Invariance of the Gibbs Measures for Periodic Generalized Korteweg-de Vries Equations

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Abstract. In this paper, we study the Gibbs measures for periodic generalized Korteweg-de Vries equations (gKdV) with quartic or higher nonlinearities. In order to bypass the analytical ill-posedness of the equation in the Sobolev support of the Gibbs measures, we establish deterministic well-posedness of the gauged gKdV equations within the framework of the Fourier-Lebesgue spaces. Our argument relies on bilinear and trilinear Strichartz estimates adapted to the Fourier-Lebesgue setting. Then, following Bourgain’s invariant measure argument, we construct almost sure global-in-time dynamics and show invariance of the Gibbs measures for the gauged equations. These results can be brought back to the ungauged side by inverting the gauge transformation and exploiting the invariance of the Gibbs measures under spatial translations. We thus complete the program initiated by Bourgain (1994) on the invariance of the Gibbs measures for periodic gKdV equations.

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1. Introduction

We study the Cauchy problem for the generalized Korteweg-de Vries equation (gKdV) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

\[
\begin{align*}
\partial_t u + \partial_x^3 u &= \pm \partial_x (u^k), \\
|u|_{t=0} &= u_0, \\
(t, x) &\in \mathbb{R} \times \mathbb{T},
\end{align*}
\]

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where \( k \geq 2 \) is an integer. When \( k = 2 \) and \( k = 3 \), (1.1) corresponds to the well-known Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations, respectively. These two equations are known to be completely integrable, therefore satisfying infinitely many conservation laws, which is no longer true for (1.1) with \( k \geq 4 \).

The gKdV equation (1.1) can be reformulated as a Hamiltonian system

\[
\partial_t u = \partial_x \frac{\delta H}{\delta u},
\]

where \( \delta H/\delta u \) denotes the Fréchet derivative and the Hamiltonian is given by

\[
H(u) := \frac{1}{2} \int_T (\partial_x u)^2 \, dx \pm \frac{1}{k+1} \int_T u^{k+1} \, dx.
\]

In particular, \( H(u) \) is conserved under the dynamics of (1.1). Note that the mean \( \int_T u \, dx \) and the mass \( M(u) = \int_T u^2 \, dx \) are also conserved quantities. Due to the conservation of the mean, we will restrict our discussion to mean zero initial data. See Remark 1.7 for further details on the non-zero mean case.

In view of the Hamiltonian structure of gKdV (1.1), we expect the Gibbs measure \( \mu \) formally defined by

\[
d\mu = Z^{-1} e^{-H(u)} \, du = Z^{-1} e^{\pm \frac{1}{k+1} \int_T u^{k+1} \, dx} e^{-\frac{1}{2} \int_T (\partial_x u)^2 \, dx} \, du,
\]

to be invariant under the dynamics of gKdV (1.1). In this paper, we complete the program initiated by Bourgain in [4] by establishing invariance of the Gibbs measure \( \mu \) in (1.2) (under suitable normalization) for any \( k \geq 4 \). Our result builds upon the work of Bourgain for KdV (\( k = 2 \)) and mKdV (\( k = 3 \)) [3, 4], and of Richards [52] for quartic gKdV (\( k = 4 \)).

The construction of Gibbs measures for Hamiltonian PDEs was initiated by Lebowitz-Rose-Speer [35] in the context of the nonlinear Schrödinger equation and has since been successfully pursued for other equations, see [4, 5, 6, 56, 57, 10, 11, 45, 43, 58, 55, 40, 8, 7, 19, 52, 48, 51, 20, 49, 21, 28] and references therein. The expression in (1.2) is only formal, but it can be made rigorous by interpreting the Gibbs measure \( \mu \) as a probability measure which is absolutely continuous with respect to the Gaussian measure \( \rho \)

\[
d\rho = Z_0^{-1} e^{-\frac{1}{2} \int_T (\partial_x u)^2 \, dx} \, du.
\]

The measure \( \rho \) can be seen as the induced probability measure under the map

\[
\omega \mapsto u^\omega(x) = \sum_{n \in \mathbb{Z}_*} g_n(\omega) e^{inx},
\]

where \( \{g_n\}_{n \in \mathbb{Z}_*}, \mathbb{Z}_* = \mathbb{Z} \setminus \{0\} \), is a sequence of complex-valued independent Gaussian random variables on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), satisfying \( g_{-n} = \overline{g_n} \). Note that \( u \) defined in (1.4) lies in \( \bigcap_{s<\frac{1}{2}} H^s(\mathbb{T}) \) almost surely. Consequently, the support of \( \rho \) and of \( \mu \) (when well-defined) is included in this set.

In order to discuss the invariance of the Gibbs measure \( \mu \), we must first construct a (globally-in-time) well-defined flow for gKdV (1.1) on the support of \( \mu \). Before proceeding, we recall some known well-posedness results of (1.1). In [3], Bourgain introduced the Fourier restriction norm method and proved local well-posedness of KdV in \( L^2(\mathbb{T}) \supset \text{supp}\, \mu \), which was immediately extended to global well-posedness due to the conservation of mass. Following the same method, for mKdV Bourgain [4] established its local well-posedness in
over, he rigorously established the invariance of the Gibbs measure measures corresponding to the truncated dynamics to globalize solutions of mKdV. More-

Instead, Bourgain used a probabilistic argument to construct global-in-time solutions of mKdV. In the seminal work [4], he exploited the invariance of the finite dimensional Gibbs measures for mKdV equation (1.7). Note that the two equations (1.7) and (1.1) are equivalent

Unfortunately, the conservation laws of mKdV were not sufficient to globalize solutions. Unfortunately, the conservation laws of mKdV were not sufficient to globalize solutions.

Here, with the solution map \( \Psi(t) \) of (1.1) given at least almost surely with respect to \( \mu \), invariance of \( \mu \) is understood as

\[
\mu(\Psi(-t)A) = \mu(A),
\]

for any measurable set \( A \) and \( t \in \mathbb{R} \). This approach is known as Bourgain’s invariant measure argument. The main breakthrough in [4] was the globalization argument, in particular, using the formal invariance of the Gibbs measure \( \mu \) as a substitute for a conservation law.

Regarding (1.1) with \( k \geq 4 \), in [3], Bourgain proved local existence of solutions (without uniqueness) in \( H^s(\mathbb{T}) \) for \( s \geq 1 \). Later, Staffilani [53] upgraded this result to local well-posedness in \( H^s(\mathbb{T}) \), \( s \geq 1 \) (and hence global well-posedness under the presence of a priori \( H^1 \)-control), which extended the local well-posedness result for \( s > \frac{3}{2} \) obtained by the classical approach not exploiting dispersion (see Kato’s results [32] [33] on quasilinear hyperbolic systems). In [17], Colliander-Keel-Staffilani-Takaoka-Tao further refined the relevant nonlinear estimate in the Fourier restriction norm to establish local well-posedness in \( H^s(\mathbb{T}) \) for \( s \geq \frac{1}{4} \) and used the \( I \)-method to construct global solutions for \( s > \frac{3}{8} \) and \( k = 4 \) (see also [31] [11]).

The common strategy in [33] [53] [17] is to study the following gauged gKdV equation (\( \mathcal{G} \)-gKdV):

\[
\partial_t u + \partial_x^3 u = \pm \partial_x (u^k - kP_0(u^{k-1})u),
\]

where \( P_0 \) denotes the mean \( P_0(f) = \int_T f \, dx \). In fact, compared with the original equation (1.1), certain problematic frequency interactions have been removed in the nonlinearity of the gauged equation (1.7). Note that the two equations (1.7) and (1.1) are equivalent\(^1\) in

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\(^1\)After [4], local well-posedness for mKdV on \( \mathbb{T} \) has been obtained in less regular Sobolev and Fourier-Lebesgue spaces including the support of the Gibbs measure. Furthermore, the local-in-time flow constructed on these spaces has been globalized (in the deterministic sense) by exploiting complete integrability of mKdV. For details, see, e.g., the recent works [13] [11] by the first author and references therein.

\(^2\)The equivalence is obvious for smooth solutions. Even for \( u \in C([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})) \), we can show the equivalence when \( s \geq 0 \) and \( s + \frac{3}{4} > 1 - \frac{1}{p} \). In fact, since the embedding \( \mathcal{F}L^{s,p}(\mathbb{T}) \hookrightarrow L^1(\mathbb{T}) \) holds in this case, the gauge transformation (1.5) can be defined in the sense of spatial translation of an \( L^1 \)-function for each \( t \), and it is continuous on \( C([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})) \), as we will show in Lemma 3.3. Note also that both of the nonlinearities \( \pm \partial_x (u^k) \) and \( \pm \partial_x (u^k - kP_0(u^{k-1})u) \) make sense as distributions in \( C([-T,T];\mathcal{F}L^{-1}\frac{1}{p} + \beta - \beta'(\mathbb{T})) \) in view of the embedding \( L^1 \hookrightarrow \mathcal{F}L^{-1}\frac{1}{p} + \beta - \beta'(\mathbb{T}) \). Then, it is not hard to verify that \( u \in C([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})) \) is a mild solution of (1.1) (i.e., \( u \) satisfies the associated integral equation in \( \mathcal{F}L^{s,p}(\mathbb{T}) \) pointwise in \( t \) if and only if \( v = G_0\partial_t u \) is a mild solution of (1.7). In Theorem 1.1 below, we only consider \( s,p \) within this range.
the following sense: $u$ is a solution of (1.1) if and only if $v = G(u)$ is a solution of (1.7), where the gauge transformation $G = G_{0,t}$ is given by

$$G_{0,t}(u)(t,x) = u\left(t, x \mp k \int_{0}^{t} P_{0}(u^{k-1}(t')) \, dt'\right).$$  \hfill (1.8)

Concerning the construction of invariant Gibbs dynamics, we first recall that the Gibbs measure is supported in $H^{s}(\mathbb{T})$ for $s < \frac{1}{2}$. However, this range is exactly where the (gauged) gKdV equation on $\mathbb{T}$ is known to be analytically ill-posed. In fact, it has been shown in $[6, 15, 17]$ that the data-to-solution map fails to be $C^{k}$-continuous, which means that one cannot use a contraction mapping argument to construct the flow. To bypass this difficulty, Richards $[52]$, following the argument in $[4, 18]$, established probabilistic local well-posedness of (1.1) with $k = 4$ in $H^{s}(\mathbb{T})$ for $s < \frac{1}{2}$, and proved invariance of the Gibbs measure under the flow of (1.7). To the knowledge of the authors, the question of invariance of $\mu$ in the sense of (1.6) for $k \geq 5$ remains open (see Remark 1.6(ii) for a result on invariance in a weaker sense).

The main difficulty in applying the probabilistic approach in $[52]$ to gKdV with higher order nonlinearities is that the number of cases involved in analyzing the nonlinearity increases with its degree. It may be possible to bypass this difficulty and implement a probabilistic well-posedness argument in a unified manner for $k \geq 4$ by adapting the method recently introduced by Deng-Nahmod-Yue in $[20]$, where they handled the nonlinear Schrödinger equations in dimension two with an arbitrarily high power. However, if we have deterministic well-posedness in the support of the Gibbs measure, we can obtain a better approximation property for the solutions, when compared to probabilistic methods. In fact, by establishing continuity of the solution map, any approximating sequence of initial data leads to a good approximating sequence for solutions of gKdV. The probabilistic methods above are usually restricted to particular approximations of initial data, such as those obtained by truncation to low frequencies.

In this paper, we shall take an approach based on deterministic well-posedness as in $[4]$ for the case $k = 3$, namely we establish deterministic local well-posedness of $G$-gKdV (1.7) in the Fourier-Lebesgue spaces containing the support of the Gibbs measure $\mu$, i.e., $FL^{s,p}(\mathbb{T})$ with $(s - 1)p < -1$. It is worthwhile to observe that the counterexample to analytical well-posedness given in $[17]$ also applies to the Cauchy problem in the Fourier-Lebesgue spaces $FL^{s,p}(\mathbb{T})$ for $s < \frac{1}{2}$ and any $p$. Note that for $p = 2$, the regularity criterion $(s - 1)p < -1$ implies $s < \frac{1}{2}$ for which a contraction argument does not work. However, by choosing $p > 2$, it is possible to take regularity $s$ satisfying both $(s - 1)p < -1$ and $s > \frac{1}{2}$. Such a choice guarantees that $FL^{s,p}(\mathbb{T})$ contains the support of the Gibbs measure while avoiding the counterexample in $[17]$.$^4$ Hence, our first result is the following local well-posedness in $FL^{s,p}(\mathbb{T})$ for some $s, p$ which meet the above requirements. Indeed, the main difficulty in our approach lies in this step.

$^3$Although $G$ is a spatial translation only when considered as a map on space-time functions, we intend to follow the literature $[17, 52, 48]$ and call it a gauge transformation. We also refer to the transformed equation (1.7) as the gauged equation. See Remark (1.8) for further discussion.

$^4$This idea of bypassing some ill-posedness results in $L^{2}$-based Sobolev spaces by instead considering alternative spaces that contain the support of the Gibbs measure can be found in the literature; see, e.g., $[44, 46, 47, 23]$. 
**Theorem 1.1.** For $2 < p < \infty$ there exists $\frac{1}{2} < s_*(p) < 1 - \frac{1}{p}$ such that $G$-gKdV \((\ref{eq:1.7})\) is locally well-posed in $FL^{s,p}(\mathbb{T})$ for any $s > s_*(p)$. Moreover, by inverting the gauge transformation, we also obtain local well-posedness of the gKdV equation \((\ref{eq:1.1})\) in $FL^{s,p}(\mathbb{T})$.

**Remark 1.2.** We start by clarifying our notion of local well-posedness of $G$-gKdV \((\ref{eq:1.7})\) in $FL^{s,p}(\mathbb{T})$: for any $u_0 \in FL^{s,p}(\mathbb{T})$ there exists $T = T(||u_0||_{FL^{s,p}}) > 0$ and a unique solution $u$ in $X^{s,\frac{1}{2}}_{p,2}(T) \cap X^{s,0}_{p,1}(T) \hookrightarrow C([-T,T];FL^{s,p}(\mathbb{T}))$ (see Definition \((\ref{def:2.1})\)) which satisfies the Duhamel formulation of \((\ref{eq:1.7})\):

$$u(t) = S(t)u_0 + \int_0^t S(t-t')\partial_x(u^k - kP_0(u^{k-1})u)(t')\,dt', \quad t \in [-T,T],$$

where $S(t)$ denotes the linear propagator. Moreover, the data-to-solution map $\Phi$ is (locally Lipschitz) continuous. Note that $G_{0,t}$ is a bijection on $C([-T,T];FL^{s,p}(\mathbb{T}))$ with inverse given by

$$G_{0,t}^{-1}(u)(t,x) = u\left(t, x \pm k \int_0^t P_0(u^{k-1}(t'))\,dt'\right).$$

Consequently, Theorem \((\ref{thm:1.1})\) asserts the following notion of local well-posedness for the original gKdV equation \((\ref{eq:1.1})\): for any $u_0 \in FL^{s,p}(\mathbb{T})$ there exist $T = T(||u_0||_{FL^{s,p}}) > 0$ and a unique solution $u \in G_{0,t}^{-1}(X^{s,\frac{1}{2}}_{p,2}(T) \cap X^{s,0}_{p,1}(T)) \subset C([-T,T];FL^{s,p}(\mathbb{T}))$ which satisfies the Duhamel formulation of \((\ref{eq:1.1})\). The data-to-solution map $\Psi$ of gKdV \((\ref{eq:1.1})\) can be defined as $\Psi(t) = R_t \circ G_{0,t}^{-1} \circ \Phi$, where $R_t$ denotes the evaluation map at time $t$. The map $\Psi(t)$ is defined on a neighborhood of the origin in $FL^{s,p}(\mathbb{T})$ and is continuous, but not Lipschitz or uniformly continuous in the $FL^{s,p}$ topology due to the properties of $G_{0,t}^{-1}$. Moreover, it satisfies the group property $\Psi(t+s) = \Psi(t)\Psi(s)$ for any $t,s$. See Appendix A for more details on this map.

**Remark 1.3.** The lower bound $s_*(p)$ in Theorem \((\ref{thm:1.1})\) is the same as that for the main multilinear estimates in Proposition \((\ref{prop:3.1})\. We see from the proof of Proposition \((\ref{prop:3.1})\) that $s_*(p)$ can be chosen as

$$s_*(p) = 1 - \frac{1}{p} - \frac{\min(p-2,2)}{\max(2(k-1)p,8p)}.$$

This lower bound is not likely to be sharp; indeed, the critical exponents $s,p$ suggested by the scale invariance of the equation satisfy $s = 1 - \frac{1}{p} - \frac{2}{k-1}$. It may be possible to improve the range of $s$ by adapting the method of \([21,14]\), for instance. However, in this paper, we do not intend to determine the optimal range of $s$ and $p$ for local well-posedness, since our focus is on constructing a global-in-time flow on the support of the Gibbs measure.

We prove Theorem \((\ref{thm:1.1})\) by applying the Fourier restriction norm method with the $X^{s,b}$ spaces adapted to the Fourier-Lebesgue setting (see Definition \((\ref{def:2.1})\). The method reduces to establishing a fundamental nonlinear estimate, where the main difficulty lies in controlling the derivative in the nonlinearity. To overcome this derivative loss, we want to exploit the multilinear dispersion by analyzing the phase function

$$\phi_k(n,n_1,\ldots,n_k) = n^3 - n_1^3 - \ldots - n_k^3.$$
on the hyperplane $n = n_1 + \ldots + n_k$. For KdV ($k = 2$) and mKdV ($k = 3$), the corresponding phase functions $\phi_2$ and $\phi_3$ are known to factorize, providing an explicit characterization of the resonant set, where $\phi_k(n, n_1, \ldots, n_k) = 0$. Unfortunately, such factorizations are no longer available for $\phi_k$ when $k \geq 4$, complicating the study of the resonant frequency regions. In fact, the failure of analyticity of the solution map in $H^s(\mathbb{T})$ and $\mathcal{F}L^{s,p}(\mathbb{T})$ with $s < \frac{1}{2}$ in [17] is due to the failure of the corresponding nonlinear estimate in the region where $|\phi_k(n, n_1, \ldots, n_k)| \ll \max(|n_1|, \ldots, |n_k|)$. Our approach is inspired by the “bilinear+multilinear” strategy in the work of Colliander-Keel-Staffilani-Takaoka-Tao [16, 17]. Instead of starting by showing a bilinear estimate, we first pursue a more careful description of the frequency space by comparing $\phi_k(n, n_1, \ldots, n_k)$ to the Gaussian measure $\mu$ to be integrable with respect to the Gaussian measure $\rho$ in (1.3). In the defocusing case, ‘+’ sign in (1.1) and odd $k \geq 3$, it follows from the Sobolev embedding that $\mu$ is a well-defined probability measure on $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \leq p \leq \infty$ and $s \in (1 - \frac{1}{p} - \frac{1}{k+1}, 1 - \frac{1}{p})$. However, in the non-defocusing case, ‘−’ sign in (1.1) or even $k \geq 2$, the quantity $e^{+\frac{1}{k+1}\int_T u^{k+1}d\tau}$ is unbounded on $\mathcal{F}L^{s,p}(\mathbb{T})$ and the measure (1.2) is not normalizable. To bypass this difficulty, Lebowitz-Rose-Speer [35] and Bourgain [4] introduced a mass cutoff and studied the following Gibbs measure instead

$$d\mu = Z^{-1}1_{\{|u|_{L^2} \leq R\}} e^{-H(u)} du. \quad (1.9)$$

They showed that the measure $\mu$ in (1.9) is only normalizable for $1 \leq k \leq 5$ and an appropriate choice of $R$. The normalizability at the optimal threshold for $k = 5$ was recently shown by Oh-Sosoe-Tolomeo [50]. See Theorem 4.2 for more details.

Following the strategy in [4], we start by proving the invariance of the Gibbs measures associated with the following truncated dynamics

$$\left\{ \begin{array}{l}
\partial_t u_N + \partial_x^2 u_N = \pm \mathcal{P}_{\leq N} \partial_x ((\mathcal{P}_{\leq N} u_N)^k - k \mathcal{P}_0 ((\mathcal{P}_{\leq N} u_N)^{k-1}) \mathcal{P}_{\leq N} u_N), \\
u_N|_{t=0} = u_0,
\end{array} \right. \quad (1.10)$$

where $\mathcal{P}_{\leq N}$ denotes the Dirichlet projection onto frequencies $\{|n| \leq N\}$. Unfortunately, the Hamiltonian structure of (1.10) is disrupted by the gauge transformation. Therefore, the invariance of the corresponding Gibbs measures does not follow immediately from Liouville’s Theorem. A similar difficulty was found by Nahmod-Oh-Rey-Bellet-Staffilani when studying the Gibbs measure for derivative nonlinear Schrödinger equation in [40]. See Remark 1.3 for additional details. As a consequence, we must establish conservation of the mass and of the Hamiltonian for (1.10) as well as the invariance of the finite dimensional

Before discussing its invariance, we must guarantee that the Gibbs measure $\mu$ is a well-defined probability measure on $\mathcal{F}L^{s,p}(\mathbb{T})$. In particular, we need the weight $e^{+\frac{1}{k+1}\int_T u^{k+1}d\tau}$ to be integrable with respect to the Gaussian measure $\rho$ in (1.3). In the defocusing case, ‘+’ sign in (1.1) and odd $k \geq 3$, it follows from the Sobolev embedding that $\mu$ is a well-defined probability measure on $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \leq p \leq \infty$ and $s \in (1 - \frac{1}{p} - \frac{1}{k+1}, 1 - \frac{1}{p})$. However, in the non-defocusing case, ‘−’ sign in (1.1) or even $k \geq 2$, the quantity $e^{+\frac{1}{k+1}\int_T u^{k+1}d\tau}$ is unbounded on $\mathcal{F}L^{s,p}(\mathbb{T})$ and the measure (1.2) is not normalizable. To bypass this difficulty, Lebowitz-Rose-Speer [35] and Bourgain [4] introduced a mass cutoff and studied the following Gibbs measure instead

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Lebesgue measures under the flow of $\mathbf{(1.10)}$. Then, using the invariance of the finite dimensional Gibbs measures for $\mathbf{(1.10)}$, we extend solutions of $\mathbf{(1.7)}$ globally-in-time and also establish the invariance of $\mu$ under its flow.

**Theorem 1.4.** Assume one of the following conditions:

(a) defocusing case: ‘$+$’ sign in $\mathbf{(1.1)}$ and $k$ odd;
(b) non-defocusing case: ‘$+$’ sign in $\mathbf{(1.1)}$ and $k = 4$, or ‘$-$’ sign in $\mathbf{(1.1)}$ and $3 \leq k \leq 5$, with mass $0 < R \leq \|Q\|_{L^2(\mathbb{R})}$ if $k = 5$ and $0 < R < \infty$ otherwise. Here, $Q$ denotes the (unique) optimizer of the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ $\mathbf{(1.3)}$ with $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|\partial_x Q\|_{L^2(\mathbb{R})}^2$.

Then, the $\mathcal{G}$-gKdV equation $\mathbf{(1.7)}$ is almost surely globally well-posed with respect to the Gibbs measure $\mu$ defined by $\mathbf{(1.2)}$ for the case (a) or $\mathbf{(1.9)}$ for the case (b): More precisely, for $2 < p < \infty$, there exists a $\mu$-measurable set $\Sigma \subset \bigcap_{s < -\frac{1}{p}} FL^{s,p}(\mathbb{T})$ of full $\mu$-measure such that for every $u_0 \in \Sigma$, the $\mathcal{G}$-gKdV equation $\mathbf{(1.7)}$ with initial data $u_0$ has a uniquely defined global-in-time solution $u \in \bigcap_{s < -\frac{1}{p}} C(\mathbb{R}; FL^{s,p}(\mathbb{T}))$. The obtained solution map $\Phi(t): u_0 \mapsto u(t)$ for $\mathcal{G}$-gKdV defined on $\Sigma$ is $\mu$-measurable and satisfies the flow property

\[
\Phi(t)\Sigma = \Sigma \text{ for all } t \in \mathbb{R}, \quad \Phi(t+s) = \Phi(t)\Phi(s) \text{ for all } t, s \in \mathbb{R}. \tag{1.11}
\]

Moreover, the Gibbs measure $\mu$ is invariant under the flow of $\mathcal{G}$-gKdV $\mathbf{(1.7)}$ in the sense that $\mu(\Phi(-t)A) = \mu(A)$ for any $\mu$-measurable set $A \subset \Sigma$ and $t \in \mathbb{R}$.

By inverting the gauge transformation and exploiting the invariance of the Gibbs measure under spatial translations, we obtain our main result.

**Theorem 1.5.** Under the assumptions of Theorem $\mathbf{1.4}$ for every $u_0$ in the set $\Sigma$ of full $\mu$-measure given in Theorem $\mathbf{1.4}$ the gKdV equation $\mathbf{(1.1)}$ with initial data $u_0$ has a uniquely defined global-in-time solution $u \in \bigcap_{s < -\frac{1}{p}} C(\mathbb{R}; FL^{s,p}(\mathbb{T}))$. Moreover, the obtained solution map $\Psi(t)$ has the same flow property as in $\mathbf{(1.11)}$, and the Gibbs measure $\mu$ is invariant under $\Psi(t)$ in the sense that $\mathbf{(1.9)}$ holds for any $\mu$-measurable set $A \subset \Sigma$ and $t \in \mathbb{R}$.

**Remark 1.6.** (i) Theorem $\mathbf{1.5}$ extends the results of Bourgain $\mathbf{41}$ and Richards $\mathbf{52}$ on the invariance of the Gibbs measure $\mu$. Our work establishes the first result on the invariance of the Gibbs measure $\mu$ in the sense of $\mathbf{(1.6)}$ for large values $k \geq 5$.

(ii) A weaker notion of invariance of $\mu$ for $k \geq 5$ was established by Oh-Richards-Thomann in $\mathbf{48}$. They constructed almost sure global dynamics for gKdV $\mathbf{(1.1)}$, without uniqueness, and established invariance in the following sense: for any $t \in \mathbb{R}$, the law $\mathcal{L}(u(t))$ of the random variable $u(t)$ which solves $\mathbf{(1.1)}$ is given by the Gibbs measure $\mu$. They followed the compactness argument introduced by Burq-Thomann-Tzvetkov $\mathbf{9}$, exploiting the invariance of the truncated measures to construct a tight sequence of space-time measures. Although their result can be easily extended to the Fourier-Lebesgue spaces in Theorem $\mathbf{1.3}$ we do not know if our solutions coincide with those in $\mathbf{48}$. Due to the lack of uniqueness of solutions in $\mathbf{48}$ and the conditional uniqueness of our result, we cannot directly compare these solutions.

(iii) In the optimal threshold for the non-defocusing case, i.e., when $k = 5$ with ‘$-$’ sign in $\mathbf{(1.1)}$ and $R = \|Q\|_{L^2(\mathbb{R})}$, we can still realize the Gibbs measure $\mu$ as a weighted Gaussian measure. However, we do not have the $L^q(\mathcal{R})$-integrability of the corresponding density for
q > 1, which prevents us from directly applying Bourgain’s invariant measure argument. As mentioned in [50], this difficulty can be bypassed by using the corresponding result for \( R = \|Q\|_{L^2(\mathbb{R})} - \delta, \delta > 0, \) and the dominated convergence theorem as \( \delta \to 0. \) See Appendix C for further details.

**Remark 1.7.** For simplicity, we have restricted our discussion to mean zero initial data. However, our results extend to general data in \( \mathcal{F}L^{s,p}(\mathbb{T}) \) without the mean zero condition with the following modifications. Instead of the Gaussian measure \( \rho \) in (1.3), we consider the measure \( \overline{\rho} \) induced by the Ornstein-Uhlenbeck loop

\[
\sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx},
\]

which has the formal density

\[
\overline{\rho} = Z_0^{-1} e^{-\frac{1}{2} \int_\mathbb{T} u^2 \, dx} \int_\mathbb{T} (\partial_x u)^2 \, dx \, du.
\]

The properties of the Gaussian measure \( \rho \) discussed in Section 3 including Lemma 4.1 also hold for \( \overline{\rho}; \) see [40]. We can then define the Gibbs measure \( \mu \) with \( \overline{\rho} \) as the underlying Gaussian measure, which is still normalizable (see Remarks 1.2 and 4.1 in [50]). The main difference in the argument comes from the local well-posedness of (1.7) without the mean zero condition, since the nonlinear estimates in Section 3 hold only for mean zero functions. This is resolved by defining the solution map \( \Phi(t) \) as a concatenation of the solution maps \( \Phi_\alpha(t) \) of (1.7) with prescribed mean \( \alpha. \) This construction is further discussed in Appendix B. Since the proof of the a.s. global well-posedness and invariance of the Gibbs measure without prescribing zero mean is analogous to the argument in Section 4, we omit the details.

**Remark 1.8.** (i) In [40], Nahmod-Oh-Rey-Bellet-Staffilani studied the derivative nonlinear Schrödinger equation (DNLS) on the one-dimensional torus. In particular, they constructed a weighted Wiener measure, invariant under the gauged dynamics, and established almost sure global well-posedness of DNLS in the support of said measure. Unlike for gKdV (1.1), local well-posedness in the support of the measure was already available in [27]. Consequently, the main difficulty arose in the globalization process. The energy associated to the gauged dynamics was no longer conserved for truncated solutions, which required an approach reminiscent of the I-method to instead establish almost invariance of the truncated measures. In our case, the main difficulty is in establishing the local well-posedness of \( G \)-gKdV (1.7) in the Fourier-Lebesgue support of the measure, which was readily available for DNLS. Although we also have to prove the invariance of the finite-dimensional Lebesgue measure with respect to the truncated dynamics in (1.10), unlike in [40], the Hamiltonian is still conserved and we can easily show invariance of the Gibbs measures associated to (1.10). (ii) One additional difficulty in establishing invariance of the Gibbs measure \( \mu \) under the flow of (1.1) was due to the gauge transformation. The map \( \tilde{G}_0, \) for \( k \geq 4 \) is only a gauge transformation when acting on space-time functions. This is a sharp contrast with DNLS, whose more involved gauge transformation is well defined as a map on \( \mathcal{F}L^{s,p}(\mathbb{T}) \), allowing the authors in [40] to consider the push-forward of the measure \( \mu \) by the gauge transformation. This topic was further explored for DNLS in a subsequent work [41].
In this paper, we bypass the difficulty associated with the gauge transformation by exploiting the invariance of the Gibbs measure under spatial translations.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2, we introduce relevant notations, linear estimates and auxiliary results needed to establish the main nonlinear estimates. Theorem 1.1 is shown in Section 3, where we decompose the nonlinearity into non-resonant and resonant contributions, and establish corresponding nonlinear estimates. In Section 4, we prove almost sure global well-posedness of (1.7) and the invariance of the Gibbs measure under the corresponding flow. Moreover, we establish invariance of $\mu$ under the flow of gKdV (1.1). Some results on properties of the gauge transformation and of the solution map are included in Appendix A. The construction of the solution map without the mean zero condition is discussed in Appendix B, and further details on the threshold case ($k = 5$ and $R = \| \mathcal{Q} \|_{L^2(\mathbb{R})}$ in Theorem 1.4 (b)) can be found in Appendix C. Proofs of some lemmas in Section 4 are given in Appendix D.

2. Notations, function spaces and linear estimates

We start by introducing some useful notation. For non-negative quantities $A, B$, let $A \lesssim B$ denote an estimate of the form $A \leq CB$ for some constant $C > 0$. Similarly, $A \sim B$ will denote $A \lesssim B$ and $B \lesssim A$, while $A \ll B$ will denote $A \leq \varepsilon B$, for some small positive constant $\varepsilon$. The notations $a^+$ and $a^-$ represent $a + \varepsilon$ and $a - \varepsilon$ for arbitrarily small $\varepsilon > 0$, respectively. Lastly, our conventions for the Fourier transform are as follows. The Fourier transform of $u$: $\mathbb{R} \times \mathbb{T} \to \mathbb{R}$ with respect to the space variable is given by

$$\mathcal{F}_x u(t, n) = \hat{u}(t, n) = \int_{\mathbb{T}} u(t, x) e^{-2\pi i n x} dx.$$ 

The Fourier transform of $u$ with respect to the time variable is given by

$$\mathcal{F}_t u(\tau, x) = \int_{\mathbb{R}} u(t, x) e^{-2\pi i \tau t} dt.$$ 

The space-time Fourier transform is denoted by $\mathcal{F}_{t,x} = \mathcal{F}_t \mathcal{F}_x$. For simplicity, we will drop the harmless factors of $2\pi$.

Let $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ denote the space of functions $u: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, with $u \in C^\infty(\mathbb{R} \times \mathbb{T})$ which satisfy

$$u(t, x + 1) = u(t, x), \quad \sup_{(t,x) \in \mathbb{R} \times \mathbb{T}} \| t^\alpha \partial_t^\beta \partial_x^\gamma u(t, x) \| < \infty, \quad \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}.$$ 

In the following, we define the $X^{s,b}$-spaces adapted to the Fourier-Lebesgue setting (see [26, 27]).

**Definition 2.1.** Let $s, b \in \mathbb{R}$, $1 \leq p,q \leq \infty$. The space $X^{s,b}_{p,q}(\mathbb{R} \times \mathbb{T})$, abbreviated as $X^{s,b}_{p,q}$, is defined as the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ with respect to the norm

$$\| u \|_{X^{s,b}_{p,q}} = \| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(\tau, n) \|_{\ell_p^r L_q^r}.$$ 

When $p = q = 2$, the $X^{s,b}_{p,q}$-spaces defined above reduce to the standard $X^{s,b}$-spaces in [3].

---

5According to our definition of the Fourier transform, the weight factor $\langle \tau - n^3 \rangle$ should be chosen as $\langle \tau - 4\pi^2 n^3 \rangle$. For simplicity of the notation, we shall ignore such an inessential power of $2\pi$ in the sequel.
Recall the following embedding. For any $1 \leq p < \infty$,
\[ X^{s,b}_{p,q}(\mathbb{T}) \hookrightarrow C(\mathbb{R}; F^{s,p}(\mathbb{T})) \] for $b > \frac{1}{q'} = 1 - \frac{1}{q}$.

Since $X^{s,\frac{1}{2}}_{p,2}$ fails to embed into $C(\mathbb{R}; F^{s,p}(\mathbb{T}))$, we will consider the smaller space $Z^{s,b}_{p}\hookrightarrow C(\mathbb{R}; F^{s,p}(\mathbb{T}))$ defined by $Z^{s,b}_{p} := X^{s,\frac{1}{2}}_{p,2} \cap X^{s,b-\frac{1}{2}}_{p,1}$. We can also define the local-in-time version of these spaces on the interval $[-T, T]$ through the following norm
\[ \|u\|_{Z^{s,b}_{p}(T)} = \inf \{\|v\|_{Z^{s,b}_{p}} : v|_{[-T,T]} = u\}, \]
where the infimum is taken over all possible extensions of $u$ on $[-T, T]$.

The following are linear estimates associated with the gKdV equation (see [27, Lemma 7.1] for an analogous proof). Let $S(t) = e^{-t\partial_x^3}$ denote the linear propagator of the Airy equation.

**Lemma 2.2.** Let $1 \leq p < \infty$ and $s, b \in \mathbb{R}$. Then, the following estimates hold:
\[ \|S(t)u_0\|_{Z^{s,b}_{p}(T)} \lesssim \|u_0\|_{F^{s,p}}, \]
\[ \left\| \int_0^t S(t-t') F(t', x) \, dt' \right\|_{Z^{s,b-\frac{1}{2}}_{p}(T)} \lesssim \|F\|_{Z^{s,-\frac{1}{2}}_{p}(T)}, \]
for any $0 < T \leq 1$.

The following lemma allows us to gain a small power of $T$ needed to close the contraction mapping argument. It can be shown by modifying the proof for $p = 2$ (see [54, Lemma 2.11]).

**Lemma 2.3.** Let $-\frac{1}{2} < b' \leq b < \frac{1}{2}$ and $1 \leq p < \infty$. The following holds:
\[ \|u\|_{X^{s,b'}_{p,2}(T)} \lesssim T^{b-b'} \|u\|_{X^{s,b}_{p,2}(T)}, \]
for any $0 < T \leq 1$.

Lastly, we include well-known results needed in the proof of the nonlinear estimate (see, e.g., [24, Lemma 4.2], [39, Lemma 5], and [17, Lemma 4.1], respectively).

**Lemma 2.4.** Let $0 \leq \alpha \leq \beta$ such that $\alpha + \beta > 1$ and $\varepsilon > 0$. Then, we have
\[ \int_{\mathbb{R}} \frac{1}{(x-a)^\alpha (x-b)^\beta} \, dx \lesssim \frac{1}{(a-b)^\gamma}, \]
where
\[ \gamma = \begin{cases} \alpha + \beta - 1, & \beta < 1, \\ \alpha - \varepsilon, & \beta = 1, \\ \alpha, & \beta > 1. \end{cases} \]

**Lemma 2.5.** Let $0 \leq \alpha, \beta < 1$ such that $\alpha + \beta > 1$. Then, we have
\[ \sum_{n_1, n_2 \in \mathbb{Z}} \frac{1}{(n_1)^\alpha (n_2)^\beta} \lesssim \frac{1}{(n)^{\alpha+\beta-1}}, \]
uniformly over $n \in \mathbb{Z}$. 

Lemma 2.6. If \(|n_1| \geq \ldots \geq |n_k|\) and \(n_1 + \ldots + n_k = 0\), then
\[
|n_1^3 + \ldots + n_k^3| \lesssim |n_1 n_2 n_3|.
\]
Recall the phase function \(\phi_k(n, n_1, \ldots, n_k) = n^3 - n_1^3 - \ldots - n_k^3\), which we will denote by \(\phi\) for simplicity. When \(k = 2\) (KdV) or \(k = 3\) (mKdV), the phase function restricted to \(n = n_1 + \ldots + n_k\) satisfies the following factorizations
\[
(n_1 + n_2)^3 - n_1^3 - n_2^3 = 3(n_1 + n_2)n_1 n_2, \quad (2.1)
\]
\[
(n_1 + n_2 + n_3)^3 - n_1^3 - n_2^3 - n_3^3 = 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3). \quad (2.2)
\]
Unfortunately, analogous factorizations no longer hold for \(k \geq 4\). Instead, we establish the following lemma.

Lemma 2.7. Let \(k \geq 4\), \(n = n_1 + \ldots + n_k\) and \(|n_1| \geq \ldots \geq |n_k| > 0\).

A. If \(|n_1| \sim |n_1| \gg |n_2|\), \(n \neq n_1\), then one of the following holds
   A.1. \(|n_1|^2 |n - n_1| \lesssim |\phi|\);
   A.2. \(|n_1|^2 |n - n_1| \lesssim |n_2 n_3 n_4|\).

B. If \(|n_1| \sim |n_2| \gg |n_3|\), \(n \neq n_1\), \(n \neq n_2\), \(n_1 + n_2 \neq 0\), then one of the following holds
   B.1. \(|n_1|^2 |n_1 + n_2| \lesssim |\phi|\);
   B.2. \(|n_1 + n_2| \ll |n_4|\).

Proof. We start by proving A. Assume that \(|n_1|^2 |n - n_1| \gg \max(|\phi|, |n_2 n_3 n_4|)\). Using (2.1), we can rewrite \(\phi\) as follows
\[
\phi = 3nn_1(n - n_1) + (n - n_1)^3 - n_2^3 - \ldots - n_k^3.
\]
Since \(|nn_1(n - n_1)| \sim |n_1|^2 |n - n_1|\) and using Lemma 2.6, we have
\[
|n_1|^2 |n - n_1| \sim |(n - n_1)^3 - n_2^3 - \ldots - n_k^3| \lesssim |n_2 n_3 n_4| \max(|n - n_1|, |n_4|).
\]
From the above estimate, we must have \(|n_1|^2 |n - n_1| \lesssim |n_2 n_3 n_4|\) which contradicts our initial assumption. To prove part B, assume that \(|n_1|^2 |n_1 + n_2| \gg |\phi|\) and \(|n_1 + n_2| \gtrsim |n_4|\).

Using (2.2), we can rewrite \(\phi\) as follows
\[
\phi = 3(n - n_1)(n - n_2)(n_1 + n_2) + (n_3 + \ldots + n_k)^3 - n_3^3 - \ldots - n_k^3.
\]
Since \(|(n - n_1)(n - n_2)(n_1 + n_2)| \sim |n_1|^2 |n_1 + n_2|\), using Lemma 2.6, we have
\[
|n_1|^2 |n_1 + n_2| \sim |(n_3 + \ldots + n_k)^3 - n_3^3 - \ldots - n_k^3| \lesssim |n_3|^2 |n_4|.
\]
From the above estimate, we must have \(|n_1 + n_2| \ll |n_4|\) which contradicts our assumption.

\[\square\]

3. Nonlinear estimates and local well-posedness

In this section we state and prove the main nonlinear estimates needed to show Theorem 1.1 as well as proving the latter theorem through a contraction mapping argument. We will establish a nonlinear estimate for the more general multilinear operator
\[
\mathcal{N}(u_0, \ldots, u_m) = \mathcal{P}(u_1 \ldots u_m) \partial_x u_0 - \sum_{j=1}^{m} \mathcal{P}_0(u_j \partial_x u_0) \prod_{i=1 \atop i \neq j}^{m} u_i, \quad (3.1)
\]
for mean zero functions $u_0, \ldots, u_m$.

The main difficulty in estimating $\mathcal{N}(u_0, \ldots, u_m)$ lies in controlling the derivative. To that end, we want to exploit the multilinear dispersion through the phase function $\phi$ and use Lemma 2.7 to guide our case separation in the nonlinearity. Due to the restrictions in Lemma 2.7, consider the following resonant regions in frequency space:

$$A_j(n) = \left\{ \begin{array}{ll}
\{(n_0, \ldots, n_m) \in \mathbb{Z}^{m+1}_*: n = n_j \}, & j = 0, \ldots, m, \\
\{(n_0, \ldots, n_m) \in \mathbb{Z}^{m+1}_*: n_0 + n_i = 0 \}, & -j = l = 1, \ldots, m.
\end{array} \right.$$ 

We can decompose the nonlinearity as $\mathcal{N} = \mathcal{N}_0 + \mathcal{R}$, where the non-resonant and resonant contributions are respectively defined as

$$F_x(\mathcal{N}_0(u_0, \ldots, u_m))(n) = \sum_{n=n_0+\ldots+n_m \atop n_0 \cdots n_m \neq 0} \mathcal{A}_j(n) \sum_{j=0}^{m} \mathcal{A}_j \hat{u}_0(n_0) \cdots \hat{u}_m(n_m),$$

$$F_x(\mathcal{R}(u_0, \ldots, u_m))(n) = \sum_{n=n_0+\ldots+n_m \atop n_0 \cdots n_m \neq 0} \left[ \sum_{J \in \mathcal{C}} (-1)^{|J|+1} \mathcal{A}_j \right] \hat{u}_0(n_0) \cdots \hat{u}_m(n_m),$$

where $J \in \mathcal{C}$ if $J = \{j\}$, $j = 1, \ldots, m$, or $J \subset \{-m, \ldots, m\}$ and $|J| \geq 2$.

The following proposition states the main nonlinear estimates.

**Proposition 3.1.** Let $u_0, \ldots, u_m$ be mean zero functions. For $2 < p < \infty$ there exists $\frac{1}{2} < s_*(p) < 1 - \frac{1}{p}$ such that for any $s > s_*(p)$ the following estimates hold

$$\|\mathcal{N}_0(u_0, \ldots, u_m)\|_{Z_p^{-\frac{1}{2}}(T)} \lesssim T^0 \prod_{j=0}^{m} \|u_j\|_{Z_p^{s-\frac{1}{2}}(T)},$$

$$\|\mathcal{R}(u_0, \ldots, u_m)\|_{Z_p^{-\frac{1}{2}}(T)} \lesssim T^0 \prod_{j=0}^{m} \|u_j\|_{Z_p^{s-\frac{1}{2}}(T)},$$

for some $0 < \theta < 1$ and any $0 < T \leq 1$.

**Remark 3.2.** It will suffice to show the above estimates for $v_0, \ldots, v_m$ extensions of $u_0, \ldots, u_m$ on $[-T, T]$. Consequently, in the remaining of this section we will show the estimates in $Z_p^{-\frac{1}{2}}$ and $Z_p^{s-\frac{1}{2}}$, instead of the time localized versions. Moreover, we will establish stronger estimates which allow us to gain a small power of $T$ by applying Lemma 2.3.

Assuming Proposition 3.1 holds, we can prove Theorem 1.1 for mean zero initial data. For details on the non-zero mean case, see Appendix B.
Proof of Theorem 1.1. Let \((s, p)\) satisfy the assumptions in Proposition 3.1. Given \(u_0 \in \mathcal{F}L^{s, p}(\mathbb{T})\) with zero mean, define the map \(\Gamma_{u_0}\) as follows
\[
\Gamma_{u_0}(u)(t) := S(t)u_0 + \int_0^t S(t-t')\mathcal{N}(u, \ldots, u)(t')\, dt'.
\]
Let \(R > 0\) and \(B_R := \{ u \in Z_p^{s, \frac{1}{2}}(T) : \|u\|_{Z_p^{s, \frac{1}{2}}(T)} \leq R \}\). Using Lemma 2.2, Proposition 3.1 and the fact that \(\mathcal{N} = \mathcal{N}_0 + \mathcal{R}\), we have
\[
\|\Gamma_{u_0}(u)\|_{Z_p^{s, \frac{1}{2}}(T)} \leq C_1\|u_0\|_{\mathcal{F}L^{s, p}} + C_2\|\mathcal{N}(u, \ldots, u)\|_{Z_p^{s, \frac{1}{2}}(T)}
\leq C_1\|u_0\|_{\mathcal{F}L^{s, p}} + C_3T^\theta\|u\|_{m}^{m+1}_{Z_p^{s, \frac{1}{2}}(T)}
\leq C_1\|u_0\|_{\mathcal{F}L^{s, p}} + C_3T^\theta\|u\|_{m+1}^{m+1}_{Z_p^{s, \frac{1}{2}}(T)}
\]
for some \(0 < \theta < 1\), \(C_1, C_2, C_3 > 0\) and any \(0 < T \leq 1\). Similarly, since \(\mathcal{N}, \mathcal{N}_0\), and \(\mathcal{R}\) are multilinear maps, we have
\[
\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{Z_p^{s, \frac{1}{2}}(T)} \leq C_4T^\theta\left(\|u\|_m^{m} + \|v\|_m^{m}_{Z_p^{s, \frac{1}{2}}(T)}\right)\|u - v\|_{Z_p^{s, \frac{1}{2}}(T)}
\]
for a constant \(C_4 > 0\) and any \(0 < T \leq 1\). Choosing \(R := 2C_1\|u_0\|_{\mathcal{F}L^{s, p}}\) and \(0 < T = T(R) \leq 1\) such that \(C_3T^\thetaR^2 \leq \frac{1}{2}\) and \(C_4T^\thetaR^2 \leq \frac{1}{2}\), it follows from (3.4) and (3.5) that \(\Gamma_{u_0}\) is a contraction on the closed ball \(B_R \subset Z_p^{s, \frac{1}{2}}(T)\). Consequently, \(\Gamma_{u_0}\) has a (unique) fixed point \(u = \Gamma_{u_0}(u)\) in \(B_R\), which gives a solution \(u \in Z_p^{s, \frac{1}{2}}(T)\) to (1.7) with initial condition \(u|_{t=0} = u_0\). The uniqueness in \(Z_p^{s, \frac{1}{2}}(T)\) can be shown using the estimate (3.5) with a suitably chosen \(T\). The local Lipschitz continuity of the data-to-solution map follows from an analogous argument. 

3.1. Bilinear and trilinear Strichartz estimates. In order to show Proposition 3.1, we first establish bilinear and trilinear Strichartz estimates adapted to the Fourier-Lebesgue setting. Let \(P\) denote the projection onto mean zero functions and \(P_0\) denote the mean. The following lemma generalizes the periodic \(L^4\)-Strichartz of Bourgain in [3] to the Fourier-Lebesgue setting.

Lemma 3.3. The following estimate holds for any \(2 \leq p \leq \infty\) and \(b > \max\{\frac{1}{3}, \frac{3p-2}{8p}\}\)
\[
\|P(Pu_1 \cdot Pu_2)\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0,0}_{p,2}}\|u_2\|_{X^{0,0}_{p,2}}.
\]

Proof. The proof is an adaptation of the standard bilinear argument for \(p = 2\) (see [5, Proposition 2.13] for instance) to the Fourier-Lebesgue setting. Let \(M_1, M_2 \geq 1\) denote dyadic numbers, \(P_{M_j}\) the projection onto space-time frequencies \(\{\langle \tau - n^3 \rangle \sim M_j\}\) and \(u_{M_1} = P_{M_1}u_1, v_{M_2} = P_{M_2}u_2\). Since
\[
\|P(Pu_1 \cdot Pu_2)\|_{X^{0,0}_{p,2}} \leq \sum_{M_1, M_2} \|P(Pu_{M_1} \cdot Pu_{M_2})\|_{X^{0,0}_{p,2}},
\]
it suffices to show that
\[
\|P(Pu_{M_1} \cdot Pu_{M_2})\|_{X^{0,0}_{p,2}} \lesssim M_1^bM_2^b\|u_{M_1}\|_{X^{0,0}_{p,2}}\|u_{M_2}\|_{X^{0,0}_{p,2}}.
\]
for any $b > \max\{\frac{1}{q}, \frac{3p-2}{8p}\}$. We assume $M_1 \leq M_2$, while the same proof applies to the other case. Using Hölder’s inequality, we get

$$\left\| P(Pu_{M_1} \cdot Pu_{M_2}) \right\|_{X^{0,0}_{p,2}} \lesssim \left\| M_1^{\frac{1}{p}} A(\tau, n) \right\|_{L^1} \left\| \left[ |\hat{u}_{M_1}|^{q'} + |\hat{v}_{M_2}|^{q'} \right]^{\frac{1}{q'}} \right\|_{L^{q'}},$$

(3.8)

for $q > 1$, where

$$A(\tau, n) = \{ n_1 \in \mathbb{Z} : 0 \neq 3nn_1(n - n_1) = -\tau + n^3 + O(M_2) \}.$$ 

Since we can rewrite the above condition as $(n_1 - \frac{n}{2})^2 = \frac{1}{3n}(\tau - \frac{n^3}{3}) + O(\frac{M_2}{n^3})$, then we conclude that there are at most $O(1 + (\frac{M_2}{n})^{\frac{2}{3}})$ elements in $A(\tau, n)$. We first consider the case when $|n| > M_2$ and thus $|A(\tau, n)| \lesssim 1$. From (3.8) with $q = 2$ and Young’s inequality, we have

$$\text{RHS of (3.8) } \lesssim M_1^{\frac{1}{p}} \left\| \left[ |\hat{u}_{M_1}|^{2} + |\hat{v}_{M_2}|^{2} \right]^{\frac{1}{q'}} \right\|_{L^{q'}},$$

$$\leq M_1^{\frac{1}{p}} \left\| |\hat{u}_{M_1}|^{2} \right\|_{L^{q'}} \left\| |\hat{v}_{M_2}|^{2} \right\|_{L^{q'}} = M_1^{\frac{1}{p}} \left\| |\hat{u}_{M_1}|^{2} \right\|_{L^{q'}} \left\| |\hat{v}_{M_2}|^{2} \right\|_{L^{q'}};$$

which implies (3.7), since $M_1 \leq M_2$. For the case when $|n| \leq M_2$ and $|A(\tau, n)| \lesssim (\frac{M_2}{|n|})^{\frac{1}{p}}$, we set

$$\left( \frac{1}{p}, \frac{1}{q'} \right) = \begin{cases} (\frac{1}{3} + \varepsilon, \frac{1}{p} - \frac{1}{6} + \varepsilon) & \text{for } 2 \leq p \leq 6, \\ (\frac{1}{2} - \frac{1}{p}, 0) & \text{for } 6 < p \leq \infty, \end{cases}$$

with sufficiently small $\varepsilon > 0$. Note that

$$\frac{1}{p} - \frac{1}{r} < \frac{1}{2q}, \quad \frac{q'}{r} + 1 = \frac{q'}{p} + \frac{q'}{2}, \quad 1 < q' \leq 2 \leq p \leq r \leq \infty.$$ 

Applying first Hölder’s inequality in $n$, then following the above computation, and using Hölder’s inequality in $\tau$, we have

$$\text{RHS of (3.8) } \lesssim M_1^{\frac{1}{p}} M_2^{\frac{1}{q'}} \left\| n \right\|_{L^{\frac{1}{r}}} \left\| \left[ |\hat{u}_{M_1}|^{q'} + |\hat{v}_{M_2}|^{q'} \right]^{\frac{1}{q'}} \right\|_{L^{q'}} \left\| |\hat{u}_{M_1}| \right\|_{L^{q'}} \left\| |\hat{v}_{M_2}| \right\|_{L^{q'}} \left\| |\hat{u}_{M_1}| \right\|_{L^{q'}} \left\| |\hat{v}_{M_2}| \right\|_{L^{q'}};$$

Since $M_1 \leq M_2$ and $\frac{1}{p} + \frac{1}{2q} \leq 2 \max\{\frac{1}{3}, \frac{3p-2}{8p}\} + \frac{1}{2}$, we obtain (3.7), from which the estimate follows.

We can then establish the following estimate.

**Lemma 3.4.** The following estimate holds for any $2 \leq p \leq \infty$

$$\left\| P(Pu_1 \cdot Pu_2) \right\|_{X^{0,0}_{p,2}} \lesssim \left\| u_1 \right\|_{X^{0,0}_{p,2}} \left\| u_2 \right\|_{X^{0,0}_{p',2}}.$$ 

(3.9)
Lemma 3.5. The following estimate holds for any \(1 \leq p, q \leq \infty\)
\[
\|P_0(u_1 u_2)u_3\|_{X_{p,q}^{0,0}} \lesssim \|u_1\|_{X_{q,2}^{0,\frac{1}{2}}} \|u_2\|_{X_{p',2}^{0,\frac{1}{2}}} \|u_3\|_{X_{p,2}^{0,\frac{1}{2}}}.
\] (3.10)

Proof. By Young’s and Hölder’s inequalities, it follows that
\[
\|P_0(u_1 u_2)u_3\|_{X_{p,q}^{0,0}} \lesssim \left\| \sum_{n_1} \int (\hat{u}_1(\tau_1, n_1) \hat{u}_2(-\tau_2, -n_1) \hat{u}_3(\tau_3, n_3)) \right\|_{\ell_{\tau}^p L_{\tau}^q}.
\]
\[
\lesssim \|u_1\|_{X_{q,2}^{0,0}} \|u_2\|_{X_{p',2}^{0,0}} \|u_3\|_{X_{p,2}^{0,0}}.
\]
\[
\lesssim \|u_1\|_{X_{q,2}^{0,\frac{1}{2}}} \|u_2\|_{X_{p',2}^{0,\frac{1}{2}}} \|u_3\|_{X_{p,2}^{0,\frac{1}{2}}}.
\]

The following trilinear estimate can be seen as a multilinear analogue of the \(L^6\)-Strichartz in \([3]\) adapted to the Fourier-Lebesgue spaces.

Lemma 3.6. Let \(2 \leq p \leq \infty\), \(n_j\) denote the spatial frequency corresponding to \(\hat{u}_j\), \(j = 1, 2, 3\), and assume that \((n_1 + n_2)(n_1 + n_3)(n_2 + n_3) \neq 0\). We have the following estimate
\[
\|u_1 u_2 u_3\|_{X_{p,2}^{0,0}} \lesssim \|u_1\|_{X_{p,2}^{0,\frac{1}{2}}} \|u_2\|_{X_{p,2}^{0,\frac{1}{2}}} \|u_3\|_{X_{p,2}^{0,\frac{1}{2}}}. \tag{3.11}
\]

Proof. Let \(\phi = 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3)\). Using Cauchy-Schwarz inequality and Lemma 2.4 we have
\[
\|u_1 u_2 u_3\|_{X_{p,2}^{0,0}} = \left\| \sum_{n_1+n_2+n_3} \int \prod_{j=1}^3 \hat{u}_j(\tau_j, n_j) \right\|_{\ell_{\tau}^p L_{\tau}^q}.
\]
\[
\lesssim \left\| \sum_{n_1+n_2+n_3} \frac{1}{(\tau - n^3 + \phi)^{1(1-\varepsilon)}} \right\|_{\ell_{\tau}^p L_{\tau}^q} \left\| \prod_{j=1}^3 (\sigma_j^{1(1-\varepsilon)} \hat{u}_j(n_j, \tau_j) \right\|_{L_{\tau}^2 L_{\tau}^3} \right\|_{\ell_{\tau}^p L_{\tau}^q},
\]
for any \(\varepsilon > 0\). Since \(\langle x + y \rangle \lesssim \langle x \rangle \langle y \rangle\) for any \(x, y\), we have the following for any \(\theta > 0\)
\[
\frac{1}{\langle n_1 \rangle^{2\theta} \langle n_2 \rangle^{2\theta} \langle n_3 \rangle^{2\theta}} = \frac{1}{\langle n - n_2 - n_3 \rangle^{2\theta} \langle n_2 \rangle^{2\theta} \langle n_3 \rangle^{2\theta}} \lesssim \frac{1}{\langle n_2 + n_3 \rangle^{\theta} \langle n - n_2 \rangle^{\theta} \langle n - n_3 \rangle^{\theta}}.
\]
By letting $\theta = 2\varepsilon$ and using Cauchy-Schwarz inequality, we obtain
\[
\|u_1 u_2 u_3\|_{X_{p,2}^{\varepsilon,0}} \lesssim \sup_{\tau,n} (\tau, n) \left( \prod_{j=1}^{3} (n_j)^{4\varepsilon} (\tau_j - n_j^3)^{2\varepsilon} \hat{u}_j(n_j, \tau_j) \right)_{L^p_t L^q_x L^2_y} ,
\]
where
\[
I(\tau, n) = \sum_{n_2, n_3} \frac{1}{(n_2 + n_3)^{4\varepsilon} (n - n_2)^{4\varepsilon} (n - n_3)^{4\varepsilon} (\tau - n^3 + \phi)^{1-\varepsilon}}.
\]
The estimate follows from Minkowski's inequality and showing a uniform bound on $I(\tau, n)$. Let $a = \tau - n^3$. Then, we can rewrite $I(\tau, n)$ and estimate it as follows
\[
I(\tau, n) \lesssim \sum_{l_2, l_3 \neq 0 \atop 2n - l_2 - l_3 \neq 0} \frac{1}{(2n - l_2 - l_3)^{4\varepsilon} l_2^{4\varepsilon} l_3^{4\varepsilon} (a + 3(2n - l_2 - l_3)l_2l_3)^{1-\varepsilon}} \left( \{ l_2, l_3 : r = (2n - l_2 - l_3)l_2l_3 \} \right).
\]
Now, we employ the divisor bound (see, e.g., [29, Theorem 278]):
\[
\forall \delta > 0, \exists C_\delta > 0 \text{ s.t. } \left| \left\{ n \in \mathbb{N} : n \text{ divides } N \right\} \right| \leq C_\delta N^\delta \quad (\forall N \in \mathbb{N}),
\]
to estimate the number of $(l_2, l_3)$'s by
\[
\left| \left\{ l_2, l_3 : r = (2n - l_2 - l_3)l_2l_3 \right\} \right| \leq \left| \left\{ l_2 : \text{divides } |r| \right\} \right| \cdot \left| \left\{ l_3 : \text{divides } |r| \right\} \right| \lesssim |r|^{\varepsilon'}
\]
for any $\varepsilon' > 0$. Choosing $\varepsilon' \leq 2\varepsilon$, for example, gives
\[
I(\tau, n) \lesssim \sum_{r \neq 0} \frac{1}{(r)^{2\varepsilon} (a + 3r)^{1-\varepsilon}} < \infty,
\]
from Lemma 2.5 and the estimate follows.

3.2. Resonant contributions. We start by considering $R$ in (3.3) where $J$ satisfies $\{0, j\} \subset J$, $\{-j, 0\} \subset J$ or $\{-j, j\} \subset J$, for some $j = 1, \ldots, m$. The intended estimate essentially follows from the stronger estimate in Lemma 3.7.

Lemma 3.7. Let $2 \leq p < \infty$ and $s > 1 - \frac{1}{p} - \frac{2}{mp}$. Then the following estimate holds
\[
\left\| \int_{\tau = \tau_0 + \cdots + \tau_m} \sum_{n = n_2 + \cdots + n_m} \langle n \rangle^{s+1} \langle \hat{u}_0(\tau_0, n) \hat{u}_1(\tau_1, n) \hat{u}_j(\tau_j, n_j) \hat{u}_j(\tau_j, n_j) \rangle_{L^p_t L^2_x} \right\|_{X_{p,2}^{s,0}} \lesssim \|u_0\|_{X_{p,2}^{s,0}} \prod_{j=1}^{m} \|u_j\|_{X_{p,1}^{s,0}}. \tag{3.12}
\]
Proof. Assume that $|n_2| \geq \ldots \geq |n_m|$, without loss of generality. Then $|n| \lesssim |n_2|$. Using Young’s and Hölder’s inequalities, we have

$$\text{LHS of (3.12)} \lesssim \|u_0\|_{X^s,0} \|u_1\|_{X^s,1} \left\| \langle n \rangle^{1-s} \sum_{n=n_2+\ldots+n_m}^m \prod_{j=2}^m \|\tilde{u}_j(n_j)\|_{L^s_p} \right\|_{p_n}$$

$$\lesssim \sup_n \left( \sum_{n=n_2+\ldots+n_m} \frac{1}{\langle n \rangle^{2s-1} \langle n_3 \rangle^s \ldots \langle n_m \rangle^s} \|u_0(\tau_0,-n_2)\|_{\ell_0^m L^s_{p_n}} \right)^{\frac{1}{p'}} \|u_0\|_{X^s,0} \prod_{j=1}^m \|u_j\|_{X^s,0}.$$  

The estimate follows from Lemma 2.5 for $s > \max(\frac{1}{2}, 1 - \frac{1}{p} - \frac{p-2}{mp}) = 1 - \frac{1}{p} - \frac{p-2}{mp}$. □

The following lemma establishes an estimate for $\mathcal{R}$ in (3.3) when $J \subset \{-m, \ldots, -1\}$.

**Lemma 3.8.** For $m \geq 3$, $2 \leq p < \infty$ and $s > 1 - \frac{1}{p} - \min(\frac{1}{p}, \frac{p-2}{mp}, \frac{2}{2m-31_{m>4}})$, we have

$$\left\| \int \sum_{n=n_2+\ldots+n_m} \frac{1}{\langle n \rangle^{2s-1} \langle n_3 \rangle^s \ldots \langle n_m \rangle^s} \right\| \lesssim \|u_0\|_{X^s,0} \prod_{j=1}^m \|u_j\|_{X^s,0}.$$  

(3.13)

**Proof.** Assume that $|n_3| \geq \ldots \geq |n_m|$, without loss of generality. We will consider two cases: $|n_2| \gtrsim |n_3|$ and $|n_2| \ll |n_3|$. If $|n_2| \gtrsim |n_3|$, using Young’s and Hölder’s inequalities gives

$$\text{LHS of (3.13)} \lesssim \sup_n \left( \sum_{n=n_2+\ldots+n_m} \frac{1}{\langle n \rangle^{2s-1} \langle n_3 \rangle^s \ldots \langle n_m \rangle^s} \right)^{\frac{1}{p'}} \|u_0\|_{X^s,0} \prod_{j=1}^m \|u_j\|_{X^s,0}.$$  

The estimate follows from Lemma 2.5 if

$$s > 1 - \frac{1}{p} - \frac{p-2}{mp}.$$  

(3.14)

If $|n_2| \ll |n_3|$, then $|n| \lesssim |n_3|$. Let $v(t,x) = F_x^{-1}(\langle \rangle \hat{u}_0(\tau, \cdot) \hat{u}_1(\tau, \cdot) \hat{u}_2(\tau, \cdot))$. If $m = 3$, from Young’s inequality, we have

$$\text{LHS of (3.13)} \lesssim \|v \cdot D^s u_3\|_{X^{0,-\frac{1}{2}}} \lesssim \|v\|_{X^{0,0}} \|u_3\|_{X^{s,0}}.$$  

Note that using Young’s inequality in time and Hölder’s in space, we have

$$\|v\|_{X^{0,0}} \lesssim \prod_{j=0}^2 \|\langle n \rangle^{\frac{1}{3}} \hat{u}_j(\tau, n)\|_{\ell^3_0 L^s_{p_n}} \lesssim \prod_{j=0}^2 \|u_j\|_{X^{s,0}},$$  

(3.15)

for $s \geq \frac{1}{2}$, $2 \leq p \leq 3$ or $s > \frac{2}{3} - \frac{1}{p}$, $p > 3$, which are less restrictive than (3.14). If $m \geq 4$ and $n_3 + n_4 \neq 0$, we use Young’s inequality to obtain

$$\text{LHS of (3.13)} \lesssim \|v \cdot D^s u_3 \cdot u_4 \cdots u_m\|_{X^{0,-\frac{1}{2}}} \lesssim \|v\|_{X^{0,0}} \|P(D^s u_3 \cdot u_4)\|_{X^{0,0}} \prod_{j=5}^m \|u_j\|_{X^{s,0}}.$$  

The first factor is estimated as in (3.15). For the remaining factors we use (3.5) and Hölder’s inequality, and the fact that $|n_4| \geq \ldots \geq |n_m|$. The estimate follows if $s > \frac{1}{2} - \frac{1}{p}$ and, for
Lemma 3.9. If \( n_3 + n_4 = 0 \), we have the following

\[
\text{LHS of (3.13)} \lesssim \left\| \sum_{n=n_2+n_5+\ldots+n_m} \right. \left. \|v(n_2)\|_{L^1_\tau} \prod_{j=5}^m \|\tilde{u}_j(n_j)\|_{L^1_\tau} \times \sum_{n_3} \langle n_3 \rangle^s \|\hat{u}_3(n_3)\|_{L^2_\tau} \|\tilde{u}_4(-n_3)\|_{L^1_\tau} \right\|_{L^p_\tilde{\tau}}.
\]

\[
\lesssim \|u_3\|_{X^{s \cdot 0}_{\tilde{2}, 2}} \|u_4\|_{X^{s \cdot 0}_{\tilde{2}, 2}} \prod_{j=5}^m \|u_j\|_{X^{s \cdot 0}_{p, 1}} \sup_n J(n)^{\frac{1}{p'}}
\]

where from Lemma 2.5 given that \( s > \frac{1}{3} \) and, when \( m > 4 \), that \( s > 1 - \frac{1}{p} - \frac{2p-3}{(m-1)p} \), we have

\[
J(n) = \sum_{n=n_2+n_5+\ldots+n_m} \left\| \frac{1}{\langle n_2 \rangle^s \langle n_5 \rangle^s \cdots \langle n_m \rangle^s} \right\|_{L^p_\tilde{\tau}} \lesssim 1.
\]

The estimate follows from Hölder’s inequality given that \( s > 1 - \frac{2}{p} \).

Lastly, we consider \( R \) restricted to \( J = J_+ \cup (-J_-) \), where \( J_+, J_- \subset \{1, \ldots, m\} \) are disjoint sets and \( |J_+| \geq 1 \). The following lemma estimates the case when \( J_+ = \{1, \ldots, m\} \).

**Lemma 3.9.** The following estimate holds for any \( 1 \leq p < \infty \) and \( s \geq \frac{1}{m} \)

\[
\left\| \int_{\tau=\tau_0+\ldots+\tau_m} \frac{\langle n \rangle^s (\langle m - 1 \rangle n)}{(\tau - n^3)^{\frac{1}{2}}} |\tilde{u}_0(\tau_0, -(m - 1)n)| \prod_{j=1}^m |\tilde{u}_j(\tau_j, n_j)| \right\|_{L^p_{\tilde{\tau}}(\mathbb{T}^d)} \lesssim \prod_{j=0}^m \|u_j\|_{X^{s \cdot 0}_{p, 2} \cap X^{s \cdot 0}_{p, 1}}.
\]

**Proof.** Using Young’s inequality in time and taking a supremum in \( n \), we can estimate the intended quantity by placing \( u_0 \) in \( X^{s \cdot 0}_{p, 2} \) and the remaining terms in \( X^{s \cdot 0}_{\infty, 1} \). The estimate therefore follows for \( s \geq \frac{1}{m} \).

For the remaining cases, we fix \( J_+ \) and gather the contributions from \( J_- \subset \{1, \ldots, m\} \setminus J_+ \). Appealing to symmetry, let \( J_+ = \{1, \ldots, l\} \) for some \( 1 \leq l \leq m - 1 \). Then, the net contribution can be rewritten as follows

\[
R_l(u_0, \ldots, u_m) := \int_{\tau=\tau_0+\ldots+\tau_m} \sum_{n=n_0+\ldots+n_m} \left[ \sum_{J_+ \subset \{l+1, \ldots, m\}} (-1)^{l+|J_-|+1} \prod_{j=1}^l (\bigcap_{i=1}^l A_i) \cap(\bigcap_{j \in J_-} A_{-j}) \right]
\]

\[
\times \|n_0\| \prod_{j=0}^m \|\tilde{u}_j(\tau_j, n_j)\|
\]

which is estimated in the following lemma.

**Lemma 3.10.** Let \( 1 \leq l \leq m - 1 \) and \( m \geq 3 \). Then, for \( 2 < p < \infty \) and \( s > 1 - \frac{1}{p} - \min\left(\frac{2}{4p}, \frac{2}{mp}, \frac{1}{2m}\right) \), the following holds

\[
\left\| \frac{\langle n \rangle^s}{(\tau - n^3)^{\frac{1}{2}}} R_l(u_0, \ldots, u_m) \right\|_{L^p_{\tilde{\tau}}(\mathbb{T}^d)} \lesssim \prod_{j=0}^m \|u_j\|_{X^{s \cdot 0}_{p, 2} \cap X^{s \cdot 0}_{p, 1}}.
\]
Proof. Fix $1 \leq l \leq m-1$ and assume without loss of generality that $|n_{l+1}| \geq \ldots \geq |n_m|$. If $l = 1$, then $0 = n_0 + n_2 + \ldots + n_m$, $n_0 + n_j \neq 0$ for $j = 2, \ldots, m$ and $|n_0| \lesssim |n_2|$. Taking a supremum in $n$ and using Young’s inequality, we obtain

$$\text{LHS of (3.16)} \lesssim \left\| \int_{\tau = \tau_0 + \ldots + \tau_m} \langle n \rangle^s \widehat{u}_1(\tau_1, n) \sum_{0=n_0+n_2+\ldots+n_m} |n_0| \widehat{u}_0(\tau_0, n_0) \prod_{j=2}^m \widehat{u}_j(\tau_j, n_j) \right\|_{\ell^p_n L^2_t}$$

$$\lesssim \|u_1\|_{X^0_{p,1}} \|P(D^s u_0 \cdot D^{1-s} u_2) \cdot u_3 \ldots u_m\|_{X^0_{\infty,2}}$$

$$\lesssim \|u_1\|_{X^0_{p,1}} \|P(D^s u_0 \cdot D^{1-s} u_2)\|_{X^0_{\infty,2}} \prod_{j=3}^m \|u_j\|_{X^0_{q,1}}$$

where $(m-2) = \frac{1}{p} + \frac{m-2}{q}$. Using (3.6) and Hölder’s inequality we have

$$\|P(D^s u_0 \cdot D^{1-s} u_2)\|_{X^0_{\infty,2}} \prod_{j=3}^m \|u_j\|_{X^0_{q,1}} \lesssim \|u_0\|_{X^{s\frac{1}{p},p}_{p,2}} \|u_2\|_{X^{1-s\frac{1}{p},p}_{p,2}} \prod_{j=3}^m \|u_j\|_{X^{1-s\frac{1}{p},p}_{p,2}}$$

and we must impose

$$s > \max \left( \frac{1}{2}, 1 - \frac{1}{p} - \frac{p - 2}{4p}, 1 - \frac{1}{p} - \frac{1}{2m} \right) = 1 - \frac{1}{p} - \min \left( \frac{p - 2}{4p}, \frac{1}{2m} \right). \quad (3.17)$$

Now let $1 < l < m$. Then, $(1-l)n = n_0 + n_{l+1} + \ldots + n_m$ and $n_0 + n_j \neq 0$ for $j = l+1, \ldots, m$. Assuming that $|n| \sim |n_0| \gg |n_{l+1}|$ and using Hölder’s and Young’s inequalities, we have

$$\text{LHS of (3.16)} \lesssim \left\| \int_{\tau = \tau_0 + \ldots + \tau_m} \langle n \rangle^s \prod_{i=1}^l \widehat{u}_i(\tau_i, n) \sum_{(1-l)n=n_0+\ldots+n_m} |n_0| \widehat{u}_0(\tau_0, n_0) \prod_{j=l+1}^m \widehat{u}_j(\tau_j, n_j) \right\|_{\ell^p_n L^2_t}$$

$$\lesssim \prod_{i=1}^l \|u_i\|_{X^{s\frac{1}{p},p}_{p,2}} \left\| \sum_{(1-l)n=n_0+\ldots+n_m} |n_0|^{1-(l-1)s} \|\widehat{u}_0(n_0)\|_{L^2_n} \prod_{j=l+1}^m \|\widehat{u}_j(n_j)\|_{L^2_n} \right\|_{\ell^p_n}$$

$$\lesssim \|u_0\|_{X^{s\frac{1}{p},p}_{p,2}} \prod_{i=1}^l \|u_i\|_{X^{s\frac{1}{p},p}_{p,2}} \prod_{j=l+1}^m \|u_j\|_{X^{1-s\frac{1}{p},p}_{p,2}},$$

where $\alpha = \frac{ls}{m-1}$ and we have assumed that $\alpha > 0$. The estimate follows from Young’s inequality if $s > \max \left( \frac{1}{2}, 1 - \frac{1}{p} - \frac{(l-1)(p-1)-1}{mp} \right)$, less restrictive than

$$s > \max \left( \frac{1}{2}, 1 - \frac{1}{p} - \frac{p-2}{mp} \right) = 1 - \frac{1}{p} - \frac{p-2}{mp}.$$
If $|n_0| \lesssim |n_{l+1}|$, then we proceed as in the case when $l = 1$,
\[
\text{LHS of (3.16)} \lesssim \prod_{i=1}^{l} \|u_i\|_{X_{p,1}^{s,0}} \|P(D^s u_0 \cdot D^{1-s} u_{l+1}) \cdot u_{l+2} \cdot \ldots \cdot u_m\|_{X_{\infty,2}^{0,0}}
\]
\[
\lesssim \|P(D^s u_0 \cdot D^{1-s} u_{l+1})\|_{X_{p,2}^{0,0}} \left( \prod_{i=1}^{l} \|u_i\|_{X_{p,1}^{s,0}} \right) \left( \prod_{j=l+2}^{m} \|u_j\|_{X_{q,1}^{0,0}} \right),
\]
where $(m-l-1) = \frac{1}{p} + \frac{m-l-1}{q}$. Using (3.6) and Hölder’s inequality, we obtain
\[
\text{LHS of (3.16)} \lesssim \|u_0\|_{X_{p,2}^{s,\frac{1}{2}}} \|u_{l+1}\|_{X_{2,2}^{s,\frac{1}{2}}} \left( \prod_{i=1}^{l} \|u_i\|_{X_{p,1}^{s,0}} \right) \left( \prod_{j=l+2}^{m} \|u_j\|_{X_{q,1}^{0,0}} \right)
\]
and the estimate follows if $s > 1 - \frac{1}{p} - \min\left(\frac{\mu-2}{4p}, 2(m-l+1)\frac{1}{2l+2} \right)$, which is less restrictive than (3.17).

### 3.3. Non-resonant contributions

In this section, we establish the estimate for the non-resonant contribution $N_0$ in (3.2). Without loss of generality, we can assume that $|n_1| \geq \ldots \geq |n_m|$. We further split the non-resonant contribution as follows
\[
N_0 = N_1 + N_3 + \ldots + N_m \quad \text{if } m \text{ is odd},
\]
\[
N_0 = N_1 + N_3 + \ldots + N_{m-1} \quad \text{if } m \text{ is even},
\]
where $N_\alpha$, for odd $1 \leq \alpha \leq m-1$, corresponds to $N_0$ further restricted to the region
\[
\Lambda_\alpha(n) = \{(n_0, \ldots, n_m) \in \mathbb{Z}_*^{m+1} : |n_1| \geq \ldots \geq |n_m|, \quad
n_j + n_{j+1} = 0, \ 1 \leq j \leq \alpha - 1 \text{ odd}, \quad
n_\alpha + n_{\alpha+1} \neq 0\}
\]
and $\Lambda_m(n) = \{(n_0, \ldots, n_m) \in \mathbb{Z}_*^{m+1} : |n_1| \geq \ldots \geq |n_m|, \ n_1+n_2 = \ldots = n_{m-2}+n_{m-1} = 0\}$.

We will start by estimating the most difficult contribution $N_1$. Guided by Lemma 2.7, we will consider the following case separation:

- **Case 1:** $|n| \sim |n_0| \gg |n_1|$
  - Case 1.1: $|n_0|^2|n-n_0| \lesssim |\phi|
  - Case 1.2: $|n_0|^2|n-n_0| \lesssim |n_1 n_2 n_3|
- **Case 2:** $|n_0| \sim |n_1| \gg |n_2|$
  - Case 2.1: $|n_0|^2|n_0+n_1| \lesssim |\phi|
  - Case 2.2: $|n_0+n_1| \ll |n_3|
- **Case 3:** $|n_0| \sim |n_1| \sim |n_2| \gg |n_3|$
  - Case 3.1: $(n-n_1)(n-n_2)(n_1+n_2) \lesssim |\phi|
  - Case 3.2: $(n-n_1)(n-n_2)(n_1+n_2) \lesssim |n_0|^2|n_3|$
- **Case 4:** $|n| \sim |n_1| \gg |n_2|, |n_1| \gg |n_0| \gg |n_3|$
  - Case 4.1: $|n_1|^2|n-n_1| \lesssim |\phi|
  - Case 4.2: $|n_1|^2|n-n_1| \lesssim |n_0 n_2 n_3|
- **Case 5:** $|n_1| \sim |n_2| \gg |n_0| \gg |n_3|$
Lemma 2.4, we have the following upper bound for the resonance relation:

- **Case 5.1:** $|n_1|^2 |n_1 + n_2| \lesssim |\phi|$
- **Case 5.2:** $|n_1 + n_2| \ll |n_3|$
- **Case 6:** $|n_0| \ll |n_3|$

We see that this covers all the cases: First, the frequency region $|n_0| \gg |n_3|$ is divided into two subregions $|n_0| \gtrsim |n_1|$ and $|n_0| \ll |n_1|$, which are further divided into **Cases 1, 2, 3** and into **Cases 4, 5, 6**, respectively. Then, Lemma 2.7 divides each of **Cases 1, 2, 4, 5** into the two subcases mentioned above, while the division of **Case 3** is based on the fact that $\phi - 3(n - n_1)(n - n_2)(n_1 + n_2) = (n_0 + n_3 + \ldots + n_m)^3 - (n_0^3 + n_3^3 + \ldots + n_m^3) = O(|n_0|^2|n_3|)$. We also observe that in **Case 3** we have

$$\max(|n - n_1|, |n - n_2|, |n_1 + n_2|) \gtrsim |n_0|.$$  

**Cases 1.1–5.1:**

Let $\sigma = \tau - n_3$ and $\sigma_j = \tau_j - n_j^3$, $j = 0, \ldots, m$, denote the modulations. Then, we have the following upper bound for the resonance relation:

$$|\phi| = |\sigma - \sigma_0 - \ldots - \sigma_m| \lesssim \max(|\sigma|, |\sigma_0|, \ldots, |\sigma_m|) = \sigma_{\max},$$

which we can use to gain a power of $\phi$. First, let $\sigma_{\max} = |\sigma|$. Using Hölder’s inequality and Lemma 2.4 we have

$$\|N_1(u_0, \ldots, u_m)\|_{X_{p,1}^{s,-1}} \lesssim \left\| \sum_{n = n_0 + \ldots + n_m} \frac{\langle n \rangle^s |n_0|}{\langle \phi \rangle \frac{1}{2}} \left( \int_{\tau, \tau_0 + \ldots + \tau_m} \frac{1}{\langle \sigma \rangle^{(\sigma_0)^{1-\ldots-(\sigma_m)^{1-}}} \prod_{j=0}^m \|\langle \sigma_j \rangle^{\frac{1}{2}} - \hat{u}_j(n_j)\|_{L^2_t} \right) \right\|_{\ell^p_n} \lesssim \left\| \sum_{n = n_0 + \ldots + n_m} \frac{\langle n \rangle^s |n_0|}{\langle \phi \rangle \frac{1}{2}} \prod_{j=0}^m \|\langle \sigma_j \rangle^{\frac{1}{2}} - \hat{u}_j(n_j)\|_{L^2_t} \right\|_{\ell^p_n}. \quad (3.18)$$

For the $X_{p,2}^{s,-\frac{1}{2}}$-norm, the same approach holds. If $\sigma_{\max} = |\sigma_j|$ for $j = 0, \ldots, m$, we need to use $\frac{1}{2}$ power of $\langle \sigma_j \rangle$ and proceed by duality in time and estimate the stronger norm $X_{p,2}^{s,-\frac{1}{2}}$. It then suffices to estimate (3.18). By using Hölder’s inequality, we obtain

$$3.18 \lesssim \sup_n \left( I_\phi(n) \right) \frac{1}{p'} \prod_{j=0}^m \|u_j\|_{X_{p,2}^{s,'}}.$$

where

$$I_\phi(n) = \sum_{n = n_0 + \ldots + n_m} \left| \frac{\langle n \rangle^s |n_0|}{\langle \phi \rangle \frac{1}{2} \langle n_0 \rangle^s \langle n_1 \rangle^s \ldots \langle n_m \rangle^s} \right|^{p'},$$

and it suffices to bound $I_\phi(n)$ uniformly in $n$. To this end, we must consider the lower bound for $\phi$. In **Case 1.1**, we have

$$I_\phi(n) \lesssim \sum_{n_1, \ldots, n_m} \frac{1}{(n_1 + \ldots + n_m)^{\frac{sp'}{p'}} \langle n_1 \rangle^{sp'} \ldots \langle n_m \rangle^{sp'}} \lesssim 1$$
by applying Lemma 2.5 given that $s > 1 - \frac{1}{p} - \frac{1}{2m}$. In Case 2.1, $|n_0| \sim |n_1| \gg |n_2|$, if $|n| \lesssim |n_0 + n_1|$, then

$$I_{\phi}(n) \lesssim \sum_{n_1, \ldots, n_m} \frac{1}{\langle n \rangle^{(s+\frac{3}{2})p'}} \lesssim \left( \sum_{n} \frac{1}{\langle n \rangle^{(s+\frac{3}{2})p'}} \right)^m \lesssim 1,$$

under the following assumption

$$s > \max \left( \frac{1}{2}, 1 - \frac{1}{p} - \frac{1}{2m} \right). \quad (3.19)$$

If $|n| \gg |n_0 + n_1|$, then $|n| \sim |n_2 + \ldots + n_m| \lesssim |n_2|$ and we have

$$I_{\phi}(n) \lesssim \sum_{n=n_0+\ldots+n_m} \frac{1}{\langle n_0 + n_1 \rangle^{2sp'}} \lesssim \left( \sum_{n_0, n_1} \frac{1}{\langle n_0 + n_1 \rangle^{(s+\frac{3}{2})p'}} \right)^{m-2} \lesssim 1,$$

if (3.19) holds. In Case 3.1, if $|n_0||n_1|n_2| \lesssim |\phi|$, we use Lemma 2.5 to obtain

$$I_{\phi}(n) \lesssim \sum_{n_0, n_2, \ldots, n_m} \frac{1}{\langle n_0 + n_2 + \ldots + n_m \rangle^{3p'}} \lesssim \left( \sum_{n_0, n_2} \frac{1}{\langle n_0 + n_2 \rangle^{3p'}} \right) \lesssim 1,$$

for $\beta = \frac{1}{m-1}(ms - \frac{1}{2})p'$ and the estimate follows from Lemma 2.5 if (3.19) holds. If $|n_0||n_1 + n_2|n - n_1| \lesssim |\phi|$, then

$$I_{\phi}(n) \lesssim \sum_{n_1, \ldots, n_m} \frac{1}{\langle n_1 + n_2 \rangle^{3p'}} \lesssim \left( \sum_{n_2, n_3, \ldots, n_m} \frac{1}{\langle n + n_2 \rangle^{2p'}} \right) \lesssim 1,$$

proceeding as in the previous cases by splitting the power of $\langle n_2 \rangle$ between the other frequencies. By exchanging the roles of $n_1$ and $n_2$, we obtain the estimate when $|n_0||n_1 + n_2||n - n_2| \lesssim |\phi|$. In Case 4.1, the estimate follows from that of Case 1.1, by exchanging the roles of $n_0$ and $n_1$. Similarly, the estimate in Case 5.1 follows from that of Case 2.1 by exchanging the roles of $(n_0, n_1)$ with $(n_1, n_2)$.

In Cases 1.2–5.2 and Case 6, we can no longer use the largest modulation. However, note that it suffices to control the stronger norm $X^{s,\frac{3}{2}+}_{p,2}$. 

**Case 1.2:**

Here, we have $|n| \sim |n_0| \gg |n_1|$ and $|n_0|^2|n - n_0| \lesssim |n_1n_2n_3|$. Thus, we can control the multiplier as follows

$$\langle n \rangle^s \langle n_0 \rangle^s \lesssim \langle n_0 \rangle^s \frac{|n_1n_2n_3|}{|n_1 + \ldots + n_m|^{\frac{1}{2}}}.$$
Using Lemma 3.4, we have
\[\|N_1(u_0, \ldots, u_m)\|_{X_{p,2}^{s-\frac{1}{2}+}} \lesssim \| (D^s u_0) \cdot D^{-\frac{1}{2}}(P(D^\frac{1}{2} u_1 \cdot D^\frac{1}{2} u_2) \cdot D^{\frac{1}{2}} u_3 \cdot u_4 \cdots u_m) \|_{X_{p,2}^{0,-\frac{1}{2}+}}\]
\[\lesssim \| u_0 \|_{X_{\rho,2}^{s+\frac{1}{2}}} \| P(D^\frac{1}{2} u_1 \cdot D^\frac{1}{2} u_2) \cdot D^{\frac{1}{2}} u_3 \cdot u_4 \cdots u_m \|_{X_{\rho,2}'^{-\frac{1}{2}+}}\]
and then using Hölder’s inequality in \(n\) (and also Young’s inequality if \(m \geq 4\)),
\[\lesssim \| u_0 \|_{X_{\rho,2}^{s+\frac{1}{2}}} \| P(D^\frac{1}{2} u_1 \cdot D^\frac{1}{2} u_2) \cdot D^{\frac{1}{2}} u_3 \cdot u_4 \cdots u_m \|_{X_{\rho,2}'^{-\frac{1}{2}+}},\]
where \(q = p\) for \(2 < p < 4\) and \(q = \frac{2p}{p-2}\) for \(4 \leq p < \infty\). Applying (3.10) or (3.11), we obtain
\[\| D^{\frac{1}{2}} u_2 \cdot P_0(D^{\frac{1}{2}} u_1 \cdot D^{\frac{1}{2}} u_3) \|_{X_{\rho,2}^{0,0}} + \| D^{\frac{1}{2}} u_1 \cdot P_0(D^{\frac{1}{2}} u_2 \cdot D^{\frac{1}{2}} u_3) \|_{X_{\rho,2}^{0,0}}\]
\[+ \| D^{\frac{1}{2}} u_1 \cdot D^{\frac{1}{2}} u_2 \cdot D^{\frac{1}{2}} u_3 \|_{X_{\rho,2}^{0,0}} \phi' \not= 0 \|_{X_{\rho,2}^{0,0}}\]
\[\lesssim \| u_1 \|_{X_{\rho,2}^{0,0}} \| u_2 \|_{X_{\rho,2}^{0,0}} \| u_3 \|_{X_{\rho,2}^{0,0}} \| u_1 \|_{X_{\rho,2}^{0,0}} \| u_2 \|_{X_{\rho,2}^{0,0}} \| u_3 \|_{X_{\rho,2}^{0,0}},\]
where \(\phi' = (n_1 + n_2)(n_1 + n_3)(n_2 + n_3)\). Using the fact that \(|n_1| \geq \ldots \geq |n_m|\), the estimate follows from Hölder’s inequality if \(2 < p < 4\) and \(s > 1 - \frac{1}{p} - \frac{2}{p m}\), or \(4 \leq p < \infty\) and \(s > 1 - \frac{1}{p} - \frac{1}{pm}\).

**Case 2.2:**
In this case we have \(|n_0| \sim |n_1| \gg |n_2|\) and \(|n_0 + n_1| \ll |n_3|\). Thus, \(|n| \ll |n_2|\), \(|n_0 + n_1 + n_3 + \ldots + n_m| \ll |n_3|\) and we can estimate the multiplier as
\[\langle n \rangle^s |n_0| \lesssim \langle n_2 \rangle^s \langle n_0 \rangle^\frac{1}{2} \langle n_1 \rangle^\frac{1}{2} \lesssim \langle n_2 \rangle^s \langle n_0 \rangle^\frac{1}{2} \langle n_1 \rangle^\frac{1}{2} \lesssim \langle n_2 \rangle^s \langle n_0 \rangle^\frac{1}{2} \langle n_1 \rangle^\frac{1}{2} \lesssim \langle n_0 + n_1 + n_3 + \ldots + n_m \rangle^\frac{1}{2}.\]
Using (3.9) since \(nn_2(n - n_2) \neq 0\) and Young’s inequality, we have
\[\| N_1(u_0, \ldots, u_m) \|_{X_{p,2}^{s-\frac{1}{2}+}} \lesssim \| (D^s u_2) D^{-\frac{1}{2}} (D^\frac{1}{2} u_0 \cdot D^\frac{1}{2} u_1 \cdot D^\frac{1}{2} u_3 \cdot u_4 \cdots u_m) \|_{X_{p,2}^{0,-\frac{1}{2}+}}\]
\[\lesssim \| u_2 \|_{X_{p,2}^{s+\frac{1}{2}}} \| D^\frac{1}{2} u_0 \cdot D^\frac{1}{2} u_1 \cdot D^\frac{1}{2} u_3 \cdot u_4 \cdots u_m \|_{X_{p,2}'^{-\frac{1}{2}+}}\]
The intended estimate follows from the argument in Case 1.2 exchanging the roles of \((u_0, u_1, u_2, u_3)\) by \((u_2, u_0, u_1, u_3)\). Note that \((n_0 + n_1)(n_0 + n_3)(n_1 + n_3) \neq 0\), with the last factor nonzero because \(|n_1| \gg |n_3|\).

**Case 3.2:**
Since \(|n_0| \sim |n_1| \sim |n_2| \gg |n_3|\) and
\[\|(n - n_1)(n - n_2)(n_1 + n_2)\| \lesssim |n_0|^2 |n_3|,\]
for \(N_{\min} = \min(|n - n_1|, |n - n_2|, |n_1 + n_2|)\), we get
\[N_{\min}^2 |n_0| \lesssim |n_0|^2 |n_3| \implies N_{\min} \lesssim |n_0 n_3|^{\frac{1}{2}}.\]
If $N_{\min} = |n_1 + n_2|$, then $|n_1 + \ldots + n_m| \lesssim |n_1 + n_2| + |n_3| \lesssim |n_0 n_3|^\frac{1}{4}$ and we can estimate the multiplier as follows

\[
\langle n \rangle^s |n_0| \lesssim \langle n_0 \rangle^s |n_1 n_2|^\frac{1}{2} \frac{|n_0 n_3|^\delta}{|n_1 + \ldots + n_m|^\delta} \sim \langle n_0 \rangle^s |n_1|^\frac{1}{4} |n_2|^\frac{1}{4} |n_3|^\frac{1}{4} |n_0|^\frac{1}{4} |n_1|^\frac{1}{4} |n_2|^\frac{1}{4} |n_3|^\frac{1}{4} \frac{|n_0 n_3|^\delta}{|n_1 + \ldots + n_m|^\delta},
\]

where $\delta = \frac{3}{2} - \frac{3}{2p}$ for $2 < p < 4$ and $\delta = 1 - \frac{1}{p}$ for $4 \leq p < \infty$. Using (3.3) and Young’s inequality, we obtain the following

\[
\|N_1(u_0, \ldots, u_m)\|_{X^{s, \frac{1}{2}}_{p, 2}} \lesssim \left\| D^{\frac{1}{4}} u_0 \cdot D^{-\delta} \left( D^{\frac{1}{4}} u_1 \cdot D^{\frac{1}{4}} u_2 \cdot D^{\frac{1}{4}} u_3 \cdot u_4 \cdot \ldots \cdot u_m \right) \right\|_{X^{0, \frac{1}{2}}_{p, 2}} \\
\lesssim \min \left( \|u_0\|_{X^{\frac{1}{2}, 1}} \|u_1\|_{X^{\frac{1}{2}, 1}} \|u_2\|_{X^{\frac{1}{2}, 1}} \|u_3\|_{X^{\frac{1}{2}, 1}} \right) \\
\lesssim \|u_0\|_{X^{\frac{1}{2}, 1}} \|u_1\|_{X^{\frac{1}{2}, 1}} \|u_2\|_{X^{\frac{1}{2}, 1}} \|u_3\|_{X^{\frac{1}{2}, 1}},
\]

using multilinear interpolation for the last inequality, where $r = p$ for $2 < p < 4$ and $r = 4$ for $4 \leq p < \infty$. The estimate follows if $2 < p < 4$ and

\[
s > \max \left( 1 - \frac{1}{p} - \frac{p - 2}{8p}, 1 - \frac{1}{3p}, (1 - \frac{1}{p} - \frac{1}{mp}) 1_{m \geq 4} \right) \\
= 1 + \frac{1}{p} - \min \left( \frac{p - 2}{8p}, \frac{1}{mp} \right),
\]

or $4 \leq p < \infty$ and

\[
s > \max \left( 1 - \frac{1}{p} - \frac{1}{4p}, 1 - \frac{1}{p} - \frac{1}{3p}, (1 - \frac{1}{p} - \frac{1}{mp}) 1_{m \geq 4} \right) \\
= 1 - \frac{1}{p} - \min \left( \frac{1}{4p}, \frac{1}{mp} \right).
\]

If $N_{\min} = |n - n_1| = |n_0 + n_2 + \ldots + n_m|$, then

\[
\langle n \rangle^s |n_0| \lesssim \langle n_1 \rangle^s |n_0 n_2|^\frac{1}{2} \frac{|n_0 n_3|^\delta}{|n_0 + n_2 + \ldots + n_m|^\delta} \sim \langle n_1 \rangle^s |n_0|^\frac{3}{4} |n_2|^\frac{3}{4} |n_3|^\frac{3}{4} |n_0|^\frac{3}{4} |n_1|^\frac{3}{4} |n_2|^\frac{3}{4} |n_3|^\frac{3}{4} \frac{|n_0 n_3|^\delta}{|n_0 + n_2 + \ldots + n_m|^\delta},
\]

and the estimate follows from the previous argument, exchanging the roles of $u_0$ and $u_1$. Similarly, if $N_{\min} = |n - n_2| = |n_0 + n_1 + n_3 + \ldots + n_m|$, we can control the multiplier as follows

\[
\langle n \rangle^s |n_0| \lesssim \langle n_2 \rangle^s |n_0 n_1|^\frac{1}{2} \frac{|n_0 n_3|^\delta}{|n_0 + n_1 + n_3 + \ldots + n_m|^\delta} \sim \langle n_2 \rangle^s |n_0|^\frac{3}{4} |n_1|^\frac{3}{4} |n_3|^\frac{3}{4} |n_0|^\frac{3}{4} |n_1|^\frac{3}{4} |n_3|^\frac{3}{4} \frac{|n_0 n_3|^\delta}{|n_0 + n_1 + n_3 + \ldots + n_m|^\delta},
\]

and the estimate follows from the same arguments.
Case 4.2:
Since $|n| \sim |n_1| \gg |n_2|$, $|n_1| \gg |n_0| \gg |n_3|$ and $|n_1|^2 |n - n_1| \lesssim |n_0 n_2 n_3|$, we estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n \rangle^s |n_0 n_2 n_3|^{\frac{1}{2}} \frac{|n_3|^{\frac{1}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\frac{1}{2}}}.$$ 

The estimate follows from the strategy in Case 1.2, exchanging the roles of $u_0$ and $u_1$, considering two possibilities $(n_0 + n_2)(n_0 + n_3)(n_2 + n_3) \neq 0$ and $n_2 + n_3 = 0$, $|n_0| \gg |n_2|$. 

Case 5.2:
In this case we have $|n_1| \sim |n_2| \gg |n_0| \gg |n_3|$ and $|n_1 + n_2| \ll |n_3|$. Then, $|n| \sim |n_0|$, $|n_1 + \ldots + n_m| \lesssim |n_3|$ and we estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n \rangle^s \frac{|n_1 n_2 n_3|^{\frac{1}{2}}}{|n_1 + \ldots + n_m|^{\frac{1}{2}}}. $$

The estimate follows from the approach in Case 1.2, since $(n_1 + n_2)(n_1 + n_3)(n_2 + n_3) \neq 0$. 

Case 6:
Let $|n_0| \lesssim |n_3|$. Let us first consider the case when $n_2 + n_3 = 0$. Using Young’s and Hölder’s inequalities, we obtain the following

$$\|N_1(u_0, \ldots, u_m)\|_{X^{s, \frac{1}{p} +}} \lesssim \left\| D^{s + \frac{1}{p} - \frac{1}{2}} u_0 \cdot D^s u_1 \cdot P_0(D^{\frac{3}{2} - \frac{1}{p} - s} u_2 \cdot u_3) \cdot u_4 \cdots u_m \right\|_{X^{0, \frac{1}{p} +}}$$

$$\lesssim \left\| D^{s + \frac{1}{p} - \frac{1}{2}} u_0 \cdot D^s u_1 \right\|_{X^{0, \frac{1}{2} +}} \left\| u_2 \right\|_{X^{s, \frac{1}{2} +}} \left\| u_3 \right\|_{X^{s, \frac{1}{2} +}} \prod_{j=4}^m \left\| u_j \right\|_{X^{1, \frac{1}{2} +}}.$$ 

Using (3.6) and Hölder’s inequality, we have

$$\left\| D^{s + \frac{1}{p} - \frac{1}{2}} u_0 \cdot D^s u_1 \right\|_{X^{0, \frac{1}{2} +}} \lesssim \left\| u_0 \right\|_{X^{s, \frac{1}{2} +}} \left\| u_1 \right\|_{X^{s, \frac{1}{2} +}} \lesssim \left\| u_0 \right\|_{X^{s, \frac{1}{2} +}} \left\| u_1 \right\|_{X^{s, \frac{1}{2} +}}.$$ 

Then, the estimate follows from $|n_3| \geq \ldots \geq |n_m|$ given that $s > 1 - \frac{1}{p} - \frac{1}{2m}$. If $n_2 + n_3 \neq 0$, note that $|n_0 + n_2 + \ldots + n_m| \lesssim |n_2|$, so we can estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n \rangle^s |n_0 n_3|^{\frac{1}{2}} \lesssim \langle n \rangle^s \frac{|n_0 n_2 n_3|^{\frac{1}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\frac{1}{2}}}. $$

Then, we can proceed as in Case 1.2, exchanging the roles of $(u_0, u_1, u_2, u_3)$ by $(u_1, u_2, u_3, u_0)$, and using the fact that $(n_0 + n_2)(n_0 + n_3)(n_2 + n_3) \neq 0$. This completes the estimate of $N_1$.

Lastly, we want to estimate $N_\alpha$ for odd $3 \leq \alpha \leq m$. Note that

$$N_\alpha(u_0, \ldots, u_m) = P_0(u_1 u_2) \cdots P_0(u_{\alpha-2} u_{\alpha-1}) N'_\alpha(u_0, u_\alpha, \ldots, u_m),$$ 

where

$$F_{n}(N'_\alpha(u_0, \ldots, u_m))(t, n) = \sum_{n=n_0+n_{\alpha}+\ldots+n_m \atop n_0+n_{\alpha}+1 \neq 0 \atop n_0+n_{\alpha}+n_{\alpha-1} \neq 0} i n_0 \tilde{u}_0(n_0) \tilde{u}_\alpha(n_{\alpha}) \cdots \tilde{u}_m(n_m).$$ 

For $\alpha = m$ or $\alpha = m - 1$, the resonance relation satisfies $|\phi| \sim |m n_0 n_m|$ and $|\phi| \sim |(n - n_0)(n - n_{m-1})(n - n_m)|$, respectively, thus we can proceed as in Cases 1.1–5.1, following
the same strategy in time and using Cauchy-Schwarz inequality in space on the terms $P_0(u_1 u_2), \ldots, P_0(u_{\alpha-2} u_{\alpha-1})$. For $3 \leq \alpha \leq m-2$, an analogous case separation holds by replacing $(n_1, n_2, n_3)$ by $(n_\alpha, n_{\alpha+1}, n_{\alpha+2})$. In Cases 1.1–5.1 we follow the strategy mentioned above. To illustrate the strategy in the remaining cases, consider Case 1.2. Following the strategy for $\mathcal{N}_1$, we have

$$
\|\mathcal{N}_\alpha(u_0, \ldots, u_m)\|_{X_{p,2}^{\frac{1}{(s-\frac{1}{2})+}}}
\lesssim \left\| (D^s u_0) \cdot D^{-\frac{1}{2}} \left( \left( \prod_{j=1}^{\alpha-2} P_0(u_j u_{j+1}) \right) \cdot P(D^{\frac{1}{2}} u_\alpha \cdot D^{\frac{1}{2}} u_{\alpha+1}) \cdot D^{\frac{1}{2}} u_{\alpha+2} \cdot \prod_{i=\alpha+3}^m u_i \right) \right\|_{X_{p,2}^{\frac{1}{s-\frac{1}{2}}+}}
\lesssim \|u_0\|_{X_{p,2}^{\frac{1}{s-\frac{1}{2}}+}} \left( \prod_{j=1}^{\alpha-1} \|u_j\|_{X_{p,1}^{\frac{1}{s-\frac{1}{2}}+}} \right) \left\| P \left( D^{\frac{1}{2}} u_\alpha \cdot D^{\frac{1}{2}} u_{\alpha+1} \right) \cdot D^{\frac{1}{2}} u_{\alpha+2} \cdot \prod_{i=\alpha+3}^m u_i \right\|_{X_{p,2}^{\frac{1}{s-\frac{1}{2}}+}}
$$

using Young’s and Cauchy-Schwarz inequalities in the last step. The last term can be estimated following the same approach as for $\mathcal{N}_1$.

4. Almost sure global well-posedness and invariance of the Gibbs measure

In this section, we extend the solutions of Theorem 1.1 globally-in-time and show invariance of the Gibbs measure under the dynamics of gKdV (1.1), for mean zero initial data. We closely follow the argument in [40].

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{g_n\}_{n \in \mathbb{Z}_+}, Z_\ast = \mathbb{Z} \setminus \{0\}$, a sequence of complex-valued standard Gaussian random variables with $g_{-n} = \overline{g_n}$. We can define the Gaussian measure $\rho$ as the induced probability measure under the map

$$
\omega \mapsto u^\omega(x) = \sum_{n \in \mathbb{Z}_+} \frac{g_n(\omega)}{|n|} e^{inx} \in \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T}) \text{ a.s.,} \tag{4.1}
$$

or equivalently, as $\rho = \mathbb{P} \circ u^{-1}$ the push-forward of the map in (4.1), with the following density

$$
d\rho = Z^{-1} e^{-\frac{1}{2} \int \partial_x u^2} du.
$$

Further details on the construction of Gaussian measures in Banach spaces can be found in [25, 34], for example. Before discussing the construction of the Gibbs measure $\mu$, we recall the following tail estimate for $\rho$.

**Lemma 4.1.** Let $(s,p)$ satisfy $(s-1)p < -1$ and $K > 0$. Then, the following estimate holds

$$
\rho(\|u\|_{\mathcal{F}L^{s,p}} > K) \leq C e^{-cK^2},
$$

for some constants $C, c > 0$ depending only on $s$ and $p$. 

This lemma follows from the fact that $\mathcal{F}L^{s,p}(\mathbb{T})$ is an abstract Wiener space for $(s-1)p < -1$ (see [2], [10]) and from Fernique’s theorem [22]. Now, we view the Gibbs measure $\mu$ in (1.2) as a weighted Gaussian measure

$$d\mu = Z^{-1}e^{\mp \frac{1}{s-1} \int \rho u_{t} u^{k+1}dx} dp(u).$$

In the defocusing case ($+$ in (1.1)) and for odd $k \geq 1$, the measure $\mu$ is well-defined as a measure in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \leq p < \infty$ and $1 - \frac{1}{p} < \frac{1}{k+1} < 1 - \frac{1}{p}$, and it is absolutely continuous with respect to $\rho$. This follows easily from Sobolev inequality. For the non-defocusing case, Lebowitz-Rose-Speer [35] and Bourgain [4] proposed the introduction of a mass cutoff and instead studied the following Gibbs measure

$$d\mu = Z^{-1} \mathbf{1}_{\{\|u\|_{L_2} \leq R\}} e^{\mp \frac{1}{s-1} \int \rho u_{t} u^{k+1}dx} dp(u).$$

This new measure is known to be normalizable as stated in the following theorem.

**Theorem 4.2** ([35], [4], [50]). Let $k \geq 2$, $R > 0$, and define $F(u)$ by

$$F(u) = e^{\pm \frac{1}{s-1} \int \rho u_{t} u^{k+1}dx} \mathbf{1}_{\{\|u\|_{L_2} \leq R\}},$$

(4.2)

where ‘$\mp$’ above corresponds to ‘$\pm$’ in the equation (1.1). Then, for $1 \leq q < \infty$, we have that $F(u) \in L^q(d\rho)$ if one of the following (a), (b) holds:

(a) $2 \leq k \leq 4$ with ‘$+$’ sign in (1.2) when $k = 3$, and any finite $R > 0$;

(b) $k = 5$ with ‘$+$’ sign in (1.2), and $0 < R < \|Q\|_{L_2(\mathbb{R})}$,

where $Q$ is the (unique) optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$:

$$\|u\|_{L^6(\mathbb{R})}^6 \leq C\|u\|_{L_2(\mathbb{R})}^4 \|\partial_x u\|_{L_2(\mathbb{R})}^2,$$

(4.3)

with $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|\partial_x Q\|_{L_2(\mathbb{R})}^2$. Moreover, if

(c) $k = 5$ with ‘$+$’ sign in (1.2), and $R = \|Q\|_{L_2(\mathbb{R})}$,

then we have $F(u) \in L^1(d\rho)$.

**Remark 4.3.** (i) Theorem 4.2 was first claimed in [35]. Unfortunately, there was a gap in the argument for (b) as stated in [12]. In [4], Bourgain presented a more analytic proof of Theorem 4.2 (a) for any finite $R$ and (b) for small enough $R$. The result for (c) was recently proved by Oh-Sosoe-Tolomeo in [50]. Since in this case we do not have the $L^q$-integrability of the density $F(u)$ for $q > 1$, we cannot directly apply Bourgain’s invariant measure argument to prove Theorem 4.2. However, using a limiting argument, we can extend almost sure global well-posedness and the invariance of the Gibbs measure to this case. We will give a proof of Theorem 1.3 in the cases (a), (b) in this section, and describe how to treat the threshold case (c) in Appendix C.

(ii) The gKdV equations (1.1) and the nonlinear Schrödinger equation (NLS) share a Hamiltonian, and consequently they have the same associated Gibbs measure. An NLS analogue of Theorem 4.2 (i.e., in the complex-valued setting and for general fractional power) was also shown in [50]. In fact, the critical threshold $R = \|Q\|_{L_2(\mathbb{R})}$ for $k = 5$ is related to the existence of finite time blow-up solutions with minimal mass $\|Q\|_{L_2(\mathbb{R})}$ for NLS on $\mathbb{T}$ due to Ogawa-Tsutsumi [42]. Although the quintic focusing gKdV equation (1.1) on the real line also exhibits finite time blow-up solutions with mass arbitrarily close to $\|Q\|_{L_2(\mathbb{R})}$ [38], [36], it is known [37] that there does not exist a blow-up solution with minimal mass (satisfying a certain condition), and moreover there are no analogous results on $\mathbb{T}$. 


Moreover, for all \( \varepsilon > 0 \) it follows from its proof that, for \( F_N(u) := F(P_{\leq N}u) \) and any \( 1 \leq q < \infty \), the estimate

\[
\| F_N \|_{L^q(\mu)} \leq C < \infty
\]  

holds uniformly in \( N \).

In the rest of this section, we focus on the cases (a), (b) of Theorem 4.2. For simplicity, we choose to take the ‘−’ sign in the definition of \( \mu \), as it will not play a role in the results. Lastly, we state the following known result on the convergence of the truncated measures \( \mu_N \) defined by

\[
d\mu_N(u) = Z_N^{-1} F_N(u)d\mu(u).
\]

**Lemma 4.4.** For all \( 1 \leq q < \infty \), we have

\[
F_N(u) \to F(u) \quad \text{in} \quad L^q(\mu) \quad \text{as} \quad N \to \infty.
\]

Moreover, for all \( \varepsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that for \( N \geq N_0 \) and any measurable set \( A \subset \mathcal{F}^{k,p}(\mathbb{T}) \), for \( 1 \leq p < \infty \) and \( s \in (1 - \frac{1}{p} - \frac{1}{k+1}, 1 - \frac{1}{p}) \), the following holds

\[
|\mu_N(A) - \mu(A)| < \varepsilon.
\]

Consider the following truncated gauged gKdV equation \((G\text{-gKdV}_N)\)

\[
\begin{aligned}
\partial_t u_N + \partial_x^3 u_N &= kP_{\leq N} \left( \partial_x (P_{\leq N} u_N) \cdot P(P_{\leq N} u_N)^{k-1} \right), \\
u_N|_{t=0} &= u_0.
\end{aligned}
\]  

(4.5)

The local well-posedness of (4.5) follows from the proof of Theorem 1.1 with the same time of existence \( \delta \sim (1 + \|u_0\|_{\mathcal{F}^{k,p}})^{-\gamma} \) as the solution \( u \) of (1.7). Moreover, as we see below, (4.5) is globally well-posed. Note that we can decompose \( u_N \) into high and low frequencies \( u_N = u_{\text{low}} + u_{\text{high}} \), which solve the following equations

\[
\begin{aligned}
\partial_t u_{\text{high}} + \partial_x^3 u_{\text{high}} &= 0, \\
\partial_t u_{\text{low}} + \partial_x^3 u_{\text{low}} &= kP_{\leq N} \left( \partial_x u_{\text{low}} \cdot P(u_{\text{low}})^{k-1} \right),
\end{aligned}
\]

allowing us to discuss the two decoupled flows \( \Phi_{\text{high}} \) and \( \Phi_{\text{low}} \), respectively. The high frequency part evolves linearly, therefore \( \Phi_{\text{high}}(t) = S(t)P_{>N} \). We can view the low frequency part as a finite-dimensional system of nonlinear ODEs on the Fourier coefficients of \( u_N \). In fact, for \( 0 < |n| \leq N \) and \( c_n = \widehat{u}_N(n) \), we want to solve the following system for \( c = \{c_n\}_{0 < |n| \leq N} \in \mathbb{C}^{2N} \) with \( c_{-n} = \overline{c}_n \),

\[
\frac{d}{dt} c_n = in^3 c_n + k \sum_{n=n_0+\ldots+n_{k-1}} \frac{\sum_{n_{n_0+\ldots+n_{k-1}}} c_{n_0} \ldots c_{n_{k-1}}}{\sum_{n_0 \neq 0}} = N_n(c).
\]  

(4.6)

Since \( N = \{N_n\}_{0 < |n| \leq N} \) is Lipschitz, we can conclude by the Cauchy-Lipschitz theorem that the system of ODEs is locally well-posed. Furthermore, we can extend these solutions
globally-in-time since the $L^2$-norm of $u_N$ is conserved:

$$
\frac{d}{dt} M(u_N)(t) = 2 \int u_N \left( - \partial_x^3 u_N + k P \leq N \left( \partial_x P \leq N u_N \cdot P(P \leq N u_N)^{k-1} \right) \right) dx
$$

$$
= \int \partial_x \left( \partial_x u_N \right)^2 dx + \frac{2k}{k+1} \int \partial_x \left( P \leq N u_N \right)^{k+1} dx
$$

$$
- k P_0 \left( P \leq N u_N \right)^{k-1} \int \partial_x \left( P \leq N u_N \right)^2 dx = 0. \quad (4.7)
$$

Thus, $M(u_N)(t) = M(u_0)$. In addition, the mass is also conserved for $u_{\text{low}}, M(u_{\text{low}})(t) = M(P \leq N u_0)$, and the solution of (4.6) exists globally-in-time, proving that $u_N$ extends to a global solution of (4.5). We denote the flow of $G$-gKdV$_N$ (4.5) by $\Phi_N(t)$.

We now focus on proving invariance of the Gibbs measure associated with (4.5). We first decompose the measure $\rho = \rho_N \otimes \rho_N^\perp$, where

$$
d\rho_N = Z_N^{-1} e^{-\frac{1}{2} \sum_{0 < |n| \leq N} |g_n|^2} \prod_{0 < |n| \leq N} dg_n,
$$

$$
d\rho_N^\perp = \tilde{Z}_N^{-1} e^{-\frac{1}{2} \sum_{|n| > N} |g_n|^2} \prod_{|n| > N} dg_n,
$$

which are probability measures in $\mathcal{F}L^{s,p}(T)$ for $s < 1 - \frac{1}{p}$. Let $\tilde{\mu}_N$ denote the finite dimensional Gibbs measure associated with density

$$
d\tilde{\mu}_N(u) = Z_N^{-1} F_N(u) d\rho_N(u).
$$

Then, $\mu_N = \tilde{\mu}_N \otimes \rho_N^\perp$ is the Gibbs measure associated with $G$-gKdV$_N$ (4.5).

**Proposition 4.5.** The finite-dimensional Gibbs measure $\tilde{\mu}_N$ is invariant under the flow $\Phi_{\text{low}}$. Moreover, the Gibbs measure $\mu_N$ is invariant under the flow $\Phi_N$ of $G$-gKdV$_N$ (4.5).

**Proof.** We follow the strategy in [40]. We start by establishing the invariance of $\tilde{\mu}_N$ under the flow of $\Phi_{\text{low}}$. The conservation of mass for $u_{\text{low}}$ follows from the calculation in (4.7) by replacing $u_N$ by $u_{\text{low}} = P \leq N u_N$. An analogous straightforward computation establishes the conservation of the Hamiltonian for $u_{\text{low}}$. It remains to show the invariance of the Lebesgue measure on $\mathbb{C}^{2N}$ with respect to the system defined in (4.6). We can rewrite the system as

$$
\frac{d}{dt} a_n = \text{Re} \left( N_n(\{a_n, b_n\}) \right), \quad \frac{d}{dt} b_n = \text{Im} \left( N_n(\{a_n, b_n\}) \right),
$$

where $c_n = a_n + i b_n$. Thus, invariance of the Lebesgue measure follows from Liouville’s theorem once we establish that the divergence of the vector field vanishes:

$$
\sum_{1 \leq |n| \leq N} \left( \frac{\partial \text{Re}(N_n)}{\partial a_n} + \frac{\partial \text{Im}(N_n)}{\partial b_n} \right) = 0. \quad (4.8)
$$
For $1 \leq |n| \leq N$, we have

\[
\frac{\partial \text{Re}(N_n)}{\partial a_n} = \frac{\partial}{\partial a_n} \left( -n^3 b_n + \frac{k}{2} \sum_{n \neq n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}} - i \rho_n c_{n_0} \cdots c_{n_{k-1}}) \right)
\]

\[
= \frac{k}{2} \sum_{n = n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}} - i \rho_n c_{n_0} \cdots c_{n_{k-1}})
\]

\[
= k(k-1) \sum_{n \neq n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}} - i \rho_n c_{n_0} \cdots c_{n_{k-1}})
\]

Similarly, we have

\[
\frac{\partial \text{Im}(N_n)}{\partial b_n} = \frac{\partial}{\partial b_n} \left( n^3 a_n + \frac{k}{2i} \sum_{n \neq n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}} + i \rho_n c_{n_0} \cdots c_{n_{k-1}}) \right)
\]

\[
= k(k-1) \sum_{n \neq n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}})
\]

Since

\[
\sum_{0=n_0, n \neq n_{k-1}} (i \rho_n c_{n_0} \cdots c_{n_{k-1}}) = \int_\mathbb{T} \partial_x u_{\text{low}} \cdot u_{\text{low}}^{k-2} \, dx = 0,
\]

we conclude (4.5) after summing up over $n$. Lastly, the invariance of $\mu_N = \tilde{\mu}_N \otimes \rho_N^{1/2}$ under the flow $\Phi_N = (\Phi_{\text{low}}, \Phi_{\text{high}})$ follows from that of $\tilde{\mu}_N$ under the flow $\Phi_{\text{low}}$ and the invariance of Gaussian measures under rotation. \hfill \Box

Let $2 < p < \infty$ and $s_s = s_s(p)$ given by Theorem 1.1 such that (1.1) and (1.7) are locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s_s < s < 1 - \frac{1}{p}$. The following two lemmas can be shown through the method in [4] (see also [27, 10, 45, 40]). The proof of Lemma 4.6 requires the tail estimate in Lemma 4.1 Theorem 1.1 Proposition 4.5 and (4.4). Lemma 4.7 is purely deterministic and follows from the local theory for $G$-gKdV (1.7). Proofs of these lemmas will be given in Appendix [D]

Lemma 4.6. Let $s_s < s < 1 - \frac{1}{p}$. Then, there exists $C_0 > 0$ (independent of $s$) and $C_s > 0$ such that: for all $N \in \mathbb{N}$, $T \geq 1$, $0 < \varepsilon \leq \frac{1}{2}$, $A \geq 1$, there exists $\Omega_N^s(T, \varepsilon, A) \subset \mathcal{F}L^{s,p}(\mathbb{T})$ such that:

(a) $\mu_N(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Omega_N^s(T, \varepsilon, A)) < \varepsilon$.

(b) For $u_0 \in \Omega_N^s(T, \varepsilon, A)$, the solution $u_N$ to (4.5) satisfies

\[
\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq AC_0 C_s \left( \frac{1}{\log T} \right)^{1/2}, \quad |t| \leq T.
\]
(c) For $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, if the solution $u_N$ to (4.5) satisfies
\[
\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq AC_0(\log \frac{T}{\varepsilon})^{\frac{1}{2}}, \quad |t| \leq T,
\]
then $u_0 \in \Omega^s_{N}(T,\varepsilon, A)$.

**Lemma 4.7.** For any $s_* < s < \sigma < 1 - \frac{1}{p}$, $T \geq 1$, and $K \geq 1$, there exists $N_0 \in \mathbb{N}$ such that:

(a) Let $N \geq N_0$ and $u_N \in C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T}))$ be the solution of $\mathcal{G}$-gKdV$_N$ (4.5) with initial data $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$. Assume that $\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq K$ for $|t| \leq T$. Then, there exists a unique solution $u \in C([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})) \cap Z^{s,\frac{1}{2}}_p(T)$ to $\mathcal{G}$-gKdV (1.7) with $u(0) = u_0$ satisfying
\[
\|u(t) - P_{\leq N}u_N(t)\|_{\mathcal{F}L^{s,p}} \leq \left(\frac{N_0}{N}\right)^{\sigma - s} K, \quad |t| \leq T.
\]
In particular, $\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2K$ for $|t| \leq T$.

(b) Let $u \in C([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})) \cap Z^{s,\frac{1}{2}}_p(T)$ be a solution of $\mathcal{G}$-gKdV (1.7) with $u(0) = u_0$ satisfying $\|u(t)\|_{\mathcal{F}L^{s,p}} \leq K$ for $|t| \leq T$. Then, for any $N \geq N_0$, the solution $u_N$ of $\mathcal{G}$-gKdV$_N$ (4.5) with initial data $u_0$ satisfies
\[
\|u(t) - P_{\leq N}u_N(t)\|_{\mathcal{F}L^{s,p}} \leq \left(\frac{N_0}{N}\right)^{\sigma - s} K, \quad |t| \leq T.
\]
In particular, $\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq 3K$ for $|t| \leq T$.

**Remark 4.8.** We can choose, for example, $N_0 \sim \exp\left(\frac{CK}{\sigma - s}\right)$ with $\gamma = \frac{k-1}{\theta}$ and $\theta > 0$ given in Proposition 3.1.

Using Lemma 4.6 and Lemma 4.7, we establish almost a.s. global well-posedness of the $\mathcal{G}$-gKdV equation (1.7).

**Proposition 4.9.** Let $s_* < s < 1 - \frac{1}{p}$, $T \geq 1$, and $0 < \varepsilon \leq \frac{1}{2}$. For any $A \geq 1$, there exists $N_1 = N_1(A) \in \mathbb{N}$ such that the set $\Sigma^s_{T,\varepsilon}(A) := \Omega^s_{N_1}(T,\varepsilon, A)$, with $\sigma = \frac{1}{2}(s + 1 - \frac{1}{p})$, satisfies:

(a) $\mu(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Sigma^s_{T,\varepsilon}(A)) < \varepsilon$;

(b) For $u_0 \in \Sigma^s_{T,\varepsilon}(A)$, there exists a unique corresponding solution $u \in C([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})) \cap Z^{s,\frac{1}{2}}_p(T)$ to $\mathcal{G}$-gKdV (1.7) on $[-T,T]$ such that
\[
\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2\sqrt{2AC_0C_\sigma(\log \frac{T}{\varepsilon})^{\frac{1}{2}}}, \quad |t| \leq T.
\]

**Proof.** Lemma 4.6(b) shows that for $u_0 \in \Sigma^s_{T,\varepsilon}(A)$ we have
\[
\|\Phi_{N_1}(t)(u_0)\|_{\mathcal{F}L^{s,p}} \leq AC_0C_\sigma(\log \frac{2T}{\varepsilon})^{\frac{1}{2}}, \quad |t| \leq T.
\]
From Lemma 4.7(a), there exists a unique solution $u$ to $\mathcal{G}$-gKdV on $[-T,T]$ with $u(0) = u_0$ satisfying
\[
\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2AC_0C_\sigma(\log \frac{2T}{\varepsilon})^{\frac{1}{2}}, \quad |t| \leq T,
\]
provided $N_1$ is large enough. The intended estimate follows from $\log(2x) \leq 2\log x$ for $x \geq 2$. Note that from Lemma 4.4 there exists $N_2 \in \mathbb{N}$ such that
\[
|\mu_N(A) - \mu(A)| < \frac{\varepsilon}{2}.
\]
for any $N \geq N_2$ and measurable set $A$. By taking $N_1$ larger so that the previous bound holds, using Lemma 4.6(a) and the fact that $\mathcal{F}L^{s,p}(\mathbb{T})$, $\mathcal{F}L^{q,p}(\mathbb{T})$ have full $\mu$-measure, we have

$$
\mu(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Sigma^{\mu}_{T,\varepsilon}(A)) \leq \mu_{N_1}(\mathcal{F}L^{q,p}(\mathbb{T}) \setminus \Omega^{N_1}_{N_1}(T,\frac{\varepsilon}{2},A)) + \frac{\varepsilon}{2} < \varepsilon,
$$

which completes the proof.

We can now show Theorem 1.4.

**Proof of Theorem 1.4.** This proof follows the approaches in [57, 40]. We first establish almost sure global well-posedness of $G$-gKdV. Define an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ by $s_1 = \frac{1}{2}(s_0 + 1 - \frac{1}{p})$ and $s_{j+1} = \frac{1}{2}(s_j + 1 - \frac{1}{p})$, which converges to $1 - \frac{1}{p}$ as $j \to \infty$. Fix $0 < \varepsilon \leq 1$ and let $T_j = 2^j$, $\varepsilon_j = 2^{-j}\varepsilon$, $j \in \mathbb{N}$. For $\Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ as defined in Proposition 4.9 with $s = s_j$ and $\sigma = s_{j+1}$, let

$$
\Sigma_\varepsilon = \bigcap_{j=1}^{\infty} \Sigma^{s_j}_{T_j,\varepsilon_j} = \bigcap_{j=1}^{\infty} \left( \bigcup_{k=1}^{\infty} \Sigma^{s_j}_{T_j,\varepsilon_j}(2^k) \right).
$$

Lastly, let $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_{\frac{1}{n}}$.

First note that $\Sigma \subset \bigcap_{s=1-\frac{1}{p}}^{\infty} \mathcal{F}L^{s,p}(\mathbb{T})$. Let $u_0 \in \Sigma$, then for any $j \in \mathbb{N}$, we have $u_0 \in \Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ for some $\varepsilon = \frac{1}{n}$ and $k \in \mathbb{N}$. Hence, by Proposition 4.9 there exists a solution $u \in C([-T_j, T_j]; \mathcal{F}L^{s_j,p}(\mathbb{T})) \cap Z^{s_j,\frac{3}{2}}_{p,j}(T_j)$ of $G$-gKdV with $u(0) = u_0$. By uniqueness of local solutions in $Z^{s_j,\frac{3}{2}}_{p,j}(T)$, we obtain a unique global solution $u \in \bigcap_{k=1-\frac{1}{p}}^{\infty} C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T}))$.

Moreover, since $\Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ is closed in $\mathcal{F}L^{s_j,p}(\mathbb{T})$ and $\mu(\Sigma_\varepsilon) \leq \sum_{j \in \mathbb{N}} \mu((\Sigma^{s_j}_{T_j,\varepsilon_j})^c) < \varepsilon$, $\Sigma$ is $\mu$-measurable and $\mu(\Sigma^c) = 0$.

We now establish that $\Phi(t)\Sigma = \Sigma$ for any $t \in \mathbb{R}$, where $\Phi(t) : u_0 \mapsto u(t)$ denotes the solution map of $G$-gKdV defined above. Fix $t \in \mathbb{R}$. It suffices to show that $\Phi(\tau)\Sigma \subset \Sigma$, as the other inclusion follows from this and the reversibility of the flow. It suffices to show that $\Phi(\tau)\Sigma_\varepsilon \subset \Sigma_\varepsilon$, $\varepsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$. We actually establish that if $|\tau| \leq T_k$ for some $k \in \mathbb{N}$, then for every $i \in \mathbb{N}$, $\Phi(\tau)\Sigma^{s_j}_{T_{j-i},\varepsilon_j} \subset \Sigma^{s_i}_{T_{j-i},\varepsilon}$ for $j = \max(i+2, k+1)$, from which the intended result follows. Let $u_0 \in \Sigma^{s_j}_{T_{j-i},\varepsilon_j}$, then there exists $A \in 2^N$ such that $u_0 \in \Sigma^{s_j}_{T_{j-i},\varepsilon_j}(A)$. From Proposition 4.9 there exists a solution $u(t)$ of $G$-gKdV for $|t| \leq T_j$ satisfying

$$
||u(t)||_{\mathcal{F}L^{s_j,p}} \leq 2\sqrt{2}AC_0C_{s_j+1}\left(\log \frac{T_i}{\varepsilon_j}\right)^{\frac{1}{2}}, \quad |t| \leq T_j.
$$

Note that $u_\tau(t) = u(\tau + t)$ is a solution of $G$-gKdV with $u_\tau(0) = u(\tau) = \Phi(\tau)u_0$, which belongs to $C([-T_{j-1}, T_{j-1}]; \mathcal{F}L^{s_j,p}(\mathbb{T})) \cap Z^{s_j,\frac{3}{2}}_{p,j}(T_{j-1})$, because $k \leq j - 1$ and then $|t + \tau| \leq T_{j-1} + T_k \leq T_j$. Since the above estimate holds for $u_\tau(t)$ if $|t| \leq T_{j-1}$, from Lemma 4.7(b), it follows that

$$
||\Phi_N(t)\Phi(\tau)u_0||_{\mathcal{F}L^{s_{j-1},p}} \leq 6\sqrt{2}AC_0C_{s_{j+1}}\left(\log \frac{T_{j-1}}{\varepsilon_j}\right)^{\frac{1}{2}}, \quad |t| \leq T_{j-1},
$$

for any $N \geq N_0$. Since $i \leq j - 2$ and $\frac{T_j}{\varepsilon_j} \leq \left(\frac{T_{j-1}}{\varepsilon_j}\right)^{j/j}$ for $0 < \varepsilon \leq 1$, we get that

$$
||\Phi_N(t)\Phi(\tau)u_0||_{\mathcal{F}L^{s_{j+1},p}} \leq 6\sqrt{2j^2}AC_0C_{s_{j+1}}\left(\log \frac{T_{j-1}}{\varepsilon_j}\right)^{\frac{1}{2}}, \quad |t| \leq T_{i+1}.
$$
Consequently, by choosing $\tilde{A} \in 2^{\mathbb{N}}$ such that $6\sqrt{2}/\pi AC_0 C_{s+1} \leq \tilde{A} C_{s+1}$ and $N_1(\tilde{A}) \geq N_0$, and applying Lemma 4.6(c), we conclude that $\Phi(\tau)u_0 \in \Sigma_{T_i,\varepsilon_i}(\tilde{A})$. The group property of $\Phi(t)$ follows from uniqueness of local solutions in $Z_{s+1}^{1/2}(T)$.

Before showing the invariance of $\mu$ under the flow map $\Phi(t)$, we show that $\Phi(t)$ is $\mu$-measurable for every $t \in \mathbb{R}$. It suffices to show the continuity of the map in the topology induced by $F L^{s_1,p}(\mathbb{T})$. Fix $t \in \mathbb{R}$ and $u_0 \in \Sigma$. Consider a sequence $\{u_{0,k}\}_{k \in \mathbb{N}} \subset \Sigma$ converging to $u_0$ in $F L^{s_1,p}(\mathbb{T})$. Let $j \in \mathbb{N}$ such that $|t| \leq T_j$. Then, $u_0 \in \Sigma_{T_j,\varepsilon_j}(A)$ for some $\varepsilon = 1/n$ and some $A$. By Proposition 4.9, we have

$$\sup_{|\tau| \leq T_j} \|\Phi(\tau)u_0\|_{F L^{s_1,p}} \leq 2\sqrt{2}AC_0 C_{s+1} \left(\log \frac{T_j}{\varepsilon_j}\right)^{1/2} =: \Lambda.$$

Let $\tau_0$ be the local time of existence for data of size $2\Lambda$ in $F L^{s_1,p}(\mathbb{T})$. From the Lipschitz continuity of the solution map, we obtain

$$\|\Phi(t)u_0 - \Phi(t)u_{0,k}\|_{F L^{s_1,p}} \leq C \frac{|n|}{n_0} \|u_0 - u_{0,k}\|_{F L^{s_1,p}},$$

as long as the right-hand side is bounded by $\Lambda$, which holds for $k$ large enough. Consequently, by taking $k \to \infty$, we conclude that $\Phi(t)u_{0,k} \to \Phi(t)u_0$ in $F L^{s_1,p}(\mathbb{T})$.

It remains to show the invariance of the Gibbs measure $\mu$ under the flow $\Phi(t)$ of GgKdV (1.7). Having established the flow property of $\Phi(t)$, it suffices to show that for all $G \in L^1(F L^{s_1,p}(\mathbb{T}), d\mu)$ and $t \in \mathbb{R}$, we have

$$\int_{\Sigma} G(\Phi(t)u) d\mu(u) = \int_{\Sigma} G(u) d\mu(u). \quad (4.9)$$

Moreover, it suffices to show (4.9) for $G$ in a dense subset of $L^1(F L^{s_1,p}(\mathbb{T}), d\mu)$. In particular, we choose this set $\mathcal{H}$ as the set of continuous and bounded functions on $F L^{s_1,p}(\mathbb{T})$. Fix $G \in \mathcal{H}$, $t \in \mathbb{R}$ and $\kappa > 0$. We have the following

$$\left| \int_{\Sigma} G(\Phi(t)u) d\mu(u) - \int_{\Sigma} G(u) d\mu(u) \right| \leq \left| \int_{\Sigma} G(\Phi(t)u) d\mu(u) - \int_{\Sigma} G(\Phi(t)u) d\mu_N(u) \right| + \left| \int_{\Sigma} G(\Phi(t)u) d\mu_N(u) - \int_{\Sigma} G(\Phi_N(t)u) d\mu_N(u) \right| + \left| \int_{\Sigma} G(\Phi_N(t)u) d\mu_N(u) - \int_{\Sigma} G(u) d\mu_N(u) \right| + \left| \int_{\Sigma} G(u) d\mu_N(u) - \int_{\Sigma} G(u) d\mu(u) \right|$$

$$= I + II + III + IV.$$

From Lemma 4.4 we have

$$\int G(u) d\mu_N(u) - \int G(u) d\mu(u) = \int \tilde{G}(u) \left( \frac{F_N(u)}{\|F_N\|_{L^1(dp)}} - \frac{F(u)}{\|F\|_{L^1(dp)}} \right) d\rho(u) \to 0, \quad N \to \infty$$

for every bounded measurable function $\tilde{G}$ on $F L^{s_1,p}(\mathbb{T})$. Consequently, since $G$ is bounded and continuous and $\Phi(t)$ is measurable, there exists $N_0 \in \mathbb{N}$ such that $I + IV < 2^{-2}$, for $N \geq N_0$. From Proposition 4.9 the measure $\mu_N$ is invariant under the flow $\Phi_N(t)$, thus $III = 0$. It only remains to estimate $II$. For $0 < \varepsilon \leq \frac{1}{2}$, consider the set $\Sigma(t, \varepsilon) = \Sigma_{T_i,\varepsilon_i}(1) \subset$
\( \mathcal{F}L^{s,p}(\mathbb{T}) \). From Lemma \([14]\) there exists \( N_1 \in \mathbb{N} \) such that \( \mu_N(\Sigma(t,\varepsilon)^c) < \mu(\Sigma(t,\varepsilon)^c) + \varepsilon \) for \( N \geq N_1 \). Since \( \mu(\Sigma(t,\varepsilon)^c) < \varepsilon \) by Proposition \([4,9]\), we see that

\[
\left| \int_{\Sigma(\Sigma(t,\varepsilon))} G(\Phi(t)u) \, d\mu_N(u) - \int_{\Sigma(\Sigma(t,\varepsilon))} G(\Phi_N(t)u) \, d\mu_N(u) \right| \leq 2 \| G \|_{L^\infty}(\mu(\Sigma(t,\varepsilon)^c) + \varepsilon) < \frac{\kappa}{4},
\]

for \( N \geq N_1 \) and by choosing \( \varepsilon \leq \frac{\kappa}{16 \| G \|_{L^\infty}} \). In order to estimate the contribution restricted to \( \Sigma(t,\varepsilon) \), we want to exploit the continuity of \( G \). For \( u_0 \in \Sigma \cap \Sigma(t,\varepsilon) \), from Proposition \([4,9]\) and uniqueness, we have

\[
\| u_0 \|_{\mathcal{F}L^{s_2,p}}, \| \Phi(s)u_0 \|_{\mathcal{F}L^{s_2,p}} \leq 2\sqrt{2}C_0 C_{s_3}(\log \frac{1 + |t|}{\varepsilon})^{\frac{3}{2}}, \quad |s| \leq 1 + |t|.
\]

In particular, the set \( \{ \Phi(t)u_0 : u_0 \in \Sigma \cap \Sigma(t,\varepsilon) \} \) is bounded in \( \mathcal{F}L^{s_2,p}(\mathbb{T}) \) and thus precompact in \( \mathcal{F}L^{s_1,p}(\mathbb{T}) \), which implies that \( G \) is uniformly continuous on this set. Next, from Lemma \([4,7]b\) we have

\[
\| \Phi(t)u_0 - P_{\lesssim N}\Phi_N(t)u_0 \|_{\mathcal{F}L^{s_1,p}} \leq C(t,\varepsilon)N^{-(s_2-s_1)}
\]

for any \( N \) large enough. Thus, it follows that

\[
\| \Phi(t)u_0 - \Phi_N(t)u_0 \|_{\mathcal{F}L^{s_1,p}} \leq \| \Phi(t)u_0 - P_{\lesssim N}\Phi_N(t)u_0 \|_{\mathcal{F}L^{s_1,p}} + \| P_{\gtrsim N}\Phi_N(t)u_0 \|_{\mathcal{F}L^{s_1,p}}
\]

\[
\leq C(t,\varepsilon)N^{-(s_2-s_1)} + N^{-(s_2-s_1)}\| u_0 \|_{\mathcal{F}L^{s_2,p}}
\]

\[
\leq C(t,\varepsilon)N^{-(s_2-s_1)}.
\]

Hence, there exists \( N_2 \in \mathbb{N} \) depending on \( t,\varepsilon \) such that

\[
| G(\Phi(t)u_0) - G(\Phi_N(t)u_0) | < \frac{\kappa}{4}
\]

for \( N \geq N_2 \) and \( u_0 \in \Sigma \cap \Sigma(t,\varepsilon) \), and we can estimate the remaining piece of \( \Pi \),

\[
\left| \int_{\Sigma(\Sigma(t,\varepsilon))} G(\Phi(t)u) \, d\mu_N(u) - \int_{\Sigma(\Sigma(t,\varepsilon))} G(\Phi_N(t)u) \, d\mu_N(u) \right| \leq \int_{\Sigma(\Sigma(t,\varepsilon))} \kappa \, d\mu_N(u) = \frac{\kappa}{4}.
\]

Consequently, we have that \( \Pi < \frac{\kappa}{2} \) for \( N \geq \max(N_1, N_2) \). Combining all the estimates, we obtain

\[
\left| \int_{\Sigma} G(\Phi(t)u) \, d\mu(u) - \int_{\Sigma} G(u) \, d\mu(u) \right| < \kappa.
\]

Since \( \kappa \) is arbitrarily small, we obtain \([4,9]\), as intended. \( \square \)

Lastly, we establish the invariance of the Gibbs measure \( \mu \) under the flow \( \Psi(t) \) of the original gKdV equation \((1.1)\).

**Proof of Theorem** \([1.2] \) Let \( \Sigma \) be the subset of \( \bigcap_{s<1-\frac{2}{p}} \mathcal{F}L^{s,p}(\mathbb{T}) \) constructed in Theorem \([1.4]\) and denote by \( T(y) \), for \( y \in \mathbb{T} \), the spatial translation operator \( f(x) \mapsto f(x-y) \). Note that \( \Sigma \) is invariant under \( T(y) \). Consequently, we can establish the global-in-time dynamics on \( \Sigma \) for the gKdV equation \((1.1)\) with the solution map \( \Psi(t) \) satisfying the flow property as \((1.1)\); see Appendix \([A]\) for the definition of \( \Psi(t) \) and the proof of the group property of it.
It remains to prove the invariance of the Gibbs measure $\mu$. Let $m$ denote the Haar measure on $\mathbb{T}$. Fix $A \subset \Sigma$ and $t \in \mathbb{R}$. Using the invariance of $\mu$ under $T(y)$, the fact that $T(y)$ and $\Psi(t)$ commute and Fubini’s Theorem, we have

$$
\mu(\Psi(-t)A) = \int_{\mathbb{T}} \mu(T(-y)\Psi(-t)A) \, dm(y)
$$

$$
= \int_{\mathbb{T}} \int_{\Sigma} \mathbb{1}_A(T(y)\Psi(t)u_0) \, d\mu(u_0) \, dm(y)
$$

$$
= \int_{\Sigma} \int_{\mathbb{T}} \mathbb{1}_A \left[ T\left( y \pm \int_0^t \mathbb{P}_0(\Phi(t')u_0)^{k-1} \, dt' \right) \Phi(t)u_0 \right] \, dm(y) \, d\mu(u_0).
$$

From the translation invariance of $m$, Fubini’s Theorem and the fact that $\Phi(t)$ commutes with $T(y)$, we have that

$$
\mu(\Psi(-t)A) = \int_{\mathbb{T}} \int_{\Sigma} \mathbb{1}_A(T(y)\Phi(t)u_0) \, d\mu(u_0) \, dm(y)
$$

$$
= \int_{\Sigma} \mu(T(-y)\Phi(-t)A) \, dm(y).
$$

Since $\mu$ is invariant under $T(y)$ and under the flow map $\Phi(t)$ of \footnote{We would like to thank Terence Tao and Rowan Killip for suggesting this argument.} from Theorem \footnote{To see this, we first observe that $T(y) \left[ \sum_{n \in \mathbb{Z}_+} g_n(\omega) \frac{e^{inx}}{|n|} \right] = \sum_{n \in \mathbb{Z}_+} e^{-i\omega n} g_n(\omega) \frac{e^{inx}}{|n|}$. Then, from the invariance of complex Gaussians under rotations, we see that the Gaussian measure $\rho$ is invariant under $T(y)$. This implies the invariance of the Gibbs measure $\mu$ under $T(y)$, since the density $F(u) = \mathbb{1}_{|u|_{L^2} \leq n} e^{-\frac{1}{\alpha} \int_0^1 u^{2k+1} \, dx}$ is invariant under $T(y)$.}, we get $\mu(\Psi(-t)A) = \mu(\Phi(-t)A) = \mu(A)$, as intended. \hfill \Box

**Appendix A. Gauge transformation and solution map for gKdV**

We start by establishing continuity of the (inverse) gauge transformation.

**Lemma A.1.** The (inverse) gauge transformation in \footnote{To see this, we first observe that $T(y) \left[ \sum_{n \in \mathbb{Z}_+} g_n(\omega) \frac{e^{inx}}{|n|} \right] = \sum_{n \in \mathbb{Z}_+} e^{-i\omega n} g_n(\omega) \frac{e^{inx}}{|n|}$. Then, from the invariance of complex Gaussians under rotations, we see that the Gaussian measure $\rho$ is invariant under $T(y)$. This implies the invariance of the Gibbs measure $\mu$ under $T(y)$, since the density $F(u) = \mathbb{1}_{|u|_{L^2} \leq n} e^{-\frac{1}{\alpha} \int_0^1 u^{2k+1} \, dx}$ is invariant under $T(y)$.} is a continuous map on $C([-T,T]; F L^{s,p}(\mathbb{T}))$ given that $1 \leq p < \infty$ and $s > 1 - \frac{1}{p} - \frac{1}{k+1}$.

**Proof.** Let $u$ be any function in $C([-T,T]; F L^{s,p}(\mathbb{T}))$. Consider a sequence $\{u_m\}_{m \in \mathbb{N}}$ in $C([-T,T]; F L^{s,p}(\mathbb{T}))$ converging to $u$ and fix $t \in [-T,T]$. Then,

$$
\left\| \mathcal{G}_{0,t}(u(t)) - \mathcal{G}_{0,t}(u_m(t)) \right\|_{FL^{s,p}} \leq 2\left\| \mathbb{1}_{|n| > N}(\frac{\hat{u}(t,n)}{\hat{u}_m(t,n)}) \right\|_{ell_n^p} + \|u(t) - u_m(t)\|_{FL^{s,p}}
$$

$$
+ \|\mathbb{1}_{|n| \leq N}(\frac{e^{ink_0^t P_0(\Phi(t')u_0)^{k-1} dt'} \hat{u}_m(t,n) - e^{ink_0^t P_0(\Phi(t')u_0)^{k-1} dt'} \hat{u}(t,n)}{\hat{u}_m(t,n)}) \right\|_{\ell^\infty}.
$$

The first two terms on the right-hand side of the estimate converge to zero as $N \to \infty$ and $m \to \infty$, thus it only remains to consider the last one. Using the mean value theorem, we have

$$
\left\| \mathbb{1}_{|n| \leq N}(\frac{e^{ink_0^t P_0(\Phi(t')u_0)^{k-1} dt'} - e^{ink_0^t P_0(\Phi(t')u_0)^{k-1} dt'}}{\hat{u}_m(t,n) - e^{ink_0^t P_0(\Phi(t')u_0)^{k-1} dt'}}) \right\|_{\ell^\infty} \leq |t|N\|u^{k-1} - u_m^{k-1}\|_{C(t)L^1}.
$$

Let $\mathcal{G}_{0,t}$ denote the gauge transformation. Then, from the invariance of complex Gaussians under rotations, we see that the Gaussian measure $\rho$ is invariant under $T(y)$. This implies the invariance of the Gibbs measure $\mu$ under $T(y)$, since the density $F(u) = \mathbb{1}_{|u|_{L^2} \leq n} e^{-\frac{1}{\alpha} \int_0^1 u^{2k+1} \, dx}$ is invariant under $T(y)$.
Since $FL^{s,p}(\mathbb{T}) \hookrightarrow L^{k-1}(\mathbb{T})$ for $s > 1 - \frac{1}{p} - \frac{1}{k-1}$, then the above quantity converges to zero for each fixed $N$, establishing the continuity of $\mathcal{G}_{0,t}$. An analogous proof works for $\mathcal{G}_{0,t}^{-1}$. □

Following the argument in [27], we establish the following result for the (inverse) gauge transformation in (1.8).

**Proposition A.2.** Let $1 \leq p < \infty$ and $s > 1 - \frac{1}{p} - \frac{1}{k-1}$. Then, the (inverse) gauge transformation in (1.8) is not uniformly continuous on arbitrarily small balls of $C([-T,T]; FL^{s,p}(\mathbb{T}))$ centered at the origin.

**Proof.** Let $R > 0$ and $N \in \mathbb{N}$. Define $\{u_{N,j}\}_{N \in \mathbb{N}}$ for $j = 1, 2$ as follows

$$u_{N,1}(t, x) = RN^{-s}(e^{iNx} + e^{-iNx}) + N^{-\frac{s}{k-1}}(e^{iMx} + e^{-iMx}),$$

$$u_{N,2}(t, x) = RN^{-s}(e^{iNx} + e^{-iNx}),$$

with $M = 0$ for $k$ even, and $M = 1$ for $k$ odd. Note that

$$\|u_{N,1}\|_{C_T FL^{s,p}} \lesssim R,$$

for $N$ large enough, and $\|u_{N,2}\|_{C_T FL^{s,p}} \sim R$. Moreover,

$$\|u_{N,1} - u_{N,2}\|_{C_T FL^{s,p}} \sim N^{-\frac{1}{k-1}} \to 0,$$

as $N \to \infty$. Using the mean value theorem, we obtain

$$\|\mathcal{G}_{0,t}(u_{N,1}) - \mathcal{G}_{0,t}(u_{N,2})\|_{C_T FL^{s,p}} \geq TN \left| \int_\mathbb{T} (u_{N,1}^{k-1}(x) - u_{N,2}^{k-1}(x)) \, dx \right|.$$ 

Calculating $\int_\mathbb{T} (u_{N,1}^{k-1} - u_{N,2}^{k-1}) \, dx$, we have

$$\sim \sum_{\substack{1 \leq j \leq k-1 \leq k-1-j \leq 0 \leq l \leq k-1-j \leq 0 \leq m \leq j}} \frac{(k-1)}{j} N^{-s(k-1-j) - \frac{s}{k-1}} \int_\mathbb{T} e^{iNx(k-j-l-1)+iMx(j-m)},$$

thus the nonzero contributions correspond to the choices of indices satisfying $k-1-j = 2l$ and $M(j-2m) = 0$, since $N \gg M$. Consequently, we see that the quantity is dominated by the contribution at $j = k-1$, therefore

$$\|\mathcal{G}_{0,t}(u_{N,1}) - \mathcal{G}_{0,t}(u_{N,2})\|_{C_T FL^{s,p}} \gtrsim 1,$$

which does not decay as $N \to \infty$. □

We now focus on the solution map of gKdV (1.1). We can define the map $\Psi(s, t)$ for $t, s \in \mathbb{R}$ as

$$\Psi(s, t)u_0 = [\Phi(t-s)u_0] \left( x \pm k \int_s^t P_0(\Phi(t' - s)u_0)^{k-1} \, dt' \right),$$

which is a solution of gKdV (1.1) at time $t$, with initial data $u_0$ at time $s$. Since $\Psi(s, t) = \Psi(0, t - s)$, we can denote the solution map of gKdV (1.1) at time $t$ as $\Psi(t) := \Psi(0, t)$. The following lemma establishes that the solution map $\Psi(t)$ satisfies the group property.

**Lemma A.3.** For any $t, s \in \mathbb{R}$ we have that $\Psi(t + s) = \Psi(t)\Psi(s)$. 

Proof. Let \( u_0 \in \mathcal{F}L^{s,p}(\mathbb{T}) \) and \( t, s \in \mathbb{R} \). From the definition of \( \Psi \), we have
\[
\Psi(s + t)u_0 = \Phi(s + t)u_0 \left( x \pm k \int_0^{s+t} P_0(\Phi(t')u_0)^{-1} dt' \right).
\]
Using the group property of \( \Phi \) and a change of variables, we obtain
\[
\Psi(s)\Psi(t)u_0 = \Psi(s) \left[ \Phi(t)u_0 \right] \left( x \pm k \int_0^t P_0(\Phi(t')u_0)^{-1} dt' \right)
\]
\[
= \Phi(s + t)u_0 \left( x \pm k \int_0^t P_0(\Phi(t')u_0)^{-1} dt' \right) \pm k \int_0^s P_0(\Phi(t + t')u_0)^{-1} dt'\],
\]
which is equal to \( \Psi(s + t)u_0 \), establishing the group property of the map. \( \square \)

Appendix B. Lifting the mean zero condition

In this section, we clarify how to construct the solution map for (1.7) without restricting to mean zero initial data. We first consider the set \( \mathcal{F}L^{s,p}_0(\mathbb{T}) \) of functions in \( \mathcal{F}L^{s,p}(\mathbb{T}) \) with prescribed mean \( \alpha \in \mathbb{R} \), and define the translation \( \tau_\alpha[u] := u - \alpha \). For \( u_0 \in \mathcal{F}L^{s,p}_0(\mathbb{T}) \), \( v_0 = \tau_\alpha[u_0] \in \mathcal{F}L^{s,p}_0(\mathbb{T}) \) and we consider the following Cauchy problem
\[
\begin{align*}
\partial_t v + \partial_x^3 v &= kP((v + \alpha)^{-1})\partial_x v, \\
v|_{t=0} &= v_0.
\end{align*}
\] (B.1)

Since conservation of mean still holds for solutions of (B.1), \( v \) has mean zero and we can apply the nonlinear estimates in Proposition 3.1 to \( v \). Following the proof of Theorem 1.1 we prove that (B.1) is locally well-posed in \( \mathcal{F}L^{s,p}_0(\mathbb{T}) \), with local time of existence \( T \sim (1 + \|v_0\|_{\mathcal{F}L^{s,p}} + |\alpha|)^{-\frac{k-1}{3}} \). Let \( \Phi_0(t) \) be the obtained solution map. We can now define the local-in-time flow \( \Phi_\alpha(t) \) of the gauged gKdV equation (1.7) on \( \mathcal{F}L^{s,p}_0(\mathbb{T}) \) as
\[
\Phi_\alpha(t) = \tau_{-\alpha} \circ \Phi_0(t) \circ \tau_\alpha,
\]
and the flow \( \Phi(t) \) of (1.7) on \( \mathcal{F}L^{s,p}(\mathbb{T}) \) as
\[
\Phi(t) = \Phi_0(t) \quad \text{on} \quad \mathcal{F}L^{s,p}_0, \quad \text{for each} \quad \alpha \in \mathbb{R}.
\]

Similarly to the mean zero case, this solution map \( \Phi(t) \) is still locally Lipschitz continuous.

Proposition B.1. For any \( R > 0 \) there exists \( T \sim (1 + R)^{-\frac{k-1}{3}} \) such that the flow \( \Phi(t) \) can be defined on \( B_R := \{u_0 \in \mathcal{F}L^{s,p} : \|u_0\|_{\mathcal{F}L^{s,p}} \leq R \} \) for \( |t| \leq T \). Moreover, for any \( u_0, \tilde{u}_0 \in B_R \) we have
\[
\left\| \Phi(\cdot)u_0 \right\|_{\mathcal{F}L^{s,p}(T)} \lesssim \|u_0\|_{\mathcal{F}L^{s,p}}, \quad \left\| \Phi(\cdot)u_0 - \Phi(\cdot)\tilde{u}_0 \right\|_{\mathcal{F}L^{s,p}(T)} \lesssim \|u_0 - \tilde{u}_0\|_{\mathcal{F}L^{s,p}},
\]
where the implicit constants are independent of \( R \) and the means of \( u_0, \tilde{u}_0 \).

\[\text{Precisely, we need the estimates of } P(v)^j \partial_x v, \ l = k - 1, k - 2, \ldots, 1, \text{ for mean zero } v. \text{ Proposition 3.1 treats the case } l \geq 3, \text{ while the } l = 2 \text{ case can be found in [13], Proposition 5, which holds for } 2 < p < \infty \text{ and } s > \max\left(\frac{3}{2}, \frac{3}{2} - \frac{1}{p}\right). \text{ For } l = 1, \text{ by adapting the proof of Proposition 3.1 we can easily see that the required estimate is available at least for } 2 < p < \infty \text{ and } s > \frac{1}{2}. \text{ In fact, there are only two frequencies } n_0, n_1, \text{ and we can treat two possibilities } |n_0| \gg |n_1| \text{ and } |n_0| \sim |n_1| \text{ by following the argument for Case 1.1 and Case 2.1, respectively. Note that the most restrictive condition on } s \text{ is that for } l = k - 1, \text{ which is the same as the one imposed in Theorem 1.1 for the mean-zero case.} \]
Proof. The flow $\Phi(t)$ is well-defined on $B(R)$ since each flow $\Phi_\alpha(t)$ is defined on $B_R \cap L^p(L^\alpha(T))$ for $|t| \leq T \sim (1 + R + |\alpha|)^{-\frac{1}{2}}$ and $|\alpha| = |P_0 u_0| \leq \|u_0\|_{L^s,p} \leq R$ on $B_R \cap \mathcal{F}L^{s,p}(\mathbb{T})$. From the local theory in $\mathcal{F}L^{s,p}(\mathbb{T})$, we have

$$
\|\Phi^\alpha_0 (\cdot) \tau_\alpha u_0\|_{Z^\cdot, \frac{1}{2}(T)} \lesssim \|\tau_\alpha u_0\|_{L^s,p} \lesssim \|u_0\|_{L^s,p}
$$

for $u_0 \in \mathcal{F}L^{s,p}_\alpha$, and hence

$$
\|\Phi(\cdot) u_0\|_{Z^\cdot, \frac{1}{2}(T)} \leq \|\Phi^\alpha_0 (\cdot) \tau_\alpha u_0\|_{Z^\cdot, \frac{1}{2}(T)} + C|\alpha| \lesssim \|u_0\|_{L^s,p},
$$

where the implicit constants are uniform in $\alpha$.

To prove the Lipschitz bound, let $u_0, \tilde{u}_0 \in B_R$ be two initial data with means $\alpha$ and $\tilde{\alpha}$, respectively. Note that $|\alpha| \leq R$, $|\tilde{\alpha}| \leq R$ and $|\alpha - \tilde{\alpha}| \leq \|u_0 - \tilde{u}_0\|_{L^s,p}$. Let $v(t) = \Phi^\alpha_0(t)[\tau_\alpha u_0]$ and $\tilde{v}(t) = \Phi^{\tilde{\alpha}}_0(t)[\tau_{\tilde{\alpha}} \tilde{u}_0]$ be the corresponding solutions of (B.1) for $\alpha$ and $\tilde{\alpha}$, respectively. From the local well-posedness of (B.1) in $\mathcal{F}L^{s,p}_\alpha$, we have $\|v\|_{Z^\cdot, \frac{1}{2}(T)} \lesssim \|\tilde{v}\|_{Z^\cdot, \frac{1}{2}(T)} \lesssim R$ for $T \sim (1 + R)^{-\frac{1}{2}}$. Let $w = v - \tilde{v}$. Then, we can use the nonlinear estimates in Proposition 3.1 to show that

$$
\|w\|_{Z^\cdot, \frac{1}{2}(T)} \leq C\|\tau_\alpha u_0 - \tau_{\tilde{\alpha}} \tilde{u}_0\|_{L^s,p} + CT^\theta R^{k-1}(|\alpha - \tilde{\alpha}| + \|w\|_{Z^\cdot, \frac{1}{2}(T)}).
$$

Replacing $T \sim (1 + R)^{-\frac{k-1}{2}}$ if necessary, we have

$$
\|w\|_{Z^\cdot, \frac{1}{2}(T)} \lesssim \|\tau_\alpha u_0 - \tau_{\tilde{\alpha}} \tilde{u}_0\|_{L^s,p} + |\alpha - \tilde{\alpha}|
\lesssim \|u_0 - \tilde{u}_0\|_{L^s,p} + 2|\alpha - \tilde{\alpha}|.
$$

Therefore,

$$
\|\Phi(\cdot) u_0 - \Phi(\cdot) \tilde{u}_0\|_{Z^\cdot, \frac{1}{2}(T)} = \|\tau_{-\alpha} v - \tau_{-\tilde{\alpha}} \tilde{v}\|_{Z^\cdot, \frac{1}{2}(T)}
\lesssim \|w\|_{Z^\cdot, \frac{1}{2}(T)} + |\alpha - \tilde{\alpha}|
\lesssim \|u_0 - \tilde{u}_0\|_{L^s,p} + |\alpha - \tilde{\alpha}|
\lesssim \|u_0 - \tilde{u}_0\|_{L^s,p}.
$$

Remark B.2. The same conclusions as in Proposition B.1 hold for the flow $\Phi_N(t)$ of the truncated equation

$$
\begin{align*}
\partial_t u_N + \partial^3_x u_N = k\mathbb{P}_{\leq N}\left[\mathbb{P}\left(\mathbb{P}_{\leq N} u_N\right)^{k-1}\right]\partial_x \mathbb{P}_{\leq N} u_N, \\
u_N|_{t=0} = u_0,
\end{align*}
$$

with $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, and the result is uniform in $N$. In fact, since the mean is still conserved for solutions to (B.2), we can define $\Phi_{N,\alpha}(t) = \Phi_N(t)|_{\mathcal{F}L^{s,p}}$ by $\Phi_{N,\alpha}(t) = \tau_{-\alpha} \circ \Phi^\alpha_{N,0}(t) \circ \tau_\alpha$, where $\Phi^\alpha_{N,0}$ is the flow of the equation for $v_N = \tau_\alpha u_N$ given by

$$
\begin{align*}
\partial_t v_N + \partial^3_x v_N = k\mathbb{P}_{\leq N}\left[\mathbb{P}\left(\mathbb{P}_{\leq N} v_N + \alpha\right)^{k-1}\right]\partial_x \mathbb{P}_{\leq N} v_N, \\
v_N|_{t=0} = v_0,
\end{align*}
$$

with $v_0 = \tau_\alpha u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$. The argument for (B.3) is analogous to that for (B.1) and uniform in $N$. 
Now, the a.s. global well-posedness of (1.7) and of (1.1) without prescribing mean zero and the invariance of the Gibbs measure follow from the approach in Section 4.

APPENDIX C. INVARIANCE IN THE THRESHOLD CASE

In this section, we outline the proof of Theorem 1.4 (b) when \( k = 5 \) and \( R = \|Q\|_{L^2(\mathbb{R})} \). For \( \delta \geq 0 \), we introduce the notation
\[
F_\delta(u) = e^{\frac{1}{\delta} \int_T u^6 \, dx} 1_{\{\|u\|_{L^2} \leq R - \delta\}},
\]
and let \( \mu_\delta \) denote the Gibbs measure given by
\[
d\mu_\delta(u) = Z_\delta^{-1} F_\delta(u) \, d\rho(u).
\]
From Theorem 4.2, we have that
\[
F_\delta(u) \in L^q(d\rho), \quad 1 \leq q < \infty, \quad \delta > 0,
\]
\[
F_0(u) \in L^1(d\rho).
\]
In addition, fix \( 2 < p < \infty \) and set \( s = s_1 = \frac{1}{2} (s_p (p) + 1 - \frac{1}{p}) \), \( B \) the Borel \( \sigma \)-field on \( FL^{s,p}(\mathbb{T}) \).

We will use the following approximation property between \( \mu_\delta \) and \( \mu_0 \). Note that \( 0 \leq F_\delta(u) \leq F_0(u) \) for \( \delta > 0 \) and \( F_\delta(u) \to F_0(u) \) for any \( u \in FL^{s,p}(\mathbb{T}) \). Therefore, by the dominated convergence theorem, we have that \( F_\delta \to F_0 \) in \( L^1(d\rho) \), from which we obtain
\[
\sup_{A \in B} |\mu_\delta(A) - \mu_0(A)| \to 0 \quad \text{as} \ \delta \to 0. \tag{C.1}
\]
For \( \delta > 0 \), the argument in Section 4 holds. Consequently, by Theorem 1.4 we can construct a set \( \Sigma_\delta \) of full \( \mu_\delta \)-measure and a global-in-time flow \( \Phi_\delta(t) \) for (1.7) on \( \Sigma_\delta \).

Now, we establish Theorem 1.4 in the threshold case \( \delta = 0 \). We first define the following set and solution map
\[
\Sigma_0 := \bigcup_{n \geq 1} \Sigma_{\frac{1}{n}}, \quad \Phi_0(t) := \Phi_{\frac{1}{n}}(t) \quad \text{on} \ \Sigma_{\frac{1}{n}}, \ n \in \mathbb{N}.
\]
From uniqueness of local-in-time solutions in \( Z_\delta^{s, \frac{3}{2}} \), \( \Phi_0(t) \) is well-defined on \( \Sigma_0 \). Similarly, the flow property (1.11) extends to \( \Phi_0(t) \) on \( \Sigma_0 \) from that of \( \Phi_{\frac{1}{n}}(t) \) on \( \Sigma_{\frac{1}{n}} \). In addition, \( \Sigma_0 \) is \( \mu_0 \)-measurable since it is a countable union of Borel sets of \( FL^{s,p}(\mathbb{T}) \). Furthermore, the \( \mu_0 \)-measurability of \( \Phi_0(t) \) follows from its continuity on \( \Sigma_0 \) in the topology induced by \( FL^{s,p}(\mathbb{T}) \), the proof of which boils down to the argument for a fixed \( \delta > 0 \) presented in Section 4.

It only remains to show that \( \mu_0(\Sigma_0) = 1 \) and \( \mu_0 \) is invariant under the solution map \( \Phi_0(t) \). For any \( n \in \mathbb{N} \), we have
\[
\mu_{\frac{1}{n}}(\Sigma_0) = \mu_{\frac{1}{n}} \left( \bigcap_{n \geq 1} \Sigma_{\frac{1}{n}}^c \right) \leq \mu_{\frac{1}{n}}(\Sigma_{\frac{1}{n}}) = 0,
\]
and for any measurable \( A \subset \Sigma_0, \ t \in \mathbb{R}, \ n \in \mathbb{N} \), we have
\[
\mu_{\frac{1}{n}}(\Phi_0(t) A) = \mu_{\frac{1}{n}}(\Phi_0(t) A \cap \Sigma_{\frac{1}{n}}) = \mu_{\frac{1}{n}}(\Phi_{\frac{1}{n}}(t) (A \cap \Sigma_{\frac{1}{n}})) = \mu_{\frac{1}{n}}(A \cap \Sigma_{\frac{1}{n}}) = \mu_{\frac{1}{n}}(A),
\]
from the invariance of \( \mu_{\frac{1}{n}} \) under \( \Phi_{\frac{1}{n}}(t) \) and the fact that \( \Sigma_{\frac{1}{n}} \) has full \( \mu_{\frac{1}{n}} \)-measure. By taking the limit as \( n \to \infty \) and using the convergence in (C.1), we conclude that \( \mu_0(\Sigma_0) = 0 \) and that \( \mu_0(\Phi_0(t) A) = \mu_0(A) \).
This concludes the proof of Theorem 4.4 (b) in the case \( k = 5 \) and \( R = \|Q\|_{L^2(\mathbb{R})} \).

Although we have considered the mean zero case, the above argument also works for the problem without the mean zero condition (see Remark 4.7 and Appendix D). Moreover, the corresponding results for gKdV (1.1) can be verified by the same argument as for the non-threshold case \( R < \|Q\|_{L^2(\mathbb{R})} \) (see the proof of Theorem 4.3 in Section D).

**Appendix D. Proof of Lemmas 4.6 and 4.7**

First, we prove Lemma 4.6 using Lemma 4.1.

**Proof of Lemma 4.6.** From Theorem 4.1, we know that for \( u_0 \in \mathcal{F}L^{s,p}(\mathbb{T}) \) with \( \|u_0\|_{\mathcal{F}L^{s,p}} \leq K \), the corresponding solution \( u_N \) to (4.5) satisfies

\[
\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq C_0 K,
\]

for \( |t| \leq \delta \sim K^{-\gamma} \), \( \gamma = \frac{k-1}{\theta} > 0 \) with \( \theta \) given in Proposition 3.1 where \( C_0 > 0 \) does not depend on \( s \). Note also that the constants can be taken uniformly in \( N \). We want to establish a bound on \( u_N(t) \) for all \( |t| \leq T \). Let \( [x] \) denote the integer part of a real number \( x \) and define

\[
\Omega_N(T, \varepsilon, A) = \bigcap_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \Phi_N(j\delta \mathbb{N}) \left( \{ \|u_0\|_{\mathcal{F}L^{s,p}} \leq K \} \right),
\]

where \( K = AC_2 \left( \log \frac{T}{\varepsilon} \right)^{\frac{1}{2}} \) with a constant \( C_2 > 0 \) to be chosen later.

We start by showing (a). Let \( B_K = \{ \|u_0\|_{\mathcal{F}L^{s,p}} \leq K \} \). From the uniqueness of solution to (4.5) in each time interval \( [j\delta, (j+1)\delta) \), we see that the solution map is invertible and

\[
\Phi_N(j\delta)(B_K^c) = \Phi_N(j\delta)(B_K^c).
\]

Consequently,

\[
\mu_N([\Omega_N(T, \varepsilon, A)]^c) = \mu_N \left( \bigcup_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \Phi_N(j\delta)(B_K^c) \right)
\leq \sum_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \mu_N(\Phi_N(j\delta)(B_K^c)) = 2\left[\frac{T}{\delta}\right] \mu_N(B_K^c)
\]

from the invariance of \( \mu_N \) under the flow \( \Phi_N(t) \) of (4.5) in Proposition 4.5. From Cauchy-Schwarz inequality, Lemma 4.1 and 4.4, we have

\[
\mu_N([\Omega_N(T, \varepsilon, A)]^c) \lesssim \frac{T}{\delta} \int_{B_K^c} F_N(u) \ d\rho(u) \sim \frac{T}{\delta} \|F_N\|_{L^2(\mathcal{F}L^{s,p})} \rho(B_K^c) \lesssim \frac{T}{\delta} e^{-cK^2} \sim TK^2e^{-cK^2}.
\]

Since \( \log \frac{T}{\varepsilon} \geq \log 2 \) by the assumption, there exists \( C_2 > 0 \) such that if \( K \geq C_2 \left( \log \frac{T}{\varepsilon} \right)^{\frac{1}{2}} \), then \( TK^2e^{-cK^2} \leq T e^{-\frac{1}{2}K^2} \ll \varepsilon \). Hence, the above estimate, for \( K = AC_2 \left( \log \frac{T}{\varepsilon} \right)^{\frac{1}{2}} \) with \( A \geq 1 \) and such a constant \( C_2 \), ensures that \( \mu_N([\Omega_N(T, \varepsilon, A)]^c) \ll \varepsilon \), establishing (a). With the invertibility of the solution map, (b) is a consequence of the local bound mentioned at the beginning, and (c) immediately follows from the definition of \( \Omega_N(T, \varepsilon, A) \). \( \square \)
Next, we derive Lemma 4.7 from the local theory.

Proof of Lemma 4.7. We only consider the positive time direction. We start by showing (a). Let $\mathcal{N}(u) := k\mathcal{P}(u^{k-1})\partial_x u$. By the local theory, with $\delta \sim (1 + K)^{-\gamma}$ the solution $u_N$ to (4.5) satisfies

$$\|u_N\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} \leq C_2 K, \quad 0 \leq j < \left[\frac{T}{\delta}\right]$$

for some $C_2 > 0$. Note that the solution to (4.5) in $C([-T, T]; FL_{\sigma}^{s,p}(\mathbb{T}))$ coincides on each interval $[j\delta, (j+1)\delta]$ with the solution constructed by the iteration argument in $Z_p^{\sigma,1/2}$, and also that

$$P_{\leq N}u_N(t) = S(t - j\delta)P_{\leq N}u_N(j\delta) + \int_{j\delta}^t S(t - t')P_{\leq N}(P_{\leq N}u_N(t')) dt', \quad t \in [j\delta, (j+1)\delta]$$

for any $0 \leq j < \left[\frac{T}{\delta}\right]$. We want to construct a solution $u$ to

$$u(t) = S(t - j\delta)u(j\delta) + \int_{j\delta}^t S(t - t')\mathcal{N}(u(t')) dt', \quad t \in [j\delta, (j+1)\delta]$$

for each $j = 0, 1, \ldots, \left[\frac{T}{\delta}\right] - 1$. This amounts to constructing $w(t) := u(t) - P_{\leq N}u_N(t)$, which solves

$$w(t) = \Xi_j[w](t) := S(t - j\delta)w(j\delta) + \int_{j\delta}^t S(t - t')P_{> N}(P_{\leq N}u_N)(t') dt'$$

$$+ \int_{j\delta}^t S(t - t')\{\mathcal{N}(w + P_{\leq N}u_N) - \mathcal{N}(P_{\leq N}u_N)\}(t') dt'. \quad (D.2)$$

By the nonlinear estimates in $Z_p^{\sigma,1/2}$ and $Z_p^{\sigma,1/2}$, together with (D.1), we have

$$\|\Xi_j[w]\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} \leq C_0\|w(j\delta)\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} + C_1\sigma^\theta \left(\|w\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} + C_2 K\right)^{k-1}\|w\|_{L^p_{\sigma}([j\delta,(j+1)\delta])}$$

$$+ C_1 N^{-(\sigma-s)}\sigma^\theta (C_2 K)^k,$$

$$\|\Xi_j[w] - \Xi_j[w]\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} \leq C_1\sigma^\theta \left(\|w\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} + \|\tilde{u}\|_{L^p_{\sigma}([j\delta,(j+1)\delta])} + C_2 K\right)^{k-1}\|w - \tilde{u}\|_{L^p_{\sigma}([j\delta,(j+1)\delta])},$$

for some $C_0 > 0$ and $C_1 = C_1(s, p) > 0$. Therefore, taking smaller $\delta \sim_{s,p} (1 + K)^{-\gamma}$ if necessary, we can show that $\Xi_j$ is a contraction on

$$\{w \in Z_p^{\sigma,1/2}([j\delta,(j+1)\delta]) : \|w\|_{Z_p^{\sigma,1/2}([j\delta,(j+1)\delta])} \leq 2C_0\|w(j\delta)\|_{L^p_{\sigma}} + N^{-(\sigma-s)} K\}$$

as long as

$$\|w(j\delta)\|_{L^p_{\sigma}} \leq K.$$

Starting from $\|w(0)\|_{L^p_{\sigma}} \leq N^{-(\sigma-s)} K$, we obtain the solution $w$ to (D.2) on $[j\delta, (j+1)\delta]$ with

$$\|w((j+1)\delta)\|_{L^p_{\sigma}} \leq \tilde{C}^{j+1}N^{-(\sigma-s)} K, \quad j = 0, 1, \ldots, \left[\frac{T}{\delta}\right] - 1,$$
for some $\tilde{C}_0 > 0$. In particular, the solution can be extended up to $t = T$ if $N$ satisfies
\[ N^{\sigma-s} \geq e^{C_3(1+K)\gamma T} (\geq \tilde{C}_0^\frac{T}{\sigma-s}) \]
for some $C_3 = C_3(s,p) > 0$. Consequently, for $N$ large enough, we obtain
\[ \max_{0 \leq t \leq T} \|w(t)\|_{F_{L^s,p}} \leq \max_{0 \leq j < \left[\frac{T}{\delta}\right]} \tilde{C}_j^{j+1} N^{-(\sigma-s)} K \leq e^{C_3(1+K)\gamma T} N^{-(\sigma-s)} K. \]
The estimate follows by further imposing $N \geq N_0$ where $N_0 \sim \exp\left(\frac{CK^\gamma T}{\sigma-s}\right)$.

To establish (b), note that we can also write $w(t)$ as follows
\[
w(t) = \tilde{\Xi}_j[w](t) := S(t-j\delta)w(j\delta) + \int_{j\delta}^t S(t-t')P_{>N}N(u)(t') dt' + \int_{j\delta}^t S(t-t')P_{\leq N}\{N(u)-N(u-w)\}(t') dt'.
\]
The estimate then follows from the same arguments as for (a). □

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