Comments, suggestions, corrections, and references welcomed!

CAPABLE GROUPS OF PRIME EXPONENT AND CLASS TWO II

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Abstract. We consider the capability of \( p \)-groups of class two and odd prime exponent. We use linear algebra and counting arguments to establish a number of new results. In particular, we settle the 4-generator case, and prove a sufficient condition based on the ranks of \( G/Z(G) \) and \( [G,G] \).

Introduction.

In this work we continue to study the capability of finite \( p \)-groups of class two and prime exponent, using the approach introduced in [6]. We restrict to odd primes, since the case of abelian groups is well understood. I owe a considerable amount of the material here to discussions with David McKinnon, who helped me clarify the ideas and helped with most of the geometry.

A group \( G \) is said to be capable if and only if there exists a group \( K \) such that \( G = K/Z(K) \). For \( p \)-groups (groups of finite prime-power order) capability is closely related to their classification. Baer characterized the capable groups which are direct sums of cyclic groups in [2]; the capable extra-special \( p \)-groups were characterized by Beyl, Felgner, and Schmid in [3] (only the dihedral group of order 8 and the extra-special groups of order \( p^3 \) and exponent \( p \) are capable); they also described the metacyclic groups which are capable. The author characterized the 2-generated capable \( p \)-groups of class two [5, 7] (for odd \( p \), independently obtained in part by Bacon and Kappe in [1]).

For the case of \( p \)-groups of class two and exponent \( p \), where \( p \) is an odd prime, some necessary and some sufficient conditions for capability are known, and so there is a hope expressed in [1] that a full characterization for this class may be tractable with current techniques. We began to study this situation in [6]; there we described a way to translate the problem into a statement of linear algebra, and obtained several results using this restatement. We will introduce what I hope is clearer notation here and obtain new results, including a new sufficient condition. We direct the reader to [6] for basic definitions and notation.

In Section 1 we recap the characterization of capability first described in [5] (along the way we will correct a slight misstatement made in the proof there), and then describe the linear algebra problem that is derived from it. In Section 2 we prove some basic facts about the linear algebra situation. In Section 3 we use a counting argument to derive a sufficient condition for the capability of \( G \), based on the ranks of \( G^{ab} \) and \( [G,G] \). Then in Section 4 we recall some results proven in [5] and obtain some new ones. We will note there the relevance of element of \( G \) which lie in \( Z(G) \) and have nontrivial image in \( G^{ab} \). In Section 5 we will therefore establish a connection between these elements and certain subspaces discussed in Section 3. Finally, in Section 6 we settle the four-generated case for this class.

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1. The set-up.

Throughout $p$ will be used to denote an odd prime. We begin by describing a "canonical witness" to the capability of a finite group $G$ of class two and odd prime exponent. For the definition of the 3-nilpotent product, see [10].

**Theorem 1.1** (Theorem 2.3 in [6]). Let $G$ be a finite noncyclic group of class at most two and exponent an odd prime $p$. Let $g_1, \ldots, g_n$ be elements of $G$ that project onto a basis for $G^{ab}$, and let $F$ be the 3-nilpotent product of $n$ cyclic groups of order $p$, generated by $x_1, \ldots, x_n$ respectively. Let $N$ be the kernel of the morphism $F \to G$ induced by mapping $x_i \mapsto g_i$ for each $i$. Then $G$ is capable if and only if

$$G \cong (F/[N,F])/Z(F/[N,F]).$$

**Proof.** Note that since $[N,F] \subset F_3$ and $N[N,F] \subset Z(F/[N,F])$, it is always the case that the central quotient of $F/[N,F]$ is a quotient of $G$.

The "if" clause of the theorem immediate. For the converse, assume that $G$ is capable, and let $K$ be a group with $K/Z(K) \cong G$. Let $k_1, \ldots, k_n$ be elements of $K$ that project onto $g_1, \ldots, g_n$, respectively. The subgroup of $K$ generated by $k_1, \ldots, k_n$ is cocentral, hence has central quotient isomorphic to $K/Z(K)$; we may thus assume that $K$ is generated by $k_1, \ldots, k_n$.

Note that $K$ is of class at most 3. We claim that $K_p^2 = \{e\}$. Indeed, since $K/Z(K)$ is of exponent $p$, it follows that the $p$-th power of any element of $K$ is in $Z(K)$. So, if $c \in K_2$ and $k \in K$, then

$$e = [c, k^p] = [c, k]^p[c, k, k]^{(\zeta)} = [c, k]^p.$$  

In particular, $K_3$ is of exponent $p$. If $h, k \in K$, then we have:

$$e = [h, k^p] = [h, k]^p[h, k, k]^{(\zeta)} = [h, k]^p,$$

so every commutator is of exponent $p$. Since $K_2$ is abelian, it follows that $K_2$ is of exponent $p$, as claimed.

Let $\mathfrak{F}$ be the 3-nilpotent product of $n$ infinite cyclic groups, and let $\mathcal{F} = \mathfrak{F}/(\mathfrak{F}/2)^p$. Denote the generators of the cyclic groups by $y_1, \ldots, y_n$. Then $\mathcal{F}$ is the relatively free group of rank $n$ in the variety of all groups of class at most 3 in which the commutator subgroup is of exponent $p$. Note that $F \cong \mathcal{F}/\langle y_1^p, \ldots, y_n^p \rangle$, and that under this isomorphism we have an identification of $F_2$ with $\mathcal{F}_2$. Let $\mathcal{N}$ be the subgroup of $\mathcal{F}_2$ corresponding to the subgroup $N$ of $F$.

The map sending $y_i$ to $k_i$ induces a morphism $\mathcal{F} \to K$. Since $\mathcal{N}$ maps to $e$ under the composite map $\mathcal{F} \to K \to K/Z(K)$, it follows that $\mathcal{N}$ maps into $Z(K)$, and hence that the map $\mathcal{F} \to K$ factors through $\mathcal{F}/[\mathcal{N},\mathcal{F}]$. As this map sends the center of the latter group into the center of $K$, it follows that $G$ is a quotient of $(\mathcal{F}/[\mathcal{N},\mathcal{F}])/Z(\mathcal{F}/[\mathcal{N},\mathcal{F}])$.

Using the normal forms for $\mathcal{F}$ (see [10]), it is easy to show that the central quotients of $\mathcal{F}/[\mathcal{N},\mathcal{F}]$ and of $\mathcal{F}/[y_1^p, \ldots, y_n^p, [\mathcal{N},\mathcal{F}]]$ are isomorphic. This latter group is of course isomorphic to $F/[N,F]$, so we conclude that if $G$ is capable, then it is a quotient of $(F/[N,F])/Z(F/[N,F])$.

So if $G$ is capable, then $G$ is isomorphic to a quotient of the central quotient of $F/[N,F]$ (which is a finite group), and in turn has the central quotient of $F/[N,F]$ as a quotient. The only way this is possible is for $G$ to be isomorphic to the central quotient of $F/[N,F]$, as claimed. 

\[\square\]
Remark 1.2. In [6] there was a slight error in the assertion just prior to Theorem 2.3. It was asserted there that \( Z(F/\langle F^p | [N, F] \rangle) \cong Z(F/\langle [N, F] \rangle/F^p \) \), but this assertion is false when \( p = 3 \) (it is correct if \( p > 3 \) by regularity). The subgroup \( F^p \) should be replaced by the subgroup generated by the \( p \)-th powers of the generators, as we did in the proof above.

Since \( G \) is isomorphic to \( F/NF_3 \) and \( F_3 \subset Z(F) \), the following result now follows:

**Theorem 1.3 (Theorem 3.6 in [6]).** Let \( G \) be a finite group of class at most two and exponent an odd prime \( p \). Let \( g_1, \dots, g_n \) be elements of \( G \) that project onto a basis for \( G^{ab} \), and let \( F \) be the 3-nilpotent product of \( n \) cyclic groups of order \( p \), generated by \( x_1, \dots, x_n \) respectively. Let \( N \) be the kernel of the canonical map \( F \to G \) induced by mapping \( x_i \mapsto g_i \) for each \( i \), and let \( C \) be the subgroup of \( F_3 \) spanned by all basic commutators of the form \([x_j, x_i] \), \( 1 \leq i < j \leq n \). Write \( N = X \oplus F_3 \), where \( X \subset C \). Then \( G \) is capable if and only if \( \{ x \in C \mid [x, F] \subset \{ X, F \} \} = X \).

It is at this point that we introduce linear algebra as a way to codify the situation we wish to study: we have abelian group homomorphisms between elementary abelian \( p \)-groups given by \([-,-]_k : F_2 \to F_3 \). Thus, we may interpret it as linear maps between vectors spaces over \( \mathbb{F}_p \). If \( X \) is a subgroup of \( C \), then \([X, F] \) will be the span of the images of \([X, x_i] \) with \( k = 1, \dots, n \). On the other hand, the elements \( x \in C \) such that \([x, F] \) is contained in \([X, F] \) correspond to those elements whose images under all maps \([-,-]_k \) lie in \([X, F] \), i.e., the intersection of the pullbacks of these maps. This suggests the following general construction and definitions:

**Definition 1.4.** Let \( V_1 \) and \( V_2 \) be vector spaces, and let \( \{ \ell_i : V_1 \to V_2 \}_{i \in I} \) be a nonempty family of linear transformations. For every subspace \( X \) of \( V_1 \), we define \( X^* \subset V_2 \) to be the subspace spanned by all the images of \( X \) under the linear transformations \( \ell_i \), that is:

\[
X^* = \left\langle \ell_i(X) \mid i \in I \right\rangle.
\]

For every subspace \( Y \) of \( V_2 \), we define \( Y^* \subset V_1 \) to be the intersection of all pullbacks of \( Y \) under the transformations \( \ell_i \), that is:

\[
Y^* = \bigcap_{i \in I} \ell_i^{-1}(Y).
\]

Note that for all subspaces of \( V_1 \), if \( X \subset X' \), then \( X^* \subset X'^* \); likewise, for subspaces of \( V_2 \), if \( Y \subset Y' \), then \( Y^* \subset Y'^* \).

**Theorem 1.5.** Let \( V_1 \) and \( V_2 \) be vector spaces, and let \( \{ \ell_i : V_1 \to V_2 \}_{i \in I} \) be a nonempty family of linear transformations. The operator on the subspaces of \( V_1 \) defined by \( X \mapsto X^{**} \) is a closure operator; that is, it is increasing, isotone, and idempotent. Moreover, if \( X \) is a subspace of \( V_1 \), then \( X^* = X^{**} \).

**Proof.** It is clear that \( X \subset X^{**} \), so the operator is increasing; since \( X \subset X' \) implies \( X^* \subset X'^* \), which in turn implies \( X^{**} \subset X'^{**} \), the operator is isotone.

To show the operator is idempotent, we want to show that \( X^{**} = (X^{**})^{**} \). For simplicity, write \( X^{**} = Z \). Since the operator is increasing, we know that \( Z \subset Z^{**} \).

By construction, we also have that \( \ell_i(Z) \subset X^* \) for each \( i \), so \( Z^* \subset X^* \). From this, we get that \( Z^{**} \subset X^{**} = Z \), as desired.

Finally, let \( X < V_1 \). Since \( X \subset X^{**} \), we must have \( X^* \subset X^{***} \). Conversely, from the definition of \( X^{***} \) it follows that \( \ell_i(X^{**}) \subset X^* \) for all \( i \), so \( X^{**} \subset X^* \), giving equality. \( \square \)
Remark 1.6. It may be worth noting that this closure operator is algebraic (meaning that the closure of a subspace $X$ is the union of the closure of all finitely generated subspaces $X'$ contained in $X$), though it is not topological (in general, the closure of the subspace generated by $X$ and $Y$ is not the subspace generated by $X''$ and $Y''$).

The dual property is true for subspaces of $V_2$:

**Theorem 1.7.** Let $V_1$ and $V_2$ be vector spaces, and let $\{\ell_i: V_1 \to V_2\}_{i \in I}$ be a nonempty family of linear transformations. The operator on subspaces of $V_2$ defined by $Y \mapsto Y^{**}$ is an interior operator; that is, it is decreasing, isotone, and idempotent. Moreover, if $Y$ is a subspace of $V_2$, then $Y^{**} = Y^{**}$.

**Proof.** As before, the operator is isotone. Since $Y^{**} = \langle \ell_i(Y^{**}) \rangle$, and $Y^{*} \subset \ell^{-1}_i(Y)$ for each $i$, it follows that $Y^{**} \subset Y$, so the operator is decreasing.

To show the operator is idempotent, set $Z = Y^{**}$. Since $Y^{*} \subset Y^{**}$, we must have $Z = Y^{**} \subset Y^{****} = Z^{**}$. The reverse inclusion always holds, so $Z = Z^{**}$, as desired.

Finally, since $Y^{**} \subset Y$, it follows that $Y^{****} \subset Y^{*}$. But since the operator $^{**}$ on subspaces of $V_1$ is increasing, we also know that $Y^{*} \subset Y^{****}$, giving equality. \hfill $\square$

**Definition 1.8.** Let $V_1$, $V_2$ be vector spaces, and let $\{\ell_i: V_1 \to V_2\}_{i \in I}$ be a family of linear transformations. We will say that a subspace $X$ of $V_1$ is $\{\varphi_i\}_{i \in I}$-closed (or simply closed if there is no danger of ambiguity) if and only if $X = X^{**}$.

To tie in this construction with our capability problem, we define specific spaces and maps. We fix an odd prime $p$ throughout. Let $n$ be an integer, $n > 1$. We define three vectors spaces with distinguished bases:

**Definition 1.9.** Let $U(n)$ be the vector space over the finite field $\mathbb{F}_p$ of $p$ elements, with basis vectors $u_1, \ldots, u_n$. Let $V(n)$ be the vector space over $\mathbb{F}_p$ of dimension $\binom{n}{2}$, with basis vectors $v_{ij}$, $1 \leq i < j \leq n$. Let $W(n)$ be the vector space over $\mathbb{F}_p$ of dimension $2\binom{n}{2} + 2\binom{n}{3} = 2\binom{n+1}{3}$, with basis vectors $w_{ijk}$, $1 \leq i < j < k \leq n$.

**Notation 1.10.** For simplicity, we will sometimes use $v_{ij}$ with $i < j$ for an element of $V(n)$; in that case, we understand it to mean that $v_{ij} = -v_{ji}$. Likewise, $v_{ii} = 0$ for each $i$.

If there is no danger of ambiguity, and $n$ is understood from context, we will simply use $U$, $V$, and $W$ instead of $U(n)$, $V(n)$, and $W(n)$. This will be the case in most of our applications.

We define two families of $n$ linear operators:

**Definition 1.11.** The linear maps $\psi_i: U \to V$, $i = 1, \ldots, n$ are defined to be:

$$\psi_i(u_j) = v_{ji} \begin{cases} v_{ji} & \text{if } j > i \\ -v_{ij} & \text{if } j < i \\ 0 & \text{if } i = j. \end{cases}$$

**Definition 1.12.** The linear maps $\varphi_k: V \to W$, $k = 1, \ldots, n$, are defined to be:

$$\varphi_k(v_{ji}) = \begin{cases} w_{jik} & \text{if } k \geq i, \\ w_{jki} - w_{ijk} & \text{if } k < i. \end{cases}$$

Note that there is a natural identification of $V$ with $U \wedge U$, and of $W$ with $(U \wedge U) \otimes U$ modulo the Jacobi identity:

$$(u_i \wedge u_j) \otimes u_k + (u_j \wedge u_k) \otimes u_i + (u_k \wedge u_i) \otimes u_j = 0.$$
Under these identifications, the maps $\psi_i$ correspond to $\psi_i(u) = u \wedge u_i$, and the maps $\varphi_k$ correspond to $\varphi_k(v \wedge w) = (v \wedge w) \otimes u_k$.

The connection between capability and this construction is now apparent: $U$ corresponds to $F_{ab}$ by identifying the generators $u_i$ with $x_i$, $i = 1, \ldots, n$. The space $V$ corresponds to the subgroup $\langle [x_j, x_i] \rangle$ by identifying $v_{ji}$ with the commutator $[x_j, x_i]$, $1 \leq i < j \leq n$. And $W$ corresponds to $F_3$ by identifying $w_{jik}$ with $[x_j, x_i, x_k]$, $1 \leq i < j \leq n$, $i \leq k \leq n$. The maps $\psi_i$ correspond to maps $[-, x_i]: F/F_2 \to F_3/F_3$, while $\varphi_k$ corresponds to the map $[-, x_k]: F_2/F_3 \to F_3$. In particular, if $X$ is a subspace of $V$, then it corresponds to a subgroup $N$ of $F$ contained in $\langle [x_j, x_i] \rangle$; and by Theorem 1.3 it follows that the group $F/NF_3$ is capable if and only if $X = X^{**}$. Therefore, we have:

**Theorem 1.13.** Let $G$ be a finite noncyclic group of class at most two and exponent an odd prime $p$. Let $g_1, \ldots, g_n$ be elements of $G$ that project onto a basis for $G_{ab}$, and let $F$ be the 3-nilpotent product of $n$ cyclic groups of order $p$, generated by $x_1, \ldots, x_n$ respectively. Let $N$ be the kernel of the morphism $F \to G$ induced by mapping $x_i \mapsto g_i$ for each $i$. Let $N = X \oplus F_3$, where $X \subseteq \langle [x_j, x_i] \rangle$ $1 \leq i < j \leq n$, and identify $X$ with the corresponding subspace of $V(n)$. Then $G$ is capable if and only if $X$ is closed with respect to $\{\varphi_i\}_{i=1}^n$.

Thus, the question of which $n$-generated $p$-groups of class at most 2 and exponent $p$ ($p$ odd) are capable is equivalent to the question of which subspaces of $V(n)$ are closed. Note, however, that distinct subspaces may correspond to isomorphic groups: for example, if we permute the indices, we obtain a different $X$ but an isomorphic group $G$.

The reason we include the maps $\psi_i$ may seem a bit more mysterious. Note that if $Z$ is a subspace of $U$, then identifying $V$ with $U \wedge U$ we have that $Z^*$ is none other than $Z \wedge U$. This corresponds to the relations necessary to make the elements of $F$ corresponding to $Z$ central in $F$; these relations have an important interplay with the morphisms $\varphi_i$ and with questions of capability, as we will see in Sections 4 and 5.

2. The linear algebra.

We collect here some observations on the spaces $V$ and $W$, and the linear transformations $\varphi_i$ defined above.

**Definition 2.1.** Let $i, j$ be integers, $1 \leq i < j \leq n$. We let $\pi_{ji}: V \to \langle v_{ji} \rangle$ be the canonical projection.

**Definition 2.2.** Let $i, j, k$ be integers, $1 \leq i < j \leq n$, $i \leq k \leq n$. We let $\pi_{jik}: W \to \langle w_{jik} \rangle$ be the canonical projection.

**Definition 2.3.** Let $i$ be an integer, $1 \leq i \leq n$. We let

$$\Pi_i: V \to \langle v_{i1}, v_{i2}, \ldots, v_{i(i-1)}, v_{i(i+1)} \rangle$$

be the canonical projection.

The following observation follows immediately from the definition of the $\varphi_i$:

**Lemma 2.4.** For each $k$, $\varphi_k$ is one-to-one.

**Lemma 2.5.** Let $w \in \varphi(V)$. If $\pi_{rst}(w) \neq 0$, then $s \leq i \leq t$; moreover, at most one of the inequalities is strict.
Lemma 2.6. Fix $n > 1$, and let $i, j$ be integers satisfying $1 \leq i < j \leq n$. Then
\[ \varphi_i(V) \cap \varphi_j(V) = \{ 0 \}. \]

Proof. Assume $\varphi_i(V) \in \varphi_j(V)$, and that $\pi_{sr}(v) \neq 0$, where $1 \leq r < s \leq n$. If $r \leq i$, then $\pi_{sr}(\varphi_i(V)) \neq 0$; but since $\varphi_i(V) \in \varphi_j(V)$, this implies that $r \leq j \leq i$, which is impossible. If, on the other hand, $i < r$, then $\pi_{sr}(\varphi_i(V)) \neq 0$ and $\pi_{rs}(\varphi_i(V)) \neq 0$; again, since $\varphi_i(V) \in \varphi_j(V)$, Lemma 2.5 implies that $s = r = j$, which is also impossible.

Lemma 2.7. Fix $n > 1$, $r \leq n$. Let $i_1, \ldots, i_r$ be integers, $1 \leq i_1 \leq \cdots \leq i_r \leq n$. Then $\varphi_{i_1}(V) \cap \langle \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \rangle$ is of dimension $(r-1)$, with basis given by all vectors $w_{a_i b} - w_{b i a}$, $a, b \in \{i_2, \ldots, i_r\}$, $a > b$. A basis for the pullback
\[ \varphi_{i_1}^{-1}(\langle \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \rangle) \]
is given by $\{v_{ab}\}$, with $a, b \in \{i_2, \ldots, i_r\}$, $a > b$.

Proof. It is enough to prove the last statement. The result is trivial when $r = 1, 2$, so assume that $r \geq 3$. Certainly each $v_{ab}$ lies in the pullback, since
\[ \varphi_{i_1}(v_{ab}) = w_{a_i b} - w_{b i a} = \varphi_b(v_{a i}) - \varphi_a(v_{b i}). \]

If $w \in \langle \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \rangle$, and $\pi_{sr}(w) \neq 0$, then we must have $s \leq i_j \leq t$ for some $j \in \{2, \ldots, r\}$, and with at most one inequality strict. Let $v \in V$ be a vector such that $\varphi_i(v) \in \langle \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \rangle$. Let $r, s$ be integers, $1 \leq r < s \leq n$, such that $\pi_{sr}(v) \neq 0$. We want to show that $s, r \in \{i_2, \ldots, i_r\}$.

If $r \leq i_1$, then $\pi_{sr}(\varphi_{i_1}(v)) \neq 0$, so we must have $r \leq i_j \leq i_1$ for some $j > 1$, but this is impossible, since $i_j > i_1$. Therefore, we must have $r > i_1$. In that case, we have $\pi_{sr}(v) \neq 0$ and $\pi_{rs}(v) \neq 0$. Thus we obtain that there is some $j > 1$ such that $i_1 \leq i_j \leq r$ and at most one inequality is strict, and there is some $k > 1$ such that $i_1 \leq i_k \leq s$ and at most one inequality is strict. Since $i_1 < i_j, i_k$, it follows that $r = i_j$ and $s = i_k$, as desired.

Corollary 2.8. Fix $n > 1$, $r \leq n$, and let $1 \leq i_1 \leq \cdots \leq i_r \leq n$ be integers. Then
\[ \dim(\langle \varphi_{i_1}(V), \ldots, \varphi_{i_r}(V) \rangle) = r \binom{n}{2} - \binom{r}{3}. \]

Proof. Each $\varphi_k$ is injective, so we have:
\begin{align*}
\dim(\langle \varphi_{i_1}(V), \ldots, \varphi_{i_r}(V) \rangle) &= \left( \sum_{k=1}^{r} \dim(\varphi_{i_k}(V)) \right) - \left( \sum_{k=1}^{r-2} \dim(\varphi_{i-k+1}(V) \cap \langle \varphi_{i_k}(V), \ldots, \varphi_{i-k}(V) \rangle) \right) \\
&= r \binom{n}{2} - \left( \sum_{k=1}^{r} \binom{k-1}{2} \right) = r \binom{n}{2} - \binom{r}{3},
\end{align*}
as claimed.

We have concentrated on the “lowest index” for simplicity. Of course, given the definitions, our treatment has symmetry; for example:

Proposition 2.9. Let $i_1, \ldots, i_r \in \{1, \ldots, n\}$ be pairwise distinct. Then
\[ \varphi_{i_1}(V) \cap \langle \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \rangle \]
has dimension \( \binom{n-1}{2} \). Moreover, a basis for the pullback is given by the vectors \( v_{ab} \), with \( a, b \in \{i_1, \ldots, i_r\} \), and \( a > b \).

**Proof.** It is easy to verify that
\[
\varphi_i(V) \cap \left( \varphi_{i_2}(V), \ldots, \varphi_{i_r}(V) \right)
\]
is generated by the vectors \( w_{ab_i} \) for \( a, b \in \{i_2, \ldots, i_r\}, a, i_1 > b \), and the vectors \( w_{ai_1b} - w_{bi_1a} \) when \( b > i_1 \). These vectors are linearly independent, and pulling them back gives the desired result. \( \square \)

A somewhat different description of the intersections will be useful in the following sections.

**Definition 2.10.** Fix \( n > 1 \). We define \( \Phi : V^n \rightarrow W \) to be
\[
\Phi(v_1, \ldots, v_n) = \varphi_1(v_1) + \cdots + \varphi_n(v_n).
\]
If there is danger of ambiguity, we use \( \Phi_n \) to denote the map corresponding to \( n \).

**Theorem 2.11.** Fix \( n > 1 \). The kernel of \( \Phi \) is of dimension \( \binom{n}{3} \). A basis for \( \ker(\Phi) \) is as follows: for each choice of \( a, b, c \), \( 1 \leq a < b < c \leq n \), the vector \( (v_1, \ldots, v_n) \in V^n \) with
\[
v_i = \begin{cases} v_{cb} & \text{if } i = a; \\ -v_{ca} & \text{if } i = b; \\ v_{ba} & \text{if } i = c; \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Denote the element corresponding to \( a, b, c \) by \( v_{(abc)} \). Note that each \( v_{(abc)} \) lies in \( \ker(\Phi) \):
\[
\Phi(v_{(abc)}) = \varphi_a(v_{cb}) + \varphi_b(-v_{ca}) + \varphi_c(v_{ba}) = (w_{cab} - w_{bac}) - w_{cab} + w_{bac} = 0.
\]
Since \( \Phi \) is surjective, \( \dim(W) = n \dim(V) - \dim(\ker(\Phi)) \); therefore
\[
\dim(\ker(\Phi)) = n \binom{n}{2} - 2 \binom{n+1}{3} = \binom{n}{3}.
\]
Thus, it is enough to show that the \( v_{(abc)} \) are linearly independent. Assume that \( \sum \beta_{abc} v_{(abc)} = 0 \), where the sum is taken over all triples of integers \( a, b, c \) that satisfy \( 1 \leq a < b < c \leq n \). Considering the \( i \)-th component alone, we obtain
\[
\sum_{1 \leq r < s < i} \beta_{rsi} v_{sr} = \sum_{1 \leq r < i} \beta_{ris} v_{sr} + \sum_{i < r < s \leq n} \beta_{irs} v_{sr} = 0.
\]
Thus, for each choice of \( a, b, c \) with \( i \in \{a, b, c\} \), we must have \( \beta_{abc} = 0 \). This proves the given \( v_{(abc)} \) are linearly independent, and hence form a basis for \( \ker(\Phi) \). \( \square \)

**Notation 2.12.** Fix \( n > 1 \), and let \( a, b, c \) be pairwise distinct integers, \( 1 \leq a, b, c \leq n \). We will let \( v_{(abc)} \) denote the element of \( \ker(\Phi) \) described in the statement of Theorem 2.11.

**Theorem 2.13.** Let \((v_1, \ldots, v_n) \in \ker(\Phi)\). Write
\[
v_k = \sum_{1 \leq i < j \leq n} \alpha_{ji}^{(k)} v_{ji},
\]
(i) If \( i = k \) or \( j = k \), then \( \alpha_{ji}^{(k)} = 0 \); i.e., \( \Pi_k(v_k) = 0 \).
(ii) If \(1 \leq a < b < c \leq n\), then \(\alpha_{ba} = \alpha_{cb} = -\alpha_{ca}\).

(iii) Fix \(i, j, 1 \leq i < j \leq n\). Then

\[
\Pi_i(v_j) = \sum_{r=1}^{i-1} (-\alpha_{jr}^{(i)}) v_{ir} + \sum_{r=i+1}^{j-1} \alpha_{jr}^{(i)} v_{ri} + \sum_{r=j+1}^{n} (-\alpha_{rj}^{(i)}) v_{ri},
\]

\[
\Pi_j(v_i) = \sum_{r=1}^{i-1} (-\alpha_{ir}^{(j)}) v_{jr} + \sum_{r=i+1}^{j-1} \alpha_{ir}^{(j)} v_{jr} + \sum_{r=j+1}^{n} (-\alpha_{ri}^{(j)}) v_{rj}.
\]

Proof. The first part follows either from Proposition 2.13 or from the description of the basis in Theorem 2.11. For part (ii), note that

\[
\pi_{bac}(\varphi_1(v_1) + \cdots + \varphi_n(v_n)) = (\alpha_{ba}^{(c)} - \alpha_{cb}^{(a)}) w_{bac},
\]

\[
\pi_{eab}(\varphi_1(v_1) + \cdots + \varphi_n(v_n)) = (\alpha_{ca}^{(b)} + \alpha_{cb}^{(a)}) w_{cab}.
\]

Since they must both be equal to zero, we obtain that \(\alpha_{ba}^{(c)} = \alpha_{cb}^{(a)}\) and \(\alpha_{ca}^{(b)} = -\alpha_{cb}^{(a)}\), as claimed. Finally, for (iii), we know that \(\Pi_i(v_i) = \Pi_j(v_j) = 0\) from (i), so we can write:

\[
\Pi_i(v_j) = \sum_{r=1}^{i-1} \alpha_{ir}^{(j)} v_{jr} + \sum_{r=i+1}^{j-1} \alpha_{ir}^{(j)} v_{ri} + \sum_{r=j+1}^{n} \alpha_{ri}^{(j)} v_{rj},
\]

and applying (ii) gives the desired identities. \(\square\)

Corollary 2.14. Let \(v \in \ker(\Phi)\). \(\Pi_j(v_i) = 0\), then \(\Pi_i(v_j) = 0\). In particular, if \(v_i = 0\), then \(\Pi_i(v_j) = 0\) for all \(j\).

Proof. If \(\Pi_j(v_i) = 0\), then \(\alpha_{jr}^{(i)} = 0\) for all \(r\), so by Theorem 2.13(iii) it follows that \(\Pi_i(v_j) = 0\). The second assertion follows immediately. \(\square\)

Corollary 2.15. Let \(v \in \ker(\Phi)\), \(v \neq 0\). If \(v = (v_1, \ldots, v_n)\) then the dimension of \(\langle v_1, \ldots, v_n \rangle\) is at least 3.

Proof. Write

\[
v = \sum_{1 \leq a < b < c \leq n} \beta_{abc}v_{abc}
\]

Fix \(a, b, c\) such that \(1 \leq a < b < c \leq n\), \(\beta_{abc} \neq 0\). We claim that \(v_a\), \(v_b\), and \(v_c\) are linearly independent. Indeed, note that \(\Pi_a(v_a) = \Pi_b(v_b) = \Pi_c(v_c) = 0\), and \(\pi_{cb}(v_a) \neq 0\). Therefore, if \(\alpha_a v_a + \alpha_b v_b + \alpha_c v_c = 0\), then we must have \(\alpha_a = 0\). A symmetric argument looking at \(\pi_{ca}\) shows that \(\alpha_b = 0\), and considering \(\pi_{ba}\) shows that \(\alpha_c = 0\). \(\square\)

In [3] we proved the following result by considering the collection of images of a basis of \(X\) under the maps \(\varphi_1, \ldots, \varphi_n\), and showing they would necessarily be linearly independent. We give a different proof here based on the considerations above.

Corollary 2.16 (Prop. 4.6 in [3]). Fix \(n > 1\), and let \(X\) be a subspace of \(V\). If \(\dim(X) = 1\), then \(\dim(X^*) = n\); if \(\dim(X) = 2\), then \(\dim(X^*) = 2n\).
Proof. We prove the contrapositive. Since \( \dim(X^*) = n \dim(X) - \dim(X^n \cap \ker(\Phi)) \), if \( \dim(X^*) < n \dim(X) \), then \( X^n \cap \ker(\Phi) \neq \{0\} \).

Let \( \mathbf{v} = (v_1, \ldots, v_n) \in X^n \cap \ker(\Phi) \), \( \mathbf{v} \neq \mathbf{0} \). Then \( v_i \in X \) for \( i = 1, \ldots, n \), so by Corollary 2.14, \( \dim(X) \geq 3 \), as claimed.

As will become apparent in Section 6 it would be useful to make Corollary 2.15 somewhat more precise. Unfortunately it appears we cannot do this easily, as the next sequence of results shows. We will show that for any \( k, 3 \leq k \leq n, k \neq 4 \), there exists \( (v_1, \ldots, v_n) \in \ker(\Phi_n) \) such that the span of \( v_1, \ldots, v_n \) is \( k \)-dimensional. I do not know whether one can also obtain a 4-dimensional span.

Lemma 2.17. Fix \( n > 1 \), and let \( k \) be an integer, \( 3 \leq k \leq n \). If there exists \( (v_1, \ldots, v_n) \in \ker(\Phi_n) \) such that \( \dim(\langle v_1, \ldots, v_n \rangle) = k \), then for any \( m \geq n \) there exists \( (w_1, \ldots, w_m) \in \ker(\Phi_m) \) such that \( \dim(\langle w_1, \ldots, w_m \rangle) = k \).

Proof. We can embed \( \ker(\Phi_n) \) into \( \ker(\Phi_{n+1}) \) by mapping \( (v_1, \ldots, v_n) \in \ker(\Phi_n) \) to \( (v_1, \ldots, v_n, \mathbf{0}) \). The dimension of the span of the components is clearly unaffected by this embedding.

Lemma 2.18. Fix \( n > 1 \). If \( (v_1, \ldots, v_n) \in \ker(\Phi_n) \), then
\[
\mathbf{w} = (v_1, \ldots, v_n, \mathbf{0}, \mathbf{0}, \mathbf{0}) + \mathbf{v}_{(abc)} \in \ker(\Phi_{n+3}),
\]
where \( a = n + 1 \), \( b = n + 2 \), and \( c = n + 3 \) (using the notation described in 2.12), and if we let \( \mathbf{w} = (w_1, \ldots, w_{n+3}) \), then
\[
\dim(\langle w_1, \ldots, w_{n+3} \rangle) = \dim(\langle v_1, \ldots, v_n \rangle) + 3.
\]

Proof. The first part follows by embedding \( \ker(\Phi_n) \) into \( \ker(\Phi_{n+3}) \) by appending three zeros; then both vectors lie in \( \ker(\Phi_{n+3}) \), hence so does their sum \( \mathbf{w} \). For the second part, simply note that \( w_i = v_i \) for \( 1 \leq i \leq n \), and \( w_{n+1} = v_{n+3,n+2} \), \( w_{n+2} = -v_{n+3,n+1} \), and \( w_{n+3} = v_{n+2,n+1} \). This adds three to the dimension of the span of \( v_1, \ldots, v_n \), since \( \Pi_k(v_i) = \mathbf{0} \) for \( 1 \leq i \leq n, k > n \).

Lemma 2.19. If \( k = 3\ell \) or \( k = 3\ell + 2, \ell \geq 1 \), and \( n \geq k \), then there exists \( (v_1, \ldots, v_n) \in \ker(\Phi_n) \) such that \( \dim(\langle v_1, \ldots, v_n \rangle) = k \). The result also holds if \( k = 3\ell + 1 \) and \( \ell \geq 2 \).

Proof. For \( k = 3 \), any nontrivial element of \( \ker(\Phi_3) \) will do. Then we can apply Lemmas 2.17 and 2.18 to obtain the result whenever \( k \) is a multiple of 3. For \( k = 5 \) and \( n = 5 \), the element \( v_{(123)} + v_{(145)} = (v_{32} + v_{54}, -v_{31}, v_{21}, -v_{51}, v_{41}) \in \ker(\Phi_5) \) gives an example of dimension 5, and again applying Lemmas 2.17 and 2.18 we obtain the result whenever \( k \equiv 2 \pmod{3}, k \geq 5 \).

If \( k = 7 \), then we have \( v_{(123)} + v_{(145)} + v_{(176)} \in \ker(\Phi_7) \) which gives an example of dimension 7, from which we can obtain any \( k \equiv 1 \pmod{3}, k \geq 7 \) as above.

I currently do not know if one can find an element of \( \ker(\Phi_n) \) which will yield a subspace of dimension exactly four for some \( n \). We can, however, show that it is impossible if \( n = 4 \):

Proposition 2.20. If \( (v_1, v_2, v_3, v_4) \) is nontrivial and lies in \( \ker(\Phi_4) \), then
\[
\dim(\langle v_1, v_2, v_3, v_4 \rangle) = 3.
\]
Proof. From Theorem 2.13 it follows that we can write the vectors $v_i$ as follows:

\[
\begin{align*}
    v_1 &= \beta_{123}v_{32} + \beta_{124}v_{42} + \beta_{134}v_{43}, \\
    v_2 &= -\beta_{123}v_{34} - \beta_{124}v_{41} + \beta_{234}v_{43}, \\
    v_3 &= \beta_{123}v_{21} - \beta_{134}v_{41} - \beta_{234}v_{42}, \\
    v_4 &= \beta_{124}v_{21} + \beta_{134}v_{31} + \beta_{234}v_{32},
\end{align*}
\]

for some choice of coefficients $\beta_{123}$, $\beta_{124}$, $\beta_{134}$, and $\beta_{234}$. If not all $\beta_{abc}$ are equal to zero, then we have the following nontrivial linear relation between the four vectors:

\[
\beta_{234}v_1 + \beta_{124}v_3 = \beta_{134}v_2 + \beta_{123}v_4.
\]

By Corollary 2.16 the four vectors span a space of dimension at least three. Thus, the subspace is of dimension exactly three, and if $\beta_{abc} \neq 0$, then a basis for the subspace is given by $v_a$, $v_b$, and $v_c$. This establishes the result for $n = 4$. \hfill \Box

3. Dimension counting.

In this section we will obtain bounds for $\dim(X^*)$ in terms of $\dim(X)$. To see why this is interesting, consider the following two observations:

Proposition 3.1. Let $X < V$. Assume that for all subspaces $Y$ of $V$, if $Y$ properly contains $X$ then $Y^*$ properly contains $X^*$. Then $X = X^{**}$.

Proof. If $X^{**}$ properly contains $X$, then $X^{***}$ would properly contain $X^*$. But $X^{***} = X^*$, a contradiction. \hfill \Box

Corollary 3.2. Fix $n > 1$, and suppose that $f(k)$ is a function such that for all $k$-dimensional subspaces of $V$, $\dim(X^*) \geq nk - f(k)$. If $f(k+1) < n$, then all subspaces of dimension $k$ are closed.

Proof. It is trivial that $\dim(X^*) \leq n \dim(X)$. If $f(k+1) < n$ then for subspaces $X$ and $Y$ with $\dim(X) = k$ and $\dim(Y) = k + 1$ we have

\[
\dim(Y^*) \geq n(k+1) - f(k+1) > nk \geq \dim(X^*),
\]

and thus, by Proposition 3.1 it follows that all $X$ of dimension $k$ are closed. \hfill \Box

The smallest possible value for $f(k)$ is given by

\[
f(k) = \max \{ \dim(X^* \cap \ker(\Phi)) \mid \dim(X) = k \}.
\]

Our objective in this section is to find an expression for $f(k)$ in terms of $k$. The main workhorse in our calculations will Lemma 3.4 below. The idea is to find the value of $\dim(X^* \cap \ker(\Phi))$ by examining the “partial intersections”; namely, the intersections of the form

\[
\ker(\Phi) \cap \langle \{0, \ldots, 0, v_i, v_{i+1}, \ldots, v_n\} \mid v_j \in X \rangle,
\]

as $i$ ranges from 1 to $n - 2$ (when $i = n - 1$ or $i = n$, the intersection is trivial by Corollary 2.15). For a fixed $i$, we can consider the subspace of $X$ consisting of all vectors $v_i$ which can be “completed” to an element of $\ker(\Phi)$ by appending 0 prior to it, and any vector of $X$ after. This is the same as considering the pullbacks $X \cap \varphi_i^{-1}(\langle \varphi_{i+1}(X), \ldots, \varphi_n(X) \rangle)$. It is easy to verify that the sum of the dimensions of these pullbacks is equal to the dimension of $X^* \cap \ker(\Phi)$. We will first use the dimension of these pullbacks to establish a lower bound for the dimension of $X$;
then we will turn around and use these calculations to give an upper bound for the dimension of the pullbacks in terms of the dimension of $X$.

Making the bounds as precise as possible, however, requires one to keep track of a lot of information; this in turn requires the use of multiple indices and subindices in the proof, for which I apologize in advance. To illustrate the ideas and help the reader navigate through the proof, we will first present an example. This is not an example in the sense of a specific $X$, but rather a run-through the main part of the analysis we will perform with specific values for the indices and some of the variables.

**Example 3.3.** Set $n = 6$, and let $X$ be a subspace of $V$. We will be interested in bounding above the dimension of $Z_i$ in terms of $\dim(X)$, where

$$Z_i = X \cap \varphi_i^{-1}\left(\langle \varphi_{i+1}(X), \ldots, \varphi_6(X) \rangle \right):$$

i.e., $Z_i$ consists of all $v \in X$ for which there exist $v_{i+1}, \ldots, v_6$ in $X$ such that

$$(0, \ldots, 0, v, v_{i+1}, \ldots, v_6) \in X^6 \cap \ker(\Phi).$$

To do this, we will obtain a lower bound for $\dim(X)$ in terms of $\dim(Z_i)$. To further fix ideas, set $i = 2$. Note that by Lemma 2.7 (or Theorem 2.13) we must have $\Pi_1(Z_2) = \Pi_2(Z_2) = 0$. Assume that the dimension of $Z_2$ is 4. Order all pairs $(j, i)$ lexicographically from right to left, so $(j, i) < (b, a)$ if and only if $i < a$, or $i = a$ and $j < b$. Then considering all pairs in order, find out which pairs $(b, a)$ have $\pi_{ba}(Z_2) \neq 0$. In this example, say that it is all possible pairs:

$$(4, 3), (5, 3), (6, 3), (5, 4), (6, 4), (6, 5).$$

Doing row reduction, we can find a basis $v_{1,2}$, $v_{2,2}$, $v_{3,2}$, and $v_{4,2}$ for $Z_2$ (the second index refers to the fact that these vectors are in the second component of an element of $\ker(\Phi)$), satisfying that the “leading pair” (smallest nonzero component) of each is strictly smaller than that of its successors, and all other vectors have zero component for that pair. For example,

$$v_{1,2} = v_{43} + \alpha_1 v_{53} + \alpha_2 v_{64}, \quad v_{3,2} = v_{54} + \gamma v_{64},$$

$$v_{2,2} = v_{63} + \beta v_{64}, \quad v_{4,2} = v_{65},$$

for some coefficients $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{F}_p$. We know there exist vectors $v_{i,3}$, $v_{i,4}$, $v_{i,5}$, $v_{i,6}$ such that $(0, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,6}) \in X^6 \cap \ker(\Phi)$ for $i = 1, 2, 3, 4$. Naturally, $X$ contains all twenty vectors, but there will normally be some linear dependencies between them: some may even be equal to 0. We want to extract a subset that we know is linearly independent in some systematic fashion. First let us consider the information we can obtain about these vectors from our knowledge of the vectors $v_{i,2}$.

Since $(0, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,6})$ lies in $\ker(\Phi)$, we can use Theorem 2.13(iii) to describe the $\Pi_2$-image of each vector $v_{i,j}$, where $i \leq 2$ and $j > 2$. The $\Pi_1$-image must be trivial, and for the $\Pi_2$ image we obtain the following:

$$\Pi_2(v_{1,3}) = v_{42} + \alpha_1 v_{52}, \quad \Pi_2(v_{2,3}) = v_{62},$$

$$\Pi_2(v_{1,4}) = -v_{32} + \alpha_2 v_{62}, \quad \Pi_2(v_{2,4}) = \beta v_{62},$$

$$\Pi_2(v_{1,5}) = -\alpha_1 v_{32}, \quad \Pi_2(v_{2,5}) = 0,$$

$$\Pi_2(v_{1,6}) = -\alpha_2 v_{42}, \quad \Pi_2(v_{2,6}) = -v_{32} - \beta v_{42}.$$
\[ \Pi_2(v_{3,3}) = 0, \quad \Pi_2(v_{4,3}) = 0, \]
\[ \Pi_2(v_{3,4}) = \gamma v_{52} + \gamma v_{62}, \quad \Pi_2(v_{4,4}) = 0, \]
\[ \Pi_2(v_{3,5}) = -v_{42}, \quad \Pi_2(v_{4,5}) = v_{62}, \]
\[ \Pi_2(v_{3,6}) = -\gamma v_{42}, \quad \Pi_2(v_{4,6}) = -v_{52}. \]

One way to obtain these without too much confusion is as follows: to find \( \Pi_2(v_{j,k}) \), go through the expression for \( v_{j,k} \) replacing all indices \( k \) by 2, remembering that \( v_{ab} = -v_{ba} \). Any \( v_{ba} \) in which neither \( a \) nor \( b \) are equal to \( k \) are simply removed.

To systematically extract from this list a set of linearly independent vectors, we proceed as follows: consider all the pairs which are leading components of the basis vectors; in this case, \((4,3), (6,3), (5,4), \) and \((6,5)\). The individual indices that occur are 3, 4, 5, and 6. For each of them, we identify the smallest pair in which it occurs. Thus, 3 first occurs in pair number one, as does 4. The index 5 first occurs in pair number three, and 6 first occurs in pair number two.

Since the first pair in which 3 appears is the first pair, \((4,3)\), if we consider the vector we obtain when we replace the index 4 (the other index in the pair we found for 3) from the vector which has \((4,3)\) as its leading component, we will obtain a vector whose first nontrivial component is \((3,2)\). That is, the vector \( v_{1,4} \) (replacing 4 in the first vector). The next index is 4, again in the first pair. If we replace the other index, 3, in the first vector, i.e., when we look at \( v_{1,3} \), we obtain a vector with nontrivial \((4,2)\), component, and for which all \((j,i)\)-components with \((j,i) < (4,2)\) are trivial. Then take the index 5: it first occurs in the third pair, paired with 4, so the vector we obtain by replacing 4 in the third vector, i.e., the vector \( v_{3,4} \), is a vector with nontrivial \((5,2)\) component and trivial \((j,i)\) component for all \((j,i) < (5,2)\). For the index 6 we will take \( v_{2,3} \) (since 6 first occurs in the second pair, paired with 3) which gives a vector with nontrivial \((6,2)\) component and trivial \((j,i)\) component for all \((j,i) < (6,2)\). In summary, we want to consider, in addition to the basis for \( Z_2 \), the vectors \( v_{1,4}, v_{1,3}, v_{3,4}, v_{2,3} \) corresponding, respectively, to the indices 3, 4, 5, and 6. The choices we have made ensure that the \( \Pi_2 \)-images of these vectors are linearly independent, and so the vectors themselves must be linearly independent. Since \( \Pi_2(Z_2) = 0 \), the subset formed of the basis for \( Z_2 \) together with these four vectors is a linearly independent subset of \( X \); so we can conclude that \( X \) must have dimension at least 8. What is more, note that none of these last four vectors will occur in a similar analysis involving \( Z_3 \): when performing a similar analysis, all vectors will have trivial \( \Pi_i \)-image when \( i < 3 \). Note as well that the number of indices, in this case 4, must satisfy \( \dim(Z_2) \leq \binom{4}{2} \), since we need to be able to obtain at least \( \dim(Z_2) \) pairs out of the indices that occur. \( \square \)

What ensures that this process will work the way we want is how we choose the vectors of the basis and the vectors that “correspond” to each index. The former count towards the value of \( \dim(X \cap \ker(\Phi)) \), while the latter may be removed from consideration when we move on to \( Z_{i+1} \). This is all done in generality in the proof of the following promised lemma:

**Lemma 3.4.** Let \( X \) be a subspace of \( V \). For each \( i, 1 \leq i \leq n, \) let
\[ Z_i = X \cap \varphi_i^{-1}\left(\langle \varphi_{i+1}(X), \ldots, \varphi_n(X) \rangle \right); \]
Proof. Fix \(d\) and \(r_i\) or \(\pi\) for which we may assume \(k \leq 1\). Again we may assume for simplicity, write \(v = r_{i_0}\). By Theorem 2.13 if \(v \in Z_{i_0}\) then \(\Pi_i(v) = 0\) for all \(i \leq i_0\).

Let \(v_{i_0}, \ldots, v_{r_{i_0}}\) be a basis for \(Z_{i_0}\). We will modify it as follows:

Order all pairs \((j, i), i_0 < i < j \leq n\) by letting \((j, i) < (b, a)\) if and only if \(i < a\) or \(i = a\) and \(j < b\) (lexicographically from right to left). Let \((j_1, i_1)\) be the smallest pair for which \(\pi_{j_1i_1}(v_{k_{i_0}}) \neq 0\) for some \(k, 1 \leq k \leq r\). Reordering if necessary we may assume \(k = 1\). Replacing \(v_{i_0}\) with a scalar multiple of itself and adding adequate multiples to the remaining \(v_{k_{i_0}}\) if necessary we may also assume that

\[
\pi_{j_1i_1}(v_{k_{i_0}}) = \begin{cases} v_{j_1i_1} & \text{if } k = 1; \\ 0 & \text{if } k \neq 1. \end{cases}
\]

Let \((j_2, i_2)\) be the smallest pair for which \(\pi_{j_2i_2}(v_{k_{i_0}}) \neq 0\) for some \(k, 2 \leq k \leq r\). Again we may assume \(k = 2\), and that

\[
\pi_{j_2i_2}(v_{k_{i_0}}) = \begin{cases} v_{j_2i_2} & \text{if } k = 2; \\ 0 & \text{if } k \neq 2. \end{cases}
\]

Proceeding in the same way for \(k = 3, \ldots, r\), we obtain an ordered list of pairs \((j_1, i_1) < (j_2, i_2) < \cdots < (j_r, i_r)\) and a basis \(v_{1_{i_0}}, \ldots, v_{r_{i_0}}\) such that

\[
\pi_{j_\ell i_\ell}(v_{k_{i_0}}) = \begin{cases} v_{j_\ell i_\ell} & \text{if } \ell = k; \\ 0 & \text{if } \ell \neq k; \end{cases}
\]

and such that \(\pi_{b_{a}}(v_{k_{i_0}}) = 0\) for all \((b, a) < (j_k, i_k)\). Write \(v_{k_{i_0}} = \sum_{i_0 < i < j \leq n} \alpha_{(k, i_0)}^{(j, i)} v_{j_1}\).

From the above we have:

\[
\alpha_{ji}^{(k, i_0)} = \begin{cases} 1 & \text{if } (j, i) = (j_k, i_k), \\ 0 & \text{if } (j, i) < (j_k, i_k). \end{cases}
\]

For \(k = 1, \ldots, r\) and \(i = i_0 + 1, \ldots, n\), let \(v_{ki}\) be vectors in \(X\) such that

\[(0, \ldots, 0, v_{k_{i_0}}, v_{k_{i_1}+1}, \ldots, v_{kn}) \in \ker(\Phi) \cap X^m.\]

By Theorem 2.13(iii) we have

\[
\Pi_{i_0}(v_{kj}) = \sum_{m=i_0+1}^{j-1} \alpha_{jm}^{(k, i_0)} v_{mi_0} - \sum_{m=1}^{n} \alpha_{mj}^{(k, i_0)} v_{mi_0}.
\]

For simplicity, set \(\alpha_{ji}^{(k, i_0)} = -\alpha_{ij}^{(k, i_0)}, \text{ and } \alpha_{jj}^{(k, i_0)} = 0; \text{ then we can rewrite the above expression as:}

\[
(3.5) \quad \Pi_{i_0}(v_{kj}) = \sum_{m=1}^{n} \alpha_{jm}^{(k, i_0)} v_{mi_0}.
\]

Let \(s\) be the cardinality of the set \(\{i_1, j_1, \ldots, i_r, j_r\}\); that is, \(s\) is the number of distinct indices that occur in the list \((j_1, i_1), \ldots, (j_r, i_r)\). Note that \(r \leq \binom{s}{2}\). Let \(a_1 < a_2 < \cdots < a_s\) be the list of these distinct indices. For each \(\ell\) with \(1 \leq \ell \leq s\), let \((j_{k(\ell)}, i_{k(\ell)})\) be the smallest pair among \((j_1, i_1), \ldots, (j_r, i_r)\) that has
Definition 3.6. Let \( d \) be a nonnegative integer. We define \( r(d) \) to be the largest integer such that \( r(d) \leq d \) and \( r(d) \leq \binom{d-r(d)}{2} \).

Theorem 3.7. Fix \( n > 1 \) and let \( X < V \). Fix \( i_0, 1 \leq i_0 \leq n - 2 \), and let

\[
Z_{i_0} = X \cap \varphi_{i_0}^{-1}\left(\varphi_{i_0+1}(X), \ldots, \varphi_n(X)\right).
\]
If \( \dim(X \cap \langle v_{ij} \mid i \leq j \leq n \rangle) = d \), then \( \dim(Z_{i_0}) \leq r(d) \). Equivalently,

\[
\dim(Z_{i_0}) \leq d - \left\lfloor \frac{\sqrt{8d + 1} - 1}{2} \right\rfloor
\]

where \( \left\lfloor x \right\rfloor \) is the smallest integer greater than or equal to \( x \).

**Proof.** Let \( \dim(Z_{i_0}) = r \). By Lemma 3.2, \( r \leq \frac{(d-r)}{2} \), so \( r \leq r(d) \), as claimed. From \( r(d) \leq \frac{(d-r(d))}{2} \) we easily obtain (3.8).

We have two other ways of describing the function \( r(d) \), which will prove useful below:

**Corollary 3.9.** Let \( d \) be a positive integer. Then \( r(d) \) is the number of nontriangular numbers strictly less than \( d \). Equivalently, if we write \( d = \binom{t}{2} + s \), with \( 0 < s \leq t \), then \( r(d) = \binom{t-1}{2} + (s-1) \).

**Proof.** Since \( r(d) \leq \frac{(d-r(d))}{2} \leq \frac{(d+1)-(r(d))}{2} \), it follows that \( r(d+1) \geq r(d) \). We also have

\[
r(d) + 2 > r(d) + 1 > \left( \frac{d - (r(d) + 1)}{2} \right) = \left( \frac{(d+1) - (r(d) + 2)}{2} \right),
\]

so \( r(d+1) < r(d) + 2 \). If \( r(d) < \frac{(d-r(d))}{2} \), then

\[
r(d) + 1 \leq \left( \frac{d - r(d)}{2} \right) = \left( \frac{(d+1) - (r(d) + 1)}{2} \right),
\]

so \( r(d+1) \geq r(d) + 1 \) and in this case we have \( r(d+1) = r(d) + 1 \). If \( r(d) = \frac{(d-r(d))}{2} \), then \( r(d) + 1 > \left( \frac{(d+1) - (r(d) + 1)}{2} \right) \), hence \( r(d+1) < r(d) + 1 \) and we conclude that \( r(d+1) = r(d) \). In summary, we have:

\[
r(d+1) = \begin{cases} 
    r(d) + 1 & \text{if } r(d) < \frac{(d-r(d))}{2}, \\
    r(d) & \text{if } r(d) = \frac{(d-r(d))}{2}.
\end{cases}
\]

We claim that \( r(d) = \frac{(d-r(d))}{2} \) if and only if \( d \) is a triangular number: when \( d = \binom{t+1}{2} \) for some \( t \geq 0 \), we have

\[
\binom{t}{2} = \binom{t+1}{2} - \binom{t}{2} = \left( \frac{d - \binom{t}{2}}{2} \right),
\]

so \( r(d) = \binom{t}{2} = \frac{(d-r(d))}{2} \). Conversely, if \( r(d) = \frac{(d-r(d))}{2} \), then solving for \( d \) we obtain \( d = \frac{(d-r(d)+1)}{2} \), proving that \( d \) is a triangular number. Therefore, we have:

\[
r(d+1) = \begin{cases} 
    r(d) + 1 & \text{if } d \text{ is not a triangular number}, \\
    r(d) & \text{if } d \text{ is a triangular number}.
\end{cases}
\]

Since \( r(1) = 0 \), we conclude that \( r(d) \) is the number of nontriangular numbers strictly smaller than \( d \), as claimed. To establish the formula, note that the value of \( r \) at \( \binom{t}{2} \) is \( \binom{t-1}{2} \), and therefore \( r(\binom{t}{2} + s) = \binom{t-1}{2} + (s-1) \) for \( 0 < s < t \), since there are exactly \( s-1 \) more nontriangular numbers strictly less than \( \binom{t}{2} + s \) than there are strictly less than \( \binom{t}{2} \). And \( \binom{t}{2} + t = \binom{t+1}{2} \), so we also get equality when \( s = t \). □
Remark 3.10. These alternate descriptions can also be obtained by examining sequence A083920 in [9]; for example, compare the closed formula there with \( (3.8) \).

In fact, the author first realized these descriptions hold by calculating the first few values of \( r(d) \) directly, and then consulting [9].

We can now obtain an upper bound for \( \sum \dim(Z_k) \) in terms of \( \dim(X) \), which in turn gives a lower bound for \( \dim(X^*) \) in terms of \( \dim(X) \).

Definition 3.11. For \( m > 0 \), we let \( f(m) \) denote the largest possible value of \( \sum \dim(Z_k) \) for a subspace \( X \) of \( V \) with \( \dim(X) = m \), for a suitable choice of \( n \). Equivalently,

\[
\begin{align*}
f(m) &= \max \left\{ \dim(X^n \cap \ker(\Phi)) \mid X < V(n), \ \dim(X) = m \right\}.
\end{align*}
\]

Remark 3.12. It would appear that this quantity should really be a function of \( m \) and \( n \), \( f(m, n) \). It is easy to verify that \( f(m, n) \leq f(m, N) \) for any \( N \geq n \): if \( X \) is a subspace of \( V(n) \), we can also consider it as a subspace of \( V(N) \) for any \( N \geq n \).

If the dimension of \( X^* \) (with respect to \( \{\varphi_i\}_{i=1}^N \)) is \( nm - k \), then the dimension of \( X^* \) (with respect to \( \{\varphi_i\}_{i=1}^N \)) is \( Nm - k \), so we have

\[
\dim(X^n \cap \ker(\Phi_n)) = \dim(X^N \cap \ker(\Phi_N)).
\]

Intuitively, the reason the reverse inequality also holds is that the largest value of \( f(m, n) \) occurs when the vectors in \( Z_i \) use fewer indices rather than more, since more indices implies a smaller value for \( \dim(X \cap \langle v_{sr} \mid i < r < s \leq n \rangle) \), and hence a smaller possible value for \( Z_j \) with \( j > i \). So the “best” strategy for larger intersection with \( \ker(\Phi) \) is to keep \( X \) confined to as small a number of indices as possible. The proof below will formalize this intuition, and show that indeed the value of \( f \) depends only on \( m \).

Theorem 3.13. Let \( m > 0 \), and write \( m = (\frac{T}{3}) + s \), \( 0 \leq s \leq T \). Then

\[
f(m) = \left( \frac{T}{3} \right) + \left( \frac{s}{2} \right).
\]

Remark 3.14. Although there is some ambiguity in the expression for \( m \), since \( (\frac{T}{3}) + T = (\frac{T+1}{3}) \), note that the values \( (\frac{T}{3}) + (\frac{T}{2}) \) and \( (\frac{T+1}{3}) + (\frac{s}{2}) \) are equal, so the given value of \( f(m) \) is well-defined.

Proof. Assume \( s > 0 \). First we show that \( f(m) \geq (\frac{T}{3}) + (\frac{s}{2}) \).

Let \( X \) be the \( m \)-dimensional coordinate subspace of \( V(T + 1) \) (the smallest allowable value of \( n \)) generated by all \( v_{ji} \) with \( 1 \leq i < j \leq T \), and the vectors \( v_{T+1,1}, \ldots, v_{T+1,s} \). Then

\[
X^* = \langle \varphi_1(X), \ldots, \varphi_T(X), \varphi_{T+1}(X) \rangle
\]

is a coordinate subspace of \( W \) generated by the vectors \( w_{ijk} \), \( 1 \leq i < j \leq T \), \( i \leq k \leq T + 1 \), and the vectors \( w_{T+1,1,i,k} \) with \( 1 \leq i \leq s \), \( i \leq k \leq T + 1 \). This gives...
a total of $2\binom{T}{3} + 3\binom{T}{2} + s(T + 1) - \binom{s}{2}$ basis vectors. Therefore,

\[
(T + 1)m - \dim(X^*) = (T + 1)\left(\binom{T}{2} + s\right) - 2\binom{T}{3} - 3\binom{T}{2} - s(T + 1) + \binom{s}{2}
\]

\[
= (T - 2)\binom{T}{2} - 2\binom{T}{3} + \binom{s}{2}
\]

\[
= \binom{T}{3} + \binom{s}{2},
\]

as claimed. As noted in Remark 3.12, this shows $f(m) \geq \binom{\ell}{3} + \binom{s}{2}$ for all $n$ which satisfies $m \leq \binom{n}{2}$.

For the reverse inequality, we will apply induction. Fix $m = \binom{T}{2} + s$ with $0 < s \leq T$. Let $X$ be a subspace of $V(n)$ of dimension $m$, where $n$ is any integer with $m \leq \binom{n}{2}$. We want to show that $\sum \dim(Z_i)$ is bounded above by $\binom{T}{3} + \binom{s}{2}$. If all $Z_i$ are trivial, this follows. Otherwise, assume $i$ is the smallest index with nontrivial $Z_i$, and that $\dim(Z_i) = k > 0$. Then $k \leq r(m)$, and if $\ell$ is the smallest positive integer such that $k \leq \binom{\ell}{2}$ then

\[
\dim(X \cap \langle v_{sr} \mid i < r < s \leq n \rangle) \leq m - \ell.
\]

Thus, $\sum \dim(Z_h) \leq k + f(m - \ell)$. We want to show that this expression is bounded above by $\binom{T}{3} + \binom{s}{2}$. Note that $k \leq r(m) \leq \binom{m - r(m)}{2}$, so $\ell \leq m - r(m) = T$.

It is easy to show that for $m = 1, 2, 3, 4, 5$, all values of the form $k + f(m - \ell)$, $k \leq r(m)$ and $\ell$ as above are less than or equal to $\binom{T}{3} + \binom{s}{2}$, which establishes the base case of the induction.

If $\ell = T = m - r(m)$, then since $k \leq r(m)$ we have

\[
k + f(m - \ell) \leq r(m) + f(r(m))
\]

\[
= \binom{T - 1}{2} + (s - 1) + f\left(\binom{T - 1}{2} + (s - 1)\right)
\]

\[
= \binom{T - 1}{2} + (s - 1) + \binom{T - 1}{3} + \binom{s - 1}{2}
\]

\[
= \binom{T}{3} + \binom{s}{2};
\]

(this holds even if $s = 1$, using induction and as noted above). If $\ell < T$, then

\[
\binom{\ell}{2} \leq \binom{T - 1}{2} \leq \binom{T - 1}{2} + (s - 1) = r(m),
\]

and since $k \leq \binom{\ell}{2}$, it is enough to consider the expression $\binom{\ell}{2} + f(m - \ell)$. When $\ell = T$ this simplification may not be possible, since we will have $r(m) < \binom{T}{2}$ whenever $s < T$. To finish the proof it is enough to show that for $1 < \ell < T$,

\[
\binom{\ell}{2} + f(m - \ell) \leq \binom{T}{3} + \binom{s}{2}.
\]
If $2 \leq \ell \leq s$, then:

$$\binom{\ell}{2} + f(m-\ell) = \binom{\ell}{2} + f\left(\binom{T}{2} + (s-\ell)\right)$$

$$\leq \binom{T}{3} + \binom{s}{2}.$$

The last inequality follows since $\binom{\ell}{2} + \binom{s-\ell}{2}$ is the number of two element subsets of \{1, \ldots, s\}, where either both elements are less than or equal to $\ell$, or both strictly larger than $\ell$.

If $s < \ell < T$, then write $\ell = s + a$, $a > 0$. We then have

$$m - \ell = \binom{T}{2} + s - (s + a) = \binom{T - 1}{2} + (T - 1 - a),$$

so

$$\binom{\ell}{2} + f(m - \ell) = \binom{\ell}{2} + \binom{T - 1}{3} + \binom{T - 1 - a}{2}.$$

Since $\ell + 1 - t \leq 0$ and $a > 0$, we must have

$$6a(s + a + 1 - T) \leq 0.$$

Rewriting and introducing suitable terms we have:

$$6as + 3a^2 - 3a - 3T^2 + 9T - 6 + 3T^2 - 9T - 6aT + 9a + 3a^2 + 6 \leq 0$$

In turn, this can be rewritten as

$$6as + 3a^2 - 3a - 3(T - 1)(T - 2) + 3(T - a - 1)(T - a - 2) \leq 0.$$

This gives:

$$3(s^2 + 2as + a^2 - s - a) - 3(T - 1)(T - 2) + 3(T - a - 1)(T - a - 2) \leq 3(s^2 - s),$$

and so

$$3((s + a)^2 - (s + a)) - 3(T - 1)(T - 2) + 3(T - a - 1)(T - a - 2) \leq 3(s^2 - s).$$

Substituting $\ell$ for $s + a$ and adding $T(T - 1)(T - 2)$ to both sides we have

$$3(\ell^2 - \ell) + (T - 3)(T - 1)(T - 1) + 3(T - a - 1)(T - a - 2) \leq T(T - 1)(T - 2) + 3(s^2 - s),$$

and dividing through by 6 yields the desired inequality:

$$\binom{\ell}{2} + \binom{T - 1}{3} + \binom{T - 1 - a}{2} \leq \binom{T}{3} + \binom{s}{2}.$$

We therefore conclude that $f(m) \leq \binom{T}{3} + \binom{s}{2}$, which completes the proof. Note that indeed, the value of $n$ is not relevant. \hfill \Box

**Theorem 3.15.** Fix $n > 1$ and let $X$ be a subspace of $V$. Write $\dim(X) = \binom{T}{2} + s$, $0 \leq s \leq T$. Then

$$n \dim(X) - \binom{T}{3} - \binom{s}{2} \leq \dim(X^*) \leq \min \left\{ n \dim(X), \ 2 \binom{n + 1}{3} \right\}.$$

**Proof.** The lower bound follows from $\dim(X^*) \geq n \dim(X) - f(\dim(X))$, and the upper bound is immediate. \hfill \Box
Corollary 3.16. Fix $n > 1$ and let $X$ be a subspace of $V$ with $\dim(X) = m$. If $\dim(X^*) = nm - k$ and $n + k > f(m + 1)$, then $X$ is closed.

Proof. Suppose $X$ is as in the statement, and let $Y$ be any subspace of $V$ of dimension $m + 1$. From the definition of $f$ we know that
\[
\dim(Y^*) \geq n(m + 1) - f(m + 1),
\]
so $\dim(Y^*) - \dim(X^*) \geq n + k - f(m + 1) > 0$. Therefore every $Y$ strictly larger than $X$ must have $\dim(X^*) < \dim(Y^*)$, which shows that $X$ is closed by Proposition 3.11.

Corollary 3.17. Fix $n > 1$ and let $X$ be a subspace of $V$ with $\dim(X) = m$. Write $m = \left(\begin{array}{c} \frac{m+1}{2} \\ s \end{array}\right) + s$, $0 \leq s < T$. If $\left(\begin{array}{c} \frac{m+1}{2} \\ s \end{array}\right) + \frac{(s+1)}{2} < n$, then $X$ is closed.

Proof. This follows from the previous corollary and the formula for $f(m + 1)$ in Theorem 3.13.

For reference, Table 1 contains the values of $f(m)$, $3 \leq m \leq 50$. Note that $f(1) = f(2) = 0$ by Corollary 2.10.

Translating back into group theory, we obtain the following:

Theorem 3.18. Let $G$ be a group of class at most two and exponent $p$, where $p$ is an odd prime. Let $\text{rank}(G^{ab}) = n$, and let $\text{rank}([G,G]) = m$. If $f\left(\begin{array}{c} n \\ 2 \end{array}\right) - m + 1 < n$, where $f(k)$ is the function in Theorem 3.13, then $G$ is capable.

Proof. The subspace $X$ of $V(n)$ corresponding to $G$ has dimension $\left(\begin{array}{c} n \\ 2 \end{array}\right) - m$; so the result follows directly from Corollary 3.17.

Remark 3.19. Below we will be able to replace $\text{rank}(G^{ab})$ with $\text{rank}(G/Z(G))$. 

| $m$ | $f(m)$ | $m$ | $f(m)$ | $m$ | $f(m)$ |
|-----|--------|-----|--------|-----|--------|
| 3   | 1      | 19  | 26     | 35  | 77     |
| 4   | 1      | 20  | 30     | 36  | 84     |
| 5   | 2      | 21  | 35     | 37  | 84     |
| 6   | 4      | 22  | 35     | 38  | 85     |
| 7   | 4      | 23  | 36     | 39  | 87     |
| 8   | 5      | 24  | 38     | 40  | 90     |
| 9   | 7      | 25  | 41     | 41  | 94     |
| 10  | 10     | 26  | 45     | 42  | 99     |
| 11  | 10     | 27  | 50     | 43  | 105    |
| 12  | 11     | 28  | 56     | 44  | 112    |
| 13  | 13     | 29  | 56     | 45  | 120    |
| 14  | 16     | 30  | 57     | 46  | 120    |
| 15  | 20     | 31  | 59     | 47  | 121    |
| 16  | 20     | 32  | 62     | 48  | 123    |
| 17  | 21     | 33  | 66     | 49  | 126    |
| 18  | 23     | 34  | 71     | 50  | 130    |

Table 1. Explicit values of $f(m)$, $3 \leq m \leq 50$
Lemma 4.2. Fix $i, j$, $1 \leq i < j \leq n$. If $\pi_{ji}(X) = 0$, then $\pi_{ji}(X^{**}) = 0$.

Proof. Note that $\pi_{ji}(\varphi_k(v)) \neq 0$ if and only if $k = i$ and $\pi_{ji}(v) \neq 0$. From the hypothesis we thus conclude that $\pi_{ji}(X^*) = 0$. This is the same as $\pi_{ji}(X^{**})$, so $\pi_{ji}(X^{**}) = 0$, as claimed. \qed

Lemma 4.4. If $\Pi_i(X) = 0$ for some $i$, $1 \leq i \leq n$, then $X$ is closed.

Proof. By Theorem 2.13(iii), if $(v_1, \ldots, v_n) \in X^n \cap \ker(\Phi)$, then for $j > i$ we have:

$$0 = \Pi_i(v_j) = \sum_{r=1}^{i-1} (-\alpha^{(i)}_{jr}) v_{ir} + \sum_{r=i+1}^{j-1} \alpha^{(i)}_{jr} v_{ir} + \sum_{r=j+1}^{n} (-\alpha^{(i)}_{rj}) v_{ri},$$

and if $j < i$ then

$$0 = \Pi_i(v_j) = \sum_{r=1}^{j-1} (-\alpha^{(i)}_{jr}) v_{ir} + \sum_{r=j+1}^{i-1} \alpha^{(i)}_{rj} v_{ir} + \sum_{r=i+1}^{n} (-\alpha^{(i)}_{rj}) v_{ri}. $$

Thus we conclude that $\alpha^{(i)}_{rs} = 0$ for all $1 \leq r < s \leq n$. Therefore, $v_i = 0$. In particular, the intersection of $\varphi_i(X)$ and $\langle \varphi_j(X) \mid j \neq i \rangle$ is trivial, so $\varphi_i^{-1}(X^*) = X$. Therefore, $X^{**} \subset \varphi_i^{-1}(X^*) = X$, proving $X$ is closed. \qed

In [6] Lemma 5.9 we proved a special case of the following result:

Lemma 4.5. Let $n > 1$ be an integer. Suppose $I$ is a proper nonempty subset of \{1, \ldots, n\}, and let $J = \{1, \ldots, n\} - I$. Let

$$V_I = \langle v_{ji} \mid 1 \leq i < j \leq n; \ i, j \in I \rangle$$

$$V_J = \langle v_{ji} \mid 1 \leq i < j \leq n; \ i, j \in J \rangle$$

$$V_{(I,J)} = \langle v_{ji} \mid 1 \leq i < j \leq n; \ \text{exactly one of} \ i, j \ \text{is in} \ I, \ \text{one in} \ J \rangle.$$

Let $X_I$ be a subspace of $V_I$, $X_J$ be a subspace of $V_J$, and let

$$X = X_I \oplus X_J \oplus V_{(I,J)}.$$

Then

$$X^{**} = \left( \bigcap_{i \in I} \varphi_i^{-1}(\langle \varphi_k(X_I) \mid k \in I \rangle) \right) \oplus \left( \bigcap_{j \in J} \varphi_j^{-1}(\langle \varphi_k(X_J) \mid k \in J \rangle) \right) \oplus V_{(I,J)}.$$  

In particular, $X$ is $\{\varphi_k\}_{k=1}^n$-closed if and only if $X_I$ is $\{\varphi_i\}_{i \in I}$-closed and $X_J$ is $\{\varphi_j\}_{j \in J}$-closed.
Proof. Let $W'$ be the subspace of $W$ generated by all $w_{ijk}$ in which exactly one or two of $i, j, k$ are in $I$. First, we claim that $W' \subset X^*$. Suppose that exactly one of $i, j, k$ lies in $I$. If $j \in I$, then $v_{ji} \in V(\iota, j)$, so $\varphi_k(v_{ji}) = w_{ijk} \in X^*$. If $i \in I$, then again $v_{ji} \in V(\iota, j)$, so again we conclude that $w_{ijk} \in X^*$. Finally, if $k \in I$, then either $v_{jk}$ or $v_{kj}$ lies in $V(\iota, j)$ (whichever is appropriate); the image under $\varphi_i$ of this vector is $\pm (w_{ijk} - w_{kj})$. Since $w_{kij} \in X^*$ by the argument above it follows that $w_{ijk} \in X^*$ as desired. The case when exactly two if $i, j, k$ lie in $I$ follows by symmetry, since exactly one of them will lie in $J$.

As a second step, we claim that

$$X^* = \langle \varphi_i(X_I) \mid i \in I \rangle \oplus \langle \varphi_j(X_J) \mid j \in J \rangle \oplus W'.$$

Indeed, the right hand side is contained in $X^*$. It suffices to show that $\varphi_j(X_I) \subset W'$ for every $j \in J$ (the symmetrical argument will show it for $\varphi_i(X_I)$, $i \in I$). Since $\pi_{abc}(\varphi_j(X_I)) \neq \{0\}$ only if exactly two of $a, b, c$ are in $I$ and either $b$ or $c$ are equal to $j$, this establishes our second claim.

Thirdly, if we let $W_I$ (resp. $W_J$) denote the subspace of $W$ generated by all $w_{ijk}$ in which all of $i, j, k$ lie in $I$ (resp. $i, j, k$ lie in $J$), then we claim that

$$X^* \cap W_I = \langle \varphi_i(X_I) \mid i \in I \rangle \quad \text{and} \quad X^* \cap W_J = \langle \varphi_j(X_J) \mid j \in J \rangle.$$

Clearly, $\varphi_i(X_I) \subset X^* \cap W_I$ for all $i \in I$. For the converse inclusion simply note that $X^* = X_I^* + X_J^* + V(\iota, J)$, and both $X_J^*$ and $V(\iota, J)$ are disjoint from $W_I$. As $\varphi_k(X_I)$ is contained in $W'$ when $k \in J$ and contained in $W_I$ when $k \in I$, the claim now follows. A symmetric argument holds for $X^* \cap W_J$.

Finally, we establish the equality in the lemma: let

$$v \in \left( \bigcap_{i \notin I} \varphi_i^{-1} \left( \langle \varphi_k(X_I) \mid k \in I \rangle \right) \right) \oplus \left( \bigcap_{j \notin J} \varphi_j^{-1} \left( \langle \varphi_k(X_J) \mid k \in J \rangle \right) \right) \oplus V(\iota, J).$$

Then we can write $v = x_I + x_{(I,J)} + x_J$, where

$$x_I \in \bigcap_{i \notin I} \varphi_i^{-1} \left( \langle \varphi_k(X_I) \mid k \in I \rangle \right),$$

$$x_J \in \bigcap_{j \notin J} \varphi_j^{-1} \left( \langle \varphi_k(X_J) \mid k \in J \rangle \right),$$

$$x_{(I,J)} \in V(\iota, J).$$

Since $V(\iota, J) \subset X \subset X^{**}$, to show that $v \in X^{**}$ it is enough to show that $x_I$ and $x_J$ are both in $X^{**}$. And indeed: notice that $\pi_{ab}(x_I) \neq 0$ only if $a, b \in I$. Therefore, $\varphi_j(x_I) \in W' \subset X^*$ for every $j \in J$. And by construction we have $\varphi_i(x_I) \in X^*$ for every $i \in I$. Thus $\varphi_k(x_I) \in X^*$ for all $k$. Symmetrically, $\varphi_k(x_J) \in X^*$ for all $k$ as well. Therefore, each of $x_I$ and $x_J$ lie in $X^{**}$, as desired.

For the converse inclusion, let $x \in X^{**}$. We can write $x = x_I + x_J + x_{(I,J)}$, with $x_I \in V_I$, $x_J \in V_J$, and $x_{(I,J)} \in V(\iota, J)$. We must have $\varphi_k(x) \in X^*$ for every $k$. If $k \in I$, then we have

$$\varphi_k(x_I) \in X^* \cap W_I = \langle \varphi_i(X_I) \mid i \in I \rangle,$$

and

$$\varphi_k(x_I), \varphi_k(x_{(I,J)}) \in W'.$$

And if $k \in J$, then

$$\varphi_k(x_J) \in X^* \cap W_J = \langle \varphi_j(X_J) \mid j \in J \rangle,$$

and

$$\varphi_k(x_I), \varphi_k(x_{(I,J)}) \in W'.$$
Corollary 4.6. Let \( G_1 \) and \( G_2 \) be two noncyclic \( p \)-groups of class at most two and exponent \( p \). Then \( G = G_1 \oplus G_2 \) is capable if and only if each of \( G_1 \) and \( G_2 \) are capable.

The corollary is clearly not true if we drop the “noncyclic” hypothesis, since a cyclic group of order \( p \) is not capable, but the direct sum of two cyclic groups of order \( p \) is capable. To understand why we must make the distinction, note that when we interpret the subspace \( V_{(I,J)} \) as the subgroup of commutators we are making trivial, this subgroup “says” that each \( x_i \) commutes with each \( x_j \) in \( G_i \), where \( i \in I \) and \( j \in J \). Translating back into group theoretic terms we obtain the following:

**Corollary 4.6.** Let \( G_1 \) and \( G_2 \) be two noncyclic \( p \)-groups of class at most two and exponent \( p \). Then \( G = G_1 \oplus G_2 \) is capable if and only if each of \( G_1 \) and \( G_2 \) are capable.

Therefore,
\[
\begin{align*}
\varphi_i^{-1}(\langle \varphi_k(X_I) \mid k \in I \rangle), \\
\varphi_j^{-1}(\langle \varphi_k(X_J) \mid k \in J \rangle),
\end{align*}
\]
as desired. This proves the equality.

The final clause of the lemma follows from the observation that \( X \) will be closed if and only if
\[
X_I = \bigcap_{i \in I} \varphi_i^{-1}(\langle \varphi_k(X_I) \mid k \in I \rangle) \quad \text{and} \quad X_J = \bigcap_{j \in J} \varphi_j^{-1}(\langle \varphi_k(X_J) \mid k \in J \rangle).
\]

\[\square\]

The usefulness of these theorems becomes apparent when we translate it back into group theoretic terms:}

**Corollary 4.8.** Let \( n > 2 \) and let \( X' \) be a subspace of \( V(n) \). If there exists a vector \( u \in U(n) \), \( u \neq 0 \), such that \( \langle u \rangle^* \) is contained in \( X' \), then there exists a subspace \( X \) of \( V(n-1) \) such that \( X' = \langle \varphi_i \rangle_{i=1}^{n-1} \) closed if and only if \( X = \langle \varphi_i \rangle_{i=1}^{n-1} \) closed, and \( \dim(X) = \dim(X') - n + 1 \).

The usefulness of these theorems becomes apparent when we translate it back into group theoretic terms:

**Corollary 4.9.** Let \( K \) be a finite noncyclic group of exponent \( p \) and class \( 2 \), and let \( G = K \oplus C_p \), where \( C_p \) is the cyclic group of order \( p \). Then \( G \) is capable if and only if \( K \) is capable.

Since every finite group of class at most two and exponent \( p \) may be written as \( K \oplus C_r^p \) for some \( r \geq 0 \) and \( K \) satisfying \([K, K] = Z(K)\), we conclude that:
Corollary 4.10. Let $G$ be a $p$-group of class at most two and exponent $p$. Write $G = K \oplus C_p^r$, where $K$ satisfies $[K, K] = Z(K)$. Then $G$ is capable if and only if (i) $K$ is nontrivial and capable, or (ii) $K$ is trivial and $r \geq 2$.

This allows us to ignore the “extra” elements in the center of $G$ that do not come from commutators. Noting that $G/Z(G) \cong K^{ab}$ and $[G, G] = [K, K]$, we can, for example, strengthen Theorem 4.12 by replacing the rank of $G^{ab}$ by the rank of $G/Z(G)$. We then have the following:

Corollary 4.11. Let $G$ be a $p$-group of class at most two and exponent $p$. Let $\text{rank}(G/Z(G)) = n$, and $\text{rank}([G, G]) = m$. If $f \left( \binom{n}{2} - m + 1 \right) < n$, where $f(k)$ is the function in Theorem 4.12, then then $G$ is capable.

In [4], the authors established a minimum size for the commutator subgroup of a capable group $G$ of class two and exponent $p$ which satisfies $[G, G] = Z(G)$; namely, they proved:

Theorem 4.12 (Theorem 1 in [4]). Suppose that $G$ is a group of exponent $p$ which satisfies $[G, G] = Z(G)$. If $G$ is capable and $[G, G]$ is of rank $m$, then $G/Z(G)$ is of rank at most $2m + \binom{m}{2}$.

Our development above now shows that if $G$ is not cyclic, then Theorem 4.12 remains true if we weaken the hypothesis that $[G, G] = Z(G)$ to $[G, G] \subset Z(G)$ (i.e., class at most two). Note also that Theorem 4.12 gives a necessary condition. Combining it with Corollary 4.11 we have:

Theorem 4.13. Let $G$ be a noncyclic group of exponent $p$ and class at most two. Let $\text{rank}(G/Z(G)) = n$ and $\text{rank}([G, G]) = m$. A necessary condition for the capability of $G$ is that $n$ and $m$ satisfy $n \leq 2m + \binom{m}{2}$. A sufficient condition for the capability of $G$ is that $n$ and $m$ satisfy $f \left( \binom{n}{2} - m + 1 \right) < n$, where $f(k)$ is the function in Theorem 4.12.

The theorem can be interpreted as saying that a group of class exactly 2 and exponent $p$ is capable only if it is “nonabelian enough”: the necessary condition shows that $m$ cannot be too small relative to $n$ (so there must be “enough” nontrivial commutators), while the sufficient condition shows that if the commutator subgroup is large enough then the group will necessarily be capable.

5. Central elements and the kernel of $\Phi$.

Let $F$ be a field, and let $k, n$ be integers. The Grassmannian $Gr(k, n)$ is the set of all $k$-dimensional subspaces of $F^n$. This set is in fact an algebraic variety with very rich structure; for example, $Gr(1, n)$ is $\mathbb{P}^{n-1}$, projective $(n - 1)$-space over $F$.

Corollary 4.8 hints at the importance of subspaces of $V$ the form $Z^*$, where $Z$ is a subspace of $U$, for the question of capability. In this section we will explore some of the connections between these subspaces and $\text{ker}(\Phi)$.

Lemma 5.1. Fix $n > 1$ and let $Z$ be a subspace of $U$. If $\text{dim}(Z) = k$, then $\text{dim}(Z^*) = k(n - k) + \binom{k}{2} = kn - \binom{k+1}{2}$.

Proof. Let $z_1, \ldots, z_k$ be a basis for $Z$. Let $u_{k+1}, \ldots, u_n$ be vectors of $U$ that extend this to a basis for $U$. Then $Z^* \cong Z \wedge U$, and a basis for $Z \wedge U$ is given by all elements of the form $z_j \wedge z_i$, with $1 \leq i < j \leq k$, and all elements of the form $z_j \wedge u_r$, with $1 \leq j \leq k$ and $k + 1 \leq r \leq n$. □
Lemma 5.2. Fix $n > 1$, and let $k$ be a positive integer. If $k + 1 < n$, then the map $Gr(k,U) \to Gr(nk - \binom{k+1}{2},V)$ given by $Z \mapsto Z^*$ is one-to-one.

Proof. If $Z^* = Z'$, then any subspace $Y \subset (Z,Z')$ satisfies $Y^* \subset Z^*$. Thus, it is enough to show that if $\dim(Y) = k + 1$, then $\dim(Y^*) > \dim(Z^*)$. This is equivalent to verifying the inequality $n(k+1) - \binom{k+2}{2} > nk - \binom{k+1}{2}$, which holds exactly when $k + 1 < n$. \qed

The special case $k = 1$ is of particular interest:

Definition 5.3. We define $\Psi: \mathbb{P}^{n-1} = Gr(1,U) \to Gr(n-1,V)$ as the map that sends $[\alpha_1 : \cdots : \alpha_n]$ to $(\alpha_1 u_1 + \cdots + \alpha_n u_n) \wedge U$; i.e., the subspace generated by
\[
\begin{align*}
\nu_1 &= \sum_{j=1}^{n} \alpha_j v_{1j}, \\
\nu_2 &= \sum_{j=1}^{n} \alpha_j v_{2j}, \\
&\quad \vdots \\
\nu_n &= \sum_{j=1}^{n} \alpha_j v_{nj},
\end{align*}
\]
with $v_{ij} = -v_{ji}$ and $v_{ii} = 0$.

The subspaces which are images under this map are closely connected to $\ker(\Phi)$. We explore this connection in the next series of results.

Lemma 5.4. Let $p = [\alpha_1 : \cdots : \alpha_n] \in \mathbb{P}^{n-1}$; if $\alpha_i \neq 0$, then a basis for $\Psi(p)$ is given by $\nu_1, \ldots, \nu_i, \ldots, \nu_n$, i.e., taking all $\nu_j$ and omitting the vector $\nu_i$.

Proof. By Lemma 5.1 it is enough to show that the given list is linearly independent. Indeed, if $\beta_1 \nu_1 + \cdots + \beta_i \nu_i + \cdots + \beta_n \nu_n = 0$ (where as usual we use $\beta_i \nu_i$ to mean we are omitting that summand), then for fixed $j \neq i$ we have:
\[
\pi_{ji} (\beta_1 \nu_1 + \cdots + \beta_i \nu_i + \cdots + \beta_n \nu_n) = \beta_j \alpha_i v_{ij} = 0,
\]
and since $\alpha_i \neq 0$ we conclude that $\beta_j = 0$. \qed

Lemma 5.5. Fix $n > 1$, let $p \in \mathbb{P}^{n-1}$, $p = [\alpha_1 : \cdots : \alpha_n]$, and let $v \in \Psi(p)$. If $\alpha_i \neq 0$ and $\Pi_i(v) = 0$, then $v = 0$.

Proof. Write $v = \beta_1 \nu_1 + \cdots + \beta_i \nu_i + \cdots + \beta_n \nu_n$. If $j \neq i$, then $\pi_{ji}(v) = \beta_j \alpha_i v_{ij}$. Since $\alpha_i \neq 0$ by hypothesis, it follows that $\beta_j = 0$. Therefore, $v = 0$, as claimed. \qed

Lemma 5.6. Fix $n > 1$ and let $p \in \mathbb{P}^{n-1}$. Then $\Psi(p)^n \cap \ker(\Phi)$ is trivial.

Proof. Let $(w_1,\ldots,w_n) \in \ker(\Phi) \cap (\Psi(p))^n$. Write $p = [\alpha_1 : \cdots : \alpha_n]$. By Theorem 2.13(ii), $\Pi_j(w_j) = 0$ for all $j$. If $\alpha_i \neq 0$, then by Lemma 5.5 we must have $w_i = 0$. If we then let $j \neq i$, then by Theorem 2.13(iii) we also have $\Pi_i(w_j) = 0$, hence $w_j = 0$ for all $j$, as claimed. \qed

Theorem 5.7. Fix $n > 1$, and let $p \in \mathbb{P}^{n-1}$, $v \in V$, and $X = (\Psi(p),v)$. Then $X^n \cap \ker(\Phi)$ is nontrivial if and only if $v \notin \Psi(p)$.

Proof. Let $p = [\alpha_1 : \cdots : \alpha_n]$ be an element of $\mathbb{P}^{n-1}$. Let $v_1,\ldots,v_n$ be the vectors
\[
v_i = \sum_{j=1}^{n} \alpha_j v_{ji};
\]
we know that if $\alpha_i \neq 0$, then $v_1,\ldots,\hat{v}_i,\ldots,v_n$ is a basis for $\Psi(p)$.\hfill \square
The "only if" clause follows from Lemma 5.4. For the "if" clause, let \( v \) be a vector of \( V \) which is not in \( \Psi(v) \), with
\[
v = \sum_{1 \leq i < j \leq n} \alpha_{ji} v_{ji}.
\]
Let \( \alpha_{ji} = -\alpha_{ij} \) and \( \alpha_{ii} = 0 \) for simplicity. We claim that
\[
w = \varphi_1 \left( \alpha_1 v + \sum_{j=1}^n \alpha_{j1} v_j \right) + \cdots + \varphi_n \left( \alpha_n v + \sum_{j=1}^n \alpha_{jn} v_n \right) = 0.
\]
In general we have:
\[
\pi_{ml} \left( \alpha_u v + \sum_{j=1}^n \alpha_{jv} v_j \right) = \pi_{ml} \left( \sum_{1 \leq i < j \leq n} \alpha_{ji} \alpha_{ji} v_{ji} + \sum_{j=1}^n \sum_{i=1}^n \alpha_{jv} \alpha_{vji} v_{ij} \right) = \alpha_{ml} \alpha_{vml} v_{vml} + (\alpha_{mu} \alpha_{ul} v_{ul} + \alpha_{lu} \alpha_{mu} v_{ml}) = (\alpha_{ml} \alpha_u + \alpha_{um} \alpha_{ul} + \alpha_{lu} \alpha_{mu}) v_{ml}.
\]
Let \( r, s, t \) be integers, \( 1 \leq r < s \leq n, s \leq t \leq n \). We want to prove that \( \pi_{rst}(w) = 0 \). The \((rst)\) coefficient of \( w \) is equal to the \((tr)\) coefficient of \( \alpha_v + \sum \alpha_{jt} v_j \) when \( s = t \); to the sum of the \((sr)\) coefficient of \( \alpha_v + \sum \alpha_{jv} v_j \) and the \((st)\) coefficient of \( \alpha_r v + \sum \alpha_{jv} v_j \) when \( s > t \); and to the difference of the \((sr)\) coefficient of \( \alpha_r v + \sum \alpha_{jv} v_j \) and the \((ts)\) coefficient of \( \alpha_r v + \sum \alpha_{jv} v_j \) when \( s < t \).

Therefore, if \( s = t \) then we have: \( \pi_{trt}(w) = (\alpha_{r+1} \alpha_r + \alpha_{r+1} \alpha_{r-1} + \alpha_{r+1} \alpha_{r-1}) w_{trt} = 0 \), (since \( \alpha_{ti} = 0 \) and \( \alpha_{rt} = -\alpha_{tr} \)). If \( s > t \) then we have:
\[
\pi_{rst}(w) = \left( (\alpha_{sr} \alpha_1 + \alpha_{sr} \alpha_2 + \alpha_{sr} \alpha_3) + (\alpha_{sr} \alpha_3 + \alpha_{sr} \alpha_5 + \alpha_{sr} \alpha_6) \right) w_{rst} = 0.
\]
And finally if \( s < t \) then we have:
\[
\pi_{rst}(w) = \left( (\alpha_{sr} \alpha_1 + \alpha_{sr} \alpha_2 + \alpha_{sr} \alpha_3) - (\alpha_{sr} \alpha_3 + \alpha_{sr} \alpha_5 + \alpha_{sr} \alpha_6) \right) w_{rst} = 0.
\]
Thus, \( w = 0 \) and so
\[
\pi_{ml} \left( \alpha_1 v + \sum_{j=1}^n \alpha_{1j} v_j, \ldots, \alpha_n v + \sum_{j=1}^n \alpha_{jn} v_j \right)
\]
lies in \( X^n \cap \ker(\Phi) \).

We claim that this element is nonzero if and only if \( v \notin \Psi(p) \). Indeed, if the element is trivial, then since \( \alpha_i \neq 0 \) for some \( i \), the \( i \)-th component will yield an expression for \( v \) as a linear combination of \( v_1, \ldots, v_n \), proving that \( v \) lies in \( \Psi(p) \). Conversely, assume that \( v \in \Psi(p) \). Assuming \( \alpha_i \neq 0 \), then we can write
\[
v = \sum_{j=1}^n \beta_j v_j = \sum_{j=1}^n \sum_{k=1}^n \beta_j \alpha_k v_{kj}, \quad \text{where} \ \beta_i = 0.
\]
Then for any pair \( a, b \), \( 1 \leq a < b \leq n \), we have \( \alpha_{ba} = \beta_a \alpha_b - \beta_b \alpha_a \), and therefore:
\[
\pi_{ml} \left( \alpha_a v + \sum_{j=1}^n \alpha_{aj} v_j \right) = (\alpha_{ml} \alpha_a + \alpha_{lu} \alpha_m + \alpha_{mu} \alpha_l) v_{ml} = (\beta_m \alpha_{lu} \alpha_u - \beta_l \alpha_m \alpha_u + \beta_l \alpha_u \alpha_m - \beta_u \alpha_l \alpha_m + \beta_u \alpha_m \alpha_l - \beta_m \alpha_u \alpha_l) v_{ml} = 0,
\]
Lemma 6.4. follows:

X is closed if and only if \( \Upsilon(q) \) exists.

Proof. From Lemma 5.6 we know that \( \Psi(p) \) is nontrivial, proving the theorem. \( \square \)

6. The case \( n = 4 \).

In this section we will settle the case of \( n \)-generator groups of class two and exponent \( p \), with \( n \leq 4 \).

From Table 1 and Corollary 4.11 we deduce that for \( n \leq 4 \), all subspaces \( X \) of \( V(n) \) are closed, with the possible exception of \( n = 4 \) and \( \dim(X) = 5 \) (in that case, we have \( f(\dim(X) + 1) = 4 \), so the corollary does not apply; if \( \dim(X) = 6 \) and \( n = 4 \), then \( X = V \) which is trivially closed).

Thus, we may restrict our attention to the case \( n = 4 \) and \( \dim(X) = 5 \). The only two possibilities are \( X^{**} = X \) and \( X^{**} = V \). Since \( X^{**} = V \) if and only if \( X^* = W \), we have:

**Proposition 6.1.** Let \( n = 4 \), and let \( X \) be a 5-dimensional subspace of \( V \). Then \( X \) is closed if and only if \( X^* \) is a proper subspace of \( W \). Equivalently, \( X \) is closed if and only if \( X^n \cap \ker(\Phi) \) is nontrivial.

**Proof.** For the last assertion, note that \( n \dim(X) = 20 = \dim(W) \), so \( X^* \neq W \) if and only if \( \dim(X^n \cap \ker(\Phi)) > 0 \).

By Proposition 2.20 if \( v = (v_1, v_2, v_3, v_4) \in \ker(\Phi) \), then either \( v = (0, 0, 0, 0) \), or else the subspace spanned by \( v_1, v_2, v_3, v_4 \) has dimension 3. Since \( \ker(\Phi) \) is of dimension \( \binom{4}{3} \), in this case dimension 4, the one-dimensional subspaces of \( \ker(\Phi) \) correspond to points in \( \mathbb{P}^3 \). Thus we obtain a map from \( \mathbb{P}^3 \) to \( Gr(3, V) \). Explicitly:

**Definition 6.2.** Let \( q = [\alpha_{123} : \alpha_{124} : \alpha_{134} : \alpha_{234}] \in \mathbb{P}^3 \). Then \( \Upsilon(q) \) is defined to be the element of \( Gr(3, V) \) spanned by

\[
\begin{align*}
v_1 &= \alpha_{123}v_{32} + \alpha_{124}v_{42} + \alpha_{134}v_{43}, \\
v_2 &= -\alpha_{123}v_{31} - \alpha_{124}v_{41} + \alpha_{234}v_{43}, \\
v_3 &= \alpha_{123}v_{21} - \alpha_{134}v_{41} - \alpha_{234}v_{42}, \\
v_4 &= \alpha_{124}v_{21} + \alpha_{134}v_{31} + \alpha_{234}v_{32}.
\end{align*}
\]

Note that \( (v_1, v_2, v_3, v_4) \in \ker(\Phi) \), and each nonzero element of \( \ker(\Phi) \) corresponds to a point in \( \mathbb{P}^3 \). We are using triples of integers as indices because we are “really” working over \( \mathbb{P}(d)^{-1} \).

Thus we have:

**Corollary 6.3.** Let \( n = 4 \) and let \( X \) be a 5-dimensional subspace of \( V \). Then \( X \) is closed if and only if \( X \) contains \( \Upsilon(q) \) for some \( q \in \mathbb{P}^3 \).

Using this notation, we can rephrase Lemma 5.6 and Theorem 5.7 for \( n = 4 \) as follows:

**Lemma 6.4.** Fix \( n = 4 \) and let \( p \in \mathbb{P}^3 \). Then \( \Psi(p) \neq \Upsilon(q) \) for any \( q \in \mathbb{P}^3 \).

**Proof.** From Lemma 5.6, we know that \( \Psi(p) \) cannot contain \( \Upsilon(q) \) for any \( q \); since \( \Upsilon(q) \) is of dimension 3, the statement follows. \( \square \)

**Lemma 6.5.** Let \( n = 4 \), and let \( p \in \mathbb{P}^3 \), \( v \in V \), and \( X = \langle \Psi(p), v \rangle \). Then there exists \( q \in \mathbb{P}^3 \) such that \( \Upsilon(q) \) is contained in \( X \) if and only if \( v \notin \Psi(p) \).

In the case \( n = 4 \) we can prove the symmetric result:
Theorem 6.6. Let \( n = 4 \) and let \( q \in \mathbb{P}^3 \), \( v \in V \), and \( X = \langle \Upsilon(q), v \rangle \). Then there exists \( p \in \mathbb{P}^3 \) such that \( \Psi(p) \) is contained in \( X \) if and only if \( v \not\in \Upsilon(q) \).

Proof. Let \( q = [\alpha_{121} : \alpha_{124} : \alpha_{134} : \alpha_{234}] \in \mathbb{P}^3 \). Then \( \Upsilon(q) \) is generated by the vectors:

\[
\begin{align*}
v_1 &= \alpha_{123}v_{32} + \alpha_{124}v_{42} + \alpha_{134}v_{43}, \\
v_2 &= -\alpha_{123}v_{31} - \alpha_{124}v_{41} + \alpha_{234}v_{43}, \\
v_3 &= \alpha_{123}v_{21} - \alpha_{134}v_{31} - \alpha_{234}v_{42}, \\
v_4 &= \alpha_{124}v_{21} + \alpha_{134}v_{31} + \alpha_{234}v_{32}.
\end{align*}
\]

The “only if” clause follows from Lemma 6.4. For the “if” clause, let \( v \in V \) be the vector given by

\[
v = \sum_{1 \leq i < j \leq n} \alpha_{ji}v_{ij},
\]

and let \( p \in \mathbb{P}^3 \) be given by \( p = [\beta_1 : \beta_2 : \beta_3 : \beta_4] \), where

\[
\begin{align*}
\beta_1 &= \alpha_{123}\alpha_{41} - \alpha_{124}\alpha_{31} + \alpha_{134}\alpha_{21} \\
\beta_2 &= \alpha_{123}\alpha_{42} - \alpha_{124}\alpha_{32} + \alpha_{234}\alpha_{21} \\
\beta_3 &= \alpha_{123}\alpha_{43} - \alpha_{134}\alpha_{32} + \alpha_{234}\alpha_{31} \\
\beta_4 &= \alpha_{124}\alpha_{43} - \alpha_{134}\alpha_{42} + \alpha_{234}\alpha_{41}.
\end{align*}
\]

(Technically, we need to show that not all \( \beta_i \) are equal to 0 to justify that \( p \in \mathbb{P}^3 \); we do this below).

Let \( w_1, w_2, w_3, \) and \( w_4 \) be the four vectors as in Definition 5.3, i.e.,

\[
w_i = \sum_{j=1}^{4} \beta_{ji}v_{ji}, \quad i = 1, 2, 3, 4,
\]

where as usual we let \( v_{ij} = -v_{ji} \) and \( v_{ii} = 0 \). It is straightforward to verify that:

\[
\begin{align*}
w_1 &= \alpha_{234}v - \alpha_{43}v_{2} + \alpha_{42}v_{3} - \alpha_{32}v_{4}, \\
w_2 &= -\alpha_{134}v + \alpha_{43}v_{1} - \alpha_{41}v_{3} + \alpha_{31}v_{4}, \\
w_3 &= \alpha_{124}v - \alpha_{42}v_{1} + \alpha_{41}v_{2} - \alpha_{21}v_{4}, \\
w_4 &= -\alpha_{123}v + \alpha_{32}v_{1} - \alpha_{31}v_{2} + \alpha_{21}v_{3}.
\end{align*}
\]

The pattern in the above is as follows: for \( w_i \), we take \( \alpha_{abc}v \), where \( a < b < c \) and all three are different from \( i \), plus the sum of all three cyclic permutations of the indices: \( \alpha_{ab}v_{c} + \alpha_{bc}v_{a} + \alpha_{ca}v_{b} \), with the usual proviso that \( \alpha_{ji} = -\alpha_{ij} \). In the case of \( w_2 \) and \( w_4 \) we further multiply everything by \( -1 \).

We claim that all \( \beta_i \) are equal to 0 if and only if \( v \in \Upsilon(q) \). Indeed: if \( \beta_i = 0 \) for each \( i \), then \( w_j = 0 \) for each \( j \). At least one \( \alpha_{abc} \) is nonzero, so if \( d \not\in \{a, b, c\} \), 1 ≤ \( d \) ≤ 4, then the formula for \( w_d \) expresses \( v \) as a linear combination of three of the \( v_j \); hence \( v \in \Upsilon(q) \).

Conversely, if \( v \in \Upsilon(q) \), we want to show that all \( \beta_i \) are equal to 0. Write:

\[
v = \gamma_1v_{1} + \gamma_2v_{2} + \gamma_3v_{3} + \gamma_4v_{4},
\]
where we fix $a, b, c$, $1 \leq a < b < c \leq 4$, such that $\alpha_{abc} \neq 0$, and set $\gamma_d = 0$, where $d$ is the element of \{1, 2, 3, 4\} not in \{a, b, c\}. With this proviso, we have:

\[
\begin{align*}
\alpha_2 &= \gamma_3 \alpha_{123} + \gamma_4 \alpha_{124}, \\
\alpha_3 &= -\gamma_2 \alpha_{123} + \gamma_4 \alpha_{134}, \\
\alpha_4 &= -\gamma_2 \alpha_{124} - \gamma_3 \alpha_{134}, \\
\alpha_{32} &= \gamma_1 \alpha_{123} + \gamma_4 \alpha_{234}, \\
\alpha_{42} &= \gamma_1 \alpha_{124} - \gamma_3 \alpha_{234}, \\
\alpha_{43} &= \gamma_1 \alpha_{134} + \gamma_2 \alpha_{234}.
\end{align*}
\]

From this, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ readily follows.

Therefore, $p$ is a well-defined point in $\mathbb{P}^3$ if and only if $v \notin \Upsilon(q)$, from which the theorem follows. \qed

So we obtain:

**Corollary 6.7.** Let $n = 4$, and let $X$ be a 5-dimensional subspace of $V(4)$. Then $X$ is closed if and only if there exists $p \in \mathbb{P}^3$ such that $\Psi(p) \subset X$.

Translated into group theory clarifies the situation:

**Theorem 6.8.** Let $G$ be a group of class two and exponent $p$, $p$ an odd prime, and assume that $G^{ab}$ is of rank 4. If $[G, G]$ is of rank 1, then $G$ is capable if and only if $Z(G)/[G, G]$ is nontrivial; that is, if and only if $G$ is not extra-special.

Recall that we say a group is $k$-generated if it can be generated by $k$ elements, though it may need fewer. We obtain:

**Theorem 6.9.** Let $G$ be a 4-generated group of class at most 2 and exponent an odd prime $p$. Then $G$ is one and only of:

(i) Cyclic and nontrivial;
(ii) Extra special of order $p^5$ and exponent $p$;
(iii) Capable.

**Remark 6.10.** Here is an alternative proof of Theorem 6.8, which is entirely geometrical, due to David McKinnon [5]. The maps $\Psi : \mathbb{P}^3 \rightarrow Gr(3, V)$ and $\Upsilon : \mathbb{P}^3 \rightarrow Gr(3, V)$ are both regular maps; that is, they are defined everywhere, and are locally (in the Zariski topology) determined by rational functions on the coordinates; in fact, in this case, by linear functions on the coordinates. We define two subsets of the variety $Gr(4, V) \times \mathbb{P}^3$ by

\[
A = \left\{ (X, p) \mid \Psi(p) \subset X \right\}, \\
B = \left\{ (X, q) \mid \Upsilon(q) \subset X \right\}.
\]

Since both $\Psi$ and $\Upsilon$ are regular, it follows that both $A$ and $B$ are closed subvarieties of $Gr(4, V) \times \mathbb{P}^3$. The projections

\[
p_1 : Gr(4, V) \times \mathbb{P}^3 \rightarrow Gr(4, V), \\
p_2 : Gr(4, V) \times \mathbb{P}^3 \rightarrow \mathbb{P}^3,
\]

induce maps from each of $A$ and $B$ into $Gr(4, V)$ and $\mathbb{P}^3$. The maps to $\mathbb{P}^3$ are surjections, and the fibers all have dimension 2 because the fiber over $p$ (resp. $q$) is the set of all 4-dimensions subspaces of $V$ which contain the 3-dimensional subspace
\( \Psi(p) \) (resp. \( \Upsilon(p) \)). This set is isomorphic to the set of lines in the quotient space \( V/\Psi(p) \) (resp. \( V/\Upsilon(p) \)), which is in turn isomorphic to \( \mathbb{P}^2 \), so it is 2-dimensional.

The maps are also smooth, so we have smooth maps of fiber dimension 2 over a smooth 3-dimensional variety, hence \( A \) and \( B \) are both of dimension \( 3 + 2 = 5 \).

Consider now the projections to \( Gr(4, 6) \): we know that \( p_1(A) \) and \( p_1(B) \) are irreducible subvarieties of \( Gr(4, 6) \) of dimension at most 5, and that \( p_1(A) \) is contained in \( p_1(B) \) (since by Theorem 5.7 if \( (X, p) \) is in \( A \), then there exists \( q \) such that \( (X, q) \in B \)). If we can show that \( p_1(A) \) has dimension exactly 5, then the irreducibility of \( p_1(B) \) will imply that \( p_1(B) = p_1(A) \), which will show that if \( X \) is any 4-dimensional subspace of \( V \), and \( (X, q) \in B \), then there exists \( p \) such that \( (X, p) \in A \), which is what Theorem 6.6 states.

To show that \( p_1(A) \) has dimension exactly 5, it is enough to show that it is generically finite; for this it is enough to show that there is at least one \( X \in Gr(4, 6) \) such that \( p_1^{-1}(X) \) is nonempty and finite. But in fact we know that \( p_1^{-1}(X) \) has at most one element, since \( p \neq q \) implies that \( \langle \Psi(p), \Psi(q) \rangle \) has dimension 5 by Lemma 5.1. Thus, Theorem 6.6 follows.

**Remark 6.11.** Unfortunately, The analogue of Theorem 6.6 does not hold for \( n = 3 \) or for \( n > 4 \). At least for odd \( n \), we can consider the point \( q \) which has components equal to 0 except for those associated to the triples \((1, 2, 3), (1, 4, 5), (1, 6, 7)\), etc. Then it is easy to verify that \( \Upsilon(q) \) is of dimension \( n \), generated by all pairs \( v_1 \) with \( j > 1 \), and the vector \( v = v_{32} + v_{54} + v_{76} + \cdots + v_{2n+1, 2n} \). Thus, the subspace \( \Upsilon(q) \) contains \( \langle u_1 \rangle^* \).

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