Abstract

We present a canonical formalism facilitating investigations of the dynamical Casimir effect by means of a response theory approach. We consider a massless scalar field confined inside of an arbitrary domain \( G(t) \), which undergoes small displacements for a certain period of time. Under rather general conditions a formula for the number of created particles per mode is derived. The perturbative approach reveals the occurrence of two generic processes contributing to the particle production: the squeezing of the vacuum by changing the shape and an acceleration effect due to motion of the boundaries. The method is applied to the configuration of moving mirror(s). Some properties as well as the relation to local Green function methods are discussed.

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1 Introduction

After the discovery of the static Casimir effect [1] (see, e.g. Refs. [2], [3] and [4] for a review), the possibility of creating particles out of the vacuum by moving one of the mirrors (see, e.g. [1]-[2]) or both plates (see, e.g. [1]) has been analyzed. Other authors calculated the radiation from one single mirror (see [3]-[7]) and the backreaction on perfectly or nonperfectly conducting boundaries (see, e.g. [21]-[29]). Many interesting studies also have been devoted to the analysis of quantum vacuum radiation induced by moving dielectrics [31]. We discuss a general Hamiltonian formalism for an arbitrary domain \( G(t) \) with Dirichlet boundary conditions that experiences small changes during a time interval \((0, T)\):

\[
G(t < 0) = G(t > T) = G_0.
\]

In our derivations we imply that the time-dependent disturbances \( \Delta G(t) = G(t) \ominus G_0 \) of the boundary can be considered as small with respect to some parameter \( \varepsilon \), i.e. \( \Delta G = \mathcal{O}(\varepsilon) \). Having introduced a proper definition of particles together with a vacuum state, we calculate the number of produced particles within the framework of response theory. The result will be applied to the special case of moving mirror(s) and the relation to results obtained by means of the adiabatic approach or local Green function methods will be indicated.
2 Canonical formulation

2.1 Equations of motion

We consider a non-interacting real massless scalar (Klein-Gordon) field in Minkowski-space-time with the Dirichlet-boundary-conditions: $\Phi = 0$ at $\partial G(t)$ and $\Box \Phi = 0$ in $G(t)$. In the following we shall not quantize the degrees of freedom assigned to the motion of the boundary. Focussing on the scalar field sector only, we consider the Lagrangian ($\hbar = c = 1$ throughout)

$$L = \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi$$  \hspace{1cm} (2)

Expanding $\Phi$ in terms of eigenfunctions $f_\alpha(\vec{r}, t)$ that fullfil $f_\alpha = 0$ at the boundary $\partial G(t)$, i.e.,

$$\Phi(\vec{r}, t) = \sum_\alpha q_\alpha(t) f_\alpha(\vec{r}, t)$$  \hspace{1cm} (3)

and calculating the Lagrangian by making use of the properties Eqs. (70)-(72) given in the appendix, we arrive at

$$L = \int_{G(t)} dV \left[ \frac{1}{2} \dot{q}_\alpha^2 - \frac{1}{2} \Omega_\alpha^2(t) q_\alpha^2 + q_\alpha \mathcal{M}_{\alpha\beta}(t) \dot{q}_\beta + \frac{1}{2} q_\alpha \mathcal{M}_{\alpha\gamma}(t) \mathcal{M}_{\beta\gamma}(t) q_\beta \right]$$  \hspace{1cm} (4)

together with the eigenvalue equation

$$\nabla^2 f_\alpha(\vec{r}, t) = -\Omega_\alpha^2(t) f_\alpha(\vec{r}, t)$$  \hspace{1cm} (5)

and the coupling matrix

$$\mathcal{M}_{\alpha\beta}(t) = \int_{G(t)} dV \frac{\partial f_\alpha(\vec{r}, t)}{\partial t} \frac{\partial f_\beta(\vec{r}, t)}{\partial t}$$  \hspace{1cm} (6)

In view of the orthonormality of the $f_\alpha$ (see Eq. (71) in the appendix; $dG = dV$; $d\mathcal{G} = \vec{v} d\vec{A}$) and the required boundary-conditions the $\mathcal{M}_{\alpha\beta}$ turn out to be antisymmetric:

$$\mathcal{M}_{\alpha\beta}(t) + \mathcal{M}_{\beta\alpha}(t) = \int_{G(t)} dG \frac{\partial}{\partial t} [f_\alpha(\vec{r}, t)f_\beta(\vec{r}, t)] = \frac{d}{dt} \delta(\alpha, \beta) - \int_{\partial G(t)} d\mathcal{G} f_\alpha(\vec{r}, t)f_\beta(\vec{r}, t) = 0$$  \hspace{1cm} (7)

In Eq. (4) and in most of the following formulae we drop the summation- and integration signs and declare that one has to sum over all multi-indices like $\alpha, \beta$ etc. that do not occur on both sides of the equation. Introducing the canonical conjugate momenta

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \dot{q}_\alpha + q_\beta \mathcal{M}_{\beta\alpha}(t)$$  \hspace{1cm} (8)
the Hamiltonian takes on the form:

$$H(t) = \frac{1}{2} \dot{p}_\alpha^2 + \frac{1}{2} \Omega^2_\alpha(t) q_\alpha^2 + p_\alpha \mathcal{M}_{\alpha\beta}(t) q_\beta .$$  \hspace{1cm} (9)$$

There are two effects which could lead to an unstable vacuum: The nonstationary eigenfrequencies \(\Omega_\alpha(t)\) due to a dynamical change of the shape of the domain \(G(t)\) – we shall refer to this effect as “squeezing” of the vacuum – and the additional \(q_\alpha-p_\beta\)-coupling \(\mathcal{M}_{\alpha\beta}\), indicating the motion of the boundaries – the “acceleration”-effect. The total energy of the \(\Phi\)-field is given as the integral over the time-dependent domain \(G(t)\) of the energy density \(T_{00}\)

$$E(t) = \int_{G(t)} dV T_{00} = \frac{1}{2} \dot{p}_\alpha^2 + \frac{1}{2} \Omega^2_\alpha(t) q_\alpha^2$$  \hspace{1cm} (10)$$
of the minimal coupled energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi .$$  \hspace{1cm} (11)$$

Comparison with Eq. (9) reveals that

$$H(t) = E(t) + W(t)$$  \hspace{1cm} (12)$$

holds. The time-dependent transformation \(\Phi(\vec{r},t) \rightarrow q_\alpha(t)\) results in the difference between the Hamiltonian, describing the time-evolution of the \(q_\alpha\), and that of \(\Phi\) (which is equal to the field energy):

$$H[q_\alpha, p_\beta, t] \neq H[\Phi, \Pi, t] = E[\Phi, \Pi, t] = E[q_\alpha, p_\beta, t] .$$  \hspace{1cm} (13)$$

### 2.2 Quantization

Now we perform the usual canonical quantization, assuming the following set of equal-time-commutation-relations:

$$[\hat{q}_\alpha(t), \hat{q}_\beta(t)] = [\hat{p}_\alpha(t), \hat{p}_\beta(t)] = 0 , \hspace{1cm} (14)$$

$$[\hat{q}_\alpha(t), \hat{p}_\beta(t)] = i\delta(\alpha, \beta) . \hspace{1cm} (15)$$

Note that, because of (3) and (72) together with the decomposition

$$\Pi(\vec{r},t) = \dot{\Phi}(\vec{r},t) = p_\alpha(t) f_\alpha(\vec{r},t)$$  \hspace{1cm} (16)$$
of the conjugate momenta these commutation relations are consistent with those between the fields:

$$[\hat{\Phi}(\vec{r},t), \hat{\Phi}(\vec{r}',t)] = [\hat{\Pi}(\vec{r},t), \hat{\Pi}(\vec{r}',t)] = 0$$  \hspace{1cm} (17)$$

and

$$[\hat{\Phi}(\vec{r},t), \hat{\Pi}(\vec{r}',t)] = i\delta(\vec{r} - \vec{r}') , \hspace{1cm} (18)$$

which are valid inside the domain \(G(t)\). Equation (16) and therefore Eq. (18) are not pointwise equalities (think of \(\partial G\)), they have to be read as identities of \(L^2(G)\)-distributions.
3 Vacuum-definition

3.1 Interaction-representation

For performing the pertubation-theory we shall adopt the interaction representation. Accordingly, the time-evolution of the operators will be governed by the undisturbed energy operator $E_0 = \hat{E}(t < 0) = \hat{E}(t > T)$ defined via

$$\hat{H}(t) = \hat{E}(t) + \hat{W}(t) = \hat{E}_0 + \Delta \hat{E}(t) + \hat{W}(t) = \hat{E}_0 + \hat{H}_1(t) = \hat{H}_0 + \hat{H}_1(t) \quad .$$ (19)

Time-dependent operators $\hat{A}(t)$ obey the equation of motion

$$\frac{d\hat{A}}{dt} = i[\hat{E}_0, \hat{A}] + \frac{\partial \hat{A}}{\partial t},$$ (20)

while the dynamics of any given quantum state $|\psi\rangle$ is described by

$$\frac{d}{dt} |\psi\rangle = -i\hat{H}_1(t) |\psi\rangle \quad ,$$ (21)

where the interaction Hamiltonian $\hat{H}_1$ is specified in Eqs. (35)–(37) below.

3.2 The number operator

A proper set of particle creation and annihilation operators should be introduced with respect to the unperturbed Hamiltonian $\hat{H}_0 = \hat{H}(t < 0) = \hat{H}(t > T) = \hat{E}_0$ according to

$$\hat{a}_\alpha(t) = (2\Omega_0^\alpha)^{-1/2}(\Omega_0^\alpha \hat{q}_\alpha(t) + i\hat{p}_\alpha(t)) \quad ,$$ (22)

together with the static frequencies $\Omega_0^\alpha = \Omega_\alpha(t < 0) = \Omega_\alpha(t > T)$.

They obey the equation of motion:

$$\frac{d\hat{a}_\alpha}{dt} = i[\hat{E}_0, \hat{a}_\alpha] = -i\Omega_0^\alpha \hat{a}_\alpha$$ (23)

and the commutation relation

$$[\hat{a}_\alpha(t), \hat{a}_\beta^+(t)] = \delta(\alpha, \beta) \quad .$$ (24)

These particle creation and annihilation operators diagonalize the unperturbed Hamiltonian, i.e., expressed in terms of the corresponding number operator

$$\hat{N}_\alpha = \hat{a}_\alpha^+(t)\hat{a}_\alpha(t)$$ (25)

it thus takes the form

$$\hat{E}_0 = \Omega_0^\alpha \left(\hat{N}_\alpha + \frac{1}{2}\right) \quad .$$ (26)

Evidently we define the vacuum $|0\rangle$ as the ground state of $\hat{E}_0$:

$$\forall \alpha : \hat{a}_\alpha |0\rangle = 0 \quad .$$ (27)

So $\hat{N}_\alpha$ counts the physical relevant particles in the mode $\alpha$ before and after the motion of the boundaries (due to $|0, in\rangle = |0, out\rangle$), a particle definition during the movement is not so easy to obtain (see section 5 and 6).
4 Particle creation

4.1 Response theory

Now we investigate the change of the state vector $|\psi\rangle$, which satisfies the initial condition

$$|\psi(t < 0)\rangle = |0\rangle \quad ,$$

(28)
due to a small time-bounded but otherwise arbitrary motion of the boundary by computing the number of created particles per mode $\alpha$ to first non-vanishing order perturbation theory. Equation (21) can be formally integrated by means of the time ordering operator $T$

$$|\psi(T)\rangle = T \left[ \exp \left\{ -i \int_0^T dt \hat{H}_1(t) \right\} \right] |0\rangle \quad ,$$

(29)

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^T dt_n \cdots \int_0^T dt_1 \ T \left[ \hat{H}_1(t_n) \cdots \hat{H}_1(t_1) \right] |0\rangle \quad .$$

Assuming small perturbations we shall keep only the lowest-order terms of the expansion above. The time-evolved vacuum state reads

$$|\psi(T)\rangle = \left[ 1 - i \int_0^T dt \hat{H}_1(t) \right] |0\rangle + \mathcal{O}(\hat{H}_1^2) \quad .$$

(30)

In view of the property

$$\forall \alpha \quad \hat{N}_\alpha |0\rangle = 0 \quad ,$$

(31)

the number operator has no linear response; the first non-vanishing order is quadratic. But other operators $\hat{A}$ with $\hat{A} |0\rangle \neq 0$ such as e.g. the components of the energy-momentum tensor $\hat{T}_{\mu\nu}$, possess a linear response:

$$\langle \psi(T) | \hat{A} | \psi(T) \rangle = \langle 0 | \hat{A} | 0 \rangle + i \int_0^T dt \langle 0 | [\hat{H}_1(t), \hat{A}] | 0 \rangle + \mathcal{O}(\hat{H}_1^2) \quad .$$

(32)

To be complete we note here the general expression for the quadratic response:

$$\langle \psi(T) | \hat{A} | \psi(T) \rangle = \langle 0 | \hat{A} | 0 \rangle + i \int_0^T dt \langle 0 | [\hat{H}_1(t), \hat{A}] | 0 \rangle$$

$$+ \int_0^T dt \int_0^T dt' \left( \langle 0 | \hat{H}_1(t) \hat{A} \hat{H}_1(t') | 0 \rangle - \frac{1}{2} \langle 0 | \left\{ \mathcal{T}[\hat{H}_1(t)\hat{H}_1(t')], \hat{A} \right\} | 0 \rangle \right) + \mathcal{O}(\hat{H}_1^3) \quad .$$

(33)

We are now in the position to calculate the number of particles $N_\alpha$ created in the specific mode $\alpha$ after the time duration $T$, when the boundaries are again at rest. We have to evaluate the matrix element

$$\langle \psi(T) | \hat{N}_\alpha | \psi(T) \rangle = \int_0^T dt \int_0^T dt' \langle 0 | \hat{H}_1(t) \hat{N}_\alpha \hat{H}_1(t') | 0 \rangle + \mathcal{O}(\hat{H}_1^3) = N_\alpha + \mathcal{O}(\hat{H}_1^3) \quad .$$

(34)
with the interaction Hamiltonian
\[ \hat{H}_1(t) = \Delta \hat{E}(t) + \hat{W}(t) , \] (35)
where
\[ \Delta \hat{E}(t) = \frac{1}{2} \hat{q}_\alpha(t) \Delta \Omega_\alpha^2(t) \] (36)
and
\[ \hat{W}(t) = \hat{q}_\alpha(t) \mathcal{M}_{\alpha\beta}(t) \hat{p}_\beta(t) . \] (37)
According to Eq. (7) \( \mathcal{M}_{\alpha\alpha}(t) = 0 \) holds and thus the cross terms vanish, i.e.
\[ \forall \alpha \langle 0 | \hat{W}(t) \hat{N}_\alpha \Delta \hat{E}(t') | 0 \rangle = 0 . \] (38)
As a consequence, one obtains
\[ \hat{N}_\alpha = \int_0^T dt \int_0^T dt' \langle 0 | \Delta \hat{E}(t) \hat{N}_\alpha \Delta \hat{E}(t') | 0 \rangle + \langle 0 | \hat{W}(t) \hat{N}_\alpha \hat{W}(t') | 0 \rangle . \] (39)
Up to quadratic order the squeezing- (first term) and the acceleration-effect (second term) decouple, so that: \( \hat{N}_\alpha = \hat{N}_\alpha^S + \hat{N}_\alpha^A \). With the aid of Eqs. (36) and (37) we get:
\[ \hat{N}_\alpha = \int_0^T dt \int_0^T dt' \frac{1}{4} \Delta \Omega_\beta^2(t) \Delta \Omega_\gamma^2(t') \langle 0 | \hat{q}_\beta^2(t) \hat{N}_\alpha \hat{q}_\gamma^2(t') | 0 \rangle + \int_0^T dt \int_0^T dt' \mathcal{M}_{\alpha\lambda}(t) \mathcal{M}_{\sigma\tau}(t) \langle 0 | \hat{q}_\alpha(t) \hat{p}_\lambda(t) \hat{N}_\alpha \hat{q}_\sigma(t') \hat{p}_\tau(t') | 0 \rangle . \] (40)
Evaluation of the expectation values by utilizing the equation of motion (in the \( \hat{E}_0 \)-dynamic) leads to
\[ \hat{N}_\alpha = \int_0^T dt \int_0^T dt' \frac{1}{4(\Omega_\alpha^0)^2} \Delta \Omega_\alpha^2(t) \Delta \Omega_\alpha^2(t') \exp(2i \Omega_\alpha^0 [t' - t]) \]
\[ + \int_0^T dt \int_0^T dt' S_{\alpha\beta}(t) S_{\alpha\beta}(t') \exp(i[\Omega_\alpha^0 + \Omega_\beta^0][t' - t]) \] (41)
with the symmetric matrix:
\[ S_{\alpha\beta}(t) = S_{\beta\alpha}(t) = \frac{1}{2} \mathcal{M}_{\alpha\beta}(t) \left( \sqrt{\frac{\Omega_\alpha^0}{\Omega_\alpha}} - \sqrt{\frac{\Omega_\beta^0}{\Omega_\beta}} \right) . \] (42)
A more compact form of Eq. (41) is derived by using the Fourier-transformation \( \mathcal{F} \) with \( \varphi(t) \rightarrow \tilde{\varphi}(\omega) = [\mathcal{F}\varphi](\omega) \) :
\[ N_\alpha = \frac{1}{4(\Omega_0^\alpha)^2} |\Delta \Omega_\alpha^2(2\Omega_0^\alpha)|^2 + |\mathcal{S}_{\alpha\beta}(\Omega_0^\alpha + \Omega_0^\beta)|^2 = N_\alpha^S + N_\alpha^A. \] (43)

This spectral representation above provides the main result of the perturbative approach: The number of particles created in the mode \( \alpha \) (with an energy \( \Omega_0^\alpha \)) decomposes into a squeezing and an acceleration contribution. The occurrence of these two distinct contributions reflects the two basic degrees of freedom that characterize dynamical changes of the boundaries: deformations in shape and motion of the boundaries. The total number of particles produced after the time interval \((0, T)\) is obtained by summing/integrating over all modes \( \alpha \):

\[ N = \sum_\alpha N_\alpha. \] (44)

However, this quantity is in general ill-defined and requires an appropriate regularization. This may be most easily achieved by introducing explicitly a frequency cut-off which simulates imperfect conducting boundary conditions. Whether a cut-off-independent contribution remains after its removal will depend on the particular configuration of boundaries under consideration. We should also note, that for “well behaving” time dependencies the spectral form (43) may provide sufficient convergence of the summation/integration in the limit of perfect conductors.

### 4.2 Discussion

Formula (43) gives the produced number of particles in a given mode \( \alpha \) in lowest order perturbation-theory in \( \hat{H}_1 \). If we want to analyze a particular situation within the perturbative approach we need to specify the very general quantity \( \hat{H}_1 \). Furthermore, we have to introduce the magnitude of the displacement of the boundaries \( \varepsilon \). By inspection, we see that all quantities entering \( \hat{H}_1 \) are at least of order \( \varepsilon \), i.e., \( \Delta \Omega_\alpha^2 = \mathcal{O}(\varepsilon) \) and \( \mathcal{M}_{\alpha\beta} = \mathcal{O}(\varepsilon) \) (and \( \Delta f_\alpha = \mathcal{O}(\varepsilon) \) as well) reveals that \( \hat{H}_1(t) = \mathcal{O}(\varepsilon) \). This already indicates that both, the squeezing- and the acceleration-effect are of the same order of magnitude. As a matter of fact, the adiabatic approximation applied frequently in studies cannot be sufficient in general. In the adiabatic approach the acceleration term (second term) is not obtained. Consequently, the dynamics reduces to a set of decoupled ordinary differential equations of the form:

\[ \ddot{x}_\alpha(t) + \Omega_\alpha^2(t)x_\alpha(t) = 0. \] (45)

They can be solved by means of a scattering-theory approach (see e.g. [15]). The adiabatic approach thus mainly accounts for the squeezing contribution. We like to stress that there are indeed situations, where \( N_\alpha^S \) is much smaller than \( N_\alpha^A \). Let \( \Omega_\alpha(t) \) have a functional form corresponding to a reflectionless potential, for example:

\[ \Delta \Omega_\alpha^2(t) = \frac{2\nu^2}{\cosh^2(\nu t)} \] (46)

one obtains \( N_\alpha^S = 0 \). Or, consider e.g. harmonic oscillations of the boundaries with the frequency \( \omega_0 \) over a long period \( T \) with \( T\omega_0 \gg 1 \). In that case only particles with \( \Omega_\alpha = \frac{1}{T}\omega_0 \) are produced by the squeezing effect, but the acceleration effect also leads to the production of
particles with other frequencies. This example already demonstrates the high-resonant character of the squeezing term, while the acceleration term does not have this property due to the summation/integration over the modes. As mentioned above, another feature of the Fourier transformation leads to the result, that all expressions like total number of particles, total energy, etc. are convergent after summation/integration over all modes, if (and only if) the time-dependent function of the displacement of the boundaries is smooth enough.

5 Squeezing effects

To investigate the pure squeezing effect it is convenient to neglect any explicit time-dependence of the system. In the following considerations we restrict to situations, where the time-dependence enters only implicitly via some global length parameter $\chi$ describing the changes of the shape of the boundary $\partial G$. In general $\chi$ could abbreviate a set of suitable parameters characterizing the dynamics of the shape. While the pure acceleration effect can appear separately e.g. as a rigid motion of the boundary as a whole, there seem to be no possible $G(t)$ that will lead in a pure squeezing effect only, since there is no shape-changing without inducing motions of the boundaries. Accordingly, the $\Omega_\alpha$ become functions of $\chi$ and therefore also the creation- and annihilation-operators as well:

$$\hat{a}_\alpha(\chi) = (2\Omega_\alpha(\chi))^{-1/2}(\Omega_\alpha(\chi)\hat{q}_\alpha + i\hat{p}_\alpha) \quad .$$

The same holds for the total energy of the field:

$$\hat{E}(\chi) = \Omega_\alpha(\chi) \left( \hat{N}_\alpha(\chi) + \frac{1}{2} \right) = \hat{E}_N + E_Z \quad ,$$

where $E_Z$ denotes the zero-point energy and $\hat{E}_N$ that of the contained particles. Equivalently,

$$\forall \alpha : \hat{a}_\alpha(\chi) \ket{0(\chi)} = 0 \quad (49)$$

for the vacuum as the ground state of $\hat{E}(\chi)$. Now we investigate the potential arising from the constrained vacuum of the quantized scalar field, which may be expanded around a fixed but arbitrary (shape) configuration $\chi = \chi_0 + \Delta \chi$:

$$V(\chi) = \bra{0(\chi_0)} \hat{E}(\chi) \ket{0(\chi_0)} = V(\chi_0) + \left( \frac{\partial V}{\partial \chi} \right)_{\chi_0} \Delta \chi + \frac{1}{2} \left( \frac{\partial^2 V}{\partial \chi^2} \right)_{\chi_0} \Delta \chi^2 + \mathcal{O}(\Delta \chi^3) \quad .$$

Formally we can introduce a force via

$$F(\chi_0) = \left( \frac{\partial V}{\partial \chi} \right)_{\chi_0} = \left( \frac{\partial E_Z}{\partial \chi} \right)_{\chi_0} = \left( \frac{\partial \Omega_\alpha}{\partial \chi} \right)_{\chi_0} \quad ,$$

which should lead to the well-known Casimir force after appropriate regularizations have been performed \[30\]. However, the second-derivative term

$$\left( \frac{\partial^2 V}{\partial \chi^2} \right)_{\chi_0} = \left( \frac{\partial^2 E_Z}{\partial \chi^2} \right)_{\chi_0} + \bra{0(\chi_0)} \left( \frac{\partial^2 \hat{E}_N}{\partial \chi^2} \right)_{\chi_0} \ket{0(\chi_0)}$$

(52)
with
\[ \langle 0(\chi_0) | \left( \frac{\partial^2 \tilde{E}_N}{\partial \chi^2} \right)_{\chi_0} | 0(\chi_0) \rangle = \Omega_\alpha \langle 0(\chi_0) | \left( \frac{\partial^2 \tilde{N}_\alpha}{\partial \chi^2} \right)_{\chi_0} | 0(\chi_0) \rangle \]
\[ = 2\Omega_\alpha \langle 0(\chi_0) | \left( \frac{\partial \tilde{a}_\alpha^+}{\partial \chi} \right)_{\chi_0} \left( \frac{\partial \tilde{a}_\alpha}{\partial \chi} \right)_{\chi_0} | 0(\chi_0) \rangle \]
\[ = \Omega \alpha \langle 0(\chi_0) | (\partial_2 \tilde{E})_{\chi_0} (\partial_2 \tilde{N})_{\chi_0} | 0(\chi_0) \rangle \]
(53)

and
\[ \frac{\partial \tilde{a}_\alpha}{\partial \chi} = \frac{1}{2\Omega_\alpha} \frac{\partial \Omega_\alpha}{\partial \chi} \hat{a}_\alpha^+ \]
(54)
gives rise to an additional parabolic potential due to the fact that
\[ \langle 0(\chi) | \frac{\partial^2 \hat{E}}{\partial \chi^2} | 0(\chi) \rangle \neq \frac{\partial^2}{\partial \chi^2} \langle 0(\chi) | \hat{E}(\chi) | 0(\chi) \rangle . \]
(55)

As a consequence, after the summation/integration over all modes \( \alpha \) is performed the additional parabolic potential has an infinite strength (easy to verify e.g. for the moving mirror example; in contrast to the parabolic potential the Casimir force turns out to be convergent because its divergent parts cancel, if both sides of the mirror are taken into account) which counteracts to any displacements of the boundary. This leads to the conclusion that for an empty closed system of perfectly conducting boundaries at zero temperature it would be impossible to observe the static Casimir effect via measurements of the Casimir forces exerted on the boundary. As we can see in (53), any finite change of \( \chi \) would lead to an infinite amount of produced particles so that the backreaction would compensate the Casimir force even after an infinitesimal displacement.

\[ \hat{W}(t) | 0 \rangle = \frac{i}{2} S_{\alpha \beta}(t) \hat{a}_\alpha^+ \hat{a}_\beta^+ | 0 \rangle \neq 0 \]
(56)

the ground state of \( \hat{E}_0 \) is not stable under the time-evolution of \( \hat{H} \), even for constant velocities. Therefore, for nonvanishing \( \tilde{\eta} \), the diagonalization of \( \hat{E}_0 \) does not yield a proper definition of creation/annihilation operators \( \hat{a}_\alpha^+ / \hat{a}_\alpha \) that describe physical (i.e. Lorentz-invariant) particles. Only in the frame where the boundaries are (globally) at rest a reasonable definition of particles by diagonalization of \( \hat{E}_0 \) is possible. One should notice that:
\[ \hat{H}(t) = \hat{E}_0 + \hat{W}(t) = \int_{G(t)} dV \left( \frac{1}{2} [ \hat{\Pi}^2 + (\nabla \hat{\Phi})^2 ] - \hat{\Pi} (\tilde{\eta} \nabla) \hat{\Phi} \right) \]
(57)

and therefore:
\[ \gamma \hat{H} = \int_{G(t)} dV \left( \hat{T}_{\mu \nu} \Lambda^\nu_{\nu}(\tilde{\eta}) \right)_{G(t)} = \int_{G(t)} dV \hat{T}_{0\mu} \Lambda^\nu(\tilde{\eta}) = \int_{G(t)} d\Sigma^\mu_G \hat{T}_{\mu \nu} u^\nu , \]
(58)

\section{6 Velocity effects}

The pure acceleration effect can be studied by choosing \( G(t) = \tilde{\eta}(t) + G_0 \). (That means a time dependent translation of the “rigid” domain \( G_0 \), another possibility could be a rotation.) In view of
\[ \hat{W}(t) | 0 \rangle = \frac{i}{2} S_{\alpha \beta}(t) \hat{a}_\alpha^+ \hat{a}_\beta^+ | 0 \rangle \neq 0 \]

the ground state of \( \hat{E}_0 \) is not stable under the time-evolution of \( \hat{H} \), even for constant velocities. Therefore, for nonvanishing \( \tilde{\eta} \), the diagonalization of \( \hat{E}_0 \) does not yield a proper definition of creation/annihilation operators \( \hat{a}_\alpha^+ / \hat{a}_\alpha \) that describe physical (i.e. Lorentz-invariant) particles. Only in the frame where the boundaries are (globally) at rest a reasonable definition of particles by diagonalization of \( \hat{E}_0 \) is possible. One should notice that:
\[ \hat{H}(t) = \hat{E}_0 + \hat{W}(t) = \int_{G(t)} dV \left( \frac{1}{2} [ \hat{\Pi}^2 + (\nabla \hat{\Phi})^2 ] - \hat{\Pi} (\tilde{\eta} \nabla) \hat{\Phi} \right) \]
(57)

and therefore:
\[ \gamma \hat{H} = \int_{G(t)} dV \left( \hat{T}_{\mu \nu} \Lambda^\nu_{\nu}(\tilde{\eta}) \right)_{G(t)} = \int_{G(t)} dV \hat{T}_{0\mu} \Lambda^\nu(\tilde{\eta}) = \int_{G(t)} d\Sigma^\mu_G \hat{T}_{\mu \nu} u^\nu , \]
(58)
where $\Lambda = \Lambda^\nu_\rho(\hat{\eta})$ denotes the Lorentz transformation with the four-velocity $u_\mu = \gamma(1, \hat{\eta}) = (1, 0, 0, \hat{\eta})/\sqrt{1 - \hat{\eta}^2}$ of the boundary.

Also for $\gamma \approx 1$ the Hamiltonian as defined above does not coincide with the energy operator:

$$\hat{H} \neq \hat{E}_0' = \int dV' (\hat{T}')_{00} = \int dV' (\Lambda^+ \hat{T} \Lambda)_{00} = \int dV' u^\mu \hat{T}_{\mu\nu} u^\nu$$

introduced by a co-moving observer. This is an indication of the fact, that $\hat{H}(t)$ describes the dynamics for the observer time $t$, and not for the time $t'$ of the co-moving frame.

## 7 Moving-mirrors-configuration

Now we are going to apply the formalism derived above to the special case of two parallel mirrors placed at $z = \eta(t)$ and $z = \eta(t) + l(t)$ or one single mirror located at $z = \eta(t)$.

(We assume $\eta(t < 0) = \eta(t > T) = 0$ and $l(t < 0) = l(t > T) = l_0$.)

With $I(t) = (\eta(t); l(t) + \eta(t))$ it follows (see appendix):

$$\Delta \Omega^2_\alpha(t) = \frac{n^2 \pi^2}{l^2(t)} - \frac{n^2 \pi^2}{l_0^2} = \frac{n^2 \pi^2}{l_0^2} \xi(t) = (\Omega_\alpha^0)^2 \xi(t)$$

And then the squeezing term has the simple form:

$$N^S_\alpha = \frac{(\Omega_\alpha^0)_0^4}{4(\Omega_\alpha^0)^2} |\tilde{\xi}(2\Omega_\alpha^0)|^2 .$$

(61)

For the acceleration term one does not obtain such a simple expression, an explicit form follows from (57), (58), (59) and (12), inserted in (11) or (13). With $G(t) = I(t) = (\eta(t); \infty)$ the squeezing term vanishes (the same as in the case $l = \text{const}$) and the acceleration term yields after some simplifications [see (90) and (91)]:

$$N_\alpha = \frac{1}{\pi^2} \int d\Omega_\beta \Omega_\alpha^0 \Omega_\beta^0 \left| \tilde{\eta}(\Omega_\alpha^0 + \Omega_\beta^0) \right|^2 .$$

(62)

(62) enables us to calculate also the total energy radiated by a mirror into the 1+1-dimensional $G(t)$:

$$E = \Omega_\alpha^0 N^A_\alpha + \mathcal{O}(\varepsilon^3) = \frac{1}{12\pi} \int dt \dot{\tilde{\eta}}^2(t) + \mathcal{O}(\eta^3) .$$

(63)

### 7.1 Mechanical properties

The force exerted upon each mirror can be calculated by computing the divergence of the symmetrized energy-momentum tensor:

$$\hat{f}_\nu = \partial_\mu \hat{T}_{\mu\nu} = \{ (\partial_\nu \hat{\Phi}) \Box \hat{\Phi} \} .$$

(64)
\( \square \Phi = 0 \) is valid only in \( G(t) \) but it is not at \( \partial G(t) \). E.g., a mirror placed at \( z = \eta(t) \) with \( \eta(t = 0) = 0 \) and \( \dot{\eta}(t = 0) = 0 \) induces the following source term (this can be verified by means of Fourier analysis):

\[
\Box \Phi = (\vec{n} \nabla \Phi) \delta(z) ,
\]

where \( \vec{n} \) denotes the normal to the plane mirror. Therefore:

\[
\hat{f}_\nu = \partial^\mu \hat{T}_{\mu\nu} = \left\{ (\partial_\nu \hat{\Phi}) \vec{n} \nabla \hat{\Phi} \right\} \delta(z) .
\] (66)

Expressions (64)–(66) describe the force acting on the mirror at only one side, for a complete examination (and for renormalization) it is necessary to take both sides into account. Taking the vacuum expectation value, we obtain the mechanical force density

\[
\vec{f} = \vec{n} \langle 0 \vert (\vec{n} \nabla \hat{\Phi})^2 \vert 0 \rangle \delta(z) .
\] (67)

The corresponding force is obtained after integration over space. So a mirror at \( t = 0 \) experiences only the static Casimir force, also for non-vanishing \( \dot{\eta} \) and \( \ddot{\eta} \). Other forces (e.g. \( \sim \eta \), see [5]-[8] and [21]-[29]) do not occur at \( t = 0 \) but possibly at later times, when the state vector describing the system has changed: \( |\psi\rangle \neq |0\rangle \).

### 8 Remaining questions

Some modifications of the formalism presented so far become necessary, if we turn to the electromagnetic field (polarizations, gauge, etc.) or if Neumann boundary conditions would be required. However, the general structure of the formalism and the results presented remain very much the same. Another possible generalization of the results of this paper is to apply them to dynamical situations leading to different vacua, i.e. \( G(t < 0) \neq G(t > T) \) and \( |0, in\rangle \neq |0, out\rangle \).

As we can see in section 5 and 6, then it is necessary to distinguish between the particle production due to the dynamical Casimir effect and the one which results already from the comparison of different vacua. The investigation of non-perfect conducting boundaries does not seem to be feasible in a straightforward manner within the canonical formalism presented above. This requires more involved studies.

#### 8.1 Comparison with other results

Applying our approach to the dynamical parallel-plate configuration we also recover most of the results obtained earlier (see [10]-[20]) provided the used approximations are taken into account carefully. E.g., for the example investigated in [19] : \( G(t) = (0, L_0[1 + \varepsilon \sin(2\omega_1 t)]) \) for \( 0 < t < T \) with \( \omega_1 = \pi/L_0 \) and \( \omega_1 T \gg 1 \) we reproduce the obtained result for \( \varepsilon \omega_1 T \ll 1 : \)

\[
N_1 = N_1^S = \frac{1}{4}(\varepsilon \omega_1 T)^2 .
\] (68)

For the radiation of a single mirror (see [3]-[8]) it is possible to compare the total radiated energy from (63) with the formula (3.15) of Ref. [5]:

\[
E = (6\pi)^{-1} \int_0^\infty \vec{V}^2 dx_0
\] (69)
with $V = \dot{\eta}$ and $dx_0 = dt$ (The $\dot{V}^2$ in the original formula is probably a printing error.) The result is the same because the additional factor of 2 in (63) is due to the fact, that (63) in contrast to (69) describes only the particles radiated into $G(t)$, i.e. the emission to the right. It could be interesting to compare the canonical formalism presented in this paper with that of Ford, Vilenkin (see [8]), Moore (see [11]) and that of Fulling, Davies (see [5], [6] and also [7]). The main differences are:
-In our approach the dynamics is governed by the Hamiltonian, but otherwise calculated using a coordinate transformation or Green’s functions.
-The regularization by means of the point splitting method, usually applied to local quantities as for deriving the renormalized energy-momentum tensor, is expected to provide results that are independent of the infinitesimal displacement vector $\epsilon^\mu$, is avoided in the canonical formalism in favour of regularization procedures (if necessary) applied on mode sums.

Under which conditions both approaches will lead to a unique result requires careful investigations for each configuration of boundaries under consideration. E.g., one needs to clarify in which coordinate system $\langle \hat{T}_{\mu\nu} \rangle$ should be calculated and renormalized (see section 6).

9 Appendix

We have to introduce a complete set of real and orthonormal eigenfunctions of the Laplace operator, satisfying the Dirichlet boundary conditions $f_\alpha = 0$ at $\partial G(t)$:

$$\int_{G(t)} dV f_\alpha f_\beta = \delta(\alpha, \beta) \ , \quad (70)$$

$$\int_{G(t)} dV (\nabla f_\alpha) (\nabla f_\beta) = (\Omega_\alpha(t))^2 \delta(\alpha, \beta) \ , \quad (71)$$

$$\sum_{\alpha} f_\alpha(\vec{r}) f_\alpha(\vec{r}') = \delta(\vec{r} - \vec{r}') \quad \text{in} \ G(t) \ . \quad (72)$$

For the moving-mirrors-configuration the domain $G(t)$ can be expressed as follows:

$$G(t) = I(t) \otimes G^\perp \quad (73)$$

with a time-dependent one-dimensional $I(t)$ determining the separation and the time-independent subdomain $G^\perp$ defining the areas of the plates.

Then the eigenfunctions $f_\alpha$ factorize:

$$f_\alpha(\vec{r}, t) = f_\parallel^\alpha(z, t) f_\perp^\alpha(\vec{r}^\perp) \quad (74)$$

with $\alpha = (n, r)$, $\beta = (m, s)$ and $z$ denotes the parallel component of the position vector $\vec{r}$. Accordingly, $f_\perp^\alpha = 0$ at $\partial G^\perp$ and $f_\parallel^\alpha = 0$ at $\partial I(t)$.

The frequencies decompose into two corresponding parts:

$$\Omega_\alpha^2(t) = [\Omega_{\parallel}^\alpha]^2 + [\Omega_{\perp}^\alpha(t)]^2 \ . \quad (75)$$
In $G^\perp$:
\[
\int_{G^\perp} dV^\perp f^\perp_r f^\perp_s = \delta(r, s) \, ,
\]
(76)
\[
\int_{G^\perp} dV^\perp (\nabla f^\perp_r) (\nabla f^\perp_s) = (\Omega_r^\perp)^2 \delta(r, s) \, ,
\]
(77)
\[
\nabla^\perp \cdot f^\perp_r \nabla^\perp \cdot f^\perp_s = \delta(\vec{a} - \vec{b}) \, .
\]
(78)

Note that, if we expand $\Phi$ in eigenfunctions only in $G^\perp$:
\[
\Phi(\vec{r}, t) = \sum_{r} \int_{I(t)} \phi_r(z, t) f^\perp_r(\vec{r}) \, ,
\]
(79)
where $\forall r \phi_r = 0$ at $\partial I(t)$, the $n$-dimensional Lagrangian becomes a sum of effective 1-dimensional Lagrangians:
\[
L = \sum_{r} \int_{I(t)} dz \frac{1}{2} (\dot{\phi}_r^2 - (\nabla \phi_r)^2 - (\Omega_r^\perp)^2 \phi_r^2) \, .
\]
(80)

In $I(t)$ we may expand the $\phi_r$ according to:
\[
\phi_r(z, t) = \sum_{n} q_{(n,r)}(t) f^\parallel_n(z, t) \, ,
\]
(81)
that fulfil the relations
\[
\int_{I(t)} dz \ f^\parallel_n(z, t) f^\parallel_m(z, t) = \delta(n, m) \, ,
\]
(82)
\[
\int_{I(t)} dz \ (\nabla f^\parallel_n(z, t))(\nabla f^\parallel_m(z, t)) = (\Omega^\parallel_n(t))^2 \delta(n, m) \, ,
\]
(83)
and
\[
\sum_{n} f^\parallel_n(a, t) f^\parallel_n(b, t) = \delta(a - b) \quad \text{in } I(t) \, .
\]
(84)

For instance, we may specify the parallel-plate configuration via $I(t) = (\eta(t); l(t) + \eta(t))$ together with the eigenmodes refering to the constrained dimension:
\[
f^\parallel_n(z, t) = \sqrt{\frac{2}{l(t)}} \sin \left( \frac{n\pi}{l(t)} [z - \eta(t)] \right)
\]
(85)
with eigenfrequencies:
\[
\Omega^\parallel_n(t) = \frac{n\pi}{l(t)} \, .
\]
(86)
The coupling matrix then reads

$$M_{\alpha\beta}(t) = -M_{\beta\alpha}(t) = \frac{i(t)}{l(t)} G_{\alpha\beta} + \frac{\dot{i}(t)}{l(t)} A_{\alpha\beta},$$

(87)

where for $n \neq m$:

$$A_{\alpha\beta} = \left[(-1)^{m+n} - 1\right] \frac{2mn}{m^2 - n^2} \delta(r, s),$$

(88)

$$G_{\alpha\beta} = (-1)^{m+n} \frac{2mn}{m^2 - n^2} \delta(r, s)$$

(89)

and with $G_{\alpha\beta} = A_{\alpha\beta} = 0$ for $n = m$.

Considering one single mirror as the limiting case $G(t) = (\eta(t); \infty)$ it follows:

$$M_{\alpha\beta}(t) = \dot{\eta}(t) \frac{2}{\pi} \mathcal{P} \left( \frac{\Omega_{\alpha}\Omega_{\beta}}{\Omega_{\alpha}^2 - \Omega_{\beta}^2} \right),$$

(90)

where $\mathcal{P}$ denotes the principal value. Special care is required in any calculation involving such distribution-like functions (e.g. order of integration); for instance to obtain the following result:

$$M_{\alpha\gamma}(t) M_{\beta\gamma}(t) = \dot{\eta}^2(t) \Omega_{\alpha}^2 \delta(\alpha, \beta).$$

(91)
References

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948)

[2] G. Plunien, B. Müller and W. Greiner, Phys. Rep. 134, 87 (1986)

[3] A. A. Grib, S. G. Mamayev, V. M. Mostepanenko, “Vacuum Quantum Effects in Strong Fields”, p. 54 ff, (Friedmann Laboratory Publishing, St. Petersburg, 1994)

[4] M. Bordag, “Quantum Field Theory Under the Influence of External Conditions” (Teubner, Stuttgart, 1996)

[5] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. A 348, 393 (1976)

[6] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. A 356, 237 (1977)

[7] N. D. Birrell and P. C. W. Davies, “Quantum Fields in Curved Space” (Cambridge University Press, Cambridge, 1982)

[8] L. H. Ford and A. Vilenkin, Phys. Rev. D 25, 2569 (1982)

[9] P. A. Maia Neto and L. A. S. Machado, Phys. Rev. A 54, 3420 (1996)

[10] A. Lambrecht, M.-T. Jaeckel and S. Reynaud, Phys. Rev. Lett. 77, 615 (1996)

[11] G. T. Moore, J. Math. Phys. 11, 2679 (1970)

[12] M. Castagnino and R. Ferraro, Ann. Phys. (N.Y.) 154, 1 (1984)

[13] M. Razavy and J. Terning, Phys Rev. D 31, 307 (1985)

[14] G. Calucci, J. Phys. A 25, 3873 (1992)

[15] E. Sassaroli, Y. N. Srivastava and A. Widom, Phys. Rev. A 50, 1027 (1994)

[16] C. K. Law, Phys. Rev. A 51, 2537 (1995)

[17] V. V. Dodonov, Phys. Lett. A 207, 126 (1995)

[18] O. Méplan and C. Gignoux, Phys. Rev. Lett. 76, 408 (1996)

[19] V. V. Dodonov and A. B. Klimov, Phys. Rev. A 53, 2664 (1996)

[20] J.-Y. Ji, H.-H. Jung, J.-W. Park and K.-S. Soh, quant-ph/9706007v2

[21] V. B. Braginsky and F. Y. Khalili, Phys Lett. A 161, 197 (1991)

[22] M.-T. Jaeckel and S. Reynaud, J. Phys. (France) I 2, 149 (1992)

[23] M.-T. Jaeckel and S. Reynaud, J. Phys. (France) I 3, 1 (1993)

[24] M.-T. Jaeckel and S. Reynaud, J. Phys. (France) I 3, 339 (1993)

[25] M.-T. Jaeckel and S. Reynaud, J. Phys. (France) I 3, 1093 (1993)
[26] M.-T. Jaeckel and S. Reynaud, Phys. Lett. A 167, 227 (1992)
[27] M.-T. Jaeckel and S. Reynaud, Phys. Lett. A 172, 319 (1993)
[28] M.-T. Jaeckel and S. Reynaud, Phys. Lett. A 180, 9 (1993)
[29] R. Golestanian and M. Kardar, Phys. Rev. Lett. 76, 3421 (1997)
[30] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini,
    “Zeta Regularization Techniques with Applications”,
    (World Scientific, Singapore, 1994)
[31] G. Barton and C. Eberlein, Ann. Phys. (N.Y.) 227, 222 (1993)
    and references therein