THE ORLICZ-MINKOWSKI PROBLEM FOR POLYTOPES

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ABSTRACT. The Orlicz-Minkowski problem for polytopes is studied, and some existence results are established by the variational method.

1. Introduction. Given a set of unit vectors $u_1, \cdots, u_N$ in $\mathbb{R}^n$ and a set of positive numbers $a_1, \cdots, a_N > 0$, the classical Minkowski theorem states that there is an $m$-faced polytope in $\mathbb{R}^n$ such that its faces have outer normal $u_1, \cdots, u_N$ and the corresponding face-areas $a_1, \cdots, a_N$ if and only if $u_1, \cdots, u_N$ are not located on a closed hemisphere and satisfy

$$a_1 u_1 + \cdots + a_N u_N = 0. \quad (1)$$

If (1) is satisfied, the polytope is unique up to a translation. The more general set up is as follows: Given a finite Borel measure $\mu$ on the $(n-1)$-dimensional unit sphere $S^{n-1}$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$, find a convex body $K$ such that the surface measure of $K$ is $\mu$. This was solved by Alexandrov, Fenchel and Jensen. A finite Borel measure $\mu$ is the surface measure of a convex body $K$ if and only if $\mu$ is not supported on a great subsphere and satisfies

$$\int_{S^{n-1}} x d\mu = 0, \quad (2)$$

where $x = (x_1, \cdots, x_n)$ is the coordinate function of $S^{n-1}$, moreover, the convex body is unique up to a translation, see [24]. The regularity of $K$ is an important and difficult problem if $\mu$ is regular w.r.t. the Hausdorff measure $H^{n-1}$, i.e., $\mu = gdH^{n-1}$. The convex body $K$ is regular if the density function $g$ is, see [5, 18, 22, 23].

The $L_p$-Minkowski problem proposed by Lutwak in [20] is a natural and important generalization of the classical Minkowski problem. In order to state the problem, we need the notion of $L_p$ surface measure of a convex body. Unlike the surface measure, it depends on the support function. Let $K$ be a convex body in $\mathbb{R}^n$ and the origin be in its interior, the support function of $K$ is defined as $h : \mathbb{R}^n \to \mathbb{R}$, $h_K(x) = \sup\{x \cdot y | y \in K\}$. It is positively homogeneous of degree one and subadditive. The convex body $K$ is uniquely determined by its support function up to a

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translation. For \( p \in \mathbb{R} \), the \( L_p \) surface measure \( S_p(K, \cdot) \) of \( K \) is a Borel measure on the unit sphere \( S^{n-1} \) defined by

\[
S_p(K, \omega) = \int_{x \in \nu^{-1}_K(\omega)} (x \cdot \nu_K(x))^{1-p} dH^{n-1}, \quad \omega \subset S^{n-1},
\]

where \( H^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure, \( \nu_K : \partial K \to S^{n-1} \) is the Gauss map, which is defined almost everywhere. If \( K \) is regular, then

\[
S_p(K, \omega) = \int_{\omega} h^{1-p} \det (\nabla_{ij} h_K + \delta_{ij} h_K) dH^{n-1},
\]

\( \nabla_{ij} \) is the covariant derivative w.r.t. an orthonormal basis \( e_i \) of the standard metric on \( S^{n-1} \); and if \( K \) is a convex polytope with the unit out normals \( \{u_1, \cdots, u_N\} \), then

\[
S_p(K, \cdot) = \sum_{i=1}^N h_i^{1-p} S_i \delta(u_i),
\]

where \( h_i = \sup \{ u_i \cdot x | x \in K \} \) and \( S_i \) are the support function and \((n-1)\)-dimensional area of the facet of \( K \) with normal \( u_i \), respectively.

The \( L_p \)-Minkowski problem for convex body can be stated as: Given a finite Borel measure \( \mu \) on \( S^{n-1} \) which is not supported in a closed hemisphere, find a convex body \( K \) in \( \mathbb{R}^n \) such that the origin is in its interior and its \( L_p \) surface measure is \( \mu \):

\[
S_p(K, \omega) = \mu(\omega), \forall \ Borel \ set \ \omega \subset S^{n-1}. \tag{3}
\]

This is equivalent to solve

\[
h^{1-p} \det (\nabla_{ij} h + \delta_{ij} h) = g(x), \quad x \in S^{n-1} \tag{4}
\]

if \( \mu = gdH^{n-1} \), and

\[
h_i^{1-p} S_i = a_i, \quad i = 1, \cdots, N \tag{5}
\]

if \( \mu = \sum_{i=1}^N a_i \delta(u_i) \). The classical Minkowski problem corresponds to the case \( p = 1 \).

This \( L_p \)-Minkowski and some related problem have been extensively studied in the last three decades by Lutwak and his collaborators and many others, see [2, 3, 4, 6, 7, 8, 11, 12, 13, 14, 19, 20, 21] and the references therein. For equation (4), the case \( p = -n \) is very special, it relates to affine geometry and is known that there are obstructions to solve (4), see [1, 6, 15, 16]. Another special and important case is \( p = 0 \), which is also called the logarithmic Minkowski problem. A necessary and sufficient condition was given by Borceczky et al [3] for even \( \mu \): an even measure \( \mu \) is the \( L_0 \) surface measure of a symmetric convex body if and only if the following subspace concentration condition is satisfied: for any subspace \( \xi \) of \( \mathbb{R}^n \) with \( 0 < \dim \xi < n \),

\[
\mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}), \tag{6}
\]

and if the equality holds, then there is a subspace \( \xi' \), which is complementary to \( \xi \) such that

\[
\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}). \tag{7}
\]

It is shown in [2, 14, 25] that the strict subspace concentration condition

\[
\mu(\xi \cap S^{n-1}) < \frac{\dim \xi}{n} \mu(S^{n-1}) \tag{8}
\]
is also sufficient for the solvability of the logarithmic Minkowski problem for discrete measure \( \mu = \sum_{i=1}^{N} a_i \delta(u_i) \) without the even condition.

A further generalization is the following Orlicz-Minkowski problem, see \([11]\): Given a function \( \phi : (0, \infty) \to (0, \infty) \), find a convex body \( K \) such that

\[
c\phi(h_K)dS_K = d\mu,
\]

for some constant \( c > 0 \), that is,

\[
c\int_{\omega} \phi(h_K)dS_K = \mu(\omega), \quad \text{for any Borel set } \omega \subset S^{n-1}.
\]

If \( \phi(h) = h^{1-p} \), this is equivalent to the \( L_p \)-Minkowski problem.

Throughout this paper we assume that the support of a measure \( \mu \) is not contained in a closed hemisphere as this is a necessary condition. The Orlicz-Minkowski problem (9) is solvable for an even measure \( \mu \) if one of the following conditions is satisfied, see \([11]\):

1. \( \phi : (0, \infty) \to (0, \infty) \) is continuous and monotone decreasing;
2. \( \phi : (0, \infty) \to (0, \infty) \) is continuous and satisfies

\[
\int_{0}^{1} \frac{ds}{\phi(s)} < \infty, \quad \int_{1}^{\infty} \frac{ds}{\phi(s)} = \infty.
\]

Without the symmetry assumption of \( \mu \), the following is proved in \([10]\),

3. \( \phi : (0, \infty) \to (0, \infty) \) is continuous and satisfies (\( \phi \)) and \( \phi(0) = \infty \), then the problem is solvable for a measure \( \mu \).

We see that case 1 and case 2 are related to the \( L_p \)-Minkowski problem for \( p > 1 \) and \( 0 < p < 1 \), respectively. For other results of the problem we refer to \([6, 10, 11]\) and references therein.

In this paper we study the polytope case \( \mu = \sum_{i=1}^{N} a_i \delta(u_i) \) with \( a_i > 0 \) and \( u_1, \ldots, u_N \) being not located in a closed hemisphere via the variational method. Our aim is twofolds, the first is to present some results complementary to the above mentioned ones, the second is to simplify some arguments for the results related to polytope \( L_p \)-Minkowski problem in \([2, 13, 14, 25, 26, 27, 28, 29, 30]\).

The main results can be stated as follows.

**Theorem 1.1.** Let \( \phi : (0, \infty) \to (0, \infty) \) be continuous and monotone increasing satisfying (\( \phi \)) and \( \phi(0) = 0 \). Then the Orlicz-Minkowski problem (9) is solvable for the measure \( \mu = \sum_{i=1}^{N} a_i \delta(u_i) \).

As pointed out before this is related to \( L_p \)-Minkowski problem for \( 0 < p < 1 \), the case \( p < 0 \) is connected to the next result.

**Theorem 1.2.** Let \( \phi : (0, \infty) \to (0, \infty) \) be continuous and monotone increasing satisfying \( \phi(0) = 0 \). Then the Orlicz-Minkowski problem (9) is solvable for the measure \( \mu = \sum_{i=1}^{N} a_i \delta(u_i) \) provided any \( n \) normals \( u_{i_1}, \ldots, u_{i_n} \) are linearly independent.

The condition that any \( n \) normals \( u_{i_1}, \ldots, u_{i_n} \) are linearly independent is called the normals are in general positions, which plays an important role in the solvability of the \( L_p \)-Minkowski problem, see \([13, 14, 25, 27, 29, 30]\).

It is known that the case \( p = 0 \) is more complicated for \( L_p \)-Minkowski problem. The following result is a slight generalization of results for the logarithmic Minkowski problem for discrete measure.
Theorem 1.3. Let $\phi : (0, \infty) \to (0, \infty)$ be continuous and monotone increasing satisfying $\phi(0) = 0$ and there exist $A > 0$ and $B \in \mathbb{R}$ such that
\[
\int_1^s \frac{d\tau}{\phi(\tau)} \geq A \log s + B, \quad s \in (0, \infty).
\] (\phi 1)

Then the Orlicz-Minkowski problem (9) is solvable for the measure $\mu = \sum_1^N a_i \delta(u_i)$ if the subspace concentration condition (8) holds for any subspace $\xi \subset \mathbb{R}^n$ with $0 < \dim \xi < n$.

The proofs of these results are based on the variational method as in [10, 11].

For $\mu = \sum_1^N a_i \delta(u_i)$, the Orlicz-Minkowski problem (9) can be stated as: find a polytope $P$ with normals $\{u_1, \cdots, u_N\}$ such that
\[
c\phi(h_i) S_i = a_i, \quad 1 \leq i \leq N,
\] (11)
or equivalently,
\[
S_i = \frac{1}{c\phi(h_i)} a_i, \quad 1 \leq i \leq N,
\] (12)
where $S_i$ is the surface area of the facet of the polytope $P$ with normal $u_i$, $h_i = h(u_i)$.

Equation (12) is a variational problem, and we show that is equivalent to a restricted variational problem and there is a maximizer which solves the problem.

The paper is organized as follows. In Section 2, the variational formulation is given and Theorem 1.1 and Theorem 1.2 are proved, in Section 2 a proof of Theorem 1.3 is provided based on the John’s Lemma for convex body.

2. Proofs of Theorem 1.1 and Theorem 1.2. In this section we first formulate equation (12) as a variational problem, and prove Theorem 1.1 and Theorem 1.2.

Given a set of unit vectors $\{u_i\}_{1}^{N}$ in $\mathbb{R}^n$ not located in a closed hemisphere and $h \in \mathbb{R}^{N}$ with $h_i > 0$, let $P(h)$ be the Alexandrov polytope:
\[
P(h) = \bigcap_{i=1}^{N} H_{u_i, h_i} = \bigcap_{i=1}^{N} \{y \in \mathbb{R}^n : y \cdot u_i \leq h_i\}.
\]

It is easy to see that $P(h)$ is a polytope in $\mathbb{R}^n$, whose normals of facets form a subset of $\{u_1, \cdots, u_N\}$.

The following is a well known fact, which is the basis of our approach, for a proof, see for instance [11].

Proposition 1. Let $V(h)$ be the volume of $P(h)$. Then the function $V(h)$ is $C^1$ in $\mathbb{R}_+^N = \{h \in \mathbb{R}^N \mid h = (h_1, \cdots, h_N), h_i > 0\}$ and
\[
\frac{\partial V(h)}{\partial h_i} = n S_i(h), \quad i = 1, \cdots, N,
\]
where $S_i(h)$ is the surface area of the facet of the polytope $P(h)$ with normal $u_i$.

Let $\Phi(s) = \int_1^s \frac{d\tau}{\phi(\tau)}$ and
\[
M = \{h \in \mathbb{R}^N \mid h_i > 0, \sum_{1}^{N} a_i \Phi(h_i) = 1\}.
\]

It is an $(N-1)$-dimensional manifold if it is nonempty. With Proposition 1, writing (12) as
\[
\frac{\partial V(h)}{\partial h_i} = \frac{1}{nc} \frac{\partial \sum_{1}^{N} a_i \Phi(h_i)}{\partial h_i}, \quad 1 \leq i \leq N,
\] (13)
and considering $\frac{1}{nc}$ as a Lagrange multiplier we have
Proposition 2. Let \( h \in M \) be a critical point of \( V(h) \) restricted to \( M \), then \( h \) is a solution of (13).

This variational formulation of the (13) is slightly different from the one used in [11], where the following function, which was introduced in [6] first

\[
F(h) = \frac{n}{\alpha} V(h)^{\frac{n}{\alpha}} - \sum_{i=1}^{N} a_i \Phi(h_i)
\]

was used. With some conditions on \( \phi \) and \( \mu \), it is showed in [11] that \( F(h) \) has a maximizer that solves (13).

We know a maximizer is a critical point, but the function \( V(h) \) may not have an upper bound in \( M \). To overcome this difficulty we consider the following restricted variational problem. Let \( f(h) = \sum_{i=1}^{N} a_i \Phi(h_i) \) and

\[
M_1 = \{ h \in M | \sum_{i=1}^{N} \frac{a_i}{\phi(h_i)} u_i = 0 \}.
\]

This restriction has the following consequence: for \( h \in M_1 \),

\[
f(h, 0) \geq f(h, \xi), \quad \xi \in P(h)
\]

and 0 is the unique maximizer of \( f(h, \xi) \) in \( \xi \in P(h) \), where \( f(h, \xi) = \sum_{i=1}^{N} a_i \Phi(h_i) - \xi \cdot u_i \).

The following proposition plays a key role in our arguments.

Proposition 3. Let \( h = (h_1, \cdots, h_N) \in M_1 \) be a critical point of \( V(h) \) restricted to \( M_1 \). Then \( h \) is also a critical point of \( V(h)|_M \).

To prove this proposition we need

Lemma 2.1. Let \( u_i = (u_{i,1}, \cdots, u_{i,n}) \in \mathbb{R}^n \) and \( U_j = (u_{1,j}, \cdots, u_{N,j}) \in \mathbb{R}^N \). Then the matrix \( (\nabla^2 f(h_j) U_j, U_i) \) is definite for \( h \in M_1 \) if \( \phi \) is strictly monotone. In particular, it is invertible.

Proof. The function \( f(h) \) is concave or convex in \( h \), it follows that the Hessian \( \nabla^2 f(h) \) is definite in \( \mathbb{R}^N \). We know \( \text{span}\{u_i | 1 \leq i \leq N\} = \mathbb{R}^n \), so the vectors \( U_1, \cdots, U_n \) are linearly independent, and \( (\nabla^2 f(h_j) U_j, U_i) \) is the Hessian with respect to the basis \( \{U_1, \cdots, U_n\} \), whence it is definite. \( \square \)

Lemma 2.2. The function \( V(h) \) satisfies

\[
\sum_{i=1}^{N} \frac{\partial V(h)}{\partial h_i} u_i = \sum_{i=1}^{N} S_i(h) u_i = 0. \quad (14)
\]

Proof. Let \( \{e_1, \cdots, e_n\} \) be the standard orthonormal basis of \( \mathbb{R}^n \) and \( P(h) \) be the polytope \( P(h) + te_1 \), the translation of \( P(h) \). It is easy to see that \( P(h^t) = P(h) \) with \( h^t = (h_1 + tu_1 \cdot e_1, \cdots, h_N + tu_N \cdot e_1) \). Clearly, \( V(h^t) = V(h) \) as the volume is translation invariant and

\[
\frac{dV(h^t)}{dt} = \sum_{i=1}^{N} \frac{\partial V(h)}{\partial h_i} u_i \cdot e_1 = 0.
\]

This holds for all \( e_i \), hence (14) follows. \( \square \)
Proof of Proposition 3: Let \( h = (h_1, \cdots, h_N) \) with \( h_i > 0 \) be a critical point of \( V(h) \) restricted to \( M_1 \). Then there are Lagrange multipliers \( \lambda, \lambda_1, \cdots, \lambda_n \) such that
\[
\nabla V(h) = \lambda \nabla f(h) + \lambda_1 \nabla^2 f(h) U_1 + \cdots + \lambda_n \nabla^2 f(h) U_n.
\]
(15)
In terms of \( U_j \), the constraint \( \sum_1^N \frac{a_i}{\phi(h_i)} u_i = 0 \) becomes
\[
(\nabla f(h), U_j) = 0, \quad j = 1, \cdots, n.
\]
Hence taking the inner product with \( U_j \) we obtain
\[
\lambda_1 (\nabla^2 f(h) U_j, U_1) + \cdots + \lambda_n (\nabla^2 f(h) U_j, U_n) = 0, \quad j = 1, \cdots, n,
\]
(16)
which imply \( \lambda_1 = \cdots = \lambda_n = 0 \) as the matrix \( (\nabla^2 f(h) U_j, U_i) \) is invertible by Lemma 2.1. Then from (15) we get
\[
\frac{\partial V}{\partial h_i} = \lambda \frac{a_i}{\phi(h_i)}, \quad j = 1, \cdots, N.
\]
(17)
This finishes the proof. \( \square \)

Given a sequence \( h^k = (h_1^k, \cdots, h_N^k) \) \( \in M_1 \), we omit the upper index \( k \) for the simplicity of notations and assume \( h \) satisfies
\[
h_1 \leq h_2 \cdots \leq h_N.
\]
(18)
This can be achieved after a subsequence.

The following elementary fact will be used frequently later.

**Lemma 2.3.** Let \( \phi : (0, \infty) \to (0, \infty) \) be monotone increasing and \( \phi(0) = 0 \) and \( h = (h_1, \cdots, h_N) \in M_1 \). If there is an \( i_0 \) such that \( h_{i_0} \to 0 \), then there are positive numbers \( b_1, \cdots, b_m \) with \( m \geq 2 \) such that
\[
b_1 u_1 + b_2 u_2 + \cdots + b_m u_m = 0.
\]
(19)

**Proof.** Because of (18) we can assume
\[
h_i \to 0, \phi(h_i) \to 0, \quad 1 \leq i \leq i_0, \quad h_i \geq \delta > 0, \phi(h_i) \geq \delta > 0, \quad i_0 + 1 \leq i \leq N.
\]
From \( \sum_1^N \frac{a_i}{\phi(h_i)} u_i = 0 \) we get
\[
a_1 u_1 + a_2 \frac{\phi(h_1)}{\phi(h_2)} u_2 + \cdots + a_N \frac{\phi(h_1)}{\phi(h_N)} u_N = 0.
\]
(20)
Taking limit in (20) we have
\[
b_1 u_1 + b_2 u_2 + \cdots + b_{i_0} u_{i_0} = 0,
\]
(21)
where \( b_1 = a_1 > 0, b_i = \lim a_i \frac{\phi(h_1)}{\phi(h_i)} \geq 0, 2 \leq i \leq i_0 \). It follows from (21) that there is \( m \geq 2 \) such that \( b_i > 0 \) for \( 1 \leq i \leq m \) and (19) holds. \( \square \)

With these preparations now we can prove our results. We will show that \( V(h) \) has a maximizer on \( M_1 \).

**Proof of Theorem 1.1:** With the assumption
\[
\int_0^1 \frac{ds}{\phi(s)} < \infty, \quad \int_1^\infty \frac{ds}{\phi(s)} = \infty,
\]
(\( \phi \))
we see that there is a constant \( C > 0 \) such that
\[
\|h\| \leq C, \quad h \in M_1,
\]
(22)
where \( \|h\| \) is the Euclidean norm on \( \mathbb{R}^N \). Hence \( V(h) \) has an upper bound on \( M_1 \). Take a sequence \( h^k \in M_1 \) such that \( V(h^k) \to \sup_{h \in M_1} V(h) = V_0 > 0 \) as \( k \to \infty \).
We can assume \( h^k \to h^0 \). It is easy to see that \( h^0 \) is a maximizer of \( V(h) \) on \( M_1 \) provided each component of \( h^k \) possesses a positive lower bound.

Suppose the \( i_0 \)-th component \( h_{i_0} \) of \( h^k \) tends to zero. Then

\[
h_i \to 0, \quad 1 \leq i \leq i_0
\]

and from the arguments in Lemma 2.3 we find that \( i_0 \geq 2 \). Hence

\[
P(h) \subset \{ y \in \xi[y \cdot u_i \leq h_i, 1 \leq i \leq i_0] \times \{ y \in \xi_+ \| y \| \leq L \}
\]

for large \( L \) by (22) and

\[
V(h) \leq C \vol_{\xi}(\{ y \in \xi[y \cdot u_i \leq h_i, 1 \leq i \leq i_0] \}) \to 0
\]

as \( h_i \to 0 \) for \( 1 \leq i \leq i_0 \), where \( \xi = \text{span}\{u_1, \ldots, u_{i_0}\} \) and \( \vol_{\xi} \) is the \( \xi \)-volume.

To prove Theorem 1.2, let \( h^k \in M_1 \) be a sequence such that

\[
V(h^k) \to \sup_{h \in M_1} V(h) \geq V_0 > 0 \quad \text{as} \quad k \to \infty.
\]

Using the assumption that \( u_1, \ldots, u_N \) are in general positions, we will show that each component \( h_i \) of \( h^k \) satisfies

\[
0 < \frac{1}{C} \leq h_i \leq C, \quad 1 \leq i \leq N
\] (23)

for some constant \( C \). With this property, the existence of a maximizer of \( V(h) \) on \( M_1 \) follows immediately.

**Lemma 2.4.** Let \( u_1, \ldots, u_N \) be in general positions and \( h \in M_1 \) such that \( V(h) \geq V_0 > 0 \). Then there is an \( \epsilon_0 > 0 \) such that \( h_i \geq \epsilon_0, \quad 1 \leq i \leq N \).

**Proof.** Suppose there is a sequence \( h^k \in M_1 \) such that \( V(h^k) \geq V_0 > 0 \) and some components of \( h^k = (h^k_1, \ldots, h^k_N) \) goes to zero as \( k \to \infty \). By Lemma 2.3, there are \( b_i > 0 \) for \( 1 \leq i \leq m \) such that

\[
b_1 u_1 + b_2 u_2 + \cdots + b_m u_m = 0.
\]

The general position condition implies \( m \geq n + 1 \) and \( \xi = \text{span}\{u_1, \ldots, u_m\} = \mathbb{R}^n \).

Hence

\[
P(h) \subset \{ y \in \xi[y \cdot u_i \leq h_i, 1 \leq i \leq m] \}
\]

and \( V(h) \to 0 \) as \( h_i \to 0 \) for \( 1 \leq i \leq m \). This contradicts to \( V(h) \geq V_0 > 0 \). \( \square \)

It remains to prove that each component of \( h^k \) has an upper bound. To this end it suffices to deal with the case \( \Phi(\infty) = \int_1^\infty \frac{dt}{\phi(t)} < \infty \) as the case \( \Phi(\infty) = \infty \) is the same as in Theorem 1.1. We assume \( \sum_1^N a_i \Phi(\infty) > 1 \). This is not a restriction for our problem.

**Lemma 2.5.** Let \( u_1, \ldots, u_N \) be in general positions and \( \Phi(\infty) = \int_1^\infty \frac{dt}{\phi(t)} < \infty \). Then there is a constant \( V_1 \) such that

\[
V(h) \leq V_1, \quad h \in M_1.
\] (24)

**Proof.** Let \( h^k \in M_1 \) be a sequence such that \( V(h^k) \to \infty \) as \( k \to \infty \). Then \( \| h^k \| \to \infty \) and there exist an integer \( m \) and a constant \( C > 0 \) such that

\[
h_i \leq C, \quad 1 \leq i \leq m, \quad h_i \to \infty, \quad m + 1 \leq i \leq N.
\]

By the assumption \( \sum_1^N a_i \Phi(\infty) > 1 \) we have \( m \geq 1 \) and from \( \Phi(\infty) < \infty \) there holds

\[
\phi(h_i) \to \infty, \quad m + 1 \leq i \leq N.
\]
Inserting this into $\sum_{i=1}^{N} \frac{a_i}{\phi(h_i)} u_i = 0$ we obtain

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m = 0,$$

where $c_i = \lim \frac{a_i}{\phi(h_i)} > 0$, $1 \leq i \leq m$. The general position condition concludes $m \geq n + 1$ and $\xi = \text{span}\{u_1, \cdots, u_m\} = \mathbb{R}^n$. Hence

$$P(h) \subset \{ y \in \mathbb{R}^n | y \cdot u_i \leq h_i, 1 \leq i \leq m \}$$

and $V(h)$ is bounded as $h_i \leq C$ for $1 \leq i \leq m$. This is impossible as $V(h) \to \infty$. \hfill \Box

By Lemma 2.4 and Lemma 2.5, there is a sequence $h^k \in M_1$ such that

$$0 < V_0 \leq V(h^k) \to \sup_{h \in M_1} V(h) = V_1 < \infty \quad k \to \infty$$

and each component $h_i$ of $h^k$ satisfies $h_i \geq \epsilon_0 > 0$ for $1 \leq i \leq N$. We use the contrary argument to get (23).

Suppose $\|h\| \to \infty$, that is,

$$h_i \leq C, \quad 1 \leq i \leq m, \quad h_i \to \infty, \quad m + 1 \leq i \leq N;$$

we will construct an $\tilde{h} \in M_1$ such that $V(\tilde{h}) > V_1$, which contradicts to (26).

Let $\epsilon_0 \leq h_0^i = \lim h_i < \infty$, $1 \leq i \leq m$. Taking the limits in $\sum_{i=1}^{N} \frac{a_i}{\phi(h_i)} u_i = 0$ and $\sum_{i=1}^{N} a_i \Phi(h_i) = 1$ we get

$$a_1 \Phi(h_1^0) u_1 + a_2 \Phi(h_2^0) u_2 + \cdots + a_m \Phi(h_m^0) u_m = 0,$$

and

$$a_1 \Phi(h_1^0) + a_2 \Phi(h_2^0) + \cdots + a_m \Phi(h_m^0) + \sum_{m+1}^{N} a_i \Phi(\infty) = 1.$$  \hfill (27)

The fact that $u_1, \cdots, u_N$ are in general positions implies $m \geq n + 1$ and $P_0 = \cap_1^{m} \{ y \in \mathbb{R}^n | y \cdot u_i \leq h_i^0 \}$ is a polytope with $0 \in \text{int}(P_0)$, the interior, and

$$V_1 = \sup_{h \in M_1} V(h) = V(P_0)$$

as $V(h)$ is continuous in $h$. The function $\xi \to \sum_{i=1}^{m} a_i \Phi(h_i - \xi \cdot u_i)$ is concave in $P_0$ and its derivative is zero by (27) at $\xi = 0$, so it is a maximizer

$$\sum_{i=1}^{m} a_i \Phi(h_i^0 - \xi \cdot u_i) \leq \sum_{i=1}^{m} a_i \Phi(h_i^0), \quad \xi \in P_0$$

and there is a $\delta_0 > 0$ such that

$$\sum_{i=1}^{m} a_i \Phi(h_i^0 - \xi \cdot u_i) \leq \sum_{i=1}^{m} a_i \Phi(h_i^0) - \delta_0, \quad \xi \in \partial P_0,$$  \hfill (29)

Now we fix an $L_0 > 0$ such that $P(h^*) = P_0$ with $h^* = (h_1^0, \cdots, h_m^0, L_0, \cdots, L_0) \in \mathbb{R}^N$ and for $L, L' \geq L_0$, there hold

$$L - \xi \cdot u_i \leq 2L, \quad |\Phi(L) - \Phi(L')| \leq \frac{\delta_0}{2}, \quad \xi \in P_0, L, L' \geq L_0$$

\hfill (31)
It follows from (28), (29), (30) and (31) that
\[
\begin{align*}
  f(h^*, \xi) &= \sum_{i=1}^{m} a_i \Phi(h_i^0 - \xi \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(L_0 - \xi \cdot u_i) \\
  &\leq \sum_{i=1}^{m} a_i \Phi(h_i^0) + \sum_{m+1}^{N} a_i \Phi(2L_0) \\
  &< \sum_{i=1}^{m} a_i \Phi(h_i^0) + \sum_{m+1}^{N} a_i \Phi(\infty) = 1, \quad \xi \in P_0,
\end{align*}
\]
and
\[
\begin{align*}
  f(h^*, \xi) &\leq \sum_{i=1}^{m} a_i \Phi(h_i^0) - \delta_0 + \sum_{m+1}^{N} a_i \Phi(2L_0) \\
  &\leq \sum_{i=1}^{m} a_i \Phi(h_i^0) - \delta_0 + \sum_{m+1}^{N} a_i \Phi(L_0) + \frac{\delta_0}{2} \\
  &= \sum_{i=1}^{m} a_i \Phi(h_i^0) + \sum_{m+1}^{N} a_i \Phi(L_0) - \frac{\delta_0}{2}, \quad \xi \in \partial P_0.
\end{align*}
\]
Therefore \(\max_{\xi \in P_0} f(h^*, \xi) < 1\) and there is \(\xi_1 \in \text{int}(P_0)\) such that
\[
  f(h^*, \xi_1) = \max_{\xi \in P_0} f(h^*, \xi) < 1 \tag{34}
\]
and \(\xi_1\) satisfies
\[
  \sum_{i=1}^{m} \frac{a_i}{\Phi(h_i^0 - \xi_1 \cdot u_i)} u_i + \sum_{m+1}^{N} \frac{a_i}{\Phi(L_0 - \xi_1 \cdot u_i)} u_i = 0.
\]
Now we have the function \(F(\lambda) : [1, \infty) \to \mathbb{R},\)
\[
  F(\lambda) = \max_{\xi \in P(h^*)} \left[ \sum_{i=1}^{m} a_i \Phi(\lambda h_i^0 - \xi \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(\lambda L_0 - \xi \cdot u_i) \right] \\
= \max_{\xi \in P(h^*)} \left[ \sum_{i=1}^{m} a_i \Phi(\lambda h_i^0 - \xi \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(\lambda (L_0 - \xi \cdot u_i)) \right].
\]
From (34) we see \(F(1) < 1.\)

**Lemma 2.6.** For \(\lambda \geq 1,\) there is a unique \(\xi_1 \in \text{int}(P(h^*)) = \text{int}(P_0)\) starting from \(\xi_1,\) continuous in \(\lambda\) and
\[
  \frac{\max_{\xi \in P(h^*)} \left[ \sum_{i=1}^{m} a_i \Phi(\lambda h_i^0 - \xi \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(\lambda (L_0 - \xi \cdot u_i)) \right]}{\sum_{i=1}^{m} a_i \Phi(\lambda h_i^0 - \xi_1 \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(\lambda (L_0 - \xi_1 \cdot u_i))} \\
= \sum_{i=1}^{m} a_i \Phi(\lambda h_i^0 - \xi_1 \cdot u_i) + \sum_{m+1}^{N} a_i \Phi(\lambda (L_0 - \xi_1 \cdot u_i)). \tag{35}
\]

**Proof.** The existence of \(\xi_1 \in P(h^*) = P_0\) satisfying (35) is clear. We show \(\xi_1 \in \text{int}(P_0)\). Indeed, if \(\xi_1 \in \partial P_0,\) then there are \(m_1\) normals, say \(u_1, \cdots, u_{m_1}\) such that
\[
h_i^0 - \xi_1 \cdot u_i = 0, \quad 1 \leq i \leq m_1, \quad h_i^0 - \xi_1 \cdot u_i > 0, \quad m_1 + 1 \leq i \leq m. \tag{36}
\]
Since $0 \in \text{int}(P_0)$, for $t \in (0, 1)$, $(1 - t)\xi_{\lambda} \in \text{int}(P_0)$ as well. The function

$$G(t) = \sum_{i=1}^{m} a_i \Phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i)) + \sum_{m+1}^{N} a_i \Phi(\lambda(L_0 - (1-t)\xi_{\lambda} \cdot u_i))$$

is continuous in $[0, 1]$, differentiable in $(0, 1)$ and

$$G'(t) = \sum_{i=1}^{m} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i))} + \sum_{m+1}^{N} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(L_0 - (1-t)\xi_{\lambda} \cdot u_i))}$$

$$= \sum_{i=1}^{m} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i))} + \sum_{m+1}^{N} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i))} + \sum_{m+1}^{N} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(L_0 - (1-t)\xi_{\lambda} \cdot u_i))}$$

$$= \sum_{i=1}^{m} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i))} + \sum_{m+1}^{N} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(L_0 - (1-t)\xi_{\lambda} \cdot u_i))}$$

For $t \to 0^+$, using $\phi(0) = 0$ and (36) we have

$$\sum_{i=1}^{m} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(h_i^0 - (1-t)\xi_{\lambda} \cdot u_i))} \to +\infty$$

and

$$\sum_{m+1}^{N} \frac{a_i \xi_{\lambda} \cdot u_i}{\phi(\lambda(L_0 - (1-t)\xi_{\lambda} \cdot u_i))} = O(1)$$

Plug (38) and (39) into (37) we get

$$\lim_{t \to 0^+} G'(t) = +\infty$$

Therefore, for $0 < t << 1$, $G(t) > G(0)$. This contradicts to (35).

$\xi_{\lambda}$ is unique since $P_0$ is convex and the function considered is concave. $\xi_{\lambda}$ is continuous in $\lambda$ follows from the implicit function theorem. □

Proof of Theorem 1.2: For the function $F(\lambda) : [1, \infty) \to \mathbb{R}$

$$F(\lambda) = \max_{\xi \in \mathcal{P}(h^*)} \left( \sum_{i=1}^{m} a_i \Phi(\lambda(h_i^0 - \xi \cdot u_i)) + \sum_{m+1}^{N} a_i \Phi(\lambda(L_0 - \xi \cdot u_i)) \right)$$

we have $F(1) < 1$ by (34) and $\lim_{\lambda \to \infty} F(\lambda) > 1$, so there is $\lambda_0 > 1$ such that $F(\lambda_0) = 1$, that is,

$$\sum_{i=1}^{m} a_i \Phi(\lambda_0(h_i^0 - \xi_{\lambda_0} \cdot u_i)) + \sum_{m+1}^{N} a_i \Phi(\lambda_0(L_0 - \xi_{\lambda_0} \cdot u_i)) = 1.$$

There also holds

$$\sum_{i=1}^{m} \frac{a_i}{\phi(\lambda_0(h_i^0 - \xi_{\lambda_0} \cdot u_i))} u_i + \sum_{m+1}^{N} \frac{a_i}{\phi(\lambda_0(L_0 - \xi_{\lambda_0} \cdot u_i))} u_i = 0.$$

Hence

$$\lambda_0 \tilde{h} = \lambda_0(h_1^0 - \xi_{\lambda_0} \cdot u_1, \ldots, h_m^0 - \xi_{\lambda_0} \cdot u_m, L_0 - \xi_{\lambda_0} \cdot u_{m+1}, L_0 - \xi_{\lambda_0} \cdot u_N) \in \mathcal{M}_1$$

and

$$V(\lambda_0 \tilde{h}) = \lambda_0^n V(\tilde{h}) = \lambda_0^n V(h^0) > V(h^0) = \sup_{h \in \mathcal{M}_1} V(h)$$
since $\lambda_0 > 1$. A contradiction.

This contradiction shows that the maximizing sequence $h^k \in M_1$ is bounded, and the existence of a maximizer of $V(h)$ on $M_1$ follows. \qed

3. **Proof of Theorem 1.3.** In this section we present a proof of the Theorem 1.3.
In view of Proposition 2, we need to find a critical point $h$ of $V(h)|_{M_1}$. It is a consequence of

**Proposition 4.** With the assumptions of Theorem 1.3, there holds

$$V(h) \to 0, \quad h \in M_1, \|h\| \to \infty.$$  (41)

Hence there is an $h_0 \in M_1$ such that $V(h_0) = \max_{h \in M_1} V(h)$, which is a solution of the Orlicz-Minkowski problem.

To prove this result we need the well known John’s Lemma for convex body, see [17].

**Lemma 3.1.** Let $K \subset \mathbb{R}^n$ be a convex and bounded set with $\text{int}(K) \neq \emptyset$. Then there is a cube $E$, centered at the origin 0 and an $\eta \in \text{int}(K)$ such that

$$E \subset K - \eta \subset n^2E.$$  

**Proof of Proposition 4:** Suppose there is a sequence $h^k = (h^k_1, \ldots, h^k_N) \in M_1$ such that $\|h^k\| \to \infty$ and $V(h^k) \geq V_0 > 0$. We omit the upper index $k$ in $h^k$ as before and assume $\sum_1^N a_i = 1$. From the condition ($\phi 1$) we find

$$A \sum_1^N a_i \log h_i + B \leq \sum_1^N a_i \Phi(h_i) \leq 1,$$  (42)

which implies $\min_i h_i \to 0$ and

$$\sum_1^N a_i \log h_i \leq C.$$  (43)

Let $m$ and $m_1$ be integers such that

$$0 < h_1, \ldots, h_m \to 0, \quad 0 < \delta \leq h_{m+1} \cdots, \quad h_{m_1} \leq L, \quad h_{m_1+1} \leq \cdots, h_N, h_{m_1+1} \to \infty.$$  

As in Section 2, multiplying $\sum_1^N \frac{a_i}{\phi(h_i)} u_i = 0$ by $\phi(h_1)$ and taking limit we have $b_1 = a_1, b_s > 0$, and $2 \leq s \leq m$ such that

$$b_1u_1 + b_2u_2 + \cdots + b_su_s = 0.$$  (44)

**Case 1:** If the diameter $d(h)$ of $P(h)$ is bounded, $d(h) \leq C$, then $P(h) \subset \cap_1^m \{ y \in \mathbb{R}^n | y \cdot u_i \leq h_i \}$ and

$$V(h) \leq \text{Vol}(\cap_1^m \{ y \in \mathbb{R}^n | y \cdot u_i \leq h_i \}) \leq CVol_{\xi}(\cap_1^m \{ y \in \xi | y \cdot u_i \leq h_i \}) \to 0,$$

where $\xi = \text{span}\{u_1, \ldots, u_m\}$ and $\text{Vol}_{\xi}$ s the $\text{dim}\xi$ volume.

**Case 2:** $d(h) \to \infty$. By John’s Lemma, there is an $\eta \in \text{int}(P(h))$ such that

$$E \subset P(h) - \eta \subset n^2E,$$

where $E$ is a cube centered at the origin 0. It follows that

$$V(h) \leq CVol(E).$$  (45)

Therefore we get a contradiction if $\text{Vol}(E) \to 0$. 
For $h = (h_1, \cdots, h_N) \in M_1$, the function
\[
G(h, \xi) = \sum_{i=1}^{N} a_i \Phi(h_i - \xi \cdot u_i), \quad \xi \in P(h)
\]
is concave in $\xi$ and $\frac{\partial G(h,0)}{\partial \xi} = 0$, so $\xi = 0$ is a maximizer, that is
\[
G(h, \xi) = G(h,0) = 1, \quad \xi \in P(h).
\] (46)

Since
\[
h_i - \eta \cdot u_i = \max_{x \in P(h)} x \cdot u_i - \eta \cdot u_i = \max_{x \in P(h) - \eta} x \cdot u_i \geq \max_{x \in E} x \cdot u_i = h^E(u_i),
\]
by (46) we have
\[
A \sum_{i=1}^{N} a_i \log h^E(u_i) + B \leq G(h^E,0) \leq G(h,\eta) \leq G(h,0) = 1.
\]

Hence
\[
\sum_{i=1}^{N} a_i \log h^E(u_i) \leq C. \tag{47}
\]

Since volume is rotational invariant we can assume
\[
E = \{y = (y_1, \cdots, y_n) | |y_i| \leq r_i, 1 \leq i \leq n\}
\]
and $r_1 \leq \cdots \leq r_n$. From $d(h) \to \infty$ we obtain $\|r\| \to \infty$, $r = (r_1, \cdots, r_n)$.

**Lemma 3.2.** Let $E$ be the above cube such that $\|r\| \to \infty$ and $\sum_{i=1}^{N} a_i \log h^E(u_i) \leq C$. Then $\text{Vol}(E) \to 0$.

This is a known result, see [3]. For the completeness we repeat the proof.

**Proof.** Let $\delta$ be a positive and small number and
\[
S^{n-1} = \{y \in \mathbb{R}^n | \sum_{i=1}^{n} |y_i|^2 = 1\}, \quad S^j = \{y \in S^{n-1} | y_i = 0, i \geq j + 1\},
\]
\[
A_{j,\delta} = \{y \in S^{n-1} | |y_j| \geq \delta, |y_i| < \delta, \forall i < j\}. \tag{48}
\]
Then $S^{n-1} = \bigcup_{j=1}^{n} A_{j,\delta}$ and we can choose $\delta$ such that
\[
\sum_{u_i \in A_{j,\delta}} a_i = \mu(A_{j,\delta}) = \mu(S^j \setminus S^{j-1}) = \sum_{u_i \in S^j} a_i - \sum_{u_i \in S^{j-1}} a_i.
\]

The strict subspace concentration condition implies $\exists t > 0$ such that
\[
\sum_{i=1}^{n} b_i = \sum_{i=1}^{N} a_i = 1, \quad S_k = \sum_{i=1}^{k} b_j \leq \frac{k}{n} - t, \quad k < n
\]
with $b_j = \mu(A_{j,\delta})$. For $u \in A_{j,\delta}$, there holds
\[
h^E(u) = \max_{x \in E} x \cdot u \geq \delta r_j.
\]
Hence
\[
\sum_{i=1}^{N} a_i \log h^E(u_i) \geq \sum_{j=1}^{n} \sum_{u_i \in A_{j, \delta}} a_i \log r_j
\]
\[
= \sum_{j=1}^{n} b_j \log r_j
\]
\[
= S_n \log r_n + \sum_{j=1}^{n-1} S_j (\log r_j - \log r_{j+1})
\]
\[
\geq \log r_n + \sum_{j=1}^{n-1} \left( \frac{2}{n} - t \right) (\log r_j - \log r_{j+1})
\]
\[
= n (\log r_n - \log r_1) + \frac{1}{n} \sum_{j=1}^{n} \log r_j
\]
\[
= C + n (\log r_n - \log r_1) + \frac{1}{n} \sum_{j=1}^{n} \log \text{Vol}(E).
\]

Hence \( \text{Vol}(E) \to 0 \) because \( r_n \to \infty \) and \( r_1 \) is bounded.

\[ \square \]

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