New Methods of finding support planes of cut-level fuzzy power sets and Geometry of the convex fuzzy power sets

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Abstract. The concepts of the convex fuzzy power sets and crisply defined membership functions in fuzzy set theory are important to deeply analyze to look further for more precise solutions of the fuzzy mathematical programming problems. The Extension principle was defined in terms of the level cuts of the membership functions. The theorem of the decomposition proves the existence of the fuzzy power set as the union of all level cuts of membership function. The Convexity of the level cut fuzzy set was determined at the points of the universe, where the membership function is approximated. The Theorems of the convex and quasi-convex fuzzy sets extend and expand the extension principle to the higher quality to look for more precise solutions of fuzzy problems beyond the support planes. The Fuzzy geometry and topological categories researched further to arrive to the newest Lemma for the cyclic quadrilateral convex fuzzy sets. The analytical and graphical representation of modeling with membership functions type I through types 6 are investigated to the point to be highly recommended in engineering problems by using heuristic methods and models.

1. ↓-level-cut fuzzy sets and their decomposition

Definition 1.1. A membership function in subset of \( \mathcal{F} \subseteq X \), where \( X \) is the universe with the elements of \( x \in X \), which is, alternatively, referred as the characteristic function, can be defined as it is:

\[
\phi_x(x) = \begin{cases} 
1 & \text{iff } x \in \mathcal{F} \\
0 & \text{iff } x \notin \mathcal{F}
\end{cases}
\]

here \( \phi_x(x): \mathcal{F} \rightarrow (0,1) \), where value 1 contributes to a membership, while 0 is a non-membership [1,2,3,4].

Definition 1.2. If \( \text{Range}(\phi_x \rightarrow [0,1]) \in \mathbb{R} \), where \( [0,1] \) is the real numbers closed interval , then \( \phi_x(x) \) is the class of membership of \( x \in \mathcal{F} \subseteq X \) in \( \mathcal{F} \).

Based on definition 1.1 we describe it as closer membership function approaches to 1, then \( x \) is more attributes to \( \mathcal{F} \).

Definition 1.3. A fuzzy set is the ordered set of pair, such as:

\( \mathcal{F} = \{(x, \phi_x(x)), x \in X\} \).

There is given geometrical representation of the graph as the bell-shaped curve which represents membership function and its corresponding fuzzy set, defined as the cartesian product of all elements of the ordered pairs of nonempty set of \( \mathcal{F} \). (Figure 1).
**Definition 1.3.** Fuzzy set as a bijective membership function or one-to-one correspondence membership function is an ordinary interval-valued fuzzy set, where an element of a fuzzy set is assigned to closed intervals of the real numbers:

\[ \phi_F(x) : F \rightarrow [0,1], \text{ where } [0,1] \text{ is the set consisting of intervals of the real numbers in } [0,1]. \]

![Figure 1. Membership function and its corresponding fuzzy set.](image)

**Definition 1.4.** The set of the elements of the fuzzy set of \( F \), where membership function is the element of the fuzzy set to the certain level of \( i \) is defined as the \( i \)-level cut set:

\[ \{x \in F \mid \phi_F(x) \geq i\}, \text{ or, if the inequality is strict: } \phi_F(x) > i, \text{ then is defined as the strict } i \text{-level cut set.} \]

The graph or geometrical representation of the \( i \)-level cut set is given at Figure 2.

**Definition 1.5.** The decomposition of the \( i \)-level cut sets is the fuzzy set defined as the union of the fuzzy sets:

\[ F_i(x) = \bigcup_{i} [0,1] F_i \text{ ... } U. \]

![Figure 2. The \( i \)-level cut fuzzy set and corresponding membership function.](image)
Theorem 1.1 (Decomposition of the fuzzy power sets). Let $\mathcal{F} = \bigcup_{i \in [0,1]} \mathcal{F}_i(x), \mathcal{F} \in \mathcal{F}(x)$ be the union of $i$–level cut sets, where $\mathcal{F}_i(x) = i \mathcal{F}^{-i}(x)$. Then $\mathcal{F}(x)$ is the fuzzy power set.

Proof: Let us denote the membership function of the fuzzy set as the corresponding membership grade of supporting cuts using summation as it is:

$$\mathcal{F}_{i_1}(x) = \frac{i_1}{x_1} + \frac{i_2}{x_2} + \ldots + \frac{i_n}{x_n} = \sum_{j=1}^{n} \Phi_{\mathcal{F}}(i_j)/x_j,$$

$$\mathcal{F}_{i_2}(x) = 0/\frac{i_2}{x_2} + \ldots + \frac{i_n}{x_n} = \sum_{j=1}^{n} \Phi_{\mathcal{F}}(i_j)/x_j,$$

$$\ldots$$

$$\mathcal{F}_{i_n}(x) = 0/\frac{i_n}{x_n} = \sum_{j=1}^{n} \Phi_{\mathcal{F}}(i_n)/x_j.$$

There is symbol $\slash$ is utilized to show relationship between an element of the universal domain and the membership grade of $x$ in the formula here.

Moreover, there $0$ is the empty set $\emptyset, x \in \mathcal{F} \subseteq X, \Phi_{\emptyset}(x) = 0, \forall x \in \mathcal{F}, \Phi_{\mathcal{F}}(x) = i_n$, and $i_1, i_2, \ldots, i_n$ are the $i$–level cut sets of $\Phi_{\mathcal{F}}(x), i \in [0,1]$. Thereafter, $\mathcal{F}_{i_1} \cup \mathcal{F}_{i_2} \cup \ldots \cup \mathcal{F}_{i_n} = \bigcup_{i \in [0,1]} \mathcal{F}_i(x) = \mathcal{F}(x). \square$

2. Convexity of the $i$–level cut fuzzy sets.

Definition 2.1. A fuzzy set is convex if and only if

$$\Phi_{\mathcal{F}}(px_1 + (1 - p)x_2) > \max(\Phi_{\mathcal{F}}(x_1), \Phi_{\mathcal{F}}(x_2)), \forall x_1, x_2 \in X \subseteq \mathbb{R}, p \in (0,1) \ (2.1)$$

Definition 2.2. A fuzzy set is convex on $X$ if and only if all its $i$–level cut sets are convex based on $2.1$.

If a fuzzy set doesn’t satisfy the conditions of $2.1$, then, this type of a fuzzy set is not convex.

The Figure 3 consists of two geometrical illustration: (a) graphical illustration of the convex fuzzy set, (b) graphical illustration of the fuzzy set, which is not convex.

![Figure 3(a). Convex level cut fuzzy set.](image-url)
Definition 2.3. If fuzzy set is non-empty convex fuzzy set based on (2.1) and $\phi_F(x): X \rightarrow \{0,1\} \in \mathbb{R}$, then a fuzzy set is quasi-convex if for $x_1, x_2 \in \mathcal{F} \subseteq X$, the following inequality holds on:

$$\phi_F(\rho x_1 + (1 - \rho)x_2) \geq \max(\phi_F(x_1), \phi_F(x_2)).$$

Theorem 2.1 (Convexity of the $i$–level cuts of the fuzzy sets). If $\mathcal{F}$ is non-empty fuzzy set and $\phi_F: X \rightarrow (0,1) \in \mathbb{R}$, then $i$–level cuts fuzzy sets $\mathcal{F}_i (x) = \{x, \phi_F(x)\}$, $\phi_F(x) \geq i$, are quasi-convex for any $i \in (0,1) \in \mathbb{R}$, $i(\rho x_1 + (1 - \rho)x_2) \geq \max(i(x_1), i(x_2))$, $x_1, x_2 \in \mathcal{F}_i \subseteq X$, $i(x): X \rightarrow (0,1) \in \mathbb{R}$.

Proof: Let us choose $\tilde{x}$ as element of the support plane of the level cut fuzzy set and suppose, there exists neighborhood around this point such as $N_{\epsilon}(\tilde{x}), \tilde{x} \in \mathcal{F}_i \subseteq N_{\epsilon}(\tilde{x})$.

Let us choose another point $\tilde{x} \neq \tilde{x}, \hat{x} \in \mathcal{F}_i$, and $\phi_{F_i}(\tilde{x}) \geq \phi_{F_i}(\hat{x})$. If we use inequality (2.2), then we arrive to the convexity: $\phi_{F_i}(\rho \tilde{x} + (1 - \rho)\hat{x}) \geq \max(\phi_{F_i}(\tilde{x}), \phi_{F_i}(\hat{x})) = \phi_{F_i}(\hat{x})$. Since $\rho \tilde{x} + (1 - \rho)\hat{x} \in \mathcal{F}_i, \hat{x} \in \mathcal{F}_i \cap N_{\epsilon}(\tilde{x})$, then we have arrived to the statement, which contradicts the condition of the convexity at the point $\tilde{x}$.

Zadeh [3] has proposed the Extension principle, which applies to the large class of the fuzzy sets and membership functions, whether a fuzzy set is convex or non-convex and membership function is a bijection or one-to-one or not a bijection.

Definition 2.4. The height or the least upper bound of the fuzzy set of $\mathcal{F}_i$ is defined as $\text{height}(\mathcal{F}_i) = \sup_{x \in \mathcal{X}} \phi_{F_i}(x)$.

In a case, where a membership function is not a bijection, the Extension Principle [1, 2, 3, 4, 5, 7] states that the fuzzy images of fuzzy arguments are located at the support planes or the heights of the fuzzy sets: $\phi_F(i) = \text{height}(\mathcal{F}_i) = \sup \phi_F(x) \geq i$.

Extension Principle: Let’s consider a function from $\mathcal{X}$ to $\mathcal{Y}$: $y = \phi_F(x): \mathcal{X} \rightarrow \mathcal{Y}, x \in \mathcal{X}, y \in \mathcal{Y}$.

Case 1 for bijection: If $\phi_F(x)$ is one-to-one mapping, then

$$\phi_F(y) = \phi_F(\phi^{-1}(y)) = \phi_F(x).$$

(1)

Case 2 for non-bijection: If $\phi$ is not one-to-one function, then there is the following rule applies:

$$\phi_F(y) = \sup_{x \mid y = \phi_F(x)} \phi_F(x) = \text{height}(\mathcal{F})$$

(2)
3. Fuzzy geometry of the fuzzy power sets
The following Lemma refers to the Fuzzy Geometry, which is considered as the elongation of the applications of the geometry properties to the fuzzy convex sets [14, 15, 16, 17, 18, 19]. Fuzzy geometry involves the geometrical properties such as topological categories as perimeter, area, length, adjacency, similarity, equivalency, compactness and so on. These geometrical categories are utilized in fuzzy logic to cogitate and reflect on spatial inexactness of the image analysis and dissemination [20].

![Geometrical illustration of the extension principle for non-bijection of the membership functions.](image1)

**Figure 4.** Geometrical illustration of the extension principle for non-bijection of the membership functions.

**Definition 2.5.** Fuzzy point $\overline{x}_0 \in \mathbb{R}$ as the kernel of the convex fuzzy sets is referred to the monotonically decreasing membership functions.

Figure 5 is the geometrical illustration of the fuzzy points with right circular cone membership function.

![Geometrical illustration of the fuzzy points with circular cone membership function.](image2)

**Figure 5.** Geometrical illustration of the fuzzy points with circular cone membership function.
**Definition 2.6.** If fuzzy bounded domain in $X \in \mathbb{R}$ contains two normal fuzzy convex sets, then this geometrical topologies refers to the fuzzy interval $[\bar{x}_1, \bar{x}_2]$ as the kernel, where the membership function is monotonically decreasing to 0.

Figure 6 is the geometrical representation of the fuzzy interval.

![Figure 6. Geometrical representation of the fuzzy intervals.](image)

The following lemma is based on the Butterfly theorem in geometry [13], which is representation of the topological categories of the geometry in fuzzy sets such as the intervals, lengths, similarity, and similar plane figures.

**Lemma:** Cyclic quadrilateral convex fuzzy sets.

Through the midpoint $M$ of the $\ell$–level cut of the membership function as the chord in any circle or fuzzy interval $I_0$, any other fuzzy intervals such as the chords $I_1, I_2$ are drawn, and the other fuzzy intervals $I_3, I_4$ such as the chords intersect $I_0$ at points $F_x, F_y$, respectively. Then, the point of $M$ is the midpoint the fuzzy interval $I_{[F_x,F_y]}$.

There is given Figure 7, which provides the geometric illustration of the cyclic quadrilateral convex fuzzy sets.

![Figure 7. Geometrical illustration of the cyclic quadrilateral convex sets.](image)
Proof: Let us denote the following:

1. \( I_0 = AB, M = \frac{1}{2} (B - A), AM = MB, \)
2. \( I_1 = CD, I_2 = EG, I_3 = CG, I_4 = ED, \)
3. \( CG, ED \) intersects \( I_0 \) at points \( F_x, F_y \) and \( M \) is the midpoint of these points: \( M = \frac{1}{2} (F_y - F_x) \).

Next, let us draw perpendiculars \( h_1 \perp I_1, h_2 \perp I_2, h_3 \perp I_2, h_4 \perp I_1 \) (Figure 8).

Figure 8 illustrates the geometrical view of the relationship between the chords of the circle as the fuzzy intervals.

As the result of these drawings, we have obtained four pairs of the similar triangles. Based on the properties of the similar triangles there are the following ratios exist there: let \( I_5 = AM = MB, I_6 = F_xM, I_7 = F_yM, I_8 = F_xC, I_9 = F_yE, I_{10} = F_xG, I_{11} = F_yD \) and \( I_5 = I_6 = I_7 = I_8 = \frac{h_1}{h_4}, I_9 = I_{10} = I_{11} = \frac{h_1}{h_4}, I_9 = I_{10} = I_{11} = \frac{h_1}{h_4} \).

The intersecting chords theorem \([13]\) yields the following products: \( I_5 \cdot I_{10} = F_xA \cdot F_yB, F_yE \cdot F_yD = F_yA \cdot F_yB, \) and let denote \( AM = MB = t \).

Thus, \( I_5^2 = \frac{h_1}{h_4}, h_2 = \frac{h_1}{h_4}, h_3 = \frac{h_1}{h_4}, I_9 = I_{10} = \frac{I_5}{I_7} = \frac{F_xA}{F_yE}F_yB = \frac{F_yA}{F_yB} \) for \( \frac{I_5}{I_7} = \frac{F_yA}{F_yB} \), which yields that \( I_6 = I_7, M = \frac{1}{2} (F_y - F_x) \).

Next theorem provides with the opportunity to assess the values of the characteristic function at two points of the universe of the domain to find the support plane of the membership functions.

This theorem assists to find the location of support planes of fuzzy sets to enhance the extension principle to the more accurate values of the approximation of the membership functions. Moreover, the theorem enables to approximate the value of membership function regardless whether a function is a bijection or not a bijection to locate the elements of the fuzzy sets at the support plane of a membership function.

**Theorem 3.1:** Let \( \mathcal{F} \) be non-empty quasi-convex fuzzy set on \( [x_1, x_2] \). Consider \( \psi, \omega \in [x_1, x_2] \), such that \( \psi < \omega \).

(a): If \( \phi_{\mathcal{F}}(\psi) > \phi_{\mathcal{F}}(\omega) \), then there exists \( \bar{\chi} \in [x_1, x_2] \) such that \( \phi_{\mathcal{F}}(\bar{\chi}) > \phi_{\mathcal{F}}(\psi) \) for \( \bar{\chi} \in [x_1, \psi] \).

Then \( \phi_{\mathcal{F}}(\bar{\chi}) = \sup \phi_{\mathcal{F}}(x) \) at \( \bar{\chi} \in [x_1, \psi] \).

(b): If \( \phi_{\mathcal{F}}(\psi) < \phi_{\mathcal{F}}(\omega) \), then \( \phi_{\mathcal{F}}(\bar{\chi}) > \phi_{\mathcal{F}}(\omega) \) for \( \bar{\chi} \in (\omega, x_2] \).

Then \( \phi_{\mathcal{F}}(\bar{\chi}) = \sup \phi_{\mathcal{F}}(x) \) for \( \bar{\chi} \in (\omega, x_2] \).
Proof. (a) Let \( \phi_F(\psi) > \phi_F(\omega) \) and \( \tilde{h} \in [x_1, \psi] \). Next, let us make conjecture that \( \phi_F(\tilde{h}) < \phi_F(\psi) \). Since \( \psi \) is the convex combination of \( \bar{\omega} \), \( \bar{\omega} \) and a membership function is strictly quasi-convex, we obtain that \( \phi_F(\psi) \leq \max\{\phi_F(\bar{\omega}), \phi_F(\bar{\omega})\} = \phi_F(\bar{\omega}) \). However, this result contradicts that \( \phi_F(\tilde{h}) > \phi_F(\bar{\omega}) \). Therefore, \( \phi_F(\tilde{h}) = \sup \phi_F(x) \).

(b) This part is proved by using similar contradiction.

The following theorem is for the strictly quasi-convex fuzzy sets over the given interval \([x_1, x_2]\) and the midpoint of this interval \( \tau = \frac{1}{2}(x_2 - x_1) \) is the convex combination of the other endpoints.

**Theorem 3.2.** Let \( F \) be non-empty strictly quasi-convex fuzzy set on \([x_1, x_2]\). Let \( \tau = \frac{1}{2}(x_2 - x_1) \) and \( u > 0 \), where \( u \) is very small. Next, let \( \sigma \in [x_1, \tau - u], \kappa \in (\tau + u, x_2] \), such that \( \sigma < \tau < \kappa \).

1. If \( \phi_F(\tau) > \phi_F(\kappa) \), then \( \phi_F(h) \geq \phi_F(\kappa) \) for \( h \in [x_1, \tau] \). If \( \phi_F(\tau) \leq \phi_F(\kappa) \), then \( \phi_F(h) \geq \phi_F(\tau) \) for \( h \in (\kappa, x_2] \).

2. If \( \phi_F(\tau) > \phi_F(\sigma) \), then \( \phi_F(h) \geq \phi_F(\sigma) \) for \( h \in (\tau, x_2] \). If \( \phi_F(\tau) \leq \phi_F(\sigma) \), then \( \phi_F(h) \geq \phi_F(\tau) \) for \( h \in [x_1, \sigma] \).

Proof. (1) Let \( \phi_F(\tau) > \phi_F(\kappa) \) and \( h \in [x_1, \tau] \). Using contradiction, let \( \phi_F(h) < \phi_F(\kappa) \). \( \tau \) is convex combination of \( h, \kappa \), then using definition of the strictly convex fuzzy sets we obtain \( \phi_F(\tau) < \max\{\phi_F(h), \phi_F(\kappa)\} = \phi_F(\kappa) \). However, this results bring us to contradiction, since, there was supposed that \( \phi_F(\tau) > \phi_F(\kappa) \). Therefore, \( \phi_F(h) \geq \phi_F(\kappa) \).

(2) Let \( \phi_F(\tau) \leq \phi_F(\kappa) \) and \( h \in (\kappa, x_2] \). Using contradiction let \( \phi_F(h) < \phi_F(\tau) \). Since \( \kappa \) is a convex combination of \( h, \tau \) and using the definition of the strictly convex fuzzy sets we have \( \phi_F(h) < \max\{\phi_F(h), \phi_F(\tau)\} = \phi_F(\tau) \), which contradicts to \( \phi_F(\tau) \leq \phi_F(\kappa) \). Therefore, \( \phi_F(h) \geq \phi_F(\tau) \).

4. Modeling of fuzzy convex sets with functional graphs of the membership functions

There are existing various of methods to estimate and construct the membership functions. The construction of the membership function is important process, especially, in engineering problems.

One of the methods are the heuristic methods for estimation of the membership functions. The heuristic methods to estimate a membership function are based on incomplete data or imprecise information. However, such ambiguity to utilize the partial data serves as the argument in favor to further construct and estimate membership functions since pro features of the heuristic models allow to discover the values of the membership functions with limited algorithmic steps in the universe of infinite continuity [7]. He fuzzy sets can be described by two forms of the representation:

1. **Form 1:** Discrete form of the fuzzy sets, where the fuzzy sets are in the form of discrete pairs \( \phi_F(x)/x \).

2. **Form 2:** Analytical representation, when the fuzzy sets are described as the parametric or piecewise functions.

Functional graphs of the important membership functions type I through type VI for modeling of the convex fuzzy sets are given below:

1. Convex membership functions with sink (Figure 9):
   \[ \phi_F(x) = 1 \text{ for } -\infty \leq x < x_1, \phi_F(x) = (x_2 - x)/(x_2 - x_1) \text{ for } x_1 \leq x \leq x_2, \phi_F(x) = 0 \text{ for } x_2 \leq x \leq x_3, \phi_F(x) = (x - x_3)/(x_4 - x_3) \text{ for } x_3 \leq x \leq x_4, \phi_F(x) = 1 \text{ for } x_4 \leq x \leq \infty. \]
II. Modeling with smoothed convex membership (Figure 10):

$$\phi_F(x) = 1 - e^{-k(x-x_1)^2} \text{ for } k > 0.$$  
Alternatively: $$\phi_F(x) = \frac{1}{1+k(x-x_1)^2} \text{ for } k > 1.$$  

Figure 10. Modeling with smoothed sink membership function.

III. Modeling with concave membership $\phi(x)$ – function with flanks for modeling (Figure 11):

$$\phi_1(x) = e^{-x}, \phi_2(x) = \frac{1}{1+x^2}, \phi_F(x) = \phi_1\left(\frac{x-x_1}{x_1-0}\right) \text{ for } x_1 > x, \phi_F(x) = \phi_2\left(\frac{x-x_2}{x_2-0}\right) \text{ for } x > x_2, \phi_F(x) = 1 \text{ for } x_1 \leq x \leq x_2.$$  

Figure 9. Modeling with sink membership functions.
IV. Modeling with Piece-wise Convex membership L-functions (Figure12):

\[ \phi_F(x) = 1 \text{ for } x < x_1, \phi_F(x) = \left( \frac{x_2 - x}{x_2 - x_1} \right) \text{ for } x_1 \leq x \leq x_2, \phi_F(x) = 0, \text{ for } x > x_2. \]

V. Modeling with Convex Fuzzy membership smoothed L-functions (Figure13):

\[ \phi_F(x) = e^{-kx^2}, k > 0. \]

Alternative smoothed L-function: \[ \phi_F(x) = \frac{1}{1 + kx^2}, k > 1. \]
VI. Modeling with the Membership S-function by Zadeh (Figure 14):

\[ \phi_F(x) = \begin{cases} 
0 & \text{for } x < x_1, \\
2\frac{x-x_1}{x_3-x_1}^2 & \text{for } x_1 \leq x \leq x_2, \\
1 - 2\left(\frac{x-x_3}{x_3-x_1}\right)^2 & \text{for } x_2 \leq x \leq x_3, \\
1 & \text{for } x > x_4.
\]

Figure 14. Modeling with S-membership function.

It worth to note that the construction and, further, utilization of membership functions with analytical formulas and graphical representations in type I-VI is the fundamental key factor to utilize various modeling techniques in the theory and applications of the control systems.

The modeling with membership function(s) type I through VI is highly recommended, especially, when the parameters of the newly constructed membership function are chosen relative to the particular problem.
In a case, if a membership function is chosen and, then approximated to match to the parameters of the particular problem by modeling with type I-VI, then such estimated model is called modeling type of human phenomena problem [5, 6, 7, 8, 9, 10].

5. Conclusion
The concavity and convex feature of the membership functions provides enormous opportunities to describe the fuzzy sets analytically and graphically for further use in fuzzy mathematical programming problems.

The Decomposition of the \( \ell \)-level cut convex fuzzy sets is defined as the union of the fuzzy sets in the article here. This union constitutes the base for the fuzzy power level cut sets. The Decomposition Theorem 1.1. was introduced to mathematically verify and prove the existence of the fuzzy power sets in the article here, too.

The convexity of the \( \ell \)-level cut fuzzy sets was determined at the point of the universe of the membership function. Moreover, the definition of the convexity was refined within the closed intervals of the universe of the fuzzy sets.

Furthermore, the definition of the quasi-convex fuzzy sets was introduced to locate the sub-cut fuzzy level sets. These sub-cut fuzzy levels sets are the solutions to the fuzzy mathematical programming problems.

The Theorem 2.1 was stated here in the article to prove that the convexity of the fuzzy power sets occurs at all cut levels of the membership function.

Zadeh’s original Extension Principle was re-introduced to serve as the platform for the Theorem 2.2. The Theorem 2.2 proves that the partitioned subintervals of the universe directed to the crisply defined membership functions approximated in terms of the level cut sets. The Theorem 2.2 is the significant outreach in terms to extend and further expand the extension Principle to the quality, where the solutions to the fuzzy mathematical programming problems can be found beyond the bijection criteria.

The Fuzzy Geometry of the fuzzy power sets was presented in terms of the topological and geometrical categories such as the length, intervals, similarity, circle, chords, diameters, quadrilaterals and etc. The Fuzzy Geometry categories were described to reflect the spatial imprecision of the image analysis and image distribution in the article here.

There was originally introduced the Lemma for the Cyclic quadrilateral convex fuzzy sets. This Lemma states that there is existing the circle of the fuzzy convex sets, where the geometrical fuzzy categories such as the midpoints of the chords and diameters can be utilized to serve as the intervals of the universe for the solution of the fuzzy mathematical programming problem.

The Theorem 3.1 states that the search for the optimal support planes can be refined in terms of finding of the convex combinations of the points of the universe, where the membership function is strictly quasi-convex. Moreover, the theorem 3.1 helps to find the level cuts of the fuzzy power sets without the use of the Extension Principle. This theorem enormously expands the extension Principle to apply Theorem 3.1 to the broader class of the membership functions.

The Theorem 3.2 states that the mid-point of the interval of the universe is the convex combination of its endpoints. This mid-point fuzzy category can be utilized further to find the optimal level cut set of the membership function, which was done in the article here.

There were presented six types (type I through type VI) of the analytical forms of the membership functions. The graphs of these types of the membership functions were introduced in the article here, too. These types of the modeling are highly recommended in heuristics methods to construct the membership function, which were described in the article here, also.

Acknowledgements
We have to note that all Theorems, Lemma, corresponding proofs and theoretical original text are all original and newly unabridged versions. Moreover, the original text and theorems have not been published or presented to the publications.
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