Statistical Representation of Spacetime

Hamidreza Simchi

Department of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran and
Semiconductor Technology Center, P.O.Box 19575-199, Tehran, Iran

(Dated: March 31, 2022)
Abstract

It is assumed that the spacetime is composed by events and can be explained by partially ordered set (causal set). The parent events born two kinds of children. Some children have a causal relation with their parents and other kinds have not. It is assumed that evolution of the population is only happen by the causal children. The assumed population can be modeled by finite (infinite) dimension Leslie matrix. In both finite and infinite cases, it is shown that the stationary state of the population always exists and the matrix has positive eigenvalues. By finding the relation between the statistical information of the population and the stationary state, a probability matrix and a Shannon-like entropy is defined. It is shown that the change in entropy is always quantized and positive and in consequence, the world is inflating. We show that the vacuum energy can be attributed to the necessary done work for preserving the causal relation between the parents and the children (cohesive energy). By assuming that the sum of cohesive energy and kinetic energy of the denumerable causal spacetime is equal to the heat, which flows across a causal horizon, we find the relation between energy-momentum tensor and discrete Ricci tensor which can be called the Einstein state equation. Finally, it is shown that the constant of proportionality $\eta$ between the entropy and the area is proportional to $\frac{k_B l_p}{l_p}$ at Planck scale which is in good agreement with the Hawking’s result.

I. INTRODUCTION

After the discovery of quantum physics, it became possible to describe the electronic and structural properties of materials based on their constituent elements, namely atoms and electrons. Using the concepts related to atomic orbitals and their hybridization, the cause of the stability of the material structure was described [1], and based on the energy band theory, and many-body theory, the electronic properties and the quantum transport properties of the majority carriers were well described [2-5] so that the computational results were consistent with what was seen in the laboratory. This process continued with the discovery of more elementary particles, and scientists tried to describe strong nuclear forces [6] and weak forces [7] based on the properties of related elementary particles. Hence, it can be said that quantum physics, many-body theory, and quantum field theory are attempts to

*simchi@alumni.iust.ac.ir
express and interpret the macroscopic properties of a physical system based on its atomic and sub-atomic microscopic components. Of course, it should be noted that many efforts were made to unifying the known forces in nature so that finally, by presenting a standard model [8], these efforts were completed, without considering the force of gravity.

In contrast, the theory of relativity is not an attempt to describe the macroscopic properties of a system based on its microscopic properties. This theory can be considered as a description of how energy and matter interact with the gravitational field (spacetime field) [9-10]. It is based on how spacetime field interacts with matter that the planets move in a certain orbit, light is deflected when it passes near the sun, the existence of black holes in the galaxy is predicted, and a scientific description is given for some other cosmic phenomena. Of course, the efforts made to determine the cause of black hole radiation [11] and the thermodynamic description of Einstein’s relativistic equation [12], which is made by using a combination of quantum physics and Einstein’s theory of relativity, must be taken into account.

If we take quantum gravity theory as a theory for describing the observable macroscopic phenomena based on the atomic and sub-atomic microscopic components that make up the spacetime field, then when the problem scale approaches to the Planck scale, the question arises as to what these microscopic components are and what are their characteristics? It is well known that one of the important features of worldline is its causal property. This means that at any point of spacetime (present) one set of events can be considered as past and another set as future event related to the present. Due to the constant and maximum speed of light in the vacuum and this causal relationship between the present, the future and the past, some events will not have any physical connection with the present. In order to mathematically represent this causal property of events, the partially ordered causal set, which has been studied by mathematicians before [13], may be used.

Attempts to use partially ordered sets as the microscopic elements that make up spacetime go back many years. By defining the null, parallel lines and planes and proving numerous theorems involving them, Robb has described the relativity using the discrete spacetime (i.e., casual structure) [14,15]. It has been shown that the casual structure of a spacetime, together with a conformal factor, determine the metric of a Lorentzian spacetime, uniquely [16,17]. Therefore, one can recover the conformal metric by using the before and after relations amongst all events [18]. Now, if one has a measure for the conformal factor, he/she
can recover the entire metric and spacetime [18]. Of course, t Hooft [19] and Myrheim [20] have independently found the causal set theory too. Extensive research is being done today on the theory of causal sets and its application in the formulation of the theory of relativity at the Planck scale [21]. In one group of this research, by focusing on the causal events, an attempt is made to study the static and dynamic state characteristics of the causal set [22, 23], while a bunch of research focuses on the causal relationship between events [24]. Of course, the third group, by assuming the irreversibility of time and explaining and using the principles of energy and momentum conservation and the absence of red shift in the momentum relation at Planck scale, have introduced energetic causal set theory that includes both classical and quantum cases [25, 26]. Ref. 21 is a fine review on the causal set theory and its outlook.

Now, the question that can be asked is whether spacetime can be considered as a population consisting of discrete members (parents) between whom there is a causal relationship (such as a partially ordered set) that has evolved with the birth of new members (children) of this set and \(i\)-th step goes to \((i+1)\)-th step and define entropy for this set and study it, statistically? In this article, we try to answer this question in the affirmative. We assume that spacetime is composed by events which is shown by the partially ordered set. At each \(i\)-th step, the ordered set is called the parent set, and at each \((i+1)\)-th step, the ordered set is called children set. Some children have causal relation with their parents and some have not. It is assumed that the evolution of the system is only done by causal children. By using finite and infinite Leslie matrix, the system is modeled and it is shown that the stationary state always exists. We show that, since for preserving the causality relation between parents and children some works should be done, the entropy of the system is quantized and increases by the birth of new children. It means that we encounter the universe inflation. Finally, by using the Jacobson’s procedure [12], we calculate the energy-momentum tensor and in consequence the Einstein state equation including discrete Ricci tensor.

The structure of the article is as follows. The basic assumptions are introduced in section II and the Leslie matrix model is introduced in section III. The quantized entropy and Einstein state equation are studied in section IV and V, respectively. The summary is provided in section VI.
II. ASSUMPTIONS

Let us to consider a causal set $C$ which is a locally finite, partially ordered set. The order relation is shown by `$\prec$' and the set has the below properties [13, 22]:

1. For all $x \in C$, $x \not\prec x$ (Irreflexive)
2. For all $x, y, z \in C$ if $x \prec y$ and $y \prec z$ then $x \prec z$ (Transitive)
3. For all $x, y \in C$ if $x \prec y$ and $y \prec x$ then $x = y$ (Antisymmetric)
4. For all $x, y \in C$ we have $|\{z \in C | x \prec z \prec y\}| < N_0$ (Locally finite)

The causal set can be considered as age-structured populations if adding elements to the causal set are considered as individuals born. One of our main goals is mixing the Sorkin’s idea [23] about the new generations in the causal set theory and the Demetrius’s idea [30] about the aged-structure populations model for developing own model. Demetrius has begun with the Leslie model and considered a population divided into $n$ age classes [30]. Sorkin has assumed that the causal sets can be formed by adjoining a single maximal element to a given causal set and the result will be called a family [23]. For doing that, we used the rules which are mentioned in table one.

| Rule | Sorkin’s(S) Or Demetrius’s(D) idea | New status (our model) | Figure |
|------|-----------------------------------|------------------------|--------|
| 1    | Timid children(S)                 | Causal children        | Red cycles in Fig.1 |
| 2    | Gregarious child(S)               | Non-causal children    | Blue cycles in Fig.1 |
| 3    | $m_i$-individuals born numbers(D) | $m_i$-causal children numbers | Green box in Fig.2 |
| 4    | $b_i$-proportion of surviving individuals(D) | $b_i$-proportion of non-causal children | Red box in Fig.2 |
| 5    | All children are next parents(S)  | Causal children are next parents | Fig.1 |

At each step of growth, two kinds of babies may be born. In the first kind, $m_i$ number of individuals born have the order relation with their parents and in the second kind, some individuals born have not. For the second kind, $b_i$ stands for the proportion of individuals born. The individuals born with order relation are placed on the causal worldline and the individuals born without order relation are not [27]. It is assumed that the evolution of the
population is only done by causal children. As Fig.1 shows, one can find the number of causal individuals born (non-causal individual born) by counting the red (blue) cycles from the beginning up to the $i$-th step. All of the red cycles may be considered as parents in the $i$-th step.

Based on the mentioned rules in table 1, the number of parents increase by increasing the growth steps. In fact, the new causal children in $i$-th step are considered as new parents of the children in $(i + 1)$-th step. Therefore, the change in age structure between step $i$ and step $(i + 1)$ can be found by using the Leslie matrix as Demetrius has assumed [30]. It means that, $m_1$ stands for the number of causal children in step one respect to itself, $m_2$ stands for the number of causal children in step two respect to step one and generally, $m_i$ stands for the number of causal children in the $i$-th step respect to step one. Also, $b_1$ stands for the proportion of non-causal children in step one surviving to step two, $b_2$ stands for the proportion of non-causal children in step two surviving to step three, and generally $b_j$ stands for the proportion of non-causal children in step $j$ surviving to step $(j + 1)$.

### III. MATRIX REPRESENTATION

Let us to define the below Leslie matrix [28]

\[ M_{ij} = \begin{cases} 
    m_j > 0 & \text{for } i = 1 \\
    0 < b_i \leq 1 & \text{for } i = j + 1 \\
    0 & \text{Otherwise}
\end{cases} \]

i.e.,

\[
M = \begin{pmatrix}
    m_1 & m_2 & m_3 & m_4 & m_5 & \cdots & \cdots & \cdots & m_n \\
    b_1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
    0 & b_2 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
    0 & 0 & b_3 & 0 & 0 & \cdots & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & 0 & b_4 & 0 & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & 0 & b_5 & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & 0 & 0 & b_6 & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & b_{n-1} & 0 \\
\end{pmatrix}
\]
FIG. 1. (Color online) An example of a causal set as a population growth. The events in red cycles are placed on the causal worldlines. The non-dashed red color cycles show an example of the casual worldline. The blue cycles show the non-causal events. One can find the number of causal individuals born (non-causal individuals born) by counting the red (blue) cycles from the beginning up to the i-th step.

Generally, based on the Perron–Frobenius theorem [29, 30], The eigenvalues of $M$ are real and it has eigenvalues with all positive elements such that $M\vec{u} = \lambda \vec{u}$ and $\vec{v}M = \lambda \vec{v}$.

It should be noted that, one of the important subjects is the relation between the elements of the Leslie matrix and the future growth of the population [28, 30]. It has been shown that there are relationships between certain elements of a population and the dominant eigenvalue, which determines growth [28, 30]. For example, If the dominant eigenvalue and, hence, all the eigenvalues are less than 1, then the population will decline. If the dominant eigenvalue is greater than one, regardless of the values of the other eigenvalues, the population will grow [28, 30]. In below, the population growth will be related to the
world inflation. It has been shown that, if \[30\]

\[
l_j = \begin{cases} 
1 & \text{for } j = 1 \\
\Pi_{r=1}^{j-1} b_r & \text{for } j \geq 2
\end{cases}
\]

Then

\[
u_i = \frac{l_i}{\lambda^i}, \quad v_i = \frac{\left(\sum_{j=i}^n m_j u_j\right)}{u_i}
\]

For example, if \( M = \begin{bmatrix} m_1 & m_2 \\ b_1 & 0 \end{bmatrix} \) then \( u_1 = 1/\lambda_i, \ u_2 = b_1/\lambda_i^2, \ v_1 = \lambda_i \) and \( v_2 = m_2 \) (Appendix A).

Now, a question can be asked: what can be the physical interpretation of \( M, \bar{u}, \lambda \)?

Fig. 2 shows the new causal and non-causal children of \((i+1)\)-th generation step when the \(n_i\) causal parents present at \(i\)-th generation step. The parents belong to the own \(i\)-th step and the all of the previous steps from the beginning. By substituting Eq. (4) in the eigenvalue equation, one can find

\[
\begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ b_1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} 1/\lambda \\ b_1/\lambda^2 \\ \vdots \\ b_1 \cdots b_{n-1}/\lambda^n \end{pmatrix} = \lambda \begin{pmatrix} 1/\lambda \\ b_1/\lambda^2 \\ \vdots \\ b_1 \cdots b_{n-1}/\lambda^n \end{pmatrix}
\]

or

\[
\begin{pmatrix} m_1 + m_2 b_1/\lambda + \cdots + m_n b_1 b_2 \cdots b_{n-1}/\lambda^n \\ b_1/\lambda \\ \vdots \\ b_1 \cdots b_{n-1}/\lambda^{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ b_1/\lambda \\ \vdots \\ b_1 \cdots b_{n-1}/\lambda^{n-1} \end{pmatrix}
\]
Therefore, he/she can write the below eigenvalue equation

\[
\begin{pmatrix}
\frac{m_1}{\lambda} & \frac{m_2 b_1}{\lambda^2} & \cdots & \frac{m_n b_1 b_2 \cdots b_{n-1}}{\lambda^{n-1}} \\
1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
0 & \cdots & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
\end{pmatrix} 
\]  

(7)

It means that he/she can define the probability element and the probability matrix as below [30]

\[ p_i = \frac{m_i l_i}{\lambda^i} \]  

(8)

\[ P = (P_{ij}) = \begin{cases} 
p_i & \text{for } i = 1 \\
1 & \text{for } i = J = 1 \\
0 & \text{Otherwise} \end{cases} \]  

(9)

Therefore, \( m_i u_i = \frac{m_i l_i}{\lambda^i} \) can be considered as the born probability of \( m_i \) new causal children related to the parent \( u_i \) and in consequence \( m_1 u_1 + m_2 u_2 + \cdots + m_n u_n \) is the total born probability of new casual children at \( (i+1) \)-step.
By attention to the above descriptions, the matrix $M$ is the evolution matrix such that

$$\vec{u}'(i+1) = M\vec{u}(i)$$

where $\vec{u}(\vec{u}')$ is the distribution of parents from the beginning up to the $i$ ($i+1$)-th step. It should be noted that if the term $M_{nn} = b_n \neq 0$ it means that the $n$-th parent born non-causal child with the probability $b_n$ [28]. The obtained general results satisfy again and one should only find the new eigenvalues and eigenvectors. For providing the physical interpretation of $\lambda$ we define the entropy in the next section.

IV. ENTROPY

Entropy is the total number of ways to rearrange the internal microstates of a system while keeping its external macrostates unaltered [31]. Here, we encounter the ensemble of parents with the boron probability $p_{ij}$ as internal microstates and the stationary population as the macrostate. For studying the effect of the variation of microstates on the macrostate, we should define the entropy.

By using the irreducibility properties of matrix $P$ and associating the graph $G(P)$ to the matrix $P$, Demetrius [30] has represented the set of all paths in the graph by $\Omega$. He also defined the shift transformation $T$ as $\Omega \rightarrow \Omega$ and called $(\Omega, T)$ the symbolic dynamical system associated with the matrix $P$. He has used the Markov matrix $P$ to introduce a probability measure $\mu$ on the space of sequences $\Omega$ and finally defined the entropy $H(T)$ [30].

By using the Eq. (4) we can define the new population vector $\vec{Z}$ as below [30]

$$Z_i = \frac{1}{\sum_{i=1}^{n} ip_i} \sum_{i=1}^{n} ip_i = \frac{1}{\sum_{i=1}^{n} ip_i}$$

Therefore, $\vec{Z}P = \vec{Z}$. Then, $\vec{Z}$ stands for the stationary distribution and in consequence, similar to the Demetrius’s method, we define the population entropy for the stationary distribution as below [30]

$$S = -\frac{\sum_{i=1}^{n} p_i \log(p_i)}{\sum_{i=1}^{n} ip_i}$$

(11)

where, $p_i = m_i l_i / \lambda^i$. It should be noted that the numerator of the Eq. (11) is similar to the Shannon entropy [32]. For providing the physical interpretation of $\lambda$ we write
\[
\log \lambda = \frac{\sum_{i=1}^{n} p_i \log \left( \frac{\lambda_i p_i}{p_i} \right)}{\sum_{i=1}^{n} i p_i} = \frac{1}{\sum_{i=1}^{n} i p_i} \sum_{i=1}^{n} p_i (\log \lambda_i p_i - \log p_i) \tag{12}
\]

or

\[
\log \lambda = \frac{\sum_{i=1}^{n} p_i \log \lambda_i p_i}{\sum_{i=1}^{n} i p_i} - \frac{\sum_{i=1}^{n} p_i \log (p_i)}{\sum_{i=1}^{n} i p_i} = \Phi + S \tag{13}
\]

or

\[
\delta = \Phi + S = \Phi + \frac{S'}{T} = \Phi + T' S' \tag{14}
\]

where \( \Phi = \sum_{i=1}^{n} p_i \log \lambda_i p_i \), \( S' = \sum_{i=1}^{n} p_i \log (p_i) \), \( T' = (T = \sum_{i=1}^{n} i p_i)^{-1} \), and \( \delta = \log \lambda \). From thermodynamic, we know, \( F = U - TS \) where \( F \) and \( U \) are Helmholtz free energy and internal energy, respectively. Also, \( T \) and \( S \) are absolute temperature and entropy, respectively. Since, \( \delta = \Phi + T' S' \) then \( \Delta \Phi + T' \Delta S' = 0 \tag{15} \)

or

\[
\frac{1}{T'} = T = \frac{\Delta S'}{-\Delta \Phi} \rightarrow \Delta S' = -T \Delta \Phi \tag{16}
\]

In thermodynamic \( \frac{1}{T} = \frac{\Delta S}{\Delta U} \) and in consequence the temperature of any object can be described by the amount of heat that must be added to it to increase its entropy by one unit. Similarly, by attention to the Eq. (16), \( T \) can be described by the amount of \( -\Delta \Phi \) that must be added to the population to increase its entropy by one unit. By comparison between Eq. (14) and the equation \( F = U - TS \), one can conclude the comparison table 1.

| Statistical Representation | Thermodynamic          |
|----------------------------|-------------------------|
| \( \delta \)              | Helmholtz Free energy   |
| \( T \)                   | Inverse of absolute temperature |
| \( -T'S' \)               | Entropy                 |
| \( \Delta \Phi \)         | Internal (mean) energy  |

Now, let us to assume that the causal relation between children and parents is energetic relation. That is, after establishing a causal relationship between a parent and a child if we assume that the amount of \( -\Delta \Phi \) increases by \( V_i \) then \( \Delta S = (\sum_{i=1}^{n} i p_i)^{-1} \sum_i V_i > 0 \). Therefore, the entropy increases by growing the population and in consequence the population
will be more stable than before. Since, the causal events are denumerable then $V_i$ is discrete and in consequence $\Delta S$ is too i.e., $\Delta S$ is quantized. It can be assumed that, the cohesive energy of the denumerable causal spacetime is the energy of vacuum which is appeared in quantum filed theory as $\sum_i \hbar \omega / 2$. In consequence, if it is assumed that $V_i = \hbar \omega / 2$, since $\sum_i V_i = m\hbar \omega / 2$. It means that the entropy is quantized and equal to

$$S = + \frac{m\hbar}{2} T = + \frac{m\hbar}{2} \left( \sum_{i=1}^{n} i p_i \right)^{-1}$$

where, $(T = \sum_{i=1}^{n} i p_i)^{-1} = cte$. For comparison, we can take into account the formation of a crystal. By bringing an element of infinity closer to one element in the crystal, the two elements are bounded to each other and the internal (cohesive) energy will be increased and stored in the crystal [33]. It means that the cohesive energy is always negative [33]. Similarly, by adding a new child to the population and establishing the causal relation the amount of $-\Delta \Phi$ increases by the quanta of energy and stored in the population. Therefore, it is expected that the energy $-\Delta \Phi = \sum_i V_i (= m\hbar \omega / 2)$ will be equal to the energy of vacuum. Of course, the causal spacetime is a disordered system and Bloch’s theorem does not satisfy here. Increment of entropy, due to the increasing the number of parents and causal children, cases the expansion of the universe (universe inflation). Why we see universe inflation because its entropy increases due to the increment of the spacetime events.

V. EINSTEIN STATE EQUATION

It has been shown that, when $M$ is an infinite-dimensional population matrix, under special conditions, a stationary distribution exists [34] (Appendix B). Also, when, the index $i, j \to \infty$, we encounter an infinite denumerable Leslie matrix, $M$. It has been shown that for $i, j \to \infty$, an essentially unique stationary distribution exists [35] and there is a generating function $G(z)$ for calculating $M^n$, where $z$ is a complex indeterminate. Now, if $u_0$ is the first parent (initial condition) then $u_n = M^n u_0$[35] (Appendix B). Therefore, the stationary distribution exists when $i, j \to \infty$ [34, 35].

Also, in spacetime dynamic, it can be assumed that the heat is energy which flows across a causal horizon and the heat flux is given by [12]

$$\delta Q = \int T_{ab} \chi^a d\Sigma^b$$

(18)
where, all integrands are defined in Appendix C. It should be noted that the past horizon of a local Rindler horizon (system) is instantaneously stationary (in local equilibrium) at spacetime point \( p \) [12]. We assume that each unit area of spacetime is composed by the denumerable causal events. Each causal event has kinetic energy \( K_i \) and is bounded to another causal event by the causal correspondence energy \( V_i \). It can be assumed that the heat which flows across a causal horizon is equal to \( \sum_i (V_i + K_i) \delta A \) where \( \delta A \) is defined in Appendix C. The assumed phenomenon is similar to the relaxed state of a crystal. In the crystal, the elements vibrate around their equilibrium with preserving the bounding between elements (preserving the cohesive energy). Here, although the unit area of the causal spacetime is a disorder system and the Bloch’s theorem does not satisfy we assume that

\[
\delta Q = - \sum_i (V_i + K_i) \delta A
\]

Since, the area \( \delta A \) is composed by the denumerable causal events by using Eq. (C2) and Eq. (19), we can write

\[
- \kappa \int \lambda R_{ab} k^a k^b d\lambda dA = \{- \sum_i (V_i + K_i)\} \delta A
\]

where, all integrands, \( \delta A \) and \( \kappa \) are defined in Appendix C.

Using Eq. (C5)

\[
\kappa \int T_{ab} k^a k^b d\lambda dA = \{- \sum_i (V_i + K_i)\} \int R_{ab} k^a k^b d\lambda dA
\]

Therefore,

\[
\kappa T_{ab} k^a k^b = \{- \sum_i (V_i + K_i)\} R_{ab} k^a k^b
\]

which is valid for all null \( k^a \). It implies that [12]

\[
T_{ab} = -\frac{1}{\kappa} \sum_i (V_i + K_i) \left\{ R_{ab} + \left( -\frac{R}{2} + \Lambda \right) \right\} g_{ab}
\]

where, \( R_{ab} \) and \( R \) are Ricci tensor and Ricci scalar, respectively and \( \Lambda \) is some constant. Therefore, we can imagine the unit area is composed by the energetic spacetime events with discrete geometry. As Eq. (23) shows, the energy-momentum tensor can be calculated by the product of the energy stored within the discrete geometry and the discrete curvature of the unit area. It means that energy-momentum and the stationary state as two macroscopic
parameters can be studied by the behavior of the energetic spacetime events which are the microscopic constitutes. Eq. (23) can be called Einstein state equation. Th Eq. (23) can be written in matrix form as follows

\[ T = \text{Trace} (\Xi) (R + \Upsilon g) \]  

(24)

where, \( \Upsilon = -\frac{R}{2} + \Lambda \) and

\[ \Xi = -\frac{1}{\kappa} \begin{pmatrix} V_1 + K_1 & 0 & \cdots \\ 0 & V_2 + K_2 & 0 \\ \vdots & \vdots & \ddots \end{pmatrix} \]  

(25)

Each \( V_i + K_i \) is attributed to a causal relation between two events, after coarse-graining. Of course, if

\[ \sum_i (V_i + K_i) = -\frac{\hbar \kappa \eta}{2\pi} \]  

(26)

where, the minus sign stands for showing the cohesive energy behavior of the causal relation between the events of the denumerable causal spacetime, then

\[ \left( \frac{2\pi}{\hbar \kappa \eta} \right) T_{ab} = \{ R_{ab} + (R/2 + \Lambda) \} g_{ab} \]  

(27)

which is the Einstein state equation [12]. Using Eq. (26), it is possible to obtain a limit that the discrete state (Eq. (23)) becomes a continuous state (Eq. (27)).

Since, \( k_B T \) has energy dimension, therefore \( k_B (\frac{h \kappa}{2\pi}) \) has energy dimension. But, \( \sum_i (V_i + K_i) \) is energy per unit area therefore the coefficient \( \eta/k_B \) has the inverse of the dimension of area and \( k_B (\frac{h \kappa}{2\pi}) \left( \frac{\eta}{k_B} \right) \) has energy per area dimension. Now, if \( \frac{\eta}{k_B} \sim \frac{1}{l_p^2} \) where \( l_p = \sqrt{G\hbar/c^3} \) is the Planck length then \( \eta \sim \frac{k_B}{l_p^2} \). The result is in good agreement with the Hawking’s result which is \( \eta = \frac{k_B}{4l_p^2} \) [36].

Also, since the spacetime area \( \delta A \) is composed by the causal events, it may be possible one calculates the discrete Ricci curvature in terms of the graph which is made by the causal events when the background-independent coarse grain technique is applied [37, 38] and then solve the matrix Eq. (24). It can be the subject of the future researches.

VI. SUMMARY

It was assumed that spacetime is composed by events (called parents). The parents born
children which some children have causal relation with their parents and some have not. It was assumed that the only causal children take part in the evolution of the system in each step of growth. The system has been modeled by the Leslie matrix with finite (infinite) dimensions. In both finite and infinite dimension, it has been shown that the stationary population of causal events exist and for the population the relation $\Delta S = -\Delta Q$ is satisfied where $S$ is the entropy of system and $Q$ is its internal energy. For preserving the causal relation between parents and children, some works should be done which is stored in the system as internal energy $\Delta Q < 0$. Since, there are denumerable casual relations, by assuming that the stored energy is equal to $\hbar \omega/2$ for each causal relation, it has been shown that the energy of vacuum is equal to $m\hbar \omega/2$ where $m$ is the number of casual parents which took part in the evolution of the system and in consequence the entropy is quantized. Also, we assumed that $\delta A$ area of spacetime is composed by the disordered causal events which have the causal bounding energy $V_i$ and the kinetic energy $K_i$. By considering the past horizon of a local Rindler horizon which is instantaneously stationary (in local equilibrium) at spacetime point $p$, the relation between energy-momentum tensor $T_{ab}$ and Ricci tensor has been found. The relation can be called the Einstein state equation. Using the discrete curvature theory and the coarse grain technique, one may solve the discrete state Einstein equation for specific shape of denumerable causal spacetime (further future research subjects). Finally, it was shown that the coefficient between entropy and area i.e., $\eta \sim \frac{k_B}{\ell_p}$ which is in good agreement with the Hawking’s result.

Appendix A

For example, if $M = \begin{bmatrix} m_1 & m_2 \\ b_1 & 0 \end{bmatrix}$ it can be easily shown that, the eigenvalues are

$$E = \frac{m_1}{2} \pm \sqrt{\frac{m_1^2}{4} + m_2 b_1} := \lambda_1, \lambda_2$$

(A1)

and if $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is eigenvector then

$$m_1 u_1 + m_2 u_2 = \lambda_i u_1$$

(A2)
\[ b_1 u_1 = \lambda_i u_2 \]

Therefore
\[
\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{\lambda_i^2 + b_1^2}} \begin{pmatrix} \lambda_i \\ b_1 \end{pmatrix} = \frac{1}{\lambda_i} \frac{1}{\sqrt{\lambda_i^2 + b_1^2}} \begin{pmatrix} 1/\lambda_i \\ b_1/\lambda_i^2 \end{pmatrix}
\]  

(A4)

and
\[
\begin{pmatrix} m_1 & m_2 \\ b_1 & 0 \end{pmatrix} \begin{pmatrix} 1/\lambda_i \\ b_1/\lambda_i^2 \end{pmatrix} = \begin{pmatrix} (m_1 \lambda_i + b_1 m_2) / \lambda_i^2 \\ b_1/\lambda_i \end{pmatrix} = \begin{pmatrix} 1 \\ b_1/\lambda_i \end{pmatrix} = \lambda_i \begin{pmatrix} 1/\lambda_i \\ b_1/\lambda_i^2 \end{pmatrix}
\]

(A5)

However, if \( \vec{v} M = \lambda \vec{v} \) and \( \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) then
\[ m_1 v_1 + b_1 v_2 = \lambda v_1 \]

(A6)

\[ m_2 v_1 = \lambda v_2 \]

(A7)

Therefore
\[
\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{\lambda_i^2 + m_2^2}} \begin{pmatrix} \lambda_i \\ m_2 \end{pmatrix}
\]

(A8)

and
\[
\begin{pmatrix} \lambda_i & m_2 \\ b_1 & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \lambda_i m_1 + m_2 b_1 \\ \lambda_i m_2 \end{pmatrix} = \begin{pmatrix} \lambda_i^2 \\ \lambda_i m_2 \end{pmatrix} = \lambda_i \begin{pmatrix} \lambda_i \\ m_2 \end{pmatrix}
\]

(A9)

Now, if \( M^* = \begin{pmatrix} 0 & -m_2 \\ -b_1 & m_1 \end{pmatrix} \) is the adjoint of \( M \) then
\[
\begin{pmatrix} 0 & -m_2 \\ -b_1 & m_1 \end{pmatrix} \begin{pmatrix} -m_2 \\ \lambda_i \end{pmatrix} = \begin{pmatrix} -m_2 \lambda_i \\ b_1 m_2 + m_1 \lambda_i \end{pmatrix} = \lambda_i \begin{pmatrix} -m_2 \\ \lambda_i \end{pmatrix}
\]

(A10)

Therefore, \( \vec{v}^* = \begin{pmatrix} -m_2 \\ \lambda_i \end{pmatrix} \) is the eigenvector of \( M^* \).
Appendix B

It has been shown that [34]

**Theorem#1:** Let $M$ be an infinite-dimensional population matrix. If (i) $m_i > 0$ for infinitely many $i$ and $m_i$, bounded, (ii) the greatest common divisor of the subscripts $i$ of the positive $m_i$ is unity, and (iii) Limit$_{i \to \infty} b_i = 0$, then the population matrix $M$ has essentially unique stationary distribution $\bar{x} = (x_1, x_2, \cdots), x_i > 0$.

And also [34]

**Theorem#2:** If (i) $m_i > 0$ for infinitely many $i$, and $m_i$ bounded; (ii) the matrix $M$ defines a compact operator $T$ on $l_2$, that is the Hilbert space of vectors $\bar{x} = (x_1, x_2, \cdots)$ for which $\sum |x_i|^2 < \infty$, then the equation $M \bar{x} = \lambda \bar{x}$ has a solution $\lambda > 0$ and $\bar{x} = (x_1, x_2, \cdots), x_i > 0$.

Therefore, for $i \to \infty$ an essentially unique stationary distribution exists [34].

However, we can write the entries of an infinite denumerable Leslie matrix as [35]

$$M_{ij} = \delta_{1j}m_j + \delta_{i,j+1}b_i$$  \hspace{1cm} (B1)

Therefore, the corresponding discrete dynamical equation is

$$u_n = M_{n-1} u_{n-1}$$  \hspace{1cm} (B2)

Now, if $u_0$ is the first parent (initial condition) then

$$u_n = M^n u_0$$  \hspace{1cm} (B3)

**Lemma:** It has been shown that, there is the generating matrix $G(z)$ such that [35]

$$G(z) = \sum_{n \geq 0} M^n z^n$$

where, $z$ is a complex indeterminate and the entries of $G$ is

$$G_{ij} = C_{j}^{i-1}z^{i-j} + \frac{C_{i}^{i-1} \sum_{n \geq 0} m_{j+n-1} C_{j+n-1}^{j+n-1} z^{n+i}}{\Delta(z)}$$  \hspace{1cm} (B4)
Here,

\[
C_{k_i}^{k_f} = \begin{cases} 
\Pi_{k=k_i}^{k_f} b_k & \text{for } k_f > k_i \\
1 & \text{for } k_f = k_i - 1 \\
0 & \text{Otherwise}
\end{cases}
\]  

which is similar to Eq. (2) and

\[
\Delta (z) = 1 - \sum_{n \geq 0} m_n C_1^n z^{n+1}
\]

It should be noted that, the real root of \( \Delta (z) \) is associated with the solution of Euler–Lotka equation which is [35]

\[
1 - \sum_{n \geq 0} m_n C_1^n \left( \frac{1}{\rho} \right)^{n+1} = 0
\]

where, \( \rho \) is the leading eigenvalue \((\rho > 0)\).

**Appendix C**

Let us to consider a black hole event horizon. The system is the degrees of freedom beyond the horizon and the outside world is separated from the system by a causality barrier. Such a system is not in equilibrium because the horizon is expanding, contracting, or shearing. A local Rindler horizon of a small spacelike 2-surface element \( P \) whose past directed null normal congruence to one side (which we call the inside) and has vanishing expansion and shear at a first order neighborhood of each spacetime point \( P \) is considered [12]. The part of spacetime beyond the Rindler horizon that is instantaneously stationary (in local equilibrium) at \( P \) is considered as system. The heat flux to the past is given by [12]

\[
\delta Q = \int T_{ab} \chi^a d\Sigma^b
\]

where, \( T_{ab} \) is the matter energy-momentum tensor and \( \chi^a \) is the killing field generating boost orthogonal to \( P \) and vanishing at \( P \). The integral is over a pencil of generators of the inside past horizon of \( P \). Of course, it is assumed that the temperature of the system is equal to the Unruh temperature which is observed by observer hovering just inside the horizon. It can be shown that [12]

\[
\delta Q = -\kappa \int \lambda T_{ab} k^a k^b d\lambda dA
\]
where, $k^a$ is the tangent vector to the horizon generators for an affine parameter $\lambda$ that vanishes at $P$ and is negative to the past of $P$, $\chi^a = -\kappa \lambda k^a$ and $d\Sigma^a = k^a d\lambda dA$, where $dA$ is the area element on a cross section of the horizon. Here, $\kappa$ is the acceleration of the Killing orbit on which the norm of $\chi^a$ is unity and it is assumed that the speed of light is equal to unity. However, according to the Unruh effect, the Minkowski vacuum state of quantum fields is a thermal state with respect to the boost Hamiltonian at temperature $T = \hbar \kappa / 2\pi$ and it can be shown that [12]

$$\delta Q = \left(\frac{\hbar \kappa}{2\pi}\right) \eta \int \lambda R_{ab} k^a k^b \lambda dA$$

(C3)

Therefore, for all null $k^a$ [12]

$$\frac{2\pi}{\hbar \eta} T_{ab} = R_{ab} + \left(-\frac{R}{2} + \Lambda\right) g_{ab}$$

(C4)

where, $R_{ab}$ and $R$ are Ricci tensor and Ricci scalar, respectively. $\Lambda$ is some constant. Here, it is assumed that [12]

$$\delta A = -\int \lambda R_{ab} k^a k^b \lambda dA$$

(C5)

which is the area variation of a cross section of a pencil of generators of the inside past horizon of $P$.

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