Master action and helicity decomposition for spin-4 models in $D = 2 + 1$

R. Schimidt Bittencourt*, Elias L. Mendonça †

UNESP - Campus de Guaratinguetá - DFI
Av. Dr. Ariberto Pereira da Cunha, 333
CEP 12516-410 - Guaratinguetá - SP - Brazil.

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Abstract

A master action interpolating four self-dual models describing massive spin-4 particles in $D = 2 + 1$ dimensions is suggested. With such action we have observed that the four descriptions are indeed quantum equivalents up to contact terms. We have demonstrated that a geometrical approach can be used to elegantly describe, the third and fourth order in derivatives models. We use the helicity decomposition technique to demonstrate the triviality of the so called mixing terms, which are essentials to construct the master action.
1 Introduction

A classical description of massive higher spin particles was introduced by Singh-Hagen \[1, 2\] in terms of totally symmetric double traceless rank-s tensors. Such description requires a considerable number of auxiliary lower rank fields in order to eliminate spurious degrees of freedom on-shell. After all the constraints to be used in the equations of motion one can demonstrate that it reduces to the Klein-Gordon equation describing two massive modes with the desired spin. Gauge invariant descriptions has been also introduced in the literature, see for example \[3, 4\] and references therein. When working with the gauge invariant descriptions the gauge parameter must be totally symmetric and traceless which by its turn can be used as a guide to the construction of “geometrical” objects like the Riemann, Ricci and Einstein tensor. As an example, see the spin-3 case in \[D = 2 + 1\] \[5\], see also a simple generalization for the spin-4 case \[6\] and interesting geometrical approaches by \[7, 8, 9\]. Some of the massive models for higher-spin particles can be elegantly written in terms of such objects.

A particular feature of \[D = 2 + 1\] space-times is the existence of the so called topologically massive gauge theories reached even for Yang-Mills as well as for gravitational descriptions by augmenting the lagrangians by gauge invariant topological mass terms. Such procedure was introduced in the literature by the seminal works \[10, 11\]. In the same line, a series of papers has been suggesting generalizations of such models for different spins, bosonic and fermionic versions, such models are well known as self-dual descriptions, see for example the bosonic examples in \[12, 13, 14\]. Despite the particle content of such models, in the recent work \[15\], the authors have demonstrated that the self-dual descriptions suggested by Aragone and Khoudeir are in fact propagating just one massive particle with the desired spin. Here, starting from such model we will obtain higher derivative, massive, gauge invariant descriptions which are quantum equivalent to the original theory. Besides, we demonstrate that a geometric description can be elegantly used to describe the third and fourth order in derivative models.

In this work we have used the first order self-dual model, suggested in \[14\], as the starting point for constructing a unique master action interpolating such model with other three self-dual descriptions of second, third and fourth orders in derivatives. Once the first-order self-dual model is free of the propagation of spurious degrees of freedom, thanks to the addition of proper auxiliary fields, one can demonstrate that the same particle content can be recovered by the higher derivative descriptions, i.e: they are quantum equivalent models by comparing their correlation functions. It turns out that, during the construction of the so called master action one needs a key ingredient, the mixing term. Such term, must be free of particle content. In order to demonstrate the triviality of these terms we use the decomposition in helicities technique \[16, 17, 18\].

2 Notation

Throughout this work we use the mostly plus flat metric \((- , + , +\)). The spin-4 self-dual descriptions in \[D = 2 + 1\] can be written in the frame-like or in the totally symmetric approach. In the so called frame-like approach \[19\], we use the spin-4 field \(\omega_{\mu(\alpha\beta\gamma)}\), where the indices
between parentheses are symmetric and traceless. However, one can define the trace, by
\[ \eta^{\mu\alpha} \omega_{\mu(\alpha\beta\gamma)} = \omega_{\beta\gamma}. \]
The 21 independent components of \( \omega_{\mu(\alpha\beta\gamma)} \) can be algebraically irreducible
decomposed as:
\[ \omega_{\mu(\beta\gamma\lambda)} = \phi_{\mu\beta\gamma\lambda} + \frac{1}{3} \eta_{\mu(\beta\phi\gamma\lambda)} - \frac{1}{3} \eta_{(\beta\gamma\phi\lambda)\mu} + c \epsilon_{\mu\rho(\beta\chi^{\rho\gamma\lambda})}. \]  

(1)

Such 21 components on the left hand side are then represented by the 14 components of the
totally symmetric field \( \phi_{\mu\beta\gamma\lambda} \) plus the 7 components of the totally symmetric traceless field
\( \tilde{\chi}_{\mu\beta\gamma} \). The numerical coefficients used in the decomposition are chosen in such a way that
the terms containing derivatives in the frame like approach coincides exactly with the ones
of the totally symmetric approach suggested by [5]. Finally, the last term in (1) is indeed a
symmetry of such terms and then we may kept \( c \) arbitrary \textit{a priori}.

The master action notation we use through this work is exactly based on the same idea we
have used in [20], then we have:
\[ \int (\omega^2) \equiv \int d^3 x \left( \omega_{\mu\beta}\omega^{\mu\beta} - \omega_{\mu(\beta\gamma\lambda)}\omega^{(\beta\mu\gamma\lambda)} \right), \]  

(2)

\[ \int \omega \cdot d\omega \equiv \int d^3 x \; \epsilon^{\mu\nu\alpha} \omega_{\mu(\beta\gamma\lambda)} \partial_\nu \omega^{(\beta\gamma\lambda)}, \]  

(3)

\[ \int \omega \cdot d\Omega(\omega) \equiv \int d^3 x \; \epsilon^{\mu\nu\alpha} \omega_{\mu(\beta\gamma\lambda)} \partial_\nu \Omega^{(\beta\gamma\lambda)} [\xi(\omega)], \]  

(4)

\[ \int \Omega(w) \cdot d\Omega(w) \equiv \int d^3 x \; \epsilon^{\mu\nu\alpha} \Omega_{\mu(\beta\gamma\lambda)} [\xi(\omega)] \partial_\nu \Omega^{(\beta\gamma\lambda)} [\xi(\omega)], \]  

(5)

where the symbol \( .d \equiv \epsilon^{\mu\nu\alpha} \partial_\nu \), while \( \Omega_{\rho(\alpha\beta\gamma)} \) has the same symmetry properties of the field
\( \omega_{\rho(\alpha\beta\gamma)} \) and is given by:
\[ \Omega_{\rho(\alpha\beta\gamma)} [\xi(\omega)] \equiv \xi_{\rho(\alpha\beta\gamma)} - \frac{1}{2} \left( \xi_{\alpha(\rho\beta\gamma)} + \xi_{\beta(\rho\alpha\gamma)} + \xi_{(\rho\beta\alpha\gamma)} \right) \]
\[ - \frac{1}{8} \left( \eta_{\rho\alpha} \xi_{\beta\gamma} + \eta_{\rho\beta} \xi_{\alpha\gamma} + \eta_{\rho\gamma} \xi_{\beta\alpha} \right) \]
\[ + \frac{1}{4} \left( \eta_{\beta\alpha} \xi_{\rho\gamma} + \eta_{\gamma\beta} \xi_{\alpha\rho} + \eta_{\alpha\gamma} \xi_{\beta\rho} \right). \]  

(6)

Here, \( \xi_{\mu(\beta\gamma\lambda)} \equiv E_{\mu\nu} \omega^{\nu(\beta\gamma\lambda)} \) and the operator \( E_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha} \partial^\alpha \). Notice that, in the master action
notation where the indices are omitted, we will use simply \( \Omega(\omega) \) instead of \( \Omega[\xi(\omega)] \), this will
turn the calculations more clean. Some times however, a field or a combination of them will
be present in the same tensorial structure offered by the symbol \( \Omega \) but without the presence of
the derivatives contained in \( \xi \). In order to avoid any kind of confusion, when this is the case,
we denote the \( \Omega \)-symbol by a tilde, i.e.: \( \tilde{\Omega}(a) \) is, for example, the expression (6) changing \( \xi \)
for \( a \). Through the manipulations with the master action it will be often used the self-adjoint
property of the symbol \( \Omega \), which is given by:
\[ \int A \cdot d\Omega(B) = \int B \cdot d\Omega(A), \]  

(7)

\[ \int A \cdot \Omega(B) = \int B \cdot \Omega(A), \]  

(8)

for any \( A \) and \( B \).
The frame-like and the totally symmetric descriptions can be related each other by mean of the algebraically irreducible decomposition (1). Substituting (1) in (4) we get:

$$\int \omega \cdot d\Omega(\omega) = -\frac{1}{2} \int d^3x \, \phi_{\mu\nu\lambda\beta} \mathbb{G}^{\mu\nu\lambda\beta}(\phi),$$

for the second order term, and:

$$\int \Omega(\omega) \cdot d\Omega(\omega) = \frac{1}{16} \int d^3x \, C_{\mu\nu\gamma\lambda}(\phi) \mathbb{G}^{\mu\nu\gamma\lambda}(\phi)$$

for the third order term. Here, we have introduced the so called “geometrical” objects $\mathbb{G}^{\mu\nu\lambda\beta}(\phi)$ and $C_{\mu\nu\gamma\lambda}(\phi)$, which we have used before in [6] and which was based on [5]. They are the second order in derivatives, Einstein tensor, and the first order in derivatives, symmetrized curl, respectively given by:

$$\mathbb{G}^{\mu\nu\lambda\beta}(\phi) \equiv \mathbb{R}^{\mu\nu\lambda\beta} - \frac{1}{2} \eta_{\mu\nu} \mathbb{R}_{\lambda\beta},$$

where:

$$\mathbb{R}^{\mu\nu\lambda\beta} = \Box \phi_{\mu\nu\lambda\beta} - \partial_{(\mu} \partial^{\alpha} \phi_{\alpha\nu\lambda\beta)} + \partial_{(\mu} \partial_{\nu} \phi_{\lambda\beta)},$$

and:

$$\mathbb{R}_{\lambda\beta} = 2 \left[ \Box \phi_{\lambda\beta} - \partial^{\mu} \partial^{\alpha} \phi_{\mu\alpha\lambda\beta} + \frac{1}{2} \partial_{(\beta} \partial^{\alpha} \phi_{\alpha\lambda)} \right].$$

The operator $C$, on the other hand, is given by:

$$C_{\mu\nu\gamma\lambda}(\phi) \equiv -E_{(\mu}^{\beta} \phi_{\beta\nu\gamma\lambda)}. $$

Both of them were constructed based on the assumptions of [5] and are equipped with the following algebraic properties: The operator $\mathbb{G}^{\mu\nu\lambda\beta}(\phi)$ is self-adjoint in the sense that $\int \psi \mathbb{G}(\phi) = \int \phi \mathbb{G}(\psi)$, for any symmetric fields $\psi$ and $\phi$. Such property will be determinant for obtaining equations of motion as well as for the interpolation with other self-dual models. Similarly, the operator $C_{\mu\nu\gamma\lambda}(\phi)$ is in the same sense also self-adjoint. Besides, one can also check that the operators $C$ and $\mathbb{G}$ commute each other i.e.: $\int \phi C[\mathbb{G}(\phi)] = \int \phi \mathbb{G}[C(\phi)]$.

### 3 Master action on the frame-like

As we have demonstrated in some of our previous works, see for example [20] and subsequent works, a key ingredient in order to construct master actions, is to have in hands trivial mixing terms. Here, we are going to use three of them, the first order Chern-Simons like term [3], the second order Einstein-Hilbert like term [4] and the third order topological Chern-Simons term. Despite the triviality of the Chern-Simons like term to be evident since it is a topological term the demonstration of absence of particle content of the second and third order terms is not that easy. So, we give an explicit demonstration of that in the section - 5.

In order to interpolate among the self-dual frame like descriptions we use as a starting point the first order self-dual model suggested by [14] as a fundamental model to construct the master action:
\[ S_M = \int \left[ \frac{m}{2} \omega \cdot d\omega + \frac{m^2}{2} \omega^2 + c_1 (\omega - g) \cdot d(\omega - g) + c_2 (h - g) \cdot d\Omega(h - g) \right. \]
\[ + \ c_3 \Omega(s - h) \cdot d\Omega(s - h) \] \[ + \ m^2 \int d^3x \ \omega_{\mu\nu} U^{\mu\nu} + S^1_{\text{aux}}[U, H, V]. \]  

In the action (15), the terms preceded by the coefficients \( c_1, c_2 \) and \( c_3 \) are the so called mixing terms, while such coefficients are a priori arbitrary. The spin-4 field is coupled to the auxiliary rank-2 field \( U^{\alpha\beta} \) and at the end of the expression we have an auxiliary action, which can be written explicitly as:

\[
S^1_{\text{aux}} = \int d^3x \left[ - \frac{3m}{4} \epsilon^\rho_{\mu} U_{\rho\alpha} \partial_\mu U^{\alpha\alpha} - \frac{3m^2}{2} \epsilon^\rho_{\mu} \epsilon^{\alpha\beta\gamma} \eta_{\alpha\beta} U_{\mu\beta} U_{\nu\gamma} - \frac{8m}{9} \epsilon^{\mu\nu\beta} H_\mu \partial_\nu H_\beta - \frac{9m}{20} \epsilon^{\mu\nu\beta} V_\mu \partial_\nu V_\beta + \frac{32m^2}{9} H_\mu H^{\mu} - \frac{9m^2}{5} V_\mu V^{\mu} + \frac{2m^2}{5} H^{\mu} V_\mu - \frac{9m}{5} U \partial_\mu V^{\mu} + \frac{22m^2}{5} U^2 \right].
\]

All the coefficients used in the auxiliary action can be obtained after a hard working calculation on the equations of motion, which was done originally by [14], see also [15] for a modern review on the particle content. While the field \( U^{\alpha\beta} \) guarantees that no spin-2 ghosts are propagating, the fields \( H^{\mu} \) and \( V_\mu \) and an additional auxiliary scalar \( U \) which is indeed the trace of \( U^{\alpha\beta} \) are responsible for killing the propagation of spins 1 and 0. It is not so difficult to see that we can integrate over the trace \( U \) in (16). Besides, after deriving the equations of motion to the spin-4 field, one can check that we end up only with the symmetric traceless part of \( U^{\alpha\beta} \) in the game which we will call it \( \tilde{U}^{\alpha\beta} \).

In order to verify the quantum equivalence of the self-dual descriptions to be interpolated by the master action (15) we proceed by adding a source term to the spin-4 field \( j_{\mu(\beta\gamma\lambda)} \) and defining the generating functional:

\[
W_M[j] = \int D\omega \ D\gamma \ D\mu D\nu D\rho D\chi \exp \left( i S_M + \int \omega_{\mu(\beta\gamma\lambda)} \right). \]  

The first equivalence one can check is done by means of the following shifts on the master action (15) in this order: \( s \rightarrow s + h, \ h \rightarrow h + g \) and \( g \rightarrow g + \omega. \) With such shifts the three mixing terms get completely decoupled, and once they are free of particle content, they can be functionally integrated out. Then, one conclude by the equivalence of the correlation functions:

\[
\langle \omega_{\mu_1(\alpha_1\beta_1\gamma_1)}(x_1) \cdots \omega_{\mu_N(\alpha_N\beta_N\gamma_N)}(x_N) \rangle_M = \langle \omega_{\mu_1(\alpha_1\beta_1\gamma_1)}(x_1) \cdots \omega_{\mu_N(\alpha_N\beta_N\gamma_N)}(x_N) \rangle_{SD(1)} \]  

On the other hand, back in (15), if we perform just the first two shifts \( s \rightarrow s + h \) and \( h \rightarrow h + g \) we can decouple only the mixing terms preceded by \( c_2 \) and \( c_3, \) while the remaining one, must be worked in order to take us to the following intermediary step:

\[
S_M[j] = \int \left[ \frac{m^2}{2} \omega^2 - \frac{m}{2} g \cdot dg \right] + \int d^3x \ \omega_{\mu(\beta\gamma\lambda)} \tilde{g}^{\mu(\beta\gamma\lambda)} + S^1_{\text{aux}}, \]  

where

\[ S^1_{\text{aux}} = \int d^3x \ \omega_{\mu(\beta\gamma\lambda)} \tilde{g}^{\mu(\beta\gamma\lambda)} + S^1_{\text{aux}}[U, H, V]. \]
where we have choose \( c_1 = -m/2 \). Besides, we have defined:

\[
\tilde{g}^{\mu(\beta\gamma\lambda)} \equiv -m \xi^{\mu(\beta\gamma\lambda)} + \frac{m^2}{3} f^{\mu(\beta\gamma\lambda)}(U) + j^{\mu(\beta\gamma\lambda)},
\]

(20)

with the partially symmetric-traceless combination of the auxiliary fields \( \tilde{U}^{(\beta\gamma)} \) explicitly given by:

\[
f^{\mu(\beta\gamma\lambda)}(\tilde{U}) \equiv \eta^{\mu\beta} \tilde{U}^{(\gamma\lambda)} + \eta^{\mu\gamma} \tilde{U}^{(\beta\lambda)} + \eta^{\mu\lambda} \tilde{U}^{(\gamma\beta)} - \frac{2}{5} (\eta^{\beta\gamma} \tilde{U}^{(\mu\lambda)} + \eta^{\beta\lambda} \tilde{U}^{(\mu\gamma)} + \eta^{\gamma\lambda} \tilde{U}^{(\mu\beta)}).
\]

(21)

Since we have a quadratic and a linear term on \( \omega \) one can functionally integrate over \( \omega \) in (19), obtaining:

\[
S_M[j] = -\frac{m}{2} \int g \cdot dg - \frac{1}{m^2} \int d^3x \tilde{g}^{\mu(\beta\gamma\lambda)}(\tilde{g}) + S^1_{aux}.
\]

(22)

After substituting back \( \tilde{g} \) from (20) in (22) we finally have:

\[
S_{SD(2)}[j] = \int \left[ -\frac{m}{2} g \cdot dg + g \cdot d\tilde{g}(g) - \frac{m}{4} g \cdot df + j^{\mu(\beta\gamma\lambda)} F^{\mu(\beta\gamma\lambda)}(g, U) + \mathcal{O}(j^2) \right] + S^2_{aux};
\]

(23)

where \( f \) is abbreviation to \( f(U) \). The model (23), is exactly the second order self-dual model with all the new corrections in the auxiliary fields and in the linking terms. For example, the term \(-m g \cdot df/4 = 3m \xi^{\mu\beta} \tilde{U}^{(\alpha\beta)} / 4\) and the corrected auxiliary action, \( S^2_{aux} \), is given by:

\[
S^2_{aux} = S^1_{aux} - \frac{21m^2}{40} \int d^3x \tilde{U}^{(\alpha\beta)} \tilde{U}^{(\alpha\beta)}
\]

(24)

Such results can be compared with those we have obtained in (6). Notice also, that we have gained a new kind of coupling with the source term, which is now given in terms of the dual (gauge invariant) combination \( F^{\mu(\beta\gamma\lambda)} \) which establish a dual map given by:

\[
\omega^{\mu(\beta\gamma\lambda)} \longleftrightarrow F^{\mu(\beta\gamma\lambda)}(g, U) = \frac{2}{m} \Omega^{\mu(\beta\gamma\lambda)}(g) - \frac{1}{4} f^{\mu(\beta\gamma\lambda)}(U),
\]

(25)

with such dual map one can recover the equations of motion of the second order model from the equations of motion of the first order self-dual model model and vice versa.

Deriving with respect the source from the master action in (17) and from the second order self-dual model in (23), we have the equivalence of the correlation functions:

\[
\langle \omega_{\mu_1(\alpha_1, \beta_1, \gamma_1)}(x_1) \ldots \omega_{\mu_N(\alpha_N, \beta_N, \gamma_N)}(x_N) \rangle_M = \langle F_{\mu_1(\alpha_1, \beta_1, \gamma_1)}(x_1) \ldots F_{\mu_N(\alpha_N, \beta_N, \gamma_N)}(x_N) \rangle_{SD(2)} + \mathcal{C.T}
\]

(26)

where \( \mathcal{C.T} \) stands for Contact Terms, which comes from the quadratic terms on the source. They are proportional to the Kronecker and Dirac deltas and do not affect the quantum equivalence at all.

As we have seen, the master action is equivalent to the second order self-dual model up to contact terms, then, considering (23) as our new starting point and adding back the mixing term proportional to \( c_2 \), we have:

\footnote{Observe we have a little mistake in \( [6] \) where the coefficient \( 21/40 \) has been written as \( 11/40 \).}
\[ S_M[j] = \int \left[ -\frac{m}{2} g \cdot dg + g \cdot d\Omega(g) - \frac{m}{4} g \cdot df + c_2 (h - g) \cdot d\Omega(h - g) \right. \\
\left. + \int_{\mu(\beta\gamma\lambda)} F_{\mu(\beta\gamma\lambda)}(g, U) + \mathcal{O}(j^2) \right] + S_{aux}^2 \]

(27)

choosing \( c_2 = -1 \), the expression (30) can be rearranged, after working with the mixing term as:

\[ S_M[j] = \int \left[ -\frac{m}{2} g \cdot dg + g \cdot dC - h \cdot d\Omega(h) - \frac{1}{4} \int_{\mu(\beta\gamma\lambda)} f^{\mu(\beta\gamma\lambda)}(U) + \mathcal{O}(j^2) \right] + S_{aux}^2 \]

(28)

where we have defined:

\[ C^{\mu(\beta\gamma\lambda)} = \frac{2}{m^2} \Omega^{\mu(\beta\gamma\lambda)}(j) - \frac{2}{m} \Omega^{\mu(\beta\gamma\lambda)}(h) + \frac{1}{4} f^{\mu(\beta\gamma\lambda)}(U) \]

(29)

once we have a quadratic and a linear term on \( g \) we can rewrite the master action as:

\[ S_M[j] = \int \left[ -\frac{m}{2} (g + C) \cdot d(g + C) + \frac{m}{2} C \cdot dC - h \cdot d\Omega(h) - \frac{1}{4} \int_{\mu(\beta\gamma\lambda)} f^{\mu(\beta\gamma\lambda)}(U) + \mathcal{O}(j^2) \right] \\
+ S_{aux}^2. \]

(30)

It is easy to see that the shift \( g \rightarrow g + C \) completely decouple the trivial Chern-Simons like term, which then can be functionally integrated out. After substituting back the combination \( C \) we have:

\[ S_{SD(3)}[j] = \int \left[ -h \cdot d\Omega(h) + \frac{2}{m} \Omega(h) \cdot d\Omega(h) - \frac{1}{2} f \cdot d\Omega(h) + \tilde{\Omega}_{\mu(\beta\gamma\lambda)}(j) H^{\mu(\beta\gamma\lambda)}(h, U) + \mathcal{O}(j^2) \right] \\
+ S_{aux}^3, \]

(31)

which is precisely the third order self-dual model we have obtained before in [6]. The new auxiliary action is given by:

\[ S_{aux}^3 = S_{aux}^2 + \frac{21 m}{80} \int d^3 x \ U_{\mu} E_{\gamma} E^\gamma_{\lambda} \]

(32)

Here, the source is coupled with the dual \( H^{\mu(\beta\gamma)} \) given by:

\[ H^{\mu(\beta\gamma)}(h, U) = \frac{1}{2m} E_{\alpha} f^{\alpha(\beta\gamma)}(U) - \frac{4}{m^2} E_{\alpha} \Omega^{\alpha(\beta\gamma\lambda)}(h) - \frac{2}{3} f^{\mu(\beta\gamma\lambda)}(U). \]

(33)

through the combination \( \tilde{\Omega} \), and then the dual map with the previous self-dual models is given by:

\[ \omega^{\mu(\beta\gamma\lambda)} \longleftrightarrow \tilde{\Omega}^{\mu(\beta\gamma\lambda)}[H(h, U)]. \]

(34)

Which, after deriving with respect to the source term from (15) and (31) give us the correspondence of the correlation functions:
\[ \langle \omega_{\mu_1(\alpha_1\beta_1\gamma_1)}(x_1)\omega_{\mu_N(\alpha_N\beta_N\gamma_N)}(x_N) \rangle_M = \langle \hat{\Omega}(H)_{\mu_1(\alpha_1\beta_1\gamma_1)}(x_1)\hat{\Omega}(H)_{\mu_N(\alpha_N\beta_N\gamma_N)}(x_N) \rangle_{SD(3)} + C.T. \]  

(35)

In the next subsection we change from the partially symmetric notation to the totally symmetric notation in order to interpolate with the fourth order-self dual model. This is not mandatory in order to observe such equivalence but just to avoid unnecessary proliferation of symbols and to make the calculations more clean. Despite the source term, which have been coupled through a very technical way to the auxiliary fields and to the spin-4 fields itself, we are going to suggest a redefinition for sake of simplicity, taking advantage of the fact that, the source term is coupled to \( H \) in (31) through the linear, partially symmetric combination \( \tilde{\Omega}(j) \) which has the same number of components of \( j \), we will make simply \( \tilde{\Omega}(j) \rightarrow J \). We will also neglect the coupling of the source with the auxiliary sector considering only the coupling to the totally symmetric fields.

4 \quad SD(3) to SD(4) - totally symmetric approach

Back to the third order self-dual model, we finally add the last mixing term preceded by the coefficient \( c_3 \), which give us the following master action:

\[
S_M[j] = \int \left[ -h \cdot \Omega(h) + \frac{2}{m} \Omega(h) \cdot d\Omega(h) - \frac{1}{2} J \cdot d\Omega(h) + c_3 \Omega(s - h) \cdot d\Omega(s - h) + \frac{4}{m^2} J_{\mu(\beta\gamma\lambda)} \Omega(\mu(\beta\gamma\lambda)) + \mathcal{O}(J^2) \right] + S_{3aux}^3. 
\]

(36)

Applying the decomposition introduced in (1) to the fields \( h, s \) and to the source \( J \) and choosing \( c_3 = -2/m \) we have:

\[
S_M = \int d^3x \left\{ \frac{1}{2} \phi_{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\phi) + \frac{1}{8m} C_{\mu\nu\gamma\lambda}(\phi) G^{\mu\nu\gamma\lambda}(\phi) - \frac{1}{8m} C_{\mu\nu\gamma\lambda}(\sigma - \phi) G^{\mu\nu\gamma\lambda}(\sigma - \phi) \right. \\
+ \left. \psi_{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\phi) - \frac{2}{m^2} \mathcal{J}_{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\phi) \right\} + S_{3aux}^3
\]

(37)

where we have defined \( \psi_{\mu\nu\lambda\beta} \equiv 3 \eta_{(\mu(\lambda\beta)} \tilde{U}(\lambda\beta)) / 40 \). The fields \( \phi \) and \( \sigma \) corresponds respectively to the totally symmetric parts of \( h \) and \( s \). The source \( \mathcal{J} \) is the totally symmetric part of the redefined source \( J \) and it is perfectly coupled to the field \( \phi \) through the Einstein tensor.

Making the shift \( \sigma \rightarrow \sigma + \phi \) it is obvious that the master action becomes the third order self-dual model. On the other hand, by opening the mixing term in (37), we have:

\[
S_M = \int d^3x \left[ -\frac{1}{2} \phi_{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\psi) - \frac{1}{8m} C_{\mu\nu\gamma\lambda}(\sigma) G^{\mu\nu\gamma\lambda}(\sigma) \right] + S_{3aux}^3,
\]

(38)

where we have defined \( \psi \), given by:

\[
\psi_{\mu\nu\lambda\beta} = \frac{1}{4m} C_{\mu\nu\lambda\beta}(\sigma) + \psi_{\mu\nu\lambda\beta} - \frac{2}{m^2} \mathcal{J}_{\mu\nu\lambda\beta},
\]

(39)
and made the shift \( \phi \to \phi - \psi \). By substituting back the field \( \psi \) we have finally:

\[
S_{SD(4)} = \int d^3x \left[ -\frac{1}{8m} \mathcal{C}_{\mu\nu\gamma\lambda}(\phi) G^{\mu\nu\gamma\lambda}(\phi) - \frac{1}{32m^2} \mathcal{C}_{\mu\nu\gamma\lambda}(\phi) G^{\mu\nu\gamma\lambda}(C) - \frac{1}{4m} \mathcal{C}_{\mu\nu\gamma\lambda}(\phi) G^{\mu\nu\gamma\lambda}(\mathcal{U}) 
- \frac{2}{m^2} \mathcal{J}^{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\chi) + \mathcal{O}(\mathcal{J}^2) \right] + S^4_{aux};
\]

where we have defined \( \chi \equiv -\mathcal{C}(\sigma)/4m - \mathcal{U} \). The corrected auxiliary action is given by:

\[
S^4_{aux} = S^3_{aux} - \int d^3x \frac{1}{2} \mathcal{U}_{\mu\nu\gamma\lambda} G^{\mu\nu\gamma\lambda}(\mathcal{U})
\]

Taking functional derivatives with respect the source from (37) and (40) we have the equivalence of the correlation functions:

\[
\langle \phi_{\mu_1\alpha_1\beta_1\gamma_1}(x_1) \ldots \phi_{\mu_N\alpha_N\beta_N\gamma_N}(x_N) \rangle_M = \langle \chi_{\mu_1\alpha_1\beta_1\gamma_1}(x_1) \ldots \chi_{\mu_N\alpha_N\beta_N\gamma_N}(x_N) \rangle_{SD(4)} + C.T.
\]

Notice that, the correlation functions are taken under the same operator \( G \) which is the Einstein tensor. In the notation we have used here, based on [5], the Einstein tensor is second order in derivatives, however in [7] the authors have done a study on the conformal geometry of higher spin bosonic gauge fields in three spacetime dimensions where the Einstein tensor is proportional to the Riemann tensor, and in this case for a rank-\( s \) field the Einstein tensor is of order \( s \) in derivatives.

5 The triviality of the mixing terms

The key ingredient to construct master actions are the mixing terms. During calculations, after some of the shifts, they can be completely decoupled and once integrated one can get rid of them from the lagrangians. Then, since they are free of particle content one can conclude by the equivalence of the interpolated models by the master action. Here, we give an explicit proof of the absence of particle content of two of them. The second order, Einstein-Hilbert like term (9) and the third order topological Chern-Simons like term (10). Such proof can be done by several other ways, here we show that the decomposition in helicities used for example in [16, 17, 18] can be used in a relatively simple way. Other possible way of analysis is through the determination of the propagator, but, it would be necessary to have in hands the complete basis of spin-projectors as we have done for example in [21, 22] in the spin-3 case.

5.1 The second order mixing term

The second order Einstein-Hilbert like term, given by:

\[
S^{(2)} = \frac{1}{2} \int d^3x \ \phi_{\mu\nu\lambda\beta} G^{\mu\nu\lambda\beta}(\phi)
\]

is written in terms of the totally symmetric double traceless field \( \phi_{\alpha\nu\lambda\beta} \), which in \( D = 2 + 1 \) dimensions, have 14 independent components. The action is invariant under the reparametrizations:

\[
\delta \phi_{\alpha\nu\lambda\beta} = \partial_{(\alpha} \tilde{\xi}_{\nu\lambda\beta)};
\]

where we have defined \( \tilde{\xi}_{\nu\lambda\beta} \) as the Lagrange multiplier. The action is invariant under the reparametrizations:

\[
\delta \phi_{\alpha\nu\lambda\beta} = \partial_{(\alpha} \tilde{\xi}_{\nu\lambda\beta)};
\]
where the gauge parameter $\xi_{\nu\lambda\beta}$ is completely symmetric and traceless, thus, in $D = 2 + 1$ it has 7 independent components. This allows us to fix $14 - 7 = 7$ constraint equations, which can be written as:

$$\partial_i \phi_{ijk\mu} = 0. \quad (45)$$

Such constraint equations can be fixed in the action level, since they are complete, in the sense of that explored by [23, 24]. To demonstrate that they are complete we must be able to invert the gauge parameter in terms of the field without any ambiguities. After some tedious calculations one can indeed verify such inversion:

$$\xi_{ijk\mu} = -\frac{\partial_i \phi_{ijk\mu}}{\nabla^2} + \frac{\partial_i \partial_n \partial_j (\phi \partial_k \phi_{mn\mu})}{2\nabla^4} - \frac{\partial_i \partial_n \partial_p \partial_j (\partial_k \phi_{mn\mu})}{3\nabla^6} + \frac{\partial_i \partial_n \partial_p \partial_q \partial_j \partial_k \partial_\mu \phi_{mnpq}}{4\nabla^8}. \quad (46)$$

Where $\nabla^2 = \partial_i \partial_i$. Noticing that the double traceless condition allows us the elimination of $\phi_{0000} = 2\phi_{00ii} - \phi_{iiij}$, one can use the following helicity decomposition:

$$\phi_{0000} = 2\varphi_3 - \nabla^4 \theta; \quad \phi_{000i} = \dot{\partial}_i \gamma + \partial_i \Gamma \quad (47)$$

$$\phi_{00ij} = (\dot{\partial}_i \partial_j + \partial_i \dot{\partial}_j)\varphi_1 + \left(\partial_i \partial_j - \frac{\delta_{ij} \nabla^2}{2}\right) \varphi_2 + \frac{\delta_{ij} \varphi_3}{2} \quad (48)$$

$$\phi_{0ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k \psi; \quad \phi_{ijkl} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k \partial_\theta \quad (49)$$

where $\dot{\partial}_i \equiv \epsilon_{ij} \partial_j$, while the traces are given by:

$$\phi_{00} = -\varphi_3 + \nabla^4 \theta; \quad \phi_{0i} = -\dot{\partial}_i \gamma - \partial_i \Gamma + \dot{\partial}_i \nabla^2 \psi \quad (50)$$

$$\phi_{ij} = -(\dot{\partial}_i \partial_j + \partial_i \dot{\partial}_j)\varphi_1 - \left(\partial_i \partial_j - \frac{\delta_{ij} \nabla^2}{2}\right) \varphi_2 - \frac{\delta_{ij} \varphi_3}{2} + \dot{\partial}_i \dot{\partial}_j \nabla^2 \theta. \quad (51)$$

One can easily verify that the number of independent components of the tensors in the left hand side of each decomposition equation coincides with the number of scalar fields we have in the right hand side of them. Opening the lagrangian [44] in components, using the constraint condition [45], and substituting the decomposition [47-51] we find two decoupled lagrangians given by:

$$L_{\varphi_3,\varphi_2,\theta,\Gamma} = -\frac{25}{2} \varphi_3 \nabla^2 \varphi_3 + 20 \varphi_3 \nabla^6 \theta - 10 \varphi_3 \nabla^2 \Gamma - 15 \varphi_3 \nabla^4 \varphi_2 - 8 \dot{\theta} \nabla^8 \dot{\theta} - 10 \theta \nabla^{10} \theta - 4 \theta \nabla^6 \dot{\Gamma} - 2 \dot{\Gamma} \nabla^2 \Gamma - 18 \Gamma \nabla^4 \Gamma - 6 \varphi_2 \nabla^4 \dot{\Gamma} - \frac{9}{2} \varphi_2 \nabla^6 \varphi_2 \quad (52)$$

and

$$L_{\gamma,\psi,\varphi_1} = -2\dot{\gamma} \nabla^2 \varphi_3 - 8 \gamma \nabla^4 \varphi_1 + 12 \gamma \nabla^4 \psi + 24 \gamma \nabla^6 \psi - 12 \dot{\gamma} \nabla^4 \varphi_1 - 18 \psi \nabla^6 \psi \nabla^8 \varphi_1 + 36 \psi \nabla^6 \varphi_1 - 18 \varphi_1 \nabla^6 \varphi_1. \quad (53)$$

As one can observe the resulting lagrangians has several couplings among the scalar fields and such couplings are involving time derivatives.
In order to diagonalize the first lagrangian \((52)\) and eliminate any dynamical content coming from time derivatives, we proceed with the following set of transformations:

\[
\varphi_2 = \dot{\varphi}_2 - \frac{5\ddot{\varphi}_3}{3\nabla^2} + \frac{2\dddot{\varphi}_3}{3\nabla^4} + \frac{\dot{\Gamma}}{\nabla^2} + \frac{2\ddot{\theta}}{3} \tag{54}
\]

\[
\varphi_3 = \dot{\varphi}_3 - \frac{2\ddot{\varphi}_3}{3\nabla^2} - \frac{\dot{\Gamma}}{2\nabla^2\ddot{\theta}} \tag{55}
\]

\[
\Gamma = \frac{2\dddot{\varphi}_3}{3\nabla^2} + \tilde{\Gamma} - \frac{2\nabla^2\ddot{\theta}}{3} \tag{56}
\]

\[
\theta = \frac{\dddot{\varphi}_3}{\nabla^4} + \ddot{\theta}; \tag{57}
\]

which take the original variables \(\Phi_I \equiv (\varphi_2, \varphi_3, \Gamma, \theta)\) to the new ones \(\tilde{\Phi}_J \equiv (\tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\Gamma}, \tilde{\theta})\) through the matrix \(M_{IJ}\) whose the determinant is unitary i.e: \(\det M = 1\). After that, we obtain:

\[
\mathcal{L}_{\tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\Gamma}, \tilde{\theta}} = 10 \tilde{\varphi}_3 \nabla^2 \tilde{\varphi}_3 - 10 \tilde{\theta} \nabla^{10} \tilde{\theta} - 18 \tilde{\Gamma} \nabla^4 \tilde{\Gamma} - \frac{9}{2} \tilde{\varphi}_2 \nabla^6 \tilde{\varphi}_2 \tag{58}
\]

which clearly does not propagate any degrees of freedom. It remains now, to demonstrate that the lagrangian \((53)\) is also free of particle content, so, we have to diagonalize it also.

In \((53)\) we make the transformations:

\[
\varphi_1 = \dot{\varphi}_1 - \frac{\ddot{\gamma}}{3\nabla^2} + \frac{\ddot{\psi}}{2}; \tag{59}
\]

\[
\gamma = \tilde{\gamma} + \frac{3\nabla^2}{2}\tilde{\psi} \tag{60}
\]

\[
\psi = \ddot{\psi}; \tag{61}
\]

taking the original variables \(\Phi_I \equiv (\varphi_1, \gamma, \psi)\) to the new ones \(\tilde{\Phi}_J \equiv (\tilde{\varphi}_1, \tilde{\gamma}, \tilde{\psi})\) through the matrix \(J_{IJ}\) whose the determinant is also unitary. After that, substituting the new variables we obtain:

\[
\mathcal{L}_{\tilde{\varphi}_1, \tilde{\gamma}, \tilde{\psi}} = -8 \ddot{\gamma} \nabla^4 \tilde{\gamma} + 10 \ddot{\psi} \nabla^8 \tilde{\psi} - 18 \varphi_1 \nabla^6 \varphi_1. \tag{62}
\]

Therefore, once the lagrangians \((58)\) and \((62)\) are completely decoupled and evidently are not propagating any of their fields it can be guaranteed that the second order lagrangian given by \((43)\) can in fact be used as a mixing term.

### 5.2 The third order mixing term

Despite the third order mixing term

\[
\mathcal{L}^{(3)} = \int d^3 x \, C_{\mu \nu \gamma \lambda}(\phi) \mathcal{G}^{\mu \nu \gamma \lambda}(\phi), \tag{63}
\]

to be invariant under a new gauge symmetry which would allow us to fix one more constraint equation, see the discussion in \([6]\), we still use only the gauge condition \((45)\). After substituting the decompositions \((47 - 51)\) and the constraint equation, we obtain:

\[
\mathcal{L} = 68 \dddot{\varphi}_3 \nabla^2 \dddot{\gamma} + 20 \varphi_3 \nabla^4 \gamma - 12 \varphi_3 \nabla^4 \dot{\psi} - 60 \varphi_3 \nabla^6 \psi + 108 \varphi_3 \nabla^4 \dot{\varphi}_1 - 52 \theta \nabla^8 \gamma - 40 \theta \nabla^8 \psi - 28 \theta \nabla^8 \gamma + 40 \theta \nabla^{10} \psi + 36 \theta \nabla^8 \dot{\varphi}_1 + 4 \varphi_2 \nabla^6 \gamma + 12 \varphi_2 \nabla^6 \dot{\psi} - 12 \varphi_2 \nabla^6 \psi + 24 \varphi_2 \nabla^4 \gamma + 12 \varphi_2 \nabla^6 \psi - 12 \varphi_2 \nabla^6 \dot{\psi} + 12 \varphi_2 \nabla^6 \psi \tag{64}
\]
which is a unique and completely coupled lagrangian. In order to decouple all the fields and eliminate the time derivatives, we will proceed with some rounds of rather technical transformations which obviously could be presented at once, but in order to be more didactical we have divided in sub-steps. The first round consists of:

\[
\begin{align*}
\varphi_3 &= \varphi_3 - \frac{6 \ddot{\varphi}_3}{\nabla^2} + \frac{\nabla^4 \bar{\theta}}{3} - 2 \ddot{\Gamma} \\
\varphi_2 &= \varphi_2 + \frac{40 \Phi_1}{3 \nabla^2} \\
\theta &= \bar{\theta} - \frac{14 \Phi_1}{\nabla^4} \\
\Gamma &= \bar{\Gamma} + \frac{3 \ddot{\varphi}_3}{\nabla^2},
\end{align*}
\]

(65)

while \((\gamma, \psi, \varphi_1) \rightarrow (\bar{\gamma}, \bar{\psi}, \bar{\varphi}_1)\). Here, we have defined the combination \(\Phi_1 \equiv \ddot{\varphi}_3/\nabla^2 + \ddot{\Gamma}/3\). This gives us the following lagrangian free of the time derivative couplings involving \((\varphi_3, \varphi_1), (\Gamma, \varphi_1), (\varphi_2, \varphi_1)\) and \((\bar{\theta}, \varphi_1)\):

\[
\mathcal{L} = -496 \ddot{\varphi}_3 \nabla^2 \bar{\gamma} + 20 \varphi_3 \nabla^4 \bar{\gamma} - 80 \ddot{\varphi}_3 \nabla^4 \bar{\psi} - 60 \varphi_3 \nabla^6 \bar{\psi} - \frac{88}{3} \ddot{\bar{\theta}} \nabla^6 \bar{\gamma} - \frac{100}{3} \bar{\theta} \nabla^8 \bar{\psi} + 24 \ddot{\bar{\theta}} \nabla^8 \bar{\psi} + 20 \bar{\theta} \nabla^{10} \bar{\psi} - 24 \ddot{\varphi}_2 \nabla^4 \bar{\gamma} + 12 \varphi_2 \nabla^6 \bar{\gamma} + 24 \ddot{\varphi}_2 \nabla^6 \bar{\psi} + 12 \varphi_2 \nabla^8 \bar{\psi} + 188 \ddot{\bar{\Gamma}} \nabla^4 \bar{\gamma} + 68 \ddot{\bar{\Gamma}} \nabla^6 \bar{\psi} + 36 \ddot{\bar{\Gamma}} \nabla^6 \bar{\varphi}_1. \tag{66}
\]

Then, we proceed with:

\[
\begin{align*}
\varphi_3 &= \bar{\varphi}_3 - \frac{\nabla^4 \bar{\theta}}{108} \\
\varphi_2 &= \bar{\varphi}_2 - \frac{167 \nabla^2 \bar{\theta}}{162} \\
\varphi_1 &= \bar{\varphi}_1 + \frac{47 \bar{\gamma}}{9 \nabla^2} + \frac{17 \bar{\psi}}{27} \tag{67}
\end{align*}
\]

while \((\bar{\gamma}, \bar{\psi}, \bar{\theta}, \bar{\Gamma}) \rightarrow (\bar{\bar{\gamma}}, \bar{\bar{\psi}}, \bar{\bar{\theta}}, \bar{\bar{\Gamma}})\). After this transformation, one can show that:

\[
\begin{align*}
\mathcal{L} &= -496 \ddot{\varphi}_3 \nabla^2 \bar{\gamma} + 20 \bar{\varphi}_3 \nabla^4 \bar{\gamma} - 80 \ddot{\varphi}_3 \nabla^4 \bar{\psi} - 60 \bar{\varphi}_3 \nabla^6 \bar{\psi} - \frac{413}{9} \ddot{\bar{\theta}} \nabla^8 \bar{\gamma} + \frac{221}{27} \ddot{\bar{\theta}} \nabla^{10} \bar{\psi} + 36 \ddot{\bar{\Gamma}} \nabla^6 \bar{\varphi}_1. \tag{68}
\end{align*}
\]

Which is now, free of time derivatives involving \((\bar{\theta}, \bar{\gamma}), (\bar{\bar{\theta}}, \bar{\bar{\gamma}})\) and \((\bar{\bar{\varphi}}, \bar{\bar{\varphi}})\). Then, introducing the following set of transformations:

\[
\begin{align*}
\bar{\theta} &= \bar{\theta} + \frac{36 \Phi_2}{a \nabla^6} + \frac{108 \Phi_3}{a \nabla^4} \\
\bar{\bar{\gamma}} &= \bar{\bar{\gamma}} + \frac{221 \nabla^2 \bar{\psi}}{3a} \tag{69}
\end{align*}
\]

while \((\bar{\gamma}, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3, \bar{\Gamma}) \rightarrow (\bar{\bar{\gamma}}, \bar{\bar{\varphi}_1}, \bar{\bar{\varphi}_2}, \bar{\bar{\varphi}_3}, \bar{\bar{\Gamma}})\) where we define the numerical coefficient \(a = 413\) and the combinations \(\Phi_2 \equiv 5 \nabla^2 \bar{\varphi}_3 + 124 \ddot{\bar{\varphi}}_3\) and \(\Phi_3 \equiv \nabla^2 \bar{\varphi}_2 + 2 \ddot{\bar{\varphi}}_2\). Such transformations take
us to the lagrangian:

\[\mathcal{L} = -\frac{1}{3a}(208736 \dot{\phi}_3 \nabla^4 \dot{\psi} + 69920 \dot{\phi}_3 \nabla^6 \dot{\psi}) + \frac{1}{a}(8144 \dot{\phi}_2 \nabla^6 \dot{\psi} + 5840 \dot{\phi}_2 \nabla^8 \dot{\psi}) - \frac{a}{9} \dot{\theta} \nabla^8 \dot{\gamma} + 36 \dot{\Gamma} \nabla^6 \dot{\phi}_1\]  

(70)

In order to eliminate by complete all time derivatives we suggest:

\[
\begin{align*}
\dot{\phi}_3 &= \dot{\phi}_3 + \frac{b}{a} \Phi_4 \\
\dot{\phi}_2 &= \dot{\phi}_2 + \frac{13046}{1527} \dot{\phi}_3
\end{align*}
\]

(71)

while \((\dot{\gamma}, \dot{\psi}, \dot{\phi}_1, \dot{\Gamma}, \dot{\theta}) \rightarrow (\dot{\gamma}, \dot{\psi}, \dot{\phi}_1, \dot{\Gamma}, \dot{\theta})\). Here, we have defined the numerical complicated coefficient \(b = 210217/13533120\) and the new combinations \(\Phi_4 \equiv 8144 \dot{\phi}_2 - 5840 \nabla^2 \dot{\phi}_2\). This will take us to the following result:

\[\mathcal{L} = \frac{1}{b} \dot{\phi}_3 \nabla^6 \dot{\psi} - \frac{a}{9} \dot{\theta} \nabla^8 \dot{\gamma} + 36 \dot{\Gamma} \nabla^6 \dot{\phi}_1\]

(72)

where one can notice that we do not have any time derivatives in game but still some couplings between the fields. Such couplings can be easily eliminated by simple rotations defined by:

\[
\begin{align*}
\Psi_1 &= \frac{\sqrt{2}}{2} (\dot{\Gamma} - \dot{\phi}) \\
\Psi_3 &= \frac{\sqrt{2}}{2} (\dot{\theta} - \dot{\gamma}) \\
\Psi_5 &= \frac{\sqrt{2}}{2} (\dot{\phi}_3 - \dot{\psi})
\end{align*}
\]

\[
\begin{align*}
\Psi_2 &= \frac{\sqrt{2}}{2} (\dot{\Gamma} + \dot{\phi}) \\
\Psi_4 &= \frac{\sqrt{2}}{2} (\dot{\theta} + \dot{\gamma}) \\
\Psi_6 &= \frac{\sqrt{2}}{2} (\dot{\phi}_3 + \dot{\psi})
\end{align*}
\]

(73)

which finally give us the completely decoupled lagrangian:

\[\mathcal{L} = \frac{1}{2b} (\Psi_6 \nabla^6 \Psi_6 - \Psi_5 \nabla^6 \Psi_5) - \frac{a}{18} (\Psi_4 \nabla^8 \Psi_4 - \Psi_3 \nabla^8 \Psi_3) + 18 (\Psi_2 \nabla^6 \Psi_2 - \Psi_1 \nabla^6 \Psi_1),\]

(74)

which is evidently free of particle content. It is remarkable that all the transformations we have done have unitary determinants as we have observed in the case of the the second order term, guaranteeing that the descriptions in terms of the new variables are canonically equivalent to the original ones. Notice however that the final set of helicity decomposition variables is one variable lower then the original set. This is a consequence of the not used gauge condition associated with the extra gauge invariance we have talked about in [6]. Besides, it is obvious that the complicated numerical coefficients \(a\) and \(b\) can be absorbed by simple redefinitions of the fields which makes the result more elegant. Notice also that an alternative way of analysis, based on the identification of gauge invariant objects could also be used to deal with this demonstration, such approach has successfully been used in the spin-1, 2, 3 and 4 cases in [25] for example.
6 Conclusion

The master action technique has been used in order to demonstrate that in $D = 2 + 1$ dimensions the massive spin-4 particle can be described by four quantum equivalent self-dual descriptions. The equivalence is verified by the comparison of the correlation functions up to contact terms. The first two self-dual descriptions can uniquely be written in terms of the so called frame-like approach, originally suggested by Vasiliev [19] while the third and fourth order models can be written in terms of a totally symmetric double traceless field which then allow the introduction of a “geometrical” description based on [5].

In order to construct the master action we have argued that we need identify mixing terms. Such terms, must be free of any particle content. Here, we have used three of them, the first order Chern-Simons like term (3), which one can easily verify that it is a trivial term, a second order Einstein-Hilbert like term (4) and finally a third order topologically Chern-Simons term (5). Although the particle content of the first one is easy to verify, the same is not true to the other two of them. Taking advantage of the so called helicity decomposition method used for example in [16, 17, 18] we have performed a didactical analysis of them by decomposing the symmetric double traceless fields and using a gauge fixing condition at the level of the action. In order to demonstrate that we are able to use such procedure, in the spirit of [23] and [24], we have demonstrated that our gauge condition is complete, i.e: without any ambiguities one can invert the gauge parameter in terms of the field itself. Performing then a set of rather technical transformations we have demonstrated that the lagrangians can be diagonalized and are free of time derivatives which would implicate in dynamical content.

Our work may contribute to the general discussion recently addressed in [25], where the authors have investigated higher derivative descriptions of spin-4 self-dual models of seventh and eight order in derivatives without any auxiliary fields and in terms of totally symmetric fields, as well as, its version on a maximally symmetric curved spacetime, suggested by [26].

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