Non-split Brauer-Severi varieties do not admit full exceptional collections

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Abstract

Recently, Novaković conjectured that non-split Brauer-Severi varieties do not admit full strong exceptional collections. In this short note, we explain how a stronger version of this conjecture follows easily from known results on noncommutative motives.

1 Introduction

For an arbitrary field $k$, Novaković stated the following as a conjecture in [3]:

Conjecture 1.1. Let $X \neq \mathbb{P}^n_k$ be a $n$-dimensional Brauer-Severi variety. Then $D^b(X)$ does not admit a full strongly exceptional collection.

He proves the conjecture in dimension $n \leq 3$ [4] by exploiting the transitivity of the braid group action on full exceptional collections for $\mathbb{P}^n_k$ to reduce to an equivalence $D^b(A) \cong D^b(k)$. If $A \cong M_l(D)$, for $D$ a division algebra over $k$, these are just the categories of $\mathbb{Z}$-graded vector spaces over $D$, respectively $k$, so there is an equivalence only if $D$ is isomorphic to $k$. Since the transitivity of the braid group action (which is only established for $n \leq 3$) is only used to be able to reduce to a single semi-orthogonal component, this suggests that noncommutative motives might provide the right framework for this conjecture. Using some results from [7] on noncommutative motives of separable algebras, we prove a slightly stronger version of Conjecture 1.1, showing that non-split Brauer-Severi varieties do not admit full étale exceptional collections.

2 Noncommutative motives of separable algebras

To any small dg-category $\mathcal{A}$, one can associate (functorially) its noncommutative motive $U(\mathcal{A})$, which takes values in a category $\mathsf{Hmo}_0(k)$. This category has as objects small dg-categories, and for two such categories $\mathcal{A}$ and $\mathcal{B}$,

$$\mathsf{Hom}_{\mathsf{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) \cong K_0\mathsf{rep}(\mathcal{A}, \mathcal{B}),$$

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where \( \text{rep}(A, B) \) is the full triangulated subcategory of \( D(A^{\text{op}} \otimes^L B) \) consisting of those \( A-B \)-bimodules \( B \) such that for every \( x \in A \), the right \( B \)-module \( B(x, -) \) is a compact object in \( D(B) \). The composition is induced by the derived tensor product of bimodules.

More details on the construction of \( U \) can be found in [6], but for the purposes of this note, we will only need that \( U \) is a “universal additive invariant”. An additive invariant is any functor \( E : \text{dgcat}(k) \to D \) taking values in an additive category \( D \) such that:

1. it sends dg-Morita equivalences to isomorphisms,
2. for any pre-triangulated dg-category \( A \), with full pre-triangulated dg-subcategories \( B \) and \( C \) giving rise to a semi-orthogonal decomposition \( H^0(A) = \langle H^0(B), H^0(C) \rangle \), the morphism \( E(B) \oplus E(C) \to E(A) \) induced by the inclusions is an isomorphism.

We now review some results from [7]. Remember that the category of noncommutative Chow motives \( \text{NChow}(k) \) is defined as the idempotent completion of the full subcategory of \( \text{Hmo}_0(k) \) containing the smooth and proper dg-categories. Now let \( \text{Sep}(k) \) (respectively \( \text{CSep}(k) \)) denote the full subcategory of \( \text{NChow}(k) \) consisting of the \( U(A) \), for \( A \) a separable (respectively commutative separable) \( k \)-algebra. Also let \( \text{CSA}(k) \oplus \) denote the closure under finite direct sums of the full subcategory of \( \text{NChow}(k) \) consisting of the \( U(A) \), for \( A \) a central simple \( k \)-algebras. Note that the \( \oplus \) is there since central simple \( k \)-algebras are not closed under products, whereas (commutative) separable algebras are. In this way \( \text{Sep}(k), \text{CSep}(k) \) and \( \text{CSA}(k) \oplus \) are additive symmetric monoidal categories.

**Theorem 2.1.** [7, Corollary 2.13] There is an equivalence of categories

\[
\{ U(k)^{\oplus n} | n \in \mathbb{N} \} \simeq \text{CSA}(k)^{\oplus} \times_{\text{Sep}(k)} \text{CSep}(k),
\]

i.e. \( \{ U(k)^{\oplus n} | n \in \mathbb{N} \} \) is a 2-pullback of categories with respect to the obvious inclusion morphisms.

For a central simple algebra \( A \) over \( k \), denote by \( \text{ind}(A) \) and \( \text{deg}(A) \) the index (respectively degree) of \( A \). Then by [2, Proposition 4.5.16], \( A \) admits a \( p \)-primary decomposition

\[
A = \bigotimes_{i=1}^{k} A^{p_i},
\]

where \( A^{p_i} \) is uniquely characterised by the property \( \text{ind}(A^{p_i}) = p_i^{n_i} \) if

\[
\text{ind}(A) = p_1^{n_1} \cdots p_k^{n_k}
\]

is the primary decomposition.
Theorem 2.2. [7, Theorem 2.19] Given central simple $k$-algebras $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, the following two conditions are equivalent:

1. There is an isomorphism of noncommutative motives:
   \[
   U(A_1) \oplus \cdots \oplus U(A_n) \simeq U(B_1) \oplus \cdots \oplus U(B_m).
   \]

2. The equality $n = m$ holds, and for all $1 \leq j \leq n$ and all $p$
   \[
   [B_j^p] = [A_{\sigma_p(j)}^p]
   \]
   holds in $\text{Br}(k)$, for some permutations $\sigma_p$ depending on $p$.

Remark 2.3. Though the isomorphism classes of objects in $\text{CSA}(k)^\oplus$ are in some sense understood by Theorem 2.2, this is not true for $\text{CSep}(k)$. In fact, using the (additive) equivalence $\text{CSep}(k) \simeq \text{Perm}(G)$, where $G = \text{Gal}(k_{\text{sep}}/k)$, and $\text{Perm}(G)$ is the category of permutation $G$-modules, interesting examples can be obtained from integral representation theory, see [7, Remark 2.5, 2.6].

3 Brauer-Severi varieties and full étale exceptional collections

Denote by $BS(A)$ the Brauer-Severi variety associated to a central simple $k$-algebra $A$. We will say (see also [5]) that an object $E \in D^b(BS(A))$ satisfying $\text{Hom}(E, E[i]) = 0$ for all $i \neq 0$ is

- semi-exceptional if $\text{Hom}(E, E) = S$ is a semisimple $k$-algebra,
- étale exceptional if $\text{Hom}(E, E) = L$ is an étale $k$-algebra.

It is well known [1] that $BS(A)$ has a full semi-exceptional collection giving rise to a semi-orthogonal decomposition
\[
D^b(BS(A)) = \langle D^b(k), D^b(A), \ldots, D^b(A^{\oplus \deg(A) - 1}) \rangle.
\]

The following theorem now provides a positive answer to Conjecture 1.1.

Theorem 3.1. Non-split Severi-Brauer varieties do not admit full étale exceptional collections.

Proof. Suppose $A$ is non-split and $\deg(A) = d$. Then if $BS(A)$ has a full étale exceptional collection, we deduce from (3.1) and additivity of $U(-)$ with respect to semi-orthogonal decompositions that there is an isomorphism
\[
U(k) \oplus U(A) \oplus \cdots \oplus U(A^{\oplus d - 1}) \simeq U(D^b(BS(A))) \cong U(L_1) \oplus \cdots \oplus U(L_d),
\]
where the $L_i$ are étale $k$-algebras. Using Theorem 2.1 and the universal property of fibre products, this isomorphism gives rise to an isomorphism

$$U(k) \oplus U(A) \oplus \cdots \oplus U(A^\otimes d-1) \simeq U(k)^\otimes d.$$ 

Now by Theorem 2.2, for all $p : [A^p] = [k]$ in Br$(k)$, so $[A] = [k]$ or in other words $A$ should split. \qed

**Remark 3.2.** This result formalizes (in this case) the intuition that for varieties defined over arbitrary fields, one should consider semi-exceptional collections instead of usual exceptional collections.

**References**

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