Quantum Lobachevsky Planes

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Abstract: We classify all $SL(2, \mathbb{R})$-covariant Poisson structures on the Lobachevsky plane with respect to all multiplicative Poisson structures on $SL(2, \mathbb{R})$ and describe Quantisations for all these Poisson structures.

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I. Introduction

A first step into the direction of a quantisation of the notions of algebraic geometry (cf. Ref. 1) is the quantisation of the three fundamental one-dimensional complex domains: the complex plane, the Riemannian sphere and the complex upper half plane. Quantum planes were considered in Ref. 2. Quantum Riemannian spheres are discussed in Ref. 3 and Ref. 4. A quantum upper half plane appears firstly in Ref. 5 (for certain other quantisations of the upper half plane see also Remark 7). In this letter we consider the case of the quantum complex upper half plane.

In the theory of quantum groups the classification of certain Poisson structures gives generally a good insight into the problem of the classification of quantum structures. At first we give the full solution of the classification problem in the classical limit (i.e. the description of all possible $SL(2, \mathbb{R})$-covariant Poisson structures for the upper half plane with respect to the action of all possible multiplicative Poisson structures on $SL(2, \mathbb{R})$). Further we describe classes of quantum structures, which reproduce all listed Poisson structures in the classical limit. We obtain two-parameter quantisations of the upper half plane for every action of one of the quantum groups $SL_q(2, \mathbb{R})$, $SU_q(1, 1)$ and $SL_h(2, \mathbb{R})$.

II. Preliminaries

In this section we introduce the basic concepts (see also Ref. 5, Ref. 6 and Ref. 7). Let $(A, \Delta, \epsilon)$ be a Hopf algebra. A left quantum space $(H, \phi)$ is an algebra $H$ along with an algebra homomorphism $\phi : H \to A \otimes H$ such that $(\Delta \otimes id)\phi = (id \otimes \phi)\phi$ and $(\epsilon \otimes id)\phi = id$. Two left quantum spaces $(H_1, \phi_1)$, $(H_2, \phi_2)$ are isomorphic if there is an algebra isomorphism $h : H_1 \to H_2$, such that $\phi_2 \circ h = (id \otimes h) \circ \phi_1$. Further consider commutative Hopf algebras and commutative quantum spaces. We say that the Poisson bracket $\{..\}$ on $A$ is multiplicative if $\{\Delta(x), \Delta(y)\}_{A \otimes A} = \Delta(\{x, y\})$, where $A \otimes A$ carries the Poisson structure of the direct product. We call a Poisson bracket $\{..\}$ on $H$ covariant if $\{\phi(x), \phi(y)\}_{A \otimes H} = \phi(\{x, y\})$, where $A \otimes H$ carries the direct product structure from $A$ and $H$. We say that two multiplicative or covariant, respectively, Poisson brackets on $A$ and, respectively, $H$ are equivalent if they intertwine with an automorphism of $A$ and, respectively, $H$.

III. The classical case

A. Poisson-Structures on $SL(2, \mathbb{R})$

Let $A = \mathbb{R}[a, b, c, d]/(ad - bc - 1)$ be the commutative unital algebra of
polynomial functions in the coordinates of

\[ SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| \ a, b, c, d \in \mathbb{R}, \ ad - bd = 1 \right\}. \]

A becomes an Hopf algebra with respect to the Hopf multiplication

\[ \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \]

and the counit \( \epsilon \) with

\[ \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

A has the coinverse \( \kappa \) with

\[ \kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \]

We consider three types of multiplicative Poisson algebras \( A^A_\lambda, A^K_\lambda \) and \( A^N \):

1. \( A^A_\lambda = (A, \{\}, \lambda), \lambda \in \mathbb{R}, \lambda \neq 0 \), with

\[
\begin{align*}
\{a, b\}_\lambda &= \lambda ab, \\
\{a, c\}_\lambda &= \lambda ac, \\
\{a, d\}_\lambda &= 2\lambda bc, \\
\{b, c\}_\lambda &= 0, \\
\{b, d\}_\lambda &= \lambda bd, \\
\{c, d\}_\lambda &= \lambda cd;
\end{align*}
\]

2. \( A^K_\lambda = (A, \{\}, \lambda), \lambda > 0 \), with

\[
\begin{align*}
\{a, b\}_\lambda &= \lambda(1 - a^2 - b^2), \\
\{a, c\}_\lambda &= \lambda(a^2 + c^2 - 1), \\
\{a, d\}_\lambda &= \lambda(a - d)(b - c), \\
\{b, c\}_\lambda &= \lambda(a + d)(b + c), \\
\{b, d\}_\lambda &= \lambda(b^2 + d^2 - 1), \\
\{c, d\}_\lambda &= \lambda(1 - c^2 - d^2)
\end{align*}
\]

and

3. \( A^N = (A, \{\}) \) with

\[
\begin{align*}
\{a, b\} &= (1 - a^2), \\
\{a, c\} &= c^2, \\
\{a, d\} &= c(d - a), \\
\{b, c\} &= c(d + a), \\
\{b, d\} &= (d^2 - 1), \\
\{c, d\} &= -c^2.
\end{align*}
\]
\( A^A_\lambda (\lambda \in \mathbb{R}, \lambda \neq 0), A^K_\lambda (\lambda > 0) \) and \( A^N \) we shortly denote by \( A^I_\lambda \) \((I = A, K, N)\).

**Remark 1** The notation \( A^I_\lambda \) with \( I = A, K, N \) is justified by the fact, that the Poisson brackets of \( A^A_\lambda, A^K_\lambda \) and \( A^N \), respectively, vanish on the subgroups of the KAN-decomposition

\[
G_A = \left\{ \left( \begin{array}{cc} e^x & 0 \\ 0 & e^{-x} \end{array} \right) \mid x \in \mathbb{R} \right\},
\]

\[
G_K = \left\{ \left( \begin{array}{cc} \cos x & \sin x \\ -\sin x & \cos x \end{array} \right) \mid x \in [0, 2\pi) \right\},
\]

and

\[
G_N = \left\{ \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \mid x \in \mathbb{R} \right\}
\]

respectively.

**Proposition 1** Every nontrivial multiplicative Poisson structure on \( A \) is equivalent to one of the structures \( A^A_\lambda (\lambda \neq 0), A^K_\lambda (\lambda > 0), A^N \). All these structures are non-equivalent.

**Proof.** We give a sketch proof. Set \( g = sl(2, \mathbb{R}) \). Because \( H^1(g, g \wedge g) = 0 \), every Poisson structure on \( SL(2, \mathbb{R}) \) arises from a classical r-matrix \( r \in g \wedge g \), which satisfies the modified classical Yang Baxter equation \( (\wedge_3 \text{ad}_x)C(r) = 0, \forall x \in g \) with a certain element \( C(r) \in g \wedge g \wedge g \) (cf. Ref. 6). Because \( g = sl(2, \mathbb{R}) \) is a simple Lie algebra, the one-dimensional representation \( \wedge_3 \text{ad} \) is trivial. That is, the MCYBE is satisfied for every \( r \in g \).

To show, that two structures are equivalent, it is enough to show that their r-matrices are connected by an automorphism of \( sl(2, \mathbb{R}) \). All automorphisms are generated by inner automorphisms (i.e. the adjoint action of \( SL(2, \mathbb{R}) \)) and the automorphism \( \alpha \) with

\[
\alpha \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & -b \\ -c & d \end{array} \right).
\]

Let \( e_{-1} = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), e_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), e_1 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \) and

\[
r = \alpha e_1 \wedge e_{-1} + \beta e_0 \wedge e_1 + \gamma e_0 \wedge e_{-1}.
\]

Studying the adjoint action \( \wedge_3^2 \text{Ad} \) of \( SL(2, \mathbb{R}) \) and the action of \( \alpha \) on \( r \in g \wedge g \) we recognize, that \( r \) is equivalent to one of the following elements

\[
r^A_\lambda = \lambda e_1 \wedge e_{-1}, \quad \lambda \in \mathbb{R}, \lambda \neq 0,
\]

\[
r^K_\lambda = \lambda (e_0 \wedge e_1 + e_0 \wedge e_{-1}), \quad \lambda > 0,
\]

\[
r^N = e_0 \wedge e_1,
\]

respectively.
All these elements are non-equivalent.

Because $A$ consists of the matrix elements of the finite dimensional representations $\rho : SL(2, \mathbb{R}) \to \mathbb{R}^n$, one can give the Poisson structures by the formula

$$\{l_1(\rho_1(g)v_1), l_2(\rho_2(g)v_2)\} = (l_1 \otimes l_2)((\rho_1 \otimes \rho_2)(r_0), (\rho_1 \otimes \rho_2)(g)v_1 \otimes v_2)$$

($v \in \mathbb{R}^d, l \in (\mathbb{R}^d)^*$, cf. Ref. 8). The structures $A^A_\lambda, A^K_\lambda, A^N$, correspond to the above normal forms $r^A_\lambda, r^K_\lambda, r^N$ for $r$. $\square$

B. Poisson structures on the upper half plane

Further consider the subalgebra $H \subseteq A$ generated by the elements $A = a^2 + b^2, B = ac + bd, D = c^2 + d^2$ (cf. Ref. 5) and the left coaction $\phi = \Delta_{|H} : H \to A \otimes H$ with

$$\phi(A) = a^2 \otimes A + b^2 \otimes D + 2ab \otimes B,$$

$$\phi(D) = c^2 \otimes A + d^2 \otimes D + 2cd \otimes B,$$

$$\phi(B) = ac \otimes A + bd \otimes D + (ad + bc) \otimes B.$$

By the next proposition we give a complete classification of all possible covariant Poisson structures on $H$ with respect to $A^I_\lambda$.

Proposition 2 1. All with respect to $A^I_\lambda$ covariant Poisson structures are given by the one-parameter series:

(i) $H^A_{\lambda, \mu}$ ($\lambda \in \mathbb{R}, \lambda \neq 0, \mu \in \mathbb{R}$)

$$\{A, B\} = 2\lambda A(B + \mu),$$

$$\{A, D\} = 4\lambda B(B + \mu),$$

$$\{B, D\} = 2\lambda D(B + \mu),$$

(ii) $H^K_{\lambda, \mu}$ ($\lambda > 0, \mu \in \mathbb{R}$)

$$\{A, B\} = 2\lambda A(A + D + \mu),$$

$$\{A, D\} = 4\lambda B(A + D + \mu),$$

$$\{B, D\} = 2\lambda D(A + D + \mu),$$

(iii) $H^N_{\mu}$ ($\mu \in \mathbb{R}$)

$$\{A, B\} = 2A(D + \mu),$$

$$\{A, D\} = 4B(D + \mu),$$

$$\{B, D\} = 2D(D + \mu).$$
for $I = n$.

2. Two Poisson algebras $H^I_{\lambda,\mu_1}$ and $H^I_{\lambda,\mu_2}$ are equivalent if and only if $|\mu_1| = |\mu_2|$.

Proof. 1. All left invariant Poisson structures (i.e. $\phi(\{x,y\}) = \{\phi(x),\phi(y)\}$, $\forall x,y \in H$, where $A \otimes H$ carries a Poisson structure which is the direct product of the zero structure on $A$ and of the structure on $H$) are given by

$\{A,B\} = 2\mu A,$

$\{A,D\} = 4\mu B,$

$\{B,D\} = 2\mu D,$

($\mu \in \mathbb{R}$). We obtain the above formulas by the calculation for a fixed $\mu$ (for example $\mu = 0$) and from the fact that the difference of two left covariant structures is left invariant (cf. Ref. 7).

2. The proof of the equivalence follows from the fact that $(H, \phi)$ has the unique automorphism $x \rightarrow -x$, $\forall x \in H$. $\square$

The following Proposition shows, that we can realize the Poisson algebras $H^I_{\lambda,\mu}$ as subalgebras of $A^I_{\lambda}$.

Consider subalgebras $H_{\alpha,\beta,\gamma} \subset A$ generated by elements $\overline{A}, \overline{B}, \overline{D} \in A$ with

$\overline{A} : = \alpha a^2 + \beta b^2 + 2\gamma ab,$

$\overline{B} : = \alpha ac + \beta bd + \gamma(ad + bc),$

$\overline{D} : = \alpha c^2 + \beta d^2 + 2\gamma cd.$

We have $H_{1,1,0} = H$ and the correspondence between $A, B, D$ and the over-lined elements, arranges an isomorphism $H_{\alpha,\beta,\gamma} \cong H$ (i.e. $\overline{A}\overline{D} - \overline{B}^2 = 1$), if and only if $\alpha \beta - \gamma^2 = 1$.

**Proposition 3** Let $\alpha \beta - \gamma^2 = 1$. Then $H_{\alpha,\beta,\gamma}$ is a Poisson subalgebra of $A^I_{\lambda}$ and we have

(i) $H_{\alpha,\beta,\gamma} \cong H^A_{\lambda,-\gamma}$,

(ii) $H_{\alpha,\beta,\gamma} \cong H^K_{\lambda,-\alpha-\beta}$ and

(iii) $H_{\alpha,\beta,\gamma} \cong H^N_{\lambda,-\beta}$.

**Proof.** The isomorphies can be verified by an explicite calculation.

**Remark 2** The preceeding Proposition admits the realisation of all $H^I_{\lambda,\mu}$ by $H_{\alpha,\beta,\gamma}$ with real $\alpha, \beta, \gamma$ except the cases $H^N_0$ and $H^K_{\lambda,\mu}$, $|\mu| < 2$. For example $H^A_{\lambda,\mu} \cong H_{1,1+\mu^2,\mu}$, $H^K_{\lambda,\mu} \cong H_{\frac{\mu + \sqrt{\mu^2 - 4}}{2},\frac{\mu - \sqrt{\mu^2 - 4}}{2},0}$ and $H^N_{\mu} \cong H_{\frac{1}{\mu},-\mu,0}$. In the other cases we have only complex realisations, for example $H^K_{\lambda,\mu} \cong H_{\frac{\mu + \sqrt{\mu^2 - 4}}{2},\frac{-\mu - \sqrt{\mu^2 - 4}}{2},0}$ and $H^N_0 \cong H_{0,0,0}$. The realisations are not unique.
C. Geometric interpretation

There is an interpretation of the parameter $\mu$ in terms of fixed points $z_0$ of the upper half plane.

Let $z \to \frac{az+b}{cz+d}$ be the usual action of $SL(2,\mathbb{R})$ on the complex coordinate $z = x + iy$, $y > 0$ of the upper half plane, i.e.

\[
\begin{align*}
x & \to \frac{ac(x^2 + y^2) + (ad + bc)x + bd}{c^2(x^2 + y^2) + 2cdx + d^2}, \\
y & \to \frac{y}{c^2(x^2 + y^2) + 2cdx + d^2}.
\end{align*}
\]

Further let $G_{z_0}$ be the subgroup of $SL(2,\mathbb{R})$, which fixes the point $z_0 = x_0 + iy_0$, $y_0 > 0$. It follows that the functions

\[
x = x(a, b, c, d) = \frac{ac(x_0^2 + y_0^2) + (ad + bc)x_0 + bd}{c^2(x_0^2 + y_0^2) + 2cdx_0 + d^2}
\]

and

\[
y = y(a, b, c, d) = \frac{y_0}{c^2(x_0^2 + y_0^2) + 2cdx_0 + d^2}
\]

are left $G_{z_0}$-invariant and can be identified with functions on $G_{z_0} \setminus G$. We recognize that $B = xy^{-1}$, $D = y^{-1}$ generate $H_{y_0 + \frac{x_0^2}{y_0}, \frac{x_0}{y_0}}$. For the parameter $\mu$ we get the interpretation

\[
\mu = \text{ctg}(\text{arg}(z_0)) = \frac{\text{Re}z_0}{\text{Im}z_0}, \quad \text{if } I = A
\]

\[
\mu = -\frac{1 + |z_0|^2}{\text{Im}z_0}, \quad \text{if } I = K
\]

\[
\mu = -\frac{1}{\text{Im}z_0}, \quad \text{if } I = N.
\]

We remark, that $x, y \notin H = \mathbb{R}[\overline{A}, \overline{B}, \overline{D}]$ and $\overline{A}, \overline{B}, \overline{D} \notin H' := \mathbb{R}[x, y]$, but $H$ and $H'$ are dense subalgebras of the algebra of smooth functions on the upper half plane ($H$ and $H'$ are dense on every compact subset). With respect to the xy-coordinates the Poisson brackets have the form

\[
\{x, y\} = -2\lambda(xy + \mu y^2)
\]

if $I = A$,

\[
\{x, y\} = -2\lambda(x^2y + y + y^3 + \mu y^2)
\]

if $I = K$ and

\[
\{x, y\} = -2(y + \mu y^2)
\]

if $I = N$.

IV. The quantum case
A. The Quantum groups $\mathcal{A}_h^I$

First we recall the definition of the three known quantum deformations of $SL(2, \mathbb{R})$.

The Poisson algebras $\mathcal{A}_h^I$ correspond to the quantum algebras $\mathcal{A}_h^A (h \in \mathbb{R}, \ h \neq 2\pi n, \ n \in \mathbb{N}), \mathcal{A}_h^K (h > 0)$ and $\mathcal{A}_h^N$. For all these algebras we write shortly $\mathcal{A}_h^I (I = A, K, N)$. The algebras $\mathcal{A}_h^I$ are generated by elements $a, b, c, d$ and relations

\[
\begin{align*}
ab &= e^{ih}ba, \\
ac &= e^{ih}ca, \\
bd &= e^{ih}db, \\
cd &= e^{ih}dc, \\
bc &= cb, \\
ad - da &= (e^{ih} - e^{-ih})bc, \\
ad - e^{ih}bc &= da - e^{-ih}cb = 1, \quad (h \neq 2\pi n),
\end{align*}
\]

for $I = A$,

\[
\begin{align*}
ab - ba &= i(1 - a^2 - b^2), \\
ac - ca &= ih(a^2 + c^2 - 1), \\
bd - db &= ih(b^2 + d^2 - 1), \\
cd - dc &= ih(1 - c^2 - d^2), \\
ad - da &= ih(ab + ba + ac + ca + bd + db + cd + dc), \\
bc - cb &= ih(ab + ba - ac - ca - bd - db + cd + dc), \\
ad + da - bc - cb &= 2 + h(2 - a^2 - b^2 - c^2 - d^2), \quad (h > 0),
\end{align*}
\]

for $I = K$ and

\[
\begin{align*}
ab - ba &= i(1 - a^2), \\
ac - ca &= ic^2, \\
bd - db &= i(d^2 - 1), \\
cd - dc &= -ic^2, \\
ad - da &= ic(d - a), \\
bc - cb &= ic(d + a), \\
ad - bc + iac &= da - cb - iac = 1.
\end{align*}
\]

for $I = N$.

The algebras $\mathcal{A}_h^I$ become Hopf algebras with respect to the Hopf multiplication

\[
\Delta \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{array} \right)
\]

and Hopf $*$-algebras with respect to the involution

$a^* := a, \ b^* := b, \ c^* := c, \ d^* := d$.

The counit $\epsilon$ is given by
\[
\epsilon \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]
and the coinverse \( \kappa \) is given by
\[
\kappa \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} d & -q^{-1}b \\ -qc & a \end{array} \right)
\]
for \( I = A \),
\[
\begin{align*}
\kappa(a) &= \frac{1}{4} \left(2 - \frac{2 - 2h^2}{1 + h^2}a + (2 + \frac{2 - 2h^2}{1 + h^2})d - \frac{2h}{1 + h^2}b - \frac{2h}{1 + h^2}c\right), \\
\kappa(b) &= \frac{1}{4} \left(\frac{2}{1 + h^2}a - \frac{2}{1 + h^2}d - (2 + \frac{2 - 2h^2}{1 + h^2})b + (2 - \frac{2 - 2h^2}{1 + h^2})c\right), \\
\kappa(c) &= \frac{1}{4} \left(\frac{2}{1 + h^2}a - \frac{2}{1 + h^2}d + (2 - \frac{2 - 2h^2}{1 + h^2})b - (2 + \frac{2 - 2h^2}{1 + h^2})c\right), \\
\kappa(d) &= \frac{1}{4} \left((2 + \frac{2 - 2h^2}{1 + h^2})a + (2 - \frac{2 - 2h^2}{1 + h^2})d + \frac{2h}{1 + h^2}b + \frac{2h}{1 + h^2}c\right)
\end{align*}
\]
for \( I = K \) and
\[
\kappa \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} d + c & -b + (d - a + c) \\ -c & a - c \end{array} \right)
\]
for \( I = N \).

**Remark 3** \( A^K_h \) is equivalent to the quantum group \( SU_q(1, 1) \), \( q \in \mathbb{R} \) which is given by elements \( \alpha, \alpha^*, \beta, \beta^* \) and relations
\[
\begin{align*}
\alpha \beta &= q \beta \alpha, \\
\alpha \beta^* &= q \beta^* \alpha, \\
\beta \beta^* &= \beta^* \beta, \\
\beta \alpha^* &= q \alpha^* \beta, \\
\beta^* \alpha^* &= q \alpha^* \beta^*, \\
\alpha^* \alpha - q^2 \beta^* \beta^* &= \alpha^* \alpha - \beta^* \beta = 1.
\end{align*}
\]
The formulas of \( SU_q(1, 1) \) and \( A^K_h \) are connected by the transformation of the deformation parameter \( h = \frac{1 - q}{1 + q} \) and by the ”quantum Cayley transformation” of the matrix elements
\[
\alpha := \frac{1}{2}(a + d + i(b - c)),
\]
\[ \alpha^* := \frac{1}{2}(a + d + i(-b + c)), \]
\[ \beta := \frac{1}{2}(a - d - i(b + c)), \]
\[ \beta^* := \frac{1}{2}(a - d + i(b + c)). \]

Remark 4 Instead of \( A^N \) one considers Hopf *-algebras \( SL_h(2, \mathbb{R}) \), \((h \neq 0)\) where the algebra structure is replaced by
\[ \begin{align*}
ab - ba &= i\hbar(1 - a^2),
ac - ca &= i\hbar c^2, bd - db &= i\hbar(d^2 - 1),
\end{align*} \]
\[ \begin{align*}
bc - cb &= i\hbar dc + ca,
ad - bc + i\hbar ac &= da - cb - i\hbar ca = 1
\end{align*} \]
(cf. Ref. 9 and Ref. 10). All these structures are equivalent to \( A^N \) by the Hopf *-algebra isomorphism \( a \rightarrow a, b \rightarrow \hbar b, c \rightarrow \frac{1}{\hbar}c, d \rightarrow d \).

B. The quantum spaces \((H^I_{h,k,d}, \phi)\)

1. The algebras \( H^I_{h,k,d} \)

The Poisson algebras \( H^I_{A,K,d} \) correspond to the algebras \( H^I_{h,k,d} \) \((h \neq 2\pi n)\), \( H^I_{h,k,d} \) \((h > 0)\), \( H^I_{k,d}, k, d \in \mathbb{R} \). For all these algebras we write shortly \( H^I_{h,k,d} \).

We define the \( H^I_{h,k,d} \) as algebras which are generated by elements \( A,B,C,D \) and relations
\[ \begin{align*}
C &= e^{-i\hbar}B + k(1 - e^{-i\hbar}), \\
AB &= e^{2i\hbar}BA + k(1 - e^{2i\hbar})A, \\
BD &= e^{2i\hbar}DB + k(1 - e^{2i\hbar})D, \\
AD - e^{2i\hbar}BC &= d + k(1 - e^{2i\hbar})B, \\
DA - e^{-2i\hbar}CB &= d + k(1 - e^{-2i\hbar})B, \quad (h \neq 2\pi n),
\end{align*} \]
for \( I = A \),
\[ \begin{align*}
C &= B - i\hbar(A + D + k), \\
AB - BA &= i\hbar(2kA + 2A^2 + AD + BC), \\
BD - DB &= i\hbar(2kD + 2D^2 + AD + CB), \\
AD - B^2 &= d + i\hbar(kB + AB + BD), \\
DA - C^2 &= d - i\hbar(kC + CA + DC), \quad (h > 0),
\end{align*} \]
for \( I = K \) and
\[ \begin{align*}
C &= B - i(D + k), \\
AB - BA &= 2iA(D + k) - 2B(D + k), \\
BD - DB &= 2iD(D + k), \\
AD - BC &= d + 2iB(D + k), \\
DA - CB &= d - 2i(D + k)C
\end{align*} \]
for $I = N$.

The algebras $\mathcal{H}_{h,k,d}^I$ become $\ast$-algebras with respect to the involutions

$$A^* := A, \quad D^* := D, \quad B^* := C = e^{-ih}B + k(1 - e^{-ih}), \quad C^* := B$$

for $I = A$,

$$A^* := A, \quad D^* := D, \quad B^* := C = B - ih(A + D + k), \quad C^* := B$$

for $I = K$ and

$$A^* := A, \quad D^* := D, \quad B^* := C = B - i(D + k), \quad C^* := B$$

for $I = N$.

**Remark 5** Formally we recover the Poisson structures of subsection III.B in the limit $t \to \infty$, if we set $B = C$, $q = \lambda t$, $k = \mu$, $d = 1$ and $\{x, y\} := \lim_{t \to \infty} \frac{1}{it}[x, y]$.

Next we show, that we can realize the $\ast$-algebras $\mathcal{H}_{h,k,d}^I$ as $\ast$-subalgebras of $\mathcal{A}_{h}^I$. Let

$$\overline{A} := \alpha a^2 + \beta b^2 + \gamma(ab + ba), \quad \overline{B} := \alpha ac + \beta bd + \gamma(ad + bc), \quad \overline{C} := \alpha ca + \beta db + \gamma(da + cb), \quad \overline{D} := \alpha c^2 + \beta d^2 + \gamma(cd + dc).$$

**Proposition 4** Let $\alpha, \beta, \gamma \in \mathbb{R}$. $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ satisfy the relations of

(i) $\mathcal{H}_{h,\gamma,\alpha\beta,\gamma^2}$,
(ii) $\mathcal{H}_{h,-\alpha,\beta,\alpha\beta,\gamma^2}$ and
(iii) $\mathcal{H}_{h,-\beta,\alpha\beta,\gamma^2}$.

**Proof.** The proof can be given by an explicite calculation.

**Remark 6** Proposition 4 admits the realisation of all $\mathcal{H}_{h,k,d}^I$, with real $\alpha, \beta, \gamma$ without the cases $\mathcal{H}_{h,0,d}^K$, $|k| < 2$ and $\mathcal{H}_{h,0,d}^N$, $d > 0$. For example $\mathcal{H}_{h,k,d}^A \cong \mathcal{H}_{1,d+k^2,k}$, $\mathcal{H}_{h,k,d}^K \cong \mathcal{H}_{k+\sqrt{4-k^2},k-\sqrt{4-k^2},0}$ and $\mathcal{H}_{h,k,d}^N \cong \mathcal{H}_{k,0,0}$. In other cases we have only complex realisations, for example $\mathcal{H}_{h,k,d}^K \cong \mathcal{H}_{k+i\sqrt{4-k^2},k-i\sqrt{4-k^2},0}$ and $\mathcal{H}_{0,0,i\sqrt{d}} \cong \mathcal{H}_{h,0,d}^N$. [11]
Remark 7 1. The special case $\mathcal{H}_{1,1,0} \cong \mathcal{H}^A_{h,0,1}$ was first mentioned in Ref. 5, p. 188.
2. The formulas for $\mathcal{H}_{h,k,d}^I$ are similar those from Podles’ sphere $C(X_{\mu,\lambda,\rho})$ (cf. Ref. 4). Both can be specified from the complex quantum space $\Xi_{q,\lambda,\rho}$ of $SL_q(2,C)$ from Ref. 11 by fixing certain involutions.
3. In Ref. 12 the quantum spaces $\mathcal{H}_{\alpha,\beta,0} \cong \mathcal{H}^A_{h,0,\alpha\beta}$, $\alpha, \beta \geq 0$ were considered.
4. The quantum spaces $\mathcal{H}_{h,k,d}^K$ correspond to one-parameter series of quantum discs $C_{\mu,q}(U)$ in Ref. 13. Formally the correspondence is arranged by the quantum Cayley transformation (cf. Remark 3).

2. The coaction of $A^I_h$ on $\mathcal{H}_{h,k,d}^I$

We will describe a coaction $\phi$, i.e. an homomorphism $\phi : \mathcal{H}_{h,k,d}^I \to A^I_h \otimes \mathcal{H}_{h,k,d}^I$ with $(\Delta \otimes \text{id})\phi = (\text{id} \otimes \phi)\phi$ and $(\epsilon \otimes \text{id})\phi = \text{id}$ such that we can call $(\mathcal{H}_{h,k,d}^I, \phi)$ a quantum space with respect to $A^I_h$.

According to Remark 6 consider $\mathcal{H}_{h,k,d}^I$ as a subalgebra of $A^I_h$, i.e. we identify $A, B, C, D$ with $\overline{A, B, C, D}$. We obtain (independent from $I, h, k, d$)

$$\Delta(A) = a^2 \otimes A + b^2 \otimes D + ab \otimes B + ba \otimes C,$$
$$\Delta(D) = c^2 \otimes A + d^2 \otimes D + cd \otimes B + dc \otimes C,$$
$$\Delta(B) = ac \otimes A + bd \otimes D + ad \otimes B + bc \otimes C,$$
$$\Delta(C) = ca \otimes A + db \otimes D + cb \otimes B + da \otimes C.$$

That is, $\mathcal{H}_{h,k,d}^I$ is a left coideal of $A^I_h$ and we have proven the following proposition.

Proposition 5 The homomorphism $\phi : \mathcal{H}_{h,k,d}^I \to A^I_h \otimes \mathcal{H}_{h,k,d}^I : \phi := \Delta|_{\mathcal{H}_{h,k,d}^I}$ defines a left coaction, that is $(\mathcal{H}_{h,k,d}^I, \phi)$ are left quantum spaces.

Remark 8 Every $\mathcal{H}_{h,k,d}^I$ is equivalent to one of the quantum spaces

$$\mathcal{H}_{h,k,\pm 1}^I, \mathcal{H}_{h,k,0}^I$$

with $k \geq 0$.

Proof. We achieve $k \in [0, \infty)$ because of the automorphism $X \to -X$, $X \in \mathcal{H}_{h,k,d}^I$, and we achieve $d = 0, \pm 1$ by the reparametrisation $X \to \frac{1}{\sqrt{|d|}}X$, $X \in \mathcal{H}_{h,k,d}^I$. $\Box$

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