ABSENCE OF CARTAN SUBALGEBRA IN CONTINUOUS CORES OF FREE PRODUCT VON NEUMANN ALGEBRAS

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Abstract. We show that the continuous core of any type III factor arising as a direct summand of a free product von Neumann algebra has no Cartan subalgebra. This is a complement to previous works due to Houdayer–Ricard and Boutonnet–Houdayer–Raum.

1. Introduction and Statement

This short note complements two recent important works on free product von Neumann algebras, due to Houdayer–Ricard [5] and Boutonnet–Houdayer–Raum [1] establishing, among others, the absence of Cartan subalgebra in any free product von Neumann algebra. Those works are based upon the so-called deformation/rigidity theory due to Popa, and the latter one generalizes Ioana’s previous important results [6] on type II_{1} factors to arbitrary factors. See [1, §1] for historical backgrounds on the subject matter.

Let \( M_1, M_2 \) be two nontrivial (i.e., \( \neq \mathbb{C} \)) von Neumann algebras with separable preduals, and \( \varphi_1, \varphi_2 \) be faithful normal states on them, respectively. Then we take their free product \((M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)\). By [11, Theorem 4.1] the free product von Neumann algebra \( M \) admits the following general structure: \( M = M_d \oplus M_c \) with finite dimensional \( M_d \) and diffuse \( M_c \) such that \( M_d \) can explicitly be calculated with possibly \( M = M_c \), and moreover, such that if \((\dim(M_1), \dim(M_2)) \neq (2, 2)\), then \( M_c \) must be a full factor of type II_{1} or III_{\lambda} (\( \lambda \neq 0 \)) with the T-set \( T(M_c) = \ker(t \mapsto \sigma^{t} \varphi_1 t \star \sigma^{t} \varphi_2 t)\); otherwise \( M_c = L^\infty[0, 1] \otimes M_2(\mathbb{C}) \). This note is devoted to establishing the following theorem:

Theorem 1. If \( M_c \) is of type III (automatically a factor of type III) or equivalently the free product state \( \varphi \) is not tracial, then the continuous core \( \tilde{M}_c = M_c \rtimes_{\varphi_c} R \) with \( \varphi_c := \varphi |_{M_c} \) of the diffuse factor part \( M_c \) does never possess any Cartan subalgebra.

It is well-known that for a von Neumann algebra \( N \) and a MASA \( A \) in \( N \) one has

\[ A \rtimes_{\psi \circ E} R = A \rtimes_{\psi} L(R) \]

with a faithful normal state \( \psi = \psi \circ E \) on \( N \), where \( E \) denotes the unique normal conditional expectation from \( N \) onto \( A \). We remark that the work [1] shows the absence of only such special Cartan subalgebra in the continuous core of an arbitrary free product von Neumann algebra. Hence Theorem [1] is seemingly stronger than the original one [1, Theorem A] (though it is unclear at the moment of this writing whether the consequence of Theorem [1] does not follow from that of [1, Theorem A] as a general principle). Therefore, Theorem [1] and [12, §§2.4] establish that all the canonical semifinite factors attached to any type III factor arising as a direct summand of a free product von Neumann algebra have no Cartan subalgebra.
Consequently, this note completes, though its contribution is small, the study of proving the absence of Cartan subalgebra for arbitrary free product von Neumann algebras.

We follow the notational rule in our previous papers [11, 12] (see the glossary at the end of the introduction of [11]), and simply refer to [10, Ch.XII] for the general structure theory for type III factors. The continuous core $\tilde{M}$ already appeared above is exactly the semifinite von Neumann algebra of the so-called associated covariant system of $M$ in [10] Definition XII.1.3]

2. Proof

The next observation is the key of our discussion below. In what follows, for a given (unital) inclusion $P \supset Q$ of von Neumann algebras we denote by $\mathcal{N}_P(Q)$ the set of all normalizers of $Q$ in $P$, i.e., all $u \in P^\prime$ with $uQu^* = Q$.

**Lemma 2.** Let $N$ be a von Neumann algebra with separable predual and $\psi$ be a faithful normal positive linear functional on it. Then the normalizers $\mathcal{N}_{\mathcal{N}_\psi}(\mathcal{C}_1 \rtimes_{\sigma_\psi} \mathbb{R})$ of $\mathcal{C}_1 \rtimes_{\sigma_\psi} \mathbb{R} = \mathcal{C}_1 \bar{\otimes} L(\mathbb{R})$ in $\mathcal{N}_\sigma \mathbb{R}$ must sit inside $\mathcal{N}_\psi \rtimes_{\sigma_\psi} \mathbb{R} = N_\psi \bar{\otimes} L(\mathbb{R})$, and hence $\mathcal{N}_{\mathcal{N}_\psi}(\mathcal{C}_1 \rtimes_{\sigma_\psi} \mathbb{R})$ is exactly the unitary group of $N_\psi \bar{\otimes} L(\mathbb{R})$. In particular, if the centralizer $N_\psi$ is trivial, then $\mathcal{C}_1 \rtimes_{\sigma_\psi} \mathbb{R}$ must be a singular MASA in $\mathcal{N}_\sigma \mathbb{R}$.

**Proof.** The discussion below follows the idea of the proof of [8, Theorem 2.1].

Let $\rho : \mathbb{R} \rightarrow L^2(\mathbb{R})$ be the ‘right’ regular representation, i.e., $\rho_t = \lambda_{-t}$, $t \in \mathbb{R}$, with the usual ‘left’ regular representation $\lambda : \mathbb{R} \rightarrow L^2(\mathbb{R})$. It is standard, see e.g. [3, Theorem 3.11], that $\mathcal{N}_\sigma \mathbb{R} \supseteq \mathcal{C}_1 \rtimes_{\sigma_\psi} \mathbb{R}$ is identical to $(\mathcal{N} \otimes B(L^2(\mathbb{R}))) (\sigma_\psi \bar{\otimes} \text{Ad}_\rho(\mathbb{R}))$, which is conjugate to

\[
(N \otimes B(L^2(\mathbb{R}))) (\sigma_\psi \bar{\otimes} \text{Ad}_\rho(\mathbb{R})) \supseteq \mathcal{C}_1 \bar{\otimes} L^\infty(\mathbb{R})
\]

by taking the Fourier transform on the second component, where $L^\infty(\mathbb{R})$ acts on $L^2(\mathbb{R})$ by multiplication and the $v_t$, $t \in \mathbb{R}$, are the unitary elements in $L^\infty(\mathbb{R})$ defined to be $v_t(s) := e^{its}$, $s \in \mathbb{R}$. Hence it suffices to work with the inclusion (1) instead of the original inclusion.

Let $u \in (N \otimes B(L^2(\mathbb{R}))) (\sigma_\psi \bar{\otimes} \text{Ad}_\rho(\mathbb{R}))$ be a unitary element such that $u(\mathcal{C}_1 \bar{\otimes} L^\infty(\mathbb{R})) u^* = \mathcal{C}_1 \bar{\otimes} L^\infty(\mathbb{R})$. By [3, Appendix IV] (together with a trick used in the proof of [7, Theorem 17.41] if necessary) one can choose a non-singular Borel bijection $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ in such a way that $u(1 \otimes f) u^* = 1 \otimes (f \circ \alpha^{-1}) = 1 \otimes u_{s\alpha} f u_{s\alpha}^*$ for every $f \in L^\infty(\mathbb{R})$, where $u_{s\alpha}(g) = [(dn \alpha^{-1}/dm)(s)]^{1/2} g(\alpha^{-1}(s))$, $g \in L^2(\mathbb{R})$ with the Lebesgue measure $m(ds) = ds$. Set $w := u(1 \otimes u_{s\alpha}^*)$, a unitary element in $(N \otimes B(L^2(\mathbb{R}))) \cap (\mathcal{C}_1 \bar{\otimes} L^\infty(\mathbb{R})) = N \bar{\otimes} L^\infty(\mathbb{R})$ by [10, Theorem IV.5.9; Corollary IV.5.10], since $L^\infty(\mathbb{R})$ is a MASA in $B(L^2(\mathbb{R}))$. Since $(\sigma_\psi \bar{\otimes} \text{Ad}_\rho)\!(u) = u$, for every $t \in \mathbb{R}$ one has $w = (\sigma_\psi \bar{\otimes} \text{id})(w)(1 \otimes e^{it(\cdot)-\alpha^{-1}(\cdot)})$, hence

\[
(\sigma_\psi \bar{\otimes} \text{id})(w)(1 \otimes e^{it(\cdot)-\alpha^{-1}(\cdot)}) = (1 \otimes e^{it(\cdot)-\alpha^{-1}(\cdot)})w.
\]

Since the standard Hilbert space $\mathcal{H} := L^2(N)$ is separable (see e.g. [13, Lemma 1.8]), we can appeal to the disintegration

\[
\mathcal{H} \otimes L^2(\mathbb{R}) = \int_\mathbb{R} \mathcal{H}(s) ds, \quad N \otimes L^\infty(\mathbb{R}) = \int_\mathbb{R} N(s) ds
\]

with the constant fields $\mathcal{H}(s) = \mathcal{H}$, $N(s) = N$ (see e.g. [4, Part II, Ch. 3, §4; Corollary of Proposition 3]). Thus we can write $w = \int_\mathbb{R} w(s) ds$ and choose $s \mapsto w(s)$ as a measurable field of unitary elements in $N$ (see e.g. [4, Part II, Ch. 2, p.183]). By the identification [4] we observe that

\[
(\sigma_\psi \bar{\otimes} \text{id})(w)(1 \otimes \Delta_\psi^{-it} \bar{\otimes} 1) \Delta_\psi^{-it} \Delta_\psi^{-it} ds = \int_\mathbb{R} \Delta_\psi^{-it} w(s) \Delta_\psi^{-it} ds = \int_\mathbb{R} \sigma_\psi^{-it}(w(s)) ds,
\]
where \( \Delta_\psi \) is the modular operator associated with \( \psi \). Therefore, the identity (2) is translated into

\[
\int_{\mathbb{R}} \sigma_1^{\psi}(w(s)) \, ds = \int_{\mathbb{R}} e^{it(\alpha^{-1}(s)-s)} w(s) \, ds.
\]

This implies, by e.g. [4] Part II, Ch. 2, §3, Corollary of Proposition 2, that there exists a co-null subset \( S \) of \( \mathbb{R} \) so that for every \( s \in S \), one has \( \sigma_1^{\psi}(w(s)) = e^{it(\alpha^{-1}(s)-s)} w(s) \) for all rational numbers \( t \) and hence for all \( t \in \mathbb{R} \) by continuity. Therefore, for all \( s \in S \), \( t \rightarrow \sigma_1^{\psi}(w(s)) \) has an entire extension \( \sigma_1^\infty(w(s)) := e^{iz(\alpha^{-1}(s)-s)} w(s) \) so that by e.g. Exercise VIII.2.2

\[
\psi(1) = \psi(w(s)w(s)) = \psi(\sigma_1^{\psi}(w(s))w(s)^*) = e(s-\alpha^{-1}(s))\psi(w(s)w(s)^*) = e(s-\alpha^{-1}(s))\psi(1),
\]

implying that \( \alpha^{-1}(s) = s \) and \( w(s) \in N_{\psi} \). Thanks to [4] Part II, Ch. 3, §1, Theorem 1 we conclude that

\[
u = w = \int_{\mathbb{R}} w(s) \, ds \in \int_{\mathbb{R}} N_{\psi}(s) \, ds = N_{\psi} \bar{\otimes} L^\infty(\mathbb{R})
\]

with the constant field \( N_{\psi}(s) = N_{\psi} \). This immediately implies the desired assertion. □

**Remark 3.** Lemma 2 above and [5] Remark 5.4 precisely show that the von Neumann subalgebras generated by the quasi-normalizers and the (groupoid) normalizers for \( N \rtimes_\alpha \mathbb{R} \supset C^*N \rtimes_\alpha \mathbb{R} \) are different in general. Thus one may further expect that the continuous core of any type III factor arising as a direct summand of a free product von Neumann algebra even has no regular diffuse hyperfinite von Neumann subalgebra similarly to free group factors. However this is unclear at the moment of this writing.

**Proof.** (Theorem 1) As explained in [12] §2.1 we may and do assume \( M = M_c \) after cutting \( M \) by a suitable central projection of either \( M_1 \) or \( M_2 \) if necessary. Suppose, on the contrary, that there exists a Cartan subalgebra \( A \) in \( \tilde{M} := M \rtimes_\alpha \mathbb{R} \).

Firstly, assume that both \( M_1 \) and \( M_2 \) are hyperfinite. As pointed out in [5] §5.2 Theorem 5.2 in the same paper still holds true in the present setup. The proof is basically same, but one needs to replace the free malleable deformation and Theorem A there by Ioana–Peterson–Popa’s original one (see [2] §4.1) and [9] Theorem 4.8, respectively. The consequence is as follows.

Let \( p \in C^*N \rtimes_\alpha \mathbb{R} \) be a non-zero projection with \( \text{Tr}(p) \) finite, where \( \text{Tr} \) denotes the canonical trace on \( \tilde{M} \), which is the so-called dual action scales. Let \( P \subset p\tilde{M}p \) be a (unital) hyperfinite von Neumann subalgebra. If \( P \nsubseteq \tilde{M} \subset C^*N \rtimes_\alpha \mathbb{R} \) in the sense of [5] Lemma 2.2, then \( \gamma(p\tilde{M}p)(P)'' \) must be hyperfinite. Since \( \text{Tr} \{ A \} \) must be semi-finite (see e.g. [10] Lemma VIII.1.11) and \( A \) diffuse, we may and do assume \( p \in A \). Remark that \( Ap \) is also a Cartan subalgebra in \( p\tilde{M}p \) (see e.g. [12] Lemma 4.1 (i) and [5] Proposition 2.7). However, we have known that [11] Theorem 4.1 \( M \) is a non-hyperfinite factor of type III, and hence it is standard, see e.g. [11] Proposition 2.8, that \( p\tilde{M}p \) is never hyperfinite. Therefore, we conclude that \( Ap \nsubseteq \tilde{M} \subset C^*N \rtimes_\alpha \mathbb{R} \) in the above sense.

Secondly, assume that at least one of the \( M_i \) is not hyperfinite. The argument below follows the proof of [11] Theorem A] (or [12] Proposition 6)). As before, after cutting \( M \) by a suitable central projection of either \( M_1 \) or \( M_2 \) if necessary we may and do assume that either \( M_1 \) or \( M_2 \) has no hyperfinite direct summand from the beginning. Choose a non-zero projection \( p \in C^*N \rtimes_\alpha \mathbb{R} \) with \( \text{Tr}(p) \) finite. Then we may and do also assume \( p \in A \) as above. By [11] Theorem 5.1, Lemma 5.2 we conclude that \( Ap \nsubseteq \tilde{M} \subset C^*N \rtimes_\alpha \mathbb{R} \) in the sense of Popa; hence \( Ap \nsubseteq \tilde{M} \subset C^*N \rtimes_\alpha \mathbb{R} \) in the sense before.

We have established that \( Ap \nsubseteq \tilde{M} \subset C^*N \rtimes_\alpha \mathbb{R} \), and the rest of our discussion will prove that this is indeed impossible in the general setup. Choose a MASA \( D \) in \( \tilde{M} \). By [5] Proposition 2.4\] (or the proof of Lemma 2\] \( B := D \rtimes_\alpha \mathbb{R} = D \bar{\otimes} L(\mathbb{R}) \) becomes a MASA in \( \tilde{M} \). Clearly,
Ap ≤ M B so that by [3] Proposition 2.3 there exists a non-zero partial isometry v ∈ M such that vv∗ ∈ Ap, v∗v ∈ B and v∗Av = Bu∗v. Choose a maximal, orthogonal family of minimal projections e1, e2, . . . of D. Set e0 := 1 − ∑k≥1 ek, and then De0 is diffuse or e0 = 0.

Assume that w := v(e0 ⊕ 1) ̸= 0 for some k ≥ 1. Remark that e k ⊕ 1 ∈ D ⊕ L( R) = B. Thus w∗Aw = Bw∗w ⊆ w∗w(M ∗ σw R)w∗w = w∗w(ekMe k ∗ σw R)w∗w, where we define νek := ν |ekek Me k so that σtνek = σt ν |ekek Me k for every t ∈ R because ek ∈ D ⊆ Mw. Note that Bw∗w = (De k ∗ σw R)w∗w = (Cek ⊕ L( R))w∗w = (Cek ∗ σw R)w∗w. By e.g. [13] Lemma 4.1 (i) and [5] Proposition 2.7 Aww∗ is a Cartan subalgebra in w w∗(M ∗ σw R) w w∗, and hence so is (Cek ∗ σw R)w∗w in w∗w(ekMe k ∗ σw R)w∗w. By [13] Lemma 4.1 (i) again one observes that (ekMe k)νw = (De k)′ ∩ (ekMe k)νw = (De k)′ ∩ ekek Me k = (D ∩ Me)ek = De k = Cek. Therefore, by Lemma 2 Cek ∗ σw R must be a singular MASA in ekMe k ∗ σw R, and thus by [5] Proposition 2.7 so is (Cek ∗ σw R)w∗w in w ∗ w(ekMe k ∗ σw R) w ∗ w, a contradiction.

We then treat with the remaining case; namely w := v(e0 ⊕ 1) ̸= 0. This case was essentially treated in the proof of [5] Theorem D). As before one observes that (De0 ∗ σw R)w∗w is a Cartan subalgebra in w∗w(e0Me0 ∗ σw R)w∗w with νw := ν |e0Me0. By the assumption here e0Me0 ≃ M (= Me0) is a non-hyperfinite factor of type III, and hence w∗w(e0Me0 ∗ σw R)w∗w is not hyperfinite by [1] Proposition 2.8. On the other hand, [5] Proposition 5.3 shows that (De0 ∗ σw R)w∗w ̸⊆ e0Me0 ∗ σw R ⊆ Ce0 ∗ σw R, a contradiction due to [5] Theorem 5.2. □

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