Upper Bounds on the Probability of Error in terms of Mean Divergence Measures *

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Abstract

In this paper we shall consider some famous means such as arithmetic, harmonic, geometric, root square mean, etc. Considering the difference of these means, we can establish [5, 6], some inequalities among them. Interestingly, the difference of mean considered is convex functions. Applying some properties, upper bounds on the probability of error are established in this paper. It is also shown that the results obtained are sharper than obtained directly applying known inequalities.

1 Introduction

Taneja [6, 7] considered the following inequality among some well-known means:

\[ H(a, b) \leq G(a, b) \leq N_1(a, b) \leq N_2(a, b) \leq N_3(a, b) \leq A(a, b) \leq S(a, b), \]

where

\[ A(a, b) = \frac{a + b}{2}, \]
\[ G(a, b) = \sqrt{ab}, \]
\[ H(a, b) = \frac{2ab}{a + b}, \]
\[ N_1(a, b) = \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right)^2, \]
\[ N_2(a, b) = \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right) \left( \frac{\sqrt{a + b}}{2} \right), \]
\[ N_3(a, b) = \frac{a + \sqrt{ab} + b}{3}. \]

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and

\[ S(a, b) = \sqrt{\frac{a^2 + b^2}{2}} \]

for all \( a, b \in (0, \infty) \).

The means, \( H(a, b) \), \( G(a, b) \), \( A(a, b) \) and \( S(a, b) \) are known in the literature as harmonic, geometric, arithmetic and root-square means respectively. For simplicity, we can call the mean, \( N_1 \) as square-root mean. The \( N_2(a, b) \) can be seen in Taneja [5] and the mean \( N_3(a, b) \) can be seen in Zhang and Wu [9]. Schur-geometric convexity of the means appearing in (1) can be seen in [10, 11].

The above measures can be written in terms of arithmetic and geometric means. See below

\[
N_1(a, b) = \left(A\left(\sqrt{a}, \sqrt{b}\right)\right)^2 = \frac{A(a, b) + G(a, b)}{2},
\]

\[
N_2(a, b) = A\left(\sqrt{a}, \sqrt{b}\right) \cdot \sqrt{A(a, b)},
\]

\[
N_3(a, b) = \frac{2A(a, b) + G(a, b)}{3},
\]

\[
H(a, b) = \frac{(G(a, b))^2}{A(a, b)}
\]

and

\[
S(a, b) = \sqrt{A(a^2, b^2)}.
\]

2 Inequalities among difference of means

Let us consider the following nonnegative difference of means:

\[
M_{SA}(a, b) = S(a, b) - A(a, b),
\]

\[
M_{SN_2}(a, b) = S(a, b) - N_2(a, b),
\]

\[
M_{SN_3}(a, b) = S(a, b) - N_3(a, b),
\]

\[
M_{SN_1}(a, b) = S(a, b) - N_1(a, b),
\]

\[
M_{SG}(a, b) = S(a, b) - G(a, b),
\]

\[
M_{SH}(a, b) = S(a, b) - H(a, b),
\]

\[
M_{AN_2}(a, b) = A(a, b) - N_2(a, b),
\]

\[
M_{AG}(a, b) = A(a, b) - G(a, b),
\]

\[
M_{AH}(a, b) = A(a, b) - H(a, b),
\]

\[
M_{N_2N_1}(a, b) = N_2(a, b) - N_1(a, b)
\]
and
\[ M_{N_2G}(a, b) = N_2(a, b) - G(a, b). \] (12)

The convexity of the means (2)-(12) can be seen in Taneja [5, 6, 7]. The Schur-
\[ \text{geometric convexity is given in [10].} \]

The mean difference \( M_{N_2N_1}(a, b) \) is not considered here, since it is not convex.

Taneja [6] proved the following inequalities among the difference of means:
\[ M_{SA}(a, b) \leq \frac{1}{3} M_{SH}(a, b) \leq \frac{1}{2} M_{AH}(a, b) \leq \frac{1}{4} M_{SG}(a, b) \leq M_{AG}(a, b), \] (13)
\[ \frac{1}{8} M_{AH}(a, b) \leq M_{N_2N_1}(a, b) \leq \frac{1}{3} M_{N_2G}(a, b) \leq \frac{1}{4} M_{AG}(a, b) \leq M_{AN_2}(a, b), \] (14)
\[ \frac{1}{4} M_{SA}(a, b) \leq \frac{1}{5} M_{SN_2}(a, b) \leq M_{AN_2}(a, b), \] (15)
\[ \frac{1}{2} M_{SH}(a, b) \leq M_{SN_1}(a, b) \leq \frac{3}{4} M_{SG}(a, b) \] (16)
and
\[ M_{SA}(a, b) \leq \frac{3}{4} M_{SN_3}(a, b) \leq \frac{2}{3} M_{SN_1}(a, b). \] (17)

The aim of this paper is to obtain bounds on the probability of error in terms of
the means given in (2)-(12).

3 \( f - \text{Divergence and Probability of Error} \)

Csiszár [2] have given a measure for the divergence between two probability density
functions, say \( p(x) \) and \( q(x) \). This so called \( f - \text{divergence given by} \)
\[ C_f(p, q) = \int_X f\left( \frac{p(x)}{q(x)} \right) q(x) dx. \] (18)

The function \( f(x) \), with \( x \in (0, \infty) \) is a convex function which has to satisfy the
conditions
\[ f(0) = \lim_{u \to 0} f(u); \quad 0 f\left( \frac{0}{0} \right) = 0; \quad 0 f\left( \frac{a}{c} \right) = \lim_{\varepsilon \to \infty} f\left( \frac{a}{c} \right) = a \lim_{x \to \infty} \frac{f(u)}{u}. \] (19)

It can be easily checked that \( C_f(p, q) \geq f(1) \) and that \( C_f(p, q) = f(1) \) only when
\( p(x) = q(x) \) a.e. Thus, \( C_f(p, q) - f(1) \) is a distance or divergence measure in the sense
that \( C_f(p, q) - f(1) \geq 0 \). However, it is not symmetric in \( p \) and \( q \) and in general does
not satisfy triangle inequality.

Boekee and Van der Lubbe [1] have introduced the average \( f - \text{divergence between two hypothesis} \ C_1 \) and \( C_2 \) in terms of their “a posteriori” probabilities. This average \( f - \text{divergence is defined as} \)
\[ C_f(C_1, C_2) = \int_X f\left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x)p(x) dx = E_X \left\{ f\left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}. \] (20)
If we introduce the function

\[ f^*(u) = u f \left( \frac{1 - u}{u} \right) \]

and set \( u = u(x) = P(C_2|x) \), it is easy to see from \( P(C_1|x) = 1 - P(C_2|x) \) that

\[ \overline{C}_f(C_1, C_2) = \int_X f^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x)p(x)dx = E_X \left\{ f^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}. \]

(21)

From Vajda [8] it follows that \( f^*(u) \) is convex on \([0,1]\) and is strictly convex iff \( f(u) \) is strictly convex.

3.1 A Class of Upper Bounds

In [1] it has been shown that the Bayesian probability of error can be upper bounds in terms of the average \( f - \text{divergence} \overline{C}_f(C_1, C_2) \). This upper bound is given by

\[ P_e \leq f_0 P(C_2) + f_\infty P(C_1) - \overline{C}_f(C_1, C_2), \]

(22)

where \( f_2 \) should be finite with

\[ f_0 = \lim_{u \to \infty} f(u); \quad f_1 = f(1); \quad f_2 = f_0 + f_\infty; \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \]

(23)

The above bound is valid only for every convex function \( f(u) \) which satisfies the conditions given in [19]. However, if \( f^*(u) = u f \left( \frac{1 - u}{u} \right) \) is symmetric with respect to \( u = \frac{1}{2} \) i.e., \( f^*(u) = f^*(1 - u) \), this bound can be written in a simpler form given in the following theorem.

**Theorem 3.1** The probability of error is upper bounds by \( \overline{C}_f(C_1, C_2) \), where \( f^*(u) = f^*(1 - u) \) is symmetric with respect to \( u = \frac{1}{2} \), as follows:

\[ P_e \leq \frac{1}{2 f_\infty - f_1} \left[ f_\infty - \overline{C}_f(C_1, C_2) \right], \]

(24)

provided \( f_\infty \) is finite.

If \( f_1 = f(1) = 0 \) and \( f_\infty \) is finite, then the above bound can be written as

\[ P_e \leq \frac{1}{2} \left[ 1 - \frac{1}{f_\infty} \overline{C}_f(C_1, C_2) \right]. \]

(25)

In this paper we shall apply the upper bound (25) for different divergences based on difference of means given by (2)-(12).
4 Bounds on the Probability of Error in terms of Difference Mean Divergences

In this section we shall give bounds on the probability of error in terms of the mean differences given by (2)-(12) based on Theorem 3.1.

Result 4.1. Let us consider the measure

\[ \overline{M}_{SA}(C_1, C_2) = E_X \left\{ f_{SA}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]

where

\[ f_{SA}^*(x) = x f_{SA} \left( \frac{1 - x}{x} \right) \]

with

\[ f_{SA}(x) = \sqrt{\frac{x^2 + 1}{2} - \frac{x + 1}{2}}, \quad \forall x \in (0, \infty). \]

The convexity of the function \( f_{SA}(x) \) can be seen in [5, 6]. In this case we have

\[ f_{SA}^*(x) = \frac{1}{2} \left[ \sqrt{2(x^2 + (1 - x)^2)} - 1 \right] = f_{SA}^*(1 - x), \quad (26) \]

\[ f_{SA\infty} = \lim_{x \to \infty} \frac{f_{SA}(x)}{x} = \frac{1}{2} \left( \sqrt{2} - 1 \right) \quad (27) \]

and

\[ f_{SA}(1) = 0. \quad (28) \]

Expression (25) together with (26)-(28) give the following upper bound on the probability of error

\[ P_e \leq \frac{1}{2} \left[ 1 - \left( \frac{2}{\sqrt{2} - 1} \right) \overline{M}_{SA}(C_1, C_2) \right]. \quad (29) \]

Result 4.2. Let us consider the measure

\[ \overline{M}_{SN_2}(C_1, C_2) = E_X \left\{ f_{SN_2}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]

where

\[ f_{SN_2}^*(x) = x f_{SN_2} \left( \frac{1 - x}{x} \right) \]

with

\[ f_{SN_2}(x) = \sqrt{\frac{x^2 + 1}{2} - \left( \frac{\sqrt{x} + 1}{2} \right)^2} \sqrt{x + 1 - \frac{x}{2}}, \quad \forall x \in (0, \infty). \]

The convexity of the function \( f_{SN_2}(x) \) can be seen in [5, 6]. In this case we have

\[ f_{SN_2}^*(x) = \frac{\sqrt{2}}{4} \left( 2\sqrt{x^2 + (1 - x)^2} - \sqrt{x} - \sqrt{1 - x} \right) = f_{SN_2}^*(1 - x), \quad (30) \]
\[ f_{(SN_2)}(x) = \lim_{x \to \infty} \frac{f_{SN_2}(x)}{x} = \frac{\sqrt{2}}{4} \] (31)

and

\[ f_{SN_2}(1) = 0. \] (32)

Expression (25) together with (30)-(32) give the following upper bound on the probability of error

\[ P_e \leq \frac{1}{2} \left[ 1 - \frac{4}{\sqrt{2}} \overline{M}_{SN_2}(C_1, C_2) \right]. \] (33)

Result 4.3. Let us consider the measure

\[ \overline{M}_{SN_3}(C_1, C_2) = E_X \left\{ f_{SN_3}^* \left( \frac{P(C_1|x)}{P(C_2|x)} P(C_1|x) \right) \right\}, \]

where

\[ f_{SN_3}^*(x) = x f_{SN_3} \left( \frac{1 - x}{x} \right) \]

with

\[ f_{SN_3}(x) = \frac{\sqrt{x^2 + 1}}{2} - \frac{x + \sqrt{x + 1}}{3}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{SN_3}(x) \) can be seen in [5, 6]. In this case we have

\[ f_{SN_3}^*(x) = \frac{\sqrt{2}}{2} \sqrt{x^2 + (1-x)^2} - \frac{1}{3} \left( 1 + \sqrt{x(1-x)} \right) = f_{SN_3}^*(1-x), \] (34)

\[ f_{(SN_3)}(x) = \lim_{x \to \infty} \frac{f_{SA}(x)}{x} = \frac{\sqrt{2}}{2} - \frac{1}{3} = \frac{3\sqrt{2} - 2}{6} \] (35)

and

\[ f_{SN_3}(1) = 0. \] (36)

Expression (25) together with (34)- (36) give the following upper bound on the probability of error

\[ P_e \leq \frac{1}{2} \left[ 1 - \frac{6}{3\sqrt{2} - 2} \overline{M}_{SN_3}(C_1, C_2) \right]. \] (37)

Result 4.4. Let us consider the measure

\[ \overline{M}_{SN_1}(C_1, C_2) = E_X \left\{ f_{SN_1}^* \left( \frac{P(C_1|x)}{P(C_2|x)} P(C_1|x) \right) \right\}, \]

where

\[ f_{SN_1}^*(x) = x f_{SN_1} \left( \frac{1 - x}{x} \right), \]

with

\[ f_{SN_1}(x) = \sqrt{\frac{x^2 + 1}{2}} - \left( \frac{\sqrt{x} + 1}{2} \right)^2, \forall x \in (0, \infty). \]
The convexity of the function $f_{SN_1}(x)$ can be seen in [5, 6]. In this case we have

$$f^*_{SN_1}(x) = \frac{\sqrt{2}}{2} \sqrt{x^2 + (1-x)^2} - \frac{1}{4} \left( 1 + 2 \sqrt{x(1-x)} \right) = f^*_{SN_1}(1-x), \quad (38)$$

$$f_{(SN_1)_\infty} = \lim_{x \to \infty} \frac{f_{SN_1}(x)}{x} = \frac{\sqrt{2}}{2} - \frac{1}{4} = \frac{2 \sqrt{2} - 1}{4} \quad (39)$$

and

$$f_{SN_1}(1) = 0. \quad (40)$$

Expression (25) together with (38)-(40) give the following upper bound on the probability of error

$$P_e \leq \frac{1}{2} \left[ 1 - \frac{2}{2 \sqrt{2} - 1} M_{SN_1}(C_1, C_2) \right]. \quad (41)$$

**Result 4.5.** Let us consider the measure

$$M_{SG}(C_1, C_2) = E_X \left\{ f^*_{SG} \left( \frac{P(C_1|X)}{P(C_2|X)} \right) P(C_1|X) \right\},$$

where

$$f^*_{SG}(x) = x f_{SG} \left( \frac{1-x}{x} \right) ,$$

with

$$f_{SG}(x) = \sqrt{\frac{x^2 + 1}{2}} - \sqrt{x}, \forall x \in (0, \infty).$$

The convexity of the function $f_{SG}(x)$ can be seen in [5, 6]. In this case we have

$$f^*_{SG}(x) = \frac{\sqrt{2}}{2} \sqrt{x^2 + (1-x)^2} - \sqrt{x(1-x)} = f^*_{SG}(1-x), \quad (42)$$

$$f_{(SG)_\infty} = \lim_{x \to \infty} \frac{f_{SG}(x)}{x} = \frac{\sqrt{2}}{2} \quad (43)$$

and

$$f_{SG}(1) = 0. \quad (44)$$

Expression (25) together with (41)-(43) give the following upper bound on the probability of error

$$P_e \leq \frac{1}{2} \left[ 1 - \frac{2}{2 \sqrt{2} M_{SG}(C_1, C_2)} \right]. \quad (45)$$

**Result 4.6.** Let us consider the measure

$$M_{SH}(C_1, C_2) = E_X \left\{ f^*_{SH} \left( \frac{P(C_1|X)}{P(C_2|X)} \right) P(C_1|X) \right\},$$

where

$$f^*_{SH}(x) = x f_{SH} \left( \frac{1-x}{x} \right)$$
with
\[ f_{SH}(x) = \sqrt{\frac{x^2 + 1}{2} - \frac{2x}{x + 1}}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{SH}(x) \) can be seen in \([5, 6]\). In this case we have
\[ f_{SH}^* (x) = \frac{\sqrt{2}}{2} \sqrt{x^2 + (1 - x)^2} - 2x(1 - x) = f_{SH}^*(1 - x), \tag{46} \]
\[ f_{(SH)} = \lim_{x \to \infty} f_{SH}(x) = \frac{\sqrt{2}}{2} \tag{47} \]
and
\[ f_{SH}(1) = 0. \tag{48} \]

Expression \((25)\) together with \((46)-(48)\) give the following upper bound on the probability of error
\[ P_e \leq \frac{1}{2} \left[ 1 - \frac{2}{\sqrt{2}} \overline{M}_{SH}(C_1, C_2) \right]. \tag{49} \]

Result 4.7. Let us consider the measure
\[ \overline{M}_{AN_2}(C_1, C_2) = \mathbb{E}_X \left\{ f_{AN_2}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]
where
\[ f_{AN_2}^*(x) = x f_{AN_2} \left( \frac{1 - x}{x} \right) \]
with
\[ f_{AN_2}(x) = \frac{x + 1}{2} - \left( \frac{\sqrt{x} + 1}{2} \right) \sqrt{\frac{x + 1}{2}}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{AN_2}(x) \) can be seen in \([5, 6]\). In this case we have
\[ f_{AN_2}^* (x) = \frac{1}{2} - \frac{\sqrt{2}}{4} \left( \sqrt{x} + \sqrt{1 - x} \right) = f_{AN_2}^*(1 - x), \tag{50} \]
\[ f_{(AN_2)} = \lim_{x \to \infty} f_{AN_2}(x) = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2 - \sqrt{2}}{4} \tag{51} \]
and
\[ f_{AN_2}(1) = 0. \tag{52} \]

Expression \((25)\) together with \((50)-(52)\) give the following upper bound on the probability of error
\[ P_e \leq \frac{1}{2} \left[ 1 - \left( \frac{4}{2 - \sqrt{2}} \right) \overline{M}_{AN_2}(C_1, C_2) \right]. \tag{53} \]

Result 4.8. Let us consider the measure
\[ \overline{M}_{AG}(C_1, C_2) = \mathbb{E}_X \left\{ f_{AG}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]

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where

\[ f_{AG}^*(x) = x f_{AG} \left( \frac{1-x}{x} \right), \]

with

\[ f_{AG}(x) = \frac{x+1}{2} - \sqrt{x}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{AG}(x) \) can be seen in [5, 6]. In this case we have

\[ f_{AG}^*(x) = \frac{1}{2} - \sqrt{x(1-x)} = f_{AG}^*(1-x), \quad (54) \]

\[ f_{(AG)_\infty} = \lim_{x \to \infty} \frac{f_{AG}(x)}{x} = \frac{1}{2} \quad (55) \]

and

\[ f_{AG}(1) = 0. \quad (56) \]

Expression (25) together with (58)-(60) give the following upper bound on the probability of error

\[ P_e \leq \frac{1}{2} \left[ 1 - 2 M_{AG}(C_1, C_2) \right]. \quad (57) \]

Result 4.9. Let us consider the measure

\[ \overline{M}_{AH}(C_1, C_2) = E_X \left\{ f_{AH}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]

where

\[ f_{AH}^*(x) = x f_{AH} \left( \frac{1-x}{x} \right), \]

with

\[ f_{AH}(x) = \frac{x+1}{2} - \frac{2x}{x+1}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{AH}(x) \) can be seen in [5, 6]. In this case we have

\[ f_{AH}^*(x) = \frac{1}{2} (2x-1)^2 = f_{AH}^*(1-x), \quad (58) \]

\[ f_{(AH)_\infty} = \lim_{x \to \infty} \frac{f_{AH}(x)}{x} = \frac{1}{2} \quad (59) \]

and

\[ f_{AH}(1) = 0. \quad (60) \]

Expression (25) together with (58)-(60) give the following upper bound on the probability of error

\[ P_e \leq \frac{1}{2} \left[ 1 - 2 \overline{M}_{AH}(C_1, C_2) \right]. \quad (61) \]

Result 4.10. Let us consider the measure

\[ \overline{M}_{N_2N_1}(C_1, C_2) = E_X \left\{ f_{N_2N_1}^* \left( \frac{P(C_1|x)}{P(C_2|x)} \right) P(C_1|x) \right\}, \]

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Where
\[ f'_{N_2N_1}(x) = xf_{N_2N_1}\left(\frac{1-x}{x}\right), \]
with
\[ f_{N_2N_1}(x) = \left(\frac{\sqrt{x} + 1}{2}\right) \sqrt{x} + 1 - \left(\frac{\sqrt{x} + 1}{2}\right)^2, \forall x \in (0, \infty). \]

The convexity of the function \( f_{N_2N_1}(x) \) can be seen in \([5, 6]\). In this case we have
\[ f'_{N_2N_1}(x) = \frac{\sqrt{2}}{4} (\sqrt{x} + \sqrt{1-x}) - \frac{1}{4} \left(1 + 2\sqrt{x(1-x)}\right) = f'_{N_2N_1}(1-x), \tag{62} \]
\[ f_{(N_2N_1)_\infty} = \lim_{x \to \infty} \frac{f_{N_2N_1}(x)}{x} = \frac{\sqrt{2}}{2} - \frac{1}{4} = \frac{2\sqrt{2} - 1}{4} \tag{63} \]
and
\[ f_{N_2N_1}(1) = 0. \tag{64} \]

Expression \((25)\) together with \((66)-(68)\) give the following upper bound on the probability of error
\[ P_e \leq \frac{1}{2} \left[1 - \frac{4}{2\sqrt{2} - 1} \frac{1}{M_{N_2N_1}(C_1, C_2)}\right]. \tag{69} \]

**Result 4.11.** Let us consider the measure
\[ \overline{M}_{N_2G}(C_1, C_2) = E_X \left\{ f'_{N_2G} \left(\frac{P(C_1|x)}{P(C_2|x)}\right) P(C_1|x)\right\}, \]
where
\[ f'_{N_2G}(x) = xf_{N_2G}\left(\frac{1-x}{x}\right), \]
with
\[ f_{N_2G}(x) = \left(\frac{\sqrt{x} + 1}{2}\right) \sqrt{x} + 1 - \sqrt{x}, \forall x \in (0, \infty). \]

The convexity of the function \( f_{N_2G}(x) \) can be seen in \([5, 6]\). In this case we have
\[ f'_{N_2G}(x) = \frac{\sqrt{2}}{4} (\sqrt{x} + \sqrt{1-x}) - \sqrt{x(1-x)} = f'_{N_2G}(1-x), \tag{66} \]
\[ f_{(N_2G)_\infty} = \lim_{x \to \infty} \frac{f_{N_2G}(x)}{x} = \frac{\sqrt{2}}{4} \tag{67} \]
and
\[ f_{N_2N_1}(1) = 0. \tag{68} \]

Expression \((25)\) together with \((66)-(68)\) give the following upper bound on the probability of error
\[ P_e \leq \frac{1}{2} \left[1 - \frac{4}{\sqrt{2} \overline{M}_{N_2N_1}(C_1, C_2)}\right]. \tag{69} \]
4.1 Final Remarks

(i) According to inequalities (13) and the result (57), we have

\[ P_e \leq \frac{1}{2} \left[ 1 - 2M_{AG}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - MG(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{AH}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SH}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SA}(C_1, C_2) \right]. \]  

(70)

From (70) and (45), we have

\[ P_e \leq \frac{1}{2} \left[ 1 - \sqrt{2} M_{SG}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SG}(C_1, C_2) \right]. \]  

(71)

Again from (70) and (61), we have

\[ P_e \leq \frac{1}{2} \left[ 1 - 2M_{AH}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{AH}(C_1, C_2) \right]. \]  

(72)

From the expressions (71) and (72) we observe that the results obtained here individually are sharper than that we get from the inequalities given in (13).

(ii) According to inequalities (16) and the result (45), we have

\[ P_e \leq \frac{1}{2} \left[ 1 - \sqrt{2} M_{SG}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SG}(C_1, C_2) \right]. \]  

(73)

From (73) and (41), we have

\[ P_e \leq \frac{1}{2} \left[ 1 - 2M_{SN_1}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SN_1}(C_1, C_2) \right]. \]  

(74)

Again from (73) and (49) we have

\[ P_e \leq \frac{1}{2} \left[ 1 - \sqrt{2} M_{SN_1}(C_1, C_2) \right] \leq \frac{1}{2} \left[ 1 - M_{SN_1}(C_1, C_2) \right]. \]  

(75)

From the expressions (74) and (75), we observe that the results obtained here individually are sharper than that we get from the inequalities given in (16).

Similarly, we can compare other results proving that the results obtained individually are sharper than applying directly the inequalities given in (13)-(17). More studies on probability of error having different entropy-type and generalized divergence measures can be seen in Taneja [3, 4].
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