POSITIVITY OF QCD AT ASYMPTOTIC DENSITY

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In this talk, I try to show that the sign problem of dense QCD is due to modes whose frequency is higher than the chemical potential. An effective theory of quasi-quarks near the Fermi surface has a positive measure in the leading order. The higher-order corrections make the measure complex, but they are suppressed as long as the chemical potential is sufficiently larger than $\Lambda_{\text{QCD}}$. As a consequence of the positivity of the effective theory, we can show that the global vector symmetries except the $U(1)$ baryon number are unbroken at asymptotic density.

1. Introduction

It is now firmly believed that quantum chromodynamics (QCD) successfully describes the strong interaction. Its prediction on the hadron interaction at high energy is well confirmed by experiments. The coupling extracted from various hadronic processes scales logarithmically with respect to energy as QCD predicted. Furthermore, the low energy dynamics of hadrons is in good agreement of the chiral symmetry breaking of QCD. It explains successfully why pions and kaons are much lighter than baryons and why hadron spectra are not in parity-doublet, though the strong interaction preserves parity.

Being the theory of strong interaction, QCD should be able to tell us how matter behaves at extreme environments, as encountered in heavy ion collisions, in early universe, or in compact stars. One salient feature of dense matter is that it undergoes phase transitions at extreme environments. QCD predicts phase transitions for hot and dense matter as one increases the temperature or the density of matter. On dimensional ground, the critical temperature and the critical chemical potential have to be of order of $\Lambda_{\text{QCD}}$, which is the only dimensional parameter that QCD dynamics depends on: $T_C \sim \Lambda_{\text{QCD}}$ and $\mu_C \sim \Lambda_{\text{QCD}}$. In fact, some of these predictions on hot matter have been confirmed by lattice calculations.
calculation shows $T_C = 175$ MeV, which is close to $\Lambda_{\text{QCD}} \approx 213$ MeV. However, little progress in lattice QCD has been made to probe the density phase transition in matter except when the chemical potential is small, since lattice QCD at finite density suffers a notorious sign problem due to the complexity of the measure.  

In this talk, I will argue that the sign problem can be solved for certain quantities at high density, thus allowing lattice calculation, and the QCD measure becomes positive at asymptotic density.

2. High density effective theory

Quark matter on lattice is described by a partition function given as

$$Z(\mu) = \int dA \det (M) e^{-S(A)},$$  \hspace{1cm} (1)

where $M = \gamma_\mu D_\mu + \mu \gamma_\mu$ is the Euclidean Dirac operator with a chemical potential $\mu$. In general the measure of dense QCD is complex, since there is no matrix $P$ that satisfies for arbitrary gauge field $A$

$$M(A) = P^{-1} M(A)^{\dagger} P.$$  \hspace{1cm} (2)

However, we claim that the complexity of the measure is due to fast modes, whose frequency is larger than the chemical potential, $\omega > \mu$. If we are interested in Fermi surface phenomena or low energy dynamics of dense matter, most of degrees of freedom in QCD are irrelevant. For instance, modes in the deep Dirac sea are hard to excite at low energy due to Pauli blocking by the states in the Fermi sea and thus decoupled to physics near the Fermi surface. On the other hand, modes near the Fermi surface are easy to excite, since it does not cost much energy to put in or remove the modes near the Fermi surface. Therefore, we need to know the energy spectra of QCD near the Fermi surface to find out the relevant modes for the low energy dynamics of dense QCD. This is in general very difficult since it amounts to solving QCD. However, the problem becomes easier at extreme density because the typical momentum transfer in the scattering of quarks near the Fermi surface is quite large compared to $\Lambda_{\text{QCD}}$.

Due to asymptotic freedom, the QCD interaction of modes near the Fermi surface can be treated perturbatively and the spectrum is determined approximately by the energy eigenvalue equation of free Dirac particles;

$$(\vec{\alpha} \cdot \vec{p} - \mu) \psi_\pm = E_\pm \psi_\pm,$$  \hspace{1cm} (3)
where $\psi_{\pm}$ are the eigenstates of $\vec{\alpha} \cdot \vec{p}$ with eigenvalues $\pm |\vec{p}|$. At low energy ($E < \mu$), the states $\psi_+$ near the Fermi surface, $|\vec{p}| \sim \mu$, are easily excited, while $\psi_-$, corresponding to the states in the Dirac sea, are completely decoupled. Therefore, the relevant modes for the low-energy QCD at high density ($\mu \gg \Lambda_{\text{QCD}}$) are $\psi_+$ modes and the soft gluons.

Consider a quark near the Fermi surface and decompose the quark momentum into the Fermi momentum and a residual momentum as

$$p_\mu = \mu v_\mu + l_\mu, \quad |l_\mu| < \mu,$$

where $v_\mu = (0, \vec{v}_F)$ and $\vec{v}_F = \vec{p}_F / \mu$ is the Fermi velocity, neglecting the quark masses. In the leading approximation in $1/\mu$ expansion, the energy of the quark near the Fermi surface depends only on the residual momentum parallel to the Fermi velocity, while the perpendicular component, $l_\perp$, labels the degeneracy on the Fermi surface. Therefore, the integration over the perpendicular component should give the area of the Fermi surface,

$$\int d^2 l_\perp = 4\pi p_F^2.$$

(4)

Now, at low energy $E < \mu$, the Fermi velocity of the quark near the Fermi surface does not change under any scattering, since any change in the Fermi velocity can be absorbed into the redefinition of the residual momentum. So, it is convenient to define a Fermi-velocity dependent field which carries the residual momentum only,

$$\psi_+ (\vec{v}_F, x) = \frac{1 + \vec{\alpha} \cdot \vec{v}_F}{2} e^{-i\mu\vec{v}_F \cdot \vec{x}} \psi(x).$$

Integrating out the irrelevant modes, $\psi_-$ and hard gluons, we get the high density effective theory (HDET) of QCD for dense quark matter at low energy.

3. Positivity at asymptotic density

The effective theory is described by

$$L_{\text{eff}} = \bar{\psi}_+ i\gamma_\mu D_\mu \psi_+ (\vec{v}_F, x) - \frac{c_1}{2\mu} \bar{\psi}_+ \gamma_5 (\mathcal{D}_\perp)^2 \psi_+ + \cdots,$$

(5)

where $c_1$ is a dimensionless constant due to loop effects of the irrelevant modes and the ellipsis denotes the higher order terms in $1/\mu$ expansion.

We note that the Dirac operator of the effective theory in Euclidean space is related to its hermitian conjugate by a similarity transformation,

$$M_{\text{eff}} = \gamma_\mu \cdot D(A) = \gamma_5 M_{\text{eff}}^1 \gamma_5.$$

(6)
Therefore, HDET has a positive measure in the leading order. Since the next-to-leading term is hermitian, while the leading term is anti-hermitian, the sign problem comes in at the next-leading order. However, the sign problem is suppressed by $1/\mu$.

To implement HDET on lattice, it is useful to introduce an operator formalism in which the velocity is realized as an operator,

$$\vec{v} = -i \frac{\partial}{\sqrt{-\nabla^2}} \vec{v},$$

since one needs to know the Fermi velocity for a given configuration of quarks. Then, the quasi-quarks near the Fermi surface are described by

$$\psi_+ = \exp(-i \mu x \cdot v \cdot \alpha \cdot v) \psi.$$  \hspace{1cm} (8)

Now, the effective Lagrangian density becomes

$$L_+ = \bar{\psi} + \gamma_\mu \partial_\mu \psi + \bar{\psi} \gamma_\mu A_\mu \psi,$$  \hspace{1cm} (9)

where $A_\mu^a = e^{-iX} A^\mu \, e^{+iX}$ and $X = \mu x \cdot v \cdot \alpha \cdot v$. Note that $\gamma_\mu \partial_\mu = \gamma_\mu \partial_\mu$, since $v \cdot \partial v \cdot \gamma = \partial \cdot \gamma$.

The partition function of dense QCD can be rewritten as

$$Z(\mu) = \int dA_+ \, \det M_{\text{eff}}(A_+) e^{-S_{\text{eff}}(A_+)}$$  \hspace{1cm} (10)

and the effective action is given as

$$S_{\text{eff}}(A) = \int d^4x \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} \sum_{\vec{v}} A_{\perp\mu} A_{\perp\mu}^a + \cdots \right)$$  \hspace{1cm} (11)

where $A_{\perp} = A - A_\parallel$, $M = \sqrt{N_f/(2\pi^2)} g_s \mu$ is the Debye mass, and the ellipsis denotes terms suppressed by $1/\mu$. Therefore, we see that at the leading order HDET has a positive measure and the lattice calculation is possible.

To estimate the size of the higher-order contributions, we calculate the correction to the vacuum energy by the naive dimensional analysis. We found

$$\frac{\delta E_{\text{vac}}}{E_{\text{vac}}} \sim \frac{\alpha_s \Lambda}{2\pi \mu},$$  \hspace{1cm} (12)

where $\Lambda \simeq \Lambda_{\text{QCD}}$ is the energy scale that we are interested in. Therefore, the positivity of HDET is good, as long as the chemical potential is much larger than $\Lambda_{\text{QCD}}$. 

As an application of the positivity of QCD at asymptotic density, one can establish a rigorous inequality like the Vafa-Witten theorem to show that the color-flavor locked (CFL) phase is in fact exact at asymptotic density.

Consider the correlator of the $SU(3)_V$ flavor currents

$$\langle J_A^\mu (\vec{v}_F, x) J_B^\nu (\vec{v}_F, y) \rangle_A = -\text{Tr} \gamma_\mu T^A S^A(x, y; \Delta) \gamma_\nu T^B S^A(y, x; \Delta),$$

(13)

where $J_A^\mu (\vec{v}_F, x) = \bar{\psi} (\vec{v}_F, x) \gamma_\mu T^A \psi (\vec{v}_F, x)$ and we have introduced an infrared cut-off $\Delta$, which breaks the $U(1)$ baryon number symmetry. The anomalous propagator can be rewritten as

$$S^A(x, y; \Delta) = \langle x| \frac{1}{M} | y \rangle = \int_0^\infty d\tau \langle x| e^{-i\tau(-iM)} | y \rangle = e^{-\Delta R} \Delta \sqrt{\langle x| x \rangle} \sqrt{\langle y| y \rangle}. (14)$$

where $D = \partial + iA$ and

$$M = \gamma_0 \begin{pmatrix} D \cdot V & \Delta \\ \Delta & D \cdot \bar{V} \end{pmatrix},$$

(15)

with $V = (1, \vec{v}_F)$, $\bar{V} = (1, -\vec{v}_F)$. Since the eigenvalues of $M$ are bound from below by $\Delta$, we have the following inequality:

$$\langle x| \frac{1}{M} | y \rangle \leq \int_R^\infty \text{d}\tau e^{-\Delta \tau} \sqrt{\langle x| x \rangle} \sqrt{\langle y| y \rangle} = e^{-\Delta R} \Delta \sqrt{\langle x| x \rangle} \sqrt{\langle y| y \rangle}. (16)$$

Since the measure of HDET is positive, the vector current correlator falls off exponentially even after integrating over the gauge fields. Therefore, there is no Nambu-Goldstone mode along the vector channel. Combining this with the Cooper theorem, we prove that the CFL phase is exact.

In conclusion, we have shown that dense QCD is positive at asymptotic density. Furthermore, a lattice calculation should be possible using HDET, an effective theory for quasi-quarks near the Fermi surface, as long as $\mu \gg \Lambda_{QCD}$. As a consequence of the positivity, we were able to show that the (global) vector symmetries except the $U(1)$ baryon number are not broken in QCD at asymptotic density.

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*aNote that any infrared regulator has to break the $U(1)$ baryon number to open a gap at the Fermi surface*
Acknowledgments

I would like to thank Steve Hsu for the collaboration on which this talk is based. This work is supported in part by the KOSEF grant number 1999-2-111-005-5 and also by the academic research fund of Ministry of Education, Republic of Korea, Project No. BSRI-99-015-DI0114.

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