THE COX RING OF A SPHERICAL EMBEDDING

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Abstract. Let $G$ be a connected reductive group and $G/H$ a spherical homogeneous space. We show that the ideal of relations between a natural set of generators of the Cox ring of a $G$-embedding of $G/H$ can be obtained by homogenizing certain equations which depend only on the homogeneous space. Using this result, we describe some examples of spherical homogeneous spaces such that the Cox ring of any of their $G$-embeddings is defined by one equation.

Introduction

Throughout the paper, we work with algebraic varieties and algebraic groups over the field of complex numbers $\mathbb{C}$.

Let $Y$ be a normal irreducible variety whose divisor class group $\text{Cl}(Y)$ is finitely generated and $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}^*$. The Cox ring of $Y$ is the $\text{Cl}(Y)$-graded $\mathbb{C}$-algebra

$$R(Y) := \bigoplus_{[D] \in \text{Cl}(Y)} \Gamma(Y, \mathcal{O}_Y(D)),$$

with the multiplication defined by the canonical maps

$$\Gamma(Y, \mathcal{O}_Y(D_1)) \otimes \Gamma(Y, \mathcal{O}_Y(D_2)) \to \Gamma(Y, \mathcal{O}_Y(D_1 + D_2)).$$

Some accuracy is required in order for this multiplication to be well-defined (cf. [Hau08] or [ADHL10] for details).

It has been shown by Cox that $R(Y)$ is a polynomial ring if $Y$ is a toric variety (cf. [Cox95]). The converse was obtained by Hu and Keel for smooth projective varieties (cf. [HK00, Corollary 2.10]). Toric varieties can be considered as examples in the more general class of spherical varieties, which are quasihomogeneous with respect to the action of a connected reductive group $G$.

The aim of this paper is to investigate the Cox ring of an arbitrary spherical variety.

We recall some standard notions from the theory of spherical varieties. Let $G$ be a connected reductive group. A closed subgroup $H \subseteq G$ is called spherical if a Borel subgroup $B \subseteq G$ has an open orbit in $G/H$. Then $G/H$ is called a spherical homogeneous space. In this case, we may always assume that $BH$ is open in $G$. A $G$-equivariant open embedding $G/H \hookrightarrow Y$ into a normal irreducible $G$-variety $Y$ is called a spherical embedding, and $Y$ is called a spherical variety. Let $p : G' \to G$ be a finite covering such that $G'$ is of simply connected type, i.e. $G' = G^{ss} \times C$ where $G^{ss}$ is semisimple simply connected and $C$ is a torus. Then $G/H \cong G'/p^{-1}(H)$, and these two spherical homogeneous spaces have exactly the same embeddings. Therefore in this paper we will always assume $G = G^{ss} \times C$. Similarly to the theory of toric varieties, spherical embeddings $G/H \hookrightarrow Y$ can be described by some combinatorial data introduced by Luna and Vust (cf. [LV83] and [Kno91]).

We denote by $\mathcal{M}$ the weight lattice of $B$-eigenvectors in the function field $\mathbb{C}(G/H)$ and by $\mathcal{N} := \text{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice with the natural pairing $\langle \cdot, \cdot \rangle : \mathcal{N} \times \mathcal{M} \to \mathbb{Z}$. If $\nu : \mathbb{C}(G/H) \to \mathbb{Q}$ is a discrete valuation, its restriction to $B$-eigenvectors induces a map $u : \mathcal{M} \to \mathbb{Q}$, which lies in the vector space $\mathcal{N}_\mathbb{Q} := \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote the set of $G$-invariant discrete valuations on $\mathbb{C}(G/H)$ by $\mathcal{V}$. The above assignment is
injective on $V$ and therefore defines an inclusion $V \subseteq \mathcal{N}_G$. The set $V$ is a polyhedral convex cone called the valuation cone of $G/H$. In fact, the cone $V$ is the intersection of half-spaces $\{u \in \mathcal{N}_G : \langle u, \gamma_i \rangle \leq 0\}$ where $\gamma_1, \ldots, \gamma_s$ are linearly independent primitive elements in the lattice $\mathcal{M}$ called the spherical roots of $G/H$.

A spherical embedding $G/H \hookrightarrow Y$ is called a wonderful completion if $Y$ is complete, smooth, and contains exactly one closed $G$-orbit. Then $Y$ is called a wonderful variety. A wonderful completion of $G/H$ exists (and is unique) if and only if the valuation cone $V$ is spanned by a basis of $\mathcal{N}$. To any spherical homogeneous space $G/H$ one can associate in a natural way a closed subgroup $H' \times C \subseteq N_G(H)$ of finite index and containing $H$ such that a wonderful completion $G/(H' \times C) \cong G^\ast/H' \hookrightarrow Y'$ exists (cf. Luna01).

Let $G/H \hookrightarrow Y$ be a spherical embedding. As the Cox ring does not depend on $G$-orbits of codimension two or greater, we may assume that $Y$ contains only non-open $G$-orbits of codimension one, which are exactly the $G$-invariant prime divisors $Y_1, \ldots, Y_n$ in $Y$. To each $Y_i$ we assign the primitive element $u_i \in \mathcal{N}$ corresponding to the discrete valuation on $\mathbb{C}(G/H)$ induced by $Y_i$. It follows from the Luna-Vust theory that the embedding $G/H \hookrightarrow Y$ can be described combinatorially by the fan $\Sigma$ in $\mathcal{N}_G$ consisting exactly of the trivial cone 0 and the one-dimensional cones $\sigma_i := \langle \mathcal{Q} \rangle u_i$, which lie in the valuation cone $V$.

The Cox ring of $Y$ has been described by Brion (cf. Bri07). In fact, he describes the equivariant Cox ring $\mathcal{R}_G(Y) \cong \mathcal{R}(Y) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{C}]$. As a first step, he computes the Cox ring $\mathcal{R}(Y')$ of the associated wonderful completion $G^\ast/H' \hookrightarrow Y'$ as the Rees algebra of a certain filtration of the ring $\Gamma(G^\ast/K', \mathcal{O}_{G^\ast/K'})$ where $K'$ is the intersection of the kernels of all characters of $H'$. The equivariant Cox ring $\mathcal{R}_G(Y)$ is then obtained from $\mathcal{R}(Y')$ by a base change. There is a natural bijection between the set $\{\gamma_1, \ldots, \gamma_s\}$ of spherical roots of $G^\ast/H'$ and the set $\{Y'_1, \ldots, Y'_n\}$ of $G$-invariant prime divisors in $Y'$. It immediately follows from the result of Brion that

$$\mathcal{R}(Y) \cong \mathcal{R}(Y') \otimes_{\mathbb{C}[Z_1, \ldots, Z_s]} \mathbb{C}[W_1, \ldots, W_n]$$

where $\mathbb{C}[Z_1, \ldots, Z_s]$ and $\mathbb{C}[W_1, \ldots, W_n]$ are polynomial rings. The homomorphism $\mathbb{C}[Z_1, \ldots, Z_s] \rightarrow \mathcal{R}(Y')$ sends $Z_i$ to the canonical section in $\Gamma(Y', \mathcal{O}_{Y'}(Y'_i))$, and the homomorphism $\mathbb{C}[Z_1, \ldots, Z_s] \rightarrow \mathbb{C}[W_1, \ldots, W_n]$ sends $Z_i \mapsto \prod_{i=1}^n W_i^{-(u_i, \gamma_i)}$.

Our main result is another description of the Cox ring $\mathcal{R}(Y)$. We will reduce the general case to the case where the spherical homogeneous space $G/H$ has trivial divisor class group and is therefore quasiaffine. We show that the Cox ring $\mathcal{R}(Y)$ is obtained by homogenizing the equations of the affine closure of $G/H$ inside an appropriately constructed embedding $G/H \hookrightarrow \mathbb{C}^d \times (\mathbb{C}^*)^m$. As a byproduct, we obtain a description of the valuation cone $V$ in terms of tropical algebraic geometry. We will compare our results with the approach of Brion in Section 5.

We now give a detailed summary of our results. We assume that the homogeneous space $G/H$ has trivial divisor class group. Let $D := \{D_1, \ldots, D_r\}$ be the set of $B$-invariant prime divisors in $G/H$, and assume that the spherical embedding $G/H \hookrightarrow Y$ satisfies $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}^\ast$. We choose prime elements $f_1, \ldots, f_r \in \Gamma(G/H, \mathcal{O}_{G/H})$ with $\forall(f_i) = D_i$ and obtain irreducible $G$-modules $V_i := \langle G \cdot f_i \rangle \subseteq \Gamma(G/H, \mathcal{O}_{G/H})$. For each $i$ we set $s_i := \dim V_i$ and choose a basis $\{f_{ij}\}_{j=1}^{s_i} \subseteq G \cdot f_i$ of $V_i$ with $f_{i1} = f_i$. We let $m$ be the rank of the finitely generated free abelian group $\mathcal{R}(G/H, \mathcal{O}_{G/H})$ tensor $\mathbb{Q}$ with $\mathbb{Q}$ to be the valuation $\mathcal{O}(G/H)$ and $\mathcal{O}^\ast(G/H)$ where $p_2 : G \rightarrow C$ denotes the projection and choose representatives $\{g_k\}_{k=1}^m \subseteq \Gamma(G/H, \mathcal{O}_{G/H})$ of a basis. The $B$-weights of
with trivial divisor class group is given in Section 2. Then we explain in Section 3
how the results can be extended to arbitrary spherical homogeneous spaces. In

The next step is to define a G-equivariant locally closed embedding
\[ G/H \hookrightarrow Z \coloneqq V^* \times T \cong C^{s_1+\ldots+s_r} \times (C^*)^m \]
where \( V^* \) is dual to \( V \coloneqq V_1 \oplus \ldots \oplus V_r \). We have \( \mathbb{C}[Z] = S(V) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{M}_T] \), where \( S(V) \)

denotes the symmetric algebra of \( V \). The coordinate ring of \( V^*_i \) is the symmetric

algebra \( S(V^*_i) \), whose generators corresponding to the above basis we denote by \( S_{ij} \)
for \( 1 \leq j \leq s_i \), i.e. \( \mathbb{C}[V^*_i] = S(V^*_i) = \mathbb{C}[S_{i1}, \ldots, S_{is_i}] \). We denote the generators of
the coordinate ring of \( T \) corresponding to the above basis of \( \mathcal{M}_T \) by \( T_k \) for \( 1 \leq k \leq m \),
i.e. \( \mathbb{C}[T] = \mathbb{C}[\mathcal{M}_T] = \mathbb{C}[T_1^{\pm 1}, \ldots, T_m^{\pm 1}] \). The locally closed embedding \( G/H \hookrightarrow Z \)
is then given by the \( G \)-equivariant surjective map \( \mathbb{C}[Z] \to \Gamma(G/H, \mathcal{O}_{G/H}) \)
sending \( S_{ij} \mapsto f_{ij} \) and \( T_k \mapsto g_k \). Its kernel is the prime ideal \( \mathcal{I}(G/H) \). Considering
the natural action of the torus \( \text{Spec}(\mathbb{C}[\mathcal{M}_T]) \) on \( Z \), we obtain a corresponding \( \mathcal{M} \)-grading
on the coordinate ring \( \mathbb{C}[Z] \). For \( f \in \mathbb{C}[Z] \) and \( \mu \in \mathcal{M} \) we denote the \( \mu \)-homogeneous
component of \( f \) by \( f^{(\mu)} \).

In order to describe the relations of \( \mathcal{R}(Y) \), we define a homogenization operation
in two steps. The first step is the map \( \alpha : \mathbb{C}[Z] \to (\mathbb{C}[Z])[W_1, \ldots, W_n] \) defined as follows. For each \( f \in \mathbb{C}[Z] \) and \( u \in \mathcal{N} \) we define
\[
\text{ord}_u(f) := \min_{\mu \in \mathcal{M}} \{ (u, \mu); f^{(u)} \neq 0 \},
\]
and set
\[ f^\alpha := \sum_{\mu \in \mathcal{M}} \left( f^{(u)} \prod_{l=1}^n W_l^\text{ord}_l(f) \right). \]
The second step is the map \( \beta : (\mathbb{C}[Z])[W_1, \ldots, W_n] \to \mathbb{C}[S(V)[W_1, \ldots, W_n] \)
sending \( T_k \mapsto 1 \) for each \( 1 \leq k \leq m \). Finally, we define the map \( h : \mathbb{C}[Z] \to \mathbb{C}[S(V)[W_1, \ldots, W_n] \)
by composing the two steps, i.e. \( h := \beta \circ \alpha \). Note that we write
the application of the maps \( \alpha \), \( \beta \), and \( h \) as exponents, for example we write
\( f^h \) instead of \( h(f) \). We can now state our main result.

**Main Theorem.** We have
\[ \mathcal{R}(Y) \cong S(V)[W_1, \ldots, W_n]/(f^h; f \in \mathcal{I}(G/H)), \]
with \( \text{Cl}(Y) \)-grading given by \( \deg(S_{ij}) = [D_i] \) and \( \deg(W_i) = [Y_i] \). If \( H \) is connected,
\( \mathcal{R}(Y) \) is a factorial ring.

In particular, if \( \mathcal{I}(G/H) = (f) \) is a principal ideal generated by \( f \), then the ideal
of relations of \( \mathcal{R}(Y) \) is generated by \( f^h \).

In the special case of a toric variety \( Y \), the Main Theorem reduces to the result of
Cox that \( \mathcal{R}(Y) \) is a polynomial ring with one variable per \( G \)-invariant prime divisor
because the homogeneous space has no \( B \)-invariant prime divisors, i.e. \( V = (0), \)
and \( \mathcal{I}(G/H) = (0) \). In the special case of a horospherical variety \( Y \), we will see
that \( \mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \ldots, W_n] \), where \( P := N_G(H) \), i.e. the Cox ring of \( Y \)
is a polynomial ring over the Cox ring of \( G/P \). This also follows directly from the
description of the Cox ring by Brion.

The paper is organized in five sections. In Section 1 we present some information
about the valuation cone \( \mathcal{V} \) and describe it using tropical algebraic geometry. The
proof of the Main Theorem in the crucial case of a spherical homogeneous space
with trivial divisor class group is given in Section 2. Then we explain in Section 3
how the results can be extended to arbitrary spherical homogeneous spaces. In
Section 4 we illustrate our results by some explicit examples. Finally, we compare our results with the approach of Brion in Section 5.

1. The valuation cone $\mathcal{V}$

We continue to use the notation and the assumptions from the introduction in this section. It was shown in [Br90] and [Kno94] that the valuation cone $\mathcal{V}$ is a fundamental chamber for the action of a crystallographic reflection group $W_{G/H}$ on $\mathcal{N}_Q$ called the little Weyl group (cf. [Tim11] Theorem 22.13). In particular, the cone $\mathcal{V}$ is cosimplicial, and the set of spherical roots is the set of simple roots of a root system with Weyl group $W_{G/H}$. We have the isotypic decomposition into $G$-modules

$$\Gamma(G/H, \mathcal{O}_{G/H}) = \bigoplus_{\mu} V_{\mu},$$

where $\mu$ runs over pairwise distinct elements of $\mathcal{M}$ and $V_\mu$ is an irreducible $G$-module of highest weight $\mu$. Let $\mathcal{T}$ be the cone in $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the elements of the form $\mu_1 + \mu_2 - \mu_3$ such that

$$V_{\mu_3} \subseteq V_{\mu_1} \cdot V_{\mu_2}.$$

It follows from [AB05] Proposition 2.13 that the cone $\mathcal{T}$ is polyhedral. The cone $\mathcal{T}$ is equal to the cone generated by the spherical roots, i.e. $-\mathcal{V}$ is the dual cone of $\mathcal{T}$ (cf. [Los09] Section 3 and [Kno91] Section 5).

In this sense, the valuation cone $\mathcal{V}$ is related to the failure of the $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ being $\mathcal{M}$-graded. We have the following extreme case: a spherical homogeneous space $G/H$ is called horospherical if $\mathcal{V} = \mathcal{N}_Q$. In our setting, this is equivalent to the $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ being $\mathcal{M}$-graded. Another characterization of horospherical homogeneous spaces is that $H$ contains a maximal unipotent subgroup of $G$ (cf. [Pan84] and [Kno91] Corollary 6.2).

We will state a result about the valuation cone which uses tropical algebraic geometry. An introduction to this subject can be found in [Mac12]. Let $X$ be the toric variety associated to a fan $\Sigma_X$ in $\mathcal{N}_Q$ with torus $\mathbb{T}$. If $S \subseteq \mathbb{T}$ is a closed subset, the tropicalization $\text{trop}(S) \subseteq \mathcal{N}_Q$ of $S$ is the support of a polyhedral fan in $\mathcal{N}_Q$. It gives an answer to the question of which $\mathbb{T}$-orbits in $X$ intersect the closure $\overline{S}$ of $S$ inside $X$: by a result of Tevelev (cf. [Tev07]), the $\mathbb{T}$-orbit corresponding to $\sigma \in \Sigma_X$ intersects $\overline{S}$ if and only if trop$(S)$ intersects relint$(\sigma)$.

Our plan is to use this machinery on the embedding $G/H \hookrightarrow Z$. First, we show that this is indeed a locally closed embedding. Along the way, we also prove some other claims which have been stated in the introduction.

Remark 1.1. By [KKV89] Proposition 1.3], every $f_i$ is a $B$-eigenvector, all elements of $\Gamma(G/H, \mathcal{O}^*_{G/H})$ are $G$-eigenvectors, and the quotient $\Gamma(G/H, \mathcal{O}^*_{G/H})/\mathbb{C}^*$ is a finitely generated free abelian group. Moreover, for each $f_i$ the $G$-module spanned by $G \cdot f_i$ in $\Gamma(G/H, \mathcal{O}_{G/H})$ is irreducible and finite-dimensional (cf. [Kra84] III.1.5]).

Proposition 1.2. The $B$-eigenvectors in $\Gamma(G/H, \mathcal{O}_{G/H})$ are given by

$$\Gamma(G/H, \mathcal{O}_{G/H})^{(B)} = \left\{ c f_1^{d_1} \cdots f_r^{d_r} \mid c \in \Gamma(G/H, \mathcal{O}_{G/H}^*), d_i \in \mathbb{N}_0 \right\}.$$

Proof. For each $f \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ all irreducible components of $\mathcal{V}(f)$ are $B$-invariant since $B$ is irreducible. \hfill \Box

Proposition 1.3. The $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ is generated by $\{ f_{ij}, g^{k \pm 1}_k \}$.
We denote by $\chi_f$ its $B$-weight. For the weight lattice $\mathcal{M}$ of $B$-eigenvectors in $\mathcal{C}(G/H)$ we then have
$$\mathcal{M} = \{\chi \in \mathfrak{X}(B); \text{there exists } f \in \mathcal{C}(G/H)^{(B)} \text{ with } \chi = \chi_f\}. $$
Recall that there is an exact sequence (cf. [Kno91, after 1.7])
$$1 \to \mathbb{C}^* \to \mathcal{C}(G/H)^{(B)} \to \mathcal{M} \to 0. $$
We define $v_i^* := \chi_{f_i}$ and $w_k^* := \chi_{g_k}$.

**Proposition 1.4.** The lattice $\mathcal{M} \subseteq \mathfrak{X}(B)$ is freely generated by
$$\{v_1^*, \ldots, v_r^*, w_1^*, \ldots, w_m^*\}. $$

**Proof.** Every $f \in \mathcal{C}(G/H)^{(B)}$ can be written as $\frac{g}{h}$ with $g, h \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ since Supp(div($f$)) is the union of $B$-invariant prime divisors. The claim then follows from Proposition 1.2, the exact sequence above, and the fact that $\{g_k\}_{k=1}^n$ is a basis of $\Gamma(G/H, \mathcal{O}_{G/H})/\mathbb{C}$. 

We denote the corresponding dual basis of $\mathcal{N}$ by $\{v_1, \ldots, v_r, w_1, \ldots, w_m\}$. We also have $\mathcal{M}_V = \langle v_1^*, \ldots, v_r^* \rangle$ and $\mathcal{M}_T = \langle w_1^*, \ldots, w_m^* \rangle$.

**Lemma 1.5.** Let $0 \neq h \in \Gamma(G/H, \mathcal{O}_{G/H})$ and $g \in G$ such that $g \cdot h = ch$ where $c$ is a unit. Then we have $c \in \mathbb{C}$.

**Proof.** We recursively define $h_0 := h$ and $h_{i+1} := g \cdot h_i$ for $i \in \mathbb{N}_0$. As $h$ is contained in a finite-dimensional $G$-module, there exists $d \in \mathbb{N}$ such that $\lambda_0 h_0 + \ldots + \lambda_d h_d = 0$ for some coefficients $\lambda_i \in \mathbb{C}$. As $c$ is a $G$-eigenvector (cf. [KKV89 Proposition 1.3]), dividing by $h$ yields a polynomial relation for $c$. Hence $c$ is algebraic over $\mathbb{C}$.

**Proposition 1.6.** The $f_{ij}$ are pairwise nonassociated prime elements.

**Proof.** For every $f_{ij}$ we have $f_{ij} \in G \cdot f_i$ and $f_i$ is prime, hence $f_{ij}$ is also prime.

Now let $f_{i_1 j_1} = c f_{i_2 j_2}$ for some unit $c$. Then $g \cdot f_{i_1} = c' f_{i_2}$ for some unit $c'$ and $g \in G$. By Proposition 1.4 we have $i_1 = i_2$, so there exists $g' \in G$ with $g' \cdot f_{i_2 j_2} = f_{i_1 j_1} = c f_{i_2 j_2}$. By Lemma 1.5 it follows that $c \in \mathbb{C}$, therefore $j_1 = j_2$. 

The map defined in the introduction
$$\Phi : S(V) \otimes \mathbb{C}[T] \to \Gamma(G/H, \mathcal{O}_{G/H})$$
$$S_{ij} \mapsto f_{ij}$$
$$T_k \mapsto g_k, $$
is surjective by Proposition 1.3 and $G$-equivariant, hence indeed induces a locally closed $G$-equivariant embedding $G/H \hookrightarrow Z$ with respect to the corresponding $G$-action on $Z$.

Finally, we have to explain how $Z$ can be naturally regarded as a toric variety. We denote by $M$ the finitely generated free abelian group with basis
$$\{S_{ij}, T_k; 1 \leq i \leq r, 1 \leq j \leq s_i, 1 \leq k \leq m\}, $$
which is isomorphic to $\mathbb{Z}^{s_1 + \ldots + s_r + m}$. We define the torus $T := \text{Spec}(\mathbb{C}[M])$ with character lattice $M$, denote the dual lattice by $N := \text{Hom}(M, \mathbb{Z})$, denote the corresponding dual basis by
$$\{v_{ij}, w_k; 1 \leq i \leq r, 1 \leq j \leq s_i, 1 \leq k \leq m\}, $$
and set $N := N \otimes \mathbb{Z} \mathbb{Q}$. The surjective map $M \rightarrow M$ sending $S_{ij} \mapsto v_i^*$ and $T_k \mapsto w_k^*$ induces an inclusion $M \rightarrow M$ sending $v_i \mapsto v_i$ and $w_k \mapsto w_k$. The action of $T$ on $Z$ makes $Z$ a toric variety.

**Theorem 1.7.** We have

$$V = \text{trop}(G/H \cap T) \cap N,$$

The proof will be given at the end of Section 2. We can compare this result to the approach Luna and Vust introduced in [LV83] using formal curves (cf. [Tim11, Chapter 24]), which shows that every $G$-invariant discrete valuation $\nu$ can be obtained up to proportionality by choosing a $C((t))$-valued point $x(t)$ of $G/H$ and defining

$$\nu(f) := \text{ord}(f \cdot g \cdot x(t))$$

where $g \in G$ is a general point depending on $f \in C(G/H)$.

On the other hand, taking into account the fundamental theorem of tropical algebraic geometry (cf. [SS04, Theorem 2.1]), Theorem 1.7 implies that any $G$-invariant discrete valuation $\nu$ comes from a $C((t))$-valued point $x(t)$ of $G/H$ satisfying

$$\text{ord}(f_{ij}(x(t))) = \text{ord}(f_i(x(t)))$$

for every $i$ and $j$ and is uniquely determined by

$$\nu(f_i) = \text{ord}(f_i(x(t)))$$

and $\nu(g_k) = \text{ord}(g_k(x(t)))$

for every $i$ and $k$.

2. Proof of the Main Theorem

We continue to use the notation and the assumptions from the previous section in this section.

**Remark 2.1.** The action of $G$ on $Z \cong C^* \times \ldots \times C^* \times C^* \times \ldots \times C^*$ is linear on each factor.

The natural action of the torus $\text{Spec}(\mathbb{C}[\mathcal{M}])$ on $Z$ defines a corresponding $\mathcal{M}$-grading on $\mathbb{C}[Z]$. We set $p := I(G/H) = \ker \Phi$.

**Proposition 2.2.** The prime ideal $p$ is $\mathcal{M}$-graded, i.e. $\Gamma(G/H, \mathcal{O}_{G/H})$ is a $\mathcal{M}$-graded $\mathbb{C}$-algebra, if and only if $G/H$ is horospherical.

**Proof.** This follows from [Tim11, Proposition 7.6].

**Lemma 2.3.** Every non-open $G$-orbit of the spherical embedding $G/H \hookrightarrow \overline{G/H} \subseteq Z$ is contained in the closure of a $B$-invariant prime divisor in $G/H$.

**Proof.** It follows from Proposition 1.3 and the definition of the map $\Phi$ that $G/H$ and $\overline{G/H}$ have the same ring of global sections. Therefore the non-open $G$-orbits are at least of codimension two. It follows from the general theory of spherical embeddings (cf. [Kno91]) that any non-open $G$-orbit lies in the closure of some $B$-invariant prime divisor in $G/H$ if there is no $G$-invariant prime divisor.

**Proposition 2.4.** Let $Z_i := \mathbb{V}(S_{ij}; 1 \leq j \leq s_i) \subseteq Z$. Then we have

$$G/H = \overline{G/H} \setminus (Z_1 \cup \ldots \cup Z_r).$$
Proof. Let $O$ be a non-open $G$-orbit in $G/H$. By Lemma 2.3 there exists a $D_i$ with $O \subseteq \overline{D_i}$. It follows that $S_{i1}$ vanishes on $O$. Since $V_i$ is an irreducible $G$-module and $O$ is $G$-invariant, the whole module vanishes on $O$, so $O \subseteq Z_i$ follows. \qed

We set

$$X_0 := Z \setminus (Z_1 \cup \ldots \cup Z_r).$$

Then $G/H \hookrightarrow X_0$ is a closed embedding, and $G$ and $T$ act on $X_0$. Consider the fan $\Sigma_X$ in $N_\mathbb{Q}$ corresponding to the toric variety $X_0$. Our plan is to define a fan $\Sigma_X$ extending the fan $\Sigma_{X_0}$ such that the closure of $G/H$ inside the toric variety $X$ associated to the fan $\Sigma_X$ is the spherical variety $Y$ corresponding to the fan $\Sigma$. We have already established the natural inclusion $N_\mathbb{Q} \hookrightarrow N_\mathbb{Q}$, so the cones of the fan $\Sigma$ in $N_\mathbb{Q}$ can as well be considered as cones in $N_\mathbb{Q}$. But extending $\Sigma_{X_0}$ by these one-dimensional cones is not enough, since it might be impossible to extend the action of $G$ on $X_0$ to the resulting toric variety $X$. The plan can be carried out, however, if we also add some higher-dimensional cones.

We construct the fan $\Sigma_X$ in $N_\mathbb{Q}$ as follows. We first define the set

$$\mathfrak{A} := \{a \subseteq \{v_{ij}\} : \text{for each } i \text{ there is exactly one } j \text{ with } v_{ij} \notin a\}.$$ 

For each $1 \leq l \leq n$ and $a \in \mathfrak{A}$ we define the cone

$$\sigma_{l,a} := \text{cone}(\{u_l\} \cup a) \subseteq N_\mathbb{Q}$$

and set

$$\Sigma_X := \text{fan}(\{\sigma_{l,a} : 1 \leq l \leq n, a \in \mathfrak{A}\}),$$

the fan generated by the $\sigma_{l,a}$ in $N_\mathbb{Q}$.

**Proposition 2.5.** The cones $\sigma_{l,a}$ are smooth and compatible. In particular, the fan $\Sigma_X$ is well-defined.

Proof. Let $\sigma_{l,a}$ be a cone as defined above. We define the complement

$$a^c := \{v_{ij}, w_{jk} : 1 \leq i \leq r, 1 \leq j \leq s_i, 1 \leq k \leq m\} \setminus a.$$ 

We have $N = (a) \oplus (a^c)$. We write $u_l = u'_l + u''_l$ with $u'_l \in (a)$ and $u''_l \in (a^c)$. Then $u''_l$ is primitive, hence $\sigma_{l,a}$ is smooth.

Now let $\sigma_{l',a'}$ be another cone as defined above, and let $v \in \sigma_{l,a} \cap \sigma_{l',a'}$. We have

$$v = \sum_{v_{ij} \in a \setminus a'} d_{ij}v_{ij} + \sum_{v_{ij} \in a \cap a'} d_{ij}v_{ij} + d_0u_l$$

and

$$v = \sum_{v_{ij} \in a \setminus a'} d'_{ij}v_{ij} + \sum_{v_{ij} \in a \cap a'} d'_{ij}v_{ij} + d'_0u_{l'}$$

for some uniquely determined coefficients $d_{ij}, d'_{ij}, d_0, d'_0 \in \mathbb{Q}_0^+$. We define

$$m_i = \min\{\langle v, v_{ij}' \rangle : 1 \leq j \leq s_i\}.$$ 

As $\langle v, v_{ij}' \rangle = m_i$ for $v_{ij} \notin a \cap a'$, the coefficients in the first sum are zero on both lines. It is then clear that $d_0 = d'_0 = 0$ if $l \neq l'$. It follows that the intersection of the cones $\sigma_{l,a}$ and $\sigma_{l',a'}$ is a face of both. Hence the cones are compatible. \qed

We denote by $X$ the toric variety associated to the fan $\Sigma_X$. As the fan $\Sigma_{X_0}$ can also be obtained by the above construction from the cones $\sigma_{0,a}$ where $u_0 = 0$, the fan $\Sigma_X$ extends $\Sigma_{X_0}$. Hence we have an open embedding $X_0 \subseteq X$. We will now show that the $G$-action on $X_0$ can be extended to $X$. We consider $\hat{N} := N \oplus \mathbb{Z}^n$ and regard $N_\mathbb{Q} \subseteq \hat{N}_\mathbb{Q} := \hat{N} \otimes \mathbb{Q}$ as naturally included. We denote the standard basis of $\mathbb{Z}^n$ by $\{e_l\}_{l=1}^n$. We set

$$\tilde{\sigma}_{l,a} := \text{cone}(\{e_l\} \cup a) \subseteq \hat{N}_\mathbb{Q}.$$
and
\[ \Sigma_\mathcal{X} := \text{fan(} \{ \sigma_{l,a}; 1 \leq l \leq n, a \in \mathbb{A} \} \). \]

The associated quasiaffine toric variety \( \mathcal{X} \) comes with a natural toric morphism
\[ p : \mathcal{X} \rightarrow X \]
defined by the lattice map \( N \rightarrow N \) sending \( v_{ij} \mapsto v_{ij}, w_k \mapsto w_k \), and \( e_l \mapsto u_l \).

**Remark 2.6.** We have
\[ \mathcal{X} \cong V^* \times T \times \mathbb{A}^n \setminus \mathcal{S}, \]
where \( \mathcal{S} \) is a closed subset of codimension at least two. According to the theory in [Świ99], the toric morphism \( p : \mathcal{X} \rightarrow X \) is a good quotient for the action of a subtorus \( \Gamma \subseteq T \times (\mathbb{C}^*)^n \). It is even a geometric quotient. The subtorus \( \Gamma \) has a natural parametrization \( \kappa : (\mathbb{C}^*)^n \rightarrow \Gamma \). Denoting by \( W_l \) for \( 1 \leq l \leq n \) the coordinates of \( (\mathbb{C}^*)^n \), the action of \( \Gamma \) on \( \mathcal{X} \) is given by
\[ \kappa(t) \cdot v = \left( \prod_{l=1}^{n} W_l(t)^{-\langle u_l, v_i \rangle} \right) \cdot v, \]
on \( v \in V_i^* \), similarly, by
\[ \kappa(t) \cdot v = \left( \prod_{l=1}^{n} W_l(t)^{-\langle u_l, w_k \rangle} \right) \cdot v, \]
on \( v \) in the \( k \)-th factor \( \mathbb{C}^* \) of \( T \cong (\mathbb{C}^*)^m \), and finally by the natural action on \( \mathbb{A}^n \).

We let \( G \) act linearly on the first factors of \( \mathcal{X} \) with the same action as in Remark 2.1 and trivially on \( \mathbb{A}^n \). Then the action of \( G \) on \( \mathcal{X} \) commutes with the action of the torus \( \Gamma \) from Remark 2.6. In particular, we obtain an action of \( G \) on \( X \).

**Remark 2.7.** The natural inclusion \( N_{\mathcal{Q}} \subseteq \hat{N}_{\mathcal{Q}} \) defines a \( G \)-equivariant toric closed embedding
\[ \psi : X_0 \rightarrow \mathcal{X}, \]
and we obtain the following commutative diagram.

\[ \begin{array}{ccc}
\mathcal{X} & \overset{p}{\longrightarrow} & X \\
\downarrow \psi & & \downarrow \\
X_0 & \longrightarrow & X
\end{array} \]

This shows that the action of \( G \) on \( X \) is an extension of the action of \( G \) on \( X_0 \).

We denote by \( Y' \) the closure of \( G/H \) inside \( X \). It will now take some time to prove that \( Y' \) is indeed the spherical embedding of \( G/H \) associated to the fan \( \Sigma \), i.e. that \( Y' \) can be identified with \( Y \).

**Proposition 2.8.** The preimage \( p^{-1}(G/H) \) with the action of \( G \times \Gamma \) is a spherical homogeneous space isomorphic to \( G/H \times \Gamma \) with the natural action of \( G \times \Gamma \). The isomorphism
\[ G/H \times \Gamma \cong p^{-1}(G/H) \]
is given by \( (x,t) \mapsto t \cdot \psi(x) \).

**Proof.** As \( p \) is a geometric quotient, it follows that \( p^{-1}(G/H) = \Gamma \cdot \psi(G/H) \). \( \square \)
Remark 2.9. If we denote by $W_l$ for $1 \leq l \leq n$ the coordinates of $\mathbb{A}^n$ as well as the coordinates of $\Gamma$ under the parametrization $\kappa$ from Remark 2.6 we have $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[W_1, \ldots, W_n]$ and $\mathbb{C}[\Gamma] = \mathbb{C}[W_1^{\pm 1}, \ldots, W_n^{\pm 1}]$. Considering the natural inclusion

$$p^{-1}(G/H) \subseteq V^* \times T \times \mathbb{A}^n,$$

the isomorphism of Proposition 2.8 is induced by the map

$$\Psi : S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n] \to (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])/p \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$$

$$S_{tij} \mapsto S_{tij} \otimes \left( \prod_{l=1}^{n} W_l^{-(w_{l}, w_{l}')} \right)$$

$$T_k \mapsto T_k \otimes \left( \prod_{l=1}^{n} W_l^{-(u_{l}, u_{l}')} \right)$$

$$W_l \mapsto W_l.$$ 

We denote the closure of $p^{-1}(G/H)$ inside $\hat{X}$ by $\hat{Y}$.

**Proposition 2.10.** $\hat{Y}$ is the quasiaffine spherical embedding of $G/H \times \Gamma$ corresponding to the fan $\hat{\Sigma}$ in $(\mathbb{N} \oplus \mathbb{Z}^n)_q$ which consists of the one-dimensional cones

$$Q_{\geq 0}(w_l + e_i)$$

for $1 \leq l \leq n$. Furthermore, the ring $\Gamma(\hat{Y}, \mathcal{O}_{\hat{Y}})$ is factorial.

**Proof.** Consider the spherical embedding associated to the fan $\hat{\Sigma}$, which is quasiaffine with factorial ring of global sections (cf. [Bri07, Proposition 4.1.1]). By possibly adding some orbits of codimension at least two, we can make it affine (cf. [Kno91, Proof of Theorem 6.7]) and denote its coordinate ring by $R$. We will show that it is the closure of $p^{-1}(G/H)$ inside the affine toric variety $V^* \times T \times \mathbb{A}^n$. Consider the following diagram.

$$S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n] \quad \longrightarrow \quad R$$

$$\downarrow$$

$$\frac{(S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])/p \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]}{\Gamma(G/H \times \Gamma, \mathcal{O}_{G/H \times T})}$$

As the ring $R$ is normal, it consists of all the elements in the ring of global sections of $G/H \times \Gamma$ which have a nonnegative value under the valuations induced by the $G$-invariant prime divisors in $\text{Spec}(R)$. It follows that the image of $\Psi$ (cf. Remark 2.9) is contained in $R$.

Additionally, all the $B \times \Gamma$-eigenvectors in $R$, and even the $G \times \Gamma$-modules generated by them, belong to the image of the horizontal map, so we obtain surjectivity of the horizontal map with the same argument as in Proposition 1.3. The composition of the horizontal map and the vertical localization is the same map as in Proposition 2.8. It follows that we have $p^{-1}(G/H) = \text{Spec}(R)$ inside $V^* \times T \times \mathbb{A}^n$.

We use the same symbols for elements of $S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n]$ and their images in $R$. As the valuations induced by the $G$-invariant prime divisors in $\text{Spec}(R)$ have values of 0 or 1 respectively on the $W_l$ and of 0 on the $S_{tij}$, it follows that the $W_l$ and the $S_{tij}$ are pairwise nonassociated prime elements of $R$. The $G$-orbits which lie in the closure of $D_{s_l} \times \Gamma$ are contained in $\mathcal{V}(S_{tij}; 1 \leq j \leq s_l)$, while the other $G$-orbits of codimension at least two are contained in $\mathcal{V}(W_{l_1}) \cap \mathcal{V}(W_{l_2})$ for some $l_1 \neq l_2$. Therefore intersecting with $\hat{X}$, i.e. removing the set $\hat{S}$, removes exactly the $G$-orbits of codimension at least two, and the result follows. \qed
We will reuse the following fact from the proof of Proposition \textbf{2.10}.

**Remark 2.11.** The $W_i$ and the $S_{ij}$ are pairwise nonassociated prime elements of $\Gamma(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$.

**Remark 2.12.** The restriction

$$p|_{\tilde{Y}} : \tilde{Y} \to Y''$$

is a good geometric quotient. In particular, $Y'$ is normal.

In the following Lemma \textbf{2.13} we will assume that $\Sigma$ contains only one one-dimensional cone $\sigma$. We also allow the cone $\sigma$ to lie outside the valuation cone $V$. In that case, the spherical variety $Y$ is not defined, but the closure $Y'$ of $G/H$ inside $X$ can still be constructed. Remark \textbf{2.12} not being available, the normality of $Y'$ is not certain. This more general setting will only be required for one direction of the proof of Theorem \textbf{1.7} at the end of this section.

**Lemma 2.13.** Any irreducible component of $Y' \setminus (G/H)$ which intersects the $T$-orbit in $X$ corresponding to $\sigma$ induces a $G$-invariant valuation which induces $\sigma$ (after possibly normalizing $Y'$).

**Proof.** We consider the affine open toric subvariety $U_{\sigma}$ of $X$ with two orbits, the torus and the orbit corresponding to the cone $\sigma$. Let $\pi \in \mathbb{C}[U_{\sigma}]$ be the prime element such that $\mathcal{V}(\pi)$ is the orbit corresponding to the cone $\sigma$. We obtain the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{C}[U_{\sigma}] & \xrightarrow{\mathfrak{M}} & (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])_p \\
\mathbb{C}[U_{\sigma}]/(P \cap \mathbb{C}[U_{\sigma}]) & \xrightarrow{\pi} & R_1 \xrightarrow{p} R_2 \xrightarrow{\pi} (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])_p/\mathfrak{p} \\
\end{array}
\]

It is sufficient to localize at the prime ideal $\mathfrak{p}$ to get the inclusion in the first row since the $S_{ij}$ are not in $\mathfrak{p}$. In the bottom row, $\mathfrak{M}$ denotes normalization, and $\Sigma$ is localization in such a way that $\mathcal{V}(\pi) = \mathcal{V}(\pi_0)$ in $\text{Spec}(R_2)$ where $\pi_0 \in R_2$ is a prime element. Each $L \in \{S_1, T_k\}$ can be written as $L = c_1 \pi d_1 / \pi d_2$ with $c_1 \in \mathbb{C}[U_{\sigma}]^*$ and $d_1, d_2 \in \mathbb{N}_0$ since there are no $T$-invariant prime divisors other than $\mathcal{V}(\pi)$ in $U_{\sigma}$. Therefore we have $L = c_2 \pi_0 d_1 / \pi_0 d_2$ for some $c_2 \in R_2^*$ and $d_3 \in \mathbb{N}_0$. It follows that $\mathcal{V}(\pi_0)$ induces the correct cone $\sigma$. \hfill $\square$

**Proposition 2.14.** $Y'$ is the spherical embedding of $G/H$ corresponding to the fan $\Sigma$.

**Proof.** As general embeddings of $G/H$ can be obtained by gluing embeddings with only one closed orbit, we may assume $n = 1$. In this case, it is clear that $Y'$ has two $G$-orbits, and Lemma \textbf{2.13} implies that the closed $G$-orbit induces the correct $G$-invariant valuation. \hfill $\square$

From now on, we will identify $Y'$ and $Y$. We denote by $X_1, \ldots, X_n$ the $T$-invariant prime divisors in $X$ corresponding to $u_1, \ldots, u_n$. Note that $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ implies $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$.

We temporarily reuse our notation in a more general setting to give an overview over the next step. Let $X$ be a toric $T$-variety with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and $T$-invariant prime divisors $X_1, \ldots, X_n$. Then we have the canonical quotient construction $\pi : \tilde{X} \to X$, where $\tilde{X}$ is a quaaffine toric variety and $\mathcal{R}(X) \cong \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ (cf. \textbf{[CLS11, Theorem 5.1.11]}). Let $i : Y \hookrightarrow X$ be a closed embedding with $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ as well as $X$ and $Y$ smooth. It follows from the work of Hausen (cf. \textbf{[Hau08]}) that $\mathcal{R}(Y) \cong \Gamma(\pi^{-1}(Y), \mathcal{O}_{\pi^{-1}(Y)})$ if $X_l \cap Y$ is an irreducible hypersurface in $Y$ intersecting
Theorem 2.16. We have 

\[ \text{Theorem 2.6} \] if we show that the ideal

\[ \text{There is a canonical toric closed embedding} \]

\[ \text{The second step is the map} \]

\[ \phi \]

\[ \text{By} \ [\text{ADHL10, Construction 5.1.4}], \text{the scheme-theoretic fiber} \]

\[ \hat{X} \times X \]

\[ \text{Y} \]

\[ \text{We now return to our setting. We identify the prime divisors} D_i \text{ in } G/H \text{ and} \]

\[ \mathcal{V}(S_{ij}) \text{ in } X_0 \text{ with their closures inside} \]

\[ Y \text{ and} \]

\[ X \text{ respectively as long as there is no danger of confusion.} \]

Proposition 2.15. The closed embedding

\[ \iota : Y \hookrightarrow X \]

is a neat embedding in the sense of [Hau08, Definition 2.5].

Proof. Consider the following commutative diagram.

\[ \xymatrix{ \hat{Y} \ar[r]^-{\hat{\iota}} & \hat{X} \ar[r]^\phi & X \ar[d]^-p \ar[r]^-{\iota} & Y \ar[d]^-p \ar@<2ex>[l]^-\pi \ar@<2ex>[l]^-{\pi} } \]

Since the diagram commutes, \( \iota^{-1}(X_l) \) and \( \iota^{-1}(\mathcal{V}(S_{ij})) \) are irreducible for each \( i, j \), and \( l \), and using Remark [2.11] as in the proof of Theorem [1.7], we obtain that they intersect the corresponding \( X \)-orbit of codimension one in \( X \).

We have a pullback map of Cartier divisors \( \iota^* : \text{Div}(X) \rightarrow \text{Div}(Y) \), and clearly \( \text{Supp}(\iota^* (X_l)) \subseteq Y_l \) and \( \text{Supp}(\iota^* (\mathcal{V}(S_{ij}))) \subseteq D_i \). Using the diagram again, we see that locally the pullbacks of \( X_l \) and \( \mathcal{V}(S_{ij}) \) are prime divisors. Therefore we have \( \iota^* (X_l) = Y_l \) and \( \iota^* (\mathcal{V}(S_{ij})) = D_i \).

Using the explicit descriptions of the divisor class groups of toric and spherical varieties (cf. [CLS11, Theorem 4.1.3] and [Bri07, Proposition 4.1.1]), we obtain that the induced pullback map \( \iota^* : \text{Cl}(X) \rightarrow \text{Cl}(Y) \) is an isomorphism. \( \square \)

We are now almost ready to describe the Cox ring \( \mathcal{R}(Y) \). We recall the homogenization operation from the introduction. The first step is the map \( \alpha : \mathbb{C}[Z] \rightarrow (\mathbb{C}[Z])[W_1, \ldots, W_n] \) defined as follows. For each \( f \in \mathbb{C}[Z] \) and \( u \in \mathcal{N} \) we define

\[ \text{ord}_u(f) := \min_{\mu \in \mathcal{M}} \{ (u, \mu); f(u) \neq 0 \}, \]

and set

\[ f^\alpha := \frac{\sum_{\mu \in \mathcal{M}} (f(u) \prod_{l=1}^n W_i^{\text{ord}_u(f)})}{\prod_{l=1}^n W_i^{\text{ord}_u(f)}}. \]

The second step is the map \( \beta : (\mathbb{C}[Z])[W_1, \ldots, W_n] \rightarrow S(V)[W_1, \ldots, W_n] \) sending \( T_k \mapsto 1 \) for each \( 1 \leq k \leq m \). Finally, we define the map \( h : \mathbb{C}[Z] \rightarrow S(V)[W_1, \ldots, W_n] \) by composing the two steps, i.e. \( h := \beta \circ \alpha \).

Theorem 2.16. We have

\[ \mathcal{R}(Y) \cong S(V)[W_1, \ldots, W_n]/(f^h ; f \in \mathfrak{p}), \]

with \( \text{Cl}(Y) \)-grading given by \( \text{deg}(S_{ij}) = [D_i] \) and \( \text{deg}(W_i) = [Y_i] \).

Proof. Let \( \pi : \hat{X} \rightarrow X \) be the canonical quotient construction. We set \( \hat{Y} := \pi^{-1}(Y) \).

By [ADHL10, Construction 5.1.4], the scheme-theoretic fiber \( \hat{X} \times X Y \) is reduced. There is a canonical toric closed embedding \( \phi : \hat{X} \hookrightarrow \hat{X} \) such that \( \pi = p \circ \phi \). Note that \( \phi^* \) sends \( f \mapsto f^\beta \). The assertion now follows from Proposition 2.15 and [Hau08, Theorem 2.6] if we show that the ideal \( \mathfrak{p} \) of \( \hat{Y} \) in \( \hat{X} \) is \( p^\alpha := (f^\alpha ; f \in \mathfrak{p}) \).
We use the map \( \psi \) from Remark 2.7. We have \( \tilde{p} = \| (\Gamma \cdot \psi(V(p))) \). Every \( f^\alpha \in p^\alpha \) vanishes on \( \psi(V(p)) \) since \( \psi^*(f^\alpha) = f \in p \) and \( f^\alpha \) is a \( \Gamma \)-eigenvector, so \( f^\alpha \in \tilde{p} \), and \( p^\alpha \subseteq \tilde{p} \) follows. Now, let \( g \in \tilde{p} \). As \( \tilde{p} \) is a homogeneous ideal with respect to the \( X(\Gamma) \)-grading, all homogeneous components \( g^{(\ell)} \) are in \( \tilde{p} \). It is not difficult to see that \( g^{(\ell)} = (\psi^*(g^{(\ell)}))^\alpha \prod_{i=1}^{\alpha} W_i^{d_i} \) for some exponents \( d_i \in \mathbb{N}_0 \). Since \( \psi^*(g^{(\ell)}) \in p \), the inclusion \( \tilde{p} \subseteq p^\alpha \) follows.

**Remark 2.17.** If \( \Gamma(G/H, O^*_G) = C^* \), we have \( \tilde{X} = \tilde{X} \), and \( \phi \) is the identity.

Cox rings are always factorially graded (cf. [BH03 Theorem 7.3]), but not factorial in general (cf. [Arz09 Example 4.2]). A sufficient condition for the Cox ring to be factorial is the divisor class group being free (cf. [BH03 Proposition 8.4], also [EKW04 Corollary 1.2]). The last part of the Main Theorem also provides a sufficient condition for the Cox ring of a spherical variety to be factorial.

**Theorem 2.18.** If \( H \) is connected, \( R(Y) \) is a factorial ring.

**Proof.** The finitely generated free abelian group \( \Gamma(G/H, O^*_G) \cap C^* \) is naturally isomorphic to the subgroup \( X(C)/H \subseteq X(C) \) consisting of \( H \)-invariant characters. The quotient group \( X(C)/X(C)/H \) is free as \( H \) is connected. Therefore there exists a decomposition \( X(C) = X(C)/H \oplus X(C)^1 \), and we can choose the \( f_{ij} \) in such a way that \( C \) acts on the \( f_{ij} \) with characters belonging to \( X(C)^1 \). It follows that for each \( 1 \leq k \leq m \) we have a one-parameter subgroup of \( C \) acting nontrivially only on the variable \( T_k \). As \( \tilde{Y} \) is invariant, we obtain \( \tilde{Y} \cong \tilde{Y} \times T \). Therefore the factoriality of \( R(Y) \cong \Gamma(\tilde{Y}, O_{\tilde{Y}}) \) follows from the factoriality of \( \Gamma(\tilde{Y}, O_{\tilde{Y}}) \) (cf. Proposition 2.10).

Finally, we provide the proof of Theorem 1.7

**Proof of Theorem 1.7.** We have to show \( V = \text{trop}(G/H \cap T) \cap X_G \). Using [Tev07 Lemma 2.2], we get the inclusion from the left to the right using Proposition 2.14 if \( Y_t \) intersects the \( T \)-orbit which is dense in \( X_t \). This is the case, since otherwise it would follow from Remark 2.11 that the codimension of \( Y_t \) in \( Y \) is at least two. If the inclusion from the right to the left did not hold, Lemma 2.15 would yield a non-existing \( G \)-invariant valuation.

3. Generalization to Arbitrary Spherical Homogeneous Spaces

We now consider the case where the spherical homogeneous space is allowed to have nontrivial divisor class group. Let \( G \) be a connected reductive group and \( H \subseteq G \) a spherical subgroup. We may again assume that \( G \) is of simply connected type, i.e. \( G = G^{ss} \times C \) where \( G^{ss} \) is semisimple simply connected and \( C \) is a torus.

We fix a Borel subgroup \( B \subseteq G \) such that the base point \( 1 \in G/H \) lies in the open \( B \)-orbit and denote by \( D := \{ D_1, \ldots, D_r \} \) the set of \( B \)-invariant prime divisors in \( G/H \). For each \( D_i \), the pullback under the quotient map \( G \to G/H \) is a divisor with equation \( f_i \in \mathbb{C}[G] \) where \( f_i \) is uniquely determined by being \( C \)-invariant with respect to the action from the left and \( f_i(1) = 1 \) (cf. [Br07 4.1]).

The group \( H \) acts from the right on each \( f_i \) with a character \( \chi_i \in X(H) \).

We define

\[
G := G \times (C^*)^P \quad H := \{(h, \chi_1(h), \ldots, \chi_r(h)); h \in H \} \subseteq G,
\]

and set \( B := B \times (C^*)^P \). We have a quotient map \( \pi : G/H \to G/H \), which is a good geometric quotient by the torus \((C^*)^P \). There is a natural isomorphism \( H \cong H \), and \( G/H \) is a spherical homogeneous space. The pullbacks of the \( B \)-invariant
prime divisors in $G/H$ under the quotient map $\pi$ are exactly the $B$-invariant prime divisors $D_1, \ldots, D_r$ in $G/H$.

We denote by $\text{Pic}_G(G/H)$ the group of isomorphism classes of $G$-linearized invertible sheaves on $G/H$. The character lattice of $(\mathbb{C}^*)^D$ is $\mathbb{Z}^D$ with standard basis $\{\eta_1, \ldots, \eta_r\}$.

**Proposition 3.1.** The following diagram of natural maps is commutative, and the top row is an exact sequence.

$$
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{M} & \longrightarrow \mathcal{X}(C) \oplus \mathbb{Z}^D & \longrightarrow & \text{Pic}_G(G/H) & \longrightarrow & 0 \\
& & \bigg\downarrow \cong & & \bigg\downarrow \cong & & \bigg\downarrow \cong \\
& & \mathcal{X}(G) & \longrightarrow & \mathcal{X}(H) & & \\
\end{array}
$$

**Proof.** The map $\mathcal{X}(C) \oplus \mathbb{Z}^D \to \text{Pic}_G(G/H)$ sends $\chi \mapsto \mathcal{O}_{G/H}(\chi)$ for $\chi \in \mathcal{X}(C)$ and $\eta_i \mapsto \mathcal{O}_{G/H}(D_i)$ as in [Bri07] Proposition 4.1.1 where $\mathcal{O}_{G/H}(D_i)$ is canonically $G$-linearized as in [Bri07] 4.1. The left-hand isomorphism is obvious, res is the restriction map, and the right-hand isomorphism sends $\chi \mapsto \mathcal{L}_{G/H}(\chi)$ for $\chi \in \mathcal{X}(H)$ where $\mathcal{L}_{G/H}(\chi)$ is the standard construction (cf. [KKLV89] 2.1 or [Tim11] after Proposition 2.4). It is not difficult to see that the diagram commutes. We denote by $\rho(D_i) \in \mathcal{N}_G$ the vector corresponding to the restriction of the discrete valuation induced by $D_i \in \mathcal{D}$ and set $d_i(\mu) := (\rho(D_i))(\mu) \in \mathbb{Z}$. Then the map $\mathcal{M} \to \mathcal{X}(C) \oplus \mathbb{Z}^D$ sends

$$
\mu \mapsto \sum_{i=1}^r d_i(\mu) \eta_i - \mu|_C.
$$

If $d_i(\mu) = 0$ for all $1 \leq i \leq r$ and $\mu$ is the $B$-weight of $f \in \mathbb{C}(G/H)^{(B)}$, it follows that $\text{div}(f) = 0$, so $f$ is a unit in $\Gamma(G/H, \mathcal{O}_{G/H})$. This means we have $f \in \mathcal{X}(C)$, so $\mu|_C = 0$ if and only if $\mu = 0$. Therefore the map is injective. Using the first statement of [Bri07] Proposition 4.1.1], which in fact does not require the assumption $\Gamma(G/H, \mathcal{O}_{G/H}) = \mathbb{C}$, we obtain the exactness of the top row. \hfill $\Box$

**Corollary 3.2.** The spherical homogeneous space $G/H$ has trivial divisor class group.

**Proof.** By a theorem of Popov, $\text{Cl}(G/H) = \text{Pic}(G/H)$ is trivial if and only if every character of $H$ is the restriction of a character of $G$ (cf. [Pop74, Tim11 Theorem 2.5]). By Proposition 3.1 res : $\mathcal{X}(G) \to \mathcal{X}(H)$ is surjective. \hfill $\Box$

We will continue to use the notation from the previous sections for the spherical homogeneous space $G/H$. Where applicable, we use the same notation for the spherical homogeneous space $G/H$, but in boldface symbols. In particular, $\mathcal{M}$ is the weight lattice of $B$-eigenvectors in the function field $\mathbb{C}(G/H)$, $\mathcal{N} = \text{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice, $\mathcal{N}_Q = \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathcal{V}$ is the valuation cone of $G/H$.

We fix a convenient choice for the prime elements $f_1, \ldots, f_r \in \Gamma(G/H, \mathcal{O}_{G/H})$. We have a natural inclusion $\epsilon : \mathbb{Z}^D \hookrightarrow \mathbb{C}[G]^*, \chi \mapsto \epsilon^\chi$. Then $f_i := f_i \epsilon^{-\eta_i} \in \mathbb{C}[G]$ is $H$-invariant under the action from the right, and $f_i$ is a prime element in $\Gamma(G/H, \mathcal{O}_{G/H})$ with $\mathcal{V}(f_i) = D_i$. The torus $(\mathbb{C}^*)^D$ acts on $f_i \in \Gamma(G/H, \mathcal{O}_{G/H})$ with weight $\eta_i$.

Recall the decomposition $\mathcal{M} = \mathcal{M}_V \oplus \mathcal{M}_T$. We have a corresponding decomposition of the dual lattice $\mathcal{N} = \mathcal{N}_V \oplus \mathcal{N}_T$.

We consider a spherical embedding $G/H \hookrightarrow Y$ which contains only non-open $G$-orbits of codimension one, given by a fan $\Sigma$ in $\mathcal{N}_Q$, and assume $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}^*$. We denote by $Y_1, \ldots, Y_n$ the $G$-invariant prime divisors in $Y$. 


The next two results will allow us to obtain a fan $\Sigma$ in $N_\mathbb{Q}$ with associated spherical embedding $G/H \hookrightarrow Y$ from the fan $\Sigma$ in $N_\mathbb{Q}$. We will then construct the ring $\mathcal{R}(Y)$ exactly as in Section 2. We may have $\Gamma(Y,\mathcal{O}_Y) \neq \mathbb{C}^*$, the ring $\mathcal{R}(Y)$ may not be the Cox ring of $Y$. This ring will, however, be the Cox ring $\mathcal{R}(Y)$ of $Y$.

The quotient map $\pi : G/H \rightarrow G/H$ induces an inclusion $\pi^* : \mathcal{M} \rightarrow \mathcal{M}$, which is the restriction of the natural inclusion $\pi^* : \mathcal{X}(B) \hookrightarrow \mathcal{X}(B)$. We also obtain the surjective dual map $\pi_* : N_\mathbb{Q} \rightarrow N_\mathbb{Q}$.

**Proposition 3.3.** We have $V = \pi_\mathbb{Q}^{-1}(V)$.

**Proof.** Let $v \in N_\mathbb{Q}$ with $\pi_*(v) = 0$. Interpreting $v$ as map $v : M \rightarrow \mathbb{Q}$, this means $v \circ \pi^* = 0$. Therefore there exists an extension $v^+ : \mathcal{X}(B) \rightarrow \mathbb{Q}$ of $v$ with $v^+ \circ \pi^+ = 0$. By [Kno91, Corollary 5.3], we obtain $v \in V$, therefore $\pi_\mathbb{Q}^{-1}(0) \subseteq V$. Finally, we use [Kno91, Corollary 1.5]. □

As the $B$-weights of the kernel of the restriction map res : $\mathcal{X}(G) \rightarrow \mathcal{X}(H)$ are exactly the lattice $M_T$, Proposition 3.1 yields an isomorphism $\gamma : \mathcal{M} \rightarrow M_T$.

**Proposition 3.4.** The restricted map $\pi_*|(N_T)_\mathbb{Q} : (N_T)_\mathbb{Q} \rightarrow N_\mathbb{Q}$ is dual to the isomorphism $\gamma : \mathcal{M} \rightarrow M_T$. In particular, it is itself an isomorphism.

**Proof.** We have to show that for each $v \in N_\mathbb{Q}$ with $v|_{M_T} = 0$ and each $\mu \in \mathcal{M}$ we have $v(\gamma(\mu)) = v(\pi^*(\mu))$. It therefore suffices to show that $\gamma(\mu) - \pi^*(\mu) \in M_V$ for each $\mu \in \mathcal{M}$.

Let $\mu \in \mathcal{M}$ be the $B$-weight of the $B$-eigenvector $f \in \mathbb{C}[G/H] \subseteq \mathbb{C}(G)$. Using the notation from the proof of Proposition 3.1, we necessarily have

$$f = c \prod_{i=1}^r f_i^{d_i(\mu)},$$

where $c \in \mathbb{C}[G]^*$ has left $B$-weight $\mu|_C$. Therefore we obtain

$$\gamma(\mu) = - \sum_{i=1}^r d_i(\mu) \eta_i + \mu|_C \quad \text{and} \quad \pi^*(\mu) = \mu|_C + \sum_{i=1}^r d_i(\mu) \omega_i$$

where $\omega_i$ is the left $B$-weight of $f_i$, hence $\gamma(\mu) - \pi^*(\mu) \in M_V$. □

We obtain the fan $\Sigma$ in $N_\mathbb{Q}$ (with associated spherical embedding $G/H \hookrightarrow Y$) as preimage under $\pi_*|(N_T)_\mathbb{Q}$ of the fan $\Sigma$ in $N_\mathbb{Q}$.

**Remark 3.5.** There is a good geometric quotient of the whole toric variety $X$ by the action of $(\mathbb{C}^*)^D$. This means that $\pi$ can be extended to a quotient $\pi : X \rightarrow X$. We obtain the following natural commutative diagram.

\[
\begin{array}{ccc}
(N_T)_\mathbb{Q} & \rightarrow & \mathcal{N}_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
N_\mathbb{Q} & \rightarrow & N_\mathbb{Q}
\end{array}
\]

Similarly to Section 2, we obtain $Y$ as closure of $G/H$ inside $X$, and the embedding $Y \hookrightarrow X$ is neat.

**Theorem 3.6.** We have $\mathcal{R}(Y) \cong S(V)[W_1,\ldots,W_n]/(f^h; f \in p)$, with $\text{Cl}(Y)$-grading given by $\deg(S_{ij}) = [D_i]$ and $\deg(W_i) = [Y_i]$. 

Proof. As in Section 2, we construct the good quotient \( p : \tilde{X} \to X \), and we have the canonical quotient construction \( \pi : \tilde{X} \to X \) as well as a canonical toric closed embedding \( \phi : \tilde{X} \to \tilde{X} \) such that \( \pi = \pi \circ p \circ \phi \). The result now follows as in Section 2.

\[ \text{Theorem 3.7. If } H \text{ is connected, } \mathcal{R}(Y) \text{ is a factorial ring.} \]

Proof. We embed \( Y \) into another toric variety \( X' \) using primes \( f'_i \in \Gamma(G/H, \mathcal{O}_{G/H}) \) satisfying the requirements of the proof of Theorem 2.18 instead of the primes \( f_i \). The \( f'_i \) can be chosen in such a way that for each \( 1 \leq i \leq r \) there is a unit \( c_i \) such that \( f'_i = c_i f_i \) and \( c_i \) is the product of elements of \( \{g_k\}_{k=1}^n \). As a basis for the \( G \)-module \( \langle G \cdot f'_i \rangle \) we then choose \( \{c_i f_{ij}\}_{j=1}^r \). In this case, there is a toric isomorphism \( X' \cong X \) which fixes \( G/H \) and therefore \( Y \) as well. We obtain the following commutative diagram.

\[
\begin{array}{ccc}
\tilde{X}' & \xrightarrow{\phi'} & X' \\
\downarrow & & \downarrow \cong \\
\tilde{X} & \xrightarrow{\phi} & X \xrightarrow{\pi} X
\end{array}
\]

The factoriality of \( \mathcal{R}(Y) \) now follows as in Theorem 2.18.

The following Theorem 3.8 follows directly from [Bri07, Theorem 4.3.2]. In order to be self-contained, we give another proof.

\[ \text{Theorem 3.8. If } G/H \text{ is horospherical, we have } \]

\[ \mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \ldots, W_n], \]

where \( P := N_G(H) \).

Proof. As \( G/H \) being horospherical implies that \( G/H \) is horospherical as well, the ideal \( \mathfrak{p} \) is \( \mathcal{M} \)-graded by Proposition 2.2. This means, for each \( f \in \mathfrak{p} \) and \( \mu \in \mathcal{M} \) we have \( f^{(\mu)} \in \mathfrak{p} \). As \( (f^{(\mu)})^\lambda = f^{(\mu)} \), the ideal \( (f^h; f \in \mathfrak{p}) \) can be generated by elements which do not contain any of the variables \( W_i \). It follows that \( \mathcal{R}(Y) \cong R[W_1, \ldots, W_n] \) for some ring \( R \) which depends only on the homogeneous space.

It only remains to show that \( R \cong \mathcal{R}(G/P) \). We have canonical maps

\[
\begin{array}{ccc}
G & \xrightarrow{=} & G/H \xrightarrow{=} (G \times T)/P \xrightarrow{=} G/P,
\end{array}
\]

where \( T := P/H \) is a torus, \( P \) acts on \( T \) via \( P \to T \), and \( P \) acts on \( G \times T \) via \( p \cdot (g,t) := (gp^{-1}, pt) \). The last map has fibers isomorphic to \( T \) and is a trivial fiberation over the open orbit of any Borel subgroup of \( G \) (cf. [BM13 after Theorem 2.2]). The pullbacks of the \( B \)-invariant prime divisors in \( G/P \) are exactly the \( B \)-invariant prime divisors in \( G/H \). In particular, \( P \) acts from the right on each \( f_i \in \mathbb{C}[G] \) with an extension of the character \( \chi_i \in \mathcal{X}(H) \) which we will also call \( \chi_i \in \mathcal{X}(P) \). We define

\[ P := \{(p, \chi_1(p), \ldots, \chi_r(p)); p \in P\} \subseteq G. \]

It is not difficult to see that \( G/H \) is isomorphic to \( (G \times T)/P \) where \( P \cong P \) acts on \( G \times T \) via \( p \cdot (g,t) := (gp^{-1}, pt) \). As all characters of \( P \) can be extended to \( G \) (cf. Proposition 3.1), we obtain \( G/H \cong G/P \times T \), and the result follows.
4. Examples

Example 4.1. Consider $G := \text{SL}(3)$ and $H := U$, the set of unipotent upper triangular matrices. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices, and let $G$ act linearly on $\mathbb{C}^3 \times \mathbb{C}^3$ by acting naturally on the first factor and with the contragredient action on the second factor. We denote the coordinates of the first factor by $S_{11}, S_{12}, S_{13}$ and the coordinates of the second factor by $S_{21}, S_{22}, S_{23}$. Then the point $((1, 0, 0), (0, 0, 1))$ has isotropy group $U$, and its orbit is $V(S_{11}S_{21} + S_{12}S_{22} + S_{13}S_{23})$. The homogeneous space $G/H$ is horospherical with $B$-invariant prime divisors $D_1 := V(S_{11})$ and $D_2 := V(S_{21})$ and has trivial divisor class group. Consider the embedding $Y$ corresponding to a fan containing exactly $n$ one-dimensional cones. By Theorem 3.8 and setting $P := N_G(H)$, we obtain

$$\mathcal{R}(G/P) \cong \mathbb{C}[S_{11}, S_{12}, S_{13}, S_{21}, S_{22}, S_{23}] / (S_{11}S_{21} + S_{12}S_{22} + S_{13}S_{23})$$

$$\mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \ldots, W_n].$$

Example 4.2. Consider $G := \text{SL}(d)$ for $d \geq 3$ and $H := \text{SL}(d - 1)$ embedded as the lower-right entries of $\text{SL}(d)$. The case $d = 3$ has been studied in [Pau83] and [Pau89]. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices, and let $G$ act linearly on $\mathbb{C}^d \times \mathbb{C}^d$ by acting naturally on the first factor and with the contragredient action on the second factor. We denote the coordinates of the first factor by $S_{11}, \ldots, S_{1d}$ and the coordinates of the second factor by $S_{21}, \ldots, S_{2d}$. Then the point $((1, 0, \ldots, 0), (1, 0, \ldots, 0))$ has isotropy group $\text{SL}(d - 1)$, and its orbit is $V(\sum_{j=1}^d S_{1j}S_{2j} - 1)$. The homogeneous space $G/H$ is spherical with $B$-invariant prime divisors $D_1 := V(S_{1d})$ and $D_2 := V(S_{21})$ as well as affine with factorial coordinate ring. By Theorem 1.7 we obtain $V = \{v^{+} + v^{-} \leq 0\}$. Consider the embedding $Y$ corresponding to the fan containing the one-dimensional cones having primitive lattice generators $(p_1, q_1), \ldots, (p_n, q_n) \in \mathcal{N}$ with respect to the basis $\{v_1, v_2\}$. The following picture illustrates $\mathcal{N}_Q$.

By Theorem 2.16, we obtain

$$\mathcal{R}(Y) \cong \mathbb{C}[S_{1j}, S_{2j}, W_1, \ldots, W_n]_{j=1}^d / \left(\sum_{j=1}^d S_{1j}S_{2j} - W_1^{-p_1} - q_1 \ldots W_n^{-p_n} - q_n\right).$$

Example 4.3. Consider $G := \text{SL}(2)$ and $H := T$, the diagonal torus. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices. Then, considering the orbit of $((1 : 0), [0 : 1])$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain $G/H \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}(\mathbb{P}^1)$, and there are two $B$-invariant prime divisors. From $C[G] = \mathbb{C}[M_{11}, M_{12}, M_{21}, M_{22}] / (M_{11}M_{22} - M_{12}M_{21} - 1)$,

we obtain $f_1 = M_{21}$ and $f_2 = M_{22}$. We can now construct $G = G \times (\mathbb{C}^*)^D$ and $H$. It is not difficult to see that $G/H$ is the orbit of the point $((1, 0), (0, 1), 1)$ in $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^*$, where $G$ acts naturally on both factors $\mathbb{C}^2$ and trivially on $\mathbb{C}^*$ while $(\mathbb{C}^*)^D$ acts with the weights $-\eta_1$ and $-\eta_2$ respectively on the factors $\mathbb{C}^2$ as well as with $-\eta_1 - \eta_2$ on $\mathbb{C}^*$. Denoting by $S_{11}, S_{12}$ and $S_{21}, S_{22}$ the coordinates of the factors $\mathbb{C}^2$ and by $T_1$ the coordinate of the factor $\mathbb{C}^*$, we obtain
We observe that which is a non-factorial ring. Using Proposition 3.4 and Proposition 3.1 we see that \( \pi_*(w_1) \) is a basis of \( \mathcal{N}_Q \) and that \( \pi_*(v_1) = \pi_*(v_2) = -\pi_*(w_1) \). By Proposition 3.3 \( \mathcal{V} = \{ \pi_*(w_1)^* \geq 0 \} \) follows. The following picture illustrates \( \mathcal{N}_Q \).

By Theorem 3.6 we have
\[
\mathcal{R}(G/H) \cong \mathbb{C}[S_{11}, S_{12}, S_{21}, S_{22}]/(S_{11}S_{22} - S_{12}S_{21} - 1).
\]

The only nontrivial embedding of \( G/H \) is the embedding \( Y := \mathbb{P}^1 \times \mathbb{P}^1 \) given by the cone \( \mathbb{Q}_{\geq 0} \pi_*(w_1) \). In this case, Theorem 3.6 yields
\[
\mathcal{R}(Y) \cong \mathbb{C}[S_{11}, S_{12}, S_{22}, S_{21}, W_1]/(S_{11}S_{22} - S_{12}S_{21} - W_1),
\]
which is isomorphic to the polynomial ring in four variables.

Example 4.4 (cf. [Arz09 Example 4.2]). Consider \( G := SL(2) \) and \( H := N_G(T) \) where \( T \) is the diagonal torus. Let \( B \subseteq G \) be the Borel subgroup of upper triangular matrices. The group \( H \) consists of two connected components, the identity component \( H^0 = \{ (\lambda I, \lambda I) : \lambda \in \mathbb{C}^* \} \) and a second component \( H^+ = \{ (-\lambda I, \lambda I) : \lambda \in \mathbb{C}^* \} \). There exists a character \( \chi \in \mathcal{X}(H) \) with \( \chi|_{H^0} = 1 \) and \( \chi|_{H^+} = -1 \). Consider the complex vector space \( \text{Sym}(2 \times 2) = \{ (s_{ij})_{ij} ; s_{ij} \in \mathbb{C} \} \) of symmetrical 2 by 2 matrices. Let \( G \times \mathbb{C}^* \) act on \( \text{Sym}(2 \times 2) \times \mathbb{C}^* \) via \((g,t) \cdot (x,y) := (t^{-1} g x g^T, t^{-2} y)\). The isotropy group of the point \( ((0,1), 1) \) is \( H' := \{ (h, \chi(h)) ; h \in H \} \) and its orbit is the closed subset \( \mathcal{V}(S_{11}S_{12} - S_{13}^2 - T_1) \).

We observe that \( (G \times \mathbb{C}^*)/H' \) is a spherical homogeneous space with one \( B \times \mathbb{C}^* \)-invariant prime divisor \( D_1 := \mathcal{V}(S_{11}) \) and that it coincides with \( G/H \) as defined in Section 3. By Theorem 1.7 we obtain \( \mathcal{V} = \{ 2v_1^* \leq w_1^* \} \). Using Proposition 3.4 and Proposition 3.1 we see that \( \pi_*(v_1) \) is a basis of \( \mathcal{N}_Q \) and that \( \pi_*(v_1) = -2\pi_*(w_1) \). By Proposition 3.3 \( \mathcal{V} = \{ \pi_*(w_1)^* \geq 0 \} \) follows. The following picture illustrates \( \mathcal{N}_Q \).

Consider the trivial embedding \( Y := G/H \). From a geometric viewpoint, this is the complement of a conic in \( \mathbb{P}^2 \). We have \( \text{Cl}(Y) \cong \mathbb{Z}/2\mathbb{Z} \) with generator \([D_1] \in \text{Cl}(Y)\). By Theorem 3.6 we obtain
\[
\mathcal{R}(Y) \cong \mathbb{C}[S_{11}, S_{12}, S_{13}]/(S_{11}S_{12} - S_{13}^2 - 1),
\]
which is a non-factorial ring.

5. Comparison with the approach of Brion

When \( Y \) is a spherical \( G \)-variety satisfying \( \Gamma(Y, \mathcal{O}_Y) = \mathbb{C} \), Brion defines the equivariant Cox ring of \( Y \), which is graded by the group \( \text{Cl}_G(Y) \) of isomorphism classes of \( G \)-linearized divisorial sheaves on \( Y \), as
\[
\mathcal{R}_G(Y) := \bigoplus_{[\mathcal{F}] \in \text{Cl}_G(Y)} \Gamma(Y, \mathcal{F}),
\]
where again some accuracy is required in order to define a multiplication law (cf. [Br07 4.2]). We have \( \mathcal{R}_G(Y) \cong \mathcal{R}(Y) \otimes \mathbb{C} [C] \), where the characters of the torus \( C \) correspond to the various choices of linearizations.

In order to obtain a description of \( \mathcal{R}_G(Y) \), Brion first computes the Cox ring of the associated wonderful variety \( Y' \). According to [Lun01 6.1], the wonderful
variety $Y'$ is obtained as follows. The equivariant automorphism group of $G/H$ can be identified with $N_G(H)/H$, hence $N_G(H)$ acts on the set $D$ of $B$-invariant prime divisors in $G/H$. The subgroup of $N_G(H)$ which stabilizes $D$ is called the spherical closure of $H$. It can be written as $H' \times C$ where $H' \subseteq G := G^c$. Then $G'/H'$ is a spherical homogeneous space admitting a wonderful completion $G'/H' \hookrightarrow Y'$.

We consider the Borel subgroup $B' := B \cap G'$ as well as the maximal torus $T' := T \cap G'$, define $\Xi := \Xi(B') = \Xi(T')$, and denote by $\mathfrak{X}^+ \subseteq \mathfrak{X}$ the subset of dominant weights. There is a natural inclusion $\epsilon : \mathfrak{X} \hookrightarrow \mathbb{C}[T']^*$, $\chi \mapsto e^{\chi}$. The set of $B'$-invariant prime divisors in $G'/H'$ can be identified with $D = \{D_1, \ldots, D_r\}$. We denote by $\omega_i$ the left $B'$-weight of the equation $f'_i$ of the pullback of $D_i$ under $G' \to G'/H'$. There is a natural bijection between the set $\{\gamma_1, \ldots, \gamma_k\}$ of spherical roots of $G'/H'$ and the set $\{Y_1', \ldots, Y_s'\}$ of $G'$-invariant prime divisors in $Y'$.

The Cox ring $\mathcal{R}(Y')$ is realized as a subring of

$$
\mathbb{C}[G' \times T'] = \mathbb{C}[G'] \otimes_{\mathbb{C}} \mathbb{C}[T'] = \bigoplus_{\chi \in \mathfrak{X}} \mathbb{C}[G'] \otimes_{\mathbb{C}} \mathbb{C} e^{\chi}.
$$

We will require the following notation: when $V$ is a $G'$-module and $\chi \in \mathfrak{X}^+$, we denote by $V(\chi)$, the corresponding isotypic component of $V$, i.e. the sum of all irreducible $G'$-submodules of highest weight $\chi$. We denote by $s_i \in \Gamma(Y', \mathcal{O}_{Y'}(f'_i))$ and $s_{D_i} \in \Gamma(Y', \mathcal{O}_{Y'}(D_i))$ the respective canonical sections. Finally, we denote by $K'$ the intersection of the kernels of all characters of $H'$. Note that $G'/K'$ is always quasifinite, but may fail to have trivial divisor class group when $H'$ is not connected.

**Theorem 5.1** (cf. [Br07, Theorem 3.2.3]). We define the following partial order on $\mathfrak{X}$: we write $\chi \preceq \eta$ if $\eta - \chi$ is a linear combination of spherical roots of $G'/H'$ with nonnegative coefficients. Then we have

$$
\mathcal{R}(Y') \cong \bigoplus_{(\chi, \eta) \in \mathfrak{X}^+ \times \mathfrak{X}} \Gamma(G'/K', \mathcal{O}_{G'/K'})(\chi) \otimes_{\mathbb{C}} \mathbb{C} e^{\chi}.
$$

This isomorphism sends $s_i \mapsto 1 \otimes e^{\chi_i}$ and $s_{D_i} \mapsto f'_i \otimes e^{\chi_i}$. Furthermore, $\mathcal{R}(Y')$ is the Rees algebra associated to the ascending filtration

$$
\mathcal{F}^n(\Gamma(G'/K', \mathcal{O}_{G'/K'})) := \bigoplus_{\chi \in \mathfrak{X}} \bigoplus_{\chi \preceq \eta} \Gamma(G'/K', \mathcal{O}_{G'/K'})(\chi)
$$

of $\Gamma(G'/K', \mathcal{O}_{G'/K'})$ indexed by the partially ordered set $\mathfrak{X}$.

**Example 5.2.** Let $G' := \text{SL}(d)$ for $d \geq 3$ act on $Y' := \mathbb{P}^d \times \mathbb{P}^d$ by acting naturally on the first factor and with the contragredient action on the second factor. Then $Y'$ is a wonderful variety and the point $((1 : 0 : \ldots : 0), (1 : 0 : \ldots : 0)) \in Y'$ with isotropy group $H' := \text{GL}(d-1)$ lies in the open $G'$-orbit. We have

$$
\mathcal{R}(Y') = \mathbb{C}[X_1, \ldots, X_d, Y_1, \ldots, Y_d].
$$

There is exactly one $G$-invariant prime divisor $D := \text{V}(X_1Y_1 + \ldots + X_dY_d)$. The intersection $K'$ of the kernels of all characters of $H'$ is $\text{SL}(d-1)$ as in Example 4.2.

In particular, we have

$$
R := \Gamma(G'/K', \mathcal{O}_{G'/K'}) = \mathbb{C}[X_1, \ldots, X_d, Y_1, \ldots, Y_d]/(X_1Y_1 + \ldots + X_dY_d - 1).
$$

With the notation from Example 4.2, $\gamma_1 := v_1^2 + v_2^2 \in \mathfrak{X}(B')$ is the unique spherical root of $G'/H'$. Theorem 5.1 yields an isomorphism

$$
\mathcal{R}(Y') \cong \bigoplus_{i,j,k=0}^{\infty} R_{(iv_1^2 + jv_2^2)} \otimes_{\mathbb{C}} \mathbb{C} e^{iv_1^2 + jv_2^2 + k\gamma_1}.
$$
sending $X_i \mapsto X_i \otimes e^{v_1}$ and $Y_i \mapsto Y_i \otimes e^{v_2}$. In particular, it sends 
\[ X_1 Y_1 + \ldots + X_d Y_d \mapsto 1 \otimes e^{v_3}. \]

Now assume that the fan associated to the spherical embedding $G/H \hookrightarrow Y$ contains exactly the trivial cone 0 and the one-dimensional cones with primitive lattice generators $u_1, \ldots, u_n \in \mathcal{N}$. The natural map $G/H \to G'/H'$ extends to a $G$-equivariant morphism $\phi : Y \to Y'$, which induces an equivariant homomorphism of graded algebras
\[ \phi^* : \mathcal{R}(Y') = \mathcal{R}_G(Y') \to \mathcal{R}_G(Y). \]

The invariant subrings
\[ \mathcal{R}_G(Y)^G' \cong \mathbb{C}[C][W_1, \ldots, W_n] \]
\[ \mathcal{R}(Y)^G' \cong \mathbb{C}[Z_1, \ldots, Z_s] \]
are polynomial rings over $\mathbb{C}[C]$ and $\mathbb{C}$ where $W_1, \ldots, W_n$ and $Z_1, \ldots, Z_s$ are identified with the canonical sections corresponding to the $G$-invariant prime divisors in $Y$ and $Y'$ respectively.

**Theorem 5.3** (cf. [Bri07, Theorem 4.3.2]). The restricted map
\[ \phi^*|_{\mathcal{R}(Y)^G'} : \mathcal{R}(Y)^G' \to \mathcal{R}_G(Y)^G' \]
sends $Z_i \mapsto W_i^{-(u_1, \gamma_1)} \cdots W_n^{-(u_n, \gamma_n)}$, and the map
\[ \mathcal{R}(Y') \otimes_{\mathcal{R}(Y)^G'} \mathcal{R}_G(Y)^G' \to \mathcal{R}_G(Y) \]
sending $s' \otimes s \mapsto \phi^*(s') s$ is an isomorphism.

**Remark 5.4.** As $\mathcal{R}(Y) \cong \mathcal{R}_G(Y)^G$ (cf. [Bri07] proof of Theorem 4.3.2]), it immediately follows that
\[ \mathcal{R}(Y) \cong \mathcal{R}(Y') \otimes_{\mathbb{C}[Z_1, \ldots, Z_s]} \mathbb{C}[W_1, \ldots, W_n]. \]
Furthermore, if $Y$ is horospherical and $P := N_G(H)$, we have $Y' \cong G/P$ and $s = 0$, hence $\mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \ldots, W_n]$ (cf. Theorem 5.8).

**Example 5.5.** Consider $G := SL(d)$ for $d \geq 3$ and $H := SL(d-1)$ as in Example 4.2 Then $G' = G$, and the spherical closure of $H$ is $H' := GL(d-1)$ as in Example 5.2. Consider the embedding $G/H \hookrightarrow Y$ corresponding to the fan containing the one-dimensional cones having primitive lattice generators $(p_1, q_1), \ldots, (p_n, q_n) \in \mathcal{N}$ with respect to the basis $\{v_1, v_2\}$ as in Example 4.2. There is exactly one spherical root $\gamma_1 = v_1^1 + v_2^2$. As $G$ is semisimple, it follows that $\mathcal{R}_G(Y) = \mathcal{R}(Y)$. We have
\[ \mathcal{R}(Y') = \mathbb{C}[X_1, \ldots, X_d, Y_1, \ldots, Y_d] \]
\[ \mathcal{R}(Y)^G = \mathbb{C}[Z_1] \]
\[ \mathcal{R}(Y)^G = \mathbb{C}[W_1, \ldots, W_n], \]
the inclusion $\mathbb{C}[Z_1] \hookrightarrow \mathbb{C}[X_1, \ldots, X_d, Y_1, \ldots, Y_d]$ is given by $Z_1 \mapsto X_1 Y_1 + \ldots + X_d Y_d$, and $\phi^* : \mathbb{C}[Z_1] \to \mathbb{C}[W_1, \ldots, W_n]$ sends $Z_1 \mapsto W_1^{-p_1 q_1} \cdots W_n^{-p_n q_n}$. From
\[ \mathcal{R}(Y) \cong \mathbb{C}[X_1, \ldots, X_d, Y_1, \ldots, Y_d] \otimes_{\mathbb{C}[Z_1]} \mathbb{C}[W_1, \ldots, W_n], \]
we obtain
\[ \mathcal{R}(Y) \cong \mathbb{C}[X_j, Y_j, W_1, \ldots, W_n]_d \]
\[ \sum_{j=1}^d X_j Y_j - W_1^{-p_1 q_1} \cdots W_n^{-p_n q_n}, \]
which is in agreement with Example 4.2.
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REFERENCES

[AB05] Valery Alexeev and Michel Brion, Moduli of affine schemes with reductive group action, J. Algebraic Geom. 14 (2005), no. 1, 83–117.

[ADHL10] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, Cox rings, arXiv:1003.4229v2.

[Arz09] I. V. Arzhantsev, On the factoriality of Cox rings, Mat. Zametki 85 (2009), no. 5, 643–651, arXiv:0802.0763v1.

[BH03] Florian Berchtold and Jürgen Hausen, Homogeneous coordinates for algebraic varieties, J. Algebra 266 (2003), no. 2, 636–670.

[BM13] Victor Batyrev and Anne Moreau, The arc space of horospherical varieties and motivic integration, Compos. Math. 149 (2013), no. 8, 1327–1352.

[Bri07] Michel Brion, Vers une généralisation des espaces symétriques, J. Algebra 313 (1990), no. 1, 115–143.

[Bri07] Michel Brion, The total coordinate ring of a wonderful variety, J. Algebra 313 (2007), no. 1, 61–99.

[Cox95] David A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), no. 1, 17–50.

[EKW04] E. Javier Elizondo, Kazuhiko Kurano, and Kei-ichi Watanabe, The total coordinate ring of a normal projective variety, J. Algebra 276 (2004), no. 2, 625–637.

[HK00] Yi Hu and Sean Keel, Mori dream spaces and GIT, Michigan Math. J. 48 (2000), 331–348.

[KKLV89] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust, Local properties of algebraic group actions, Algebraic Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 63–75.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust, The Picard group of a G-variety, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 77–87.

[Kno91] Friedrich Knop, The Luna-Vust theory of spherical embeddings, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989) (Madras), Manoj Prakashan, 1991, pp. 225–249.

[Kno94] Friedrich Knop, The asymptotic behavior of invariant collective motion, Invent. Math. 116 (1994), no. 1-3, 309–328.

[Kra84] Hanspeter Kraft, Geometrische Methoden in der Invariantentheorie, Aspects of Mathematics, D1, Friedr. Vieweg & Sohn, Braunschweig, 1984.

[Lus98] Ivan V. Losev, Proof of the Knop conjecture, Ann. Inst. Fourier (Grenoble) 59 (2009), no. 3, 1105–1134.

[Lun01] D. Luna, Variétés sphériques de type A, Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 161–226.

[LV83] D. Luna and Th. Vust, Plongements d’espaces homogènes, Comment. Math. Helv. 58 (1983), no. 2, 186–245.

[Mac12] Diane Maclagan, Introduction to tropical algebraic geometry, Tropical geometry and integrable systems, Contemp. Math., vol. 580, Amer. Math. Soc., Providence, RI, 2012, pp. 1–19.

[Pau83] Franz Pauer, Plongements normaux de l’espace homogène SL(3)/SL(2), C. R. du 108ème Congrès Nat. Soc. Sav. (Grenoble), vol. 3, 1983, pp. 87–104.

[Pau84] Franz Pauer, “Caracterisation valuative” d’une classe de sous-groupes d’un groupe algébrique, C. R. du 109ème Congrès Nat. Soc. Sav. (Grenoble), vol. 3, 1984, pp. 159–166.

[Pau89] Franz Pauer, Normale Einbettungen von sphärischen homogenen Räumen, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 145–155.
[Pop74] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 294–322.

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411.

[Świ99] Joanna Święcicka, *Quotients of toric varieties by actions of subtori*, Colloq. Math. **82** (1999), no. 1, 105–116.

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104.

[Tim11] Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. 138, Springer, Heidelberg, 2011, Invariant Theory and Algebraic Transformation Groups, 8.

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