

INFINITE RANK SURFACE CLUSTER ALGEBRAS

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Abstract. We generalise surface cluster algebras to the case of infinite surfaces where the surface contains finitely many accumulation points of boundary marked points. To connect different triangulations of an infinite surface, we consider infinite mutation sequences.

We show transitivity of infinite mutation sequences on triangulations of an infinite surface and also give a characterisation for different types of mutation sequences. Moreover, we use a hyperbolic structure on an infinite surface to extend the notion of surface cluster algebras to infinite rank by giving cluster variables as lambda lengths of arcs. Furthermore, we study the structural properties of infinite rank surface cluster algebras in combinatorial terms, namely we extend “snake graph combinatorics” to give an expansion formula for cluster variables. We also show skein relations for infinite rank surface cluster algebras.

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1. Introduction

We introduce cluster algebras associated to infinite bordered surfaces. Our work extends cluster algebras from (finite) marked surfaces [FST, FT] and generalises surface cluster algebra combinatorics advanced in [MSW, CS].

Cluster algebras were introduced by a groundbreaking work of Fomin and Zelevinsky [FZ1] and further developed in [BFZ, FZ2, FZ4] with the original motivation to give an algebraic framework for the study of dual canonical basis in Lie theory. Cluster algebras are combinatorially defined commutative algebras given by generators, cluster variables, and relations which are iteratively defined via a process called mutation.

This work was partially supported by EPSRC grant EP/N005457/1.
An important class of cluster algebras, called cluster algebras associated to surfaces or surface cluster algebras, was introduced in [FST] by using combinatorics of (oriented) Riemann surfaces with marked points and consequently, building on an earlier work of [FG1, FG2], an intrinsic formulation of surface cluster algebras was given in [FT] where lambda lengths of curves serve as cluster variables (see also [P]). Surface cluster algebras also constitute a significant class in terms of classification of cluster algebras since it was shown in [FeSTu, FeSTu2, FeSTu3] that all but finitely many cluster algebras of finite mutation type are those associated to triangulated surfaces (or to orbifolds for the skew-symmetrizable case). An initial data to construct a cluster algebra corresponds to a triangulation of the surface, namely a maximal collection of non-crossing arcs.

Combinatorial and geometric aspects of surface cluster algebras have been studied remarkably widely. For instance, an expansion formula for cluster variables was given by Musiker, Schiffler and Williams [MSW] extending the work of [S, ST, S2, MS] and bases for surface cluster algebras were constructed in [MSW2, T, FeTu]. Further combinatorial aspects of surface cluster algebras were studied in [CS, CS2, CS3, CLS]. Moreover, it was shown in [M] that the quantum cluster algebra coincides with the quantum upper cluster algebra for the surface type under certain assumptions. In addition, combinatorial topology of triangulated surfaces and surface cluster algebra combinatorics have been influential in representation theory, e.g. in cluster categories associated to marked surfaces [BZ, QZ, CaSc, ZZZ] and in gentle algebras associated to marked surfaces [ABC, LF, Lad, CaSc].

Infinite rank cluster algebras have appeared in a number of different contexts. In particular, in the study of cluster categories associated to the infinitely-gon and the infinite double strip in [IT, IT2, HJ, LP]. Moreover, triangulations of the infinity-gon were used in [BHJ, HL] to give full classification of $SL_2$-tilings and also in [GG] to associate subalgebras of the coordinate ring $\mathbb{C}[Gr(2, \pm \infty)]$ of an infinite rank Grassmannian $Gr(2, \pm \infty)$. Furthermore, infinite rank cluster algebras were considered in [HL, HL2] in the context of quantum affine algebras and also in [FK] to show that the equations of generalised $Q$-systems can be given as mutations. However so far, to the best of our knowledge, infinite rank cluster algebras were considered with the focus only on finite mutation sequences. One exception to this restriction was given in an independent recent work of Baur and Gratz in [BG] which considers infinite triangulations of unpunctured surfaces and classifies those in the same equivalence class. [BG] concentrates on two examples, namely, on triangulations of the infinity-gon as well as of the completed infinity-gon where triangulations are completed with strictly asymptotic arcs (those connecting to limit points). The first is motivated to overcome finiteness constraints in the corresponding representation theory whereas the latter is motivated to introduce a combinatorial model to capture the representation theory of the polynomial ring in one variable. [BG] does not study the associated cluster algebras though.

In this paper, we aim to consider infinite sequences of mutations and subsequently we introduce cluster algebras for infinite surfaces allowing infinite mutation sequences. More precisely, we have a three-fold goal for this paper:

- **topological**: not to have any restrictions on the type of triangulations, i.e. being able to mutate between any two triangulations of a fixed surface;
- **geometric**: to introduce cluster algebras from infinite surfaces by associating a hyperbolic structure to the surface and considering lambda lengths of arcs as cluster variables without having to deal with combinatorics of (infinite) quiver mutations;
- **combinatorial**: to extend further main properties of surface cluster algebras from finite case using “snake graph calculus”, in particular:
  - to give an expansion formula for cluster variables in terms of intersection pattern of arcs with a fixed initial triangulation;
  - to show skein relations, i.e. certain identities in the cluster algebra associated to generalised Ptolemy relations in the surface, for infinite surface cluster algebras.

Our setting is the following. By an infinite surface we mean a connected oriented surface of a finite genus, with finitely many interior marked points (punctures), with finitely many boundary
components, and with infinitely many boundary marked points, located in such a way that they have only finitely many accumulation points of boundary marked points.

We start with investigating the combinatorics of infinite mutations in Section 2. In order to be able to mutate between any two triangulations, we need to introduce infinite sequences of mutations, called infinite mutations, and moreover, infinite sequences of infinite mutations. When we refer to infinite mutation sequences, we only permit sequences “converging” in a certain sense. Our main result establishes the transitivity of infinite sequences of infinite mutations on triangulations of a given infinite surface.

**Theorem A** (Theorem 2.24). For any two triangulations $T$ and $T'$ of an infinite surface $S$, there exists a mutation sequence $\mu^*$ such that $\mu^*(T) = T'$, where $\mu^*$ is a “finite mutation”, a “finite sequence of infinite mutations” or an “infinite sequence of infinite mutations” (see also Definition 2.11).

Furthermore, we give a characterisation for each type of mutation sequences between two triangulations $T$ and $T'$ of a surface $S$ in terms of intersection numbers of the arcs in these triangulations. We show that the type of the mutation $\mu^*$ required in Theorem A depends on these intersection numbers, in particular, there are some pairs of triangulations requiring infinite mutations and there are pairs of triangulations for which infinite sequences of infinite mutations are unavoidable. More precisely, we have the following result.

**Theorem B** (Theorem 2.47). Let $T$ and $T'$ be triangulations of an infinite surface. Then the mutation sequence $\mu^*$ satisfying $\mu^*(T) = T'$ in Theorem A can be chosen as

1. “finite” if and only if $T'$ crosses $T$ in finitely many places;
2. “finite sequence of infinite mutations” if and only if $T'$ contains at most finitely many arcs crossing $T$ infinitely many times;
3. “infinite sequence of infinite mutations” otherwise, namely, if and only if $T'$ contains infinitely many arcs each of which crosses $T$ infinitely many times.

In Section 3 we follow [FT] and [P] and consider triangulated surfaces with hyperbolic metrics such that every triangle of $T$ is an ideal triangle in this metric. We choose horocycles at every marked point (including accumulation points). Doing so, we require that if a sequence of marked points converge to an accumulation point $p_*$, then the corresponding horocycles converge to the horocycle at $p_*$. Under these assumptions, the lambda lengths of the arcs satisfy some “limit conditions”. On the other hand, it turns out that as soon as these necessary conditions are satisfied, there exists a unique hyperbolic structure and a unique choice of horocycles leading to the prescribed values of lambda lengths. In other words, lambda lengths of the arcs in an infinite surface can serve as coordinates on the decorated Teichmüller space. In order to achieve these results, we need to consider triangulations without infinite zig-zag (or leapfrog) pieces and we refer to such triangulations as fan triangulations. In this setting, the following version of the Laurent phenomenon holds.

**Theorem C** (Theorem 3.38). Lambda lengths of the arcs with respect to an initial fan triangulation are absolutely converging Laurent series in terms of initial (infinite) set of variables corresponding to the fan triangulation.

Based on the result of this theorem, we can introduce infinite rank surface cluster algebras associated to fan triangulations. Subsequently, we show that the definition of cluster algebra associated to a fan triangulation is independent of the choice of an initial fan triangulation which allows us to speak about cluster algebras associated to a given infinite surface.

Section 4 is devoted to the generalisation of combinatorial results known for (finite) surface cluster algebras. We extend the notion of snake graphs introduced in [MSW, MS] to infinite snake graphs and give an expansion formula for cluster variables when fan triangulations of the surface are considered by generalising the [MSW] formula subject to some “limit” conditions. This formula also manifestly give cluster variables as Laurent series (with positive integer coefficients) in an (infinite) initial set of cluster variables associated to fan triangulations.
**Theorem D** (Theorem 4.18). Let $T$ be a fan triangulation of an infinite surface $S$ and let $\gamma$ be an arc. Let $x_\gamma$ be the cluster variable associated to $\gamma$, $G_\gamma$ be the snake graph of $\gamma$, and $x_{G_\gamma}$ is the expansion formula for $G_\gamma$. Then $x_\gamma = x_{G_\gamma}$.

We also generalise the skein relations of [MW, CS] by extending the technique of [CS] for cluster algebras from unpunctured infinite surfaces.

**Theorem E** (Theorem 4.23). Let $S$ be an infinite unpunctured surface, $T$ be a fan triangulation of $S$, and $\gamma_1, \gamma_2$ be crossing generalised arcs. Let $\gamma_3, \ldots, \gamma_6$ be the (generalised) arcs obtained by smoothing a crossing of $\gamma_1$ and $\gamma_2$ and $x_1, \ldots, x_6$ be the corresponding elements in the associated infinite surface cluster algebra $\mathcal{A}(S)$ in terms of the initial cluster corresponding to $T$. Then the identity

$$x_1 x_2 = x_3 x_4 + x_5 x_6$$

holds in $\mathcal{A}(S)$.

Finally, in Section 5 we collect properties of infinite surface cluster algebras.

**Theorem F** (Theorem 5.1). Let $S$ be an infinite surface and $\mathcal{A}(S)$ be the corresponding cluster algebra. Then

- seeds are uniquely determined by their clusters;
- for any two seeds containing a particular cluster variable $x_\gamma$ there exists a mutation sequence $\mu^*$ (where $\mu^*$ is a finite mutation, a finite sequence of infinite mutations or an infinite sequence of infinite mutations) such that $x_\gamma$ belongs to every cluster obtained in the course of mutation $\mu^*$;
- there is a cluster containing a collection of cluster variables $\{x_i \mid i \in I\}$, where $I$ is a finite or infinite index set, if and only if for every choice of $i, j \in I$ there exists a cluster containing $x_i$ and $x_j$.

Moreover, if $T$ is a fan triangulation of $S$ then

- the “Laurent phenomenon” holds, i.e. any cluster variable in $\mathcal{A}(S)$ is an absolutely converging Laurent series in cluster variables corresponding to the arcs (and boundary arcs) of $T$;
- “Positivity” holds, i.e. the coefficients in the Laurent series expansion of a cluster variable in $\mathcal{A}(S)$ are positive.

If in addition the surface $S$ is unpunctured then

- the exponents of initial variables $x_i$ in the denominator of a cluster variable $x_\gamma$ corresponding to an arc $\gamma$ is equal to the intersection numbers $|\gamma \cap \tau_i|$ of $\gamma$ with the arc $\tau_i$ of $T$.

We would expect that our construction can be used as a combinatorial model to generalise finite dimensional representation theory (e.g. surface cluster categories, surface gentle algebras, tilting theory, etc.) to the infinite case beyond the Dynkin types $A_\infty$ and $A_\infty^\infty$.

The paper is organised as follows. Section 2 is devoted to combinatorics of infinite triangulations, in particular transitivity of infinite mutations and a characterisation of different types of mutation sequences. Section 3 deals with hyperbolic geometry and introduces surface cluster algebras associated to infinite surfaces. Section 4 concerns with infinite snake graphs and introduces an expansion formula for cluster variables as well as giving skein relations in this context. Finally in Section 5 we establish some properties of infinite rank surface cluster algebras.

**Acknowledgements:** We would like to thank Peter Jørgensen for stimulating discussions; in particular, the current version of Theorem 2.47 is due to his constructive comments. We would like to thank Karin Baur and Sira Gratz for sharing their preprint [BG] shortly before publishing it on the arXiv. We would also like to thank Peter Jørgensen and Robert Marsh for insightful discussions regarding potential representation theory associated to our combinatorial model.
2. Triangulations and mutations of infinite surfaces

In this section, we introduce our setting for infinite surfaces, infinite triangulations, and infinite sequences of mutations. The main result of this section presents transitivity of infinite mutation sequences.

2.1. Infinite triangulations. We first fix our setting for infinite surfaces.

Definition 2.1 (Infinite surface). Throughout the paper by an infinite surface $S$ we mean a connected oriented surface
- of a finite genus,
- with finitely many interior marked points (punctures),
- with finitely many boundary components,
- with infinitely many boundary marked points, located in such a way that they have only finitely many accumulation points of boundary marked points.

Accumulation points themselves are also considered as boundary marked points.

The following definition coincides with the one for finite surfaces.

Definition 2.2 (Arc, compatible arcs, triangulation, boundary arc). An arc on an infinite surface $S$ is a non-self-intersecting curve with both endpoints at the marked points of $S$, considered up to isotopy. As usual, we assume that an arc $\gamma$ is disjoint from the boundary of $S$ except for the endpoints, and that it does not cut an unpunctured monogon from $S$. The two endpoints of $\gamma$ may coincide. Two arcs are called compatible if they do not intersect (i.e. if there are representatives in the corresponding isotopy classes which do not intersect). A triangulation of $S$ is a maximal (by inclusion) collection of mutually compatible arcs. A boundary arc is a non-self-intersecting curve $\alpha \in \partial S$ such that the endpoints of $\alpha$ are boundary marked points and no other point of $\alpha$ is a boundary marked point.

Obviously, any triangulation of an infinite surface contains infinitely many arcs and infinitely many triangles.

Remark 2.3. In case of an infinite punctured surface we will also consider tagged triangulations as in [FST]. We will not present this definition here as we will only use this notion occasionally and in a very straightforward way.

Notation 2.4. Denote by $T_{S}$ the set of all triangulations of an infinite surface $S$. When the surface is clear from the context, we will simply abbreviate this notation by $T$.

Definition 2.5 (Convergence of arcs, limit of arcs, limit arc).

- We say that a sequence of arcs $\{\gamma_i\}_{i \in \mathbb{N}}$, converges to an arc (or to a boundary arc) $\gamma_s$ in an infinite surface $S$ if
  1. the endpoints of $\{\gamma_i\}$ converge to the endpoints of $\gamma_s$ and
  2. for $i$ large, the curve $\{\gamma_i\}$ is “almost isotopic” to $\gamma_s$ in the following sense:
     - Let $\{p_i\} \to p_s$ and $\{q_i\} \to q_s$ be the endpoints of $\{\gamma_i\}$, $i \in \mathbb{N}$, converging to the endpoints of $\gamma_s$. Let $U_{p_s} \subset \partial S$ and $U_{q_s} \subset \partial S$ be neighbourhoods of $p_s$ and $q_s$ such that neither $U_{p_s}$ nor $U_{q_s}$ contains a connected component of $\partial S$. Let $\tilde{\gamma_i}$ be the arc obtained from $\gamma_i$ by shifting the endpoints to $p_s$ and $q_s$ without leaving the sets $U_{p_s}$ and $U_{q_s}$, respectively. We say that $\{\gamma_i\}$ converge to $\gamma_s$ if $\tilde{\gamma_i}$ is isotopic to $\gamma_s$ for all $i > N'$ for some $N' > N$.

If $\{\gamma_i\}$ converge to $\gamma_s$ we also say that $\gamma_s$ is a limit of the arcs $\gamma_i$ and write $\gamma_i \to \gamma_s$ as $i \to \infty$.

- Given a triangulation $T$ of $S$, if $\{\gamma_i\} \to \gamma_s$ as $i \to \infty$ and $\gamma_i, \gamma_s \in T$ for all $i \in \mathbb{N}$, we say that $\gamma_s$ is a limit arc of $T$. Graphically, we will show limit arcs (as well as boundary limit arcs) by dashed lines, see Fig. 2.1.
Furthermore, if $p_* = q_*$ and the curves $\tilde{\gamma}_i$ (obtained from $\gamma_i$ by shifting the endpoints to $p_* = q_*$ inside $U_{p_*}$) are contractible to the accumulation point $p_*$, we say that $\gamma_i$ converges to the accumulation point $p_*$ and write $\gamma_i \to p_*$. 

**Figure 2.1.** A limit arc in a triangulation may arise in two different ways: in an infinite fan (a) and as a limit of an infinite zig-zag (b).

**Proposition 2.6.** Let $T$ be a triangulation of $S$ containing a sequence of arcs $\gamma_i$, $i \in \mathbb{N}$. If $\{\gamma_i\} \to \gamma_*$ as $i \to \infty$ then $\gamma_* \in T$.

**Proof.** Suppose that $\gamma_* \notin T$. Then $T$ contains an arc $\alpha$ which intersects $\gamma_*$, otherwise $T$ would not be a maximal set of compatible arcs. However, it is easy to see that if $\alpha$ intersects $\gamma_*$, then it also intersects $\gamma_i$ for large enough $i$, which contradicts the assumption that $\alpha$ and $\gamma_i$ lie in the same triangulation $T$. 

**Remark 2.7.** Notice that an arc with an endpoint at an accumulation point or an arc connecting two accumulation points is not necessarily a limit arc, see Fig. 2.2 for an example.

**Figure 2.2.** Triangulation of an infinite surface: marked points are denoted by dots, accumulation points are indicated by white dots and limit arcs are shown by dashed lines.

**Proposition 2.8.** Let $S$ be an infinite surface. Every triangulation of $S$ contains only finitely many limit arcs.

**Proof.** Every accumulation point is an endpoint of at most two limit arcs since each of the right and left limit at this point gives rise to at most one limit arc. The result follows since $S$ has finitely many accumulation points. 

**Proposition 2.9.** Two arcs on an infinite surface $S$ have only finitely many intersections.

**Proof.** Fix two arcs $\gamma, \gamma'$ on $S$. Each accumulation point of $S$ has a right and a left neighbourhood containing no endpoints of $\gamma$ and $\gamma'$. We remove these neighbourhoods from $S$ as shown in Fig. 2.3. The obtained surface is finite, and hence the result follows.

Propositions 2.8 and 2.9 imply the following corollary.

**Corollary 2.10.** For any arc $\gamma \in S$ and a triangulation $T$ of $S$, there are only finitely many crossings of $\gamma$ with limit arcs of $T$. 

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2.2. Infinite mutation sequences. As in [FST], we are going to use flips of arcs to change a triangulation (see Fig. 2.4). Our aim is to be able to transform every triangulation of an infinite surface $S$ to any other triangulation of $S$. Observe that a limit arc cannot be flipped, and moreover, one cannot make it flippable in finitely many steps. To fix this, we will need to introduce infinite mutations (see Definition 2.11). However, even that would not be enough for transitivity of the action of our moves on all triangulations of $S$, and therefore we will also need to introduce infinite sequences of infinite mutations.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2_2_4}
\caption{Flip of an arc $\gamma \in T$: $\mu(\gamma) = \gamma'$, $\mu(\alpha) = \alpha$ $\forall \alpha \in T$, $\alpha \neq \gamma$.}
\end{figure}

**Definition 2.11** (Infinite mutation, infinite sequence of infinite mutations). Consider a triangulation $T = \{\gamma_i \mid i \in \mathbb{N}\}$ of an infinite surface $S$.

1. An elementary mutation $\mu$ is a flip of an arc in $T$, see Fig. 2.4.
2. A finite mutation $\mu_n \circ \ldots \circ \mu_1$ is a composition of finitely many flips for some $n \in \mathbb{N}$. We will use the notation $\mu^{(0)}$ to refer to a finite mutation.
3. An infinite mutation is the following two step procedure:
   - apply an admissible infinite sequence of elementary mutations $\ldots \circ \mu_2 \circ \mu_1$, where
     a sequence is admissible if for every $\gamma \in T$ there exists $n = n(\gamma) \in \mathbb{N}$ such that
     $\forall i > n$, we have $\mu_i \circ \ldots \circ \mu_2 \circ \mu_1(\gamma) = \mu_n \circ \ldots \circ \mu_2 \circ \mu_1(\gamma)$;
   - complete the resulting collection of arcs by all limit arcs (if there are any).
     We will use the notation $\mu^{(1)}$ to specify an infinite mutation.
4. A finite sequence of infinite mutations $\mu_n^{(1)} \circ \ldots \circ \mu_1^{(1)}$ is a composition of finitely many admissible infinite mutations $\mu_i^{(1)}$ for $i = 1, \ldots, n$ for some $n \in \mathbb{N}$. We will use the notation $\mu^{(n)}$ to specify a sequence of $n$ infinite mutations.
5. An infinite sequence of infinite mutations is the following two step procedure:
   - apply an admissible infinite composition of infinite mutations $\ldots \circ \mu_2^{(1)} \circ \mu_1^{(1)}$, where
     a composition is admissible if the orbit of every individual arc converges, i.e. the sequence of arcs $\{\mu_i^{(1)} \circ \ldots \circ \mu_2^{(1)} \circ \mu_1^{(1)}(\gamma)\}$ converges for every arc
     $\gamma \in \bigcup_{i=0}^{\infty} \left[\mu_i^{(1)} \circ \ldots \circ \mu_2^{(1)} \circ \mu_1^{(1)}(T)\right]$;
   - complete the resulting collection of arcs by all limit arcs (if there are any).
     We will use the notation $\mu^{\infty}$ to specify an infinite sequence of infinite mutations.

If an arc $\gamma'$ is obtained from $\gamma$ by a mutation sequence $\mu$ (defined above) we will say that $\gamma'$ lies in the orbit of $\gamma$ for the mutation sequence $\mu$. 

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Example 2.12.  
(a) In Fig. 2.5, 2.6 and 2.7 we see examples of finite sequences of infinite mutations.
(b) The infinite mutation in Fig. 2.5 allows to shift the source of a fan in one direction, namely in the direction of the accumulation point. However, to shift it back we would need to apply an infinite sequence of infinite mutations (this will follow from Theorem 2.47).
(c) Applying the shift as in Fig. 2.6 infinitely many times we get an infinite sequence of infinite mutations which may serve as an “inverse” to the infinite mutation shown in Fig. 2.5 (again, Theorem 2.47 shows that there is no way to mutate back with a finite sequence of infinite mutations). Notice that in this example the orbit of every arc of the initial triangulation converges to the accumulation point.

Remark 2.13.  
• Our notion of infinite mutation is similar to the notion of mutation along admissible sequences in [BG]. More precisely, an infinite mutation is a mutation along admissible sequences completed by all limit arcs. However, this does not coincide precisely with the notion of completed mutation in [BG], as we only add limit arcs while [BG] sometimes adds more arcs.
• Our notion of infinite sequence of infinite mutations is a bit more restrictive than the notion of transfinite mutation in [BG]: we require every individual orbit to converge, while [BG] considers a collection of orbits which stabilise allowing the others to diverge.

Remark 2.14. Note that not every mutation sequence (even admissible ones!) give rise to a complete triangulation, see Fig. 2.8.

2.3. Elementary domains. For the proofs of main results of this section, we will need to cut the surface into smaller pieces. Each of the pieces will be a disc triangulated in one of the five ways classified according to the local behaviour around an accumulation point.

In what follows, by disc, we always mean an unpunctured disc with finitely or infinitely many boundary marked points.
Figure 2.7. Transforming an infinite zig-zag into a union of two fans via composition of two infinite mutations (where the second infinite mutation coincides with the one in Fig. 2.6).

Figure 2.8. Mutating an infinite zig-zag to an incomplete triangulation: there is no diagonal in the resulting triangulation as every individual arc moves to a lower fan (for even indices) or to an upper fan (for odd indices) and we cannot add a diagonal at the end since it does not appear as a limit arc.

Definition 2.15 (Elementary domains). Triangulations of discs combinatorially equivalent to the ones shown in Fig. 2.9 will be called elementary domains. Moreover, the elementary domains shown in Fig. 2.9(a)-(e) will be called

(a) finite fan (may contain any finite number of triangles);
(b) infinite incoming fan;
(c) infinite outgoing fan;
(d) infinite zig-zag around an accumulation point;
(e) infinite zig-zag converging to a limit arc.

Remark 2.16. In the literature, infinite fan triangulations are also called fountains. Zig-zag triangulations around an accumulation point are also called leap-frogs.

Remark 2.17. Any domain in Definition 2.16 has at most one left accumulation point and at most one right accumulation point, and these points may coincide.

If the two limit points of an infinite zig-zag of type (e) coincide (i.e. when the limit arc is contracted to a point and therefore vanishes), we obtain exactly an infinite zig-zag of type (d). Therefore, we will understand a zig-zag around an accumulation point as a zig-zag converging to a (vanishing) limit arc.

Similarly, when a limit arc of an incoming fan is contracted and vanishes, we obtain an outgoing fan. Hence, in many cases it makes sense to speak about fans in general, without specifying the type.
Figure 2.9. Five types of elementary domains: (a) finite fan, (b) incoming fan, (c) outgoing fan, (d) zig-zag around an accumulation point, (e) zig-zag converging to a limit arc. Notice that the incoming and outgoing fans may lie on the right or on the left of the accumulation point.

There are triangulations of a disc with at most two accumulation points which are not exactly of the form given as elementary domains, but have very similar underlying behaviour, see Fig. 2.10. This leads us to the following definitions.

**Definition 2.18** (Total number of accumulation points $\text{Acc}(S)$ of $S$). By a *total number of accumulation points in $S$* we mean the total number of left and right accumulation points on $S$ (where two-sided accumulation points are counted twice). Denote this number by $\text{Acc}(S)$.

**Definition 2.19** (Almost elementary domains). Let $D$ be a disc with at least one accumulation point, let $T$ be a triangulation of $D$. Suppose that it is possible to remove an infinite sequence of marked points from $D$ together with all arcs incident to them so that

(a) the total number of accumulation points of the obtained surface $S'$ is the same as the one of $S$, i.e. $\text{Acc}(S') = \text{Acc}(S)$;

(b) the obtained triangulation is an infinite elementary domain.

Then we say that $D$ together with $T$ is an *almost elementary domain* $(D, T)$. The type of an almost elementary domain is determined according to the type of the corresponding elementary domain (i.e. almost elementary incoming/outgoing fan, almost elementary zig-zag).

Figure 2.10. Examples of almost elementary domains.

**Definition 2.20** (Source of fan, base of fan, bases of zig-zag).

- By a *source* of a fan $F$ we mean the marked point incident to infinitely many arcs of $F$.
- Let $\gamma$ be the (possibly vanishing) limit arc of a fan $F$ and let $\alpha \neq \gamma$ be another boundary arc of $F$ incident to the source of $F$. Then by the *base* of $F$ we mean $\partial F \setminus \{\gamma \cup \alpha\}$ (see the horizontal part of the boundary of the fan in Fig. 2.11).
- Similarly, for an almost elementary zig-zag $Z$, let $\gamma$ be the (possibly vanishing) limit arc and let $\alpha$ be the first arc of the zig-zag. Then by the *bases* of $Z$ we mean the connected components of $\partial Z \setminus \{\gamma \cup \alpha\}$ (see the two horizontal parts of the boundary of the zig-zag in Fig. 2.11).

Our next aim is to prove Proposition 2.22 which shows that almost elementary domains may serve as building blocks of triangulated infinite surfaces and that every triangulation requires
Lemma 2.21. Let $D$ be a disc with at least one accumulation point of marked points on the boundary. Let $T$ be a triangulation of $D$ such that it contains

1. no limit arc, and
2. no arc $\gamma$ cutting $D$ into two discs $D_1$ and $D_2$ such that $\text{Acc}(D_i) < \text{Acc}(D)$ for $i = 1, 2$.

Then $(D, T)$ is an almost elementary fan or an almost elementary zig-zag.

Proof. We will consider two cases: either $D$ has at least two accumulation points or $D$ has a unique accumulation point.

Case 1: Suppose that $D$ contains at least 2 accumulation points. We will identify the disc $D$ with the upper half-plane $\{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \cup \{ \infty \}$, and we will assume that 0 and $\infty$ are successive accumulation points, i.e. there is no accumulation point at any $r > 0$. Furthermore, there are two possibilities: either there is a sequence of marked points decreasing to 0 or not. We will consider these cases.

Case 1.a: Suppose 0 is not an accumulation point from the right. The marked points on the positive ray can be labeled by $q_i$, $i \in \mathbb{N}$, so that $q_i < q_j$ if $i < j$, $q_i \to \infty$ as $i \to \infty$ (see Fig. 2.12, left). Assumption (2) implies that there is no arc of $T$ connecting $q_i$ to any point $s < 0$; and moreover, there is no arc from $q_i$ to $\infty$. Similarly, there is no arc from 0 to any point $s \leq 0$ (since 0 is an accumulation point from the left).

Consider the point $q_1$. Denote $u_1 = q_1$. The triangle $t_1$ in $T$ containing the boundary arc $0q_1$ has a third vertex. From the discussion above we derive that it is one of $q_i$ ($i > 1$). Denote $u_2 = q_i$, and let $\gamma_{0,2}$ be the arc connecting $0$ with $u_2$. There is another triangle $t_2 \neq t_1$ containing the arc $\gamma_{0,2}$. Let $u_3 = q_j$, where $j > i$, be the third vertex of $t_2$. Denote by $\gamma_{0,3}$ the arc connecting 0 with $u_3$. We may continue in this way constructing an infinite growing sequence of the points $u_i$, an infinite sequence of triangles $t_i$ all having 0 as a vertex, and an infinite sequence of arcs $\gamma_{0,i} \in T$. However, this implies that the arc connecting 0 with $\infty$ is a limit arc, which contradicts assumption (1).

Case 1.b: Suppose 0 is an accumulation point from the right. The marked points on the positive ray maybe labelled by $q_i$, $i \in \mathbb{Z}$, so that $q_i < q_j$ if $i < j$, $q_i \to \infty$ as $i \to \infty$ and $q_i \to 0$
as \( i \to -\infty \) (see Fig. 2.12, right). Assumption (2) implies that every arc emanating from \( q_i \) have the other endpoint at some \( q_j \) (as it cannot terminate at any point \( r \leq 0 \) or \( r = \infty \)).

Let \( u_0 = q_0 \). There are finitely many triangles of \( T \) incident to \( q_0 \) (as \( q_0 \) is not an accumulation point), and exactly one of them has vertices \( a_0q_0b_0 \) such that \( 0 < a_0 < q_0 < b_0 < \infty \). Denote this triangle by \( t_0 \). The side \( a_0b_0 \) of \( t_0 \) belongs to some other triangle \( t_1 \in T \), with the third vertex either at a point \( a_1 < a_0 \) or at a point \( b_1 > b_0 \). Denote the leftmost vertex of \( t_1 \) by \( a_1 \) and the rightmost vertex of \( t_1 \) by \( b_1 \) (with either \( a_1 = a_0 \) or \( b_1 = b_0 \)). Similarly, this new side \( a_1b_1 \) of \( t_1 \) also belongs to the next triangle \( t_2 \). Proceeding in this way we obtain a sequence of triangles \( t_i \) and a sequence of arcs \( a_i b_i \). Notice that all \( a_i, b_i \) are positive, the sequence \( \{a_i\} \) is monotone decreasing and the sequence \( \{b_i\} \) is monotone increasing. Hence both sequences are converging, which implies that there exists a limit arc \( \gamma \) (a limit of the arcs \( a_i b_i \)). This contradicts assumption (1) unless \( \gamma \) is actually a boundary segment with endpoints 0 and \( \infty \). In the latter case, we see that there are no marked points in the negative ray, and the arcs \( a_i b_i \) define an almost elementary zig-zag of \( D \). Moreover, the set of vertices \( \{a_i, b_i\} \) is obtained from the set \( \{q_i\} \) by removing some marked points (so that the total number of accumulation points in the disc remains equal to 2), removing some further vertices from the set \( \{a_i, b_i\} \) we can obtain an elementary zig-zag. By definition, this implies that the original triangulation of \( D \) is an almost elementary zig-zag.

This completes the consideration of Case 1.

**Case 2:** Suppose that \( D \) contains a unique accumulation point, say at \( \infty \). If it has a two-sided accumulation point, then the case is considered exactly in the same way as in Case 1.b. So, suppose it is a one-sided limit, say from the right. Then we may assume that the marked points \( q_i, i \in \mathbb{N} \), on \( D \) satisfy \( q_1 < q_2 < q_3 < \ldots \). Using the same construction as in Case 1.a, we arrive to an almost elementary incoming or outgoing fan. Notice that in this setting there may be an arc from \( q_i \) to \( \infty \), and if there are infinitely many of such arcs, the fan under consideration is outgoing.

**Proposition 2.22.** Fix a triangulation \( T \) of \( S \). There exists a finite set of arcs \( \gamma_i \) in \( T \) with \( i = 1, \ldots, n \) such that \( S \setminus \{\gamma_1, \ldots, \gamma_n\} \) is a union of almost elementary domains.

**Proof.** For each puncture on \( S \), we cut along an arc coming to this puncture to get an unpunctured surface. Then, we cut along each of the finitely many limit arcs to obtain finitely many surfaces. On each connected component, there may be 3 different types of arcs, see Fig. 2.13:

- A. arcs connecting two boundary components;
- B. arcs with two endpoints on the same boundary component, which cut a disc from \( S \) (in other words, these arcs are contractible to the boundary of \( S \));
- C. a non-trivial arc with endpoints at the same boundary but not contractible to this boundary.

![Figure 2.13. Examples of arcs of types A (left), B (middle), C (right).](image)

If there is an arc of type A, we cut along this arc and reduce the number of boundary components. If there is an arc of type C, we cut along it either to reduce the genus or to split the surface into two connected components each having either a smaller genus or smaller number...
of boundary components. If there is no arc of type A or C, then the connected component is a disc.

Now we are left with finitely many discs $D_i$ each of them having finitely many accumulation points. The discs having no accumulation points on the boundary may be cut into finitely many separate triangles. Hence, it is left to consider a disc $D_i$ containing at least one accumulation point. If there is an arc $\gamma \in T$ cutting $D_i$ into two parts $D_i^+$ and $D_i^-$ such that $\text{Acc}(D_i^+) < \text{Acc}(D_i)$ and $\text{Acc}(D_i^-) < \text{Acc}(D_i)$, then we cut $D_i$ along $\gamma$. Otherwise, Lemma 2.21 implies that $D_i$ is an almost elementary domain (see Fig. 2.14). Clearly, this process terminates in finitely many steps and results in finitely many discs all triangulated as almost elementary domains, as required.

\[ \square \]

\[ \text{Figure 2.14. Example of cutting along thick arcs into almost elementary fans and zig-zags.} \]

\textbf{Remark 2.23.} The choice of cuts splitting $S$ into almost elementary domains in Proposition 2.22 is not unique. In particular, one can always cut a finite number of triangles from any infinite domain.

\[ \text{2.4. Transitivity of infinite mutations.} \text{ The main result of this section is that any two infinite triangulations of } S \text{ are connected by a sequence of mutations. This generalises the result of [BG] which is obtained in a slightly different setting and which establishes transitivity of transfinite mutations for the case of completed infinity-gon.} \]

\textbf{Theorem 2.24.} For any two triangulations $T$ and $T'$ of $S$, there exists a mutation sequence $\mu^*$ such that $\mu^*(T) = T'$, where $\mu^*$ is a finite mutation, a finite sequence of infinite mutations or an infinite sequence of infinite mutations.

We will first prove a couple of technical lemmas and the proof of the theorem will be presented at the end of this section.

\textbf{Lemma 2.25.} (1) Let $F$ be an almost elementary fan. Then there exists an infinite mutation $\mu^{(1)}$ such that $\mu^{(1)}(F)$ is an elementary fan.

(2) Let $Z$ be an almost elementary zig-zag. Then there exists an infinite mutation $\mu^{(1)}$ such that $\mu^{(1)}(Z)$ is an elementary zig-zag with (possibly infinitely many) finite polygons attached along its bases.

\textbf{Proof.} Let $D$ be an almost elementary fan or zig-zag. By Definition 2.19, one can remove from $D$ a (possibly infinite) number of boundary marked points (together with the arcs incident to them), so that the resulting surface $D'$ is an elementary domain and $\text{Acc}(D') = \text{Acc}(D)$. From each triangle $t_i$ of $D'$, we have removed at most finitely many marked points (otherwise we would get a contradiction to condition (a) of Definition 2.19). Hence, each triangle $t_i$ of $D'$
is subdivided in $D$ into finitely many triangles constituting a finite triangulated polygon $\hat{t}_i$ in $D$. We can transform the triangulation of $\hat{t}_i$ to a finite fan as shown in Fig. 2.15 in finitely many flips. Doing so for $\hat{t}_i, i \in \mathbb{N}$, successively, we will get an infinite mutation transforming the triangulation of $D$ to the required pattern.

Figure 2.15. To the proof of Lemma 2.25.

**Definition 2.26** (Domain of $\gamma$ in $T$, almost elementary domains of $\gamma$). Let $\gamma \in S$ be an arc and $T$ be a triangulation of $S$.

1. A domain $D^T_\gamma$ of $\gamma$ in $T$ is an open set which consists of triangles intersected by $\gamma$ given by

$$D^T_\gamma = \left( \bigcup_{i \in I} \Delta^\gamma_i \right) \bigcup \left( \bigcup_{j \in J} \gamma_j \right),$$

where $J$ is the index set of all arcs in $T$ intersected by $\gamma$, $I$ is the index set of triangles of $T$ having at least one of their boundary arcs in the set $\{\gamma_j \mid j \in J\}$, and $\Delta^\gamma$ is the interior of a triangle $\Delta$.

2. As it is the case for any surface, the domain $D^T_\gamma$ of $\gamma$ may be cut into almost elementary domains (see Proposition 2.22). These domains for $D^T_\gamma$ will be called almost elementary domains of $\gamma$.

Examples of elementary domains of an arc $\gamma$ are shown in Fig. 2.16.

Remark 2.27. In the case of a punctured surface we need to use the notion of crossing of tagged arcs, see [FST, Definition 7.4]. In particular, the arc $\gamma$ in Fig. 2.16(c) (having an end at a puncture $p$ and tagged oppositely at $p$ with respect to the tagging of $T$) crosses every arc of the fan, so the whole fan will be an almost elementary domain for $\gamma$.

Remark 2.28. Almost elementary domains of $\gamma$ in $T$ are not uniquely defined (compare to Remark 2.23).

Remark 2.29. If $D$ is an almost elementary fan domain of an arc $\gamma$, then (after removing finitely many triangles) $D$ is actually an elementary fan domain. Indeed, as $\gamma$ intersects every triangle in the domain $D^T_\gamma$, $\gamma$ crosses only the arcs of the triangulation of $D$ incident to the source of the fan (except for finitely many triangles at the end).
Lemma 2.30. For any arc $\gamma \in S$ and a triangulation $T$, the domain $D^T_\gamma$ of $\gamma$ is a finite union of disjoint almost elementary domains $R_1, \ldots, R_k$ such that for every domain $R_i$ one of the following holds:

- either $R_i$ is a single triangle,
- or all parts of $\gamma$ crossing $R_i$ are parallel to each other (see Fig. 2.17).

\[\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig217}
\end{array}\]

**Figure 2.17.** Parallel crossings of elementary fan and zig-zag domains.

Proof. Applying Proposition 2.22 to $D^T_\gamma$ we can find a finite set of arcs $\alpha_1, \ldots, \alpha_k \in T$ such that $D^T_\gamma \setminus \{\alpha_1, \ldots, \alpha_k\}$ is a finite union of unpunctured discs $R_i$ with at most two accumulation points. In each of these discs the triangulation is an almost elementary fan or almost elementary zig-zag. There are only finitely many pieces $\gamma_j^i$ of the arc $\gamma$ in each $R_i$ since the discs $R_i$ are obtained by finitely many cuts, each cutting $\gamma$ into finitely many pieces (as any two arcs on $D^T_\gamma$ have finitely many intersections by Proposition 2.22).

Let $\gamma_j^i \in R_i$ be a piece of $\gamma$ crossing finitely many triangles in $R_i$. Then we can cut these triangles out of $R_i$ and consider each of them as an individual elementary domain. Therefore, we can assume that $R_i$ is an almost elementary fan or zig-zag and every part $\gamma_j^i \in R_i$ of $\gamma$ crosses the limit arc (or terminates at the accumulation point in the case of a zig-zag around an accumulation point).

Furthermore, as there are finitely many pieces $\gamma_j^i$ in $R_i$, there are finitely many endpoints of $\gamma_j^i$ leaving $R_i$ through the base/bases. Thus, we can also remove finitely many triangles containing these endpoints, see Fig. 2.18. This will cut $R_i$ into finitely many subdomains (one infinite and several finite ones). We will cut the finite domains into finitely many triangles. The remaining infinite domain then is crossed in a parallel way as there is a unique way to cross a fan or a zig-zag not crossing the bases.

Therefore, $D^T_\gamma$ is a finite union of disjoint triangles, infinite fan domains and infinite zig-zag domains, and each infinite domain is crossed by pieces of $\gamma$ in a parallel way. \[\square\]

\[\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig218}
\end{array}\]

**Figure 2.18.** Cutting fan and zig-zag domains to isolate parallel crossings by $\gamma$.

Lemma 2.31. For any arc $\gamma \in S$ and triangulation $T$ of $S$, there exists a finite sequence of infinite mutations $\mu^{(n)}$ of $T$ such that $T' = \mu^{(n)}(T) \in T$, $\gamma \in T'$ and $T|_{S \setminus D^T_\gamma} = T'|_{S \setminus D^T_\gamma}$. Moreover, $\mu^{(n)}$ can be chosen so that no elementary mutation inside $\mu^{(n)}$ flips any arc $\alpha \in S \setminus D^T_\gamma$. 

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Proof. By Lemma 2.30 the domain $D^T_\gamma$ of $\gamma$ is a finite union of disjoint almost elementary domains $R_1, \ldots, R_k$ such that for every infinite domain $R_i$ the crossing of $R_i$ by $\gamma$ looks like a parallel pencil. More precisely, among $R_1, \ldots, R_k$ there are finitely many triangles, finitely many elementary fans and finitely many almost elementary zig-zags.

Given an elementary fan domain $R_i$, let $\mu_i^{(1)}$ be an infinite mutation shifting the source of the fan as in the left of Fig. 2.19 (see Fig. 2.6 for the construction of $\mu_i^{(1)}$). Denote $\mu_{R_i} := \mu_i^{(1)}$.

![Figure 2.19. $\mu_{R_i}$: resolving infinitely many crossings in a fan and zig-zag domains.](image)

Similarly, for every almost elementary zig-zag domain $R_j$, we first apply an infinite mutation $\mu_j^{(1)}$ to turn it to an elementary zig-zag (with finite polygons attached along its boundary arcs, see Lemma 2.26). Then we apply a composition of two infinite mutations $\mu_j^{(2)}$ which turns the zig-zag into two fans as in Fig. 2.7. Denote $\mu_{R_j} := \mu_j^{(2)} \circ \mu_j^{(1)}$, see Fig. 2.19 (right).

Notice that as the domains $R_1, \ldots, R_k$ are disjoint, the above mutations $\mu_{R_i}$ do not interact, i.e. they commute. Applying the composition $\mu^{(n)} = \mu_{R_n} \circ \cdots \circ \mu_{R_1}$ to $T$, we obtain a triangulation $T^* := \mu^{(n)}(T)$ having finitely many crossings with $\gamma$ inside every $R_i$. This means that $D^T_{\gamma}$ is a finite surface, and in view of [H] there exists a finite mutation $\mu^{(0)}$ such that $\gamma \in \mu^{(0)}(T^*) \subseteq T$. Hence, $\mu^{(0)} \circ \mu^{(n)}$ is a finite sequence of infinite mutations of the required form.

Since we only consider arcs lying inside the domain $D^T_\gamma$ of $\gamma$ in the argument of the proof, no elementary mutation inside $\mu^{(0)} \circ \mu^{(n)}$ flips any arc $\alpha \in S \setminus D^T_\gamma$.

Remark 2.32. Applying Lemma 2.31 we can try to prove Theorem 2.24 (transitivity of sequences of infinite mutations) in the following way:

1. label the arcs of $T'$ with positive integers, so that $T' = \{\gamma_i \mid i \in \mathbb{N}\}$;
2. apply Lemma 2.31 repeatedly, to obtain first a triangulation $T_1$ containing $\gamma_1$, then a triangulation $T_2$ containing $\gamma_1, \gamma_2$; after $n$ steps we get a finite sequence of infinite mutations transforming $T$ to a triangulation $T_n$ containing $\gamma_1, \ldots, \gamma_n$;
3. applying Lemma 2.31 infinitely many times we get a triangulation containing all arcs $\gamma_i \in T'$, i.e. we obtain $T''$.

The only problem with this algorithm is that in general it is not clear whether the infinite sequence of infinite mutations is admissible (i.e. whether the orbit of every arc converges).

To prove Theorem 2.24 we will first show the statement for almost elementary domains in Lemma 2.34. Both the proof of Lemma 2.31 and the proof of Theorem 2.24 will be based on Lemma 2.33 and Observation 2.33.

Observation 2.33. Let $S$ be an infinite surface, $T$ be a triangulation of $S$, and $\alpha, \gamma$ be two arcs on $S$. If $\alpha \cap \gamma = \emptyset$ then $\alpha \cap D^T_\gamma = \emptyset$.

Proof. By Definition 2.26 of $D^T_\gamma$, an arc $\alpha$ have common points with $D^T_\gamma$ if and only if $\alpha \cap \gamma \neq \emptyset$.

Lemma 2.34. Let $(D, T')$ be an almost elementary domain, and let $T$ be another triangulation of $D$. Then there exists a mutation sequence $\mu^*$ such that $\mu^*(T) = T'$, where $\mu^*$ is a finite mutation, a finite sequence of infinite mutations or an infinite sequence of infinite mutations.


Proof. First, suppose that $D$ is an elementary domain. Label the arcs of $T'$ by $\gamma_i$, $i \in \mathbb{N}$ so that the sequence $\{\gamma_i\}$ converges as $i \to \infty$ (in other words, label the arcs from top to bottom in Fig. 2.9(c)-(e), and from bottom to top in Fig. 2.9(b)). By Lemma 2.31, we can find a finite sequence of infinite mutations $\mu_{(n)}$ such that $\gamma_1 \in T_1 := \mu_{(n)}(T)$, then we find a sequence $\mu_{(n)}$ such that $\gamma_2 \in T_2 := \mu_{(n)} \circ \mu_{(n)}(T)$. As $\gamma_2$ does not intersect $\gamma_1$ (lying in the same triangulation $T'$), we see from Observation 2.33 that $\gamma_1$ is disjoint from $D_{\gamma_2}^i$, which implies that $\gamma_1$ is not touched by any elementary mutation inside $\mu_{(n)}$. In particular, we have $\gamma_1, \gamma_2 \in T_2$. Repeatedly applying Lemma 2.31 for $\gamma_i$, we obtain a finite sequence of infinite mutations $\mu_{(n)} \circ \cdots \circ \mu_{(n)}(\gamma_1) \cap \gamma_i = \emptyset$ (this follows by induction from Observation 2.33 and the fact that $\mu_{(n)}$ acts in the domain $D_{\gamma_1}^{i-1}$). Hence, the orbit of every arc $\alpha \in T$ crossing a subsequence converging to the limit arc and another subsequence converging to one of the accumulation points.

1: arcs with finitely many crossings. Let $\alpha \in T$ be an arc. Notice that if $\alpha \cap \gamma_i = \emptyset$ then we obtain that $\mu_{(n)} \circ \cdots \circ \mu_{(n)}(\alpha) \cap \gamma_i = \emptyset$ (this follows by induction from Observation 2.33 and the fact that $\mu_{(n)}$ acts in the domain $D_{\gamma_1}^{i-1}$). Hence, the orbit of every arc $\alpha \in T$ crossing a subsequence converging to the limit arc and another subsequence converging to one of the accumulation points.

2: arcs with infinitely many crossings. Notice that the arcs $\gamma_1, \ldots, \gamma_i$ are not touched by any of the mutations $\mu_{(n)}$ with $j > i$. This implies that as $j$ grows, all elementary mutations contained in $\mu_{(n)}$ are performed inside smaller and smaller regions, which shrink in the limit to the accumulation point (when $D$ is as in Fig. 2.9 (c) or (d)) or to the limit arc (in cases when $D$ as is in Fig. 2.9 (b) or (e)). In the former case this immediately implies that the orbit of every arc either stabilises or converges. The latter case (i.e. the cases of an incoming fan and of a zig-zag converging to a limit arc) require more work as the orbit of an arc may have a subsequence converging to the limit arc and another subsequence converging to one of the accumulation points.

2.a: ($D, T'$) is an incoming fan. In this case an arc $\alpha \in T$ crossing infinitely many arcs of the incoming fan $T'$ have one of its endpoints at the accumulation point and another at some marked point in the base of the fan. If after several mutations the arc $\alpha$ transforms to an arc $\alpha'$ connecting the source of the fan to the base, then none of the further mutations will change $\alpha'$ anymore (since $\alpha'$ does not cross any of $\gamma_i$ and so does not lie in any of domains $D_{\gamma_1}^{i-1}$), hence the orbit of $\alpha$ will stabilise. If every arc in the orbit of $\alpha$ has one end at the accumulation point and another end at the base, then the orbit of $\alpha$ converges to the accumulation point.

2.b: ($D, T'$) is a zig-zag converging to a limit arc. In this case, an arc $\alpha \in T$ crossing infinitely many arcs of $T'$ connects a marked point $p_i$ in one of the bases of the zig-zag to an accumulation point $q$ lying either on the other or on the same base of the zig-zag (see Fig. 2.20 (a) and (b)). We may assume that the mutations $\mu_{(n)}$ are already applied, so that all arcs $\gamma_j = p_j p_{j+1}$ for $j < i$ are already in the triangulation $T_i$. Also, we assume $\alpha = p_i q \in T_i$. We need to construct the mutation $\mu_{(n)}$ clearing the arc $\gamma_{i+1} = p_{i+1} p_{i+2}$ from the intersections. In both cases of Fig. 2.20 (a) and (b), all almost elementary domains of $\gamma_{i+1}$ in $T_i$ are fans (as the the arc $\alpha = p_i q$ separates the left limit from the right one in $T_i$, so that the zig-zags are impossible). We will first remove infinite number of intersections (if they exist) by shifting the source of fans as in Fig. 2.6 then we remove all the other intersections in order of their appearance on the (oriented) path $p_{i+1} p_{i+2}$. Then the arc $\alpha = p_i q$ will be flipped either to an arc having no endpoints at an accumulation point (which would imply its orbit will stabilise as shown in Case 1) or to an arc connecting $p_{i+1}$ with the accumulation point on the other base. The latter may only occur in the case of Fig. 2.20 (a), and if this configuration repeats infinitely many times, then the orbit of $\alpha$ converges to the limit arc.
Figure 2.20. Arcs crossing infinitely many arcs of a zig-zag: if an arc $\alpha = p_iq$ is the first arc crossed by the (oriented) arc $p_{i+1}p_{i+2}$ then $\alpha$ flips to an arc crossing only finitely many arcs of the zig-zag.

So, in each of the cases the orbits of all arcs either stabilise or converge, hence the infinite mutations $\mu_i^{(n)}$ compose an admissible infinite sequence of infinite mutations.

Finally, if $D$ is not an elementary domain but an almost elementary one, then exactly the same reasoning works for appropriate numeration of arcs of $T'$ (i.e. for the numeration where the arcs forming the underlying elementary domain triangulation have increasing numbers and additional arcs lying “between” $\gamma_i$ and $\gamma_j$ have the numbers between $i$ and $j$).

Remark 2.35. We expect that Lemma 2.34 can be derived from the results of [BG], but it is not a straightforward task due to differences in the definitions, see Remark 2.13.

Corollary 2.36. Let $D$ with triangulation $T'$ be an almost elementary domain, and let $T$ be another triangulation of $D$. If every arc of $T'$ intersects only finitely many arcs of $T$, then there exists an infinite mutation $\mu(1)$ such that $\mu(1)(T) = T'$.

Proof. We construct the sequence of mutations as in the proof of Lemma 2.34 but notice that when each arc of $T$ intersects only finitely many arcs of $T'$, one can substitute each of $\mu_i^{(n)}$ by a finite mutation. Moreover, the proof of Lemma 2.34 also shows that the orbit of every arc stabilises.

We are now ready to prove the main theorem of this section.

Proof of Theorem 2.24. We will start by choosing arcs $\gamma_1, \ldots, \gamma_s$ in $T'$, $s \in \mathbb{N}$, as in Proposition 2.22 which split the surface $S$ into finitely many almost elementary domains $D_1, \ldots, D_k$ for some $k \in \mathbb{N}$. Then we apply Lemma 2.31 repeatedly for each of $\gamma_1, \ldots, \gamma_s$ in the same way as we did in the proof of Lemma 2.34. We obtain a finite sequence of infinite mutations $\mu^{(n)}$ such that $T^* := \mu^{(n)}(T)$ is a triangulation containing the arcs $\gamma_1, \ldots, \gamma_s$.

As $D_1, \ldots, D_k$ are almost elementary domains of $T'$, Lemma 2.34 implies that there exist finite or infinite sequences of infinite mutations $\mu_i^{(n)}$ transforming $T^*|_{D_i}$ to $T'|_{D_i}$. All mutations in these sequences only flip the arcs in the corresponding almost elementary domains, so the infinite mutations contained in $\mu_i^{(n)}$ and $\mu_j^{(n)}$ commute for $i \neq j$. Which means that we can apply the first mutations from each of $\mu_1^{(n)}, \ldots, \mu_k^{(n)}$, then the second mutations from each of $\mu_1^{(n)}, \ldots, \mu_k^{(n)}$, then the third ones and so on, forming an infinite sequence of infinite mutations and obtaining all arcs of $T'$ in the limit (and this is an admissible sequence of infinite mutations since $\mu_i^{(n)}$ is admissible sequence for each $i$).

2.5. Mutation hierarchy. In this section, we will discuss how finite mutations, infinite mutations and infinite sequences of infinite mutations are related.

Notation 2.37 (Intersection number). Denote by

- $|\gamma \cap \alpha|$ the intersection number of two arcs $\gamma$ and $\alpha$ on $S$. 

- \(|\gamma \cap T| = \sum_{\tau \in T} |\gamma \cap \tau|\) the sum of intersection numbers \(|\gamma \cap \tau|\) for all \(\tau \in T\) (this may be infinite);
- \(|T \cap T'| = \sum_{\gamma \in T'} |\gamma \cap T|\) the sum of all intersection numbers of all arcs in the triangulation \(T\) with the arcs of the triangulation \(T'\) (this may also be infinite).

**Remark 2.38.** In a punctured surface, the intersection numbers for the tagged arcs are computed as defined in [FST1 Definition 8.4].

**Proposition 2.39.** Two triangulations \(T\) and \(T'\) of \(S\) are related by a finite sequence of (finite) mutations if and only if \(|T \cap T'| < \infty\).

**Proof.** Resolve each of the finitely many crossings one by one to switch from one triangulation to the other. Conversely, a finite mutation is completely contained inside a finite union of finite polygons, so it cannot create infinitely many crossings. \(\square\)

**Definition 2.40** (Bad arcs for a given triangulation). For an arc \(\gamma \in S\) and a triangulation \(T\) of \(S\), we say \(\gamma\) is a bad arc for \(T\) if \(|\gamma \cap T| = \infty\). We denote by \(\text{Bad}_T T'\) the set of all arcs of \(T'\) which are bad for the triangulation \(T\). We also denote by \(|\text{Bad}_T T'|\) the number of elements in this set (which may be infinite).

**Proposition 2.41.** For two triangulations \(T\) and \(T'\) of \(S\), if \(|\text{Bad}_T T'| = 0\), then there exists an infinite mutation \(\mu^{(1)}\) such that \(\mu^{(1)}(T) = T'\).

**Proof.** As in the proof of Theorem 2.24, we first use Proposition 2.22 to choose finitely many arcs \(\gamma_1, \ldots, \gamma_s \in T', s \in \mathbb{N}\), splitting the surface \(S\) into finitely many almost elementary domains \(D_1, \ldots, D_k\) for some \(k \in \mathbb{N}\). As \(|\text{Bad}_T T'| = 0\), we can find a finite sequence \(\mu^{(0)}\) of elementary mutations first transforming \(T\) to a triangulation containing \(\gamma_1\), then to a triangulation containing \(\gamma_1, \gamma_2\), and finally to a triangulation \(T_*\) containing all of the arcs \(\gamma_1, \ldots, \gamma_s\).

Now, when \(\gamma_1, \ldots, \gamma_s\) are in the triangulation \(T'' = \mu^{(0)}(T)\), we are left to sort the question in each of the almost elementary domains \(D_1, \ldots, D_k\) separately. By Corollary 2.36 there exists an infinite mutation \(\mu^{(1)}_i\) such that \(\mu^{(1)}_i(T_*|D_i) = T'|D_i\). Composing the first elementary mutations of \(\mu^{(1)}_1, \ldots, \mu^{(1)}_k\), then the second elementary mutations of \(\mu^{(1)}_1, \ldots, \mu^{(1)}_k\), then the third ones an so on, we obtain an infinite composition \(\tilde{\mu}^{(1)}\) of elementary mutations transforming \(T_*\) to \(T'\). Moreover, this composition is an admissible infinite mutation, i.e. the orbit of every arc stabilise, since each of \(\mu^{(1)}_i\) is an infinite mutation and distinct mutations act in distinct elementary domains \(D_i\). Hence, \(\tilde{\mu}^{(1)} \circ \mu^{(0)}\) is an infinite mutation transforming \(T\) to \(T'\). \(\square\)

**Lemma 2.42.** For two triangulations \(T\) and \(T'\) of \(S\), if \(0 < |\text{Bad}_T T'| < \infty\), then there exists a finite sequence of mutations \(\mu^{(n)}\) such that

- \(\mu^{(n)}(T) = \tilde{T} \in T,\) and
- \(|\text{Bad}_\tilde{T} T'| < |\text{Bad}_T T'|\).

**Proof.** Let \(\gamma \in \text{Bad}_T T'\). Consider the domain \(D_i^T\) of \(\gamma\) in \(T\). By Proposition 2.22 \(D_i^T = \{R_1, \ldots, R_m\}, m \in \mathbb{N}\), is a finite union of disjoint almost elementary fans and zig-zags. We will show that there is a finite sequence of infinite mutations \(\mu^{(k_i)}\) inside each of \(R_i\) such that

- \(\tilde{T}_i := \mu^{(k_i)}(T) \in \tilde{T}\);
- \(\gamma\) crosses only finitely many arcs of \(\tilde{T}_i\) inside \(R_i\);
- if \(\alpha \in \text{Bad}_\tilde{T}_i T'\) then \(\alpha \in \text{Bad}_T T'\).

The composition of these sequences \(\mu^{(k_m)} \circ \cdots \circ \mu^{(k_1)}\) is a finite sequence of infinite mutations \(\mu^{(n)}\) such that \(\tilde{T} := \mu^{(n)}(T) \in \tilde{T}, \gamma \notin \text{Bad}_\tilde{T} T'\) and if \(\alpha \in \text{Bad}_T T'\) then \(\alpha \in \text{Bad}_T T'\). Therefore, \(|\text{Bad}_\tilde{T} T'| \leq |\text{Bad}_T T'| - 1\) as required.

To find a sequence \(\mu^{(k_i)}\) for a given almost elementary domain \(R_i\), we first apply an infinite mutation (or a sequence of two infinite mutations in the case of a zig-zag) which takes the triangulation of \(R_i\) to the form shown in Fig. 2.21 (as described in Fig. 2.6 and Fig. 2.7). We
denote this infinite mutation or this sequence of infinite mutations by $\mu^{(r_1)}$ or $\mu^{(r_2)}$, respectively. This transformation removes all but finitely many crossings of $\gamma$ with the triangulation inside $R_i$, however, this could possibly imply appearance of some new bad arcs in $T'$ with respect to the newly constructed triangulation $\tilde{T}_i$. The aim of the next step is to ensure that we do not create any new bad arcs in $T'$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.21.png}
\caption{$\mu^{(r_1)}(T)$: shift of a source of a fan (left) and a zig-zag transforming into a union of two fans (right).}
\end{figure}

First, we will consider the case when $R_i$ is an infinite fan (considered after the shift of the source of the fan given at the previous step). Denote by $\tau_i$ the limit arc of the initial fan triangulation and by $p_i$ the endpoint of $\tau_i$ contained in the base of the fan (see Fig. 2.22). As $\gamma$ crosses infinitely many arcs of $R_i$, $\gamma$ either crosses $\tau_i$ or crosses the corresponding fan and runs into $p_i$. We will see that in the former case there is another arc $\gamma' \in T'$ crossing the fan and landing at $p_i$. Indeed, if $\gamma$ crosses $\tau_i$ then $\tau_i$ intersects (non-empty) finite set of arcs of $T'$ (as $|\text{Bad}_T T'| < \infty$), which implies that the domain $D_{\tau_i}^{T'}$ of $\tau_i$ in $T'$ is a polygon $P_i$. In particular, the triangle $\Delta_i$ of $T'$ lying in $D_{\tau_i}^{T'}$ and incident to $p_i$ will have a side $\gamma'$ crossing the fan and incident to $p_i$. So, either $\gamma \in T'$ or $\gamma' \in T'$ crosses the fan and runs into $p_i$. This implies that an almost elementary domain of $T'$ containing the left limit at $p_i$ cannot be a zig-zag. Furthermore, it cannot form an outgoing fan with a source at $p_i$ (otherwise the arcs of this fan will form an infinite set of bad arcs of $T'$ with respect to $T$, see Fig. 2.22 (b)). Hence, the triangulation $T'$ restricted to $R_i$ contains an incoming fan; denote its source by $q_i$ (see Fig. 2.22 (c)). Let $\mu^{(s_i)}$ be a finite sequence of infinite mutations which shifts the starting point of the fan in $\mu^{(r_i)}(T)|_{R_i}$ to $q_i$. The constructed triangulation $\tilde{T}_i = \mu^{(s_i)} \circ \mu^{(r_1)}(T)$ has only finitely many crossings with $\gamma$ inside $R_i$. Moreover, the only limit arc of $\tilde{T}_i$ contained inside $R_i$ is the arc $q_i p_i \in T'$ (it is also a limit arc of $T'$). This implies that no limit arc of $\tilde{T}_i$ can be crossed by an arc of $T'$ inside $R_i$, so if $\alpha \in T'$ is a bad arc for $\tilde{T}_i$ then it is also a bad arc for $T$. This completes the proof for the case when the domain $R_i$ is a fan.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.22.png}
\caption{To the proof of Lemma 2.42.}
\end{figure}

In the case of a zig-zag domain $R_i$ (transformed at the first step to a union of two triangles and two fans, see Fig. 2.21 right) we proceed in a similar way. More precisely, we use exactly the same reasoning for the fan lying below the diagonal. For the fan above the diagonal, notice that any arc of $T'$ crossing the diagonal above $\gamma$ also crosses the limit arc of the zig-zag (see Fig. 2.23); so, there are finitely many of these arcs as $|\text{Bad}_T T'| < \infty$. Hence, we can consider a finite polygon $P_i^+$ formed by all triangles of $T'$ crossing the diagonal above $\gamma$ and we can shift the source of the upper fan away from $P_i^+$. This completes the proof of the lemma. \qed
Proposition 2.43. For two triangulations $T$ and $T'$ of $S$, if $0 < |\text{Bad}_T T'| < \infty$ then there exists a finite sequence of infinite mutations $\mu^{(n)}$ for some $0 < n < \infty$ such that $\mu^{(n)}(T) = T'$.

Proof. Let $\text{Bad}_T T' = \{\gamma_1, \ldots, \gamma_k\}$. By Lemma 2.42 there exists a finite sequence of infinite mutations $\mu_1^{(n_1)}$ such that $\mu_1^{(n_1)}(T) = T_1 \in T$ and $|\text{Bad}_{T_1} T'| < |\text{Bad}_T T'|$. Again by Lemma 2.42 there exists a finite sequence of infinite mutations $\mu_2^{(n_2)}$ such that $\mu_2^{(n_2)}(T_1) = T_2 \in T$ and $|\text{Bad}_{T_2} T'| < |\text{Bad}_{T_1} T'|$. Proceeding in this manner for at most $k = |\text{Bad}_T T'|$ steps, we obtain a finite sequence of infinite mutations $\mu_k^{(n_k)}$ such that $\mu_k^{(n_k)}(T_{k-1}) = T_k \in T$ and $|\text{Bad}_{T_k} T'| = 0$. Then by Proposition 2.41 there exists an infinite sequence $\mu^{(1)}$ such that

$$\mu^{(1)}(T_k) = \mu^{(1)}(\mu_k^{(n_k)} \circ \cdots \circ \mu_1^{(n_1)}(T)) = T',
$$

which completes the proof.

The following lemma asserts that if $|\text{Bad}_T T'| = \infty$ then $T'$ cannot be obtained from $T$ by any finite sequence of infinite mutations.

Proposition 2.44. For two triangulations $T$ and $T'$ of $S$, if $|\text{Bad}_T T'| = \infty$ then $\mu^{(n)}(T) \neq T'$ for any finite sequence of infinite mutations $\mu^{(n)}$.

Proof. Enough to prove the statement for one infinite mutation $\mu^{(1)}$. Suppose that $\mu^{(1)}(T) = T'$. Then every non-limit arc of $T'$ arises after finitely many flips, i.e. intersects with finitely many arcs in $T'$. However by Corollary 2.10 there are only finitely many limit arcs in $T'$. This contradicts the assumption that $|\text{Bad}_T T'| = \infty$.

Collecting results from Theorem 2.24 and Propositions 2.41, 2.43, 2.44, we will now present a characterisation of different kinds of mutation sequences.

Notation 2.45. Let $S$ be an infinite surface. Fix a triangulation $T$ of $S$.

- Hierarchy in $T$ by mutation distance from $T$: denote by
  $M(T) = \{T' \in T \mid \exists \mu^{(0)}, \mu^{(0)}(T) = T'\}$,
  $M^{<\infty}(T) = \{T' \in T \mid \exists \mu^{(n)} \text{ for some } 0 < n < \infty, \mu^{(n)}(T) = T'\}$,
  $M^\infty(T) = \{T' \in T \mid \exists \mu^\infty, \mu^\infty(T) = T'\}$;

the set of triangulations obtained from the triangulation $T$ by
- finite mutation sequences,
- finite sequences of infinite mutations, and
- infinite sequences of infinite mutations, respectively.
• Hierarchy in $T$ in terms of intersections with $T$ (more precisely, in terms of bad arcs with respect to $T$):

$$I(T) = \{T' \in T \mid |T \cap T'| < \infty\};$$

$$I^{< \infty}(T) = \{T' \in T \mid 0 \leq |\text{Bad}_T T'| < \infty, |T \cap T'| = \infty\};$$

$$I^\infty(T) = \{T' \in T \mid |\text{Bad}_T T'| = \infty\}. $$

Note that $T = I(T) \cup I^{< \infty}(T) \cup I^\infty(T)$.

**Remark 2.46.** We have $M(T) \subset M^{< \infty}(T)$ as any finite mutation may be represented as a composition of two infinite mutations in the following way: first apply a finite mutation and then in infinitely many disjoint quadrilaterals in the surface, first flip all the diagonals (this is an infinite mutation) and then flip them all back (again an infinite mutation). Similarly we have $M(T)^{< \infty} \subset M^\infty(T)$.

**Theorem 2.47.** For any triangulation $T$ of an infinite surface $S$ the following holds:

(a) $I(T) = M(T)$;
(b) $I^{< \infty}(T) = M^{< \infty}(T)$;
(c) $I^\infty(T) = T \setminus (M^{< \infty}(T));$
(d) $T = M^\infty(T)$.

**Proof.** Part (a) is a reformulation of Proposition 2.39, part (b) follows from Propositions 2.41, 2.43 and 2.44. In view of Remark 2.46, part (d) is equivalent to Theorem 2.24. Finally, part (c) follows from parts (a), (b), (d) and Remark 2.46. □

**Example 2.48** (Infinity-gon). In Fig. 2.24 and 2.25, we illustrate Theorem 2.47 for some triangulations of the one-sided infinity-gon $I_1$ and the two-sided infinity-gon $I_2$, respectively, where

- solid arrows indicate finite sequences of infinite mutations,
- dashed arrows indicate infinite sequences of infinite mutations.

Moreover, these figures can be considered as “underlying exchange graphs” for $I_1$ and $I_2$:

- vertices of the graphs correspond to classes of triangulations of $I_1$ and $I_2$, respectively, where classes are composed of triangulations having similar combinatorics, which roughly speaking means that two triangulations are in the same class if they have the same set of almost elementary domains attached to each other in the same way.

The graphs shown in Fig. 2.24 and 2.25 agree with those presented in Fig. 8 and 9 of [BG].

**Figure 2.24.** “Underlying exchange graph” for the one-sided infinity-gon.

**Remark 2.49.** It is shown in [BG] that mutations along admissible sequences induce a partial order on the triangulations of a given surface (where $T < T'$ when there exists an admissible sequence of elementary mutations $\mu$ such that $\mu(T) = T'$). In our settings of infinite mutations
completed with all limit arcs, this property obviously holds for the relation $<_n$ (where $T <_n T'$ if there exists a finite sequence of infinite mutations $\mu^{(n)}$ such that $\mu^{(n)}(T) = T'$). On the other hand, the relation $<_1$ (where $T <_1 T'$ if there exists an infinite mutation $\mu^{(1)}$ such that $\mu^{(1)}(T) = T'$) does not induce a partial order as a composition of two infinite mutations is not necessarily an infinite mutation.

In the following proposition we will see that for every infinite surface there is a minimal element with respect to the partial order defined by the relation $<_n$.

**Proposition 2.50.** For every infinite surface $S$, there exists a triangulation $T$ such that

- for any arc $\gamma \in S$, $|\gamma \cap T| < \infty$, and
- for any $T' \in \mathbb{T}_S$ there exists an infinite mutation $\mu^{(1)}$ satisfying $T' = \mu^{(1)}(T)$.

**Proof.** For each accumulation point $p_i$, choose a small disc neighbourhood $D_i$ so that these neighbourhoods do not intersect. Inside each $D_i$, set an outgoing fan triangulation (see Fig. 2.26). Choose any triangulation on the rest of the surface (the surface $S \setminus \bigcup D_i$ has finitely many boundary marked points, so it has a finite triangulation).

Denote the triangulation constructed above by $T$ and observe that any arc $\gamma$ in $S$ crosses $T$ only finitely many times (indeed, $\gamma$ crosses at most finitely many arcs from each $D_i$). Thus, for any triangulation $T'$, we have $|\text{Bad}_T T'| = 0$ which in view of Proposition 2.41 proves the result. \[\square\]

**Remark 2.51.** Proposition 2.50 states that among all the triangulations there is a class of particularly good ones composed of almost elementary outgoing fans. We will call them *outgoing fan triangulations* (see Definition 3.26).

In the next section, we will introduce cluster algebras associated to infinite surfaces, and we will see that for an outgoing fan triangulation every cluster variable can be expressed as a Laurent polynomial in the initial cluster, see also Remark 5.2.

**Remark 2.52.** In Fig. 2.24 and 2.25 one can find examples of outgoing fan triangulations for the infinity-gon: these are those triangulations shown in the middle of the figure and connected to every other triangulation by a solid arrow.
3. Infinite rank surface cluster algebras

In this section we will use hyperbolic structures on infinite surfaces to define infinite rank surface cluster algebras. The spirit of this section follows the one of Fomin and Thurston [FT].

3.1. Hyperbolic structure and converging horocycles. Consider an infinite surface $S$ with a triangulation $T$. We will understand the triangles of $T$ as ideal hyperbolic triangles, i.e. hyperbolic triangles with all three vertices on the boundary of the hyperbolic plane. To each marked point $p_i$, we will assign a horocycle $h_i$ with an additional requirement that if marked points $p_j, j \in \mathbb{N}$, converge to an accumulation point $p_*$ then the corresponding horocycles converge to the horocycle $h_*$ at $p_*$ (see Definition 3.3).

Similarly to the case of finite surfaces, hyperbolic triangles together with a choice of horocycles induce a lambda length assigned to each arc of the triangulation. We will show in Section 3.2 that these lambda lengths satisfy certain conditions (as long as the horocycles are converging). Conversely, in Section 3.3 we will see that given a set of positive numbers associated with the arcs of $T$ and satisfying the above-mentioned conditions, there exists a hyperbolic surface with a choice of converging horocycles such that the lambda lengths of the arcs of $T$ coincide with the prescribed numbers, see Theorem 3.29.

3.1.1. Admissible hyperbolic structures.

Definition 3.1 (Admissible hyperbolic structure). By an admissible hyperbolic structure on an infinite surface $S$ we mean a locally hyperbolic metric such that
- all boundary arcs are complete (i.e. bi-infinite) geodesics, and
- all interior marked points are cusps.

A surface with admissible hyperbolic structure will be called an admissible hyperbolic surface.

This means that having all vertices of triangles at interior or boundary marked points of a triangulation of $S$, every triangle is isometric to a hyperbolic triangle with all vertices at the boundary of the hyperbolic plane.

Lemma 3.2. Let $S$ be an infinite surface with an admissible hyperbolic structure. Then the universal cover of $S$ is contained in the hyperbolic plane $\mathbb{H}^2$ as a proper subset.

Proof. We start by choosing an outgoing fan triangulation $T$ of $S$ as in Proposition 2.50. Let $t_1$ be a triangle in $T$. We can embed $t_1$ isometrically into the hyperbolic plane $\mathbb{H}^2$ (uniquely up to hyperbolic isometry) and we denote the obtained triangle $\tilde{t}_1$. The triangle $t_1$ has at most 3 adjacent triangles $t_2, t_3, t_4$ in $S$ (some of them may coincide with others or with $t_1$). The gluing of two adjacent triangles is determined by the choice of hyperbolic structure on $S$ and it may be described by shear coordinates as in Fig 3.1 (a): we drop the perpendiculars from the vertices of the adjacent triangles to the common side and look at the hyperbolic distance between the feet of the two perpendiculars. This dictates how the triangles $t_2, t_3, t_4$ will lift to the universal
cover. We add the triangles to the universal cover one by one. Notice that we will always be able to embed the next triangle $\tilde{t}_{n+1}$ (attached to the previous triangle $\tilde{t}_n$ by the common side $\gamma$, see Fig 3.1 (b)) into the hyperbolic plane without intersecting the previous ones. Indeed, the finite union of already placed triangles $\tilde{t}_1, \ldots, \tilde{t}_n$ lies on one side from the line $\gamma = \tilde{t}_n \cap \tilde{t}_{n+1}$, and the new triangle $\tilde{t}_{n+1}$ lies on the other side. Lifting adjacent triangles of each triangle $t_i \in T$ we will reach every triangle of $T$ in finitely many steps: indeed $T$ was chosen as an outgoing fan triangulation, in particular, it contains no limit arcs. The fact that the boundary of $S$ is non-empty implies that the lifts will not cover the whole hyperbolic plane.

**Figure 3.1.** Lift of $S$ to the universal cover: (a) gluing of adjacent triangles (shear coordinates); (b) attaching the next triangle.

In this section, we will often consider the surface $S$ as lifted to the hyperbolic plane $\mathbb{H}^2$. More precisely, we will often lift $S$ to the upper half-plane model $\mathcal{H}$ of the hyperbolic plane $\mathbb{H}^2$, i.e. to the set of points \( \{z \in \mathbb{C} \mid \text{Im } z > 0\} \) with hyperbolic distance defined by

\[
\cosh d(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2 \text{Im}(z_1) \text{Im}(z_2)}.
\]

In this model the lines are represented by half-lines and half-circles orthogonal to $\partial \mathcal{H}$. The cusps, as well as the boundary marked points, lift to the points on the boundary $\partial \mathcal{H}$.

**3.1.2. Converging horocycles.** A horocycle centred at a cusp $p$ is a curve perpendicular to every geodesic incident to $p$. Lifted to the universal cover $\mathcal{H}$, a horocycle centred at $p \neq \infty$ is represented by a circle tangent to $\partial \mathcal{H}$ at $p$. A horocycle centred at $p = \infty$ is represented by a horizontal line $\text{Im } z = \text{const}$. A horocycle centred at $p$ may be understood as a set of points "on the same (infinite) distance from $p''", and it is also a limit of a sequence of circles through a fixed point with the centres converging to $p$.

**Figure 3.2.** Converging horocycles in the upper half-plane.

**Definition 3.3** (Converging horocycles in $\mathcal{H}$). Let $\mathcal{H}$ be the upper half-plane model of the hyperbolic plane and let $p_i \in \partial \mathcal{H}, i \in \mathbb{N},$ be a sequence of points on the boundary. Suppose that the points $p_i$ converge to some point $p_*$, denote it by $p_i \to p_*$ Let $h_i$ be a horocycle centred at
The relation between infinite surfaces and hyperbolic metrics.

Definition 3.6. Let $\lambda$ be a lambda length in $S$ (smooth or punctured).

Remark 3.7. For punctured or smooth surfaces, the authors of [FT] introduced the notion of lambda lengths.

Definition 3.8. (Ptolemy relation) Let $S_{ij}$ be a quadrilateral (where each of $p,q,r,s$ is a puncture or a boundary marked point), see Fig. [3.3]. It is shown in [P] that the lambda lengths of the arcs connecting these marked points satisfy the Ptolemy relation

$$\lambda_{pr}\lambda_{qs} = \lambda_{pq}\lambda_{rs} + \lambda_{qr}\lambda_{ps},$$

where $\lambda_{ij}$ is the lambda length of the arc connecting $i$ to $j$ for $i,j \in \{p,q,r,s\}$.

In the case of an infinite surface, the same relation obviously holds, as all elements of the relation are just a part of one quadrilateral.

Part (c) below.

Part (a) of Remark 3.4 allows us to introduce the notion of converging horocycles in any infinite hyperbolic surface.

Definition 3.5 (Converging horocycles in $S$, surface with converging horocycles). Let $S$ be an infinite surface with an admissible hyperbolic metric. Fix a horocycle at every marked point.

(a) Let $p_i$ be a sequence of boundary marked points converging to an accumulation point $p_\ast$. We say that the horocycles $h_i$ at $p_i$ converge to the horocycle $h_\ast$ at $p_\ast$ if there are lifts of $h_i$ to the universal cover converging to a lift of $h_\ast$.

(b) We say that the horocycles $h_i$ at $p_i$ converge to the horocycle $h_\ast$ at $p_\ast$ if they pointwise converge in the hyperbolic plane. More precisely, for every point $q \in \partial H$, let $\xi_i(q) = qp_i \cap h_i$ be the point of intersection of the horocycle $h_i$ and the hyperbolic line $qp_i$ connecting $q$ to $p_i$. Then $\xi_i(q)$ converges to $\xi_\ast(q) = qp_\ast \cap h_\ast$ as $p_i \to p_\ast$.

Similarly to the case of finite surfaces, we will use horocycles to define lambda lengths of arcs and boundary segments.

Remark 3.4.

(a) If a point $p_\ast$ in Definition 3.3 is represented by $\infty$ in $H$, we apply an isometry of the hyperbolic plane (i.e. any linear-fractional transformation $f(z) = \frac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$) which takes $\infty$ to another point of $\partial H$ and then use the definition above.

(b) Definition 3.3 refers to the Euclidean shapes in the upper half-plane model. So, we need to check the convergence of horocycles (i.e. the property that they can be represented by a converging sequence of Euclidean circles) does not change under hyperbolic isometries. This follows since the property can also be reformulated in purely hyperbolic terms (see part (c) below).

(c) Hyperbolic interpretation of convergence of horocycles: the horocycles $h_i$ at $p_i$ converge to the horocycle $h_\ast$ at $p_\ast$ if they pointwise converge in the hyperbolic plane. More precisely, for every point $q \in \partial H$, let $\xi_i(q) = qp_i \cap h_i$ be the point of intersection of the horocycle $h_i$ and the hyperbolic line $qp_i$ connecting $q$ to $p_i$. Then $\xi_i(q)$ converges to $\xi_\ast(q) = qp_\ast \cap h_\ast$ as $p_i \to p_\ast$.

Part (c) of Remark 3.4 allows us to introduce the notion of converging horocycles in any infinite hyperbolic surface.
Choose marked points

Proposition 3.12. that \( h \) is a unique horocycle \( H \) half-plane \( s \) let with corresponding horocycles at both endpoints of \( \gamma \) length of the arc \( 01 \) is determined by the horocycles \( (\[P\], Corollary 4.8) \)

Observation 3.10 based on the following simple observation (which will also play the key role for infinite surfaces).

In the case of a finite surface, the construction (implying both existence and uniqueness) was based on the following simple observation (which will also play the key role for infinite surfaces).

**Definition 3.9** (Decorated Teichmüller space). Let \( S \) be an infinite surface. A point in the decorated Teichmüller space \( \hat{T}(S) \) of \( S \) is an admissible hyperbolic structure on \( S \) together with a choice of converging horocycles.

It is shown in [FT] that given a triangulation \( T \) of a finite hyperbolic surface, lambda lengths of the arcs of \( T \) uniquely define a point of the decorated Teichmüller space \( \hat{T}(S) \). We will obtain a similar result for triangulations containing no infinite zig-zag domains, see Theorem 3.29.

In the case of a finite surface, the construction (implying both existence and uniqueness) was based on the following simple observation (which will also play the key role for infinite surfaces).

**Observation 3.10** ([FT], Corollary 4.8). For every \( x_1, x_2, x_3 \in \mathbb{R}_+ \), there exists a unique hyperbolic triangle \( \Delta \) with all vertices at \( \partial H \) (modulo isometry of hyperbolic plane) and a unique choice of horocycles at its vertices such that \( x_1, x_2, x_3 \) are the lambda lengths of the sides of \( \Delta \).

**Proof.** Up to an isometry of hyperbolic plane (i.e. up to a linear-fractional map of \( \mathcal{H} \)) all triangles with vertices on \( \partial H \) are equivalent to the triangle \( 01\infty \). Choose any horocycle \( h_\infty \) at \( \infty \) (it is represented in \( \mathcal{H} \) by a horizontal line \( \text{Im } z = k \) for some \( k \in \mathbb{R} \)). Then there exists a unique horocycle \( h_0 \) at 0 such that the lambda length of the arc \( 0\infty \) is \( x_1 \). Similarly, there is a unique horocycle \( h_1 \) at 1 such that the lambda length of the arc \( 1\infty \) is \( x_2 \). The lambda length of the arc \( 01 \) is determined by the horocycles \( h_0 \) and \( h_1 \), but in most cases is not equal to \( x_3 \). However, it changes monotonically depending on the initial choice of \( h_\infty \), and it tends to 0 or \( \infty \) when the constant \( k \) in the definition of the horocycle \( h_\infty \) is very large or very small, respectively. This implies that for every value of \( x_3 \) there exists a unique choice of suitable horocyle \( h_\infty \), which in turn gives a unique choice of the horocycles \( h_0 \) and \( h_1 \). \( \Box \)

3.2. Conditions induced by the geometry of the surface. We will now analyse the behavior of infinite sequences of arcs in an infinite surface \( S \). As a first easy result in this direction we will show that the lambda length of a limit arc equals to the limit of the lambda lengths of the arcs.

**Proposition 3.11.** Let \( S \) be an infinite surface with an admissible hyperbolic structure and with converging horocyles. Let \( \{\gamma_i\} \) be a sequence of arcs in \( S \) such that \( \gamma_i \to \gamma_\ast \) as \( i \to \infty \). Let \( x_i \) be the lambda length of \( \gamma_i \) for \( i \in \mathbb{N} \cup \{\ast\} \). Then \( x_i \to x_\ast \) as \( i \to \infty \).

**Proof.** Since the horocycles on \( S \) are converging by construction, the intersection points of \( \gamma_i \) with corresponding horocycles at both endpoints of \( \gamma_i \) converge to the intersection points of \( \gamma_\ast \) with horocycles at both endpoints of \( \gamma_\ast \). \( \Box \)

**Proposition 3.12.** Choose marked points \( \{p_i \mid i \in \mathbb{N} \cup \{\ast\}\} \) on the boundary of the upper half-plane \( \mathcal{H} \) so that \( p_i \to p_\ast \) with \( p_\ast \neq \infty \), and assign horocycles \( h_i \) at each marked point \( p_i \) so that \( h_i \to h_\ast \) as \( i \to \infty \). Let \( x_i,i+1 \) be the lambda length of the geodesic arc from \( p_i \) to \( p_{i+1} \) and let \( s_{i,i+1} \) be the Euclidean distance between \( p_i \) and \( p_{i+1} \), for each \( i \). Then

\[
(1) \quad \frac{x_{i,i+1}}{s_{i,i+1}} \to \frac{1}{2}.
\]

![Figure 3.3. Ptolemy relation in a quadrilateral: \( \lambda_{pq}\lambda_{qs} = \lambda_{pq}\lambda_{rs} + \lambda_{qr}\lambda_{ps} \).](image)
Proof. Part (1) is a straightforward computation in the upper half-plane (we write the equation of the hyperbolic line explicitly, find its intersections with horocycles, compute the lambda length, and take the limit). Parts (2) and (3) are direct consequences of part (1). □

Combining Propositions 3.11 and 3.12(3), we obtain the following corollary.

**Corollary 3.13.** If \( \{ \gamma_i \} \) is a converging sequence of arcs in \( \mathcal{S} \), then the corresponding lambda lengths \( \{ x_{\gamma_i} \} \) converge. Moreover,
- if \( \{ \gamma_i \} \) converges to a limit arc \( \gamma_* \) on \( \mathcal{S} \) then \( x_{\gamma_i} \to x_{\gamma_*} \), and
- if \( \{ \gamma_i \} \) converges to an accumulation point, then \( x_{\gamma_i} \to 0 \).

**Proposition 3.14.** Let \( \mathcal{F} = \{ \gamma_i, \gamma_{i,i+1} \mid i \in \mathbb{N} \} \cup \{ \gamma_* \} \) be an elementary incoming fan with converging horocycles. Let \( x_i \) be the lambda lengths of the arcs \( \gamma_i, i \in \mathbb{N} \), and \( x_{i,j} \) be the lambda length of the geodesic arc between marked points \( p_i \) and \( p_j \), for \( i \neq j \), where \( i, j \in \mathbb{N} \cup \{ * \} \), see Fig. 3.4(a), (b). Then

1. \( x_i \to x_* \) as \( i \to \infty \);
2. \( \sum_{i=1}^{\infty} x_{i,i+1} < \infty \);
3. \( x_{s,n} = x_s x_n \sum_{i=s}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \) for \( s \in \mathbb{N}, n > s \);
4. \( x_{s,*} = x_{s} x_{*} \sum_{i=s}^{\infty} \frac{x_{i,i+1}}{x_i x_{i+1}} \) for \( s \in \mathbb{N} \); in particular, \( x_{s,*} \) is an absolutely converging Laurent series in \( \{ x_i, x_{i,i+1} \mid i \in \mathbb{N} \} \).
Figure 3.5. An arc crossing an infinite incoming fan (left: viewed with the source at \( \infty \); right: viewed with the source at a finite point).

**Proof.**

1. Follows from Proposition 3.11.

2. Follows from Proposition 3.12 (2).

3. Without loss of generality, we will set \( s = 1 \). We will show by induction on \( n \) that

\[
x_{1,n} = x_1 x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}}.
\]

The base step is evident since

\[
x_{1,2} = x_1 x_2 \sum_{i=1}^{1} \frac{x_{i,i+1}}{x_i x_{i+1}}.
\]

Assume now the identity holds for all \( n < k \) for some \( k \in \mathbb{N} \). By the Ptolemy relation in the quadrilateral \( p_1 p_{k-1} p_k p_s \),

\[
x_{1,k} = \frac{x_k x_{1,k-1} + x_1 x_{k-1,k}}{x_{k-1}}
\]

\[
= \frac{x_k}{x_{k-1}} x_1 x_{k-1} \sum_{i=1}^{k-2} \frac{x_{i,i+1}}{x_i x_{i+1}} + \frac{x_1 x_{k-1,k}}{x_{k-1}}
\]

\[
= x_1 x_k \sum_{i=1}^{k-1} \frac{x_{i,i+1}}{x_i x_{i+1}}.
\]

4. It remains to show \( x_{1,n} \to x_1 x_\ast \sum_{i=1}^{\infty} \frac{x_{1,i+1}}{x_1 x_{i+1}} \) as \( n \to \infty \), or in other words that

\[
\delta_k := |x_1 x_\ast \sum_{i=1}^{\infty} \frac{x_{1,i+1}}{x_1 x_{i+1}} - x_{1,k}| \to 0 \quad \text{as} \quad k \to \infty.
\]

By part (1) we have \( x_i \to x_\ast \), and by part (2) we know that \( \sum_{i=1}^{\infty} x_{i,i+1} \) converges.

Therefore, for any \( \varepsilon > 0 \), we can choose \( k \in \mathbb{N} \) such that \( |x_i - x_\ast| < \varepsilon \), and \( \sum_{i=k}^{\infty} x_{i,i+1} < \varepsilon \) for all \( i > k \). Also, as \( x_i > 0 \) for all \( i \) and \( x_i \to x_\ast \) with \( x_i > 0 \) and \( x_\ast > 0 \), there exists \( K \in \mathbb{R} \) such that and \( \frac{1}{x_\ast} < K \) for all \( j \). Thus,

\[
\delta_k = |x_1 x_\ast \sum_{i=1}^{\infty} \frac{x_{1,i+1}}{x_1 x_{i+1}} - x_{1,k}| = \left| x_1 (x_\ast - x_k) \sum_{i=1}^{k-1} \frac{x_{i,i+1}}{x_i x_{i+1}} + x_1 x_\ast \sum_{i=k}^{\infty} \frac{x_{1,i+1}}{x_i x_{i+1}} \right|
\]

\[
< x_1 \varepsilon K^2 \sum_{i=1}^{\infty} x_{i,i+1} + x_1 x_\ast \left( \frac{1}{x_\ast} + \varepsilon \right)^2 \sum_{i=k}^{\infty} x_{i,i+1}.
\]
By Proposition 3.12 (2), \( \sum_{i=1}^{\infty} x_{i,i+1} \) converges, say to \( C \). Therefore,

\[
\delta_k < x_1 \varepsilon K^2 C + x_1 x_* (1 + \varepsilon)^2 \varepsilon.
\]

Hence, \( \delta_k \) is bounded by a constant times \( \varepsilon \), which completes the proof. \( \square \)

**Proposition 3.15.** Let \( F = \{ \gamma_i, \gamma_i, i+1 \mid i \in \mathbb{N} \} \) be an elementary outgoing fan with converging horocycles and let \( \{ x_i, x_i, i+1 \mid i \in \mathbb{N} \} \) be the corresponding lambda lengths, \( i \in \mathbb{N} \) (see Fig. 3.4 (c)-(d)). Then

1. \( x_i \to 0 \) as \( i \to \infty \);
2. \( x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \to 1 \) as \( n \to \infty \).

**Proof.** Part (1) follows from Proposition 3.12 (3). To show part (2), notice that \( x_1, n \to x_1 \) as \( n \to \infty \) in view of Proposition 3.11. On the other hand, by Proposition 3.14 (3), we have \( x_1, n = x_1 x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \) for \( n \in \mathbb{N} \). Hence, in order to have \( x_1, n \to x_1 \) as \( n \to \infty \), we need \( x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \to 1 \).

The proof of part (2) of Proposition 3.15 implies the following corollary.

**Corollary 3.16.** Let \( F = \{ \gamma_i, \gamma_i, i+1 \mid i \in \mathbb{N} \} \) be an elementary outgoing fan on the hyperbolic plane. Let \( \{ p_i \}, i \in \mathbb{N} \), be the marked points in \( F \) such that \( p_i \to p_* \) for some \( p_* \), and let \( h_i \) be a horocycle at \( p_i \) and \( h_* \) be a horocycle at \( p_* \). Let \( \{ x_i, x_i, i+1 \mid i \in \mathbb{N} \} \) be the corresponding lambda lengths. Then \( h_i \to h_* \) if and only if \( x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \to 1 \) as \( n \to \infty \).

**Proof.** If \( h_i \to h_* \) then \( x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \to 1 \) by Proposition 3.15 (2).

Conversely, if \( x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}} \to 1 \), then \( x_1, n \to x_1 \), and hence the horocycles converge. \( \square \)

Now, we will introduce a metric characteristic of a fan which will allow us to distinguish incoming fans from outgoing ones.

**Definition 3.17** (Width of fan). Let \( F \) (resp. \( \overline{F} \)) be an elementary outgoing (resp. incoming) fan with the source \( p_* \) (resp. \( p_0 \)) and the arcs \( \gamma_n = p_* p_n \), for \( n \in \mathbb{N} \) (resp. the arcs \( \gamma_n = p_0 p_n \), for \( n \in \mathbb{N} \), and a limit arc \( \gamma_* = p_0 p_* \)). We will define the width of \( F \) (resp. \( \overline{F} \)) as follows:
- drop a perpendicular from \( p_1 \) to every arc \( \gamma_n \) for \( n \in \mathbb{N} \) and denote by \( q_n \) the foot of the perpendicular;
- let \( l_n \) be the (signed) hyperbolic distance from \( q_n \) to the horocycle \( h_1 \) centred at \( p_1 \) (in other words, \( l_n \) is the hyperbolic length of the portion of the perpendicular lying outside of the horocycle \( h_1 \));
- \( w_n = \exp(l_n) \) will be called the width of a finite fan with \( n - 1 \) triangles;
- the width \( w_F \) of an infinite fan \( F \) (resp. \( \overline{F} \)) is defined by the (finite or infinite) limit of \( w_n \) as \( n \to \infty \) (see Lemma 3.18 for the proof of the existence of the limit).

See Fig. 3.8 for an illustration. We will simply use the notation \( w \) for the width of a fan \( F \) when the fan is clear from the context.

**Lemma 3.18.** The width of an infinite elementary fan is well-defined, i.e. the (finite or infinite) limit in Definition 3.17 does exist. Moreover, the width of an incoming fan is finite and the width of an outgoing fan is infinite.

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Proof. In the case of an incoming fan, the length $l_n$ tends to the distance between the horocycle $h_1$ centred at $p_1$ and the limit arc, and this distance is finite (see Fig. 3.6 (a)-(b)). In the case of an outgoing fan, the length $l_n$ is the distance between the horocycle $h_1$ and the line $\gamma_n$, which grows as $n$ gets larger and tends to the infinite distance between the horocycle $h_1$ and the accumulation point $p_*$, see Fig. 3.6 (c)-(d).

\[ w_n = 2x_1^2 \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}}. \]

Lemma 3.19. Let $\mathcal{F}_n$ be a finite fan $\{\gamma_i, \gamma_{i,i+1} \mid 1 \leq i \leq n-1\} \cup \{\gamma_n\}$, let $x_i$ and $x_{i,i+1}$ be the corresponding lambda lengths of arcs and boundary arcs of $\mathcal{F}$. Then the width $w_n$ of $\mathcal{F}_n$ satisfies

$$w_n = x_{1,2n-1} = x_1 x_{2n-1} \sum_{i=1}^{2n-2} \frac{x_{i,i+1}}{x_i x_{i+1}} = 2x_1^2 \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i+1}},$$

where the last equality holds since $x_{n+i} = x_{n-i}$ and $x_{i,i+1} = x_{2n-i-1,2n-i}$ for $i = 1, \ldots, n-1$. \[\square\]
3.3. Geometry induced by compatible data. In this section, we will start with combinatorial data (triangulation $T$ of $S$) and numerical data (numbers associated to arcs of $T$) and show that under certain conditions the data is realisable by a decorated hyperbolic surface with converging horocycles. The data sets $(\mathcal{F}, \mathcal{D})$ and $(\overline{\mathcal{F}}, \overline{\mathcal{D}})$ will be considered for an outgoing and incoming elementary fans and will have the following ingredients:

- a collection of arcs
  \[ \mathcal{F} = \{ \gamma_i, \gamma_{i+1} \mid i \in \mathbb{N} \} \quad \text{or} \quad \overline{\mathcal{F}} = \{ \gamma_i, \gamma_{i+1} \mid i \in \mathbb{N} \} \cup \{ \gamma_s \} \]
  forming an elementary outgoing or incoming fan (together with its base), respectively;

- a collection of positive real numbers
  \[ \mathcal{D} = \{ x_i, x_{i+1} \in \mathbb{R}_{>0} \mid i \in \mathbb{N} \} \quad \text{or} \quad \overline{\mathcal{D}} = \{ x_i, x_{i+1} \in \mathbb{R}_{>0} \mid i \in \mathbb{N} \} \cup \{ x_s \in \mathbb{R}_{>0} \} \]
  associated to each arc in the outgoing or incoming fan, respectively, see Fig. 3.3.

**Definition 3.20** (Compatibility of combinatorial and numerical data). We say that $\mathcal{D}$ is compatible with $\mathcal{F}$ (resp. $\overline{\mathcal{D}}$ is compatible with $\overline{\mathcal{F}}$) if there is an admissible hyperbolic surface $S$ with converging horocycles triangulated according to $\mathcal{F}$ (resp. $\overline{\mathcal{F}}$) and such that the lambda lengths of the geodesic arcs in $S$ coincide with the numerical data given in $\mathcal{D}$ (resp. $\overline{\mathcal{D}}$).

**Proposition 3.21.** Let $\mathcal{F} = \{ \gamma_i, \gamma_{i,i+1} \mid i \in \mathbb{N} \} \cup \{ \gamma_s \}$ be an elementary incoming fan and let $\overline{\mathcal{D}} = \{ x_i, x_{i+1} \in \mathbb{R}_{>0} \mid i \in \mathbb{N} \} \cup \{ x_s \in \mathbb{R}_{>0} \}$, see Fig. 3.4 (a)-(b). Then $\mathcal{F}$ is compatible with $\overline{\mathcal{D}}$ if and only if the following two conditions hold:

1. $x_n \to x_s$ as $n \to \infty$, and
2. $\sum_{i=1}^{\infty} x_{i,i+1} < \infty$.

**Proof.** By Proposition 3.14, the conditions are necessary for the existence of the corresponding admissible hyperbolic structure with converging horocycles. To show they are also sufficient, we will assume that conditions (1) and (2) hold and construct an embedding of the incoming fan $\mathcal{F}$ to the upper half-plane. We will map the source of the fan to $\infty$.

We start with embedding the first triangle of the fan by mapping it to $(0, 1, \infty) = (p_0, p_1, p_\infty)$. Then we can glue each successive triangles $\Delta_i$ in $\mathcal{F}$ one by one so that each triangle satisfy $\Delta_i = p_ip_{i+1}p_\infty$. Indeed, Observation 3.10 implies that given $p_i \in \partial H$ (together with the choice of horocycles at $p_i$ and at $p_\infty$) there exists a unique position for $p_{i+1}$ and a unique choice of horocycle $h_{i+1}$ at $p_{i+1}$ allowing to match the numerical data assigned to $\gamma_{i,i+1}$ and $\gamma_i$. It is left to show that the constructed points $p_i$ converge to some $p_s \neq \infty$ (i.e. we obtain an incoming fan as required and not an outgoing one) and that the constructed horocycles $h_i$ at $p_i$ converge to a horocycle at $p_s$.

By condition (1), $x_i \to x_s$ as $i \to \infty$, which implies that the Euclidean sizes of horocycles $h_i$ converge as $i \to \infty$. Condition (2) implies that $x_{i,i+1} \to 0$ as $i \to \infty$, which (as Euclidean sizes of horocycles converge) is only possible if $(p_{i+1} - p_i) \to 0$. 

![Figure 3.7. Computing the width of a finite fan by using two symmetric copies of it.](image)
Now, shift each of the triangles $\Delta_i$ to $p_1$ by the map $f_i(x) = x - p_i + p_1$, $i \in \mathbb{N}$ (the corresponding horocycles are shifted together with the triangles). Since $p_{i+1} - p_i \to 0$, we have $f_i(p_{i+1}) \to p_1$ as $i \to \infty$. Note also that the shifted horocycles $f_i(h_{i+1})$ at $f_i(p_{i+1})$ converge. Hence, we can use Proposition 3.12 (1) to see that $\sum x_{i,i+1} < \infty$ by assumption, this implies that $\sum s_{i,i+1} < \infty$. Thus, $p_i \to p_*$ for some point $p_* \neq \infty$. As Euclidean sizes of the horocycles $h_i$ converge, we conclude that the horocycles $h_i$ converge.

Proposition 3.22. Let $\mathcal{F} = \{\gamma_i, \gamma_{i,i+1} \mid i \in \mathbb{N}\}$ be an elementary outgoing fan and let $\mathcal{D} = \{x_i, x_{i,i+1} \in \mathbb{R}_{>0} \mid i \in \mathbb{N}\}$, see Figure 3.4 (c)-(d). Then $\mathcal{F}$ is compatible with $\mathcal{D}$ if and only if the following two conditions hold:

1. $x_n \to 0$ as $n \to \infty$, and
2. $x_n \sum_{i=1}^{n-1} \frac{x_{i,i+1}}{x_i x_{i-1}} \to 1$ as $n \to \infty$.

Proof. The conditions are necessary in view of Proposition 3.15. To prove they are also sufficient, we suppose that the assumptions (1) and (2) hold and construct an embedding of the outgoing fan to the upper half-plane (with the source of the fan mapped to $\infty$).

We start by embedding the first triangle of the fan and then we glue each successive triangle while assigning the appropriate horocycles at its vertices at each step. We construct some infinite fan in $\mathcal{H}$. Combining assumptions (1) and (2), we have $\sum_{i=1}^{\infty} x_{i,i+1} \to \infty$ as $n \to \infty$. In view of Lemma 3.19 this implies that the width of the constructed fan is infinite, so the fan is outgoing. This implies $p_i \to \infty$ as $i \to \infty$. Assumption (2) together with Corollary 3.16 implies that horocycles at marked points $p_i$ converge, as required.

We collect compatibility conditions for elementary fan domains in Table 3.1.

Remark 3.23. For each of the two types of fans it is easy to check that none of the two conditions is sufficient alone without the other condition.

| Domain             | Conditions                           |
|--------------------|--------------------------------------|
| incoming fan       | $x_n \to x_*$                        |
|                    | $\sum_{i=1}^{\infty} x_{i,i+1} < \infty$ |
| outgoing fan       | $x_n \to 0$                          |
|                    | $x_n \sum_{i=1}^{\infty} \frac{x_{i,i+1}}{x_i x_{i-1}} \to 1$ |

Proposition 3.24. The compatibility conditions for almost elementary incoming and outgoing fans are exactly the same as the ones given in Table 3.1.

Proof. We use Propositions 3.21 and 3.22 to construct elementary incoming and outgoing fans embedded into almost elementary fans. We also construct separately all polygons attached to almost elementary fans, see Fig. 3.8. Then we glue them to elementary fans to construct a surface for almost elementary incoming and outgoing fans. □
Figure 3.8. An almost elementary fan as a union of an elementary fan and an infinite number of finite polygons, attached to its base.

Remark 3.25. We do not know how to derive compatibility conditions for elementary zig-zag pieces. This leads us to the following definition.

Definition 3.26 (Fan triangulations, incoming/outgoing fan triangulations). We say a triangulation $T$ of $S$ is

- a \textit{fan triangulation} if $T$ is a finite union of almost elementary fan domains;
- an \textit{incoming} (resp. \textit{outgoing}) fan triangulation if all almost elementary domains are incoming (resp. outgoing) fans.

Example 3.27. The triangulation used in the proof of Theorem 2.50 is an outgoing fan triangulation (see Fig. 2.26).

Remark 3.28. Let $T$ be a fan triangulation of $S$ and $\gamma \in S$ be an arc. Then the domain $D^T_\gamma$ of $\gamma$ in $T$ is a finite union of elementary fans (compare to Remark 2.29).

Theorem 3.29. Given a fan triangulation $T$ of an infinite surface $S$ together with a collection of positive real numbers associated to arcs and boundary arcs of $T$, if the compatibility conditions listed in Table 3.1 are satisfied for all almost elementary domains of $T$ then the numerical data is compatible with the combinatorial data in the whole surface $S$.

Proof. We cut the surface $S$ into finitely many almost elementary domains and realise each domain using Proposition 3.24. We then glue these domains in finitely many steps. \hfill $\square$

3.4. Teichmüller Space.

Definition 3.30. We say numerical values \( \{x_i \in \mathbb{R}_{>0}\mid i \in \mathbb{N}\} \) are \textit{compatible} with a fan triangulation $T = \{\gamma_i \mid i \in \mathbb{N}\}$ of an infinite surface $S$ if compatibility conditions hold for every almost elementary domain in $T$.

In view of Definition 3.30 we reformulate Theorem 3.29 as follows.

Corollary 3.31. Let $T$ be a fan triangulation of an infinite surface $S$. Collections of values compatible with $T$ are in bijection with points in the decorated Teichmüller space $\tilde{T}(S)$.

Since the compatibility conditions for incoming fan triangulations are linear, the Teichmüller space has the following nice properties.

Theorem 3.32. The set of points in the decorated Teichmüller space $\tilde{T}$ of an infinite surface $S$ forms a convex cone.

Proof. Let $T = \{\gamma_i \mid i \in \mathbb{N}\}$ be an incoming fan triangulation and $\mathbb{x}_T = \{x_i \mid i \in \mathbb{N}\}$ be the associated collection of values. Since the compatibility conditions are linear in the case of an incoming fan, it is immediate that, given $\mathbb{x} \in \tilde{T}$ and $\lambda > 0$, we have $\lambda \mathbb{x} \in \tilde{T}$ and given $x, y \in \tilde{T}$ we have $\mathbb{x} + y \in \tilde{T}$. Also $\lambda \mathbb{x} + (1 - \lambda)y \in \tilde{T}$ for any scalar $\lambda$, where $0 < \lambda < 1$. Hence, $\tilde{T}$ is a convex cone. \hfill $\square$
3.5. Lambda lengths as Laurent series. Given a triangulated infinite surface \( S \) with an admissible hyperbolic structure and with converging horocycles, consider the lambda lengths \( \{ x_i \} \) of the arcs \( \gamma_i \) of \( T \) as variables (depending on the point in \( \tilde{T} \)).

**Proposition 3.33.** Let \( S \) be an unpunctured infinite surface. Let \( T \) be a fan triangulation of \( S \) and \( \mathbf{x} = \{ x_i \in \mathbb{R}_{>0} \mid i \in \mathbb{N} \} \) be variables (including boundary variables) compatible with \( T \). If \( \gamma \) is an arc in \( S \), then the lambda length of \( \gamma \) is a Laurent series in \( \{ x_{\beta_1}, x_{\beta_2}, \ldots \} \) converging absolutely for any \( \mathbf{x} \) compatible with \( T \).

**Proof.** Consider the lift \( D \) of the domain \( D_T^\gamma \) of \( \gamma \) to the universal cover. By Proposition 2.22 together with Remark 2.29, \( D \) is a finite union of elementary fans (as \( T \) contains no infinite zig-zags). The proof of the statement is by induction on the number of elementary fans in \( D \).

The base case is when \( \gamma \) lies in a single fan and it follows from Proposition 3.14. To show that the statement holds for an arc crossing \( k \) elementary fans, we assume that it holds for all arcs crossing less than \( k \) fans and apply the Ptolemy relation (see Fig. 3.9 for notation):

\[
x_{\gamma} x_{\gamma'} = x_{\beta_1} x_{\beta_3} + x_{\beta_2} x_{\beta_4},
\]

where the \( k \)-th elementary fan of \( D \) is attached to the \((k - 1)\)-th one along the arc \( \gamma' \in T \). Notice that the arcs \( \beta_1, \beta_2, \beta_3, \beta_4 \) satisfy the induction assumption, which implies that the variables \( x_{\beta_j} \) are converging Laurent series in \( \{ x_i \} \). As \( \gamma' \in T \), dividing both sides by \( x_{\gamma'} \) we obtain the required expression for \( x_{\gamma} \). \( \square \)

![Figure 3.9. Induction on the number of fans crossed by \( \gamma \): Ptolemy relation.](image)

**Remark 3.34.** In the proof of Proposition 3.33, the fans adjacent to the arc \( \gamma' \) may be finite or infinite, they may either share the same source or have the sources at different endpoints of \( \gamma' \); however, the proof does not depend on any of these factors.

We will extend the result of Proposition 3.33 to the case of punctured surfaces in Theorem 3.38. For that matter we will need the following definitions and a technical lemma.

**Definition 3.35 (Ordinary and conjugate pair punctures with respect to \( T \)).** Let \( T \) be a triangulation of an infinite surface \( S \). We say that a puncture \( p \) of \( S \)

- is an **ordinary puncture** with respect to \( T \) if all arcs of \( T \) incident to \( p \) are tagged the same way at \( p \);
- otherwise, \( p \) is the endpoint of two copies of the same arc with different taggings at \( p \) (these arcs are called a **conjugate pair**). In this case, we will call \( p \) a **conjugate pair puncture** for \( T \) (see Fig. 3.10(a) for an example).

**Definition 3.36 (Right and wrong tagging).** Let \( T \) be a triangulation of a punctured surface and let \( p \) be an ordinary puncture with respect to \( T \). An arc \( \gamma \) with an end point at \( p \) has a **right tagging** (resp. wrong tagging) at \( p \) if it is tagged the same (resp. tagged oppositely) at \( p \) as the arcs of \( T \).

**Lemma 3.37.** Let \( T \) be a triangulation of an infinite surface \( S \) and let \( p \) be a puncture on \( S \). Then there exists an arc \( \psi \in T \) having exactly one endpoint at \( p \).
Theorem 3.38. The statement of Proposition 3.33 holds for punctured infinite surfaces, i.e. if \( T \) is a fan triangulation of an infinite surface \( S \) and \( z = \{ x_i \mid i \in \mathbb{N} \} \) are variables (including boundary variables) compatible with \( T \), and \( \gamma \) is an arc in \( S \), then the lambda length of \( \gamma \) is a Laurent series in \( \{ x_1, x_2, \ldots \} \) converging absolutely for any \( x \) compatible with \( T \).

Proof. The proof remains the same for arcs with no endpoints at punctures and for arcs having one or two endpoints at ordinary punctures with the right tagging at these punctures. This implies that we are left to prove the statement in two cases:

- when \( \gamma \) has a wrongly tagged endpoint at an ordinary puncture \( p \);
- when \( \gamma \) has an endpoint at a conjugate pair puncture \( p \).

Case 1: wrong tagging at an ordinary puncture. Let \( \gamma \) be an arc with a wrongly tagged endpoint at an ordinary puncture \( p \). We will assume that the other endpoint of \( \gamma \) is not a conjugate puncture. We will consider the three cases given below:

1.a: only one endpoint of \( \gamma \) has wrong tagging. Let \( \gamma' \) be the same arc as \( \gamma \) but with the opposite tagging at \( p \). As it is shown above, \( x_{\gamma'} \) is a Laurent series in \( \{ x_1, x_2, \ldots \} \). We will use \( x_{\gamma'} \) to find \( x_{\gamma} \).

Notice that given two curves \( \varphi \) and \( \psi \) both with exactly one endpoint at \( p \) and tagged the same at \( p \), and given the arcs \( \varphi' \) and \( \psi' \) which coincide with \( \psi \) and \( \varphi \) except that they are tagged oppositely at \( p \), one has

\[
\frac{x_\varphi}{x_{\varphi'}} = \frac{x_\psi}{x_{\psi'}}
\]

since the ratio above is \( e^{\pm d/2} \), where \( d \) is the distance between the two conjugate horocycles at \( p \), see [FT, Section 7]. Hence, it is sufficient to show that the ratio \( x_{\varphi}/x_{\varphi'} \) is a Laurent series for some specifically chosen arc \( \psi' \in T \) and get

\[
x_\varphi = x_{\gamma'} x_\psi / x_{\psi'}
\]

which is a Laurent series as \( x_{\gamma'} \) is.

If the triangulation \( T \) is locally finite at \( p \) (i.e. there are finitely many triangles incident to \( p \)), then we choose \( \psi' \in T \) to be an arc having exactly one end at \( p \) (there is such an arc in view of Lemma [3.37]). Notice that \( \psi' \) is tagged oppositely to \( \gamma \) at \( p \) by the assumption. Then \( x_\psi \) is a Laurent polynomial in view of classical theory for finite surfaces, and hence \( x_{\varphi}/x_{\varphi'} \) is also a Laurent polynomial in \( \{ x_1, x_2, \ldots \} \).

If \( T \) is not locally finite around \( p \), then there are infinitely many arcs of \( T \) incident to \( p \). Hence, there is a limit arc \( \psi' \in T \) incident to \( p \), i.e. an arc with one endpoint at \( p \) and another endpoint \( q \in \partial S \) (since a limit arc should have at least one endpoint at the boundary). Again by the assumption, \( \psi' \) is tagged oppositely to \( \gamma \) at \( p \). Let \( \psi \) be the arc parallel to \( \psi' \) but tagged the same as \( \gamma \) at \( p \), and let \( \mu \) be the loop going around \( \psi' \), see Fig. [8.10 b). Then \( x_{\mu} \) is a Laurent series in \( \{ x_1, x_2, \ldots \} \) as it is shown above.
(since this arc has no endpoints at punctures). By [FT, Lemma 7.10], \(x_\mu = x_\psi x_\psi'\). This implies that \(x_\psi / x_\psi' = x_\mu / (x_\psi')^2\) is also a converging Laurent series since \(\psi' \in T\).

1.b: two endpoints of \(\gamma\) are distinct ordinary punctures, both with wrong taggings.
In this case we apply the same reasoning as in Case 1.a, changing the tagging of \(\gamma\) first in one endpoint and then at the other.

1.c: both endpoints of \(\gamma\) are at the same ordinary puncture \(p\), tagged wrongly at \(p\). In this case, \(x_\gamma / x_\gamma = (e^{\pm d/2})^2\) since we add or remove a segment of length \(d\) from both ends of \(\gamma\) which implies that
\[
x_\gamma = x_\gamma (x_\psi / x_\psi')^2,
\]
where \(\psi' \in T\) is chosen in the same way as in Case 1.a.

Case 2: conjugate pair puncture. Let \(p\) be a puncture and let \(\rho, \rho' \in T\) be two curves forming a conjugate pair incident to \(p\). Let \(\alpha, \beta \in T\) be the sides of the digon containing the conjugate pair, and let \(\gamma\) be an arc coming to \(p\) through the arc \(\beta\) (see Fig. 3.11(a)). Without loss of generality, we may assume that \(\gamma\) is tagged at \(p\) in the same way as \(\rho\).

We will consider the following three cases:

2.a: only one endpoint of \(\gamma\) is a conjugate pair puncture. Let \(\sigma\) be the arc inside the digon tagged the same as \(\rho\) at \(p\). By the Ptolemy relation, we have
\[
x_\gamma x_\beta = x_\sigma x_\psi + x_\rho x_\varphi,
\]
see Fig. 3.11(a). Notice that none of the endpoints of the arcs \(\alpha\) and \(\beta\) can be a conjugate pair puncture in \(T\). This implies that \(x_\psi\) and \(x_\varphi\) are Laurent series in \(\{x_1, x_2, \ldots\}\) by Case 1. As \(\rho \in T\), \(x_\rho\) is one of \(\{x_1, x_2, \ldots\}\). The lambda length \(x_\sigma\) may be computed by the Ptolemy relation in a (self-folded) quadrilateral lying inside the digon:
\[
x_\sigma x_\mu = x_\rho (x_\alpha + x_\beta),
\]
where \(\mu\) is a loop around \(\rho\). By [FT, Lemma 7.10], \(x_\mu = x_\rho x_\rho'\), which implies
\[
x_\sigma = \frac{x_\alpha + x_\beta}{x_\rho'},
\]
Finally, as \(\beta \in T\) we see that
\[
x_\gamma = \frac{x_\sigma x_\psi + x_\rho x_\varphi}{x_\beta}
\]
is a converging Laurent series.

![Figure 3.11. Arc \(\gamma\) with an endpoint in a conjugate puncture.](image)

2.b: two endpoints of \(\gamma\) are at two distinct conjugate pair punctures \(p\) and \(q\). The proof is the same as in Case 2.a. We apply the Ptolemy relation as in Fig. 3.11(b), and since the arcs \(\psi\) and \(\varphi\) have only one of their endpoints at a conjugate pair puncture, \(x_\psi\) and \(x_\varphi\) are converging Laurent series by Case 2.a.
2.c: both endpoints of \( \gamma \) are at the same conjugate pair puncture \( p \). First suppose that \( \gamma \) intersects finitely many arcs of \( T \). Then the domain \( D_{T}^p \) of \( \gamma \) for \( T \) consists of finitely many triangles, which implies that \( x_\gamma \) is a Laurent polynomial in terms of the lambda lengths of the finitely many arcs contained in the finite surface \( D_{T}^p \).

Now, suppose that \( \gamma \) intersects infinitely many arcs. As both endpoints of \( \gamma \) are at the conjugate puncture \( p \) (i.e. there are finitely many arcs intersecting \( \gamma \) at the endpoints), there exists a limit arc \( \psi \in T \) crossing \( \gamma \). Notice that both endpoints of \( \psi \) are distinct from \( p \) (as \( p \) is a conjugate pair puncture), i.e. for any arc \( \varphi \) starting from \( p \) and terminating at an endpoint of \( \psi \), the lambda length \( x_\varphi \) is a Laurent series in \( \{x_1,x_2,\ldots\} \) in view of Cases 2.a and 2.b. Consider a lift of \( \gamma \) to the universal cover (together with a lift of \( \psi \) intersecting the lift of \( \gamma \)). Applying the Ptolemy relation as in Fig. 3.11(c), we obtain

\[
x_\gamma x_\psi = x_{\delta_1} x_{\delta_3} + x_{\delta_2} x_{\delta_4},
\]

which implies that \( x_\gamma \) is a Laurent series as each of \( x_{\delta_i} \) is for \( i = 1,2,3,4 \); and thus, \( \psi \in T \) (here \( \delta_i, i = 1,\ldots,4 \), is obtained by resolving the crossing of \( \psi \) and \( \gamma \)).

\[\square\]

**Corollary 3.39.** The coefficients of the Laurent series obtained in Theorem 3.38 are positive integers.

**Proof.** The Laurent series were constructed by applying Ptolemy relations repeatedly, where the only division performed was by the variable corresponding to an arc lying in the triangulation. Also, we have not used subtraction in the construction. So, the coefficients of the terms are positive integers by induction.

\[\square\]

**Theorem 3.40.** Let \( T \) be a fan triangulation of an infinite surface \( S \). Let \( \beta, \gamma \in S \) be two distinct arcs and \( x_\beta \) and \( x_\gamma \) be the Laurent series associated to \( \beta \) and \( \gamma \) with respect to the triangulation \( T \). Then \( x_\gamma \neq x_\beta \).

**Proof.** First, notice that given a fan triangulation \( T \), the Laurent series (in terms of the initial lambda lengths \( \{x_1,x_2,\ldots\} \) of the internal and boundary arcs of \( T \)) does not depend on the choice of initial numerical data compatible with the combinatorial structure of the triangulation. Hence, it is sufficient to show that for any two arcs \( \beta \) and \( \gamma \) in the combinatorial infinite surface \( S \), there is an admissible hyperbolic structure with converging horocycles such that the lambda lengths of \( \beta \) and \( \gamma \) are different. In what follows, we construct such a hyperbolic structure on the (combinatorial) surface \( S \).

Fix two arcs \( \beta, \gamma \in S \), and let \( T \) be an outgoing fan triangulation (as in Definition 3.26). Then each of the arcs \( \beta \) and \( \gamma \) crosses only finitely many arcs of \( T \). Hence, it is possible to choose finitely many arcs \( \alpha_1,\ldots,\alpha_n \) (where \( n = Acc(S) \) is the total number of accumulation points, see Definition 2.18) so that

- none of the arcs \( \alpha_i \) crosses \( \beta \) or \( \gamma \), and
- \( \{\alpha_i\} \) cuts \( S \) into finitely many pieces one of which is a finite subsurface \( S' \) containing only finitely many triangles of \( T \) and all the others being elementary outgoing fans.

Without loss of generality (i.e. by changing the choice of \( \alpha_i \) if needed), we may assume that \( \beta, \gamma \in S' \). Then the lambda lengths of \( \gamma \) and \( \beta \) are expressed through finitely many of \( \{x_i\} \) (more precisely, both of them are Laurent polynomials in \( \{x_i | i \in I(S')\} \), where \( I(S') \) is a finite index set of the arcs lying inside or on the boundary of \( S' \)). In particular, \( x_\gamma \neq x_\beta \) in view of the corresponding result for finite surfaces (see [FST], proof of Proposition 9.21). This implies that for some choice of \( \{x_i | i \in I(S')\} \) the corresponding lambda lengths of \( \beta \) and \( \gamma \) are different.

Now, fix a hyperbolic structure and a choice of horocycles on \( S' \) such that the lambda lengths of \( \beta \) and \( \gamma \) are different. To get the required structure on \( S \), we are left to attach finitely many outgoing fans with a unique condition that the lambda lengths of arcs \( \alpha_i \) cutting the fans are given from the boundary of \( S' \). This can be easily done by constructing the corresponding fans in the upper half-plane.

\[\square\]
3.6. **Cluster algebras from infinite surfaces.** The definition of cluster algebras from infinite surfaces is an immediate generalisation of the one associated to finite surfaces.

**Definition 3.41** (Cluster algebra from infinite triangulated surface). Let $S$ be an infinite surface together with an admissible hyperbolic structure and a choice of converging horocycles, and let $T$ be a fan triangulation of $S$. Denote by $\{\gamma_i\}$, $i \in \mathbb{N}$, the set of arcs of $T$ including infinitely many boundary arcs $\{\beta_j\}$ as a subset. Let $\{x_i\}$ be the lambda lengths of $\{\gamma_i\}$. Then
- for any (internal) arc $\gamma$ in $S$, the Laurent series giving the lambda length of $\gamma$ in terms of $\{x_i\}$ will be called the cluster variable associated to $\gamma$;
- denote by $\mathfrak{X}$ the set of all cluster variables;
- the lambda lengths (considered as independent variables) of the boundary arcs will be called boundary coefficients;
- by a cluster we mean a collection of cluster variables associated to arcs lying in one triangulation together with the boundary coefficients;
- by an initial cluster we mean the set $\{x_i\}$ corresponding to the arcs of an initial triangulation $T$ (notice that we require an initial triangulation to be a fan triangulation);
- by a seed we mean a triangulation of $S$ together with the corresponding cluster;
- mutations of triangulations are defined as in Section 2 and mutations of clusters are corresponding transformation of the set of functions.

Finally,
- the cluster algebra $\mathcal{A}(S, T)$ associated to $(S, T)$ is the $\mathbb{Z}$-algebra generated by all cluster variables, i.e.

$$
\mathcal{A}(S, T) = \left\{ \sum_{t=1}^{t} \prod_{k=1}^{n_k} x_{j_i}^{m_{i_j}} \mid n_k, j_i, t \in \mathbb{N}, m_i \in \mathbb{Z}, x_{j_i} \in \mathfrak{X} \right\}.
$$

**Remark 3.42.** A cluster variable may be defined as an infinite Laurent series, however when generating a cluster algebra we only consider finite sums of finite products of cluster variables.

**Remark 3.43.** As usual, in the case of punctured surfaces by “triangulations” we mean “tagged triangulations”.

**Remark 3.44** (Independence of triangulation). The definition of cluster algebra $\mathcal{A}(S, T)$ given above depends on the choice of a fan triangulation, however one can check that given two fan triangulations $T_1$ and $T_2$ of the surface $S$, there exists a homomorphism $f$ of algebras $\mathcal{A}_1 = \mathcal{A}(S, T_1)$ and $\mathcal{A}_2 = \mathcal{A}(S, T_2)$.

Indeed, by Theorem 3.40 we see that a cluster variable is uniquely determined by the corresponding arc $\gamma \in S$. Construct the bijection between the sets of cluster variables of $\mathcal{A}_1$ and $\mathcal{A}_2$ by mapping the cluster variable of $\mathcal{A}_1$ associated to an arc $\gamma$ to the cluster variable of $\mathcal{A}_2$ associated to $\gamma$.

To see that the bijection $f$ defines an algebra homomorphism, notice that cluster variables of both algebras satisfy the same relations coming from the geometry of $S$, i.e.
- Ptolemy relations, and
- limit relations (see Corollary 3.13 stating that the lambda lengths of the converging sequence of arcs converge).

Moreover, for any given initial fan triangulation $T$, every cluster variable may be obtained from the initial cluster variables in finitely many steps (where at every step we apply either a Ptolemy relation or a limit relation to the variables obtained in the previous steps).

**Remark 3.44** leads to the following definition.

**Definition 3.45** (Cluster algebra from an infinite surface). By a cluster algebra $\mathcal{A}(S)$ from an infinite surface $S$ we mean the cluster algebra obtained from any fan triangulation of $S$. 

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4. Infinite staircase snake graphs

In this section, we will generalise the construction of snake graphs, introduced by Musiker, Schiffler and Williams in [MSW], for arcs in an infinite surface and use them to give an expansion formula for cluster variables. Since we consider cluster algebras with initial variables associated to fan triangulations, we will introduce the construction of snake graphs in this setting for simplicity, though infinite snake graphs associated to non-fan triangulations can be given in the same way.

4.1. Construction of snake graphs associated to fan triangulations. Let \( S \) be an infinite surface with a fan triangulation \( T \) and \( \gamma \) be an arc in \( S \). Consider the domain \( D_T^\gamma \) of \( \gamma \) lifted to the universal cover. The domain \( D_T^\gamma \) is a finite union of elementary fans \( F_i \) where \( 1 \leq i \leq k \) for some \( k \in \mathbb{N} \).

Fix an orientation on \( \gamma \). We will construct the snake graph of \( \gamma \) in several steps depending on the complexity of the collection of crossings of \( \gamma \) with \( T \). In the case of finitely many crossings, our construction coincides with that of [MSW]. Suppose \( \{ \tau_i \mid i \in I \} \) is the collection of arcs in \( T \) that \( \gamma \) crosses in the universal cover (remark that \( \tau_i \) may be a lift of the same arc as \( \tau_j \), for some \( i \) and \( j \)).

**Step I. Finitely many crossings of \( \gamma \) with \( T \):**
- to each crossing of \( \gamma \) in \( T \), associate a weighted tile which is a square of a fixed side length:
  - if \( \gamma \) crosses an arc \( \tau_i \), consider the quadrilateral \( Q_i \) in \( T \) which contains \( \tau_i \) as a diagonal; the corresponding tile \( G_i \) will have \( \tau_i \) in the diagonal and weights on the edges induced by the labelling on \( Q_i \), see Figure 4.1;
- glue successive tiles as follows: the tiles \( G_i \) and \( G_{i+1} \) are glued along the edge with weight \( \tau_b \), where \( \tau_b \) is the third side of the triangle bounded by \( \tau_i \) and \( \tau_{i+1} \): arrange the gluing so that the diagonals \( \tau_i \) and \( \tau_{i+1} \) of the tiles \( G_i \) and \( G_{i+1} \) connect the top-left to the bottom-right corners of the tiles, see Figure 4.1.

**Step II. Crossing an infinite elementary fan:**
- associate a tile to the first crossing of an almost elementary fan;
- glue each tile associated to successive crossings as in the finite case;
- obtain an infinite graph \( \overline{G} \);
- associate a limit tile \( G_{lim} \) at the end of \( \overline{G} \) to indicate that we approach to the limit arc \( \gamma_s \) of the fan:
  - \( G_{lim} \) is a dotted square of the same side length;
  - \( \overline{G} \) approaches \( G_{lim} \) from the bottom;
  - associate the weight * to the top edge of \( G_{lim} \), see left of Fig. 4.2.

**Step III. Crossing a limit arc:**
- if an arc $\gamma$ crosses a limit arc $\gamma^*$, we associate a regular tile to this crossing and glue it to $\mathcal{G}$ according to the local configuration of $\gamma$ and $\gamma^*$ in $(S, T)$, see Fig. 4.2 (right).

**Step IV. Crossing a finite union of infinite elementary fans and polygons:**
- construct finitely many (finite or one-sided infinite zig-zag) snake graphs $\{\mathcal{G}_i\}_{i=1}^k$ for elementary fans $\mathcal{F}_i$ in $\mathcal{D}_\gamma$ by Step I and II;
- glue each $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ at the common ends according to the local configuration of the change of the fans from $\mathcal{F}_i$ to $\mathcal{F}_{i+1}$, see Fig. 4.3;

**Step V. Completing the construction of snake graph $\mathcal{G}_\gamma$ associated to $\gamma$:**
- remove the diagonals in all tiles.

After this process, we obtain an infinite snake graph $\mathcal{G}_\gamma$ associated to $\gamma$, see Fig. 4.3 for an example.

**Remark 4.1.**
- Changing the orientation of an arc $\gamma$ gives rise to isomorphic snake graphs, one obtained from the other by a reflection. On the other hand, the choice of orientation of $\gamma$ induces a total order on the tiles of $\mathcal{G}_\gamma$.
- Limit tile of a snake graph has no face weight and only one of its edges has a weight. This edge is labelled by the limit arc.

**Observation 4.2.**
(a) If an arc $\gamma \in S$ crosses arcs forming a zig-zag in a triangulation, then the corresponding piece in the snake graph is straight and if it forms a fan, then the corresponding piece in the snake graph is zig-zag, see Fig. 4.3 (we acknowledge that this terminology is a bit confusing but it will be clear from the context which meaning of zig-zag we refer to).
By construction, there is no tile in a snake graph which is a limit tile for two consecutive infinite subgraphs of an infinite staircase snake graph (approaching the tile from two distinct sides). Indeed, if an arc $\gamma$ crosses away from an elementary fan, then $\gamma$ crosses at least a limit arc $\gamma_*$ for which we associate a regular tile $G_*$ in $G_\gamma$. Hence a tile associated to a limit arc separates the limit tile from the rest of the snake graph. Therefore, $G_\gamma$ cannot have a two-sided limit tile.

\[ \text{Figure 4.4. Crossing a fan in a surface gives rise to a zig-zag in the snake graph and crossing a zig-zag in a surface gives rise to a straight piece in the snake graph.} \]

The next proposition follows directly from the construction of snake graphs.

**Proposition 4.3.** Let $T$ be a fan triangulation of $S$ and $\gamma$ be an arc. Then $G_\gamma$ is either
(a) a finite snake graph, or
(b) a finite union of finite and infinite one-sided zig-zag snake graphs.

**Definition 4.4 (Infinite staircase snake graph).** Let $T$ be a fan triangulation and $\gamma$ be an arc such that $|\gamma \cap T| = \infty$. We will call the infinite snake graph $G_\gamma$ associated to the arc $\gamma$, the infinite staircase snake graph $G_\gamma$ associated to $\gamma$ (see Fig. 4.3 and 4.7 (d) for examples).

We now extend the notion of sign function of [CS] to infinite staircase snake graphs.

**Definition 4.5 (Sign function).** A sign function $f$ of an infinite staircase snake graph $G$ is a map from the edges of $G$ to $\{+,-\}$ such that
- the bottom and the right edges of a tile $G_i$ have the same sign which is opposite to that of the top and the left edges;
- fixing a sign function on a single edge of a snake graph induces a sign function on the whole snake graph by extending with the rule $\begin{array}{c} - \varepsilon \\ \varepsilon \end{array}$ where $\varepsilon \in \{+,-\}$ and applying the limit rules when needed; in particular, the sign function on a limit tile is induced by the sign of the interior edges of the zig-zag approaching the limit tile;
- the sign function of a limit tile induces a sign function in the infinite zig-zag following this tile.

**Remark 4.6.** A sign function of an infinite zig-zag snake graph $G$ has either all $+$ or all $-$ assigned to the interior edges of $G$.

By construction, two successive tiles of $G$ have different orientations relative to the orientation of the surface $S$. We will use this to define edge positions in the tiles relative to the orientation of the surface and to the sign function.

**Definition 4.7 (Orientation of a limit tile).** The orientation of a limit tile is induced by the orientation of the tiles in the zig-zag approaching it, i.e. the orientation of the limit tile coincides with the orientation of the tiles having the previous tile attached at the bottom edge, see Fig. 4.5.

**Definition 4.8 (Edge positions in the tiles).** Let $G_i$ be a tile of an infinite staircase snake graph $G$ together with its orientation induced from $G$ and $f$ be a sign function of $G$. The edges $a,b,c,d$ of $G_i$ are in positions I, II, III, IV, respectively if
- $a,b,c,d$ follow each other in the direction of the orientation of $G_i$, and
- $a$ is the first edge of $G_i$ with sign “$-$” in $f(G)$ (when walked in the direction of the orientation).
**Figure 4.5.** The induced orientation of the limit tile: two different shadings represent two different orientations of the tiles with respect to the surface.

**Figure 4.6.** Edge positions defined depending on the orientation and the sign function.

See Fig. 4.6.

**Convention 4.9** (On drawing snake graphs). We always assume that the first tile is oriented clockwise and that the second tile of a snake graph is attached to the first from the right.

Next, we will extend the notion of perfect matchings to infinite staircase snake graphs.

**Definition 4.10.** A perfect matching $P$ of an infinite staircase snake graph $G$ is a set of edges of $G$ such that each vertex in $G$ is incident to precisely one edge in $P$ and an edge of a limit tile is in the perfect matching if one of the following applies:

- a limit tile $G_{\text{lim}}$ is followed by a regular tile $G_*$ and the edge between $G_{\text{lim}}$ and $G_*$ is in the matching of $G_*$;
- limit edge condition: if $G_{\text{lim}}$ is a limit tile of the tiles $G_i$, for $i \in \mathbb{N}$, and there exists $k \in \mathbb{N}$ such that an edge of $G_i$ at the position $Y \in \{I, II, III, IV\}$ is in $P$ for all $i > k$, then the edge of $G_{\text{lim}}$ at the position $Y$ is also in $P$, see Example 4.12.

**Remark 4.11.**

- By definition, the vertices of a limit tile are either matched by the limit condition or by an edge of a successive tile (if such tile exists). We cannot add any other edge of a limit tile to complete a set of edges to a perfect matching of an infinite snake graph.
- By Observation 4.2 (b), a limit tile cannot be a two-sided limit which implies that two opposite boundary edges of a limit tile are never in a matching of an infinite staircase snake graph, compare with Example 4.12 (b).

**Example 4.12.**

(a) The set of edges $P$ in Fig. 4.7 (a) is a perfect matching of the snake graph $G$ since the top edges of every odd tile and the right edge of every even tile is in the matching which induces the top edge of the limit tile in the matching (since it has the same orientation as the odd tiles). Indeed, comparing to Fig. 4.6 we see that the edge at position II is in the matching of $G_i$ for all $i$. Therefore, by the limit edge condition the edge at position II of the limit tile is also in the matching, i.e. the top edge of the limit tile is in the matching. Hence, every vertex of $G$ is incident to precisely one edge in $P$ and thus $P$ is a perfect matching.

(b) (A non-perfect matching example) We consider the set of edges $P'$ consisting of the edge at position I of $G_i$ for each $i$ together with the edge at position I of the limit tile (obtained by the limit rule) as illustrated in Fig. 4.7 (b). Then the top-right vertex of the limit tile $G_{\text{lim}}$ is not incident to any edge in $P'$, and hence $P'$ is not a perfect matching of $G$. 

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(c) The set of edges of $P''$ in Fig. 4.7(c) is a perfect matching of the bi-infinite zig-zag snake graph $G'$ since two vertices of each limit tiles are matched by the limit condition and other vertices of the limit tiles are matched by an edge in the adjacent tile $G_*$.

(d) The set of edges of $P'''$ in Fig. 4.7(d) forms a perfect matching of an infinite staircase snake graph.

Notation 4.13. Given a snake graph $G$,
- we denote by $\text{Match } G$ the set of all perfect matchings of $G$;
- we sometimes use the shorthand notation $P \models G$ to indicate that $P \in \text{Match } G$.

Lemma 4.14. Let $G = \cup_{i \in \mathbb{N}} G_i$ be an infinite one-sided zig-zag snake graph (as in Fig. 4.7) and let $P \in \text{Match } G$ be a perfect matching. Then there exists a unique $i \in \mathbb{N}$ such that $i$ is the first tile of $G$ with the property that the restriction $P|_{G_i}$ is a perfect matching of $G_i$. Moreover, the matching $P$ is uniquely determined by the matching $P|_{G_i}$ of $G_i$.

Proof. Suppose that no tile of $G$ is completely matched. The left edge of the first tile $G_1$ must be in the matching because otherwise in order to cover the bottom-left vertex we would have to take the bottom edge in the matching of $G_1$ which would require to take the top edge giving rise to a complete matching on $G_1$. Similarly, this forces the bottom edge of $G_2$ to be in the matching. Continuing in this manner successively, we construct the pattern in Fig. 4.7(b) which does not lead to a perfect matching as it is shown in Example 4.12 (b). Hence, there exists $i \in \mathbb{N}$ such that $P|_{G_i}$ is a perfect matching of $G_i$. We can choose $i$ so that no tile $G_k$ is completely matched for $k < i$.

![Figure 4.8. Corner snake graph.](image)

Suppose that $i > 1$. Without loss of generality, suppose $G_i$ is placed on the top-left of the corner piece $(G_{i-1}, G_i, G_{i+1})$, see Fig. 4.8 (a) (bottom-right case may by considered similarly). Then the matching $P$ contains either the two vertical or the two horizontal edges of $G_i$ as in Fig. 4.8 (b) and (e). First, assume that the two vertical edges of $G_i$ are in the matching. This enforces to have the bottom edge of the tile $G_{i-1}$ and the right edge of $G_{i+1}$ in the matching, see Fig. 4.8 (c). The matching on $G_{i-1}$ enforces the left edge of $G_{i-2}$ and the matching on $G_{i+1}$ enforces the top edge of $G_{i+2}$, see Fig. 4.8 (d). Continuing in this manner, all the bottom edges of the tiles $G_{i-2k+1}$ and all the left edges of the tiles $G_{i-2k}$ for $1 \leq k \leq \left\lfloor \frac{i}{2} \right\rfloor$ are in the matching of $G$. Similarly, all the right edges of the tiles $G_{i+2j-1}$ and all the top edges of the tiles $G_{i+2j}$ for
\( j \in \mathbb{N} \) are in the matching of \( \mathcal{G} \). Moreover, the limit tile \( G_{\text{lim}} \) have the top edge in the matching by the limit condition, hence this is a perfect matching of \( \mathcal{G} \) which is uniquely determined by a matching on the tile \( G_i \).

If \( G_i \) has two horizontal edges in the matching (see Fig. 4.8 (f)), then this forces the bottom of the tile \( G_{i-1} \) in the matching, see Fig. 4.8 (f). But then \( P|G_{i-1} \) is a matching of \( G_{i-1} \) and \( i \) is not the first position with a complete matching, hence we argue as above for \( i-1 \).

In the case of \( i = 1 \), each of two possible perfect matchings of \( G_i = G_1 \) (see Fig. 4.9) extends uniquely to a perfect matching of \( \mathcal{G} \).

![Figure 4.9. Initial segments of two perfect matchings of infinite one-sided zig-zag snake graph with two complete matchings of the first tile \( G_1 \).](image)

**Definition 4.15** (Height of a perfect matching). Let \( \mathcal{G} \) be a one-sided zig-zag snake graph, let \( P \) be a matching of \( \mathcal{G} \), and let \( i \) be the first position of a tile of \( \mathcal{G} \) such that \( P|G_i \) is a perfect matching of \( G_i \) (this tile exists and is unique by Lemma 4.14). If \( i > 1 \), we say that \( P \) is a perfect matching of height \( i \). If \( i = 1 \), we will associate height 0 and height 1 to the perfect matchings of the form shown in Fig. 4.9 (a) and (b) respectively.

4.2. **Laurent series associated to snake graphs.** [MSW] introduced an explicit formula for cluster variables of surface cluster algebras which is parametrised by perfect matchings of snake graphs associated to arcs in a finite surface. In this section, we aim to generalise this formula for infinite surfaces with fan triangulations.

**Definition 4.16** (Laurent series associated to \( \mathcal{G}_\gamma \)). Let \( \mathcal{S} \) be an infinite surface and \( T \) be a fan triangulation of \( \mathcal{S} \). Let \( \gamma \) be an arc in \( \mathcal{S} \) and \( \mathcal{G}_\gamma \) its snake graph. Let \( P \) be a perfect matching of \( \mathcal{G}_\gamma \). Then

- assign a formal variable \( x_\gamma \) to every arc \( \tau_\gamma \) (including the limit arcs and the boundary arcs) of the triangulation \( T \);
- the (possibly infinite) **weight monomial** \( x(P) \) of \( P \) is given by
  \[
  x(P) = \prod_{e_i \in P} x_{e_i},
  \]
  where \( x_{e_i} = x_j \) for an edge \( e_i \in P \) with weight \( \tau_j \);
- the **crossing monomial** of \( \gamma \) with \( T \) is given by
  \[
  \text{cross} (\gamma, T) = \prod_{j \in J} x_j,
  \]
  where \( J \) is the index set of the tiles in \( \mathcal{G}_\gamma \), i.e. \( \mathcal{G}_\gamma = \bigcup_{j \in J} G_j \) and
  \[
  x_j = \begin{cases} x_k & \text{if } G_j \text{ has weight } \tau_k \\ 1 & \text{if } G_j \text{ is a limit tile} \end{cases}
  \]
- the **Laurent series associated to \( \mathcal{G}_\gamma \)** is defined by
  \[
  x_{\mathcal{G}_\gamma} = \sum_{P \in \text{Match } \mathcal{G}_\gamma} x(P) \text{cross} (\gamma, T),
  \]
  where \( x(P) \) is the weight monomial of \( P \) and \( \text{cross} (\gamma, T) \) is the crossing monomial of \( \gamma \) with \( T \).
As in the finite case, $x_G$ gives an explicit formula for cluster variables. We will first show this for arcs lying inside a single fan.

**Proposition 4.17.** Let $\mathcal{F} = \{\gamma_i, \gamma_{i+1} \mid i \in \mathbb{Z}_{\geq 0}\} \cup \{\gamma_0\}$ be an infinite incoming elementary fan and $\gamma$ be the arc which crosses $\gamma_i$ for $i \in \mathbb{N}$, see Fig. 4.10. Let $x_\gamma$ be the cluster variable associated to $\gamma$ and $G_\gamma$ be the snake graph of $\gamma$. Then $x_\gamma = x_G$.

**Proof.** The snake graph $G_\gamma$ associated to $\gamma$ is an infinite one-sided zig-zag. By Definition 4.15, perfect matchings of $G_\gamma$ are indexed by $\mathbb{Z}_{\geq 0}$ corresponding to the height $i$ of a perfect matching $P_i$. We will show $x(P_i)$ cross $(\gamma, T)$ = $x_0 x_1 / x_2 / \ldots / x_{i-1} / x_i, x_i, x_{i+1} / \ldots / x_0$, for $i \in \mathbb{Z}_{\geq 0}$, which will imply $x_G = x_0 x_1 \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{x_{i+1}}{x_i x_{i+1}} = x_\gamma$, where the last equality follows from Proposition 3.14 (4). To show the claim, consider the perfect matching $P_i$ given in the Fig. 4.10. By definition, we have

\[
x(P_i) \bigg| \text{cross} (\gamma, T) = \frac{x_0 x_1 \ldots x_i x_{i+1} \ldots x_k}{x_0 x_1 \ldots x_i x_{i+1} \ldots x_k} = \frac{x_0 x_1 x_{i+1}}{x_i x_{i+1}},
\]

as required. \hfill \square

**Figure 4.10.** Arc $\gamma$ inside an incoming fan (left) and the corresponding one-sided infinite snake graph $G_\gamma$ (right). $G_\gamma$ is given with a perfect matching of height $i$.

We are now ready to present the main theorem of this section.

**Theorem 4.18.** Let $T$ be a fan triangulation of an infinite unpunctured surface $S$ and let $\gamma$ be an arc. Let $x_\gamma$ be the cluster variable associated to $\gamma$ and $G_\gamma$ be the snake graph of $\gamma$. Then $x_\gamma = x_G$.

**Proof.** Abusing the notation, we will denote by $\gamma$ the lift of an arc $\gamma$ to the universal cover. Let $D^T_\gamma$ be the domain of $\gamma$ in the universal cover. By Remark 2.29, the domain $D^T_\gamma$ is a finite union of elementary fans. Similarly to the proof of Theorem 3.38, we will argue by induction on the number of elementary fans in $D^T_\gamma$.

For the base of the induction, we will consider the case when $D^T_\gamma$ is a fan. Then it is either a finite fan or an infinite elementary fan. In the first case, the snake graph is finite, hence the
statement follows from [MSW]. In the latter, the snake graph is a one-sided infinite zig-zag and $x_\gamma = x_{\gamma^*_s}$ by Proposition 4.17.

Assume that $x_{\gamma^*_s} = x_\delta$ for every arc $\delta$ crossing strictly less than $k$ fans. Now suppose $\gamma$ is an arc crossing $k$ fans in the universal cover. We will denote these fans by $F_i^{(i)}$ for $1 \leq i \leq k$. Fix an orientation on $\gamma$ and let $\gamma_s$ be the arc separating the fans $F_i^{(k-1)}$ and $F_i^{(k)}$. Note that $\gamma_s$ might be a limit arc. We have one of the following main configurations in the neighbourhood of $\gamma_s$ in $D_\gamma$ (see also Fig. 4.11):

(I) the fans $F_i^{(k-1)}$ and $F_i^{(k)}$ are attached to the same end of $\gamma_s$;

(II) the fans $F_i^{(k-1)}$ and $F_i^{(k)}$ have a source at opposite ends of $\gamma_s$.

Figure 4.11. Two principle ways to change an elementary fan (each of the two fans in the figures may be finite or infinite).

**Case (I):** Suppose that the last change of fan in $D_\gamma$ is of type (I). We will also assume that both of the fans $F_{k-1}$ and $F_k$ are infinite. Then the snake graph of $\gamma$ is of the form given in Fig. 4.12.

Denote the arcs in $D_\gamma$ as shown in the figure, denote also the snake subgraph corresponding to the arc $\gamma$ by $G$ and the snake subgraph corresponding to the crossing of $\gamma$ with the fan $F_i$ by $G_i$.

Figure 4.12. Snake graph associated to $\gamma$ in Case (I).

The snake graphs $G_{k-1}$ and $G_k$ are one-sided infinite zig-zag snake graphs since they are associated to infinite zig-zag fans $F_{k-1}$ and $F_k$. Note that since the fans $F_{k-1}$ and $F_k$ have
the same source, the corresponding glueing in \( G_\gamma \) between \( F_{k-1}, \gamma_s \) and \( F_k \) is a zig-zag and \( G = G_{\gamma,T} \cup G_{k-1} \cup G_s \cup G_k \).

By Ptolemy relations, we have the identity

\[
x_{\gamma} x_s = x_{\beta_1} x_{\nu} + x_{\beta_2} x_{\tau}
\]

in terms of lambda lengths of the arcs \( \gamma_s, \beta_1, \gamma_s, \beta_2, \) and \( \tau \), see Fig. 4.12. We will show that the corresponding identity

\[
x_{G} x_{G_s} = x_{G_{\beta_1}} x_{G_{\nu}} + x_{G_{\beta_2}} x_{G_{\tau}}
\]

in terms of expansion formula for the associated snake graphs holds. Observe that we have \( x_{\beta_i} = x_{G_{\beta_i}} \) for \( i = 1, 2 \), \( x_{\tau} = x_{G_{\tau}} \), \( x_s = x_{G_s} \) and \( x_{\nu} = x_{G_{\nu}} \) by the induction hypothesis.

In order to show the identity (2), we will reorganise the set of perfect matchings of \( G_* \) by considering those that have the vertical boundary edge \( a \) or the horizontal boundary edge \( b \) of the tile \( G_s \) in a matching of \( G_\gamma \):

\[
\text{Match } G_\gamma = \text{Match } \left( \begin{array}{c}
\text{Match } G_{\beta_1} \\
\text{Match } G_{\nu}
\end{array} \right) \cup \text{Match } \left( \begin{array}{c}
\text{Match } G_{\beta_2} \\
\text{Match } G_{\tau}
\end{array} \right)
\]

\[
(3) \quad = ( \text{Match } G_{\beta_1} \cup P) \cup ( \text{Match } G_{\nu} \cup G_{\beta_2}).
\]

Note that we obtain unique matchings \( \tilde{P} \) and \( \bar{P} \) on \( G^{(k)} \cup \{a\} \) and \( (G^{(k-1)} \setminus G_1) \cup \{b\} \), respectively. Indeed, when the edge \( a \) is in the matching, we must have the top of the successive limit tile in the matching as well. Hence the top of every even (or odd) tile in \( G^{(k-1)} \setminus G_1 \) must be in the matching. This also requires that the right edge of every odd (resp. even) tile is in the matching, compared to Example 4.12. Similarly, if the edge \( b \) is in the matching, this gives rise to the unique matching \( \bar{P} \).

We will also use the following rearrangement of cross \( (\gamma, T) \):

\[
\text{cross } (\gamma, T) = \text{cross } (\beta_1, T) \cdot x_s \cdot \text{cross } (\beta_2, T)
\]

\[
= \text{cross } (\gamma, T) \cdot x_1 x_2 \ldots x_s \cdot \text{cross } (\beta_2, T)
\]

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Applying equation (3) of perfect matchings of $G_\gamma$, we obtain:

$$x_{G_\gamma} \overset{(def)}{=} \sum_{P \in G_\gamma} x(P) \quad \text{cross} (\gamma, T)$$

$$= \left( \sum_{P_1 \in G_{\beta_1}} x(P_1) \right) x(\bar{P}) + \left( \sum_{P_2 \in G_{\beta_2}} x(P_2) \right) x(\bar{P}) \quad \text{cross} (\gamma, T)$$

$$= \left( \sum_{P_1 \in G_{\beta_1}} x(P_1) \right) \frac{1}{x_*} \left( \frac{x(\bar{P})}{x_1 x_2 x_3 \cdots} \right) + \left( \sum_{P_2 \in G_{\beta_2}} x(P_2) \right) \frac{1}{x_*} \left( \frac{x(\bar{P})}{x_1 x_2 x_3 \cdots} \right)$$

$$\overset{(def)}{=} (x_{\beta_1}) \left( \frac{1}{x_*} \right) \frac{x_0}{x_1} \frac{x_3}{x_4} \cdots \frac{x_{\gamma}}{x_{\beta_2}} \cdots + (x_{\gamma}) \frac{x_1}{x_2} \frac{x_3}{x_4} \cdots \frac{1}{x_*} \frac{x_{\beta_2}}{x_{\gamma}}$$

$$= \frac{x_{\beta_1} x_{0'}}{x_*} + \frac{x_0 x_{\beta_2}}{x_*}$$

$$= x_{\gamma}$$

where the last equality follows from (1) since $x_*$ is an initial variable.

The proof of the remaining cases where only one of the fans is infinite is given in the same way. The key consideration is that the snake graph $G_\gamma$ will be exactly as in Fig. 4.12 except that one of the zig-zag subgraphs $G^{(k-1)}$ or $G^{(k)}$ will be finite.

**Case (II):** Consideration of this case follows similar steps to Case (I) except that the local configuration at $G_\gamma$ will be different. Now suppose that last change of fan in the domain of $\gamma$ is of Type (II) and that both fans around $\gamma_\ast$ are infinite. Then the snake graph of $\gamma$ is of the form given in Fig. 4.13. Denote the arcs of $D_\gamma$ as in the figure, denote also the snake subgraph of $G_\gamma$ corresponding to the arc $\gamma$ by $G_{\gamma}$ and the snake subgraph corresponding to the crossing of $\gamma$ with the fan $F_1$ by $G_{\gamma}$.

![Figure 4.13. Snake graph associated to $\gamma$ in Case (II).](image)

The snake graphs $G_{k-1}$ and $G_k$ are one-sided infinite zig-zag snake graphs since they are associated to the infinite zig-zag fans $F_{k-1}$ and $F_k$. Note that since the fans $F_{k-1}$ and $F_k$ have
distinct sources, the corresponding glueing in $G_{\gamma}$ between $G_{k-1}$, $G_*$ and $G_k$ in $G = \overline{G} \cup G_{k-1} \cup G_* \cup G_k$ is straight.

Using the notation as in Fig. 4.13 and applying the Ptolemy relation, we have the identity

$$x_{\gamma}, x_* = x_{\beta_1}, x_{\beta_2} + x_{\gamma'} x_{\tau}$$

in terms of lambda lengths of the arcs $\gamma, \gamma, \beta_1, \beta_2, \gamma_0$, and $\tau$. We will show the identity

$$x_{\gamma}, x_* = x_{\beta_1} x_{\beta_2} + x_{\gamma_0} x_{\tau_0}.$$

Observe that we have $x_{\beta_i} = x_{\tilde{G}_i}^t$, for $i = 1, 2$, and $x_{\tau} = x_{\tilde{G}_c}$ by the induction hypothesis.

To show the identity (7), we will reorganise the set of perfect matchings of $G_{\gamma}$ by considering those having the two boundary edges of the tile $G_*$ in the matching or not:

$$\text{Match } G_{\gamma} = \text{Match } \left( \begin{array}{cc}
\text{Match } G_{\gamma_1} & \text{Match } G_{\gamma_2}
\end{array} \right) \cup \text{Match } \left( \begin{array}{cc}
\text{Match } G_{\gamma_3} & \text{Match } G_{\gamma_4}
\end{array} \right)$$

$$= \text{Match } \left( \begin{array}{cc}
\text{Match } G_{\beta_1} \cup \text{Match } G_{\beta_2}
\end{array} \right) \cup \text{Match } \left( \begin{array}{cc}
\text{Match } G_{\tau} \cup \tilde{P}
\end{array} \right)$$

Observe that we obtain the unique matching $\tilde{P}$ of $G_{\gamma} \setminus \overline{G}$ as in Case (I). Applying equation (8) of perfect matchings of $G_{\gamma}$, we get:

$$x_{\gamma} = \sum_{P \in \overline{G}} \frac{x(P)}{\text{cross } (\gamma, T)}$$

$$= \left( \sum_{P \in \overline{G}_{\beta_1}} x(P_1) \right) \left( \sum_{P \in \overline{G}_{\beta_2}} x(P_2) \right) + \left( \sum_{P \in \overline{G}_{\tau}} x(P) \right) x(\tilde{P})$$

$$= \left( \sum_{P \in \overline{G}_{\beta_1}} x(P_1) \right) \left( \sum_{P \in \overline{G}_{\beta_2}} x(P_2) \right) \times \frac{x(\tilde{P})}{x_1 x_2 x_3 \ldots x_* \ldots x_2 x_1}$$

$$= (x_{\gamma_1}) (x_{\gamma_2}) + (x_{\gamma}) \frac{x_{\gamma_0} x_{\tau_0}}{x_*}$$

$$= x_{\gamma}$$

where the last equality follows from (6) as $x_*$ is an initial variable.
4.3. Skein relations. In this section we will consider fan triangulations of unpunctured infinite surfaces. The goal is to generalise skein relations to infinite case under this setting. To establish these relations, we will generalise the snake graph calculus of \([CS]\) to the infinite case.

For a finite unpunctured surface \(\mathcal{S}\), given two arcs \(\gamma_1, \gamma_2 \subset \mathcal{S}\) and a crossing of them, one can resolve the crossing by locally replacing the crossing \(\times\) with the pair of segments \(\yup{\down}{\down}{\up}\) or \(\ydown{\updown}{\updown}\). This gives rise to four new (generalised) arcs \(\gamma_3, \gamma_4, \gamma_5, \gamma_6\) (see Definition 4.19) corresponding to the two different ways of smoothing the crossing. As shown initially in \([MS]\), the corresponding elements \(x_{\gamma_1}, x_{\gamma_2}, \ldots, x_{\gamma_6}\) in the cluster algebra satisfy the skein relations given by

\[
x_{\gamma_1}x_{\gamma_2} = x_{\gamma_3}x_{\gamma_4} + x_{\gamma_5}x_{\gamma_6}.
\]

Again in the finite case, another proof of skein relations is given combinatorially in \([CS]\) in terms of snake graphs associated to a pair of crossing arcs and those obtained by smoothing a crossing. The main idea of the proof is given by

- detecting the region of a crossing in the surface in terms of the associated snake graphs (this notion is called crossing overlap in \([CS]\)):
  - crossing in a (finite or infinite) surface occurs exactly in one of the 3 ways given in Fig. 4.14. In the infinite surface case, the same configuration remains fixed and the only difference is that the region where the crossing occurs might be a triangulated infinite polygon in the universal cover of the surface. However, the crossing condition only depends on the local configuration on the boundary of this region, and therefore, the whole argument in terms of snake graphs immediately generalises when infinite surfaces are considered;

- giving a bijection between perfect matchings of \(\mathcal{G}_1 \sqcup \mathcal{G}_2\) and \((\mathcal{G}_3 \sqcup \mathcal{G}_4) \cup (\mathcal{G}_5 \sqcup \mathcal{G}_6)\) where \(\mathcal{G}_i\) for \(i = 1, \ldots, 6\) are the snake graphs associated to the arcs obtained by smoothing a crossing: this notion is called resolution or grafting in \([CS]\):
  - the essential idea of the proof of the bijection between perfect matchings of a pair of snake graphs is to find a switching position in an overlap under consideration. When infinite staircase snake graphs are considered, one might need to determine the switching position in an infinite snake graph. Although it may not be obvious at first sight, this could be done in a similar manner to that of the finite version. This is due to the fact that we only consider infinite staircase snake graphs (Proposition 4.13) and perfect matchings of infinite zig-zag snake graphs are characterised explicitly (Lemma 3.1.4).

**Definition 4.19 (Generalised arc).** By a generalised arc we mean any curve on \(\mathcal{S}\) with endpoints in the marked points of \(\mathcal{S}\) that is not contractible to a boundary segment and not cutting out an unpunctured monogon. As usual, the curves are considered up to isotopy.

**Definition 4.20 (Laurent series associated to generalised arcs).** Let \(\mathcal{S}\) be an infinite surface and \(T\) be a fan triangulation of \(\mathcal{S}\). Let \(\gamma\) be a generalised arc in \(\mathcal{S}\) and \(\mathcal{G}_\gamma\) be the infinite snake...
graph associated to the lift of $\gamma$ in the universal cover. The *Laurent series* $x_\gamma$ associated to $\gamma$ is given by the Laurent series associated to $G_\gamma$.

**Remark 4.21.** The assumption that a generalised arc does not cut out an unpunctured monogon can be removed from the definition, and in this case one could use smoothings of self-crossings to associate Laurent series as in [MSW2].

If $\gamma$ is a generalised arc, then $x_\gamma$ is not a cluster variable; however, applying induction on the number of fans crossed by the lift of $\gamma$ as in Theorem 4.18 one can show that $x_\gamma$ is an element of the associated cluster algebra.

**Proposition 4.22.** The Laurent series associated to a generalised arc is an element of the cluster algebra $A(S)$. By following the same arguments as in [CS], we obtain skein relations for cluster algebras associated to unpunctured infinite surfaces.

**Theorem 4.23.** Let $S$ be an infinite unpunctured surface and $\gamma_1, \gamma_2$ be crossing generalised arcs. Let $\gamma_3, \ldots, \gamma_6$ be the (generalised) arcs obtained by smoothing a crossing of $\gamma_1$ and $\gamma_2$ and $x_1, \ldots, x_6$ be the corresponding elements in the associated infinite surface cluster algebra $A(S)$. Then the identity

$$x_1x_2 = x_3x_4 + x_5x_6$$

holds in $A(S)$.

5. Properties of infinite rank cluster algebras

In this section we generalise some properties of (surface) cluster algebras to the setting of cluster algebras from infinite surfaces.

**Theorem 5.1.** Let $S$ be an infinite surface and $A(S)$ be the corresponding cluster algebra. Then

- seeds are uniquely determined by their clusters;
- for any two seeds containing a particular cluster variable $x_\gamma$ there exists a mutation sequence $\mu^* \in \{\mu(0), \mu(\infty)\}$ such that $x_\gamma$ belongs to every cluster obtained in the course of mutation $\mu^*$;
- there is a cluster containing cluster variables $\{x_i | i \in I\}$, where $I$ is a finite or infinite index set, if and only if for every choice of $i, j \in I$ there exists a cluster containing $x_i$ and $x_j$.

Moreover, if $T$ is a fan triangulation of $S$ then

- the “Laurent phenomenon” holds, i.e. any cluster variable in $A(S, T)$ is a Laurent series in cluster variables corresponding to the arcs (and boundary arcs) of $T$;
- “Positivity” holds, i.e. the coefficients in the Laurent series expansion of a cluster variable in $A(S, T)$ are positive.

If in addition the surface $S$ is unpunctured then

- the exponents in the denominator of a cluster variable $x_\gamma$ corresponding to an arc $\gamma$ are equal to the intersection numbers $|\gamma \cap \tau_i|$ of $\gamma$ with arcs $\tau_i$ of $T$.

**Proof.** First three properties follow from the bijection between cluster variables and arcs in $S$ established in Theorem 3.40.

The other three properties are immediate consequences of Theorem 4.18 since the expansion formula is a Laurent series by definition and the coefficients of the terms are parametrised by perfect matchings of snake graphs. The coefficients are positive integers in view of Corollary 3.39. In addition, in the case of unpunctured surface, the denominator of the expansion formula is given by the crossing pattern of the arc.

**Remark 5.2.** For clusters associated to outgoing fan triangulations, the Laurent phenomenon coincides with the classical form: in this case the cluster variables are Laurent polynomials as every arc on $S$ crosses only finitely many arcs of a given outgoing fan triangulation.
Remark 5.3. All results (and the proofs) of Sections 2 and 3 hold also in presence of finite number of orbifold points (in the sense of [FeSTu3]).

Remark 5.4. In principle, one could also introduce laminations (hence principal coefficients) and give skein relations for infinite rank surface cluster algebras with principal coefficients. Moreover, one could associate infinite adjacency quivers of infinite triangulations of surfaces, discuss quiver mutations for cluster variables and also generalise [MSW] expansion formula for snake graphs to give Laurent series expansions for cluster variables with principal coefficients.

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