THE WAŻEWSKI METHOD AND FEEDBACK CONTROL

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Abstract. In this short paper we consider a possible application of the Ważewski topological method to feedback control systems. We show how this method can be efficiently used to prove the impossibility of global stabilization in such problems.

1. Introduction

The problem of feedback control design is among the major applied mathematical problems. In applications we often face the task of making some configuration of the system asymptotically stable in the sense of Lyapunov. For this we usually can use a feedback control and this control is constrained by the general design of our system. At the same time, we are not only interested in the stability, but also it is often required to make the corresponding basin of attraction as large as possible.

In [1] it was proved that, given a continuous semi-flow on a manifold $M$, such that there exists a vector bundle $\pi: M \to N$, $N$ is a closed manifold, it is impossible for the system to have a globally asymptotically stable equilibrium. In particular, if we consider a dynamical system acting on the tangent bundle $TN$ of a closed manifold $N$, then this system cannot have such an equilibrium.

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As a simple example of such a system one can consider a planar pendulum with feedback control: the system defined on $TS^1$ cannot be globally asymptotically stable. However, if we impose constraints on this pendulum system, the phase space of the corresponding system can change. If the pendulum is placed on a horizontal plane of support, then its phase is $\mathbb{R} \times [0, \pi]$ and the result from [1] can’t be applied here.

The main idea of the Ważewski topological method [5, 7] can be efficiently applied if we want to prove the impossibility of global stabilization in systems with feedback control. To be more precise, we will show that if an uniformly Lyapunov stable equilibrium is located in a larger set such that solutions of the system intersect its boundary transversely (or — in a more general case — the set of strictly egress points coincides with the set of egress points, if we use the terminology of the Ważewski method), then there is a solution that is separated from the equilibrium.

In the paper, we present the above in a more formal form. The paper can be considered as a more accurate generalization of the result in [4]. We also present some mechanical examples. Comparing to the result in [1], we might say that [1] gives a solution to the problem of global stabilization on a manifold without boundary and in our paper we outline a possible approach to the same problem on a manifold with boundary.

2. Results

Let $M$ be a smooth ($C^\infty$) manifold. Let $\Phi$ be a continuous semi-process (see, for instance, [6]):

\begin{equation}
\Phi: M \times \mathbb{R} \times [0, \infty) \to M.
\end{equation}

In other words, $\Phi$ is a continuous map such that

\[ \varphi: (x, t_0, t) \mapsto (\Phi(x, t_0, t), t_0 + t) \in M \times \mathbb{R} \]

is a continuous semi-flow on $M \times \mathbb{R}$. Below we will use the following notation $\Phi(x, t_0, t) = \Phi_{t_0,t}(x)$. Note that any continuous semi-flow on $M$ can be considered as a continuous semi-process on $M \times \mathbb{R}$ with no dependence on $t_0$.

DEFINITION 2.1. As usual, we say that $x_0$ is an equilibrium for semi-process (2.1), if $\Phi_{t_0,t}(x_0) = x_0$ for all $t_0$ and $t \geq 0$.

DEFINITION 2.2. We say that an equilibrium $x_0$ is uniformly Lyapunov stable if for any open set $U \subset M$ such that $x_0 \in U$, there exists an open set $V \subset M$ such that

\[ \Phi_{t_0,t}(x) \in U \]

for any $x \in V$ and all $t_0$ and $t \geq 0$. 

Definition 2.3. We say that point \( x_0 \) is globally attractive if \( \Phi_{0,t}(x) \rightarrow x_0 \) as \( t \rightarrow \infty \).

Here, and everywhere below, we fix the initial moment of time to be zero.

Let \( W \subset M \times \mathbb{R} \) be an open set such that
\[
W \cap \{ t = 0 \} \neq \emptyset.
\]

For any point \((x,0) \in M \times \mathbb{R}\), let us consider the half trajectory of the semi-process:
\[
\gamma_{\tau}(x) = \bigcup_{t \in [0,\tau)} (\Phi_{0,t}(x), t).
\]

Definition 2.4. For point \((x,0) \in W\), we say that
\[
\sigma(x) = \sup\{ \tau \geq 0 : \gamma_{\tau}(x) \subset W \}
\]
is the time of egress from \( W \). If \( \sigma(x) = \infty \), we say that the half trajectory starting at \( x \) does not leave \( W \).

Definition 2.5. We say that point \((x_1,t_1) \in \partial W, t_1 > 0\) is an egress point for \( W \), if there exists a point \((x,0) \in W\) such that \((x_1,t_1) = (\Phi_{0,\sigma(x)}(x), \sigma(x))\).

Point \((x_1,t_1) \in \partial W, t_1 = 0\) is an egress point if for some \( \varepsilon > 0 \) we have
\[
(\Phi_{0,t}(x_1), t) \notin W \cup \partial W
\]
for all \( t \in (0, \varepsilon) \). The set all egress point we denote by \( W^- \).

Definition 2.6. We say that egress point \((x_1,t_1) \in \partial W\) is a strictly egress point for \( W \) if for some \( \varepsilon > 0 \) we have
\[
(\Phi_{0,\sigma(x)+t}(x), \sigma(x) + t) \notin W \cup \partial W
\]
for all \( t \in (0, \varepsilon) \). Here \((x_1,t_1) = (\Phi_{0,\sigma(x)}(x), \sigma(x))\). If \((x_1,t_1) = (x_1,0) \in \partial W\), we put \( \sigma(x) = 0 \). The set all strictly egress point we denote by \( W'^- \).

Definition 2.7. Let \( S \subset M \times \mathbb{R} \). By \( S_0 \) we will denote the following subset of \( M \times \mathbb{R} \)
\[
S_0 = \{(x,t) \in M \times \mathbb{R} : t = 0\}.
\]

Theorem 2.8. Let \( x_0 \) be an uniformly stable equilibrium, \( U \subset M \) be an open subset, \( x_0 \in U \). Let \( W \) be another open set such that \( U \times \mathbb{R} \subset W \), \( W^+ = W'^+ \) and \( W_0'^+ \neq \emptyset \). Suppose that \( W_0'^+ \) can be connected with the equilibrium by a continuous path \( \Gamma : [0,1] \rightarrow M \times \mathbb{R} \) such that \( \Gamma(s) \in W_0' \) for \( s \in (0,1) \) and \( \Gamma(0) = (x_0,0), \Gamma(1) \in \partial W \cap W_0'^+ \). Then \( x_0 \) cannot be globally attractive.
Proof. Assume that \( x_0 \) is globally attractive. Since the equilibrium \( x_0 \) is uniformly stable, there exist an open set \( V \subset M \), \( x_0 \in V \) such that for any \( t_0 \geq 0 \) and any \( x \in V \) for all \( t > 0 \) we have \( \Phi_{t_0,t}(x) \in U \).

For any half trajectory of the semi-process \( \Phi_{0,t}(x) \) starting at \( \Gamma \) we have either \( \sigma(x) < \infty \), or \( \sigma(x) = \infty \) and \( \lim_{t \to \infty} \Phi_t(x) = x_0 \). Let us consider the following map \( \Omega \) from \( \Gamma \) to its boundary points \( \Gamma(0) \) and \( \Gamma(1) \):

\[
\Omega(x) = \begin{cases} 
\Gamma(0), & \text{if } \sigma(x) = \infty, \\
\Gamma(1), & \text{if } \sigma(x) < \infty.
\end{cases}
\]

(2.4)

Now we will prove that \( \Omega \) is continuous provided our assumption holds. As it is usually proved in the Ważewski method, if \( \Omega(x) \to \Gamma(1) \), then for all \( y \), sufficiently close to \( x \), we also have \( \Omega(y) \to \Gamma(1) \). This fact follows from the assumption that \( W^+ = W^{++} \). Also we use here that our semi-process is continuous. Similarly, if \( \Omega(x) \to \Gamma(0) \), then \( \Omega(y) \to \Gamma(0) \) provided \( y \) is close to \( x \): for some \( \tau \) we have \( \Phi_{0,\tau}(x) \in V \). Hence, \( \Phi_{0,\tau}(y) \in V \) and \( \Phi_{0,t}(y) \in U \) for all \( t \geq \tau \).

And finally, we have constructed a continuous map between a line segment and its boundary: the contradiction proves the theorem.

Remark 2.9. From the proof, we have that there exists a point \( x \) such that \( (\Phi_{0,t}(x), t) \in W \) for all \( t \geq 0 \) and \( \Phi_{0,t}(x) \to x_0 \) as \( t \to \infty \). Indeed, our assumption in the proof is equivalent to the following: there are just two types of points in \( \Gamma \), for one \( \sigma(x) < \infty \) (solution leaves \( W \)) and for other points \( \sigma(x) = \infty \) and \( \Phi_{0,t}(x) \to x_0 \). Both sets are not empty. And we have shown that our assumption leads to the contradiction. Therefore, there exists a point \( x \) with the above properties.

Similar results can be easily proved for the case when our system have a stable invariant manifold. For instance, the following direct generalization can be considered.

Let \( M = S \times N \) where \( S \) and \( N \) are smooth manifolds. Let \( x_0 \in S \), we say that \( \{x_0\} \times N \) is an invariant manifold for \( \Phi \) if for any \( (x_0, y) \in \{x_0\} \times N \), \( t_0 \in \mathbb{R} \) and \( t \geq 0 \) we have

\[
\Phi_{t_0,t}(x_0, y) \in \{x_0\} \times N.
\]

We say that invariant manifold \( \{x_0\} \times N \) is uniformly Lyapunov stable if for any open set \( U \subset S \) such that \( x_0 \in U \), there exists an open set \( V \subset S \), \( x_0 \in V \) such that

\[
\Phi_{t_0,t}(x, y) \in U \times N
\]

for any \( x \in V \), \( y \in N \) and all \( t_0 \) and \( t \geq 0 \).

We say that invariant manifold \( \{x_0\} \times N \) is globally attractive if for any \( (x, y) \in S \times N \) we have \( \Phi_{0,t}(x, y) \to \{x_0\} \times N \) as \( t \to \infty \).
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Theorem 2.10. Let \( \{x_0\} \times N \subset M \) be an uniformly stable invariant manifold, \( U \subset S \) be an open subset, \( x_0 \in U \). Let \( W \subset M \times \mathbb{R} \) be another open set such that \( \bar{U} \times N \times \mathbb{R} \subset W, W^+ = W^{++} \) and \( W_0^{++} \neq \emptyset \). Suppose that \( W_0^{++} \) can be connected with the invariant manifold by a continuous path \( \Gamma: [0, 1] \to M \times \mathbb{R} \) such that \( \Gamma(s) \in W_0 \) for \( s \in (0, 1) \) and \( \pi_M(\Gamma(0)) \in \{x_0\} \times N, \Gamma(1) \in \partial W \cap W_0^{++} \).

Then \( \{x_0\} \times N \) cannot be globally attractive. Moreover there exists a point \((x, y) \in M\) such that \( \Phi_{0,t}(x, y) \in W \) for all \( t \geq 0 \) and \( \Phi_{0,t}(x, y) \not\to \{x_0\} \times N \) as \( t \to \infty \).

The proof is the same as in Theorem 2.3.

3. Examples

Let us consider the following equation
\[
\ddot{\varphi} = u(\varphi, \dot{\varphi}) \sin \varphi - \cos \varphi + v(\varphi, \dot{\varphi}) = f(\varphi, \dot{\varphi}).
\]
We assume that this equation defines a continuous semi-flow on \( \mathbb{R}^2 \). Here \( u, v \in C^1(\mathbb{R}^2, \mathbb{R}), |v(0, 0)| < 1 \) and \( |v(\pi, 0)| < 1 \) and \( \varphi = \pi/2 \) is a Lyapunov stable equilibrium. Then this equilibrium cannot be globally attractive. Moreover, there exists a one-parameter family of solutions such that for any solution \( \varphi(t) \) from this family we have

1. \( \varphi(t) \in (0, \pi) \) for all \( t \geq 0 \),
2. \( (\varphi(t), \dot{\varphi}(t)) \not\to (\pi/2, 0) \) as \( t \to \infty \).

For this system, set \( W \) has the following simple form
\[
W = \{\varphi, \dot{\varphi}, t: 0 < \varphi < \pi\}.
\]
We have \( W^+ = W^{++} \). This immediately follows from the Taylor expansion of \( \varphi(t) \) in points \( \varphi = 0, \dot{\varphi} = 0 \) and \( \varphi = \pi, \dot{\varphi} = 0 \). Absolutely similar considerations can be found, for instance, in [3, 2] and we omit them here.

Equation (3.1) describes motion of a controlled inverted pendulum in a gravitational field. The feedback control is given by functions \( u \) and \( v \). Function \( u \) defines the horizontal acceleration of the pivot point and \( v \) is a control torque.

Therefore, \( (\pi/2, 0) \) cannot be a globally attractive uniformly stable equilibrium in the system where the pendulum moves along the plane of support (the horizontal line) and the rod can hit this plane. Moreover, the above is true for any model of impact.

Let us now consider the following system
\[
\begin{align*}
\dot{q} & = p, \\
\dot{p} & = \frac{u(q, p, x, y) \sin q + p^2 \sin q \cos q - (1 + m) \cos q}{m + \cos^2 q}, \\
\dot{x} & = y, \\
\dot{y} & = (m + \cos^2 q)^{-1} \left( u(q, p, x, y) + p^2 \cos q - \sin q \cos q \right).
\end{align*}
\]
This system describes motion of an inverted pendulum on a cart. Suppose that these equations define a continuous semi-flow on $\mathbb{R}^4$. The feedback control is given by the horizontal force applied to the cart. Here $m > 0$ is the mass of the cart, $x$ is the coordinate of the pivot point on the horizontal line, $u \in C^1(\mathbb{R}^4, \mathbb{R})$ is the horizontal force applied to the cart. We assume that the mass of the pendulum, its length and the gravity acceleration equal 1.

If $u(\pi/2, 0, x, y) \equiv 0$ for all $x$ and $y$, then $\varphi = \pi/2$ is an invariant manifold for this system. This invariant manifold cannot be globally attractive and uniformly stable (in the sense of the above definition).

To prove this, one can consider the following set $W$:

$$W = \{q, p, x, y, t : 0 < q < \pi\}.$$  

Again, we have $W^+ = W^{++}$ and this immediately follows from the following inequalities

$$\dot{p} \big|_{q=0, p=0} < 0, \quad \dot{p} \big|_{q=\pi, p=0} > 0.$$

As the last example, let us consider the following system:

$$\ddot{\varphi}_1 + \cos \varphi_1 = (u_1(\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2) - 2dk) \sin \varphi_1 - k \sin(\varphi_1 - \varphi_2),$$

$$\ddot{\varphi}_2 + \cos \varphi_2 = (u_2(\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2) + 2dk) \sin \varphi_2 - k \sin(\varphi_2 - \varphi_1).$$

These equations describe the following system of two interacting pendulums: the pivot points of the pendulums are located in points $(-d, 0)$ and $(d, 0)$ of the vertical plane, the lengths and masses of pendulums and the gravity acceleration equal 1, $u_i \in C^1(\mathbb{R}^4, \mathbb{R})$ is the horizontal force acting on the $i$-th pendulum. The pendulums are connected by a spring and $k \geq 0$ is the stiffness.

Suppose that $k < 1$ and the above equations define a semi-flow on $\mathbb{R}^4$. Then point $\varphi_1 = \varphi_2 = \pi/2$, $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$ cannot be globally asymptotically stable even if we consider possible impacts on the horizontal plane. For this system, $W$ can be chosen as follows:

$$W = \{\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2 : 0 < \varphi_1 < \pi, 0 < \varphi_2 < \pi\}.$$

References

[1] S. P. Bhat and D. S. Bernstein, A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon, Systems & Control Letters 39 (2000), 63–70.

[2] I. Polekhin, Periodic and falling-free motion of inverted spherical pendulum with moving pivot point, arXiv:1411.1585 (2014).

[3] I. Polekhin, Examples of topological approach to the problem of inverted pendulum with moving pivot point, Nelineinaya Dinamika [Russian Journal of Nonlinear Dynamics] (Russian) 10 (2014), 465–472.

[4] I. Polekhin, On topological obstructions to global stabilization of an inverted pendulum, Systems & Control Letters 113 (2018), 31–35.
[5] G. Reissig, G. Sansone, and R. Conti, Theorie nichtlinearer Differentialgleichungen. Edizioni Cremonese, 1963 (German).
[6] R. Srzednicki, W. Wójcik, and P. Zgliczyński, Fixed point results based on the Ważewski method, in: Handbook of topological fixed point theory, Springer (1988), 905-943.
[7] T. Ważewski, Sur un principe topologique de lexamen de l’allure asymptotique des intégrales des équations différentielles ordinaires, Ann. Soc.Polon. Math 20 (1947), 279–313.

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