FUNCTION FIELD GENUS THEORY FOR NON-KUMMER EXTENSIONS

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ABSTRACT. In this paper we first obtain the genus field of a finite abelian non-Kummer $l$-extension of a global rational function field. Then, using that the genus field of a composite of two abelian extensions of a global rational function field with relatively prime degrees is equal to the composite of their respective genus fields and our previous results, we deduce the general expression of the genus field of a finite abelian extension of a global rational function field.

1. INTRODUCTION

The concepts of Hilbert Class Field (HCF) $K_H$ and narrow or extended Hilbert Class Field $K_{H^*}$, of a fixed number field $K$, are canonically defined as the maximal abelian unramified abelian extension of $K$ and as the maximal abelian extension unramified at the finite primes of $K$, respectively. The function field case is different because the direct definition of the HCF over a global function field $K$ as the maximal unramified abelian extension has the inconvenience of being of infinite degree over $K$.

There are several different possible definitions of HCF of a global function field $K$. The one we will be using is that for a fixed finite nonempty set $S$ of places of $K$, the HCF $K_H$ of $K$ is defined as the maximal unramified abelian extension of $K$ where all the places of $S$ decompose fully. The extension $K_H/K$ is a finite extension with Galois group isomorphic, via the Artin Reciprocity Law, to the class group $C_{l_S}$ of the Dedekind domain consisting of the elements of $K$ regular away from $S$. The genus field of $K$ over a subfield $k$ is defined as $K_{ge} = Kk^*$, where $k^*$ is the maximal abelian extension of $k$ contained in $K_H$.

We are interested in the case of the rational function field $k = \mathbb{F}_q(T)$ and $K/k$ a finite abelian extension. In [3], H. Leopoldt studied the extended genus field $K_{ge}$ of a finite abelian extension of the field of rational numbers $\mathbb{Q}$, by means of Dirichlet characters. Using Leopoldt’s technique we applied Dirichlet characters to the function field case and found a general description of $K_{ge}$ ([1, 4, 5]). In these papers it was also provided an explicit description of $K_{ge}$ in the cases of a Kummer cyclic extension of prime degree $l$ and of an abelian $p$-extension where $p$ is the characteristic of $k$. In [2, 6, 8], the explicit description of $K_{ge}$ was given when $K/k$ is a finite Kummer $l$-extension with $l$ a prime number.

In this paper, we study the explicit description of the genus field $K_{ge}$ of the remaining case: $K/k$ a finite abelian non-Kummer $l$-extension with $l \neq p$, the characteristic of $K$. Using this explicit description and the results of [1] and [8],...
we have the explicit description of $K_{ge}$ of any finite abelian extension $K/k$. By explicit description, we mean to give $K_{ge,w}$ in terms of radical extensions, where $K_{ge,w}/K_{ge}$ is the extension of constants $K_{ge,w} = K_{ge}^{\sqrt[qw]{}}$.

The main tool to find $K_{ge,w}$ is that, given $K/k$ a finite $l$-cyclic non-Kummer extension with $l$ a prime other than the characteristic, if $k_w$ denotes the extension of constants adjoining all the relevant roots of unity, then we find explicitly $D$ in the ring of integers of $k_w$ such that $k_w = k_w(\sqrt[l]{D})$. Then we generalize the technique to a general finite abelian $l$-extension. Our main result is Theorem 4.2.

Theorem 5.2 gives the general description of $K_{ge}$ for a finite abelian extension $K/k$. One crucial result that allows us to be able to give explicitly this general description is that if $K_1/k$ and $K_2/k$ are two finite abelian extensions of relatively prime degrees, we have $(K_1)_{ge}(K_2)_{ge} = (K_1K_2)_{ge}$.

2. Antecedents and General Notations

Let $k = \mathbb{F}_q(T)$ be a global rational function field, where $\mathbb{F}_q$ is the finite field of $q$ elements, $R_T = \mathbb{F}_q[T]$ denotes the polynomial ring that may be considered as the ring of integers of $k$. Let $R_T^+$ be the set of the monic irreducible elements of $R_T$ or, equivalently, the “finite” primes of $k$.

For the Carlitz–Hayes theory of cyclotomic function fields, we will be using [9, Ch. 12] and [7, Cap. 9]. For $N \in R_T$, $k(\Lambda_N)$ denotes the $N$-th cyclotomic function field where $\Lambda_N$ is the $N$-th torsion of the Carlitz module. For $D \in R_T$ we denote $D^* := (-1)^{\deg D}D$.

The results on genus fields of function fields can all be found in [1, 4, 5] and [7, Cap. 14]. For the explicit description of genus fields, we refer to [1, 6, 8].

We denote the infinite prime of $k$ by $P_\infty$. That is, $P_\infty$ is the pole divisor of $T$ and $1/T$ is a uniformizer for $P_\infty$. The ramification index of $P_\infty$ in $k(\Lambda_N)/k$ is equal to $q - 1$ and the inertia degree of $P_\infty$ in every cyclotomic function field is always equal to 1.

For any extension $L/K$ with $L/k$ a finite abelian extension, $e_\infty(L/K)$ denotes the ramification index of the infinite primes of $K$, that is, the primes of $K$ dividing $P_\infty$ and $f_\infty(L/K)$ denotes the inertia degree of the infinite primes. Similarly $e_P$ and $f_P$ for a finite prime $P$ of $k$.

Let $F$ be any cyclotomic function field, that is, $F \subseteq k(\Lambda_N)$ for some $N \in R_T$. Then $F_{ge} = M^+F$, where $M$ is the maximal cyclotomic extension of $F$ unramified at the finite primes, and $M^+$ denotes the “real subfield” of $M$, that is, the decompositon field of $P_\infty$. We denote $F_{g_{\infty}} = M$ the extended genus field of $F/k$. Then $F_{ge} = F_{g_{\infty}}^+F$. We have $F_{ge} \subseteq F_{g_{\infty}} \subseteq k(\Lambda_N)$ and $F_{g_{\infty}}/F_{ge}$ is totally ramified at the infinite primes. We have that $[F_{g_{\infty}} : F_{ge}] = e_\infty(F_{g_{\infty}}/F_{ge}) = e_\infty(F_{g_{\infty}}/F)$. Therefore, to obtain $F_{ge}$ we need to compute a suitable subextension of $F_{g_{\infty}}$ of degree $e_\infty(F_{g_{\infty}}/F)$.

When $K/k$ is a finite abelian extension, it follows from the Kronecker–Weber Theorem that there exist $N \in R_T$, $n' \in \mathbb{N} \cup \{0\}$ and $m' \in \mathbb{N}$ such that $K \subseteq n'k(\Lambda_N)_{m'}$, where for any field $F$ containing $k$, $F_{m'} := F^{q^{m'}}$ is the extension of constants, and $n'F := FL_m$, with $L_m$ the maximal subfield of $k(\Lambda_{1/T^{m'}})$, where $P_\infty$ is totally and wildly ramified. Then we define

$$E := K \mathcal{M} \cap k(\Lambda_N),$$

(2.1)
where $\mathcal{M} = L_n, k_m$. We have that $K_{ge} = E_h^r K$, where $H$ is the decomposition group of the infinite primes in $E_h K/K$ and of $E K/K$ (see [1]).

We only consider geometric extensions $K/k$, that is, $\mathbb{F}_q$ is the field of constants of $K$.

For $v \in \mathbb{N}$, $C_v$ will denote the cyclic group of $v$ elements and $\zeta_{v^n}$ will denote a primitive $v^n$-th root of unity in a finite field. We will use the notation $con_{F/E}$ for the conorm map from $F$ to $E$.

3. Extensions of $k$ of prime power degree

In this section we consider, for a prime number $l$ a finite abelian $l$-extension $K/k$ of exponent $l^n$, $n \in \mathbb{N}$.

3.1. Case $l = p$. In [1, Corollary 6.6] we found the explicit genus field $K_{ge}$ of a finite abelian $p$-extension $K/k$. Let $\diver, \cdot$ and $\der$ be the Witt operations. Let $P_1, \ldots, P_r \in \mathcal{P}_k$ be the finite primes in $k$ ramified in $K$. Given a Witt vector $\xi_0$, we may decompose $\xi_0$ as

$$\bar{\xi}_0 = \bar{\delta}_1 + \ldots + \bar{\delta}_r + \bar{\gamma},$$

where $\delta_{i,j} = \frac{Q_{i,j}}{P_i}$, $e_{i,j} \geq 0$, $Q_{i,j} \in \mathcal{R}_T$ and if $e_{i,j} > 0$, then $e_{i,j} = \lambda_{i,j} p^{m_{i,j}}$, $\gcd(\lambda_{i,j}, p) = 1$, $0 \leq m_{i,j} < n$, $\gcd(Q_{i,j}, P_i) = 1$ and $\deg(Q_{i,j}) < \deg(P_i^{\lambda_{i,j}})$, and $\gamma_j = f_j(T) \in \mathcal{R}_T$ with $\deg f_j = \nu_j p^{m_j}$ and $\gcd(q, \nu_j) = 1$, $0 \leq m_j < n$ when $f_j \not\in k_0$.

**Theorem 3.1.** Let $K/k$ be a finite abelian $p$-extension with Galois group $\text{Gal}(K/k) = G \cong G_1 \times \cdots \times G_s$ with $G_i \cong C_{p^{m_i}}, 1 \leq i \leq s$. Let $K$ be the composite $K = K_1 \cdots K_s$ such that $\text{Gal}(K_i/k) \cong G_i$. Let $P_1, \ldots, P_r$ be the finite primes ramified in $K/k$. Let $K_i = k(\bar{w}_i)$ be given by the equation

$$\bar{w}_i^p \cdot \bar{w}_i = \bar{\xi}_i, \quad 1 \leq i \leq s.$$

Write each $\bar{\xi}_i$ as in (3.1) that is,

$$\bar{\xi}_i = \bar{\delta}_{i,1} + \ldots + \bar{\delta}_{i,r} + \bar{\gamma}_i,$$

such that all the components of $\bar{\delta}_{i,j}$ are written so that the degree of the numerator is less than the degree of the denominator, the support of the denominator is at most $\{P_j\}$ and the components of $\bar{\gamma}_i$ are polynomials. Let

$$\bar{w}_i^p \cdot \bar{w}_{i,j} = \bar{\delta}_{i,j}, \quad 1 \leq i \leq s, \quad 1 \leq j \leq r$$

and

$$\bar{w}_i^p \cdot \bar{z}_i = \bar{\gamma}_i, \quad 1 \leq i \leq s.$$

Then, the genus field $K_{ge}$ of $K$ is given by

$$K_{ge} = k(\bar{w}_{i,j}, \bar{z}_i | 1 \leq i \leq s, 1 \leq j \leq r).$$
3.2. Case \( l \neq p \), \( K/k \) Kummer. We now consider the case of a finite abelian Kummer \( l \)-extension \( K/k \) of exponent \( t^\mu \). That is \( t^\mu | q - 1 \) or, equivalently, \( \zeta_{t^\mu} \in \mathbb{F}_q \).

Let 
\[
K = k\left(\sqrt[\alpha_1]{D_1}, \ldots, \sqrt[\alpha_s]{D_s}\right) = K_1 \cdots K_s,
\]
where \( K_\varepsilon = k\left(\sqrt[\alpha_1]{D_1}, \ldots, \sqrt[\alpha_s]{D_s}\right) \), \( D_\varepsilon \in R_T \) monic, \( \gamma_\varepsilon \in \mathbb{F}_q^* \), \( 1 \leq \varepsilon \leq s \) and \( n = n_1 \geq \cdots \geq n_s \). Let \( P_1, \ldots, P_r \) be the finite primes ramified in \( K/k \) and let
\[
D_\varepsilon = P_{1,\varepsilon}^{\alpha_1} \cdots P_{s,\varepsilon}^{\alpha_s} \quad \text{with} \quad 0 \leq \alpha_{j,\varepsilon} \leq l^{\varepsilon} - 1, \quad 1 \leq j \leq r, \quad 1 \leq \varepsilon \leq s,
\]
where \( \alpha_{j,\varepsilon} = b_{j,\varepsilon}l^{\varepsilon} - 1 \) with \( \gcd(l, b_{j,\varepsilon}) = 1 \) and \( \deg P_j = c_j l^{d_j} \) with \( \gcd(l, c_j) = 1, 1 \leq j \leq r \). We have, for \( P_j \), that
\[
e_{P_j}(K/k) = \text{lcm}_{1 \leq \varepsilon \leq s} \left[ e_{P_j}(K_\varepsilon/k) \right] = l^{\beta_j}
\]
with
\[
\beta_j = \max_{1 \leq \varepsilon \leq s} \left\{ n_\varepsilon - v_l(\alpha_{j,\varepsilon}) \right\} = \max_{1 \leq \varepsilon \leq s} \left( n_\varepsilon - \alpha_{j,\varepsilon} \right).
\]  
(3.2)

Also, we have
\[
l^{\mu} = e_{\infty}(K/k) = \text{lcm}_{1 \leq \varepsilon \leq s} \left[ e_{\infty}(K_\varepsilon/k) \right] = \text{lcm}_{1 \leq \varepsilon \leq s} \left[ \frac{l^{n_\varepsilon}}{\gcd(l^{n_\varepsilon}, \deg D_\varepsilon)} \right] = \text{lcm}_{1 \leq \varepsilon \leq s} \left[ l^{n_\varepsilon} - \min\left\{ n_\varepsilon, v_l(\deg D_\varepsilon) \right\} \right]
\]
(3.3)

that is, \( t' = \max_{1 \leq \varepsilon \leq s} \left\{ n_\varepsilon - \min\left\{ n_\varepsilon, v_l(\deg D_\varepsilon) \right\} \right\} \).

**Theorem 3.2** (Kummer case). Let \( K/k \) be a finite Kummer \( l \)-extension of \( k \). Let \( \text{Gal}(K/k) \cong C_{n_1} \times \cdots \times C_{n_s} \) with \( n = n_1 \geq n_2 \geq \cdots \geq n_s \) and \( t^\mu | q - 1 \). We have \( K = K_1 \cdots K_s \). Let \( K_{\varepsilon} = k\left(\sqrt[\alpha_1]{D_1}, \ldots, \sqrt[\alpha_s]{D_s}\right) \), \( D_\varepsilon \in R_T \) monic and \( \gamma_\varepsilon \in \mathbb{F}_q^* \), \( 1 \leq \varepsilon \leq s \). Let \( P_1, \ldots, P_r \) be the finite primes in \( k \) ramified in \( K \) with \( P_1, \ldots, P_r \) \( R_T \) distinct. Let
\[
e_{P_j}(K/k) = l^{\beta_j}, \quad 1 \leq \beta_j \leq n, \quad 1 \leq j \leq r, \quad \text{and} \quad e_{\infty}(K/k) = l^{t'}, \quad 0 \leq t' \leq n
\]
given by (3.2) and (3.3) and let \( \deg P_j = c_j l^{d_j} \) with \( \gcd(c_j, l) = 1, 1 \leq j \leq r \).

We order \( P_1, \ldots, P_r \), so that \( n = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_r \).

Let \( E \) be given by (2.1). Then \( E = k\left(\sqrt[i_1]{D_1}, \ldots, \sqrt[i_s]{D_s}\right) \). The maximal cyclotomic extension \( M \) of \( E \), unramified at the finite primes, is given by \( M = E_{\text{fin}} = \prod_{i=1}^r k\left(\sqrt[i_1]{D_1}, \ldots, \sqrt[i_s]{P_{2i-1}}\right) \). Let \( l^{m'} = e_{\infty}(M/k) \). Then \( m' = \max_{1 \leq j \leq r} \left\{ \beta_j - \min\left\{ \beta_j, d_j' \right\} \right\} \).

Choose \( i \) such that \( m' = \beta_i - \min\left\{ \beta_i, d_i' \right\} \) and such that for \( j > i \) we have \( m' > \beta_j - \min\left\{ \beta_j, d_j' \right\} \). That is, \( i \) is the largest index obtaining \( l^{m'} \) as the ramification index of \( P_\infty \).

In case \( m' = t' = 0 \) we have \( M = E_{\text{fin}} = \prod_{i=1}^r k\left(\sqrt[i_1]{D_1}, \ldots, \sqrt[i_s]{P_{2i-1}}\right) \) and \( K_{\text{fin}} = MK \).

In case \( m' > t' \geq 0 \) or \( m' < t' > 0 \), we have \( \min\left\{ \beta_i, d_i' \right\} = d_i' \) and \( m' = \beta_i - d_i' \).

Let \( a, b \in \mathbb{Z} \) be such that \( a \deg P_i + bl^{m' - d_i'} = l^{d_i'} = \gcd(l^{n + d_i'}, \deg P_i) \). Set \( z_j = -a l^{\deg P_i} = -a l^{d_i'} - d_i' \in \mathbb{Z} \) for \( 1 \leq j \leq i - 1 \). For \( j > i \), consider \( y_j \in \mathbb{Z} \) with
\[ y_j \equiv -c_j' \cdot c_j \mod l^n \equiv -ac_j' \cdot c_j \mod l^n. \]

Let

\[
E_j = \begin{cases} 
  k(\sqrt{l^j P_j P_i^{d_j}}) & \text{if } j < i, \\
  k(\sqrt{l^{d_j - d_i} P_i}) & \text{if } j = i, \\
  k(\sqrt{l^j P_j P_i^{d_j - d_i}}) & \text{if } j > i \text{ and } d_j' \geq d_i', \\
  k(\sqrt{l^j P_j^{d_i'} - d_j'} P_i) & \text{if } j > i \text{ and } d_j' > d_i',
\end{cases}
\]

Then \( K_{ge} = E_1 \cdots E_{i-1} E_i E_{i+1} \cdots E_r K \), where \( l^{u''} = \left[ \prod_{i=1}^{l^r} \left( \frac{\zeta_i^{\varphi(i)} \cdot e_{j_i}}{\zeta_i^{\varphi(i)}} \right) \right] \) and \( \varepsilon_j = (-1)^{\deg \delta_j} \gamma_j, 1 \leq j \leq s. \)

\[ \text{Proof. See [8, Theorems 3.4 and 3.6].} \]

3.3. Case \( l \neq p \), \( K/k \) non-Kummer. In this case, we have that the extension \( K/k \) is a finite abelian \( l \)-extension of exponent \( l^n \) such that \( l^n \mid q - 1 \). This case is treated in the following section.

4. The non-Kummer case

Now we consider a finite abelian non-Kummer \( l \)-extension \( K/k \) of exponent \( l^n \). Therefore \( l^n \mid q - 1 \) and \( \zeta_l \not\in \mathbb{F}_q \). The non-explicit description of \( K_{ge} \) is given in [1, Theorem 2.2]. Now, the description of the subfields of a cyclotomic function field \( k(\Lambda_N) \) is not explicit except in very few cases. That is, if \( F = k(\delta) \), it is hard to describe \( \delta \) in terms of roots of polynomial equations. Our objective is to give explicitly the field \( K_{ge,w} \), where \( w := [\mathbb{F}_q(\zeta_{l^n}) : \mathbb{F}_q] | l^{n-1}(l-1) \). We have that \( K_w/K_w \) and \( K_{ge,w}/K_w \) are Kummer extensions and therefore we may use Theorem 3.2 to give \( K_{ge,w} \) explicitly.

First, we recall the following non-explicit result.

**Proposition 4.1.** Let \( F/k \) be a cyclic non-Kummer extension of prime degree \( l \). Let \( \xi \in \mathcal{O}_F \), the integral closure of \( R_T \) in \( F \), such that \( F = k(\xi) \) and

\[ \chi = \sum_{i=0}^{l-1} \zeta_l^i \varphi(i)(\xi) \neq 0, \]

where \( \operatorname{Gal}(F/k) = \langle \varphi \rangle \). Then \( \mu = \chi^{l} \in \mathbb{F}_{q^l}[T] \) and \( F_w = k_w(\sqrt{\mu}) = k_w(\chi) \), with \( w = [\mathbb{F}_q(\zeta_{l^n}) : \mathbb{F}_q] \).

**Proof.** See [10, Proposition 4.1].

Our first goal is to give in Theorem 4.2 an explicit generalization of Proposition 4.1.

As a first step, we consider the case of only one finite prime ramified. Let \( P \in R_T^+ \) and consider a cyclic extension \( K/k \) of degree \( l^n \) with \( l^n \mid q - 1 \) and such that \( P \) is the only finite prime of \( k \) ramified in \( K \) and it is fully ramified. Let \( w = [\mathbb{F}_q(\zeta_{l^n}) : \mathbb{F}_q] | l^{n-1}(l-1) \). Let \( d_P = \deg_k P \). Then \( l^n \mid q^{d_P} - 1 \). We have that \( w = \operatorname{ord}_k(q \mod l^n) \). Hence \( w|d_P \). In the extension of constants \( k_{w}/k \) we have that \( \operatorname{con}_{k_{w}/k} P = \prod_{i=1}^{l^n} \mathcal{P}_i \), where \( h = \gcd(d_P, w) \). That is, \( P \) decomposes fully in \( k_w \), and \( \deg_{k_{w}} \mathcal{P}_i = d_P/w, 1 \leq i \leq w \) (see [9, Theorem 6.2.1]).

Then \( K_{w}/k_w \) is a Kummer extension of degree \( l^n \) and the finite primes ramified are precisely \( \mathcal{P}_i, 1 \leq i \leq w \). All of them are fully ramified. Therefore there exist \( \alpha_i \)
such that $1 \leq \alpha_i \leq l^n - 1$ and $\gcd(l, \alpha_i) = 1$, $1 \leq i \leq w$, with

$$K_w = k_w \left( \sqrt[l^n]{\gamma P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w}} \right) = k_w \left( \sqrt[l^n]{\gamma \mathcal{D}} \right) = k_w(\delta)$$

for some $\gamma \in \mathbb{F}_q^n$, $\mathcal{D} = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w} \in \mathbb{F}_q[T]$ and $\delta = \sqrt[l^n]{\gamma \mathcal{D}}$.

Let $\text{Gal}(K_w/K) \cong \text{Gal}(k_w/k) = \langle \tau \rangle$. Then $\sigma(\tau) = w$ and we may assume that $\sigma(\delta) = \zeta^n l \delta$. Let $\tau(\zeta^n) = \zeta_n^n$, where $\eta = \text{ord}(\eta \mod l^n) = w$. Since $\text{Gal}(K_w/k)$ is an abelian extension, $\text{Gal}(K_w, k) = \langle \sigma, \tau \rangle$ and $\sigma \tau = \tau \sigma$. Therefore

$$\tau(\sigma(\delta)) = \tau(\zeta_n \delta) = \zeta_n^w \eta \tau, \quad \tau(\zeta_{n}^n) = \zeta_{n}^n \tau = \sigma(\delta) = \sigma(\zeta_{n}^n) \epsilon = \sigma(\tau) = \sigma(\sigma(\delta)) = \sigma(\delta),$$

where $\epsilon := \tau(\delta)$. Therefore we have that $\sigma(\epsilon) = \zeta_n^w \epsilon$. It follows that $\sigma(\delta - \eta \epsilon) = (\zeta_n \delta)^{-n} \zeta_n^w \epsilon = \delta - \eta \epsilon$. Thus $\delta - \eta \epsilon \in k_w$ and $\epsilon = \lambda \delta^n$ for some $\lambda \in k_w$.

On the other hand, since $\text{Gal}(k_w/k) = \langle \tau \rangle$, we have that $\tau$ acts transitively on the set $\{P_1, \ldots, P_w\}$. Hence the only finite prime divisors dividing $\mathcal{D}^n$ are $P_1, \ldots, P_w$. Without loss of generality, we may assume that $\langle \tau \rangle$ acts as $(1, w)$ on the set $\{P_1, \ldots, P_w\}$. That is, $\tau(P_i) = P_{i+1}$ for $i = 1, 2, \ldots, w-1$ and $\tau(P_w) = P_1$. Thus

$$e^{l^n} = \tau(\delta^n) = \tau(\gamma \mathcal{D}) = \tau(\gamma P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w}) = \gamma P_2^{\alpha_1} P_3^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w} = \lambda^{l^n}(\gamma \mathcal{D})^n = \lambda^{l^n}(\gamma P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w})^n. \quad (4.1)$$

It follows that, if for some finite prime divisor $P$ we have $v_P(\lambda) \neq 0$, then $P \in \{P_1, \ldots, P_w\}$. Set $\xi_i := v_P(\lambda), 1 \leq i \leq w$. Then, from (4.1) we have

$$\alpha_{i-1} = \eta \alpha_i + l^n \xi_i, \quad 2 \leq i \leq w,$$

$$\alpha_w = \eta \alpha_1 + l^n \xi_1. \quad (4.2)$$

From (4.2) we obtain

$$\alpha_{w-1} = \eta \alpha_w \mod l^n,$$

$$\alpha_{w-2} = \eta \alpha_{w-1} \mod l^n = \eta^2 \alpha_w \mod l^n,$$

$$\vdots$$

$$\alpha_2 = \eta \alpha_3 \mod l^n = \ldots = \eta^{w-2} \alpha_w \mod l^n,$$

$$\alpha_1 = \eta \alpha_2 \mod l^n = \ldots = \eta^{w-1} \alpha_w \mod l^n.$$

Since $l \nmid \alpha_w$, we obtain that

$$K_w = k_w \left( \sqrt[l^n]{\gamma \mathcal{D}} \right) = k_w(\delta) = k_w \left( \sqrt[l^n]{\gamma (\mathcal{D})^{\alpha_w}} \right) = k_w \left( \sqrt[l^n]{\gamma P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_{w-1}^{\alpha_{w-1}} P_w^{\alpha_w}} \right). \quad (4.3)$$

The extension given in (4.3) is determined by the class of $\gamma \in \mathbb{F}_q^n$ modulo $(\mathbb{F}_q^n)^l$. In particular, $K$ is cyclotomic if and only if $(-1)^{\text{deg}(\mathcal{D})} \gamma \in (\mathbb{F}_q^n)^l$.

Let us obtain the ramification of the infinite primes in $K/k$. Note that since $K/k$ is an abelian extension, if $e_\infty(K/k)$ denotes the ramification index of $P_\infty$ in $K$, 

$$e_\infty(K/k) = \text{ord}(\eta \mod l^n) = w.$$
then we have \(e_\infty(K/k)|q^{\deg P_\infty} - 1 = q - 1\). In particular \(P_\infty\) is not fully ramified.

On the other hand, since \(k_w/k\) is unramified, we obtain from (3.3) that

\[
e_\infty(K/k) = e_\infty(K_w/k_w) = \frac{l^n}{\gcd(\deg_{k_w}(D), l^n)}.
\]

(4.4)

Now, \(\deg_{k_w}(D) = \sum_{i=1}^{w} \eta^{w-i} \deg_{k_w}(P_i) = \sum_{i=1}^{w} \eta^{w-i} d_P / w = \frac{d_P}{w} \frac{n^{w-1}}{\eta-1} \) (recall that \(d_P = \deg_k(P))\), so that

\[
e_\infty(K/k) = \frac{l^n}{\gcd \left( \frac{d_P}{w} \frac{n^{w-1}}{\eta-1}, l^n \right)}.
\]

(4.5)

4.1. General non-Kummer abelian \(l\)-extensions. Now we may consider the general case. Let \(K/k\) be a finite abelian \(l\)-extension, where \(l\) is a prime number other than the characteristic of \(k\). Let \(\text{Gal}(K/k) \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_s}\), with \(n = n_1 \geq n_2 \geq \cdots \geq n_s\). Then \(\text{Gal}(K/k)\) is of exponent \(l^n\). We assume that \(K/k\) is a non-Kummer extension. However, what we will obtain, could be applied to Kummer extensions, see Remark 4.3.

We assume that \(l^n \nmid q - 1\). Let \(w = [k(\zeta_n) : k] > 1\). We have that \(w = \text{ord}(q \mod l^n)\). The Kummer case is when \(w = 1\). We are assuming that \(w \geq 2\).

Since \(K/k\) is abelian and \(\deg_k(P_\infty) = 1\), \(P_\infty\) is not fully ramified in \(K/k\).

Let \(K = K_1 \cdots K_e \cdots K_s\) where \(K_i/k\) is a cyclic extension of degree \(l^{n_i}, 1 \leq i \leq s\). Let \(K_{e,w} = K_\gamma k_w = k_w \left( \frac{w}{\gamma} \sqrt{\eta_{e,w} D_e} \right), \gamma \in F_{q^w}^*\) and \(D_e \in F_{q^w}[T], 1 \leq e \leq s\).

Let \(P_1, \ldots, P_r \in R_{p}^T\) be the finite primes ramified in \(K/k\) and \(\text{deg}_{k_w} P_j = c_j l^{n_j}\) with \(l \nmid c_j\).

Let \(\text{con}_{k/w} P_j = \mathcal{P}_{j,1} \cdots \mathcal{P}_{j,s_j}\) where \(s_j = \gcd(\text{deg}_{k_w} P_j, w), 1 \leq j \leq r\). Then

\[
\text{deg}_{k_w}(\mathcal{P}_{j,v}) = \frac{\text{deg}_{k_w} P_j}{s_j} = c_j l^{n_j}
\]

for all \(1 \leq v \leq s_j\). Let \(e_{P_j}(K/k) = l^{\beta_j}, 1 \leq \beta_j \leq n, 1 \leq j \leq r\) and \(e_\infty(K/k) = l^t, 0 \leq t \leq n\).

We have that \(e_{P_j}(K_{e,w}/k_w) = e_{P_j}(K_{e,w}/k) = l^{n_j}, 1 \leq e \leq s\) where \(\beta_j = \max_{1 \leq s_j \leq s}\{g_{j,e}\}\).

We have \(K_{e,w} = k_w \left( \frac{w}{\gamma} \sqrt{\eta_{e,w} D_e} \right) = k_w \left( \frac{w}{\gamma} \sqrt{\eta_{e,w} T_{e,1}T_{e,2} \cdots T_{e,r}} \right), D_e \in F_{q^w}[T]\) and

\[
T_{j,e} = \mathcal{P}_{j,1}^{\alpha_{j,1}} \mathcal{P}_{j,2}^{\alpha_{j,2}} \cdots \mathcal{P}_{j,s_j - 1}^{\alpha_{j,s_j - 1}} \mathcal{P}_{j,s_j}^{\alpha_{j,s_j}}
\]

\[
0 \leq \alpha_{j,v,e} \leq l^n - 1, 1 \leq e \leq s, 1 \leq j \leq r, 1 \leq v \leq s_j.
\]

Then \(g_{j,e} = 0\) if \(\alpha_{j,v,e} = 0\) for some \(1 \leq v \leq s_j\), and \(g_{j,e} = n_s - v(\alpha_{j,v,e})\) when \(\alpha_{j,v,e} \neq 0\) for all \(1 \leq v \leq s_j\).

Set \((\sigma_v) \cong \text{Gal}(K_{e,w}/k) \cong \text{Gal}(K_{e,w}/k_w) \cong C_{l^{n_s}}\). We have \((\tau) \cong \text{Gal}(K_{e,w}/K_e) \cong \text{Gal}(k_w/k) \cong C_w\).

\[
\begin{array}{ccc}
K_e & \tau & K_{e,w} \\
\sigma_v & \downarrow & \downarrow \\
K & \tau & k_w
\end{array}
\]

Since \(\tau(\zeta_{l^n}) = \zeta_{l^n}\) and \(\text{ord}(\eta \mod l^n) = w\), we have that \(\tau\) acts transitively on the set \(\{\mathcal{P}_{j,1}, \mathcal{P}_{j,2}, \ldots, \mathcal{P}_{j,s_j}\}\) and we may assume that \(\tau(P_j,\nu) = P_{j,\nu+1}, 1 \leq \nu \leq s_j - 1\) and \(\tau(P_j,s_j) = P_{j,1}\).
Let $\delta_\varepsilon = \sqrt[n]{\varepsilon D_{\varepsilon}}$, $\delta_\varepsilon^{m_\varepsilon} = \gamma_\varepsilon D_{\varepsilon}$. Let $\epsilon_\varepsilon = \tau(\delta_\varepsilon)$. From (4.1), (4.2) and (4.3) we obtain

$$\sigma(\epsilon_\varepsilon) = \gamma_\varepsilon^{m_\varepsilon} \epsilon_\varepsilon, \quad \epsilon_\varepsilon = \lambda_\varepsilon \delta_\varepsilon^{m_\varepsilon},$$

and

$$T_{j,\varepsilon} = (P_{j,1}^{\eta_j^{-1}} P_{j,2}^{\eta_j^{-2}} \cdots P_{j,s_{j-1}}^{\eta_j} P_{j,s_j})^{\alpha_{j,\varepsilon}},$$

where $\alpha_{j,\varepsilon} := \alpha_{j,s_{j-1},\varepsilon}, 1 \leq j \leq r$ and $1 \leq \varepsilon \leq s$, and for some $\lambda_\varepsilon \in k_w$. Hence

$$\deg_{k_w}(D_{\varepsilon}) = \sum_{j=1}^{r} \deg_{k_w}(T_{j,\varepsilon}) = \sum_{j=1}^{r} \sum_{\nu=1}^{s_j} \eta_j^{j-\nu} \alpha_{j,\varepsilon} \frac{\epsilon_j^{l_{j,\varepsilon}^\prime}}{s_j} = \sum_{j=1}^{r} \alpha_{j,\varepsilon} \frac{\epsilon_j^{l_{j,\varepsilon}^\prime} \eta_j^{j-1} - 1}{\eta - 1}.$$ (4.6)

Therefore

$$e_\infty(K_{\varepsilon}/k) = e_\infty(K_{\varepsilon,w}/k_w) = \frac{l_{n_\varepsilon}}{\gcd(\deg_{k_w}(D_{\varepsilon}),l_{n_\varepsilon})} = l_{n_\varepsilon} - \min\{n_\varepsilon,v_l(\deg_{k_w}(D_{\varepsilon}))\}$$

where $\deg_{k_w}(D_{\varepsilon})$ is given by (4.6). Hence,

$$e_\infty(K/k) = e_\infty(K_w/k_w) = l^t$$

with

$$t = \max_{1 \leq \varepsilon \leq s} \{n_\varepsilon - \min\{n_\varepsilon,v_l(\deg_{k_w}(D_{\varepsilon}))\}\}.$$ (4.7)

Now, since $e_{P_j}(K/k) = e_{P_j}(E/k) = l^{\beta_j},$ where $E$ is given by (2.1), we have that $E_{q_j} = \prod_{j=1}^{r} F_j$ with $k \subseteq F_j \subseteq k(\Lambda_{P_j})$ and $[F_j : k] = l^{\beta_j}$. From (4.3), we obtain that $F_j,w = k_w(\sqrt{\prod_{j=1}^{r} P_{j,1}^{\eta_j^{-1}} P_{j,2}^{\eta_j^{-2}} \cdots P_{j,s_{j-1}}^{\eta_j} P_{j,s_j}}) = k_w(\sqrt{Q_j}),$ where

$$Q_j := P_{j,1}^{\eta_j^{-1}} P_{j,2}^{\eta_j^{-2}} \cdots P_{j,s_{j-1}}^{\eta_j} P_{j,s_j}, \quad 1 \leq j \leq r.$$ (4.8)

We have

$$e_\infty(F_j/k) = e_\infty(F_j,w/k_w) = \frac{l_{\beta_j}}{\gcd(\deg_{k_w}(Q_j),l_{\beta_j})} = \frac{l_{\beta_j}}{\gcd(\min\{\beta_j,v_l(\deg_{k_w}(Q_j))\},l_{\beta_j})},$$

where $\deg_{k_w}(Q_j) = \epsilon_j^{l_{j,\varepsilon}^\prime} \eta_j^{j-1} - 1$ (see (4.6)). Therefore

$$l_m := e_\infty(K_{q_j,f}/k) = e_\infty(K_{q_j,f,w}/k_w) = e_\infty(E_{q_j,f}/k) = e_\infty(E_{q_j,f,w}/k_w) = \frac{\text{lcm}}{\text{lcm}} e_\infty(F_j/k).$$

Hence

$$m = \max_{1 \leq j \leq r} \{\beta_j - \min\{\beta_j,v_l(\deg_{k_w}(Q_j))\}\}.$$ (4.9)

Now, $K_{q_j,f}$ is the extension $K \subseteq K_{q_j,f} \subseteq K_{q_j,f,w}$ such that $e_\infty(K_{q_j,f}/K) = 1$ and $[K_{q_j,f,w} : K_{q_j,f}] = l_{m-t}$. Thus, by the Galois correspondence, $K_{q_j,f,w}$ is the extension $K_w \subseteq K_{q_j,f,w} \subseteq K_{q_j,f,w}$ such that $e_\infty(K_{q_j,f,w}/K_w) = 1$ and $[K_{q_j,f,w} : K_{q_j,f,w}] = l_{m-t}$. Our main result is the explicit description of $K_{q_j,f,w}$. 
Theorem 4.2. Let $K/k$ be a finite non-Kummer $l$-extension of $k$ with Galois group $\text{Gal}(K/k) \cong C_{n_1} \times \cdots \times C_{n_s}$, where $n = n_1 \geq n_2 \geq \cdots \geq n_s$ and $l^n \nmid q - 1$.

Let $K = K_1 \cdots K_s$ be such that $\text{Gal}(K_{j}/k) \cong C_{n_j}$, $1 \leq j \leq s$.

Let $P_1, \ldots, P_r$ be the finite primes in $K$ ramified in $K$ with $P_1, \ldots, P_r \in \mathcal{R}_F$ distinct.

Let $\text{deg } P_j = c_j l^{d_j}$ with $\text{gcd}(c_j, l) = 1$, $1 \leq j \leq r$.

Let $\text{deg } P_j(K/k) = l^{\beta_j}$, $1 \leq \beta_j \leq n$, $1 \leq j \leq r$, and $e_\infty(K/k) = l^t$, $0 \leq t \leq n$ given by (4.7). We order $P_1, \ldots, P_r$ so that $n_j = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_r$.

Let $E$ be given by (2.1). The maximal cyclotomic extension $M$ of $E$, unramified at the finite primes, is given by $M = E_{\text{cycl}} = \prod_{j=1}^r F_j$ where $F_j$ is the only field satisfying $k \subseteq F_j \subseteq k(\zeta_{n_j})$ and $[F_j : k] = l^{\beta_j}$.

Let $w = [F_q(q^n) : F_q] > 1$, $(\tau) = \text{Gal}(k_w/k)$ with $\tau(\zeta_n) = \zeta_{n_j}$, where ord($\eta$ mod $n_j$) = $u$. Let $\text{con}_{k/k_w} P_j = \mathcal{P}_{j,1} \cdots \mathcal{P}_{j,s_j}$, $1 \leq j \leq r$ and

$$Q_j := \mathcal{P}_{j,1} \mathcal{P}_{j,2} \cdots \mathcal{P}_{j,s_j} \in F_q[T].$$

Set $\text{deg } g_{n_j}(Q_j) = c_j l^{d_j}$ with $l \mid c_j$. Then we have $F_{j,w} = k_w(\sqrt[i]{\tau})$ and $K_{\text{cycl}} = k_w(\sqrt[l]{\tau})$ for some $\gamma_\varepsilon \in F_q^{-1}$ and $\mathcal{D}_\varepsilon = \prod_{j=1}^r T_{j,\varepsilon}$ where $T_{j,\varepsilon}$ is given by $T_{j,\varepsilon} = (\mathcal{P}_{j,1} \mathcal{P}_{j,2} \cdots \mathcal{P}_{j,s_j-1}) (\ell^\varepsilon)^d$ where $0 \leq \alpha, \varepsilon \leq l^n - 1$.

Let $l^n = e_\infty(K_{\text{cycl}}/k) = e_\infty(K_{\text{cycl}}/k_w)$.

Then $m = \max_{1 \leq j \leq r} \{\beta_j - \min(\beta_j, d_j)\}$, (see (4.9)).

Choose $i$ such that $m = \beta_i - \min(\beta_i, d_i)$ and such that for $j > i$ we have $m > \beta_j - \min(\beta_j, d_j)$. That is, $i$ is the largest index obtaining $l^n$ as the ramification index of $\text{con}_{k/k_w} P_\infty = \mathcal{P}_\infty$, the infinite prime of $F_q[T]$.

In case $m = t = 0$ we have $M = E_{\text{cycl}} = \prod_{j=1}^r F_j$, $E_{\text{cycl}} = \prod_{j=1}^r k_w(\sqrt[l]{\tau})$ and $K_{\text{cycl}} = E_{\text{cycl}} K_w$.

In case $m > t \geq 0$ or $m = t > 0$, we have $\min(\beta_i, d_i) = d_i$ and $m = \beta_i - d_i$.

Let $a, b \in \mathbb{Z}$ be such that $a \text{deg } K_w(\mathcal{Q}_j) + b l^{n+d_i} = l^{d_i} \equiv \text{gcd}(l^{n+d_i}, \text{deg } K_w(\mathcal{Q}_j))$. Set $z_j = -a l^{d_i} \equiv -a c_j l^{d_i} \equiv z_j \in \mathbb{Z}$ for $1 \leq j \leq i - 1$. For $j > i$, consider $y_j \in \mathbb{Z}$ with $y_j \equiv -c_j^{-1} l^{d_i} \equiv -a c_j l^{d_i} \equiv y_j$.

$$L_j = \begin{cases} k_w(\sqrt[l]{\mathcal{Q}_j}) & \text{if } j < i, \\ k_w(\sqrt[l]{\mathcal{Q}_j} \mathcal{Q}_{y_j}) & \text{if } j = i, \\ k_w(\sqrt[l]{\mathcal{Q}_j} \mathcal{Q}_{y_j} l^{d_j-i}) & \text{if } j > i \text{ and } d_j \geq d_i, \\ k_w(\sqrt[l]{\mathcal{Q}_j} \mathcal{Q}_{y_j} l^{d_j-i}) & \text{if } j > i \text{ and } d_i > d_j, \end{cases}$$

where $l^u = [F_{q^n}(\sqrt[l]{\mathcal{Q}_1}, \ldots, \sqrt[l]{\mathcal{Q}_i}) : F_{q^n}]$, $\mu_{\varepsilon} = (-1)^{\text{deg } K_w(\mathcal{Q}_j)}$, $1 \leq \varepsilon \leq s$, $\text{deg } K_w(\mathcal{P}_\infty) = \frac{\text{deg } K_w(\mathcal{P}_\infty, w)}{\text{gcd}(\text{deg } K_w(\mathcal{P}_\infty, w))}$ and $l^v = \frac{\text{gcd}([E K : K], \mu_{\varepsilon})}{\text{gcd}(\text{deg } K_w(\mathcal{P}_\infty)))}$. Equivalently,

$$l^u + v = \frac{[F_{q^n}(\sqrt[l]{\mathcal{Q}_1}, \ldots, \sqrt[l]{\mathcal{Q}_i}) : F_{q^n}] \cdot \text{gcd}([E K : K], \mu_{\varepsilon})}{\text{deg } K_w(\mathcal{P}_\infty)}.$$
value of $u$. That is $l^n = \left[\frac{\varphi_u(n)}{\varphi_u(n_k)}\right]$. Set $\xi = u + v$, that is, $|H| = |H'|^n$, for some $v \geq 0$.

From [1, proof of Theorem 2.2] we have that $E_K/K$ is an extension of constants and since we are assuming that the field of constants of $K$ is $\mathbb{F}_q$, we have that the field of constants of $E_K$ is $\mathbb{F}_q$, where $\chi = [E_K : K] = [E_K : K]|H|$. By the same reason, $E_K/E$ is an extension of constants and $[E_K : E] = f_{\infty}(E_K/k) = \deg_{E_K}(P_{\infty}) = f_{\infty}(E_K/K)f_{\infty}(K/k) = |H|\deg_{K}(P_{\infty})$.

Therefore, $[E_H : K] = \deg_{K}(P_{\infty})w [E_K : K] = [E_K : E]$ and $\mathbb{F}_{q^{|H|\deg_{K}(P_{\infty})}}$ is the field of constants of $E_K$.

Since $E_wK_w = (E_K)_w = E_K\mathbb{F}_{q^w}$, the field of constants of $E_wK_w$ is

$$\mathbb{F}_{q^{lcm[|H|\deg_{K}(P_{\infty}), w]}} = \mathbb{F}_{q^{lcm[|E_K : K], w}}}.$$ (4.10)

On the other hand, $E_wK_w/K_w$ is an extension of constants of degree $[E_wK_w : K_w] = |H'|\deg_{K_w}(P_{\infty})$. Since the field of constants of $K_w$ is $\mathbb{F}_{q^w}$, we have that the field of constants of $E_wK_w$ is

$$\mathbb{F}_{q^{|H'|\deg_{K_w}(P_{\infty})}} = \mathbb{F}_{q^{|H'|lcm[\deg_{K}(P_{\infty}), w]}}.$$ (4.11)

From (4.10) and (4.11), we obtain

$$lcm[|H|\deg_{K}(P_{\infty}), w] = |H'|lcm[\deg_{K}(P_{\infty}), w].$$

Now

$$lcm[|H|\deg_{K}(P_{\infty}), w] = \frac{|H|\deg_{K}(P_{\infty})w}{gcd(|H|\deg_{K}(P_{\infty}), w)} = |H|\frac{lcm[\deg_{K}(P_{\infty}), w]}{gcd(|H|\deg_{K}(P_{\infty}), w)}.$$ 

Therefore

$$|H'| = |H|\frac{gcd[\deg_{K}(P_{\infty}), w]}{gcd(|H|\deg_{K}(P_{\infty}), w)} = |H|\frac{gcd[\deg_{K}(P_{\infty}), w]}{gcd([E_K : K], w)}.$$ 

Hence $l^n = \frac{gcd([E_K : K], w)}{gcd[\deg_{K}(P_{\infty}), w]}$.

The rest of the proof follows the lines of the proof of Theorem 3.2 (see [8, Theorem 3.4]).

Remark 4.3. In Theorem 4.2, the Kummer case occurs when we allow $w = 1$ and Theorem 3.2 in this sense can be considered the special case $w = 1$ in Theorem 4.2.

5. Genus fields of finite abelian extensions of $k$

In [8, Theorem 4.1], we obtained the following result.

Theorem 5.1. Let $J_i/k$, $i = 1, 2$ be two finite abelian extensions such that $gcd([J_1 : k], [J_2 : k]) = 1$. Then $(J_1)_{ge} (J_2)_{ge} = (J_1J_2)_{ge}$. □

As a consequence of the previous results, we have the explicit description of any finite abelian $K/k$. For a finite non-trivial $l$-group $S$, we denote by $\exp(S) = l^n$ where $n$ is the minimum natural number $n$ such that $S^n = \{1\}$.
Theorem 5.2 (Genus field of an abelian extension). Let $K/k$ be a finite abelian extension with Galois group $G = \text{Gal}(K/k)$. Let $S_1, S_2, \ldots, S_t$ be the different Sylow subgroups of $G$ with $S_j$ the $l_j$-Sylow subgroup of $G$. Let $K = K_1 \cdots K_t$ be such that $\text{Gal}(K_j/k) \cong S_j, 1 \leq j \leq t$. Then

$$K_{ge} = \prod_{j=1}^{t} (K_j)_{ge},$$

where $(K_j)_{ge}$ is given by

$$\begin{cases} 
\text{Theorem 3.1} & \text{if } l_j = p, \\
\text{Theorem 3.2} & \text{if } l_j \neq p \text{ and } \exp(S_j)|q - 1, \\
\text{Theorem 4.2} & \text{if } l_j \neq p \text{ and } \exp(S_j) \nmid q - 1.
\end{cases}$$

Proof. The result is an immediate consequence of Theorems 3.1, 3.2, 4.2 and 5.1. \qed

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