IMPROVING E. CARTAN CONSIDERATIONS ON THE INVARIANCE OF NONHOLONOMIC MECHANICS

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Abstract. This paper concerns an intrinsic formulation of nonholonomic mechanics. Our point of departure is the paper [6], by Koiller et al., revisiting E. Cartan’s address at the International Congress of Mathematics held in 1928 at Bologna, Italy ([3]). Two notions of equivalence for nonholonomic mechanical systems \((M, g, \mathcal{D})\) are introduced and studied. According to [6], the notions of equivalence considered in this paper coincide. A counterexample is presented here showing that this coincidence is not always true.

1. Introduction and objectives. Let \((M, g)\) be a Riemannian manifold and \(\mathcal{D}\) a distribution on \(M\), i.e. a smooth vector subbundle of the tangent bundle \(\tau_M : TM \to M\) (all objects in this paper are assumed to be in the \(C^\infty\)-setting). We denote by \(P, P^\perp : TM \to TM\) the orthogonal projections on \(\mathcal{D}\) and \(\mathcal{D}^\perp\), respectively. The nonholonomic trajectories of a linearly constrained mechanical system \((M, g, \mathcal{D})\) are the base integral curves of the second order vector field \(X_{\mathcal{D}} \in \mathfrak{X}(\mathcal{D})\) given by \((\forall v_q \in \mathcal{D}) X_{\mathcal{D}}(v_q) = TP \cdot S(v_q)\), where \(S\) is the geodesic spray of \((M, g)\). Our primary goals are: (i) to establish conditions for the invariance of the nonholonomic vector field \(X_{\mathcal{D}}\) under a change of metric tensor \(g \rightsquigarrow G\) and (ii) the study of geometrical objects which remain invariant under such a change of metric tensors.

We organise the paper as follows. In section 2, we define a notion of equivalence for linearly constrained mechanical systems \((M, g, \mathcal{D})\) and \((M', g', \mathcal{D}')\): they are equivalent if they have the same nonholonomic trajectories. That is the case if, and only if, their difference tensor (see definition 2.1) is antisymmetric. We show that, for each equivalence class, there exists a well defined “nonholonomic connection”

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(see definition 2.4) which, despite not being a proper “affine connection”, has many properties in common with such a connection (see proposition 3), and may be used to define the nonholonomic trajectories in a similar manner to which the Levi-Civita connection defines the geodesics of Riemannian geometry — see proposition 4, part (d). It seems that E. Cartan, in §5 of [3], by stating that “(...) Il faut donc restreindre le choix des coefficients $B_\lambda$ de la substitution linéaire (4) par la condition que, les coefficients $C_{ik}$ étant fixes, les formes $\omega_\iota$ restent les mêmes, à des combinaisons linéaires près des $\omega_\alpha (...)”$, he did not want to impose the systems to keep the same nonholonomic trajectories, as we require in our weak equivalence notion in section 2.

We believe he had in mind the stronger condition for equivalence mentioned above. What was not elucidated by Cartan, or appreciated in [6], is that this strong notion of equivalence is sufficient, but not necessary to maintain the nonholonomic geodesics, as our counterexample in section 4 shows. Proposition 4.1 and theorem 6.1 in [6] are not valid in general for our weak notion of equivalence. Of course, this minor mistake does not invalidate what is done in that paper — one just has to consider the appropriate stronger notion of equivalence in their statements.

2. Invariant nonholonomic mechanics. Henceforth, we use superscripts $g$ or $G$ to denote objects defined in terms of the corresponding metric tensor. For instance, we denote by $\nabla^g$ the Levi-Civita connection of the Riemannian manifold $(M, g)$. We omit the superscripts if the discussion concerns only one of the metric tensors.

**Definition 2.1.** The difference tensor is the smooth tensor field $D : \mathcal{D} \oplus_M \mathcal{D} \to \mathcal{D}$ defined by, $\forall X, Y \in \Gamma^\infty(\mathcal{D})$:

$$D(X,Y) = P^g \nabla_X^g Y - P^G \nabla_Y^G Y$$

**Definition 2.2** (equivalence of nonholonomic mechanical systems). We say that the nonholonomic mechanical systems $(M, g, \mathcal{D})$ and $(M, G, \mathcal{D})$ are equivalent if they have the same nonholonomic trajectories, i.e. if $X^g_\iota = X^G_\iota$.

**Proposition 1.** With the notation above, $(M, g, \mathcal{D})$ and $(M, G, \mathcal{D})$ are equivalent if, and only if, the difference tensor $D$ is antisymmetric.

**Proof.** It is immediate from the fact that the base integral curves of $X^g_\iota$ are the geodesics of the tangent connection $\Gamma^\infty(TM) \times \Gamma^\infty(\mathcal{D}) \to \Gamma^\infty(\mathcal{D})$, $(X,Y) \mapsto P^g \nabla_X^g Y$, i.e. the solutions of $P^g \nabla_{[X,Y]} = 0$. \qed

**Definition 2.3.** Let $B_\mathcal{D} : TM \oplus_M \mathcal{D} \to \mathcal{D}^\perp$ be the total second fundamental form of $(M, g, \mathcal{D})$, i.e. the smooth tensor field defined by $(X,Y) \in \Gamma^\infty(TM) \times \Gamma^\infty(\mathcal{D}) \mapsto P^\perp \nabla_X Y \in \Gamma^\infty(\mathcal{D}^\perp)$. We denote by $B^g_\mathcal{D}$ and $B^G_\mathcal{D}$, respectively, the antisymmetric and symmetric parts of the restriction of $B_\mathcal{D}$ to $\mathcal{D} \oplus_M \mathcal{D}$.

Note that, $\forall X, Y \in \Gamma^\infty(\mathcal{D})$, $P \nabla_X Y = \nabla_X Y - B_\mathcal{D}(X,Y)$, so that $P \nabla_X Y + P \nabla_Y X = 2(P \nabla_X Y + B^g_\mathcal{D}(X,Y)) - [X,Y]$, where $[\cdot,\cdot]$ is the Lie bracket. Since the Lie bracket does not depend on the metric tensor, we have proven the following:
Proposition 2. With the notation above, the difference tensor $D$ is antisymmetric if, and only if, the map $\nabla^{nh} : \Gamma^\infty(\mathcal{D}) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(TM)$ defined by $(X,Y) \mapsto P\nabla_X Y + B^\mathcal{D}_\emptyset(X,Y)$ is invariant under the change of metric tensors.

Therefore, the object introduced in the definition below depends only on the nonholonomic vector field $X_\mathcal{D}$, i.e. it does not depend on the particular metric tensor which induces $X_\mathcal{D}$.

Definition 2.4 (the nonholonomic connection). We call nonholonomic connection the map introduced in the previous proposition, i.e.

$$\nabla^{nh} : \Gamma^\infty(\mathcal{D}) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(TM)$$

$$(X,Y) \mapsto P\nabla_X Y + B^\mathcal{D}_\emptyset(X,Y)$$

Despite not being a proper “affine connection”, we have used the nomenclature and notation above because $\nabla^{nh}$ has properties which are quite similar to those of an affine connection. We list some of these properties in the proposition below.

Definition 2.5 (1st derived system). We call first derived system the $C^\infty$-module $\mathcal{D}^{(1)}$ generated by the smooth sections of $\mathcal{D}$ along with their Lie brackets.

For the sake of simplicity, we assume henceforth that $\mathcal{D}^{(1)}$ is regular; i.e. it coincides with the $C^\infty$-module of the smooth sections of a smooth vector subbundle of $TM$, denoted by the same notation $\mathcal{D}^{(1)}$.

Proposition 3 (properties of the nonholonomic connection). With the notation above, the following properties hold:

a) $\nabla^{nh}$ is $C^\infty$-linear in the first factor and a derivation on the second factor, i.e. $\forall f \in C^\infty(M), \forall X,Y \in \Gamma^\infty(\mathcal{D})$:

i) $\nabla^{nh}_{fX} Y = f\nabla^{nh}_X Y$

ii) $\nabla^{nh}_X fY = X[f]Y + f\nabla^{nh}_X Y$

b) $\forall v_q \in \mathcal{D}_q, \forall Y \in \Gamma^\infty(\mathcal{D})$, $\nabla^{nh}_v Y$ depends only on $Y \circ \gamma_{v_q}$, for a given smooth path $\gamma_{v_q} : (-,\epsilon) \rightarrow M$ such that $\gamma_{v_q}(0) = v_q$.

c) $\nabla^{nh}$ is symmetric, i.e. $\forall X,Y \in \Gamma^\infty(\mathcal{D})$, $\nabla^{nh}_X Y - \nabla^{nh}_Y X = [X,Y]$.

d) $\nabla^{nh}$ is compatible with the metric tensor, i.e. $\forall X,Y,Z \in \Gamma^\infty(\mathcal{D})$, $X(Y,Z) = (\nabla^{nh}_X Y, Z) + (Y, \nabla^{nh}_X Z)$.

e) $\nabla^{nh}$ takes values in $\Gamma^\infty(\mathcal{D}^{(1)})$. Moreover, its image lies in $\Gamma^\infty(\mathcal{D})$ if, and only if, $\mathcal{D}$ is involutive, in which case it coincides with the Levi-Civita connection of the integral leaves of $\mathcal{D}$ (that is, its restriction to the smooth sections of the tangent bundle of a given leaf is the Levi-Civita connection of that leaf).

In view of parts (a) and (b) of the previous proposition, it makes sense to define a “horizontal lift” induced by the nonholonomic connection:

Definition 2.6 (nonholonomic horizontal lift). With the notation above, given $v_q \in \mathcal{D}$, we define the nonholonomic horizontal lift at $v_q$:

$$H^\mathcal{D}_{v_q} : \mathcal{D}_q \longrightarrow TM_{v_q}\mathcal{D}^{(1)}$$

$$y_q \mapsto TV : y_q - \lambda_{v_q}(\nabla^{nh}_{y_q} V)$$

where $V$ is any smooth local section of $\mathcal{D}$ such that $V(q) = v_q$ and $\lambda_{v_q}$ denotes the vertical lift at $v_q$, i.e. the natural isomorphism $T_qM \rightarrow TM_{v_q}$.

Note that the nonholonomic horizontal lift at $v_q$ is linear and well-defined: the definition does not depend on the smooth local section $V$ of $\mathcal{D}$ such that $V(q) = v_q$. 
(as a consequence of parts (a) and (b) of the previous proposition). Moreover, since \( \mathcal{D}^{(1)} \) is regular, the nonholonomic horizontal lift indeed takes values in the tangent space at \( v_q \) of \( \mathcal{D}^{(1)} \) (as a consequence of part (e) of the previous proposition). Besides, the following properties hold:

**Proposition 4** (properties of the nonholonomic horizontal lift). With the notation above, for all \( v_q, w_q \in \mathcal{D} \):

a) \( H_{v_q}^\mathcal{D} \) is a linear isomorphism onto a subspace of \( T_{v_q} \mathcal{D}^{(1)} \) which is transverse (but not complementary, except for the involutive case) to the vertical subspace \( V_{v_q} \mathcal{D}^{(1)} \). We call this subspace nonholonomic horizontal subspace at \( v_q \) and we denote it with the same notation \( H_{v_q}^\mathcal{D} \).

b) The horizontal lift and the horizontal subspace at \( v_q \) depend only on \( X_{\mathcal{D}} \), i.e., they are invariant under a change of metric tensor \( g \sim G \) which preserves the nonholonomic vector field.

c) \( H_{v_q}^\mathcal{D} \cdot v_q = X_{\mathcal{D}}(v_q) \).

d) \( H_{v_q}^\mathcal{D}(w_q) \) is tangent to \( \mathcal{D} \) if, and only if, \( B_{\mathcal{D}}(v_q, w_q) = 0 \).

**Proof.**

a) It is an immediate consequence of the fact that \( T_{\mathcal{D}} \circ H_{v_q}^\mathcal{D} = \text{id}_{\mathcal{D}} \).

b) It is an immediate consequence of the fact that the nonholonomic connection depends only on the nonholonomic vector field.

c) Let \( \kappa : TTM \to TM \) be the connector induced by the Levi-Civita connection \( \nabla \) of the Riemannian manifold \( (M, g) \) (that is, \( \kappa \) is the projection on the vertical subbundle followed by the inverse of the vertical lift), and \( H_{v_q} : T_q M \to T_{v_q} \mathcal{D} \), the horizontal lift at \( v_q \) (that is, the inverse of the restriction of the tangent map \( T_{\mathcal{D}} \) to the horizontal subspace of \( TM \) at \( v_q \)). Taking a smooth section \( V \) of \( \mathcal{D} \) such that \( V(q) = v_q \), it follows that \( \kappa \cdot H_{v_q}^\mathcal{D}(w_q) = \nabla_{w_q} V - \nabla_{w_q} V = \nabla_{w_q} V - P\nabla_{w_q} V = B_{\mathcal{D}}(v_q, w_q) \). On the other hand, \( T_{\mathcal{D}} \cdot H_{v_q}^\mathcal{D}(v_q) = v_q \). It then follows that \( H_{v_q}^\mathcal{D}(v_q) = H_{v_q}(v_q) + \lambda_{v_q} \cdot B_{\mathcal{D}}(v_q, v_q) = S(v_q) + \lambda_{v_q} \cdot B_{\mathcal{D}}(v_q, v_q) = X_{\mathcal{D}}(v_q) \).

d) Using the connector induced by the Levi-Civita connection of \( (M, g) \), the parallel derivative (see [9]) of \( P^\perp \) is given by \( (\forall y_q, \ z_q \in T_q M) \mathcal{P} \mathcal{P}^\perp \cdot (y_q, \ z_q) = B_{\mathcal{D}}(y_q, P^\perp \cdot : y_q) - B_{\mathcal{D}}(z_q, P^\perp \cdot y_q) \), where \( B_{\mathcal{D}} \) denotes the total second fundamental form of the orthogonal subbundle \( \mathcal{D}^\perp \). Therefore, denoting by \( \mathcal{P} \mathcal{P}^\perp \) the fiber derivative of \( P^\perp \) (that is, the restriction of its tangent map to the vertical subbundle of \( TTM \)), given \( v_q \in T_{v_q} \mathcal{D} \), we have \( X_{v_q} \in T_{v_q} \mathcal{D} \iff 0 = \kappa \cdot P^\perp \cdot X_{v_q} = \mathcal{P} \mathcal{P}^\perp \cdot (v_q) \cdot X_{v_q} + \mathcal{P} \mathcal{P}^\perp (v_q) \cdot T_{\mathcal{D}} \cdot X_{v_q} = P^\perp \cdot \kappa X_{v_q} + B_{\mathcal{D}}(T_{\mathcal{D}} \cdot X_{v_q}, P^\perp \cdot v_q) - B_{\mathcal{D}}(T_{\mathcal{D}} \cdot X_{v_q}, P^\perp \cdot v_q) = P^\perp \cdot \kappa X_{v_q} - B_{\mathcal{D}}(T_{\mathcal{D}} \cdot X_{v_q}, v_q) \). By the formula obtained in the previous item, we have \( T_{\mathcal{D}} \cdot H_{v_q}^\mathcal{D}(w_q) = w_q \) and \( \kappa \cdot H_{v_q}^\mathcal{D}(w_q) = B_{\mathcal{D}}(w_q, v_q) \); it then follows that \( H_{v_q}^\mathcal{D}(w_q) \in T_{v_q} \mathcal{D} \iff P^\perp \cdot B_{\mathcal{D}}(v_q, w_q) - B_{\mathcal{D}}(w_q, v_q) = 0 \iff B_{\mathcal{D}}(v_q, w_q) = 0 \).

**Proposition 5.** The nonholonomic connection depends only on the restriction of the metric tensor to \( \mathcal{D} \) and on the orthogonal subbundle of \( \mathcal{D} \) in \( \mathcal{D}^{(1)} \), i.e., on \( \mathcal{D}^\perp \cap \mathcal{D}^{(1)} \).

**Proof.** We have, for all \( X, Y \in \Gamma^\infty(\mathcal{D}) \), \( \nabla_{\mathcal{D}} Y = P \nabla X Y + B_{\mathcal{D}}(X, Y) = P \nabla X Y + \frac{1}{2} P^\perp \cdot [X, Y] \). It is clear that the second summand depends only on the orthogonal subbundle of \( \mathcal{D} \) in \( \mathcal{D}^{(1)} \). The fact that the first summand depends only on the restriction of the metric tensor to \( \mathcal{D} \) and on the orthogonal subbundle of \( \mathcal{D} \) in \( \mathcal{D}^{(1)} \)
is a consequence of the Koszul formula that defines the Levi-Civita connection (see [7], eq. 3.19, page 31).

\[ \square \]

**Corollary 1.** The nonholonomic connection, as well as the nonholonomic horizontal lift, depend only on the restriction of the metric tensor to \( \mathcal{D}(1) \).

3. **Strongly invariant nonholonomic mechanics.** We consider in this section a stronger notion of equivalence:

**Definition 3.1** (strong equivalence of nonholonomic mechanical systems). We say that the nonholonomic mechanical systems \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) are strongly equivalent if the following conditions hold:

i) they are equivalent in the sense of definition 2.2, i.e. if \( X^g_\mathcal{D} = X^G_\mathcal{D} \).

ii) the orthogonal complement of \( \mathcal{D} \) in \( \mathcal{D}(1) \) is the same for both \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\).

**Proposition 6.** For nonholonomic mechanical systems \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\), the following conditions are equivalent:

a) \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) are strongly equivalent.

b) Both \( B^G_\mathcal{D} : \Gamma^\infty(\mathcal{D}) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(\mathcal{D}^\perp) \) and \( P\nabla : \Gamma^\infty(\mathcal{D}) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(\mathcal{D}) \) are invariant under the change of metric tensor \( g \rightarrow G \).

c) The difference tensor \( D : \Gamma^\infty(\mathcal{D}) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(\mathcal{D}) \) is null.

**Proof.** a) \( \Rightarrow \) b): It follows from a) that both \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) have the same nonholonomic connection \( \nabla^{nh} \), and \( \mathcal{D}^{1*} \cap \mathcal{D}(1) = \mathcal{D}^{1*} \cap \mathcal{D}(1) \). For all \( X, Y \in \Gamma^\infty(\mathcal{D}) \), the tangential and orthogonal components of \( \nabla_X^{nh} Y \in \mathcal{D}(1) \) with respect to either metric tensor and the decomposition \( \mathcal{D}(1) = \mathcal{D} \oplus \mathcal{D}^{1*} \cap \mathcal{D}(1) \), are, respectively, \( P^g \nabla_X^{nh} Y = P^G \nabla_X^{nh} Y \) and \( B^G_\mathcal{D}(X, Y) = B^G_\mathcal{D}(X, Y) \).

b) \( \Rightarrow \) c): is clear by the definition of the difference tensor.

c) \( \Rightarrow \) a): the difference tensor is null, hence antisymmetric, i.e. condition i) in definition 3.1 holds, so that the nonholonomic connection \( \nabla^{nh} \) is the same for both \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\). Moreover, the fact that the difference tensor is null clearly implies the invariance of \( P\nabla \), hence the invariance of \( B^G_\mathcal{D} = \nabla^{nh} - P\nabla \). Since \( \forall X, Y \in \Gamma^\infty(\mathcal{D}), B^G_\mathcal{D}(X, Y) = P^{\perp}[X, Y], \) the invariance of \( \mathcal{D}^{1*} \cap \mathcal{D}(1) \) by the change of metric tensor \( g \rightarrow G \) follows, i.e. the second condition in definition 3.1 is fulfilled.

\[ \square \]

**Proposition 7.** If \( g \) and \( G \) tame the same subriemannian metric on \( \mathcal{D} \) and \( \mathcal{D}^{1*} \cap \mathcal{D}(1) = \mathcal{D}^{1*} \cap \mathcal{D}(1) \), then \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) are strongly equivalent.

**Proof.** As a corollary of proposition 5, both \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) have the same nonholonomic connection.

\[ \square \]

In the sequel we consider, until the end of this section, the particular case in which both metric tensors \( g \) and \( G \) tame the same subriemannian metric on \( \mathcal{D} \), i.e. \( g_{\mathcal{D}} = G_{\mathcal{D}} \). In order to make a comparison of our results with the ones from [6] and [3], we do computations locally by means of adapted orthonormal frame fields.

Let \( \text{rk } \mathcal{D} = m \), \( \dim M = n \). With respect to the metric tensor \( g \), we fix an orthonormal frame field \( \mathcal{E} = (X_1, \ldots, X_m, X_{m+1}, \ldots, X_n) \) defined on an open set \( \mathcal{U} \subset M \) and adapted to the Whitney sum \( TM = \mathcal{D} \oplus M \mathcal{D}^\perp \). We denote by \( (\omega^1, \ldots, \omega^n) \) the corresponding dual coframe field. Finally, for \( I, J \in \{1, \ldots, n\} \), let
\(\omega_j^I\) be the connection 1-forms associated to the frame field \(E\) for the Levi-Civita connection \(\nabla\) of the metric tensor \(g\), i.e. \(\forall X \in \mathfrak{X}(U), \langle \omega_j^I, X \rangle = \langle \omega^I, \nabla X \rangle \).

We adopt the following index convention: capital roman letters \(I, J, K, \ldots\) run from 1 to \(n\), lower roman letters \(i, j, k, \ldots\) run from 1 to \(m\) and greek letters \(\alpha, \beta, \gamma, \ldots\) run from \(m + 1\) to \(n\). Repeated indices mean “sum”, unless explicitly stated otherwise.

We use a bar to denote the corresponding objects with respect to the metric tensor \(G\). Since \(g|_\mathcal{D} = G|_\mathcal{D}\), we may take a \(G\)-orthonormal frame field \(\bar{E} = (\bar{X}_1, \ldots, \bar{X}_n)\) on \(U\) adapted to the Whitney sum \(TM = \mathcal{D} \oplus \mathcal{D}^{1-6}\) such that \(\bar{X}_i = X_i\) for \(1 \leq i \leq m\). It then follows that \(\omega_i\) and \(\bar{\omega}_i\) coincide on tangent vectors to \(\mathcal{D}\), for \(1 \leq i \leq m\).

Following the definitions and notation in [6], we say that two 1-forms \(\theta^1, \theta^2 \in \Omega_1(M)\) are \(\mathcal{D}\)-equivalent, and we write \(\theta^1 \sim_\mathcal{D} \theta^2\), if these 1-forms coincide on tangent vectors to \(\mathcal{D}\). Part (b) of the following proposition corrects proposition 4.1 in the aforementioned paper.

**Proposition 8.** With the notation above, we have:

a) \((U, g, \mathcal{D}|_U)\) and \((U, G, \mathcal{D}|_U)\) are equivalent if, and only if, \(\forall 1 \leq k \leq m\), the matrices \([\langle \omega_j^k, X_i \rangle]_{1 \leq i, j \leq m}\) and \([\langle \bar{\omega}_j^k, X_i \rangle]_{1 \leq i, j \leq m}\) have the same symmetric part.

b) \((U, g, \mathcal{D}|_U)\) and \((U, G, \mathcal{D}|_U)\) are strongly equivalent if, and only if, \(\forall 1 \leq k, j \leq m\), \(\omega_j^k \sim_\mathcal{D} \bar{\omega}_j^k\).

**Proof.** Let \(X = \xi^i X_i, Y = \eta^i X_i \in \Gamma^\infty(\mathcal{D}|_U)\). We compute:

\[
P^g\nabla_X^g Y = \langle \omega^k, \nabla_X^g Y \rangle X_k = \{\xi^i \eta^j (\omega_j^k, X_i) + X[\eta^k]\} X_k.
\]

Likewise:

\[
P^G\nabla_X^G Y = \langle \bar{\omega}^k, \nabla_X^g Y \rangle \bar{X}_k = \{\xi^i \eta^j (\bar{\omega}_j^k, X_i) + X[\eta^k]\} \bar{X}_k.
\]

Therefore, \(D(X, Y) = P^g\nabla_X^g Y - P^G\nabla_X^G Y = \{\xi^i \eta^j [\langle \omega_j^k, X_i \rangle - \langle \bar{\omega}_j^k, X_i \rangle]\} X_k\). Since, on each fiber, \((\xi^i)_{1 \leq i \leq m}\) and \((\eta^j)_{1 \leq j \leq m}\) are arbitrary \(m\)-tuples of real numbers, we conclude that the difference tensor \(D\) on \(U\) is antisymmetric (respectively, null) if, and only if, \(\forall 1 \leq k \leq m\), the matrix \([\langle \omega_j^k, X_i \rangle - \langle \bar{\omega}_j^k, X_i \rangle]_{1 \leq i, j \leq m}\) is antisymmetric (respectively, null), what proves both assertions. \(\square\)

4. **A counterexample.** The following is an example of two maximally nonholonomic mechanical systems which are equivalent, but not strongly equivalent.

Let \(M = \text{SO}(4)\) and consider the following basis for its Lie algebra \(\mathfrak{so}(4)\):

\[
(A_i)_{j,k} = \begin{pmatrix}
-\epsilon_{i,j,k} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad 1 \leq i, j, k \leq 3
\]

\[
B_1 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad B_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

so that \([A_i, A_j] = \epsilon_{i,j,k} A_k, [B_i, B_j] = \epsilon_{i,j,k} A_k\) and \([A_i, B_j] = \epsilon_{i,j,k} B_k\). We consider the left invariant distribution \(\mathcal{D}\) on \(M\) whose fiber at the identity is \(\mathcal{D}_d = \langle B_1, B_2, B_3 \rangle\). It then follows that \(\mathcal{D}^{(1)} = TM\), i.e. \(\mathcal{D}\) is maximally nonholonomic.

We now introduce two metric tensors \(g\) and \(G\) on \(M\) such that \((M, g, \mathcal{D})\) and \((M, G, \mathcal{D})\) be equivalent, but not strongly equivalent.
We take \( g \) as the bi-invariant metric on \( M \) induced by the Cartan-Killing form on \( \mathfrak{so}(4) \), i.e. \( \forall X, Y \in \mathfrak{so}(4), \ g(X, Y) = -\frac{1}{2} \text{tr}(XY) \). With respect to this metric tensor, \( \mathcal{E} = (B_1, B_2, B_3, A_1, A_2, A_3) \) is a global left invariant orthonormal frame field adapted to the Whitney sum \( TM = \mathcal{D} \oplus M \mathcal{D}^{-1} \).

As the metric tensor \( G \), we take the left invariant metric which is given at \( T_{id}M = \mathfrak{so}(4) \), with respect to the ordered basis \( \mathcal{E} = (B_1, B_2, B_3, A_1, A_2, A_3) \), by the matrix:

\[
[G(id)] = \begin{pmatrix}
1 & 0 & 0 & \delta & 0 & 0 \\
0 & 1 & 0 & 0 & \delta & 0 \\
0 & 0 & 1 & 0 & 0 & \delta \\
\delta & 0 & 0 & 1 & 0 & 0 \\
0 & \delta & 0 & 0 & 1 & 0 \\
0 & 0 & \delta & 0 & 0 & 1
\end{pmatrix},
\]

where \( \delta > 0 \) is chosen small enough in order for \( [G(id)] \) to be positive definite. It is clear that \( g \) and \( G \) tame the same subriemannian metric on \( \mathcal{D} \), and that \( (B_1, B_2, B_3) \) is a global left invariant orthonormal frame field for \( \mathcal{D} \).

Note that \( \mathcal{D}^{-1} = (A_1, A_2, A_3) \). Besides, by the bi-invariance of \( g \) and the Koszul formula for the Levi-Civita connection, for all left invariant vector fields \( X, Y \in \mathfrak{X}(M) \), \( \nabla^g_X Y = \frac{1}{2}[X, Y] \). It then follows that, for \( 1 \leq i, j \leq 3 \):

\[
P^g \nabla^g_{B_i} B_j = 0. \tag{1}
\]

On the other hand, from the left invariance of the \( B_i \)'s and the Koszul formula for \( \nabla^G \), denoting by \( \bar{G} = [\bar{G}_{i,j} \equiv G(B_i, A_j)]_{1 \leq i, j \leq 3} \), i.e. the diagonal matrix \( \bar{G} = \delta \text{id} \), we compute:

\[
2G(\nabla^G_{B_i} B_j, B_k) = -G(B_i, [B_j, B_k]) + G(B_j, [B_k, B_i]) + G(B_k, [B_i, B_j]) = -\epsilon_{jkl} \bar{G}_{il} + \epsilon_{kil} \bar{G}_{jl} + \epsilon_{ijl} \bar{G}_{kl} = \delta(-\epsilon_{jki} + \epsilon_{kij} + \epsilon_{ijk}) = \epsilon_{ijk} \delta
\]

It then follows that, for \( 1 \leq i, j \leq 3 \):

\[
P^G \nabla^G_{B_i} B_j = \frac{1}{2} \epsilon_{ijk} \delta B_k. \tag{2}
\]

From equations (1) and (2), we compute the difference tensor, for \( 1 \leq i, j \leq 3 \):

\[
D(B_i, B_j) = P^g \nabla^g_{B_i} B_j - P^G \nabla^G_{B_i} B_j = -\frac{1}{2} \epsilon_{ijk} \delta B_k. \tag{3}
\]

Therefore, for \( 1 \leq i, j, k \leq 3 \), \( D(B_i, B_j) + D(B_j, B_i) = -\frac{1}{2} \epsilon_{ijk} \delta B_k = -\frac{1}{2} (\epsilon_{ijk} + \epsilon_{jik}) B_k = 0 \), since \( \epsilon_{ijk} + \epsilon_{jik} = 0 \). We conclude that \( D \) is antisymmetric, i.e. \( (M, g, \mathcal{D}) \) and \( (M, G, \mathcal{D}) \) are equivalent. However, since \( D(B_1, B_2) = -\frac{1}{2} B_3 \neq 0 \) (since we have chosen \( \delta > 0 \)), the difference tensor \( D \) is not null, i.e. \( (M, g, \mathcal{D}) \) and \( (M, G, \mathcal{D}) \) are not strongly equivalent, as asserted.

**Remark 1.** The referee has observed that finding examples that are weakly but not strongly equivalent leads to an overdetermined system of equations, indicating that such examples should be rare. He claims that the notions of strongly equivalent and weakly equivalent coincide for rank two distributions ([5]). This implies for instance that in Cartan’s example of a sphere rolling rolling on a plane without slipping or twisting considered in paragraph §11 of [3], a \((2,3,5)\) distribution, the two notions of equivalence coincide. He also claims that for a rank 3 (or higher) distribution, the corank of \( \mathcal{D} \) in \( \mathcal{D}^{(1)} \) must be at least 3 in order to find examples where the two notions of equivalence do not coincide. This implies that the “minimal example”
occurs with a (3,6) distribution such as the one presented in the current paper. This rules out the possibility of finding examples on (3,5) distributions such as Chaplygin’s marble sphere.

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