Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory

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Abstract — In a first part we propose an introduction to multisymplectic formalisms, which are generalisations of Hamilton’s formulation of Mechanics to the calculus of variations with several variables: we give some physical motivations, related to the quantum field theory, and expound the simplest example, based on a theory due to T. de Donder and H. Weyl. In a second part we explain quickly a work in collaboration with J. Kouneiher on generalizations of the de Donder–Weyl theory (known as Lepage theories). Lastly we show that in this framework a perturbative classical field theory (analog of the perturbative quantum field theory) can be constructed.

1 Introduction

The main question investigated in this text concerns the construction of a Hamiltonian description of classical fields theory compatible with the principles of special and general relativity, or more generally with any effort towards understanding gravitation like string theory, supergravity or Ashtekar’s theory: since space-time should merge out from the dynamics we need a description which does not assume any space-time/field splitting a priori. This means that there is no space-time structure given a priori, but space-time coordinates should instead merge out from the analysis of what are the observable quantities and from the dynamics. From this point of view, as we will see, the Lepage–Dedecker theory seems to be much more appropriate than the de Donder–Weyl one. This is the philosophy that we have followed in [21]. Here a caveat is in order, in the classical one-dimensional Hamiltonian formalism: we start with a Lagrangian action functional

\[ \mathcal{L}[c] := \int_{t_0}^{t_1} L(t, c(t), \dot{c}(t)) \, dt, \]

defined on a set of smooth\(^1\) paths \( \{t \mapsto c(t) \in \mathcal{Y}\} \). Here \( \mathcal{Y} \) is a smooth \( k \)-dimensional manifold and \( L \) is a smooth function on \([t_0, t_1] \times T \mathcal{Y}\) (\( T \mathcal{Y} \) is the tangent bundle to \( \mathcal{Y} \): we

\(^1\)here we assume for instance that \( c \) is \( C^2 \)
denote by \( q \) a point in \( Y \) and by \( v \in T_q Y \) a vector tangent to \( Y \) at \( q \). The critical points of \( L \) satisfy the Euler–Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v}(t, c(t), \dot{c}(t)) \right) = \frac{\partial L}{\partial q}(t, c(t), \dot{c}(t)).
\]

For all fixed time \( t \in [t_0, t_1] \), the Legendre transform is the mapping

\[
T Y \rightarrow T^* Y \quad (q, v) \mapsto (q, p) = (q, \partial L(t, q, v)/\partial v),
\]

where \( q \in Y, \ v \in T_q Y \) and \( p \in T^*_q Y \). In cases where, for all time \( t \), this mapping is a diffeomorphism, we define the Hamiltonian function \( H : [t_0, t_1] \times T^* Y \rightarrow \mathbb{R} \) by

\[
H(t, q, p) := \langle p, V(t, q, p) \rangle - L(t, q, V(t, q, p)),
\]

where \( (q, p) \mapsto (q, V(t, q, p)) \) is the inverse mapping of the Legendre mapping. Then it is well-known that \( t \mapsto c(t) \) is a solution of the Euler–Lagrange equations if and only if \( t \mapsto (c(t), \partial L(t, c(t), \dot{c}(t))/\partial v) =: (c(t), \pi(t)) \) is a solution of the Hamilton equations

\[
\frac{dc^i}{dt}(t) = \frac{\partial H}{\partial p_i}(t, c(t), \pi(t)), \quad \text{and} \quad \frac{d\pi_i}{dt}(t) = -\frac{\partial H}{\partial q_i}(t, c(t), \pi(t)).
\]

Thus this converts the second order Euler–Lagrange equations into the flow equation of the non autonomous vector field \( X_{H,t} \) defined over \( T^* Y \) by

\[
X_{H,t} \cdot \Omega + dH_t = 0.
\]

where \( \Omega := \sum_{i=1}^k dp_i \wedge dq^i \) is the symplectic form over \( T^* Y \), \( H_t(q, p) := H(t, q, p) \) and “\( \cdot \)” denote the interior product, i.e. for any tangent vector \( \xi \in T_{(q,p)}(T^* Y) \), \( \xi \cdot \Omega \) is the 1-form such that \( \xi \cdot \Omega(V) = \Omega(\xi, V), \forall V \in T_{(q,p)}(T^* Y) \)

Instead of viewing the dynamics as the motion of a point in some space, like for instance the phase space \( T^* Y \), we can use another approach which consists in determining how an observable quantity, such as the position or the momentum of a particle, evolves. This is achieved by using the Poisson bracket operation

\[
C^\infty(T^* Y) \times C^\infty(T^* Y) \rightarrow C^\infty(T^* Y) \quad (F, G) \mapsto \{F, G\},
\]

where

\[
\{F, G\} := \sum_{i=1}^k \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} \right).
\]

Then, for all Hamiltonian trajectory \( t \mapsto (c(t), \pi(t)) \), and for all \( F \in C^\infty(T^* Y) \), we have

\[
\frac{dF(c(t), \pi(t))}{dt} = \{H, F\}(c(t), \pi(t)).
\]
For example in the particular case where the variational problem is autonomous (i.e. \( L \) does not depend on \( t \)) then \( H \) does not depend on time and we deduce from the skewsymmetry of the Poisson bracket that the energy is conserved along the trajectories, a special case of Noether theorem when the problem is invariant by time translation.

Eventually this formulation of the dynamics is a good preliminary for modelling the evolution of the quantum version of our problem: for instance by replacing the functions in \( C^\infty(T^*Y) \) by Hermitian self-adjoint operators and the Poisson bracket by the commutator \([\cdot,\cdot]\) we “guess” the Heisenberg evolution equation

\[
\frac{i\hbar}{\partial t} \hat{F} = [\hat{H}, \hat{F}],
\]

consequently the commutation relations \([\hat{p}_i, \hat{q}^j] = i\hbar \delta^j_i\), is nothing but a formalisation of Heisenberg uncertainty principle.

All that leads to a now “well understood” strategy of building a mathematical description of a quantum particle governed by a Hamiltonian functions \( H \) (with the restriction that, among other things, the correspondence \( H \mapsto \hat{H} \) is far from being uniquely defined).

Starting from a variational formulation of Newton’s law of mechanics, it is thus possible to formally derive the Schrödinger equation more or less by following the steps discussed above.

Now the more challenging question is to produce a similar analysis for fields theories.

Quantum fields theory\(^2\) results from the efforts of physicists in order to cure the shortcoming of the Schrödinger equation. Indeed, this latter equation is not invariant by the group

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\(^2\)Starting from the framework of classical physics, the concept of a field at first might evoke ideas about macroscopic systems, for example velocity fields or temperature fields in fluids and gases, etc. Fields of this kind will not concern us, however; they can be viewed as derived quantities which arise from an averaging of microscopic particle densities. Our subjects are the fundamental fields that describe matter on a microscopic level: it is the quantum-mechanical wave function \( \psi(x,t) \) of a system which can be viewed as a field from which the observable quantities can be deduced. In quantum mechanics the wave function is introduced as an ordinary complex-valued function of space and time. In Dirac’s terminology it has the character of a “\( c \) number”. Quantum field theory goes one step further and treats the wave function itself as an object which has to undergo quantization. In this way the wave function \( \psi(x,t) \) is transmuted into a field operator \( \hat{\psi}(x,t) \), which is an operator-valued quantity (a “\( q \) number”) satisfying certain commutation relations. This process, often called “second quantization”, is quite analogous to the route that in ordinary quantum mechanics leads from a set of classical coordinates \( q_i \) to a set of quantum operators \( \hat{q}_i \). There is one important technical difference, though, since \( \hat{\psi}(x,t) \) is a field, i.e.an object which depends on the coordinate \( x \). The latter plays the role of a “continuous-valued” index, in contrast to the discrete index \( i \), which labels the set \( q_i \). Field theory therefore is concerned with systems having an infinite number of degrees of freedom. The concept of field quantization has far-reaching consequences and is one of the cornerstones of modern physics. Field quantization provides an elegant language to describe particle systems. Moreover the theory naturally leads to the insight that there are field quanta which can be created and annihilated. These field quanta come in many guises and are found in virtually all areas of physics.

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of special relativity, the Poincaré group, but by the (projective) Galilean group. This was one of the motivations which led Dirac to its famous equation. Another concern was the interactions of charged particles and a relativistic field, namely the electro-magnetic field governed by Maxwell equations. A second reason for fields theory is that, unlike the Schrödinger equation, they allow interactions of a variable number of particles. The simplest example is the Klein–Gordon equation for scalar fields $\varphi : \mathbb{R}^{1,3} \rightarrow \mathbb{R}$ (or $\mathbb{C}$):

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + m^2 \varphi = 0.$$  

This is the Euler–Lagrange equation for the variational problem

$$\mathcal{L}[\varphi] := \int_{\mathbb{R}^{1,3}} \frac{1}{2} \left( \frac{1}{c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 - |\nabla \varphi|^2 - m^2 |\varphi|^2 \right) dt dx^1 dx^2 dx^3.$$  

Here $\mathbb{R}^{1,3}$ is the Minkowski space. Note that the integral $\mathcal{L}[\varphi]$ may not be defined, but if $\varphi$ smooth or in $H^1_{\text{loc}}$ then, for any compact subset $K \subset \mathbb{R}^{1,3}$, the integral $\mathcal{L}_K[\varphi]$ of the Lagrangian density over $K$ is defined and we say that $\varphi$ is a critical point of $\mathcal{L}$ if and only if, for any $K$ the restriction of $\varphi$ to $K$ is a critical point of $\mathcal{L}_K$. At this level we can address the following questions: is it possible to follow the same lines as for a 1-dimensional variational problem and build a Hamiltonian formulation of the Klein–Gordon equation? And can we deduce a quantization procedure for that equation?

The answer is positive using an approach developed by physicists and is known as the canonical quantization of fields: one chooses a time coordinate $t$ over the Minkowski space-time and looks, for any value of $t_0$, at the instantaneous state of the field, i.e. the restriction of $\varphi$ on the Cauchy hypersurface $\{t = t_0\}$. Then we picture the whole history of the field as an evolution, parametrized by $t$, of a point in the infinite dimensional space $\{(x^1, x^2, x^3) \mapsto \varphi(x^1, x^2, x^3)\}$. One associates to each path in this infinite dimensional space an action, which is just the one obtained by using Fubini theorem through the splitting $\mathbb{R}^{1,3} \simeq \mathbb{R} \times \mathbb{R}^3$. Then we follow the same procedure as the one that we described at the beginning of this paper, but this time in some infinite dimensional manifold: we consider the cotangent bundle to the manifold $\{(x^1, x^2, x^3) \mapsto \varphi(x^1, x^2, x^3)\}$ and, using the Legendre transform obtained from the Lagrangian, we build a conjugate (momentum) variable $[(x^1, x^2, x^3) \mapsto \pi(t, x^1, x^2, x^3)]$ (which in this case is just $[(x^1, x^2, x^3) \mapsto \frac{\partial \varphi}{\partial t}(t, x^1, x^2, x^3)]$). Then one can write Hamilton equations, Poisson bracket and deduce the canonical quantization.

Note that all that is relatively formal but, after each step, it is possible to formulate a theory which makes sense mathematically. This is because of the very particular structure of the Klein–Gordon equation, being hyperbolic and linear. Similarly one can perform a quantization of Maxwell equations (with some extra work due to the gauge invariance). But everything breaks down as soon as the equation is nonlinear, even if the nonlinearity is very mild (the best that we can do is to compute physically relevant quantities by perturbation, even if there is no mathematical bases). So one can quantize only extremely
particular variational problems. Beside this outstanding difficulty we are faced with a further critic which is that all this procedure does not respect relativistic invariance: indeed, we were obliged to choose a time coordinate from the beginning in order to perform a Legendre transform and then to write down Hamilton’s equations, and so on. Commonly, we say that this theory is not covariant, i.e. it does not respect the (general) relativity principle which implies the independence toward a used coordinate or reference system. Consequently, this description is not quite satisfactory, even if actually one can check that the resulting quantum field does not depend on the time coordinate which was used.

We want here to consider seriously this critic: is it possible to follow a more covariant path? A method is well-known: it is the Feynman integral approach, based on the Lagrangian formulation, without using the Hamiltonian framework. It is much more suitable than the canonical approach for computing “correlations” for nonlinear (i.e. interacting) theories. However it is less constructive than the canonical approach, which has the advantage of providing us with a scheme to construct the Hilbert Fock space and the operators. So is there a covariant Hamiltonian approach? In principle it should be possible and this was suggested independently by M. Born [3] and H. Weyl [37] in 1935: it would be based on using a covariant Hamiltonian formalism for variational problems with several variables quite different from the one that we described above, which used a slicing of space-time.

The first example of such a formalism was built by C. Carathéodory [4] and another version was proposed later independently by H. Weyl [37] and T. de Donder [8]. Note that in constrast with the 1-dimensional calculus of variations, there are actually infinitely many possible theories. There were described by T.H.J. Lepage [30] and H. Boerner [4]. We shall see later how to understand within a global picture this multiplicity of formalisms. Before that we will expound the de Donder–Weyl theory, since it is the simplest one.

2 The de Donder–Weyl theory

The historical background of the formalism expounded in this section is deeply rooted in the work of C. Carathéodory, H. Weyl and T. de Donder, followed by the observation by E. Cartan [5] (who called $g_{dW}$ the de Donder form) in 1933. But it seems really to have been developped under the impulsion of W. Tulczyjew [36] in 1968 and the Polish school of mathematical physics: J. Śniatycki [34], K. Gawędski [12], J. Kijowski [26], J. Kijowski and W. Szczyrba [28], J. Kijowski and W.M. Tulczyjew [29], and through the papers of P.L. García and A. Pérez-Rendón [11] and H. Goldschmidt and S. Sternberg [14]. Various descriptions and points of view about the foundations of these theories can also be found in the books [1], [13], [32] and [33] or in the papers [15], [16], [17], [25], [31].

Let us consider the 4-dimensional space-time $\mathbb{R}^{1,3}$ as a source domain and $\mathbb{R}^k$ as a target space. A first order action functional on maps $u : \mathbb{R}^{1,3} \rightarrow \mathbb{R}^k$ is defined by means of a Lagrangian density $L$: it is a function on the variables $(x, y, v)$, where $x \in \mathbb{R}^{1,3}$, $y \in \mathbb{R}^k$.
and $v \in (\mathbb{R}^{1,3})^* \otimes \mathbb{R}^k$ is a linear map from $\mathbb{R}^{1,3}$ to $\mathbb{R}^k$ (alternatively we may consider $v$ as a $n \times k$ real matrix). Then we define the functional

$$\mathcal{L}[u] := \int_{\mathbb{R}^{1,3}} L(x, u(x), du(x))\omega,$$

where $\omega := dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The Euler–Lagrange equation for critical points is

$$\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial v^i_\mu}(x, u(x), du(x)) \right) = \frac{\partial L}{\partial y^i}(x, u(x), du(x)).$$

The de Donder–Weyl theory is simply based on the change of variable

$$\pi^\mu_\nu(x) := \partial L/\partial v^i_\mu(x, u(x), du(x)),$$

i.e. exchanging the variables $(x, y, v)$ with $(x, y, p)$ where $p \in (\mathbb{R}^{1,3})^* \otimes \mathbb{R}^k$ is given by

$$p^\mu_\nu := \partial L/\partial v^i_\mu(x, y, v).$$

This works of course if we make the assumption that $(x, y, v) \mapsto (x, y, p)$ is a diffeomorphism, an analog of the Legendre condition. In the following we shall suppose that this Legendre hypothesis is satisfied (but very interesting situations occur when precisely this Legendre condition fails, as for instance in the case of gauge theories). Then we can consider the Hamiltonian function on $\mathbb{R}^{1,3} \times \mathbb{R}^k \times (\mathbb{R}^{1,3} \otimes (\mathbb{R}^k)^*)$

$$H(x, y, p) := p^\mu_\nu v^i_\mu - L(x, y, v), \quad \text{where } v \text{ is defined implicitely by } p^\mu_\nu = \partial L/\partial v^i_\mu(x, y, v).$$

A simple computation shows that the Euler–Lagrange equations are equivalent to the system of generalized Hamilton equations

$$\sum_\mu \frac{\partial \pi^\mu_\nu(x)}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}(x, u(x), \pi(x)), \quad \frac{\partial u^i(x)}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_\nu}(x, u(x), \pi(x)). \quad (2)$$

This is a simple example of a more general situation. We can replace for example $\mathbb{R}^{1,3}$ by a smooth $n$-dimensional oriented manifold $\mathcal{X}$ and $\mathbb{R}^k$ by another $k$-dimensional manifold $\mathcal{Y}$. Then the Lagrangian density $L$ is a function on variables $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $v \in T_y\mathcal{Y} \otimes T^*_x\mathcal{X}$. Thus $L$ can be seen as a smooth function defined on the bundle over $\mathcal{X} \times \mathcal{Y}$ with fiber over $(x, y)$ equal to $T_y\mathcal{Y} \otimes T^*_x\mathcal{X}$. We denote by $T\mathcal{Y} \otimes_{\mathcal{X} \times \mathcal{Y}} T^*\mathcal{X}$ this bundle. Using some volume $n$-form $\omega$ on $\mathcal{X}$ we hence define the functional $\mathcal{L}[u] := \int_{\mathcal{X}} L(x, u(x), du(x))\omega$ on the set of maps $u : \mathcal{X} \longrightarrow \mathcal{Y}$. A similar Legendre transform can be defined, leading to a Hamiltonian function defined on the \textit{multisymplectic manifold} $T^*\mathcal{Y} \otimes_{\mathcal{X} \times \mathcal{Y}} T\mathcal{X}$.

We shall see now that this manipulation has some geometrical interpretation in a way analogous to the 1-dimensional calculus of variations. The more naive way to formulate it consists to associate to any pair of maps $x \mapsto (u(x), \pi(x))$ its graph $\Gamma$ in $T^*\mathcal{Y} \otimes_{\mathcal{X} \times \mathcal{Y}} T\mathcal{X}$, i.e. the image of $\mathcal{X} \ni x \mapsto (x, u(x), \pi(x))$. $\Gamma$ is the $n$-dimensional analog of a curve in
a symplectic manifold. Then at each point \((x, u(x), \pi(x))\) of \(\Gamma\) we attach the tangent \(n\)-multivector \(X \in \Lambda^n T_{(x,u(x),\pi(x))} (T^*\mathcal{Y} \otimes X \times Y \text{T}\mathcal{X})\) defined by

\[
X := X_1 \wedge \cdots \wedge X_n, \quad \text{where} \quad X_\mu := \frac{\partial u^i}{\partial x^\mu} \frac{\partial}{\partial y^i} + \frac{\partial \pi^j}{\partial x^\mu} \frac{\partial}{\partial p^j}.
\]

Now we define on the multisymplectic manifold \(T^*\mathcal{Y} \otimes X \times Y \text{T}\mathcal{X}\) the multisymplectic \((n+1)\)-form

\[
\Omega^* := \sum_\mu \sum_i dp^\mu_i \wedge dy^i \wedge \omega_\mu,
\]

where \(\omega_\mu := \frac{\partial}{\partial x^\mu} \mathcal{J} \omega\), i.e. \(\omega_\mu\) is the unique \((n+1)\)-form such that \(\forall V_1, \cdots, V_{n-1} \in T_{(x,y,p)} (T^*\mathcal{Y} \otimes X \times Y \text{T}\mathcal{X}), \omega_\mu(V_1, \cdots, V_{n-1}) = \omega(\frac{\partial}{\partial x^\mu}, V_1, \cdots, V_{n-1})\).

We also define the interior product of \(X\) by \(\Omega^*\) to be the unique 1-form \(X \mathcal{J} \Omega^*\) such that \(\forall V \in T_{(x,y,p)} (T^*\mathcal{Y} \otimes X \times Y \text{T}\mathcal{X}), X \mathcal{J} \Omega^*(V) = \Omega^*(X_1, \cdots, X_n, V)\). Then we can compute that

\[
X \mathcal{J} \Omega^* = (-1)^n \left( \frac{\partial u^i(x)}{\partial x^\mu} dp^\mu_i - \frac{\partial \pi^j(x)}{\partial x^\mu} dy^i \right).
\]

Hence, by comparing this last expression with the value of \(dH = \frac{\partial H}{\partial x^\mu} dx^\mu + \frac{\partial H}{\partial y^i} dy^i + \frac{\partial H}{\partial p^j} dp^j\) at \((x, u(x), \pi(x))\), we see that \(\Gamma\) is the graph of a solution of the Hamilton system of equations (2) if and only if

\[
X \mathcal{J} \Omega^* = (-1)^n dH_{(x,u(x),\pi(x))} \mod dx^\mu. \tag{3}
\]

Here “mod \(dx^\mu\)” means that the equality holds between the coefficients of \(dy^i\) and \(dp^\mu_i\) in both sides. This looks like the analog of the classical Hamilton equation (1) (except that here we do have the “mod \(dx^\mu\)” restriction). This suggests us to use \(\Omega^*\) as a replacement for the standard symplectic form. Note that \(\Omega^*\) is the differential of the generalized Poincaré-Cartan \(n\)-form \(\theta^* := \sum_\mu \sum_i p^\mu_i dy^i \wedge \omega_\mu\). Moreover, as in the case of classical mechanics, the generalized Poincaré-Cartan form encodes all the informations that we need in order to perform the Legendre transform.

Let us discuss now the second question — about the Poisson bracket. It has to be defined on functionals on the space of solutions to the generalized Hamilton equations. Thus we first consider the space \(\mathcal{E}\) of all smooth \(n\)-dimensional submanifolds \(\Gamma\) of the multisymplectic manifold which are graphs of the generalised Hamilton equations, i.e. such that for any \(m \in \Gamma\), there exists an \(n\)-multivector \(X\) which is tangent to \(\Gamma\) at \(m\) and which satisfies equation (3). We call a Hamiltonian \(n\)-curve any such submanifold. Among the functionals \(\mathcal{E} \rightarrow \mathbb{R}\) we shall restrict ourself to a particular class: this choice is motivated by the particular observable quantities used by physicists in quantum field theory. Indeed the observable functionals in the canonical field theory are integrals (smeared with smooth test functions) over a spacelike hypersurface of the space-time of either the values of fields components or the value of their time first derivative (in the latter case the time derivative...
appear because we are actually considering momenta). And it turns out that (almost) all such observable functionals (see the next section) can be described by a pair \((\Sigma, F)\), where \(\Sigma\) is a codimension 1 submanifold of the multisymplectic manifold and \(F\) is a \((n-1)\)-differential form on the multisymplectic manifold such that there exists a tangent vector field \(\xi_F\) such that

\[
dF + \xi_F \lrcorner \Omega^* = 0.
\]

\(F\) is then called an \textit{algebraic observable} \((n-1)\)-form. Then, if \(\Sigma\) satisfies suitable transversality conditions (see \cite{21}), \(\Sigma \cap \Gamma\) is a \((n-1)\)-dimensional manifold and we define the functional \(\int_{\Sigma} F\) to be

\[
\mathcal{E} \quad \to \quad \mathbb{R} \\
\Gamma \quad \mapsto \quad \int_{\Sigma \cap \Gamma} F.
\]

Then a bracket can be defined between two observable functionals \(\int_{\Sigma} F\) and \(\int_{\Sigma} G\) by

\[
\left\{ \int_{\Sigma} F, \int_{\Sigma} G \right\} := \int_{\Sigma} \{F, G\}, \quad \text{where} \quad \{F, G\} := \xi_F \wedge \xi_G \lrcorner \Omega^*.
\]

Here \(\xi_F \wedge \xi_G \lrcorner \Omega^*\) is the unique \((n-1)\)-form such that for all tangent vectors \(V_1, \ldots, V_{n-1}\),

\[
\xi_F \wedge \xi_G \lrcorner \Omega^* (V_1, \ldots, V_{n-1}) = \Omega^* (\xi_F, \xi_G, V_1, \ldots, V_{n-1}).
\]

It can be checked that this Poisson bracket satisfies “good” properties: first of all it coincides with the standard Poisson bracket of the canonical theory of fields, second it satisfies the Jacobi identity, if for instance the observable \((n-1)\)-forms decrease to zero at infinity.

The relatively naive description presented here deserves some critics:

- Equation (3) holds only “mod \(dx^\mu\)”, which is not very aesthetic: this reflects a disymmetry between the space-time variables and the field component variables.

- Very important “observable” quantities are the components of the energy-momentum tensor \(T_{\mu\nu}\). This tensor is linked through the Noether theorem to space-time translations symmetries in special relativity or to diffeomorphism invariance in general relativity. It is important in quantum field theory, where it helps to construct the Hamiltonian in the standard canonical picture, but also crucial in general relativity, since it models the way the distribution of energy and matter bend the space-time geometry through the Einstein equation \(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu}\). In the previous setting there is no clear representation of the energy-momentum tensor.

- Most important variational problems in physics involve fiber bundles: the Maxwell–Dirac, the Yang–Mills–Dirac or the Yang–Mills–Higgs theories for the electrodynamics, the electroweak and the strong forces and also the general relativity. They does not fit into the above formalism: indeed the field components cannot be treated as coordinates on an independant manifold and one needs a more general framework.

\footnote{nevertheless a notable difference with the Poisson bracket of classical mechanics is that we cannot easily make sense of the product of two such forms. Attempts in this direction are proposed in \cite{22}; we suggest an alternative point of view at the end of Paragraph 5.4 in this text.}
The two first critics can be cured by adding to the set of variable \((x, y, p)\) a further variable \(e \in \mathbb{R}\), canonically conjugate to the space-time volume form \(\omega\). Then we consider on \((T^*\mathcal{Y} \otimes X \times \mathcal{Y} T\mathcal{X}) \times \mathbb{R}\) the multisymplectic form

\[
\Omega^{\text{dDW}} := de \wedge \omega + \sum_{\mu} \sum_{i} dp^\mu_i \wedge dy^i \wedge \omega_\mu = de \wedge \omega + \Omega^*.
\]

Any solution of the Hamilton equations can be represented by a smooth \(n\)-dimensional submanifold \(\Gamma\) in \((T^*\mathcal{Y} \otimes X \times \mathcal{Y} T\mathcal{X}) \times \mathbb{R}\) such that, instead of equation (3) we have

\[
X \lrcorner \Omega^{\text{dDW}} = (-1)^n d\mathcal{H}(x,u(x),\epsilon(x),\pi(x)), \quad \text{where} \quad \mathcal{H}(x, y, e, p) := e + H(x, y, p). \tag{4}
\]

Indeed it can be achieved by choosing \(\epsilon(x)\) such that \(\mathcal{H}(x, u(x), \epsilon(x), \pi(x))\) is constant along \(\Gamma\). We can hence avoid the “mod \(dx^\mu\)” restriction. Moreover \(\Omega^{\text{dDW}}\) is the differential of a Poincaré–Cartan form \(\theta^{\text{dDW}} := e \omega + p^\mu_i dy^i \wedge \omega_\mu\) and the component of the stress-energy tensor can be recovered as coefficients of the observable \((n - 1)\)-forms \(\frac{\partial}{\partial x^\mu} \lrcorner \theta^{\text{dDW}}\).

Thus it remains to understand better the third point: to find a geometrical framework for generalizing the above construction to more general variational problems. This question is even more accurate now since we added a further variable \(e\) whose geometrical sense needs to be understood. For all variational problems on fields which are sections of a bundle \(\mathcal{F}\), the usual approach is to consider the affine first jet bundle associated to these sections and to build the multisymplectic manifold as a dual of this affine jet bundle. References concerning this approach are [18] or [10].

3 A general framework: multisymplectic manifolds

The above theory is an example of a multisymplectic manifold \((\mathcal{M}, \Omega)\): a differential manifold \(\mathcal{M}\) equipped with a multisymplectic \((n + 1)\)-form \(\Omega\). An \((n + 1)\)-form \(\Omega\) is multisymplectic if and only if

- it is closed: \(d\Omega = 0\)
- it is nondegenerate: \(\forall \xi \in T_m \mathcal{M}, \xi \lrcorner \Omega = 0 \implies \xi = 0\).

Given a multisymplectic manifold \((\mathcal{M}, \Omega)\) we can define the notion of algebraic observable \((n - 1)\)-forms and use them, together with hypersurfaces \(\Sigma\) to define observable functionals as in the preceding section. Poisson brackets are defined in a similar way.

There is also a notion which leads to a generalization of the standard relation \(\frac{dF}{dt} = \{H, F\}\) of classical mechanics. Given the Hamiltonian function \(\mathcal{H}\), for any algebraic observable \((n - 1)\)-form \(F\) we let

\[
\{\mathcal{H}, F\} := -d\mathcal{H}(\xi_F). \tag{5}
\]
Although this definition and the notation used here are reminiscent from those of Poisson bracket, it is not clear whether we may consider this operation a Poisson bracket (because in particular, $H$ is not an observable form nor any observable quantity). Thus we prefer to call this operation a Poisson pseudobracket. Anyway it is related to the following useful result: assume that $\Gamma$ is a Hamiltonian $n$-curve and $F$ is an algebraic observable $(n-1)$-form, then we have

$$dF|_{\Gamma} = \{H, F\} \omega|_{\Gamma},$$

where $dF|_{\Gamma}$ is the restriction of $dF$ to $\Gamma$ (see [22], [20]). The proof is straightforward: for all $q \in \Gamma$ we let $(X_1, \ldots, X_n)$ be a basis of $T_q\Gamma$ such that $\omega(X_1, \ldots, X_n) = 1$. Then

$$dF(X_1, \ldots, X_n) = -\xi_F \lrcorner \Omega(X_1, \ldots, X_n) = -(-1)^n X_1 \wedge \cdots \wedge X_n \lrcorner \Omega(\xi_F) = -dH(\xi_F) = \{H, F\}.$$  

However in a work in collaboration with J. Kouneiher in [21] we point out that the class of algebraic observable $(n-1)$-forms can be enlarged as follows: an $(n-1)$-form $F$ is called an observable $(n-1)$-form if and only if, $\forall q \in M$, if $X$ and $\tilde{X}$ are two decomposable $n$-multivectors in $\Lambda^n T_q M$ such that $X \lrcorner \Omega = \tilde{X} \lrcorner \Omega$, then $dF(X) = dF(\tilde{X})$ (see [21] for details). One can then define the pseudobracket $\{H, F\}$ to be equal to $dF(X)$, where $X$ is any decomposable $n$-multivector such that $X \lrcorner \Omega = (-1)^n dH$. It is easy to show that this pseudobracket agrees with the previous one (5) for algebraic observable $(n-1)$-forms and that the dynamical relation (6) can hence be generalized to (non necessarily algebraic) observable $(n-1)$-forms. We believe that this point of view is more natural, being directly related to the fundamental identity (6). When we shall follow this point of view in the next Section to the question of observable $(p-1)$-forms, for $1 \leq p < n$, it will lead also to a new definition of these observable $(p-1)$-forms.

Now what is the difference between the two definitions? On the one hand every algebraic observable $(n-1)$-form $F$ is an observable $(n-1)$-form since $X \lrcorner \Omega = \tilde{X} \lrcorner \Omega$ implies $dF(X) = -\xi_F \lrcorner \Omega(X) = -\xi_F \lrcorner \Omega(\tilde{X}) = dF(\tilde{X})$. On the other hand the converse is false: consider for example the de Donder–Weyl theory expounded above for maps $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$, for $k > 1$. Then $y^1 dy^2 \wedge dx^3 \wedge \cdots \wedge dx^n$ is observable but not algebraic observable. We propose to call a multisymplectic manifold on which the set of algebraic observable $(n-1)$-forms coincide with the set of observable $(n-1)$-forms a pataplectic manifold. We shall see examples in the next section.

## 4 Generalizations

The formalism that we have presented in Section 2 is based on the de Donder–Weyl theory, which is a particular case among infinitely many theories which were classified in 1936 by T.H.J. Lepage [30]. A geometric and universal setting for describing simultaneously all these theories was expounded first by P. Dedecker in 1953 [7]. Here the idea is that
we can view each first order variational problem as a variational problem on a class of $n$-
-dimensional submanifolds $G$ of some manifold $\mathcal{N}$ of dimension $n + k$: $G$ could be the graph
of a map between two manifolds, the image of a section of a fiber bundle or something
more general. Then the Lagrangian density can be identified with a function $L$ defined on
the Grassmannian bundle $Gr^n\mathcal{N}$, i.e. the bundle over $\mathcal{N}$ whose fiber at any point $q \in \mathcal{N}$
is the set of oriented $n$-dimensional vector subspaces of $T_q\mathcal{N}$. This is a kind of analog of the
(projective) tangent bundle of a manifold that is used in Lagrangian Mechanics. Then
the analog of the cotangent bundle is the bundle of differential $n$-forms on $\mathcal{N}$, $\Lambda^n T^*\mathcal{N}$.

Note that, in contrast with classical mechanics (which corresponds to the de Donder–Weyl theory, $1 + \dim Gr$, the Grassmannian bundle
the analog of the cotangent bundle is the bundle of differential $n$-dimensional submanifolds
the set of oriented $n$-dimensional vector subspaces of $T_q\mathcal{N}$. This is a kind of analog of the
(projective) tangent bundle of a manifold that is used in Lagrangian Mechanics. Then
the analog of the cotangent bundle is the bundle of differential $n$-forms on $\mathcal{N}$, $\Lambda^n T^*\mathcal{N}$.

Note that, in contrast with classical mechanics (which corresponds to $n = 1$) or with the de Donder–Weyl theory, $1 + \dim Gr^n\mathcal{N} = 1 + n + k + nk$ is in general strictly less than $\dim \Lambda^n T^*\mathcal{N} = n + k + \frac{(n + k)!}{n!k!}$. So the Legendre transform is replaced by a Legendre

The Legendre correspondence which — generically — associates to each “multivelocity” $T \in Gr^n\mathcal{N}$ an
affine subspace of $\Lambda^n T^*\mathcal{N}$ of dimension $\frac{(n + k)!}{n!k!} - nk - 1$ called pseudofiber by Dedecker.
Now each Lepage theory (one instance being the de Donder–Weyl one, when it makes
sense) corresponds to choosing a submanifold of $\Lambda^n T^*\mathcal{N}$ which intersects transversally all pseudofibers through exactly one point (if the Legendre condition holds).

Note that, to my knowledge, almost all the literature on the subject focuses on the de
Donder–Weyl theory, excepted [7] and [27]. It seems however important (if in particular
we are interested in gravitation theories) to understand all the development expounded
in the previous paragraph in the Lepage–Dedecker framework. This has been addressed
in collaboration with J. Kouneiher in our papers [20] and [21].

In order to understand the difference let us see a very simple example: variational
problems on maps $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Let us denote by $\omega := dx^1 \wedge dx^2$ the volume form on
the domain space $\mathbb{R}^2$. A map is pictured by its graph, a 2-dimensional submanifold $G$ of
$\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, whose projection on the first factor $\mathbb{R}^2$ is a diffeomorphism (or equivalently s.t. $\omega|_G \neq 0$). Given a point $(x, y) \in G \subset \mathbb{R}^4$ the tangent plane to $G$ at $(x, y)$ is spanned by
vectors $X_1 = \frac{\partial}{\partial x^1} + v_1 \frac{\partial}{\partial y^1}$ and $X_2 = \frac{\partial}{\partial x^2} + v_2 \frac{\partial}{\partial y^2}$, where $(v_1, v_2, v_1, v_2) \in \mathbb{R}^4$. So the set of all possible tangent planes to such $G$’s is parametrized by the variables $v_i^j$ and we can use the local coordinates $x^\mu$, $y^i$ and $v_i^j$ on $Gr^2\mathbb{R}^4$. Now the analog of the cotangent bundle in this situation is $\Lambda^2 T^*\mathcal{N}$. Using coordinates $x^\mu$ and $y^i$ on $\mathbb{R}^4$, a basis of the 6-dimensional space $\Lambda^2 T^*_{(x,y)} \mathbb{R}^4$ is $(dx^1 \wedge dx^2, dy^1 \wedge dx^i, dy^1 \wedge dy^2)$. Thus any 2-form $P \in \Lambda^2 T^*_{(x,y)} \mathbb{R}^4$ can be identified with the coordinates $(\epsilon, p_i^\mu, r)$ such that $P = \epsilon dx^1 \wedge dx^2 + \epsilon_{\mu\nu} p_i^\mu dy^i \wedge dx^\nu + r dy^1 \wedge dy^2$, where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$.

Now the Legendre correspondence is obtained by the following. Given some $P \simeq (\epsilon, p_i^\mu, r) \in 
\Lambda^2 T^*_{(x,y)} \mathbb{R}^4$ and some tangent space $T \simeq (v_i^\mu) \in Gr^2_{(x,y)} \mathcal{N}$, we define the pairing $(T, P) := P(X_1, X_2)$, where $(X_1, X_2)$ forms a basis of $T$ such that $\omega(X_1, X_2) = 1$. Using local coordinates we have here $(T, P) = \epsilon + p_i^\mu v_i^\mu + r(v_1^2 v_2^3 - v_2^1 v_1^3)$. We also define the function
$W(x, y, T, P) = \langle T, P \rangle - L(x, y, T)$, where $L$ is the Lagrangian density (identified here with a function on $Gr^2\mathbb{R}^4$). Then we say that $T$ is in correspondence with $P$ if and only
if $\frac{\partial W}{\partial T}(x, y, T, P) = 0$. This relation writes in local coordinates

$$p^\mu_i + \epsilon^{\mu\nu} \epsilon_{ij} v^j_\nu r = \frac{\partial L}{\partial v^i_\mu}(x^\mu, y^i, v^i_\mu).$$

As announced previously, given some $(x^\mu, y^i, v^i_\mu)$ the solution to this equation is in general not unique, but it is actually an affine plane (the pseudofiber) inside $\Lambda^2 T^*_x \mathbb{R}^4$, parallel to the vector plane spanned by

$$dx^1 \wedge dx^2 \text{ and } (v^1_1 v^2_2 - v^1_2 v^2_1) \; dx^1 \wedge dx^2 - \epsilon_{ij} v^j_\nu dy^\nu \wedge dx^\nu + dy^1 \wedge dy^2.$$

Hence, inside $\Lambda^2 T^*_x \mathbb{R}^4$, pseudofibers form a 4-parameters family of non parallel affine planes. The subset $\mathcal{M}_{(x,y)}$ of $\Lambda^2 T^*_x \mathbb{R}^4$ filled by all these pseudofibers is always dense — meaning that the Legendre correspondence is “almost surjective” — and it can inverted on a dense subset. This contrasts strongly with situations in classical mechanics (where the Legendre transform may degenerate, i.e. its image may be reduced to a strict submanifold, in particular if the variational problem is parametrization invariant) or in fields theory, if we restrict ourself to the de Donder–Weyl theory or if we use the standard canonical approach (here the Legendre transform degenerates as soon as we have a gauge invariance, a phenomenon known as Dirac’s constraints). A Hamiltonian function $\mathcal{H}$ can be defined on $\mathcal{M} := \bigcup_{x, y} \mathcal{M}_{(x,y)}$ by setting $\mathcal{H}(x, y, P) := W(x, y, T, P)$, where $T$ is an implicit function of $(x, y, P)$ through the relation $\frac{\partial W}{\partial T}(x, y, T, P) = 0$.

The more convincing example is the trivial variational problem: We just take $L = 0$, so that any map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a critical point of our variational problem ! Then the image $\mathcal{M}_{(x,y)}$ of the Legendre correspondence is the union of the complementary of the hyperplane $r = 0$ and of $\{(e, p^\mu_i, r) = (e, 0, 0) / e \in \mathbb{R}\}$. The Hamiltonian function is given by $\mathcal{H}(x, y, e, 0, 0) = e$ and

$$\mathcal{H}(x, y, e, p^\mu_i, r) = e - \frac{p^1_1 p^2_2 - p^1_2 p^2_1}{r},$$

if $r \neq 0$. One can then check that all Hamiltonian 2-curves are of the form

$$\Gamma = \{(x, u(x), e(x)) dx^1 \wedge dx^2 + \epsilon_{\mu\nu} p^\mu_i(x) dy^i \wedge dx^\nu + r(x) dy^1 \wedge dy^2) / x \in \mathbb{R}^2\},$$

where $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an arbitrary smooth function, $r : \mathbb{R}^2 \rightarrow \mathbb{R}^*$ is also an arbitrary smooth function and

$$e(x) = r(x) \left( \frac{\partial u^1}{\partial x^1}(x) \frac{\partial u^2}{\partial x^2}(x) - \frac{\partial u^1}{\partial x^2}(x) \frac{\partial u^2}{\partial x^1}(x) \right) + h,$$

for some constant $h \in \mathbb{R}$, and

$$p^\mu_i(x) = -r(x) \epsilon^{\mu\nu} \epsilon_{ij} \frac{\partial u^j}{\partial x^\nu}(x).$$
Note that in this setting the de Donder–Weyl theory corresponds to the further constraint \( r = 0 \), which here implies that \( p_1 = 0 \): we thus recover the fact that the Legendre transform completely degenerates.

Other examples can be studied like for instance:

- The harmonic map Lagrangian \( L(x, y, v) = \frac{1}{2} |v|^2 \) where \( |v|^2 := (v_1^2 + (v_2^2 + (v_3^2)^2) \). Then one finds that \( H(q, p) = e + \frac{1}{1-r} \left( \frac{1}{2} p_1^2 + r(p_1 p_2^2 - p_1 p_3^2) \right) \).

- The Maxwell equations in two dimensions. We take \( L(x, y, v) = -\frac{1}{2} (v_1^2 - v_2^2)^2 \). Then \( H(q, p) = e + \frac{(v_1^2 + p_2^2)^2 - 4p_1^2 p_2^2}{4r} - \frac{1}{2} \frac{(p_1 - p_3)^2}{2 + r} \).

Beside the Legendre correspondence, the Lepage–Dedecker theory differs from the de Donder–Weyl theory in many other aspects, sometime relatively subtle. For example, on \( M := \Lambda^n T^* N \), with its standard multisymplectic form \( \Omega \), the set of algebraic observable \((n-1)\)-forms coincides with the set of observable \((n-1)\)-forms. Hence \((M, \Omega)\) is an example of a pataplectic manifold. I refer to our paper [21] for a complete exposition.

Another question which is discussed in [20] and [21] is the possibility and the relevance of considering observable forms of degree \( p - 1 \), where \( p < n \). This was proposed first by I. Kanatchikov in [22], [23]. For instance in the Hamiltonian system [2], one sees a symmetry between \( \pi_i^\mu \) and \( u^i \) (which disappears when \( n = 1 \)): the equations on \( \pi_i^\mu \) involve a divergence whereas equations on \( u^i \) prescribe its derivatives in all direction. This reflects the fact that the \( \pi_i^\mu \)'s are actually the components of an observable \((n-1)\)-form, whereas \( u^i \) could be considered as an observable 0-form. Another beautiful example holds for gauge theories (see [23], [24]): where the gauge potential \( A_\mu dx^\mu \) can be considered as an observable 1-form, whereas the Faraday form \( *(dA + A \wedge A) \) as an observable \((n-2)\)-form. Moreover these forms are canonically conjugate (in a sense similar to the duality between position and momentum variables in classical mechanics). The first definition proposed in [22] (a \((p-1)\)-form \( F \) is observable if and only if there exists an \((n-p)\)-multivector \( \xi \) such that \( dF + \xi \we \Omega = 0 \)) leads to a quite interesting notion of graded Poisson bracket, but causes some difficulties when one tries to generalize relation [3] to \((p-1)\)-forms, for \( 1 \leq p < n \). In [21], starting from another point of view, namely by characterising the property which seems to be relevant in order to generalize relation [3], we proposed the following alternative definition. We define collectively the set of all observable \((p-1)\)-forms, for \( 1 \leq p < n \): it is a vector subspace \( \mathfrak{P}^* M \) of the set of sections of \( \bigoplus_{p=1}^n \Lambda^{p-1} T^* M \) such that \( \forall F_1, \ldots, F_k \in \mathfrak{P}^* M, \) if \( dF_1 \wedge \cdots \wedge dF_k \) is a section of \( \Lambda^k T^* M \), then there exists a vector field \( \xi \) on \( M \) such that \( dF_1 \wedge \cdots \wedge dF_k + \xi \we \Omega = 0 \). This definition seems slightly unpleasant at first glance (there could be several systems of observable forms, i.e. several choices for \( \mathfrak{P}^* M \)) but it provides the right hypothesis in order to prove generalizations of [3] (see [21]). Note also that it has some physical meaning (see the next paragraph). One drawback is that it is now more delicate to define a notion of Poisson bracket between such forms. However we were able to propose a partial definition which works for the
most important cases (see [21]).

Note that the above definition of observable \((p - 1)\)-forms collectively is an example of a mechanism by which observable quantities (and in particular space-time coordinates) could merge out from intrinsic properties. In this spirit we also remarked in [21] that the dynamical relation (3) can be replaced by a more general one:

\[
\{\mathcal{H}, F\}dG|_\Gamma = \{\mathcal{H}, G\}dF|_\Gamma,
\]

(a generalisation for observable forms of lower degrees also exists). The underlying idea, that nothing can be measured in an absolute sense and that we can only compare the results of two different measures, is very much in the spirit of general relativity.

5 Dynamical observable forms and perturbation theory

5.1 Dynamical observable \((n - 1)\)-forms and their motivations

When building a quantum field theory, in order to achieve a relativistic invariance and in particular to be free from any choice of time coordinate, one is led to use the Heisenberg point of view. There a vector in the Hilbert (Fock) space of quantum states represents the complete history over all space-time of a quantized field, and observable operators act on this Hilbert space with eigenvalues which are smeared integrals of functions of the values of the fields and their space-time derivatives on space-time. The classical counterpart of this point of view, sometime called Einstein point of view, is to consider the set \(\mathcal{E}\) of solutions to the Euler–Lagrange equations of motion over all space-time to be the set of physical states and to consider the set of functionals defined on \(\mathcal{E}\) to be the set of observables. If we wish to understand in a covariant way how to quantize these fields it seems to be crucial to be able to define a Poisson bracket between observable functionals and in particular between two observable functionals of the type \(\int_\Sigma F : \Gamma \mapsto \int_{\Gamma \cap \Sigma} F\) and \(\int_{\tilde{\Sigma}} G : \Gamma \mapsto \int_{\Gamma \cap \tilde{\Sigma}} G\), even if \(\Sigma\) and \(\tilde{\Sigma}\) are two different hypersurfaces. This is however not clear in general. One possibility arises when, for instance, \(F\) is such that

\[
\{\mathcal{H}, F\} = -d\mathcal{H}(\xi_F) = 0.
\]

We then say that \(F\) is a dynamical observable \((n - 1)\)-form. Assume furthermore that \(\Sigma\) and \(\tilde{\Sigma}\) are homologous hypersurfaces, so that \(\tilde{\Sigma} - \Sigma\) is the boundary of an open subset \(D\). Then by applying first Stokes’ theorem and second (3) we obtain that

\[
\int_{\Gamma \cap \Sigma} F - \int_{\Gamma \cap \tilde{\Sigma}} F = \int_{\Gamma \cap D} dF = \int_{\Gamma \cap D} \{\mathcal{H}, F\}\omega = 0.
\]

Hence the two functionals \(\int_\Sigma F\) and \(\int_{\tilde{\Sigma}} F\) coincide on \(\mathcal{E}\). We can thus pose

\[
\left\{ \int_{\Sigma} F, \int_{\tilde{\Sigma}} G \right\} := \left\{ \int_{\Sigma} F, \int_{\Sigma} G \right\} = \int_{\Sigma} \{F, G\}.
\]
Thus it remains to find dynamical observable forms. Here comes a surprise and a relative deception. We quote H. Goldschmidt and S. Sternberg in [14]: "For the free fields (i.e. quadratic Lagrangians) that arise in quantum field theory, the algebra $P$ is infinite dimensional and provides enough elements to yield the operators of the associated free quantum fields. However computations done jointly with S. Coleman to whom we are very grateful, seem to indicate that of $n \geq 3$, then for “interacting Lagrangians”, i.e. those containing higher order terms, the algebra $P$ is finite dimensional, and hence does not provide enough operators for quantization”. A more detailed computation with basically the same conclusion can be found in J. Kijowski’s paper [26]. It is interesting here that this question meet the same kind of difficulties as the quantization problem for fields: the quantization procedure works when the classical equation is linear but fails as soon as the problem become nonlinear (interacting fields in the language of physicists). This is perhaps an indication that the two questions are related (although there is no doubt that the quantization problem is much more difficult).

There are however some ways to escape from this dead end. One is to remark that the set of dynamical observable forms is roughly speaking in correspondence with the set of symmetries of the problem (Noether theorem). In particular in the presence of a gauge symmetry we can produce an infinite dimensional family of dynamical observable forms, which corresponds to the set of all current densities smeared with any test function. We have discussed this approach in [21]. It suggests the question whether a gauge symmetry could improve the quantization of a field theory. In the same spirit it could be interesting to explore integrable systems, which possesses infinitely many symmetries. A third possibility is when the problem is nonlinear but close to a linear one: one can then hope to build observable functionals by perturbations. We expound here a simple example which illustrates this idea.

5.2 The interacting field: obstructions to dynamical observables forms

We let $\eta_{\mu\nu}$ be a constant metric on $\mathbb{R}^n$ (which could be Euclidean or Minkowskian), with inverse $\eta^{\mu\nu}$, and we consider the following functional on the set of maps $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$$
\mathcal{L}[\varphi] := \int_{\mathbb{R}^n} \left( \frac{1}{2} \eta^{\mu\nu} \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu} + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{3} \varphi^3 \right) \omega,
$$

where $\omega = dx^1 \wedge \cdots \wedge dx^n$ and $\lambda$ is a scalar constant, that we suppose to be small. Denoting by $\Delta := -\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu}$, the Euler–Lagrange equation is

$$
\Delta \varphi + m^2 \varphi + \lambda \varphi^2 = 0.
$$

Since here the target space is 1-dimensional there is no difference between the de Donder–Weyl theory and the other Lepage theories. The multisymplectic manifold $\mathcal{M}$ can hence

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$P$ is here the quotient of the set of algebraic observable $(n-1)$-forms by the subset of exact $(n-1)$-forms
either be constructed as the dual of the first affine jet bundle or with $\Lambda^* T^* (\mathbb{R}^n \times \mathbb{R})$. We identify it with $\mathbb{R}^{2n+2}$, with the coordinates $(x^\mu, \phi, e, p^\mu)$ and with the multisymplectic (or pataplectic) form

$$\Omega := de \wedge \omega + dp^\mu \wedge d\phi \wedge \omega_\mu,$$

where $\omega_\mu := \frac{\partial}{\partial x^\mu} \omega$. Then Hamiltonian $n$-curves are $n$-dimensional submanifolds $\Gamma$ of $\mathcal{M} \cong \mathbb{R}^{2n+2}$ such that, for any point $p \in \Gamma$ there exists a unique $n$-multivector $X$ tangent to $\Gamma$ at $p$ such that $X \lrcorner \Omega = (-1)^n d\mathcal{H}$, where

$$\mathcal{H}(x, \phi, e, p) := e + \frac{1}{2} \eta_{\mu\nu} p^\mu p^\nu - \frac{m^2}{2} \phi^2 - \frac{\lambda}{3} \phi^3.$$

The search for all dynamical algebraic observable $(n-1)$-forms consists in the following: one looks at vector fields $\xi$ on $\mathcal{M}$ such that $d(\xi \lrcorner \Omega) = 0$, (8) which will implies that there exists an $(n-1)$-form $F$ such that $\xi = \xi_F$, i.e. $dF + \xi \lrcorner \Omega = 0$, and such that

$$\{\mathcal{H}, F\} = -d\mathcal{H}(\xi) = 0.$$ (9)

This has been done in [26]. The results are that: if $\lambda \neq 0$ the only solutions are $\xi = X^\mu \frac{\partial}{\partial x^\mu}$, where $X^\mu$ are constants, if $\lambda = 0$ the solutions are $\xi = X^\mu \frac{\partial}{\partial x^\mu} + \eta_{\mu\nu} \frac{\partial \Phi}{\partial x^\nu}(x) \frac{\partial}{\partial p^\nu} + (m^2 \phi \Phi(x) - p^\mu \frac{\partial \Phi}{\partial x^\mu}(x)) \frac{\partial}{\partial e} + \Phi(x) \frac{\partial}{\partial \phi}$, where $\Phi$ is a solution of $\Delta \Phi + m^2 \Phi = 0$.

Let us revisit partially this analysis: we assume for simplicity that $dx^\mu(\xi) = 0$ (i.e. we throw away the $X^\mu$’s which correspond to parts of the stress-energy-tensor). First one finds that such a $\xi$ satisfies (8) if and only if

$$\xi = \left( P^\mu(x, \phi) - p^\mu \frac{\partial \Phi}{\partial \phi}(x, \phi) \right) \frac{\partial}{\partial p^\mu} + \left( E(x, \phi) - p^\mu \frac{\partial \Phi}{\partial x^\mu}(x, \phi) \right) \frac{\partial}{\partial e} + \Phi(x, \phi) \frac{\partial}{\partial \phi},$$ (10)

where $\Phi$, $E$ and $P^\mu$ are arbitrary functions of $(x, \phi)$ subject to the condition

$$\frac{\partial E}{\partial \phi} - \frac{\partial P^\mu}{\partial x^\mu} = 0.$$ (11)

Second the substitution of the value of $\xi$ in (8) leads to the system of equations

$$\frac{\partial \Phi}{\partial \phi}(x, \phi) = 0, \quad \frac{\partial \Phi}{\partial x^\mu}(x, \phi) - \eta_{\mu\nu} P^\nu(x, \phi) = 0,$$ (12)

(which implies that $\Phi$ and $P^\mu$ depend only on $x$ and $P^\mu(x) = \eta_{\mu\nu} \frac{\partial \Phi}{\partial x^\nu}(x)$) and

$$(m^2 \phi + \lambda \phi^2) \Phi(x) - E(x, \phi) = 0.$$ (13)
The system \([12], \ [13]\) has no nontrivial solution when \(\lambda \neq 0\), because by \([11]\) it would contradict the fact that \(P^\mu(x) = \eta^{\mu\nu} \partial \Phi / \partial x^\nu(x)\).

So let us forget condition \([13]\) and assume only \([10], [11] \) and \([12] \). By \([11] \) and \([12] \) we deduce that \(\partial E / \partial \phi(x, \phi) = -\Delta \Phi(x)\) and so there exists a function \(A : \mathbb{R}^n \to \mathbb{R}\) such that 
\[
E(x, \phi) = A(x) - \phi \Delta \Phi(x).
\]
We shall assume \(A = 0\) in the following (since \(A\) does not help in anything), we deduce that
\[
\xi = \eta^{\mu\nu} \partial \Phi / \partial x^\nu(x) \frac{\partial}{\partial p^\mu} - \left( \phi \Delta \Phi(x) + p^\mu \partial \Phi / \partial x^\mu(x) \right) \frac{\partial}{\partial e} + \Phi(x) \frac{\partial}{\partial \phi},
\]
(then \(\mathcal{J} \Omega = -dF\), where \(F = (p^\mu \Phi(x) - \eta^{\mu\nu} \phi \partial \Phi / \partial x^\mu(x)) \omega_\mu\) and 
\[
d\mathcal{H}(\xi) = -\phi \left( \Delta \Phi(x) + m^2 \Phi(x) \right) - \lambda \phi^2 \Phi(x).
\]
We now suppose that \(\Phi = \Phi^{(1)}\), a solution of the equation \(\Delta \Phi^{(1)} + m^2 \Phi^{(1)} = 0\). We denote by \(\xi^{(1)}\) the corresponding vector field and \(F^{(1)}\) the associated observable \((n - 1)\)-form: 
\[
F^{(1)} := \left( p^\mu \Phi^{(1)}(x) - \eta^{\mu\nu} \phi \partial \Phi^{(1)} / \partial x^\mu(x) \right) \omega_\mu.
\]
Then, instead of \([3]\), we have 
\[
\{ \mathcal{H}, F^{(1)} \} = -d\mathcal{H}(\xi^{(1)}) = \lambda \phi^2 \Phi^{(1)}(x), \tag{14}
\]
and thus
\[
\int_{\Gamma \cap \partial D} F^{(1)} = \int_{\Gamma \cap D} dF^{(1)} = \int_{\Gamma \cap D} \lambda \phi^2 \Phi^{(1)}(x) \omega.
\]

### 5.3 Second order correction

We now add to the functional \(\int_{\partial D} F^{(1)}\) another functional of the form

\[
\lambda \int_{\partial D} \int_{\partial D} F^{(2)} : \mathcal{E} \to \mathbb{R}, \quad \Gamma \mapsto \lambda \int_{\Gamma \cap \partial D} \int_{\Gamma \cap \partial D} F^{(2)},
\]
where \(F^{(2)} \in \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M}) \otimes \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M})\) (here \(\Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M})\) is the set of sections over \(\mathcal{M}\) of the bundle \(\Lambda^{n-1} T^* \mathcal{M}\), i.e. the set of \((n - 1)\)-forms over \(\mathcal{M}\)). We shall make the following hypotheses on \(F^{(2)}\): we let \(\Omega^{\otimes 2} := \Omega \otimes \Omega\) be in \(\Gamma(\mathcal{M}, \Lambda^{n+1} T^* \mathcal{M}) \otimes \Gamma(\mathcal{M}, \Lambda^{n+1} T^* \mathcal{M})\) and we assume that there exists a “bivector” \(\xi^{(2)}\), i.e. an element of \(\Gamma(\mathcal{M}, T \mathcal{M}) \otimes \Gamma(\mathcal{M}, T \mathcal{M})\), such that
\[
d^{\otimes 2} F^{(2)} = (-1)^2 \xi^{(2)} \mathcal{J} \Omega \otimes \Omega. \tag{15}
\]
It deserves some definitions about notations. Let us abbreviate by \(z^\prime\) the system of coordinates \((x^\mu, y^i, e, p^\mu)\) on \(\mathcal{M}\) and by \((z^I_1, z^I_2) = (x^\mu_1, y^i_1, e_1, p^\mu_1, x^\mu_2, y^i_2, e_2, p^\mu_2)\) coordinates on \(\mathcal{M} \times \mathcal{M}\). Then any \(d^{\otimes 2}\) is the unique linear operator from \(\Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M}) \otimes \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M})\)
to $\Gamma(\mathcal{M}, \Lambda^n T^* \mathcal{M}) \otimes \Gamma(\mathcal{M}, \Lambda^n T^* \mathcal{M})$ such that if $\alpha^{(2)} \in \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M}) \otimes \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M})$ writes
\[
\alpha^{(2)} = \alpha(z^I_1, z^J_2) dz^I_1 \wedge \cdots \wedge dz^{I_n}_1 \otimes dx^J_2 \wedge \cdots \wedge dx^{J_n}_2,
\]
then
\[
d^{\otimes 2} \alpha^{(2)} := \sum_{I,J} \frac{\partial^2 \alpha}{\partial z^I_1 \partial z^J_2}(z^I_1, z^J_2) dz^I_1 \wedge dx^J_2 \wedge \cdots \wedge dz^{I_n}_1 \otimes dx^{J_n}_2.
\]
(Remark: similar tensor product of differential forms and operators $d^{\otimes 2}$ were used in [19] for the purpose of proving isoperimetric inequalities through calibrations.) Similarly $2\bar{\partial}$ is an operation with standard linear properties such that if $\xi^{(2)} \in \Gamma(\mathcal{M}, T \mathcal{M}) \otimes \Gamma(\mathcal{M}, T \mathcal{M})$ writes
\[
\xi^{(2)} = \xi(z^I_1, z^J_2) \frac{\bar{\partial}}{\partial z^I_1} \otimes \frac{\bar{\partial}}{\partial z^J_2},
\]
then
\[
\xi^{(2)} \bar{\partial} \Omega^{\otimes 2} := \xi(z^I_1, z^J_2) \left( \frac{\bar{\partial}}{\partial z^I_1} \bar{\partial} \Omega \right) \otimes \left( \frac{\bar{\partial}}{\partial z^J_2} \bar{\partial} \Omega \right).
\]
Then, denoting by $d_1 := d \otimes I d$ and $d_2 := I d \otimes d$, so that $d^{\otimes 2} = d_1 \circ d_2$, we have
\[
\int_{\Gamma \cap \partial D} \int_{\Gamma \cap \partial D} F^{(2)} = \int_{\Gamma \cap D} \int_{\Gamma \cap D} d_1 F^{(2)} = \int_{\Gamma \cap D} \int_{\Gamma \cap D} d^{\otimes 2} F^{(2)} = \int_{\Gamma \cap D} \int_{\Gamma \cap D} \xi^{(2)} \bar{\partial} \Omega \otimes \Omega = \int_{\Gamma \cap D} \int_{\Gamma \cap D} (\xi^{(2)} \bar{\partial} \Omega \otimes \Omega) (X(z_1) \otimes X(z_2)) \omega \otimes \omega
\]
(X(z) is there a $n$-multivector tangent to $\Gamma$ at $z$, s.t. $\omega(X(z)) = 1$)
\[
= \int_{\Gamma \cap D} \int_{\Gamma \cap D} (-1)^{2n} (X(z_1) \otimes X(z_2) \bar{\partial} \Omega \otimes \Omega) (\xi^{(2)} \omega \otimes \omega
\]
\[
= \int_{\Gamma \cap D} \int_{\Gamma \cap D} d\mathcal{H}_{z_1} \otimes d\mathcal{H}_{z_2} (\xi^{(2)} \omega \otimes \omega,
\]
where we have used the fact that $(X(z_1) \otimes X(z_2) \bar{\partial} \Omega \otimes \Omega)|_{\Gamma \times \Gamma} = (-1)^{2n} (d\mathcal{H}_{z_1} \otimes d\mathcal{H}_{z_2})|_{\Gamma \times \Gamma}$. The idea is to look for an $F^{(2)}$ such that
\[
\{ \mathcal{H}^{\otimes 2}, F^{(2)} \} := d\mathcal{H}_{z_1} \otimes d\mathcal{H}_{z_2} (\xi^{(2)}) = -\Phi^{(1)}(x_1) \delta(x_1 - x_2) \phi_1 \phi_2 + \mathcal{O}(\lambda),
\]
where $\delta$ is the Dirac distribution on $\mathbb{R}^n$. Then
\[
\lambda \int_{\Gamma \cap D} \int_{\Gamma \cap D} F^{(2)} = -\lambda \int_{\Gamma \cap D} \Phi^{(1)}(x) \phi^2 \omega + \mathcal{O}(\lambda^2),
\]
so that $\int_{\Gamma \cap D} F^{(1)} + \lambda \int_{\Gamma \cap D} \int_{\Gamma \cap D} F^{(2)} = \mathcal{O}(\lambda^2)$.
We choose $F^{(2)}$ of the form

$$F^{(2)}(x_1, x_2) = \left( p_1^\mu - \eta^{\mu\lambda} \phi_1 \frac{\partial}{\partial x_1^\lambda} \right) \left( p_2^\nu - \eta^{\nu\sigma} \phi_1 \frac{\partial}{\partial x_2^\sigma} \right) \Phi^{(2)}(x_1, x_2) \omega^\mu \otimes \omega^\nu,$$

where $\Phi^{(2)} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function to be precised later. One can check that such an $F^{(2)}$ satisfies (13) with

$$\xi^{(2)} = \left( \left( p_1^\mu \frac{\partial}{\partial x_1^\mu} + \phi_1 \Delta_1 \right) - \eta^{\mu\lambda} \frac{\partial}{\partial x_1^\lambda} \frac{\partial}{\partial p_1^\mu} \right) \otimes \left( \left( p_2^\nu \frac{\partial}{\partial x_1^\nu} + \phi_2 \Delta_2 \right) - \eta^{\nu\sigma} \frac{\partial}{\partial x_2^\sigma} \frac{\partial}{\partial p_2^\nu} \right) \Phi^{(2)}(x_1, x_2).$$

Here we denote by $\Delta_1 := \eta^{\mu\nu} \frac{\partial^2}{\partial x_1^\mu \partial x_1^\nu}$ and $\Delta_2 := \eta^{\mu\nu} \frac{\partial^2}{\partial x_2^\mu \partial x_2^\nu}$ and we have introduced a symbolic notation in order to shorten the expression of $\xi^{(2)}$ (which is quite long to write): one should understand that this expression should be developp using the rule

$$\left( K_1 \frac{\partial}{\partial z_1^\nu} \otimes K_2 \frac{\partial}{\partial z_2^\nu} \right) A(x_1, x_2) = \left( K_1 K_2 A(x_1, x_2) \right) \frac{\partial}{\partial z_1^\nu} \otimes \frac{\partial}{\partial z_2^\nu},$$

for all linear differential operator $K_1$ (resp. $K_2$) acting on the variables $x_1^\mu$ (resp. $x_2^\nu$) (for instance $K_1$ can be $p_1^\mu \frac{\partial}{\partial x_1^\mu} + \phi_1 \Delta_1$, $\frac{\partial}{\partial x_1^\mu}$ or 1).

Then one compute that

$$\{ \mathcal{H}^{(2)}, F^{(2)} \} = d\mathcal{H}_{z_1} \otimes d\mathcal{H}_{z_2} \left( \xi^{(2)} \right) = \phi_1 \phi_2 \left( \Delta_1 + m^2 \right) \left( \Delta_2 + m^2 \right) \Phi^{(2)}(x_1, x_2)\Phi^{(2)}(x_1, x_2) + \lambda \left( \phi_1^2 \phi_2^2 \left( \Delta_1 + m^2 \right) \Phi^{(2)}(x_1, x_2)\Phi^{(2)}(x_1, x_2) + \lambda^2 \phi_1^2 \phi_2^2 \Phi^{(2)}(x_1, x_2)\Phi^{(2)}(x_1, x_2).$$

Thus in order to achieve (14) it suffices to choose $\Phi^{(2)}$ such that

$$\left( \Delta_1 + m^2 \right) \left( \Delta_2 + m^2 \right) \Phi^{(2)}(x_1, x_2) = -\Phi^{(1)}(x_1)\delta(x_1 - x_2).$$

A solution to this equation is formally

$$\Phi^{(2)}(x_1, x_2) = -\int_{\mathbb{R}^n} \Phi^{(1)}(t) G(t, x_1) G(t, x_2) dt,$$

where $G$ is the Green function on $\mathbb{R}^n$ of the operator $\Delta + m^2$.

### 5.4 Perturbation series

We have produced two observables, $\int_{\partial D} F^{(1)}$ and $\int_{\partial D} \Phi^{(1)} + \lambda \left( \int_{\partial D} \right)^2 F^{(2)}$, which are approximately vanishing, up to order $\lambda$ (resp. $\lambda^2$). This looks clearly as the beginning of an infinite expansion series defining the functional

$$\Gamma \mapsto \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{k!} \left( \int_{\Gamma \cap \partial D} \right)^k F^{(k)}.$$ (17)
Here we let \( F^{(k)} := \prod_{j=1}^{k} \left( p_{j}^{\mu} - \eta^{\mu\lambda} \phi_{j} \frac{\partial}{\partial x_{j}^{\lambda}} \right) \) \( \Phi^{(k)}(x_{1}, \ldots, x_{k}) \in \Gamma(\mathcal{M}, \Lambda^{n-1}T^{*}\mathcal{M})^{\otimes k} \), where \( \Phi^{(k)} \) is a function on \((\mathbb{R}^{n})^{k}\). We need to choose functions \( \Phi^{(k)} \) in such a way that

\[
\sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Gamma^{\cap} \partial D} \right)^{k} \{ \mathcal{H}^{\otimes k}, F^{(k)} \} \omega^{\otimes k} = 0,
\]

where \( \{ \mathcal{H}^{\otimes k}, F^{(k)} \} := (-1)^{k} d^{\otimes k} \mathcal{H}^{\otimes k}(\xi^{(k)}) = \left[ \prod_{j=1}^{k} \left( \phi_{j}(\Delta_{j} + m^{2}) + \lambda \phi_{j}^{2} \right) \right] \Phi^{(k)}(x_{1}, \ldots, x_{k}) \).

Of course the computation of the further terms should more and more complicated but is possible in principle and should be described by graphs analog to Feynman’s diagramm. Note that these graphs should all be trees (i.e. without loops), since they describe classical observable functionals.

Then (17) provides us with a vanishing functional, if \( \lambda \) is sufficiently small. This was not exactly our original motivation, which was to find non vanishing dynamical functionals. These can be obtained as follows. Assume for instance that \( \eta^{\mu\nu} \) is Minkowskian, i.e. \( \Delta \) is hyperbolic, and \( \partial D = \tilde{\Sigma} - \Sigma \), where \( \Sigma \) is a fixed space-like hypersurface (say \( \{ x^{0} = t_{0} \} \) for some fixed \( t_{0} \)) and \( \tilde{\Sigma} \) is a parallel space-like hypersurface (say \( \{ x^{0} = t \} \), where \( t \neq t_{0} \)). Then we prescribe all functions \( \Phi^{(k)} \), for \( k \geq 2 \), in such a way that they vanish and their first time derivative vanish along \( \Sigma \). This implies that

\[
\sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Gamma^{\cap} \partial D} \right)^{k} F^{(k)} = \sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Gamma^{\cap} \Sigma} \right)^{k} F^{(k)} - \int_{\Gamma^{\cap} \Sigma} F^{(1)}.
\]

Hence we get the coincidence of the two functionals \( \sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Sigma} \right)^{k} F^{(k)} \) and \( \int_{\Sigma} F^{(1)} \) on \( \mathcal{E} \). Then it remains of course to define the bracket \( \left\{ \sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Sigma} \right)^{k} F^{(k)}, \int_{\Sigma} G \right\} \), in order to set

\[
\left\{ \int_{\Sigma} F^{(1)}, \int_{\Sigma} G \right\} := \left\{ \sum_{k=1}^{\infty} \lambda^{k-1} \left( \int_{\Sigma} \right)^{k} F^{(k)}, \int_{\Sigma} G \right\}.
\]

In principle there should be no difficulty in defining the above Poisson bracket, either by coming back to the Poisson bracket obtained by the standard canonical theory of physicists or by using for instance the theory expounded in [9]. Note that in order to make connection with quantum field theory, which provides us quantum scattering amplitudes, it is more appropriate to choose the slice \( \Sigma \) at infinity, i.e. such that \( t_{0} = -\infty \) or \( t_{0} = \infty \).

Lastly we also remark that replacing the set of functionals \( \int_{\Sigma} F \) by the set of functionals of the type (17) has another advantage: it is then possible to define a product law between such functionals, by the rule

\[
\left( \sum_{k=1}^{\infty} \left( \int_{\Sigma} \right)^{k} F^{(k)} \right) \left( \sum_{k=1}^{\infty} \left( \int_{\Sigma} \right)^{k} G^{(k)} \right) = \sum_{k=1}^{\infty} \sum_{l=0}^{k} \left( \int_{\Sigma} \right)^{k} F^{(l)} \otimes G^{(k-l)}.
\]
6 Conclusion

We have tried here to introduce the Reader to multisymplectic formalisms, mainly through the de Donder–Weyl theory, we have explained quickly a more general framework based on Lepage–Dedecker developed in [20] and [21] (that is, we believe, supported by a more relativistic point of view) and we have explained how a perturbative theory analog to the theory of Feynman and Schwinger could be be built for classical solutions. This theory need of course to be developed and it should be interesting to understand whether it could lead to the perturbative quantum field theory by a direct quantization.

We have not discussed many other important questions like the construction of a non-perturbative quantum field theory in this framework (interesting results have been obtained by I. Kanatchikov, see [24]), or how such theories could help in understanding hyperbolicity of variational nonlinear hyperbolic partial differential equations as done in D. Christodoulou’s book [6]. We should also mention the existence of other covariant theories as for example F. Takens’ one [35] (see also [38] or [9]).

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