Cryptanalysis and improvement of Wu-Cai-Wu-Zhang’s quantum private comparison protocol

Guang Ping He
School of Physics, Sun Yat-sen University, Guangzhou 510275, China

In a recent paper (Int. J. Quantum Inf. 17 (2019) 1950026), the authors discussed the shortcomings in the security of a quantum private comparison protocol that we previously proposed (Int. J. Quantum Inf. 15 (2017) 1750014). They also proposed a new protocol aimed to avoid these problems. Here we analyze the information leaked in their protocol, and find that it is even less secure than our protocol in certain cases. We further propose an improved version which has the following advantages: (1) no entanglement needed, (2) quantum memory is no longer required, and (3) less information leaked. Therefore, better security and great feasibility are both achieved.

keywords: Quantum cryptography; Quantum private comparison; Bell states.

I. INTRODUCTION

Private comparison [1] is a two-party cryptographic problem where Alice has a private data a and Bob has a private data b. They want to determine whether a and b are equal, without revealing any extra information on their values other than what can be inferred from the comparison result.

It is well-known that unconditionally secure quantum two-party secure computations are impossible [2–7]. Therefore, most existing quantum private comparison (QPC) protocols added a third party to accomplish the task (see Ref. [8] and the references therein). In 2016, we proposed a QPC protocol [9] which involves two parties only. Although it is not unconditionally secure, the loose upper bound of the average amount of information leaked is 14 bits only. It is also very feasible because quantum memory and entanglement are not required. Later, we further proposed the device-independent version of the protocol [10].

Recently, Wu, Cai, Wu and Zhang [11] reported that they found our protocol in Ref. [9] contains two problems: (1) it is insecure against external eavesdropping, and (2) it is not suitable for comparing short strings. They also proposed a new one (called as the WCWZ protocol thereafter) aiming to fix these problems. We feel that the criticism is improper. But Ref. [11] was submitted as a Comment paper, so we did not aware of it until it was published, and we did not have a chance to make a formal reply. Also, Ref. [11] has not provided a rigorous calculation on the amount of information leaked in their own protocol as we did in Ref. [9], making it hard to judge whether their protocol is more suitable for comparing short strings.

In the current paper, we will reply to the criticism, and study the amount of information leaked in the WCWZ protocol so that the performance of the protocols can be compared clearly. Moreover, we will also propose an improved protocol, which not only has all the advantages of the WCWZ protocol (e.g., secure against external eavesdropping, and low amount of information leaked without the need of a third party), but also requires much less quantum resources (e.g., quantum memory and entanglement) so that it becomes much more feasible.

The paper is organized as follows. In the next section, we will review our previous protocol in Ref. [9]. Then in section III, we will present the criticism made by Ref. [11] and our reply. The WCWZ protocol [11] will be reviewed in section IV, and we will analyze its amount of information leaked and related problems in section V. In section VI, we will propose our improved protocol. Then we will prove its security in section VII and compare it with the protocols in Refs. [11] and [9] in section VIII.

II. OUR PREVIOUS PROTOCOL

Let $H(x)$ be a classical hash function which is a 1-to-1 mapping between the n-bit strings $x$ and $y = H(x)$ (i.e., $H : \{0, 1\}^n \rightarrow \{0, 1\}^n$). Denote the two orthogonal states of a qubit as $|0\rangle$ and $|1\rangle$ respectively, and define $|0\rangle_1 \equiv (|0\rangle_0 + |1\rangle_0)/\sqrt{2}$, $|1\rangle_1 \equiv (|0\rangle_0 - |1\rangle_0)/\sqrt{2}$. That is, the subscript $\sigma = 0, 1$ in $|\gamma\rangle_\sigma$ stands for two incompatible measurement bases, while $\gamma = 0, 1$ distinguishes the two states in the same basis. The two-party QPC protocol that we proposed in Ref. [9] is as follows.

Our previous QPC Protocol (for comparing Alice’s n-bit string $a = a_1a_2...a_n$ and Bob’s n-bit string $b = b_1b_2...b_n$):

(1) Using the hash function $H(x)$, Alice calculates the n-bit string $h^A \equiv h^A_1h^A_2...h^A_n = H(a)$, and Bob calculates the n-bit string $h^B \equiv h^B_1h^B_2...h^B_n = H(b)$.

(2) From $i = 1$ to $n$, Alice and Bob compare $h^A_i$ and $h^B_i$ bit-by-bit as follows.

If $i$ is odd, then:

(2.1A) Alice randomly picks a bit $\gamma^A_i \in \{0, 1\}$ and sends Bob a qubit in the state $|\gamma^A_i\rangle_{h^A_i}$.

(2.2A) Bob measures it in the $h^B_i$ basis and
obtains the result $\gamma_i^B$ he announces $\gamma_i^B$ while keeping $h_i^B$ secret.

(2.3A) Alice announces $\gamma_i^A$.

If $i$ is even, then:

(2.1B) Bob randomly picks a bit $\gamma_i^B \in \{0, 1\}$ and sends Alice a qubit in the state $|\gamma_i^B\rangle_{h_B}$.

(2.2B) Alice measures it in the $h_A$ basis and obtains the result $|\gamma_i^A\rangle_{h_A}$. She announces $\gamma_i^A$ while keeping $h_A^B$ secret.

(2.3B) Bob announces $\gamma_i^B$.

(2.4) If $\gamma_i^A \neq \gamma_i^B$, then they conclude that $a \neq b$, and abort the protocol immediately without comparing the rest bits of $h_A$ and $h_B$. Otherwise they continue with the next $i$.

(3) If Alice and Bob find $\gamma_i^A = \gamma_i^B$ for all $i = 1, ..., n$ then they conclude that $a = b$.

As shown by Eqs. (7) and (8) of Ref. [2], the loose upper bound of the average amount of mutual information leaked in this protocol is

$$I_A = \sum_{k=1}^{[n/2]} 2k \cos^2(k-1)(\pi/8) \sin^2(\pi/8)$$ (1)

for dishonest Alice, and

$$I_B = \sum_{k=1}^{[(n+1)/2]} (2k-1) \cos^2(k-1)(\pi/8) \sin^2(\pi/8)$$ (2)

for dishonest Bob, where $[x]$ denotes the integer part of $x$.

The purpose of using the hash function in the protocol is to change the direct information leaked into mutual information. For example, suppose that $n = 3$ and Bob’s secret string is $b = 011$. Consider that he compares $b$ with Alice’s secret string $a$ bit-by-bit directly in the above protocol without using the hash function. If the protocol aborts at the $i = 1$ round, then Bob knows immediately that the first bit of $a$ must be different from that of $b$, i.e., $a_1 = b_1 = 1$. Thus, Bob knows that the possible choices of the value of $a$ must be limited to the set $\{100, 101, 110, 111\}$. Note that before running the protocol, from Bob’s view, all the $2^3 = 8$ possible values of an arbitrary 3-bit string may be taken by $a$. Therefore, the amount of information leaked to Bob in the protocol is $\log_2 8 = \log_2 4 = 1$ bit. On the contrary, consider that they use the hash function $H(x)$ shown in Fig.1 and run the above protocol faithfully, i.e., they compare $h_A^A = H(a)$ and $h_B^B = H(b)$ instead of $a$ and $b$ directly. If the protocol also aborts at the $i = 1$ round, since $H(b) = H(011) = 010$, Bob knows that there must be $h_A^A = h_B^B = 1$, so that $H(a)$ must be limited to $\{100, 101, 110, 111\}$. Then from Fig.1 he knows that $a$ must be limited to $\{000, 110, 101, 010\}$. Again, the amount of information leaked is also 1 bit. But in this case, we can see that Bob can no longer be sure whether the first bit of $a$ is 0 or 1. That is, when using the hash function, while the amount of information leaked remains the same, the type of this information is changed from the direct information of the secret string of the other party into the mutual information.

This could be useful in practical applications. For instance, suppose that two companies want to compare whether their bids $a$ and $b$ to a project is equal or not, while they do not want to reveal whose bid is higher. Without using the hash function, after running the protocol the first bit will be leaked to the other party, so that they both know who had placed a higher bid. But when the hash function is used, generally they merely learn that their bids are different, without revealing which is the higher one.

III. WCWZ’S CRITICISM AND OUR REPLY

In Ref. [11], Wu, Cai, Wu and Zhang claimed that our above protocol has two security loopholes. Namely, in their own words, (1) “outside party (or called Eve) can obtain Alice’s and Bob’s partial private information before aborting the protocol without being detected”, and (2) “He’s protocol cannot ensure fairness perfectly... this cheating strategy leads to the average amount of mutual information leaked as 13 bits at most in Ref. [2]”. So He’s original protocol is not suitable for a smaller bit-length comparison protocol”.

We do not agree with these criticism. First, in literature two-party cryptographic protocols (including QPC) are required to be secure against internal cheatings only, while the security against external attacks are not mandatory. Thus it is improper to call it a security loophole if the attack does not come from internal legitimate participants, i.e., Alice and Bob. Second, the 13 bits of mutual information leaked is merely the loose upper bound for very long strings. When comparing strings with a smaller bit-length, the amount of mutual information leaked will drop significantly. Now let us elaborate these two points in details.

![FIG. 1: An example of the hash function $H(x)$ used in the protocols.](image-url)
(1) As Kilian pointed out [12], “the reason two-party protocol problems are so difficult is due to a simple symmetry condition on what players know about each other’s data”. Therefore, most previous studies on two-party cryptography merely interested in the security against internal cheating from legitimate participants (i.e., Alice and Bob). The security against the attack from external eavesdroppers is not considered as an obligated task of two-party secure computation protocols. For example, in various no-go proofs of unconditionally secure quantum bit commitment [13, 29] and cheat sensitive bit commitment [30, 32], coin flipping [17, 33, 35], quantum seals [40, 41], oblivious transfer [42, 43] and two-party secure computations [2–7], the security of the protocols is always discussed with internal attacks only. Especially, there were proposals on relativistic bit commitment [44–52], spacetime-constrained oblivious transfer [53, 55] and quantum token [56] which are all accepted as unconditionally secure, but the security proofs against external attacks are not presented either.

That is, in literature two-party secure computation protocols were regarded as secure as long as it can defeat internal attacks. It is not considered as a “loophole” even if the external party Eve can cheat in these protocols. This is also the case of our previous QPC protocol in Ref. [9].

(2) In section 3.2 of the WCWZ paper [11], the authors claimed that “this subsection points that He’s protocol cannot ensure fairness perfectly. The result of comparison can be manipulated partially by either party”. But as we were fully aware that unconditionally secure quantum two-party secure computations are considered impossible [2–7] we already presented clearly in the abstract of Ref. [9] that our purpose was to “study how far we can go with two parties only”, and in our protocol “the average amount of information leaked cannot be made arbitrarily small”. We were surprised that the WCWZ paper made it sound like as if they found a new loophole that we missed to mention.

The second paragraph of section 3.2 of Ref. [11] went on to stated that “the cheating strategy is simple... this cheating strategy leads to the average amount of mutual information leaked as 13 bits at most in Ref. [9]. So He’s original protocol is not suitable for a smaller bit-length comparison protocol”. However, their cheating strategy is exactly what we already described in the security analysis in section 4 of Ref. [9]. We also clearly presented that the average amount of mutual information leaked is below 13 bits for dishonest Bob (14 bits for dishonest Alice), which was obtained from Eqs. (7) and (8) of Ref. [9] (i.e., Eqs. (11) and (2) of the current paper). Thus this is not a new result either. But as we also elaborated in section 4 of Ref. [9], this amount of mutual information leaked is merely the loose upperbound so that it can be “sufficiently general to cover any kind of cheating strategies potentially existed”. Any currently known cheating strategy, including the one that section 3.2 of Ref. [11] cited from us, cannot actually saturate this bound. In fact, as we shown in the last paragraph of page 6 of Ref. [9], when using the cheating strategy that Ref. [11] cited, the cheater can optimally discriminate $\gamma_i$ (in Ref. [11] it was written as $x_i$) but has absolutely zero knowledge on the value of the hash bit $h_i$ of the other party in half of the rounds. Therefore, in Eqs. (11) and (2) the first terms “$2k$” and “$2k - 1$” will be replaced by “$k$” and “$k - 1$”, respectively. Then the bound for this specific cheating strategy will drop to 7 bits for dishonest Alice and 6 bits for dishonest Bob. More importantly, the general bound “13 bit” is for dishonest Bob in the limit $n \to \infty$ only, where $n$ denotes the length of the bit-strings being compared. When $n < 60$ bit, the average amount of mutual information leaked will be significantly reduced, as shown in Fig.1 of Ref. [9] (see also point (3) of section V.C of the current paper).

Meanwhile, Ref. [11] has not provided a rigorous calculation on the amount of information leaked in their own protocol. In the next section, we will review the WCWZ protocol. Then a rigorous calculation on its amount of information leaked and comparison with our protocol will be provided at the end of section V.C and in section VIII.A, where we will see that our protocol is actually more suitable for comparing short strings than their protocol does.

IV. THE WCWZ PROTOCOL

In section 3.3 of Ref. [11], Wu, Cai, Wu and Zhang proposed the following protocol.

The WCWZ Protocol:

Step 1. Using a 1-to-1 classical hash function $H : \{0,1\}^n \to \{0,1\}^n$, Alice computes the $n$-bit string $H(a) = h_1^A...h_n^A$ of secret information $a$, and Bob computes the $n$-bit string $H(b) = h_1^B...h_n^B$ of secret information $b$.

Step 2. Alice divides the value $H(a)$ into $\lceil n/m \rceil$ ($m \geq 2$) groups, which are

$$X_0 = \{h_1^A, ..., h_m^A\}$$
$$X_1 = \{h_{m+1}^A, ..., h_{2m}^A\}$$
$$\vdots$$
$$X_{\lceil n/m \rceil - 1} = \{h_{n-m+1}^A, ..., h_n^A\}. \quad (3)$$

Bob does the same operation as Alice and obtains

$$Y_0 = \{h_1^B, ..., h_m^B\}$$
$$Y_1 = \{h_{m+1}^B, ..., h_{2m}^B\}$$
$$\vdots$$
$$Y_{\lceil n/m \rceil - 1} = \{h_{n-m+1}^B, ..., h_n^B\}. \quad (4)$$

(While we believe that the last terms $h_{n-m}^B$ and $h_{n-1}^B$ in these two equations should be $h_n^A$ and $h_n^B$, respectively, here we present the original form of the protocol in Ref. [11] as is.)
Step 3. Alice (Bob) prepares $m$ Bell states as initial states, every Bell state is randomly chosen from $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Alice (Bob) records these initial states as $S_A$ ($S_B$). The first particles of all Bell states $S_A$ ($S_B$) form the sequence $S_{A_1}$ ($S_{B_1}$), and the rest form the sequence $S_{A_2}$ ($S_{B_2}$).

Step 4. Alice (Bob) prepares decoy states $D_A$ ($D_B$), randomly in states $|0\rangle$, $|1\rangle$, $(|0\rangle + |1\rangle)/\sqrt{2}$, $(|0\rangle - |1\rangle)/\sqrt{2}$. Alice (Bob) randomly inserts $D_A$ ($D_B$) in $S_{A_1}$ ($S_{B_1}$) to form a new sequence $S'_{A_1}$ ($S'_{B_1}$), then sends it to Bob (Alice).

Step 5. After confirming that Bob (Alice) has received the quantum sequence $S'_{A_1}$ ($S'_{B_1}$), Alice (Bob) informs the positions and the measurement bases of $D_A$ ($D_B$) to Bob (Alice). Subsequently, Bob (Alice) extracts the particles in $D_A$ ($D_B$) from $S'_{A_1}$ ($S'_{B_1}$), and gets the sequences $S_{A_1}$ ($S_{B_1}$). Therefore, Alice and Bob can check the existence of an Eve by a predetermined threshold of error rate. If the error rate is limited to the predetermined threshold, there is no Eve and the protocol continues. Otherwise, Alice and Bob abort the protocol and restart from step 1.

Step 6. Bob (Alice), respectively, performs $X = |1\rangle \langle 1| + |0\rangle \langle 0|$ or $I = |0\rangle \langle 0| + |1\rangle \langle 1|$ operation on the $i$th particle of sequence $S_{A_1}$ ($S_{B_1}$) when $h_A^B = 1$ ($h_A^B = 1$) or $h_A^B = 0$ ($h_A^B = 0$), and obtains the sequence $S''_{A_1}$ ($S''_{B_1}$). Then, Bob (Alice) randomly inserts $D''_A$ ($D''_B$) in $S''_{A_1}$ ($S''_{B_1}$) and forms a new sequence $S'''_{A_1}$ ($S'''_{B_1}$).

Step 7. Bob sends sequence $S'''_{A_1}$ to Alice, and Alice and Bob publish the positions and the measurement bases of $D''_A$ and $D''_B$ (we believe that the authors meant the decoy states $D''_A$ and $D''_B$ simultaneously). Alice and Bob recover the sequences $S'''_{A_1}$ and $S'''_{B_1}$ through discarding the decoy states, individually. Alice and Bob check for the existence of an Eve, as described in step 5. If there is no Eve and the protocol continues. Otherwise, Alice and Bob abort the protocol and restart from step 1.

Step 8. Bob (Alice) performs $X = |1\rangle \langle 1| + |0\rangle \langle 0|$ or $I = |0\rangle \langle 0| + |1\rangle \langle 1|$ operation on the $i$th particle of sequence $S''_{A_1}$ ($S''_{B_1}$) when $h_A^B = 1$ ($h_A^B = 1$) or $h_A^B = 0$ ($h_A^B = 0$), and obtains the new state $S'_B$ ($S'_A$). If $S'_B = S_B$ ($S'_A = S_A$), then Bob (Alice) announces that the compared secret information are identical after measurements. Otherwise, Bob (Alice) announces the comparison which are regarded as different.

V. CRYPTANALYSIS OF THE WCWZ PROTOCOL

A. A trivial problem

This protocol contains an obvious problem, probably came from typos. That is, the secret information $a$ and $b$ that Alice and Bob want to compare are $n$-bit strings, but in step 3 they exchange $m$ Bell states only (where $m < n$, as can be seen from step 2), i.e., only the first $m$ bits of the hash values $H(a)$ and $H(b)$ are compared. Consequently, there will be the problem that if $H(a) \neq H(b)$ while the first $m$ bits of $H(a)$ and $H(b)$ happen to be identical, their protocol will mistakenly output $H(a) = H(b)$ in step 8 as the final result.

There is a trivial fix to this problem. In step 8 when Alice and Bob found the $m$ bits being compared are identical, they should repeat steps 3-8 again, exchanging another set of $m$ Bell states to compare the next $m$ bits of $H(a)$ and $H(b)$. They repeat this procedure over and over until there is a run in which step 8 shows that some bits of $H(a)$ and $H(b)$ are different, or until all bits are compared and shown to be identical. With this modification, the protocol will always output the correct result. Actually we believe that this is exactly what the authors had in mind. But without this modification explicitly written, the original protocol in Ref. [11] cannot be regarded as correct.

B. Simultaneity problem

In step 7 of the WCWZ protocol, Alice and Bob are required to “publish the positions and the measurement bases of $D''_A$ and $D''_B$ simultaneously”. But such a requirement is generally considered inappropriate in nonrelativistic cryptography. This is because it is well-accepted that “the standard nonrelativistic cryptographic scenario for two mistrustful parties is as follows ... In particular, neither of them has any way of ensuring that a message sent by the other was sent a certain time before receipt, and so an effectively simultaneous exchange of messages cannot be arranged. A standard cryptographic protocol thus prescribes a sequential exchange of messages between A and B, in which message $i + 1$ is not sent until the sender has received message $i$. Indeed, if the security of a nonrelativistic protocol relies on simultaneity, then it leaves room for potential cheats. For example, consider that the distance between Alice’s and Bob’s sites is $L$, and they are supposed to receive messages from each other at time $t_1$ which is measured in the same reference frame stationary to both of them. If they communicate with methods in which the message carrier travels with the speed of light $c$, then each of them should start sending her/his message at time $t_0 \equiv t_1 - L/c$. But dishonest Bob may set up an “agent” site secretly, which is $L/3$ away from Alice. Then if honest Alice sends her message at time $t_0$, Bob’s agent site will receive it at time $t' \equiv t_0 + (1/3)L/c < t_1$. In this case, Bob has a time interval with the duration $(1/3)L/c$ to analysis the message received from Alice, and decide the content of the message to be sent which could benefit his cheating, and send it at time $t'' \equiv t_0 + (2/3)L/c$. His message will still reach Alice at time $t_1$ so that this cheating cannot be detected. But he manages to delay the sending of his message until he receives Alice’s message so that the simultaneity is broken.”
On the contrary, in relativistic cryptography there is countermeasure against the above cheating, so that simultaneous exchange of messages can be available. Refs. \cite{14, 13, 57} described such an arrangement. Alice and Bob agree on a frame and two locations $x_1$, $x_2$. Honest Alice (Bob) is supposed to erect her (his) laboratory near $x_1$ ($x_2$), within an agreed distance $\delta \ll |x_1 - x_2|$. To verify that Alice is indeed there, in relativistic cryptography Bob is allowed to have an “agent” $B_1$ near $x_1$. At any time $B_1$ can send a test signal to Alice and expect to receive a response within time $2\delta/c$, so that he can confirm to Bob about Alice’s location. Similarly, Alice can test whether Bob is indeed near $x_2$ via her own agent. Given that superluminal signaling is impossible, in this setup once Alice sends a message $m_A$ to Bob and then receives Bob’s message $m_B$ within time $t < 2(|x_1 - x_2| - 2\delta)/c$, she can be sure that Bob had sent $m_B$ before he received $m_A$. The same criterion also holds for Bob. In this case, we can take $m_A$ and $m_B$ as messages being exchanged simultaneously even if their actual sending times may differ slightly, because it makes no difference to the security analysis.

Therefore, if a protocol assumes that there is an approach to force both Alice and Bob to send messages simultaneously, then it is actually a relativistic protocol. In this scenario, it is not surprising that hard cryptographic tasks will become much easier. For example, coin flipping (CF) can be easily realized even without quantum methods, as long as simultaneity becomes available \cite{57} (please see the appendix for a brief review on the protocol). On the other hand, it is a widely-accepted result that nonrelativistic unconditionally secure CF with an arbitrarily small bias is impossible \cite{15, 38}. This is another evidence showing that in literature, simultaneity is not accepted in nonrelativistic cryptography.

For this reason, it is unfair to compare the WCWZ protocol (which relies on simultaneity) with our nonrelativistic protocol in Ref. \cite{9} (where simultaneity is not needed). In section VI, we will propose another protocol in which simultaneity is also assumed to be available, so that it can be compared with the WCWZ protocol in the same scenario.

### C. Information leaked

In Ref. \cite{11} the authors did not give a rigorous evaluation on the amount of information leaked in their protocol. Here we provide such an evaluation.

Since Alice and Bob compare $m$ bits of $H(a)$ and $H(b)$ all at the same time, they will always know $m$ bits of the other’s data no matter the comparison result is identical or not. Therefore, even when nobody cheats, the protocol leaks at least $m$ bits of information to the other party, with or without including the modification in section V.A.

Now consider the case where the above modification is included, i.e., if the first $m$ bits of $H(a)$ and $H(b)$ are found to be identical, Alice and Bob continue to compare the rest bits. Given that the hash function $H(x)$ ($x \in \{a, b\}$) is a random mapping between $x$ and $y = H(x)$, each pair of the hash bits $h_i^a$ and $h_i^b$ stands probability 1/2 to be identical. Consequently, when Alice and Bob compare the first $m$ bits of $H(a)$ and $H(b)$ using the WCWZ protocol, the probability for finding all these $m$ bits to be identical in step 8 (so that the protocol continues) is

$$p_m = \left(\frac{1}{2}\right)^m,$$

and the probability for finding at least one of these $m$ bits to be different in step 8 (so that the protocol aborts) is then

$$p_a = 1 - p_m = 1 - \frac{1}{2^m}.$$

If the protocol indeed aborts, then both Alice and Bob know these $m$ bits of mutual information on the other’s private data. That is, there is probability $p_a$ that the amount of information leaked is $m$ bits.

Else if these $m$ bits are identical (which occurs with probability $p_m$) and Alice and Bob continue to compare the next $m$ bits by repeating steps 3-8 for the 2nd round, then again there will be probability $p_a$ that the next $m$ bits are different so that the protocol aborts after the 2nd round. In this case both parties know the first $2m$ bits. That is, with probability $p_m p_a$ the protocol will abort at the 2nd round, and the amount of information leaked is $2m$ bits.

Continue with this analysis, we can see that the probability for the protocol to abort at the $i$-th round is

$$p_i = (p_m)^i p_a = \frac{1}{2^{m(i-1)}} \left(1 - \frac{1}{2^m}\right),$$

and the amount of information leaked is $im$ bits.

For simplicity, suppose that $n/m$ is an integer. Then summing over all possible $i$ values, we finally yield the average amount of mutual information leaked

$$I = \sum_{i=1}^{\lfloor n/m \rfloor} (im \times p_i) = m \sum_{i=1}^{\lfloor n/m \rfloor} i(2^m - 1) \frac{2^{2m-1}}{2^{2m}}.$$

Note that this value does not included the amount of information leaked when the protocol never aborts in the middle, but continues until all bits are compared and found to be identical instead. This is correct, because when $a$ and $b$ are identical, both Alice and Bob surely know all the $n$ bits, and this is allowed since by definition a QPC protocol is secure as long as it does not reveal any extra information on the compared values “other than what can be inferred from the comparison result”.

Fig.2 shows $I$ as a function of the value of $m$, as calculated from Eq. \cite{58}, with the length of the compared strings $a$ and $b$ is fixed as $n = 360360$ (so that $n/m$ is
always an integer for \( m = 2, ..., 15 \). From the figure we can find the following results:

1. The value of \( I \) grows as \( m \) increases. It indicates that introducing \( m \) in the WCWZ protocol is completely unnecessary, and taking \( m \geq 2 \) in step 2 is not a wise choice. The smaller the value of \( m \) is, the less amount of information will be leaked to Alice and Bob. Thus the optimal choice is \( m = 1 \). That is, Alice and Bob should not divide \( H(a) \) and \( H(b) \) into \([n/m]\) groups in step 2. Instead, they should better compare them bit-by-bit, like we did in our protocol in Ref. 3.

2. If Alice and Bob choose \( m \geq 14 \), then there is \( I \geq 14 \) bits. In this case the WCWZ protocol is always less secure than ours in Ref. 3 for any length of the strings being compared.

3. When comparing very short strings, the WCWZ protocol is also less secure than ours in Ref. 3 even for \( m < 14 \). For example, when \( n = 6 \), Eq. 8 gives that \( I \approx 2.53 \) bits when \( m = 2 \), and \( I \approx 1.88 \) bits when \( m = 1 \). On the contrary, when using our protocol in Ref. 3, Eqs. 10 and 11 (i.e., Eqs. (7) and (8) of Ref. 3) show that the average amount of information leaked for \( n = 6 \) is merely \( I \approx 1.43 \) bits for dishonest Alice and \( I \approx 1.05 \) bits for dishonest Bob, both are smaller than these of the WCWZ protocol.

D. Feasibility problems

The WCWZ protocol is also very costly in terms of quantum resource.

First, it takes a great amount of quantum memory. As can be seen from steps 5 and 7, whenever Alice and Bob receive the sequences \( S'_{A_k} \), \( S''_{B_k} \), \( S''_{A_k} \), and \( S''_{B_k} \), they need to wait for the other party to publish the positions of the decoy states \( D_A, D_B, D'_A \) and \( D''_B \) before they perform measurements on them. Otherwise, they may accidentally measure the Bell states, while they should have performed unitary operations \( X \) or \( I \) on them in steps 6 and 8 instead. Suppose that each of the sequences \( D_A, D_B, D'_A \) and \( D''_B \) contains \( k \) decoy qubits, then the protocol totally needs the quantum memory for storing \( 2(k + 2m) \) qubits (i.e., \( (k + 2m) \) for Alice to store \( S'_{A_k} \) and \( S''_{B_k} \) (or \( S''_{A_k} \)), and the other \( (k + 2m) \) for Bob to store \( S_{B_k} \) and \( S'_{A_k} \) (or \( S''_{B_k} \))).

Second, it requires quantum entanglement. Step 3 shows that for each compared bit, both Alice and Bob need to prepare a pair of Bell state. Thus, to compare two \( n \)-bit strings \( a \) and \( b \), the protocol totally needs \( 2n \) pairs of Bell states.

To this day, the technology for handling entangled states is still far from perfect. Long-term storage for quantum states is even more challenging. Thus, the above requirements make the WCWZ protocol very infeasible.

VI. OUR IMPROVED PROTOCOL

From the above analysis, we can see that the WCWZ protocol can be improved in many ways. Here we propose the following one.

Our Improved Protocol:

Step i. Alice and Bob perform a quantum key distribution (QKD) protocol (e.g., the BB84 protocol [58]) to share two classical random key strings \( k^A = k_{1A} ... k_{nA} \) and \( k^B = k_{1B} ... k_{nB} \). That is, at the end of this process, they both know \( k^A \) and \( k^B \) while the QKD protocol can keep \( k^A \) and \( k^B \) secret from any external eavesdropping.

Step ii. Using a 1-to-1 classical hash function \( H : \{0, 1\}^n \rightarrow \{0, 1\}^n \), Alice computes the \( n \)-bit hash value \( H(a) = h_1^A ... h_n^A \) of her secret information \( a \), and Bob computes the \( n \)-bit hash value \( H(b) = h_1^B ... h_n^B \) of his secret information \( b \). Then they compare \( H(a) \) and \( H(b) \) bit-by-bit, i.e., for each single pair of \( h_i^A \) and \( h_i^B \) (\( i = 1, ..., n \)):

- Step iii. Alice announces \( c_i^A = h_i^A \oplus k_i^A \) and Bob announces \( c_i^B = h_i^B \oplus k_i^B \) simultaneously.

- Step iv. Alice calculates \( h_i^B = c_i^B \oplus k_i^B \) and Bob calculates \( h_i^A = c_i^A \oplus k_i^A \).

- Step v. Now both Alice and Bob know \( h_i^A \) and \( h_i^B \).

If they find \( h_i^A = h_i^B \), they repeat steps iii-iv to compare the next pair of \( h_i^A \) and \( h_i^B \). If all pairs were compared and found to be identical, they both know that \( a = b \).

Else if they find \( h_i^A \neq h_i^B \), they both know that \( a \neq b \) (but do not announce this comparison result publicly). Unlike previous protocols, they do not abort the protocol at this stage. Instead, they replace the rest bits of \( H(a) \) and \( H(b) \) that have not been compared (i.e., \( h_{i+1}^A, h_{i+2}^A, ..., h_{n+1}^A \) and \( h_{i+1}^B, h_{i+2}^B, ..., h_{n+1}^B \)) with random meaningless bits irrelevant with \( a \) or \( b \), \( H(a) \) and \( H(b) \), and repeat step iii to compare these bits until step iii was totally repeated for \( n \) times. The comparison result of these meaningless bits is not important. The purpose of doing so is merely to puzzle potential eavesdropper Eve, so
that she cannot learn whether \( a = b \) or \( a \neq b \) by observing whether the protocol aborts in the middle before all the \( n \) bits are compared.

Note that in step iii we assumed that Alice and Bob can announce information simultaneously, just like they did in step 7 of the WCWZ protocol. If this simultaneity is not available, the protocol needs a minor modification. That is, similar to our protocol in Ref. [9], when \( i \) is odd (even), let Alice announce the information after (before) Bob does. With this modification, the amount of information leaked to internal cheaters will become a little higher, because in the odd (even) rounds, dishonest Alice (Bob) may alter her (his) announcement basing on what the other party already announced, thus increase the probability for finding \( h_i^A = h_i^B \) in this round, like it was elaborated in section 4 of Ref. [9].

As the WCWZ protocol made use of the existence of simultaneity, for a fair comparison, in the following security analysis we also take simultaneity as available, i.e., we only study our protocol with the original form of its step iii without including the above modification.

VII. SECURITY OF OUR IMPROVED PROTOCOL

A. External eavesdropping

It is easy to prove that an external eavesdropper Eve cannot learn the hash values \( h_i^A \) and \( h_i^B \), so that she cannot know Alice’s and Bob’s secret information \( a \) and \( b \). This is because \( c_i^A \) and \( c_i^B \) were publicly announced in step iii so that Eve surely knows them. If she can have a strategy to learn either \( h_i^A \) or \( h_i^B \), then she can learn the secret key \( k_i^A \) or \( k_i^B \) by calculating \( k_i^A = c_i^A \oplus h_i^A \) or \( k_i^B = c_i^B \oplus h_i^B \). However, \( k_i^A \) and \( k_i^B \) were generated in step i by the QKD protocol. It is well known that QKD protocols can be unconditionally secure, as proven in Ref. [69]. Therefore, the existence of any strategy for Eve to learn \( h_i^A \) or \( h_i^B \) will conflict with the security proof of QKD. Thus we know that our protocol is unconditionally secure against external eavesdropping.

Also, Eve cannot spoil the protocol (i.e., mislead Alice and Bob to a wrong comparison result). This is also because the QKD protocol ensures that the key \( k_i^A \) and \( k_i^B \) cannot be altered. Meanwhile, \( c_i^A \) and \( c_i^B \) were publicly announced classically, so that Eve cannot change them either. Consequently, Alice’s and Bob’s calculation results of \( h_i^A \) and \( h_i^B \) in step iv will always be correct, leaving no chance for external eavesdroppers to turn them into the opposite values.

B. Internal attack

Obviously, when the protocol outputs \( h_i^A \neq h_i^B \) in the \( i \)-th round, both Alice and Bob surely know that the first \( i - 1 \) bits of \( h_i^A \) and \( h_i^B \) are identical, while the \( i \)-th bits are different. There is no secret in these bits. Therefore, the goal of a dishonest internal party is to try to make the case \( h_i^A \neq h_i^B \) occur as late as possible, so that he can learn more bits of data of the other party. Now we show that this is impossible.

When both parties are honest and the hash function \( H(x) \) is a random mapping between \( x \) and \( y = H(x) \), the average probability for \( h_i^A = h_i^B \) (so that the protocol does not abort) is \( 1/2 \). To show that a dishonest party cannot cheat, we need to show that this probability cannot be increased.

As the protocol is symmetrical, without loss of generality, let us assume that Alice is dishonest. Before step iii of the protocol, Bob has not announced \( c_i^B \), so that Alice does not know the value of \( h_i^B \). Therefore, by the time that Alice needs to announce \( c_i^A \) in step iii, she does not know which value can stand a higher probability to make the calculation of the other party in step iv result in \( h_i^A = h_i^B \). Also, in step iii we assumed that Alice and Bob can announce information simultaneously, like the WCWZ protocol did. Then by the time that Alice knows Bob’s \( h_i^B \) from his announced \( c_i^B \), she has also announced \( c_i^A \) to Bob so that she cannot change it anymore.

Consequently, no matter she announces \( c_i^A \) honestly or not, the probability for \( h_i^A = h_i^B \) will still be \( 1/2 \) for any \( i \). For the same reason, dishonest Bob cannot increase the probability for \( h_i^A = h_i^B \) either. Then repeating the reasoning in section V.C, we find that the average amount of mutual information leaked in our protocol is also described by Eq. (5), where \( m = 1 \) since \( h_i^A \) and \( h_i^B \) are compared bit-by-bit in our protocol. The relationship between \( I \) (the average amount of information leaked) and \( n \) (the length of the strings being compared) is shown as the green line in Fig. 3.

As we mentioned in the previous section, when si-
multaneity is not available, the protocol needs modification. Then the amount of information leaked will become higher because dishonest party may increase the probability for finding $h^A_i = h^B_i$ in half of the rounds. But since step 7 of the WCWZ protocol also makes use of the existence of simultaneity, for a fair comparison between the protocols, here we assume that simultaneity is available for our improved protocol too, without taking the above mentioned modification into account.

VIII. ADVANTAGES OF OUR IMPROVED PROTOCOL

A. Security

Fig. 3 illustrated the comparison between the three protocols. The green line represents the performance of our above improved protocol, where the average amount of mutual information leaked $I$ is calculated from Eq. (5) by taking $m = 1$. The black solid (dashed) line is corresponding to the $m = 2$ ($m = 13$) case of the WCWZ protocol. The red (blue) line indicates the loose upper bound of the average amount of mutual information leaked to Alice (Bob) in our previous protocol in Ref. [9], which is calculated from Eq. (7) (Eq. (8)) of Ref. [9]. From the comparison we find the following results.

(I) Since the WCWZ protocol suggested to take $m \geq 2$ in its step 2, it is always less secure (i.e., the amount of information leaked is higher) than our improved protocol for any value of the length $n$ of the strings being compared.

(II) Comparing with our previous protocol in Ref. [9], as we mentioned in section V.C, when taking $m \geq 14$, the WCWZ protocol is always less secure than ours in Ref. [9] for any length of the strings being compared. Thus the WCWZ protocol may be valuable only when $2 \leq m \leq 13$, which is covered by the area between the black solid line and the black dashed line in Fig. 3. Even in this range, when $m = 2$ ($m = 13$), we can see that the amount of information leaked in the WCWZ protocol is still higher than that in our previous one in Ref. [9] for $n \leq 10$ ($n \leq 60$). In fact, even our improved protocol in the current paper (where $m = 1$) cannot be less secure than our previous one when $n \leq 8$. Furthermore, as we emphasized in section III, $I_A$ and $I_B$ in Fig. 3 are merely the loose bounds of the protocol in Ref. [9]. Thus the claim in Ref. [11] that “He’s original protocol is not suitable for a smaller bit-length comparison protocol” is obviously wrong. Instead, the WCWZ protocol is even worst for comparing short strings.

(III) The simultaneity problem. Both step 7 of the WCWZ protocol and step iii of our improved protocol require the existence of simultaneity. Thus our previous protocol in Ref. [9] wins again as it does not have this requirement. In case simultaneously publishing informations is impossible, both the WCWZ protocol and our improved one need modification. Then the amount of information leaked in these two protocols could be even higher than what is shown in Fig. 3.

B. Feasibility

This is where our protocols really shine. As we stated in section V.D, the WCWZ protocol not only requires the quantum memory for storing $2(k + 2m)$ qubits, but also $2n$ pairs of Bell states for comparing two $n$-bit strings $a$ and $b$.

On the contrary, in our previous protocol in Ref. [9], Alice and Bob merely need to send qubits prepared in the pure states $|0\rangle, |1\rangle, |+\rangle$ and $|−\rangle$ non-entangled with other systems. Also, no quantum memory is required because once Alice and Bob receive the qubits, they can measure them immediately without the need to wait for the other party to announce further information.

Similarly, our improved protocol has the same low requirement on quantum resource too. This is because only step i is quantum as it involves a QKD process. All other steps are completely classical. As it is known, there exist QKD protocols which can be run without entangled states and quantum memory. The original BB84 protocol Ref. [58] is exactly such an example.

Therefore, our two protocols are both much more feasible than the WCWZ protocol, and can be implemented with currently available technology. Especially, the improved protocol proposed in the current paper can be realized directly using existing QKD systems.

IX. SUMMARY

We analyzed the amount of information leaked in the WCWZ protocol, and found that it is less secure than our previous protocol in Ref. [9] against internal attacks when $m \geq 14$. For comparing short bit-strings with the length $n \leq 10$, the WCWZ protocol is always less secure than ours no matter which $m$ value it chooses, in contrast to the claim in Ref. [11].

We also proposed an improved protocol, which is more secure than the WCWZ protocol for any length of the strings being compared. Moreover, the WCWZ protocol has to rely on the use of quantum memory and entanglement, while these resources are not needed in both of our improved protocol and the previous one.

Acknowledgements

The work was supported in part by Guangdong Basic and Applied Basic Research Foundation under grant No. 2019A1515011048.
Appendix

According to Ref. [57], if simultaneity is available, then perfectly secure coin flipping (CF) a.k.a. coin tossing can be achieved.

CF is aimed to provide a method for two separated parties Alice and Bob to generate a random bit value $c = 0$ or $1$ remotely, while they do not trust each other. A CF protocol is considered secure if neither party can bias the outcome, so that $c = 0$ and $c = 1$ will both occur with the equal probability $1/2$, just as if they are tossing an ideal fair coin.

Assume that there is an approach to ensure both Alice and Bob to send messages simultaneously. The CF protocol in Ref. [57] can be described in plain words as follows.

**CF Protocol:**

I) Alice picks a random bit $a \in \{0, 1\}$ and Bob picks a random bit $b \in \{0, 1\}$.

II) Alice and Bob publish $a$ and $b$ simultaneously.

III) They take $c = a \oplus b$ as the result of the protocol.

The simultaneous publishing of information in step II can be done using the approach described in the second paragraph of section V.B, where both Alice and Bob should have agents to test and confirm that they are located in the prescribed regions. This enables them to deduce the sending time of the other party from the time they receive the other’s published information, so that none of them can delay the publishing of information without being discovered. This step is the key to the security of the protocol. It guarantees that as long as the two parties are separated spatially, by the time when Alice is forced to publish her bit $a$, she has not received Bob’s bit $b$ yet because the latter information has to spend a non-vanishing period of time to travel the distance between Alice and Bob due to the impossibility of superluminal signaling. Consequently, the possible value of $b$ is unknown and looks completely random to Alice at this stage, so that no matter she publishes $a = 0$ or $a = 1$, the final result $c$ can be 0 or 1 with equal probability $1/2$. Bob’s case is exactly the same. Therefore, the bias in this protocol is absolutely zero [57]. Thus we can see that the existence of simultaneity can easily evade the no-go proofs of nonrelativistic unconditionally secure CF with an arbitrarily small bias [15, 53, 54].

[1] M. Jakobsson and M. Yung, Proving without knowing: On oblivious, agnostic and blindfolded provers, in Advances in Cryptology: CRYPTO ’96, Lecture Notes in Computer Science, Vol. 1109 (Springer-Verlag, 1996), p. 186.
[2] H.-K. Lo, *Phys. Rev. A* **56** (1997) 1154.
[3] T. Rudolph, arXiv: quant-ph/0202143.
[4] R. Colbeck, *Phys. Rev. A* **76** (2007) 062308.
[5] L. Salvail, C. Schaffner, and M. Sotakova, On the power of two-party quantum cryptography, in ASIACRYPT 2009, Lecture Notes in Computer Science, Vol. 5912 (Springer-Verlag, 2009), p. 70.
[6] L. Salvail and M. Sotakova, arXiv:0906.1671.
[7] H. Buhrman, M. Christandl, and C. Schaffner, *Phys. Rev. Lett.* **109** (2012) 160501.
[8] G. P. He, *Int. J. Quantum Inf.* **11** (2013) 1350025.
[9] G. P. He, *Int. J. Quantum Inf.* **15** (2017) 1750014.
[10] G. P. He, *Phys. Scr.* **93** (2018) 095001.
[11] W. Q. Wu, Q. Y. Cai, S. M. Wu and H. G. Zhang, *Int. J. Quantum Inf.* **17** (2019) 1950026.
[12] J. Kilian, Founding cryptography on oblivious transfer, in Proc. 1988 ACM Annual Symposium on Theory of Computing (ACM, New York, 1988), p. 20.
[13] D. Mayers, *Phys. Rev. Lett.* **78** (1997) 3414.
[14] H.-K. Lo and H. F. Chau, *Phys. Rev. Lett.* **78** (1997) 3410.
[15] H.-K. Lo and H. F. Chau, *Physica D* **120** (1998) 177.
[16] A. Kent, *Phys. Rev. A* **61** (2000) 042301.
[17] J. Buhrman, *Found. Phys.* **31** (2001) 735.
[18] R. W. Spekkens and T. Rudolph, *Phys. Rev. A* **65** (2001) 012310.
[19] R. W. Spekkens and T. Rudolph, *Quant. Inf. Comput.* **2** (2002) 66.
[20] H. Halvorson, *J. Math. Phys.* **45** (2004) 4920.
[21] A. Kitaev, D. Mayers, and J. Preskill, *Phys. Rev. A* **69** (2004) 052326.
[22] C.-Y. Cheung, *Int. J. Mod. Phys. B* **21** (2007) 4271.
[23] G. M. D’Ariano, D. Kretschmann, D. Schlingemann, and R. F. Werner, *Phys. Rev. A* **76** (2007) 032328.
[24] L. Magnin, F. Magniez, A. Leverrier, and N. J. Cerf, *Phys. Rev. A* **81** (2010) 010302(R).
[25] G. Chiribella, G. M. D’Ariano, and P. Perinotti, *Phys. Rev. A* **81** (2010) 062348.
[26] A. Kent, *Quantum Inf. Process.* **11** (2012) 493.
[27] Q. Li, C.-Q. Li, D.-Y. Long, W. H. Chan, and C.-H. Wu, *Quantum Inf. Process.* **11**, (2012) 519.
[28] G. Chiribella, G. M. D’Ariano, P. Perinotti, D. M. Schlingemann, and R. F. Werner, *Phys. Lett. A* **377** (2013) 1076.
[29] M. Nagy and N. Nagy, *Entropy* **20** (2018) 193.
[30] L. Hardy and A. Kent, *Phys. Rev. Lett.* **92** (2004) 157901.
[31] S. Ishizaka, *Phys. Rev. Lett.* **100** (2008) 070501.
[32] G. P. He, *Sci. Rep.* **5** (2015) 9398.
[33] A. Nayak and P. Shor, *Phys. Rev. A* **67** (2003) 012304.
[34] A. Kitaev, A negative result about quantum coin flipping, in Lecture delivered at the 2003 Annual Quantum Information Processing (QIP) Workshop, Mathematical Sciences Research Institute, Berkeley, CA (unpublished). Available online at http://www.msri.org/publications/ln/msri/2002/qip/kitaev/1/meta/aux/kitaev.pdf.
[35] A. Ambainis, *J. Comput. Syst. Sci.* **68** (2004) 398.
[36] A. Ambainis, H. Buhrman, Y. Dodis, and H. Röhrig, Multiparty quantum coin flipping, in CCC ’04: Proc. 19th Annual IEEE Conference on Computational Complexity (IEEE, 2004), p. 250.
[37] C. Mochon, *Phys. Rev. A* **72** (2005) 022341.

[38] G. Gutoski and J. Watrous, Toward a general theory of quantum games, in *Proc. the Thirty-Ninth Annual ACM Symposium on Theory of Computing (STOC 2007)* (ACM New York, 2007), p. 565.

[39] E. Hänggi and J. Wullschleger, Tight bounds for classical and quantum coin flipping, in *Proc. TCC 2011* (Brown University, 2011), p. 468.

[40] H. Bechmann-Pasquinucci, G. M. D’Ariano, and C. Macchiavello, *Int. J. Quantum Inf.* **3** (2005) 435.

[41] G. P. He, *Phys. Rev. A* **71** (2005) 054304.

[42] S. Winkler and J. Wullschleger, On the efficiency of classical and quantum oblivious transfer reductions, in *Advances in Cryptology: CRYPTO 2010*, T. Rabin (Eds.), Lecture Notes in Computer Science, Vol. 6223 (Springer-Verlag, 2010), p. 707.

[43] A. Chailloux, I. Kerenidis, and J. Sikora, arXiv:1007.1875.

[44] A. Kent, *Phys. Rev. Lett.* **83** (1999) 1447.

[45] A. Kent, *J. Cryptol.* **18** (2005) 313.

[46] A. Kent, *New J. Phys.* **13** (2011) 113015.

[47] A. Kent, *Phys. Rev. Lett.* **109** (2012) 130501.

[48] S. Croke and A. Kent, *Phys. Rev. A* **86** (2012) 052309.

[49] J. Kaniewski, M. Tomamichel, E. Hänggi, and S. Wehner, *IEEE Trans. Inf. Theory* **59** (2013) 4687.

[50] T. Lunghi, et al., *Phys. Rev. Lett.* **111** (2013) 180504.

[51] Y. Liu, et al., *Phys. Rev. Lett.* **112** (2014) 010504.

[52] T. Lunghi, J. Kaniewski, F. Bussières, R. Houlmann, M. Tomamichel, S. Wehner, and H. Zbinden, *Phys. Rev. Lett.* **115** (2015) 030502.

[53] D. Pitalúa-García, *Phys. Rev. A* **93** (2016) 062346.

[54] D. Pitalúa-García and I. Kerenidis, *Phys. Rev. A* **98** (2018) 032327.

[55] D. Pitalúa-García, *Phys. Rev. A* **100** (2019) 012302.

[56] A. Kent and D. Pitalúa-García, *Phys. Rev. A* **101** (2020) 022309.

[57] A. Kent, *Phys. Rev. Lett.* **83** (1999) 5382.

[58] C. H. Bennett and G. Brassard, Quantum cryptography: public key distribution and coin tossing, in *Proc. IEEE International Conference on Computers, Systems, and Signal Processing* (IEEE, New York, 1984), p. 175.

[59] P. W. Shor and J. Preskill, *Phys. Rev. Lett.* **85** (2000) 441.