THE NO-$\beta$ MCMULLEN GAME AND THE PERFECT SET PROPERTY

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Abstract. Given a target set $A \subseteq \mathbb{R}^d$ and a real number $\beta \in (0, 1)$, McMullen [6] introduced the notion of $A$ being an absolutely $\beta$-winning set. This involves a two player game which we call the $\beta$-McMullen game. We consider the version of this game in which the parameter $\beta$ is removed, which we call the no-$\beta$ McMullen game. More generally, we consider the game with respect to arbitrary norms on $\mathbb{R}^d$, and even more generally with respect to general convex sets. We show that for strictly convex sets in $\mathbb{R}^d$, polytopes in $\mathbb{R}^d$, and general convex sets in $\mathbb{R}^2$, that player $I$ wins the no-$\beta$ McMullen game iff $A$ contains a perfect set and player $II$ wins iff $A$ is countable. So, the no-$\beta$ McMullen game is equivalent to the perfect set game for $A$ in these cases. The proofs of these results use a connection between the geometry of the game and techniques from logic. Because of the geometry of this game, this result has strong implications for the geometry of uncountable sets in $\mathbb{R}^d$. We also present an example of a compact, convex set in $\mathbb{R}^3$ to which our methods do not apply, and also an example due to D. Simmons of a closed, convex set in $\ell_2(\mathbb{R})$ which illustrate the obstacles in extending the results further.

1. Introduction

In McMullen’s original game we have a target set $A \subseteq \mathbb{R}^d$ and a parameter $\beta \in (0, 1/3)$. The two players, which we call $I$ and $II$, alternate playing closed balls in $\mathbb{R}^d$, with $I$ making moves $B_{2^n} = B(x_{2^n}, \rho_{2^n})$ and $II$ making moves $B_{2^n+1} = B(x_{2^n+1}, \rho_{2^n+1})$. The rules of the game are that $B_{2^n+2} \subseteq B_{2^n} \setminus B_{2^n+1}$ (this is only a requirement on $I$, note that we do not require that $B_{2^n+1} \subseteq B_{2^n}$). Also, we require that $\rho_{2^n+2} \geq \beta \rho_{2^n}$ and $\rho_{2^n+1} \leq \beta \rho_{2^n}$. In a run of the game following the rules, a unique point $x = \lim_n x_n$ is determined (so $\{x\} = \bigcap_n B_{2^n}$). Then $I$ wins the run of the game iff $x \in A$.

In the no-$\beta$ McMullen game, we change only the rule restricting the radii. We instead require only that $\rho_{2n} > 0$ and that $\rho_{2n+1} < \rho_{2n}$. In this variation of the original game, we may not have that $\rho_n \to 0$, and so we declare $I$ the winner just in case $\bigcap_n B_{2n} \cap A \neq \emptyset$.

More generally, given a convex, compact set $P \subseteq \mathbb{R}^d$, we define another game in which the players alternate playing translated and scaled copies of $P$ (say by playing pairs $(x_n, \rho_n)$ to play the set $P_n = x_n + \rho_n P$). Again we require that $P_{2n+2} \subseteq P_{2n} \setminus P_{2n+1}$ and $\rho_{2n} > 0$, and $\rho_{2n+1} < \rho_{2n}$. If the players follow these rules, we declare $I$ the winner if $\bigcap_n P_{2n} \cap A \neq \emptyset$. A particular instance of this would be when the convex compact sets $P$ are the closed unit balls with respect to...
some norm \( \| \cdot \| \) on \( \mathbb{R}^n \). This gives the no-\( \beta \) McMullen game as a special instance of this more general game. We let \( G_P(A) \) denote this no-\( \beta \) McMullen game just defined. See Figure 1 for an illustration of the moves of the game.

We work throughout in the base theory \( \text{ZF} + \text{DC} \). This will allow our main results to be presented both in the \( \text{ZFC} \) context for Borel target sets and in the \( \text{AD} \) context for arbitrary target sets. In fact, our main Theorems 2, 3, 4, 8 hold as stated for arbitrary sets with no determinacy hypothesis but, for example, to apply Theorem 8 to general uncountable sets in \( \mathbb{R}^d \) we need have the perfect set property, a consequence of \( \text{AD} \).

![Figure 1. The no-\( \beta \) McMullen game](image)

A fundamental theorem of Martin [4] (see also [5]) asserts that every Borel game on an arbitrary set \( X \) is determined (without assuming \( \text{AC} \) one only gets a quasistrategy for one of the players; see §6F of [7] for further discussion). It follows easily that for any Borel target set \( A \subseteq X \), where \( X \) is a separable Banach space, that the no-\( \beta \) game is determined (or without \( \text{AC} \), quasidetermined). In Remark 1 we note that for the no-\( \beta \) McMullen game that in fact we do not need full \( \text{AC} \) but just \( \text{DC} \) to get a strategy in the case that the target set \( A \) is Borel. In Remark 6 we will note that for any \( P \subseteq \mathbb{R}^d \) for which we can prove our main theorem we have that the no-\( \beta \) game \( G_P(A) \) is determined for all \( A \subseteq \mathbb{R}^d \) from just the axiom of determinacy, \( \text{AD} \), even though it is a real game.

**Remark 1.** For Borel target sets \( A \subseteq X \) (\( X \) any separable Banach space), the determinacy of the game \( G_P(A) \) follows from Martin’s theorem if we assume \( \text{AC} \). Assuming only \( \text{DC} \), the determinacy of integer games, we also get the determinacy of \( G_P(A) \) as follows. First, it is not hard to see that the game \( G_P(A) \) is equivalent to the version \( G_P'(A) \) in which \( \text{II} \) must play scaled copies \( s + tP \) with \( s \in \mathbb{Q}^d, \ t \in \mathbb{Q} \) ("equivalent" here means that if one of the players has a winning strategy in one of the games, then that same player has a winning strategy in the other game). Without \( \text{AC} \), Martin’s theorem gives that one of the players has a winning quasistrategy for \( G_P'(A) \). If \( \text{II} \) has a winning quasistrategy for \( G_P'(A) \), then (since \( \text{II} \)’s moves are coming from a countable set) \( \text{II} \) actually has a winning strategy for \( G_P'(A) \) and thus a winning strategy for \( G_P(A) \) (using a fixed wellordering of the countable set of \( \text{II} \)’s moves, \( \text{II} \) plays at each round the least move consistent with the quasistrategy). If \( \text{I} \) has a winning quasistrategy in \( G_P'(A) \) then, using \( \text{DC} \) and the fact that \( \text{II} \)’s moves are coming from a countable set, we can turn the
quasistrategy into a strategy. Similarly, if $A$ is closed, then the game $G_P(A)$ is easily a closed game and the by Gale-Stewart and so by the argument above the game $G_P(A)$ is determined.

Removing the $\beta$ restrictions from the McMullen game gives the game some of the flavor of the Banach-Mazur game. In [2] an intermediate form of the game was considered where the $\beta$ requirement on player $I$ was removed, but the $\beta$ requirement for player $II$ is kept. In this case it was shown that the game is equivalent to the perfect set game. Recall the perfect set game, which was introduced in [1] (see also §6A of [7] for further discussion), is a game defined for any set $A \subseteq X$, for $X$ a Polish space, and has the property that $II$ wins iff $A$ is countable, and $I$ wins iff $A$ contains a perfect set. Removing the $\beta$ requirements from player $II$ as well superficially seems to give player $II$ much more control. It is perhaps somewhat surprising then that our main results say that the no-$\beta$ game is still equivalent to the perfect set game, at least for the classes of $P$ for which we have been able to prove the theorem. This result has consequences for the geometric structure of sets in $\mathbb{R}^d$, which we state below.

We say a norm $\| \cdot \|$ on $\mathbb{R}^d$ with closed unit ball $P$ is strictly convex (or we say the convex compact set $P$ is strictly convex) if for every $x, y \in P$ every point of the line segment $\ell$ between $x$ and $y$ except for $x$ and $y$ lies in the interior of $P$. Equivalently, $P$ is strictly convex if there is no proper (i.e., not a point) line segment contained in $\partial P$.

In the following Theorems, when we say that a game $G$ is “equivalent to the perfect set game” we mean simply that $I$ has a winning strategy for the game $G$ iff the target set $A$ contains a perfect set, and $II$ has a winning strategy for $G$ iff $A$ is countable.

We note that the hypotheses of the following Theorems all imply that the convex set $P \subseteq \mathbb{R}^d$ has a non-empty interior. This is necessary as the theorem does not hold in general without this assumption.

The first class of $P$ for which we are able to prove our result is the class of strictly convex sets.

**Theorem 2.** Let $P$ be a non-degenerate (i.e., more than one point) compact, strictly convex set in $\mathbb{R}^d$. Then the no-$\beta$ McMullen game $G_P(A)$ with respect to $P$ and target set $A$ is equivalent to the perfect set game.

In particular, if $\| \cdot \|$ is a strictly convex norm on $\mathbb{R}^n$ then the no-$\beta$ McMullen game with respect to the closed unit ball for the norm is equivalent to the perfect set game.

The second class of $P \subseteq \mathbb{R}^d$ for which we can establish our result are the convex $d$-polytopes (the compact convex sets $P$ which are the intersections of finitely many closed half-spaces and have non-empty interior). For example, convex polygons in $\mathbb{R}^2$ and convex polyhedra in $\mathbb{R}^3$.

**Theorem 3.** Let $P$ be a compact, convex $d$-polytope in $\mathbb{R}^d$. Then the no-$\beta$ McMullen game $G_P(A)$ is equivalent to the perfect set game.

Finally, we are able to establish the result for arbitrary compact, convex set in $\mathbb{R}^2$.

**Theorem 4.** Let $P$ be a compact, convex set in $\mathbb{R}^2$ with non-empty interior. Then the no-$\beta$ McMullen game $G_P(A)$ is equivalent to the perfect set game.
In showing the equivalence of the game \(G_P(A)\) with the perfect set game, two of the four implications are easy. If \(A\) is countable, then easily \(\mathbf{II}\) as a winning strategy by consecutively deleting the members of \(A\) in each round. If \(\mathbf{I}\) has a winning strategy, then \(A\) contains a perfect set. In fact, this holds in a general Banach space which we state in the next fact.

**Fact 5.** Let \(P\) be a bounded, closed, convex set in a Banach space \(X\), and let \(A \subseteq X\). If \(\mathbf{I}\) has a winning strategy for the no-\(\beta\) McMullen game \(G_P(A)\), then \(A\) contains a perfect set.

*Proof.* Let \(\sigma\) be a winning strategy for \(\mathbf{I}\) in \(G_P(A)\). Without loss of generality we may assume that \(0 \in P\) (where \(0\) is the zero vector of the Banach space). We make use of the following observation. If \(Q = y + \delta P \subseteq P\setminus (1 - \epsilon)P\), then \(\delta \leq \epsilon\). Let \(L\) be the line in \(X\) through \(0\) and \(y\). Let \(m = |L \cap P|\). Then \(|L \cap (1 - \epsilon)P| = (1 - \epsilon)m\) and \(|L \cap Q| = \delta m\). The second equality follows as \(Q \cap L = Q \cap (y + L) = (y + \delta P) \cap (y + L) = y + (\delta P \cap L)\). But then \(m = |L \cap P| \geq |L \cap Q| + |L \cap (1 - \epsilon)P| = \delta m + (1 - \epsilon)m\), and so \(\delta \leq \epsilon\).

We build a perfect set contained as \(A\) as follows. Let \(\mathbf{I}\) play \(P_0 = x_0 + \rho_0 P\) according to \(\sigma\). In general suppose we have \(P_s = x_s + \rho_s P\) defined for some \(s \in 2^{<\omega}\). Let \(P_{s,0}^s \subseteq P_s\) have a scaling factor \(> \frac{1}{2}\rho_s\). Let \(P_{s,0} = x_{s,0} + \rho_{s,0} P\) be \(\mathbf{I}\)'s response to \(P_{s,0}^s\). By the above observation we have that \(\rho_{s,0} < \frac{1}{2}\rho_s\). Let \(P_{s,1} = P_{s,0}\), and let \(P_{s,1}^s\) be \(\mathbf{I}\)'s response to \(P_{s,1}^s\). Let \(P_{s,1}^s \subseteq P_{s,1}\) have a scaling factor greater than one-half of that for \(P_{s,1}^s\). Finally, let \(P_{s,1} = P_{s,1}\) be \(\mathbf{I}\)'s response to \(P_{s,1}^s\). Again by the observation, \(P_{s,1}^s\) has scaling factor \(< \frac{1}{2}\rho_s\). So, \(P_{s,0}^s, P_{s,1}^s \subseteq P_s, P_{s,0} \cap P_{s,1} = \emptyset\), and \(\rho_{s,0}, \rho_{s,1} < \frac{1}{2}\rho_s\). For each \(\zeta \in 2^{<\omega}\), let \(\{x_\zeta; \zeta \in 2^{<\omega}\}\) is a perfect set contained in \(A\).

In the proofs of Theorems 2, 3 and 4 we will actually show that if \(\mathbf{II}\) has a winning strategy in \(G_P(A)\) then \(A\) is countable. The following remark shows that this is enough to obtain the full statements of these theorems.

**Remark 6.** Let \(P \subseteq \mathbb{R}^d\) be as in Theorems 2, 3 or 4. Suppose we have shown that for any target set \(A\) if \(\mathbf{II}\) wins \(G_P(A)\) then \(A\) is countable. If \(A\) is countable, easily \(\mathbf{II}\) wins \(G_P(A)\) and so \(\mathbf{II}\) wins \(G_P(A)\) iff \(A\) is countable. If \(\mathbf{I}\) has a winning strategy then by Fact 3 \(A\) contains a perfect set. On the other hand, if \(A\) contains a perfect set \(F\), then by Remark 1 the game \(G_P(F)\) is determined. Since \(F\) is uncountable, \(\mathbf{II}\) cannot have a winning strategy in \(G_P(F)\) by assumption, and so \(\mathbf{I}\) has a winning strategy for \(G_P(F)\). A winning strategy for \(G_P(F)\) is easily a winning strategy for \(G_P(A)\) as well.

Also, for any target set \(A \subseteq \mathbb{R}^d\) the determinacy of the game \(G_P(A)\) follows just from AD, even though these are real games. To see this, let \(A \subseteq \mathbb{R}^d\). By AD, either \(A\) is countable or else \(A\) contains a perfect set. If \(A\) is countable, then as above easily \(\mathbf{II}\) wins \(G_P(A)\). If \(A\) contains a perfect set \(F\), then also as above \(\mathbf{I}\) wins \(G_P(F)\) and thus \(G_P(A)\).

To state the geometric consequences of these results, we introduce the notion of the derivative of a set \(A \subseteq \mathbb{R}^d\) with respect to the set \(P\). The notion is somewhat analogous to the usual Cantor-Bendixson derivative for sets, which is obtained by iterating the operation of taking limit points (and taking intersections at limit stages).
Definition 7. Let \( A \subseteq \mathbb{R}^d \), and \( P \subseteq \mathbb{R}^d \) a compact, convex set. Then the derivative \( A'_p \) of \( A \) with respect to \( P \) is defined by: \( A \setminus A'_p \) is the union of all open balls which do not contain a good copy of \( P \). We say \( Q = x + \rho P \) (\( Q \) is a translated, scaled copy of \( P \)) is a good copy of \( P \) if the set of points \( B \subseteq A \cap \partial Q \) which are limit points of either \( A \cap Q^\circ \) or \( A \cap \partial Q \cap H \) where \( H \) is a supporting hyperplane for \( Q \) in \( \mathbb{R}^d \), cannot be covered by a strictly smaller copy of \( P \) (i.e., some \( x' + \rho' P \) where \( \rho' < \rho \)).

Note that the derivative operation is monotone, that is, if \( A \subseteq B \) then \( A'_p \subseteq B'_p \).

As usual, we iterate this notion of derivative, taking intersections at limit ordinals. This produces a sequence of sets \( A = A^0_p \supseteq A^1_p \supseteq \cdots \supseteq A^\infty_p \) where \( A^\infty_p = (A^\infty_p)' \). The usual argument shows that starting from any set \( A \subseteq \mathbb{R}^d \), the derivative stabilizes at a countable ordinal, that is, there is a countable ordinal \( \alpha \) so that \( A^\infty_p = A^\alpha_p \). As usual, if we start with a closed set \( A \), then all of the \( A^\alpha_p \) are closed.

Note that for any set \( A, A'_p \subseteq A' \), where \( A' \) here denotes the usual Cantor-Bendixson derivative. So if \( F \) is a closed set and \( F^\infty_p = F \) then not only is \( F \) perfect in the usual sense, but \( F \) has strong geometric regularity with respect to the shape \( P \).

In section 4 we get the following geometric consequence of the above theorems and a derivative analysis.

Theorem 8. Suppose \( P \subseteq \mathbb{R}^d \) is a compact, convex set such that the hypothesis of Theorems 2, 3, or 4 is satisfied. Then if \( A \subseteq \mathbb{R}^d \) contains a perfect set then \( A^\infty_p \neq \emptyset \).

As a consequence, if \( A \subseteq \mathbb{R}^d \) is any uncountable Borel set, then \( A \) contains a perfect set \( F \) such that \( F^\infty_p = F \). So, for every point \( x \in F \) and every neighborhood \( U \) of \( x \), \( U \) contains a good copy of \( P \). Similarly, if we assume AD then this result holds for any \( A \subseteq \mathbb{R}^d \).

To illustrate this statement with an example, consider the case where \( P \) is the standard closed ball in \( \mathbb{R}^2 \) (i.e., we use the standard Euclidean norm on \( \mathbb{R}^2 \)). Let \( A \subseteq \mathbb{R}^2 \) be any uncountable Borel set. Then \( A \) contains a perfect set \( F \) such that for any \( x \in F \) and any neighborhood \( U \) of \( x \), there is a good closed ball \( B = B(y,r) \subseteq U \) for \( F \), that is, the points of \( F \cap B^\circ \) which are limit points of \( F \cap B^\circ \) are not contained in any strict half of \( B \). More precisely, these points do not lie strictly on one side of a line through \( y \), or equivalently, \( y \) is in the convex hull of these points. See Figure 2.

We introduce some notation that we will use for the rest of the paper. Let \( P \) be a compact, convex set in \( \mathbb{R}^d \) with non-empty interior.

Definition 9. Let \( \ell \subseteq P \) be a line segment in \( \mathbb{R}^d \). We say \( \ell \) is maximal with respect to \( P \) if there is no translated enlarged copy \( \ell' = x + s \ell \), with \( s > 1 \), and \( \ell' \subseteq P \).

Note that \( \ell \) being maximal is equivalent to saying that \( \ell \not\subseteq P^\circ \) for any translated smaller copy \( P' = x + sP \), where \( s < 1 \), of \( P \).

Remark 10. Note the simple but important fact that there is a lower bound \( \epsilon(P) > 0 \) on the lengths of the maximal line segments with respect to \( P \). This is immediate from the fact that \( P^\circ \neq \emptyset \).

Given distinct points \( x, y \in \mathbb{R}^d \), let \( [x,y]_P = w + sP \) be a scaled copy of \( P \) containing both \( x, y \) and such that no copy \( w' + s'P \) with \( s' < s \) contains both \( x \)
and $y$. In general, $[x, y]_P$ is not unique, and we just fix a choice for it (this can be done without any form of AC; the relation $R((x, y), (w, s))$ satisfying this definition is a Borel relation with compact sections). In the case where $P$ is strictly convex, there is a unique choice for $[x, y]_P$. In section 4, however, we will be more particular in our choice of $[x, y]_P$. Note that $x, y$ are always in the boundary of $[x, y]_P$, and $\ell_{x,y}$ is maximal in $[x, y]_P$.

**Figure 2. A good copy of the unit ball in $\mathbb{R}^2$**

2. The strictly convex case

In this section we prove our main result, Theorem 2, in the case where $P$ is a compact, non-degenerate (at least two points) strictly convex set in $\mathbb{R}^d$. The proof will make use of an idea from logic involving elementary substructures, and this idea will be used in the other cases as well.

Throughout the rest of this section $P$ will denote a compact, non-degenerate, strictly convex set in $\mathbb{R}^d$. This implies $P^o \neq \emptyset$.

For $x \in \partial P$, let $\mathcal{H}_P(x)$ denote the collection of supporting hyperplanes for $P$ at $x$. That is, $H \in \mathcal{H}_P(x)$ iff $H \subset \mathbb{R}^d$ is an affine hyperplane containing $x$ with $P$ entirely contained (not-strictly) on one side of $H$. If $H$ is an affine hyperplane in $\mathbb{R}^d$ and $u$ a vector in $\mathbb{R}^d$, then by $|H \cdot u|$ we mean the absolute value of the dot-product $|u \cdot n_H|$ where $n_H$ is a unit normal vector to $H$. If $\ell$ is a line segment, we also use $|\ell \parallel H|$ to denote the dot-product where we interpret $\ell$ as a vector (the sign ambiguity doesn’t matter due to the absolute value).

**Definition 11.** Let $x \in \partial P$. Let $M_P(x)$ be the collection of line segments $\ell$ which are maximal with respect to $P$ and which have an endpoint equal to $x$ (note that both endpoints of $\ell$ lie in $\partial P$). Then $\delta_p(x) = \inf\{|\ell \parallel H|: \ell \in M_P(x) \land H \in \mathcal{H}_P(x)\}$. Let $M_P = \bigcup_{x \in \partial P} M_P(x)$, and $\mathcal{H}_P = \bigcup_{x \in \partial P} \mathcal{H}_P(x)$. Let $\delta_P = \inf\{\delta_P(x): x \in \partial P\}$.

We topologize $M_P$ using the Hausdorff metric on the set of line segments in $\mathbb{R}^d$. Similarly, we topologize $\mathcal{H}_P(x)$ using the Hausdorff metric on the unit inward (toward $P$) normal vectors for the hyperplanes.

**Lemma 12.** $M_P$ is a closed set, and in particular $M_P(x)$ is closed for all $x \in \partial P$. 
is a line segment \( \ell \) large enough \( n \) \( \ell \subseteq P \) is contained in \( M \). But then \( C^\circ \) contains a translated (not enlarged) copy of \( \ell \). For \( n \) large enough \( \ell_n \) will then also lie in \( C^\circ \), a contradiction to \( \ell_n \) being maximal. \( \Box \)

**Lemma 13.** For any compact, convex set \( P \subseteq \mathbb{R}^d \), and any point \( x \in \partial P \), there is a line segment \( \ell \in M_P(x) \) and a supporting hyperplane \( H \in \mathcal{H}_P(x) \) such that \( \delta_P(x) = \| \ell \cdot H \| \).

**Proof.** This follows immediately from the fact that \( M_P(x) \) and \( \mathcal{H}_P(x) \) are compact and the map \( (\ell, H) \mapsto \frac{\ell}{\| \ell \|} \cdot H \) is continuous. \( \Box \)

**Lemma 14.** Let \( P \) be a compact, non-degenerate, strictly convex set. Then \( \delta_P > 0 \).

**Proof.** Suppose \( \delta_P = 0 \). Let \( \ell_n = \ell_{x_n, y_n} \subseteq P \) be maximal line segments with respect to \( P \), and \( H_n \in \mathcal{H}_P(x_n) \) be supporting hyperplanes with \( \frac{\ell_n}{\| \ell_n \|} \cdot H_n \to 0 \). Without loss of generality we may assume that \( \ell_n \to \ell = \ell_{x,y} \) (with \( x \neq y \)) and the \( H_n \) converge to \( H \), a supporting hyperplane for \( P \) through the point \( x \). By Lemma 13, \( \ell \in M_P(x) \). Also, \( H \in \mathcal{H}_P(x) \) and so by continuity of the dot product we have that \( \frac{\ell}{\| \ell \|} \cdot H = 0 \). So, \( \ell \subseteq H \cap P \), but \( H \cap P \subseteq \partial P \), and so \( \ell \subseteq \partial P \). This contradicts the strict convexity of \( P \). \( \Box \)

Suppose now \( P \subseteq \mathbb{R}^d \) is a compact, strictly convex set. Consider the no-\( \beta \) McMullen game for target set \( A \subseteq \mathbb{R}^d \) with respect to the set \( P \). If player I wins this game, then by Fact 5 we have that \( A \) contains a perfect set. For the rest of the argument we assume that II has a winning strategy \( \tau \). We must show that \( A \) is countable. Suppose towards a contradiction that \( A \) is uncountable. Let \( m = \min \{ k \in \omega : \exists H \text{ an affine subspace, } \dim H = k, \text{ and } A \cap H \text{ is uncountable} \} \).

Let \( H \) be a \( m \)-dimensional affine subspace of \( \mathbb{R}^d \) so that \( H \cap A \) is uncountable. Without loss of generality, we can assume that \( A \subseteq H \).

Next consider the tree \( T \) of all even length positions in the game which are consistent with \( \tau \), and for each \( x \in A \), let \( T_x \) be the subtree of \( T \) consisting only of positions \( p = (B_0, B_1, \ldots, B_{n+1}) \) so that \( x \in B_n \setminus B_{n+1} \), i.e., the point \( x \) is well-inside player I’s last move, and was not yet deleted by \( \tau \). Since \( \tau \) is a winning strategy for II, no full infinite run of the game can exist which results in a point \( x \in A \), and thus for each \( x \in A \), the tree \( T_x \) is wellfounded.

Let \( N \) be a countable elementary substructure containing \( \tau \) and \( H \), and let \( T^N = T \cap N \) and for each \( x \in A \) let \( T^N_x = T_x \cap N \). Note that \( T \) is definable from \( N \) and so \( T \in N \), but \( T_x \notin N \) in general. The set of all positions in \( N \) is countable, and so the set of all positions which occur as terminal nodes in some tree \( T^N_x \) for \( x \in A \) is also countable. Since \( A \) is uncountable by hypothesis, we can find some fixed position \( p = (B_0, \ldots, B_{n+1}) \) and some uncountable subset \( A' \subseteq A \) so that \( p \) is terminal in \( T^N_x \) for every \( x \in A' \).

Note that by the choice of \( A' \), we have that \( A' \subseteq B^n \), and since \( A' \) is uncountable, it must have a strong limit point \( x_0 \in A' \). Since \( x_0 \in B^n \setminus B_{n+1} \), there is some small enough \( \epsilon \) so that \( B(x_0, \epsilon) \subseteq B^n \setminus B_{n+1} \). Next, we note that since \( A' \subseteq A \subseteq H \), and \( x_0 \) is a strong limit point of \( A' \), we know that \( B(x_0, \epsilon) \cap H \cap A' \) is uncountable,
and \( p^\tau B(x_0, \epsilon) \) is a legal position in the game. Let \( D \) be a small enough rational ball containing \( x_0 \) so that for any \( x, y \in D, [x, y] \subseteq B(x_0, \epsilon) \). Note that \( \tau \) induces a one-round strategy in the no-\( \beta \) McMullen game on \( D \).

We will need the following two lemmas to complete the proof. For the first lemma, we introduce the following terminology.

Given hyperplanes \( H_1, \ldots, H_k \) in \( \mathbb{R}^d \), we define the family of cones which they induce as follows. For each hyperplane \( H \), we let \( H^+ \) and \( H^- \) be the two closed half-spaces in \( \mathbb{R}^d \) determined by \( H \). For each \( p \in \{\pm 1\}^k \), let \( C(p) = \bigcap_i H_i^{p(i)} \), where by \( H_i^{p+1} \) mean \( H^+ \) and by \( H_i^{-1} \) we mean \( H^- \). We let \( \tau \) be the sequence \( -p \) and \( \tau \) be the “positive cone” associated to the \( H_i \), and \( C(-p) = C(\tau) \) be the “negative cone” associated to the \( H_i \). In this manner every collection \( H_1, \ldots, H_k \) of hyperplanes induces a collection \( C_1, \ldots, C_\ell \) of cones in \( \mathbb{R}^d \), and for each of these cones \( C_i \) we have defined the positive part \( C_i^+ \) and negative part \( C_i^- \). The following “cone lemma” will be proved in the next section.

**Lemma 15.** Suppose \( A \subseteq \mathbb{R}^d \) is uncountable. Suppose that \( H_1, \ldots, H_k \) are distinct hyperplanes inducing the family of cones \( C_1 \ldots C_\ell \). Then there is a cone \( C_i \) and an uncountable subset \( A' \subseteq A \) so that for every point \( x \in A', x \) is a strong limit both from \( C_i^+(x) \cap A' \) and \( C_i^-(x) \cap A' \).

Using Lemma 15, we next prove the following lemma concerning elementary substructures and their relation to the no-\( \beta \) McMullen game.

**Lemma 16.** Suppose \( \tau \) is a one-round strategy in the no-\( \beta \) McMullen game for player II on the ball \( B \subseteq \mathbb{R}^d \). Then for any elementary substructure \( N \) containing \( \tau \) and \( P \), and for any uncountable set \( A \subseteq B \) which contains at most countably many points from any affine hyperplane, there is some legal move (for I) \( Q = w + sP \subseteq B \) from \( N \) so that \( A \cap Q \neq \emptyset \).

**Proof of Lemma 16.** Let \( D \subseteq B \) be a rational ball so that \( A \cap D \) is uncountable and for every \( x, y \in D, [x, y] \subseteq B \). Recall \( P \) is compact and strictly convex. Let \( \delta = \delta_P > 0 \) be as in Lemma 15. Let \( H_1, \ldots, H_k \) be distinct hyperplanes in \( \mathbb{R}^d \) with corresponding cones \( C_1, \ldots, C_\ell \), such that for any of the cones \( C_i \) and any two unit vectors \( u, v \in C_i \), \( 1 - |u \cdot v| < \frac{1}{2}\delta^2 \). This is easily done by taking sufficiently many of the hyperplanes. Fix the cone \( C = C_i \) from Lemma 15 along with the uncountable set \( A' \subseteq A \). Let \( x \in A' \cap D \). Let \( y \in A' \cap D \) be such that \( v \) is the vector from \( x \) to \( y \). Then \( v \in C_i^+ \). Let \( R = [x, y]_P \), so \( R \subseteq B \). Let \( R' = \tau(R) \). Since \( R' \) is a smaller scaled copy than \( R \), it follows from the definition of \( [x, y]_P \) that at least one of \( x, y \) is not in \( R' \). The rest of the argument is symmetrical between \( x \) and \( y \), so we assume that \( x \in R \setminus R' \).

Say without loss of generality that \( v \in C_+ \). By definition of \( [x, y]_P \), \( \ell_{x,y} \in M_R(x) \). Since \( \delta_P(x) \geq \delta_P = \delta \), it follows from the definition of \( \delta_P(x) \) that for any supporting hyperplane \( H \) for \( R \) at \( x \), we have that \( \frac{\ell_{x,y} - e_x}{\|e_x\|} \cdot H > \delta \). This implies that for any point \( \ell_{x,w} \in \ell_{x,y} \) that if \( \ell' = \ell_{x,w} \), then \( \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \cdot H > 0 \). This is because

\[
\left( \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \cdot H \right) > 1 - \frac{1}{2}\delta^2, \quad \text{and} \quad \left| \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \cdot H \right| \geq \left| \frac{\ell_{x,y} - e_x}{\|e_x\|} \cdot H \right| - \left| \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \right|, \quad \text{and} \quad \left( \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \right)^2 = 2(1 - \frac{1}{2}\delta^2) < \delta^2.
\]

So, \( \frac{\ell_{x,y} - \ell'}{\|\ell_{x,y} - \ell'\|} \cdot H > 0 \).

In particular, for any point \( z \in C_+(x) \) which is sufficiently close to \( x \) we have that \( z \in R^0 \). To see this, we first observe that for every \( z \in C_+(x) \) there is a \( z' \in \ell_{x,z} \),
such that $\ell_{x,z'} \subseteq R$. If not, then $\ell_{x,z} \cap R = \{x\}$. There is then a supporting hyperplane $H$ for $R$ which contain $x$ and $\ell_{x,z} \setminus \{x\}$ lies (non-strictly) on the other side of $H$ from $R$. Since $y,z \in C^+(x)$ and $C^+(x)$ is convex, the line segment $\ell_{y,z}$ lies in $C^+(x)$. Thus there is a point $w \in \ell_{y,z} \subseteq C^+(x)$ with $w \in H$. This contradicts the above fact. Next we observe that there is a $z'' \in \ell_{x,z}$ with $\ell_{x,z''} \subseteq R^\circ$. It is enough to see that there is a $z'' \in \ell_{x,z}$ with $z'' \in R^\circ$. If this failed, then $\ell_{x,z'}$ is contained in $\partial R$. Then there is a supporting hyperplane $H$ for $R$ which contains $\ell_{x,z'}$, a contradiction as $z' \in C^+(x)$.

Fix a rational ball $B_1$ about $x$ so that $R' \cap B_1 = \emptyset$. Let $z \in B_1 \cap C^+(x) \cap A'$, with $z \in R^\circ$, which we can do by the choice of $C$ as $x \in A'$. Let $B_2 \subseteq B_1$ be a small enough rational ball about $z$ so that $B_2 \subseteq [x,y]_P = R$. See Figure 3 for an illustration of these objects.

Consider the statement

$$\varphi(x,y) = (x,y \in D) \land (\tau([x,y]_P) \cap B_1 = \emptyset) \land (B_2 \subseteq [x,y]_P).$$

By elementarity of $N$ we get that

$$\exists x', y' \in N \ (x', y' \in D) \land (\tau([x', y']_P) \cap B_1 = \emptyset) \land (B_2 \subseteq [x', y']_P).$$

Let $Q = [x', y']_P$. Then we have $z \in [x', y']_P$ since $z \in B_2 \subseteq [x', y']_P$ and also $z \notin \tau([x', y']_P)$ since $z \in B_2 \subseteq B_1$ and $\tau([x', y']_P) \cap B_1 = \emptyset$. Also, $x', y' \in D$ so $Q = [x', y']_P \subseteq B$. Since $x', y' \in N$, $Q \in N$. As $z \in A' \cap (Q \setminus \tau(Q))$, this proves the lemma.

By Lemma 16, we know that there is some move $Q \subseteq D$, $Q \in N$, so that $A' \cap Q \setminus \tau(Q)$ is nonempty. This contradicts that the position $p$ was terminal for every $x \in A'$. This completes the proof of Theorem 2.

\[ \square \]
3. Cone Lemma

In this section we first prove the cone lemma, Lemma 14, which for convenience we restate. We then prove another variation of the theorem which abstracts part of the argument and does not mention cones. We will make use of this other variation as well in the proof of Theorem 3.

**Lemma.** Suppose $A \subseteq \mathbb{R}^d$ is uncountable. Suppose that $H_1, \ldots, H_k$ are distinct hyperplanes inducing the family of cones $C_1, \ldots, C_\ell$. Then there is a cone $C_i$ and an uncountable subset $A' \subseteq A$ such that for every point $x \in A'$, $x$ is a strong limit both from $C_i^+(x) \cap A'$ and $C_i^-(x) \cap A'$.

**Proof.** Suppose $A \subseteq \mathbb{R}^d$ is uncountable, and the hyperplanes $H_1, \ldots, H_k$ and given which induce the cones $C_1, \ldots, C_\ell$ in $\mathbb{R}^d$.

Without loss of generality we may assume that $A \cap H$ is countable for every affine hyperplane $H \subseteq \mathbb{R}^d$. To see this, let $S \subseteq \mathbb{R}^d$ be an affine subspace of $\mathbb{R}^d$ of minimal dimension such that $A \cap S$ is uncountable. By an affine translation we may identify $S$ with $\mathbb{R}^m$ for some $m = \dim(S) < d$. This maps the hyperplanes $H_1, \ldots, H_k$ to new hyperplanes $H'_1, \ldots, H'_k$ and gives new cones $C'_1, \ldots, C'_\ell$. By the same affine translation the set $A$ moves to the uncountable set $A'$. $A'$ now intersects every hyperplane in $\mathbb{R}^m$ in a countable set. Proving the result for $A'$ then easily gives the result for $A$ by moving back under the affine translation. So, we assume for the rest of the argument that $A \cap H$ is countable for every affine hyperplane $H \subseteq \mathbb{R}^d$.

Let $A_0 = A$. Consider the first cone $C_1$. We ask if there is an uncountable $A_0' \subseteq A_0$ such that for every $x \in A_0'$ there is a neighborhood $U$ of $x$ such that either $U \cap C_1^+(x) \cap A_0'$ is countable or $U \cap C_1^-(x) \cap A_0'$ is countable. If this is the case then we may thin $A_0'$ out to an uncountable set $A_1 \subset A_0'$ and fix an integer $p(1) \in \{\pm 1\}$ so that for all $x \in A_1$, there is a neighborhood $U$ of $x$ such that $U \cap C_1^{p(1)}(x) \cap A_1$ is countable. That is, by thinning from $A_0'$ to $A_1$ we fix the side of the cone $C_1$ which is weakly isolating $x$ in $A_1$ with respect to $C_1$. We continue in this manner for $\ell$ stages, assuming that we can at each stage go from $A_k$ to $A_{k+1} \subset A_k$ using the cone $C_k$. We then define $A_{k+1} \subset A_k$ and $p(k)$ exactly as in the first step. Assuming this construction continues for all $\ell$ steps, we end with an uncountable set $A_{\ell} \subset A$ and a “cone pattern” $p \in \{\pm 1\}^\ell$.

In this case, for each $x \in A_{\ell}$ we let $B_x$ be a rational ball containing $x$ such that $\forall i \in \mathbb{N}, B_x \cap C_i^{p(i)}(x) \cap A_{\ell}$ is countable. We may fix a rational ball $B$ so that $B_x = B$ for an uncountable subset $A_{\ell}'$ of $A_{\ell}$. Let $x, y \in A_{\ell}'$ be distinct strong limit points of $A_{\ell}'$, so in particular $x, y \in B = B_x = B_y$. Since the set of points of $A_{\ell}'$ which are strong limit points of $A_{\ell}'$ is uncountable, from our assumption on $A$ we may assume that the line $\ell_{x, y}$ between $x$ and $y$ does not lie on any of the $H_1, \ldots, H_k$. For one of the cones, say $C_1$, we have that $y \in C_1^-(x)$ (and so also $x \in C_1^+(y)$). Say without loss of generality $y \in C_1^+(x)$, and so $x \in C_1^-(y)$. If $p(i) = +1$, then we have a contradiction from $y \in C_1^+(x)$, and so $y$ is in the interior of $C_1^-(x)$, and the facts that $x \in A_{\ell}'$ and $y$ is a strong limit point of $A_{\ell}'$. Similarly, if $p(i) = -1$ we have a contradiction from the fact that $x \in C_1^-(y)$ and $y \in A_{\ell}'$.

Thus, the above construction cannot proceed through all $\ell$ cones. Let $i$ be the least stage where the construction fails. This means that there is an uncountable set $A' = A_{i-1} \subset A$ such that for any uncountable $A'' \subset A'$ there is an $x \in A''$ such that $x$ is a strong limit of both $C_i^+(x) \cap A''$ and $C_i^-(x) \cap A''$. Let $E \subseteq A'$ be the set of $x \in A'$ such that there is a neighborhood $U$ of $x$ such that either $U \cap C_i^+(x) \cap A'$
is countable or \( U \cap C_+^i(x) \cap A' \) is countable. \( E \) is countable, as otherwise we could fix a rational ball \( B \) and an uncountable \( E' \subseteq E \cap B \) such that for all \( x \in E' \), either \( C_+^i(x) \cap E' \) is countable or \( C_-^i(x) \cap E' \) is countable. This contradicts that the construction failed at step \( i \), as we could take \( A_i = E' \). So, \( A' \setminus E \) is uncountable. From the definition of \( A' \) and the fact that \( E \) is countable we also have that for any \( x \in A' \setminus E \), \( x \) is a strong limit point of both \( (A' \setminus E) \cap C_+^i(x) \) and \( (A' \setminus E) \cap C_-^i(x) \) (note that being a strong limit point of \( A' \setminus C_+^i \) implies being a strong limit point of \( (A' \setminus E) \cap C_+^i \) as \( E \) is countable). So, the set \( A' \setminus E \) and the cone \( C_i \), verify the statement of the theorem.

\[\square\]

The next theorem is the abstraction of Lemma \[15\] In the statement of the theorem, by \( A^2 \) we mean the set of unordered pairs from the set \( A \).

**Theorem 17.** Let \( X \) be a second countable space, \( A \subseteq X \) uncountable, and \( c: A^2 \to \{1, \ldots, n\} \) a partition of the (unordered) pairs from \( A \) into finitely many colors. Then there is an uncountable \( B \subseteq A \) and an \( i \leq n \) such that \( B \) is partially homogeneous for \( i \). By this we mean that for every \( x \in B \), \( x \) is a strong limit point of the set \( \{y \in B: c(x, y) = i\} \).

**Proof.** The proof is similar to that of Lemma \[15\], so we only give a sketch. Starting with \( A_0 = A \), and given at stage \( i \) the sets \( A_{i-1} \subseteq A_{i-2} \subseteq \cdots \subseteq A_0 \), we ask if there is an uncountable \( A_i \subseteq A_{i-1} \) such that for every \( x \in A_i \), there is a neighborhood \( U \) of \( x \) such that \( \{y \in U \cap A_i: c(x, y) = i\} \) is countable. If we do this for all \( i \leq n \), let \( A_n \) be the ending set, which is uncountable and such that for every \( i \leq n \) and every \( x \in A_n \), there is a neighborhood \( U \) of \( x \) such that the set \( \{y \in A_n: c(x, y) = i\} \) is countable. Since \( X \) is second countable we may fix the neighborhood \( U \) which has this property for every point in an uncountable subset \( A'_n \subseteq A_n \). Fix an \( x \in A'_n \). Then for each \( i \leq n \) there are only countably many \( y \in U \cap A'_n \) with \( c(x, y) = i \), and thus only countably many points of \( U \cap A'_n \), a contradiction.

So, the construction must stop with some set \( A_{n-1} \) (that is, \( A_i \) is not defined), for some \( \ell \leq n \). Let \( C \) be the set of \( x \in A_{n-1} \) such that there is a neighborhood \( U \) of \( x \) with \( \{y \in A_{n-1} \cap U: c(x, y) = \ell\} \) is countable. We have that \( C \) is countable, otherwise we could use \( C = A_{n-1} \) and continue the construction. So, \( C \) is countable and if we let \( B = A_{n-1} \setminus C \), then easily \( B \) is a partially homogeneous set for \( \ell \). \[\square\]

4. Polytopes in \( \mathbb{R}^d \)

We begin this section with some notation and statement of some results concerning polytopes which we need for the theorem. These geometric results on polytopes will be proved in the next section.

**Definition 18.** Let \( x, y \in \partial P \) be such that \( \ell_{x,y} \) is maximal. We say \( \ell_{x,y} \) is an extreme line segment if for every \( t \in \mathbb{R}^d \) and every \( \epsilon > 0 \), either \( \ell_{x,y} + \epsilon t \) or \( \ell_{x,y} - \epsilon t \) does not lie in \( P \).

**Definition 19.** Let \( \mathcal{F} \) be a subset of the faces of \( P \). We let \( G(x, \mathcal{F}) \) be the set of points \( y \) such that there is a scaled, translated copy \( P' \) of \( P \) with \( \ell_{x,y} \) extreme and maximal in \( P' \) and for every face \( F' \) of \( P' \), \( \ell_{x,y} \subset F' \) if the corresponding face \( F \) of \( P \) is in \( \mathcal{F} \).

Given an affine subspace \( S \subseteq \mathbb{R}^d \) and an intersection \( \mathcal{F} \) of faces of \( P \), we say \( S \) is parallel to \( \mathcal{F} \) if a translation of \( S \) is contained in the affine extension of \( \mathcal{F} \). By the
affine extension of a subset $F$ we mean the set of points of the form $x + t(y - x)$ where $x, y \in F$ and $t \in \mathbb{R}$.

The next lemma is a generalization of the cone lemma, Lemma 15, which we need for the main proof. The proof will be given in the next section.

**Lemma 20.** Let $P$ be a polytope in $\mathbb{R}^d$. Suppose $A \subseteq S$ is uncountable where $S \subseteq \mathbb{R}^d$ is an affine subspace, and assume that every hyperplane in $S$ intersects $A$ in a countable set. Suppose that $H_1, \ldots, H_k$ are distinct hyperplanes in $S$ inducing the family of cones $C_1, \ldots, C_k$. Then there is a cone $C_i$, a subset $\mathcal{F}$ of the faces of $P$ and an uncountable subset $A' \subseteq A$ so that for every uncountable $B \subseteq A'$ there is an uncountable $D \subseteq B$ such that for every point $x \in D$, $x$ is a strong limit both from $C_i^+(x) \cap D$ and $C_i^-(x) \cap D$ of points $y$ in $G(x, \mathcal{F})$.

In §2 we gave the proof of the main theorem in the case where $P$ is a compact, strictly convex set in $\mathbb{R}^d$, Theorem 2. In this section we modify the argument to handle the case where $P$ is a $d$-polytope in $\mathbb{R}^d$. By a $d$-polytope (generalized polyhedron) we mean a compact set $P$ in $\mathbb{R}^d$ which is the intersection of a finite set of closed half-spaces, and which has non-empty interior. Such a set is necessarily convex, but not strictly convex. Equivalently, $P$ is the convex hull of a finite set in $\mathbb{R}^d$ which does not lie in a $< d$ dimensional affine subspace. For the rest of this section $P$ will denote a fixed $d$-polytope in $\mathbb{R}^d$, and we prove our main theorem, Theorem 3, in this case.

As in the proof of Theorem 2 we assume $A \subseteq \mathbb{R}^d$ is uncountable and $\tau$ is a winning strategy for $\eta$ in the no-$\beta$ McMullen game and we proceed to get a contradiction. Fix an affine subspace $S \subseteq \mathbb{R}^d$ of minimal dimension such that $A \cap S$ is uncountable. So for every affine hyperplane $H \subseteq \mathbb{R}^d$, either $S \subseteq H$ or $H \cap A$ is countable. Replacing $A$ by $A \cap S$, we may assume that $A \subseteq S$.

Let $H_1, \ldots, H_k$ be the affine hyperplanes determined by the faces of $P$ such that a translation of $H_i$ intersect $S$ properly. Applying Lemma 20 gives a set of faces $\mathcal{F}$, a cone $C$ defined by the $H_i$, and an uncountable $A' \subseteq A$. To save notation, let us rename the set $A'$ to $A$.

For $x, y \in A$ let $[x, y]_P$ be a scaled, translated copy of $P$ for which $\ell_{x,y}$ is maximal, and if $y \in G(x, \mathcal{F})$ (equivalently, $x \in G(y, \mathcal{F})$), then we take $[x, y]_P$ so that $\ell_{x,y}$ is on exactly the faces in $\mathcal{F}$.

We define the trees $T$ and $T_x$ as in Theorem 2 with a few changes. As before, $p = (P_0, \ldots, P_{n+1})$ of even length is in $T$ iff it is consistent with $\tau$, and we also require that every move $P_{2i}$ is of the form $[x, y]_P$ for some $x, y \in A$ with $y \in G(x, \mathcal{F})$. Such a $p \in T$ is in $T_x$ if in addition $x$ is in the relative interior $P_n^0(S)$ of $P_n \cap S$ with respect to the relative topology on $S$, and $x$ is not in $P_{n+1}$. So again we have that for every $x \in A$ the tree $T_x$ is wellfounded.

We let $N$ be a countable elementary substructure containing $\tau$, $P$, $A$, $S$, and the map $x, y \mapsto [x, y]_P$. As before let $T^N = T \cap N$ and $T_x^N = T_x \cap N$. Also as before we can find a position $p = (P_0, P_1, \ldots, P_{n+1})$ of even length; recall all of the moves $P_i$ are translated, scaled copies of $P$ such that for some uncountable $A' \subseteq A$ and all $x \in A'$, $p$ is a terminal position in $T_x^N$. In particular, for all $x \in A'$ we have $x \in P_n^0(S) \setminus P_{n+1}$. From Lemma 20 we may assume that $A'$ has the property that every $x \in A'$ is a strong limit point from both of $A' \cap C^+(x) \cap G(x, \mathcal{F})$ and $A' \cap C^-(x) \cap G(x, \mathcal{F})$.

Let $D$ be a rational ball in $\mathbb{R}^d$ such that $D \cap S \subseteq P_n^0(S) \setminus P_{n+1}$, $A' \cap D$ is uncountable. We may take $D$ to be sufficiently small so that for any distinct
Let \( x, y \in D \cap S \), if \( y \in G(x, \mathcal{F}) \) (so \( \ell_{x,y} \) lies on exactly the faces in \( \mathcal{F} \)), then \( [x,y]_P \) is disjoint from any face of \( P_{n+1} \) and any face of \( P_n \) not in \( \mathcal{F} \). To see this, let \( w \) be a strong limit point of \( A' \) in \( P_n^*(S) \setminus P_{n+1} \). Let \( D \) be a small enough neighborhood of \( w \) such that \( D \) is disjoint from \( P_{n+1} \) and also disjoint from any face \( F \) of \( P_n \) which \( w \) does not lie on. If there is a face \( F \) of \( P_n \) not in \( \mathcal{F} \) which \( w \) lies on, then the affine extension \( H \) of \( F \) must contain \( S \) as otherwise its intersection with \( S \) is proper, and since \( w \in P_n^*(S) \) we would have \( w \notin F \). If \( P_n = [x,y]_P \) with \( y \in G(x, \mathcal{F}) \), then \( \ell_{x,y} \notin F \) and so also \( \ell_{x,y} \) is not a subset of \( H \). Since \( x, y \in S \), this shows that \( H \) properly intersects \( S \), a contradiction.

It follows that for any \( x, y \in A' \cap D \), if \( y \in G(x, \mathcal{F}) \) then \([x,y]_P \) is disjoint from all faces of \( P_{n+1} \) and those of \( P_n \) not in \( \mathcal{F} \), and for those faces \( F \) in \( \mathcal{F} \) we have that the affine extension of the face \( F' \) of \([x,y]_P \) corresponding to \( F \) is the affine extension of the face \( F \) (note that the affine extension \( H \) of \( F \) and the affine extension \( H' \) of \( F' \) cannot be disjoint as \( H \) contains \( S \) and \( x, y \in S \cap F' \)). So, for any such \( x, y \in A' \cap D \) (with \( y \in G(x, \mathcal{F}) \)) we have that \([x,y]_P \subseteq P_n \setminus P_{n+1} \). Thus, \([x,y]_P \) is a valid next move for \( I \) in the game \( G_P(A) \).

Fix \( x, y \in A' \cap D \) with \( y \in G(x, \mathcal{F}) \). This is possible since \( A' \cap D \) is uncountable and each of the co-dimension one affine hyperplanes \( H(x) \) from Lemma \ref{lemma} has a countable intersection with \( A' \). Again, \( R = [x,y]_P \subseteq P_n \setminus P_{n+1} \) is a valid move for \( I \). Let \( R' = \tau(R) \) be \( I \)'s response by \( \tau \). By maximality of \( \ell_{x,y} \) in \( R, R' \) cannot delete both \( x \) and \( y \), so without loss of generality assume \( x \in R' \). Also without loss of generality assume \( y \in C^+(x) \). Since \( x \in A' \), \( x \) is a strong limit point of points \( z \in A' \cap C^+(x) \). For such \( z \) close enough to \( x \) we have that \( z \in R \). This is because if \( F \) is a face of \( R \) which does not contain \( x \), and \( z \) is close enough to \( x \) then \( z \) is on the same side of \( F \) that \( x \) is on. If \( F \) is a face of \( R \) containing \( x \), and the affine extension of \( F \) contains \( S \) then since \( z \in A' \subseteq S \), we have that \( z \) is also on the same side of \( F \) that \( x \) is on. Finally, if \( x \in F \) and the affine extension \( H \) of \( F \) does not contain \( S \), then \( H \cap S \) is one of the co-dimension one subsets of \( S \) used in forming the cone \( C^+(x) \), and so \( z \) lies on the same side of \( F \) as \( R \) does (we use here that \( y \in C^+(x) \)). So, \( z \in R \), and we may assume \( z \notin R' \) by taking \( z \) close enough to \( x \).

Let \( B_1 \) be a rational ball about \( x \) so that \( R' \cap B_1 = \emptyset \). We may assume that \( z \in B_1 \). Let \( B_2 \subseteq B_1 \) be a rational ball about \( z \) with \( B_2 \cap S \subseteq R \). Let \( \varphi(x,y) \) be the statement

\[
\varphi(x,y) = (x, y \in D \cap S) \land (y \in G(x, \mathcal{F})) \land (B_2 \cap S \subseteq [x,y]_P) \\
\land (\tau([x,y]_P) \cap B_1 = \emptyset) \land ([x,y]_P \subseteq P_n \setminus P_{n+1}) .
\]

Since \( S, D, B_1, P_n, P_{n+1} = \tau(P_n) \), and the map \( x, y \mapsto [x,y]_P \) are all in \( N \), by elementarity we have that

\[
\exists x', y' \in N \cap D \cap S \land (y' \in G(x', \mathcal{F})) \land (B_2 \cap S \subseteq [x',y']_P) \\
\land (\tau([x',y']_P) \cap B_1 = \emptyset) \land ([x',y']_P \subseteq P_n \setminus P_{n+1}) .
\]

Let \( P_{n+2} = [x',y']_P \). Since \([x',y']_P \subseteq P_n \setminus P_{n+1} \), \( P_{n+2} \) is a legal move for \( I \) in \( G_P(A) \) extending the position \( p \). Since \( x', y' \in N \), we have that \( P_{n+2} \in N \). Since \( z \in B_2 \cap S \) and \( B_2 \cap S \subseteq [x',y']_P \), we have that \( z \in P_{n+2} \). Since \( z \in B_1 \) and \( \tau([x',y']_P) \cap B_1 = \emptyset \) we have that \( z \notin \tau([x',y']_P) = \tau(P_{n+2}) \). As \( P_{n+2} \in N \) and \( z \in A' \), and since \( y' \in G(x', \mathcal{F}) \), this contradicts the fact that \( p \) was a terminal node of \( T^N \).

This completes the proof of Theorem \ref{theorem}.
5. Lemmas on Polytopes and a Generalized Cone Lemma

Notation 21. Let $P$ be a $d$-polytope in $\mathbb{R}^d$. For $x \in \partial P$, let $F_x$ denote the intersection of all faces of $P$ which $x$ lies on.

Definition 22. Let $x, y \in \partial P$ be such that $\ell_{x,y}$ is maximal. Let $\mathcal{F}$ be the intersection of a (finite) subset of the faces of $P$. We say $\ell_{x,y}$ is critical for $\mathcal{F}$ of $P$ if there is a maximal line segment parallel to $\ell_{x,y}$ which lies in $\mathcal{F}$, and for every neighborhoods $U$ of $x$ and $V$ of $y$ there are points $u \in U$ and $v \in V$ such that some translated scaled copy of $\ell_{u,v}$ lies in $\mathcal{F}$ but no such copy is maximal in $P$.

Note that a line segment being critical for faces $F$ is invariant under translation, provided the translated segment still lies in $P$.

Lemma 23. Let $P \subseteq \mathbb{R}^d$ be a $d$-polytope, and let $\ell_{x,y}$ be an extreme maximal line segment in $P$. Let $S \subseteq \mathbb{R}^d$ be an affine linear subspace. Then $\dim (\mathcal{F}_x \cap S) + \dim (\mathcal{F}_y \cap S) \leq \dim S - 1$.

Proof. Suppose not, so that $\dim (\mathcal{F}_x \cap S) + \dim (\mathcal{F}_y \cap S) > \dim S$. Let $B_x, B_y$ be bases for the linear subspaces corresponding to $\mathcal{F}_x \cap S$, $\mathcal{F}_y \cap S$ respectively, and let $B$ be a basis for the translate of $S$ which contains the origin.

First suppose $\text{span} B_x \cap \text{span} B_y \neq \{0\}$. If we let $v \in \text{span} B_x \cap \text{span} B_y$, then for all small enough $\epsilon > 0$, $x \pm \epsilon v \in \mathcal{F}_x \cap S$ and $y \pm \epsilon v \in \mathcal{F}_y \cap S$. In particular, $x \pm \epsilon v \in P$ and $y \pm \epsilon v \in P$. This contradicts $\ell_{x,y}$ being extreme.

So we must have $\text{span} B_x \cap \text{span} B_y = \{0\}$, but then we have that $B_x \cup B_y$ is linearly independent and contains at least as many vectors as $B$ does, and so we have $\text{span}(B_x \cup B_y) = \text{span}(B)$. In particular, since $x, y \in S$, we have $y - x \in \text{span}(B_x \cup B_y)$, and so there are vectors $u \in \text{span} B_x$, $v \in \text{span} B_y$ so that $u + v = y - x$. For any $\epsilon > 0$ small enough, we have that $x - \epsilon u \in \mathcal{F}_x \cap S$ and $y + \epsilon v \in \mathcal{F}_y \cap S$, and by convexity, we have that the line segment $\ell_{x-\epsilon u, y+\epsilon v}$ is in $P$, but note that $(y + \epsilon v) - (x - \epsilon u) = y - x + \epsilon(u + v) = (1 + \epsilon)(y - x)$, and so the line segment $\ell_{x-\epsilon u, y+\epsilon v}$ is strictly longer than $\ell_{x,y}$, contradicting the maximality of $\ell_{x,y}$ in $P$.

Lemma 24. Let $\ell_{x,y}$ be maximal in $P$. Suppose $u_n \to x$ and $v_n \to y$. Then there are maximal line segments $\ell_{u'_n, v'_n}$ parallel to $\ell_{u_n,v_n}$ and a subsequence $i_n$ with $u'_{i_n} \to x$ and $v'_{i_n} \to y$.

Proof. Let $\ell_{u_n,v_n}$ be maximal in $P$ and parallel to $\ell_{u_n,v_n}$. Let $u'_{i_n}, v'_{i_n}$ be a subsequence which converges to some $\ell_{x', y'}$ which is necessarily maximal in $P$ as well. Note that $\ell_{x', y'}$ is parallel to $\ell_{x,y}$, and in fact is of the same length as well by maximality of $\ell_{x,y}$ and $\ell_{x', y'}$. Let $\ell_{x', y'} = \ell_{x', y'} + t$. By compactness, for any $\epsilon > 0$ there is a $\delta > 0$ such that if $u'_{i_n}, v'_{i_n}$ are in $B_\delta(x'), B_\delta(y')$ respectively, then for any $\alpha \leq 1 - \epsilon$ we have that if $u'_{i_n} + \alpha t$ is on a face $F$ of $P$, then so is $x' + \alpha t$, and likewise for $v'_{i_n}$ and $y'$. Since the $x' + \alpha t$ can be translated further in the direction of $t$ and remain in $P$, it follows that the same is true for $u'_{i_n} + \alpha t$, and like wise for $y' + \alpha t$ and $v'_{i_n} + \alpha t$. So, for all $\alpha \leq 1 - \epsilon$, we have that $u'_{i_n} + \alpha t, v'_{i_n} + \alpha t$ are in $P$. In particular $u'' = u'_{i_n} + (1 - \epsilon)t, v'' = v'_{i_n} + (1 - \epsilon)t \in P$. Note that $\ell_{u'', v''}$ is also maximal in $P$. Letting $\epsilon$ go to 0 now gives the result.

Lemma 25. Suppose $\ell_{x,y}$ is an extreme maximal line segment. Let $U', V'$ be neighborhoods of $x$ and $y$ respectively. Then there are neighborhoods $U \subseteq U'$, $V \subseteq V'$...
V' of x, y such that if u ∈ U, v ∈ V and ℓ_u,v is maximal in P, then there is a translation ℓ_u',v' of ℓ_u,v with u' ∈ U', v' ∈ V' and ℓ_u',v' is extreme and maximal in P.

Proof. Let ε > 0 be such that the ε neighborhoods of x and y are contained in U', V' respectively. Consider a translation vector t ∈ ℝ^d. For at least one of the points x, y we have that this point cannot be translated in the +t or −t direction and stay inside P. Without loss of generality, say x cannot be translated in the +t direction and stay inside of P. So, there is a face F of P such that x ∈ F and for every η > 0, x + ηt is on the side of the affine hyperplane H defined by F not containing P. Note that if n is the outward normal vector for F, then t · n > 0. Thus there is a neighborhood W_t of t and a δ_t > 0 such that if t' ∈ W_t then x + δ_t' is more than δ_t away from H and on the side of H not containing P. It follows that if u is within distance δ_t of x and t' ∈ W_t, then u + δ_t' is on the side of H not containing P. This defines a cover {W_t} of the set of translation vectors with corresponding δ_t > 0. By compactness, there is a δ > 0 such that for all t we have that if u ∈ B(x, δ) and v ∈ B(y, δ) then at least one of u + δt, v + δt is outside of P.

Suppose now ℓ_u,v is maximal in P with u ∈ B(x, δ), v ∈ B(y, δ). If ℓ_u,v is not extreme, there is a vector t such that ℓ_u,v + ηt is in P for some η > 0. By the definition of δ we have, without loss of generality, that ℓ_u,v + ηt is not contained in P. So there is a δ < η' < δ such that ℓ_u,v + η't is contained in P and the total number of faces the endpoints lie on has strictly increased (note that since ℓ_u,v is maximal, if ℓ_u,v + ηt ⊆ P then u ± ηt, v ± ηt lie on at least the same faces of P that u, v respectively lie on).

Let δ_0 = δ/2. We have shown that there is a δ_1 > 0 such that if u ∈ B(x, δ_1), v ∈ B(y, δ_1) and ℓ_u,v is maximal, then either ℓ_u,v is extreme there is a translation ℓ_u_1,v_1 of ℓ_u,v in P such that the total number of faces the endpoints lie on has strictly increased Let N be the number of faces of P. Repeating this argument at most N times gives δ_k = δ_0 > δ_1 > · · · > δ_N > 0 such that if ℓ_u,v is maximal in P with u ∈ B(x, δ_N), v ∈ B(y, δ_N), then there is a translation ℓ_u',v' of ℓ_x,y in P which is extreme and with u' ∈ B(x, δ_k), v' ∈ B(y, δ_k).

Lemma 26. Let F be an intersection of faces of the d-polytope P ⊆ ℝ^d. Let S be a proper affine linear subspace of ℝ^d which is parallel to F. Suppose x, y ∈ S ∩ F are such that ℓ_x,y is extreme and critical for F. Then dim (F_x ∩ S) + dim (F_y ∩ S) ≤ dim S − 2.

Proof. Let u_n → x, v_n → y witness the criticality of ℓ_x,y. In particular, there is a scaled, translated copy of ℓ_u_n,v_n which lies in F, but no such maximal copy of ℓ_u_n,v_n lies in F. Let u'_n, v'_n ∈ ∂F be such that ℓ_u'_n,v'_n is parallel to ℓ_u_n,v_n and is maximal in P. From Lemma 24 we may assume that u'_n → x and v'_n → y. From Lemma 25 we may assume that the ℓ_u'_n,v'_n are extreme and maximal. Renaming the points, we may assume that u_n → x, v_n → y, ℓ_u_n,v_n is extreme and maximal, and by criticality that ℓ_u_n,v_n does not lie in F. Since there is a scaled, translated copy of ℓ_u_n,v_n which lies in F, neither endpoint u_n, v_n lies in F.

Let B be a basis for the translate of S which contains the origin. Define S' = x + span (B ∪ {u_0 − x}) = y + span (B ∪ {v_0 − y}) (these are equal because u_0 − v_0, x − y are both in S). Note that x, y, v_0, u_0 are all in S'. By Lemma 23 dim (F_{u_0} ∩ S') +
dim(\(F_{v_0} \cap S'\)) ≤ dim(S') - 1. By the choice of the vector \(u_0 - x\) which we used to extend \(S\) to \(S'\), and the fact that \(x \in F\), we have that \(F_x \cap S' = F_x \cap F \cap S' = F_x \cap S\). Likewise \(F_y \cap S' = F_y \cap S\).

Since all the faces of \(P\) are closed, we may assume that \(u_0, v_0\) are close enough to \(x, y\) so that \(F_x \subseteq F_{u_0} \cap F\) and \(F_y \subseteq F_{v_0} \cap F\). Now, \(x, y \in F\) but none of the points \(u_n, v_n\) can lie in \(F\), so in particular, \(u_0 - x\) is not in the subspace defined by the affine extension of \(F\) (that is, the translation of the affine space which contains \(\vec{0}\)), and so \(dim(F_x \cap S) ≤ dim(F_{u_0} \cap F \cap S') < dim(F_{u_0} \cap S')\). Likewise \(dim(F_y \cap S) < dim(F_{v_0} \cap S')\), and so overall we have \(dim(F_x \cap S) + dim(F_y \cap S) ≤ dim(F_{u_0} \cap S') + dim(F_{v_0} \cap S') - 2 ≤ dim(S') - 3 = dim(S) - 2\). □

**Lemma 27.** Suppose \(S_1, S_2\) are affine linear subspaces of \(\mathbb{R}^d\). Let \(T\) be the affine extension of \(S_1 \cup S_2\). Then \(dim(T) \leq dim(S_1) + dim(S_2) + 1\)

**Proof.** We may assume without loss of generality that \(S_1\) contains the origin. Let \(B_1\) be a basis for \(S_1\). Let \(B_2\) be a basis for the translate of \(S_2\) which contains the origin, and let \(x \in S_2\). We claim \(T' = \text{span}(B_1 \cup B_2 \cup \{x\})\) contains \(S_1 \cup S_2\). Indeed, any vector \(v \in S_1\) is in \(\text{span}(B_1)\), whereas any vector \(v \in S_2\) has \(v - x \in \text{span}(B_2)\), and so \(v \in \text{span}(B_2 \cup \{x\})\). Since \(T \subseteq T'\), and \(dim(T') \leq dim(S_1) + dim(S_2) + 1\), this gives the required estimate on the dimension. □

The above analysis of the critical line segments now allows us to give the proof of the following Lemma 28 which we use for the proof of the cone lemma, Lemma 20.

**Lemma 28.** Let \(P\) be a \(d\)-polytope in \(\mathbb{R}^d\), and let \(S \subseteq \mathbb{R}^d\) be an affine subspace, with \(dim(S) ≥ 2\). Then there are finitely many co-dimension 1 affine subspaces \(S_1, \ldots, S_k \subseteq S\) such that if \(x, y \in S\) and \(\ell_{x,y}\) is critical for some intersection \(F\) of faces of \(P\), then for some \(i\) we have \(y \in S_i(x)\). Recall \(S_i(x) = S_i + (x - v_i)\) denotes the translation of \(S_i\) containing \(x\).

**Proof.** Consider all possible pairs \((A, B)\) where \(A, B\) are subsets of the faces of \(P\) and if \(A', B'\) denote the corresponding sets of hyperplanes then \(dim(\bigcap A' \cap S(\vec{0})) + dim(\bigcap B' \cap S(\vec{0})) ≤ dim(S) - 2\). For each such pair \((A, B)\), let \(x\) be any point in \(\bigcap A\), and note that by Lemma 27 if \(T\) is the smallest linear subspace containing both \((\bigcap A - x) \cap S(\vec{0})\) and \((\bigcap B - x) \cap S(\vec{0})\), then \(dim(T) ≤ dim(S) - 1\). The resulting space \(T(A, B) = T\) does not depend on the choice of \(x\).

By the proof of Lemma 28 we may assume that \(\ell_{x,y}\) is extreme. Now by Lemma 29 if \(\ell_{x,y}\) is extreme and critical for an intersection \(F\) of faces of \(P\), then if \(A\) is the set of faces of \(P\) which \(x\) lies on and \(B\) is the set of faces of \(P\) that \(y\) lies on, then \((A, B)\) is a pair satisfying \(dim(\bigcap A' \cap S(\vec{0})) + dim(\bigcap B' \cap S(\vec{0})) ≤ dim(S) - 2\), and so \(T(A, B) \subseteq S(\vec{0})\) has co-dimension at least 1 in \(S(\vec{0})\). We also have that \(y - x \in T(A, B)\), and so \(y \in T(A, B)(x)\), so that the collection of finitely many \(T(A, B)\), translated to lie inside of \(S\) are as desired. □

**Lemma 29.** Let \(P\) be a \(d\)-polytope in \(\mathbb{R}^d\), and let \(S \subseteq \mathbb{R}^d\) be an affine subspace. Suppose \(x, y \in P\) and \(\ell_{x,y}\) is extreme and maximal for \(P\) with \(\ell_{x,y}\) lying in exactly the set \(\mathfrak{F}\) of faces of \(P\). Suppose \(\ell_{x,y}\) is not critical for \(F = \cap \mathfrak{F}\). Then there are neighborhoods \(U, V\) of \(x, y\) respectively such that for any \(u \in U, v \in V\), if \(\ell_{u,v}\) can be translated and scaled to be in \(F\) then \(\ell_{u,v}\) can be translated and scaled to be extreme and maximal for \(P\) and lie in exactly the faces \(\mathfrak{F}\).
Proof. Since $\ell_{x,y}$ is not critical for $\mathcal{F}$, there are neighborhood $U$, $V$ of $x$, $y$ such that for any $u \in U$, $v \in V$, if $\ell_{u,v}$ can be scaled and translated to be in $\mathcal{F}$, then $\ell_{u,v}$ can be scaled and translated to be maximal in $P$ and lying in $\mathcal{F}$. Furthermore, Lemma 24 showed that we may take $U$, $V$ so that for such $u \in U$, $v \in V$, we may find a maximal copy of $\ell_{u,v} \subseteq \mathcal{F}$ in small enough neighborhoods of $x$, $y$ so that $u$ does not lie on any faces that $x$ does not lie on, and likewise for $v$. Also, by Lemma 25 we may further assume that $\ell_{u,v}$ is extreme and maximal. Also, $\ell_{u,v} \subseteq \mathcal{F}$ (as translating a maximal segment to make it extreme does not decrease the set of faces it lies on) but also from the assumption on $U$, $V$ we have that $\ell_{u,v}$ does not lie on more faces than $\ell_{x,y}$. □

Now we are ready to prove the generalized cone lemma, Lemma 20, which we restate.

**Lemma.** Let $P$ be a polytope in $\mathbb{R}^d$. Suppose $A \subseteq S$ is uncountable where $S \subseteq \mathbb{R}^d$ is an affine subspace, and assume that every hyperplane in $S$ intersects $A$ in a countable set. Suppose that $H_1, \ldots, H_k$ are distinct hyperplanes in $S$ inducing the family of cones $C_1 \ldots C_\ell$. Then there is a cone $C_i$, a subset $\mathcal{F}$ of the faces of $P$ and an uncountable subset $A' \subseteq A$ so that for every uncountable $B \subseteq A'$ there is an uncountable $D \subseteq B$ such that for every point $x \in D$, $x$ is a strong limit both from $C_i^+(x) \cap D$ and $C_i^-(x) \cap D$ of points $y$ in $G(x, \mathcal{F})$.

Proof. We first note that it suffices to prove the following weaker version of the theorem: for any uncountable $A \subseteq S$ and cones $C_1, \ldots, C_\ell$, there is an uncountable $A' \subseteq A$, a $C_i$ and a collection of faces $\mathcal{F}$ such that every $x \in A'$ is a strong limit point of $C_i^+(x) \cap A'$ and an $A'$ in $\mathcal{F}$ and a strong limit point of $C_i^-(x) \cap A'$ of points $y$ such that there is a scaled, translated copy $P'$ of $P$ with $\ell_{x,y}$ extreme and maximal in $P'$ and for every face $F'$ of $P'$, $\ell_{x,y} \subseteq F'$ if the corresponding face $F$ of $P$ is in $\mathcal{F}$.

To see this, suppose the conclusion of Lemma 20 fails. We let $G(x, \mathcal{F})$ be the set of such points $y$ as in the statement of the theorem for a given $x$ and set of faces $\mathcal{F}$. Let $(C_{p_i}, \mathcal{F}_{q_i})$, for $1 \leq i \leq m$, enumerate the pairs of cones and subsets of faces. Let $A_0 = A$, and supposing $A_{i-1}$ has been defined, let $A_i \subseteq A_{i-1}$ be uncountable so that for all uncountable $E \subseteq A_i$ it is not the case that for every $x \in E$, $x$ is a strong limit both from $C_{p_i}(x) \cap E$ and $C_{p_i}^-(x) \cap E$ of points $y \in G(x, \mathcal{F}_{q_i})$. We can get $A_i$ from the assumption that the theorem fails for the pair $(C_{p_i}, \mathcal{F}_{q_i})$ and $A_{i-1}$. Then $A_m$ violates the weaker version of the theorem.

To prove the weaker version, fix the uncountable set $A \subseteq S$, the hyperplanes $H_1, \ldots, H_k$ and corresponding cones $C_1, \ldots, C_\ell$. We may assume that the hyperplanes $H_1, \ldots, H_k$ includes the hyperplanes from Lemma 28.

As in the proof of Theorem 15 we define an operation thinning the uncountable set $A$ out, but in this case we use the enumeration of the pairs $(C_{p_i}, \mathcal{F}_{q_i})$ to do so. Let $A_0 = A$. Consider the first pair $(C_{p_1}, \mathcal{F}_{q_1})$. We ask if there is an uncountable $A'_0 \subseteq A_0$ such that for every $x \in A'_0$ there is a neighborhood $U$ of $x$ such that either the set $U \cap C_{p_1}^+(x) \cap A'_0 \cap G(x, \mathcal{F}_{q_1})$ or the set $U \cap C_{p_1}^-(x) \cap A'_0 \cap G(x, \mathcal{F}_{q_1})$ is countable. If this is the case then we may thin $A'_0$ out to an uncountable set $A_1 \subseteq A'_0$ and fix a symbol $s(1) \in \{+, -\}$ so that for all $x \in A_1$, there is a neighborhood $U$ of $x$ such that $U \cap C_{p_1}^{s(1)}(x) \cap A_1 \cap G(x, \mathcal{F}_{q_1})$ is countable. That is, by thinning from $A'_0$ to $A_1$ we fix the side of the cone $C_{p_1}$ which is weakly isolating $x$ in $A_1$ with respect this property. We continue in this manner for $m$ stages, assuming that we can at each stage go from $A_k$ to $A'_k \subseteq A_k$ using the pair $(C_{p_k}, \mathcal{F}_{q_k})$. We then define $A_{k+1} \subseteq A'_k$
and $s(k)$ exactly as in the first step. Assuming this construction continues for all $m$ steps, we end with an uncountable set $A_m \subseteq A$ and for each pair $(C_p, \mathfrak{F}_q)$, we have a side $s(i)$ of the cone $C_p$.

In this case, for each $x \in A_m$ we let $B_x$ be a rational ball containing $x$ such that $\forall i$, the set $B_x \cap C_\alpha(i)(x) \cap A_m \cap G(x, \mathfrak{F}_q)$ is countable. We may fix a rational ball $B$ so that $B_x = B$ for an uncountable subset $A'_m$ of $A_m$. Let $x, y \in A'_m$ be distinct strong limit points of $A'_m$, so in particular $x, y \in B = B_x = B_y$. Since the set of points of $A'_m$ which are strong limit points of $A'_m$ is uncountable, from our assumption on $A$ we may assume that the line $\ell_{x,y}$ between $x$ and $y$ does not lie on any of the $H_1, \ldots, H_k$, and in particular, by Lemma 28 is not a critical line segment for any intersection $\mathcal{F}$ of faces of any copy $P'$ of $P$ in which it is extreme and maximal. Let $C$ be the (unique) cone such that $y \in C^\pm(x)$ (and so also $x \in C^\pm(y)$). Note that $y$ is in the interior of $C^\pm(x)$, and so also $x$ is in the interior of $C^\pm(y)$. Let $\mathfrak{F}$ be a subset of the faces of $P$ such that some extreme maximal (translated and scaled) copy of $\ell_{x,y}$ lies on exactly the faces $\mathfrak{F}$ of $P$. So, $y \in G(x, \mathfrak{F})$ (and so also $x \in G(y, \mathfrak{F})$). Let $i$ be such that $(C, \mathfrak{F}) = (C_p, \mathfrak{F}_q)$. Without loss of generality we may assume $y \in C_\alpha(i)(x)$. Let $V$ be a neighborhood of $y$ contained in the interior of $B \cap C_\alpha(i)(x)$. On the one hand, since $V \subseteq B$ there are only countably many $y' \in V$ such that $y' \in G(x, \mathfrak{F})$. On the other hand, from Lemma 28 since $\ell_{x,y}$ is extreme and maximal in some scaled translated copy $P'$ of $P$ with $\ell_{x,y}$ lying on exactly the faces of $P'$ corresponding to $\mathfrak{F}$ and since $\ell_{x,y}$ is not critical for $\mathcal{F} = \cap \mathfrak{F}$, by Lemma 28 we have that for all $y'$ in a small enough neighborhood of $y$ that $y' \in G(x, \mathfrak{F})$. This contradiction completes the proof.

6. Compact, convex sets in $\mathbb{R}^2$

In this section we consider the case where $P$ is a compact, convex (not necessarily strictly convex) set in $\mathbb{R}^2$ with non-empty interior, and prove the corresponding version of our main result, Theorem 2. For the rest of this section we fix the compact, convex set $P \subseteq \mathbb{R}^2$. The argument in this case will require a generalization of the cone lemma Lemma 10.

Since $P$ is not necessarily strictly convex, Lemma 14 does not apply and there may be points $x \in \partial P$ with $\delta_P(x) = 0$. The following lemma will serve as a replacement for Lemma 14 in the current case.

**Lemma 30.** Let $P$ be a compact, convex set set $\mathbb{R}^2$. Then there is a $\delta > 0$ and a finite set $F \subseteq \partial P$ such that $\delta_P(x) > \delta$ for all $x \in \partial P \setminus F$, and $\delta_P(x) = 0$ for all $x \in F$.

**Proof.** If the conclusion fails then there is a sequence of distinct points $\{x_n\} \subseteq \partial P$ with $x_n \to x \in \partial P$ and with $\delta_P(x) = 0$ and $\delta_P(x_n) \to 0$. We use here the facts that $\partial P$ is compact and if $\{x_n\} \subseteq \partial P$, $x_n \to x$, and $\delta_P(x_n) \to 0$, then $\delta_P(x) = 0$. To see this, suppose towards a contradiction that $\{x_n\} \subseteq \partial P$ are distinct points, $x_n \to x$ (so $x \in \partial P$), $\delta_P(x_n) \to 0$, but $\delta_P(x) > 0$. For each $n$, let $y_n \in \partial P$ be such that $y_n \in M_P(x_n)$ and if $\ell_n$ is the vector from $x_n$ to $y_n$ then $\frac{\ell_n}{\|\ell_n\|} \cdot H = \delta_P(x_n)$ for some supporting hyperplane (line) for $P$ which contains $x_n$. Likewise, let $y \in \partial P$ be in $M_P(x)$. By passing to a subsequence we may assume that $y_n \to y \in \partial P$. From Lemma 12 we have that $y \in M_P(x)$. From Remark 10 we have that $x \neq y$, that is, the vector $\ell$ from $x$ to $y$ is non-degenerate. Since $\delta_P(x) > 0$ by assumption, there
is a separation between \( \ell \) and any supporting line for \( P \) at \( x \). This easily implies that the only points of \( \ell \) which are not in \( P^o \) are \( x,y \). For we cannot have \( \ell \subseteq \partial P \) and so there is a point \( z \in \ell \) with \( z \in P^o \). From the convexity of \( P \) it then follows that every point on \( \ell \) except \( x \) and \( y \) is in \( P^o \). Let \( z \in \ell \) be a point distinct from \( x \) and \( y \). Let \( B \) be a ball about \( z \) with \( B \subseteq P \). Then for \( n \) large enough, so \( x_n,y_n \) are sufficiently close to \( x,y \) we have that \( \ell \) intersects \( B \) and in fact since \( \delta_P(x_n) \to 0 \) we will have that for large enough \( n \) that some supporting line \( H_n \) for \( P \) at \( x_n \) will intersect \( B \). This contradicts the fact that \( B \subseteq P^o \).

So, assume \( \{ x_n \} \subseteq \partial P \) are distinct points, \( x_n \to x \), \( \delta_P(x_n) \to 0 \), and \( \delta_P(x) = 0 \). As above, for each \( n \) let \( y_n \in \partial P \) be such that \( y_n \in M_P(x_n) \) and if \( \ell_n \) again denotes the vector from \( x_n \) to \( y_n \) then for some supporting hyperplane \( H_n \) for \( P \) at \( x_n \) we have that \( \frac{x_n}{\|x_n\|} \cdot H_n = \delta_P(x_n) \). As before we may assume \( y_n \to y \) for some \( y \in \partial P \). So \( y \in M_P(x) \), and the above argument shows that \( \ell \subseteq \partial P \), where \( \ell \) is the vector from \( x \) to \( y \). Also, \( x \neq y \) so \( \ell \) is non-degenerate. Consider a sufficiently large \( n \), so \( x_n,y_n \) are sufficiently close to \( x,y \). Let \( z_n \) be the midpoint of \( \ell_n \). We cannot have \( z_n \) on the line segment \( \ell \) from \( x \) to \( y \) as otherwise if \( x_n \notin \ell \) then we contradict that the line determined by \( \ell \) is a supporting line for \( P \), and if \( x_n \in \ell \), then by maximality of \( \ell \) and \( \ell_n \) we would have \( \ell = \ell_n \) and so \( x = x_n \) which could only happen for one \( n \). Thus, the triangle \( T = \delta_m(x,y) \) determined by the points \( z_n, x, y \) is non-degenerate. By convexity \( T \subseteq P \). For \( m \) sufficiently large, \( \ell_m \cap T^o \neq \emptyset \). For \( m \) sufficiently large, if \( x_m \notin \ell \), then we have \( H_m \cap T^o \neq \emptyset \), which contradicts the fact that \( H_m \) is a supporting line for \( P \) and \( T^o \subseteq P \).

So, we may suppose that for large enough \( n \) we have \( x_n \in \ell \), and so \( y_n \notin \ell \). Fix an \( n \) and consider the cone \( C \) from \( x \) with sides \( \ell \) and \( \ell_{x,y_n} \). For \( n' > n \) large enough, we have that \( y_{n'} \) lies in \( C^o \). Note that by convexity the convex hull \( Q \) of the points \( y,x_{y_n},y_{n'} \) is contained in \( P \). Since \( y_{n'} \in C^o \), the ray from \( y_{n'} \) parallel to \( \ell \) in the direction towards \( x \) contains a non-degenerate interval inside \( Q \). Thus the line segment \( \ell_{x,y} \) can be translated in the direction \( x - y \) to stay inside \( P \) and with an endpoint in \( P^o \). This contradicts the maximality of \( \ell_{n'} \).

Given the compact, convex set \( P \subseteq \mathbb{R}^2 \), let \( \delta > 0 \) and \( F \subseteq \partial P \) be from Lemma \ref{lema-lower}.

Consider a point \( x \in F \), so \( \delta_P(x) = 0 \). Let \( L \) be a supporting hyperplane (line) for \( P \) at \( x \). We may identify \( L \) with the \( x \)-axis for convenience of notation, with \( x \) identified with the origin and with \( P \) in the upper half-space. Consider the left and right tangents to \( P \) at \( x \). For example, the right tangent is the ray \( r_0 \) from the origin with angle \( \theta \) (measured counterclockwise from the \( x \)-axis as usual) where \( \theta \) is largest so that \( P \) lies on the half-space which is determined the line \( L_\theta \) extending \( r_\theta \) and which is disjoint from the sector determined by the positive \( x \)-axis and the ray \( r_0 \). We let \( r_x \) denote the unit right tangent to \( P \) at \( x \). We similarly define the unit left tangent \( l_x \) to \( P \) at \( x \). We may suppose that \( l_x \neq r_x \) as otherwise \( P \) lies on a line segment in \( \mathbb{R}^2 \), which violates the assumption that \( P \) has non-empty interior.

Since \( \delta_P(x) = 0 \) (as \( x \in F \)), we have that at least one of \( l_x, r_x \) lies in the direction of a line segment \( \ell \in M_P(x) \).

**Claim 31.** Suppose that only one of these tangents, say \( r_x \), lies in the direction of a vector in \( M_P(x) \). There is an \( \eta_x > 0 \) such that if the directed line segment \( l \) from \( x \) to a point \( y \in \partial P \) lies in \( M_P(x) \) and \( l \) is not coincident with \( r_x \), then

\[
\left| \frac{l}{\|l\|} \cdot N(l_x) \right| > \eta_x,
\]

where \( N(l_x) \) is the normal vector to the vector \( l_x \).
Proof: If the claim fails then there is a sequence of points \( \{y_n\} \subseteq \partial P \) with the line segments \( \ell_n \) from \( x \) to \( y_n \) in \( MP(x) \) and such that \( \frac{\ell_n \cdot N(\ell_n)}{\|\ell_n\|} \to 0 \). The lengths of the line segments \( \ell_n \) are bounded away from 0, and we may assume that \( y_n \to y \in \partial P \) with \( x \neq y \). By convexity, the line segment \( \ell \) from \( x \) to \( y \) lies in \( P \). We also have that \( \frac{\ell \cdot N(\ell)}{\|\ell\|} = 0 \), and so \( \ell \) is parallel to \( \ell_x \). From the definition of \( \ell_x \) we easily have that \( \ell \) points in the same direction as \( \ell_x \) (as \( \ell_x \), \( \ell \) both point into the same half-space determined by the line \( L \) as in the definition of \( \ell_x \) above).

From Lemma [12] we have that \( \ell \in MP(x) \). This contradicts our assumption that \( \ell_x \) does not lie in the direction of a vector in \( MP(x) \).

Let \( F' \subseteq F \) be the points \( x \in F \) such that exactly one of the \( \ell_x \), \( r_x \) directions has a maximal line segment with endpoint \( x \) which lies in \( \partial P \). Let \( \eta = \min\{\eta_1 \mid x \in F'\} \), and let \( \delta' = \min\{\delta, \eta\} \). Let \( H_1, \ldots, H_k \) be a finite set of hyperplanes (lines) in \( \mathbb{R}^2 \) containing all of the \( \ell_x \) and \( r_x \) (translated to pass through the origin) for each \( x \in F \), and also such that if \( C_1, \ldots, C_k \) is the set of corresponding cones (in this case \( \ell = k \)) then the angle between \( \delta \) and \( C_j \) is less than \( \delta' \).

Let \( A \subseteq \mathbb{R}^2 \) be such that \( \mathbf{II} \) has a winning strategy \( \tau \) in the no-\( \beta \) McMullen game for \( A \). Suppose towards a contradiction that \( A \) is uncountable. As in the proof of Theorem [2] we may assume that \( \mathcal{A} \cap \ell \) is countable for every line \( \ell \subseteq \mathbb{R}^2 \) (as otherwise we may replace \( A \) with \( \mathcal{A} \cap \ell \), and the argument is a much easier version of the argument to follow). Let \( N \) be a countable elementary substructure (of a large \( V_\gamma \)) containing \( \mathcal{T} \) and \( H_1, \ldots, H_k \) and (so the cones \( C_1, \ldots, C_k \)). We define the trees \( T, T^N, T_x, T_x^N \) as before. As before, we get a position \( p = (B_0, B_1, \ldots, B_{n+1}) \) (so \( n \) is even) in the tree \( T^N \) such that for all \( x \) is some uncountable \( \mathcal{A}' \subseteq A \), \( p \) is terminal in \( T_x^N \). As before we get a rational ball \( D \subseteq B_{n+1}^0 \setminus B_{n+1} \) such that \( \mathcal{A}' \cap D \) is uncountable and for any \( x, y \in D \), \( [x, y] \cap p \subseteq B_{n+1}^0 \setminus B_{n+1} \).

We apply Lemma [13] to \( \mathcal{A}' \cap D \) and the hyperplanes \( H_1, \ldots, H_k \). Let \( A'' \subseteq A' \) be the uncountable set and \( C_i \in \{C_1, \ldots, C_k\} \) be the cone from Lemma [15].

Let \( x \in A'' \). We have that \( x \) is a strong limit point of \( A'' \cap C_i^+ (x) \), and since \( A \cap \ell \) is countable for every line \( \ell \), we can choose \( y \in A'' \) with \( y \in \text{the interior of } C_i^+ (x) \). So, \( x \) is in the interior of \( C_i^+ (y) \). Consider \( [x, y] \cap p \). By definition the line-segment \( \ell_{x,y} \) from \( x \) to \( y \) is maximal in \( [x, y] \cap p \). Without loss of generality assume \( x \notin \tau([x, y] \cap p) \). Let \( B_1 \) be a rational ball containing \( [x, y] \cap p \). By \( \delta' \leq \delta \) (and the cone angle of \( C_i \) is \( \leq \delta' \) it follows that if \( z \) is in the interior of \( C_i^+ (x) \) is close enough to \( x \) then \( z \in [x, y] \cap p \). Suppose now that \( \delta_{[x, y] \cap p} (x) = 0 \). Note that \( y \notin \ell_x \cup r_x \). We consider cases as to whether both or just one of \( \ell_x, r_x \) is contained in \( \partial P \). In the case that both are contained in \( \partial P \) we claim that the cone \( C_i^+ (x) \) is contained in the cone with vertex at \( x \) and sides given by \( \ell_x, r_x \). Thus this follows from the facts that \( \ell_{x,y} \subseteq C_i^+(x) \), and the lines \( \ell \) and \( r \) through the origin parallel to \( \ell_x \) and \( r_x \) were included among the \( H_1, \ldots, H_k \). On the other hand, if either \( \ell_x \cap [x, y] = \{x\} \) or \( r_x \cap [x, y] = \{x\} \), then from Claim [11] we have that the angle between \( \ell_{x,y} \) and \( \ell_x \) (or \( r_x \) respectively) is at least \( \eta_{\ell_x} \geq \delta' \). Thus, in both cases if \( z \) is in the interior of \( C_i^+ (x) \) is close enough to \( x \), then \( z \in [x, y] \cap p \). Let \( z \in A'' \) be a point close enough to \( x \).

So, in all cases we have \( z \in A'' \cap [x, y] \cap p \). Let \( B_2 \subseteq B_1 \cap [x, y] \cap p \) be a rational ball containing \( z \). The argument now finishes exactly as in Theorems [2] and [3].
7. Ordinal Analysis

Throughout this section we fix a compact, convex set $P \subseteq \mathbb{R}^d$ for which the no-$\beta$ McMullen game is equivalent to the perfect set game. By Theorems 2, 3, and 4 this includes all strictly convex sets in $\mathbb{R}^d$, all compact, convex sets in $\mathbb{R}^2$, and all polytopes in $\mathbb{R}^d$.

Recall from Definition 7 of §1 that for a set $A \subseteq \mathbb{R}^d$ we have defined a derivative notion $A^*_p \subseteq A$. Iterating this defined the sets $A_{\omega}^\alpha$. This process stops at a countable ordinal $\alpha$, and we set $A_\omega^\alpha = A^\omega_p$.

Our geometric consequences result, Theorem 8, follows immediately from Theorems 2, 3, and 4 and the following Theorem 32, which relates the ordinal analysis of the derivative for closed sets to the no-$\beta$ McMullen game.

We note that (1) of the following theorem would follow easily if in Definition 7 we did not have the clause, in the definition of a good copy, involving limits along hyperplanes $H$ but only consider limits from points in the interior of the copy of $P$. For in this case, $I$ could win by playing good copies for $A_\omega^\alpha$. However, this clause of Definition 7 is necessary for the proof of the following theorem in the polytope case (but not in the strictly convex case).

**Theorem 32.** Let $P \subseteq \mathbb{R}^d$ be a compact, convex set and assume either $P$ is strictly convex, or $d = 2$, or $P$ is a polytope. Let $A \subseteq \mathbb{R}^d$ be a closed set and consider the sets $A^\alpha_p$ and $A_\omega^\alpha$ defined above.

1. If $A_\omega^\alpha$ is nonempty, then player $I$ has a winning strategy in the no-$\beta$ McMullen game $G_P(A)$.

2. If $A_\omega^\alpha$ is empty, then player $II$ has a winning strategy in the no-$\beta$ McMullen game $G_P(A)$.

**Proof.** We first show that (1) follows from (2). Assuming (2), let $A \subseteq \mathbb{R}^d$ be a closed set with $A_\omega^\alpha \neq \emptyset$. This implies that $A^\omega \neq \emptyset$, where we recall that $A^\omega$ refers to the ordinary Cantor-Bendixson derivative. Since $A$ is closed, this means that $A$ is uncountable and in fact $A^\omega$ is a perfect set. By Theorems 2, 3, and 4, $II$ does not have a winning strategy in the game $G_P(A)$. The game $G_P(A)$ is a closed game for $I$, and so is determined by Gale-Stewart (see [3] or §6A of [7]). So, $I$ has a winning strategy for $G_P(A)$.

Now we show that (2) holds. Assume that $A_\omega^\alpha = \emptyset$. Since $P$ is fixed for the rest of the argument and we have no need for the ordinary Cantor-Bendixson derivative, we drop the subscripts $P$ for the rest of the argument, so we write $A^\omega$ for $A_\omega^\alpha$ and $A^\alpha$ for $A^\alpha_p$. If $A^\omega$ is empty, and player $I$ plays some copy $P_0$, then every point $x$ of $A$ on the boundary of $P_0$ has some rank $\alpha$ so that $x \in A^\alpha \setminus A^{\alpha+1}$.

Let $\alpha_0$ be minimal so that $P_0 \cap A_{\alpha_0+1} = \emptyset$. Such an $\alpha_0$ exists: first note that if there are no points of $A = A_0$ on the boundary of $P_0$, then we can immediately win the game, since by a compactness argument, there are no points of $A$ within some fixed $\epsilon$ of the boundary. Next note that each of the $A_{\alpha}$ are closed, and so $A_{\alpha} \cap \partial P_0$ is compact, and so the minimal $\alpha$ so that $\partial P_0 \cap A_\alpha = \emptyset$ cannot be a limit ordinal, as we would have a sequence of descending nonempty compact sets whose intersection is empty.

The next claim will essentially finish the proof by allowing us to construct a decreasing sequence of ordinals starting from $\alpha_0$, which will give a strategy for $II$. 

Claim 33. Suppose $A_{n+1} \cap P_n = \emptyset$ for $P_n$ some latest ball by player $I$ in the no-$\beta$ McMullen game. There is a finite-round strategy by player $II$ which eliminates all of $A_{n}$.

Proof. Suppose $A_{n+1} \cap P_n = \emptyset$. Note $A_n \cap \partial P_n$ is compact, and every point in it is not in $A_{n+1}$, so there is some rational ball around each point which contains no good copy of $P$ for $A_n$. For each point in $\partial P_n \setminus A_n$, there is some rational neighborhood which contains no points of $A_n$. By the compactness of $\partial P_n$, there is some finite subcover, which covers some closed $\epsilon$-neighborhood of $\partial P_n$. This closed strip is compact and so by the Lebesgue covering lemma, there’s some $\delta > 0$ small enough so that any ball of diameter less than $\delta$ inside this $\epsilon$-neighborhood must be contained in one of these rational neighborhoods, and thus contain no good copy of $P$ for $A_n$. Play a move $P_{n+1}$ as $II$ to delete enough of $P_n$ so that any move played by $I$ must lie in this neighborhood of $\partial P_n$ and must be of diameter less than $\delta$. Now if player $I$ plays $P_{n+2}$, then $P_{n+2}$ is not a good copy of $P$ for $A_n$.

Lemma 34. Let $P$ be a compact, convex set in $\mathbb{R}^d$ with non-empty interior. Let $B \subseteq \partial P$ be compact and such that there is a smaller copy of $P$, that is $Q = s + tP$ for $t < 1$, with $B \subseteq Q$. Then there is a neighborhood $U$ of $B$ so that for any $\epsilon \in (0, 1)$, $U \cap P$ is in the interior of $\epsilon s + P$.

Proof. Note that $-s + B \subseteq -s + Q = tP \subseteq P^\circ$. Let $V$ be a neighborhood of $-s + B$ with $-s + B \subseteq V \subseteq P$. Let $U = s + V$, so $B \subseteq U$. Let $\epsilon \in (0, 1)$. Let $x \in U \cap P$. Consider the line segment $L$ between $x$ and $-s + x \in V \subseteq P$. By convexity, $L \subseteq P$. Also by convexity, since $-s + x \in P^\circ$ we have that all points of $L$ except possibly $x$ are in $P^\circ$. In particular $-\epsilon s + x \in P^\circ$. Thus $x$ is in the interior of $\epsilon s + P$. So, $U \cap P$ is in the interior of $\epsilon s + P$ for any $\epsilon \in (0, 1)$.

Let $B \subseteq A_n \cap \partial P_{n+2}$ the set of points which are limit points of either $A_n \cap P_{n+2}^\circ$ or $A_n \cap \partial P_{n+2} \cap H$ for some supporting hyperplane $H$ in $\mathbb{R}^d$. Since $P_{n+2}$ is not good for $A_n$ there is a legal move $Q$ for $II$ which covers $B$. So, $Q = s + tP_{n+2}$ where $t < 1$. Let $U$ be a neighborhood of $B$ as in Lemma 34 for $P_{n+2}$. Let $V$ be a neighborhood of $B$ with $B \subseteq V \subseteq \partial V \subseteq U$. Let $C = A_n \cap P_{n+2}^\circ \setminus V$. By a simple compactness argument, there is an $\epsilon > 0$ such that $C \cap N(\partial P_{n+2}, \epsilon) = \emptyset$, where $N(\partial P_{n+2}, \epsilon) = \{x : \rho(x, \partial P_{n+2}) < \epsilon\}$ denotes the $\epsilon$-neighborhood of $\partial P_{n+2}$. So $C$ is actually closed. It follows that there is a $\delta > 0$ such that if we shrink and translate $P_{n+2}$ by less than $\delta$, then that copy of $P_{n+2}$ will still contain $C$. By Lemma 34 and the definition of $\delta$, there is a $0 < \delta' < \delta$ and a $\delta'$-translation $P'$ of $P_{n+2}$ such that $P'$ contains $U \cap P$ and hence $\partial V \cap P$. By compactness, there is a $\delta'' < \delta'$ such that the $\delta''$ shrinking of $P'$, say $P''$, also contains $\partial V \cap P$. So, $P''$ contains all of $B$ as well as $A_n \cap P_{n+2}^\circ$. It follows for the definition of $B$ that $A_n \cap (P_{n+2} \setminus P'')$ is a subset of $\partial P_{n+2} \cap H$ for any hyperplane $H$ in $\mathbb{R}^d$. Have $II$ play the move $P_{n+3} = P''$, and let $I$ respond by $P_{n+4} \subseteq P_{n+2} \setminus P_{n+3}$. We have $A_n \cap P_{n+4} \subseteq \partial P_{n+2} \cap H$.

Lemma 35. Let $P$ be a compact, convex set in $\mathbb{R}^d$ with non-empty interior. Let $Q = s + tP$ where $t < 1$ be a smaller, translated copy of $P$, and assume $Q \subseteq P$ and $\partial Q \cap \partial P \neq \emptyset$. Then the following hold. If $P$ is strictly convex, then $|\partial Q \cap \partial P| = 1$. If $d = 2$, then $\partial Q \cap \partial P$ is contained in the union of two line segments which are supporting hyperplanes for $P$. 


Proof. Suppose $Q \subseteq P$ and $x \in \partial P \cap \partial Q$. Let $x' = s + tx$ be the point “corresponding” to $x$ on $\partial Q$. Let $H(x, P)$ denote the set of hyperplanes for $P$ at $x$, that is, $x + H(x, P)$ is the set of (affine) supporting hyperplanes for $P$ at $x$. We clearly have that $H(x, P) = H(x', Q)$. Let $H \in H(x, P)$. Then $x + H$ is a supporting hyperplane for $P$ at $x$ and $x' + H$ is a supporting hyperplane for $Q$ at $x'$. We also have that the side of the $x + H$ that $P$ lies on must be the same side of $x' + H$ that $Q$ lies on. So we must have that $x + H = x' + H$ as $x'$ must lie on the “good” side of $x + H$ (as $x' \in Q \subseteq P$) and $x$ is in the “good” side of $x' + H$ (as $x \in Q$). Thus, $x' \in x + H$. Thus $x + H = x' + H$ is a supporting hyperplane for both $P$ and $Q$ at both $x$ and $x'$. It follows that if $x' \neq x$, then $x + H$ is a supporting hyperplane for any point on the line segment $L_{x,x'}$ between $x$ and $x'$ for both $P$ and $Q$, and so $L_{x,x'} \subseteq \partial P \cap \partial Q$.

Consider the map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be given by $f(x) = s + tx$. This is a contraction map, so there is a unique fixed point $x_\infty$ for $f$. Suppose $x \in \partial P \cap \partial Q$ is any point in the common boundary of $P$ and $Q$. Let $x_0 = x$ and $x_{i+1} = f(x_i)$. By the above we have that each line segment $L_{x_i,x_{i+1}} \subseteq \partial P \cap \partial Q$. Also, we have that all of the line segments $L_{x_i,x_{i+1}}$ lie on a single line $L_x$. To see this, simply note that for any point $y$, the points $y$, $f(y) = s + ty$ and $f(f(y)) = s + st + t^2y$ lie on a line. So we have that the line segment $L_{x,x_\infty}$ is contained in $\partial P \cap \partial Q$.

Suppose first that $P$ is strictly convex, and suppose $x \in \partial P \cap \partial Q$. Then we must have $x = x_\infty$, as otherwise we have a proper line segment $L_{x,x_\infty} \subseteq \partial P$. So, $\partial P \cap \partial Q = \{x\}$.

Suppose now $d = 2$, so $P \subseteq \mathbb{R}^2$. Suppose there is a point $x \neq x_\infty$ with $x \in \partial P \cap \partial Q$. So, $L_{x,x_\infty} \subseteq \partial P \cap \partial Q$. Suppose there is another point $y \in \partial P \cap \partial Q$ with $y$ not on the line $\ell_x$ containing $x$ and $x_\infty$. Then we likewise have that $L_{y,x_\infty} \subseteq \partial P \cap \partial Q$. Let $\ell_y$ be the line containing $y$ and $x_\infty$. We then claim that $\partial P \cap \partial Q \subseteq \ell_x \cup \ell_y$. For suppose $z \in \partial P \cap \partial Q$ with $z \notin \ell_x \cup \ell_y$. Consider the region $R$ of points between the two rays $L_{x,x_\infty}$ and $L_{y,x_\infty}$ emanating from $x_\infty$. Since $\ell_x$ and $\ell_y$ are supporting hyperplanes for $P$, we have $P \subseteq R$. Since $z \notin \ell_x \cup \ell_y$, $z$ must lie in the interior of $R$, and in fact $L_{x,x_\infty} \setminus \{x_\infty\} \subseteq R^c$. But, points on $L_{x,x_\infty} \setminus \{x_\infty\}$ close enough to $x_\infty$ are then in the interior of $P$, a contradiction.

We claim that $C = A_\alpha \cap \partial P_{n+2} \cap \partial P_{n+4}$ is finite. In the case where $P$ is strictly convex, $|A_\alpha \cap \partial P_{n+2} \cap \partial P_{n+4}| \leq 1$ by Lemma 33. Suppose $d = 2$. By Lemma 33 there are at most two line segments $L_1, L_2$ which are supporting hyperplanes for $P_{n+2}$ such that $\partial P_{n+2} \cap H \subseteq L_1 \cup L_2$. If $C$ were infinite, then $C \cap L_1$ or $C \cap L_2$ would be infinite and so have a limit point, which would be a point in $B$. Here $B$ (as before) denotes the set of points which are limit points of either $A_\alpha \cap P_{n+2}$ or $A_\alpha \cap \partial P_{n+2} \cap H$ for some supporting hyperplane $H$ in $\mathbb{R}^d$. This contradicts the fact that $P_{n+4} \cap B = \emptyset$. If the case where $P$ is a polytope, then $C$ is a subset of finitely many supporting hyperplanes for $P_{n+2}$. By the same argument, if $C$ were infinite we would get a point of $C$ in $B$, a contradiction.

In finitely many moves player II can delete all of $C$. This results in a play with final position $P_{n+4+2k}$ with $P_{n+4+2k} \cap A_\alpha = \emptyset$.

This completes the proof of Claim 33.

We have now defined a strategy $\tau$ for player II such for any run $(P_0, P_1, P_2, P_3, \ldots)$ according to $\tau$ then if we let $\alpha_{2n}$ be the largest ordinal such that $P_{2n} \cap A_{\alpha_{2n}} \neq \emptyset$
(assuming there is an ordinal such that \( P_{2n} \cap A_\alpha \neq \emptyset \), the largest such ordinal is well-defined by a simple compactness argument), then we have \( \alpha_0 \geq \alpha_2 \geq \cdots \), and for each \( n \) such that \( \alpha_{2n} > 0 \) there is an \( m > n \) such that \( \alpha_{2m} < \alpha_{2n} \). So, following \( \tau \) results in a move \( P_{2n} \) with \( P_{2n} \cap A = \emptyset \) (assuming \( I \) has followed the rules). Thus, \( \tau \) is a winning strategy for \( II \).

This completes the proof of Theorem 32. □

**Remark 36.** This proof of Theorem 32 does not require the axiom of choice, AC.

In the proof of (4) we invoked the Gale-Stewart theorem for the closed real game \( G_P(A) \). It is not hard to see that the game \( G_P(A) \) is equivalent to the version \( G_P'(A) \) in which \( II \) must play scaled copies \( s+tP \) with \( s \in \mathbb{Q}^d, t \in \mathbb{Q} \) (equivalent here means that if one of the players has a winning strategy in one of the games, then that same player has a winning strategy in the other game). Without AC, Gale-Stewart gives that one of the players has a winning quasistrategy (the interested reader can consult [7] for the definition) for \( G_P'(A) \). If \( II \) has a winning quasistrategy for \( G_P'(A) \), then (since \( II \)'s moves are coming from a countable set) \( II \) actually has a winning strategy for \( G_P'(A) \) and thus a winning strategy for \( G_P(A) \). So we get in this case that \( A \) is countable, a contradiction. So, \( I \) has a winning quasistrategy in \( G_P'(A) \). But again (using DC this time) this gives a winning strategy for \( I \) in \( G_P(A) \).

### 8. Some Examples and Questions

In this section we present an example of a compact, convex set \( P \subseteq \mathbb{R}^3 \) for which the methods of the previous theorems do not seem to apply. We also present an example due to David Simmons which shows that even in the case of \( \ell_2(\mathbb{R}) \) with \( P \) being the closed unit ball, the conclusion of Theorem 3 does not hold. These examples naturally motivate some questions which we pose.

Recall that Theorem 3 established the main result for the no-\( \beta \) McMullen game for the case of polytopes \( P \) in \( \mathbb{R}^3 \) in particular. The proof of this relied on the fact that there were only finitely many faces of \( P \), since these there were taken as some of the hyperplanes in Lemma 15. Thus, if there are infinitely many pairwise non-coplanar maximal line segments in \( \partial P \) we cannot use this argument. The authors have shown that in the case of \( P \) being a cone in \( \mathbb{R}^3 \) that the main theorem (the analog of Theorems 2, 5, 6) still holds. In this case, the set of maximal line segments in \( \partial P \) does not lie in the union of finitely many hyperplanes, however these line segments are pairwise coplanar in this case. When there are infinitely many pairwise non-coplanar line segments in \( \partial P \) then our arguments seem to break down, though we do not know of any counterexamples to our main result. We now present an example of a compact, convex set \( P \subseteq \mathbb{R}^3 \) such that the set of maximal line segments in \( \partial P \) contains an infinite, pairwise non-coplanar set.

Let \( C \) be a cone in \( \mathbb{R}^3 \) with vertex \( v \), and circular base \( B \). Let \( P_0 \) be the part of the cone between the base \( B \) and a parallel plane \( H \). Let \( L_0, L_1, \ldots \) be line segments with one endpoint on the circular edge of \( B \), and the other endpoint on \( H \cap \partial P_0 \). We choose these line segments so that the line \( \ell_i \) extending \( L_i \) passes through \( v \). We choose them so that the \( L_i \) have a unique limit segment \( L_\infty \) and the angles between \( \ell_i \) and \( \ell_\infty \) decreases monotonically to 0. There are easily neighborhoods \( U_0, U_1, \ldots \) of the line segments so that for each \( i \), \( U_i \) is disjoint from the convex hull of \( \bigcup_{j \neq i} U_j \). In fact, by choosing the \( U_i \) sufficiently small we may assume that for
each $i$ there is a hyperplane $H_i$ containing $L_i$ such that the convex hull of $\bigcup_{j \neq i} U_j$ lies strictly on one side of $H_i$.

We now choose line segments $L_i' \in U_i$ such that no distinct pair $L_i', L_j'$ are coplanar, and we may in fact choose them so that their endpoints are on the circles defined by $B$ and $H$. We may also choose the $L_i'$ close enough to the $L_i$ so that there is a hyperplane $H_i'$ containing $L_i'$ such that the convex hull of $\bigcup_{j \neq i} U_j$ lies strictly on one side of $H_i'$. In particular, the convex hull of $\bigcup_{j \neq i} L_j'$ lies strictly on one side of $H_i'$. Let $P$ be the convex hull of $\bigcup_i L_i'$. We have that each of the line segments $L_i'$ lies in the boundary of $P$, since $H_i'$ is a supporting hyperplane for $P$ containing $L_i'$.

Note also that each of the line segments $L_i'$ is also maximal with respect to $P$ (as in Definition 9). See Figure 4 for an illustration of the region.

There are two main results of this paper. The first is that for a compact, convex set $P \subseteq \mathbb{R}^d$ satisfying the hypotheses of Theorem 2, 3, or 4 (that is either $P$ is strictly convex, $d = 2$, or $P$ is a polytope), then the no-$\beta$ McMullen game is equivalent to the perfect set game. The second result is that, under these same hypotheses on $P$, for any closed set $A \subseteq \mathbb{R}^d$ we have that $A$ is countable iff the limiting derivative $A^\infty_P = \emptyset$. The proofs of both of these results needed the extra hypothesis on $P$. We do not know if either of these results is true for general compact, convex set $P \subseteq \mathbb{R}^d$ (for $d > 2$).

**Question 1.** If $P$ is a general compact, convex set in $\mathbb{R}^d$, then is the no-$\beta$ McMullen game equivalent to the perfect set game?

**Question 2.** If $P$ is a general compact, convex set in $\mathbb{R}^d$, and $A \subseteq \mathbb{R}^d$ a closed set, then is it the case that $A$ is countable iff $A^\infty_P = \emptyset$?

Another question one could ask is whether the results of this paper extend to infinite dimensional spaces. We now present a result due to David Simmons which shows that in the case of $\ell_2(\mathbb{R})$ with $P$ being the closed unit ball, the results do not. In particular, the conclusion of Theorem 8 does not hold in this case. The closed unit ball, of course, is not compact, so one could still ask if the main theorem holds for compact, strictly convex sets $P$ in $\ell_2(\mathbb{R})$. 

![Figure 4. The convex set $P \subseteq \mathbb{R}^3$](image-url)
Example 37 (David Simmons). Let $X = \{-1, 1\}^\omega$, and let
\[
\{e_u : u \in \{-1, 1\}^\omega\}
\]
be a basis for an infinite dimensional Hilbert space $\mathcal{H}$. Consider the map $\pi : X \to \mathcal{H}$
\[
\pi(x) = \sum_{n \in \omega} 2^{-n} x_n e_{x|n}
\]
Let $K = \pi(X)$ and note that $K$ is compact.

Claim 38. If $y$ is in the convex hull of $K \cap \partial B(y, \rho)$ for some $\rho$, then $K \cap B(y, \rho)^\circ = \emptyset$.

Proof. Since $y$ is in the convex hull of $K \cap \partial B(y, \rho)$, there is a measure $\mu$ on $X$, the support of which is a subset of $\pi^{-1}(K \cap \partial B(y, \rho))$, so that
\[
y = \int \pi(x) \, d\mu(x)
\]
Note that
\[
y_u = y \cdot e_u = 2^{-|u|-1} (\mu([u^{-1}]) - \mu([u^{-1}]))
\]
Suppose for the sake of a contradiction that there was some $z \in K \cap B(y, \rho)^\circ$, and let $\alpha \in X$ so that $\pi(\alpha) = z$. Since $z$ is not on the boundary of the ball, which contains the support of $\mu$, there must be some shortest initial segment $\alpha|(m + 1)$ of $\alpha$ so that $\mu([\alpha|(m + 1)]) = 0$. Define $\beta$ inductively to be the extension of $\alpha|m$ where every initial segment has the larger $\mu$-measure, i.e.
\[
\beta|m = \alpha|m
\]
and
\[
\beta(m + k) = \begin{cases} 1 & \mu([\beta|(m - 1 + k)^\omega]) \geq \mu([\beta|(m - 1 + k)^\omega]) \\ -1 & \text{otherwise} \end{cases}
\]
Now we compute
\[
d(y, \pi(\alpha))^2 - d(y, \pi(\beta))^2
\]
and we will show this quantity is nonnegative, contradicting the fact that $\pi(\beta)$ is on the boundary of the ball, while $\pi(\alpha)$ is in the interior. Note that the terms coming from the initial segments $\beta|m$ and $\alpha|m$ cancel, and by our choice of basis for $\mathcal{H}$, the rest of the terms are coming from orthogonal basis vectors. Note that $y_{\alpha|k} = 0$ for all $k > m$ as $\mu([\alpha|k]) = 0$, using Equation (1).
\[= (-2\alpha(m)y_{\alpha|m} + 2\beta(m)y_{\beta|m}) + \sum_{u \leq \alpha \atop |u| > m} (y_u - \pi(\alpha)_u)^2 + \sum_{u \leq \beta \atop |u| > m} (y_u)^2\]

\[- - \sum_{u \leq \alpha \atop |u| > m} 0 - \sum_{u \leq \beta \atop |u| > m} (y_u - \pi(\beta)_u)^2\]

\[\geq 4\beta(m)y_{\beta|m} + \sum_{u \leq \alpha \atop |u| > m} (y_u - \pi(\alpha)_u)^2 - \sum_{u \leq \beta \atop |u| > m} (y_u - \pi(\beta)_u)^2\]

\[= 4\beta(m)^2\mu([\beta](m + 1)) + \sum_{u \leq \alpha \atop |u| > m} (y_u - \pi(\alpha)_u)^2 - \sum_{u \leq \beta \atop |u| > m} (y_u - \pi(\beta)_u)^2\]

\[\geq 0 \text{ since } \operatorname{sgn}(\beta_n) = \operatorname{sgn}(\mu([\beta](n^\circ1)) - \mu([\beta](n^\circ(-1))) - 2^{-n}\beta_n)^2\]

From Claim \[\] it follows that there are no good balls in \(\ell_2(\mathbb{R})\) for the set \(A\) (recall the definition of a good copy of \(P\) in Definition \[\) here \(P\) is the closed unit ball in \(\ell_2(\mathbb{R})\)). For suppose \(B(y, \rho)\) were a good ball for the set \(A\). Let \(B \subseteq A \cap \partial B(y, \rho)\) be the points of \(A \cap \partial B(y, \rho)\) which are limits of \(A \cap B(y, \rho)^c\). Note here that \(B(y, \rho)\) is strictly convex so no points \(x\) of \(A \cap \partial B(y, \rho)\) are limits of points in supporting hyperplanes for \(B(y, \rho)\) containing \(x\). Since \(B(y, \rho)\) is a good ball, there is no translated smaller copy of \(B(y, \rho)\) which contains \(B\). If \(y\) is in the closed convex hull \(C\) of \(B\), then from Claim \[\] we have that \(A \cap B(y, \rho)^c = \emptyset\), and so \(B\) is empty, a contradiction. So, \(y\) is not in the closed convex hull of \(B\), and thus there is a hyperplane \(H\) in \(\ell_2(\mathbb{R})\) which strictly separates \(C\) from an \(\epsilon\) neighborhood \(B(y, \epsilon)\) of \(y\). An easy argument now shows that if \(\vec{n}\) denotes the normal vector to \(H\) in the direction of \(C\), then for all small enough \(\delta > 0\), the translated ball \(B(y + \delta\vec{n}, \rho)\) contains a neighborhood of \(C\). So for small enough \(\eta\), \(B(y + \delta\vec{n}, (1 - \eta)\rho)\) contains \(C\). This shows that \(B(y, \rho)\) is not good.

So in fact we have that in \(\ell_2(\mathbb{R})\) there is a closed, strictly convex set \(P\) (the closed unit ball) and a perfect set \(A\) (the set \(A = \pi(X)\) of Claim \[\) such that \(A_P = \emptyset\). Since \(P\) is not compact, this doesn’t show that Theorem \[\) cannot be extended to infinite dimensional spaces, but does that the current methods encounter difficulties in trying to do so. Even supposing the strictly convex set \(P\) is compact, although some of the arguments of Theorem \[\) go through, Lemma \[\) would no longer suffice as we would need a version of Lemma \[\) for infinitely many cones \(C_i\).

**Question 3.** Do Theorems \[\) and \[\) hold for strictly convex, compact \(P \subseteq \ell_2(\mathbb{R})\)?

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