THE RECOVERY OF A PARABOLIC EQUATION FROM MEASUREMENTS AT A SINGLE POINT

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ABSTRACT. By measuring the temperature at an arbitrary single point located inside an unknown object or on its boundary, we show how we can uniquely reconstruct all the coefficients appearing in a general parabolic equation which models its cooling. We also reconstruct the shape of the object. The proof hinges on the fact that we can detect infinitely many eigenfunctions whose Wronskian does not vanish. This allows us to evaluate these coefficients by solving a simple linear algebraic system. The geometry of the domain and its boundary are found by reconstructing the first eigenfunction.

1. Introduction. We are concerned with the reconstruction of a linear parabolic equation defined by

\[
\begin{cases}
\frac{\partial}{\partial t} u(x,t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} u(x,t) \right) - q(x) u(x,t), & x \in \Omega \subset \mathbb{R}^n, \ t > 0, \\
u(x,t) = 0, & x \in \partial \Omega, \\
u(x,0) = \psi(x) \in L^2(\Omega),
\end{cases}
\]

(1)

from the measurement of the temperature, here given by \( u(x,t) \), taken at a single point. Standard methods dealing with uniqueness and stability, such as Calereman estimates are well known in the literature, [6, 15, 16], however these methods do not provide any practical reconstruction algorithm for the recovery of coefficients that can be used in engineering or control theory, such as minimizing the use of sensors and the data acquisition, [21]. Equation (1) describes the evolution of heat in an anisotropic media with thermal coefficients \( a_{ij}(x) \) and an internal heat sink coefficient \( q(x) \), and arises from the following problem: Consider a body \( \Omega \), whose composition and shape are both unknown, that is stored in a cool blackbox \( B \). The body can be given any initial temperature \( \psi(x) \) and be left there to cool down. Assume that we can measure its temperature at any single point \( x = b \), i.e. read \( u(b,t) \), from inside the box \( B \) for different initial temperature profiles \( u(x,0) = \psi(x) \). Our question is: What can be said about the composition of the body \( \Omega \), its geometry, the thermal coefficients \( a_{ij}(x) \), and the internal heat sink coefficient \( q(x) \)? This is a typical question faced by physicists working on Tokamak with plasma inside a closed chamber, [10]. The range of the coefficients \( a_{ij}(x) \)

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will unveil the nature of the medium. In applied control theory one would like to know, for example, how a combustion process is evolving, also how a medium is changing under a chemical process, and how the diffusion process is taking place on or close to the boundary. It is also important to observe and identify, if possible, these evolution processes in finite time, which would help predict the long term behavior of $u(x,t)$. Note that measuring a solution at all points inside or all over the boundary is not possible to realize in practice. Thus, in applications one would like to use the least possible, and certainly a finite number of sensors.

In [8], the inverse problem (1) was studied for the simple case when $a_{ij}(x) = \delta_{ij}$ and the domain $\Omega \subset \mathbb{R}^n$ is known. The idea was to reconstruct the first eigenfunction $\varphi_1$ by recovering its Fourier coefficients from the measurements of heat flux at a single boundary point $b \in \partial\Omega$ under different initial temperatures. Once $\varphi_1$ is reconstructed, the only unknown $q(x)$ is recovered from the facts that $-\Delta \varphi_1(x) + q(x) \varphi_1(x) = \lambda_1 \varphi_1(x)$ and $\varphi_1(x) \neq 0$ for all $x \in \Omega$. To extend this idea to equation (1) and reconstruct all the coefficients $a_{ij}$ and $q$ we need more than just the first eigenfunction, but at least $(n(n+1)/2 + 1)$ eigenfunctions to match the number of unknown coefficients. The first key idea in this paper is to prove the existence of infinitely many eigenfunctions that would not vanish at any given point $b \in \Omega$ and so can be observed. After reconstructing enough eigenfunctions, the next step is to show that we can still solve for the unknowns $a_{ij}(x)$ and $q(x)$ uniquely from systems of linear equations. This question reduces to showing that certain generalized Wronskians of eigenfunctions are nonsingular. To the best of our knowledge results and tools used in this work are new.

2. Auxiliary results. In all that follows we assume that $B$ is a given cube in $\mathbb{R}^n$, $n \geq 2$, and that $\Omega$, $a_{ij}$, and $q$ are unknown, but satisfy the following standard assumption:

**Assumption H.** for $0 < \alpha < 1$,  

a) $\Omega$ is an open connected bounded set in $\mathbb{R}^n$, $\partial\Omega \in C^{2+\alpha}$, and $\Omega \subset B$,

b) $a_{ij}(x) = a_{ji}(x) \in C^{1+\alpha}(\overline{\Omega})$ and $\exists \beta > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \beta \|\xi\|^2$

for any $\xi \in \mathbb{R}^n$,

c) $q(x) \in C^\alpha(\overline{\Omega})$.

Under assumption H, the operator $A$ defined by

\[
\begin{cases}
A\phi(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \phi(x) + q(x)\phi(x), & x \in \Omega, \\
\phi(x) = 0, & x \in \partial\Omega,
\end{cases}
\]

(2)

is then self-adjoint and uniformly elliptic, acts in $L^2(\Omega)$, and has a discrete spectrum with eigenvalues $\lambda_m \nearrow \infty$,

\[
\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_m \leq \cdots
\]

(3)

Moreover, only a finite number of eigenvalues can be negative. We normalize the eigenfunctions $\varphi_m \in C^{2+\alpha}(\overline{\Omega})$, $A\varphi_m = \lambda_m \varphi_m$, by $\|\varphi_m\| = 1$. Since $\Omega$ is assumed to be connected, the first eigenvalue $\lambda_1$ is simple, [13, Theorem 1.2.5], and by the maximum principle, [12], the first eigenfunction is known to have a constant sign, say $\varphi_1(x) > 0$, for all $x \in \Omega$. 

The solution of (1) exists for all \( t > 0 \), is smooth, and can be expanded in a Fourier series as
\[
u(x,t) = \sum_{m \geq 1} c_m(\psi) \varphi_m(x)e^{-\lambda_m t}, \quad x \in \Omega, \quad t > 0,
\] (4)
where the Fourier coefficients are given by
\[
c_m(\psi) = \int_{\Omega} \psi(x)\varphi_m(x) \, dx.
\] (5)

Observe that since \( \Omega \subset B \), initial conditions \( \psi \) in (5) can be seen as restrictions to \( \Omega \) of functions from \( L^2(B) \). Also since the solution \( u(x,t) \) and the eigenfunctions \( \varphi_m \) vanish on \( \partial\Omega \), they can be extended by zero outside \( \Omega \), i.e., \( u(x,t) = 0 \) for \( x \in B \setminus \Omega \). Now if \( u(x,0) = \psi(x)|_{\Omega} \) is the initial condition in (1) and \( b \in \Omega \), we can define a map \( \Gamma_b : L^2(B) \to \ell^\infty \) by
\[
\Gamma_b : \psi \mapsto \{u(b,k)\}_{k \geq 1}.
\] (6)

Note that if \( \Gamma_b(\psi) = 0 \) for all \( \psi \in L^2(B) \), that is we have only the trivial reading, then the point \( b \) must lie outside of \( \Omega \), i.e., \( b \notin \Omega \). In other words, \( \Gamma_b \) acts like a thermometer that detects heat coming from \( \Omega \). Thus to observe the solution we have to vary \( b \in \Omega \) until \( \Gamma_b \neq 0 \). Then the point \( b \in \Omega \) is fixed. Due to the decay of the exponentials, the series (4) converges in \( C^{2+\alpha}(\overline{\Omega}) \) for any fixed \( t > 0 \).

Assume now that \( \Gamma_b \neq 0 \), i.e., \( b \in \Omega \). Then we can measure or read from (4), the sequence of values
\[
u(b,k) = \sum_{m \geq 1} c_m(\psi) \varphi_m(b)e^{-\lambda_m k}, \quad k = 1,2,\ldots.
\] (7)

It is important to see that we can extract from the series (7), as we shall see in Section 4, only the nonzero Fourier coefficients \( c_m(\psi)\varphi_m(b) \neq 0 \). Otherwise they will not appear in the sum. Thus, for example as done in [8], we can always recover \( c_1(\psi)\varphi_1(b) \) since \( \varphi_1(b) \neq 0 \) for any \( b \in \Omega \). We now show that there are infinitely many \( \varphi_m(b) \neq 0 \).

**Proposition 1.** For any \( b \in \Omega \) there are infinitely many \( \varphi_m(b) \neq 0 \).

**Proof.** Otherwise if only a finite number of \( \varphi_m \), say \( N \), are nonzero at \( x = b \), then we can rearrange them, and change their sign if necessary, to get
\[
\varphi_m(b) > 0 \quad \text{for all } m \leq N \quad \text{and} \quad \varphi_m(b) = 0 \quad \text{for all } m > N.
\] (8)

It follows from their continuity that for any \( \epsilon > 0 \) there exists a ball \( B_{\delta}(b) \subset \Omega \) with center at \( b \) and radius \( \delta \) such that
\[
\varphi_m(x) > \epsilon > 0, \quad \forall x \in B_{\delta}(b), \quad m = 1,\ldots,N.
\] (9)

We now construct \( \psi \in C_0^\infty(\Omega) \) with \( \psi(b) = -1 \) such that its Fourier coefficients, see (5), satisfy
\[
c_m(\psi) > 0, \quad m = 1,\ldots,N.
\] (10)

To this end use the cut-off function
\[
\rho_{a,r}(x) := \begin{cases} \exp\left(1 + \frac{r^2}{||x-a||^2-r^2}\right) & \text{if } x \in B_r(a), \\ 0 & \text{otherwise}, \end{cases}
\] (11)

to define
\[
\psi(x) = -\rho_{b,\frac{1}{4}}(x) + M\rho_{b,\frac{1}{4}}, \quad \xi(x),
\] (12)
where $M$ is a positive constant to be determined later. Clearly, supp $\psi(x) \subset B_b(b)$ and $\psi(b) = -1$. It follows from (9) and (12) that for any sufficiently large $M > 0$

$$c_m(\psi) = Mc_m\left(\rho_{b+\frac{\pi}{4}}\right) - c_m\left(\rho_{b}\right) > 0,$$

$m = 1, \ldots, N$.  

(13) 

On the other hand, according to [1, 14] the Fourier expansion of $\psi \in C_0^\infty(\Omega)$

$$\psi(x) = \sum_{m \geq 1} c_m(\psi) \varphi_m(x)$$

holds not only in $L^2(\Omega)$, but also converges uniformly on any compact subset of $\Omega$. In particular, at $x = b \in \Omega$ the pointwise convergence yields

$$-1 = \psi(b) = \sum_{m = 1}^\infty c_m(\psi) \varphi_m(b) = \sum_{m = 1}^N c_m(\psi) \varphi_m(b) > 0,$$

which is impossible. Hence, there are infinitely many $\varphi_m(b) \neq 0$. 

Remark 1. The above proposition is best possible, as the presence of nodal sets prevents the argument to be extended to almost all eigenfunctions. A simple counter example is given by the Dirichlet Laplacian on a disk. Clearly its eigenfunctions, $\varphi_{ml}(r, \theta) = J_m(r \rho_m l) e^{i\theta}$ vanish at the origin, i.e. $\varphi_{ml}(0, \theta) = 0$ for all $m \geq 1$ and $l \in \mathbb{Z}$. Only the remaining sequence $\{\varphi_{0l}(0, \theta)\}_{l \in \mathbb{Z}}$ is nonzero as predicted. Thus if we choose $b = 0$, out of all the entries in the matrix $[\lambda_{ml}]_{m \geq 0, l \in \mathbb{Z}}$, we can observe only the first row $\{\lambda_{0l}\}_{l \in \mathbb{Z}}$. In other words, the values $\varphi_m(b)$ and the Fourier coefficients $c_m(\psi)$ filter which spectral data can be detected from the observation (7). 

3. Wronskians. Ignoring the eigenvalues and eigenfunctions that vanish at $b$, and so cannot be observed, we rename only those visible at hand again by $\{\lambda_m, \varphi_m(x)\}_{m=1}^\infty$. Assume for now that we have recovered infinitely many eigenfunctions $\varphi_m(x)$, for $x \in \Omega$. To find $q(x)$, we solve

$$-q(x)\varphi_m(x) + \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \sum_{i=1}^n a_{ij}(x) \right) \frac{\partial \varphi_m}{\partial x_j}(x) + \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \varphi_m}{\partial x_i \partial x_j}(x) = -\lambda_m \varphi_m(x).$$

(15) 

We now deal with generalized Wronskians of functions of several variables. Since

$$\frac{\partial^2 \varphi_m}{\partial x_i \partial x_j} = \frac{\partial^2 \varphi_m}{\partial x_j \partial x_i},$$

the system (15) can be further reduced by using the $N$-vectors $\Phi_m(x)$, with $N = \frac{1}{2} (n + 1) (n + 2)$,

$$\Phi_m(x) := \left( \varphi_m(x), \frac{\partial \varphi_m}{\partial x_1}(x), \ldots, \frac{\partial \varphi_m}{\partial x_n}(x), \frac{\partial^2 \varphi_m}{\partial x_1^2}(x), \ldots, \frac{\partial^2 \varphi_m}{\partial x_1 \partial x_n}(x), \ldots, \frac{\partial^2 \varphi_m}{\partial x_n^2}(x) \right),$$

(16) 

where $(\varphi_m)_{m=1}^\infty$ are the eigenfunctions of the operator $A$. It is proved in [7] that if $\varphi_1, \varphi_2, \ldots, \varphi_N$ are analytic functions, then one of the generalized Wronskians is not identically zero. This is clearly insufficient to prove the uniqueness of the solution of (15). We now show that for a dense subset of $\Omega$, we can always find a nonzero generalized Wronskian, whose rows are (16). The main idea here is to use
the constant rank theorem to solve locally (15). To this end we proceed through the following propositions, where $\mathbf{M}_m$ is a matrix function

$$
\mathbf{M}_m(x) := \begin{pmatrix}
c_{11}(x) & c_{12}(x) & \cdots & c_{1N}(x) \\
c_{21}(x) & c_{22}(x) & \cdots & c_{2N}(x) \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1}(x) & c_{m2}(x) & \cdots & c_{mN}(x)
\end{pmatrix}, \quad m \geq N,
$$

(17)

and whose entries $c_{ij}(x)$ are bounded and measurable functions defined on a measurable nonzero set $D$.

**Proposition 2.** Assume that all the entries of $\mathbf{M}_m(x)$ in (17) are bounded and measurable functions on $D$ and that

$$
\text{rank} \mathbf{M}_m(x) \leq N - 1, \quad \forall x \in D.
$$

Then, there exists a nonzero, bounded, and measurable vector function $\mathbf{y}(x) := (y_1(x), y_2(x), \ldots, y_N(x))^T$ such that

$$
\mathbf{M}_m(x)\mathbf{y}(x) = 0, \quad \forall x \in D, \quad \forall m \geq N.
$$

(18)

**Proof.** Consider the system of equations

$$
\mathbf{M}_m(x)\mathbf{y}_m(x) = 0, \forall x \in D, m \geq N, \mathbf{y}_m(x) := (y_{m1}(x), y_{m2}(x), \ldots, y_{mN}(x))^T.
$$

(19)

Let

$$
\mathbf{N}(x) := \mathbf{N}_m(x) := \mathbf{M}_m^T(x)\mathbf{M}_m(x).
$$

Then $\mathbf{N}(x)$ is an $N \times N$ matrix function, $\text{rank} \mathbf{N}(x) = \text{rank} \mathbf{M}_m(x) \leq N - 1, \forall x \in D$, and system of equations (19) is equivalent to the system

$$
\mathbf{N}(x)\mathbf{y}_m(x) = 0, \quad \forall x \in D.
$$

(20)

Since all entries of $\mathbf{M}_m(x)$ are bounded and measurable functions on $D$, so are all entries of $\mathbf{N}(x)$.

Define

$$
P_x(\lambda) := \alpha_N(x)\lambda^N + \alpha_{N-1}(x)\lambda^{N-1} + \cdots + \alpha_1(x)\lambda + \alpha_0(x) := \text{det}(\lambda I - \mathbf{N}(x)), \quad x \in D.
$$

(21)

It is clear that the coefficients $\alpha_i(x)$, $i = 0, \ldots, N$, are bounded and measurable functions on $D$. Since $\text{rank} \mathbf{N}(x) \leq N - 1, \forall x \in D$, we have $\text{det} \mathbf{N}(x) = 0, \forall x \in D$. Thus, $\lambda = 0$ is a root of $P_x(\lambda), \forall x \in D$. This implies that $\alpha_0(x) = 0, \forall x \in D$. Therefore,

$$
P_x(\lambda) = \lambda Q_x(\lambda), \quad Q_x(\lambda) := \alpha_N(x)\lambda^{N-1} + \alpha_{N-1}(x)\lambda^{N-2} + \cdots + \alpha_1(x).
$$

(22)

It follows from (21) that $P_x(\lambda)$ is the characteristic polynomial of $\mathbf{N}(x)$, i.e., $P_x(\mathbf{N}(x)) = 0, \forall x \in D$. Thus, we have

$$
0 = P_x(\mathbf{N}(x)) = \mathbf{N}(x)Q_x(\mathbf{N}(x)), \quad \forall x \in D.
$$

(23)

Let us consider the following cases:

**Case 1.** $Q_x(\mathbf{N}(x)) \not\equiv 0$ on $D$. Then there exists a nonzero vector function $\mathbf{z}(x) = (z_1(x), z_2(x), \ldots, z_N(x))^T$ with bounded and measurable entries such that $Q_x(\mathbf{N}(x))\mathbf{z}(x) \not\equiv 0, x \in D$. Then we have

$$
0 = P_x(\mathbf{N}(x))\mathbf{z}(x) = \mathbf{N}(x)Q_x(\mathbf{N}(x))\mathbf{z}(x), \quad \forall x \in D.
$$

(24)
This implies that $y(x) := Q_x(N(x))z(x)$ is a nonzero, bounded, and measurable solution to equation (20).

**Case 2.** $Q_x(N(x)) \equiv 0$ on $D$. This implies that $Q_x(\lambda)$ is a multiple of the minimal polynomial of $N(x)$ for any fixed $x \in D$. Note that eigenvalues of $N(x)$ are zeros of the minimal polynomial of $N(x)$. Since $\lambda = 0$ is an eigenvalue of $N(x)$, $\forall x \in D$, $\lambda = 0$ is also a zero of the minimal polynomial of $N(x)$ for all $x$ in $D$. Therefore, $\lambda = 0$ is also a zero of $Q_x(\lambda)$ for all $x \in D$. Thus, $\alpha_1(x) = 0$, $\forall x \in D$, and we have

$$Q_x(\lambda) = \lambda R_x(\lambda), \quad R_x(\lambda) := \alpha_N(x)\lambda^{N-2} + \alpha_{N-1}(x)\lambda^{N-3} + \cdots + \alpha_2(x).$$

It is clear that all coefficients of $R_x(\lambda)$ are bounded and measurable functions on $D$. We have

$$0 = Q_x(N(x)) = N(x)R_x(N(x)).$$

Now, there are two cases which are $R_x(N(x)) \neq 0$ on $D$ and $R_x(N(x)) \equiv 0$ on $D$. Using similar arguments as those above with $Q_x(\lambda)$ one obtains after a finite number of steps a nonzero, bounded, and measurable vector function $y_m(x) = (y_{m1}(x), y_{m2}(x), \cdots, y_{mN}(x))^T$ which solves system (20). By scaling if necessary, we can assume that

$$\left\| \sqrt{y_{m1}^2(x) + y_{m2}^2(x) + \cdots + y_{mN}^2(x)} \right\|_{L^\infty(D)} = 1. \quad (25)$$

Since systems (19) and (20) are equivalent, this $y_m(x)$ also solves system (19).

Let $S_m$ denote the set of all measurable nontrivial solutions to (19) which satisfy equation (25). It follows from the arguments above that $y_m(x) \in S_m$, i.e., $S_m$ is not empty. Moreover, the set $S_m$ is closed for all $m \geq N$. Since $M_{m+1}(x)$ is obtained by adding one more row to $M_m(x)$, $m \geq n$, one has $S_N \supset S_{N+1} \supset S_{N+2} \cdots$. According to Cantor’s Intersection Theorem, the intersection $\cap_{m=N}^{\infty} S_m$ is not empty, i.e., there exists a vector function $y(x) = (y_1(x), y_2(x), \cdots, y_N(x))^T$ which belongs to all $S_m$, $m \geq N$. Thus, the vector function $y(x)$ solves system (18) for all $m \geq N$. It is clear that $y_m(x)$ satisfies (25), and, therefore, is a nonzero vector function. Proposition 2 is proved.

**Remark 2.** It follows from the proof of Proposition 2 that the vector function $y(x)$ is as smooth as the entries of $M_m(x)$. If all the entries of $M_m(x)$ are piecewise continuous functions on $D$, then all the entries of $y(x)$ are also piecewise continuous functions.

To proceed further let us denote the nodal set of $\varphi_k$ by

$$\gamma_k := \{ x \in \Omega : \varphi_k(x) = 0 \}. \quad (26)$$

**Proposition 3.** Given a set of indices $k_1 < k_2 < \cdots < k_N$, such that $\lambda_{k_j}$, $j = 1, \cdots, N$, are distinct and nonzero. Then the set

$$E := \{ x \in \Omega : \det [\Phi_{k_1}^T(x), \Phi_{k_2}^T(x), \cdots, \Phi_{k_N}^T(x)] \neq 0 \} \quad (27)$$

is dense in $\Omega$.

**Proof.** Consider the $N \times N$ system of linear equations generated by (15), where $m = k_1, k_2, \ldots, k_N$. Assume the contrary, that the system is not full rank in a neighborhood of $x_0 \in \Omega$, say $B(x_0)$, it means that the rows $\{\Phi_{k_j}(x)\}_{j=1}^N$ are linearly
dependent in $B_ε(x_0)$, i.e. there exist constants $\{α_j\}_{1 ≤ j ≤ N}$, not all zero, such that
$$\sum_{j=1}^{N} α_j Φ_k (x) = 0, \quad x ∈ B_ε(x_0).$$

Looking at the augmented system we also have
$$\sum_{j=1}^{N} α_j λ_k φ_k (x) = 0$$
in $B_ε(x_0)$. Applying the operator $A$, defined by (2), successively $N − 1$ times to (28) in $B_ε(x_0)$, we end up with a Vandermonde homogeneous system
$$\sum_{j=1}^{N} λ_k^r α_j φ_k (x) = 0 \quad \text{for } r = 1, 2, \cdots, N, \quad x ∈ B_ε(x_0),$$
which implies that
$$α_j φ_k (x) = 0 \quad \text{for } x ∈ B_ε(x_0), \quad j = 1, \cdots, N.$$
Since $α_j$ are not all zero, it means that $x ∈ γ_k$ for some $j ∈ \{1, 2, \cdots, N\}$. Therefore,
$$B_ε(x_0) ⊂ \bigcup_{j=1}^{N} γ_k.$$
Since nodal sets have measure zero, their finite union also has measure zero and so we deduce that the set $B_ε(x_0)$ has measure zero, which is a contradiction. Thus, $E$ is dense in $Ω$.

4. Extracting the spectral data. It follows from Proposition 1 that there are infinitely many $φ_m (b) \neq 0$ present in the Fourier series of $u(b, k)$ in (7). Thus the series (7) does not terminate. As the eigenvalues may be multiple, we regroup the terms in series (7) as follows: First, we drop all eigenvalues $λ_m$ from the sequence $\{λ_m\}_{m=1}^{∞}$, if $φ_m (b) = 0$. Next, we rename the remained sequence in the strictly increasing order, not counting their multiplicity
$$\hat{λ}_1 < \hat{λ}_2 < \cdots < \hat{λ}_m < \cdots.$$
Since there are infinitely many $φ_m (b) \neq 0$, and all eigenvalues have finite multiplicity, the newly ordered sequence of eigenvalues $\hat{λ}_1, \hat{λ}_2, \cdots$, is also infinite. Denote by $I_m$ the set of indices $j$ such that $φ_j (b) \neq 0$, and $λ_m = λ_j$. In other words, $I_m$ contains all indices of eigenfunctions associated with the newly ordered eigenvalue $\hat{λ}_m$ and appearing in the series (7). Since $λ_m$ has a finite multiplicity, and $\hat{λ}_1$ is a simple eigenvalue, $I_m$ is finite for each $m$, and $|I_1| = 1$.

Denote by
$$\tilde{φ}_m (x) = \sum_{j ∈ I_m} φ_j(x) φ_j (b)$$
a non normalized eigenfunction associated with $\hat{λ}_m$ and its Fourier coefficient
$$c_m (ψ) = \int_{Ω} ψ(x) φ_m (x) dx = \sum_{j ∈ I_m} c_j (ψ) φ_j (b).$$
Clearly, from (30)
$$\tilde{φ}_m (b) = \sum_{j ∈ I_m} φ_j^2 (b) > 0.$$
From (7) we arrive at the following new series representation for the observation $u(b, k)$

$$u(b, k) = \sum_{m=1}^{\infty} \tilde{c}_m(\psi) e^{-\tilde{\lambda}_m k}, \quad k = 1, 2, \ldots.$$  \(31\)

To simplify notations we drop the tildes in (31), so we come back to the representation similar to (7),

$$u(b, k) = \sum_{m=1}^{\infty} c_m(\psi) e^{-\lambda_m k}, \quad k = 1, 2, \ldots,$$  \(32\)

but with some distinct properties due to the observable eigenfunctions at $x = b$

1. $\lambda_1 < \lambda_2 < \cdots$ are now distinct eigenvalues of $A$.
2. $\varphi_m(x)$ is a (non normalized) eigenfunction of $A$ associated with $\lambda_m$, $\varphi_1(x) > 0$ on $\Omega$, and $\varphi_m(b) > 0$ for any $m > 1$.
3. For any $m \geq 1$ there is $\psi \in L^2(B)$ such that $c_m(\psi) \neq 0$.

**Remark 3.** Recall that, by Weyl asymptotics, $\lambda_m \asymp m^\frac{2}{n}$ and so $\sum 1/\lambda_m = \infty$, when $n \geq 2$. Then by Muntz’s theorem, the system $\{e^{-\lambda_m t}\}_{m \geq 1}$ is complete in $L^2(0, T)$, but not minimal, [5, 11]. Therefore we cannot use the finite time spectral estimation methods in [3, 4].

To overcome this difficulty we can use classical methods, such as the method of limits or the Laplace transform. Now we show how to extract the corresponding eigenvalues $\lambda_m$ and factors $c_m(\psi)$ of the exponentials $e^{-\lambda_m k}$ in (32) from $u(b, k)$ under conditions a1-a3. Note that we do not assume minimality of the family $\{e^{-\lambda_m t}\}_{m \geq 1}$.

**Proposition 4** (method of limits). Given the sequence $\{u(b, k)\}_{k \geq 1}$, and under conditions a1-a3, we can extract all the pairs $(c_m(\psi), \lambda_m)$ for $m \geq 1$.

**Proof.** To begin with, the first eigenvalue $\lambda_1$ is found from the relation

$$\lambda_1 = -\sup_{\psi} \lim_{k \to \infty} \ln \left( \frac{u(b, k+1)}{u(b, k)} \right),$$  \(33\)

which follows from the asymptotic formula $u(b, k) = c_1(\psi) e^{-\lambda_1 k} + o(e^{-\lambda_1 k})$ as $k \to \infty$, and from the fact that $c_1(\psi) \neq 0$ for some $\psi \in L^2(B)$. Next, use $\lambda_1$ found in (33) to evaluate

$$c_1(\psi) = \lim_{k \to \infty} u(b, k) e^{\lambda_1 k}.$$  \(34\)

Removing the obtained first term, $c_1(\psi) e^{-\lambda_1 t}$ (that can be trivial if $c_1(\psi) = 0$), and the subsequent ones, we can look at the new series

$$u_m(b, k) = \sum_{l=m}^{\infty} c_l(\psi) e^{-\lambda_l k}.$$  \(35\)

Repeating (33) and (34) we obtain $\lambda_m$ and $c_m(\psi)$. Thus, for any $\psi \in L^2(B)$ the following map is known

$$F_\psi : \psi \rightarrow \{\lambda_m, c_m(\psi)\}_{m=1}^{\infty}.$$  \(36\)

In case we are given a continuous reading, $u(b, t)$ for $t > 0$, then we can use the Laplace transform.
Proposition 5 (Laplace transform). Given the function \( r(t) = \sum_{m=1}^{\infty} c_m e^{-\lambda_m t} \) for \( t > 0 \), where \( c_m \neq 0 \), \( \sum_{m \geq 1} |c_m| < \infty \), and let condition a1 hold. Then we can extract all \((c_m, \lambda_m)\) by using the Laplace transform.

Proof. Since \( r(t) = O\left( e^{-\lambda t} \right) \) and \( \sum_{m \geq 1} |c_m| < \infty \) it follows that its Laplace transform \( \mathcal{L}(r)(s) = \sum_{m=1}^{\infty} \frac{c_m}{s + \lambda_m} \) is a meromorphic function. The poles \( -\lambda_m \), are simply the zeros of \( 1/\mathcal{L}(r)(s) \) and \( c_m \) are their residues.

The main advantage of the Laplace transform method is that the computation of the \( \lambda_m \) is done independently from each other, while in the method of limits one has to compute the \( \lambda_m \) sequentially, i.e. \( \lambda_1 \) first, update the new series and then \( \lambda_2, \ldots \) and so on.

5. Constructing \( \varphi_m \). Recall, by assumption Ha, that although \( \Omega \) is unknown, it is contained in a larger given cube \( B \). Thus, we search for \( b \in B \) until \( \Gamma \Omega \neq \emptyset \), i.e., \( b \neq \Omega \). For now let \( \{\psi_i\}_{i=1}^{\infty} \) be an arbitrary orthonormal basis of \( L^2(B) \) whose elements are used to generate different initial conditions for (1), i.e., \( \psi(x) = \psi_j(x) \), \( i \geq 1 \), when \( x \in \Omega \). By the previous section, we can extract from (32) the Fourier coefficients \( c_m(\psi_i) \) for all \( m \geq 1 \) as long as \( c_m(\psi_i) \neq 0 \). Since \( \varphi_m \neq 0 \), at least one of its Fourier coefficient, say the \( i \)-th coefficient \( c_m(\psi_i) \), is not zero. Thus, \( \lambda_m \) and \( c_m(\psi_i) \) can be extracted. If \( \psi(x) = \psi_j(x) \) is an initial condition such that the term \( e^{-\lambda_m k} \) in (32) is missing, then it means that \( c_m(\psi_j) = 0 \). Hence, by property a2, for any given \( i \) we can always read \( c_m(\psi_i) \) (\( \neq 0 \) or \( = 0 \)).

Note that \( c_m(\psi_i) \) is not only the \( m \)-th Fourier coefficient of \( \psi_i \) in the basis \( \{\varphi_j\}_{j=1}^{\infty} \) but also the \( i \)-th Fourier coefficient of \( \varphi_m \) in the basis \( \{\psi_j\}_{j=1}^{\infty} \). Thus, from the readings (32) we extract all the Fourier coefficients \( c_m(\psi_i) \) of \( \varphi_m \) in any given orthogonal basis \( \{\psi_i\}_{i=1}^{\infty} \). Therefore, we can reconstruct the eigenfunction \( \varphi_m \) from the equation

\[
\varphi_m(x) = \sum_{i \geq 1} c_m(\psi_i) \psi_i(x), \quad \forall m \geq 1. \tag{35}
\]

Although \( \varphi_m \in C^{2+\alpha}(\overline{B}) \) due to being an eigenfunction of (2), the convergence of (35) holds only in the \( L^2(B) \) sense. Obviously, the type of convergence of series (35) depends on the rate of decay of \( c_m(\psi_i) \) and the smoothness of the functions \( \{\psi_i\}_{i=1}^{\infty} \). This leads to questions on summability and localization of the Fourier series in (35). A simple choice for a “good” basis \( \{\psi_i\}_{i=1}^{\infty} \) is provided by the eigenfunctions of the Dirichlet Laplacian. As eigenfunctions of an operator close to operator \( A \) in (2), the new basis can provide us with uniform convergence wherever \( \varphi_m \) is continuous. We now study in detail the various types of convergence of (35).

5.1. Uniform convergence in \( B \). For simplicity, let us assume that \( \Omega \subset B \) where

\[
B = \{ x = (x_k)_{k=1}^{n} \in \mathbb{R}^n : -\pi < x_k < \pi, \quad k = 1, \ldots, n \}.
\]

Let \( \psi_l(x) = e^{i l \cdot x} \), where \( l \in \mathbb{Z}^n \) and \( x \in B \). For \( \alpha = (\alpha_k)_{k=1}^{n} \in \mathbb{N}^n \), define the rectangular partial sum of \( \varphi_m \) by

\[
S_\alpha(\varphi_m)(x) = \sum_{|l|_1 \leq \alpha_1} \cdots \sum_{|l|_n \leq \alpha_n} c_m \left( e^{-i l \cdot x} \right) e^{i l \cdot x}, \quad l = (l_k)_{k=1}^{n}, \quad \tag{36}
\]

where

\[
c_m \left( e^{-i l \cdot x} \right) = \frac{1}{(2\pi)^n} \int_B e^{-i l \cdot x} \varphi_m(x) dx.
\]

By \( \alpha \to \infty \) we mean \( \min_j \alpha_j \to \infty \).
Proposition 6. Under Assumption \( H \), the rectangular partial sum \( S_\alpha (\varphi_m) (x) \) converges to \( \varphi_m \) uniformly in \( x \) in \( B \).

**Proof.** The result follows from the well known Zhizhiashvili’s result, [1, section 2] and [20]. We only need to show that \( \varphi_m \) is Lipschitz continuous in \( B \). Indeed since \( \varphi_m \in C^{2+\alpha} (\overline{\Omega}) \) and \( \varphi_m = 0 \) on \( \partial \Omega \), its extension by \( 0 \) to \( B \setminus \Omega \) defines a continuous function in \( B \). Obviously, it is Lipschitz continuous on \( \overline{\Omega} \) and \( B \setminus \overline{\Omega} \). It is easy to see that it remains Lipschitz across the boundary. To this end let \( x \in \overline{\Omega} \) and \( y \in B \setminus \overline{\Omega} \), then the line segment joining the points \( x \) and \( y \) will intersect \( \partial \Omega \) at least one point, say \( p \in \partial \Omega \), and so \( \| x - p \| \leq \| x - y \| \).

We have

\[
|\varphi_m(x) - \varphi_m(y)| = |\varphi_m(x) - \varphi_m(p)| \leq k_m \| x - p \| \leq k_m \| x - y \| ,
\]

since \( \varphi_m(y) = \varphi_m(p) = 0 \). This implies that the extension of \( \varphi_m \) remains Lipschitz continuous with the same constant and so its modulus of continuity

\[
\omega(\delta, \varphi_m) = \sup_{\| x - y \| \leq \delta} |\varphi_m(x) - \varphi_m(y)| = O(\delta). \tag{37}
\]

Since \( \delta \log^n \left( \frac{1}{\delta} \right) \to 0 \) as \( \delta \to 0 \), it follows from Zhizhiashvili’s theorem that the trigonometric rectangular partial sums (36) converge uniformly in \( x \) to \( \varphi_m \) in the whole cube \( B \).

Now let us consider the case when neither \( \{ \varphi_m \}_{m=1}^\infty \) are Lipschitz continuous in \( B \) nor the estimate of the modulus of continuity \( \omega(\delta, \varphi_m) \) in (37) is valid. If this is the case, then Zhizhiashvili’s theorem is not applicable. This maybe the case when instead of Assumption \( H \) we have the following weaker Assumption \( H' \) on the smoothness of the boundary and the coefficients.

**Assumption \( H' \).** For \( 0 < \alpha < 1 \),

\( a' \) \( \Omega \) is an open connected bounded set in \( \mathbb{R}^n \), \( \Omega \subset B \), and \( \Omega \) satisfies an exterior sphere condition at every boundary point, that is, for every \( \xi \in \partial \Omega \) there exists a ball \( B_\epsilon(x) \) satisfying \( B_\epsilon(x) \cap \overline{\Omega} = \{ \xi \} \).

\( b' \) \( a_{ij}(x) = a_{ij}(x) \in C^{1+\alpha}(\Omega) \) and \( \exists \beta > 0 \) such that \( \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \beta \| \xi \|^2 \), for any \( \xi \in \mathbb{R}^n \),

\( c' \) \( q(x) \in C^{\alpha}(\Omega) \).

If Assumption \( H' \) holds, then \( \varphi_m \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega) \), [12, Theorem 6.13]. Although its extension by zero outside \( \Omega \) is still continuous on \( B \), no further smoothness can be expected. In this case we choose \( \{ \psi_i \}_{i=1}^\infty \) as normalized eigenfunctions of the Dirichlet Laplacian on a ball \( B \) (to avoid corners of a cube), i.e.,

\[
\begin{cases}
    \Delta \psi_i = -\mu_i \psi_i & \text{in } B, \\
    \psi_i = 0 & \text{on } \partial B. \tag{38}
\end{cases}
\]

Since no uniform convergence can be expected in (35), we shall use the partial sums of the Riesz means

\[
E_\mu f(x) = \sum_{\mu_i < \mu} \left( 1 - \frac{\mu_i}{\mu} \right)^s (f, \psi_i) \psi_i(x), \quad x \in B. \tag{39}
\]

It is proved in [1, 14] that if \( f \in C_0^s(B) \), \( 0 \leq s \leq \frac{n-1}{2} \), and \( s + \alpha \geq \frac{n-1}{2} \), then the Riesz means converge uniformly to \( f \) on any compact subset of \( B \). Applying this
result to \( f(x) = \varphi_m(x) \in C^0(B) \) with \( s = \frac{n-1}{2} \), we get
\[
\varphi_m(x) = \lim_{\mu \to \infty} \sum_{\mu_i < \mu} \left( 1 - \frac{\mu_i}{\mu} \right)^{\frac{s}{2}} \psi_i(x), \quad m \geq 1, \tag{40}
\]
where the convergence is uniform on any compact subset of \( B \). Using (40) we recover \( \varphi_m \) pointwise, and from \( \varphi_1(x) \) we get \( \bar{\Omega} = \text{supp} \varphi_1(x) \). Thus, we have the following result.

**Proposition 7.** Let \( s = \frac{n-1}{2} \) and Assumption \( H' \) hold. Then, the Riesz partial sums \( \tilde{E}_n^\mu \varphi_m(x) \) converge to \( \varphi_m \) uniformly in \( x \) on any compact subset of \( B \).

There are three crucial consequences of Propositions 6 and 7.

- \( \varphi_1 \) is obtained as a continuous function. Clearly, \( \bar{\Omega} = \text{supp} \varphi_1(x) \), since \( \varphi_1(x) > 0 \) for \( x \in \Omega \), and \( \varphi_1(x) = 0 \) in \( B \setminus \Omega \). Thus, the domain \( \Omega \) is uniquely determined from our observations.
- We can evaluate \( \varphi_m(x) \) at any point inside \( B \), and then compute directional derivatives \( \frac{\partial \varphi_m}{\partial \nu}(x) \), \( \nu \in \mathbb{R}^n \), \( ||\nu|| = 1 \), by simply evaluating the limits
  \[
  \frac{\partial \varphi_m}{\partial \nu}(x) = \lim_{h \to 0} \frac{1}{h} [\varphi_m(x + h\nu) - \varphi_m(x)], \quad \forall x \in \Omega.
  \]
These limits exist since \( \varphi_m \in C^{2+\alpha}(\bar{\Omega}) \) under Assumption \( H \) or \( \varphi_m \in C(\bar{\Omega}) \cap C^{2+\alpha}(\Omega) \) under Assumption \( H' \). This helps us to avoid differentiating the Fourier series in (35) or (40).
- Once \( \Omega \) is determined, we can restrict our analysis to \( L^2(\Omega) \) instead of \( L^2(B) \).

We can use eigenfunctions defined by (38) but with \( B \) replaced by \( \Omega \) to find the remaining \( \varphi_m \) with \( m \geq 2 \). We shall show below, under certain smoothness conditions, that the series (35) converges in \( C^2(\bar{\Omega}) \) and, therefore, is twice differentiable.

**Remark 4.** When both series (35) and (40) converge, the Riesz means (40) usually converge at a faster rate.

### 5.2. Convergence in \( C^2(\bar{\Omega}) \)

Having determined \( \varphi_1 \) and the domain \( \Omega \), we can use \( \Omega \) to generate a new basis \( \{\psi_i\}_{i=1}^\infty \) in \( L^2(\Omega) \). Let us consider (35) with eigenfunctions \( \{\psi_i\}_{i=1}^\infty \) defined by
\[
\begin{align*}
\Delta \psi_i &= -\mu_i \psi_i \quad \text{in} \quad \Omega, \\
\psi_i &= 0 \quad \text{on} \quad \partial \Omega. 
\end{align*} \tag{41}
\]
Observe that if \( \varphi_m \in \text{Dom}(\Delta^p) \), then \( \mu_i \psi_i \in \ell^2 \) and this implies a faster decay of the \( c_m(\psi_i) \) in (35).

**Remark 5.** It follows from Sobolev embeddings, [17, Theorem 9], that
\[
\text{Dom}(\Delta^p) \subset H^{2p}(\Omega) \hookrightarrow C^j(\bar{\Omega}), \quad \text{where} \quad 0 \leq j < 2p - \frac{n}{2},
\]
and so convergence in \( H^{2p}(\Omega) \) implies convergence in \( C^j(\bar{\Omega}) \).

For simplicity, we prove the following proposition for \( \Omega \subset \mathbb{R}^n \) with \( n = 2,3 \). Higher dimensions require more smoothness conditions in Assumption \( H \).

**Proposition 8.** Let Assumption \( H \) hold with \( \alpha = 2 \) and \( n = 2 \) or \( n = 3 \). Assume that on \( \partial \Omega \) we have \( a_{ij}(x) = \delta_{ij} \) and \( \sum_{i=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) = 0 \). Then the Fourier series in (35) with \( \{\psi_i\}_{i=1}^\infty \) defined by (41) converges to \( \varphi_m \) in \( C^2(\bar{\Omega}) \).
Proof. In Remark 5 if \( p = 2 \) and \( n \leq 3 \), then one can take \( j = 2 \). Thus we only need to show that \( \varphi_m \in \text{Dom} (\Delta^2) \) and so \( \varphi_m(x) = \Delta \varphi_m(x) = 0 \) when \( x \in \partial \Omega \). Since \( \alpha = 2 \), we have \( \varphi_m \in C^4 (\Omega) \subset H^4 (\Omega) \). By our assumptions we already have \( \varphi_m(x) = 0, \forall x \in \partial \Omega \) and \( a_{ij}(x) = \delta_{ij}, \forall x \in \partial \Omega \). This and equation (15) imply \( \Delta \varphi_m(x) = 0 \) for all \( x \in \partial \Omega \). Therefore, \( \varphi_m \in \text{Dom} (\Delta^2) \) which implies that (35) holds in \( C^2 (\Omega) \).

6. Main result. From the previous sections, we have now identified \( \Omega \), reconstructed the eigenfunctions \( \varphi_k \in C^{2+\alpha} (\Omega) \), evaluated their \( \frac{\partial}{\partial x_j} \varphi_k(x) \) and \( \frac{\partial^2}{\partial x_i \partial x_j} \varphi_k(x) \) in \( \Omega \). We are now ready to state and prove the main result.

**Theorem 6.1.** Let Assumption \( \text{H} \) hold. Then we can reconstruct uniquely all the coefficients \( a_{ij}, q \), and the domain \( \Omega \), of (1) from measurements of the solution at any single point \( b \in \Omega \) and at evenly spaced times, i.e., from \( \{u(b,k)\}_{k \in \mathbb{N}} \).

**Proof.** To recover the coefficients \( a_{ij} \) and \( q \) we consider the following system of equations in \( \Omega \)

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \hat{a}_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \varphi_k(x) + \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \right) \frac{\partial}{\partial x_j} \varphi_k(x) - q(x) \varphi_k(x) = -\lambda_k \varphi_k(x),
\]

(42)

where

\[
\hat{a}_{ij} := 2a_{ij} \quad \text{for} \quad j < i, \quad \hat{a}_{ii} := a_{ii}.
\]

It follows from Proposition 3 that for all \( x \in \Omega \) except for those in a subset of measure zero of \( \Omega \), there exists a set of indices \( 1 \leq k_1 < k_2 < \cdots < k_N \), depending on \( x \), such that the determinant of the matrix

\[
\begin{pmatrix}
\frac{\partial}{\partial x_1} \varphi_{k_1} & \cdots & \frac{\partial}{\partial x_n} \varphi_{k_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_1} \varphi_{k_N} & \cdots & \frac{\partial}{\partial x_n} \varphi_{k_N}
\end{pmatrix}
\]

is nonzero at \( x \). Thus, one can compute \( \{a_{ij}(x)\}_{i,j=1}^{n} \) and \( q(x) \) uniquely from the linear system (42) almost everywhere in \( \Omega \).

**Remark 6.** Observations at equally spaced points \( t = k \) can be replaced by more general and arbitrary sequence of times, \( t_k \uparrow \infty \). In that case, \( \lambda_1 \) and \( c_1(\psi) \varphi_1(b) \) can be recovered by the formulas

\[
\lambda_1 = - \sup_{\psi \in L^2(B)} \lim_{k \to \infty} \ln \left( \frac{u(b, t_{k+1})}{u(b, t_k)} \right) / (t_{k+1} - t_k),
\]

\[
c_1(\psi) = \lim_{k \to \infty} u(b, t_k) e^{\lambda_1 t_k}.
\]

Similarly, one can recover all \( \lambda_m \) and \( c_m(\psi) \) from \( \{u(b, t_k)\}_{k \geq 1} \).

**Minimality by filtering**

Since the family \( \{e^{-\lambda_m t}\} \) is not minimal in higher dimension, see Remark 6, we need to collect data over \( t_k \uparrow \infty \) or over an infinite time, i.e. \( t > 0 \) for the Laplace transform. However in control theory and identification problems, finite and real time recovery are crucial. We now explain, why although the full system \( \{e^{-\lambda_m t}\} \)
may not be minimal, the observed one in sum (4) might be. Denote the observed set of indices by
\[ O_b (\psi) = \{ m \in \mathbb{N} : c_m (\psi) \varphi_m (b) \neq 0 \}. \]
Note that \( O_b (\psi) \) can be finite, and we have a double filtering, namely by both \( \varphi_m (b) \) and \( c_m (\psi) \) that can vanish. Most importantly, if
\[ \sum_{m \in O_b (\psi)} \frac{1}{\lambda_m} < \infty, \]
then the observed set of exponentials \( \{ e^{-\lambda_m t} \}_{m \in O_b (\psi)} \) is minimal in \( L^2 (0, T) \), in the sense that any \( e^{-\lambda_m t} \) for \( m \in O_b (\psi) \), does not belong to the closure of linear spans of the remaining exponentials. Then obviously we can use the methods in [3, 4] and identify the parabolic equation in a finite time.

We now illustrate the above idea by a simple example. Let \( \Omega = [0, \pi] \times [0, \pi] \), then \( \varphi_{ij} (x, y) = \frac{2}{\pi} \sin (ix) \sin (jy) \) are normalized Dirichlet eigenfunctions of \( -\Delta \). The eigenvalues are \( \lambda_{ij} = i^2 + j^2 \) for \( i, j \geq 1 \). It is readily seen that
\[ \sum_{i,j \geq 1} \frac{1}{\lambda_{ij}} = \sum_{i,j \geq 1} \frac{1}{i^2 + j^2} = \infty \]
as expected, and means that the complete family \( \{ e^{-(i^2+j^2)t} \}_{i,j \geq 1} \) is not minimal in \( L^2 (0, T) \). However if we take \( \psi_l (x, y) = f(x) \sin (ly) \) say, then
\[ c_{ij} (\psi_l) = \frac{2}{\pi} \int_0^\pi \sin (ix) f(x) dx \int_0^\pi \sin (jy) \sin (ly) dy = \delta_{jl} \int_0^\pi \sin (ix) f(x) dx, \]
and so \( O_b (\psi_l) \subset \{(i, l) : i \in \mathbb{N} \} \) and since \( \lambda_{il} = i^2 + l^2 \), we have
\[ \sum_{i,j \in O_b (\psi_l)} \frac{1}{\lambda_{ij}} \leq \sum_{i \geq 1} \frac{1}{i^2 + l^2} < \infty, \]
which means the set \( \{ e^{-\lambda_{ij} t} \}_{O_b (\psi_l)} \) is minimal in \( L^2 (0, T) \), and so we can use a finite time extraction method to read off the Fourier coefficients. The rest of the procedure is the same.

**Corollary 1.** Assume that Assumption H holds and the set \( \{ e^{-\lambda_{ij} t} \}_{m \in O_b (\psi_l)} \) is minimal in \( L^2 (0, T) \) for all \( \{ \psi_l \} \) of a certain basis in \( L^2 (B) \). Then the observations of the solution at the single point \( b \in \Omega \) and over a finite interval of time \( (0, T) \) are sufficient to uniquely reconstruct all the coefficients \( a_{ij} \), \( q \), and the domain \( \Omega \).

7. **Reading from the boundary.** In certain applications we cannot get inside the domain \( \Omega \) and only the boundary, yet unknown, is accessible. Examples include measuring the temperature with a laser or a thermoscan. Thus, the issue at hand is how to solve the above problem when \( b \in \partial \Omega \). In case \( b \in \partial \Omega \), we have \( u(b, t) = 0 \) and so the previous reading is trivial. In this case we need to measure \( \frac{\partial u}{\partial \nu} (b, t) \), i.e. the heat flow in direction \( \nu \) through the map \( \Gamma_{\partial \Omega} : L^2 (B) \rightarrow \ell^\infty \)
\[ \Gamma_{\partial \Omega} : a \rightarrow \left\{ \frac{\partial u}{\partial \nu} (b, k) \right\}_{k \geq 1}. \]
(43)
Since the boundary \( \partial \Omega \) is unknown, the normal direction at \( b \) is also unknown. To search for the heat flow direction \( \nu \), observe that if \( \nu \) is an inward or tangent vector, then the map \( \Gamma_{\partial \Omega} \) is trivial. In that case we have to change the direction of \( \nu \) until we get a nontrivial map \( \Gamma_{\partial \Omega} \). This guarantees that \( \nu \) is a non-tangent outward
vector. It follows from Assumption $H$ that $u(x,t) \in C^{2+\alpha}({\Omega})$ for any $t > 0$. From (4) we get,

$$\frac{\partial u}{\partial \nu}(b,k) = \sum_{m \geq 1} c_m(\psi) e^{-\lambda_m k} \frac{\partial \varphi_m}{\partial \nu}(b), \quad k = 1, 2, \ldots$$  \hfill (44)

We already have that $\frac{\partial \varphi_1}{\partial \nu}(b) \neq 0$. We now show that a result similar to Proposition 1 also holds for the outward directional derivatives.

**Proposition 9.** For any $b \in \partial \Omega$ there are infinitely many $\frac{\partial \varphi_m}{\partial \nu}(b) \neq 0$.

**Proof.** Assume that we have a finite number of non vanishing non-tangent outwards directional derivatives only, then after changing the sign and rearranging them if necessary, we would have

$$\frac{\partial \varphi_i}{\partial \nu}(b) < 0 \quad \text{for all} \quad i \leq N \quad \text{and} \quad \frac{\partial \varphi_i}{\partial \nu}(b) = 0 \quad \text{for all} \quad i > N. \hfill (45)$$

Since $\varphi_i \in C^{2+\alpha}({\Omega})$, it follows from their differentiability and the linear approximation at $b \in \partial \Omega$, $\varphi_i(b) = 0$, that near the boundary we have

$$\varphi_i(x) \approx \nabla \varphi_i(b) \cdot (x - b), \quad x \in \Omega \quad \text{and} \quad x \text{ close to } \partial \Omega. \hfill (46)$$

Let

$$\psi(x) := \rho_{\beta,\delta/2}(x) \hfill (47)$$

be the initial condition (cf. (11)). Then $\psi \in C^\infty_0(\Omega)$ and, by (11) and (46), its Fourier coefficients satisfy

$$c_i(\psi) = \int_{\Omega} \psi(x) \varphi_i(x) \, dx > \varepsilon \int_{\Omega} \rho_{\beta,\delta/2}(x) \, dx > 0. \hfill (48)$$

Now consider the parabolic equation (1) with the initial condition (47). Since $\psi(x) \otimes 1(t) \in C^{\infty,\infty}({Q_T})$, where $Q_T = \Omega \times [0,T]$, it follows, under assumption $H$, that the solution $u(x,t)$ is in $C^{2+\alpha,1+\alpha/2}({Q_T})$ and is classical. Therefore, its uniqueness, (4), and (45), imply that for any $b \in \partial \Omega$

$$\frac{\partial u}{\partial \nu}(b,t) = \sum_{i \geq 1} c_i(\psi) e^{-\lambda_i t} \frac{\partial \varphi_i}{\partial \nu}(b) = \sum_{i=1}^N c_i(\psi) e^{-\lambda_i t} \frac{\partial \varphi_i}{\partial \nu}(b) \quad \text{for} \quad 0 < t < T. \hfill (49)$$

By continuity of the solution, as $t \to 0^+$, and $\psi \in C^\infty_0(\Omega)$, we have a contradiction at the boundary point $x = b$,

$$0 = \frac{\partial \psi}{\partial \nu}(b) = \lim_{t \to 0^+} \frac{\partial u}{\partial \nu}(b,t) - \sum_{i=1}^N c_i(\psi) \frac{\partial \varphi_i}{\partial \nu}(b) < 0. \hfill (50)$$

Proposition 9 is proved.
By using similar procedures as done before for the case when \( b \) is inside \( \Omega \) we can extract from measurements on boundary \( \{ \frac{\partial \varphi_j}{\partial \nu}(b, k) \}_{k \geq 1} \) the sequence of values

\[
\lambda_m, \quad \sum_{\lambda_i = \lambda_m} c_j(\psi_i) \frac{\partial \varphi_j(b)}{\partial \nu}, \quad m = 1, 2, \ldots, \psi_i \in L^2(B),
\]

where \( \{ \psi_i \}_{i \geq 1} \) is the same basis of \( L^2(B) \), as long as \( \frac{\partial \varphi_j(b)}{\partial \nu} \neq 0 \). As done in Section 6, we have extended the results of Theorem 6.1 to the case of observation from the boundary:

**Theorem 7.1.** Let Assumption \( H' \) hold. Then we can reconstruct all the coefficients \( a_{ij}, q \), and the domain \( \Omega \) from measurements of the heat flow at any single point \( b \in \partial \Omega \) in any onwards nontangent direction \( \nu \), and at evenly spaced times, i.e., from \( \{ \frac{\partial u}{\partial \nu}(b, k) \}_{k \in \mathbb{N}} \).

8. **Neumann boundary condition.** Now we can consider the parabolic equation with a Neumann boundary condition, and under the same Assumption \( H \),

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j}(x, t) \right) - q(x)u(x, t), \quad x \in \Omega \subset \mathbb{R}^n, \ t > 0, \\
\frac{\partial u}{\partial \nu}(x, t) &= 0, \quad x \in \partial \Omega, \\
u(x, 0) &= \psi(x) \in L^2(\Omega),
\end{aligned}
\]

(49)

where \( \mathbf{n} \) is the unit exterior normal vector. In this case, we can measure

\[
u(b, k) = \sum_{m \geq 1} c_m(\psi) e^{-\lambda_m k} \varphi_m(b), \quad k = 1, 2, \ldots, \text{ if } b \in \overline{\Omega},
\]

where \( \varphi_m \) are the eigenfunctions of

\[
\begin{align*}
A\varphi(x) &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \varphi(x) + q(x)\varphi(x), \quad x \in \Omega, \\
\frac{\partial \varphi}{\partial \mathbf{n}}(x) &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

and \( \lambda_m \) are their associated eigenvalues, while \( c_m \) are the Fourier coefficients

\[
c_m(\psi) = \int_{\Omega} \psi(x) \varphi_m(x) \, dx.
\]

According to [12, Theorem 6.14], \( \varphi_m \in C^{2+\alpha}(\overline{\Omega}) \). Repeating the above analysis we can recover the Fourier coefficients \( c_m(\psi)\varphi_m(b) \) of \( \varphi_m(b)\varphi_m(x) \). Because \( \varphi_m \) does not vanish on the boundary \( \partial \Omega \), its extension by zero outside \( \Omega \) is not continuous in \( B \). Nevertheless, their integral moduli of continuity satisfy \( \omega_p(\delta, \varphi_m) = O(\delta^\alpha) \), that guarantee the pointwise convergence almost everywhere of (35) to \( \varphi_m \) in \( B \). To get uniform convergence we have to use again the Riesz means (39). Although \( \varphi_m \) is not continuous on \( B \), it is infinitely differentiable on \( B \setminus \overline{\Omega} \) and is twice differentiable on \( \Omega \). Moreover, it has a compact support in \( B \). Hence, the localization principle of Riesz means [14] can be applied to \( \varphi_m \) that says that the Riesz means (40) of the critical order \( s = n+1 \) converge uniformly to \( \varphi_m \) on any compact subset of \( B \setminus \overline{\Omega} \) and on any compact subset of \( \Omega \). Consequently, (40) determines \( \varphi_m \), that are \( C^{2+\alpha}(\Omega) \) in \( \Omega \), and zero on \( B \setminus \overline{\Omega} \). By similar arguments we arrive at
Theorem 8.1. Let Assumption $H$ hold. Then all the thermal coefficients $a_{ij}$, the internal sink coefficient $q$, and the domain $\Omega$ of (49) can be uniquely reconstructed from measurements $\{u(b,k)\}_{k \in \mathbb{N}}$ at any single point $b \in \Omega$.

9. Example. Consider the simple parabolic equation
\[
\begin{align*}
\frac{\partial u}{\partial t}(x,y,t) &= a \frac{\partial^2 u}{\partial x^2}(x,y,t) + b \frac{\partial^2 u}{\partial y^2}(x,y,t) + cu(x,y,t), & (x,y) \in \Omega, \\
u(x,y,t) &= 0 & \text{for } (x,y) \in \partial \Omega, \\
u(x,y,0) &= \psi_{ij}(x,y) & \text{for } (x,y) \in \Omega.
\end{align*}
\]
We shall search the unknown $\Omega$ inside the bigger square $B = [0, 10\pi] \times [0, 10\pi]$ and also the coefficients $\{a, b, c\}$. An obvious basis for $L^2(B)$ is given by
\[
\psi_{ij}(x,y) = \frac{1}{5\pi} \sin \left(\frac{i}{10} x\right) \sin \left(\frac{j}{10} y\right) & \text{ for } (x,y) \in B \text{ and } i, j \geq 1.
\]
Consider the solution generated with $\Omega = (7\pi, 8\pi) \times (6\pi, 8\pi)$, $a = 9$, $b = 1$, $c = 8$, and initial condition $\psi_{ij}$. Note that although the boundary of the rectangle $\Omega$ is not smooth, as required by assumption $H$, the solution due to the constant coefficients is a strong solution and so we can measure its values, say at the point $(7.3\pi, 7.4\pi)$, and at times $k \in \mathbb{N}$. Thus we can obtain $\{u_{ij}(7.3\pi, 7.4\pi, k)\}_{k \in \mathbb{N}}$, numerically using CAS, Maple 16, say the first 41 terms of $\{u_{11}(7.3\pi, 7.4\pi, k)\}_{1 \leq k \leq 41}$ are

Then in order to extract the first eigenvalue of the operator $A$, i.e. $\lambda_1$ as prescribed by (33), we look at the sequence of ratios
\[
\lim_{k \to \infty} \frac{u_{11}(7.3\pi, 7.4\pi, k+1)}{u_{11}(7.3\pi, 7.4\pi, k)} = e^{\lambda_1}.
\]
We get the following sequence for $k = 1, 2, \ldots, 41$
approximations by Fourier series. 

would shrink these outside "pot holes" but would increase their numbers, 

been changed to a new

Thus we deduce that

\[ e^{\lambda_1} = 0.28650479686019 \quad \text{or} \quad \lambda_1 := \ln(0.28650479686019) = -1.25000000. \]

Next using this last result we extract the coefficients by looking at the sequence

\[ u_{11}(7.3\pi, 7.4\pi, k) \ast e^{0.25k}, \quad \text{for} \quad k = 1, 2, \cdots, 41; \quad (50) \]

Thus we deduce that, the value of the Fourier coefficient in (34) is

\[ c_1(\psi_{1,1}) = 0.038192502975723 \]

Once we have \( \lambda_1 \), we repeat the same operation, where the initial condition has been changed to a new \( \psi_{i,j} \). We shall fill in the matrix of Fourier coefficients \( c_1(\psi_{i,j}) \) for \( 1 \leq i, j \leq 15 \) and then summing them up provides the following plot

To find the domain we can plot the level lines of \( \varphi_1 \), see Fig. 2. Despite the corners of the non-smooth boundary rectangle, and the few data we used, the smooth level lines give a good approximation of the domain \( \Omega \). Obviously because we are truncating a Fourier series, we expect few bumps generated by Gibbs phenomenon. However they can be easily discarded since we know that the domain \( \Omega \) is connected and the first eigenfunction has a constant sign. Thus the domain is in \( (7\pi, 8\pi) \times (6\pi, 8\pi) \), as it can be seen from the contour plot. Taking more initial conditions, would shrink these outside "pot holes" but would increase their numbers, leaving a closer approximation of the main domain. This is a well known issue in approximations by Fourier series.
Figure 1. $\varphi_1$

Figure 2. $\Omega$

Figure 3. $\varphi_2$
Repeating the same procedure we successively get \( \varphi_2 \) and \( \varphi_{12} \) which correspond to the eigenvalue \( \lambda_2 = -2 \) while \( \lambda_{12} = -29 \). Again the graphs of \( \varphi_i \) exhibit the well known Gibbs phenomena of a truncated Fourier series.

If we could reconstruct the full Fourier series, we would then get the rescaled eigenfunctions

\[
\varphi_1(x, y) = \sin(x - 7\pi)\sin((y - 6\pi)/2),
\]
\[
\varphi_2(x, y) = \sin(x - 7\pi)\sin((y - 6\pi)),
\]
\[
\varphi_{12}(x, y) = \sin(2(x - 7\pi))\sin((y - 6\pi))
\]

At the point \((7.3\pi, 7.4\pi)\) we would get the system

\[
x_1 := 7.3 \cdot \pi; y_1 := 7.4 \cdot \pi; l_1 := -1.25; l_2 := -2; l_3 := -29;
\]
\[
eq 1 := (\text{diff}(f_1, xx)) \cdot a + (\text{diff}(f_1, yy)) \cdot b + f_1 \cdot c = l_1 \cdot f_1;
\]
\[
eq 2 := (\text{diff}(f_2, xx)) \cdot a + (\text{diff}(f_2, yy)) \cdot b + f_2 \cdot c = l_2 \cdot f_2;
\]
\[
eq 3 := (\text{diff}(f_3, xx)) \cdot a + (\text{diff}(f_3, yy)) \cdot b + f_3 \cdot c = l_3 \cdot f_3;
\]
\[
solve\{\{eq1, eq2, eq3\}, \{a, b, c\}\};
\]

at the chosen point, which then yields back the values of the sought coefficients

\[
(a = 8.999999999, b = 1., c = 7.999999999).
\]

Before we end we would like to point out that the above algebraic system with \( \{\varphi_1, \varphi_2, \varphi_3\} \) is not full rank, but \( \{\varphi_1, \varphi_2, \varphi_{12}\} \) is, as predicted by Proposition 3.

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REFERENCES

[1] Sh. A. Alimov, V. A. Il’in and E. M. Nikishin, Questions on the convergence of multiple trigonometric series and spectral expansions I, Russian Mathematical Surveys, 31 (1976), 28–83.
[2] H. Ammari, An Introduction to Mathematics of Emerging Biomedical Imaging, Mathématiques & Applications 62, Springer, Berlin, 2008.

[3] S. A. Avdonin and A. Bulanova, Boundary control approach to the spectral estimation problem: The case of multiple poles, Math. Control Signals Systems, 22 (2011), 245–265.

[4] S. A. Avdonin, F. Gesztesy and A. Makarov, Spectral estimation and inverse initial boundary value problems, Inverse Probl. and Imaging, 4 (2010), 1–9.

[5] S. A. Avdonin and S. A. Ivanov, Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems, Cambridge University Press, Cambridge, 1995.

[6] L. Beilina and M. V. Klibanov, Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems, Springer, New York, 2012.

[7] A. Bostan and P. Dumas, Wronskians and linear independence, Amer. Math. Monthly, 117 (2010), 722–727, arXiv:1301.6598v1.

[8] A. Boumenir and V. K. Tuan, Inverse problems for multidimensional heat equations by measurements at a single point on the boundary, Numer. Funct. Anal. Optim., 30 (2009), 1215–1230.

[9] A. Boumenir and V. K. Tuan, An inverse problem for the wave equation, Journal of Inverse and Ill-posed Problems, 19 (2011), 573–592.

[10] A. S. Demidov and M. Moussaoui, An inverse problem originating from magnetohydrodynamics, Inverse Problems, 20 (2004), 137–154.

[11] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, Archive for Rational Mechanics and Analysis, 43 (1971), 272–292.

[12] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer–Verlag, Berlin, 2001.

[13] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics, Birkhäuser, Basel, 2006.

[14] V. A. Il’in and Sh. A. Alimov, Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators, V, Differential Equations, 10 (1974), 360–377.

[15] V. Isakov, Inverse Problems for Partial Differential Equations, 2nd Ed, Applied Mathematical Sciences 127, Springer, New York, 2006.

[16] M. V. Klibanov and A. Timonov, Carleman Estimates for Coefficient Inverse Problems and Numerical Applications, VSP, Utrecht, The Netherlands, 2004.

[17] V. Mikhailov, Equations aux Derivees Partielles, French transl., Mir, Moscow, 1980.

[18] K. Wolsson, Linear dependence of a function set of m variables with vanishing generalized Wronskians, Linear Algebra and its Applications, 117 (1989), 73–80.

[19] Z. Wu, J. Yin and C. Wang, Elliptic and Parabolic Equations, World Scientific, Singapore, 2006.

[20] L. V. Zhizhialeshvili, Some problems in the theory of simple and multiple trigonometric series, Russian Mathematical Surveys 28 (1973), 65–119.

[21] E. Zuazua, Controllability and observability of partial differential equations: Some results and open problems, in Handbook of Differential Equations: Evolutionary Equations, Vol. III, Elsevier/North-Holland, Amsterdam, 3 (2007), 527–621.

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