Charged fermion tunnelling from electrically and magnetically charged rotating black hole in de Sitter space

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Abstract

Thermal radiation of electrically charged fermions from rotating black hole with electric and magnetic charges in de Sitter space is considered. The tunnelling probabilities for outgoing and incoming particles are obtained and the Hawking temperature is calculated. The relation for the classical action for the particles in the black hole’s background is also found.

1 Introduction

Hawking radiation has been attracting a lot of attention since it was proposed [1]. To consider it different methods have been applied [2]. The semi-classical tunnelling approach that was proposed by Kraus and Wilczek [3, 4] has gained considerable interest recently. It was shown that Hawking temperature is defined by the imaginary part of the emitted particle’s action for the classically forbidden region near the horizon. To calculate it two methods were proposed. The first one is so-called null geodesic method proposed by Parikh and Wilczek [5] is based on the fact that imaginary part of the action is caused by the integration of radial momentum $p_r$ for the emitted particles. The second method is based on relativistic Hamilton-Jacobi equation, and the imaginary part of the action can be obtained after integration of that equation [6]. This second approach can be treated as an extension of complex path method proposed by Padmanabhan with collaborators [7]-[9].

Tunnelling approach based on Hamilton-Jacobi equation at first was applied to the emission of scalar particles. Then it was successfully applied to vast area of well-known and exotic space-times, in particular Kerr and Kerr-Newman ones [10, 11], Taub-NUT space-time [12] Gödel space-time [13], BTZ black holes [14], dynamical black holes [15]. The review of tunnelling method is considered in paper [16] where further references can be found.

Tunnelling approach was also successfully applied to tunnelling of fermions. In their seminal work Kerner and Mann used Dirac equation instead of Hamilton-Jacobi one to obtain temperature of emitted fermions and showed that for a chosen type of space-time temperature of emitted fermions would be the same as the temperature of scalar particles.

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That method was later applied to different kinds of black hole space-times including Reissner-Nordström one [18], Kerr-Newman [19, 20], dilatonic black holes [21], BTZ black hole [22], black holes in Hořava-Lifshitz gravity [23, 24], accelerating and rotating black hole [25, 26], rotating black strings [27].

In our work we consider Kerr-Newman-de Sitter black hole which carries both electric and magnetic charges. Using Kerner-Mann procedure we consider emission of charged spin 1/2 particles. We show that in the presence of electric and magnetic charges of the black hole the variables of Dirac equation can also be separated and as a consequence the temperature can be found. We also find the quasiclassical action of emitted particle.

2 Charged spin 1/2 particle tunnelling from Kerr-Newman-de Sitter black hole

Emission of charged particles with spin 1/2 was considered independently in works [19, 20]. We consider emission of charged spin 1/2 from Kerr-Newman-de Sitter black hole which carries both electric and magnetic charges (dyonic black hole). Kerr-Newman-de Sitter black hole’s metrics in Boyer-Lindquist coordinates takes form:

$$ds^2 = -\frac{\Delta}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( \frac{R^2}{\Xi} d\varphi - adt \right)^2,$$  \hspace{1cm} (1)

where $R^2 = r^2 + a^2$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = R^2 (1 - r^2/l^2) + Q_e^2 + Q_m^2 - 2Mr$, $\Xi = 1 + a^2/l^2$, $\Delta_\theta = 1 + a^2/l^2 \cos^2 \theta$. Here $a = J/M$ is the angular momentum parameter and another parameter $l^2 = 3/\Lambda$ is defined by the cosmological constant, parameters $Q_e$ and $Q_m$ are electric and magnetic charges respectively. It is known that black hole’s horizons can be found from equation $\Delta = 0$ and in the case of Kerr-Newman-de Sitter metrics we have four roots. The largest one is the cosmological horizon $r_c$ (CH), the minimal positive root is the Cauchy horizon $r_i$ and the intermediate root is the event horizon $r_+$. The last root is negative and it is not taken into consideration.

Components of electromagnetic potential takes the form:

$$A_t = -\frac{1}{\rho^2} (Q_e r + Q_m a \cos \theta),$$ \hspace{1cm} (2)

$$A_\varphi = \frac{1}{\Xi \rho^2} (Q_e r a \sin^2 \theta + Q_m R^2 \cos \theta)$$ \hspace{1cm} (3)

The Dirac equation for electrically charged particle takes form:

$$i\gamma^\mu \left( D_\mu - \frac{iq}{\hbar} A_\mu \right) \psi + \frac{m}{\hbar} \psi = 0$$ \hspace{1cm} (4)

where $D_\mu = \partial_\mu + \Omega_\mu$, $\Omega_\mu = \frac{1}{8} \Gamma^\alpha_{\mu\nu} [\gamma^\beta, \gamma^\alpha]$ and $\gamma^\mu$ matrices are obeyed to commutation relation:

$$[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu} \mathbf{1}$$ \hspace{1cm} (5)

The representation for the $\gamma^\mu$ can be taken as follows:

$$\gamma^t = \sqrt{\frac{K(r, \theta)}{\rho \sqrt{\Delta \Delta_\theta}}} \gamma^0, \quad \gamma^r = \frac{\sqrt{\Delta}}{\rho} \gamma^3, \quad \gamma^\theta = \frac{\sqrt{\Delta_\theta}}{\rho} \gamma^1,$$ \hspace{1cm} (6)

$$\gamma^\varphi = \frac{\Xi}{\sqrt{K(r, \theta)}} \left( \frac{\rho}{\sin^2 \theta} \gamma^2 - \frac{a(\Delta - R^2 \Delta_\theta)}{\rho \sqrt{\Delta \Delta_\theta}} \gamma^0 \right)$$ \hspace{1cm} (7)
where we denoted $K(r, \theta) = R^4 \Delta_\theta - \Delta a^2 \sin^2 \theta$ and matrices $\gamma^a$ take following form:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}
\end{align*}
\]  

(8)

and $\sigma_i$ are the Pauli matrices. We note that representation similar to (6) was used in the work [20].

We choose the wave functions with spin up and down in the form:

\[
\psi_\uparrow = \begin{pmatrix} A(t, r, \theta, \varphi) \\ 0 \\ B(t, r, \theta, \varphi) \end{pmatrix}, \quad \psi_\downarrow = \begin{pmatrix} 0 \\ C(t, r, \theta, \varphi) \\ 0 \end{pmatrix}
\]

(10)

where $A$ and $B$ are the Pauli matrices. We note that representation similar to (6) was used in the work [20].

Then we substitute (10) into Dirac equation (4) and performing quasiclassical approximation we obtain

\[
B \left[ -\frac{\sqrt{K(r, \theta)}}{\rho \sqrt{\Delta_\theta}} \partial_t I_\uparrow - \frac{\sqrt{\Delta_\theta}}{\rho \sqrt{\Delta_\theta}} \partial_\rho I_\uparrow + \frac{\Xi a (\Delta - R^2 \Delta_\theta)}{\rho \sqrt{\Delta_\theta} K(r, \theta)} \partial_\varphi I_\uparrow - \frac{q \sqrt{K(r, \theta)}}{\rho^3 \sqrt{\Delta_\theta}} (Q_e r + Q_m a \cos \theta) \right] + mA = 0;
\]

(11)

\[
B \left[ -\frac{\sqrt{\Delta_\theta}}{\rho \sqrt{K(r, \theta) \sin \theta}} \partial_\rho I_\uparrow - \frac{i \Xi a}{\rho \sqrt{K(r, \theta) \sin \theta}} \partial_\varphi I_\uparrow + \frac{q \sqrt{K(r, \theta)}}{\rho^3 \sqrt{\Delta_\theta}} (Q_e r + Q_m a \cos \theta) \right] = 0;
\]

(12)

\[
\begin{align*}
A \left[ \frac{\sqrt{K(r, \theta)}}{\rho \sqrt{\Delta_\theta}} \partial_t I_\uparrow - \frac{\sqrt{\Delta_\theta}}{\rho \sqrt{\Delta_\theta}} \partial_r I_\uparrow - \frac{\Xi a (\Delta - R^2 \Delta_\theta)}{\rho \sqrt{\Delta_\theta} K(r, \theta)} \partial_\varphi I_\uparrow + \frac{q \sqrt{K(r, \theta)}}{\rho^3 \sqrt{\Delta_\theta}} (Q_e r + Q_m a \cos \theta) \right] + mB = 0;
\end{align*}
\]

(13)

\[
B \left[ -\frac{\sqrt{\Delta_\theta}}{\rho \sqrt{K(r, \theta) \sin \theta}} \partial_\rho I_\uparrow - \frac{i \Xi a}{\rho \sqrt{K(r, \theta) \sin \theta}} \partial_\varphi I_\uparrow + \frac{q \sqrt{K(r, \theta)}}{\rho^3 \sqrt{\Delta_\theta}} (Q_e r + Q_m a \cos \theta) \right] = 0.
\]

(14)

Note, that in the first order WKB approximation the terms proportional to $\Omega_\mu$ are omitted.

Suppose that the action $I_\uparrow$ takes the form:

\[
I_\uparrow = -Et + J_\varphi + W(r, \theta)
\]

(15)

Now we substitute (15) into (11)-(14). In order to make these equations simpler they are decomposed into the series near the horizon surface [19, 20] (here the decomposition in the vicinity of the event horizon is represented):

\[
B \left[ \frac{R^2 E - \Xi a J - qQ_e r_+}{\rho_+ \sqrt{\Delta_\theta(r_+)}(r - r_+)} \right] + mA = 0;
\]

(16)
Here $\Delta_r(r_+) = 2 \left( r_+ \left( 1 - a^2/l^2 \right) - 2r_+^3/l^2 - M \right), R_+^2 = r_+^2 + a^2$ and $\rho_+^2 = r_+^2 + a^2 \cos^2 \theta$.

For the massless case equations (16) and (18) decouple and can be solved. It is easy to see that variables $r$ and $\theta$ can be separated. So the function $W(r, \theta)$ is represented in form:

$$W(r, \theta) = W(r) + \Theta(\theta)$$

When $A = 0$ equation (16) leads to:

$$W_r(r, \theta) = \frac{R_+^2 E - \Xi a J - qQe r_+}{\Delta_r(r_+)(r - r_+)}$$

Having integrated around the pole and taking the imaginary part of the action we obtain:

$$\operatorname{Im} W_+ = \frac{\pi R_+^2 \left( E - \Xi \Omega_+ J - qQe r_+ \right)}{2 \Delta_r(r_+)}$$

where $\Omega_+ = a/R_+^2$ is the angular velocity at the horizon.

Similarly when $B = 0$ we write:

$$W_r(r, \theta) = -\frac{R_+^2 E - \Xi a J - qQe r_+}{\Delta_r(r_+)(r - r_+)}$$

And after integration the result is as follows:

$$\operatorname{Im} W_- = -\frac{\pi R_+^2 \left( E - \Xi \Omega_+ J - qQe r_+ \right)}{2 \Delta_r(r_+)}$$

As it was argued in [7, 8, 9, 20] that probabilities of crossing the horizon are defined by the imaginary part of the action:

$$P_{\text{out}} \propto \exp[-2(\operatorname{Im} W_+ + \operatorname{Im} \Theta)], \quad P_{\text{in}} \propto \exp[-2(\operatorname{Im} W_- + \operatorname{Im} \Theta)]$$

The resulting tunnelling probability is represented as the ratio of probabilities (25):

$$\Gamma \propto \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\exp[-2\operatorname{Im} W_+]}{\exp[-2\operatorname{Im} W_-]} = \exp[-4\operatorname{Im} W_+]$$

Substituting relation (22) into (26) we obtain:

$$\Gamma = \exp \left( -4\pi \frac{R_+^2 \left( E - \Xi \Omega_+ J - qQe r_+ \right)}{2 \Delta_r(r_+)} \right)$$
As a result the temperature takes form:

\[ T = \frac{\Delta_r(r_+)}{4\pi R_+^2} = \frac{(r_+ \left(1 - a^2/l^2\right) - 2r_+^3/l^2 - M)}{2\pi(r_+^2 + a^2)}. \]  

(28)

In the massive case \((m \neq 0)\) one has to solve the system of equations (16) and (18). So we obtain:

\[ 2AB \frac{\rho_+^2}{\Delta_r(r_+)(r - r_+)} \left( \frac{R_+^2 E - \Xi a J - qQ_e r_+}{\sqrt{\Delta_r(r_+)(r - r_+)}} \right) + m(A^2 - B^2) = 0 \]

(29) and as a result:

\[ \frac{A}{B} = \frac{-(R_+^2 E - \Xi a J - qQ_e r_+) \pm \sqrt{(R_+^2 E - \Xi a J - qQ_e r_+)^2 + m^2\rho_+^2 \Delta_r(r_+)(r - r_+)}}{m\rho_+ \sqrt{\Delta_r(r_+)(r - r_+)}} \]

(30)

When \(r \to r_+\) ratio \(A/B\) can tend to 0 or \(-\infty\) (see also [20]). When \(A/B \to 0\) then \(A \to 0\) the equation (18) is solved in terms of \(m\) and the result is inserted into (16). It is easy to see that resulting expression does not depend on the variable \(\theta\). So we obtain:

\[ W_r(r, \theta) = \frac{R_+^2 \left(E - \Xi \Omega_+ J - qQ_e \frac{r_+}{R_+^2}\right)}{\Delta_r(r_+)(r - r_+)} \frac{1 + \frac{A^2}{B^2}}{1 - \frac{B^2}{A^2}} \]

(31)

The result that is found after integration around the pole is the same as in the massless case since \(A/B \to 0\) at the horizon. When \(B \to 0\) the relation (31) can be rewritten as follows:

\[ W_r(r, \theta) = \frac{R_+^2 \left(E - \Xi \Omega_+ J - qQ_e \frac{r_+}{R_+^2}\right)}{\Delta_r(r_+)(r - r_+)} \frac{1 + \frac{A^2}{B^2}}{1 - \frac{B^2}{A^2}} \]

(32)

Similarly the result after integration gets no correction in addition to the massless case. So as a consequence the temperature for emitted massive fermions is the same as for the massless and given by the relation (28).

It is known that Hawking radiation in de Sitter space can also appear at the cosmological horizon. In contrast to the emission of particles at the event horizon Hawking radiation at the cosmological horizon is caused by incoming particles whereas outgoing particles move along classically permitted trajectories. In spite of that qualitative difference calculations of the Hawking temperature at the cosmological horizon can be made in the same way as at the event horizon. To obtain the expression for the Hawking temperature at the cosmological horizon one should replace the event horizon radius in formula (28) by the cosmological one \((r_+ \to r_c)\). So we have:

\[ T = \frac{\Delta_r(r_c)}{4\pi R_c^2} = \frac{(r_c \left(1 - a^2/l^2\right) - 2r_c^3/l^2 - M)}{2\pi(r_c^2 + a^2)}. \]

(33)

Relations for temperature at the event horizon (28) as well as at the cosmological one take the same form as in case of the hole with only the electric charge [19]. It should be noted that radii of horizons depend on the both electric and magnetic charges so our formulas are consistent with relations given in [19] when \(Q_m \to 0\).

## 3 Action for the emitted particles

Equations (16)-(19) allow one to obtain explicit expression for action of emitted particles. As it was already shown angle and radial variables are separated near the horizon. So the
radial and angular parts of the action can be obtained independently. Having used the relation (20) the equation (16) can be rewritten in the form:

\[ W'(r) = \frac{R_+^2 E - \Xi a J - q Q e r_+}{\Delta r(r_+)(r - r_+)} + \frac{A m \rho_+}{B \sqrt{\Delta r(r_+)(r - r_+)}} \]  

(34)

After integration we obtain:

\[ W_+(r) = \frac{R_+^2 E - \Xi a J - q Q e r_+}{\Delta r(r_+)} \ln(r - r_+) + \int \frac{A m \rho_+}{B \sqrt{\Delta r(r_+)(r - r_+)}} dr. \]  

(35)

In order to find relation for radial part of the action for incoming particles one should integrate equation (18). After integration we arrive at the following relation:

\[ W_-(r) = -\frac{R_+^2 E - \Xi a J - q Q e r_+}{\Delta r(r_+)} \ln(r - r_+) + \int \frac{B m \rho_+}{A \sqrt{\Delta r(r_+)(r - r_+)}} dr. \]  

(36)

For the angular part relation (17) can be represented in the form:

\[ \Theta'(\theta) = -\frac{i \Xi \rho_2^2 J}{R_+^2 \sin \theta \Delta \theta} + \frac{iq(Q e r_+ a \sin^2 \theta + Q_m R_+^2 \cos \theta)}{\Xi R_+^2 \Delta \theta} \]  

(37)

Having integrated the last equation we obtain:

\[ \Theta(\theta) = \frac{iq Q m l^2}{\Xi a \sqrt{l^2 - a^2}} \arctan \left[ \frac{l \sin \theta}{\sqrt{l^2 - a^2}} \right] + \frac{i \Xi J a l}{2 (a^2 - l^2)} \left( \ln \left| \frac{l + a \cos \theta}{l - a \cos \theta} \right| - \frac{l}{a} \ln \left| \frac{1 + \cos \theta}{1 - \cos \theta} \right| \right) - \frac{i \Xi J a l}{2 R_+^2} \ln \left| \frac{l + a \cos \theta}{l - a \cos \theta} \right| + \frac{iq Q_m a r_+}{\Xi R_+^2} \left( \frac{l^2}{a^2} - \frac{1}{a} \sqrt{l^2 - a^2} \arctan \left[ \frac{l \tan \theta}{\sqrt{l^2 - a^2}} \right] \right) \]  

(38)

Using relations (35) and (38) one can write the action for outgoing massive particles. Similarly equations (36) and (38) allow one to get the action for incoming particles.

4 Conclusions

In this paper we considered charged fermion tunnelling form the electrically and magnetically charged Kerr-Newman-de Sitter black hole. Using Kerner-Mann approach [20] we successfully recovered black hole’s temperature. It was shown that similarly to the case when black carries only the electric charge inclusion additional magnetic charge does not spoil separability of the Dirac equation in the vicinity of the horizons. So relations for the temperature is obtained in the same manner and take almost the same form as it was in the case of electrically charged black hole. We also note that for temperature explicit dependence on the magnetic charge is hidden in definition of horizons radii.

We also obtained relations for radial and angular part of the action. Those relations might be helpful if one tries to find corrections to the spectrum of emitted particles. Here we have explicit dependence on electric as well as magnetic charges so these terms might have different influence on the spectrum of emitted particles.

Another issue that still remains open is taking into account higher order of WKB corrections. This problem is connected with calculation of terms caused by spin connection. These terms can affect on the separability and tractability of the Dirac equation and this problem requires additional careful consideration.
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