Marginal Deformations of Non-Relativistic Field Theories

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Abstract

We construct the supergravity duals of marginal deformations of a (0,2) Landau-Ginsburg theory that describes the supersymmetric lowest Landau level. These deformations preserve supersymmetry and it is proposed that they are associated with the introduction of a phase in the (0,2) superpotential. We also consider marginal deformations of various field theories that exhibit Schrödinger symmetry and Lifshitz scaling. This includes countably-infinite examples with dynamical exponent $z = 2$ based on the Sasaki-Einstein spaces $Y^{p,q}$ and $L^{p,q,r}$, as well as an example with general dynamical exponent $z \geq 1$. 
1 Introduction

The AdS/CFT correspondence provides a technique for studying certain strongly-coupled conformal field theories in terms of string theory on a weakly curved spacetime (see [1] for a review). In recent years, holographic techniques have been used to study strongly-coupled condensed matter systems, such as atomic gases at ultra-low temperature. Gravitational backgrounds that provide descriptions of these Lifshitz and Schrödinger-like fixed points have been proposed in [2] and [3,4], respectively. In these systems, rather than obeying conformal scale invariance, the temporal and spatial coordinates scale anisotropically. Another example of a condensed matter system for which a dual gravitational description has been proposed is the effective field theory of the supersymmetric lowest Landau level, which is a $(0,2)$ Landau-Ginsburg model [5].

All of these systems can undergo marginal deformations, in the sense that the deformed theories preserve the non-relativistic symmetries of the undeformed theory. We will employ a solution-generating technique that is based on U-duality in order to find gravitational backgrounds in type IIB theory that correspond to marginal deformations of various non-relativistic theories. The procedure can be outlined as follows. Begin with a theory that has a global symmetry group that includes $U(1) \times U(1)$, which corresponds to a two-torus subspace of the type IIB background. T-dualize along one of the $U(1)$ directions to type IIA theory. Lifting the solution to eleven dimensions provides a third direction which is associated with a $U(1)$ symmetry. One can now apply an $SL(3, R)$ transformation along these $U(1)^3$ directions. Dimensionally reducing to type IIA theory and T-dualizing back to type IIB theory along the shifted directions yields a new type IIB solution.

This method for finding new solutions has been widely used, one instance of which was to generate the type IIB supergravity background that corresponds to marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory [6]. In addition to the $U(1) \times U(1)$ symmetry, the theory has a $U(1)$ R-symmetry. Since the corresponding direction in the gravitational background was not involved in the solution-generating procedure, the deformed theory has $\mathcal{N} = 1$ supersymmetry. The deformation of the supergravity background was matched to an exactly marginal operator in the field theory, thereby providing a holographic test of the methods of Leigh and Strassler [7].

For the case of $\mathcal{N} = 4$ super Yang-Mills theory, one can perform a chain of T-duality-shift-T-duality transformations which involve the direction that corresponds to the $U(1)$ R-symmetry, which results in additional marginal deformations that do not preserve supersymmetry [8]. In contrast, some of the Lifshitz and Schrödinger-like fixed points that we consider have a global symmetry group that contains $U(1)^3$, which enables us to explore various marginal deformations that preserve a minimal amount of supersymmetry.

This paper is organized as follows. In section 2, we construct the type IIB supergravity dual background that describes the marginal deformations of the effective theory of the supersymmetric lowest Landau level, which is a $(0,2)$ Landau-Ginsburg theory. In section 3, we construct the supergravity duals of deformations of field theories which preserve Schrödinger symmetry. The Sasaki-Einstein spaces $L^{p,q,r}$ provide countably-infinite examples of such theories that can be de-
formed in this manner. In section 4, we consider the marginal deformations of theories that exhibit Lifshitz scaling. The Sasaki-Einstein spaces $Y^{p,q}$ are used to construct a countably-infinite family of such marginally deformed theories that all have dynamical exponent $z = 2$. We also consider a massive type IIA background that describes a marginally deformed theory with general dynamical exponent $z \geq 1$. Conclusions are presented in section 5.

2 Marginal deformations of $(0, 2)$ Landau-Ginsburg theory

The bosonic sector of $U(1)^3$ truncation of $D = 5$ SO(6) gauged supergravity [9] that keeps two neutral scalar fields $\phi_a$ has the Lagrangian

$$e^{-1}L_5 = R - \frac{1}{2}(\partial \varphi_1)^2 - \frac{1}{2}(\partial \varphi_2)^2 + 4 \sum_{i=1}^{3} X_i^{-1} - \frac{1}{4} \sum_{i=1}^{3} X_i^{-2}(F^{i}_{(2)})^2 + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\lambda} F^1_{\mu\nu}F^2_{\rho\sigma}A^3_{\lambda},$$

(2.1)

where the two scalars are expressed in terms of three constrained scalars $X_i$ via

$$X_1 = e^{-\frac{1}{\sqrt{2}}(\varphi_1 - \sqrt{2}\varphi_2)}, \quad X_2 = e^{-\frac{1}{\sqrt{2}}(\varphi_1 + \sqrt{2}\varphi_2)}, \quad X_3 = e^{\frac{1}{\sqrt{6}}\varphi_1},$$

(2.2)

with $X_1X_2X_3 = 1$. Also, we are taking $g = 1$. A family of magnetic AdS$_3 \times \mathbb{R}^2$ solutions found in [10] and studied further in [5, 11] are given by

$$ds^2_5 = L^2 \ ds^2_{\text{AdS}_3} + dy_1^2 + dy_2^2,$$

$$F^{i}_{(2)} = 2q^i \ dy_1 \wedge dy_2,$$

$$\varphi_1 = f_1, \quad \varphi_2 = f_2,$$

(2.3)

where $f_1$ and $f_2$ are constants and

$$L^{-2} = \sum_{i=1}^{3} X_i^{-1}, \quad (q^i)^2 = X_i.$$

(2.4)

For $f_2 = 0$, these solutions reduce to a “magnetovac” solution of Romans’ $D = 5$ gauged supergravity [12], which was uplifted to ten and eleven dimensions in [13]. The case in which both $f_1$ and $f_2$ vanish is a non-supersymmetric solution of minimal gauged supergravity, which arises in the near-horizon limit of magnetic black brane solutions at zero temperature [14]. There is a subset of solutions which preserve supersymmetry provided that they satisfy the constraint

$$\sum_{i=1}^{3} q^i = 0,$$

(2.5)

as well as

$$L^{-1} = \frac{1}{2} \sum_{i=1}^{3} X_i, \quad 2 \sum_{i=1}^{3} X_i^2 = \left( \sum_{i=1}^{3} X_i \right)^2.$$

(2.6)

Since the solutions degenerate if any of the $q_i$ vanish, we require all of them to be nonzero.
It has been proposed that the above supersymmetric subset of solutions provide a supergravity dual description of the effective field theory of the supersymmetric lowest Landau level \([5]\). This is a \((0, 2)\) Landau-Ginsburg theory which has \(\nu_1 = |q_1| N_{\Phi} N^2\) chiral multiplets \(\Theta_{1i}\), \(\nu_2 = |q_2| N_{\Phi} N^2\) chiral multiplets \(\Theta_{2j}\) and \(\nu_3 = \nu_1 + \nu_2\) fermionic multiplets \(\Psi_{3k}\), where \(N_{\Phi} = BV_2/2\pi\), \(B\) is the magnitude of the magnetic field, \(V_2\) is the volume of the magnetic plane and \(q_1\) and \(q_2\) are the charges of the chiral multiplets. The \((0, 2)\) superpotential is given by

\[
c_{ijk} \Theta_{1i} \Theta_{2j} \Psi_{3k},
\]

where \(c_{ijk}\) is from the overlap of the lowest Landau level wavefunctions. The \((0, 2)\) Landau-Ginsburg theory can arise in the low energy limit of a flow from four-dimensional \(\mathcal{N} = 4\) super Yang-Mills theory, which is described by a supergravity solution that smoothly interpolates from \(\text{AdS}_5\) to \(\text{AdS}_3 \times \mathbb{R}^2\).

These solutions can be lifted to ten-dimensional type IIB supergravity on a 5-sphere using the Kaluza-Klein reduction ansatz in \([9]\) to give

\[
ds_{10}^2 = \sqrt{\Delta} \, ds_5^2 + \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{3} X_i^{-1} \left( d\mu_i^2 + \mu_i^2 D\phi_i^2 \right),
\]

\[
F_{(5)} = L^3 \epsilon_{(3)} \wedge \sum_{i=1}^{3} \left[ 2(X_i^2 \mu_i^2 - \Delta X_i) \, dy_1 \wedge dy_2 + q X_i^{-1} d(\mu_i^2) \wedge D\phi_i \right] + \text{dual},
\]

where \(\epsilon_{(3)}\) is the volume-form for \(\text{AdS}_3\),

\[
D\phi_i = d\phi_i + A^i, \quad \Delta = \sum_{i=1}^{3} X_i \mu_i^2, \quad dA^i = F_{(2)}^i,
\]

and the \(\mu_i\) obey the constraint \(\sum_{i=1}^{3} \mu_i^2 = 1\).

We will consider deformations of these backgrounds which leave the \(\text{AdS}_3 \times \mathbb{R}^2\) subspace intact. Such deformations can be considered to be marginal in the sense that they maintain the two-dimensional conformal symmetry of the undeformed field theory. In order to study deformations which preserve a \(U(1) \times U(1)\) global symmetry as well as the \((0, 2)\) supersymmetry, it is convenient to define the coordinates

\[
\phi_1 = \psi - \tilde{\phi}_2, \quad \phi_2 = \psi + \tilde{\phi}_1 + \tilde{\phi}_2, \quad \phi_3 = \psi - \tilde{\phi}_1.
\]

We T-dualize to type IIA theory along the \(\tilde{\phi}_1\) direction and lift the resulting solution to eleven dimensions. Next, we perform the coordinate transformation

\[
\tilde{\phi}_2 \rightarrow \tilde{\phi}_2 + \gamma \tilde{\phi}_1 + \sigma x_{11},
\]

where \(x_{11}\) is the eleventh direction. Reducing along the transformed \(x_{11}\) direction and T-dualizing
back along the transformed \( \tilde{\phi}_1 \) direction yields the deformed type IIB solution

\[
\begin{align*}
\text{ds}_{10}^2 &= \frac{1}{G^2 \Delta} \left[ \Delta \text{ds}_5^2 + \sum_{i=1}^{3} X_i^{-1} \left( d\mu_i^2 + G\mu_i^2 D\phi_i^2 \right) + (\gamma^2 + \sigma^2) \frac{G}{\Delta} \prod_{i=1}^{3} \frac{\mu_i^2}{X_i} \left( \sum_{i=1}^{3} D\phi_i \right)^2 \right], \\
F_{(5)} &= L^3 \epsilon_{(5)} \Delta \sum_{i=1}^{3} \left[ 2(X_i^2 \mu_i^2 - \Delta X_i) \, dy_1 \wedge dy_2 + q^i X_i^{-1} d(\mu_i^2) \wedge D\phi_i \right] + \text{dual}, \\
F_{(3)}^{RR} &= \sigma \, dB_{(2)} - \gamma \, GHC_{(3)}, \quad F_{(3)}^{NS} = \gamma \, d[GHB_{(2)}] + \sigma \, C_{(3)}, \\
e^{2\phi} &= GH^2, \quad \chi = -\gamma \sigma gH^{-1},
\end{align*}
\]

where

\[
\begin{align*}
B_{(2)} &= \frac{1}{\Delta H} \sum_{j<k} (-1)^{j+k} \frac{\mu_j^2 \mu_k^2}{X_j X_k} \, D\phi_j \wedge D\phi_k, \\
* C_{(3)} &= L^3 \epsilon_{(3)} \wedge \sum_{i=1}^{3} \left[ 2(X_i^2 \mu_i^2 - \Delta X_i) \, dy_1 \wedge dy_2 + q^i X_i^{-1} d(\mu_i^2) \wedge D\phi_i \right] \wedge B_{(2)},
\end{align*}
\]

and

\[
\begin{align*}
H &= 1 + \sigma^2 g, \quad G^{-1} = 1 + (\gamma^2 + \sigma^2) g, \quad g = \frac{1}{\Delta} \sum_{j<k} \frac{\mu_j^2 \mu_k^2}{X_j X_k}.
\end{align*}
\]

We propose that the \((0,2)\) superpotential of the deformed theory changes as follows:

\[
c_{ijk} \Theta_1 \Theta_2 \Psi_{3k} \rightarrow e^{i\pi \beta} c_{ijk} \Theta_1 \Theta_2 \Psi_{3k},
\]

where \( \beta = \gamma - i\sigma \). This can arise in the low-energy limit of a flow from a marginal deformation of \( \mathcal{N} = 4 \) super Yang-Mills theory, whose supergravity dual is described by the Lunin-Maldacena background [6].

3 Marginal deformations of theories with Schrödinger symmetry

3.1 An example with a five-sphere

We will now consider gravity duals of theories which exhibit Schrödinger symmetry, which can be used to study non-relativistic systems such as atomic gases at ultra-low temperature [3, 4]. An example of such a solution in type IIB theory is given by [15, 16]

\[
\begin{align*}
\text{ds}_{10}^2 &= r^2 (-2 dt dy - r^2 dt^2 + dx^2) + \frac{dr^2}{r^2} + \text{ds}_{S^5}^2, \\
F_{(3)} &= 4 (\Omega_{(3)} + \ast \Omega_{(3)}), \\
B_{(2)}^{NS} &= -r^2 dt \wedge (dp\psi + A_{(1)}),
\end{align*}
\]

where \( F_{(3)}^{NS} = dB_{(2)}^{NS} \) and \( \Omega_{(3)} \) is the volume-form on a unit \( S^5 \). The metric on \( S^5 \) has been expressed as a \( U(1) \) bundle over \( \mathbb{CP}^2 \):

\[
ds_{S^5}^2 = (d\psi + A_{(1)})^2 + ds_{\mathbb{CP}^2}^2,
\]

with the metric for the $\mathbb{CP}^2$ base space
\[ ds_{\mathbb{CP}^2} = d\sigma^2 + \frac{1}{4} s_\sigma^2 (d\theta^2 + s_\theta^2 d\phi^2) + \frac{1}{4} s_\sigma^2 c_\sigma^2 (d\beta + c_\theta d\phi)^2, \tag{3.3} \]
and the Kähler potential
\[ A_{(1)} = \frac{1}{4} s_\sigma^2 (d\beta + c_\theta d\phi). \tag{3.4} \]

The solution (3.3) can be obtained by applying a null Melvin twist to the AdS$_5 \times S^5$ background, which enables the dual field theory interpretation of the result to be $\mathcal{N} = 4$ super Yang-Mills twisted by an R-charge \cite{15, 16}. Rather than obeying conformal scale invariance, the temporal and spatial coordinates in the theory scale anisotropically. This corresponds to the metric in (3.1) obeying the scaling relation
\[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad r \rightarrow \lambda^{-1} r, \quad y \rightarrow \lambda^{2-z} y, \tag{3.5} \]
where in this case the dynamical exponent $z = 2$. If the momentum along the $y$ direction is interpreted as rest mass, then this geometry describes a system which exhibits time and space translation invariance, spatial rotational symmetry and invariance under the combined operations of time reversal and charge conjugation.

One can apply U-duality to generate deformations of the AdS$_5 \times S^5$ solution using the $\beta$ and $\phi$ directions, which correspond to marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory that preserve conformal symmetry and $\mathcal{N} = 1$ supersymmetry \cite{6}. A null Melvin twist of this deformed solution has been presented in \cite{17}. Since this null Melvin twist does not involve the $\beta$ and $\phi$ directions, one can obtain the same result by generating the marginal deformations directly on the solution (3.1). The final solution can be interpreted as describing marginally deformed super Yang-Mills theory twisted by an R-charge. Although the conformal symmetry has been broken by the null Melvin twist, the deformations are marginal with respect to the Schrödinger symmetry.

Alternatively, one can generate marginal deformations using the $U(1)$ symmetry associated with the $y$ direction. Working directly from the solution given by (3.1), we perform T-duality along the $\beta$ direction and lift to eleven dimensions. Then we perform the coordinate transformation $y \rightarrow y + \gamma_1 \beta + \gamma_2 x_{11}$, where $x_{11}$ is the eleventh direction. Upon reducing to type IIA theory along the transformed $x_{11}$ direction and T-dualizing along the transformed $\beta$ direction, we obtain the deformed type IIB solution
\begin{align*}
  ds_{10}^2 &= r^2 (-2 dt dy - r^2 H dt^2 + d\vec{x}^2) + \frac{dr^2}{r^2} + ds_{S^5}^2, \\
  F_{(5)} &= 4 (\Omega_{(5)} + *\Omega_{(5)}), \\
  C_{(2)}^{RR} &= \gamma_2 r^2 dt \wedge (s_\sigma^2 d\psi + A_{(1)}), \\
  B_{(2)}^{NS} &= -r^2 dt \wedge [(1 + \gamma_1^2 s_\sigma^2) d\psi + (1 + \gamma_1) A_{(1)}], \\
  \phi &= \chi = 0, 	ag{3.6} \end{align*}
where
\[ H = 1 + (\gamma_1 + \gamma_1^2 + \gamma_2^2) s_\sigma^2, \tag{3.7} \]
and $F_{(3)}^{RR} = dC_{(2)}^{RR}$. Since the Schrödinger portion of the geometry remains intact for constant $\sigma$ slicings, the dual field theory retains the Schrödinger symmetry.

In contrast to the previously-mentioned case, the procedure for obtaining these marginal deformations does not commute with the null Melvin twist, since the latter operation entails taking a lightlike boost in the $y$ direction. In fact, the order of operations has a qualitative effect on the field theory interpretation of the deformations. Namely, if the above deformations had been generated on the AdS$_5 \times S^5$ background, the result would have been a nonlocal field theory [18].

3.2 Countably-infinite examples with the $L^{p,q,r}$ spaces

The above construction can be generalized to gravity duals that involve the countably-infinite five-dimensional cohomogeneity two Sasaki-Einstein spaces $L^{p,q,r}$ [19]. The metric for the $L^{p,q,r}$ spaces can be written in the canonical form

$$ds^2_{L^{p,q,r}} = (d\tau + A_{(1)})^2 + ds^2_4,$$

(3.8)

where the metric of the four-dimensional Einstein-Kähler base space is

$$ds^2_4 = \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2} \left( \frac{s_\theta^2}{\alpha} d\phi + \frac{c_\theta^2}{\beta} d\psi \right)^2 + \frac{\Delta_\theta s_\theta^2 c_\theta^2}{\rho^2} \left( \frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right)^2,$$

(3.9)

the Kähler potential is

$$A_{(1)} = \frac{\alpha - x}{\alpha} s_\theta^2 d\phi + \frac{\beta - x}{\beta} c_\theta^2 d\psi,$$

(3.10)

and the various functions are given by

$$\Delta_x = x(\alpha - x)(\beta - x) - \mu, \quad \Delta_\theta = \alpha c_\theta^2 + \beta s_\theta^2, \quad \rho^2 = \Delta_\theta - x.$$

(3.11)

Details regarding the conditions that ensure that the $L^{p,q,r}$ metric extends smoothly onto a complete and non-singular manifold are given in [19]. The result is that the parameters $\alpha$ and $\beta$ as well as the roots of $\Delta_x$ can be expressed in terms of coprime integer triples $p$, $q$ and $r$ which satisfy $0 < p \leq q$ and $0 < r < p + q$, with $p$ and $q$ each coprime to $r$ and to $s = p + q - r$. The $L^{p,q,r}$ spaces have $U(1)^3$ isometry in general, which is enlarged to $SU(2) \times U(1)^2$ for $p + q = 2r$, which corresponds to the subset of $Y^{p,q} = L^{p-q,p+q,p}$ spaces found in [20][21]. Note that the $L^{p,q,r}$ metric reduces to that of $S^5$ for $\mu = 0$, and $\mu$ can otherwise be rescaled to $\mu = 1$.

Consider the type IIB solution

$$ds^2_{10} = r^2 \left( -2dt dy - r^2 dt^2 + r^2 dx^2 \right) + \frac{dr^2}{r^2} + ds^2_{L^{p,q,r}},$$

$$F_{(5)} = 4(\epsilon_{(5)} + \ast \epsilon_{(5)}),$$

$$B_{NS}^{(2)} = -r^2 dt \wedge (d\tau + A_{(1)}),$$

(3.12)

which can be obtained by applying a null Melvin twist to the AdS$_5 \times L^{p,q,r}$ background. $\epsilon_{(5)}$ denotes the volume-form on the $L^{p,q,r}$ space. We will first consider marginal deformations which involve the $U(1)$ symmetries associated with the $\phi$ and $\psi$ directions. We T-dualize the solution (3.12) along the
φ direction, lift to eleven dimensions and perform the coordinate transformation \( \psi \rightarrow \psi + \gamma \phi + \sigma x_{11} \), where \( x_{11} \) is the eleventh direction. Upon reducing and T-dualizing back to type IIB theory along the transformed \( x_{11} \) and \( \phi \) directions, respectively, we obtain the deformed solution

\[
\begin{align*}
 ds_{10}^2 &= G^{-1/4} \left[ \rho^2 \left( -2 dt dy - r^2 d\phi^2 + dx^2 \right) + \frac{dr^2}{r^2} + \frac{\rho^2}{4\Delta_x} dx^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + f d\tau^2 + \frac{Gc_\theta^2}{\beta^2 \rho^2} D\psi^2 \\
 &\quad + \frac{Gc_\theta^2}{\alpha^2 \rho^2 a} (d\phi + B_{(1)})^2 \right], \\
 F_{(5)} &= \frac{2\rho^2 s_\theta c_\theta}{\alpha \beta} d\tau \wedge dx \wedge d\theta \wedge \left(G d\psi + \gamma \frac{\alpha - x}{\alpha} r^2 s_\theta^2 dt \right) \wedge (d\phi + B_{(1)}) + \text{dual}, \\
 F_{RR}^{(3)} &= \frac{2\gamma}{\alpha \beta \rho^2} s_\theta c_\theta d\tau \wedge dx \wedge d\theta + \frac{\sigma}{H} \left( \frac{\gamma g}{\alpha} d[(\alpha - x) r^2 s_\theta^2 dt] - G^{-1} d[g G D\psi] \right) \wedge (d\phi + B_{(1)}), \\
 F_{NS}^{(3)} &= -d[r^2 dt \wedge (d\tau + A_{(1)})] + \gamma d[g G D\psi \wedge (d\phi + B_{(1)})] + \frac{2\sigma}{\alpha \beta} \rho^2 s_\theta c_\theta d\tau \wedge dx \wedge d\theta, \\
 e^{2\phi} &= GH^2, \quad \chi = -\gamma \sigma g H^{-1}, \quad \text{(3.13)}
\end{align*}
\]

where

\[
\begin{align*}
 B_{(1)} &= \alpha \rho^2 (\alpha - x) a d\tau - \frac{\alpha}{\beta} a c_\theta^2 d\psi - \frac{\gamma}{\beta} r^2 c_\theta^2 dt, \\
 D\psi &= d\psi + \beta \rho^2 be d\tau + \gamma r^2 s_\theta^2 \left( \frac{\alpha - x}{\alpha} \right) dt, \quad \text{(3.14)}
\end{align*}
\]

\[
H = 1 + \sigma^2 g, \quad G^{-1} = 1 + (\gamma^2 + \sigma^2) g, \quad g = \frac{s_\theta^2 c_\theta^2}{\alpha^2 \beta^2 \rho^4 a b}, \quad \text{(3.15)}
\]

and we have defined the functions

\[
\begin{align*}
 a^{-1} &= \alpha (\alpha - x) \rho^2 - \mu s_\theta^2, \\
 b^{-1} &= \beta (\beta - x) \rho^2 - \mu c_\theta^2 - a s_\theta^2 c_\theta^2, \\
 e &= \beta - x + (\alpha - x) a s_\theta^2, \\
 f &= 1 - (\alpha - x)^2 \rho^2 a s_\theta^2 - \rho^2 be^2 c_\theta^2. \quad \text{(3.16)}
\end{align*}
\]

Since the above procedure for generating marginal deformations commutes with the null Melvin twist, the same result can be obtained by applying the null Melvin twist to the marginal deformations of \( \text{AdS}_5 \times L^{p,q,r} \) that are described in [22]. This enables us to interpret the final solution as describing marginal deformations of the corresponding quiver gauge theories twisted by an R-charge, for which conformal symmetry is broken but the Schrödinger symmetry is preserved.

The field theories have \( p + 3q \) chiral fields which come in six different types: \( q \) \( Y \), \( (p + q - r) \) \( U_1 \), \( r \) \( U_2 \), \( p \) \( Z \), \( (r - p) \) \( V_1 \) and \( (q - r) \) \( V_2 \) fields. For the \( Y^{p,q} \) subset, the \( U_i \) fields become a doublet under the \( SU(2) \) flavor symmetry. For the undeformed \( L^{p,q,r} \) theories, a superpotential can be built out of these fields which has the following schematic form [23]:

\[
W = 2p \text{ Tr}(YU_1 ZU_2) + 2(q - r) \text{ Tr}(YU_1 V_1) + 2(r - p) \text{ Tr}(YU_2 V_2). \quad \text{(3.17)}
\]
For the above marginal deformations, the quartic portion of the superpotential is altered as follows:

$$\text{Tr}(YU_1ZU_2 - YU_2ZU_1) \rightarrow \text{Tr}\left(e^{i\tilde{\beta}} YU_1ZU_2 - e^{-i\tilde{\beta}} YU_2ZU_1\right),$$  \hspace{1cm} (3.18)

where the complex deformation parameter $\tilde{\beta} = \gamma - i\sigma$. This interpretation of the marginal deformations survives taking the null Melvin twist.

Now we consider marginal deformations which involve the $U(1)$ symmetries associated with the $y$ and $\phi$ directions. We T-dualize along the $\phi$ direction, lift to eleven dimensions and perform the coordinate transformation $y \rightarrow y + \gamma_1\phi + \gamma_2 x_{11}$. Upon reducing to type IIA theory along the transformed $x_{11}$ direction and T-dualizing along the transformed $\phi$ direction, we obtain the deformed type IIB solution

$$ds_{10}^2 = r^2(-2dtdy - r^2H dt^2 + d\vec{x}^2) + \frac{dr^2}{r^2} + ds_{S^5}^2,$$

$$F(5) = 4(\epsilon(5) + *\epsilon(5)),$$

$$C_{(2)}^{RR} = \gamma_2 \frac{r^2 s_\theta^2}{\alpha^2 \rho^2 b} dt \wedge (d\phi + B_{(1)}),$$

$$B_{(2)}^{NS} = -r^2 dt \wedge \left((d\tau + A_{(1)}) + \frac{\gamma_1 s_\theta^2}{\alpha^2 \rho^2 b}(d\phi + B_{(1)})\right),$$

$$\phi = \chi = 0,$$  \hspace{1cm} (3.19)

where

$$B_{(1)} = \frac{\alpha \rho H}{\beta} \left(\beta(\alpha - x) \rho \ d\tau - c_\theta^2 \ d\psi\right),$$  \hspace{1cm} (3.20)

and

$$H = 1 + s_\theta^2 \left[\gamma_1^2 + \frac{\gamma_2^2}{\alpha^2 \rho^2 b} + 2\gamma_1 \left(\frac{x - \alpha}{\alpha}\right)\right].$$  \hspace{1cm} (3.21)

For constant slicings of $\theta$ and $x$, the Schrödinger portion of the geometry remains intact, and so the dual field theory still has Schrödinger symmetry.

4 Marginal deformations of Lifshitz vacua

4.1 Lifshitz-Chern-Simons gauge theories

We will now consider some examples of gravity dual descriptions of marginal deformations of field theories with Lifshitz scaling \[2\]. A candidate gravity dual for a class of $2 + 1$ dimensional Lifshitz-Chern-Simons field theories with dynamical exponent $z = 2$ was constructed in \[24\]. The type IIB solution is given by

$$ds_{10}^2 = r^2(2dtdx_3 + d\vec{x}^2) + f \ dx_3^2 + \frac{dr^2}{r^2} + ds_{S^5}^2,$$

$$F(5) = 2r^3 d^4x \wedge dr + \text{dual},$$

$$\chi = \frac{Q x_3}{L_3}, \quad \phi = 0,$$  \hspace{1cm} (4.1)

\[1\]We put a tilde in order to distinguish this from the parameter $\beta$ in the $L_p,q,r$ metric.
where
\[ f = \frac{Q^2}{4L_3^2} - \frac{r_0^4}{r^2}, \] (4.2)
and \( d\bar{x}^2 = dx_1^2 + dx_2^2 \). It has been proposed that this background describes non-Abelian Lifshitz Chern-Simons gauge theories which can be realized as deformations of DLCQ \( N = 4 \) super Yang-Mills theory. A Chern-Simons term explicitly breaks parity and time-reversal symmetries. The background (4.1) asymptotically exhibits the following scaling symmetry:
\[ t \to \lambda^2 t, \quad \vec{x} \to \lambda \vec{x}, \quad r \to \lambda^{-1} r, \quad x_3 \to x_3. \] (4.3)

\( x_3 \) is a compact direction and therefore cannot scale, since scaling would change the compactification radius. The \( x_3 \) circle shrinks to zero size at
\[ r = r_\ast = \frac{2L_3 r_0^2}{Q}. \] (4.4)
The metric is geodesically incomplete in the region \( r \geq r_\ast \), and a straightforward way of extending geodesics past \( r_\ast \) leads to closed timelike curves [25]. The implications of this hidden singularity on the dual field theory as well as its resolutions are discussed in [24].

The metric on \( S^5 \) can be expressed as
\[ ds_{S^5}^2 = \sum_{i=1}^{3} (d\mu_i^2 + \mu_i^2 d\phi_i^2), \] (4.5)
where \( \sum_{i=1}^{3} \mu_i^2 = 1 \). Before applying the solution-generating technique, we perform the coordinate transformation (2.10). We T-dualize along the \( x_3 \) direction to obtain a solution in massive type IIA theory and perform the transformation
\[ \bar{\phi}_1 \to \bar{\phi}_1 + \gamma_1 x_3. \] (4.6)

Then we T-dualize back to type IIB theory along the transformed \( x_3 \) direction to obtain the deformed type IIB solution
\[
\begin{align*}
ds_{10}^2 &= \frac{1}{G^{1/4}} \left[ -\frac{r^4}{f} dt^2 + \frac{r^2}{r^2} + G f \left( dx_3 + \frac{r^2}{f} dt \right)^2 + ds_{S^5}^2 - \gamma_1^2 f G d\phi_{23} \right], \\
F_{(5)} &= 2r^3 d^4 x \wedge dr + \text{dual}, \\
F_{(3)}^{RR} &= \gamma_1 \frac{QGr^2}{L_3} dt \wedge dx_3 \wedge d\phi_{23}, \\
e^{2\phi} &= G, \\
\chi &= \frac{Qx_3}{L_3},
\end{align*}
\] (4.7)
where
\[ d\phi_{23} \equiv \mu_2^2 d\phi_2 - \mu_3^2 d\phi_3, \quad G^{-1} = 1 + \gamma_1^2 f (\mu_2^2 + \mu_3^2). \] (4.8)

While the solution (4.1) has a null Killing vector, the deformed solution (4.7) has a timelike Killing vector \( \partial_t \) generating the time translations of the dual field theory for \( r > 0 \) and \( \mu_2^2 + \mu_3^2 > 0 \).
A different deformation can be obtained by T-dualizing along the \( \tilde{\phi}_1 \) direction to type IIA theory, performing the transformation

\[
\tilde{\phi}_2 \rightarrow \tilde{\phi}_2 + \gamma_2 \tilde{\phi}_1,
\]

and T-dualizing back to type IIB theory along the transformed \( \tilde{\phi}_1 \) direction. The resulting deformed type IIB solution is given by

\[
ds_{10}^2 = \frac{1}{G^{1/4}} \left[ r^2 (2 dt dx_3 + dx^2) + f dx_3^2 + \frac{dr^2}{r^2} + \sum_{i=1}^3 (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \gamma_2^2 G \prod_{j=1}^3 \mu_j^2 \left( \sum_{k=1}^3 d\phi_k \right)^2 \right],
\]

\[
F_{(5)} = 2 r^3 d^4 x \wedge dr + \text{dual},
\]

\[
F_{(3)}^{RR} = \gamma_2 \frac{QG}{L_3} dx_3 \wedge \sum_{j<k} (1)^{j+k} \mu_j^2 \mu_k^2 d\phi_j \wedge d\phi_k + \frac{3}{2} \gamma_2 d(\mu_2^2) \wedge d(\mu_3^2) \wedge d\psi,
\]

\[
F_{(3)}^{NS} = \gamma_2 d \left[ G \sum_{j<k} (1)^{j+k} \mu_j^2 \mu_k^2 d\phi_j \wedge d\phi_k \right],
\]

\[
e^{2\phi} = G,
\]

where

\[
G^{-1} = 1 + \gamma_2^2 \sum_{j<k} \mu_j^2 \mu_k^2.
\]

Note that the Killing vector \( \partial_t \) remains null for this deformation. If the \( \gamma_1 \) and \( \gamma_2 \) deformations are turned on at the same time, then the fiber structure of the resulting solution implies that the \( x_3 \) direction is periodic. However, we will restrict ourselves to the scenario in which the \( x_3 \) direction is extended.

### 4.2 Countably-infinite Lifshitz vacua with dynamical exponent \( z = 2 \)

Infinite families of Lifshitz solutions of ten and eleven-dimensional supergravity with dynamical exponent \( z = 2 \) were constructed in [26]. As the starting point for applying the solution-generating technique, we will consider solutions in eleven-dimensional supergravity. Before turning to a countably infinite family of solutions involving the \( Y^{p,q} \) spaces, we first consider the eleven-dimensional solution involving the space \( T^{1,1} \), which is given by [26]

\[
ds_{11}^2 = ds_4^2 + \left( d\sigma + \frac{1}{\sqrt{18}} (c_1 d\phi_1 - c_2 d\phi_2) \right)^2 + dx_{11}^2 + \frac{1}{9} (d\psi - c_1 d\phi_1 - c_2 d\phi_2)^2
\]

\[
+ \frac{1}{6} (d\theta_1^2 + s_1^2 d\phi_1^2 + d\theta_2^2 + s_2^2 d\phi_2^2),
\]

\[
G_{(4)} = dt \wedge d \left[ r^4 dx_1 \wedge dx_2 + r^2 dx_{11} \wedge \left( d\sigma + \frac{1}{\sqrt{18}} (c_1 d\phi_1 - c_2 d\phi_2) \right) \right],
\]

where

\[
ds_4^2 = -r^4 dt^2 + r^2 (dx_1^2 + dx_2^2) + \frac{dr^2}{r^2},
\]

\[
e^{2\phi} = G,
\]

\[
\chi = \frac{Q x_3}{L_3},
\]

\[
G^{-1} = 1 + \gamma_2^2 \sum_{j<k} \mu_j^2 \mu_k^2.
\]
and we denote \( c_1 \equiv \cos \theta_1, s_2 \equiv \sin \theta_2 \), etc. We perform the coordinate transformation

\[
\phi_2 \to \phi_2 + \gamma x_{11},
\]

(4.14)

reduce to type IIA theory along the transformed \( x_{11} \) direction and T-dualize along the transformed \( \sigma \) direction to obtain the type IIB solution

\[
ds_{10}^2 = G^{-1/4} \left[ ds^2_4 + \frac{1}{6} (d\theta_1^2 + s_1^2 d\phi_1^2 + d\theta_2^2 + G K s_2^2 d\phi_2^2) + \frac{G}{9K} (d\psi - c_1 d\phi_1 - K c_2 d\phi_2)^2 \right. \\
+ \left. G (d\sigma + r^2 dt)^2 \right],
\]

(4.15)

where

\[
F_{(5)} = 4r^3 d\sigma \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 - \gamma hr^2 \left( 1 + \frac{1}{18} \gamma G c_2^2 \right) d\sigma \wedge dt \wedge C_2(2) \wedge B_{(1)} + \text{dual},
\]

\[
F^{RR}_{(3)} = -\gamma H d \left[ B_{(1)} \wedge (d\sigma + r^2 dt) \right] - \frac{1}{\sqrt{18} \gamma H c_2} C_2(2) \wedge d\sigma + \frac{1}{2} HG dG^{-1} \wedge B_{(1)} \wedge (d\sigma + r^2 dt),
\]

\[
F^{NS}_{(3)} = C_2(2) \wedge d\sigma - \frac{1}{\sqrt{18} \gamma} d \left[ G c_2 B_{(1)} \wedge (d\sigma + r^2 dt) \right],
\]

\[
e^{2\phi} = Gh^{-2}, \quad \chi = \frac{1}{\sqrt{18} \gamma} H c_2,
\]

and

\[
G^{-1} = 1 + \frac{\gamma^2}{9}, \quad h^{-1} = 1 + \frac{\gamma^2}{6}, \quad K^{-1} = 1 + \frac{\gamma^2}{6} s_2^2.
\]

(4.17)

Note that while the Killing vector \( \partial_t \) is null for \( \gamma = 0 \), it is timelike for \( \gamma \neq 0 \) and \( r > 0 \).

The above construction can be generalized to gravity duals that involve the Sasaki-Einstein spaces \( Y^{p,q} \), which are characterised by two coprime positive integers \( p \) and \( q \) with \( q < p \) [20,21]. We begin with the eleven-dimensional solution [26]

\[
ds_{11}^2 = f^{1/3} ds^2_4 + f^{-2/3} \left[ \left( d\sigma - \frac{D\beta}{\sqrt{72}(1-y)} \right)^2 + dx_{11}^2 \right] \\
+ \frac{1}{9} \left( d\psi + Y D\beta - c_2 d\phi \right)^2 + \frac{1-y}{6} (d\theta^2 + s_2^2 d\phi^2) + \frac{dy^2}{g} + \frac{g}{36} D\beta^2, \]

\[
G_4 = dt \wedge dx_1 \wedge dx_2 + \frac{r^2}{f} dx_{11} \wedge \left( d\sigma - \frac{D\beta}{\sqrt{72}(1-y)} \right),
\]

(4.18)

where

\[
ds^2_4 = -\frac{r^4}{f} dt^2 + r^2(dx_1^2 + dx_2^2) + \frac{dr^2}{r^2},
\]

\[
D\beta = d\beta + c_2 d\phi, \quad g = \frac{2(a - 3y^2 + 2y^3)}{1-y},
\]

(4.19)

and \( c_2 \equiv \cos \theta, s_2 \equiv \sin \theta \). We have expressed the metric of the \( Y^{p,q} \) subspace in canonical form as a \( U(1) \) bundle over an Einstein-Kähler metric. The function \( f \) satisfies

\[
-4f + \frac{2}{1-y} \partial_y \left[ (a - 3y^2 + 2y^3) \partial_y f \right] + \frac{1}{(1-y)^4} = 0.
\]

(4.21)
Upon performing the coordinate transformation

$$\beta \to \beta + \gamma x_{11}, \quad (4.22)$$

reducing to type IIA theory along the transformed $x_{11}$ direction and T-dualizing along the transformed $\sigma$ direction, we obtain the type IIB solution

$$ds_{10}^2 = G^{-1/4} \left[ ds_4^2 + \frac{1 - y}{6} (d\theta^2 + s_9^2 d\phi^2) + \frac{dy^2}{g} + \frac{Kg}{36} D\beta^2 + \frac{G}{9 K} \left( D\psi - \frac{\gamma^2}{36} K f g D\beta \right) \right]^2$$

$$+ f G \left( d\sigma + \frac{r^2}{f} dt \right)^2,$$

$$F^{(5)} = 4 r^3 \left( d\sigma \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 \right) + \gamma \frac{G r^2}{f} \left( d\sigma \wedge dt \wedge dB_{(1)} \wedge C_{(1)} \right) + \text{dual},$$

$$F^{RR}_{(3)} = \gamma d \left[ \frac{H}{\sqrt{72(1-y)}} \right] \wedge A_{(1)} \wedge \left( d\sigma + \frac{r^2}{f} dt \right) + \gamma d \left[ H C_{(1)} \wedge \left( d\sigma + \frac{r^2}{f} dt \right) \right]$$

$$- \gamma \frac{H}{\sqrt{72(1-y)}} dB_{(1)} \wedge d\sigma,$$

$$F^{NS}_{(3)} = d \left[ B_{(1)} \wedge d\sigma + A_{(1)} \wedge \left( d\sigma + \frac{r^2}{f} dt \right) \right],$$

$$e^{2\phi} = G H^{-2}, \quad \chi = \frac{\gamma H}{\sqrt{72(1-y)}}, \quad (4.23)$$

where

$$A_{(1)} = \gamma^2 G \frac{1}{\sqrt{72(1-y)}} C_{(1)}, \quad B_{(1)} = - \frac{D\beta}{\sqrt{72(1-y)}}, \quad C_{(1)} = \frac{f y}{9} D\psi + \frac{f g}{36} D\beta, \quad (4.24)$$

$$D\sigma = d\sigma + B_{(1)}, \quad D\psi = d\psi + y D\beta - c_6 d\phi, \quad (4.25)$$

and

$$K^{-1} = 1 + \frac{\gamma^2}{36} f g, \quad G^{-1} = K^{-1} + \frac{\gamma^2}{9} f y^2, \quad H^{-1} = G^{-1} + \frac{\gamma^2}{72(1-y)^2}. \quad (4.26)$$

The Killing vector $\partial_t$ is null for $\gamma = 0$ and timelike for $\gamma \neq 0$ and $r > 0$.

Alternatively, one can perform the coordinate transformation

$$\phi \to \phi + \gamma x_{11}. \quad (4.27)$$

Then reducing along the transformed $x_{11}$ direction and T-dualizing along the transformed $\sigma$ direc-
tion yields the type IIB solution

\[
\begin{align*}
\text{ds}_{10}^2 &= G^{-1/4} \left[ \text{ds}_4^2 + \frac{1-y}{6} (d\theta^2 + L_s^2 d\phi^2) + \frac{dy^2}{g} + \frac{Kg}{36L} \left( D\beta - \frac{\gamma^2}{9} L f(1-y)s_\theta^2 c_\theta d\phi \right)^2 \right. \\
& \quad + \left. \frac{G}{9K} \left[ D\psi + \frac{\gamma^2}{36} K f(1-y) \left( gc_\theta D\beta + 6(1-y)s_\theta^2 d\phi \right) \right]^2 + Gf \left( ds_4^2 + \frac{r^2}{f} dt \right)^2 \right], \\
F_{(5)} &= 4r^3 \left[ ds_4 \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 - \frac{\gamma}{9} Gr^2 (1-y)c_\theta \left( ds_4 \wedge dt \wedge dB^{(1)} \wedge D\psi + \text{dual} \right) \right], \\
F_{RR}^{(3)} &= \gamma d \left[ \frac{H c_\theta}{\sqrt{72}(1-y)} \right] \wedge A_{(1)} \wedge \left( ds + \frac{r^2}{f} dt \right) - \frac{\gamma}{9} d \left[ H f (1-y)c_\theta D\psi \wedge \left( ds + \frac{r^2}{f} dt \right) \right], \\
e^{2\phi} &= GH^{-2}, \quad \chi = \frac{\gamma H c_\theta}{\sqrt{72}(1-y)},
\end{align*}
\] (4.28)

where now we take

\[
A_{(1)} = -\frac{\gamma^2 G f c_\theta^2}{9\sqrt{72}} D\psi, \quad B_{(1)} = -\frac{\sqrt{72}(1-y)}{\sqrt{72}(1-y)},
\] (4.29)

\[
D\psi = D\psi - \frac{g D\beta}{4(1-y)} - \frac{3s_\theta^2}{2c_\theta} d\phi,
\] (4.30)

and we are now defining

\[
L^{-1} = 1 + \frac{\gamma^2}{6} f(1-y)s_\theta^2, \quad K^{-1} = L^{-1} + \frac{\gamma^2}{36} f g c_\theta^2, \\
G^{-1} = K^{-1} + \frac{\gamma^2}{9} f(1-y)^2 c_\theta^2, \quad H^{-1} = G^{-1} + \frac{\gamma^2 c_\theta^2}{72(1-y)^2}.
\] (4.31)

As with the previous deformation, the Killing vector \( \partial_t \) is null for \( \gamma = 0 \) and timelike for \( \gamma \neq 0 \) and \( r > 0 \).

### 4.3 An example with general dynamical exponent

Lifshitz solutions of Romans’ six-dimensional gauged, massive, \( \mathcal{N} = 4 \) supergravity [27] were found in [28]. These solutions have with general dynamical exponent \( z \geq 1 \) and break supersymmetry. The geometry is a direct product of a four-dimensional Lifshitz geometry and a two-dimensional hyperboloid. The metric is given by

\[
ds_6^2 = L^2 \left( -r^{2z} dt^2 + r^2 (dx_1^2 + dx_2^2) + \frac{dr^2}{r^{2z}} \right) + a^2 dH_2^2,
\] (4.32)

where the metric for a hyperboloid \( dH_2^2 \) can be made compact by modding out a non-compact discrete subgroup of the isometry group. There is a topological restriction on \( z \) in terms of the gauge coupling \( g \) and the mass parameter \( m \) of the six-dimensional theory, due to flux quantization on the compact hyperbolic space.
These solutions can be lifted to massive type IIA theory [29] by using the Kaluza-Klein reduction given in [30]. The resulting ten-dimensional solution is given by

\[
\begin{align*}
\text{ds}_{10}^2 &= S^{1/12} k_0^{1/8} \Delta^{3/8} \left[ ds_6^2 + k_1 d\rho^2 + k_2 \Delta^{-1} C^2 \left[ d\theta^2 + s_0^2 d\phi^2 + (d\psi + c_0 d\phi - g A_{(1)})^2 \right] \right], \\
F_{(4)} &= k_3 S^{1/3} C^3 \Delta^{-2} U s_\theta \rho d\theta \wedge d\phi \wedge (d\psi + c_6 d\phi - g A_{(1)}) + \beta a^2 L k_4 r^z S^{1/3} C \ dt \wedge H_{(2)} \wedge d\rho \\
&+ k_5 S^{1/3} C \ G_{(2)} \wedge (d\psi + c_6 d\phi - g A_{(1)}) \wedge d\rho + k_6 S^{4/3} C^2 \Delta^{-1} s_\theta \ G_{(2)} \wedge d\theta \wedge d\phi, \\
F_{(3)} &= k_7 \beta L^3 r^z S^{2/3} \ dx_1 \wedge dx_2 \wedge dr, \\
e^{2\phi} &= S^{-5/3} \Delta^{1/2} k_0^{-5/2}, \\
\end{align*}
\tag{4.33}
\]

where

\[
dA_{(1)} = G_{(2)} = \alpha L^2 r^{z-1} \ dt \wedge dr + \gamma a^2 \ H_{(2)},
\tag{4.34}
\]

and \(H_{(2)}\) is the volume-form of a unit hyperboloid. The type IIA mass parameter is given by

\[
M = \left( \frac{2 mg^3}{27} \right)^{1/4}.
\tag{4.35}
\]

The various functions are given by

\[
\begin{align*}
\Delta &= k_0 \ C^2 + k_0^{-3} S^2, \\
U &= k_0^{-6} S^2 - 3k_0^2 \ C^2 + 4k_0^{-2} C^2 - 6k_0^{-2}, \\
C &= \cos \rho, \\
S &= \sin \rho, \\
c_\theta &= \cos \theta, \\
s_\theta &= \sin \theta,
\end{align*}
\tag{4.36}
\]

and the various constants are

\[
\begin{align*}
k_0 &= e^{\phi_0/\sqrt{2}} \left( \frac{g}{3m} \right)^{1/4}, \\
k_1 &= \frac{8}{3mg} \ e^{\sqrt{2} \phi_0}, \\
k_2 &= \frac{2}{g^2} \left( \frac{g}{3m} \right)^{1/4} e^{-\phi_0/\sqrt{2}}, \\
k_3 &= -\frac{4\sqrt{2}}{3g^3} \left( \frac{g}{3m} \right)^{3/4}, \\
k_4 &= 3g^2 \ e^{2\sqrt{2} \phi_0} k_3, \\
k_5 &= 3g \ k_3, \\
k_6 &= -\frac{\sqrt{2}}{g^2} \ e^{-3\phi_0/\sqrt{2}}, \\
k_7 &= \sqrt{\frac{12m}{g}}, \tag{4.37}
\end{align*}
\]

and

\[
\begin{align*}
L^2 \beta^2 \ e^{\sqrt{8} \phi_0} &= z - 1, \\
\alpha^2 &= \gamma^2 (z - 1), \\
L^2 \gamma^2 \ e^{-\sqrt{2} \phi_0} &= \frac{(2 + z)(z - 3) \pm 2 \sqrt{2} (z + 4)}{2z}, \\
L^2 g^2 \ e^{\sqrt{2} \phi_0} &= 2z (4 + z), \\
\frac{1}{2} L^2 m^2 \ e^{-3\sqrt{2} \phi_0} &= \frac{6 + z \mp 2 \sqrt{2} (z + 4)}{z}, \\
\frac{L^2}{a^2} &= 6 + 3z \mp 2 \sqrt{2} (z + 4). \tag{4.38}
\end{align*}
\]

Since the metric in (4.33) is singular at \(\rho = 0\) and \(\pi\) and the string coupling diverges there, we will consider this solution away from these regions.

T-dualizing to type IIB theory along the \(\psi\) direction using the extended T-duality transformation rules in [31], performing the transformation

\[
\phi \rightarrow \phi + \sigma \psi, \tag{4.39}
\]

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and T-dualizing back to massive type IIA theory along the transformed $\psi$ direction yields the deformed solution

\begin{align}
 ds_{10}^2 &= G^{-1/4} S^{1/12} k_0^{1/8} \Delta^{3/8} \left[ ds_6^2 + k_1 \, d\rho^2 + k_2 \, \Delta^{-1} C^2 \left( d\theta^2 + s_0^2 d\phi^2 + G(d\psi + c_0 d\phi - gA_{(1)})^2 \right) \right], \\
 F_{(4)} &= k_3 G S^{1/3} C^2 \Delta^{-2} U s_0 \, d\rho \wedge d\theta \wedge d\phi \wedge (d\psi + c_0 d\phi - gA_{(1)}) + \beta a^2 L k_4 r^2 S^{1/3} C \, dt \wedge H(2) \wedge d\rho + k_5 \, S^{1/3} C \, G(2) \wedge (d\psi + c_0 d\phi - gA_{(1)}) \wedge d\rho + k_6 \, S^{4/3} C^2 \Delta^{-1} s_0 \, G(2) \wedge d\theta \wedge d\phi \\
 &+ \sigma \frac{\beta L^3 S^{1/3} k_4 r s_0}{k_0 \Delta^{3/4} S^{1/3}} \frac{G C^7}{k_1} \, dx_1 \wedge dx_2 \wedge d\theta \wedge dr, \\
 F_{(3)} &= k_7 L^3 S^{2/3} \, dx_1 \wedge dx_2 \wedge dr + d \left[ \frac{\sigma G k_3^2 C^4 s_0^2}{\Delta S^{2/3} k_0} \, d\phi \wedge (d\psi - gA_{(1)}) \right], \\
 F_{(2)} &= \frac{\sigma M G}{\Delta S^{2/3} k_0} \frac{k_2^3 C^4 s_0^2}{s_0} \, d\phi \wedge (d\psi - gA_{(1)}) + \sigma \frac{k_3}{k_3} S^{1/3} C^3 \Delta^{-2} U s_0 \, d\theta \wedge d\rho, \\
 e^{2\phi} &= G S^{-5/3} \Delta^{1/2} k_0^{-5/2}, \tag{4.40}
\end{align}

where

\begin{equation}
 G^{-1} = 1 + \sigma^2 \frac{k_3^2 C^4 s_0^2}{k_0 S^{2/3} \Delta}. \tag{4.41}
\end{equation}

### 5 Conclusions

A solution-generating technique based on U-duality has been used to construct supergravity backgrounds that holographically describe the marginal deformations of various non-relativistic field theories which preserve a $U(1) \times U(1)$ global symmetry. For a $(0, 2)$ Landau-Ginsburg theory describing the supersymmetric lowest Landau level, we have proposed that the marginal deformations are associated with the introduction of a phase in the $(0, 2)$ superpotential. This can arise in the low-energy limit of a flow from marginal deformations of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory described by the Lunin-Maldacena background [6].

We have generated the supergravity duals of marginally deformed field theories with Schrödinger symmetry or Lifshitz scaling. This includes a class of Lifshitz-Chern-Simons gauge theories, as well as countably-infinite families of field theories whose dual gravity description involves the Sasaki-Einstein spaces $Y^{p,q}$ and $L^{p,q,r}$. These theories all have dynamical exponent $z = 2$. We have also considered massive type IIA backgrounds which are dual to marginal deformations of Lifshitz theories with general dynamical exponent $z \geq 1$.

With the exception of the last example, we have focused on marginal deformations that preserve supersymmetry; these constructions can be straightforwardly generalized to scenarios in which supersymmetry is not preserved. For instance, one could begin with a supersymmetric gravity background and perform a chain of T-duality-shift-T-duality transformations to generate multiple deformations which do not preserve supersymmetry. If this is done for the case of the $(0, 2)$ Landau-Ginsburg theory, then one obtains the description of a non-supersymmetric system that arises in the low-energy limit of a flow from the $6 + 2$-parameter deformation of super Yang-Mills theory found in [3]. Another possibility is to consider cases for which the undeformed field theory itself
does not preserve any supersymmetry. For instance, one could study gravity duals of field theories with Schrödinger symmetry or Lifshitz scaling that involve five-dimensional Einstein spaces which are not Sasakian. Examples of such spaces include the \( T^{p,q} \) spaces, as well as the \( \Lambda^{p,q,r} \) spaces which encompass the Sasaki-Einstein spaces \( L^{p,q,r} \) \[32\].

While examples without supersymmetry are more realistic, one then has less control over their behavior and must worry about instabilities. However, there are cases of non-supersymmetric string states that may be completely stable. For example, the gravity dual description for the effective field theory of the lowest Landau level presented in \[5\] has a regime of parameter space in which all known instabilities are apparently absent.

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