ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH POISSON DISTRIBUTION SERIES

B.A. FRASIN

Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series $K(m, z) = \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n$ belonging to the subclasses $S(k, \lambda)$ and $C(k, \lambda)$ of analytic functions with negative coefficients. Further, we consider the integral operator $G(m, z) = \int_0^z F(m, z) \varsigma d\zeta$ belonging to the above classes.

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1. Introduction and definitions

Let $A$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $T$ be a subclass of $A$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U.$$

A function $f$ of the form (2) is in $S(k, \lambda)$ if it satisfies the condition

$$\left| \frac{zf'(z)}{1-\lambda f'(z) + \lambda zf(z)} - 1 \right| < k, \quad (0 < k \leq 1, \ 0 \leq \lambda < 1, \ z \in U)$$

and $f \in C(k, \lambda)$ if and only if $zf' \in S(k, \lambda).$ The class $S(k, \lambda)$ was introduced by Frasin et al. [5].

We note that $S(k, 0) = S(k)$ and $C(k, 0) = C(k)$, where the classes $S(k)$ and $C(k)$ were introduced and studied by Padmanabhan [4] (see also, [2], [3]).

A function $f \in A$ is said to be in the class $R^\tau(A, B), \tau \in \mathbb{C} \setminus \{0\}, \ -1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in U.$$

This class was introduced by Dixit and Pal [1].
A variable $x$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, ...$ with probabilities $e^{-m}$, $m e^{-m}$, $m^2 e^{-m}$, $m^3 e^{-m}$, ... respectively, where $m$ is called the parameter. Thus

$$P(x = r) = \frac{m^r e^{-m}}{r!}, \quad r = 0, 1, 2, 3, ...$$

Very recently, Porwal [6] (see also, [13, 14]) introduce a power series whose coefficients are probabilities of Poisson distribution

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U,$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. In [6], Porwal also defined the series

$$F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U.$$

Using the Hadamard product, Porwal and Kumar [8] introduced a new linear operator $I(m, z) : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$I(m, z) f = K(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad z \in U,$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see [9, 10, 11, 12]) and the recent investigations of Porwal ([6, 8, 7]), in the present paper we determine the necessary and sufficient conditions for $F(m, z)$ to be in our new classes $S(k, \lambda)$ and $C(k, \lambda)$ and connections of these subclasses with $R^{\tau}(A, B)$. Finally, we give conditions for the integral operator $G(m, z) = \int_{0}^{z} \frac{F(m, \zeta)}{\zeta} d\zeta$ belonging to the classes $S(k, \lambda)$ and $C(k, \lambda)$.

To establish our main results, we will require the following Lemmas.

**Lemma 1.1.** [5] A function $f$ of the form (3) is in $S(k, \lambda)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k) \right] |a_n| \leq 2k$$

where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.

**Lemma 1.2.** [5] A function $f$ of the form (3) is in $C(k, \lambda)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k) \right] |a_n| \leq 2k$$

where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.
Lemma 1.3. If \( f \in R^\tau(A, B) \) is of the form \( a_n \), then
\[
|a_n| \leq (A - B)\frac{\tau}{n}, \quad n \in \mathbb{N} - \{1\}.
\]

2. The necessary and sufficient conditions

Theorem 2.1. If \( m > 0, 0 < k \leq 1 \) and \( 0 \leq \lambda < 1 \), then \( \mathcal{F}(m, z) \) is in \( S(k, \lambda) \) if and only if
\[
((1 - \lambda) + k(1 + \lambda))me^m \leq 2k.
\]

Proof. Since
\[
\mathcal{F}(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}e^{-m}z^n
\]
according to (3) of Lemma 1.1, we must show that
\[
\sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m.
\]

Writing \( n = (n - 1) + 1 \), we have
\[
\sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!}
= \sum_{n=2}^{\infty} [(n - 1)((1 - \lambda) + k(1 + \lambda)) + 2k] \frac{m^{n-1}}{(n-1)!}
= [(1 - \lambda) + k(1 + \lambda)] \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2k \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}
= ((1 - \lambda) + k(1 + \lambda))me^m + 2k(e^m - 1).
\]

But this last expression is bounded above by \( 2ke^m \) if and only if \( 5 \) holds.

Theorem 2.2. If \( m > 0, 0 < k \leq 1 \) and \( 0 \leq \lambda < 1 \), then \( \mathcal{F}(m, z) \) is in \( C(k, \lambda) \) if and only if
\[
((1 - \lambda) + k(1 + \lambda))m^2e^m + 2(1 + 2k + k\lambda - \lambda)me^m \leq 2k
\]

Proof. In view of Lemma 1.1, it suffices to show that
\[
\sum_{n=2}^{\infty} n[(1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m.
\]

Now
\[
\sum_{n=2}^{\infty} n[(1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!}
= \sum_{n=2}^{\infty} n^2[(1 - \lambda) + k(1 + \lambda)) + n(1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!}.
\]
Writing $n = (n - 1) + 1$ and $n^2 = (n - 1)(n - 2) + 3(n - 1) + 1$, in (10) we see that

$$\sum_{n=2}^{\infty} n^2((1 - \lambda) + k(1 + \lambda)) + n(1 - \lambda)(k - 1) \frac{m^{n-1}}{(n-1)!}$$

$$= \sum_{n=2}^{\infty} (n - 1)(n - 2)((1 - \lambda) + k(1 + \lambda)) \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} \frac{2k m^{n-1}}{(n-1)!}$$

$$= ((1 - \lambda) + k(1 + \lambda)) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} + 2(1 + 2k + k\lambda - \lambda) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!}$$

$$+2k \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}$$

$$= ((1 - \lambda) + k(1 + \lambda)) m^2 e^m + 2(1 + 2k + k\lambda - \lambda) m e^m + 2k (e^m - 1).$$

But this last expression is bounded above by $2ke^m$ if and only if (9) holds. □

By specializing the parameter $\lambda = 0$ in Theorems 2.1 and 2.2, we have the following corollaries.

**Corollary 2.3.** If $m > 0$ and $0 < k \leq 1$, then $F(m, z)$ is in $S(k)$ if and only if

$$(1 + k) m e^m \leq 2k.$$ (11)

**Corollary 2.4.** If $m > 0$ and $0 < k \leq 1$, then $F(m, z)$ is in $C(k)$ if and only if

$$(1 + k) m^2 e^m + 2(1 + 2k + k\lambda - \lambda) me^m \leq 2k.$$ (12)

### 3. Inclusion Properties

**Theorem 3.1.** Let $m > 0$, $0 < k \leq 1$ and $0 \leq \lambda < 1$. If $f \in R^+(A, B)$, then $I(m, z) f$ is in $S(k, \lambda)$ if and only if

$$(A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda))(1 - e^{-m}) + \frac{(1 - \lambda)(k - 1)}{m}(1 - e^{-m}(1 + m)) \right] \leq 2k.$$ (13)

**Proof.** In view of Lemma 1.1, it suffices to show that

$$\sum_{n=2}^{\infty} |n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)| \frac{m^{n-1}}{(n-1)!} |a_n| \leq 2ke^m.$$

Since $f \in R^+(A, B)$, then by Lemma 3.3, we get

$$|a_n| \leq \frac{(A - B) |\tau|}{n}.$$ (14)
Thus, we have
\[
\sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{(n-1)!} |a_n| \leq (A - B) |\tau| \sum_{n=2}^{\infty} \left[ n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1) \right] \frac{m^{n-1}}{n!}.
\]

Then
\[
= (A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda))(e^m - 1) + \frac{1 - \lambda}{m}(e^m - 1 - m) \right].
\]

But this last expression is bounded above by \(2ke^m\) if and only if (13) holds. \(\square\)

**Theorem 3.2.** Let \(m > 0\), \(0 < k \leq 1\) and \(0 \leq \lambda < 1\). If \(f \in \mathcal{R}^r(A, B)\), then \(\mathcal{F}(m, z)f\) is in \(\mathcal{C}(k, \lambda)\) if and only if
\[
(A - B) |\tau| \left[ ((1 - \lambda) + k(1 + \lambda)m + 2k(1 - e^{-m})) \right] \leq 2k.
\]

**Proof.** In view of Lemma 1.2, it suffices to show that
\[
\sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} |a_n| \leq 2ke^m.
\]

Using (13), we have
\[
\sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} |a_n| \leq \sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \frac{(A - B) |\tau|}{n}
\]

Then
\[
= (A - B) |\tau| \sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \leq (A - B) |\tau| \sum_{n=2}^{\infty} ((n - 1)((1 - \lambda) + k(1 + \lambda)) + 2k) \frac{m^{n-1}}{(n-1)!}
\]

But this last expression is bounded above by \(2ke^m\) if and only if (15) holds. \(\square\)

By taking \(\lambda = 0\) in Theorems 3.1 and 3.2, we obtain the following corollaries.

**Corollary 3.3.** Let \(m > 0\) and \(0 < k \leq 1\). If \(f \in \mathcal{R}^r(A, B)\), then \(\mathcal{T}(m, z)f\) is in \(\mathcal{S}(k)\) if and only if
\[
(A - B) |\tau| \left[ (1 + k)(1 - e^{-m}) + \frac{(k - 1)}{m}(1 - e^{-m}(1 + m)) \right] \leq 2k.
\]
Corollary 3.4. Let $m > 0$ and $0 < k \leq 1$. If $f \in \mathcal{R}^\tau(A, B)$, then $\mathcal{I}(m, z)f$ is in $\mathcal{C}(k)$ if and only if
\[
(A - B) |\tau| [(1 + k)m + 2k(1 - e^{-m})] \leq 2k.
\]

4. AN INTEGRAL OPERATOR

Theorem 4.1. If $m > 0$, $0 < k \leq 1$ and $0 \leq \lambda < 1$, then
\[
\mathcal{G}(m, z) = \int_0^z \frac{\mathcal{F}(m, t)}{t} dt
\]
is in $\mathcal{C}(k, \lambda)$ if and only if (6) is satisfied.

Proof. Since
\[
\mathcal{G}(m, z) = z - \sum_{n=0}^{\infty} \frac{e^{-m}m^{n-1}}{n!} z^n
\]
then by Lemma 1.2, we need only to show that
\[
\sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m.
\]
or, equivalently
\[
\sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} \leq 2ke^m.
\]
From (6) it follows that
\[
\sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) + (1 - \lambda)(k - 1)] \frac{m^{n-1}}{(n-1)!} = ((1 - \lambda) + k(1 + \lambda))me^m + 2k(e^m - 1)
\]
and this last expression is bounded above by $2ke^m$ if and only if (6) holds. $\square$

The proof of Theorem 4.2 below is much similar to that of Theorem 4.1 and so the details are omitted.

Theorem 4.2. If $m > 0$, $0 < k \leq 1$ and $0 \leq \lambda < 1$, then the integral operator defined by (18) is in $\mathcal{S}(k, \lambda)$ if and only if
\[
((1 - \lambda) + k(1 + \lambda))(1 - e^{-m}) + \frac{(1 - \lambda)(k - 1)}{m} (1 - e^{-m} - me^{-m}) \leq 2k.
\]

By taking $\lambda = 0$ in Theorems 4.1 and 4.2, we obtain the following corollaries.

Corollary 4.3. If $m > 0$ and $0 < k \leq 1$, then the integral operator defined by (18) is in $\mathcal{C}(k)$ if and only if (17) is satisfied.

Corollary 4.4. If $m > 0$ and $0 < k \leq 1$, then the integral operator defined by (18) is in $\mathcal{S}(k)$ if and only if
\[
(1 + k)(1 - e^{-m}) + \frac{(k - 1)}{m} (1 - e^{-m} - me^{-m}) \leq 2k.
\]
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FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, AL AL-BAYT UNIVERSITY, MAFRAQ, JORDAN

E-mail address: bafrasin@yahoo.com