Development of calibration estimator of population mean under non-response

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DOI: https://doi.org/10.22271/chemi.2020.v8.i5s.10491

Abstract

Using the calibration approach, the Hanson and Hurwitz (1946) technique-based estimator is developed for the situation where the information on auxiliary variable is assumed known for the entire sample units. Expressions for the estimator of population mean/total, its variance, and variance estimator were developed. The theoretical results are illustrated with the help of empirical studies. Empirical results showed that proposed calibration approach-based estimator outperforms the Hansen and Hurwitz technique.

Keywords: Calibration approach, Hansen and Hurwitz estimator, non-response, population mean

Introduction

In many human surveys, it generally is not possible to obtain information from all the units in the surveyed population. The problem of non-response persists even after call backs. The estimates obtained from incomplete data may be biased particularly when the respondents differ from the non-respondents. To address the problem of bias, Hansen and Hurwitz (1946) proposed a technique essentially to adjust for non-response. The technique consists of selecting a sample from the population, identifying the non-respondents in the sample and selecting a sub sample of non-respondent. When the auxiliary information on auxiliary variable related to study variate is available, Deville and Särndal (1992) have developed calibration approach based estimator of the finite population mean/total of the study variate by calibrating sampling design weight using certain calibration equation which involve known population mean/total of the Auxiliary variable. Methods of estimation of finite population mean under non-response in sample surveys using auxiliary information have been developed by various research workers in the past which have been explained in the previous chapter. Raman et al. (2013 a, 2013 b) have developed calibration estimator of the finite population total under non-response depending upon the different situation of availability of the auxiliary information. However, before we present a brief account of their work, we will first describe the Hansen and Hurwitz (1946) estimator under general sampling design.

It may be mentioned that the weighting and imputation procedures aim at elimination of bias caused by non-response. However, these procedures are based on certain assumptions on the response mechanism. When these assumptions do not hold good the resulting estimate may be seriously biased. Further, when the non-response is confounded, i.e. the response probability is dependent on the survey character. It becomes difficult to eliminate the bias entirely.

Theoretical developments

Consider that the finite population \( U = \{1, 2, \ldots, k, \ldots, N\} \) consists of \( N \) identifiable sampling units. Consider that a sample \( s_a \) of size \( n_a \) is drawn by sampling design \( P(.) \) from \( U \) with first and second order inclusion probabilities \( \pi_{ak} \) and \( \pi_{akl} \), \( k \neq l \) we also define

\[
\Delta_{a kl} = \pi_{akl} \pi_{ak} \pi_{al}, \quad k \neq l \subset U
\]  

...(1.1)
Suppose that the responding sample \( s_1 \) is of size \( n_1 \) and non-responding sample \( s_2 \) of size \( n_2 \). Such that \( n_1 + n_2 = n \). A subsample \( s_2 \) of size \( n_2 \) is drawn from \( s_1 \) by design \( P(\cdot | s_1) \), with first and second order inclusion properties \( n_1' \) and \( n_2' \) and

\[
\Delta_{k|s_2} = \pi_{k|s_2} - \pi_{k|s_1} \pi_{s_2}, \quad k \neq 1 \in s_2
\]  

...(1.2)

Hansen & Hurwitz (HH) estimator for the total \( T \) is given by

\[
\hat{T}_x = \sum_{s_1} \frac{y_k}{\pi_k}
\]

with

\[
\pi_k^* = \begin{cases} 
\pi_{ak} & \text{if } k \in s_{a1} \\
\pi_{ak} \pi_k/s_{a2} & \text{if } k \in s_{a2}
\end{cases}
\]  

...(1.3)

The above estimator can also be expressed as

\[
\hat{T}_x = \sum_{s_{a1}} \frac{y_k}{\pi_{ak}} + \sum_{s_{a2}} \frac{y_k}{\pi_{ak} \pi_k/s_{a2}}
\]  

...(1.4)

\[
\hat{T}_x = \sum_{s_{a1}} d_{ak} y_k + \sum_{s_{a2}} d^*_{ak} y_k
\]  

...(1.5)

Where

\[
d_{ak} = \frac{1}{\pi_{ak}}, \quad d^*_{ak} = \frac{1}{\pi_{ak} \pi_k/s_{a2}}
\]

with \( E_{RD}(\cdot/s_a) \) denoting the expectation with respect to (unknown) Random Distribution (RD), given \( s_a \), \( E_2 \) denoting the expectation of all possible samples of size \( n_2 \) from \( n_{a2} \), with \( E_2(\hat{T}_x/s_{a1}, s_{a2}) = \hat{T}_x/s_{a1} \), and \( E_1 \) denoting the expectation of all possible samples of size \( n_1 \) from \( N \). That is, the estimator \( \hat{T}_{n1} \) given by (1.5) is unbiased for population total \( T \).

Under simple random sampling without replacement (SRSWOR, hereafter SI), the estimation is given by

\[
\hat{T}_{SI} = N \left( \bar{w}_{a1} \bar{y}_{n_{a1}} + \bar{w}_{a2} \bar{y}_{n_{a2}} \right)
\]  

...(1.6)

where \( w_{a1} = n_{a1} n_{-a1}^{-1}, \quad w_{a2} = n_{a2} n_{a1}^{-1}, \quad \bar{y}_{n_{a1}} = n_{a1} \sum_{k=1}^{n_{a1}} y_k, \quad \text{and} \quad \bar{y}_{n_{a2}} = n_{a2} \sum_{k=1}^{n_{a2}} y_k
\]

In this case, the variance of estimator for the population total (1.5) is given by (Särndal et al. 1992).

\[
\text{V}(\hat{T}_x) = \sum_{k=1}^{N} \sum_{i=1}^{N} \Delta_{ai} \frac{y_k}{n_{ai}} \frac{y_i}{n_{ai}} + \sum_{k=1}^{N} \sum_{i=1}^{N} \Delta_{ai} \frac{y_k}{n_{ai}} \frac{y_i}{n_{ai}} + 2 \sum_{k=1}^{N} \sum_{i=1}^{N} \Delta_{ai} \frac{y_k}{n_{ai}} \frac{y_i}{n_{ai}} + E_1 E_{RD} \left\{ \sum_{k=1}^{N} \sum_{i=1}^{N} \Delta_{k/i} \frac{y_k}{n_{ai}} \frac{y_i}{n_{ai}} \frac{y_i}{s_{a2}} \right\}
\]  

...(1.7)

Under SRSWOR, the variance expression reduces to

\[
\text{V}(\hat{T}_{SI}) = N(f_a - 1) S_y^2 + f_a N_x (f_2 - 1) S_{2y}
\]  

...(1.8)

\[
f_a = \frac{N}{n_a}, \quad f_2 = \frac{n_2}{N}, \quad S_y^2 = \frac{1}{(N-1)} \sum_{k=1}^{N} (y_k - \bar{y}_N)^2, \quad S_{2y} = \frac{1}{(N-1)} \sum_{k=1}^{N} (y_k - \bar{y}_{N_2})^2
\]

where

An unbiased variance estimator of variance (1.8) is
\[ \hat{V}(T_{kl}) = \sum_{k=1}^{N} \sum_{l=1}^{n_k} \Delta_{kl} \frac{y_k}{n_k} \frac{y_l}{n_l} + \sum_{k=1}^{N} \sum_{l=1}^{n_k} \Delta_{kl} \frac{y_k}{n_k} \frac{y_l}{n_l} + 2 \sum_{k=1}^{N} \sum_{l=1}^{n_k} \Delta_{kl} \frac{y_k}{n_k} \frac{y_l}{n_l} \]

Where

\[ \pi^*_k = \begin{cases} \frac{\pi_{ak} \pi_{kl/s2}}{\pi_{ak} \pi_{kl/s2}} & \text{if } k, l \in s_{a2} \\ \frac{\pi_{ak} \pi_{kl/s2}}{\pi_{ak} \pi_{kl/s2}} & \text{if } k \in s_{a2}, l \in s_{a1} \\ \frac{\pi_{ak} \pi_{kl/s2}}{\pi_{ak} \pi_{kl/s2}} & \text{if } k \in s_{a1}, l \in s_{a2} \\ \pi_{kl} & \text{if } k, l \in s_{a1} \end{cases} \]

Under SRSWOR, the estimator of variance (1.9) is

\[ \hat{V}(\hat{T}_{st}) = \frac{N(N-1)n_a}{n_a - 1} \left[ \frac{(N-n_a)}{n_a (N-1)} \left( \bar{G}_w - \bar{y}_w^2 \right) + \left( f_2 - 1 \right) \frac{w_a^2}{n_a} s_{n_2}^2 \right] \]

where

\[ \bar{G}_w = \frac{1}{n_a} \sum\limits_{k=1}^{n_a} y_k^2 + \frac{n_a}{n_2} \sum\limits_{k=1}^{n_2} y_k^2 \]

\[ \bar{y}_w = \frac{1}{n_a} \left( n_a \bar{y}_{n_a} + n_2 \bar{y}_{n_2} \right) \]

and

\[ s_{n_2}^2 = \frac{1}{(n_2 - 1)} \left( \sum\limits_{k=1}^{n_2} y_k^2 - n_2 \bar{y}_{n_2}^2 \right) \]

When auxiliary variable x and study variable y is positively correlated, Cochran (1977) suggested the ratio type extension of HH estimator of population total of form

\[ \hat{T}_{r}^* = N \left( \frac{\bar{y}}{\bar{x}} \right) \bar{X} = N(\bar{r}^* \bar{X}) \]

\[ \bar{r}^* = \frac{\bar{y}^*}{\bar{X}} \]

where \( \bar{X} \) and \( \bar{Y} \) are the HH estimators for population mean \( \bar{X} \) and \( \bar{Y} \), respectively.

A large sample first-order approximation to the variance of estimator \( \hat{T}_{r}^* \) given in (1.11), obtained by using the Taylor linearization, is given by

\[ \hat{V}(\hat{T}_{r}^*) = N^2 \left\{ \frac{1}{n_a} - \frac{1}{N} \right\} S_r^2 + \frac{N_2}{N} \frac{(f_2 - 1)}{n_a} S_{2r}^2 \]

Here

\[ S_r^2 = S_y^2 + R^2 S_x^2 - 2RS_{xy} \]

\[ S_{2r}^2 = S_{2y}^2 + R^2 S_{2x}^2 - 2RS_{2xy} \]

and

\[ S_{2x}^2 = \frac{1}{(N_2 - 1)} \sum\limits_{k=1}^{N_2} \left( x_k - \bar{X}_{N_2} \right)^2 \]

where R is the population ratio of \( \bar{Y} \) to \( \bar{X} \). Again \( S_{y}^2 \) and \( S_{2y}^2 \) are respectively, the variance for the whole population and population variance for the stratum of non-respondents of the variable \( u \). Similarly \( S_{xy}^2 \) and \( S_{2xy}^2 \) are the covariance for the whole population and the population of non-respondents, respectively.

An approximate estimate of variance of ratio estimator \( \hat{T}_{r}^* \) (1.12) is given by

\[ \hat{V}(\hat{T}_{r}^*) = N^2 \left\{ \frac{1}{n_a} - \frac{1}{N} \right\} S_r^2 + w_a \left( f_2 - 1 \right) S_{2r}^2 \]
where
\[ s_r^2 = \frac{1}{(n_a - 1)} \sum_{k=1}^{n_a} \left( y_k - r^* x_k \right)^2 \]
and
\[ s_{2r}^2 = \frac{1}{(n_a - 1)} \sum_{k=1}^{n_a} \left( y_k - r^* x_k \right)^2 \]
are the sample estimates of \( r^* \) respectively.

Raman et al. (2013) proposed a calibration estimator when auxiliary information is available for the sample. In this case, the variance \( x \) is known for both the subset \( S_{a1} \) and \( S_{a2} \) because non-response is assumed absent for the auxiliary variable. Let us define
\[ \hat{X}_{S_{a2}} = \sum_{k \in S_{a2}} d_{ak} x_k \]
\[ d_{ak} = \frac{1}{n_{ak}} \]
with \( n_{ak} \). The calibration estimator of population total \( T \) given by Raman et al. (2013) is defined as
\[ \hat{T}_{cal} = \sum_{s_{a2}} d_{ak} y_k + \sum_{s_{a2}} w_k y_k \]
\[ \text{...}(1.14) \]
where
\[ w_k = d_{ak} d_k / s_{a2} + \left( \hat{X}_{S_{a2}} - \sum_{s_{a2}} x_k d_{ak} d_k / s_{a2} \right) \frac{d_{ak} d_k / s_{a2} q_k x_k}{\sum_{s_{a2}} d_{ak} d_k / s_{a2} q_k x_k^2} \]
are the calibrated weights and \( d_{ak} d_k / s_{a2} \) the original weights with
\[ d_{k/s_{a2}} = \frac{1}{\pi_k / s_{a2}} \]
\[ q_k = \frac{1}{x_k} \]
for \( x_k \), the estimator (1.14) as
\[ \hat{T}_{cal} = \sum_{s_{a2}} y_k d_{ak} + \sum_{s_{a2}} x_k d_{ak} d_{k/s_{a2}} \hat{X}_{S_{a2}} \]
\[ \text{...}(1.15) \]
Note that the estimator \( \hat{T}_{cal} \) given in (1.15) has from of Hajek type estimator, and therefore it is based. Under SRSWOR, the estimator \( \hat{T}_{cal} \) reduces to
\[ \hat{T}_{cal} = N \left[ \sum_{s_{a1}} w_{a1} \bar{y}_{a1} + \sum_{s_{a2}} w_{a2} \frac{\bar{y}_{a2}}{\bar{X}_{S_{a2}}} \bar{X}_{S_{a2}} \right] \]
\[ \bar{X}_{S_{a2}} = \frac{1}{n_{a2}} \sum_{k=1}^{n_{a2}} x_k \]
\[ \text{...}(1.16) \]
Raman et al. (2013) \([1, 4]\) have considered simple constraint, i.e. \( \sum_{s_{a2}} d_{k/s_{a2}} = 1 \) in developing calibration estimator of finite population total. However, we should have another calibration constraint, i.e. \( \sum_{s_{a2}} w_k = \sum_{s_{a2}} d_{k/s_{a2}} \). Singh & Sedery (2013) have considered calibrated weight \( w_k \) proportional to the design weight \( d_k \), i.e.
\[ w_k \propto d_{k/s_{a2}} \]
\[ \text{...}(1.17) \]
or
\[ w_k = g_k d_{k/s_{a2}} \]
The leads to some other calibration constraints, i.e.
\[ \sum_{s_{a2}} w_k = \sum_{s_{a2}} g_k d_{k/s_{a2}} \]
\[ \text{...}(1.18) \]
Motivated by these works, an attempt has been made in the present paper to develop calibration estimators of finite population total of the study variate \( y \) depending upon the different situations of availability of information on the auxiliary variable related to \( y \). The properties of the calibration estimators are derived and are discussed. Empirical studies have been conducted to study relative performance of the estimators.
2. Development calibration estimator under non-response when auxiliary information is available only on sampled units.

Recall from equation (1.5) that the HH estimator of T under general sampling design \( P(\cdot) \) is given by

\[
T_g = \sum_{s_1} d_{ak} y_k + \sum_{s_2} d^*_k y_k
\]  

...(2.1)

The calibration estimator of t can be expressed as

\[
\hat{T}_{\text{cal}(1)} = \sum_{s_1} d_{ak} y_k + \sum_{s_2} W_k y_k
\]  

...(2.2)

where \( W_k \) is calibration weight of sampling design weight \( d^*_k \) obtained by minimizing the distance measure

\[
\sum_{s_2} \left( W_k - d^*_k \right)^2 / q_k d^*_k^2
\]

subject to the following calibration equations

\[
\sum_{s_2} W_k x_k = \hat{X}_{s_2}
\]  

...(2.3)

\[
\sum_{s_2} W_k = \sum_{s_2} d^*_k
\]  

...(2.4)

The following function is minimized with regression \( W_k \)

\[
\varphi = \sum_{s_2} \left( W_k - d^*_k \right)^2 / q_k d^*_k^2 + 2\lambda_1 \left( \hat{X}_{s_2} - \sum_{s_2} W_k x_k \right) + 2\lambda_2 \left( \sum_{s_2} d^*_k - \sum_{s_2} W_k \right)
\]  

...(2.5)

Differentiating \( \varphi \) with respect to \( W_k \), and equating it to zero and solving for \( W_k \), we get

\[
W_k = d^*_k + \lambda_1 x_k q_k d^*_k + \lambda_2 q_k d^*_k
\]  

...(2.6)

Substituting \( W_k \) from equation (2.6) into equation (2.3) and equation (2.4), we get the following two equations

\[
\sum_{s_2} d^*_k x_k + \lambda_1 \sum_{s_2} q_k d^*_k x_k^2 + \lambda_2 \sum_{s_2} q_k d^*_k x_k = \hat{X}_{s_2}
\]  

...(2.7)

\[
\sum_{s_2} d^*_k + \lambda_1 \sum_{s_2} q_k d^*_k x_k + \lambda_2 \sum_{s_2} q_k d^*_k = \sum_{s_2} d^*_k
\]  

...(2.8)

From equation (2.8), we get \( \lambda_2 \) as

\[
\lambda_2 = -\frac{\lambda_1 \sum_{s_2} q_k d^*_k x_k}{\sum_{s_2} q_k d^*_k}
\]  

...(2.9)

Substituting \( \lambda_2 \) from above into equation (2.7), we get
\[
\sum_{s_2} d_{*k}^s + \lambda_1 \sum_{s_2} q_k d_{*k}^s x_k^2 - \lambda_1 \frac{\left( \sum_{s_2} q_k d_{*k}^s \right)^2}{\sum_{s_2} q_k d_{*k}^s} = \hat{X}_{s_2}
\]

\[
\lambda_1 = \frac{\left( \hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k \right)}{\sum_{s_2} q_k d_{*k}^s - \left( \sum_{s_2} q_k d_{*k}^s \right)^2 / \sum_{s_2} q_k d_{*k}^s}
\]

or

\[
\lambda_2 = -\frac{\left( \hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k \right) \sum_{s_2} q_k d_{*k}^s x_k}{A \sum_{s_2} q_k d_{*k}^s}
\]

\[
A = \sum_{s_2} q_k d_{*k}^s x_k - \left( \sum_{s_2} q_k d_{*k}^s \right)^2 / \sum_{s_2} q_k d_{*k}^s
\]

From equation (2.9) and equation (2.10), we get

\[
\lambda_2 = -\frac{\left( \hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k \right) \sum_{s_2} q_k d_{*k}^s x_k}{A \sum_{s_2} q_k d_{*k}^s}
\]

...(2.11)

From equation (2.6), (2.10) and (2.11), we get the calibration weight \( W_k \) as follows

\[
W_k = d_{*k}^s + \left( \frac{\hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k}{A} \right) q_k d_{*k}^s
\]

\[
W_k = d_{*k}^s + \left[ x_k q_k d_{*k}^s - \frac{q_k d_{*k}^s \sum_{s_2} q_k d_{*k}^s x_k}{\sum_{s_2} q_k d_{*k}^s} \right] \frac{\hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k}{A}
\]

...(2.12)

Now, calibration estimator given in (2.2) can be expressed as

\[
\hat{T}_{cal} = \sum_{s_2} d_{k} y_k + \sum_{s_2} W_k y_k
\]

\[
\hat{T}_{cal} = \sum_{s_2} d_{k} y_k + \sum_{s_2} d_{*k} y_k + \sum_{s_2} q_k d_{*k}^s x_k - \frac{q_k d_{*k}^s \sum_{s_2} q_k d_{*k}^s x_k}{\sum_{s_2} q_k d_{*k}^s} \frac{\hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k}{A}
\]

\[
\hat{T}_{cal} = \sum_{s_2} d_{k} y_k + \sum_{s_2} d_{*k} y_k + \hat{B} \left( \hat{X}_{s_2} - \sum_{s_2} d_{*k}^s x_k \right)
\]

...(2.13)
\[
\begin{aligned}
\sum_{s_2} q_{k} d_{s}^* x_{k} y_{k} - \left( \sum_{s_2} q_{k} d_{s}^* x_{k} \right) \left( \sum_{s_2} q_{k} d_{s}^* y_{k} \right) \\
\hat{B} = \frac{\sum_{s_2} q_{k} d_{s}^* x_{k} y_{k} - \left( \sum_{s_2} q_{k} d_{s}^* x_{k} \right) \left( \sum_{s_2} q_{k} d_{s}^* y_{k} \right)}{N n_{a}} \\
\sum_{s_2} q_{k} d_{s}^* x_{k}^2 - \left( \sum_{s_2} q_{k} d_{s}^* x_{k} \right)^2
\end{aligned}
\]

where

If the sampling design is sample random sampling without replacement (SRSWOR, saySI), then and

\[
d_{ak} = \frac{1}{\pi_{sk}} = \frac{N}{n_{a}} \quad d_{s}^* = d_{ak} / s_{a2} = \frac{1}{\pi_{sk}} = \frac{N n_{a2}}{n_{a} n_{2}}
\]

For \( q_{k} = 1 \), the calibration estimator \( \hat{T}_{\text{cal}} \) under SRSWOR is deduced to

\[
\hat{T}_{\text{cal}(SI)} = N \left[ n_{a1} \bar{y}_{s_{a1}} + n_{a2} \bar{y}_{s_{a2}} \right] + \frac{1}{n_{a}} \sum_{s_{a1}} y_{k} x_{k} - \frac{1}{n_{a}} \sum_{s_{a2}} y_{k} \sum_{s_{a2}} x_{k} / n_{2} \left( \bar{x}_{s_{a2}} - \bar{x}_{s_{a2}} \right)
\]

\[
= N \left[ W_{1} \bar{y}_{s_{a1}} + W_{2} \bar{y}_{s_{a2}} \right] + W_{2} \beta_{2} \left( \bar{x}_{s_{a2}} - \bar{x}_{s_{a2}} \right)
\]

\[
\bar{y}_{s_{a1}} = \frac{1}{n_{a1}} \sum_{s_{a1}} y_{k} \quad \bar{y}_{s_{a2}} = \frac{1}{n_{a2}} \sum_{s_{a2}} y_{k}
\]

where

\[
\bar{x}_{s_{a2}} = \frac{1}{n_{a2}} \sum_{s_{a2}} x_{k} \quad \bar{x}_{s_{a2}} = \frac{1}{n_{a2}} \sum_{s_{a2}} x_{k}
\]

\[
\beta_{2} = \frac{\sum_{s_{a2}} y_{k} x_{k} - \sum_{s_{a2}} y_{k} \sum_{s_{a2}} x_{k}}{n_{2}}
\]

\[
W_{1} = \frac{n_{a1}}{n_{a}} \quad W_{2} = \frac{n_{a2}}{n_{a}}
\]

\[
\hat{\beta}_{2}
\]

\[
\text{Infect} \quad \hat{\beta}_{2}
\]

The estimator \( \hat{T}_{\text{cal}(SI)} \) in equation (2.14) can also be expressed as

\[
\hat{T}_{\text{cal}(SI)} = N \bar{y}^* + W_{2} \beta_{2} \left( \bar{x}_{s_{a2}} - \bar{x}_{s_{a2}} \right)
\]

\[
\bar{y}^* \quad \text{is the usual unbiased estimator of the population mean under non-response due to Hansen & Hurwitz (1946) [2].}
\]

The second part of the estimator \( \hat{T}_{\text{cal}(SI)} \) makes its biased estimator of population total. In the following sub-section, we derive the bias and mean square error (MSE) of \( \hat{T}_{\text{cal}(SI)} \).
4.2.1 Bias and MSE of $\hat{T}_{\text{cal}(1)(\text{SI})}$

The bias of $\hat{T}_{\text{cal}(1)(\text{SI})}$ is given by

$$\text{B}[\hat{T}_{\text{cal}(1)(\text{SI})}] = \mathbb{E}[\hat{T}_{\text{cal}(1)(\text{SI})}] - T$$

\[
\begin{align*}
\mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) (\bar{x}_{x_2} - \bar{x}) \right] &= \frac{1}{n_2 - 1} \sum_{s_2} (x_k - \bar{x}_{x_2}) (y_k - \bar{y}_{x_2}) s_{x_2}^2
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) (\bar{x}_{x_2} - \bar{x}) \right] &= \frac{1}{n_2 - 1} \sum_{s_2} (x_k - \bar{x}_{x_2}) (y_k - \bar{y}_{x_2}) s_{x_2}^2
\end{align*}
\]

where,

Now, we first derive the second part of the expectation of (2.1.1).

Then, we have

Assume that

$$\bar{x}_{x_2} = \bar{x} + \epsilon_1$$

$$\bar{x}_{x_2} = \bar{x} + \epsilon_2$$

Then, we have

\[
\begin{align*}
\mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) (\bar{x}_{x_2} - \bar{x}) \right] &= \mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) \right] \left( \bar{x}_{x_2} - \bar{x} \right)
\end{align*}
\]

Ignoring the higher term of order $\epsilon_1^2, \epsilon_2^2$, and their product more than 2, we get

\[
\begin{align*}
\mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) (\bar{x}_{x_2} - \bar{x}) \right] &= \mathbb{E}_W \left[ \left( \frac{s_{xy}(2)}{s_{x_2}^2} \right) \right] \left( \bar{x}_{x_2} - \bar{x} \right)
\end{align*}
\]

$$\mathbb{E}[{\epsilon_1}\eta_2] = \text{cov}(\bar{x}_{x_2}, S_{x_2}^2)$$

$$= \frac{h - 1}{n_2} + \frac{k - 1}{n_2} \mu_{30(2)}$$

$$\text{cov}(\bar{x}_{x_2}, S_{x_2}^2)$$

$$= \frac{1}{N_2 - 1} \sum_{k=1}^{N_2} (x_k - \bar{x}_{x_2}) (y_k - \bar{y}_{x_2})$$

where

$$h = \frac{N_2}{n_2}, \quad k = \frac{n_2}{n_2}$$

and

$$\text{cov}(\bar{x}_{x_2}, S_{x_2}^2)$$

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\[
E(\varepsilon_i \eta_i) = \text{cov}[\bar{x}_{s_2}, s_{y(2)}] = h - 1 \frac{\mu_{3(2)}}{N_2} \quad \ldots (2.1.4)
\]

\[
E(\varepsilon_i \eta_i) = \text{cov}[\bar{x}_{s_2}, s_{y(2)}] = \left(\frac{h - 1}{N_2} + \frac{k - 1}{n_{s_2}}\right) \mu_{21} \quad \ldots (2.1.5)
\]

Now, we have from the equation (2.1.3), (2.1.4), (2.1.5), and (2.1.6), putting these values in equation (2.1.2). We get

\[
E_1 \left[ \beta_1 (\bar{x}_{s_2} - \bar{x}_{s_2}) \right] = \frac{k - 1}{n_{s_2}} \beta_2 \left[ \mu_{30(2)} - \mu_{21(2)} \right] \quad \ldots (2.1.7)
\]

Mean square error (MSE) of \( \hat{T}_{\text{cal(SI)}} \) is derived as follows

\[
\text{MSE} [\hat{T}_{\text{cal(SI)}}] = E[\hat{T}_{\text{cal(SI)}} - T]^2 = \left[ N(\bar{y}_w - \bar{Y}) + NW_2 \beta_1 (\bar{x}_{s_2} - \bar{x}_{s_2}) \right]^2
\]

Where, \( T = N\bar{Y}, \bar{Y} \) the population mean of \( Y \).

\[
= N^2 \left[ V(\bar{y}_w) + \beta_2^2 E(W_2^2 (\epsilon_2^2 - \epsilon_1^2 - 2\epsilon_1\epsilon_2)) + 2\beta_2 E(W_2 (\bar{y}_w - \bar{Y}) (\epsilon_2 - \epsilon_1)) \right] \quad \ldots (2.1.9)
\]

Now, we drive the first various expectations as follows.

\[
E(\epsilon_1^2) = V(\bar{x}_{s_2}) = V[E(\bar{x}_{s_2}/s_{s_2})] + E[V(\bar{x}_{s_2}/s_{s_2})]
\]

\[
= \left( \frac{h - 1}{N_2} + \frac{k - 1}{n_{s_2}} \right) s_{s_2}^2 \quad \ldots (2.1.10)
\]

\[
E(\epsilon_2^2) = V(\bar{x}_{s_2}) = V[E(s_{s_2}/\bar{x}_{s_2})] + E[V(\bar{x}_{s_2}/s_{s_2})]
\]
\[
E(\varepsilon_i \varepsilon_j) = \text{cov}\left(\bar{x}_{s_2}, \bar{x}_{s_2}\right) = \text{cov}\left[E\left(\bar{x}_{s_2} / s_{a_2}\right)E\left(\bar{x}_{s_2} / s_{a_2}\right)\right] + E\left[\text{cov}\left(\bar{x}_{s_2}, \bar{x}_{s_2} / s_{a_2}\right)\right] \\
= \text{cov}\left[\bar{x}_{s_2}, \bar{x}_{s_2}\right] \\
= \text{var}(\bar{x}_{s_2}) \\
= \frac{h-1}{N_2} S_{x^2} 
\]

\[\text{(2.1.11)}\]

Substituting the value of \(E(\varepsilon_i^2), E(\varepsilon_i^2)\) and \(E(\varepsilon_i \varepsilon_j)\) from the equation (2.1.10), (2.1.11) and (2.1.12) in to the equation (2.1.9). we have

\[
\text{MSE}\left[\hat{T}_{(a)(si)}\right] = N^2 \left[\text{var}(\bar{y}_w) + \beta_2^2 E_{RD}(W_2) \left(\frac{k-1}{n_{a_2}} S_{x^2}\right) + 2\beta_2 E_{RD}(W_2) E_2(\bar{y}_w - \bar{y})(\varepsilon_2 - \varepsilon_1)\right] 
\]

\[\text{(2.1.12)}\]

We now drive the expectation of third term in equation (2.1.13) as follows

\[
E_2(\bar{y}_w - \bar{y})(\varepsilon_2 - \varepsilon_1) = E_2\left[\left(W_2 \bar{y}_{s_2} + W_2 \bar{y}_{s_2} - \bar{y}\right)(\varepsilon_2 - \varepsilon_1)\right] 
\]

and get

\[
= W_2 E_2 \bar{y}_{s_2} (\varepsilon_2 - \varepsilon_1) 
\]

\[\text{(2.1.14)}\]

Now, we derive the expectation of equation (2.1.14).

Assume that

\[
\bar{y}_{s_2} = \left(1 + \frac{\varepsilon_0}{\bar{y}_2}\right) \bar{y}_2 \\
= \bar{y}_2 E(W_2) E\left[\frac{\varepsilon_0 \varepsilon_2}{\bar{y}_2} - \frac{\varepsilon_0 \varepsilon_1}{\bar{y}_2}\right] 
\]

\[\text{(2.1.15)}\]

\[
E(\varepsilon_0 \varepsilon_2) = \text{cov}\left(\bar{y}_{s_2}, \bar{x}_{s_2}\right) \\
= \frac{h-1}{N_2} S_{y^2} 
\]

\[\text{(2.1.16)}\]

\[
E(\varepsilon_0 \varepsilon_1) = \text{cov}\left(\bar{y}_{s_2}, \bar{x}_{s_2}\right) \\
= \left[\frac{h-1}{N_2} + \frac{k-1}{n_{a_2}}\right] S_{y^2} 
\]

\[\text{(2.1.17)}\]

Now, we have from (2.1.15), (2.1.16) and (2.1.17) putting these values in equation (2.1.14). we get

\[
\text{EW}_2 E(\bar{y}_w - \bar{y})(\varepsilon_2 - \varepsilon_1) = \bar{y}_2 E(W_2) \left[\frac{h-1}{N_2 \bar{y}_2} - \frac{h-1}{N_2 \bar{y}_2} - \frac{k-1}{n_{a_2} \bar{y}_2}\right] S_{y^2} \\
= -E(W_2) \frac{k-1}{n_{a_2}} S_{y^2} 
\]
From the equation of (2.1.18), (2.1.19) and (2.1.20), we get

\[
\text{MSE} \left[ \hat{T}_{\text{cal}}(\text{SI}) \right] = N^2 \left[ V \left( \bar{y}_w \right) + \beta_2^2 E_{RD} \left( W_2^2 \left( k - 1 \right) S_{xy}^2 \right) - 2\beta_2 E_{RD} \left( W_2^2 \left( k - 1 \right) S_{xy}^2 \right) \right]
\]

\[
V \left[ \hat{T}_{\text{cal}}(\text{SI}) \right] = N^2 \left[ \frac{N - n_a}{N n_a} \frac{S_{\xi}^2}{S_{\gamma}^2} + \frac{k - 1}{n_a} W_2 \left[ 1 - \rho_2^2 S_{\gamma}^2 \right] \right]
\]

\[
\beta_2^2 = \left( \frac{S_{xy}^2}{S_{\xi}^2} \right)^2
\]

\[
= \rho_2^2 S_{\xi}^2 S_{\gamma}^2
\]

\[
= \rho_2^2 S_{\gamma}^2
\]

An empirical study has been conducted with real data to examine the relative efficiency of the proposed and existing estimators in comparison to Hansen & Hurwitz (1946) estimator. We have considered the municipality level data of two populations from the Appendix B (MU284 population) of Sarndal et al. (2003), the details of which are given below.

| Population | Study variate | Auxiliary variable |
|------------|--------------|--------------------|
| (I)        | RMT85 Revenues from the 1985 municipal taxation (in millions of kronor). | REV 84 Real estate values according to 1984 assessment (in millions of kronor). |

The population consists of N=284 municipalities. It is assumed that the last 25 percent municipalities fall under non-response. That means response class consists of first 213 (N₁) municipalities and non-response class consist of last 71 (N₂) municipalities. The various population parameters of the both populations have been computed and are presented in the Table 1. It is also considered that a sample size of n=57 was drawn by SRSWOR from the population of size N=284. It is also assumed that 42 units respondent while 15 units did not respond while conducting the surveys. A sub-sample of size \( n_2 = 9 \) from non-respondent units (15) was drawn by SRSWOR, and efforts were made to get response from there 9 units.

\[
k = \frac{n_2}{h_2} = 1.67, \text{ i.e. } k - 1 = 0.67 \text{ and } f = \frac{n}{N} = 0.20
\]

We have considered the some real population -1 as described below. We have also considered the following value of the sample size in reference to the Table 1 of the below
\[ n_a = 57, \ n_{a1} = 42, \ n_{a2} = 15 \quad \text{and} \quad n_2 = 9 \]

The variance of HH estimator of the population total and MSE of calibration estimator \( \hat{T}_{\text{cal}(l)(SI)} \) using auxiliary information on only sample units belonging to non-response class have been computed for the population. The relative efficiency of the \( \hat{T}_{\text{cal}(l)(SI)} \) over HH estimator has also been computed to the 115.22 and percent for the population -1.

| Table 1: Results of population parameters |
|------------------------------------------|
| Population parameters | Population |
|------------------------|------------|
| \( S_y^2 \)            | 355612.50  |
| \( S_{y2}^2 \)         | 31736.44   |
| \( S_x^2 \)            | 22521693.33|
| \( S_{x2}^2 \)         | 5698777.027|
| \( S_{xy} \)           | 2648372.57 |
| \( S_{y2} \)           | 360961.938 |
| \( \rho \)             | 0.936      |
| \( \rho_2 \)           | 0.848      |
| \( \beta_{2x1} \)      | 85.0496    |
| \( \beta \)            | 0.117592   |
| \( \beta_2 \)          | 0.063340   |
| \( C_x \)              | 1.54       |
| \( C_{x2} \)           | 0.9131     |
| \( C_y \)              | 2.433      |
| \( C_{y2} \)           | 1.0727     |
| \( \bar{X} \)          | 3074.355   |
| \( \bar{Y} \)          | 245.088    |

This shows that if partial auxiliary information is available, then calibration approach can be applied to get more precise estimate of population mean/ total as compared to HH estimator.

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