On multi-soliton solutions to a generalized inhomogeneous nonlinear Schrödinger equation for the Heisenberg ferromagnetic spin chain

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Abstract A generalized inhomogeneous higher-order nonlinear Schrödinger (GIHNLS) equation for the Heisenberg ferromagnetic spin chain system in (1+1)-dimensions under zero boundary condition at infinity is taken into account. The spectral analysis is first performed to generate a related matrix Riemann–Hilbert problem on the real axis. Then, through solving the resulting matrix Riemann–Hilbert problem by taking the jump matrix to be the identity matrix, the general bright multi-soliton solutions to the GIHNLS equation are attained. Furthermore, the one-, two-, and three-soliton solutions are written out and analyzed by figures.

Keywords Higher-order nonlinear Schrödinger equation · Riemann–Hilbert problem · Soliton solutions

1 Introduction

Solitons are stable, nonlinear pulses which show a fine balance between nonlinearity and dispersion. They often arise from some real physical phenomena described by integrable nonlinear partial differential equations (NLPDEs) modeling shallow water waves, nonlinear optics, electrical network pulses and many other applications in mathematical physics [1–3]. Both theoretical and experimental investigations [4–6] have been made on solitons. The derivation of abundant soliton solutions [7,8] to NLPDEs has been closely concerned by scholars from mathematics and physics, and a variety of approaches and their extensions have been established and applicable to NLPDEs up to present, such as the Hirota’s bilinear method [9,10], the Darboux transformation [11,12], the Riemann–Hilbert method [13–15], and the Lie symmetry method [16,17]. In the past years, a considerable literature has grown up around the applications of the Riemann–Hilbert technique to solve integrable NLPDEs with zero or nonzero boundary condition, some of which include the coupled NLS equation [18], the Kundu–Eckhaus equation [19], the six-component fourth-order AKNS system [20], the multicomponent mKdV system [21], the N-coupled Hirota equation [22], and the fifth-order NLS equation [23].

In this paper, we focus on a generalized inhomogeneous higher-order nonlinear Schrödinger (GIHNLS) equation for the Heisenberg ferromagnetic spin system [26] in (1+1)-dimensions

\[

t_{t} + \epsilon u_{xxxx} + 8\epsilon |u|^2 u_{xx} + 2\epsilon u^2 u_{x}^2 \\
+ 4\epsilon u_{x} u_{x}^* u + 6\epsilon u^* u_{x}^2 + 6\epsilon |u|^4 u \\
+ \left( \frac{1}{2} - 3\epsilon \right) u_{xx} + (1 - 6\epsilon) u^2 u^* - ih u_{x} = 0,
\]

where \( u \) denotes the complex function of the scaled spatial variable \( x \) and temporal variable \( t \), the real number \( \epsilon \) is a perturbation parameter, the real number \( h \) stands
for the inhomogeneities in the medium [24, 25], and the asterisk and subscripts mean the complex conjugation and partial derivatives, respectively. Equation (1) is an integrable model. When \( h = 0 \), Eq. (1) reduces to the fourth-order NLS equation, which governs the Davydov solitons in the alpha helical protein with higher-order effects [27]. In the past, many studies have been conducted on Eq. (1). The Lax pair [24] was first presented. The gauge transformation was used to construct soliton solutions [26]. The generalized Darboux technique was applied to generate some higher-order rogue wave solutions [28]. In a follow-up study, some solutions were computed by Hirota’s bilinear method, and infinitely many conservation laws were derived based upon the AKNS system [29].

The rest of the paper is arranged as follows. In Sect. 2, we formulate a matrix Riemann–Hilbert problem by carrying out the spectral analysis and obtain the reconstruction formula of potential. In Sect. 3, we gain soliton solutions from a specific Riemann–Hilbert problem on the real axis, in which the jump matrix is taken as the identity matrix. The final section is a brief conclusion.

## 2 Matrix Riemann–Hilbert problem

What we intend to describe in this section is a matrix Riemann–Hilbert problem. We start by considering the Lax pair [28] for Eq. (1)

\[
\begin{align*}
\phi_x &= U \phi, \\
\phi_t &= V \phi,
\end{align*}
\]

(2) (3)

where \( \phi = (\phi_1, \phi_2)^T \) is the spectral function, the symbol T stands for the vector transpose, and \( \lambda \in \mathbb{C} \) is a spectral parameter. And

\[
U = \begin{pmatrix} -i \lambda & 0 \\ 0 & i \lambda \end{pmatrix}, \quad V = \begin{pmatrix} V_{11} & V_{12} \\ -V_{21} & -V_{11} \end{pmatrix},
\]

\[
\begin{align*}
V_{11} &= 2 \epsilon \lambda u_x u^* - i \epsilon u_x u^* + i \epsilon u_{xx} u^* - 4 i \epsilon \lambda^2 u u^* \\
&\quad -2 \epsilon u u_x^* + 3 i \epsilon u^2 (u^*)^2 - 3 i \epsilon u u^*_x + i \epsilon u u^*_x \\
&\quad + \frac{1}{2} i u u^* - i h \lambda - i \lambda^2 + 8 i \epsilon \lambda^4 + 6 i \epsilon \lambda^2, \\
V_{12} &= 6 i \epsilon u u_x u^* + 4 e \lambda^2 u u^* + h u - 4 i \epsilon \lambda^2 u_x + 2 \epsilon u u_{xx} \\
&\quad -3 i \epsilon u_x + i \epsilon u_{xxx} + \frac{1}{2} i u x + \lambda u - 8 i \epsilon \lambda^3 u - 6 \epsilon \lambda u, \\
V_{21} &= h u^* + \lambda u^* - 8 i \epsilon \lambda^3 u^* + 4 i \epsilon \lambda^2 u^*_x \\
&\quad + 4 \epsilon u (u^*)_x - 6 i \epsilon u u^*_x + i \epsilon u u^*_x \\
&\quad + 6 i \epsilon u^* u_{xx} - \frac{1}{2} i u^*_x.
\end{align*}
\]

Equivalently, the Lax pair (2) and (3) reads

\[
\begin{align*}
\phi_x &= (-i \lambda \Lambda + Q) \phi, \\
\phi_t &= \left( (8 i \epsilon \lambda^4 + 6 i \epsilon \lambda^2 - i h \lambda) \Lambda + V_1 \right) \phi,
\end{align*}
\]

(4) (5)

in which \( \Lambda = \text{diag}(1, -1) \) and

\[
\begin{align*}
Q &= \begin{pmatrix} 0 & \epsilon \lambda \\ -\epsilon \lambda & 0 \end{pmatrix}, \\
V_1 &= \begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ -\tilde{V}_{21} & -\tilde{V}_{11} \end{pmatrix}.
\end{align*}
\]

In our analysis, we suppose the potential \( u \) to be vanished rapidly at infinity. It is evident to see from (4) and (5) that \( \phi \sim e^{-(i \lambda \Lambda \Delta t + (8 i \epsilon \lambda^4 + 6 i \epsilon \lambda^2 - i h \lambda) \Delta t)} \). Thus, we introduce the transformation

\[
\begin{align*}
\psi_+ &= e^{-i \lambda \Delta t + (8 i \epsilon \lambda^4 + 6 i \epsilon \lambda^2 - i h \lambda) \Delta t},
\end{align*}
\]

which enable us to convert the Lax pair (4) and (5) into

\[
\begin{align*}
\psi_x &= -i \lambda [\Lambda, \psi] + Q \psi, \\
\psi_t &= (8 i \epsilon \lambda^4 + 6 i \epsilon \lambda^2 - i h \lambda) [\Lambda, \psi] + V_1 \psi,
\end{align*}
\]

(6) (7)

where the square brackets denote the usual matrix commutator, namely \([\Lambda, \psi] = \Lambda \psi - \psi \Lambda\).

In what follows, the spectral problem (6) will be analyzed, and \( t \) will be treated as a constant. We represent the matrix Jost solutions \( \psi_\pm(x, \lambda) \) as

\[
\psi_\pm(x, \lambda) = \left( [\psi_\pm][x], [\psi_\pm]_2(x, \lambda) \right),
\]

(8)

with the boundary conditions

\[
\psi_\pm(x, \lambda) \rightarrow I_2, \quad x \rightarrow \pm \infty.
\]

(9)

The above subscripts in \( \psi \) refer to which end of the \( x \)-axis the boundary conditions are required for, and
\( I_2 \) is the identity matrix of size 2. Using the boundary conditions (9), one obtains Volterra-type integral equations

\[
\psi_-(x, \lambda) = I_2 + \int_{-\infty}^{x} e^{-i\lambda \Lambda (x-z)} Q(z) \psi_-(z, \lambda) e^{i\lambda \Lambda (x-z)} dz, \tag{10}
\]

\[
\psi_+(x, \lambda) = I_2 - \int_{x}^{+\infty} e^{-i\lambda \Lambda (x-z)} Q(z) \psi_+(z, \lambda) e^{i\lambda \Lambda (x-z)} dz. \tag{11}
\]

From (10) and (11), we find that \([\psi_+]_1, [\psi_-]_2\) and \([\psi_-]_1, [\psi_+]_2\) are analytic for \(\lambda \in \mathbb{C}_-\) and continuous for \(\lambda \in \mathbb{C}_- \cup \mathbb{R}\), while \([\psi_-]_1, [\psi_+]_2\) are analytic for \(\lambda \in \mathbb{C}_+\) and continuous for \(\lambda \in \mathbb{C}_+ \cup \mathbb{R}\), where \(\mathbb{C}_-\) and \(\mathbb{C}_+\) are the lower and upper half \(\lambda\)-plane. Applying the Abel’s identity, we reveal that \(\det \psi_\pm\) are independent of \(x\), since \(\text{tr} Q = 0\). Evaluating \(\det \psi_-\) at \(x = -\infty\) and \(\det \psi_+\) at \(x = +\infty\), we have \(\det \psi_\pm = 1\) for \(\forall x \in \mathbb{R}\). Due to matrix solutions of (4), \(\psi_- e^{-i\lambda \Lambda x}\) and \(\psi_+ e^{-i\lambda \Lambda x}\) are linearly associated by the scattering matrix \(S(\lambda)\)

\[
\psi_- e^{-i\lambda \Lambda x} = \psi_+ e^{-i\lambda \Lambda x} S(\lambda),
\]

\[
S(\lambda) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{12}
\]

We notice that \(\det S(\lambda) = 1\) due to \(\det \psi_\pm(x, \lambda) = 1\).

A matrix Riemann–Hilbert problem is associated with two matrix analytic functions. In view of the analytic properties of \(\psi_\pm\), the analytic function in \(\mathbb{C}_+\) is given by

\[
P_1(x, \lambda) = ([\psi_-]_1, [\psi_+]_2)(x, \lambda)
= \psi_- A_1 + \psi_+ A_2, \tag{13}
\]

in which

\[
A_1 = \text{diag}(1, 0), \quad A_2 = \text{diag}(0, 1). \tag{14}
\]

Because \(P_1\) solves (6), we make an asymptotic expansion for \(P_1\) at large-\(\lambda\)

\[
P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty,
\]

and substitute the asymptotic expansion into (6). Comparing the coefficients of the same powers of \(\lambda\) yields

\[
O(1): P_1^{(0)} = -i[\Lambda, P_1^{(1)}] + Q P_1^{(0)}; \quad O(\lambda): -i[\Lambda, P_1^{(0)}] = 0.
\]

Thus, we see that \(P_1^{(0)} = I_2\), namely \(P_1 \to I_2\) as \(\lambda \in \mathbb{C}_+ \to \infty\).

For construction of a matrix Riemann–Hilbert problem, we still need the analytic counterpart of \(P_2\) in \(\mathbb{C}_-\). Consider the adjoint equation of (6)

\[
\kappa_x = -i\lambda [\Lambda, \kappa] - \kappa Q. \tag{15}
\]

One can verify that the inverse matrices

\[
\psi_\pm^{-1}(x, \lambda) = \begin{pmatrix} [\psi_\pm^{-1}]_1 \\ [\psi_\pm^{-1}]_2 \end{pmatrix} (x, \lambda) = A_1 \psi_\mp^{-1} + A_2 \psi_\mp^{-1}, \tag{16}
\]

where \(A_1\) and \(A_2\) are given by (14). One can find that the asymptotic behavior of \(P_2\) turns out to be \(P_2 \to I_2\) as \(\lambda \to \infty\).

Inserting \(\psi_\pm(x, \lambda)\) into (12) gives

\[
[\psi_-]_1 = s_{11} [\psi_+]_1 + s_{21} e^{2i\lambda x} [\psi_+]_2.
\]

The \(\psi_\pm^{-1}(x, \lambda)\) are then substituted into (17) yielding

\[
[\psi_-^{-1}]_1 = r_{11} [\psi_+]^{-1}_1 + r_{12} e^{-2i\lambda x} [\psi_+]^{-1}_2.
\]

Thus, \(P_1\) and \(P_2\) can be represented as

\[
P_1 = ([\psi_-]_1, [\psi_+]_2)
= ([\psi_+]_1, [\psi_+]_2) \begin{pmatrix} s_{11} & 0 \\ s_{21} e^{2i\lambda x} & 1 \end{pmatrix},
\]

\[
P_2 = \begin{pmatrix} [\psi_-^{-1}]_1 \\ [\psi_+^{-1}]_2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} e^{-2i\lambda x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\psi_+]_1 \\ [\psi_+]_2 \end{pmatrix}.
\]

Having presented two matrix functions \(P_1\) and \(P_2\) which are analytic in \(\mathbb{C}_+\) and \(\mathbb{C}_-\), respectively, a matrix Riemann–Hilbert problem on the real axis can be formed below.
we find that

\[
P^-(x, \lambda)P^+(x, \lambda) = G(x, \lambda)
\]

\[
= \begin{pmatrix}
1 & r_{12}e^{-2i\lambda x} \\
\bar{r}_{12}e^{2i\lambda x} & 1
\end{pmatrix}, \quad \lambda \in \mathbb{R}, \quad (19)
\]

in which we have denoted that \( P_1, P_\ast \) as \( \lambda \in \mathbb{C}_+ \to \mathbb{R} \) and \( P_2, P^*_2 \) as \( \lambda \in \mathbb{C}_- \to \mathbb{R} \). And the canonical normalization conditions are given by

\[
P_1(x, \lambda) \to I_2, \quad \lambda \in \mathbb{C}_+ \to \infty; \quad P_2(x, \lambda) \to I_2, \quad \lambda \in \mathbb{C}_- \to \infty.
\]

Next, we are going to present the reconstruction formula of the potential. Since \( P_1(x, \lambda) \) solves (6), expanding \( P_1(x, \lambda) \) at large-\( \lambda \) as

\[
P_1 = I_2 + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty,
\]

and inserting this expansion into (6), we see that

\[
Q = i[A, P_1^{(1)}]
\]

\[
= \begin{pmatrix}
0 & i(P_1^{(1)})_{12} \\
2i(P_1^{(1)})_{21} & 0
\end{pmatrix}
\]

\[
\Rightarrow u = 2i(P_1^{(1)})_{12},
\]

where \((P_1^{(1)})_{12}\) is the (1,2)-entry of \( P_1^{(1)} \). By now, we have achieved the reconstruction for the potential.

### 3 Soliton solutions

For calculation of soliton solutions to Eq. (1), we make an assumption that \( \det P_1(\lambda) \) and \( \det P_2(\lambda) \) can be zeros at certain discrete locations in analytic domains. Based on \( \det \psi_\pm = 1 \), (13) and (18), as well as the scattering relation (12), we reveal that \( \det P_1(\lambda) = s_{11}(\lambda) \) and \( \det P_2(\lambda) = r_{11}(\lambda) \). That is to say, \( \det P_1(\lambda) \) and \( \det P_2(\lambda) \) have the same zeros as \( s_{11}(\lambda) \) and \( r_{11}(\lambda) \). We now need the locations of zeros. Notice that the potential matrix \( Q \) satisfies the anti-Hermitian property \( Q^\dagger = -Q \), where \( ^\dagger \) means the matrix Hermitian. Taking advantage of this property in \( Q \), one has

\[
\psi^\pm_\pm(x, \lambda^* ) = \psi^\pm_\pm(x, \lambda).
\]

After taking the Hermitian to (13) and considering (18), we find that

\[
P_1^\dagger(\lambda^*) = P_2(\lambda), \quad \lambda \in \mathbb{C}_-.
\]

and

\[
S^\dagger(\lambda^*) = S^{-1}(\lambda) \Rightarrow s^*_{11}(\lambda^*) = r_{11}(\lambda).
\]

From this, it is found that each zero \( \lambda_j \) of \( \det P_1 \) produces each zero \( \lambda_j^* \) of \( \det P_2 \). Let \( N \) be a free natural number. Generally, we assume that \( \det P_1 \) and \( \det P_2 \) have some simple zeros at \( \lambda_j \in \mathbb{C}_+ \) and \( \tilde{\lambda}_j = \lambda_j^* \in \mathbb{C}_- \), respectively. For this case, each of the kernel of \( P_1(\lambda_j) \) and \( P_2(\tilde{\lambda}_j) \) contains a single basis column vector \( v_j \) or row vector \( \hat{v}_j \):

\[
P_1(\lambda_j)v_j = 0, \quad (22)
\]

\[
\hat{v}_j P_2(\tilde{\lambda}_j) = 0, \quad (23)
\]

By taking the Hermitian to (22) and utilizing (21), we get

\[
\hat{v}_j = v_j^\dagger, \quad 1 \leq j \leq N.
\]

Then, computing \( x \)-derivative and \( t \)-derivative in (22), respectively, and using (6) and (7) yields

\[
P_1(\lambda_j) \left( \frac{\partial v_j}{\partial x} + i\lambda_j \Lambda v_j \right) = 0,
\]

\[
P_1(\lambda_j) \left( \frac{\partial v_j}{\partial t} - (8i\epsilon x)^j + 6i\epsilon x^2 - i\lambda_j^2 - ih\lambda_j)\Lambda v_j \right) = 0.
\]

Therefore, we get

\[
v_j = e^{-\left( -i\lambda_j x + (8i\epsilon x)^j + 6i\epsilon x^2 - i\lambda_j^2 - ih\lambda_j \right)t} \Lambda v_{j0}.
\]

In view of the relation (24), we see that

\[
\hat{v}_j = v_j^\dagger e^{\left( i\lambda_j x - (8i\epsilon x)^j + 6i\epsilon x^2 - i\lambda_j^2 - ih\lambda_j \right)t} \Lambda,
\]

where \( v_{j0} \) and \( v_j^\dagger \) are constants.

For presenting soliton solutions, we consider the reflectionless case, namely \( G(x, \lambda) = I_2 \). This resulting special Riemann–Hilbert problem [30] possesses the solutions

\[
P_1(\lambda) = I_2 - \sum_{k=1}^{N} \sum_{j=1}^{N} v_k \hat{v}_j \left( M^{-1} \right)_{kj},
\]

\[
P_2(\lambda) = I_2 + \sum_{k=1}^{N} \sum_{j=1}^{N} v_k \hat{v}_j \left( M^{-1} \right)_{kj},
\]

\[
\text{where } M = (m_{kj})_{N \times N} \text{ and }
\]

\[
m_{kj} = \frac{\hat{v}_k v_j}{\lambda_j - \lambda_k}, \quad 1 \leq k, j \leq N.
\]

From Eq. (25), we derive

\[
P_1^{(1)} = -\sum_{k=1}^{N} \sum_{j=1}^{N} v_k \hat{v}_j \left( M^{-1} \right)_{kj}.
\]
Combining the established results with $v_j = (a_j, b_j)^T$ and $\vartheta_j = -i\lambda_j x + (8i \epsilon \lambda_j^4 + 6i \epsilon \lambda_j^2 - i h \lambda_j) t$, the general $N$-soliton solution to Eq. (1) can be written as

$$u(x, t) = -2i \sum_{k=1}^{N} \sum_{j=1}^{N} a_k b_j e^{\vartheta_k - \vartheta_j} (M^{-1})_{kj},$$

where $M_{jk} = \frac{1}{\lambda_j - \lambda_k} (a_k^* a_j e^{\vartheta_k + \vartheta_j} + b_k^* b_j e^{-\vartheta_k - \vartheta_j}).$

In what follows, we intend to discuss one-, two-, and three-soliton solutions graphically.

(i) For $N = 1$, Eq. (1) possesses one-soliton solution

$$u(x, t) = \frac{4a_1b_1 m_1 e^{\vartheta_1 - \vartheta_1^*} - a_1b_1^* m_1 e^{\vartheta_1 - \vartheta_1^*} - a_2b_2^* m_2 e^{\vartheta_2 - \vartheta_2^*} + a_2b_2 m_2 e^{\vartheta_2 - \vartheta_2^*}}{m_1 m_2 - m_2 m_1},$$

where we have assumed $\vartheta_1 = -i\lambda_1 x + (8i \epsilon \lambda_1^4 + 6i \epsilon \lambda_1^2 - i h \lambda_1) t$ and $\lambda_1 = \lambda_{11} + i \lambda_{12}$. Further, upon assuming $b_1 = 1$ and $|a_1|^2 = e^{2\xi_1}$, then the solution (26) can be written as

$$u(x, t) = 2a_1 \lambda_{12} e^{-\xi_1} \vartheta_1 e^{\vartheta_1^*} \text{sech}(\vartheta_1^* + \vartheta_1 + \xi_1),$$

where $\vartheta_1^* + \vartheta_1 = 2\lambda_{12} x + (64 \epsilon \lambda_{11}^3 \lambda_{12} - 64 \epsilon \lambda_{11}^3 + 24 \epsilon \lambda_{11} + 2h \lambda_{12} + 4i \lambda_{11} \lambda_{12}) t$ and $\vartheta_1 - \vartheta_1^* = -2i\lambda_{11} x - 2i (-8 \epsilon \lambda_{11}^4 + 48 \epsilon \lambda_{12} - 6 + 1) \lambda_{11} + h \lambda_{11} + (-8 \epsilon \lambda_{12}^4 + 6 \epsilon \lambda_{12}) e^{-\lambda_{12}^2 t}$. Equivalently, the solution (27) reads

$$u(x, t) = 2a_1 \lambda_{12} e^{-\xi_1} \vartheta_1 e^{\vartheta_1^*} \text{sech}[2\lambda_{12} x + (64 \epsilon \lambda_{11}^3 \lambda_{12} - 64 \epsilon \lambda_{11}^3 + 24 \epsilon \lambda_{11} + 2h \lambda_{12} + 4i \lambda_{11} \lambda_{12}) t + \xi_1].$$

According to expression (28), we know that the amplitude function $|u(x, t)|$ has a sech profile. This soliton with peak amplitude $2|a_1| \lambda_{12} e^{-\xi_1}$ travels at velocity $-32 \epsilon \lambda_{11} \lambda_{12}^2 + 32 \epsilon \lambda_{11}^3 + 12 \epsilon \lambda_{11} - h - 2 \lambda_{11}$ depending on both real and imaginary parts of the spectral parameter $\lambda_1$, unlike the basic NLS equation. The phase relies on the spatial variable $x$ and temporal variable $t$ linearly. The soliton in Fig. 1 is formed for $a_1 = 1, \lambda_1 = \frac{1}{6} + i \frac{1}{2}, \xi_1 = 0, \epsilon = 1, h = 1$. From Fig. 1b, it is seen that the wave travels from right to left along the $x$-axis over time.

(ii) For $N = 2$, two-soliton solution is given by

$$u(x, t) = \frac{2i(a_1 b_2 m_{22} e^{\vartheta_1 - \vartheta_1^*} - a_1 b_2^* m_{12} e^{\vartheta_1 - \vartheta_1^*} - a_2 b_1^* m_{21} e^{\vartheta_2 - \vartheta_2^*} + a_2 b_1 m_{11} e^{\vartheta_2 - \vartheta_2^*})}{m_{11} m_{22} - m_{12} m_{21}},$$

where

$$m_{11} = \frac{1}{\lambda_1 - \lambda_2} (|a_1|^2 e^{\vartheta_1 + \vartheta_1^*} + |b_1|^2 e^{-\vartheta_1^* - \vartheta_1}),$$

$$m_{12} = \frac{1}{\lambda_2 - \lambda_1} (a_1^* a_2 e^{\vartheta_1 + \vartheta_2^*} + b_1^* b_2 e^{-\vartheta_1^* - \vartheta_2^*}),$$

$$m_{21} = \frac{1}{\lambda_1 - \lambda_2^*} (a_2^* a_1 e^{\vartheta_2 + \vartheta_1^*} + b_2^* b_1 e^{-\vartheta_2^* - \vartheta_1^*}),$$

$$m_{22} = \frac{1}{\lambda_2 - \lambda_2^*} (|a_2|^2 e^{\vartheta_2 + \vartheta_2^*} + |b_2|^2 e^{-\vartheta_2^* - \vartheta_2^*}),$$

and $\vartheta_1 = -i\lambda_1 x + (8i \epsilon \lambda_1^4 + 6i \epsilon \lambda_1^2 - i h \lambda_1) t, \lambda_1 = \lambda_{11} + i \lambda_{12}, t = 1, 2$. Through assuming $a_1 = a_2, b_1 = b_2 = 1, |a_1|^2 = e^{2\xi_1}$, the solution (29) reads

$$u(x, t) = \frac{2i(a_1 m_{22} e^{\vartheta_1 - \vartheta_1^*} - a_1 m_{12} e^{\vartheta_1 - \vartheta_1^*} - a_2 m_{21} e^{\vartheta_2 - \vartheta_2^*} + a_2 m_{11} e^{\vartheta_2 - \vartheta_2^*})}{m_{11} m_{22} - m_{12} m_{21}},$$

in which

$$m_{11} = -\frac{i e^{\xi_1}}{\lambda_2} \cosh(\vartheta_1^* + \vartheta_1 + \xi_1),$$

$$m_{12} = \frac{2e^{\xi_1}}{(\lambda_2 - \lambda_1) + i (\lambda_2 + \lambda_2^*)} \cosh(\vartheta_1^* + \vartheta_2 + \xi_1),$$

$$m_{22} = -\frac{i e^{\xi_1}}{\lambda_2} \cosh(\vartheta_2^* + \vartheta_2 + \xi_1),$$

$$m_{21} = \frac{2e^{\xi_1}}{(\lambda_1 - \lambda_2) + i (\lambda_2 + \lambda_2^*)} \cosh(\vartheta_2^* + \vartheta_1 + \xi_1).$$

In order to show interaction behaviors between two solitons, some graphs are plotted and two cases are under consideration here.

We first consider the case of two solitons traveling at different velocities. In this case, the solution parameters in (30) are first chosen as $a_1 = 1, a_2 = 1, \lambda_1 = \frac{1}{10} + i \frac{1}{2}, \lambda_2 = \frac{1}{10} + i \frac{1}{2}, \xi_1 = 0, \epsilon = 1, h = 1$. According
to these values, some plots are made to shed light on the localization and dynamical behaviors. Figure 2a shows the localized structure of this solution on $(x, t)$-plane clearly, which is a typical cross two bright solitons. It can be observed that the overtaking collision between the solitons takes place as depicted in Fig. 2b, where two solitons with different velocities move together toward the same direction along the $x$-axis. The (taller) soliton with a larger amplitude travels much faster than the other (shorter) soliton with a smaller amplitude, and the taller soliton catches up with the shorter soliton over time. Both solitons then continue to proceed in the same direction. At the moment $t = 0$, the amplitude value for two solitons reaches the maximum. Before and after the collision, their speeds and shapes are unchanged. In other words, the overtaking is an elastic interaction.

In Fig. 3, we show the head-on collision between two solitons with the parameters as $a_1 = 1, a_2 = 1, \lambda_1 = \frac{1}{10} + i \frac{1}{7}, \lambda_2 = \frac{i}{6} + i \frac{1}{7}, \xi_1 = 0, \epsilon = 1, h = 1$. The taller soliton crashes the shorter one in the opposite direction of the $x$-axis. After the collision, their amplitudes, widths, speeds, and directions are same as those before only except phase shifts, see Fig. 3b. Evidently, the head-on interaction of two solitons is also elastic.

With regard to the second case, we consider that two solitons travel at same speeds. The solution parameters in (30) are specified as $a_1 = 1, a_2 = 1, \lambda_1 = \frac{i}{7}, \lambda_2 = \frac{i}{7}, \xi_1 = 0, \epsilon = 1, h = 1$. The bound state of two solitons is shown on $(x, t)$-plane in Fig. 4, in which two solitons are localized spatially and keep together in propagation. Indeed, this solution indicates the breather, namely when two solitons propagate, the amplitude function is periodic in oscillation over time.

(iii) For $N = 3$, Eq. (1) admits three-soliton solution

$$u(x, t) = -\frac{2i}{\varrho}[a_1 b_1^* (m_{22} m_{33} - m_{23} m_{32}) e^{\varphi_1 - \varphi_1^*} - a_1 b_2^* (m_{12} m_{33} - m_{13} m_{32}) e^{\varphi_2 - \varphi_2^*} + a_1 b_3^* (m_{12} m_{23} - m_{13} m_{22}) e^{\varphi_3 - \varphi_3^*} - a_2 b_1^* (m_{21} m_{33} - m_{23} m_{31}) e^{\varphi_2 - \varphi_2^*} + a_2 b_3^* (m_{11} m_{33} - m_{13} m_{31}) e^{\varphi_2 - \varphi_2^*} - a_3 b_1^* (m_{21} m_{32} - m_{22} m_{31}) e^{\varphi_3 - \varphi_3^*} + a_3 b_2^* (m_{11} m_{32} - m_{12} m_{31}) e^{\varphi_3 - \varphi_3^*} + a_3 b_3^* (m_{11} m_{22} - m_{12} m_{21}) e^{\varphi_3 - \varphi_3^*}],$$

where $\varrho = m_{11} m_{22} m_{33} - m_{12} m_{23} m_{33} - m_{13} m_{21} m_{33} + m_{12} m_{23} m_{31} + m_{13} m_{21} m_{32} - m_{12} m_{23} m_{31}$. 

$$m_{11} = \frac{1}{\lambda_1 - \lambda_1^*} (|a_1|^2 e^{\varphi_1 + \varphi_1^*} + |b_1|^2 e^{-\varphi_1 - \varphi_1^*}),$$

$$m_{12} = \frac{1}{\lambda_2 - \lambda_1^*} (a_1^* a_2 e^{\varphi_1 + \varphi_2} + b_1^* b_2 e^{-\varphi_1 - \varphi_2}),$$

$$m_{13} = \frac{1}{\lambda_3 - \lambda_1^*} (a_1^* a_3 e^{\varphi_1 + \varphi_3} + b_1^* b_3 e^{-\varphi_1 - \varphi_3}),$$

$$m_{21} = \frac{1}{\lambda_1 - \lambda_2^*} (a_1^* a_2 e^{\varphi_2 + \varphi_1} + b_1^* b_2 e^{-\varphi_2 - \varphi_1}),$$

$$m_{22} = \frac{1}{\lambda_2 - \lambda_2^*} (|a_2|^2 e^{\varphi_2 + \varphi_2^*} + |b_2|^2 e^{-\varphi_2 - \varphi_2^*}),$$

$$m_{23} = \frac{1}{\lambda_3 - \lambda_2^*} (a_2^* a_3 e^{\varphi_2 + \varphi_3} + b_2^* b_3 e^{-\varphi_2 - \varphi_3}).$$
Fig. 2 Profiles of two-soliton solution (30) with $a_1 = 1, a_2 = 1, \lambda_1 = \frac{1}{5} + \frac{i}{2}, \lambda_2 = \frac{1}{5} + \frac{i}{2}, \xi_1 = 0, \epsilon = 1, h = 1$. a 3D plot; b $x$-curves

Fig. 3 Profiles of two-soliton solution (30) with $a_1 = 1, a_2 = 1, \lambda_1 = \frac{1}{5} + \frac{i}{2}, \lambda_2 = \frac{1}{5} + \frac{i}{2}, \xi_1 = 0, \epsilon = 1, h = 1$. a 3D plot; b $x$-curves

Fig. 4 Profiles of two-soliton solution (30) with $a_1 = 1, a_2 = 1, \lambda_1 = \frac{i}{2}, \lambda_2 = \frac{i}{2}, \xi_1 = 0, \epsilon = 1, h = 1$. a 3D plot; b $x$-curves
Fig. 5 Profiles of three-soliton solution (31) with \( a_1 = 1, a_2 = 1, a_3 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \lambda_1 = \frac{1}{10} + \frac{i}{2}, \lambda_2 = \frac{1}{10} + \frac{2i}{3}, \lambda_3 = \frac{1}{10} + \frac{i}{6}, \epsilon = 1, h = 1. \) a x-curve at \( t = -18; \) b x-curve at \( t = 0; \) c x-curve at \( t = 18. \)

Fig. 6 Profiles of three-soliton solution (31) with \( a_1 = 1, a_2 = 1, a_3 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \lambda_1 = \frac{1}{10} + \frac{i}{2}, \lambda_2 = \frac{1}{10} + \frac{2i}{3}, \lambda_3 = \frac{1}{10} + \frac{i}{6}, \epsilon = 1, h = 1. \) a x-curve at \( t = -8; \) b x-curve at \( t = 0; \) c x-curve at \( t = 18. \)

Based on these values, a special solution can be gained at once. And we can know the velocity relation for the three solitons \( S_1 < S_2 < S_3. \) Here, we have denoted that the solitons from left to right in Fig. 5a are \( S_1, S_2, \) and \( S_3. \) Figure 5 presents an elastic overtaking process among three solitons moving together toward the negative direction of the x-axis. Ultimately as time evolves, \( S_2 \) overtakes \( S_1, \) and \( S_3 \) overtakes \( S_1 \) and \( S_2. \) When \( t = 0, \) the peak amplitude is maximum.

Then, we take the parameters as \( a_1 = 1, a_2 = 1, a_3 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \lambda_1 = \frac{1}{10} + \frac{i}{2}, \lambda_2 = \frac{1}{10} + \frac{2i}{3}, \lambda_3 = \frac{1}{10} + \frac{i}{6}, \epsilon = 1, h = 1. \) Denoting that the solitons from left to right in Fig. 6a are \( S_1, S_2, \) and \( S_3. \)
On multi-soliton solutions to a generalized inhomogeneous

Fig. 7 Profiles of three-soliton solution (31) with $a_1 = 1, a_2 = 1, a_3 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \lambda_1 = \frac{i}{2}, \lambda_2 = \frac{1}{6} + \frac{i}{5}, \lambda_3 = \frac{i}{5}, \epsilon = 1, h = 1$. a 3D plot; b x-curve at $t = -15$; c x-curve at $t = -15 \frac{1}{2}$; d x-curve at $t = 0$; e x-curve at $t = 15 \frac{1}{2}$; f x-curve at $t = 15$

respective, it is found in Fig. 6 that $s_1$ moves toward the positive direction of the $x$-axis, which is opposite to the propagation direction of $s_2$ and $s_3$. As time goes on, $s_1$ collides head-on with $s_2$ and $s_3$, while $s_3$ overtakes $s_2$. After the interactions, the three solitons $s_1, s_2$, and $s_3$ continue to move along their original directions. Both head-on and overtaking interactions in the process are elastic. Additionally, the head-on interaction of two solitons in bound state with another soliton during propagation can be observed in Fig. 7. And Fig. 8 shows the evolution of bound state of three solitons.

4 Conclusion

In this study, a generalized inhomogeneous higher-order nonlinear Schrödinger equation for the Heisenberg ferromagnetic spin chain system in (1+1) -dimensions with the zero boundary condition was taken into account. A matrix Riemann–Hilbert problem was built, based on which multi-bright-soliton solutions to the examined equation were explored eventually. Moreover, the explicit forms of one-, two-, and three-bright-soliton solutions were given, and a few vivid plots were made to exhibit their spatial struc-
Fig. 8 Profiles of three-soliton solution (31) with $a_1 = 1, a_2 = 1, a_3 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \lambda_1 = \frac{i}{2}, \lambda_2 = \frac{i}{4}, \lambda_3 = \frac{3i}{4}, \epsilon = 1, h = 1$. a 3D plot; b $x$-curve at $t = -15$; c $x$-curve at $t = -\frac{5}{4}$; d $x$-curve at $t = 0$; e $x$-curve at $t = \frac{5}{4}$; f $x$-curve at $t = 15$

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