A Note on Higher Dimensional Instantons and Supersymmetric Cycles

Hiroaki Kanno

Department of Mathematics, Hiroshima University
Higashi-Hiroshima, 739-8526, Japan

Abstract

We discuss instantons in dimensions higher than four. A generalized self-dual or anti-self-dual instanton equation in \( n \)-dimensions can be defined in terms of a closed \( (n-4) \) form \( \Omega \) and it was recently employed as a topological gauge fixing condition in higher dimensional generalizations of cohomological Yang-Mills theory. When \( \Omega \) is a calibration which is naturally introduced on the manifold of special holomony, we argue that higher dimensional instanton may be locally characterized as a family of four dimensional instantons over a supersymmetric \( (n-4) \) cycle \( \Sigma \) with respect to the calibration \( \Omega \). This is an instanton configuration on the total space of the normal bundle \( N(\Sigma) \) of the submanifold \( \Sigma \) and regarded as a natural generalization of point-like instanton in four dimensions that plays a distinguished role in a compactification of instanton moduli space.

\* Talk presented at the workshop on “Gauge Theory and Integrable Models” (YITP, Kyoto), January 26-29, 1999.
1 Introduction

Instantons or (anti-)self-dual connections in four dimensions are interesting and important objects both in mathematics and physics. In this article we discuss higher dimensional generalization. One of motivations to consider higher dimensional instantons comes from recent developments in string dualities and $M$-theory, where we obtain low energy effective (supersymmetric) gauge theories in diverse dimensions. We are also motivated by its use in topological (or more precisely cohomological) gauge theories in higher dimensions. We can employ higher dimensional instanton equation as a topological gauge fixing condition in BRST quantization of topological action in eight and other dimensions [1],[2],[3]. Consequently, we expect such cohomological gauge theory explores the moduli space of higher dimensional instantons. Supersymmetry plays a key role in all of these subjects.

Our higher dimensional instanton equation is

$$\Omega \wedge F \pm (\ast F) = 0,$$

where $\Omega$ is a closed $(n - 4)$ form on an $n$ dimensional Riemannian manifold. (See the next section for details.) We will call a solution either anti-self-dual or self-dual according to the choice of the sign in (1). We argue that, when $\Omega$ defines a calibration, higher dimensional instantons are locally characterized as a family of four dimensional instantons over an $(n - 4)$ dimensional cycle $\Sigma$. The generalized instanton equation (1) requires that the cycle $\Sigma$ should be supersymmetric. (The idea of supersymmetric (SUSY) cycles and the relation to the calibration $\Omega$ are reviewed in section 3.) More precisely, by rescaling the transverse coordinates to $\Sigma$, we will see that higher dimensional instantons can be reduced to those on the total space of the normal bundle $N(\Sigma)$ of the SUSY cycle $\Sigma$. Note that the fibre of $N(\Sigma)$ is the four dimensional Euclidean space $\mathbb{R}^4$. Such a scaled instanton is the most natural generalization of point-like (or ideal) instanton in four dimensions where the SUSY cycle is just a set of points. We expect that it will play a crucial role in an attempt at compactifying the moduli space of higher dimensional instantons [4],[5],[6].

We already know a trivial example of a family of instantons. It is the gauge five-brane solution in the heterotic string theory, where four dimensional Yang-Mills instanton lives in the transverse direction to the world-volume of flat five-brane [7]. This is a four dimensional instanton trivially embedded to ten dimensional space-time and should not be regarded as a genuine higher dimensional instanton. Our proposal on higher dimensional instantons modifies it in two aspects. Firstly, our supersymmetric cycle $\Sigma$ is curved in general and, especially, can be compact, while the world-volume of the gauge five-brane solution is flat and non-compact. Secondly and more significantly, the moduli (or the size and shape) of instanton in general depends on the coordinates of the base space $\Sigma$ and hence our solution is not a direct product of four dimensional instanton and the world volume of the brane.
The connection of cohomological Yang-Mills theories based on the generalized instanton equations to the D-branes and the calibrated geometry has been discussed in e.g. [8], [9]. In these literatures the entire manifold (e.g. a manifold of $\text{Spin}(7)$ holonomy for the octonionic instanton) was identified with the world volume of D-brane or a supersymmetric cycle in ten or eleven dimensional space-time. This is very natural viewpoint to see the fact that the cohomological Yang-Mills theory in eight dimensions is in fact a dimensional reduction of ten-dimensional super Yang-Mills theory. However, we emphasize that our viewpoint on supersymmetric cycles is rather different from the above story. In this paper the supersymmetric cycle is not the entire manifold, but a calibrated submanifold in the manifold of special holonomy on which the gauge theory is defined. The idea of branes within branes might reconcile these two pictures [10].

2 Generalized Self-Dual Instanton Equation

The instanton equation in four dimensions is the (anti-)self-dual condition on the curvature two-form $F$, which is quite intrinsic to four dimensions. We can generalize the (anti-)self-duality to higher dimensions in the following way. Let $(M, g)$ be an $n$ dimensional manifold with a Riemannian metric $g$. We assume there exists a closed $(n - 4)$ form $\Omega$. Then the generalization in the self-dual case is

$$\Omega \wedge F = (\ast F), \quad (2)$$

where $\ast$ is the Hodge dual operator defined by the metric. In [11] this equation was first appeared in the form

$$F^\mu_\nu = \frac{1}{2} T^\mu_\nu_\rho_\sigma F_\rho_\sigma, \quad (3)$$

with a totally anti-symmetric tensor $T^\mu_\nu_\rho_\sigma$. If we think of $T^\mu_\nu_\rho_\sigma$ as components of a four-form $T$, then the relation of these conditions is simply given by $T = (\ast \Omega)$. We take the equation (2) as our definition of self-duality, since $\Omega$ is geometrically more natural object as we will see in the following. A solution to the generalized self-dual equation is necessarily a solution to the Yang-Mills equation, because

$$D_A * F = D_A (\Omega \wedge F) = \pm \Omega \wedge D_A F = 0, \quad (4)$$

where we have used $d\Omega = 0$ and the Bianchi identity. For a configuration that satisfies the generalized self-dual condition, the total action is given by a topological density;

$$\int \text{Tr}(\ast F \wedge F) = \int \Omega \wedge \text{Tr}(F \wedge F). \quad (5)$$

A typical example of the generalized instanton equation is the equation of Donaldson-Uhlenbeck-Yau (DUY) on a Kähler manifold in six dimensions, where $\Omega$ is the Kähler
two form $\omega$. To reproduce the DUY equation, it is appropriate to take the anti-self-dual condition. Since $\omega^3$ gives the volume form up to the normalization, we have $*\omega = \omega \wedge \omega$. The generalized anti-self-dual equation implies
\[
\omega \wedge (*F) = -(\omega \wedge \omega) \wedge F = -(\omega) \wedge F .
\] (6)
Hence, we have the DUY equation;
\[
\omega \wedge (*F) = 0 , \quad \text{or} \quad \omega_{\mu\nu} F^{\mu\nu} = 0 .
\] (7)
We have other examples of the $G_2$ instanton equation in seven dimensions where $\Omega$ is the associative three form and also the octonionic instanton equation in eight dimensions where $\Omega$ is the Cayley four form $\Phi$.

When one first tries to construct an example of higher dimensional instantons on the flat Euclidean space, he will soon encounter the following problem. Namely it has been argued that there are no finite action solutions to the Yang-Mills equation in dimensions other than four. One of the reasonings goes as follows\footnote{This type of argument is known as the Derrick’s theorem \cite{12,13,14}.} The energy-momentum tensor of the Yang-Mills theory
\[
T_{\mu\nu} = \text{Tr} \left( F_{\mu\lambda} F_{\nu}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \right) ,
\] (8)
is conserved due to the equation of motion and the Bianchi identity. Hence we have
\[
\int d^n x \ T_{\mu\nu}(x) = 0 ,
\] (9)
where no surface term appears, if the action is supposed to be finite. Taking trace, we obtain
\[
(4 - n) \int d^n x \ \text{Tr} \ (F_{\mu\nu} F^{\mu\nu}) = 0 .
\] (10)
Since the integral is nothing but the standard Yang-Mills action itself, we come to the conclusion. This argument shows that the problem is closely related to the fact that the standard Yang-Mills action is conformal invariant only in four dimensions. We have to accept that we can find no self-dual instanton with finite action on the flat Euclidean space $\mathbb{R}^n$ or its conformal compactification, the $n$-dimensional sphere $S^n$. Thus there are no higher dimensional analogues of the basic (BPST) instanton on $S^4$. Point-like (ideal) instantons are important objects in the compactification of the instanton moduli space in four dimensions \cite{13}. They are modeled after a superposition of BPST instantons. Therefore, to consider a possible compactification of the moduli space of higher dimensional instantons, it is desirable to have a substitute in higher dimensions for the BPST instanton in four dimensions. To look for such an object we will try to make a local splitting of
the total space into a four dimensional part and the remaining \((n-4)\) dimensional one. This splitting is motivated by the conformal invariance of the four dimensional Yang-Mills action. Point-like instantons can be obtained by a conformal rescaling in four dimensions. Taking this fact into account, we make use of the rescaling in the four dimensional part in the splitting. In section 4 we will work out a possible configuration of instanton that is stable under this scaling and find the four dimensional self-dual instantons together with an \((n-4)\) dimensional supersymmetric cycle.

To elucidate the above idea of splitting, let us look at the case of five dimensions, where \(\Omega\) is a one form
\[
\Omega = n = n_\mu dx^\mu .
\] (11)

Since
\[
n \wedge (\ast F) = n \wedge n \wedge F = 0 ,
\] (12)
we see that the curvature \(F\) is transverse to \(n\) in the sense that
\[
n_\mu F^{\mu \nu} = 0
\] (13)

Furthermore, the generalized self-dual instanton equation reduces to four dimensional instanton equation on the transverse subspace to the vector \(n_\mu\). Thus in five dimensions \(\Omega\) defines the normal direction to the four dimensional plane where an instanton “lives”. In general we have to make it clear what is the geometrical meaning of \(\Omega\). In the above argument the normal direction is defined at each point. On a curved manifold we would have a curve whose cotangent vector is given by the one form \(\Omega = n_\mu dx^\mu\). We may think of this curve as a world line of a superparticle. Thus in higher dimensions we are naturally led to the idea of the world volume of a super \(p\)-brane and supersymmetric cycles.

Before concluding this section we would like to point out an additional issue in the energy-momentum tensor of higher dimensional self-dual instantons. In four dimensions both self-duality and anti-self-duality of the curvature imply the vanishing of the energy-momentum tensor. This property allows us to consider instantons on Ricci flat manifolds such as \(K3\) surface and ALE spaces without disturbing Ricci-flatness. They are consistent with the Einstein equation. Unfortunately, this nice feature is lost in higher dimensions. If the energy momentum vanishes, we have
\[
g_{\mu \nu} T^{\mu \nu} = \frac{1}{4} (4-n) \text{Tr} (F_{\mu \nu} F^{\mu \nu}) = 0 .
\] (14)

If \(n \neq 4\), this means the action density must vanish and hence it gives a trivial solution. We should examine a physical consistency of higher dimensional instanton on a Ricci-flat manifold.
3 Supersymmetric Cycles

In this section we review the idea of supersymmetric cycles following [16] and [17]. Consider a world volume $\Sigma$ of $p$-brane with local coordinates $(\sigma^a)$ and an embedding of $\Sigma$ into the superspace $\mathcal{SM}$ with bosonic and fermionic coordinates $(X^\mu, \theta^a)$. When $p = 1$, this is a standard framework of the Green-Schwarz formulation of superstring theory. We ask what is a supersymmetric configuration that is purely bosonic. A supersymmetry transformation of $X(\sigma)$ is automatically vanishing, since we have no fermionic backgrounds. The non-trivial condition arises from the requirement of the vanishing of SUSY variation of fermionic coordinates;

$$\delta_{\text{SUSY}} \theta = \epsilon,$$

where $\epsilon$ is a global SUSY parameter of the space-time $\mathcal{SM}$. As was pointed out in [16] we have to take into account the kappa-symmetry of super $p$-brane action;

$$\delta_\kappa \theta = P_+ \kappa(\sigma).$$

The projection operator $P_\pm$ is defined by

$$P_\pm = \frac{1}{2} \left( 1 \pm \frac{1}{(p+1)!\sqrt{h}} \epsilon^{a_1 \cdots a_{p+1}} \Pi_{a_1}^\mu \cdots \Pi_{a_{p+1}}^\mu \Gamma_{\mu_1 \cdots \mu_{p+1}} \right),$$

where

$$\Pi_\mu^a = \partial_\mu X - i \left( \bar{\theta} \Gamma_\mu \partial_a \theta - \partial_a \bar{\theta} \Gamma_\mu \theta \right)$$

is a supercovariant vielbein and $\Gamma_{\mu_1 \cdots \mu_{p+1}}$ is the totally anti-symmetric product of space-time gamma matrices $\Gamma_\mu$. The induced metric on the world volume is $h_{\mu \nu} = \Pi_\mu^a \Pi_\nu^a g_{\mu \nu}$ and $h$ is its determinant. Since we are interested in purely bosonic background, we can neglect the fermion bilinear terms in (18).

Now a $(p + 1)$ cycle $\Sigma$ is supersymmetric, if it is invariant under supersymmetry transformation modulo $\kappa$-symmetry. For purely bosonic configurations this means that the translation in fermionic direction has to be compensated by $\kappa$-symmetry;

$$\frac{3}{2} \epsilon \in \text{Im } P_+ \iff \frac{3}{2} \epsilon \in \text{Ker } P_-.$$

We have

$$P_- \epsilon = 0,$$

for a SUSY cycle $\Sigma$. Once the homology class of $(p+1)$ cycle is fixed, the supersymmetric cycles are volume minimizing. We can see this property as follows; for any $(p + 1)$ cycle $\Sigma$, or a world volume of $p$-brane, we have

$$\int_{\Sigma} d^{p+1} \sigma \sqrt{h} (P_- \epsilon) (P_- \epsilon) = \int_{\Sigma} d^{p+1} \sigma \sqrt{h} \tau (P_- \epsilon) \geq 0,$$

5
where the equality holds only for SUSY cycles. Substituting the form of the projection $P_-$, we obtain a lower bound

$$\text{vol}(\Sigma) = \int_{\Sigma} d^{p+1}\sigma \sqrt{h} \tau \epsilon \geq \int_{\Sigma} X^*(\Omega)$$

(22)

where we have used a normalization $\tau \epsilon = 1$ and

$$\Omega = \left(\tau \Gamma_{\mu_1\cdots\mu_{p+1}}\epsilon\right) dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_{p+1}},$$

(23)

is a $(p + 1)$ form on the space-time. The right hand side of eq. (22) depends only on the homology class of $\Sigma$ and it gives the lower bound of the volume of the $(p + 1)$ cycles with a fixed homology class. In supersymmetric field theory a bound of this type is called BPS bound. We see the SUSY cycles are BPS saturated states. For SUSY cycles that saturate the bound the volume form $\text{vol}(\Sigma)$ is given by the pull-back $X^*(\Omega)$ of the space-time $(p + 1)$ form $\Omega$ and this means

$$\Omega|_{\Sigma} = \text{vol}(\Sigma),$$

(24)

if we identify the $(p + 1)$ cycle with its image in the space-time.

According to [18, 19, 20], a closed form that satisfies the inequality of the type (22) defines a calibration on the manifold. Then, a cycle with the property (24) is called $\Omega$-calibrated submanifold. The geometry of SUSY cycles is the theory of calibrated submanifolds in differential geometry. The space-time $(p + 1)$ form $\Omega$ defined by eq. (23) is closed, if $\epsilon$ is covariantly constant. Thus we are led to consider the manifolds of special holonomy such as Calabi-Yau, hyperKähler, $G_2$ and $Spin(7)$ (or Joyce) manifolds that allow a covariantly constant spinor. It is known that all manifolds in this class are Ricci-flat. In the following we assume $\epsilon$ is a covariantly constant spinor, so that we may use the $(p + 1)$ form $\Omega$ to define the generalized self-dual equation (2).

4 Topology of Higher Dimensional Instantons

Let us consider the generalized self-dual equation (2) with a calibration $\Omega$ introduced in the last section. The $(n - 4)$ form $\Omega$ is used to define SUSY cycles on an $n$-dimensional manifold $M^n$ of special holonomy. Following the argument in section 2, we will make a splitting of local coordinates around a point $x \in M^n$ into “transverse” four dimensional ones $(y^1, \cdots, y^4)$ and the remaining coordinates $(z^1, \cdots, z^{n-4})$. The transverse direction may be regarded as a fibre in the normal bundle $N(\Sigma)$ over the $(n - 4)$ dimensional base space $\Sigma$ with the coordinates $(z^1, \cdots, z^{n-4})$. At this stage we do not assume that the cycle $\Sigma$ is supersymmetric. It will be derived from the generalized self-dual condition (2).

We then consider a conformal rescaling along the transverse directions. This rescaling may be equivalent to a rescaling of the metric, which is a standard trick in topological
dimensional reduction \cite{21,22}. But here we will not assume that the manifold $M^n$ is a direct product, which is a usual ansatz in the procedure of dimensional reduction. Though our consideration will be restricted to a neighborhood of the point $x$ we chose, as we will discuss later our reduction is definitely different from the dimensional reduction. This partial rescaling of four dimensional fibre of $N(\Sigma)$ is made possible by the conformal invariance of the Yang-Mills action in four dimensions. In four dimensional case such a rescaling produces a point-like instanton. Therefore, it is expected that the partial rescaling enables us to see an analogue of point-like instanton that is localized on the $(n-4)$ dimensional cycle $\Sigma$. Under the rescaling of the transverse coordinates

$$y^i \longrightarrow \lambda y^i ,$$

the scaling of the curvature is

$$F = \lambda^{-2}F_{yy} + \lambda^{-1}F_{yz} + F_{zz} ,$$

and the Yang-Mills action is transformed into

$$\int |F|^2dydz = \int |F_{yy}|^2dydz + \lambda^2 \int |F_{yz}|^2dydz + \lambda^4 \int |F_{zz}|^2dydz .$$

Thus to keep the action finite in the limit $\lambda \rightarrow \infty$, both the components $F_{yz}$ and $F_{zz}$ should be suppressed and we have a reduced self-dual condition

$$\Omega \wedge F_{yy} = *F_{yy} .$$

in this limit. We can think of the equation (28) as the condition satisfied by the higher dimensional analogue of point-like instanton that is stable under the partial rescaling. To make the meaning of eq.(28) more transparent, we write the $(n-4)$ form $\Omega$ as

$$\Omega = \alpha(dz^1 \wedge \cdots \wedge dz^{n-4}) + \tilde{\Omega} .$$

The first term is proportional to the volume form of the base space $\Sigma$ and

$$\tilde{\Omega}|_{\Sigma} = 0 .$$

By the definition of *-operator the right hand side of (28) has to be proportional to the volume form of $\Sigma$. Hence, we must have

$$\alpha F_{yy} = *_4 F_{yy} ,$$

where *$_4$ stands for the Hodge dual on the transverse four dimensional fibre. Since the square of the Hodge dual operator is one in four dimensions, $\alpha = 1$ and we finally obtain

$$\Omega|_{\Sigma} = dz^1 \wedge \cdots \wedge dz^{n-4} .$$
Hence, $\Sigma$ has to be a SUSY $(n - 4)$ cycle with respect to the $(n - 4)$ form $\Omega$. Moreover, we have

$$F_{yy} = *_4 F_{yy},$$

(33)

the four dimensional self-dual condition in the transverse direction!

We have made a partial four dimensional rescaling to get an idea on the higher dimensional analogue of point like instantons. The picture we have obtained is a family of four dimensional instantons over a $(n - 4)$ dimensional SUSY cycle. Note that this is not a direct product of the SUSY cycle $\Sigma$ and self-dual instantons. The moduli (e.g. the size and shape) of the instantons may in principle depend on the coordinates of the SUSY cycle $\Sigma$. To be mathematically more precise, the four dimensional transverse direction should be identified with the fibre in the normal bundle $N(\Sigma)$ of the SUSY cycle $\Sigma$. Thus what we have seen is that higher dimensional instantons can be topologically reduced to those on the normal bundle $N(\Sigma)$ over an $(n - 4)$ dimensional submanifold $\Sigma$, which is necessarily an $\Omega$-calibrated submanifold. Note that the dimensions of the total space of the normal bundle $N(\Sigma)$ is the same as the original manifold $M^n$. Our gauge theory is still defined in $n$-dimensions. It is not to be confused with the dimensional reduction to the submanifold $\Sigma$.

To illustrate the above idea let us look at an explicit example. Some time ago Fubini-Nicolai and Fairlie-Nuyts constructed an octonionic instanton [23],[24]. (See also [25].) Their solution is to be regarded as an instanton on the flat eight dimensional Euclidean space $\mathbb{R}^8$ and its action is divergent as follows from a general argument in section 2. However the octonionic instanton of FNFN well illustrates some of the features we have discussed. The octonionic instanton equation is the generalized self-dual instanton equation (2) in eight dimensions with the Cayley four form

$$\Omega = \frac{1}{4!} C_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d,$$

(34)

where $C_{abcd}$ is defined in terms of the structure constants of the algebra of octonions and $\{ e^a = e^a_\mu dx^\mu \}$ is an orthonormal frame (veilbein). The four form $\Omega$ is self-dual ($*\Omega = \Omega$) in eight dimensions and closed on a $Spin(7)$ manifold [26]. On such a manifold there is a unique covariantly constant spinor $\epsilon$ and $\Omega$ is also expressed by

$$C_{\mu\nu\rho\sigma} = C_{abcd} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma = (\gamma \Gamma_{\mu\nu\rho\sigma} \epsilon).$$

(35)

The constant fourth rank tensor $C_{abcd}$ satisfies the following identity [27],[28]:

$$C_{abcd}C^{cdst} = 6(\delta^c_a \delta^d_b - \delta^d_a \delta^c_b) - 4C_{ab}^{\\ cd}.$$

(36)

Hence the linear map

$$C : F \mapsto *(\Omega \wedge F),$$

(37)
on the space of two-forms $\Lambda^2(\mathbb{R}^8)$ satisfies
\[
(C - 2E)(C + 6E) = 0 ,
\]
and the eigenvalues of $(1/2)C$ are 1 and $-3$. We obtain the eigenspace decomposition;
\[
\Lambda^2(\mathbb{R}^8) = E(1) \oplus E(-3) .
\]
Since the linear map $C$ is traceless and $\dim \Lambda^2(\mathbb{R}^8) = 28$, we have
\[
\dim E(1) = 21, \quad \dim E(-3) = 7 .
\]
The octonionic instanton equation
\[
F_{\mu\nu} = \frac{1}{2} C_{\mu\nu\rho\sigma} F^{\rho\sigma}
\]
means the curvature has no components in $E(-3)$.

Let $G_{\mu\nu}$ be generators of $Spin(7)$. From an ansatz for the gauge field;
\[
A_{\mu} = G_{\mu\nu} \partial_\nu f ,
\]
we obtain the following solution to (41);
\[
A_{\mu} = - \frac{2}{3} \frac{1}{\rho^2 + x^2} G_{\mu\nu} x_\nu ,
\]
where $x^\mu$ are Euclidean coordinates of $\mathbb{R}^8$ and $\rho$ is a parameter in integration that represents the “size” of the instanton. Using the commutation relation of $Spin(7)$ generators, we can compute the curvature
\[
F_{\mu\nu} = \frac{2}{3} \frac{2\rho^2 + x^2}{(\rho^2 + x^2)^2} G_{\mu\nu} + \frac{4}{3} \frac{x_{[\mu} G_{\nu]\lambda} x^\lambda}{(\rho^2 + x^2)^2} - \frac{2}{9} \frac{x^\lambda x^\sigma C_{\mu\nu\lambda\kappa} G_{\kappa}^\rho}{(\rho^2 + x^2)^2} .
\]
Then the density of the second Chern class is
\[
\text{Tr} F_{[\mu\nu} F_{\lambda\kappa]} = \frac{3}{(\rho^2 + x^2)^2} C_{\mu\nu\lambda\kappa} + \frac{4(4\rho^2 + x^2)}{(\rho^2 + x^2)^4} x_{[\mu} C_{\nu\omega\kappa] x^\omega} .
\]
this special combination of four indices, the density of the second Chern class is reduced to
\[
\text{Tr } F_{[MNFPQ]} = \left(3\rho^4 + z^2(4\rho^2 + y^2 + z^2)/(\rho^2 + y^2 + z^2)^4\right).
\]
(46)

Note that if we chose a four dimensional plane that is not calibrated, then \(C_{MNFPQ} = 0\) on the plane and we could not see such a topological density. Finally the integration on the transverse coordinates gives
\[
\int \text{Tr } F_{[MNFPQ]} d^4y = \frac{\pi}{2} C_{MNFPQ}.
\]
(47)

We find that the second Chern class is independent of the scale parameter \(\rho\) and moreover of the coordinates \(z^i\) of the SUSY cycle \(\Sigma\). Hence the topological type of the self-dual instantons over the SUSY cycle is universal. This fact suggests that, if we could take a compact base space \(\Sigma\), there would be a chance for the total action to be finite. From \(46\) we see that the “shape” of the instantons does depend on the coordinates \(z^i\) and the octonionic instanton does not have a direct product structure. But the dependence of the instanton moduli on the coordinates of the SUSY cycle \(\Sigma\) cannot be arbitrary. From the result in \[22\] on a topological dimensional reduction of octonionic instantons, we expect the triholomorphic condition would control such dependence. In the limit of the scaling of the transverse coordinates \(y^a \rightarrow \lambda y^a\) together with the scaling of the parameter \(\rho \rightarrow \lambda \rho\), we recover the standard action density of one instanton solution;
\[
\text{Tr } F_{[MNFPQ]} = \frac{3\rho^4}{(\rho^2 + y^4)^4}.
\]
(48)

Thus after the rescaling of the four dimensional fibre, we find exactly a four dimensional instanton.

5 Conclusion and Outlook

Higher dimensional instanton equation can be used in constructing cohomological Yang-Mills theories and the geometry of the moduli space may lead a new development in mathematics. However, as was discussed in section 2, there are a few issues in a physically acceptable interpretation of these instantons as solutions in the higher dimensional gauge theory. These are closely related to the fact that the standard Yang-Mills action is conformal invariant only in four dimensions. Motivated by this fact, we have paid attention to an \((n - 4)\) dimensional submanifold \(\Sigma\) and made a scaling transformation along the four dimensional fibre of the normal bundle \(N(\Sigma)\). We have shown that \(\Sigma\) has to be supersymmetric and reduced higher dimensional instantons to configurations on the total space of \(N(\Sigma)\). Though this picture of a fibration of four dimensional instantons
over a SUSY cycle gives only a local characterization of higher dimensional instantons, we believe that this is a good starting point to discuss the moduli problem of higher dimensional instantons. As we have emphasized before, this is a natural analogue of point-like (ideal) instantons in four dimensions.

There is a nice formulation of small-size instantons in the framework of D-branes and this also explains the ADHM construction of four dimensional instantons [29]. It is an interesting challenge to develop a similar formulation in terms of D-brane for a family of instantons over a SUSY cycle. Eventually a resolution of the issues mentioned above might be obtained by embedding the instantons in the brane picture of string theory or M theory.

In algebraic geometry a compactification of the instanton moduli space can be discussed as the problem of holomorphic vector bundles. It is interesting to compare the construction discussed in this paper, which is differential geometrical, with the idea from algebraic geometry. The case of holomorphic vector bundles on Calabi-Yau three-folds or four-folds seems to be a good example for the comparison.

Acknowledgements

I am grateful to the organizers of the workshop for providing a chance to give a talk. Among the participants I would like to thank E. D’Hoker, T. Eguchi, M. Mariño and Y. Yasui for discussions and comments. The initial stage of this work was done, when I visited MIT last summer. I would like to thank G. Tian for illuminating discussion and I.M. Singer for warm hospitality.

This work is supported in part by the Grant-in-Aid for Scientific Research on Priority Area 707 ”Supersymmetry and Unified Theory of Elementary Particles” and No. 10640081, from Japan Ministry of Education.
References

[1] L. Baulieu, H. Kanno and I.M. Singer, Commun. Math. Phys. 194 (1998) 149, hep-th/9704167.

[2] L. Baulieu, H. Kanno and I.M. Singer, Cohomological Yang-Mills Theory in Eight Dimensions, In: Dualities in Gauge and String Theories, Eds. Y. M. Cho and S. Nam, World Scientific, (1998) 365, hep-th/9705127.

[3] B.S. Acharya, M. O'Loughlin and B. Spence, Nucl. Phys. B503 (1997) 657, hep-th/9705138.

[4] H. Nakajima, J. Math. Soc. Japan 40 (1988) 383.

[5] S.K Donaldson and R.P. Thomas, Gauge Theory in Higher Dimensions, In: The Geometric Universe (Oxford, 1996), Oxford University Press (1998) 31.

[6] G. Tian, Gauge Theory and Calibrated Geometry I, MIT preprint 1998.

[7] A. Strominger, Nucl. Phys. B343 (1990) 167; Erratum, B353 (1991) 565; J.A. Harvey and A. Strominger, Commun. Math. Phys. 151 (1993) 221.

[8] B.S. Acharya, J.M. Figueroa-O’Farrill, M. O’Loughlin and B. Spence, Nucl. Phys. B514 (1998) 583, hep-th/9707118.

[9] M. Blau and G. Thompson, Phys. Lett. B415 (1997) 242, hep-th/9706225.

[10] M.R. Douglas, Branes within Branes, hep-th/9512077; J. Geom. Phys. 28 (1998) 255, hep-th/9604198.

[11] E. Corrigan, C. Devchand, D.B. Fairlie and J.Nuyts, Nucl. Phys. B214 (1983) 452.

[12] G.H. Derrick, J. Math. Phys. 5 (1964) 1252.

[13] S. Coleman, Classical lumps and their quantum descendants, Erice Lecture (1975); In Aspects of Symmetry, Cambridge University Press (1985).

[14] S. Weinberg, The Quantum Theory of Fields II, section 23.1, Cambridge University Press (1996).

[15] S.K. Donaldson and P.B. Kronheimer, The Geometry of Four-Manifolds, Oxford University Press (1990).

[16] K. Becker, M. Becker and A. Strominger, Nucl. Phys. B456 (1995) 130, hep-th/9507158.
[17] M. Bershadsky, V. Sadov and C. Vafa, Nucl. Phys. B463 (1996) 420, hep-th/9511222.

[18] R. Harvey and H.B. Lawson, Jr., Acta. Math. 148 (1982) 47.

[19] R. Harvey, Spinors and Calibrations, Academic Press (1990).

[20] R.C. McLean, Comm. Anal. Geom. 6 (1998) 705.

[21] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, Nucl. Phys. B448 (1995) 166, hep-th/9501096.

[22] J.M. Figueroa-O’Farrill, C. Köhl and B. Spence, Nucl. Phys. B521 (1998) 419, hep-th/9710082.

[23] S. Fubini and H. Nicolai, Phys. Lett. B155 (1985) 369.

[24] D.B. Fairlie and J. Nuyts, J. Phys. A17 (1984) 431.

[25] T.A. Ivanova, Phys. Lett. B315 (1993) 277; T.A. Ivanova and A.D. Popov, Lett. Math. Phys. 24 (1992) 85; Theor. Math. Phys. 94 (1993) 225.

[26] D.D. Joyce, Invent. Math. 123 (1996) 507.

[27] R. Dünderer, F. Gürsey and C.-H. Tze, J. Math. Phys. 25 (1984) 1496.

[28] B. de Wit and H. Nicolai, Nucl. Phys. B231 (1984) 506.

[29] E. Witten, Some comments on string dynamics, hep-th/9507121, Nucl. Phys. B460 (1996) 541, hep-th/9511030.