ON THE SLICING GENUS OF LEGENDRIAN KNOTS

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Abstract. We apply Heegaard-Floer homology theory to establish generalized slicing Bennequin inequalities closely related to a recent result of T. Mrowka and Y. Rollin proved using Seiberg-Witten monopoles.

1. Introduction

Let $\xi$ be an oriented 2-plane distribution on an oriented 3-manifold $M$. (Unless otherwise specified, all 3-manifolds in this paper are closed, connected and oriented.) $\xi$ is said to be a contact structure on $M$ if there is a 1-form $\alpha$ on $M$ so that $\xi = \ker \alpha$, $d\alpha|_\xi > 0$ and $\alpha \wedge d\alpha > 0$. Such a 1-form is called a contact form for $\xi$. And $(M, \xi)$ is called a contact 3-manifold. A knot $K$ in a contact 3-manifold $(M, \xi)$ is called a Legendrian knot if it's tangent to $\xi$. (Unless otherwise specified, all the knots in this paper are oriented.) $(M, \xi)$ is said to be overtwisted if there is an embedded disk $D$ in $M$ s.t. $\partial D$ is Legendrian, but $D$ is transverse to $\xi$ along $\partial D$. If $(M, \xi)$ is not overtwisted, then it's called tight. For example, the standard contact structure $\xi_{st}$ on $S^3$ given by the complex tangencies of the unit 3-sphere in $\mathbb{C}^2$ is tight. Overtwisted contact structures are kind of "soft", and are completely classified up to isotopy by the homotopy type of the underlying 2-plane distribution. (See [2].) Tight contact structures display more rigidity, and possess more interesting properties.

There are two "classical" invariants, the Thurston-Bennequin number $tb(K)$ and the rotation number $r(K)$, for a Legendrian knot $K$ in $(S^3, \xi_{st})$. These are generalized to null-homologous Legendrian knots in any contact 3-manifold (c.f. [4]). Let $K$ be a null-homologous Legendrian knot in a contact 3-manifold $(M, \xi)$, and $\Sigma \subset M$ a Seifert surface of $K$. Let $K'$ be a knot obtained by pushing $K$ slightly in the direction of a vector field that is transverse to $\xi$ along $K$. Then the Thurston-Bennequin number $tb(K, \Sigma)$ is defined to be the intersection number $\#(K' \cap \Sigma)$. Let $u$ be the positive unit tangent vector field of $K$. Then the rotation number $r(K, \Sigma)$ is defined to be the pairing $\langle c_1(\xi, u), [\Sigma] \rangle$, where $[\Sigma] \in H_2(M, K)$ is the relative homology class represented by $\Sigma$. If we reverse the orientation of $K$, then $tb(K, \Sigma)$ is unchanged, and $r(K, \Sigma)$ changes sign. Note that $tb(K, \Sigma)$ and $r(K, \Sigma)$ depend on $\Sigma$ only through the relative homology class $[\Sigma]$. If $M$ is a homology sphere, then $H_2(M, K) = \mathbb{Z}$, and $tb$, $r$ are independent of $\Sigma$. In this case, we suppress $\Sigma$ from the notation.

In [1], D. Bennequin proved the following Bennequin inequality:

For any Legendrian knot $K$ in $(S^3, \xi_{st})$, 

\begin{equation}
    tb(K) + |r(K)| \leq 2g(K) - 1,
\end{equation}

where $g(K)$ is the genus of $K$. 

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In [4], Y. Eliashberg generalized (1) to any tight contact 3-manifold, and get:

For any null-homologous Legendrian knot $K$ in a tight contact 3-manifold, and any Seifert surface $\Sigma$ of $K$,

$$tb(K, \Sigma) + |r(K, \Sigma)| \leq -\chi(\Sigma).$$

In [16], L. Rudolph strengthened (1) to the slicing Bennequin inequality:

For any Legendrian knot $K$ in $(S^3, \xi_{st})$,

$$tb(K) + |r(K)| \leq 2g_s(K) - 1,$$

where $g_s(K)$ is the slicing genus of $K$.

Let $W$ be an oriented 4-manifold with connected boundary $\partial W = M$, and $\xi$ a contact structure on $M$. Assume that $K$ is a Legendrian knot in $(M, \xi)$, and $\Sigma$ is an embedded surface in $W$ bounded by $K$. In [9], T. Mrowka and Y. Rollin extended the definitions of $tb$ and $r$ to this situation. The following is their construction. Let $v$ be a vector field on $M$ transverse to $\xi$. Extend $v$ to a vector field on $W$, and denote by $\{\varphi_t\}$ the flow of this extended vector field. For a small $\varepsilon > 0$, let $K' = \varphi_\varepsilon(K)$ and $\Sigma' = \varphi_\varepsilon(\Sigma)$. Then the intersection number $\#(\Sigma \cap \Sigma')$ is well defined. The Thurston-Bennequin number is defined to be $tb(K, \Sigma) = \#(\Sigma \cap \Sigma')$. Note that $tb(K, \Sigma)$ depends only through the relative homology class $[\Sigma] \in H_2(W, K)$, and, when $\Sigma \subset M$, this definition coincide with the previous definition of $tb$. Assume $s \in Spin^C(W)$, and there is an isomorphism $h : s|_M \rightarrow t_\xi$, where $t_\xi$ is the canonical $Spin^C$-structure on $M$ associated to $\xi$. Choose a complex structure on $\xi$. Then $det(t_\xi)$ is canonically isomorphic to $\xi$, and $h$ induces an isomorphism $det(h) : det(s)|_M \rightarrow \xi$. Let $u$ be the positive unit tangent vector field of $K$. The rotation number is defined to be $r(K, \Sigma, s, h) = \langle c_1(det(s), det(h)^{-1}(u)), [\Sigma] \rangle$. Note that $r(K, \Sigma, s, h)$ depends on $\Sigma$ only through the relative homology class $[\Sigma] \in H_2(W, K)$, depends on the pair $(s, h)$ only through the isomorphism type of it in $Spin^C(W, \xi)$, and, again, when $\Sigma \subset M$, $r$ is independent of $(s, h)$ and coincide with the previous definition of the rotation number. As before, under the reversal of the orientation of $K$, $tb$ is unchanged, and $r$ changes sign. In the special case that there is a symplectic form $\omega$ on $W$ such that $(W, \omega)$ is a weak symplectic filling of $(M, \xi)$, i.e., $\omega|_\xi > 0$, this symplectic form $\omega$ determines a canonical $Spin^C$-structure $s_\omega$ on $W$ and a canonical isomorphism $h_\omega : s_\omega|_M \rightarrow t_\xi$. Write $r(K, \Sigma, \omega) = r(K, \Sigma, s_\omega, h_\omega)$.

In [9], T. Mrowka and Y. Rollin prove the following generalized slicing Bennequin inequality using Seiberg-Witten monopole invariants.

**Theorem 1.1** (9, Theorem A). Let $W$ be an oriented 4-manifold with connected boundary $\partial W = M$, and $\xi$ a contact structure on $M$. Let $K$ be a Legendrian knot in $(M, \xi)$, and $\Sigma$ an embedded surface in $W$ bounded by $K$. Assume there is an element $(s, h) \in Spin^C(W, \xi)$, such that $SW(s, h) \neq 0$. Then

$$tb(K, \Sigma) + |r(K, \Sigma, s, h)| \leq -\chi(\Sigma).$$

Specially, when $(W, \omega)$ is a weak symplectic filling of $(M, \xi)$ (c.f. [8], Theorem 1.1), we have

$$tb(K, \Sigma) + |r(K, \Sigma, \omega)| \leq -\chi(\Sigma).$$
There are two approaches in the study of 3-dimensional gauge theory: the Seiberg-Witten-Floer approach by counting solutions to the Seiberg-Witten equation; and the Heegaard-Floer approach by counting holomorphic curves. Though the techniques used in these two approaches are quite different, it is conjectured that these give equivalent theories as their 4-dimensional counterparts do. In this paper, we use Heegaard-Floer homology to prove the following generalizations of the slicing Bennequin inequality, which further demonstrates the similarity between the two theories.

**Theorem 1.2.** Let $W$ be an oriented 4-manifold with connected boundary $\partial W = M$, $\xi$ a contact structure on $M$, and $K$ a Legendrian knot in $(M, \xi)$.

(a) If there is a $\text{Spin}^C$-structure $s$ on $W$ with $F_{W \setminus B, s|W \setminus B}(c^+(\xi)) \neq 0$, where $B$ is an embedded 4-ball in the interior of $W$, then there is an isomorphism $h : s|_M \to t_\xi$ such that, for any embedded surface $\Sigma$ in $W$ bounded by $K$,

$$tb(K, \Sigma) + |r(K, \Sigma, s, h)| \leq -\chi(\Sigma).$$

(b) If $(W, \omega)$ is a weak symplectic filling of $(M, \xi)$, then, for any embedded surface $\Sigma$ in $W$ bounded by $K$,

$$tb(K, \Sigma) + |r(K, \Sigma, \omega)| \leq -\chi(\Sigma).$$

2. **Heegaard-Floer Homology**

In this section, we review aspects of the Heegaard-Floer theory necessary for the proof of Theorem 1.2.

2.1. **Heegaard-Floer homology.** In [14], P. Ozsváth and Z. Szabó defined the Heegaard-Floer homology groups of 3-manifolds. Given a connected oriented closed 3-manifold $M$ and a $\text{Spin}^C$-structure $t$ on $M$, there are four Heegaard-Floer homology groups associated to $M$: $HF^\infty(M, t)$, $HF^-(M, t)$, $HF^+(M, t)$, and $HF(M, t)$. The first three are $\mathbb{Z}[U]$-modules, and the last one is a $\mathbb{Z}$-module. In this paper, we will mostly use $HF^+(M, t)$. Moreover, given a $\mathbb{Z}[H^1(M)]$-module $\mathcal{M}$, there is the notion of $\mathcal{M}$-twisted Heegaard-Floer homology $HF^+(M, t; \mathcal{M})$, which is a $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(M)]$-module (c.f. [15]).

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two $\mathbb{Z}[H^1(M)]$-modules, and $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism, then $\theta$ naturally induces a homomorphism

$$\Theta : HF^+(M, t; \mathcal{M}_1) \to HF^+(M, t; \mathcal{M}_2).$$

If we consider $\mathbb{Z}$ as a $\mathbb{Z}[H^1(M)]$-module, then $HF^+(M, t; \mathbb{Z})$ is the (untwisted) Heegaard-Floer homology $HF^+(M, t)$ defined with the appropriate coherent orientation system, and the $2^{b_1(M)}$ choices of $\mathbb{Z}[H^1(M)]$-module structures on $\mathbb{Z}$ correspond to the $2^{b_1(M)}$ coherent orientation systems on the moduli spaces (c.f. [14] [15]).

In [13], P. Ozsváth and Z. Szabó introduced the Heegaard-Floer homology twisted by a 2-form. More precisely, consider the polynomial ring

$$\mathbb{Z}[\mathbb{R}] = \{ \sum_{i=1}^k c_i T^{s_i} \mid k \in \mathbb{Z}_{\geq 0}, c_i \in \mathbb{Z}, s_i \in \mathbb{R} \}. $$
Let $[\omega] \in H^2(M; \mathbb{R})$. The action $e^{[\nu]} \cdot T^s = T^{s + f_1 \nu \wedge \omega}$, where $[\nu] \in H^1(M)$, gives $\mathbb{Z}[\mathbb{R}]$ a $\mathbb{Z}[H^1(M)]$-module structure. Denote the module by $\mathbb{Z}[\mathbb{R}][\omega]$. Then the Heegaard-Floer homology of $M$ twisted by $[\omega]$ is defined to be

$$HF^+(M, t; [\omega]) = HF^+(M, t; \mathbb{Z}[\mathbb{R}][\omega]).$$

In [15], P. Ozsváth and Z. Szabó deduced the following adjunction inequality:

**Theorem 2.1** ([15], Theorem 7.1). Let $\Sigma$ be a close oriented surface embedded in a 3-manifold $M$ with $g(\Sigma) \leq 1$, $t$ a Spin$^C$-structure on $M$, and $\mathcal{M}$ a $\mathbb{Z}[H^1(M)]$-module. If $HF^+(M, t; \mathcal{M}) \neq 0$, then

$$\langle c_1(t), [\Sigma] \rangle \leq -\chi(\Sigma).$$

Note that, although the adjunction inequality is only prove for untwisted Heegaard-Floer homology in [15], the proof there readily adapts to the twisted case.

### 2.2. Homomorphisms induced by cobordisms.

Let $W$ be a cobordism from a 3-manifold $M_1$ to another 3-manifold $M_2$, and $s$ a Spin$^C$-structure on $W$. Then $W$ and $s$ induce a homomorphism

$$F^+_{W,s} : HF^+(M_1, s|_{M_1}) \to HF^+(M_2, s|_{M_2}).$$

Let $\mathcal{M}$ be a $\mathbb{Z}[H^1(M_1)]$-module, and $\delta : H^1(\partial W) \to H^2(W, \partial W)$ the connecting map in the long exact sequence of the pair $(W, \partial W)$. Define

$$\mathcal{M}(W) = \mathcal{M} \otimes_{\mathbb{Z}[H^1(M_1)]} \mathbb{Z}[\delta H^1(\partial W)],$$

where the action of $\mathbb{Z}[H^1(M_1)]$ on $\mathbb{Z}[\delta H^1(\partial W)]$ is induced by $e^{[\nu]} \mapsto e^{\delta([\nu])}$. Then $W$ and $s$ also induce a homomorphism

$$F^+_{W,s} : HF^+(M_1, s|_{M_1}; \mathcal{M}) \to HF^+(M_2, s|_{M_2}; \mathcal{M}(W)).$$

The definition of this homomorphism depends on some auxiliary choices. So it’s only define up to right action by units of $\mathbb{Z}[H^1(M_1)]$ and left action by units of $\mathbb{Z}[H^1(M_2)]$. Alternatively, we consider it as an equivalence class of homomorphisms from $HF^+(M_1, s|_{M_1}; \mathcal{M})$ to $HF^+(M_2, s|_{M_2}; \mathcal{M}(W))$, and denote this equivalence class by $[F^+_{W,s}]$.

Specially, for an $[\omega] \in H^2(W; \mathbb{R})$, let $\mathcal{M} = \mathbb{Z}[\mathbb{R}][\omega|_{M_1}]$. There is a natural homomorphism $\theta : \mathcal{M}(W) \to \mathbb{Z}[\mathbb{R}][\omega|_{M_2}]$ induced by $T^s \otimes e^{\delta([\nu])} \mapsto T^{s + f_1 \nu \wedge \omega}$, where $[\nu] \in H^1(M_2)$. This map induces a homomorphism $\Theta$ between the Heegaard-Floer homologies of $M_2$ twisted by these two $\mathbb{Z}[H^1(M_2)]$-modules. Composing it with $F^+_{W,s}$, we get a homomorphism

$$F^+_{W,s}[\omega] = \Theta \circ F^+_{W,s} : HF^+(M_1, s|_{M_1}; [\omega|_{M_1}]) \to HF^+(M_2, s|_{M_2}; [\omega|_{M_2}]).$$

Again, $F^+_{W,s}[\omega]$ is only defined up to multiplication by $\pm T^s$, and is consider as an equivalent class $[F^+_{W,s}[\omega]]$ (c.f. [13]).

The homomorphisms defined here satisfy the following composition laws:
Theorem 2.2 ([10], Theorems 3.4, 3.9). Let \( W_1 \) be a cobordism from a 3-manifold \( M_1 \) to another 3-manifold \( M_2 \), and \( W_1 \) a cobordism from \( M_2 \) to a third 3-manifold \( M_3 \). Then \( W = W_1 \cup_{M_2} W_2 \) is a cobordism from \( M_1 \) to \( M_3 \). We have:

(a) For any \( \text{Spin}^C \)-structures \( s_i \in \text{Spin}^C(W_i) \), \( i = 1, 2 \),

\[
F_{W_2, s_2}^+ \circ F_{W_1, s_1}^+ = \sum_{\{s \in \text{Spin}^C(W) \mid s|_{W_i} \cong s_i\}} \pm F_{W, s}^+.
\]

(b) Let \( s \in \text{Spin}^C(W) \), and \( s_i = s|_{W_i} \). For any \( \mathbb{Z}[H^1(M)] \)-module \( \mathfrak{M} \), there are representatives \( F_{W_1, s_1}^+ \in \mathfrak{F}_{W_1, s_1}^+ \) and \( F_{W_2, s_2}^+ \in \mathfrak{F}_{W_2, s_2}^+ \) such that

\[
[F_{W, s}] = [\Pi \circ F_{W_1, s_1}^+ \circ F_{W_2, s_2}^+],
\]

where \( \Pi \) is induced by the natural homomorphism from \( \mathfrak{M}(W_1)(W_2) \) to \( \mathfrak{M}(W) \).

Combine Theorem 2.2 and the blow-up formula ([10], Theorem 3.7), we have the following theorem.

Theorem 2.3 ([10], Theorems 3.4, 3.7). Let \( W \) be a cobordism from a 3-manifold \( M_1 \) to another 3-manifold \( M_2 \), and \( s \) a \( \text{Spin}^C \)-structure on \( W \). Blow up an interior point of \( W \). We get a new cobordism \( \hat{W} \) from \( M_1 \) to \( M_2 \). Let \( \hat{s} \) be the lift of \( s \) to \( \hat{W} \) with \( \langle c_1(\hat{s}), [E] \rangle = -1 \), where \( E \) is the exceptional sphere. Then \( F_{W, s}^+ = F_{\hat{W}, \hat{s}}^+ \).

2.3. The contact invariant. In [12], P. Ozsváth and Z. Szabó defined the Ozsváth-Szabó invariants for contact 3-manifolds. For each contact 3-manifold \( (M, \xi) \), it is an element \( c(\xi) \) of the quotient \( \widehat{HF}(-M, t_\xi)/\{\pm 1\} \), where \( t_\xi \) is the \( \text{Spin}^C \)-structure associated to \( \xi \). Let \( \iota : \widehat{HF}(-M) \to HF^+(M) \) be the natural map (c.f. [14]). We set \( c^+(\xi) = \iota(c(\xi)) \in HF^+(M, t_\xi)/\{\pm 1\} \). This version of the Ozsváth-Szabó contact invariants is easier to use for our purpose. Given a \( \mathbb{Z}[H^1(M)] \)-module \( \mathfrak{M} \), one can similarly define the twisted Ozsváth-Szabó contact invariant \( c^+(\xi; \mathfrak{M}) \in HF^+(M, t_\xi; \mathfrak{M})/\mathbb{Z}[H^1(M)]^\times \), where \( \mathbb{Z}[H^1(M)]^\times \) is the set of units of \( \mathbb{Z}[H^1(M)] \).

Specially, if \( [\omega] \in H^2(M; \mathbb{R}) \), then we have the \( [\omega] \)-twisted invariant \( c^+(\xi; [\omega]) \in HF^+(M, t_\xi; [\omega])/\{\pm T_s \mid s \in \mathbb{R} \} \) (c.f. [13]). These contact invariants vanish when \( \xi \) is overtwisted. Following properties of the Ozsváth-Szabó contact invariants are needed for the proof of Theorem 2.2.

Proposition 2.4 ([5], Proposition 3.3). Suppose that \( (M', \xi') \) is obtained from \( (M, \xi) \) by Legendrian surgery on a Legendrian link. Then we have \( F_{W, s_0}^+(c^+(\xi')) = c^+(\xi) \), where \( W \) is the cobordism induced by the surgery and \( s_0 \) is the canonical \( \text{Spin}^C \)-structure associated to the symplectic structure on \( W \). Moreover, \( F_{W, s}^+(c^+(\xi')) = 0 \) for any \( \text{Spin}^C \)-structure \( s \) on \( W \) with \( s \not\cong s_0 \).

Theorem 2.5 ([13], Theorem 4.2). Let \( (M, \xi) \) be a contact 3-manifold with a weak symplectic filling \( (W, \omega) \). Let \( B \) be an embedded 4-ball in the interior of \( W \). Consider \( W \setminus B \) as a cobordism from \( -M \) to \( -\partial B \). Then \( \sum_{W \setminus B, s_\omega} \langle c^+(\xi; [\omega|_{M}]) \rangle \neq 0 \), where \( s_\omega \) is the \( \text{Spin}^C \)-structure on \( W \) associated to \( \omega \).
In this section, we adapt T. Mrowka and Y. Rollin’s idea into the Heegaard-Floer setting, and prove Theorem 1.2.

Lemma 3.1 ([9]). Let $W$ be an oriented 4-manifold with connected boundary $\partial W = M$, $\xi$ a contact structure on $M$, and $s$ a Spin$^c$-structure on $W$ with an isomorphism $h: s|_M \to t_\xi$. Assume $K$ is a Legendrian knot in $(M, \xi)$, and $\Sigma \subset W$ is an embedded surface bounded by $K$. Then there are a Legendrian knot $K'$ in $(M, \xi)$ and an embedded surface $\Sigma' \subset W$ bounded by $K'$, such that $tb(K', \Sigma') \geq 1$, $\chi(\Sigma') \leq -1$, and $tb(K', \Sigma') + |r(K', \Sigma', s, h)| + \chi(\Sigma') = tb(K, \Sigma) + |r(K, \Sigma, s, h)| + \chi(\Sigma)$.

Proof. Let $p$ be a point on $K$. There is a neighborhood $U$ of $p$ so that $(U, \xi|_U) \cong (\mathbb{R}^3, \xi_0)$, where $\xi_0$ is the standard contact structure on $\mathbb{R}^3$ defined by $dz - ydx$. By the following Legendrian Reidemeister move, we create a pair of cusps on the front projection of $K \cap U$ (c.f. [5]).

![Figure 1. Creating cusps](image)

Near a cusp, connect sum $K$ with a Legendrian righthand trefoil knot $T_r$ in $U$ with $tb(T_r) = 1$. We get a new Legendrian knot $K_1$ and an embedded surface $\Sigma_1$ in $W$ bounded by $K_1$, s.t., $tb(K_1, \Sigma_1) = tb(K, \Sigma) + 1$, $\chi(\Sigma_1) = \chi(\Sigma) - 1$, and $|r(K_1, \Sigma_1, s, h)| = |r(K, \Sigma, s, h)|$. Repeat this process, we will find a $K'$ and a $\Sigma'$ with the properties specified in the lemma. □

![Figure 2. Connect summing with $T_r$](image)

Proof of Theorem 1.2 By Lemma 3.1 we only need prove the theorem for $K$ and $\Sigma$ with $tb(K, \Sigma) \geq 1$ and $\chi(\Sigma) \leq -1$. We assume these are true throughout the proof.

We prove part (a) first. Performing Legendrian surgery along $K$ gives a symplectic cobordism $(V, \omega')$ from $(M, \xi)$ to another contact 3-manifold $(M', \xi')$ (c.f. [17][18]). By Proposition 2.4 $F^+_{V, s_\omega}(c^+(\xi')) = c^+(\xi)$. Let $\bar{W} = W \cup_V V$. Then, by Theorem 2.2...
we have
\[ \sum_{\{ \tilde{s} \in Spin^C(\tilde{W}) \mid \tilde{s}\mid W \cong \tilde{s}, \tilde{s}\mid V \cong \tilde{s}_W \}} \pm F^+_{W, \tilde{s}}(c^+(\xi')) = F^+_{W, \tilde{s}} \circ F^+_{V, \tilde{s}_W}(c^+(\xi')) = F^+_{W, \tilde{s}}(c^+(\xi)) \neq 0. \]

Thus, there is an \( \tilde{s} \in Spin^C(\tilde{W}) \) with \( \tilde{s}\mid W \cong \tilde{s} \), and \( \tilde{s}\mid V \cong \tilde{s}_W \), such that
\[ F^\pm_{\tilde{W} \setminus \tilde{B}, \tilde{s}\mid \tilde{W} \setminus \tilde{B}}(c^+(\xi')) \neq 0. \]

Let \( h_1 : \tilde{s}\mid W \to \tilde{s} \), \( h_2 : \tilde{s}\mid V \to \tilde{s}_W \) be the above isomorphisms, and \( h_3 : \tilde{s}_W \mid M \to t_\xi \) the natural projection. And define \( h : \tilde{s}|_M \to t_\xi \) by \( h = h_3 \circ h_2 \circ h^{-1}_1 \).

Capping off \( \Sigma \) by the core of the 2-handle, we get an embedded closed surface \( \hat{\Sigma} \) satisfying \( \chi(\hat{\Sigma}) = \chi(\Sigma) + 1 \leq 0 \), \( [\hat{\Sigma}] \cdot [\hat{\Sigma}] = tb(K, \Sigma) - 1 \geq 0 \), and \( \langle c_1(\hat{\Sigma}), [\hat{\Sigma}] \rangle = r(K, \Sigma, s, h) \). Next, blow up \( tb(K, \Sigma) - 1 \) points on the core of the 2-handle, we get a new 4-manifold \( \hat{\tilde{W}} \) with a natural projection \( \pi : \hat{\tilde{W}} \to \tilde{W} \). Let \( \hat{s} \) be the lift of \( \tilde{s} \) to \( \hat{\tilde{W}} \) whose evaluation on each exceptional sphere is \( -1 \), and \( \hat{\Sigma} \) be the lift of \( \Sigma \) to \( \hat{\tilde{W}} \) obtained by removing the exceptional spheres from \( \pi^{-1}(\hat{\Sigma}) \). Then \( \chi(\hat{\Sigma}) = \chi(\Sigma) + 1 \), \( [\hat{\Sigma}] \cdot [\hat{\Sigma}] = 0 \), and \( \langle c_1(\hat{\Sigma}), [\hat{\Sigma}] \rangle = r(K, \Sigma, s, h) + tb(K, \Sigma) - 1 \). Also, by Theorem 2.8,
\[ F^\pm_{\hat{\tilde{W}} \setminus \hat{\tilde{B}}, \hat{s}\mid \hat{\tilde{W}} \setminus \hat{\tilde{B}}}(c^+(\xi')) = F^\pm_{\hat{\tilde{W}} \setminus \hat{\tilde{B}}, \hat{s}\mid \hat{\tilde{W}} \setminus \hat{\tilde{B}}}(c^+(\xi')) \neq 0, \]

where \( \hat{\tilde{B}} \subset \hat{\tilde{W}} \) is the pre-image of \( B \subset W \subset \tilde{W} \) under \( \pi \). Since \( [\hat{\Sigma}] \cdot [\hat{\Sigma}] = 0 \), there is a neighborhood \( U \) of \( \hat{\Sigma} \) in \( \hat{\tilde{W}} \) diffeomorphic to \( \hat{\Sigma} \times D^2 \). Since the location of \( \hat{\tilde{B}} \) does not affect the map \( F^\pm_{\hat{\tilde{W}} \setminus \hat{\tilde{B}}, \hat{s}\mid \hat{\tilde{W}} \setminus \hat{\tilde{B}}} \), we assume that \( \hat{\tilde{B}} \) is in the interior of \( U \). Let \( W_1 = \tilde{W} \setminus U \), and \( W_2 = U \setminus \hat{\tilde{B}} \). Then, by Theorem 2.2, there are maps
\[ F^\pm_{W_1, \tilde{s}\mid W_1} : HF^+(-M', t_\xi') \to HF^+(-\partial U, \tilde{s}\mid \partial U; \mathbb{Z}(W_1)), \]
\[ F^\pm_{W_2, \tilde{s}\mid W_2} : HF^+(-\partial U, \tilde{s}|_{\partial U}; \mathbb{Z}(W_1)) \to HF^+(-\partial B, \tilde{s}|_{\partial B}; \mathbb{Z}(W_1)(W_2)), \]

such that \( F^\pm_{\hat{\tilde{W}} \setminus \hat{\tilde{B}}, \hat{s}\mid \hat{\tilde{W}} \setminus \hat{\tilde{B}}}(\Theta) = \theta \circ F^\pm_{W_2, \tilde{s}\mid W_2} \circ F^\pm_{W_1, \tilde{s}\mid W_1}, \) where
\[ \Theta : HF^+(-\partial B, \tilde{s}|_{\partial B}; \mathbb{Z}(W_1)(W_2)) \to HF^+(-\partial B, \tilde{s}|_{\partial B}) \]
is induced by the natural projection \( \theta : \mathbb{Z}(W_1)(W_2) \to \mathbb{Z} \). Specially, this implies \( HF^+(-\partial U, \tilde{s}|_{\partial U}; \mathbb{Z}(W_1)) \neq 0 \). Note that \( \partial U \cong \hat{\Sigma} \times S^1 \). Hence, by Theorem 2.1 we have
\[ \langle c_1(\tilde{\xi}), [\hat{\Sigma}] \rangle \leq -\chi(\hat{\Sigma}), \]
that is
\[ tb(K, \Sigma) + r(K, \Sigma, s, h) \leq -\chi(\Sigma). \]

Reverse the orientations of \( K \) and \( \Sigma \), and repeat the whole argument. We get
\[ tb(K, \Sigma) - r(K, \Sigma, s, h) \leq -\chi(\Sigma). \]

Thus,
\[ tb(K, \Sigma) + |r(K, \Sigma, s, h)| \leq -\chi(\Sigma). \]

Now we use twisted Heegaard-Floer homology to prove part (b). Again, perform Legendrian surgery along \( K \). This gives a new contact 3-manifold \( (M', \xi') \) with a weak
symplectic filling \((\tilde{W},\tilde{\omega})\) (c.f. [14, 15]). Define \(\tilde{\Sigma}\) and \(\tilde{W}\) as above, i.e., by capping off \(\Sigma\) with the core of the 2-handle, and then blowing up \(tb(K, \Sigma) - 1\) points on the core the of two handle. Let \(\tilde{\omega}\) be the blown-up symplectic form on \(\tilde{W}\). Then \((\tilde{W}, \tilde{\omega})\) is also a weak symplectic filling of \((M', \xi')\). Denote by \(\hat{s}\) the canonical \(Spin^c\)-structure associated to \(\tilde{\omega}\). We have \(\chi(\tilde{\Sigma}) = \chi(\Sigma) + 1, [\tilde{\Sigma}] : [\tilde{\Sigma}] = 0\), and \((c_1(\tilde{s}), [\tilde{\Sigma}]) = r(K, \Sigma, \omega) + tb(K, \Sigma) - 1\). Also, by Theorem 2.5, we have \(\mathcal{F}_{\tilde{\Sigma}}(\tilde{\omega})\) factors through \(HF^+(-\partial U, \tilde{s}|_{\partial U}; \mathbb{Z}[\mathbb{R}][\omega|_{M'}](W_1))\), where \(W_1 = \tilde{W} \setminus U\). So
\[
\mathcal{F}_{\tilde{\Sigma}}(\tilde{\omega}) = 0.
\]
and we apply Theorem 2.1 as above to prove part (b). 

\[\Box\]

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