A Performance Bound for Model Based Online Reinforcement Learning

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Abstract—Model based reinforcement learning (RL) refers to an approximate optimal control design for infinite-horizon (IH) problems that aims at approximating the optimal IH controller and associated cost parametrically. In online RL, the training process of the respective approximators is performed along the de facto system trajectory (potentially in addition to offline data). While there exist stability results for online RL, the IH controller performance has been addressed only fragmentarily, rarely considering the parametric and error-prone nature of the approximation explicitly even in the model based case. To assess the performance for such a case, this work utilizes a model predictive control framework to mimic an online RL controller. More precisely, the optimization based controller is associated with an online adapted approximate cost which serves as a terminal cost function. The results include a stability and performance estimate statement for the control and training scheme and demonstrate the dependence of the controller’s performance bound on the error resulting from parameterized cost approximation.

I. INTRODUCTION

Reinforcement learning (RL) is a particular control strategy based on a reinforcement signal that relates the control signal to a penalty over an infinite-horizon (IH) via various approaches [24]. One specific class of RL methods utilizes the actor-critic structure [4] which can be found, e.g., in approximate dynamic programming (ADP), in which the critic assesses the controller’s IH cost approximately. An iteration scheme then proposes an improved control and cost estimate by updates, usually consecutively (see [24], [30]). Convergence of the approximate IH cost and the corresponding controller to the optimal one is commonly guaranteed for an infinite number of iterations with ideal update (in the sense that the update rule can be solved exactly over the entire state space) [1], [26]. For discounted stage costs, performance statement were obtained as early as, e.g., [34] where the derived bounds however fail to exist whenever the undiscounted case is considered. Lately, some effort has been put into assessing the resulting controller properties for a finite number of iterations [9] as well as establishing boundedness results for parameters when explicit approximation structures are included [27], [35] and providing stability for data-based updates [42].

One of the main remaining challenges in learning-based optimal control, specifically online RL, is therefore the assessment of the controller performance when online data is used and specific structures are utilized for the cost approximation. Online training can be useful whenever the domain of interest is variable and not captured entirely by offline training. Additionally, online adaptation extends the finite offline phase by virtually infinite iterations, making further controller improvement possible.

Recently, learning methods have experienced much attention in the model predictive control (MPC) community [10], [14], [33], [36], [40]. Particularly due to its model-free control characteristics, RL has been used for MPC on uncertain dynamics [5], [28], [40].

Simultaneously, cost shaping has become an attractive subject in MPC with known dynamics and can be found in several economic MPC studies (see, e.g., [3], [29]) as well as stabilizing, so-called set point MPC. Particular focus of the latter on stage cost and terminal cost adjustment, which may be governed by RL, can be found in, e.g., [2], [8] and [22], [33], respectively. Given an exact model, the MPC framework allows to estimate the performance of the implicit controller w.r.t. the optimal IH cost as well as to establish stability when no (terminal) stabilizing ingredients are used (see, e.g., [6], [11], [25], [32], [38], [39]). Analyses of such MPC schemes with terminal cost – which ideally reflect the optimal IH tail or at least an approximation thereof – have been performed in, e.g., [6], [20], [21], [25]. In [20], the authors work with a finite-horizon (FH) terminal cost under a local controller and rigorously provide a stability and performance analysis based on the properties of the control within a sublevel set of the stage cost.

Assuming an exact model, this work tackles the aforementioned challenge and derives a performance estimate for a learning-based optimal controller that is associated with an error-prone approximation of an optimal IH cost. To assess the performance, an MPC framework is utilized to exploit existing analyses while the approximate IH cost is used as terminal cost function, acting as the IH tail. Specifically, the controller minimizes a FH cost with adapting terminal cost based on ADP. To circumvent practically intractable perfect approximators, error-prone approximations through, e.g., parametric structures or neural nets, are considered explicitly which entail errors in the cost update. Beside stability, this work highlights the relation between the performance bound and the approximation update error and demonstrates possible performance improvement via error reduction.

The remainder of this work is structured as follows. In the next Sec. III the MPC setting is introduced and the utilized ADP variant as well as its properties are reviewed. Then, in Sec. IIII stability and a performance mark associated with a prediction horizon are given. A case study demonstrates the results in Sec. IV followed by a conclusion.
Consider the nonlinear discrete-time system
\[ x^+ = f(x, u), \]  
with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \) and dynamic \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), where \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{U} \subset \mathbb{R}^m \) denote compact state and input constraints, respectively. For brevity, \( \mathbb{Z} := \mathbb{R} \times \mathbb{U} \).

Given a state \( x \in \mathcal{X} \) and an input sequence \( u(j, x) \in \mathcal{U} \), \( j = 0, 1, \ldots, x_u(j + 1, x) \) denotes the associated state sequence along \( \mathcal{I} \) with \( x_u(0, x) = x \).

**A. MPC Setting with Adapted Terminal Cost**

The herein time-variant MPC is characterized by solving
\[ V_N(x, n) = \min_{u(x, n) \in \mathcal{U}} \sum_{j=0}^{N-1} l(x_u(j, x), u(j, x)) + \hat{V}_n(x_u(N, x)) \]  
for all \( \mathcal{X} \subset \mathbb{X} \), where \( \mathcal{X} \to \mathbb{R} \) is an initial offset. If \( \varepsilon_1 = 0 \), denote \( V_0(x) := \hat{V}_0(x) - \) the distinction will be useful in the later analysis. The following assumption, which is related to MPC bounds as in, e.g., [13], is important and provides a local bound:

**Assumption 2:** There exists a \( \gamma_0 \in \mathbb{R}_{>0} \) such that \( V_0(x) \leq \gamma_0 l^\ast(x) \), for all \( x \in \mathcal{X} \).

The preceding statement implies that assuming one could obtain \( V_0 \) from \( \mathcal{I} \) without error \( \varepsilon_1 \), e.g., through rich parametrization or in case of LQR problems, a constant \( \gamma_0 \) can be found for \( V_0 \). If \( |\varepsilon_1| > 0 \), \( \gamma_0 \) is increased under Asm. [3] below.

The error-prone value iteration is characterized by iterating through
\[ \mu_i(x) = \arg \min_{u \in \Omega} \left\{ l\left(x, \mu_i(x)\right) + \hat{V}_i\left(f\left(x, \mu_i(x)\right)\right) \right\} \]  
and subsequently the value function update
\[ \hat{V}_{i+1}(x) = l(x, \mu_i(x)) + \hat{V}_i\left(f\left(x, \mu_i(x)\right)\right) + \varepsilon_i(x) \]  
for all \( x \in \hat{\Omega} \) and \( \varepsilon_i : \mathbb{X} \to \mathbb{R} \), \( i \in \mathbb{N}_0 \), starting with \( \mu_\ast(x) \) and \( \hat{V}_0\left(x, \mu_\ast(x)\right) \) according to (5).\n
**Assumption 3:** The error satisfies \( |\varepsilon_i(x)| \leq \delta l^\ast(x) \) for some \( \delta \in [0, 1) \), for all \( x \in \hat{\Omega} \) and \( i \in \mathbb{N}_0 \cup \{-1\} \).

**Remark 1:** Given a data set \( \Omega_p \subset \mathbb{X} \) of \( p \in \mathbb{N} \) state input samples \( (x^s, \mu_\ast(x^s)) \), \( s = 1, \ldots, p \), representing \( \hat{\Omega} \), errors w.r.t. to the nominal VI value function update (refer to [1], [24]) are to be expected. Regarding the error bound, further discussion is done in Rem. [5].

First, several useful properties of the iterated approximate value function are reviewed. In [17, Thm. 3] it is observed that, under Asm. [3]

i) \( \hat{V}_{i+1}(x) - \hat{V}_{i}(x) + \frac{\delta}{1-\delta} V^0(x), \)

ii) \( l^\ast(x) \leq \hat{V}_i(x), \)

as well as
\[ \hat{V}_i\left(f\left(x, \mu_i(x)\right)\right) - \hat{V}_i(x) \leq l\left(x, \mu_i(x)\right) - \varepsilon_i(x) + \frac{4\delta}{1-\delta} V^0(x) \]  
for all \( x \in \hat{\Omega} \) and \( i \in \mathbb{N}_0 \). Additionally
\[ l^\ast(x) \leq \hat{V}_i(x) \leq (1 - \delta)^{-1} \hat{V}_i(x), \]  
where \( \hat{V}_i \) is a well-defined auxiliary function, for all \( x \in \hat{\Omega}, i \in \mathbb{N}_0 \), which can be used to relate stage costs along the iteration to all previous values.

Given that \( \delta > 0 \), in general due to approximations, a critical difference to nominal stabilizing VI (and also PI) is for the approximate value function not to be non-increasing each iteration step, i.e., \( \hat{V}_{i+1}(x) \nless \hat{V}_i(x), x \in \hat{\Omega}, i \in \mathbb{N}_0 \) (refer to, e.g., [16], [26] for such analyzes). With Asm. [3] as well as (3) and (6), by which \( V^0 \) need merely be a relaxed local CLF, it follows, e.g., that \( V_1(x, 0) \nless V_0(x) \). One consequence of this is that the optimal FH cost will in general not satisfy monotonicity, i.e., \( V_{M+1}(x, i) \nless V_M(x, i), M, i \in \mathbb{N}_0 \) and \( x \in \mathcal{X}_M \) (refer to [31, Prop. 2.18]). Therefore a classical reachability analysis in the spirit of,
e.g., [25], exploiting monotonicity, is not applicable without further adjustment.

However, in order to exploit properties of the iterated terminal cost, the terminal state must lie in \( \hat{\Omega} \). It appears that a description as in [20] is useful, based on the following observation:

**Proposition 1:** Assuming Asm. 2 and 3 hold, the set \( \{ x \in \hat{\Omega} : l^*(x) \leq \frac{1}{2\gamma_0} \} \) is a subset of \( B_t(\hat{\Omega}) := \{ x \in \hat{\Omega} : \hat{V}_i(x) \leq \varepsilon^* \} \), \( \varepsilon^* > 0 \).

**Proof:** The statement follows directly from the assumptions and relation ii) above, yielding \( \hat{V}_i(x) \leq 2V^0(x) \leq 2\gamma_0 l^*(x) \), \( x \in \hat{\Omega}, i \in \mathbb{N} \).

Oftentimes the (hypothetical) terminal set for the terminal predicted state, used for stability analyses, is related to a sublevel set of the terminal cost (refer to, e.g., [19], [22], [25]), by which choosing \( B_t \) is advisable. Yet, due to adaptation of \( \hat{V}_i \) (or in the online case \( \hat{V}_n \)), \( B_t(X) \subset \hat{\Omega} \) with fixed \( \varepsilon^* \) is hard to satisfy for all \( n \in \mathbb{N}_0 \), a priori. Therefore, the subsequent analysis is based on utilizing a sublevel of \( l^* \) around the origin.

III. ANALYSIS

This section provides a performance description for learning-based optimal controller with adapted \( IH \) cost. By the nature of the iterative approximation of the infinite tail, this work exploits the analysis of [20] leading to the following relations:

\[
\sum_{n=0}^{\infty} l(x_{u^*(n)}(n), n) u^*_n(x, n)) \leq \frac{1}{\alpha_1(N, \delta)} V_N(x, 0) \quad (8a)
\]

\[
V_N(x, 0) \leq \alpha_2(N) V_{\infty}(x, 0). \quad (8b)
\]

Observe that the performance and stability will be dependent on the horizon length \( N \) as well as the approximation error characteristic \( \delta \) of the terminal cost adaptation. Specifically, the factor \( \alpha_1(N, \delta) \) arises from a relaxed decay property of \( \Delta V_N(x, n) := V_N(x^+, n + 1) - V_N(x, n) \) over \( x \in \hat{X} \) and all times steps \( n \in \mathbb{N}_0 \), where the time-variance is an uncommon element in comparable MPC analyses. Based on the properties depicted in Sec. II-B it is possible however to relate to \( V_N(x, n) \) to \( V_N(x, 0) \) in the following sense:

**Proposition 2:** Let Asm. 2 and 3 hold. Further, let \( \delta \in (0, 1) \) satisfy

\[
0 \leq \delta < 1 + 2\gamma_0 - \sqrt{4\gamma_0^2 + 4\gamma_0} \quad (9)
\]

and be \( N \in \mathbb{N}, \varepsilon^* > 0 \) such that \( x_{u^*}(N, x) \in X_0(\varepsilon^*) := \{ x \in \hat{\Omega} : 2\gamma_0 l^*(x) \leq \varepsilon^* \} \subset \hat{\Omega} \) for all \( (x, n) \in \hat{X} \times \mathbb{N}_0 \).

Then the difference \( \Delta V_N(x, n) \) is bounded by a function of \( V_n(x_{u^*}(N, x)) \) and \( l(x, u^*(0, x)) \) with respective \( u^*(\cdot, x) \) for step \( n \), for all \( (x, n) \in \hat{X} \times \mathbb{N}_0 \).

**Proof:** By optimality, it can be argued that

\[
\Delta V_N(x, n) \leq -l(x, u^*_n(0, x)) + l(x_{u^*}(N, x), \hat{u})
\]

\[
+ \hat{V}_{n+1}(f(x_{u^*}(N, x), \hat{u})) - \hat{V}_n(x_{u^*}(N, x))
\]

for some feasible \( \hat{u} \in \mathcal{U} \). With \( y := x_{u^*}(N, x) \in \hat{\Omega} \), by assumption, let \( \hat{u} = \mu_n(y) \). For brevity, \( f^\dagger = f(y, \mu_n(y)) \).

First \( \hat{V}_{n+1} \) is inspected: Since \( \hat{V}_i(y) \leq 2\gamma_0 l^*(y) \leq \varepsilon^* \), \( f^\dagger \in \hat{\Omega} \) if \( \hat{V}_i(f^\dagger) \leq \hat{V}_i(y) \) which is true under 9 [17]. Then

\[
\hat{V}_{n+1}(f^\dagger) \leq \hat{V}_i(f^\dagger) + \frac{4\delta}{1 - \delta} V^0(f^\dagger)
\]

\[
= \hat{V}_n(f^\dagger) + \frac{4\delta}{1 - \delta} \left( V^0(y) - l(y, \mu_n(y)) \right)
\]

where additionally the definition of \( V^0 \) is used.

With 5, the above inequality becomes

\[
\Delta V_N(x, n) \leq -l(x, u^*_n(0, x)) - \varepsilon_n(y) + \frac{4\delta}{1 - \delta} V^0(y)
\]

\[
+ \frac{4\delta}{1 - \delta} \left( V^0(y) - l(y, \mu_n(y)) \right)
\]

**Asm. 3**

\[
\Delta V_N(x, n) \leq -l(x, u^*_n(0, x)) + \frac{8\delta}{1 - \delta} V^0(y) + \frac{\delta(1 - \delta) - 4\delta}{1 - \delta} l^*(y)
\]

\[
\leq -l(x, u^*_n(0, x)) + \frac{\delta(1 - \delta) - 4\delta + 4\gamma_0 \delta}{1 - \delta} l^*(y)
\]

\[
\leq -l(x, u^*_n(0, x)) + \delta(1 - \delta) - 4 + 4\gamma_0 \frac{\hat{V}_n(y)}{(1 - \delta)^2}
\]

(10)

since \( -l(y, \mu_n(y)) \leq -l^*(y) \) and due to 4.

The respective \( \varepsilon^* \) can be selected according to the local control set \( \hat{\Omega} \) and \( \gamma_n \) whereas a suitable \( N \) will be provided through the subsequent analysis. Note that \( \delta = 0 \) implies that \( \Delta V_N(x, n) \leq -l(x, u^*_n(0, x)) \) as well as \( V_{\infty}(x, 0) \leq V_N(x, n) \).

In the subsequent analysis, an extension to Asm. 1 is required in accordance with [20]:

**Assumption 4:** There exist \( C \geq 1, \sigma \in (0, 1), \varepsilon^* > 0 \) such that \( X_0(\varepsilon^*) \subset \hat{\Omega} \) and for all \( x \in X_0(\varepsilon^*) \),

\[
l(x_{\mu_{-1}(j, x), \mu_{-1}(x_{\mu_{-1}(j, x)})}) \leq C\sigma l^*(x), \quad j \in \mathbb{N}_0,
\]

and \( (x_{\mu_{-1}(j, x), \mu_{-1}(x_{\mu_{-1}(j, x)})}) \in \mathbb{Z} \).

**Proposition 3:** Let Asm. 1, 2, 3 hold. Then \( V_N \) is bounded for all \( (x, n) \in \mathcal{X}_N(\varepsilon^*) \times \mathbb{N}_0 \) by

\[
V_N(x, n) \leq B(l^*(x)) \quad (12)
\]

with \( B \in K, N \in \mathbb{N} \).

**Proof:** For all \( x_n \in \mathcal{X}_N(\varepsilon^*) \times \mathbb{N}_0 \),

\[
V_N(x, n) \leq \sum_{j=0}^{N-1} l(x_{\mu_{-1}(j, x), \mu_{-1}(x_{\mu_{-1}(j, x)})}) + \hat{V}_n(x_{\mu_{-1}(N, x)})
\]

\[
\leq 2\gamma_0 l^*(x_{\mu_{-1}(N, x)}) \leq 2\gamma_0 l^*(x_{\mu_{-1}(N, x)})
\]

\[
\leq C \frac{1 - \sigma^N}{1 - \sigma} l^*(x) + 2\gamma_0 l(x_{\mu_{-1}(N, x)}) \mu_{-1}(x_{\mu_{-1}(N, x)})
\]

\[
\leq \left( C \frac{1 - \sigma^N}{1 - \sigma} + 2\gamma_0 C \sigma^N \right) l^*(x)
\]

With \( \lambda := (2\gamma_0 C) / \gamma_\infty \in \mathbb{R}_0, \gamma_N \leq \gamma_\infty \) and \( \sigma < 1 \),

\[
V_N(x, n) \leq (1 + \lambda) \gamma_\infty l^*(x). \quad (13)
\]
Remark 2: A similar bound for time-variant MPC was suggested in, e.g., [12, Asm. 6.29], where the stage cost is a time-variant entity, as well as [12, Chap. 10.2]. Computing \( \alpha_1(N, \delta) = \alpha_1(N) \) for \( x \in X_t(\varepsilon^*) \) is thus straightforward using existing results.

The above bound (13) does not relate \( \gamma_0 \) to properties under Asm. 3 directly, but only assess its relation to \( \gamma_\infty \). Assuming that for any \( N \in \mathbb{N} \) there exists a \( M_N \in \mathbb{N} \) such that with \((C, \sigma)\) from Asm. 4

\[
2\gamma_0 \leq 1 + \frac{1}{C\sigma N} \sum_{j=N+1}^{M_N} C\sigma^j,
\]

another (potentially tighter) bound is detectable as

\[
V_N(x, n) \leq C_0 \frac{1 - \sigma^{M_N}}{1 - \sigma} l^*(x) \leq \gamma_\infty l^*(x).
\]

It is worthwhile to note that further studies may incorporate adaptive horizons by specifically considering \( \delta = \delta(n) \) and adapting the prediction horizon adequately (see, e.g., [21] for adaptive horizon MPC); a particular RL based horizon adaptation has been considered in, e.g., [7].

IV. Case Study

This section demonstrates the result on an orbital rendezvous maneuver which has been previously addressed in the context of approximate dynamic programming [18]. The considered system, with state \( x = [X, Y, X_\perp, Y_\perp]^T \), is given by

\[
x^+ = x + \Delta t \left( \begin{bmatrix} X \\ Y \\ 2Y_t - (1 + X) \left( \frac{\delta}{\alpha} - 1 \right) \\ -2X_t - Y \left( \frac{\delta}{\alpha} - 1 \right) \end{bmatrix} \right)
\]

where \( \Delta t = \sqrt{(1 + X)^2 + Y^2} \) and \( \Delta t = 0.05 \). The stage cost is given by \( l(x, u) = x^T Q x + u^T R u \), where \( l^*(x) = l(x, 0) = x^T Q x \) and \( R = 1 \) and \( Q = 5I_2 \). For the system, Asm. 4 can be verified with \( C = 10.7068 \) and \( \sigma = 0.9201 \) under an LQ controller \( \mu_\perp(x) = K x \) in the domain \( x \in \mathbb{X} \) satisfying \( l^*(x) \leq 1.5185 \). As for the terminal cost in 3, a linear combination of basis functions

\[
\hat{V}_i(x) = w_i^T \phi_i(x), \quad \phi_i(x) = (x \otimes x)
\]

is used, for which \( w_0 \) corresponding to 3 is found by least-squares on a random box grid \( \Omega = [-0.3, 0.3]^2 \) with 200 samples. For \( x \in \mathbb{X} \) satisfying \( l^*(x) \leq 0.1485 = \bar{\varepsilon} \), the terminal cost under the trained \( w_0 \) satisfies \( |\varepsilon_{-1}(x)| \leq 0.3058 l^*(x) = \delta l^*(x) \) in 3 while, for comparison, the LQR cost \( V_0(x) = x^T P x \) gives \( |\varepsilon_{-1}(x)| \leq 0.7288 l^*(x) \). The adaptive controller associated to \( \hat{V}_i \) is designed as \( \mu_i(x) = v_i^T \phi_i(x) \), where \( \phi_i(x) = [x^T \otimes x] \). Recall that the adaptive controller is never applied to the system but only used in the update of the terminal cost.

The terminal cost update 5 is performed via gradient descent on the squared temporal difference (TD) \( e_{c,n}(x, w) \), in which \( e_{c,n}(x, w) := w^T \phi_c(x) - (l(x, \hat{u}) - w_0^T \phi_0(f(x, \hat{u}))) \) at \( x = x_{\text{i}n}(N, x(n)) \) and with \( \hat{u} \) a feedback resulting from a so-called inner loop. Specifically,

\[
w_{n+1}^j = w_n^j - \eta e_{c,n}(x, w_n^j) \eta \phi_j(x), \quad \eta = 0.1 > 0
\]

is iterated over \( j = 0, 1, \ldots \), with \( w_0^j = w_n \) until \( w_{n+1}^j = w_n^j \) satisfies \( V_{n+1} - \hat{V}_n \leq \bar{m} \) with precision margin \( \bar{m} > 0 \) after some \( j \). As mentioned in Rem. 5, the update should be such that \( \delta = \delta(0, 1) \) is a valid bound for all times (and states). This could be achieved by optimization with suitable constraints over \( \Omega \) where in practice a gradient based update is commonly employed [37], [41] – Fig. 10 confirms local validity of \( \delta \) for all updates along \( x_{\text{i}n}(N, x(n)) \). The update of the controller is done with inner loops in each iteration \( j \) of \( w_n^j \), as described in, e.g., [15]. In this study, the inner loop seeks to iteratively adapt \( \hat{u} \) to the closed-form feedback \( k(x, w_n^j) = -0.5 B^T \nabla \hat{V}_j(x^+) \) under \( w_n^j \) at each sample \( x = x_{\text{i}n}(N, x(n)) \).
The applied algorithm is sketched in Alg. 1.

**Setup** select basis functions $\phi_c$, sample grid on $\hat{\Omega}$, controller $\mu$, setup feedback $k(x,w)$; compute $(C,\sigma)$ and associated $\tilde{\varepsilon}$ s. t. Asm. 4 holds; compute $\hat{V}_0$.

**while** $\delta \notin [0,1)$ of $\varepsilon$ for all samples **do**

| reduce $\varepsilon$; |
|-----------------|

**Init** state $x^0$, weight $w_0$ from offline, horizon $N$;

**for** $n \in \mathbb{N}_0$ **do**

| get $x(n)$; |
|-----------------|

**while** $|V^{n+1} - \hat{V}^n| > \bar{m}$ **do**

| perform inner loop until $\bar{u}$ is found; |
|-----------------|

| update $w_{n}^j \rightarrow w_{n+1}^j$; |
|-----------------|

**end**

**end**

**Result:** $(u, w)$ trajectories

**Algorithm 1:** Sketch of a practical algorithm.

Fig. 1 depicts the state evolution (Fig. 1a, 1b) and associated input (Fig. 1c) generated by the MPC with adaptive terminal cost for the initial state $x^0 = \begin{bmatrix} -0.5 & 0.7 & 0.1 & 0.1 \end{bmatrix}^T$ and horizon $N = 10$, whose weights $w_n$ are shown in Fig. 1c. With $N$, a suitable value $c$ for $\mathcal{X}_l$ can be determined. It can be seen that the weights are adapted primarily during the first 25 time steps as then the TD becomes very small (please refer also to Fig. 1f for $a$).

It turns out that $\lambda$ has a significant impact on the minimal horizon length which can be reduced by finding a tighter bound in Prop. 2. Lastly, Fig. 2 shows the significance of the error margin $\delta$ in $\alpha_1 = \alpha_1(N, \delta)$. It can be observed that the error margin affects the relaxed decay of $V_N$ as in (8a) (for a fixed $N = N_0$) and a decrease of error results in a decrease of suboptimality.

**V. CONCLUSION**

This work provides an approach to assess the performance of online RL by relating the control scheme to an MPC framework and exploiting performance analyses. The derived bounds show the dependence of the performance on the quality of the approximation, i.e., margin of the approximation (update) error, and provide a prediction horizon dependent performance and stability statement. Though the analysis concludes a potentially conservative bound, it aims to lay the foundation for further investigations in online RL. As was addressed in the work, extensions may include tighter bounds and less conservative estimates. Regarding the computational complexity, a parametric feedback mimicking the FH cost minimizing open-loop controls could be introduced to obtain a fully parametric setup.

**APPENDIX**

**Proof:** (of Prop. 3) In its core, the proof is resembles that of [20, Thm.5, Part II], however with $\varepsilon = \varepsilon$ and $(1 + \lambda)\gamma_\infty$ substituted for $\gamma$. Yet, dependence of the decay condition and required horizon length on the update error $\delta$ differ from the existing result and need certain adjustment.

First observe the lower boundedness as in $l^\ast(x) \leq V_N(x,n), \text{ for all } (x,n)$. From dynamic programming,

$$ V_N(x, n) = V_N(x, n) - \sum_{j=0}^{i-1} l(x_u^\ast(j,x), u^\ast(j,x)) $$

(16)

for any $i \in \{0, \ldots, N\}$. Starting with $V_N(x(0),0) \leq c$, assume for now that $V_N(x(n),n) \leq c$ for all $n \in \mathbb{N}_0$, independent from $N \in \mathbb{N}$, which is then to be shown to hold using a minimal horizon length $\bar{N}$.

Denote $\gamma_c = \max\{1 + \lambda\gamma_\infty, \bar{\gamma}\}$ as well as $N' = \max\{0, \frac{\varepsilon}{(1 + \lambda)\gamma_\infty - \bar{\gamma}}\}$ for $\varepsilon \leq c$. Assuming $V_N(x_u^\ast(i,x), n) > (1 + \lambda)\gamma_\infty \bar{\varepsilon}$ for some $i \in \{0, \ldots, N'\}$, which implies $x_u^\ast(i, x) \notin \mathcal{X}_l(\varepsilon)$ and thus $\bar{\varepsilon} < l^\ast(x_u^\ast(i,x)) \leq l(x_u^\ast(i,x), u^\ast(i,x))$ by Prop. 3 the same.
reasoning as in [20] can be used to argue that there exists a $j^* \in \{0, \ldots, N\}$ such that $V_{N-i}(x_u^*, (i, x), n) \leq V_{N-j^*}(x_u^*(j^*, x), n) \leq (1 + \lambda)\gamma_{\infty}l$ for all $i \in \{j^*, \ldots, N\}$, with the first inequality following from DP. Hence also $l^*(x_u^*, (i, x)) \leq \bar{\varepsilon}$ for $i \in \{j^*, \ldots, N\}$. This further implies that if $l(x_u^*(0, x), u^*(0, x)) \leq \bar{\varepsilon}$ then $j^* = 0$.

Next, note that with $V_{N-i}(x_u^*, (i, x), n) \leq (1 + \lambda)\gamma_{\infty}l$ from before, $l(x_u^*(i, x), u^*(i, x)) > \bar{\varepsilon}$ for $i \in \{j^*, \ldots, N\}$ (which may occur even though $l^*(x_u^*, (i, x)) \leq \bar{\varepsilon}$) directly implies $V_{N-i}(x_u^*, (i, x), n) \leq (1 + \lambda)\gamma_{\infty}l(x_u^*(i, x), u^*(i, x))$. Contrary to if $l(x_u^*(i, x), u^*(i, x)) \leq \bar{\varepsilon}$ then also $l^*(x_u^*, (i, x)) \leq \bar{\varepsilon}$ and thus $x_u^*, (i, x) \in \mathcal{X}_i(\varepsilon^*)$, where from Prop. 3 $V_{N-i}(x_u^*, (i, x), n) \leq (1 + \lambda)\gamma_{\infty}l^*(x_u^*(i, x)) \leq (1 + \lambda)\gamma_{\infty}l(x_u^*(i, x), u^*(i, x))$. Consequently, $V_{N-j^*}(x_u^*, (j^*, x), n) \leq (1 + \lambda)\gamma_{\infty}l\min\{l(x_u^*(0, x), u^*(0, x)), \bar{\varepsilon}\}$, knowing that the case $l(x_u^*(i, x), u^*(i, x)) \leq \bar{\varepsilon}$, $i \geq j^*$, implies the case $j^* = 0$, which is manifested the first term in the bracket on the right-hand side. Denoting $\gamma = \min\{1 + \lambda\gamma_{\infty} - 1\}$, $V_{N-j^*}(x_u^*, (j^*, x), n) \leq c$ also gives rise to $V_{N-j^*}(x_u^*, (j^*, x), n) \leq \gamma \bar{\varepsilon}$. Utilizing the lower bound on $(1 + \lambda)\gamma_{\infty}l(x_u^*, (i, x), u^*(i, x))$ in (10) for all $i \in \{j^*, \ldots, N\}$, it follows that $V_{N-i}(x_u^*, (i, x), n) \leq \rho_i^{j-i} V_{N-j^*}(x_u^*, (j^*, x), n)$ with $\rho_i = ((1 + \lambda)\gamma_{\infty} - 1)/(1 + \lambda)\gamma_{\infty}$. Together, with the previous bound and the fact that $N - i \geq N - N'$, $i \in \{j^*, \ldots, N\}$, evaluating the inequality for $i = N$ yields $V_{N-N}(x_u^*, (N, x), n) \leq \rho_{N-N'}^{N-N'} \min\{1 + \lambda\gamma_{\infty}l(x, u^*(0, x)), \gamma \bar{\varepsilon}\}$.

Observe that for at least $x_u^*(N, x) \in \mathcal{X}_i(\varepsilon^*)$ and the preceding bounds to hold, $\frac{1}{1 - \rho_i} V_{N-N}(x_u^*(N, x), n) \leq \bar{\varepsilon}$ is required due to (7) and property of $V$ which gives $(1 - \delta)l^*(x_u^*(N, x)) \leq V_{N-N}(x_u^*(N, x), n) \leq \bar{\varepsilon}$ with $V_{N-N}(x_u^*(N, x), n) = \hat{V}_n(x_u^*(N, x))$. Thus $x_u^*(N, x) \in \mathcal{X}_i(\varepsilon^*)$ if

$$\frac{1}{1 - \rho_i} \min\{(1 + \lambda)\gamma_{\infty}l(x, u^*(0, x)), \gamma \bar{\varepsilon}\} \leq \bar{\varepsilon},$$

yielding

$$N \geq N' + \frac{\max\{0, \ln\gamma - \ln(1 - \delta)\}}{\ln((1 + \lambda)\gamma_{\infty} - \ln((1 + \lambda)\gamma_{\infty} - 1))} =: N''.$$

Here, the first entry of the bracket covers the case where $N = N'$ is required to reach the set $\mathcal{X}_i(\varepsilon^*)$, observing that $\ln\gamma \leq 0$. Finally, with $\hat{V}_n(x_u^*(N, x), n) \leq \rho_i^{N-N'}(1 + \lambda)\gamma_{\infty}l(x, u^*(0, x))$ for all $n \in \mathbb{N}_{N'}$, (10) yields

$$\Delta V_N(x, n) \leq -\left(1 - \frac{\delta(1 - \delta) - 4 + 4\rho_i}{(1 - \delta)^2}\rho_i^{N-N'}(1 + \lambda)\gamma_{\infty}l(x, u^*(0, x))\right) = \alpha_1(x, n).$$

Observe that for $\delta = 0$, $\alpha_1 = 1$. A horizon satisfying

$$N \geq N' + \frac{\ln\left(\frac{\delta(1 - \delta) + 4 + 4\rho_i}{(1 - \delta)^2}(1 + \lambda)\gamma_{\infty}\right)}{\ln((1 + \lambda)\gamma_{\infty}) - \ln((1 + \lambda)\gamma_{\infty} - 1)}$$

gives $\alpha_1 > 0$. To ensure this, from which $V_N(x(n), n) \leq C$ due to decay, as well as $V_{N-N}(x_u^*(N, x), n) \leq \bar{\varepsilon}(1 - \delta)$ for all $(x, n)$, the minimal horizon is given by

$$N \geq N'' := N' + \max\left\{\ln\gamma - \ln(1 - \delta), \ln\left(\frac{\delta(1 - \delta) + 4 + 4\rho_i}{(1 - \delta)^2}(1 + \lambda)\gamma_{\infty}\right)\right\}.$$

Lastly, note that $V_N$ is upper bounded by $V_N(x, n) \leq \gamma \bar{\varepsilon}(x)$. Asymptotic stability follows.

**Proof:** (of Prop. 5) Let $\kappa_{\infty}(\cdot)$ and $x^*(\cdot, x)$ denote the IH optimal controller and the associated state trajectory, respectively. Since $V_{\infty}(x) := V\infty(0, 0) \leq \gamma_{\infty}$ for all $x \in \mathcal{X}_i(\varepsilon^*)$, which is less conservative than $(1 + \lambda)\gamma_{\infty}$, some minor extension to the proof of Prop. 4 is required. By analogy, observe that there exists $j^* \in \{0, \ldots, N''\}$, where $N'' = \max\{0, \frac{\ln\gamma}{\gamma_{\infty}}\}$, such that $V_{\infty}(x^*(i, x), 0) \leq V_{\infty}(x^*(j^*, x), 0) \leq \gamma \bar{\varepsilon}$ for $i \in \{j^*, \ldots, N''\}$. Hence with

$$V_{\infty}(x^*(N, x), n) \leq \frac{\gamma_{\infty} - 1}{\gamma_{\infty}} V_{\infty}(x^*(i, x), n) \leq V_{\infty}(x^*(N, x)), n \leq \bar{\varepsilon},$$

for $i \in \{j^*, \ldots, N''\}$, which again comes from (10), and $N - i \geq N - N''$.

$$V_{\infty}(x^*(N, x), n) \leq \frac{\gamma_{\infty} - 1}{\gamma_{\infty}} V_{\infty}(x^*(N, x), n).$$

Repeating the steps from Prop. 4 it also holds that

$$V_{\infty}(x^*(N, x)) \leq \left(\frac{\gamma_{\infty} - 1}{\gamma_{\infty}}\right)^{N''} \min\{\gamma_{\infty}l(x, \kappa_{\infty}(x)), \gamma \bar{\varepsilon}\}$$

with $\gamma = \min\{\gamma_{\infty}, \frac{\gamma_{\infty} - 1}{\gamma_{\infty}}\}$, from which $l^*(x^*(N, x)) \leq V_{\infty}(x^*(N, x), n) \leq \bar{\varepsilon}$ for

$$N \geq N'' + \frac{\max\{0, \ln\gamma - \ln(1 - \delta)\}}{\ln((1 + \lambda)\gamma_{\infty}) - \ln((1 + \lambda)\gamma_{\infty} - 1)} =: N''.$$

This is equivalent to [20]. Now, for any horizon $N \geq N' = \max\{N', N''\}$, it additionally holds that $\frac{1}{1 - \rho_i} V_{N-N}(x_u^*(N, x), n) \leq \bar{\varepsilon}$ and thus $x^*(N, x) \in \mathcal{X}_i(\varepsilon^*)$.
$\mathcal{V}_l(x)$. Since the latter implies that $\hat{V}_0(x^*(N,x)) \leq 2\gamma_0 l^N x^*(N,x)$ by Asm. 2 and the properties detailed in Sec. II-B, it follows that

$$\hat{V}_0(x^*(N,x)) \leq 2\gamma_0 l^N x^*(N,x) \leq 2\gamma_0 V_{\infty}(x^*(N,x)) \leq 2\gamma_0 \left( \frac{\gamma_0 - 1}{\gamma_0} \right) N^{-N''} V_{\infty}(x).$$

Because $x^*(i,x), i \in \{0, \ldots, N\}$, is a feasible candidate,

$$V_N(x,0) = \sum_{j=0}^{N-1} l(x^*(j,x), \kappa_0(x^*(j,x))) + \hat{V}_0(x^*(N,x)) \leq V_{\infty}(x) + \hat{V}_0(x^*(N,x)) \leq \left( 1 + 2\gamma_0 \left( \frac{\gamma_0 - 1}{\gamma_0} \right) N^{-N''} \right) V_{\infty}(x,0).$$

### REFERENCES

[1] A. Al Tamimi, F. L. Lewis, and M. Abu Khalaf. Discrete-time nonlinear HJB solution using approximate dynamic programming: Convergence proof. *IEEE Trans. Syst., Man, and Cyb., Part B (Cybernetics)*, 38(4):943–949, 2008.

[2] M. Alamir. Stability proof for nonlinear MPC design using monotonically increasing weighting profiles without terminal constraints. *Automatica*, 87:455–459, 2018.

[3] D. Angeli, A. Casavola, and F. Tedesco. Theoretical advances on nonlinear model predictive control. *IFAC-PapersOnLine*, 51(5):832–836, 2018.

[4] D. Liu and Q. Wei. Policy iteration adaptive dynamic programming algorithm for discrete-time nonlinear systems. *IEEE Trans. Neural Netw. Learn. Syst.*, 25(6):621–634, 2014.

[5] F. Liu, J. Sun, I. Si, W. Guo, and S. Mei. A boundedness result for the direct heuristic dynamic programming. *Neural Networks*, 32:229–235, 2012.

[6] J. E. Morinelli and B. E. Ydstie. Dual MPC with reinforcement learning. *IFAC-PapersOnLine*, 49(7):226–271, 2016.

[7] M. A. Müller, D. Angeli, and F. Allgöwer. On the performance of economic model predictive control with self-tuning terminal cost. *J. Proc. Control*, 24(8):1179–1186, 2014.

[8] W. B. Powell. What you should know about approximate dynamic programming. *NRL*, 56(3):239–249, 2009.

[9] J. B. Rawlings, D. Q. Mayne, and M. Diehl. *Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, 2nd edition, 2017.

[10] M. Reble and F. Allgöwer. Unconstrained nonlinear model predictive control and suboptimality estimates for continuous-time systems. In *Proc. of the 18th IFAC World Congress*, pages 5733–6738, 2011.

[11] U. Rosolia and F. Borrelli. Learning model predictive control for iterative tasks. A data-driven control framework. *IEEE Trans. Automat. Control*, 63(7):1883–1896, 2018.

[12] S. Singh and R. C. Yee. An upper bound on the loss from approximate value functions. *Machine Learning*, 16(3):227–233, 1994.

[13] Y. Sokolov, R. Kozma, L. D. Werbos, and P. J. Werbos. Complete stability analysis of a heuristic approximate dynamic programming control design. *Automatica*, 59:9–18, 2015.

[14] R. Soloperto, M. A. Müller, S. Tinne, and F. Allgöwer. Learning-based robust model predictive control with state-dependent uncertainty. *IFAC-PapersOnLine*, 51(20):442–447, 2018.

[15] Q. Wei, R. Song, B. Li, and X. Lin. *Self-Learning Optimal Control of Nonlinear Systems Adaptive Dynamic Programming Approach*, volume 103 of *Studies in Systems, Decision and Control*. Science Press, Beijing and Springer, 2018.

[16] K. Worthmann. Estimates on the prediction horizon length in MPC. In *Proc. of the 20th Int. Symp. Math. Theory Netw. Syst.*, 2012.

[17] K. Worthmann, M. W. Mehrez, G. K. I. Mann, R. G. Gosine, and J. Pannek. Interaction of open and closed loop control in MPC. *Automatica*, 82:243–250, 2017.

[18] M. Zanon and S. Gros. Safe reinforcement learning using robust MPC. *IEEE Trans. Automat. Control*, 2020. Early Access.

[19] Y. Zhu, D. Zhao, and H. He. Invariant adaptive dynamic programming for discrete-time optimal control. *IEEE Trans. Syst. Man Cyb. Systems*, 50(11):3959–3971, 2020.