NONLOCAL DIFFUSION OF SMOOTH SETS

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ABSTRACT. We consider normal velocity of smooth sets evolving by the \( s \)-fractional diffusion. We prove that for small time, the normal velocity of such sets is nearly proportional to the mean curvature of the boundary of the initial set for \( s \in \left( \frac{1}{2}, 1 \right) \) while, for \( s \in \left( 0, \frac{1}{2} \right) \), it is nearly proportional to the fractional mean curvature of the initial set. Our results shows that the motion by (fractional) mean curvature flow can be approximated by fractional heat diffusion and by a diffusion by means of harmonic extension of smooth sets.

1. Introduction

For \( N \geq 2 \), we let \( \Omega_0 \) be a bounded open set of \( \mathbb{R}^N \) with boundary \( \Gamma_0 \). Consider the heat equation with initial data the indicator function of the set \( \Omega_0 \):

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in } \mathbb{R}^N \times (0, t_1] \\
u(x, 0) &= \mathbf{1}_{\Omega_0}(x) \quad \text{on } \mathbb{R}^N.
\end{aligned}
\]

for some time \( t_1 > 0 \). In 1992, Bence-Merriman-Osher [7] provided a computational algorithm for tracking the evolution in time of the set \( \Omega_0 \) whose boundary \( \Gamma_0 \) moves with normal velocity proportional to its classical mean curvature. At time \( t_1 > 0 \), they considered

\[ \Omega_1 = \{ x \in \mathbb{R}^N : u(x, t_1) \geq 1/2 \} . \]

Bence-Merriman-Osher [7] applied iteratively this procedure to generate a sequence of sets \( (\Omega_j)_{j \geq 0} \) and conjectured in [7] that their boundaries \( \Gamma_j \) evolved by mean curvature flow. Later Evans [20] provided a rigorous proof for the Bence-Merriman-Osher algorithm by means of the level-set approach to mean curvature flow developed by Osher-Sethian [42], Evans-Spruck [21–24] and Chen-Giga-Goto [13]. For related works in this direction, we refer the reader to [5, 31, 37, 38, 40, 43, 47] and references therein.

Recently Caffarelli and Souganidis considered in [12] nonlocal diffusion of open sets \( E \subset \mathbb{R}^N \) given by

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (\Delta)^s u &= 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) \\
u(x, 0) &= \tau_E(x) \quad \text{in } \mathbb{R}^N \times \{ t = 0 \},
\end{aligned}
\]

where

\[ \tau_E(x) = \mathbf{1}_E(x) - \mathbf{1}_{\mathbb{R}^N \setminus E}(x) . \]

We consider the fractional heat kernel \( K_s \) with Fourier transform given by \( \tilde{K}(\xi, t) = e^{-t|\xi|^{2s}} \). It satisfies

\[
\begin{aligned}
\frac{\partial K_s}{\partial t} + (\Delta)^s K_s &= 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) \\
K_s &= \delta_0 \quad \text{on } \mathbb{R}^N \times \{ t = 0 \} .
\end{aligned}
\]

It follows that the unique bounded solution to (1.2) is given by

\[
u(x, t) = K_s(\cdot, t) \ast \tau_E(x) = \int_{\mathbb{R}^N} K_s(x - y, t) \tau_E(y) dy.
\]

Keywords: Motion by fractional mean curvature flow; Fractional heat equation; Fractional mean curvature; Harmonic extension.
By solving a finite number of times \[12\] for a small fixed time step \(\sigma_s(h)\), the authors in \[12\] find a discrete family of sets

\[ E_0^h = E, \quad E_{nh}^h = \{ x \in \mathbb{R}^N : K_s(\cdot, \sigma_s(h)) \star \tau E_{(n-1)h}^h(x) > 0 \}, \]

for a suitable scaling function \(\sigma_s\) to be defined below. It is proved in \[12\] that as \(nh \to t\), \(\partial E_{nh}^h\) converges, in a suitable sense, to \(\Gamma_t\). Here, the family of hypersurface \(\{\Gamma_t\}_{t>0}\), with \(\Gamma_0 = \partial E\), evolves under generalized mean curvature flow for \(s \in (\frac{1}{2}, 1)\) and under generalized fractional mean curvature flow for \(s \in (0, \frac{1}{2})\). We refer the reader to \[12, 20, 36\] for the notion generalized (nonlocal) mean curvature flow which considers the level sets of viscosity solutions to quasilinear parabolic integro-differential equations.

In the present paper, we are interested in the normal velocity of the sets

\[ E_t := \{ x \in \mathbb{R}^N : K_s(\cdot, \sigma_s(t)) \star \tau E(x) > 0 \} \]

as they depart from a sufficiently smooth initial set \(E_0 := E\). We consider here and in the following

\[ K_s(x, t) = t^{-\frac{N}{s}} P_s(t^{-\frac{1}{s}}, x), \quad \text{for some radially symmetric function} \quad P_s \in C^1(\mathbb{R}^N). \]

We make the following assumptions:

\[ \frac{C_{N,s}}{1 + |y|^{N+2s}} \leq P_s(y) \leq \frac{C_{N,s}}{1 + |y|^{N+2s}}, \quad |\nabla P_s(y)| \leq \frac{C_{N,s}}{1 + |y|^{N+2s+1}}. \]

and

\[ \lim_{t \to 0} t^{-1} K_s(y, t) = \frac{C_{N,s}}{|y|^{N+2s}} \quad \text{locally uniformly in} \quad \mathbb{R}^N \setminus \{0\}, \]

for some constants \(C_{N,s}, C_{N,s} > 0\). In the Section \[1.1\] below, we provide examples of valuable kernels \(K_s\) satisfying the above properties.

Now, as we shall see below (Lemma \[2.1\]), for \(t > 0\) small, \(\nabla_x u(x, \sigma_s(t)) \neq 0\) for all \(x \in B(y, \sigma_s(t) \frac{1}{2})\) and \(y \in \partial E\). Hence \(\partial E_t\) is a \(C^1\) hypersurface, for small \(t > 0\). For \(t > 0\) and \(y \in \partial E\), we let \(v = v(t, y)\) be such that

\[ y + tv \nu(y) \in \partial E_t \cap B(y, \sigma_s(t) \frac{1}{2}), \]

where \(\nu(y)\) is the unit exterior normal of \(E\) at \(y\). In the spirit of the work of Evans \[20\] on diffusion of smooth sets, we provide in this paper an expansion of \(v(t, y)\) as \(t \to 0\). It turns out that \(v(0, y)\) is proportional to the fractional mean curvature of \(\partial E\) at \(y\) for \(s \in (0, 1/2)\) and \(v(0, y)\) is proportional to the classical mean curvature of \(\partial E\) at \(y\) for \(s \in [1/2, 1)\).

We notice that it is not a priori clear from \[1.8\], that \(v\) remains finite as \(t \to 0\). This is where the (unique) appropriate choice of \(\sigma_s(t)\) enters during our estimates. Here and in the following, we define

\[ \sigma_s(t) = \begin{cases} t^{\frac{1}{s}} & \text{for } s \in (0, 1/2), \\ t^s & \text{for } s \in (1/2, 1) \end{cases} \]

and for \(s = 1/2\), \(\sigma_s(t)\) is the unique positive solution to

\[ t = \sigma_{1/2}^2(t) |\log(\sigma_{1/2}(t))|. \]

Before stating our main result, we recall that for \(s \in (0, \frac{1}{2})\) and \(\partial E\) is of class \(C^{1, \beta}\) for some \(\beta > 2s\), the fractional mean curvature of \(\partial E\) is defined for \(x \in \partial E\) as

\[ H_s(x) := P.V. \int_{\mathbb{R}^N} \frac{\tau E(y)}{|x-y|^{N+2s}}dy = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\tau E(y)}{|x-y|^{N+2s}}dy. \]

On the other hand, if \(\partial E\) is of class \(C^2\) then the normalized mean curvature of \(\partial E\) is given, for \(x \in \partial E\), by

\[ H(x) := \frac{N-1}{N+1} \lim_{\epsilon \to 0} \frac{1}{|B_\epsilon(x)| \int_{B_\epsilon(x)} \tau E(y)dy}, \]

see also \[2.4\] and \[25\]. Having fixed the above definitions, we now state our main result.
Theorem 1.1. We let \( s \in (0, 1) \) and \( E \subset \mathbb{R}^N, \ N \geq 2 \). We assume, for \( s \in (0, 1/2) \), that \( \partial E \) is of class \( C^{1, \beta} \) for some \( \beta > 2s \) and that \( \partial E \) is of class \( C^3 \), for \( s \in [1/2, 1) \). Then, as \( t \to 0 \), the expansion of \( v(t, y) \), defined in (1.8), is given, locally uniformly in \( y \in \partial E \), by

\[
v(t, y) = \begin{cases} 
  a_{N,s} H_s(y) + o(1) & \text{for } s \in (0, 1/2) \\
  b_N(t) H(y) + O\left(\frac{1}{\log(\sigma_{1/2}(t))}\right) & \text{for } s = 1/2 \\
  c_{N,s} H_s(y) + O\left(\frac{2s-1}{2}\right) & \text{for } s \in (1/2, 1),
\end{cases}
\]

where \( H_s \) and \( H \) are respectively the fractional and the classical mean curvatures of \( \partial E \) and the positive constants \( a_{N,s}, b_{N,1/2} \) and \( c_{N,s} \) are given by

\[
a_{N,s} = \frac{C_{N,s}}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0) \, dy'}, \quad b_N(t) = \frac{\int_{B^{N-1}_{\frac{1}{(1+|y'|^2)^{ \frac{s}{2}}} - 1}} |y'|^2 P_{1/2}(y', 0) \, dy'}{2 |\log(\sigma_{1/2}(t))| \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) \, dy'}, \quad c_{N,s} = \frac{\int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) \, dy'}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0) \, dy'}
\]

and \( P_s(y) := K_s(y, 1) \).

Some remarks are in order. The assumption of \( E \) being of class \( C^3 \) in Theorem 1.1 is motivated by the result of Evans in [20], where in the case \( s = 1 \) and \( K_1 \), the heat kernel, he obtained \( v = (N - 1)H(0) + O(t^{\frac{2}{3}}) \). We notice that from our argument below, we cannot improve the error term \( o(1) \) in the case \( s \in (0, 1/2) \) even if \( E \) is of class \( C^\infty \). This is due to the definition of the fractional mean curvature \( H_s \) as a principal value integral.

We finally remark, in the particular case, that \( K_{1/2}(y, t) = C_N \frac{t}{(t^2 + |y|^2)^{\frac{N+1}{2}}} \), we have that

\[
b_N(t) = \frac{|s^{N-2}| C_N}{2 \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) \, dy'} + O\left(\frac{1}{\log(\sigma_{1/2}(t))}\right).
\]

1.1. Some applications of Theorem 1.1. We next put emphasis on two valuable examples where Theorem 1.1 applies.

1) Fractional heat diffusion of smooth sets. We recall, see e.g. [8490], that the fractional heat kernel \( K_s \) satisfies (1.5), (1.6) and (1.7) with

\[
C_{N,s} = s 2^{2s} \sin(s\pi) \Gamma\left(\frac{N}{2} + s\right) \Gamma(s) \pi^{1+\frac{N}{2}}.
\]

We recall that \( K_s \) is known explicitly only in the case \( s = 1/2 \), where \( K_{1/2}(y, t) = C_{N,1/2} \frac{t}{(t^2 + |y|^2)^{\frac{N+1}{2}}} \).

In this case Theorem 1.1 provides an approximation of the (fractional) mean curvature motion by fractional heat diffusion of smooth sets, thereby extending, in the fractional setting, Evan’s result in [20] on heat diffusion of smooth sets.

2) Diffusion of smooth sets by Harmonic extension. We consider the Poisson kernel on the half space \( \mathbb{R}_+^{N+1} := \mathbb{R}^N \times (0, \infty) \), given by

\[
\mathcal{K}_s(x, t) := t^{-N} P_s(x/t), \quad P_s(x) = \frac{P_{N,s}}{(1 + |x|^2)^{\frac{N+2s}{2}}},
\]

where \( P_{N,s} := \int_{\mathbb{R}^N (1+|y|^2)^{\frac{N+2s}{2}}} \, dy \). Thanks to the result of Caffarelli and Silvestre in [11], the function

\[
w(x, t) = \mathcal{K}_s(\cdot, t) * \tau_E(x) = P_{N,s} t^{2s} \int_{\mathbb{R}^N} \frac{\tau_E(y)}{(t^2 + |y - x|^2)^{\frac{N+2s}{2}}} \, dy
\]
We also write \( f, g \). In the following, for \( \gamma \) with \( \epsilon > 0 \), we may assume that \( E_t := \{ x \in \mathbb{R}^N : \mathbf{K}_s(\cdot, \sigma_s(t)) \ast \tau_E(x) > 0 \} \), where \( \sigma_s(t) \) is given by \( (1.9) \) and \( (1.10) \). Therefore this Harmonic extension yields an approximation of (fractional) mean curvature motion of smooth sets.

We conclude Section \( 1 \) by noting that the notion of nonlocal curvature appeared for the first time in \( [12] \). Later on, the study of geometric problems involving fractional mean curvature has attracted a lot of interest, see e.g. \( [26, 27, 45] \), the survey paper \( [25] \) and the references therein. While the mean curvature flow is well studied, see e.g. \( [24, 49, 29, 30, 35] \), its fractional counterpart appeared only recently in the literature, see e.g. \( [14, 18, 36, 45] \).

We finally remark that the changes of normal velocity of the nonlocal diffused sets as \( s \) varies in \((0, 1/2)\) and \([1/2, 1)\), appeared analogously in phases transition problems, see e.g. \( [26, 27, 45] \).

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### 2. Preliminary results and notations

Unless otherwise stated, we assume for the following that \( E \) is an open set of class \( C^{1,\beta} \), with \( 0 \in \partial E \) and the unit normal of \( \partial E \) at 0 coincides with \( e_N \). We denote by \( Q_r = B_r^{-1} \times (-r, r) \) the cylinder of \( \mathbb{R}^N \) centred at the origin with \( B_r^{-1} \) the ball of \( \mathbb{R}^{N-1} \) centred at the origin with radius \( r > 0 \). Decreasing \( r \), if necessary, we may assume that

\[
E \cap Q_r = \{ (y', y_N) \in B_r^{-1} \times \mathbb{R} : y_N > \gamma(y') \},
\]

with \( \gamma \in C^{1,\beta}(B_r^{-1}) \) satisfying

\[
\gamma(y') = O \left( |y'|^{1+\beta} \right).
\]

In the following, for \( f, g : \mathbb{R} \to \mathbb{R} \), we write \( g(t) := O(f(t)) \) if

\[
|g(t)| \leq C|f(t)|.
\]

We also write \( g(t) = o(f(t)) \) if \( g(t) = O(f(t)) \) and moreover when \( f(t) \neq 0 \), we have

\[
\lim_{t \to 0} \frac{|g(t)|}{|f(t)|} = 0.
\]

We denote by \( o_t(1) \) any function that tends to zero when \( t \to 0 \). If in addition, \( \partial E \) is of class \( C^3 \), then for \( y' \in B_r^{-1} \), we have

\[
\gamma(y') = \frac{1}{2} D^2 \gamma(0)[y', y'] + O \left( |y'|^3 \right)
\]

and the normalized mean curvature of \( \partial E \) at 0 is given by

\[
H(0) = \frac{\Delta \gamma(0)}{N - 1} = \frac{N - 1}{N + 1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{Q_\varepsilon(y)} \tau_E(y) \, dy.
\]

Recall that the unit exterior normal \( \nu(y') := \nu(y', \gamma(y')) \) of \( E \) and the volume element \( d\sigma(y') \) on \( \partial E \cap Q_r \) are given by

\[
\nu(y') = \frac{(-\nabla \gamma(y'), 1)}{\sqrt{1 + |\nabla \gamma(y')|^2}} \quad \text{and} \quad d\sigma(y') = \sqrt{1 + |\nabla \gamma(y')|^2} \, dy'.
\]
Lemma 2.1. Let $s \in (0,1)$ and $E$ be a $C^{1,\beta}$ hypersurface satisfying (2.1). Define

$$w(z, t) = \int_{\mathbb{R}^N} K_s(z - y, t) \tau_{E}(y)dy.$$  

Then there exist $t_0, C > 0$, only depending on $N, s, \beta$ and $E$, such that for all $t \in (0, t_0)$ and $z \in B_{\frac{1}{12}}$, 

$$\frac{\partial w}{\partial z_N}(z, t) \geq Ct^{-1/2s}. \tag{2.7}$$

As a consequence, for all $t \in (0, t_0)$, the set 

$$\{z \in \mathbb{R}^N : w(z, t) = 0\} \cap B_{\frac{1}{12}}$$

is of class $C^1$. \tag{2.8}

Proof. We fix $t > 0$ small so that $t^{\frac{1}{3s}} < \frac{c}{8}$ and let $z \in B_{\frac{1}{12}}$. We write

$$\frac{\partial w}{\partial z_N}(z, t) = \int_{\mathbb{R}^N} \frac{\partial K_s}{\partial z_N}(z - y, t) \tau_{E}(y)dy = \int_{B_{\frac{1}{2}}(z)} \frac{\partial K_s}{\partial z_N}(z - y, t) \tau_{E}(y)dy + \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(z)} \frac{\partial K_s}{\partial z_N}(z - y, t) \tau_{E}(y)dy. \tag{2.9}$$

By a change of variable, (1.5) and (1.6), we have

$$\int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(z)} \frac{\partial K_s}{\partial y_N}(z - y, t)dy = t^{\frac{1}{3s}} \int_{\mathbb{R}^N \setminus t^{\frac{1}{3s}} B_{\frac{1}{2}}(z)} \frac{\partial P_s}{\partial y_N}(t^{\frac{1}{3s}} z - y)dy$$

$$= O\left(t^{\frac{1}{3s}} \int_{\mathbb{R}^N \setminus t^{\frac{1}{3s}} B_{\frac{1}{2}}(z)} \frac{1}{|t^{\frac{1}{3s}} z - y|^{N+2s}}\right) = O(t). \tag{2.10}$$

Integrating by parts, we have

$$\int_{B_{\frac{1}{2}}(z) \cap E} \frac{\partial K_s}{\partial y_N}(z - y, t) \tau_{E}(y)dy = \int_{B_{\frac{1}{2}}(z) \cap \partial E} \frac{\partial K_s}{\partial y_N}(z - y, t) \tau_{E}(y)dy - \int_{B_{\frac{1}{2}}(z) \cap E^c} \frac{\partial K_s}{\partial y_N}(z - y, t)dy$$

$$= 2 \int_{B_{\frac{1}{2}}(z) \cap \partial E} K_s(z - y, t) \nu_E(y) d\sigma(y) + \int_{B_{\frac{1}{2}}(z) \cap E^c} K_s(z - y, t) \nu_{B_{\frac{1}{2}}}(y) \tau_{E}(y) d\sigma(y).$$

By a change of variable, (1.5), (1.6) and the fact that $Q_{r/8} \subset B_{r/4} \subset B_{\frac{1}{2}}(z) \subset Q_r$, we have

$$\int_{B_{\frac{1}{2}}(z) \cap \partial E} K_s(z - y, t) \nu_E(y) d\sigma(y) \geq C \int_{B_{\frac{1}{8}}(z)} K_s(z' - y', z_N - \gamma(y'), t) dy'$$

$$= Ct^{\frac{1}{3s}} \int_{B_{\frac{1}{8}}(z)} P_s(t^{\frac{1}{3s}} z' - y', t^{\frac{1}{3s}} z_N - t^{\frac{1}{3s} N} \gamma(t^{\frac{1}{3s}} y')) dy'$$

$$\geq Ct^{\frac{1}{3s}} \int_{B_{\frac{1}{8}}(z) \setminus B_{\frac{1}{2}}} \frac{1}{|y|^{N+2s}} dy' \tag{2.11}$$

provided $r$. Next, using (2.6) and recalling that $z \in B_{\frac{1}{12}}$, we then have

$$\left|\int_{\partial B_{\frac{1}{2}}(z)} K_s(z - y, t) \nu_{B_{\frac{1}{2}}}(y) d\sigma(y)\right| \leq \int_{\partial B_{\frac{1}{2}}(z)} K_s(z - y, t) d\sigma(y) \leq C \int_{\partial B_{\frac{1}{2}}(z)} \frac{1}{|y|^{N+2s}} d\sigma(y) = O(t).$$
From this and (2.11), we deduce that
\[ \int_{B_r(z)} \frac{\partial K_s(y)}{\partial y_N}(z - y, t) \tau_E(y) dy \geq Ct^{-1/2s}. \]
Combining this with (2.9) and (2.10), we get
\[ \frac{\partial w}{\partial z_N}(z, t) \geq Ct^{-1/2s}. \]
Therefore (2.7) follows. Finally (2.8) follows from the inverse function theorem and the fact that \( w \) is of class \( C^1 \) on \( \mathbb{R}^N \times (0, \infty) \).

In the sequel, we will need the following lemmas to estimate some error terms.

**Lemma 2.2.** For \( s \in (0, 1) \), we let \( E \subset \mathbb{R}^N \) be a set of class \( C^{1, \beta} \), for some \( \beta > 2s \), as in Section 2. For \( r > 0 \), we set
\[ J_r(t) := \int_{Q_{r,t}} K_s(y, t) \tau_E(y) dy \quad \text{and} \quad I_r(t) = \frac{C_{N,s}}{|y|^{N+2s}} \int_{Q_{r,t}} t^{-1} K_s(y, t) \tau_E(y) dy. \]
Then we have
\[ |J_r(t)| \leq Ctr^{\beta-2s} \quad \text{and} \quad \lim_{t \to 0} I_r(t) = 0, \]
where \( C \) is a positive constant depending only on \( N, \beta, s \) and \( E \).

**Proof.** Since \( \tau_E = 1_E - 1_{\mathbb{R}^N \setminus E} \), we get
\[ J_r(t) = \int_{Q_{r,t} \cap E} K_s(y, t) dy - \int_{Q_{r,t} \cap E^c} K_s(y, t) dy \]
\[ = \int_{B_{r}^{N-1}} \int_{\gamma(y')} K_s((y', y_N), t) dy_N dy' - \int_{B_{r}^{N-1}} \int_{\gamma(y')} K_s((y', y_N), t) dy_N dy' \]
\[ = \int_{B_{r}^{N-1}} \left( \int_{\gamma(y')} K_s((y', y_N), t) dy_N - \int_{-\gamma(y')} K_s((y', y_N), t) dy_N \right) dy' \]
\[ = \int_{B_{r}^{N-1}} \left( \int_{\gamma(y')} K_s((y', y_N), t) dy_N + \int_{\gamma(y')} K_s((y', -y_N), t) dy_N - \int_{-\gamma(y')} K_s((y', y_N), t) dy_N \right) dy' \]
\[ = \int_{B_{r}^{N-1}} \left( \int_{\gamma(y')} K_s((y', y_N), t) dy_N + \int_{\gamma(y')} K_s((y', -y_N), t) dy_N - \int_{-\gamma(y')} K_s((y', y_N), t) dy_N \right) dy'. \]
Since the map \( y \mapsto K_s(y, t) \) is radial, we have \( K_s((y', y_N), t) = K_s((y', -y_N), t) \) so that
\[ \int_{\gamma(y')} K_s((y', y_N), t) dy_N + \int_{\gamma(y')} K_s((y', -y_N), t) dy_N = 0. \]
Therefore
\[ J_r(t) = -2 \int_{B_{r}^{N-1}} \int_{-\gamma(y')} K_s((y', y_N), t) dy_N dy' = -2 \int_{B_{r}^{N-1}} \int_{0}^{\gamma(y')} K_s((y', y_N), t) dy_N dy'. \]
Then, by (2.6),
\[ |J_r(t)| \leq 2C_{N,s} t \left| \int_{B_r^{N-1}} \int_0^\gamma(y') \frac{1}{|(y',y_N)|^{N+2s}} dy_N dy' \right| \leq Ct_r^{\beta - 2s}. \]

where \( C \) is a positive constant depending on \( N, \beta \) and \( s \) and which may change from a line to another. Next, using (2.6), (1.7) and the dominate convergence theorem, we obtain
\[ \lim_{t \to 0} I_r(t) = 0. \]

This then ends the proof. \( \square \)

**Lemma 2.3.** Let \( s \in (0,1) \) and let \( x = vte_N \in \partial E_t \cap B_{\frac{1}{2s}}(\frac{1}{2s}) \), with \( E_t \) given by (1.4). Then
\[ vt = O \left( (\sigma_s(t))^{-\frac{1+2s}{2s}} \right) \quad \text{as } t \to 0. \]

**Proof.** Since \( x = vte_N \in \partial E_t \), we have that \( u(x, \sigma_s(t)) = 0 \). By the fundamental theorem of calculus, we have
\[ u(x, \sigma_s(t)) = u(0, \sigma_s(t)) + vt \int_0^1 \frac{\partial u}{\partial x_N}(\theta vte_N, \sigma_s(t)) d\theta = 0 \]
so that
\[ vt \int_0^1 \frac{\partial u}{\partial x_N}(\theta vte_N, \sigma_s(t)) d\theta = -u(0, \sigma_s(t)). \] (2.12)

We write
\[ u(0, \sigma_s(t)) = \int_{E^N} K_s(y, \sigma_s(t)) \tau_E(y) dy = \int_{Q_r} K_s(y, \sigma_s(t)) \tau_E(y) dy + \int_{Q_r^c} K_s(y, \sigma_s(t)) \tau_E(y) dy. \]

Then by Lemma 2.2 and (2.6), we have
\[ \left| \int_{Q_r} K_s(y, \sigma_s(t)) \tau_E(y) dy \right| \leq C \sigma_s(t) \quad \text{and} \quad \int_{Q_r^c} K_s(y, \sigma_s(t)) \tau_E(y) dy = O(\sigma_s(t)) \] (2.13)
for some constant \( C \) depending on \( r \). Furthermore by (2.7), we have
\[ \frac{\partial u}{\partial x_N}(\theta vte_N, \sigma_s(t)) \geq C(\sigma_s(t))^{-1/2s}. \] (2.14)

Therefore, the result immediately follows from (2.12), (2.13) and (2.14). \( \square \)

**Lemma 2.4.** Under the assumptions of Lemma 2.3, we have
\[ \int_0^1 \int_{B_r^{N-1}} \gamma(y') - vt \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t)) dy_N d\theta = O(\sigma_s(t)) \quad \text{as } t \to 0. \]
Applying Lemma 2.3, we get

$$
\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dy
$$

Then, for all $\theta \in [0,1]$, By (1.5), (1.6) and a change of variable, we have

$$
\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial P_s}{\partial y_N}(y'(\sigma_s(t)) - \frac{1}{2}, \theta y_N(\sigma_s(t)) - \frac{1}{2})dy
$$

Thus, for all $\theta \in [0,1]$, we have

$$
\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta = O(\sigma_s(t))
$$

as $t \to 0$. This then ends the proof.

$$\square$$

3. Proof of Theorem 1.1 in the case $s \in (0, \frac{1}{2})$

In this section, we start by the following preliminary result.

**Lemma 3.1.** Let $s \in (0, \frac{1}{2})$. We assume that $E$ is of class $C^{1,\beta}$ for some $\beta > 2s$ satisfying (2.1). Then, for all $\theta \in [0,1]$, we have

$$
\int_{\mathbb{R}^N} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^N} P_s(y', 0)dy' + O((\sigma_s(t))^{\frac{2s-1}{2}}).
$$

**Proof.** We have

$$
\int_{\mathbb{R}^N} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = \int_{B_r} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy
$$

$$
+ \int_{B_r^c} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy,
$$

where $B_r$ is the ball of $\mathbb{R}^N$ centered at the origin and of radius $r > 0$. By integration by parts, we have

$$
\int_{B_r} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = 2 \int_{\partial E \cap B_r} K_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y)d\sigma(y)
$$

$$
+ \int_{\partial B_r} K_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y)\tau_E(y)d\sigma'(y).
$$

Therefore

$$
\int_{\mathbb{R}^N} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = 2 \int_{\partial E \cap B_r} K_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y')d\sigma(y')
$$

$$
+ \int_{B_r^c} \frac{\partial K_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy + \int_{\partial B_r} K_s(y', y_N - vt\theta, \sigma_s(t))\frac{y_N}{t}\tau_E(y)d\sigma'(y).
$$

(3.2)
Then by a change of variable and (2.5), we have
\[
\int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y)d\sigma(y) = \int_{B^{N-1}_r} \mathcal{K}_s(y', \gamma(y') - vt\theta, \sigma_s(t))dy'.
\]
By the Fundamental Theorem of calculus, we can write
\[
\mathcal{K}_s(y', \gamma(y') - vt\theta, \sigma_s(t)) = \mathcal{K}_s(y', 0, \sigma_s(t)) + (\gamma(y') - vt\theta) \int_0^1 \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta' (\gamma(y') - vt\theta)), \sigma_s(t))d\theta'.
\] (3.3)

In the following, we let
\[
\varepsilon(y') := \gamma(y') - vt\theta.
\] (3.4)

Then we have
\[
\int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y)d\sigma(y) = \int_{B^{N-1}_r} \mathcal{K}_s(y', \varepsilon(y'), \sigma_s(t))dy' = \int_{B^{N-1}_r} \mathcal{K}_s(y', 0, \sigma_s(t))dy' + \int_0^1 \int_{B^{N-1}_r} \varepsilon(y') \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta' \varepsilon(y'), \sigma_s(t))dy'd\theta'.
\] (3.5)

Therefore, by a change of variable, (1.5) and (1.6), we have
\[
\int_{B^{N-1}_r} \mathcal{K}_s(y', 0, \sigma_s(t))dy' = \int_{B^{N-1}_r} (\sigma_s(t))^{-1/2s} P_s(y'(\sigma_s(t))^{-1/2s}, 0)dy'
\]
\[
= (\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + (\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1} \setminus B^{N-1}_r(\sigma_s(t))^{-1/2s}} P_s(y', 0)dy'
\]
\[
= (\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O \left( (\sigma_s(t))^{-1/2s} \right)
\]
(3.6)

By a change of variable, (3.6) and (3.4), we have
\[
\int_{B^{N-1}_r} \varepsilon(y') \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta' \varepsilon(y'), \sigma_s(t))dy'
\]
\[
= (\sigma_s(t))^{-1/s} \int_{B^{N-1}_r(r(\sigma_s(t))^{-1/2s})} \varepsilon(y'(\sigma_s(t))^{1/2s}) \frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}, \varepsilon(y'(\sigma_s(t))^{1/2s}))dy'.
\]

We use (2.2), (3.4) and Lemma 2.3 to get
\[
(\sigma_s(t))^{-1/s} \varepsilon(y'(\sigma_s(t))^{1/2s}) = O \left( |y'|^{1+s} (\sigma_s(t))^{s-1/s} \right) - vt\theta(\sigma_s(t))^{-1/s}
\]
\[
= O \left( |y'|^{1+s} (\sigma_s(t)^{2s-1}) \right) + O \left( (\sigma_s(t))^{2s-1} \right)
\]
in $B^{N-1}_r(r(\sigma_s(t))^{-1/2s})$.

Then by (1.6), we have
\[
\int_{B^{N-1}_r} \varepsilon(y') \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta' \varepsilon(y'), \sigma_s(t))dy'
\]
\[
= O \left( (\sigma_s(t))^{2s-1} \int_{B^{N-1}_r(r(\sigma_s(t))^{-1/2s})} |y'|^{1+s} \frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}, \varepsilon(y'(\sigma_s(t))^{1/2s}))dy' \right)
\]
\[
+ O \left( (\sigma_s(t))^{2s-1} \int_{B^{N-1}_r(r(\sigma_s(t))^{-1/2s})} \frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}, \varepsilon(y'(\sigma_s(t))^{1/2s}))dy' \right)
\]
\[
= O \left( (\sigma_s(t))^{2s-1} \int_{\mathbb{R}^{N-1}} \frac{1 + |y'|^{1+s}}{1 + |y'|^{N+s+1}}dy' \right) = O \left( (\sigma_s(t))^{2s-1} \right).
\]
By a change of variable and the fact that
tH

Under the assumptions of Lemma 3.1, we have

where

\tilde{t}

By the fundamental theorem of calculus, we have

We put with

Proof.

Then

\int_{\partial E \cap B_r} K_s(y', y_N - vt \theta, \sigma_s(t)) \nu_N(y) d\sigma(y) = (\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O \left( (\sigma_s(t))^{2s-1/2s} \right) \text{ as } t \to 0.

(3.8)

By a change of variable and the fact that \(|\tau_E(y)| \leq 1\), we have

\int_{B_r} \frac{\partial K_s}{\partial y_N}(y', y_N - vt \theta, \sigma_s(t)) \tau_E(y) dy = O \left( (\sigma_s(t))^{-1/2s} \int_{B_r} \frac{\partial P_s}{\partial y_N}(y', y_N - vt(\sigma_s(t))^{-1/2s} \theta) dy \right)

= O \left( (\sigma_s(t))^{-1/2s} \int_{B_r(\sigma_s(t))^{-1/2s}} \frac{1}{1 + \|y\|^{N+2s+1}} dy \right) = O(\sigma_s(t)).

(3.9)

We use (1.7) to get, as \( t \to 0 \),

\left| \int_{\partial B_r} K_s(y', y_N - vt \theta, \sigma_s(t)) \frac{\nu_N}{r} \tau_E(y) d\sigma'(y) \right| \leq \int_{\partial B_r} K_s(y', y_N - vt \theta, \sigma_s(t)) d\sigma'(y)

\leq \sigma_s(t) \int_{\partial B_r} \frac{C_{N,s}}{|(y', y_N - vt \theta)|^{N+2s}} d\sigma(y) = O(\sigma_s(t)).

Therefore, the expansion (3.1) follows immediately from (3.2), (3.8), (3.9) and the above estimate. This ends the proof.

The following result completes the proof of Theorem 1.1 in the case \( s \in (0, 1/2) \).

Proposition 3.2. Under the assumptions of Lemma 3.1, we have

\[ v = a_{N,s} H_s(0) + o_t(1) \quad \text{as } t \to 0, \]

(3.10)

where \( H_s(0) \) is the fractional mean curvature of \( \partial E \) at the point 0 and the positive constant \( a_{N,s} \) is given by

\[ a_{N,s} = \frac{C_{N,s}}{\int_{\mathbb{R}^{N-1}} P_s(y', 0) dy'}. \]

Proof. We put with \( x = v t e_N \) and we recall that

\[ u(x, \sigma_s(t)) = \int_{\mathbb{R}^N} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = 0. \]

By the fundamental theorem of calculus, we have

\[ K_s(y - x, \sigma_s(t)) = K_s(y, \sigma_s(t)) - vt \int_0^1 \frac{\partial K_s}{\partial y_N}(y', y_N - vt \theta, \sigma_s(t)) d\theta. \]

Then

\[ u(x, \sigma_s(t)) = \bar{J}_r(t) + \sigma_s(t) \tilde{I}_r(t) + \sigma_s(t) C_{N,s} \int_{\mathbb{R}^N \setminus Q_r} \frac{\tau_E(y)}{|y|^{N+2s}} dy - vt \int_{\mathbb{R}^N} \int_0^1 \frac{\partial K_s}{\partial y_N}(y', y_N - vt \theta, \sigma_s(t)) \tau_E(y) d\theta dy, \]

(3.11)

where \( \bar{J}_r(t) = J_r(\sigma_s(t)), \tilde{I}_r(t) = I_r(\sigma_s(t)) \), while \( I_r(t) \) and \( J_r(t) \) are given by Lemma 2.2 Moreover, by Lemma 3.1 we have

\[ \int_{\mathbb{R}^N} \frac{\partial K_s}{\partial y_N}(y', y_N - vt \theta, \sigma_s(t)) \tau_E(y) dy = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O \left( (\sigma_s(t))^{2s-1/2s} \right). \]
Therefore
\[ \int_0^1 \int_{\mathbb{R}^N} \frac{\partial K_s}{\partial y_N}(y', y_N - v\theta, \sigma_s(t)) \tau_E(y) dy d\theta = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O \left( (\sigma_s(t))^{2s-\frac{1}{2s}} \right) \quad \text{as } t \to 0. \] (3.12)

Putting (3.12) in (3.11), we obtain that
\[ u(x, \sigma_s(t)) = \tilde{J}_r(t) + \sigma_s(t) \tilde{I}_r(t) + \sigma_s(t) C_{N,s} \int_{\mathbb{R}^N \setminus Q_r} \frac{\tau_E(y)}{|y|^{N+2s}} dy 
- vt \left( 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O \left( (\sigma_s(t))^{2s-\frac{1}{2s}} \right) \right) 
= \sigma_s(t) \left[ (\sigma_s(t))^{-1} \tilde{J}_r(t) + \tilde{I}_r(t) + C_{N,s} \int_{\mathbb{R}^N \setminus Q_r} \frac{\tau_E(y)}{|y|^{N+2s}} dy - 2vt(\sigma_s(t))^{\frac{2s-1}{2s}} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O(\sigma_s(t)) \right]. \]

Recalling that \( \sigma_s(t) = t^{\frac{2s}{1+2s}} \) and using the fact that \( u(x, \sigma_s(t)) = 0 \), we have
\[ 0 = t^{\frac{2s}{1+2s}} \tilde{J}_r(t) + \tilde{I}_r(t) + C_{N,s} \int_{\mathbb{R}^N \setminus Q_r} \frac{\tau_E(y)}{|y|^{N+2s}} dy - 2t^{\frac{2s}{1+2s}} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O(t^{\frac{2s}{1+2s}}) \quad \text{as } t \to 0. \]

As a consequence,
\[ \left| C_{N,s} H_s(0) - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' \right| \leq C_{N,s} H_s(0) - C_{N,s} \int_{\mathbb{R}^N \setminus Q_r} \frac{\tau_E(y)}{|y|^{N+2s}} dy + |t^{\frac{2s}{1+2s}} \tilde{J}_r(t)| + \tilde{I}_r(t) + O(t^{\frac{2s}{1+2s}}). \]

Therefore by Lemma 2.2 taking the limsup as \( t \to 0 \) and as \( r \to 0 \) respectively, we obtain
\[ \limsup_{t \to 0} \left| C_{N,s} H_s(0) - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' \right| = 0. \]

Hence
\[ v = \frac{C_{N,s} H_s(0)}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy'} + o_t(1) \quad \text{as } t \to 0. \]

This then ends the proof. \( \square \)

4. Proof of Theorem 1.1 in the case \( s \in (\frac{1}{2}, 1) \)

We have the following result.

**Proposition 4.1.** We consider \( E \) a hypersurface of class \( C^3 \) satisfying the condition in Section 2. For \( s \in (1/2, 1) \), we have
\[ v = c_{N,s} H(0) + O \left( t^{\frac{2s-1}{2s}} \right), \quad \text{as } t \to 0. \] (4.1)

where \( H(0) \) is the normalized mean curvature of \( \partial E \) at \( 0 \) and the positive constant \( c_{N,s} \) is given by
\[ c_{N,s} = \int_{\mathbb{R}^{N-1}} \frac{|y'|^2 P_s(y', 0) dy'}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy'}. \]

**Proof.** We let \( x = vte_N \in \partial E_t \) and we expand
\[ u(x, \sigma_s(t)) = \int_{\mathbb{R}^N} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = \int_{Q_r} K_s(y - x, \sigma_s(t)) \tau_E(y) dy + \int_{Q_r^c} K_s(y - x, \sigma_s(t)) \tau_E(y) dy. \] (4.2)
By (2.6) and Lemma 2.3 we have

$$\int_{Q_r^c} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = O(\sigma_s(t)) \quad \text{as } t \to 0.$$  

Therefore

$$u(x, \sigma_s(t)) = \int_{Q_r} K_s(y - x, \sigma_s(t)) \tau_E(y) dy + O(\sigma_s(t)). \quad (4.3)$$

By a change of variable, the fact that \(\tau_E = 1_E(x) - 1_{\mathbb{R}^N \setminus \mathcal{P}}(x)\) and \(x = vt \sigma_N\), we have

$$\int_{Q_r} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = \int_{E \cap Q_r} K_s(y - x, \sigma_s(t)) dy - \int_{E^c \cap Q_r} K_s(y - x, \sigma_s(t)) dy$$

$$= \int_{B_x^{N-1}} \int_{-r}^{\gamma(y')} K_s(y - x, \sigma_s(t)) dy - \int_{B_x^{N-1}} \int_{\gamma(y')}^{-r} K_s(y - x, \sigma_s(t)) dy$$

$$= \int_{B_x^{N-1}} \int_{-r-vt}^{\gamma(y')-vt} K_s(y, \sigma_s(t)) dy - \int_{B_x^{N-1}} \int_{\gamma(y')-vt}^{-r-vt} K_s(y, \sigma_s(t)) dy$$

$$= 2 \int_{B_x^{N-1}} \int_{0}^{\gamma(y')-vt} K_s(y, \sigma_s(t)) dy + \int_{B_x^{N-1}} \int_{-r-vt}^{-r+vt} K_s(y, \sigma_s(t)) dy.$$

The last line is due to the fact that the map \(y_N \to K_s(y, \sigma_s(t))\) is even so that

$$\int_{0}^{r-vt} K_s(y, \sigma_s(t)) dy_N = - \int_{0}^{-r+vt} K_s(y, \sigma_s(t)) dy_N.$$

Therefore we have

$$\int_{Q_r} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = 2 \int_{B_x^{N-1}} \int_{0}^{\gamma(y')-vt} K_s(y, \sigma_s(t)) dy + \int_{B_x^{N-1}} \int_{-r-vt}^{-r+vt} K_s(y, \sigma_s(t)) dy. \quad (4.4)$$

By (2.6) and the fact that \(vt = o_t(1)\), we have

$$\int_{B_x^{N-1}} \int_{-r-vt}^{-r+vt} K_s(y, \sigma_s(t)) dy = O(\sigma_s(t)). \quad (4.5)$$
By a change of variable, the Fundamental Theorem of Calculus, (1.5) and (2.3), we have

\[
\int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} K_s(y, \sigma_s(t))dy
\]

\[
= \int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} K_s(y', 0, \sigma_s(t))dy + \int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} \int_{0}^{1} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta
\]

\[
= \int_{B_{N-1}^{n-1}} K_s(y', 0, \sigma_s(t))\left(\frac{1}{2} \gamma y_j y_j + O(|y'|^3 - vt)\right)dy'
\]

\[
+ \int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} \int_{0}^{1} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta
\]

\[
= \frac{\Delta \gamma(0)}{2(N-1)} \int_{B_{N-1}^{n-1}} |y'|^2 K_s(y', 0, \sigma_s(t))dy' - vt \int_{B_{N-1}^{n-1}} K_s(y', 0, \sigma_s(t))dy'
\]

\[
+ \int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} \int_{0}^{1} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta + O\left(\int_{B_{N-1}^{n-1}} |y'|^3 K_s(y', 0, \sigma_s(t))dy'\right).
\]

Therefore, recalling (2.4),

\[
\int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} K_s(y, \sigma_s(t))dy = \frac{H(0)}{2} \int_{B_{N-1}^{n-1}} |y'|^2 K_s(y', 0, \sigma_s(t))dy' - vt \int_{B_{N-1}^{n-1}} K_s(y', 0, \sigma_s(t))dy'
\]

\[
+ \int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} \int_{0}^{1} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta
\]

\[
+ O\left(\int_{B_{N-1}^{n-1}} |y'|^3 K_s(y', 0, \sigma_s(t))dy'\right).
\]

(4.6)

By a change of variable and (1.5), we have

\[
\int_{B_{N-1}^{n-1}} |y'|^2 K_s(y', 0, \sigma_s(t))dy' = (\sigma_s(t))^{\frac{1}{2\alpha}} \int_{B_{N-1}^{n-1}} |y'|^2 P_s(y', 0)dy'
\]

(4.7)

and

\[
\int_{B_{N-1}^{n-1}} K_s(y', 0, \sigma_s(t))dy' = (\sigma_s(t))^{-\frac{1}{2\alpha}} \int_{B_{N-1}^{n-1}} P_s(y', 0)dy'.
\]

(4.8)

Moreover by (1.6), we get

\[
(\sigma_s(t))^{\frac{1}{\alpha}} \int_{\mathbb{R}^{N-1} \setminus B_{N-1}^{n-1}} |y'|^2 P_s(y', 0)dy' + (\sigma_s(t))^{\frac{1}{2\alpha}} \int_{\mathbb{R}^{N-1} \setminus B_{N-1}^{n-1}} P_s(y', 0)dy' = O(\sigma_s(t)) \quad \text{as} \quad t \to 0
\]

(4.9)

and

\[
\int_{B_{N-1}^{n-1}} |y'|^3 K_s(y', 0, \sigma_s(t))dy' = O(\sigma_s(t)).
\]

(4.10)

By Lemma 2.4, we get

\[
\int_{B_{N-1}^{n-1}} \int_{0}^{\gamma(y')-vt} \int_{0}^{1} y_N \frac{\partial K_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta = O(\sigma_s(t)).
\]

(4.11)
Combining (4.1), (4.6), (4.7), (4.8), (4.9) and (4.11), we obtain
\[
\int_{Q_r} K_s(y - x, \sigma_s(t)) \tau_E(y) dy = (\sigma_s(t))^{1/2} H(0) \frac{1}{2} \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) dy' - vt(\sigma_s(t))^{1/2} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O(\sigma_s(t)).
\]

By (4.3) and (4.12), we obtain
\[
u(x, \sigma_s(t)) = (\sigma_s(t))^{1/2} H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) dy' - 2vt(\sigma_s(t))^{-1/2} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O(\sigma_s(t))
\]
\[
= (\sigma_s(t))^{-1/2} \left[ (\sigma_s(t))^{1/2} H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) dy' - 2vt \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O((\sigma_s(t))^{1/2}) \right].
\]

Since \( x = vt \nu \in \partial E_t \), we have \( u(x, \sigma_s(t)) = 0 \). Now, from the definition of \( \sigma_s(t) = t^s \), we deduce that
\[
H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) dy' - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O \left( t^{\frac{2s+1}{s}} \right) = 0.
\]

Thus
\[
v = c_{N,s} H(0) + O(t^{\frac{2s-1}{s}}),
\]
where
\[
c_{N,s} = \frac{\int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0) dy'}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy'}.
\]

This then ends the proof.

5. Proof of Theorem 1.1 in the case \( s = \frac{1}{2} \)

As usual, we consider the function
\[
u(x, t) = K_{1/2}(\cdot, t) \star \tau_E(x)
\]
and recall that
\[E_t := \{ x \in \mathbb{R}^N : u(x, \sigma_{1/2}(t)) \geq 0 \}.
\]
To alleviate the notations, for the following of this section, we write \( \sigma_{1/2}(t) := \sigma(t) \).

**Proposition 5.1.** For \( s = 1/2 \), we have
\[
v = b_N(t) H(0) + O \left( \frac{1}{\log(\sigma_{1/2}(t))} \right) \quad \text{as } t \to 0,
\]
where \( H(0) \) is the mean curvature of \( \partial E \) at 0.

**Proof.** Recall that \( x = vt \nu \to 0 \) as \( t \to 0 \), thanks to Lemma 2.3. We write
\[
u(x, \sigma(t)) = \int_{\mathbb{R}^N} K_{1/2}(y - x, \sigma(t)) \tau_E(y) dy = \int_{Q_r} K_{1/2}(y - x, \sigma(t)) \tau_E(y) dy + \int_{Q_r^c} K_{1/2}(y - x, \sigma(t)) \tau_E(y) dy,
\]
where \( Q_r = B_r^{N-1} \times (-r, r) \). By (2.6), we have
\[
\int_{Q_r^c} K_{1/2}(y - x, \sigma(t)) \tau_E(y) dy = O(\sigma(t)) \quad \text{as } t \to 0.
\]

Then, we have
\[
u(x, \sigma(t)) = \int_{Q_r} K_{1/2}(y - x, \sigma(t)) \tau_E(y) dy + O(\sigma(t)).
\]
By a change of variable and (2.3), we have

$$\int_{Q_r} K_{1/2}(y - x, \sigma(t)) \tau_{E}(y) dy = \int_{\mathbb{R}^n \cap Q_r} K_{1/2}(y - x, \sigma(t)) dy - \int_{E \cap Q_r} K_{1/2}(y - x, \sigma(t)) dy$$

$$= \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y - x, \sigma(t)) dy - \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y - x, \sigma(t)) dy$$

$$= \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy - \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy$$

$$= \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy + \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy - \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy$$

The last line is due to the fact that the map $y_N \to K_{1/2}(y, \sigma(t))$ is even so that

$$\int_{0}^{r} K_{1/2}(y, \sigma(t)) dy_N = - \int_{0}^{-r} K_{1/2}(y, \sigma(t)) dy_N.$$

Therefore we have

$$\int_{Q_r} K_{1/2}(y - x, \sigma(t)) \tau_{E}(y) dy = 2 \int_{B_r^n}^{r} \int_{0}^{r} K_{1/2}(y, \sigma(t)) dy + \int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy. \quad (5.4)$$

Using (2.3), we find that

$$\int_{B_r^n}^{r} \int_{-r}^{r} K_{1/2}(y, \sigma(t)) dy = O(\sigma(t)). \quad (5.5)$$

By a change of variable, the fundamental theorem of calculus, (1.5) and (2.3), we have

$$\int_{B_r^n}^{r} \int_{0}^{r} K_{1/2}(y, \sigma(t)) dy$$

$$= \int_{B_r^n}^{r} \int_{0}^{r} y_N \frac{\partial K_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t)) dyd\theta$$

$$= \int_{B_r^n}^{r} \int_{0}^{r} \left( \frac{1}{2} y_N (1 - (\theta^2 - 1)) \right) dy'$$

$$+ \int_{B_r^n}^{r} \int_{0}^{r} \left( y_N \frac{\partial K_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t)) dyd\theta \right)$$

$$= \frac{\Delta \gamma(0)}{2(N - 1)} \int_{B_r^n}^{r} |y'|^2 K_{1/2}(y', 0, \sigma(t)) dy' - vt \int_{B_r^n}^{r} K_{1/2}(y', 0, \sigma(t)) dy'$$

$$+ \int_{B_r^n}^{r} \int_{0}^{r} y_N \frac{\partial K_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t)) dyd\theta + O \left( \int_{B_r^n}^{r} |y'|^3 K_{1/2}(y', 0, \sigma(t)) dy' \right).$$
Therefore
\[
\int_{B_{r}^{N-1}} \gamma(y') \ K_{1/2}(y, \sigma(t)) dy = \frac{H(0)}{2} \int_{B_{r}^{N-1}} |y'|^2 K_{1/2}(y', 0, \sigma(t)) dy' 
\]
\[- vt \int_{B_{r}^{N-1}} K_{1/2}(y', 0, \sigma(t)) dy' + \int_{B_{r}^{N-1}} \gamma(y') \ dt \int_{B_{r}^{N-1}} \int_{0}^{1} yN \frac{\partial K_{1/2}}{\partial yN}(y', \theta yN, \sigma(t)) dyd\theta + O(\sigma(t)).
\]
(5.6)

By (1.5), (1.6) and a change of variable, we have
\[
\int_{B_{r}^{N-1}} \gamma(y') \ yN \frac{\partial K_{1/2}}{\partial yN}(y', \theta yN, \sigma(t)) dy = O \left( \int_{B_{r}^{N-1}} \frac{\sigma(t)^{-2} (\gamma(y' \sigma(t)) - vt)^2}{(1 + |y'|^2)^{\frac{N+2}{2}}} dy \right)
\]
\[
= O \left( \sigma^{-2}(t) \int_{B_{r}^{N-1}} \frac{(\sigma(t)^2 |y'|^2 - vt)^2}{(1 + |y'|^2)^{\frac{N+2}{2}}} dy \right) = O(\sigma(t)) + O(vt) + O(v^2 t^2(\sigma(t))^{-2}).
\]

Now Lemma 2.3 yields $vt = O(\sigma(t)^2)$ and thus
\[
\int_{B_{r}^{N-1}} \gamma(y') \ dt \int_{B_{r}^{N-1}} \int_{0}^{1} yN \frac{\partial K_{1/2}}{\partial yN}(y', \theta yN, \sigma(t)) dyd\theta = O(\sigma(t)).
\]
(5.7)

We get from (5.3), (5.4), (5.5), (5.6) and (5.7) that
\[
u(x, \sigma(t)) = \sigma(t) H(0) \int_{B_{r}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy' - 2vt \int_{B_{r}^{N-1}} P_{1/2}(y', 0) dy' + O(\sigma(t)) \quad \text{as } t \to 0.
\]

Thanks to (1.6), we have
\[
\int_{\mathbb{R}^{N-1} \setminus B_{r}^{N-1}} P_{1/2}(y', 0) dy' = O(\sigma(t)^2) \quad \text{and} \quad \int_{B_{r}^{N-1} \setminus B_{\sigma(t)}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy' = O(1) \quad \text{as } t \to 0.
\]

This implies that
\[
u(x, \sigma(t)) = \sigma(t) H(0) \int_{B_{\sigma(t)}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy' - 2vt(\sigma(t))^{-1} \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy' + O(\sigma(t)).
\]
(5.8)

Using polar coordinates and (1.6), we then have
\[
\int_{B_{\sigma(t)}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy' \asymp C_{N,1/2}\omega_{N-2} \int_{0}^{1/\sigma(t)} \frac{m^N}{(1 + m^2)^{\frac{N+2}{2}}} dm,
\]
(5.9)

where $\omega_{N-2} := |S^{N-2}|$. By the change of variable $\rho = \frac{1}{m}$, we have
\[
\int_{0}^{1/\sigma(t)} \frac{m^N}{(1 + m^2)^{\frac{N+2}{2}}} dm = \int_{\sigma(t)}^{+\infty} \frac{1}{\rho (1 + \rho^2)^{\frac{N+2}{2}}} d\rho
\]
\[
= \int_{\sigma(t)}^{1} \frac{1}{\rho (1 + \rho^2)^{\frac{N+2}{2}}} d\rho + \int_{1}^{+\infty} \frac{1}{\rho (1 + \rho^2)^{\frac{N+2}{2}}} d\rho
\]
\[
= \int_{\sigma(t)}^{1} \frac{1}{\rho (1 + \rho^2)^{\frac{N+2}{2}}} d\rho + O(1) = - \log(\sigma(t)) + O(1).
\]
Letting \( b_N(t) := \frac{\int_{B_{\sigma(t)}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy'}{-2 \log(\sigma(t)) \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy'} \), by (5.8), (5.9) and the above estimate, we obtain, as \( t \to 0 \),

\[
0 = \sigma(t) \log(\sigma(t)) \left[ H(0) \frac{\int_{B_{\sigma(t)}^{N-1}} |y'|^2 P_{1/2}(y', 0) dy'}{\log(\sigma(t))} - 2v \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy' + O \left( \frac{1}{\log(\sigma(t))} \right) \right].
\]

Hence

\[
v = b_N(t) H(0) + O \left( \frac{1}{\log(\sigma(t))} \right) \quad \text{as} \quad t \to 0.
\]

The proof is then ended. \( \square \)

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