ZETA FUNCTIONS OF COMPLEXES ARISING FROM $\text{PGL}(3)$

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Abstract. In this paper we obtain a closed form expression of the zeta function $Z(X_\Gamma, u)$ of a finite quotient $X_\Gamma = \Gamma \backslash \text{PGL}_3(F)/\text{PGL}_3(OF)$ of the Bruhat-Tits building of $\text{PGL}_3$ over a non-archimedean local field $F$. Analogous to a graph zeta function, $Z(X_\Gamma, u)$ is a rational function and it satisfies the Riemann hypothesis if and only if $X_\Gamma$ is a Ramanujan complex.

1. Introduction

First introduced by Ihara [Ih] for groups and later reformulated by Serre for regular graphs, the zeta function of a finite, connected, undirected graph $X$ is defined as

$$Z(X, u) = \prod_{[C]} (1 - u^{l([C])})^{-1},$$

where the product is over equivalence classes $[C]$ of backtrackless tailless primitive cycles $C$, and $l([C])$ is the length of a cycle in $[C]$. Taking the logarithmic derivative of $Z(X, u)$, one gets

$$Z(X, u) = \exp \left( \sum_{n \geq 1} \frac{N_n}{n} u^n \right),$$

where $N_n$ counts the number of backtrackless and tailless cycles in $X$ of length $n$.

Not only formally analogous to a curve zeta function, the graph zeta function is also a rational function. This can be seen in two ways. The first is the result of Ihara:

**Theorem 1 (Ihara [Ih]).** Suppose $X = (V, E)$ with vertex set $V$ and edge set $E$ is $(q + 1)$-regular. Then its zeta function is a rational function of the form

$$Z(X, u) = \frac{(1 - u^2)^\chi(X)}{\det(I - Au + qu^2I)},$$

where $\chi(X) = \#(V) - \#(E)$ is the Euler characteristic of $X$ and $A$ is the adjacency matrix of $X$.

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If $X$ is not regular, the same expression holds with $qI$ replaced by the diagonal degree matrix of $X$ minus the identity matrix. This was proved by Bass [Ba] and Hashimoto [Ha2], Stark and Terras provided several proofs in [ST], while Hoffman [Ho] gave a cohomological interpretation. The reader is referred to [ST] and the references therein for the history and various zeta functions attached to a graph. Endow two orientations on each edge of $X$. Denote by $A_e$ the associated edge adjacency matrix. Hashimoto [Ha] observed that $N_n = \text{Tr} A_e^n$ so that

$$Z(X, u) = \frac{1}{\det(I - A_e u)}.$$ 

This gives the second viewpoint of the rationality of the graph zeta function.

A $(q + 1)$-regular graph $X$ is called Ramanujan if all eigenvalues $\lambda$ of its adjacency matrix $A$ other than $\pm(q + 1)$ satisfy $|\lambda| \leq 2\sqrt{q}$ (cf. [LPS]). The Ramanujan graphs are optimal expanders with extremal spectral property. It is easily checked that $X$ is Ramanujan if and only if its zeta function $Z(X, u)$ satisfies the Riemann hypothesis, that is, the poles of $Z(X, u)$ other than $\pm 1$ and $\pm q^{-1}$, called nontrivial poles, all have absolute value $q^{-1/2}$ (cf. [ST]).

When $q$ is a prime power, the universal cover of a $(q + 1)$-regular graph can be identified with the $(q + 1)$-regular tree on $\text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F)$ for a nonarchimedean local field $F$ with ring of integers $\mathcal{O}_F$ and $q$ elements in its residue field. Let $\pi$ be a uniformizer of $F$. The vertices of the tree are $\text{PGL}_2(\mathcal{O}_F)$-cosets and the directed edges are $\mathcal{I}$-cosets, where $\mathcal{I}$ is the Iwahori subgroup of $\text{PGL}_2(\mathcal{O}_F)$. Moreover, the (vertex) adjacency operator $A$ on the tree is the Hecke operator on the double coset $\text{PGL}_2(\mathcal{O}_F)\text{diag}(1, \pi)\text{PGL}_2(\mathcal{O}_F)$ and the edge adjacency operator $A_e$ is the Iwahori-Hecke operator on the double coset $\mathcal{I}\text{diag}(1, \pi)\mathcal{I}$. One obtains a $(q + 1)$-regular graph by taking a left quotient by a torsion-free discrete cocompact subgroup of $\text{PGL}_2(F)$.

This set-up has a higher dimensional extension to the Bruhat-Tits building $\mathcal{B}_n$ associated to $\text{PGL}_n(F)/\text{PGL}_n(\mathcal{O}_F)$, which is a simply connected $(q + 1)$-regular $(n - 1)$-dimensional simplicial complex. Its vertices are $\text{PGL}_n(\mathcal{O}_F)$-cosets, naturally partitioned into $n$ types, marked by $\mathbb{Z}/n\mathbb{Z}$. There are $n - 1$ Hecke operators $A_i$, for $1 \leq i \leq n - 1$, supported on $\text{PGL}_n(\mathcal{O}_F)$-double cosets represented by $\text{diag}(\pi, ..., \pi, 1, ..., 1)$ with determinant $\pi^i$. A finite quotient $X_\Gamma = \Gamma\backslash\mathcal{B}_n$ of $\mathcal{B}_n$ by a torsion-free cocompact discrete subgroup $\Gamma$ preserving the types of vertices is again a $(q + 1)$-regular finite complex. It is called a Ramanujan complex if all the nontrivial eigenvalues of $A_i$ on $X_\Gamma$ fall within the spectrum of $A_i$ on the universal cover $\mathcal{B}_n$. See [Li] for more details. Three explicit constructions of infinite families of Ramanujan complexes are given in Li [Li], Lubotzky-Samuels-Vishne [LSVI] and Sarveniazi [Sa], respectively, using deep results on the Ramanujan conjecture over function fields for automorphic representations of the multiplicative group of a
division algebra by Laumon-Rapoport-Stuhler [LRS] and of $GL_n$ by Lafforgue [La]. Further, the paper [LSV2] discusses what kind of $\Gamma$ would fail to yield a Ramanujan complex.

To extend the results from graphs to complexes, one seeks a similarly defined zeta function of closed geodesics in $X_\Gamma$ with the following properties:

1. it is a rational function with a closed form expression;
2. it captures both topological and spectral information of $X_\Gamma$; and
3. it satisfies the Riemann hypothesis if and only if $X_\Gamma$ is a Ramanujan complex.

Questions of this sort were previously considered in Deitmar [De1], [De2], and Deitmar-Hoffman [DH], where partial results were obtained. The cycles we consider are on the 1-skeleton of $X_\Gamma$, up to homotopy in $X_\Gamma$. We are compelled to take homotopy into account in order to eliminate the obvious cycles arising from triangles. This is a major difference from graphs.

The purpose of this paper is to present a zeta function with the asserted properties for the case $n = 3$. In what follows, we fix a local field $F$ with $q$ elements in its residue field as before. Write $G$ for $PGL_3(F)$, $K$ for its maximal compact subgroup $PGL_3(O_F)$, and $B$ for the Bruhat-Tits building $B_3$. Let $\sigma = \begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}$. The Iwahori subgroup of $K$ is

$$B := K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma,$$

which is contained in the subgroup $E := K \cap \sigma K \sigma^{-1} \subset K$. Similar to a tree, the geometric objects in the building $B$ can be parametrized algebraically: the vertices and type one edges are parametrized by the $K$- and $E$-cosets, respectively, while the $B$-cosets parametrize the directed chambers $(C, e)$, where $C$ is a chamber and $e$ an edge of $C$. Write $L_B$ for the Iwahori-Hecke operator on the $B$-double coset $Bt_2\sigma^2 B$, where $t_2 = \begin{pmatrix} \pi^{-1} \\ 1 \end{pmatrix}$.

The zeta function of $X_\Gamma$ is defined as

$$Z(X_\Gamma, u) = \prod_{[C]} (1 - u^{l_A([C])})^{-1},$$

where $[C]$ runs through the equivalence classes of tailless primitive closed geodesics of types one and two in $X_\Gamma$ up to homotopy, and $l_A([C])$ is the algebraic length of any geodesic in $[C]$.

**Main Theorem.** Let $\Gamma$ be a discrete cocompact torsion-free subgroup of $G$ such that
While the parallel identity of operators on a (\(q + 1\))-regular finite complex \(X_G = \Gamma \setminus \mathcal{B}\) is a rational function

\[
Z(X_G, u) = \frac{(1 - u^3)^{\chi(X_G)}}{\det(I + L_B u) \det(I - A_1 u + q A_2 u^2 - q^3 u^3 I)},
\]

where \(\chi(X_G)\) is the Euler characteristic of \(X_G\).

\(Z(X_G, u)\) clearly has properties (1) and (2). Now we discuss its connection with the Riemann hypothesis. The trivial zeros of \(\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I)\) arise from the trivial eigenvalues of \(A_1\) and \(A_2\) on \(X_G\); they are 1, \(q^{-1}\), \(q^{-2}\) and their multiples by cubic roots of unity. An equivalent statement for \(X_G\) being Ramanujan is that the nontrivial zeros of \(\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I)\) all have absolute value \(q^{-1}\) (cf. [Li]), which is the Riemann hypothesis for \(Z(X_G, u)\).

Similar to the graph zeta function, our complex zeta function can be expressed as

\[
Z(X_G, u) = \frac{1}{\det(I - L_E u) \det(I - (L_E)^t u^2)} = \frac{1}{\det(I - L_B u) \det(I - L_E u^2)},
\]

where \(L_E\) is the operator on the double coset \(E(t_2 \sigma^2)^2 E\), which is also the adjacency matrix of type one edges in \(X_G\). Its transpose \((L_E)^t\) is the adjacency matrix for type two edges. Likewise, \(L_B\) is the adjacency operator of directed chambers in \(X_G\). Consequently, the identity (1.1) can be expressed in terms of operators on \(X_G\) as

\[
(1 - u^3)^{\chi(X_G)} \quad \frac{\det(I + L_B u)}{\det(I - L_E u) \det(I - (L_E)^t u^2)} = \frac{\det(I + L_B u)}{\det(I - L_E u) \det(I - (L_E)^t u^2)},
\]

while the parallel identity of operators on a \((q + 1)\)-regular graph \(X\) reads

\[
(1 - u^2)^{\chi(X)} \quad \frac{\det(I + Lu)}{\det(I - Au + qu^2 I)} = \frac{1}{\det(I - Au + qu^2 I)}.
\]

The similarity is reminiscent of the zeta functions attached to a surface and a curve over a finite field. Since (1.2) is expressed in terms of the operators on the finite complex, it is likely to be the prototype of complex zeta functions in general. The zeros of each determinant in (1.2) are computed in a forthcoming paper [LKW], where it is shown that \(X_G\) being Ramanujan is equivalent to the nontrivial eigenvalues of \(\det(I + L_B u)\) having absolute values 1, \(q^{-1/2}\) and \(q^{-1/4}\), which in turn is equivalent to the nontrivial eigenvalues of \(\det(I - L_E u)\) having absolute values \(q^{-1}\) and \(q^{-1/2}\).

It is also worth pointing out that the right hand side of (1.2) is equal to \(Z(X_G, u)/Z_2(X_G, -u)\), which can be expressed as an infinite product. Here \(Z_2(X_G, u)\) is the zeta function of tailless type one closed galleries in \(X_G\), so the quotient \(Z(X_G, u)/Z_2(X_G, -u)\) itself can also be interpreted as
a zeta function on type one and type two tailless cycles modulo homotopy equivalence given by
tailless type one closed galleries.

This paper is organized as follows. In §2 the actions of the Hecke operators, including their
recursive relations, are summarized. The types and lengths of geodesics in $B$ are introduced in
§3. Basic concepts concerning cycles in a finite quotient $X_\Gamma$ of $B$ are laid out in §4. The based
homotopy classes of closed geodesics in $X_\Gamma$ are partitioned into sets indexed by the conjugacy
classes $[\gamma]$ of $\Gamma$, with each set consisting of based homotopy classes which are base-point free
homotopic to the path from $K$ to $\gamma K$. Each set $[\gamma]$ has a type, algebraic length and geometric
length, defined in terms of the eigenvalues of the conjugacy class $[\gamma]$. Theorem 5 says that the
lengths of the set $[\gamma]$ are the minimal respective lengths of the homotopy cycles contained in the
set. Cycles achieving minimal geometric (resp. algebraic) length in a set are called tailless (resp.
algebraically tailless). In other words, among the cycles base-point free homotopic to each other,
the shortest ones are called tailless. This definition also applies to graphs. Algebraically tailless
cycles afford an explicit algebraic characterization, as shown in §5 and §6 according as $\gamma$ is split
or rank-one split, and hence are more amenable to computation. We shall see in §5 and §6 that,
for type one and type two cycles, there is no distinction between algebraic tailless and tailless.

While the zeta function only concerns tailless cycles of types one and two, to find its closed form,
we have to consider all cycles up to homotopy. Indeed, we shall compute the number of cycles, as
well as those of type one, in a set $[\gamma]$ with given algebraic length. This is carried out in §5 and
§6. As shown in §9, where the Main Theorem is proved, these numbers can be put together to
show that the logarithmic derivative of the left hand side of (1.2) counts the number of type one
tailless closed geodesics in $X_\Gamma$, namely, those from the logarithmic derivative of $1/\det(I - L_E u)$,
and some extra terms arising from sets represented by rank-one split $\gamma$’s.

Sections 7 and 8 are devoted to explaining these extra terms. In §7 we discuss type one tailless
closed galleries and define chamber zeta function $Z_2(X_\Gamma, u)$, while the zeta function on type one
tailless closed geodesics, $Z_1(X_\Gamma, u)$, is discussed in §8. The boundary of a type one tailless closed
gallery is analyzed in §8.2, where it is shown that the boundary of an even/odd length gallery
consists of two/one tailless type one cycle(s). The information on the boundary further leads to
a criterion on the chambers occurring in a type one tailless closed gallery. This in turn allows us
to compute the logarithmic derivative of $\frac{\det(I + L_E u)}{\det(I - (L_E)^r u^2)} = \frac{Z_1(X_\Gamma, u^2)}{Z_2(X_\Gamma, -u)}$, which gives the aforementioned
extra terms.
2. Hecke operators on $B$

2.1. Hecke operators. The group $G$ is the disjoint union of the $K$-double cosets

$$T_{n,m} = K \text{ diag}(1, \pi^m, \pi^{m+n})K$$

as $m, n$ run through all non-negative integers. We shall also regard $T_{n,m}$ as the Hecke operator acting on functions $f \in L^2(G/K)$ via

$$T_{n,m}f(gK) = \sum_{\alpha K \in T_{n,m}/K} f(g\alpha K).$$

In particular,

$$A_1 = T_{1,0} \quad \text{and} \quad A_2 = T_{0,1}.$$

2.2. Recursive relations among Hecke operators. It is well-known that each Hecke operator is a polynomial in $A_1$ and $A_2$. Tamagawa [Ta] obtained a recursive relation on Hecke operators:

$$(\sum_{n,m \geq 0} T_{n,m} u^{n+2m})(I - A_1 u + q A_2 u^2 - q^3 u^3 I) = (1 - u^3)I.$$  \hspace*{1cm} (2.1)

We prove a version of the recursive relation which will be used later.

**Theorem 2.**

$$(q \sum_{k=1}^{\infty} T_{k,0} u^k - (q-1)(\sum_{k=1}^{\infty} \sum_{n+2m=k} T_{n,m} u^k) \frac{1 - q^2 u^3}{1 - u^3}) = u \frac{d}{du} \log \frac{(1 - u^3)^r I}{I - A_1 u + A_2 q u^2 - q^3 u^3 I},$$

where $r = \frac{(q+1)(q-1)^2}{3}$. \hspace*{1cm} (2.2)

**Proof.** The Satake isomorphism $\psi$ from the algebra of Hecke operators to the polynomial ring $\mathbb{C}[z_1, z_2, z_3]^{S_3}/(z_1 z_2 z_3 - 1)$ sends $A_1$ to $q(z_1 + z_2 + z_3)$ and $A_2$ to $q(z_1 z_2 + z_2 z_3 + z_3 z_1)$ (cf. [Sat]).

To describe the image of $T_{n,m}$, define a quasi-character $\chi$ on the Borel subgroup $P$ of $G$ by

$$\chi \left( \begin{bmatrix} b_1 & * & * \\ b_2 & * \\ b_3 \end{bmatrix} \right) = z_1^{\text{ord}_P b_1} z_2^{\text{ord}_P b_2} z_3^{\text{ord}_P b_3},$$

and regard it as a homomorphism from $G/K$ to $\mathbb{C}[z_1, z_2, z_3]/(z_1 z_2 z_3 - 1)$. Denote by $\delta_P$ the modular character on $P$. Let $\phi$ be the function on $G$ given by

$$\phi(bk) = \chi(b) \delta_P^{1/2}(b) \quad (b \in P, k \in K).$$
Then
\[ \psi(T_{n,m}) = \int_G T_{n,m}(g) \phi(g) dg, \]
where \( dg \) is the Haar measure on \( G \) so that \( K \) has volume 1.

For convenience, let \( T_k = \sum_{n+2m=k} T_{n,m} \), and denote by \( \sigma_{k,i}(z_1, z_2, z_3) \) the sum of all degree \( k \) monomials in \( \mathbb{C}[z_1, z_2, z_3] \) in \( i \) variables, where \( 1 \leq i \leq 3 \). Our strategy is to show that the identity holds after applying the Satake isomorphism \( \psi \). For this, we compute the coefficient of \( z_1^{a_1} z_2^{a_2} z_3^{a_3} \) in \( \psi(T_k) \) for \( a_1 \geq a_2 \geq a_3 \). As \( z_1 z_2 z_3 = 1 \), we may further assume \( a_3 \geq 0 \) and \( a_1 + a_2 + a_3 = k \). Since \( \psi(T_k) \) is a symmetric polynomial, it can be easily determined using symmetry.

It is straightforward to check that the number of elements \( gK \in G/K \) mapped to \( z_1^{a_1} z_2^{a_2} z_3^{a_3} \) by \( \chi \) is equal to \( q^{2a_1+a_2} \) if \( a_3 = 0 \), and \( (q^3 - 1)q^{2a_1+a_2-3} \) if \( a_3 > 0 \). Moreover, for such \( gK \) we have \( \delta_P(gK)^{1/2} = q^{a_3-a_1} \). Therefore the coefficient of \( z_1^{a_1} z_2^{a_2} z_3^{a_3} \) in \( \psi(T_k) \) is equal to \( q^{a_1+a_2+a_3} \) or \( q^{a_1+a_2+a_3-3}(q^3 - 1) \) according to \( a_3 = 0 \) or \( a_3 > 0 \). By symmetry, this yields
\[ \psi(T_k) = q^k(\sigma_{k,1} + \sigma_{k,2} + \frac{q^3-1}{q^3} \sigma_{k,3}). \]

Noting that
\[ \sum_{k=1}^{\infty} \sigma_{k,3} u^k = ((z_1 z_2 z_3) u^3 + (z_1 z_2 z_3)^2 u^6 + \cdots) \sum_{k=0}^{\infty} (1 + \sigma_{k,1} + \sigma_{k,2}) u^k = \frac{u^3}{1-u^3} \sum_{k=0}^{\infty} (1 + \sigma_{k,1} + \sigma_{k,2}) u^k, \]
we obtain
\[ \psi(\sum_{k=1}^{\infty} T_k u^k) = \sum_{k=1}^{\infty} (\sigma_{k,1} + \sigma_{k,2} + \frac{q^3-1}{q^3} \sigma_{k,3})(qu)^k \]
\[ = \frac{(q^3-1)u^3}{1-q^3 u^3} + \frac{1-u^3}{1-q^3 u^3} \sum_{k=1}^{\infty} (\sigma_{k,1} + \sigma_{k,2})(qu)^k. \]

On the other hand, put \( G_0 = \bigsqcup_{k=1}^{\infty} T_{k,0} \). One verifies that the number of elements in \( G_0/K \) mapped to \( z_1^{a_1} z_2^{a_2} z_3^{a_3} \) by \( \chi \) is \( q^{2a_1} \) if \( a_2 = a_3 = 0 \), \( (q-1)q^{2a_1+a_2-1} \) if \( a_2 > a_3 = 0 \), and \( (q-1)^2 q^{2a_1+a_2-2} \) if \( a_2 \geq a_3 > 0 \). Therefore,
\[ \psi(\sum_{k=1}^{\infty} T_{k,0} u^k) = \sum_{k=1}^{\infty} (\sigma_{k,1} + \frac{q-1}{q} \sigma_{k,2} + \frac{(q-1)^2}{q^2} \sigma_{k,3})(qu)^k \]
\[ = \frac{q(q-1)^2 u^3}{1-q^3 u^3} + \frac{1+qu^3-2q^2 u^3}{1-q^3 u^3} \sum_{k=1}^{\infty} \sigma_{k,1}(qu)^k + \frac{(q-1)(1-q^2 u^3)}{q(1-q^3 u^3)} \sum_{k=1}^{\infty} \sigma_{k,2}(qu)^k. \]

Consequently,
\[
\psi\left( q(\infty \sum_{k=1} T_{k,0}u^k) - (q - 1)(\infty \sum_{k=1} T_{k}u^k) \frac{1 - q^2u^3}{1 - u^3} \right) = \sum_{k=0}^{\infty} \sigma_k,1(qu)^k + \frac{(q - 1)(q^2 - 1)u^3}{1 - u^3}
\]

\[
= \frac{z_1qu}{1 - z_1qu} + \frac{z_2qu}{1 - z_2qu} + \frac{z_2qu}{1 - z_2qu} - \frac{3ru^3}{1 - u^3} = u \frac{d}{du} \log \frac{(1 - u^3)^r}{(1 - z_1qu)(1 - z_2qu)(1 - z_3qu)}
\]

\[
= \psi\left( u \frac{d}{du} \log \frac{(1 - u^3)^r}{I - A_1u + A_2qu^2 - q^3u^3I} \right).
\]

\[
\square
\]

2.3. Description of type 1 and type 2 edges. The type of a vertex \( gK \) in \( \mathcal{B} \) is \( \tau(gK) := \text{ord}_n \det g \mod 3 \). Adjacent vertices do not have the same type. The type of a directed edge \( gK \to g'K \) is \( \tau(g'K) - \tau(gK) = i \), which is 1 or 2. Out of each vertex there are \( q^2 + q + 1 \) edges of a given type. The type 1 edges out of \( gK \) have terminal vertices \( g\alpha K \), where \( \alpha K \) are the \( K \)-cosets contained in the double coset \( A_1 = T_{1,0} \):

\[
A_1 = T_{1,0} = K \begin{pmatrix} 1 & \pi \\ 1 & \pi \end{pmatrix} K = \bigcup_{a,b \in \mathcal{O}_F / \pi \mathcal{O}_F} \begin{pmatrix} \pi & a & b \\ 1 & \pi & c \\ 1 & \pi \end{pmatrix} K \bigcup \begin{pmatrix} 1 & \pi \\ 1 & \pi \end{pmatrix} K.
\]

Similarly, the terminal vertices of type 2 edges out of \( gK \) can be described using the following \( q^2 + q + 1 \) left \( K \)-coset representatives of \( A_2 = T_{0,1} \):

\[
\begin{pmatrix} \pi & b \\ \pi & c \\ 1 & \pi \end{pmatrix}, \begin{pmatrix} \pi & a \\ 1 & \pi \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ \pi \end{pmatrix}, \text{ where } a, b, c \in \mathcal{O}_F / \pi \mathcal{O}_F.
\]

3. Geodesics and lengths in \( \mathcal{B} \)

Since \( \mathcal{B} \) is simply connected, all paths between two vertices are homotopic. By a geodesic between two vertices of \( \mathcal{B} \) we mean a path with shortest length in the 1-skeleton of the building \( \mathcal{B} \), which is the (undirected) graph with vertex set \( G/K \) and adjacency matrix \( A_1 + A_2 \).

It can be shown that all geodesics between two vertices \( g_1K \) and \( g_2K \) with \( g_1^{-1}g_2 \in T_{n,m} \) lie in the same apartment, and they use \( n \) type one edges and \( m \) type two edges. We say that they have type \((n, m)\). When \( m = 0 \) (resp. \( n = 0 \)), the path is called type one (resp. type two) for short. Define \( n + m \) to be the geometric length of the homotopy class in \( \mathcal{B} \) of the geodesics from \( g_1K \) to \( g_2K \). The algebraic length of this homotopy class is \( n + 2m = \text{ord}_n \det g_1^{-1}g_2 \). Note that the same path traveled backwards has algebraic length \( m + 2n \). Further, when the path has type one or two, there is only one geodesic between the two vertices.
4. Finite quotients of \( B \)

4.1. The group \( \Gamma \). Let \( \Gamma \) be a discrete cocompact torsion-free subgroup of \( G \) which acts on \( B \) by left translations. Then \( \Gamma \) intersects any compact subgroup of \( G \) trivially. In particular, \( \Gamma \) acts on \( B \) free of fixed points. Denote by \( X_\Gamma = \Gamma \backslash B \) the finite quotient, whose vertices are the double cosets \( \Gamma \backslash G/K \). We shall assume that \( \Gamma \) satisfies the following two additional conditions:

(I) \( \text{ord}_\pi \det \Gamma \subset 3\mathbb{Z} \) so that \( \Gamma \) identifies vertices of the same type, and consequently \( X_\Gamma \) is a finite connected \((q + 1)\)-regular 2-dimensional simplicial complex.

(II) \( \Gamma \) is regular, that is, the centralizer in \( G \) of any non-identity element \( \gamma \in \Gamma \) is a torus.

As a consequence of (II), the non-identity elements in \( \Gamma \) have distinct eigenvalues. An element in \( \Gamma \) having three distinct eigenvalues in \( F \) is called split; it is called rank-one split if it has exactly one eigenvalue in \( F \). In the latter case we say it is unramified/ramified rank-one split if its eigenvalues generate an unramified/ramified quadratic extension of \( F \). Note that \( \Gamma \) does not contain elements with no eigenvalue in \( F \). Indeed, if \( \gamma \) is such an element, then its characteristic polynomial is irreducible over \( F \). As \( \text{ord}_\pi(\det \gamma) = 3m \) for some integer \( m \), the eigenvalues of \( \gamma' = \pi^{-m}\gamma \) are units in a cubic extension of \( F \), which implies that \( \gamma' \) lies in the intersection of \( \Gamma \) with a conjugate of \( K \), and hence is the identity element.

We exhibit below some examples of such \( \Gamma \).

**Example 3.** Let \( M \) be a number field or a function field such that \( F \) is the completion of \( M \) at a nonarchimedean place \( v \). Let \( \infty \neq v \) be an archimedean place if \( M \) is a number field or a place at infinity if \( M \) is a function field. Let \( H \) be a division algebra of dimension 9 over \( M \) which is unramified at \( v \) and ramified at \( \infty \), and \( D = H^\times/\text{center} \). Then for any congruence subgroup \( \mathcal{K} \) of \( \prod_{\text{finite places } w \neq v, \infty} D(\mathcal{O}_w) \), \( \Gamma = D(M) \cap \mathcal{K} \prod_{\text{remaining places } w} D(M_w) \) is a discrete cocompact regular subgroup of \( G \). Since the torsion subgroup of \( \Gamma \) is finite, by choosing a smaller congruence subgroup if necessary, we may assume that \( \Gamma \) is also torsion free.

4.2. Homotopy classes of closed paths on \( X_\Gamma \). The 1-skeleton of \( X_\Gamma \) is an undirected graph with the adjacency matrix \( A_1 + A_2 \). We study cycles on this graph which are homotopic in \( X_\Gamma \).

A closed geodesic in \( X_\Gamma \) starting at the vertex \( \Gamma gK \) can be lifted to a path in \( B \) starting at \( gK \) and ending at \( \gamma gK \) for some \( \gamma \in \Gamma \). Denote by \( \kappa_\gamma(gK) \) the homotopy class of the geodesic paths from \( gK \) to \( \gamma gK \) in \( B \). Note that, when projected to \( X_\Gamma \), \( \kappa_\gamma(gK) \) has shortest geometric length among all cycles in its homotopy class in \( X_\Gamma \). By abuse of notation, we also use \( \kappa_\gamma(gK) \) to denote
its homotopy class in $X_{\Gamma}$. Thus the fundamental group of $X_{\Gamma}$ based at $\Gamma gK$ is

$$\pi_1(X_{\Gamma}, \Gamma gK) = \{\kappa_\gamma(gK) : \gamma \in \Gamma\}.$$ 

Since $\Gamma$ has no fixed points, all $\kappa_\gamma(gK)$ are distinct and $\pi_1(X_{\Gamma}, \Gamma gK)$ is isomorphic to $\Gamma$.

When all base points are taken into account, the set of based homotopy classes of all closed geodesics in $X_{\Gamma}$ is parametrized by

$$\Gamma \times \Gamma \backslash G/K \cong \bigsqcup_{\gamma \in \Gamma} \pi_1(X_{\Gamma}, \Gamma gK).$$

For each conjugacy class of $\Gamma$ fix a representative $\gamma$ and denote that class by $[\gamma]$. Let $[\Gamma] = \{\gamma\}$ be the set of representatives of conjugacy classes. Since the conjugacy class of $\gamma$ in $\Gamma$ corresponds bijectively to $\Gamma$ modulo the centralizer $C_{\Gamma}(\gamma)$ of $\gamma$ in $\Gamma$, we have

$$\Gamma \times \Gamma \backslash G/K \cong \bigsqcup_{\gamma \in \Gamma} (C_{\Gamma}(\gamma) \backslash G/K) \cong \bigsqcup_{\gamma \in \Gamma} (C_{\Gamma}(\gamma) \backslash G/K) \cong \bigsqcup_{\gamma \in \Gamma} C_{\Gamma}(\gamma) \backslash G/K.$$

Letting, for each $\gamma \in [\Gamma]$,

$$\begin{align*}
[\gamma] &= \{\kappa_\gamma(gK) : g \in C_{\Gamma}(\gamma) \backslash G/K\},
\end{align*}$$

we can express the set of all based homotopy classes of $X_{\Gamma}$ as the disjoint union of $[\gamma]$ over $\gamma \in [\Gamma]$.

Two based homotopy classes of $X_{\Gamma}$ are said to be base-point free homotopic if a cycle in one class is obtained from a cycle in the other class by repeated applications the following procedures:

(H1) Shifting the starting vertex to another vertex on the cycle;

(H2) Replacing the cycle by a homotopic cycle while holding the end points fixed.

The set $[\gamma]$ has the following geometric interpretation.

**Proposition 4.** Let $\gamma \in [\Gamma]$. The set $[\gamma]$ defined by (4.1) has the following properties:

(i) It is closed under base-point free homotopy;

(ii) Any two classes in $[\gamma]$ are base-point free homotopic;

(iii) The set $[\gamma]$ is independent of the choice of representative $\gamma$ in the conjugacy class $[\gamma]_\Gamma$.

Consequently, $[\gamma]$ consists of all based homotopic classes which are base-point free homotopic to $\kappa_\gamma(K)$.

**Proof.** (i) Take any homotopy cycle $\kappa_\gamma(gK)$ in $[\gamma]$. It is represented by the path $gK \to g_2K \to \cdots \to \gamma gK$ in $B$. If we shift the starting vertex to $g_2K$, the resulting cycle in $X_{\Gamma}$ is represented by the path $g_2K \to \cdots \to \gamma gK$ from $\kappa_\gamma(gK)$ followed by $\gamma gK \to \gamma g_2K$ in $B$, which is homotopic to $\kappa_\gamma(g_2K)$. This shows that (H1) is satisfied. (H2) is obvious. This proves (i).
(ii) Let $\kappa_\gamma(gK) : gK \to \cdots \to \gamma gK$ and $\kappa_\gamma(hK) : hK \to \cdots \to \gamma hK$ be two homotopy cycles in $[\gamma]$. Then $\kappa_\gamma(gK)$ is homotopic in $\mathcal{B}$ and hence in $X_\Gamma$ to the path $C$ which is a path $P(gK, hK)$ from $gK$ to $hK$ followed by $\kappa_\gamma(hK)$ then followed by $P(\gamma hK, \gamma gK)$, the left translation of $P(gK, hK)$ by $\gamma$. Next, shifting the vertex on $C$ from $gK$ to $hK$, we obtain a new path $C'$, which is $\kappa_\gamma(hK)$ followed by $P(\gamma hK, \gamma gK)$ and then by $P(gK, hK)$. Note that on $X_\Gamma$ the last two paths are reverse of each other so that $C'$ is homotopic to $\kappa_\gamma(hK)$. This proves that $\kappa_\gamma(gK)$ is base-point free homotopic to $\kappa_\gamma(hK)$, which establishes (ii).

(iii) This is because the classes in $[\gamma]$ are base-point free homotopic to the cycles in $\mathcal{B}$ from $K$ to $g^{-1}\gamma gK$, with ending vertices represented by all elements in the conjugacy class of $\gamma$ in $G$. 

4.3. The type and lengths of a homotopy class. The type, geometric length and algebraic length of a homotopy class $\kappa_\gamma(gK)$ of $X_\Gamma$ are those of $\kappa_\gamma(gK)$ in $\mathcal{B}$. In other words, If $g^{-1}\gamma g \in T_{n,m}$, then $\kappa_\gamma(gK)$ has algebraic length $l_A(\kappa_\gamma(gK)) = n + 2m$, geometric length $l_C(\kappa_\gamma(gK)) = n + m$, and type $(n, m)$. By assumption, $\kappa_\gamma(gK)$ has positive length if and only if $\gamma$ is not identity.

4.4. The type and lengths of $[\gamma]$. Let $\gamma \in [\Gamma]$ not equal to identity. If $\gamma$ is split, we shall assume that the eigenvalues are $1, a, b \in F^\times$ with $\text{ord}_\pi b \geq \text{ord}_\pi a \geq 0$. Then $\gamma$ is conjugate to $r_\gamma := \text{diag}(1, a, b)$. If $\gamma$ is rank-one split, then its characteristic polynomial has the form $(x - a)(x^2 - b'x - c')$ with $x^2 - b'x - c'$ irreducible over $F$. The splitting field of $x^2 - b'x - c'$ is a quadratic extension $L = F(\lambda)$ of $F$. We fix the choice of $\lambda$ so that it is a unit if $L$ is unramified over $F$ and it is a uniformizing element if $L$ is ramified over $F$. Let $x^2 - bx - c$ be the irreducible polynomial of $\lambda$ over $F$ and let $\bar{\lambda}$ be the Galois conjugate of $\lambda$. Then $\text{ord}_\pi c = 0$ or $1$ according as $L$ is unramified or ramified over $F$ and $\text{ord}_\pi b \geq \frac{1}{2}\text{ord}_\pi c$. There are elements $e, d \in F$ such that $e + d\lambda$ and $e + d\bar{\lambda}$ are the roots of $x^2 - b'x - c'$ in $L$. Consequently $\gamma$ is conjugate to $r_\gamma := \begin{pmatrix} a & e & de \\ e & d & e + db \\ d & e + db \end{pmatrix}$. We shall assume that all eigenvalues of $\gamma$ are integral and minimally so. In other words, $a, e, d$ are in $\mathcal{O}_F$ and at least one of them is a unit. Fix a choice of $P_\gamma \in G$ such that $r_\gamma = (P_\gamma)^{-1}\gamma P_\gamma$. Denote by $C_G(g)$ the centralizer of $g \in G$ in $G$. As $C_G(\gamma) = P_\gamma C_G(r_\gamma)P_\gamma^{-1}$, we have $C_{\Gamma}(\gamma)P_\gamma = P_\gamma C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$, and hence we can express $[\gamma]$ in two ways:

$$[\gamma] = \{ \kappa_\gamma(gK) \mid g \in C_{\Gamma}(\gamma)\backslash G/K \}$$

$$= \{ \kappa_\gamma(P_\gamma gK) \mid g \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash G/K \}.$$
The assumption (II) on $\Gamma$ implies that $C_G(r_\gamma)$ is a torus in $G$ containing the discrete cocompact subgroup $C_{P_\gamma^-1}\Gamma P_\gamma (r_\gamma)$. Hence the double coset $C_{P_\gamma^-1}\Gamma P_\gamma (r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap K)$ is finite. Denote its cardinality by

$$c([\gamma]) = \#(C_{P_\gamma^-1}\Gamma P_\gamma (r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap K)).$$

For any integer $m \neq 0$, the eigenvalues of $\gamma^m$ are the $m$-th power of those of $\gamma$, hence $r_{\gamma^m} = (r_\gamma)^m$ up to a central element (due to normalization), and thus we may assume $P_{\gamma^m} = P_\gamma$. Consequently, $P_{\gamma^m}^{-1}\Gamma P_{\gamma^m} = P_\gamma^{-1}\Gamma P_\gamma$ for all $m \neq 0$. On the other hand, since $C_G(r_{\gamma^m}) = C_G(r_\gamma)$, we conclude $c([\gamma]) = c([\gamma^m])$ for all $m \neq 0$.

Suppose $r_\gamma \in T_{n,m}$. We say that $[\gamma]$ has type $(n,m)$, algebraic length $l_A([\gamma]) = n + 2m$ and geometric length $l_G([\gamma]) = n + m$. We shall prove

**Theorem 5.** Let $\gamma \in [\Gamma]$ and $\gamma \neq id$. Then

$$l_A([\gamma]) = \min_{g \in G} l_A(\kappa_\gamma(gK)) \text{ and } l_G([\gamma]) = \min_{g \in G} l_G(\kappa_\gamma(gK)).$$

Moreover, for $g \in C_G(r_\gamma)$, we have $l_A(\kappa_\gamma(P_\gamma gK)) = l_A([\gamma])$, $l_G(\kappa_\gamma(P_\gamma gK)) = l_G([\gamma])$ and the type of $\kappa_\gamma(P_\gamma gK)$ coincides with the type of $[\gamma]$.

The second assertion is obvious since $(P_\gamma g)^{-1}\kappa_\gamma P_\gamma g = g^{-1}r_\gamma g = r_\gamma$ for $g \in C_G(r_\gamma)$. The proof of the first assertion is contained in Theorem 6 for $\gamma$ split and Theorem 14 for $\gamma$ rank-one split.

Note that $l_A(\kappa_\gamma(gK)) \equiv \text{ord}_a \det \gamma \pmod{3}$, hence $l_A(\kappa_\gamma(gK)) = l_A([\gamma]) + 3m$ for some non-negative integer $m$.

4.5. Tailless cycles. In view of Theorem 6, a homotopy class $\kappa_\gamma(gK)$ is called *algebraically tailless* if its algebraic length agrees with $l_A([\gamma])$. It is called *tailless* if its geometric length is $l_G([\gamma])$. Hence a tailless based homotopy cycle has shortest geometric length among all cycles base-point free homotopic to it. We shall count such cycles of type one.

5. Homotopy cycles in $[\gamma]$ for $\gamma$ split

Let $|\ |$ be the valuation on $F$ such that $|\pi| = q^{-1}$. In this section we fix a split $\gamma \in [\Gamma], \gamma \neq id$, with $r_\gamma = \text{diag}(1,a,b)$, where $\text{ord}_a b \geq \text{ord}_a a \geq 0$.

5.1. Minimal lengths of homotopy cycles in $[\gamma]$. We begin by proving the first assertion of Theorem 6 for the split case.

**Theorem 6.** Suppose $\gamma \in \Gamma$ is split with $r_\gamma = \text{diag}(1,a,b)$, where $\text{ord}_a b \geq \text{ord}_\pi a \geq 0$. Then
(1) \( l_A([\gamma]) = \text{ord}_\pi a + \text{ord}_\pi b = \min_{\kappa_\gamma(gK) \in [\gamma]} l_A(\kappa_\gamma(gK)) \) and
\( l_G([\gamma]) = \text{ord}_\pi b = \min_{\kappa_\gamma(gK) \in [\gamma]} l_G(\kappa_\gamma(gK)) \).

Proof. The centralizer \( C_G(r_\gamma) \) consists of the diagonal matrices in \( G \). It is well-known that
\[
U = \left\{ \begin{pmatrix} 1 & x & y \\ 1 & z \\ 1 \end{pmatrix} \mid x, y, z \in F/\mathcal{O}_F \right\}
\]
represents the double coset \( C_G(r_\gamma) \backslash G/K \). It suffices to consider the lengths of \( \kappa_\gamma(P_\gamma gK) \) with \( g \in U \). Write \( g = \begin{pmatrix} 1 & x & y \\ 1 & z \\ 1 \end{pmatrix} \). Then
\[
(P_\gamma g)^{-1} \kappa_\gamma P_\gamma g = g^{-1} r_\gamma g = \begin{pmatrix} 1 & x(1-a) & y(1-b) + xz(b-a) \\ a & z(a-b) \\ b \end{pmatrix} \in K \begin{pmatrix} \pi^{e_1} \\ \pi^{e_2} \\ \pi^{e_3} \end{pmatrix} K
\]
for some integers \( e_1 \leq e_2 \leq e_3 \). In fact, for \( 1 \leq i \leq 3 \), \( e_1 + \cdots + e_i = \text{min}_y \text{ord}_\pi y \) where \( y \) runs through the determinant of all \( i \times i \) minors of \( g^{-1} r_\gamma g \).

Consequently,

\[(5.1) \quad e_1 = \text{min}\{0, \text{ord}_\pi x(1-a), \text{ord}_\pi z(a-b), \text{ord}_\pi (y(1-b) + xz(b-a))\} \leq 0,
\]
\[(5.2) \quad e_1 + e_2 = \text{min}\{\text{ord}_\pi a, \text{ord}_\pi [x(1-a)z(a-b) - a(y(1-b) + xz(b-a))]\} \leq \text{ord}_\pi a,
\]
and
\[(5.3) \quad e_1 + e_2 + e_3 = \text{ord}_\pi a + \text{ord}_\pi b.
\]
In particular, \( e_3 \geq \text{ord}_\pi b \) from the last two inequalities. Moreover, we have, for any \( g \in G \),
\[(5.4) \quad l_A(\kappa_\gamma(P_\gamma gK)) = e_3 + e_2 + e_1 - 3e_1 = \text{ord}_\pi a + \text{ord}_\pi b - 3e_1 \geq \text{ord}_\pi a + \text{ord}_\pi b = l_A([\gamma])
\]
and
\[(5.5) \quad l_G(\kappa_\gamma(P_\gamma gK)) = e_3 - e_1 \geq \text{ord}_\pi b - e_1 \geq \text{ord}_\pi b = l_G([\gamma]).
\]
Corollary 7. Suppose $\gamma \in [\Gamma]$ is split. Then all tailless cycles in $[\gamma]$ are also algebraically tailless, and they have the same type as $[\gamma]$. Furthermore, if $[\gamma]$ has type one, then the algebraically tailless cycles in $[\gamma]$ are tailless.

5.2. Counting homotopy cycles in $[\gamma]$ in algebraic length. Let

$$\Delta_A([\gamma]) = \{gK \in G/K \mid l_A(\kappa_\gamma(P_\gamma gK)) = l_A([\gamma])\}.$$ 

As noted before, $\Delta_A([\gamma]) \supset C_G(r_\gamma)K/K$ and is invariant under left multiplication by $C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma)$. So the number of algebraically tailless cycles in $[\gamma]$ is the cardinality of $C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash \Delta_A([\gamma])$.

The following theorem, stated in terms of a formal power series, gives the number of homotopy cycles of a given algebraic length in $[\gamma]$.

Theorem 8. Suppose $\gamma \in [\Gamma]$ is split with $r_\gamma = \text{diag}(1,a,b)$. Then

$$\sum_{\kappa_\gamma(gK) \in [\gamma]} u^{l_A(\kappa_\gamma(gK))} = \#(C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash \Delta_A([\gamma])) \, u^{l_A([\gamma])} \frac{1 - u^3}{1 - q^3 u^3}$$

$$= c([\gamma])(|1 - a||a - b||b - 1|)^{-1} \, u^{l_A([\gamma])} \frac{1 - u^3}{1 - q^3 u^3},$$

where $c([\gamma])$ is given by (4.2).

Proof. In view of the decomposition $G = C_G(r_\gamma)UK$, the left hand side can be expressed as

$$\sum_{\kappa_\gamma(P_\gamma gK) \in [\gamma]} u^{l_A(\kappa_\gamma(P_\gamma gK))} = c([\gamma]) \sum_{v \in U} u^{l_A(\kappa_\gamma(P_\gamma vK))}.$$ 

To proceed, we compute the sum on the right hand side.
Proposition 9. Let $\gamma$ be split with $r_\gamma = \text{diag}(1, a, b)$. Then

$$\sum_{v \in U} u^{l_A(\kappa_\gamma(P_\gamma vK))} = \frac{u^{l_A([\gamma])}}{|1-a||a-b||b-1|} \left(1 - \frac{u^3}{1 - q^3u^3}\right).$$

Proof. Given $v \in U$, write $v = \begin{pmatrix} 1 & x & y \\ 1 & z & 1 \end{pmatrix}$. As computed in the proof of Theorem 6, we have

$$P_\gamma^{-1} \gamma P_\gamma v = v^1r_\gamma v = \begin{pmatrix} 1 & x(1-a) & y(1-b) + xz(b-a) \\ a & z(a-b) & b \end{pmatrix} = (v_{ij}).$$

For fixed $m \geq 0$, we count the number of $v$’s such that $l_A(\kappa_\gamma(P_\gamma vK)) \leq l_A([\gamma]) + 3m$. By (5.4), the constraints are $|v_{ij}| \leq q^m$ for all $1 \leq i, j \leq 3$. In other words,

$$|x(1-a)| \leq q^m, \quad |z(a-b)| \leq q^m \quad \text{and} \quad |y(1-b) + xz(b-a)| \leq q^m.$$

This implies

$$|x| \leq q^m|1-a|^{-1} \quad \text{and} \quad |z| \leq q^m|a-b|^{-1}$$

so that the numbers of $x$ and $z$ in $F/O_F$ are $q^m|1-a|^{-1}$ and $q^m|a-b|^{-1}$, respectively. Further, for chosen $x$ and $z$, there are $q^m|1-b|^{-1}$ choices of $y$ satisfying the above constraint. We have shown

$$\|v \in U \mid l_A(\kappa_\gamma(P_\gamma vK)) = l_A([\gamma])\| = (|1-a||a-b||b-1|)^{-1}$$

and, for $m > 0$,

$$(5.7) \#\{v \in U \mid l_A(\kappa_\gamma(P_\gamma vK)) = l_A([\gamma]) + 3m\} = (q^{3m} - q^{3m-3})(|1-a||a-b||b-1|)^{-1}.$$

Put together, this gives

$$\sum_{v \in U} u^{l_A(\kappa_\gamma(P_\gamma vK))} = \frac{u^{l_A([\gamma])}}{|1-a||a-b||b-1|} \left(1 + \sum_{m \geq 1} (q^{3m} - q^{3m-3})u^{3m}\right) \left(1 - \frac{u^3}{1 - q^3u^3}\right).$$

The argument above shows that the number of algebraically tailless homotopy classes in $[\gamma]$ is $c([\gamma])$ times the number of elements in $U$ with $m = 0$, which is given by (5.6). This proves

Proposition 10. Suppose $\gamma \in \Gamma$ is split with $r_\gamma = \text{diag}(1, a, b)$. Then

$$\#(C_{P_\gamma^{-1} \Gamma P_\gamma}(r_\gamma) \setminus \Delta_A([\gamma])) = c([\gamma])(|1-a||a-b||b-1|)^{-1}.$$
Theorem 11. Suppose \( \gamma \in \Gamma \) is split with \( r_\gamma = \text{diag}(1, a, b) \). The following assertions hold.

(i) If \( [\gamma] \) does not have type one, then

\[
\sum_{\kappa_\gamma(gK) \in [\gamma], \text{type one}} u^I_A(\kappa_\gamma(gK)) = c([\gamma])(|1 - a||a - b||b - 1|)^{-1} u^I_A([\gamma])(1 - q^{-1})(\frac{1 - q^2u^3}{1 - q^3u^3}).
\]

Moreover, no type one cycles in \( [\gamma] \) are tailless.

(ii) If \( [\gamma] \) has type one, then

\[
\sum_{\kappa_\gamma(gK) \in [\gamma], \text{type one}} u^I_A(\kappa_\gamma(gK)) = c([\gamma])(|1 - a||a - b||b - 1|)^{-1} u^I_A([\gamma]) \left( q^{-1} + (1 - q^{-1})(\frac{1 - q^2u^3}{1 - q^3u^3}) \right).
\]

Proof. Since \( r_\gamma = \text{diag}(1, a, b) \), \( [\gamma] \) has type \( (\text{ord}_\pi b - \text{ord}_\pi a, \text{ord}_\pi a) \) and \( l_A([\gamma]) = \text{ord}_\pi b + \text{ord}_\pi a \). It has type one if and only if \( \text{ord}_\pi a = 0 \). The argument is similar to the proof of Theorem 8, the difference is that we only need to consider those \( v \in U \) such that \( \kappa_\gamma(P_\gamma vK) \) has type one. So we count the number of

\[
\{ v \in U \mid l_G(\kappa_\gamma(P_\gamma vK)) = l_A(\kappa_\gamma(P_\gamma vK)) = l_A([\gamma]) + 3m = \text{ord}_\pi b + \text{ord}_\pi a + 3m \}
\]

for each \( m \geq 0 \). As before, writing \( v \) as

\[
\begin{pmatrix}
1 & x & y \\
1 & z \\
1
\end{pmatrix}
\]

and following the proofs of Proposition 9 and Theorem 8, we arrive at the following constraints on \( x, y, z \in F/\mathcal{O}_F \):

1. \( \min\{0, \text{ord}_\pi x(1 - a), \text{ord}_\pi z(a - b), \text{ord}_\pi (y(1 - b) + xz(b - a))\} = -m \), and

2. \( \min\{\text{ord}_\pi a, \text{ord}_\pi [x(1 - a)z(a - b) - a(y(1 - b) + xz(b - a)))]\} = -2m \).

For \( m > 0 \), the two constraints are equivalent to

3. \( \text{ord}_\pi x(1 - a) = -m = \text{ord}_\pi z(a - b) \) and \( \text{ord}_\pi (y(1 - b) + xz(b - a)) \geq -m \).

Hence the number of \( x \) is \((1 - q^{-1})q^m|1 - a|^{-1}\), the number of \( z \) is \((1 - q^{-1})q^m|a - b|^{-1}\), and the number of \( y \) is \( q^m|1 - b|^{-1} \) so that the total number of \( v \) is \((1 - q^{-1})q^m(1 - a||a - b||b - 1||)^{-1} \). For \( m = 0 \) and \( \text{ord}_\pi a > 0 \), the same constraint (3) holds. In this case the number of \( x \) is \( |1 - a|^{-1} = 1 \), the number of \( y \) is \(|1 - b|^{-1} = 1 \) and the number of \( z \) is \((1 - q^{-1})|a - b|^{-1} \) so that the total number of \( v \) is \((1 - q^{-1})(|1 - a||a - b||b - 1||)^{-1} \). Finally, when \( m = \text{ord}_\pi a = 0 \), the constraints (1) and (2) are equivalent to
(4) $\text{ord}_\pi x(1-a) \geq 0$, $\text{ord}_\pi z(a-b) \geq 0$ and $\text{ord}_\pi(y(1-b) + xz(b-a)) \geq 0$.

Hence the numbers of $x$, $y$ and $z$ are $|1-a|^{-1}$, $|1-b|^{-1}$ and $|a-b|^{-1}$, respectively, so that the number of $v$ is $(|1-a||a-b||b-1|)^{-1}$.

Since $c([\gamma])((1-a||a-b||b-1)|^{-1}$ is present in both cases, it suffices to compute

$$1 \sum_{\kappa, (gK) \in [\gamma], \text{type one}} \text{u}^A(\kappa, (gK)).$$

In case $\text{ord}_\pi a > 0$, this sum is equal to

$$u^A(\gamma)(1 - q^{-1} + \sum_{m \geq 1} (1 - q^{-1})^2 q^{3m} u^{3m}) = u^A(\gamma)(1 - q^{-1})(\frac{1 - q^2 u^3}{1 - q^3 u^3}),$$

and in case $\text{ord}_\pi a = 0$, it is equal to

$$u^A(\gamma)(1 + \sum_{m \geq 1} (1 - q^{-1})^2 q^{3m} u^{3m}) = u^A(\gamma)\left(q^{-1} + (1 - q^{-1})(\frac{1 - q^2 u^3}{1 - q^3 u^3})\right).$$

This proves the theorem. \qed

Contained in the proof above is the following statement.

**Corollary 12.** Suppose $\gamma \in \Gamma$ is split with $r_\gamma = \text{diag}(1, a, b)$. Assume that $\gamma$ has type one. Let $\delta = \delta([\gamma]) = \text{ord}_\pi(1-a)$ and $n = \text{ord}_\pi b$. Then

$$\Delta_A([\gamma]) = \{hv_xK \mid h \in C_G(r_\gamma)/(C_G(r_\gamma) \cap K), \ v_x = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \text{ with } x \in \pi^{-\delta}O_F/O_F\}$$

and for $hv_xK \in \Delta_A([\gamma])$, the geodesic $\kappa_\gamma(P_\gamma hv_xK) \text{ in } \mathcal{B}$ is

$$P_\gamma hv_xK \rightarrow P_\gamma hv_x \text{diag}(1, 1, \pi)K \rightarrow \cdots \rightarrow P_\gamma hv_x \text{diag}(1, 1, \pi^n)K = \gamma P_\gamma hv_xK.$$

Here we used $P_\gamma hv_x \text{diag}(1, 1, \pi^n)K = P_\gamma hv_x r_\gamma K = P_\gamma r_\gamma hv_{ax} K = \gamma P_\gamma hv_x K$ since $v_{ax-x} \in K$.

**6. Homotopy cycles in $[\gamma]$ for $\gamma$ rank-one split**

In this section we fix a rank-one split $\gamma \in [\Gamma]$ whose eigenvalues $a, e+d\lambda, e+d\bar{\lambda}$, where $a, e, d \in O_F$ and at least one of them is a unit, generate a quadratic extension $L = F(\lambda)$ of $F$. Here $\lambda$ is a unit or uniformizer in $L$ according as $L$ is unramified or ramified over $F$, in other words, $\gamma$ is unramified or ramified rank-one split. Using the coefficients of the irreducible polynomial $x^2 - bx - c$ of $\lambda$, in §4.4 we defined $r_\gamma = \begin{pmatrix} a & e & dc \\ e & d & e + db \end{pmatrix}$. A matrix $P_\gamma$ was chosen so that $P_\gamma^{-1} \gamma P_\gamma = r_\gamma$. 
6.1. The centralizers of \( r_\gamma \) for \( \gamma \) rank-one split. Embed \( L^x \) in \( \text{GL}_2(F) \) as the subgroup

\[
\begin{pmatrix}
  u & vc \\
  v & u + vb
\end{pmatrix} \mid u, v \in F, \text{ not both zero},
\]

which is further imbedded in \( \text{GL}_3(F) \) as \( \begin{pmatrix} 1 & u & vc \\ u & v & u + vb \end{pmatrix} \). Embed \( F^x \) into \( \text{GL}_3(F) \) as the diagonal matrices \( \text{diag}(F^x, 1, 1) \). Note that \( r_\gamma \) lies in \( F^x \times L^x \), and \( F^x \times L^x \) modulo the diagonal embedding of \( F^x \) in this product is the centralizer of \( r_\gamma \) in \( G \). Recall from (4.2) that \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash C_G(r_\gamma)/(C_G(r_\gamma) \cap K) \) has cardinality \( c(\gamma) \).

Observe that the group of units \( U_L \) of \( L^x \) is contained in \( K \). If \( L \) is unramified over \( F \), then \( L^x =< \pi > U_L \) so that \( C_G(r_\gamma)K/K \) is represented by the vertices \( \text{diag}(\pi^n, 1, 1)K, n \in \mathbb{Z} \), on a line in \( \mathcal{B} \), and \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash C_G(r_\gamma)/(C_G(r_\gamma) \cap K) \) by \( \text{diag}(\pi^n, 1, 1)K, n \mod c(\gamma) \). If \( L \) is ramified over \( F \), then \( L^x =< \pi_L > U_L \), where the uniformizer \( \pi_L \) does not lie in \( F \) and \( \pi_L^2 \) differs from \( \pi \) by a unit multiple. In this case \( C_G(r_\gamma)K/K \) is represented by the vertices \( \text{diag}(\pi^n, 1, 1)K \) and \( \text{diag}(\pi^n, 1, 1)\pi_L K, n \in \mathbb{Z} \), lying on two lines in \( \mathcal{B} \). There are two possibilities for \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \):

Case (i). The vertices in \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash K/K \) are contained in the line \( \text{diag}(\pi^n, 1, 1)K, n \in \mathbb{Z} \). Then \( c(\gamma) \) is even so that \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash C_G(r_\gamma)/(C_G(r_\gamma) \cap K) \) is represented by the vertices \( \text{diag}(\pi^n, 1, 1)K \) and \( \text{diag}(\pi^n, 1, 1)\pi_L K, n \mod c(\gamma)/2 \).

Case (ii). \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash K/K \) contains a vertex on the line \( \text{diag}(\pi^n, 1, 1)\pi_L K, n \in \mathbb{Z} \). Let \( y \in C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \) be such that \( yK = \text{diag}(\pi^N, 1, 1)\pi_L K \) has the least non-negative \( N \). Then \( y \) generates the group \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \), \( y^2K = \text{diag}(\pi^{2N-1}, 1, 1)K \), \( c(\gamma) = 2N - 1 \) is odd, and \( C_{P^{-1}_\gamma \Gamma P_\gamma}(r_\gamma) \backslash C_G(r_\gamma)/(C_G(r_\gamma) \cap K) \) is represented by the vertices \( \text{diag}(\pi^n, 1, 1)K, 0 \leq n \leq N - 1 = (c(\gamma) - 1)/2 \), and \( \text{diag}(\pi^n, 1, 1)\pi_L K, 0 \leq n \leq N - 2 = (c(\gamma) - 3)/2 \).

6.2. Double coset representatives of \( C_G(r_\gamma) \backslash G/K \).

**Proposition 13.** The set

\[
S = \left\{ \begin{pmatrix} 1 & x & y \\ 1 & 0 & 1 \\ \pi^n \end{pmatrix} \mid x, y \in F/\mathcal{O}_F, n \geq 0 \right\}
\]

represents the double coset \( C_G(r_\gamma) \backslash G/K \).
Proof. Write an element \( g \in G \) as \( wk \) for some upper triangular \( w \) and some \( k \in K \). Since \( C_G(r_\gamma) = F^x \times L^x \) modulo the diagonal embedding of \( F^x \), we may assume that \( w = \begin{pmatrix} 1 & x & y \\ 1 & z \\ \pi^n \end{pmatrix} \), where \( x, y, z \in F/O_F \) and \( n \in \mathbb{Z} \). We are reduced to proving

\[
(6.2) \quad GL_2(F) = \prod_{n \geq 0} L^x \begin{pmatrix} 1 \\ \pi^n \end{pmatrix} GL_2(O_F),
\]

where \( L^x \) is given by (6.1).

First we check the disjoint union. Suppose otherwise. Then there exist \( m \neq n \) and \( g \) satisfying

\[
g \in L^x \cap \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} GL_2(O_F) \begin{pmatrix} 1 \\ \pi^{-n} \end{pmatrix}.
\]

Replacing \( g \) by its inverse if necessary, we may assume \( m > n \). Write \( g = \begin{pmatrix} x & y \pi^{-n} \\ \pi^m z & w \pi^{m-n} \end{pmatrix} = \begin{pmatrix} u & vc \\ v & u + vb \end{pmatrix} \) for some \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(O_F) \) and \( u, v \in F \). Comparing entries, we find \( y \pi^{-n} = c \pi^m \) and \( w \pi^{m-n} = x + zb \pi^m \). Since \( x, y, z, w, b, c \) are all integral, we conclude that \( x \) is a nonunit and hence \( z \) and \( y \) should both be units, but then \( y \pi^{-n} = c \pi^m \) cannot hold by checking the order of both sides.

Next we prove equality. Let \( w = \begin{pmatrix} 1 & z \\ \pi^m \end{pmatrix} \in GL_2(F) \). Observe that for \( m \geq 0 \),

\[
\begin{pmatrix} 0 & c \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi^m \end{pmatrix} = \begin{pmatrix} 0 & c \pi^m \\ 1 & b \pi^m \end{pmatrix} = \begin{pmatrix} c \pi^m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \pi b \pi^m \end{pmatrix},
\]

showing that \( \begin{pmatrix} 1 & 0 \\ \pi^m \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-m-\text{ord}_z c} \end{pmatrix} \) represent the same double coset. Since \( \text{ord}_z c = 0 \) or 1, only such diagonal matrices with \( m \geq 0 \) are needed as double coset representatives. Thus we assume \( \text{ord}_z c < 0 \). It suffices to reduce \( w \) to a diagonal matrix via left multiplication by elements in \( L^x \) and right multiplication by elements in \( GL_2(O_F) \).

Case (I). \( 0 > \text{ord}_z c \geq m + \text{ord}_z c \). Choose \( v \in O_F \) with \( \text{ord}_z v + m + \text{ord}_z c = \text{ord}_z c \) and \( u \) a unit in \( O_F \) satisfying \( uz = -cv \pi^m \). Then \( \begin{pmatrix} u & vc \\ v & u + vb \end{pmatrix} w = \begin{pmatrix} u & 0 \\ v & vz + (u + vb) \pi^m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi^m \end{pmatrix} k \) for some \( k \in GL_2(O_F) \). Here we used the fact that \( u(u + vb) - v^2 c \) is a unit. It is obvious if \( v \) or \( c \)
(and hence $b$) is not a unit; when $v$ and $c$ are both units, this results from the irreducibility of $x^2 - bx - c$.

Case (II). $m + \text{ord}_\pi c > \text{ord}_\pi z$. Choose $u \in \mathcal{O}_F$ with $\text{ord}_\pi u + \text{ord}_\pi z = m + \text{ord}_\pi c$ and $v$ a unit such that $uz = -vcd_\pi^m$. Then 
\[
\begin{pmatrix} u & vc \\
v & u + vb \end{pmatrix} w = \begin{pmatrix} u & 0 \\
v & vz + (u + vb)\pi^m \end{pmatrix} = \begin{pmatrix} u & 0 \\
v & 0 \end{pmatrix} k \text{ for some } k \in \text{GL}_2(\mathcal{O}_F).
\]

In both cases we have shown that $w$ lies in the right hand side of (6.2), therefore (6.2) holds. This proves the proposition. 

\[\square\]

6.3. Minimal lengths of cycles in $[\gamma]$. First we discuss the type of $[\gamma]$, which is defined in §4.4 to be $(n, m)$ such that $r_\gamma \in T_{n,m} = K\text{diag}(1, \pi^m, \pi^{m+n})K$. Observe that $\text{ord}_\pi \det \gamma = \text{ord}_\pi \det r_\gamma = \text{ord}_\pi a(e + d\lambda)(e + d\bar{\lambda}) \in 3\mathbb{Z}$ by assumption (I) on $\Gamma$. Hence if $e + d\lambda$ is a unit in $L$, then at least one of $e, d$ is a unit and $a$ is not a unit. Consequently, $[\gamma]$ has type $(\text{ord}_\pi a, 0)$. Next assume $e + d\lambda$ is not a unit. We distinguish two cases. If $L$ is unramified over $F$ (hence $\lambda$ is a unit), then both $e$ and $d$ are non-units and $a$ is a unit; in this case $[\gamma]$ has type $(0, \min(\text{ord}_\pi e, \text{ord}_\pi d))$. If $L$ is ramified over $F$ (hence $\lambda$ is a uniformizer of $L$), then there are two possibilities:

(i) $\text{ord}_\pi (e + d\lambda)(e + d\bar{\lambda}) = 1$. This happens if and only if $e$ is a non-unit, $d$ is a unit, and $\text{ord}_\pi e > 2$; in this case $[\gamma]$ has type $(\text{ord}_\pi a - 1, 1)$.

(ii) $\text{ord}_\pi (e + d\lambda)(e + d\bar{\lambda}) > 1$. This happens if and only if both $e$ and $d$ are non-units and $a$ is a unit; in this case $[\gamma]$ has type $(0, \text{ord}_\pi e)$ if $\text{ord}_\pi e \leq \text{ord}_\pi d$, and type $(1, \text{ord}_\pi d)$ if $\text{ord}_\pi e > \text{ord}_\pi d$.

This proves the first assertion of

**Theorem 14.** Let $\gamma$ be a rank-one split element in $[\Gamma]$ with $r_\gamma = \begin{pmatrix} a & e & dc \\
& d & e + db \end{pmatrix}$. Suppose that $r_\gamma \in K\text{diag}(1, \pi^m, \pi^{m+n})K$. Then

1. The type $(n, m)$ of $[\gamma]$ is as follows.
   (1.i) If $\text{ord}_\pi c = 0$, then $(n, m) = (\text{ord}_\pi a, \min\{\text{ord}_\pi e, \text{ord}_\pi d\})$.
   (1.ii) If $\text{ord}_\pi c = 1$, then $(n, m) = (\text{ord}_\pi a, \text{ord}_\pi e)$ provided that $\text{ord}_\pi e \leq \text{ord}_\pi d$, otherwise $(n, m) = (\max\{\text{ord}_\pi a - 1, 1\}, \max\{\text{ord}_\pi d, 1\})$.

2. $l_A([\gamma]) = \min_{\kappa, (gK) \in [\gamma]} l_A(\kappa_\gamma(gK)) = \text{ord}_\pi a(e^2 + edb - cd^2) = n + 2m$.

3. $l_G([\gamma]) = \min_{\kappa, (gK) \in [\gamma]} l_G(\kappa_\gamma(gK)) = n + m$.

This theorem combined with Theorem [4] completes the proof of Theorem [5].

**Remark.** If $\gamma$ is ramified rank-one split and $[\gamma]$ has type $(n, 1)$, then $[\gamma^2]$ has type $(2n - 1, 0)$. 

Proof. It remains to show that the algebraic and geometric lengths of the cycles in $[\gamma]$ are at least those of $[\gamma]$ since, as observed before, the cycles $\kappa_\gamma(P_\gamma gK)$ with $g \in C_G(r_\gamma)$ have the same algebraic and geometric lengths as $[\gamma]$. By Proposition 13 it suffices to compute $(P_\gamma g)^{-1}P_\gamma g = g^{-1}r_\gamma g$ for $g \in S$. Let $g = \begin{pmatrix} 1 & x & y \\ 1 & 0 & \pi^i \end{pmatrix}$, where $x, y \in F/O_F$ and $i \geq 0$. Then

$$g^{-1}r_\gamma g = \begin{pmatrix} 1 & -x & -y\pi^{-i} \\ 1 & 0 & \pi^{-i} \end{pmatrix} \begin{pmatrix} a & e & dc \\ d & e + db & \pi^i \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 1 & 0 & \pi^i \end{pmatrix}$$

$$= \begin{pmatrix} a & (a - e)x - dy\pi^{-i} & (a - e - db)y - cdx\pi^i \\ e & dc\pi^i & d\pi^{-i} \end{pmatrix} \in K \begin{pmatrix} \pi^{e_1} & \pi^{e_2} \\ \pi^{e_3} \end{pmatrix} K.$$

Here $e_1 \leq e_2 \leq e_3$, and as in the proof of Theorem 6, we have

$$e_1 \leq \min\{\text{ord}_\pi a, -i + \text{ord}_\pi d, \text{ord}_\pi e\} \leq \min\{\text{ord}_\pi a, \text{ord}_\pi d, \text{ord}_\pi e\} = 0,$$

$$e_1 + e_2 \leq \min\{\text{ord}_\pi ae, -i + \text{ord}_\pi ad, \text{ord}_\pi (e^2 + bed - cd^2)\}$$

$$\leq \min\{\text{ord}_\pi ae, \text{ord}_\pi ad, \text{ord}_\pi (e^2 + bed - cd^2)\} = m,$$

and

$$e_1 + e_2 + e_3 = \text{ord}_\pi a(e^2 + bed - cd^2) = n + 2m,$$

in which the last upper bound for $e_1 + e_2$ can be verified using the statement (1). Therefore $l_A(\kappa_\gamma(P_\gamma gK)) = e_1 + e_2 + e_3 - 3e_1 \geq e_1 + e_2 + e_3 = n + 2m = l_A([\gamma])$ since $e_1 \leq 0$. The inequalities (6.4) and (6.5) together give the lower bounds $e_3 \geq n + 2m - m = n + m$, which in turn implies $l_G(\kappa_\gamma(P_\gamma gK)) = e_3 - e_1 \geq n + m$. This proves the theorem. \hfill \Box

As shown in the above proof, an algebraically tailless cycle in $[\gamma]$ satisfies the condition $e_1 = 0$, while a tailless cycle $\kappa_\gamma(P_\gamma gK)$ in $[\gamma]$ should satisfy $e_1 + e_2 = m$ and $e_1 = 0$. This shows that $\kappa_\gamma(P_\gamma gK)$ is also algebraically tailless. Moreover, it also satisfies $e_2 = m$, which shows that $\kappa_\gamma(P_\gamma gK)$ has the same type as $[\gamma]$. If furthermore, $[\gamma]$ has type zero, then an algebraically tailless cycle in $[\gamma]$ satisfies $e_1 = 0$, which implies $e_1 + e_2 \geq 0$ and hence $e_1 + e_2 = 0 = m$ by (6.4) and $e_3 = n + m$. This shows that in this case an algebraically tailless cycle in $[\gamma]$ is also tailless. We record this discussion in
Corollary 15. Suppose $\gamma \in [\Gamma]$ is rank-one split. Then all tailless cycles in $[\gamma]$ are also algebraically tailless, and they have the same type as $[\gamma]$. Moreover, if $[\gamma]$ has type one, then the algebraically tailless and tailless cycles in $[\gamma]$ coincide.

We have shown that as long as $[\gamma]$ has type one, there is no distinction between algebraically tailless and tailless, regardless whether $\gamma$ is split or rank-one split.

6.4. Counting the number of cycles in $[\gamma]$ in algebraic length. In order to count the number of cycles $\kappa_\gamma(P_\gamma gK)$ in $[\gamma]$ of a given length, we need to determine the cardinality of $C_{P_\gamma^{-1}P_\gamma}(r_\gamma)\setminus C_{G}(r_\gamma)gK/K$. For this, we may take as representatives the product of representatives of $C_{P_\gamma^{-1}P_\gamma}(r_\gamma)\setminus C_{G}(r_\gamma)$ by the representatives of $(C_{G}(r_\gamma)\cap K)gK/K$. The number of the former representatives is $c([\gamma])$, defined by (4.2).

It remains to compute the cardinality of the latter. Recall that $L^\times \cap K$ consists of the units in $L^\times$, which we shall identify as the set of matrices

$$U_L = \left\{ \begin{pmatrix} u & vc \\ v & u + vb \end{pmatrix} \mid u, v \in \mathcal{O}_F, u^2 + uvb - cv^2 \text{ is a unit} \right\}.$$

Denote by $K'$ the group $GL_2(\mathcal{O}_F)$. As analyzed in the proof of Proposition 13, we are reduced to counting, for given $m \geq 0$, the cardinality of $U_L \left( \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} \right) K'/K'$.

Proposition 16.

$$\# [U_L \left( \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} \right) K'/K'] = \begin{cases} 1 & \text{when } m = 0, \\ q^m & \text{when } m \geq 1 \text{ and } \text{ord}_c \pi = 1, \\ q^m + q^{m-1} & \text{when } m \geq 1 \text{ and } \text{ord}_c \pi = 0. \end{cases}$$

Proof. It is clear that the cardinality is 1 when $m = 0$. Thus assume $m \geq 1$.

Case (I) $\text{ord}_c \pi = 1$. Then any $\begin{pmatrix} u & vc \\ v & u + vb \end{pmatrix} \in U_L$ satisfies $u \in \mathcal{O}_F^\times$. For $n \geq 0$, let

$$U_L(n) = \left\{ \begin{pmatrix} u & v\pi^n \\ v\pi^n & u + vb\pi^n \end{pmatrix} \in U_L \mid u, v \in \mathcal{O}_F^\times \right\}$$

so that

$$U_L = U_L(\infty) \cup_{n \geq 0} U_L(n),$$
where

\[ \mathcal{U}_L(\infty) = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in \mathcal{O}_F^\times \right\}. \]

One verifies that

\[ \mathcal{U}_L(n) \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K' = \bigcup_{u \in \mathcal{O}_F^\times/\pi^{m-n} \mathcal{O}_F} \begin{pmatrix} \pi^{m-n} & u \\ \pi^n \end{pmatrix} K' \]

for \(0 \leq n < m\), and

\[ \mathcal{U}_L(n) \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K' = \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K' \]

for \(n \geq m\) and \(n = \infty\). Therefore

\[ \#[\mathcal{U}_L \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K'/K'] = 1 + \sum_{0 \leq n < m} (q^{m-n} - q^{m-n-1}) = q^m. \]

Case (II) \(\text{ord}_\pi c = 0\). Let

\[ \mathcal{U}'_L = \left\{ \begin{pmatrix} u & v c \\ v & u + v b \end{pmatrix} \in \mathcal{U}_L \mid u \in \mathcal{O}_F^\times \right\} \]

and

\[ \mathcal{U}''_L = \left\{ \begin{pmatrix} u & v c \\ v & u + v b \end{pmatrix} \in \mathcal{U}_L \mid u \in \pi \mathcal{O}_F \right\} \]

so that

\[ \mathcal{U}_L = \mathcal{U}'_L \cup \mathcal{U}''_L. \]

As in Case (I), we have

\[ \mathcal{U}'_L \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K' = \bigcup_{\substack{m \geq n \geq 0 \\ u \in \mathcal{O}_F^\times/\pi^{m-n} \mathcal{O}_F}} \begin{pmatrix} \pi^{m-n} & u \\ \pi^n \end{pmatrix} K'. \]

One checks that

\[ \mathcal{U}''_L \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K' = \bigcup_{z \in \pi \mathcal{O}_F/\pi^m \mathcal{O}_F} \begin{pmatrix} \pi^m & z \\ 1 \end{pmatrix} K'. \]

Therefore

\[ \#[\mathcal{U}_L \begin{pmatrix} 1 \\ \pi^m \end{pmatrix} K'/K'] = q^m + q^{m-1} \]

for \(m \geq 1\). \(\Box\)

We summarize the above discussion in
Corollary 17. For each $g = \begin{pmatrix} 1 & x & y \\ 1 & 0 & \pi^n \end{pmatrix} \in S$, the cardinality of $C_{P_{\gamma}^1 \Gamma P_{\gamma}}(r_{\gamma}) \setminus C_G(r_{\gamma})gK/K$ is
\[
c([\gamma]) = \begin{cases} 1 & \text{when } n = 0, \\ q^n & \text{when } n \geq 1 \text{ and } \text{ord}_n c = 1, \\ q^n + q^{n-1} & \text{when } n \geq 1 \text{ and } \text{ord}_n c = 0. \end{cases}
\]

Now we are ready to state the main result of this section.

Theorem 18. Suppose $\gamma \in \Gamma$ is rank-one split with $r_{\gamma} = \begin{pmatrix} a & e & dc \\ e & dc & \end{pmatrix}$. Set $\delta = \text{ord}_n d$ and $\mu = \mu([\gamma]) = \text{ord}_n ((a - e)^2 - db(a - e) - cd^2)$.

(A) $\gamma$ is unramified rank-one split. Then the following hold.

(A1) \[ \sum_{\kappa_{\gamma}(gK) \in [\gamma]} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( \frac{q^{\delta+1} + q^{\delta} - 2}{q-1} + \frac{(q+1)q^{\delta+2}u^3}{1 - q^3u^3} \right) \left( 1 - u^3 \right). \]

(A2) If $[\gamma]$ does not have type one, then
\[ \sum_{\kappa_{\gamma}(gK) \in [\gamma], \text{ type one}} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( q^{\delta} + q^{\delta-1} + \frac{(q^2 - 1)q^{\delta+1}u^3}{1 - q^3u^3} \right). \]

(A3) If $[\gamma]$ has type one, then
\[ \sum_{\kappa_{\gamma}(gK) \in [\gamma], \text{ type one}} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( \frac{q^{\delta+1} + q^{\delta} - 2}{q-1} + \frac{(q^2 - 1)q^{\delta+1}u^3}{1 - q^3u^3} \right). \]

(B) $\gamma$ is ramified rank-one split. Then the following hold.

(B1) \[ \sum_{\kappa_{\gamma}(gK) \in [\gamma]} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( \frac{q^{\delta+1} - 1}{q-1} + \frac{q^{\delta+3}u^3}{1 - q^3u^3} \right) \frac{1 - u^3}{1 - q^2u^3}. \]

(B2) If $[\gamma]$ does not have type one, then
\[ \sum_{\kappa_{\gamma}(gK) \in [\gamma], \text{ type one}} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( q^{\delta}(q^\mu - \mu) + \frac{(q-1)q^{\delta+2}u^3}{1 - q^3u^3} \right). \]

(B3) If $[\gamma]$ has type one, then
\[ \sum_{\kappa_{\gamma}(gK) \in [\gamma], \text{ type one}} u^{\mu}(\kappa_{\gamma}(gK)) = c([\gamma])u^{\mu}(\gamma) \left( \frac{q^{\delta+1} - 1}{q-1} + \frac{(q-1)q^{\delta+2}u^3}{1 - q^3u^3} \right). \]
Moreover, in each case, if $[\gamma]$ does not have type one, none of the type one cycles in $[\gamma]$ are tailless.

 Remarks. 1. $\mu = 0$ unless $a, e, c$ are all nonunit, in which case it is 1 and $\delta = 0$.

 2. $\mu = 0$ when $[\gamma]$ has type one.

 3. $\delta > 0$ in case (A2), while $\delta$ may be zero in case (A3).

 Proof. Recall that the algebraic length of a cycle in $[\gamma]$ is equal to $l_A([\gamma]) + 3m$ for some $m \geq 0$. We shall follow the same notation and computation as in the proof of Theorem [14] letting $g$ run through all elements in the double coset representatives $S$ and computing, for each $m \geq 0$, the number of cycles $\kappa_\gamma(P_\gamma gK)$ with $l_A(\kappa_\gamma(P_\gamma gK)) \leq l_A([\gamma]) + 3m$ using Corollary [17]. As $g = \begin{pmatrix} 1 & x & y \\ 1 & 0 & -
\pi^i \end{pmatrix}$, this amounts to computing the number of $x, y \in F/\mathcal{O}_F$ and $i \geq 0$ such that

$$e_1 = \min\{\text{ord}_\pi((a - e)x - d\pi^{-i}y), \text{ord}_\pi(-cd\pi^i x + (a - e - db)y), -i + \text{ord}_\pi d\} \geq -m.$$ 

This is equivalent to $0 \leq i \leq m + \text{ord}_\pi d$, $(a - e)x - d\pi^{-i}y \in \pi^{-m}\mathcal{O}_F$ and $-cd\pi^i x + (a - e - db)y \in \pi^{-m}\mathcal{O}_F$. Denote $\text{ord}_\pi d$ by $\delta$ for short. So for each $0 \leq i \leq m + \delta$, we solve the following system of linear equations

$$(6.6) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a - e & -d\pi^{-i} \\ -cd\pi^i & a - e - db \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$$

for $\alpha, \beta \in \pi^{-m}\mathcal{O}_F$ and count the distinct pairs $(x, y) \in F/\mathcal{O}_F \times F/\mathcal{O}_F$. Recall that $a, e, d$ are integral, at least one of them is a unit, and $a$ and $e$ cannot be both units since $\text{ord}_\pi \det r_\gamma > 0$. Let

$$\mu := \text{ord}_\pi \det M = \text{ord}_\pi((a - e)^2 - db(a - e) - cd^2),$$

which is 0 unless $a, e$ and $c$ are all nonunits, in which case it is 1. Put

$$\varepsilon := \min\{\text{ord}_\pi(a - e), -i + \delta, \text{ord}_\pi(a - e - bd)\},$$

which is equal to $-i + \delta$ if $\delta \leq i \leq m + \delta$, and 0 if $0 \leq i < \delta$. Then the coefficient matrix $M = k_1 \text{diag}(\pi^{\varepsilon}, \pi^{\mu - \varepsilon})k_2$ for some $k_1, k_2 \in GL_2(\mathcal{O}_F)$. Thus system (6.6) has the same number of solutions as the system

$$(6.7) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \pi^{\varepsilon} \\ \pi^{\mu - \varepsilon} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
for $\alpha, \beta \in \pi^{-m}O_F$ and $(x, y) \in F/O_F \times F/O_F$. We get the solutions $x \in \pi^{-m-\varepsilon}O_F/O_F$ and $y \in \pi^{-m-\mu+\varepsilon}O_F/O_F$ so that there are $q^{2m+\mu}$ different pairs $(x, y)$ for each $0 \leq i \leq m + \delta$. To proceed, we distinguish two cases.

Case (A) $\text{ord}_\pi c = 0$, that is, $\gamma$ is unramified rank-one split. Then $\mu = 0$. By Corollary 17, the number of classes in $[\gamma]$ with algebraic length at most $l_A([\gamma]) + 3m$ is

$$c([\gamma])q^{2m}(1 + \sum_{1 \leq n \leq m + \delta} q^n + q^{n-1}) = c([\gamma])q^{2m}\left(\frac{q^{2m+\delta} - 1}{q - 1} + \frac{q^{m+\delta+1} - 1}{q - 1}\right) = \frac{c([\gamma])}{q - 1}(q^{3m+\delta+1} + q^{3m+\delta} - 2q^{2m}).$$

Therefore

$$\sum_{\gamma \in [\gamma]} u_{\lambda_A(\gamma)} = \sum_{\gamma \in [\gamma]} u_{\lambda_A(\gamma)} = \frac{c([\gamma])}{q - 1}\left(q^{\delta+1} + q^\delta - 2 + \sum_{m \geq 1} (q^{3m+\delta+1} + q^{3m+\delta} - 2q^{2m} - q^{3m+\delta-2} - q^{3m+\delta-3} + 2q^{2m-2})u^{3m}\right).$$

Among the cycles with $l_A(\gamma) = l_A([\gamma]) + 3m$, we compute the number of those with type one. First consider the case $m \geq 1$. In order that $l_A(\gamma) = l_A([\gamma]) + 3m$ and $\gamma$ has type one, two conditions must be satisfied:

$$\epsilon_1 = \text{min}\{\text{ord}_\pi((a - e)x - d\pi^{-i}y), \text{ord}_\pi(-cd\pi^i x + (a - e - db)y), -i + \delta\} = -m,$$

and

$$\epsilon_1 + \epsilon_2 = \text{ord}_\pi[((a - e)x - d\pi^{-i}y)(e + db) - d\pi^{-i}(-cd\pi^i x + (a - e - db)y)] = -2m.$$

These two conditions are equivalent to $i = \delta + m$, $\text{ord}_\pi(-cd\pi^i x + (a - e - db)y) = -m$, and $\text{ord}_\pi((a - e)x - d\pi^{-i}y) \geq -m$. This amounts to solving system (3.6) with $\alpha \in \pi^{-m}O_F$ and $\beta \in \pi^{-m}O_F^\alpha$, hence we obtain $(q - 1)q^{2m-1}$ distinct pairs $(x, y)$. Combined with Corollary 17, we see that the number of rank one cycles $\gamma$ with $l_A(\gamma) = l_A([\gamma]) + 3m$ is $c([\gamma])(q - 1)q^{2m-1}(q^{\delta+m} + q^{\delta+m-1})$.

Next consider the case $m = 0$. Under the assumption $\text{ord}_\pi c = 0$, we know from Theorem 14 that $[\gamma]$ has type $(\text{ord}_\pi a, \text{min}\{\text{ord}_\pi e, \text{ord}_\pi d\})$. Therefore it has type one if and only if $\text{ord}_\pi a > 0$, in which case all cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ have type one, and the number of such cycles is $c([\gamma])\frac{q^{\delta+1} - 1}{q - 1}$, as computed above. If $[\gamma]$ does not have type one, then $\delta = \text{ord}_\pi d > 0$; the condition $\epsilon_1 = \epsilon_2 = 0$ implies $i = \delta$ and only one solution $(x, y) = (0, 0)$. In this case the number
of type one cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ is $q^\delta + q^{\delta-1}$ by Corollary 17. Put together, we have shown the following:

If $[\gamma]$ has type one, then

$$
\sum_{\kappa_\gamma(gK) \in [\gamma], \text{type one}} u^{l_A(\kappa_\gamma(gK))} = c([\gamma])u^{l_A([\gamma])} \left( \frac{q^{\delta+1} + q^{\delta-2}}{q-1} + \sum_{m \geq 1} (q-1)q^{2m-1}(q^{\delta+m} + q^{\delta+m-1})u^{3m} \right)
$$

$$= c([\gamma])u^{l_A([\gamma])} \left( \frac{q^{\delta+1} + q^{\delta-2}}{q-1} + \frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3} \right),
$$

while if $[\gamma]$ does not have type one, then

$$
\sum_{\kappa_\gamma(gK) \in [\gamma], \text{type one}} u^{l_A(\kappa_\gamma(gK))} = c([\gamma])u^{l_A([\gamma])} \left( q^\delta + q^{\delta-1} + \frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3} \right).
$$

Case (B) ord$_e c = 1$, that is, $\gamma$ is ramified rank-one split. Then $\mu = 0$ or 1. The same computation as in Case (A) together with Corollary 17 shows that the number of classes in $[\gamma]$ with algebraic length at most $l_A([\gamma]) + 3m$ is

$$c([\gamma])q^{2m+\mu} \sum_{0 \leq n \leq m+\delta} q^n = c([\gamma])q^{2m+\mu} \frac{q^{m+\delta+1} - 1}{q-1} = c([\gamma]) \frac{q^\mu}{q-1} (q^{3m+\delta+1} - q^{2m}).$$

Therefore

$$
\sum_{\kappa_\gamma(gK) \in [\gamma]} u^{l_A(\kappa_\gamma(gK))} = c([\gamma]) \frac{q^\mu}{q-1} u^{l_A([\gamma])} \left( \sum_{m \geq 0} (q^{3m+\delta+1} - q^{2m})u^{3m} - \sum_{m \geq 1} (q^{3m+\delta-2} - q^{2m-2})u^{3m} \right)
$$

$$= c([\gamma]) \frac{q^\mu}{q-1} u^{l_A([\gamma])} \left( \frac{q^{\delta+1}}{1-q^3u^3} - \frac{1}{1-q^2u^3} \right) (1-u^3)
$$

$$= c([\gamma])q^\mu u^{l_A([\gamma])} \left( \frac{q^{\delta+1} - 1}{q-1} + \frac{q^{\delta+3}u^3}{1-q^2u^3} \right) \frac{1-u^3}{1-q^2u^3}.
$$

Now we compute the number of type one cycles $\kappa_\gamma(P_\gamma gK)$ with algebraic length $l_A(\kappa_\gamma(P_\gamma gK)) = l_A([\gamma]) + 3m$. First consider the case $m \geq 1$. Following the same argument as in Case (A) and applying Corollary 17, we see that the number of such cycles is $c([\gamma]) (q-1)q^{2m+\mu-1}q^{\delta+m}$.

Next we discuss the remaining case $m = 0$. By Theorem 14, $[\gamma]$ has type one if and only if ord$_e a > 0$ and ord$_e e = 0$, in which case all cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ are of type one, and the number of such cycles is $c([\gamma])q^\mu \frac{q^{\delta+1} - 1}{q-1}$. When $[\gamma]$ does not have type one, we have ord$_e e > 0$; the condition $e_1 = e_2 = 0$ implies $i = \delta$. Moreover, if $\mu = 0$, in which case $a$ is a unit, then there is only one pair $(x, y) = (0, 0)$; while if $\mu = 1$, in which case $a$ is not a unit, then there are $q-1$ pairs $(x, y) = (0, y)$ with $y \in \pi^{-1}\mathcal{O}_F/\mathcal{O}_F$ so that ord$_e (-cd\pi^i x + (a - e - db)y) = 0$. Consequently, when $[\gamma]$ does not have type one, the number of type one cycles in $[\gamma]$ with algebraic
length equal to \( l_\mathcal{A}(\{\gamma\}) \) is \( c(\gamma)q^\delta \) if \( \mu = 0 \), and \( c(\gamma)(q - 1)q^\delta \) if \( \mu = 1 \). In other words, it is \( c(\gamma)q^\delta(q^\mu - \mu) \). Summing up, we have proved the following:

If \( [\gamma] \) has type one, then

\[
\sum_{\kappa, (gK) \in [\gamma], \text{ type one}} u^{l_\mathcal{A}(\kappa, (gK))} = c(\gamma)u^{l_\mathcal{A}(\gamma)}q^\mu \left( \frac{q^{\delta+1} - 1}{q - 1} + \sum_{m \geq 1} (q - 1)q^{3m+\delta-1}u^{3m} \right)
\]

\[
= c(\gamma)u^{l_\mathcal{A}(\gamma)}q^\mu \left( \frac{q^{\delta+1} - 1}{q - 1} + \frac{(q - 1)q^{\delta+2}u^3}{1 - q^3u^3} \right),
\]

while if \( [\gamma] \) does not have type one, then

\[
\sum_{\kappa, (gK) \in [\gamma], \text{ type one}} u^{l_\mathcal{A}(\kappa, (gK))} = c(\gamma)u^{l_\mathcal{A}(\gamma)} \left( q^\delta(q^\mu - \mu) + \frac{(q - 1)q^{\delta+\mu+2}u^3}{1 - q^3u^3} \right).
\]

This completes the proof of the theorem. \( \square \)

As before, let

\[
(6.8) \quad \Delta_\mathcal{A}(\{\gamma\}) = \{gK \in G/K \mid l_\mathcal{A}(\kappa, (P_\gamma gK)) = l_\mathcal{A}(\{\gamma\})\}.
\]

Then \( \Delta_\mathcal{A}(\{\gamma\}) \) contains \( C_G(r_\gamma)K/K \) and it is invariant under left multiplication by \( C_{P_\gamma^{-1}P_\gamma}(r_\gamma) \). Moreover, \( C_{P_\gamma^{-1}P_\gamma}(r_\gamma) \backslash \Delta_\mathcal{A}(\{\gamma\}) \) is finite, and its cardinality is the number of algebraically tailless cycles in \( [\gamma] \). Contained in the proofs of Corollary 17 and Theorem 18 is the first assertion of the proposition below. Let

\[
(6.9) \quad g_{i,j,u} = \begin{pmatrix}
1 \\
\pi^{i-j} & u \\
\pi^j
\end{pmatrix}
\quad \text{and} \quad g_{i,z} = \begin{pmatrix}
1 \\
\pi^i & z \\
1
\end{pmatrix}.
\]

**Proposition 19.** Let \( \gamma \in \Gamma \) be rank-one split with \( r_\gamma = \begin{pmatrix}
a \\
e & dc \\
d & e + db
\end{pmatrix} \). Set \( \delta = \delta([\gamma]) = \text{ord}_\pi d \).

Suppose that \( [\gamma] \) has type one with \( n = \text{ord}_\pi a \). Then

\[
\Delta_\mathcal{A}(\{\gamma\}) = \{hg_{i,j,u}K \mid h \in C_G(r_\gamma)/(C_G(r_\gamma) \cap K), \ 0 \leq j \leq i \leq \delta, \\
u \in \mathcal{O}_F/\pi^{i-j}\mathcal{O}_F \text{ for } j < i, \text{ and } u = 0 \text{ for } j = i\}
\]

if \( \gamma \) is ramified rank-one split, and

\[
\Delta_\mathcal{A}(\{\gamma\}) = \{hg_{i,j,u}K \mid h \text{ and } g_{i,j,u} \text{ as above} \} \cup \{hg_{i,z}K \mid h \text{ as above}, \ 1 \leq i \leq \delta, z \in \pi\mathcal{O}_F/\pi^i\mathcal{O}_F\}.
\]
if $\gamma$ is unramified rank-one split. Consequently, the number of algebraically tailless cycles in $[\gamma]$ is

$$
\#(C^{-1}_P(\gamma)P_{\gamma}(r_{\gamma})\backslash \Delta_A([\gamma])) = c([\gamma]) \begin{cases} 
\frac{q^{\delta+1}+q^{\delta}-2}{q-1} & \text{if } [\gamma] \text{ is unramified rank-one split}, \\
\frac{q^{\delta+1}-1}{q-1} & \text{if } [\gamma] \text{ is ramified rank-one split}.
\end{cases}
$$

Moreover, for $g = hg_{i,j,u}$ or $hg_{i,z}$ such that $gK \in \Delta_A([\gamma])$, the geodesic $\kappa_{\gamma}(P_{\gamma}gK)$ in $B$ is given by $P_{\gamma}gK \to P_{\gamma}g\text{diag}(\pi,1,1)K \to \cdots \to P_{\gamma}g\text{diag}(\pi^n,1,1)K = \gamma P_{\gamma}gK$. The last assertion follows from $P_{\gamma}g\text{diag}(\pi^n,1,1)K = P_{\gamma}g\text{diag}(\pi^n,1,1)K = \gamma P_{\gamma}gK$ since $g^{-1}r_{\gamma}g \in K$ by choice.

6.5. Tailless type one primitive cycles. A cycle in $X_\Gamma$ (resp. an element in $\Gamma$) is primitive if it is not obtained by repeating a cycle (resp. multiplying an element) more than once. Suppose that $\kappa_{\gamma}(gK)$ is $\kappa_{\beta}(gK)$ repeated $m$ times in $X_\Gamma$. Then $\gamma gK = \beta^m gK$ in $B$. As the action of $\Gamma$ on $B$ is fixed point free, this implies $\gamma = \beta^m$. In other words, a necessary condition for a cycle $\kappa_{\gamma}(gK)$ to be non-primitive is that $\gamma$ is a non-primitive element in $\Gamma$.

On the other hand, suppose $\gamma \in \Gamma$ is non-primitive and of type one. Since $\Gamma$ is torsion-free, we may write $\gamma = \beta^m$ for a unique primitive $\beta \in \Gamma$ and $m > 1$. Then $r_{\gamma}$ and $r_{\beta}$ have the same centralizers in $\Gamma$, and $c([\gamma]) = c([\beta^j])$ for all $j \geq 1$. Moreover, $\beta^j$ also has type one and $\delta([\beta^j]) \leq \delta([\gamma])$ for all positive divisors $j$ of $m$. Combining Corollary 12 and Proposition 19, we conclude that $\Delta_A([\beta^j]) \subset \Delta_A([\gamma])$ for $j|m$, and the cycles $\kappa_{\gamma}(gK)$ with $gK$ in $\Delta_A([\gamma]) \setminus \cup_{j|m, \ 0<j<m} \Delta_A([\beta^j])$ are the tailless type one primitive closed geodesics in $[\gamma]$. Further, by shifting vertices on such a cycle we obtain $l_A([\gamma])$ distinct cycles.

This is different from the case of graphs arising from $PGL_2(F)$ where all tailless cycles contained in the conjugacy class of a non-primitive element are non-primitive.

7. Type one gallery Zeta function of $X_\Gamma$

7.1. Chambers and Iwahori-Hecke algebra on the building $B$. A chamber of the building $B = G/K$ is a 2-simplex with three mutually adjacent vertices $v_1, v_2, v_3$. The group $G$ acts on the vertices of $B$ transitively, and it preserves edges and chambers of $B$. Let

$$
\sigma = \begin{pmatrix}
1 \\
\pi \\
1
\end{pmatrix}.
$$

Denote by $C_0$ the fundamental chamber with vertices $v_1 = K, v_2 = \sigma K,$ and $v_3 = \sigma^2 K$. The Iwahori subgroup $B$ of $K$ consisting of elements $k \in K$ congruent to upper triangular matrices
mod \( \pi \) is the largest subgroup of \( G \) stabilizing each vertex of \( C_0 \), while \( \sigma \) rotates the vertices of \( C_0 \). Denote by \( \sigma' \) the permutation \((1 \ 2 \ 3)\) in \( S_3 \) such that \( \sigma(v_i) = v_{\sigma'(i)} \). Since \( G \) acts transitively on the chambers of \( B \), the assignment \( gB \mapsto gC_0 \) is a three-to-one map from \( G/B \) to the set of chambers such that \( gB, g\sigma B \) and \( g\sigma^2 B \) all correspond to the same chamber \( gC_0 \). The matrices

\[
t_1 = \begin{pmatrix} 1 & \pi^{-1} \\ 1 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

act as reflections which fix the edges \( \{v_1, v_2\}, \{v_2, v_3\} \) and \( \{v_3, v_1\} \) of \( C_0 \), respectively. We have \( \sigma t_i = t_{\sigma'(i)} \sigma \) for \( i = 1, 2, 3 \).

Note that \( t_1, t_2, t_3 \) generate the Weyl group \( W \) of \( \text{PSL}_3(F) \) subject to the relations \( t_i^2 = Id \) and \( (t_i t_j)^3 = Id \) for \( i \neq j \). The Bruhat decomposition of \( G \) is

\[
G = \bigsqcup_{w \in W \ltimes <\sigma>} BwB.
\]

Each element \( w \in W \ltimes <\sigma> \) defines an operator \( L_w \) on \( L^2(G/B) \) by sending a function \( f \) to \( L_w f \) given by

\[
L_w f(gB) = \sum_{w_i B \in BwB/B} f(gw_i B) \quad \text{for all } gB.
\]

These operators form a generalized Iwahori-Hecke algebra satisfying the following relations (cf. \text{[Ga]}):

1. \( L_{t_i} \cdot L_{t_i} = (q - 1) L_{t_i} + qId \),

2. \( L_{t_i} \cdot L_{t_j} = L_{t_i t_j} \) for \( i \neq j \),

3. \( L_{t_i} \cdot L_w = L_{t_i w} \) if the length of \( t_i w \) is 1 plus the length of \( w \),

4. \( L_\sigma \cdot L_{t_i} = L_{t_i} L_{t_{\sigma'(i)} \sigma} \) for \( i = 1, 2, 3 \).

Let

\[
(7.1) \quad L_B = L_{t_2 \sigma^2}.
\]

Then the above properties imply \( (L_B)^{3n} = (L_{t_2 t_1 t_3})^n \) for \( n \geq 1 \).

### 7.2. Galleries in \( B \)

Paths formed by the edge-adjacent chambers are called galleries. A geodesic gallery between two chambers is a gallery containing the least number of intermediate chambers. To get geodesic galleries from \( g_1 B \) to \( g_2 B \), we find the element \( w \in W \ltimes <\sigma> \) such that \( g_1^{-1} g_2 \in BwB \) and write \( w = t_{i_1} \cdots t_{i_n} \sigma^j \) as a word using least number of reflections \( t_1, t_2, t_3 \); call \( n \) the \textit{length} of the gallery. All geodesic galleries from \( g_1 B \) to \( g_2 B \) have length \( n \); different
galleries arise from different expressions of \( w \) as a product of generators, and they are regarded as *homotopic*. Like the case of paths, given two distinct chambers \( g_1B \) and \( g_2B \), there is only one homotopic class of geodesic galleries in \( B \) from \( g_1B \) to \( g_2B \).

Observe that a geodesic gallery arising from \( w = t_{i_1} \cdots t_{i_n} \sigma^j \) is a strip if and only if the difference \( i_k - i_{k+1} \) remains the same mod 3 for \( 1 \leq k \leq n - 1 \). It is said to have type one or two according to the common difference being 1 or 2. Note that the homotopy class of a gallery of type one or two contains only one geodesic gallery, thus we shall drop the word “homotopy” in this case.

### 7.3. Closed galleries in \( X_\Gamma \).

A closed gallery in \( X_\Gamma \) starting at the chamber \( \Gamma gB \) of \( X_\Gamma \) can be lifted to a gallery in \( B \) starting at \( gB \) and ending at \( \gamma gB \) for some \( \gamma \in \Gamma \). Denote by \( \kappa_\gamma(gB) \) the homotopy class of geodesic galleries in \( B \) from \( gB \) to \( \gamma gB \). By abuse of notation, it also represents a homotopy class of closed geodesic gallery in \( X_\Gamma \) starting at \( \Gamma gB \). The argument in §4.2 holds with \( K \) replaced by \( B \). Let, for \( \gamma \in [\Gamma] \),

\[
[\gamma]_B = \{ \kappa_\gamma(gB) : g \in C_\Gamma(\gamma) \setminus G/B \}.
\]

Then the union of \([\gamma]_B\) over \( \gamma \in [\Gamma] \) is the set of all based homotopy classes of closed geodesic galleries in \( X_\Gamma \).

A closed gallery \( \kappa_\gamma(gB) \) of length \( n \) in \( X_\Gamma \) is called *tailless* if the geodesic gallery in \( B \) from \( gB \) to \( \gamma gB \) followed by the geodesic gallery from \( \gamma gB \) to \( \gamma^2 gB \) is a geodesic gallery from \( gB \) to \( \gamma^2 gB \) of length \( 2n \). Note that the condition (I) imposed on \( \Gamma \) in §4.1 implies that \( g^{-1}\gamma g \in BWB \) for all \( g \in G \) and \( \gamma \in \Gamma \). So if \( \kappa_\gamma(gB) \) has length \( n \), then \( g^{-1}\gamma g \in BWB \) for some \( w = t_{i_1} \cdots t_{i_n} \in W \) of length \( n \). Since \( g^{-1}\gamma^2 g \in BWB \cdot BWB \), then \( \kappa_{\gamma^2}(gB) \) has length \( 2n \) if and only if the word \( w^2 \) has length \( 2n \), which is equivalent to \( BWB \cdot BWB = BW^2B \).

**Proposition 20.** Let \( \kappa_\gamma(gB) \) be a type one tailless closed gallery in \( X_\Gamma \). Write \( g^{-1}\gamma g \in BWB \) for some \( w \in W \). Then its length \( n = 3m \) is a multiple of 3 and \( w \in \{(t_3t_2t_1)^m,(t_2t_1t_3)^m,(t_1t_3t_2)^m\} \).

**Proof.** Write \( w = t_{i_1} \cdots t_{i_n} \). Since \( \kappa_\gamma(gB) \) has type one, \( w \) is one of the three length \( n \) words: \( t_3t_2t_1 \) or \( t_2t_1t_3 \), or \( t_1t_3t_2 \). One checks easily that if \( n \) is not a multiple of 3, then the length of \( w^2 \) is less than \( 2n \), while if \( n \) is a multiple of 3, the length of \( w^2 \) is \( 2n \). \( \square \)

We want to count the number of type one tailless closed geodesic galleries in \( X_\Gamma \) of length \( 3n \). Before doing this, some remark is in order. Note that \( \sigma t_1 t_3 t_2 \sigma^{-1} = t_3 t_1 t_3 \) and \( \sigma^2 t_1 t_3 t_2 \sigma^{-2} = t_3 t_2 t_1 \). By applying suitable powers of \( t_2 t_1 t_3 \) to \( gB \), \( g \sigma B \) and \( g \sigma^2 B \), we obtain all tailless type one geodesic galleries starting at the chamber \( gC_0 \). In what follows, we shall use the three \( B \)-coset representatives
for each chamber, but call \( \kappa_\gamma(gB) \) type one tailless of length \( 3m \) if and only if \( g^{-1}\gamma g \in B(t_2t_1t_3)^mB \).

Recall the operator \( L_B = L_{t_2t_1t_3} \) defined by (7.1). Further, \( L_B \) on \( X_\Gamma \) can be interpreted as the adjacency matrix on directed chambers \((C,e)\), where \( e \) is a type one edge of the chamber \( C \) in \( X_\Gamma \).

**Theorem 21.** For \( n \geq 1 \), \( \text{Tr} L_B^{3n} \) counts the number of type one tailless closed galleries in \( X_\Gamma \) of length \( 3n \).

**Proof.** Write \( Bt_2t_1t_3B = \bigsqcup_{1 \leq i \leq M} w_iB \) as a disjoint union. As \( B(t_2t_1t_3)^nB = (Bt_2t_1t_3B)^n \) and the length of \( (t_2t_1t_3)^n \) is \( 3n \), we have \( B(t_2t_1t_3)^nB = \bigsqcup_{1 \leq l_1,...,l_n \leq M} w_{l_1} \cdots w_{l_n}B \). Consequently, \( \kappa_\gamma(gB) \) is a type one tailless closed gallery of length \( 3n \) in \( X_\Gamma \) if and only if \( g^{-1}\gamma g \) lies in \( w_{l_1} \cdots w_{l_n}B \) for some \( 1 \leq l_1,...,l_n \leq M \), that is, \( \gamma gB = gw_{l_1} \cdots w_{l_n}B \). As we vary \( \gamma \) and \( gB \), this amounts to counting, for each double coset \( \Gamma gB \), the number of \( w_{l_1} \cdots w_{l_n} \)'s such that \( \Gamma gB = \Gamma gw_{l_1} \cdots w_{l_n}B \), and then total over all double cosets \( \Gamma \setminus G/B \).

On the other hand, represent \( L_B^3 = L_{t_2t_1t_3} \) by a square matrix with rows and columns parametrized by the characteristic functions of \( \Gamma \setminus G/B = \bigsqcup_{1 \leq i \leq N} \Gamma g_iB \). Then the \( ij \) entry of \( L_B^3 \) is one if \( \Gamma g_jB = \Gamma g_iw_lB \) for some \( 1 \leq l \leq M \), and zero otherwise. Therefore the trace of the \( n \)th power of \( L_B^3 \) gives the number of type one tailless closed galleries in \( X_\Gamma \) of length \( 3n \). \( \square \)

**7.4. The type one gallery zeta function of \( X_\Gamma \).** A type one tailless closed gallery \( \kappa_\gamma(gB) \) is called **primitive** if it is not a repetition of another closed gallery of shorter length. If \( \kappa_\gamma(gB) \) is a primitive tailless type one closed gallery of length \( n \), then so is the same closed gallery with a different starting chamber. These galleries are said to be **equivalent**. Denote by \( \left[ \kappa_\gamma(gB) \right] \) the collection of the \( n \) galleries equivalent to \( \kappa_\gamma(gB) \).

The type one gallery zeta function of \( X_\Gamma \) is defined as an Euler product:

\[
Z_2(X_\Gamma, u) = \prod_{\gamma \in [\Gamma]} \prod_{\left[ \kappa_\gamma(gB) \right]} (1 - u^{(\kappa_\gamma(gB))})^{-1}
\]

where \( \left[ \kappa_\gamma(gB) \right] \) runs through the equivalence classes of primitive, tailless, type one galleries in \( [\gamma]_B \).

**Theorem 22.** The type one gallery zeta function of \( X_\Gamma \) is a rational function, given by

\[
Z_2(X_\Gamma, u) = \frac{1}{\det(I - L_Bu)}.
\]
Proof. We compute

\[ u \frac{d}{du} \log Z_2(X_\Gamma, u) = u \frac{d}{du} \left( \sum_{\gamma \in [\Gamma]} \sum_{[\kappa_\gamma(gB)]} \sum_{m \geq 1} u^{l(\kappa_\gamma(gB))m} \right) \]

\[ = \sum_{\gamma \in [\Gamma]} \sum_{[\kappa_\gamma(gB)]} \sum_{m \geq 1} l(\kappa_\gamma(gB)) u^{l(\kappa_\gamma(gB))m} \]

\[ = \sum_{\gamma \in [\Gamma]} \sum_{\kappa_\gamma(gB) \text{primitive, tailless, type one}} \sum_{m \geq 1} u^{l(\kappa_\gamma(gB))m} \]

since there are \( l(\kappa_\gamma(gB)) \) galleries in \([\gamma]_B\) equivalent to \( \kappa_\gamma(gB) \). As we get all tailless type one galleries by repeating the primitive ones, the above can be rewritten as

\[ u \frac{d}{du} \log Z_2(X_\Gamma, u) = \sum_{\gamma \in [\Gamma]} \sum_{\kappa_\gamma(gB) \text{tailless, type one}} u^{l(\kappa_\gamma(gB))} \]

\[ = \sum_{m \geq 1} \text{Tr} L_B^m u^m \quad \text{by Proposition 21} \]

\[ = \text{Tr}((1 - L_Bu)^{-1}L_Bu) = \text{Tr} \left( -u \frac{d}{du} \log(I - L_Bu) \right). \]

Therefore \( \log Z_2(X_\Gamma, u) \) differs from \( -\text{Tr} \log(1 - L_Bu) \) by a constant. Exponentiating both functions, using Lemma 3 of [ST] and comparing the constants, we get the desired conclusion. \( \square \)

Remark. By Proposition 20, the lengths of the closed galleries occurring in the gallery zeta function are multiples of 3, so \( \det(1 - L_Bu) \) is a polynomial in \( u^3 \).

8. Type one edge zeta function of \( X_\Gamma \)

8.1. The type one edge zeta function of \( X_\Gamma \). The intersection of the stabilizers in \( G \) of \( v_1 = K \) and \( v_2 = \sigma K \) is the group \( E \) consisting of elements \( k \in K \) whose third row is congruent to \( (0, 0, *) \) mod \( \pi \). Therefore \( E \) stabilizes the type one edge \( E_0 : v_1 \rightarrow v_2 \). Further, \( gE_0 \mapsto gE \) is a bijection between the type one edges on \( \mathcal{B} \) and the coset space \( G/E \).

We have

\[ (t_2 \sigma^2)^2 = \left( \begin{array}{c} \pi \\ \pi \\ \pi^2 \end{array} \right) = \left( \begin{array}{c} 1 \\ 1 \\ \pi \end{array} \right) \quad \text{(in } G) \]

and

\[ E(t_2 \sigma^2)^2 E = E \left( \begin{array}{c} 1 \\ 1 \\ \pi \end{array} \right) E = \prod_{x, y \in \mathcal{O}_F/\pi \mathcal{O}_F} \left( \begin{array}{c} 1 \\ \pi \end{array} \right). \]
Let $L_E$ be the operator which sends a function $f$ in $L^2(G/E)$ to the function $L_E f$ whose value at $gE$ is given by

$$L_E f(gE) = \sum_{g' \in E((t, \pi^2)E)} f(gg'E) = \sum_{x, y \in \mathcal{O}/\pi \mathcal{O}} f\left(g \begin{pmatrix} 1 & 1 \\ x\pi & y\pi \pi \end{pmatrix} \right).$$

Observe that left multiplications by the elements

$$\begin{pmatrix} 1 & 1 \\ x\pi & y\pi \pi \end{pmatrix}$$

map the vertex $v_1 = K$ to $v_2 = \sigma K = \text{diag}(1, 1, \pi)K$ and $v_2K$ to its type one neighbors which are not adjacent to $v_1$. In other words, $L_E$ may be interpreted as the “edge adjacency operator” on the set of type one edges $G/E$ of $B$ such that the neighbors of a type one edge $v \rightarrow v'$ are the $q^2$ type one edges $v' \rightarrow v''$ with $v''$ not adjacent to $v$.

Regard $L_E$ as an operator on the type one edges in $X_\Gamma$. Then $\text{Tr}L_E^n$ counts the number of type one tailless cycles of length $n$ in $X_\Gamma$. Similar to the type one gallery zeta function, we define the type one edge zeta function on $X_\Gamma$ to be

$$Z_1(X_\Gamma, u) = \prod_{\gamma \in [\Gamma]} \prod_{[\kappa_{\gamma}(gK)]} (1 - u^{l_A(\kappa_{\gamma}(gK))})^{-1},$$

where $[\kappa_{\gamma}(gK)]$ runs through the classes of equivalent primitive tailless type one cycles in $X_\Gamma$. The same argument as the proof of Theorem 22 shows

**Theorem 23.** The type one edge zeta function of $X_\Gamma$ is a rational function, given by

$$Z_1(X_\Gamma, u) = \frac{1}{\det(I - L_Eu)}.$$  

### 8.2. Boundaries of tailless type one closed galleries.

We characterize the boundary of a type one tailless closed gallery. Recall from Proposition 20 that the length of such a gallery is a multiple of 3. For $\gamma \in [\Gamma]$, let

$$\Delta_G([\gamma]) = \{gK \in G/K \mid l_G(\kappa_{\gamma}(P_{\gamma}gK)) = l_G([\gamma])\}.$$

By Corollaries 7 and 15 tailless cycles in $[\gamma]$ are algebraically tailless, thus $\Delta_G([\gamma]) \subseteq \Delta_A([\gamma])$; furthermore, the two sets agree when $[\gamma]$ has type one.

**Proposition 24.** Let $\kappa_{\gamma}(gB)$ be a type one tailless closed gallery of length $3m$ in $X_\Gamma$ with the chamber sequence

$$gB = g_1B \rightarrow g_2B \rightarrow \cdots \rightarrow g_{3m}B \rightarrow g_{3m+1}B = \gamma g_1B.$$
Remark. The element $\gamma$ in case (2) is ramified rank-one split, in view of Theorem 14, (1).

Proof. Since the edge sequences we are considering come from every other term of the chamber sequence, they are obtained by right multiplications by suitable $B$-coset representatives of $B(t_2\sigma^2)^2B = \sum_{1 \leq t \leq q^2} w_l B$. If the closed gallery has even length $6n$, then there are $w_{l_1}, \ldots, w_{l_{3n}}$ with $1 \leq l_1, \ldots, l_{3n} \leq q^2$ so that for $1 \leq j \leq 3n$ we have $g_{2j+1}B = g_{2j-1}w_{l_j}B$. As explained at the beginning of the previous section, each $g_jE$ is a type one edge of the chamber $g_jB$, and $g_{2j+1}E = g_{2j-1}w_{l_j}E$ is adjacent to $g_{2j-1}E$. Therefore $g_1E \to g_3E \to \cdots \to g_{6n-1}E \to g_{6n+1}E = \gamma g_1E$ is a type one tailless edge cycle in $X_\Gamma$. The same holds for $g_2E \to g_4E \to \cdots \to g_{6n}E \to \gamma g_2E$.

To see the type of the vertex cycles $\kappa_\gamma(gK)$ and $\kappa_\gamma(g_2K)$, note that $g_1w_{l_1} \cdots w_{l_{3n}}B = \gamma g_1B$ implies that $g_1^{-1}\gamma g_1 \in w_{l_1} \cdots w_{l_{3n}}B \subset B(t_2\sigma^2)^{6n}B = B(t_2t_1t_3)^{2n}B \subset K(t_2t_1t_3)^{2n}K$. Similarly, we also have $g_2^{-1}\gamma g_2 \in K(t_2t_1t_3)^{2n}K$. A straightforward computation gives

$$t_2t_1t_3 = \begin{pmatrix} \pi^{-1} \\ 1 \pi \end{pmatrix} \text{ and } (t_2t_1t_3)^2 = \begin{pmatrix} 1 \\ \pi^3 \end{pmatrix} \text{ in } G.$$  

(8.4)  

This shows that $\kappa_\gamma(gK)$ and $\kappa_\gamma(g_2K)$ both have type $(3n, 0)$. As they are tailless type one cycles, we know that $[\gamma]$ has the same type and the vertices on $\kappa_\gamma(gK)$ and $\kappa_\gamma(g_2K)$, that is, the vertices contained in the gallery $\kappa_\gamma(gB)$, all belong to $\Delta_G([\gamma])$.

If, however, the gallery has odd length $3m = 3(2n+1)$, then the boundary sequence is $g_1E \to g_3E \to \cdots \to g_{6n+1}E \to g_{6n+3}E \to g_{6n+5}E = \gamma g_2E \to \gamma g_4E \to \cdots \to \gamma g_{6n+2}E \to \gamma g_{6n+4}E = \gamma^2 g_1E$. The same argument shows that it is a tailless type one edge cycle in $X_\Gamma$, and as a vertex cycle, it is $\kappa_{\gamma^2}(gK)$. Further, we have $g^{-1}\gamma^2g \in K(t_2t_1t_3)^{2m}K$. Therefore $\kappa_{\gamma^2}(gK)$ has type...
(3m, 0) by (8.4). Since \( m \) is odd, \( g^{-1}\gamma g \in K(t_2t_1t_3)^mK \) has type \((3n + 1, 1)\). If \( \kappa_\gamma(gK) \) is not tailless in \([\gamma]\), then \( l_G([\gamma]) \leq 3n + 1 \), which in turn implies \( l_G([\gamma^2]) \leq 6n + 2 \), contradicting \( l_G(\kappa_\gamma^2(gK)) = l_G([\gamma^2]) = 6n + 3 \) since \( \kappa_\gamma(gK) \) is tailless. Thus \( \kappa_\gamma(gK) \) is tailless so that \([\gamma] \) has type \((3n + 1, 1)\). This also shows that the vertices in the gallery \( \kappa_\gamma(gB) \) lie in \( \Delta_G([\gamma]) \).

Finally, the unique type one edge of each chamber which starts a cycle in \([\gamma] \) is the one which shows up in the edge sequences in (1) and (2), respectively. \( \square \)

The proposition above says that if \([\gamma]_B \) contains a tailless type one closed gallery, then either \([\gamma] \) has type \((3n, 0)\), or it has type \((3n + 1, 1)\). Further, each chamber of such a gallery has its vertices contained in the set \( \Delta_G([\gamma]) \) with a unique type one edge which starts a tailless cycle in \([\gamma] \). Now we show that the last statement characterizes the chambers which start a tailless type one closed gallery in \( X_T \).

Given \([\gamma] \) with type as described above, let \( C \) be a chamber whose three vertices are contained in the set \( \Delta_G([\gamma]) \) with a unique type one edge \( E' \) which is the starting edge of a tailless cycle in \([\gamma] \). Initially, the chamber \( C \) has three possible labels: \( gB, g\sigma B \) and \( g\sigma^2 B \). The edge \( E' \) then determines the unique labeling, say, \( gB \) so that \( E' \) is labeled as \( gE \). The three vertices of \( gB \) are \( gK, g\sigma K \) and \( g\sigma^2 K \). Denote by \( gA \) the apartment containing \( gB \) and \( \gamma gB \). Up to translation by an element in \( B \), we may assume that \( A \) is the standard apartment whose chambers are represented by \( DS_3B \), where \( D \) is the group of diagonal matrices in \( G \) and \( S_3 \) is the subgroup of permutation matrices in \( G \). Therefore \( g^{-1}\gamma g = Msb \) for some \( M \in D, s \in S_3 \) and \( b \in B \). The cycles in \([\gamma] \) starting at the vertices of \( C \) are tailless and have the same type and length as \([\gamma] \).

Case (I). \([\gamma] \) has type \((3n, 0)\). We have, by assumption, that \( g^{-1}\gamma g, \sigma^{-1}g^{-1}\gamma g\sigma \) and \( \sigma g^{-1}g\sigma^{-1} \) all lie in \( K\text{diag}(1, 1, \pi^{3n})K \). Therefore \( M = \text{diag}(1, 1, \pi^{3n}) \) from \( g^{-1}\gamma g \in K\text{diag}(1, 1, \pi^{3n})K \). Writing \( \sigma = \text{diag}(1, 1, \pi)s_3 \) with \( s_3 \in S_3 \), we proceed to determine \( s \) using the other two conditions. Since

\[
\sigma^{-1}g^{-1}\gamma g\sigma = s_3^{-1}\text{diag}(1, 1, \pi^{-1})\text{diag}(1, 1, \pi^{3n})s\sigma b' \quad \text{since } B\sigma = \sigma B \\
\quad = s_3^{-1}\text{diag}(1, 1, \pi^{-1})\text{diag}(1, 1, \pi^{3n})s \text{ diag}(1, 1, \pi)s_3b'
\]

and \( s\text{diag}(1, 1, \pi) \) is \( \text{diag}(\pi, 1, 1)s \) or \( \text{diag}(1, \pi, 1)s \) or \( \text{diag}(1, 1, \pi)s \) depending on the the first, second, or third row of \( s \) is \((0 0 1)\), in order that \( \sigma^{-1}g^{-1}\gamma g\sigma \in K\text{diag}(1, 1, \pi^{3n})K \), we must have the third row of \( s \) being \((0 0 1)\). Similarly, \( \sigma g^{-1}g\sigma^{-1} \in K\text{diag}(1, 1, \pi^{3n})K \) implies the first row of \( s \) should be \((1 0 0)\). Therefore \( s \) is the identity matrix and hence \( g^{-1}\gamma g = \text{diag}(1, 1, \pi^{3n})b \), showing that \( \kappa_\gamma(gB) \) is a tailless type one closed gallery of length \( 6n \).
Case (II) $[\gamma]$ has type $(3n + 1, 1)$. Since $\Delta_G([\gamma]) \subset \Delta_G([\gamma^2])$ and $[\gamma^2]$ has type one, we may use the result above to conclude that there is a labeling of $C$ by $gB$ such that $\kappa_{\gamma^2}(gB)$ is a tailless type one gallery of length $2(6n + 3)$. In other words, $g^{-1}\gamma^2g \in B(t_2t_1t_3)^{2(2n+1)}B$. Since the vertices of $gB$ are in $\Delta_G([\gamma])$, we know $g^{-1}\gamma g \in K(t_2t_1t_3)^{2n+1}K$. This condition allows us to write $g^{-1}\gamma g = Msb$ with $M = (t_2t_1t_3)^{2n+1}$, $s \in S_3$ and $b \in B$. A similar argument as in Case (I) shows that the remaining two conditions force $s$ to be the identity matrix. Therefore $g^{-1}\gamma g \in B(t_2t_1t_3)^{2n+1}B$, implying that $\kappa_{\gamma}(gB)$ is a tailless type one closed gallery.

We record the above result in

**Proposition 25.** Suppose $[\gamma]$ has type $(3n, 0)$ or $(3n + 1, 1)$. Then for any chamber $C$ whose vertices belong to $\Delta_G([\gamma])$ with a unique type one edge which starts a tailless cycle in $[\gamma]$, there is a unique labeling of $C$ by $gB$ such that $\kappa_{\gamma}(gB)$ is a tailless type one closed gallery of even length $6n$ if $[\gamma]$ has type $(3n, 0)$, or odd length $3(2n + 1)$ if $[\gamma]$ has type $(3n + 1, 1)$.

### 8.3. Comparison between type one chamber zeta function and type two edge zeta function.

The type two cycles are obtained from the type one cycles traveled in reverse direction, hence their algebraic length is doubled while the geometric length remains the same. Consequently the type two edge zeta function of $X_\Gamma$ is equal to $Z_1(X_\Gamma, u^2)$.

The following theorem compares the difference between the numbers of type two tailless edge cycles and type one tailless closed galleries.

**Theorem 26.**

$$ u \frac{d}{du} \log Z_1(X_\Gamma, u^2) - u \frac{d}{du} \log Z_2(X_\Gamma, -u) = \sum_{n \geq 1} \sum_{[\gamma] \text{ unramified rank one split of type } (3n, 0)} 2c([\gamma])u^{2A([\gamma])} $$

$$ + \sum_{[\gamma] \text{ ramified rank one split of type } (3n, 0)} c([\gamma])u^{2A([\gamma])} $$

$$ + \sum_{[\gamma] \text{ ramified rank one split of type } (3n+1, 1)} c([\gamma])u^{4A([\gamma^2])}. $$

**Proof.** Combining Propositions 24 and 25 as well as the proof of Theorem 22, we have

$$ u \frac{d}{du} \log Z_2(X_\Gamma, -u) = \sum_{\gamma \in \Gamma} \sum_{\kappa_{\gamma}(gB) \text{ tailless, type one}} u^{(\kappa_{\gamma}(gB))} $$

$$ = \sum_{n \geq 1} \sum_{\gamma \in \Gamma, [\gamma] \text{ of type } (3n, 0)} N_B(\gamma)u^{6n} - \sum_{\gamma \in \Gamma, [\gamma] \text{ of type } (3n+1, 1)} N_B(\gamma)u^{6n+3}, $$

where $N_B(\gamma)u^{6n}$ counts the number of type one cycles of length $6n$ and $N_B(\gamma)u^{6n+3}$ counts the number of type one cycles of length $6n+3$. The sum over $n \geq 1$ gives the contribution from the unramified split case, while the sum over $n > 1$ gives the contribution from the ramified split cases.
where \( N_B(\gamma) \) is the number of chambers with vertices \( P_\gamma gK \), where \( gK \in C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_G([\gamma])} \), and containing a unique type one edge which starts a tailless cycle in \([\gamma]\). On the other hand, for type one cycles we have

\[
\frac{d}{du} \log Z_1(X_{\Gamma}, u^2) = \sum_{\gamma \in [\Gamma]} \sum_{\kappa_\gamma(gK) \text{ tailless, type one}} 2u^{2\lambda(\kappa_\gamma(gK))} = \sum_{n \geq 1} \sum_{\gamma \in [\Gamma], [\gamma] \text{ of type } (3n, 0)} 2N_K(\gamma)u^{6n},
\]

where the number \( N_K(\gamma) \) of tailless type one cycles in \([\gamma]\) was calculated in \$5\) and \$6\). We shall compare this with the number \( N_B(\gamma) \). Recall that for \( \gamma \) of type one, we have \( \Delta_G([\gamma]) = \Delta_A([\gamma]) \).

Case I. \( \gamma \) is split with type \((3n, 0)\). Then \( r_\gamma = \text{diag}(1, a, b) \), where \( 1, a, b \) are distinct with \( \text{ord}_a(a) = 0 \) and \( \text{ord}_a(b) = 3n \). Put \( \delta = \text{ord}_a(1 - a) \). The centralizer \( C_G(r_\gamma) \) consists of diagonal elements in \( G \). By Corollary \[12\] \( C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_A([\gamma])} \) has cardinality \( N_K(\gamma) = c([\gamma])q^\delta \) and is represented by vertices \( h_{i,j}v_xK \), where \( h_{i,j} = \text{diag}(1, \pi^i, \pi^j) \in C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus C_G(r_\gamma)/(C_G(r_\gamma) \cap K)} \)

and \( v_x = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \) with \( x \in \pi^{-\delta}O_F/O_F \).

The type one tailless cycle \( \kappa_\gamma(P_\gamma h_{i,j}v_xK) \) is \( P_\gamma h_{i,j}v_xK \rightarrow P_\gamma h_{i,j+1}v_xK \cdots \rightarrow P_\gamma h_{i,j+3n}v_xK = \gamma P_\gamma h_{i,j}v_xK \) by Corollary \[12\].

There are \( q + 1 \) chambers sharing the type one edge \( K \rightarrow \text{diag}(1, 1, \pi)K \) with the third vertex being \( u_cK := \begin{pmatrix} \pi & c \\ 1 & \pi \end{pmatrix} K \) with \( c \in O_F/\pi O_F \) and \( u_\infty K := \begin{pmatrix} 1 \\ \pi \end{pmatrix} K \). Left multiplication by \( h_{i,j}v_x \) sends the type one edge to \( h_{i,j}v_xK \rightarrow h_{i,j+1}v_xK \) and the third vertex to \( h_{i,j}v_xu_cK = \begin{pmatrix} 1 & (c + x)/\pi \\ \pi^{i-1} & \pi^i \end{pmatrix} K \) and \( h_{i,j}v_xu_\infty K = \begin{pmatrix} 1 & x\pi \\ \pi^{i+1} & \pi^{i+1} \end{pmatrix} K \). We count the number of such vertices belonging to \( C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_A([\gamma])} \).

There is only one integral \( x \), namely, \( x = 0 \). When \( \delta = 0 \), each type one edge \( h_{i,j}v_0K \rightarrow h_{i,j+1}v_0K \) forms a chamber with only two vertices \( h_{i+1,j+1}v_0K \) and \( h_{i-1,j}v_0K \) in \( C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_A([\gamma])} \). Hence the number of type one tailless galleries in \([\gamma]_B\) is \( N_B(\gamma) = 2\#(C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_A([\gamma])) = 2N_K(\gamma) \).

Next assume \( \delta \geq 1 \). In this case, each type one edge \( h_{i,j}v_0K \rightarrow h_{i,j+1}v_0K \) forms a chamber with the \( q + 1 \) vertices \( h_{i,j}v_0u_cK \) and \( h_{i,j}v_0u_\infty K \) in \( C_{F_{\gamma^{-1}TP_\gamma}(r_\gamma)\setminus \Delta_A([\gamma])} \). The same holds when \( h_{i,j}v_0 \) is replaced by \( h_{i,j}v_x \) for \(-1 \geq \text{ord}_a x \geq -\delta + 1 \). This gives rise to \((q + 1)(q^{\delta-1} - 1)\) chambers.
Finally, when $\text{ord}_\pi x = -\delta$, each type one edge $h_{i,j}v_x K \to h_{i,j+1}v_x K$ forms a chamber with only one vertex $h_{i,j}v_x u_{\infty} K$ in $C_{P_{\gamma}^{-1}G}(r_{\gamma}) \backslash \Delta_A([\gamma])$, so there are $(q - 1)q^{\delta - 1}$ chambers. Put together, we get $N_B(\gamma) = c([\gamma]) (q + 1 + (q + 1)(q^{\delta - 1} - 1) + (q - 1)q^{\delta - 1}) = c([\gamma]) 2q^\delta = 2N_K(\gamma)$.

Hence there is no contribution from split type one $[\gamma]$ in $u \frac{d}{du} \log Z_1(X_{\Gamma}, u^2) - u \frac{d}{du} \log Z_2(X_{\Gamma}, -u)$.

Case II. $\gamma$ is unramified rank-one split with type $(3n, 0)$. In this case $r_{\gamma} = \begin{pmatrix} a & e & dc \\ d & e + db \end{pmatrix}$, and the eigenvalues $a$, $e + d\lambda$ and $e + d\bar{\lambda}$ of $\gamma$ generate an unramified quadratic extension $L$ over $F$. The type assumption on $\gamma$ implies that $\text{ord}_\pi a = 3n$ and $\min(\text{ord}_\pi e, \text{ord}_\pi d) = 0$ so that $e + d\lambda$ and $e + d\bar{\lambda}$ are units in $L$. Let $\delta = \text{ord}_\pi d$.

As discussed in §6.1, the double cosets $C_{P_{\gamma}^{-1}G}(r_{\gamma}) \backslash C_G(r_{\gamma}) / C_G(r_{\gamma}) \cap K$ are represented by $h_m = \text{diag}(\pi^m, 1, 1)$, $m \mod c([\gamma])$. By Proposition 19, $C_{P_{\gamma}^{-1}G}(r_{\gamma}) \backslash \Delta_A([\gamma])$ has cardinality $N_K(\gamma) = c([\gamma]) \frac{q^{\frac{\delta + 1}{2}} - q^{\frac{\delta - 1}{2}}}{q - 1}$ and is represented by $h_mg_{i,j,u} K$ and $h_mg_{i,z} K$, where $m \mod c([\gamma])$,

$$g_{i,j,u} = \begin{pmatrix} \pi^{-j} & u \\ \pi^j & 1 \end{pmatrix} \text{ with } 0 \leq j \leq i \leq \delta, \quad u \in \mathcal{O}_F^\times / \pi^{i-j}\mathcal{O}_F \text{ for } j < i \text{ and } u = 0 \text{ for } j = i,$$

and $g_{i,z} = \begin{pmatrix} 1 \\ \pi^{i} & z \end{pmatrix} \text{ with } 1 \leq i \leq \delta \text{ and } z \in \pi \mathcal{O}_F / \pi^i \mathcal{O}_F$. Let $g = h_mg_{i,j,u}$ or $h_mg_{i,z}$. Then, by Proposition 19, the type one tailless closed geodesic $\kappa_{\gamma}(P_\gamma gK)$ is given by $P_\gamma gK \to P_\gamma g\text{diag}(\pi, 1, 1)K \to \cdots \to P_\gamma g\text{diag}(\pi^{3n}, 1, 1)K = \gamma P_\gamma gK$.

It remains to count the number of chambers with vertices in $C_{P_{\gamma}^{-1}G}(r_{\gamma}) \backslash \Delta_A([\gamma])$ containing a given type one edge $gK \to g\text{diag}(\pi, 1, 1)K$ for $g = h_mg_{i,j,u}$ or $h_mg_{i,z}$. When $\delta = 0$, there are no $g_{i,z}$ and only one $g_{i,j,u}$, equal to the identity matrix, hence the vertices in $C_{P_{\gamma}^{-1}G}(r_{\gamma}) \backslash \Delta_A([\gamma])$ are $h_mK$, $m \mod c([\gamma])$. It is clear that there are no chambers formed by these vertices. Hence $N_K(\gamma) = c([\gamma])$ and $N_B(\gamma) = 0$ when $\delta = 0$.

Next assume $\delta \geq 1$. There are $q + 1$ chambers in $\mathcal{B}$ sharing the type one edge $K \to \text{diag}(\pi, 1, 1)K$ with the third vertex being $w_x K := \begin{pmatrix} \pi & x \\ \pi & 1 \end{pmatrix} K$ with $x \in \mathcal{O}_F / \pi \mathcal{O}_F$ and $w_\infty K := \text{diag}(1, \pi^{-1}, 1)K$, respectively. Left multiplication by $g = h_mg_{i,j,u}$ or $h_mg_{i,z}$ sends the edge $K \to \text{diag}(\pi, 1, 1)K$ to the type one edge $gK \to g\text{diag}(\pi, 1, 1)K$, so we need to count the number of distinct vertices among
\( gw_x K \) and \( gw_\infty K \) which fall in \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus \Delta_A([\gamma]) \). Observe that

\[
h_{m}g_{i,j,u}w_x K = \begin{pmatrix} \pi^{m+1} & x\pi^i + u \\ \pi^i & 1 \end{pmatrix} K, \quad h_{m}g_{i,j,u}w_\infty K = \begin{pmatrix} \pi^m & u \\ \pi^i & 1 \end{pmatrix} K,
\]

\[
h_{m}g_{i,z}w_x K = \begin{pmatrix} \pi^{m+1} & x\pi^i + z \\ \pi^i & 1 \end{pmatrix} K, \quad \text{and} \quad h_{m}g_{i,z}w_\infty K = \begin{pmatrix} \pi^m & z \\ \pi^i & 1 \end{pmatrix} K.
\]

It is straightforward to check that, for \( 0 \leq i \leq \delta - 1 \), all \( gw_x K \) and \( gw_\infty K \) are distinct vertices in \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus \Delta_A([\gamma]) \), thus there are \( c([\gamma])(q+1)^{\frac{d+q^{\delta-1}+2}{q-1}} \) chambers. When \( i = \delta \), for each \( g \) above, only \( gw_\infty K \) lies in \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus \Delta_A([\gamma]) \), hence there are \( c([\gamma])(q^\delta + q^{\delta-1}) \) chambers. Altogether, \( N_B(\gamma) \) is equal to \( 2N_K(\gamma) - 2c([\gamma]) \) for \( \delta \geq 0 \).

In conclusion, the contribution of an unramified rank-one split \([\gamma]\) of type one in \( u \frac{d}{du} \log Z_1(X_\Gamma, u^2) - u \frac{d}{du} \log Z_2(X_\Gamma, -u) \) is \( 2c([\gamma])u^{2\lambda([\gamma])} \).

Case III. \( \gamma \) is ramified rank-one split with type \((3n, 0)\). Then \( r_\gamma = \begin{pmatrix} a & e & dc \\ e & d & e + db \\ d & e + d\bar{\lambda} & e + d\bar{\lambda} \end{pmatrix} \) and the eigenvalues \( a, e + d\lambda \) and \( e + d\bar{\lambda} \) of \( \gamma \) generate a ramified quadratic extension \( L \) over \( F \). In this case, \( \text{ord}_\pi a = 3n \) and \( \text{ord}_\pi e = 0 \) so that \( e + d\lambda \) and \( e + d\bar{\lambda} \) are units in \( L \). Let \( \delta = \text{ord}_\pi d \).

As discussed in §6.1, \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus C_G(r_\gamma) \cap K \) has cardinality \( c([\gamma]) \), and it is represented by \( h = \text{diag}(\pi^m, 1, 1) \) with \( 0 \leq m \leq (c([\gamma]) - 1)/2 \) and \( \text{diag}(\pi^m, 1, 1)\pi_L \) with \( 0 \leq m \leq (c([\gamma]) - 3)/2 \) if \( c([\gamma]) \) is odd, and by \( h = \text{diag}(\pi^m, 1, 1) \) and \( \text{diag}(\pi^m, 1, 1)\pi_L \) with \( m \mod c([\gamma])/2 \) if \( c([\gamma]) \) is even. Here \( \pi_L = \begin{pmatrix} 1 & c \\ c & \frac{1}{b} \end{pmatrix} \) is imbedded in \( G \).

It follows from Proposition 19 that \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus \Delta_A(\gamma) \) is represented by \( hg_{i,j,u}K \) for \( g_{i,j,u} \) as in Case II and \( h \) as above, so the total number of vertices is \( c([\gamma])(q^{\delta+1}-1)/(q-1) = N_K(\gamma) \). Now, for any \( gK = hg_{i,j,u}K \) in \( \Delta_A([\gamma]) \), the type one tailless cycle \( \kappa(\gamma, P_\gamma gK) \) is \( P_\gamma gK \rightarrow P_\gamma g\text{diag}(\pi, 1, 1)K \rightarrow \cdots \rightarrow P_\gamma g\text{diag}(\pi^{3n}, 1, 1)K = \gamma P_\gamma gK \) by Proposition 19.

To count the number of chambers we proceed as in Case II by counting, for each \( g = hg_{i,j,u} \), the number of \( gw_x K \) and \( gw_\infty K \) which lie in \( C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_\gamma) \setminus \Delta_A(\gamma) \).
We first discuss the case $\delta = 0$. Then there is only one $g_{0,0,u}$, equal to the identity matrix. All representatives are given by $hK$. Observe that $\text{diag}(\pi^m, 1, 1)\pi_LK = \begin{pmatrix} \pi^m & 0 \\ \pi & 1 \end{pmatrix}K$. So there is only one vertex $gw_0K$ which will form a chamber containing the type one edge $gK \to g\text{diag}(\pi, 1, 1)K$. Hence the number of chambers is $N_B(\gamma) = c([\gamma]) = 2N_K(\gamma) - c([\gamma])$ for $\delta = 0$.

Now assume $\delta \geq 1$. One sees from the explicit computation in Case II that for $g = hg_{i,j,u}$, all $q + 1$ vertices $gw_xK$ and $gw_\infty K$ are distinct vertices in $C_{P_{\gamma}^{-1}\Gamma P_{\gamma}}(r_\gamma)\backslash \Delta_A([\gamma])$ provided that $0 \leq i \leq \delta - 1$; when $i = \delta$, only one vertex, $gw_\infty K$, lies in $C_{P_{\gamma}^{-1}\Gamma P_{\gamma}}(r_\gamma)\backslash \Delta_A(\gamma)$. This gives $c([\gamma])((q^\delta - 1)(q + 1)/(q - 1) + q^\delta) = c([\gamma])(2(q^{\delta+1} - 1)/(q - 1) - 1)$ chambers. Therefore $N_B(\gamma) = 2N_K(\gamma) - c([\gamma])$ for $\delta \geq 1$.

This shows that the contribution of a ramified rank-one split $[\gamma]$ of type one in $u\frac{d}{du}\log Z_1(X, u^2) - u\frac{d}{du}\log Z_2(X, -u) = c([\gamma])u^{2A([\gamma])}$.

Finally we consider $[\gamma]$ of type $(3n + 1, 1)$. This happens only when $\gamma$ is ramified rank-one split with eigenvalues $a, e + d\lambda_1, e + d\lambda_2$, where $a, e, d \in F$, $\text{ord}_a a = 3n + 2$, $\text{ord}_e e \geq 1$ and $\delta = \text{ord}_a d = 0$ by the analysis above Theorem 14. As noted before, such $[\gamma]$ has no contribution to $L_1(X, u^2)$ and the length of a type one tailless gallery in $[\gamma]_B$ is $6n + 3$. Its contribution in $u\frac{d}{du}\log Z_2(X, -u)$ is $-N_B(\gamma)u^{6n+3}$ with $N_B(\gamma) = \#C_{P_{\gamma}^{-1}\Gamma P_{\gamma}}(r_\gamma)\backslash \Delta_G([\gamma])$. Since $\delta = 0$ and $\mu = 0$ by the remark following Theorem 18 we have $\Delta_G([\gamma]) = \Delta_A([\gamma])$ such that $N_B(\gamma) = c([\gamma])$ by Corollary 17.

This completes the proof of the theorem. \(\square\)

9. The proof of the Main Theorem

9.1. Type one zeta function. As defined in §8.1, the type one vertex zeta function of the quotient $X$ is

\[
Z_1(X, u) = \prod_{\gamma \in \Gamma, \text{type one}} \prod_{\text{primitive, tailless up to equivalence}} (1 - u^{l_A(\kappa_\gamma(gK))})^{-1}.
\]

(9.1)

Note that $l_A(\kappa_\gamma(gK)) = l_G(\kappa_\gamma(gK)) = l_A([\gamma]) = l_G([\gamma])$ is the length of $[\gamma]$. We proceed to investigate its logarithmic derivative.

Although the zeta function only concerns type one tailless cycles, to describe it we shall involve all homotopy cycles. First we introduce the numbers $P_{n,m}$, $Q_{n,m}$, and $R_{n,m}$ which count the algebraically tailless homotopy cycles of type $(n, m)$ arising from split, unramified rank-one split,
and ramified rank-one split $\gamma$’s, respectively:

\[
P_{n,m} = \sum_{\gamma \in [\Gamma] \text{ split}} \#(C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_{\gamma}) \Delta A([\gamma])),
\]

(9.2)

\[
Q_{n,m} = \sum_{\gamma \in [\Gamma] \text{ unram. rank-one split}} \#(C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_{\gamma}) \Delta A([\gamma])),
\]

(9.3)

\[
R_{n,m} = \sum_{\gamma \in [\Gamma] \text{ ram. rank-one split}} \#(C_{P_{\gamma}^{-1} \Gamma P_{\gamma}}(r_{\gamma}) \Delta A([\gamma])),
\]

(9.4)

The following expression describes the type one edge zeta function in terms of the number of tailless type one homotopy cycles in $X_{\Gamma}$.

**Proposition 27.**

\[
ud \frac{d}{du} \log Z_1(X_{\Gamma}, u) = \sum_{n>0} (P_{n,0} + Q_{n,0} + R_{n,0}) u^n.
\]

*Proof.* By definition,

\[
\log Z_1(X_{\Gamma}, u) = \sum_{\gamma \in [\Gamma], l_A([\gamma])>0} \sum_{\kappa_{\gamma}(gK) \text{ primitive, tailless up to equivalence}} \sum_{m \geq 1} \frac{u^{ml_A(\kappa_{\gamma}(gK))}}{m}
\]

so that

\[
u \frac{d}{du} \log Z_1(X_{\Gamma}, u) = \sum_{\gamma \in [\Gamma], l_A([\gamma])>0} \sum_{\kappa_{\gamma}(gK) \text{ primitive, tailless up to equivalence}} \sum_{m \geq 1} l_A(\kappa_{\gamma}(gK)) u^{ml_A(\kappa_{\gamma}(gK))}
\]

\[
= \sum_{\gamma \in [\Gamma], l_A([\gamma])>0} \sum_{\kappa_{\gamma}(gK) \text{ primitive, tailless up to equivalence}} \sum_{m \geq 1} u^{ml_A(\kappa_{\gamma}(gK))}
\]

since, as discussed in §6.5, there are $l_A([\gamma])$ type zero tailless homotopy cycles equivalent to a given primitive tailless type zero homotopy cycle in $[\gamma]$. Observe that the $\kappa_{\gamma}(gK)$ above runs through all primitive tailless type zero homotopy cycles on $X_{\Gamma}$, hence their repetitions give all type zero tailless homotopy cycles. The proposition follows by noting that when a cycle is repeated $m$ times, the length is multiplied by $m$. \qed
9.2. The number of homotopy cycles of type \((n, m)\). In order to gain information on \(P_{n,0}\), \(Q_{n,0}\) and \(R_{n,0}\), we extend the summation to include homotopy cycles of type \((n, m)\). For non-negative integers \(n\) and \(m\), denote by \(B_{n,m}\) the matrix whose rows and columns are parametrized by vertices of \(X_{\Gamma}\) such that the \((\Gamma gK, \Gamma g'K)\) entry records the number of homotopy classes of geodesic paths from \(\Gamma gK\) to \(\Gamma g'K\) in \(X_{\Gamma}\) of type \((n, m)\). Alternatively, this is the number of \(\gamma \in \Gamma\) such that the homotopy classes of the geodesics from \(gK\) to \(\gamma gK\) have type \((n, m)\). The trace of \(B_{n,m}\) then gives the number of homotopy geodesic cycles of type \((n, m)\). In view of the analysis in §4.2, we have

\[
\text{Tr}(B_{n,m}) = \# \left\{ \kappa_\gamma(gK) \mid \gamma \in [\Gamma], \kappa_\gamma(gK) \in [\gamma] \text{ has type } (n, m) \right\}.
\]

To facilitate our computations, form two kinds of formal power series:

\[
\sum_{n,m \geq 0 \atop (n,m) \neq (0,0)} \text{Tr}(B_{n,m})u^{n+2m} = \sum_{\gamma \in [\Gamma], \gamma \neq id} \sum_{\kappa_\gamma(gK) \in [\gamma]} u^{I_A(\kappa_\gamma(gK))}, \tag{9.5}
\]

and

\[
\sum_{n > 0} \text{Tr}(B_{n,0})u^n = \sum_{\gamma \in [\Gamma], \gamma \neq id} \sum_{\kappa_\gamma(gK) \in [\gamma] \text{ has type one}} u^{I_A(\kappa_\gamma(gK))}. \tag{9.6}
\]

Note that \(\text{Tr}B_{n,m}\) may include cycles with tails. Their relation with algebraically tailless cycles is given in the following proposition.

**Proposition 28.** With the same notation as in Theorem 18, we have

\[
\sum_{n,m \geq 0 \atop (n,m) \neq (0,0)} \text{Tr}(B_{n,m})u^{n+2m} = \left( \sum_{n,m \geq 0 \atop (n,m) \neq (0,0)} P_{n,m}u^{n+2m} \right) \frac{1 - u^3}{1 - q^3u^3} + \sum_{[\gamma] \text{ unram. rank-one split}} c([\gamma])u^{I_A([\gamma])} \left( \frac{q^{\delta([\gamma])} + 1}{q - 1} - \frac{2}{1 - q^3u^3} \right) \left( \frac{1 - u^3}{1 - q^2u^3} \right) + \sum_{[\gamma] \text{ ram. rank-one split}} c([\gamma])q^{\mu([\gamma])}u^{I_A([\gamma])} \left( \frac{q^{\delta([\gamma])} + 1}{q - 1} + \frac{q^{\delta([\gamma])} + 3}{1 - q^3u^3} \right) \left( \frac{1 - u^3}{1 - q^2u^3} \right).
\]

**Proof.** Break the right side of (9.5) into three parts, over split, unramified rank-one split, and ramified rank-one split \(\gamma\)'s, respectively. Applying Theorem 8 to the split part and Theorem 18 to the unramified and ramified rank-one split parts, and using the definition of \(P_{n,m}\), we get the desired formula.

Next we compute the number of type one homotopy cycles on \(X_{\Gamma}\).
Proposition 29. With the same notation as in Theorem 18, we have
\[
\sum_{n>0} \text{Tr}(B_{n,0})u^n = (1 - q^{-1}) \left( \sum_{(n,m) \neq (0,0)} P_{n,m} u^{n+2m} \right) \frac{1 - q^2 u^3}{1 - q^3 u^3} + q^{-1} \left( \sum_{n>0} P_{n,0} u^n \right).
\]

By definition,
\[
\text{Proposition 29.}
\]

We split the sum over \(\gamma\) (9.7) of Theorem 18 to get
\[
\sum_{\gamma \in [\Gamma]} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} + \frac{(q^2 - 1)q^{\delta([\gamma]) + 1} u^3}{1 - q^3 u^3}\right) + q^{-1} \left( \sum_{\gamma \in [\Gamma]} c([\gamma]) u^{l_A(\gamma)} \right)
\]

For the unramified (resp. ramified) rank-one split part, we add (A2) and (A3) (resp. (B2) and (B3)) of Theorem 18 to arrive at the sum
\[
\sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} + \frac{(q^2 - 1)q^{\delta([\gamma]) + 1} u^3}{1 - q^3 u^3}\right) + \sum_{\gamma \in [\Gamma]} c([\gamma]) u^{l_A(\gamma)}
\]

For the unramified (resp. ramified) rank-one split part, we add (A2) and (A3) (resp. (B2) and (B3)) of Theorem 18 to get
\[
\sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} + \frac{(q^2 - 1)q^{\delta([\gamma]) + 1} u^3}{1 - q^3 u^3}\right) + \sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} - \frac{2}{q - 1}\right)
\]

We split the sum over \(\gamma\) into three parts according to \(\gamma\) split, unramified rank-one split, or ramified rank-one split. For the split part, we add (i) and (ii) of Theorem 11 and use the definition of \(P_{n,m}\) to arrive at the sum
\[
(1 - q^{-1}) \left( \sum_{(n,m) \neq (0,0)} P_{n,m} u^{n+2m} \right) \frac{1 - q^2 u^3}{1 - q^3 u^3} = q^{-1} \left( \sum_{n>0} P_{n,0} u^n \right).
\]

For the unramified (resp. ramified) rank-one split part, we add (A2) and (A3) (resp. (B2) and (B3)) of Theorem 18 to get
\[
\sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} + \frac{(q^2 - 1)q^{\delta([\gamma]) + 1} u^3}{1 - q^3 u^3}\right) + \sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} - \frac{2}{q - 1}\right)
\]

Proof. By definition,
\[
\sum_{n>0} \text{Tr}(B_{n,0})u^n = \sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} \sum_{\kappa \gamma (gK) \in [\gamma], \text{type one}} u^{l_A(\kappa \gamma (gK))}.
\]

We split the sum over \(\gamma\) into three parts according to \(\gamma\) split, unramified rank-one split, or ramified rank-one split. For the split part, we add (i) and (ii) of Theorem 11 and use the definition of \(P_{n,m}\) to arrive at the sum
\[
(1 - q^{-1}) \left( \sum_{(n,m) \neq (0,0)} P_{n,m} u^{n+2m} \right) \frac{1 - q^2 u^3}{1 - q^3 u^3} + q^{-1} \left( \sum_{n>0} P_{n,0} u^n \right).
\]

For the unramified (resp. ramified) rank-one split part, we add (A2) and (A3) (resp. (B2) and (B3)) of Theorem 18 to get
\[
\sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} + \frac{(q^2 - 1)q^{\delta([\gamma]) + 1} u^3}{1 - q^3 u^3}\right) + \sum_{\gamma \in [\Gamma], \gamma \neq \text{id}} c([\gamma]) u^{l_A(\gamma)} \left(q^{\delta([\gamma])} + q^{\delta([\gamma]) - 1} - \frac{2}{q - 1}\right)
\]
It follows from Proposition 19 and the definitions of $Q_{n,0}$ and $R_{n,0}$ that

$$
\sum_{\gamma \in [\Gamma], \text{type one}} c([\gamma]) u^{l_A(\gamma)} q^{\delta([\gamma])} = q^{-1} \sum_{n>0} Q_{n,0} u^n - 2q^{-1} \sum_{\gamma \in [\Gamma], \text{type one}} c([\gamma]) u^{l_A(\gamma)}
$$

and

$$
\sum_{\gamma \in [\Gamma], \text{type one}} c([\gamma]) u^{l_A(\gamma)} q^{\mu([\gamma])} q^{\delta([\gamma])} = q^{-1} \sum_{n>0} R_{n,0} u^n + \sum_{\gamma \in [\Gamma], \text{type one}} c([\gamma]) u^{l_A(\gamma)} (-q^{\mu([\gamma])} - 1 - 2 \mu([\gamma]) q^{\delta([\gamma])}).
$$

Finally, plug (9.8) and (9.9) into (9.7) to complete the proof. 

9.3. Proof of the Main Theorem. Combining Propositions 29 and 28 we obtain

$$
q \left( \sum_{n>0} \operatorname{Tr}(B_{n,0}) u^n \right) - (q-1) \left( \sum_{n,m>0} \operatorname{Tr}(B_{n,m}) u^{n+2m} \right) \left( 1 - q^2 u^3 \right) = \sum_{n>0} (P_{n,0} + Q_{n,0} + R_{n,0}) u^n + \sum_{\gamma \in [\Gamma], \text{type one}} c([\gamma]) u^{l_A(\gamma)} q^{\mu([\gamma])} - \mu([\gamma]) q^{\delta([\gamma])} + 1.
$$

As before, to a rank-one split $\gamma$, we associate $r_\gamma = \begin{pmatrix} a & e & dc \\ d & e + db \end{pmatrix}$. First assume $\gamma$ is unramified rank-one split. By Theorem 14, $[\gamma]$ has type $(n, m) = (\operatorname{ord}_\gamma a, \min(\operatorname{ord}_\gamma e, \operatorname{ord}_\gamma d))$, hence $[\gamma]$ is not of type one if and only if $a$ is a unit, which is equivalent to its inverse $[\gamma^{-1}]$ having type $(m, 0)$. Note that $l_A([\gamma]) = 2m = 2l_A([\gamma^{-1}])$ by Theorem 14. Next assume that $[\gamma]$ is ramified rank-one split. Since $\mu([\gamma]) = 1$ implies $\delta([\gamma]) = 0$, we have $q^{\mu([\gamma])} - \mu([\gamma]) q^{\delta([\gamma])} = 0$ in this case. Thus we need only consider the case $\mu([\gamma]) = 0$ so that $q^{\mu([\gamma])} - \mu([\gamma]) q^{\delta([\gamma])} = 1$. Then $[\gamma]$ is not of type one if and only if $a$ is a unit, in which case it has type $(0, \operatorname{ord}_\gamma e)$ if $\operatorname{ord}_\gamma e \leq \operatorname{ord}_\gamma d$, and type $(1, \operatorname{ord}_\gamma d)$ if $\operatorname{ord}_\gamma d < \operatorname{ord}_\gamma d$ by Theorem 14. Further, we see that $[\gamma^{-1}]$ has type $(\operatorname{ord}_\gamma e, 0)$ so that $l_A([\gamma]) = 2l_A([\gamma^{-1}]) = 2\operatorname{ord}_\gamma e$ in the former case, and in the latter case, $[\gamma^{-1}]$ has type $(\operatorname{ord}_\gamma d, 1)$, $[\gamma^{-2}]$ has type $(2\operatorname{ord}_\gamma d + 1, 0)$ and $l_A([\gamma]) = 1 + 2\operatorname{ord}_\gamma d = l_A([\gamma^{-2}])$. As noted before,
\( c(\gamma) = c(\gamma^{-1}) = c(\gamma^{-2}) \) for \( \gamma \) rank-one split. Consequently, we may replace \( \gamma \) by \( \gamma^{-1} \) and rewrite

\[
\begin{align*}
2c(\gamma)u^{\Lambda(\gamma)} + c(\gamma)u^{\Lambda(\gamma)}(q^{\mu(\gamma)} - \mu(\gamma))q^{\delta(\gamma) + 1} + \\
\sum_{\gamma \in \Gamma, \text{type one}} c(\gamma)u^{2\Lambda(\gamma)} + \sum_{\gamma \in \Gamma, \text{type one}} c(\gamma)u^{2\Lambda(\gamma)} + \\
\sum_{\gamma \in \Gamma, [\gamma] \text{ of type } (m,1)} c(\gamma)u^{\Lambda(\gamma^2)},
\end{align*}
\]

which can be expressed as the difference of the logarithmic derivatives of \( Z_1(X_\Gamma, u^2) \) and \( Z_2(X_\Gamma, -u) \) by Theorem 26.

Together with Proposition 27, this proves

**Proposition 30.**

\[
q \left( \sum_{n>0} \text{Tr}(B_{n,0})u^n \right) - (q - 1) \left( \sum_{n,m \geq 0 \atop (n,m) \neq (0,0)} \text{Tr}(B_{n,m})u^{n+2m} \right) \left( \frac{1 - q^2u^3}{1 - u^3} \right) = u \frac{d}{du} \log Z_1(X_\Gamma, u) + u \frac{d}{du} \log Z_1(X_\Gamma, u^2) - u \frac{d}{du} \log Z_2(X_\Gamma, -u).
\]

\[
(9.10)
\]

It remains to identify a function whose logarithmic derivative yields the left hand side of (9.10). Notice that the matrix \( B_{n,m} \) is the action of the Hecke operator \( T_{n,m} \) on the quotient \( X_\Gamma \). Hence (2.2) with \( B_{n,m} \) replacing \( T_{n,m} \) is an identity on \( X_\Gamma \). Taking trace of this identity, we see that the left hand side of (9.10) is equal to

\[
u \frac{d}{du} \log \left( \frac{(1 - u^3)^rV}{\det(I - A_1u + A_2qu^2 - q^3u^3I)} \right),
\]

where \( r = \frac{(q+1)(q-1)^2}{3} \) and \( V \) is the number of vertices in \( X_\Gamma \). Each vertex has \( q^2 + q + 1 \) type 1 edges and \( q^2 + q + 1 \) type 2 edges so that the total number of edges in \( X_\Gamma \) is \( \frac{2(q^2+q+1)}{2}V \). Since there are \( q + 1 \) chambers containing a given edge, so the number of chambers in \( X_\Gamma \) is \( \frac{(q+1)}{3}(q^2 + q + 1)V \). Therefore the Euler characteristic of \( X_\Gamma \) is

\[\chi(X_\Gamma) = V - (q^2 + q + 1)V + \frac{(q+1)}{3}(q^2 + q + 1)V = \frac{(q-1)^2(q+1)}{3}V = rV.\]

We have shown

**Proposition 31.** The left hand side of (9.10) is

\[
u \frac{d}{du} \log \left( \frac{(1 - u^3)^\chi(X_\Gamma)}{\det(I - A_1u + qA_2u^2 - q^3u^3I)} \right).
\]
Combined with Proposition 30 we obtain
\[
\frac{(1 - u^3)^\chi_{(X_\Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 I u^3)} = c \frac{Z_1(X_\Gamma, u)Z_1(X_\Gamma, u^2)}{Z_2(X_\Gamma, -u)} = c \frac{\det(1 + L_B u)}{\det(I - L_{E} u)\det(I - L_{E} u^2)}
\]
for some constant \(c\). Here the last equality comes from Theorems 22 and 23. Since both sides are formal power series with constant term 1, we find \(c = 1\). This concludes the proof of the Main Theorem.

**Remark.** Theorem 26 can be restated as
\[
\frac{\det(1 + L_B u)}{\det(I - L_{E} u^2)} = \prod_{\gamma \in \Gamma} \frac{1}{1 - u^{L_{A}((\gamma^2)^n)}} \prod_{\gamma \in \Gamma} \frac{1}{1 - u^{L_{A}((\gamma^2)^n)}}
\]
so that the right hand side of (1.2) is an infinite product.

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