INTRANSITIVE SELF-SIMILAR GROUPS

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Abstract. A group is said to be self-similar provided it admits a faithful state-closed representation on some regular \( m \)-tree and the group is said to be transitive self-similar provided additionally it induces transitive action on the first level of the tree. A standard approach for constructing a transitive self-similar representation of a group has been by way of a single virtual endomorphism of the group in question. Recently, it was shown that this approach when applied to the restricted wreath product \( \mathbb{Z} \wr \mathbb{Z} \) could not produce a faithful transitive self-similar representations for any \( m \geq 2 \) (see, [8]). In this work we study state-closed representations without assuming the transitivity condition. This general action is translated into a set of virtual endomorphisms corresponding to the different orbits of the action on the first level of the tree. In this manner, we produce faithful self-similar representations, some of which are also finite-state, for a number of groups such as \( \mathbb{Z}^\omega \), \( \mathbb{Z} \wr \mathbb{Z} \) and \( (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{C} \).

1. Introduction

Self-similar groups have been given directly as automorphism groups of some \( m \)-tree, as is the case of the infinite torsion group of Grigorchuk [11] and those of Gupta-Sidki [13], or constructed to act on such trees by way of a single virtual endomorphism. However these constructions necessarily produce transitive self-similar groups, in the sense that the corresponding state-closed group of automorphisms of the \( m \)-tree satisfies the additional property of acting transitively on the first level of the tree.

Transitive state-closed representations have been studied for the family of abelian groups, finitely generated nilpotent groups, as well as for metabelian groups, affine linear groups and arithmetic groups; see [2, 3, 15, 16] for more details.

It was shown recently that the group \( G = \mathbb{Z} \wr \mathbb{Z} \) fails to have a faithful transitive state-closed representation on an \( m \)-tree for any \( m \) [8]. Yet as we will prove, this group admits a faithful intransitive state-closed

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representation on the 3-tree. Indeed, in this representation the group is a 3-letter, 3-state automata group; it has the following diagram

![Diagram 1](image)

Given a self-similar group $G$ we will prove a number of results about self-similarity of its over-groups from the following types:

1. restricted direct product $G^{(\omega)}$ of countably many copies of $G$;
2. restricted wreath product $G \wr K$ where $K$ is finite; restricted wreath product $A \wr G$ where $A$ is abelian, both finitely and infinitely generated, and for particular cases where $G$ is abelian.

The two types of groups in item (2) are in accord with Gruenberg’s dichotomy of residually-finite wreath products [12]. Our results extend ones which have appeared in [2] and [8].

Given a state-closed subgroup $G$ of automorphisms of the $m$-tree indexed by strings from a set $Y$ of size $m$, a $G$-data $(m, H, F)$ is obtained as follows: suppose $G$ have $s$ orbits $Y_i = \{y_{i1}, ..., y_{im_i}\}$ in its action on $Y$ then $m = (m_1, ..., m_s)$. Define the set of subgroups

$$H = \{H_i \mid [G : H_i] = m_i \ (1 \leq i \leq s)\}$$

where $H_i = \text{Fix}_G(y_{i1})$; define the set of projections

$$F = \{f_i : H_i \to G \mid 1 \leq i \leq s\}.$$ 

On the other hand, given a group $G$, an $s$-set of subgroups of $G$

$$H = \{(H_i \mid [G : H_i] = m_i \ (1 \leq i \leq s))\},$$

$$m = (m_1, ..., m_s), \ m = m_1 + ... + m_s$$

and a set of virtual endomorphisms

$$F = \{f_i : H_i \to G \mid 1 \leq i \leq s\},$$

we have an abstract $G$-data $(m, H, F)$. We prove reciprocally
Proposition A. Given a group $G$, $m \geq 1$ and a $G$-data $(m, H, F)$. Then the data provides a state-closed representation of $G$ on the $m$-tree with kernel

$$\langle K \leq \cap_{i=1}^s H_i \mid K \triangleleft G, K^{f_i} \leq K, \forall i = 1, \ldots, s \rangle,$$

called the $F$-core of $H$.

The partition $m = (m_1, \ldots, m_s)$ is called the orbit-type of the representation.

We apply the Proposition A to obtain families of self-similar groups.

Theorem B. Let $G$ be a self-similar group of degree $m$ and orbit-type $(m_1, \ldots, m_s)$. Then the following hold.

1) $G^{(\omega)}$ admits a faithful state-closed representation of degree $m+1$ and orbit-type $(m_1, \ldots, m_s, 1)$; in particular, for $G = \mathbb{Z}$, the representation of the group $\mathbb{Z}^{(\omega)}$ is of orbit-type $(2, 1)$, and is in addition finite-state.

2) Let $K$ be a regular subgroup of $\text{Sym}([1, \ldots, s])$. Then the restricted wreath product $G \wr K$ admits a faithful state-closed representation of degree $(m_1 \cdot m_2 \cdot \ldots \cdot m_s) \cdot s$.

With respect to the first item, for any positive integer $k$, the direct product group $G^k$ is self-similar of degree $m$. It was shown in [2] that the group $\mathbb{Z}^{(\omega)}$ has a faithful transitive self-similar representation of degree 2, moreover and importantly, no such representation can also be finite-state for any degree. As a consequence of the tree-wreathing operation defined by Brunner and Sidki [7] and by using $k$-inflation, the group $\mathbb{Z} \wr \mathbb{Z}$ is self-similar of orbit-type $(2, 1)$; see Diagram 1. Then, by the second item of Theorem B, it follows that $(\mathbb{Z} \wr \mathbb{Z}) \wr C_2$ is a self-similar group of degree 4.

Restricted wreath products $A \wr G$ for $A$ abelian and $G = \mathbb{Z}^d$ have been a good source for automata groups. The first instance in this family is the classical Lamplighter group $C_2 \wr \mathbb{Z}$. It was shown in [8] that if $A \wr \mathbb{Z}^d$ admits a faithful transitive self-similar representation then $A$ is necessarily a torsion group of finite exponent. Also, it was proven in [2, Proposition 6.1] that when $B$ is a finite abelian group, then $B \wr \mathbb{Z}^d$ is an automata group of degree $2|B|$.

We generalize both results as follows

Theorem C. Let $A$ be a finitely generated abelian group and $B = \text{Tor}(A)$. Then $G = A \wr \mathbb{Z}^d$ is an automata group of degree $2|B| + 4$. In particular, for $A = \mathbb{Z}^l$, the degree can be reduced to 4.

The theorem will follow from a general process of concatenation (see Proposition 5.2) of the two cases $G_1 = B \wr \mathbb{Z}^d$ and $G_2 = \mathbb{Z}^l \wr \mathbb{Z}^d$. We
note that the result for $\mathbb{Z} \wr \mathbb{Z}$ answers positively a question posed by A. Woryna in [23, page 100].

It was shown in [9] that the group $C_p \wr \mathbb{Z}^d$ where $C_p$ is cyclic of prime order $p$ and $d \geq 2$ is self-similar of degree $p^2$, but that such a group does not have a faithful transitive state-closed representation of prime degree. In this context we prove:

**Theorem D.** Let $p$ a prime number then $C_p \wr \mathbb{Z}^2$ is a self-similar group of degree $p + 1$ of orbit-type $(p, 1)$. Indeed, $C_p \wr \mathbb{Z}^2$ is generated by $\alpha = (\alpha, \alpha \sigma, \ldots, \alpha \sigma^{p-1}, \alpha \beta)$, $\sigma = (e, \ldots, e, \sigma)(0 \ldots p - 1)$ and $\beta = (e, \ldots, e, \alpha)$. In particular, the group $C_2 \wr \mathbb{Z}^2$ is self-similar of degree 3.

Let $f : H \to G$ be a virtual endomorphism. Set $G_0 = G$ and $G_n = G_{n-1}^{f-1}$ for all $n \geq 1$. Define the parabolic subgroup $G_\omega = \cap_{j \geq 0} G_j$. Let $G_\omega \setminus G$ denote the set of right cosets $G_\omega$ in $G$, let $A^{(G_\omega \setminus G)} = \{ \phi : G_\omega \setminus G \to A \text{ of finite support} \}$ and have $g \in G$ act on it by translation. The following result was proven for transitive self-similar groups in [2].

**Proposition.** (Proposition 6.1) Let $G$ be a transitive self-similar group of degree $m$ and parabolic subgroup $G_\omega$, and let $B$ be a finite abelian group. Then the extension $B^{(G_\omega \setminus G)} \rtimes G$ is transitive self-similar of degree $|B|m$, and is finite-state whenever $G$ is.

Let $G$ be a state-closed group with respect the data $(m, H, F)$, where $m = (m_1, \ldots, m_s)$, with $m_1 \geq 2, \ldots, m_s \geq 2$. For each $i = 1, \ldots, s$ define $G_{i0} = G$, $G_{ij} = (G_{i(j-1)}^{f-1})^{f-1}$ ($j > 0$) and $G_{i\omega} = \cap_{j \geq 0} G_{ij}$. With this notation we have:

**Theorem E.** Let $B$ be a finite abelian group and $G$ be a self-similar group of orbit-type $(m_1, \ldots, m_s)$, with $m_1 \geq 2, \ldots, m_s \geq 2$. Then the group $B^{(G_{\omega_1} \setminus G) \times \cdots \times (G_{\omega_s} \setminus G)} \rtimes G^s$ is self-similar of orbit-type $(|B|m_1 \ldots m_s, 1)$.

A question which has remained open is whether the group $C_2 \wr (\mathbb{Z} \wr \mathbb{Z})$ is self-similar.

2. Preliminaries

2.1. Groups acting on rooted $m$-trees. The vertices of a rooted $m$-tree $\mathcal{T}_m$ are indexed by strings from an alphabet $Y$ of $m \geq 1$ letters, ordered by $u < v$ provided the string $v$ is a prefix of $u$. The tree $\mathcal{T}_m$ is also denoted as $\mathcal{T}(Y)$; normally, we chose to take $Y = \{0, 1, \ldots, m - 1\}$. Given a group and a representation of it on $\mathcal{T}_m$ we say both the group and its representation have degree $m$. 
The automorphism group $A_m$, or $A(Y)$, of $T_m$ is isomorphic to the restricted wreath product recursively defined as $A_m = A_m \wr S_m$, where $S_m$ is the symmetric group of degree $m$. An automorphism $\alpha$ of $T_m$ has the form $\alpha = (\alpha_0, \ldots, \alpha_{m-1})\sigma(\alpha)$, where the state $\alpha_i$ belongs to $A_m$ and where $\sigma : A_m \to S_m$ is the permutational representation of $A_m$ on $Y$, the first level of the tree $T_m$. Successive developments of the automorphisms $\alpha_i$ produce $\alpha_u$ for all vertices $u$ of the tree. For $k \geq 1$, the action of $\alpha$ on a string $y_1 y_2 \ldots y_k \in Y^k$ is as follows

$$\alpha : y_1 y_2 \ldots y_k \mapsto (y_1)^{\sigma(\alpha)} (y_2 \ldots y_k)^{\alpha_y}.$$ 

This implies that $\alpha$ induces an automorphism $\alpha_k$ on the $m^k$-tree $T(Y^k)$ and the above action on $k$-stings gives us a group embedding $A(Y) \to A(Y^k)$ which we call $k$-inflation.

For $\alpha \in A(Y)$, the set of automorphisms

$$Q(\alpha) = \{\alpha, \alpha_0, \ldots, \alpha_{m-1}\} \cup_{i=0}^{m-1} Q(\alpha_i)$$

is called the set of states of $\alpha$ and this automorphism is said to be finite-state provided $Q(\alpha)$ is finite. A subgroup $G$ of $A_m$ is state-closed (or, self similar) if $Q(\alpha)$ is a subset of $G$ for all $\alpha$ in $G$. More generally, a subgroup $G$ of $A_m$ is $k$th-level state-closed if for all $\alpha \in G$, the states $\alpha_u$ belong to $G$ for all strings $u$ of length $k$; we see then that the $k$-inflation of $G$ is state-closed. A group which is finitely generated, state-closed and finite-state is called an automata group.

2.2. Virtual endomorphisms. Given a subgroup $H$ of $G$ of finite index $m$, a homomorphism $f : H \to G$ is called a virtual endomorphism of $G$. A subgroup $K$ of $H$ is $f$-invariant provided $K^f \subseteq K$. The maximal subgroup of $H$ which is both $f$-invariant and normal in $G$ is called the $f$-core of $H$; if this subgroup is trivial then $f$ is said to be simple. If $G$ is a self-similar subgroup of $A_m$, then for $H = \text{Fix}_G(0)$, the subgroup stabilizer of the vertex $0 \in Y$, we have the projection $f : H \to G$ which can be seen to be simple.

Let $G$ be a group, $H$ a subgroup of $G$ of finite index $m$ with right transversal $T = \{t_0, \ldots, t_{m-1}\}$ in $G$. Then the induced permutation representation $\sigma$ of $G$ on $T$ is transitive and we define the Scheier function $\theta : G \times T \to H$ by $\theta(g, t_i) = t_i g(t_j)^{-1}$ where $Ht_j = H t_i g$.

2.3. Recursive Kaloujnine-Krasner. The Kaloujnine-Krasner Theorem [14] provides us with an extension of $\sigma$ to a representation $\varphi : G \to H \wr G^\sigma$ defined by

$$\varphi : g \mapsto (\theta(g, t_0), \ldots, \theta(g, t_{m-1})) g^\sigma.$$
On applying the virtual endomorphism \( f \) to \( \theta(g, t_i) \), we obtain \( \theta(g, t_i)^f \in G \) which allows us to repeat the above representation \( \varphi \) to this element. Thus the Kaloujnine-Krasner representation extends recursively to a representation on the \( m \)-tree \( T_m \), indicated by the same symbol, as follows

\[
\varphi : g \mapsto \left( \theta(g, t_0)^{f^{\sigma}}, ..., \theta(g, t_{m-1})^{f^{\sigma}} \right) g^{\sigma}.
\]

The kernel of the representation \( \varphi \) is the \( f \)-core of \( H \) \cite{15}.

Given a group \( G \), \( m \geq 1 \) and \( G \)-data \( (m, H, F) \) we re-work the recursive representation in the transitive case as follows. For each \( H_i \), choose a right transversal \( T_i = \{t_{i1}, ..., t_{im_i} \} \) and let \( \theta_i \) be the corresponding Schreier function. Then define \( \varphi : G \to A_m \) by

\[
\varphi : g \mapsto \left( \theta_i(g, t)^{f_i^{\sigma}} \mid 1 \leq i \leq s, t \in T_i \right) g^{\sigma}.
\]

3. Proof of Proposition A

We reformulate the proposition more concretely as follows

**Proposition A.** Given a group \( G \), \( m \geq 1 \) and \( G \)-data \( (m, H, F) \), the function \( \varphi : G \to A_m \) defined by

\[
g \mapsto \left( \theta_i(g, t)^{f_i^{\sigma}} \mid 1 \leq i \leq s, t \in T_i \right) g^{\sigma}.
\]

is a state-closed representation of \( G \) on the \( m \)-tree with kernel the \( F \)-core of \( H \).

**Proof.** Consider \( g \) and \( h \) elements of \( G \). Clearly \( (gh)^{\sigma} = g^{\sigma}h^{\sigma} \), so if \( i \in Y \), then

\[
\varphi(gh) = \varphi^i = \varphi^i = \varphi^h = \varphi^{gh}.
\]

By induction on the length of word in \( Y \), we have that

\[
(iu)(gh) = i(g^h)^{\sigma} u^{(gh)^{\sigma_i}} = i(g^h)^{\sigma} u^{(gh)^{\sigma_i}} = (iu)^{gh}.
\]

Therefore \( \varphi \) is a homomorphism. By construction, \( G^\varphi \) is state-closed.

Let \( K \) be the \( F \)-core of \( H \) and \( x \in K \). Then \( x^{\sigma} = 1 \); in fact, \( k^{x^{\sigma}} = l \) if and only if \( H_t x = H_t x^{\sigma} \) for some \( i = 1, ..., s \). But \( H_t x = H_t x^{\sigma} \), so \( k = 1 \) and \( x^{\sigma} = 1 \). Since \( K^{f_i} \) is a trivial permutation for all \( i = 1, ..., s \) and \( t \in T_i \). So \( x \in \ker \varphi \). Clearly \( \ker \varphi \leq \cap_{i=1}^s H_i \), \( \ker \varphi \triangleleft G \) and \( \ker \varphi^{f_i} \leq \ker \varphi \), \( \forall i = 1, ..., s \).

**Proposition 3.1.** The group \( \mathbb{Z} \wr \mathbb{Z} \) is a 3-state and 3-letter automata.
Proof. First we prove that \( \mathbb{Z} \wr \mathbb{Z} \) has a faithful state-closed representation of degree 4 which we then reduced to degree 3.

By Brunner and Sidki [7, Theorem 1], if \( H \) is an abelian subgroup of \( \mathcal{A}_2 \) and \( \alpha = ((e, e), (e, e)(0, 1)) \), then \( \langle \tilde{H}, \alpha \rangle \simeq \tilde{H} \wr \langle \alpha \rangle \), where \( \tilde{H} = \{ \tilde{h} = ((\tilde{h}, h), (e, e)) \mid h \in H \} \). Note that \( \tilde{H} \simeq H \). If \( H = \langle \alpha \rangle \), then \( \tilde{H} \wr \langle \alpha \rangle = \langle \tilde{\alpha} \rangle \simeq \mathbb{Z} \wr \mathbb{Z} \). Since \( \langle \tilde{\alpha} \rangle \simeq \mathbb{Z} \wr \mathbb{Z} \), it follows that \( \langle \tilde{\alpha} \rangle \) is level two, state-closed group.

Thus, by 2-inflation, \( \Psi : G \rightarrow \mathcal{A}_4 \)

\[
(\langle \tilde{\alpha} \rangle \wr \langle \alpha \rangle)^\Psi = \langle \tilde{\alpha} \rangle^\Psi = \langle \tilde{\alpha}, \alpha^\Psi, e, e \rangle, \alpha^\Psi = (e, e, (e, e))(0, 2)(1, 3) \simeq \mathbb{Z} \wr \mathbb{Z}
\]

is a faithful state-closed representation of \( \mathbb{Z} \wr \mathbb{Z} \) of degree 4.

Now note that the map

\[
\alpha^\Psi \mapsto \alpha_1 = (e, \alpha_1, e)(0, 1) \\
\tilde{\alpha}^\Psi \mapsto \beta = (\beta, e, \alpha_1)
\]

extends to an isomorphism \( \phi \) from

\[
\langle \tilde{\alpha}^\Psi = \langle \tilde{\alpha}, \alpha^\Psi, e, e \rangle, \alpha^\Psi = (e, e, (e, e))(0, 2)(1, 3) \rangle
\]

to

\[
\langle \beta = (\beta, e, \alpha_1), \alpha_1 = (e, \alpha_1, e)(0, 1) \rangle.
\]

Therefore \( \mathbb{Z} \wr \mathbb{Z} \) is a 3-state and 3-letter automata, as in Diagram 1. \( \square \)

We prove in Section 5 a more general form of this proposition.

4. PROOF OF THEOREM B

**Theorem B.** Let \( G \) be a self-similar group of degree \( m \) and orbit-type \( (m_1, \ldots, m_s) \). Then the following hold.

1) \( G^{(\omega)} \) admits a faithful state-closed representation of degree \( m + 1 \) and orbit-type \( (m_1, \ldots, m_s, 1) \); in particular, for \( G = \mathbb{Z} \), the representation of the group \( \mathbb{Z}^{(\omega)} \) is of orbit-type \( (2, 1) \), and is in addition finite-state.

2) Let \( K \) be a regular subgroup of \( \text{Sym} \{1, \ldots, s\} \). Then the restricted wreath product \( G \wr K \) admits a faithful state-closed representation of degree \( m_1 \cdot m_2 \cdot \ldots \cdot m_s \cdot s \); in particular, the group \( (\mathbb{Z} \wr \mathbb{Z}) \wr C_2 \) is finite-state and self-similar of degree 4.

**Proof.** Let \( G \) be a state-closed group. By Proposition A, there is data \( (m, H, F) \) such that \( H \) is \( F \) core-free.
(1) The subgroup \( L_i = \{(h, g_2, g_3, \ldots) \in G(\omega) \mid h \in H_i\} \) has index \( m_i \) in \( G(\omega) \). For each \( i = 1, \ldots, s \) define the homomorphism \( \bar{f}_i : L_i \to G(\omega) \) by

\[
(h, g_2, g_3, \ldots)\bar{f}_i = (h^{h_i}, g_2, g_3, \ldots)
\]

and the homomorphism \( \bar{f}_{s+1} : L_{s+1} = G(\omega) \to G(\omega) \) by

\[
(g_1, g_2, \ldots)\bar{f}_{s+1} = (g_2, g_3, \ldots).
\]

It is clear that

\[
(L \leq \bigcap_{i=1}^{s+1} L_i \mid L < G(\omega), L\bar{f}_i \leq L, \forall i = 1, \ldots, s + 1)
\]

is trivial. By Proposition A, the group \( G(\omega) \) is state-closed with respect to the data \((\mathbf{m}, \mathbf{H}, \mathbf{F})\), where \( \mathbf{m} = (m_1, \ldots, m_s, 1) \), \( \mathbf{H} = \{L_1, \ldots, L_s, L_{s+1}\} \) and \( \mathbf{F} = \{\bar{f}_1, \ldots, \bar{f}_s, \bar{f}_{s+1}\} \).

Note that \( f : 2\mathbb{Z} \to \mathbb{Z} \) defined by \( 2n \mapsto n \) is a simple virtual endomorphism and therefore \( \mathbb{Z} \) is self-similar of degree 2. Thus the group \( \mathbb{Z}(\omega) \) is self-similar with respect to the data \((\{2, 1\}, \{L_1, L_2\}, \{f_1, f_2\})\) where \( f_1 : (2n_1, n_2, \ldots) \mapsto (n_1, n_2, \ldots) \) and \( f_2 : (n_1, n_2, \ldots) \mapsto (n_2, n_3, \ldots) \). On defining the transversals \( T_1 = \{e, (1, 0, 0, \ldots)\} \) and \( T_2 = \{e\} \) of \( L_1 \) and \( L_2 \) respectively, we obtain the following representation of \( \mathbb{Z}(\omega) \)

\[
\langle \alpha_1 = (e, \alpha_1, e)(0 1), \alpha_i = (\alpha_i, \alpha_i, \alpha_{i-1}) \mid i = 2, 3, 4, \ldots \rangle
\]

which is faithful and finite-state.

\[
\begin{array}{cccc}
0|0, 1|1, 2|2 & 1|0 & 0|0, 1|1 & 0|0, 1|1 \\
\alpha_1 & 2|2 & \alpha_2 & 2|2 \\
0|1, 2|2 & 2|2 & \alpha_3 & 2|2 \\
\end{array}
\]

Diagram 2

(2) Let \( l \) be an integer such that \( H_i \neq G \) for \( 1 \leq i \leq l \) and \( H_i = G \) for \( l + 1 \leq i \leq s \). Define \( H = H_1 \times \ldots \times H_s \) and \( W = G \triangleright K \). Then \([W : H] = s(m_1 \ldots m_l)\) and the endomorphism \( f : H \to W \) given by

\[
(h_1, \ldots, h_s) \mapsto (h_1^{f_1}, \ldots, h_s^{f_s})
\]

is well-defined. Let \( L \) be a subgroup of \( H \), normal in \( W \), and \( f \)-invariant, and let \( g = (g_1, \ldots, g_s) \in L \). Since \( K \) is a transitive group of degree \( s \) follows that for each \( 1 \leq i \neq j \leq s \) there exists \( h \in K \) such that \((i)h = j\). But

\[
g^{hf} = (g_{(1)h}, \ldots, g_{(s)h}) = (g_{(1)h}, \ldots, g_{(s)h}) \in L.
\]
Thus \( g_r \in \langle K \leq \cap_{i=1}^s H_i \mid K \triangleleft G, K_f^r \leq K, \forall i = 1, \ldots, s \rangle = \{1\} \) for each \( r = 1, \ldots, s \). Therefore \( L = \{1\} \) and \( G \wr K \) is self-similar of degree \( s.(m_1 \ldots m_l) \).

On applying Proposition 3.1 and Proposition A we obtain:
\[
(Z \wr Z) \wr C_2 \simeq \langle \sigma = (0, 2)(1, 3), \gamma = (\gamma, e, \alpha^\sigma, \alpha^\sigma), \alpha = (e, \alpha, e, e)(0, 1) \rangle;
\]
that is, the group \((Z \wr Z) \wr C_2\) is generated by the following 5-state and 4-letter automaton:

![Diagram 3](image)

5. PROOF OF THEOREM C

First we will prove the second case of Theorem C.

**Proposition 5.1.** Let \( G = Z^l \wr Z^d \). Then \( G \) is an automata group of degree 4. In case \( d = 1 \), the degree is 3.

**Proof.** Denote \( Z^l \) by \( A \) and \( Z^d \) by \( X \). Then, the normal closure of \( A \) in \( G \) is \( A^X \) and we have a semi-direct product form for the group \( G = A^X \cdot X \).

Define the subgroups
\[
H_1 = A^X \langle x_1^2, x_2, \ldots, x_d \rangle,
\]
\[
H_2 = H_3 = G, \bigcap_{i=1}^3 H_i = H_1.
\]
Also, for \( i = 1, 2, 3 \), define the homomorphisms \( f_i : H_i \to G \) which extend the maps:

\[
\begin{align*}
  f_1 & : x_i^2 \mapsto x_1, \quad x_i \mapsto x_i (2 \leq i \leq d), \\
  a_i^{x_2^a q(x_2, x_3, \ldots, x_d)} & \mapsto a_i^{x_2^a q(x_2, x_3, \ldots, x_d)} , \quad a_i^{x_1^a q(x_2, x_3, \ldots, x_d)} \mapsto e ;
\end{align*}
\]

\[
\begin{align*}
  f_2 & : x_i \mapsto x_{i-1} (1 \leq i \leq d), \\
  a_i^{p(x_1, x_2, \ldots, x_d)} & \mapsto a_i^{p(x_d, x_1, \ldots, x_{d-1})} (1 \leq i \leq l);
\end{align*}
\]

and

\[
\begin{align*}
  f_3 & : X \mapsto \{ e \} , \\
  a_i^{p(x_1, x_2, \ldots, x_d)} & \mapsto x_i^{p(1, 1, \ldots)}, \\
  a_i^{p(x_1, x_2, \ldots, x_d)} & \mapsto e (2 \leq i \leq l)
\end{align*}
\]

Then, \( A^X \langle x_2, \ldots, x_d \rangle \) is the \( f_1 \)-core of \( H_1 \) and both \( H_2, H_3 \) are their own \( f \)-cores.

Let \( K \) be the \( F \)-core of \( H \). Then \( K \leq A^X \langle x_2, \ldots, x_d \rangle \) and by applying \( f_2 \) we find \( K \leq A^X \).

Suppose \( K \) is non-trivial. Then, as \( K \) is normal in \( G \) it follows that \( K_+ = K \cap A^{Z[X]} \) is non-trivial. Elements \( h \) of \( K_+ \) have an unique form

\[ h = a_1^{p_1} a_2^{p_2} \ldots a_l^{p_l} \]

where \( p_i = p_i(x_1, x_2, \ldots, x_d) \in \mathbb{Z}[X] \). Define \( \delta_{x_j}(h) \) to be the maximum \( x_j \)-degree of \( p_i \) for all \( i \). If the maximum of all \( \delta_{x_j}(h) \) occurs for \( j = k \) then by applying an adequate power of \( f_2 \) to \( h \), we may assume \( j = 1 \).

Choose an \( h \in K_+ \) such that \( h \neq e \) and which involves a minimum number of variables from \( \{ x_1, x_2, \ldots, x_d \} \).

(1) Suppose \( p_i \) is constant for all \( i \). There exists \( j \) such that \( p_j \neq 0 \). Then on applying \( f_3 \) to \( h \) we get \( x_1^{p_1} \neq e \in K \) which is impossible.

(2) Write \( \delta_{x_1}(h) = n \) then \( n \neq 0 \).

(2.1) Suppose \( n = 2k \). Then on applying \( f_1 \) to \( h \) we obtain \( h' \neq e \) and \( \delta_{x_1}(h') = k \) which is absurd.

(2.2) Suppose \( \delta_{x_1}(h) = 2k + 1 \). Then conjugate by \( x_1 \) to get

\[
\begin{align*}
  h' &= h x_1 = a_1^{p_1} a_2^{p_2} \ldots a_l^{p_l} \in K_+, \\
  p_i' &= p_i x_1, \quad \delta_{x_1}(h') = 2k + 2.
\end{align*}
\]
On applying $f_1$ to $h'$ we get $h'' \in K_+$ with $\delta_{x_1}(h'') = k + 1$. Now, $k + 1 < 2k + 1$, unless $k = 0$; that is, we have $\delta_{x_1}(h) = 1$ and for all $i$,

$$p_i = p_{i0} + p_{i1}x_1$$

where $p_{i0}, p_{i1} \in \mathbb{Z}[x_2, \ldots, x_d]$. Then, as before, for $h' = h^{x_1}$ we have $p'_i = p_{i0}x_1 + p_{i1}x_1^2$ and $h'' = (a^{p_{i1}}_1a^{p_{i2}}_2 \ldots a^{p_{i1}}_l)^{x_1} \in K_+ \setminus \{e\}$. Since

$$h'' = (h'')^{x_1^{-1}} = a^{p_{i1}}_1a^{p_{i2}}_2 \ldots a^{p_{i1}}_l \in K_+ \setminus \{e\}$$

which involves a lesser number of variables than $h$, we have a contradiction.

With notation of Proposition 5.1, a faithful self-similar representation of $G = \mathbb{Z}^l \wr \mathbb{Z}^d$ with respect the data

$$((2, 1), \{H_1, H_2 = G, H_3 = G\}, \{f_1, f_2, f_3\})$$

and the transversals $T_i$ of $H_i$ in $G$ defined by

$$T_1 = \{e, x_1\}, T_2 = T_3 = \{e\},$$

is

$$G^e = \langle \gamma_1, \ldots, \gamma_l \rangle \wr \langle \alpha_1, \ldots, \alpha_d \rangle$$

where

$$\gamma_1 = (\gamma_1, e, \gamma_1, \alpha_1), \ \gamma_2 = (\gamma_1, e, \gamma_2, e), \ \gamma_t = (\gamma_t, e, \gamma_t, e),$$

$$\alpha_1 = (e, \alpha_1, \alpha_d, e)(0 1), \ \alpha_2 = (\alpha_2, \alpha_2, \alpha_1, e), \ \alpha_d = (\alpha_d, \alpha_d, \alpha_d, e).$$

Therefore, $G^e$ is finitely generated, finite-state and self-similar; that is, $G$ is an automata group. We note that if $d = 1$ then $H_2$ and $f_2$ are superfluous. \hfill \Box

5.1. General concatenation. Let $G_1 = A_1 \wr U$, $G_2 = A_2 \wr U$ and $G = (A_1 \oplus A_2) \wr U$. For $i = 1, 2$, define the $G_i$-data $(m_i, H_i, F_i)$, where $m_i = (m_{i1}, \ldots, m_{is_i})$, $H_i = \{H_{i1}, \ldots, H_{is_i}\}$ and $F_i = \{f_{i1}, \ldots, f_{is_i}\}$. Furthermore, define the data $G$-data $(m, H, F)$, where $m$ is the concatenation $(m_1, m_2)$,

$$H = \left\{ H_{1j} = (A_2^U) \circ H_{1j} \ (1 \leq j \leq s_1) \right\}$$

$$\cup \left\{ H_{2k} = (A_1^U) \circ H_{2k} \ (1 \leq k \leq s_2) \right\},$$

$$F = \{ f_{1j}, \ldots, f_{s_1}, f_{21}, \ldots, f_{s_2} \}$$

where $f_{1j} : H_{1j} \to G$, $1 \leq j \leq s_1$, is defined by

$$f_{1j} : ah \mapsto h^{f_{1j}}, \text{ for } a \in A_2^U, h \in H_{1j}$$

and $f_{2k} : H_{2k} \to G$, $1 \leq k \leq s_2$, is defined by

$$f_{2k} : ah \mapsto h^{f_{2k}}, \text{ for } a \in A_1^U, h \in H_{2k}.$$
For $i = 1, 2$, let $G_i$ has its state-closed representation with respect to $(m_i, H_i, F_i)$ with $F_i$ core $K_i$. Likewise, let $G$ has its state-closed representation with respect to $(m, H, F)$ with $F$-core $K$.

**Proposition 5.2.** Maintaining the above notation:

1. $K \cap (A_1 \oplus A_2)^U = K_1 (A_2)^U \cap K_2 (A_1)^U$;
2. Suppose the above state-closed representations of $G_1$ and $G_2$ are faithful. Then so is the corresponding state-closed representation of $G$;
3. If $G_1$ and $G_2$ are finite-state then $G$ is also finite-state.

**Proof.** Define for $i = 1, 2$,

$$R_i = \langle S \leq \cap_{j=1}^{s_i} H_{ij} \mid S \triangleleft G, \ S^{j_{ij}} \leq S, \forall j = 1, ..., s_i \rangle,$$

(1) We have

- $K = R_1 \cap R_2$,
- $R_1 \cap (A_1 \oplus A_2)^U = K_1 (A_2)^U$,
- $R_2 \cap (A_1 \oplus A_2)^U = K_2 (A_1)^U$,
- $K \cap (A_1 \oplus A_2)^U = K_1 (A_2)^U \cap K_2 (A_1)^U$.

(2) As $K_1 = K_2 = \{e\}$, we find

$$K \cap (A_1 \oplus A_2)^U = (A_2)^U \cap (A_1)^U = \{e\}.$$

Now since both $K$ and $(A_1 \oplus A_2)^U$ are normal subgroups of $G$, it follows that $K$ centralizes $(A_1 \oplus A_2)^U$. However $(A_1 \oplus A_2)^U$ contains its own centralizer in $G$. Thus, $K = K \cap (A_1 \oplus A_2)^U$ which is trivial by (1).

(3) There exist transversals of $H_{11}, ..., H_{1s_1}$ in $G_1$ and of $H_{21}, ..., H_{2s_2}$ in $G_2$ such that $G_1$ and $G_2$ are finite-state. These transversals also induce a finite-state representation of $G$. 

\[ \square \]

Now Theorem C follows directly from the above proposition and Proposition 5.1.

**Question 1.** Is there a faithful state-closed representation of degree 3 for the group $\mathbb{Z}^l \wr \mathbb{Z}^d$ when $l, d \geq 2$?
We start with the following observation. Let $k$ be a field and $X = \langle x, y \rangle \simeq \mathbb{Z}^2$. Consider the following equivalence relation on the $k$-algebra $k[X]$ defined by:

$$p(x, y) \sim q(x, y) \text{ iff } q(x, y) = up(x, y), \text{ where } u \text{ is a unit of } k[X].$$

Let $p'(x, y) = x^s y^t p(x, y)$, where $s, t \geq 0$ minimal such that $p'(x, y)$ in $k[x, y]$. Let $m, n$ be respectively the $x$-degree and $y$-degree of $p'(x, y)$ and define $\delta(p(x, y)) = (m, n)$. Then, for $p(x, y), q(x, y)$ non-invertible elements of $k[X], \delta(p(x, y)q(x, y)) \geq \delta(p(x, y)), \delta(q(x, y))$. Let $p_1(x, y)$ be a sequence of non-invertible elements of $k[X]$ where $i \geq 0$ and let $\delta(p_i(x, y)) = (m_i, n_i)$. Suppose $m_i, n_i \to \infty$ as $i \to \infty$. Then, the ideal $\bigcap_{i=0}^{\infty} \langle p_i(x, y) \rangle$ is null.

**Theorem D.** Let $p$ a prime number then $C_p \wr \mathbb{Z}^2$ is a self-similar group of degree $p + 1$ of orbit-type $(p, 1)$. Indeed, $C_p \wr \mathbb{Z}^2$ is generated by $\alpha = (\alpha, \alpha \sigma, ..., \alpha \sigma^{p-1}, \alpha \beta), \sigma = (e, ..., e, \sigma) (01..p-1)$ and $\beta = (e, ..., e, \alpha)$. In particular, the group $C_2 \wr \mathbb{Z}^2$ is self-similar of degree 3.

**Proof.** Let $C_p = \langle a \rangle, \mathbb{Z}^2 = \langle x, y \rangle$ and $G = C_p \wr \mathbb{Z}^2$. Define the subgroups $H_1 = G' \langle x, y \rangle$ and $H_2 = G$. Note that $[G : H_1] = p$. Elements of $H_1$ have the unique form $a^{s(x,y)} \cdot x^i y^j$ where $s(x, y)$ is an element of the ideal $\mathcal{I}$ of $\mathbb{Z} \langle x, y \rangle$, generated by $x - 1$ and $y - 1$. Also, elements of $\mathcal{I}$ have the unique form

$$s(x, y) = p(x)(x - 1) + q(y)(y - 1) + r(x, y)(x - 1)(y - 1).$$

Define $f_1 : H_1 \to G$ by $a^{s(x,y)} x^i y^j \mapsto a^q(y) y^j$. Furthermore define $f_2 : H_2 \to G$ by $a^{r(x,y)} x^i y^j \mapsto a^r(x, y) x^i y^i + j$. It can be checked directly that $f_1$ and $f_2$ are homomorphisms.

Suppose by contradiction that $K$ is a non-trivial subgroup of $H_1 \cap H_2 = H_1$, normal in $G$ and $\{f_1, f_2\}$-invariant. Note that the subgroup $L = K \cap G'$ is trivial if and only if $K$ is trivial. Let $g = a^{s(x,y)}$ be a nontrivial element in $L$. So $a^{s(x,y)} f_1 = a^q(y)$ and successive applications of $f_1$ in $a^q(y)$ results $q(y) = 0$, thus $x - 1$ divides $s(x, y)$.
Since \( a^{s(x,y)f_2} = a^{s(y,xy)} \in L \), it follows that \( x-1 \) also divides \( s(y,xy) \), this is, \( s(y,xy) = (x-1)t(x,y) \). Then
\[
a^{s(x,y)} = a^{(s(y,xy))f_2^{-1}}
= a^{((x-1)t_1(x,y))f_2^{-1}}
= a^{(x^{-1}y-1)t_2(x^{-1}y,x)}
= a^{x^{-1}(y-x)t_2(x^{-1}y,x)}
= a^{(x-y)t_3(x,y)}
\]
and \( x-y \) divides \( s(x,y) \). Iterating this argument, we find that \( x^{m_i} - y^{n_i-1} \) divides \( s(x,y) \) for all \( n_i \), where \( n_i \) is the Fibonacci sequence defined by \( n_i = n_{i-1} + n_{i-2} \) and \( n_0 = 0, n_1 = 1, n_2 = 1, i \geq 0 \). Hence \( s(x,y) \in \bigcap_{i=0}^{\infty} (x^{m_i} - y^{n_i-1}) = \{0\} \); a contradiction. Therefore \( G \) is state-closed of degree \( p + 1 \) and orbit-type \((p,1)\).

Choose the transversals \( T_1 = \{e, a, ..., a^{p-1}\} \) and \( T_2 = \{e\} \) of \( H_1 \) and \( H_2 \), respectively. So we have a state-closed representation of \( G \) generated by the automorphisms
\[
\sigma = (e, e, ..., e, \sigma)(0 1), \quad \alpha = (\alpha, \alpha\sigma, ..., \alpha\sigma^{p-1}, \alpha\beta), \quad \beta = (e, e, ..., e, \alpha).
\]
Note that \( \alpha^m\beta^n = (\alpha^m, (\alpha\sigma)^m, ..., (\alpha\sigma^{p-1})^m, \alpha^{m+n}\beta^m) \), for \( m, n \in \mathbb{Z} \), hence \( \{\alpha, \alpha\beta, \alpha^2\beta, ..., \alpha^m(n-1), ..., \} \subset Q(\alpha) \) and this representation is not finite-state.

**Question 2.** Is there a faithful finite-state and state-closed representation for the group \( C_2 \wr \mathbb{Z}^2 \) of degree 3?

### 7. Proof of Theorem E

Let \( G \) be a state-closed group with respect the data \((\mathbf{m}, \mathbf{H}, \mathbf{F})\), where \( \mathbf{m} = (m_1, ..., m_s) \), with \( m_1 \geq 2, ..., m_s \geq 2 \). For each \( i = 1, ..., s \) define \( G_{i0} = G, \ G_{ij} = (G_{i(j-1)})f^{-1} \) \((j > 0)\) and \( G_{\omega_i} = \cap_{j \geq 0} G_{ij} \). With this notation we have:

**Theorem E.** Let \( B \) be a finite abelian group and \( G \) be a self-similar group of orbit-type \((m_1, ..., m_s)\), with \( m_1 \geq 2, ..., m_s \geq 2 \). Then the group
\[
B^{(\cap (G_{\omega_1}\setminus G) \times ... \times (G_{\omega_s}\setminus G))} \times G^s
\]
is self-similar of orbit-type \((|B| m_1...m_s, 1)\).
Proof. Let $\mathcal{G}$ be the group $\bigtimes_{i=1}^{s} B((G_{\omega_i}\setminus G) \times \cdots \times (G_{\omega_s}\setminus G)) \rtimes G^s$ and let $H$ be the subgroup $\prod_{i=1}^{s} H_i$ of $G^s$. Define the set of $s$-tuples of cosets
\[ U = \prod_{i=1}^{s} (G_{\omega_i} \setminus G) \]
and define the group extension
\[ \mathcal{H} = \left\{ \phi : U \to B \text{ finitely supported, } \prod_{\bar{g} \in U} \phi(\bar{g}) = 1 \right\} \rtimes H. \]

Note that the index $[\mathcal{G} : \mathcal{H}]$ is $|B| \cdot m_1 \cdots m_s$. Now consider the map defined on the $s$-tuples of cosets.
\[ \lambda : \prod_{i=1}^{s} (G_{\omega_i} \setminus H_i) \to \prod_{i=1}^{s} (G_{\omega_i} \setminus G) \]
\[ (G_{\omega_1} h_1, \ldots, G_{\omega_s} h_s) \mapsto (G_{\omega_1} h_i^f, \ldots, G_{\omega_s} h_s^f). \]
If $(G_{\omega_i} h_i)^\lambda = (G_{\omega_i} h_i')^\lambda$, then $(h_i(h_i')^{-1})^f \in G_{\omega_i}$ for each $1 \leq i \leq s$; therefore $\lambda$ is injective.

Define
\[ \lambda' : \prod_{i=1}^{s} (G_{\omega_i} \setminus G) \to \prod_{i=1}^{s} (G_{\omega_i} \setminus G) \]
by
\[ \bar{y} = (G_{\omega_1} y_1, \ldots, G_{\omega_s} y_s) \mapsto \bar{x} = (G_{\omega_1} x_1, \ldots, G_{\omega_s} x_s), \]
provided $\lambda(\bar{x}) = \bar{y}$; otherwise, define it as $\bar{e} = (G_{\omega_1}, \ldots, G_{\omega_s})$.

Furthermore, define the map $\chi_1 : \mathcal{H} \to \mathcal{G}$
\[ (\phi, (h_i^s)_{i=1}^{s}) \mapsto \left( \bar{x} = (G_{\omega_1} x_1, \ldots, G_{\omega_s} x_s) \mapsto \phi(\lambda'(\bar{x})), (h_i^f)_{i=1}^{s} \right), \]
Then it is direct to show that $\chi_1$ is a homomorphism.

Further, let $\chi_2 : \mathcal{G} \to \mathcal{G}$ be the homomorphism defined by
\[ (\phi, (g_1, \ldots, g_s)) \mapsto (G_{\omega_i} x_i)_{i=1}^{s} \mapsto \phi((G_{\omega_i} x_{i+1})_{i=1}^{s}, (g_{i+1})_{i=1}^{s}). \]

We claim that the state-closed representation of $\mathcal{G}$ defined by the data $((|B|\prod_{i=1}^{s} |G : G_{\omega_i}\setminus H_i|, 1), \{\mathcal{H}, \mathcal{G}\}, \{\chi_1, \chi_2\})$ is faithful.

Consider $K \leq \mathcal{H}$ with $K \triangleleft \mathcal{G}$ and $K^{\chi_r} \leq K$, $r = 1, 2$. Firstly, the assumption that $F$ is core-free and the definition of $\chi_2$ imply that $K \leq B^{(K, \mathcal{G})^s}$. Let $\phi : U \to B$ be a non-trivial element in $K$, then $\prod_{\bar{g} \in U} \phi(\bar{g}) = 1$ and so $|\text{Supp}(\phi)| \geq 2$. Choose $\phi$ with support $S$, etc.
of minimal cardinality. Since \( \phi \) is non-trivial we can assume by \( G \)-conjugation that \( \phi(G_{\omega_1}, ..., G_{\omega_s}) \neq 1 \), and so \( (G_{\omega_1}, ..., G_{\omega_s}) \in S \) and \( S \) contains at least two elements.

Define \( \phi_j = \varphi^j \) for \( j \geq 0 \); it is given by \( \phi_j(\omega_{\omega_1}a_{\omega_1}^{f_1}, ..., \omega_{\omega_s}a_{\omega_s}^{f_s}) = \phi_{j-1}(\omega_{\omega_1}a_{\omega_1}, ..., \omega_{\omega_s}a_{\omega_s}) \) for all \((\omega_{\omega_1}, ..., \omega_{\omega_s}) \in G_{1j} \times ... \times G_{sj}\), extended by the identity away from \((G_{1j} \times ... \times G_{sj})^\lambda \). So

\[
\phi_1 : G_{\omega_1} \setminus H_1^{f_1} \times ... \times G_{\omega_s} \setminus H_s^{f_s} \to B
\]

\[
(G_{\omega_1}a_{\omega_1}^{f_1}, ..., G_{\omega_s}a_{\omega_s}^{f_s}) \mapsto \phi(G_{\omega_1}a_{\omega_1}, ..., G_{\omega_s}a_{\omega_s})
\]

and

\[
Supp(\phi_1) = \{(\omega_{\omega_1}a_{\omega_1}^{f_1}, ..., \omega_{\omega_s}a_{\omega_s}^{f_s}) \mid (\omega_{\omega_1}a_{\omega_1}, ..., \omega_{\omega_s}a_{\omega_s}) \in S\}
\]

\[
= G_{\omega_1}(\pi_{\omega_1} \cap H_1)^{f_1} \times ... \times G_{\omega_s}(\pi_{\omega_s} \cap H_s)^{f_s},
\]

where \( \pi_i, i = 1, ..., s \), is the projection on the \( i \)-th coordinate.

If \((\omega_{\omega_1}a_{\omega_1}^{f_1}, ..., \omega_{\omega_s}a_{\omega_s}^{f_s}) \in Supp(\phi_1)\) then \((\omega_{\omega_1}a_{\omega_1}, ..., \omega_{\omega_s}a_{\omega_s}) \in S\) and so \(|Supp(\phi_1)| \leq |S|\). By minimality of the cardinality of \( S \) we have that \(|Supp(\phi_1)| = |S|\). Continuing in this manner, the support of \( \phi_j \) is

\[
G_{\omega_1}(\pi_{\omega_1} \cap G_{1j})^{f_1} \times ... \times G_{\omega_s}(\pi_{\omega_s} \cap G_{sj})^{f_s},
\]

which has the same cardinality as \( S \). Therefore, we have \( S \subset (G_{1j}, ..., G_{sj}) \) for all \( j \) and thus \( S \subset \{(\omega_{\omega_1}, ..., \omega_{\omega_s})\} \); a contradiction.

\( \square \)

**Corollary 7.1.** Let \( B \) be a finite abelian group and \( G \) be a self-similar group with orbit-type \((m_1 > 1, ..., m_l > 1, m_{l+1} = 1, ..., m_s = 1)\). Then there exist proper subgroups \( R_{l+1}, ..., R_s \) of \( G \) such that

\[
B^l(G_{\omega_1} \setminus G) \times ... \times G_{\omega_s}(G \setminus G) \times \times (R_{l+1} \setminus G) \times ... \times (R_s \setminus G) \not\subset G^s
\]

is a self-similar group of orbit-type \((|B|, m_1, ..., m_s^{s-l+1}, 1)\).

**Proof.** For each \( l + 1 \leq j \leq s \) the restriction \( \hat{f}_j = f_j \) : \( \hat{H}_j = H_l \to G \) is well-defined. Define the triple \((\bar{m}, \bar{H}, \bar{F})\) by

\[
\bar{m} = (m_1, ..., m_l, m_{l+1}, ..., m_s), \bar{H} = \{H_1, ..., H_l, \hat{H}_{l+1} = H_l, ..., \hat{H}_s = H_l\},
\]

\[
\bar{F} = \{f_1, ..., f_l, \hat{f}_{l+1}, ..., \hat{f}_s\}.
\]

and \( R_j \) the parabolic subgroup of \( \hat{f}_j \) for \( j = l + 1, ..., s \). Since \( G \) has a faithful state-closed representation with respect to the data \((\bar{m}, \bar{H}, \bar{F})\), the same holds for the representation with respect to the data \((\bar{m}, \bar{H}, \bar{F})\); in fact the \( F \)-core of \( \bar{H} \) and the \( \bar{F} \)-core of \( \bar{H} \) are the same. Since \( m_i > 1 \) for each \( m_i \) in \( \bar{m} \), we apply Theorem E to obtain the result. \( \square \)
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