Iterating free-field AdS/CFT: higher spin partition function relations

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Abstract

We find a simple relation between a free higher spin partition function on the thermal quotient of $\text{AdS}_{d+1}$ and the partition function of the associated $d$-dimensional conformal higher spin field defined on the thermal quotient of $\text{AdS}_d$. Starting with a conformal higher spin field defined in $\text{AdS}_d$, one may also associate to with another conformal field in $d-1$ dimensions, thus iterating AdS/CFT. We observe that in the case of $d=4$, this iteration leads to a trivial 3d higher spin conformal theory with parity-even non-local action: it describes a zero total number of dynamical degrees of freedom and the corresponding partition function is equal to 1.

Keywords: AdS/CFT duality, higher spins, conformal symmetry

1. Introduction

Kinematical anti-de Sitter/conformal field theory (AdS/CFT) correspondence relates a field $\varphi$ in $\text{AdS}_{d+1}$ (e.g., with standard 2-derivative action with some mass parameter $M^2$ or associated dimension $\Delta$) to a conformal field (CF) $\phi$ at the boundary $\mathcal{M}^d = \partial(\text{AdS}_{d+1})$ with canonical dimension $\Delta = d - \Delta$. The value of the AdS mass parameter and thus of $\Delta$ determines the number of derivatives in the kinetic term in the action for $\phi$:

$$S_d = \int d^dx \, \phi \partial^k \phi, \quad k = d - 2\Delta = 2\Delta - d. \quad (1.1)$$

For example, a massless totally symmetric higher spin (HS) field $\varphi_S$ in $\text{AdS}_{d+1}$ is associated with a conformal HS (CHS) field $\phi_S$ with the action $S_d = \int d^dx \, \phi_S P_d \partial^{2d-4} \phi_S$ where

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$P_{s}$ is traceless transverse projector (see [1] for a review and references). From the standard AdS/CFT perspective a massless AdS field $\phi_0$ is a counterpart of a bilinear conserved current $J_s$ of a free (e.g., scalar $\Phi$) boundary CFT$_d$ while $\phi_s$ is associated with a shadow field or a source for $J_s$; thus the action for $\phi_s$ may be interpreted as an induced action found upon integrating out $\Phi$ coupled to $\phi_s$ via $J_s(\Phi)$.

There are other relations between the two free theories $S_{d+1}(\phi)$ and $S_d(\phi)$ in $d + 1$ and $d$ dimensions beyond just kinematic $SO(2, d)$ representation theory correspondence. First, the AdS$_{d+1}$ action for $\phi$ evaluated on the solution of the Dirichlet problem $\varphi|_0 = \phi$ gives an induced action for $\phi$. For even $d$ and, e.g., for a massless field $\varphi$ the AdS$_{d+1}$ action contains a logarithmically divergent local term, which is identified with a local action for $\phi$. For odd $d$ one gets, in general, a non-local action as the power $k$ in the kinetic operator in (1.1) may be a half-integer or negative. In addition to this free-level relation there is also a 1-loop relation between the ratio of partition functions for an HS field $\varphi$ in AdS$_{d+1}$ with D and N (or $+$ and $-$) boundary conditions and for the dual CF $\phi$ at the boundary:

$$Z_{\text{HS}}(\text{AdS}_{d+1}; q) = Z_{\text{CF}}(\text{AdS}_{d+1}; q).$$

This relation is true, e.g., for global AdS$_{d+1}$ with boundary $S^d$ and also for a thermal quotient of AdS$_{d+1}$ with boundary $S^1 \times S^{d-1}$. In the latter case equation (1.2) translates into a relation between one-particle partition functions $Z$ as functions of the $q = e^{-\beta}$ variable:

$$Z^-_{\text{HS}}(\text{AdS}_{d+1}; q) = Z^+_{\text{HS}}(\text{AdS}_{d+1}; q) = Z_{\text{CF}}(S^1 \times S^{d-1}; q),$$

$$Z = \exp \sum_{n=1}^{\infty} Z(q^n),$$

$$Z^-(\text{AdS}_{d+1}; q) = Z^+(\text{AdS}_{d+1}; q) = 0,$$

$$Z^-(\text{AdS}_{d+1}; q) = (-1)^d Z^+(\text{AdS}_{d+1}; q^{-1}).$$

Equation (1.3) may be interpreted in terms of counting of operators in the boundary CFT or as a group-theoretic relation for characters of the conformal group. More generally, equation (1.2) is expected to be true even for asymptotically AdS space and its generic curved boundary (provided the corresponding $d + 1$ and d-dimensional theories can be consistently defined) and should thus provide, in particular, an AdS theory based way to compute not only the conformal anomaly $a$-coefficients [2] but also the c-coefficients [3].

Having identified a CF $\phi$ in $\mathbb{R}^d$ associated to a field $\varphi$ in AdS$_{d+1}$ we may attempt to repeat this step one more time. Namely, we may first define this $\phi$ not on $\mathbb{R}^d$ (or $S^d$ or $S^1 \times S^{d-1}$) but on AdS$_d$ and then associate to it another conformal field $\tilde{\varphi}$ in $d - 1$ dimensions. We will then have the following dimensional $(d + 1 \rightarrow d \rightarrow d - 1)$ digression$^6$

$$\varphi(\text{AdS}_{d+1}) \rightarrow \phi(\partial \text{AdS}_{d+1} \sim \text{AdS}_d) \rightarrow \tilde{\varphi}(\partial \text{AdS}_d).$$

If $\varphi$ is a gauge field with 2-derivative action in AdS$_{d+1}$, then $\phi$ is also a single (gauge) CF with, in general, higher derivative action. The latter can be represented as a collection of

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5 Here $c_d(q)$ is a finite polynomial in $q + q^{-1}$ that represents a contribution of a finite number of zero modes related to the gauge invariance of the conformal (shadow) field [1, 2].

6 For standard global AdS$_{d+1}$ we have $\partial \text{AdS}_{d+1} = \mathbb{R} \times S^{d-1}$. This space is equivalent to two copies of AdS$_d$ glued along their boundary identified with the equator of $S^{d-1}$. The middle step in (1.7) means that we start with the conformal action on $\partial \text{AdS}_{d+1}$ and then translate it into AdS$_d$ (taking also into account the freedom in the choice of boundary conditions, see below).
second-derivative fields in AdS$_d$ and hence $\hat{\phi}$ in $d - 1$ dimensions will be given by set of several CFs, each being dual to an individual second-derivative field in $d$ dimensions.

Our aim below will be to explore some implications of this iterated AdS/CFT correspondence (1.7) at the level of relations between partition functions such as (1.2) and (1.3). We shall find that for a generic HS field in AdS$_{d+1}$ and its dual CF in $d$ dimensions one also gets

$$ Z^-_{\text{HS}}(\text{AdS}_{d+1}; q) - Z^+_{\text{HS}}(\text{AdS}_{d+1}; q) = Z^-_{\text{CFT}}(\text{AdS}_d; q) + Z^+_{\text{CFT}}(\text{AdS}_d; q), \tag{1.8} $$

where $Z^-$ may be replaced by $\tilde{Z}^-$ in equation (1.6) as one also finds that the $\sigma$ terms in equation (1.5) match, $\sigma_{\text{HS},d+1} = \sigma_{\text{CFT},d+1}$. Equation (3.16) follows from (1.3) and

$$ Z^-_{\text{CFT}}(\text{AdS}_d; q) + Z^+_{\text{CFT}}(\text{AdS}_d; q) = Z_{\text{CFT}}(S^1 \times S^{d-1}; q), \tag{1.9} $$

which may be related to the fact that AdS$_d$ is conformal to half of $\mathbb{R} \times S^{d-1}$ so that the respective partition functions are related provided one sums over the two possible boundary conditions at the boundary of AdS$_d$. Applying equation (1.3) to the CF in AdS$_d$ and its counterpart CF CFT in $d - 1$ dimensions (cf equation (1.7)) we get also

$$ Z^-_{\text{CFT}}(\text{AdS}_d; q) - Z^+_{\text{CFT}}(\text{AdS}_d; q) = Z_{\text{CFT}}(S^1 \times S^{d-2}; q). \tag{1.10} $$

We shall find that the case of $d = 4$ is special: starting with a HS field in AdS$_4$, the resulting 3 d conformal theory represented by $\hat{\phi}$ is effectively topological, having a zero number of dynamical d.o.f. and trivial partition function. This may be related to the equivalence of ± modes with non-zero spins in AdS$_4$ [4], implying

$$ Z^-_{\text{CFT}}(\text{AdS}_4; q) = Z^+_{\text{CFT}}(\text{AdS}_4; q) \Rightarrow Z_{\text{CFT}}(S^1 \times S^2; q) = 0. \tag{1.11} $$

Very loosely, this may be interpreted as a version of the boundary of boundary = 0 relation, or as $(\text{AdS/CFT})^2 = 0$.

We shall start in section 2 with a review of some general definitions and relations. Then in section 3 we shall demonstrate the validity of (3.1) on several examples, in particular for massless higher spin fields in AdS$_{d+1}$ related to CHS fields in AdS$_d$.

In section 4 we shall first analyse the detailed structure of the relation (3.1) on the example of the totally symmetric field in AdS$_3$ with generic mass parameter and mention its possible group-theoretic interpretation and then justify the $Z^- = Z^+$ equality in equation (1.11). We shall then discuss in detail the corresponding 3 d conformal theory with non-local linearized action describing total of zero degrees of freedom and leading to trivial partition function. We shall use spin 1 Maxwell and spin 2 conformal graviton fields as examples.

Section 5 will contain some concluding remarks. In appendix A we shall discuss the algebraic structure of the partition functions appearing in relation (3.1) and then in appendix B argue for the equality of the corresponding $\sigma$-terms in equation (1.5). In appendices C–E the relation (3.1) will be further illustrated on the examples of conformal higher derivative scalars, fermionic CHS fields and conformal antisymmetric tensor field in 4 d.

7 Let us note that our interpretation and examples will be different from previous discussions of sequential AdS/CFT such as AdS$_d$/CF$_1$ → AdS$_{d-1}$/CF$_2$ in [5, 6] (for related work discussing AdS$_d$ foliations of AdS$_{d+1}$ see also [7–11]). In particular, in contrast to [5] the 3 d CHS theory that will naturally appear in our context is not of local Chern–Simons type but has parity-even non-local action. Let us also mention for completeness that discussions of dimensional reduction from AdS$_{d+1}$ to AdS$_d$ appeared in [12, 13].
2. Some general relations

Let us consider a conformally invariant action in \( \text{AdS}_d \). This space is conformally equivalent to one half of static Einstein Universe \( S^1 \times S^{d-1} \), with the boundary of \( \text{AdS}_d \) being mapped to the equator of \( S^{d-1} \) [4, 14, 15]. One can consider the single particle partition function \( Z(\text{AdS}_d; q) \) on thermal \( \text{AdS}_d \) where we identify \( t \sim t + \beta \). This can be compared with the partition function in Einstein Universe \( Z(S^1 \times S^{d-1}; q) \) where \( S^1 \) is the thermal circle with length \( b \).

The calculation of the total partition function \( Z(\text{AdS}_d; q) \) (and thus of \( Z(\text{AdS}_d; q) \)) is straightforward assuming that the kinetic operator of a conformal field factorizes, i.e. the action in \( \text{AdS}_d \) can be written as a sum of second-derivative terms (as, e.g., in [16]). For example, let us consider

\[
\log Z(\text{AdS}_d) = - \frac{1}{2} \sum_{i=1}^{N} n_i \log \text{det} \Delta_{\perp}(M^2), \quad \Delta_{\perp}(M^2) \equiv (-\nabla^2 - M^2)_{\perp} \tag{2.1}
\]

where \( \Delta_{\perp} \) is defined on transverse traceless symmetric tensors of rank \( s \), and the integers \( n_i \) are field multiplicities positive for physical fields and negative for ghost fields. For each operator in equation (2.1) the value of the mass term then determines possible ground state energies \( \Delta_{\perp} \) that are solutions of the quadratic equation [17–19]

\[
\Delta_{\perp}(\Delta_{\perp} - d + 1) - s = -M^2, \quad \Delta_{\perp} = d - 1 - \Delta_{\perp}, \quad \Delta_{\perp} \leq \Delta_{\perp}, \tag{2.2}
\]

and are associated with classical solutions of the wave equation \( \Delta_{\perp}(M^2)\varphi_{\perp} = 0 \) with two different boundary conditions. Taking the thermal quotient of \( \text{AdS}_d \), we then get from (2.1) the following two possibilities for the corresponding single particle partition function (\( q = e^{-\beta} \))

\[
Z_+(\text{AdS}_d; q) = \sum_{i=1}^{N} n_i g_{\perp}^{(d)} \frac{q^{\Delta_{\perp}}}{(1 - q)^{d-1}}, \quad Z_-(\text{AdS}_d; q) = \sum_{i=1}^{N} n_i g_{\perp}^{(d)} \frac{q^{\Delta_{\perp}}}{(1 - q)^{d-1}}. \tag{2.3}
\]

In equation (2.3) \( g_{\perp}^{(d)} \) is the multiplicity that counts the number of off-shell degrees of freedom\(^8\)

\[
g_{\perp}^{(d)} = (2s + d - 3) \frac{(s + d - 4)!}{(d - 3)! s!}. \tag{2.4}
\]

Using that \( \Delta_{\perp} = d - 1 - \Delta_{\perp} \) we find

\[
\check{Z}_-(\text{AdS}_d; q) = (-1)^{d-1} Z_+(\text{AdS}_d; q^{-1}). \tag{2.5}
\]

In the presence of gauge invariance the proper \( Z_+(\text{AdS}_d; q) \) partition function differs from \( \check{Z}_+(\text{AdS}_d; q) \)

\[
Z_+^{\text{HS}}(\text{AdS}_d; q) = \check{Z}_+^{\text{HS}}(\text{AdS}_d; q) + \sigma(q), \tag{2.6}
\]

where \( \sigma(q) \) is a finite polynomial in \( q + q^{-1} \) related to the missing gauge transformations discussed in [1].

The calculation of \( Z(S^1 \times S^{d-1}; q) \) on the Einstein Universe background is \textit{a priori} unrelated to the one on \( \text{AdS}_d \). If the action on generic \( M^d \) is known, one may just specialize it

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\(^8\) In general, we define \( \Delta_{\perp}(M^2) = (-\nabla^2 + M^2)_{\perp} \), where \( \epsilon = -1 \) for \( \text{AdS}_d \) and \( \epsilon = +1 \) for \( S^d \) (here we set the curvature scale to 1).

\(^9\) Special cases are \( g_{\perp}^{(0)} = 2s + 1 \) and \( g_{\perp}^{(0)} = \frac{1}{6}(s + 1)(s + 2)(2s + 3) \).
to $S^l \times S^{d-l}$, factorize the kinetic operator and use the methods discussed in [1]10. Alternatively, one can make use of the conformal map to flat space $\mathbb{R}^d$ (radial quantization) and use flat space operator counting techniques.

At the same time $Z(S^l \times S^{d-l}; \phi)$ can also be computed starting with the dual theory in AdS$_{d+1}$ (cf equation (1.3)). If $S^l \times S^{d-l}$ is interpreted as the boundary of AdS$_{d+1}$ the corresponding conformally invariant action on $S^l \times S^{d-l}$ can be interpreted as induced from an action of a dual field in the bulk (see, e.g., [1, 2] and references there). Let us call a generic tensor bulk field an HS one; this name will include the cases of a massive or partially massless or exactly massless higher spin field in AdS$_{d+1}$. The dual conformal field at the boundary will be denoted as CF. Then [1]

$$Z_{\text{HS}}(\text{AdS}_d; q) - Z_{\text{HS}}^+(\text{AdS}_{d+1}; q) = Z_{\text{CF}}(S^l \times S^{d-l}; q). \quad (2.7)$$

In the case of a massless higher spin (MHS) field in AdS$_{d+1}$ having maximal gauge invariance the associated conformal field at the boundary is a CHS one and $a_{d+1}$ in equation (2.6) is non-trivial [1]11.

### 3. Partition functions on AdS$_{d+1}$ and AdS$_d$

Let us now propose and check on several examples a general relation between the partition functions of an HS field in AdS$_{d+1}$ and associated CF originally induced on $\partial\text{AdS}_{d+1} = \mathbb{R} \times S^{d-1}$ or $S^d$ but that can then also be defined on AdS$_d$. This relation is (3.16) that we rewrite here for the reader’s convenience

$$Z_{\text{HS}}^-(\text{AdS}_d; q) - Z_{\text{HS}}^+(\text{AdS}_{d+1}; q) = Z_{\text{CF}}^-(\text{AdS}_d; q) + Z_{\text{CF}}^+(\text{AdS}_d; q). \quad (3.1)$$

Heuristically, the relation (3.1) may be motivated as follows. AdS$_d$ is conformal to half of the Einstein Universe $S^1 \times S^{d-1}$ with two possible choices of the boundary conditions at the equator; thus defining the partition function on $S^1 \times S^{d-1}$ in terms of AdS$_d$ one we may need to sum over the two boundary condition choices,

$$Z_{\text{CF}}^-(\text{AdS}_d; q) + Z_{\text{CF}}^+(\text{AdS}_d; q) = Z_{\text{CF}}(S^1 \times S^{d-1}; q). \quad (3.2)$$

Combining this with equation (2.7) then gives equation (3.5).

Note that starting with a CF field in AdS$_d$ we may also associate it with another conformal field $\tilde{\text{CF}}$ at the $d-1$ boundary and then the analogue of equation (2.7) will read

$$Z_{\text{CF}}^-(\text{AdS}_d; q) - Z_{\text{CF}}^+(\text{AdS}_d; q) = Z_{\tilde{\text{CF}}}(S^1 \times S^{d-2}; q). \quad (3.3)$$

Furthermore, the $\sigma(q)$ terms in equation (2.6) for $Z_{\text{HS}}^+(\text{AdS}_{d+1}; q)$ and $Z_{\text{CF}}^-(\text{AdS}_d; q)$ appear to match (see appendix B)

$$\sigma_{\text{HS}, d+1}(q) = \sigma_{\text{CF}, d}(q) \quad (3.4)$$

10 If one knows the set of masses $M^2$ in equation (2.1) for an action on AdS$_d$, this is not enough to compute the partition function for the same theory on $S^l \times S^{d-1}$. The reason is that $M^2$ values come from the specialization to AdS$_d$ of the action on a generic curved background $M^d$ where certain combinations of curvature tensor terms lead to mass terms. Specification of this action to $S^l \times S^{d-1}$ will then lead to different kinetic term structures, i.e. to different mass terms in the corresponding second order operators.

11 In addition to the quantum one-loop relation (2.7) the quadratic actions for HS and CF also have a classical relation: evaluating the action of the HS field in AdS$_{d+1}$ on the solution with boundary data being equal to the CF field one gets the action of the CF field as an induced one. In the even $d$ case the local CF action is the coefficient of the leading logarithmic IR divergence while in the odd $d$ case it is finite but non-local.
so that relation (3.1) may also be written as
\[ \tilde{Z}_{	ext{HS}}^-(\text{AdS}_{d+1}; q) - \tilde{Z}_{	ext{HS}}^+(\text{AdS}_{d+1}; q) = \tilde{Z}_{\text{CF}}^-(\text{AdS}_d; q) + \tilde{Z}_{\text{CF}}^+(\text{AdS}_d; q). \] (3.5)

Using equation (2.5) this can also be put in the following more symmetric form
\[ - \tilde{Z}_{	ext{HS}}^+(\text{AdS}_{d+1}; q) + (-1)^d \tilde{Z}_{	ext{HS}}^+(\text{AdS}_{d+1}; q^{-1}) = \tilde{Z}_{\text{CF}}^+(\text{AdS}_d; q) - (-1)^d \tilde{Z}_{\text{CF}}^+(\text{AdS}_d; q^{-1}). \] (3.6)

Below we will demonstrate the validity of equations (3.1), (3.2) and (3.5) on several examples of conformal fields (for some consequences of equation (3.5) see also appendix A).

### 3.1. Conformal scalar

Let us start with the case of a particular scalar field in $\text{AdS}_{d+1}$ with the mass term $M^2 = \frac{1}{4}d^2 + 1$, i.e. with $\Delta_{d+1}^+ = \frac{1}{2}(d \pm 2)$ (cf (2.2)). The corresponding partition function is
\[ \tilde{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \frac{q^{\frac{d}{2}(d+2)}}{(1-q)^d}. \] (3.7)

This scalar in $\text{AdS}_{d+1}$ induces a spin 0 field $\varphi$ at the boundary with canonical dimension $\Delta_{d+1}^+ = \frac{1}{2}(d - 2)$, i.e. which thus represents a conformally coupled scalar. The corresponding kinetic operator in a curved $d$-dimensional space specified to the case of the unit-scale $\text{AdS}_d$ (with $R = -d(d-1)$ is
\[ -\nabla^2 + \frac{d-2}{4(d-1)}R = -\nabla^2 - \frac{1}{4}d(d-2). \] (3.8)

Thus defining $\varphi$ on $\text{AdS}_d$ we find that the mass term (cf equation (2.1)) is $M^2 = \frac{1}{4}d(d-2)$ and thus from equation (2.2)
\[ \Delta_+^+ = \frac{d}{2}, \quad \Delta_+^d = \frac{d-2}{2}. \] (3.9)

From (2.3) the partition functions corresponding to this conformal scalar (cs) are then
\[ \tilde{Z}_{\text{cs}}^+(\text{AdS}_d; q) = \frac{q^{d/2}}{(1-q)^d}, \quad \tilde{Z}_{\text{cs}}^-(\text{AdS}_d; q) = (-1)^{d+1} \tilde{Z}_{\text{cs}}^+(\text{AdS}_d; q^{-1}). \] (3.10)

Comparing to equation (3.7) one can then check that
\[ \tilde{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \tilde{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \tilde{Z}_{\text{cs}}^-(\text{AdS}_d; q) + \tilde{Z}_{\text{cs}}^+(\text{AdS}_d; q). \] (3.11)

which is a particular spin 0 case of equation (3.1).

To demonstrate equation (3.2) we recall that the partition function on $S^1 \times S^{d-1}$ can be found, e.g., by the operator counting method. For a scalar $\varphi$ with canonical dimension $\frac{1}{2}(d-2)$ and equations of motion $\partial^2 \varphi = 0$, which gives
\[ \tilde{Z}_{\text{cs}}(S^1 \times S^{d-1}; q) = \frac{q^{\frac{1}{2}(d-2)} - q^{\frac{1}{2}(d+2)}}{(1-q)^d}. \] (3.12)

Then one can check that equation (2.7) is satisfied (there is no gauge invariance so $\sigma(q) = 0$ in equation (2.6)). As a result, we verify a special case of equation (3.2)
As already mentioned above, this relation means that one needs to sum over both $\pm$ scalar modes in $\text{AdS}_d$ in order to match the conformal scalar partition function on $S^1 \times S^{d-1}$ space, which is conformally equivalent to a double copy of $\text{AdS}_d$.

The above discussion can be extended to higher derivative GJMS conformal scalars with higher derivative kinetic operators, see appendix C, again verifying the general relations (3.1)—(3.5).

### 3.2. CHSs

Let us now consider a totally symmetric spin $s$ CHS field in $d$ dimensions. The CHS theory defined on $\text{AdS}_d$ has the following partition function [20, 21]

$$Z_{\text{CHS},s}(\text{AdS}_d) = \frac{1}{(1-q)^{d+1}} \left( \sum_{k=0}^{n-1} [s^{(d)}_k q^{d+k-2} - s^{(d)}_{k-1} q^{d+k-2}] + \sum_{k'=\frac{1}{2}(d-4)}^{n-1} g^{(d)}_{s_{k'}} q^{d+k'-2} \right),$$

(3.14)

$$M^2_{n,k} \equiv n - (k-1)(k+d-2).$$

(3.15)

For each determinant here (cf equation (2.1)) we may then compute the corresponding contribution to the one-particle partition function using the general relations (2.2) and (2.3).

As a result, we get

$$Z^{+}_{\text{CHS},s}(\text{AdS}_d; q) = \frac{1}{(1-q)^{d+1}} \left( \sum_{k=0}^{n-1} [s^{(d)}_k q^{d+k-2} - s^{(d)}_{k-1} q^{d+k-2}] + \sum_{k'=\frac{1}{2}(d-4)}^{n-1} g^{(d)}_{s_{k'}} q^{d+k'-2} \right),$$

(3.16)

Doing the sum, we find

$$Z^{+}_{\text{CHS},s}(\text{AdS}_d; q) = \frac{\Gamma(d+s-3)}{\Gamma(d-1)\Gamma(s+1)} \frac{q^{d-2}}{1-q} [(d-2)(d+2s-3)q^2 - (d+s-3)(d+2s-2)q^{d+s} + s(d+2s-4)q^{d+s+1}].$$

(3.17)

In the special case of the $s=2$ CHS field or, equivalently, of Weyl gravity on the thermal quotient of $\text{AdS}_4$ and $\text{AdS}_6$, this partition function was also independently computed in [22, 23]

$$Z^{+}_{\text{CHS},2}(\text{AdS}_4; q) = \frac{q^2(5 + 5q - 4q^2)}{(1-q)^3},$$

$$Z^{+}_{\text{CHS},2}(\text{AdS}_6; q) = \frac{2q^3(7 + 7q + 7q^2 - 3q^3)}{(1-q)^3}.\quad (3.18)$$

The CHS field in $d$ dimensions is naturally associated to the MHS field in $\text{AdS}_{d+1}$ with $\Delta_{d+1} = d+s-2$. It has the one-particle partition function [24–26]

$$Z^{+}_{\text{MHS},s}(\text{AdS}_{d+1}; q) = \frac{g^{(d+1)}_{s} q^{d+s-2} - g^{(d+1)}_{s-1} q^{d+s-1}}{(1-q)^d},$$

(3.19)

$$Z^{+}_{\text{MHS},s}(\text{AdS}_{d+1}; q) = (-1)^d \frac{Z^{+}_{\text{MHS},s}(\text{AdS}_{d+1}; q^{-1})}{(1-q)^d}.\quad (3.20)$$

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where $g_0^{(d)}$ is given by equation (2.4). One can then check that

$$\hat{Z}_{\text{MHS},j}^{-}(\text{AdS}_{d+1}; q) = \hat{Z}_{\text{MHS},j}(\text{AdS}_{d+1}; q) = \hat{Z}_{\text{CHS},j}^{-}(\text{AdS}_d; q) + \hat{Z}_{\text{CHS},j}^{+}(\text{AdS}_d; q),$$

(3.21)

which is another special case of equation (3.5). One can also verify the validity of relation (3.1) or, equivalently, (3.4) (see appendix B).

### 3.3. Conformal symmetric tensors in $d = 4$

Next, let us discuss the conformal symmetric rank $s$ tensor field (CST) in $d = 4$ considered in [27, 28]. This is a non-unitary theory that may be viewed as a maximal depth $r = s$ representative of the family of FT type [29] CHS fields with rank $s-r$ tensor gauge invariance [30–32]. The CHS theory is the minimal depth case (i.e. the case of maximal gauge invariance) when $r = 1$. The CST field has a second-derivative Lagrangian with scalar gauge invariance and corresponds to a short representation of $SO(2, 4)$ given by

$$\text{CST}_s = \left(1; \frac{s}{2}, \frac{s}{2}\right) - (1 - s; 0, 0).$$

(3.22)

The partition function for a CST field defined on AdS$_4$ is found to be [28] (cf equation (2.1))

$$Z_{\text{CST},s}(\text{AdS}_4) = \prod_{k=1}^{s} \left[ \frac{\det \Delta_0(2 - k^2)}{\det \Delta_{k+1}(2 + k)} \right]^{s/2}.$$  

(3.23)

Using equations (2.2) and (2.3) we then find for the one-particle partition function on thermal AdS$_4$

$$Z_{\text{CST},s}^{+}(\text{AdS}_4; q) = -\hat{Z}_{\text{CST},s}^{-}(\text{AdS}_4; q^{-1}) = \frac{1}{(1 - q)^s} \sum_{k=1}^{s} [(2k + 1)q^2 - q^{k+2}]$$

$$= \frac{q^2s(s + 2) - (s + 1)^2q^2 + q^{s+1}}{(1 - q)^2}. $$

(3.24)

This 4d CST field corresponds to the maximal-depth partially massless (PM) totally symmetric spin $s$ field in AdS$_5$ associated with the following combination of $SO(2, 4)$ representations [31]

$$\text{PM}^{(s)}_r = \left(3; \frac{s}{2}, \frac{s}{2}\right) - (3 + s; 0, 0),$$

(3.25)

for which equation (3.23) is a shadow counterpart. Then from equation (2.3) we get

$$Z_{\text{PM}^{(s)}_r}(\text{AdS}_5; q) = \hat{Z}_{\text{PM}^{(s)}_r}(\text{AdS}_5; q^{-1}) = \frac{(s + 1)^2q^3 - q^{s+3}}{(1 - q)^2}. $$

(3.26)

Comparing equations (3.24) and (3.26) we conclude that

$$\hat{Z}_{\text{PM}^{(s)}_r}(\text{AdS}_5; q) - Z_{\text{PM}^{(s)}_r}(\text{AdS}_5; q) = \hat{Z}_{\text{CST},s}(\text{AdS}_4; q) + Z_{\text{CST},s}(\text{AdS}_4; q),$$

(3.27)

in agreement with equation (3.5). As in the CHS case, one can also verify the validity of relation (3.1) in the CST case. We shall further discuss the properties of the 4d CST field partition functions in the next section.

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12 The superscript $r$ in $\text{PM}^{(s)}_r$ denotes the depth. Here, we consider only the maximal case $r = s$. 
4. From five to four to three dimensions

In this section we shall consider a special case of \( d = 4 \) where some relations simplify. We shall discuss a further descent to three dimensions, thus getting a triple of related fields: HS in five, \( \text{CF} \) in four, and \( \overline{\text{CF}} \) in three dimensions. In particular, starting with a massless HS field in \( \text{AdS}_5 \) one gets a CHS in 4 d and then defining it on \( \text{AdS}_4 \) can further associate it with another CHS field in 3d. The latter turns out to have a non-local action describing a zero number of dynamical degrees of freedom, i.e. giving a trivial partition function.

4.1. \( \text{AdS}_5 \rightarrow \text{AdS}_4 \)

Let us first consider the \( d = 4 \) version of the relation (3.1) between partition functions of some higher spin field in \( \text{AdS}_5 \) and the corresponding 4 d conformal field defined on \( \text{AdS}_4 \). Starting with a totally symmetric spin \( s \) HS field in \( \text{AdS}_5 \) corresponding to \( SO(2, 4) \) representation \( \left( \Delta_5; \frac{s}{2}, \frac{s}{2} \right) \) we may associate it with a (in general, higher derivative) \( \text{CF} \) in 4 d that may also be represented (when defined on \( \mathbb{R}^4 \) or \( \text{AdS}_4 \)) as a collection of second-derivative fields with a particular values of masses. Our proposal for such a general relation is \(^{13}\)

\[
\left( \Delta_5; \frac{s}{2}, \frac{s}{2} \right)_{\text{AdS}_5} \rightarrow \tilde{Z}_{\text{CF}, \beta}(\text{AdS}_4) = \prod_{\ell'=0}^{\Delta_5-3} \prod_{k=0}^{\Delta_4-3} \left[ \det \Delta_{s', \ell'}(M_{s', \ell'}^{\Delta_4}) \right]^{1/2}, \tag{4.1}
\]

where \( M_{s', \ell'}^{\Delta_4} = s' - 2 - k(k + 1) \) as in equation (3.15). Special HS fields with gauge invariance will require combinations of the above building blocks to take into account ghost field contributions. One can check that equation (4.1) is consistent with all special CFs in 4 d that we have analysed directly: CHS, CST and also GJMS scalar fields (see appendix C).

Given equation (4.1) one can then demonstrate the validity of the relation (3.1) (equivalent to (3.5) in the absence of gauge invariance) between the partition functions in \( \text{AdS}_5 \) and \( \text{AdS}_4 \). For each factor in the r.h.s. of equation (4.1) we indeed verify relation (3.1), i.e.

\[
\tilde{Z}_{(\Delta_5; \frac{s}{2}, \frac{s}{2})_{\text{AdS}_5}}(\text{AdS}_4; q) - \tilde{Z}_{(\Delta_4; \frac{s}{2}, \frac{s}{2})_{\text{AdS}_4}}(\text{AdS}_4; q) = \mathcal{Z}_{\text{CF}}(\text{AdS}_4; q) + \mathcal{Z}_{\overline{\text{CF}}}(\text{AdS}_4; q). \tag{4.4}
\]

To provide additional support for the correspondence rule (4.1) let us consider the \( \text{CF} \) partition function defined on \( S^4 \) instead of \( \text{AdS}_4 \), which may be viewed as a boundary of global \( \text{AdS}_5 \). In this case we should get an analogue of equation (2.7), i.e. the relation (1.2)

\[
\log \mathcal{Z}_{\text{CHS}}(\text{AdS}_5) - \log \mathcal{Z}_{\text{CHS}}(\text{AdS}_5) = \log \mathcal{Z}_{\text{CF}}(S^4). \tag{4.5}
\]

One may check this relation by comparing the coefficient of the IR divergent term on the l.h.s. to the coefficient of the UV divergent term on the r.h.s., i.e. to the 4 d conformal anomaly a-coefficient \([1, 2]\). According to equation (3.3) of [1], we get for the coefficient in the l.h.s.

\(^{13}\) A similar correspondence rule in 6d was discussed in appendix A of [33].
\[ a(\Delta s; \frac{x}{2}, \frac{y}{2}) = -\frac{1}{720}(\Delta s - 2)^3 (s + 1)^2 [3(\Delta s - 2)^2 - 5s^2 - 10s - 5]. \]  

(4.6)

On the other hand, each \( \det (-\nabla^2 + M^2)_{\perp} \) in the product in equation (4.1) defined on \( S^4 \) gives the contribution (see equation (3.37) of [20])

\[ a_{\perp}(M^2) = \frac{1}{720}(2s + 1)[30s^3 + 85s^2 + 10s - 58 - 30(s^2 - 2)M^2 - 15M^4]. \]  

(4.7)

Then for the particular combination of the operators in equation (4.1) we get indeed

\[ a(\Delta s; \frac{x}{2}, \frac{y}{2}) = \sum_{k=0}^{\Delta s - 3} \sum_{s'=0}^{s} a_{\perp}(M^2_{\perp, k}). \]  

(4.8)

Let us note that the relation (4.4), implied by the correspondence rule (4.1), should have a group-theoretic interpretation. To see an indication of this, let us consider the non-blind characters \( \chi_4 \) and \( \chi_3 \) of massive representations of \( SO(4, 2) \) and \( SO(3, 2) \), respectively [34]

\[ \chi_4(\Delta; j_1, j_2|q, x, y) = \frac{q^\Delta f_{SU(2)}(j_1|x)f_{SU(2)}(j_2|y)}{(1 - q x^z y^z)(1 - q x^{z-\frac{1}{2}} y^{z-\frac{1}{2}})(1 - q x^{-\frac{1}{2}} y^{-\frac{1}{2}})(1 - q x^{-\frac{1}{2}} y^{-\frac{1}{2}})}, \]

\[ \chi_3(\Delta; j_1|q, x) = \frac{q^\Delta f_{SU(2)}(j_2|x)}{(1 - q)(1 - q x)(1 - q x^{-1})}, \]

\[ f_{SU(2)}(j|x) \equiv \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \]  

(4.9)

Let us generalize (4.3) and define

\[ \chi(AdS_5|q, x) \equiv \chi_4(\Delta s; \frac{x}{2}, \frac{y}{2}|q, x, x), \]

\[ \chi(AdS_4|q, x) \equiv \sum_{s'=0}^{s} \sum_{k=0}^{\Delta s - 3} \chi_3(k + 2; s'|q, x). \]  

(4.10)

Let us also denote by a tilde the charge conjugation, i.e. the replacement \( q \to q^{-1}, \ x \to x^{-1}. \) One can then check that

\[ \tilde{\chi}(AdS_4|q, x) = \chi(AdS_4|q, x) = \chi(AdS_5|q, x) = \chi(AdS_4|q, x). \]  

(4.11)

This reduces to equation (4.4) in the blind limit \( x \to 1. \) The fact that equation (4.11) also holds for a generic argument \( x \) suggests that equation (4.4) has a group-theoretic interpretation in terms of a map between representations of the corresponding 5d and 4d isometry groups.

### 4.2. Relation between partition functions on AdS_4 and S^1 \times S^3

As already mentioned above, given a CF in AdS_4 we may make a Weyl transformation to replace AdS_4 by half of the Einstein Universe \( \mathbb{R} \times S^{3-1} \), and then represent the partition function in \( \mathbb{R} \times S^{3-1} \) in terms of the partition function in AdS_4 with two possible choices of boundary conditions. In the case of thermal quotients this leads to the relation (3.2).

In the special case of AdS_4 it was observed in [4] that the two choices (+ and −) of the possible boundary conditions are equivalent, i.e. the corresponding higher spin representations

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14 Here \( x \) and \( y \) are chemical potentials for charges corresponding to other Cartan generators in addition to the dilatation operator.
are equivalent for \( s > 0 \), with the spin 0 (scalar) case being an exception.\(^{15}\) This suggests that for any conformal field not containing a scalar component we should have the equality between partition functions corresponding to the two alternative boundary conditions

\[
Z_{\text{CF}}(\text{AdS}_4; q) = Z_{\text{CF}}(\text{AdS}_4; q),
\]

\[
Z_{\text{CF}}(\text{AdS}_4; q) = Z_{\text{CF}}(\text{AdS}_4; q) + \sigma_4(q), \quad Z_{\text{CF}}(\text{AdS}_4; q) = -Z_{\text{CF}}(\text{AdS}_4; q^{-1}).
\]

Then the relation (3.2) should simplify in the \( d = 4 \) case to

\[
Z_{\text{CF}}(S^1 \times S^3; q) = 2 Z_{\text{CF}}(\text{AdS}_4; q).
\]

This identity can be verified directly for the CHS or CST fields as follows.

The CHS partition function on \( \text{AdS}_4 \) is a special case of equation (3.14):\(^{16}\)

\[
Z_{\text{CHS},s}(\text{AdS}_4) = \prod_{k=0}^{s-1} \frac{\det \Delta_{k+1}(k - (s - 1)(s + 2))}{\det \Delta_{k+1}(s - (s - 1)(s + 2))}^{1/2}.
\]

From equation (2.2) we see that the fields corresponding to terms in the numerator have \( \Delta_{k+1} = s + 2 \), while the terms in the denominator give \( \Delta_{k+1} = k + 2 \). The corresponding one-particle partition function is then a \( d = 4 \) case of equation (3.17)

\[
Z_{\text{CHS},s}(\text{AdS}_4; q) = \frac{1}{(1 - q)^3} \sum_{k=0}^{s-1} \left(2s + 1\right)q^{k+2} - \frac{(2k + 1)q^{t+2}}{(1 - q)^3}.
\]

Comparing this to the CHS partition function on \( S^1 \times S^3 \) given in equation (4.8) of [1] we find that indeed

\[
Z_{\text{CHS},s}(S^1 \times S^3; q) = 2 Z_{\text{CHS},s}(\text{AdS}_4; q).
\]

In the case of the CST field the partition function on \( S^1 \times S^3 \) was found in [28]. Comparing to equation (3.24) we conclude again that

\[
Z_{\text{CST},s}(S^1 \times S^3; q) = 2 Z_{\text{CST},s}(\text{AdS}_4; q).
\]

The 4 d relation (4.14) may also be extended to the fermionic CHS fields, see appendix D.

This relation (4.14) is not, however, true for a conformal scalar and thus also for any conformal theory with a \( \text{AdS}_4 \) partition function containing a conformal scalar factor. In particular, it is not true for the 4-derivative conformal scalar field as follows from the comparison of (C.12) and (C.8) in appendix C. Another counter-example is the conformal theory of an antisymmetric rank 2 tensor, as discussed in appendix E.\(^{17}\)

4.3. Further descent: \( \text{AdS}_4 \rightarrow \mathbb{R} \times S^2 \)

Given a conformal field in 4 d related to some higher spin field in \( \text{AdS}_5 \) we may define it on \( \text{AdS}_4 \) and then further associate it with another CF \( \text{CF} \) at the \( \text{AdS}_4 \) boundary \( \mathbb{R} \times S^2 \). This

\(^{15}\) More generally, the fact that the highest weight unitary representation of \( so(d, 2) \) algebra that admits extension to \( so(d + 1, 2) \) conformal algebra has two inequivalent extensions was demonstrated in [35]; for a scalar field these representations are not equivalent as representations of \( so(d, 2) \) while for spin \( s > 0 \) fields they are.

\(^{16}\) While here we have a scalar field contribution at \( k = 0 \) this is not a conformal massless scalar but a ghost field needed to guarantee the conformal invariance of the spin \( s \) CHS field.

\(^{17}\) Here the presence of a scalar components is also apparent in the approach developed in [36].
gives a triplet of fields

\[ \text{HS on AdS}_4 \longrightarrow \text{CF on AdS}_4 \longrightarrow \tilde{\text{CF}} \text{ on } \mathbb{R} \times S^2. \]

Then the thermal partition functions of CF and \( \tilde{\text{CF}} \) are related by equations (1.3) or (2.7), i.e. for the second step we get

\[ Z_{\text{CF}}(\text{AdS}_4; q) = Z_{\tilde{\text{CF}}}(\text{AdS}_4; q) = Z_{\tilde{\text{CF}}}(S^4 \times S^2; q). \]  \hspace{1cm} (4.19)

Combining this with equation (4.12) we conclude that \( \tilde{\text{CF}} \) should have zero partition function,

\[ Z_{\tilde{\text{CF}}}(S^4 \times S^2; q) = 0. \] \hspace{1cm} (4.20)

Then the total partition function of the theory defined on the boundary of the boundary is \( Z = 1 \), i.e. the resulting 3d conformal theory \( \tilde{\text{CF}} \) should be trivial or topological. We will also define \( \tilde{\text{CF}} \) on \( \text{AdS}_4 \) and then find, in agreement with equation (3.3), that

\[ Z_{\tilde{\text{CF}}}(\text{AdS}_4; q) = Z_{\tilde{\text{CF}}}(\text{AdS}_4; q) = 0. \] \hspace{1cm} (4.21)

Before giving some explicit examples let us first recall that a totally symmetric spin \( s \) CHS field \( \phi_s \) in \( d \) dimensions has the action

\[ S_{\text{CHS}} = \int d^d x \phi_s P_s \partial^{2s+d-4} \phi_s = \int d^d x \partial^{d-4} C_s, \] \hspace{1cm} (4.22)

where \( P_s \) is a projector onto transverse traceless tensors and \( C_s = \partial^2 \phi_s \) is gauge-invariant field strength (generalized Maxwell or Weyl tensor). The number of the corresponding dynamical degrees of freedom is [16, 21]

\[ \nu_{s,d} = \frac{(d-3)(2s+d-2)(2s+d-4)(s+d-4)!}{2(d-2)! s!}. \] \hspace{1cm} (4.23)

Equation (4.22) is local for even \( d \geq 4 \) (where \( \partial^2 = \Box \) enters in positive power) but can also be formally defined for odd \( d \). The case of \( d = 3 \) is special in that the number of dynamical degrees of freedom (4.23) vanishes, while equation (4.22) takes a non-local form

\[ S_{\text{CHS}} = \int d^3 x \phi_s P_s \Box^{s-1/2} \phi_s = \int d^3 x C_s \Box^{-1/2} C_s. \] \hspace{1cm} (4.24)

Let us also recall that the CHS action in \( d \) dimensions may be viewed as an induced one [37] from a free CFT if \( \phi_s \) is coupled to a spin \( s \) conserved current \( J_s \) then the kinetic term of \( \phi_s \) is determined by the two-point function \( \langle J(s) J(\xi) \rangle \).\(^{18}\) Insisting on locality one may consider a Chern–Simons type action for the corresponding 3d CHS field that may be induced from chiral 3d fermions (see [40–45] for \( s = 2 \)). For a more natural parity-even case induced from a free 3d scalar CFT we get in momentum space \( \langle J_s J_s \rangle = \frac{i}{q^d} \tilde{P}_s (p) \), where \( \tilde{P}_s (p) \) is Fourier transform of the transverse traceless projector in equation (4.22) (i.e. a symmetrized and traceless product of \( s \) factors of \( \tilde{P}_s (p) = \tilde{\phi}_s (p) = 0 \). The corresponding parity-even 3d CHS action is then given by equation (4.22).

4.3.1. Spin 1. Let us now illustrate equation (4.20) turning to some special cases and start with a massless spin 1 gauge field in AdS\(_4\) that has \( \Delta_+^s = 2 + s = 3 \) and is associated with the following combination of \( SO(2, 4) \) representations \( \text{MHS}_4(\text{AdS}_4) = (3, \frac{1}{2}, \frac{1}{2}) - (4, 0, 0) \). The corresponding 4d boundary field is the \( s = 1 \) CHS field, i.e. the standard Maxwell theory (cf equation (4.22)). Its partition function when defined on AdS\(_4\)

18 In general, in 3d there are two possible conformally invariant tensor structures that may appear in a two-point function of a conserved current \( J_s \); a non-local parity-even and a local parity-odd one (see, e.g., [2, 38, 39]).
is a special case of equation (3.14)
\[
Z_{\text{CHS},1}(\text{AdS}_4) = \left[ \frac{\det \Delta_0(0)}{\det \Delta_{s\perp}(3)} \right]^{1/2}.
\] (4.25)

Here each operator is in turn associated with a conformal field at the \( R \times S^2 \) boundary (we get a scalar with \( \Delta_+^s = 3 \) and a transverse vector with \( \Delta_-^s = 2 \)). The corresponding combination of \( SO(2, 3) \) representations \((\Delta_+^s; j)\) is\(^{19}\)
\[
\text{CHS}_1(\text{AdS}_4) = (2; 1) - (3; 0) = \text{MHS}_1(\text{AdS}_4).
\] (4.26)

Thus the field at the 3d boundary should be the \( s = 1 \) member of the 3d CHS family (4.24) with a non-local action (cf [2, 46])
\[
S_{\text{CHS}} = \int d^3 x \ F_{\mu \nu} \Box^{-1/2} F_{\mu \nu}.
\] (4.27)

This theory is effectively topological, having no dynamical degrees of freedom (in agreement with equation (4.23)). One can see this explicitly, e.g., by computing the corresponding partition function in flat 3d space\(^{20}\)
\[
Z_{\text{CHS}}(\mathbb{R}^3) = \left[ \frac{\det \Box}{\det (\Box^{1/2})_{\perp}} \right]^{1/2} = \left[ \frac{\det \Box}{\det (\Box^{1/2})_{\perp}} \right]^{1/2} = 1.
\] (4.28)

As there is no conformal anomaly in odd dimensions the same should also be true for all conformally flat spaces, e.g., \( \text{AdS}_3 \)
\[
Z_{\text{CHS}}(\text{AdS}_3) = 1.
\] (4.29)

Defining the 4d Maxwell field on \( \text{AdS}_4 \) we get from equation (A.6)
\[
\tilde{Z}_{\text{CHS}}(\text{AdS}_4; q) - Z_{\text{CHS}}(\text{AdS}_4; q) = -Z_{\text{CHS}}(\text{AdS}_4; q^{-1}) - Z_{\text{CHS}}(\text{AdS}_4; q)
\]
\[
= -3q^2 - 1/q^3 - 3q^2 - q^3
\]
\[
= -1.
\] (4.30)

This \(-1\) is precisely what is removed by the \( \sigma_{\text{CHS},s}(q) \) term in equation (2.6) in agreement with equation (4.22) so that we get \( Z_{\text{CHS}}(S^3 \times S^2; q) = 0 \) as a special case of equation (4.20)\(^{21}\).

We can also explicitly check the equality (3.4) of the \( \sigma \)-terms. For the MHS\(_1(\text{AdS}_3)\) theory the \( \sigma_{\text{CHS},s}(q) \) term for a general spin \( s \) may be found in equation (5.5) of [1] and for \( s = 1 \) it is equal to 1, i.e. it is indeed the same as the above \( \sigma_{\text{CHS},s}(q) \).

\(^{19}\) Here \( j \) is the \( SO(3) \) angular momentum. In general, \( \text{MHS}_1(\text{AdS}_3) = (1 + s; s) - (2 + s; s - 1) \), i.e. it corresponds to a spin \( s \) field with gauge invariance with a spin \( s - 1 \) parameter. The partition function for a massive \( SO(2, 3) \) representation \((\Delta_+; s)\) is given by \( Z_{\Delta_+,s}(q) = (2s + 1) \frac{\Delta_+^{1/2}}{1 - q^s} \).

\(^{20}\) Here the measure contribution from the decomposition \( A_+ = A_{s\perp} + \partial_\phi \phi \) cancels against the kinetic operator contribution. Note also that in 3d one can dualize \( F_{\mu \nu} \to F_{\mu \nu} \) to a scalar with kinetic term \( \Box^{1/2} \phi \), but the corresponding partition function is still 1 as the scalar determinant is cancelled by the measure contribution coming from integrating out the auxiliary field \( F_{\mu \nu} \).

\(^{21}\) Indeed, the partition function on \( S^1 \times S^{d-1} \) for the CHS\(_1\) Maxwell field with the action in equation (4.22), i.e. \( \int d^dx \ F^{\mu \nu} \Box^{-1/2} F_{\mu \nu} \), was already computed in [1] with the general expression being
\[
Z_{\text{CHS}}(S^1 \times S^{d-1}; q) = 1 - \frac{d - q + dq^{d-1} - q^d}{(1 - q)^d}
\]
This vanishes for \( d = 3 \).
4.3.2. Spin 2. Let us now consider the $s = 2$ case, i.e. start with the MHS theory in $\text{AdS}_4$ describing massless rank 2 tensor with $\Delta^+ = 2 + s = 4$ and spin 1 gauge invariance parameter, i.e. associated with the following combination of $SO(2, 4)$ representations

$$\text{MHS}_2(\text{AdS}_4) = (4; 1, 1) - \left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right).$$

The dual conformal field in 4d is the $s = 2$ CHS theory, i.e. Weyl gravity. Its partition function on $\text{AdS}_4$ is a special case of equation (3.23), i.e. [29, 47, 48]

$$Z_{\text{CHS}}(\text{AdS}_4) = \left[\frac{\det \Delta_0(-4)\det \Delta_{1\perp}(-3)}{\det \Delta_{2\perp}(4)\det \Delta_{2\perp}(2)}\right]^{1/2}.$$  \hspace{1cm} (4.31)

Using equation (2.2) the values of the scaling dimensions $\Delta^+$ corresponding to each factor in equaiton (4.31) are (cf equation (2.1))

$$\Delta^+ \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$  \hspace{1cm} (4.32)

This means that the equivalent combination of $SO(2, 3)$ representations is

$$\text{CHS}_2(\text{AdS}_4) = \text{MHS}_2(\text{AdS}_4) \oplus \text{PM}^{(2)}_s(\text{AdS}_4) = [(3; 2) - (4; 1)] \oplus [(2; 2) - (4; 0)].$$  \hspace{1cm} (4.33)

Indeed, the Weyl graviton on $\text{AdS}_4$ is a combination of an Einstein graviton and a partially massless spin 2 field with scalar gauge invariance [20, 49, 50].

The 3d conformal theory induced by a 4d Weyl graviton at the boundary of $\text{AdS}_4$ thus contains two parts. From the Einstein graviton $\text{MHS}_2(\text{AdS}_4)$ we get a conformally invariant 3d CHS or Weyl theory with parity-even non-local linearized action (4.24), i.e. \( \int d^3x \, C_2 \Box^{-1/2} C_2 \) (see also [39])23. From the partially massless field $\text{PM}^{(2)}_s(\text{AdS}_4)$ we get a CST$_2$ field representing a non-unitary 3d symmetric tensor $\varphi_{\mu\nu}$ with scalar gauge invariance $\delta \varphi_{\mu\nu} = \partial_{(\mu} \partial_{\nu)} \epsilon$. This theory has a non-local action \( \int d^3x \, \varphi_2 P_2 \Box^{3/2} \varphi_2 \) where $P_2$ is an appropriate projector ensuring scalar gauge invariance24. In summary, the CF combination corresponding to equation (4.33) on a flat 3d boundary is25

$$\tilde{\mathcal{C}F}(\mathbb{R}^3) = \text{CHS}_2(\mathbb{R}^3) \oplus \text{CST}_2(\mathbb{R}^3).$$  \hspace{1cm} (4.34)

Let us now show that this system has a zero total number $\nu$ of dynamical degrees of freedom. Indeed, the $\text{CHS}_2$ field has $\nu = 0$ according to equation (4.23). For CST$_2$ the flat space partition function is

22 In general, for a partially massless spin $s$ field we have $\text{PM}^{(s)}_s(\text{AdS}_4) = (2; s) - (2 + s; 0)$.

23 The full non-linear action of Weyl-invariant gravity in 3d with quadratic part given by $\int d^3x \, C_2 \Box^{-1/2} C_2$ where $C_2$ is a linearized Weyl tensor can be obtained as an induced one corresponding to a conformally coupled scalar in 3d, i.e. as $\log\det\left\{ -\nabla^2 + \frac{4}{3} \mathcal{R} \right\}$. Since there is no Weyl anomaly in three dimensions this non-local functional of the metric will be both reparametrization and Weyl invariant.

24 Indeed, from equation (4.33) we find that the canonical dimension of $\varphi_{\mu\nu}$ is $d - \Delta^+ = 3 - 2 = 1$.

25 An alternative way of obtaining a (quadratic) action for this set of 3d fields is to start with the (linearized) Weyl gravity in $\text{AdS}_4$ space, specify separate boundary conditions for the 4d graviton and partially massless mode (in terms of 3d graviton and 3d CST$_2$ field, respectively) and then evaluate the 4d action on the solution of the equations of motion. This procedure may have non-linear generalization if one starts with the full non-linear 4d Weyl gravity action and considers a generic asymptotically $\text{AdS}_4$ background.
\[
Z_{\text{CST}}(\mathbb{R}^3) = \left[ \frac{\det \Box_{1,\perp} (\det \Box_{0,\perp})^2}{(\det \Box_{1,\perp}^2 (\det \partial \Box_{1,\perp}^2 \partial_{1,\perp}))^{1/2}} \right]^{1/2} = \left[ \frac{\det \Box_{1,\perp} (\det \Box_{0,\perp})^2}{(\det \Box_{2,\perp})^2 (\det \Box_{1,\perp})^{3/2}} \right]^{1/2} = 1. 
\] (4.35)

Here the numerator is the Jacobian for the change of variables \( \varphi_{\mu
u} = \varphi_{\mu
u}^\prime + \partial_{\mu} V_{\nu}^\prime + \left( \partial_{\mu} \partial_{\nu} - \frac{1}{2} \delta_{\mu\nu} \partial^2 \right) \gamma \). The denominator is from the action \( \int d^3 \varphi_2 P_2 \Box_{1/2} \varphi_2 \) where the scalar component \( \gamma \) drops out due to gauge invariance. Thus \( \nu(C\text{ST}_2)|_{\nu=3} = \frac{3}{2} \times 2 + \frac{1}{2} \times 2 - (2 + 2) = 0 \). We have used that in three dimensions a contribution \( (\det \Box_{1,\perp})^{-1/2} \) in the partition function is equivalent\(^{26}\) to \( (\det \Box_{0})^{-1} \) (i.e. \( \nu = 2 \)) for \( s > 0 \).

Let us note for completeness that equation (4.35) may be generalized to any spin \( s > 0 \) as follows
\[
Z_{\text{CST}}(\mathbb{R}^3) = \left[ \frac{\det \Box_{s-1,\perp} \det \Box_{s-2,\perp} \cdots \det \Box_{1,\perp}}{(\det \Box_{s-1,\perp}^2 \det \Box_{s-2,\perp}^2 \cdots \det \Box_{1,\perp}^2)^{1/2}} \right]^{1/2} = 1. 
\] (4.36)

Thus, as for a CHS field, the total number of d.o.f. of a 3d CST field is zero for any \( s: \nu = 2 \sum_{n=1}^{s} \left( \frac{n-1}{2} \right) - 2 \sum_{n=1}^{s} n = 0 \).

At the level of the partition function, the decomposition (4.33) implies
\[
Z_{\text{CHS}}^+(\text{AdS}_4; q) = \frac{5q^3 - 3q^4}{(1 - q)^3} + \frac{5q^2 - q^3}{(1 - q)} = \frac{q^2(5 + 5q - 4q^2)}{(1 - q)^3}. 
\] (4.37)

We may compute the partition function of \( \widehat{\text{CF}} = \text{CHS}_2 \oplus \text{CST}_1 \) as in equations (2.6) and (2.7)
\[
Z_{\text{CF}}(S^1 \times S^2; q) = -Z_{\text{CHS}}^-(\text{AdS}_4; q^{-1}) - Z_{\text{CHS}}^+(\text{AdS}_4; q) + \sigma_{\text{CHS}_5}(q) 
= -4(q + q^{-1}) - 7 + \sigma_{\text{CHS}_5}(q) = 0, 
\] (4.38)

which is in agreement with equation (4.20) (see also appendix B). We used that, as one can check, \( \sigma_{\text{CHS}_5}(q) = 4(q + q^{-1}) + 7 \). Again, we can compare the expression for \( \sigma_{\text{CHS}_5}(q) \) with the \( \sigma_{\text{ch}_{5,s}}(q) \) term in equation (5.5) of [1] for \( s = 2 \) and thus verify the relation (3.4).

5. Concluding remarks

The \( (\text{AdS}_5/\text{CFT})^2 = 0 \) relation (4.20) is special to the \( d = 4 \) case because we used equation (4.22) to obtain it. In the \( d > 4 \) case we expect to find a more complicated picture. For example, suppose we start with a massless HS field in AdS\(_7\). Then at the boundary we get a CHS field in 6d and can define it on AdS\(_6\), thus associating to it some other conformal field CF\(_5\) at the boundary of AdS\(_6\). We can then continue the descent, i.e. define CF\(_5\) on AdS\(_5\) and associate it with another CF, CF\(_4\), at the boundary of AdS\(_5\), etc. One may then look for some new identities between partition functions of these fields in addition to equations (2.7) and (3.1). For example, one can check that in the spin 1 case, CF\(_5\) appears to be represented by a

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\(^{26}\) The number of components of a totally symmetric traceless rank \( s \) tensor \( \phi_{\ell} \) in \( d \) dimensions is \( N_s = \left( \frac{s + d - 1}{s} \right) - \left( \frac{s + d - 3}{s - 2} \right) \), i.e. \( N_s|_{\nu=3} = 2s + 1 \). The number of components of a transverse \( (\Box - \phi_{\ell,\perp} = 0) \) traceless rank \( s > 0 \) tensor is \( N_s|_{\nu=3} = N_s - N_s|_{\nu=1} \frac{3d - 2}{d} \).

\(^{27}\) Here we can use the expression for \( Z_{\text{CHS}}(S^1 \times S^{d-1}; q) \) in equation (5.17) of [1] and check that it vanishes for \( d = 3 \).
combination of a 5d CHS1 field (with non-local action \( \int d^5 x \, F_{\mu \nu} \square^{1/2} F_{\mu \nu} \) as in equation (4.22)) and an extra field, such that one ends up with CF4 being just the standard CHS1 Maxwell field. Details of the corresponding relations between partition functions remain to be studied.

Given the general relations (1.3) and (3.16) between partition functions of particular higher spin fields one may apply them to theories containing an infinite number of spins. For example, the Vasiliev-type theory in AdS\(_{d+1}\) (containing a scalar and totally symmetric MHS spin 1, 2, ... fields and dual to the singlet sector of free \( U(N) \) scalar theory in 4d) is naturally associated with the CHS theory of all conformal spins \( s = 0, 1, 2, ... \) in \( d \) dimensions (with linearized action (4.22)). Summing over all spins the relation (3.16) should trivialize. Indeed, for the MHS theory we find [26] (spin 0 field here has \( D^2 = d^2 \))

\[
Z^{+}_{\text{MHS}}(\text{AdS}_{d+1}; \ q) = \sum_{s=0}^{\infty} Z^{+}_{\text{MHS},s}(\text{AdS}_{d+1}; \ q) = \frac{q^{d-2}(1 + q)^2}{(1 - q)^{2d-2}},
\]

(5.1)

Thus, e.g., for \( d = 4 \) the lhs of equation (3.16) vanishes after summing over all spins (the \( \sigma \) term in \( Z^{-} \) in equation (1.5) drops out, being symmetric under \( q \to q^{-1} \)). At the same time, the summed CHS partition function on thermal AdS\(_{d}\) in equation (3.17) appears to be divergent (cf [1]). In four dimensions \( Z^{+}_{\text{CHS},s}(\text{AdS}_{4}; \ q) \) is given by equation (4.16) and the divergence is due to the term \( \sim (2s + 1)q^2 \) in the numerator, which is not suppressed at large \( s \). Nevertheless, applying the analogue of the standard \( \zeta \)-function regularization (i.e. \( \sum_{s=1}^{\infty} = -\frac{1}{2}, \) etc.), we obtain for the regularized expression of the sum over spins

\[
Z^{+}_{\text{CHS}}(\text{AdS}_{4}; \ q) = \lim_{\zeta \to 0} \sum_{s=1}^{\infty} s^{\zeta} Z^{+}_{\text{CHS},s}(\text{AdS}_{4}; \ q) = -\frac{2}{3} \left( q^2 + 4q + 1 \right),
\]

(5.3)

Thus the r.h.s. of equation (3.16) also vanishes. The same conclusion is also reached in general even dimension \( d > 4 \) once the non-trivial spin 0 contribution is included in the sum in equation (5.3).

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Appendix A. Reconstructing \( Z^{+}_{\text{CHS}}(\text{AdS}_{d}) \) from \( Z^{+}_{\text{MHS}}(\text{AdS}_{d+1}) \)

Here we shall reverse the logic: assume that the relation (3.5) is true and use it to determine the partition function \( Z^{+}_{\text{CHS}}(\text{AdS}_{d}) \) from the knowledge of \( Z^{+}_{\text{MHS}}(\text{AdS}_{d+1}) \) just by doing algebraic manipulations.

These partition functions have the following general form

\[
Z^{+}_{\text{MHS}}(\text{AdS}_{d+1}; \ q) = \frac{P(q)q^{d^2}}{(1 - q)^{d^4}}, \quad Z^{+}_{\text{CHS}}(\text{AdS}_{d}; \ q) = \frac{F(q)q^{d-1}}{(1 - q)^{d-1}},
\]

(A.1)
where \( P(q) \) and \( F(q) \) are finite sums of non-negative powers of \( q \) (in AdS, we have \( \Delta_j \geq \frac{1}{2}(d - 1) \), cf equation (2.3)). Equation (3.5) or (3.6) implies that

\[
F(q) + F(q^{-1}) = \frac{\sqrt{q}}{1 - q} [P(q^{-1}) - P(q)]. \tag{A.2}
\]

The r.h.s. of equation (A.2) may be expanded in a Laurent series around \( q = 0 \) and comparing with the l.h.s., we may then determine \( F(q) \).

Let us consider some examples. Let us start with the conformal scalar in AdS, which corresponds to a massive scalar in AdS with \( \Delta_5 = 3 \), i.e. (cf equation (3.7))

\[
Z_{5}(\text{AdS}; q) = \frac{q^3}{(1 - q)^4}, \quad P(q) = q. \tag{A.3}
\]

Then equation (A.2) gives

\[
F(q) + F(q^{-1}) = \frac{q + 1}{\sqrt{q}} = q^{-1/2} + q^{1/2} \quad \rightarrow \quad F(q) = q^{1/2}, \tag{A.4}
\]

and therefore, in agreement with equation (3.10), \( Z_{5}(\text{AdS}; q) = \frac{q^2}{(1 - q)^2} \).

Another example is the spin 1 field in AdS corresponding to the spin 1 CHS field (i.e. Maxwell field) in AdS. From equation (3.19) we have

\[
Z_{\text{Max}}(\text{AdS}; q) = \frac{4q^3 - q^4}{(1 - q)^4}, \quad P(q) = 4q - q^2. \tag{A.5}
\]

Then, equation (A.2) gives

\[
F(q) + F(q^{-1}) = -q^{1/2} - q^{-3/2} + 3q^{1/2} + 3q^{-1/2} \quad \rightarrow \quad F(q) = 3q^{1/2} - q^{3/2}, \tag{A.6}
\]

and therefore we get, in agreement with equation (3.17), \( Z_{\text{Max}}(\text{AdS}; q) = \frac{3q^2 - q^4}{(1 - q)^2} \).

Our third example is a non-unitary CFT represented by a vector \( V \) in 6d with a second-derivative kinetic term. This is a special case of a CST family of CFs described by rank \( s \) symmetric tensors \( \varphi_{\mu_1 \ldots \mu_s} \), which in \( d = 6 \) have no gauge invariance. As discussed in [33], this CF is induced by a HS field in AdS transforming in the \( (\Delta; h_1, h_2, h_3) = (4; 1, 0, 0) \) representation of \( SO(2, 6) \). Taking into account that \( \text{dim}[1, 0, 0] = 6 \), we get (cf equation (2.3))

\[
Z_{\text{HS}}(\text{AdS}; q) = \frac{6q^4}{(1 - q)^6}, \quad P(q) = q. \tag{A.7}
\]

Then from equation (A.2) we have again \( F(q) = q^{1/2} \) as in equation (A.4) and thus we predict that

\[
Z_{\text{HS}}(\text{AdS}; q) = \frac{6q^3}{(1 - q)^5}. \tag{A.8}
\]

This expression follows indeed from the general expression for the \( V \) partition function on \( S^6 \) or on AdS given in equation (A.4) of [33]

\[
Z_{V}(\text{AdS}) = [\det \hat{\Delta}_{4}(7)\det \hat{\Delta}_{6}(6)]^{-1/2}. \tag{A.9}
\]

Applying equation (2.2) to the operators here we find that we have the same \( \Delta_5 = 3 \) and \( \Delta_6 = 2 \) for both factors. However, the number of degrees of freedom of a transverse vector in
6d is $6 - 1 = 5$ while the scalar contributes only one. Thus the numerator of $Z_\text{HS}(\text{AdS}_d; q)$ should be $(5 + 1)q^3 = 6q^3$, in agreement with equation (A.8).

### Appendix B. $\sigma$-term relation in equation (3.4)

Let us now use the general structure equations (A.1) and (A.2) of the higher spin partition functions in $\text{AdS}_{d+1}$ and the corresponding CF partition functions in $\text{AdS}_d$ to justify the equality in equation (3.4).

According to equations (2.6) and (2.7) we have for the partition function of the CF on $S^1 \times S^d$ and the partition of another conformal field $\text{CF}$ (the one associated with the CF on $\text{AdS}_d$) on $S^1 \times S^{d-1}$

$$Z_{\text{CF}}(S^1 \times S^d; q) = Z_{\text{HS}}^-(\text{AdS}_{d+1}; q) - Z_{\text{HS}}^+(\text{AdS}_{d+1}; q) + \sigma_{\text{HS},d+1}(q),$$  \hspace{1cm} (B.1)

$$Z_{\text{CF}}(S^1 \times S^{d-1}; q) = Z_{\text{CF}}^-(\text{AdS}_d; q) - Z_{\text{CF}}^+(\text{AdS}_d; q) + \sigma_{\text{CF},d}(q).$$  \hspace{1cm} (B.2)

Using $Z^-(\text{AdS}_n; q) = (-1)^{n+1} Z^+(\text{AdS}_n; q^{-1})$, and also equation (A.2), we find

$$Z_{\text{CF}}(S^1 \times S^d; q) = \frac{q^{d+1}}{1 - q^{d-1}} [F(q^{-1}) + F(q)] + \sigma_{\text{HS},d+1}(q),$$  \hspace{1cm} (B.3)

$$Z_{\text{CF}}(S^1 \times S^{d-1}; q) = \frac{q^{d+1}}{1 - q^{d-1}} [F(q^{-1}) - F(q)] + \sigma_{\text{CF},d}(q).$$  \hspace{1cm} (B.4)

The role of the $\sigma$-terms is to remove the negative powers of $q$ in the expansion around $q = 0$ of the r.h.s. of equations (B.3) and (B.4) as such terms cannot be present on the l.h.s that can be computed using the operator counting method and thus should have only positive powers of $q$. Such terms come only from the $F(q^{-1})$ term that is the same in the two lines of equation (B.3). This then implies that

$$\sigma_{\text{HS},d+1}(q) = \sigma_{\text{CF},d}(q).$$  \hspace{1cm} (B.5)

To give an example, let us consider the CHS$_2$ field in 4d for which from equation (4.37) we have

$$Z_{\text{CHS}_2}(\text{AdS}_4; q) = \frac{q^2(5 + 5q - 4q^2)}{(1 - q)^3} \rightarrow F(q) = 5q^{1/2} + 5q^{3/2} - 4q^{5/2}. \hspace{1cm} (B.6)$$

Then equation (B.3) contains

$$\frac{q^{3/2}F(q^{-1})}{(1 - q)^3} = \frac{-4 + 5q + 5q^2}{q(1 - q)^3} = -4q^{-1} - 7 - 4q + 5q^2 + \cdots,$$  \hspace{1cm} (B.7)

leading to the same result as in [1]

$$\sigma_{\text{CHS}_2}(q) = 4q^{-1} + 7 + 4q,$$  \hspace{1cm} (B.8)

with the same expression also for $\sigma_{\text{CHS}_2}(q)$.\[18\]
Appendix C. Higher derivative conformal scalar fields

Here we will illustrate the relations (3.1), (3.2) and (3.5) on the example of the Weyl-covariant scalar theory with kinetic operator $\Delta_{(2r)} = -(\nabla^2)^r + ...$ where the dots denote curvature dependent terms.

The GJMS operators $\Delta_{(2r)}$ naturally exist in a technical sense for $r \leq d/2$ (see [51] and references therein).

This means that their definition in generic dimension $d$ involves terms whose coefficients have poles in $d$ when $r > d/2$. For instance, for $r = 3$, there are tensor structures with coefficient $\frac{1}{d-4}$ forbidding a naive extension to 4d case. These obstructions are proportional to the Bach tensor and vanish for the Einstein spaces with $R_{mn} = \frac{R}{d} g_{mn}$. In this case it is possible to construct generalized Gover–GJMS operators (defined beyond critical order) but the resulting expressions are non-natural in a technical sense [52]. For all orders (below, at, and beyond critical order) the Gover–GJMS operators factorize as in equation (C.10) below.

In the conformally flat spaces (that need not be Einstein in general), there is no obstruction in going beyond the critical order $r = d/2$.

### C.1. Partition function on $S^1 \times S^3$

Let us first consider the general case where the space $S^d \times S^p$ is conformally flat if defined with indefinite $(p, q)$ signature metric (here the spheres have unit radius and $d = p + q$). Then one can show that for $r = 2N$ [53]

$$\Delta_{(4N)} = \prod_{k=1}^{N} [(\mathcal{O}_p - \mathcal{O}_q)^2 - 2(2k - 1)^2(\mathcal{O}_p + \mathcal{O}_q) + (2k - 1)^4], \quad (C.1)$$

$$\Delta_{(4N + 2)} = (\mathcal{O}_p - \mathcal{O}_q) \prod_{k=1}^{N} [(\mathcal{O}_p - \mathcal{O}_q)^2 - 2(2k)^2(\mathcal{O}_p + \mathcal{O}_q) + (2k)^4] \quad (C.2)$$

where $\mathcal{O}_p \equiv -\nabla^2_p + \frac{1}{4}(p - 1)^2$. For example, in the special case of $\Delta_{(4)}$ in $d = 4$ we have for a general curved background [49, 54]

$$\Delta_{(4)} = -(\nabla^2)^2 + 2\left(R^m - \frac{1}{3}g^{mn}R\right)\nabla_m \nabla_n. \quad (C.3)$$

Then for $S^2 \times S^2$ with $(++--)$ signature we get

$$\Delta_{(4)} = -(\nabla^2_{S^2} + \nabla^2_{S^2})^2 + 2(\nabla^2_{S^2} + \nabla^2_{S^2}) = (\mathcal{O}_2 - \mathcal{O}_2')^2 - 2(\mathcal{O}_2 + \mathcal{O}_2') + 1, \quad (C.4)$$

in agreement with equation (C.1) where $\mathcal{O}_2 = -\nabla^2_{S^2} + \frac{1}{4}$.

For conformally flat but not Einstein space $S^1 \times S^3$ with $(-+++)$ signature we have for the spectra of $\mathcal{O}_p$ in equation (C.1)

$$\mathcal{O}_1 \rightarrow w, \quad \mathcal{O}_3 \rightarrow n(n + 2) + 1 = (n + 1)^2, \quad n = 0, 1, 2, ... \quad (C.5)$$

For $r = 2N$ the factorization of $\Delta_{(4N)}$ in equation (C.1) leads to the energy eigenvalues $w$ represented by

28 Here $R_{mn} = \pm g^{mn}$, where $g^{(0)}_{mn}$ is the metric of a standard 2-sphere and the sign depends on whether $(mn)$ are in the first or second sphere. Thus $\nabla^2 = -\nabla^2_{S^2} + \nabla^2_{S^2}$ and $R^m \nabla_m \nabla_n = \nabla^2_{S^2} + \nabla^2_{S^2}$.
so that the final one-particle partition function for a GJMS scalar field is
\[
\mathcal{Z}_{\text{GJMS}}(S^l \times S^3; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{l-1} (n+1)^2 q^{n+2-r+2k} = \frac{q^{2-r} - q^{2+r}}{(1-q)^3}.
\]
This is the same as the partition function that counts descendants of a conformal scalar operator in flat space modulo its equation of motion. In general dimension \(d\), for a GJMS scalar field \(\phi\) with canonical dimension \(-\frac{d}{2}\) and equations of motion \(\partial^{2r} \phi = 0\) of complementary dimension \(-\frac{d}{2}\) we get
\[
\mathcal{Z}_{\text{GJMS}}(S^l \times S^{d-1}; q) = \frac{q^{2-r} - q^{2+r}}{(1-q)^d}.
\]

C.2. Partition function on AdS

The discussion in section 3.1 may be generalized by considering a massive scalar field in AdS\(_{d+1}\) with \(\Delta_{d+1}^\pm = \frac{d}{2}(d \pm 2r)\) (with \(r = 2, 3, \ldots\)) for which the associated \(d\)-dimensional boundary conformal field is the higher derivative conformal scalar with canonical dimension \(\Delta_{d+1}^\pm = \frac{d}{2}(d - 2r)\) and the kinetic operator \(\Delta_{(2r)}\). Here equation (3.7) is replaced by
\[
\mathcal{Z}_{0,r}(\text{AdS}_{d+1}; q) = \frac{q^{\frac{d+2r}{2}}}{(1-q)^d}.
\]
On a generic \(d\)-dimensional Einstein space \(\Delta_{(2r)}\) factorizes as follows [52]
\[
\Delta_{(2r)} = \prod_{k=1}^{r} \left( -\nabla^2 + \frac{d}{2} - k \left( \frac{d}{2} + k - 1 \right) \right) R.
\]
For the AdS\(_d\) case we have \(R = -d(d-1)\) and using equation (2.2) then find that for each massive Laplacian factor in equation (C.10) the associated dimensions are
\[
\Delta_{+,k} = \frac{d + 2k}{2} - 1, \quad \Delta_{-,k} = \frac{d - 2k}{2}, \quad k = 1, \ldots, r.
\]
Hence, equation (3.10) is generalized to (there is no gauge invariance so that \(\alpha_d = 0\) and \(\Delta^- = \Delta^+\))
\[
\mathcal{Z}_{\text{GJMS}}^+(\text{AdS}_d; q) = \sum_{k=1}^{r} \frac{q^{\Delta_{+,k}}}{(1-q)^{d-1}} = \frac{q^d(1-q^d)}{(1-q)^d},
\]
\[
\mathcal{Z}_{\text{GJMS}}^-(\text{AdS}_d; q) = (-1)^{d+1} \mathcal{Z}_{\text{GJMS}}^+(\text{AdS}_d; q^{-1}).
\]
Comparing equation (C.9) to equations (C.12) and (C.13) one checks again the relations (3.1), (3.5) and (3.1)
\[
\mathcal{Z}_{0,r}(\text{AdS}_{d+1}; q) - \mathcal{Z}_{0,r}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{GJMS}}^+(\text{AdS}_d; q) + \mathcal{Z}_{\text{GJMS}}^-(\text{AdS}_d; q). \tag{C.14}
\]
We can also demonstrate the relation (3.2) by using equations (C.8), and equations (C.12) and (C.13) to check that
\[
\mathcal{Z}_{\text{GJMS}}(S^l \times S^{d-1}; q) = \mathcal{Z}_{\text{GJMS}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{GJMS}}^+(\text{AdS}_d; q). \tag{C.15}
\]
Appendix D. Fermionic CHS fields

The discussion of partition function relations for the bosonic CHS fields may be extended to the 4d fermionic ones (see [20]). These are boundary counterparts for the massless spin $s$ fermionic HS fields in $\text{AdS}_5$ to the $\text{SO}(2, 4)$ representation

$$\text{MHS}_s = \left( s + \frac{5}{2}; \frac{s}{2}, \frac{s+1}{2} \right) + \left( s + \frac{5}{2}; \frac{s+1}{2}, \frac{s}{2} \right), \quad s \equiv s - \frac{1}{2} = 0, 1, 2, \ldots.$$  \hspace{1cm} (D.1)

Here $s = 0$ corresponds to a spin $\frac{1}{2}$ fermion, $s = 1$ to a conformal gravitino, etc. Recalling that for the massless $\text{SO}(2, 4)$ representation $(2 + j_1 + j_2; j_1, j_2)$ we have\footnote{The gauge subtraction in equation (D.2) is present only for $s > 0$ but since it happens to vanish for $s = 0$ this formula is general.}

$$Z^+_{(2+j_1+j_2;j_1,j_2)}(\text{AdS}_5; q) = \frac{q^{2+j_1+j_2}}{(1 - q)^4} [(2j_1 + 1)(2j_2 + 1) - 4 j_1 j_2], \hspace{1cm} (D.2)$$

we obtain (see equation (2.17) in [3])

$$Z^+_{\text{MHS}}(\text{AdS}_5; q) = \frac{2(s + 1)(s + 2)q^{s+s} - 2s(s + 1)q^{s+s}}{(1 - q)^4}. \hspace{1cm} (D.3)$$

Applying the reconstruction algorithm in appendix A or using the explicit factorized form of the fermionic CHS partition function on $S^4$ [20] and thus also on $\text{AdS}_4$ one may find that

$$Z^+_{\text{CHS}}(\text{AdS}_4; q) = 2 \frac{(s + 1)(q^2 + q^2) - (s + 1)(s + 2)q^{2+s} + s(s + 1)q^{2+s}}{(1 - q)^4}. \hspace{1cm} (D.4)$$

One then concludes that the relation (3.5) is again satisfied

$$Z^+_{\text{MHS}}(\text{AdS}_5; q) - Z^+_{\text{MHS}}(\text{AdS}_5; q) = Z^+_{\text{CHS}}(\text{AdS}_4; q) + Z^+_{\text{CHS}}(\text{AdS}_4; q). \hspace{1cm} (D.5)$$

In addition, the partition function $Z_{\text{CHS}}(S^1 \times S^3; q)$ was found in [3] (see equation (2.26) there). Comparing it with (D.4), we conclude that the relation (4.14) also holds for the fermionic CSH$_s$ family, i.e.

$$Z_{\text{CHS}}(S^1 \times S^3; q) = 2 Z^+_{\text{CHS}}(\text{AdS}_4; q). \hspace{1cm} (D.6)$$

Appendix E. Conformal antisymmetric tensor fields in 4d

The Weyl-covariant Lagrangian for the conformal antisymmetric tensor field $T_{\mu\nu}$ on a generic curved 4d background is [29]

$$\mathcal{L} = (\nabla^\mu T_{\mu\nu})^2 - \frac{1}{4} (\nabla_{\mu} T_{\nu\rho})^2 - R_{\mu\nu} T^{\mu\nu} T_{\lambda} + \frac{1}{8} R T_{\mu\nu}^2 + \frac{1}{2} R_{\mu\lambda\nu\rho} T^{\mu\nu} T^{\lambda\rho}. \hspace{1cm} (E.1)$$

This conformal field in flat 4d space corresponds in $\text{AdS}_5$ to a massive spin 1 theory with representation content $\text{HS} = (3; 1, 0) \oplus (3; 0, 1)$ and no gauge invariance. The Lagrangian (E.1) restricted to $\text{AdS}_4$ background gives the kinetic operator that factorizes into vector operators as discussed in [48]. The thermal partition function on $S^1 \times S^3$ may be found in equation (B.26) of [1]. As a result,
\[ Z^+(\text{AdS}_5; q) = \frac{6q^3}{(1-q)^2}, \quad Z^+_{\hat{\text{T}}} (\text{AdS}_4; q) = \frac{6q^3}{(1-q)^3}, \]
\[ Z_T(S^1 \times S^3; q) = \frac{6q - 6q^3}{(1-q)^4}. \quad (E.2) \]

One finds then that equation (3.5) is satisfied
\[ \tilde{Z}^+_{1\text{H}} (\text{AdS}_5; q) - \tilde{Z}^+_{1\text{H}} (\text{AdS}_5; q) = \tilde{Z}^-_{\text{T}} (\text{AdS}_4; q) + Z^+_{\hat{\text{T}}} (\text{AdS}_4; q), \quad (E.3) \]

but there is no analogue of equation (4.14).

Let us elaborate on the derivation of \( Z^+_{\hat{\text{T}}} (\text{AdS}_4; q) \) in equation (E.2). The antisymmetric tensor partition function on \( S^4 \) is [48]
\[ Z_T(S^4) = [\det \tilde{\Delta}_{(1,0)}(4)\det \tilde{\Delta}_{(0,1)}(4)]^{-1/2}, \quad (E.4) \]

where the second order operator \( \tilde{\Delta}_{(1,0)}(M^2) \) (cf equation (2.1)) acts on a field in an irreducible \( SO(1, 3) \) representation \((j_1, j_2)\) (see, e.g., [20, 55]). A similar partition function is found on \( \text{AdS}_4 \) where the mass term is related to the corresponding conformal dimension \( \Delta^+_2 \) by the following generalization of equation (2.2)
\[ \Delta^+_2 (\Delta^+_2 - 3) = j_1 (j_1 + 1) - j_2 (j_2 + 1) = -M^2. \quad (E.5) \]

For \( M^2 = 4 \) and \((j_1, j_2) = (0, 1) \) or \((1, 0) \) as in equation (E.4) this gives \( \Delta^+_2 = 2 \) and \( \Delta^-_1 = 1 \). Therefore, \( Z^+_{\hat{\text{T}}} (\text{AdS}_4; q) = \frac{6q}{(1-q)^2} \), in agreement with equation (E.2) (the factor 6 is the dimension of the \((1, 0) \oplus (0, 1) \) representation).

Note that the partition function in (E.4) leads to the correct value of the conformal a-anomaly coefficient for \( T_{\mu\nu} \). Using equations (3.34) and (3.35) of [20], the contribution to the a-anomaly from \( \tilde{\Delta}_{(1,0)}(M^2) \) is found to be
\[ a_{(1,0)}(M^2) = \frac{1}{720} (2j_1 + 1)(2j_2 + 1)[10(j_1(j_1 + 1) + j_2 (j_2 + 1)) - 15 M^4 + 60 M^2 - 58], \quad (E.6) \]

so that \( a(T) = \tilde{a}_{(1,0)} + \tilde{a}_{(0,1)} = -\frac{19}{60} \), in agreement with [3, 48]. Indeed, the general form of equation (4.6) is [1]
\[ a(\Delta_5; j_1, j_2) = \frac{1}{720} (2j_1 + 1)(2j_2 + 1)(\Delta_5 - 2) \]
\[ \times \left[ -3(\Delta_5 - 2)^4 + 10\left(j_1^2 + j_2^2 + j_1 + j_2 + \frac{1}{2}\right)(\Delta_5 - 2)^2 \right. \]
\[ \left. - 15(j_1 - j_2)^2(j_1 + j_2 + 1)^2 \right], \quad (E.7) \]

and we again get \( a(3; 1, 0) + a(3; 0, 1) = \frac{19}{60} \).

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30 The explicit dependence on \( j_1, j_2 \) through the Casimir of \( SO(4) \) comes from the particular definition of the operator, see equation (3.5) of [20].
31 As explained in [56], it is also possible to express \( T_{\mu\nu} \) in terms of two spin 1 vector fields and thus write the partition function in the form (see equation (5) of [48], cf equation (4.25)) \( Z_T(S^4) = C[\det \tilde{\Delta}_{\hat{\text{T}}}(3)]^{-1} \), where \( C \) accounts for the zero mode contributions (cf discussion after equation (6.18) in [29]). This zero mode factor is essential to reproduce the correct value for the a-anomaly giving extra \( -\frac{1}{2} \) shift: using equation (4.7) to find \( \tilde{a}_{\hat{\text{T}}}(3) \) we get \( \tilde{a}(T) = 2 \tilde{a}_{\hat{\text{T}}}(3) - \frac{1}{2} = -\frac{19}{60} \).
One can repeat the above discussion for a conformal 4d field $T_p$ transforming in the $(p, 0) \oplus (0, p)$ representation of the $SO(1, 3)^{32}$. We may start in AdS$_5$ with a 5d field in $(\Delta_5; p, 0) \oplus (\Delta_5; 0, p)$ representation (to be denoted as HS$_p$). It should correspond to a CF in AdS$_4$ with the canonical dimension $4 - \Delta_5$ and thus with the kinetic term $T_p \Box^{\Delta_5 - 2} T_p + \ldots$. The correspondence rule (4.1) here reads as

$$\Delta_5; p, 0) \oplus (\Delta_5; 0, p) \rightarrow Z_{T_p}(\text{AdS}_4)$$

$$= \prod_{k=1}^{\Delta_5 - 2} \left[ \det \tilde{\Delta}_{(p, 0)\oplus(0, p)}(2 + p(p + 1) - k(k - 1)) \right]^{-1/2}. \quad (E.8)$$

Using equations (E.6) and (E.7) one finds that the equality of the a-anomaly coefficients implied by equation (E.8) indeed holds

$$a(\Delta_5; p, 0) + a(\Delta_5; 0, p) = 2 \sum_{k=1}^{\Delta_5 - 1} a_{(p, 0)}(2 + p(p + 1) - k(k - 1)). \quad (E.9)$$

From equation (E.5) we find that the dimension corresponding according to equation (2.2) to the kth operator in the r.h.s. of equation (E.8) is $\Delta_k = k + 1$, so that from equation (2.3) we get

$$Z_{\text{HS}}^{+,+}(\text{AdS}_5; q) = \frac{2(2p + 1)q^{\Delta_s}}{(1 - q)^2},$$

$$Z_{\text{T}_p}^{+,+}(\text{AdS}_4; q) = \frac{2(2p + 1)(1 + 2k + 1)}{(1 - q)^3} q^{k+1} = \frac{2(2p + 1)}{(1 - q)^4} (q^2 - q^{\Delta_5}). \quad (E.10)$$

Thus once again we get the relation (3.5)

$$Z_{\text{HS}}^{+,+}(\text{AdS}_5; q) - Z_{\text{HS}}^{-,+}(\text{AdS}_5; q) = Z_{\text{T}_p}^{+,+}(\text{AdS}_4; q) + Z_{\text{T}_p}^{-,+}(\text{AdS}_4; q). \quad (E.11)$$

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32 Fields transforming in the $(p, 0) \oplus (0, p)$ representation are Weyl-like tensors (see, e.g., [57]). They may be represented as rank 2p tensors $T_{\mu_1\nu_1\ldots\mu_p\nu_p}$ antisymmetric in each pair $\mu_i$,$\nu_i$, totally symmetric with respect to the exchange of the pairs $(\mu_i,\nu_i)$ and $(\mu_j,\nu_j)$, traceless, and obeying a generalized Bianchi identity $T_{\ldots [\mu

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