FLAT SEMILATTICES

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Abstract. Let $A$, $B$, and $S$ be $\{\lor, 0\}$-semilattices and let $f: A \hookrightarrow B$ be a $\{\lor, 0\}$-semilattice embedding. Then the canonical map, $f \otimes \text{id}_S$, of the tensor product $A \otimes S$ into the tensor product $B \otimes S$ is not necessarily an embedding.

The $\{\lor, 0\}$-semilattice $S$ is flat, if for every embedding $f: A \hookrightarrow B$, the canonical map $f \otimes \text{id}$ is an embedding. We prove that a $\{\lor, 0\}$-semilattice $S$ is flat if and only if it is distributive.

Introduction

Let $A$ and $B$ be $\{\lor, 0\}$-semilattices. We denote by $A \otimes B$ the tensor product of $A$ and $B$, defined as the free $\{\lor, 0\}$-semilattice generated by the set $(A - \{0\}) \times (B - \{0\})$ subject to the relations

$\langle a, b_0 \rangle \lor \langle a, b_1 \rangle = \langle a, b_0 \lor b_1 \rangle,$
for $a \in A - \{0\}$, $b_0, b_1 \in B - \{0\}$; and symmetrically,

$\langle a_0, b \rangle \lor \langle a_1, b \rangle = \langle a_0 \lor a_1, b \rangle,$
for $a_0, a_1 \in A - \{0\}$, $b \in B - \{0\}$.

$A \otimes B$ is a universal object with respect to a natural notion of bimorphism, see [2], [4], and [6]. This definition is similar to the classical definition of the tensor product of modules over a commutative ring. Thus, for instance, flatness is defined similarly: The $\{\lor, 0\}$-semilattice $S$ is flat, if for every embedding $f: A \hookrightarrow B$, the canonical map $f \otimes \text{id}_S: A \otimes S \rightarrow B \otimes S$ is an embedding.

Our main result is the following:

Theorem. Let $S$ be a $\{\lor, 0\}$-semilattice. Then $S$ is flat iff $S$ is distributive.

1. Background

1.1. Basic concepts. We shall adopt the notation and terminology of [4]. In particular, for every $\{\lor, 0\}$-semilattice $A$, we use the notation $A^- = A - \{0\}$. Note that $A^-$ is a subsemilattice of $A$.

A semilattice $S$ is distributive, if whenever $a \leq b_0 \lor b_1$ in $S$, then there exist $a_0 \leq b_0$ and $a_1 \leq b_1$ such that $a = a_0 \lor a_1$; equivalently, iff the lattice $\text{Id}_S$ of all ideals of $S$, ordered under inclusion, is a distributive lattice; see [4].
1.2. The set representation. In [3], we used the following representation of the tensor product.

First, we introduce the notation:

\[ \nabla_{A,B} = (A \times \{0\}) \cup (\{0\} \times B). \]

Second, we introduce a partial binary operation on \( A \times B \): let \( \langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B \); the lateral join of \( \langle a_0, b_0 \rangle \) and \( \langle a_1, b_1 \rangle \) is defined if \( a_0 = a_1 \) or \( b_0 = b_1 \), in which case, it is the join, \( (a_0 \lor a_1, b_0 \lor b_1) \).

Third, we define bi-ideals: a nonempty subset \( I \) of \( A \times B \) is a bi-ideal of \( A \times B \), if it satisfies the following conditions:

(i) \( I \) is hereditary;
(ii) \( I \) contains \( \nabla_{A,B} \);
(iii) \( I \) is closed under lateral joins.

The extended tensor product of \( A \) and \( B \), denoted by \( A \boxtimes B \), is the lattice of all bi-ideals of \( A \times B \).

It is easy to see that \( A \boxtimes B \) is an algebraic lattice. For \( a \in A \) and \( b \in B \), we define \( a \otimes b \in A \boxtimes B \) by

\[ a \otimes b = \nabla_{A,B} \cup \{ \langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leq \langle a, b \rangle \} \]

and call \( a \otimes b \) a pure tensor. A pure tensor is a principal (that is, one-generated) bi-ideal.

Now we can state the representation:

Proposition 1.1. The tensor product \( A \otimes B \) can be represented as the \( \{\lor, 0\} \)-sub-semilattice of compact elements of \( A \boxtimes B \).

1.3. The construction of \( A \boxtimes B \). The proof of the Theorem uses the following representation of the tensor product, see J. Anderson and N. Kimura [1].

Let \( A \) and \( B \) be \( \{\lor, 0\} \)-semilattices. Define

\[ A \boxtimes B = \text{Hom}(\langle A^{-}; \lor \rangle, (\text{Id } B; \cap)) \]

and for \( \xi \in A \boxtimes B \), let

\[ \varepsilon(\xi) = \{ \langle a, b \rangle \in A^{-} \times B^{-} \mid b \in \xi(a) \} \cup \nabla_{A,B}. \]

Proposition 1.2. The map \( \varepsilon \) is an order preserving isomorphism between \( A \boxtimes B \) and \( A \boxtimes B \) and, for \( H \in A \boxtimes B \), \( \varepsilon^{-1}(H) \) is given by the formula

\[ \varepsilon^{-1}(H)(a) = \{ b \in B \mid \langle a, b \rangle \in H \}, \]

for \( a \in A^{-} \).

If \( a \in A \) and \( b \in B \), then \( \varepsilon(a \otimes b) \) is the map \( \xi: A^{-} \rightarrow \text{Id } B \):

\[ \xi(x) = \begin{cases} \{0\}, & \text{if } x \leq a; \\ \{b\}, & \text{otherwise}. \end{cases} \]

If \( A \) is finite, then a homomorphism from \( \langle A^{-}; \lor \rangle \) to \( (\text{Id } B; \cap) \) is determined by its restriction to \( \text{J}(A) \), the set of all join-irreducible elements of \( A \). For example, let \( A \) be a finite Boolean semilattice, say \( A = \text{P}(n) \) \((n \) is a non-negative integer, \( n = \{0, 1, \ldots, n - 1\}) \), then \( A \boxtimes B \cong (\text{Id } B)^{n} \), and the isomorphism from \( A \boxtimes B \) onto \( (\text{Id } B)^{n} \) given by Proposition 1.2 is the unique complete \( \{\lor, 0\} \)-homomorphism sending every element of the form \( \{i\} \otimes b \) \((i < n \text{ and } b \in B) \) to \( \{\delta_{ij}b \mid j < n\} \).
(where $\delta_{ij}$ is the Kronecker symbol). If $n = 3$, let $\beta : P(3) \boxtimes S \to (\text{Id} S)^3$ denote the natural isomorphism.

Next we compute $A \otimes B$, for $A = M_3$, the diamond, and $A = N_5$, the pentagon (see Figure 1). In the following two subsections, let $S$ be a $\{\lor, 0\}$-semilattice. Furthermore, we shall denote by $\tilde{S}$ the ideal lattice of $S$, and identify every element $s$ of $S$ with its image, $(s)$, in $\tilde{S}$.

![Figure 1](image_url)

1.4. **The lattices $M_3 \boxtimes S$ and $M_3[\tilde{S}]$: the map $i$.** Let $M_3 = \{0, p, q, r, 1\}$, $J(M_3) = \{p, q, r\}$ (see Figure 1). The nontrivial relations of $J(M_3)$ are the following:

$$p < q \lor r, \quad q < p \lor r, \quad \text{and} \quad r < p \lor q. \quad (1)$$

Accordingly, for every lattice $L$, we define

$$M_3[L] = \{(x, y, z) \in L^3 \mid x \land y = x \land z = y \land z\} \quad (2)$$

(this is the Schmidt’s construction, see [9] and [10]). The isomorphism from $M_3 \boxtimes S$ onto $M_3[\tilde{S}]$ given by Proposition 1.2 is the unique complete $\{\lor, 0\}$-homomorphism $\alpha$ such that, for all $x \in S$,

$$\alpha(p \otimes x) = (x, 0, 0),$$
$$\alpha(q \otimes x) = (0, x, 0),$$
$$\alpha(r \otimes x) = (0, 0, x).$$

We shall make use later of the unique $\{\lor, 0\}$-embedding

$$i : M_3 \hookrightarrow P(3)$$

defined by

$$i(p) = \{1, 2\},$$
$$i(q) = \{0, 2\},$$
$$i(r) = \{0, 1\}.$$
1.5. The lattices $N_5 \otimes S$ and $N_5[\overline{S}]$; the map $i'$. Let $N_5 = \{0, a, b, c, 1\}$, $J(N_5) = \{a, b, c\}$ with $a > c$ (see Figure 1). The nontrivial relations of $J(N_5)$ are the following:

$$c < a \quad \text{and} \quad a < b \lor c.$$  

(3)

Accordingly, for every lattice $L$, we define

$$N_5[L] = \{ \langle x, y, z \rangle \in L^3 \mid y \land z \leq x \leq z \}. $$

(4)

The isomorphism from $N_5 \otimes S$ onto $N_5[\overline{S}]$, given by Proposition 1.12, is the unique complete $\{\vee, 0\}$-homomorphism $\alpha'$ such that, for all $x \in S$,

$$\alpha'(a \otimes x) = \langle x, 0, x \rangle, $$

$$\alpha'(b \otimes x) = \langle 0, x, 0 \rangle, $$

$$\alpha'(c \otimes x) = \langle 0, 0, x \rangle. $$

We shall make use later of the unique $\{\vee, 0\}$-embedding

$$i' : N_5 \hookrightarrow P(3)$$

defined by

$$i'(a) = \{0, 2\},$$

$$i'(b) = \{1, 2\},$$

$$i'(c) = \{0\}.$$ 

1.6. The complete homomorphisms $f \otimes g$. The proof of the following lemma is straightforward:

**Lemma 1.3.** Let $A$, $B$, $A'$, and $B'$ be $\{\vee, 0\}$-semilattices, let $f : A \to A'$ and $g : B \to B'$ be $\{\vee, 0\}$-homomorphisms. Then the natural $\{\vee, 0\}$-homomorphism $h = f \otimes g$ from $A \otimes B$ to $A' \otimes B'$ extends to a unique complete $\{\vee, 0\}$-homomorphism $\overline{h} = f \otimes g$ from $A \otimes B$ to $A' \otimes B'$. Furthermore, if $h$ is an embedding, then $\overline{h}$ is also an embedding.

We refer to Proposition 3.4 of [6] for an explicit description of the map $\overline{h}$.

2. Characterization of flat $\{\vee, 0\}$-semilattices

Our definition of flatness is similar to the usual one for modules over a commutative ring:

**Definition.** A $\{\vee, 0\}$-semilattice $S$ is flat, if for every embedding $f : A \to B$ of $\{\vee, 0\}$-semilattices, the tensor map $f \otimes \text{id}_S : A \otimes S \to B \otimes S$ is an embedding.

In this definition, $\text{id}_S$ is the identity map on $S$.

In Lemmas 2.1 and 2.3, let $S$ be a $\{\vee, 0\}$-semilattice and we assume that both homomorphisms $f = i \otimes \text{id}_S$ and $f' = i' \otimes \text{id}_S$ are embeddings.

As in the previous section, we use the notation $\overline{S} = \text{Id}S$, and identify every element $s$ of $S$ with the corresponding principal ideal $(s)$.

We define the maps $g : M_3[\overline{S}] \to \overline{S^3}$ and $g' : N_5[\overline{S}] \to \overline{S^3}$ by the following formulas:

- For all $\langle x, y, z \rangle \in M_3[\overline{S}]$, $g(\langle x, y, z \rangle) = \langle y \lor z, x \lor z, x \lor y \rangle$.
- For all $\langle x, y, z \rangle \in N_5[\overline{S}]$, $g'(\langle x, y, z \rangle) = \langle z, y, x \lor y \rangle$. 

Note that $g$ and $g'$ are complete $\{\lor, 0\}$-homomorphisms. The proof of the following lemma is a straightforward calculation.

**Lemma 2.1.** The following two diagrams commute:

\[
\begin{array}{c}
\begin{array}{c}
M_3 \boxtimes S \xrightarrow{f} P(3) \boxtimes S \\
\downarrow \alpha \hspace{2cm} \downarrow \beta \\
M_3[S] \xrightarrow{g} \tilde{S}^3 \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
N_5 \boxtimes S \xrightarrow{f'} P(3) \boxtimes S \\
\downarrow \alpha' \hspace{2cm} \downarrow \beta' \\
N_5[S] \xrightarrow{g'} \tilde{S}^3 \\
\end{array}
\end{array}
\]

Therefore, both $g$ and $g'$ are embeddings.

**Lemma 2.2.** The lattice $\tilde{S}$ does not contain a copy of $M_3$.

*Proof.* Suppose, on the contrary, that $\tilde{S}$ contains a copy of $M_3$, say $\{o, x, y, z, i\}$ with $o < x < y, z < i$. Then both elements $u = \langle x, y, z \rangle$ and $v = \langle i, i, i \rangle$ of $L^3$ belong to $M_3[S]$, and $g(u) = g(v) = \langle i, i, i \rangle$. This contradicts the fact, proved in Lemma 2.1 that $g$ is one-to-one. \hfill \Box

**Lemma 2.3.** The lattice $\tilde{S}$ does not contain a copy of $N_5$.

*Proof.* Suppose, on the contrary, that $\tilde{S}$ contains a copy of $N_5$, say $\{o, x, y, z, i\}$ with $o < x < z < i$ and $o < y < i$. Then both elements $u = \langle x, y, z \rangle$ and $v = \langle z, y, z \rangle$ of $L^3$ belong to $N_5[S]$, and $g'(u) = g'(v) = \langle z, y, i \rangle$. This contradicts the fact, proved in Lemma 2.1 that $g'$ is one-to-one. \hfill \Box

Lemmas 2.2 and 2.3 together prove that $\tilde{S}$ is distributive, and therefore $S$ is a distributive semilattice. Now we are in position to prove the main result of this paper in the following form:

**Theorem 1.** Let $S$ be a $\{\lor, 0\}$-semilattice. Then the following are equivalent:

(i) $S$ is flat.

(ii) Both homomorphisms $i \otimes \text{id}_S$ and $i' \otimes \text{id}_S$ are embeddings.

(iii) $S$ is distributive.

*Proof.*

(i) implies (ii). This is trivial.

(ii) implies (iii). This was proved in Lemmas 2.2 and 2.3.

(iii) implies (i). Let $S$ be a distributive $\{\lor, 0\}$-semilattice; we prove that $S$ is flat. Since the tensor product by a fixed factor preserves direct limits (see Proposition 2.6 of [8]), flatness is preserved under direct limits. By P. Pudlák [8], every distributive join-semilattice is the direct union of all its finite distributive subsemilattices; therefore, it suffices to prove that every finite distributive $\{\lor, 0\}$-semilattice $S$ is flat. Since $S$ is a distributive lattice, it admits a lattice embedding into a finite Boolean lattice $B$. We have seen in Section 1.3 that if $B = P(n)$, then $A \otimes B = A^n$ (up to a natural isomorphism), for every $\{\lor, 0\}$-semilattice $A$. It follows that $B$
is flat. Furthermore, the inclusion map \( S \hookrightarrow B \) is a lattice embedding; in particular, with the terminology of [6], an \( L \)-homomorphism. Thus, the natural map from \( A \otimes S \) to \( A \otimes B \) is, by Proposition 3.4 of [6], a \( \{ \lor, 0 \} \)-semilattice embedding. This implies the flatness of \( S \).

\[ \square \]

3. Discussion

It is well-known that a module over a given principal ideal domain \( R \) is flat if and only if it is torsion-free, which is equivalent to the module being a direct limit of (finitely generated) free modules over \( R \). So the analogue of the concept of torsion-free module for semilattices is be the concept of distributive semilattice. This analogy can be pushed further, by using the following result, proved in [3]: a join-semilattice is distributive iff it is a direct limit of finite Boolean semilattices.

**Problem 1.** Let \( V \) be a variety of lattices. Let us say that a \( \{ \lor, 0 \} \)-semilattice \( S \) is in \( V \), if \( \text{Id} \ S \) as a lattice is in \( V \). Is every \( \{ \lor, 0 \} \)-semilattice in \( V \) a direct limit (resp., direct union) of finite join-semilattices in \( V \)?

If \( V \) is the variety of all lattices, we obtain the obvious result that every \( \{ \lor, 0 \} \)-semilattice is the direct union of its finite \( \{ \lor, 0 \} \)-subsemilattices. If \( V \) is the variety of all distributive lattices, there are two results (both quoted above): P. Pudlák’s result and K. R. Goodearl and the second author’s result.

**Problem 2.** Let \( V \) be a variety of lattices. When is a \( \{ \lor, 0 \} \)-semilattice \( S \) flat with respect to \( \{ \lor, 0 \} \)-semilattice embeddings in \( V \)? That is, when is it the case that for all \( \{ \lor, 0 \} \)-semilattices \( A \) and \( B \) in \( V \) and every semilattice embedding \( f : A \rightarrow B \), the natural map \( f \otimes \text{id}_S \) is an embedding?

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