtosis is a decreasing function of \( \beta \).

Under \( \alpha > \beta > 1 \), the PDF distribution is skewed to the left, and as \( q \) increases, the degree of right skewness increases. Under \( \beta > \alpha > 1 \), the PDF distribution is skewed to the right, and as \( p \) increases, the degree of right skewness increases.

When \( \alpha = \beta > 1 \), the PDF distribution follows a symmetric normal distribution curve. Under \( \alpha < 1 \), \( \beta < 1 \) and \( \alpha = \beta \), the distribution is symmetric and platykurtic. With decreasing \( p \) and \( q \), the tail becomes heavier until bimodality occurs.

The gamma function requires introduction before the BF infiltration model can be deduced. Euler first proposed the gamma function in 1729, which can be used to define positive integers and can be extended to other values [27]. The classical gamma function is given by:

\[
\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt
\]  
(5)

The BF [28] in the interval \( (0, \infty) \) is given by:

\[
B(p, q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx
\]  
(6)

Equation (6) can be rewritten as the following segmented integral function in the intervals [0,1] and \([1, +\infty]\):

\[
B(\alpha, \beta) = \int_0^1 \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} dx + \int_1^\infty x^{\alpha-1} \frac{1}{(1+x)^{\alpha+\beta}} dx
\]  
(7)

The present study defined \( x = \mu^{-1} \) and transformed the integral in equation (7) from the interval \([1, +\infty]\). Thus, equation (7) can be converted to the following BF in the interval [0,1]:

\[
B(\alpha, \beta) = \int_0^1 x^{\alpha-1} \cdot (1-x)^{\beta-1} dx
\]  
(8)

When \( \alpha > 0 \) and \( \beta > 0 \), using equation (8), the following can be obtained:
\[ \frac{\Gamma(\alpha)}{e^a} = a^a \int_0^{+\infty} y^{a-1} e^{-ay} dy = \int_0^{+\infty} y^{a-1} e^{-ty} dy \]  

(9)

By substituting equation (5) into equation (9), the following can be obtained:

\[ \Gamma(\alpha)\Gamma(\beta) = \int_0^{+\infty} x^{a-1} e^{-dx} \int_0^{+\infty} y^{\beta-1} e^{-y} dy \]  

(10)

By defining \( x = ty \) and substituting into equation (10), the following can be obtained:

\[ \Gamma(\alpha)\Gamma(\beta) = \int_0^{+\infty} (ty)^{a-1} e^{-ty} dy \int_0^{+\infty} y^{\beta-1} e^{-y} dy = \int_0^{+\infty} t^{a-1} dt \int_0^{+\infty} y^{a+\beta-1} e^{-y(1+t)} dy \]

\[ = \int_0^{+\infty} \left( \frac{a-1}{(1+t)^{a+\beta}} \right) dt = \int_0^{+\infty} (1+t)y^{a+\beta-1} e^{-y(1+t)} d(1+t) y = B(\alpha, \beta) \]  

(11)

The PDF of equation (12) is:

\[ f(t|\alpha, \beta) = B(\alpha, \beta) t^{\alpha-1} (1-t)^{\beta-1} \]  

(13)

By assigning the coefficient \( \mu \) and the constant \( \varphi \) to equation (13) and defining \( i(t) = f(t|\alpha, \beta) \), the IR of water-repellent soil is given by:

\[ i(t) = \mu \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha-1} (1-t)^{\beta-1} + \varphi \]  

(14)

Although the beta distribution is defined on the interval \([0, 1]\) [26], water infiltration often extends over a long time \((t > 1)\). By defining \( t_s = t/t_e \) to normalize the \( t \) in the range \([0, 1]\), equation (14) can be written as:

\[ i(t_s) = \mu \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} t_s^{\alpha-1} (1-t_s)^{\beta-1} + \varphi \]  

(15)

Hence, the IR of the SBF can be obtained as:

\[ i(t_s) = \begin{cases} 
\mu_f \frac{\Gamma(\alpha_f)\Gamma(\beta_f)}{\Gamma(\alpha_f+\beta_f)} t_s^{\alpha_f-1} (1-t_s)^{\beta_f-1} + \varphi_f, & 0 < t < t_m \\
\mu_b \frac{\Gamma(\alpha_b)\Gamma(\beta_b)}{\Gamma(\alpha_b+\beta_b)} t_s^{\alpha_b-1} (1-t_s)^{\beta_b-1} + \varphi_b, & t \geq t_m 
\end{cases} \]  

(16)

In equation (16), \( t_e \) is the time taken for termination of the experiment (min), \( t_s \) is the normalization time, \( \alpha \) and \( \beta \) are shape parameters characterizing the kurtosis, skewness and bimodality of the distribution, \( \mu \) is a coefficient, \( \varphi \) is a constant, \( \mu_f, \alpha_f, \beta_f \), and \( \varphi_f \) are parameters applicable pre-turning point and \( \mu_b, \alpha_b, \beta_b \) and \( \varphi_b \) are parameters applicable post-turning point.

3. Results and Discussion

3.1. CI and IR Changes over Time

ISWC affects soil matric potential [29], with matric potential decreasing with increasing ISWC. consequently, there is also a decrease in the forces associated with the hydraulic gradient. Hence, it is believed that IR decreases over time, primarily due to the decrease in the potential gradient, and the drier the soil and lower the ISWC, the greater the IR. However, figure 4 a and b show that an increase in ISWC results in an increase in CI in both hydrophilic and water-repellent soil. In other words, the greater the ISWC, the more rapid the infiltration of water into the soil. While this conclusion contradicts established assumptions, it is supported by the findings of Wangemann et al. [30] who showed that IR
corresponds positively to ISWC. Wangemann et al. [30] concluded that the decline in infiltration rates at all soil water contents is due to constricted air being squeezed out of tiny pores through absorption of water into larger conducting pores. Since dryer soil would experience greater displacement of air, this displaced air can result in the blockage of conducting pores. The difference between hydrophilic and water-repellent soil is that the increase in CI over time shows a smooth curve for the former (figure 4a), whereas a two-stage growth curve (ISWC = 6.2% and ISWC = 9.6%) or a more variable increase (ISWC = 4.7%) occurs in the latter. The above observations for water-repellent soil are due to SWR having a dramatic impact on CI. This is consistent with the results of a previous study[18] where CI showed a turning point and demonstrated growth of a smooth monotonic nature at the tipping point (figure 4b).

Figure 5(a) shows that the IR of hydrophilic soil generally decreased monotonically over time due to the effect of the capillary-driven part of the hydraulic gradient that developed as the distance of the wetting front from the surface increased [31], following which the IR stabilized. In contrast, the IR of water-repellent soil (figure 5b) exhibited a slow decline in its primary stage (0 min–20 min), following which it transformed to curve with a single peak that increased before declining. Figure 5a also shows that the turning points of the higher ISWC values (ISWC =6.2% and ISWC =9.6%, ≅ 310 min) appeared earlier than that of the smaller ISWC (ISWC =4.7%, ≅ 568 min). Under all three ISWC conditions, the wave crests of the IR were relatively narrow, the process leading up to the maximum IR was more dramatic and the IR decay process exhibited a delayed post-peak curve. In other words, the IR presented a curve with an asymmetric single peak with a distribution skewed to the left. In addition, the stable IR post-peak was larger than that of pre-peak. The pre- and post-peaks of the infiltration curve were consistent with the soil affected by hydrophobicity and the same soil after dissipation of repellency [13]. The delay in the infiltration pre-turning point could be due to hydrophobicity preventing the infiltration of water into the soil, resulting in a reduced IR. The delayed IR of pre-peak could be due to the impact on the matric potential by the hydrophobic material, preventing pores in which capillary strength was overcome by hydrophobicity from participating in the infiltration process. Hydrophobicity lowers matric potential and consequently the volume of water-conductive pores, thereby facilitating a decline in IR. The soil pores were shown to start participating in infiltration as hydrophobicity dissipated, which facilitated an increase in IR [18]. Vogelmann et al. [13] concluded that the hydraulic gradient is predominantly a function of the soil matric potential in unsaturated flow, that SWR reduces matric potential and that there is a reduction in the volume of water-conductive pores, thereby facilitating a decline in IR. The repellency effect gradually dissipated after the wetting of hydrophobic compounds and extended contact with water [32], thereby facilitating an increase in IR as the SWR reduced [22]. Alagna et al. [22] proposed some explanations of the mechanisms of increasing IR, which were generally the same as those stated in terms of CA. Zhao et al. [17] proposed three reasons why CI shows a turning point and why IR shows a single peak curve, namely the dissipation of SWR, soil volume expansion and the development of plant roots and increase in soil temperature. The dissipation of SWR can result in an increase in the water adsorbing capacity of soil particles, expansion of soil volume can increase soil porosity which will facilitate easier transport of water in soil, and during abundant water at the soil surface, the size of the conducting macropore is the factor having the greatest effect on water flow and gravity is the dominant driver [33]. The development of plant roots leads to faster soil particle absorption of water, and an increase in soil temperatures decreases the water viscous coefficient, thereby reducing the resistance of soil to water. In other words, the mutation phenomena of CI (double-slope shapes) and IR (single peak curve) result from the physical structure of soil in combination with biological and chemical factors.
3.2. Statistical Regression Analysis

Eqs. (1), (3) and (15) were used to calculate the IR of the hydrophilic soil and Eqs. (2), (4) and (16) were used to calculate the IR of the water-repellent soil. The determination coefficient ($R^2$) and root mean square error (RMSE) were applied as a measure of the applicability of the model, with the larger and smaller the $R^2$ and RMSE, respectively, the higher the fitting precision. For the three models, table 2 and table 3 show the $R^2$ and RMSE between simulated and observed IR and figure 7–10 show a visual comparison of the observed and simulated values.

Table 2 summarizing the model fitting precision for hydrophilic soil shows that the average $R^2$ values for the BF, KF and GF models were 0.946, 0.931 and 0.787, respectively, whereas the average RMSE values were 0.0164, 0.0144 and 0.0138, respectively. These results illustrate that the KF and GF models obtained relatively good fitting accuracy for hydrophilic soil compared to that of the BF model.

Table 3 summarizing the model fitting precision for water-repellent soil shows that both the SKF and SBF models achieved relatively high average $R^2$ values between simulated and measured IR of 0.659 and 0.657, respectively compared to that of the SGF model at 0.606. The average RMSE values of the SGF, SBF and SKF models were 0.0185, 0.0174 and 0.0151, respectively. These results show that the simulations of IR by the SKF and SBF models provided a relatively good fit to measured values for water-repellent soil compared to that of the SGF model.

The time taken for the turning point in CI to appear is known as the “hydrophobicity cessation time”.

Moret-Fernández et al. [31] calculated hydrophobicity cessation time using the method of point of intersection of two straight lines, which represents the $I = \sqrt{t}$ relationships between hydrophobic and nearly wettable states. Table 3 (row 6–9, columns 3 and 4) shows that the value of the parameter $t_m$ of the GF model matches exactly with the infiltration time at the turning point (581 min, 332 min and 326 min). In other words, the GF model provides an additional method to measure hydrophobicity cessation.
Table 2. The infiltration rate (IR) fitting equations and beta function (BF), determination coefficient ($R^2$) and root mean square error (RMSE) of hydrophilic soil.

| Model | ISWC | The infiltration rate fitting formula | $R^2$ | RMSE  |
|-------|------|---------------------------------------|-------|-------|
| KF    | 4%   | $i(t) = 0.0804t^{-0.2805}$             | 0.9281| 0.00639|
|       | 8%   | $i(t) = 0.2121t^{-0.4149}$             | 0.9246| 0.01253|
|       | 12%  | $i(t) = 0.5920t^{-0.5843}$             | 0.9408| 0.02236|
|       | 4%   | $i(t) = 0.0695 \cdot \exp\left(-\frac{t + 864}{1027}\right) + 0.0065$ | 0.8251| 0.00700|
| GF    | 8%   | $i(t) = 0.3069 \cdot \exp\left(-\frac{t + 817}{1510}\right) + 0.0087$ | 0.7694| 0.01273|
|       | 12%  | $i(t) = 0.0444 \cdot \exp\left(-\frac{t + 737}{1138}\right) + 0.0091$ | 0.7672| 0.02345|
| BF    | 4%   | $i(t) = 0.1842 \cdot \frac{\Gamma(0.35)\Gamma(1.2)}{\Gamma(1.55)} \frac{t}{800}^{-0.55} \left(1 - \frac{t}{800}\right)^{-2} + 0.0107$ | 0.9176| 0.00712|
|       | 8%   | $i(t) = 0.3227 \cdot \frac{\Gamma(0.45)\Gamma(1.2)}{\Gamma(1.65)} \frac{t}{800}^{-0.55} \left(1 - \frac{t}{800}\right)^{-2} + 0.0053$ | 0.9637| 0.01711|
|       | 12%  | $i(t) = 0.4796 \cdot \frac{\Gamma(0.55)\Gamma(1.2)}{\Gamma(1.75)} \frac{t}{800}^{-0.45} \left(1 - \frac{t}{800}\right)^{-2} + 0.0012$ | 0.9580| 0.02486|

Abbreviations: KF- Kostiakov function; GF- Gaussian function; ISWC- initial soil water content

Table 3. The infiltration rate (IR) equations, determination coefficient ($R^2$) and root mean square error (RMSE) of water-repellent soil.

| Model | ISWC | The infiltration rate fitting formula | $R^2$ | RMSE  |
|-------|------|---------------------------------------|-------|-------|
| SKF   | 4.7% | $i(t) = 0.0440t^{-0.228}$             | 0.458 | 0.00740|
|       | 6.2% | $i(t) = 0.0012t^{-0.997}$             | 0.799 | 0.01419|
|       | 9.6% | $i(t) = 0.0059t^{-0.911}$             | 0.742 | 0.02369|
| SBF   | 4.7% | $i(t) = 0.0475 \cdot \exp\left(-\frac{t + 581}{127}\right) + 0.0052$ | 0.536 | 0.01079|
|       | 6.2% | $i(t) = 0.0736 \cdot \exp\left(-\frac{t + 332}{47}\right) + 0.0048$ | 0.764 | 0.01697|
|       | 9.6% | $i(t) = 0.1317 \cdot \exp\left(-\frac{t + 326}{60}\right) + 0.0057$ | 0.517 | 0.02780|
| SGF   | 4.7% | $i(t) = 0.0062 \cdot \frac{\Gamma(27/10)\Gamma(11)}{\Gamma(27/10)} \frac{t}{792}^{-1/10} \left(1 - \frac{t}{792}\right)^{-1/10} + 0.0047$ | 0.669 | 0.01034|
|       | 6.2% | $i(t) = 0.0099 \cdot \frac{\Gamma(0.55)\Gamma(0.55)}{\Gamma(0.55)} \frac{t}{792}^{-0.55} \left(1 - \frac{t}{792}\right)^{-0.55} + 0.0017$ | 0.746 | 0.01531|
|       | 9.6% | $i(t) = 0.0241 \cdot \frac{\Gamma(0.10)\Gamma(0.10)}{\Gamma(0.10)} \frac{t}{792}^{-0.55} \left(1 - \frac{t}{792}\right)^{-0.55} + 0.0022$ | 0.616 | 0.02652|

Abbreviations: SKF- segmented Kostiakov function; SBF- segmented beta function; SGF- segmented Gaussian function; ISWC- initial soil water content

3.3. Mechanism of Infiltration in Water-Repellent Soil

The infiltration process in hydrophilic soil can be categorized into three stages: (1) rapid infiltration; (2) a gradual decline in infiltration and; (3) stabilization of infiltration. Figure 5b suggests that the process of infiltration in water-repellent soil should be divided into five stages as shown in figure 6 in which the area under the curve is divided into 5 sections representing each stage. During Section I, there is a decrease in IR during which there is a localized IR maximum at $i_{start}$ in which the air pressure of soil is closer to zero at the starting point. Infiltration shows a stage of temporary stabilization pre-turning point during Section II. During this stage, a dynamic balance between the deep soil layer air-resistance, the water gravitational and matric potential and the repulsive force of hydrophobicity exists. The minimum IR $i_{bottom}$ appears during this stage as IR gradually decreases due to the compact layer.
During Section III, there is an increase in IR, possibly due to a gradual decrease in SWR. The maximum IR $i_{peak}$ appears when the critical value for the dissipation of SWR is exceeded by the soil water content. During Section IV, there is a decrease in IR during which infiltration is controlled by the pressure from the head gradient and gravity, and the IR declines as the wetting front expands. There is a decline in the water pressure gradient across the transmission zone due to a dynamic balance between soil air pressure and the capillary suction at the wetting front. Declines in both the pressure gradient and hydraulic conductivity result in a reduction in IR. During Section V, there is a stabilization of the infiltration process post-turning point and the IR reaches the $i_{stable}$ value. This phenomenon may be due free water gravitational potential in the upper saturated soil layer overwhelming the air resistance of soil. The curve of the IR of water-repellent soil shows a right-skewed single peak when ignoring the rapid water infiltration process at the initial section.

As shown in figure 7, the simulations of IR by the GF model for hydrophilic soil overestimated measured values during the early stage but were representative of observed values in the later stage. Furthermore, the simulations of IR by the KF and BF models showed good fits to observed data during both the early and late stages.

As shown in figure 8, the application of the SKF model to water-repellent soil generated a simultaneous minimum and maximum simulated IR at the turning point during which the mutation in IR was almost instantaneous. However, the dissipation of SWR is in fact a slow process since SWR is affected by the ISWC [32]. As there is an increase in soil water content and once SWR completely dissipates, the participation of all soil pores in the infiltration process is initiated, producing an increase in IR [18].

As shown in figure 9 the simulations of IR by the SGF model for water-repellent soil first increased and then decreased with a single peak. The SGF model was able to mimic the observed single-peak process of an increase followed by a decrease in IR (Sections III to IV of figure 6), and was also able to represent the larger observed stable IR post-turning point (Section V of figure 6) compared to the pre-turning point (Section II of figure 6). However, the model struggled to reflect the gradual decrease in IR in the initial section (Section I of figure 6). These results show that the simulations by the SGF model will better represent observed data under a situation in which the gradual decrease in IR during the initial section (Section I of figure 6) occurs over a shorter duration or if the IR during the initial section can be ignored.

As shown in figure 10, simulations by the SBF model not only reflected the change in the single-peak curve (Sections III to IV of figure 6), but also was able to represent the observed greater stable IR post-turning point (Section V of figure 6) compared to that of the pre-turning point (Section II of figure 6). The SBF model was also able to represent the observed gradual decrease in IR during initial water infiltration (Section I of figure 6).

In conclusion, when applied to water-repellent soil, The SBF model was not only able to reflect the simple decline in IR ($0 < \alpha \leq 1, \beta > 1$) and the U-shaped change of IR pre- and post-turning point, but was also able to recreate the right-skewed distribution curve of IR pre- and post-turning point, thereby indicating good applicability of the model.
Figure 7. The infiltration rate (IR) of hydrophilic soil as simulated by the Kostiakov function (KF), Gaussian function (GF) and beta function (BF). (a) initial soil water content (ISWC) = 4%, (b) ISWC = 8% and (c) ISWC = 12%.

Figure 8. The infiltration rate (IR) of water-repellent soil was calculated by the segmented Kostiakov function (SKF). (a) initial soil water content (ISWC) = 4.7%, (b) ISWC = 6.2% and (c) ISWC = 9.6%.
Figure 9. The infiltration rate (IR) of water-repellent soil was calculated by the segmented Gaussian function (SGF). (a) initial soil water content (ISWC) = 4.7%, (b) ISWC = 6.2% and (c) ISWC = 9.6%.

Figure 10. The infiltration rate (IR) of water-repellent soil was calculated by the segmented beta function (SBF). (a) initial soil water content (ISWC) = 4.7%, (b) ISWC = 6.2% and (c) ISWC = 9.6%.

3.4. Model Comparison
As demonstrated in Section 2.4.2, the BF parameters $\alpha$ and $\beta$ can affect the skewness, kurtosis and bimodality of the curve shape. This sub-section describes the assignment of different values to the parameters $\alpha$ and $\beta$ so as to identify the relationship and difference between the traditional water infiltration model and the BF model.

When $\alpha = 1, \Gamma(1) = 1$, equation (15) can be rewritten as:

$$i(t) = \mu \frac{\Gamma(\beta)}{\Gamma(1+\beta)} (1 - t\varsigma)^{\beta - 1} + \varphi$$  (17)
Given the properties of BF [26], \( \Gamma(1 + \beta) = \beta \Gamma(\beta) \), equation (17) can be simplified as:

\[
i(t) = \frac{\mu}{\beta} (1 - t_s)^{\beta - 1} + \phi
\]  
(18)

When \( \beta > 1 \), equation (18) is a monotonically increasing function, and when \( 0 < \beta < 1 \), Eq (17) is a descending function and \( i(t) > 0 \), which can be used to fit the IR of hydrophilic soil. If \( \beta < 0 \), equation (18) is a descending function and \( i(t) > 0 \).

When \( \beta = 1, \Gamma(1) = 1 \) and \( (1 - t_s)^{\beta - 1} = 1 \), the following can be derived:

\[
i(t) = \mu \frac{\Gamma(\beta)}{\Gamma(\beta + 1)} t_s^{\beta - 1} + \phi
\]  
(19)

As \( \Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \), equation (19) can be expressed as:

\[
i(t) = \frac{\mu}{\alpha} t_s^{\alpha - 1} + \phi
\]  
(20)

By comparing equation (20) and Kostiakov’s model [34], the results of the present study found that the coefficient \( \mu/\alpha \) was equivalent to Kostiakov’s model coefficient. In other words, \( \mu/\alpha = \text{sol} \), which is similar to Kostiakov’s infiltration model. When \( 0 < \alpha < 1 \), equation (20) is a monotonically decreasing function, which can be used to fit the IR of hydrophilic soil. When \( \alpha > 1 \), equation (20) is a monotonically increasing function. When \( \alpha = 1 \), equation (19) is a constant \( \phi \).

When \( \alpha = 0.5 \), equation (20) can be used to derive:

\[
i(t) = 2\mu t^{-0.5} + \phi
\]  
(21)

The form of equation (21) was found to be more similar to the power function of the Philip’s [35] water infiltration model, with the values of the coefficients in the Philip’s model [35] and equation (21) being \( 0.5 \) and \( 2\mu \), respectively, and both showing a decreasing function.

4. Conclusions

The present study conducted an indoors one-dimensional vertical water infiltration experiment for hydrophilic and water-repellent soil under controlled laboratory conditions. The CI and IR of hydrophilic soil showed a monotonous increase and a monotonous decrease over time, respectively. Unlike that in hydrophilic soil, CI in water-repellent soil featuring two stages showed an overall growth and IR showed a mutation characteristic. IR in water-repellent soil showed a single peak when neglecting the initial infiltration process, and the stabilized IR value post-peak was higher than that of the pre-peak in water-repellent soil.

(1) The currently study found that the water infiltration process of water-repellent soil could be divided into five stages, which could be represented as sections of the area under the curve in figure 6. Section I showed a decrease in IR with a local maximum in IR at the start. Section II showed a temporary period of stable infiltration pre-turning point during which the minimum IR occurred. Section III showed an increase in IR, possibly due to the gradual decrease in SWR, and the maximum IR occurred at the turning point. Section IV showed an increase in IR, possibly due to the decrease in pressure from both the gradient and hydraulic conductivity. Section V showed a stabilization of the infiltration process post-turning point due to the gravitational potential of free water in the upper saturated soil layer, which overwhelmed the soil air-resistance. If the initial section of rapid water infiltration is ignored, the IR of the water-repellent soil presented as a right-skewed single-peak curve.

(2) The BF model was not only able to describe the monotonous decrease in IR in hydrophilic soil during which model parameters \( 0 < \alpha < 1 \), or \( \beta = 1 \) and \( 0 < \beta < 1 \), but was also able to reflect the increase in IR of water-repellent soil followed by a decrease with a single peak when \( \alpha > 1 \) and \( \beta > 1 \). The BF/SBF models achieved the highest simulation accuracies among the models assessed and also had the widest applicability.
Acknowledgments
This project was funded by the National Natural Science Foundation of China (No. 51909116), State Key Laboratory of Simulation and Regulation of Water Cycle in River Basin, China Institute of Water Resources and Hydropower Research (IWHR Key Laboratory of Simulation and Regulation of Water Cycle in River Basin, China Institute of Water Resources and Hydropower Research) National Key Research and Development Program of China (No. 2016YFC0401407), and the Natural Science Foundation of Jiangxi Province, China (No.20202BABL204068).

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