THE STRUCTURE OF QUASI-COMPLETE INTERSECTION IDEALS

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ABSTRACT. We prove that every quasi-complete intersection (q.c.i.) ideal is obtained from a pair of nested complete intersection ideals by way of a flat base change. As a by-product we establish a rigidity statement for the minimal two-step Tate complex associated to an ideal \( I \) in a local ring \( R \). Furthermore, we define a minimal two-step complete Tate complex \( T \) for each ideal \( I \) in a local ring \( R \); and prove a rigidity result for it. The complex \( T \) is exact if and only if \( I \) is a q.c.i. ideal; and in this case, \( T \) is the minimal complete resolution of \( R/I \) by free \( R \)-modules.

1. INTRODUCTION.

Let \( R \to S \) be a homomorphism of commutative Noetherian rings. Quillen [14, 5.6] conjectured that if the André-Quillen homology functors \( D_i(S|R, -) \) vanish for all large \( i \), then they vanish for \( 3 \leq i \). We investigate the structure of ideals \( I \) in a local Noetherian ring \( R \) for which the natural quotient map \( R \to S = R/I \) satisfies the conclusion of the Quillen conjecture. Such ideals are called quasi-intersection (q.c.i.) ideals; see [3]. (Other equivalent definitions are given in Section 2.B.) The title of the present paper refers to Corollary 6.7 which states that every q.c.i. ideal in a local Noetherian ring is obtained from a pair of nested complete intersection ideals by way of a “flat base change”, in the sense of [3, 8.7].

Given a particular q.c.i. ideal \( I \), the proof of Corollary 6.7 for \( I \) involves creating a generic pair of nested complete intersection ideals for \( I \) in \( R \). As such, we view the two-step Tate complex for \( I \) as being obtained from the generic case by way of a base change. The generic case is the prototype of a module of finite complete intersection dimension (CI-dim). Theorems due to Jorgensen [10] and Avramov and Buchweitz [1] prove that if \( M \) is an \( R \)-module of finite CI-dim, then \( \operatorname{Tor}_i^R(M,N) \) can not vanish for many consecutive values of \( i \) unless it vanishes for all large values of \( i \). We use these Theorems to prove a strong rigidity result, Theorem 6.4, about the two-step Tate complex associated to an ideal. Corollary 7.1 recovers and extends a rigidity result of Jason Lutz [12] using apparently different methods. The rigidity results of Theorem 6.4 and Corollary 7.1 are prettiest when they are expressed in...
terms of Tate homology rather than ordinary homology. As a consequence, we give an explicit form for the complete resolution of each q.c.i. ideal.

It was not possible to explain the q.c.i. ideal $I$ of the ring $R$ in [11, Sect. 4] using any of the techniques that appeared in [11]. Indeed, this example seemed to indicate that q.c.i. ideals could be arbitrarily complicated. However, we show in Example 6.8.(c) that $I$ is obtained from a pair of nested complete intersection ideals $A \subseteq B$ in a local ring $\mathfrak{R}$ by way of a base change $\mathfrak{R}/A \to R$ and that $R$ has finite projective dimension as a module over $\mathfrak{R}/A$. We apply the rigidity portion of Theorem 6.4 in order to conclude that the base change is flat; hence, the unexplained example in [11] is accounted for by Corollary 6.7.

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2. NOTATION, CONVENTIONS, AND PRELIMINARY RESULTS.

2.A. Terminology.

2.1. When it is clear that “$R$” is the ambient ring, we use $(-)^*$ to mean $\text{Hom}_R(-, R)$.

2.1.1. “Let $(R, m, k)$ be a local ring” identifies $m$ as the unique maximal ideal of the commutative Noetherian local ring $R$ and $k$ as the residue class field $k = R/m$. If $M$ is a finitely generated $R$-module, we denote by $\mu(M)$ the minimal number of generators of $M$.

2.1.2. We use “im”, “ker”, “mult”, and “pd” as abbreviations for “image”, “kernel”, “multiplication”, and “projective dimension”, respectively.

2.1.3. If $M$ is a finitely generated module over a local ring $(R, m)$ and $\phi : M \to N$ is an $R$-module homomorphism, then $\phi$ is a minimal homomorphism if $\ker \phi \subseteq mM$. 

2.1.4. The grade of an ideal $I$ in a commutative Noetherian ring $R$ is the length of a maximal regular sequence on $R$ which is contained in $I$.

2.1.5. If $Y$ is a complex, then we use $Z_i(Y)$, $B_i(Y)$, and $H_i(Y)$ to represent the modules of $i$-cycles, $i$-boundaries, and $i$th-homology of $Y$, respectively. If $z$ is an $i$-cycle, then $\text{cls}(z)$ is the homology class of $z$ in $H_i$. In a similar manner, $Z^i$, $B^i$, and $H^i$ represent co-cycles, co-boundaries, and cohomology, respectively. A quasi-isomorphism is a homomorphism of complexes that induces an isomorphism on homology.

2.1.6. If $\Phi$ is a matrix (or a homomorphism of finitely generated free $R$-modules), then $I_r(\Phi)$ is the ideal generated by the $r \times r$ minors of $\Phi$ (or any matrix representation of $\Phi$).

2.1.7. Let $F$ be a free $R$-module of finite rank, $F^* = \text{Hom}_R(F, R)$, $ev : F \otimes F^* \to R$ be the evaluation map, and $ev^* : R \to F^* \otimes F$ be the dual of the evaluation map. If $(\{x_i\}, \{x_i^*\})$ is a pair of dual bases for $F$ and $F^*$, respectively, then

$$ev^*(1) = \sum_i x_i^* \otimes x_i.$$  

Of course, $\sum_i x_i^* \otimes x_i$ is a canonical element of $F^* \otimes F$.

2.1.8. If $F$ is a free $R$-module of finite rank $f$, then the exterior algebra $\bigwedge \cdot F$ is a module over the graded-commutative ring $\bigwedge \cdot F^*$. In particular, if $\xi \in F^*$, then $(\bigwedge \cdot F, \xi)$ is the Koszul complex

$$0 \to \bigwedge^1 F \xrightarrow{\xi} \bigwedge^{-1} F \xrightarrow{\xi} \cdots \xrightarrow{\xi} \bigwedge^2 F \xrightarrow{\xi} \bigwedge^1 F \xrightarrow{\xi} R.$$  

We refer to $(\bigwedge \cdot F, \xi)$ as the Koszul complex associated to $\xi$. In a similar manner, the divided power algebra $D_\bullet F^*$ is a module over the polynomial ring $\text{Sym}_\bullet F$.

2.1.9. If $I$ is an ideal in a local ring $R$ and $\xi : F \to R$ is a minimal homomorphism with $\text{im} \xi = I$, then $\mu(H_1(\bigwedge \cdot F, \xi))$ does not depend on $\xi$ ([7, 1.6.21]) and we denote this number by $\mu(K_1(I))$. We refer to this number as “the minimal number of generators of the first Koszul homology module associated to a minimal generating set for $I$.”

2.B. Quasi-complete intersections and the two-step Tate Complex.

**Data 2.2.** Let $I$ be an ideal in a local ring $(R, m)$, $F$ be a free $R$-module of finite rank, $\xi : F \to R$ be an $R$-module homomorphism with the image of $\xi$ equal to $I$, and $\bigwedge \cdot F$ be the Koszul complex associated to $\xi$.

**Definition 2.2.1.** In the setup of 2.2, the ideal $I$ is a quasi-complete intersection (q.c.i.) if

(a) $H_1(\bigwedge \cdot F)$ is a free $R/I$-module and
(b) the natural map $\bigwedge \cdot (H_1(\bigwedge \cdot F)) \to H_\bullet(\bigwedge \cdot F)$ is an isomorphism of graded $R/I$-algebras.
Remarks 2.2.2. (a) The defining conditions 2.2.1.(a) and 2.2.1.(b) of q.c.i. ideals do not depend on the choice of presentation for \( R/I \) because the ambient ring \( R \) is local; see, for example, [7, 1.6.21].
(b) The transition from q.c.i. ideals as defined in the introduction to q.c.i. ideals as defined in 2.2.1 is contained in [3, 8.5].
(c) Ideals which satisfy properties 2.2.1.(a) and 2.2.1.(b) were first studied in [15]; they were named \textit{ideals with free exterior homology} in [5].

The two-step Tate complex detects q.c.i. ideals.

Definition 2.2.3. Adopt the setup of 2.2. Let \( G \) be a free \( R \)-module and \( \phi: G \to Z_1(\bigwedge F) \) be an \( R \)-module homomorphism with the property that the composition
\[
G \xrightarrow{\phi} Z_1(\bigwedge F) \xrightarrow{\text{natural quotient map}} H_1(\bigwedge F)
\]
is a minimal surjection. Then the Divided Power Algebra
\[
P = \langle \bigwedge^* F \otimes_R D_* G, \partial \rangle,
\]
where the restriction of the differential \( \partial \) to \( F \) is given by \( \xi \) and the restriction of \( \partial \) to \( G \) is given by \( \phi \), is called the \textit{two-step Tate complex} associated to the data \((\xi, \phi)\).

Remark 2.2.4. A coordinate-dependent formulation of the two-step Tate complex may be found in [3, 1.5] and many other places. In this alternate language, \( P \) is called a “Tate construction” and is written
\[
R\langle v_1, \ldots, v_r; w_1, \ldots, w_g \rangle,
\]
where the exterior variables \( v_1, \ldots, v_r \) are a basis for \( F \) and the divided power variables \( w_1, \ldots, w_g \) are a basis for \( G \).

Proposition 2.2.5. ([6, Thm. 1]) In the language of Definition 2.2.3, the two-step Tate complex associated to the data \((\xi, \phi)\) is acyclic if and only if \( I \) is a q.c.i. ideal.

Remark. Observe that in Proposition 2.2.5 the homomorphism \( \xi \) need not be minimal, but the homomorphism \( \phi \) must be minimal. On the other hand, if \( \xi \) is a minimal homomorphism and the conditions of 2.2.5 hold, then the two-step Tate complex associated to the data \((\xi, \phi)\) is a minimal resolution of \( R/I \) by free \( R \)-modules.

Proposition 2.2.6. Adopt the setup of Definition 2.2.1. Let \( \bar{x} = x_1, \ldots, x_r \) be the beginning of a minimal generating set for \( I \) which is also a regular sequence on \( R \), \( I' = I/(\bar{x}) \), and \( R' = R/(\bar{x}) \).
(a) Natural data \((\xi', \phi')\) for \( I' \) in \( R' \) can be constructed from the given data
\[
(\xi: F \to R, \phi: G_1 \to Z_1(\bigwedge^* F))
\]
for \( I \) in \( R \).
There is a quasi-isomorphism from the two-step Tate complex associated to the data \((\xi, \phi)\) to the two-step Tate complex associated to the data \((\xi', \phi')\).

**Proof.** (a) Let \(X\) be a free summand of \(F\) with \(\xi(X) = (x)\). Define
\[
F' = (F/X) \otimes_R R',
\]
\(\xi' : F' \rightarrow R'\) to be the map induced by
\[
F \otimes_R R' \xrightarrow{\xi \otimes_R 1} R \otimes_R R',
\]
\(G' = G \otimes_R R',\) and \(\phi' : G' \rightarrow F'\) to be the composition
\[
G' = G \otimes_R R' \xrightarrow{\phi \otimes 1} F \otimes_R R' \xrightarrow{\text{natural quotient map}} F/X \otimes_R R' = F'.
\]
The proof of [3, Lem 1.3] shows that the natural map of Koszul complexes
(2.2.7)
\[
(\bigwedge^* F, \xi) \rightarrow (\bigwedge^* F', \xi')
\]
is a quasi-isomorphism. It follows, in particular, that the composition
\[
G' \xrightarrow{\phi'} Z_1(\bigwedge^* F') \xrightarrow{\text{natural quotient map}} H_1(\bigwedge^* F')
\]
is a minimal surjection.

(b) The quasi-isomorphism (2.2.7) can be extended to a quasi-isomorphism of the two-step Tate complexes by [9, 1.3.5].

**2.C. Complete resolutions and Tate homology.**

**2.3.** Let \(R\) be a commutative Noetherian ring, \((-)^*\) represent \(\text{Hom}_R(-, R)\), and \(M\) be a finitely generated \(R\)-module.

**2.3.1.** A complete resolution of \(M\) is a complex \(T\) of finitely generated projective \(R\)-modules, such that \(H_i(T) = 0 = H^i(T^*)\) for all integers \(i\), and \(T_{\geq r} = F_{\geq r}\) for some projective resolution \(F\) of \(M\) and some integer \(r\). If \(M\) has complete resolutions, then any two of them are homotopy equivalent, see, for example [8, Lem. 2.4]. In particular, in this case, the modules \(\widetilde{\text{Tor}}_i^R(M, N) = H_i(T \otimes_R N)\) are well defined for all \(R\)-modules \(N\); we refer to these modules as Tate Tor modules.

**2.3.2.** The module \(M\) is totally reflexive if \(M \cong M^{**}\) and
\[
\text{Ext}_R^i(M, R) = \text{Ext}_R^i(M^*, R) = 0,
\]
for all positive \(i\).

**2.3.3.** If the module \(M\) is not zero, then the \(G\)-dimension of \(M\) is the length of the shortest resolution of \(M\) by totally reflexive \(R\)-modules.

**2.3.4.** The module \(M\) has a complete resolution if and only if the \(G\)-dimension of \(M\) is finite; see, for example, [1, 4.4.4].
2.3.5. If the ring \((R, \mathfrak{m})\) is local and the complete resolution \((T, d)\) of \(M\) satisfies \(d(T) \subseteq \mathfrak{m}T\), then \(T\) is a minimal complete resolution of \(M\). It is shown in [4, Thm. 8.4] that any two minimal complete resolutions of \(M\) are isomorphic.

2.D. Complete intersection dimension.

2.4. A quasi-deformation (of codimension \(c\)) of a local ring \(R\) is a diagram of local homomorphisms \(R \to R' \leftarrow Q\), in which the left-most map is faithfully flat and the right-most map is surjective with kernel generated by a regular sequence on \(Q\) (of length \(c\)).

2.4.1. Let \(M\) be a non-zero finitely generated module over a Noetherian ring \(R\). If \(R\) is local, then

\[
\text{CI-dim}_R M = \inf \{ \text{pd}_Q(M \otimes_R R') - \text{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation} \};
\]

in general, the complete intersection dimension of \(M\) over \(R\) is defined by

\[
\text{CI-dim}_R M = \sup \{ \text{CI-dim}_R M_\mathfrak{m} \mid \mathfrak{m} \in \text{Max}(R) \} \quad \text{and} \quad \text{CI-dim}_R 0 = 0,
\]

where \(\text{pd}\) means “projective dimension” and \(\text{Max}\) means “maximal spectrum”.

2.4.2. It is shown in [2, Thm. 1.4] that if \(M\) is a finitely generated module \(M\) over a Noetherian ring \(R\) then

\[
\text{G-dim}_R M \leq \text{CI-dim}_R M \leq \text{pd}_R M.
\]

Furthermore, if any of these dimensions is finite, then this dimension is equal to all dimensions to its left. Also, if \(R\) is local and \(\text{CI-dim}_R M < \infty\), then

\[
\text{CI-dim}_R M = \text{depth} R - \text{depth}_R M.
\]

2.E. Complexity.

2.5. Let \(M\) be a finitely generated module over the local ring \((R, \mathfrak{m}, k)\). The complexity of \(M\) is equal to

\[
\text{cx}_R M = \inf \left\{ \text{non-negative integers } d \left| \begin{array}{l}
\text{there exists a positive real number } \gamma \\
\text{with } b^R_i(M) \leq \gamma^{d-1} \text{ for } 0 \ll i
\end{array} \right. \right\},
\]

where \(b^R_i(M)\) is the \(i^{th}\)-Betti number \(\dim_k \text{Tor}_R^i(k, M)\) of the \(R\)-module \(M\). If the CI-dimension of \(M\) is finite, then [2, Thm. 5.3] proves that the complexity of \(M\) is finite and is equal to the order of the pole at \(t = 1\) of the Poincaré series

\[
P^R_M(t) = \sum_{i=0}^{\infty} b^R_i(M) t^i.
\]
2.F. The vanishing theorem for homology of modules of finite CI-dimension. The following theorem of Avramov and Buchweitz plays a central role in this paper. It should be noted that the hypothesis that $M$ has finite CI-dimension guarantees that $M$ has finite G-dimension (2.4.2) and that the Tate homology modules $\hat{\Tor}_i(M, -)$ are defined (2.3.4) and (2.3.1). A version of the equivalence of the first three conditions was shown by Jorgensen [10, Thm. 2.1]. The final three conditions are much less fussy than the first three conditions; consequently, they serve as an advertisement for Tate homology.

**Theorem 2.6.** [1, Thm. 4.9] If $R$ is a Noetherian ring, and $M$ is a finitely generated $R$-module of finite CI-dimension, then for each $R$-module $N$ the following conditions are equivalent:

(i) $\Tor^R_i(M, N) = 0$ for $\text{cx}_R M + 1$ consecutive values of $i$ provided each of these values $i$ satisfies $\text{CI-dim}_R M < i$;

(ii) $\Tor^R_i(M, N) = 0$ for $0 \ll i$;

(iii) $\Tor^R_i(M, N) = 0$ for all $i$ with $\text{CI-dim}_R M < i$;

(iv) $\hat{\Tor}^R_i(M, N) = 0$ for $\text{cx}_R M + 1$ consecutive values of $i$;

(v) $\hat{\Tor}^R_i(M, N) = 0$ for $i \ll 0$; and

(vi) $\hat{\Tor}^R_i(M, N) = 0$ for all integers $i$.

3. The two-step complete Tate complex associated to an ideal in a local ring.

Let $I$ be an ideal in a local ring $R$ with $\mu(K_1(I)) \leq \mu(I)$. In 3.2 we define the minimal two-step complete Tate complex

$$T: \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow T_{-1} \rightarrow \cdots$$

for $I$ in $R$. In Corollary 7.1 we prove that $T$ is exact if and only if $I$ is a q.c.i.; furthermore, in this case, $T$ is the minimal complete resolution of $R/I$ by free $R$-modules.

**Data 3.1.** Let $(R, m)$ be a local ring, $I$ be a proper ideal of $R$ which is minimally generated by $\bar{f}$ elements, $F$ be a free $R$-module of rank $\bar{f}$, and $\check{\xi}: F \rightarrow R$ be an $R$-module homomorphism with $\text{im}(\check{\xi}) = I$. Let $\wedge^\bullet F$ be the Koszul complex associated to $\check{\xi}$, $\bar{g}$ be the minimal number of generators of $H_1(\wedge^\bullet F)$, $G$ be a free $R$-module of rank $\bar{g}$, and

$$\phi: G \rightarrow Z_1(\wedge^\bullet F)$$

be an $R$-module homomorphism with the property that the composition

$$G \xrightarrow{\phi} Z_1(\wedge^\bullet F) \xrightarrow{\text{natural quotient map}} H_1(\wedge^\bullet F)$$

(3.1.1) is a surjection.
Remark 3.1.2. We emphasize that the parameters $\xi$ and $\eta$ of Data 3.1 are equal to $\mu(I)$ and $\mu(K_1(I))$, respectively; and that the homomorphisms $\xi : F \to R$ and $G \to H_1(\wedge^g F)$ of (3.1.1) are minimal homomorphisms.

Definition 3.2. Adopt the data of 3.1. Let $(\cdot)^\vee$ be the functor $\operatorname{Hom}_R(\cdot, \wedge^l F)$ and

$$P = (\wedge^* F \otimes_R D_* G, \partial),$$

be the two-step Tate complex associated to the data $(\xi, \phi)$ of 2.2.3. For $0 \leq i \leq f - g$, define an $R$-module homomorphism

$$\alpha_i : P_i \otimes_R \wedge^g G \to (P_{f - g - i})^\vee$$
as follows. Every component of $\alpha_i$ is zero except the component

$$\wedge^i F \otimes_R \wedge^g G \to (\wedge^{f-g-i} F)^\vee;$$

and, if $\theta_i \in \wedge^i F$ and $\omega_g \in \wedge^g G$, then

$$\alpha_i(\theta_i \otimes \omega_g) : \wedge^{f-g-i} F \to \wedge^i F$$
is the homomorphism which sends $\theta_{f-g-i} \in \wedge^{f-g-i} F$ to

$$(-1)^{\frac{i(i-1)}{2} + ig} \theta_i \wedge (\wedge^g \phi) \omega_g \wedge \theta_{f-g-i} \in \wedge^i F.$$

Let $\alpha : P \otimes_R \wedge^g G \to P^\vee \left[ g - f \right]$ represent the picture

$$\cdots \xrightarrow{\partial_{i-g+1}} P_{f-g} \otimes_R \wedge^g G \xrightarrow{\partial_{i-g}} \cdots \xrightarrow{\partial_2} P_1 \otimes_R \wedge^g G \xrightarrow{\partial_1} P_0 \otimes_R \wedge^g G \xrightarrow{\partial_0} 0$$

and let $T$ be the (formal) mapping cone of the picture $\alpha$. In other words, $T$ is the collection of maps

$$T : \cdots \xrightarrow{\tau_2} T_1 \xrightarrow{\tau_1} T_0 \xrightarrow{\tau_0} T_{-1} \xrightarrow{\tau_{-1}} \cdots,$$

where

$$T_i = \begin{cases} P_i \otimes_R \wedge^g G & \text{and} \quad \tau_i = \begin{bmatrix} \partial_i & 0 \\ \alpha_i & -\partial_{i-g-i} \end{bmatrix}, \\ 0 & \end{cases}$$

for each integer $i$.

Remark 3.2.1. It is not difficult to see that $\alpha$ is a map of complexes and that $\operatorname{im} \tau_i$ is contained in $mT_{i-1}$ for each $i$. It follows that $T$ is a complex of free $R$-modules. We call $T$ the minimal two-step complete Tate complex associated to the data $(\xi, \phi)$. 
4. THE COMPLETE RESOLUTION ASSOCIATED TO A PAIR OF NESTED COMPLETE INTERSECTIONS.

The main result in this section is Proposition 4.3. We prove that if \( A \subseteq B \) is a pair of complete intersection ideals in the local ring \((R, \mathfrak{M})\), with \( A \subseteq \mathfrak{M}B \), then the “minimal two-step complete Tate complex” of Section 3, for the ideal \( B/A \) of the ring \( R/A \), is exact.

**Data 4.1.** Let \((R, \mathfrak{M})\) be a local ring and \( A \subseteq B \) be ideals in \( R \), with \( A \subseteq \mathfrak{M}B \). Assume that each of the ideals \( A \) and \( B \) is generated by a regular sequence. Let \( g = \text{grade} A \), \( f = \text{grade} B \), \( \mathfrak{F} \) and \( \mathfrak{G} \) be free \( R \)-modules of rank \( f \) and \( g \), respectively, and \( \Xi : \mathfrak{F} \to R \) and \( \Phi : \mathfrak{G} \to \mathfrak{F} \) be \( R \)-module homomorphisms with \( \text{im} \Xi = B \) and \( \text{im}(\Xi \circ \Phi) = A \). Let \( \eta \) represent the functor \((R/A) \otimes R - \) and \((-)^{\vee} \) represent the functor \( \text{Hom}_{R}(-, \Lambda^{1} \mathfrak{F}) \).

**4.2.** Adopt Data 4.1. Let \( \mathfrak{P} \) be the two-step Tate complex associated to \((\Xi, \Phi)\) as given in 2.2.3. Tate [16, Thm. 4] proved that \( \mathfrak{P} \) is a minimal resolution of \( R/B \) by free \( R/A \)-modules. Define the map of complexes \( \alpha : \mathfrak{P} \otimes_{R} \Lambda^{g} \mathfrak{G} \to \mathfrak{P}^{\vee} \mathfrak{G}[g - f] \):

\[
\cdots \to \mathfrak{P}_{f-g} \otimes_{R} \Lambda^{g} \mathfrak{G} \xrightarrow{\text{\( \partial_{1}\)}} \cdots \to \mathfrak{P}_{1} \otimes_{R} \Lambda^{g} \mathfrak{G} \xrightarrow{\text{\( \partial_{1}\)}} \mathfrak{P}_{0} \otimes_{R} \Lambda^{g} \mathfrak{G} \xrightarrow{\text{\( \partial_{1}\)}} 0
\]

\[
0 \xrightarrow{\text{\( \partial_{1}\)}} \mathfrak{P}_{0}^{\vee} \xrightarrow{\text{\( \partial_{1}\)}} \cdots \to \mathfrak{P}_{f-g-1}^{\vee} \xrightarrow{\text{\( \partial_{1}\)}} \mathfrak{P}_{f-g-2}^{\vee} \xrightarrow{\text{\( \partial_{1}\)}} \cdots
\]

exactly as was done in 3.2. The mapping cone \( \mathfrak{C} \) of \( \alpha \) is the minimal two-step complete Tate complex associated to the data \((\Xi, \Phi)\) as described in 3.2.

**Remark 4.2.1.** The data of Section 4 is analogous to the data of Section 3 in the sense of 3.1.2. In particular, \( \mu(B/A) = f, \mu(K_{1}(B/A)) = g \), and the homomorphisms \( \Xi : \mathfrak{F} \to R \) and \( \Phi : \mathfrak{G} \to H_{1}(\Lambda^{\bullet} \mathfrak{F}) \) are minimal. The inclusion \( A \subseteq B \) guarantees that \( g \leq f \) because \( \text{grade} A \) and \( \text{grade} B \) are also equal to \( g \) and \( f \), respectively.

**Proposition 4.3.** Adopt the data of 4.1. Then the complex \( \mathfrak{C} \) of 4.2 is the minimal complete resolution of \( R/B \) by free \( R \)-modules.

Definition 4.4 is used in our proof of Proposition 4.3.

**Definition 4.4.** Retain the notation of 4.1 and 4.2. Let \( \eta \in (\Lambda^{g} \mathfrak{G} \otimes_{R} \mathfrak{P}_{f-g})^{\vee} \) be the following homomorphism. The restriction of \( \eta \) to

\[
\sum_{p+2q+1 \leq g} \Lambda^{p} \mathfrak{G} \otimes_{R} \Lambda^{q} \mathfrak{F} \otimes_{R} D_{q} \mathfrak{G} \to \Lambda^{1} \mathfrak{F}
\]

is equal to zero and the restriction \( \eta \) to \( \Lambda^{0} \mathfrak{G} \otimes_{R} \Lambda^{f-g} \mathfrak{F} \otimes_{R} D_{0} \mathfrak{G} \) is given by

\[
\eta(\omega_{g} \otimes \theta_{f-g}) = (\Lambda^{0} \Phi)(\omega_{g}) \wedge \theta_{f-g},
\]

for \( \omega_{g} \in \Lambda^{g} \mathfrak{G} \) and \( \theta_{f-g} \in \Lambda^{f-g} \mathfrak{F} \).
Proof of Proposition 4.3. Fix a generator $\omega_\theta$ of $\wedge^s F$. Observe that

$$\omega_\theta = \eta(\omega_\theta \otimes -) \in (\mathfrak{P}^{i-g})^\vee;$$

furthermore, $\eta(\omega_\theta \otimes -)$ is a co-cycle in the complex $\mathfrak{P}^\vee$. It suffices to prove that

$$(4.4.1) \quad H^{f-g}(\mathfrak{P}^\vee) \text{ is generated by } \eta(\omega_\theta \otimes -).$$

Indeed, it is shown in [3, 2.5(4)] that

$$H^i(\mathfrak{P}^\vee) \simeq \begin{cases} 0 & \text{if } i \neq f-g, \\ \mathfrak{R}/B & \text{if } i = f-g. \end{cases}$$

Thus, once (4.4.1) is established, then the map of complexes of 4.2 is a quasi-isomorphism and the proof is complete. We prove (4.4.1). Let $c_{i-g}$ be a co-cycle in $Z^{f-g}(\mathfrak{P}^\vee)$. It follows that

$$c_{i-g} = \sum_{2p+q = i-g} \zeta_{p,q},$$

with $\zeta_{p,q} \in (D_p \mathfrak{F} \otimes \mathfrak{R} \wedge q F)^\vee$, and

$$\Phi^\vee(\zeta_{p,q}) + \Xi^\vee(\zeta_{p+1,q-2}) = 0 \quad \text{in} \quad (D_{p+1} \mathfrak{F} \otimes \mathfrak{R} \wedge q-1 F)^\vee,$$

for all $(p,q)$ with $2p+q = i-g$.

We first show that $\zeta_{0,i-g}$ is a scalar multiple of $\eta(\omega_\theta \otimes -)$.

The natural quotient map $\mathfrak{R}/A \to \mathfrak{R}/B$ is a q.c.i. homomorphism; see [3, 1.4]. Hence, the $\mathfrak{R}$-module homomorphism $\Phi: \mathfrak{F} \to \mathfrak{F}$ induces an isomorphism of exterior algebras $\wedge^\bullet_{\mathfrak{R}/B}(\mathfrak{F} \otimes \mathfrak{R}/B) \to H_\bullet(\wedge^\bullet_{\mathfrak{R}/B} F)$. In particular, the class of the cycle $(\wedge^0 \Phi)(\omega_\theta)$ in $Z_\theta(\wedge^\bullet_{\mathfrak{R}} F)$ generates the homology module $H_\theta(\wedge^\bullet_{\mathfrak{R}} F)$. On the other hand, there is an isomorphism of complexes

$$(\wedge^\bullet_{\mathfrak{R}} F, \mathfrak{E}) \xrightarrow{\sim} ( (\wedge^\bullet_{\mathfrak{F}})^\vee, \mathfrak{E}^\vee)[ -f],$$

which is induced by the map which sends $\theta_i$ in $\wedge_i^\bullet F$ to $\pm \theta_i \wedge - (\wedge^\bullet_{\mathfrak{F}})^\vee$. It follows that

$$(4.4.2) \quad H^i(\mathfrak{P}^\vee) = 0 \quad \text{for } 0 \leq i \leq f-g-1,$$

and the cohomology class of the co-cycle $(\wedge^0 \Phi)(\omega_\theta) \wedge -$ in $Z^{f-g}(\wedge^\bullet_{\mathfrak{F}})^\vee$ generates the cohomology module $H^{f-g}(\wedge^\bullet_{\mathfrak{F}})^\vee$. Thus, $\zeta_{0,i-g} = \lambda \eta(\omega_\theta \otimes -)$, for some $\lambda \in \mathfrak{R}$. We prove that $c_{i-g} = \lambda \eta(\omega_\theta \otimes -)$, which is equal to

$$\sum_{2p+q = i-g}^{\lambda \eta(\omega_\theta \otimes -)} \zeta_{p,q},$$

is a boundary in $(\mathfrak{P}^\vee)_{i-g}$.

Let $p_0$ be the smallest index with $\zeta_{p_0,q_0} \neq 0$. Observe that $\zeta_{p_0,q_0}$ is a co-cycle in the complex

$$\begin{align*}
(D_{p_0} \mathfrak{F} \otimes \mathfrak{R} \wedge q F_{p_0}^{-1} F)^\vee & \xrightarrow{\mathfrak{E}^\vee} (D_{p_0} \mathfrak{F} \otimes \mathfrak{R} \wedge q F_{p_0}^0 F)^\vee \xrightarrow{\mathfrak{E}^\vee} (D_{p_0} \mathfrak{F} \otimes \mathfrak{R} \wedge q F_{p_0}^{q_0-1} F)^\vee.
\end{align*}$$
This complex is exact because $H_i(P^\vee) = 0$ for $0 \leq i \leq \ell - g - 1$ (see, for example, (4.4.2)) and $q_0 = \ell - g - 2p_0 \leq \ell - g - 2$. Thus, $c_{\ell-g} - \lambda(g \otimes -)$ is congruent, mod $B^{\ell-g}(P^\vee)$, to

$$
\sum_{2p+q = \ell-g} \zeta_{p,q}^{'}
$$

for some $\zeta_{p,q}^{'} \in (D_p \overline{G} \otimes \wedge^q \overline{G})^\vee$. Iterate this procedure to conclude that $c_{\ell-g}$ and $\lambda\eta(g \otimes -)$ represent the same class in $H^{\ell-g}(P^\vee)$. This completes the proof of (4.4.1). □

5. THE GENERIC TATE CONSTRUCTION.

Given an (almost arbitrary) ideal $I$ in a local ring $R$, we produce a generic pair of nested complete intersection ideals that can be used in Theorem 6.4 to determine if $I$ is a q.c.i.. There is a small restriction imposed on $I$; it must satisfy

$$
\mu(K_1(I)) \leq \mu(I),
$$

where $K_1(I)$ is the first Koszul homology associated to a minimal generating set for $I$, as described in 2.1.9. This hypothesis is benign, in the situation of interest, because if $I$ is a q.c.i., then

$$
\text{grade } I = \mu(I) - \mu(K_1(I));
$$

see [3, Lem. 1.2], and of course grade $I$ is always non-negative.

Data 5.1. Let $(R, m)$ be a local ring, $I$ be a proper ideal of $R$ which is minimally generated by $\ell$ elements, $F$ be a free $R$-module of rank $\ell$, and $\xi : F \rightarrow R$ be a $R$-module homomorphism with $\text{im}(\xi) = I$. Let $\wedge^\bullet F$ be the Koszul complex associated to $\xi$, $g$ be the minimal number of generators of $H_1(\wedge^\bullet F)$, $G$ be a free $R$-module of rank $g$, and

$$
\phi : G \rightarrow Z_1(\wedge^\bullet F)
$$

be an $R$-module homomorphism with the property that the composition

$$
G \xrightarrow{\phi} Z_1(\wedge^\bullet F) \xrightarrow{\text{natural quotient map}} H_1(\wedge^\bullet F)
$$

is a surjection. Assume $g \leq \ell$.

Construction 5.2. Begin with the data of 5.1. Consider the polynomial ring

$$
\text{Sym}_R^\bullet(X_1 \oplus X_2),
$$

where $X_1$ and $X_2$ are the free $R$-modules

$$
X_1 = F \quad \text{and} \quad X_2 = F^* \otimes_R G.
$$

Let $\mathfrak{M}$ be the maximal ideal

$$
\mathfrak{M} = m + \sum_{1 \leq i} \text{Sym}_R^i(X_1 \oplus X_2)
$$
of $\operatorname{Sym}^{R}(X_{1} \oplus X_{2}); \tilde{R}$ be the local ring $(\operatorname{Sym}^{R}(X_{1} \oplus X_{2}))_{M};$ and $\sim$ be the functor $- \otimes_{R} \tilde{R}$. Define
\[
\Xi : \tilde{F} \to \tilde{R} \quad \text{and} \quad \Phi : \tilde{G} \to \tilde{F}
\]
to be the compositions
\[
\tilde{F} = F \otimes_{R} \tilde{R} = X_{1} \otimes_{R} \tilde{R} \xrightarrow{\text{mult}} \tilde{R}
\]
and
\[
\tilde{G} = G \otimes_{R} \tilde{R} \xrightarrow{\text{ev} \otimes 1 \otimes 1} F \otimes F^{*} \otimes G \otimes \tilde{R} = F \otimes X_{2} \otimes \tilde{R} \xrightarrow{1 \otimes \text{mult}} F \otimes \tilde{R} = \tilde{F},
\]
respectively. Define $\rho : \tilde{R} \to R$ to be the $R$-algebra homomorphism induced by
\[
\rho(\theta_{1}) = \xi(\theta_{1}), \quad \text{for } \theta_{1} \in F = X_{1}, \text{ and}
\]
\[
\rho(\Theta_{1} \otimes g) = (\Theta_{1} \circ \phi)(g), \quad \text{for } \Theta_{1} \otimes g \in F^{*} \otimes G = X_{2}.
\]

Remark. Observe that $\rho(M) \subseteq m$. Indeed,
\[
\rho(X_{1}) \subseteq \text{im}(\xi) \subseteq I \subseteq m \quad \text{and} \quad \rho(X_{2}) \subseteq I_{1}(\Lambda^{*} F) \subseteq m.
\]
The final inclusion holds because the rank of $F$ is the minimal number of generators of $I$; hence, $\xi : F \to R$ is a minimal homomorphism.

Proposition 5.3. Given the data of 5.1, apply Construction 5.2 to produce $\tilde{R}, \Xi, \Phi,$ and $\rho$. Let $B$ be the image of $\Xi$ in $\tilde{R}$ and $A$ be the image of $\Xi \circ \Phi$ in $\tilde{R}$. The following statements hold:

(a) $B$ is generated by a regular sequence on $\tilde{R}$ of length $f$,
(b) $A$ is generated by a regular sequence on $\tilde{R}$ of length $g$,
(c) $BR = \text{im} \xi$,
(d) $AR = 0$,
(e) the $\tilde{R}$-module homomorphisms $\Xi : \tilde{F} \to \tilde{R}$ and $\Phi : \tilde{G} \to \tilde{F}$ are minimal,
(f) the $R$-module homomorphism $\Xi \otimes 1 : \tilde{F} \otimes_{\tilde{R}} R \to \tilde{R} \otimes_{\tilde{R}} R$ is equal to $\xi : F \to R$,
and
(g) the $R$-module homomorphism $\Phi \otimes 1 : \tilde{G} \otimes_{\tilde{R}} R \to \tilde{F} \otimes_{\tilde{R}} R$ is equal to $\phi : G \to F$.

Remark. The homomorphism $\rho$ makes $R$ an $\tilde{R}$-algebra; this $\tilde{R}$-algebra structure on $R$ is used in (c), (d), (f), and (g).

Proof. (a) The ideal $B$ is generated by $f$ distinct indeterminates; these generators form a regular sequence.

(b) The ideal $A$ is generated by the entries of the product $bc$, where $b$ and $c$ are matrices of distinct indeterminates, $b$ has shape $1 \times f$, $c$ has shape $f \times g$, and $g \leq f$. These generators form a regular sequence; see, for example, [13, 6.13].

(c) The ideal $B$ of $\tilde{R}$ is generated by $\Xi(F) = X_{1}$; so, $\rho(B)$ is generated by
\[
(\rho \circ \Xi)(\tilde{F}) = \xi(F).
\]
(d) Let \( \{x_i\}, \{x^*_i\} \) be a pair of dual bases for \( F \) and \( F^* \), respectively. The ideal \( A \) of \( \tilde{R} \) is generated by
\[
\left\{ \sum_i x_i(x^*_i \otimes g) \in X_1 \cdot X_2 \subseteq \tilde{R} \mid g \in G \right\}.
\]

It follows that \( \rho(A) \) is generated by
\[
\sum_i \xi(x_i) \cdot x^*_i(\phi(g)) = (\xi \circ \phi)(g) \in \xi(Z_1(\wedge^\bullet F, \xi)) = 0.
\]

(e) If one expresses either of these maps as a matrix, then the entries of this matrix form a regular sequence. It follows that the kernel of the map is in \( \mathfrak{M} \).

(f) and (g) The composition
\[
R \xrightarrow{\text{inclusion}} \tilde{R} \xrightarrow{\rho} R
\]
is the identity map. It follows that \( \tilde{R} \otimes_R R \cong R \). The assertions are now obvious. \( \square \)

6. THE MAIN THEOREM.

Data 6.1 has 3 parts. Part (a) concerns an ideal \( I \) in a local ring \( R \); this part of the data is exactly the same as Data 3.1, except that hypothesis 6.1.2 has now been added. Part (b) is about a pair of nested complete intersection ideals \( A \subseteq B \) in a local ring \( \mathfrak{M} \). Finally, part (c) is about a surjection \( \rho : \mathfrak{M} \to R \) which carries \( A \) to 0 and \( B \) to \( I \). Proposition 5.3 guarantees that for every ideal \( I \) which satisfies the hypotheses of (a), the rest of Data 6.1 can be created generically. The subsequent results in the paper may be applied to the generic data built in Proposition 5.3 or any other data which satisfies the hypotheses of Data 6.1.

**Data 6.1.** (a) Let \( (R, \mathfrak{m}) \) be a local ring, \( I \) be a proper ideal of \( R \) which is minimally generated by \( \ell \) elements, \( F \) be a free \( R \)-module of rank \( \ell \), and \( \xi : F \to R \) be an \( R \)-module homomorphism with \( \text{im}(\xi) = I \). Let \( \wedge^\bullet F \) be the Koszul complex associated to \( \xi \), \( g \) be the minimal number of generators of \( H_1(\wedge^\bullet F) \), \( G \) be a free \( R \)-module of rank \( g \), and
\[
\phi : G \to Z_1(\wedge^\bullet F)
\]
be an \( R \)-module homomorphism with the property that the composition
\[
G \xrightarrow{\phi} Z_1(\wedge^\bullet F) \xrightarrow{\text{natural quotient map}} H_1(\wedge^\bullet F)
\]
is a surjection. Assume
\[
0 \leq \ell - g \leq \text{grade} I.
\]

(b) Let \( \mathfrak{M} \) be a local ring and \( A \subseteq B \) be ideals in \( \mathfrak{M} \), each of which is generated by a regular sequence. Let \( g = \text{grade} A \), \( \ell = \text{grade} B \), \( \mathfrak{F} \) and \( \mathfrak{G} \) be free \( \mathfrak{M} \)-modules of rank \( \ell \) and \( g \), respectively, and \( \Xi : \mathfrak{F} \to \mathfrak{M} \) and \( \Phi : \mathfrak{G} \to \mathfrak{F} \) be minimal \( \mathfrak{M} \)-module homomorphisms with \( \text{im} \Xi = B \) and \( \text{im}(\Xi \circ \Phi) = A \). Let \( \mathfrak{P} \) and \( \Xi \) be the Tate
resolution and the complete Tate resolution of $\mathcal{R}/B$ by free $\mathcal{R}/A$-modules as described in 4.2 and 4.3.

(c) Let $\rho: \mathcal{R} \rightarrow R$ be a surjective ring homomorphism with $A \subseteq \ker \rho$. Assume that $\mathfrak{F} \otimes_{\mathcal{R}} R = F$, $\mathfrak{G} \otimes_{\mathcal{R}} R = G$, the composition

$$F = \mathfrak{F} \otimes_{\mathcal{R}} R \xrightarrow{\xi} \mathcal{R} \otimes_{\mathcal{R}} R = R$$

is $\xi$ and the composition

$$G = \mathfrak{G} \otimes_{\mathcal{R}} R \xrightarrow{\Phi \otimes_{\mathcal{R}} 1} \mathfrak{F} \otimes_{\mathcal{R}} R = F$$

is $\phi$.

Remark. We use Data 6.1 as we state and prove conditions which are equivalent to the statement “$I$ is a q.c.i.”. Recall from [3, Lem. 1.2] that if $I$ is a q.c.i., then $\text{grade } I = f - g$; and therefore, the inequality (6.1.2) holds automatically in this case.

**Proposition 6.2.** Adopt the data of 6.1. Then the following statements hold:

(a) $\Phi$ and $\Sigma$ are the minimal resolution and the minimal complete resolution of $\mathcal{R}/B$ by free $\mathcal{R}/A$-modules, respectively;

(b) $\Phi \otimes_{\mathcal{R}/A} R$ and $\Sigma \otimes_{\mathcal{R}/A} R$ are the minimal two-step Tate complex and the minimal two-step complete Tate complex associated to the data $(\xi, \phi)$ in the sense of 2.2.3 and 3.2.1, respectively;

(c) the following statements are equivalent:

(i) $I$ is a q.c.i. ideal of $R$;

(ii) $\Phi \otimes_{\mathcal{R}/A} R$ is a resolution of $R/I$ by free $R$-modules; and

(iii) $\text{Tor}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R) = 0$ for all positive $i$;

(d) the following statements are equivalent:

(i) $\Sigma \otimes_{\mathcal{R}/A} R$ is a complete resolution of $R/I$ by free $R$-modules; and

(ii) $\widetilde{\text{Tor}}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R) = 0$ for all integers $i$.

Proof. Assertion (a) is established in [16, Thm. 4] (for $\Phi$) and Proposition 4.3 (for $\Sigma$). Assertion (b) follows from the definition of $\rho$. Assertions (ci) and (cii) are equivalent because of (2.2.5). Assertions (cii) and (ciii) are equivalent because the homology of $\Phi \otimes_{\mathcal{R}/A} R$ is $\text{Tor}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R)$. Similarly, assertions (di) and (dii) are equivalent because the homology of $\Sigma \otimes_{\mathcal{R}/A} R$ is $\widetilde{\text{Tor}}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R)$.

The main result of the paper, Theorem 6.4, is an extension of Proposition 6.2.(c), by way of Theorem 2.6. The two-step Tate complex associated to the data $(\xi, \phi)$ exhibits significant rigidity (see assertions (i) and (ii) of Theorem 6.4 and also Corollary 7.1) and one can use Tate homology in place of ordinary homology when determining if $I$ is a q.c.i. (see assertions (iv), (v), and (vi) of Theorem 6.4). Furthermore, if $I$ is a q.c.i., then an explicit complete resolution for $R/I$ is given.

The first step in the transition from Proposition 6.2 to Theorem 6.4 is given in Observation 6.3.
Observation 6.3. Adopt the data of 6.1. The following statements hold:

(a) $\text{CI-dim}_{\mathcal{R}/A}(\mathcal{R}/B) = \ell - g$, and
(b) $\text{cx}_{\mathcal{R}/A}(\mathcal{R}/B) = g$.

Proof. First consider the quasi-deformation

\[ \mathcal{R}/A \xrightarrow{\text{natural quotient map}} \mathcal{R}. \]

Observe that $\text{pd}_{\mathcal{R}}(\mathcal{R}/B) - \text{pd}_{\mathcal{R}}(\mathcal{R}/A) = \ell - g$.

It follows from (2.4.1) that $\text{CI-dim}_{\mathcal{R}/A}(\mathcal{R}/B)$ is finite (and at most $\ell - g$). Furthermore, it follows from (2.4.2) that

\[ \text{CI-dim}_{\mathcal{R}/A}(\mathcal{R}/B) = \text{depth}(\mathcal{R}/A) - \text{depth}(\mathcal{R}/B) \]
\[ = \text{depth}(\mathcal{R}/A) - \text{depth}(\mathcal{R}/B) \]
\[ = (\text{depth} \mathcal{R} - g) - (\text{depth} \mathcal{R} - \ell) = \ell - g. \]

At this point, 2.5 guarantees that $\text{cx}_{\mathcal{R}/A}(\mathcal{R}/B)$ is the order of the pole of the Poincaré series $P_{\mathcal{R}/B}$. The complex $P$ is the minimal resolution of $\mathcal{R}/B$ by free $\mathcal{R}/A$ modules; thus $P_{\mathcal{R}/B} = \frac{(1 + t)^f}{(1 - t)^g}$. It follows that $\text{cx}_{\mathcal{R}/A}(\mathcal{R}/B) = g$. \qed

Theorem 6.4. Adopt the data of 6.1. Then the following statements are equivalent:

(i) $\text{Tor}_{\mathcal{R}/A}^i(\mathcal{R}/B, R) = 0$ for $g + 1$ consecutive values of $i$ with $0 < i$;
(ii) $\text{Tor}_{\mathcal{R}/A}^i(\mathcal{R}/B, R) = 0$ for $0 \ll i$;
(iii) $\text{Tor}_{\mathcal{R}/A}^i(\mathcal{R}/B, R) = 0$ for all $i$ with $0 < i$;
(iv) $\text{Tor}_{\mathcal{R}/B}^i(\mathcal{R}/B, R) = 0$ for $g + 1$ consecutive values of $i$;
(v) $\text{Tor}_{\mathcal{R}/B}^i(\mathcal{R}/B, R) = 0$ for $i \ll 0$;
(vi) $\text{Tor}_{\mathcal{R}/B}^i(\mathcal{R}/B, R) = 0$ for all integers $i$; and
(vii) $I$ is a q.c.i. ideal of $R$.

Furthermore, if the above statements hold, then $P \otimes_{\mathcal{R}/A} R$ and $\Sigma \otimes_{\mathcal{R}/A} R$ are the minimal resolution and the minimal complete resolution of $R/I$ by free $R$-modules, respectively.

Proof. We saw in Proposition 6.2.(c) that

\[ (6.4.1) \quad (\text{vii}) \iff (\text{iii}). \]

Apply Theorem 2.6 with $R$ replaced by $\mathcal{R}/A$, $M$ by $\mathcal{R}/B$, and $N$ by $R$. Use the results from Observation 6.3:

\[ \text{CI-dim}_{\mathcal{R}/A} \mathcal{R}/B = \ell - g \quad \text{and} \quad \text{cx}_{\mathcal{R}/A} \mathcal{R}/B = g. \]

It follows that the statements

\[ (6.4.2) \quad (i'), (ii), (iii'), (iv), (v), (vi) \]
are equivalent, where (i′) and (iii′) are

(i′) $\text{Tor}_i^{R/A}(R/B, R) = 0$ for $g + 1$ consecutive values of $i$ provided each of these values $i$ satisfies $\hat{j} - g < i$; and

(iii′) $\text{Tor}_i^{R/A}(R/B, R) = 0$ for all $i$ with $\hat{j} - g < i$.

It is clear that

(iii) $\Rightarrow$ (iii′) and (i) $\Rightarrow$ (i′).

To complete the proof, we show

(6.4.3) (iii′) $\Rightarrow$ (iii) and (i′) $\Rightarrow$ (i).

Let $r = \hat{j} - g$ and $\underline{x} = x_1, \ldots, x_r$ be the beginning of a minimal generating set for $I$ which is also a regular sequence in $I$ on $R$. (Hypothesis (6.1.2), together with the prime avoidance lemma, guarantees that $\underline{x}$ exists.) Let $R'$, $I'$, $\hat{j}'$ and $g'$ denote $R/(\underline{x}), I/(\underline{x}), \mu(I')$ and $\mu(H_1(\wedge^\bullet(F \otimes_R R')))$, respectively. The fact that $\underline{x}$ begins a minimal generating set for $I$ ensures that $\hat{j}' = \hat{j} - r$. Create the data $(\xi', \phi')$ for $I'$ in $R'$ as described in Proposition 2.2.6. The proof of Proposition 2.2.6 demonstrates that there is a quasi-isomorphism from the Koszul complex associated to $\xi$ to the Koszul complex associated to $\xi'$. It follows that $g' = g$. The statement of Proposition 2.2.6 asserts that there is a quasi-isomorphism from the two-step Tate complex for $I$ in $R$ to the two-step Tate complex for $I'$ in $R'$. On the other hand, we know from Proposition 6.2.(b) that $\mathcal{P} \otimes_{R/A} R$ is the minimal two-step Tate complex for $I$ in $R$ and $\mathcal{P} \otimes_{R/A} R'$ is the minimal two-step Tate complex for the ideal $I'$ of $R'$. Thus,

$$H_i(\mathcal{P} \otimes_{R/A} R) \cong H_i(\mathcal{P} \otimes_{R/A} R'), \text{ for all } i;$$

hence,

(6.4.4) $\text{Tor}_i^{R/A}(R/B, R) \cong \text{Tor}_i^{R/A}(R/B, R')$, for all $i$.

We prove (6.4.3). Assume that either (i′) or (iii′) holds for $I$. It follows from (6.4.2) that (ii) holds for $I$. Apply (6.4.4) to see that (ii) holds for $I'$. Hence, (iii′) holds for $I'$ by (6.4.2), again. On the other hand, (iii′) for $I'$ is the same as (iii) for $I'$ because

$$\hat{j}' - g' = (\hat{j} - r) - g = 0.$$

Use (6.4.4), again, to see that (iii) holds for $I$. It is clear that (iii) implies (i). □

Corollaries 6.5 and 6.7 are reformulations of (vii) implies (iii) from Theorem 6.4. Corollary 6.5 is easier to apply than the full statement of Theorem 6.4.

**Corollary 6.5.** If $I$ is a q.c.i. ideal in a local ring $R$, then there exists a local ring $\mathfrak{R}$ and ideals $A \subseteq B \subseteq C$ in $\mathfrak{R}$ such that

(a) $A$ is generated by a regular sequence of length $\mu(I) - \text{grade} I$,
(b) $B$ is generated by a regular sequence of length $\mu(I)$,
(c) $R = \mathfrak{R}/C$,
(d) $\text{Tor}_i^\mathfrak{R}/A(\mathfrak{R}/B, R) = 0$, for $1 \leq i$, and

(e) $BR = I$.

**Proof.** Apply Theorem 6.4 to the generic data built in Proposition 5.3 for the ideal $I$ in $R$. 

The following observation-definition has been adapted from [3, 8.7].

**Theorem 6.6.** Let $\rho : Q \to R$ be a surjective homomorphism of Noetherian local rings, and $\mathcal{I}$ be a q.c.i. ideal of $Q$. If $\text{Tor}_i^Q(Q/\mathcal{I}, R) = 0$ for $1 \leq i$, then $\mathcal{I}R$ is a q.c.i. ideal of $R$. Furthermore, one says that $\mathcal{I}R$ is obtained from $\mathcal{I}$ by flat base change.

**Corollary 6.7.** Every q.c.i. ideal in a local Noetherian ring is obtained from a pair of nested complete intersection ideals by way of a flat base change.

**Proof.** Apply Corollary 6.5 with $Q = \mathfrak{R}/A$, $\mathcal{I} = B/A$, and $\ker \rho = C/A$. 

**Examples 6.8.** Examples (a) and (b) were the well-understood examples of q.c.i. ideals as described in [11]. On the other hand, the Example (c) is also given, but was not well-understood, in [11].

(a) If the ideal $I$ is generated by a regular sequence in the local ring $R$, then, in the language of Corollary 6.5, one can take $\mathfrak{R} = R$, $A = C = 0$, and $B = I$.

(b) If $a$ and $b$ are a pair of exact zero divisors in the local ring $(R, m)$ and $I$ is generated by $a$, then, in the language of Corollary 6.5, one can take $\mathfrak{R} = R[x_1, x_2](m, x_1, x_2)$, $A = (x_1x_2)$, and $B = (x_1)$.

Define $\rho : \mathfrak{R} \to R$ to be the $R$-algebra homomorphism with $\rho(x_1) = a$, and $\rho(x_2) = b$. Thus, $C = (x_1 - a, x_2 - b)$.

(c) It was not possible to explain the q.c.i. ideal $I$ of the ring $R$ in [11, Sect. 4] using any of the techniques that appeared in [11].

Let $k$ be a field, $\mathfrak{H}$ be the polynomial ring $\mathfrak{R} = k[x_1, x_2, x_3, x_4, x_5]$, $C$ be the ideal

$$C = (x_1^2 - x_2x_3, x_2^2 - x_3x_5, x_3^2 - x_1x_4, x_4^2, x_5^2, x_3x_4, x_2x_5, x_4x_5)$$

of $\mathfrak{R}$, $f_1$ and $f_2$ be the elements $f_1 = x_1 + x_2 + x_4$ and $f_2 = x_2 + x_3 + x_5$ of $\mathfrak{R}$, $R$ be the ring $R = \mathfrak{R}/C$ and $I$ be the ideal $(f_1, f_2)R$ of $R$.

The following explanation of $I$ lead to Corollary 6.7. Define $A$ to be the ideal of $\mathfrak{R}$ generated by the entries of the product

$$[f_1 \quad f_2] \begin{bmatrix} x_1 - x_2 \\ -x_3 + x_4 + 2x_5 \\ x_2 - x_3 - x_4 \end{bmatrix}$$

and $B = (f_1, f_2)\mathfrak{R}$. Observe that $A \subseteq B$ are complete intersections. Observe further that $\text{pd}_{\mathfrak{R}/A} R$ is finite; indeed, the minimal resolution of $R$ is

$$0 \to \mathfrak{R}/A(-4)^3 \to \mathfrak{R}/A(-3)^8 \to \mathfrak{R}/A(-2)^6 \to \mathfrak{R}/A.$$
Thus,
\[ \text{Tor}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R) = 0, \quad \text{for } 0 \ll i; \]

Apply (ii) implies (iii) from Theorem 6.4 to conclude that
\[ \text{Tor}_{i}^{\mathcal{R}/A}(\mathcal{R}/B, R) = 0, \quad \text{for } 0 < i; \]

hence \( I \) is obtained from the q.c.i. ideal \( B(\mathcal{R}/A) \) by way of flat base change. (In this example, one should localize as needed.)

(d) We sketch a coordinate dependent argument for Corollary 6.7 in the general case. Let \( b_1, \ldots, b_f \) be a minimal generating set for the q.c.i. ideal \( I \) in the local ring \( (R, m) \), let \( (E, \partial) \) be the Koszul complex on this generating set, and \( v_1, \ldots, v_f \) be a basis for \( E_1 \) with \( \partial(v_i) = b_i \).

Consider a set of cycles
\[ z_j = \sum_{i=1}^{f} c_{i,j} v_i, \]
in \( E_1 \), with \( c_{i,j} \in m \) and \( 1 \leq j \leq g \), such that the homology classes
\[ \{ \text{cls}(z_j) \mid 1 \leq j \leq g \} \]
minimally generate \( H_1(E) \). According to [3, 1.2], \( g \leq f \); indeed, \( \text{grade}_R(I) \) is equal to \( f - g \). Let
\[ \{ \tilde{b}_i \mid 1 \leq i \leq f \} \cup \{ \tilde{c}_{i,j} \mid 1 \leq i \leq f, 1 \leq j \leq g \} \]
represent new indeterminates, \( \mathfrak{M} \) be the maximal homogeneous ideal of the polynomial ring
\[ R[\{ \tilde{b}_i \} \cup \{ \tilde{c}_{i,j} \}], \]
\( \tilde{R} \) be the local ring
\[ \tilde{R} = R[\{ \tilde{b}_i \} \cup \{ \tilde{c}_{i,j} \}]_{\mathfrak{M}}, \]
and \( \rho : \tilde{R} \to R \) be the surjective local \( R \)-algebra homomorphism with \( \rho(\tilde{b}_i) = b_i \) and \( \rho(\tilde{c}_{i,j}) = c_{i,j} \). (The \( b_i \) are in the maximal ideal of \( R \) because \( I \) is a proper ideal of \( R \); the \( c_{i,j} \) are in the maximal ideal of \( R \) because \( b_1, \ldots, b_f \) minimally generate \( I \).

Consider the ideals \( A \subseteq B \) in \( \tilde{R} \),
\[ A = \left( \{ \sum_{i} \tilde{b}_i \tilde{c}_{i,j} \mid 1 \leq j \leq g \} \right) \quad \text{and} \quad B = (\tilde{b}_1, \ldots, \tilde{b}_f). \]
The ideals \( A \) and \( B \) are both complete intersections; \( B/A \) is a q.c.i. ideal of \( R/A \); and the two step Tate complex
\[ \mathfrak{P} = (\tilde{R}/A)<V_1, \ldots, V_f, W_1, \ldots, W_g \mid \partial(V_i) = \tilde{b}_i, \partial(W_j) = \sum_{i} \tilde{c}_{i,j} V_i> \]
is a resolution of \( \tilde{R}/B \) by free \( (\tilde{R}/A) \)-modules. This notation is explained in Remark 2.2.4.
Notice that
\[ \rho(\sum_i \tilde{b}_i \tilde{c}_{i,j}) = \sum_i b_i c_{i,j} = \partial(\sum_i c_{i,j} v_i) = \partial(z_j) = 0; \]
so \( R \) is a \( \tilde{R}/A \)-algebra. Notice also that \( R \otimes_{\tilde{R}/A} \Psi \) is the two-step Tate complex
\[ R \otimes_{\tilde{R}/A} \Psi = R < v_1, \ldots, v_l, w_1, \ldots, w_g | \partial(v_i) = b_i, \partial(w_j) = z_j >, \]
which is a resolution of \( R/I \) by free \( R \)-modules. It follows that
\[ \text{Tor}^R_{i}(R, \tilde{R}/B) = 0, \quad \text{for } 1 \leq i. \]
Thus, the q.c.i. ideal is obtained from a pair of nested complete intersection ideals by way of a flat base change.

7. Application: Rigidity of the Two-Step Tate Complex and the Two-Step Complete Tate Complex.

In this section, we record our rigidity result Corollary 7.1 and compare it to the rigidity result of Jason Lutz.

**Corollary 7.1.** Let \( I \) be an ideal in a local ring \( R \). Assume that
\[ 0 \leq \mu(I) - \mu(K_1(I)) \leq \text{grade} I. \]
Let \( P \) be the minimal two-step Tate complex associated to \( I \) in \( R \) as described in Definition 2.2.3 and \( T \) be the minimal two-step complete Tate complex associated to \( I \) in \( R \) as described in Definition 3.2. The following statements are equivalent:

(a) \( H_i(P) = 0 \) for \( \mu(K_1(I)) + 1 \) consecutive values of \( i \) with \( 0 < i \);
(b) \( H_i(P) = 0 \) for \( 0 \ll i \);
(c) \( H_i(P) = 0 \) for all \( i \) with \( 0 < i \);
(d) \( H_i(T) = 0 \) for \( \mu(K_1(I)) + 1 \) consecutive values of \( i \);
(e) \( H_i(T) = 0 \) for \( i \ll 0 \);
(f) \( H_i(T) = 0 \) for all integers \( i \); and
(g) \( I \) is a q.c.i. ideal of \( R \).

Furthermore, if the above statements hold, then \( P \) and \( T \) are the minimal resolution and the minimal complete resolution of \( R/I \) by free \( R \)-modules, respectively.

**Proof.** Apply Theorem 6.4 to the generic data built in Proposition 5.3 for the ideal \( I \) in \( R \). Recall from Proposition 6.2.(b) that \( \Psi \otimes_{\tilde{R}/A} R = P \) and \( \Sigma \otimes_{\tilde{R}/A} R = T \).

**Corollary 7.2.** If \( I \) is a q.c.i. ideal in a local ring \( R \) and \( T \) is the minimal complete resolution of \( R/I \) by free \( R \)-modules, then \( T \) is isomorphic to \( \text{Hom}_R(T,R) \).

**Proof.** The minimal complete resolution of \( R/I \) is shown in Corollary 7.1 to be the minimal two-step complete Tate complex associated to \( I \) in \( R \) of 3.2. This complex has the stated property.

Theorem 7.3 is Jason Lutz’s rigidity result.
Theorem 7.3. ([12, Thm. 3.1]) Let $I$ be an ideal in a local ring $R$ and let $P$ be the two-step Tate complex for $I$. Assume $\mu(K_1(I)) \leq \mu(I) - \text{grade}(I)$. If $H_i(P) = 0$ for $q \leq i \leq q + \mu(I) - \text{grade}(I)$, for some integer $q$, with $2 \leq q$, then $I$ is a quasi-complete intersection.

Remark 7.4. The results 7.1 and 7.3 agree in that they both show that if $H_i(P) = 0$ for an appropriate collection of consecutive integers $i$, then $I$ is a q.c.i. The two results differ in three aspects:

1. the technical assumption on the acceptable inequalities relating $\mu(I), \mu(K_1(I))$, and $\text{grade}(I)$ appear to be different;
2. our result allows one to be the beginning of the band of vanishing homology, but Lutz insists that band begin at some integer which is at least two; and
3. our result needs $\mu(K_1(I)) + 1$ consecutive integers $i$ with $H_i(P) = 0$; whereas Lutz’s result needs $H_i(P)$ to vanish for $\mu(I) - \text{grade}(I) + 1$ consecutive values of $i$.

Notice, however, that if $I$ is a q.c.i. then $\text{grade}(I) = \mu(I) - \mu(K_1(I))$, see [3, 1.2]. In this case, both technical assumptions from (1) hold and the parameters from (3) are equal.

8. Application: The Dimension Theorem for Quasi-Homogeneous Q.C.I. Ideals.

In this section, we reprove [3, Thm. 4.1(c)] using different methods. The ring $R$ in the following result is non-negatively graded over a field; this ring does not have to be standard graded.

Proposition 8.1. Let $R = \bigoplus_{0 \leq i} R_i$ be a local graded ring with $R_0$ equal to a field. If $I$ is a homogeneous q.c.i. ideal in $R$, then
\[
\text{grade}(I) = \dim R - \dim R/I.
\]

Proof. Apply Corollary 6.5 and identify a local ring $R$ and ideals $A \subseteq B \cap C$ in $R$ with $A$ generated by a regular sequence of length $\mu(I) - \text{grade}(I)$, $B$ generated by a regular sequence of length $\mu(I)$, $\text{Tor}_i^{R/A}(R/B, R/C) = 0$ for $1 \leq i$, $R = R/C$, and $I = BR$. If $J$ is generated by a regular sequence of length $r$ in a local ring $Q$, then
\[
\dim Q/J = \dim Q - r = \dim Q - \text{grade} J.
\]

Therefore,
\[
\dim R/A = \dim R - \mu(I) + \text{grade} I; \quad \dim R/B = \dim R - \mu(I);
\]
and $\dim R/A - \dim R/B = \text{grade} I$.

On the other hand, the fact that $\text{Tor}_i^{R/A}(R/B, R/C) = 0$, for $1 \leq i$, ensures that a resolution of $R/B \otimes_{R/A} R/C$ by free $R/A$-modules may be obtained by forming
the tensor product of a resolution of $\mathcal{R}/B$ with a resolution of $\mathcal{R}/C$; hence, the Hilbert series of these rings are related by the following identity:

$$H_{(\mathcal{R}/B) \otimes_{\mathcal{R}/A}(\mathcal{R}/C)}(t) = \frac{H_{\mathcal{R}/B}(t)H_{\mathcal{R}/C}(t)}{H_{\mathcal{R}/A}(t)}.$$ 

We conclude that

$$\dim(\mathcal{R}/B \otimes \mathcal{R}/C) = \dim \mathcal{R}/B + \dim \mathcal{R}/C - \dim \mathcal{R}/A.$$ 

The ring $\mathcal{R}/C$ is equal to $R$; the ring $\mathcal{R}/B \otimes \mathcal{R}/C$ is equal to $\mathcal{R}/(B + C) = R/I$; and $\dim \mathcal{R}/A - \dim \mathcal{R}/B = \text{grade } I$. It follows that

$$\dim R/I = \dim R - \text{grade } I.$$ 

□

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