SURFACES OF REVOLUTION WITH POINTWISE 1-TYPE GAUSS MAP IN PSEUDO-GALILEAN SPACE

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Abstract. In this paper, we study surfaces of revolution in the three dimensional pseudo-Galilean space. We classify surfaces of revolution generated by a non-isotropic curve in terms of the Gauss map and the Laplacian of the surface. Furthermore, we give the classification of surfaces of revolution generated by an isotropic curve satisfying pointwise 1-type Gauss map equation.

1. Introduction

In late 1970’s B.-Y. Chen introduced the notion of Euclidean immersions of finite type. Essentially these are submanifolds whose immersion into the \(m\)-dimensional Euclidean space \(\mathbb{E}^m\) is constructed by making use of a finite number of \(\mathbb{E}^m\)-valued eigenfunctions of their Laplacian. The first results on this subject have been collected in the book [2]. Many works were done to characterize or classify submanifolds in terms of finite type. In a framework of the theory of finite type, B.-Y. Chen and P. Piccini [4] made a general study on submanifolds of Euclidean spaces with finite type Gauss map. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map.

From the above definition one can see that a submanifold has 1-type Gauss map \(G\) if and only if \(G\) satisfies the equation

\[
\Delta G = \lambda (G + C)
\]

for a constant \(\lambda\) and a constant vector \(C\), where \(\Delta\) denotes the Laplace operator on a submanifold. A plane, a circular cylinder and a sphere are surfaces with 1-type Gauss map.

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Similarly, a submanifold is said to have pointwise 1-type Gauss map if the Laplacian of its Gauss map takes the form
\begin{equation}
\Delta G = F(G + C)
\end{equation}
for a non-zero smooth function $F$ and a constant vector $C$. More precisely, a pointwise 1-type Gauss map is said to be of the first kind if (1.2) is satisfied for $C = 0$, and of the second kind if $C \neq 0$. A helicoid, a catenoid and a right cone are the typical examples of surfaces with pointwise 1-type Gauss map. Many results of submanifolds with pointwise 1-type Gauss map were obtained in [1], [3], [5], [6], [7], [9], [12], etc, when the ambient spaces are the Euclidean space, Minkowski space and Galilean space.

In this paper, we study surfaces of revolution in the three dimensional pseudo-Galilean space \( G_3^1 \) in terms of their Gauss map. In Sections 2 and 3, we introduce pseudo-Galilean space and construct surfaces of revolution in \( G_3^1 \) by non-isotropic and isotropic rotations. In Section 4, we obtain the complete classification of surfaces of revolution generated by non-isotropic curve with pointwise 1-type Gauss map. In the last section, we focus on surfaces of revolution generated by isotropic curve with pointwise 1-type Gauss map and give the complete classification of such surfaces.

2. Pseudo-Galilean space

Let us recall the basic facts about the three dimensional pseudo-Galilean space \( G_3^1 \). The geometry of the pseudo-Galilean space has been firstly explained in [10]. The pseudo-Galilean space \( G_3^1 \) is a Cayley-Klein space with the absolute figure consisting of an ordered triple \( \{ \omega, f, I \} \), where \( \omega \) is the ideal (absolute) plane in the three dimensional real projective space \( \mathbb{R}P^3 \), \( f \) the line (the absolute line) in \( \omega \) and \( I \) the fixed hyperbolic involution of points of \( f \).

Homogenous coordinates in \( G_3^1 \) are introduced in such a way that the absolute plane \( \omega \) is given by \( x_0 = 0 \), the absolute line \( f \) by \( x_0 = x_1 = 0 \) and the hyperbolic involution \( \eta \) by \( \eta : (x_0 : x_1 : x_2 : x_3) \rightarrow (0 : 0 : x_3 : x_2) \).

Let \( \mathbf{x} = (x_1, y_1, z_1) \) and \( \mathbf{y} = (x_2, y_2, z_2) \) be two vectors in \( G_3^1 \). A vector \( \mathbf{x} \) is called isotropic if \( x_1 = 0 \), otherwise it is called non-isotropic. The pseudo-Galilean scalar product of \( \mathbf{x} \) and \( \mathbf{y} \) is defined by
\begin{equation}
\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases}
 x_1x_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\
y_1y_2 - z_1z_2, & \text{if } x_1 = 0 \text{ and } x_2 = 0.
\end{cases}
\end{equation}

From this, the pseudo-Galilean norm of a vector \( \mathbf{x} \) in \( G_3^1 \) is given by \( ||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \) and all unit non-isotropic vectors are the form \((1, y_1, z_1)\). There are four types of isotropic vectors: spacelike \((y_1^2 - z_1^2 > 0)\), timelike \((y_1^2 - z_1^2 < 0)\) and the two types of lightlike \((y_1 = \pm z_1)\) vectors. A non-lightlike isotropic vector is a unit vector if \( y_1^2 - z_1^2 = \pm 1 \).
A plane of the form \( x = \text{constant} \) is called a pseudo-Euclidean plane, otherwise it is called isotropic. An isotropic plane \( ax + by + cz + d = 0 \) is called light-like if \( b^2 - c^2 = 0 \).

The pseudo-Galilean cross product of \( \mathbf{x} \) and \( \mathbf{y} \) on \( G^1_3 \) is defined by

\[
\mathbf{x} \times \mathbf{y} = \begin{vmatrix}
0 & -e_2 & e_3 \\
-x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2
\end{vmatrix},
\]

where \( e_2 = (0,1,0) \) and \( e_3 = (0,0,1) \).

Consider a \( C^2 \)-surface \( M (r \geq 1) \) in \( G^1_3 \) parameterized by

\[
\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).
\]

Let us denote \( g_i = \frac{\partial x}{\partial u_i} \), \( h_{ij} = (\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}) (i, j = 1, 2) \), where \( \sim \) stands for the projection of a vector on the pseudo-Euclidean \( yz \)-plane. A surface \( M \) is called admissible if it does not have Euclidean tangent planes. Therefore a surface \( M \) is admissible if and only if \( x_i \neq 0 \) for some \( i = 1, 2 \).

Let \( M \) be an admissible surface in \( G^1_3 \). Then, the corresponding matrix of the first fundamental form \( ds^2 \) of a surface \( M \) is given by (cf. [11])

\[
\frac{\text{d}s^2}{\text{d}u_1^2} = \begin{pmatrix}
g_{11} & 0 \\
0 & g_{22}
\end{pmatrix},
\]

where \( g_{11} = (g_1du_1 + g_2du_2)^2 \) and \( g_{22} = h_{11}du_1^2 + 2h_{12}du_1du_2 + h_{22}du_2^2 \). Here \( g_i = x_i \) and \( h_{ij} = (\mathbf{x}_i, \mathbf{x}_j) \) \((i, j = 1, 2)\). In such case, we denote the coefficients of \( \text{d}s^2 \) by \( g_{ij} \).

On the other hand, the unit normal vector field \( U \) of a surface \( M \) is defined by

\[
U = \frac{1}{W}(0, x, 1z, 2 - x, 2z, 1, x, 1y, 2 - x, 2y, 1),
\]

where

\[
W = \sqrt{(x, 1y, 2 - x, 2y, 1)^2 - (x, 1z, 2 - x, 2z, 1)^2}.
\]

The Gaussian curvature \( K \) of a surface \( M \) is defined by means of the coefficients \( L_{ij} (i, j = 1, 2) \) of the second fundamental form, which are the normal components of \( \mathbf{x}_{,ij} (i, j = 1, 2) \), that is,

\[
L_{ij} = \frac{1}{g_i}(g_{1j}\mathbf{x}_{,ij} - g_{i,j}\mathbf{x}_1, U) = \frac{1}{g_j}(g_{2j}\mathbf{x}_{,ij} - g_{i,j}\mathbf{x}_2, U).
\]

Thus, the Gaussian curvature \( K \) of \( M \) is defined by

\[
K = -\frac{L_{11}L_{22} - L_{12}^2}{W^2},
\]

and the mean curvature \( H \) is given by

\[
H = \frac{\epsilon}{2W^2}(g_2^2L_{11} - 2g_1g_2L_{12} + g_1^2L_{22}),
\]

where \( \epsilon = (\pm 1) \) is the sign of the unit normal vector field.
For the coefficients $g^*_{ij}$ of the first fundamental form on $M$ we denote by $(g^*_{ij})$ the inverse matrix of the matrix $(g^*_{ij})$. In terms of a local coordinate system $\{x_i\}$, the Laplacian $\Delta f$ of a smooth function $f$ is given by

$$
\Delta f = -\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^*_{ij} \frac{\partial f}{\partial x_j}),
$$

where $g$ denotes the determinant of the matrix $(g^*_{ij})$.

### 3. Surfaces of revolution in $G^1_3$

In the three dimensional pseudo-Galilean space $G^1_3$, there are two types of rotations: pseudo-Euclidean rotations given by the normal form

$$
\bar{x} = x, \quad \bar{y} = y \cosh t + z \sinh t, \quad \bar{z} = y \sinh t + z \cosh t
$$

and isotropic rotations with the normal form

$$
\bar{x} = x + bt, \quad \bar{y} = y + xt + \frac{bt^2}{2}, \quad \bar{z} = z,
$$

where $t \in \mathbb{R}$ and $b$ is a positive constant.

First of all, we consider a non-isotropic curve $\alpha$ parameterized by

$$
\alpha(u) = (f(u), g(u), 0) \quad \text{or} \quad \alpha(u) = (f(u), 0, g(u))
$$

around the $x$-axis by pseudo-Euclidean rotation (3.1), where $g$ is a positive function and $f$ is a smooth function on an open interval $I$. Then the surface of revolution can be written as

$$
x(u, v) = (f(u), g(u) \cosh v, g(u) \sinh v)
$$

or

$$
x(u, v) = (f(u), g(u) \sinh v, g(u) \cosh v)
$$

for any $v \in \mathbb{R}$.

Next, we consider the isotropic rotations. By an isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the $z$-axis by an isotropic rotation (3.2), we obtain a surface

$$
x(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right),
$$

where $f$ and $g$ are smooth functions and $b \neq 0$ [11].
4. Surfaces of revolution generated by non-isotropic curve

Let \( M \) be a surface of revolution generated by non-isotropic curve \( \alpha(u) = (u, g(u), 0) \) in \( G^3_1 \). Then \( M \) is parameterized by
\[
(4.1) \quad x(u, v) = (u, g(u) \cosh v, g(u) \sinh v),
\]
where \( g \) is a positive function.

The coefficients of the first fundamental form on \( M \) are given by
\[
g_{11}^* = 1, \quad g_{12}^* = 0, \quad g_{22}^* = -g(u)^2.
\]
We see that \( M \) is a time-like surface. By a direct computation with the help of (2.5), the Laplacian \( \Delta \) on \( M \) is given by [12]
\[
(4.2) \quad \Delta = -\frac{g'(u)}{g(u)} \frac{\partial}{\partial u} - \frac{\partial^2}{\partial u^2} + \frac{1}{g(u)^2} \frac{\partial^2}{\partial v^2}.
\]

Also, the Gauss map \( G \) of \( M \) becomes
\[
(4.3) \quad G = (0, \cosh v, \sinh v).
\]

From (4.2) and (4.3), the Laplacian \( \Delta G \) of the Gauss map \( G \) can be expressed as
\[
(4.4) \quad \Delta G = \frac{1}{g(u)^2} G.
\]

Thus, we have the following theorems.

**Theorem 4.1.** There is no surfaces of revolution generated by a non-isotropic curve in \( G^3_1 \) with harmonic Gauss map.

**Proof.** Let \( M \) be a surface of revolution defined by (4.1) in \( G^3_1 \). If \( M \) has harmonic Gauss map, that is, \( M \) satisfies \( \Delta G = 0 \), then \( g^{-2}(u)G = 0 \). It is impossible because \( g(u) \) is a positive function and \( G \) is the unit normal vector field of \( M \). \( \square \)

**Theorem 4.2.** Let \( M \) be a surface of revolution generated by a non-isotropic curve in the three dimensional pseudo-Galilean space \( G^3_1 \). Then \( M \) has pointwise 1-type Gauss map of the first kind.

**Proof.** Let \( M \) be a surface of revolution generated by a non-isotropic curve in \( G^3_1 \). Suppose that \( M \) has pointwise 1-type Gauss map. Combining (1.2) and (4.4), one gets \( F(u) = g^{-2}(u) \) and \( C = 0 \). Thus the Gauss map \( G \) of \( M \) is of pointwise 1-type of the first kind. \( \square \)

**Theorem 4.3.** There is no surfaces of revolution generated by a non-isotropic curve in \( G^3_1 \) with pointwise 1-type Gauss map of the second kind.

**Proof.** Let \( M \) be a surface of revolution defined by (4.1) in \( G^3_1 \). By Theorem 4.2, \( M \) has only pointwise 1-type Gauss map of the first kind. Thus, the theorem is proved. \( \square \)
Remark. We consider a surface defined by
\[ \mathbf{x}(u, v) = (u, (a^2 u + b^2) \cosh v, (a^2 u + b^2) \sinh v), \]
where \( a, b \in \mathbb{R} \) and \( u > -\frac{b^2}{a^2} \). The surface is a Lorentzian cone satisfying the equation \((a^2 x + b^2)^2 = y^2 - z^2\). From (4.4) the Laplacian \( \Delta G \) of the Gauss map \( G \) of the surface is obtained by \( \Delta G = \frac{1}{(a^2 u + b^2)^2} G \). Thus, a Lorentzian cone in \( G_3^1 \) has pointwise 1-type Gauss map of the first kind. On the other hand, a Lorentzian cone in the three dimensional Minkowski space \( \mathbb{E}_3^1 \) has pointwise 1-type Gauss map of the second kind (see [8]).

5. Surfaces of revolution generated by isotropic curve

In this section, we consider the isotropic rotations. By rotating an isotropic curve \( \alpha(u) = (0, f(u), g(u)) \) about the z-axis, we obtain a surface of revolution \( M \) parameterized by
\[ \mathbf{x}(u, v) = (v, f(u) + \frac{v^2}{2b}, g(u)), \]
where \( b \) is a non-zero constant. We assume that the isotropic curve is parameterized by arc-length, that is,
\[ f'(u)^2 - g'(u)^2 = -\epsilon(= \pm 1). \]
By using (5.2), the coefficients of the first fundamental form \( ds^2 \) on \( M \) are given by
\[ g_{11} = 1, \ g_{12} = 0, \ g_{22} = -\epsilon. \]
On the other hand, the Gauss map \( G \) and the Laplacian \( \Delta \) on \( M \), respectively, are given by
\[ G = (0, -g', -f') \]
and
\[ \Delta = \epsilon \frac{\partial^2}{\partial u^2} + \epsilon \frac{\partial^2}{\partial v^2}. \]
Hence the Laplacian \( \Delta G \) of the Gauss map \( G \) is obtained by [12]
\[ \Delta G = (0, -\epsilon g'', -\epsilon f''). \]

In terms of the harmonic Gauss map, we have:

**Theorem 5.1.** Let \( M \) be a surface of revolution generated by an isotropic curve \( \alpha(u) = (0, f(u), g(u)) \) in \( G_3^1 \). Then \( M \) has a harmonic Gauss map if and only if the functions \( f \) and \( g \) are quadric.
5.1. Surfaces of revolution with pointwise 1-type Gauss map of the first kind

Let $M$ be a surface of revolution in $G^3_1$ satisfying $\Delta G = FG$ for a non-zero smooth function $F$. Then from (5.4) we have

\begin{align*}
\epsilon g''' &= F g', \\
\epsilon f''' &= F f'.
\end{align*}

(5.5)

If $f' = 0$, from (5.2) $g(u) = \pm u + a$ with $a \in \mathbb{R}$ and from the first equation of (5.5) $F = 0$, a contradiction. Similarly, the case of $g' = 0$ is also impossible.

Now we suppose that $f' g' \neq 0$. If $\epsilon = -1$, then we put

\begin{align*}
f'(u) &= \cosh \theta(u), \quad g'(u) = \sinh \theta(u),
\end{align*}

where $\theta$ is a smooth function.

By substituting (5.6) into (5.5) and calculating we obtain $\theta'' = 0$, that is, $\theta(u) = a_1 u + a_2$, $a_1, a_2 \in \mathbb{R}$. From this and (5.5) we can show that $F = -a_1^2$. Therefore, the Gauss map $G$ of $M$ is of 1-type. In the case of $\epsilon = 1$ we have the same result.

\textbf{Theorem 5.2 (The Classification Theorem).} \textit{Let $M$ be a surface of revolution generated by an isotropic curve in the three dimensional Galilean space $G^3_1$. If $M$ has pointwise 1-type Gauss map of the first kind, then the Gauss map of $M$ is of usual 1-type.}

Furthermore, $M$ is parameterized as

\begin{align*}
\mathbf{x}(u, v) &= \left( v, \frac{1}{a_1} \sinh(a_1 u + a_2) + a_3 + \frac{v^2}{2b} \cdot \frac{1}{a_1} \cosh(a_1 u + a_2) + a_3 \right) \\
or \\
\mathbf{x}(u, v) &= \left( v, \frac{1}{d_1} \cosh(d_1 u + d_2) + d_3 + \frac{v^2}{2b} \cdot \frac{1}{d_1} \sinh(d_1 u + d_2) + d_3 \right),
\end{align*}

where $a_i, d_i \in \mathbb{R}$, $i = 1, 2, 3$.

5.2. Surfaces of revolution with pointwise 1-type Gauss map of the second kind

Suppose that $M$ has pointwise 1-type Gauss map of the second kind, that is, $M$ satisfies $\Delta G = F(G + C)$. Then we easily see that the first component $c_1$ of a constant vector $C = (c_1, c_2, c_3)$ is zero and we have a system of differential equations as follows:

\begin{align*}
-\epsilon g''' &= F(-g' + c_2), \\
-\epsilon f''' &= F(-f' + c_3).
\end{align*}

(5.7)

If $f' = 0$, then $\epsilon = 1$ and $g(u) = \pm u + k_1$, where $k_1$ is constant. Hence from (5.7) we have

\begin{align*}
F(\pm 1 + c_2) = 0 \quad \text{and} \quad Fc_3 = 0.
\end{align*}
It implies that \( c_2 = \pm 1 \) and \( c_3 = 0 \), i.e., \( C = (0, \pm 1, 0) \). It follows that \( C = -G \). Thus, \( M \) is parameterized by

\[
x(u, v) = \left( v, \frac{v^2}{2b} + k_2, \pm u + k_1 \right),
\]

where \( k_1 \) and \( k_2 \) are constant. The surface is a time-like parabolic cylinder and it has pointwise 1-type Gauss map of the second kind.

Similarly as above, if \( g' = 0 \), then \( f(u) = \pm u + k_1 \) for some constant \( k_1 \) and so \( \epsilon = -1 \). Hence, \( C = (0, 0, \pm 1) \) and moreover \( C = -G \). Therefore \( M \) has pointwise 1-type Gauss map of the second kind and its parametrization is given by

\[
x(u, v) = \left( v, \pm u + \frac{v^2}{2b} + k_1, k_2 \right)
\]

for some constants \( k_1 \) and \( k_2 \). We see that it is a space-like plane.

Suppose that \( f'g' \neq 0 \). In this case, we first consider \( \epsilon = -1 \) and put

\[
f'(u) = \cosh \theta(u) \quad \text{and} \quad g'(u) = \sinh \theta(u).
\]

Then, (5.7) can be rewritten as

\[
(\theta')^2 \sinh \theta + \theta'' \cosh \theta = F(-\sinh \theta + c_2), \quad (\theta')^2 \cosh \theta + \theta'' \sinh \theta = F(-\cosh \theta + c_3),
\]

where \( \theta \) is a smooth function. It follows that

\[
(5.8) \quad -\theta'' = F(1 - c_2 \sinh \theta + c_3 \cosh \theta)
\]

and

\[
(5.9) \quad \theta'' = F(\cosh \theta - c_3 \sinh \theta).
\]

If \( \theta' = 0 \) identically, then \( f' \) and \( g' \) are constants, say \( a_1 \) and \( b_1 \). It follows that we have \( \Delta G = 0 \) and \( C = -G \). Thus \( M \) has pointwise 1-type Gauss map of the second kind and it is parameterized by

\[
x(u, v) = \left( v, a_1 u + a_2 + \frac{v^2}{2b}, b_1 u + b_2 \right)
\]

for some constants \( a_1, a_2, b_1 \) and \( b_2 \).

Next, suppose that \( \theta' \neq 0 \). From (5.8), \( F \) depends only on the parameter \( u \), i.e., \( F(u, v) = F(u) \). Differentiating equation (5.8) with respect to \( u \), we get

\[
-2\theta' \theta'' = F'(1 - c_2 \sinh \theta + c_3 \cosh \theta) + F(-c_2 \cosh \theta + c_3 \sinh \theta) \theta'.
\]

By using (5.9), it implies that

\[
(5.10) \quad -\theta' \theta'' = F'(1 - c_2 \sinh \theta + c_3 \cosh \theta).
\]

Combining (5.8) and (5.10), we have the following equation

\[
\frac{\theta''}{\theta'} = \frac{F'}{F}.
\]
which implies

(5.11) \[ \theta' = kF \]

for a non-zero constant \( k \).

Applying the composition of trigonometric function in (5.8) and (5.9), we obtain the following differential equation

\[
\left( \frac{\theta'^2}{F} + 1 \right)^2 - \left( \frac{\theta''}{F} \right)^2 = -c_2^2 + c_3^2.
\]

With the help of (5.11), it becomes

(5.12) \[ (k^2 F + 1)^2 - k^2 \left( \frac{F'}{F} \right)^2 = -c_2^2 + c_3^2. \]

Without loss of generality, we assume that \( k = \pm 1 \). In order to solve the above equation, we put

\[ p = \ln F. \]

Then, (5.12) can be rewritten as the following equation:

(5.13) \[ (e^p + 1)^2 - \left( \frac{dp}{du} \right)^2 = -c_2^2 + c_3^2. \]

Let us distinguish three cases according to the constant vector \( C \).

Case 1. \( C = (0, c_2, c_3) \) is null, that is, \( c_2^2 - c_3^2 = 0 \).

In this case, we can easily obtain a general solution given by

\[ F(u) = \frac{d_1 e^{\pm u}}{1 - d_1 e^{\pm u}}, \]

where \( d_1 \) is non-zero constant. Therefore, from (5.11) we have

\[ \theta(u) = \mp \ln \left| 1 - d_1 e^{\pm u} \right| + d_2, \]

where \( d_2 \) is constant. Thus, \( M \) is parameterized as

\[
x(u, v) = \left( v, \int \cosh \left( \ln \left| 1 - d_1 e^{\pm u} \right| + d_2 \right) \, du + \frac{v^2}{2b}, \right.
\]

\[ - \int \sinh \left( \ln \left| 1 - d_1 e^{\pm u} \right| + d_2 \right) \, du \bigg). \]

Case 2. \( C \) is time-like, that is, \( c_2^2 - c_3^2 < 0 \).

We assume that \( c_2^2 - c_3^2 = -1 \). Then (5.13) becomes

\[
\left( \frac{dp}{du} \right)^2 = (e^p + 1)^2 - 1
\]

and its general solution is given by

\[ F(u) = e^p = \frac{2}{(u \pm d_1)^2 - 1}, \]
where $d_1$ is constant. Thus a first integration of (5.11) implies
\[
\theta(u) = \ln \left| \frac{u \pm d_1 - 1}{u \pm d_1 + 1} \right| + d_2,
\]
where $d_2$ is constant. Thus, the parametrization of $M$ is given by
\[
x(u, v) = \left( v, \int \cosh \left( \ln \left| \frac{u \pm d_1 - 1}{u \pm d_1 + 1} \right| + d_2 \right) du + \frac{v^2}{2b}, \right.
\]
\[
\left. \int \sinh \left( \ln \left| \frac{u \pm d_1 - 1}{u \pm d_1 + 1} \right| + d_2 \right) du \right).
\]

Case 3. $C$ is space-like, that is, $c_2^2 - c_3^2 > 0$.
In the case, we assume that $c_2^2 - c_3^2 = 1$. (5.13) becomes
\[
(5.14) \quad \frac{dp}{du} = \pm \sqrt{(ep + 1)^2 + 1}.
\]
In order to solve (5.14), we put
\[
h(p) = \frac{e^p + 2}{\sqrt{2}\sqrt{(ep + 1)^2 + 1}}.
\]
Then, by a direct computation, we show that
\[
(5.15) \quad \frac{1}{\sqrt{(ep + 1)^2 + 1}} = -\frac{\dot{h}(p)}{\sqrt{2}(1 - h(p)^2)},
\]
where "·" denotes the derivative with respect to $p$. Thus, a direct integration of (5.14) yields
\[
-\frac{1}{\sqrt{2}} \tanh^{-1} h(p) = \pm u + d_1
\]
or, equivalently
\[
-\frac{1}{\sqrt{2}} \tanh^{-1} \frac{F + 2}{\sqrt{2}(F^2 + 2F + 2)} = \pm u + d_1.
\]
The last equation can be written as the form:
\[
(5.16) \quad \left( 1 - \sinh^2(\sqrt{2}u + d_1) \right) F^2 + 4F + 4 = 0,
\]
where $d_1$ is constant.
This implies
\[
F(u) = \frac{2}{1 \pm \sinh(\sqrt{2}u + d_1)}.
\]
From here, we have two values for $F$. First, by taking the sign $+$ and using (5.11) with $k = \pm 1$ we get
\[
(5.17) \quad \theta(u) = \pm 2 \tanh^{-1} \left( \frac{\sqrt{2}}{e^{\sqrt{2}u + d_1} + 1} \right) + d_2.
\]
Finally, by taking the sign $-\,$ we get

\begin{equation}
\theta(u) = \pm 2 \tanh^{-1} \left( \frac{\sqrt{2e^{\sqrt{2u+d_1}}}}{e^{\sqrt{2u+d_1}} + 1} \right) + d_2,
\end{equation}

where $d_2$ is constant. In this case, the parametrization of $M$ is given by

\begin{equation}
x(u, v) = \left( v, \int \cosh \theta(u) du + \frac{v^2}{2b}, \int \sinh \theta(u) du \right),
\end{equation}

where $\theta(u)$ is given by (5.17) or (5.18). Consequently, we have

**Theorem 5.3 (The Classification Theorem).** Let $M$ be a surface of revolution generated by isotropic curve in the three dimensional Galilean space $G^3_1$. If $M$ has pointwise 1-type Gauss map of the second kind, then $M$ is one of the following surfaces:

1. $x(u, v) = \left( v, \frac{u^2}{2b} + d_2, \pm u + d_1 \right)$.
2. $x(u, v) = \left( v, \pm u + \frac{u^2}{2b} + d_1, d_2 \right)$.
3. $x(u, v) = \left( v, d_1 u + d_2 + \frac{u^2}{2b}, d_3 u + d_4 \right)$.
4. $x(u, v) = \left( v, \int \cosh(\ln |1 - d_1 e^{\pm u}| + d_2) du + \frac{u^2}{2b}, - \int \sinh(\ln |1 - d_1 e^{\pm u}| + d_2) du \right)$.
5. $x(u, v) = \left( v, \int \cosh(\ln \frac{|u+d_1-1|}{u_2 d_1 + 1} + d_2) du + \frac{u^2}{2b}, \int \sinh(\ln \frac{|u+d_1-1|}{u_2 d_1 + 1} + d_2) du \right)$.
6. $x(u, v) = \left( v, \int \cosh \theta(u) du + \frac{v^2}{2b}, \int \sinh \theta(u) du \right)$, where $\theta(u)$ is given by (5.17) or (5.18).

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