The analysis of the wear process in building constructions with the use of the theory of stochastic functions.

Zhiharev Fedor Karpovich¹,³ and Al’ Malul’ Rafik Muhamedovich²
¹Candidate of Technical Sciences, Professor of the "Reinforced Concrete and Stone Structures" Department, Moscow State University of Civil Engineering (MGSU), Russia,129337,Yaroslavskoe Shosse St,26.
²Dr. of Tech.Sci., of «Construction Mechanics» Department, Moscow State University of Civil Engineering (MGSU), Russia,129337,Yaroslavskoe Shosse St,26.
³Phone: 89859968211
E-mail: fzh@mgsu.ru

Abstract. One of the main causes of reduced reliability and durability of constructions, machinery, mechanisms, reinforced concrete structures, metal parts of the equipment is the wear. To develop the methods of the estimation of the reliability of the building constructions exposed to wear, the parametric failure models are used. The impact on the construction (the load coefficient) and the parameters of the load capacity are presented as the stationary or non-stationary Gaussian stochastic processes. By the proposed formulas for the reliability function, the example calculations are performed here below.

Key words: wear of the elements of constructions, wear of building structures, decrease of the load capacity during time, the increase in the probability of failure.

The irreversible processes in constructions, which are the effects of wear, make it necessary to change the way of description of many parameters specifying work capacity of the construction with the use of stochastic process.

The criterion of the non-failure work could be formulated as the following:

\[ R_0 \varphi(t) = Z(t) > S(t) \]  \hspace{1cm} (1)

- \( R_0 \) — the initial value of the load capacity
- \( S(t) \) — the load effect (caused by the stress or the tension)
- \( \varphi(t) \) — the wear function
- \( Z(t) \) — the wear process

The expression (1) could be formulated as

\[ \varphi(t) > \frac{S(t)}{R_0} \]  \hspace{1cm} (2)

Or, if we represent the impact on the construction as the stochastic value, then:
\[
\varphi(t) > \frac{1}{\xi} = \alpha
\]

Here \( \xi = \frac{R_0}{S} \) — the reserve of strength coefficient.

According to (3), when the process \( Z(t) \) exceeds the \( \alpha \)-level at first time, the failure happens. The probability of the non-failure work could be expressed as

\[
\Phi(t) = \Phi[x(\tau) > \alpha, \tau \in (0, t)]
\]

Taking after [6], we can write down:

\[
\Phi(t) \geq 1 - g(\alpha) - Na(t)
\]

Here \( g(\alpha) \) — the value of the distribution function of the ordinate of the stochastic function, at the moment \( t=0 \).

\( Na(t) \) — the average number of the excesses of the level.

For the stationery process:

\[
\Phi(t) \geq 1 - g(\alpha) - n_a t
\]

Here \( n_a \) — the average number of the excesses of the \( \alpha \)-level of the process \( Z(t) \) per the unit of time.

The wear process can be represented reliably enough as the normal stationery process; then the average number of the excesses of the \( \alpha \)-level per unit of time is written as the following sentence:

\[
n_a = \frac{1}{2\pi} \frac{\sigma_{1z}}{\sigma_z} \exp \left[ -\frac{(a - \bar{Z})^2}{2\sigma_z^2} \right]
\]

Here \( \sigma_{1z}^2, \sigma_z^2 \) — the dispersions of the wear process and its’ rate, \( \bar{Z} \) — mathematical expectation.

The dispersion of the rate of the wear process:

\[
\sigma_{1z}^2 = -\left. \frac{d^2 K_z}{dt^2} \right|_{t=0} = -K_0''
\]

Here \( K_z(\tau) \) — correlation function.

If \( K_z(\tau) = \sigma_z^2 (\tau) \), then

\[
n_a = \sqrt{\frac{z_0}{2\pi}} \exp \left[ -\frac{(a - \bar{Z})^2}{2\sigma_z^2} \right]
\]

Here \( z(\tau) \) — the normalized correlation function

\[
z_0 = \left. \frac{d^2 z(\tau)}{dt^2} \right|_{t=0}
\]

Let us represent the wear process \( Z(t) \) as the sum of the normal stationary process \( y(t) \) and the normally distributed random value \( x \) [4]

\[
z(t) = y(t) + x = \bar{y}(t) + \bar{x}
\]
\[ k_x(t) = \sigma_x^2 r(t) + \sigma_x^2 \] (11)

The average number of excesses of the process (10) per unit of time could be found out from that formula (the level is a constant):

\[ n_a = \frac{1}{2\pi} \left( \frac{\sigma_x^2 r''_0}{\sigma_x^2 + \sigma_x^2} \right)^{1/2} \exp \left[ - \frac{(a - \bar{y} - \bar{x})^2}{2(\sigma_y^2 + \sigma_x^2)} \right] \] (12)

If the condition of non-failure work is (1), and S is the normally distributed random value, and the wear process \( Z(t) \) is the normal stationary process, then the average number of excesses (\( S \) and \( Z(t) \)) could be written as

\[ n_a = n_0 \exp \left[ - \frac{(\bar{S} - \bar{Z})^2}{2(\sigma_S^2 + \sigma_Z^2)} \right] \] (13)

here \( n_0 = \frac{1}{2\pi} \left( \frac{\sigma_Z^2 r''_0}{\sigma_S^2 \sigma_Z^2} \right)^{1/2} \)

The lower estimate of the probability for that the process (10) wouldn’t exceed the level \( \alpha = 1/\zeta \) during the time, could be learned from the following sentence:

\[ F(t) \geq \phi \left( \frac{a - (y + x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right) \frac{t}{2\pi} \left( \frac{\sigma_x^2 r''_0}{\sigma_y^2 + \sigma_x^2} \right)^{1/2} \exp \left[ - \frac{(a - (y + x))^2}{2(\sigma_y^2 + \sigma_x^2)} \right] \] (14)

The estimate (14) is wright for

\[ t \leq \phi \left( \frac{a - (y + x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right) \exp \left\{ - \frac{[a - (y + x)]^2}{2(\sigma_y^2 + \sigma_x^2)} \right\} \frac{1}{n_0} \] (15)

For the positive correlative wear process

\[ F(t) \geq \phi \left( \frac{a - (y + x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right) - n_1 \exp \left\{ - \frac{[a - (y + x)]^2}{2(\sigma_y^2 + \sigma_x^2)} \right\} \] (16)

\[ n_1 = n_0 \Delta t; \quad \Delta t = 1; \quad t \geq 1 \]

The upper estimate of the probability \( F(t) \) is given by this formula:

\[ F(t) \leq \phi \left( \frac{a - (y - x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right) \exp \left\{ - n_0 \exp \left[ - \frac{[a - (y + x)]^2}{2(\sigma_y^2 + \sigma_x^2)} \right] \right\} \] (17)

with the parameter \( n_0 \) being calculated by the formula (13)

The corresponding estimates of the probability of that the normal stationary process \( Z(t) \) wouldn’t exceed the level \( S \) are the random values, so we can write down that

\[ F(t) \geq \phi \left( \frac{(S + Z)}{\sqrt{\sigma_S^2 + \sigma_Z^2}} \right) - n_0 \exp \left\{ - \frac{(S - Z)^2}{2(\sigma_S^2 + \sigma_Z^2)} \right\} \]

\[ F(t) \leq \phi \left( \frac{(S - Z)}{\sqrt{\sigma_S^2 + \sigma_Z^2}} \right) \exp \left\{ - n_0 \exp \left[ - \frac{(S - Z)^2}{2(\sigma_S^2 + \sigma_Z^2)} \right] \right\} \] (18)

For practical application, the probability of the non-failure work is interesting, then the load coefficient \( S(t) \) and generalized strength \( R(t) \) are the stochastic process. If they are represented as normal stationary processes, then \( n \) — the average number of the normal stationary process \( R(t) \)
excesses of the level, the level is represented as the normal stationary process \( S(t) \), per unit of time, is performed by the formula

\[
n = n_0 \exp \left( \frac{(\tilde{s} - \tilde{R})^2}{2\sigma^2} \right)
\]

(19)

Here the parameters are:

\[
n_0 = \frac{1}{2\pi \alpha} \left[ -\left( \sigma_R^2 r''(0) + \sigma_S^2 r_S''(0) - 2r''_{RS}(0)\sigma_s\sigma_R \right) \right]^{1/2}
\]

(20)

\[
\sigma^2 = \sigma_R^2 + \sigma_S^2 - 2r_{RS}\sigma_R\sigma_S
\]

Here the parameter \( r_{RS} \) means the value of the mutual normalized correlation function. We can use the estimation of the probability \( F(t) \) to make the reliability margin

\[
P(t) \geq \phi \left( \frac{\zeta - 1}{\sqrt{\frac{\sigma^2}{\sigma_R^2} + \xi\frac{\sigma^2}{\sigma_S^2}}} \right) - n_0 \exp \left[ -\left( \frac{(\zeta - 1)^2}{2(\frac{\sigma^2}{\sigma_R^2} + \xi\frac{\sigma^2}{\sigma_S^2})} \right) \right]
\]

(22)

Here \( \zeta = \frac{\tilde{R}}{S} \); \( \vartheta_R = \frac{\sigma_R}{R} \); \( \vartheta_S = \frac{\sigma_S}{S} \)

Let us assume that the processes \( S(t) \) and \( R(t) \) are not correlated, then

\[
\sigma = \sigma_R^2 + \sigma_S^2 \text{ and } n_0 = \frac{1}{2\pi \sqrt{\sigma_R^2 + \sigma_S^2}} \left[ -\left( \sigma_R^2 r_R''(0) + \sigma_S^2 r_S''(0) \right) \right]^{1/2}
\]

(23)

Setting the normal level of reliability \( P^* \) from (22), we can depict the relationship between the reliability factor and time

\[
t = \frac{1}{n_0} \left[ \phi \left( \frac{\zeta - 1}{\sqrt{\frac{\sigma^2}{\sigma_R^2} + \xi\frac{\sigma^2}{\sigma_S^2}}} \right) - P^* \right] \exp \left[ \frac{(\zeta - 1)^2}{2(\frac{\sigma^2}{\sigma_R^2} + \xi\frac{\sigma^2}{\sigma_S^2})} \right]
\]

(24)

The expression (23) can be converted into the following expression:

\[
n_0 = \frac{1}{2\pi \sqrt{\vartheta_R^2 + \zeta^2\vartheta_R^2}} \left[ -\vartheta_R^2 \xi^2 r_R''(0) - \vartheta_S^2 r_S''(0) \right]^{1/2}
\]

(25)

The correlative functions — the load function and the function of generalized strength, could be performed as that:

\[
K_S(\tau) = \sigma_S^2 r_S(\tau) = \sigma_S^2 \exp(-\alpha\tau^2)
\]

\[
K_R(\tau) = r_R(\tau) = \sigma_R^2 \left( \cos \beta \tau + \frac{\gamma}{\beta} \sin \beta |\tau| \right) \exp(-\gamma |\tau|)
\]

(26)

It follows from the expression (26) that \( r_S''(0) = -2\alpha \); \( r_R''(0) = -(\gamma^2 + \beta^2) \)

The average number of the excesses can be written down as

\[
n_0 = \frac{1}{2\pi \sqrt{\vartheta_S^2 + \zeta^2\vartheta_R^2}} \left[ \xi^2\vartheta_R^2 (\gamma^2 + \beta^2) + 2\vartheta_S^2 \alpha \right]^{1/2}
\]
With the use of the formula (22), taking into consideration of (27), we can graph the relationship between the probability of the non-failure work $P$ and the factor of safety $\zeta$. Set the time in years: $t=1, 2, 10, 20, 50, 100$ (Fig.1)

If $\vartheta_S = 0.2; \vartheta_R = 0.1; \alpha = 0.98 \frac{1}{\text{year}}; \gamma = 0.1 \frac{1}{\text{year}}; \beta = 0.7 \frac{1}{\text{year}}$

Investigating the pict.1, we can find out the influence of the exploitation period on the decrease of the factor of safety in conditions of the wear with the given level of the normative reliability.

The irreversible processes in constructions, due to the wear, make it necessary to perform the variation of many parameters specifying work capacity with the unstationary processes.

Let us consider the normal unstationary process, the process can be represented as

$$Z(t) = X(t) + \varphi(t)$$

where $X(t)$ — the normal stationary process with its’ mathematical expectation and the correlation function both are zero.

$\sigma^2_X(t), \varphi(t)$ — some determined monotonous functions depending on the time.

For the non-stationary process (28), the approximate estimate of the probability $F(t)$ can be made with the help of the relationship given above, for the stationary processes. To make that, using the expressions for the stationary process, we should find the probability of that the stationary process will not exceed the level $a = a - \varphi(t)$, the value of the probability $F(t)$ is made lower then its’ real value.

But this method simplify the procedure of the calculation of the probability $F(t)$ dramatically, and the method could be recommended for the estimation of the probability of non-failure work of the fairly safe elements of the constructions. For the areas of the failure, bounded from above and below, the estimation of that the process (28) will not excess the $\alpha$-level is that:

$$F(t) \geq \phi \left( \frac{a - \varphi(t)}{\sigma_x} \right) - \frac{t}{2\pi} \sqrt{-r''_x(0)} \exp \left[ -\frac{1}{2} \left( \frac{a - \varphi(t)}{\sigma_x} \right)^2 \right]$$

$$F(t) \geq \phi \left( \frac{\varphi(t) - a}{\sigma_x} \right) - \frac{t}{2\pi} \sqrt{-r''_x(0)} \exp \left[ -\frac{1}{2} \left( \frac{a - \varphi(t)}{\sigma_x} \right)^2 \right]$$

(29)

Herewith that is not obligatory to define the function $\varphi(t)$ analytically. For different wear functions $\varphi(t)$ and for the wear area is bounded from below in accordance to the formula (3), the expression for the estimate of the non-failure work probability is

$$\varphi(t) = b_0 - b_1 t \quad (b_1 > 0)$$

(30)

$$F(t) \geq \phi \left( \frac{b_0 - a - b_1 t}{\sigma_x} \right) - \frac{t}{2\pi} \sqrt{-r''_x(0)} \exp \left[ -\frac{1}{2} \left( \frac{a - b_0 - b_1 t}{\sigma_x} \right)^2 \right]$$

(31)

If

$$\varphi(t) = b_0 - b_1 t - b_2 t^2$$

(32)

then

$$F(t) \geq \phi \left( \frac{b_0 - a - b_1 t - b_2 t^2}{\sigma_x} \right) - \frac{t}{2\pi} \sqrt{-r''_x(0)} \exp \left[ -\frac{1}{2} \left( \frac{a - b_0 - b_1 t - b_2 t^2}{\sigma_x} \right)^2 \right]$$

(33)

If
\[
\varphi(t) = b_0 - b_1 \exp(\alpha t)
\]
\[
F(t) \geq 1 - \varphi \left( \frac{a - b_0 + b_1 \exp(\alpha t)}{\sigma_x} \right) - t \sqrt{-r''(0)} \exp \left\{ - \frac{1}{2} \left[ \frac{a - b_0 + b_1 \exp(\alpha t)}{\sigma_x} \right]^2 \right\}
\]

(35)

According to [4], the wear process can be represented as the independent sum of two components:

\[ Z(t) = \alpha(t) + Y(t) \]

where \( \alpha(t) \) — the normal stationary process with its mathematical expectation is zero and its correlation function \( \sigma^2_Y(r) \); \( \alpha(t) \) — some function of time which is already known, its parameters are stochastic variables. For example,

\[ \alpha(t) = b_0 + b_1 t \quad Z(t) = b_0 + b_1 t + X(t) \]

here \( b_0, b_1 \) — stochastic variables. The distribution density of the process \( \alpha(t) \) at the moment \( t \), with \( b_0 \) and \( b_1 \) are independent normally distributed values, can be written down as

\[
P(\alpha) = \frac{1}{\sigma_y \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\alpha - \bar{\alpha}}{\sigma_y} \right)^2 \right]
\]

(38)

Here the assumption is that the probability of negative values of \( \alpha \) is close to zero. Parameters \( \sigma_y \) and \( \bar{\alpha} \) are defined with the following expressions

\[
\bar{\alpha} = \bar{b}_0 + t \bar{b}_1
\]

\[
\sigma_y^2 = \sigma^2_{b_0} + 2rt \sigma_{b_0} \sigma_{b_1} + \sigma_{b_1}^2 t^2
\]

(39)

here \( r \) — the correlation coefficient between the values \( b_0 \) and \( b_1 \). The task of the calculation the probability of that the process \( Z(t) \) will not exceed the \( \alpha \)-level can be reduced to the task of finding the distribution function for the sum of two stochastic values \( \alpha(t) \) and \( \alpha(t) \). Then the probability of that the given level will not be exceeded is

\[
F(t) = \int_{-\infty}^{\alpha} \int_{-\infty}^{\alpha} g_x(Z-Y) dy dz
\]

(40)

\( F(t) \) is determined by the expression

\[
F(t) \geq \int_{-\infty}^{\alpha} P(\alpha - Y) d\alpha
\]

The expression for \( P(t) \) can be found with the use of the formulas (5)

\[
F(t) \geq \int_{-\infty}^{\alpha} \left[ \phi \left( \frac{\alpha - \bar{\alpha}}{\sigma_x} \right) - n_0 t \exp \left[ -\frac{1}{2} \left( \frac{\alpha - \bar{\alpha}}{\sigma_x} \right)^2 \right] \right] d\alpha
\]

(42)

Using the transformations (4), we can write that:

\[
F(t) \geq \phi \left( \frac{a - \bar{\alpha}}{\sqrt{\sigma_x^2 + \sigma_y^2}} \right) - \frac{n_0 t \sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}} \exp \left[ -\frac{(a - \bar{\alpha})^2}{2(\sigma_x^2 + \sigma_y^2)} \right]
\]

(43)

Put (39) in (43), then we get

\[
F(t) \geq \phi \left( \frac{a - \bar{b}_0 - t \bar{b}_1}{\sqrt{\sigma_x^2 + \sigma_{b_0}^2 + 2rt \sigma_{b_0} \sigma_{b_1} + t^2 \sigma_{b_1}^2}} \right) - \frac{n_0 t \sigma_x}{\sqrt{\sigma_x^2 + \sigma_{b_0}^2 + 2rt \sigma_{b_0} \sigma_{b_1} + t^2 \sigma_{b_1}^2}} \exp \left[ -\frac{(a - \bar{b}_0 - t \bar{b}_1)^2}{(\sigma_x^2 + \sigma_{b_0}^2 + 2rt \sigma_{b_0} \sigma_{b_1} + t^2 \sigma_{b_1}^2)} \right]
\]

(44)

Meanwhile, the average number of the excesses of the process per unit of time is calculated with the use of the formula
\[ n_0 = \frac{1}{2\pi\sigma_X} \sqrt{-\sigma_X^2 r''(0)} = \frac{1}{2\pi} \sqrt{-r''(0)} \]  \hspace{1cm} (45)

**Conclusions**

As an example, let us consider the reliability of the pressure pipeline. The wall thickness of the pipe is \( h = 12 \text{mm} \) (millimeters). The corrosive wear leads to the formation of the through hole and the failure is that the product flows out, that is \( h = a \). The reduction of wall thickness occurs by law

\[ Y = b_0 + b_1 t, \]  

where \( b_0 \) and \( b_1 \) are normally distributed independent stochastic values:

\[ r = 0; \quad b_0 = 0,002 \text{ mm}; \]

\[ b_1 = 0,15 \text{ mm \, year}^{-1}; \quad \sigma_{b_0} = 0,0004 \text{ mm \, year}^{-1}; \quad \sigma_{b_1} = 0,004 \text{ mm \, year}^{-1}; \quad n_0 = 0,09 \frac{1}{\text{ year}}. \]

The substitution of the given numerical values into (44), makes it possible to graph the relationship between the reliability function and time (Fig. 2).

This graph (a curve) makes it possible to determine the necessary wall thickness of the pipeline at a given value of probability of failure-free operation and operating life of the pipeline.
Fig. 2

Bibliography

[1] Raizer V. D. *Reliability theory of structures*. M., The Publishing house of the Association of Building Universities, 2010
[2] Bolotin V. V. *Statistical methods in structural mechanics*. M. Stroyizdat, 1965, 272 p.
[3] Tikhonov, V. I. *The excesses of stochastic processes*. M. Nauka 1970, 392 p.
[4] Pereverzev, S. V. *Stochastic processes in parametric models of reliability*. Kiev. Naukova Dumka, 1987., 237 p.
[5] Ventuel E. S., Ovcharov L. A. *Theory of stochastic process and its engineering applications*. M. Nauka 1991 p. 380 p.
[6] Rzhansky A. R. *Theory of calculation of building structures on the reliability*. M.: Stroyizdat, 1978, 240 p.
[7] Tamrazyan A. G. *The calculation of eccentrically compressed reinforced concrete elements under dynamic loading in terms of fire effects*. Industrial and Civil Construction (Promyshlennoe and Grazhdanskoe Stroitelstvo, PGS), 2015, 3, page 29-35
[8] Mueller M., Charypar D., Gross M. *Particle-based fluid simulation for interactive applications*. Eurographics, SIGGRAPH Symposium on Computer Animation, D. Breen, M. Lin (Editors) 2003
[9] Raizer V.D. *Reliability optimization for pipelines to corrosion wear*, *Proc. of ASCE conference advances in Pipeline Engineering and Construction*. Pipelines 2001. — San Diego, CA, US, 2001