HYPERKÄHLER TORSION STRUCTURES INVARIANT BY NILPOTENT LIE GROUPS

ISABEL G. DOTTI AND ANNA FINO

Abstract. We study HKT structures on nilpotent Lie groups and on associated nilmanifolds. We exhibit three weak HKT structures on $\mathbb{R}^8$ which are homogeneous with respect to extensions of Heisenberg type Lie groups. The corresponding hypercomplex structures are of a special kind, called abelian. We prove that on any 2-step nilpotent Lie group all invariant HKT structures arise from abelian hypercomplex structures. Furthermore, we use a correspondence between abelian hypercomplex structures and subspaces of $\mathfrak{sp}(n)$ to produce continuous families of compact and noncompact of manifolds carrying non isometric HKT structures. Finally, geometrical properties of invariant HKT structures on 2-step nilpotent Lie groups are obtained.

1. Introduction

Metric connections having totally skew-symmetric torsion arise in a natural way in theoretical and mathematical physics. For example, the geometry of such connections is present on the target space of supersymmetric sigma models with the Wess-Zumino term [14, 15, 19] and, in the supergravity theories, on the moduli space of a class of black holes [13]. Moreover, the geometry of NS-5 brane solution of type II supergravity theories is generated by such connection [26, 27, 25].

Let $M$ be a smooth manifold with a hypercomplex structure $\{J_i\}_{i=1,2,3}$ and a riemannian metric $g$. $M$ is said to be a hyperhermitian manifold if it is hermitian with respect to every $J_i$, $i=1,2,3$. A given hyperhermitian manifold $(M, \{J_i\}_{i=1,2,3}, g)$ is an HKT (hyperkähler torsion) manifold ([18]) if there is a connection $\nabla$ such that

\[ \nabla g = 0, \quad \nabla J_i = 0, \quad i=1,2,3, \quad c(X, Y, Z) = g(X, T(Y, Z)) \text{ a three form}. \]

Such a connection is called by physicists a KT connection; among mathematicians this connection is known as the Bismut connection ([3]).

On any hermitian manifold $(M, J, g)$ there exists a unique connection $\nabla$ satisfying $\nabla g = 0$, $\nabla J = 0$ and whose torsion tensor $c(X, Y, Z) = g(X, T(Y, Z))$ is totally skew-symmetric (i.e a three form). The torsion tensor of this connection is given by $c = -JdJF$, where $F = g(J., .)$ is the Kähler form for $J$ ([4]). The geometry of such a connection is called by physicists a KT connection; among mathematicians this connection is known as the Bismut connection ([3]).

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1
associated to \((J_i, g_i)\) or equivalently if \(\overline{\partial}_J (F_2 - iF_3) = 0\). If this connection exists, it is unique \([11]\). Moreover, by \([20]\) the associated Lee forms \(\theta_i = J_i d^* F_i\) coincide for \(i = 1, 2, 3\).

Every 4-dimensional hyperhermitian manifold is HKT. If the dimension is 8 we obtained in \([9]\) all simply connected nilpotent Lie groups which carry invariant abelian hypercomplex structures. There are three such groups and they are central extensions of Heisenberg type Lie groups (see Example 1 in Section 2). The abelian hypercomplex structures give rise to weak HKT structures on these groups (see Proposition 2.1), with respect to any compatible and invariant riemannian metric. These groups are diffeomorphic to \(\mathbb{R}^8\). In coordinates \((x_1, ..., x_4, y_1, ..., y_4)\) the corresponding HKT metrics are given by:

\[
\begin{align*}
g_1 &= \sum dx_i^2 + (dy_1 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3))^2 + \sum_{j\geq 2} dy_j^2, \\
g_2 &= \sum dx_i^2 + dy_1^2 + (dy_2 - \frac{1}{2}(x_1 dx_3 - x_3 dx_1 + x_2 dx_4 - x_4 dx_2))^2 + (dy_3 - \frac{1}{2}(x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2))^2 + dy_4^2, \\
g_3 &= \sum dx_i^2 + (dy_1 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3))^2 + (dy_2 - \frac{1}{2}(x_1 dx_3 - x_3 dx_1 + x_2 dx_4 - x_4 dx_2))^2 + (dy_3 - \frac{1}{2}(x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2))^2 + dy_4^2.
\end{align*}
\]

These metrics have a transitive nilpotent group of isometries (hence they are complete) and they are non isometric to each other.

The 8-dimensional HKT structures obtained above are associated to abelian hypercomplex structures. One of the main goals of this paper is to prove that on any 2-step nilpotent Lie groups all invariant HKT structures arise this way (see Theorem 3.1).

On the other hand, the correspondence given in \([4]\) between abelian hypercomplex structures on 2-step nilpotent Lie groups and subspaces of \(\mathfrak{sp}(n)\), gives a method to construct infinitely many compact and non compact families of manifolds carrying non isometric HKT structures. By using this construction we show in Section 4 that there exist non trivial deformations of homogeneous HKT structures on \(\mathbb{R}^{4l}, l \geq 3\). Moreover, for rational parameters one obtains infinitely many HKT compact quotients of nilpotent Lie groups by discrete subgroups. This is in contrast with results in \([4, 17]\) in the Kähler case.

In the last section we analyze some geometrical properties of invariant HKT structures on 2-step nilpotent Lie groups. We show that in this class, and with respect to the Bismut connection, the Ricci tensor is symmetric, hence by \([20]\) the torsion 3-form \(c\) is co-closed, every one form in the dual of the center is parallel and all Lee forms are zero. This last assertion says that the corresponding riemannian manifolds are hermitian semi-kähler \([2]\) or hermitian balanced \([13]\). These seem to be the first examples of this type. In the particular case of dimension 8, using the explicit description of the Bismut connection we show that its Ricci tensor has two distinct eigenvalues \((0, -\lambda, \lambda > 0)\) and only one of the groups carrying invariant HKT structure has parallel torsion.

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2. HYPERKÄHLER TORSION STRUCTURES ON GROUPS

A hypercomplex structure on a Lie algebra \(\mathfrak{g}\) is a triple of endomorphisms \(\{J_i\}_{i=1,2,3}\) satisfying the quaternion relations \(J_i^2 = -I, \ i = 1, 2, 3, \ J_1 J_2 = J_3\).
\(-J_2J_1 = J_3\), together with the vanishing of the Nijenhuis tensor

\[ N_i(X, Y) = J_i([X, Y] - [J_iX, J_iY]) - ([J_iX, Y] + [X, J_iY]), \]

where \(X, Y \in \mathfrak{g}\) and \(i = 1, 2, 3\).

The hypercomplex structure will be called abelian if

\[ [J_iX, J_iY] = [X, Y], \]

for all \(X, Y \in \mathfrak{g}\), \(i = 1, 2, 3\). Abelian hypercomplex structures were previously considered in \([3], [8], [9]\); they can only occur on solvable Lie algebras \([10]\).

Let \(\mathfrak{g}\) be a Lie algebra endowed with a hypercomplex structure \(\{J_i\}_{i=1,2,3}\) and an inner product \(g\), compatible with the hypercomplex structure, that is

\[
g(X, Y) = g(J_1X, J_1Y) = g(J_2X, J_2Y) = g(J_3X, J_3Y).\tag{2}
\]

for all \(X, Y \in \mathfrak{g}\). Assume furthermore that the hypercomplex structure together with the inner product given on \(\mathfrak{g}\) satisfy the extra condition

\[
g([J_1X, J_1Y], Z) + g([J_1Y, J_1Z], X) + g([J_1Z, J_1X], Y) = 0, \tag{3}\]

for all \(X, Y, Z \in \mathfrak{g}\). Note that if one substitutes \(X, Y, Z\) by \(J_3X, J_3Y, J_3Z\) in the second and third row of (3), then one obtains

\[
g(J_3[X, Y], Z) + g(J_3[Y, Z], X) + g(J_3[Z, X], Y) = 0, \tag{4}\]

and conversely, (4) implies that the last two rows in (3) are equal. Since the equality of any two rows in (3) gives equality of the three rows, one has in particular that (3) and (4) are equivalent.

An HKT structure \(\{\{J_i\}_{i=1,2,3}, g\}\) on a Lie algebra \(\mathfrak{g}\) consists of a hypercomplex structure \(\{J_i\}_{i=1,2,3}\) together with an inner product \(g\) on \(\mathfrak{g}\) satisfying conditions (2) and (3) or conditions (2) and (4). If \(G\) is a Lie group with Lie algebra \(\mathfrak{g}\) carrying an HKT structure, by left translating the \(J_i\), \(i = 1, 2, 3\) and the inner product \(g\), one obtains in \(G\) an invariant HKT structure. Indeed, in this case one finds that the Bismut connection is defined by the equation

\[
g(\nabla_XY, Z) = \frac{1}{2}(g([X, Y] - [J_iX, J_iY], Z) - g([Y, Z] + [J_iY, J_iZ], X) + g([Z, X] - [J_iZ, J_iX], Y)), \tag{5}\]

for \(X, Y, Z\) left invariant vector fields. A verification shows that this connection satisfies (4).

When the hypercomplex structure is abelian, (3) is always satisfied and moreover, the HKT structure is weak. Indeed, to prove the last assertion, we note that the Bismut connection \(\nabla\) and its torsion \(c\) are given by

\[
g(\nabla_XY, Z) = -g([Y, Z], X),
\]

\[
c(X, Y, Z) = g(X, T(Y, Z)) = (-1)(g([X, Y], Z) + g([Y, Z], X) + g([Z, X], Y)).
\]

Since

\[
dc(X, Y, Z, W) = -2g([X, Y], [Z, W]) + 2g([X, Z], [Y, W]) - 2g([X, W], [Y, Z]),
\]

one obtains in particular

\[
dc(X, J_1X, J_2X, J_3X) = 2||[X, J_1X]||^2 + 2||[X, J_2X]||^2 + 2||[X, J_3X]||^2.	ag{6}\]
and

$$dc(X, J_1X, Y, J_1Y) = -2g([X, J_1X], [Y, J_1Y])$$

$$(7)$$

$$+2g([X, Y], [X, J_1X]) + 2g([X, J_1Y], [X, J_1Y])$$

Equations (6) and (7) imply that $dc \neq 0$ unless $\mathfrak{g}$ is abelian. Indeed, the condition $dc = 0$ in (6) implies $g([X, J_1X], [X, J_1X]) = 0$ and substituting this in (7) gives $g([X, Y], [X, Y]) = 0$. Hence, we have proved

**Proposition 2.1.** Every abelian hypercomplex structure on a non abelian Lie group $G$ give rise to an invariant weak HKT structure on $G$.

**2. Examples.**

1. Let $H_i(n)$ for $1 \leq i \leq 3$, denote respectively the real, complex or quaternionic Heisenberg groups. The hypercomplex structures on the 8-dimensional nilpotent Lie groups $N_1 = \mathbb{R}^3 \times H_1(2)$, $N_2 = \mathbb{R}^2 \times H_2(1)$ $N_3 = \mathbb{R}^1 \times H_3(1)$ constructed in [8] are abelian and give rise, with respect to any compatible and invariant riemannian metric, to weak HKT-structures on these groups. Moreover, as proved in [10], the Obata connection associated to any hypercomplex structure on $N_i$, $i = 1, 2, 3$ is flat. Their corresponding Lie algebras are $\mathfrak{n}_i = \mathfrak{v} \oplus \mathfrak{z}$, $i = 1, 2, 3$, $\mathfrak{v} = \text{span}\{e_1, e_2, e_3, e_4\}$ and $\mathfrak{z} = \text{span}\{e_5, e_6, e_7, e_8\}$, with non zero brackets

$$[e_1, e_2] = -[e_3, e_4] = e_5$$

in $\mathfrak{n}_1$, 

$$[e_1, e_3] = [e_2, e_4] = e_6; \quad [e_1, e_4] = -[e_2, e_3] = e_7$$

in $\mathfrak{n}_2$ and

$$[e_1, e_2] = -[e_3, e_4] = e_5; \quad [e_1, e_3] = [e_2, e_4] = e_6; \quad [e_1, e_4] = -[e_2, e_3] = e_7$$

in $\mathfrak{n}_3$. The hypercomplex structure is given by $J_i e_1 = e_{i+1}$, $J_i e_5 = e_{5+i}$, $i = 1, 2, 3$, $J_1^2 = -I$, $J_1 J_2 = -J_2 J_1 = J_3$. We note that these nilpotent Lie groups do admit lattices, hence we also obtain compact examples.

2. The 8-dimensional Lie group $H_1(2) \times SU(2)$ has an invariant weak HKT structure (see [28]). This group does not admit invariant abelian hypercomplex structures since it is not solvable ([10])

3. The 12-dimensional 3-step nilpotent Lie group with non zero brackets

$$[e_1, e_2] = -[e_5, e_6] = -e_{10}, [e_2, e_5] = -[e_1, e_6] = -e_{11},$$

$$[e_1, e_4] = [e_2, e_{10}] = [e_5, e_8] = [e_6, e_{11}] = -e_{12}$$

admits an abelian hypercomplex structure $\{J_i\}_{i=1,2,3}$ given by ([10])

$$J_1 e_1 = e_2, J_1 e_3 = e_{12}, J_1 e_4 = e_{10}, J_1 e_5 = e_6, J_1 e_7 = e_9, J_1 e_8 = e_{11},$$

$$J_2 e_1 = e_6, J_2 e_2 = e_5, J_2 e_3 = e_9, J_2 e_4 = e_{11}, J_2 e_8 = -e_{10}, J_2 e_7 = -e_{12}$$

whose associated Obata connection is not flat, by [10]. On the other hand the hypercomplex structure together with the metric such that the above basis is orthonormal give a weak invariant HKT structure.
3. HKT structures on nilpotent Lie groups

In this section we will restrict to the case of invariant HKT structures on nilpotent Lie groups.

Let \( \mathfrak{n} \) be a nilpotent Lie algebra, that is a Lie algebra satisfying \( n^k = 0 \) for some \( k \geq 1 \), where \( n^i \) is the chain of ideals defined inductively by \( n^0 = \mathfrak{n} \) and

\[
n^i = [n^{i-1}, n], \quad i \geq 1.
\]

One says that \( \mathfrak{n} \) is \( k \)-step nilpotent if \( n^k = 0 \) and \( n^{k-1} \neq 0, k \geq 1 \).

Let \( \{J_i\}_{i=1,2,3} \) be a hypercomplex structure on a nilpotent Lie algebra \( \mathfrak{n} \). In order to prove the main result of this section we first prove two useful lemmas.

**Lemma 3.1.** If \( \{J_i\}_{i=1,2,3} \) is a hypercomplex structure on an \( s \)-step nilpotent Lie algebra \( \mathfrak{n} \) then the inclusion

\[
n_Q^{s-1} = n^{s-1} + J_1 n^{s-1} + J_2 n^{s-1} + J_3 n^{s-1} \subset \mathfrak{n}
\]

is proper.

**Proof.** Suppose it is not. Let \( X \in \mathfrak{n} \) and write \( X = X_0 + J_1 X_1 + J_2 X_2 + J_3 X_3 \), \( X_i \in n^{s-1}, i = 0, 1, 2, 3 \). If \( Y \in \mathfrak{n} \), then

\[
[X, Y] = [J_i X_1, Y] + [J_2 X_2, Y] + [J_3 X_3, Y],
\]

since \( \mathfrak{n} \) is \( s \)-step. Write \( Y = Y_0 + J_1 Y_1 + J_2 Y_2 + J_3 Y_3 \), \( Y_i \in n^{s-1}, i = 0, 1, 2, 3 \) and substitute in the previous expression obtaining

\[
[X, Y] = \sum_{i=1}^{3} [J_i X_i, J_i Y_i] + [J_1 X_1, J_2 Y_2 + J_3 Y_3] +
\]

\[
[J_2 X_2, J_3 Y_3 + J_1 Y_1] + [J_3 X_3, J_1 Y_1 + J_2 Y_2].
\]

Denote by \( \mathfrak{z} \) the center of \( \mathfrak{n} \) (note that \( n^{s-1} \subset \mathfrak{z} \)) and observe that the integrability of \( J_i \) gives, for \( l = 1, 2, 3 \),

\[
[J_i U, J_i V] = 0, \quad \text{if } U, V \in \mathfrak{z},
\]

\[
J_i [J_i U, V] = [J_i U, J_i V] \text{ if } U \in \mathfrak{z} \text{ and } V \in \mathfrak{n}.
\]

Note that (11) above implies

\[
[J_1 U, J_3 V] = [J_1 J_1 U, J_2 V], \quad [J_2 U, J_3 V] = [J_2 J_2 U, J_1 (-V)], \quad U, V \in \mathfrak{z}.
\]

Hence using (10) and (12) in the expression of \([X, Y]\) and setting \( u = [J_1 n^{s-1}, J_2 n^{s-1}] \) one obtains

\[
n^1 = u + J_1 u + J_2 u.
\]

Then

\[
n = n_Q^{s-1} \subset n^1 + J_1 n^1 + J_2 n^1 + J_3 n^1 \subset u + J_1 u + J_2 u + J_3 u
\]

and as a consequence \( n = n^1 + J_3 n^1 \) contradicting the fact that for any invariant complex structure on a nilpotent Lie group there exists a closed (1, 0) form (9).

**Remark 1.** We observe that \( \mathfrak{z} = \mathfrak{z} + J_1 \mathfrak{z} + J_2 \mathfrak{z} + J_3 \mathfrak{z} \) can be all of \( \mathfrak{n} \). Indeed in \( \mathfrak{g} \) such an example is given of a hypercomplex nilpotent Lie algebra of dimension 8 having a 5-dimensional center. Also, when the hypercomplex structure is abelian one has (see 3) that the inclusion

\[
n_Q^i = n^i + J_1 n^i + J_2 n^i + J_3 n^i \subset n^{i-1} + J_1 n^{i-1} + J_2 n^{i-1} + J_3 n^{i-1} = n_Q^{i-1},
\]

\( i \geq 1 \), is proper.
Lemma 3.2. Let \( \mathfrak{n} \) be a 2-step nilpotent Lie algebra with an HKT structure. If \( n_\mathfrak{Q}_1 = n_1 + J_1 n_1 + J_2 n_1 + J_3 n_1 \) and \( \mathfrak{m} = (n_\mathfrak{Q}_1)^{\perp} \) then

\[ i) \ [n_\mathfrak{Q}_1, n_\mathfrak{Q}_1^\perp] = 0. \]

\[ ii) \ [n_\mathfrak{Q}_1, \mathfrak{m}] = 0. \]

Proof. Note first that by the previous lemma the subspace \( n_\mathfrak{Q}_1 \) is proper. To prove assertion i) we observe first that \( n_1 + J_1 n_1 \) is an abelian subalgebra, for any \( l = 1, 2, 3 \), since \( n_1 \) is contained in the center of \( \mathfrak{n} \) (see (10)). We show next that \([J_1 n_1, J_2 n_1] = 0\). Take \( X, Z \in \mathfrak{n}^1 \) and substitute in (4) obtaining

\[ 0 = g(J_3 [J_1 X, -J_2 Y'], J_3 Z') + g(J_3 [-J_2 Z', J_1 X], J_3 Y') \]

or equivalently \( \text{ad}_{J_1 X} J_2 \) is a self-adjoint transformation of \( n^1 \). Since its square is zero by (11), \([J_1 X, J_2 Y] = 0, X, Y \in n^1 \). Using (12) one shows \([J_1 X, J_1 Y] = 0, X, Y \in n^1 \) in the remaining cases. We next prove assertion ii). Substituting \( X, Z \in \mathfrak{n}^1 \) and \( Y \in \mathfrak{m} \) into the last two lines of (10) implies that \( \text{ad}_{J_2 Y} J_2 - \text{ad}_{J_1 Y} J_3 \) is a self-adjoint transformation of \( n^1 \). Since it is also nilpotent (see (11)) it must be zero hence

\[ [J_2 X, J_2 Y'] = [J_3 X, J_3 Y], \quad X \in \mathfrak{n}^1, \quad Y \in \mathfrak{m}. \]

Take now \( X, Z \in \mathfrak{n}^1, \ Z = J_2 \tilde{Z} \) and substitute in (12) obtaining

\[ g([J_1 X, J_1 Z], \tilde{Z}) = g(J_3 ([Y, J_3 \tilde{Z}] + [J_1 Y, J_2 \tilde{Z}]), X). \]

Since \([J_2 \tilde{Z}, J_1 Y] = [J_2 \tilde{Z}, J_2 (J_1 J_2 Y')] = [J_3 \tilde{Z}, J_3 (J_1 J_2 Y')] = -[J_3 \tilde{Z}, Y] \) by (12) it follows that \([J_1 X, J_1 Y] = 0 \) or equivalently \([J_1 n^1, \mathfrak{m}] = 0 \). The proof in the remaining cases is similar.

Theorem 3.1. The hypercomplex structure of an invariant HKT structure on any 2-step nilpotent Lie group is abelian.

Proof. From i) and ii) of the previous lemma it follows that \( n_\mathfrak{Q}_1^1 \subset \mathfrak{z} \) where \( \mathfrak{z} \) stands for the center of \( \mathfrak{n} \). If \( Y, Z \in \mathfrak{m} \) then (8) implies \([J_1 Y, J_1 Z] = [J_2 Y, J_2 Z] = [J_3 Y, J_3 Z] \). Since \( \mathfrak{m} = J_1 \)-invariant, it follows that for any \( Y, Z \in \mathfrak{m} \) then \([Y, Z] = [J_1 Y, J_1 Z], \ i = 1, 2, 3 \). Now it is straightforward to show that the hypercomplex structure is abelian, by decomposing any given \( U, V \in \mathfrak{n} \), as \( U = U_1 + U_2, \ V = V_1 + V_2 \) according to \( \mathfrak{n} = n_\mathfrak{Q}_1^1 \oplus \mathfrak{m} \).

Remark 2. If we restrict to the 8-dimensional case it was proved in [8] that the only 2-step nilpotent Lie groups carrying abelian hypercomplex structures are the groups \( N_i, i = 1, 2, 3 \) considered in Example 1. As remarked in [8] \( N_1, N_2 \) can only carry abelian hypercomplex structures. On the other hand, \( N_3 \) does admit non-abelian hypercomplex structures, thus by Theorem 3.1, \( N_3 \) endowed with a non-abelian hypercomplex structure admits no invariant metric such that it becomes an HKT manifold.

4. Deformations of HKT structures

In [6] M. L. Barberis proved that there is a one to one correspondence between injective linear maps \( j : \mathbb{R}^m \rightarrow \mathfrak{sp}(k) \) \((m \leq k(2k + 1))\) and 2-step nilpotent Lie algebras \( \mathfrak{n} \) with dimension \(([\mathfrak{n}, \mathfrak{n}]) = k \) carrying an abelian hypercomplex structure. Using Theorem 3.1 one can rephrase the above result saying that the correspondence
is between injective linear maps $j : \mathbb{R}^m \to \mathfrak{sp}(k)$ ($m \leq k(2k + 1)$) and 2-step nilpotent Lie algebras $\mathfrak{n}$ with dimension $([\mathfrak{n}, \mathfrak{n}] = k)$ carrying an HKT structure.

We reproduce the construction given in [1] with the only modification introduced by Theorem 3.1.

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an HKT structure ($\{J_i\}_{i=1,2,3}, g$) and consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, with $\mathfrak{z}$ the center of $\mathfrak{n}$. Note that since $\mathfrak{n}$ is 2-step nilpotent one has that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}$. Since $\{J_i\}_{i=1,2,3}$ is an abelian hypercomplex structure then it preserves $\mathfrak{z}$, hence $\mathfrak{v}$, since it is hyperhermitian with respect to $g$. Let $j : [\mathfrak{n}, \mathfrak{n}] \to \mathfrak{so}(\mathfrak{v})$ be defined by

$$g(j_z X, Y) = g([X, Y], z), \quad X, Y \in \mathfrak{v}, \quad z \in \mathfrak{z}.$$ 

Then $j$ is one to one and if $\mathfrak{sp}(\mathfrak{v}) = \{T \in \mathfrak{so}(\mathfrak{v}) : TJ_i = J_i T, i = 1, 2, 3\}$ one has $j_z \in \mathfrak{sp}(\mathfrak{v})$ for all $z \in [\mathfrak{n}, \mathfrak{n}]$ since

$$g(j_z J_i X, J_i Y) = g([J_i X, J_i Y], z) = g([X, Y], z) = g(j_z X, Y).$$

Conversely, given $j : \mathbb{R}^m \to \mathfrak{sp}(k)$, fix $0 \leq s \leq 3$ with $s + m \equiv 0 \mod(4)$ and set $\mathfrak{n} = \mathbb{R}^k \oplus \mathbb{R}^s \oplus \mathbb{R}^m$ with $g$ the canonical inner product. Define the bracket such that $\mathbb{R}^s \oplus \mathbb{R}^m$ is central and $g([X, Y], z) = g(j_z X, Y)$ if $X, Y \in \mathbb{R}^k$, $z \in \mathbb{R}^m$.

Let $\{J_i\}_{i=1,2,3}$ be the endomorphisms of $\mathbb{R}^k$ defining $\mathfrak{sp}(k)$ extended to all of $\mathfrak{n}$ by anticommuting complex endomorphisms on $\mathbb{R}^s \oplus \mathbb{R}^m$ compatible with the metric. It is easy to verify that the resulting hypercomplex structure is abelian, hence it is an HKT structure on $\mathfrak{n}$.

**Remark 3.** By applying the previous construction to the case $m = 1$ and $j_z$ any complex structure commuting with the complex structures defining $\mathfrak{sp}(k)$ (for a fixed $z \neq 0$ in $\mathbb{R}$), the resulting algebra is an extension of the Heisenberg algebra and the HKT-structure is that obtained in [10] (5.2).

We next give some non trivial deformations of HKT structures.

Fix in $\mathbb{R}^{4l}, l \geq 2$, identified with $\mathbb{H}^l$, the quaternions, the hypercomplex structure $\{J_1, J_2, J_3\}$ given by right multiplication by $(i, \ldots, i)$, $(j, \ldots, j)$, and $(-k, \ldots, -k)$ respectively. Let $t > 0$ and $j^t : \mathbb{R}^2 \to \mathfrak{sp}(l)$, with

$$\mathfrak{sp}(l) = \{T \in \mathfrak{so}(4l) : TJ_i = J_i T, i = 1, 2, 3\},$$

be given by

$$j^t_{e_1} = L_{(i, \ldots, i, t)}, \quad j^t_{e_2} = L_{(j, \ldots, j, t)}$$

where $L$ stands for left multiplication, and $e_1, e_2$ denotes a basis of $\mathbb{R}^2$. It is clear that $j^t$ is a mapping into $\mathfrak{sp}(l)$ since left and right multiplication commute.

Similarly, if $l \geq 3$, let $(t, s)$ be such that $0 < t < s < 1$ and $j^{t,s} : \mathbb{R}^3 \to \mathfrak{sp}(l)$, be given by

$$j^{t,s}_{e_1} = L_{(i, \ldots, i, t)}, \quad j^{t,s}_{e_2} = L_{(j, \ldots, j, t)}, \quad j^{t,s}_{e_3} = L_{(k, \ldots, k, s)},$$

where $L$ stands for left multiplication, and $e_1, e_2, e_3$ denotes a basis of $\mathbb{R}^3$.

Let $\mathfrak{n}_t (\text{resp. } \mathfrak{n}_{t,s}) = \mathbb{R}^{4l} \oplus \mathbb{R}^4$ be the 2-step nilpotent Lie algebras with the HKT structure constructed as above and let $N_t$ (resp. $N_{t,s}$) be the simply connected Lie group with Lie algebra $\mathfrak{n}_t$ (resp. $\mathfrak{n}_{t,s}$) and invariant HKT structure induced by left translating the inner product and hypercomplex structure on $\mathfrak{n}_t$ (resp. $\mathfrak{n}_{t,s}$).

**Claim 1.** The riemannian manifolds $N_t$ and $N_{t'}$ (resp. $N_{t,s}$ and $N_{t',s}$) are isometric if and only if $t = t'$ (resp. $(t, s) = (t', s')$).
According to E. Wilson [30], if two nilpotent Lie groups with left invariant metric are isometric there exists an isomorphism which is also an isometry. Hence, its derivative is an orthogonal Lie algebra isomorphism between the corresponding Lie algebras. Assume then that \( f \) denotes an orthogonal isomorphism from \( \mathfrak{n} \) onto \( \mathfrak{n}' \). Using the description of the Lie brackets given by \( j \) and \( j' \) respectively, it follows that \( j'_t f^{-1} z = f^{-1} j'_t z f \) for all \( z \in \mathbb{R}^2 \). Squaring both sides in the previous equality and carrying out a tedious but straightforward computation one finds that \( t = t' \). Using similar arguments one can show that the riemannian manifolds \( N_{t,s} \) and \( N'_{t',s'} \) are isometric if and only if \( (t, s) = (t', s') \).

Claim 2. The riemannian manifolds \( N_q, q \in \mathbb{Q}, q > 0 \), do admit discrete subgroups \( \Gamma_q \) such that \( T_q = N_q/\Gamma_q \) is compact. Furthermore, the (non homogeneous) \( T_q \) with the induced metrics, are not isometric to each other for different \( q \)’s.

We recall that according to [22] a nilpotent Lie group \( N \) admits a discrete subgroup \( \Gamma \) such that \( N/\Gamma \) is compact if and only if its Lie algebra \( \mathfrak{n} \) admits a basis with rational structure constants. But this is clear in the case \( t \) rational in the definition of \( \mathfrak{n} \). Furthermore, an isometry between \( T_q \) and \( T'_q \) lifts to an isometry between \( N_q \) and \( N'_q \), which is impossible by Claim 1.

A compact quotient of a nilpotent Lie group \( N \) by a discrete subgroup is called a nilmanifold.

Theorem 4.1. There exists a one parameter (resp. two parameter family) of homogeneous HKT structures on \( \mathbb{R}^4_l, l \geq 3 \) (resp. \( \mathbb{R}^4_l, l \geq 4 \)). Moreover, for rational parameters there exists infinitely many non isometric HKT nilmanifolds.

Proof. To prove the existence of a one parameter family of non isometric HKT structures on \( \mathbb{R}^4_l, l \geq 3 \) (respectively a two parameter family of HKT structures on \( \mathbb{R}^4_l, l \geq 4 \)) one uses the fact that the exponential map \( \exp_t \) (resp. \( \exp_{t,s} \)) is a diffeomorphism from \( \mathbb{R}^4_l \rightarrow N_t \) (resp. \( \mathbb{R}^4_l \rightarrow N_{t,s} \)). The pullback by \( \exp_t \) (resp. \( \exp_{t,s} \)) of the invariant HKT-structures on \( N_t \) (resp. \( N_{t,s} \)) together with Claim 1 gives the asserted deformation. The second statement in Theorem 4.1 follows from Claim 2.

Remark 4. The existence of infinitely many nilmanifolds carrying HKT structures is in contrast with the Kähler case (compare [4], [17]).

5. Geometrical consequences

Let \( \mathfrak{n} \) be a 2-step nilpotent Lie algebra with an HKT structure \( \{J_i\}_{i=1,2,3}, g \), \( \nabla \) the Bismut connection and \( c \) the torsion 3-form. We show next that the Ricci tensor of \( \nabla \) is symmetric (hence \( c \) is co-closed by [20]) and the Lee forms are zero, hence the corresponding riemannian manifolds are hermitian semikähler (according to [12]) or hermitian balanced [13].

In Section 2 we observed that the Bismut connection associated to an abelian hypercomplex structure and its torsion 3-form were given respectively by

\[
\begin{align*}
g(\nabla_X Y, Z) &= -g([Y, Z], X), \\
c(X, Y, Z) &= (-1)(g([X, Y], Z) + g([Y, Z], X) + g([Z, X], Y)),
\end{align*}
\]

for any \( X, Y, Z \in \mathfrak{n} \).

Decompose \( \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{j} \) where \( \mathfrak{j} \) is the center of \( \mathfrak{n} \) and \( \mathfrak{v} \) its orthogonal complement. It follows easily that
i) $\nabla_X Z = 0, \ Z \in \mathfrak{z}, X \in \mathfrak{n}$.
ii) $\nabla_V X = 0, \ V \in \mathfrak{v}, X \in \mathfrak{n}$.
iii) $\nabla_X Y \in \mathfrak{v}, \ X, Y \in \mathfrak{n}$.

The curvature tensor associated to $\nabla$ is $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. As a consequence of i) and ii) above it follows that

iii) $g(R(X,Z)Y, Z') = 0, \ Z, Z' \in \mathfrak{z}, X, Y \in \mathfrak{n}$.
iv) $g(R(X,V)Y, V') = g([X,V], [Y, V']), \ V, V' \in \mathfrak{v}, X, Y \in \mathfrak{n}$.

In particular, the Ricci tensor $\rho$ of $\nabla$ is symmetric, given by

$$\rho(X, Y) = \sum_{j} g(R(X, V_j)Y, V_j) = \sum_{j} g([Y, V_j], [X, V_j]),$$

where $V_j$ is an orthonormal basis of $\mathfrak{v}$. It is worth to point out that the Ricci tensor of the Bismut connection $\nabla$ is not symmetric in general. By \cite{20} Corollary 3.2 the Ricci tensor of $\nabla$ is symmetric if and only the torsion 3-form $c$ is co-closed. In particular, on a 2-step nilpotent Lie algebra with an HKT structure, the torsion 3-form $c$ is always co-closed. Moreover, it follows from iii) that for any 1-form $\alpha$ in the dual of the center $\mathfrak{z}^*$ is parallel with respect to the Bismut connection, thus giving a reduction of its holonomy group.

One can also verify, using the expression of $\rho$ above that

v) $\rho(Z, X) = 0, \ Z \in \mathfrak{z}, X \in \mathfrak{n}$.
vi) $\rho(V, J_l V) = 0, V \in \mathfrak{v}, l = 1, 2, 3$.
vii) $\rho(V, V) = \rho(J_l V, J_l V), \ V \in \mathfrak{v}, l = 1, 2, 3$.

The last two assertions follow from

$$\rho(V, J_l V) = \sum_{j} g([V, V_j], [J_l V, V_j]) = -\sum_{j} g([J_l V, J_l V_j], [V, J_l V_j]) = -\rho(V, J_l V),$$

and

$$\rho(V, V) = \sum_{j} g([V, V_j], [V, V_j]) = \sum_{j} g([V, J_l V_j], [V, J_l V_j]) = \rho(J_l V, J_l V).$$

Finally, to show that the Lee forms are trivial we need to recall that by \cite{1, 20}

$$\theta(X) = -1/2 \sum_{i=1}^{2n} c(J_i X, e_i, J_l e_i), \tag{17}$$

where $c$ is the torsion 3-form and $\{e_i\}$ is an orthonormal basis of $\mathfrak{n}$.

In general, one has that if $Y \in \mathfrak{z},$

$$c(J_i X, Y, J_l Y) = 0,$$

and if $Y \in \mathfrak{v},$

$$c(J_i X, Y, J_l Y) = -g([Y, J_l Y], J_i X).$$

Then, using a basis $V_j, J_1 V_j, J_2 V_j, J_3 V_j$ of $\mathfrak{v}$ and letting $l = 1$ in (17)

$$\theta(X) = -1/2 \sum_{j} 2g([V_j, J_1 V_j], J_1 X) + 2g([J_2 V_j, J_3 V_j], J_1 X) = 0$$

since $J_1$ is abelian.
5.1. Geometry of 8-dimensional examples. We restrict next to the case of an 8-dimensional 2-step nilpotent Lie group. In [30] we showed that the only nilpotent 8-dimensional Lie groups carrying abelian hypercomplex structures were the groups $N_i, i = 1, 2, 3$ described in example 2 of 2.1. The groups $N_i, i = 1, 2, 3$ are diffeomorphic to $\mathbb{R}^8$ via the inverse of the exponential map. Using this diffeomorphism $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) : N \rightarrow \mathbb{R}^8 \oplus \mathfrak{z}$ as a coordinate system one can write the three complete HKT metrics on $\mathbb{R}^8$ as follows

\[ g_1 = \sum dx_i^2 + (dy_1 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3))^2 + \sum_{j \geq 2} dy_j^2, \]
\[ g_2 = \sum dx_i^2 + dy_1 + (dy_2 - \frac{1}{2}(x_1 dx_3 - x_3 dx_1 + x_2 dx_4 - x_4 dx_2))^2 + \]
\[ (dy_3 - \frac{1}{2}(x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2))^2 + dy_3^2, \]
\[ g_3 = \sum dx_i^2 + (dy_1 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3))^2 + \]
\[ (dy_2 - \frac{1}{2}(x_1 dx_3 - x_3 dx_1 + x_2 dx_4 - x_4 dx_2))^2 + \]
\[ (dy_3 - \frac{1}{2}(x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2))^2 + dy_4^2. \]

The metrics are not isometric since they come from non isomorphic groups (see [31]). The Ricci tensor associated to the Bismut connection, in these cases is given by

\[ \rho(Z, X) = 0, \quad Z \in \mathfrak{z}, X \in \mathfrak{n} \quad \rho(V, V) = c, c < 0, \quad V \in \mathfrak{v}, ||V|| = 1. \]

Indeed, in this case dim $\mathfrak{v} = 4$ and one can consider $\{V, J_1 V, J_2 V, J_3 V\}$ as basis of $\mathfrak{v}$ and apply v),vi),vii) above.

We show next that the torsion 3-form $c$ is parallel with respect to the Bismut connection $\nabla$ only in the case of $N_1$. The Bismut connection $\nabla$ is given on $N_1$ by

\[ \nabla e_5 e_1 = -e_2, \quad \nabla e_5 e_2 = e_1, \quad \nabla e_5 e_3 = e_4, \quad \nabla e_5 e_4 = -e_3, \]

on $N_2$ by

\[ \nabla e_5 e_1 = -e_3, \quad \nabla e_5 e_2 = -e_4, \quad \nabla e_5 e_3 = e_1, \quad \nabla e_5 e_4 = e_2, \]
\[ \nabla e_5 e_1 = -e_4, \quad \nabla e_5 e_2 = e_3, \quad \nabla e_5 e_3 = -e_2, \quad \nabla e_5 e_4 = e_1 \]

and on $N_3$ by

\[ \nabla e_5 e_1 = -e_2, \quad \nabla e_5 e_2 = e_1, \quad \nabla e_5 e_3 = e_4, \quad \nabla e_5 e_4 = -e_3, \]
\[ \nabla e_5 e_1 = -e_3, \quad \nabla e_5 e_2 = -e_4, \quad \nabla e_5 e_3 = -e_1, \quad \nabla e_5 e_4 = e_2, \]
\[ \nabla e_5 e_1 = -e_4, \quad \nabla e_5 e_2 = e_3, \quad \nabla e_5 e_3 = -e_2, \quad \nabla e_5 e_4 = e_1, \]

respectively.

On $N_1$ the torsion 3-form $c$ is given by

\[ e^3 \wedge e^4 \wedge e^5 - e^1 \wedge e^2 \wedge e^5 \]

and it is parallel with respect to the Bismut connection. On $N_2$ the torsion 3-form $c$ is given by

\[ -e^2 \wedge e^4 \wedge e^6 + e^1 \wedge e^3 \wedge e^6 + e^7 \wedge e^2 \wedge e^3 - e^7 \wedge e^1 \wedge e^4 \]

and it is not parallel, since for example $\langle \nabla e_6 c \rangle (e_1, e_2, e_7) \neq 0$. On $N_3$ the torsion 3-form $c$ is given by

\[ e^5 \wedge e^2 \wedge e^1 \wedge e^3 - e^5 \wedge e^6 \wedge e^3 - e^2 \wedge e^4 \wedge e^6 - e^1 \wedge e^3 \wedge e^6 + e^7 \wedge e^2 \wedge e^3 - e^7 \wedge e^1 \wedge e^4 \]

and it is not parallel, since for example $\langle \nabla e_6 c \rangle (e_1, e_3, e_8) \neq 0$.

Concluding remarks

In [13] it is shown that the geometry of the moduli space of a class of black holes in five dimension is HKT and the relation between the number of supersymmetries of a sigma model and the geometry of its target space is examined. Moreover it is
found that any weak HKT manifold solves all the conditions required by $N = 4B$
one dimensional supersymmetry. The understanding of HKT geometries requires
the investigation of various examples. In this note we present a class of invariant hypercomplex structures on nilpotent Lie groups which give rise always to weak HKT structures. Moreover, they are Obata flat when restricted to the 2-step nilpotent case ([8]) but not in general (see Example 3.3 in [11]). In the 8-dimensional case there are only 3 possible groups and in dimension $4k, k > 2$ there are continuous families of weak HKT structures (see Section 4). The HKT metrics in dimension 8 and their properties are well understood (see 5.1). We note that they have directions of positive Ricci curvature and directions of negative Ricci curvature [24] and their geodesics can be given explicitly [24]. It would be of interest to un-
derstand the geodesic behaviour on compact quotients. In [23] J. Michelson and A. Strominger proved that any weak HKT structure which is quaternionic integrable (equivalently Obata flat) can be constructed from a potential $L$ and ask whether generically, one can do without the integrability condition. In particular all HKT structures on the 2-step case considered in this note are associated to a potential. It would be of interest to see if the 12-dimensional example given in [10] having a weak non integrable HKT structure can be constructed from a potential.

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