SPECTRAL PERTURBATION THEORY AND THE TWO WEIGHTS PROBLEM

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Abstract. The famous two weights problem consists in characterising all possible pairs of weights such that the Hardy projection is bounded between the corresponding weighted $L^2$ spaces. Koosis’ theorem of 1980 gives a way to construct a certain class of pairs of weights. We show that Koosis’ theorem is closely related to (in fact, is a direct consequence of) a spectral perturbation model suggested by de Branges in 1962. Further, we show that de Branges’ model provides an operator-valued version of Koosis’ theorem.

1. Introduction and main result

1.1. Introduction. Let $P_{\pm}$ be the Hardy projections in $L^2(T)$ ($T$ is the unit circle parameterised by $(0, 2\pi)$):

$$ (P_{\pm}f)(e^{i\theta}) = \pm \lim_{r \to 1 \mp 0} \int_0^{2\pi} \frac{f(e^{it})}{1 - re^{i(\theta - t)}} \frac{dt}{2\pi}. \quad (1.1) $$

In its simplest form, the two weights problem consists in the characterisation of all pairs of weights $v_j : T \to [0, \infty)$, $j = 0, 1$, such that

$$ P_+ : L^2(T, v_0(e^{it})dt) \mapsto L^2(T, v_1(e^{it})dt) \quad (1.2) $$

is a bounded operator. (Of course, one could equally speak of $P_-$. If $v_1 = v_0$, then the characterisation of such weights is given by the celebrated Muckenhoupt condition [8]:

$$ \sup_{\Delta} \left( \frac{1}{|\Delta|} \int_{\Delta} v_0(e^{it})dt \cdot \frac{1}{|\Delta|} \int_{\Delta} v_0(e^{it})^{-1}dt \right) < \infty, $$

where the supremum is taken over the set of all intervals $\Delta \subset (0, 2\pi)$ and $|\Delta|$ is the length of the interval $\Delta$. If there is no a priori relation between $v_0$ and $v_1$, the two weights problem is open, despite many years of efforts. Some necessary and some sufficient conditions are known but no effective complete description of all pairs of weights $v_0$, $v_1$ was available available till recently. The recent news at the time of writing is that the conjunction of three preprints [9], [7], [6] proved a long-standing conjecture of Nazarov–Treil–Volberg (see [14]), stating that for the Hilbert transform the so-called two-weight $T1$ theorem is valid. However, the conditions of $T1$ theorem are not easily translated (if at all) into conditions on weights.

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Under these circumstances, any partial information on the problem is valuable. One such piece of information is Koosis’ theorem [3]:

**Theorem** (Koosis). For every weight \( v_0 \geq 0 \) such that \( 0 < v_0(e^{it}) < 1 \) for a.e. \( t \in (0, 2\pi) \) and \( v_0^{-1} \in L^1(\mathbb{T}) \), one can find another weight \( v_1 \), \( 0 \leq v_1 \leq v_0 \), such that \( \log v_1 \in L^1(\mathbb{T}) \) and such that the Hardy projection \( P_+ \) is bounded between the weighted spaces \( L^2(v_0) \). (Koosis, version 2)

Koosis’ proof (see also [10, Appendix]) is an ingenious calculation, but one can argue that it has a rather ad hoc flavour. The purpose of this note is to point out that Koosis’ theorem follows naturally from the formalism of spectral perturbation theory (more precisely, scattering theory) in the form suggested by de Branges in [1]. In fact, the statement we get in this way is more general than the original Koosis’ theorem; we obtain an operator-valued analogue. That is, our \( L^2 \) spaces consist of functions on \( \mathbb{T} \) with values in a Hilbert space \( \mathcal{K} \) and our weights are functions with values in the Schatten classes of compact operators in \( \mathcal{K} \).

We hope that this note will attract the attention of experts to the connection between the two weights problem and scattering theory. We believe that this connection is yet to be thoroughly explored.

1.2. Preliminaries. First we would like to rewrite the two weights problem in an equivalent form. Let \( f \in L^2(\mathbb{T}, v_0(e^{it})dt) \) and suppose that the weight \( v_0 \) vanishes on some open set. Then the function \( f \) is not defined on this open set, and therefore it is not clear how to define the projections \( P_{\pm}f \) by (1.1). This suggests that the integration in the definition (1.1) of the projections \( P_{\pm} \) should be performed with respect to the weighted measure \( v_0(e^{it})dt \). Thus, for a weight \( w_0 : \mathbb{T} \to [0, \infty) \), we define the weighted Hardy projections \( P_{\pm}^{(w_0)} \) by

\[
(P_{\pm}^{(w_0)}f)(e^{i\theta}) = \pm \lim_{r \to 1^\pm} \int_0^{2\pi} \frac{w_0(e^{it})f(e^{it})}{1 - r e^{i(\theta - t)}} \frac{dt}{2\pi};
\]

the existence of the limits will be discussed separately.

If \( v_0(e^{it}) > 0 \) for a.e. \( t \), then a simple argument with replacing \( f \) by \( v_0f \) shows that \( P_+ \) is a bounded operator between the spaces (1.2) if and only if

\[
P_{+}^{(w_0)} : L^2(\mathbb{T}, w_0(e^{it})dt) \to L^2(\mathbb{R}, w_1(e^{it})dt)
\]

is bounded, where \( w_1 = v_1 \) and \( w_0 = v_0^{-1} \). Thus, we obtain

**Theorem** (Koosis, version 2). For every weight \( w_0 \geq 0 \) such that \( w_0(e^{it}) > 0 \) for a.e. \( t \in (0, 2\pi) \) and \( w_0 \in L^1(\mathbb{T}) \), one can find another weight \( w_1 \geq 0 \) with \( w_1w_0 \leq 1 \) and \( \log w_1 \in L^1(\mathbb{T}) \) such that the weighted Hardy projection \( P_{+}^{(w_0)} \) is a bounded operator between the spaces (1.4).

It is this second version of Koosis’ theorem that we will discuss in this paper.

1.3. Operator valued functions. Let \( \mathcal{K} \) be a Hilbert space; the case \( \dim \mathcal{K} < \infty \) is not excluded, neither it is trivial. We denote by \( \langle \cdot, \cdot \rangle \) the inner product in \( \mathcal{K} \) and by \( \| \cdot \| \) the norm in \( \mathcal{K} \). Notation \( \mathcal{B}(\mathcal{K}) \) stands for the set of all bounded linear operators on \( \mathcal{K} \) and \( \mathcal{S}_p, 1 \leq p < \infty \), denotes the Schatten class of compact operators in \( \mathcal{K} \);
in particular, $S_1$ is the trace class. We denote by $\|\cdot\|_p$ the norm in $S_p$ and by $\|\cdot\|_B$ the norm in $B(K)$. As usual, for $w \in B(K)$, notation $w \geq 0$ means that $(w\chi, \chi) \geq 0$ for all elements $\chi \in K$, and in the same way $w \leq C$, where $C$ is a constant, means $(w\chi, \chi) \leq C\|\chi\|^2$ for all $\chi \in K$. For any $w \in B(K)$ such that $w \geq 0$, the square root $w^{1/2}$ is defined via the functional calculus for self-adjoint operators.

Below we work with “nice” $K$-valued functions of the form

$$f(\mu) = \sum_i (\mu - z_i)^{-1} \chi_i, \quad \mu \in \mathbb{T}, \quad \chi_i \in K, \quad |z_i| \neq 1,$$

where the sum has finitely many terms. We will denote by $L$ the set of all such “nice” functions $f$.

Let $w : \mathbb{T} \to B(K)$ be a Borel measurable function. Suppose that $w$ is non-negative i.e. $w(e^{it}) \geq 0$ for a.e. $t \in (0, 2\pi)$, and that $w$ satisfies

$$\int_0^{2\pi} (w(e^{it})\chi, \chi) \frac{dt}{2\pi} \leq C\|\chi\|^2$$

for some constant $C$ and all $\chi \in K$. Then for any $f \in L$ we can define the quasi-norm

$$\|f\|_{L^2(w)} = \int_0^{2\pi} (w(e^{it})f(e^{it}), f(e^{it})) \frac{dt}{2\pi}.$$  

After taking the quotient over the subspace of functions $f$ with $\|f\|_{L^2(w)} = 0$, we obtain a norm on the quotient space; the space obtained by taking the closure is, by definition, the weighted space $L^2(w)$. Thus, by construction, $L$ is dense in $L^2(w)$.

1.4. Main result and discussion. Let $1 \leq p < \infty$, and let $w_0 : \mathbb{T} \to S_p$ be a Borel measurable non-negative (i.e. $w_0 \geq 0$ a.e.) weight function which satisfies (1.8). Then for all $f \in L$ (i.e. for all $f$ of the form (1.5)) and for a.e. $\theta \in (0, 2\pi)$, the limits in (1.3) exist in the norm of $K$. Further, there exists a non-trivial Borel measurable non-negative weight function $w_1 : \mathbb{T} \to B(K)$, which satisfies

$$\int_0^{2\pi} (w_1(e^{it})\chi, \chi) \frac{dt}{2\pi} \leq \|\chi\|^2, \quad \forall \chi \in K,$$

For such weight $w_0$ and for $f \in L$, we define the weighted Hardy projections $P_{\pm}^{(w_0)}$, as in the scalar case, by (1.3). It is clear that for every $r \neq 1$, the integrals in (1.3) converge absolutely in the norm of $K$.

**Theorem 1.1.** Let $1 \leq p < \infty$, and let $w_0 : \mathbb{T} \to S_p$ be a Borel measurable non-negative (i.e. $w_0 \geq 0$ a.e.) weight function which satisfies (1.8). Then for all $f \in L$ (i.e. for all $f$ of the form (1.5)) and for a.e. $\theta \in (0, 2\pi)$, the limits in (1.3) exist in the norm of $K$. Further, there exists a non-trivial Borel measurable non-negative weight function $w_1 : \mathbb{T} \to B(K)$, which satisfies
and there exist contractions (i.e. operators of norm $\leq 1$) $X, Y_+, Y_-$, acting from $L^2(w_0)$ to $L^2(w_1)$, such that the weighted Hardy projections $P_{\pm}^{(w_0)}$ can be represented as

$$P_{\pm}^{(w_0)} = \pm \frac{i}{2} (X - Y_\pm).$$

(1.9)

In particular, 

$$P_{\pm}^{(w_0)} : L^2(w_0) \to L^2(w_1)$$

are contractions.

Let us discuss this result.

1. It is easy to see that the sum $P_+^{(w_0)} + P_-^{(w_0)}$ is simply the operator of multiplication by $w_0$:

$$(P_+^{(w_0)} f)(e^{i\theta}) + (P_-^{(w_0)} f)(e^{i\theta}) = w_0(e^{i\theta}) f(e^{i\theta}).$$

By (1.9), it follows that this operator of multiplication has norm $\leq 1$. From this it follows that

$$w_0(e^{i\theta})^{1/2} w_1(e^{i\theta}) w_0(e^{i\theta})^{1/2} \leq 1$$

(1.10)

for a.e. $\theta \in (0, 2\pi)$; see the end of Section 4 for the details of this argument.

2. In fact, more than (1.10) is true; we note without proof that the boundedness of $P_{\pm}^{(w_0)}$ implies that

$$(P_r * w_0)^{1/2}(P_r * w_1)(P_r * w_0)^{1/2} \leq C$$

for all $r < 1$ with some constant $C$; here $P_r * w_{0,1}$ is the convolution with the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}.$$  

(1.11)

3. Of course, the boundedness of $P_{\pm}^{(w_0)}$ implies that the weighted Hilbert transform

$$(H^{(w_0)} f)(e^{i\theta}) = \lim_{r \to 1} \int_0^{2\pi} w_0(e^{it}) \frac{2 \sin(\theta - t)}{1 + r^2 - 2r \cos(\theta - t)} f(e^{it}) \frac{dt}{2\pi}$$

is a bounded map from $L^2(w_0)$ to $L^2(w_1)$.

4. If $p = 1$, the weight $w_1$ can be chosen to satisfy

$$\int_0^{2\pi} \|w_1(e^{it})\|_1 \frac{dt}{2\pi} < \infty;$$

see the end of Section 4.

5. The weight function $w_1$ constructed in the Koosis theorem is non-degenerate in the sense that $\log w_1 \in L^1(\mathbb{T})$. The weight function $w_1$ that we construct in Theorem 1.1 is also non-degenerate in the following sense. One has

$$w_0(\mu) = D_0^+(\mu)^* w_1(\mu) D_0^+(\mu), \quad \text{a.e. } \mu \in \mathbb{T},$$

where $D_0^+$ is an operator valued function to be constructed below (see (1.28)). The function $D_0^+$ satisfies $\|D_0^+(\cdot)\|_{B} \in L^{1,\infty}(\mathbb{T})$ and $D_0^+(\mu)$ has a bounded inverse for a.e. $\mu \in \mathbb{T}$. In particular,

$$\operatorname{rank} w_0(\mu) = \operatorname{rank} w_1(\mu), \quad \text{a.e. } \mu \in \mathbb{T},$$

(1.13)
and
\[ \|w_1(\mu)\|_B \geq \frac{\|w_0(\mu)\|_B}{\|D_0^+(\mu)\|_B^2}, \quad \text{a.e. } \mu \in \mathbb{T}. \tag{1.14} \]

By (1.14), we have
\[ \log \|w_1(\mu)\|_B \geq \log \|w_0(\mu)\|_B - 2 \log^+ \|D_0^+(\mu)\|_B, \]
and \( \|D_0^+(\cdot)\|_B \in L^{1,\infty}(\mathbb{T}) \) implies \( \log^+ \|D_0^+(\cdot)\|_B \in L^p(\mathbb{T}) \) for all \( p < \infty \).

1.5. **The outline of the proof.** We consider the absolutely continuous (a.c.) operator valued measure on \( \mathbb{T} \) given by
\[ d\nu_0(e^{i\theta}) = w_0(e^{i\theta}) \frac{d\theta}{2\pi}. \tag{1.15} \]

For this measure \( \nu_0 \), we exhibit (see Lemma 2.1) a Hilbert space \( \mathcal{H} \), a unitary operator \( U_0 \) in \( \mathcal{H} \) and a contraction \( G : \mathcal{H} \to \mathcal{K} \) such that
\[ \nu_0(\delta) = GE_{U_0}(\delta)G^*, \quad \delta \subset \mathbb{T}, \tag{1.16} \]
where \( E_{U_0} \) is the projection-valued spectral measure of \( U_0 \), and \( \delta \subset \mathbb{T} \) is any Borel set. Next, we construct (see (2.2)) a unitary operator \( U_1 \) in \( \mathcal{H} \) such that the identities
\[ (\alpha + \psi_0(z))(\alpha - \psi_1(z)) = I, \tag{1.17} \]
\[ (\alpha - \psi_1(z))(\alpha + \psi_0(z)) = I, \tag{1.18} \]
hold true for all \( |z| \neq 1 \); here \( \alpha \) is the auxiliary bounded self-adjoint operator given by
\[ \alpha = \sqrt{I - (GG^*)^2}, \tag{1.19} \]
and
\[ \psi_j(z) = iG \frac{U_j + z}{U_j - z} G^*. \tag{1.20} \]

Further, similarly to (1.16), we set
\[ \nu_1(\delta) = GE_{U_1}(\delta)G^*, \quad \delta \subset \mathbb{T}. \tag{1.21} \]

We will be able to prove (in Lemma 3.2) that the a.c. part of the measure \( \nu_1 \) can be represented as
\[ d\nu_1^{(ac)}(e^{i\theta}) = w_1(e^{i\theta}) \frac{d\theta}{2\pi} \]
with some operator valued non-negative weight function \( w_1 \). Note that this is not automatic: the Radon-Nikodym theorem for operator valued measures in general fails; to see this, consider the spectral measure of a self-adjoint or unitary operator with a non-trivial a.c. component.

Key to our construction is the connection between the weighted Hardy projections \( P_{\pm}^{(w_0)} \) and certain operators appearing in scattering theory for the pair \( U_0, U_1 \). We use the formalism suggested by de Branges [1] with some simplifications due to Kuroda [4]. This formalism makes use of the weighted Hilbert spaces \( L^2(\nu_j), j = 0, 1 \) of
\( K \)-valued functions on \( \mathbb{T} \). They are defined, similarly to (1.7), starting from the quasi-norm
\[
\| f \|_{L^2(v_\alpha)}^2 = \int_0^{2\pi} d(\nu)e^{it}f(e^{it}), f(e^{it})
\]
on the set \( L \), by taking a quotient and then a closure. We note that \( \nu_0 = \nu_0^{(ac)} \) and
\[
L^2(\nu_1) \subset L^2(\nu_1^{(ac)}) \quad \text{and} \quad \| f \|_{L^2(\nu_1^{(ac)})} \leq \| f \|_{L^2(\nu_1)}.
\]
(1.22)

Following de Branges, we define some auxiliary bounded operators \( X, Y_+ \) and \( Y_- \) acting from \( L^2(\nu_0) \) to \( L^2(\nu_1^{(ac)}) \). First we denote (cf. (1.17), (1.18))
\[
D_0(z) = \alpha + \psi_0(z), \quad D_1(z) = -\alpha + \psi_1(z).
\]
(1.23)

By (1.17), (1.18) we have
\[
D_0(z)D_1(z) = D_1(z)D_0(z) = -I, \quad |z| \neq 1.
\]
(1.24)

Let
\[
X : L^2(\nu_0) \to L^2(\nu_1)
\]
be the linear operator, defined on the dense set \( L \) by
\[
(Xf)(\mu) = \sum (\mu - z_i)^{-1}D_0(z_i)\chi_i, \quad f(\mu) = \sum (\mu - z_i)^{-1}\chi_i.
\]
(1.25)

It turns out (see Lemma 2.4) that \( X \) is a unitary operator between the spaces (1.25). Moreover, this is true for any operators \( U_0, U_1, G, \alpha \), related by (1.17)–(1.20); assumption (1.8) is not relevant here. This fact is part of de Branges’ construction [1]. Bearing in mind the embedding (1.22), we see that \( X \) is a contraction as a map from \( L^2(\nu_0) \) to \( L^2(\nu_1^{(ac)}) \).

Further, by the spectral theorem for the unitary operator \( U_0 \), we have
\[
\psi_0(z) = i \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\nu_0(e^{it}) = i \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} w_0(e^{it}) \frac{dt}{2\pi}.
\]
(1.27)

Thus, \( \psi_0 \) is the Cauchy transform of \( w_0 \). Using assumption (1.8) on the weight \( w_0 \) and the UMD property (see e.g. [12]) of the space \( S_p, 1 < p < \infty \), we check (in Lemma 3.1) that the limits
\[
D_0^\pm(e^{i\theta}) = \lim_{r \to 1^\pm} D_0(re^{i\theta}), \quad D_1^\pm(e^{i\theta}) = \lim_{r \to 1^\pm} D_1(re^{i\theta})
\]
exist for a.e. \( \theta \in (0,2\pi) \) in the operator norm. For \( p = 1 \), this was proven in [1]; for \( p > 1 \), this fact is borrowed from our related work [1].

Again following de Branges, we consider the operators
\[
Y_\pm : f(\mu) \mapsto D_0^\pm(\mu)f(\mu), \quad \mu \in \mathbb{T},
\]
(1.29)
defined initially on the set \( L \), and show that \( Y_\pm \) extend as isometric operators
\[
Y_\pm : L^2(\nu_0) \to L^2(\nu_1^{(ac)}).
\]
Finally, a simple calculation (see Section 1) shows that \( P_\pm(w_0) \), \( X, Y_\pm \) are related by (1.9). We note that \( Y_\pm \) are unitarily equivalent to the wave operators \( W_\pm(U_1, U_0) \) (see [5]), although we will not need this fact.
2. Identities (1.17), (1.18) and the map $X$

2.1. The construction of $G$, $U_0$, $U_1$, $\alpha$. Let $\mathcal{H}$ be the Hilbert space of all Borel measurable $\mathcal{K}$-valued functions on $\mathbb{T}$ with the norm

$$
\|f\|_{\mathcal{H}}^2 = \int_0^{2\pi} \|f(e^{it})\|^2 \frac{dt}{2\pi}.
$$

Let $U_0$ be the operator of multiplication by $e^{it}$ in $\mathcal{H}$. Let $G : \mathcal{H} \to \mathcal{K}$ be defined by

$$
Gf = \int_0^{2\pi} w_0(e^{it})^{1/2} f(e^{it}) \frac{dt}{2\pi}.
$$

Then our assumption (1.8) implies that $G$ is a contraction:

$$
\|Gf\| \leq \left( \int_0^{2\pi} \|w_0(e^{it})\|^{1/2} \frac{dt}{2\pi} \right)^{1/2} \left( \int_0^{2\pi} \|f(e^{it})\|^2 \frac{dt}{2\pi} \right)^{1/2}
= \left( \int_0^{2\pi} \|w_0(e^{it})\| \frac{dt}{2\pi} \right)^{1/2} \|f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}.
$$

It is clear that setting $\nu_0(\delta) = G E_{U_0}(\delta) G^*$ (see (1.16)) yields (1.15). Next, let

$$
\Theta = 2 \sin^{-1}(G^* G);
$$

thus, $\Theta$ is a bounded self-adjoint operator in $\mathcal{H}$ with $\sigma(\Theta) \subset [0, \pi)$ and

$$
G^* G = \sin(\frac{1}{2} \Theta). \tag{2.1}
$$

Set

$$
U_1 = \exp\left(\frac{i}{2} \Theta\right) U_0 \exp\left(\frac{i}{2} \Theta\right) \tag{2.2}
$$

and let $\alpha$ be defined by (1.19).

**Lemma 2.1.** Let $U_0$, $U_1$, $G$, $\alpha$ be as described above. Then identities (1.17), (1.18) hold true. The measure $\nu_1$, defined by (1.21), satisfies $\nu_1(\mathbb{T}) = \nu_0(\mathbb{T})$ and

$$
\|\nu_1(\mathbb{T})\| \leq 1. \tag{2.3}
$$

**Proof.** Denote

$$
\beta = \sqrt{I - (G^* G)^2};
$$

clearly, we have

$$
\alpha G = G \beta. \tag{2.4}
$$

Comparing (2.1) and the definition of $\beta$, we find that

$$
\beta = \cos(\frac{1}{2} \Theta).
$$

Using this and a little algebra, we obtain

$$
U_1 G^* G + G^* G U_0 + i(U_1 \beta - \beta U_0) = 0.
$$

From here by straightforward manipulation we obtain the identity

$$(U_1 - z) G^* G (U_0 - z) + i((U_1 + z) \beta (U_0 - z) - (U_1 - z) \beta (U_0 + z)) - (U_1 + z) G^* G (U_0 + z) = 0$$
for any \( z \in \mathbb{C} \). Taking \(|z| \neq 1\) and multiplying by \((U_1 - z)^{-1}\) on the left and by \((U_0 - z)^{-1}\) on the right, we get
\[
G^*G + i \left( \frac{U_1 + z}{U_1 - z} - \beta \frac{U_0 + z}{U_0 - z} \right) - \frac{U_1 + z}{U_1 - z} G^*G \frac{U_0 + z}{U_0 - z} = 0.
\]

Multiplying this by \( G \) on the left and by \( G^* \) on the right and using that (by (1.19))
\[
(GG^*)^2 = I - \alpha^2,
\]
we obtain
\[
-\alpha^2 + iG \frac{U_1 + z}{U_1 - z} \beta G^* - iG \beta \frac{U_0 + z}{U_0 - z} G^* - G \frac{U_1 + z}{U_1 - z} G^* \frac{U_0 + z}{U_0 - z} G^* = -I.
\]

Finally, using (2.4), this transforms into (1.18). The relation (1.17) is obtained by taking adjoints in (1.18) and changing \( z \) to \( \bar{z}^{-1} \). By (1.16), (1.21), we have \( \nu_0(\mathbb{T}) = \nu_1(\mathbb{T}) = GG^* \). The estimate (2.3) follows from the inequality \( \|G\| \leq 1 \).

**Remark 2.2.** In fact, the construction of [1, 5] allows for a whole family of possible choices for operators \( G, U_0, U_1 \), suitable for our argument. For simplicity, we have chosen only one representative of this family.

**Remark 2.3.** In order to clarify the ideas behind Lemma 2.1, let us sketch the analogous argument for the case of the weights \( w_0, w_1 \) on the real line. In this case the construction naturally leads to self-adjoint (rather than unitary) operators and the algebra is somewhat more transparent. Let a non-negative weight \( w_0 : \mathbb{R} \to \mathcal{B}(\mathcal{K}) \) satisfy
\[
\int_{\mathbb{R}} \|w_0(t)\| dt < \infty.
\]

Let \( \mathcal{H}_\mathbb{R} \) be the \( L^2 \) space of \( \mathcal{K} \)-valued functions on \( \mathbb{R} \) with the norm
\[
\|f\|_{\mathcal{H}_\mathbb{R}}^2 = \int_{\mathbb{R}} \|f(t)\|^2 dt.
\]

Let \( A_0 \) be the operator of multiplication by the independent variable \( t \) in \( \mathcal{H}_\mathbb{R} \) and let \( G_\mathbb{R} : \mathcal{H}_\mathbb{R} \to \mathcal{K} \) be given by
\[
G_\mathbb{R}f = \int_{\mathbb{R}} w_0(t)^{1/2} f(t) dt.
\]

We set \( A_1 = A_0 + G_\mathbb{R}^* G_\mathbb{R} \). Then from the standard resolvent identity we get
\[
(I + G_\mathbb{R}(A_0 - z)^{-1} G_\mathbb{R}^*)(I - G_\mathbb{R}(A_1 - z)^{-1} G_\mathbb{R}^*)
= (I - G_\mathbb{R}(A_1 - z)^{-1} G_\mathbb{R}^*)(I + G_\mathbb{R}(A_0 - z)^{-1} G_\mathbb{R}^*) = I;
\]
this is the analogue of (1.17), (1.18). One sets
\[
\nu_j^\mathbb{R}(\delta) = G_\mathbb{R} E_{A_j}(\delta) G_\mathbb{R}^*, \quad j = 0, 1, \quad \delta \subset \mathbb{R},
\]
and the rest of the construction is very similar to the case of measures on \( \mathbb{T} \).
2.2. The map $X$. Let the map $X$ be defined by (1.25), (1.26).

**Lemma 2.4.** The map $X$ is unitary between the spaces $L^2(\nu_0)$ and $L^2(\nu_1)$.

**Proof.** For $j = 0, 1$, the functions $\psi_j$ (see (1.20)) can be expressed as

$$\psi_j(z) = i \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\nu_j(e^{it}).$$

(2.5)

We note two identities for $\psi_j$:

$$\frac{\psi_j(z_1) - \psi_j(z_2)}{z_1 - z_2^{-1}} = 2i \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z_1)(e^{it} - z_2^{-1})} d\nu_j(e^{it}),$$

(2.6)

$$\psi_j(z)^* = \psi_j(z^{-1}).$$

(2.7)

Next, using (1.23), (1.24), we have for $|z_{1,2}| \neq 1$:

$$\psi_0(z_1) - \psi_0(z_2) = D_0(z_1) - D_0(z_2)^* = -D_0(z_2)^* D_1(z_2)^* D_0(z_1) + D_0(z_2)^* D_1(z_1) D_0(z_1)$$

$$= D_0(z_2)^* (-\psi_1(z_2)^* + \psi_1(z_2)) D_0(z_1).$$

Combining this with (2.6), (2.7), we get

$$\int_0^{2\pi} \frac{d\nu_0(e^{it})}{(e^{-it} - z_2)(e^{it} - z_1)} = D_0(z_2)^* \int_0^{2\pi} \frac{d\nu_1(e^{it})}{(e^{-it} - z_2)(e^{it} - z_1)} D_0(z_1).$$

(2.8)

Now let

$$f_1(\mu) = (\mu - z_1)^{-1} \chi_1, \quad f_2(\mu) = (\mu - z_2)^{-1} \chi_2, \quad \mu \in \mathbb{T},$$

(2.9)

where $|z_{1,2}| \neq 1$ and $\chi_{1,2} \in \mathcal{K}$. Then from (2.8) we get

$$(f_1, f_2)_{L^2(\nu_0)} = (X f_1, X f_2)_{L^2(\nu_1)}.$$ 

This extends to all $f_1, f_2 \in \mathcal{L}$. It follows that $X$ is an isometry. By considering an operator $X_1$ defined in a similar way with $D_1$ instead of $D_0$, and using (1.24), we obtain $X X_1 = -I$, hence $X$ is a surjection. Thus, $X$ is a unitary operator. \hfill \Box

3. The boundary values of $D_0$ and $D_1$

3.1. Existence of boundary values of $D_0$ and $D_1$.

**Lemma 3.1.** The limits $D_0^\pm(e^{i\theta})$, $D_1^\pm(e^{i\theta})$ (see (1.28)) exist for a.e. $\theta \in (0, 2\pi)$ in the operator norm.

For $p = 1$, this was proven in [1].

**Proof.** 1. First we consider the limits $D_0^\pm$. We have

$$D_0(z) = \alpha + \psi_0(z),$$

where $\psi_0(z)$ is given by (2.5). Thus, it suffices to consider the limits of $\psi_0$. By (2.7), it suffices to consider the limits as $z$ approaches the unit circle from inside the unit.
disk. Without loss of generality assume $p > 1$ in (1.8). In fact, we will prove the existence of the non-tangential limits

$$\lim_{z \to e^{i\theta}} \psi_0(z)$$

in the norm of $S_p$. Here $S_\theta$ is the appropriate sector of opening $\pi/2$ with the vertex at $e^{i\theta}$ (see e.g. [4, Section VIII:C3]). The argument below is presented in more detail in our related work [11].

The function $\psi_0$ is the Cauchy transform of the weight function $w_0$ (see (1.27)). Consider the non-tangential maximal function

$$(Tw_0)(e^{i\theta}) = \sup \left\{ \| \psi_0(z) \|_p : z \in S_\theta \right\}.$$  

The key fact is that for $1 < p < \infty$, the Banach space $S_p$ possesses the UMD property, see [12]; that is, the Hilbert transform and many other integral transforms are bounded as operators in $L^2$ spaces of $S_p$-valued functions. Using this, one can prove that the (non-linear) operator $T$ is of the weak $1$-$1$ type, i.e. $Tw_0$ belongs to the weak $L^1,\infty(T)$ class.

Next, using this fact and repeating the classical construction of Privalov’s uniqueness theorem (see e.g. [4, Section III:D]), for any $\varepsilon > 0$ one constructs a simply connected domain $D$ in the unit disk such that $\| \psi_0 \|_p$ is bounded in $D$ and the boundary of $D$ contains the unit circle $\mathbb{T}$ up to a set of measure $\varepsilon$. Let $\varphi$ be a conformal map of the unit disk onto $D$. Then $F(z) = \psi_0(\varphi(z))$ is a bounded $S_p$-valued analytic function on the unit disk. By standard results on Banach space valued analytic functions (see e.g. [2]), $F(z)$ attains non-tangential boundary values in $S_p$ norm a.e. on the unit circle. It follows that the function $\psi_0$ attains non-tangential boundary values in $S_p$ norm on the unit circle minus a set of measure $\varepsilon$. Sending $\varepsilon \to 0$, one obtains the desired result.

2. Let us consider the limits of $D_1$. Since $D_1(z) = -D_0(z)^{-1}$, it suffices to prove that the limiting operators $D_0^\pm(e^{i\theta})$ have bounded inverses for a.e. $\theta$. We do this by employing an argument from [13]. We have

$$D_0(z) = D_0(0) \left( I + D_0(0)^{-1}(D_0(z) - D_0(0)) \right),$$

and therefore it suffices to check that the operators

$$I + D_0(0)^{-1}(D_0^\pm(e^{i\theta}) - D_0(0))$$

have a bounded inverse for a.e. $\theta$. By (1.8), we have $\psi_0(z) \in S_p$ for all $|z| \neq 1$. Let $q \geq p$ be any integer; consider the regularised determinant

$$d(z) = \text{Det}_q(I + D_0(0)^{-1}(\psi_0(z) - \psi_0(0))).$$

The functional $A \mapsto \text{Det}_q(I + A)$ is continuous (in fact, analytic) on $S_q$. Thus, $d(z)$ is analytic in $z$ and by the previous step of the proof, $d(z)$ has non-tangential boundary values a.e. on the unit circle. Applying Privalov’s uniqueness theorem, we obtain that these boundary values are non-zero a.e. on the unit circle. Now since $\text{Det}_q(I + A) \neq 0$ if and only if $I + A$ has a bounded inverse, we conclude that the operators (3.1) have bounded inverses for a.e. $\theta$. \qed
3.2. The a.c. part of $\nu_1$. Taking $z_1 = z_2 = re^{i\theta}$ in (2.6), one obtains

$$\psi_j(re^{i\theta}) - \psi_j(re^{i\theta})^* = 2i \int_0^{2\pi} P_r(\theta - t) d\nu_j(e^{it}),$$

(3.2)

where $P_r$ is the Poisson kernel (1.11) on $\mathbb{T}$. From the existence of the boundary values of $\psi_j$ on $\mathbb{T}$ (see Lemma 3.1) it follows that the r.h.s. of (3.2) attains a limit (in the operator norm) as $r \to 1$ for a.e. $\theta \in (0, 2\pi)$. Of course, by the definition (1.15) of $\nu_0$ we have

$$w_0(e^{i\theta}) = \lim_{r \to 1} \int_0^{2\pi} P_r(\theta - t) d\nu_0(e^{it})$$

(3.3)

for a.e. $\theta$. Similarly, we define the weight function $w_1$ by

$$w_1(e^{i\theta}) = \lim_{r \to 1} \int_0^{2\pi} P_r(\theta - t) d\nu_1(e^{it})$$

(3.4)

for a.e. $\theta$. In Lemmas 3.2 and 3.3 we follow de Branges’ work [1].

**Lemma 3.2.** The a.c. part of the measure $\nu_1$ is given by

$$d\nu_1^{(ac)}(e^{i\theta}) = w_1(e^{i\theta}) \frac{d\theta}{2\pi}, \quad \text{a.e. } \theta \in (0, 2\pi).$$

(3.5)

**Proof.** Of course, in the scalar case $\dim K < \infty$ formula (3.5) follows directly from (3.4); the point here is to consider the general case. Let $\chi_1, \chi_2 \in \mathcal{K}$; consider the scalar (complex-valued) measure $\langle \nu_1(\cdot)\chi_1, \chi_2 \rangle$. If $\nu_1^{(ac)}$ and $\nu_1^{(sing)}$ are the a.c. and the singular parts of $\nu_1$ with respect to the Lebesgue measure on $\mathbb{T}$, then

$$\langle \nu_1(\cdot)\chi_1, \chi_2 \rangle = \langle \nu_1^{(ac)}(\cdot)\chi_1, \chi_2 \rangle + \langle \nu_1^{(sing)}(\cdot)\chi_1, \chi_2 \rangle$$

gives the unique decomposition of the scalar measure $\langle \nu_1(\cdot)\chi_1, \chi_2 \rangle$ into the a.c. and singular parts. By the scalar theory, we have

$$\lim_{r \to 1} \int_0^{2\pi} P_r(\theta - t) d\nu_1^{(sing)}(e^{it}) \chi_1, \chi_2 = 0$$

for a.e. $\theta$. Thus, using (3.4), we obtain

$$\langle w_1(e^{i\theta}) \chi_1, \chi_2 \rangle = \lim_{r \to 1} \int_0^{2\pi} P_r(\theta - t) d\nu_1^{(ac)}(e^{it}) \chi_1, \chi_2. \quad (3.6)$$

Now take $f_1, f_2$ as in (2.9); multiplying (3.6) by $(e^{i\theta} - z_1)^{-1}(e^{-i\theta} - \bar{z}_2)^{-1}$ and integrating, we get

$$\int_0^{2\pi} \langle w_1(e^{i\theta}) f_1(e^{i\theta}), f_2(e^{i\theta}) \rangle \frac{d\theta}{2\pi} = \int_0^{2\pi} d\nu_1^{(ac)}(e^{i\theta}) \langle f_1(e^{i\theta}), f_2(e^{i\theta}) \rangle.$$

By linearity, this extends to all $f_1, f_2 \in \mathcal{L}$. This yields (3.5). □
3.3. The operators $Y_{\pm}$. Next, we consider the operators $Y_{\pm}$ of multiplication by $D_{0}^{\pm}$, see (1.29).

**Lemma 3.3.** The operators $Y_{\pm}$ are unitary maps from $L^{2}(\nu_{0})$ to $L^{2}(\nu_{1}^{(ac)})$.

*Proof.* Taking $z_{1} = z_{2} = re^{i\theta}$ in (2.8), we obtain

$$
\int_{0}^{2\pi} P_{r}(\theta - t)d\nu_{0}(e^{it}) = D_{0}(re^{i\theta})\ast \int_{0}^{2\pi} P_{r}(\theta - t)d\nu_{1}(e^{it}) D_{0}(re^{i\theta}).
$$

Taking $r \to 1 \pm 0$ and using Lemma 3.2, we get

$$
w_{0}(\mu) = D_{0}^{\pm}(\mu)w_{1}(\mu)D_{0}^{\pm}(\mu), \quad \text{a.e. } \mu \in \mathbb{T}.
$$

This shows that $Y_{\pm}$ are isometries. Considering the operators of multiplication by the boundary values of $D_{1}$ and using the identity (1.24), it is easy to prove that $Y_{\pm}$ are surjections, so they are unitary operators. \(\square\)

4. The proof of Theorem 1.1

*Proof of Theorem 1.1.* The weight function $w_{1}$ has been defined by (3.4). By construction, it is non-negative. It is Borel measurable as a pointwise norm limit of continuous weight functions. Let us prove that the limits in (1.1) exist in $K$ and

$$
(P_{\pm}^{(w_{0})} f)(e^{i\theta}) = \pm \frac{i}{2}((X f)(e^{i\theta}) - (Y_{\pm} f)(e^{i\theta})) \quad (4.1)
$$

for a.e. $\theta$. Take $f(\mu) = (\mu - z)^{-1} \chi, \chi \in K, |z| \neq 1$. We have

$$
D_{0}(z) = \alpha + i \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{e^{it} + z}{e^{it} - z} = \alpha - i \int_{0}^{2\pi} d\nu_{0}(e^{it}) + 2i \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{e^{it}}{e^{it} - z},
$$

and therefore, by the definition (1.26) of $X$,

$$
(X f)(e^{i\theta}) = \left(\alpha - i \int_{0}^{2\pi} d\nu_{0}(e^{it})\right) f(e^{i\theta}) + 2i \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{e^{it}}{(e^{i\theta} - z)(e^{i\theta} - z)} \chi.
$$

For the second term in the above sum, we have

$$
\int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{e^{it}}{(e^{i\theta} - z)(e^{i\theta} - z)} \chi = \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} e^{it}
$$

$$
= \lim_{r \to 1} \left\{ \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{f(e^{i\theta})}{1 - re^{i(\theta - t)}} - \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{f(e^{it})}{1 - re^{i(\theta - t)}} \right\},
$$

where the limits exist in the norm of $K$. Putting this together, after a little algebra we get

$$
(X f)(e^{i\theta}) = \lim_{r \to 1} \left\{ D_{0}(re^{i\theta}) f(e^{i\theta}) - 2i \int_{0}^{2\pi} d\nu_{0}(e^{it}) \frac{f(e^{it})}{1 - re^{i(\theta - t)}} \right\} \quad (4.2)
$$

By Lemma 3.1 the limits

$$
\lim_{r \to \pm 1} D_{0}(re^{i\theta}) f(e^{i\theta})
$$

exist in the norm of $K$. Thus, the limits of the integral in (1.2) also exist. Recalling the definition (1.1) of $P_{\pm}^{(w_{0})}$, we obtain (4.1). \(\square\)
Proof of (1.10). By (3.2) and (3.3), we have

\[ w_0(e^{i\theta}) = \lim_{r \to 1} \int_0^{2\pi} P_r(\theta - t) d\nu_0(e^{it}) \]

\[ = \frac{1}{2i} \lim_{r \to 1} (\psi_0(re^{i\theta}) - \psi_0(\frac{1}{r}e^{i\theta})) = \frac{1}{2i} \lim_{r \to 1} (D_0(re^{i\theta}) - D_0(\frac{1}{r}e^{i\theta})). \]

Thus, if we denote by \( Y_0 \) the operator of multiplication by \( w_0(e^{i\theta}) \), acting from \( L^2(\nu_0) \) to \( L^2(\nu_1^{(ac)}) \), we obtain

\[ Y_0 = \frac{1}{2i} (Y_+ - Y_-), \]

and therefore \( \|Y_0\| \leq 1 \). This yields

\[ \int_0^{2\pi} (w_1(e^{i\theta})w_0(e^{i\theta})f(e^{i\theta}), w_0(e^{i\theta})f(e^{i\theta})) \frac{d\theta}{2\pi} \leq \int_0^{2\pi} (w_0(e^{i\theta})f(e^{i\theta}), f(e^{i\theta})) \frac{d\theta}{2\pi}, \]

which implies (1.10). \qed

Proof of (1.12). Suppose \( p = 1 \). Then

\[ \text{Tr}(GG^*) = \int_0^{2\pi} \text{Tr}(w_0(e^{i\theta})) \frac{d\theta}{2\pi} \leq 1, \]

hence \( G \) is Hilbert-Schmidt. Then

\[ \int_0^{2\pi} \|w_1(e^{i\theta})\|_1 \frac{d\theta}{2\pi} = \int_0^{2\pi} \text{Tr}(w_1(e^{i\theta})) \frac{d\theta}{2\pi} = \text{Tr}(\nu_1^{(ac)}(\mathbb{T})) \leq \text{Tr}(\nu_1(\mathbb{T})) = \text{Tr}(GG^*) \leq 1, \]

i.e. \( \|w_1(\cdot)\|_1 \in L^1(\mathbb{T}) \). \qed

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