Computing numerically the functional derivative of an effective action

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Abstract

The functional derivative of the effective action with respect to an external field is part of the equation of motion of this field if one-loop effects induced by quantum fluctuations or thermal fluctuations are included in minimizing the action of this field. Examples occur in all field theories displaying classical solutions or - as the Nambu-Jona-Lasinio model - selfconsistent field configurations. We describe here a numerical method for computing such functional derivatives; we use a fermion field with Yukawa interaction to an external field as an example which is sufficiently simple and sufficiently general. We compare the computed functional derivative to analytical estimates.


1 Introduction

The one-loop effective action plays an important rôle in various contexts of quantum field theory. It leads to a correction to the energy of classical solutions (see e.g. [1]) and to corrections to quantum tunneling [2] and thermal transition rates [3-5]; it is the basis of Hartree type selfconsistent field equations. If in the former case the corrections are important the effective action should be included in determining the extremal classical field configuration. In Hartree type computations as e.g. in the nontopological soliton model of Friedberg and Lee [6, 7] or in the Nambu-Jona-Lasinio model [8-11] the classical action may even have no solution and the classical field may have no genuine action, it may be just a parametrization of condensates. Denoting the “classical” field as $\phi$ the equation of motion reads

$$\delta \Gamma \frac{\delta S_{\text{cl}}}{\delta \phi(x)} + \delta S_{\text{eff}}^{1-1} \frac{\delta S_{\text{eff}}^{1-1}}{\delta \phi(x)} = 0$$ (1.1)

where $S_{\text{cl}}$ may be absent. The one-loop effective action takes usually the form

$$S_{\text{eff}}^{1-1} = \pm \frac{1}{2} \ln \frac{\det(-\Delta + m^2 + V(\phi(x)))}{\det(-\Delta + m^2)}$$ (1.2)

where the sign + or − refers to bosonic and fermionic actions respectively. The Klein-Gordon type operators will in general be multichannel operators, mass term and potential being given by matrices. Two methods for computing the effective action numerically have been proposed recently in [12,13] and in [14-19] and have been applied in various contexts [20-24]. Both methods are closely related. In both of them the usual problems of renormalization and of removing zero and unstable modes are easy to handle. While the second method, based on a theorem on functional determinants [25], is considerably faster the first of them is more convenient in the present context. It can be adapted easily for computing the functional derivative $\delta S_{\text{eff}}^{1-1}/\delta \phi(x)$. Renormalization remains easy to handle and will be discussed. We will not consider zero or unstable modes here. These are particular to specific models and here we would like to keep the presentation of the method general. For reasons of economy we restrict ourselves furthermore to the case of one scalar external field; this restriction can easily be removed. For the fluctuating field we choose a Dirac field as the simplest case of a coupled channel problem for the fluctuation operator.

2 The fermion determinant and its functional derivative

We consider a fermion field in 4 Euclidean dimensions which receives its mass $m_F = g v$ from its coupling to a Higgs field $\phi(x)$ which takes the vacuum expec-
The Dirac operator has the form
\[ D = \partial - g\phi(x) = D^{(0)} - g(\phi(x) - v), \] (2.1)
where \( D^{(0)} = \partial - m_F \) is the free Dirac operator in the ‘broken symmetry phase’. \( D \) is the Euclidean Dirac operator, the gamma matrices are hermitean and satisfy \( \{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} \). Since \( D \) is not a positive definite operator we introduce the operator
\[ M = \gamma_5 D \gamma_5 = -\partial^2 + m_F^2 + V(x), \] (2.2)
with the potential
\[ V(x) = g^2(\phi^2(x) - v^2) + g\partial\phi(x). \] (2.3)
The Euclidean effective action induced by the fermion field is then obtained as
\[ S_{\text{eff}}[\phi] = -\ln \det \left( \frac{D}{D^{(0)}} \right) = -\frac{1}{2} \ln \det \left( \frac{-\partial^2 + m_F^2 + V(x)}{-\partial^2 + m_F^2} \right) \]
\[ = -\frac{1}{2} \ln \det \left( 1 + (-\partial^2 + m_F^2)^{-1}V(x) \right). \] (2.4)

As a next step we will derive an explicit expression for the functional derivative of \( S_{\text{eff}}[\phi] \) with respect to \( \phi(x) \). As a starting point we use the perturbative expansion of the effective action with respect to the potential \( V(x) \). We define the free Green function \( G_0(x - x') \) as the solution of the equation
\[ (-\partial^2 + m_F^2)G_0(x - x') = \delta^{(4)}(x - x') \] (2.5)
where \( \partial^2 \) is the Euclidean Laplace operator in 4 dimensions. Then the effective action can be expanded as
\[ S_{\text{eff}}[\phi] = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \text{tr} \int d^4x \prod_{i=1}^{n} G_0(x_i - x_{i-1}) V(x_i) \quad \text{(with } x_0 := x_n). \] (2.6)
Obviously only \( V(x) \) depends on \( \phi(x) \), so the functional derivative is obtained in a straightforward way as:
\[ \frac{\delta S_{\text{eff}}}{\delta \phi(x)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \text{tr} \int d^4x_1 \frac{\delta V(x_1)}{\delta \phi(x)} \int \prod_{i=2}^{n} d^4x_i G_0(x_i - x_{i-1}) V(x_i) G_0(x_1 - x_n). \] (2.7)
If one looks for static solutions, the actual variation will be with respect to fields depending only on the three dimensional vector \( x \). We will restrict our considerations to this case in the following. For instanton computations (see e.g. [22]) one introduces an extra Euclidean time variable, so that the present restriction represents no loss of generality. Obviously the time integrations can be carried

\[ ^3 \text{We omit the superscript } 1-1 \text{ in the following.} \]
out explicitly. In order to do so we introduce the Green function \( G_0(x - x', \nu) \) which is related to \( G_0(x - x') \) via a Fourier transform with respect to Euclidean time \( \tau \). It satisfies

\[
(\Delta - m_F^2 - \nu^2) G_0(x - x', \nu) = -\delta^{(3)}(x - x')
\]  

(2.8)

and is given explicitly as

\[
G_0(x - x', \nu) = \int \frac{d^3p_i}{(2\pi)^3} \frac{\exp(ip_i(x - x'))}{p_i^2 + \kappa^2}
\]  

(2.9)

with \( \kappa^2 := m_F^2 + \nu^2 \). The time integration leads to delta functions for the differences of the Euclidean energies \( \nu_i \), leaving just one integration with respect to \( \nu \). Explicitly we find

\[
\frac{\delta S_{\text{eff}}}{\delta \phi(x)} = 2\pi \delta(0) \left( -\frac{1}{2} \right) \text{tr} \int d^3x_1 \frac{\delta V(x_1)}{\delta \phi(x)} 
\]

\[
\times \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \left( \prod_{i=2}^{n} d^3x_i G_0(x_i - x_{i-1}, \nu) V(x_i) \right) G_0(x_1 - x_n, \nu).
\]

(2.10)

The factor \( 2\pi \delta(0) \) is due to a delta function \( 2\pi \delta(\nu_n - \nu_0) \) whose argument vanishes due to the cyclic identification of \( \nu_n \) with \( \nu_0 \). It is to be identified, as usual, with Euclidean time via \( 2\pi \delta(0) \rightarrow \tau \). Indeed, for time independent external potentials the effective action is proportional to \( \tau \) and in fact we are considering \( S_{\text{eff}}/\tau \), the zero point energy. As \( G_0(x - x', \nu) \) is an even function of \( \nu \) the integral over this variable may be replaced as

\[
\int_{-\infty}^{\infty} d\nu \rightarrow 2 \int_{0}^{\infty} d\nu.
\]

(2.11)

Furthermore the perturbative series in the potential \( V(x) \) can be recollected into an exact Green function

\[
G(x, x', \nu) \sum_{n=1}^{\infty} (-1)^{n+1} \left( \prod_{i=2}^{n} d^3x_i G_0(x_i - x_{i-1}, \nu) V(x_i) \right) G_0(x' - x_n, \nu)
\]

(2.12)

with \( x_1 = x \). \( G(x, x', \nu) \) satisfies

\[
(\Delta - \kappa^2 - V(x)) G(x, x', \nu) = -\delta^{(3)}(x - x').
\]

(2.13)

With these replacements the derivative of the effective action becomes

\[
\frac{1}{\tau} \frac{\delta S_{\text{eff}}}{\delta \phi(x)} = -\text{tr} \int d^3x_1 \frac{\delta V(x_1)}{\delta \phi(x)} \int_{0}^{\infty} \frac{d\nu}{2\pi} G(x_1, x_1, \nu).
\]

(2.14)
It remains to evaluate the derivative of the potential $V(x)$ with respect to $\phi$. One obtains

$$\frac{\delta V(x_1)}{\delta \phi(x)} = \frac{\delta}{\delta \phi(x)} \left( g^2 (\phi^2(x_1) - \nu^2) + g \gamma \nabla_{x_1} \phi(x_1) \right) = (2g^2 \phi(x_1) + g \gamma \nabla_{x_1}) \delta(x_1 - x).$$

Inserting this expression into (2.14) yields the result

$$\frac{1}{\tau} \frac{\delta S_{\text{eff}}}{\delta \phi(x)} = \frac{g^2 \phi(x)}{\pi} \int_0^\infty d\nu \text{tr} G(x, x, \nu) + \frac{g}{2\pi} \int_0^\infty d\nu \gamma \nabla G(x, x, \nu). \quad (2.15)$$

Here the gradient in the last term acts on the Green function at equal arguments, i.e., one takes the gradient after taking the limit of equal arguments. The expression we have obtained is a formal one, yet. The Green functions are not defined at equal arguments, they have to be regularized and a renormalization has to be performed. This will be discussed below; we prepare this here, however, by introducing a convenient notation. As we have seen $G(x, x', \nu)$ can be expanded with respect to $V(x)$ via (2.12). The trace with respect to $x$ and with respect to internal indices is equivalent to a summation of one loop Feynman graphs. In the following we denote by a superscript $(k)$ a contribution of order $k$ in the potential $V(x)$ and by the superscript $(\bar{k})$ the sum of all such contributions starting with order $k$. With this convention we can define a finite part of the functional derivative via

$$\frac{1}{\tau} \frac{\delta S_{\text{eff}}^{(3)}}{\delta \phi(x)} = \frac{g^2 \phi(x)}{\pi} \int_0^\infty d\nu \text{tr} \bar{G}^{(2)}(x, x, \nu) + \frac{g}{2\pi} \int_0^\infty d\nu \gamma \nabla \bar{G}^{(2)}(x, x, \nu). \quad (2.16)$$

This quantity can be evaluated numerically; the first and second order contributions are divergent and will be considered later.

In some applications the effective action is computed in a quantum state in which some of the fluctuations are excited, e.g., in the form of occupied valence quark states. Then the “sea quark” Green function has to be modified by adding the “valence quark” contributions

$$\int \frac{d\nu}{2\pi} G^{\text{val}}(x, x, \nu) = \sum_\alpha \frac{1}{2E_\alpha} \psi^\dagger_\alpha(x) \psi_\alpha(x) \quad (2.17)$$

where $\alpha$ runs over the occupied levels. Since the operator $\gamma \nabla - g \phi(x)$ occurring in (2.15) is just the static Dirac operator and

$$(-\gamma \nabla + g \phi(x)) \psi_\alpha(x) = E_\alpha \gamma_0 \psi_\alpha(x) \quad (2.18)$$

this means that we obtain an additional valence contribution

$$\frac{1}{\tau} \frac{\delta S_{\text{eff}}^{\text{val}}}{\delta \phi(x)} = -g \sum_\alpha \bar{\psi}_\alpha(x) \psi_\alpha(x) \quad (2.19)$$

as to be expected.
3 Computation of the Green function

The computation of the Green function $G(x, x', \nu)$ has been considered previously in Ref. [12, 13]. In order to proceed we have to introduce a further restriction on the classical configuration $\phi(x)$. We have to assume that it is spherically symmetric. For most cases of interest this is the case, as minimal or extremal solutions have usually the highest possible symmetry. Then we may expand the exact Green function into partial wave Green functions. These can be obtained by solving the radial Schrödinger equation numerically.

3.1 Partial wave reduction

For the partial wave reduction we have to use an appropriate spherical spinor basis. It can be found in [26]. One introduces the 2-spinors

$$\varphi^{(+)} (\hat{x}) = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l + 1/2 + m} Y_l^{m-1/2}(\hat{x}) \\ \sqrt{l + 1/2 - m} Y_l^{m+1/2}(\hat{x}) \end{pmatrix} \quad \text{for } j = l + 1/2, \quad (3.1)$$

$$\varphi^{(-)} (\hat{x}) = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l + 1/2 - m} Y_l^{m-1/2}(\hat{x}) \\ -\sqrt{l + 1/2 + m} Y_l^{m+1/2}(\hat{x}) \end{pmatrix} \quad \text{for } j = l - 1/2. \quad (3.2)$$

which are eigenspinors of $J^2, J_z, L^2$ and $S^2$. Further they satisfy

$$\sigma \hat{x} \varphi^{(\pm)} = \varphi^{(\mp)} \quad (3.3)$$

Using these spinors one can construct 4-spinors and reduce the Dirac equation into two coupled radial equations for the partial waves. In order to obtain a similar Ansatz for the Green function we introduce the matrices $\Omega^{(\pm)} (\hat{x}, \hat{x}')$ via

$$\Omega^{(\pm)} (\hat{x}, \hat{x}') := \sum_{m=-j}^j \varphi^{(\pm) \dagger}(\hat{x}) \varphi^{(\pm)}(\hat{x}')$$

in terms of which the Green function can be expanded as

$$G^{(\pm)}(x, x', \nu) = \sum_j \begin{pmatrix} g_{j,11}^{(+)}(r, r', \nu) \Omega_j^{(+)}(\hat{x}, \hat{x}') & -g_{j,12}^{(+)}(r, r', \nu) \Omega_j^{(+)}(\hat{x}, \hat{x}) i\sigma \hat{x}' \\ g_{j,21}^{(+)}(r, r', \nu) i\sigma \hat{x} \Omega_j^{(+)}(\hat{x}, \hat{x}') & g_{j,22}^{(+)}(r, r', \nu) \Omega_j^{(\mp)}(\hat{x}, \hat{x}') \end{pmatrix}.$$ \quad (3.5)

The partial wave Green functions $g_{j,km}^{(\pm)}(r, r', \nu)$ satisfy

$$\left( \delta_{nk} (-D_n + \kappa^2) + v_{nk}(r) \right) g_{km}^{(\pm)}(r, r', \nu) = \delta_{nm} \frac{\delta(r - r')}{r^2}. \quad (3.6)$$

Here $D_n$ is the radial operator

$$D_n = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l_n(l_n + 1)}{r^2} \quad (3.7)$$

5
with $l_1 = j - 1/2$ and $l_2 = j + 1/2$ for $g_j^{(+)}$ and $l_1 = j + 1/2$ and $l_2 = j - 1/2$ for $g_j^{(-)}$. The potential has been reduced to a $(2 \times 2)$ matrix

$$v(r) = \begin{pmatrix} g^2 (\phi^2 - v^2) & -g \phi' \\ -g \phi' & g^2 (\phi^2 - v^2) \end{pmatrix}.$$  

(3.8)

One sees easily that the (+) and (−) systems at equal $j$ are equivalent upon identifying

$$g_j^{(-)} \rightarrow g_j^{(+)} , \quad g_j^{(+)} \rightarrow g_j^{(-)} , \quad g_j^{(−)} \rightarrow g_j^{(+)} , \quad g_j^{(+)} \rightarrow g_j^{(−)} .$$  

(3.9)

### 3.2 Partial wave Green functions

The partial wave Green function can be expressed in standard way by regular and singular solutions of the homogeneous equation

$$\sum_k \left[ \delta_{nk} (-D_n + \kappa^2) + v_{nk}(r) \right] f_k(r) = 0 .$$  

(3.10)

This system has a fundamental set of two independent two-component solutions regular as $r \rightarrow \infty$ which will be denoted as $f_k^{\alpha+}$ ($\alpha = 1, 2$) and two such solutions regular at $r = 0$, denoted as $f_k^{\alpha-}$. In terms of these solutions the partial wave Green function can be expressed as

$$g_{km}(r, r') = \sum_{\alpha, \beta} \left\{ \Theta(r - r') f_k^{\alpha+}(r) \omega_{\alpha \beta}^{-1} f_m^{\beta-}(r') + \Theta(r' - r) f_k^{\alpha-}(r) \omega_{\alpha \beta}^{-1} f_m^{\beta+}(r') \right\} .$$  

(3.11)

where $\omega_{\alpha \beta}$ is related to the Wronskians of the two sets of solutions via

$$\omega_{\alpha \beta} = r^2 \sum_n \left( f_n^{\alpha+} \frac{d}{dr} f_n^{\beta-} - f_n^{\beta-} \frac{d}{dr} f_n^{\alpha+} \right) = r^2 W(f^{\alpha+}(r), f^{\beta-}(r)) .$$  

(3.12)

A short derivation of this formula is given in the Appendix. It is convenient to choose the two sets of solutions in the form

$$f_n^{\alpha\pm} = \left( \delta_n^{\alpha} + h_n^{\alpha\pm}(r) \right) b_n^{\pm}(\kappa r) .$$  

(3.13)

Here $b_n^+(z) = k_l(z)$ and $b_n^-(z) = i_l(z)$ are the modified spherical Bessel functions regular for $z \rightarrow \infty$ and as $z \rightarrow 0$, respectively. The functions $h_n^{\alpha\pm}$ are regular both at $r =$ and as $r \rightarrow \infty$. They can be chosen to vanish as $r \rightarrow \infty$ as a boundary and normalization condition. Then the Wronskian simplifies and we have

$$\omega_{\alpha \beta} = \delta_{\alpha \beta} / \kappa .$$  

(3.14)

Footnote: We suppress the angular momentum index $j$, the superscript $(\pm)$ and the argument $\nu$ in the following.
We will need the Green function only at \( r = r' \). The limits \( r' \searrow r \) and \( r' \nearrow r \) are equal by construction, they involve however different sets of functions. The numerical differences are found to be unimportant. We have chosen to take the symmetric limit

\[
g_{mn}(r, r) = \sum_{\alpha} \frac{\kappa}{2} \left\{ f_m^{\alpha+}(r) f_n^{\alpha-}(r) + f_m^{\alpha-}(r) f_n^{\alpha+}(r) \right\} .
\]

(3.15)

### 3.3 Removing the leading orders in the external potential

As mentioned in section 2 we will have to remove the leading orders in the external potential \( \mathcal{V}(x) \) in order to obtain the finite part \( \delta S_{\text{eff}}^{(3)}/\delta \phi(x) \); this implies obtaining the Green function in order (2). We note that our ansatz (3.13) for the solutions \( f_n^{\alpha \pm} \) represents a separation into the zeroth order part proportional to \( \delta_n^{\alpha} \) and the order (1) contribution proportional to the functions \( h_n^{\alpha \pm} \). These functions satisfy the coupled differential equations

\[
\left( \frac{d^2}{dr^2} + 2 \left( \frac{1}{r} + \kappa \frac{l_{n \pm}}{b_{n \pm}} \right) \frac{d}{dr} \right) h_n^{\alpha \pm}(r) = \sum_m v_{nm} \left( \delta_m^\alpha + h_m^{\alpha \pm} \right) \frac{b_{m \pm}}{b_{n \pm}} .
\]

(3.16)

which can be written in symbolic form as

\[
\mathcal{D} h^{\pm} = V \left( 1 + h^{\pm} \right) .
\]

(3.17)

If one expands the functions \( h \) with respect to powers of \( V \) as

\[
h^{\pm} := h^{\pm(1)} = h^{\pm(1)} + h^{\pm(2)} + \ldots ,
\]

(3.18)

one obtains the following sequence of equations

\[
\begin{align*}
\mathcal{D} h^{\pm(1)} &= V \left( 1 + h^{\pm(1)} \right), \\
\mathcal{D} h^{\pm(1)} &= V, \\
\mathcal{D} h^{\pm(2)} &= V h^{\pm(1)}
\end{align*}
\]

(3.19) – (3.21)

which can be solved numerically. In terms of these functions the partial wave Green function at \( r = r' \) can be expressed as

\[
g_{mn}(r, r') = \frac{\kappa}{2} \left\{ \left( \delta_n^m + h_n^{m+(1)} + h_m^{m-(1)} \right) i_n k_m + \left( \delta_n^m + h_n^{m-(1)} + h_m^{m+(1)} \right) i_m k_n \\
+ \left( h_n^{m-(2)} + h_n^{m+(2)} + \sum_{\alpha} h_n^{\alpha+(1)} h_n^{\alpha-(1)} \right) i_n k_m \\
+ \left( h_n^{m+(2)} + h_n^{m-(2)} + \sum_{\alpha} h_n^{\alpha-(1)} h_n^{\alpha+(1)} \right) i_m k_n \right\} .
\]

(3.22) – (3.23)
The third and fourth term represent the contribution of order $(2)$
\[
g_{mn}^{(2)}(r, r) = \kappa \left\{ \left( h_{m-(2)}^{n} + h_{m+(2)}^{n} + \sum_{\alpha} h_{m}^{(\alpha-1)} h_{n}^{(\alpha+1)} \right) i_{m} k_{n} \right. \\
+ \left. \left( h_{m+(2)}^{n} + h_{m-(2)}^{n} + \sum_{\alpha} h_{m}^{(\alpha-1)} h_{n}^{(\alpha+1)} \right) i_{m} k_{n} \right\}. \tag{3.24}
\]

### 3.4 The derivative of the effective action

Having at our disposal a numerical method to compute the Green function it is straightforward, using (2.16), to derive an expression which allows the numerical computation of the functional derivative of the effective action. We first have to take some traces of the Green function and its derivatives. Using the identities
\[
\Omega_{j}^{(\pm)}(\hat{x}, \hat{x}) = \frac{j + 1/2}{4\pi} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \tag{3.25}
\]
and
\[
\sum_{m=-l}^{l} |Y_{l}^{m}|^2 = \frac{2l+1}{4\pi}, \\
\sum_{m=-l}^{l} m|Y_{l}^{m}|^2 = 0, \\
\sum_{m=-j}^{j} \sqrt{l + \frac{1}{2} - m} \sqrt{l + \frac{1}{2} + m} Y_{l}^{m+1/2} Y_{l}^{m-1/2*} = 0
\]
one finds
\[
\text{tr} G^{(2)}(x, x, \nu) = \frac{1}{4\pi} \sum_{j=1/2}^{\infty} (2j+1) \left[ g_{j,11}^{(2)}(r, r) + g_{j,22}^{(2)}(r, r) \right], \tag{3.26}
\]
\[
\text{tr} \gamma \nabla G^{(2)}(x, x, \nu) = -\frac{1}{4\pi} \sum_{j=1/2}^{\infty} 2(2j+1) \left[ g_{j,12}^{(2)}(r, r) + \frac{2}{r} g_{j,12}^{(2)}(r, r) \right]. \tag{3.27}
\]

Supplying a factor of 2 in order to take into account both the $l = j + 1/2$ and the $l = j - 1/2$ modes we obtain the final result
\[
\frac{1}{\tau} \delta S_{\text{eff}}^{(3)}(\phi(x)) = -\frac{g^2 \phi(r)}{2\pi^2} \int_{0}^{\infty} d\nu \sum_{j=1/2}^{\infty} (2j+1) \left[ g_{j,11}^{(2)}(r, r) + g_{j,22}^{(2)}(r, r) \right] \\
-\frac{g}{2\pi^2} \int_{0}^{\infty} d\nu \sum_{j=1/2}^{\infty} (2j+1) \left[ g_{j,12}^{(2)}(r, r) + \frac{2}{r} g_{j,12}^{(2)}(r, r) \right]. \tag{3.28}
\]
This expression is finite and can be computed numerically. However, we have to include the renormalized divergent parts as well. This will be done in the next section. Here we will discuss some details of the numerical evaluation and we present an analytic approximation, based on the derivative expansion. This analytic approximation will be compared to the numerical results.
3.5 Some details of the numerical evaluation

The numerical evaluation of the effective action has been described in extenso in previous publications \[12, 13\]. If one computes the effective action the first operation is to integrate the trace of the Green function over the radial variable \(r\). Thereby one obtains the partial wave contributions at fixed \(\nu\) which are then summed using an asymptotic extrapolation. Finally one has to do the integration over \(\nu\). Here the information we want to draw from the Green function is a local one, we want to determine \(\delta S^{(3)}_{\text{eff}}/\delta \phi(x)\) as a function of \(r\). This requires a different order of steps. Let us write (3.28) in the form

\[
\frac{1}{\tau} \frac{\delta S^{(3)}_{\text{eff}}}{\delta \phi(x)} = - \frac{g}{2\pi^2} \int_0^\infty d\nu \sum_{j=1/2}^{\infty} (2j + 1)t_j(\nu, r)
\]

where now

\[
t_j(\nu, r) = g\phi(r) \left[ g_{j,11}^{(2)}(r, r) + g_{j,22}^{(2)}(r, r) \right] + \left[ g_{j,12}^{(2)}(r, r) + \frac{2}{r} g_{j,12}^{(2)}(r, r) \right].
\]

The fact that the method for computing the effective action developed in \[12, 13\] is very fast is due to the fact that after only two Runge-Kutta runs for computing the functions \(h_0^{(2)}(r)\) one knows the partial wave Green function at all mesh points \(r_n\). This advantage can be maintained in the following way: the Green function is computed, at fixed angular momentum \(j\) for all \(r_n\) in one sweep and at each of the (roughly 2000) mesh points \(r_n\) the appropriate traces \(t_j(\nu, r)\) are computed and stored as functions of \(\nu\) and \(j\).

The computation was extended to \(j - 1/2 \leq 25\) and to \(\nu \leq 2\). The partial wave sums \(\sigma_n(\nu) = \sum_{j=1/2}^{\infty} t_j(\nu, r_n)\) were computed using an extrapolation of the form \(a/j^4 + b/j^5\) to include the terms with \(j - 1/2 > 25\) and the integration over \(\nu\) was extended beyond \(\nu = 2\) by using an extrapolation of the form \(a/\nu^2 + b/\nu^4\).

The form of the extrapolations can be derived rigorously, the convergence was checked for several values of \(r\).

The total time for one evaluation of \(\delta S^{(3)}_{\text{eff}}/\delta \phi(x)\) at all values of \(|x|\) takes about 5 minutes on a workstation IBM Risc 6000/340.

3.6 Analytic approximation

In \[15\] we presented an analytic approximation to the finite part \(S^{(3)}_{\text{eff}}\) of the action, using an expansion with respect to gradients of the external field \(\phi\). It reads

\[
S^{(3)}_{\text{eff}}[\phi] \approx - \frac{g^4}{16\pi^2} \int d^4x \left[ \phi^4 \ln \frac{\phi^2}{v^2} - \frac{3}{2}(\phi^2 - v^2)^2 - v^2(\phi^2 - v^2) \right] - \frac{g^2}{16\pi^2} \int d^4x (\partial_\mu \phi)^2 \left[ \ln \frac{\phi^2}{v^2} - \frac{2}{3v^2}(\phi^2 - v^2) \right].
\]

(3.31)
This approximation was found to be excellent at sufficiently high fermion masses. It is interesting to compare exact results and this analytic approximation also on the level of the functional derivative. Taking the fields to be time independent and dependent only on \( r = |x| \) we obtain

\[
\frac{1}{\tau} \frac{\delta \overline{S}_{\text{eff}}^{(3)}}{\delta \phi(x)} = -\frac{g^4 \phi}{4\pi^2} \left[ \phi^2 \ln \frac{\phi^2}{v^2} - \phi^2 + v^2 \right] - \frac{g^2}{16\pi^2} \left[ -(\phi')^2 \left( \frac{2}{\phi} - \frac{4}{3v^2} \phi \right) 
- 2 \left( \phi'' + \frac{2}{r} \phi' \right) \left( \ln \frac{\phi^2}{v^2} - \frac{2}{3v^2} (\phi^2 - v^2) \right) \right].
\] (3.32)

This analytic approximation is compared to the numerical results in Figs. 1 to 4. For the numerical evaluation we relate the physical quantities to dimensionless ones by \( x = \hat{x}/v \) and \( \phi = v \hat{\phi} \). Here and in the following we use for \( \hat{\phi} \) the profile function

\[
f(r) = (1 + \exp(-\sqrt{r^2 + 1} + \sqrt{R^2 + 1}))^{-1}
\] (3.33)

with \( R = 2 \) as in [15]. It has the typical shape of a medium-sized bubble. The approximation is found to reproduce the numerical results quite well; as to be expected the approximation improves with increasing fermion mass.

To the finite part of order \( \overline{(3)} \) we have to add the finite parts of the first and second order contributions. They are known analytically and their numerical evaluation, involving only a numerical Fourier transform and simple integrations, can be performed with high precision. For this reason we have not included them here into the comparison between numerical and approximate analytical results.

### 4 Renormalization of the effective action

Having discussed the computation of the finite contributions of third and higher order in the external potential \( \mathcal{V}(x) \) we have to come back now to the computation of the first and second order contributions to the effective action. These contributions are divergent. As a first step they have to be regularized, e.g. by dimensional or Pauli-Villars regularization, introducing thereby an ultraviolet cutoff \( \Lambda \). If there is no genuine action for the field \( \phi \), as e.g. in Nambu-Jona-Lasinio type models where \( \phi \) parametrizes fermion condensates, the theory will depend on this cutoff - the latter will be a physically significant quantity. If the \( \phi \) field has a genuine action of the \( \phi^4 \) type then the divergences can be absorbed into counterterms and the cutoff dependence disappears via renormalization of the physical masses and couplings. We will consider the latter case here; to be specific we choose a Higgs type potential for the field \( \phi \).

The bare Lagrangian of a Higgs-Yukawa theory reads

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi) + \bar{\psi} \left( i \gamma^{\mu} \partial_{\mu} - g \phi \right) \psi.
\] (4.1)
with
\[ U(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2. \] (4.2)

If the Higgs field is assumed to take its vacuum expectation value \( v \) at large \( r \) it is convenient to introduce the field shift
\[ \phi(x) = v + \varphi(x) \] (4.3)
and to express the Lagrangian in terms of \( \varphi \) as
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m_\varphi^2 \varphi^2 - \lambda v \varphi^3 - \frac{\lambda}{4} \varphi^4 + \bar{\psi} (i \gamma - (m_F + g \varphi)) \psi. \] (4.4)

Here we have introduced the boson mass \( m_\varphi = \sqrt{2}\lambda v \) and the fermion mass \( m_F = g v \).

The counterterm Lagrangian will have to include a fourth order polynomial in the field \( \varphi \) and also a wave function renormalization counterterm. We do not have to include here corresponding counterterms for the fermion part of the action. Such terms would be made necessary only if boson loops were included. The counterterm Lagrangian reads then
\[
\mathcal{L}_{c.t.}(\varphi) = -A \varphi - \frac{1}{2} B \varphi^2 - \frac{1}{6} C \varphi^3 - \frac{1}{24} D \varphi^4 + \frac{1}{2} \delta Z \partial_\mu \varphi \partial^\mu \varphi. \] (4.5)

For the Yukawa - Higgs Lagrangian we note that the bare Lagrangian depends only on the square of the unshifted field \( \phi^2 \) and its derivative. This holds also true for the divergent terms of the one-loop effective action generated by the fermion field, due to the structure \([1,2]\) of the external potential \( V(x) \). Therefore, only two of the four constants \( A, B, C \) and \( D \) are independent. This implies the relations
\[
A - B v + \frac{1}{3} D v^3 = 0 \quad \text{and} \quad C = v D. \] (4.6)

The choice of the independent quantities depends on the renormalization conditions which will be given below.

The first and second order contributions to the one-loop effective action are given - first as formal definitions and then in dimensionally regularized form - by
\[
S_{\text{eff}}^{(1)} = -\frac{1}{2} \text{tr} \int d^4x G_0(0) \mathcal{V}(x) = \frac{2m_\varphi^2}{(4\pi)^2} \left\{ \frac{1}{\epsilon} + \psi(1) + 1 + \ln \frac{4\pi \mu^2}{m_F^2} \right\} \int d^4x (g^2 \varphi^2 + 2m_F g \varphi),
\]
\[
S_{\text{eff}}^{(2)} = \frac{1}{4} \text{tr} \int d^4x \int d^4y G_0(x-y) \mathcal{V}(y) G_0(y-x) \mathcal{V}(x)
\]
Here $\rho(q)$ is defined as $\rho(q) := q/\sqrt{q^2 + 4m_F^2}$. We choose the renormalization conditions in analogy to Coleman and Weinberg [27]. In order to do so we introduce at first the one-loop effective potential. It is given essentially by the one-loop action induced by constant fields $\varphi(x) = \bar{\varphi}$. Then the external potential takes the form

$$\tilde{V}(q)g^2 \begin{bmatrix} \bar{\varphi}^2 - v^2 & 0 \\ 0 & \bar{\varphi}^2 - v^2 \end{bmatrix} \delta^4(q) = \hat{V}(\varphi) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \delta^4(q)$$

with

$$\hat{V}(\varphi) = g^2(\varphi^2 + 2v\varphi)$$

and the effective action can be written as

$$S_{\text{eff}}[\varphi(x) = \bar{\varphi}] = V_{\text{eff}}^{1-1}\Omega^{(4)}$$

where $\Omega^{(4)}$ is the volume of space-time. The renormalized effective potential including the bare potential and the counterterms is given by

$$V_{\text{eff,ren}} : U(\phi = v + \bar{\varphi}) + V_{\text{eff}}^{1-1} + \left\{ A\bar{\varphi} + B\bar{\varphi}^2 + C\bar{\varphi}^3 + D\bar{\varphi}^4 \right\}.$$  

Here $V_{\text{eff}}^{1-1}$ includes the one loop contributions to all orders, i.e. the regularized divergent parts of first and second order and the $O(3)$ contribution of all higher orders. The first and second order terms can be found easily from (4.10); they read

$$V_{\text{eff,reg}}^{1-1(1)} = 2\hat{V}(\bar{\varphi})\frac{m_F^2}{(4\pi)^2} \left( \frac{1}{\varepsilon} + \psi(1) + 1 + \ln\frac{4\pi\mu^2}{m_F^2} \right).$$

and

$$V_{\text{eff,reg}}^{1-1(2)} = \frac{\hat{V}^2(\bar{\varphi})}{(4\pi)^2} \left( \frac{1}{\varepsilon} + \psi(1) + \ln\frac{4\pi\mu^2}{m_F^2} \right).$$

The $O(3)$ contribution has been given in [15], it reads

$$V_{\text{eff}}^{1-1(3)} = -2 \int \frac{d^4p}{(2\pi)^4} \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\hat{V}(\bar{\varphi})}{p^2 + m_F^2} \right)^n$$

$$= -\frac{2m_F^4}{(4\pi)^2} \left\{ \ln \left( 1 + \frac{\hat{V}(\bar{\varphi})}{m_F^2} \right) \frac{1}{2} + \frac{\hat{V}(\bar{\varphi})}{m_F^2} + \frac{1}{2} \frac{\hat{V}^2(\bar{\varphi})}{m_F^2} \right\} - \frac{\hat{V}(\bar{\varphi})}{2m_F^2} - \frac{3}{4} \frac{\hat{V}^2(\bar{\varphi})}{m_F^2}. $$
We require the effective potential to retain the following properties of the tree-level potential $U(\phi = v + \varphi)$:

\[
\left. \frac{d^2 V_{\text{eff, ren}}(\varphi)}{d\varphi^2} \right|_{\varphi=0} = m_{\varphi}^2 \quad \text{(physical mass)}, \quad (4.15)
\]

\[
\left. \frac{d V_{\text{eff, ren}}(\varphi)}{d\varphi} \right|_{\varphi=0} = 0 \quad \langle \varphi \rangle = 0. \quad (4.16)
\]

The first of these conditions fixes the constant $B$ of the counterterm action, the second one the constant $A$. The remaining constants are fixed by the constraints (4.6). One finds after some algebra

\[
A = -\frac{g m_F^3}{4\pi^2} \left( \frac{1}{\varepsilon} + \psi(1) + 1 + \ln \frac{4\pi\mu^2}{m_F^2} \right),
\]

\[
B = -\frac{3g^2 m_F^2}{4\pi^2} \left( \frac{1}{\varepsilon} + \psi(1) + \frac{1}{3} + \ln \frac{4\pi\mu^2}{m_F^2} \right),
\]

\[
C = -\frac{3g^3 m_F}{2\pi^2} \left( \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_F^2} \right),
\]

\[
D = -\frac{3g^4}{2\pi^2} \left( \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_F^2} \right).
\]

With these counterterms the total effective potential becomes

\[
V_{\text{eff, ren}}(\varphi) = \frac{g m_F^3}{2(2\pi)^2} \varphi^2 + \frac{1}{2} m_{\varphi}^2 \left( 1 + \frac{7g^2 m_F^2}{2(2\pi)^2 m_{\varphi}^2} \right) \varphi^2
\]

\[
+ \lambda v \left( 1 + \frac{3}{2(2\pi)^2} \frac{g^3 m_F}{\lambda v} \right) \varphi^3 + \frac{1}{4} \lambda \left( 1 + \frac{3}{2(2\pi)^2} \frac{g^4}{\lambda} \right) \varphi^4
\]

\[
- \frac{2}{(4\pi)^2} (g_{\varphi} + m_F) m_{\varphi} \ln \frac{g_{\varphi} + m_F}{m_F}.
\]

and is independent of $\varepsilon$ and $\mu$. In order to determine the wave function renormalization we have to consider $S_{\text{eff}}^{(2)}$ again. Inserting the explicit expression for $\text{tr}(\tilde{V}(q))^2$ into (4.7) we find

\[
S_{\text{eff}}^{(2)} = \frac{1}{16\pi^2} \int \frac{d^4 q}{(2\pi)^4} \left( |\tilde{V}(\varphi)(q)|^2 + g^2 q^2 |\tilde{\varphi}(q)|^2 \right) \left\{ \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_F^2} \right\}
\]

\[
+ 2 - \frac{1}{\rho(q)} \ln \frac{1 + \rho(q)}{1 - \rho(q)} \right\}.
\]

where $\tilde{V}(\varphi)$ has been defined in (4.8). In the integrand the contribution

\[
|\tilde{V}(\varphi)(q)|^2 \left( \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_F^2} \right) (4.19)
\]
has been absorbed already into the renormalized effective potential. (see (4.13) and (4.17)). The contribution
\[
\frac{1}{16\pi^2} \int \frac{d^4q}{(2\pi)^4} g^2 q^2 |\tilde{\phi}(q)|^2 \left( \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi \mu^2}{m^2} \right)
\]
(4.20)
will be absorbed into the wave function renormalization \(\delta Z\) (see below). This leaves a finite part
\[
S_{\text{eff, finite}}^{(2)} = \frac{1}{16\pi^2} \int \frac{d^4q}{(2\pi)^4} \left( |\tilde{V}(\varphi)(q)|^2 + g^2 q^2 |\tilde{\phi}(q)|^2 \right) \left\{ 2 - \frac{1}{\rho(q)} \ln \frac{1 + \rho(q)}{1 - \rho(q)} \right\}.
\]
(4.21)
The expression in curly brackets can be expanded as
\[
2 - \frac{1}{\rho(q)} \ln \frac{1 + \rho(q)}{1 - \rho(q)} \simeq - \frac{q^2}{6m^2} + O(q^4).
\]
(4.22)
Therefore the term \(g^2 q^2 |\tilde{\phi}(q)|^2\) in the first parenthesis will be multiplied at least with \(q^2\) and yield terms with at least four derivatives of the fields. On the other hand the term \(|\tilde{V}(\varphi)(q)|^2\) will yield another second derivative term when multiplied with \(q^2/6m^2\). We denote this contribution as \(S_{\text{eff, analytic}}^{(2)}\). It can be rewritten as
\[
S_{\text{eff, analytic}}^{(2)} = - \frac{1}{16\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{q^2}{6m^2} \left( \frac{g^2}{m^2} \varphi^2 + g^2 + 2 \frac{g^3}{m^2} \varphi \right) (\partial_{\mu} \varphi)^2.
\]
(4.23)
(4.24)
Since we renormalize at \(\phi = v\), i.e. \(\varphi = 0\) we find here another contribution to the wave function renormalization:
\[
- \frac{g^2}{24\pi^2} \int d^4x (\partial_{\mu} \varphi)^2,
\]
(4.25)
to be absorbed into \(\delta Z\). Other second derivative terms appear in \(S_{\text{eff}}^{(3)}\) given in (3.31), their coefficient vanishes at \(\phi = v\). We find therefore, collecting the contributions (4.20) and (4.24)
\[
\delta Z = - \frac{2g^2}{(4\pi)^2} \left( \frac{1}{\varepsilon} + \psi(1) + \ln \frac{4\pi \mu^2}{m^2} \right) + \frac{g^2}{12\pi^2}.
\]
(4.26)
The effective action
\[ \Gamma = S_{cl} + S_{\text{eff}}^{1-1} \]  
becomes therefore
\[ \Gamma = \int d^4x \left( \frac{1}{2} (1 + \frac{g^2}{12\pi^2})(\partial_\mu \phi)^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 \right) + S_{\text{eff,finite}}^{(2)} + S_{\text{eff}}^{(3)}. \]  

While taking the functional derivative of the first term is trivial and the numerical computation of the term \( S_{\text{eff}}^{(3)} \) has been described in section 2, we have to discuss, as a final step, the computation of the functional derivative of \( S_{\text{eff,finite}}^{(2)} \). Restricting ourselves again to time-independent configurations we find, using the definition \( (4.21) \),
\[ \frac{1}{\tau} \frac{\delta S_{\text{eff,finite}}^{(2)}}{\delta \phi(x)} = \frac{g^2}{4\pi^2} \left( 2g^2 \phi(x)(\phi^2(x) - v^2) - \Delta \phi(x) \right) - \frac{g^2}{8\pi^2} \int \frac{d^3q}{(2\pi)^3} \exp(iqx) \left( 2g^2 \phi(x)(\phi^2 - v^2) \right) \sim (q)^ \frac{1}{\rho(q)} \ln \frac{1 + \rho(q)}{1 - \rho(q)}. \]  

Here \((..)\sim\) denotes the Fourier transform of the expression in parentheses. Taking into account of the fact that the fields depend only on \( r = |x| \) and therefore their Fourier transforms only on \( q = |q| \), this expression can be further simplified. Its computation involves simple one-dimensional Fourier-Bessel transforms and integrations.

We compare the numerical results for the renormalized second order contribution \((1/\tau)\delta S_{\text{eff,finite}}^{(2)}/\delta \phi(x) - g^2 \Delta \phi/12\pi^2\) with those for \((1/\tau)\delta S^{(3)}/\delta \phi\) in Figure 5. The renormalized second order contribution is found to be considerably smaller than the contributions of order \((3)\).

5 Conclusions

We have presented here a method of computing the functional derivative of an exact effective action \( \Gamma \) for spherically symmetric background field configurations. The vanishing of this functional derivative determines field configurations that minimize the action or, for time independent configurations, the energy of a quantum system in one-loop order. We have considered the exact computation of the finite part as well as the implementation of renormalization, whenever it is necessary. We have considered here a fermionic loop contribution, the method is of course more general.

For the special case considered here we have also presented an analytic estimate
\[ \frac{1}{\tau} \frac{\delta S_{\text{eff}}^{(3)}}{\delta \phi(x)} = -\frac{m_F^4}{4\pi^2 v} \left[ (\ln \phi^2 - 1) + 1 \right] - \frac{m_F^2}{16\pi^2 v} \left[ -\left( \phi' \right)^2 \left( \frac{2}{\phi} - \frac{4}{3} \right) \right]. \]
\(-2 \left( \hat{\phi}'' + \frac{2}{r} \hat{\phi}' \right) \left( \ln \hat{\phi}^2 - \frac{2}{3} \left( \hat{\phi}^2 - 1 \right) \right) \] (5.30)

for the finite part of order (3) and compared it to the exact numerical results. The approximation is found to be good at large \( m_F \), as to expected. This local expression, as well as some local terms occuring in \( \delta S_{\text{eff, finite}}^{(2)} / \delta \phi(x) \) (see (4.29)) can be introduced explicitly into the classical equation of motion (1.1) thus obtaining an improved differential equation for the classical solution. Its solution should represent a good starting point for an iterative procedure which takes into account the remaining parts that have to be computed numerically.
Appendix

Equations (3.11) and (3.12) are relations one expects to find in some textbook, therefore in [13] they were given without proof. Since we have not found such a textbook yet, we give here a heuristic derivation. The Green function we want to determine satisfies

\[
\delta_{nk} \left( -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ln(l_n+1)}{r^2} + \kappa^2 + v_{nk}(r) \right) g_{km}(r, r') = \frac{1}{r^2} \delta(r - r') . \quad (A.1)
\]

From the fact that it is a solution of the homogeneous differential equation for \( r \neq r' \), both as a function of \( r \) and \( r' \), and imposing regularity at \( r = 0 \) and \( \infty \) we conclude that it must have the form

\[
g_{mn}(r, r') = \sum_{\alpha,\beta} \left\{ \Theta(r - r') f_{\alpha}^{+}(r) c_{\alpha\beta} f_{\beta}^{-}(r') + \Theta(r' - r) f_{\alpha}^{-}(r) d_{\alpha\beta} f_{\beta}^{+}(r') \right\} \quad (A.2)
\]

in terms of the functions \( f_n^{\alpha \pm}(r) \) defined in section 3.2. The Green function has to be continuous at \( r = r' \). Its first derivative has to have a discontinuity so that the second derivative produces the appropriate \( \delta \) function. These two conditions lead to the relations

\[
\sum_{\alpha,\beta} \left( c_{\alpha\beta} f_{\alpha}^{+}(r) f_{\beta}^{-}(r) - d_{\alpha\beta} f_{\alpha}^{-}(r) f_{\beta}^{+}(r) \right) = 0 \quad (A.3)
\]

\[
r^2 \sum_{\alpha,\beta} \left( c_{\alpha\beta} f_{\alpha}^{+}(r) f_{\beta}^{-}(r) - d_{\alpha\beta} f_{\alpha}^{-}(r) f_{\beta}^{+}(r) \right) = -\delta_{mn} . \quad (A.4)
\]

We contract the first of these equations with \( r^2 f_{\gamma}^{-}(r) \) and the second one with \( f_{\alpha}^{+}(r) \), then we subtract the resulting equations from each other. We obtain

\[
\sum_{\alpha,\beta} \left( c_{\alpha\beta} r^2 W(f_{\gamma}^{+}, f_{\alpha}^{+}) f_{\beta}^{-} - d_{\alpha\beta} r^2 W(f_{\gamma}^{+}, f_{\alpha}^{-}) f_{\beta}^{+} \right) = -f_{\alpha}^{+}(r) \quad (A.5)
\]

where we have defined the Wronskians as

\[
W(f^\alpha, g^\beta) = \sum_m \left( f_m^\alpha(r) g_m^\beta(r) - f_m^\alpha(r) g_m^\beta(r) \right) . \quad (A.6)
\]

If the potential \( v_{nk}(r) \) is symmetric the differential equation satisfied by the functions \( f_n^{\alpha \pm}(r) \) implies that \( r^2 W(f^\alpha, g^\beta) \) is independent of \( r \). Since the functions \( f_n^{\alpha}(r) \) vanish as \( r \to \infty \) the Wronskians \( W(f_{\gamma}^{+}, f_{\alpha}^{+}) \) vanish. Using the fact that the solutions \( f_{\gamma}^{+}(r) \) have been chosen to form a linearly independent set, we conclude that

\[
\sum_{\alpha} d_{\alpha\beta} r^2 W(f_{\gamma}^{+}, f_{\alpha}^{-}) = \sum_{\alpha} d_{\alpha\beta} \omega_{\gamma\alpha} = \delta_{\beta} \quad (A.7)
\]
or
\[ d_{\alpha\beta} = \omega_{\alpha\beta}^{-1}. \]  
(A.8)

In order to obtain the coefficients \( c_{\alpha\beta} \) we contract (A.3) with \( r^2 f^\gamma_m (r) \) and (A.4) with \( f^\gamma_m (r) \). Using the fact that \( r^2 W(f^\gamma_m, f^\alpha_m) \) vanishes because it does so as \( r \to 0 \), we find
\[ c_{\alpha\beta} = \omega_{\beta\alpha}^{-1}. \]  
(A.9)

This closes the demonstration of (3.11) and (3.12).

References

[1] see e. g. R. Rajaraman, *Solitons and Instantons*, North-Holland Publ. Comp., Amsterdam 1982

[2] G. Callan and S. Coleman, Phys. Rev. **D16**, 1762 (1977).

[3] J. S. Langer, Ann. Phys. (N. Y.) **41**, 108 (1967); *ibid*. **54**, 258 (1969).

[4] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).

[5] A. D. Linde, Nucl. Phys. **B216**, 421 (1983).

[6] R. Friedberg and T. D. Lee, Phys. Rev. **D 15**, 1694 (1977).

[7] J. Baacke, Z. Phys. **C2**, 63 (1979).

[8] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); *ibidem* **124**, 246 (1961).

[9] H. Reinhardt and R. Wünsch, Phys. Lett. **B215**, 577 (1988).

[10] T. Meissner, F. Grümmer and K. Goeke, Phys. Lett. **B227**, 296 (1989).

[11] R. Alkofer, Phys. Lett. **B236**, 310 (1990)

[12] J. Baacke, Z. Phys. **C47** 263 (1990).

[13] J. Baacke, Z. Phys. **C49** 619 (1990).

[14] J. Baacke and V. G. Kiselev, Phys. Rev. **D48**, 5648 (1993).

[15] J. Baacke, H. So und A. Sürig, Z. Phys. **C63** 689 (1994).

[16] V. G. Kiselev and K. G. Selivanov, Pis’ma Zh. Eksp. Teor. Fiz. **39**, 72 (1984) [Sov. Phys. JETP Lett. **39**, 85 (1984)].

[17] V. G. Kiselev and K. G. Selivanov, Yad. Phys. **43**, 239 (1986) [Sov. J. Nucl. Phys. **43**, 153 (1986)].
[18] K. G. Selivanov, Z. Exp. Teor. Fiz. 94, 57 (1988) [Sov. Phys. JETP 67, 1548 (1988)].

[19] A. Bochkarev, Phys. Rev. D 46, 5550 (1992).

[20] J. Baacke, Z. Phys C53, 407 (1992).

[21] J. Baacke, S. Junker, Phys. Rev D49, 2055 (1994); ibid. D50, 4227 (1994).

[22] J. Baacke and T. Daiber, Phys. Rev D 51, 795 (1995).

[23] J. Baacke, “Fluctuation corrections to bubble nucleation”, Dortmund preprint DO-TH 95/04 (1995), to appear in Phys. Rev. D.

[24] J. Baacke and A. Sürig, “Fermionic fluctuation corrections to bubble nucleation”, Dortmund preprint DO-TH 95/08.

[25] S. Coleman, The Uses of Instantons, in The Aspects of Symmetry, Cambridge University Press (1985)

[26] J. D. Bjorken and S. D. Drell: Relativistic Quantum Mechanics; McGraw-Hill, London 1964.

[27] S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).
Figure Captions

**Fig. 1** The functional derivative of the finite part of the effective action \((1/\tau)\delta S^{(3)}_{\text{eff}}/\delta \phi(x)\) in units \(v^3\), calculated for a spherically symmetric time independent profile \(\phi(x) = f(r)\) (see (3.33)) with \(R = 2\). Here we choose the fermion mass in units of \(v\) to be \(\hat{m}_F \equiv g = 1.5\). The solid line is the exact numerical result and the dashed line is the analytic estimate given in Eq. (3.32).

**Fig. 2** The same as in Figure 1 for \(g = 1.7\).

**Fig. 3** The same as in Figure 1 for \(g = 2.0\).

**Fig. 4** The same as in Figure 1 for \(g = 2.2\).

**Fig. 5** The various contributions to the full one-loop correction of the functional derivative of the effective action in units of \(v^3\) for \(g = 2.0\): The dotted line represents the renormalized second order contribution \((1/\tau)\delta S^{(2)}_{\text{eff.finite}}/\delta \phi(x) - g^2\Delta \phi/12\pi^2\); the dashed line is the exact numerical result of \((1/\tau)\delta S^{(3)}_{\text{eff}}/\delta \phi\) which is also presented in Figure 3. The solid line is the sum of both contributions.
Figure 1

Figure 1
Figure 2
Figure 3
Figure 4
Figure 5