Advancing Space-Time Simulation of Random Fields: From Storms to Cyclones and Beyond

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Abstract
Realistic stochastic simulation of hydro-environmental fluxes in space and time, such as rainfall, is challenging yet of paramount importance to inform environmental risk analysis and decision making under uncertainty. Here, we advance random field simulation by introducing the concepts of general velocity fields and general anisotropy transformations. This expands the capabilities of the so-called Complete Stochastic Modeling Solution (CoSMoS) framework enabling the simulation of random fields preserving: (1) any non-Gaussian marginal distributions, (2) any spatiotemporal correlation structure, (3) general advection expressed by velocity fields with locally varying speed and direction, and (4) locally varying anisotropy. We also introduce new copula-based spatiotemporal correlation structures and provide conditions guaranteeing their positive definiteness. To illustrate the potential of CoSMoS, we simulate random fields with complex patterns and motion mimicking rainfall storms moving across an area, spiraling fields resembling weather cyclones, fields converging to (or diverging from) a point, and colliding air masses. The proposed methodology is implemented in the freely available CoSMoS R package.

Key Points
- Introducing more realistic random field simulation of hydro-environmental fluxes
- Introducing general velocity fields and general anisotropy transformations
- Introducing new copula-based spatiotemporal correlation functions
  Simulation preserving marginals, spatiotemporal correlation, anisotropy, and advection

1. Introduction: The Rainfall Paradigm

“The little reed, bending to the force of the wind, soon stood upright again when the storm had passed over.” ~ Aesop

The complexity of hydro-environmental fluxes, such as weather systems, often prevents deterministic modelling via numerical models discretizing systems of partial differential equations, especially when focusing on high spatial and temporal resolution required in water resources planning and management. In these cases, stochastic methods become a viable alternative to model hydro-environmental processes of interest under uncertainty, and simulate scenarios preserving the main properties of the observations. However, building effective stochastic models is not easy if one aims to obtain simulations mimicking...
the observed records as closely as possible. Therefore, realistic simulations should reproduce characteristics such as: (1) generally skewed non-Gaussian marginal distributions of the process intensity, (2) spatiotemporal dependence, (3) anisotropy, that is, the inhomogeneous variability of process characteristics in different directions, and (4) advection, which is meant here as transport of a quantity by bulk motion of fluid (e.g., horizontal transport of some property of the atmosphere such as humidity).

Most of the literature on the topic under consideration refers to modelling precipitation dynamics in space and time. Indeed, at several spatiotemporal scales of practical interest, this hydro-meteorological process exhibits discrete-continuous marginal and joint distributions, whereby the discrete part describes zero-inflation related to transition rain/no rain, while the continuous part describes precipitation intensity, which is generally skewed (non-Gaussian). Precipitation is also characterized by spatiotemporal dependence, anisotropy and advection related to the physics and the spatiotemporal evolution of the weather systems. Moreover, precipitation is of great interest as it is the main driver of surface and underground hydrological processes, and therefore one of the main input variables in every study concerning water resources management and risk assessment of water-related hazards, such as droughts and floods.

The four above-mentioned properties are related and influence one another in several ways at different spatiotemporal scales. Synoptic weather systems, such as (extra) tropical cyclones, comprise large scale air masses rotating around a strong center of low atmospheric pressure exhibiting spiral-like spatial structure directly related to rotational motion. Much of the significant weather observed in association with such systems tends to be concentrated within narrow bands called frontal zones containing large mesoscale precipitation areas (rain bands) roughly oriented parallel to the front lines (Houze et al., 1976). Mesoscale precipitation areas are composed by regions of cumulus convective precipitation, commonly known as convective cells (Amorocho & Wu, 1977; Gupta & Waymire, 1979; Houze, 2014; Houze et al., 1976; McMurdie & Houze, 2006). Bands are nearly perpendicular to the direction of movement of the front, and the motion of these bands is sometimes faster than the motion of the general storm system (Amorocho & Wu, 1977; Ippolito, 1989). Cells within a rain band tend to move along or slightly ahead of the direction of the front, and the more intense cells tend to elongate in their direction of motion (Houze, 2018; Houze et al., 1990; Ippolito, 1989). This relationship between the shape of intense cells and advection direction was also recognized by Moszkowicz (2000), studying the spatial correlation functions of radar data for scales from two to tens of kilometers.

The interconnection among correlation, advection, and anisotropy plays a key role also in the quantification of these properties. Rainfall fields show an apparent motion resulting from the combined effect of the winds at some steering level and the systematic precipitation growth and dissipation (Germann et al., 2006). This suggests the possibility of analyzing the rainfall process on either Eulerian or Lagrangian coordinates, whereby the former refer to a fixed reference system (e.g., the ground), while the latter to a reference system moving with the precipitation field. Taking the motion into account, one can define Eulerian and Lagrangian variants of temporal correlation of radar rainfall fields (Zawadzki, 1973). Thus, the maximum of the cross-correlation function or the maximum of the lag-correlation structure can be used as recognizable features of the rainfall pattern. These features are then followed as they move in space across the study area (i.e., the Eulerian reference system),
quantifying thus storm velocity and direction (Bacchi & Kottegoda, 1995; Niemczynowicz, 1987; Rinehart & Garvey, 1978; Shaw, 1983; Vischel et al., 2011; Zawadzki, 1973).

Concerning the link between anisotropy and correlation, anisotropy of geophysical fields is often studied by using two-dimensional spatial correlation functions, which allow for the analysis of both the scale over which patterns occur and the direction of the pattern (Niemi et al., 2014). Two-dimensional spatial correlation functions and the corresponding Fourier power spectra (where the latter are the Fourier transform of former according to the Wiener-Khinchin theorem) have been extensively used to study the spatial structure of precipitation fields (e.g., Zawadzki, 1973; Krajewski, 1987; Sinclair & Pegram, 2005; Mandapaka and Qin, 2013; Niemi, Kokkonen and Seed, 2014; Gyasi-Agyei, 2016 Cassiraga et al., 2020). They are also the basis of methods, such as the Generalized Scale Invariance (GSI; Lovejoy and Schertzer, 1985), that allow one to quantify the scaling of anisotropic systems, thus accounting for the different anisotropy of cells and rain bands (see also Ramanathan & Satyanarayana, 2019).

The foregoing discussion highlights the importance of developing effective stochastic models for environmental/geophysical flows, such as rainfall, that account for possibly complex (non-Gaussian) marginal distributions, spatiotemporal correlation, anisotropy, and advection. However, building models incorporating these properties is challenging. Recent attempts concerning wind speed and lightning are provided for example by Gneiting, Genton and Guttorp (2006), Youngman and Stephenson (2016), and North et al. (2020), while the problem of modeling rainfall was tackled, for example, by Paschalis et al., (2013), Leblois and Creutin (2013), Niemi et al. (2016), Nerini et al. (2017), as discussed below in more depth.

However, the existing methods generally focus on a subset of those characteristics, thus lacking some accuracy generality in the description of the remaining features. Building on Papalexiou (2018) and Papalexiou and Serinaldi (2020), this study proposes a general framework that allows consistent modelling of all the foregoing properties along with extensions and additional features. We extend the CoSMoS (Complete Stochastic Modeling Solution) framework, which allows for an accurate reproduction of marginal distributions and spatiotemporal correlations, to incorporate general advection velocity fields and anisotropy. CoSMoS is a general stochastic modeling framework that can be applied to any geophysical process, yet rainfall is the most natural example for this type of models.

2. Random Fields and Space-Time Correlations

Random fields (RF’s) offer an integrated approach to model the spatial variability and temporal evolution of environmental processes. Natural processes are continuous in space and time, yet in practice RF’s are typically implemented in discrete space and time. Let Ω be the sample space of a random experiment; a spatiotemporal RF is a stochastic process \( X(\omega; s, t) \) with \( \omega \in \Omega \) and \( (s, t) \in \mathbb{R}^d \times \mathbb{R} \). Thus, an RF can be represented as a collection of random variables (rv’s) \( \{X_1, \ldots, X_n\} \) at points (coordinates) \( (s_1, t), \ldots, (s_n, t) \) with \( t \) taking values in the discretized timeline. Most applications deal with 2- or 3-dimensional spaces; here all demonstrations use \( d = 2 \) and thus \( s = (x, y)^T \) represents a point in the Cartesian plane (note, rv’s and their values are denoted with script letters to avoid ambiguity with Cartesian coordinates; see Table A1 in Annex A for abbreviations and notation).

The space and time interaction of rv’s forming an RF is typically expressed by spatiotemporal correlation structures (STCS’s; for a review on STCS’s see Porcu et al., 2020)). Here, we start with STCS’s that are spatially isotropic and temporally symmetric. That is, the
STCS $\rho(\delta, \tau)$ gives the correlation between two rv's $X(s_i, t_k)$ and $X(s_j, t_l)$ having Euclidean distance $\delta$ in the 2-dimensional plane

$$
\delta = \|s_i - s_j\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}
$$

with $\delta \in [0, \infty)$, and lagged in time by $\tau = |t_k - t_l|$ with $\tau \geq 0$.

Here, we use STCS's that are built based on the framework proposed by Papalexiou and Serinaldi (2020) and have the form

$$
\rho(\delta, \tau; \theta) = C(F_S(\delta; \theta_S), F_T(\tau; \theta_T); \theta_C), \quad \delta, \tau \geq 0,
$$

where $C$ is a bivariate copula function with parameter vector $\theta_C$, and $F_S(\delta; \theta_S)$ and $F_T(\tau; \theta_T)$ are survival functions (sf's) with parameter vectors $\theta_S$ and $\theta_T$, respectively. Using this approach, a STCS was formed by combining the Clayton ($C$) Copula with Weibull sf's leading to a so-called Clayton-Weibull STCS (Papalexiou & Serinaldi, 2020)

$$
\rho_{CW}(\delta, \tau) := \rho(\delta, \tau; \theta) = \left( \exp \left( \theta \left( \frac{\delta}{b_S} \right)^{c_S} + \exp \left( \theta \left( \frac{\tau}{b_T} \right)^{c_T} \right) - 1 \right) \right)^{-\frac{1}{\theta}}
$$

Here, we combine the Ali-Mikhail-Haq (AMH) copula and Weibull sf's to form an Ali-Mikhail-Haq-Weibull (AMHW) STCS

$$
\rho_{AMHW}(\delta, \tau) := \rho(\delta, \tau; \theta) = \frac{\exp(-\delta/b_S)^{c_S} \exp(-\tau/b_T)^{c_T}}{1 - \theta(1 - \exp(-\delta/b_S)^{c_S})(1 - \exp(-\tau/b_T)^{c_T})}
$$

The parameter vector of both STCS's is $\theta = (b_S, c_S, b_T, c_T, \theta)^T$ with $\delta, \tau \geq 0$. Details and mathematical proofs showing the admissible values of $\theta$ for the previous two STCS's to be positive definite are provided in the Supplementary Material. Note that a positive definite STCS guarantees positive variance for every linear combination of the rv's defining the RF. In general, the parameters $\theta_S$ and $\theta_T$ control the correlation properties in space and time, respectively, while $\theta_C$ controls the space-time interaction. For example, the AMHW STCS with parameters $\theta = (10, 1, 30, 1, 1)$ is non-separable for $\theta \neq 0$ (Figure 1a), and with same parameters but $\theta = 0$ is separable (Figure 1b) with no interaction is space and time.

**Figure 1.** Examples of Ali-Mikhail-Haq-Weibull spatiotemporal correlation structures (STCS) giving the correlation between two points in space distanced by $\delta$ and lagged by time $\tau$. (a) Non-separable for $\theta \neq 0$, (b) separable for $\theta = 0$. 

3. Lagrangian Gaussian Random Fields and the Dimple Effect

As mentioned in the introduction, environmental, atmospheric, and geophysical processes are often influenced by prevailing winds or ocean currents; such cases can be modelled using a framework based on a Lagrangian reference system (Gneiting et al., 2007; Rodriguez-Iturbe et al., 1987). As mentioned in Porcu et al. (2019) random rotation or movement (advection) might represent a prevailing wind as in Gupta and Waymire (1987), a westerly wind as considered by Haslett and Raftery (1989), or again, it might be updated dynamically according to the current state of the atmosphere.

In a Lagrangian model, the advection of a field in the 2-dimensional space is expressed by a velocity vector \( \mathbf{v}_{x,y} = (v_x, v_y) \), where \( v_x \) and \( v_y \) are the orthogonal components of the velocity at each time step. Since the rv's forming the field move with velocity \( \mathbf{v}_{x,y} \) an rv \( X(s, t) \) lagged by \( \tau \) time was located at the point \( s_i - \tau \mathbf{v}_{x,y} \) at time \( t_i = t_k - \tau \). This implies that the distance between \( X(s_i, t_k) \) and \( X(s_j, t_i) \) is

\[
\delta_L = \| s_i - \tau \mathbf{v}_{x,y} - s_j \| = \sqrt{(x_i - \tau v_x - x_j)^2 + (y_i - \tau v_y - y_j)^2}
\]

(5)

In turn, the Lagrangian distance \( \delta_L \) allows for expressing the STCS \( \rho_L \) of a Lagrangian field moving with velocity \( \mathbf{v}_{x,y} \) in terms of the untransformed \( \rho \) as

\[
\rho_L(\delta, \tau; \theta) = \rho(\delta_L, \tau; \theta)
\]

(6)

In a plain language, in Lagrangian perspective, the observer follows an individual fluid parcel as it moves through space and time. Conversely, from Eulerian standpoint, the same observer focuses on specific locations in the space through which the fluid flows as time passes. In the context of rainfall fields, records of rain gauges are Eulerian point images of the ground-level rain fields produced by the moving bands (Amorocho & Wu, 1977), while radar echoes enable Lagrangian analysis by tracking the field motion in a sequence of radar images at subsequent time steps. Therefore, taking the motion into account, the Eulerian version of spatiotemporal correlation of radar fields depends on the motion of the storm and changes in structure that occur within it, while the Lagrangian variant is defined relative to the storm coordinates and therefore is independent of the storm motion, which indeed filtered out in equation (5) (Zawadzki, 1973). An important consequence of the equivalence in equation (5) is that a Lagrangian STCS \( \rho_L \) does not peak at \( \delta = 0 \) for \( \tau > 0 \). A first description of this characteristic of Lagrangian covariance functions, which is called dimple effect, is due to Kent et al. (2011), while Cuevas et al. (2017) provided a discussion for the Gneiting class of STCFs. In essence, a STCS has a dimple if the rv \( X(s_{\text{here}}, t_{\text{now}}) \) is more correlated with \( X(s_{\text{there}}, t_{\text{then}}) \) than with \( X(s_{\text{there}}, t_{\text{now}}) \).

Here we offer further insights on the dimple effect and show, for a constant advection vector, how the STCS varies depending on the direction considered in space. The STCS values in a Lagrangian framework depend not only on the distance and time lag between two rv’s, but also on their direction in relevance to the field’s movement. In the Cartesian plane the direction from point \( A \) to \( B \) is the angle of the line connecting the two points (see Figure A1). Thus, a STCS expressing an advected field depends on three independent variables (distance, time, direction) and can be visualized in a 3D plot if a variable is fixed.

Let a space-time process, for example a storm, be observed over an \( n \times n \) spatial grid comprising \( n^2 \) rv’s. For example, let the field be Lagrangian with advection velocity \( \mathbf{v}_{x,y} = \)}
(νₓ, νᵧ) = (4,4). Thus, the field is moving over the 45° diagonal covering a 4√2 distance every time step. If the untransformed STCS ρ is the non-separable AMHW (Figure 1a), it can be modified based on equation (6) to become a Lagrangian STCS ρₓ to account for any advection speed. In this case, the Lagrangian STCS shows a dimple in space and time (Figure 2 and Figure 3). The dimple in space is apparent as for temporal lags τ > 0 the STCS peaks at nonzero distance (Figure 2) in contrast to the static STCS (Figure 1). Note that the Lagrangian STCS, if the advection vector is constant, peaks in the opposite direction of movement (Figure 2a,b), that is at 225°, and for lag τ = 1 and distance 4√2 (the distance the field covers in one time step). For temporal lag τ = 2, it peaks at distance 8√2 (Figure 2c,d) since this is the distance the RF covers in two time steps. These peaks are the “global” maxima of the STCS for given lags and are observed in the opposite direction of the RF’s movement and for specific distances; for different directions, the peaks are observed in shorter distances (Figure 2).

Figure 2. Dimple in space. Example of a spatiotemporal correlation structure (STCS), the Ali-Mikhail-Haq-Weibull shown in Figure 1a, modified for a moving field with speed νₓ,y = (4,4). For temporal lags τ > 0 the STCS peaks at nonzero distance in the opposite direction of the movement: (a) values of the STCS (for fixed lag τ = 1) vs. distance and direction and (b) tomography for specific directions; (c,d) similar to (a,b) but for temporal lag τ = 2.

Lagrangian STCS’s have a dimple effect also in time (Figure 3). For a fixed distance δ > 0 the STCS peaks at nonzero time lags, as opposed to the static STCS. The global maxima for any fixed distance are always observed in the opposite direction of the movement for constant advection. If we set δ = 4√2 the STCS peaks for τ = 1 at the 225° direction (Figure 3a,b), while it peaks at τ = 3 for δ = 12√2 (Figure 3a,b). Note that the Lagrangian STCS is symmetric over the direction of movement. In this example, since νₓ,y = (4,4) and the RF’s move at 45°, the STCS values for directions 45° ± ω° coincide (e.g., 135° and 315°; Figure 2 and Figure 3). As mentioned above, the “global” maximum occurs in the opposite direction of movement (225° here) at a distance δ = τν²ₓ + ν²ᵧ for time lag τ, while the minimum for
fixed $\delta$ and $\tau$ is observed over the same direction (45° here). In other words, the STCS curves formed for the opposite and same directions of movement envelope the curves for all other directions (Figure 2b,d and Figure 3b,d).

**Figure 3.** Dimple in time. Example of a spatiotemporal correlation structure (STCS), the Ali-Mikhail-Haq-Weibull shown in Figure 1a, modified for a moving field with speed $\mathbf{v}_{x,y} = (4,4)$. For distance $\delta > 0$ the STCS peaks at nonzero time lags in the opposite direction of the movement: (a) values of the STCS (fixed distance $\delta = 4\sqrt{2}$) vs. time lag and direction and (b) tomography for specific directions; (c,d) similar to (a,b) but for fixed distance $\delta = 12\sqrt{2}$.

To simulate Lagrangian Gaussian RF’s we use multivariate autoregressive models of order $p$ (MAR($p$)) modifying the approach for static RF’s simulation described in Papalexiou and Serinaldi (2020). Specifically:

1. An RF is described by a set of rv’s $Z_{i,j}$ over an $n \times n$ grid (see the example in Figure A2a in Annex A).
2. Given the Cartesian coordinates of the $Z_{i,j}$ rv’s we compute distance matrices $D_L(\tau)$ based on the Lagrangian distance $\delta_L$ in equation (5). For $\tau = 0$ the distance matrix is the Euclidean distance matrix (Figure A2b), but for $\tau > 0$ distances differ based on the velocity $\mathbf{v}_{x,y}$ (Figure A2c) and the value of $\tau$ (for example see the $D_L(1)$ in Figure A2d).
3. The elements $\delta_L$ of the distance matrices $D_L(\tau)$ are plugged into the $\rho$ expression to obtain the STCS correlation values $\rho_L = \rho(\delta_L, \tau; \Theta)$ forming the Lagrangian correlation matrices $K(\tau)$.
4. The $K(\tau)$ matrices for $\tau = 0, \ldots, p - 1$ are therefore plugged into the multivariate Yule-Walker equations (e.g., Lütkepohl, 2005) to estimate the MAR($p$) parameters.
5. The MAR($p$) model allows the simulation of vectors of $n^2$ rv’s that preserve the desired STCS and the velocity vector.

We demonstrate a simulation of Lagrangian Gaussian RF’s (Figure 4; Movie 4) over a 75 × 75 grid with advection velocity $\mathbf{v}_{x,y} = (4,4)$ described by the Lagrangian Ali-Mikhail-Haq-Weibull STCS (Figure 2 and Figure 3). The movement over the 45° axis is apparent with a high intensity region moving from the lower left corner towards northeast.
Figure 4. Demonstration of Lagrangian Gaussian random fields with advection velocity \( \mathbf{v}_{x,y} = (4,4) \) described by the Ali-Mikhail-Haq-Weibull spatiotemporal correlation structure (see Movie 4).

4. Simulating Non-Gaussian Random Fields

4.1 Non-advective Non-Gaussian Random Fields

Here, we summarize the CoSMoS framework of Papalexiou and Serinaldi (2020) for simulating non-Gaussian RFs that preserve any probability distribution and spatiotemporal correlation structure. A continuous spatiotemporal RF is discretized and represented over an \( n \times n \) grid by \( n^2 \) rv's \( \{X(s,t): s \in \mathbb{R}^2, t \in \mathbb{R}\} \). The RF is characterized by the probability distributions \( F_X(x) \) of \( X(s,t) \) and the STCS \( \rho_X(\delta, \tau) \) that defines the correlations between pairs of \( X(s,t) \). Such a field is generated by transforming a Gaussian RF \( \{Z(s,t): s \in \mathbb{R}^2, t \in \mathbb{R}\} \), with \( Z(s,t) \) following the standard Gaussian distribution \( \mathcal{N}(0,1) \) and a specific STCS \( \rho_Z(\delta, \tau) \) describing the correlations between \( Z(s,t) \) pairs. The \( \rho_Z(\delta, \tau) \) is specified based on the desired target distribution \( F_X(x) \) and STCS \( \rho_X(\delta, \tau) \) (Papalexiou, 2018; Papalexiou et al., 2018; Papalexiou & Serinaldi, 2020).

Briefly, an rv \( Z \sim \mathcal{N}(0,1) \) can be transformed to \( X \sim F_X(x) \) by applying \( X := Q_X(F_Z(Z)) \), where \( Q_X \) is the quantile function of the rv \( X \) and \( F_Z(z) \) is the standard Gaussian cdf. Yet this transformation is non-linear and when applied to a bivariate Gaussian rv \( (Z_i, Z_j) \) with correlation \( \rho_{Z_i,Z_j} \), leads to a bivariate rv \( (X_i, X_j) \) with correlation \( |\rho_{X_i,X_j}| < |\rho_{Z_i,Z_j}| \) due to the maximal property of the bivariate Gaussian distribution (Gebelein, 1941; Lancaster, 1957; Maung, 1941). Analytical expressions linking \( \rho_{X_i,X_j} \) and \( \rho_{Z_i,Z_j} \) exist only for specific cases including the uniform (Baum, 1957) and the Lognormal (Matalas, 1967) distributions. In the past, numerical or asymptotic approximations were used based on Hermite-Chebyshev polynomial expansions (van der Geest, 1998; Lancaster, 1957).

Here, we use the scheme introduced in Papalexiou (2018) and extended to STCS in Papalexiou and Serinaldi (2020). The link between correlations of marginally Gaussian and non-Gaussian processes is formed by using a correlation transformation function (CTF)

\[
\rho_z = T(\rho_x; b, c) = \frac{(1 + b\rho_x)^{1-c} - 1}{(1 + b)^{1-c} - 1}
\]  

(7)

where \( b > 0 \) and \( c \geq 0 \) are parameters (estimation details are given in Papalexiou (2018), and software implementation in CoSMoS-R package (Papalexiou et al., 2019)). This function
yields the values $\rho_{z_i,z_j}$ corresponding to the desired $\rho_{x_i,x_j}$ of the target variables resulting from the transformation $X := Q_X(\Phi_Z(Z))$.

Papalexiou and Serinaldi (2020) proposed two variations on using the CTF; here, the second one is used, that assumes a well-defined STCS $\rho_Z(\delta, \tau)$ for the Gaussian RF’s, and defines the STCS of the target process as $\rho_X(\delta, \tau) = T^{-1}(\rho_Z(\delta, \tau))$. This approach has two advantages: (1) it guarantees that the Gaussian RF’s are described by positive definite STCS that enables their simulation; and (2), in the case of Lagrangian RF’s discussed in the next section, it allows the characterization of the target process by the STCS of the parent non-Lagrangian Gaussian RF’s. This overcomes the inconvenience of the STCS’s of the target process, which are spatially varying, and are not described by simple STCS expressions in a Lagrangian coordinate system.

Therefore, hereafter, we characterize a space-time process (in its full extent) by four components: (1) the STCS of a parent Gaussian RF, which is a positive definite parametric STCS, (2) the marginal probability distribution of the target RF’s, (3) the velocity vector field describing the advection, and (4) the anisotropy of the target RF’s.

We illustrate a step-by-step example (Figure 5) of non-Gaussian RF’s simulation with any probability distribution and STCS. This scheme is generalized next to incorporate general velocity fields of advection and anisotropy. We generate RF’s having a J-shaped $Br\times II$ distribution (equation A1) with parameters $(\beta, y_1, y_2) = (3, 0.8, 0.25)$ and a Clayton-Weibull STCS with parameters $\theta = (20, 1, 10, 1, -0.5)$ describing the Gaussian RF’s. We highlight the following steps:

1. Figure 5a: estimate the correlation tranformation function $\rho_Z = T(\rho_X; b, c)$ for the target distribution $F_X(x)$ of the non-Gaussian RF’s; the estimated parameters of equation (7) corresponding to the target $Br\times II$ are $(b, c) = (2.57, 0.99)$. This function creates the link between non-Gaussian and Gaussian correlations.
2. Figure 5b: define the STCS of the non-Gaussian RF’s as $\rho_X(\delta, \tau) = T^{-1}(\rho_Z(\delta, \tau))$, where $\rho_Z(\delta, \tau)$ is a positive definite parametric STCS; here, $\rho_Z(\delta, \tau)$ is the Clayton-Weibull STCS.
3. Figure 5c: obtain the inflated STCS of the Gaussian RF’s $\rho_Z(\delta, \tau)$ corresponding to $\rho_X(\delta, \tau)$ (Figure 5b). For example, two non-Gaussian rv’s distanced by $\delta = 10$ and lagged by $\tau = 3$ have correlation $\rho_X(10,3) = 0.49$ while the corresponding Gaussian rv’s have $\rho_Z(10,3) = 0.63$.
4. Figure 5d: Gaussian RF’s are generated with the STCS in Figure 5c. We fit a multivariate autoregressive model MAR(1) to the STCS and generate RF’s in a $75 \times 75$ grid, thus, the multivariate model comprises $75^2$ rv’s.
5. Figure 5e,f,g: the values of the Gaussian RF’s are transformed to non-Gaussian values by applying the transformation $X := Q_X(\Phi_Z(Z))$. Figure 5f demonstrates how a Gaussian value is transformed to a non-Gaussian one. The arrows show a Gaussian $z = 0.8$ transformed to $x = 4.8$. As anticipated, the values of the Gaussian RF’s follow the standard Gaussian distribution (Figure 5e), while the transformed ones follow the desired $Br\times II$ distribution (red line in Figure 5g).
6. Figure 5h: the non-Gaussian RF’s, transformed from the Gaussian ones (Figure 5d), have the desired STCS (Figure 5b) and the desired marginal distribution (Figure 5g).
Figure 5. Generating non-Gaussian random fields (RFs): (a) correlation transformation function (CTF) linking non-Gaussian and Gaussian correlation; (b) Target spatiotemporal correlation structure (STCS) of RF’s with non-Gaussian marginal distribution; (c) Gaussian STCS after applying the CTF to the non-Gaussian STCS; (d) five generated $75 \times 75$ Gaussian RF’s with Gaussian STCS; (e) histogram of the values comprising the Gaussian RF’s; (f) transforming the Gaussian rv $Z$ to any non-Gaussian $X$ through the transformation $X = Q_X(\Phi(Z))$; (g) histogram of the transformed values compared with the target (red line) distribution; (h) non-Gaussian fields having the desired STCS in (a) and the desired marginal distribution in (g).

4.2 Lagrangian and Non-Gaussian Intermittent Random Fields
The previous non-Gaussian RF’s (Figure 5h) evolve in time without a systematic movement towards a specific direction, or else, their velocity vector is $\mathbf{v}_{x,y} = (0,0)$. We can generate Lagrangian non-Gaussian RF’s by using Lagrangian Gaussian RF’s with the aforementioned CoSMoS framework. In particular, the parent Gaussian RF’s are simulated (see section 3) based on Lagrangian STCS’s to account for advection velocity; plugging these RF’s into the CoSMoS algorithm leads to RF’s that move in time with constant velocity $\mathbf{v}_{x,y}$.

Here we demonstrate simulation of intermittent Lagrangian RF’s with mixed-type marginal distributions with probability mass at zero $p_0$ and continuous distributions describing positive values. Fields of precipitation, wind and so forth, are typically intermittent in space and time, and thus, can be described by mixed-type distributions. The
method of using any mixed-type marginal to generate time series with prescribed correlation structure was introduced in Papalexiou (2018) and generalized for RF’s in Papalexiou and Serinaldi (2020). In particular, Papalexiou (2010) discussed the case of mixed-type marginals with probability mass at arbitrary points, while Papalexiou and Serinaldi (2020) highlighted the case of mixed-type marginals with probability mass at the boundaries of the sample space and continuous distribution in-between. Here, we recall the expressions of the mixed-type cdf \( F_X(x) \) and quantile function \( Q_X(u) \) with probability zero \( p_0 \), that is,

\[
F_X(x) = (1 - p_0)F_{X|X>0}(x) + p_0 \quad x \geq 0
\]

\[
x_u = Q_X(u) = \begin{cases} 
0 & 0 \leq u \leq p_0 \\
Q_{X|X>0}(u - p_0) & p_0 < u \leq 1
\end{cases}
\]

where \( X|X > 0 \) denotes the conditional non-Gaussian rv which is defined in \((0, \infty)\) in this case, and \( u = \Phi(z) \) are probabilities corresponding to standard Gaussian values. Specifically, the mixed-type quantile function is used to transform the Gaussian RF’s.

We demonstrate simulation of Lagrangian intermittent non-Gaussian RF’s with the following settings: (1) the Gaussian RF’s are described by the Ali-Mikhail-Haq-Weibull STCS with \( \theta = (28,1.2,25,1,0.5) \), (2) the target distribution is a mixed-type \( \mathcal{B}_r\mathcal{XII} \) with \( (\beta, \gamma_1, \gamma_2) = (3,0.9,0.1) \) and probability zero \( p_0 = 0.4 \), and (3) the advection velocity is \( \mathbf{v}_{x,y} = (10,10) \). The rationale of the mixed-type marginal transformation is sketched in Figure 6a,b,c, displaying how values from Gaussian RF’s are transformed to zeros and non-zeros. Probability zero \( p_0 \) corresponds to \( z_{p_0} = \Phi(p_0) \) and thus, all generated \( z \leq z_{p_0} \) are transformed to zeros, while \( z > z_{p_0} \) are transformed to positive values (Figure 6b) following the target probability distribution (here, the \( \mathcal{B}_r\mathcal{XII} \); Figure 6c). The simulated RF’s (Figure 6) reveal the north-eastward advection with a characteristic high-intensity cell moving from the lower-left corner towards the upper right corner (see Movie 6). The same approach can be used to generate RF’s having any type of marginal, including mixtures of continuous distributions, discrete-continuous with probability masses at arbitrary points, as well as discrete and binary.
Figure 6. Generating intermittent non-Gaussian random fields (RFs) with constant velocity. (a) histogram of Gaussian RF's values; (b) sketch on how Gaussian values are transformed to intermittent non-Gaussian using the mixed-type distribution; Gaussian values smaller than $z_{p_0}$ are transformed to zeros; (c) histogram of transformed non-zero values compared with the target (red line) $BrXII$ distribution; (d) intermittent non-Gaussian fields with desired velocity, STCS and mixed-type probability distribution (see Movie 6).

5. Advancing Random Fields Simulation

Over the next sections we advance spatiotemporal modelling of RF's by coupling them with velocity fields and general forms of anisotropy. We show that advection velocity (defined as speed and direction of mass) can vary in space (or even in time) and described by a velocity field. A velocity field associates a vector describing the direction and the speed of the velocity to each point of space. We use these vectors to represent the advection velocity at each grid node of the discretized domain. Velocity fields are ubiquitous in nature and describe, for example, the movement of fluid or gas particles in space and time. Here, we focus on static velocity fields that could express for example a steady state system, yet we can also extend the simulations by using velocity fields changing in time. Our approach to couple RF with velocity fields is based on the idea that the velocity speed and direction at each point of the field modifies the STCS based on the Lagrangian distance $\delta_L$ in equation (6). In this case, and in contrast to constant velocity (fixed speed and direction), the Lagrangian distance $\delta_L$ depends both on the direction between two points and their absolute coordinates; yet $\delta_L$ is expressed by the same formula.

Next, we demonstrate RF's simulations coupled with velocity fields and anisotropy having the potential to model a wide-range of natural phenomena including fluid or gas
motion. We show cases of RF’s that rotate, have low and high speeds in different regions, converge from opposite directions, converge to a point, or, diverge from a point, and mimic cyclones and storms. Each RF is described by four components: (1) the STCS of the parent Gaussian process, (2) the marginal distribution, (3) the velocity field, and (4) the anisotropy. Our aim in the following demonstrations is to show the potential of the modelling framework and not to model actual cases based on observed data. Consistent and reliable estimation of advection, anisotropy, and STCS from real data is challenging and out of this study’s scope. For brevity, we summarize the above four components for each experiment (Table 1) and avoid repeating details in the next sections.

**Table 1.** Summary of the 11 simulation experiments. For the marginal distribution formulas see equations A1-A3 in Annex A.

| Section | Potential application | Type | Parameters $b_0, c_0, b_0, c_0, \theta$ | Marginal distribution | Advection | Anisotropy | Fig.# | Mov.# |
|---------|-----------------------|------|----------------------------------|----------------------|-----------|-----------|-------|-------|
| 6.1     | Rotating fields       | CW   | 30,1,40,1,0,1                    | $p_0 = 0.5$          | Clockwise rotation | Isotropic |       |       |
| 6.2     | Fields with low-high speeds | AMHW | 25,1,10,1,0                      | $Br XII(3,1,0,1)$   | Variable speed and direction | Isotropic |       |       |
| 6.3     | Colliding air masses  | AMHW | 25,1,10,1,0                      | $Br XII(3,1,0,1)$   | Variable speed and direction over hyperbolic lines | Isotropic |       |       |
| 6.4     | Mass source, heating gas | AMHW | 30,1,17,1,0                      | $\xi(5)$            | Radiant velocity diverging from the center | Isotropic |       |       |
| 6.4     | Mass sink, cooling gas | AMHW | 25,1,20,1,0.3                    | $\xi(5)$            | Radiant velocity converging to the center | Isotropic |       |       |
| 7.1     | Fields with low-high speeds | AMHW | 25,1,20,1,0                      | $W(3,1,2)$          | Variable velocity | Affine |       |       |
| 7.2     | Cyclones              | CW   | 25,1,20,1,0                      | $Br XII(3,1,2,0.1)$ | Variable velocity converging to the center | Swirl |       |       |
| 7.3     | Wavy patterns         | CW   | 25,1,20,1,0                      | $Br XII(3,1,2,0.1)$ | Constant horizontal | Wavy |       |       |
| 8.2     | Storms, clouds        | AMHW | 25,1,20,1,0.3                    | $p_0 = 0.5$          | Constant diagonal | Isotropic |       |       |
| 8.2     | Storms                | AMHW | 25,1,20,1,0.3                    | $p_0 = 0.5$          | Constant diagonal | Affine |       |       |
| 8.2     | Storms, air masses    | AMHW | 25,1,20,1,0.3                    | $p_0 = 0.5$          | Constant speed varying direction | Affine |       |       |

6. Introducing General Advection based on Velocity Fields

6.1 Rotating Random Fields.

The first demonstration regards rotating fields over a center point with velocity vectors described by uniform curl. The velocity vector at a point $(x, y)$, in an $n \times n$ field, is given by

$$\mathbf{v}_{x,y} = \left( w_x (y - y_0), -w_y (x - x_0) \right)$$

where $w_x$ and $w_y$ are scaling factors, and $(x_0, y_0) = (n/2, n/2)$ is the center of the velocity field. The velocity speed for $w_x = w_y$ is equal over fixed-radius circles; this leads to clockwise
rotating fields with zero speed in the center and maximum speed in the corners (see velocity vectors in Figure 7a). Note, for \( w_x \neq w_y \) the vectors’ magnitudes are equal over fixed ellipses. Here, we simulate intermittent, rotating, non-Gaussian RF’s (see Table 1 for details) described by the velocity field in equation (10) for \( w_x = w_y = 1/3 \). Inspection of a few frames (Figure 7) shows that the intermittent fields rotate clockwise; for example, the high intensity region spotted on the lower left (Figure 7.42) moves in circular way and ends up in the lower right corner (Figure 7.58) after completing a full circle.

**Figure 7.** Simulation of rotating, intermittent and non-Gaussian random fields in a \( 75 \times 75 \) grid. (a) Velocity vectors with uniform curl describing the advection of the random fields; (42-58) a sequence of generated fields (Movie 7 clearly demonstrates the rotation).

### 6.2 Low-high Velocity Speed

A field with potential applications is one where the advection has low speed in a region and high in another. We demonstrate RF’s with low-high speeds that are described by the velocity field

\[
v_{x,y} = (w_x y, w_y x)
\]

(11)

For \( w_x = w_y \) velocity vectors are symmetric with respect to the main diagonal \( y = x \). The speed increases progressively from zero in the lower left corner to its maximum in the upper right corner (see vectors in Figure 8a). The velocity direction progressively changes from \( 90^\circ \) in the lower right corner to \( 45^\circ \) in the upper right corner, and from \( 0^\circ \) in upper left to \( 45^\circ \) in the upper right. We generate 100 RF’s (see Table 1 for details) based on the previously described velocity field using \( w_x = w_y = 7/\sqrt{2} \times 75^2 \) (the highest advection speed is 7 and spotted in the upper corner in an \( 75 \times 75 \) grid). The simulation shows horizontal advection from left to right in the upper part (see e.g., the high intensity region in Figure 8.4-7), and vertical advection from the lower right corner towards the top (see e.g. Figure 8.9-12). The almost zero speed in the lower left corner and the high speed in the upper right corner are clearly demonstrated in Movie 8.
6.3 Hyperbolic Velocity Fields

There are cases where mass (e.g., gas, fluid, etc.) moves from opposite directions towards a point (colliding) and must escape from it in different directions to preserve mass balance. Such a velocity field could be described by

$$\mathbf{v}_{x,y} = (w_x(y - y_0), w_y(x - x_0))$$  \hspace{1cm} (12)

with \((x_0, y_0) = (n/2, n/2)\). For \(w_x = w_y\) the velocity vectors are symmetric over the diagonals \(y = x\) and \(y = -x\). This case could fit to mass moving from northwest and southeast corners towards the center; mass speeds cancel out in the center and to preserve mass balance (if accumulation is not allowed) mass leaves the region from the southwest and northeast corners (see velocity vectors in Figure 9a). The advection “streamlines” are hyperbolas and their conjugates; the maximum speed, observed in the corners, is \(\|\mathbf{v}_{n,n}\| = w\sqrt{2}n\) for an \(n \times n\) field and \(w = w_x = w_y\). We simulate fields (see Table 1) coupled with this parabolic velocity field (Figure 9a) for \(w_x = w_y = 3\sqrt{2}/n\); thus, the highest advection speed is 3. Inspection of the simulated fields (Figure 9) reveals the advection over the parabolas; yet the movement is clearly shown in Movie 9.
Figure 9. Simulation of non-Gaussian random fields with hyperbolic advection. (a) Velocity vectors describing advection converging over the $y = -x$ diagonal and diverging over the $y = x$ diagonal; (73-89) a sequence of generated fields (see Movie 9).

6.4 Radial Velocity Fields: Divergence and Convergence
Velocity fields with positive or negative divergence are omnipresent in nature and relate with movement of fluid or gas. For example, heated air expands towards all directions and the movement of air mass forms a velocity field. In such fields, there is a point that acts like a source as the flux moves outwards from it in all directions, or else, the velocity vectors point outwards as shown in Figure 10a. In physics this is called positive divergence and the velocity vectors can be described by

$$v_{x,y} = \left( w_x (x - x_0), w_y (y - y_0) \right)$$  \hspace{1cm} (13)

Here the divergence point is set in the center of the RF, that is, $(x_0, y_0) = (n/2, n/2)$. For $w_x = w_y$ the velocity vectors are symmetric over the center and have equal speed over circles with same radius (Figure 10a). Here we demonstrate generation of RF’s having positive divergence for $w_x = w_y = 6\sqrt{2}/75$. A characteristic sequence of simulated RF’s (Figure 10.1-5) shows the divergent advection. A region of low intensity in the center of the field, surrounded by clusters of higher intensity (Figure 10.1), expands outwards over the next frames pushing the high intensity clusters out of the field (see Movie 10).
Figure 10. Simulation of non-Gaussian random fields with positive divergence from the center. (a) Velocity vectors describing the positive divergence from the center towards all directions; (1-11) a sequence of generated fields (Movie 10 clearly shows the expansion).

In contrast to the previous demonstration, velocity fields of gas or fluid can have negative divergence, or else, converge to a point. For example, cooling air contracts; the local decrease of the gas volume due to cooling combined with the external pressure, forces the gas particles to move inward or towards the low-pressure region (negative divergence). In general, such velocity fields can occur when a point acts like a sink causing flux to move towards it from all directions (Figure 11a). Velocity vectors with negative divergence can be described by

\[ \mathbf{v}_{x,y} = \left( w_x(x_0 - x), w_y(y_0 - y) \right) \]  

Similarly, to positive divergence, the velocity vectors in negative divergence for \( w_x = w_y \) are symmetric over the convergence point and have equal speed over fixed-radius circles (Figure 11a). Here, we simulate convergence RF’s for \( w_x = w_y \) having the same properties as the divergence case (see Table 1). Note that in the Supplementary Material we report general and compact representation of the velocity fields’ equations with the examples presented here being special cases. A representative sequence of generated RF’s (Figure 11.45-49) shows the negative divergent advection. A few clusters of high intensity, surrounding a large low-intensity region in the center of the field (Figure 11.45) move progressively towards the center decreasing eventually its extent (e.g., Figure 11.49). The convergence is better demonstrated in Movie 11.
Figure 11. Simulation of non-Gaussian random fields with negative divergence (convergence) to the center. (a) Velocity vectors convergence to the center from all directions; (45-55) sequence of generated fields (Movie 11 shows clearly the convergence).

7. Introducing General Anisotropy Coupled with Velocity Fields

7.1 Affine Anisotropy

Referring to Allard, Senoussi and Porcu (2016) for a detailed review of anisotropy in geostatistics, we can identify three elementary models called geometric, zonal, and separable, which can be combined to provide more complex anisotropies (Journel & Froidevaux, 1982). Among these elementary models, the geometric one is probably the most used in literature as discussed in section 1. It involves an affine transformation of the Cartesian coordinate space according to the relationship

\[
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix} = \begin{pmatrix}
\kappa_x & 0 \\
0 & \kappa_y
\end{pmatrix} \begin{pmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

(15)

where \( \kappa_x \) and \( \kappa_y \) scale the coordinates of an \( s = (x, y)^T \) point, and \( \omega \) is the angle of counterclockwise rotation around the origin (e.g., Chilès & Delfiner, 2009).

In the previous section, we demonstrated RF’s with advection described by velocity fields. This method can be combined also with anisotropy. Here, we combine affine anisotropy with the velocity field described in equation (13) with \((x_0, y_0) = (0,0)\) and \(w_x = w_y = 10/(75\sqrt{2})\). This leads to a velocity field with minimum speed in the lower-left corner and maximum in the upper right while the velocity direction is 0°, 45°, and 90° in the lower-right, upper-right, and upper-left corners respectively (Figure 12a). Particularly, we generate non-Gaussian RF’s that combine the velocity field in Figure 12a and affine anisotropy with parameters \( \kappa_x = 1/\sqrt{3}, \kappa_y = \sqrt{3}, \) and \( \omega = \pi/4 \) (Figure 12b). The affine anisotropy is apparent in the generated RF’s (Figure 12), while Movie 12 better shows that the 75 × 75 generated RF’s follow the prescribed velocity field. We clarify that Figure 12b shows how a regular rectangular grid is transformed after applying affine anisotropy (equation (15)) to its coordinates (see also Figure A2c,d for an intuitive explanation).
Figure 12. Simulation of random fields with advection velocity field and affine anisotropy. (a) Velocity vectors describing the advection; (b) affine anisotropy; (10-25) sequence of generated fields (see Movie 12).

7.2 Simulation Mimicking Cyclones

The rigid “stretching-and-rotation” geometric transformation can be generalized in the spirit of velocity fields by introducing locally varying scaling and rotation factors, that is, using deformation fields based on local affine transformations (Ligas et al., 2019). This method enables the simulation of fields with convenient anisotropy mimicking for instance cyclonic shapes via a swirl–like coordinate transformation given by

\[
\begin{align*}
\tilde{x} &= (x - x_0)\cos\left(e^{-\frac{(r^2)}{\kappa}} \omega\right) - (y - y_0)\sin\left(e^{-\frac{(r^2)}{\kappa}} \omega\right) + x_0 \\
\tilde{y} &= (x - x_0)\sin\left(e^{-\frac{(r^2)}{\kappa}} \omega\right) + (y - y_0)\cos\left(e^{-\frac{(r^2)}{\kappa}} \omega\right) + y_0
\end{align*}
\]

where \( r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \) is the distance of a point \((x, y)\) from the center of the deformation \((x_0, y_0)\), \(\omega\) is a rotation angle, and \(\kappa\) is a scaling parameter controlling the swirl extension. Figure 13b provides an example of this deformation by transforming \((x, y)\) points lying over a rectangular 22 × 22 grid to \((\tilde{x}, \tilde{y})\) points for \(\omega = 3\pi/2, \kappa = 18\), and \(x_0 = y_0 = 0\).

Velocity fields can be combined to form new ones with desired properties. Here we combine rotating (Figure 7a) and converging to the center (Figure 11a) velocity fields leading to vectors described by

\[
\mathbf{v}_{x,y} = (-w_1(x - x_0) + w_2(y - y_0), -w_1(y - y_0) - w_2(x - x_0))
\]

where \(w_1\) and \(w_2\) are weight factors for each one of the two velocity fields. This velocity field (Figure 13a) when combined with swirl anisotropy (Figure 13b) could mimic the behavior of cyclones.

Here, we demonstrate a non-Gaussian (see Table 1) cyclone-mimicking simulation. The velocity field (Figure 13a) is specified by \(w_1 = 2\sqrt{2}/(15\sqrt{5})\) and \(w_2 = \sqrt{2}/(15\sqrt{5})\) so the
maximum speed in the corner is 5, and the swirl anisotropy has parameters $\omega = 3\pi/2$ and $b = 18$ (Figure 13b). The simulation shows clear cyclonic advection and swirl anisotropy (Figure 13). The RF’s rotate but at the same time masses move towards the center as it is clearly shown in Movie 13.

**Figure 13.** Simulation random fields resembling cyclones. (a) Velocity vectors describing the rotation and the convergence of the advection to the center; (b) swirl anisotropy; (82-97) a sequence of generated fields (Movie 13 shows clearly the cyclonic movement).

### 7.3 Wavy Anisotropy

We demonstrate the potential of coupling advection with new forms of anisotropy. We experiment here by creating a form of wavy anisotropy described by

\[
\tilde{x} = \frac{x}{\kappa_1} \cos(\omega) - y \sin(\omega) \quad (19)
\]

\[
\tilde{y} = x \sin(\omega) + \frac{y}{\kappa_2} \cos(\omega) + \kappa_3 \sin(x) \quad (20)
\]

where the parameters $\kappa_1$, $\kappa_2$, $\kappa_3$ and $\omega$ control the characteristics of the “wave”. Wavy patterns are omnipresent in nature and are observed in sand and soil, wood texture, rock formations, surface of liquids, etc. Figure 14b provides an example of this wavy deformation by showing a transformed rectangular grid of $(x, y)$ points to $(\tilde{x}, \tilde{y})$ points using $\omega = 0$, $\kappa_1 = 2$, $\kappa_2 = 1/2$, $\kappa_3 = 3$. We simulate non-Gaussian wavy RF’s fields with constant velocity vector $v_{x,y} = (3,0)$ (Figure 14a) and the previously described wavy anisotropy (Figure 14b). The simulation shows the wavy patterns and the horizontal advection (Figure 14; Movie 14).
8. Simulating Realistic Rainfall Storms at any Spatiotemporal Scale

8.1 Existing Space-time Rainfall Models Featuring Advection and Anisotropy

Paschalis et al., (2013) proposed a rainfall generator called STREAP (Space-Time Realizations of Areal Precipitation), which is based on the so-called “String of Beads” model (Pegram & Clothier, 2001), and forms one of the components of the so-called Advanced WEather GENerator for a two-dimensional grid (AWE-GEN-2d; Peleg et al., 2017). STREAP assumes lognormal marginal distributions and relies on the simulation of latent Gaussian RF’s to account for spatiotemporal correlation. As the RF’s are simulated by Fast Fourier Transform (FFT), FFT symmetries are exploited to fold the simulated fields and mimic advection. The model seems not to incorporate anisotropy, FFT generation requires the simulation of much larger fields to avoid numerical artifacts due to field symmetries, and marginal back-transformation from Gaussian to Lognormal space introduces bias in the spatiotemporal correlation structures. Leblois and Creutin (2013) proposed a rainfall generator called SAMPO (Simulation of Advect mesoscale Precipitations and their Occurrence), which relies on the simulation of latent Gaussian RF’s similar to STREAP. In this case, latent fields are generated by a three-dimensional turning band method, which allows the introduction of spatiotemporal correlation, space-time anisotropy (to fulfil the so-called Taylor hypothesis (Taylor, 1938)), and advection (by using the concept of back-trajectories). Unlike STREAP, SAMPO reproduces the target spatio-temporal correlation by inflating the correlation structure of the latent Gaussian process via an anamorphosis function based on standardized Chebyshev-Hermite polynomials. SAMPO yields rainfall fields with inverse-Gaussian marginal distribution, which is skewed but asymptotically exponential, whereas rainfall tends to exhibit subexponential (heavy) tails at daily and subdaily scales of interest (Cavanaugh et al., 2015; Moustakis et al., 2021; Nerantzaki & Papalexiou, 2019; Papalexiou et al., 2013; Papalexiou & Koutsoyiannis, 2016; Papalexiou et al., 2018; Rajulapati et al., 2020; Serinaldi & Kilsby, 2014). Similar to STREAP, SAMPO requires the simulation of fields...
covering larger areas to simulate advection effects. Benoit, Allard and Mariethoz (2018) proposed a model conceptually similar to SAMPO, in which the field generation is based on Cholesky decomposition and sequential simulation, while the skewed distribution of positive rainfall is approximated by a power transformation of latent Gaussian fields.

Latent Gaussian fields are also the core of the model proposed by Niemi et al. (2016), which generalized the so-called STEPS model (Short Term Ensemble Prediction System; Bowler, Pierce and Seed, 2004; Seed, Pierce and Norman, 2013) introducing scale-varying anisotropy by GSI (hereinafter STEPS-GSI). Similar to STREAP, the STEPS-GSI model assumes lognormal distribution for positive rainfall, requires the generation of larger fields (as it uses FFT simulation), a post-processing of simulated rainfall to adjust field mean and standard deviation. The model does not explicitly address the problem of correlation bias due to logarithmic back-transformation of the latent (approximately) Gaussian fields. This problem also affects the nonstationary stochastic rainfall generator based on the short-space Fourier transform (SSFT) described by Nerini et al. (2017). The SSFT-based model enables performing a local moving window Fourier analysis to account for changes in the anisotropy and extent of correlation structures across a field. This approach differs from STEPS-GSI, which accounts for changes across scales (e.g., local cell structures within rain bands) instead of changes across a geographic area. Similar to STREAP and STEPS-GSI, the SSFT-based model assumes approximately lognormal distribution for positive rainfall, requires post-processing of simulated fields to adjust field mean and standard deviation, and does not incorporate advection.

8.2 Storm Simulation

The concepts introduced in the previous sections can be used to advance stochastic simulation of storms and weather systems. Combining such concepts results in a framework that offers several advantages and advances:

1. It allows generating storm fields described by any non-Gaussian distribution and spatiotemporal correlation structure preserving intermittency.
2. It is scale free, that is, it can reproduce the characteristics of storms at any temporal and spatial scale given that these characteristics are specified.
3. It allows general advection expressed by velocity fields with locally varying speed and direction.
4. It supports the use of general anisotropies beyond affine transformations.
5. It enables the generation of any number of RF’s since the scheme is not based on pre-generated large fields limiting the length of simulation, and
6. since the proposed method relies on multivariate AR models, it is free from the constraint of generating RF’s over a square region ($n \times n$ grids). Indeed, the procedure generates only rv’s at specified spatial coordinates, which can define a regular or irregular grid as well as the nodes of a gauging network.

Here we demonstrate storm simulation by progressively adding more components in the stochastic model, showing three characteristic cases.

**Constant-velocity, isotropic storms.** Over large regions, storms might have varying velocity (both in speed and direction). Yet in small regions, constant velocity could be a valid assumption. Additionally, while radar observations typically show some form of affine anisotropy there are cases that storms can have isotropic patterns. Here, we mimic an isotropic and constant velocity storm by generating intermittent non-Gaussian RF’s (see
Table 1) with constant advection velocity $\mathbf{v}_{x,y} = (7,7)$. The strong space-time correlation is apparent in the simulation (Figure 15; Movie 15) and enables clusters of high intensity cells to persist in time, and intermittency (zero and non-zero regions) to persist in space. A characteristic cluster of high intensity cells (Figure 15.30-39) reveals the north-eastward advection velocity entering the region from lower-left corner and exiting from the upper-right corner (see also Movie 15).

**Figure 15.** Simulation of constant-velocity, isotropic storm. 100 × 100 storm fields described by the Ali-Mikhail-Haq-Weibull STCS, the $B_r$-XII distribution, and movement from south-west to north east (see Movie 15).

**Constant-velocity, anisotropic storms.** As mentioned above, most storm radar observations at fine spatiotemporal scales (e.g., 1 km and 10 min), or even satellite observations at larger spatial scales, indicate that storm cells or precipitation patterns exhibit some form of affine anisotropy (see section 7.1). Assuming constant velocity, we can easily add affine anisotropy in the simulated intermittent RF’s. Here, we simulate a constant-velocity and anisotropic storm by adding affine anisotropy to the previous simulation with anisotropy parameters $\kappa_x = 2, \kappa_y = 1$, and $\omega = -\pi/4$ (see equation (15)). The anisotropy is apparent in the generated RF’s (Figure 16; Movie 16) and clearly observed in the cluster of high intensity cells. The selected values of $\kappa_x$ and $\kappa_y$ create elliptical anisotropy with the ratio of major to minor axis equal to two, while the rotation parameter $\omega$ makes the major anisotropy axis perpendicular to the advection velocity which in this case is $\mathbf{v}_{x,y} = (7,7)$.
Varying-velocity, anisotropic storms. The speed and direction of storm velocity can vary in space. In section 5, we showed examples of RF’s coupled with velocity fields. Similarly, we can use intermittent RF’s coupled with velocity fields (with or without anisotropy) to simulate storms. Keeping the same STCS and marginal distribution as in previous two examples, we demonstrate a storm simulation with varying-velocity and anisotropy. The affine anisotropy parameters in this case are set to $\kappa_x = 1/\sqrt{2}$, $\kappa_y = \sqrt{2}$, and $\omega = -\pi/4$ placing the major axis over the $y = x$ diagonal (see Figure 17b showing how a regular square grid is transformed). In contrast to the prior examples, where velocity fields had constant speed and direction, we create a velocity field with constant speed and varying direction (Figure 17a). This, in general, is achieved by dividing the horizontal ($v_x$) and vertical ($v_y$) velocity components with the velocity magnitude $\|v_{x,y}\| = \sqrt{v_x^2 + v_y^2}$ leading to

$$v_{x,y} = \left( w_x \frac{v_x}{\sqrt{v_x^2 + v_y^2}}, w_y \frac{v_y}{\sqrt{v_x^2 + v_y^2}} \right)$$

Here, we set $v_x = -(y - 2n - 1)$, $v_y = -(x - n - 1)$, and $w_x = w_y = 8$. This velocity field has constant speed at all points and curved “streamlines” with curvature and directions that vary over the region (Figure 17a). The generated RF’s (Figure 17; Movie 17) show the anticipated affine anisotropy and the movement of the fields over the “streamlines”. The advection is more clearly visualized in Movie 17.
Figure 17. Simulation and constant speed, varying direction, and anisotropic storm. 100 × 100 storm fields (a) Velocity vectors of constant speed and varying direction describing the advection; (b) transformed grid showing the affine anisotropy applied in this simulation; (57-72) sequence of generated fields (see Movie 17).

9. Discussion and Conclusions

This study advances the simulation of spatiotemporal random fields (RF’s) providing more realistic representations of rainfall, weather systems, or other hydro-environmental fluxes. We generalized the Complete Stochastic Modeling Solution (CoSMoS) framework, which allows the simulation of RF’s with specified non-Gaussian marginal distributions and spatiotemporal correlation structures. The proposed generalization is based on the concepts of general velocity fields and general anisotropy transformations. The former allows the introduction of locally varying advection, thus extending the commonly used uniform motion across an area or pure random motion. On the other hand, general anisotropy transformations yield space deformations going beyond the simple affine anisotropy transformation. Combining these two concepts enables the simulation of RF’s with complex structures/patterns and motion across a specified spatial domain, mimicking for instance spiraling vortices, sink or source effects, or colliding air masses. As proof of concept, we provided a set of simulations introducing progressively additional elements of complexity. We also proposed a new copula-based spatiotemporal correlation functions enabling great flexibility in modelling spatiotemporal dependence, and provide theoretical conditions for their validity, that is, their positive definiteness.

As the saying goes “Rome wasn’t built in a day”; this study focuses on model development and simulation strategies and offers a step forward towards a general and powerful stochastic modeling framework for environmental applications. Several topics require further investigation. Specifically, model inference, a crucial topic for real world applications, is not tackled here. As mentioned in the introduction, links among correlation, anisotropy, and advection, can be exploited to estimate the required parameters. Existing methods should be assessed in detail to form an estimation procedure that is consistent with the framework presented here; this explains why we did not report real-world examples.
Spatially varying marginal distributions and STCS (spatial non-stationarity) is another feature that can improve low-high pressure regions, and orographic effects on weather systems, for instance. Similarly, temporal non-stationarity that would allow deterministic evolution of the statistical properties of the RF’s was not considered. Temporal evolution of velocity fields is another aspect related to real-world applications that can improve the simulation of storm patterns or wind fields, for instance. Lastly, simulation at synoptic scales (more than 1000 km) at very fine spatiotemporal scales (e.g., 1 km) is challenging due to computational limitations. Some of these features are technically easy to implement, while others need extensive effort.

Extensions enabling the simulation over non-planar surfaces, such as quadratic surfaces (spheres and hyperboloids), are also relevant to open the door to stochastic simulation of geophysical fluxes at global scale. Furthermore, non-Gaussian dependence structures (e.g., using the t copula) can also be considered to account for quantile-varying spatial correlation, and upper tail dependence, thus better representing the different joint behavior of moderate and extreme values.

Finally, since this study extends the capabilities of CoSMoS, one of its natural applications is to generalize the DiPMaC algorithm (Precise Temporal Disaggregation Preserving Marginals and Correlations; Papalexiou et al. (2018)) that breaks down coarse-scale time series into finer scales. The improved field-compliant CoSMoS represents the core of an enhanced version of DiPMaC to allow downscaling weather products available at coarse spatiotemporal resolutions, such as Earth System Model and Regional Climate Model outputs, or gridded reanalysis and satellite products. All previous aspects are under investigation and the topic of future communications.

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Competing interests

The authors declare no competing interests.

Data Availability

The authors declare that no datasets were used in this study.

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Annex A: Abbreviations and Notation, Additional Equations and Figures

Table A1. Abbreviations and notation used in this study.

| Abbreviation | Description |
|--------------|-------------|
| AMHW | Ali-Mikhail-Haq-Weibull |
| CTF | Correlation transformation function |
| CW | Clayton-Weibull |
| RF | Random field |
| rv | Random variable |
| STCS | Spatiotemporal correlation structure |
| \( b, c \) | CTF parameters |
| \( b_s, c_s, b_T, c_T, \theta \) | STCS parameters |
| \( D_2(\tau) \) | Lagrangian distance matrix for time lag \( \tau \) |
| \( F_\xi(x) \) | Distribution function |
| \( K(\tau) \) | Correlation matrix for lag \( \tau \) |
| \( p_0 \) | Probability zero |
| \( Q_\xi(u) \) | Quantile function of the rv \( \xi \) |
| \( s = (x, y) \) | Point in the Cartesian plane |
| \( \mathcal{T} \) | CTF |
| \( \mathbf{v}_{x,y} = (v_x, v_y) \) | Velocity vector |
| \( w_x, w_y, \omega_1, \omega_2 \) | Scaling factors in velocity fields |
| \( \mathcal{X}, \mathcal{Z}, \mathcal{U} \) | Capital script letters denote rv’s (exceptions are the symbols \( \mathcal{T} \) and \( \mathcal{L} \)) |
| \( \phi(z) \) | Standard Gaussian pdf |
| \( \Phi(Z) \) | Standard Gaussian cdf |
| \( (x_0, y_0) \) | Center of velocity field or center of the swirl anisotropy |
| \( \rho_L(\delta, \tau) \) | Lagrangian STCS |
| \( \rho_L(\delta, \tau) \) | Transformed coordinate system due to anisotropy |
| \( \delta \) | Euclidean distance |
| \( \delta_L \) | Lagrangian distance |
| \( \kappa_x, \kappa_y, \omega \) | Affine anisotropy parameters |
| \( \kappa_1, \kappa_2, \kappa_y, \omega \) | Wavy anisotropy parameters |
| \( \kappa, \omega \) | Swirl anisotropy parameters |
| \( \rho(\delta, \tau) \) | STCS |

In this study, we used the Burr type XII \( \mathcal{B}r\mathcal{XII}(\beta, \gamma_1, \gamma_2) \), Exponential \( \mathcal{E}(\beta) \) and Weibull \( \mathcal{W}(\beta, \gamma) \) probability distributions with cdf’s given, respectively, by

\[
F_{\mathcal{B}r\mathcal{XII}}(x; \beta, \gamma_1, \gamma_2) = 1 - \left( 1 + \gamma_2 \left( \frac{x}{\beta} \right)^{\gamma_1} \right) \frac{1}{\gamma_1 \gamma_2}
\]

(A1)

\[
F_{\mathcal{E}}(x) = 1 - \exp \left( - \frac{x}{\beta} \right)
\]

(A2)

\[
F_{\mathcal{W}}(x) = 1 - \exp \left( - \left( \frac{x}{\beta} \right)^{\gamma} \right)
\]

(A3)

where \( \beta > 0 \) and \( \gamma \) denote scale and shape parameters. The \( \mathcal{B}r\mathcal{XII} \) has two shape parameters \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) that control, respectively, the left and right tail.
Figure A1. Demonstration of rv’s placed in a 3 × 4 grid and the definition of direction between two rv’s. For simplicity, the subscripts denote also the coordinates in space, e.g., the rv $X_{23}$ is located at the point $(x, y) = (2, 3)$.

Figure A2. Random variables in space and their corresponding distance matrices. (a) Nine rv’s $X_{i,j}$ in the Cartesian plane with $i, j$ indicating for their coordinates, and (b) the corresponding Euclidean distances among the rv’s. (c) In a Lagrangian framework the rv’s move, here for example with $v_{x,y} = (1, 1)$, (d) thus the corresponding distance matrix for lag $\tau > 0$ is modified based on the Lagrangian distance $\delta^L$; here distances for $\tau = 1$ are shown. (e) The rv’s variables in a transformed coordinate system $\overline{x}, \overline{y}$ with affine anisotropy $(\kappa_x, \kappa_y, \omega) = (2, 1, \pi/4)$ and their (f) corresponding distance matrix; since the anisotropy transformation modifies the distance matrix it leads to different correlation matrices.