ROUTH’S THEOREM FOR TETRAHEDRA REVISITED

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Abstract. A geometric proof of the Routh’s theorem for tetrahedra was given in [27]. In this paper we give another geometric proof of Routh’s theorem for tetrahedra. This proof is then generalized to calculate the volume of the "small" inner Routh’s \((n-1)\)–dimensional simplex. A comparison of the result with the formula obtained using vector analysis yields an interesting algebraic identity.

Introduction

The classical Routh’s theorem, see p.82 of [38], states the following.

**Theorem 1.** Let \(ABC\) be an arbitrary triangle of area 1, a point \(K\) lie on the line segment \(BC\), a point \(L\) lie on the line segment \(AC\) and a point \(M\) lie on the line segment \(AB\) such that the ratio \(\frac{|AM|}{|MB|} = x\), \(\frac{|BK|}{|KC|} = y\) and \(\frac{|CL|}{|LA|} = z\). Denote by \(P\) the point of intersection of lines \(AK\) and \(CM\), by \(Q\) the point of intersection of lines \(BL\) and \(AK\), and by \(R\) the point of intersection of lines \(CM\) and \(BL\) - see Figure 1.

\[
\frac{1 + xyz}{(1 + x)(1 + y)(1 + z)},
\]

Then the area of the triangle \(KLM\) is

**Figure 1.** Routh’s Triangles
and the area of the triangle PQR is
\[
\frac{(1 - xyz)^2}{(1 + x + xy)(1 + y + yz)(1 + z + zx)}.
\]

Routh’s theorem implies the theorem of Ceva:

**Theorem 2.** The lines AK, BL and CM intersect at one point if and only if \(xyz = 1\).

Routh’s theorem is also closely related to the following theorem of Menelaus.

**Theorem 3.** Let \(K\) be an arbitrary point on the line \(BC\), \(L\) on line \(BC\) and \(M\) on line \(AB\). Denote \(\frac{KM}{MB} = x\), \(\frac{BR}{KC} = y\) and \(\frac{CL}{LA} = z\). Then the points \(K, L, M\) are colinear if and only if \(xyz = 1\).

After we had written this paper, we conducted an extensive search of the literature on theorems of Routh, Ceva and Menelaus and their generalizations to higher dimensions within the context of Euclidean geometry (there are generalizations in other geometries but we did not include them here) and listed them in the bibliography to the present paper. We believe that this list of articles is interesting from the historical perspective (although we cannot guarantee its completeness) and it is valuable since it represents the wide range of generalizations of these classical theorems. We have been able to find only two papers, \([20]\) and \([43]\), where Routh’s theorem is generalized to tetrahedra and higher dimensions. Note that the statement in \([43]\) is missing an absolute value and the statement in \([20]\) needs to be reformulated to fit our notation. Unfortunately, both papers are not readily accessible to most of the readers since they are written in Chinese and Slovak languages, respectively. In \([27]\), we gave a geometric proof of Routh’s theorem for tetrahedra. Keeping in mind that we are going to generalize this theorem to simplices, we need to adjust the notation as follows.

**Theorem 4.** Let \(A_1A_2A_3A_4\) be an arbitrary tetrahedron of volume 1. Choose a point \(P_1\) on the edge \(A_1A_2\), a point \(P_2\) on the edge \(A_2A_3\), a point \(P_3\) on the edge \(A_3A_4\), and a point \(P_4\) on the edge \(A_4A_1\) such that \(\frac{|P_1A_1|}{|P_4A_1|} = x_1\), \(\frac{|P_2A_2|}{|P_3A_2|} = x_2\), \(\frac{|P_3A_3|}{|P_4A_3|} = x_3\), and \(\frac{|P_4A_4|}{|P_2A_4|} = x_4\). Then

\[
V_{P_1P_2P_3P_4} = \frac{|1 - x_1x_2x_3x_4|}{(1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4)}.
\]

The four planes given by the points \(A_1, A_2, P_3\), points \(A_2, A_3, P_4\), points \(A_3, A_4, P_1\), and points \(A_4, A_1, P_2\) enclose the tetrahedron \(R_1R_2R_3R_4\) (see Figure 3) of the volume

\[
V_{R_1R_2R_3R_4} = \frac{|1 - x_1x_2x_3x_4|^3}{(1 + x_1 + x_1x_2 + x_1x_2x_3)(1 + x_2 + x_2x_3 + x_2x_3x_4)(1 + x_3 + x_3x_4 + x_3x_4x_1)(1 + x_4 + x_4x_1 + x_4x_1x_2)}.
\]

The formulas in the above theorem correspond to the cycle \((A_1A_2A_3A_4)\). Opposite to \([27]\), we will assume that \(x_1x_2x_3x_4 > 1\). If \(x_1x_2x_3x_4 < 1\), then we can change the orientation of the cycle \((A_1A_2A_3A_4)\). As a consequence, the product
Figure 2. Notation, \(x_1x_2x_3x_4 > 1\)

\(x_1x_2x_3x_4\) will change to \(\frac{1}{x_1x_2x_3x_4} > 1\), and a simple evaluation leads to the same result.

Formula (2) will be proved using geometric considerations together with the principle of inclusion-exclusion.

In the last section of the paper we will extend our considerations to the cycle \((A_1 \ldots A_n)\) corresponding to a general \((n-1)\)-dimensional simplex \(A_1 \ldots A_n\). Comparison of the result with the formula given in [43] produces an interesting algebraic identity.

1. ROUTH’S THEOREM FOR TETRAHEDRA: PROOF OF (2)

Let us assume that \(x_1x_2x_3x_4 > 1\). To the cutting plane \(\sigma_1\) given by points \(P_1, A_3, A_4\) we assign the half-space \(S_1\) containing \(A_2\), to the cutting plane \(\sigma_2\) given by points \(A_1, P_2, A_4\) we assign the half-space \(S_2\) containing \(A_3\), to the cutting plane \(\sigma_3\) given by points \(A_1, A_2, P_3\) we assign the half-space \(S_3\) containing \(A_4\), and to the cutting plane \(\sigma_4\) given by points \(P_4, A_2, A_3\) we assign the half-space \(S_4\) containing \(A_1\). For \(i = 1, 2, 3, 4\) denote by \(T_i\) the tetrahedron that is the intersection of \(S_i\) with the tetrahedron \(A_1A_2A_3A_4\), and by \(V_i\) the volume of the tetrahedron \(T_i\).

If \(x_1x_2x_3x_4 > 1\), then the intersection \(T_1 \cap T_2 \cap T_3 \cap T_4 = S_1 \cap S_2 \cap S_3 \cap S_4\) is the tetrahedron \(R_1R_2R_3R_4\). (If \(x_1x_2x_3x_4 = 1\), then this intersection is a single point and if \(x_1x_2x_3x_4 < 1\), this intersection if empty).

In what follows we denote the volume of a tetrahedron \(T = ABCD\) by \(V_{ABCD}\) or \(V_T\) and analogously for other tetrahedra. For the convenience of the reader we will now restate Lemmas 6, 7, and 8 of [27].
Lemma 5. In the notation of Figure 3, we have
\[ V_{AMCD} = V_{ABCD} \frac{|AM|}{|AB|} \]

Lemma 6. In the notation of Figure 3, we have
\[ V_{AKCM} = V_{ABCD} \frac{|AM|}{|AB|} \frac{|CK|}{|CD|} \quad \text{and} \quad V_{AMKD} = V_{ABCD} \frac{|AM|}{|AB|} \frac{|DK|}{|DC|}. \]

Lemma 7. Consider the triangle ABC as in Figure 4. If \( \frac{|AM|}{|MB|} = v \) and \( \frac{|BK|}{|KC|} = u \), then \( \frac{|AP|}{|PK|} = v(1 + u) \).

Using Lemma 5 we see immediately that
\[ V_{T_1} = \frac{x_1}{1 + x_1}, \quad V_{T_2} = \frac{x_2}{1 + x_2}, \quad V_{T_3} = \frac{x_3}{1 + x_3} \quad \text{and} \quad V_{T_4} = \frac{x_4}{1 + x_4}. \]
It follows immediately from Lemma 6 that
\[ V_{T_1 \cap T_3} = \frac{x_1}{1 + x_1} \frac{x_3}{1 + x_3} \quad \text{and} \quad V_{T_2 \cap T_4} = \frac{x_2}{1 + x_2} \frac{x_4}{1 + x_4}. \]
Lemma 8. Consider the triangle in Figure 4. Then
\[ \frac{|MP|}{|PC|} = \frac{vu}{1 + v} \quad \text{and} \quad \frac{|MP|}{|MC|} = \frac{vu}{1 + v + vu}. \]

Proof. Lemma 9 applied to the triangle CBA yields
\[ \frac{|CP|}{|PM|} = \frac{1}{u} \left(1 + \frac{1}{v}\right) = \frac{1 + v}{uv}, \]
and the desired ratios follow. \(\square\)

Lemma 9. The volume of \(T_1 \cap T_2\) is
\[ x_1^2 x_2 \frac{1}{(1 + x_1)(1 + x_1 + x_2)}. \]

Proof. It can be observed from Figure 5 that \(T_1 \cap T_2\) is the tetrahedron \(A_1 P_1 Q_1 A_4\).

Using Lemma 5 we obtain
\[ V_{A_1 P_1 A_3 A_4} = \frac{x_1}{1 + x_1} \quad \text{and} \quad V_{A_1 P_1 Q_1 A_4} = V_{A_1 P_1 A_3 A_4} \frac{|P_1 Q_1|}{|Q_1 A_3|}. \]

Since, by Lemma 8
\[ \frac{|P_1 Q_1|}{|Q_1 A_3|} = \frac{x_1 x_2}{1 + x_1 + x_1 x_2}, \]
the formula follows. \(\square\)

It is clear that Lemma 9 also implies that
\[ V_{T_2 \cap T_3} = \frac{x_2^2 x_3}{(1 + x_2)(1 + x_2 + x_2 x_3)}, \]
\[ V_{T_3 \cap T_4} = \frac{x_3^2 x_4}{(1 + x_3)(1 + x_3 + x_3 x_4)}, \]
\[ V_{T_4 \cap T_1} = \frac{x_4^2 x_1}{(1 + x_4)(1 + x_4 + x_4 x_1)}. \]
Lemma 10. The volume of $T_1 \cap T_2 \cap T_3$ is
\[ x_1^3 x_2 x_3 \frac{(1 + x_1)(1 + x_1 + x_1 x_2)(1 + x_1 + x_1 x_2 + x_1 x_2 x_3)}{1 + x_1 + x_1 x_2 + x_1 x_2 x_3}. \]

Proof. In the notation of Figure 2, the intersection $T_1 \cap T_2 \cap T_3$ is the tetrahedron $A_1 P_1 Q_1 R_1$. Looking at Figure 6 and using Lemma 5, we determine that

\[ V_{A_1 P_1 Q_1 R_1} = V_{A_1 P_1 Q_1 A_4} \frac{|Q_1 R_1|}{|Q_1 A_4|} \]

by Lemma 9. To find the remaining ratio $\frac{|Q_1 R_1|}{|Q_1 A_4|}$, consider the triangle $A_1 P_2 A_4$ as depicted in Figure 7. Here we have $v = x_1(1 + x_2)$ by Lemma 7 applied to the triangle $A_1 A_2 A_3$ and $u = \frac{x_1 x_2 x_3}{1 + x_2}$ by Lemma 8 applied to the triangle $A_2 A_3 A_4$. Therefore Lemma 8 applied to the triangle $A_1 P_2 A_4$ yields

\[ \frac{|Q_1 R_1|}{|Q_1 A_4|} = \frac{vu}{1 + v + vu} = \frac{x_1 x_2 x_3}{1 + x_1 + x_1 x_2 + x_1 x_2 x_3}. \]
and the result follows. □

It is clear that Lemma 10 also implies that
\[
V_{T_2 \cap T_3 \cap T_4} = \frac{x_2^3 x_3^2 x_4}{(1 + x_2)(1 + x_2 + x_2 x_3)(1 + x_2 + x_2 x_3 + x_2 x_3 x_4)},
\]
\[
V_{T_3 \cap T_4 \cap T_5} = \frac{x_3^3 x_4^2 x_1}{(1 + x_3)(1 + x_3 + x_3 x_4)(1 + x_3 + x_3 x_4 + x_3 x_4 x_1)},
\]
\[
V_{T_4 \cap T_5 \cap T_6} = \frac{x_4^3 x_1^2 x_2}{(1 + x_4)(1 + x_4 + x_4 x_1)(1 + x_4 + x_4 x_1 + x_4 x_1 x_2)}.
\]

**Proof of (2) in Theorem 4** Assume \(x_1 x_2 x_3 x_4 > 1\). Using the principle of inclusion-exclusion we get
\[
V_{R_1 R_2 R_3 R_4} = V_{A_1 A_2 A_3 A_4} - V_{T_1} - V_{T_2} - V_{T_3} - V_{T_4}
+ V_{T_1 \cap T_3} + V_{T_2 \cap T_4} + V_{T_1 \cap T_2} + V_{T_3 \cap T_4} + V_{T_4 \cap T_1}
- V_{T_1 \cap T_2 \cap T_3} - V_{T_2 \cap T_1 \cap T_4} - V_{T_3 \cap T_1 \cap T_2} - V_{T_4 \cap T_1 \cap T_2}.
\]

The formula (2) now follows from the previous formulas for the volumes of the above tetrahedra together with the following identity (3).

\[
(3) \quad 1 - \frac{x_1}{1 + x_1} - \frac{x_2}{1 + x_2} - \frac{x_3}{1 + x_3} - \frac{x_4}{1 + x_4} + \frac{x_2^2 x_3}{1 + x_2 + x_2 x_3} + \frac{x_3^2 x_4}{1 + x_3 + x_3 x_4} + \frac{x_4^2 x_1}{1 + x_4 + x_4 x_1} + \frac{x_1 x_3}{1 + x_1 + x_1 x_2 + x_1 x_2 x_3} + \frac{x_2 x_4}{1 + x_2 + x_2 x_3 + x_2 x_3 x_4} + \frac{x_3 x_1}{1 + x_3 + x_3 x_4 + x_3 x_4 x_1} + \frac{x_4 x_2}{1 + x_4 + x_4 x_1 + x_4 x_1 x_2}
\]
\[
= \frac{(x_1 x_3 x_4 x_1 - 1)^3}{(1 + x_1 + x_1 x_2 + x_1 x_2 x_3)(1 + x_2 + x_2 x_3 + x_2 x_3 x_4) \times (1 + x_3 + x_3 x_4 + x_3 x_4 x_1)(1 + x_4 + x_4 x_1 + x_4 x_1 x_2)}.
\]

that can be verified either manually or using a software like Mathematica or Maple.

Case \(x_1 x_2 x_3 x_4 < 1\) can be treated similarly to [27] by reversing the orientation of the cycle \((A_1 A_2 A_3 A_4)\) to \((A_1 A_4 A_3 A_2)\), using the substitution \(x_1 \rightarrow \frac{1}{x_1}, x_2 \rightarrow \frac{1}{x_2}, x_3 \rightarrow \frac{1}{x_3}, x_4 \rightarrow \frac{1}{x_4}\) that reduces it to the case \(x_1 x_2 x_3 x_4 > 1\). □
2. Comments on Routh’s theorem for $n$-simplexes and related algebraic identities

Reviewing the formulas for volumes of various tetrahedra appearing in application of the inclusion-exclusion principle it is easy to observe the pattern that might hold in the general case of $(n-1)$-dimensional simplex $S = A_1 \ldots A_n$.

We will fix an orientation of the cycle $(A_1 \ldots A_n)$ and for simplicity of notation we will consider indices modulo $n$, that is, we identify the index $n + 1$ with 1. For each $i = 1, \ldots, n$ choose a point $P_i$ on the edge $A_iA_{i+1}$ of $S$ and denote the ratio \( |A_iP_i|/|P_iA_{i+1}| \), by $x_i$. Denote by $\sigma_i$ the half-space given by the hyperplane containing points $A_1 \ldots A_{i-1}P_iA_{i+1} \ldots A_n$ in the direction of the point $A_i$ and by $T_i$ the intersection of $\sigma_i$ with the original simplex $A_1 \ldots A_n$. We will assume that $x_1 \ldots x_n > 1$. In this case the intersection of all half-spaces $\sigma_i$ for $i = 1, \ldots, n$ is the $(n-1)$-dimensional simplex $\bigcap_{i=1}^n T_i$.

When determining the volume of the intersection $\bigcap_{i \in I} T_i$, where $I \subset \{1, \ldots, n\}$, the most important role is played by the distribution of elements $i \in I$ along the cycle $C = (1 \ldots n)$. Assume that the set $I$ is listed by blocks of consecutive elements along the cycle $C$, keeping in mind that a block containing $n$ can start before $n$ and continue through $n$ to 1 and further. Denote by $B(I)$ the set of blocks of $I$ along the cycle $C$. To each block of $I$, say $B = \{k,k+1,\ldots,k+l\}$, we assign the following expression

\[
e_B = \prod_{j=k}^{k+l} \frac{\prod_{a=k}^{l} x_a}{1 + \sum_{b=k}^{l} \prod_{a=k}^{b} x_a}.
\]

For example,

\[
e_{\{2,3,4,5\}} = \frac{x_2}{1+x_2} \frac{x_3}{1+x_2+x_3} \frac{x_4}{1+x_2+x_3+x_4} \frac{x_5}{1+x_2+x_3+x_4+x_5}.
\]

Then we claim that $V_{\bigcap_{i \in I} T_i} = \prod_{B \in B(I)} e_B$. This is verified for $n = 4$ using the formulas in the previous section. A proof of this statement in general is not important at this time and we will leave it for the consideration of the reader.

What is interesting is that if we believe the above formula for $V_{\bigcap_{i \in I} T_i}$, an application of inclusion-exclusion principle gives us the volume of the $(n-1)$ simplex $\bigcap_{i=1}^n T_i$ and a generalization of Routh’s theorem to simplexes.

Since the volume of $\bigcap_{i=1}^n T_i$ can be determined using vector analysis and determinants and by \cite{3} it is equal to

\[
\frac{(\prod_{i=1}^n x_i - 1)^{n-1}}{\prod_{k=1}^n (1 + \sum_{b=k}^{k+n-1} \prod_{a=k}^{b} x_a)}
\]

provided $\prod_{i=1}^n x_i \geq 1$, we obtain an interesting algebraic identity generalizing equation \cite{3} as follows.

\[
1 + \sum_{\emptyset \subseteq I \subseteq \{1, \ldots, n\}} (-1)^{|I|} \prod_{B \in B(I)} e_B = \frac{(\prod_{i=1}^n x_i - 1)^{n-1}}{\prod_{k=1}^n (1 + \sum_{b=k}^{k+n-1} \prod_{a=k}^{b} x_a)},
\]

where $|I|$ is the parity of the number of elements in $I$. Here both sides of the above equality equals $V_{\bigcap_{i=1}^n T_i}$ provided $\prod_{i=1}^n x_i \geq 1$.

We have checked this identity for additional values of $n$ but we are not aware of general algebraic proof of these identities, which would be interesting to see. For an amusement of the reader we will now write the identity related to Routh’s theorem for simplexes in the case when $n = 5$. 

The analogous identity for $V_n = x_1 + x_2 + \cdots + x_n = k$ was proven in [27] as a consequence of the identity

$$1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \frac{x_3}{1+x_3} - \frac{x_4}{1+x_4} - \frac{x_5}{1+x_5} + \frac{x_1^2 x_2}{(1+x_1)(1+x_1+x_2)} + \frac{x_2^2 x_3}{(1+x_2)(1+x_2+x_3)} + \frac{x_3^2 x_4}{(1+x_3)(1+x_3+x_4)} + \frac{x_4^2 x_5}{(1+x_4)(1+x_4+x_5)} + \frac{x_5^2}{1+x_5}$$

Finally, the formula [1] in Theorem 4 was proven in [27] as a consequence of the identity

$$1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \frac{x_3}{1+x_3} - \frac{x_4}{1+x_4} - \frac{x_5}{1+x_5} + \frac{x_1 x_2}{1+x_1+x_2} + \frac{x_2 x_3}{1+x_2+x_3} + \frac{x_3 x_4}{1+x_3+x_4} + \frac{x_4 x_5}{1+x_4+x_5} + \frac{x_5}{1+x_5}$$

$$= \frac{1}{(1+x_1)(1+x_2)(1+x_3)(1+x_4)(1+x_5)}$$

It would be interesting to show similar identities for higher dimensions $n-1$.

The analogous identity for $n = 5$ is

$$1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \frac{x_3}{1+x_3} - \frac{x_4}{1+x_4} - \frac{x_5}{1+x_5} + \frac{x_1 x_2}{1+x_1+x_2} + \frac{x_2 x_3}{1+x_2+x_3} + \frac{x_3 x_4}{1+x_3+x_4} + \frac{x_4 x_5}{1+x_4+x_5} + \frac{x_5}{1+x_5}$$

The reader is encouraged to find geometrical interpretation of this identity and its generalization to higher dimensions.

We finish by stating the formulas for the volumes of the previously considered tetrahedra in the special case when $x_1 = x_2 = \cdots = x_n = k$. In this case the volume $V = V_{x_1=\cdots=x_n} = \frac{k}{k-1}$. In particular, if $k = 2$ and $n = 3$, then $V = \frac{1}{2}$; if $k = 2$ and $n = 4$, then $V = \frac{5}{27}$. The volume of the tetrahedra $V_{P_1P_2P_3P_4}$ in the special case when $x_1 = x_2 = \cdots = x_n = k$ equals $\frac{k^n}{(k+1)^n}$. In particular, if $k = 2$ and $n = 3$, then $V_{P_1P_2P_3} = \frac{7}{27}$; if $k = 2$ and $n = 4$, then $V_{P_1P_2P_3P_4} = \frac{15}{27} = \frac{5}{9}$. 
Acknowledgement. The authors are indebted to Professor Jose Alfredo Jimenez for his encouragement and help with the images presented in this article.

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