STABLE MULTIVARIATE \( W \)-EULERIAN POLYNOMIALS

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Abstract. We prove a multivariate strengthening of Brenti’s result that every root of the Eulerian polynomial of type \( B \) is real. Our proof combines a refinement of the descent statistic for signed permutations with the notion of real stability—a generalization of real-rootedness to polynomials in multiple variables. The key is that our refined multivariate Eulerian polynomials satisfy a recurrence given by a stability-preserving linear operator.

Our results extend naturally to colored permutations, and we also give stable generalizations of recent real-rootedness results due to Dilks, Petersen, and Stembridge on affine Eulerian polynomials of types \( A \) and \( C \). Finally, although we are not able to settle Brenti’s real-rootedness conjecture for Eulerian polynomials of type \( D \), nor prove a companion conjecture of Dilks, Petersen, and Stembridge for affine Eulerian polynomials of types \( B \) and \( D \), we indicate some methods of attack and pose some related open problems.

1. Introduction

In this paper, we study the real-rootedness property of Eulerian polynomials for Coxeter groups from a combinatorial perspective. There is a well-known combinatorial interpretation of the Eulerian polynomial \( A_n(x) \) as the descent generating polynomial for permutations in the Coxeter group \( A_n \), the group \( \text{Sym}(n+1) \) of all permutations on \( n+1 \) letters (see [13, 22]). The notion of a descent can be extended to elements of all finite Coxeter groups as follows: for an element \( \sigma \) of the Coxeter group \( W \) the descents are exactly those generators \( s \) of \( W \) whose action on \( \sigma \) reduces its length. In [9], Brenti used this interpretation of a descent to define the \( W \)-Eulerian polynomial, denoted \( W(x) \), for any finite Coxeter group \( W \).

Brenti showed that many classical results about \( A_n(x) \) hold for the other Eulerian polynomials as well. In this paper we will investigate the remarkable property that \( A_n(x) \) has only real roots, a result due to Frobenius [15]. Brenti proved the analogous result for type \( B \), and checked by computer that it also held for the exceptional cases, but left type \( D \)—the only remaining case—as a conjecture.

Conjecture 1.1 (Conjecture 5.2 in [9]). For every finite Coxeter group \( W \), the descent generating polynomial \( W(x) \) has only real roots.

Dilks, Petersen, and Stembridge later extended the definition of Eulerian polynomials to include affine descents, and proposed the following companion to Brenti’s conjecture.

Conjecture 1.2 (Conjecture 4.1 in [12]). For every finite Weyl group \( W \), the affine descent generating polynomial \( \tilde{W}(x) \) has only real roots.

Again, this was not completely proved—\( \tilde{A}_n(x) \) and \( \tilde{C}_n(x) \) were shown to have only real roots, the exceptional cases were verified, but the real-rootedness of the affine Eulerian polynomials of types \( B \) and \( D \) remains an open problem.
In this paper, we build on the idea of real stability—a generalization of the notion of real-rootedness to multivariate polynomials. We combine this with simple recurrences for multivariate refinements of certain Eulerian polynomials to provide simple proofs of multivariate generalizations of known real-rootedness results. Specifically, we give a general framework to show that the recurrence relations satisfied by multivariate W- and \(\tilde{W}\)-Eulerian polynomials (for certain finite Coxeter groups \(W\)) are stability-preserving. We then use properties of stability to show that this implies that the univariate counterparts of these polynomials are also stable, which is equivalent the statement that they have only real roots.

The remainder of this paper is structured as follows. In Section 2, we introduce notation, define the W-Eulerian and the affine \(\tilde{W}\)-Eulerian polynomials for finite Coxeter groups and finite Weyl groups, respectively. We also review the required definitions and results related to real stability. For clarity and completeness, we begin in Section 3.1 with a proof due to Brändén of the stability of the multivariate Eulerian polynomial of type A. In Sections 3.2 and 3.3 we generalize this idea in several directions simultaneously, to type B (signed permutations) and the generalized symmetric group (colored permutations), and also to multiple \(q\) variables. Sections 4.1 and 4.2 then address the affine Eulerian polynomials for types A and C. The unresolved cases of Conjecture 1.1 (type D) and Conjecture 1.2 (types B and D) are examined within our multivariate framework in Section 5. We conclude with a discussion about the connection between our statistics with Catalan numbers, Motzkin paths and Laguerre polynomials.

2. Preliminaries

We begin by introducing some notation. For a positive integer \(n\), let \([n]\) be the set \([1, \ldots, n]\) and let \(x\) be the \(n\)-tuple \((x_1, \ldots, x_n)\); for example, \(x + y = (x_1 + y_1, \ldots, x_n + y_n)\). For \(\mathcal{T}\) a set (or multiset) with entries from \([n]\), we let \(x^{\mathcal{T}} = \prod_{i \in \mathcal{T}} x_i\); for example, \((x + y)^{[n]} = \prod_{i=1}^n (x_i + y_i)\). The cardinality of \(\mathcal{T}\) is written \(|\mathcal{T}|\). The concatenation of \(x\) and \(y\) is denoted by \((x, y)\). We apply a function \(f\) of \(n\) variables to an \(n\)-tuple \(x\) by writing \(f(x) = f(x_1, \ldots, x_n)\). We will often deal with functions that have \(x\) and \(y\) as variables, and so we define the special symbol \(\partial = \sum_{i=1}^n (\partial/\partial x_i + \partial/\partial y_i)\) as a shorthand for the sum of partial derivatives with respect to all \(x_i\) and \(y_i\) variables.

Finally, the theorems and propositions that are taken from previous works are clearly marked by a reference (indicating the source); as far as we know, all other results are new.

2.1. W-Eulerian Polynomials. Let \(S\) be a set of Coxeter generators, \(m\) be a Coxeter matrix, and

\[
W = \left\langle (s) : m(s,s') = e, \text{ for } s, s' \in S, \ m(s,s') < \infty \right\rangle
\]

be the corresponding Coxeter group (see [3]). Given such a Coxeter system \((W, S)\) and \(\sigma \in W\), we denote by \(\ell_W(\sigma)\) the length of \(\sigma\) in \(W\) with respect to \(S\).

**Definition 2.1.** For \(W\) a finite Coxeter group, with generator set \(S = \{s_1, \ldots, s_n\}\), the descent set of \(\sigma \in W\) is

\[
\mathcal{D}_W(\sigma) = \{i \in [n] : \ell_W(\sigma s_i) < \ell_W(\sigma)\}.
\]
Definition 2.2. For $W$ a finite Coxeter group, the $W$-Eulerian polynomial is the descent generating polynomial

$$W(x) = \sum_{\sigma \in W} x^{|D_W(\sigma)|}.$$ 

The above definitions were extended for a subset of finite Coxeter groups in [12] to include affine descents.

Definition 2.3. For $W$ a finite Weyl group, the affine descent set of $\sigma \in W$ is

$$\tilde{D}_W(\sigma) = D_W(\sigma) \cup \{0 : \ell_W(\sigma s_0) > \ell_W(\sigma)\},$$

where $s_0$ is the reflection corresponding to the lowest root in the underlying crystallographic root system. See [12] for further details and the motivation behind this definition.

Definition 2.4. For $W$ a finite Weyl group, the $\tilde{W}$-Eulerian polynomial is the affine descent generating polynomial (over the corresponding finite Weyl group $W$)

$$\tilde{W}(x) = \sum_{\sigma \in W} x^{|\tilde{D}_W(\sigma)|}.$$ 

2.2. Real Stable Polynomials. We define real stability, which generalizes the notion of real-rootedness from real univariate polynomials to real multivariate polynomials.

Let $\mathcal{H}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the open upper complex half-plane and similarly let $\mathcal{H}_- = \{z \in \mathbb{C} : \Im(z) < 0\}$.

Definition 2.5. A polynomial $f \in \mathbb{R}[x]$ is (real) stable if $f \equiv 0$ or for any $z \in \mathcal{H}_+$, $f(z) \neq 0$.

Note that a univariate polynomial $f(x) \in \mathbb{R}[x]$ has real roots if and only if it is stable. Following [27], we let $\mathcal{S}_{\mathbb{R}}[x]$ denote the set of stable polynomials in $\mathbb{R}[x]$.

In this paper, we have a fixed template for our proofs. We argue by induction, first checking stability (by hand) for the base case. Next, we establish recursive formulas of the following form

$$W_n = T(W_{n-1}),$$

where $W_n$ is the multivariate $W$-Eulerian polynomial of a group $W$ of rank $n$, and $T$ is some linear operator. Finally, we show that the linear operator $T$ is stability-preserving, using the following theorem.

Recall that a polynomial $f(x)$ is multiaffine if the power of each indeterminate $x_i$ is at most one. For a set $\mathcal{P}$ of polynomials, let $\mathcal{P}^{\text{MA}}$ be the set of multiaffine polynomials in $\mathcal{P}$.

Theorem 2.1 (Part of Theorem 3.5 in [27]). Let $T : \mathbb{R}[x]^{\text{MA}} \to \mathbb{R}[x]$ be a linear operator acting on the variables $x$. If the polynomial $T([x+y]^{[n]}) \in \mathcal{S}_{\mathbb{R}}[x,y]$ is stable, then $T$ maps $\mathcal{S}_{\mathbb{R}}[x]^{\text{MA}}$ into $\mathcal{S}_{\mathbb{R}}[x]^{\text{MA}}$.

Once the multivariate $W$-Eulerian polynomials are shown to be stable, we can then reduce them to real stable univariate polynomials using the following operations.

Lemma 2.2 (Part of Lemma 2.4 in [27]). Given $i, j \in [n]$, the following operations preserve real stability of $f \in \mathbb{R}[x]$:

1. Differentiation: $f \mapsto \partial f / \partial x_i$.
2. Diagonalization: $f \mapsto f|_{x_i = x_j}$. 

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(3) Specialization: for \( a \in \mathbb{R} \), \( f \mapsto f|_{x_i=a} \).

Finally, there is an easy-to-check condition for real stability that we will use to show that certain polynomials are not real stable.

**Theorem 2.3** (Theorem 5.6 in [6]). Let \( f \in \mathbb{R}[x]^M \). Then \( f \) is real stable if and only if for all \( i, j \in [n] \) and for all \( a \in \mathbb{R}^n \),

\[
\frac{\partial f}{\partial x_i}(a) \frac{\partial f}{\partial x_j}(a) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a)f(a) \geq 0.
\]

We note that most of these results have a complex counterpart, but for our purposes real stability suffices—all the polynomials we consider have positive integer coefficients. For this reason we will sometimes refer to real stable polynomials simply as stable polynomials.

### 3. Stable W-Eulerian Polynomials

For clarity and completeness, we begin with a proof of the stability of the multivariate Eulerian polynomial of type \( A \) due to Brändén.

#### 3.1. Eulerian Polynomials of Type \( A \)

Let \( A_n \) denote the Coxeter group of type \( A \) of rank \( n \). We can regard \( A_n \) as \( \text{Sym}(n+1) \), the group of all permutations on \( [n+1] \) with generators \( S = \{s_1, \ldots, s_n\} \), where \( s_i \) is the transposition \( (i, i+1) \) for \( 1 \leq i \leq n \).

**Proposition 3.1** (Proposition 1.5.3 in [3]). Given \( \sigma = \sigma_1 \ldots \sigma_{n+1} \in A_n \) written in the one-line notation, the descent set of \( \sigma \) is

\[
D_A(\sigma) = \{i \in [n] : \sigma_i > \sigma_{i+1}\}.
\]

The following theorem is well-known. Frobenius already mentioned that it follows from the recurrence these polynomials satisfy:

\[
A_n(x) = (n+1)xA_{n-1}(x) + (1-x)(xA_{n-1}(x))'.
\]

**Theorem 3.2** (p. 829 of [15]).

\[
A_n(x) = \sum_{\sigma \in A_n} x^{\left| D_A(\sigma) \right|}
\]

has only real roots.

Theorem 3.2 can also be proven using Rolle’s theorem (see proof of Theorem 1.34 in [4]).

**Definition 3.1.** Given \( \sigma \in A_n \), define the type \( A \) descent top set to be

\[
D_A(\sigma) = \{\max(\sigma_i, \sigma_{i+1}) : 1 \leq i \leq n, \sigma_i > \sigma_{i+1}\},
\]

and similarly, let the type \( A \) ascent top set be

\[
A_A(\sigma) = \{\max(\sigma_i, \sigma_{i+1}) : 1 \leq i \leq n, \sigma_i < \sigma_{i+1}\}.
\]

For example, when \( \sigma = 31452 \in A_4 \), \( D_A(\sigma) = \{3, 5\} \) and \( A_A(\sigma) = \{4, 5\} \). Note that the seemingly superfluous notation \( \max(\sigma_i, \sigma_{i+1}) \) simply reduces to \( \sigma_i \) and \( \sigma_{i+1} \) in the case of type \( A \) descent top and ascent top sets, respectively. Its significance will become apparent when we introduce the type B descent top and ascent top sets.

**Theorem 3.3** (Brändén [7]).

\[
A_n(x, y) = \sum_{\sigma \in A_n} x^{D_A(\sigma)} y^{A_A(\sigma)}
\]

is stable.
Proof. We proceed by induction. Note that $A_0(x_1, y_1) = 1$ is stable. By observing the effect of inserting $n + 1$ into a permutation $\sigma \in A_{n-1}$ on the type A ascent top and descent top sets, we obtain the following recursion. For $n > 0$, we have

(3) \[ A_n(x, y) = (x_{n+1} + y_{n+1})A_{n-1}(x, y) + x_{n+1}y_{n+1} \partial A_{n-1}(x, y). \]

We remind the reader here that $\partial = \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right)$. It is easy to check using Theorem 2.1 that the linear operator $T = (x_{n+1} + y_{n+1}) + x_{n+1}y_{n+1} \partial$ is stability-preserving, because

\[ T((x + u)^{[n]}(y + v)^{[n]}) = \]

\[ x_{n+1}y_{n+1} \left( \frac{1}{y_{n+1}} + \frac{1}{x_{n+1}} + \sum_{i=1}^{n} \left( \frac{1}{x_i + u_i} + \frac{1}{y_i + v_i} \right) \right) (x + u)^{[n]}(y + v)^{[n]} \]

is in $\mathcal{S}[x, y, u, v]$. The result follows. □

Specializing the $y_i$ variables to 1, it follows that

Corollary 3.4. \[ A_n(x) = \sum_{\sigma \in A_n} x^{D_A(\sigma)} \]
is stable.

Diagonalizing $x$, we obtain

Corollary 3.5. \[ A_n(x) = \sum_{\sigma \in A_n} x^{|D_A(\sigma)|} = \sum_{\sigma \in A_n} x^{D_A(\sigma)} \]
is stable.

Since $A_n(x)$ is univariate, this corollary is equivalent to the statement that $A_n(x)$ has only real roots (Theorem 3.2).

We refer to [18] for a proof of the stability of a (slightly different) multivariate refinement of the classical Eulerian polynomials. That refinement has close connections with the affine Eulerian polynomial of type C, which we will address in more detail in Section 4.2.

Next, we present our results. We start by defining the multivariate Eulerian polynomial of type B and proving that it is stable.

3.2. Eulerian Polynomials of Type B. Let $B_n$ denote the Coxeter group of type B of rank $n$. We regard $B_n$ as the group of all signed permutations on $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$ with generators $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0$ is the transposition $(-1, 1)$ and $s_i = (i, i+1)$ for $1 \leq i \leq n-1$.

Type B descents have a simple combinatorial description that we will exploit.

Proposition 3.6 (Corollary 3.2 of [9], also Proposition 8.1.2 of [3]). Given a signed permutation $\sigma = (\sigma_1, \ldots, \sigma_n) \in B_n$, written in its window-notation, let

$D_B(\sigma) = \{i \in [n] : \sigma_{i-1} > \sigma_i\},$
where $\sigma_0 \overset{\text{def}}{=} 0$.

Analogously to type A, the type B Eulerian polynomials have only real roots.

**Theorem 3.7** (Brenti [9]).

\[
B_n(x) = \sum_{\sigma \in B_n} x^{D_B(\sigma)}
\]

has only real roots.

In [9], Brenti also introduced a “$q$-analog” of the (univariate) type B Eulerian polynomials using the following signed permutation statistic. For $\sigma \in B_n$, let

\[
N(\sigma) = |\{i \in [n] : \sigma_i < 0\}|
\]

denote the number of negative entries in the signed permutation $\sigma$.

**Theorem 3.8** (Corollary 3.7 of [9]). For $q \geq 0$,

\[
B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{D_B(\sigma)}
\]

has only real roots.

These $B_n(x; q)$ polynomials specialize to Eulerian polynomials $A_{n-1}(x)$ and $B_n(x)$—when $q = 0$ and $q = 1$, respectively—so that Theorem 3.8 simultaneously generalizes Theorems 3.2 and 3.7.

We will proceed in the same way that the multivariate Theorem 3.3 extends the univariate Theorem 3.2. Recall that the stability of the multivariate refinement of the type A Eulerian polynomials in (2) came from the careful choice of the statistic. Choosing the maximum of the two values $\sigma_i$ and $\sigma_{i+1}$ for both ascents and descents allowed for the simple stability-preserving recursion. We apply this idea to signed permutations in such a way that the definitions remain consistent with the definitions for ordinary permutations.

**Definition 3.2.** Given $\sigma \in B_n$, define the type B descent top set to be

\[
D_B(\sigma) = \{\max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leq i \leq n-1, \sigma_i > \sigma_{i+1}\}.
\]

Analogously, we define the type B ascent top set to be

\[
A_B(\sigma) = \{\max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leq i \leq n-1, \sigma_i < \sigma_{i+1}\}.
\]

For example, when $\sigma = (3, 1, -4, -5, 2) \in B_5$, $D_B(\sigma) = \{3, 4, 5\}$ and $A_B(\sigma) = \{3, 5\}$.

**Theorem 3.9.** For $q \geq 0$,

\[
B_n(x; y; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{D_B(\sigma)} y^{A_B(\sigma)}
\]

is stable.
Proof. As in the proof of Theorem 3.3, we proceed by induction. \( B_1(x_1, y_1; q) = qx_1 + y_1 \) is stable when \( q \geq 0 \), which settles the base case. By observing the effect on the ascent top and descent top sets of type B of inserting \( n + 1 \) or \(- (n + 1)\) into a signed permutation \( \sigma \in B_n \), we obtain the following recursion. For \( n > 0 \), we have

\[
B_{n+1}(x, y; q) = (qx_{n+1} + y_{n+1})B_n(x, y; q) + (1 + q)x_ny_n \partial B_n(x, y; q).
\]

To complete the proof, we note that for a fixed \( q \geq 0 \), the linear operator acting on the right hand side,

\[
T = (qx + y) + (1 + q)xu + yv
\]

preserves stability by Theorem 2.1, since

\[
T((x + u)^n(y + v)^n) = x^{n+1}y^{n+1} + \left(1 + \frac{1}{x_i + u_i} + \frac{1 + q}{y_i + v_i}\right) (x + u)^n(y + v)^n
\]

is in \( \mathbb{S}_n[x, y, u, v] \) whenever \( q \geq 0 \).

Our theorem has some noteworthy consequences. The value of \( B_n(x, y; q) \) at \( q = -1 \) is immediate from the recursion.

\[ B_n(x, y; -1) = (y - x)^n \]

This refines the following theorem of Reiner.

**Corollary 3.10.**

\[ B_n(x, y; -1) = (y - x)^n \]

This refines the following theorem of Reiner.

**Corollary 3.11 (Theorem 3.2 of [20]).**

\[
\sum_{\sigma \in B_n} \delta(\sigma)x^{D_B(\sigma)} = \prod_{i=1}^{n}(1 - x_i),
\]

where \( \delta \) is a one-dimensional character of \( B_n \) defined by \( \delta(\sigma) = (-1)^N(\sigma) \).

If we set \( q = 1 \), we obtain the analogue of Theorem 3.3 for type B.

**Corollary 3.12.**

\[ B_n(x, y; 1) = \sum_{\sigma \in B_n} x^{D_T(\sigma)} y^{A_T(\sigma)} \]

is stable.

We would like to point out that when we plug in \( q = 0 \) into (6), we get a homogenized polynomial that is not equal to the polynomial \( A_{n-1}(x, y) \) from (2), since their recursions differ. Rather, \( B_n(x, y; 0) \) is the permanent of the following \( n \times n \) matrix \( M = (m_{ij}) \) considered in [8]. For \( i, j \in [n] \), let \( m_{ij} = x_i \), when \( i < j \) and \( m_{ij} = y_j \), otherwise. When we expand the permanent by the last column, we obtain the recurrence in (7) with \( q = 0 \) (see Lemma 3.3 in [8] for a proof).

Specializing the \( y \) variables in \( B_n(x, y; q) \) to 1, it follows that

**Corollary 3.13.** For \( q \geq 0 \),

\[ B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{D_T(\sigma)} \]

is stable.
Finally, observe that (the non-homogeneous) $B_n(x; q)$ does reduce to $A_{n-1}(x)$ and $B_n(x)$—the multivariate Eulerian polynomial of type $A$ and type $B$—when $q = 0$ and $q = 1$, respectively. Diagonalizing $x$ in $B_n(x; q)$ yields the polynomial $B_n(x; q)$ defined in (5). We therefore recover Theorem 3.8 as a corollary.

**Corollary 3.14.** For $q \geq 0$, $B_n(x; q)$ is stable.

### 3.3. Eulerian Polynomials for Colored Permutations.

Theorem 3.9 can be extended in two directions simultaneously: from signed permutations to colored permutations, and from a single $q$ variable to several.

Let $\mathbb{Z}_r$ denote the cyclic group of order $r$ with generator $\zeta$. We will take $\zeta$ to be an $r$th primitive root of unity. The wreath product $G_n^r = \mathbb{Z}_r \wr A_{n-1}$ is the semidirect product $(\mathbb{Z}_r)^n \rtimes A_{n-1}$. Its elements can be thought of as $\sigma = (\zeta e_1 \tau_1, \ldots, \zeta e_n \tau_n)$, where $e_i \in \{0, 1, \ldots, r-1\}$ and $\tau \in A_{n-1}$. The group $G_n^r$ is sometimes called the generalized symmetric group, since $A_{n-1} \cong \text{Sym}(n)$. Its elements are also known as $r$-colored permutations, which reduce to signed permutations and ordinary permutations when $r = 2$ and $r = 1$, respectively. In other words, $B_n = G_n$ and $A_{n-1} = G_n^1$.

**Definition 3.3.** Given $\sigma = (\zeta e_1 \tau_1, \ldots, \zeta e_n \tau_n) \in G_n^r$, let $\mathcal{N}(\sigma)$ be the multiset in which each $i \in [n]$ appears $e_i$ times.

Note that for $\sigma \in B_n$, $|\mathcal{N}(\sigma)| = \mathcal{N}(\sigma)$, the number of negative entries in $\sigma = (\sigma_1, \ldots, \sigma_n)$.

We adopt the following total order on the elements of $(\mathbb{Z}_r \times [n]) \cup \{0\}$ (see [1, 2], for example):

$$\zeta^{r-1} n < \cdots < \zeta n < \cdots < \zeta^{r-1} 2 < \cdots < \zeta 2 < \zeta^{r-1} 1 < \cdots < \zeta 1 < 0 < 1 < 2 < \cdots < n.$$  

While other total orders are also being used in the literature (e.g., [24, 28, 11]) our choice allows for similar stability-preserving recurrences as in the previous cases. Using this ordering, the definitions of descent top set and ascent top set all extend verbatim from $B_n$ to $G_n^r$. We shall use $D_T(\sigma)$ and $A_T(\sigma)$ to denote them for $\sigma$ in $G_n^r$. For example, when $\sigma = (3, \zeta^1, \zeta^2, \zeta^4, \zeta^5, \zeta^2) \in G_n^5$, then we have $0 < 3 > \zeta^2 1 > \zeta^2 4 > \zeta^4 5 < \zeta 2$ and hence, $D_T(\sigma) = \{3, 4, 5\}$, and $A_T(\sigma) = \{3, 5\}$.

Brenti’s $B_n(x; q)$ polynomial, defined in (5), to multiple $q$ variables, and proved the following.

**Theorem 3.15 (Corollary 6.5 in [5]).** Let $q = (q_1, \ldots, q_n)$. If $q_i \geq 0$, for $1 \leq i \leq n$, then

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{\mathcal{N}(\sigma)} x^{D_T(\sigma)}.$$  

has only simple real roots.

Next, we extend this result simultaneously to $G_n^r$ and to multiple $x$ variables.

**Theorem 3.16.** If $q_i \geq 0$, for all $1 \leq i \leq n$, then the multivariate $q$-Eulerian polynomial for the generalized symmetric group, $G_n^r$, defined as

$$G_n^r(x, y; q) = \sum_{\sigma \in G_n^r} q^{\mathcal{N}(\sigma)} x^{D_T(\sigma)} y^{A_T(\sigma)}$$  

is stable.
Proof. \( G_1^r(x_1, y_1; q_1) = (q_1 + \cdots + q_1^{r-1})x_1 + y_1 \) is clearly stable when \( q_1 \geq 0 \). The theorem follows immediately from the following recursion. For \( n > 1 \),

\[
G_n^r(x, y; q) = \left[ ((q_n + \cdots + q_n^{r-1})x_n + y_n) + (1 + \cdots + q_n^{r-1})x_n y_n \right] G_{n-1}^r(x, y; q). \square
\]

As a consequence, we obtain a generalization of Corollary 3.10 to \( G_n^r \).

**Corollary 3.17.** Let \( r \geq 2 \). For an \( r \)th root of unity, \( \zeta \neq 1 \), we have

\[
G_n^r(x, y; \zeta, \ldots, \zeta) = (y - x)^n.
\]

Letting \( r = 2 \) also generalizes Theorem 3.11 to multiple \( q \) variables.

**Corollary 3.18.** If \( q_i \geq 0 \), for all \( 1 \leq i \leq n \), then \( B_n(x, y; q) \) is stable.

Diagonalizing \( q \) gives us a result for \( G_n^r \) with a single \( q \) variable.

**Corollary 3.19.** If \( q \geq 0 \), then \( G_n^r(x, y; q) \) is stable.

By diagonalizing \( x \) and specializing \( y_i \) to 1 for all \( 1 \leq i \leq n \), we obtain a result of Steingrimsson.

**Corollary 3.20** (Theorem 17 of [24]).

\[
G_n^r(x) = \sum_{\sigma \in G_n^r} x^{[\widetilde{D}_r(\sigma)]}
\]

has only real roots.

### 4. Stable \( \widetilde{W} \)-Eulerian Polynomials

Dilks, Petersen and Stembridge studied Eulerian-like polynomials associated to affine Weyl groups. They defined the so-called “affine” \( \widetilde{W} \)-Eulerian polynomials as the “affine descent”-generating polynomials over the corresponding finite Weyl group. In [12], they showed that the (univariate) \( \widetilde{W} \)-Eulerian polynomials have only real roots for types A and C, and also for the exceptional types. We strengthen these results for types A and C by giving multivariate stable refinements of these polynomials as well.

#### 4.1. Affine Eulerian Polynomials of Type A

Let \( A_n \) denote the Coxeter group of type A of rank \( n \). The affine descents of type A contain the (ordinary) descents of type A and an extra “affine” descent at 0 if and only if, \( \sigma_{n+1} > \sigma_1 \), where \( \sigma = (\sigma_1, \ldots, \sigma_{n+1}) \in A_n \). Formally,

\[
\widetilde{D}_A(\sigma) = D_A(\sigma) \cup \{0 : \sigma_{n+1} > \sigma_1\}.
\]

See Section 5.1 in [12] for further details.

The definitions of descent top and ascent top sets for type A can be extended in the obvious way. For \( \sigma \in A_n \),

\[
\widetilde{D}_A(\sigma) = D_A(\sigma) \cup \{0 : \sigma_{n+1} > \sigma_1\},
\]

\[
\widetilde{A}_A(\sigma) = A_A(\sigma) \cup \{\sigma_1 : \sigma_{n+1} < \sigma_1\}
\]

and we obtain the following result.
Theorem 4.1.

\[ \tilde{A}_n(x, y) = \sum_{\sigma \in A_n} x^{\tilde{D}_A(\sigma)} y^{\tilde{A}_A(\sigma)} \]

is stable.

Proof. This statement is immediate once we establish the Lemma 4.2. Stability follows, since \( A_n(x, y) \) is stable and the operator on the right-hand side is clearly stability-preserving. □

Lemma 4.2. For \( n > 0 \), we have

\[ \tilde{A}_n(x, y) = (n + 1)x_{n+1}y_{n+1}A_{n-1}(x, y). \]

Proof. Consider a permutation \( \sigma \in A_{n-1} \) with ascent top set \( A \) and descent top set \( D \). We will modify it to obtain a permutation in \( A_n \) with affine ascent top set \( A \cup \{n+1\} \) and affine descent top set \( D \cup \{n+1\} \). Append \( n+1 \) to the end of \( \sigma \) and pick a cyclic rotation of the newly obtained permutation. The new permutation will have the same affine ascent set and affine descent top sets as the ascent and descent top sets \( \sigma \) had and in addition it will have \( n+1 \) both as an affine ascent top and as an affine descent top. To conclude the proof, note that there are exactly \( n+1 \) cyclic rotations. This is essentially a refinement of the proof of Proposition 1.1 of [19]. □

By diagonalizing \( x \) and specializing \( y \) to 1, Lemma 4.2 reduces to an identity discovered by Fulman (Corollary 1 in [16]) and we recover a result of Dilks, Petersen, and Stembridge:

Corollary 4.3 (see Section 4 of [12]).

\[ \tilde{A}_n(x) = \sum_{\sigma \in A_n} x^{|\tilde{D}_A(\sigma)|} \]

has only real roots.

One can construct a recurrence from Lemma 4.2 for the affine Eulerian polynomials \( \tilde{A}_n(x) \) as well, but this recurrence will not preserve stability.

4.2. Affine Eulerian Polynomials of Type C. Let \( C_n \) denote the Coxeter group of type C of rank \( n \). Affine descents of type C consist of the ordinary descent set of type C, which coincides with the descent set of type B (see Proposition 3.6 for type B descents) and an extra “affine” descent at 0 when \( \sigma_n > 0 \). Formally,

\[ \tilde{D}_C(\sigma) = D_C(\sigma) \cup \{0 : \sigma_n > 0\}. \]

See Section 5.2 of [12] for further details.

As in type A, the definition of the type C affine ascent and descent top sets can be adapted from those of type C (equivalently, type B). For \( \sigma \in C_n \), let

\[ \tilde{D}_T_C(\sigma) = D_T_B(\sigma) \cup \{\sigma_n : \sigma_n > 0\}, \]
\[ \tilde{A}_T_C(\sigma) = A_T_B(\sigma) \cup \{\sigma_n : \sigma_n < 0\}. \]

Theorem 4.4.

\[ \tilde{C}_n(x, y) = \sum_{\sigma \in C_n} x^{\tilde{D}_T_C(\sigma)} y^{\tilde{A}_T_C(\sigma)} \]

is stable.
Proof. \( \tilde{C}_1(x_1, y_1) = 2x_1y_1 \) and the following recurrence holds for \( n > 1 \):

\[
\tilde{C}_n(x, y) = 2x_ny_n\partial \tilde{C}_{n-1}(x, y).
\]

We note that a very similar recurrence also appeared in [18] (without the factor of two and with a different initial value), in connection with stable Eulerian polynomials over Stirling permutations.

There is also a direct connection between the polynomials \( \tilde{C}_n(x, y) \) and \( A_n(x, y) \).

**Proposition 4.5.**

\[
\tilde{C}_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 2^n x_ny_n A_{n-1}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}).
\]

*Proof.* Follows by a similar argument as Lemma 4.2.

Once again, this gives a refinement of the univariate identity by Fulman [17]. We mention that multivariate refinements (different from the above) and respective refinements for the identities for \( \tilde{A}_n(x, y) \) and \( \tilde{C}_n(x, y) \) have appeared in [12].

5. **Towards Stable Refinements of** \( D(x), B(x), \tilde{D}(x) \)

### 5.1. Eulerian Polynomials of Type D

Let \( D_n \) denote the Coxeter group of type D of rank \( n \). Recall that \( D_n \) can be thought of as the (order 2) subgroup of \( B_n \) consisting of all the signed permutations with an even number of negative entries. Specifically, \( D_n \) has generators \( S = \{s_0, s_1, \ldots, s_n\} \), where \( s_0 = (-2, 1)(-1, 2) \) and \( s_i = (i, i + 1) \) for \( 1 \leq i \leq n \).

**Proposition 5.1** (Proposition 8.22 of [3]). Given \( \sigma = (\sigma_1, \ldots, \sigma_n) \in D_n \) written in its window notation,

\[
\mathcal{D}_D(\sigma) = \{i \in [n] : \sigma_i > \sigma_{i+1}\},
\]

where \( \sigma_0 \overset{\text{def}}{=} -\sigma_2 \).

Based on the real-rootedness results for types A, B, and the exceptional types (and some computer evidence) Brenti conjectured the following.

**Conjecture 5.2** (Conjecture 5.1 in [9]). *The type D Eulerian polynomial has only real roots.*

This was—and still is—the remaining unproven part of Conjecture 1.1. Unfortunately, extending the type B ascent top and descent top definitions in the naive way does not work.

If we let

\[
\mathcal{D}_D^*(\sigma) = \max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leq i \leq n-1, \sigma_i > \sigma_{i+1},
\]

and

\[
\mathcal{A}_D^*(\sigma) = \max(|\sigma_i|, |\sigma_{i+1}|) : 0 \leq i \leq n-1, \sigma_i < \sigma_{i+1},
\]

then the type D Eulerian polynomial is not multiaffine (for example, the monomial corresponding to \( \sigma = 123 \) is \( y_2y_3^2 \)). Furthermore, it fails to be stable. For \( n = 3 \), we have

\[
D_3^*(x, y) = x_2^2x_3 + 2x_2x_3y_2 + x_3y_2^2 + x_2^2y_3 + 4x_2x_3y_3 + 4x_2^2y_3^2 + 2x_2y_2y_3 + 4x_3y_2y_3 + y_2^2y_3 + 4x_3y_3^2.
\]

When \( y_2 = y_3 = x_3 = 2 + i \), and \( x_2 = (-1 + 2i)(2i + \sqrt{3}) \approx -5.7321 + 1.4641i \) (which are all in the upper half plane), \( D_3^*(x, y) = 0 \). (We use the \( D^*(x, y) \) notation for the multivariate Eulerian polynomial of type D to avoid confusion with a different multivariate generalization which will appear later on.)
Brändén used the refinement given in [8] to prove an intriguing result about enumerating type B descents over type D permutations.

**Theorem 5.3** (Corollary 6.10 in [5]).

\[ \sum_{\sigma \in D_n} x^{\left| B_\rho(\sigma) \right|} \]

has only real roots.

It would be interesting to find a multivariate analog of this result. Again, the straightforward application of our method with the descent top set definition in (10) results in a non-stable polynomial already for \( n = 3 \).

Table 1 gives the type B and type D descents for permutations in \( D_3 \). As demonstrated above, our choice of \( X = x_{|\sigma_0|} \) and \( Y = y_{|\sigma_0|} \) does not result in stable polynomials, though it is possible that some other choice will.

| \( \sigma \) | \( x^{B_\rho} \) | \( x^{D_\rho} \) | \( \sigma \) | \( x^{B_\rho} \) | \( x^{D_\rho} \) | \( \sigma \) | \( x^{B_\rho} \) | \( x^{D_\rho} \) |
|---|---|---|---|---|---|---|---|---|
| 123 | \( Y_y y_3 \) | \( Y_y y_3 \) | 123 | \( X_x y_3 \) | \( X_x y_3 \) | 123 | \( Y_y y_3 \) | \( Y_y y_3 \) |
| 132 | \( Y_y y_3 \) | \( Y_y y_3 \) | 132 | \( X_x y_3 \) | \( X_x y_3 \) | 132 | \( Y_y y_3 \) | \( Y_y y_3 \) |
| 213 | \( Y_x y_3 \) | \( Y_x y_3 \) | 213 | \( X_x y_3 \) | \( X_x y_3 \) | 213 | \( Y_x y_3 \) | \( Y_x y_3 \) |
| 231 | \( Y_y y_3 \) | \( Y_y y_3 \) | 231 | \( X_x y_3 \) | \( X_x y_3 \) | 231 | \( Y_y y_3 \) | \( Y_y y_3 \) |
| 312 | \( Y_y y_3 \) | \( Y_y y_3 \) | 312 | \( X_x y_3 \) | \( X_x y_3 \) | 312 | \( Y_y y_3 \) | \( Y_y y_3 \) |
| 321 | \( Y_y y_3 \) | \( Y_y y_3 \) | 321 | \( X_x y_3 \) | \( X_x y_3 \) | 321 | \( Y_y y_3 \) | \( Y_y y_3 \) |

**Table 1.** Type B and type D descents over the permutations in \( D_3 \).

Another result related to resolving Conjecture 5.2 came from [10], in which Chow found a recurrence for \( D_n(x) \). Sadly, the resulting formula is much more complicated than its counterparts in types A and B, and does not seem amenable to a multivariate generalization. We reproduce the recurrence here, fixing a typo from the original paper (the term in the box was mistakenly written with a minus sign).

**Theorem 5.4** (cf. Theorem 5.3 in [10]). Let \( D_{-1}(x) = D_0(x) = D_1(x) = 1 \) and for \( n \geq 0 \)

\[
D_{n+2}(x) = (n(1 + 5x) + 4x) D_{n+1}(x) \\
+ 4x(1 - x) D'_{n+1}(x) \\
+ ((1 - x)^2 - n(1 + 3x)^2 - 4n(n - 1)x(1 + 2x)) D_n(x) \\
- (4nx(1 - x)(1 + 3x) + 4x(1 - x)^2) D'_n(x) \\
- 4x^2(1 - x)^2 D''_n(x) \\
+ \left( 2n(n - 1)x(3 + 2x + 3x^2) + 4n(n - 1)(n - 2)x^2(1 + x) \right) D_{n-1}(x) \\
+ (2nx(1 - x)^2(3 + x) + 8n(n - 1)x^2(1 - x)(1 + x)) D'_{n-1}(x) \\
+ 4nx^2(1 - x)^2(1 + x) D''_{n-1}(x).
\]
Encouraged by the simplicity of our methods for types A and B, we propose a new line of attack. Stembridge showed that the Eulerian polynomials of types A, B, and D are related via the following identity. This result was discovered independently by Brenti (see Corollary 4.8 and Theorem 4.7 in [9] for a “q-analog”), who also points out that if follows from Theorem 4.2 of [20].

**Theorem 5.5** (Lemma 9.1 of [23]). For \( n \geq 2 \),

\[
D_n(x) = B_n(x) - n2^{n-1}x A_{n-2}(x),
\]

where \( A_n(x) \), defined in (11), is the descent generating function in \( A_n \cong \text{Sym}(n+1) \).

By replacing the univariate polynomials \( B_n(x) \) and \( A_n(x) \) in (11) by their stable multivariate generalizations given in (2) and (3), respectively, we obtain the following multivariate refinement of \( D_n(x) \):

\[
D_n(x, y) = B_n(x, y; 1) - n2^{n-1}x_n y_n A_{n-2}(x, y).
\]

For example, when \( n = 2 \), we obtain

\[
D_2(x_1, x_2, y_1, y_2) = B_2(x_1, x_2, y_1, y_2; 1) - 4x_2 y_2 A_0(x_1, y_1) = (x_1 + y_1)(x_2 + y_2),
\]

and when \( n = 3 \), the polynomial is

\[
D_3(x, y) = x_1 x_2 x_3 + x_2 x_3 y_1 + x_1 x_3 y_2 + x_3 y_1 y_2 + x_1 x_2 y_3 + x_2 y_1 y_3 + x_1 y_2 y_3 + y_1 y_2 y_3
\]
\[+ 4(x_2 x_3 y_2 + x_1 x_3 y_3 + x_2 y_2 y_3 + x_3 y_1 y_3).\]

These polynomials fail to be stable, even for \( n = 3 \). This follows from Theorem [23] since specializing the \( y \) variables to 1 gives

\[
D_3(x) = 1 + x_1 + x_1 x_2 + x_1 x_2 x_3 + 5(x_2 + x_3 + x_1 x_3 + x_2 x_3)
\]

and

\[
\frac{\partial D_3(x)}{\partial x_1} \cdot \frac{\partial D_3(x)}{\partial x_3} - \frac{\partial^2 D_3(x)}{\partial x_1 \partial x_3} \cdot D_3(x) = -16x_2,
\]

which fails to be nonnegative for \( x_2 > 0 \). It is possible that a different refinement would result in a family of stable polynomials, from which real-rootedness would follow.

It does seem to be the case, however, that the coefficients remain non-negative.

**Conjecture 5.6.** \( D_n(x, y) \in \mathbb{N}[x, y] \) for all \( n \geq 2 \).

We have verified this conjecture by computer for \( n \leq 11 \). If true, then (12) suggests a refinement of the descent statistic of type D which in turn could lead to a new recursion for the type D Eulerian polynomials.

5.2. **Affine Eulerian Polynomials of Types B and D.** Dilks, Petersen, and Stembridge noted that the only missing cases from Conjecture [12] are types B and D. They considered multivariate refinements of these polynomials, but they indexed the variables by descents and not by descent tops. It was noted in [18] that polynomials indexed by the descents fail to be stable for type A.

Two new identities relating ordinary and affine Eulerian polynomials also appeared in [12].

**Proposition 5.7** (Proposition 6.1 in [12]). For \( n \geq 2 \),

\[
2\widetilde{C}_n(x) = \widetilde{B}_n(x) + 2nx C_{n-1}(x).
\]
This identity could be used to obtain a multivariate refinement for the unsettled type B case in a similar way as suggested for type D above. Let

\[
\tilde{B}_n(x, y) = 2\tilde{C}_n(x, y) - 2nx_ny_nB_{n-1}(x, y; 1).
\]

In contrast to the type D case, these polynomials turn out to be stable for \( n \leq 4 \) (we have verified this by computer calculations). Therefore, we suggest to investigate whether this is a stable multivariate of the affine Eulerian polynomial of type B.

**Conjecture 5.8.** \( \tilde{B}_n(x, y) \in \mathbb{G}_R[x, y] \), for \( n \geq 2 \).

Also, this multivariate refinement seems to be monomial positive.

**Conjecture 5.9.** \( \tilde{B}_n(x, y) \in \mathbb{N}[x, y] \), for \( n \geq 2 \).

Finally, for sake of completeness, we give another identity by Dilks, Petersen and Stembridge which relates the following polynomials.

**Proposition 5.10** (Proposition 6.2 in [12]). For \( n \geq 3 \),

\[
\tilde{B}_n(x) = \tilde{D}_n(x) + 2nxD_{n-1}(x).
\]

This identity might be helpful for finding a multivariate refinement of the affine Eulerian polynomial of type D.

### 6. Conclusion and further remarks on the statistics \( AT, DT \)

In this paper, we have extended the stability results for the multivariate type A Eulerian polynomial to Eulerian polynomials (and affine Eulerian polynomials) of some other Coxeter groups and the generalized permutation group \( G_{n,r} \). A crucial step in our proofs was to find a suitable generalization of the descent top and ascent top statistics to these groups.

The descent statistic \( \{i : \sigma_i > \sigma_{i+1}\} \) for permutations has a long history, going back to the works of Carlitz and Riordan. Recall that for a permutation of length \( n \) there are \( 2^{n-1} \) possible descent (and ascent) sets, as each position \( i \in [n-1] \) can either be a descent or an ascent. In particular, for any subset \( S \subseteq [n-1] \) there is a permutation whose descent set is \( S \). Finding the number of permutations with a given descent (and ascent) set is a classical problem—one uses a standard inclusion-exclusion argument (see Example 2.2.4 in [22]).

We now find the possible descent and ascent top sets (of type A), which corresponds to the possible monomials in \( A_n(x, y) \). (From here on, \( DT \) and \( AT \) refers to the descent top and ascent top sets of type A.)

**Proposition 6.1.** Given \( (DT, AT) \subset [n] \times [n] \) define \( P = DT \cap AT \) and \( V = [n] \setminus (DT \cup AT) \). The pair \( (DT, AT) \) is the descent top and ascent top set of some permutation of \([n]\) if and only if, for all \( i = 1, \ldots, n \), \( |[i] \cap V| > |[i] \cap P| \).

**Proof.** Clear from the recurrence given in (3). \( \square \)

In fact, the \((DT, AT)\) pair of statistics turns out to be equivalent to a well-known statistic studied by Françon and Viennot [14], called “type”—the quadruple of statistics consisting of the set of peaks, valleys, double descents, and double ascents. In particular, the set of peaks \( \{\sigma_i : \sigma_{i-1} < \sigma_i > \sigma_{i+1}\} \) is \( P \), defined above, and similarly, the set of valleys \( \{\sigma_i : \sigma_{i-1} > \sigma_i < \sigma_{i+1}\} \) is \( V \). For the set of double descents \( DD := \{\sigma_i : \sigma_{i-1} > \sigma_i > \sigma_{i+1}\} \) we have that \( DD = DT \setminus AT \) (and similarly for the double ascents we have \( DA = AT \setminus DT \)).

In light of the above, the following theorem should not be a surprise.
Theorem 6.2. The number of distinct monomials in the expansion of $A_{n-1}(x, y)$ is counted by $C_n = \frac{1}{n+1} \binom{2n}{n}$, the $n$th Catalan number.

Proof. We give a bijection from $(\mathcal{D} \mathcal{T}, \mathcal{A} \mathcal{T}) \in [n] \times [n]$ to 2-colored Motzkin paths of length $n - 1$. A 2-colored Motzkin path of length $n - 1$ is a lattice path of $\mathbb{N}^2$ running from $(0, 0)$ to $(n - 1, 0)$ that never goes below the $x$-axis and whose allowed steps are:

- NE steps $(1, 1)$,
- SE steps $(1, -1)$, and
- Two different colors of E steps $(1, 0)$, denoted $\mathcal{E}$ and $\overline{\mathcal{E}}$.

For $j = 2, \ldots, n$, let the $(j - 1)$st step of the Motzkin path be SE, NE, $\mathcal{E}$, or $\overline{\mathcal{E}}$ depending on which set $(\mathcal{P}, \mathcal{V}, \mathcal{D} \mathcal{D}, \text{ or } \mathcal{D} \mathcal{A}$, respectively) $j$ belongs to. The 2-colored Motzkin paths of length $n - 1$ are known to be counted by the Catalan numbers $C_n$ (see, for example, [23]).

Exploring this connection, we can also give a formula for the coefficient on a particular monomial $x^{2\mathcal{D} \mathcal{T}} y^{4\mathcal{A} \mathcal{T}}$—that is, the number of permutations with a given descent and ascent top set. Following Viennot, we define a valuation on these 2-colored Motzkin paths. NE and SE steps starting at height $k \geq 0$ are given a weight $a_k = c_k = k + 1$. We give $\mathcal{E}$ steps a weight of $b_k = k + 1$ and $\overline{\mathcal{E}}$ steps a weight of $\overline{b}_k = k + 1$, which is equivalent to forgetting that we had two different types of E steps and weighting an (uncolored) east step by $b_k = 2k + 2$. Viennot’s theory of Laguerre histories [26], which gives a combinatorial interpretation for the moments of the Laguerre polynomials, allows us to then recover the permutations with given descent and ascent top sets. In particular, the number of permutations associated to a path is simply the product of the weights of the steps in the path.

Similar reasoning works for type B as well.

Theorem 6.3. The number of distinct monomials in the expansion of $B_n(x, y; 1)$ is counted by $C_n$, the $n$th Catalan number.

Likewise, the coefficients on a particular monomial in $B_n(x, y; 1)$ are given by Viennot’s theory by using the weights $a_k = 2(k + 1), b_k = 4k + 2$, and $c_k = 2k$. It is a simple extension to show that a monomial $x^{2\mathcal{D} \mathcal{T}} y^{4\mathcal{A} \mathcal{T}}$ in $B_n(x, y; q)$ has a $q$-coefficient given by weighting NE steps by $a_k = (k + 1)(1 + q), \mathcal{E}$ steps by $\overline{b}_k = 1 + k(1 + q)$, $\overline{\mathcal{E}}$ steps by $\overline{b}_k = q + k(1 + q)$, and SE steps by $c_k = k(1 + q)$. It is also easy to extend these results to $G_n(x, y; q)$.

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Note added in proof. C. D. Savage and the first author have resolved Conjecture 5.2 (and thus Conjecture 1.1) and part of Conjecture 1.2 in [21].
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Appendix A. List of stable multivariate Eulerian polynomials

\[ A_0(x, y) = 1 \]
\[ A_1(x, y) = x_2 + y_2 \]
\[ A_2(x, y) = x_2x_3 + x_3y_2 + x_2y_3 + 2x_3y_3 + y_2y_3 \]
\[ B_1(x, y) = x_1 + y_1 \]
\[ B_2(x, y) = x_1x_2 + x_2y_1 + x_1y_2 + 4x_2y_2 + y_1y_2 \]
\[ \tilde{B}_1(x, y; q) = qx_1 + y_1 \]
\[ \tilde{B}_2(x, y; q) = q^2x_1x_2 + qx_2y_1 + q^2y_2 + (1 + q)^2x_2y_2 + y_1y_2 \]
\[ \tilde{A}_1(x, y) = 2x_2y_2 \]
\[ \tilde{A}_2(x, y) = 2x_2x_3y_3 + 2x_3y_2y_3 \]
\[ \tilde{A}_3(x, y) = 2x_2x_3x_4 + 2x_3x_4y_2y_4 + 2x_2x_4y_3y_4 + 4x_3x_4y_3y_4 + 2x_4y_2y_3y_4 \]
\[ \tilde{C}_1(x, y) = 2x_1y_1 \]
\[ \tilde{C}_2(x, y) = 4x_1x_2y_2 + 4x_2y_1y_2 \]
\[ \tilde{C}_3(x, y) = 8x_1x_2x_3y_3 + 8x_2x_3y_1y_3 + 8x_1x_3y_2y_3 + 16x_2x_3y_2y_3 + 8x_3y_1y_2y_3 \]