A model of the effect of dry friction on the behaviour of a dynamical system

F-C Ciornei¹, S Alaci¹ and C Bujoreanu²
¹ Mechanics and Technologies Department, „Stefan cel Mare” University of Suceava, Suceava, Romania
² „Mechanical Engineering, Mechatronics and Robotics Department, “Gheorghe Asachi” Technical University of Iasi, Iasi, Romania
E-mail: florina.ciornei@usm.ro

Abstract. A dynamical system with one degree of freedom is modelled in two manners: the first one considers that only the dynamical sliding friction is present and the second considers the presence of both dynamic friction and static friction. It is difficult to apply the model of revealed static friction only for the zero relative velocity. For this reason, an approximate model for the static friction was identified, that accepts that the static friction occurs when the magnitude of relative velocity in contact decreases under a certain value, the critical value. The integration of the equations of motion of the two models reveals that for important values of the relative velocity, the two models estimate the same behaviour of the system. An experimental set-up was used to validate the theoretical results.

1. Introduction

Friction is a phenomenon occurring whenever two solid bodies make contact and its main characteristic is the tendency to oppose to the relative motion between the two bodies. The implications and effects of the friction forces can be met in the most unexpected domains. Taking into account that the magnitude of the friction forces is responsible for the efficiency of any equipment, nowadays, when the subject of rational resources management is more severe, the accomplishment of machines and equipments with higher efficiency is a main concern [1-2]. Additionally, the energy to be introduced into a system should be obtained with lesser costs and with minimum effects upon the environment [3-4]. A major criterion of the classification of the friction forces concerns the presence/absence of a lubricant between the contacting surfaces and thus the lubricated friction and dry friction can be named. But further dissimilarities are identified among the two major friction types. As example, in the range of lubricated friction, a subsequent classification regarding the quantity of lubricant between the surfaces was made, so usually lubricated friction is divided into three regimes, fluid-film, mixed and boundary lubrication [5]. Regarding dry friction, in a recent paper, Flores [6] presents a comprehensive work upon the models and their evolution and identifies at least 11 dry friction models. The simplest model is the one proposed by Amonton-Coulomb, [7-8], that stipulates that the friction force is proportional to the normal force and opposite to the tendency of motion. A common sense remark referring to the fact that a nonzero force is necessary to act upon a body supported by a rough surface for taking it off from the state of rest, led to the idea of static friction
concept. The mathematical description of the phenomenon of static friction requests inequalities and this fact complicates substantially the study of the phenomenon [9-10]. The dependency of the dry friction forces, both static and dynamic, on the relative velocity, conducts to equation which describe the motion of a system that are nonlinear differential equations [11], where the chaos phenomenon may happen [12].

2. Theoretical considerations

In the preset work two models are presented and compared, regarding the effect of dry friction upon the behaviour of a dynamical system: the first considers only dynamical friction, figure 1 and the second considers both static and dynamic friction, figure 2. For the case when only dynamic friction exists, that is when relative motion exists, the dependency of the coefficient of friction on the relative velocity is presented in figure 1. For the situation when both static and dynamic friction occur, the coefficient of friction versus relative velocity plot shows the presence of friction even when the relative motions is zero. The challenge of the model from figure 2 resides in the fact that when the relative motion stops, the force of static friction which manifests in the system cannot be expressed as a function of normal reaction with the well-known relation:

\[ F_f = \mu_d N \]  

The force of static friction can have any value from the range \([-\mu_{st}N, \mu_{st}N]\). To surpass this drawback, recalling some models presented by Flores [6], it is considered that the static friction appears not only when:

\[ v_{rel} = 0 \]  

but that there is a transition accomplished when the magnitude of the relative velocity decreases below a certain value \( v_{cr} \).

\[ |v_{rel}| < v_{cr} \]  

For the domain defined by the relation (3) it is considered that the coefficient of static friction presents a continuous variation from \(-\mu_{st}\) to \(\mu_{st}\). As shown by Flores, one of the most employed models is the model due to Therfall [13], presented in figure 3, which ensures a continuous variation of the coefficient of friction along the entire domain of relative velocity variation. The disadvantage of the model resides in the necessity of stipulation of two characteristic values of relative velocity, \(v'_{cr}\) and \(v''_{cr}\). In the present work it is assumed that the variation of the coefficient of friction is described by the graph from figure 4.
3. The proposed dynamical model

The dynamical system to be analyzed is presented in figure 5. Inside a cylinder of $R$ radius, a small body of mass $M$ slides without rolling. The cylinder rotates at an angular velocity of stipulated time variation. Upon the small body, assimilated to a point, acts the weight (gravity force) $G$, the normal reaction $N$, and the tangential (friction) force $T$. The position of the body is completely determined by the $\theta$ angle made by the vector radius with the vertical direction.

The equation of motion of the body is:

$$ma = G + N + T$$

The body moves on circular trajectory and the vector equation of motion (4) can be easily projected on the axes of a system of versors $\nu$ and $\tau$ [14]:

$$a = mR\dot{\theta}\tau + mR\dot{\theta}^2\nu$$
The forces acting upon the body are also expressed using the versors $\mathbf{v}$ and $\mathbf{r}$.

$$ G = -mg \sin \theta \mathbf{r} - mg \cos \theta \mathbf{v} $$

(6)

$$ N = N \mathbf{v} $$

(7)

$$ T = -\text{sign}(v_{rel})\mu(|v_{rel}|)N \mathbf{r} $$

(8)

Where the $\text{sign}(x)$ function is the notation for the *signum* function

$$ \text{sign}(x) = \begin{cases} 
1, & x > 0 \\
0, & x = 0 \\
-1, & x < 0 
\end{cases} $$

(9)

The relative velocity in the point of contact is given by the difference between the absolute velocity $R\dot{\theta}$ and the velocity $\omega R$ of the point on the cylinder

$$ v_{rel} = R\dot{\theta} - \omega R = (\dot{\theta} - \omega)R $$

(10)

Next, the variation of the coefficient of friction with velocity is aimed as a function with unique expression instead of piecewise function. For the dependency shown in figure 4, coefficient of friction versus velocity, the next form is proposed:

$$ \mu(v) = \frac{\mu_d}{v_{cr}} v \Phi(v_c - |v|) + \mu_d \Phi(|v| - v_c) \text{sign}(v), $$

(11)

Where $\Phi(x)$ is the Heaviside step function and $(-v_{cr},v_{cr})$ is the velocity range amid which it is considered that static friction acts between the two bodies. The equation (11) is plotted in figure 6 and the remark that the graph is similar to the plot from figure 4 can be made.

![Figure 6. Graphical representation of the equation (11) - coefficient of friction vs relative velocity.](image)

The expressions (5) ÷ (11) are introduced into the vector equation (4) and the next system is obtained:

$$ \begin{cases} 
  mR\ddot{\theta} + mg \sin \theta + \text{sign}(\theta - \omega)\mu(|\dot{\theta} - \omega|)N = 0 \\
mR\dot{\theta}^2 + mg \cos \theta - N = 0 
\end{cases} $$

(12)

The unknowns of the system are $N$ and $\dot{\theta}$. The solutions of the system are:

$$ N = mR \dot{\theta}^2 + mg \cos \theta $$

(13)
The equation (14) is a differential nonlinear equation which must be integrated via a numerical procedure. The Runge-Kutta 4 algorithm [15] was the option for the present case.

4. Solutions of the equation of motion
The equation of motion (14) was integrated for several situations which are synthetically represented in figure 7 and figure 8. The integration was run for the two situations and the results are presented comparatively, for constant angular velocity \( \omega = \omega_0 \) of the cylinder. The integration was made for two values of \( v_{cr} \) and three values of angular velocity \( \omega_0 \), specified in the respective plots. The angular amplitude \( \theta_{st, d} \) (plotted with red) is for the case when both static and dynamic friction is considered and \( \theta_d \) (plotted blue) is for the case when only dynamic friction is considered.

\[
\dot{\theta} = \left[ \frac{g}{R} \sin \theta + \text{sign}(\dot{\theta} - \omega) \mu(R \mid \dot{\theta} - \omega) \right] \left( \dot{\theta}^2 + \frac{g}{R} \cos \theta \right)
\]

(14)

**Figure 7.** The angular elongation versus time for the two models of friction and for different \( v_{cr} \) for constant angular velocity.
Figure 8. The angular elongation versus time for different $v_{cr}$ and $A$ parameter.
From figure 7 the conclusions that can be drawn are: for large values of transition (the critical velocity $v_{cr} = 0.1$) from the static regime to the dynamic regime, the model with two types of friction predicts that for small velocities of the cylinder, the body reaches a final position where it remains immobile. When the value of $v_{cr}$ decreases and the operating velocity of the cylinder is increased, both models envisage an oscillatory motion of the body. This motion has one of the extremes on the vertical of the centre of the cylinder, $\theta_{min} = 0$. Characteristic to all six plots is the fact that at initial moment, between the cylinder and the body dynamic friction occurs. In figure 8 there are presented the results for the integration of equation (14) when the angular velocity of the cylinder takes values from zero and tends asymptotically to a constant value. The angular velocity of the cylinder varies obeying the law:

$$\omega(t) = A \cdot \tan(t)$$

The value of regime velocity (steady motion) of the cylinder is:

$$\lim_{t \to \infty} \omega(t) = A \cdot \lim_{t \to \infty} \tan(t) = \frac{\pi}{2} A$$

The integration was made for two values of critical velocity $v_{cr}$ and four values of the $A$ parameter. Similar to the previous case, it is noticed that for small values of regime angular velocity the results for the two models differ for small values of the angular velocity of the cylinder, the model without static friction predicts oscillatory motion of small amplitudes while the model with static friction predicts an oscillatory motion with much larger amplitudes and period, motion that degenerates into rest when the parameter $v_{cr}$ is increased. When the $A$ parameter is increased, the results are independent on the value of the transition velocity $v_{cr}$. Additionally, for larger values of the angular velocity, the obtained signals are identical and independent of $A$ and $v_{cr}$. Moreover, one can see that for large values of the steady velocity, the results from the figures 7c, 7f are identical to the figure 8g, 8h respectively, proving that for large values of the angular velocity of the cylinder, the motions premised by the two models are identical, being oscillatory motions about a position of equilibrium and having one of the extreme positions placed on the vertical passing through the centre of the cylinder.

5. Experimental validation

The experimental set-up used for the experimental corroboration of theoretical results has as main part an outer ring of a radial bearing, with the diameter 110mm, as seen in figure 9.

Figure 9. The extreme positions of the mobile body for large values of angular velocity.
The moving body cannot be a bearing ball since when contacting the raceway it would present a rotation motion about an axis parallel to the axis of the ring. To avoid this aspect, as mobile body an assembly made from two identical balls glued together was employed. The ring was fixed with the 3-jaw chuck on a lathe. For small angular velocities of the ring it was noticed that the assembly of the two balls performs an oscillatory motion but the centre of mass doesn’t reach the vertical of the centre of the ring. When the rotation velocity of the ring increases, the two-balls element present an oscillatory motion and the centre of mass (the joint point) reaches the vertical of the centre of the ring, as predicted in the last figures 7e, f and 8g, h.

Figure 10. For increased values of angular velocities of the ring, the amplitudes of the oscillations increase and the mobile body leaves the raceway.

6. Conclusions
The paper presents a comparison between theoretical results of two models of dynamical behaviour of a 1 DOF system where dry friction is present. The dry friction is modelled in two ways: the simple case considers only dynamic friction and the second case considers both dynamic and static friction, with the assumption that the static friction occurs when the relative velocity in contact has the module smaller than a critical value. The differential equation of motion of the mobile point is obtained and afterwards it is numerically integrated, assuming that the point is initially at rest in the lowest point of the circular trajectory. A first case analyzed considers that the angular velocity of the cylinder is constant. A second case, more realistic, considers that the angular velocity of the cylinder increases continuously from zero up to a constant value. It was observed that in both cases, for small values of relative velocity, the results of the two models of friction lead to results differing in time while for large values both friction models give the same solution. Furthermore, for both types of motions of the cylinder, the two friction models provide identical motions. The experimental validation was accomplished only from qualitative point of view. The mobile cylinder was materialized by the outer ting of a radial ball bearing fixed in the 3-jaw lathe chuck. The mobile body was constructed from two identical bearing balls, soldered with glue. A single ball should have a rolling with sliding motion which is more difficult to study. For small values of rotational speed of the ring, the two balls present a forced oscillatory motion, motion performed strictly on one side of the vertical diameter passing through the centre of the ring, as predicted by both models. For large rotational speeds, the motion remains oscillatory, with one of the extremes placed on the vertical passing through the centre of the ring, as predicted by both models. With increasing the rotational speed, it is observed the initiation of a process of continuous increase of amplitude of oscillation of the two-balls assembly, process that runs till the balls lose the contact with the raceway. This disconnection cannot be explained by the models presented and it may have as cause the assembling eccentricity of the ring and the deformations
produced by the jaws of the lathe. For qualitative validations of theoretical results, the experimental set-up requires higher precision of execution. Explicitly, the deviation of mounting of the ring should be rigorously controlled, the law of motion of the lathe chuck must be precisely determined, the position of the centre of mass (the contact point of the balls) have to be precisely stipulated, the vibration level of the lathe should be as reduced as possible (new machine with vibration isolators) etc.

7. References

[1] Wilson D G and Korakianitis T 2014 *The design of high efficiency turbomachinery and gas turbines* 2nd ed (The MIT Press)

[2] Atanasoe P and Pentiu R D 2014 Indices for the Power Quality Monitoring in the Romanian Power Transmission System *16th Int. Conf. on Harmonics and Quality of Power (ICHQP)* Bucharest 2014 pp 68-71

[3] Fox M 2017 *Green Power: Perspectives on Sustainable Energy Generation* (Callisto Reference)

[4] Atanasoe P and Pentiu R D 2017 Considerations on the Green Certificates Support System for Electricity Production from Renewable Energy Sources *10-th Int Conf Interdisciplinarity in Engineering Inter-Eng 2016* Procedia Engineering 181 pp 796-803

[5] Bujoreanu C, Cretu S and Nelias D 2003 An Investigation of Scuffing Failure in Angular Contact Ball-Bearings *TribologFy in Industry* 25 1&2 pp 20-26

[6] Flores P 2016 A Survey and Comparison of Several Friction Force Models for Dynamic Analysis of Multibody Mechanical Systems *Nonlinear Dynamics* August 2016 DOI: 10.1007/s11071-016-2999-3

[7] Amontons G 1699 De la resistance cause’e dans les machines *Mémoires de l’Academie Royale des Sciences* pp 206–226 https://gallica.bnf.fr/ark:/12148/bpt6k35013/f344.image.langEN

[8] Bowden F B and Tabor D 1974 *Introduction to Tribology* (Heinemann Educational Publishers)

[9] Alaci S, Ciornei F C, Romanu I C and Ciornei M C 2018 The importance of correct specification of tribological parameters in dynamical systems modelling *IOP Conference Series: Materials Science and Engineering* 294(1) 012039 Doi: 10.1088/1757-899X/294/1/012039

[10] Alaci S, Bujoreanu C and Ciornei F C 2018 Theoretical and Experimental Aspects Regarding Nonlinear Effects of Dry Friction and Unbalanced Rotational Mass in a Dynamical System *Mechanics* 24 (6) pp 811-816 https://doi.org/10.5755/j01.mech.24.6.22470

[11] Tenenbaum M and Pollard H 1985 *Ordinary Differential Equations, An Elementary Textbook for Students of Mathematics, Engineering and Sciences* (Dover Publications)

[12] Ott E 2002 *Chaos in Dynamical Systems* (Cambridge University Press)

[13] Threlfall D 1978 The inclusion of Coulomb friction in mechanisms programs with particular reference to DRAM *Mech Mach Theory* 13(4) pp 475–483

[14] Ionescu G D 1984 *Differential theory of curves and surfaces* in Romanian *Teoria diferenţială a curbelor şi suprafeţelor* (Editura Dacia Cluj Napoca)

[15] Butcher J C 2016 *Numerical Methods for Ordinary Differential Equations* 3rd ed (Willey)