Efimov effect in the two-body problem on the half-line

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The Efimov effect (in a broad sense) refers to the onset of a geometric sequence of many-body bound states as a consequence of the breakdown of continuous scale invariance to discrete scale invariance. While originally discovered in three-body problems in three dimensions, the Efimov effect has now been known to appear in a wide spectrum of many-body problems in various dimensions. Here we introduce a simple, exactly solvable toy model of two identical bosons in one dimension that exhibits the Efimov effect. In this paper, we consider the situation where the bosons reside on the half-line and interact with each other through a pairwise $\delta$-function potential with particular position-dependent coupling strength that makes the system scale invariant. We show that, for sufficiently attractive interaction, continuous scale invariance can be broken down to discrete scale invariance and there appears a geometric sequence of boundary-localized two-body bound states, where the boundary plays the role of the (infinitely heavy) third particle. We also study the two-body scattering off the boundary and derive the exact reflection amplitude that exhibits a log-periodicity. This article is intended for students and non-specialists interested in discrete scale invariance.

I. INTRODUCTION

In his seminal paper\(^1\) in 1970, Efimov pointed out that, when two-body scattering length diverges in three-body problems of identical bosons under short-range pairwise interactions, there generally appear infinitely many three-body bound states whose energy levels \{\textit{E}_n\} form a geometric sequence. This phenomenon—generally known as the Efimov effect—has been attracted much attention because the common ratio \(E_{n+1}/E_n \approx 1/(22.7)\)\(^2\) is independent of the details of interactions as well as particle species; that is, its prediction is universal. More than thirty-five years after its discovery, this universal prediction was finally observed in cold atom experiments,\(^2\)\(-6\) which has triggered an explosion of theoretical and experimental research on the Efimov effect. For more details, see the reviews \(^7\)\(-\)\(^11\). (See also Refs. \(^12\)\(-\)\(^\)\(^14\) for more elementary exposition.)

Aside from this universality, the Efimov effect takes its place among the greatest theoretical discoveries in modern physics because it was the first example in quantum many-body problems that realizes discrete scale invariance.\(^15\) Here discrete scale invariance refers to invariance under enlargement or reduction in size by a single scale factor. If a quantum system enjoys this invariance, discrete energy levels (if exist) must form a geometric sequence. It is now known that the emergence of a geometric sequence of bound states is associated with the breakdown of continuous scale invariance to discrete scale invariance\(^16\) and can be found in a wide spectrum of quantum many-body problems in various dimensions.\(^17\)\(-\)\(^23\) The notion of Efimov effect has now been broadened to include those generalizations so that its precise meaning varies in the literature. In the present paper, we will use the term “Efimov effect” to simply refer to the onset of a geometric sequence of many-body bound states as a consequence of the breakdown of continuous scale invariance to discrete scale invariance.

To date, there exist several theoretical approaches to study the Efimov effect. A main approach is to directly analyze the many-body Schrödinger equation, which normally involves the techniques of the Jacobi coordinates, the hyperspherical coordinates, the adiabatic approximation, and the Faddeev equation.\(^7\) Another main approach is to use the second quantization, or quantum field theory.\(^8\) Though the problem itself is conceptually simple, it is hard for students and non-specialists to acquire these techniques and to work out how the breakdown of continuous scale invariance to discrete scale invariance results in the onset of a geometric sequence of bound states. The essential part of the Efimov effect, however, can be understood from undergraduate-level quantum mechanics with no fancy techniques.

This paper is aimed at introducing a simple toy model of two-body problems that exhibits the Efimov effect. In this paper, we consider two identical bosons on the half-line \(\mathbb{R}_+ = \{x : x > 0\}\) that interact with each other through a pairwise $\delta$-function potential. The Hamiltonian of such a system is generally given by

\[
H = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + g(x_1, x_2)\delta(x_1 - x_2),
\]

where \(m\) is the mass of the particles and \(x_j \in \mathbb{R}_+ (j = 1, 2)\) is the coordinates of the \(j\)th particle. Here \(g\) is a coupling strength which, in general, can depend on the coordinates. In this paper, we will focus on the position-dependent coupling strength that satisfies the permutation invariance \(g(x_1, x_2) = g(x_2, x_1)\) and the scaling law \(g(\lambda x_1, \lambda x_2) = \lambda^n g(x_1, x_2)\), where \(\lambda\) is an arbitrary real. Any such functions have the unique behavior at the two-body coincidence point \(x_1 = x_2\) because the condition \(g(\lambda x_1, \lambda x_1) = \lambda^n g(x_1, x_1)\) has the unique solution \(g(x_1, x_1) \propto 1/x_1\). Hence, without any loss of generality, we can focus on the following coupling strength:

\[
g(x_1, x_2) = \frac{\hbar^2}{m} \frac{\sqrt{2g_0}}{\sqrt{x_1^2 + x_2^2}},
\]

This is the article is intended for students and non-specialists interested in discrete scale invariance.
where \( g_0 \) is a dimensionless real that can be both positive and negative. The factor \( \sqrt{2\hbar^2/m} \) is introduced just for later convenience. Physically, Eq. (2) models the situation where the interaction strength of two-body scattering becomes stronger as the particles come closer to the boundary \( x_1 = x_2 = 0 \); see Fig. 1. We note that this two-body interaction is essentially equivalent to the so-called scaling trap introduced in Ref. 19, where the Efimov effect was discussed in the context of two non-identical particles on the whole line \( \mathbb{R} \). As we will see shortly, our two-identical-particle problem on \( \mathbb{R}_+ \) enjoys simple solutions and is more tractable than the corresponding two-non-identical-particle problem on \( \mathbb{R} \).

The rest of the paper is devoted to detailed analysis of the spectral and scattering problems of \( H \). Before going into details, however, it is worth summarizing here symmetry properties of the model. Of particular importance are the following:

- **Permutation invariance.** Thanks to the relations \( g(x_1, x_2) = g(x_2, x_1) \) and \( \delta(x_1 - x_2) = \delta(x_2 - x_1) \), the Hamiltonian (1) is invariant under the permutation of coordinates, \( (x_1, x_2) \mapsto (x_2, x_1) \). Note that this permutation invariance is necessary for Eq. (1) to be a Hamiltonian of indistinguishable particles, where for bosons the two-body wavefunction should satisfy \( \psi(x_1, x_2) = \psi(x_2, x_1) \). We will see in Sec. IIIA that this invariance greatly simplifies the analysis.

- **Scale invariance.** Thanks to the relations \( g(e^t x_1, e^t x_2) = e^{-t} g(x_1, x_2) \) and \( \delta(e^t x_1 - e^t x_2) = e^{-t} \delta(x_1 - x_2) \), the Hamiltonian (1) transforms as \( H \mapsto e^{-2t} H \) under the scale transformation \( (x_1, x_2) \mapsto (e^t x_1, e^t x_2) \). This transformation law has significant implications for the spectrum of \( H \). To see this, let \( \psi_E(x_1, x_2) \) be a solution to the eigenvalue equation \( H \psi_E(x_1, x_2) = E \psi_E(x_1, x_2) \). Then, \( \psi_E(e^t x_1, e^t x_2) \) automatically satisfies \( H \psi_E(e^t x_1, e^t x_2) = e^{2t} E \psi_E(e^t x_1, e^t x_2) \); that is, \( \psi_E(e^t x_1, e^t x_2) \) is proportional to the eigenfunction \( \psi_{e^2t} E(x_1, x_2) \) with the eigenvalue \( e^{2t} E \). The proportional coefficient can be determined by requiring that \( \psi_E \) and \( \psi_{e^2t} E \) should satisfy the same normalization condition. The result is the following scaling law:

\[
\psi_{e^2t} E(x_1, x_2) = e^t \psi_E(e^t x_1, e^t x_2). \tag{3}
\]

If this indeed holds for any continuous \( t \in \mathbb{R} \), the spectrum of \( H \) must consist of only continuous spectrum, because \( e^{2t} E \) can take any arbitrary (positive) value. As we will see in Sec. III B, however, if \( g_0 \) is below a critical value \( g_* \), Eq. (3) holds only for some discrete \( t \in t_* \mathbb{Z} = \{0, \pm t_0, \pm 2t_0, \cdots\} \); that is, continuous scale invariance is broken to discrete scale invariance, where \( t_* \) defines one particular scale. As a consequence, there appears a geometric sequence of (negative) energy eigenvalues, \( \{E_0, E_0 e^{2t_*}, E_0 e^{4t_*}, \cdots\} \), where \( E_0(< 0) \) is a newly emergent energy scale. One of the goal of this paper is to show this by just using undergraduate-level calculus.

It should be noted that there is no translation invariance in our model: it is explicitly broken by the boundary as well as by the position-dependent coupling strength (2). This non-invariance means that the total momentum—the canonical conjugate of the center-of-mass coordinates—is not a well-defined conserved quantity. In other words, the two-body wavefunction cannot be of the separation-of-variable form \( \psi(x_1, x_2) = e^{iP X/\hbar} \phi(x) \), where \( X = (x_1 + x_2)/2 \) is the center-of-mass coordinates, \( P \) the total momentum, and \( x = x_1 - x_2 \) the relative coordinates with \( \phi \) being the wavefunction of relative motion. In the next section, we will first introduce an alternative coordinate system that is more suitable for the two-body problem on the half-line.

### II. TWO-BODY PROBLEM WITHOUT TRANSLATION INVARIANCE

To begin with, let us first introduce a new coordinate system in the \( (x_1, x_2) \)-space. In what follows, we will work with the polar coordinate system \((r, \theta)\) defined as follows (see Fig. 2):

\[
\begin{align}
x_1 &= r \cos(\theta + \frac{\pi}{4}), \tag{4a} \\
x_2 &= r \sin(\theta + \frac{\pi}{4}), \tag{4b}
\end{align}
\]

or, equivalently,

\[
\begin{align}
r &= \sqrt{x_1^2 + x_2^2}, \tag{5a} \\
\theta &= \frac{1}{2t} \log \left( \frac{x_1 + ix_2}{x_1 - ix_2} \right) - \frac{\pi}{4}, \tag{5b}
\end{align}
\]

where \( r \in (0, \infty) \) and \( \theta \in (-\pi/4, \pi/4) \). Note that \( \theta = 0 \) and \( \theta = \pm \pi/4 \) correspond to the two-body coincidence point \( x_1 = x_2 \) and the boundaries \( x_1 = 0 \) and \( x_2 = 0 \). Note also that the permutation \((x_1, x_2) \mapsto (x_2, x_1)\) corresponds to the parity transformation \((r, \theta) \mapsto (r, -\theta)\); see Fig. 2.
As we will see shortly, and is known to support a geometric equations:

\[ E\psi \]

\[ H\psi \]

line represents the set of two-body coincidence points.

Now, in the coordinate system \((r, \theta)\), the kinetic energy part of the two-body Hamiltonian (1) takes the following form:

\[
H_0 = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)
\]

\[
= -\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)
\]

\[
= \frac{\hbar^2}{2m} r^{-\frac{1}{2}} \left( -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \right) \frac{r}{2}. \tag{6}
\]

Likewise, the potential energy part is rewritten as

\[
V(x_1, x_2) = \frac{\hbar^2}{m} \frac{\sqrt{2g_0}}{\sqrt{x_1^2 + x_2^2}} \delta(x_1 - x_2)
\]

\[
= \frac{\hbar^2}{m} \frac{\sqrt{2g_0}}{r} \delta(r \cos(\theta + \frac{\pi}{4}) - r \sin(\theta + \frac{\pi}{4}))
\]

\[
= \frac{\hbar^2}{m} g_0 \delta(\theta) \quad \text{for} \quad \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}). \tag{7}
\]

The total Hamiltonian \(H = H_0 + V\) can then be cast into the following form:

\[
H = \frac{\hbar^2}{2m} r^{-\frac{1}{2}} \left( -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \right) \frac{r}{2}, \tag{8}
\]

where

\[
\Delta_\theta = -\frac{\partial^2}{\partial \theta^2} + 2g_0 \delta(\theta). \tag{9}
\]

Next we are ready to analyze the Schrödinger equation by means of the separation of variables. Suppose that the two-body wavefunction is of the following form:

\[
\psi(x_1, x_2) = r^{-\frac{d}{4}} R(r) \Theta(\theta). \tag{10}
\]

Then, the time-independent Schrödinger equation \(H\psi = E\psi\) can be reduced to the following set of differential equations:

\[
\left( -\frac{d^2}{d\theta^2} + 2g_0 \delta(\theta) \right) \Theta(\theta) = \lambda \Theta(\theta), \tag{11a}
\]

\[
\left( -\frac{d^2}{dr^2} + \frac{\lambda - \frac{1}{2}}{r^2} \right) R(r) = \frac{2mE}{\hbar^2} R(r). \tag{11b}
\]

It is now clear that the energy eigenvalue is determined by the inverse-square potential, which has been vastly studied over the years in the context of anomaly or renormalization, and is known to support a geometric sequence of bound states if \(\lambda < 0.32\). As we will see shortly, if \(g_0\) is below a critical value \(g_\ast\), the lowest eigenvalue \(\lambda_0\) in the eigenvalue equation (11a) becomes negative. Hence, in such a \(\lambda_0\)-channel, continuous scale invariance can be broken down to discrete scale invariance and there appears a geometric sequence of two-body bound states. Let us next see this by solving the differential equations (11a) and (11b) explicitly.

### III. TWO-BODY EFIMOV EFFECT WITH BOUNDARY

#### A. Solution to the angular equation

Let us first solve the angular equation (11a). To this end, we need to specify the connection conditions at \(\theta = 0\) and the boundary conditions at \(\theta = \pm \pi/4\). Let us first start with the connection conditions that describe the \(\delta\)-function potential.

As is well-known, the \(\delta\)-function potential system is described by the differential equation \(-\Theta'' = \lambda \Theta\) for \(\theta \neq 0\) and the following connection conditions at \(\theta = 0\):

\[
-\Theta'(0_+) + \Theta'(0_-) + g_0 (\Theta(0_+) + \Theta(0_-)) = 0, \tag{12a}
\]

\[
\Theta(0_+) = \Theta(0_-). \tag{12b}
\]

where the prime (’) indicates the derivative with respect to \(\theta\).

Let us next take into account the symmetry of the two-body wavefunction. Since we are dealing with identical bosons, the wavefunction must be symmetric under the permutation, \(\psi(x_1, x_2) = \psi(x_2, x_1)\). In the polar coordinate system, this is equivalent to \(\Theta(\theta) = \Theta(-\theta)\), whose derivative gives \(\Theta'(\theta) = -\Theta'(-\theta)\). Hence, at the two-body coincidence point \(\theta = 0\), there must hold the following additional conditions:

\[
\Theta(0_+) = \Theta(0_-) \quad \text{and} \quad \Theta'(0_+) = -\Theta'(0_-). \tag{13}
\]

Thus, for identical bosons, Eq. (12a) can be reduced to the following Robin boundary conditions:

\[
\mp \Theta'(0_{\pm}) + g_0 \Theta(0_{\pm}) = 0. \tag{14}
\]

In addition to the connection conditions at \(\theta = 0\), we also have to specify the boundary conditions at \(\theta = \pm \pi/4\). For simplicity, we will impose the following Dirichlet boundary conditions: \(\Theta(\pm \frac{\pi}{4}) = 0\).

\[
\Theta(\pm \frac{\pi}{4}) = 0. \tag{15}
\]

Now it is straightforward to solve the angular equation (11a). Thanks to the property \(\Theta(\theta) = \Theta(-\theta)\), it is enough to solve the differential equation \(-\Theta'' = \lambda \Theta\) in
the region $0 < \theta < \pi/4$ under the boundary conditions $-\Theta'(0_+) + g_0\Theta(0_+) = 0$ and $\Theta(\pi/4) = 0$. The resulting solution for $\lambda \neq 0$ is

$$\Theta_\lambda(\theta) = A_\lambda \sin \left( \sqrt{\lambda} \left( \frac{\pi}{4} - \left| \theta \right| \right) \right), \quad (16)$$

where $A_\lambda$ is a normalization constant. Here $\lambda$ is a root of the following transcendental equation:

$$g_0 = -\sqrt{\lambda} \cot \left( \frac{\pi}{4} \sqrt{\lambda} \right). \quad (17)$$

Note that, for $\lambda < 0$, the square root should be understood as $\sqrt{\lambda} = i \sqrt{|\lambda|}$. Note also that, for $\lambda < 0$, the angular wavefunction $\Theta_\lambda(\theta) \propto \sin(\sqrt{|\lambda|}(\pi/4 - |\theta|))$ sharply localizes to the two-body coincidence point $\theta = 0$; that is, it describes a two-body bound state; see Fig. 3.

Though the transcendental equation (17) cannot be solved analytically, the $g_0$ dependence of its solutions can be easily seen by plotting the graph of $g_0 = -\sqrt{\lambda} \cot(\pi \sqrt{\lambda}/4)$ and then by reflecting about the line $g_0 = \lambda$; see Fig. 3. As can be observed from this figure, the lowest eigenvalue $\lambda_0$ becomes negative for $g_0 < g_*$, where $g_*$ is the critical value given by

$$g_* := \lim_{\lambda \to 0^+} \left[ -\sqrt{\lambda} \cot \left( \frac{\pi}{4} \sqrt{\lambda} \right) \right] = -\frac{4}{\pi}. \quad (18)$$

Hence, in the $\lambda_0$-channel, continuous scale invariance must be broken down to discrete scale invariance for $g_0 < g_*$. Let us next see this by solving the radial equation (11b).

### B. Boundary-localized two-body Efimov states

Now let us focus on the case $E < 0$ in the channel $\lambda = \lambda_0 < 0$. In this case, there exists a square-integrable solution to the differential equation (11b) whose asymptotic behavior as $r \to \infty$ is $R_\kappa(r) \to N_\kappa e^{-\kappa r}$, where $N_\kappa$ is a normalization constant and $\kappa = \sqrt{2m|E|/\hbar^2} > 0$. The full solution is given by

$$R_\kappa(r) = N_\kappa \sqrt{\frac{2\kappa r}{\pi}} K_{i\kappa}(\kappa r), \quad \nu = \sqrt{|\lambda_0|}, \quad (19)$$

where $K_{i\kappa}$ is the modified Bessel function of the second kind whose asymptotic behavior is $K_{i\kappa}(\kappa r) \to \sqrt{\pi/(2\kappa r)} e^{-\kappa r}$. Note that Eq. (19) together with Eq. (16) describes the two-body wavefunction that localizes to $\theta = 0$ and $r = 0$; that is, it describes the two-body bound state localized to the boundary.

It should be noted that at this stage $\kappa$ is an arbitrary positive real. To determine its possible values, we follow the argument in Ref. 32 and require the orthonormality of the radial wavefunction. Let $R_\kappa$ and $R_{\kappa'}$ be two distinct solutions to Eq. (11b). Then we have

$$-R''_\kappa + \frac{\lambda_0 - \frac{1}{4}}{r^2} R_\kappa = -\kappa^2 R_\kappa, \quad (20a)$$

$$-R''_{\kappa'} + \frac{\lambda_0 - \frac{1}{4}}{r^2} R_{\kappa'} = -\kappa'^2 R_{\kappa'}, \quad (20b)$$

where the overline ($\overline{\cdot}$) stands for the complex conjugate and the prime here indicate the derivative with respect to $r$. By multiplying $\overline{R_{\kappa'}}$ to Eq. (20a) and $R_\kappa$ to Eq. (20b) and then subtracting the both sides, we get

$$(-\kappa^2 + \kappa'^2) R_\kappa R_{\kappa'} = \overline{R_{\kappa'}} R''_{\kappa'} - R_{\kappa'} R''_\kappa = \frac{d}{dr} \left( \overline{R_{\kappa'}} R_{\kappa'} - R_{\kappa'} R_{\kappa'} \right). \quad (21)$$

By integrating the both sides from $r = 0$ to $\infty$, we find

$$(-\kappa^2 + \kappa'^2) \int_0^\infty dr \overline{R_{\kappa'}}(r) R_\kappa(r)$$

$$= \int_0^\infty \frac{d}{dr} \left( \overline{R_{\kappa'}}(r) R_{\kappa}'(r) - R_{\kappa'}(r) R_{\kappa}'(r) \right)$$

$$= -\lim_{r \to 0} \left( R_{\kappa'}(r) R_{\kappa}'(r) - R_{\kappa'}(r) R_{\kappa}'(r) \right)$$

$$= \frac{2\sqrt{\kappa\kappa'} \sin(\nu \log \frac{\kappa}{\kappa'})}{\sinh(\nu \pi)} \frac{N_{\kappa'} N_\kappa}{N_{\kappa} N_{\kappa}}, \quad (22)$$

where the second equality follows from $R_\kappa, R_{\kappa'} \to 0$ in the limit $r \to \infty$ and the last equality follows from the short-distance behavior of the modified Bessel function; see Eq. (A.4) in Appendix.

Now, Eq. (22) enables us to determine the normalization constant as well as the energy eigenvalues. First, the normalization constant is determined by requiring that $R_\kappa$ should have the unit norm:

$$1 = \int_0^\infty dr |R_\kappa(r)|^2$$

$$= \lim_{\kappa' \to \kappa} \int_0^\infty dr \overline{R_{\kappa'}}(r) R_\kappa(r)$$

$$= \frac{2\sqrt{\kappa\kappa'} \sin(\nu \log \frac{\kappa}{\kappa'})}{\sinh(\nu \pi)(\kappa^2 - \kappa'^2)} N_{\kappa'} N_\kappa$$

$$= \frac{\nu}{\kappa \sinh(\nu \pi)} |N_\kappa|^2, \quad (23)$$

where the last equality follows from $\sin(\nu \log(\kappa/\kappa')) = \sin(\nu \log(1 + (\kappa - \kappa')/\kappa')) = \nu(\kappa - \kappa')/\kappa' + O((\kappa - \kappa')^2/\kappa'^2)$.
as $\kappa' \to \kappa$. Thus we find

$$|N_\kappa| = \sqrt{\frac{\kappa \sinh(\nu \pi)}{\nu}},$$

(24)

Second, the energy eigenvalues are determined by requiring that $R_\kappa$ and $R_{\kappa'}$ should be orthogonal for $\kappa \neq \kappa'$; that is, $\int_0^\infty dr R_{\kappa'}(r)R_\kappa(r) = 0$ for $\kappa \neq \kappa'$, which is attained if and only if $\sin(\nu \log(\kappa/\kappa')) = 0$. Thus, $\nu \log(\kappa/\kappa')$ must be an integer multiple of $\pi$:

$$\nu \log \frac{\kappa}{\kappa'} = -n\pi, \quad n \in \mathbb{Z},$$

(25)

where the minus sign on the right hand side is just the convention. The solution to this condition is given by

$$\kappa_n = \kappa_\ast \exp\left(-\frac{n\pi}{\nu}\right),$$

(26)

where $\kappa_\ast(>0)$ is an arbitrary reference scale with the dimension of inverse length, which must be introduced on the dimensional grounds. Putting these together, we obtain the following infinitely many negative energy eigenvalues:

$$E_n = -\frac{\hbar^2 \kappa_n^2}{2m} \exp\left(-\frac{2n\pi}{\nu}\right), \quad n \in \mathbb{Z}.$$  

(27)

These are the binding energies of the boundary-localized two-body bound states. Note that the spatial extent of these bound states is about $r \approx 1/\kappa_n = \kappa_\ast^{-1} e^{n\pi/\nu}$, which follows from the asymptotic behavior $R_{\kappa_n}(r) \to N_{\kappa_n} e^{-\kappa_n r}$ as $r \to \infty$; see Fig. 4. Note also that $E_n$ and $R_{\kappa_n}(r)$ fulfill the relations $E_{n-1} = E_n e^{2\pi/\nu}$ and $R_{\kappa_{n-1}}(r) = e^{\pi/(2\nu)} R_{\kappa_n}(e^{\pi/\nu} r)$, which, through Eq. (10), guarantee the scaling law (3) discussed in the introduction with the scaling factor $e^\delta = e^{\pi/\nu}$.

C. Two-body scattering off the boundary

Let us finally consider the case $E > 0$ in the channel $\lambda = \lambda_0 < 0$. In this case, we are interested in the solution to the radial equation (11b) whose asymptotic behavior as $r \to \infty$ is the linear combination of plane waves, $R_k(r) \to e^{-ikr} + S(k) e^{ikr}$, where $S(k)$ is a linear combination coefficient and $k = \sqrt{2mE/\hbar^2} > 0$. The full solution is given by

$$R_k(r) = \sqrt{\frac{2kr}{\pi}} \left( e^{\frac{i\pi}{4}} K_{i\nu}(e^{\frac{i\pi}{4}} kr) + S(k) e^{\frac{-i\pi}{4}} K_{i\nu}(e^{\frac{-i\pi}{4}} kr) \right).$$  

(28)

Note that Eq. (28) together with Eq. (16) describes the superposition of incoming wave to $r = 0$ and outgoing wave from $r = 0$, both of which localize to $\theta = 0$: that is, it describes the two-body bound state scattered off the boundary, where $S(k)$ plays the role of the reflection amplitude; see Fig. 4.

It should be noted that at this stage $S(k)$ is an arbitrary constant. In order to determine $S(k)$, we require that the scattering solution (28) should be orthogonal to all the bound-state solutions (19). In exactly the same way to arrive at Eq. (22), one can obtain the following relation:

$$(-\kappa_n^2 - k^2) \int_0^\infty dr R_{\kappa_n}(r)R_k(r) = -\lim_{r \to 0} \left( R_{\kappa_n}(r)R'_k(r) - R'_\kappa_n(r)R_k(r) \right)$$

$$= \frac{2\sqrt{\kappa_n k}}{\pi \sinh(\nu \pi)} N_{\kappa_n} \left[ e^{i\nu\pi} \sin\left( \nu \log \left( \frac{k}{\kappa_n} \right) + \frac{i\nu\pi}{2} \right) + S(k) e^{-i\nu\pi} \sin\left( \nu \log \left( \frac{k}{\kappa_n} \right) - \frac{i\nu\pi}{2} \right) \right].$$

(29)

Hence, in order to guarantee the orthogonality relation $\int_0^\infty dr R_{\kappa_n}(r)R_k(r) = 0$ for any $k > 0$, the coefficient $S(k)$ must be of the following form:

$$S(k) = -e^{\frac{i\nu\pi}{2}} \sin\left( \frac{\nu \log \left( \frac{k}{\kappa_n} \right) + \frac{i\nu\pi}{2}}{\nu \log \left( \frac{k}{\kappa_n} \right) - \frac{i\nu\pi}{2}} \right).$$

(30)

This is the reflection amplitude off the boundary for the two-body bound state. It should be noted that this amplitude, which satisfies the unitarity $S(k)\overline{S}(k) = 1$, is a periodic function of $\log k$ with the period $\pi/\nu$. This log-periodicity is a manifestation of discrete scale invariance $S(e^{\pi/\nu} k) = S(k)$ in the scattering problem. We also note that Eq. (30) has simple poles at $k = i\kappa_n = \kappa_\ast e^{\pi/2-2n\pi/\nu}$ in the complex $k$-plane. In fact, it behaves as follows:

$$S(k) \to \frac{i|N_{\kappa_n}|^2}{k - i\kappa_n} + O(1) \quad \text{as} \quad k \to i\kappa_n.$$  

(31)

These simple poles are the manifestation of the presence of infinitely many bound states that satisfy the geometric scaling $E_{n+1}/E_n = \kappa_n^2/\kappa_n^2 = e^{-2\pi/\nu}$.
IV. CONCLUSION

In this paper, we have introduced a toy scale-invariant model of two identical bosons on the half-line, where interparticle interaction is described by the pairwise $\delta$-function potential with the particular position-dependent coupling strength (2). We have seen that, if the two-body interaction is sufficiently attractive, continuous scale invariance is broken down to discrete scale invariance. In the bound-state problem, this discrete scale invariance manifests itself in the onset of boundary-localized two-body bound states whose energy levels satisfy the geometric scaling. In the scattering problem, on the other hand, this discrete scale invariance manifests itself in the log-periodic behavior of the reflection amplitude off the boundary. In this way, by breaking translation invariance, we can construct a two-body model that exhibits the Efimov effect. In contrast to the ordinary Efimov effect in three-body problems in three dimensions, our model can be solved exactly by just using undergraduate-level calculus.

Finally, it should be mentioned the stability issue of the model and its cure. As is evident from Eq. (27), there is no lower bound in the energy spectrum $\{E_n\}$ for $g < g_s$. This absence of ground state is inevitable if the system is invariant under the full discrete scale invariance that forms the group $Z$. In order to make the spectrum lower-bounded, we therefore have to break this invariance under $Z$. The easiest way to do this is to replace the short-distance singularity of inverse-square potential by, e.g., a square-well potential. Such regularization procedures have been vastly studied over the years in the context of renormalization of the inverse-square potential.

Appendix: Modified Bessel functions of imaginary order

In this section, we summarize the short- and long-distance behaviors of the modified Bessel functions. For details, we refer to the treatise 35.

First of all, the modified Bessel function of the second kind with imaginary order is defined as follows:

$$K_{i\nu}(z) = \frac{i\pi}{2} \frac{I_{\nu}(z) - I_{-\nu}(z)}{\sinh(\nu\pi)}, \quad \nu \in \mathbb{R} \setminus \{0\},$$

where $I_\nu$ is the modified Bessel function of the first kind given by the following series:

$$I_{i\nu}(z) = e^{i\nu\log z} \sum_{n=0}^{\infty} \frac{1}{n\Gamma(1+n+i\nu)} \left( \frac{z}{2} \right)^{2n}.$$  \hspace{1cm} (A.2)

It follows immediately from the definition (A.1) that $K_{-i\nu}(z) = K_{i\nu}(z)$. It also follows from Eqs. (A.1) and (A.2) that $\overline{K_{i\nu}(z)} = K_{i\nu}(z)$ for $z > 0$. The short-distance behavior of $K_{i\nu}$ is governed by the $n = 0$ term in Eq. (A.2). By using the polar form of the Gamma function,

$$\Gamma(1 + i\nu) = |\Gamma(1 + i\nu)| e^{i\arg \Gamma(1 + i\nu)}$$

$$= \sqrt{\frac{\nu\pi}{\sinh(\nu\pi)}} e^{i\arg \Gamma(1 + i\nu)},$$

where $\arg \Gamma(1 + i\nu)$ stands for the argument of $\Gamma(1 + i\nu)$, we see that $K_{i\nu}(z)$ behaves as follows:

$$K_{i\nu}(z) \rightarrow -\sqrt{\frac{\pi}{\nu\sinh(\nu\pi)}} \sin \left( \nu \log \frac{z}{2} - \arg \Gamma(1 + i\nu) \right) + O(z^2) \quad \text{as} \quad |z| \rightarrow 0.$$  \hspace{1cm} (A.4)

The long-distance behavior, on the other hand, is known to be of the following form:

$$K_{i\nu}(z) \rightarrow \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + O\left( \frac{1}{z} \right) \right] \quad \text{as} \quad |z| \rightarrow \infty.$$  \hspace{1cm} (A.5)

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1. V. Efimov, “Energy levels arising from resonant two-body forces in a three-body system,” Phys. Lett. B 33, 563–564 (1970).

2. T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nagerl, and R. Grimm, “Evidence for Efimov quantum states in an ultracold gas of caesium atoms.” Nature 440, 315–318 (2006), arXiv:cond-mat/0512394 [cond-mat.other].

3. M. Zaccanti, B. Deissler, C. D’Errico, M. Fattori, M. Jona-Lasinio, S. Müller, G. Roati, M. Inguscio, and G. Modugno, “Observation of an Efimov spectrum in an atomic system,” Nature Phys. 5, 586–591 (2009), arXiv:0904.4453 [cond-mat.quant-gas].

4. N. Gross, Z. Shotan, S. Kokkelmans, and L. Khaykovich, “Observation of Universality in Ultracold 7Li Three-Body Recombination,” Phys. Rev. Lett. 103, 163202 (2009), arXiv:0906.4731 [cond-mat.other].

5. C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nagerl, and R. Grimm, “Evidence for Efimov quantum states in an ultracold gas of caesium atoms.” Nature 440, 315–318 (2006), arXiv:cond-mat/0512394 [cond-mat.other].

6. B. Huang, L. A. Sidorenkov, R. Grimm, and Jeremy M. Hutson, “Observation of the Second Triatomic Resonance in Efimov’s Scenario,” Phys. Rev. Lett. 112, 190401 (2014), arXiv:1402.6161 [cond-mat.quant-gas].

7. E. Nielsen, D. V. Fedorov, A. S. Jensen, and E. Garrido, “The three-body problem with short-range interactions,” Phys. Rep. 347, 373–459 (2001).
