The rigid limit in Special Kähler geometry for $SU(2)$ SYM with a massive quark hypermultiplet

Chris Van Den Broeck†
Instituut voor Theoretische Fysica,
Katholieke Universiteit Leuven, B-3001 Leuven, Belgium

Abstract

We study the rigid limit of type IIB string theory, compactified on a $K3$ fibration, which, near its conifold limit, contains the Seiberg–Witten curve for $N = 2$ $SU(2)$ Super-Yang-Mills with a massive hypermultiplet in the fundamental representation. Instead of working with an ALE approximation, we treat the $K3$ fibration globally. The periods we get in this way, allow for an embedding of the field theory into a supergravity model.

† E-mail: chris.vandenbroeck@fys.kuleuven.ac.be
1 Introduction

In the past few years, a large number of quantum field theories have been solved by the method of geometric engineering [1]. Type IIB string theory compactified on a Calabi–Yau manifold, leads to an $N = 2$ supersymmetric theory in 4 dimensions. The vector multiplet moduli space of these theories receives no quantum corrections, and by going to a rigid limit of this classical moduli space and identifying the corresponding rigid low energy quantum theory (usually a field theory), one is able to obtain an exact solution for the two derivative low energy effective action [2, 3].

A type IIB compactification is mapped onto a type IIA compactification by mirror symmetry [4, 5]. This maps the non-quantum corrected vector moduli space of the type IIB model to the corresponding moduli space of the IIA model. The latter does receive quantum corrections from world-sheet instantons. When one needs to solve a quantum field theory, one can look for a type IIA model which yields this theory in a rigid limit, and map it to a classical type IIB model by mirror symmetry.

When the Calabi–Yau has a conifold singularity, branes wrapped around cycles which shrink to zero, become massless, which explains the singularities in the moduli space of the effective quantum field theory [6]. Accordingly, the limit where the Calabi–Yau develops a conifold singularity is to be identified as a rigid limit. In principle, it is only necessary to study a neighbourhood of the singularity, and the Calabi–Yau can be approximated by an $ALE$ fibration [7, 8]. These fibrations make clear how the Seiberg–Witten curve of the rigid model originates in string theory. However, this is not sufficient if we also want to embed the low-energy theory into a supergravity model. The supergravity theories we get in this way from Calabi–Yau manifolds which are $K3$ fibrations are important from a phenomenological point of view.

The two derivative action of the scalars in vector multiplets of $N = 2, D = 4$ theories defines a geometric structure known as special Kähler geometry. There are two kinds: local special geometry [9, 10] applies to locally supersymmetric theories, i.e. supergravity and strings, while rigid special geometry [11, 12] is associated to rigid supersymmetry, i.e. $N = 2$ supersymmetric gauge theories in flat spacetime.

In [13], a detailed study was made of the way in which the rigid degrees of freedom decouple from the rest of the action in the rigid limit of the local geometry. The Calabi–Yau manifolds under investigation were $K3$ fibrations, which greatly facilitated the computations. The symplectic period vector of the Calabi–Yau, which is given in terms of integrals of the holomorphic 3-form over a basis of 3-cycles, could be written as a vector of $K3$ periods integrated over cycles in the base space of the fibration.
In the present paper, we exploit this split-up to study the rigid limit of a Calabi–Yau that has the same $K3$ fibre as one of the models studied in [13]. It describes Seiberg–Witten theory with a single massive quark hypermultiplet.

The structure of this paper is as follows. In section 2, we introduce the polynomial whose vanishing locus describes the Calabi–Yau manifold we are going to study. Near the conifold limit, it can be written as an ALE fibration containing the Seiberg–Witten curve of $SU(2)$ SYM with a single massive hypermultiplet in the doublet representation. In section 3, we define a basis of cycles to calculate the periods of the Calabi–Yau. In section 4, we set up an expansion around the conifold singularity and show that the Kähler potential of the supergravity theory reproduces the rigid Kähler potential when the Planck mass is taken to infinity. Conclusions are presented in section 5.

## 2 The Calabi–Yau as a $K3$ and ALE fibration

The complex manifold we are concerned with, is described as a hypersurface in a weighted projective ambient space with homogeneous coordinates $(x_1, x_2, x_3, x_4, x_5)$ carrying weights $(1, 2, 3, 3, 3)$. In addition, we impose the following global identifications:

$$x_j \cong \exp\left(\frac{2\pi i}{12}n_j\right)x_j,$$

with

$$(n_1, n_2, n_3, n_4, n_5) = m_0(1, 2, 3, 3, 3) + m_1(2, -2, 0, 0, 0) + m_2(0, 0, 3, 0, -3) + m_3(0, 0, 3, -3, 0),$$

where $m_1, m_2, m_3 \in \mathbb{Z}$. $m_0$ can be complex as well. We then define the manifold $X_{12}^*[1, 2, 3, 3, 3]$ by the polynomial constraint $W = 0$, where

$$W = -\frac{\Lambda^6}{64}Bx_1^{12} + \frac{\Lambda^3}{8}Bmx_1^8x_2^2 - \frac{1}{4}\psi_0x_1^4x_2^4 + \frac{1}{8}Bx_2^6 + \frac{1}{4}x_3^4 + \frac{1}{4}x_4^4 + \frac{1}{4}x_5^4 - \psi_0x_1x_2x_3x_4x_5.$$  

This polynomial has weight $12 = 1 + 2 + 3 + 3 + 3$, which ensures the vanishing of the first Chern class, so we are dealing with a Calabi–Yau threefold (CY3). Up to some rescalings, it is the most general polynomial of degree 12 in these variables. The coefficients of $x_3^4$, $x_4^4$ and $x_5^4$ have already been fixed by rescaling the corresponding variables. We could also have fixed the prefactor of the first and fourth monomials,
but it will prove useful to keep them in place for the moment. $\psi_s$, $\psi_0$ and $m$ are coordinates on the moduli space.

It is easy to see that our CY3 is a $K3$ fibration. To this end, we introduce the new coordinates $\zeta$ and $x_0$, the first of which is invariant under the identifications (1). It will play the role of base space variable. The coordinate transformation is

$$\zeta = \frac{x_2^2}{x_1^4}, \quad x_0 = x_1x_2. \quad (4)$$

The polynomial (3) can now be written as

$$W_{K3} = \frac{1}{4} B' x_0^4 + \frac{1}{4} x_3^4 + \frac{1}{4} x_4^4 - \psi_0 x_0 x_3 x_4 x_5, \quad (5)$$

with

$$B' = \frac{B}{2} (\zeta + \frac{m}{\zeta} - \frac{1}{8} \frac{\Lambda^6}{\zeta^2}) - \psi_s. \quad (6)$$

Another $K3$ fibration, but with the same fibre, namely $X_8^*[1,1,2,2,2]$, has been studied extensively in [13]. There it was shown that the $K3$ manifold (4) develops a conifold singularity in $B' = \psi_0^4$ and a large complex structure singularity in $B' = 0$. Therefore, the CY3 we are studying here, generically has six points in base space where the $K3$ fibre becomes singular. One can easily check that the CY3 itself becomes singular for $B = 0$. In this case, it contains a curve of singularities, parametrized by the base space coordinate $\zeta$.

We can now expand the CY3 around this singularity. By setting $B = 2\epsilon$, $\psi_s + \psi_0^4 = 2\epsilon u$ (thereby keeping $u = \frac{\psi_s + \psi_0^4}{B}$ finite) and using the same expansion for the $K3$ coordinates as in [13], we arrive at the following local ALE fibration [7, 8]:

$$W_{ALE} = \frac{1}{2} \epsilon \left[ \frac{1}{2} \left( \zeta + \frac{m}{\zeta} - \frac{1}{8} \frac{\Lambda^6}{\zeta^2} \right) + y_1^2 - u + y_2^2 + y_3^2 \right] = 0, \quad (7)$$

which means we are dealing with an $A_1$ singularity [14, 15]. Apart from the terms in $y_2$ and $y_3$, we have found the Seiberg-Witten curve for $SU(2)$ SYM with one massive hypermultiplet in the fundamental representation of the gauge group:

$$W_{SW} = \frac{1}{2} \left( \zeta + \frac{m}{\zeta} - \frac{1}{8} \frac{\Lambda^6}{\zeta^2} \right) + y_1^2 - u = 0. \quad (8)$$

It can be transformed into a more familiar form [16, 17] by setting

$$y_1 = x + \frac{1}{4} \frac{\Lambda^3}{\zeta}; \quad \zeta = y - (x^2 - u)^2. \quad (9)$$
We then get
\[ y^2 = (x^2 - u)^2 - \Lambda^3(x + m). \]  
(10)
The Seiberg–Witten meromorphic 1-form that comes with this equation is
\[ \lambda_{SW} = \frac{1}{2\pi i} \frac{x}{y} \left( -\frac{1}{2} \frac{P(x)\Lambda^3}{y^2 - P(x)^2} - 2x \right) dx, \]  
(11)
where \( P(x) = x^2 - u \). The 1-form we will use in the context of the Calabi–Yau is
\[ \lambda = y_1(\zeta) \frac{d\zeta}{2\pi i \zeta} \]  
(12)
\[ = x(\zeta) \frac{d\zeta}{2\pi i \zeta} - \frac{\Lambda^3}{8\pi i} d \left( \frac{1}{\zeta} \right), \]  
(13)
It will arise as an integral over a K3 cycle of the Calabi–Yau holomorphic 3-form. One has
\[ x(\zeta) \frac{d\zeta}{2\pi i \zeta} = \lambda_{SW} - \frac{1}{8\pi i} \frac{mdx}{x + m}. \]  
(14)

The form (11) has a double pole at infinity and a first order pole with residue proportional to \( m \) in \( x = -m \), with opposite residues on the two sheets. As a consequence, the periods
\[ a_D = \int_\alpha \lambda_{SW}, \quad a = \int_\beta \lambda_{SW}, \]  
(15)
with \( \alpha \) and \( \beta \) the homology 1-cycles of the hyperelliptic surface, are not invariant under deformations of the cycles across the pole of \( \lambda_{SW} \), so \( a_D \) and \( a \) can make jumps proportional to \( m \) when a monodromy transformation is performed. We can just as well choose (12) as meromorphic 1-form, since the second term in (14) is moduli-independent and does not affect the condition that the derivative w.r.t. the modulus should be the holomorphic form. The 1-form \( \lambda \) has a first order pole with residue proportional to \( m \) in \( \zeta = 0 \) and is regular at \( \zeta = \infty \) and at the branch points of the hyperelliptic curve.

### 3 Cycles, monodromies and periods

We begin this section with a brief review of the results of [13] concerning the K3 fibre (5). There the periods of the holomorphic 2-form \( \hat{\Omega}^{(2,0)} \) were given in terms of solutions to the Picard-Fuchs equations depending on the moduli space parameter
\[ z = -\frac{B'(\zeta)}{\psi_0^4}. \]  
(16)
The $K3$ large complex structure limit is at $z = 0$, while the conifold singularities are at $z = -1$. The non-zero periods are

$$
\hat{\theta}_0 = \frac{1}{4\pi^2} (U_1 - U_2)^2, \quad \hat{\theta}_1 = \frac{i}{4\pi^2} (U_1 - iU_2)^2,
$$

$$
\hat{\theta}_2 = -\frac{1}{4\pi^2} (U_1 + U_2)^2, \quad \hat{\theta}_3 = -\frac{i}{4\pi^2} (U_1 + iU_2)^2.
$$

(17)

The functions $U_i(z), i = 1, 2$ have the following form in the neighbourhood of $z = \infty$:

$$
U_1(z) = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{5}{8})}{\Gamma(\frac{3}{8})} \left( \frac{1}{z} \right)^{\frac{3}{8}} F\left( \frac{1}{8}, \frac{1}{8}, \frac{3}{4}; -\frac{1}{z} \right),
$$

$$
U_2(z) = \frac{\Gamma(\frac{3}{8})\Gamma(\frac{7}{8})}{\Gamma(\frac{5}{8})} \left( \frac{1}{z} \right)^{\frac{5}{8}} F\left( \frac{3}{8}, \frac{3}{8}, \frac{5}{4}; -\frac{1}{z} \right).
$$

(18)

In order to analytically continue them, such that they are defined on the whole $z$ space, one needs a cut running from 0 to $-1$, and another one from $-1$ to $\infty$. The four solutions (17) satisfy $\sum_{k=0}^3 \hat{\theta}_k = 0$, so only three of them are independent; we will use the first three. (A priori, one might have expected 22 periods instead of three, corresponding to the 22 2-cycles of a $K3$ manifold, but most of them are zero, namely those that correspond to the algebraic cycles.)

A more convenient basis of periods regarding their behaviour in the neighbourhood of the singularities is given by

$$
\hat{\theta}' = F\hat{\theta}, \quad F = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

(19)

The intersection matrix of the associated 2-cycles is

$$
I' = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

(20)

The monodromies around the singularities at $z = 0, z = -1, z = \infty$ are given by

$$
M'_0 = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}, \quad M'_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},
$$

$$
M'_\infty = \begin{pmatrix} -3 & -1 & -2 \\ 4 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix}.
$$

(21)
We now turn to the Calabi–Yau threefold $X_{12}^{[1, 2, 3, 3, 3]}$, the $K3$ fibration that was defined in (3). We will consider the periods of the (rescaled) holomorphic 3-form, which we define using the Griffiths map $[18]$ and then split up into a $K3$ holomorphic 2-form and a base space 1-form:

$$
\hat{\Omega}^{(3,0)} = \frac{\psi_0 |G|}{(2\pi i)^4} \int_{\Gamma} \frac{\omega}{W} = \hat{\Omega}^{(2,0)} \frac{d\zeta}{2\pi i \zeta}. \tag{22}
$$

Here $|G|$ stands for the order of the group of identifications (1); $\Gamma$ is a cycle running around the surface $W = 0$ in the ambient space, and $\omega$ is the volume form

$$
\omega = w_1 x^1 dx^2 \ldots dx^5 - w_2 x^2 dx^1 \ldots dx^5 + \ldots + w_5 x^5 dx^1 \ldots dx^4, \tag{23}
$$

with $(w_1, \ldots, w_5)$ the weights of $(x_1, \ldots, x_5)$.

Starting from the $K3$ periods, we can define CY 3 periods along two kinds of cycles.

- $S^1 \times S^2$ cycles: a $K3$ cycle fibred over a closed path in base space running around point(s) where the $K3$ cycle shrinks to zero;
- $S^3$ cycles: a $K3$ cycle fibred over a path running between two points where the cycle vanishes.

As can be seen from (16), the branch cuts in the $z$ plane induce cuts in the base space of the Calabi–Yau. In our case, $z$ is given by

$$
z = -\frac{B(\zeta + \Lambda^3 \frac{m}{\zeta} - \frac{\Lambda^6}{8 \zeta^2})}{\psi_0^4} - \psi_s. \tag{24}
$$

We already mentioned that there are three points in base space where the $K3$ fibre develops a conifold singularity, while in three other points there is a large complex structure singularity. The large complex structure points are solutions of

$$
\frac{B}{2} \left[ \zeta + \Lambda^3 \frac{m}{\zeta} - \frac{1}{8 \zeta^2} \right] - \psi_s = 0. \tag{25}
$$

Let us call $e_0^1$ and $e_0^2$ the two points that go to zero and $e_{\infty}$ the point that goes to $\infty$ as $\psi_s + \psi_0^4 \to 0, B \to 0$. The conifold points satisfy

$$
\frac{B}{2} \left[ \zeta + \Lambda^3 \frac{m}{\zeta} - \frac{1}{8 \zeta^2} \right] - (\psi_s + \psi_0^4) = 0, \tag{26}
$$

and we shall call $f_0^1, f_0^2$ the solutions that converge to $e_0^1, e_0^2$ respectively, and $f_{\infty}$ the point that goes to $e_{\infty}$, as $\psi_0 \to 0$. 

6
There are six branch cuts in the CY3 base space. Two of them run from \(e_0^{1,2}\) to \(f_0^{1,2}\), another two connect \(f_0^{1,2}\) to zero, and finally there are cuts between \(e_\infty\) and \(f_\infty\) and from \(f_\infty\) to \(\infty\).

We can now construct the CY3 periods (see fig. 1). First we define the periods associated to cycles of topology \(S^1 \times S^2\). They are given by integrating \(\hat{\theta}'_0\), \(\hat{\theta}'_1\), and \(\hat{\theta}'_2\) around a path \(C\) encircling the points zero, \(f_1^{1,2}\), and \(e_1^{1,2}\):

\[
T_0 = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta} \hat{\theta}'_0, \\
T_1 = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta} \hat{\theta}'_1, \\
T_2 = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta} \hat{\theta}'_2. \quad (27)
\]

Next, we construct the periods along the \(S^3\) cycles. From the monodromy matrices \((21)\), we can read off that \(\hat{\theta}'_2\) gets a contribution \(\hat{\theta}'_0\) when running around the conifold singularity \(z = -1\). From the Picard–Lefschetz formula, it follows that the only extra cycles that can appear are vanishing cycles, so \(\hat{\theta}'_0\) must vanish at the conifold points \(f_0^{1,2}, f_\infty\). Thus, we obtain an \(S^3\) cycle as follows:

\[
V_0 = \frac{1}{2\pi i} \int_{f_0^{1,2}}^{f_\infty} \frac{d\zeta}{\zeta} \hat{\theta}'_0. \quad (28)
\]

Now look at the monodromy at \(z = 0\). By the same reasoning as in the case of the conifold point, we find that \(\hat{\theta}'_1\) and \(\hat{\theta}'_2\) must vanish at the large complex structure points. So we define the periods

\[
V_1^\infty = \frac{1}{2\pi i} \int_{e_0^{1,2}}^{e_\infty} \frac{d\zeta}{\zeta} \hat{\theta}'_1, \\
\tilde{V}_1^\infty = \frac{1}{2\pi i} \int_{e_0^{1,2}}^{e_\infty} \frac{d\zeta}{\zeta} \hat{\theta}'_1, \\
V_2^\infty = \frac{1}{2\pi i} \int_{e_0^{1,2}}^{e_\infty} \frac{d\zeta}{\zeta} \hat{\theta}'_2, \\
\tilde{V}_2^\infty = \frac{1}{2\pi i} \int_{e_0^{1,2}}^{e_\infty} \frac{d\zeta}{\zeta} \hat{\theta}'_2. \quad (29)
\]

We let the paths of \(V_{1,2}^\infty\) intersect that of \(V_0\). Later on, it will be useful not to consider these periods but the sums and differences:

\[
V_1^+ = V_1^\infty + \tilde{V}_1^\infty \\
V_2^+ = V_2^\infty + \tilde{V}_2^\infty \\
V_1^- = V_1^\infty - \tilde{V}_1^\infty \\
V_2^- = V_2^\infty - \tilde{V}_2^\infty. \quad (30)
\]
The eight cycles associated to the periods $T_0$, $V_0$, $T_1$, $T_2$, $V_{1,2}^-$, $V_{1,2}^+$ together yield an invertible intersection matrix for the Calabi–Yau. This is precisely the number of cycles we need for a basis, since $h^{2,1} = 3$, so that the third Betti number is $b_3 = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 8$. We can bring the intersection matrix in block-diagonal form by defining

\[
\begin{align*}
V_2^{+'} &= 2V_2^+ + V_0 + \tilde{V}_0 - \frac{3}{2}T_0; \\
T_2' &= 2T_2 + T_0, \\
V_1^{+'} &= V_1^+ + 2T_2', \\
V_1^{-'} &= V_1^- - \frac{1}{2}T_1, \\
V_2^{-'} &= 2V_2^- - \frac{1}{2}T_2',
\end{align*}
\]

where $\tilde{V}_0$ is the integral of $\hat{\theta}_0'$ over a base space path running from $f_0^2$ to $f_\infty$.

In the basis

\[
\mathcal{C} = \{T_0, V_0, T_1, V_1^{+'}, T_2', V_2^{+'}, V_1^{-'}, V_2^{-'}\}
\]
the intersection matrix is given by

\[
I_{CY3} = \begin{pmatrix}
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0
\end{pmatrix}.
\] (37)

For the cycles that do not intersect in a branch point, the intersection is simply equal to the sum of the intersections of the base space paths, each multiplied by the intersection of the fibred \( K^3 \) cycles. A little care is needed in calculating the intersection of, for example, \( V_{-1} \) with \( V_{+1} \). To do this, one can replace \( V_{-1} \) by an 8-shaped path running around the large complex structure points \( e_1 \) and \( e_2 \). The \( K^3 \) cycle associated to \( \hat{\theta}_0' - \frac{1}{2} \hat{\theta}_1' \) is transported in counter-clockwise direction around \( e_1 \). When crossing the cut, it is transformed into \( \hat{\theta}_0' + \frac{1}{2} \hat{\theta}_1' \). This is transported around \( e_2 \) in clockwise direction, changing it into \( \hat{\theta}_0' - \frac{1}{2} \hat{\theta}_1' \) again. \( V_{+1} \) intersects this cycle in two base space points, which together give the result 4 for the intersection.

4 The rigid limit

As it is well-known, the complex scalars of vector multiplets in \( N = 2 \) rigid supersymmetry as well as in supergravity behave as coordinates on a manifold with special Kähler geometry [9, 10]. The metric representing the couplings of these scalars is given in terms of a Kähler potential, which can be expressed as

\[
K(u, \bar{u}) = i \langle V(u), \bar{V}(\bar{u}) \rangle; \quad \mathcal{K}(z, \bar{z}) = -\log(-i \langle v(z), \bar{v}(\bar{z}) \rangle)
\] (38)

for the rigid and local geometry respectively. The period vector \( V(u) \) (resp. \( v(z) \)) is a holomorphic function of \( r \) (resp. \( n \)) complex scalars \( \{u^i\} \) (resp. \( \{z^\alpha\} \)). It has \( 2r \) (resp. \( 2(n + 1) \)) components. The symplectic inner product \( \langle \cdot, \cdot \rangle \) is defined by

\[
\langle V, W \rangle = V^T Q^{-1} W; \quad \langle v, w \rangle = v^T q^{-1} w,
\] (39)

where \( Q \) and \( q \) are real, invertible, antisymmetric matrices. \( q \) will be the Calabi–Yau intersection matrix.

To find the rigid limit of the local geometry, one divides the local coordinates \( \{z^\alpha\} \) in a set \( \{u^i\} \), which will become the rigid coordinates, and a parameter \( \epsilon \) such
that $\epsilon \to 0$ in the conifold limit. Assume that the period vector can be decomposed as

$$ v = v_0(\epsilon) + \epsilon^a v_1(u) + v_2(\epsilon, u), \quad (40) $$

where $v_0$ contains the dominant and constant pieces and is independent of the surviving moduli, while $v_1$ does not depend on $\epsilon$ and is such that the derivatives w.r.t. the moduli form a matrix of rank $r$, the number of $\{u^i\}$. For the examples of [13], it was found that

$$ \langle v, \bar{v} \rangle = iM^2(\epsilon) + f(\epsilon, u) + |\epsilon|^{2a} \langle v_1(u), \bar{v}_1(\bar{u}) \rangle + R(\epsilon, \bar{\epsilon}, u, \bar{u}), \quad (41) $$

where the function $f$ is holomorphic in $u$ and

$$ iM^2(\epsilon) = \langle v_0(\epsilon), \bar{v}_0(\bar{\epsilon}) \rangle, \quad (42) $$

with

$$ \frac{|\epsilon|^{2a}}{M^2} \to 0; \quad \frac{R}{|\epsilon|^{2a}} \to 0 \quad (43) $$

as $\epsilon \to 0$. $a$ is some real number which could be normalized to one. Assuming this structure is realized, one has

$$ \mathcal{K} = -\log(M^2) + \frac{i}{M^2} F(\epsilon, u) - \frac{i}{M^2} \tilde{F}(\bar{\epsilon}, \bar{u}) + i|\epsilon|^{2a} \langle v_1(u), \bar{v}_1(\bar{u}) \rangle + \ldots $$

$$ = -\log(M^2) + \frac{i}{M^2} F(\epsilon, u) - \frac{i}{M^2} \tilde{F}(\bar{\epsilon}, \bar{u}) + |\epsilon|^{2a} K(u, \bar{u}), + \ldots \quad (44) $$

with $K(u, \bar{u})$ the rigid Kähler potential. The $F$ terms amount to a Kähler transformation, provided the $u$-dependent parts of the function $f$ are of higher order than $|\epsilon^a|$.

Let us consider the period vector. We again use the results of [13], where the exact expressions for the $K3$ periods can be found. Expanding in powers of $1+z \sim \epsilon$, one has

$$ \hat{\theta}_0' = \eta \sqrt{1+z} + \mathcal{O}(\epsilon^2), $$

$$ \hat{\theta}_1' = k_1 + \frac{1}{2} l_1 (1+z) + \mathcal{O}(\epsilon^2), $$

$$ 2\hat{\theta}_2' + \hat{\theta}_0' = k_2 + \frac{1}{2} l_2 (1+z) + \mathcal{O}(\epsilon^2), \quad (45) $$

where $k_1$ and $k_2$ are constants, and the parameter $z$ was defined in [24]. In the rigid limit, $z$ has the expansion

$$ z = -1 + 2\bar{\epsilon}(u - \xi) + \mathcal{O}(\bar{\epsilon}^2), \quad (46) $$
where we have defined
\[ \tilde{\epsilon} = -\frac{\epsilon}{\psi_s}; \quad \xi = \frac{1}{2} \left( \zeta + \Lambda^3 \frac{m}{\zeta} - \frac{\Lambda^6}{8} \frac{1}{\zeta^2} \right). \] (47)

Instead of directly evaluating integrals over the $K_3$ periods, it is useful to first get a view on the general structure of the period vector in terms of $\tilde{\epsilon}$, by calculating the $\tilde{\epsilon}$ monodromies of the CY3 periods. From the way the periods transform under $\tilde{\epsilon} \to e^{2\pi i} \tilde{\epsilon}$, one can deduce what the expansion in $\tilde{\epsilon}$ looks like [14].

When we turn $\tilde{\epsilon}$ in the complex plane by $\tilde{\epsilon} \to e^{i\theta} \tilde{\epsilon}$, where $\theta$ runs from 0 to $2\pi$, the conifold points $f_1^{1,2}$ and $f_\infty$ remain fixed. $e_0^{1,2}$ and $e_\infty$ to lowest order in $\tilde{\epsilon}$ are given by
\[ e_0^1 = -i \frac{3}{2} \frac{\Lambda^3}{2^{3/2}} \tilde{\epsilon}^{1/2}; \quad e_0^2 = i \frac{3}{2} \frac{\Lambda^3}{2^{3/2}} \tilde{\epsilon}^{1/2}; \quad e_\infty = -\frac{1}{\epsilon}. \] (48)

This means that under a full rotation of $\tilde{\epsilon}$, $e_0^1$ and $e_0^2$ will get interchanged by turning around 0 in counter-clockwise direction, while $e_\infty$ makes a full circle around $\infty$. When calculating the transformation of a CY3 cycle, one also has to take into account the $K_3$ cycle fibres. A $K_3$ cycle $c$ gets transformed into $M'_{-1}c$.

In the basis (36), the $\tilde{\epsilon}$ monodromy is given by
\[ v \to M_{\text{CY3}}^{\tilde{\epsilon}} v, \] (49)
with
\[
M_{\text{CY3}}^{\tilde{\epsilon}} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}. \] (50)

A slight subtlety about $V_2^{-\prime}$ is that it transforms as
\[ V_2^{-\prime} \to -V_2^{-\prime} + W_0, \] (51)
where $W_0$ is the integral of $\dot{\theta}_0$ over two paths that run from $f_0^1$ to $f_0^2$ on opposite sides of the cuts. By checking the intersections of the associated CY3 cycle with the basis of CY3 cycles or by directly deforming it, one finds that it is equal to $-V_1^{-\prime}$.  

11
The above expressions for the $\hat{\epsilon}$ monodromy are almost sufficient to derive the structure of the period vector in terms of $\hat{\epsilon}$. It looks like

$$
\begin{pmatrix}
T_0 \\
V_0 \\
T_1 \\
V_1^{+} \\
T_2^{+} \\
V_2^{+} \\
V_1^{-} \\
V_2^{-}
\end{pmatrix}
= 
\begin{pmatrix}
\hat{\epsilon}^2 \left( V_1(u) + \mathcal{O}(\hat{\epsilon}) \right) \\
\hat{\epsilon}^2 \left( V_2(u) + \mathcal{O}(\hat{\epsilon}) \right) \\
-k_1 - \epsilon_l u + \mathcal{O}(\hat{\epsilon}^2) \\
-\frac{3}{2\pi} \log \hat{\epsilon} \left( k_1 + \epsilon u l_1 + \mathcal{O}(\hat{\epsilon}^2) \right) + k'_1 + l'_1 u \hat{\epsilon} + \mathcal{O}(\hat{\epsilon}^2) \\
-k_2 - \epsilon_l u + \mathcal{O}(\hat{\epsilon}^2) \\
-\frac{3}{2\pi} \log \hat{\epsilon} \left( k_2 + l_2 \epsilon u + \mathcal{O}(\hat{\epsilon}^2) \right) + k'_2 + l'_2 \epsilon u + \mathcal{O}(\hat{\epsilon}^2) \\
\frac{1}{2\pi} \log \hat{\epsilon} \left( n \epsilon^2 \right) + V(u) \epsilon^{3/2} + \mathcal{O}(\hat{\epsilon}^2)
\end{pmatrix}.
$$

(52)

The constants $k_{1,2}$ and $l_{1,2}$ are the same as the ones in (45). The expressions for $T_1$ and $T_2$ can easily be found by direct integration.

The periods $V_1^{-}$, $V_2^{-}$, $V_1^{+}$, $V_2^{+}$ need some explanation. From the $\hat{\epsilon}$ monodromy of $V_1^-$, one finds that it must have the structure

$$V_1^- = \epsilon^{3/2} n u + \mathcal{O}(\epsilon^{5/2}).$$

(53)

We can also evaluate it directly. Writing $\hat{\theta}'(\zeta)=f(\zeta)$, we have, using (48), (15) and (16),

$$
\int_{e_0^1}^{e_0^2} \frac{d\zeta}{2\pi i \zeta} f(\zeta) - \frac{1}{2} T_1 = \frac{1}{2} \left( f(0) + f'(0) 2 \epsilon u + \mathcal{O}(\epsilon^2) \right) + \frac{1}{2} \left( k_1 + l_1 \epsilon u + \mathcal{O}(\epsilon^2) \right)
$$

$$+ \sum_{i=1}^\infty \sum_{j=1}^k \sum_{k=0}^l A_{ijkl} u^{i-j} \left[ (\epsilon^{3/2})^{2i-2j+k+2l} - (-\epsilon^{3/2})^{2i-2j+k+2l} \right]
$$

$$= n \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}),$$

(54)

where $n$ is indeed a $u$-independent constant. The minus sign in the first line arises because we are integrating on the ‘lower’ side of zero.

$V_2^-$ is an integral of $\frac{1}{2}(\hat{\theta}'_2 + \hat{\theta}'_0)$ over a path from $e_0^1$ to $e_0^2$ running around the outer side of $f_0^1$, plus $\frac{1}{2}(\hat{\theta}'_2 + \hat{\theta}'_0)$ integrated over a path between the same points around the outer side of $f_0^2$ (for the definition of $V_2^-$, see (34)). The integrals of $\frac{1}{2}(\hat{\theta}'_2 + \hat{\theta}'_0)$ add up to something of the form $\tilde{n} \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2})$, where $\tilde{n}$ is a constant. The integral of $-\hat{\theta}'_0$ over a path from $e_0^1$ to $e_0^2$ around $f_0^2$ can be expected to depend on $u$ in a non-trivial way, and it transforms to minus itself plus $W_0$ under an $\hat{\epsilon}$ monodromy. As we mentioned before, $W_0$ is nothing but $-V_1^-$. Putting everything together, we find that $V_2^-$ is of the form

$$V_2^- = \frac{1}{2\pi i} \log \hat{\epsilon} \left( n \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}) \right) + V(u) \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}).$$

(55)
Figure 2: The base space paths of the cycles that cancel in the rigid limit in the calculation of $V_2^{+'}$. 

By direct integration, it is easy to see that the constants $k'_1$ and $l'_1$ in the expansion of $V_1^{+'}$ are $u$-independent. To check this for the constants $k'_2$ and $l'_2$ in $V_2^{+'}$, one writes

$$V_2^{+'} = \left( \int_{e_0}^{\infty} + \int_{f_0}^{\infty} \right) \left( 2\hat{\theta}_2' + \hat{\theta}_0' \right) + \left( -\int_{e_1}^{e_0} - \int_{e_0}^{e_2} + \int_{f_0}^{f_1} + \int_{f_1}^{f_2} - \frac{3}{2} \int_C \right) \hat{\theta}_0'. \quad (56)$$

(See eq. (31) for the definition of $V_2^{+'}$.) Only the first two integrals need to be considered; the rest are associated to cycles that, after some deformations, can be seen to cancel against each other in the rigid limit (see fig. 2).

Now that we have determined the $\tilde{\epsilon}$ expansions of the periods, we can bring the intersection matrix of the $CY3$ into a particularly simple form by performing another basis transformation (which would, however, have spoiled the Jordan form of the $\tilde{\epsilon}$ monodromy matrix):

$$V_2^{+''} = V_2^{+'} - \frac{1}{4} V_1^{-'} \quad (57)$$

In the basis

$$C' = \{ T_0, V_0, T_1, V_1^{+'}, T_2', V_2^{+''}, V_1^{-'}, V_2^{-} \} \quad (58)$$
we now get the following expression for the inverse intersection matrix:

\[
q^{-1} = \frac{1}{8} \begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
\end{pmatrix}.
\] (59)

From (52), we find

\[
\begin{align*}
 v_0 &= \begin{pmatrix}
 0 \\
 0 \\
 -\frac{3}{2\pi i} \log \tilde{\epsilon} k_1 + k_1' \\
 -\frac{3}{2\pi i} \log \tilde{\epsilon} k_2 + k_2'' \\
 0 \\
 0
\end{pmatrix} \\
 &\quad; \quad v_1 = \begin{pmatrix}
 V_1(u) \\
 V_2(u) \\
 0 \\
 0 \\
 -\frac{1}{2} n \\
 \frac{1}{2\pi i} \log n + V(u)
\end{pmatrix} \\
&\quad; \quad v_2 = \begin{pmatrix}
 \mathcal{O}(\tilde{\epsilon}^2) \\
 \mathcal{O}(\tilde{\epsilon}^2) \\
 -\tilde{\epsilon} l_1 u + \mathcal{O}(\tilde{\epsilon}^2) \\
 -\tilde{\epsilon} l_2 u + \mathcal{O}(\tilde{\epsilon}^2) \\
 -\frac{3}{2\pi i} \log (\tilde{\epsilon} l_1 u + \mathcal{O}(\tilde{\epsilon}^2)) + l_1' u \tilde{\epsilon} + \mathcal{O}(\tilde{\epsilon}^2) \\
 \mathcal{O}(\tilde{\epsilon}^2) \\
 \mathcal{O}(\tilde{\epsilon}^2)
\end{pmatrix}.
\] (60)

It will be clear that \(i \langle v_0, \bar{v}_1 \rangle = 0\), up to a constant, which will lead to a constant Kähler transformation. On the other hand,

\[
\langle v_1, \bar{v}_1 \rangle = \eta^2 \left( \int_C \lambda \int_{f_0^1}^{f_0^\infty} \tilde{\lambda} - \int_{f_0^1}^{f_0^\infty} \lambda \int_C \tilde{\lambda} \right),
\] (61)

up to a \(u\)-dependent expression, which will again lead to a Kähler transformation. (61) is equal to the rigid Kähler potential \(K(u, \bar{u})\), up to a prefactor which can be absorbed into the holomorphic 3-form of the CY3 by a rescaling. As we already noted in section 2, the form \(\lambda\) can be used as Seiberg-Witten meromorphic 1-form
It is the analogue of what one gets from the $CY3$ holomorphic $3$-form in the case of gauge theories without matter \[1, 3\].

Because of the $l$ terms in $v_2$, the products $\langle v_0, \bar{v}_2 \rangle$ and $\langle v_1, \bar{v}_2 \rangle$ also give a contribution that amounts to a Kähler transformation. The product $\langle v_2, \bar{v}_2 \rangle$ gives non-holomorphic contributions in $u$ that are, however, of higher order than $\tilde{\epsilon}$. Thus, we indeed find the structure

$$K = -\log(M^2) + \frac{|\tilde{\epsilon}|}{M^2} K(u, \bar{u}) + \frac{i}{M^2} F(\tilde{\epsilon}, u) - \frac{i}{M^2} \bar{F}(\tilde{\epsilon}, \bar{u}) + O(|\tilde{\epsilon}|^2).$$

(62)

5 Conclusions

We considered a type IIB compactification on a $K3$ fibration which near the conifold singularity contained the SW curve for $SU(2)$ SYM with a massive quark hyper-multiplet. We performed the rigid limit and explicitly showed how the field theory degrees of freedom decouple from the gravitational ones.

The $K3$ fibre was already encountered in \[13\], where a $CY3$ for pure $SU(2)$ SYM was studied. We introduced matter by changing the way the $K3$ was fibred over base space. We then showed how to treat the $K3$ fibration globally, instead of resorting to an ALE approximation. This allowed for an explicit embedding into a supergravity model. The structure that had been found for the local Kähler potential of pure gauge supergravity theories in terms of the rigid Kähler potential, was not spoiled by the introduction of matter.

Our results could easily be extended to introduce matter in any $K3$ fibration. It should not be difficult to repeat the procedure in the case of a larger gauge group and more matter. Thus, knowing the periods and monodromies of the $K3$ fibre should allow to quickly obtain information about a variety of models.

Acknowledgements

I would like to thank M. Billó, F. Denef and A. Van Proeyen for very helpful discussions.

References

[1] S. Katz, A. Klemm and C. Vafa, Nucl. Phys B497 (1997) 173; hep-th/9609239
[2] A. Klemm, in Trieste 1996, High energy physics and cosmology; hep-th/9705131

15
[3] W. Lerche, Nucl. Phys. Proc. Suppl. **55B** (1997) 83; hep-th/9611190

[4] L. Dixon, in *Superstrings, Unified Theories and Cosmology 1987*, G. Furlan et al., eds., World Scientific 1988

[5] W. Lerche, C. Vafa and N. Warner, Nucl. Phys. **B324** (1989) 427

[6] A. Strominger, Nucl. Phys. **B451** (1995) 96; hep-th/9504090

[7] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. **B459** (1996) 537; hep-th/9508155

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. **B477** (1996) 746; hep-th/9604034

[9] B. de Wit, P.G. Lauwers, R. Philippe, Su S.-Q. and A. Van Proeyen, Phys. Lett. **B134** (1984) 37

[10] B. de Wit and A. Van Proeyen, Nucl. Phys. **B245** (1984) 89

[11] G. Sierra and P.K. Townsend, in *Supersymmetry and Supergravity 1983*, ed. B. Milewski, World Scientific, Singapore, 1983, p. 396

[12] S.J. Gates, Nucl. Phys. **B238** (1984) 349

[13] M. Billó, F. Denef, P. Frè, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, to be published in Class. Quant. Grav.; hep-th/9803228

[14] V.I. Arnold, *Singularity Theory, selected papers*, Cambridge University Press, 1981

[15] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps*, volume I, Birkhauser, 1985

[16] A. Hanany and Y. Oz, Nucl. Phys. **B452** (1995) 283; hep-th/9505073

[17] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484; hep-th/9408099

[18] P. Griffiths, Ann. Math. **90** (1969) 460, 496

[19] A. Marshakov, A. Mironov, A. Morozov, hep-th/9701123