Wirtinger numbers
and holomorphic symplectic immersions

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For any subvariety of a compact holomorphic symplectic Kähler manifold, we define the symplectic Wirtinger number $W(X)$. We show that $W(X) \leq 1$, and the equality is reached if and only if the subvariety $X \subset M$ is trianalytic, i.e. compatible with the hyperkähler structure on $M$.

For a sequence $X_1 \rightarrow X_2 \rightarrow \ldots X_n \rightarrow M$ of immersions of simple holomorphic symplectic manifolds, we show that $W(X_1) \leq W(X_2) \leq \ldots \leq W(X_n)$.

Contents

1 Introduction 1
2 Hyperkähler manifolds 2
3 Wirtinger numbers and trianalytic subvarieties 5
4 Wirtinger numbers and symplectic immersions 8

1 Introduction

It is well known that every hyperkähler manifold is holomorphic symplectic (see [2.1]). In other words, we have a natural functor $F$ from the category of hyperkähler manifolds (morphisms – all immersions compatible with metric and quaternionic action) to the category of holomorphic symplectic manifolds (morphisms – all symplectic immersions). As follows from Yau’s solution of the Calabi conjecture (Theorem 2.3), every compact holomorphic symplectic Kähler manifold admits a hyperkähler metric, which is uniquely determined by its Kähler class. Thus, $F$ establishes a bijection on objects of the appropriate categories. However, $F$ is not an equivalence of categories, in any reasonable sense: there are holomorphic symplectic immersions which are not compatible with a hyperkähler structure. One of such examples was found in [V2]. Consider an embedding $\iota$ from a K3 surface $M$ to its 3-rd Hilbert scheme of points $M^{[3]}$, mapping a point $x \in M$ to the subscheme
Wirtinger numbers

M. Verbitsky, December 7, 1998

with ideal sheaf $\mathcal{O}_M/m_x^2$, where $m_x \subset \mathcal{O}_M$ is the ideal of $x$. It was shown in [V2] that $i$ is not compatible with any hyperkähler structure on $M^{[3]}$. A number of similar symplectic immersions can be easily constructed, but it was shown that for a generic K3 surface $M$, the Hilbert scheme $M^{[n]}$ does not contain non-trivial hyperkähler submanifolds. This series of examples makes it natural to ask the following question.

**Question 1.1:** Let $\varphi : X \hookrightarrow M$ be a holomorphic symplectic immersion. Is there any way to establish whether $\varphi$ is compatible with a hyperkähler structure, using the holomorphic symplectic geometry of $X$ and $M$?

Let $M$ be a compact holomorphic symplectic Kähler manifold endowed with the unique hyperkähler structure compatible with the holomorphic symplectic structure and the Kähler class (see Remark 2.4). In the present paper, we associate to any complex subvariety $X \subset M$ a non-negative real number, called the symplectic Wirtinger number $W(X)$ (Definition 3.5). We show that $W(X) \leq 1$, and that the equality is reached if and only if $X$ is compatible with the hyperkähler structure on $M$ (Proposition 3.4). For a sequence of holomorphic symplectic immersions

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \ldots \hookrightarrow M$$

(dim $H^{2,0}(X_i) = 1$, dim $H^1(X_i) = 0$ for all $i > 1$, dim $X_1 > 0$), we prove the following inequality of Wirtinger numbers:

$$W(X_1) \leq W(X_2) \leq W(X_3) \leq \ldots$$

(Theorem 4.2) In particular, if for some $k$ the submanifold $X_k$ is compatible with the hyperkähler structure, then for all $j > k$, the manifold $X_j$ is also compatible with the hyperkähler structure (Corollary 4.3).

2 Hyperkähler manifolds

This subsection gives some of the basic and well known results and definitions from hyperkähler geometry, most of which can be found in [Bes] and in [Bea].

**Definition 2.1:** (Bes) A hyperkähler manifold is a Riemannian manifold $M$ endowed with three complex structures $I$, $J$ and $K$, such that the following holds.
(i) the metric on $M$ is Kähler with respect to these complex structures and

(ii) $I, J$ and $K$, considered as endomorphisms of the real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

Clearly, a hyperkähler manifold has a natural action of the quaternion algebra $\mathbb{H}$ on its real tangent bundle $TM$. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of $TM$ is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

**Definition 2.2:** Let $M$ be a hyperkähler manifold, and $L$ a quaternion satisfying $L^2 = -1$. The corresponding complex structure on $M$ is called an induced complex structure. The $M$, considered as a Kähler manifold, is denoted by $(M, L)$. In this case, the hyperkähler structure is called compatible with the complex structure $L$.

We say that the Kähler metric on $(M, L)$ is hyperkähler if it is compatible with a hyperkähler structure.

Let $M$ be a hyperkähler manifold; denote the Riemannian form on $M$ by $(\cdot, \cdot)$. Let the form $\omega_I := (\cdot, I(\cdot))$ be the usual Kähler form which is closed and parallel (with respect to the Levi-Civita connection). Analogously defined forms $\omega_J$ and $\omega_K$ are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form

$$\Omega := \omega_J + \sqrt{-1} \omega_K \quad (2.1)$$

is of type $(2,0)$ and, being closed, it is also holomorphic. In addition, the form $\Omega$ is nowhere degenerate, as another linear algebraic argument shows. It is called the canonical holomorphic symplectic form of the manifold $M$. Thus, for each hyperkähler manifold $M$, and an induced complex structure $L$, the underlying complex manifold $(M, L)$ is holomorphically symplectic. Calabi’s conjecture, proved by Yau in [Y], gives the converse statement.

**Theorem 2.3:** ([Bea], [Bes]) Let $X$ be a compact complex manifold equipped with a holomorphic symplectic form, and let $\omega$ be an arbitrary Kähler form on $X$. Then there exists a unique hyperkähler metric on $X$ with the Kähler form cohomologous to $\omega$. ■
**Remark 2.4:** The metric does not determine uniquely the hyperkähler structure: there might by many hyperkähler structures compatible with a given metric. However, from 2.1 it is clear that the metrics together with the holomorphic symplectic form determine the hyperkähler structure uniquely.

**Definition 2.5:** ([Bea]) A connected simply connected compact hyperkähler manifold $M$ is called simple if $M$ cannot be represented as a product of two hyperkähler manifolds:

$$M \neq M_1 \times M_2, \text{ where } \dim M_1 > 0 \text{ and } \dim M_2 > 0$$

Bogomolov proved that every compact hyperkähler manifold has a finite covering which is a product of a compact torus and several simple hyperkähler manifolds. Bogomolov’s theorem implies the following result ([Bea]):

**Theorem 2.6:** Let $M$ be a compact hyperkähler manifold. Then the following conditions are equivalent.

(i) $M$ is simple

(ii) $M$ satisfies $H^1(M, \mathbb{R}) = 0$, $H^{2,0}(M) = \mathbb{C}$, where $H^{2,0}(M)$ is the space of $(2,0)$-classes taken with respect to some induced complex structure.

Calabi-Yau theorem can be stated in a more precise fashion, using Theorem 2.6.

**Theorem 2.7:** ([Bea], [Bes]) Let $M$, $\dim \mathbb{C} M = n$, be a compact holomorphic symplectic Kähler manifold with the holomorphic symplectic form $\Omega$, a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure $I$. Assume that $\dim H^{2,0}(M) = 1$, $H^1(M, \mathbb{R}) = 0$, and

$$\int_M \omega^n = \int_M (\text{Re} \Omega)^n. \quad (2.2)$$

Then there exists a unique hyperkähler structure $(I, J, K, (\cdot, \cdot))$ on $M$ such that the cohomology class of the symplectic form $\omega_I = (\cdot, I \cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1} \omega_K$ is equal to $\Omega$.

**Proof:** By Calabi-Yau (Theorem 2.3), there exists a unique hyperkähler metric on $M$ with the Kähler class equal to $[\omega]$. Let $\mathcal{H}_1 = (I, J_1, K_1)$ be
Wirtinger numbers

M. Verbitsky, December 7, 1998

the corresponding hyperkähler structure, and \( \Omega_1 := \omega_J + \sqrt{-1} \omega_K \) its holomorphic symplectic form. Since \( \dim H^{2,0}(M) = 1 \), there exists a number \( \lambda \in \mathbb{C} \) such that \( \Omega_1 = \lambda \Omega \). A simple calculation shows that

\[
\int_M \omega^n = \int_M (\text{Re} \Omega_1)^n = \frac{1}{2^{n/2}} \binom{n}{n/2} \int_M (\Omega_1 \wedge \overline{\Omega_1})^{\frac{n}{2}}
\]

(see Lemma 3.1) and

\[
\int_M (\text{Re} \Omega)^n = \frac{1}{2^{n/2}} \binom{n}{n/2} \int_M (\Omega \wedge \overline{\Omega})^{\frac{n}{2}}.
\]

From (2.2), we obtain

\[
\int_M (\Omega \wedge \overline{\Omega})^{\frac{n}{2}} = \int_M (\Omega_1 \wedge \overline{\Omega_1})^{\frac{n}{2}}.
\]

Therefore, \( |\lambda| = 1 \). Write \( \lambda \) in form \( a + b\sqrt{-1} \), \( a^2 + b^2 = 1 \). Consider a hyperkähler structure \( \mathcal{H} \) with the same metrics, and quaternionic action given by the triple \( (I, J, K) \): \( J = aJ_1 + bK, K = -bJ_1 + aK_1 \). Clearly, the corresponding form \( \omega_J + \sqrt{-1} \omega_K \) is equal to \( \Omega \). This proves Theorem 2.7.

\[\Box\]

**Definition 2.8:** \((\text{VI})\) Let \( X \subset M \) be a closed subset of a hyperkähler manifold \( M \). Then \( X \) is called **trianalytic** if \( X \) is a complex analytic subset of \((M, L)\) for every induced complex structure \( L \).

Trianalytic subvarieties were a subject of a long study. Most importantly, consider a generic induced complex structure \( L \) on \( M \). Then all closed complex subvarieties of \((M, L)\) are trianalytic. Moreover, a trianalytic subvariety can be canonically desingularized \((\text{VIII})\), and this desingularization is hyperkähler.

## 3 Wirtinger numbers and trianalytic subvarieties

Let \( M \) be a compact complex manifold equipped with a hyperkähler structure \( \mathcal{H} = (I, J, K, (\cdot, \cdot)) \) and \( X \subset M \) a closed complex subvariety of even dimension. As usually, we denote by \( \omega = \omega_I = (\cdot, I \cdot) \) the Kähler form of \( M \), by

\[
\Omega = \omega_J + \sqrt{-1} \omega_K = (\cdot, J \cdot) + \sqrt{-1} (\cdot, K \cdot)
\]
the holomorphic symplectic form, and by $\Omega$ the complex adjoint form $\omega_J - \sqrt{-1}\omega_K$.

Later on, we shall need the following lemma.

**Lemma 3.1:** Let $M$ be a compact complex manifold equipped with a hyperkähler structure and $X \subset M$ a closed complex subvariety of even dimension $d$. Then

$$\left(\frac{d}{2}\right) \int_X \left(\frac{\Omega \wedge \overline{\Omega}}{2}\right)^{\frac{d}{2}} = \int_X \omega_J^d = \int_X \omega_K^d \quad (3.1)$$

**Proof:** The proof is pure linear algebra. By definition, $\omega_J = \frac{1}{2}(\Omega + \overline{\Omega})$, and $\omega_K = -\frac{\sqrt{-1}}{2}(\Omega - \overline{\Omega})$, Therefore,

$$\int_X \omega_J^d = \frac{1}{2^d} \sum_i \binom{d}{i} \int_X \Omega^i \wedge \overline{\Omega}^{d-i}$$

Since $X$ is complex analytic, $\int_X \eta = 0$ unless $\eta$ is of type $(d, d)$. Therefore,

$$\int_X \omega_J^d = \frac{1}{2^d} \left(\frac{d}{2}\right) \int_X \Omega^\frac{d}{2} \wedge \overline{\Omega}^\frac{d}{2}.$$ 

The proof of the second equation is analogous. 

Consider the natural $SU(2)$-action on $H^*(M)$. Since the multiplication on cohomology of $M$ is $SU(2)$-invariant, we have

$$\int_M \omega_J^{\dim_C M} = \int_M \omega_J^{\dim_C M} = \int_M \omega_K^{\dim_C M}.$$ 

By Lemma 3.1,

$$\int_M \omega_K^{\dim_C M} = \int_M \omega_J^{\dim_C M} = \left(\frac{\dim_C M}{\frac{1}{2} \dim_C M}\right) \int_M \left(\frac{\Omega \wedge \overline{\Omega}}{2}\right)^{\frac{\dim_C M}{2}}.$$ 

Therefore,

$$\int_M \omega_J^{\dim_C M} = \left(\frac{\dim_C M}{\frac{1}{2} \dim_C M}\right) \int_M \left(\frac{\Omega \wedge \overline{\Omega}}{2}\right)^{\frac{\dim_C M}{2}}.$$ 

By the same reasoning, for any trianalytic subvariety $X \subset M$, we have

$$\int_X \omega_J^{\dim_C X} = \left(\frac{\dim_C X}{\frac{1}{2} \dim_C X}\right) \int_X \left(\frac{\Omega \wedge \overline{\Omega}}{2}\right)^{\frac{\dim_C X}{2}}.$$
For an arbitrary complex subvariety $X \subset M$ of even dimension, consider the numbers
\[
\deg_\omega X := \int_X \omega^{\dim \mathbb{C} X}.
\]
and
\[
\deg_\Omega X := \left( \frac{\dim \mathbb{C} X}{2} \right)^{\dim \mathbb{C} X} \int_X \left( \frac{\Omega \wedge \overline{\Omega}}{4} \right)^{\dim \mathbb{C} X/2}.
\]

**Definition 3.2:** The number $\deg_\omega X$ is called the Kähler degree of $X$, and $\deg_\Omega X$ is called the symplectic degree of $X$.

**Remark 3.3:** Notice that
\[
\deg_\omega X = (\dim \mathbb{C} X)! \text{Vol}(X),
\]
where Vol$(X)$ denotes the volume of $X$ taken with respect to the Riemannian structure on $X \subset M$.

The following fundamental inequality lays the groundwork for all manipulations with trianalytic subvarieties. Its Kähler version, the original Wirtinger’s inequality, is well known (see, e.g. [Sto]).

**Proposition 3.4:** (Wirtinger inequality in holomorphic symplectic setting) Let $M$ be a compact complex manifold equipped with a hyperkähler structure and $X \subset M$ a closed complex subvariety of even dimension $d$. Then the following inequality holds
\[
\deg_\omega X \geq |\deg_\Omega X|.
\] (3.2)

Moreover, (3.2) is strict unless $X$ is trianalytic.

**Proof:** This proof is essentially contained in [V1]. Let $\deg_{\omega,j} X$ denote the integral
\[
\int_X \omega_j^{\dim \mathbb{C} X}.
\]
By Lemma 3.1,
\[
\deg_\Omega X = \deg_{\omega,j} X.
\]
Therefore, (3.2) can be written in the form

$$|\deg_{\omega_J} X| \leq \deg_{\omega} X.$$  

By Remark 3.3,

$$\deg_{\omega} X = 2^d \operatorname{Vol}(X)$$

By the classical Wirtinger inequality (\cite{Sto}, page 7),

$$|\deg_{\omega_J} X| \leq d! \operatorname{Vol}(X),$$

and the equality is reached only if the subset $X \subset M$ is complex analytic with respect to $J$. This proves (3.2). Finally, if (3.2) is not strict, then $X$ is complex analytic with respect to $J$ and $I$. It is easy to show that such subvarieties are trianalytic. \hfill \blacksquare

**Definition 3.5:** In assumptions of Proposition 3.4, we define the Wirtinger's number $W(X)$ of the subvariety $X \subset M$ as

$$W(X) := \sqrt{\frac{|\deg_\Omega(X)|}{\deg_\omega(X)}}.$$  \hfill (3.3)

Proposition 3.4 shows that $W(X) \leq 1$, and this inequality is strict unless $X$ is trianalytic.

**Remark 3.6:** Notice that (3.3) can be used to define the Wirtinger number for any compact holomorphically symplectic Kähler manifold.

### 4 Wirtinger numbers and symplectic immersions

Let $M$ be a complex manifold. We say that $M$ is **holomorphically symplectic** if $M$ is equipped with nowhere degenerate holomorphic symplectic form $\Omega_M$.

**Definition 4.1:** Let $\varphi : X \hookrightarrow M$ be an immersion of holomorphic symplectic manifolds. The map $\varphi$ is called a **holomorphic symplectic immersion** if $\varphi^*\Omega_M = \Omega_X$, where $\varphi^*$ denotes the pullback of differential forms.
Wirtinger numbers, defined in Section 3, give an interesting invariant of holomorphic symplectic immersions.

**Theorem 4.2:** Let

\[ X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n \]

be a sequence of holomorphic symplectic immersions of Kähler manifolds, \( \dim X_1 > 0 \). Assume that all manifolds \( X_i \) are compact and Kähler, and the Kähler structures are compatible with the immersions. Assume also that for all \( i > 1 \), the manifold \( X_i \) is simple. Let \( W(X_i) \) be the Wirtinger numbers of \( X_i \), in the sense of Remark 3.6. Then

\[ W(X_1) \leq W(X_2) \leq \ldots \leq W(X_n). \]

**Corollary 4.3:** Let \( M \) be a compact hyperkähler manifold, and \( X \subset M \) a holomorphic symplectic subvariety which is simple. Assume that \( X \) contains a non-trivial subvariety which is trianalytic in \( M \). Then \( X \) is trianalytic.

**Proof:** Denote the trianalytic subvariety of \( X \) by \( Y \). By Proposition 3.4, \( W(Y) = W(M) \). By Theorem 4.2, \( W(Y) \leq W(X) \leq W(M) \). Therefore, \( W(Y) = W(X) = W(M) \). Applying Proposition 3.4 again, we obtain that \( X \) is trianalytic.

**Proof:** The proof of Theorem 4.2 uses the methods of hyperkähler geometry. Let \( X_1 \hookrightarrow X_2 \) be a holomorphic symplectic immersion. It suffices to show that \( W(X_1) \leq W(X_2) \). From the definition of Wirtinger numbers, it follows that

\[ \deg_{\Omega} X_2 = \deg_{\omega'} X_2, \tag{4.1} \]

where \( \omega' \) is the Kähler form on \( X_2 \) given by \( \omega' = W(X_2) \omega \). By Theorem 2.7, the equality (4.1) implies that there exists a hyperkähler structure on \( X_2 \) such that \( [\omega'] \) is its Kähler class and \( \Omega \) its holomorphic symplectic form. Using Proposition 3.4, we obtain that

\[ |\deg_{\Omega} X_1| \leq \deg_{\omega'} X_1. \tag{4.2} \]

On the other hand, by the definition of \( \omega' \), we have

\[ \deg_{\omega'} X_1 = \deg_{\omega} X_1 \cdot W(X_2)^{\dim C X_1}. \tag{4.3} \]
Dividing both sides of (4.2) by \(\deg_{\omega} X_1\) and using (4.3), we obtain

\[
W(X_2)^{\dim C_{X_1}} \geq \frac{\deg_{\Omega} X_1}{\deg_{\omega} X_1} = W(X_1)^{\dim C_{X_1}}
\]

This proves Theorem 4.2. 

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