Benford or not Benford: new results on digits beyond the first

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Abstract

In this paper, we will see that the proportion of \( d \) as \( p \)th digit, where \( p > 1 \) and \( d \in [0,9] \), in data (obtained thanks to the hereunder developed model) is more likely to follow a law whose probability distribution is determined by a specific upper bound, rather than the generalization of Benford’s Law to digits beyond the first one. These probability distributions fluctuate around theoretical values determined by Hill in 1995. Knowing beforehand the value of the upper bound can be a way to find a better adjusted law than Hill’s one.

Introduction

Benford’s Law is really amazing: according to it, the first digit \( d, d \in [1,9] \), of numbers in many naturally occurring collections of data does not follow a discrete uniform distribution; it rather follows a logarithmic distribution. Having been discovered by Newcomb in 1881 ([11]), this law was definitively brought to light by Benford in 1938 ([2]). He proposed the following probability distribution: the probability for \( d \) to be the first digit of a number is equal to \( \log(1 + \frac{1}{d}) \).

Most of the empirical data, as physical data (Knuth in [10] or Burke and Kincanon in [4]), demographic and economic data (Nigrini and Wood in [12]) or genome data (Friar et al. in [6]), follow approximately Benford’s Law. To such an extent that this law is used to detect possible frauds in lists of socio-economic data ([15]) or in scientific publications ([5]).

In [3], Blondeau Da Silva, building a rather relevant representative model, showed that, in this case, the proportion of each \( d \) as leading digit, \( d \in [0,9] \), structurally fluctuates. It strengthens the fact that, concerning empirical data sets, this law often appears to be a good approximation of the reality, but no more than an approximation ([7]). We can note that there also exist distributions known to disobey Benford’s Law ([13] and [1]).

Generalizing Benford’s Law, Hill ([8]) extends the law to digits beyond the first one: the probability for \( d, d \in [0,9] \), to be the \( p \)th digit of a number is equal to \( \sum_{j=10^{p-2}}^{10^{p-1}-1} \log(1 + \frac{1}{10j+d}) \).

Building a very similar model to that described in [3], the naturally occurring data will be considered as the realizations of independent random variables following the hereinafter constraints: (a) the data is strictly positive and is upper-bounded by an integer \( n \), constraint which is often valid in data sets, the
physical, biological and economical quantities being limited: (b) each random variable is considered to follow a discrete uniform distribution whereby the first strictly positive $p$-digits integers ($p > 1$) are equally likely to occur ($i$ being uniformly randomly selected in $[10^{p-1}, n]$). This model relies on the fact that the random variables are not always the same.

Through this article we will demonstrate that the predominance of 0 over 1 (and of 1 over 2, and so on), as $p$th, ($p > 1$) digit is all but surprising. Hill’s probabilities became standard values that should exactly be followed by most of naturally occurring collections of data. However the reality is that the proportion of each $d$ as leading digit structurally fluctuates. There is not a single law but numerous distinct laws that we will hereafter examine.

1 Notations and probability space

Let $p$ and $d$ be two strictly positive integers such that $p > 1$ and $d \in [0, 9]$. Let $m$ be a strictly positive integer such that $m \geq 10^{p-1}$. Let $\mathcal{U}\{10^{p-1}, m\}$ denote the discrete uniform distribution whereby integers between $10^{p-1}$ and $m$ are equally likely to be observed.

Let $n$ be a strictly positive integer such that $n \geq 10^{p-1}$. Let us consider the random experiment $E_n$ of tossing two independent dice. The first one is a fair $(n+1-10^{p-1})$-sided die showing $n+1-10^{p-1}$ different numbers from 1 to $n+1-10^{p-1}$. The number $i$ rolled on it defines the number of faces on the second die. It thus shows $i$ different numbers from $10^{p-1}$ to $i+10^{p-1}-1$.

Let us now define the probability space $\Omega_n$ as follows: $\Omega_n = \{(i,j) : i \in [1,n+1-10^{p-1}]$ and $j \in [10^{p-1},i+10^{p-1}-1]\}$. Our probability measure is denoted by $P$.

Let us denote by $D_{(n,p)}$ the random variable from $\Omega_n$ to $[0,9]$ that maps each element $\omega$ of $\Omega_n$ to the $p$th digit of the second component of $\omega$.

As our aim is to determine the probability that the $p$th digit of the integer obtained thanks to the second throw is $d$, it can be considered with no consequence on our results that we first select an integer $i$ equal to or less than $n$ among at least $p$-digits integers (following the $\mathcal{U}\{10^{p-1}, n\}$ discrete uniform distribution); afterwards we select an other at least $p$-digits integer equal to or less than $i$ (following the $\mathcal{U}\{10^{p-1}, i\}$ discrete uniform distribution).

2 Proportion of $d$

Through the below proposition, we will express the value of $P(D_{(n,p)} = d)$ i.e. the probability that the $p$th digit of our second throw in the random experiment $E_n$ is $d$.

**Proposition 2.1.** Let $k$ denote the integer such that:

$$k = \max \{i \in \mathbb{N} : 10^{i+p} \leq n\}.$$ 

Let $l$ denote the positive integer such that:

$$l = \left\lfloor \frac{n-(10^{p-1}+1)10^{k+1}}{10^{p+2}} \right\rfloor + 10^{p-2}.$$ 

2
The value of $P(D_{n,p} = d)$ is:

$$
\frac{1}{n + 1 - 10^{p-1}} \left( \sum_{i=0}^{k} \left( \sum_{j=10^{p-2}}^{10^{p-1}-1} \frac{b = ((9j + d)10^k + 10^{p-2} - 1)}{b + 1 - 10^{p-1}} \right) + \sum_{j=10^{p-2}-1}^{\min(10^{p+1}-1,10(j+1)+d)10^p} \frac{10^i(j + 1) - 10^{p-2}}{a + 1 - 10^{p-1}} \right) + r(n,d,p),
$$

where $r(n,d,p)$ is, if the $p$th digit of $n$ is $d$:

$$
\sum_{j=10^{p-2}}^{l} \frac{\min(n,(10j+(d+1))10^{k+1}-1)}{b = ((9j + d)10^{k+1} + 10^{p-2} - 1)} \frac{b - ((9j + d)10^{k+1} + 10^{p-2} - 1)}{b + 1 - 10^{p-1}} + \sum_{j=10^{p-2}-1}^{\min(n,(10(j+1)+(d+1))10^{k+1})} \frac{10^{k+1}(j + 1) - 10^{p-2}}{a + 1 - 10^{p-1}}
$$

or where $r(n,d,p)$ is, if the $p$th digit of $n$ is all but $d$:

$$
\sum_{j=10^{p-2}}^{l} \frac{(10j+(d+1))10^{k+1}-1}{b = ((9j + d)10^{k+1} + 10^{p-2} - 1)} \frac{b - ((9j + d)10^{k+1} + 10^{p-2} - 1)}{b + 1 - 10^{p-1}} + \sum_{j=10^{p-2}-1}^{\min(n,(10(j+1)+(d+1))10^{k+1})} \frac{10^{k+1}(j + 1) - 10^{p-2}}{a + 1 - 10^{p-1}}
$$

Proof. Let us denote by $F_{n,p}$ the random variable from $\Omega_n$ to $[1, n+1-10^{p-1}]$ that maps each element $\omega$ of $\Omega_n$ to the first component of $\omega$. It returns the number obtained on the first throw of the unbiased $(n+1-10^{p-1})$-sided die. For each $q \in [1, n+1-10^{p-1}]$, we have:

$$
P(F_{n,p} = q) = \frac{1}{n + 1 - 10^{p-1}}. \quad (1)
$$

According to the Law of total probability, we state:

$$
P(D_{n,p} = d) = \sum_{q=1}^{n+1-10^{p-1}} P(D_{n,p} = d|F_{n,p} = q) P(F_{n,p} = q). \quad (2)
$$

Thereupon two cases appear in determining the value, for $q \in [1, n+1-10^{p-1}]$, of $P(D_{n,p} = d|F_{n,p} = q)$. Let $k_q$ be the integer such that $k_q = \max\{k \in \mathbb{N} : 10^{p+k} \leq q + 10^{p-1} - 1\}$ in both cases.

Let us study the first case where the $p$th digit of $q + 10^{p-1} - 1$ is $d$. For all $i \in k_q$, there exist $9 \times 10^{p-2}$ sequences of $10^i$ consecutive integers from $(10j + d)10^i$ to $(10j + (d + 1))10^i - 1$, where $j \in [10^{p-2}, 10^{p-1} - 1]$, whose $p$th digit is $d$. The higher of these integers is $(10(10^{p-1} - 1) + (d + 1))10^k - 1$, the
last \((p + k_q)\)-digit number in this case. Thus, from \(10^p - 1\) to \(10^{p+k_q} - 1\), the number of integers whose \(p^{th}\) digit is \(d\) is:

\[
\sum_{i=0}^{k_q} \sum_{j=10^p-2}^{10^{p+k_q} - 1} \sum_{t=0}^{k_q} \sum_{d=10^p-1}^{10^{p+k_q} - 1} 1 = 9 \times 10^{p-2} \sum_{i=0}^{k_q} 10^i = 10^{p-2}(10^{k_q+1} - 1).
\]

This equality still holds true for \(k_q = -1\). Such types of sum would be considered null in the rest of the article. From \(10^{p+k_q}\) to \(q+10^{p-1} - 1\), there exist \(t\) sequences of \(10^{k_q+1}\) consecutive integers from \((10j + d)10^{k_q+1}\) to \((10j + (d+1))10^{k_q+1} - 1\), where \(j \in [10^{p-2}, 10^{p-2} + t - 1]\), whose \(p^{th}\) digit is \(d\). There also exist \(q + 10^{p-1} - 1 - (10(10^{p-2} + t) + d)10^{k_q+1} + 1\) additional integers in this case between \((10(10^{p-2} + t) + d)10^{k_q+1}\) and \(q + 10^{p-1} - 1\). Finally the total number of integers whose \(p^{th}\) digit is \(d\) is:

\[
10^{p-2}(10^{k_q+1} - 1) + t \times 10^{k_q+1} + q + 10^{p-1} - 1 - (10(10^{p-2} + t) + d)10^{k_q+1} + 1
\]

i.e. \(q + 10^{p-1} - 1 - (9(10^{p-2} + t) + d)10^{k_q+1} - 1)\).

It may be inferred that:

\[
P(D_{(n,p)} = d|F_{(n,p)} = q) = \frac{q + 10^{p-1} - 1 - (9(10^{p-2} + t) + d)10^{k_q+1} - 1)}{q},
\]

the \(p^{th}\) digit of \(q + 10^{p-1} - 1\) being here \(d\).

In the second case, we consider the integers \(q + 10^{p-1} - 1\) whose \(p^{th}\) digits are different from \(d\). On the basis of the previous case, the total number of integers whose \(p^{th}\) digit is \(d\) is, where \(t\) is the number of sequences of consecutive integers lower than \(q + 10^{p-1} - 1\):

\[
10^{p-2}(10^{k_q+1} - 1) + t \times 10^{k_q+1}
\]

i.e. \(10^{k_q+1}(10^{p-2} + t) - 10^{p-2}\).

It can be concluded that:

\[
P(D_{(n,p)} = d|F_{(n,p)} = q) = \frac{10^{k_q+1}(10^{p-2} + t) - 10^{p-2}}{q},
\]

the \(p^{th}\) digit of \(q + 10^{p-1} - 1\) being here different from \(d\).

Using equalities \((1), (2), (3)\) and \((4)\), we get our result. \(\Box\)

For example, we get:

**Examples 2.2.** Let us first determine the value of \(P(D_{(10003,5)} = 2)\). The probability that the fifth digit of a randomly selected number in \([10000, 10000]\) is \(2\) is \(\frac{9}{10}\), those in \([10000, 10001]\) is \(\frac{9}{10}\), those in \([10000, 10002]\) is \(\frac{1}{3}\) and those in \([10000, 10003]\) is \(\frac{1}{4}\). Hence we have:

\[
P(D_{(10003,5)} = 2) = \frac{1}{4}(0 + 0 + \frac{1}{3} + \frac{1}{4}) \approx 0.1458
\]

It is the second case of Proposition \((2.1)\) where \(n = 10003, d = 2, p = 5, k = -1\) and \(l = 1000\).
Let us now determine the value of \( P(D_{(113,3)} = 1) \) (first case of Proposition 2.1); in this case, we have \( k = 0 \) and \( l = 11 \).

\[
P(D_{(113,3)} = 1) = \frac{1}{10^{11}} \left( \sum_{j=0}^{99} \frac{j - 9}{10 j - 99} + \sum_{j=0}^{98} \frac{10 (j+1)}{a - 99} + \sum_{j=0}^{99} \frac{10 (j+1)}{a - 99} + \sum_{a=1000}^{90} \frac{10 (j+1)}{a - 99} + \sum_{k=110}^{1010} \frac{b - 919}{a - 99} + \sum_{b=1110}^{1113} \frac{b - 109}{a - 99} \right)
\]

\[
= \frac{1}{10^{11}} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{11} + \frac{2}{12} + \frac{2}{13} + \ldots + \frac{89}{90} + \frac{90}{90} + \frac{90}{91} + \ldots + \frac{90}{910} + \frac{91}{911} + \ldots + \frac{91}{910} \right)
\approx 0.1028
\]

Let us determine the value of \( P(D_{(212,2)} = 9) \) (second case of Proposition 2.1); in this case, we have \( k = 0 \) and \( l = 1 \).

\[
P(D_{(212,2)} = 9) = \frac{1}{203} \left( \sum_{j=0}^{9} \frac{10 (j+1)+a}{a - 9} + \sum_{a=100}^{99} \frac{b - 109}{a - 9} + \sum_{b=190}^{191} \frac{b - 99}{a - 9} + \sum_{a=200}^{199} \frac{9}{a - 9} \right)
\]

\[
= \frac{1}{203} \left( \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{19} + \frac{2}{20} + \frac{2}{21} + \ldots + \frac{8}{89} + \frac{9}{90} + \frac{9}{91} + \ldots + \frac{9}{180} + \frac{10}{181} + \frac{11}{182} + \ldots + \frac{19}{203} \right)
\approx 0.0759
\]

3 Study of a particular subsequence

It is natural that we take a specific look at the values of \( n \) positioned one rank before the integers for which the number of digits has just increased.

To this end we will consider the sequence \( (P(D_{n,p} = d))_{n \in \mathbb{N} \cap [0,10^{p-1} - 1]} \). In the interests of simplifying notation, we will denote by \( (P(d,n,p))_{n \in \mathbb{N} \cap [0,10^{p-1} - 1]} \) this sequence. Let us study the subsequence \( (P(d,\phi(d,p),n,p))_{n \in \mathbb{N} \cap [0,10^{p-1} - 1]} \) where \( \phi(d,p) \) is the function from \( \mathbb{N} \setminus [0, p-1] \) to \( \mathbb{N} \) that maps \( n \) to \( 10^n - 1 \). We get the below result:

**Proposition 3.1.** The subsequence \( (P(d,\phi(d,p),n,p))_{n \in \mathbb{N} \cap [0,10^{p-1} - 1]} \) converges to:

\[
10^{-1} + \frac{n(d,p) + m(d,p) - 9 (n(d,p) - d \times k(d,p) + 1)}{9 \times 10^{p-1}} + \frac{1}{90} \ln \left( \frac{10^{p-1} + d}{10^{p-1}} \right) + \frac{1}{9} \ln \left( \frac{10^{p}}{10^{p} - 10 + d + 1} \right),
\]

where:

\[
\left\{
\begin{aligned}
k(d,p) &= \sum_{j=10^{p-1}}^{10^{p-1} - 1} \ln(10j+d+1) \\
l(d,p) &= \sum_{j=10^{p-1}}^{10^{p-1} - 1} j \ln(10j+d+1) \\
m(d,p) &= \sum_{j=10^{p-2}}^{10^{p-1} - 2} \ln(10j+1+d) \\
n(d,p) &= \sum_{j=10^{p-2}}^{10^{p-1} - 2} j \ln(10j+1+d+1).
\end{aligned}
\right.
\]

**Proof.** Let \( n \) be a positive integer such that \( n \geq p \). According to Proposition 2.1 we have \( P(d,\phi(d,p),n,p) = P(d,10^{p-1},p) \) i.e., knowing that in this case \( k =
Recall that for all integers $(p, q)$, such that $1 < p < q$:  
\[
\ln\left(\frac{q + 1}{p}\right) \leq \sum_{k=p}^{q} \frac{1}{k} \leq \ln\left(\frac{q}{p - 1}\right). 
\] (5)

Consequently, we obtain, for $i \geq 1$:

\[
b_{i(p,d)} \geq 9 \times 10^{p+i-2} - \sum_{j=10^{p-2}}^{10^{p+i-1}} (9j + d)10^i \ln\left(\frac{(10j + (d + 1))10^i}{10j + d}\right) - 9 \times 10^{p+i-2} - \sum_{j=10^{p-2}}^{10^{p+i-1}} (9j + d)10^i \ln\left(\frac{10^i}{10j + d}\right). 
\]
Let us denote by $k_{(d,p)}$ the positive number $\sum_{j=10^{p-2}}^{10^{p-1}-1} \ln \left( \frac{10^{j} + (d+1)}{10^{j} + d} \right)$ and $l_{(d,p)}$ the positive number $\sum_{j=10^{p-2}}^{10^{p-1}-1} j \ln \left( \frac{10^{j} + (d+1)}{10^{j} + d} \right)$. Knowing that for all $x \in [-1; +\infty[$, we have $\ln(1 + x) \leq x$, we obtain:

$$b_{(i,d,p)} \geq 9 \times 10^{p+i-2} - d \times 10^i k_{(d,p)} - 9 \times 10^i l_{(d,p)} - \sum_{j=10^{p-2}}^{10^{p-1}-1} (9j + d) 10^j \frac{10^{p-1}}{10^j (10^{j} + d) - 10^{p-1}}$$

$$\geq 9 \times 10^{p+i-2} - d \times 10^i k_{(d,p)} - 9 \times 10^i l_{(d,p)} - \sum_{j=10^{p-2}}^{10^{p-1}-1} 10^j \frac{10^{p-1}}{10^j \times 10^{p-1} - 10^{p-1}}$$

$$\geq 9 \times 10^{p+i-2} - d \times 10^i k_{(d,p)} - 9 \times 10^i l_{(d,p)} - 9 \times 10^{p-i} \frac{10^{p-1}}{10^j - 1}.$$

Similarly, we have thanks to inequalities [5]:

$$a_{(i,d,p)} \geq \sum_{j=10^{p-2}}^{10^{p-1}-1} (10^j (j + 1) - 10^{p-2}) \ln \left( \frac{10^j (j + 1) + d}{10^j (j + d + 1)} \right) + 10^{p-i} \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}}$$

$$+ (10^{p-2+i} - 10^{-p-2}) \ln \left( \frac{10^{p-1} - d}{10^j + (d + 1)} \right) + 10^{p-i} \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}}$$

$$\geq 10^i \sum_{j=10^{p-2}}^{10^{p-1}-1} j \ln \left( \frac{10^j (j + 1) + d}{10^j (j + d + 1)} \right) + 10^{p-i} \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}}$$

$$+ (10^{p-2+i} - 10^{-p-2}) \left( \ln \left( \frac{10^{p-1} - d}{10^j + (d + 1)} \right) + 10^{p-i} \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}} \right)$$

$$+ (10^{p-2+i} - 10^{-p-2}) \left( \ln \left( \frac{10^{p-1} - d}{10^j + (d + 1)} \right) + 10^{p-i} \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}} \right).$$

Let us denote by $m_{(d,p)}$ the positive number $\sum_{j=10^{p-2}}^{10^{p-1}-1} \ln \left( \frac{10^j + (d+1)}{10^j + d} \right)$ and $n_{(d,p)}$ the positive number $\sum_{j=10^{p-2}}^{10^{p-1}-1} j \ln \left( \frac{10^j + (d+1)}{10^j + d} \right)$:

$$a_{(i,d,p)} \geq 10^i n_{(d,p)} + (10^i - 10^{-p-2}) \left( m_{(d,p)} + \sum_{j=10^{p-2}}^{10^{p-1}-2} \ln \left( 1 + \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}} \right) \right)$$

$$+ (10^{p-2+i} - 10^{-p-2}) \ln \left( \frac{10^{p-1} - d}{10^j + (d + 1)} \right) + (10^{p-1+i} - 10^{-p-2}) \ln \left( \frac{10^p}{10^p - 10^j + d + 1} \right).$$

Hence we have:

$$P_{(d,a_{(d,p)}(n,d))} \geq \frac{1}{10^p} \left( a_{(0,d,p)} + b_{(0,d,p)} + \sum_{i=1}^{n-p} \left( 9 \times 10^{p+i-2} - d \times 10^i k_{(d,p)} - 9 \times 10^i l_{(d,p)} \right) \right)$$

$$+ 10^i n_{(d,p)} + 10^m_{(d,p)} + 10^{p-2+i} \ln \left( \frac{10^{p-1} - d}{10^{p-1} - 10^j + d} \right) + 10^{p-1+i} \ln \left( \frac{10^p}{10^p - 10^j + d + 1} \right)$$

$$- 9 \times 10^{p-2} \frac{10^p - 10^{p-2}}{9} + 10^{p-2+i} \left( m_{(d,p)} + \ln \left( \frac{10^{p-1} - d}{10^{p-1} - 10^j + d} \right) + \ln \left( \frac{10^p}{10^p - 10^j + d + 1} \right) \right)$$

$$+ (10^i - 10^{-p-2}) \sum_{j=10^{p-2}}^{10^{p-1}-2} \ln \left( 1 + \frac{9 x (10^{p-1}-1)}{10^j (10^{j} + d) + 1 - 10^{p-1}} \right).$$
In light of the following equality \( \sum_{i=1}^{n} 10^i = \frac{10^n - 1 - 10}{9} \), we have:

\[
P_{(d, \alpha(d,p), n)} \geq 10^{-1} + \frac{10^{-1} + m_{(d,p)} - 9 \delta_{(d,p)}}{9} + \frac{10^{-1} \ln(\frac{10^n - 1 + d}{10^n - 1})}{9} + \frac{1}{9} \ln\left( \frac{10^n}{10^n - 10 + d + 1} \right) + \epsilon_{(d,n,p)},
\]

where \( \epsilon_{(d,n,p)} \) is:

\[
\begin{align*}
\alpha_{(d,p)} + b_{(d,p)} &- \frac{10^{p-1}}{10^n} + \frac{dk_{(d,p)} + 9\delta_{(d,p)} - m_{(d,p)} - m_{(d,p)}}{9 \times 10^{n-1}} - \frac{10^{p-1}(n - p)}{10^n} - \frac{10^{p-2}(n - p)}{10^n} (m_{(d,p)} + \ln(\frac{10^n - 1 + d}{10^n - 1}))
\end{align*}
\]

Knowing that for all \( x \in [-1, +\infty] \), we have \( \ln(1 + x) \leq x \), we obtain, for all \( i \in \{1, \ldots, p-3\} \):

\[
\sum_{i=10^n-2}^{10^{p-1}-2} \ln(1 + \frac{9 \times (10^{n-1}-1)}{10^j + (d + 1)10^i + 1 - 10^n - 1}) \leq \sum_{j=10^n-2}^{10^{p-1}-2} \ln(1 + \frac{9 \times (10^{n-1}-1)}{10^j + (d + 1)10^i + 1 - 10^n - 1}) \leq 10^n \leq \frac{10^p}{d + 2} \leq 10^p
\]

From the above upper bound and the definition of \( \epsilon_{(d,n,p)} \), it may be deduced that \( \lim_{n \to +\infty} \epsilon_{(d,n,p)} = 0 \).

Let us now find an appropriate upper bound of \( P_{(d, \alpha(d,p), n)} \). Thanks to inequalities (3):

\[
b_{(d,p)} \leq 9 \times 10^{p+1-2} - \sum_{j=10^n-2}^{10^{p-1}-1} ((9 + d)10^j + 10^{p-2} - 10^{p-1})
\]

\[
\leq 9 \times 10^{p+1-2} - \sum_{j=10^n-2}^{10^{p-1}-1} ((9 + d)10^j + 10^{p-2} - 10^{p-1})(\ln(\frac{10^j + (d + 1)}{10^j + d})
\]

\[
+ \ln(1 + \frac{10^n - 1}{10^{j + (d + 1)}})
\]

\[
\leq 9 \times 10^{p+1-2} - d \times 10^j k_{(d,p)} - 9 \times 10^j \delta_{(d,p)} + 10^{p-1} k_{(d,p)}.
\]
Similarly, we have thanks to inequalities (5):

\[
a_{i,d,p} \leq \sum_{j=10^{p-2}}^{10^{p-1}-2} 10^j (j + 1) \ln \left( \frac{10(j + 1) + d}{10(j + d + 1)} \right) + 10^{p-2-i} \ln \left( \frac{10^{p-1} + d}{10^{p-1-i} + 1} \right) + 10^{p-1-i} \ln \left( \frac{10^{p-1} + 1}{10^{p-1-i} + 1} \right)
\]

\[
\leq 10^i n(d,p) + 10^i \sum_{j=10^{p-2}}^{10^{p-1}-2} j \ln (1 + \frac{9 \times 10^{p-1} \ln(1 + \frac{10^{p-1} + d}{10^{p-1-i} + 1})}{(10j + d + 1)10^y - 10^{-i}})
\]

\[
+ 10^i (m(d,p) + \sum_{j=10^{p-2}}^{10^{p-1}-2} \ln (1 + \frac{9 \times 10^{p-1} \ln(1 + \frac{10^{p-1} + d}{10^{p-1-i} + 1})}{(10j + d + 1)10^y - 10^{-i}}))
\]

Hence we have:

\[
P(d,a_{(d,p)}(n),p) \leq \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} (9 \times 10^{p+i-2} - d \times 10^i k_{(d,p)} - 9 \times 10^i l_{(d,p)} + 10^i m_{(d,p)}
\]

\[
+ 10^i n(d,p) + 10^p-2i \ln \left( \frac{10^{p-1} + d}{10^{p-1-i} + 1} \right) + 10^{p-1-i} \ln \left( \frac{10^{p-1} + 1}{10^{p-1-i} + 1} \right)
\]

\[
+ 10^p-1 k_{(d,p)} + 10^i \sum_{j=10^{p-2}}^{10^{p-1}-2} j \ln (1 + \frac{9 \times 10^{p-1} \ln(1 + \frac{10^{p-1} + d}{10^{p-1-i} + 1})}{(10j + d + 1)10^y - 10^{-i}})
\]

\[
+ 10^p-1 \ln (1 + \frac{9 \times 10^{p-1} \ln(1 + \frac{10^{p-1} + d}{10^{p-1-i} + 1})}{(10j + d + 1)10^y - 10^{-i}})
\]

In light of the following equality \( \sum_{i=0}^{n-p} 10^p = \frac{10^{p+1}-1}{9} \), we have:

\[
\lim_{n \to +\infty} \left( \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} 9 \times 10^{p+i-2} \right) = 10^{-1}
\]

\[
\lim_{n \to +\infty} \left( -\sum_{i=0}^{n-p} d \times 10^i k_{(d,p)} \right) = -\lim_{n \to +\infty} k_{(d,p)}
\]

\[
\lim_{n \to +\infty} \left( -\sum_{i=0}^{n-p} 9 \times 10^i l_{(d,p)} \right) = -\lim_{n \to +\infty} l_{(d,p)}
\]

\[
\lim_{n \to +\infty} \left( \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} 10^i n_{(d,p)} \right) = \lim_{n \to +\infty} n_{(d,p)}
\]

\[
\lim_{n \to +\infty} \left( \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} 10^p-2i \ln \left( \frac{10^{p-1} + d}{10^{p-1-i} + 1} \right) \right) = \frac{1}{9} \ln \left( \frac{10^{p-1} + d}{10^{p-1-i} + 1} \right)
\]

\[
\lim_{n \to +\infty} \left( \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} 10^{p-1-i} \ln \left( \frac{10^{p-1} + 1}{10^{p-1-i} + 1} \right) \right) = \frac{1}{9} \ln \left( \frac{10^{p-1} + 1}{10^{p-1-i} + 1} \right)
\]

\[
\lim_{n \to +\infty} \left( \frac{1}{10^n - 10^{p-1}} \sum_{i=0}^{n-p} 10^p-1 \ln \left( \frac{10^{p-1} + d}{10^{p-1-i} + 1} \right) \right) = 0
\]
Knowing that for all \( x \in [-1; +\infty[ \), we have \( \ln(1 + x) \leq x \), we obtain, for \( i \geq 1 \):

\[
10^{i} \sum_{j=0}^{10^{i-1}-2} \alpha_{j} \ln(1 + \frac{9 \times 10^{i-1}}{(10j + d + 1)10^{i} - 10^{i-1}}) \leq 10^{i+1} \frac{10^{i}}{10^{i+1}10^{i} - 10^{i-1}} = 10^{i+1} - 1 \leq \frac{100}{9}
\]

\[
10^{i} \sum_{j=0}^{10^{i-1}-2} \alpha_{j} \ln(1 + \frac{9 \times 10^{i-1}}{(10j + d + 1)10^{i} - 10^{i-1}}) \leq 10^{i} \frac{10^{i}}{10^{i+1}10^{i} - 10^{i-1}} \leq \frac{100}{9 \times 10^{i-1}}
\]

\[
10^{i+2} \alpha_{1} \ln(1 + \frac{d \times 10^{i-1}}{(10^{i+1}10^{i} - 10^{i-1})}) \leq 10^{i+2} \frac{d \times 10^{i-1}}{10^{i+1}10^{i} - 10^{i-1}} \leq \frac{10^{i+1}}{10^{i+1}10^{i} - 10^{i-1}} \leq \frac{10}{9}
\]

Thanks to \( P_{(d,\phi(d,p))(n,p)} \) upper bound and the above inequalities, the result follows.

Let us denote by \( \alpha_{(d,p)} \) the limit of \( (P_{(d,\phi(d,p))(n,p)})_{n \in \mathbb{N} \cap \{0,p-1\}} \). Here is a few values of \( P_{(d,\phi(d,p))(n,p)} \):

| \( d \) | \( P_{(d,\phi(d,p))(2,2)} \) | \( P_{(d,\phi(d,p))(3,2)} \) | \( P_{(d,\phi(d,p))(4,2)} \) | \( P_{(d,\phi(d,p))(5,2)} \) | \( \alpha_{(d,2)} \) |
|---|---|---|---|---|---|
| 0 | 0.1330 | 0.1144 | 0.1123 | 0.1121 | 0.1121 |
| 1 | 0.1190 | 0.1103 | 0.1092 | 0.1091 | 0.1091 |
| 2 | 0.1107 | 0.1068 | 0.1063 | 0.1062 | 0.1062 |
| 3 | 0.1044 | 0.1037 | 0.1035 | 0.1035 | 0.1035 |
| 4 | 0.0991 | 0.1007 | 0.1009 | 0.1009 | 0.1009 |
| 5 | 0.0945 | 0.0979 | 0.0983 | 0.0984 | 0.0984 |
| 6 | 0.0903 | 0.0953 | 0.0958 | 0.0959 | 0.0959 |
| 7 | 0.0865 | 0.0927 | 0.0935 | 0.0936 | 0.0936 |
| 8 | 0.0829 | 0.0902 | 0.0912 | 0.0913 | 0.0913 |
| 9 | 0.0796 | 0.0879 | 0.0889 | 0.0891 | 0.0891 |

Table 1: Values of \( P_{(d,\phi(d,p))(n,2)} \) and \( \alpha_{(d,2)} \), for \( n \in [2,5] \). These values are rounded to the nearest ten-thousandth.

| \( d \) | \( P_{(d,\phi(d,p))(3,3)} \) | \( P_{(d,\phi(d,p))(4,3)} \) | \( P_{(d,\phi(d,p))(5,3)} \) | \( \alpha_{(d,3)} \) |
|---|---|---|---|---|
| 0 | 0.1045 | 0.1015 | 0.1012 | 0.1012 |
| 1 | 0.1028 | 0.1011 | 0.1009 | 0.1009 |
| 2 | 0.1017 | 0.1008 | 0.1007 | 0.1006 |
| 3 | 0.1008 | 0.1004 | 0.1004 | 0.1004 |
| 4 | 0.1000 | 0.1001 | 0.1001 | 0.1001 |
| 5 | 0.0993 | 0.0998 | 0.0999 | 0.0999 |
| 6 | 0.0986 | 0.0995 | 0.0996 | 0.0996 |
| 7 | 0.0980 | 0.0992 | 0.0993 | 0.0994 |
| 8 | 0.0974 | 0.0989 | 0.0991 | 0.0991 |
| 9 | 0.0968 | 0.0986 | 0.0988 | 0.0989 |

Table 2: Values of \( P_{(d,\phi(d,p))(n,3)} \) and \( \alpha_{(d,3)} \), for \( n \in [3,5] \). These values are rounded to the nearest ten-thousandth.
4 Graphs of \((P_{(d,n,p)})_{n\in\mathbb{N}\setminus[0,10^{p-1}-1]}\)

Let us plot graphs of sequences \((P_{(d,n,2)})_{n\in\mathbb{N}\setminus[0,10^{p-1}-1]}\) for values of \(n\) from 10 to 1000 (Figure 1). Then we plot graphs of \((P_{(d,n,3)})_{n\in\mathbb{N}\setminus[0,10^{p-1}-1]}\), for \(n\in[100,20000]\) (Figure 2).

Let us plot two additional graphs of \(P_{(d,n,2)}\) versus \(\log(n)\) and \(P_{(d,n,3)}\) versus \(\log(n)\) for values of \(n\) from 10 to 2000000:

Through Figures 3 and 4, the proportion of each \(d\) as leading digit, \(d\in[0,9]\), seems to fluctuate and consequently not follow Benford’s Law. Each “pseudo cycle” seems to be composed of \(9\times10^{p-2}\) short waves. Note that these observations were not obvious in view of Figures 1 and 2.

We can also prove the following result:

**Proposition 4.1.** For all \(n\in\mathbb{N}\setminus[0,10^{p-1}-1]\) such that \(n\geq10^{p-1}+9\) and for all \((a,b)\in[0,9]^2\) such that \(a<b\), we have:

\[ P_{(a,n,p)} > P_{(b,n,p)}. \]

The relative position of graphs of \(P_{(d,n,p)}\), for \(d\in[0,9]\), can be observed on Figures 2, 3 and 4.

**Proof.** \((a,b)\in[0,9]^2\) such that \(a<b\). For all \(m\in[10^{p-1},n]\), let us denote by \(\delta_{(a,m)}\) the subset of \(\mathbb{N}\) such that \(\delta_{(a,m)} = \{j \leq m : \text{the } p^{th} \text{ digit of } j \text{ is } a\}.\)
For all \( e \in \mathcal{E}_{b,m} \), we consider \( e' = e - (b - a) \times 10^{d_g - p} \) where \( d_g \) is the number of digits of the integer \( e \). It is clear that \( e' \in \mathcal{E}_{a,m} \). Thus we get: 
\[
|E_{(a,m)}| \geq |E_{(b,m)}|.
\]

We also have \( P_{(a,10^{p-1}+a,p)} = \frac{1}{n+1} > P_{(b,10^{p-1}+a,p)} = 0 \). The result follows.

\[\square\]

**Remark 4.2.** For \( n \in \mathbb{N} \setminus [0, 10^{p-1} - 1] \), we have, if \( n < 10^{p-1} + d \), \( P_{(d,p,n)} = 0 \).

Hence for all \( n \in \mathbb{N} \setminus [0, 10^{p-1} - 1] \) and for all \( (a, b) \in [0, 9]^2 \) such that \( a < b \), we have:

\[
P_{(a,n,p)} \geq P_{(b,n,p)}.
\]

Let us henceforth provide the following equality:

**Proposition 4.3.**

\[
P_{(d,n,p)} = \frac{1}{n+1 - 10^{p-1}} \left( P_{(d,10^{p-1},n,p)} \times (10^{k+p} - 10^{p-1}) + r(n,d,p) \right),
\]

where:

\[
k = \max \{ i \in \mathbb{N} : 10^{i+p} \leq n \}.
\]

**Proof.** Results are directly derived from Proposition 2.1. \[\square\]

5 Study of \( 9 \times 10^{p-2} \) additional subsequences

To definitively bring to light the fact that the sequence \( (P_{(d,n,p)})_{n \in \mathbb{N} \setminus [0,10^{p-1} - 1]} \) does not converge, we will show that there exist additional subsequences that converge to limits different from those of \( (P_{(d,\psi_{(d,p,i)}(n),p)})_{n \in \mathbb{N} \setminus [0,p-1]} \).

For \( i \in [10^{p-2}, 10^{p-1} - 1] \), let us in this way study the \( 9 \times 10^{p-2} \) subsequences \( (P_{(d,\psi_{(d,p,i)}(n),p)})_{n \in \mathbb{N} \setminus [0,p-1]} \) where \( \psi_{(d,p,i)} \) is the function from \( \mathbb{N} \setminus [0, p - 1] \) to \( \mathbb{N} \) that maps \( n \) to \( (10i + (d + 1))10^{n-p+1} - 1 \). We get the below result:

**Proposition 5.1.** \( i \in [10^{p-2}, 10^{p-1} - 1] \). The subsequence \( (P_{(d,\psi_{(d,p,i)}(n),p)})_{n \in \mathbb{N} \setminus [0,p-1]} \) converges to:

\[
\alpha_{(d,p)}10^{p-1} + i + 1 - 10^{p-2} - k_{(d,p,i)} - 9l_{(d,p,i)} + m_{(d,p,i)} + n_{(d,p,i)} + 10^{p-2} \ln(10^{p-1}+d).
\]

where:

\[
\begin{align*}
  k_{(d,p,i)} &= \sum_{j=10^{p-2}}^{i} \ln \left( \frac{10j+(d+1)}{10j+d} \right), \\
  l_{(d,p,i)} &= \sum_{j=10^{p-2}}^{i} j \ln \left( \frac{10j+d}{10j+(d+1)} \right), \\
  m_{(d,p,i)} &= \sum_{j=10^{p-2}}^{i-1} \ln \left( \frac{10(j+1)+d}{10j+(d+1)} \right), \\
  n_{(d,p,i)} &= \sum_{j=10^{p-2}}^{i-1} j \ln \left( \frac{10(j+1)+d}{10j+(d+1)} \right).
\end{align*}
\]

**Proof.** \( i \in [10^{p-2}, 10^{p-1} - 1] \). Thanks to Proposition 4.3, we have, for \( n \in \mathbb{N} \setminus [0,p-1] \):

\[
P_{(d,\psi_{(d,p,i)}(n),p)} = \frac{1}{(10i + (d + 1))10^{n-p+1} - 10^{p-1}} \left( P_{(d,10^{n-1}+p)} \times (10^{n} - 10^{p-1}) + r\left(\psi_{(d,p,i)}(n),d,p\right)\right).
\]
The first term of $r(\psi(d,p,i))$ can be simplified as follows:

$$
\sum_{j=10^n-2}^{i+1} \frac{(10j+d)10^n-p+1 \cdot 10^{n-p+1} - 10^n-1}{b+1 - 10^{n-1}} 
= 10^{n-p+1}(10^n-2 + 1) - \sum_{j=10^n-2}^{i+1} (9j+d)10^n-p+1 \cdot 10^{n-p+1} - 10^n-1 
\sum_{b=(10j+d)10^n-p+1}^{i+1} \frac{1}{b+1 - 10^{n-1}}
\sim_{n \to +\infty} 10^n-1(10^n-2 + 1) - \sum_{j=10^n-2}^{i+1} (9j+d)10^n-p+1 \cdot 10^{n-p+1} \ln(\frac{10j+d}{10j+d+1}),
$$
thanks to inequalities [3].

The second term of $r(\psi(d,p,i))$ can be simplified as follows:

$$
\sum_{j=10^n-2}^{i+1} \frac{(10(j+1)+d)10^n-p+1 - 10^n-p+2}{a+1 - 10^{p-1}} 
= 10^{n-p+1}(10^n-2 - 10^n-p+2) \sum_{a=10^n}^{i+1} \frac{1}{a+1 - 10^{p-1}} 
+ (10^{n-p+1}(j+1) - 10^n-p+2) \sum_{j=10^n-2}^{i+1} \frac{1}{a+1 - 10^{p-1}}
\sim_{n \to +\infty} 10^n-1 \ln(\frac{10^n-p+1+d}{10^{p-1}}) + \sum_{j=10^n-2}^{i+1} 10^n-p+1(j+1) \ln(\frac{10(j+1)+d}{10j+d+1}),
$$
thanks to inequalities [3].

Knowing that $P(d,10^n-1,p) \sim_{n \to +\infty} \alpha(d,p)$ (see Proposition 3.1), the result follows.

Let us denote by $\alpha(d,p,i)$ the limit of $(P(d,\psi(d,p,i),p))_{n \in \mathbb{N} \setminus [0,p-1]}$. Here is a few values of $P(d,\psi(d,p,i))(n,p)$:

| $d$ | $P(d,\psi(d,2,7),(2,2))$ | $P(d,\psi(d,2,7),(3,2))$ | $P(d,\psi(d,2,7),(4,2))$ | $P(d,\psi(d,2,7),(5,2))$ | $\alpha(d,2,7)$ |
|-----|------------------------|------------------------|------------------------|------------------------|----------------|
| 0   | 0.1112                 | 0.1112                 | 0.1114                 | 0.1114                 | 0.1114         |
| 1   | 0.0131                 | 0.0131                 | 0.0131                 | 0.0131                 | 0.0131         |
| 2   | 0.0762                 | 0.0764                 | 0.0769                 | 0.0769                 | 0.0769         |
| 3   | 0.1042                 | 0.1040                 | 0.1040                 | 0.1040                 | 0.1040         |
| 4   | 0.1046                 | 0.1038                 | 0.1038                 | 0.1038                 | 0.1038         |
| 5   | 0.0974                 | 0.0974                 | 0.0974                 | 0.0974                 | 0.0974         |
| 6   | 0.0954                 | 0.0950                 | 0.0951                 | 0.0951                 | 0.0951         |
| 7   | 0.0923                 | 0.0924                 | 0.0924                 | 0.0924                 | 0.0924         |
| 8   | 0.0886                 | 0.0889                 | 0.0889                 | 0.0889                 | 0.0889         |
| 9   | 0.0860                 | 0.0874                 | 0.0876                 | 0.0876                 | 0.0876         |

Table 3: Values of $P(d,\psi(d,2,7),(n,i))$ and $\alpha(d,2,7)$, for $n \in \mathbb{N} \setminus [0,p-1]$ and $i = 7$. These values are rounded to the nearest ten-thousandth.

As a result, the sequence $(P(d,n,p))_{n \in \mathbb{N} \setminus [0,10^n-1]}$ does not converge. The $9 \times 10^{p-2}$ convergent subsequences confirm the remarks raised by Figures 3 and 4 about the existence of “pseudo cycles” in the graph of $(P(d,n,p))_{n \in \mathbb{N} \setminus [0,10^n-1]}$. 

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| d | $P_{(d,\psi,(d,3,23),(n),3)}$ | $P_{(d,\psi,(d,3,23),(4),3)}$ | $P_{(d,\psi,(d,3,23),(5),3)}$ | $\alpha_{(d,3,23)}$ |
|---|---|---|---|---|
| 0 | 0.1017 | 0.1021 | 0.1022 | 0.1024 |
| 1 | 0.1026 | 0.1018 | 0.1011 | 0.1017 |
| 2 | 0.1017 | 0.1012 | 0.1012 | 0.1012 |
| 3 | 0.1027 | 0.1007 | 0.1007 | 0.1007 |
| 4 | 0.1000 | 0.1002 | 0.1002 | 0.1002 |
| 5 | 0.0995 | 0.0997 | 0.0997 | 0.0997 |
| 6 | 0.0988 | 0.0992 | 0.0993 | 0.0993 |
| 7 | 0.0982 | 0.0987 | 0.0988 | 0.0988 |
| 8 | 0.0976 | 0.0983 | 0.0983 | 0.0983 |
| 9 | 0.0969 | 0.0978 | 0.0979 | 0.0979 |

Table 4: Values of $P_{(d,\psi,(d,3,23),(n),3)}$ and $\alpha_{(d,3,23)}$, for $n \in [3, 5]$ and $i = 23$. These values are rounded to the nearest ten-thousandth.

5.1 Central values

From Figures 3 and 4, we notice that there exist fluctuations in the graph of $(P_{(d,n,p)})_{n \in \mathbb{N} \setminus [10^{p-1} - 1]}$. We define $C_{(d,p)}$ as follows:

**Definition 5.2.**

$$C_{(d,p)} = \frac{1}{9 \times 10^{p-2}} \sum_{i=10^{p-2}}^{10^{p-1}-1} \alpha_{(d,p,i)}.$$ 

Figure 5 below shows the different values of $\alpha_{(0,2,i)}$, for $i \in [1, 9]$ and also the values of $P_{(0,n,2)}$ versus $\log(n)$ for $n \in [10, \text{2000000}]$:

![Graph of $P_{(0,n,2)}$ versus $\log(n)$](image)

Figure 5: Graph of $P_{(0,n,2)}$ versus $\log(n)$. Note that points have not been all represented. Lines whose equation is $y = \alpha_{(0,2,i)}$, for $i \in [1, 9]$, have also been plotted. Note that those of equations $y = \alpha_{(0,2,1)}$ and $y = \alpha_{(0,2,7)}$ are almost coincident. We have $C_{(0,2)} \approx 0.1170$.

These means values are very close to the theoretic value highlighted in [8] as can be seen in below tables (Tables 5 and 6, where $p = 2$ and $p = 3$, respectively). According to Hill ([9]), it is absolutely normal.

We furthermore note, thanks to Table 5, that $C_{(0,2)}$ slightly underestimates $\sum_{j=1}^{9} \log(1 + \frac{1}{10^j})$ as can be inferred from Figure 5.

**Conclusion**

To conclude, through our model, we have seen that the proportion of $d$ as $p^{th}$ digit, $d \in [0, 9]$, in certain naturally occurring collections of data is more likely
ε & |d| − 2
\sum_{j=1}^{9} \log(1 + \frac{1}{10^{j+d}})
\begin{array}{|c|c|c|}
\hline
d & C(d,2) & \sum_{j=1}^{9} \log(1 + \frac{1}{10^{j+d}}) \\
\hline
0 & 0.1170 & 0.1197 \\
1 & 0.1122 & 0.1139 \\
2 & 0.1079 & 0.1088 \\
3 & 0.1039 & 0.1043 \\
4 & 0.1001 & 0.1003 \\
5 & 0.0967 & 0.0967 \\
6 & 0.0935 & 0.0934 \\
7 & 0.0905 & 0.0904 \\
8 & 0.0878 & 0.0876 \\
9 & 0.0851 & 0.0850 \\
\hline
\end{array}

Table 5: Values of $C(d, p)$ and probabilities associated to the second digit ([8]), for $p = 2$. These values are rounded to the nearest thousandth.

d & C(d,3) & \sum_{j=10}^{99} \log(1 + \frac{1}{10^{j+d}})
\begin{array}{|c|c|c|}
\hline
d & 0.1016 & 0.1018 \\
1 & 0.1013 & 0.1014 \\
2 & 0.1009 & 0.1010 \\
3 & 0.1005 & 0.1006 \\
4 & 0.1002 & 0.1002 \\
5 & 0.0998 & 0.0998 \\
6 & 0.0994 & 0.0994 \\
7 & 0.0991 & 0.0990 \\
8 & 0.0987 & 0.0986 \\
9 & 0.0984 & 0.0983 \\
\hline
\end{array}

Table 6: Values of $C(d, p)$ and probabilities associated to the third digit ([8]). These values are rounded to the nearest thousandth.

to follow a law whose probability distribution is $(d, P(d, n, p), d \in [0.9])$, where $n$ is the smaller integer upper bound of the physical, biological or economical quantities considered, rather than the generalized Benford’s Law. Knowing beforehand the value of the upper bound $n$ can be a way to find a better adjusted law than Benford’s one.

The results of the article would have been the same in terms of fluctuations of the proportion of $d \in [0, 9]$ as $p^1$ digit, of limits of subsequences, or of results on central values, if our discrete uniform distributions uniformly randomly selected were lower bounded by a positive integer different from $10^{p-1}$; first terms in proportion formulas become rapidly negligible. Through our model we understand that the predominance of 0 as $p^1$ digit (followed by those of 1 and so on) is all but surprising in experimental data: it is only due to the fact that, in the lexicographical order, 0 appears before 1, 1 appears before 2, etc.

However the limits of our model rest on the assumption that the random variables used to obtain our data are not the same and follow discrete uniform distributions that are uniformly randomly selected. In certain naturally occurring collections of data it cannot conceivably be justified. Studying the cases where the random variables follow other distributions (and not necessarily
randomly selected) sketch some avenues for future research on the subject.

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Appendix: Python script

Using Proposition 2.1 we can determine the terms of \( (P_{d,n,p})_{n \in \mathbb{N} \setminus \{0,10^p-1\}} \) for \( d \in [0,9] \). To this end, we have created a script with the Python programming language (Python Software Foundation, Python Language Reference, version 3.4, available at [http://www.python.org](http://www.python.org), see [14]). The implemented function \texttt{expvalProp} has three parameters: the rank \( n \) of the wanted term of the sequence, the position \( p \) of the considered digit and the value \( d \) of this digit. Here is the used algorithm:

```python
def expvalProp(n, d, p):
    k=-1;
    while(10**(k+p+1)<n):
        k=k+1
    l=math.floor((n-(10**(p-1)+d)*10**(k+1))/10**(k+2))+10**(p-2);S=0;T=0;
    if (k!=-1):
        for i in range(0,k+1):
            for j in range(10**(p-1)+d,10**(p+1)+1):
                T=T+(j-10**(p-2))/(j+1-10**(p-1))
        for j in range(10**(p-2),l):
            for a in range(max(10**(p+i-1),(10*(j+1)+d)*10**(k+i)),min(n,(10*(j+1)+d)*10**(k+i)-1)+1):
                S=S+((j+1)*10**(k+i)-10**(p-2))/(a+1-10**(p-1))
    else:
        for j in range(10**(p-2),l+1):
            for a in range(max(10**(p+i),(10*(j+1)+d)*10**(k+i)),min(n,(10*(j+1)+d)*10**(k+i)-1)+1):
                S=S+((j+1)*10**(k+i)-10**(p-2))/(a+1-10**(p-1))
    return((S+T)/(n+1-10**(p-1)))
```

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