DYADIC WEIGHTS ON $\mathbb{R}^n$ AND REVERSE HÖLDER INEQUALITIES

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Abstract: We prove that for any weight $\phi$ defined on $[0,1]^n$ that satisfies a reverse Hölder inequality with exponent $p > 1$ and constant $c \geq 1$ upon all dyadic subcubes of $[0,1]^n$, its non increasing rearrangement $\phi^*$, satisfies a reverse Hölder inequality with the same exponent and constant not more than $2^n c - 2^n + 1$, upon all subintervals of $[0,1]$ of the form $[0,t]$, $0 < t \leq 1$. This gives as a consequence, according to the results in [8], an interval $[p, p_0(p,c)) = I_{p,c}$, such that for any $q \in I_{p,c}$, we have that $\phi \in L^q$.

1. Introduction

The theory of Muckenhoupt’s weights has been proved to be an important tool in analysis. One of the most important facts about these is their self improving property. A way to express this is through the so called reverse Hölder inequalities (see [2], [3] and [7]).

Here we will study such inequalities on a dyadic setting. We will say that the measurable function $g : [0,1] \to \mathbb{R}^+$ satisfies the reverse Hölder inequality with exponent $p > 1$ and constant $c \geq 1$ if the inequality

$$\frac{1}{b-a} \int_a^b g^p(u)du \leq c \left( \frac{1}{b-a} \int_a^b g(u)du \right)^p,$$

holds for every subinterval of $[0,1]$.

In [1] it is proved the following

Theorem A. Let $g$ be a non-increasing function defined on $[0,1]$, which satisfies (1.1) on every interval $[a,b] \subseteq [0,1]$. Then if we define $p_0 > p$ as the root of the equation

$$\frac{p}{p_0} - p_0 \left( \frac{p_0}{p_0 - 1} \right)^p \cdot c = 1,$$

we have that $g \in L^q([0,1])$, for any $q \in [p,p_0)$. Additionally $g$ satisfies for every $q$ in the above range a reverse Hölder inequality for possibly another real constant $c'$. Moreover the result is sharp, that is the value $p_0$ cannot be increased.

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Now in [4] or [5] it is proved the following

**Theorem B.** If \( \phi : [0, 1] \to \mathbb{R}^+ \) is measurable satisfying (1.1) for every \([a, b] \subseteq [0, 1]\), then it’s non-increasing rearrangement \( \phi^* \), satisfies the same inequality with the same constant \( c \).

Here by \( \phi^* \) we denote the non-increasing rearrangement of \( \phi \), which is defined on \((0, 1]\) by

\[
\phi^*(t) = \sup_{E \subseteq [0, 1], |E|=t} \left\{ \inf_{x \in E} |\phi(x)| \right\}, \quad t \in (0, 1].
\]

This can be defined also as the unique left continuous, non-increasing function, equimeasurable to \(|\phi|\), that is, for every \( \lambda > 0 \) the following equality holds:

\[
| \{ \phi > \lambda \} | = | \{ \phi^* > \lambda \} |,
\]

where by \(| \cdot |\) we mean the Lebesgue measure on \([0, 1]\).

An immediate consequence of Theorem B, is that Theorem A can be generalized by ignoring the assumption of the monotonicity of the function \( g \).

Recently in [8] it is proved the following

**Theorem C.** Let \( g : (0, 1] \to \mathbb{R}^+ \) be non-increasing which satisfies (1.1) on every interval of the form \((0, t]\), \( 0 < t \leq 1 \). That is the following holds

\[
\frac{1}{t} \int_0^t g^p(u)du \leq c \cdot \left( \frac{1}{t} \int_0^t g(u)du \right)^p,
\]

for every \( t \) in \((0, 1]\). Then if we define \( p_0 \) by (1.2), we have that for any \( q \in [p, p_0) \) the following inequality is true

\[
\frac{1}{t} \int_0^t g^q(u)du \leq c' \left( \frac{1}{t} \int_0^t g(u)du \right)^q,
\]

for every \( t \) in \((0, 1]\) and some constant \( c' \geq c \). Thus \( g \in L^q((0, 1]) \) for any such \( q \). Moreover the result is sharp, that is we cannot increase \( p_0 \).

A consequence of Theorem C is that under the assumption that \( g \) is non-increasing, the hypothesis that (1.1) is satisfied only on the intervals of the form \((0, t]\) is enough for one to realize the existence of a \( p' > p \) for which \( g \in L^{p'}([0, 1]) \).

In several dimensions, as far as we know, there does not exists any similar result as Theorems A, B and C. All we know is the following, which can be seen in [3].

**Theorem D.** Let \( Q_0 \subseteq \mathbb{R}^n \) be a cube and \( \phi : Q_0 \to \mathbb{R}^+ \) measurable that satisfies

\[
\frac{1}{|Q|} \int_Q \phi^p \leq c \cdot \left( \frac{1}{|Q|} \int_Q \phi \right)^p,
\]

for fixed constants \( p > 1 \) and \( c \geq 1 \) and every cube \( Q \subseteq Q_0 \). Then there exists \( \varepsilon = \varepsilon(n, p, c) \) such that the following inequality holds:

\[
\frac{1}{|Q|} \int_Q \phi^q \leq c' \left( \frac{1}{|Q|} \int_Q \phi \right)^q
\]
for every $q \in [p, p + \varepsilon)$, any cube $Q \subseteq Q_0$ and some constant $c' = c'(q, p, n, c)$.

In several dimensions no estimate of the quantity $\varepsilon$, has been found. The purpose of
this work is to study the multidimensional case in the dyadic setting. More precisely
we consider a measurable function $\varphi$, defined on $[0, 1]^n = Q_0$, which satisfies (1.5) for
any $Q$, dyadic subcube of $Q_0$. These cubes can be realized by bisecting the sides of
$Q_0$, then bisecting it’s side of a resulting dyadic cube and so on. We define by $T_{2^n}$ the
respective tree consisting of those mentioned dyadic subcubes of $[0, 1]^n$. Then we will
prove the following:

**Theorem 1.** Let $\varphi : Q_0 = [0, 1]^n \to \mathbb{R}^+$ be such that

\begin{equation}
\frac{1}{|Q|} \int_Q \varphi^p \leq c \cdot \left( \frac{1}{|Q|} \int_Q \varphi \right)^p,
\end{equation}

for any $Q \in T_{2^n}$ and some fixed constants $p > 1$ and $c \geq 1$. Then, if we set $h = \varphi^*$ the
non-increasing rearrangement of $\varphi$, the following inequality is true

\begin{equation}
\frac{1}{t} \int_0^t h^p(u)du \leq (2^n c - 2^n + 1) \left( \frac{1}{t} \int_0^t h(u)du \right)^p, \quad \text{for any} \quad t \in [0, 1].
\end{equation}

As a consequence $h = \varphi^*$ satisfies the assumptions of Theorem C, which can be
applied and produce an $\varepsilon_1 = \varepsilon_1(n, p, c) > 0$ such that $h$ belongs to $L^q([0, 1])$ for any
$q \in [p, p + \varepsilon_1)$. Thus $\varphi \in L^q([0, 1]^n)$ for any such $q$. That is we can find an explicit
value of $\varepsilon_1$. This is stated as Corollary 3.1 and is presented in the last section of this
paper.

As a matter of fact we prove Theorem 1 in a much more general setting. More
precisely we consider a non-atomic probability space $(X, \mu)$ equipped with a tree $T_k$, that is a $k$-homogeneous tree for a fixed integer $k > 1$, which plays the role of dyadic
sets as in $[0, 1]^n$ (see the definition of Section 2).

As we shall see later, Theorem 1 is independent of the shape of the dyadic sets and
depends only on the homogeneity of the tree $T_k$. Additionally we need to mention that
the inequality (1.8) cannot necessarily be satisfied, under the assumptions of Theorem
1, of one replaces the intervals $(0, t]$ by $(t, 1]$. That is $\varphi^*$ is not necessarily a weight on
$(0, 1]$ satisfying a reverse Hölder inequality upon all subintervals of $[0, 1]$ (see [5]).

Additionally we mention that in [6] the study of the dyadic $A_1$-weights appears,
where one can find for any $c > 1$ the best possible range $[1, p)$, for which the following
holds: $\varphi \in A_1^q(c) \Rightarrow \varphi \in L^q$, for any $q \in [1, p)$. All last results that are connected with
$A_1$ dyadic weights $\varphi$ and the behavior of $\varphi^*$ as an $A_1$-weight on $\mathbb{R}$, can be seen in [9].

2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability space. We give the notion of a $k$-homogeneous
tree on $X$.

**Definition 2.1.** Let $k$ be an integer such that $k > 1$. A set $T_k$ will be called a $k$-

homogeneous tree on $X$ if the following hold
(i) $X \in \mathcal{T}_k$
(ii) For every $I \in \mathcal{T}_k$, there corresponds a subset $C(I) \subseteq \mathcal{T}_k$ consisting of $k$ subsets of $I$ such that

(a) the elements of $C(I)$ are pairwise disjoint
(b) $I = \bigcup C(I)$
(c) $\mu(J) = \frac{1}{k} \mu(I)$, for every $J \in C(I)$.

For example one can consider $X = [0, 1]^n$, the unit cube of $\mathbb{R}^n$. Define as $\mu$ the Lebesgue measure on this cube. Then the set $\mathcal{T}_k$ of all dyadic subcubes of $X$ is a tree of homogeneity $k = 2^n$, with $C(Q)$ being the set of $2^n$-subcubes of $Q$, obtained by bisecting it’s sides, for every $Q \in \mathcal{T}_k$, starting from $Q = X$.

Let now $(X, \mu)$ be as above and a tree $\mathcal{T}$ on $X$ as in Definition 2.1. From now on, we fix $k$ and write $\mathcal{T} = \mathcal{T}_k$. For any $I \in \mathcal{T}$, $I \neq X$ we set $I^*$ the smallest element of $\mathcal{T}$ such that $I \supseteq I^*$. That is $I^*$ is the unique element of $\mathcal{T}$ such that $I \in C(I^*)$. We call $I^*$ the father of $I$ in $\mathcal{T}$. Then $\mu(I^*) = k \mu(I)$.

**Definition 2.2.** For any $(X, \mu)$ and $\mathcal{T}$ as above we define the dyadic maximal operator on $X$ with respect to $\mathcal{T}$, noted as $M_{\mathcal{T}}$, by

\[ M_{\mathcal{T}} \phi(X) = \sup \{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T} \}, \tag{2.1} \]

for any $\phi \in L^1(X, \mu)$.

**Remark 2.1.** It is not difficult to see that the maximal operator defined by (2.1) satisfies a weak-type $(1,1)$ inequality, which is the following:

\[ \mu(\{M_{\mathcal{T}} \phi > \lambda\}) \leq \frac{1}{\lambda} \int_{\{M_{\mathcal{T}} \phi > \lambda\}} \phi \, d\mu, \quad \lambda > 0. \]

It is not difficult to see that the above inequality is best possible for every $\lambda > 0$, and is responsible for the fact that $\mathcal{T}$ differentiates $L^1(X, \mu)$, that is the following holds:

\[ \lim_{\mu(I) \to 0} \frac{1}{\mu(I)} \int_I \phi \, d\mu = \phi(x), \quad \mu\text{-almost everywhere on } X. \]

This can be seen in [4].

We will also need the following lemma which can be also seen in [4].

**Lemma 2.1.** Let $\phi$ be non-negative function defined on $E \cup \hat{E} \subseteq X$ such that

\[ \frac{1}{\mu(E)} \int_E \phi \, d\mu = \frac{1}{\mu(\hat{E})} \int_{\hat{E}} \phi \, d\mu \equiv A, \tag{2.2} \]

Additionally suppose that

\[ \phi(x) \leq A, \quad \text{for every } x \notin E \cap \hat{E}, \tag{2.3} \]

and

\[ \phi(x) \leq \phi(y), \quad \text{for every } X \in \hat{E} \setminus E, \quad \text{and } y \in E, \tag{2.4} \]
Then, for every $p > 1$ the following inequality holds

$$\frac{1}{\mu(E)} \int_E \phi^p d\mu \leq \frac{1}{\mu(E)} \int_E \phi^p d\mu,$$

(2.5)

3. Weights on $(X, \mu, \mathcal{T})$

We proceed now to the

**Proof of Theorem 1.** We suppose that $\phi$ is non-negative defined on $(X, \mu)$ and satisfies a reverse H"older inequality of the form

$$\frac{1}{\mu(I)} \int_I \phi^p d\mu \leq c \cdot \left( \frac{1}{\mu(I)} \int_I \phi d\mu \right)^p,$$

(3.1)

for every $I \in \mathcal{T}$, where $c, p$ are fixed such that $p > 1$ and $c \geq 1$. We will prove that for any $t \in (0, 1]$ we have that

$$\frac{1}{t} \int_0^t [\phi^*(u)]^p du \leq (kc - k + 1) \left( \frac{1}{t} \int_0^t \phi^*(u) du \right)^p,$$

(3.2)

where $\phi^*$ is the non-increasing rearrangement of $\phi$, defined as in Remark ??, on $(0, 1]$, and $k$ is the homogeneity of $\mathcal{T}$. Fix a $t \in (0, 1]$ and set

$$A = A_t = \frac{1}{t} \int_0^t \phi^*(u) du.$$

Consider now the following subset of $X$ defined by

$$E_t = \{ x \in X : \mathcal{M}_T \phi(x) > A \},$$

(3.3)

Then for any $x \in E_t$, there exists an element of $\mathcal{T}$, say $I_x$, such that

$$x \in I_x \quad \text{and} \quad \frac{1}{\mu(I_x)} \int_{I_x} \phi d\mu > A.$$

(3.4)

For any such $I_x$ we obviously have that $I_x \subseteq E_t$. We set $S_{\phi,t} = \{ I_x : x \in E_t \}$. This is a family of elements of $\mathcal{T}$ such that $U\{ I : I \in S_{\phi,t} \} = E_t$. Consider now those $I \in S_{\phi,t}$ that are maximal with respect to the relation of $\subseteq$. We write this subfamily of $S_{\phi,t}$ as $S'_{\phi,t} = \{ I_j : j = 1, 2, \ldots \}$ which is possibly finite. Then $S'_{\phi,t}$ is a disjoint family of elements of $\mathcal{T}$, because of the maximality of every $I_j$ and the tree structure of $\mathcal{T}$. (see Definition 2.1).

Then by construction, this family still covers $E_t$, that is $E_t = \bigcup_{j=1}^{\infty} I_j$. For any $I_j \in S'_{\phi,t}$ we have that $I_j \neq X$, because if $I_j = X$ for some $j$, we could have from (3.4) that

$$\int_0^1 \phi^*(u) du = \int_X \phi d\mu = \frac{1}{\mu(I_j)} \int_{I_j} \phi d\mu > A = \frac{1}{t} \int_0^t \phi^*(u) du,$$

which is impossible, since $\phi^*$ is non-increasing on $(0, 1]$. Thus, for every $I_j \in S'_{\phi,t}$ we have that $I_j^*$ is well defined, but may be common for any two or more elements of $S'_{\phi,t}$. We may also have that $I_j^* \subseteq I_i^*$ for some $I_j, I_i \in S'_{\phi,t}$.
We consider now the family
\[ L_{\phi,t} = \{ I_j^* : j = 1, 2, \ldots \} \subseteq \mathcal{T}. \]
As we mentioned above, this is not necessarily a pairwise disjoint family. We choose a pairwise disjoint subcollection, by considering those \( I_j^* \) that are maximal, with respect to the relation \( \subseteq \).

We denote this family as
\[ L'_{\phi,t} = \{ I^*_j \} \subseteq \mathcal{T}. \]
Then of course
\[ \bigcup J : J \in L_{\phi,t} = \bigcup J : J \in L'_{\phi,t}. \]
Since, each \( I_j^* \in S'_{\phi,t} \) is maximal we should have that
\[ \frac{1}{\mu(I^*_j)} \int_{I^*_j} \phi d\mu \leq A, \]  
(3.5)

Now note that every \( I^*_j \) contains at least one element of \( S'_{\phi,t} \), such that \( I \in C(I^*_j) \). Consider for any \( s \) the family of all those \( I \) such that \( I^* \subseteq I^*_j \). We write it as
\[ S'_{\phi,t,s} = \{ I \in S'_{\phi,t} : I^* \subseteq I^*_j \}. \]
For any \( I \in S'_{\phi,t,s} \) we have of course that
\[ \frac{1}{\mu(I)} \int_I \phi d\mu > A, \quad \text{so if we set} \quad K_s = U \{ I : I \in S'_{\phi,t,s} \}. \]
We must have, because of the disjointness of the elements of family \( S'_{\phi,t} \), that
\[ \frac{1}{\mu(K_s)} \int_{K_s} \phi d\mu > A. \]  
(3.6)
Additionally, \( K_s \subseteq I^*_j \) and by (3.5) and the comments stated above we easily see that
\[ \frac{1}{k} \mu(I^*_j) < \mu(K_s) \leq \mu(I^*_j), \]  
(3.7)
By (3.5) and (3.6) we can now choose (because \( \mu \) is non-atomic) for any \( s \), a measurable set \( B_s \subseteq I^*_j \setminus K_s \), such that if we define \( \Gamma_s = K_s \cup B_s \), then \( \frac{1}{\mu(\Gamma_s)} \int_{\Gamma_s} \phi d\mu = A. \)

We set now \( E^*_t = \bigcup_s I^*_j \) \[ \Gamma = \bigcup_s \Gamma_s, \quad \Delta = \bigcup_s \Delta_s, \]
where \( \Delta_s = I^*_j \setminus \Gamma_s \), for any \( s = 1, 2, \ldots \).

Then by all the above, we have that
\[ \Gamma \cup \Delta = E^*_t \quad \text{and} \quad \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi d\mu = A_t, \]
which is true in view of the pairwise disjointness of \( (I^*_j)_{s=1}^{\infty} \).
Define now the following function

\[ h := (\phi/\Gamma)^* : (0, \mu(\Gamma)] \to \mathbb{R}^+. \]

Then obviously

\[ \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u) du = \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi d\mu = A_t. \]

By the definition of \( h \) we have that \( h(u) \leq \phi^*(u) \), for any \( u \in (0, \mu(\Gamma)] \). Thus we conclude:

\[ (3.8) \quad \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} \phi^*(u) du \geq \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u) du = A_t = \frac{1}{t} \int_0^t \phi^*(u) du, \]

From (3.8), we have that \( \mu(\Gamma) \leq t \), since \( \phi^* \) is non-increasing.

We now consider a set \( E \subseteq X \) such that \((\phi/E)^* = \phi^*/(0, t] \), with \( \mu(E) = t \) and for which \( \{ \phi > \phi^*(t) \} \subseteq E \subseteq \{ \phi \geq \phi^*(t) \} \).

It’s existence is guaranteed by the equimeasurability of \( \phi \) and \( \phi^* \), and the fact that \((X, \mu)\) is non-atomic. Then, we see immediately that

\[ \frac{1}{\mu(E)} \int_E \phi d\mu = \frac{1}{t} \int_0^t \phi^*(u) du = A_t. \]

We are going now to construct a second set \( \hat{E} \subseteq X \). We first set \( \hat{E}_1 = \Gamma \).

Let now \( x \notin \hat{E}_1 \). Since \( \Gamma \supseteq \{ \mathcal{M}_\Gamma \phi > A_t \} \), we must have that \( \mathcal{M}_\Gamma \phi(x) \leq A_t \).

But since \( \mathcal{T} \) differentiates \( L^1(X, \mu) \) we obviously have that for \( \mu \)-almost every \( y \in X : \phi(y) \leq \mathcal{M}_\Gamma \phi(y) \). Then the set \( \Omega = \{ x \notin \hat{E}_1 : \phi(x) > \mathcal{M}_\Gamma \phi(x) \} \) has \( \mu \)-measure zero.

At last we set \( \hat{E} = \hat{E}_1 \cup \Omega = \Gamma \cup \Omega \).

Then \( \mu(\hat{E}) = \mu(\Gamma) \) and for every \( x \notin \hat{E} \) we have that \( \phi(x) \leq \mathcal{M}_\Gamma \phi(x) \leq A_t \).

Let now \( x \notin E \). By the construction of \( E \) we immediately see that \( \phi(x) \leq \phi^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) du = A_t \). Thus, if \( x \notin E \) or \( x \notin \hat{E} \), we must have that \( \phi(x) \leq A_t \), that is (2.3) of Lemma 2.1 is satisfied for these choices of \( E \) and \( \hat{E} \). Let now \( x \in \hat{E} \setminus E \) and \( y \in E \). Then we obviously have by the above discussion that \( \phi(x) \leq \phi^*(t) \leq \phi(y) \).

That is \( \phi(x) \leq \phi(y) \). Thus (2.4) is also satisfied. Also since \( \hat{E} = \Gamma \cup \Omega \), we obviously have \( \frac{1}{\mu(E)} \int_E \phi d\mu = A_t \), so as a consequence (2.2) is satisfied also.

Applying Lemma 2.1 we conclude that

\[ \frac{1}{\mu(E)} \int_E \phi^p d\mu \leq \frac{1}{\mu(\hat{E})} \int_{\hat{E}} \phi^p d\mu, \]

or by the definitions of \( E \) and \( \hat{E} \) that

\[ \frac{1}{t} \int_0^t [\phi^*(u)]^p du \leq \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi^p d\mu, \]

Our aim is now to show that the right integral average in (3.9) is less or equal that \((kc - k + 1)(A_t)^p \). We proceed to this as follows:
We set \( \ell_\Gamma = \frac{1}{\mu(\Gamma)} \int_\Gamma \phi^p d\mu \). Then by the notation given above, we have that:

\[
\ell_\Gamma = \frac{1}{\mu(\Gamma)} \left( \int_{E_t^*} \phi^p d\mu - \int_\Delta \phi^p d\mu \right) \\
= \frac{1}{\mu(\Gamma)} \left( \sum_{s=1}^\infty \int_{I_{js}} \phi^p d\mu - \sum_{s=1}^\infty \int_{\Delta_s} \phi^p d\mu \right) \\
= \frac{1}{\mu(\Gamma)} \sum_{s=1}^\infty p_s,
\]

(3.10)

where the \( p_s \) are given by

\[
p_s = \int_{I_{js}} \phi^p d\mu - \int_{\Delta_s} \phi^p d\mu, \quad \text{for any} \quad s = 1, 2, \ldots.
\]

We find now an effective lower bound for the quantity \( \int_\Delta \phi^p d\mu \). By Hölder’s inequality:

\[
\int_\Delta \phi^p d\mu \geq \frac{1}{\mu(\Delta_s)^{p-1}} \left( \int_\Delta \phi d\mu \right)^p,
\]

(3.11)

Since \( \Delta_s = I_{js}^* \setminus \Gamma_s \), (3.11) can be written as

\[
\int_\Delta \phi^p d\mu \geq \frac{\left( \int_{I_{js}^*} \phi d\mu - \int_{\Gamma_s} \phi d\mu \right)^p}{\mu(I_{js}^*) - \mu(\Gamma_s))^{p-1}},
\]

(3.12)

We now use Hölder’s inequality in the form

\[
\frac{(\lambda_1 + \lambda_2)^p}{\sigma_1 + \sigma_2} \leq \frac{\lambda_1^p}{\sigma_1^{p-1}} + \frac{\lambda_2^p}{\sigma_2^{p-1}}, \quad \text{for} \quad \lambda_i \geq 0 \quad \text{and} \quad \sigma_i > 0
\]

which holds since \( p > 1 \). Thus (3.12) gives

\[
\int_\Delta \phi^p d\mu \geq \frac{1}{\mu(I_{js}^*)^{p-1}} \left( \int_{I_{js}^*} \phi d\mu \right)^p - \frac{1}{\mu(\Gamma_s)^{p-1}} \left( \int_{\Gamma_s} \phi d\mu \right)^p.
\]

(3.13)

Since \( \frac{1}{\mu(I_s)} \int_{I_s} \phi d\mu = A_t \), (3.13) gives

\[
\int_\Delta \phi^p d\mu \geq \frac{1}{\mu(I_{js}^*)^{p-1}} \left( \int_{I_{js}^*} \phi d\mu \right)^p - \mu(\Gamma_s) \cdot (A_t)^p,
\]

so we conclude, by the definition of \( p_s \), that

\[
p_s \leq \int_{I_{js}^*} \phi^p d\mu - \frac{1}{\mu(I_{js}^*)^{p-1}} \left( \int_{I_{js}^*} \phi d\mu \right)^p + \mu(\Gamma_s) \cdot (A_t)^p,
\]

(3.14)

Using now (3.1) for \( I = I_{js}^* \), \( s = 1, 2, \ldots \) we have as a consequence that:

\[
p_s \leq (c - 1) \frac{1}{\mu(I_{js}^*)^{p-1}} \left( \int_{I_{js}^*} \phi d\mu \right)^p + \mu(\Gamma_s)(A_t)^p.
\]

(3.15)
Summing now (3.15) for \( s = 1, 2, \ldots \) we obtain in view of (3.10) that

\[
\ell_{\Gamma} \leq \frac{1}{\mu(\Gamma)} \left( (c - 1) \sum_{s=1}^{\infty} \mu(I_{js}^{*}) (A_t)^p + \left( \sum_{s=1}^{\infty} \mu(\Gamma_s) \right) (A_t)^p \right)
\]

(3.16)

Now from

\[
\frac{1}{\mu(I_{js}^{*})} \int_{I_{js}^{*}} \phi d\mu \leq A_t,
\]

we see that

\[
\ell_{\Gamma} \leq \frac{1}{\mu(\Gamma)} \left[ (c - 1) \sum_{s=1}^{\infty} \mu(I_{js}^{*}) \cdot (A_t)^p + \mu(\Gamma) \cdot (A_t)^p \right]
\]

(3.17)

Since now \( E_t^{*} \supseteq \Gamma \supseteq E_t \), by (3.7) we have that

\[
\mu(E_t^{*}) \leq k \mu(E_t) \leq K \mu(\Gamma).
\]

Thus (3.17) gives

\[
\frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi^p d\mu \leq [k(c - 1) + 1] (A_t)^p.
\]

Using now (3.9) and the last inequality we obtained the desired result. \( \square \)

**Corollary 3.1.** If \( \phi \) satisfies (3.1) for every \( I \in \mathcal{T} \), then \( \phi \in L^q \), for any \( q \in [p, p_0) \), where \( p_0 \) is defined by

\[
p_0 - \frac{p}{p_0} \cdot \left( \frac{p_0}{p_0 - 1} \right)^p \cdot (kc - k + 1) = 1.
\]

**Proof.** Immediate from Theorem 1 and A. \( \square \)

**Remark 3.1.** All the above hold if we replace the condition (3.1), by the known Muckenhoupt condition of \( \phi \) over the dyadic sets of \( X \). Then the same proof as above gives that the Muckenhoupt condition should hold for \( \phi^{*} \), for the intervals of the form \((0, t]\), and for the constant \( kc - k + 1 \). This is true since there exists analogous lemma as Lemma 2.1 for this case (as can be seen in [4]). Also the inequality that is used in order to produce (3.13) from (3.12) is true even for negative exponent \( p < 0 \). We omit the details. \( \square \)

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