A NOTE ON PAIR-DEPENDENT LINEAR STATISTICS WITH A SLOWLY INCREASING VARIANCE

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We prove Gaussian fluctuations for pair-counting statistics of the form \( \sum_{1 \leq i < j \leq N} f(\theta_i - \theta_j) \) for the circular unitary ensemble of random matrices in the large-\( N \) limit under the condition that the variance increases slowly as \( N \) increases.

Keywords: random matrix, circular unitary ensemble, pair-counting statistics, central limit theorem

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1. Introduction

The study of random matrix theory goes back to the principal component method of J. Wishart (1920s–1930s) and the revolutionary ideas of E. Wigner in quantum physics in the 1950s relating the statistical properties of the energy levels of heavy-nuclei atoms on one hand and the spectral properties of Hermitian random matrices with independent components on the other hand. In the 1960s, F. Dyson introduced three basic types of matrix ensembles: the circular unitary ensemble (CUE), circular orthogonal ensemble (COE), and circular symplectic ensemble (CSE) (see, e.g., [1], [2]).

The CUE corresponds to the joint distribution of the eigenvalues of an \( N \times N \) random unitary matrix \( U \) distributed according to the Haar measure. The COE corresponds to the joint distribution of the eigenvalues of \( U^Tu \). Finally, if \( UD \) denotes the quaternion dual, then \( UD^U \) gives the CSE for even \( N \). Details can be found in [3].

The probability density of the eigenvalues \( \{e^{i\theta_j}\}_{j=1}^N \) is given by
\[
p_N(\theta) = \frac{1}{Z_N(\beta)} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^\beta, \tag{1.1}
\]
where \( \beta = 1, \beta = 2, \) and \( \beta = 4 \) correspond to COE, CUE, and CSE. For arbitrary \( \beta > 0 \), a (sparse) block-diagonal random matrix model with an eigenvalue distribution following (1.1) was introduced in [4]. In particular, the block-diagonal random matrix in [4] has about \( 4N \) nonzero elements. Ensemble (1.1) for arbitrary \( \beta > 0 \) is called the circular beta ensemble (C\( \beta \)E).

Here, we focus on the CUE. The partition function for \( \beta = 2 \) is given by \( Z_N(2) = (2\pi)^N \times N! \).

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In [5], pair-counting statistics of the form

\[ S_N(f) = \sum_{1 \leq i \neq j \leq N} f(L_N(\theta_i - \theta_j)) \] (1.2)

were studied for a CBE with \( 1 \leq L_N \leq N \) under certain assumptions about the smoothness of \( f \). The study in [5] was motivated by a classical result of Montgomery on pair correlation of zeros of the Riemann zeta function [6], [7]. Assuming the Riemann hypothesis, Montgomery studied the distribution of the “nontrivial” zeros on the critical line \( 1/2 + iR \). Rescaling the zeros \( 1/2 \pm i\gamma_n \) as \( \tilde{\gamma}_n = \gamma_n/2\pi \log \gamma_n \), Montgomery considered the statistic

\[ \sum_{0 < \tilde{\gamma}_j \neq \tilde{\gamma}_k < T} f(\tilde{\gamma}_j - \tilde{\gamma}_k) \] for large \( T \) and sufficiently fast decaying \( f \) with \( \text{supp} F(f) \subset [-\pi, \pi] \), where \( F(f) \) denotes the Fourier transform of \( f \). The results in [6], [7] imply that in the limit, the two-point correlations of the (rescaled) critical zeros coincide with the local two-point correlations of the eigenvalues of a CUE random matrix.

The results in [5] concern the limit behavior of (1.2) in three different regimes: macroscopic (\( L_N = 1 \)), mesoscopic (\( 1 \ll L_N \ll N \)), and microscopic (\( L_N = N \)). In the unscaled \( L_N = 1 \) case, it was shown that

\[ S_N(f) = \sum_{1 \leq i \neq j \leq N} f(\theta_i - \theta_j) \] (1.3)

has a non-Gaussian fluctuation in the limit \( N \to \infty \) if \( f \) is a sufficiently smooth function on the unit circle \( \mathbb{T} \). Namely, let \( f \) be a real even integrable function on the unit circle \( \mathbb{T} \), and let

\[ \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-ikx} \, dx \] (1.4)

denote the Fourier coefficients of \( f \). We assume that \( f' \in L^2(\mathbb{T}) \) for \( \beta = 2 \),

\[ \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \cdot |k| < \infty, \quad \beta < 2, \]

\[ \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \cdot |k| \log(|k| + 1) < \infty, \quad \beta = 4, \]

\[ \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \cdot |k|^2 < \infty, \quad \beta \in (2, 4) \cup (4, \infty). \]

Then we have the convergence in distribution as \( N \to \infty \)

\[ S_N(f) - \mathbb{E}S_N(f) \xrightarrow{D} \frac{4}{\beta} \sum_{k=1}^{\infty} \hat{f}(k)(\varphi_k - 1), \] (1.5)

where \( \varphi_m \) are independent identically distributed exponential random variables with \( \mathbb{E}(\varphi_m) = 1 \). The result was proved for \( \beta = 2 \) under the optimum condition \( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 |k|^2 < \infty \) (i.e., \( f' \in L^2(\mathbb{T}) \)).

Our goal here is to study the fluctuation of pair-counting statistic (1.3) (\( L_N = 1 \)) under the condition that \( \text{Var}(S_N(f)) \) slowly increases as \( N \) goes to infinity.

**Definition 1.1.** A positive sequence \( \{V_N\} \) is said to be slowly varying in the sense of Karamata [8] if

\[ \lim_{N \to \infty} \frac{V_{\lfloor \lambda N \rfloor}}{V_N} = 1 \quad \text{for all } \lambda > 0, \] (1.6)

where \( \lfloor m \rfloor \) denotes the integer part of \( m \).
Everywhere in what follows, we use the notation

\[ V_N = \sum_{k=-N}^{k=N} |\hat{f}(k)|^2 |k|^2. \]  

(1.7)

In particular, in the proof of Theorem 1.1, we show that \( \text{Var}(S_N(f))/2V_N \to 1 \) as \( N \to \infty \) if \( \{V_N\} \) is a slowly varying sequence.

**Theorem 1.1.** Let \( f \in L^2(\mathbb{T}) \) be a real even function such that

\[ V_N = \sum_{k=-N}^{k=N} |\hat{f}(k)|^2 |k|^2, \quad N = 1, 2, \ldots, \]

is a slowly varying sequence that diverges to infinity as \( N \to \infty \). Then we have the convergence in distribution

\[ \frac{S_N(f) - ES_N(f)}{\sqrt{2 \sum_{k=-N}^{\infty} |\hat{f}(k)|^2 |k|^2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \]

where \( S_N(f) \) is defined in (1.3).

The pair-counting statistics \( S_N(f) \) can be rewritten as the linear statistics of eigenvalues:

\[ S_N(f) = \sum_{1 \leq j \neq l \leq N} f(\theta_j - \theta_l) = 2 \sum_{k=1}^{\infty} \hat{f}(k) \left| \sum_{j=1}^{N} e^{ik\theta_j} \right|^2 + \hat{f}(0)N^2 - Nf(0). \]  

(1.8)

Linear statistics of the eigenvalues of the random matrices \( \sum_{j=1}^{N} f(\lambda_j) \) have been extensively studied in the literature. For (1.1) with an arbitrary \( \beta > 0 \) and a sufficiently smooth real-valued \( f \), Johansson [9] proved that

\[ \frac{\sum_{j=1}^{N} f(\theta_j) - N\hat{f}(0)}{\sqrt{\frac{2}{\beta} \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 |k|^2}}, \quad N = 1, 2, \ldots, \]

converges in distribution to a standard Gaussian random variable. In particular, for \( \beta = 2 \), he proved the result under the optimum conditions on \( f \), namely,

\[ \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 |k| < \infty. \]

Under the condition that the variance of the linear statistic goes to infinity as \( N \to \infty \), the central limit theorem was proved in [10] in the case where \( \beta = 2 \) and the sequence \( \{\sum_{m=-N}^{N} |\hat{f}(m)|^2 |m|\}_{N \in \mathbb{N}} \) is slowly varying.

The results for the linear eigenvalue statistics in the mesoscopic regime \( \sum_{j=1}^{N} f(L_N \theta_j), 1 \ll L_N \ll N \) can be found in [11]–[13] and the references therein. Additional results for the spectral properties of \( \mathbb{C}/\beta E \) can be found in [14]–[22] and the references therein.

We prove our main result (Theorem 1.1) in the next section. Everywhere in the paper, \( a_N = O(b_N) \) means that the ratio \( a_N/b_N \) is bounded from above in absolute value, and \( a_N = o(b_N) \) means that \( a_N/b_N \to 0 \) as \( N \to \infty \). In this case, we also occasionally use the notation \( a_N \ll b_N \) for nonnegative quantities.
2. Proof of Theorem 1.1

We start by recalling the formula for the variance of $S_N(f)$ in Proposition 4.1 in [5]:

$$\text{Var}(S_N(f)) = 4 \sum_{1 \leq s \leq N-1} s^2|\hat{f}(s)|^2 + 4(N^2 - N) \sum_{N \leq s} |\hat{f}(s)|^2 - 4 \sum_{1 \leq s, t \leq N-1, \frac{N}{2} \leq n} (N - |s - t|)\hat{f}(s)\hat{f}(t) - 4 \sum_{1 \leq s, t \leq N-1, \frac{N}{2} \leq n} ((s + t) - N)\hat{f}(s)\hat{f}(t). \quad (2.1)$$

Our first goal is to show that the last two (off-diagonal) terms in variance expression (2.1) are much smaller than $V_N = \sum_{s=1}^N s^2|\hat{f}(s)|^2$ for large $N$ if (1.6) is satisfied.

**Lemma 2.1.** Let \{\{V_N\}\} in (1.7) be a slowly varying sequence diverging to infinity as $N \to \infty$. Then as $N \to \infty$, we have

$$\sum_{1 \leq s, t \leq N, \frac{N}{2} \leq n} s|\hat{f}(s)| \cdot |\hat{f}(t)| = o(V_N), \quad (2.2a)$$

$$(N + 1) \sum_{s-t \leq N, \frac{N}{2} \leq n} |\hat{f}(s)| \cdot |\hat{f}(t)| = o(V_N), \quad (2.2b)$$

$$N \sum_{|s-t| \leq N-1, \frac{N}{2} \leq n} |\hat{f}(s)| \cdot |\hat{f}(t)| = o(V_N). \quad (2.2c)$$

The proofs of relations (2.2a) and (2.2b) are somewhat similar to the proofs given in Lemma 4.4 in [5]. For the reader’s convenience, we give the proofs below in detail.

**Proof.** We consider (2.2a). Let $x_s = s|\hat{f}(s)|$ for $1 \leq s \leq N$ and $X_N = \{x_s\}_{s=1}^N$. We define a vector $Y_N := X_N1_{s>N/2}$ such that the first $\lfloor N/2 \rfloor$ coordinates are zero and the others coincide with the corresponding coordinates of $X_N$. We note that

$$2\|X_N\|_2^2 = V_N, \quad \|Y_N\|_2^2 = o(V_N), \quad (2.3)$$

where $\|X\|_2$ denotes the Euclidean norm of a vector $X \in \mathbb{R}^N$. The last bound follows from condition (1.6) for the slow increase of $V_N$.

We now write the off-diagonal term in (2.2a) as a bilinear form:

$$\sum_{1 \leq s, t \leq N, \frac{N}{2} \leq n} s|\hat{f}(s)| \cdot |\hat{f}(t)| = \sum_{t=1}^N x_t \cdot \left( \frac{1}{t} \sum_{s=N-t+1}^N x_s \right) = \sum_{t=1}^N x_t \cdot \left( \frac{1}{t} \sum_{s=1}^t (U_NX_N)_s \right) = \langle X_N, A_NX_N \rangle \quad (2.4)$$
with \( A_N = B_N U_N \), where \( U_N \) is a unitary permutation matrix given by \((U_N)_{s,t} = 1_{t=N-s+1} \) and \( B_N \) is a lower-triangular matrix given by \((B_N)_{s,t} = 1_{t \leq s} \). The matrix \( A_N \) is given by

\[
A_N = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & 1/2 & 1/2 \\
0 & \ddots & \ddots & \ddots \\
1/N & \cdots & 1/N & 1/N
\end{pmatrix}.
\]

Taking into account that the upper-left \([N/2] \times [N/2] \) block of \( A_N \) is zero, we can find an upper estimate: \( \langle X_N, A_N X_N \rangle \leq \langle X_N, A_N Y_N \rangle + \langle Y_N, A_N X_N \rangle \). It was shown in [5] that \( \|A_N\|_\text{op} \leq 3 \), where \( \|A\|_\text{op} \) denotes the spectral norm of an operator \( A \). It hence follows with (2.3) taken into account that expression (2.4) is bounded from above by \( 3\|X_N\|_2\|Y_N\|_2 = o(V_N) \). This completes the proof of (2.2a).

To prove (2.2b), we assume that \( B_N \) is defined the same as above. Similarly, we set \( x_s = s|\hat{f}(s)| \), \( 1 \leq s \leq 2N \), and \( X_N = \{x_s\}_{s=1}^{2N} \). Now \( X_N \) is a \( 2N \)-dimensional vector such that \( \|X_N\|_2^2 \) increases slowly as \( N \) increases. We define \( Y_N := X_N 1_{\{s > N\}} \) such that the first \( N/2 \) coordinates are zero and the others coincide with the corresponding coordinates of \( X_N \). We can see that

\[
N \sum_{s-t \leq N} |\hat{f}(s)| \cdot |\hat{f}(t)| \leq \sum_{t=1}^N x_t \left( \frac{1}{t} \sum_{s=N+1}^{N+t} x_s \right) = (C_N X_N, D_N X_N),
\]

where

\[
C_N = \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix}, \quad D_N = \begin{pmatrix} 0 & B_N \\ 0 & 0 \end{pmatrix}.
\]

We note that \( \|D_N\|_\text{op} \leq 3 \) and \( \|C_N\|_\text{op} = 1 \). We again have

\[
\langle C_N X_N, D_N X_N \rangle = \langle C_N X_N, D_N Y_N \rangle \leq 3\|X_N\|_2\|Y_N\|_2 = o(V_N).
\]

The proof of (2.2b) is complete.

We consider (2.2c). We start by estimating the left-hand side of (2.2c) from above by

\[
2N \sum_{t-N+1 \leq s \leq N+t-1} \sum_{t \geq s \geq 2N} |\hat{f}(s)| \cdot |\hat{f}(t)|. \tag{2.5}
\]

As above, we set \( x_s = s|\hat{f}(s)| \) for \( s \geq 1 \). We define the vectors \( X = \{x_s\}_{s=1}^\infty \) and \( X^{(j)} = X 1_{\{jN \leq s < (j+1)N\}} \), \( j = 0, 1, 2, \ldots \). We can bound (2.5) from above by the sum

\[
2 \sum_{t=N}^\infty x_t \left( \frac{1}{t} \sum_{s=t-N+1}^t x_s \right) = 2 \sum_{j=1}^\infty x_j^{(j+1)N-1} \sum_{t=jN}^t x_t \left( \frac{1}{t} \sum_{s=t-N+1}^t x_s \right). \tag{2.6}
\]

We can write the second sum in the right-hand side of (2.6) as

\[
\sum_{t=jN}^{(j+1)N-1} x_t \left( \frac{1}{t} \sum_{s=t-N+1}^t x_s \right) = \langle X^{(j)}, R_{N,j}(X^{(j-1)} + X^{(j)}) \rangle, \tag{2.7}
\]
where $R_{N,j}$ is a bounded linear operator such that
\begin{equation}
(R_{N,j})_{t,s} = \frac{1}{t} 1_{(t-N+1 \leq s \leq t)} 1_{(jN \leq t < (j+1)N)}.
\end{equation}

The operator norm of $R_{N,j}$ is bounded from above by its Hilbert–Schmidt norm,
\begin{equation}
\|R_{N,j}\|_{op} \leq \|R_{N,j}\|_{2} = \sqrt{\sum_{N=1}^{N+1} \frac{1}{t^2} \leq \sqrt{\frac{N^2}{j^2N^2}} = \frac{1}{j}}.
\end{equation}

Using the Cauchy–Schwarz inequality, we can now write an upper estimate for the right-hand side of (2.7):
\begin{equation}
\langle X(j), R_{N,j}(X^{(j-1)} + X^{(j)}) \rangle \leq \|X(j)\|_{2} \|R_{N,j}\|_{op}(\|X^{(j-1)}\|_{2} + \|X^{(j)}\|_{2}) \leq \frac{1}{j} \|X(j)\|_{2}^{2} + \frac{1}{j} \|X^{(j)}\|_{2} \|X^{(j-1)}\|_{2}.
\end{equation}

Summing the right-hand side of the last inequality over $j \geq 1$ gives $o(V_N)$. Indeed, $2\|X^{(0)}\|_{2}^{2} = V_{N}$, and summation by parts gives
\begin{equation}
\sum_{j=1}^{\infty} \frac{1}{j} \|X^{(j)}\|_{2}^{2} \leq \sum_{j=1}^{\infty} \frac{1}{j} (V_{jN} - V_{N}).
\end{equation}

It follows from condition (1.6) for the slow increase of $V_{N}$ that the right-hand side of (2.11) is of the order $o(V_N)$. To calculate the sum of the second terms in (2.10), we write
\begin{equation}
\sum_{j=1}^{\infty} \frac{1}{j} \|X^{(j)}\|_{2} \|X^{(j-1)}\|_{2} = \|X^{(1)}\|_{2} \|X^{(0)}\|_{2} + \sum_{j=2}^{\infty} \frac{1}{j} \|X^{(j)}\|_{2} \|X^{(j-1)}\|_{2}.
\end{equation}

The first term in the right-hand side is $\sqrt{(V_{2N} - V_{N})V_{N}} = o(V_N)$. To treat the second term, we use the Cauchy–Schwarz inequality and proceed as in (2.11). This completes the proof of the lemma.

**Proof of Theorem 1.1.** We finish the proof of Theorem 1.1. We recall that
\begin{equation}
S_N(f) = \sum_{1 \leq i \neq j \leq N} f(\theta_i - \theta_j) = 2 \sum_{k=1}^{N} \hat{f}(k) \left| \sum_{j=1}^{N} e^{ik\theta_j} \right|^2 + \hat{f}(0)N^2 - N f(0).
\end{equation}

We let $t_{N,k}$ denote the trace of the $k$th power of a CUE matrix:
\begin{equation}
t_{N,k} := \sum_{j=1}^{N} e^{ik\theta_j}, \quad k = 0, \pm 1, \pm 2, \ldots.
\end{equation}

Then
\begin{equation}
\frac{S_N(f) - \mathbb{E}S_N(f)}{\sqrt{V_N}} = \frac{2}{\sqrt{V_N}} \sum_{k=1}^{[N/M_N]} \hat{f}(k) (|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2) = \frac{2}{\sqrt{V_N}} \sum_{k=1}^{[N/M_N]} \hat{f}(k) (|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2) + \frac{2}{\sqrt{V_N}} \sum_{k=1}^{[N/M_N]+1} \hat{f}(k) (|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2),
\end{equation}

1687
where \( \{M_N\}_{N=1}^\infty \) is a positive integer-valued sequence increasing sufficiently slowly to infinity as \( N \to \infty \) such that
\[
\lim_{N \to \infty} \frac{V_{\lfloor N/M_N \rfloor}}{V_N} = \lim_{N \to \infty} \frac{V_N}{V_{\lfloor N/M \rfloor}} = 1. \tag{2.16}
\]
The existence of such a sequence follows from (1.6).

It follows from Lemma 2.1 that the second moment of the second term in the right-hand side of (2.15) tends to zero as \( N \to \infty \). We formulate this result in the next lemma.

**Lemma 2.2.** We have the relation
\[
\text{Var} \sum_{\lfloor N/M \rfloor + 1}^\infty \hat{f}(k)|t_{N,k}|^2 = o(V_N). \tag{2.17}
\]

**Proof.** It follows from (2.1) that
\[
\text{Var} \sum_{\lfloor N/M \rfloor + 1}^\infty \hat{f}(k)|t_{N,k}|^2 = 4 \sum_{N/M_N < s \leq N-1} s^2|\hat{f}(s)|^2 + 4(N^2 - N) \sum_{N \leq s} |\hat{f}(s)|^2 - 4 \sum_{N/M_N \leq s,t,1 \leq |s-t| \leq N-1,N \leq \max(s,t)} (N - |s-t|)\hat{f}(s)\hat{f}(t) - 4 \sum_{N/M_N \leq s,t \leq N-1,N+1 \leq s+t} ((s + t) - N)\hat{f}(s)\hat{f}(t). \tag{2.18}
\]
The first term in the right-hand side of this equation is equal to \( 2(V_N - V_{\lfloor N/M \rfloor}) \) and has the order \( o(V_N) \) by virtue of (2.16). The last two terms in the right-hand side of (2.18) have the order \( o(V_N) \) by Lemma 2.1. Finally, the second term in the right-hand side is bounded from above by
\[
4N^2 \sum_{N \leq s} |\hat{f}(s)|^2 \leq 2 \sum_{j=1}^{\infty} \frac{1}{j^2}(V_{(j+1)N} - V_{jN}) = o(V_N), \tag{2.19}
\]
where the last estimate follows from (1.6). The lemma is proved.

To finish the proof of the theorem, we must show that the first term in the right-hand side of (2.15) converges in distribution to a standard Gaussian random variable. For this, we first show that the first \( \lfloor M_N/2 \rfloor \) moments of
\[
\frac{2}{\sqrt{V_N}} \sum_{k=1}^{\lfloor N/M_N \rfloor} \hat{f}(k)(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2) \tag{2.20}
\]
coincide with the first \( \lfloor M_N/2 \rfloor \) moments of
\[
\frac{2}{\sqrt{V_N}} \sum_{k=1}^{\lfloor N/M_N \rfloor} \hat{f}(k)(\varphi_k - 1), \tag{2.21}
\]
where \( \varphi_k, k \geq 1 \), are independent identically distributed exponential random variables.
Lemma 2.3. Let $m$ be a positive integer such that $1 \leq m < M_N/2$. Then
\[
\mathbb{E}\left(\frac{2}{\sqrt{V_N}} \left| \sum_{k=1}^{[N/M_N]} \hat{f}(k)(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2) \right|^m\right) = \mathbb{E}\left(\frac{2}{\sqrt{V_N}} \sum_{k=1}^{[N/M_N]} \hat{f}(k)(\varphi_k - 1) \right)^m. 
\] (2.22)

**Proof.** Formula (2.22) follows from the identity
\[
\mathbb{E}\prod_{i=1}^l |t_{N,k_i}|^2 = \mathbb{E}\prod_{i=1}^l k_i \varphi_{k_i}, 
\] (2.23)
if
\[
2 \sum_{i=1}^l |k_i| \leq N, \quad k_1, \ldots, k_l > 0. 
\] (2.24)

This was established in [14] and [11] (also see [23]), where it was shown that very many joint moments/cumulants of
\[
k^{-1/2}t_{N,k} = k^{-1/2} \text{Tr}U^k, \quad k \geq 1, 
\] (2.25)
coincide with the corresponding joint moments/cumulants of a sequence of independent identically distributed standard complex Gaussian random variables. Namely, if we let $\kappa(t_{N,k_1}, \ldots, t_{N,k_n})$ denote the joint cumulant of $\{t_{N,k_j}, 1 \leq j \leq n\}$, then $\kappa(t_{N,k_1}, \ldots, t_{N,k_n}) = 0$ if at least one of the two conditions
\[
n \geq 1, \quad \sum_{j=1}^n k_j \neq 0, \\
n > 2, \quad \sum_{j=1}^n k_j = 0, \quad \sum_{j=1}^n |k_j| \leq N 
\]
is satisfied. In addition, $\kappa(t_{N,k}, t_{N,-k}) = \min(|k|, N)$ (see Lemma 5.2 in [5] for the details). Taking into account that the squared absolute value of a standard complex Gaussian random variable is distributed according to the exponential law, we obtain (2.23) and (2.24). The lemma is proved.

The proof of Theorem 1.1 now follows immediately from the following standard lemma.

**Lemma 2.4.** For any $t \in \mathbb{R}$,
\[
\mathbb{E}\exp\left(\frac{t}{\sqrt{\sum_{k=1}^N a_k^2}} \sum_{k=1}^N a_k(\varphi_k - 1)\right) \rightarrow e^{t^2/2}, \quad N \rightarrow \infty, 
\] (2.26)
if
\[
\sum_{k=1}^\infty a_k^2 = \infty, \quad \max_{1 \leq k \leq N} |a_k| = o\left(\sqrt{\sum_{k=1}^N a_k^2}\right). 
\] (2.27)

**Proof.** The proof follows from standard direct computations using the independence of the $\varphi_k$.

Setting $a_k = \hat{f}(k)k$, $k \geq 1$, and applying Lemma 2.4, we obtain the convergence of the exponential moment (and hence all moments) of
\[
\sqrt{\frac{2}{V_N}} \sum_{k=1}^{[N/M_N]} \hat{f}(k)(\varphi_k - 1) 
\]

1689
to that of a standard real Gaussian random variable. Therefore,

\[
\sqrt{\frac{2}{V_N}} \sum_{k=1}^{\lfloor N/M \rfloor} \hat{f}(k)(\varphi_k - 1), \quad \sqrt{\frac{2}{V_N}} \sum_{k=1}^{\lfloor N/M \rfloor} \hat{f}(k)(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2)
\]

converge in distribution to a standard real Gaussian random variable. Because the second term in (2.15) goes to 0 in \(L^2\), we conclude that Theorem 1.1 is proved.

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