ON THE CONVERGENCE AND SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

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Abstract. The convergence of partial sums and Cesáro means of negative order of double Walsh-Fourier series of functions of bounded generalized variation is investigated.

1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [17] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc., see [2, 18, 29, 27]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [16].

Let $f$ be a real and measurable function of two variables on the unit square. Given intervals $\Delta = (a, b), J = (c, d)$ and points $x, y$ from $I := [0, 1)$ we denote

$$f(\Delta, y) := f(b, y) - f(a, y), \quad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(\Delta, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from $I$ ordered in arbitrary way and let $\Omega$ be the set of all such collections $E$. Denote by $\Omega_n$ the set of all collections of $n$ nonoverlapping intervals $I_k \subset I$.

For the sequences of positive numbers

$$\Lambda^1 = \{\lambda_n^1\}_{n=1}^{\infty}, \quad \Lambda^2 = \{\lambda_n^2\}_{n=1}^{\infty}$$

and $I^2 := [0, 1)^2$ we denote

$$\Lambda^1V_i(f; I^2) = \sup_{y} \sup_{E \in \Omega} \sum_{i} \frac{|f(\Delta_i, y)|}{\lambda_i^1} \quad (E = \{\Delta_i\}),$$

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\[ \Lambda^2 V_2(f; I^2) = \sup_{x \in F \in \Omega} \sum_j \frac{|f(x, J_j)|}{\lambda^2_j} \quad (F = \{J_j\}), \]

\[ (\Lambda^1 \Lambda^2) V_{1,2}(f; I^2) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(\Delta_i, J_j)|}{\lambda_i \lambda^2_j}. \]

**Definition 1.1.** We say that the function \( f \) has Bounded \((\Lambda^1, \Lambda^2)\)-variation on \( I^2 \) and write \( f \in (\Lambda^1, \Lambda^2) BV \) \((I^2)\), if

\[(\Lambda^1, \Lambda^2) V(f; I^2) := \Lambda^1 V_1(f; I^2) + \Lambda^2 V_2(f; I^2) + (\Lambda^1 \Lambda^2) V_{1,2}(f; I^2) < \infty.\]

If \( \Lambda^1 = \Lambda^2 = \Lambda \), we say \( \Lambda \)-variation and use the notation \( \Lambda BV(I^2) \).

We say that the function \( f \) has Bounded Partial \( \Lambda \)-variation and write \( f \in P\Lambda BV ) (I^2) \) if

\[ P\Lambda BV(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty. \]

If \( \Lambda = \{\lambda_n\} \) with \( \lambda_n \equiv 1 \) (or if \( 0 < c < \lambda_n < C < \infty \), \( n = 1, 2, \ldots \)) the classes \( \Lambda BV \) and \( P\Lambda BV \) coincide, respectively, with the Hardy class \( BV \) and with the class \( PBV \) functions of bounded partial variation introduced by Goginava \([6]\). Hence it is reasonable to assume that \( \lambda_n \to \infty \) and since the intervals in \( E = \{\Delta_i\} \) are ordered arbitrarily, we will suppose, without loss of generality, that the sequence \( \{\lambda_n\} \) is increasing. Thus, we assume that

\[(1.1) \quad 1 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty, \quad \sum_{n=1}^{\infty} (1/\lambda_n) = +\infty.\]

In the case when \( \lambda_n = n \), \( n = 1, 2 \ldots \) we say Harmonic Variation instead of \( \Lambda \)-variation and write \( H \) instead of \( \Lambda \), i.e. \( HBV, PHBV, HV(f) \), etc.

The notion of \( \Lambda \)-variation was introduced by Waterman \([27]\) in one dimensional case, by Sahakian \([23]\) in two dimensional case. The notion of bounded partial \( \Lambda \)-variation \( (P\Lambda BV) \) was introduced by Goginava and Sahakian \([12]\).

Dyachenko and Waterman \([5]\) introduced another class of functions of generalized bounded variation. Denoting by \( \Gamma \) the set of finite collections of nonoverlapping rectangles \( A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset I^2 \), we define

\[ \Lambda^* V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}. \]

**Definition 1.2** (Dyachenko, Waterman). Let \( f \) be a real function on \( I^2 \). We say that \( f \in \Lambda^* BV \), if

\[ AV(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^* V(f) < \infty. \]

In \([13]\), the authors introduced a new classes of functions of generalized bounded variation and investigate the convergence of Fourier series of function of that classes.
For the sequence $\Lambda = \{ \lambda_n \}_{n=1}^{\infty}$ we denote

$$
\Lambda^\#_1 V(f) = \sup_{\{y_i\} \subseteq I} \sup_{\{I_i\} \in \Omega} \sum_{i} \frac{|f(I_i, y_i)|}{\lambda_i},
$$

$$
\Lambda^\#_2 V(f) = \sup_{\{x_j\} \subseteq J} \sup_{\{J_j\} \in \Omega} \sum_{j} \frac{|f(x_j, J_j)|}{\lambda_j}.
$$

**Definition 1.3.** We say that the function $f$ belongs to the class $\Lambda^\# BV$, if

$$
\Lambda^\# V(f) := \Lambda^\#_1 V(f) + \Lambda^\#_2 V(f) < \infty.
$$

The notion of continuity of function in $\Lambda$-variation plays an important role in the investigation of convergence Fourier series of functions of bounded $\Lambda$-variation.

**Definition 1.4.** We say that the function $f$ is continuous in $(\Lambda^1, \Lambda^2)$-variation on $I^2$ and write $f \in C (\Lambda^1, \Lambda^2) V$, if

$$
\lim_{n \to \infty} \Lambda^1_n V_1(f) = \lim_{n \to \infty} \Lambda^2_n V_2(f) = 0
$$

and

$$
\lim_{n \to \infty} (\Lambda^1_n, \Lambda^2) V_{1,2}(f) = \lim_{n \to \infty} (\Lambda^1_n, \Lambda^2) V_{1,2}(f) = 0,
$$

where $\Lambda^i_n := \{ \lambda_k^i \}_{k=n}^{\infty} = \{ \lambda_{k+n}^i \}_{k=0}^{\infty}$, $i = 1, 2$.

**Definition 1.5.** We say that the function $f$ is continuous in $\Lambda^\#$-variation on $I^2$ and write $f \in C \Lambda^\# V$, if

$$
\lim_{n \to \infty} \Lambda^\#_n V(f) = 0
$$

where $\Lambda^\#_n := \{ \lambda_k^\# \}_{k=n}^{\infty}$.

**Definition 1.6.** We say that the function $f$ is continuous in $\Lambda^*$-variation on $I^2$ and write $f \in C \Lambda^* V$, if

$$
\lim_{n \to \infty} \Lambda^1_n V_1(f) = \lim_{n \to \infty} \Lambda^2_n V_2(f) = 0
$$

and

$$
\lim_{n \to \infty} \Lambda^* V(f) = 0
$$

Now, we define

$$
v_1^\#(n, f) := \sup_{\{y_i\}_{i=1}^{n}} \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} |f(I_i, y_i)|, \quad n = 1, 2, \ldots,
$$

$$
v_2^\#(m, f) := \sup_{\{x_j\}_{j=1}^{m}} \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^{m} |f(x_j, J_j)|, \quad m = 1, 2, \ldots.
$$

The following theorems hold.
Theorem 1.1 (Goginava, Sahakian [13]). \( \{ \frac{n}{\log n} \}^\# BV \subset HBV. \)

Theorem 1.2 (Goginava, Sahakian [13]). Suppose
\[
\sum_{n=1}^{\infty} v_s^\#(f;n) \frac{\log(n+1)}{n^s} < \infty, \quad s = 1, 2.
\]
Then \( f \in \{ \frac{n}{\log(n+1)} \}^\# BV. \)

Theorem 1.3 (Goginava [10]). Let \( \alpha, \beta \in (0, 1) \), \( \alpha + \beta < 1 \) and
\[
\sum_{j=1}^{\infty} v_s^\#(f;2^j) \frac{1 - (\alpha + \beta)}{2^j(j-1)} < \infty, \quad s = 1, 2.
\]
Then \( f \in C \{ n^{1-(\alpha + \beta)} \}^\# V. \)

Theorem 1.4 (Goginava [10]). Let \( \alpha, \beta \in (0, 1) \) and \( \alpha + \beta < 1 \). Then
\[
C \{ i^{1-(\alpha + \beta)} \}^\# V \subset C \{ i^{1-\alpha} \} \{ j^{1-\beta} \} V.
\]

The next theorem shows, that for some sequences \( \Lambda \) the classes \( \Lambda^\# V \) and \( C\Lambda^\# V \) coincide.

Theorem 1.5. Let the sequence \( \Lambda = \{ \lambda_n \} \) be as in (1.1) and
\[
\lim_{n \to \infty} \frac{\lambda_{2n}}{\lambda_n} = q > 1.
\]
Then \( \Lambda^\# V = C\Lambda^\# V. \)

Proof. Suppose to the contrary, that there exists a function \( f \in \Lambda^\# V \) for which (see Definition [1.5]) \( \liminf_{n \to \infty} \Lambda_n^\# V(f) > 0 \). Without loss of generality, we can assume that \( \liminf_{n \to \infty} \Lambda_n^\# V_1(f) = \delta > 0 \) and that \( \delta = 1 \). Then, taking into account that the sequence \( \{ \Lambda_n^\# V_1(f) \} \) is decreasing, we have
\[
\lim_{n \to \infty} \Lambda_n^\# V_1(f) = 1.
\]
Let a natural \( k \) and a numbers \( \varepsilon > 0, q_0 \in (1, q) \) be fixed.

According to (1.2) and (1.3) there exist a natural \( N' > k \) such that
\[
\frac{\lambda_{2n}}{\lambda_n} > q_0, \quad \Lambda_n^\# V(f) > 1 - \varepsilon \quad \text{for} \quad n \geq N'.
\]
Then for a natural \( N > 2N' \) there are a set of points \( \{ y_i \}_{i=1}^{2i_0} \) and a set of nonoverlapping intervals \( \{ \delta_i \}_{i=1}^{2i_0} \in \Omega \) such that
\[
I := \sum_{i=1}^{2i_0} \frac{|f(\delta_i, y_i)|}{\lambda_{N+i}} \geq 1 - \varepsilon.
\]
Adding, if necessary, new summands in (1.5) we can assume that
\[ \bigcup_{i=1}^{2i_0} \delta_i = (0, 1). \]

Denote
(1.6) \[ I_1 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}}, \quad I_2 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}}. \]

Since \( N > 2N' \) implies that \( N + 2i - 1 \geq 2(N' + i) \), from (1.4) and (1.6) we have
(1.7) \[ I'_1 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}} \cdot \frac{\lambda_{N+2i-1}}{\lambda_{N'+i}} > q_0 I_1 \]
and
(1.8) \[ I'_2 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}} \cdot \frac{\lambda_{N+2i}}{\lambda_{N'+i}} > q_0 I_2. \]

Consequently, by (1.5),
(1.9) \[ I' := I'_1 + I'_2 \geq q_0 (I_1 + I_2) = q_0 I \geq q_0 (1 - \varepsilon). \]

Now, we take natural \( M \) such that,
(1.10) \[ M > N + 2(i_0 + 1) \quad \text{and} \quad \frac{2(2i_0 + 1)}{\lambda_M} \sup_{x \in [0, 1]} |f(x)| < \varepsilon, \]
and using (1.4), we find a set of points \( \{z_j\}_{j=1}^{j_0} \) of nonoverlapping intervals \( \{\Delta_j\}_{j=1}^{j_0} \in \Omega \) such that
(1.11) \[ \sum_{j=1}^{j_0} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1 - \varepsilon. \]

Denote by \( Q \) the set of indices \( j = 1, 2, \ldots, j_0 \) for which the corresponding interval \( \Delta_j \) does not contain an endpoint of intervals \( \delta_i, i = 1, 2, \ldots, 2i_0 \), i.e. \( \Delta_j \) lies in one of intervals \( \delta_i, i = 1, 2, \ldots, 2i_0 \). Then the number of indices in \( [1, j_0] \setminus Q \) does not exceed \( 2i_0 + 1 \) and by (1.10),
\[ \sum_{j \in [1, j_0] \setminus Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \leq \varepsilon. \]

Consequently, by (1.11),
(1.12) \[ J := \sum_{j \in Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1 - 2\varepsilon. \]
Denoting
\[ Q_1 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i-1} \right\}, \quad Q_2 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i} \right\} \]

and
\[ J_1 := \sum_{j \in Q_1} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}}, \quad J_2 := \sum_{j \in Q_2} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \]

from (1.9) and (1.12) we obtain
\[ (I'_1 + J_2) + (I'_2 + J_1) = I' + J \geq q_0(1 - \varepsilon) + 1 - 2\varepsilon \geq q_0 + 1 - 3\varepsilon. \]

Therefore,
\[ I'_1 + J_2 \geq \frac{q_0 + 1 - 3\varepsilon}{2} \quad \text{or} \quad (I'_2 + J_1) \geq \frac{q_0 + 1 - 3\varepsilon}{2}, \]

which means that
\[ \Lambda_{N'}^\# V_1(\alpha) \geq \frac{q_0 + 1 - 3\varepsilon}{2} \]

and hence
\[ \Lambda_k^\# V_1(\alpha) \geq \frac{q_0 + 1}{2}, \]

since \( \varepsilon \) is any positive number and \( N' > k \). Taking into account that \( k \) is an arbitrary natural number, the last inequality implies
\[ \lim_{n \to \infty} \Lambda_n^\# V_1(\alpha) \geq \frac{q_0 + 1}{2} > 1, \]

which is a contradiction to the assumption (1.3). Theorem 1.5 is proved. \( \square \)

It is easy to see, that for any \( \gamma > 0 \) the sequence \( \lambda_n = n^\gamma, \ n = 1, 2, \ldots \) satisfies the condition (1.2) with \( q = 2^\gamma \). Hence Theorem 1.5 implies

**Corollary 1.1.** If \( 0 < \gamma \leq 1 \), then \( \{n^\gamma\}^\# V = C \{n^\gamma\}^\# V \).

This, combined with Theorem 1.4 implies

**Corollary 1.2.** Let \( \alpha, \beta \in (0, 1) \) and \( \alpha + \beta < 1 \). Then
\[ \left\{ i^{1-(\alpha+\beta)} \right\}^\# V \subset C \left\{ i^{1-\alpha} \right\} \left\{ i^{1-\beta} \right\} V. \]
2. Walsh functions

Let \(\mathbb{P}\) be the set of positive integers, and \(\mathbb{N} := \mathbb{P} \cup \{0\}\). We denote the set of all integers by \(\mathbb{Z}\) and the set of dyadic rational numbers in the unit interval \(I := [0, 1)\) by \(\mathbb{Q}\). Each element of \(\mathbb{Q}\) is of the form \(\frac{p}{2^n}\) for some \(p, n \in \mathbb{N}\), \(0 \leq p \leq 2^n\). By a dyadic interval in \(I\) we mean an interval of the form \(I^l_N := [l2^{-N}, (l + 1)2^{-N})\) for some \(l \in \mathbb{N}\), \(0 \leq l < 2^N\). Given \(N \in \mathbb{N}\) and \(x \in I\), we denote by \(I_N(x)\) the dyadic interval of length \(2^{-N}\) that contains \(x\). Finally, we set \(I_N := [0, 2^{-N})\) and \(I_N := I \setminus I_N\).

Let \(r_0(x)\) be the function defined on the real line by

\[
 r_0(x) = \begin{cases} 
 1, & \text{if } x \in [0, 1/2) \\
 -1, & \text{if } x \in [1/2, 1)
\end{cases},
 r_0(x + 1) = r_0(x), \quad x \in \mathbb{R}.
\]

The Rademacher system is defined by

\[
 r_n(x) = r_0(2^n x) \quad x \in I, \quad n = 1, 2, \ldots.
\]

The Walsh functions \(w_0, w_1, \ldots\) are defined as follows. Denote \(w_0(x) = 1\) and if \(k = 2^{n_1} + \cdots + 2^{n_s}\) is a positive integer with \(n_1 > n_2 > \cdots > n_s\), then

\[
 w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).
\]

The Walsh-Dirichlet kernel is defined by

\[
 D_n(x) = \sum_{k=0}^{n-1} w_k(x), \quad n = 1, 2, \ldots
\]

Recall that [15, 25]

\begin{equation}
 (2.1) \quad D_{2^n}(x) = \begin{cases} 
 2^n, & \text{if } x \in [0, 2^{-n}) \\
 0, & \text{if } x \in [2^{-n}, 1)
\end{cases}
\end{equation}

and

\begin{equation}
 (2.2) \quad D_{2^n + m}(x) = D_{2^n}(x) + w_{2^n}(x) D_m(x), \quad 0 \leq m < 2^n, \quad n = 0, 1, \ldots
\end{equation}

It is well known that [25]

\begin{equation}
 (2.3) \quad D_n(t) = w_n(t) \sum_{j=0}^{\infty} n_j w_{2^j}(t) D_{2^j}(t), \quad \text{if } n = \sum_{j=0}^{\infty} n_j 2^j
\end{equation}

and

\begin{equation}
 (2.4) \quad |D_{q_n}(x)| \geq \frac{1}{4x}, \quad 2^{-2n-1} \leq x < 1,
\end{equation}

where

\begin{equation}
 (2.5) \quad q_n := 2^{2n-2} + 2^{2n-4} + \cdots + 2^2 + 2^0.
\end{equation}

Given \(x \in I\), the expansion
where each $x_k = 0$ or 1, is called a dyadic expansion of $x$. If $x \in I \setminus \mathbb{Q}$, then (2.6) is uniquely determined. For $x \in \mathbb{Q}$ we choose the dyadic expansion with $\lim_{k \to \infty} x_k = 0$.

The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of $x$ and $y$ is defined by

$$x + y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$ 

We say that $f(x, y)$ is continuous at $(x, y)$ if

$$\lim_{h, \delta \to 0} f(x + h, y + \delta) = f(x, y).$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on the unit square $I^2 = [0, 1) \times [0, 1)$.

If $f \in L^1(I^2)$, then

$$\hat{f}(n, m) = \int_{I^2} f(x, y) w_n(x)w_m(y) dxdy$$

is the $(n, m)$-th Walsh-Fourier coefficient of $f$.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x, y; f) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) w_m(x)w_n(y).$$

The Cesàro $(C; \alpha, \beta)$-means of double Walsh-Fourier series are defined as follows

$$s_{n,m}^{\alpha,\beta}(x, y; f) = \frac{1}{A_n^\alpha A_m^{-1}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}(x, y; f),$$

where

$$A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, ....$$

It is well-known that [30]

(2.8) \hspace{1cm} A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},

(2.9) \hspace{1cm} A_n^\alpha \sim n^\alpha
and

\[
\sigma_{n,m}^{\alpha,\beta}(x,y;f) = \int_{I^2} f(s,t) K_n^\alpha(x+s) K_m^\beta(y+t) dsdt,
\]

where

\[
K_n^\alpha(x) := \frac{1}{A_n^\alpha - 1} \sum_{k=1}^{n-1} A_{n-k}^\alpha D_k(x).
\]

3. Convergence of two-dimensional Walsh-Fourier series

The well known Dirichlet-Jordan theorem (see [30]) states that the Fourier series of a function \(f(x), x \in T\) of bounded variation converges at every point \(x\) to the value \([f(x + 0) + f(x - 0)]/2\).

Hardy [16] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function \(f(x,y)\) has bounded variation in the sense of Hardy \((f \in BV)\), then \(S[f]\) converges at any point \((x,y)\) to the value \(\frac{1}{4} \sum f(x \pm 0, y \pm 0)\). Here and below we consider the convergence of only rectangular partial sums of double Fourier series.

Convergence of \(d\)-dimensional trigonometric Fourier series of functions of bounded variation was investigated in details by Sahakian [23], Dyachenko [3, 4, 5], Bakhvalov [1], Sablin [22], Goginava, Sahakian [12, 13], etc.

For the \(d\)-dimensional Walsh-Fourier series the convergence of partial sums of functions of bounded Harmonic variation and other bounded generalized variation were studied by Moricz [19, 20], Onnewer, Waterman [21], Goginava [7].

In the two-dimensional case Sargsyan has obtained the following result.

**Theorem 3.1** (Sargsyan [24]). If \(f \in H_{BV}(I^2)\), then the double Walsh-Fourier series of \(f\) converges to \(f(x,y)\) at any point \((x,y) \in I^2\), where \(f\) is continuous.

The authors investigated convergence of multiple Walsh-Fourier series of functions of partial \(A\)-bounded variation. In particular, the following result was proved.

**Theorem 3.2** (Goginava, Sahakian [13]). a) If \(f \in P\left(\frac{\varepsilon}{\log^{1+\varepsilon} n}\right)BV(I^2)\) for some \(\varepsilon > 0\), then the double Walsh-Fourier series of \(f\) converges to \(f(x,y)\) at any point \((x,y)\), where \(f\) is continuous.

b) There exists a continuous function \(f \in P\left(\frac{\varepsilon}{\log n}\right)BV(I^2)\) such that the quadratic partial sums of its Walsh-Fourier series diverge at some point.

In the next theorem we obtain a similar result for functions of bounded \(A^\#\)-variation.
Theorem 3.3. a) If \( f \in \left\{ \frac{n}{\log n} \right\} \# BV \), then the double Walsh-Fourier series of \( f \) converges to \( f(x,y) \) at any point \((x,y)\), where \( f \) is continuous.

b) For an arbitrary sequence \( \alpha_n \to \infty \) there exists a continuous function \( f \in \left\{ \frac{n\alpha_n}{\log(n+1)} \right\} \# BV \) such that the quadratic partial sums of its Walsh-Fourier series diverge unboundedly at \((0,0)\).

Proof. Part (a) immediately follows from Theorems 1.1 and 3.1.

To prove part (b) observe that for any sequence \( \Lambda = \{ \lambda_n \} \) satisfying (1.1), the class \( C(I^2) \cap \Lambda \# BV \) is a Banach space with the norm

\[
\|f\|_{\Lambda \# BV} := \|f\|_C + \Lambda \# BV (f),
\]

and \( S_{N,N}(0,0) \), \( n = 1, 2, \ldots \), is a sequence of bounded linear functionals on that space. Denote

\[
\varphi_{N,j} (x) = \begin{cases} 
2^{2N+1}x - 2j, & \text{if } x \in \left[ j2^{-2N}, (2j + 1) 2^{-2N-1} \right] \\
-(2^{2N+1}x - 2j - 2), & \text{if } x \in \left( (2j + 1) 2^{-2N-1}, (j + 1) 2^{-2N} \right], \\
0, & \text{if } x \in I \setminus \left[ j2^{-2N}, (j + 1) 2^{2N} \right],
\end{cases}
\]

(3.1) \( \varphi_N(x) = \sum_{j=1}^{2^{2N-1}} \varphi_{N,j} (x), \quad x \in I, \)

\( g_N(x,y) = \varphi_N(x) \varphi_N(y) \text{sgn} D_{q_N} (x) \text{sgn} D_{q_N} (y), \quad x, y \in I, \)

where \( q_N \) is defined in (2.5).

Suppose \( \Lambda = \left\{ \lambda_n = \frac{n\alpha_n}{\log(n+1)} \right\}_{n=1}^{\infty} \), where \( \alpha_n \to \infty \). It is easy to show that

\[
\Lambda \# V_s (g_N) \leq c 2^{2N-1} \sum_{i=1}^{\log (i+1)} i \alpha_i = o \left( N^2 \right) \text{ as } N \to \infty,
\]

for \( s = 1, 2 \). Hence

\[
\|g_N\|_{\Lambda \# BV} = o \left( N^2 \right) = \eta_N N^2,
\]

where \( \eta_N \to 0 \) as \( N \to \infty \), and denoting

\[
G_N := \frac{g_N}{\eta_N N^2},
\]

we conclude that \( G_N \in \Lambda \# BV \) and

(3.2) \( \sup_N \|G_N\|_{\Lambda \# BV} < \infty. \)
By construction of the function $G_N$ we have

$$S_{q_N,q_N} (0,0; G_N) = \int\int_{I^2} G_N (x,y) D_{q_N} (x) D_{q_N} (y) \, dx \, dy$$

(3.3)

$$= \frac{1}{N^{2\eta N}} \int\int_{I^2} \varphi_N (x) \varphi_N (y) |D_{q_N} (x)| |D_{q_N} (y)| \, dx \, dy$$

$$= \frac{1}{N^{2\eta N}} \left( \int_{I^2} \varphi_N (x) |D_{q_N} (x)| \, dx \right)^2$$

Using (2.4) we can write

$$\int_{I} \varphi_N (x) |D_{m_N} (x)| \, dx = \sum_{j=1}^{2^{2N-1}} \left( \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j} (x) |D_{m_N} (x)| \, dx \right)$$

$$= \sum_{j=1}^{2^{2N-1}} \left| D_{m_N} \left( \frac{j}{2^N} \right) \right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j} (x) \, dx$$

$$\geq \frac{1}{2^{2N+1}} \sum_{j=1}^{2^{2N-1}} \frac{2^{2N}}{4j} \geq cN.$$

Consequently, from (3.3) we obtain

(3.4) \[ |S_{q_N,q_N} (0,0; G_N)| \geq \frac{c}{\eta N} \rightarrow \infty \quad \text{as} \quad N \rightarrow \infty.\]

According to the Banach-Steinhaus Theorem, (3.2) and (3.4) imply that there exists a continuous function $f \in \{ \frac{\text{max}_{n \geq 1} \log(n+1)}{\text{log}(n+1)} \}^\# BV$ such that

$$\sup_N |S_{N,N} (0,0; f)| = +\infty.$$

Theorem 3.3 is proved. \[ \square \]

Theorem 1.2 and Theorem 3.3 imply

**Theorem 3.4.** Let the function $f (x,y)$, $(x,y) \in I^2$, satisfies the condition

$$\sum_{n=1}^{\infty} \frac{v_s^\# (f,n) \log(n+1)}{n^s} < \infty, \quad s = 1, 2.$$

Then the double Walsh-Fourier series of $f$ converges to $f (x,y)$ at any point $(x,y)$, where $f$ is continuous.
4. Cesàro means of negative order two-dimensional Walsh-Fourier series

The problems of summability of Cesàro means of negative order for one dimensional Walsh-Fourier series were studied in the works [26], [8]. In the two-dimensional case the summability of Walsh-Fourier series by Cesàro method of negative order for functions of partial bounded variation was investigated by the first author in [9], [11]. In particular, the following results were obtained.

**Theorem 4.1** (Goginava [9]). Let $f \in C_w(I^2) \cap \text{PBV}$ and $\alpha + \beta < 1$, $\alpha, \beta > 0$. Then the double Walsh-Fourier series of the function $f$ is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.

**Theorem 4.2** (Goginava [9]). Let $\alpha + \beta \geq 1$, $\alpha, \beta > 0$. Then there exists a continuous function $f_0 \in \text{PBV}$ such that the Cesàro $(C; -\alpha, -\beta)$-means $\sigma_{n,m}^{-\alpha, -\beta}(0,0; f_0)$ of the double Walsh-Fourier series of $f_0$ diverges.

**Theorem 4.3** (Goginava [11]). Let $f \in C(\{i^{1-\alpha}\}, \{i^{1-\beta}\}) V(I^2), \alpha, \beta \in (0, 1)$. Then $(C, -\alpha, -\beta)$-means of double Walsh-Fourier series converges to $f(x,y)$, if $f$ is continuous at $(x,y)$.

**Theorem 4.4** (Goginava [11]). Let $\alpha, \beta \in (0, 1), \alpha + \beta < 1$.

a) If $f \in P\left\{n^{1-(\alpha+\beta)}\right\} \text{BV}(I^2)$ for some $\varepsilon > 0$, then the double Walsh-Fourier series of the function $f$ is $(C; -\alpha, -\beta)$ summable to $f(x,y)$, if $f$ is continuous at $(x,y)$.

b) There exists a continuous function $f \in P\left\{n^{1-(\alpha+\beta)}\right\} \text{BV}(I^2)$ such that $\sigma_{n,n}^{-\alpha, -\beta}(0,0; f)$ diverges.

In this paper we prove that the following are true.

**Theorem 4.5.** a) Let $\alpha, \beta \in (0,1), \alpha + \beta < 1$ and $f \in \left\{n^{1-(\alpha+\beta)}\right\}^\# \text{BV}$. Then $\sigma_{n,m}^{-\alpha, -\beta}(x,y; f)$ converges to $f(x,y)$, if $f$ is continuous at $(x,y)$.

b) Let $\Lambda := \left\{n^{1-(\alpha+\beta)}\xi_n\right\}$, where $\xi_n \uparrow \infty$ as $n \to \infty$. Then there exists a function $f \in C(I^2) \cap C\Lambda^\#V$ for which $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverges unboundedly at $(0,0)$.

**Proof.** Part a) immediately follows from Corollary 1.2 and Theorem 4.3. To prove part b) observe that

$$\left\{n^{1-(\alpha+\beta)}\sqrt{\xi_n}\right\}^\# \text{BV} \subset C\left\{n^{1-(\alpha+\beta)}\xi_n\right\}^\# V,$$

and since $\xi_n \uparrow \infty$ is arbitrary, it is enough to show that there exists a continuous function $f \in \Lambda^\#BV$ for which $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverges unboundedly at $(0,0)$. 
Denote
\[ h_N(x, y) := \varphi_N(x) \varphi_N(y) \text{sgn}K_{22N}^{-\alpha}(x) \text{sgn}K_{22N}^{-\beta}(y), \]
where \( \varphi_N \) is defined in (3.1), and the kernel \( K_n^\alpha \) is defined in (2.11). It is easy to show that for \( s = 1, 2, \)
\[ \left\{ n^{1-(\alpha+\beta)} \xi_n \right\} \# V_s(h_N) \leq c(\alpha, \beta) \sum_{i=1}^{2^{2N}-1} \frac{1}{i^{1-(\alpha+\beta)}\xi_i} \]
\[ = o\left(2^{2N(\alpha+\beta)}\right), \quad \text{as} \quad N \to \infty, \]
hence
\[ \|h_N\|_{\Lambda^\# BV} = o\left(2^{2N(\alpha+\beta)}\right) =: \eta_N2^{2N(\alpha+\beta)}, \]
where \( \eta_N = o(1) \) as \( N \to \infty. \) Consequently, denoting
\[ H_N(x, y) := \frac{h_N(x, y)}{\eta_N2^{2N(\alpha+\beta)}}, \]
we conclude that \( H_N \in C(I^2) \cap \Lambda^\# BV \) and
\[ (4.1) \quad \sup_N \|H_N\|_{\Lambda^\# BV} < \infty. \]

By construction of the function \( H_N, \) we have
\[ \sigma_{22N, 22N}^{-\alpha, -\beta}(0, 0; H_N) = \iint_{I^2} H_N(x, y) K_{22N}^{-\alpha}(x) K_{22N}^{-\beta}(y) \, dx \, dy \]
(4.2)
\[ = \frac{1}{\eta_N2^{2N(\alpha+\beta)}} \iint_{I^2} h_N(x, y) K_{22N}^{-\alpha}(x) K_{22N}^{-\beta}(y) \, dx \, dy \]
\[ = \frac{1}{\eta_N2^{2N(\alpha+\beta)}} \int_I \varphi_N(x) \left| K_{22N}^{-\alpha}(x) \right| \, dx \int_I \varphi_N(y) \left| K_{22N}^{-\beta}(y) \right| \, dy. \]

Now, using the following estimate from [25]:
\[ \int_{2^{-N}-1}^{2^{-N}} \left| K_{2N}^{-\alpha}(x) \right| \, dx \geq c(\alpha) 2^{m\alpha}, \quad N \in \mathbb{N}, \quad m = 1, \ldots, N, \quad 0 < \alpha < 1, \]
we can write
\[ (4.3) \quad \int_I \varphi_N(x) \left| K_{22N}^{-\alpha}(x) \right| \, dx = \sum_{j=1}^{2^{2N}-1} \int_{2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x) \left| K_{22N}^{-\alpha}(x) \right| \, dx \]
\[
\begin{align*}
&= \sum_{j=1}^{2^{2N-1}} \left| K_{22N}^{-\alpha} \left( \frac{j}{2^{2N}} \right) \right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x) \, dx \\
&= \frac{1}{2} \sum_{j=1}^{2^{2N-1}} \left| K_{22N}^{-\alpha} \left( \frac{j}{2^{2N}} \right) \right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} \, dx \\
&= \frac{1}{2} \sum_{m=0}^{2N-1} \int_{2m2^{-2N}}^{(2m+1)2^{-2N}} \left| K_{22N}^{-\alpha} (x) \right| \, dx \\
&= \frac{1}{2} \sum_{m=0}^{2N-1} 2^{m+1} \int_{2m2^{-2N}}^{(2m+1)2^{-2N}} \left| K_{22N}^{-\alpha} (x) \right| \, dx \\
&\geq c(\alpha) \sum_{m=0}^{2N-1} 2^{m\alpha} \geq c(\alpha) 2^{2N\alpha}.
\end{align*}
\]

Analogously, we can prove that
\[
(4.4) \quad \int_{I} \varphi_{N}(x) \left| K_{22N}^{-\beta} (x) \right| \, dx \geq c(\beta) 2^{2N\beta}, \quad N \in \mathbb{N}, \quad 0 < \beta < 1.
\]

Combining (4.3) and (4.4) we get
\[
(4.5) \quad \left| \sigma_{22N}^{-\alpha,-\beta} (0,0; H_N) \right| \geq \frac{c(\alpha,\beta)}{\eta_N} \to \infty \quad \text{as} \quad N \to \infty.
\]

Applying the Banach-Steinhaus Theorem, from (4.3) and (4.5) we obtain that there exists a continuous function \( f \in \Lambda^\# BV \) such that
\[
\sup_{N} \left| \sigma_{N,N}^{-\alpha,-\beta} (0,0; f) \right| = +\infty.
\]

Theorem 4.5 is proved. \( \square \)

Since
\[
\Lambda^* BV \subset \Lambda^\# BV
\]
from Theorem 4.5 we conclude that the following is true.

**Corollary 4.1.** Let \( \alpha, \beta \in (0,1) \), \( \alpha + \beta < 1 \) and \( f \in \left\{ n^{1-(\alpha+\beta)} \right\}^* BV \). Then \( \sigma_{n,m}^{-\alpha,-\beta} (x,y; f) \) converges to \( f(x,y) \), if \( f \) is continuous at \( (x,y) \).

Theorem 4.3 and Theorem 1.3 imply.

**Theorem 4.6.** Let \( \alpha, \beta \in (0,1) \), \( \alpha + \beta < 1 \) and
\[
\sum_{j=1}^{\infty} \frac{v_{s}^# (f;2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1,2.
\]

Then \( \sigma_{n,m}^{-\alpha,-\beta} (x,y; f) \) converges to \( f(x,y) \), if \( f \) is continuous at \( (x,y) \).
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