Frame-like gauge invariant formulation for mixed symmetry fermionic fields

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Abstract

In this paper we consider frame-like formulation for mixed symmetry spin-tensors corresponding to arbitrary Young tableau with two rows. First of all, we extend Skvortsov formulation [24] for massless mixed symmetry bosonic fields in flat Minkowski space to the case of massless fermionic fields. Then, using such massless fields as building blocks, we construct gauge invariant formulation for massive spin-tensors with the same symmetry properties. We give general massive theories in \((A)dS\) spaces with arbitrary cosmological constant and investigate all possible massless and partially massless limits.

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1 Introduction

As is well known, in $d = 4$ dimensions for the description of arbitrary spin particles it is enough to consider completely symmetric (spin-)tensor fields only. At the same time, in dimensions greater than four in many cases like supergravity theories, superstrings and higher spin theories, one has to deal with mixed symmetry (spin-)tensor fields [1, 2, 3, 4]. There are different approaches to investigation of such fields both light-cone [5, 6], as well as explicitly Lorentz covariant ones (e.g. [7, 8, 9, 10, 11, 12, 13]). For the investigation of possible interacting theories for higher spin particles as well as of gauge symmetry algebras behind them it is very convenient to use so-called frame-like formulation [14, 15, 16] (see also [17, 18, 19]) which is a natural generalization of well-known frame formulation of gravity in terms of vielbein $e_\mu^a$ and Lorentz connection $\omega^{ab}_\mu$.

There are two different frame-like formulations for massless mixed symmetry bosonic fields. For simplicity, let us restrict ourselves with mixed symmetry tensors corresponding to Young tableau with two rows. Let us denote $Y(k, l)$ a tensor $\Phi^{a_1...a_k,b_1...b_l}$ which is symmetric both on first $k$ as well as last $l$ indices, completely traceless on all indices and satisfies a constraint $\Phi^{a_1...a_k,b_1...b_l} = 0$, where round brackets mean symmetrization. In the first approach [20, 21, 22, 23] for the description of $Y(k, l)$ tensor ($k \neq l$) one use a one-form $e_\mu^Y(k-1,l)$ as a main physical field. In this, only one of two gauge symmetries is realized explicitly and such approach is very well adapted for the $AdS$ spaces. Another formulation [24] uses two-form $e_{\mu\nu}^Y(k-1,l-1)$ as a main physical field in this, both gauge symmetries are realized explicitly. Such formalism works in flat Minkowski space while deformations into $AdS$ space requires introduction of additional fields [25].
In Section 2 of our paper we extend the formulation of [24] to the case of mixed-symmetry spin-tensors $Y(k+\frac{1}{2}, l+\frac{1}{2})$ corresponding to arbitrary Young tableau with two rows. Similarly to the bosonic case both gauge transformations will be realized explicitly and formulation will work in flat Minkowski space only while deformation into $AdS$ space turns out to be impossible (the only exception is a spin-tensor $Y(k+\frac{1}{2}, k+\frac{1}{2})$ corresponding to rectangular Young tableau).

Then in Section 3 we construct gauge invariant frame-like formulation for massive mixed symmetry spin-tensors corresponding to arbitrary Young tableau with two rows (examples for bosonic fields were considered in [26]). There are two general approaches to gauge invariant description of massive fields. One of them uses powerful BRST approach [27, 28, 29, 30, 11, 12, 31, 32]. Another one, which we will follow in this work, [33, 34, 35, 36, 19, 26, 37] is a generalization to higher spin fields of well-known mechanism of spontaneous gauge symmetry breaking. In this, one starts with appropriate set of massless fields with all their gauge symmetries and obtain gauge invariant description of massive field as a smooth deformation. One of the nice feature of gauge invariant formulation for massive fields is that it allows us effectively use all known properties of massless fields serving as building blocks. As we have already seen in all cases considered previously and we will see again in this paper, gauge invariant description of massive fields always allows smooth deformation into $(A)dS$ space without introduction of any additional fields besides those that are necessary in flat Minkowski space so that restriction mentioned above will not be essential for us.

As we will see in all models constructed in Section 3, gauge invariance completely fixes all parameters in the Lagrangian and gauge transformations leaving us only one free parameter having dimension of mass. It is hardly possible to give meaningful definition of what is mass for mixed symmetry (spin)-tensor fields in $(A)dS$ spaces (see e.g. [38]) and we will not insist on any such definition. Instead, we will simply use this parameter to analyze all possible special limits that exist in $(A)dS$ spaces. In this, only fields having the same number of degrees of freedom as massless one in flat Minkowski space we will call massless ones, while all other special limits that appear in $(A)dS$ spaces will be called partially massless [39, 40, 41, 34, 17].

2 Massless case

In this Section we consider frame-like formulation for massless mixed symmetry fermionic fields in flat Minkowski space. We begin with some simple concrete examples and then consider their generalization up to spin-tensors corresponding to arbitrary Young tableau with two rows. In all cases we also consider a possibility to deform such theories into $AdS$ space. As is well known, most of mixed symmetry (spin)-tensors do not admit such deformation without introduction of some additional fields [25], but the structure of possible mass terms and corresponding corrections to gauge transformations will be heavily used in the next Section where we consider massive theories.
2.1 \( Y(k + \frac{3}{2}, \frac{1}{2}) \)

In what follows we will need frame-like formulation for completely symmetric spin-tensors [15, 16, 26]. For completeness we reproduce here all necessary formulas. Main object — one-form \( \Phi_\mu^{a_1 \ldots a_k} = \Phi_\mu^{(a_k)} \) completely symmetric on local indices and satisfying a constraint 
\[ \gamma^{a_1} \Phi^{a_1(ak-1)} = (\gamma \Phi)^{(ak-1)} = 0. \]

To describe correct number of physical degrees of freedom the free massless theory have to be invariant under the following gauge transformations:
\[ \delta_0 \Phi_\mu^{(ak)} = \partial_\mu \zeta^{(ak)} + \eta_\mu^{(ak)} \]

where parameters \( \zeta \) and \( \eta \) have to satisfy:
\[ (\gamma \zeta)^{ak-1} = 0, \quad \eta^{(a, ak)} = 0, \quad \gamma^a \eta^a{, (ak)} = (\gamma \eta)^{a, (ak-1)} = 0. \]

In what follows round brackets denote symmetrization. The free Lagrangian describing massless particle in flat Minkowski space can be written as follows:
\[
L_0 = -i(-1)^k \left\{ \mu^{\nu \alpha} \right\} \left[ \Phi^{(a_k)} \Gamma^{\nu \alpha} \Phi^{(a_k)} + 6k \Phi^{(a_k-1)} \gamma^b \partial_\nu \Phi^{(a_k-1)} \right]
\]

where relative coefficients are fixed by the invariance under \( \eta \) shifts. Here and further:
\[ \left\{ \mu^{\nu \alpha} \right\} = e^{[\mu}_{\nu} e^{\alpha}_{\nu} \right\}, \quad \Gamma^{\mu \nu} = \frac{1}{6} \gamma^{[a \gamma b \gamma c]} \]

and so on. It is not hard to construct a deformation into \( AdS \) space. If we replace ordinary partial derivatives in the Lagrangian and gauge transformations by the \( AdS \) covariant ones, the Lagrangian cease to be gauge invariant:
\[ \delta_0 L_0 = i(-1)^{k+1} \frac{(d + 2k - 1)(d + 2k - 2)}{2} \kappa \]

Note that the Lagrangian is completely antisymmetric on world indices, so covariant derivatives effectively act on local indices (including implicit spinor one) only, e.g.:
\[
[D_\mu, D_\nu] \zeta^{(ak)} = -\kappa [e_{[\mu}^{a_1} \zeta_{a_k]}^{ak-1} + \frac{1}{2} \Gamma_{\mu \nu} \zeta^{(ak)}], \quad \kappa = \frac{2 \Lambda}{(d - 1)(d - 2)}
\]

But gauge invariance could be restored by adding appropriate mass-like terms to the Lagrangian as well as corresponding corrections to gauge transformations:
\[
L_1 = (-1)^k b_k \left\{ \mu^{ab} \right\} \left[ \Phi^{(a_k)} \Gamma^{ab} \Phi^{(a_k)} + 2k \Phi^{a(ak-1)} \Phi^{b(ak-1)} \right]
\]

\[ \delta_1 \Phi_\mu^{(ak)} = i\beta_k\left[ \gamma_\mu \zeta^{(ak)} - \frac{2}{(d + 2k - 2)} \gamma^{(a_1} \zeta_{a_k-1)} \right] \]

provided:
\[ \beta_k = -b_k \frac{3}{3(d - 2)} \]
\[ b_k^2 = -\frac{9}{4}(d + 2k - 2)^2 \kappa \]

Note that relative coefficients in the mass-like terms are again fixed by the invariance under \( \eta \) shifts, while the structure of variations are chosen so that they are \( \gamma \)-transverse.
2.2 $Y(\frac{5}{2}, \frac{3}{2})$

Let us begin with the simplest example of mixed symmetry fermionic field. Frame-like description requires two-form $\Psi_{\mu\nu}^a$ which is $\gamma$-transverse $\gamma^a\Psi_{\mu\nu}^a = 0$. Free massless theory in flat Minkowski space has to be invariant under the following gauge transformations:

$$\delta_0 \Psi_{\mu\nu}^a = \partial_{[\mu} \xi_{\nu]}^a + \eta_{\mu\nu}^a$$

where parameter $\xi_{\mu}^a$ is $\gamma$-transverse $\gamma^a\xi_{\mu}^a = 0$, while parameter $\eta_{abc}$ is completely antisymmetric and $\gamma$-transverse $\gamma^a\eta_{abc} = 0$. The Lagrangian can be written in the following form:

$$\mathcal{L}_0 = i \left\{ \mu, \nu \right\} [\bar{\Psi}^\mu_{a} \Gamma_{abcde} \partial_{\mu} \Psi_{\nu}^d - 10 \bar{\Psi}^\mu_{a} \Gamma_{bc} \partial_{\mu} \Psi_{\nu}^e]$$

Being completely antisymmetric on world indices, both terms are separately invariant under the $\xi$ transformations, while the relative coefficients are fixed by the the invariance under the $\eta$ shifts.

As is well known it is impossible to deform such massless theory into AdS space without introduction of additional fields. Indeed, after replacement of ordinary partial derivatives by the AdS covariant ones, we could try to restore broken gauge invariance by adding mass-like terms to the Lagrangian and corresponding corrections to gauge transformations:

$$\mathcal{L}_1 = a_1 \left\{ \mu, \nu \right\} [\bar{\Psi}^\mu_{a} \Gamma_{ab} \Psi_{\nu}^a + 6 \bar{\Psi}^\mu_{a} \Gamma_{bd} \partial_{\mu} \Psi_{\nu}^d]$$

$$\delta \Psi_{\mu\nu}^a = i \alpha_1 [\gamma_{[\mu} \xi_{\nu]}^a + \frac{2}{d} \gamma^a \xi_{[\mu} \nu]}$$

In this, variations with one derivative cancel provided:

$$\alpha_1 = -\frac{a_1}{5(d - 4)}$$

but it is impossible to cancel variations without derivatives by adjusting the only free parameter $a_1$.

2.3 $Y(k + \frac{3}{2}, \frac{3}{2})$

It is pretty straightforward to generalize the example of previous Subsection to the case corresponding to Young tableau with $k + 1$ boxes in the first row and only one box in the second row. Frame-like formulation requires two-form $\Psi_{\mu\nu}^{(a_k)}$ completely symmetric on its $k$ local indices and $\gamma$-transverse $\gamma^{a_1} \Psi_{\mu\nu}^{a_1(a_{k-1})} = 0$. Free massless theory has to be invariant under the following gauge transformations:

$$\delta_0 \Psi_{\mu\nu}^{(a_k)} = \partial_{[\mu} \xi_{\nu]}^{(a_k)} + \eta_{\mu\nu}^{(a_1,a_{k-1})}$$

where parameter $\xi_{\mu}^{(a_k)}$ is $\gamma$-transverse, while parameter $\eta_{abc}^{(a_{k-1})}$ completely antisymmetric on first three indices, completely symmetric on the last $k - 1$ ones and satisfies:

$$\eta^{[abc,a_1]}(a_{k-2}) = 0, \quad \gamma^{a_1} \eta^{abc,(a_{k-1})} = (\gamma \eta)^{abc,(a_{k-2})} = 0$$
Corresponding massless Lagrangian has the form:
\[ L_0 = i(-1)^{k+1} \left\{ \mu_{abcd} \right\} \left[ \bar{\Psi}_{\mu}^{\nu}(a_{k})\Gamma_{\alpha\beta\gamma}^{\nu} \partial_{\alpha} \Psi_{\beta\gamma}^{\nu}(a_{k}) - 10k \bar{\Psi}_{\mu}^{\nu}(a_{k-1})\Gamma_{\alpha\beta\gamma}^{\nu} \partial_{\alpha} \Psi_{\beta\gamma}^{\nu}(a_{k-1}) \right] \] (10)

Exactly as in the previous case an attempt to deform such theory into AdS space without introduction of additional fields fails. Again, after replacement of ordinary derivatives by the covariant ones, we could try to restore broken gauge invariance by adding mass-like terms to the Lagrangian as well as corresponding corrections to gauge transformations:
\[ L_1 = (-1)^{k} a_k \left\{ \mu_{abcd} \right\} \left[ \bar{\Psi}_{\mu}^{\nu}(a_{k})\Gamma_{\alpha\beta\gamma}^{\nu} \Psi_{\alpha\beta\gamma}^{\nu}(a_{k}) + 6k \bar{\Psi}_{\mu}^{\nu}(a_{k-1})\Gamma_{\alpha\beta\gamma}^{\nu} \Psi_{\alpha\beta\gamma}^{\nu}(a_{k-1}) \right] \] (11)
\[ \delta_1 \Psi_{\mu}^{\nu}(a_{k}) = i\alpha_k \left[ \gamma[\mu_{\xi\nu}]^{\alpha}_{\beta\gamma}(a_{k}) + \frac{2}{(d+2k-2)} \gamma^{(a_1}_{\xi[\mu_{\nu}]}{^\alpha_{k-1}]} \right] \] (12)

In this, variations with one derivative cancel provided:
\[ \alpha_k = \frac{a_k}{5(d-4)} \]
but it is impossible to achieve the cancellation of variations without derivatives.

2.4 \( Y(\frac{5}{2}, \frac{5}{2}) \)

Among all mixed symmetry (spin)-tensors corresponding to Young tableau with two rows whose with equal number of boxes in both rows turn out to be special and require separate consideration. Let us begin with the simplest example — \( Y(\frac{5}{2}, \frac{5}{2}) \). Frame-like formulation requires two-form \( R_{\mu\nu}^{ab} \) which is antisymmetric on \( ab \) and \( \gamma \)-transverse \( \gamma^{a}_{R_{\mu\nu}^{ab}} = 0 \). Free massless theory has to be invariant under the following gauge transformations:
\[ \delta_0 R_{\mu\nu}^{ab} = \partial_{[\mu_{\xi\nu}]}^{ab} + \eta_{[\mu_{\nu}]}^{ab} \] (13)

where parameters \( \xi_{ab} \) and \( \eta_{abc} \) are antisymmetric on their local indices and \( \gamma \)-transverse. It is not hard to construct gauge invariant Lagrangian:
\[ L_0 = -i \left\{ \mu_{abcd} \right\} \left[ \bar{R}_{\mu\nu}^{fg} \Gamma_{\alpha\beta\gamma}^{fg} \partial_{\alpha} R_{\beta\gamma}^{fg} - 20 \bar{R}_{\mu\nu}^{af} \Gamma_{\alpha\beta\gamma}^{fg} \partial_{\alpha} R_{\beta\gamma}^{ef} - 60 \bar{R}_{\mu\nu}^{ab} \gamma^{c} \partial_{\alpha} R_{\beta\gamma}^{de} \right] \] (14)

where again each term is separately invariant under the \( \xi \) transformations, while relative coefficients are fixed by the invariance under the \( \eta \) shifts.

One of the main special features of such (spin)-tensors is the fact that they admit deformation into AdS space without introduction of any additional fields. Indeed, let us replace all derivatives in the Lagrangian and gauge transformations by the AdS covariant ones. As usual, the initial Lagrangian cease to be invariant:
\[ \delta_0 L_0 = -10i(d-1)(d-2)\kappa \left\{ \mu_{abcd} \right\} \left[ \bar{R}_{\mu\nu}^{de} \Gamma_{\alpha\beta\gamma}^{de} \xi_{\alpha}^{dc} - 6 \bar{R}_{\mu\nu}^{ab} \gamma^{b} \xi_{a}^{cd} \right] \]

In this, broken gauge invariance can be restored by adding mass-like terms to the Lagrangian and corresponding corrections to gauge transformations:
\[ L_1 = a_{1,1} \left\{ \mu_{abcd} \right\} \left[ \bar{R}_{\mu\nu}^{ef} \Gamma_{\alpha\beta\gamma}^{ef} R_{\alpha\beta}^{de} + 12 \bar{R}_{\mu\nu}^{ae} \Gamma_{\alpha\beta}^{bc} R_{\alpha\beta}^{de} - 12 \bar{R}_{\mu\nu}^{ab} \gamma^{c} R_{\alpha\beta}^{de} \right] \] (15)
\[ \delta_1 R_{\mu\nu}^{ab} = i\alpha_{1,1} \left[ \gamma[\mu_{\xi\nu}]^{ab} + \frac{2}{(d-2)} \gamma^{(a}_{\xi[\mu_{\nu}]}{^b]} \right] \] (16)

provided:
\[ \alpha_{1,1} = \frac{a_{1,1}}{5(d-4)}, \quad a_{1,1} = -\frac{25(d-2)^2}{4}\kappa \]
2.5 $Y(k + \frac{3}{2}, k + \frac{3}{2})$

It is straightforward to construct a generalization of previous example for arbitrary $k > 1$. For this we need a two-form $R_{\mu\nu}^{(a_k),(b_k)}$ which is symmetric and $\gamma$-transverse on both groups of local indices and satisfies $R_{\mu\nu}^{(a_k,b_1),(b_k-1)} = 0$. Moreover, $R_{\mu\nu}^{(a_k),(b_k)} = R_{\mu\nu}^{(b_k),(a_k)}$. Free massless theory has to be invariant under the following gauge transformations:

$$\delta_0 R_{\mu\nu}^{(a_k),(b_k)} = D_{[\mu} \xi_{\nu]}^{(a_k),(b_k)} + \eta_{\mu}^{(a_k),(b_k),\nu}$$

(17)

where parameter $\xi_{\mu}^{(a_k),(b_k)}$ has the same properties on local indices as $R_{\mu\nu}^{(a_k),(b_k)}$, while parameter $\eta_{\mu}^{(a_k),(b_k),\nu}$ satisfies:

$$(\gamma \eta)_{\mu}^{(a_k-1),(b_k),\nu} = (\gamma \eta)_{\mu}^{(a_k),(b_k-1),\nu} = \gamma^a \eta_{\mu}^{(a_k),(b_k),\nu} = 0$$

Massless Lagrangian can be constructed out of three terms separately invariant under $\xi_{\mu}$ transformations:

$$\mathcal{L}_0 = -i \left\{ \mu \alpha \beta \gamma \delta \right\} \left[ R_{\mu\nu}^{(a_k),(b_k)} \Gamma^{abce} \partial_{\alpha} R_{\beta\gamma}^{(a_k),(b_k)} - 20k R_{\mu\nu}^{(a_k),(b_k)} a^{(b_k-1)} \Gamma^{bcde} \partial_{\alpha} R_{\beta\gamma}^{(a_k),(b_k),\nu} + 60k^2 R_{\mu\nu}^{(a_k-1),(b_k-1)} c^{(b_k-1),\nu} \partial_{\alpha} R_{\beta\gamma}^{(a_k),(b_k-1),\nu} \right]$$

(18)

where relative coefficients are fixed by the invariance under the $\eta$ transformations.

As in the previous case, such massless theory could be deformed into $AdS$ space without introduction of any additional fields. Gauge invariance broken by the replacement of ordinary derivatives by the $AdS$ covariant ones can be restored if we add to the Lagrangian mass-like terms of the form:

$$\mathcal{L}_1 = a_{k,k} \left\{ \mu \alpha \beta \right\} \left[ R_{\mu\nu}^{(a_k),(b_k)} \Gamma^{abce} R_{\alpha\beta}^{(a_k),(b_k)} + 12k R_{\mu\nu}^{(a_k),(b_k)} a^{(b_k-1)} \Gamma^{bcde} R_{\alpha\beta}^{(a_k),(b_k),\nu} - 12k^2 R_{\mu\nu}^{(a_k-1),(b_k-1)} c^{(b_k-1),\nu} R_{\alpha\beta}^{(a_k),(b_k-1),\nu} \right]$$

(19)

as well as corresponding corrections to gauge transformations:

$$\delta_1 R_{\mu\nu}^{(k),(k)} = i a_{k,k} \left[ \gamma_{\mu} \xi_{\nu}^{(a_k),(b_k)} \right] + \frac{2}{(d + 2k - 4)} (\gamma (a_k) \xi_{\mu\nu}^{(a_k-1),(b_k)} + \gamma (b_k) \xi_{\mu\nu}^{(a_k),(b_k-1),\nu})$$

(20)

provided:

$$a_{k,k} = \frac{a_{k,k}}{5(d-4)}, \quad a_{k,k} = -\frac{25}{4} (d + 2k - 4)^2 \kappa$$

2.6 $Y(k + \frac{3}{2}, l + \frac{3}{2})$

Now we are ready to consider general case of $Y(k + \frac{3}{2}, l + \frac{3}{2})$ with $k > l \geq 1$. This time we need a two-form $\Psi_{\mu\nu}^{(a_k),(b_k)}$ which is symmetric and $\gamma$-transverse on both groups of local indices and satisfies $\Psi_{\mu\nu}^{(a_k,b_1),(b_k-1)} = 0$. Gauge transformations for free massless theory have the form:

$$\delta \Psi_{\mu\nu}^{(a_k),(b_k)} = D_{[\mu} \xi_{\nu]}^{(a_k),(b_k)} + \eta_{\mu}^{(a_k),(b_k),\nu}$$

(21)

where parameter $\xi_{\mu}^{(a_k),(b_k)}$ has the same properties on local indices as $\Psi_{\mu\nu}^{(a_k),(b_k)}$, while parameter $\eta_{\mu}^{(a_k),(b_k),\nu}$ satisfies:

$$\eta_{\mu}^{(a_k),(b_k),(b_k-1),c} = \eta_{\mu}^{(a_k),(b_k),c} = 0, \quad (\gamma \eta)_{\mu}^{(a_k-1),(b_k),c} = (\gamma \eta)_{\mu}^{(a_k),(b_k-1),c} = \gamma^c \eta_{\mu}^{(a_k),(b_k),c} = 0$$
This time we have four terms separately invariant under $\xi_\mu$ transformations to construct massless Lagrangian:

$$i(-1)^{k+l}L_0 = \left\{ \mu\nu\alpha\beta\gamma \right\} \left\{ \bar{\Psi}_{\mu\nu}^{(a_k),(b_l)} \Gamma^{abcd\gamma} \partial_\alpha \bar{\Psi}_{\beta\gamma}^{(a_k),(b_l)} - 
- 10k\bar{\Psi}_{\mu\nu}^{a(a_{k-1}),(b_l)} \Gamma^{bcd\gamma} \partial_\alpha \bar{\Psi}_{\beta\gamma}^{e(a_{k-1}),(b_l)} + 
- 10l\bar{\Psi}_{\mu\nu}^{a(a_k),(b_{l-1})} \Gamma^{bcd\gamma} \partial_\alpha \bar{\Psi}_{\beta\gamma}^{e(a_k),(b_{l-1})} - 
- 60kl\bar{\Psi}_{\mu\nu}^{a(a_{k-1}),b(b_{l-1})} \gamma^c \partial_\alpha \bar{\Psi}_{\beta\gamma}^{d(a_{k-1}),e(b_{l-1})} \right\} (22)$$

where as usual relative coefficients are fixed by the invariance under $\eta$ transformations.

It is not possible to deform this massless theory into $AdS$ space without introduction of additional fields. Indeed, possible mass-like terms look as follows:

$$(-1)^{k+l}L_1 = a_{k,l} \left\{ \mu\nu\alpha\beta \right\} \left\{ \bar{\Psi}_{\mu\nu}^{(a_k),(b_l)} \Gamma^{abcd\gamma} \Psi_{\alpha\beta}^{(a_k),(b_l)} + 
+ 6l\bar{\Psi}_{\mu\nu}^{a(a_{k-1}),(b_l)} \Gamma^{bcd\gamma} \Psi_{\alpha\beta}^{d(b_{l-1})} + 
+ 6k\bar{\Psi}_{\mu\nu}^{a(b_{l-1}),(b_l)} \Gamma^{bcd\gamma} \Psi_{\alpha\beta}^{d(a_{k-1})} + 
- 12kl\bar{\Psi}_{\mu\nu}^{a(a_{k-1}),b(b_{l-1})} \Psi_{\alpha\beta}^{d(a_{k-1}),e(b_{l-1})} \right\} (23)$$

In this, their non-invariance under the initial gauge transformations can be compensated by corresponding corrections to gauge transformations:

$$\delta_1 \Psi_{\mu\nu}^{(a_k),(b_l)} = i\alpha_{k,l} \left\{ \gamma_{[\mu\xi_\nu]}^{(a_k),(b_l)} + \frac{2}{(d+2k-2)} \gamma^{(a_1 \xi_{[\mu\nu]} a_{k-1}),(b_l)} + \right. 
+ \left. \frac{2}{(d+2l-4)} \gamma^{(b_l \xi_{[\mu\nu]} a_{k-1}),\nu]} \right. 
- \frac{4}{(d+2k-2)(d+2l-4)} \gamma^{(a_1 \xi_{[\mu\nu]} a_{k-1}),(b_l,\nu]}) \right\} (24)$$

provided:

$$\alpha_{k,l} = \frac{a_{k,l}}{5(d-4)}$$

but it is not possible to cancel variations without derivatives by adjusting the value of $a_{k,l}$.

### 3 Massive case

In this Section we construct gauge invariant frame-like formulation for massive mixed symmetry fermionic fields. Once again we begin with some simple concrete examples and then construct their generalizations. In all cases our general strategy will be the same. First of all we determine a set of massless fields which are necessary for gauge invariant description of massive field. Then we construct the Lagrangian as a sum of kinetic and mass terms for all fields involved as well as all possible cross terms without derivatives and look for the necessary corrections to gauge transformations. As we have already mentioned in the Introduction, such gauge invariant formalism works equally well both in flat Minkowski space as well as in $(A)dS$ space with arbitrary value of cosmological constant. This, in turn, allows us to investigate all possible massless and partially massless limits that exist in $(A)dS$ spaces.
3.1 $Y(\frac{5}{2}, \frac{3}{2})$

Let us begin with the simplest case — $Y(\frac{5}{2}, \frac{3}{2})$. To construct gauge invariant description of massive particle we, first of all, have to determine a set of massless fields which are necessary for such description. In general, for each gauge invariance of main gauge field we have to introduce corresponding primary Goldstone field. Usually, these fields turn out to be gauge fields themselves with their own gauge invariances, so we have to introduce secondary Goldstone fields and so on. But in the mixed symmetry (spin)-tensor case we have to take into account reducibility of their gauge transformations. Let us illustrate on this simplest case. Our main gauge field $Y(\frac{5}{2}, \frac{3}{2})$ has two gauge transformations (combined into one $\xi_\mu^a$ transformation in the frame-like approach) with the parameters $Y(\frac{5}{2}, \frac{1}{2})$ and $Y(\frac{3}{2}, \frac{3}{2})$ and reducibility corresponding to $Y(\frac{3}{2}, \frac{1}{2})$. Thus we have to introduce two primary Goldstone fields corresponding to $Y(\frac{5}{2}, \frac{1}{2})$ and $Y(\frac{3}{2}, \frac{3}{2})$. Both have its own gauge transformations with parameters $Y(\frac{5}{2}, \frac{1}{2})$ but due to reducibility of gauge transformations for the main gauge field, it is enough to introduce one secondary Goldstone field $Y(\frac{3}{2}, \frac{1}{2})$ only. This field also has its own gauge transformation with parameter $Y(\frac{3}{2}, \frac{1}{2})$ but due to reducibility of gauge transformations for the field $Y(\frac{3}{2}, \frac{1}{2})$ the procedure stops here. Thus we need four fields: $Y(\frac{5}{2}, \frac{3}{2})$, $Y(\frac{5}{2}, \frac{1}{2})$, $Y(\frac{3}{2}, \frac{3}{2})$ and $Y(\frac{3}{2}, \frac{1}{2})$. It is natural to use frame-like formalism for all fields in question so we will use $\Psi_{\mu a}^\gamma$, $\Phi_{\mu}^a$, $\Psi_{\mu}$ and $\Phi_{\mu}$ respectively.

In general, gauge invariant Lagrangian for massive fermionic field contains kinetic and mass terms for all the components as well as a number of cross terms without derivatives. Moreover, it is necessary to introduce such cross terms for the nearest neighbours only, i.e. main gauge field with the primary ones, primary with secondary and so on. Thus we will look for gauge invariant Lagrangian in the form:

$$\mathcal{L} = \frac{1}{2} \{ \mu a \} \{ \nu b \} [\bar{\Psi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Psi_{\beta \gamma} - 10 \bar{\Psi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Psi_{\beta \gamma} + \bar{\Psi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Psi_{\beta \gamma}] + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma} + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma} + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma} + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma} + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma} + \bar{\Phi}_{\mu \nu} \Gamma^{abcde} D_\alpha \Phi_{\beta \gamma}$$

(25)

where all derivatives are AdS covariant ones. In order to compensate the non-invariance of these mass and cross terms under the initial gauge transformations we have to introduce corresponding corrections to gauge transformations. And indeed all variations with one derivative cancel with the following form of gauge transformations:

$$\delta \Psi_{\mu \nu}^a = D_\mu \xi_\nu^a - \frac{i a_0}{5(d - 4)} [\gamma_{[\mu} \xi_{\nu]}^a + \frac{2}{d} \gamma^a \xi_{[\mu} \gamma_{\nu]}] - \frac{b_2}{10(d - 2)} [e_{[\mu}^a \xi_{\nu]} - \frac{1}{d} \gamma^a \gamma_{[\mu} \xi_{\nu]}]$$

$$\delta \Phi_{\mu}^a = D_\mu \xi^a - 2 b_1 \xi_\mu^a + \frac{i a_1}{3(d - 2)} [\gamma_{\mu} \xi^a - \frac{2}{d} \gamma^a \xi_\mu] + \frac{b_3}{6(d - 1)} [e_{\mu}^a \xi^a - \frac{1}{d} \gamma^a \gamma_{\mu} \xi^a]$$
\[ \delta \Psi_{\mu \nu} = D_{[\mu} \xi_{\nu]} + \frac{b_2}{20} \xi_{[\mu,\nu]} + \frac{ia_2}{5(d-4)} \gamma_{[\mu} \xi_{\nu]} - \frac{b_4}{20(d-3)(d-4)} \Gamma_{\mu \nu} \xi \]  

\[ \delta \Phi_{\mu} = D_{\mu} \zeta + \frac{b_3}{6} \zeta_{\mu} + 2b_4 \xi_{\mu} - \frac{ia_3}{3(d-2)} \gamma_{\mu} \zeta \]

where all coefficients are expressed in terms of Lagrangian parameters \( a_{1,2,3,4} \) and \( b_{1,2,3,4} \). Now we calculate all variations without derivatives (including contribution of kinetic terms due to non-commutativity of covariant derivatives) and require their cancellation. This gives us:

\[ a_1 = \frac{3(d-2)}{5(d-4)} a_0, \quad a_2 = -\frac{(d+2)}{d} a_0, \quad a_3 = \frac{3(d^2-4)}{5d(d-4)} a_0 \]

\[ a_0^2 = \frac{5(d-4)}{3(d-3)} b_1^2 - \frac{25}{4} (d-4)^2 \kappa, \]

\[ 20(d+1)(d-4) b_1^2 - 3d(d-3) b_2^2 = -600(d+1)(d-2)(d-3) \kappa \]

\[ b_3^2 = \frac{9(d-1)}{50(d-2)} b_2^2, \quad b_4^2 = \frac{2(d-1)}{(d-2)} b_1^2 \]

Let us analyze the results obtained. First of all recall that there is no strict definition of what is mass in \( (A)dS \) spaces. Working with gauge invariant description of massive particles it is natural to define massless limit as the one where all Goldstone fields decouple from the main gauge one. For the case at hands, such a limit requires that both \( b_1 \to 0 \) and \( b_2 \to 0 \) simultaneously. As the third relation above clearly shows such a limit is possible in flat Minkowski space \( \kappa = 0 \) only. For the non-zero values of cosmological constant we obtain one of the so called partially massless limits (depending on the sign of \( \kappa \)). To clarify subsequent discussion, let us give here a Figure 1 illustrating the roles of cross terms \( b_{1,2,3,4} \). In \( AdS \)

![Diagram](image-url)

Figure 1: General massive theory for \( Y(\frac{2}{3}, \frac{3}{2}) \) spin-tensor space \( (\kappa < 0) \) one can put \( b_2 = 0 \) (and thus \( b_3 = 0 \)). In this, the whole system decomposes into two disconnected subsystems as Figure 2 shows. One of them, with the fields \( \Psi_{\mu \nu}^a, \Phi_{\mu}^a \) and with the Lagrangian:

\[
\mathcal{L} = i \{ \mu \nu \rho \sigma \} \left[ \bar{\Psi}_{\mu \nu} \Gamma^{\alpha \beta \gamma \delta} D_{\alpha} \Psi_{\beta \gamma} + 10 \bar{\Psi}_{\mu \nu} \Gamma^{\alpha \beta \gamma \delta} D_{\alpha} \Psi_{\beta \gamma} \right] + \\
+ i \{ \mu \nu \alpha \beta \} \left[ \bar{\Phi}_{\mu} D_{\alpha} \Phi_{\beta} + 6 \bar{\Phi}_{\mu} \Gamma^{\alpha \beta} D_{\alpha} \Phi_{\beta} \right] + \\
+ a_0 \{ \mu \nu \rho \sigma \} \left[ \bar{\Psi}_{\mu \nu} \Gamma^{\alpha \beta \gamma \delta} \Psi_{\alpha \beta} + 6 \bar{\Psi}_{\mu \nu} \Gamma^{\alpha \beta \gamma \delta} \Psi_{\alpha \beta} \right] + a_1 \{ \mu \nu \} \left[ \bar{\Phi}_{\mu} \Gamma^{\alpha \beta \gamma \delta} \Phi_{\alpha \beta} + 2 \bar{\Phi}_{\mu} \Gamma^{\alpha \beta \gamma \delta} \Phi_{\alpha \beta} \right] + \\
+ b_1 \{ \mu \nu \} \left[ \bar{\Psi}_{\mu \nu} \Gamma^{\gamma \delta} \Phi_{\gamma \delta} + 6 \bar{\Psi}_{\mu \nu} \Gamma^{\gamma \delta} \Phi_{\gamma \delta} \right] + \bar{\Phi}_{\mu} \Gamma^{\gamma \delta} \Psi_{\gamma \delta} + 6 \bar{\Phi}_{\mu} \Gamma^{\gamma \delta} \Psi_{\gamma \delta} \right]
\]

(27)
which is invariant under the following gauge transformations:

\[
\delta \Psi_{\mu\nu}^a = D_{[\mu} \xi_{\nu]}^a - \frac{ia_0}{5(d-4)}[\gamma_{[\mu} \xi_{\nu]}^a + \frac{2}{d} \gamma^a \xi_{[\mu,\nu]}] + \frac{b_1}{15(d-3)(d-4)}[\Gamma_{\mu\nu} \zeta^a - \frac{(d^2 - 7d + 16)}{4(d-2)} \xi_{[\mu,\nu]} + \frac{(d+1)}{4(d-2)} \gamma^a \gamma_{[\mu,\nu]}]
\]

\[
\delta \Phi_{\mu}^a = D_{\mu} \zeta^a - 2b_1 \xi_{\mu}^a + \frac{ia_1}{3(d-2)}[\gamma_{\mu} \zeta^a - \frac{2}{d} \gamma^a \xi_{\mu}]
\]

describes unitary partially massless theory corresponding to irreducible representation of AdS group. At the same time, two other fields \( \Psi_{\mu\nu} \) and \( \Phi_{\mu} \) with the Lagrangian:

\[
\mathcal{L} = -i \left\{ \frac{\mu\nu\alpha\beta\gamma}{abcde} \bar{\Psi}_{\mu\nu} \Gamma_{abcde} D_{\alpha} \Psi_{\beta\gamma} - i \left\{ \frac{\mu\nu\alpha}{abc} \bar{\Phi}_{\mu} \Gamma_{abc} D_{\nu} \Phi_{\alpha} + a_2 \left\{ \frac{\mu\nu\alpha}{abc} \right\} \bar{\Psi}_{\mu\nu} \Gamma_{abc} \Psi_{\alpha\beta} + a_3 \left\{ \frac{\mu\nu}{ab} \right\} \bar{\Phi}_{\mu} \Gamma_{ab} \Phi_{\nu} + ib_2 \left\{ \frac{\mu\nu\alpha}{abc} \right\} \left[ \bar{\Psi}_{\mu\nu} \Gamma_{abc} \Phi_{\alpha} + \bar{\Phi}_{\mu} \Gamma_{abc} \Psi_{\alpha\beta} \right] \right\}
\]

invariant under the following gauge transformations:

\[
\delta \Psi_{\mu\nu} = D_{[\mu} \xi_{\nu]} + \frac{ia_2}{5(d-4)}[\gamma_{[\mu} \xi_{\nu]} - \frac{b_4}{20(d-3)(d-4)} \Gamma_{\mu\nu} \zeta]
\]

\[
\delta \Phi_{\mu} = D_{\mu} \zeta + 2b_4 \xi_{\mu} - \frac{ia_3}{3(d-2)} \gamma_{\mu} \zeta
\]

give gauge invariant description of massive antisymmetric second rank spin-tensor \([42]\).

Let us turn to the \( dS \) space (\( \kappa > 0 \)). First of all, from the equation for the \( a_0^2 \) above we see that there is a unitary forbidden region \( b_1^2 < \frac{15}{4}(d-3)(d-4)\kappa \). Inside this region "lives" one more example of partially massless theory corresponding to the limit \( b_1 \to 0 \) (and hence \( b_4 \to 0 \)) as Figure 3 shows. In this, the main field \( \Psi_{\mu\nu}^a \) together with \( \Psi_{\mu\nu} \) describe this non-unitary partially massless theory with the Lagrangian:

\[
\mathcal{L} = i \left\{ \frac{\mu\nu\alpha\beta\gamma}{abcdef} \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} D_{\gamma} \Psi_{\alpha\beta} - 10 \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} D_{\gamma} \Psi_{\alpha\beta} - \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} D_{\gamma} \Psi_{\alpha\beta} + a_0 \left\{ \frac{\mu\nu\alpha}{abcdef} \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} \Psi_{\alpha\beta} + 6a_0 \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} \Psi_{\alpha\beta} + a_2 \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} \Psi_{\alpha\beta} + ib_2 \left\{ \frac{\mu\nu\alpha}{abcdef} \right\} \left[ \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} \Psi_{\alpha\beta} - \bar{\Psi}_{\mu\nu} \Gamma_{abcdef} \Psi_{\alpha\beta} \right] \right\}
\]

(31)
which is invariant under the following gauge transformations:

\[
\begin{align*}
\delta \Psi_{\mu\nu}^a &= D_{[\mu} \xi_{\nu]}^a - \frac{ia_0}{5(d-4)} [\gamma_{[\mu} \xi_{\nu]}^a + \frac{2}{d} \gamma^a \xi_{[\mu,\nu]}] - \frac{b_2}{10(d-2)} [e_{[\mu}^a \xi_{\nu]} - \frac{1}{d} \gamma^a \gamma_{[\mu} \xi_{\nu]}] \\
\delta \Psi_{\mu\nu} &= D_{[\mu} \xi_{\nu]} + \frac{b_2}{20} \xi_{[\mu,\nu]} + \frac{ia_2}{5(d-4)} \gamma_{[\mu} \xi_{\nu]} 
\end{align*}
\]

At the same time, two other fields \(\Phi_{\mu}^a\), \(\Phi_{\mu}\) provides gauge invariant description for partially massless spin 5/2 particle [26] with the Lagrangian:

\[
\mathcal{L} = i \{_{\mu \nu \alpha bc} \} \left[ \bar{\Phi}_{\mu}^{d \Gamma^{abc}} D_{\nu} \Phi_{\alpha}^d - 6 \bar{\Phi}_{\mu}^{a \gamma^b} D_{\nu} \Phi_{\alpha}^{c} - \bar{\Phi}_{\mu}^{d \Gamma^{abc}} D_{\nu} \Phi_{\alpha}^d \right] + \\
+ \{_{\mu \nu} \} \left[ a_1 \left( \bar{\Phi}_{\mu}^{\Gamma^{abc}} \Phi_{\nu}^{c} + 2 \bar{\Phi}_{\mu}^{a \Phi_{\nu}^{d} b} + a_2 \bar{\Phi}_{\mu}^{d \Phi_{\nu}} \right) \right] + \\
+ib_3 \{_{\mu \nu} \} \left[ \bar{\Phi}_{\mu}^{a \gamma^b} \Phi_{\nu}^c - \bar{\Phi}_{\mu}^{d \gamma^a} \Phi_{\nu}^d \right]
\]

which is invariant under the following gauge transformations:

\[
\begin{align*}
\delta \Phi_{\mu}^a &= D_{\mu} \zeta^a + \frac{ia_1}{3(d-2)} [\gamma_{\mu} \zeta^a - \frac{2}{d} \gamma^a \gamma_{\mu}] \\
\delta \Phi_{\mu} &= D_{\mu} \zeta + \frac{b_3}{6(d-1)} \gamma_{\mu} 
\end{align*}
\]

3.2 \(Y(k + \frac{3}{2}, \frac{3}{2})\)

We proceed with the construction of massive theory for spin-tensor \(\Psi_{\mu\nu}^{(a_k)}\) with arbitrary \(k \geq 1\). Again our first task to determine a set of fields necessary for gauge invariant description of such massive field. Main gauge field \(Y(k + \frac{3}{2}, \frac{1}{2})\) has two gauge transformations with parameters \(Y(k + \frac{3}{2}, \frac{1}{2})\) and \(Y(k + \frac{1}{2}, \frac{3}{2})\) so we need two corresponding primary fields. The first of them has one gauge transformation with parameter \(Y(k + \frac{1}{2}, \frac{1}{2})\), while the second one has two gauge transformations with parameters \(Y(k - \frac{1}{2}, \frac{3}{2})\) and \(Y(k + \frac{1}{2}, \frac{1}{2})\). Taking into account reducibility of gauge transformations of the main gauge field we have to introduce two secondary fields \(Y(k - \frac{1}{2}, \frac{3}{2})\) and \(Y(k + \frac{1}{2}, \frac{1}{2})\) only. It is not hard to check that the procedure again stops at the \(Y(k, \frac{3}{2})\) and we need totally \(Y(l + \frac{3}{2}, \frac{3}{2})\) and \(Y(l + \frac{3}{2}, \frac{1}{2})\) with \(0 \leq l \leq k\). Thus we introduce the following fields: \(\Psi_{\mu\nu}^{(a_l)}\) and \(\Phi_{\mu}^{(a_l)}\), \(0 \leq l \leq k\).

As we have already noted gauge invariant Lagrangian for massive fermionic field contains kinetic and mass terms for all components as well as cross terms without derivatives for all
nearest neighbours. Thus we will look for massive Lagrangian in the form:

$$\mathcal{L} = \sum_{l=0}^{k} [\mathcal{L}(\Psi_{\mu\nu}^{(a_l)}) + \mathcal{L}(\Phi_{\mu}^{(a_l)})] + \sum_{l=0}^{k-1} \mathcal{L}_{\text{cross}}(l)$$

where

$$(-1)^l \mathcal{L}(\Psi_{\mu\nu}^{(a_l)}) = -i \{ \mu^a \beta \gamma \} \left[ \bar{\Psi}_{\mu\nu}^{(a_l)} \Gamma^{abcd} D_\alpha \Psi_{\beta\gamma}(a_l) - 10i \bar{\Psi}_{\mu\nu}^{(a_l-1)} \Gamma^{abcd} D_\alpha \Psi_{\beta\gamma}(a_l) \right] + a_l \{ \mu^a \alpha \beta \} \left[ \bar{\Psi}_{\mu\nu}^{(a_l)} \Gamma^{abcd} \Psi_{\alpha\beta}(a_l) + 6i \bar{\Psi}_{\mu\nu}^{(a_l-1)} \Gamma^{abc} \Psi_{\alpha\beta}(a_l) \right]$$

$$(-1)^l \mathcal{L}(\Phi_{\mu}^{(a_l)}) = -i \{ \mu^a \alpha \} \left[ \bar{\Phi}_{\mu}^{(a_l)} \Gamma^{abc} D_\nu \Phi_{\alpha}(a_l) - 6i \bar{\Phi}_{\mu}^{(a_l-1)} \Gamma^{abc} \Phi_{\alpha}(a_l) \right] + b_l \{ \mu^a \alpha \} \left[ \bar{\Phi}_{\mu}^{(a_l)} \Gamma^{abc} \Phi_{\alpha}(a_l) + 2i \bar{\Phi}_{\mu}^{(a_l-1)} \Phi_{\alpha}(a_l) \right]$$

As usual, to compensate non-invariance of all mass terms (both diagonal as well as cross terms) under the initial gauge transformations, we have to introduce corresponding corrections to gauge transformations. We have already introduced such corrections for diagonal mass terms with coefficients $a_l$ and $b_l$ in Subsection 2.3 and Subsection 2.1 respectively. Let us consider three possible type of cross terms in turn.

$$\Psi_{\mu\nu}^{(a_l+1)} \Leftrightarrow \Psi_{\mu\nu}^{(a_l)}$$. In this case cross terms look as:

$$\Delta \mathcal{L} = -i(-1)^l a_l \{ \mu^a \alpha \beta \gamma \} \left[ \bar{\Psi}_{\mu\nu}^{(a_l)} \Gamma^{abcd} \Psi_{\alpha\beta\gamma}(a_l) - 6i \bar{\Psi}_{\mu\nu}^{(a_l-1)} \Gamma^{abc} \Psi_{\alpha\beta\gamma}(a_l) \right]$$

and to compensate for their non-invariance we have to introduce:

$$\delta' \Psi_{\mu\nu}^{(a_l+1)} = \frac{c_l}{10(l+1)(d+2l-2)} \left[ \epsilon_{\mu\nu}^{(a_l+1)} \xi_{\mu\nu}^{(a_l)} \right] - \frac{1}{(d+2l)} \gamma^{(a_l+1)} \gamma_{\mu\nu}^{(a_l)} + \frac{2}{(d+2l)} \delta^{(a_l+1)} \gamma_{\mu\nu}^{(a_l-1)}$$

$$\delta' \Psi_{\mu\nu}^{(a_l)} = \frac{c_l}{10(l+2)} \delta_{\mu\nu}^{(a_l)}$$

$$\Psi_{\mu\nu}^{(a_l)} \Leftrightarrow \Phi_{\mu}^{(a_l)}$$. Here the cross terms have the following form:

$$\Delta \mathcal{L} = -i(-1)^l a_l \{ \mu^a \alpha \beta \} \left[ \bar{\Psi}_{\mu\nu}^{(a_l)} \Gamma^{abc} \Phi_{\alpha}(a_l) - 6i \bar{\Psi}_{\mu\nu}^{(a_l-1)} \Gamma^{abc} \Phi_{\alpha}(a_l) \right] + h.c.$$
\( \Phi^{(a_{l+1})} \leftrightarrow \Phi^{(a_l)} \). The last possible type of cross terms have the form:

\[
\Delta \mathcal{L} = -i(-1)^l e_l \left\{ \mu \nu \right\} [\Phi^{a(a_l)}_\mu \gamma^b \Phi^{(a_l)}_\nu - \Phi^{(a_l)}_\mu \gamma^a \Phi^{b(a_l)}_\nu]
\]

while corrections to gauge transformations can be written as follows:

\[
\delta' \Phi^{(a_{l+1})}_\mu = \frac{e_l}{6(l + 1)(d + l - 1)} [\epsilon^{(a_1)}_\mu \zeta^{(a_l)}_\mu - \frac{1}{(d + 2l)} \gamma^{(a_1)}_\mu \zeta^{(a_l)}_\mu - \frac{2}{(d + 2l)} g^{(a_1 a_2)}_\mu \zeta^{(a_{l-1})}_\mu]
\]

\[
\delta' \Phi^{(a_l)}_\mu = \frac{e_l}{6(l + 1)} \zeta^{(a_l)}_\mu
\]

Collecting all pieces together we obtain the following complete set of gauge transformations (for simplicity we omit here complicated terms which are necessary to ensure that all variations are \( \gamma \)-transverse):

\[
\delta \Psi^{(a_l)}_{\mu \nu} = D_{[\mu \xi_{\nu}]^{(a_l)}} + \frac{ia_l}{5(d - 4)} [\gamma_{[\mu \xi_{\nu}]^{(a_l)}} + \ldots] + \frac{c_l}{10(l + 2)} [\xi_{\mu \nu}]^{(a_l)} - \frac{c_l - 1}{10l(d + l - 3)} [\epsilon^{(a_1)}_\mu \zeta^{(a_l-1)}_\mu + \ldots] - \frac{(l + 1) d_l}{10(l + 2)(d - 3)(d - 5)} [\Gamma_{\mu \nu}]^{(a_l)} + \ldots
\]

\[
\delta \Phi^{(a_l)}_\mu = D_{\mu \zeta^{(a_l)}} - \frac{ib_l}{3(d - 2)} [\gamma_{\mu \zeta^{(a_l)}} + \ldots] + 2d_l \xi^{(a_l)}_\mu + \frac{e_l}{6(l + 1)} \zeta^{(a_l)}_\mu + \frac{c_l - 1}{6l(d + l - 2)} [\epsilon^{(a_1)}_\mu \zeta^{(a_{l-1})} + \ldots]
\]

At this stage we have complete Lagrangian and gauge transformations, in this all parameters in gauge transformations are expressed in terms of the Lagrangian ones \( a, b, c, d \) and \( e \) so that all variations with one derivative cancel. Our next task — calculate all variations without derivatives (including contribution of kinetic terms due to non-commutativity of covariant derivatives) and require their cancellation. We will not give here these lengthy but straightforward calculations presenting final results only. First of all we obtain a number of recurrent relations on diagonal mass parameters \( a_l \) and \( b_l \) which allows us to express all of them in terms of the main one \( a_k = M \):

\[
a_l = \frac{(d + 2k)}{(d + 2l)} M, \quad b_l = -\frac{3(d - 2)}{5(d - 4)} a_l
\]

Then we obtain recurrent relations on the parameters \( d_l \) which allows us to express all of them in terms of main one \( d_k = m \) (it is not a mass, just notation):

\[
d_l^2 = \frac{(k + 1)(d + k - 2)}{(l + 1)(d + l - 2)} m^2
\]

Further we get the following expressions for the parameters \( c_l \) and \( e_l \):

\[
c_l^2 = \frac{10(k - l)(l + 1)(d + k + l)}{(d + 2l)} \left[ \frac{(k + 1)(d - 4)}{(k + 2)(d - 3)} m^2 + 10(l + 2)(d + l - 2) \right]
\]

\[
e_l^2 = \frac{9(l + 1)(d + l - 1)}{25(l + 2)(d + l - 2)} c_l^2
\]
At last we obtain an important relation on parameters $M$ and $m$:

$$M^2 = \frac{5(k+1)(d-4)}{2(k+2)(d-3)}m^2 - \frac{25}{4}(d-4)^2 \kappa$$

Now we are ready to analyze the results obtained. To clarify the roles played by parameters $c$, $d$ and $e$ we give here Figure 4. First of all note that to obtain massless limit we have

$$M \to 0 \quad \text{and} \quad c_{k-1} \to 0$$

simultaneously. But as the expression on $c_{k-1}$ clearly shows such limit is possible in flat Minkowski space ($\kappa = 0$) only. For non-zero values of $\kappa$ we can obtain a number of partially massless limits. Let us consider $AdS$ space ($\kappa < 0$) first. The most physically interesting limit appears then $c_{k-1} \to 0$ (and hence $c_{k-1} \to 0$). In this the whole system decomposes into two disconnected subsystems as shown on the Figure 5. In

$$(-1)^k \mathcal{L} = -i \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Psi}_\mu (a_k) \Gamma^{abcde} D_\alpha \Psi_\gamma (a_k) - 10k \bar{\Psi}_\mu (a_{k-1}) \Gamma^{abcd} D_\alpha \Psi_\gamma (a_{k-1}) \right] +$$

$$-i \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Phi}_\mu (a_k) \Gamma^{abc} D_\nu \Phi_\alpha (a_k) - 6k \bar{\Phi}_\mu (a_{k-1}) \gamma^b D_\nu \Phi_\alpha (a_{k-1}) \right] +$$

$$+ a_k \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Psi}_\mu (a_k) \Gamma^{abc} \Psi_\alpha (a_k) + 6k \bar{\Psi}_\mu (a_{k-1}) \Gamma^{abcd} \Psi_\alpha (a_{k-1}) \right] +$$

$$+ b_k \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Phi}_\mu (a_k) \Gamma^{abc} \Phi_\alpha (a_k) - 6k \bar{\Phi}_\mu (a_{k-1}) \gamma^b \Phi_\alpha (a_{k-1}) \right] +$$

$$+ d_k \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Psi}_\mu (a_k) \Gamma^{abc} \Psi_\nu (a_k) - 6k \bar{\Psi}_\mu (a_{k-1}) \gamma^b \Psi_\nu (a_{k-1}) \right] +$$

$$+ d_k \left\{ \mu^\alpha \nu^\beta \gamma \right\} \left[ \bar{\Phi}_\mu (a_k) \Gamma^{abc} \Phi_\nu (a_k) - 6k \bar{\Phi}_\mu (a_{k-1}) \gamma^b \Phi_\nu (a_{k-1}) \right]$$

(42)
while gauge transformations leaving it invariant look as follows:

\[
\delta \Psi_{\mu\nu}^{(a_k)} = D_{[\mu} \xi_{\nu]}^{(a_k)} + \frac{ia_k}{5(d-4)} [\gamma_{[\mu} \xi_{\nu]}^{(a_k)} + \ldots] - \frac{(l+1)d_k}{10(k+2)(d-3)(d-5)} [\Gamma_{\mu\nu} \zeta^{(a_k)} + \ldots]
\]

\[
\delta \Phi^{(a_k)} = D_{\mu} \zeta^{(a_k)} - \frac{ib_k}{3(d-2)} [\gamma_{\mu} \zeta^{(a_k)} + \ldots] + 2d_k \xi^{(a_k)}
\]

At the same time all other fields just give massive theory for the \(\Psi_{\mu\nu}^{(a_{k-1})}\) spin-tensor. Besides a number of non-unitary partially massless limits exists. Indeed, each time when one of the \(c_l \to 0\) (and hence \(c_l \to 0\)) the whole system also decomposes into two disconnected subsystems. One of them with the fields \(\Psi_{\mu\nu}^{(a_m)}\) and \(\Phi^{(a_m)}\) with \(l \leq m \leq k\) describes a non-unitary partially massless theory, while remaining fields just give massive theory for the \(\Psi_{\mu\nu}^{(a_{l-1})}\) spin-tensor.

Let us turn to the \(dS\) space \((\kappa > 0)\). From the last relation on parameters \(M\) and \(m\) we see that there is a unitary forbidden region \(m^2 < \frac{5(k+2)(d-3)(d-4)}{2(k+1)} \kappa\). Inside this region lives the only partially massless limit possible. It appears then when we put \(m \to 0\) (and this puts all \(d_l \to 0\) simultaneously). Once again the whole system decomposes into two disconnected parts as shown on the Figure 6.

Figure 6: Non-unitary partially massless limit in \(dS\) space

more example of partially massless theory in \(dS\) space with the Lagrangian

\[
\mathcal{L} = \sum_{l=0}^{k} (-1)^l \left[ -i \{ \mu_{\alpha_{a_b}} \gamma_{c d e f} \} [\Psi_{\mu\nu}^{(a_l)} \Gamma_{a_b} \Psi_{\beta}^{(a_l)} - 10l \Psi_{\mu\nu}^{(a_{l-1})} \Gamma_{a_b} D_{a} \Psi_{\beta}^{(a_{l-1})}] + \right.

+ a_l \left[ \psi_{\mu\nu}^{(a_l)} \Gamma_{a_b} \psi_{\beta}^{(a_l)} + 6l \Psi_{\mu\nu}^{(a_{l-1})} \Gamma_{a_b} \psi_{\beta}^{(a_{l-1})} \right]

+ \left. i \sum_{l=0}^{k-1} (-1)^l c_l \{ \psi_{\mu\nu}^{(a_l)} \Gamma_{a_b} \psi_{\beta}^{(a_l)} - \Psi_{\mu\nu}^{(a_l)} \Gamma_{a_b} \psi_{\beta}^{(a_l)} \right]
\]

which is invariant under the following gauge transformations:

\[
\delta \Psi_{\mu\nu}^{(a_l)} = D_{[\mu} \xi_{\nu]}^{(a_l)} + \frac{ia_l}{5(d-4)} [\gamma_{[\mu} \xi_{\nu]}^{(a_l)} + \ldots] + \frac{c_l}{10(l+2)} [\xi_{[\mu}^{(a_l)} \xi_{\nu]}^{(a_{l-1})} + \ldots]
\]

In this, remaining fields \(\Phi_{\mu}^{(a_l)} \) \(0 \leq l \leq k\) realize partially massless theory constructed earlier [19].
3.3 $Y(\frac{5}{2}, \frac{5}{2})$

As we have already noted, (spin)-tensors corresponding to Young tableau with equal number of boxes in both rows are special and require separate consideration. Here we consider simplest example — massive theory for $R_{\mu\nu}^{ab}$ spin-tensor. First of all we have to find a set of fields necessary for gauge invariant description of such massive spin-tensor. Our main gauge field $Y(\frac{5}{2}, \frac{5}{2})$ has one own gauge transformation with the parameter $Y(\frac{5}{2}, \frac{3}{2})$ only (and this is a main feature making this (spin)-tensors special). Thus we need one primary Goldstone field $Y(\frac{5}{2}, \frac{3}{2})$ only. This field has two gauge transformations with parameters $Y(\frac{5}{2}, \frac{1}{2})$ and $Y(\frac{3}{2}, \frac{3}{2})$ but due to reducibility of our main field gauge transformations we need one secondary field $Y(\frac{5}{2}, \frac{1}{2})$ only. This field also has one own gauge transformation with the parameter $Y(\frac{3}{2}, \frac{1}{2})$ but due to reducibility of primary field gauge transformations the procedure stops here. Thus we need three fields $R_{\mu\nu}^{ab}$, $\Psi_{\mu\nu}^a$ and $\Phi_{\mu}^a$ only.

As in all previous cases, we will construct a massive gauge invariant Lagrangian as the sum of kinetic and mass terms for all three fields as well as cross terms without derivatives:

$$
\mathcal{L} = -i \left\{ \muab \right\} \left[ \tilde{R}_{\mu\nu} \Gamma^{ab} D_\alpha R_{\beta\gamma} - 20 \tilde{R}_{\mu\nu} a b c d e f - 60 \tilde{R}_{\mu\nu} ab c d e f \right] + \\
+ i \left\{ \muab \right\} \left[ \tilde{\Psi}_{\mu\nu} \Gamma^{ab} D_\alpha \Psi_{\beta\gamma} - 10 \tilde{\Psi}_{\mu\nu} a b c d e f \right] + \\
+ i \left\{ \muab \right\} \left[ \tilde{\Phi}_{\mu} \Gamma^{ab} D_\alpha \Phi_{\beta\gamma} - 6 \tilde{\Phi}_{\mu} a b c d e f \right] + \\
+ a_0 \left\{ \muab \right\} \left[ R_{\mu\nu} \Gamma^{ab} R_{\alpha\beta} - 12 \tilde{R}_{\mu\nu} a b c d e f \right] + \\
+ a_1 \left\{ \muab \right\} \left[ \tilde{\Psi}_{\mu\nu} \Gamma^{ab} \Psi_{\alpha\beta} - 6 \tilde{\Psi}_{\mu\nu} a b c d e f \right] + \\
+ b_1 \left\{ \muab \right\} \left[ \tilde{\Phi}_{\mu} \Gamma^{ab} \Phi_{\alpha\beta} - 2 \tilde{\Phi}_{\mu} a b c d e f \right] + \\
+ b_2 \left\{ \muab \right\} \left[ \tilde{\Phi}_{\mu} \Gamma^{ab} \Phi_{\alpha\beta} - 6 \tilde{\Phi}_{\mu} a b c d e f \right]
$$

(46)

Now following our usual strategy we calculate variations with one derivative to find appropriate corrections to gauge transformations. Really most of them we are already familiar with, the only new ones are related with the cross terms $R_{\mu\nu}^{ab} \Leftrightarrow \Psi_{\mu\nu}^a$. Calculating these new corrections and collecting previously known results we obtain:

$$
\delta R_{\mu\nu}^{ab} = D_{[\mu} \xi_{\nu]}^{ab} + \frac{i a_0}{5(d-4)} \left[ \gamma_{[\mu} \xi_{\nu]}^{ab} + \frac{2}{(d-2)} \gamma^{[a} \xi_{\mu \nu] b} \right] + \\
\quad + \frac{b_1}{20(d-3)} \left[ e_{[\mu}^{a} \xi_{\nu]}^{b} - \frac{1}{(d-2)} \gamma^{[a} \xi_{\mu \nu] b} - \frac{2}{(d-1)(d-2)} \Gamma^{ab} \xi_{\mu \nu]} \right]
$$

$$
\delta \Psi_{\mu}^a = D_{[\mu} \xi^{a} - \frac{b_1}{10} \xi_{[\mu}^{a} = - \frac{i a_1}{5(d-4)} \left[ \gamma_{[\mu} \xi_{\nu]}^{a} + \frac{2}{d} \gamma^{a} \xi_{\mu \nu]} \right] + \\
\quad + \frac{b_2}{15(d-3)(d-4)} \left[ \Gamma_{\mu\nu} \xi^{a} - \frac{(d^2 - 7d + 16)}{4(d-2)} e_{[\mu}^{a} \xi_{\nu]} + \frac{(d+1)}{4(d-2)} \gamma^{a} \gamma_{[\mu} \xi_{\nu]} \right]
$$

$$
\delta \Phi_{\mu}^a = D_{\mu} \xi^{a} - 2 b_2 \xi_{\mu}^{a} + \frac{a_2}{3(d-2)} \left[ \gamma_{\mu}^{a} - \frac{2}{d} \gamma^{a} \xi_{\mu} \right]
$$

(47)

Now we proceed with the variations without derivatives (including contribution of kinetic terms due to non-commutativity of covariant derivatives). First of all, their cancellation
leads to the relations on the diagonal mass terms:

\[
a_1 = -\frac{d}{(d-2)}a_0, \quad a_2 = -\frac{3(d-2)}{5(d-4)}a_1 = \frac{3d}{5(d-4)}a_0
\]

Also we obtain two important relations:

\[
a_0^2 = \frac{(d-2)}{8(d-1)}b_1^2 - \frac{25(d-2)^2}{4}\kappa
\]

\[
b_2^2 = \frac{3(d-2)}{20(d-1)(d-4)}[(d-2)b_1^2 - 200(d-1)(d-3)\kappa]
\]

Simple linear structure of this theory \( R_{\mu\nu}^{ab} \leftrightarrow \Psi_{\mu\nu}^{a} \leftrightarrow \Phi_{\mu}^{a} \) makes an analysis also simple. First of all we see that massless limit (i.e. decoupling of \( \Psi_{\mu\nu}^{a} \) in complete agreement with the fact that massless theory for \( R_{\mu\nu}^{ab} \) admits deformation into \( AdS \) space without introduction of any additional fields. In this, two other fields \( \Psi_{\mu\nu}^{a} \) and \( \Phi_{\mu}^{a} \) describe partially massless theory we already familiar with. In the \( dS \) space we once again face an unitary forbidden region \( b_1^2 < \frac{25}{2}(d-1)(d-2)\kappa \). Inside this region we find one more example of non-unitary partially massless theory. It appears then \( b_2 \rightarrow 0 \), in this the field \( \Phi_{\mu}^{a} \) decouples, while two other fields \( R_{\mu\nu}^{ab} \) and \( \Psi_{\mu\nu}^{a} \) describe partially massless theory. The Lagrangian and gauge transformations for this theory can be easily obtained from the general formulas simply omitting the field \( \Phi_{\mu}^{a} \) and all terms in the gauge transformations containing \( \zeta^{a} \).

3.4 \( Y(k + \frac{3}{2}, k + \frac{3}{2}) \)

Let us consider now general case — spin-tensor \( Y(k + \frac{3}{2}, k + \frac{3}{2}) \) with arbitrary \( k \geq 1 \). Again it is crucial that the main field \( R_{\mu\nu}^{(ak), (bk)} \) has one gauge transformation with parameter \( Y(k + \frac{3}{2}, k + \frac{1}{2}) \) only so we need one primary field. This field has two gauge transformations with parameters \( Y(k + \frac{3}{2}, k - \frac{1}{2}) \) and \( Y(k + \frac{1}{2}, k + \frac{1}{2}) \) but due to reducibility of gauge transformations of main field we need one secondary field \( Y(k + \frac{3}{2}, k + \frac{1}{2}) \) only. It is not hard to check that complete set of fields necessary for gauge invariant description contains \( Y(k + \frac{3}{2}, l + \frac{3}{2}) \) \( 0 \leq l \leq k \) and \( Y(k + \frac{3}{2}, \frac{1}{2}) \).

Following our general procedure we will look for massive gauge invariant Lagrangian as the sum of kinetic and mass terms for all fields as well as cross terms for nearest neighbours:

\[
\mathcal{L} = \mathcal{L}(R_{\mu\nu}^{(ak) (bk)}) + \sum_{l=0}^{k-1} \mathcal{L}((\Psi_{\mu\nu}^{(ak) (bl)}) + \mathcal{L}(\Phi_{\mu}^{(ak)}) + \mathcal{L}_{cross} \quad (48)
\]

where Lagrangian \( \mathcal{L}(R_{\mu\nu}^{(ak) (bk)}) \) is given by formulas (15) and (19) of Subsection 2.5, while Lagrangian \( \mathcal{L}(\Psi_{\mu\nu}^{(ak) (bl)}) \) is given by formulas (22) and (23) of Subsection 2.6. Here

\[
\mathcal{L}_{cross} = id_{k,k} \{ \frac{\mu\nu\alpha\beta}{abcd} \} [ R_{\mu\nu}^{(ak), (a(bk-1))} \Gamma^{bcd} \Psi_{\alpha\beta}^{(ak), (b(k-1))} + 6k R_{\mu\nu}^{a(ak-1), b(bk-1)} \gamma^{c} \Psi_{\alpha\beta}^{d(ak-1), (bk-1)} - \bar{\Psi}_{\mu\nu}^{(ak), (ak-1)} \Gamma^{abc} R_{\alpha\beta}^{a(ak), (bk-1)} + 6k \bar{\Psi}_{\mu\nu}^{a(ak-1), (bk-1)} \gamma^{b} R_{\alpha\beta}^{c(ak-1), (dk-1)} ] +
\]
and to compensate for their non-invariance we have to introduce the following corrections:

\[ \delta R_{\mu\nu}^{(a_k),(b_k)} = \Psi_{\mu\nu}^{(a_k),(b_k-1)} + \delta' R_{\mu\nu}^{(a_k),(b_k)} \]

As usual, to compensate for non-invariance of cross terms under the initial gauge transformations, we have to introduce corresponding corrections to gauge transformations. Let us consider different cross terms in turn.

\[ R_{\mu\nu}^{(a_k),(b_k)} \leftrightarrow \Psi_{\mu\nu}^{(a_k),(b_k-1)} \]

In this case cross terms look like:

\[ \Delta L = i d_{k,l} \{ \frac{\mu^\alpha\beta}{\omega} \} [\bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{\lambda\rho\sigma\tau} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + 6k \bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{abc} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + +6k \bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{abc} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + +6k \delta' R_{\mu\nu}^{(a_k),(b_k)} \}

and to compensate for their non-invariance we have to introduce:

\[ \delta' R_{\mu\nu}^{(a_k),(b_k)} = - \frac{d_{k,l}}{20k(d + k - 4)} [\xi_{(a_k),(b_k-1)}^{(a_k-1),(b_k-1)} + \ldots] \]

where again dots stand for the additional terms which are necessary for variations to be \( \gamma \)-transverse.

\[ \Psi_{\mu\nu}^{(a_k),(b_k)} \leftrightarrow \Phi_{\mu\nu}^{(a_k),(b_k-1)} \]

Corresponding cross terms have the following form:

\[ \Delta L = i d_{k,l} \{ \frac{\mu^\alpha\beta}{\omega} \} [\bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{\lambda\rho\sigma\tau} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + 6k \bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{abc} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + +6k \bar{\Psi}_{\mu\nu}^{(a_k),(b_k-1)} \Gamma^{abc} \Psi_{\lambda\rho\sigma\tau}^{(a_k),(b_k-1)} + +6k \delta' R_{\mu\nu}^{(a_k),(b_k)} \}

and to compensate for their non-invariance we have to introduce the following corrections:

\[ \delta' \Phi_{\mu\nu}^{(a_k),(b_k)} = - \frac{d_{k,l}}{10k} \{ \frac{\mu^\alpha\beta}{\omega} \} [\Phi_{\mu\nu}^{(a_k),(b_k-1)} (b_k-1) + \ldots] \]

We have already considered in Subsection 3.2, so we will not repeat corresponding formulas here.

Collecting all pieces together we obtain the following complete set of gauge transformations:

\[ \delta R_{\mu\nu}^{(a_k),(b_k)} = D_{[\mu} \xi_{\nu]}^{(a_k),(b_k)} + \frac{i d_{k,l}}{5(d-4)} [\gamma_{[\mu} \xi_{\nu]}^{(a_k),(b_k)} + \ldots] \]
express all of them in terms of the main one

\[ a \]

where we introduced a notation:

all we obtain a number of recurrent relations on diagonal mass terms

we omit these lengthy but straightforward calculations and give final results only. First of
to non-commutativity of covariant derivatives) and require their cancellation. Once again
to calculate all variations without derivatives (including contributions of kinetic terms due
to mass-

Let us analyze the results obtained. We have already seen in Subsection 2.5 that mass-

At last we obtain an important relation on parameters

\[ M \]

At this point we have complete Lagrangian as well as complete set of gauge transforma-
tions, in this all parameters in gauge transformations are expressed in terms of Lagrangian
parameters

At this point we have complete Lagrangian as well as complete set of gauge transforma-
tions, in this all parameters in gauge transformations are expressed in terms of Lagrangian
parameters

and

which allow us to express all of them in terms of the main one \( a_{k,k} = M \):

Then we obtain recurrent relations on parameters \( d_{k,l} \) which allow us to express all of them in terms of the main one:

\[ d_{k,l}^2 = \frac{l(k-l+2)(d+k+l-3)}{(d+2l-2)} \frac{[m^2 - 100(k-l)(d+k+l-4)\kappa]}{(d+2l-4)}, \quad 1 \leq l \leq k \]

\[ d_{k,0}^2 = \frac{(k+2)(d+k-3)}{10(d-4)} \frac{[m^2 - 100k(d+k-4)\kappa]}{(d-4)} \]

where we introduced a notation:

\[ m^2 = \frac{(d+2k-4)}{2k(d+2k-3)} d_{k,k}^2 \]

At last we obtain an important relation on parameters \( M \) and \( m \):

\[ 4M^2 = m^2 - 25(d+2k-4)^2\kappa \]

Let us analyze the results obtained. We have already seen in Subsection 2.5 that mass-
less spin-tensor \( R_{\mu\nu}^{(a_k), (b_k)} \) admits deformation into \( AdS \) space without introduction of any

\[ \delta \Psi_{\mu\nu}^{(a_k), (b_k)} = D_{[\mu} \xi_{\nu]}^{(a_k), (b_k)} + \frac{ia_{k,l}}{5(d-4)} [\gamma_{\mu} \xi^{(a_k), (b_k)}(b_k) + \ldots] - \frac{d_{k,l}}{10l(k-l+2)(d+l-4)} [(k-l+1) \xi^{(a_k), (b_k)}(b_k) + \ldots] - \frac{d_{k,l+1}}{10(l+1)} \xi^{(a_k), (b_k)}(b_k), \quad 1 \leq l \leq k-1 \]

\[ \delta \Psi_{\mu\nu}^{(a_k)} = D_{[\mu} \xi_{\nu]}^{(a_k)} + \frac{ia_{k,0}}{5(d-4)} [\gamma_{\mu} \xi^{(a_k)}(b_k) + \ldots] - \frac{d_{k,1}}{10(k+2)(d-3)(d-4)} [\Gamma_{\mu\nu} \xi^{(a_k)} + \ldots] \]

\[ \delta \Phi_{\mu}^{(a_k)} = D_{\mu} \xi^{(a_k)} - \frac{ib_k}{3(d-2)} [\gamma_{\mu} \xi^{(a_k)} + \ldots] + 2d_{k,0} \xi^{(a_k)} \]
additional fields. And indeed, as the last relation clearly shows, in AdS space ($\kappa < 0$) nothing prevent us from considering a limit $m \to 0$ when all Goldstone fields decouple from the main one. From the other hand, in the dS space we again obtain unitary forbidden region $m^2 < 25(d + 2k - 4)\kappa$. At the boundary of this region all diagonal mass terms become equal to zero so that the theory greatly simplifies (though the number of physical degrees of freedom remains to be the same). Inside forbidden region we find a number of (non-unitary) partially massless theories. They appear each time when one of the parameters $d_k, l \to 0$. In this, the whole system decomposes into two disconnected sub systems containing the fields $R_{\mu\nu}^{(a_k),(b_l)}$, $\Psi_{\mu\nu}^{(a_k),(b_n)}$ $l \leq n \leq k - 1$ and $\Psi_{\mu\nu}^{(a_k),(b_n)}$ $0 \leq n \leq l - 1$, $\Phi^{(a_k)}$, correspondingly.

3.5 $Y(k + \frac{3}{2}, l + \frac{3}{2})$

Now we are ready to consider general case — massive spin-tensor $Y(k + \frac{3}{2}, l + \frac{3}{2})$ with $k > l \geq 1$. Our usual procedure (consider gauge transformations for all fields and take into account their reducibility) leads to the following set of fields which are necessary for gauge invariant description: $Y(m + \frac{3}{2}, n + \frac{3}{2})$ and $Y(m + \frac{3}{2}, \frac{1}{2})$ where $l \leq m \leq k$ and $0 \leq n \leq l$. These fields as well as parameters determining appropriate cross terms (see below) are shown on Figure 7.

![Figure 7: General massive $Y(k + \frac{3}{2}, l + \frac{3}{2})$ theory](image)

As in all previous cases, the total Lagrangian contains kinetic and diagonal mass terms for all fields as well as cross terms without derivatives:

$$\mathcal{L} = \sum_{m-l}^{k} \sum_{n=0}^{l} \mathcal{L}(\Psi_{\mu\nu}^{(a_m),(b_n)}) + \sum_{m=l}^{k} \mathcal{L}(\Phi^{(a_m)}) + \mathcal{L}_{cross}$$  (53)
Recall that cross terms appear for nearest neighbours only, i.e., main gauge field with primary fields, primary with secondary ones and so on. Thus, general $\Psi_{\mu\nu}^{(a_m),(b_n)}$ field has cross terms with four other fields as shown on Figure 8. In the previous Subsection we have already considered cross terms for the pair $\Psi_{\mu\nu}^{(a_m),(b_n)} \leftrightarrow \Psi_{\mu\nu}^{(a_m),(b_{n-1})}$, thus the only new terms we need are cross terms for the pair $\Psi_{\mu\nu}^{(a_m),(b_n)} \leftrightarrow \Psi_{\mu\nu}^{(a_m-1),(b_n)}$. They look as follows:

$$ (-1)^{m+n} \Delta \mathcal{L} = i c_{m,n} \left\{ \frac{\mu \nu \alpha \beta}{abcd} \right\} \left[ \bar{\Psi}_{\mu\nu} a_{(a_m-1),(b_n)} \Gamma^{bcd} \Psi_{\alpha\beta}^{(a_m-1),(b_n)} + \right. $$

$$ - 6n \bar{\Psi}_{\mu\nu} a_{(a_m-1),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m-1),(d_{n-1})} - $$

$$ - \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_n)} \Gamma^{abc} \Psi_{\alpha\beta}^{(a_m-1),(b_n)} + $$

$$ - 6n \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_{n-1})} a_{(b_{n-1})} b \Psi_{\alpha\beta}^{c(a_m-1),(d_{n-1})} ] $$

In this, to compensate for their non-invariance we have to introduce the following corrections to gauge transformations:

$$ \delta' \Psi_{\mu\nu}^{(a_m),(b_n)} = \frac{c_{m,n}}{10m(d + m - 3)} \left[ e_{(a_1} \xi_{\nu) a_{(m-1),(b_n)}} \right] + \ldots $$

$$ \delta' \Psi_{\mu\nu}^{(a_{m-1}),(b_n)} = - \frac{c_{m,n}}{10(m - n + 1)(m + 1)} \left[ (m - n + 1) \xi_{(\mu \nu) \xi (a_{m-1}),(b_n)} + \xi_{(a_{m-1})(b_1,b_{n-1})} \right] $$

Note also that the last two rows on Figure 7 are to those on the Figure 4 in Subsection 3.2, so all necessary terms have already been considered there.

Collecting all pieces together we obtain the following complete set of cross terms:

$$ \mathcal{L}_{\text{cross}} = i \sum_{m=1}^{k} \sum_{n=1}^{l} (-1)^{m+n} \left\{ \frac{\mu \nu \alpha \beta}{abcd} \right\} \left[ c_{m,n} \left[ \bar{\Psi}_{\mu\nu} a_{(a_m-1),(b_n)} \Gamma^{bcd} \Psi_{\alpha\beta}^{(a_m-1),(b_n)} + \right. $$

$$ - 6n \bar{\Psi}_{\mu\nu} a_{(a_m-1),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m-1),(d_{n-1})} - $$

$$ - \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_n)} \Gamma^{abc} \Psi_{\alpha\beta}^{(a_m-1),(b_n)} + $$

$$ - 6n \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_{n-1})} a_{(b_{n-1})} b \Psi_{\alpha\beta}^{c(a_m-1),(d_{n-1})} ] + $$

$$ d_{m,n} \left[ \bar{\Psi}_{\mu\nu} a_{(a_m),(b_{n-1})} \Gamma^{bcd} \Psi_{\alpha\beta}^{(a_m),(b_{n-1})} + $$

$$ + 6m \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m-1),(d_{n-1})} - $$

$$ + 6m \bar{\Psi}_{\mu\nu} a_{(a_m),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m),(d_{n-1})} ] $$

$$ + 6m \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_{n-1})} a_{(b_{n-1})} b \Psi_{\alpha\beta}^{c(a_m-1),(d_{n-1})} $$

$$ + 6m \bar{\Psi}_{\mu\nu}^{(a_m-1),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m-1),(d_{n-1})} ] $$

$$ + 6m \bar{\Psi}_{\mu\nu} a_{(a_m),(b_{n-1})} c \Psi_{\alpha\beta}^{(a_m),(d_{n-1})} $$
\[ -\bar{\Psi}_{\alpha\beta}^{(a_m, (b_{n-1})} \Gamma^{abc} \Psi_{\alpha\beta}^{(a_m, d(b_{n-1})} + 6m \bar{\Psi}_{\alpha\beta}^{(a_m, (b_{n-1})} \gamma^b \Psi_{\alpha\beta}^{(a_m, d(b_{n-1})} ] \]
\[ + i \sum_{m=1}^k (-1)^m \left\{ \frac{\mu \nu}{abcd} \right\} \{ c_{m,0} \bar{\Psi}_{\alpha\beta}^{(a_m)} \Gamma^{bcd} \Psi_{\alpha\beta}^{(a_m)} - \bar{\Psi}_{\alpha\beta}^{(a_m)} \Gamma^{abcd} \Psi_{\alpha\beta}^{(a_m)} \} + 
\[ + d_{m,0} \bar{\Psi}_{\alpha\beta}^{(a_m)} \Gamma^{abc} \Phi_{\alpha\beta}^{(a_m)} - 6m \bar{\Psi}_{\alpha\beta}^{(a_m)} \gamma^b \Phi_{\alpha\beta}^{(a_m)} + 
\[ + \bar{\Psi}_{\alpha\beta}^{(a_m)} \Gamma^{abc} \Phi_{\alpha\beta}^{(a_m)} - 6m \bar{\Psi}_{\alpha\beta}^{(a_m)} \gamma^b \Phi_{\alpha\beta}^{(a_m)} + 
\[ + e_m \bar{\Phi}_{\mu}^{(a_m)} \gamma^b \Phi_{\nu}^{(a_m)} - \bar{\Phi}_{\mu}^{(a_m)} \gamma^b \Phi_{\nu}^{(a_m)} \} \] \tag{55}

Similarly, combining results of this and previous Subsections, we obtain complete set of gauge transformations for all fields involved:

\[ \delta \Psi_{\mu\nu}^{(a_m, (b_n)} = D_{[\mu \xi_{\nu}] (a_m, (b_n)} + \frac{i a_{m,n}}{5(d-4)} [\gamma_{[\mu \xi_{\nu}]}^{(a_m, (b_n)} + \ldots ] - 
\[ - \frac{c_{m+1,n}}{10(m - n + 2)(m + 2)} (m - n + 2) \xi_{[\mu, \nu]}^{(a_m, (b_n)} + \xi_{[\mu, (b_1, b_{n-1}) \nu]}^{(a_m, (b_n)} - 
\[ - \frac{c_{m,n}}{10m(d + m - 3)} e_{[\mu (a_1 \xi_{\nu})^{a_{m-1}, (b_n)} + \ldots ] - \frac{d_{m,n+1}}{10(n + 1)} e_{[\mu (a_1 \xi_{\nu})^{a_{m-1}, (b_n)} + \ldots ] - e_{[\mu (a_1 \xi_{\nu})^{a_{m-1}, (b_1, b_{n-1}) \nu]} + \ldots ]}, \quad 1 \leq n \leq l 
\]

\[ \delta \Psi_{\mu}^{(a_m)} = D_{[\mu \xi_{\nu}]}^{(a_m)} + \frac{i a_{m,0}}{5(d-4)} [\gamma_{[\mu \xi_{\nu}]}^{(a_m)} + \ldots ] + \frac{c_{m,0}}{10(m + 2)} \xi_{[\mu, \nu]}^{(a_m)} - 
\[ - \frac{c_{m-1,0}}{10_l (d + m - 3)} e_{[\mu (a_1 \xi_{\nu})^{a_{m-1}} + \ldots ] - 
\[ - \frac{(m + 1) d_{m,0}}{10(m + 2)(d - 3)(d - 5)} [\Gamma_{\mu\nu}^{(a_m)} + \ldots ] 
\]

\[ \delta \Phi_{\mu}^{(a_m)} = D_{[\mu \xi_{\nu}]}^{(a_m)} - \frac{i b_{m}}{3(d-2)} [\gamma_{[\mu \xi_{\nu}]}^{(a_m)} + \ldots ] + 2d_{m,0} \xi_{[\mu, \nu]}^{(a_m)} + \frac{e_{m}}{6(m + 1)} \xi_{[\mu, \nu]}^{(a_m)} + 
\[ + \frac{e_{m-1}}{6m(d + m - 2)} e_{[\mu (a_1 \xi_{\nu})^{a_{m-1}} + \ldots ]} 
\]

Having in our disposal total Lagrangian and complete set of gauge transformations where all variations with one derivative cancel, we proceed with variations without derivatives. After lengthy but straightforward calculations we obtain the following results.

First of all we obtain a number of relations on diagonal mass terms \(a_{m,n}\) and \(b_m\) which allow us to express all them in terms of main one \(a_{k,l} = M\):

\[ a_{m,n} = \frac{(d + 2k)(d + 2l - 2)}{(d + 2m)(d + 2n - 2)} M \]

Similarly, we get a number of relations on the parameters \(d_{m,n}, c_{m,n}\) and \(e_m\) (determining cross terms) so that all of them can be expressed in terms of one main parameter. We choose \(d_{k,l}\) as such main parameter and introduce a notation:

\[ m^2 = \frac{(k - l + 1)(d + 2l - 4)}{l(k - l + 2)(d + 2l - 3)} d_{k,l}^2 \]
Then we obtain the following important expressions for the parameters $d_{k,n}$ corresponding to leftmost column on Figure 7:

$$d_{k,n}^2 = \frac{n(l - n + 1)(k - n + 2)(d + l + n - 3)}{(k - n + 1)(d + 2n - 4)}[m^2 - 100(l - n)(d + l + n - 4)\kappa], \quad n \geq 1$$

$$d_{k,0}^2 = \frac{(k + 2)(l + 1)(d + l - 3)}{10(k + 1)(d - 4)}[m^2 - 100(d + l - 4)\kappa]$$

as well as for parameters $c_{m,l}$ corresponding to topmost row on Figure 7:

$$c_{m,l}^2 = \frac{m(k - m + 1)(d + k + m - 1)}{(d + 2m - 2)}[m^2 + 100(m - l + 1)(d + m + l - 3)\kappa]$$

It is very important (and this gives a nice check for all calculations) that all parameters $d_{m,n}$ corresponding to the same row on Figure 7 turn out to be proportional to the leftmost one $d_{k,n}$:

$$d_{m,n}^2 = \frac{(k - n + 1)(d + k + n - 2)}{(m - n + 1)(d + m + n - 2)}d_{k,n}^2$$

Similarly, all parameters $c_{m,n}$ and $e_{m}$ corresponding to the same column turn out to be proportional to the topmost one $c_{m,l}$:

$$c_{m,n}^2 = \frac{(m - l)(d + m + l - 2)}{(m - n)(d + m + n - 2)}c_{m,l}^2$$

$$e_{m}^2 = \frac{9(m - l)(d + m + l - 2)}{25(m + 1)(d + m - 3)}c_{m,l}^2$$

At last but not least, we obtain an important relation on two main parameters $M$ and $m$:

$$4M^2 = m^2 - 25(d + 2l - 4)^2\kappa$$

We have already mentioned in Subsection 2.6 that massless spin-tensor $\Psi_{\mu\nu}^{(a_k),(b_l)}$ does not admit deformation into AdS space without introduction of additional fields. In the gauge invariant formulation for massive spin-tensor such a limit would require that both $d_{k,l} \to 0$ and $c_{k,l} \to 0$ simultaneously and such possibility exists in flat Minkowski space ($\kappa = 0$) only. For non-zero values of cosmological constant we obtain a number of partially massless limits instead.

Let us consider AdS space ($\kappa < 0$) first. The most physically interesting limit arises when $m^2 = -100(k - l + 1)(d + k + l - 3)\kappa$. In this, parameter $c_{k,l}$ (and hence all parameters $c_{k,n}$ and $c_{k}$) becomes equals to zero and fields $\Psi_{\mu\nu}^{(a_k),(b_n)}$ $0 \leq n \leq l$ and $\Phi_{\mu}^{(a_k)}$ (corresponding to leftmost column in Figure 7) decouple and describe (the only) unitary partially massless theory. The Lagrangian for this theory has the form:

$$\mathcal{L} = \sum_{n=0}^{l} \mathcal{L}(\Psi_{\mu\nu}^{(a_k),(b_n)}) + \mathcal{L}(\Phi_{\mu}^{(a_k)}) + \mathcal{L}_{\text{cross}}$$

(57)
\[ \mathcal{L}_{\text{cross}} = i \sum_{n=1}^{l} (-1)^{k+n} d_{k,n} \left\{ \frac{\mu\nu\alpha\beta}{abcd} \right\} \left[ \bar{\Psi}_{\mu\nu}^{(a_k)}(a(b_{n-1})) \Gamma^{abcd} \Psi_{\alpha\beta}^{(a_k),(b_{n-1})} + 6k \bar{\Psi}_{\mu\nu}^{(a_k-1)}(b_{n-1}) \gamma^c \Psi_{\alpha\beta}^{(a_k),(b_{n-1})} - \bar{\Psi}_{\mu\nu}^{(a_k),(b_{n-1})} \Gamma^{abc} \Psi_{\alpha\beta}^{(a_k),(b_{n-1})} + 6k \bar{\Psi}_{\mu\nu}^{(a_k-1)}(b_{n-1}) \gamma^b \Psi_{\alpha\beta}^{(a_k-1),(b_{n-1})} \right] \]

and is invariant under the following gauge transformations:

\[
\delta \Psi_{\mu\nu}^{(a_k),(b_n)} = D_{[\mu} \xi_{\nu]}^{(a_k),(b_n)} + \frac{i a_{k,n}}{5(d-4)} [\gamma [\mu \xi_{\nu}]^{(a_k),(b_n)} + ...] - \frac{d_{k,n+1}}{10(n+1)} \xi_{[\mu}^{(a_k),(b_{n-1})})_{\nu]} - \frac{d_{k,n}}{10n(k-n+2)(d+n-4)} [(k-n+1) \xi_{[\mu}^{(a_k),(b_{n-1})} e_{\nu]}^{b_1} - e_{[\nu}^{(a_1 \xi_{\mu}]^{(a_k-1),(b_1,b_{n-1})} + ...}], \quad 1 \leq n \leq l
\]

\[
\delta \Psi_{\mu\nu}^{(a_k)} = D_{[\mu} \xi_{\nu]}^{(a_k)} + \frac{i a_{k,0}}{5(d-4)} [\gamma [\mu \xi_{\nu}]^{(a_k)} + ...] - \frac{(k+1)d_{k,0}}{10(k+2)(d-3)(d-5)} [\Gamma_{\mu\nu} \zeta^{(a_k)} + ...]
\]

\[
\delta \Phi_{\mu}^{(a_k)} = D_{\mu} \zeta^{(a_k)} - \frac{i b_k}{3(d-2)} [\gamma_{\mu} \zeta^{(a_k)} + ...] + 2d_{k,0} \xi_{\mu}^{(a_k)}
\]

All other fields just give massive theory for the spin-tensor \( \Psi_{\mu\nu}^{(a_{k-1}),(b_l)} \). Besides, a number of non-unitary partially massless limits exist. It happens each time then one of the \( c_{m,l} \) (and hence all \( c_{m,n} \) with \( 0 \leq n \leq l \) and \( c_m \)) goes to zero. In this, the whole system decomposes into two disconnected subsystems (and diagram on Figure 7 splits horizontally into two blocks as shown on Figure 9). The left block describes a non-unitary partially massless theory, while

\[ \begin{array}{ccc}
\Psi_{k,l} & \cdots & \Psi_{m,l} \\
\Phi_k & \cdots & \Phi_m \\
\Psi_{m-1,l} & \cdots & R_{l,l} \\
\Phi_{m-1} & \cdots & \Phi_l \\
\end{array} \]

\text{Figure 9: Example of non-unitary partially massless limit in AdS}

the right one gives massive theory for spin-tensor \( \Psi_{\mu\nu}^{(a_{m-1}),(b_l)} \). Recall that our definition
of masslessness is bounded to flat Minkowski space. From the anti de Sitter group point of view each vertical column on Figure 7 corresponds to unitary irreducible representation which can be called massless \[25\]. In this, all other representations (massive or partially massless) can be constructed out of appropriate set of massless ones as it should be.

Let us turn to the $dS$ space ($\kappa > 0$). Here we once again face an unitary forbidden region $m^2 < 25(d + 2l - 4)^2\kappa$ (which follows from the relation between $M$ and $m$). Inside this forbidden region we obtain a number of partially massless limits (but all of them lead to the non-unitary theories). The first one arises then parameter $d_{k,l}$ (and hence all parameters $d_{k,n}$) becomes zero. In this, the fields $\Psi_{\mu\nu}^{(a_m),(b_l)}$ with $l \leq m \leq k$ (corresponding to upper row on Figure 7, see Figure 10) decouple and describe partially massless theory which corresponds to irreducible representation of the de Sitter group (and from the de Sitter group point of view can be called massless). Contrary to what we have seen in $AdS$ case, all other fields also gives partially massless theory. The reason is that to describe complete massive theory for spin-tensor $\Psi_{\mu\nu}^{(a_k),(b_{l-1})}$ we need one more column of fields as also shown on Figure 10.

Similarly, partially massless limits happens each time when one of the parameters $d_{k,n}$ (and hence all parameters $d_{m,n}$ with $l \leq m \leq k$) becomes zero. Once again the whole system decomposes into two disconnected subsystems (and diagram on Figure 7 splits vertically into two blocks). In this, both upper and bottom blocks describe non-unitary partially massless theories. The reason again is that bottom block does not have enough fields for description of massive spin-tensor $\Psi_{\mu\nu}^{(a_k),(b_{n-1})}$.

![Figure 10: Partially massless limit in $dS$ space](image)

4 Conclusion

Once again we have seen that frame-like formalism gives a simple and elegant way for description of (spin)-tensors with different symmetry properties. The formulation for massless mixed symmetry spin-tensors constructed here turns out to be natural and straightforward
generalization of Skvortsov formulation for massless mixed symmetry tensors [24] as well as Vasiliev formulation for completely symmetric spin-tensors [15, 16]. Similarly, all results on massive mixed symmetry spin-tensors obtained here appear as natural extension of previous results on massive (spin)-tensors [19, 26, 42].

As a byproduct of our investigations, we obtain a generalization of the results [25] for anti de Sitter group to the case of de Sitter one. Recall, that in [25] it was shown that massless (from anti de Sitter group point of view) representations contain more degrees of freedom then corresponding Minkowski one and in the flat space limit decompose into sum of massless Minkowski fields. For the (spin)-tensors corresponding to Young tableau with two rows a necessary pattern of massless fields can be obtain by cutting boxes from the second row until we end up with the tableau with one row corresponding to completely symmetric (spin)-tensor. Similarly, we have seen in Subsection 3.5 that massless (from the de Sitter group point of view) representations correspond to a number of massless Minkowski ones, in this necessary pattern can be obtained by cutting boxes from the first row until we end up with the rectangular tableau. Note that in both cases the procedure stops at the field having one its own gauge transformation only.

Let us stress once again that one of the nice features of gauge invariant formulation for massive fields is that it nicely works both in flat Minkowski space as well as in \((A)dS\) space with arbitrary value of cosmological constant. In particular, this allows us to investigate all possible special partially massless limits that exist both in \(AdS\) as well as in \(dS\) spaces. As we have seen, most of these partially massless theories turn out to non-unitary. Thus besides general massive theories the most/only physically interesting cases correspond to massless (in anti de Sitter sense) fields in \(AdS\) space, in this general massive theory can be considered as smooth deformation for appropriate collection of such massless fields. Recall that till now most of results on higher spin interactions (see e.g. recent reviews [43, 44, 45, 46]) were obtained for massless fields in \(AdS\) space. Thus it seems very interesting and important to understand how such interacting theories for massless fields in \(AdS\) space could be deformed into the ones for massive fields in flat Minkowski space. Some first very modest but nevertheless encouraging results in this direction were obtained recently [47, 48].

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