TOTALLY COMPATIBLE ASSOCIATIVE AND LIE DIALGEBRAS,
TRIDENDRIFORM ALGEBRAS AND POSTLIE ALGEBRAS

YONG ZHANG, CHENGMING BAI, AND LI GUO

Abstract. This paper studies the concepts of a totally compatible dialgebra and a totally compatible Lie dialgebra, defined to be a vector space with two binary operations that satisfy individual and mixed associativity conditions and Lie algebra conditions respectively. We show that totally compatible dialgebras are closely related to bimodule algebras and semi-homomorphisms. More significantly, Rota-Baxter operators on totally compatible dialgebras provide a uniform framework to generalize known results that Rota-Baxter related operators give tridendriform algebras. Free totally compatible dialgebras are constructed guided by the general principle of universal algebra. We also show that a Rota-Baxter operator on a totally compatible Lie dialgebra gives rise to a PostLie algebra, generalizing the fact that a Rota-Baxter operator on a Lie algebra gives rise to a PostLie algebra.

1. Introduction

In recent years, there have been quite much interests on (linearly) compatible products from both mathematics and physics. For example two Lie brackets \([ , ]_1\) and \([ , ]_2\) on a vector space are called compatible if all linear combination of the two brackets are still Lie brackets. Such structures are studied in \([13, 14, 15, 24]\) in the contexts of the classical Yang-Baxter equation and principal chiral field, loop algebras over Lie algebras and elliptic theta functions. The corresponding operad is obtained in \([9]\). Compatible associative products are studied in \([22, 23, 25]\) in connection with Cartan matrices of affine Dynkin diagrams, integrable matrix equations, infinitesimal bialgebras and quiver representations. In this case, the corresponding operad and free objects are obtained in \([8]\). More general compatible products are defined in \([27]\).

Date: November 11, 2011.
In this paper we consider the Koszul dual of compatible associative dialgebra and compatible Lie dialgebras. These dual structures have been introduced in [9, 27], called totally compatible associative and Lie dialgebras respectively. Our motivation in studying such structures comes from the facts [1, 10] that Rota-Baxter algebras give rise to dendriform algebras and tridendriform algebras, and the facts that Rota-Baxter Lie algebras give the structures of pre-Lie and PostLie algebras. This connection has been studied by several subsequent papers such as [3, 5, 7, 11, 28] in the associative algebra case and [4, 10, 12, 20] in the Lie algebra case. It has been found that some other structures, such as TD-algebras [19], also give tridendriform algebras. See Section 3 for details. Given the importance of the concepts of Rota-Baxter algebras and tridendriform algebras, it is interesting to ask what kind of Rota-Baxter related structures can give rise to a tridendriform algebra. We show that totally compatible algebra is a more general structure on which a Rota-Baxter operator gives a tridendriform algebra, allowing us to give a uniform approach that combine known results such as those mentioned above. We also show that a Lie algebraic analogue holds.

In Section 2, we begin with the concept of a totally compatible dialgebra and its relationship with $A$-bimodule $k$-algebras and semi-homomorphisms. We then investigate in Section 3 Rota-Baxter operators on totally compatible dialgebras and establish their close relationship with tridendriform algebras. Free totally compatible dialgebras are constructed in Section 4 and are used to give examples of Rota-Baxter totally compatible dialgebras. The concepts of various compatible Lie dialgebras and Rota-Baxter totally compatible Lie dialgebras are introduced in Section 5. Their relationship with Rota-Baxter compatible dialgebras and PostLie algebras is established. This relationship refines the already established relationship [4] of Rota-Baxter Lie algebras and PostLie algebras.

2. Totally compatible dialgebras and their Rota-Baxter operators

In this section, we consider totally compatible dialgebras and their Rota-Baxter operators. The relations of Rota-Baxter totally compatible dialgebras with tridendriform algebras, $A$-bimodule $k$-algebras and semi-homomorphisms are established.

2.1. Totally compatible dialgebras. Let $k$ be a commutative unitary ring. The tensor product over $k$ is denoted by $\otimes_k$ or simply by $\otimes$ if it causes no confusion.

Definition 2.1. (a) A totally compatible (associative) dialgebra (TCDA) is a $k$-module $R$ with two binary operations:

$$\ast, \odot : R \otimes R \longrightarrow R,$$

satisfying the TCDA axioms:

(i) $\ast$ and $\odot$ are associative.

(ii) $$(a \ast b) \odot c = a \ast (b \odot c) = (a \odot b) \ast c = a \odot (b \ast c), \quad \forall a, b, c \in R.$$ (1)

(b) Let $(R, \ast, \odot)$ and $(R', \ast', \odot')$ be two totally compatible dialgebras. A linear map $f : R \rightarrow R'$ is a homomorphism of totally compatible dialgebras if $f$ is a $k$-module homomorphism and, for all $a, b \in R$,

$$f(a \ast b) = f(a) \ast' f(b) \quad \text{and} \quad f(a \odot b) = f(a) \odot' f(b).$$ (2)
A totally compatible disemigroup is a set $S$ with two binary operations $\ast, \odot$ on $S$ that satisfy the TCDA axioms in Items (a).

The concept of a homomorphism of totally compatible disemigroups is defined in the same way.

The operad of totally compatible dialgebras is denoted by $^2As$ in [27]. It is the Koszul dual of the operad $As^2$ of compatible dialgebras, defined to be $k$-modules $V$ with binary operations $\ast$ and $\odot$ that are associative and satisfy the relation

$$(x \odot y) \ast z + (x \ast y) \odot z = x \ast (y \odot z) + x \odot (y \ast z), \quad \forall x, y, z \in V.$$

A compatible dialgebra is also called an algebra with two compatible associative products [8, 27]. The term compatible comes from the fact that, for a $k$-module $V$ with two associative products $\ast$ and $\odot$, $(V, \ast, \odot)$ is a compatible dialgebra if and only if linear combinations of $\ast$ and $\odot$ are still associative.

It is easy to see that if $(R, \ast, \odot)$ is a totally compatible dialgebra, then, for any $r, s \in k$, the triple $(R, r \ast, s \odot)$ is again a totally compatible dialgebra. It is also easy to check that the tensor product of two totally compatible dialgebras is naturally a totally compatible dialgebra. Similarly, let $M_n(R)$ be the $k$-module of $n \times n$-matrices $\alpha := (\alpha_{ij})$ with entries $\alpha_{ij}, 1 \leq i, j \leq n$, in a totally compatible dialgebra $(R, \ast, \odot)$. For $\alpha = (\alpha_{ij}), \beta = (\beta_{ij}) \in M_n(R)$, define

$$\alpha \ast \beta := ((\alpha \ast \beta)_{ij}), \quad \alpha \odot \beta := ((\alpha \odot \beta)_{ij}),$$

where

$$(\alpha \ast \beta)_{ij} = \sum_{k=1}^n \alpha_{ik} \ast \beta_{kj} \quad \text{and} \quad (\alpha \odot \beta)_{ij} = \sum_{k=1}^n \alpha_{ik} \odot \beta_{kj}.$$

Then $(M_n(A), \ast, \odot)$ is a totally compatible dialgebra.

### 2.2. Totally compatible dialgebras, $A$-bimodule $k$-algebras and semi-homomorphisms.

We now study the relationship between totally compatible dialgebras and $A$-bimodule $k$-algebra introduced in [3] and semi-homomorphisms.

#### 2.2.1. $A$-bimodule $k$-algebras.

**Definition 2.2.** Let $(A, \ast)$ be a $k$-algebra with multiplication $\ast$. Let $(R, \odot)$ be a $k$-algebra with multiplication $\odot$. Let $\ell, r : A \to \text{End}_k(R)$ be two linear maps. We call $(R, \odot, \ell, r)$ or simply $R$ an $A$-bimodule $k$-algebra if $(R, \ell, r)$ is an $A$-bimodule that is compatible with the multiplication $\odot$ on $R$ in the sense that the following equations hold.

1. $\ell(x \ast y)v = \ell(x)(\ell(y)v), \quad vr(x \ast y) = (vr(x))r(y), \quad (\ell(x)v)r(y) = \ell(x)(vr(y))$,
2. $\ell(x)(v \odot w) = (\ell(x)v) \odot w, \quad (v \odot w)r(x) = v \odot (wr(x)), \quad (vr(x)) \odot w = v \odot (\ell(x)w),$

for all $x, y \in A, v, w \in R$.

**Proposition 2.3.** ([2, 3]) Let $(A, \ast)$ be a $k$-algebra. Then $(R, \odot, \ell, r)$ is an $A$-bimodule $k$-algebra if and only if the direct sum $A \oplus R$ of $k$-modules is turned into a $k$-algebra (the semidirect sum) by defining multiplication in $A \oplus R$ by

$$(x_1, v_1) \ast (x_2, v_2) = (x_1 \ast x_2, \ell(x_1)v_2 + v_1r(x_2) + v_1 \odot v_2), \quad \forall x_1, x_2 \in A, v_1, v_2 \in R.$$
We denote this algebra by $A \rtimes_{r, R} R$ or simply $A \rtimes R$.

**Proposition 2.4.** Let $(A, *, \odot)$ be a totally compatible dialgebra. Define left and right actions by
\[
L_*(x) : A \to A, L_*(x)(y) := x * y, \\
R_*(x) : A \to A, R_*(x)(y) := y * x, \ \forall x, y \in A.
\]
Then the quadruple $(A, \odot, L_*, R_*)$ is an $A$-bimodule $k$-algebra. Conversely, if $(A, *)$ is a $k$-algebra and $(A, \odot, L_*, R_*)$ is an $A$-bimodule $k$-algebra with the left and right actions $L_*$ and $R_*$ defined above, then the triple $(A, *, \odot)$ is a totally compatible dialgebra.

**Proof.** By the definition of $A$-bimodule $k$-algebra, it suffices to verify the equations in Definition 2.2. It is well-known that, for the associative $k$-algebra $(A, *)$, $(A, L_*, R_*)$ is an $A$-bimodule. Hence the equations in Eq. (3) hold. For the equations in Eq. (4), we check that
\[
L_*(x)(y \odot z) = x * (y \odot z) = (x * y) \odot z = (L_*(x)y) \odot z,
\]
\[
(x \odot y)R_*(z) = (x \odot y) * z = x \odot (y * z) = x \odot (yR_*(z)),
\]
\[
(xR_*(y)) \odot z = (x * y) \odot z = x \odot (y * z) = x \odot (L_*(y)z).
\]
Conversely, if $A$ is an $A$-bimodule $k$-algebra, then we have
\[
x * (y \odot z) = L_*(x)(y \odot z) = (L_*(x)y) \odot z = (x * y) \odot z,
\]
\[
(x \odot y) * z = (x \odot y)R_*(z) = x \odot (yR_*(z)) = x \odot (y * z),
\]
\[
(x * y) \odot z = (xR_*(y)) \odot z = x \odot (L_*(y)z) = x \odot (y * z).
\]
Since $*$ and $\odot$ are already associative, $(A, *, \odot)$ is a totally compatible dialgebra. \qed

**Corollary 2.5.** With the conditions as above, the following conditions are equivalent:

(a) $(A, *, \odot)$ is a totally compatible dialgebra.

(b) $(A, \odot, L_*, R_*)$ is an $A$-bimodule $k$-algebra.

(c) There is a $k$-algebra structure $A \rtimes_{L_*, R_*} A$ on the $k$-module $A \oplus A$ defined by
\[
(x, y) \star (z, w) = (x * z, x * w + y * z + y \odot w), \ \forall x, y, z, w \in A.
\]

**Proof.** The equivalence between Item (a) and Item (b) is just Proposition 2.4. The equivalence between Item (b) and Item (c) follows from Proposition 2.3. \qed

2.2.2. Semi-homomorphisms.

**Definition 2.6.** Let $(A, \cdot)$ be a $k$-algebra. A linear transformation $f : A \to A$ is called **semi-homomorphism** of $A$ if $f$ satisfies
\[
f(x \cdot y) = x \cdot f(y) = f(x) \cdot y, \ \forall x, y \in A.
\]

The set of all semi-homomorphisms is called the centroid of $A$.

**Proposition 2.7.** Let $(A, \cdot)$ be a $k$-algebra and let $f, g$ be commuting semi-homomorphisms on $A$: $fg = gf$. Define
\[
x \cdot y := f(x) \cdot y (= f(x \cdot y) = x \cdot f(y)) \ \text{and} \ x \odot y := g(x) \cdot y (= g(x \cdot y) = x \cdot g(y)) \ \forall x, y \in A.
\]
Then \((A, *, \odot)\) is a totally compatible dialgebra. In particular, for any semi-homomorphism \(f\) on \(A\), define

\[
x * y := x \cdot y, \quad x \odot y := f(x) \cdot y, \quad \forall x, y \in A.
\]

Then \((A, *, \odot)\) is a totally compatible dialgebra.

**Proof.** It is straightforward to check that both \((A, *)\) and \((A, \odot)\) are \(k\)-algebras. Let \(x, y, z\) be in \(A\). Then we have

\[
\begin{align*}
(x * y) \odot z &= g(f(x) \cdot y) \cdot z = f(x) \cdot g(y) \cdot z = x * (y \odot z); \\
(x \odot y) * z &= f(g(x) \cdot y) \cdot z = f(x) \cdot g(y) \cdot z = x \odot (y \odot z);
\end{align*}
\]

\[
x * (y \odot z) = f(x) \cdot g(y) \cdot z = x \cdot g(y) \cdot z = x \cdot g(f(y) \cdot z) = g(x) \cdot f(y) \cdot z = x \odot (y \odot z).
\]

Therefore \((A, *, \odot)\) is a totally compatible dialgebra.

To prove the last statement, we just note that the identity map \(id\) and \(f\) are commuting semi-homomorphisms on \(A\).

**Example 2.8.** Let \((A, *)\) be a \(k\)-algebra.

(a) The identity map \(id\) on \((A, *)\) is obviously a semi-homomorphism. The corresponding totally compatible dialgebra is \((A, *, *, *)\).

(b) Let \(w\) be in the center of \(A\) and define \(f(x) = x \ast w\) for all \(x \in A\). Then \(f\) is a semi-homomorphism. Thus by Proposition \(2.7\), for

\[
x \odot y := x \ast w \ast y (= w \ast x \ast y = x \ast y \ast w), \quad \forall x, y \in A,
\]

the triple \((A, *, \odot)\) is a totally compatible dialgebra. This gives another way to obtain the totally compatible dialgebra in Lemma \(3.4\)(b). See Remark \(3.5\).

3. Rota-Baxter totally compatible dialgebras and tridendriform algebras

In this section we study the close relationship between Rota-Baxter totally compatible dialgebra and tridendriform algebras. We begin with recalling the basis concepts and results.

**Definition 3.1.** (a) \([6,16,17,26]\) Let \(\lambda \in k\) be given. A linear operator \(P\) on a \(k\)-algebra \(R\) is called a **Rota-Baxter operator of weight** \(\lambda\) if

\[
P(x)P(y) = P(xP(y)) + P(P(x)y) + P(\lambda xy), \quad \forall x, y \in R,
\]

Then the pair \((R, P)\) is called a **Rota-Baxter algebra of weight** \(\lambda\).

(b) \([21]\) A **tridendriform algebra** is a quadruple \((T, \prec, \succ, \cdot)\) consisting of a \(k\)-module \(T\) and three bilinear products \(\prec, \succ\) and \(\cdot\) such that

\[
\begin{align*}
(x \prec y) \prec z &= x \prec (y \ast z), \quad (x \succ y) \prec z = x \succ (y \ast z), \\
(x \ast y) \succ z &= x \succ (y \ast z), \quad (x \ast y) \cdot z = x \cdot (y \ast z), \\
(x \prec y) \cdot z &= x \cdot (y \succ z), \quad (x \succ y) \cdot z = x \cdot (y \cdot z)
\end{align*}
\]

for all \(x, y, z \in T\). Here \(\ast = + \succ + \cdot\).

It has been established \([1,10]\) that if \(P\) is a Rota-Baxter operator of weight \(\lambda\) on a \(k\)-algebra \(R\), then

\[
x \prec_P y := xP(y), \quad x \succ_P y := P(x)y, \quad x \cdot_P y := \lambda xy, \quad \forall x, y \in R,
\]
define a tridendriform algebra structure on $R$. It has been found later that some other Rota-Baxter algebra type structures also give tridendriform algebras. For example, a TD operator $P$ on a unitary $k$-algebra $R$ is a linear operator $P : R \rightarrow R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) - P(xP(1)y), \forall x,y \in R.$$ 

By [19] Proposition 2.5, a TD operator $T$ also gives a dendriform trialgebra by

$$x <_p y := xP(y), \ x >_p y := P(x)y, \ x \cdot_p y := -xP(1)y, \ \forall x,y \in R.$$ 

As another example, for a given $w \in R$, define a Rota-Baxter operator of weight $w$ to be a linear operator $P : R \rightarrow R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + P(xwy), \ \forall x,y \in R.$$ 

Suppose $w$ is in the center of $R$. Then

$$x <_p y := xP(y), \ x >_p y := P(x)y, \ x \cdot_p y := xwy, \ \forall x,y \in R,$$

define a tridendriform algebra structure on $R$.

In each of the above cases, the products $xP(1)y$ (resp. $xwy$) are derived from the default product of $R$. Our motivation is to consider a new product on $R$ and study its “compatibility” conditions with the default product on $R$ in order to still obtain a tridendriform algebra from a Rota-Baxter operator. As we will see in Theorem [3.6], such a structure is exactly the totally compatible dialgebra.

We now consider variations of a totally compatible dialgebra that allow a Rota-Baxter operator.

**Definition 3.2.** (a) A restricted Rota-Baxter totally compatible dialgebra is a quadruple $(R, *, \circ, P)$ where $R$ is a $k$-module, $*, \circ$ are associative multiplications on $R$, $P : R \rightarrow R$ is a linear operator, satisfying the following compatibility conditions.

11) $$(P(x) * y) \circ z = P(x) * (y \circ z), \ (x * P(y)) \circ z = x \circ (P(y) * z), \ (x \circ y) * P(z) = x \circ (y * P(z)),$$

12) $$(P(x) * P(y)) = P(x * P(y)) + P(P(x) * y) + P(x \circ y), \ \forall x,y,z \in R.$$

(b) Let $(R, *, \circ)$ be a totally compatible dialgebra. A linear operator $P : R \rightarrow R$ is called a Rota-Baxter operator if

$$P(x) * P(y) = P(x * P(y)) + P(P(x) * y) + P(x \circ y), \ \forall x,y \in R.$$ 

If the equation holds, then $(R, *, \circ, P)$ is called a Rota-Baxter totally compatible dialgebra.

**Remark 3.3.** (a) Note that we did not assign a weight to the Rota-Baxter operator in Definition [3.2]. This is because the effect of a weight can be achieved by a variation of the product $\circ$: instead of considering $\lambda x \circ y$, we could consider $x \circ' y$ with $x \circ' y := \lambda x \circ y$. Such an instance can be found in Proposition [4.3].

(b) It follows from the definitions that a Rota-Baxter totally compatible dialgebra is a restricted Rota-Baxter totally compatible dialgebra. If $P$ is surjective, then the two concepts agree with each other.

The following lemma shows that the concept of restricted Rota-Baxter totally compatible dialgebra gives a suitable context to combine known cases of Rota-Baxter type operators that give tridendriform algebras. See also Corollary [3.7].
Lemma 3.4. (a) Let $R$ be a $k$-algebra, let $w$ be in $R$ and let $P : R \to R$ be a linear operator. Suppose $wP(x) = P(x)w$ for all $x \in R$ and
\[ P(x)P(y) = P(xP(y) + P(x)y + xwy), \quad \forall x, y \in R. \]

Define
\[ x \ast y := xy, \quad x \odot y := xwy, \quad \forall x, y \in R. \tag{13} \]

Then the quadruple $(R, \ast, \odot, P)$ is a restricted Rota-Baxter totally compatible dialgebra.

(b) Let $R$ be a $k$-algebra and let $w \in R$ be in the center of $R$. Let $P : R \to R$ be a Rota-Baxter operator of weight $w$. Then the quadruple $(R, \ast, \odot, P)$ with $\ast, \odot$ defined in Eq. (13) is a restricted Rota-Baxter totally compatible dialgebra.

(c) Let $R$ be a unitary $k$-algebra and let $P : R \to R$ be a TD operator. Then the quadruple $(R, \ast, \odot, P)$ where
\[ x \ast y := xy, \quad x \odot y := xP(1)y, \quad \forall x, y \in R, \]
is a restricted Rota-Baxter totally compatible dialgebra.

Remark 3.5. The triple $(R, \ast, \odot)$ in Item (b) is in fact a totally compatible dialgebra, as observed in Example 2.8(b), from semi-homomorphisms.

Proof. (a) The multiplications $\ast$ and $\odot$ are associative by their definitions. To verify Eq. (11) we compute
\[
(P(x) \ast y) \odot z = (P(x)y)wz = P(x)(ywz) = P(x) \ast (y \odot z), \\
(x \ast P(y)) \odot z = xP(y)wz = xwP(y)z = x \odot (P(y) \ast z), \\
(x \odot y) \ast P(z) = (xwy)P(z) = xw(yP(z)) = x \odot (y \ast P(z)), \quad \forall x, y, z \in R.
\]

Eq. (12) is automatic. Therefore $(R, \ast, \odot, P)$ is a restricted Rota-Baxter totally compatible dialgebra.

(b) This is a direct consequence of Item (a).

(c) For a TD operator $P$, we have [19]
\[ P(x)P(1) = P(xP(1) + P(x) - xP(1)) = P^2(x) = P(P(x) + P(1)x - P(1)x)) = P(1)P(x), \quad \forall x \in R. \]
Thus we just need to take $w = P(1)$ in Item (a). \qed

The following result shows the close relationship between restricted Rota-Baxter totally compatible dialgebras and tridendriform algebras.

Theorem 3.6. Let $(R, \ast, \odot, P)$ be a $k$-module with two associative multiplications $\ast, \odot$ and a linear map $P : R \to R$ such that
\[ P(x) \ast P(y) = P(x \ast P(y) + P(x) \ast y + x \odot y), \quad \forall x, y \in R. \]

Define
\[
\begin{align*}
x \prec_P y & := x \ast P(y), \\
x \succ_P y & := P(x) \ast y, \\
x \cdot_P y & := x \odot y.
\end{align*}
\tag{14}
\]

Then $(R, \prec_P, \succ_P, \cdot_P)$ is a tridendriform algebra if and only if $(R, \ast, \odot, P)$ is a restricted Rota-Baxter totally compatible dialgebra.
Proof. \((\iff)\). We just need to verify the seven axioms for the operations \(<_p, >_p\) and \(\cdot_p\) in the definition of a tridendriform algebra in Definition \ref{definition:tridendriform-algebra}. Denote \(\star_p = <_p + >_p + \cdot_p\). From the associativity of \(*\), we obtain
\[
(x \star_p y) > z = (P(x) * P(y)) * z = P(x) * (P(y) * z) = x >_p (y >_p z),
\]
\[
(x <_p y) <_p z = (x * P(y)) * P(z) = x * (P(y) * P(z)) = x <_p (y \star_p z)
\]
and
\[
(x >_p y) <_p z = (P(x) * y) * P(z) = P(x) * (y * P(z)) = x >_p (y <_p z).
\]
From Eq. \ref{eq:associativity}, we have
\[
(x \cdot_p y) <_p z = (x \odot y) * P(z) = x \odot (y * P(z)) = x \cdot_p (y <_p z),
\]
\[
(x >_p y) \cdot_p z = (P(x) * y) \odot z = P(x) * (y \odot z) = x >_p (y \cdot_p z)
\]
and
\[
(x <_p y) \cdot_p z = (x * P(y)) \odot z = x \odot (P(y) * z) = x \cdot_p (y >_p z).
\]
Finally from the associativity of \(\odot\), we obtain
\[
(x \cdot_p y) \cdot_p z = x \cdot_p (y \cdot_p z).
\]
Thus we have verified all the axioms for a tridendriform algebra.

\((\Rightarrow)\). Let \((R, *, \odot, P)\) be as given in the theorem and suppose \((R, <_p, >_p, \cdot_p)\) defined by Eq. \ref{eq:Rota-Baxter-compatible-tridendriform} is a tridendriform algebra. Then we also have
\[
P(x) * P(y) = P(x * P(y) + P(x) * y + x \odot y) = P(x \star y).
\]
Then the axioms of the dendriform algebra \((R, <_p, >_p, \cdot_p)\) imply
\[
\begin{align*}
\text{(a)} & \quad (P(x) * y) \odot z = P(x) * (y \odot z), \\
\text{(b)} & \quad (x * P(y)) \odot z = x \odot (P(y) * z), \\
\text{(c)} & \quad (x \odot y) * P(z) = x \odot (y * P(z)),
\end{align*}
\]
giving Eq. \ref{eq:associativity}. \(\square\)

The following corollary follows directly from Lemma \ref{lemma:Rota-Baxter-compatible-tridendriform} and Theorem \ref{thm:Rota-Baxter-tridendriform}.

**Corollary 3.7.**

(a) If \((R, *, \odot, P)\) is a Rota-Baxter totally compatible dialgebra, then \((R, <_p, >_p, \cdot_p)\) defined in Eq. \ref{eq:Rota-Baxter-compatible-tridendriform} is a tridendriform algebra.

(b) If \(P\) is surjective, then the converse of Item (a) holds.

(c) Let \(R\) be a \(k\)-algebra and let \(w \in R\) be such that \(wP(x) = P(x)w\) for all \(x \in R\). Let \((R, P)\) be a Rota-Baxter algebra of weight \(w\). Then
\[
x <_p y := xP(y), x >_p y := P(x)y, x \cdot_p y := xwy, \quad \forall x, y \in R,
\]
define a tridendriform algebra structure on \(R\).

(d) \([\ref{eq:Rota-Baxter-compatible-tridendriform}]\) Let \((R, P)\) be a TD algebra. Then
\[
x <_p y := xP(y), x >_p y := P(x)y, x \cdot_p y := xP(1)y, \quad \forall x, y \in R,
\]
define a tridendriform algebra structure on \(R\).

We end this section by showing that a Rota-Baxter operator on a totally compatible dialgebra is equivalent to a Rota-Baxter operator on a larger \(k\)-algebra.
Theorem 3.8. Let \((R, *, \odot)\) be a totally compatible dialgebra. Let \((R \ltimes_{L,R} R, \star)\) be the \(k\)-algebra in Corollary 2.5 with the product \(\star\) defined by Eq. (5). Then a linear operator \(P : R \to R\) is a Rota-Baxter operator on the totally compatible dialgebra \((R, *, \odot)\), in the sense of Definition 3.2, if and only if
\[
\hat{P} : R \ltimes_{L,R} R \to R \ltimes_{L,R} R, \quad \hat{P}(x, y) = (-x + P(y), 0), \quad \forall x, y \in R,
\]
is a Rota-Baxter operator of weight 1 on the \(k\)-algebra \(R \ltimes_{L,R} R\).

Proof. Let \(x, y, z, w \in R\). Then
\[
\hat{P}(x, y) \star \hat{P}(z, w) = \hat{P}((-x + P(y), 0) \star (-z + P(w), 0)) = \hat{P}((x * z - x * P(w) - P(y) * z + P(y) * P(w), 0)) = \hat{P}((-x * z + P(y) * z, -x * w + P(y) * w)) = (x * z - P(y) * z - P(x * w) + P(P(y) * w), 0);
\]
\[
\hat{P}((x, y) \star \hat{P}(z, w)) = \hat{P}((x, y) \star ((-z + P(w), 0))) = \hat{P}((-x * z + x * P(w), -y * z + y * P(w))) = (x * z - x * P(w) - P(y * z) + P(y * P(w)), 0);
\]
\[
\hat{P}((x, y) \star (z, w)) = \hat{P}(x * z, x * w + y * z + y \odot w) = (x * z + P(x * w) + P(y * z) + P(y \odot w), 0).
\]
Therefore \(P(y) \star P(w) = P(y * P(w) + P(y) * w + y \odot w)\) if and only if
\[
\hat{P}(x, y) \star \hat{P}(z, w) = \hat{P}(\hat{P}(x, y) \star (z, w)) + \hat{P}((x, y) \star \hat{P}(z, w)) + \hat{P}((x, y) \star (z, w)).
\]
That is, \(P : R \to R\) is a Rota-Baxter operator on \((R, *, \odot)\) if and only if \(\hat{P}\) is a Rota-Baxter operator of weight 1 on \(R \ltimes_{L,R} R\).

4. Free totally compatible dialgebras

In this section, we construct free totally compatible dialgebras and use them to provide examples of Rota-Baxter totally compatible dialgebras.

Let \(X\) be a set. Let
\[
S(X) = \left\{ x_1 \cdots x_n \mid x_i \in X, 1 \leq i \leq n, n \geq 1 \right\}
\]
be the free semigroup generated by \(X\) and let \(M(X)\) be the free monoid generated by \(X\). Then the semigroup algebra \(k(X)^0 := kS(X)\) is the noncommutative nonunitary polynomial algebra generated by \(X\) and the semigroup algebra \(k(X) := kM(X)\) is the noncommutative unitary polynomial algebra generated by \(X\). Consider the \(k\)-module underlying the tensor product
\[
F(X) := k(X)^0 \otimes k(X) \cong kS(X) \times M(X),
\]
with \(S(X) \times M(X)\) as a canonical basis. As such, \((u, v) \in S(X) \times M(X)\), with \(u \in S(X)\) and \(v \in M(X)\), corresponds to \(u \otimes v\) in \(F(X)\). Thus we will use \(u \otimes v\) to denote \((u, v)\).

With these notations, we define two binary operations \(\bar{\otimes}\) and \(\odot\) on \(S(X) \times M(X)\) as follows.

Consider two elements
\[
a = (x_1 \cdots x_m) \otimes (x_{m+1} \cdots x_{m+n}), \quad b = (x_{m+n+1} \cdots x_{m+n+k}) \otimes (x_{m+n+k+1} \cdots x_{m+n+k+l})
\]
denote by \(\bar{\cdot}\) and \(\cdot\) of totally compatible disemigroups such that

\[
\begin{align*}
(a \circ b) \circ c &= (a \circ (b \circ c)),
\end{align*}
\]

proving Eq. (\ref{eq:11}). In particularly, when \(n = \ell = 0\), we define

\[
\begin{align*}
((x_1 \cdots x_m) \otimes 1) \bar{\circ}((x_{m+1} \cdots x_{m+k}) \otimes 1) &= (x_1 \cdots x_{m+k}) \otimes 1,
\end{align*}
\]

These two binary operations expand to binary operations on \(F(X)\) by \(k\)-bilinearity that we still denote by \(\bar{*}\) and \(\bar{\circ}\).

**Theorem 4.1.** Let \(X\) be a set.

(a) The set \(S(X) \times M(X)\), with the multiplications \(\bar{*}\) and \(\bar{\circ}\) defined in Eqs. (\ref{eq:10}) – (\ref{eq:12}), and the embedding

\[
\begin{align*}
i_X : X \to S(X) \times M(X), \quad x \mapsto x \otimes 1, x \in X,
\end{align*}
\]

is the free totally compatible disemigroup on \(X\).

(b) The \(k\)-module \(F(X) = k(X)^0 \otimes k(X)\), with the multiplications \(\bar{*}\) and \(\bar{\circ}\) defined in Eqs. (\ref{eq:10}) – (\ref{eq:12}), and the embedding

\[
\begin{align*}
i_X : X \to F(X), \quad x \mapsto x \otimes 1, x \in X,
\end{align*}
\]

is the free totally compatible dialgebra on \(X\).

**Proof.** (a). We first check that the triple \((S(X) \times M(X), \bar{*}, \bar{\circ})\) is a totally compatible disemigroup. For \(a, b\) in Eq. (\ref{eq:11}) and

\[
\begin{align*}
c &= (x_{m+n+k+\ell+1} \cdots x_{m+n+k+\ell+p}) \otimes (x_{m+n+k+\ell+p+1} \cdots x_{m+n+k+\ell+p+q}),
\end{align*}
\]

with \(p \geq 1\) and \(q \geq 0\). We have

\[
\begin{align*}
(a \bar{*} b) \bar{*} c &= (x_1 \cdots x_{m+k+p}) \otimes (x_{m+k+p+1} \cdots x_{m+k+p+n+\ell+q}) = a \bar{*} (b \bar{*} c).
\end{align*}
\]

Hence \(\bar{*}\) is associative. Similarly,

\[
\begin{align*}
(a \bar{\circ} b) \bar{\circ} c &= (x_1 \cdots x_{m+k+p-2}) \otimes (x_{m+k+p-1} \cdots x_{m+k+p+n+\ell+q}) = a \bar{\circ} (b \bar{\circ} c),
\end{align*}
\]

showing that \(\bar{\circ}\) is associative.

On the other hand, for these choices of \(a, b, c\), each term in Eq. (\ref{eq:11}) equals to

\[
(x_1 \cdots x_{m+k+p-1}) \otimes (x_{m+k+p} \cdots x_{m+n+k+\ell+p+q}),
\]

proving Eq. (\ref{eq:11}).

We next show that the totally compatible disemigroup \(S(X) \times M(X)\) is free on \(X\) by checking that \((S(X) \times M(X), \bar{*}, \bar{\circ})\) satisfies the universal property of a free totally compatible disemigroup over \(X\); Let \((S, *, \circ)\) be a totally compatible disemigroup with operations \(*\) and \(\circ\). Let \(f : X \to S\) be a set map. Then there exists a unique homomorphism

\[
\overline{f} : S(X) \times M(X) \longrightarrow S
\]

of totally compatible disemigroups such that \(f = \overline{f} \circ i_X\) for the \(i_X\) defined in Eq. (\ref{eq:11}).
Let a set map $f : X \rightarrow S$ be given. Define

$$f^* : S(X) \otimes M(X) \rightarrow S,$$

(23) \hspace{0.5cm} (x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n})
\mapsto (f(x_1) \cdot \cdot \cdot f(x_m)) \otimes (f(x_{m+1}) \cdot \cdot \cdot f(x_{m+n}))
\mapsto (x_1 \cdot \cdot \cdot x_m) \otimes 1
\mapsto f(x_1) \cdot \cdot \cdot f(x_m).

We will show that $\bar{f}$ is the unique extension of $f$ that is a homomorphism of totally compatible disemigroups.

We first show that $\bar{f}$ is indeed a homomorphism of totally compatible disemigroups. For any

$a = (x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n}), \quad b = (x_{m+1} \cdot \cdot \cdot x_{m+n+k}) \otimes (x_{m+n+k+1} \cdot \cdot \cdot x_{m+n+k+l})

in S(X) \otimes M(X)$ as expressed in Eq. (16), we have

$$\bar{f}(a) \ast \bar{f}(b) = ((f(x_1) \cdot \cdot \cdot f(x_m)) \otimes (f(x_{m+1}) \cdot \cdot \cdot f(x_{m+n})))
\ast ((f(x_{m+n+1}) \cdot \cdot \cdot f(x_{m+n+k})) \otimes (f(x_{m+n+k+1}) \cdot \cdot \cdot f(x_{m+n+k+l})))
= (f(x_1) \cdot \cdot \cdot f(x_{m+k})) \otimes (f(x_{m+k+1}) \cdot \cdot \cdot f(x_{m+k+n+l}))
= \bar{f}((x_1 \cdot \cdot \cdot x_{m+k}) \otimes (x_{m+k+1} \cdot \cdot \cdot x_{m+k+n+l}))
= \bar{f}(a \ast b). \quad \text{(by Eq. (17))}$$

Similarly, we have

$$\bar{f}(a) \odot \bar{f}(b) = ((f(x_1) \cdot \cdot \cdot f(x_m)) \otimes (f(x_{m+1}) \cdot \cdot \cdot f(x_{m+n})))
\odot ((f(x_{m+n+1}) \cdot \cdot \cdot f(x_{m+n+k})) \otimes (f(x_{m+n+k+1}) \cdot \cdot \cdot f(x_{m+n+k+l})))
= (f(x_1) \cdot \cdot \cdot f(x_{m+k-1})) \odot (f(x_{m+k}) \cdot \cdot \cdot f(x_{m+k+n+l}))
= \bar{f}((x_1 \cdot \cdot \cdot x_{m+k-1}) \otimes (x_{m+k} \cdot \cdot \cdot x_{m+k+n+l}))
= \bar{f}(a \odot b). \quad \text{(by Eq. (18))}$$

Thus it remains to show that any homomorphism of totally compatible disemigroups from

$S(X) \otimes M(X)$ that extends $f$ is uniquely determined by Eq. (23). Suppose

$$f^* : S(X) \otimes M(X) \rightarrow S$$

is a homomorphism of totally compatible disemigroups that extends $f$. We will prove $f^* = \bar{f}$.

Note that the set

$$\mathcal{X} := S(X) \times M(X) := \{(x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n}) | x_i \in X, 1 \leq i \leq m + n, m \geq 1, n \geq 0\}$$

is a disjoint union

(24) \hspace{0.5cm} \mathcal{X} = \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \mathcal{X}_3,

where

(25) \hspace{0.5cm} \mathcal{X}_1 := \{(x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n}) \in \mathcal{X} | m \geq 1, n = 0\}
= \{(x_1 \cdot \cdot \cdot x_m) \otimes 1 | x_i \in X, 1 \leq i \leq m, m \geq 1\},

(26) \hspace{0.5cm} \mathcal{X}_2 := \{(x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n}) \in \mathcal{X} | m = 1, n \geq 1\}
= \{x_1 \otimes (x_2 \cdot \cdot \cdot x_{n+1}) | x_i \in X, 1 \leq i \leq n + 1, n \geq 1\},

(27) \hspace{0.5cm} \mathcal{X}_3 := \{(x_1 \cdot \cdot \cdot x_m) \otimes (x_{m+1} \cdot \cdot \cdot x_{m+n}) \in \mathcal{X} | m \geq 2, n \geq 1\}.\]
Thus in order to prove $\bar{f}' = \bar{f}$, we just need to show that $\bar{f}'$ agrees with $\bar{f}$ on the three subsets $\mathcal{X}_i$, $1 \leq i \leq 3$, of $\mathcal{X}$.

**Case 1.** $\bar{f}' = \bar{f}$ on $\mathcal{X}_1$. Note that, by Eq. (19), $\mathcal{X}_1 = S(X) \otimes 1$ is the free semigroup on $X$ with respect to the product $\bar{\ast}$. Since $\bar{f}'$ is in particular a semigroup homomorphism with respect to $\bar{\ast}$, $\bar{f}'$ must agree with the unique extension of $f$ to a semigroup homomorphism 

$$(S(X) \otimes 1, \bar{\ast}) \to (R, \ast).$$

Thus we must have

$$\bar{f}'((x_1 \cdots x_m) \otimes 1) = \bar{f}(x_1 \otimes 1) \bar{\ast} \cdots \bar{\ast} \bar{f}(x_m \otimes 1) = f(x_1) \ast \cdots \ast f(x_m).$$

This agrees with the definition of $\bar{f}$ in Eq. (23).

**Case 2.** $\bar{f}'$ agrees with $\bar{f}$ on $\mathcal{X}_2$. We first prove a lemma.

**Lemma 4.2.** For any $n \geq 1$, we have

$$(x_1 \otimes (x_2 \cdots x_{n+1})) = (x_1 \otimes 1) \bar{\circ} \cdots \bar{\circ} (x_{n+1} \otimes 1).$$

**Proof.** We prove Eq. (29) by induction on $n \geq 1$. When $n = 1$, this follows from Eq. (20). Assume that Eq. (29) has been proved for $n = k \geq 1$ and consider the case when $n = k + 1$. So we have $x_1 \otimes (x_2 \cdots x_{k+2})$. By Eq. (20) and the induction hypothesis, we have

$$x_1 \otimes (x_2 \cdots x_{k+1}) = (x_1 \otimes 1) \bar{\circ} (x_2 \otimes (x_3 \cdots x_{k+2})) = (x_1 \otimes 1) \bar{\circ} ((x_2 \otimes 1) \bar{\circ} \cdots \bar{\circ} (x_{k+1} \otimes 1)),$n

as needed. □

Thus, in order to get a totally compatible disemigroup homomorphism, the restriction of $\bar{f}'$ on $\mathcal{X}_2$ must satisfy

$$\bar{f}'(x_1 \otimes (x_2 \cdots x_{n+1})) = \bar{f}'((x_1 \otimes 1) \bar{\circ} \cdots \bar{\circ} (x_{n+1} \otimes 1)) = \bar{f}'(x_1 \otimes 1) \circ \cdots \circ \bar{f}'(x_{n+1} \otimes 1) = f(x_1) \circ \cdots \circ f(x_{n+1}). \text{ (by Eq. (28))}$$

This again agrees with $\bar{f}$ in Eq. (23).

**Case 3.** $\bar{f}'$ agrees with $\bar{f}$ on $\mathcal{X}_3$. By Eq. (17), we have

$$(x_1 \cdots x_m) \otimes (x_{m+1} \cdots x_{m+n}) = ((x_1 \cdots x_{m-1}) \otimes 1) \bar{\bar{\ast}} (x_m \otimes (x_{m+1} \cdots x_{m+n})).$$

Thus the $\bar{f}'$ in Eq. (30) can be uniquely defined on

$$\bar{f}' : \mathcal{X}_3 \to S$$
by
\[
\tilde{f}^\ast((x_1 \cdots x_m) \otimes (x_{m+1} \cdots x_{m+n})) = f^\ast(((x_1 \otimes 1) \otimes 1) \bar{\circ}(x_m \otimes (x_{m+1} \cdots x_{m+n}))) = (f_1 \otimes 1) \ast \cdots \ast f_1 \otimes 1) \bar{\circ} ((f_m \otimes 1) \otimes \cdots \otimes (f_{m+n} \otimes 1))
\]
This again agrees with \(\tilde{f}\). Therefore \(\tilde{f}^\ast = f\) and the uniqueness of \(f\) is proved.

Let \(TCDA\), \(TCDS\) and \(Sets\) be the category of totally compatible dialgebras, the category of totally compatible semigroups and the category of sets, respectively. Then the forgetful functor from \(TCDA\) to \(Sets\) is the composition of the forgetful functor from \(TCDA\) to \(TCDS\) followed by the forgetful functor from \(TCDS\) to \(Sets\). Thus the corresponding adjoint (free) functor from \(Sets\) to \(TCDA\) is the composition of the free functor from \(Sets\) to \(TCDS\) followed by the free functor from \(TCDS\) to \(TCDA\). The former is given by Item (a) and the latter is given by taking the disemigroup algebra:
\[
k(S(X) \times M(X)) \cong (kS(X)) \otimes (kM(X)) = k(X)^0 \otimes k(X),
\]
with the operations \(\bar{\ast}\) and \(\bar{\circ}\) extended by bilinearity.

As an application of Theorem 4.1, we show that free totally compatible algebras provide examples of Rota-Baxter totally compatible dialgebras as well as totally compatible dialgebras.

**Proposition 4.3.** Let \((F(X), \bar{\ast}, \bar{\circ})\) be the free totally compatible dialgebra on \(X\) in Theorem 4.1. Then \((F(X), \bar{\ast}, \bar{\circ})\) has a Rota-Baxter operator \(P\) given by
\[
P((x_1 \cdots x_m) \otimes (x_{m+1} \cdots x_{m+n})) = (x_1 \cdots x_{m+n}) \otimes 1,
\]
\[
P((x_1 \cdots x_m) \otimes 1) = (x_1 \cdots x_m) \otimes 1, \quad \forall x_i \in X, 1 \leq i \leq m + n.
\]

**Proof.** Let \(x = (x_1 \cdots x_m) \otimes (x_{m+1} \cdots x_{m+n})\) and \(y = (x_{m+n+1} \cdots x_{m+n+k}) \otimes (x_{m+n+k+1} \cdots x_{m+n+k+\ell})\). Then from the definitions of \(\bar{\ast}\), \(\bar{\circ}\) and \(P\), we directly check that each of \(P(x)\bar{\ast}P(y)\), \(P(x\bar{\ast}y)\), \(P(P(x)\bar{\circ}y)\) and \(P(x\bar{\circ}y)\) equals to \((x_1 \cdots x_{m+n+k+\ell}) \otimes 1\). Hence,
\[
P(x)\bar{\ast}P(y) = P(x\bar{\ast}P(y)) + P(P(x)\bar{\circ}y) + P(x(-\bar{\circ})y).
\]
Therefore, \((F(X), \bar{\ast}, \bar{\circ}, P)\) is Rota-Baxter totally compatible dialgebra.

5. **Totally compatible Lie dialgebras, Rota-Baxter operators and PostLie algebras**

In this section we study variations of compatible Lie dialgebra and Rota-Baxter operators on them. We generalize the relationship between associative algebras and Lie algebras to the relationship between totally compatible dialgebras and totally compatible Lie dialgebras. We also generalize the relationship between Rota-Baxter Lie algebras and PostLie algebras to the relationship between Rota-Baxter totally compatible Lie dialgebras and PostLie algebras.
5.1. **Compatible Lie dialgebras.** The origin of compatible Lie dialgebras is the following definition.

**Definition 5.1.** [13, 14, 18, 24] Let \((V, [ , ]_1)\) and \((V, [ , ]_2)\) be two Lie algebras. They are called compatible if for any \(\alpha, \beta \in k\), the following product
\[
[x, y] = \alpha[x, y]_1 + \beta[x, y]_2, \forall x, y \in V,
\]
defines a Lie algebra.

**Proposition 5.2.** [18, 13, 14, 24]. Let \((V, *)\) and \((V, \odot)\) be two Lie algebras. Then the following conditions are equivalent:

(a) \((V, [ , ]_1)\) and \((V, [ , ]_2)\) are compatible.

(b) The following equation holds:
\[
([x, y]_1, z)_2 + ([z, x]_1, y)_2 + ([x, y]_2, z)_1 + ([z, x]_2, y)_1 + ([y, z]_2, x)_1 = 0, \quad \forall x, y, z \in V.
\]

Following [27], we give the following definitions.

**Definition 5.3.** Consider a triple \((V, [ , ]_1, [ , ]_2)\) where \([ , ]_1\) and \([ , ]_2\) are Lie brackets.

(a) The triple is called a compatible Lie dialgebra if Eq. (33) holds.

(b) The triple is called a totally compatible Lie dialgebra if
\[
([x, y]_1, z)_2 = ([x, y]_2, z)_1,
\]
\[
([x, y]_1, z)_2 + ([z, x]_1, y)_2 + ([x, y]_2, z)_1 + ([z, x]_2, y)_1 + ([y, z]_2, x)_1 = 0, \quad \forall x, y, z \in V.
\]

A compatible Lie dialgebra is a Lie^2-algebra for the operad \(\text{Lie}^2\) in [27] Proposition 1.15 and a totally compatible Lie dialgebra is a \(\text{2Lie}\)-algebra for the operad \(\text{2Lie}\) in [27] Definition 1.13.

**Remark 5.4.** If \((V, [ , ]_1, [ , ]_2)\) is a totally compatible Lie dialgebra, then for any choice of \(i_1, i_2 = j_1, j_2 = k_1, k_2 = 1, 2\), we have
\[
([x, y]_{i_1}, z)_{i_2} + ([z, x]_{j_1}, y)_{j_2} + ([y, z]_{k_1}, x)_{k_2} = 0.
\]

**Proposition 5.5.** Let \((A, *, \odot)\) be a totally compatible dialgebra. Define
\[
[x, y]_1 := x * y - y * x, \quad [x, y]_2 := x \odot y - y \odot x, \forall x, y \in A.
\]

Then \((A, [ , ]_1, [ , ]_2)\) is a totally compatible Lie dialgebra.

**Proof.** Since \(*\) and \(\odot\) are associative, the brackets \([ , ]_1\) and \([ , ]_2\) are Lie brackets. Note that for any \(x, y, z \in A\), we have
\[
([x, y]_1, z)_2 = (x * y) \odot z - (y * x) \odot z - z \odot (x * y) + z \odot (y * x),
\]
\[
([x, y]_2, z)_1 = (x \odot y) * z - (y \odot x) * z - z * (x \odot y) + z * (y \odot x).
\]

Hence Eq. (34) holds. Eq. (35) is verified by a similar calculation. \(\square\)

5.2. **Rota-Baxter totally compatible Lie dialgebras and PostLie algebras.** We recall the concept of a PostLie algebra [29].

**Definition 5.6.** A PostLie algebra is a \(k\)-module \(L\) with two bilinear operations \(\circ\) and \([ , ]\) that satisfy the relations:
\[
[x, y] = -[y, x],
\]
\[
[[x, y], z] + [[z, x], y] + [[y, z], x] = 0,
\]
\[
\text{brackets (39)}
\]
which is zero by Eq. (41)

\[ z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0, \forall x, y \in L. \]

**Theorem 5.7.** Consider a triple \((V, [\ , \ ], 1, [\ , \ ], 2, P)\), where

(a) \([\ , \ ]_1\) and \([\ , \ ]_2\) are Lie brackets,

(b) \(\text{(42)}\)

\[[x, y], z]\_2 + [[z, x], y]_1 + [[y, z], x]_2 = 0, \forall x, y, z \in V.\]

(c) \(\text{(43)}\)

\[ [P(x), P(y)]_1 = P[P(x), y]_1 + P[x, P(y)]_1 + P[x, y]_2, \forall x, y \in V.\]

Then the operations

\(\text{(44)}\)

\[ x \circ y := [P(x), y]_1, \quad [x, y] := [x, y]_2, \quad \forall x, y \in V, \]

define a PostLie algebra \((V, \circ, [\ , \ ])\). In particular, if \((V, [\ , \ ]_1, [\ , \ ]_2)\) is a totally compatible Lie dialgebra, then the operations in Eq. (44) define a PostLie algebra.

**Proof.** We only need to verify Eqs. (40) and (41).

By Eq. (44), the left hand side of Eq. (40) is

\[ [P(x), [P(y), z]]_1 - [P[P(x), y], z]_1 - [P(y), [P(x), z]]_1 + [P[P(y), x], z]_1 + [P[y, x], z]_1 \]

\[ = [P(x), [P(y), z]]_1 + [[P(y), P(x)]_1 - P([y, x])_2]_1, \quad [P(y), [P(x), z]]_1 + [P[y, x], z]_1 \]

\[ - [P(y), [P(x), z]]_1 + [P[y, x], z]_1 \quad \text{(by Eq. (43))} \]

\[ = [P(x), [P(y), z]]_1 + [[P(y), P(x)]_1 - P([y, x])_2]_1 \quad \text{[P(y), [P(x), z]]}_1 \]

\[ = [z, P(y)]_1, [P(x)]_1 + [[P(y), P(x)]_1 - P([y, x])_2]_1 \quad [P(x), z]_1, [P(y)]_1 \]

which is zero by the Jacobi identity of the Lie bracket \([\ , \ ]_1\).

On the other hand, the left hand side of Eq. (41) is

\[ [P(z), [x, y]]_2 - [[P(z), x], y]_2 - [x, [P(z), y]]_2 = -[[x, y], P(z)]_1 - [[P(z), x], y]_2 - [[y, P(z)]_1, x]_2 \]

which is zero by Eq. (42) where \(x, y, z\) are replaced by \(y, P(z), x\) respectively. \(\square\)

The following commutative diagram of categories has been established in [4, 5].

\[
\begin{array}{ccc}
\text{Rota-Baxter algebras} & \xrightarrow{[x, y] := x \circ y - y \circ x} & \text{Rota-Baxter Lie algebras} \\
of weight 1 (R, *, P) & & of weight 1 (R, [\ , \ ], P) \\
\text{Tridendriform algebras} & \xrightarrow{x \circ y := x * P(y)} & \text{PostLie algebras} \\
(R, <, >, \cdot) & & (R, \circ, [\ , \ ]) \\
\end{array}
\]
To summarize, we have the following refinement of the above commutative diagram.

\[
\begin{array}{ccc}
\text{Rota-Baxter algebras} & \xrightarrow{[x,y]=x\star y-y\circ x} & \text{Rota-Baxter Lie algebras} \\
of weight 1 \,(R,*,P) & & \text{of weight 1} \,(R,[\ ,\ ],P)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Rota-Baxter TC dialgebras} & \xrightarrow{[x,y]=x\star y-y\circ x} & \text{Rota-Baxter TC Lie dialgebras} \\
of weight 1 \,(R,*,\ominus,P) & & \text{of weight 1} \,(R,[\ ,\ ],[\ ,\ ]_2,P)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Tridendriform algebras} & \xrightarrow{x \circ y = x \star y - y \circ x} & \text{PostLie algebras} \\
(R,\prec,\succ,\cdot) & & (R,\circ,[\ ,\ ])
\end{array}
\]

Here the top two vertical maps are inclusions of categories and TC is the abbreviation of totally compatible.

**Acknowledgement** C. Bai would like to thank the support by NSFC (10920161) and SRFDP (200800550015). L. Guo thanks NSF grant DMS-1001855 for support.

**References**

[1] M. Aguiar, Infinitesimal Hopf algebras, *Contemp. Math.* **267** (2000) 1-29.
[2] C. Bai, Double constructions of Frobenius algebras, Connes cocycles and their duality, *J. Noncommut. Geom.* **4** (2010) 476-530.
[3] C. Bai, L. Guo and X. Ni, 0-Operators on associative algebra and dendriform algebras, arXiv:1003.2432.
[4] C. Bai, L. Guo and X.Ni, Nonabelian Generalized Lax Pairs, the Classical Yang-Baxter Equation and PostLie Algebras, *Commun. Math. Phys.* **297** (2010) 553-596.
[5] C. Bai, L. Guo and X. Ni, 0-operators on associative algebras, associative Yang-Baxter equations and dendriform algebras (a survey), to appear in “Proceeding of International Workshop on Quantized Algebra and Physics”, World Scientific, 2011.
[6] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960) 731-742.
[7] Y. Chen and Q. Mo, Embedding dendriform dialgebra into its universal enveloping Rota-Baxter algebra, *Proc. Amer. Math. Soc.* **139** (2011), 4207-4216.
[8] V. Dotsenko, Compatible associative products and trees, *Algebra and Number Theory* **3** (2009) 567586.
[9] V. V. Dotsenko and A. S. Khoroshkin, Character formulas for the operad of a pair of compatible brackets and for the bi-Hamiltonian operad, *Funct. Anal. Appl.* **41** (2007) 117.
[10] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, *Lett. Math. Phys.* **61** (2002) 139-147.
[11] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, *J. Pure Appl. Algebra* **212** (2008) 320-339.
[12] I. Z. Golubchik and V.V. Sokolov, Generalized operator Yang-Baxter equations, integrable ODES and nonassociative algebras, *J. Nonlinear Math. Phys.* **7** (2000) 184-197.
[13] I. Z. Golubchik and V.V. Sokolov, Compatible Lie brackets and integrable equations of the principal chiral model type, *Funct. Anal. Appl.* **36** (2002) 172-181.
[14] I. Z. Golubchik and V.V. Sokolov, Compatible Lie brackets and the Yang-Baxter equation, *Theor. Math. Phys.* **146** (2006) 159-169.
[15] I. Z. Golubchik and V.V. Sokolov, Factorization of the loop algebras and compatible Lie brackets, *J. Nonlinear Math. Phys.* **12** (2005), suppl. 1, 343-350.

[16] L. Guo, An Introduction to Rota-Baxter Algebras, book to be published by Higher Education Press (China) and International Press (US), http://andromeda.rutgers.edu/liguo/rbabook.pdf.

[17] L. Guo and W. Keigher, Baxter algebras and shuffle products, *Adv. Math.* 150 (2000), 117–149.

[18] B.A. Kupershmidt, Phase spaces of algebras, Mathematics Publications and Other Works (2010). http://trace.tennessee.edu/utk-mathpubs/2.

[19] P. Leroux, Construction of Nijenhuis operators and dendriform trialgebras, *Int. J. Math. Math. Sci.* **49-52** (2006) 2595-2615.

[20] X. Li, D. Hou and C. Bai, Rota-Baxter operators on pre-Lie algebras, *J. Nonlinear Math. Phys.* **14** (2007) 269-289.

[21] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes. Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, *Contemp. Math.* **346** (2004) 369-398.

[22] A.V. Odesskii and V.V. Sokolov, Algebraic structures connected with pairs of compatible associative algebras, *Int. Math. Res. Notices* (2006), doi: 10.1155/IMRN/2006/43734.

[23] A.V. Odesskii and V.V. Sokolov, Integrable matrix equations related to pairs of compatible associative algebras, *J. Phys. A: Math. Gen.* **39** (2006) 12447-12456.

[24] A.V. Odesskii and V.V. Sokolov, Compatible Lie brackets related to elliptic curve, *J. Math. Phys.* **47** (2006) 013506.

[25] A.V. Odesskii and V.V. Sokolov, Pairs of compatible associative algebras, classical Yang-Baxter equation and quiver representations, *Commun. Math. Phys.* **278** (2008) 83-99.

[26] G. Rota, Baxter algebras and combinatorial identities I. Bull. Amer. Math. Soc., 5, 1969, 325-329.

[27] H. Strohmayer, Operads of compatible structures and weighted partitions, *J. Pure Appl. Algebra* **212** (2008) 2522-2534.

[28] K. Uchino, Quantum analogy of Poisson geometry, related dendriform algebras and Rota-Baxter operators, *Lett. Math. Phys.* **85** (2008) 91-109.

[29] B. Vallette, Homology of generalized partition posets, *J. Pure Appl. Algebra* **208** (2007) 699-725.

**Department of Mathematics, Zhejiang University, Hangzhou 310027, China**

*E-mail address*: tangmeng@zju.edu.cn

**Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China**

*E-mail address*: baicm@nankai.edu.cn

**Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102**

*E-mail address*: liguo@newark.rutgers.edu