Harnack inequalities and Bounds for Densities of Stochastic Processes

Gennaro Cibelli ∗ Sergio Polidoro †

February 6, 2017

Dedicated to Valentin Konakov in occasion of his 70th birthday.

Abstract

We consider possibly degenerate parabolic operators in the form

\[ L = \sum_{k=1}^{m} X_k^2 + X_0 - \partial_t, \]

that are naturally associated to a suitable family of stochastic differential equations, and satisfying the Hörmander condition. Note that, under this assumption, the operators in the form \( L \) have a smooth fundamental solution that agrees with the density of the corresponding stochastic process. We describe a method based on Harnack inequalities and on the construction of Harnack chains to prove lower bounds for the fundamental solution. We also briefly discuss PDE and SDE methods to prove analogous upper bounds. We eventually give a list of meaningful examples of operators to which the method applies.

1 Introduction

Let \( (W_t)_{t \geq 0} \) denote an \( m \)-dimensional Brownian motion, \( W_t = (W_t^1, \ldots, W_t^m) \) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). We consider a collection of space-time functions \((\sigma_{ij})_{(i,j) \in \{1, \ldots, N\} \times \{1, \ldots, m\}}, (b_i)_{i \in \{1, \ldots, N\}}\) such that the following SDE

\[ dZ_i^t = \sum_{j=1}^{m} \sigma_{ij}(Z_t, t) \circ dW_j^t + b_i(Z_t, t)dt, \quad i = 1, \ldots, N, \quad t \geq 0 \quad (1.1) \]

is well posed at least in the weak sense. Here \( \circ dW_t \) stands for the Stratonovich integral. We denote by \( Z_{t}^{x_0} \) the solution of the SDE \((1.1)\) with initial condition \( Z_{0}^{x_0} = x_0 \). The equation \((1.1)\)

\*Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/b, 41125 Modena (Italy). E-mail: gennaro.cibelli@unimore.it

†Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/b, 41125 Modena (Italy). E-mail: sergio.polidoro@unimore.it
is associated to the Kolmogorov operator
\[ \mathcal{L} = \sum_{i=1}^{m} X_i^2 + X_0 - \partial_t, \]
where
\[ X_i(x, t) = \frac{1}{\sqrt{2}} \sum_{j=1}^{m} \sigma_{ij}(x, t) \partial_{x_j}, \quad i = 1, \ldots, m, \quad X_0(x, t) = \sum_{i=1}^{N} b_i(x, t) \partial_{x_i}. \quad (1.2) \]

In this note we describe a general method to prove upper and lower bounds for the fundamental solution of \( \mathcal{L} \). Specifically, we say that a non-negative function \( \Gamma(x, t; y, s) \) defined for \( x, y \in \mathbb{R}^N \) and \( t > s \), is a fundamental solution for \( \mathcal{L} \) if:

i) in the weak sense, \( \mathcal{L} \Gamma(\cdot, \cdot; y, s) = 0 \) in \( s, +\infty \times \mathbb{R}^N \) and \( \mathcal{L}^* \Gamma(t, \cdot; \cdot) = 0 \) in \( ]-\infty, t[ \times \mathbb{R}^N \) where \( \mathcal{L}^* \) denotes the formal adjoint operator of \( \mathcal{L} \);

ii) for any bounded function \( \varphi \in C(\mathbb{R}^N) \) and \( x, y \in \mathbb{R}^N \), we have
\[ \lim_{(x, t) \to (y, s)} u(x, t) = \varphi(y), \quad \lim_{(y, s) \to (x, t)} v(y, s) = \varphi(x), \quad (1.3) \]
where
\[ u(x, t) := \int_{\mathbb{R}^N} \Gamma(x, t; y, s) \varphi(y) dy, \quad v(y, s) := \int_{\mathbb{R}^N} \Gamma(x, t; y, s) \varphi(x) dx. \quad (1.4) \]

Note that the functions in (1.4) are weak solutions of the following backward and forward Cauchy problems:
\[ \begin{cases} \mathcal{L} u(t, x) = 0, & (x, t) \in [s, +\infty) \times \mathbb{R}^N, \\ u(x, s) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \quad \begin{cases} \mathcal{L}^* v(y, s) = 0, & (y, s) \in ]-\infty, t[ \times \mathbb{R}^N, \\ v(y, t) = \varphi(y), & y \in \mathbb{R}^N. \end{cases} \]

We introduce the \( N \times N \) matrix \( A(x, t) = (a_{ij}(x, t))_{i,j=1,\ldots,N} \) whose elements are
\[ a_{ij}(x, t) = \frac{1}{2} \sum_{k=1}^{m} \sigma_{ik}(x, t) \sigma_{jk}(x, t), \quad i, j = 1, \ldots, N, \]
and we note that
\[ \langle A(x, t) \xi, \xi \rangle = \frac{1}{2} \| \sigma(t, x) \xi \|_2^2 \geq 0. \quad \text{for every } \xi \in \mathbb{R}^N. \]
If the smallest eigenvalue of \( A(t, x) \) is uniformly positive we say that the operator \( \mathcal{L} \) is uniformly parabolic.

A keystone result in the theory of parabolic partial differential equations reads as follows. Assume that there exist two positive constants \( \lambda, \Lambda \) such that
\[ \lambda |\xi|^2 \leq \langle A(x, t) \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \text{for every } (x, t) \in \mathbb{R}^N \times ]0, T[, \text{ and } \xi \in \mathbb{R}^N. \quad (1.5) \]
If $\Gamma = \Gamma(x,t,\xi,\tau)$ denotes the fundamental solution of the PDE
\[
\frac{\partial}{\partial t} u(x,t) = \sum_{i,j=1}^{N} \partial_{x_i} \left( a_{ij}(x,t) \partial_{x_j} u(x,t) \right), \quad (x,t) \in \mathbb{R}^N \times ]0,T[,
\]
then there exist positive constants $c^-,C^-,c^+,C^+$ only depending on $N,\Lambda,\lambda$ such that
\[
\frac{c^-}{(t-\tau)^{N/2}} \exp \left( -C^- \frac{|x-\xi|^2}{t-\tau} \right) \leq \Gamma(x,t;\xi,\tau) \leq \frac{C^+}{(t-\tau)^{N/2}} \exp \left( -c^+ \frac{|x-\xi|^2}{t-\tau} \right),
\]
for every $(x,t), (\xi,\tau) \in \mathbb{R}^N \times ]0,T[$ with $\tau < t$. We emphasize that the constants in (1.7) do not depend on $T$. This upper bound has been proved by Aronson [1] for operators with bounded measurable coefficients $a_{ij}$'s, while the lower bound has been proved by Moser [30, 31]. The results by Aronson and by Moser improve the earliest estimates given by Nash in his seminal work [32]. We also refer to the article of Krylov and Safonov [24] for non-divergence form operators.

The results described above have been extended by several authors to possibly degenerate operators in the form
\[
\mathcal{L} := \sum_{k=1}^{m} X_k^2 + Y, \quad Y := X_0 - \partial_t,
\]
where $X_0, X_1, \ldots, X_m$ are smooth vector fields on $\mathbb{R}^{N+1}$, that is
\[
X_i(x,t) = \sum_{j=1}^{N} c_{i,j}(x,t) \partial_{x_j}, \quad i = 0, \ldots, m.
\]
for some smooth functions $c_{i,j}$'s. In particular, upper bounds have been proved by a PDE approach that goes back to Aronson’s work [1], or by an approach based on Lyapunov functions (see [28] and the references therein). Several authors prove bounds analogous to (1.7) in the framework of stochastic processes. We refer to the works of Malliavin [26], Kusuoka and Stroock [25], where a general method to prove upper bounds for density is introduced and to the work of Ben Arous and Léandre [6], where the Malliavin Calculus is further developed. We also refer to the monograph of Nualart [33] for a comprehensive presentation of this subject.

In general, lower bounds have been proved by following the idea introduced by Moser in [30]. In this note we briefly describe this method for uniformly parabolic partial differential equations, then we give an overview of more recent articles where it has been adapted to the study of degenerate parabolic equations in the form (1.8). This idea is also used in the works where lower bounds are proved by probabilistic methods. We refer to Kohatsu-Higa [22], Bally [3], Bally and Kohatsu-Higa [4].

We now give a list of examples of operators considered in this note. Each one of them is the prototype of a wide family of differential operators.

- **Heat operator on the Heisenberg group** $\mathcal{L} = X_1^2 + X_2^2 - \partial_t$, where
  \[
  X_1 = \partial_x - \frac{1}{2} y \partial_w, \quad X_2 = \partial_y + \frac{1}{2} x \partial_w.
  \]
Note that $\mathcal{L}$ acts on the variable $(x, y, w, t) \in \mathbb{R}^4$, and writes in the form (1.8) with $X_0 = 0$. The degenerate elliptic operator $\Delta_H = X_1^2 + X_2^2$ is said sub-Laplacian on the Heisenberg group.

- Kolmogorov Operator $\mathcal{L} = \partial_{xx} + x \partial_y - \partial_t$, $(x, y, t) \in \mathbb{R}^3$. In this case $\mathcal{L} = X^2 + Y$ with $X = \partial_x$, $Y = x \partial_y - \partial_t$.

- More Degenerate Kolmogorov Operators $\mathcal{L} = \partial_{xx} + x^2 \partial_y - \partial_t$, $(x, y, t) \in \mathbb{R}^3$. In this case $\mathcal{L} = X^2 + Y$ with $X = \partial_x$, $Y = x^2 \partial_y - \partial_t$.

- Asian Option Operator $\mathcal{L} = x^2 \partial_{xx} + x \partial_x + x \partial_y - \partial_t$, $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2$. In this case $\mathcal{L} = X^2 + Y$ with $X = x \partial_x$, $Y = x \partial_y - \partial_t$.

All the operators in the above list are strongly degenerate, since the smallest eigenvalue of the characteristic form is zero for all the above examples. In general, operators in the form (1.8) cannot be uniformly parabolic if $m < N$. On the other hand, all the examples do satisfy the following condition:

**Hypothesis [H]** $\mathcal{L} = \sum_{k=1}^m X_k^2 + Y$ satisfies the Hörmander condition if

$$\text{rank } (\text{Lie}\{X_1, \ldots, X_m, Y\}(x, t)) = N + 1, \quad \text{for every } (x, t) \in \mathbb{R}^{N+1}.$$

In the sequel we only consider operators $\mathcal{L}$ satisfying the Hörmander condition. It is known that, for this family of operators, the law of the stochastic process (1.1) is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$, and that its density is smooth. Moreover, for every pairs $(\xi, \tau)$, $(x, t) \in \mathbb{R}^N \times [0, T]$ with $\tau > t$, the density $p(\xi, \tau; x, t)$ is linked with the fundamental solution $\Gamma$ of $\mathcal{L}$. Precisely, if $p$ denotes the density of the process

$$\left\{ \begin{array}{l}
  dZ_i^t = \sum_{i, j=1}^m \sigma_{ij}(Z_s, T - s) \circ dW_s + b_i(Z_s, T - s) ds, \quad i = 1, \ldots, N, \quad t < s \leq T; \\
  Z_i^t = x_i, \\
\end{array} \right.$$

then $\Gamma$ is defined by the relation

$$\Gamma(x, t; \xi, \tau) = p(\xi, T - \tau; x, T - t).$$

It is known that the regularity properties of the operators satisfying the Hörmander condition are related to a Lie group structure that replaces the usual Euclidean one. In the proof of the lower bounds for positive solutions the geometric aspects of this non Euclidean structure will be explicitly used. To make the exposition clear, in Section 2 we recall the method used by Moser in [30] to prove the lower bound in (1.7) for uniformly parabolic operators. In Section 3 we describe how the method outlined in Section 2 is adapted to the degenerate ones, satisfying the Hörmander condition [H]. The remaining Sections 4, 5, 6 and 7 are devoted to the examples listed above.

## 2 Uniformly parabolic equations

In this section we describe the method introduced by Moser [30] to prove the lower bound (1.7) of the fundamental solution for uniformly parabolic equations. The main ingredient of the method
is the parabolic Harnack inequality, first proved by Hadamard [17] and, independently, by Pini [34] in 1954 for the heat equation, then by Moser [30, 31] for uniformly parabolic equations in divergence form (1.6). Its statement requires some notation (see Fig. 1). Let

$$Q_r(x,t) = B(x,r) \times [t-r^2,t]$$

denote the parabolic cylinder whose upper basis is centered at $(x,t)$. Let $\alpha, \beta, \gamma, \delta \in [0,1]$ be given constants, with $\alpha < \beta < \gamma < 1$,

$$Q^-_r(x,t) = B(x,\delta r) \times [t-\gamma r^2,t-\beta r^2] \quad Q^+_r(x,t) = B(x,\delta r) \times [t-\alpha r^2,t].$$

Theorem 2.1 (Parabolic Harnack inequality) Let $Q_r(x,t) \subset \mathbb{R}^{N+1}$, and let $\alpha, \beta, \gamma, \delta \in [0,1]$ be given constants, with $\alpha < \beta < \gamma < 1$. Then there exists $C = C(\alpha, \beta, \gamma, \delta, \lambda, \Lambda, N)$ such that

$$\sup_{Q^-_r(x,t)} u \leq C \inf_{Q^+_r(x,t)} u$$

for every $u : Q_r(x,t) \to \mathbb{R}, u \geq 0$, satisfying (1.6). Here $\lambda, \Lambda$ are the constants in (1.3).

Remark 2.2 Note that $C$ does not depend on the point $(x,t)$ and on $r$, then the Harnack inequality is invariant with respect to the Euclidean translation $(x,t) \mapsto (x+x_0,t+t_0)$, and to the parabolic dilation $(x,t) \mapsto (rx,r^2t)$. For this reason, the above statement is often referred to as invariant Harnack inequality.

In the sequel we will use the following version of the parabolic Harnack inequality (see Fig. 2). For any given $c \in [0,1]$ we denote by

$$P_r(x,t) = \{(y,s) \in Q_r(x,t) \mid 0 < t-s \leq cr^2 < t, |y-x|^2 \leq t-s\}.$$
Corollary 2.3 Let $Q_r(x,t) \subset \mathbb{R}^{N+1}$, and let $c \in [0,1]$ be a given constant. Then there exists $C = C(c,\lambda,\Lambda,N)$ such that
\[
\sup_{P_r(x,t)} u \leq Cu(x,t)
\]
for every $u : Q_r(x,t) \to \mathbb{R}, u \geq 0$, satisfying (1.6). Here $\lambda,\Lambda$ are the constants in (1.5).

Proof. For every positive $\rho$ we denote
\[
S_\rho(x,t) = B(x,\rho) \times \{t - \rho^2\}.
\]
Let $\alpha,\beta,\gamma \in [0,1]$ be such that $\alpha < \beta \leq c \leq \gamma < 1$, and let $\delta = \sqrt{c}$. Then, for every $\rho \in [0,r]$ we have that $u$ is a non-negative solution of (1.6) in the domain $Q_\rho(x,t)$. Since $S_\rho(x,t) \subset Q_\rho^c(x,t)$, from Theorem 2.1 we obtain
\[
\sup_{S_\rho(x,t)} u \leq \sup_{Q_\rho^c(x,t)} u \leq \inf_{Q_\rho^c(x,t)} u \leq Cu(x,t),
\]
and the conclusion follows from the fact that $P_r(x,t) = \bigcup_{0 < \rho \leq r} S_\rho(x,t)$. □

With Corollary 2.3 in hand, we can easily obtain the following non local Harnack inequality, first proved by Moser (Theorem 2 in [30]). We also refer to Aronson & Serrin [2] for more general uniformly parabolic differential operators.

Theorem 2.4 Let $u : \mathbb{R}^N \times ]0,T[\to \mathbb{R}$ be a non-negative solution of the parabolic equation (1.6). Then there exists a positive constant $C = C(c,\lambda,\Lambda,N)$ such that
\[
u(x,t) \leq C^{1+\frac{|x|}{t_0-t}} u(x_0,t_0),
\]
for every $(x_0,t_0), (x,t) \in \mathbb{R}^N \times ]0,T[$ with $t_0 - t < ct_0$.

Proof. Let $(x_0,t_0), (x,t) \in \mathbb{R}^N \times ]0,T[$, with $t_0 - t < ct_0$, choose $r = \sqrt{t}$ and note that the cylinder $Q_r(x_0,t_0)$ is contained in $\mathbb{R}^N \times ]0,T[$. If $(x,t) \in P_r(x_0,t_0)$ we simply apply Corollary 2.3 and the proof is complete. If otherwise $(x,t) \notin P_r(x_0,t_0)$, we consider the segment whose end points are $(x_0,t_0)$ and $(x,t)$, and denote by $(x_1,t_1)$ the point where it intersects the boundary of $P_r(x_0,t_0)$. Note that $t_1 \geq t > (1-c)t_0$, then $(x_1,t_1)$ belongs to the lateral part of the boundary of $P_r(x_0,t_0)$. By Corollary 2.3 we have
\[
u(x_1,t_1) \leq Cu(x_0,t_0).
\]
We then iterate the argument. We define a finite sequence $(x_j,t_j)$, with $j = 2,\ldots,k$ such that $(x_j,t_j)$ belonging to the boundary of $P_r(x_{j-1},t_{j-1})$ for $j = 2,\ldots,k$, and $(x,t) \in P_r(x_k,t_k)$ (see Fig. 3). By applying $k$ times Corollary 2.3 we then find
\[
u(x,t) \leq Cu(x_k,t_k) \leq C^2 u(x_{k-1},t_{k-1}) \leq \cdots \leq C^{k+1} u(x_0,t_0).
\]
To conclude the proof it is sufficient to note that the integer $k$ only depends on the slope of the line connecting $(x_0, t_0)$ to $(x, t)$ and that a simple computation gives $k < \frac{|x_0 - x|^2}{t_0 - t}$.

The set $\{(x_0, t_0), (x_1, t_1), \ldots (x_k, t_k), (x, t)\}$ appearing in the above proof is often referred to as Harnack chain. By using the following property of the fundamental solution $\Gamma$ of the differential operator appearing in (1.6)

$$\Gamma(0, t) \geq C t^{N/2}, \quad \text{for every } t > 0,$$

(2.10)

for some positive constant $C = C(\lambda, \Lambda, N)$. We refer to Nash [32] and to Fabes-Strook [16] Lemma 2.6 for a derivation of (2.10). By choosing $\epsilon = \frac{1}{2}$ in Theorem 2.4, we conclude that there exist two positive constants $C^-, c^-$ such that

$$\Gamma(x, t, y, s) \geq \frac{C^-}{(t - s)^{N/2}} \exp \left( -c^\epsilon \frac{|x - y|^2}{t - s} \right),$$

for every $(x, t), (y, s) \in \mathbb{R}^{N+1}$ with $0 < s < t < T$.

We explicitly note that the method described above also applies to non-divergence uniformly parabolic operators, if we rely on the Harnack inequality proved by Krylov and Safonov [24]. In this setting the inequality (2.10) holds for $t$ belonging to any bounded interval $]0, T[$ and the constant $C$ may depend on $T$. We refer to the manuscript of Konakov [29] for the derivation of (2.10) by using the a parametrix expansion and to the monograph of Bass [5] for uniformly parabolic operators with bounded measurable coefficients.

**Remark 2.5** Before considering degenerate parabolic operators, we point out that the method used in the proof of Theorem 2.4 only relies on the following two ingredients.

i) The invariance with respect to the Euclidean translation and to the parabolic dilation $(x, t) \mapsto (x_0 + \rho x, t_0 + \rho^2 t)$ are the properties that allows us to obtain Corollary 2.3 from Theorem 2.1.

ii) Segments are very simple supports for the construction of Harnack chains. In the study of degenerate parabolic operators a more sophisticated construction will be needed.
3 Degenerate hypoelliptic operators

Consider a linear second order differential operator in the form

\[ L = \sum_{k=1}^{m} X_k^2 + X_0 - \partial_t. \]

satisfying the Hörmander condition \([H]\). We introduce a definition based on the vector fields \(X_1, ..., X_m, Y\).

**Definition 3.1** We say that \(\gamma\) is an \(L\)-admissible path starting from \(z_0 \in \mathbb{R}^{N+1}\) if it is an absolutely continuous solution of the following ODE

\[ \dot{\gamma}(\tau) = \sum_{k=1}^{m} \omega_k(\tau) X_k(\gamma(\tau)) + Y(\gamma(\tau)), \]

\(\gamma(0) = z_0\).

with \(\omega_1, ..., \omega_m \in L^1([0, T])\).

Let \(\Omega\) be an open subset of \(\mathbb{R}^{N+1}\) and \(z_0 \in \Omega\). The attainable set of \(z_0\) in \(\Omega\) is

\[ \mathcal{A}_{z_0}(\Omega) = \{ z \in \Omega \mid \text{there exists an } L\text{-admissible path } \gamma \text{ such that } \}
\]

\[ \gamma(0) = z_0, \gamma(T) = z \text{ and } \gamma(\tau) \in \Omega \text{ for } 0 \leq \tau \leq T \}. \]

The following version of the Harnack inequality is based on the definition of attainable set. It has been introduced in \([10, 11]\) and in its general form in \([21]\) for operators in the form \([13]\).

**Theorem 3.2** Let \(u\) be a non negative solution of \(Lu = 0\) in some bounded open set \(\Omega \subset \mathbb{R}^{N+1}\), and let \(z_0 \in \Omega\). Suppose that \(\text{Int} \left( \mathcal{A}_{z_0}(\Omega) \right) \neq \emptyset\). Then, for every compact set \(K \subset \text{Int} \left( \mathcal{A}_{z_0}(\Omega) \right)\) there exists a positive constant \(C_K\), only depending on \(\Omega, K, z_0\) and \(L\), such that

\[ \sup_{K} u(z) \leq C_K u(z_0). \]

If the operator \(L\) is also invariant with respect to suitable non Euclidean translations and dilations, then Theorem 3.2 restores an invariant Harnack inequality useful for the construction of Harnack chains.

**Hypothesis [G1]** There exists a Lie group \(G = (\mathbb{R}^{N+1}, \circ)\) such that \(X_1, ..., X_m, Y\) are left invariant on \(G\), i.e.: given \(\xi \in \mathbb{R}^{N+1}\) and denoting by \(\ell_\xi(z) = \xi \circ z\), the left translation of \(z \in \mathbb{R}^{N+1}\) it holds

\[ X_i(u(\ell_\xi(z))) = (X_i u)(\ell_\xi(z)), \quad i = 1, ..., m, \]

\[ Y(u(\ell_\xi(z))) = (Yu)(\ell_\xi(z)), \]

for every smooth function \(u\).

As we will see in the next sections, all the examples listed in the Introduction do satisfy the above assumption, that replaces the usual invariance with respect to the Euclidean translation. For some operators \(L\) considered in this note, the vector fields \(X_1, ..., X_m, Y\) are also invariant.
with respect to a rescaling property \((\delta_\lambda)_{\lambda>0}\) of the Lie group \(G\), which replaces the multiplication by a positive scalar in a vector space.

**Hypothesis [G2]** There exists a dilation \((\delta_\lambda)_{\lambda>0}\) on the Lie group \(G\) such that the vector fields \(X_1, \ldots, X_m\) are \(\delta_\lambda\)-homogeneous of degree one and \(Y\) is \(\delta_\lambda\)-homogeneous of degree two. i.e.:

\[
X_i(u(\delta_\lambda(z))) = \lambda(X_i(u))(\delta_\lambda(z)), \quad i = 1, \ldots, m,
\]

\[
Y(u(\delta_\lambda(z))) = \lambda^2(Y(u))(\delta_\lambda(z)),
\]

for every smooth function \(u\).

When both of assumptions [G1] and [G2] are satisfied, we say that 

\[G = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda>0})\]

is a homogeneous Lie group and the operator \(\mathcal{L}\) is invariant with respect to the left translations of \(G\), and homogeneous of degree 2 with respect to the dilation of \(G\). In this case we easily obtain from Theorem 3.2 an invariant Harnack inequality analogous to Corollary 2.3. Consider any bounded open set \(\Omega \subset \mathbb{R}^{N+1}\) with \(0 \in \Omega\) and suppose that it is star-shaped with respect to \((\delta_\lambda)_{\lambda>0}\), that is \(\delta_r(\Omega) := \{\delta_r(z) \mid z \in \Omega\} \subset \Omega\), for every \(r \in [0, 1]\).

If \(\text{Int} \left( \overline{\mathcal{A}_0(\Omega)} \right) \neq \emptyset\), we choose any compact set \(K \subset \text{Int} \left( \overline{\mathcal{A}_0(\Omega)} \right)\). For every \(r > 0\) and \(z_0 \in \mathbb{R}^{N+1}\) we set

\[\Omega_r(z_0) = z_0 \circ \delta_r(\Omega) := \{z_0 \circ \delta_r(z) \mid z \in \Omega\}.
\]

Note that we also have \(z_0 \circ \delta_r(K) \subset \text{Int} \left( \overline{\mathcal{A}_{z_0}(\Omega_r(z_0))} \right)\) for every \(\rho \in [0, r]\), since \(\Omega\) is star-shaped. We define

\[\mathcal{P}_r(z_0) = \bigcup_{0 < \rho \leq r} z_0 \circ \delta_r(K).
\]

**Theorem 3.3** Let \(\mathcal{L}\) be an operator in the form (1.8) satisfying assumptions [G1] and [G2] and let \(\Omega_r(z_0)\) as above. Suppose that \(\text{Int} \left( \overline{\mathcal{A}_{z_0}(\Omega_r(z_0))} \right) \neq \emptyset\), then

\[
\sup_{\mathcal{P}_r(z_0)} u(x,t) \leq C_K u(z_0)
\]

for every positive solution \(u\) of \(\mathcal{L}u = 0\) in \(\Omega_r(z_0)\). Here \(C_K\) is the same constant appearing in Theorem 3.2.

Theorem 3.3 is the Harnack inequality that replaces Corollary 2.3 in the non Euclidean setting that is natural for the study of degenerate operators \(\mathcal{L}\). In accordance with Remark 2.5, this is the first ingredient for the construction of Harnack chains. It turns out that the second ingredient is the \(\mathcal{L}\)-admissible path, which is the natural substitute of the segment used in the Euclidean setting. To replicate the construction made in the proof of Theorem 2.4 we only need to choose \(\gamma\), with \(\gamma(0) = (x_0, t_0)\), and \(\mathcal{P}_{(x_0, t_0)}\) with the following property:

there exists \(s_0 \in [0, t_0 - t]\) such that \(\gamma(s) \in \mathcal{P}_{(x_0, t_0)}\) for \(s \in [0, s_0]\). \hspace{1cm} (3.11)
All the examples in this note satisfy (3.11). Thus we have what we need to construct a Harnack chain \( \{(x_0, t_0), (x_1, t_1), \ldots (x_k, t_k), (x, t)\} \) with starting point at \((x_0, t_0)\) and end point at \((x, t)\).

In order to find an accurate bound of the positive solutions of \( Lu = 0 \) we need to control the length \( k \) of the Harnack chain. It is possible to prove that there exists a positive constant \( h \) such that, if we construct the Harnack chain by using the \( \mathcal{L} \)-admissible path \( \gamma \) as in Definition 3.1 with \( z_0 = (x_0, t_0) \) and \( z = (x, t) \), then \( T = t_0 - t \) and we have

\[
k \leq \frac{1}{h} \Phi(\omega) + 1, \quad \Phi(\omega) := \int_0^{t_0 - t} \| \omega(s) \|^2 ds. \tag{3.12}
\]

In the sequel we will refer to the integral appearing in (3.12) as the cost of the path \( \gamma \) associated to the control \((\omega_1, \ldots, \omega_m)\). We then conclude that there exist three positive constants \( \theta, h \) and \( M \), with \( \theta < 1 \) and \( M > 1 \), only depending on the operator \( \mathcal{L} \) such that for every positive solution \( u \) of \( Lu = 0 \) it holds

\[
u(x, t) \leq M^{1 + \frac{1}{h} \Psi(x_0, t_0; x, t)} u(x_0, t_0), \tag{3.13}
\]

where the infimum is taken in the set of all the \( \mathcal{L} \)-admissible paths \( \gamma \) steering \((x_0, t_0)\) to \((x, t)\), and satisfying (3.11). We summarize this construction in the following general statement. 

**Let** \( \mathcal{L} \) **be an operator in the form (1.8) satisfying conditions \([H], [G1] \) and \([G2] \), and assume that there is a positive \( r \) and an open star-shaped set \( \Omega \) with \( 0 \in \Omega \) such that \( \text{Int}(\mathcal{A}_0(\Omega_r(0))) \neq \emptyset \). **Moreover**, if all the \( \mathcal{L} \)-admissible paths \( \gamma \) steering \((x_0, t_0)\) to \((x, t)\) satisfy (3.11), then there exist three positive constants \( \theta, h \) and \( M \), with \( \theta < 1 \) and \( M > 1 \), only depending on the operator \( \mathcal{L} \) such that the following property holds.

Let \((x_0, t_0), (x, t) \in \mathbb{R}^N \times [0, T]\) of \( \mathcal{L} u = 0 \) it holds

\[
u(x, t) \leq M^{1 + \frac{1}{h} \Psi(x_0, t_0; x, t)} u(x_0, t_0). \tag{3.15}
\]

Inequality (3.15) is the main step in the proof of our lower bound for the fundamental solution. All the examples considered in this note satisfy conditions \([H], [G1] \). Some examples also satisfy \([G2] \), some examples do not. However, in this case, a scale invariant Harnack inequality still holds true, then the method provides us with a lower bound of the fundamental solution.

**4 Degenerate hypoelliptic operators on homogeneous groups**

The Heat operator on the Heisenberg group

\[
\mathcal{L} = X_1^2 + X_2^2 - \partial_t
\]
where
\[ X_1 = \partial_x - \frac{1}{2} y \partial_w, \quad X_2 = \partial_y + \frac{1}{2} x \partial_w \]
are vector fields acting on the variable \((x, y, w, t) \in \mathbb{R}^4\), is the simplest example of degenerate operator built by a sub-Laplacian on a stratified Lie group. The vector fields \(X_1, X_2\) are invariant with respect to the left translation on the Heisenberg group on \(\mathbb{R}^3\), whose operation is defined as
\[ (x_0, y_0, w_0) \circ (x, y, w) = (x_0 + x, y_0 + y, w_0 + w + \frac{1}{2}(x_0 y - y_0 x)). \]
The above operation is extended to \(\mathbb{R}^4\) by setting
\[ (x_0, y_0, w_0, t_0) \circ (x, y, w, t) = (x_0 + x, y_0 + y, w_0 + w + \frac{1}{2}(x_0 y - y_0 x), t_0 + t). \]
Moreover \(\mathcal{L}\) is invariant with respect to the following dilation
\[ \delta_r(x, y, w, t) = (rx, ry, r^2 w, r^2 t), \]
then the hypotheses \([G1]\) and \([G2]\) are fulfilled by \(\mathcal{L}\). Furthermore, it satisfies the following property.

\[ \text{[C]} \quad \text{For every } x_0, x \in \mathbb{R}^N, \text{ and for every positive } T \text{ there exists an absolutely continuous path } \gamma_0: [0, T] \to \mathbb{R}^N \text{ such that} \]
\[ \gamma_0(t) = \sum_{k=1}^{m} \omega_k(\tau) X_k(\gamma_0(\tau)), \quad \gamma_0(0) = x_0, \quad \gamma_0(T) = x. \] (4.16)

Note that, for operators \(\mathcal{L}\) in the form (1.8) with \(X_0 = 0\), condition \([C]\) is equivalent to the strong Hörmander condition
\[ \text{rank Lie}\{X_1, \ldots, X_m\}(x) = N, \quad \forall x \in \mathbb{R}^N. \]
Moreover, for every \(\Omega \subset \mathbb{R}^{N+1}\) and for every \((x_0, t_0) \in \Omega\) there exist a positive \(\varepsilon\) and a neighborhood \(U\) of \(x_0\) such that \(U \times ]t_0, 0 - \varepsilon[ \subset \mathcal{A}(x_0, t_0)(\Omega)\). This particular geometric property of the attainable set implies that an invariant Harnack inequality analogous to the standard parabolic one holds for this operator. The only difference is that the Euclidean translation and the parabolic dilations are replaced by the operations used to satisfy hypotheses \([G1]\) and \([G2]\).

In conclusion, the hypotheses we need to prove (3.15) are satisfied by the heat operator on the parabolic one holds for this operator. The only difference is that the Euclidean translation and radius \(r\) such that \(U \times ]t_0, 0 - \varepsilon[ \subset \mathcal{A}(x_0, t_0)(\Omega)\). This particular geometric property of the attainable set implies that an invariant Harnack inequality analogous to the standard parabolic one holds for this operator. The only difference is that the Euclidean translation and the parabolic dilations are replaced by the operations used to satisfy hypotheses \([G1]\) and \([G2]\).

In conclusion, the hypotheses we need to prove are satisfied by the heat operator on the Heisenberg group. In particular, this method leads us to the lower bound of the following version of (1.7): there exist positive constants \(c^-, C^-, c^+, C^+\) such that
\[ \frac{\varepsilon^-}{\sqrt{|\mathcal{B}_{\varepsilon^-}(x)|}} \exp\left(-C^- \frac{d_{\text{CC}}(x, \xi)^2}{\varepsilon^-}\right) \leq \Gamma(x, t, \xi, \tau) \leq \frac{C^+}{\sqrt{|\mathcal{B}_{\varepsilon^+}(x)|}} \exp\left(-c^+ \frac{d_{\text{CC}}(x, \xi)^2}{\varepsilon^+}\right), \] (4.17)
where \(d_{\text{CC}}\) denotes the Carnot-Carathéodory distance
\[ d_{\text{CC}}(x_0, x) = \inf\{\ell(\gamma_0) \mid \gamma_0 \text{ is as in (4.16)}\}, \quad \ell(\gamma) := \int_0^T \|\omega(s)\|ds. \]
and \(|\mathcal{B}_r(x)|\) is the volume of the metric ball with center at \(x\) and radius \(r\). To make more precise the analogy between (1.7) and (4.17), we recall that if \(\mathbb{H}\) is a homogeneous Lie group on \(\mathbb{R}^N\), then
\[ |\mathcal{B}_r(x)| = r^Q|\mathcal{B}_1(0)|, \]
where $Q$ is an integer called \textit{homogeneous dimension} of $\mathbb{H}$. We recall that the upper bound was proved by Davies in \cite{13}, and the upper and lower bounds are due to Jerison and Sánchez-Calle \cite{19} and to Varopoulos, Saloff-Coste and Coulhon \cite{42}. Note that $\Psi(x_0,t_0) = \frac{d_{CC}(x_0,x)^2}{t_0-t}$.

Indeed, if we consider the path $\gamma(s) = (\gamma_0(s), t_0-s)$ with $0 \leq s \leq t_0-t$, then by the Cauchy-Schwartz inequality, we obtain $\ell(\gamma_0) \leq \sqrt{\Phi(\omega)}\sqrt{t_0-t}$. Moreover the equality occurs only if the norm of the control $\omega$ is constant, that is $\ell(\gamma_0) = \sqrt{\Phi(\omega)}\sqrt{t_0-t} \iff (\omega_1^2 + ... + \omega_m^2)(s) = \frac{\Phi(\omega)}{t_0-t}$ for every $s \in [0,t_0-t]$.

We refer to the article \cite{8} for the study of a more general class of operator satisfying [G1], [G2] and [C], that includes heat operators on Carnot groups and also operators $L$ with $X_0 \neq 0$. We also recall that in the article \cite{12} the analogous upper bound has been proved by using a PDE method combined with the Optimal Control Theory.

\section{Degenerate Kolmogorov equations}

The simplest degenerate example of degenerate Kolmogorov operator is

$$L := \partial_x^2 + x \partial_y - \partial_t, \quad (x,y,t) \in \mathbb{R}^2 \times ]0, T],$$

it writes in the form \cite{13}, if the vector fields $X, Y$ are

$$X(x,y,t) = \partial_x \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y(x,y,t) = x \partial_y - \partial_t \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix}.$$  

$L$ is related to the following stochastic process

$$\begin{cases} X_t = x_0 + W_t, \\ Y_t = y_0 + \int_0^t (x_0 + W_s) \, ds. \end{cases}$$

which satisfies the \textit{Langevin equation} $dX_t = dW_t, dY_t = X_t dt$. We recall that this kind of stochastic process appears in several research areas. For instance, in Kinetic Theory, $(X_t)_{t \geq 0}$ describes the velocity of a particle, while $(Y_t)_{t \geq 0}$ is its position. We note that

\begin{itemize}
  \item[i)] $X$ and $Y$ are invariant with respect to the left translation of the group defined by the following operation
  $$(x_0,y_0,t_0) \circ (x,y,t) = (x + x_0, y + y_0 - tx_0, t + t_0), \quad (x,y,t), (x_0,y_0,t_0) \in \mathbb{R}^3,$$

  \item[ii)] $X$ and $Y$ are homogeneous of degree 1 and 2, respectively, with respect to the dilation
  $$(\delta_\rho)_{\rho > 0} : (x,y,t) \mapsto (\rho x, \rho^3 y, \rho^2 t) = \text{diag}(\rho, \rho^3, \rho^2) \cdot \begin{pmatrix} x \\ y \\ t \end{pmatrix}.$$  

\end{itemize}

In particular, $L$ satisfies the Hypotheses [G1] and [G2].
iii) The \( \mathcal{L} \)-admissible paths are the solutions \( \gamma(s) = (x(s), y(s), t(s)) \) of the following equation
\[
\begin{align*}
\dot{x}(s) &= \omega(s), \quad x(0) = x_0, \\
\dot{y}(s) &= x(s), \quad y(0) = y_0, \\
\dot{t}(s) &= -1, \quad t(0) = t_0.
\end{align*}
\]
It is easy to check that the attainable set of the point \((0, 0, 0)\) in the open set \( \Omega = \mathbb{R}^3 \) is \( \mathcal{A}_{(0,0,0)}(\Omega) = \{ (x, y, t) \in \Omega \mid t < -|y| \} \), (see Fig. 4).

As the interior of \( \mathcal{A}_{(0,0,0)}(\Omega) \) is not empty, Theorem 3.3 gives an invariant Harnack inequality for \( \mathcal{L} \), and we can apply (3.15) to prove lower bounds for positive solutions defined on the domain \( \mathbb{R}^2 \times [0, T] \). The Optimal Control Theory provides us with an explicit expression of the value function \( \Psi_0 \) for \( \mathcal{L} \) in (5.18)
\[
\Psi_0(x, y, t; \xi, \eta, \tau) = \frac{(x - \xi)^2}{t - \tau} + \frac{12}{(t - \tau)^3} \left( y - \eta - (t - \tau) \frac{(x + \xi)}{2} \right)^2.
\] (5.22)

This is a remarkable fact, as it is known that the explicit expression of the fundamental solution of \( \mathcal{L} \) was written by Kolmogorov (1934) and is
\[
\Gamma_0(x, y, t; \xi, \eta, \tau) = \frac{\sqrt{3}}{2\pi(t - \tau)^2} \exp \left( -\frac{(x - \xi)^2}{4(t - \tau)} - \frac{3}{4(t - \tau)^2} \left( y - \eta - (t - \tau) \frac{(x + \xi)}{2} \right)^2 \right).
\] (5.23)

We briefly discuss here the anisotropic dilation (5.21). We first note that the Hörmander condition is satisfied since
\[
[X, Y] = XY - YX = \partial_y \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]
and that \( \partial_y \) is homogeneous of degree three as \( XY \) and \( YX \) are both homogeneous of degree three. This explain the exponent 3 appearing in (5.21). Moreover, since
\[
\det(\text{diag}(\rho, \rho^3)) = \rho^4,
\]
then \( Q = 4 \) is the spatial homogeneous dimension of \( \mathbb{R}^2 \) with respect to the dilation (5.21). Furthermore, in view of (5.19), such dilation has a natural probabilistic meaning as one has \( \text{Var}(X_t) = t \) and \( \text{Var}(Y_t) = t^3/3 \).
The lower bound based on the value function $\Psi$ is useful as we consider Kolmogorov equations in the form

$$
\partial_t u(x, t) = \sum_{i,j=1}^{m} a_{ij}(x, t) \partial_{x_i x_j}^2 u(x, t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x, t), \quad (x, t) \in \mathbb{R}^N \times ]0, T[. 
$$

(5.24)

with bounded Hölder continuous coefficients $a_{ij}$’s. In the study of this family of operators, we assume that $m < N$, the matrix $(a_{ij}(t,x))_{i,j=1,...,m}$ is uniformly positive in $\mathbb{R}^m$. Moreover, the Hörmander condition is satisfied for the operator $L(\xi, \tau)$ frozen at some point $(\xi, \tau) \in \mathbb{R}^{N+1}$, that is obtained from the equation in (5.24) by replacing every function $a_{ij} = a_{ij}(x, t)$ with $a_{ij}(\xi, \tau)$. It turns out that this condition does not depend on the choice of the point $(\xi, \tau)$, that $L(\xi, \tau)$ is invariant with respect to a Lie group $G$ on $\mathbb{R}^{N+1}$ which does not depend on $(\xi, \tau)$. In this case the parametrix method provides us with the existence of a fundamental solution $\Gamma$ of the operator introduced in (5.24). The method also gives an upper bound of the form

$$
\Gamma(x, t; \xi, \tau) \leq \frac{C^+}{(t - \tau)^{Q/2}} \exp \left( -C^+ \Psi(x, t; \xi, \tau) \right) \quad (\xi, \tau), (x, t) \in \mathbb{R}^N \times ]0, T[, \quad t > \tau,
$$

where $Q$ is the homogeneous dimension of the space $\mathbb{R}^N$ with respect to the underlying Lie Group in $\mathbb{R}^{N+1}$, and $C^+, C^-$ are constants depending on the operator. The method described in this section gives the analogous lower bound for $\Gamma$

$$
\frac{C^-}{(t - t_0)^{Q/2}} \exp \left( -C^- \Psi(x, t; x_0, t_0) \right) \leq \Gamma(x, t; x_0, t_0) \quad (x_0, t_0), (x, t) \in \mathbb{R}^N \times ]0, T[.
$$

We conclude this section with a discussion on another meaningful example of operator which writes in the form (5.24) and is somehow more degenerate than (5.18). It is

$$
L = \partial_{x_1}^2 + x_1 \partial_{x_2} + \ldots + x_{N-1} \partial_{x_N} - \partial_t ,
$$

(5.25)

which is related to the following stochastic process

$$
dX_1^t = dW_t, \quad dX_2^t = X_1^t dt, \quad \ldots , \quad dX_i^t = X_{i-1}^t dt, \quad t \geq 0 .
$$

(5.26)

As the operator defined in (5.18), the one in (5.25) can be written as $L = X^2 + Y$ with:

$$
X(x, t) = \partial_{x_1} \sim \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad Y(x, t) = \sum_{j=1}^{N-1} x_j \partial_{x_{j+1}} - \partial_t \sim \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ \vdots \\ -1 \end{pmatrix} .
$$

Note that, in this case, $\partial_{x_{j+1}} = [\partial_{x_j}, Y]$ for $j = 1, \ldots, N - 1$. As a consequence, $L$ is invariant with respect to the dilation defined by the following matrix:

$$
\text{diag}(\rho, \rho^3, \ldots, \rho^{2N-1}, \rho^2),
$$
then its homogeneous dimension $Q$ is equal to $N^2$. Accordingly, we have that $\text{Var}(X_j^t) = c_j t^{2j-1}$, $j = 1, ..., N$, where $c_j$ is a positive constant.

We recall that the parametrix method has been used by several authors for the study of degenerate Kolmogorov equations. We recall the works of Weber [43], Il’In [18], Sonin [40], Polidoro [35, 36], Di Francesco and Polidoro [15]. In particular, the lower bound of the fundamental is proved in [36] and in [15].

More recently, Delarue and Menozzi [14] extended the above bounds to a class of Degenerate Kolmogorov Operator with possibly non-linear drifts satisfying Hörmander condition, under spatial Hölder continuity assumptions on the coefficients $a_{ij}$’s. They obtained analogous bounds by combining stochastic control methods with the parametrix representation of the fundamental solution given by McKean and Singer in [27].

6 More degenerate equations

In this section we consider a stochastic process studied by Cinti, Menozzi and Polidoro in [10]. It is similar to the one considered in Section 4 as it writes as follows

$$L := \partial_x^2 + x^2 \partial_y - \partial_t, \quad (x, y, t) \in \mathbb{R}^2 \times (0, T),$$

and is related to the following stochastic differential equation

$$\begin{cases}
X_t = x_0 + W_t, \\
Y_t = y_0 + \int_0^t (x_0 + W_s)^2 \, ds.
\end{cases}$$

A representation of the density of this process has been obtained from the seminal works of Kac [20] in terms of the Laplace transform of the process $(Y_t)_{t \geq 0}$. We also refer to the monograph of Borodin and Salminen [7] for an expression in terms of special functions. We also quote the works of Smirnov [39] and Tolmatz [41] on the distribution function of the square of the Brownian bridge.

We give explicit upper and lower bounds for the density of the process $(X_t, Y_t)_{t \geq 0}$ by the approach described in Section 3. Note that new difficulties appear in the study of the operator $L$ defined in (6.27). Indeed, if we write $L$ as follows

$$L = X^2 + Y,$$

then the commutator $[X, Y](x, y, t) = 2x \partial_y$ vanishes in the set $\{x = 0\}$, and we need a second commutator $[X, [X, Y]](x, y, t) = 2 \partial_y$ to satisfy the Hörmander condition at every point of $\mathbb{R}^3$. As a consequence, a Lie group leaving invariant the equation $L u = 0$ cannot exist. This problem is overcome by a lifting procedure (see Rothshild and Stein [38]). Specifically, we consider the following operator

$$\tilde{L} := \partial_x^2 + x \partial_w + x^2 \partial_y - \partial_t, \quad (x, y, w, t) \in \mathbb{R}^3 \times (0, T),$$

and we consider any solution of $\tilde{L} u = 0$ as a function that does not depend on $w$, and that solves the equation $\tilde{L} u = 0$. The lifting procedure allows us to rely on the Lie group invariance of $\tilde{L}$ in the study of the positive solutions of $\tilde{L} u = 0$. Indeed, we have
i) The operator $\mathcal{L}$ is invariant with respect to the following Lie group operation

$$(x_0, y_0, w_0, t_0) \circ (x, y, w, t) = (x + x_0, y + y_0 + 2x_0w - tx_0^2, w + w_0 - tx_0, t + t_0),$$

defined for every $(x, y, w, t), (x_0, y_0, w_0, t_0) \in \mathbb{R}^4$. In particular, it holds

$$(\mathcal{L}u)(z_0 \circ z) = \mathcal{L}(u(z_0 \circ z)),$$

for every $z_0 = (x_0, y_0, w_0, t_0)$ and $z = (x, y, w, t) \in \mathbb{R}^4$.

ii) The operator $\mathcal{L}$ is invariant with respect to the following dilation

$$(\delta_\rho)_{\rho \geq 0} : (x, y, w, t) \mapsto (\rho x, \rho^4 y, \rho^3 w, \rho^2 t).$$

That is, it holds:

$$\rho^2 (\mathcal{L} u)(\rho x, \rho^4 y, \rho^3 w, \rho^2 t) = \mathcal{L}(u(\rho x, \rho^4 y, \rho^3 w, \rho^2 t)).$$

iii) The attainable set of the origin in the box $\Omega = \{-1, 1\}^4$ is

$$\mathcal{A}_{(0,0,0,0)}(\Omega) = \{(x, w, y, t) \in [-1, 1]^4 \mid 0 \leq y \leq -t, w^2 \leq -ty\}.$$

Figure 4 describes the projection on the hyperplane $\{x = 0\}$ of the set $\mathcal{A}_{(0,0,0,0)}$

\[\text{Fig. 5 - Projection of } \mathcal{A}_{(0,0,0,0)}(\Omega) \text{ on the set } \{x = 0\}.\]

Then, an invariant Harnack inequality needed to construct Harnack chains for the positive solutions of $\mathcal{L}u = 0$ is available. The main result of the article is the following

**Theorem 6.1** Let $\Gamma$ denote the fundamental solution of $\partial_{xx} + x^2 \partial_y - \partial_t$.

- If $\eta - y \leq 0$, then $\Gamma(x, y, t, \xi, \eta, \tau) = 0$;
- if $\frac{\eta - y}{(t - \tau)^2} > \frac{x^2 + \xi^2}{t - \tau} + 1$, then

$$\Gamma(x, y, t, \xi, \eta, \tau) \approx \frac{1}{(t - \tau)^{5/2}} \exp \left( -C \left( \frac{(x - \xi)^2}{t - \tau} + \frac{\eta - y}{(t - \tau)^2} \right) \right);$$
if \(0 < \frac{\eta - y}{(t - \tau)} < \frac{1}{2}\), then
\[
\Gamma(x, y, t, \xi, \eta, \tau) \approx \frac{1}{(t - \tau)^{3/2}} \exp \left( -C \left( \frac{x^4 + \xi^4 + (t - \tau)^2}{\eta - y} \right) \right).
\]

We conclude this section with some remarks. We first note that, because of the particular form of the attainable set \(A_{(0,0,0,0)}(\Omega)\), it is not true that all the \(L^\infty\)-admissible paths \(\gamma\) steering \(z_0\) to \(z\) satisfy (3.11). For this reason, in the proof of our main result we do not solve any optimal control problem. We prove our lower bound by choosing smart admissible paths. This construction does not guarantee the optimality of the lower bounds. However, the comparison with the upper bound, that has the same asymptotic behavior, shows the optimality of both of them. The diagonal bounds and the upper bounds have been obtained by using probabilistic methods, and Malliavin Calculus in particular.

We eventually recall that more general operators and stochastic processes are studied in [10]. Precisely, we consider for every positive integer \(k\) the process \((X_t, Y_t)_{t \geq 0}\), with value in \(\mathbb{R}^n \times \mathbb{R}\)
\[
\begin{cases}
X_t = x + W_t, & (k \text{ even}) \\
Y_t = y + \int_0^t \sum_j (x + W_s)_j^k \, ds,
\end{cases}
\]
whose Kolmogorov equation is
\[
\mathcal{L} := \frac{1}{2} \Delta_x + (x_1^k + \cdots + x_n^k) \partial_y - \partial_t
\]
and
\[
\begin{cases}
X_t = x + W_t, & (k \text{ even}) \\
Y_t = y + \int_0^t |x + W_s|^k \, ds,
\end{cases}
\]
whose Kolmogorov equation is
\[
\mathcal{L} := \frac{1}{2} \Delta_x + |x|^k \partial_y - \partial_t.
\]
We refer to the article [10] for the precise statement of our achievements and for further details.

7 Operators related to Arithmetic Average Asian Options

In this section we consider the operator
\[
\mathcal{L} = x^2 \partial_{xx} + x \partial_x + x \partial_y - \partial_t
\]
with \((x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T)\). It appears in the Black and Scholes setting when we consider the pricing problem for Arithmetic Average Asian Option. Specifically, we assume that the price of an asset \((X_t)_{t \geq 0}\) is described by a Geometric Brownian Motion and that the option depends on the arithmetic average of \((X_t)_{t \geq 0}\). Then, according to the Black and Scholes theory, the value of the option \(v\) is modeled by a function \(v = v(t, X_t, Y_t)\) where
\[
\begin{cases}
X_t = x_0 e^{\sqrt{2} W_t}, \\
Y_t = y_0 + x_0 \int_0^t e^{\sqrt{2} W_s} \, ds.
\end{cases}
\]
This system was widely studied by Yor who wrote in [44] Chapter 6] its joint density (see equation (6.e) therein)

\[
p(x, y, t; x_0, y_0) = \frac{\sqrt{x_0}}{2\sqrt{(y - y_0)^2}} \frac{e^{\frac{x^2}{2(y - y_0)}}}{\sqrt{\pi}} \exp \left( - \frac{x + x_0}{2(y - y_0)} \right) \phi \left( \frac{\sqrt{x_0}}{y - y_0} \right), \tag{7.30}
\]

where

\[
\psi(z, t) = \int_0^\infty e^{-\frac{z^2}{4t}} e^{-z \cosh(\xi)} \sinh(\xi) \sin \left( \frac{\pi \xi}{t} \right) d\xi. \tag{7.31}
\]

As in the previous example, the density of the stochastic process \((X_t, Y_t)_{t \geq 0}\) is not strictly positive in the whole set \(\mathbb{R}^+ \times \mathbb{R} \times (0, T)\). In particular, its support is \(\mathbb{R}^+ \times (y_0, +\infty) \times (t_0, T)\).

Monti and Pascucci observe in [29] that \(\mathcal{L}\) is invariant with respect to the following group operation on \(\mathbb{R}^+ \times \mathbb{R}^2\):

\[
(x_0, y_0, t_0) \circ (x, y, t) = (x_0 x, y_0 y + x_0 y + t_0 + t). \tag{7.32}
\]

Indeed, if we set

\[
v(x, y, t) = u(x_0 x, y_0 y + x_0 y + t_0 + t), \tag{7.33}
\]

then \(\mathcal{L}v = 0\) if, and only if \(\mathcal{L}u = 0\).

Note that \(\mathcal{L}\) is not invariant with respect to any dilation group \((\delta_\rho)_{\rho \geq 0}\). On the other hand, as

\[
\mathcal{L} = X^2 + Y, \quad \text{with} \quad X(x, y, t) = x \partial_x, \quad Y(x, y, t) = x \partial_y - \partial_t,
\]

we have that \(\mathcal{L}\) can be approximated by the Kolmogorov operator \((5.18)\) defined in Section 6. Indeed, we can consider the coefficient \(x\) of the vector field \(X\) as a smooth function that is bounded and bounded by below on every compact set \(K \subset \mathbb{R}^+ \times \mathbb{R} \times (0, T)\). For this reason, the Harnack inequality introduced in Section 5 also applies to \(\mathcal{L}\).

The \(\mathcal{L}\) admissible paths are the solutions of the following differential equation

\[
\begin{cases}
\dot{x}(s) = \omega(s)x(s), & x(0) = x_0, \\
\dot{y}(s) = x(s), & y(0) = y_0, \\
\dot{t}(s) = -1, & t(0) = t_0,
\end{cases}
\]

and we denote by \(\Psi(x_0, y_0, t_0, x, y, t)\) the value function of the relevant optimal control problem with quadratic cost. The main result for the fundamental solution \(\Gamma(x, y, t; x_0, y_0, t_0)\) of the operator \(\mathcal{L}\) is the following

**Theorem 7.1** Let \(\Gamma\) be the fundamental solution of \(\mathcal{L}\). Then, for every \((x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]\) we have

\[
\Gamma(x, y, t; x_0, y_0, t_0) = 0 \quad \forall (x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2 \setminus \{x \in \mathbb{R} | x \leq y_0 \times t_0, T\}. \tag{7.34}
\]

Moreover, for arbitrary \(\varepsilon \in [0, 1]\), there exist two positive constants \(c^-_\varepsilon, C^+_\varepsilon\) depending on \(\varepsilon\), on \(T\) and on the operator \(\mathcal{L}\), and two positive constants \(C^-\varepsilon, c^+\varepsilon\), only depending on the operator \(\mathcal{L}\) such that

\[
\frac{c^-_\varepsilon}{x_0^2(t - t_0)^2} \exp \left( -C^-\varepsilon \Psi(x, y + x_0 \varepsilon(t - t_0), t - \varepsilon(t - t_0); x_0, y_0, t_0) \right) \leq \Gamma(x, y; t; x_0, y_0, t_0) \leq \frac{C^+_\varepsilon}{x_0^2(t - t_0)^2} \exp \left( -c^+\varepsilon \Psi(x, y - x_0 \varepsilon, t + \varepsilon; x_0, y_0, t_0) \right), \tag{7.35}
\]

18
for every \((x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [\gamma_0 - y_0, \gamma_0 + y_0] \times [0, T]\).

Note that, since the proof of Theorem 7.1 is based on local estimates of the solution of \(L u = 0\) and \(L\) is locally well approximated by the operator introduced in (5.18), the diagonal bound in (7.35) agrees with the diagonal term of \(\Gamma_0\) in (5.23). Furthermore, the diagonal estimate corresponds to the product of the standard deviations of the random variables \(X_t\) and \(Y_t\) defined in (7.29). Indeed,

\[
\text{Var}(X_t) = x_0^2 e^{2t} (e^{2t} - 1) = 2x_0^2 t + o(t), \quad \text{as } t \to 0,
\]

\[
\text{Var}(Y_t) = x_0^2 \left( \frac{1}{4} (e^{4t} - 1) - \frac{2}{3} (e^t - 1)^2 \right) = \frac{2}{3} x_0^2 t^3 + o(t^3), \quad \text{as } t \to 0.
\]

Clearly, the knowledge of the asymptotic behavior of the function \(\Psi\) is crucial for the application of our Theorem 7.1. In [9], it is shown that one can write the function \(\Psi\) in terms of the function \(g\) defined as follows

\[
g(r) = \begin{cases} 
\frac{\sinh(\sqrt{r})}{\sqrt{r}}, & r > 0, \\
1, & r = 0, \\
\frac{\sin(\sqrt{-r})}{\sqrt{-r}}, & -\pi^2 < r < 0,
\end{cases}
\]

and it is proven the following proposition

**Proposition 7.2** For every \((x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2\), with \(t_0 < t\) and \(y_0 > y\), we have

\[
\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1 + x_0)}{y_0 - y_1} - 4\sqrt{E + \frac{4x_1 x_0}{(y_0 - y_1)^2}},
\]

if \(E \geq -\frac{x^2}{t_1 - t_0}\);

\[
\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1 + x_0)}{y_0 - y_1} + 4\sqrt{E + \frac{4x_1 x_0}{(y_0 - y_1)^2}},
\]

if \(-\frac{\pi^2}{t_1 - t_0} < E < -\frac{x^2}{t_1 - t_0}\).

where

\[
E = \frac{4}{(t - t_0)^2} g^{-1} \left( \frac{y_0 - y}{(t - t_0) \sqrt{x_0 x}} \right).
\]

Moreover,

\[
\frac{4}{(t - t_0)^2} \log^2 \left( \frac{y_0 - y}{(t - t_0) \sqrt{x_0 x}} \right) + \frac{4(x_0 + x)}{y_0 - y} \to 1, \quad \text{as} \quad \frac{y_0 - y}{(t - t_0) \sqrt{x_0 x}} \to +\infty;
\]

\[
\frac{4}{(t - t_0)^2} \log^2 \left( \frac{y_0 - y}{(t - t_0) \sqrt{x_0 x}} \right) + \frac{4(x_0 + x)}{y_0 - y} \to 1, \quad \text{as} \quad \frac{y_0 - y}{(t - t_0) \sqrt{x_0 x}} \to 0.
\]

The above expression for the value function \(\Psi\) has been obtained by using the Pontryagin Maximum Principle [37], the upper bound in (7.35) is a consequence of the fact that \(\Psi\) satisfies the Hamilton-Jacobi-Bellman equation \(Y \Psi + \frac{1}{4} (X \Psi)^2 = 0\).

To our knowledge, it is not easy to compare the integral expression of \(p\) in (7.30) with the estimates given in Proposition 7.2, then Theorem 7.1 provides us with an alternative explicit
information on the asymptotic behavior of $p$. Moreover, the method described in this section also applies to the divergence form operator $\tilde{L}$ defined as

$$\tilde{L}u = x \partial_x(a x \partial_x u) + b x \partial_x u + x \partial_y u - \partial_t u,$$

where $a$ and $b$ are smooth bounded coefficients, with $a$ bounded by below and $x \partial_x a$ bounded. Note that, in this case, an expression of $\Gamma$ analogous to (7.30) is not available. A further consequence of (7.35) is the following result. By applying (7.35) to $\Gamma$ and to the fundamental solutions $\Gamma^\pm$ of the operators

$$\mathcal{L}^\pm u = \lambda^\pm x^2 \partial_{xx} u + x \partial_x u + x \partial_y u - \partial_t u, \quad (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times ]0, T[, \quad (7.36)$$

we obtain

$$k^- \Gamma^-(x, y + \varepsilon(t + 1), t - \varepsilon(t + 1)) \leq \Gamma(x, y, t) \leq k^+ \Gamma^+(x, y + \varepsilon(t + 1), t + \varepsilon(t + 1)), \quad (7.37)$$

for every $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times ]0, T[$ with $y + \varepsilon(t + 1) < 0$ and $t > \varepsilon/(1 - \varepsilon)$. Hence, we obtain lower and upper bounds for the fundamental solution $\Gamma$ of the variable coefficients operator $\tilde{L}$ in terms of the fundamental solutions $\Gamma^\pm$ of the constant coefficients operators $\mathcal{L}^\pm$, whose expressions, up to some scaling parameters, agree with the function $p$ in (7.30). We refer to the article [9] for the precise statement of the results of this section and for further details.

Acknowledgement

We thank the anonymous referee for his/her careful reading of our manuscript and for several suggestions that have improved the exposition of our work.

References

[1] D. G. Aronson, Bounds for the fundamental solution of a parabolic equation, Bull. Amer. Math. Soc., 73 (1967), pp. 890–896.

[2] D. G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, Arch. Rational Mech. Anal., 25 (1967), pp. 811/2.

[3] V. Bally, Lower bounds for the density of locally elliptic Itô processes, Ann. Probab., 34 (2006), pp. 24062440.

[4] V. Bally and A. Kohatsu-Higa, Lower bounds for densities of Asian type stochastic differential equations, J. Funct. Anal., 258 (2010), pp. 3134–3164.

[5] R. Bass, Diffusions and elliptic operators, Springer-Verlag, New York, 1998.

[6] G. Ben Arous and R. Léandre, Décroissance exponentielle du noyau de la chaleur sur la diagonale, Probab. Theory Related Fields, 90 (1991), pp. 175202.
[7] A. N. Borodin and P. Salminen, Handbook of Brownian motion facts and formulae, Probability and its Applications, Birkhauser Verlag, Basel, second ed., 2002.

[8] U. Boscain and S. Polidoro, Gaussian estimates for hypoelliptic operators via optimal control, Rend. Lincei Mat. Appl., 18 (2007), pp. 333–342.

[9] G. Cibelli, S. Polidoro, and F. Rossi, Sharp estimates for Geman-Yor Processes and Application to Arithmetic Average Asian Option, Submitted.

[10] C. Cinti, S. Menozzi, and S. Polidoro, Two-sides bounds for degenerate processes with densities supported in subsets of $\mathbb{R}^n$, Potential Analysis, (2014), pp. 1577–1630.

[11] C. Cinti, K. Nyström, and S. Polidoro, A note on Harnack inequalities and propagation sets for a class of hypoelliptic operators, Potential Anal., 33 (2010), pp. 341354.

[12] C. Cinti and S. Polidoro, Pointwise local estimates and Gaussian upper bounds for a class of uniformly subelliptic ultraparabolic operators, J. Math. Anal. Appl., 338 (2008), pp. 946–969.

[13] E. B. Davies, Explicit constants for Gaussian upper bounds on heat kernels, Amer. J. Math., 109 (1987), pp. 319–333.

[14] F. Delarue and S. Menozzi, Density estimates for a random noise propagating through a chain of differential equations, J. Funct. Anal., 259 (2010), pp. 1577–1630.

[15] M. Di Francesco and S. Polidoro, Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form, Adv. Differential Equations, 11 (2006), pp. 1261–1320.

[16] E.B. Fabes and D.W. Stroock, A new proof of Moser’s parabolic Harnack inequality using the old ideas of Nash, Arch. Rational Mech. Anal., 96, (1986), pp 327–338.

[17] J. Hadamard, Extension à l’équation de la chaleur d’un théorème de A. Harnack, Rend. Circ. Mat. Palermo (2), 3 (1954), pp. 337346 (1955).

[18] A. M. Il’in, On a class of ultraparabolic equations, Dokl. Akad. Nauk SSSR, 159 (1964), pp. 12141217.

[19] D. S. Jerison and A. Sánchez-Calle, Estimates for the heat kernel for a sum of squares of vector fields, Indiana Univ. Math. J., 35 (1986), pp. 835–854.

[20] M. Kac, On distributions of certain Wiener functionals, Trans. Amer. Math. Soc., 65 (1949), pp. 113.

[21] A. Kogoj and S. Polidoro, Harnack Inequality for Hypoelliptic Second Order Partial Differential Operators, (to appear on Potential Analysis), (2016).

[22] A. Kohatsu Higa, Lower bounds for densities of uniformly elliptic random variables on Wiener space, Probab. Theory Related Fields, 126 (2003), pp. 421457.
[23] V. Konakov, *Parametrix method for diffusion and Markov chains*, Russian preprint available on, [https://www.hse.ru/en/org/persons/22565341](https://www.hse.ru/en/org/persons/22565341)

[24] N. V. Krylov and M. V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat., 44 (1980), pp. 161–175.

[25] S. Kusuoka and D. Stroock, *Applications of the Malliavin calculus. III*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 34 (1987), pp. 391–442.

[26] P. Malliavin, *Stochastic calculus of variation and hypoelliptic operators*, in Proceedings of the International Symposium on Stochastic Differential Equations, Wiley, New York-Chichester-Brisbane, 1978, pp. 195–263.

[27] H. P. McKeane, Jr. and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry, 1 (1967), pp. 4369.

[28] G. Metafune, D. Pallara and A. Rhandi, *Global properties of invariant measures*, J. Funct. Anal., 223 (2005), pp. 396424.

[29] L. Monti and A. Pascucci, *Obstacle problem for arithmetic Asian options*, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 14431446.

[30] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.

[31] ——, *Correction to: “A Harnack inequality for parabolic differential equations”*, Comm. Pure Appl. Math., 20 (1967), pp. 231–236.

[32] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.

[33] D. Nualart, *The Malliavin Calculus and related topics*, Probability and its Applications. Springer-Verlag, New York.

[34] B. Pini, *Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico*, Rend. Sem. Mat. Univ. Padova, 23 (1954), pp. 422434.

[35] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), pp. 53–105 (1995).

[36] ——, *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations*, Arch. Rational Mech. Anal., 137 (1997), pp. 321–340.

[37] L. S. Pontryagin, E. Mishchenko, V. Boltyanskii, and R. Gamkrelidze, *The mathematical theory of optimal processes*, Wiley, 1962.

[38] L. Rothschild, and E.M Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math.,137, 1976, pp. 247–320.

[39] N. Smirnov, *Sur la distribution de 2 (criterium de M. von Mises)*, C. R. Acad. Sci. Paris, 202 (1936), pp. 449452.
[40] I.M. Sonin, *A class of degenerate diffusion processes*, Teor. Verojatnost. i Primenen., 12 (1967), pp. 540–547.

[41] L. Tolmatz, *Asymptotics of the distribution of the integral of the absolute value of the Brownian bridge for large arguments*, Ann. Probab., 28 (2000), pp. 132–139.

[42] N.T. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, vol. 100 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1992.

[43] M. Weber, *The fundamental solution of a degenerate partial differential equation of parabolic type*, Trans. Amer. Math. Soc., 71 (1951), pp 24-37.

[44] M. Yor, *On some exponential functionals of Brownian motion*, Adv. in Appl. Probab., 24 (1992), pp. 509–531.