Long-range interaction of magnetic moments in a coupled system of S/F/S Josephson junctions with anomalous ground state phase shift

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Nowadays, heterostructures consisting of superconducting and magnetic materials are being actively studied both theoretically and experimentally [1,6]. The reason for this interest is the possibility to realize in such hybrids properties and effects that are not possible in individual materials. In particular, one of the actively developing directions is the search and study of physical principles that can provide an indirect long-range interaction of magnetic moments through a superconductor. The indirect exchange interaction between magnetic moments carried by conduction electrons in a metal (RKKY interaction) is well known [7]. It has been studied in various materials [8–13]. However, the strongly oscillating and decaying nature of this interaction at the atomic scale makes it possible to achieve interaction between magnetic moments at characteristic distances not exceeding a few nanometers in layered structures.

In recent years, experimental and theoretical studies have been actively carried out, in which the nonmagnetic interlayer between the magnets in spin valves is replaced by a superconductor [13–21]. As was first pointed out by de Gennes, a superconductor makes the antiferromagnetic configuration of magnets more favorable [22]. The reason for this is that with such a mutual orientation of magnets, superconductivity in the interlayer is less suppressed as a result of partial compensation of paramagnetic depairing. The characteristic scale of such an interaction is the superconducting coherence length $\xi_S$, at which the effect of proximity to a magnet manifests itself in a superconductor. It is tens to hundreds of nanometers, depending on the specific superconductor used. For the case of a $d$-wave superconductor the interaction length can be strongly enhanced due to the presence of nodal quasiparticles [17]. In recent work [25] it was also proposed to use not the proximity effect to establish a coupling between magnets, but the so-called electromagnetic proximity effect [24], the essence of which is the appearance of Meissner currents in a superconductor in response to the presence of an adjacent magnetic material. The characteristic scale of this coupling is the penetration depth of the magnetic field.

The interaction between localized magnetic moments through superconductors has also been studied [25,30] and an additional to RKKY contribution decaying exponentially over $\xi_S$ and with a weaker power-law suppression, which favors an antiferromagnetic alignment, has been reported. Further, it has been shown [31] that in superconductors with SOC the superconducting condensate is coupled to the impurity spins, what results in more long-range non-exponential power-law suppression of the interaction between magnetic impurities.

Here we propose a fundamentally different way of using the superconducting state to establish a long-range indirect interaction between magnetic moments, which does not use proximity effects in superconductors. The physical principle is based on the fact that superconductivity is a macroscopic quantum state with a single phase of the condensate wave function. The idea is that the ground state energy of a system of two coupled Josephson S/F/S junctions at a given phase difference between the leads depends on the mutual orientation of the magnetizations of the ferromagnetic interlayers. We have found that in such a system a very long-range coupling between the magnetizations of the ferromagnetic interlayers is realized and the coupling constant can be controlled by the superconducting phase between the leads.

The effect can be observed in the systems, where a coupling between the direction of the magnetization of the magnet and the Josephson phase occurs. It is known that such a coupling physically manifests itself as the presence of an anomalous phase shift in the ground state of a Josephson junction and is realized in systems with a strong spin-orbit coupling [32,43]. The strongest effect can be achieved in Josephson junctions on a topological insulator [44,48], because in these materials the coupling between the electron spin and its momentum is maximally strong (spin-momentum locking) [49,52]. Josephson junctions with anomalous phase shift have already
been implemented experimentally by several groups [53–56], including those on a topological insulator. In existing experiments on anomalous phase shift, Josephson junctions do not yet contain ferromagnetic elements, and the effective exchange field, required for the realization of the anomalous phase shift, is generated by the external magnetic field. However, modern materials and techniques allow for realization of the anomalous ground state phase in S/F/S JJs. One of the possibilities is to use for the interlayers 2D or quasi 2D ferromagnets, where the Rashba spin-orbit coupling can be strong due to the structural inversion symmetry breaking. The other way in to exploit the ferromagnetic insulator/3D topological insulator (TI) hybrids as interlayers. Several works on the creation of structures containing ferromagnetic materials and topological insulators are actively and successfully carried out [57–64]. In particular, it has been experimentally demonstrated that a ferromagnetic insulator induces an effective exchange field in the surface states of a topological insulator.

**FIG. 1.** Sketch of the coupled system of two S/F/S JJs.

**System and model.** We consider two coupled S/F/S JJs, where S means a conventional superconductor and F means that the interlayer of each of the JJs consists of a spatially homogeneous ferromagnet with Rashba-type spin-orbit coupling. The spin-orbit coupling can be intrinsic or due to the structural inversion symmetry breaking, or it can be a hybrid interlayer consisting of a ferromagnet and a spin-orbit material, or it can be a ferromagnetic insulator on top of the 3D TI. The last model is investigated in detail in the Supplemental Material [65]. If the ferromagnet is an insulator, it is assumed that the magnetization \( \mathbf{M} \) of the ferromagnet induces an effective exchange field \( \mathbf{h} \sim \mathbf{M} \) in the underlying conductive layer. The sketch of the system is represented in Fig. 1. The superconducting phase difference \( \chi \) between the leads is an external controlling parameter. First of all we investigate the energy of the system as a function of \( \chi \) and \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \), where \( \mathbf{m}_i \) is the unit vector along the direction of the corresponding magnetization. It is assumed that the ferromagnets are easy-axis magnets with the easy axis along the \( y \)-direction.

The current-phase relation (CPR) of a separate S/F/S junction takes the form \( I = I_c \sin(\chi - \chi_0) \), where \( \chi_0 \) is the anomalous phase shift. For the S/F/S JJs with spin-orbit coupling (SOC) it has been calculated in the framework of different models [35, 41, 45, 46, 66] and it has been found that for Rashba-type SOC and ferromagnets on top of the 3D TI

\[
\chi_0 = rm_y
\]

irrespective of the particular model. This is the term that couples the superconducting phase to the magnetization direction. \( r \) is a constant quantifying this coupling strength and is proportional to the exchange field of the ferromagnetic interlayer or the effective exchange field \( h \), induced in the conducting layer of the interlayer due to the proximity to the ferromagnetic insulator. The critical current depends crucially on the particular model. For example, it can be independent on the magnetization direction, as it has been reported for the ferromagnets with SOC [35], or it can depend strongly on the \( x \)-component of the magnetization, as it takes place for the ferromagnetic interlayers on top of the 3D TI [46, 66].

Here we focus on the model where \( I_c \) does not depend on the magnetization direction. The influence of the dependence \( I_c(m) \) on the results is considered in details in the Supplemental Material [65]. We have obtained that this dependence brings no new physics and does not affect qualitatively the results. The energy of the system consists of the Josephson energies of the both junctions and the easy-axis anisotropy energies of the both magnets:

\[
E = \frac{\hbar}{2e} \left[ I_c \left( 1 - \cos(\psi - \chi_{0,1}) \right) + \right. \\
L_c \left( 1 - \cos(\chi - \psi - \chi_{0,2}) \right) \left] - \frac{KV_F}{2} (m_{y1}^2 + m_{y2}^2), \right.
\]

where \( K \) is the anisotropy constant, \( V_F \) is the volume of the ferromagnet. \( \psi \) is the phase of the middle superconductor relative to the left one. This phase is determined by the current conservation

\[
I_c \sin(\psi - \chi_{0,1}) = I_c \sin(\psi - \chi_{0,2}).
\]

\( \psi \) is assumed to be constant along the middle superconductor, therefore we neglect the phase difference acquired between its left and right ends due to the Josephson current flowing through the system.

**Phase-dependent stable magnetic configurations.** The energy \( E \) is a function of \( (m_{y1}, m_{y2}) \) at a given phase difference \( \chi \) is shown in Figs. 2(a)-(b). It consists of two branches (yellow and blue), which differ by the phase \( \pi \) at the middle superconductor. The upper energy value at a given magnetic configuration is unstable. The particular structure of the stable states is determined by two parameters: the ratio of the magnetic anisotropy and Josephson energies \( E_M/E_J = KV_F/2|\hbar I_c/2e| \) and the magnetization-phase coupling parameter \( r \). There are two types of the stable states. The first type is the "cor-ner states" corresponding to \( m_{y1(2)} = \pm 1 \), which can be further classified as \( \uparrow \downarrow, \downarrow \uparrow \) and \( \uparrow \uparrow, \downarrow \downarrow \) or \( \downarrow \downarrow, \uparrow \uparrow \). The second type is "non-aligned" states, where one of the magnets is misoriented from the \( y \)-axis. The larger the parameter \( E_M/E_J \) the higher the energy barrier between the different stable states. \( r \) removes the degeneracy between the...
FIG. 2. (a)-(b) Energy $E$ is a function of $(m_{g1}, m_{g2})$ at a given phase difference $\chi$. Yellow and blue surfaces represent two branches, which differ by the phase difference $\pi$ at the middle superconductor. (a) $r = 0.7$, (b) $r = 3$, $E_M/E_J = 0.2$ for the both panels. Stable states are shown by red arrows. (c) Energy as a function of $\chi$ for the "corner" states; $r = 0.7$, $E_M/E_J = 0.2$. Parts of the energy branches, which are unstable with respect to the $\pi$-jump of $\psi$, are shown by thin lines. Solid (dashed) lines represent stable (unstable) magnetic configurations. (d) Phase diagram representing all possible states of the system: (I)-non-controllable, "corner"; (II)-controllable, "corner"; (III)-non-controllable, "corner"+"non-aligned"; (IV)-controllable, "corner"+"non-aligned", see text for further details.

"corner" states, thus making some of them stable and the others unstable at a given phase difference. It allows for the control of the magnetic configuration by variations of the superconducting phase $\chi$. At $E_M/E_J < r/2$ the controllability of the magnetic configuration by the Josephson phase is easily possible, while at $E_M/E_J > r/2$ the system does not respond to the phase manipulation because the energy barrier between the stable states is too high and the phase variations cannot remove the barrier. At small values of $r$ the minima of the energy only correspond to the "corner" states [Fig. 2(a)], while at larger $r$ the "non-aligned" states are also possible [Fig. 2(b)]. The exact criterion of the "non-aligned" states existence is $(r/2)^2 > E_M/E_J$. The two lines $E_M/E_J = r/2$ and $E_M/E_J = (r/2)^2$ divide the phase diagram of the system into four regions, which are marked by numbers I-IV in Fig. 2(d). In regions II and IV the magnetic state can be controlled and manipulated by the phase difference, while in regions I and III it cannot. In regions I and II the "non-aligned" states do not exist, while in regions III and IV they can exist.

We have estimated the parameters $E_M/E_J$ and $r$ for the model of the insulating ferromagnet on top of the 3D TI. We take the parameters corresponding to Nb/Bi$_2$Te$_3$/Nb Josephson junctions [67]; the junction length $d = 50\text{nm}$, $I_c = 40A/m$, $v_F = 10^6\text{m/s}$. We assume $E_M \sim (10 - 10^2) \times d_F \text{erg/sm}^2$ for YIG thin films [68], where $d_F = 10\text{nm}$ is the F width along the $z$-direction. It gives $E_M/E_J \sim 10^{-2} - 10^{-1}$. Basing on the experimental data on the Curie temperature of the magnetized TI surface states [63], where the Curie temperature in the range $20 - 150\text{K}$ was reported, we can roughly estimate $h \lesssim 0.01 - 0.1h_F\text{YIG}$. It corresponds to the dimensionless parameter $r = 2hd/v_F \lesssim 2 - 13$.

Now we can investigate the energies of the stable states of the system as a function of $\chi$. The results are represented in Fig. 2(c). We focus here on the case of small $r$ when only "corner" stable states are possible, the case of "non-aligned" states is physically similar and considered in the Supplemental Material [65]. Due to the reflection symmetry with respect to the $(x, y)$-plane $E_{\uparrow\uparrow}(\chi) = E_{\downarrow\downarrow}(\chi)$ and $E_{\uparrow\downarrow}(\chi) = E_{\downarrow\uparrow}(\chi)$. It is seen from Fig. 2(c) that at $r \neq 0$ $E_{\uparrow\uparrow}(\chi)$ and $E_{\downarrow\uparrow}(\chi)$ are asymmetric functions of $\chi$ and, therefore, the degeneracy between them is removed. At the same time $E_{\uparrow\downarrow}(\chi)$ is a symmetric function of $\chi$ and, consequently, the states $\uparrow\downarrow$ and $\downarrow\uparrow$ remain degenerate and we refer to them as the antiparallel (AP) state. $\uparrow\uparrow$, $\downarrow\downarrow$ and AP states can be stable (solid) or unstable (dashed) depending on $\chi$. Each of the states has two energy branches, which differ by the phase difference $\pi$ at the middle superconductor. Each of the states represent the ground state of the system for the particular range of $\chi$.

Therefore, we can conclude that in the system a long-range coupling between the moments is realized and the value of the coupling is controlled by the phase difference $\chi$ between the leads. The physical origin of the effect is that (i) the Josephson energy of a separate JJ at a given phase difference depends on the magnetization orientation through the anomalous phase shift and (ii) the phase of the middle superconductor is adjusted by the system in order to provide the constant electric current through the system and simultaneously to minimize the total Josephson energy of the both junctions. Therefore, this phase depends on the both magnetizations, thus mediating interaction between them. If the orientation of one of the magnets is changed due to some reasons, $\psi$ also changes, what leads to the change of the favorable state of the second magnet. Further we investigate dynamics of the switching between different magnetization states of the system. We consider two physically different dynamical problems. The first one is the evolution of the magnetic state of the system under the adiabatic phase variations $\chi = 2eVt$. The second problem is the remote switching of one of the moments due to external impact on the other one.

Dynamics. The dynamics of each of the magnets $i = 1, 2$ is described by the Landau-Lifshitz-Gilbert (LLG) equation:

$$\frac{\partial m_i}{\partial t} = -\gamma m_i \times H_{eff} + \alpha m_i \times \frac{\partial m_i}{\partial t} + \frac{J_{ex}}{dF} m_i \times \langle s_i \rangle.$$  (4)
where $\gamma$ is the gyromagnetic ratio, $H_{\text{eff}} = (K/M)m_y e_y$ is the local effective field in the ferromagnet induced by the easy-axis magnetic anisotropy and $\alpha$ is the Gilbert damping constant. Due to the SOC an electric current $I$ flowing via the JJ induces a local spin polarization of the conduction electrons $\langle s \rangle$, which exerts a torque on the ferromagnet magnetization [69]-[71]. The last term in Eq. (4) represents the torque, averaged over the ferromagnet thickness $d_F$ along the $z$-direction. For the case of Rashba SOC or F/TI hybrids the expression for the torque takes the form:

$$ N = \frac{J_s}{d_F} \mathbf{m} \times \langle \mathbf{s} \rangle = -\frac{\gamma T I}{2eMdd_F} [\mathbf{m} \times e_y], \quad (5) $$

The total current flowing through each of the JJs consists of the supercurrent and the normal quasiparticle current contributions [72]-[73]:

$$ I_i = I_c \sin(\chi_i - \chi_0,i) + \frac{1}{2eR_N} (\dot{\chi}_i - \dot{\chi}_0,i), \quad (6) $$

where $\chi_1 = \psi(t)$ and $\chi_2 = \chi(t) - \psi(t)$. The current conservation dictates $I_1 = I_2$, what determines the phase $\psi(t)$ of the middle superconductor. The dynamics of the magnetizations $m_{1,2}$ is calculated numerically from Eqs. (4), (5) and (6). The equations for both JJs are coupled via the phase $\psi(t)$.

If the normal current, represented by the second term in Eq. (6) is small, the torque is mainly determined by the supercurrent and can be calculated via the additional contribution to the effective field in Eq. (4) $\delta H_{\text{eff}} = -(1/Mdd_F) dE/dm$ [60]-[74], [76], what leads to Eq. (5) with $I \to I_c \sin(\chi - \chi_0)$.

$$ \chi_1 = \psi(t) \quad \text{and} \quad \chi_2 = \chi(t) - \psi(t). $$

We have also investigated another dynamic problem when the external phase $\chi$ is fixed, and at $t = 0 \mathbf{m}_1$ is reversed. The results are demonstrated in Fig. 3(c)-(d). The phase $\chi$ is chosen in such a way that the equilibrium magnetic configuration is AP. Then two different situations are considered. The first one is when $\mathbf{m}_1$ is fixed in the new position by external means (for example, by the applied magnetic field). This case is plotted in Fig. 3(c). It is seen that $\mathbf{m}_2$ also switches in order to make the magnetic configuration AP, which is energetically favorable at the given $\chi$. The characteristic time of the reversal is not determined by $t_0 = M/\gamma K$, which is the characteristic time of magnetization dynamics, but depends essentially on the particular value of $\chi$.

At first we consider the adiabatic variation of the external phase $\chi = 2eVt$. The resulting coupled dynamics of $m_{y,1,2}(t)$ is demonstrated in Fig. 3. Fig. 3(a) represents $m_{y,1,2}$ as functions of $\chi \propto t$ starting from the initial AP configuration. It is seen that the magnetic state of the system can be manipulated by the adiabatic phase change. In Fig. 3(b) we match the dynamic magnetic configuration of the system with the energy of the equilibrium state at the same phase difference. The matching is performed for the phase interval $\chi \in (\pi, 3\pi)$ in Fig. 3(a). The jumps of the magnetic configuration occur according to two physically different scenarios. The first jump is realized at the crossing point of two energy branches corresponding to the same $\uparrow\uparrow$ state, where $\psi$ experiences a $\pi$-jump. But the $\uparrow\uparrow$ magnetic configuration is unstable at this new branch with lower energy. For this reason the system jumps to the stable $\downarrow\downarrow$ configuration. The second and third jumps are not accompanied by the $\pi$-shift of the middle superconductor phase. They just correspond to the transitions from the absolutely unstable to the stable energy branch.

The second physical situation is when $\mathbf{m}_1$ is reversed, but not fixed and can evolve freely. The system also switches to the energetically favorable AP configuration, but it can be reached by the back reversal of $\mathbf{m}_1$ or by the switch of $\mathbf{m}_2$. The both processes are equiprobable and are represented in Figs. 3(d) and (e), respectively.

In conclusion, we have proposed a mechanism of long-range magnetic interaction via the superconducting phase in coupled S/F/S JJs. It is based on (i) the magnetoelectric coupling between the condensate phase difference and the magnetization in the weak link of the JJs with anomalous ground state phase and (ii) the macroscopic character of the superconducting phase in the middle superconductor, which interacts with both magnets thus mediating the interaction between them. The range of the interaction is not restricted by the proximity length scales in the superconductor. It is demonstrated that the mutual orientation of the ferromagnetic interlayers can be controlled and manipulated via the superconducting phase.
SUPPLEMENTAL MATERIAL

A. Detailed discussion of the phase diagram.

In this section we give further details on the phase diagram in $(E_M/E_J, r)$-plane, which classifies all possible stable magnetic configurations of the system at an arbitrary phase difference $\chi$. We still work in the framework of the model, when $I_c$ is a constant and does not depend on the magnetization direction of the interlayer. The phase diagram is presented in the middle of Fig. 4. As it was described in the main text, it is divided into four physically different regions, marked by numbers I,II,III,IV. The regions are separated by the lines $E_M/E_J = r/2$ and $E_M/E_J = (r/2)^2$. The first line separates the parameter regions, where the magnetic configuration can and cannot be manipulated by $\chi$. The equation of this line can be obtained if we consider the energy of the system as a function of $m_{y,1}$ at a fixed $m_{y,2} = 1$:

$$E = E_J(1 - \cos(\psi - \chi_{0,1})) + E_J(1 - \cos(\chi - \psi - \chi_{0,2})) - E_M(m_{y1}^2 + m_{y2}^2). \tag{7}$$

Taking into account the current conservation one obtains that $\psi_+ = \chi/2 + (\chi_{0,1} - \chi_{0,2})/2$ or $\psi_- = \chi/2 + (\chi_{0,1} - \chi_{0,2})/2 + \pi$. Substituting these values into Eq. (7) one gets:

$$E_\pm = 2E_J[1 + \cos(\chi/2 - r(m_{y1} + m_{y2})/2)] - E_M(m_{y1}^2 + m_{y2}^2). \tag{8}$$

The magnetic part of the energy has a minimum at $m_{y,1} = \pm 1$. Let us consider the energy in the vicinity of $m_{y,1} = 1$. The situations corresponding to the other corner states lead to the same result. At $E_J(r/2) < E_M$ the corner point $m_{y,1} = 1$ is always a minimum of the energy Eq. (8) at any phase difference. Therefore, under this condition it is not possible to change the magnetic configuration by adjusting the superconducting phase $\chi$. On the contrary, at $E_J(r/2) > E_M$ it can become a maximum of the energy Eq. (8) at a particular value of $\chi$. Consequently the corresponding magnetic configuration can be made absolutely unstable by varying the phase. The condition $E_M/E_J < (r/2)^2$ represents the condition of appearance of additional minima of the energy Eq. (8), which differ from the corner states $m_{y,1} = \pm 1$. The minima corresponding to $m_{y,1} \neq 1$ and $m_{y,2} \neq 1$ do not occur in this model.

In Fig. 4 the energies of the system are demonstrated for all four qualitatively different regions. As discussed above, in regions I and III the magnetic configuration cannot be manipulated by the phase. It leads to the absence of unstable parts of the energy branches in Figs. 4b) and (f). At the same time the analogous figures (c) and (g), corresponding to the controllable regions II and IV, respectively, have unstable parts.

Right column of Fig. 4 corresponds to regions III and IV, where “non-aligned” stable magnetic states are possible at particular values of $\chi$. The “non-aligned” stable states are demonstrated in Figs. 4e) and (h) by points and arrows. The energy of these stable “non-aligned” states as a function of $\chi$ is represented in Figs. 4f) and (g) by black lines. The ranges of $\chi$ values, where the “non-aligned” stable states exist, are small and for this reason the corresponding parts of the energy branches are shown on larger scale. Therefore, in region IV the magnetic configuration of the system can be switched between $\uparrow\uparrow, \downarrow\downarrow$, AP and “non-aligned” states by varying the phase difference.

The dynamics of the magnetic configuration in region IV under the adiabatic phase variations $\chi = 2eVt$ is shown in Fig. 5. Fig. 5a) demonstrates that the switching between $\downarrow\downarrow$ and AP configurations occurs via the non-aligned states, where one of the $y$-components of the magnetization is less than unity. Fig. 5b) illustrates the dynamical trajectory of the time evolution of the magnetic configuration.

B. Role of the dependence of the critical current on the magnetization direction.

Here by considering the particular model of the S/F/S JJs on top of the 3D TI we investigate the role of the critical current dependence on the magnetization direction $I_c(m)$. The interlayer region of a S/3D TI/S JJ is covered by a ferromagnet. We believe that our results can be of potential interest for systems based on $Be_2Se_3/YIG$ or $Be_2Se_3/EuS$ hybrids, which were realized experimentally. It is assumed that the ferromagnet induces an effective
FIG. 4. Middle: the phase diagram representing four physically different regions corresponding to the "corner" uncontrollable magnetic configuration (I); "corner" controllable (II); "corner"+"non-aligned" uncontrollable (III) and "corner"+"non-aligned" controllable (IV) stable states. (a) Energy of the system for the pink circle from region I as a function of \((m_{y1}, m_{y2})\) at \(\chi = 2.3\). (b) Energy branches of stable magnetic configuration for the pink circle from region I as functions of \(\chi\). The thin part of each branch is unstable with respect to a \(\pi\)-shift of \(\psi\) and jump to the corresponding bold part. (c)-(d) The same as (a)-(b) but for the pink circle from region II. (e)-(f) the same as (a)-(b) but for the pink circle from region III. (g)-(h) (a)-(b) but for the pink circle from region IV.

The Josephson current assuming the ballistic limit for the 3D TI surface states and in the vicinity of the critical exchange field \(h \propto M\) (where \(M\) is the ferromagnet magnetization) in the underlying 3D TI surface states, as it has been reported experimentally [63]. The sketch of the setup is shown in Fig. 6(a).
FIG. 5. (a) Time evolution of $m_{x,1,2}$ under the adiabatic variation of the phase $\chi = 2\pi Vt$ in region IV. $eVt_0 = 5 \times 10^{-4}$. (b) Matching the dynamical magnetic configuration, presented in (a) to the energy of the system. Black line is the dynamical trajectory of the system. (c) Energy in range $\chi \in (-1.2, 0)$, where the system switches from $\downarrow\downarrow$ to AP configuration via the non-aligned state, on a larger scale. The yellow line is the equilibrium non-aligned energy branch and the thin black line is the dynamical trajectory of the system. $r = 0.7$, $E_M/E_F = 0.05$.

FIG. 6. (a) Sketch of the system of two coupled S/F/S JJs on top of a 3D TI. (b) $I_c$ as a function of $m_x$ for $r = 13.2$, $d/\xi_N = 4.1$ (solid blue); $r = 2.6$, $d/\xi_N = 4.1$ (solid red); $r = 13.2$, $d/\xi_N = 0.74$ (dashed blue); $r = 2.6$, $d/\xi_N = 0.74$ (dashed red). $I_c$ is normalized to $I_c(m_x = 0)$.

temperature takes the form [66]:

$$I_s = I_c \sin(\chi - \chi_0),$$

$$I_c = I_b \int_{-\pi/2}^{\pi/2} d\phi \cos \phi \exp[-\frac{2\pi Td}{v_F \cos \phi}] \cos[r m_x \tan \phi],$$

$$\chi_0 = 2h_y d/v_F = rm_y,$$

where $r = 2hd/v_F$ for the 3D TI and $I_b = ev_F N_F \Delta^2/(\pi^2 T)$, $v_F$ and $N_F$ are the Fermi velocity and the normal state density of states at the 3D TI surface. Here the critical Josephson current is only suppressed by the $x$-component of the exchange field. The $y$-component of the field does not lead to the suppression, instead it gives rise to the anomalous phase shift. This statement is also valid for the diffusive case. The Josephson current in 3D TI-based Josephson current has been considered in Ref. [66] and exactly the same expression for the anomalous phase shift $\chi_0$ has been obtained. The result for the critical current is different in the diffusive case, but it still only depends on the $x$-component of the exchange field. The suppression of the critical current as a function of $m_x \equiv M_x/M_s$ is presented in Fig. 6(b). For estimates we take $d = 50nm$, $v_F = 10^6 m/s$ and $T_c = 10K$, what corresponds to the parameters of Nb/Bi2Te3/Nb Josephson junctions [67]. In this case $\xi_N = v_F/2\pi T_c \approx 12 nm$. We have also plotted $I_c(m_x)$ for $T_c = 1.8K$, what corresponds to the Josephson junctions with Al leads.

Making use of the current conservation condition Eq. (3) of the main text, the energy of the system can be expressed in the form:

$$E_{\pm} = \frac{\hbar}{2e} \left[ I_{c1}(m_{x,1}) + I_{c2}(m_{x,2}) + I(m_1, m_2) \right] - \frac{KV_F}{2} (m_{y1}^2 + m_{y2}^2),$$
where \( I(m_1, m_2) = \sqrt{I_{c1}^2(m_x, 1) + I_{c2}^2(m_x, 2)} + 2I_{c1}(m_x, 1)I_{c2}(m_x, 2) \cos(\chi - \chi_{0,1} - \chi_{0,2}) \). Eq. (12) is exploited to calculate the phase diagrams, presented in Fig. 7 and the energy surfaces in Fig. 8. Physically different regions of the phase diagram are marked by the same numbers as for the previous model with constant critical current. Fig. 7(b) is the bottom left corner of the phase diagram, presented in Fig. 7(a) on a larger scale. It demonstrates region II, which is very small in Fig. 7(a). The black curves in this figure represent the lines \( E_M/E_J = r/2 \) and \( E_M/E_J = (r/2)^2 \), which separate the different regions in the framework of the previous model. It is seen that the boundaries between the different regions are changed due to the dependence of the critical current on the magnetization direction. Region III disappears in this model, and region IV is expanded. It is also seen from Fig. 7(a) that region IV can be divided into two subregions. Only "edge" non-aligned states with \( m_y, 1 = \pm 1 \) or \( m_y, 2 = \pm 1 \) are possible in subregion IVa, analogously to the previous model. At the same time, additional non-aligned states, corresponding to \( m_y, 1 \neq \pm 1 \) and \( m_y, 2 \neq \pm 1 \) appear in subregion IVb. The reason is the suppression of the critical current by \( m_x \). The lower critical current means the smaller Josephson energy at a given phase difference, which is more energetically favorable. The suppression is also controlled by the parameter \( r \), as it can be seen from Eq. (10). Consequently, from the point of view of the Josephson energy it is favorable to enhance \( m_x \). This tendency competes with the magnetic anisotropy energy, which tends to enhance \( m_y \). Therefore, at large enough values of \( r \) and, simultaneously, small enough \( E_M/E_J \) the non-edge states \( m_y, 1 \neq \pm 1 \) and \( m_y, 2 \neq \pm 1 \) can become energetically favorable, what is realized in region IVb.

The described above competition between the Josephson and magnetic energies is further illustrated in Fig. 8. It demonstrates the system energy as function of \((m_y, 1, m_y, 2)\) for the same parameters, which are used for Figs. 4(d) and (h). The only difference between the corresponding figures is that \( I_c \) does not depend on \( m_x \) in Fig. 8(a) and it
depends on $m_x$ in Fig. 8. It is seen that at small $r = 0.7$ the difference between the corresponding Figs. 5(d) and 8(a) is not essential. At the same time at $r = 2.5$ Figs. 5(h) and 8(b) are qualitatively different. The reason is connected to the suppression of the critical current at nonzero $m_x$ and the resulting energy gain, as it is described above.

![Figure 9](image)

**FIG. 9.** (a) Time evolution of $m_{y,1,2}$ under the adiabatic variation of the phase $\chi = 2eVt$ for the S/F/S JJ on top of the 3D TI. $eVt_0 = 5 \times 10^{-4}$. (b) Matching the dynamical magnetic configuration, presented in (a) to the energy of the system. Black line is the dynamical trajectory of the system. $r = 0.7$, $E_M/E_J = 0.2$.

Further in Fig. 9 we demonstrate the influence of $I_c(m_x)$ on the dynamics of the magnetic configuration under the adiabatic phase variations. This figure can hardly be differed from Fig. 3(a)-(b) of the main text. First of all, the energy branches of the corner states do not differ at all. It is natural because $m_x = 0$ for the corner states and, therefore, the dependence $I_c(m_x)$ does not influence them. Moreover, the dynamical trajectory is also very similar. It is valid for small enough $r$, because in this case the non-aligned states are energetically close to the corner states and only exist in the narrow regions of the superconducting phase $\psi$. For this reason the system practically does not occur in the non-aligned states. At larger $r$ the regions of the non-aligned states existence expand and the dynamics can be modified. However, these regions probably are not of great interest for studying because of the strong Josephson current suppression at the magnetization orientations $m_x \neq 0$. The suppression strongly weakens the interaction between the magnets, mediated by the Josephson coupling.

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