A Perron iteration for the solution of a quadratic vector equation arising in Markovian Binary Trees

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Abstract

We propose a novel numerical method for solving a quadratic vector equation arising in Markovian Binary Trees. The numerical method consists in a fixed point iteration, expressed by means of the Perron vectors of a sequence of nonnegative matrices. A theoretical convergence analysis is performed. The proposed method outperforms the existing methods for close-to-critical problems.

1 Introduction

In this paper we study the quadratic vector equation

\[ x = a + B(x \otimes x), \]  

(1)

where \( a \in \mathbb{R}^n \), \( B \in \mathbb{R}^{n \times n^2} \) have nonnegative entries, the symbol \( \otimes \) denotes the Kronecker product, and the unknown \( x \) is an \( n \)-dimensional vector. The coefficients \( a \) and \( B \) are such that the vector \( e = (1, 1, \ldots, 1)^T \) is a solution of (1).

Equation (1) arises in the study of Markovian Binary Trees (MBT), which are a particular family of branching processes used to model the growth of populations consisting of several types of individuals, who may produce offsprings during their lifetime. MBTs have applications in biology, epidemiology and also in telecommunication systems. We refer to [2, 4] for definitions, properties and applications.

One important issue related to MBTs is the computation of the extinction probability of the population, which is the minimal nonnegative solution \( x^* \in \mathbb{R}^n \) of the quadratic vector equation (1).

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The MBT is called subcritical, supercritical or critical if the spectral radius \( \rho(R) \) of the matrix \( R = B(e \otimes I + I \otimes e) \) is strictly less than one, strictly greater than one, or equal to one, respectively. In the subcritical and critical cases the minimal nonnegative solution is the vector of all ones, while in the supercritical case \( x^* \leq e, x^* \neq e \) (see [4] and [1]). Thus, only the supercritical case is of interest for the computation of \( x^* \).

Several numerical methods have been proposed for computing the vector \( x^* \). In [2] the authors propose two linearly convergent algorithms, called depth and order algorithms. The thickness algorithm, still linearly convergent, is proposed in [4]. In [3] the authors apply the Newton method, which has quadratic convergence. A modification of Newton’s method has been proposed in [5]. All these methods have a probabilistic interpretation, and each of them provides a sequence \( \{x_k\}_k \) of nonnegative vectors, with \( x_0 = (0, \ldots, 0)^T \), which converges monotonically to the minimal nonnegative solution \( x^* \). A common feature of all these methods is that their convergence speed slows down when the problem, while being supercritical, is close to critical, i.e., the spectral radius of \( R \) is close to one and \( x^* \approx e \). Moreover, the accuracy of the approximation deteriorates.

In this paper we write equation (1) in the form
\[
y = b(y, e) + b(e, y) - b(y, y).
\]
(2)
The sought solution \( y^* \) of (2) is \( y^* = e - x^* \), where \( x^* \) is the minimal nonnegative solution of (1). In the probability interpretation of Markovian Binary Trees, since \( x^* \) is the extinction probability, then \( y^* = e - x^* \) is the survival probability.

Applying a functional iteration directly to (2), like Newton’s method, gives nothing new, since (2) differs from (1) by a linear change of variable. However, the new equation (2) can be rewritten as
\[
y = Hyy,
\]
(3)
where \( H_y := b(\cdot, e) + b(e - y, \cdot) \). The matrix \( H_y \) is nonnegative and irreducible if \( y < e \). In particular the solution \( y^* \) is such that \( \rho(H_{y^*}) = 1 \) and \( y^* \) is the Perron vector of the matrix \( H_{y^*} \).

This interpretation allows to design a new algorithm for computing \( y^* \). To this purpose, define the map \( \text{PV}(M) \) as the Perron vector of a nonnegative irreducible matrix \( M \), so that we may rewrite (3) as
\[
y = \text{PV}(H_y).
\]
(4)
The idea is to apply a fixed-point iteration to solve (4), thus generating a sequence \( \{y_k\}_k \) of positive vectors such that \( y_{k+1} = \text{PV}(H_{y_k}) \) and \( y_k \) converges to \( y^* \). A suitable normalization of the Perron vector, consistent with the solution, is needed to obtain a well-posed iteration. In this way we obtain a new iterative scheme, which is completely different from classical functional iterations. Indeed, the proposed algorithm, unlike known methods, fully exploits the fact
that the solution $x = e$ of the equation (2) is known. Moreover, the fixed-point iteration at the basis of our algorithm relies on a new interpretation of the solution $y^*$ in terms of the Perron vector. These differences with respect to classical methods lead to great improvements in the numerical solution of MBTs which are close to critical.

We perform a convergence analysis of the fixed point iteration $y_{k+1} = PV(H_y)$, by giving an expression to the Jacobian of the map $y \rightarrow PV(H_y)$. This expression allows to derive a local convergence result. Moreover, most importantly, we prove that, although the convergence of the method is linear, the speed of convergence increases as the problem gets close to critical. In the limit case of a critical problem, the convergence becomes superlinear. This nice behavior is opposite to the one of Newton’s method, whose speed of convergence is sublinear in the supercritical case, and becomes linear in the critical case.

A wide numerical experimentation confirms our theoretical analysis. For far-from-critical problems the standard techniques are preferable, while for close-to-critical problems our method outperforms the existing ones.

The paper is organized as follows. In Section 2 we state our assumptions on the problem. In Section 3 we rewrite the vector equation in terms of an equation for the vector $y = e - x$ and discuss the properties of the equation obtained in this way. The new algorithm, based on a Perron iteration, is presented in Section 4. The theoretical convergence analysis is performed in Section 5. Finally, in Section 6 we present the results of the numerical experiments. Conclusions and open issues are addressed in Section 7.

2 Assumptions on the problem

Let $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n^2}$ have nonnegative entries, and consider the quadratic vector equation (1) where it is assumed that the vector $e = (1, 1, \ldots, 1)^T$ is a solution. Let $x^* \in \mathbb{R}^n$ be the minimal nonnegative solution of (1), i.e., $x^* \leq x$ for any other nonnegative solution, where the semi-ordering is component-wise. A unique solution $x^*$ exists, according to the results of [1, Section V.3].

We assume that $\rho(R) > 1$, where

$$R = B(e \otimes I + I \otimes e).$$

Under this assumption $x^* \leq e$, $x^* \neq e$ (see [3] and [1]). It is worth pointing out that if $\rho(R) = 1$, then $x^* = e$, therefore as $\rho(R)$ is greater than 1 and gets closer to 1, then $x^*$ approaches to the vector of all ones.

We introduce the bilinear operator

$$b(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

defined as

$$b(u, v) = B(u \otimes v)$$

and rewrite (1) as

$$x = a + b(x, x).$$

(5)
We assume that for the minimal solution $x^*$ of (5) it holds $0 < x^* < e$, and that the Jacobian of the map $x \rightarrow x - a - b(x, x)$ at $x^*$, i.e., $I - b(x^*, \cdot) - b(\cdot, x^*)$, is a nonsingular irreducible M-matrix. Since irreducibility is only determined by the nonnegativity pattern of $b(\cdot, \cdot)$, the irreducibility condition is equivalent to requiring that $b(e, \cdot) + b(\cdot, e)$ is irreducible. Notice that the latter is just another notation to represent the matrix $R$.

Moreover, we may assume that $e^T b(e - x^*, e - x^*) > 0$, otherwise $b(e - x^*, e - x^*) = 0$ (since $B \geq 0$), and the problem is trivial since it becomes a linear problem.

### 3 The optimistic equation

A property of equation (5) which has not been exploited so far in the existing literature is that $x = e$ is a solution. If we set $x = e - y$, by using the bilinearity of the operator $b(\cdot, \cdot)$ and the property that $e = a + b(e, e)$, equation (5) can be rewritten as

$$y = b(y, e) + b(e, y) - b(y, y).$$

The trivial solution is $y = 0$, which corresponds to $x = e$. We are interested in the nontrivial solution $0 < y^* < e$, which gives the sought solution $x^* = e - y^*$.

In the probabilistic interpretation of Markovian Binary Trees, $x^*$ is the extinction probability, thus $y^* = e - x^*$ is the survival probability, i.e., $y^*_i$ is the probability that a colony starting from a single individual in state $i$ does not become extinct in a finite time. For this reason, we refer to (6) as to the optimistic equation.

Notice that (6) admits the following probabilistic interpretation. The term $b(y, e)$ represents the probability that the original individual $M$ (for “mother”) spawns an offspring $F$ (for “first-born”), and after that the colony generated by the further offsprings of $M$, excluding $F$, survives. The term $b(e, y)$ represents the probability that $M$ spawns $F$, and the colony generated by $F$ survives. The term $b(y, y)$ represents the probability that $M$ spawns $F$, and after that both their colonies survive. Thus (6) follows by the well-known inclusion-exclusion principle

$$\mathbb{P}[M \text{ or } F \text{ survives}] = \mathbb{P}[M \text{ survives}] + \mathbb{P}[F \text{ survives}] - \mathbb{P}[\text{both } M \text{ and } F \text{ survive}],$$

where $\mathbb{P}[X]$ denotes the probability of the event $X$.

Equation (6) can be rewritten as

$$y = H_y y$$

where

$$H_y = b(\cdot, e) + b(e, \cdot) - b(y, \cdot).$$

Notice that $H_y$ is the sum of a fixed matrix and a matrix that depends linearly on $y$. Therefore the quadratic operator on the right-hand side of (6) is “factored” as the product of a matrix which depends on $y$, and $y$.
An important property is that \( H_y \) is a nonnegative irreducible matrix, whenever \( y < e \). Therefore, by the Perron-Frobenius theorem [7], if \( y < e \), \( H_y \) has a positive eigenvalue \( \lambda_y = \rho(H_y) \), the so-called Perron value, and to \( \lambda_y \) corresponds a positive eigenvector \( w_y \), unique up to a multiplicative constant, the so-called Perron vector, so that \( H_y w_y = \lambda_y w_y \). Therefore the sought solution \( y^* \) can be interpreted as the vector \( 0 < y^* < e \) such that \( \rho(H_{y^*}) = 1 \) and \( y^* \) is a Perron vector of \( H_{y^*} \).

It is worth pointing out that this interpretation of \( y^* \) in terms of the Perron vector allows to keep away from the trivial solution \( y = 0 \) of (7), since the Perron vector has strictly positive elements.

The formulation of the quadratic vector equation in terms of the Perron vector allows to design a new algorithm for its solution.

4 The Perron iteration

If we set up a fixed-point iteration or a Newton method for \( y \) based on (6), we get the traditional fixed-point iterations and Newton methods for MBTs [2], since what we have done is simply a linear change of variables. Instead, we exploit the fact that \( y^* \) is a Perron vector of the nonnegative irreducible matrix \( H_{y^*} \) (compare (7)).

To this purpose we introduce the operator

\[
u = PV(X)
\]

which returns the Perron vector \( u \) of the irreducible nonnegative matrix \( X \).

Thus, we can devise a fixed-point iteration to compute the solution \( y^* \) by defining the sequence of vectors

\[
y_{k+1} = PV(H_{y_{k}}), \quad k = 0, 1, 2, \ldots, \tag{9}
\]

starting from an initial approximation \( y_0 \). In order to define uniquely the sequence \( \{y_k\} \), we need to impose a normalization for the Perron vector, which is uniquely defined up to a multiplicative constant. A possible choice for the normalization is imposing that the residual of (6) is orthogonal to a suitable vector \( w \in \mathbb{R}^n \), i.e.,

\[
w^T(y_{k+1} - b(y_{k+1}, e) - b(e, y_{k+1}) + b(y_{k+1}, y_{k+1}))) = 0. \tag{10}
\]

Clearly, this normalization is consistent with the solution of (6). We choose \( w \) as the left Perron vector of the matrix \( b(\cdot, e) + b(e, \cdot) \); the rationale for this choice is discussed in Section 5.

Given a Perron vector \( u \) of \( H_{y_k} \), the equation to compute the normalization factor \( \alpha \) such that \( y_{k+1} = \alpha u \) satisfies (10) reduces to

\[
\alpha w^T u = \alpha w^T b(u, e) + \alpha w^T b(e, u) - \alpha^2 w^T b(u, u),
\]
whose only non-zero solution is
\[ \alpha = -\frac{w^T(u - b(u, e) - b(e, u))}{w^T b(u, u)}. \]

Notice that the solution \( \alpha = 0 \) corresponds to the trivial solution \( y = 0 \) (\( x = e \)), which we want to avoid.

The PV(\( \cdot \)) operator is defined on the set of irreducible nonnegative matrices. If \( y < e \), then the matrix \( H_y \) is nonnegative irreducible, therefore the sequence \( y_k \) generated by (9) is well defined if \( y_k < e \) for any \( k \).

In Section 5, we show that the iteration (9) is locally convergent. Therefore, if \( y_0 \) is quite close to \( y^* \), one can expect that \( y_k < e \) for any \( k \). In the case where \( H_{y_k} \) is not a nonnegative irreducible matrix, we can define \( y_{k+1} \) as an eigenvector corresponding to the eigenvalue of \( H_{y_k} \) having maximal real part. We call maximal eigenvector this eigenvector. Clearly if \( H_{y_k} \) is a nonnegative irreducible matrix, the maximal eigenvector is the Perron vector. We see in Section 6 that this concern is not necessary in practice.

As a starting approximation \( y_0 \) we may choose the null vector. For close to critical problems, where \( y^* \) is close to zero, this choice should guarantee the convergence, according to the results of Section 5.

The resulting iterative process is summarized in Algorithm 1.

\begin{algorithm}[h]
\caption{The Perron iteration}
\begin{algorithmic}
\Procedure{}{}
\State Set \( k \leftarrow 0 \)
\State Set \( y_0 \leftarrow 0 \)
\State Set \( w \leftarrow \) the Perron vector of \( b(e, \cdot) + b(\cdot, e) \)
\While {\( \|H_{y_k}y_k - y_k\|_1 \geq \varepsilon \)}
\State Set \( u \leftarrow \) the maximal eigenvector of \( H_{y_k} \)
\State Compute the normalization factor \( \alpha = -\frac{w^T(u - b(u, e) - b(e, u))}{w^T b(u, u)} \)
\State Set \( y_{k+1} \leftarrow \alpha u \)
\State Set \( k \leftarrow k + 1 \)
\EndWhile
\State \textbf{return} \( x = e - y_k \)
\EndProcedure
\end{algorithmic}
\end{algorithm}

5 Convergence analysis of the Perron iteration

In this section, we show that the Perron iteration (9) is locally convergent, and its convergence is linear. Moreover, the convergence speed gets faster as the problem gets closer to critical.

5.1 Derivatives of eigenvectors

It is well known [8] that the eigenvalues and eigenvectors of a matrix are analytical functions of the matrix entries in a neighborhood of a simple eigenpair.
The following formula for an analytical expression of their first derivatives is from Meyer and Stewart [6, Theorem 1].

**Theorem 1.** Let $A = A(z)$, $\lambda = \lambda(z)$, $u = u(z)$ be a matrix, eigenvalue and associated eigenvector depending on a parameter $z \in \mathbb{C}$. Let us suppose that $\lambda(z_0)$ is simple and $A'(z_0)$, $\lambda'(z_0)$, $u'(z_0)$ each exist. Let $w = w(z)$ be another vector such that $w'(z_0)$ exists and let $\sigma(u, w)$ be a function whose value is a real scalar constant for all $z$. Let $\sigma^H_1$ and $\sigma^H_2$ be the partial gradients of $\sigma(., .)$ seen as a function respectively of its first and second vector argument only.

If $\sigma^H_1 u \neq 0$ for $z = z_0$, then the derivative $u'$ of $u$ at $z = z_0$ is given by

$$u' = \frac{\sigma^H_1(A - \lambda I)^\dagger A' u - \sigma^H_2 w'}{\sigma^H_1 u} - (A - \lambda I)^\dagger A' u.$$

Here $X^\#$ denotes the so-called group inverse of a singular matrix $X$, i.e., the inverse of $X$ in the maximal multiplicative subgroup containing $X$. We refer the reader to the abovementioned paper for more details on group inverses.

In fact, very little is needed on group inverses, and the formula can be modified slightly in order to replace it with the Moore-Penrose pseudoinverse $X^\dagger$, which is a more canonical tool in matrix computations.

**Theorem 2.** With the same hypotheses as Theorem 1 let $v(z)$ be the left eigenvector of $A(z)$ corresponding to the eigenvalue $\lambda(z)$. If $\sigma^H_1 u \neq 0$ for $z = z_0$, then the derivative $u'$ of $u$ at $z = z_0$ is given by

$$u' = \frac{\sigma^H_1(A - \lambda I)^\dagger (A' - \lambda'I) u - \sigma^H_2 w'}{\sigma^H_1 u} - (A - \lambda I)^\dagger (A' - \lambda'I) u,$$

with $\lambda' = \frac{v^H A' u}{v^H u}$.  

**Proof.** The proof is a minor modification of the original proof of Theorem 1. By differentiating the identity $Au = \lambda u$ we get $A'u + Au' = \lambda' u + \lambda u'$, i.e.,

$$ (A - \lambda I)u' = -(A' - \lambda'I)u. \tag{12}$$

By left-multiplying everything by $v^H$, and noting that $v^H A = \lambda v^H$, we get the required expression for the eigenvalue derivative $\lambda'$. Moreover, since $u$ is a simple eigenvector at $z = z_0$, the kernel of $(A - \lambda I)$ is span($u$). Thus from (12) we can determine $u'$ up to a scalar multiple of $u$:

$$ u' = -(A - \lambda I)^\dagger (A' - \lambda'I) u + \delta u. \tag{13}$$

We shall now use the normalization condition $\sigma(u, w) = k$ to determine the value of $\delta$. By differentiating it, we get

$$ \sigma^H_1 (u, w)u' + \sigma^H_2 (u, w)w' = 0. \tag{14}$$

Plugging (13) into (14) yields a linear equation for $\delta$. \hfill \square
5.2 Jacobian of the Perron iteration

The Perron iteration is a fixed-point iteration for the function \( F(y) := PV(H_y) \), where the function \( u = PV(X) \) returns the Perron vector \( u \) of the nonnegative irreducible matrix \( X \), normalized such that \( w^T(u - H_u) = 0 \), where \( w \) is a fixed positive vector. We can use Theorem 2 to compute the Jacobian of this map \( F \).

**Theorem 3.** Let \( y \) be such that \( H_y \) is nonnegative and irreducible. Let \( u = F(y) \), and let \( v \) such that \( v^T H_y = \lambda v^T \), where \( \lambda = \rho(H_y) \). Then the Jacobian of the map \( F \) at \( y \) is

\[
JF_y = \left( I - \frac{u\sigma_1^T}{\sigma_1^T u} \right) (H_y - \lambda I)^\dagger \left( I - \frac{uv^T}{v^Tu} \right) b(\cdot, u) \tag{15}
\]

where

\[
\sigma_1^T = w^T(I - b(e - u, \cdot) - b(\cdot, e - u)).
\]

**Proof.** We shall compute first the directional derivative of \( F \) at \( y \) along the direction \( a \). To this purpose, let us set \( y(z) := y + az \), for any \( z \in \mathbb{C} \), and \( A(z) = H_y \). We have

\[
A'(z) = \frac{d}{dz}H_y = -b(a, \cdot).
\]

Moreover, set

\[
\sigma(u, w) = w^T(u - b(e, u) - b(u, e) + b(u, u)),
\]

where \( w(z) = w \) for each \( z \) (so that \( w' = 0 \)). The partial gradient of \( \sigma(\cdot, \cdot) \) with respect to the first argument is \( \sigma_1^T = w^T(I - b(e - u, \cdot) - b(\cdot, e - u)) \). Plugging everything into (11), we get

\[
\sigma_1^T (A - \lambda I)^\dagger (A' - \lambda I) u = \sigma_1^T (A - \lambda I)^\dagger (A') u
\]

\[
= - \left( I - \frac{u\sigma_1^T}{\sigma_1^T u} \right) (A - \lambda I)^\dagger \left( A' - \frac{uv^T}{v^Tu} \right) u
\]

\[
= - \left( I - \frac{u\sigma_1^T}{\sigma_1^T u} \right) (A - \lambda I)^\dagger \left( I - \frac{uv^T}{v^Tu} \right) A'u
\]

\[
= - \left( I - \frac{u\sigma_1^T}{\sigma_1^T u} \right) (A - \lambda I)^\dagger \left( I - \frac{uv^T}{v^Tu} \right) (-b(a, u))
\]

\[
= \left( I - \frac{u\sigma_1^T}{\sigma_1^T u} \right) (A - \lambda I)^\dagger \left( I - \frac{uv^T}{v^Tu} \right) b(\cdot, u) a.
\]

From this expression for the directional derivative, it is immediate to recognize that the Jacobian is (15).
5.3 Local convergence of the iteration

The fixed-point iteration \( y_{k+1} = F(y_k) \) is locally convergent in a neighborhood of \( y^* \) if and only if the spectral radius of \( JF_{y^*} \) is strictly smaller than 1. First notice that it makes sense to compute the Jacobian using (15) in a neighborhood of the solution \( y^* \). In fact, \( \sigma^*_1 y^* = w^T(y^* - b(e - y^*, y^*) - b(y^*, e - y^*)) = w^T b(y^*, y^*) \) and the latter quantity is positive as \( w > 0 \) and \( b(y^*, y^*) \geq 0 \), as stated in Section 2. Moreover, since \( \lambda = 1 \) is a simple eigenvalue, the left and right eigenvectors \( v = y^* \) and \( u = y^* \) cannot be orthogonal.

By evaluating (15) at \( y = y^* \), we get

\[
JF_{y^*} = \left( I - \frac{y^* \sigma^*_1^T}{w^T b(y^*, y^*)} \right) A^\dagger \left( I - \frac{y^* v^* T}{v^* T y^*} \right) b(\cdot, y^*), \tag{16}
\]

where we have set \( \sigma^*_1 = w^T (I - b(e - y^*, \cdot) - b(\cdot, e - y^*)) \) and \( A = (b(e - y^*, \cdot) + b(\cdot, e) - I) \).

Let us try to understand what happens to the spectral radius \( \rho(JF_{y^*}) \) when the problem is close to critical.

**Theorem 4.** Let \( b_t(\cdot, \cdot) \), \( t \in [0,1] \) be an analytical one-parameter family of Markovian binary trees, which is supercritical for \( t \in [0,1) \) and critical for \( t = 1 \), and let us denote with an additional subscript \( t \) the quantities defined above for this family of problems. Let us suppose that \( R_t := b_t(e, \cdot) + b_t(\cdot, e) \) is irreducible for every \( t \in [0,1] \), and let \( \rho(JF_{y^*_t,t}) \) be the spectral radius of the Jacobian of the Perron iteration as defined in (16). Then

\[
\lim_{t \to 1} \rho(JF_{y^*_t,t}) = \rho(JF_{y^*}) = 1 - \frac{\tilde{v}^T b_t(\hat{y}_1, \hat{y}_1) w^T \hat{y}_1}{\tilde{v}^T b_t(\hat{y}_1, \hat{y}_1) \tilde{v}^T \hat{y}_1},
\]

where \( \hat{y}_1 \) and \( \tilde{v}^T_1 \) are left and right Perron vectors of \( R_1 \). As a special case, if the vector \( w \) is a scalar multiple of \( \hat{v}_1 \), the limit is 0.

**Proof.** Let us define \( \hat{y}_1 \) as the Perron vector of \( H^{y^*}_{y^*_t,t} \) normalized so that \( \|\hat{y}_1\|_1 = 1 \), and similarly \( \tilde{v}^T_1 \) as the left Perron vector of the same matrix, normalized so that \( \|\tilde{v}_1\|_1 = 1 \). Since \( H^{y^*}_{y^*_t,t} \) is irreducible, its left and right Perron vectors are analytical functions of \( t \), and thus \( \hat{y}_1 \) and \( \tilde{v}_t \) converge to \( \hat{y}_1 \) and \( \hat{v}_1 \) respectively. We have \( H^{y^*}_{y^*_t,t} = H_{0,1} = R_1 \). Notice that \( \sigma^*_1 \to w^T (I - R_1) \), and that

\[
A^\dagger_1 = (b(e - y^*, \cdot) + b(\cdot, e) - I)^\dagger \to (R_1 - I)^\dagger.
\]

Moreover, since \( \hat{y}_1 \) and \( \tilde{v}_1 \) span the right and left kernel of \( I - R_1 \), we have

\[
(I - R_1)(I - R_1)^\dagger = I - \frac{\hat{y}_1 \tilde{v}_1^T}{\hat{v}_1^T \hat{y}_1}.
\]

Additionally, we shall make use of the relation \( \rho(AB) = \rho(BA) \), valid for any \( A \) and \( B \) such that \( AB \) and \( BA \) are square matrices, in the first and second-to-last step of the following computation.
Putting all together, we get

$$\rho(JF_{y_e,t}) = \rho \left( \left( I - \frac{y_e^* \sigma_{t,e}^T}{w^T b_t(y_e^*, y_t^*)} \right) A_t^l \left( I - \frac{y_e^* \sigma_{t,e}^T}{v_t^T y_t^*} \right) b_t(\cdot, y_t^*) \right)$$

$$\approx \rho \left( b_t(\cdot, y_t^*) - \frac{b_t(y_e^*, y_t^*)}{w^T b_t(y_e^*, y_t^*)} \sigma_{t,e}^T \right) A_t^l \left( I - \frac{y_e^* \sigma_{t,e}^T}{v_t^T y_t^*} \right).$$

Therefore, we obtain

$$\lim_{e \to 1} \rho(JF_{y_e,t}) = \rho \left( \left( I - \frac{b_t(\hat{y}_1, \hat{y}_1) w^T (I - R_1)}{w^T b_t(\hat{y}_1, \hat{y}_1)} (R_1 - I)^T \left( I - \frac{\hat{v}_1 \hat{v}_1^T}{v_1^T y_1} \right) \right) \right)$$

$$= \rho \left( \frac{b_t(\hat{y}_1, \hat{y}_1) w^T}{w^T b_t(\hat{y}_1, \hat{y}_1)} \left( I - \frac{\hat{v}_1 \hat{v}_1^T}{v_1^T y_1} \right) \right)$$

$$= \rho \left( \frac{b_t(\hat{y}_1, \hat{y}_1) w^T}{w^T b_t(\hat{y}_1, \hat{y}_1)} \left( I - \frac{\hat{v}_1 \hat{v}_1^T}{v_1^T y_1} \right) \right)$$

$$= \rho \left( \frac{1}{w^T b_t(\hat{y}_1, \hat{y}_1)} w^T \left( I - \frac{\hat{y}_1 \hat{v}_1^T}{v_1^T y_1} \right) b_t(\hat{y}_1, \hat{y}_1) \right)$$

$$= \left| 1 - \frac{\hat{v}_1^T b_t(\hat{y}_1, \hat{y}_1)}{w^T b_t(\hat{y}_1, \hat{y}_1) \frac{v_1^T y_1}{v_1^T y_1}} \right|. \qed$$

For the normalization condition, the above result suggests taking \( w \) as the left Perron vector of \( b(e, \cdot) + b(\cdot, e) \). Indeed, this choice guarantees the local convergence of the Perron iteration for close-to-critical problems. Moreover we point out that, even though the convergence is linear, the speed of convergence increases as the MBT gets closer to critical; in particular, the convergence is superlinear in the critical case.

6 Numerical experiments

We compared the Perron iteration (PI) with the Newton method (NM) \(^3\) and with the thicknesses algorithm (TH) \(^4\). As stated before, TH and PI are linearly convergent algorithms, while NM is a quadratically convergent one. All the experiments were performed using Matlab 7 (R14) on an Intel Xeon 2.80Ghz bi-processor.

We applied the algorithms to the two test cases reported in \(^3\). The first one (E1) is an MBT of size \( n = 9 \) depending on a parameter \( \lambda \), which is critical for \( \lambda \approx 0.85 \) and supercritical for larger values of \( \lambda \). The second one (E2) is a MBT of size \( n = 3 \) depending on a parameter \( \lambda \), which is critical for \( \lambda \approx 0.34 \) and \( \lambda \approx 0.84 \), and supercritical for the values inbetween.

The only noteworthy issue in the implementation of PI is the method used for the computation of the maximal eigenvector. The classical methods are
Figure 1: CPU time in sec. (and number of iterations in brackets) for TH, NM and PI on several choices of $\lambda$ for E1 (top) and E2 (bottom).

| $\lambda$ | TH         | NM         | PI          |
|-----------|------------|------------|-------------|
| 0.86      | 2.3935e+00 (11879) | 5.0932e-03 (14) | 4.9267e-03 (7) |
| 0.9       | 6.5353e-01 (3005)  | 4.2859e-03 (12) | 5.5756e-03 (8)  |
| 1         | 2.8049e-01 (1149)  | 3.9009e-03 (11) | 5.5090e-03 (8)  |
| 2         | 9.2644e-02 (191)   | 2.8453e-03 (8)  | 5.5125e-03 (8)  |
| 0.5       | 7.7003e-02 (132)   | 2.3305e-03 (8)  | 5.6983e-03 (11) |
| 0.7       | 7.6503e-02 (135)   | 2.1842e-03 (8)  | 5.6081e-03 (11) |
| 0.8       | 9.3603e-02 (313)   | 2.4543e-03 (9)  | 4.6166e-03 (9)  |
| 0.84      | 7.3060e-01 (4561)  | 4.0001e-03 (13) | 4.1090e-03 (8)  |

usually optimized for matrices of much larger size; however, here we deal with matrices of size $n = 3$ and $n = 9$, for which the complexity constants matter. We compared several candidates ($\text{eigs}$, $\text{eig}$, the power method, a power method accelerated by repeated squaring of the matrix), and found that in our examples the fastest method to find the maximal eigenvector is computing the full eigenvector basis with $[V,\text{Lambda}] = \text{eig}(P)$ and then selecting the maximal eigenvector. The picture should change for problems of larger size: $\text{eig}$ takes $O(n^3)$ operations, while for instance $\text{eigs}$ should take only $O(n^2)$ in typical cases. On the other hand, we point out that in absence of any structure (such as sparsity) in $b(\cdot, \cdot)$, forming the matrix $b(v, \cdot)$ or $b(\cdot, v)$ for a new vector $v$, an operation which is required at every step in all known iterative algorithms, requires $O(n^3)$ operations. Therefore, the CPU times are somehow indicative of the real complexity of the algorithms, but should be taken with a grain of salt.

The stopping criterion was chosen to be $\|x - a + b(x, x)\| \leq n\varepsilon$, with $\varepsilon = 10^{-13}$, for all algorithms.

The table in Figure 1 shows the results for several choices of $\lambda$. The algorithm TH is clearly the slowest, taking far more CPU time than the two competitors. The different behavior of PI when approaching the critical cases is apparent: while the iterations for TH and NM increase, PI seems to be unaffected by the near-singularity of the problem, and in fact the iteration count decreases slightly.

To show further results on the comparison between NM and PI, we report a number of graphs comparing the iteration count and CPU times of the two algorithms. The graphs are not cut at the critical values, but they extend to subcritical cases as well. It is an interesting point to note that when the MBT is subcritical, and thus the minimal solution $x^*$ (extinction probability) is $e$, the two algorithms have a different behavior: NM (and TH as well) converges to $e$, while PI skips this solution and converges to a different solution $x > e$. This is because in the derivation of the Perron iteration we chose the solution $\alpha \neq 0$ for the normalization equation, thus explicitly excluding the solution $y = 0$ (i.e., $x = e$).
Figure 2: Iteration count vs. parameter $\lambda$ for E1 (top) and E2 (bottom)
Figure 2 shows a plot of the iteration count of the two methods vs. different values of the parameter \( \lambda \). While in close-to-critical cases the iteration count for NM has a spike, the one for PI seems to decrease. However, the iteration count comparison is not fair since the steps of the two iterations require a different machine time. Figure 3 shows a similar plot, considering the CPU time instead of the iteration count. In order to achieve better accuracy, the plotted times are averages over 100 consecutive runs.

The results now favor the Newton method in most experiments, but in close-to-critical cases the new method achieves better performance. The results are very close to each other, though, so it is to be expected that for larger input sizes or different implementations the differences in the performance of the eigensolver could lead to significant changes in the results.

In order to highlight the performance difference in close-to-critical cases, we report in Figure 4 a plot with the CPU times sampled at a larger number of points around the most “interesting” regions of the previous graphs.

The Jacobian (16) had spectral radius less than 1 in all the above experiments, a condition which is needed to ensure the convergence of PI. However, this is not true for all possible MBTs. In fact, by setting the parameter \( \lambda \) for E1 to much larger values, we encountered problematic cases in which PI did not converge. Specifically, starting from \( \lambda \approx 78 \) the Jacobian (16) is larger than 1 and PI does not converge. However, such cases are of little practical interest since they are highly supercritical MBTs, distant from the critical case, and thus they are easily solved with the traditional methods (NM or the customary functional iterations [2]) with a small number of iterations.

The problem E2 is well-posed only for \( 0 \leq \lambda \leq 1 \), otherwise negative entries appear in \( b \), thus the above discussion does not apply.

Along all the experiments reported above, all the matrices \( H_y \) appearing in the PI steps always turned out to have positive entries, even in the subcritical problems; thus their Perron vector and values were always well-defined and real.

### 7 Conclusions and open issues

We have proposed a new algorithm for solving the quadratic vector equation (1), based on a Perron iteration. The algorithm performs well, both in terms of speed of convergence and accuracy, for close-to-critical problems where the classical methods are slower.

Along the framework that we have exposed, several different choices are possible in the practical implementation of the new algorithm.

One of them is the choice of the bilinear form \( b(\cdot, \cdot) \). Equation (6) and its solution depend only on the quadratic form \( b(t,t) \); however, there are different ways to extend it to a bilinear form \( b(s,t) \). This choice ultimately reflects a modeling aspect of the problem: when an individual spawns, it is transformed into two individuals in different states, and we may choose arbitrarily which of them is called the mother and which the child.

As an example of how this choice affects the solution algorithms, changing
Figure 3: CPU time (in sec.) vs. parameter $\lambda$ for E1 (top) and E2 (bottom)
Figure 4: Detailed views from Figure 3: CPU time (in sec.) vs. parameter $\lambda$ for E1 (top) and E2 (middle and bottom)
the bilinear form may transform the depth algorithm into the order one and vice versa. The algorithms we proposed depend on the actual choice of the bilinear extension of the quadratic form \( b(t, t) \), and the convergence speed is affected by this decision.

A second choice is the normalization of the computed Perron vector: different approaches may be attempted — for instance, minimization of the 1-norm, of the 2-norm, or orthogonality of the residual of \([7]\) with respect to a suitably chosen vector — although it is not clear whether we can improve the results of the normalization presented here.

A third choice, crucial in the computational experiments, is the method used to compute the Perron vector. For moderate sizes of the problem, it is cheaper to do a full eigendecomposition of the matrix and extract the eigenvalue with maximum modulus, but for larger problems it pays off to use different specific methods for its computation.

All these variants deserve to be better understood, and are now under our investigation.

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