Controllability of semi-infinite rod heating
by a point source

A Khurshudyan
Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
E-mail: khurshudyan@mechins.sci.am

Abstract.
The possibility of control over heating of a semi-infinite thin rod by a point source concentrated at an inner point of the rod, is studied. Quadratic and piecewise constant solutions of the problem are derived, and the possibilities of solving appropriate problems of optimal control are indicated. Determining of the parameters of the piecewise constant solution is reduced to a problem of nonlinear programming. Numerical examples are considered.

1. Introduction
Controllability in unbounded domains, i.e., the ability to change the state of an a system defined in an unbounded domain as desired in given finite time, is one of the difficultly implementable features of control systems. However, in investigations of specific control systems, it turns out that, for a given initial state, it is impossible to find a control function providing (generally, approximately) the desired state in a finite time. More often, it is more difficult to determine the set of implementable terminal states rather than the control functions implementing them.

In [1], it is shown that, for arbitrary initial temperature, there do not exist $L^2$-boundary heating regimes $u$ for unbounded three-dimensional bodies providing the null distribution of temperature in the body, i.e., the system

\[ \frac{\partial \Theta(x,t)}{\partial t} - \Delta \Theta(x,t) = 0, \quad x \in \Omega, \quad t \in (0, T), \]

\[ \Theta(x,t) = u(t) \chi_{\partial \Omega_0}(x), \quad (x,y,z) \in \partial \Omega, \quad t \in [0, T], \]

\[ \Theta(x,0) = \Theta_0(x), \quad x \in \Omega, \]

is not exact null-controllable in a given finite time $T$. Here $\Omega \subset \mathbb{R}^3$ is a semi-infinite domain of the body, $\partial \Omega$ is its boundary, $\Theta$ characterizes the heat distribution in the body, and $\chi_{\partial \Omega_0}$ is the characteristic function of $\partial \Omega_0 \subseteq \partial \Omega$. It is important to note that heat is propagating in space with an infinite velocity. We also refer to [2–4] and the references therein.

In [5], a necessary and sufficient condition on the initial and terminal states is derived, for
which there exists an $L^1$-control providing formal exact controllability of

$$\frac{\partial}{\partial x} \left[ N(x) \frac{\partial w(x,t)}{\partial x} \right] - \rho(x) \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad x \in (0, \infty), \quad t \in (0, T),$$

$$w(0,t) = u(t), \quad \lim_{x \to \infty} w(x,t) = 0, \quad t \in [0, T],$$

$$w(x,0) = w_0(x), \quad \frac{\partial w(x,t)}{\partial t} \bigg|_{t=0} = w_1^0(x), \quad x \in [0, \infty),$$

in given finite $T$, for $0 \leq N \in C^1_p[0, \infty)$, $0 < \rho \in C^p_p[0, \infty)$. A numerical analysis allows one to find the corresponding initial and terminal data. In this case, the finiteness of wave speed plays the key role.

For most of the systems defined in unbounded domains, it is generally impossible to provide given terminal state in given finite time exactly, i.e.,

$$\mathbf{x}(T) = \mathbf{x}_T,$$

where $\mathbf{x}$ is the state of the system and $\mathbf{x}_T$ is the desired terminal state, even if the set of admissible controls is large. In other words, the systems defined in unbounded domains are often not exactly controllable. More often, the terminal state is implemented approximately. In such cases, it is required to provide a terminal state, which is in a small neighborhood of the desired state [8, 9]:

$$|\mathbf{x}(T) - \mathbf{x}_T| \leq \epsilon$$

for a given tolerance $\epsilon > 0$. Then, the system is called approximately controllable.

The aim of the paper is the approximate implementation of the desired terminal heat distribution in a semi-infinite insulated rod in given finite time, by means of a heat source concentrated at an internal point of the rod.

2. Statement of the problem

Let a semi-infinite rod of sufficiently small thickness\footnote{It is basically assumed that all points of each cross-section of the rod have the same temperature at any fixed time instant.} be heated by a point source with controllable intensity $u$, bounded by $|u| \leq u_0$, and concentrated at a given point $x = x_0 \in (0, \infty)$ of the rod. The heat propagation in the rod obeys the one-dimensional heat equation

$$\frac{c_p \rho}{\kappa} \frac{\partial \Theta(x,t)}{\partial t} = \frac{\partial^2 \Theta(x,t)}{\partial x^2} + \frac{u(t)}{\kappa} \delta(x-x_0), \quad x > 0, \quad t > 0,$$

(4)

where $\Theta$ is the temperature, $c_p$ is the specific heat capacity, $\rho$ is the density, $\kappa$ is the thermal conductivity of the rod, and $\delta$ is the Dirac delta function.

Let, for simplicity, the $x = 0$ section and infinitely far sections of the rod be insulated, i.e.,

$$\Theta(0,t) = 0, \quad \lim_{x \to \infty} \Theta(x,t) = 0, \quad t \geq 0.$$  

(5)

The initial distribution of temperature in the rod is known:

$$\Theta(x,0) = \Theta_0(x), \quad x \geq 0.$$  

(6)

Our aim is to find a heating regime $u$ such that, in for given finite $T$, the state

$$\Theta(x,T) = \Theta_T(x), \quad x \geq 0,$$

(7)
is implemented. For simplicity, we assume that \( \Theta_0 \) and \( \Theta_T \) are continuous functions in \([0, \infty)\).

We introduce dimensionless variables and functions

\[
\tilde{x} = \frac{x}{x_0}, \quad \tilde{t} = \frac{c_p \rho}{\kappa x_0^2} t, \quad \tilde{\Theta} = \frac{\Theta}{\Theta_0}, \quad \tilde{u} = \frac{x_0}{\Theta_0} u,
\]

where \( \Theta_0 \) is the intensity of the initial temperature distribution in the rod, i.e., \( \Theta_0(x) = \Theta_0(\tilde{x}) \). Note that \( \tilde{T} \) then characterizes the ratio of the control time to the time required for the heat propagation at the distance \( x_0 \) from the source. It is taken into account that \( x_0 \delta(x - x_0) = \delta(\tilde{x} - 1) \). We do not introduce new symbols and simply omit \( \tilde{\cdot} \) over the corresponding characters.

The boundary and initial data are supposed to be consistent, i.e., the conditions

\[
\Theta_0(0) = \Theta_T(0) = 0, \quad \Theta_0(1) = u(0), \quad \Theta_T(1) = u(T), \quad \lim_{x \to \infty} \Theta_0(x) = \lim_{x \to \infty} \Theta_T(x) = 0,
\]

are satisfied.

3. Solution of the problem

Then, the solution of the problem is given by the following theorem.

**Theorem 1.** For distributed controllability of (4)–(6), i.e., for the fulfilment of (7), in a finite \( T \), it is necessary and sufficient that

\[
\int_0^T K(x, T - \tau) u(\tau) \, d\tau = F(x) \quad \text{for all } x \geq 0, \tag{8}
\]

where \( K(x, t) = G(x, 1, t) \),

\[
F(x) = \Theta_T(x) - \int_0^\infty G(x, \xi, T) \Theta_0(\xi) \, d\xi,
\]

\[
G(x, \xi, t) = \frac{1}{\sqrt{4\pi t}} \left\{ \exp \left[ -\frac{(x - \xi)^2}{4t} \right] - \exp \left[ -\frac{(x + \xi)^2}{4t} \right] \right\}.
\]

At this, the control function satisfies

\[
\int_0^T K(1, T - \tau) u(\tau) \, d\tau = F(1), \tag{9}
\]

\[
u(0) = \Theta_0(1), \quad u(T) = \Theta_T(1), \tag{10}
\]

**Proof.** The proof is straightforward. Since the general solution of (4)–(6) is given by [6]

\[
\Theta(x, t) = \int_0^t G(x, 1, t - \tau) u(\tau) \, d\tau + \int_0^\infty G(x, \xi, t) \Theta_0(\xi) \, d\xi, \quad x \geq 0, \quad t \geq 0, \tag{11}
\]

we arrive at (8) evaluating it at \( t = T \). Further, (9) is a consequence of the assumption that the point \( x_0 \) of the rod is heated by the source within \([0, T]\), and (10) are the consistency conditions between the boundary and initial and terminal data.

In general, it is impossible exactly to satisfy condition (7) for all \( x \geq 0 \). Indeed, if a control function satisfying (9), (10) is found, then (8) is satisfied exactly only for \( x = 1 \). Evidently, (8) is satisfied for \( x = 0 \) and as \( x \to \infty \).
It is also evident that the solution of (9), (10) is non-unique. Indeed, it can be represented in quadratic form
\[ u(t) = \Theta_0(1) + \left[ \Theta_T(1) - \Theta_0(1) \right] t + u_1(t(T - t)), \]
where \( u_1 \) is determined by substituting this form into (9) and by satisfying the equality. Another explicit form of the control function is
\[ u(t) = F(1) \frac{L(t)}{K(1, T - t)}, \]
where
\[ L(0) = \frac{\Theta_0(1)}{F(1)} K(1, T), \quad L(T) = \frac{\Theta_T(1)}{F(1)} K(1, 0), \quad \int_0^T L(t) \, dt = 1. \]

The non-uniqueness of the solution allows one to choose the following controls optimizing some criteria: the heating time, the error between the implemented and desired states (in the case of approximate controllability), etc.

4. Piecewise constant solution
The non-uniqueness of solution of (9), (10) allows one to consider piecewise constant intensities [5, 7, 10, 11]:
\[ u(t) = \Theta_0(1) + \sum_{k=1}^n u_k \cdot \theta(t - t_k), \quad t \in [0, T], \]
where \( \theta(t) \) is the Heaviside function, \( u_k, t_k (t_0 = 0, t_n = T) \) are determined by substituting (12) into (9):
\[ \sum_{k=1}^n u_k [1 - K_k(t_k)] = \int_0^\infty G(1, \xi, T) \Theta_0(\xi) \, d\xi, \]
\[ u(T) = \Theta_0(1) + \sum_{k=1}^n u_k = \Theta_T(1), \quad \sum_{k=1}^n |u_k| \leq u_0, \quad 0 \leq t_k < t_{k+1} \leq T, \]
where
\[ K_k(t_k) = \int_{t_{k-1}}^{t_k} K(1, 1 - \tau) \, d\tau, \]
which can be computed exactly. It follows from (13) that \( |\Theta_T(1)| \leq |\Theta_0(1)| + u_0. \)

If we require minimization of some functional like [12]
\[ \max_k |u_k|, \quad \int_0^1 |u(t)| \, dt, \quad \sum_{k=1}^n |u_k|, \]
then the methods of nonlinear programming can be involved to determine \( u_k \) and \( t_k \).

5. Numerical analysis
It is evident that it is practically impossible to satisfy (8) exactly for all \( x \geq 0 \). Therefore, our aim is to establish its approximate fulfillment.

Example 1. Let us begin with the quadratic control \( u(t) = \Theta_0(1) + \left[ \Theta_T(1) - \Theta_0(1) \right] t + u_1(t(T - t)). \) It is given that \( T = 8, \Theta_0(x) = x \exp(-x), \) and \( \Theta_T(x) = x^2 \exp(-0.5x). \) Then
\[ u_1 = \frac{1}{\kappa_1 T - \kappa_2} \left[ F(1) - \Theta_0(1) \kappa_0 - \frac{\Theta_T(1) - \Theta_0(1)}{T} \right] \kappa_1, \quad \kappa_k = \int_0^T K(1, T - \tau) \tau^k \, d\tau. \]
Figure 1 shows the graphs of the implemented and desired terminal states $\Theta(x, T)$ and $\Theta_T(x)$ for $u_1 = -0.086$. It turned out that the maximal distance is $\max_{0 \leq x \leq 35} |\Theta(x, 8) - \Theta_T| \leq 0.005$. For larger $x$, $|\Theta(x, 8) - \Theta_T| \to 0$.

**Example 2.** Now we consider the case where the heating regime is piecewise constant.
Let (12) contain only 3 terms. Then, from (13), (14) we obtain \( u_1 = 0, u_2 = 0.1, u_2 = 0.138, \) and \( t_2 = 1 (t_1 = 0, t_3 = 8). \)

Figure 2 presents the graphs of the implemented state \( \Theta(x, T) \) and the desired state \( \Theta_T(x) \). Evidently, the maximal distance is \( \max_{0 \leq x \leq 35} |\Theta(x, 8) - \Theta_T| \leq 0.005 \). For larger \( x \), \( |\Theta(x, 8) - \Theta_T| \to 0. \)

It is evident that, in the case of quadratic control, the two states have significant gap for \( 0 \leq x \leq 2 \) and \( 10 \leq x \leq 15 \); however, in the case of piecewise constant control, they differ only for \( 10 \leq x \leq 15 \). Adding more terms into (12) or considering minimization problem

\[
\max_{x \geq 0} |\Theta(x, T) - \Theta_T| \rightarrow \min
\]

allow us to decrease the distance between the two states.

**Example 3.** It turns out that there is another possibility for improving the approximation rate. It is done by considering the case

\[
u_s(t) = u(t)\chi_{[0,t_0]}(t),
\]

where the source is switched off at \( 0 < t_0 < T \). The improvement of approximation is shown in figure 3 for the quadratic control with \( u_1 = 0.41 \) and \( t_0 = 2 \). In this case, \( \max_{0 \leq x \leq 35} |\Theta(x, 8) - \Theta_T| \leq 10^{-4}. \)
Conclusions
Under some assumptions, it is shown that sufficiently thin semi-infinite rod can be heated from some initial temperature distribution to some final one using a point source of heat. By the method of Green’s function, the problem of finding a suitable heating regime is reduced to the problem of solving an integral equation with a singular kernel. Several regular solutions satisfying necessary restrictions are derived. A piecewise constant solution of the integral equation whose the parameters are determined by solving nonlinear programming problem with constraints of equality and inequality type is obtained. The non-uniqueness of the solution of the integral equation allows considering the optimization problems.

The results can be generalized to other PDEs, coupled systems of differential equations, etc., using the Green’s function of which is known or can be easily found.

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