The Berry phase in inflationary cosmology

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Abstract
We derive an analogue of the Berry phase associated with inflationary cosmological perturbations of a quantum-mechanical origin by obtaining the corresponding wavefunction. We have further shown that the cosmological Berry phase can be completely envisioned through the observable parameters, namely spectral indices. Finally, the physical significance of this phase is discussed from the point of view of theoretical and observational aspects with some possible consequences of this quantity in inflationary cosmology.

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1. Introduction

Since its inception, the Berry phase has drawn a lot of attention in physics community. Although the geometric phase was known long ago in the manner of Aharonov–Bohm effect, the general context of a quantum-mechanical state developing adiabatically in time under a slowly varying parameter-dependent Hamiltonian has been analyzed by Berry [1], who argued that when the parameters return adiabatically to their initial values after traversing a closed path, the wavefunction acquires a geometric phase factor depending on the path, in addition to the well-known dynamical phase factor. The existence of a geometric phase in an adiabatic evolution is not only confined to the quantum phenomenon, but the classical analogue of it also exists and is referred to as the Hannay angle [2]. Berry established a semi-classical relation between the quantum and classical geometric phases in adiabatic evolution [3]. The Berry phase has been the subject of a variety of theoretical and experimental investigations [4]; possible applications range from quantum optics and molecular physics to fundamental quantum mechanics and quantum computation. Some analyses have been made to study this phase in the area of cosmology and gravitation also. In particular, the Berry phase has been calculated in the context of relic gravitons [5]. In [6], a covariant generalization of the Berry phase has been obtained. Investigations were also made to study the behavior of a scalar particle in a class of stationary spacetime backgrounds and the emergence of Berry phases in
the dynamics of a particle in the presence of a rotating cosmic string [7]. The gravitational analogue of the Aharonov–Bohm effect in the spinning cosmic string spacetime background was also obtained [8]. Within a typical framework of the cosmological model, the Berry phase has been shown to be associated with the decay width of the state in the case of some well-known examples of vacuum instability [9].

The inflationary scenario [10]—so far the most physically motivated paradigm for early universe —is also in vogue for quite some time now. Among other motivations, the inflationary scenario is successful to a great extent in explaining the origin of the cosmological perturbation seeds [11]. The accelerated expansion converts the initial vacuum quantum fluctuations into macroscopic cosmological perturbations. So, the measurement of any quantum property which reflects on classical observables serves as a supplementary probe of inflationary cosmology, complementing the well-known cosmic microwave background (CMB) polarization measurements [12, 13]. Although this is an important issue, we notice that there has been very little study in the literature which deals with proposing measurable quantities which may measure the genuine quantum property of the seeds of classical cosmological perturbations. The only proposition that has drawn our attention is via violation of Bell’s inequality [14]. This has led us to investigate for the potentiality of the Berry phase in providing a measurable quantum property which is inherent in the macroscopic character of classical cosmological perturbations.

The paradigm of inflation provides a well-motivated mechanism for the origin of fluctuations observed in the large-scale structure of the matter and in the CMB. In order to quantify these fluctuations, the theory of general relativistic cosmological perturbations has to be employed. However, in practice, the field equations cannot be solved with their full generality; some approximation has to be used. The linear-order approximation to the cosmological perturbation theory has been developed to a high degree of sophistication during the last few decades [17, 15, 16]. Although attempts have been made to develop the second-order perturbation theory, which provide results with better accuracy. There are discussions on the deviation from the first-order approximation from the observation [18] and theoretical approaches [19] through the non-Gaussianity, non-adiabaticity and so on. There are pieces of work which focus [20] on the development of the second-order gauge-invariant cosmological perturbation technique. Attention has also been paid [21] to investigate the importance of second-order corrections to the linearized cosmological perturbation theory in the inflationary scenario, which suggests that for many parameters of slow-roll inflation, the second-order contributions may dominate over the first-order effects during the super-Hubble evolution. All these recent efforts indicate the importance of the study of the second-order cosmological perturbations, but at a first go, linear-order perturbation theory has remained as a primary tool to investigate the behavior of fluctuations during inflation. The main motivation behind this development was to clarify the relation between the scenarios of the early universe and cosmological data, such as the CMB anisotropies. The developments in the observations were also supported by the theoretical sophistication of the linear-order cosmological perturbation theory. Recently, the first-order approximation of our Universe from a homogeneous isotropic one has also been revealed through the observation of the CMB by the Wilkinson Microwave Anisotropy Probe (WMAP) [12]. At this juncture, we also recall that our objective here is to see whether a link can be established between the quantum property of the seeds of classical cosmological perturbations and inflationary cosmological parameters through the derivation of the associated Berry phase, for which the consideration of the sub-Hubble modes is appropriate. Keeping all these points under consideration, we concentrate on the evolution of the sub-Hubble perturbation modes within the framework of the linearized perturbation theory.
In this paper, our primary purpose is to demonstrate the effect of the curved spacetime background in the dynamical evolution of the quantum fluctuations during inflation through the derivation of the associated Berry phase and search for the possible consequences via observable parameters. The quantum fluctuations in inflaton are realized by the Mukhanov–Sasaki equation which is analogous to the time-dependent harmonic oscillator equation. The associated physical mechanism for cosmological perturbations can be reduced to the quantization of a parametric oscillator leading to particle creation due to the interaction with the gravitational field and may be termed as the cosmological Schwinger effect [22]. The relation between the Berry phase and the Hannay angle has been studied for the generalized time-dependent harmonic oscillator [23]. This relation is also extended [24] from an adiabatic to a non-adiabatic time-dependent harmonic oscillator. Stimulated by these, one may expect to derive the cosmological analogue of the Berry phase in the context of inflationary perturbations and search for possible consequences via observable parameters. To this end, we first find an exact wavefunction for the system of inflationary cosmological perturbation by solving the associated Schrödinger equation. The relation [25, 26] between the dynamical invariant [28, 29, 27] and the geometric phase has then been utilized to derive the corresponding Berry phase. For slow-roll inflation, the total accumulated phase gained by each of the modes during sub-Hubble oscillations (adiabatic limit) is found to be a new parameter made of corresponding (scalar and tensor) spectral indices. So, in principle, the measurement of the Berry phase of the quantum cosmological perturbations provides us an indirect route to estimate spectral indices and other observable parameters therefrom. Furthermore, since the tensor spectral index is related to the tensor-to-scalar amplitude ratio through the consistency relation, the Berry phase can indeed be utilized to act as a supplementary probe of inflationary cosmology.

2. Linear cosmological perturbation as a time-dependent harmonic oscillator

In cosmology, inhomogeneities grow because of the attractive nature of gravity. So inhomogeneities were smaller in the past. Since we are interested in inflationary cosmological perturbations, inhomogeneities can be treated as linear perturbations around the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker universe, the metric of which can be written as

$$ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij} dx^i dx^j].$$

During inflation, the energy density of the universe was dominated by the potential energy, $V$, of scalar field(s). Therefore, the total action of the system can be cast into the form of

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R - \int d^4\sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]$$

where $\phi$ is the scalar field driving inflation known as the inflaton field. Also, the most general linear perturbation, without the vector modes, about the homogeneous metric can be expressed as [17]

$$ds^2 = a^2(\eta)[-d\eta^2(1 - 2A) + 2\partial_i B dx^i d\eta + (1 - \psi)\delta_{ij} + 2\partial_i \partial_j E + h_{ij}] dx^i dx^j$$

where the functions $A, B, \psi$ and $E$ represent the scalar sector, whereas the tensor $h_{ij}$, satisfying $h^i_j = 0 = (\delta^{jm} \partial_n)h_{ij}$, represents the gravitational waves. Here, we are interested in cosmological perturbations induced by a single scalar field; as a consequence, there will not be any vector perturbations. At the linear level, scalar and tensor perturbations decouple and can therefore be studied separately.
The scalar fluctuations of geometry can be characterized by a single quantity, with the gauge-invariant Bardeen potential \([17]\) defined by
\[
\Phi_B \equiv A + \frac{1}{a} \frac{d}{d\eta} \left[ a \left( B - \frac{dE}{d\eta} \right) \right].
\]  
(2.4)

The fluctuations in the inflaton field are characterized by the following gauge-invariant quantity:
\[
\delta \phi_{GI} \equiv \delta \phi + \frac{d\phi}{d\eta} \left( B - \frac{dE}{d\eta} \right).
\]  
(2.5)

These two gauge-invariant quantities are coupled through the perturbed Einstein equations, and in the scalar sector, everything can be reduced to the study of a single-gauge-invariant variable defined by \([15]\)
\[
v \equiv a \left[ \delta \phi_{GI} + \frac{d\phi}{d\eta} \Phi_B \right].
\]  
(2.6)

Here, the variable \(v\) is related to the co-moving curvature perturbation \(\mathcal{R}\) via the relation \(v = -\zeta \mathcal{R}\), where \(\zeta \equiv \frac{\dot{\phi} \eta(\eta)}{H(\eta)}\) and \(H \equiv \dot{\phi}/\dot{\phi}\) being the conformal Hubble parameter and the prime (’) denotes derivative w.r.t. conformal time \(\eta\). The action for the scalar perturbation only can then be written as
\[
S_S = \frac{1}{2} \int d\eta d^3 x \left[ v' \right] \left[ v'^2 - \delta^{ij} \partial_i v \partial_j v + \frac{z''}{z} v^2 + M^2 \Pi^2 D \right].
\]  
(2.7)

To arrive at the action (2.7), one needs to expand the action (2.2) up to the second order in the metric perturbations and the scalar field fluctuations and has to use the background Einstein equations.

As an aside, a point to note here is that in our calculations we use flat slicing. To reduce the action for the curvature perturbation to its simplest form, equation (2.7) in terms of a single-gauge-invariant variable \(v\) utilizes the constraint equations obtained by varying the action with respect to the first-order perturbation variables \([16]\). The complete expression for the second-order action is found by taking into account the background and constraint equation obtained by varying the action with respect to \(B - E'\) has the form
\[
S_S = \frac{1}{2} \int d\eta d^3 x \left[ v'^2 - \delta^{ij} \partial_i v \partial_j v + \frac{z''}{z} v^2 + M^2 \right].
\]  
(2.7)

Now promoting the fields to operators and taking the following Fourier decompositions
\[
\hat{\nu}(x, \eta) = \int \frac{d^3 k}{(2\pi)^3} \hat{\nu}_k e^{ikx} \]
\[
\hat{\Pi}(x, \eta) = \int \frac{d^3 k}{(2\pi)^3} \hat{\Pi}_k e^{ikx},
\]  
(2.9)
we found the Hamiltonian density operator corresponding to the above action (2.7) to be
\[ \mathbf{H}_k^s = \frac{1}{2} \left\{ \hat{\mathbf{P}}_{1k}^2 + \left( k^2 - \frac{\alpha''}{\alpha} \right) \hat{\mathbf{a}}_{1k}^2 + \frac{1}{2} \left[ \hat{\mathbf{P}}_{2k}^2 + \left( k^2 - \frac{\alpha''}{\alpha} \right) \hat{\mathbf{a}}_{2k}^2 \right] \right\} \]
\[ = \mathbf{H}_k^s + \mathbf{H}_k^t \]  
(2.10)
where we have decomposed \( \hat{\mathbf{a}}_k \equiv \hat{\mathbf{a}}_{1k} + i\hat{\mathbf{a}}_{2k} \) and \( \hat{\mathbf{P}}_k \equiv \hat{\mathbf{P}}_{1k} + i\hat{\mathbf{P}}_{2k} \) into their real and imaginary parts.

Similarly, considering the tensor perturbation only, we have the corresponding action
\[ S^T = \frac{M_p^2}{2} \int d\eta \, dx \, \frac{\partial^2}{2} \left[ \mathbf{H}^2 - \delta^{ij} \mathbf{a}_j \partial \mathbf{a}_j \right]. \]
(2.11)
By means of the substitution \( u = \frac{M_p^2}{2} \), promoting the fields to operators and taking the Fourier decomposition, the Hamiltonian operator corresponding to the above action (2.11) turns out to be
\[ \mathbf{H}_k^T = \frac{1}{2} \left\{ \hat{\mathbf{P}}_{1k}^2 + \left( k^2 - \frac{\alpha''}{\alpha} \right) \hat{\mathbf{a}}_{1k}^2 + \frac{1}{2} \left[ \hat{\mathbf{P}}_{2k}^2 + \left( k^2 - \frac{\alpha''}{\alpha} \right) \hat{\mathbf{a}}_{2k}^2 \right] \right\} \]
\[ = \mathbf{H}_k^s + \mathbf{H}_k^t \]  
(2.12)
where we have decomposed \( \hat{\mathbf{a}}_k \equiv \hat{\mathbf{a}}_{1k} + i\hat{\mathbf{a}}_{2k} \) and \( \hat{\mathbf{P}}_k \equiv \hat{\mathbf{P}}_{1k} + i\hat{\mathbf{P}}_{2k} \) similarly.

Thus, for both the scalar and tensor modes, the Hamiltonians are the sum of two time-dependent harmonic oscillators, each of them having the following form:
\[ \mathbf{H}_k = \frac{1}{2} \left[ \mathbf{P}_{2k}^2 + \omega^2 \mathbf{a}_{2k}^2 \right] \]
(2.13)
where \( \mathbf{q}_k \equiv \hat{\mathbf{q}}_{1k}, \mathbf{p}_k \equiv \hat{\mathbf{p}}_{1k}, \mathbf{q}_{1k} \equiv \hat{\mathbf{q}}_{2k}, \mathbf{p}_{1k} \equiv \hat{\mathbf{p}}_{2k} \) and \( \omega = \sqrt{k^2 - \frac{\alpha''}{\alpha}}, \sqrt{k^2 - \frac{\alpha''}{\alpha}} \) for scalars and tensors, respectively. One may note here that for the complete solution of the Schroedinger equation for the Hamiltonian (2.13), we have to deal with two situations:
(i) \( k^2 > \frac{\alpha''}{\alpha} \), where \( \omega \) is real and corresponds to the sub-Hubble modes;
(ii) \( k^2 < \frac{\alpha''}{\alpha} \), \( \frac{\alpha''}{\alpha} \) corresponds to the super-Hubble modes with imaginary \( \omega \). For the later case, the Hamiltonian can be rewritten as
\[ \mathbf{H}_k = \frac{1}{2} \left[ \mathbf{p}_{2k}^2 - \omega^2 \mathbf{q}_{2k}^2 \right] \]
(2.14)
which represents an inverted harmonic oscillator with the time-dependent frequency given by \( i\omega = i\sqrt{\frac{\alpha''}{\alpha}} - k^2 \), \( i\sqrt{\frac{\alpha''}{\alpha}} - k^2 \) for scalars and tensors, respectively.

Although in further derivations we are not concerned with the super-Hubble modes, but at this stage, we are ready to provide the solutions for the whole spectrum.

We find the solution to the Schroedinger equation for the Hamiltonians (2.13) and (2.14) using the dynamical invariant operator method [27], i.e., the Lewis–Risenfeld invariant formulation which has now become evident that it can be applied to the treatment of the time-dependent quantum system if a invariant can be found. To be precise, we want to analyze the situation by solving the associated Schroedinger equation
\[ \mathbf{H}_k \Psi = (\mathbf{H}_{1k} + \mathbf{H}_{2k}) \Psi = i \frac{\partial}{\partial \eta} \Psi. \]
(2.15)
In this invariant method, we first look for a nontrivial Hermitian operator \( I_k(\eta) \) satisfying the Liouville–von Neumann equation
\[ \frac{dI_k}{d\eta} = -i [I_k, \mathbf{H}_k] + i \frac{\partial I_k}{\partial \eta} = 0. \]  
(2.16)
Whenever such an invariant operator exists provided it does not contain time-derivative operator, one can write down the solutions of the Schroedinger equation in the following form:

$$\Psi_n = e^{i\alpha_n(\eta)}\Theta_n, \quad n = 0, 1, 2, \ldots$$  \(2.17\)

where \(\Theta_n\) are the eigenfunctions of the operator \(I_k\) and \(\alpha_n(\eta)\) are known as the Lewis phase. Here, \(H_k = H_{1k} + H_{2k}\), and the invariant operator associated with this Hamiltonian can be expressed as

$$I_k(\eta) \equiv I_1(q_{1k}, \eta) + I_2(q_{2k}, \eta).$$  \(2.18\)

In the following, we shall first derive these invariant operators and will find the solution to the Schroedinger equation for two different cases.

2.1. Modes with \(k^2 > \frac{\zeta}{z}, \frac{d^2}{d\eta^2}\)

The modes in this regime, known as sub-Hubble modes, oscillate with real frequencies and the corresponding Hamiltonian has the form \(2.13\). Following the usual technique \(27–31\), we obtain

$$I_k = \frac{1}{2} \left[ q_{1k}^2 + (\rho_k p_{1k} - \rho_k' q_{1k})^2 \right] + \frac{1}{2} \left[ q_{2k}^2 + (\rho_k p_{2k} - \rho_k' q_{2k})^2 \right]$$

$$= I_1 + I_2$$  \(2.19\)

where \(\rho_k\) is a time-dependent real function satisfying the following Milne–Pinney equation:

$$\rho_k'' + \alpha^2(\eta, k)\rho_k = \frac{1}{\rho_k'}(\eta).$$  \(2.20\)

To find the solutions of the Schroedinger equation \(2.15\), we also need to know the eigenstates of the operator \(I_k\) governed by the eigenvalue equation

$$H_k\Theta_{n_{1},n_{2}}(q_{1k}, q_{2k}, \eta) = \lambda_{n_{1},n_{2}}\Theta_{n_{1},n_{2}}(q_{1k}, q_{2k}, \eta).$$  \(2.21\)

The eigenstates of the operator \(I_k\) turn out to be \([5]\)

$$\Theta_{n_{1},n_{2}} = \frac{\tilde{H}_n\left[\frac{q_{1k}}{\rho_k}\right]\tilde{H}_n\left[\frac{q_{2k}}{\rho_k}\right]}{\sqrt{\pi^2 2^n n_{1}! n_{2}! \rho_k^4}} \exp \left[ \frac{i}{2} \left( \frac{\rho_k'}{\rho_k} + \frac{i}{\rho_k^2} \right) (q_{1k}^2 + q_{2k}^2) \right]$$  \(2.22\)

where \(\tilde{H}_n\) are the Hermite polynomials of order \(n\) and the associated eigenvalues are given by

$$\lambda_{n_{1},n_{2}} = \left( n_{1} + \frac{1}{2} \right) + \left( n_{2} + \frac{1}{2} \right).$$  \(2.23\)

The Lewis phase can be found from its definition

$$\frac{d\alpha_{n_{1},n_{2}}}{d\eta} = \left. \Theta_{n_{1},n_{2}} \right| \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta} \tilde{H}_n \Theta_{n_{1},n_{2}}$$  \(2.24\)

resulting in \([29]\)

$$\alpha_{n_{1},n_{2}} = -(n_{1} + n_{2} + 1) \int \frac{d\eta}{\rho_k^2}.$$  \(2.25\)

The eigenstates of the Hamiltonian are now completely known and are given by

$$\Psi_{n_{1},n_{2}} = \frac{e^{i\alpha_{n_{1},n_{2}}(\eta)}\tilde{H}_n\left[\frac{q_{1k}}{\rho_k}\right]\tilde{H}_n\left[\frac{q_{2k}}{\rho_k}\right]}{\sqrt{\pi^2 2^n n_{1}! n_{2}! \rho_k^4}} \exp \left[ \frac{i}{2} \left( \frac{\rho_k'}{\rho_k} + \frac{i}{\rho_k^2} \right) (q_{1k}^2 + q_{2k}^2) \right].$$  \(2.26\)

Thus, one can find the eigenstates of the Hamiltonian for cosmological perturbations for the modes having wavelength less than the horizon, provided it possesses a dynamical invariant containing no time-derivative operation.
2.2. Modes with \( k^2 < \frac{z''}{z}, \frac{a''}{a} \)

Now to find the solution of the Schrödinger equation (2.15) for the modes with \( k^2 < \frac{z''}{z}, a'' a \), we shall make use of the Hamiltonian given by (2.14) with the imaginary frequency \( i\omega_I \) which is nothing but an inverted harmonic oscillator which possesses a continuous energy spectrum. The time-dependent inverted harmonic oscillator is exactly solvable just like the standard time-dependent harmonic oscillator. However, the physics of the time-dependent inverted oscillator is very different \([32–34]\): it has a continuous energy spectrum varying from minus to plus infinity; its energy eigenstates are no longer square-integrable, and they are doubly degenerate with respect to either the incident direction or, alternatively, the parity.

In this case also the invariant operator can be worked out in a similar way as before which turns out to be \([33]\):

\[
I_k = \frac{1}{2} \left[ -\frac{q_1^2}{\rho_k^2} + (\rho_k p_{1k} - \rho_k' q_{1k})^2 \right] + \frac{1}{2} \left[ -\frac{q_2^2}{\rho_k^2} + (\rho_k p_{2k} - \rho_k' q_{2k})^2 \right]
\]

where \( \rho_k \) now satisfies the following auxiliary equation:

\[
\rho_k'' - i\omega^2(\eta, k)\rho_k = -\frac{1}{\rho_k^3} \rho_k.
\]

Now the eigenstates of the operator \( I_k \) are governed by the eigenvalue equation

\[
I_k(\Theta_{\lambda_1, \lambda_2}, q_{1k}, q_{2k}, \eta) = \lambda_{\lambda_1, \lambda_2} I_k(\Theta_{\lambda_1, \lambda_2}, q_{1k}, q_{2k}, \eta).
\]

Following the steps as in \([33, 34]\), the eigenstates of the operator \( I_k \) turn out to be

\[
\Theta_{\lambda_1, \lambda_2} = \frac{1}{\rho_k} \exp \left[ i\frac{\rho_k'}{2\rho_k} (q_{1k}^2 + q_{2k}^2) \right] W_{\lambda_1} \left( \frac{\sqrt{2} q_{1k}}{\rho_k}, \lambda_1 \right) W_{\lambda_2} \left( \frac{\sqrt{2} q_{2k}}{\rho_k}, \lambda_2 \right).
\]

As a result, the solution to the Schrödinger equation (2.15) for the Hamiltonian (2.14) is now completely known and given by

\[
\Psi_{\lambda_1, \lambda_2} = e^{i\alpha_{\lambda_1, \lambda_2}} \Theta_{\lambda_1, \lambda_2}
\]

where \( W_{\lambda_1} \) and \( W_{\lambda_2} \) are the parabolic cylinders or Weber functions and \( \alpha_{\lambda_1, \lambda_2} \) are the phase factors, i.e. the Lewis phases which are given by

\[
\alpha_{\lambda_1, \lambda_2} = - (\lambda_1 + \lambda_2) \int \frac{d\rho}{\rho_k^2}.
\]

The above equation clearly shows that in this framework, the calculation of the Berry phase for the super-Hubble modes is not tractable, since in this case the system does not possess any well-defined ground state. As in this paper, our primary intention is to show the significance of the geometric phase in the context of inflationary cosmological perturbations; from now on, we restrict our attention to the sub-Hubble modes only and try to derive expressions connecting the Berry phase and cosmological observables.

3. Berry phase for the sub-Hubble modes

To calculate the Berry phase for the sub-Hubble modes, we shall first make use of the following identity:

\[
\frac{z''}{z} v^2 = \left( \frac{z'}{z} \right)^2 v^2 - 2 \frac{z'}{z} v v' + \frac{d}{d\eta} \left[ \frac{z'}{z} v^2 \right],
\]

(3.1)
and then the above action (2.7) can be expressed as [35]

\[
S^T = \frac{1}{2} \int d\eta \ dx \left[ \nu^2 - \delta^j_i \partial_j \nu \partial_j \nu - 2 \left( \frac{z'}{z} \right)^2 \nu \nu' + \frac{\nu''}{z} \nu^2 \right] \tag{3.2}
\]

which we find more convenient to work with. Constructing the Hamiltonian, we obtain

\[
H = \frac{1}{2} \int d^3 x \left[ \Pi^2 + \delta^j_i \partial_j \nu \partial_j \nu + 2 \frac{\nu''}{z} \nu^2 \right] \tag{3.3}
\]

where now \( \Pi = \nu' - \frac{z'}{z} \nu \). Now promoting the fields to operators and taking the Fourier decomposition, we find the Hamiltonian density operator corresponding to the above action (3.2) to be

\[
\hat{H}^S_{jk} = \frac{1}{2} \left[ \hat{\Pi}_{jk}^2 + \frac{z'}{z} (\hat{\Pi}_{jk} \hat{v}_{jk} + \hat{v}_{jk} \hat{\Pi}_{jk}) + k^2 \hat{v}_{jk}^2 \right] \tag{3.4}
\]

Similarly, the Hamiltonian operator corresponding to the tensor perturbations is found to be

\[
\hat{H}^T_{jk} = \frac{1}{2} \left[ \hat{\Pi}_{jk}^2 + \frac{z'}{z} (\hat{\Pi}_{jk} \hat{\nu}_{jk} + \hat{\nu}_{jk} \hat{\Pi}_{jk}) + k^2 \hat{\nu}_{jk}^2 \right] \tag{3.5}
\]

In a compact general form, the Hamiltonians can be written as a sum of two generalized time-dependent harmonic oscillators as

\[
\hat{H}_{jk} = \frac{1}{2} \left[ k^2 \hat{q}_{jk}^2 + Y(\eta) \left( \hat{p}_{jk} \hat{q}_{jk} + \hat{q}_{jk} \hat{p}_{jk} \right) + \hat{\rho}_{jk}^2 \right] \tag{3.6}
\]

where \( \hat{q}_{jk} = \hat{v}_{jk} - \hat{\nu}_{jk}; \hat{p}_{jk} = \hat{\Pi}_{jk}, \hat{\nu}_{jk} \) and \( Y = \frac{z'}{z}, \frac{\nu''}{\nu} \) for the scalar and tensor modes, respectively, and \( j = 1, 2 \) with the frequency given by \( \omega = \sqrt{k^2 - Y^2} \).

Following the same trail [28, 29, 27], we find

\[
I_k = \frac{1}{2} \left[ \frac{\hat{q}_{jk}^2}{\rho_k^2} + \left( \rho_k [p_{jk} + Y q_{jk}] - \rho_k' q_{jk} \right)^2 \right] + \frac{1}{2} \left[ \frac{\hat{\rho}_{jk}^2}{\rho_k^2} + \left( \rho_k [p_{jk} + Y q_{jk}] - \rho_k' q_{jk} \right)^2 \right] = I_1 + I_2 \tag{3.7}
\]

where \( \rho_k \) now satisfies the following equation:

\[
\rho_k'' + \Omega^2 \rho_k(\eta, k) = \frac{1}{\rho_k^3(\eta)} \tag{3.8}
\]

with \( \Omega^2 = \omega^2 - \frac{\nu''}{\nu} \).

The eigenstates of the operator \( I_k \) turn out to be

\[
\Theta_{n_1, n_2}^k = \frac{\tilde{H}_n \left[ \frac{\omega_{n_1}}{\rho_k} \right] \tilde{H}_m \left[ \frac{\omega_{n_2}}{\rho_k} \right]}{\sqrt{\pi^2 2^{(n_1+n_2)(n_1+n_2)!} n_1! n_2!}} \exp \left[ -\frac{i}{2} \left( \frac{\rho_k' (\rho_k - Y(\eta)) + \frac{i}{\rho_k} \rho_k''}{\rho_k^3} \right) (\hat{q}_{jk}^2 + \hat{\rho}_{jk}^2) \right]. \tag{3.9}
\]

As a consequence, the eigenstates of the Hamiltonian are now given by

\[
\Psi_{n_1, n_2} = e^{i \omega_{n_1, n_2}(\eta)} \Theta_{n_1, n_2}^k \tag{3.10}
\]
where the Lewis phases are given by (2.25). The phase $\alpha_{\eta, \eta_{\eta}}(\eta)$ is the combination of the dynamical phase and the geometric phase which can be well understood from equation (2.24). Once the Lewis phase is calculated, this can be utilized in deriving the geometric phase associated with the system corresponding to the particle creation through the vacuum quantum fluctuations during inflation.

But before proceeding in this direction, we would like to present the general wavefunction for the vacuum state of the inflationary cosmological perturbations. To this end, let us consider the parametric harmonic oscillator Hamiltonian

$$\hat{H}^S_k = \frac{1}{2} \left[ \hat{\pi}^2_k + \frac{\zeta'}{\zeta} (\hat{\pi}_k \hat{v}_k + \hat{v}_k \hat{\pi}_k) + k^2 \hat{v}_k^2 \right]$$

(3.11)

which refers to (3.4) and can be solved analytically for the vacuum. By the following similarity transformation:

$$\hat{H}_k \equiv \hat{\Lambda} \hat{H} \hat{\Lambda}^{-1}, \quad \text{where} \quad \hat{\Lambda} = e^{-i\frac{\pi}{4} \hat{v} \hat{\pi}},$$

(3.12)

the Hamiltonian can be reduced to the following form:

$$\hat{H}^S_k = \frac{1}{2} \left[ \hat{\pi}^2_k + \left( k^2 - \frac{\zeta'^2}{\zeta^2} \right) \hat{v}_k^2 \right].$$

(3.13)

The wavefunction for the Hamiltonian (3.13) is quite well known and has the form [36]

$$\hat{\psi}_k = N_k e^{-\Omega_k \hat{v}^2}$$

(3.14)

where

$$|N_k| = \left( \frac{2 \text{Re} \Omega_k}{\pi} \right)^{1/4}, \quad \Omega_k = -\frac{i}{2} \frac{f'_k}{f_k}$$

(3.15)

and $f_k$ satisfies

$$f''_k + \left( k^2 - \frac{\zeta'^2}{\zeta^2} \right) f_k = 0$$

(3.16)

For the vacuum state we know

$$f_k = \frac{1}{\sqrt{2k}} e^{ik\eta}$$

(3.17)

which gives us

$$\hat{\psi}_k = \left( \frac{k}{\pi} \right)^{1/4} e^{-i\frac{\pi}{4} \hat{v}^2}.$$ 

(3.18)

Hence the vacuum state wavefunction for the inflationary cosmological scalar perturbations turns out to be

$$\psi^S_k = \left( \frac{k}{\pi} \right)^{1/2} \exp \left( - \left( \frac{\zeta'}{\zeta} + k \right) \left( v_{1k}^2 + v_{2k}^2 \right) \right).$$

(3.19)

Similarly, we can write the vacuum state wavefunction for the inflationary cosmological tensor perturbation as

$$\psi^T_k = \left( \frac{k}{\pi} \right)^{1/2} \exp \left( - \left( \frac{\alpha'}{\alpha} + k \right) \left( v_{1k}^2 + v_{2k}^2 \right) \right)$$

(3.20)

The phase part, which is the combination of the dynamical phase and the geometric phase, of the wavefunction is now explicit, but it is not easy to separate out the geometric phase from this expression. But in our present framework, this can be done using the expressions already
derived, which is what we do next. With the help of equations (3.9) and (2.25), we obtain the corresponding Berry phase

\[ \gamma_{n_1, n_2, k} = i \int_0^\tau \left( \Theta_{n_1, n_2} \frac{\partial}{\partial \eta} \Theta_{n_1, n_2} \right) d\eta \]

\[ = -\frac{1}{2} (n_1 + n_2 + 1) \int_0^\tau \left( \frac{1}{\rho_k^2} - \rho_k^2 \omega^2 - (\rho_k')^2 \right) d\eta \]  
(3.21)

where it has been assumed that the invariant \( I_k(\eta) \) is \( \Gamma \) periodic and its eigenvalues are non-degenerate.

To get a deeper physical insight, the quantitative estimation of the Berry phase is very important. Equation (3.21) tells us that for this estimation, the knowledge of \( \rho_k \) is essential, but the solution of equation (2.20) is difficult to obtain. Another point to be carefully handled is to set the value of the parameter \( \Gamma \). Keeping all these in mind and considering compatible physical conditions we proceed as follows.

First, we note that in the adiabatic limit (which is quite justified for sub-Hubble modes), equation (2.20) can be solved [29] by a series of powers in the adiabatic parameter, \( \delta (\ll 1) \). To this end, we define a slowly varying time variable as \( \tau = \delta \eta \) and write the solution to the Milne–Pinney equation as

\[ \rho_k = \rho_0 + \delta \rho_1 + \delta^2 \rho_2 + \cdots \]  
(3.22)

Inserting this expansion into the Milne–Pinney equation, we obtain

\[ \delta^2 \rho_0 \rho_0' + \rho_0^2 \left[ 1 + 2 \delta \rho_0 \rho_1 + \delta^2 \rho_1^2 + 2 \delta^2 \rho_0 \rho_2 \right] \left( \omega^2 - \delta Y \right) \]

\[ = \frac{1}{\rho_0^2} + 2 \delta \rho_0 \rho_1 + \delta^2 \rho_1^2 + 2 \delta^2 \rho_0 \rho_2 + O(\delta^3) \]  
(3.23)

Here, dot represents derivative w.r.t. the new time variable \( \tau \). Collecting the zeroth-order terms from both sides, we obtain \( \rho_0^2 = \omega^{-2} \). Now the integrand of equation (3.21) can be rewritten as

\[ \frac{1}{\rho_k^2} - \rho_k^2 \omega^2 - (\rho_k')^2 = \rho_k \rho_k'' - (\rho_k')^2 - \rho_k^2 Y' \]

\[ \approx \delta \rho_0^2 \frac{d}{d\eta} \delta Y + \delta^3 \left[ \rho_0 \rho_0' - \rho_0^2 - 2 \rho_0 \rho_1 \frac{d}{d\eta} \right] + O(\delta^3) \]

\[ = \delta \omega^{-1} \frac{d}{d\eta} \delta Y + O(\delta^3) \]  
(3.24)

Thus, for the ground state of the system, in the adiabatic limit, the Berry phase for a particular perturbation mode can be evaluated up to the first order in \( \delta \), which is given by

\[ \gamma_k^{S,T} = -\frac{1}{2} \int_0^\tau \frac{\delta Y}{\sqrt{k^2 - Y^2}} d\eta \approx -\frac{1}{2} \int_0^\tau \frac{Y'}{\sqrt{k^2 - Y^2}} d\eta \]  
(3.25)

where the superscripts \( S \) and \( T \) stand for the scalar and tensor modes, respectively. One may note that our result (3.25) coincides with that of Berry [3].

Our next task is to fix the value of the parameter \( \Gamma \). To this end, we shall calculate the total Berry phase accumulated by each mode during sub-Hubble evolution in the inflationary era. For the ground state of the system, this turns out to be

\[ \gamma_k^{S,T}_{\text{sub}} = -\frac{1}{2} \lim_{\eta_0 \to -\infty} \int_{\eta_0}^{\eta_0'} \frac{Y'}{\sqrt{k^2 - Y^2}} d\eta \]  
(3.26)

where \( \eta_0 \) is the conformal time which satisfies the relation \( k^2 = \left[ Y \left( \eta_0^{S,T} \right) \right]^2 \) so that the modes are within the horizon and oscillating with real frequencies. A non-zero value of the parameter \( \gamma_k^{S,T}_{\text{sub}} \) will ensure that there are some nontrivial effects of the curved spacetime background on the evolution of the quantum fluctuations and may play an important role in the growth of inflationary cosmological perturbations.
4. Berry phase and the cosmological parameters

Let us now set up the link between this cosmological analogue of the Berry phase and the cosmological observables. In the adiabatic limit, the *accumulated Berry phase* during sub-Hubble oscillations of the each mode is given by (3.26). Formula (3.26) is adopted to derive the relations between the accumulated Berry phase and the cosmological observable parameters. From now on, we shall drop the subscript ‘*sub*’ keeping in mind that the calculations are for sub-horizon modes only.

Now the variable $Y(\eta)$ can be expressed in terms of the slow-roll parameters. If we neglect the time variation in the slow-roll parameters, then equation (3.26) can be integrated analytically. Then the *accumulated Berry phase* during the sub-Hubble evolution of the scalar modes, in terms of the slow-roll parameters, turns out to be

$$\gamma_k^S = \frac{1}{2} \lim_{\eta' \to -\infty} \int_{\eta'}^{\eta_0} \frac{\eta'' - \left( \frac{\eta'}{\eta} \right)^2}{\sqrt{k^2 - \left( \frac{\eta'}{\eta} \right)^2}} \, d\eta$$

$$= \frac{1}{2} \lim_{\eta' \to -\infty} \int_{\eta'}^{\eta_0} \frac{\eta'' - \left( \frac{\eta'}{\eta} \right)^2}{\sqrt{k^2 - \left( \frac{\eta'}{\eta} \right)^2}} \, d\eta + O(\epsilon_1^2, \epsilon_2^2, \epsilon_1 \epsilon_2)$$

$$\approx -\frac{\pi}{4} \frac{1 + 3 \epsilon_1 - \epsilon_2}{\sqrt{1 + 6 \epsilon_1 - 2 \epsilon_2}}$$

(4.1)

For brevity, we have restricted our analysis up to the first order in the *slow-roll* parameters and have neglected any time variation in $\epsilon_1, \epsilon_2$. And for the tensor modes, we have

$$\gamma_k^T = \frac{1}{2} \lim_{\eta' \to -\infty} \int_{\eta'}^{\eta_0} \frac{\eta'' - \left( \frac{\eta'}{\eta} \right)^2}{\sqrt{k^2 - \left( \frac{\eta'}{\eta} \right)^2}} \, d\eta$$

$$= \frac{1}{2} \lim_{\eta' \to -\infty} \int_{\eta'}^{\eta_0} \frac{\eta'' - \left( \frac{\eta'}{\eta} \right)^2}{\sqrt{k^2 - \left( \frac{\eta'}{\eta} \right)^2}} \, d\eta + O(\epsilon_1^2, \epsilon_2^2, \epsilon_1 \epsilon_2)$$

$$\approx -\frac{\pi}{4} \frac{1 + \epsilon_1}{\sqrt{1 + 2 \epsilon_1}}$$

(4.2)

Here also we have neglected any time variation in $\epsilon_1, \epsilon_2$ and restricted our attention to the first order in them. In the above derivations, we have made use of the standard definition of the slow-roll parameters [37].

$$\epsilon_1 = \frac{M_p^2}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \epsilon_2 = \frac{M_p^2}{2} \left( \frac{V''(\phi)}{V(\phi)} \right),$$

(4.3)

with $V(\phi)$ being the inflaton potential. For the estimation of $\gamma_k^{S,T}$, the slow-roll parameters are to be evaluated at the start of inflation. But during inflation, the slow-roll parameters do not evolve significantly from their initial values for first few $e$-folds, which is relevant for the present-day observable modes as they are supposed to leave the horizon during first ten $e$-folds. So, in the above estimates for $\gamma_k^{S,T}$, we can consider $\epsilon_1$ and $\epsilon_2$ as their values at horizon crossing without committing any substantial error.

We are now in a position to relate this phase to the observable parameters. At the horizon exit, the fundamental observable parameters can be expressed in terms of the slow-roll
parameters (up to the first order in $\epsilon_1, \epsilon_2$) as [37–39]

\[ P_R = \frac{V}{24\pi^2 M^4 \epsilon_1}, \quad n_S = 1 + 2\epsilon_2 - 6\epsilon_1 \]

\[ n_T = -2\epsilon_1, \quad r = 16\epsilon_1 \]  

(4.4)

where $P_R$ is the scalar power spectrum, $n_S$ and $n_T$ are the scalar and tensor spectral indices, respectively, $r$ is the tensor to scalar ratio. As a consequence, the accumulated Berry phase associated with the sub-Hubble oscillations of the scalar fluctuations during inflation can be expressed in terms of the observable parameters (and vice versa) using (4.1) and (4.4) as follows:

\[ \gamma_S^k \approx -\frac{\pi}{8} \left[ \frac{3 - n_S(k)}{\sqrt{2 - n_S(k)}} \right] \]  

(4.5)

\[ n_S(k) \approx 3 - 8\frac{\gamma_S^k}{\pi} \left( \frac{4\gamma_S^k}{\pi} - \sqrt{\frac{16[\gamma_S^k]^2}{\pi^2} - 1} \right) \]  

(4.6)

Therefore, the accumulated Berry phase for the scalar modes is related to the scalar spectral index. From the above relation, it is also very clear how the Berry phase is related to the cosmological curvature perturbations. For the tensor modes using (4.2) and (4.4), the corresponding expressions turn out to be

\[ \gamma_T^k \approx -\frac{\pi}{8} \left[ \frac{2 - n_T(k)}{\sqrt{1 - n_T(k)}} \right] \]  

(4.7)

\[ n_T(k) \approx 2 - 8\frac{\gamma_T^k}{\pi} \left( \frac{4\gamma_T^k}{\pi} - \sqrt{\frac{16[\gamma_T^k]^2}{\pi^2} - 1} \right) \]  

(4.8)

Equations (4.5) and (4.7) reveal that the Berry phase due to the scalar and tensor modes basically correspond to a new parameter made of corresponding spectral indices. Here, we note that relations (4.5) and (4.7) are not exact in general, but they are in the linearized theory of cosmological perturbation. Had we taken into account the second-order and higher order contributions of the cosmological fluctuations, relations (4.5) and (4.7) would have been different.

Furthermore, the accumulated Berry phase associated with the total gravitational fluctuations (a sum total of $\gamma_S^k$ and $\gamma_T^k$) can be expressed in terms of the other observable parameter as well, giving

\[ \gamma_k \equiv \gamma_S^k + \gamma_T^k \approx -\frac{\pi}{8} \left[ \frac{3 - n_S(k)}{\sqrt{2 - n_S(k)}} + \frac{2 + \frac{r}{8}}{\sqrt{1 + \frac{r}{8}}} \right] \]  

(4.9)

\[ \approx \frac{\pi}{8} \left[ \frac{3 - n_S(k)}{\sqrt{2 - n_S(k)}} + \frac{2 + \frac{\sqrt{V}}{12\pi^2 M^4 P_R}}{\sqrt{1 + \frac{\sqrt{V}}{12\pi^2 M^4 P_R}}} \right]. \]  

(4.10)

Therefore, the accumulated Berry phase for the sub-Hubble oscillations of the perturbation modes during inflation can be completely envisioned through the observable parameters. Here also we see that the total Berry phase of a single mode can be characterized by the curvature perturbations. The estimation of the Berry phase gives a deeper physical insight of the quantum property of the inflationary perturbation modes. As a result, at least in principle, we can claim that the measurement of the Berry phase can serve as a probe of quantum properties reflected on classical observables.
5. Physical significance of cosmological Berry phase

The physical implication of the Berry phase in cosmology is already transparent from our above analysis. In a nutshell, the classical cosmological perturbation modes (both scalar and tensor) having a quantum origin pick up a phase during their advancement through the curved spacetime background that depends entirely on the background geometry and may be, at least in principle, estimated quantitatively by measuring the corresponding spectral indices. So, the Berry phase for the quantum counterpart of the classical cosmological perturbations endows us with the measure of the spectral index.

Also, the existing literature suggests that there may be an intriguing direct link of the cosmological Berry phase with the CMB. The interpretation of the Wigner rotation matrix as the Berry phase [40] has already been elaborated by the proposal of an optical demonstration [41]. On the other hand, the Wigner rotation matrix can be represented as a measure of the statistical isotropy violation of the temperature fluctuations in the CMB [42]. These results motivate us to investigate whether the cosmological analogue of the Berry phase may be thought of as a measure of violation of statistical isotropy in the CMB as our future project.

The current observations from WMAP7 [12] have put stringent constraints on $n_S$ (0.948 < $n_S$ < 1), but only an upper bound for $r$ has been reported so far ($r < 0.36$ at 95% C.L.), with PLANCK [13] expecting to survey up to the order of $10^{-2}$. Given this status, any attempt toward the measurement of the cosmological Berry phase may thus reflect observational credentials of this parameter in inflationary cosmology. For example, it is now well known that any conclusive comment on the energy scale of inflation ($V$ in equation (4.9)) provides crucial information about fundamental physics. However, in CMB polarization experiments, the energy scale cannot be conclusively determined because there is a degeneracy between E and B modes via the first slow-roll parameter $\epsilon_1$ (equation (4.4)), which can only be sorted out once $r$ is measured conclusively. But B-mode polarized states can be contaminated with cosmic strings, primordial magnetic field, etc, thereby making it difficult to measure $r$ conclusively (for a lucid discussion see [43]). So, the cosmological Berry phase may have the potentiality to play some important role in inflationary cosmology, since it is related to $r$ and $V$ via equations (4.9) and (4.10).

6. Conclusion

In this paper, we have demonstrated how the exact expression for the wavefunction of the quantum cosmological perturbations can be analytically obtained by solving the associated Schroedinger equation following the dynamical invariant technique. This helps us to derive an expression for cosmological analogue of the Berry phase. Finally, we demonstrate how this quantity is related to cosmological parameters and show the physical significance of the cosmological Berry phase.

So far, as the detection of the cosmological Berry phase is concerned, we are far away from quantitative measurements. A possible theoretical aspect of detection [44] of the analogue of the cosmological Berry phase may be developed in squeezed state formalism [35]. In principle, the Berry phase can be measured from an experiment dealing with phase difference (e.g. interference). Recently, an analogy between phonons in an axially time-dependent ion trap and quantum fields in an expanding/contracting universe has been derived and the corresponding detection scheme for the analogue of cosmological particle creation has been proposed which is feasible with present-day technology [45]. Besides, there exists [35] a scheme for measuring the Berry phase in the vibrational degree of freedom of a trapped ion. We hope that these types
of detection schemes may be helpful for the observation of the cosmological analogue of the Berry phase in laboratory in future.

It is hoped that further research in this direction may help us to anticipate how these relations can be utilized in extracting further information related to the theoretical and observational aspects of inflationary perturbations.

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