A VON NEUMANN TYPE INEQUALITY FOR AN ANNULUS

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Abstract. Let $A_r = \{ r < |z| < 1 \}$ be an annulus. We consider the class of operators

$$
\mathcal{F}_r := \{ T \in \mathcal{B}(H) : r^2 T^{-1} (T^{-1})^* + T T^* \leq r^2 + 1, \sigma(T) \subset A_r \}
$$

and show that for every bounded holomorphic function $\phi$ on $A_r$:

$$
\sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq \sqrt{2} ||\phi||_{\infty},
$$

where the constant $\sqrt{2}$ is the best possible. We do this by characterizing the calculus norm induced on $H^\infty(A_r)$ by $\mathcal{F}_r$ as the multiplier norm of a suitable holomorphic function space on $A_r$.

1. INTRODUCTION

Let $T$ be a contractive operator on a Hilbert space. A famous inequality due to von Neumann ([9]) asserts that

$$
||p(T)|| \leq \sup_{D} |p|
$$

whenever $p$ is a polynomial in one variable. As an immediate consequence, we find that for every one-variable polynomial $p$:

$$
\sup_{||T|| \leq 1} ||p(T)|| = ||p||_{\infty}.
$$

An extraordinary amount of work has been done by the operator theory community extending the inequality of von Neumann. We mention two well-known generalisations due to Andô and Drury. Andô’s inequality ([4]) states that if $T = (T_1, T_2)$ is a contractive commuting pair of operators on a Hilbert space, then

$$
||p(T)|| \leq \sup_{D^2} |p|
$$

whenever $p$ is a polynomial in two variables. Now, while the corresponding analogue of Andô’s theorem and the von Neumann inequality fails for three or more contractions (see e.g. [11]), Drury’s generalization ([5]) does tell us that every row $d$-contraction $T$ on a Hilbert space (i.e. every $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting operators such that $T_1 T_1^* + \cdots + T_d T_d^* \leq 1$) satisfies

$$
||\phi(T)|| \leq ||\phi||_{\text{Mult}(H^2_2)}
$$

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whenever \( \phi \) is a polynomial in \( d \) variables. Here, \( \|\phi\|_{\text{Mult}(H^2_d)} \) denotes the norm of the multiplication operator \( M_\phi f := \phi f \) acting on the Drury-Arveson space \( H^2_d \) over the (open) \( d \)-dimensional complex unit ball \( \mathbb{B}_d \). Since the operators \( M_z \) of multiplication by the independent variables form a row \( d \)-contraction \( M_z := (M_{z_1}, \ldots, M_{z_d}) \) acting on \( H^2_d \), Drury’s inequality is actually equivalent to the statement

\[
\sup_{T_1, T_1' + \cdots + T_d T_d' \leq 1} \|\phi(T_1, \ldots, T_d)\| = \|\phi\|_{\text{Mult}(H^2_d)},
\]

for every \( d \)-variable polynomial \( \phi \).

In this note, we deduce a von Neumann type inequality for the class of operators

\[
\mathcal{F}_r := \{ T \in \mathcal{B}(H) : r^2 T^{-1} (T^{-1})^* + T T^* \leq r^2 + 1, \sigma(T) \subset A_r \}
\]

associated with the annulus \( A_r = \{ r < |z| < 1 \} \) by applying standard positivity arguments to model formulas in an appropriate function space setting. This setting is introduced in Section 3; it is the holomorphic function space \( \mathcal{H}^2(A_r) \) induced on \( A_r \) by the kernel

\[
k_{A_r}(\lambda, \mu) := \frac{1 - r^2}{(1 - \lambda \bar{\mu})(1 - r^2/\lambda \bar{\mu})}, \quad \forall \lambda, \mu \in A_r.
\]

Our main result is the following theorem, the proof of which is contained in Section 4.

**Theorem 1.1.** For every \( \phi \in H^\infty(A_r) \),

\[
\sup_{T \in \mathcal{F}_r} \|\phi(T)\| = \|\phi\|_{\text{Mult}(\mathcal{H}^2(A_r))} \leq \sqrt{2} \|\phi\|_{\infty},
\]

where the constant \( \sqrt{2} \) is the best possible.

The fact that \( k_{A_r} \) is a complete Pick kernel (to be defined below) also allows us to show the following extension result.

**Theorem 1.2.** Let \( 0 < r < 1 \). For every \( \phi \in H^\infty(A_r) \), the quantity

\[
\min\{\|\psi\|_{\text{Mult}(H^2_2)} : \psi \in \text{Mult}(H^2_2) \text{ and } \psi\left(\frac{z}{\sqrt{r^2 + 1}}, \frac{r}{\sqrt{r^2 + 1}}\right) = \phi(z), \forall z \in A_r\}
\]

lies in the interval \([\|\phi\|_{\infty}, \sqrt{2} \|\phi\|_{\infty}]\). Moreover, the constant \( \sqrt{2} \) is the best possible.

Here, \( H^2_2 \) denotes the 2-dimensional Drury-Arveson space on the open unit ball \( \mathbb{B}_2 \subseteq \mathbb{C}^2 \).

2. PRELIMINARIES

Let \( X \) be a nonempty set. A function \( k : X \times X \to \mathbb{C} \) is called positive semi-definite, if whenever \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X \) and \( w_1, \ldots, w_n \in \mathbb{C} \), then \( \sum_{i,j=1}^n k(x_j, x_i) w_i \overline{w_j} \geq 0 \). We also say that \( k \) is a kernel. For each \( x \in X \), define a function \( k(\cdot, x) \) on \( X \) by \( k(\cdot, x)(y) = k(y, x) \). Define also an inner product on the linear span of these functions by

\[
\langle \sum_i a_i k(\cdot, x_i), \sum_j b_j k(\cdot, x_j) \rangle = \sum_{i,j} a_i \overline{b_j} k(x_j, x_i).
\]
Let $\mathcal{H}_k$ denote the Hilbert space obtained by completing the linear span of the functions $k(\cdot, x)$ with respect to the previous inner product. We may regard vectors $f$ in $\mathcal{H}_k$ as functions on $X$, with $f(x) = \langle f, k(\cdot, x) \rangle$.

The multiplier algebra $\text{Mult}(\mathcal{H}_k)$ is defined as the collection of functions $\phi : X \to \mathbb{C}$ such that $(M_\phi f)(x) = \phi(x)f(x)$ defines a bounded operator $M_\phi : \mathcal{H}_k \to \mathcal{H}_k$. The multipliers $\phi$ with $||M_\phi|| \leq C$ are characterized by

\[(2.1) \quad (C^2 - \phi(y)\phi(x))k(y, x) \geq 0,
\]

since it is equivalent to $||M_\phi f||_{\mathcal{H}_k} \leq C||f||_{\mathcal{H}_k}$, for a dense subset of $\mathcal{H}_k$.

Now, suppose $k$ is a kernel on $X$. We say that $k$ is a (normalized) complete Pick kernel (and the space $\mathcal{H}_k$ it induces we call a complete Pick space) if there exists some (possibly infinite) cardinal $d$ and an injection $b : X \to \mathbb{B}_d$ ($\mathbb{B}_d$ being the (open) unit ball in some $d$- dimensional Hilbert space $K$) such that

$$1 - \frac{1}{k(y, x)} = \langle b(y), b(x) \rangle_K$$

for all $x$ and $y$ in $X$. The standard definition is actually based on finite interpolation problems (its equivalence to the one we gave is a theorem of Agler and McCarthy [1]), however we will not be needing it for the purposes of this note. As a consequence of our definition, $\mathcal{H}_k$ can be identified with the $*$-invariant subspace

$$\mathcal{H}_k = \text{closed linear span of } \{k_x : x \in \text{ran } b\} \subset H^2_d.$$ Here, $H^2_d$ denotes the $d$-dimensional Drury-Arveson space, which is the reproducing kernel Hilbert space induced by the kernel

$$a_d(\lambda, \mu) = \frac{1}{1 - \langle \lambda, \mu \rangle_K}$$

and defined on $\mathbb{B}_d$.

The class of complete Pick kernels is well-known and extensively studied. A comprehensive treatment can be found in [2]. In the sequel, we will only have occasion to use the following basic result. Suppose

$$k(y, x) = \frac{1}{1 - \langle b(y), b(x) \rangle}$$

is a complete Pick kernel on a set $X$ embedding into the Drury-Arveson space $H^2_d$. Then, for every $\phi \in \text{Mult}(\mathcal{H}_k)$ we obtain that $||\phi||_{\text{Mult}(\mathcal{H}_k)}$ is equal to

\[(2.2) \quad \min\{||\psi||_{\text{Mult}(H^2_d)} : \psi \in \text{Mult}(H^2_d) \text{ and } \psi(b(x)) = \phi(x), \forall x \in X\}.
\]

For the proof of Theorem 1.1 we will be making use of the Riesz-Dunford functional calculus in the setting of the annulus $A_r$. Instead of employing the standard Cauchy integral formula, we will adopt the equivalent Laurent series definition. Let $T \in \mathcal{B}(H)$ and suppose that the spectrum $\sigma(T)$ of $T$ is contained in $A_r$. If $f = \sum_{n \in \mathbb{Z}} a_n z^n$ is any function holomorphic on $A_r$, then $f(T) \in \mathcal{B}(H)$ is defined as

$$f(T) = \sum_{n \in \mathbb{Z}} a_n T^n.$$
Observe that since $\sigma(T) \subseteq A_r$, the convergence of the above Laurent series is guaranteed. See e.g. \[3\] for the basic properties of the Riesz-Dunford functional calculus.

We now set up the \textit{hereditary functional calculus} on $A_r$. We say that $h$ is a hereditary function on $A_r$ if $h$ is a mapping from $A_r \times A_r$ to $C$ and has the property that

$$ (\lambda, \mu) \mapsto h(\lambda, \bar{\mu}) \in C $$

is a holomorphic function on $A_r \times A_r$. The set $\text{Her}(A_r)$ of hereditary functions on $A_r$ forms a complete metrizable locally convex topological vector space when equipped with the topology of uniform convergence on compact subsets of $A_r \times A_r$.

If $T \in \mathcal{B}(H)$ with $\sigma(T) \subseteq A_r$ and $h$ is a hereditary function on $A_r$, then we may define $h(T) \in \mathcal{B}(H)$ by the following procedure. Expand $h$ into a double Laurent series

$$ h(\lambda, \mu) = \sum_{m,n \in \mathbb{Z}} c_{mn} \lambda^m \bar{\mu}^n $$

for all $\lambda, \mu \in A_r$.

and then define $h(T)$ by substituting $T$ for $\lambda$ and $T^*$ for $\bar{\mu}$:

$$ h(T) = \sum_{m,n \in \mathbb{Z}} c_{mn} T^m (T^*)^n. $$

There is a natural involution $h \mapsto h^*$ on $\text{Her}(A_r)$, defined by

$$ h^*(\lambda, \mu) = \overline{h(\mu, \lambda)}, \quad \forall \lambda, \mu \in A_r. $$

It is easy to see that $h^*(T) = h(T^*)$.

Finally, we record the following fundamental lemma (the counterpart of Theorem 2.88 in \[2\] for the annulus) which is essentially the holomorphic version of Moore’s theorem on the factorisation of positive semi-definite kernels. It will allow us to decompose positive semi-definite hereditary functions as sums of dyads.

**Lemma 2.1.** Suppose $U$ is a positive semi-definite hereditary function on $A_r$, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions holomorphic on $A_r$ such that

$$ U(\lambda, \mu) = \sum_{n=1}^{\infty} f_n(\lambda) \overline{f_n(\mu)}, \quad \forall \lambda, \mu \in A_r, $$

the series converging uniformly on compact subsets of $A_r \times A_r$.

3. THE SPACE $\mathcal{H}^2(A_r)$

Fix $r < 1$ and let $A_r = \{ r < |z| < 1 \}$. Denote by $H^2(A_r)$ the classical Hardy space on an annulus. This is the Hilbert function space

$$ H^2(A_r) = \{ f \in \text{Hol}(A_r) : \sup_{r < \rho < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{it})|^2 \, dt < \infty \} $$

equipped with the norm (for $f = \sum_{n \in \mathbb{Z}} a_n z^n$)

$$ \|f\|_{H^2(A_r)}^2 = \sum_{n=-\infty}^{\infty} (r^{2n} + 1) |a_n|^2. $$
An important observation is that the multiplier algebra \( \text{Mult}(H^2(A_r)) \) is isometrically isomorphic to the algebra \( H^\infty(A_r) \) of bounded holomorphic functions on \( A_r \).

Now, we define the space \( \mathcal{H}^2(A_r) \) by equipping \( H^2(A_r) \) with a different norm:

\[
||f||^2_{\mathcal{H}^2(A_r)} = \sum_{-\infty}^{-1} r^{2n}|a_n|^2 + \sum_{0}^{\infty} |a_n|^2.
\]

These two norms are equivalent, as

\[
||f||^2_{\mathcal{H}^2(A_r)} \leq ||f||^2_{H^2(A_r)} = \sum_{-\infty}^{\infty} (r^{2n} + 1)|a_n|^2
\]

\[
\leq 2 \sum_{-\infty}^{-1} r^{2n}|a_n|^2 + 2 \sum_{0}^{\infty} |a_n|^2 = 2||f||^2_{\mathcal{H}^2(A_r)}.
\]

Hence,

\[
(3.1) \quad ||f||_{\mathcal{H}^2(A_r)} \leq ||f||_{H^2(A_r)} \leq \sqrt{2}||f||_{\mathcal{H}^2(A_r)},
\]

for every \( f \in \mathcal{H}^2(A_r) \). Notice also that the set

\[
\left\{ \frac{z^n}{r^n} \right\}_{n \leq -1} \cup \left\{ \frac{z^n}{r^n} \right\}_{n \geq 0}
\]

is an orthonormal basis for \( \mathcal{H}^2(A_r) \). Applying Parseval’s identity, we can then calculate the kernel function for \( \mathcal{H}^2(A_r) \) as follows

\[
k_{A_r}(\lambda, \mu) = \langle k_{A_r}(\cdot, \mu), k_{A_r}(\cdot, \lambda) \rangle
\]

\[
= \sum_{-\infty}^{-1} \langle k_{A_r}(\cdot, \mu), z^n/r^n \rangle \langle z^n/r^n, k_{A_r}(\cdot, \lambda) \rangle + \sum_{0}^{\infty} \langle k_{A_r}(\cdot, \mu), z^n \rangle \langle z^n, k_{A_r}(\cdot, \lambda) \rangle
\]

\[
= \sum_{-\infty}^{-1} \frac{\lambda^n \mu^n}{r^n} + \sum_{0}^{\infty} \lambda^n \mu^n
\]

\[
= (1 - r^2) \frac{1}{(1 - \lambda \mu)(1 - \lambda \mu)} , \quad \forall \lambda, \mu \in A_r.
\]

There are a few interesting observations we can make here. Recall that \( a_2(\lambda, \mu) \) (where \( \lambda = (\lambda_1, \lambda_2) \) and \( \mu = (\mu_1, \mu_2) \) denotes the reproducing kernel of the Drury-Arveson space \( H^2_2 \) defined on the 2-dimensional complex unit ball \( \mathbb{B}_2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\} \). Denote also by \( s_2(\lambda, \mu) \) the kernel of the classical Hardy space \( H^2(\mathbb{D}^2) \) defined on the bidisk \( \mathbb{D}^2 = \{(z_1, z_2) : |z_1|, |z_2| < 1\} \);

\[
s_2(\lambda, \mu) = \frac{1}{(1 - \lambda_1 \mu_1)(1 - \lambda_2 \mu_2)}.
\]

A short calculation then leads us to the equalities

\[
k_{A_r}(\lambda, \mu) =
\]

\[
(3.2) \quad = \left( \frac{1 - r^2}{1 + r^2} \right) a_2 \left( \left( \frac{\lambda}{\sqrt{r^2 + 1}}, \frac{r}{\sqrt{r^2 + 1}}, \frac{1}{\lambda} \right), \left( \frac{\mu}{\sqrt{r^2 + 1}}, \frac{r}{\sqrt{r^2 + 1}}, \frac{1}{\mu} \right) \right)
\]
for every $\lambda$ and $\mu$ in $A_r$. We can now apply the pull-back theorem for reproducing kernels (see e.g. Theorem 5.7 in [10]) to obtain two new descriptions of the norm of $\mathcal{H}^2(A_r)$. By (3.2), we obtain that for every $f \in \mathcal{H}^2(A_r)$:

$$\|f\|_{\mathcal{H}^2(A_r)} = \sqrt{1 + r^2 \min\{||g||_{H^2_2} : g \in H^2_2 \text{ and } g\left(\frac{z}{\sqrt{1+r^2}}, \frac{r}{\sqrt{1+r^2}}z\right) = f(z), \forall z \in A_r\}},$$

while (3.3) gives us:

$$\|f\|_{\mathcal{H}^2(A_r)} = \sqrt{1 + r^2 \min\{||g||_{H^2_2(D^2)} : g \in H^2_2(D^2) \text{ and } g(z, \frac{r}{\sqrt{1+r^2}}z) = f(z), \forall z \in A_r\}}.$$

We will now use (3.1) to compare the norm of the multipliers of $\mathcal{H}^2(A_r)$ with the supremum norm on $H^\infty(A_r)$.

**Proposition 3.1.** For every $\phi \in H^\infty(A_r)$,

$$||\phi||_\infty \leq ||\phi||_{\text{Mult}(H^2(A_r))} \leq \sqrt{2}||\phi||_\infty.$$

Moreover, the constant $\sqrt{2}$ is the best possible.

**Proof.** Since $\mathcal{H}^2(A_r)$ is a reproducing kernel Hilbert space, the inequality $||\phi||_\infty \leq ||\phi||_{\text{Mult}(H^2(A_r))}$ is automatic, for all $\phi$ in $\text{Mult}(\mathcal{H}^2(A_r))$.

Now, fix $\phi \in H^\infty(A_r)$. (3.1) allows us to write

$$||\phi f||_{\mathcal{H}^2(A_r)} \leq ||\phi||_{H^2(A_r)} \leq ||\phi||_\infty ||f||_{H^2(A_r)} \leq \sqrt{2}||\phi||_\infty ||f||_{\mathcal{H}^2(A_r)},$$

for every $f \in \mathcal{H}^2(A_r)$. Hence, $||\phi||_{\text{Mult}(H^2(A_r))} \leq \sqrt{2}||\phi||_\infty$, as desired.

Now, to prove that the constant $\sqrt{2}$ is the best possible, we define

$$g_n(z) = \frac{r^n}{z^n} + z^n, \ \forall z \in A_r, \ \forall n \geq 1.$$

Then, for every $z \in A_r$:

$$|g_n(z)| \leq \frac{r^n}{|z|^n} + |z|^n \leq 1 + r^n.$$

Hence, $||g_n||_\infty = 1 + r^n$ for all $n \geq 1$. Notice also that

$$\frac{||g_n||_{\text{Mult}(H^2(A_r))}}{||g_n||_\infty} = \frac{||g_n||_{\text{Mult}(H^2(A_r))}}{1 + r^n} \geq \frac{||g_n \cdot 1||_{\mathcal{H}^2(A_r)}}{1 + r^n} = \frac{\sqrt{2}}{1 + r^n} \overset{n \to \infty}{\rightarrow} \sqrt{2}.$$

This concludes our proof. □
We return to equality (3.2). This can be written equivalently as
\[ k_{A_r}(\lambda, \mu) = \frac{1 - r^2}{1 + r^2} \frac{1}{1 - \langle b_r(\lambda), b_r(\mu) \rangle}, \quad \forall \lambda, \mu \in A_r, \]
where
\[ b_r(\lambda) := \left( \frac{\lambda}{\sqrt{r^2 + 1}}, \frac{r}{\sqrt{r^2 + 1}} \frac{1}{\lambda} \right) \]
and \(|b_r(\lambda)| < 1 \) in \( A_r \).
Hence, \( k_{A_r} \) is a complete Pick kernel. This allows us to draw an interesting connection between the supremum norm of \( H_\infty(A_r) \) and the multiplier norm of \( \text{Mult}(H^2_r) \), formulated as an extension result of holomorphic functions off a subvariety of \( \mathbb{B}_2 \). It is the content of Theorem 1.2, which we now prove.

\textbf{Proof of Theorem 1.2}. Since \( \frac{1 + r^2}{1 - r^2} k_{A_r} \) is a (normalized) complete Pick kernel, we can use (2.2) to deduce that for every \( \phi \in \text{Mult}(H^2_r(A_r)) = H_\infty(A_r) \),
\[ ||\phi||_{\text{Mult}(H^2_r)} = \min \{ ||\psi||_{\text{Mult}(H^2_r)} : \psi \in \text{Mult}(H^2_r) \text{ and } \psi(b_r(z)) = \phi(z), \forall z \in A_r \}. \]
Applying Proposition 3.1 then concludes the proof. \( \square \)

\textbf{Remark 1}. A rescaled version of the space \( H^2_r(A_r) \) was also considered by Arcozzi, Rochberg, Sawyer in [5]. There, the authors proved the much more general fact that every Hardy space over a finitely connected domain with smooth boundary curves admits an equivalent norm originating from a complete Pick kernel.

4. PROOF OF THEOREM 1.1

We will now prove Theorem 1.1, which shows that the class of operators \( \mathcal{F}_r = \{ T \in B(H) : r^2 T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1, \sigma(T) \subset A_r \} \) can be naturally associated with the space \( \mathcal{H}^2_r(A_r) \).

\textbf{Proof of Theorem 1.1}. Since we have already established Proposition 3.1, it remains to show that
\[ \sup_{T \in \mathcal{F}_r} ||\phi(T)|| = ||\phi||_{\text{Mult}(H^2_r(A_r))}, \]
for every \( \phi \in H_\infty(A_r) \).
First, suppose \( ||\phi||_{\text{Mult}(H^2_r(A_r))} \leq 1 \). By (2.1), we obtain the existence of a positive-semidefinite kernel \( U : A_r \times A_r \to \mathbb{C} \) such that
\[ (1 - \phi(\lambda)\bar{\phi}(\mu))k_{A_r}(\lambda, \mu) = U(\lambda, \mu), \]
hence we have the model formula
\[ 1 - \phi(\lambda)\bar{\phi}(\mu) = \frac{U(\lambda, \mu)}{1 - r^2} (1 + r^2 - r^2/\lambda\bar{\mu} - \lambda\bar{\mu}). \]
Evidently, \( U \) is a positive semi-definite element of \( \text{Her}(A_r) \). Applying Lemma 2.1, we obtain the existence of a sequence \( \{ f_n \}_{n \in \mathbb{N}} \) of elements of \( \text{Hol}(A_r) \) such that
\[ 1 - \phi(\lambda)\bar{\phi}(\mu) = \frac{1}{1 - r^2} \sum_{n=1}^{\infty} f_n(\lambda)(1 + r^2 - r^2/\lambda\bar{\mu} - \lambda\bar{\mu}) \bar{f_n}(\mu), \]
Thus, we also have that \( \sigma \) satisfies Lemma 4.1.

We now show the reverse inequality. First, we prove a lemma.

Let \( T \in \mathcal{F}_r \). We have showed that Lemma 4.2.

Our next step will be to prove the following special case.

By (4.1), we can then conclude that

\[
1 - \phi(T)\phi(T)^* = \frac{1}{1 - r^2} \sum_{n=1}^{\infty} f_n(T)(1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^*)f_n(T)^*.
\]

However, observe that since \( T \in \mathcal{F}_r \), we can write

\[
1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^* \geq 0
\]

\[
\Rightarrow f_n(T)(1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^*)f_n(T)^* \geq 0
\]

\[
\Rightarrow \sum_{n=1}^{\infty} f_n(T)(1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^*)f_n(T)^* \geq 0.
\]

By (4.1), we can then conclude that

\[
1 - \phi(T)\phi(T)^* \geq 0
\]

\[
\Rightarrow ||\phi(T)|| \leq 1.
\]

We have showed that

\[
\sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq ||\phi||_{\text{Mult}(\mathcal{H}^2(\mathbb{A}_r))}.
\]

We now show the reverse inequality. First, we prove a lemma.

**Lemma 4.1.** Every \( T \in \mathcal{B}(H) \) such that \( r^2 T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1 \) satisfies

\[
r^2 \leq TT^* \leq 1.
\]

Thus, we also have that \( \sigma(T) \subset \mathbb{A}_r \) for every such operator.

**Proof of Lemma 4.1.** Suppose instead that \( ||T^*|| = ||T|| = 1 + \delta > 1 \). Thus, for any \( \epsilon \in (0, \delta) \) there exists \( y \in H \) with \( ||y|| = 1 \) such that \( ||T^*y|| > 1 + \delta - \epsilon \). Since \( ||T^*x|| \leq (1 + \delta)||x|| \), we also obtain \( \frac{1}{1 + \delta}||x|| \leq ||(T^{-1})^*x|| \), for every \( x \in H \). We can now write

\[
\frac{r^2}{(1 + \delta)^2} + (1 + \delta - \epsilon)^2 < r^2 ||(T^{-1})^*y||^2 + ||T^*y||^2
\]

\[
= (r^2 T^{-1}(T^{-1})^* + TT^*)y, y) \leq r^2 + 1.
\]

Letting \( \epsilon \to 0 \), we obtain

\[
\frac{r^2}{(1 + \delta)^2} + (1 + \delta)^2 \leq r^2 + 1,
\]

a contradiction. Hence, \( ||T|| \leq 1 \) and an analogous argument shows that \( ||r T^{-1}|| \leq 1 \) as well.

Our next step will be to prove the following special case.

**Lemma 4.2.** Let \( \phi \) be holomorphic in a neighborhood of \( \mathbb{A}_r \) with the property that \( ||\phi(T)|| \leq 1 \) for all \( T \in \mathcal{B}(H) \) such that \( r^2 T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1 \). Then, \( ||\phi||_{\text{Mult}(\mathcal{H}^2(\mathbb{A}_r))} \leq 1 \).
Proof of Lemma 4.2. If $T \in \mathcal{B}(H)$ is such that $r^2T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1$, then by Lemma 4.1, $T$ also satisfies $r^2 \leq TT^* \leq 1$. In particular, $\sigma(T)$ has to lie in $\overline{A_r}$ and so the operator $\phi(T)$ is indeed well-defined whenever $\phi \in \text{Hol}(\overline{A_r})$.

Now, suppose that $\phi$ satisfies the given hypotheses and consider the bilateral shift operator $(Sf)(z) = zf(z)$ defined on $\mathcal{H}^2(A_r)$. A standard computation shows that $S^*k_{A_r}(\cdot, \lambda) = \bar{\lambda}k_{A_r}(\cdot, \lambda)$, for every $\lambda \in A_r$. Notice also that for every $\lambda, \mu$ in $A_r$,

$$\langle (r^2 + 1 - r^2S^{-1}(S^{-1})^* - SS^*)k_{A_r}(\cdot, \mu), k_{A_r}(\cdot, \lambda) \rangle$$

$$= (r^2 + 1)\langle k_{A_r}(\cdot, \mu), k_{A_r}(\cdot, \lambda) \rangle - r^2\langle (S^{-1})^*k_{A_r}(\cdot, \mu), (S^{-1})^*k_{A_r}(\cdot, \lambda) \rangle - \langle S^*k_{A_r}(\cdot, \mu), S^*k_{A_r}(\cdot, \lambda) \rangle$$

$$= (r^2 + 1)k_{A_r}(\lambda, \mu) - \frac{r^2}{\lambda \mu}k_{A_r}(\lambda, \mu) - \lambda \bar{\mu}k_{A_r}(\lambda, \mu)$$

$$= 1 - r^2,$$

a (trivial) positive semi-definite kernel on $A_r \times A_r$. Since linear combinations of kernel functions are dense in $\mathcal{H}^2(A_r)$, our previous equality implies that

$$r^2 + 1 - r^2S^{-1}(S^{-1})^* - SS^* \geq 0.$$ But our hypotheses on $\phi$ then allow us to deduce that

$$||\phi||_{\text{Mul}(\mathcal{H}^2(A_r))} = ||\phi(S)|| \leq \sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq 1,$$ which concludes the proof of the lemma. □

To complete our main proof, we will apply an approximation argument to extend the previous special case to every multiplier of $\mathcal{H}^2(A_r)$.

Suppose $\phi \in \text{Hol}(A_r)$ is such that $\sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq 1$. For $n > 2/(1 - r)$, define $A_{r,n} := \{r + 1/n < |\lambda| < 1 - 1/n\}$. We will be needing the following lemma, the proof of which is just a simple calculation.

Lemma 4.3. The following two inequalities hold for all $n > 2/(1 - r)$:

$$\frac{(r + \frac{1}{n})^2}{1 + (\frac{r + \frac{1}{n}}{1 - \frac{1}{n}})^2} \geq \frac{r^2}{1 + r^2},$$

$$\frac{1}{(1 - \frac{1}{n})^2 + (r + \frac{1}{n})^2} \geq \frac{1}{1 + r^2}.$$

Now, define the classes of operators

$$\mathcal{F}_{r,n} = \{T \in \mathcal{B}(H) : [r + (1/n)]^2T^{-1}(T^{-1})^* + [1/(1 - n)]^2TT^* \leq 1 + [(r + (1/n))/(1 - (1/n))]^2\}$$

and also the family of kernels

$$k_{r,n}(\lambda, \mu) := \frac{1}{\left(1 - \frac{(r + 1/n)^2}{\bar{\mu} \lambda} \right) \left(1 - \frac{\mu \lambda}{(1 - 1/n)^2}\right)}.$$
To sum up, we have proved (for every $m$, $\varphi$, $\sigma$, $r$, $\mu$, $\lambda$) that

$$
(4.2) = \left( 1 + \frac{(r + 1/n)^2}{(1 - 1/n)^2} - \frac{(r + 1/n)^2}{\mu \lambda} - \frac{\mu \lambda}{(1 - 1/n)^2} \right) h_n(\lambda, \mu)
$$

Each $k_{r,n}$ is a positive-semidefinite kernel on $A_{r,n} \times A_{r,n}$, simply a rescaled version of $k_{A_r}$. Denote by $H^2(A_{r,n})$ the corresponding Hilbert space of holomorphic functions on $A_{r,n}$.

Now, let $T \in \mathcal{F}_{r,n}$. By the appropriately rescaled version of Lemma 4.1, we obtain $\sigma(T) \subseteq \bar{A}_{r,n} \subseteq A_r$. Observe also that

$$
[r + (1/n)]^2 T^{-1} (T^{-1})^* + [1/(1 - n)]^2 TT^* \leq 1 + [(r + (1/n))/(1 - (1/n))]^2
$$

for every $x \in H$. Using our inequalities from Lemma 4.3, we obtain

$$
\frac{r^2}{r^2 + 1} ||(T^{-1})^* x||^2 + \frac{1}{r^2 + 1} ||T^* x||^2
\leq \frac{(r + 1/n)^2}{1 + \left( \frac{r + 1/n}{1 - 1/n} \right)^2 ||(T^{-1})^* x||^2 + \frac{1}{1 + \left( \frac{r + 1/n}{1 - 1/n} \right)^2} ||T^* x||^2}
$$

Thus, $r^2 T^{-1} (T^{-1})^* + TT^* \leq r^2 + 1$, which means that $T \in \mathcal{F}_r$. By our assumptions on $\phi$, we then obtain that $||\phi(T)|| \leq 1$.

To sum up, we have proved (for every $n > 2/(1 - r)$) that $\phi$, a function holomorphic on a neighborhood of $\bar{A}_{r,n}$, satisfies $||\phi(T)|| \leq 1$ for all $T \in \mathcal{B}(H)$ such that

$$
[r + (1/n)]^2 T^{-1} (T^{-1})^* + [1/(1 - n)]^2 TT^* \leq 1 + [(r + (1/n))/(1 - (1/n))]^2.
$$

The appropriately rescaled version of Lemma 4.2 now allows us to conclude that $||\phi||_{\mathcal{M}(H^2(A_{r,n}))} \leq 1$ and so there exists a positive semi-definite hereditary function $h_n : A_{r,n} \times A_{r,n} \to \mathbb{C}$ such that

$$
1 - \phi(\lambda) \overline{\phi(\mu)} = (1/k_{r,n}(\lambda, \mu)) h_n(\lambda, \mu)
$$

for all $\lambda, \mu \in A_{r,n}$ and for every $n > 2/(1 - r)$.

Let $K \subseteq A_r \times A_r$ be compact and fix $N > 2/(1 - r)$ large enough so that $K \subseteq A_{r,n} \times A_{r,n}$ for every $n \geq N$. Then, for every such $n$ and for every $\lambda \in K$ we have

$$
|h_n(\lambda, \lambda)| = h_n(\lambda, \lambda) = (1 - ||\phi(\lambda)||^2) k_{r,n}(\lambda, \lambda)
\leq \sup_{z \in K} \left( \frac{1}{(1 - (r + 1/n)^2/||z||^2) (1 - ||z||^2/(1 - 1/n)^2)} \right)
$$
\[
\leq \sup_{z \in K} \left[ (1 - |\phi(z)|^2) \frac{1}{(1 - (r + 1/N)^2/|z|^2)(1 - |z|^2/(1 - 1/N)^2)} \right] \\
= M < \infty,
\]
where \( M \) is independent of \( n \). Notice now that by Lemma 2.1 there exists (for every \( n \in \mathbb{N} \)) a function \( u_n : A_{r,n} \to l^2 \) with the property that
\[
h_n(\lambda, \mu) = \langle u_n(\lambda), u_n(\mu) \rangle_{l^2}, \text{ in } A_{r,n} \times A_{r,n}.
\]
Hence, using the Cauchy-Schwarz inequality and the bound (4.3) we can write
\[
|h_n(\lambda, \mu)|^2 \leq |h_n(\lambda, \lambda)||h_n(\mu, \mu)| \leq M^2,
\]
for every \( \lambda, \mu \in K \) and for every \( n \geq N \). In other words, the sequence of holomorphic functions \( \{(\lambda, \mu) \mapsto h_n(\lambda, \mu)\}_{n \geq N} \) is uniformly bounded on \( K \).
By Montel’s theorem and the completeness of \( \text{Her}(A_r) \), we can then deduce the existence of an element \( h \in \text{Her}(A_r) \) with the property that \( h_{n_k} \to h \) uniformly on compact subsets of \( A_r \times A_r \) for some subsequence \( \{n_k\} \). Since every \( h_{n_k} \) is positive semi-definite, the same must be true for \( h \) as well. Now, equality (4.2) combined with the convergence \( h_{n_k} \to h \) gives
\[
1 - \phi(\lambda)\overline{\phi(\mu)} = (1 + r^2 - r^2/\lambda\bar{\mu} - \lambda\bar{\mu})h(\lambda, \mu) \quad \text{on } A_r \times A_r,
\]
and so
\[
(1 - \phi(\lambda)\overline{\phi(\mu)})k_{A_r}(\lambda, \mu) \geq 0 \quad \text{on } A_r \times A_r.
\]
Thus, \( ||\phi||_{\text{Mult}(\mathcal{K}(A_r))} \leq 1 \) and our proof is complete. \( \square \)

Remark 2. The class \( \mathcal{F}_r \) was also considered in the recent preprint [3] of Bello, Yakubovich, where the authors obtained, with an alternative approach, that \( A_r \) is a complete \( \sqrt{r} \)-spectral set for every \( T \in \mathcal{F}_r \).

Remark 3. By Lemma 4.1, every \( T \in \mathcal{B}(H) \) such that \( r^2T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1 \) also satisfies \( r^2 \leq TT^* \leq 1 \). The converse assertion is not true, even if we restrict ourselves to \( 2 \times 2 \) matrices. Indeed, define \( A \in \mathcal{B}(\mathbb{C}^2) \) by
\[
A = \begin{bmatrix} \sqrt{T} & 1 - r \\ 0 & \sqrt{r} \end{bmatrix}.
\]
A short computations shows that \( ||A|| = ||rA^{-1}|| = 1 \). However, notice that
\[
\left\langle \begin{pmatrix} r^2 + 1 - r^2A^{-1}(A^{-1})^* - AA^* \\ 1/\sqrt{r} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{r} \\ 1/\sqrt{r} \end{pmatrix} \right\rangle = r^2(r + 1 - 1/r - (2 - 1/r)^2),
\]
which is negative for all \( r \in (0, 1) \).

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