A new exponential upper bound for the Erdős-Ginzburg-Ziv constant

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Abstract

Naslund used Tao’s slice rank bounding method to give new exponential upper bounds for the Erdős–Ginzburg-Ziv constant of finite Abelian groups of high rank. In our short manuscript we improve slightly Naslund’s upper bounds. We extend Naslund’s results and prove new exponential upper bounds for the Erdős–Ginzburg-Ziv constant of arbitrary finite Abelian groups. Our main results depend on a conjecture about Property D.

1 Introduction

Let $A$ denote an additive finite Abelian group. Let $\exp(A)$ denote the exponent of $A$.

We denote by $\eta(A)$ the smallest integer $\ell \in \mathbb{N}$ such that every sequence $S$ over $G$ of length $|S| \geq \ell$ has a zero–sum sub-sequence of length $1 \leq |T| \leq \exp(A)$.

We denote by $s(A)$ the smallest integer $\ell \in \mathbb{N}$ such that every sequence $S$ over $G$ of length $|S| \geq \ell$ has a zero–sum sub-sequence of length $|T| = \exp(A)$.

Then $s(A)$ is the Erdős-Ginzburg-Ziv constant of $A$.

We use frequently the following result (see [2] Proposition 3.1).

**Theorem 1.1** Let $G$ be a finite Abelian group and let $H \leq G$ be a subgroup such that $\exp(G) = \exp(H)\exp(G/H)$. Then

$$s(G) \leq \exp(G/H)(s(H) - 1) + s(G/H).$$
The following Lemma will be useful in our proofs (see [5] Lemma 3.5).

**Lemma 1.2** Let $A$ be a finite Abelian group. Let us write $A$ as

$$A \cong A(p_1) \times \ldots \times A(p_m)$$

where each $A_i := A(p_i)$ is a $p_i$-group. Then each $A_i$ can be written as a product of cyclic groups whose orders are power of $p_i$. Let $n_i$ denote the number of these cyclic factors. Then

$$s(A) < \exp(A) \left( \sum_{j=1}^{m} \frac{s(Z_{p_j}^{n_j})}{p_j - 1} \right).$$

The following inequality is well-known, see [7].

**Theorem 1.3** Let $k \geq 2$, $n \geq 1$ be arbitrary integers. Let $A := (Z_k)^n$. Then

$$(k - 1)2^n + 1 \leq s(A) \leq (k - 1)k^n + 1.$$ 

Harborth determined $s(A)$ in the following special case in [7].

**Theorem 1.4** Let $a \geq 1$, $n \geq 1$ be arbitrary integers. Let $k := 2^a$ and $A := (Z_k)^n$. Then

$$s(A) = (k - 1)2^n + 1.$$  

Let $A := (Z_k)^n$ with $k, n \in \mathbb{N}$ and $k \geq 2$. We can ask for the structure of sequences of length $s(A) - 1$ that do not have a zero-sum sub-sequence of length $k$. The following conjecture is well-known: every group $A := (Z_k)^n$ satisfies Property D (see [?] Conj. 7.2).

**Property D:** Every sequence $S$ over $A$ of length $|S| = s(A) - 1$ that has no zero-sum sub-sequence of length $k$ has the form $S = T^{k-1}$ for some subset $T$ over $A$.

In the following Theorem we collected all known groups satisfying Property D.

**Theorem 1.5** The following groups has Property D:

(i) $A = (Z_k)^n$, where $k = 2^\alpha$, $\alpha, n \geq 1$ is arbitrary;

(ii) $A = (Z_k)^n$, where $k = 3$, $n \geq 1$ is arbitrary (see [7], Hilfsatz 3);

(iii) $A = (Z_k)^n$, where $n = 1$, $k \geq 2$ is arbitrary;
(iv) \( A = (\mathbb{Z}_k)^n \), where \( n = 2, k \) is not divisible a prime greater than 7 (see [?]);

(v) \( A = (\mathbb{Z}_k)^n \), where \( n = 3, k = 5^\alpha, \alpha > 0 \) (see [?], Theorem 1.9);

(vi) \( A = (\mathbb{Z}_k)^n \), where \( n = 3, k = 3^\alpha, \alpha > 0 \) (see [?] Corollary 1.1).

Elsholtz proved the following lower bound for \( s(A) \) in [1], where \( A := (\mathbb{Z}_k)^n \).

**Theorem 1.6** Let \( k \) be an odd integer. The following inequality holds:

\[
s((\mathbb{Z}_k)^n) \geq (1.125)^{\frac{|\Phi|}{2}}(k - 1)2^n + 1.
\]

These remarkable lower bounds appeared in [3]:

**Theorem 1.7** Let \( k \) be an odd integer. Then \( \eta((\mathbb{Z}_k)^3) \geq 8k - 7 \) and \( s((\mathbb{Z}_k)^3) \geq 9k - 8 \).

**Theorem 1.8** Let \( k \) be an odd integer with \( k \geq 3 \). Then \( \eta((\mathbb{Z}_k)^4) \geq 19k - 18 \) and \( s((\mathbb{Z}_k)^4) \geq 20k - 19 \).

Let \( A := (\mathbb{Z}_k)^r \). Let \( \mathcal{P} \) denote the set of all prime factors of \( k \). One of our main results is a better bound for \( s(A) \) if we suppose that Property D is satisfied for all groups \( (\mathbb{Z}_p)^n \), where \( p \in \mathcal{P} \). We give also new exponential upper bounds for the numbers \( s((\mathbb{Z}_q)^n)) \), where \( q \) is an arbitrary prime power.

Naslund achieved the following breakthrough in [8] Theorem 2.

**Theorem 1.9** Let \( k \geq 2 \) be a fixed integer. Let \( q \) denote the largest prime power dividing \( k \). Suppose that \( A := (\mathbb{Z}_k)^n \) satisfies Property D. Then

\[
s(A) \leq (k - 1)(\gamma_{k,q})^n + 1,
\]

where

\[
\gamma_{k,q} = \inf_{0 < x < 1} \frac{1 - x^q}{1 - x} x^{-\frac{q - 1}{k}}.
\]

In particular, if \( q \) is a prime power and \( A := (\mathbb{Z}_q)^n \) satisfies Property D, then

\[
s(A) \leq (q - 1)4^n + 1.
\]
Remark. The manuscript [8] contains some typos in Theorem 2.

Remark. It is easy to check that
\[ \gamma_{q,q} \sim \left( \frac{2^q - 1}{2^q} \right) 2^{\frac{2q-1}{q}}, \]
when \( q \) is a sufficiently large prime power.

Finally we use the following well–known Lemma in the proofs of our main results.

**Lemma 1.10** Consider the set of monomials
\[ B(n, k) := \{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \sum_i \alpha_i \leq k \}. \]
Then
\[ |B(n, k)| = \binom{n+k}{n}. \]

# 2 Main results

We state here our main results.

**Theorem 2.1** Let \( p \) be a prime. Let \( n \geq 1 \) be an integer. Suppose that Property D is satisfied for the group \( (\mathbb{Z}_p)^n \). Then
\[ s((\mathbb{Z}_p)^n) \leq p(p-1) \left( \frac{n(2p-1)}{p} \right). \]

**Corollary 2.2** Let \( p \) be a fixed prime. Let \( n \geq 1 \) be an integer. Suppose that Property D is satisfied for the group \( (\mathbb{Z}_p)^n \). Then
\[ s((\mathbb{Z}_p)^n) \leq (p-1) \left( (2 + \frac{1}{p-1}) \left( \frac{n}{p} \right) \left( 2 - \frac{1}{p} \right) \right)^n + 1. \]

**Corollary 2.3** Let \( p \) be a fixed prime. Let \( n \geq 1 \) be an integer. Suppose that Property D is satisfied for the group \( (\mathbb{Z}_p)^n \). Then
\[ s((\mathbb{Z}_p)^n) \leq p(p-1) \binom{2n}{n} + 1. \]
Remark. It is easy to check from Stirling’s formula that
\[
\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}},
\]
when \(n\) is sufficiently large.

**Theorem 2.4** Let \( q = p^\alpha \geq 3 \) be an odd prime power. Let \( n \geq 1 \) be an integer. Suppose that Property D is satisfied for the group \((\mathbb{Z}_p)^n\). Then
\[
s((\mathbb{Z}_q)^n) \leq p(q - 1)\left(\frac{2n}{n}\right) + 1.
\]

We can extend Theorem 2.4 from a prime power to an arbitrary composite number.

**Theorem 2.5** Let \( k \geq 2 \) be a fixed odd integer. We can factorize \( k \) as
\[
k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}
\]
where \( p_i \) are distinct primes.

Let \( \mathcal{P} \) denote the set of all prime factors of \( k \). Suppose that Property D is satisfied for each groups \((\mathbb{Z}_p)^n\), where \( p \in \mathcal{P} \). Then
\[
s((\mathbb{Z}_k)^n) \leq (p_1 \cdots p_r)(k - 1)\left(\frac{2n}{n}\right) + 1.
\]

**Theorem 2.6** Let \( A \) be a finite Abelian group. We can write \( A \) as
\[
A \cong A(p_1) \times \cdots \times A(p_m)
\]
where each \( A_i := A(p_i) \) is a \( p_i \)-group. Then each \( A_i \) is a product of cyclic groups whose orders are power of \( p_i \). Let \( n_i \) denote the number of these cyclic factors. Suppose that Property D is satisfied for each groups \((\mathbb{Z}_{p_i})^n\), where \( 1 \leq i \leq m \). Then
\[
s(A) < \exp(A)\left(\sum_{j=1}^{m} p_j \left(\frac{2n_j}{n_j}\right) + \sum_{j=1}^{m} \frac{1}{p_j - 1}\right).
\]

**Theorem 2.7** Let \( k := 3^\alpha 5^\beta \), where \( \alpha, \beta \geq 0, \alpha + \beta \geq 1 \) are integers. Then
\[
s((\mathbb{Z}_k)^3) \leq 300k - 299
\]
and
\[
\eta((\mathbb{Z}_k)^3) \leq 299k - 298.
\]
3 Proofs of the main results

Proof of Theorem 2.1:

First we prove the following Theorem.

**Theorem 3.1** Suppose that $A \subseteq (\mathbb{Z}_p)^n$ satisfies

$$|A| > p \left( \frac{n(2p-1)}{p} \right).$$

Then $A$ contains $p$ not necessarily distinct but not all equal elements $v_1, \ldots, v_p$ such that

$$\sum_i v_i = 0.$$

**Proof.** Indirectly, suppose that $A$ doesn’t contain $p$ not necessarily distinct but not all equal elements $v_1, \ldots, v_p$ such that

$$\sum_i v_i = 0.$$

Then it follows from Tao’s slice rank bounding method (see [9], [8] Proposition 1 and inequality 4.2) that

$$|A| \leq p \cdot |\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : 0 \leq \alpha_i \leq p - 1 \text{ for each } i, \sum_i \alpha_i \leq \frac{n(p-1)}{p}\}|.$$

But

$$|\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : 0 \leq \alpha_i \leq p - 1 \text{ for each } i, \sum_i \alpha_i \leq \frac{n(p-1)}{p}\}| \leq$$

$$|\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \sum_i \alpha_i \leq \frac{n(p-1)}{p}\}| = \left( \frac{n(2p-1)}{p} \right).$$

by Lemma 1.10, hence

$$|A| \leq p \left( \frac{n(2p-1)}{p} \right).$$

$\square$
Theorem 2.1 follows easily from the assumption that Property D is satisfied for the group \((\mathbb{Z}_p)^n\) and Theorem 3.1. Namely let \(S\) be an arbitrary sequence in \((\mathbb{Z}_p)^n\) of length \(s((\mathbb{Z}_p)^n) - 1\) for which there exist no \(p\) elements that sum to zero. Then Property D implies that we can write \(S\) as a multi-set in the form

\[ S = \bigcup_{i=1}^{p-1} B, \]

where \(B \subseteq (\mathbb{Z}_p)^n\) is a subset. Clearly \(B\) doesn’t contain \(p\) not necessarily distinct but not all equal elements that sum to zero.

We get from Theorem 3.1 that

\[ |B| \leq p \left( \frac{n(2p-1)}{p} \right), \]

consequently

\[ s(\mathbb{Z}_p^n) \leq p(p-1) \left( \frac{n(2p-1)}{p} \right) + 1. \]

Proof of Corollary 2.2:
First we prove:

**Theorem 3.2** Suppose that \(A \subseteq (\mathbb{Z}_p)^n\) satisfies

\[ |A| > \left( \left( 2 + \frac{1}{p-1} \right)^{\frac{p-1}{p}} \left( 2 - \frac{1}{p} \right) \right)^n. \]

Then \(A\) contains \(p\) not necessarily distinct but not all equal elements \(v_1, \ldots, v_p\) such that

\[ \sum_i v_i = 0. \]

**Proof.** Indirectly, suppose that \(A\) doesn’t contain \(p\) not necessarily distinct but not all equal elements \(v_1, \ldots, v_p\) such that

\[ \sum_i v_i = 0. \]
Then we get from the proof Theorem of 3.1 that

$$|A| \leq p\left(\frac{n(2p-1)}{n}\right).$$

Sondow and Zudilin proved the following simple upper bound for the binomial coefficients in [10]:

**Theorem 3.3** Let $m \geq 1$ be a positive integer and $r \in \mathbb{R}$ be an arbitrary real. Then

$$\binom{(r + 1)m}{m} \leq \left(\frac{(r + 1)^{r+1}}{r^r}\right)^m.$$

It follows from Theorem 3.3 with the choices $m := n$ and $r := \frac{n(p-1)}{p}$ that

$$|A| \leq p\left(\left(2 + \frac{1}{p-1}\right)^{\frac{p-1}{p}}\left(2 - \frac{1}{p}\right)\right)^n$$

Finally it follows from a standard amplification argument, that

$$|A| \leq \left(\left(2 + \frac{1}{p-1}\right)^{\frac{p-1}{p}}\left(2 - \frac{1}{p}\right)\right)^n,$$

which gives a contradiction.

Namely let $m \geq 1$ be arbitrary and consider the set $A^m \subseteq (\mathbb{Z}_p)^{nm}$. Then

$$|A|^m = |A^m| \leq p\left(\left(2 + \frac{1}{p-1}\right)^{\frac{p-1}{p}}\left(2 - \frac{1}{p}\right)\right)^{nm}.$$  

Consequently

$$|A| \leq p^{1/m}\left(\left(2 + \frac{1}{p-1}\right)^{\frac{p-1}{p}}\left(2 - \frac{1}{p}\right)\right)^n,$$

and if $m$ tends to infinity, then we get the inequality (1).

Corollary 2.2 is a clear consequence of the assumption that Property D is satisfied for the group $(\mathbb{Z}_p)^n$ and Theorem 3.2.
Proof of Corollary 2.3:

Corollary 2.3 follows obviously from Theorem 2.1.

Proof of Theorem 2.4:
Let $G := (\mathbb{Z}_q)^n$ and $H := (\mathbb{Z}_p)^n$. Clearly

$$G/H \cong (\mathbb{Z}_{p^\alpha - 1})^n$$

and $\exp(G/H) = p^{\alpha - 1}$.

By the induction hypothesis we get

$$s(G/H) \leq p(p^{\alpha - 1} - 1) \left(\frac{2n}{n}\right) + 1.$$ 

It follows from Theorem 2.3 that

$$s(H) \leq p(p - 1) \left(\frac{2n}{n}\right) + 1.$$ 

We can apply Theorem 1.1 for $G$ and $H$, since $\exp(G) = \exp(H)\exp(G/H)$:

$$s(G) \leq \exp(G/H)(s(H) - 1) + s(G/H) \leq p^{\alpha - 1} \cdot \left(p(p - 1) \left(\frac{2n}{n}\right)\right) + (p^\alpha - p) \left(\frac{2n}{n}\right) + 1 = p(p - 1) \left(\frac{2n}{n}\right) + 1.$$

Proof of Theorem 2.5:

We can derive Theorem 2.5 easily from Theorem 1.1. This proof is very similar to the proof of Theorem 2.4 so we omit it.

Proof of Theorem 2.6:

It follows from Theorem 2.3 that

$$s(\mathbb{Z}_{p^i}^{n_j}) \leq p_j(p_j - 1) \left(\frac{2n_j}{n_j}\right) + 1.$$
for each \( 1 \leq i \leq m \). If we combine this result with Lemma 1.2 then we get our result.

\[ \square \]

**Proof of Theorem 2.7**

Theorem 2.7 follows clearly from Theorem 1.5 (v) and (vi) and Theorem 2.5.

\[ \square \]

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