Finite-size effects from higher conservation laws for the one-dimensional Bose gas

Erik Eriksson\textsuperscript{1} and Vladimir Korepin\textsuperscript{2}

\textsuperscript{1} Department of Physics, University of Gothenburg, SE-412 96 Gothenburg, Sweden
\textsuperscript{2} CN Yang Institute for Theoretical Physics, State University of New York at Stony Brook, NY 11794-3840, USA

E-mail: erik.eriksson@physics.gu.se and korepin@insti.physics.sunysb.edu

Received 21 February 2013, in final form 22 April 2013
Published 21 May 2013
Online at stacks.iop.org/JPhysA/46/235002

Abstract

We consider a generalized Lieb–Liniger model, describing a one-dimensional Bose gas with all its conservation laws appearing in the density matrix. This will be the case for the generalized Gibbs ensemble, or when the conserved charges are added to the Hamiltonian. The finite-size corrections are calculated for the energy spectrum. Large-distance asymptotics of correlation functions are then determined using methods from conformal field theory.

PACS numbers: 02.30.Ik, 05.30.Jp

1. Introduction

There has recently been much interest in so-called generalized Gibbs ensembles (GGEs), where the density matrix of a system is given by

$$\hat{\rho}_{\text{GGE}} = Z^{-1}_{\text{GGE}} e^{-\sum \beta_n \hat{Q}_n},$$

(1)

generalizing the usual Gibbs ensemble for generic (non-integrable) systems where only the Hamiltonian and a particle number operator appear in the exponent. Instead, for an integrable system, \(\{\hat{Q}_n\}\) is the complete set of local conserved charges, with \(\{\beta_n\}\) the generalized inverse temperatures (Lagrange multipliers) and \(Z_{\text{GGE}} = \text{Tr} e^{-\sum \beta_n \hat{Q}_n}\) the generalized partition function [1]. For such quantum ensembles, the expectation values of observables are obtained as

$$\langle \hat{O} \rangle_{\text{GGE}} = \text{Tr} \hat{\rho}_{\text{GGE}} \hat{O},$$

(2)

whereas the time evolution is governed by the usual Schrödinger Hamiltonian.

Recent experimental advances with quantum non-equilibrium dynamics of cold atoms [2], where the GGE was proposed [3] to describe local large-time behavior, have spurred a great interest in the possible equilibrium ensembles for integrable systems (see e.g. [4–6] and references therein, and in particular [7–10] for treatments of the one-dimensional Bose gas).
But effects from higher conservation laws in integrable systems were also studied longer ago in the context of competing interactions in spin chains [11, 12].

It is therefore a timely question to now ask in general what sort of effects one can anticipate when incorporating higher conservation laws into the density matrix for an integrable model. In this paper, we study finite-size effects and obtain conformal dimensions for the one-dimensional Bose gas (Lieb–Liniger model), when governed by a density matrix of the form (1). In particular, using the exact Bethe Ansatz solution we calculate the finite-size corrections for the energy and momentum. Comparing this to the expressions given by conformal field theory, we are able to obtain the large-distance asymptotics of correlation functions.

2. Lieb–Liniger model

The Lieb–Liniger model describes a one-dimensional Bose gas with point-like interaction through the Hamiltonian

\[ \hat{H} = \int dx [\partial_x \hat{\Psi}_1(x) \partial_x \hat{\Psi}_1(x) + c \hat{\Psi}_1(x) \hat{\Psi}_1(x) \hat{\Psi}_1(x) \hat{\Psi}_1(x)], \]

with the coupling constant \( c > 0 \) and Bose fields \( \hat{\Psi}_1(x) \) with equal-time commutation relations \( [\hat{\Psi}_1(x), \hat{\Psi}_1(y)] = i \delta(x-y) \) and \( [\hat{\Psi}_1(x), \hat{\Psi}_1(x)] = [\hat{\Psi}_1(x), \hat{\Psi}_1(y)] = 0 \). The model is solved through the Bethe Ansatz [13–15], yielding eigenstates \( |\{\lambda_j\}\rangle \) given in terms of rapidities \( \lambda_j \) satisfying the Bethe equations

\[ e^{i\lambda_j L} = -\prod_{k=1}^N \frac{\lambda_j - \lambda_k + i c}{\lambda_j - \lambda_k - i c}, \quad j = 1, \ldots, N, \]  

for a system of \( N \) particles in a box of length \( L \) with periodic boundary conditions. The Bethe equations (4) can equivalently be written as

\[ L \lambda_j + \sum_{k=1}^N \theta(\lambda_j - \lambda_k) = 2\pi n_j, \quad j = 1, \ldots, N, \]

with \( n_j \) integer when \( N \) odd and half-integer when \( N \) even and \( \theta(\lambda) = \text{iln}[(-ic+\lambda)/(ic-\lambda)] \).

The conserved charges \( \hat{Q}_n \) have eigenvalues \( Q_n \) given by

\[ Q_n = \sum_{j=1}^N \lambda_j^n, \]

the three lowest being particle number \( N = Q_0 \), momentum \( P = Q_1 \) and energy \( E = Q_2 \). Explicit expressions for some of the higher conserved charges \( \hat{Q}_n \) can be found in [16]. Now, let us consider a generalized Hamiltonian \( \hat{H}_G \) where all the conserved charges \( \hat{Q}_n \) have been added to the Lieb–Liniger Hamiltonian (3),

\[ \hat{H}_G = \hat{H} + \sum_{n \neq 2} b_n \hat{Q}_n, \]

with coefficients \( b_n \). Extremizing the entropy gives a density matrix of the form (1),

\[ \hat{\rho}_G = Z_G^{-1} e^{-\beta \hat{H}_G}, \]

with \( \beta = 1/T \) the inverse temperature and \( Z_G \) the generalized partition function. Hence, we can analyze both the situations of a system described by a generalized Gibbs ensemble (1) as well as a system where the Schrödinger Hamiltonian itself is given by \( \hat{H}_G \) in equation (7).
The eigenvalues $E(\{\lambda_j\})$ of the generalized Hamiltonian $\hat{H}_G$ are given by
\begin{equation}
E(\{\lambda_j\}) = \sum_{j=1}^{N} \varepsilon_0(\lambda_j),
\end{equation}
where the bare one-particle energy $\varepsilon_0(\lambda)$ is given by the polynomial function
\begin{equation}
\varepsilon_0(\lambda) = \sum_{n=0}^{\infty} b_n \lambda^n,
\end{equation}
with $b_2 = 1$ and $b_0 = -\hbar$ the chemical potential. It was shown in [17] that the generalized Yang–Yang thermodynamic Bethe Ansatz equation becomes
\begin{equation}
\varepsilon(\lambda) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu K(\lambda, \mu) \ln(1 + e^{-\beta \varepsilon(\mu)}) = \varepsilon_0(\lambda),
\end{equation}
and that a solution exists provided that $\varepsilon_0(\lambda)$ is bounded from below and $\lim_{\lambda\to\pm\infty} \varepsilon_0(\lambda) = +\infty$. The kernel $K(\lambda, \mu)$ is given by
\begin{equation}
K(\lambda, \mu) = \theta'(\lambda - \mu) = \frac{2c}{c^2 + (\lambda - \mu)^2},
\end{equation}
and the equilibrium particle distribution function $\rho(\lambda)$ by
\begin{equation}
\rho(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \theta(\lambda - \mu) K(\lambda, \mu) \rho(\mu),
\end{equation}
with the Fermi weight
\begin{equation}
\theta(\lambda) = \frac{1}{1 + e^{\beta \varepsilon(\lambda)}}.
\end{equation}
This can also be written as $\rho(\lambda) = \theta(\lambda) \rho_1(\lambda)$, where
\begin{equation}
\rho_1(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \theta(\mu) K(\lambda, \mu) \rho_1(\mu) = \frac{1}{2\pi}
\end{equation}
is the density of states. Similarly, the so-called dressed charge $Z(\lambda)$ is defined by
\begin{equation}
Z(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \theta(\mu) K(\lambda, \mu) Z(\mu) = 1,
\end{equation}
reflecting that for the Lieb–Liniger model $Z(\lambda) = 2\pi \rho_1(\lambda)$.

3. Finite-size corrections at zero temperature

3.1. Energy

Let us now investigate the finite-size corrections to the generalized energy (9), focusing on the zero-temperature limit. Then the set of numbers $n_j$ in equation (5) that minimizes the generalized energy (9) is such that all $\lambda_j$ with $\varepsilon(\lambda_j) < 0$ are occupied and the rest empty of particles. In the thermodynamic limit, this gives Fermi points $q_i^\pm$ wherever $\varepsilon(q_i^\pm) = 0$, with a filled Fermi sea $i$ for $q_i^- \leq \lambda \leq q_i^+$. The ground-state energy in the thermodynamic limit can then be written as
\begin{equation}
E_0 = L \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \varepsilon_0(\lambda) = L \sum_i \int_{q_i^-}^{q_i^+} d\lambda \rho_1(\lambda) \varepsilon_0(\lambda),
\end{equation}
with finite-size corrections coming from the replacement of the sum in equation (9) with an integral. These are obtained using the Euler–Maclaurin formula in complete analogy with the usual case [18, 19], yielding for the leading corrections
\begin{equation}
E_0 = L \sum_i \int_{q_i^-}^{q_i^+} d\lambda \rho_1(\lambda) \varepsilon_0(\lambda) - \frac{\pi}{12L} \sum_{i,j} |v_i|^2,
\end{equation}
where $v_i$ are the...
where $i$ is the Fermi sea index and $v = \pm$ the right/left index. Here, $v_i^\pm$ is the Fermi velocity at the Fermi point $q_i^\pm$, given by

$$v_i^\pm = \frac{1}{2\pi\rho_0(q_i^\pm)} \frac{\partial E}{\partial q_i^\pm}.$$  (19)

Now we investigate the finite-size corrections of low-energy excited states, using the techniques found in [20–25]. First, we expand the energy $E(q_i^\pm) = E_0 + \delta E$ to second order around the ground-state energy $E_0$ in equation (18),

$$\delta E = \sum_{i,v} \left( \frac{\partial E}{\partial q_i^v} \right) \delta q_i^v + \frac{1}{2} \sum_{i,j,v,v'} \left( \frac{\partial^2 E}{\partial q_i^v \partial q_j^{v'}} \right) \delta q_i^v \delta q_j^{v'}$$  (20)

with $\delta q_i^\pm$ the change in Fermi momentum $q_i^\pm$ with respect to the ground state. Now,

$$\frac{\partial E}{\partial q_i^v} = \pm L \rho_v(q_i^\pm) e_v(q_i^\pm) + L \sum_j \int_{q_i^j}^{q_i^j} d\lambda \frac{\partial \rho_v(\lambda)}{\partial q_i^v} e_0(\lambda) = \pm L \rho_v(q_i^\pm) e(q_i^\pm),$$  (21)

where $e$ is the dressed energy (11). Since the ground state minimizes $E$, one has

$$e(q_i^\pm) = 0.$$  (22)

Then, equation (20) becomes

$$\delta E = \frac{L}{2} \sum_{i,v,v'} v \rho_v(q_i^\pm) e'(q_i^v) (\delta q_i^v)^2 = \pi L \sum_{i,v} |v_v|^2 \left[ \rho_v(q_i^\pm) \delta q_i^v \right]^2.$$  (23)

Now, we will express $\delta q_i^\pm$ in terms of the quantum numbers $\Delta N_i$ and $\Delta D_j$, defined as the changes in

$$N_i = L \int_{q_i^-}^{q_i^+} d\mu \rho_v(\mu),$$  (25)

$$D_j = L \left( \int_{-\infty}^{q_i^-} - \int_{q_i^+}^{\infty} \right) d\mu \rho_v(\mu),$$  (26)

compared to the ground state. Then,

$$\delta q_i^\pm = \sum_j \frac{\partial q_i^\pm}{\partial N_j} \Delta N_j + \sum_j \frac{\partial q_i^\pm}{\partial D_j} \Delta D_j.$$  (27)

The Jacobian is found from

$$\frac{\partial N_j}{\partial q_i^\pm} = \pm L \rho_v(q_i^\pm) \left[ \delta_{ij} \pm \int_{q_i^j}^{q_i^j} d\mu \ g(\mu|q_i^\pm) \right],$$  (28)

$$\frac{\partial D_j}{\partial q_i^\pm} = L \rho_v(q_i^\pm) \left[ \delta_{ij} + \left( \int_{-\infty}^{q_i^-} - \int_{q_i^+}^{\infty} \right) d\mu \ g(\mu|q_i^\pm) \right],$$  (29)

where the function $g(\lambda|q_i^\pm)$ is defined through

$$g(\lambda|q_i^\pm) = \frac{1}{2\pi} \sum_k \int_{q_k^-}^{q_k^+} d\mu \ K(\lambda, \mu) g(\mu|q_i^\pm) = \pm \frac{1}{2\pi} K(\lambda, q_i^\pm).$$  (30)
Equation (30) is equivalent to $g(\lambda|q_i^\pm) = [\partial \rho_i(\lambda)/\partial q_i^\pm]/\rho_i(q_i^\pm)$. For the general case, we need to consider the vectors

$$\vec{n} = \begin{pmatrix} N_1 \\ \vdots \\ N_L \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} q_1^- \\ q_1^+ \\ q_2^- \\ \vdots \end{pmatrix}$$

so that we can write

$$\Delta n_i = L \sum_j M_{ij} \rho_i(p_j) \delta p_j, \quad M_{ij} = \frac{1}{L \rho_i(p_j)} \frac{\partial n_i}{\partial p_j},$$

with $\partial n_i/\partial p_j$ given by equations (28)—(29), and hence obtain $\rho_i(q_i^\pm)\delta q_i^\pm$ from

$$\rho_i(p_i)\delta p_i = \frac{1}{L} \sum_j (M^{-1})_{ij} \Delta n_j.$$  

Inserting this into equation (24), and including the numbers $N_j^-$ and $N_j^+$ of particle–hole excitations at $q_i^-$ and $q_i^+$, respectively, gives the general expression

$$\delta E = \frac{\pi}{L} \sum_{k,v} |v_k^v|^2 \left[ 2N_j^v + \left( \sum_j (M^{-1})_{kj} \Delta n_j \right)^2 \right].$$

This expression is simplified when the model has parity symmetry, e.g. around $\lambda = 0$ so that only even powers of $\lambda$ appear in equation (10) and hence $\varepsilon_0(\lambda) = \varepsilon_0(-\lambda)$. If this is the case, we can consider the new pairs of Fermi points $q_j^\pm = \pm q_j (q_1 > q_2 > \cdots > 0)$, with a sea of either particles or holes between them and Fermi velocities $|v_i^\pm| = v_i$. Now, define

$$N_j = L \int_{-q_j}^{q_j} d\mu \rho_i(\mu),$$

$$D_j = L \left( \int_{-\infty}^{-q_j} - \int_{q_j}^{\infty} \right) d\mu \rho_i(\mu),$$

i.e. $\Delta N_j$ being the number of particles/holes added to Fermi sea $j$ of particles/holes, and $d_j = \Delta D_j/2$ the number jumping from $q_j$ to $-q_j$. Let us now write

$$\frac{\partial N_j}{\partial q_j^\pm} = \pm L \rho_i(q_j) \frac{2 \left[ \delta_{ij} - (-1)^j \int_{-q_j}^{q_j} d\mu \, g(\mu|q_j) \right]}{2} \equiv \pm L \rho_i(q_j) 2Z_{ij},$$

$$\frac{\partial D_j}{\partial q_j^\pm} = L \rho_i(q_j) \left[ \frac{2 \left[ \delta_{ij} - (-1)^j \left( \int_{-\infty}^{-q_j} - \int_{q_j}^{\infty} \right) d\mu \, g(\mu|q_j) \right]}{2} \right] \equiv L \rho_i(q_j) 2Y_{ij},$$

with $g(\lambda|q_i^\pm)$ given by equation (30). By expressing the matrix elements of $Z$ as $Z_{ij} = \delta_{ij} + \int_{-q_j}^{q_j} d\mu \, g_{ij}(\mu)$, with $g_{ij}(\mu) \equiv (-1)^{i+j} g(\mu|q_j)$, one finds $Z_{ij} = \xi_{ij}(q_j)$, where $\xi(\lambda)$ is the dressed charge matrix defined through

$$\xi_{ij}(\lambda) = -\frac{1}{2\pi} \sum_k \int_{-\infty}^{\infty} d\mu \, K_{ik}(\lambda, \mu) \xi_{kj}(\mu) = \delta_{ij},$$

so that here, the kernel matrix element is $K_{ij}(\lambda, \mu) = K(\lambda, \mu)$ when $-q_j \leq \mu \leq q_j$, otherwise zero. Now, from

$$Z_{ij} = \delta_{ij} = \int_{-\infty}^{+\infty} d\lambda \left. \frac{\partial \xi_{ij}(\lambda)}{\partial q_i} \right|_{\lambda = q_i} = -\sum_k \int_{-\infty}^{+\infty} d\lambda \left[ g_{ik}(\lambda) - g_{ik}(\lambda) \right] Z_{kj},$$
it follows that
\[ Y_{ij} = \delta_{ij} + \left( \int_{-\infty}^{-q_i} - \int_{q_i}^{\infty} \right) d\mu g_{ij}(\mu) = \delta_{ij} - \sum_k [Z_{jk} - \delta_{jk}] (Z^{-1})_{jk} = (Z^{-1})_{ji}. \] (41)

Hence, \( Y = (Z^T)^{-1} \). Putting this into equations (27) and (24) finally gives
\[ \delta E = \frac{2\pi}{L} \sum_j v_j [\Delta^+_j + \Delta^-_j], \] (42)
where
\[ \Delta^\pm = N^\pm + \frac{1}{2} \left( \sum_k (Z^{-1})_{jk} \frac{\Delta N_k}{2} \pm \sum_k Z_{kj} \delta_k \right)^2. \] (43)

Obviously, when the bare single-particle dispersion in equation (10) is given by \( \epsilon_0(\lambda) = \lambda^2 \), there is just a single Fermi sea, between \( \lambda = -q \) and \( \lambda = q \), and equations (42) and (43) reduce to those for the usual Lieb–Liniger model,
\[ \delta E = \frac{2\pi}{L} v [\Delta^+ + \Delta^-], \quad \Delta^\pm = N^\pm + \frac{1}{2} \left( \frac{\Delta N}{2Z(q)} \pm Z(q) d \right)^2. \] (44)

### 3.2. Momentum

Since the momentum \( P = \sum_i \lambda_i = \frac{2\pi}{L} \sum_i n_i \), as is easily seen from equation (5), it is trivially obtained as
\[ P = P_0 + \sum_i \left[ \tilde{P}_i \Delta N_i - k_i d_i + \frac{2\pi}{L} \left\{ N^+_i - N^-_i + d_i \Delta N_i \right\} \right]. \] (45)
for the excited states, where \( P_0 \) is the ground-state momentum. In the thermodynamic limit, \( \tilde{P}_i = (q^+_i + q^-_i)/2 \) and \( k_i = q^+_i - q^-_i \). In the case of parity symmetry, \( \tilde{P}_j = 0 \) and \( k_j = 2q_j \).

### 4. Conformal dimensions and correlation functions

It is now clear that the finite-size corrections in equations (18), (42) and (45) can be written in the form
\[ E_0 - E_0(L \to \infty) = -\frac{\pi}{6L} \sum_j c_j |v_j| + \text{h.o.c.} \] (46)
\[ E - E_0 = \frac{2\pi}{L} \sum_j |v_j| [\Delta^+_j + \Delta^-_j] + \text{h.o.c.} \] (47)
\[ P - P_0 = \sum_j [\tilde{P}_j \Delta N_j - k_j d_j] + \frac{2\pi}{L} \sum_j [\Delta^+_j - \Delta^-_j] + \text{h.o.c.} \] (48)
where h.o.c. denotes higher-order corrections. This tells us that the low-energy physics is given by a sum of conformal field theories [26, 27], where \( \Delta^\pm \) are the scaling dimensions of the scaling operators of the theories and \( c_j = 1 \) the central charges. We will now use this to obtain the large-distance asymptotics of the equal-time correlation functions [21]. The field correlation function
\[ \langle \Psi(x) \Psi^\dagger(0) \rangle = \text{Tr} \tilde{\rho}_0 \Psi(x) \Psi^\dagger(0), \] (49)
for which the total number of added particles is $\Delta N = \sum_j (\Delta N_{2j-1} - \Delta N_{2j}) = 1$, should then have the asymptotic form

$$
\langle \Psi(x) \Psi^\dagger(0) \rangle \sim \sum_{Q_\psi} A(Q_\psi) e^{i \sum_j d_j k_j x - 2 \sum_j (\Delta_j^+ + \Delta_j^-)},
$$

(50)

where $Q_\psi = \{ \Delta N_j, \{ d_j \}, \{ N_j^\pm \} | \Delta N = 1 \}$ is the set of quantum numbers for the excitations, and $A(Q_\psi)$ the amplitudes. The leading term is given by

$$
\langle \Psi(x) \Psi^\dagger(0) \rangle \sim x^{-\alpha},
$$

(51)

with $\alpha$ the smallest sum of scaling dimensions $2 \sum_j (\Delta_j^+ + \Delta_j^-)$ given the constraint $\Delta N = 1$, i.e. $\alpha$ is the smallest of the numbers $(\sum_j (Z^{-1}) j) \frac{1}{2} j^2$. Similarly, density–density correlators are obtained with $\Delta N = 0$,

$$
\langle j(x) j(0) \rangle - \langle j(0) \rangle^2 \sim \sum_{Q_d} A(Q_d) e^{i \sum_j d_j k_j x - 2 \sum_j (\Delta_j^+ + \Delta_j^-)},
$$

(52)

where $j(x) = \Psi^\dagger(x) \Psi(x)$ and $Q_d = \{ \Delta N_j, \{ d_j \}, \{ N_j^\pm \} | \Delta N = 0 \}$. The leading terms are

$$
\langle j(x) j(0) \rangle - \langle j(0) \rangle^2 \sim A_1 x^{-2} + A_2 \cos(d k_0 x) x^{-\theta},
$$

(53)

where the first term comes from the processes with $d_k = 0$ and one single number $N_k^h$ equal to one, and the second term is from the two processes with the $d_k = \pm 1$ giving smallest $\theta$, i.e. with $\theta$ the smallest of the numbers $2(\sum_j Z_{kj})^2$.

Importantly, the finite-temperature correlation functions for mixed states are obtained by the standard conformal mapping $z(u) = e^{2\pi T w/i}$ in the complex plane $z = x - iv/t$, so that in the formulas above

$$
x^{-2\Delta_j^h} \rightarrow \{ (\pi T/v_j)/\sinh(\pi T x/v_j) \}^{2\Delta_j^h}.
$$

(54)

5. Discussion

We have obtained the finite-size corrections for the energy and momentum of a one-dimensional Bose gas with delta-function interaction (Lieb–Liniger model) when additional conservation laws are present in the density matrix, or the conserved charges are added to the Hamiltonian. The results show that the low-energy physics is described by a sum of conformal field theories each with central charge $c = 1$, where each may have its own specific speed of sound. The picture is most clear when the bare dispersion is parity symmetric. In this case, we derived the asymptotic behavior of the correlation functions using standard arguments.

The finite-temperature mapping (54) provides a possible connection to present studies of quantum non-equilibrium dynamics. In the standard setup, an isolated system in a pure state is time evolved by, but not an eigenstate of, the usual Schrödinger Hamiltonian. The density matrix of a subsystem will however be in a mixed state, presumably approaching the form (1) with a finite effective temperature. It remains to be seen whether the new types of correlation effects in the generalized model studied here may appear in such systems.

It is interesting to note that the generalized one-dimensional Bose gas with a dispersion relation with many Fermi points for the bare particles gives similar types of finite-size effects as in well-studied examples of multicomponent (a.k.a. nested) Bethe Ansatz solvable models [29], such as integrable quantum spin chains [11, 12, 28, 30–34] and the one-dimensional Hubbard model [24, 35], even though the one-dimensional Bose gas only contains a single species of particles.
Acknowledgments

We wish to thank Holger Frahm for helpful discussions. EE acknowledges financial support from the Swedish Research Council (grant no. 621-2011-3942) and from STINT (grant no. IG2011-2028), and VK from NSF (grant no. DMS-1205422).

References

[1] Jaynes E T 1957 Information theory and statistical mechanics Phys. Rev. 106 620
[2] Kinoshita T, Wenger T and Weiss T D 2006 A quantum Newton’s cradle Nature 440 900
[3] Rigol M, Dunjko V, Yurovsky V and Olshnani M 2007 Relaxation in a completely integrable many-body quantum system: an ab initio study of the dynamics of the highly excited states of 1D lattice hard-core bosons Phys. Rev. Lett. 98 050405
[4] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Nonequilibrium dynamics of closed interacting quantum systems Rev. Mod. Phys. 83 863
[5] Calabrese P, Essler F H L and Fagotti M 2012 Quantum quenches in the transverse field Ising chain: II. Stationary state properties J. Stat. Mech. P07022
[6] Caux J-S and Essler F H L 2013 Time evolution of local observables after quenching to an integrable model arXiv:1301.3806
[7] Caux J-S and Konik R M 2012 Constructing the generalized Gibbs ensemble after a quantum quench Phys. Rev. Lett. 109 175301
[8] Kormos M, Shashi A, Chou Y-Z and Imambekov A 2012 Interaction quenches in the Lieb–Liniger model arXiv:1204.3889
[9] Mossel J and Caux J-C 2012 Exact time evolution of space- and time-dependent correlation functions after an interaction quench in the one-dimensional Bose gas New J. Phys. 14 075006
[10] Brandino G, Caux J-C and Konik R 2013 Relaxation dynamics of conserved quantities in a weakly non-integrable one-dimensional Bose gas arXiv:1301.0308
[11] Tsvelik A M 1990 Incommensurate phases of quantum one-dimensional magnetics Phys. Rev. B 42 779
[12] Frahm H 1992 Integrable spin-1/2 XXZ Heisenberg chain with competing interactions J. Phys. A: Math. Gen. 25 1417
[13] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas: I. The general solution and the ground state Phys. Rev. 130 1605
[14] Lieb E H 1963 Exact analysis of an interacting Bose gas: II. The excitation spectrum Phys. Rev. 130 1616
[15] Korepin V E, Bogoliubov N M and Izergin A G 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)
[16] Davies B and Korepin V E 1989 Higher conservation laws for the quantum non-linear Schrödinger equation Report No. CMA-R33-89, Centre for Mathematical Analysis Australian National University, Canberra (arXiv:1109.6604)
[17] Mosser J and Caux J-S 2012 Generalized TBA and generalized Gibbs J. Phys. A: Math. Theor. 45 255001
[18] Woynarovich F and Eckle H-P 1987 Finite-size corrections and numerical calculations for long spin-1/2 Heisenberg chains in the critical region J. Phys. A: Math. Gen. 20 L97
[19] Berkovich A and Murthy G 1988 Finite-size corrections in the non-linear Schrödinger model J. Phys. A: Math. Gen. 21 L395
[20] Korepin V E 1970 Direct calculation of the S matrix in the massive Thirring model Theor. Math. Phys. 41 953
[21] Bogoliubov N M, Izergin A G and Reshetikhin N Y 1987 Finite-size effects and infrared asymptotics of the correlation functions in two dimensions J. Phys. A: Math. Gen. 20 5361
[22] Woynarovich F, Eckle H-P and Trauernicht T T 1989 Non-analytic finite-size corrections in the one-dimensional Bose gas and Heisenberg chain J. Phys. A: Math. Gen. 22 4027
[23] Woynarovich F 1989 Finite-size effects in a non-half-filled Hubbard chain J. Phys. A: Math. Gen. 22 4243
[24] Essler F H L, Frahm H, Göhmann F, Klümper A and Korepin V E 2005 The One-Dimensional Hubbard Model (Cambridge: Cambridge University Press)
[25] Zvyagin A A 2005 Finite Size Effects in Correlated Electron Models: Exact Results (London: Imperial College Press)
[26] Cardy J L 1984 Conformal invariance and universality in finite-size scaling J. Phys. A: Math. Gen. 17 L385
[27] Blöte H W, Cardy J L and Nightingale M P 1986 Conformal invariance, the central charge, and universal finite-size amplitudes at criticality Phys. Rev. Lett. 56 742
[28] Pokrovskii S V and Tsvelik A M 1987 Conformal dimension spectrum for lattice integrable models of magnets
Zh. Eksp. Teor. Fiz. 93 2232
Pokrovskii S V and Tsvelik A M 1987 Conformal dimension spectrum for lattice integrable models of magnets
Sov. Phys.—JETP 66 1275 (Engl. transl.)

[29] Izergin A G, Korepin V E and Reshetikhin N Y 1989 Conformal dimensions in Bethe ansatz solvable models
J. Phys. A: Math. Gen. 22 2615

[30] Frahm H and Yu N-C 1990 Finite-size effects in the integrable XXZ Heisenberg model with arbitrary spin
J. Phys. A: Math. Gen. 23 2115

[31] Frahm H and Rödenbeck C 1997 Properties of the chiral spin liquid state in generalized spin ladders J. Phys. A: Math. Gen. 30 4467

[32] Zvyagin A A 2000 Commensurate-incommensurate phase transitions for multicell quantum spin models: exact results Low Temp. Phys. 26 134

[33] Zvyagin A A, Klümper A and Zittartz J 2001 Integrable correlated electron model with next-nearest-neighbour interactions Eur. Phys. J. B 19 25

[34] Zvyagin A A and Klümper A 2003 Quantum phase transitions and thermodynamics of quantum antiferromagnets with next-nearest-neighbor couplings Phys. Rev. B 68 144426

[35] Frahm H and Korepin V E 1990 Critical exponents for the one-dimensional Hubbard model Phys. Rev. B 42 10553