Generalized modulation theory for nonlinear gravity waves in a compressible atmosphere

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This study investigates nonlinear gravity waves in the compressible atmosphere from the Earth’s surface to the deep atmosphere. These waves are effectively described by Grimshaw’s dissipative modulation equations which provide the basis for finding stationary solutions such as mountain lee waves and testing their stability in an analytic fashion. Assuming energetically consistent boundary and far-field conditions, that is no energy flux through the surface, free-slip boundary, and finite total energy, general wave solutions are derived and illustrated in terms of realistic background fields. These assumptions also imply that the wave-Reynolds number must become less than unity above a certain height. The modulational stability of admissible, both non-hydrostatic and hydrostatic, waves is examined. It turns out that, when accounting for the self-induced mean flow, the wave-Froude number has a resonance condition. If it becomes $1/\sqrt{2}$, then the wave destabilizes due to perturbations from the essential spectrum of the linearized modulation equations. However, if the horizontal wavelength is large enough, waves overturn before they can reach the modulational stability condition.

1 Introduction

Gravity waves are an omnipresent oscillation mode in the atmosphere. They redistribute energy vertically but also laterally and thereby affect the dynamics relevant for weather and climate prediction [Fritts and Alexander, 2003, Becker, 2012]. Usually excited in the troposphere, gravity waves may persist deep into the upper layers of the atmosphere [Fritts et al., 2016, 2018, 2019] where they interact with the mean flow. They
exert drag onto the horizontal mean-flow momentum, produce heat when dissipating [Becker, 2004], and cause increased mixing of tracer constituents such as green-house gases [Schlutow et al., 2014].

The dominant excitation mechanism is background flow over mountain ranges resulting in mountain lee waves that can be considered stationary as their horizontal phase speed is exact opposite of the mean-flow horizontal wind. In the troposphere wave amplitudes are often small such that linear wave theory is applicable [Eliassen and Palm, 1961, Pätz et al., 2019]. When mountain lee waves extend into the higher layers they get anelastically amplified due to the decreasing background density which is an effect of the compressibility of the atmosphere. Amplitudes get indeed so large that linear theory becomes invalid.

Weakly nonlinear theory for gravity waves was studied by Whitham [1974], Grimshaw [1977], Sutherland [2001, 2006], Tabaei and Akylas [2007]. Here, nonlinear effects such as Doppler shift of the frequency and interaction with the mean flow appear as higher-order corrections to the linear model. In the asymptotic limit the model therefore approaches linear theory. An important effect of the weak nonlinearity is the occurrence of modulational instabilities: plane non-hydrostatic Boussinesq waves become modulationally unstable if the second derivative of the dispersion relation with respect to the wavenumber becomes negative. Schlutow et al. [2019] showed that this holds true even for nonlinear waves of the same kind. In the nonlinear theory, Doppler shift and wave-mean-flow interaction appear to leading order such that the amplitude is finite in the asymptotic limit. However, Boussinesq theory does not account for the anelastic amplification. Furthermore, the plane waves extend to the infinities experiencing no lower boundary conditions and no dissipative effects. In Schlutow [2019], a particular flow regime, where anelastic amplification and dissipative forces are exactly balanced, was investigated with respect to modulational stability.

This study aims to generalize the modulation theory incorporating nonlinearity, anelastic amplification, dissipative damping and lower boundary conditions in a comprehensive fashion. Pioneering work on the modulation theory of nonlinear gravity waves was accomplished by Grimshaw [1972, 1974].

In the section 2 of this paper we will introduce Grimshaw’s dissipative modulation equations as our governing equations and link them to the asymptotic solution to the compressible Navier-Stokes equations. Boundary conditions and limit behavior as derived from physical arguments will be shown in sections 3 and 4. After the introduction of the antitriptic flow assumption in section 5 stationary solutions will be found and illustrated in terms of observational data in section 6. In section 7 the modulational stability of the stationary solution will be investigated followed by some concluding remarks in section 8.

2 Model equations

The foundation for our investigations is established by Grimshaw’s dimensionless dissipative modulation equations for horizontally periodic two-dimensional gravity waves
modulated only in the vertical \( (Z-) \) direction

\[
\frac{\partial k_z}{\partial T} + \frac{\partial}{\partial Z} (\tilde{\omega} + k_x u) = 0 \quad (1a)
\]

\[
\rho \frac{\partial a}{\partial T} + \frac{\partial}{\partial Z} (\tilde{\omega}' k_x \rho a) = -\Lambda |k|^2 \rho a \quad (1b)
\]

\[
\rho \frac{\partial u}{\partial T} + \frac{\partial}{\partial Z} (\tilde{\omega}' k_x \rho a) = -\frac{\partial p}{\partial X} \quad (1c)
\]

in the domain \( Z \in [0, \infty) \), from the Earth’s surface into the deep atmosphere [Grimshaw, 1974, Achatz et al., 2010]. This set of coupled nonlinear partial differential equations governs the evolution of vertical wavenumber \( k_z \), wave action density \( A = \rho a \), and mean-flow horizontal wind \( u \). Horizontal wavenumber \( k_x \) is, without loss of generality, a positive constant. Intrinsic frequency \( \tilde{\omega} \) and linear vertical group velocity \( \tilde{\omega}' \) are functions of \( k_z \). They are determined by the dispersion relation for either hydrostatic \( (h) \) or non-hydrostatic \( (nh) \) gravity waves

\[
\tilde{\omega}(k_z) = \begin{cases} 
Nk_x/|k|, & (nh), \\
Nk_x/|k|, & (h),
\end{cases}
\]

\[
\tilde{\omega}'(k_z) = \begin{cases} 
-Nk_x k_z/|k|^3, & (nh), \\
-Nk_x k_z/|k|^3, & (h),
\end{cases}
\]

with \( |k|^2 = k_x^2 + k_z^2 \). Three coefficients appear being functions of \( Z \) only: \( \Lambda, \rho \) and \( N \). They denote the kinematic viscosity, background density and the Brunt-Väisälä frequency, respectively. \( \rho \) and \( N \) are computed assuming a hydrostatic atmosphere being an ideal gas. Given a background temperature profile \( T(Z) \) we obtain

\[
N^2(Z) = \frac{1}{T} \left( \frac{dT}{dZ} + 1 \right), \quad (3a)
\]

\[
\rho(Z) = \exp \left( \int_0^Z \eta(Z') \, dZ' \right) \quad \text{with} \quad \eta(Z) = -\frac{1}{T} \left( \frac{dT}{dZ} + \frac{1}{\kappa} \right) \quad (3b)
\]

where \( \kappa = (\gamma - 1)/\gamma = 2/7 \) and \( \gamma \) the heat capacity ratio for ideal gases.

In Whitham’s modulation theory [Whitham, 1974], equation (1a) represents conservation of waves. The second equation (1b) gives conservation of wave action plus a sink due to dissipation. Finally, (1c) describes the acceleration of mean-flow horizontal momentum due to horizontal pseudo-momentum flux convergence and the horizontal gradient of mean-flow kinematic pressure \( p \). The latter is unspecified at the moment and needs further assumptions to close the system.

All prognostic fields generally depend on the compressed dimensionless coordinates

\[
(X, Z, T) = (\varepsilon^\alpha x, \varepsilon z, \varepsilon^\alpha t) \quad \text{where} \quad \alpha = \begin{cases} 
1, & (nh), \\
2, & (h),
\end{cases} \quad (4)
\]

that originate from a rescaling of the dimensionless coordinates

\[
(x, z, t) = (x^*/L_r, z^*/L_r, t^*/t_r).
\]

3
The dimensionless coordinates are obtained by the non-dimensionalization with the reference length scale $L_r$ and reference time scale $t_r$. Note that throughout this work, dimensioned variables are labeled with an asterisk. Scale separation between the slow variation of the modulational fields and the rapidly oscillating wave field is determined by $0 < \varepsilon \ll 1$. The modulation equations can be derived from the compressible Navier-Stokes equations (NSE) in an asymptotic fashion as $\varepsilon \to 0$ which is shown in a comprehensive manner in Schlutow et al. [2017]. For this derivation, a distinguished limit on the dimensionless NSE is assumed that maps $\varepsilon$ to the Mach, Froude, Reynolds and Prandtl number, individually. The background field is separated from the perturbing wave field by a scaling appropriate for either hydrostatic or non-hydrostatic waves obtained from linear theory. Next, the state vector $UUU = (\nu, \beta, \phi)^T$ of the NSE is expanded in terms of a nonlinear Wentzel-Kramers-Brillouin (WKB) ansatz

$$UUU(x, z, t; \varepsilon) = U_{0,0}(X, Z, T) + \left(U_{0,1}(X, Z, T)e^{i\Phi(X, Z, T)/\varepsilon} + \text{c.c.}\right) + \text{h.h.} + O(\varepsilon) \quad (6)$$

where c.c. stands for the complex conjugate and h.h. for higher harmonics. Here, $\nu$, $N\beta = b$ and $\phi$ denote the two-dimensional velocity vector, buoyancy force and kinematic pressure, respectively. By construction, the WKB ansatz is indeed a nonlinear approach—in the asymptotic limit the amplitudes are finite. In fact, the ansatz converges to the nonlinear plane wave of Boussinesq theory which is known to be an analytical solution. This ansatz is substituted into the NSE and terms are ordered in powers of $\varepsilon$ and harmonics, i.e. integer multiples of the phase $\Phi$. Then, the leading-order solution of the NSE as $\varepsilon \to 0$ is given in terms of the modulation fields governed by (1) as

$$\left(\frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial Z}\right)^T = k, \quad (7a)$$

$$-\frac{\partial \Phi}{\partial T} = \dot{\omega} + k_x u, \quad (7b)$$

$$U_{0,0} = (u, 0, 0, p)^T, \quad (7c)$$

$$U_{0,1} = BU^\dagger \quad (7d)$$

where

$$U^\dagger = \begin{pmatrix} -i k_z \dot{\omega} N & i \dot{\omega} N & 1 & -i k_z \dot{\omega} N \end{pmatrix}^T \quad (8)$$

represents the polarization vector and $B = \sqrt{\dot{\omega}a/2}$ the amplitude.

Furthermore, the modulation equations exhibit a total energy density being the sum of mean-flow kinetic energy density and wave energy density,

$$\rho e = \frac{1}{2} \rho u^2 + \rho a \dot{\omega}. \quad (9)$$

It evolves in time as governed by

$$\rho \frac{\partial e}{\partial T} + \frac{\partial}{\partial Z} (\dot{\omega}' \rho a (\dot{\omega} + k_x u)) + \frac{\partial}{\partial X} (pu) = -\dot{\omega} \Lambda |k|^2 \rho a. \quad (10)$$

In conclusion, (2) is a locally conserved quantity in the inviscid limit, i.e. $\Lambda \to 0$.  

4
3 Surface boundary condition

The leading-order asymptotic solution to the NSE as defined by (6) governed by the modulation equations already determines the shape of the anticipated wave solution. However, it has some degrees of freedom to specify physically motivated surface boundary conditions. Given a mean-flow horizontal wind, three additional conditions are needed to set the horizontal and vertical wavenumber as well as the specific wave action density at $Z = 0$. We will assume that close to the surface the viscosity is negligible. Thus, a free-slip boundary is justified. Let us carve a mountain to the shape of the lowest stream line. Then, this particular, prototypical mountain will define two of the surface boundary conditions. For the third, we will assume that there shall be no energy flux through the boundary and therefore the extrinsic frequency will be zero at the mountain.

3.1 No-energy-flux boundary condition

According to (10), if no energy flux through the boundary but finite wave action and wavenumber at the boundary are presumed, then

$$\hat{\omega} + k_x u = 0 \quad \text{at} \quad Z = 0 \quad (11)$$

must hold which coincides with the absence of wavenumber flux (cf. (1a)). Let us remark that therefore $u < 0$ as $k_x > 0$ and $\hat{\omega} > 0$ at $Z = 0$.

3.2 Free-slip boundary condition: carving a mountain to the wave

To leading order the solution as given by (6) is solenoidal and therefore entirely determined by a stream function. From the polarization vector (8), the stream function can be written in terms of the fast variables as

$$\Psi(x, z) = uz - 2B \frac{\hat{\omega}}{Nk_x} \cos(k_x x + k_z z) + O(\varepsilon). \quad (12)$$

Stream functions of solenoidal flows are constant on stream lines $h(x)$ which can be formulated mathematically by

$$\Psi(x, h(x)) = \text{const.} \quad (13)$$

Considering the stream line in an $O(\varepsilon)$-neighborhood of the boundary $Z = 0$, (13) defines a parametrization of $h$ implicitly via

$$h(x) = \frac{2B}{u} \frac{\hat{\omega}}{Nk_x} \cos(k_x x + k_z h(x)) + O(\varepsilon). \quad (14)$$

Here, we have essentially presented the boundary condition of Lilly and Klemp [1979, Eq. 7], who define the condition in terms of the vertical displacement of an air parcel $\delta = -\Psi/u$, but in the framework of Grimshaw’s dissipative modulation equations. Note that (14) is of the form

$$y = f(y) \quad (15)$$
with \( y(\xi) = k_x h, \xi = k_x x \) and \( f = q \cos(\xi + y) \) where

\[
q = \frac{2B \hat{\omega} k_z}{u N k_x},
\]

Thus, the stream line can be interpreted as a fixed point of \( f \). For every fixed \( \xi \) the differentiable function \( f \) of \( y \) has a Lipschitz constant

\[
L = \left| \sup_y f'(y) \right| = |q|.
\]

Banach’s fixed point theorem states if \( L = |q| < 1 \), then \( f \) is a contraction and has hence a fixed point. The results of a fixed-point iteration \( y_{n+1} = F(y_n) \) with \( y_0 = 0 \) are plotted in figure 1. It can be observed how the stream lines steepen when \( q \) is increased whereas for small values the stream lines approximate the sinusoidal (linear) profile. Exploiting the polarization, we can write

\[
q = \frac{u'}{u}, \quad u' = 2|u_{0,1}| = \frac{2B \hat{\omega}|k_z|}{N k_x}
\]

which provides a measure for nonlinearity. If it is small one may assume linear wave excitation. The factor 2 appears simply because of the definition of amplitude due to (6). Also, using polarization and no-energy-flux in combination with the convergence condition for the fixed-point iteration, we obtain

\[
N^2 > |k_z|b', \quad b' = 2BN
\]

which confirms the classical condition for static stability, lines of constant potential temperature must not overturn [Bökönyi et al., 2016].
A boundary condition close to the surface but sufficiently far away to be considered free-slip assuming \( \Lambda(0) = 0 \) can be prescribed by a periodic mountain ridge with period \( P \) and maximum mountain height \( H_m \). In terms of the nonlinearity parameter \( k_x = 2\pi/P \) and the no-energy-flux condition \( (11) \), we obtain that

\[
H_m = \frac{2B}{N} \quad \text{at} \quad Z = 0. \tag{20}
\]

The combined boundary condition from \( (20) \) and \( (11) \) may be written in vector form as

\[
B(y) = \left( \frac{\dot{\omega} + k_x u}{H_m N^2 - 2\omega \dot{\omega}} \right) = 0 \quad \text{at} \quad Z = 0. \tag{21}
\]

4 Far-field condition - limit behavior

Additionally to the lower boundary, the limit of the solution as \( Z \to \infty \) must be specified in order to obtain a physical wave solution. For this argument, we exploit the global energy as derived from \( (9) \) being

\[
\int_0^\infty \rho e \, dZ = \int_0^\infty \frac{1}{2} \rho u^2 \, dZ + \int_0^\infty \rho a \dot{\omega} \, dZ. \tag{22}
\]

A physical wave must be such that the global energy is finite.

In the thermosphere temperature approaches an equilibrium \( T_\infty \) as altitude increases \cite{Schmidtke1984}. Hence, it is save to assume that \( N \to N_\infty \) and \( \eta \to \eta_\infty < 0 \) as \( Z \to \infty \) (cf. \( (3a) \) and \( (3b) \)). Due to the high viscosity in the thermosphere and above, we can also assume that \( u, a \to 0 \) as \( Z \to \infty \). These assumptions suffice that the integrals in \( (22) \) converge and also that the energy flux in \( (10) \) vanishes.

5 The antitriptic flow assumption and momentum conservation

In the inviscid limit the modulation equations \( (1) \) assume stationary solutions where \( \partial p/\partial X = 0 \) which can be computed analytically by a formula of \cite{Schlutow2017}, Eq. 5.20]. When we multiply \( (1b) \) by \( k_x \) and subtract \( (1c) \), we obtain the evolution equation for total momentum density,

\[
\rho \frac{\partial}{\partial T} (k_x a - u) = \frac{\partial p}{\partial X} - k_x \Lambda |k|^2 \rho a. \tag{23}
\]

Thus, to be consistent with the inviscid limit, the dissipative modulation equations assume stationary solutions if the right hand side of \( (23) \) vanishes which provides eventually a closure for the horizontal gradient of mean-flow kinematic pressure,

\[
\frac{\partial p}{\partial X} = \Lambda |k|^2 \rho k_x a. \tag{24}
\]
This result implicates that the mean-flow horizontal kinematic pressure gradient balances the dissipation of horizontal pseudo-momentum, \( \rho k_x a \). Thereby, total momentum is locally conserved. Also, let us remark that there is no vertical momentum flux. Gravity waves carry only pseudo-momentum [cf. McIntyre, 1981].

A flow configuration where pressure gradient balances viscous forces is referred to as antitriptic flow in the literature [Jeffreys, 1922]. Under this antitriptic flow assumption the modulation equations degenerate as we can integrate (23) with respect to time to obtain the mean-flow horizontal wind,

\[
u(Z, T) = k_x a(Z, T) + \bar{U}(Z), \tag{25}\]

reducing to a diagnostic variable. We call \( \bar{U} \) the background horizontal wind since it is time-independent and we emphasize that in the absence of a wave \( (a = 0) \), mean-flow and background horizontal wind coincide. The difference between mean-flow and background horizontal wind can be identified as specific horizontal pseudo-momentum. Note that in weakly nonlinear wave theory, the pseudo-momentum is of higher order and would not appear in a leading-order equation as shown here.

Substituting (25) into the governing equations (1) reduces the dimension of the system. We may reformulate the reduced system in vector form

\[
\frac{\partial y}{\partial T} + \frac{\partial F(y)}{\partial Z} = G(y) \tag{26}
\]

where we call \( y = (k_z, a)^T : [0, \infty)^2 \to \mathbb{R}^2 \) the prognostic state vector. The nonlinear flux and inhomogeneity are determined by

\[
F = (\dot{\omega} + k_x u, \dot{\omega}' \rho a)^T, \tag{27a}
\]

\[
G = (0, -(\Lambda |k| + \eta \dot{\omega}') a)^T. \tag{27b}
\]

### 6 Stationary solution

This section will examine stationary solutions to Grimshaw’s dissipative modulation equations. They describe typical mountain lee waves which are exited by a background flow over a mountain.

#### 6.1 Derivation for stationary waves

A stationary solution \( y = Y(Z) = (K_z, A)^T \) must fulfill

\[
\frac{\partial F(Y)}{\partial Z} = G(Y). \tag{28}
\]

Note that we label the stationary solution with capital letters. The first component of (28) can readily be integrated, so

\[
\text{const.} = \Omega = \dot{\Omega} + K_x U. \tag{29}
\]
with \( U(A) = K_x A + \bar{U} \) and \( \hat{\Omega}^{(n)}(n) = \dot{\hat{\omega}}^{(n)}(K_x) \). We can solve (29) explicitly exploiting (2) for

\[
K_x(A) = \begin{cases} 
-K_x\sqrt{N^2/(K_x \Omega) - 1}, & \text{(nh)}, \\
K_x N/(K_x \Omega), & \text{(h)}.
\end{cases}
\] (30)

The second component of (28) becomes an explicit, non-autonomous, scalar, ordinary differential equation when we insert (30),

\[
\frac{\partial A}{\partial Z} = \Gamma(A, Z) A \quad \text{in} \quad (0, \infty),
\] (31a)
\[
\Gamma = \eta\left(Re_{\text{wave}} - 1\right) - (\alpha\hat{\Omega})\partial Z \ln(N) + \alpha K_x \partial Z \bar{U} - K_x^2 \alpha A,
\] (31b)
\[
\alpha = \frac{\hat{\Omega}''}{\hat{\Omega}^2},
\] (31c)
\[
Re_{\text{wave}} = \frac{\eta [\hat{\Omega}']^2}{\Lambda |K|^2}.
\] (31d)

Definition (31d) can be interpreted as a wave-Reynolds number. It was introduced and discussed by Schlutow [2019] and measures, roughly speaking, the damping of wave amplitude.

From the boundary condition (21), \( B(Y) = 0 \) at \( Z = 0 \), we obtain

\[
\Omega = 0,
\] (32)
\[
A^\pm = \frac{|\bar{U}|}{2K_x} \left( 1 \mp \sqrt{1 - 2Fr_{\text{wave}}^2} \right) \quad \text{at} \quad Z = 0
\] (33)

where

\[
Fr_{\text{wave}} = \frac{H_m N}{|\bar{U}|} \bigg|_{Z=0} \leq \frac{1}{\sqrt{2}}.
\] (34)

We can rule out \( A^- \) immediately as non-physical solution as there must be no wave if \( H_m = 0 \). The constant \( Fr_{\text{wave}} \) possesses an upper bound in order to obtain real-valued wave action. It is often referred to as “non-dimensional mountain height” in the literature. Other names are also common. However, we follow a recent discussion by Mayer and Fringer [2017] who argue to call it wave-Froude number.

To meet the requirements in the deep-atmosphere limit from section 4, \( A \to 0 \), we must find \( \Gamma < 0 \) above a certain height. By close inspection of (31b), we obtain therefore

\[
Re_{\text{wave}} < 1 \quad \text{for} \quad Z > Z_{\text{turbo}}.
\] (35)

\( Z_{\text{turbo}} \) marks the turbopause, i.e. the transition zone from turbulent-dominated damping to energy diffusion by molecular viscosity [Nicolet, 1954].
6.2 Illustrative example

An illustrative example for a typical mountain wave is plotted in figure 2. The purpose of this plot is to demonstrate the level of generality of the stationary solution defined by (28). We took observation data from the zonal mean COSPAR International Reference Atmosphere (CIRA-86, [Fleming et al. 1990]) for the dimensional temperature profile $T^*$ (Panel a) and the dimensional background horizontal wind $U^*$ (Panel d). The values are taken for March in the northern hemisphere at 50° as here conditions for waves that extend deep into the atmosphere are optimal. We defined the dimensional kinematic viscosity profile $\Lambda^*(z^*) \propto \tanh$ in extrapolation of [Walterscheid and Hickey 2011, Fig. 1] since viscosity is unavailable in the CIRA-86 data set. In contrast to the common definition, we span the zonal $x$-axis from East to West due to our sign convention ($k_x > 0$).

Background and wave fields are computed accordingly (see figure caption for equation references). The data set contains important features of the atmosphere such as clearly pronounced troposphere, stratosphere, mesosphere and thermosphere due to the temperature inflection points. Also, the polar and mesospheric jets are distinctly visible. A typical wave envelope profile is generated (panel d) that grows exponentially with height as the background density decreases until viscosity starts to dominate damping the wave to disappearance.

7 Modulational stability of the stationary solution

This section is dedicated to the stability of the stationary solution of the modulation equations. In order to assess stability we linearize the governing equations (26) and the boundary condition (21) around the stationary solution (28) and apply the ansatz for the perturbation

$$y(Z, T) = \hat{y}(Z)e^{\lambda T}.$$ 

(36)

This transforms the problem of stability into a boundary eigenvalue problem (BEVP),

$$\lambda \hat{y} + \frac{\partial}{\partial Z} \left[ D\hat{F}(Y)\hat{y} \right] = D\hat{G}(Y)\hat{y} \quad \text{for} \quad Z \in (0, \infty),$$

(37a)

$$DB(Y)\hat{y} = 0 \quad \text{at} \quad Z = 0$$

(37b)

with the Jacobian matrices

$$D\hat{F}(Y) = \begin{pmatrix} \hat{\Omega}'K_x & K_x^2 \\ \hat{\Omega}'A & \hat{\Omega}' \end{pmatrix},$$

(38a)

$$D\hat{G}(Y) = \begin{pmatrix} 0 & 0 \\ -\eta\hat{\Omega}'A - 2\Lambda K_x A & -\eta\hat{\Omega}' - \Lambda|K|^2 \end{pmatrix},$$

(38b)

$$DB(Y) = \begin{pmatrix} \hat{\Omega}' & K_x^2 \\ -2A\hat{\Omega}' & -2\hat{\Omega}' \end{pmatrix}.$$
Figure 2: Illustrative example. CIRA-86 zonally averaged temperature (panel a) and background horizontal wind (panel d, thick blue line) for March at 50° N. Kinematic viscosity (panel b), Brunt-Väisälä frequency (panel c) computed by (3a). Vertical wave number (panel e) and specific wave action density (panel f) computed by (30) and (31a), respectively, assuming horizontal wavenumber $K_x^* = 0.2 \text{ km}^{-1}$. Wave-induced mean flow (panel d, dashed blue line).
of the complex plane. We can decompose the spectrum into the essential (continuous) and the point (matrix-like) spectrum. A comprehensive introduction in this method can be found in Sandstede [2002]. In the following sections, we will study each part of the spectrum individually.

7.1 Essential spectrum

The linear operator $L$ of the BEVP can be approximated by an asymptotic differential operator having constant coefficients, $L_\infty$. Utilizing Fredholm operator theory one can prove that it has the same essential spectrum as the original operator [Kapitula and Promislow, 2013]. The BEVP of the asymptotic operator can be written as an ordinary differential equation (ODE)

$$\frac{\partial \hat{y}}{\partial Z} = C_\infty(\lambda)\hat{y},$$

(39)

where the constant coefficient matrix is given by

$$C_\infty(\lambda) = \lim_{Z \to +\infty} C(Z, \lambda),$$

(40)

$$C(Z, \lambda) = DF(Y)^{-1}(DG(Y) - \frac{\partial DF(Y)}{\partial Z} - \lambda).$$

(41)

Existence of the limit is granted due to the assumptions of section 4. The asymptotic operator $L_\infty - \lambda$, and hence the original operator $L - \lambda$, are Fredholm if $C_\infty$ is hyperbolic, i.e. all its eigenvalues have non-zero real part. We find two distinct spatial eigenvalues

$$\nu_1(\lambda) = -\frac{\lambda}{\hat{\Omega}_\infty'},$$

(42a)

$$\nu_2(\lambda) = -\frac{\lambda + \eta_\infty'\hat{\Omega}_\infty' + \Lambda_\infty|K_\infty|^2}{\hat{\Omega}_\infty'}.$$

(42b)

Thus, the Morse index, which is defined as the dimension of the unstable subspace of a hyperbolic matrix, is

$$i_\infty(\lambda) = \begin{cases} 0 & \text{if } 0 < \Re(\lambda), \\ 1 & \text{if } -\eta_\infty'\hat{\Omega}_\infty' - \Lambda_\infty|K_\infty|^2 < \Re(\lambda) < 0, \\ 2 & \text{if } \Re(\lambda) < -\eta_\infty'\hat{\Omega}_\infty' - \Lambda_\infty|K_\infty|^2. \end{cases}$$

(43)

Note that $-\eta_\infty'\hat{\Omega}_\infty' - \Lambda_\infty|K_\infty|^2 < 0$ due to (35). On lines in the complex plane where $\Re(\lambda) = 0$ and $\Re(\lambda) = -\eta_\infty'\hat{\Omega}_\infty' - \Lambda_\infty|K_\infty|^2$ the matrix $C_\infty$ is not hyperbolic and hence the operator is not Fredholm. Eventually, the Fredholm index tells us where the essential spectrum lies. According to [Ben-Artzi et al., 1993, their formula 1.10] and also [Gohberg et al., 1990, p 391], it can be written as

$$\text{ind} = \dim(\ker DB(Y_0)) - i_\infty(\lambda)$$

(44)
where \( Y(0) = Y_0 \). The essential spectrum is the set of \( \lambda \)'s for which the operator \( L - \lambda \) is Fredholm but \( \text{ind} \neq 0 \) or it is not Fredholm. The point spectrum, on the other hand, lies where the operator is Fredholm and \( \text{ind} = 0 \) but the operator is not invertible. We will investigate the point spectrum in section 7.2. Having a closer look on (43), it turns out that

\[
\dim(\ker DB(Y_0)) = 0
\]  

must be true in order to obtain a stable essential spectrum, i.e. no essential spectrum on the right hand side of the complex plane. In particular, if the kernel is non-empty, the waves are unstable and the operator is even ill-posed as the complete right plane is in the essential spectrum. The criterion can be rephrased: stable waves necessitate Dirichlet boundary conditions. This is violated and hence the wave destabilizes due to perturbations from the essential spectrum if

\[
\frac{H_{mN}}{|U|} = \sqrt{2} \quad \text{at } Z = 0
\]

which can be called the gross wave-Froude number. This modulational instability criterion is not determined by an inequality like most fluid dynamical stability criteria. In fact the instability condition has to be fulfilled by equality. It is therefore more similar to a catastrophic resonance condition.

### 7.2 Point spectrum

In this section we will prove the non-existence of unstable point spectrum. Let us assume a stable essential spectrum and the existence of an unstable eigenvalue. For this eigenvalue the Fredholm index is zero and hence it may belong to the point spectrum. Then, the eigenfunction solves the associated ODE

\[
\frac{\partial \hat{y}}{\partial Z} = C(Z, \lambda) \hat{y}.
\]

By assumption, the essential spectrum is stable and hence the kernel of the Jacobian of the boundary condition is empty or, in other words, we get a Dirichlet boundary condition, \( \hat{y} = 0 \) at \( Z = 0 \). The Dirichlet boundary is the initial condition for the ODE which then assumes the trivial solution, so there is no eigenfunction. This contradiction completes the proof that there is no unstable point spectrum. In conclusion, modulational instabilities, given that the resonance condition is true, originate from the essential spectrum.

### 7.3 Summary

We will summarize all results regarding stability of the stationary solution in this section. The gross wave-Froude number depending on the wave itself due to the induced mean
flow is specified in terms of (33) and (34) by

\[
\frac{H_m N}{|U|} = \frac{2Fr_{\text{wave}}}{\sqrt{1 - 2Fr_{\text{wave}}^2 + 1}} \quad \text{at } Z = 0.
\] (48)

In conclusion, stationary waves extending from the surface to the deep atmosphere, that experience

\[Re_{\text{wave}} < 0 \quad \text{above } Z_{\text{turbo}},\] (49)

are

- non-evanescent and statically stable if

\[
\begin{cases}
0, & (h), \\
\frac{H_m N}{|U|}, & (nh) < \frac{H_m N}{|U|} < \frac{1}{\sqrt{1 + H_m K_x^2}}, & (h), \\
\frac{1}{\sqrt{1 + H_m K_x^2}}, & (nh) < \frac{H_m N}{|U|}.
\end{cases}
\] (50)

Here, the lower bound originates from (30). It provides a real-valued and negative vertical wavenumber. The upper bound is due to (19) and guarantees static stability.

- modulationally stable if

\[
\frac{H_m N}{|U|} \neq \sqrt{2} \quad \text{at } Z = 0.
\] (51)

This non-resonance criterion stems from the investigation of the essential spectrum in section 7.1.

An illustration combining these criteria for the wave-Froude number computed by inversion of (48) is plotted in figure 3. It turns out that the resonance condition for the wave-Froude number is the same as the threshold that guarantees real-valued wave action in (33). Combing this result with the upper bound for real-valued wave action (34), we obtain a strict inequality,

\[Fr_{\text{wave}} < \frac{1}{\sqrt{2}} \] (52)

for the existence and modulational stability of the wave.

Also, beyond \(H_m K_x = \sqrt{2}\) waves cease to exist. Only solutions where \(H_m K_x > 1\) may become modulationally unstable since waves in the region \(H_m K_x < 1\) overturn before they ever reach the resonance condition.

8 Discussion

The main result of this paper is that the stability of stationary nonlinear gravity waves with respect to Grimshaw’s dissipative modulation equations depends on three characteristic parameters: The wave-Reynolds number, the wave-Froude number and the
Figure 3: Admissible wave-Froude number as function of height-period ratio for non-hydrostatic waves. For hydrostatic waves, it is the same result but $P \to \infty$ ($K_x = 0$). Only values in the blue area correspond to wave solutions.

mountain’s height-period ratio. Considering horizontally periodic waves from the surface to the deep atmosphere, we find that the stability is completely determined by the boundary and far-field conditions. These results are valid for fairly general wave solutions that posses only minor restrictions on the background fields: the background is hydrostatic and exhibits a physical far-field behavior. Other than this, the background temperature and horizontal wind are unconditioned.

We want to give some useful remarks on the characteristic parameters. When reformulated in dimensional variables, the wave-Reynolds number reads

$$R_{\text{wave}} = \frac{C_{gz}^* D^*}{\Lambda^*},$$

where $C_{gz}^*$ represents the vertical group velocity and $D^* = H_p^* |K^*|^{-2}$ defines a length scale with $H_p^*$ the local pressure scale height.

The net wave-Froude number as defined in this work is readily redimensionalized, so

$$F_{\text{wave}} = \frac{H_m^* N^*}{|U^*|}.$$
the background flow of a mountain wave which really is the flow without the mountain. However, knowing the wave parameters of the excited wave, it is possible to compute the background wind by the total momentum equation (25). Also, in weakly nonlinear theory they are indeed the same as the induced mean flow is a higher-order correction. Still, the nonlinear description in this paper reduces the mean-flow wind and therefore gains wave energy from the mean flow when excited. Hence, from a theoretical point of view the net wave-Froude number should not depend a priori on the wave that is excited by the background flow over the mountain.

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