Inflationary perturbations in anisotropic backgrounds and their imprint on the cosmic microwave background

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Abstract. We extend the standard theory of cosmological perturbations to homogeneous but anisotropic universes. We present an exhaustive computation for the case of a Bianchi I model, with a residual isotropy between two spatial dimensions, which is undergoing complete isotropization at the onset of inflation; we also show how the computation can be further extended to more general backgrounds. In the presence of a single inflaton field, there are three physical perturbations (precisely as in the isotropic case), which are obtained (i) by removing gauge and non-dynamical degrees of freedom, and (ii) by finding the combinations of the remaining modes in terms of which the quadratic action of the perturbations is canonical. The three perturbations, which later in the isotropic regime become a scalar mode and two tensor polarizations (gravitational wave), are coupled to each other already at the linearized level during the anisotropic phase. This generates non-vanishing correlations between different modes of the cosmic microwave background (CMB) anisotropies, \( \langle a_{\ell m} a_{\ell' m'} \rangle \propto \delta_{\ell \ell'} \delta_{mm'} \), which can be particularly relevant at large scales (and, potentially, be related to the large scale anomalies in the WMAP (Wilkinson Microwave Anisotropy Probe) data). As an example, we compute the spectrum of the perturbations in this Bianchi I geometry, assuming that the inflaton is in a slow roll regime also in the anisotropic phase. For this simple set-up, fixing the initial conditions for the
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB perturbations appears more difficult than in the standard case, and additional assumptions seem to be needed to provide predictions for the CMB anisotropies.

**Keywords:** CMBR theory, cosmological perturbation theory, inflation

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### Contents

1. Introduction .................................................. 2
2. The $a_{\ell m}$ covariance ..................................... 6
3. The background .................................................. 9
4. Cosmological perturbations....................................... 12
   4.1. Anisotropic background .................................... 12
   4.2. Initial conditions ......................................... 13
   4.3. Curvature power spectrum .................................. 16
5. Discussion ...................................................... 19
   Acknowledgments .................................................. 21
   Appendix A. Explicit computation of $C_{\ell \ell' m m'}$ ............. 21
   Appendix B. Explicit computation of the perturbations .............. 22
      B.1. Gauge choice .............................................. 23
      B.2. Comparison with the standard modes in the isotropic limit ($b = a$) .. 24
      B.3. 2D vectors .............................................. 26
      B.4. 2D scalars .............................................. 27
   Appendix C. Dependence of the frequency, and of the adiabaticity condition on the choice of conformal time .......................... 30
      C.1. Positive branch .......................................... 32
      C.2. Negative branch .......................................... 33
6. References ..................................................... 34

### 1. Introduction

The study of cosmic microwave background (CMB) full sky maps from the WMAP experiment [1] has led to some intriguing anomalies which seem to suggest that the assumption of statistical isotropy is broken on the largest angular scales [2]. These anomalies include an alignment of the moments in the lowest multipoles dubbed the ‘axis of evil’, an asymmetry in the power between the northern and southern ecliptic hemispheres and an apparently non-Gaussian excursion in the southern galactic hemisphere known as the ‘cold spot’. Another well-known puzzle in the observations has been the lack of power in the quadrupole which had also been noted in the COBE-DMR maps [3] and is still present in the latest WMAP data [4].
The significance of the anomalies has been debated extensively in the literature (see e.g. [5]) with some reported effects more significant than others. The difficulty in quantifying exactly the importance of any effect is due to how correctly the \textit{a posteriori} probability of observing it is estimated. It is clear however that the anomalies implying an overall anisotropy in the data are more significant than the lack of power in the CMB quadrupole. A natural explanation for the observed anomalies may be some form of as yet undetermined systematic or foreground signal which is not being taken into account properly in the data reduction producing the final maps [6]. However, a conclusive explanation along these lines has not been put forward yet. It is therefore legitimate to ask whether the observed anomalies may be an indication of a departure from the standard cosmological model. In order to find convincing evidence on such deviations, one should study the CMB properties predicted by any specific model, with a particular emphasis on those which go beyond the ones characterizing the standard picture. For instance, if the signal \textit{is} statistically anisotropic, crucial information will be encoded in the off-diagonal correlations of the spherical harmonic modes with \( \langle a_{\ell m} a_{\ell' m'}^{*} \rangle \propto \delta_{\ell \ell'} \delta_{mm'} \), which are predicted to vanish in the standard case.

Since, aside the anomalies [2], CMB data strongly support the general theory of inflation, we focus here on departures from the standard picture which can however be reconciled with the inflationary framework. For example, the low quadrupole power has motivated a number of studies to explain the observed cut-off within the context of inflation [7]–[10] (reference [11] further extended the model [7], to account for a spatial asymmetry of the CMB perturbations at large scales). These models assume a short period of inflation, leading to a fine tuning problem in the standard inflationary picture since the slow roll regime responsible for the observed \( N \sim 60 \) or so e-folds is an attractor in the inflaton phase space and thus the probability of inflating for a large number of e-folds is much larger than that of inflating for only 60. Recently, other groups have argued that inflation should be short-lived [13] and in particular models of inflation on the string theory landscape seem to predict a highly suppressed probability of large \( N \) [14]. Coupled to the fact that our observed universe requires \( N \approx 60 \), these arguments imply that inflation must have lasted ‘just long enough’. Theoretical arguments aside, a limited amount of inflation would certainly lead to a richer phenomenology than a prolonged one. The reason why inflation was originally postulated is that it leads to the isotropic and homogeneous Friedmann–Robertson–Walker (FRW) universe starting from rather generic initial conditions. If inflation lasted many e-folds, then any trace of the pre-existing universe is inflated away to scales much larger than the present day horizon. On the other hand, a limited amount of inflation could have left some trace in the data, particularly at large scales, that would appear as anomalies within the standard inflationary picture.

With these considerations in mind, in this work we perform an initial step towards the study of the CMB anisotropies taking into account a pre-existing phase before inflation, in which the universe is not yet of the FRW type. Motivated by the suggested alignment of the large scale multipoles along a common axis, we focus on models which are anisotropic at the onset of inflation. For simplicity, we still assume that the universe is homogeneous. As we will see, even this very minor departure from standard cosmology requires a significant extension of the well consolidated framework for the computation of CMB

\footnote{See however other inflation models that naturally suppress the overall number of e-folds, e.g. [12].}
anisotropies in a FRW universe. Homogeneous but anisotropic cosmologies have long ago been classified into equivalence classes known as the Bianchi types [15]. With the possible exception of type IX, Bianchi universes with a positive cosmological constant evolve towards an asymptotic (isotropic) de Sitter stage [16]. The situation is more complicated when the cosmological constant is replaced by an inflaton field (which one has to do, in order to have a finite amount of inflation). Also in this case, however, one finds that inflation takes place for a wide variety of circumstances, leading to the isotropization of the space \(^4\).

It was shown in [18] that a late time contribution of the Bianchi VII\(_h\) form to the temperature anisotropies can account for several of the WMAP anomalies; more in general, Bianchi models with generalized backgrounds driven by anisotropic stresses have been studied extensively in the literature (see e.g. [19, 20]). These models require a particular mechanism actively driving the anisotropy, which can for this reason survive for a much longer period than in the case considered here (and, possibly, result in stronger observational effects). In this work we consider only the case consistent with the simplest models of inflation where the background is initially anisotropic and is isotropized at the onset of inflation \(^5\). We expect to recover a standard power spectrum at the scales that leave the horizon after the isotropization has been achieved, but non-conventional results at larger scales \(^6\).

Thus, the anisotropy is restricted to the initial conditions, in the form of a power spectrum of the primordial perturbations which depends also on the directionality of the modes, and not only on the magnitude of their momentum \(k\) (as in the isotropic case). On the contrary, the transfer functions, relating the late time temperature anisotropies to the primordial perturbations, are the standard ones, since the propagation of the modes, soon after the onset of inflation, occurs in an isotropic background. This simplifies greatly the calculation of the correlations induced in the CMB \(^7\). We present this computation in section 2, in the case in which there is a residual anisotropy between two of the three spatial directions.

Namely, we compute the correlation between different CMB modes, under the assumption that (i) the primordial perturbations are Gaussian, and (ii) their power spectrum depends on the magnitude \(|k|\) and on the angle between the momentum and the anisotropic direction. We do not impose any further assumption in this computation, so that it holds for any Bianchi model with a residual 2D isotropy and an asymptotic standard inflationary behaviour. As we anticipated a few paragraphs above, the main result is

\(^4\) A list of works discussing this issue, and several aspects of inflation in Bianchi models can be found in the review [17].

\(^5\) We also assume that the only source is the inflaton field. Reference [21] studied the background evolutions in cases in which an anti-symmetric tensor is also dynamically relevant.

\(^6\) Reference [22] studied the perturbations of a scalar field assuming that a small anisotropy (corresponding to the ratio \(h/H\) in our equation (14)) is present during inflation, and that it then decays away at the end of inflation. Such perturbations are then converted into metric perturbations through the mechanism of modulated perturbations [23]. This work presents analytical results for the perturbations using an expansion series in the anisotropy parameter, which, in our context, is valid for modes leaving the horizon towards the end of the anisotropic phase.

\(^7\) This is in contrast to the case where the anisotropy is present during the evolution of perturbations at any time in the radiation through to the present epoch. In this case the Einstein–Boltzmann system must be modified to account for the anisotropic evolution of the modes [24]–[26].
a non-vanishing correlation between off-diagonal modes, see equation (7). The specific relations found between the different non-vanishing correlators can in principle allow for a detailed data analysis within any given model. We do not perform this analysis in the present work (since, as we mentioned, our main result is the extension of the computation of the primordial perturbations to such non-FRW geometries); however we include the relations (7), since they offer a model oriented way of extracting information from the CMB data. This information is lost in the standard diagonal $C_\ell$ statistic analysis. The observed correlations can be fit with model-independent templates (see e.g. [27, 1]) but these are not calculated a priori from any particular theory and thus the information contained in the CMB will not constrain any fundamental parameter.

In the remainder of the work, we present the detailed computation of the primordial perturbations within a Bianchi I model with a residual 2D isotropy, which is the simplest deviation from the FRW geometry. The line element is of the form

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 (dy^2 + dz^2),$$

(1)

and late time isotropy ($a = b$) is achieved due to an inflaton scalar field $\phi$. In section 3 we study the background evolution for this model, under the assumption that the inflaton is initially in a slow roll regime. As it is well known, the anisotropy is rapidly damped away, on a timescale corresponding to the inverse Hubble rate due to the potential energy of $\phi$. For this reason, in order to have a sizable effect, we need to start from a significant anisotropy. The model then admits two possible background solutions at early times, according to whether the expansion rate of the scale factor $a$ is greater or smaller than the one of the scale factor $b$. In the first case (which we call positive branch), $a \propto t$ at asymptotically early times, while $b$ is constant. In the second case (which we call negative branch), the anisotropic direction is initially contracting, $a \propto t^{-1/3}$, while the other two directions expand as $b \propto t^{2/3}$. In both cases, the metric is singular at $t = 0$, and the early time evolution is of the Kasner [28] type.

In section 4 we compute the primordial perturbations about this geometry, by extending to this case the Mukhanov–Sasaki [29] linearized computation and quantization procedure for the perturbations valid for a FRW background. For clarity, we summarize at various stages the results of the isotropic computation [34], and we then show how each of them extends to our case. We start by showing that also in the present case there are three physical modes of the perturbations. We see this with a simple counting of modes, which holds in the standard case, but which does not rely on the isotropy (nor homogeneity) of the background. For a FRW geometry, the three modes are encoded in a scalar perturbation, and in the two polarizations of a tensor mode (gravitational wave). These modes are decoupled at the linearized level, due to the homogeneity and isotropy of the background. In our case, only one mode is decoupled, due to the residual 2D isotropy, while the two remaining ones are coupled to each other. As the background becomes isotropic, the latter two become the scalar mode, and one of the two tensor polarizations. To extract the physical modes, we choose a non-conventional gauge that preserves all the $g_{0\mu}$ metric perturbations. Such modes are non-dynamical, and can be

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8 Perturbations in Bianchi models have been previously studied in [30] (through the formalism of [31]) and in [32] (through the formalism of [33]). These analyses do not compute the canonical variables of the actions of the perturbations, as instead done in [29] for the FRW case, which, as we shall see, is a necessary step to compute the initial conditions for these modes.
readily integrated out. Since there are initially 11 perturbations (symmetric $\delta g_{\mu\nu}$ and $\delta \phi$), we are left with the $11 - 4(gauge) - 4(non-dynamical) = 3$ physical modes. Although we perform the explicit computation only for the Bianchi I model with a residual 2D isotropy, we emphasize that this procedure can be further extended to more general geometries. In the general case, one expects that all the three physical modes are coupled to each other at the linearized level.

To compute the power spectrum of these modes, we need to study their early times frequency. In the standard case, the physical frequency is initially dominated by the physical momentum, and it changes only very slowly (adiabatically) due to the expansion of the universe. In conformal time $\eta$, related to the physical time $t$ by $dt = a\, d\eta$ the (conformal) frequency is actually constant at early times. Therefore, one can consistently start from an adiabatic vacuum in the asymptotic past (this, in turn, leads to a nearly scale invariant primordial power spectrum). The situation is more complicated for the two Bianchi I backgrounds that we have described above; first of all there is some ambiguity in the choice of conformal time (since one may use different powers of the two scale factors in the definition). This has lead us to study the condition under which two different conformal times can be employed, and result in the same prescription for the initial conditions. We find that, in the negative branch, the frequencies of two coupled modes are not adiabatically evolving at early times; it is possible to find a time variable in which the frequency of the decoupled mode is adiabatically evolving; however, the mode is tachyonic, leading to an instability and to a non-linear regime that prevents us from computing firm predictions for the CMB anisotropies in it. For the positive branch, $(a \propto t, b \rightarrow \text{const})$, all the three modes are decoupled at asymptotically early times, and their three frequencies are adiabatically evolving in the conformal time variable $\eta$, defined as $dt = a\, b^\gamma\, d\eta$ (where $\gamma$ any constant; for simplicity we then choose $\gamma = 0$ in the computation). This allows us to consistently start from an adiabatic vacuum. The power spectrum that we obtained approaches isotropy for the modes which exit the horizon right after the universe becomes isotropic. At larger scales, however, the spectrum exhibits an angular dependence, and it actually diverges for momenta that are aligned along the directions which are not expanding at asymptotically early times. As a consequence, a computation of the temperature anisotropies requires that we resolve this singularity. In the concluding section 5, we discuss several ways in which this can be done (at the very least, this will occur due to non-linear effects, since the singularity of the modes indicates a breakdown of the linearized computation). However, all of them require additional input in the computation. We therefore conclude that the Bianchi I model we have studied is too simple to provide firm predictions for the anisotropy, at least in the assumption of initial slow roll of the inflaton that we have made in this analysis.

The paper is concluded by some appendices, where we outline several intermediate steps of the computations.

2. The $a_{\ell m}$ covariance

In this section we compute the statistical correlation between different multipoles of the CMB temperature anisotropies, relaxing the usual assumption of statistical isotropy. The starting point is the power spectrum

$$\langle R(k) R^*(k') \rangle \equiv \frac{(2\pi)^3}{k^3} \delta(k - k') P(k)$$

(2)
of the primordial comoving curvature perturbations (we assume that the perturbations are Gaussian). If the background is isotropic, the fluctuations are statistically isotropic, and the power spectrum $P$ depends only on the magnitude of $k$. In general, it will depend also on its direction.

In the following sections, we restrict our attention to a Bianchi I cosmology. The present computation is instead valid for more general anisotropic backgrounds, with the assumptions that (i) there is a residual isotropy of two of the three spatial dimensions, and (ii) this anisotropy is damped away during inflation. It is straightforward to generalize the calculation if condition (i) is relaxed. Under these assumptions, the primordial power spectrum depends on the magnitude of $k$, and on the angle between $k$ and the anisotropic direction. We denote by $\xi$ the cosine of this angle.

We denote by $\delta T(\hat{p}, \eta_0, x_0)$ the temperature perturbation in the direction $\hat{p}$, as measured by an observer at position $x_0$ and at conformal time $\eta_0$. It is customary to decompose the temperature perturbations that we observe into spherical harmonics:

$$\delta T(\hat{p}, \eta_0, x_0) = \sum_{l,m} a_{lm} Y_{lm}(\hat{p}).$$

(3)

Since we are assuming that the perturbations are Gaussian, their statistical properties are still encoded in the second order correlations

$$C_{\ell \ell'}^{m m'} \equiv \langle a_{\ell m} a_{\ell' m'}^* \rangle.$$  (4)

The correlations however are not diagonal as in the isotropic case.

In the linear regime, the temperature anisotropy (3) is given by

$$\delta T(\hat{p}, \eta_0, x_0) = \int \frac{d^3k}{(2\pi)^3} R(k, \eta_i) \Delta(k, \hat{k} \cdot \hat{p}, \eta_0) e^{i k \cdot x_0},$$

(5)

where the transfer function $\Delta(k, \hat{k} \cdot \hat{p})$ describes the change in amplitude of the radiation perturbation from an initial time $\eta_i$, which can be taken deep in the radiation dominated era, to today. Since the background is isotropic soon after the onset of inflation, the transfer function is also isotropic (in real space). As a consequence $\Delta$ depends only on the magnitude of the wavenumber $k = |k|$ of the mode, and on the angle between $k$ and the line of sight at which the photon is being observed. It is convenient to expand it in a basis of Legendre polynomials:

$$\Delta(k, \hat{k} \cdot \hat{p}, \eta_0) = \sum_{\ell} (-i)^\ell (2\ell + 1) P_\ell(\hat{k} \cdot \hat{p}) \Delta_\ell(k, \eta_0).$$

(6)

We rotate our coordinate system such that the anisotropic direction lies on the $z$-axis. Starting from the above expressions, after the algebra outlined in appendix A, we find

$$C_{\ell \ell' m m'} = \frac{\delta_{m m'}}{\pi} (-i)^{\ell - \ell'} \frac{(2\ell + 1)(2\ell' + 1)(\ell - m)!(\ell' - m)!}{(\ell + m)!(\ell' + m)!}$$

$$\times \int \frac{dk}{k} \Delta_\ell(k, \eta_0) \Delta_{\ell'}(k, \eta_0) \int_{-1}^{1} d\xi P_\ell^m(\xi) P_{\ell'}^m(\xi) P(k, \xi),$$

(7)

where the $P_\ell^m$ are associated Legendre polynomials. The above expression conveys all the statistical information present in the CMB in the axisymmetric anisotropic case. When constraining anisotropic models the full covariance should be compared to the data.
Some general properties of the correlations follow from equation (7).

(i) The correlations reduce to the standard result in the isotropic case, $P(k, \xi) \equiv P(k)$

\[ C_{\ell \ell' m m'}^{(i o)} = \frac{2}{\pi} \delta_{\ell \ell'} \delta_{m m'} \int \frac{dk}{k} P(k) \Delta_\ell(k, \eta_0)^2 \]  

in which different multipoles are uncorrelated. In any realistic model, the power spectrum becomes isotropic at small scales (large $k$), since such modes exit the horizon when the background is isotropic. Small scales correspond to large $l$ (mathematically, this is due to the transfer functions, which are peaked at increasingly large $k$ as $l$ increases). Therefore, off-diagonal correlators become progressively smaller as $l$ and $l'$ increase, and the result (7) reduces to the isotropic one. The scale at which this happens is strongly sensitive to the amount of inflation which takes place after the universe has become isotropic. If this stage is too prolonged, the power spectrum is anisotropic only at too large scales to be observed today, and the isotropic result (8) is recovered for all multipoles.

(ii) The residual 2D isotropy of the background restricts the number of non-vanishing correlators. The restriction is mostly manifest in the coordinate system considered here, where the anisotropic direction coincides with the $z$-axis, and the multipoles are uncorrelated unless $m = m'$. For a general orientation of the anisotropic direction we find a greater number of non-vanishing correlators. However (obviously) the number of linearly independent correlators remains the same.

(iii) Bianchi I cosmologies have planar reflection symmetries ($x \rightarrow -x$), and, as a consequence, $P(k, -\xi) = P(k, \xi)$. In this case, the correlation (7) also vanishes whenever the difference between $l$ and $l'$ is odd (mathematically, this follows from the parity properties of the associated Legendre polynomials).

Naturally, the orientation of the anisotropic direction is not expected to be correlated with any local preferred direction; for any coordinate system, the orientation can be given in terms of the Euler angles ($\alpha, \beta, \gamma$); these are extra free parameters in anisotropic models, which need to be fit to the data in addition to the usual physical parameters $P$ determining the radiation transfer function and the spectral parameters $S$ determining the primordial power spectrum.

In principle, the anisotropic models can be fit to the observed correlations in the data by evaluating a likelihood of the form

\[ L[a|C(P, S, \alpha, \beta, \gamma)] = \frac{1}{\sqrt{(2\pi)^N|C|}} \exp \left( -\frac{1}{2} a^\dagger \cdot C^{-1} \cdot a \right), \]  

where $N$ is the number of modes in the expansion, $a$ is the vector of observed spherical harmonic coefficients and $C$ is the model correlation (equation (7)) between $\ell, m$ and $\ell', m'$ pairs. The orientation of the anisotropic direction can be rotated using the spherical harmonic rotation operators $D^{\ell}_{m m'}(\alpha, \beta, \gamma)$ [35]. For the axisymmetric case one of the Euler angles is degenerate and can be marginalized out; this is not the case for general anisotropy.

A full maximum likelihood search of (9) is computationally intensive since the dimensionality of the matrices involved scale as $(\ell + 1)^2$. In addition, cut-sky effects complicate the rotation of the data vector which is computationally much faster than multiple rotations of the model correlations $C$. For this reason we postpone any detailed analysis of the likelihood surface and focus here on the framework required to compute
the correlations in the axisymmetric anisotropic case. As we will see below, this requires a careful treatment of both the theory of perturbation evolution in an anisotropic background and the early time initial conditions of the perturbations.

3. The background

Here and in the remainder of the paper we focus on a Bianchi I model, with equal expansion rate in two of the three spatial dimensions:

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 (dy^2 + dz^2)$$

(10)

(contrary to the previous section, we take the anisotropic direction to coincide with the $x$-axis). This model describes an anisotropic universe as long as the ratio between the two different scale factors evolves with time (since any constant ratio can be rescaled to one).

The two physical quantities characterizing the geometry are the two expansion rates

$$H_a \equiv \frac{\dot{a}}{a}, \quad H_b \equiv \frac{\dot{b}}{b},$$

(11)

where, as usual, dot denotes derivative with respect to physical time $t$.

Since we want to study inflation in this background, we consider a scalar field $\varphi$, with the potential $V$. The action of the system is

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V \right].$$

(12)

We decompose the scalar field as $\varphi = \phi + \delta \phi$, namely in a background quantity $\phi$ (that we take to be homogeneous, as the underlying geometry), plus perturbations $\delta \phi$. In the present section we discuss the background evolution, so we only include $\phi$ here. The scalar field perturbations will be considered in the remainder of the paper, when we also perturb the background metric (10).

The background evolves according to the three independent equations

$$\dot{H} + 3H^2 = V/M_p^2,$$

$$\ddot{\phi} + 3H \dot{\phi} + V' = 0,$$

$$3H^2 - h^2 = \frac{1}{M_p^2} \left[ \frac{1}{2} \dot{\phi}^2 + V \right],$$

(13)

where

$$H \equiv \frac{H_a + 2H_b}{3}, \quad h \equiv \frac{H_a - H_b}{\sqrt{3}}$$

(14)

($M_p$ is the reduced Planck mass, while a prime on the potential denotes differentiation with respect to $\varphi$). The first two equations are identical to the ones obtained in the isotropic case, in terms of the ‘average’ expansion rate $H$. The third equation, appearing

9 The first of (13) is obtained by combining the three non-trivial Einstein equations (namely, $3\text{Eq}_0 - \text{Eq}_1 - 2\text{Eq}_2$, where Eq$_i$ refers to the $ij$ Einstein equation), while the third one is the 00 Einstein equation. The second one is the equation for the scalar field (any other linear combination of Eq$_0$, Eq$_1$, and Eq$_2$ can be obtained from (13), as a consequence of the 0 component Bianchi identity).
as a ‘modified Friedmann equation’, can be then used as an algebraic equation for the difference $h$ between the two expansion rates. By differentiating this equation, and by combining it with the other two, one can also find

$$h \left( \dot{h} + 3 H h \right) = 0,$$

which is solved either by $h = 0$ (isotropic case), or by a time evolving $h$ which is decreasing by a rate set by the average expansion parameter $H$.

We want to study the possible background solutions for this system under the assumption of initial slow roll of $\phi$. Let us first study the approximate case in which $V$ is constant, and the inflaton is at rest (vacuum energy). The first and third of (13) are then solved by

$$H = \frac{\sqrt{V}}{\sqrt{3} M_p} \frac{e^{2 \sqrt{3} V t/M_p} + 1}{e^{2 \sqrt{3} V t/M_p} - 1}, \quad h = \pm \frac{2 \sqrt{V}}{M_p} \frac{e^{\sqrt{3} V t/M_p}}{e^{2 \sqrt{3} V t/M_p} - 1}, \quad V \text{ constant}. \quad (16)$$

We note the presence of two distinct branches of solutions, characterized by either positive or negative values of $h$. In the following, we refer to them as the positive and the negative branch, respectively.

We see that the metric has a singularity at $t = 0$. Close to the singularity, the system approaches a Kasner [28] ‘vacuum solution’, where the presence of the source ($V$) can be neglected. The line element of Kasner geometries is

$$ds^2_{\text{Kasner}} = -dt^2 + t^{2\alpha} dx^2 + t^{2\beta} dy^2 + t^{2\gamma} dz^2,$$

where the exponents satisfy the two properties $\alpha + \beta + \gamma = \alpha^2 + \beta^2 + \gamma^2 = 1$. In the present case ($\beta = \gamma$), we can invert the relations (14), to find, close to the singularity,

$$H_a \rightarrow \frac{1}{t}, \quad H_b \rightarrow 0, \quad \text{positive branch},$$

$$H_a \rightarrow -\frac{1}{3t}, \quad H_b \rightarrow \frac{2}{3t}, \quad \text{negative branch}. \quad (18)$$

Integrating these rates reproduces the line element (17) with $\alpha = 1, \beta = \gamma = 0$ in the positive branch, and $\alpha = -1/3, \beta = \gamma = 2/3$ in the negative branch.

We also see from (16) that, for both branches, the system quickly reaches isotropy on a timescale $M_p/\sqrt{V}$ (which parametrically coincides with the inverse of the Hubble rate due to the vacuum energy), after which one quickly approaches the de Sitter values $H = \sqrt{V/3M_p^2}, h = 0$. We note that, in the negative branch, the scale factor of the anisotropic direction experiences a bounce, since it contracts close to the singularity, while at late times it expands with the asymptotic de Sitter rate.

Let us now return to the complete system (13). The only restriction that we impose is that the inflaton $\phi$ is initially in a slow roll regime. Since $h$ decreases very rapidly, we need to start from a strong anisotropy in order to have a measurable deviation from the standard (isotropic) inflationary results. To do so, we choose $|h| \gg \sqrt{V}/M_p$ at the start. In this regime, the average rate $H$ is much greater than in the isotropic case, so that, due to the higher Hubble friction, the inflaton is initially rolling even more slowly than in the standard case. For this reason, the results obtained above for constant $V$ describe very accurately also the evolution of the complete system during the isotropization stage.
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

Figure 1. Time evolution of the two expansion rates $H_a$ (red solid) and $H_b$ (green dashed), in units of $m$, and of the inflaton (blue dotted), in units of $M_p$, for a choice of chaotic inflaton potential $V = m^2 \phi^2/2$. The left panel refers to the positive branch, while the right one to the negative branch. $V_0$ denotes the initial potential of the inflaton, with $\phi_0 = 16 M_p$ in this example. Notice that for the negative branch $H_a$ starts from negative values, indicating that this direction is initially contracting.

The degree of accuracy can be checked for any given inflaton potential, by solving the system (13) with an expansion series in time. For massive chaotic inflation, with $V = m^2 \phi^2/2$, we find

$$H = \frac{1}{3} t \left[ 1 + \frac{m^2 \phi_0^2 t^2}{2 M_p^2} + O \left( m^4 t^4 \right) \right], \quad h = \pm \frac{1}{\sqrt{3} t} \left[ 1 - \frac{m^2 \phi_0^2 t^2}{4 M_p^2} + O \left( m^4 t^4 \right) \right],$$

$$\phi = \phi_0 \left[ 1 - \frac{m^2 t^2}{4} + O \left( m^4 t^4 \right) \right]$$

(19)

(these expansions do not assume $m \ll M_p$, although this is the regime of phenomenological interest). In figure 1 we plot the time evolution of the two expansion rates $H_a$ and $H_b$, and of the inflaton field; the left panel refers to the positive branch, while the right one to the negative branch; $V_0$ denotes the initial potential of the inflaton. The evolutions shown (obtained by a numerical integration of the system (13)), are in perfect agreement with the above discussion. The background has an initial strong anisotropy (we choose $h_0 = \pm 10^9 m$ in the two branches, respectively), which is rapidly damped away, and it is then followed by a stage of standard isotropic inflation. Since the inflaton is nearly static in the early anisotropic stage, its initial value controls the duration of the isotropic inflationary expansion. The value $\phi_0 = 16 M_p$ chosen in the figure leads to about 60 e-folds of standard inflation.
4. Cosmological perturbations

We now compute the perturbations about the backgrounds presented in the previous section.

It is useful to recall first the standard results. In the presence of a single scalar field (the inflaton) in a homogeneous and isotropic FRW universe one finds three physical perturbations. These three modes are always decoupled from each other, due to the symmetry properties of the background. The actual computation can be found, among other works, in the comprehensive review [34]. The two polarizations of the tensor mode are denoted by $h_x$ and $h_+$. The scalar degree of freedom is instead encoded in the so-called Mukhanov–Sasaki variable $v$ [29], which is in turn related to the comoving curvature perturbation $\mathcal{R}$ as

$$\mathcal{R} = \frac{H}{a} \dot{\phi} v.$$  \hspace{1cm} (20)

These quantities are gauge invariant (they do not change under coordinate transformations), and $\mathcal{R}$ represents the gravitational potential on comoving hypersurfaces where the inflaton field is homogeneous (this property defines these hypersurfaces).

In the remainder of this section we discuss the anisotropic case. We do so in three parts. In the first part, we show how the isotropic computation can be extended to the anisotropic background presented in the previous section. We also outline the procedure to further generalize this computation to arbitrary anisotropic spaces. In the second part we discuss the initial conditions for the perturbations both in the positive and negative branch. We conclude with the computation and the discussion of the curvature power spectrum.

4.1. Anisotropic background

The number of the physical modes can be obtained by a simple counting. We start from 11 perturbations in the (symmetric) metric, plus 1 perturbation of the inflaton field. Of these 11 perturbations, 4 are gauge modes, which can be set to zero once we completely fix the freedom of general coordinate reparametrization. Of the remaining modes, 4 are non-dynamical (namely, they enter with at most one time derivative in the action), and 3 physical. A convenient method to extract the 3 physical modes is to choose a set of conditions for the perturbations which fixes the gauge completely (thus removing any $\delta g_{0\mu}$ mode). These modes are non-dynamical, as can be most easily seen in the ADM formalism [36], where the metric elements $g_{00}$ and $g_{0i}$ are written as Lagrange multipliers. We can therefore integrate them out of the action, as we explicitly show in appendices B.3 and B.4\(^{10}\). We remark that this counting does not assume any symmetry of the four-dimensional background. Therefore, the procedure that

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\(^{10}\)This is analogous to what happens in electromagnetism. There, one starts from the 4 components of the vector potential. One degree of freedom is eliminated by fixing the $U(1)$ gauge, while the $A_\mu$ component is non-dynamical (it can be fixed by Gauss' law). One is then left with the 2 physical polarizations of the massless photon. Clearly, one can also choose different gauges; for example, a gauge often chosen in the isotropic gravity computation is the longitudinal one, which removes the $\delta g_{0i}$ modes. In general, however, the non-dynamical modes will be linear combinations of the remaining perturbations, and it may be less straightforward to find them.
we have just outlined can be applied to more general geometries than the simple Bianchi I considered here.

We first performed the computation in conformal time $\eta$, defined as

$$ds^2 = a^2(\eta) \left(-d\eta^2 + dx^2\right) + b^2(\eta) \left(dy^2 + dz^2\right).$$

(21)

We explain the reason for this choice in the next subsection, where we also discuss how different choices of time can affect the quantization procedure.

We denote the physical perturbations about this background by $H_x$, $H_+$, and $V$, with the understanding that, when the background becomes isotropic, each quantity becomes the corresponding lower case perturbation introduced in the previous subsection. The explicit definition of the physical modes in terms of the starting metric and inflaton perturbations is presented in appendix B. Here we only outline the equations satisfied by these modes, while in the next subsection we give their initial conditions. The form of these equations is the same for both branches. The mode $H_x$ is always decoupled from the other two, as a consequence of the residual isotropy in the $y$–$z$ plane. The other two modes are coupled to each other. Denoting by prime a derivative with respect to conformal time, we find

$$H''_x + \omega^2_x H_x = 0,$$

$$\left(\begin{array}{c} V \\ H_+ \end{array}\right)'' + \left(\begin{array}{cc} \omega^2_{11} & \omega^2_{12} \\ \omega^2_{12} & \omega^2_{22} \end{array}\right) \left(\begin{array}{c} V \\ H_+ \end{array}\right) = 0.$$

(22)

The explicit expressions for the frequency elements are given in equations (B.15) and (B.24).

Correspondingly, the quadratic actions for these modes are formally identical to the action of oscillators in Minkowski space with time evolving frequencies. This generalizes the Mukhanov–Sasaki [29] computation, valid in the FRW case: we can then proceed to the quantization of the modes, and provide their initial conditions (see the next subsection).

As the background becomes isotropic, $b \rightarrow a$, we find

$$\omega^2_x, \omega^2_{22} \rightarrow k^2 - \frac{a''}{a},$$

$$\omega^2_{11} \rightarrow k^2 - \frac{z''}{z}, \quad z \equiv \frac{a^2 \phi'}{a'},$$

$$\omega^2_{12} \rightarrow 0,$$

(23)

where $k$ is the comoving momentum. These are the standard evolution equations for the FRW case. All the modes decouple, and the equations for the two tensor polarizations become identical.

4.2. Initial conditions

The goal of this subsection is to set the initial conditions for the perturbations at asymptotically early initial times. The key quantities are the frequencies (B.15) and (B.24), which we evaluate at asymptotically early times through the background relations (19).
Again, it is useful to start by summarizing the standard isotropic computation. The physical modes \( h_x, h_+, v \) are those that diagonalize the quadratic action for the perturbations. In conformal time, defined as 
\[
\text{d}s^2 = a^2 (\eta) \left[ -\text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \text{d}z^2 \right],
\]
the action for each of these modes (after Fourier transforming the spatial coordinates) is of the form
\[
S_2 = \frac{i}{2} \int \text{d}\eta \, \text{d}^3k \left[ |\delta'|^2 - \omega^2 |\delta|^2 \right],
\]
where \( \delta' \) denotes any of the modes, and \( \mathcal{F} \) is a time evolving function which is of the order of \( H^2 \). The term \( a^2 \mathcal{F} \) is exponentially small at asymptotically early times. The net result is that at early times the action of the mode (once written in conformal time) approaches the one of a simple harmonic oscillator in Minkowski spacetime, with the frequency equal to the comoving momentum \( k \). In the Minkowski case we would have
\[
\delta_{\text{Mink}} = e^{-ik\eta} \sqrt{2k},
\]
for a choice of positive frequency modes. In the cosmological context, this is replaced by the adiabatic vacuum
\[
\delta_{\text{in}} \approx \exp \left(-i \int^{\eta} \text{d}\eta' \frac{\omega}{\sqrt{2\omega}} \right),
\]
which is a solution of the evolution equation as long as the frequency is adiabatically changing, \( \omega' \ll \omega^2 \). The fact that \( k \) dominates \( \omega \) at early times, can be also restated as \( k/a \gg H \). Going backwards in time, the universe is nearly exponentially contracting during inflation, and the physical momentum \( p = k/a \) becomes the dominant quantity for the evolution of the mode.

Let us now discuss the anisotropic situation. We separate the comoving momentum \( k = k_1 + k_2 \), where \( k_1 \) denotes the component along the privileged direction \( x \), and \( k_2 \) the component in the \( y-z \) plane. We also denote by \( k_1 \) and \( k_2 \) the magnitudes of the two components. The magnitude of the physical momentum is therefore
\[
p^2 = \frac{k_1^2}{a^2} + \frac{k_2^2}{b^2}.
\]
For the positive branch, during the anisotropic phase, \( a \propto t \), while \( b \) is nearly constant. As we go backwards in time, only the \( x \) direction is ‘squeezing’, while the other two become frozen. Correspondingly, \( k_1/a \gg k_2/b \) at sufficiently early times, and we therefore expect that only the component of the momentum in that direction sets the initial condition. To be able to set an initial vacuum as in the isotropic case, we therefore use the scale factor \( a \) in the definition of the conformal time, cf equations (21) and (24). Indeed, by inserting the early time asymptotic behaviours (19) into the various frequency elements of equation (22), we find
\[
\omega_x^2 = k_1^2 + O \left( m^2 t^2 \right),
\]
\[
\begin{pmatrix}
\omega_{11}^2 & \omega_{12}^2 \\
\omega_{12}^2 & \omega_{22}^2
\end{pmatrix} =
\begin{pmatrix}
k_1^2 + O \left( m^2 t^2 \right) & O \left( m^2 t^2 \right) \\
O \left( m^2 t^2 \right) & k_2^2 + O \left( m^2 t^2 \right)
\end{pmatrix},
\]
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

so that $k_1$ always becomes dominant provided we go sufficiently early in time\(^\text{11}\). Moreover, we see that the modes $H_+$ and $V$ decouple also at asymptotically early times, so that we can set the initial conditions directly on them (rather than on the modes which would diagonalize the frequency matrix). Therefore, as long as

\[
\frac{\omega_1'}{\omega_2}, \quad \frac{\omega_{11}'}{\omega_{11}}, \quad \frac{\omega_2'}{\omega_2}, \quad \frac{\omega_1'}{\omega_1}, \quad \frac{\omega_{12}}{\omega_{22}} \ll 1,
\]

we can consistently start from the adiabatic vacuum (27) for the perturbations. This always happens, provided we go sufficiently early in time (the smaller $k_1$ is, the earlier we have to go in time to satisfy these conditions).

In appendix C.1, we study whether other choices of time are possible to set the initial conditions. In the isotropic case, the conformal time is related to the physical one by $d\eta = a \, dt$. The most obvious generalization is to consider some arbitrary power of the two scale factors in the definition of the conformal time. We therefore discuss conformal times $\tau$ related to $\eta$ by

\[
d\eta = a^{-2\alpha} b^{-2\beta} \, d\tau, \tag{31}\]

where $\alpha$ and $\beta$ are two constant parameters. We show in appendix C.1 that this leads to an adiabatically evolving frequency only if $\alpha = 0$. Since $b$ is constant at asymptotically early times, we conclude that the time $\eta$ used here is the only possible one for the quantization, up to a trivial constant rescaling. In particular, this rules out the most ‘symmetric’ choice $d\tau = a^{1/3} b^{2/3} \, dt$, where the scale factors along the three coordinates are used with an equal weight.

Let us now turn to the negative branch. In conformal time $\eta$, the frequency elements appearing in equation (22) are the same as for the positive branch (namely, equations (B.15) and (B.24)). However, the early time asymptotics are now different (due to the fact that we now take the negative solution for $h$ in the asymptotic equation (19)):

\[
\begin{aligned}
\omega_\times^2 &= a^2 \left[ -\frac{5}{9t^2} + \frac{k_2^2}{b^2} + O(1) \right], \\
\begin{pmatrix}
\omega_{11}^2 & \omega_{12}^2 \\
\omega_{12}^2 & \omega_{22}^2
\end{pmatrix} &= a^2 \left[ \begin{pmatrix}
\frac{4}{9t^2} + \frac{k_2^2}{b^2} & 0 \\
0 & -\frac{4}{9t^2} + \frac{k_2^2}{b^2}
\end{pmatrix} + O(1) \right],
\end{aligned}
\]

where we recall that in the negative branch $a \propto t^{-1/3}$, $b \propto t^{2/3}$ at asymptotically early times.

The situation is now significantly worse than for the positive branch. Firstly, the frequencies are not adiabatically evolving at early times:

\[
\begin{aligned}
\left| \frac{\omega_\times'}{\omega_\times} \right| \rightarrow \frac{4}{\sqrt{5}}, & \quad \frac{\omega_{11}'}{\omega_{11}}, \quad \frac{\omega_{22}'}{\omega_{22}} \rightarrow -2.
\end{aligned}
\]

For this reason, we are not able to start from an adiabatic vacuum at early times, as we could do for the positive branch. Secondly, the 2D vector is tachyonic.

As we show in appendix C.2, it is not possible to find a conformal time in which the frequency of the 2D scalars is initially adiabatically evolving. The situation is somewhat

\(^{11}\) The only exception is when $k_1$ is strictly zero, that is when the momentum of the mode lies in the $y$–$z$ plane. This is a set of measure zero in the integral (7) for the correlators $C_{\ell\ell' m m'}$, so we can disregard this exceptional case. Notice also that we used physical time to indicate the residual entries in the frequency element, while the evolution equations (22) are in conformal time.
better for the 2D vector modes; by performing a time redefinition of the form (31), with \( \alpha = 2(1 + \beta) \), the frequency \( \omega_\times^2 \) approaches a constant negative value in the asymptotic past. In this case, we may choose to start from an adiabatic vacuum (using the absolute value of \( \omega_\times^2 \)). However, the fact that \( \omega_\times^2 \) is negative makes the mode diverge. To see this, we note that, for \( \alpha = 2(1 + \beta) \), the transformation between the physical time \( t \) and the comoving time \( \tau \) reads

\[
dt = a \ d\eta = a^{-2\alpha+1} b^{-2\beta} \ dt = c \ dt, \tag{34}
\]

where \( c \) is a positive constant. Integrating this relation, we see that the initial singularity \( (t = 0 \text{ in physical time}) \) occurs at \( \tau = -\infty \). Since \( \omega_\times^2 \) approaches a negative constant as \( \tau \to -\infty \), the 2D vector experiences a tachyonic growth which makes it diverge. More appropriately, we conclude that the 2D vector becomes non-linear, and the perturbative computation breaks down. Modes with different scales and directions of momenta are coupled at the non-linear level; this will likely 'isotropize' the effective Hubble drag felt by the non-linear modes. Intuitively this would lead to a feedback resulting in the suppression of the their growth. Nonetheless, the breakdown of the linearized computation (together with the lack of adiabaticity in the 2D scalar sector) prevents us from making trustworthy and firm predictions for the temperature anisotropies in the negative branch, and for this reason we do not consider this branch further in the remainder of the analysis.

### 4.3. Curvature power spectrum

We want to compute the power spectrum for the (late time) scalar mode. According to the discussion of the previous subsection, we restrict our attention to the positive branch. We proceed as follows. We fix an initial value for the inflaton (\( \phi_0 = 16 M_p \) in all the cases studied in this work, so to have about 60 e-foldings of isotropic inflation), and we scan over several values for the two components \( k_{1,2} \) of the comoving momentum of the mode. We choose a sufficiently early time such that the conditions (30) are satisfied, and we set the initial conditions for the modes and their derivatives as given by the adiabatic solution (27). We then numerically evolve the evolution equations (22) for the coupled \( \{ V, H_+ \} \) system, until a final time for which the background is isotropic (so that \( V \equiv v, H_+ \equiv h_+ \)) and the modes of interest are well outside the horizon. In figure 2 we show a contour plot with the value of \( k^3 |v|^2 \) at the end of inflation. This quantity is the curvature power spectrum, up to an overall normalization factor. The horizontal axis in the figure is the magnitude \( k = \sqrt{k_1^2 + k_2^2} \) of the comoving momentum, in units of a reference momentum \( k_{\text{iso}} \). The latter is defined as the comoving momentum of the modes which exit the horizon when the universe becomes isotropic; namely, \( k_{\text{iso}} \equiv aH \) at \( t = t_{\text{iso}} \), which we conventionally define to be the time at which \( h = 10^{-3} m \). The vertical axis is instead the cosine of the angle between the comoving momentum and the \( x \)-axis (namely, \( k_1 = k_\xi \)). In figure 3 we show instead some sections of the power spectrum at fixed values of \( \xi \).

Modes with \( k \gg k_{\text{iso}} \) leave the horizon during the isotropic phase; at early times, their frequency changes adiabatically, and, as a consequence, they evolve according to the adiabatic solution (27). As the background becomes isotropic, the frequency \( \omega \) becomes the standard one, so that the mode adiabatically evolves to the adiabatic solution that one would have also found if the background had always been isotropic. For this reason, we expect to recover an isotropic power spectrum (no \( \xi \) dependence) in this limit. This
Figure 2. Power spectrum of the comoving curvature perturbation $R$ in the inflationary model $V = m^2 \phi^2 / 2$. The inflaton field starts with $\phi = 16 M_p$, and is evolved until the moment shown in the figure, when $\phi = M_p$. Modes with $k > k_{\text{iso}}$ leave horizon during the later isotropic stage of inflation. The standard result is recovered for these modes.

The behaviour is manifest in the results shown. Larger scale modes ($k \lesssim k_{\text{iso}}$) are instead more sensitive to the background evolution in the anisotropic phase, and we expect non-standard results in this regime. Indeed, at any fixed $\xi$, the power spectrum exhibits an oscillatory behaviour for $k \lesssim k_{\text{iso}}$, and it then sharply decreases for $k \to 0$.

We also observe that the power spectrum increases at low $\xi$. The numerical results indicate that the power spectrum actually diverges as $1/\xi$ there:

$$P (k, \xi) \simeq \frac{P (k)}{\xi} \quad \text{for } 0 < \xi < \xi (k).$$

The region where this happens shrinks to smaller and smaller size as $k$ grows (namely, $\xi (k)$ is a decreasing function of $k$ in the region we have probed), in agreement with the fact that the power spectrum approaches the isotropic result for $k \to \infty$. Nonetheless, the growth of the power spectrum takes place at all values of $k$ we have computed, provided $\xi$ is sufficiently small.
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

Figure 3. Sections of the power spectrum (same evolution as in the previous figure) at fix values of $\xi$. We see that the power spectrum becomes isotropic (no $\xi$ dependence) at large momenta; at lower momenta it presents some oscillations, whose amplitude increases as $\xi$ decreases (this leads to the $1/\xi$ divergency mentioned in the main text). Finally, the power spectrum is suppressed as $k \to 0$. Although we do not have an analytical solution of the system (22), the large power at small $\xi$ is most probably due to the initial normalization $|v_{in}| \propto k_1^{-1/2} \propto \xi^{-1/2}$. In the isotropic case, modes with very small values of $k$ start with a large amplitude $|v_{in}| \propto k^{-1/2}$, but this does not lead to a large power spectrum $P \propto k^3|v|^2$, since the prefactor $k^3$ is small in this regime. For the anisotropic case, the problem arises when $k$ is finite, but $\xi \to 0$. In this regime, $k_1 \ll k_2$, so that the physical momentum is almost lying in the $y-z$ plane in the later isotropic regime. However, the initial condition for the mode is set at sufficiently early times, for which $k_1/a_{in} \gg k_2/b_{in}$. At the initial time, the mode has the large amplitude $|v_{in}| \propto \xi^{-1/2}$, while the prefactor $k^3$ in the late time power spectrum $k^3|v|^2$ equals $k_2^3$, which is not suppressed in the $\xi \to 0$ limit.

The divergency in the power spectrum results in a logarithmic divergency in the $C_{\ell^\prime m^\prime n^\prime}$ correlators, due to the fact that the associated Legendre polynomials approach a finite constant value for $\xi \to 0$ in the angular integral of equation (7), while the power spectrum diverges there. We do not expect that this singularity will actually occur, but it rather indicates the breakdown of the linearized computation for the metric/inflaton perturbations. This means that the precise value of the correlators is sensitive to the non-linear dynamics of the modes, which cannot be accounted for in the present linearized computation.

In addition to this, we note that the growth (35) occurs at small $k_1$. We recall that the early time frequency of the modes is $\omega^2 = k_1 + O(t^2)$, see equation (19), and that the metric is singular at $t = 0$. The adiabaticity condition, $\omega'/\omega^2 \ll 1$ is satisfied by all modes at sufficiently early times. However, the smaller $k_1$ is, the closer one needs to go to the singularity for the adiabaticity condition to hold. Suppose that the background solution
can be trusted only from some time on. Then, we can consistently start from the adiabatic vacuum only for modes with sufficiently high momentum $k_1$. The value of the modes with lower momentum may be simply provided as an (arbitrary) initial condition, or may be controlled by the additional dynamics (possibly, by additional degrees of freedom), which may be relevant at earlier times.

To conclude, we expect that the singularity in the power spectrum is actually absent once the non-linear effects and/or the complete early dynamics of the system are taken into account. In either case, however, this means that additional inputs are needed to provide firm predictions for the temperature anisotropies in this simple model.

5. Discussion

The main result of this work is the extension of the computation of primordial perturbations in a FRW geometry to more general backgrounds. We have done this in detail, only for the simplest case of a Bianchi I model with residual isotropy in two spatial directions, and with one scalar inflaton field: as in the isotropic case, there are three physical modes. However, in contrast to this case, two of the modes are coupled to each other already at the linearized level, due to the fact that the background has less symmetries. At late times, when the background becomes isotropic, one of these two modes becomes the scalar perturbation, while the other one becomes one of the two perturbations of the tensor mode. The key step in the computation is to completely fix the gauge for the perturbations without removing the $\delta g_{\mu\nu}$ entries. Such modes are non-dynamical, and can be readily integrated out of the quadratic action of the perturbations. In this way, one is left with only the physical degrees of freedom. It is straightforward to extend this computation to a general Bianchi I model, without the residual 2D spatial isotropy. In this case, all the three modes will be coupled to each other at the linearized level. Moreover, this procedure can be performed also for more general backgrounds, or for cases in which more fields are present.

Such anisotropic backgrounds have a potentially very interesting phenomenology; for instance the early time coupling between the scalar and tensor modes produces an off-diagonal correlation between the CMB modes (equation (7)), and, possibly a non-standard tensor-to-scalar ratio. Moreover the two polarizations of the tensor modes behave differently at early times, and acquire different values. Therefore, a future detection of the tensor perturbations (for instance, through the B polarization modes) can provide extremely useful information to test this possibility.

The computation that we have outlined is a necessary first step for such studies, since it must be done in order to find the physical modes, and to fix their initial conditions. In standard isotropic inflation, the frequency of the modes is controlled by their physical momentum $p$ at early times, which is varying adiabatically due to the expansion of the universe, $\omega'/\omega^2 \ll 1$. It is customary to start from the adiabatic vacuum (the immediate generalization of the de Sitter Bunch–Davies [37] vacuum), which is a good solution of the equations of motion as long as the adiabaticity condition holds. Due to this fact, the final result (a nearly scale invariant spectrum of the perturbations, in perfect agreement

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12 The choice of the initial vacuum is however object of intense debate, see for instance [38], mostly due to the fact that the physical momentum is trans-Planckian close to the initial singularity, and unknown UV physics may leave some imprint in the mode.
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

with observations) is insensitive to the ‘initial time’ at which the initial vacuum is set, provided that the mode is deeply inside the horizon \((p \gg H)\) at this time.

Models which go beyond the standard FRW geometry have additional parameters. To have some predictive power, one should hope that their phenomenology depends on as few inputs as possible. For instance, if we find that also in this case the frequency of the physical perturbations varies adiabatically at early times, then the problem of setting the initial vacuum is no worse than for standard inflation. We have seen that this is not the case for the simple Bianchi I backgrounds that we have studied here. The main reason is that the anisotropy parameter \(h\) (the difference of the two expansion rates) decreases very rapidly at the onset of inflation. Clearly, this geometry is continuously connected to FRW, and one can take \(h\) arbitrary small at the initial time. In this case, however, one recovers standard FRW inflation plus negligible corrections. If one hopes instead to find sizable deviations from the isotropic case, the initial value of \(h\) must be large. In this work we have studied the early evolution of the model starting from \(|h| \gg \sqrt{V}/M_p\) (where \(V\) is the potential energy of the inflaton), assuming that the inflaton is in a slow roll regime also during the anisotropic phase. This leads to two distinct possibilities: in one case, two dimensions become static at asymptotically early times; in the other case, the anisotropic direction is initially contracting. In the second case, the frequency of two of the modes is not adiabatically evolving at early times, while for the other one is actually tachyonic. In the first case, we do find an initial adiabatic regime; however, the amplitude of the modes increases and eventually diverges as their momentum is more and more aligned towards the non-expanding directions.

As we have mentioned, this divergence is merely an indication of the breakdown of the linear approximation taken in our treatment. It is probable that non-linear dynamics will stop the growth of the modes at this stage. However, even if this is the case, the predictive power of the model is reduced greatly, since the much simpler linearized treatment is invalid. There may be other physical reasons why the run-off growth in the mode may not occur. A better behaving vacuum may result if all the directions are expanding at asymptotically early times. For a Bianchi I geometry, even dropping the assumption of residual 2D isotropy, this cannot be achieved if the inflaton is initially in a slow roll regime (this can be understood from the 00 Einstein equation; close to the initial singularity the three expansion rates cannot be all positive, if the energy density of the inflaton remains finite). Alternatively, it is possible that the Bianchi I geometry is only the final stage of the anisotropic phase, and that additional degrees of freedom are relevant at earlier times. Both these possibilities require additional input with respect to the minimal set-up studied here.

One may also study completely different anisotropic models from the start. For instance, it may be very interesting to study models which evade Wald’s theorem on the isotropization of Bianchi geometries [16]. This can happen if one modifies Einstein gravity, for instance through the Kalb–Ramond action of string theory [39], or through quadratic curvature invariants [40]. In such case, one can find attractor solutions characterized by anisotropic inflationary expansion. This may avoid the run-off growth problem found in the model we have studied, since the difference \(h\) between the expansion rates decreases slower, or not at all, in these models, and therefore does not need to be extremely large in the asymptotic past. The challenge for such models is to find solutions for which the anisotropy decays at late times, or, alternatively, can be kept at a very small but
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

controllable level. This condition itself might lead to smaller, subdominant effects in the data which are yet to be observed.

We have taken a top-down approach in suggesting a model oriented origin for the broken isotropy observed in the CMB. We have examined a particular form of anisotropy in the inflating universe but the treatment presented here can be readily generalized to other forms of anisotropy. If the intriguing CMB anomalies are verified to be of cosmological origin through more accurate, future observations, including the polarization data, then they may well indicate a departure from the simplest picture of inflation. In that case models such as the one presented here offer a minimal modification of the inflationary paradigm which could predict the anomalous correlations required to explain the data.

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Appendix A. Explicit computation of \(C_{\ell\ell'} m m'\)

We want to prove the result (7) appearing in the main text. A full treatment of the line of sight calculation of CMB anisotropies [43] can be found in [44]. Inverting the relation (3), the expression (4) becomes

\[
C_{\ell\ell'} m m' = \int d\Omega \hat{p} d\Omega \hat{p}' \langle \delta T(\hat{p}, \eta_0, x_0) \delta T(\hat{p}', \eta_0, x_0) \rangle Y_{\ell m}^*(\hat{p}) Y_{\ell' m'}(\hat{p}')
\]

(A.1)

The temperature anisotropy is related to the primordial curvature perturbation as in (5). The statistical correlation is therefore encoded in the primordial power spectrum (2) of the curvature perturbation. This gives

\[
C_{\ell\ell'} m m' = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^3} P(k) \int d\Omega \hat{p} d\Omega \hat{p}' \Delta(k, \hat{k} \cdot \hat{p}, \eta_0) \Delta^*(k, \hat{k} \cdot \hat{p}', \eta_0) Y_{\ell m}^*(\hat{p}) Y_{\ell' m'}(\hat{p}').
\]

(A.2)

Note that the \textit{directional} dependence of the radiation transfer functions \(\Delta(k, \hat{k} \cdot \hat{p}, \eta_0)\) at conformal time today \(\eta_0\) is restricted to the cosine of the angle subtended by the plane wave unit vector \(\hat{k}\) and the unit vector in the direction on the sky \(\hat{p}\). Since the universe is already isotropic when the initial conditions for the Einstein–Boltzmann system are set, the standard line of sight calculation can be used in the calculation of the multipole expanded transfer functions \(\Delta_L(k, \eta_0)\), which depend only on the magnitude of the wave vector.

We expand the transfer functions as in (6), and we decompose the Legendre polynomials entering in (6) into spherical harmonics

\[
P_L(\hat{k} \cdot \hat{p}) = \frac{4\pi}{(2L+1)} \sum_{M=-L}^{L} Y_{LM}^*(\hat{k}) Y_{LM}(\hat{p}),
\]

(A.3)
so that the correlator can be rewritten as
\[ C_{\ell' m m'} = \int \frac{d^3k}{(2\pi)^3} \frac{(4\pi)^2}{k^3} P(k) \sum_{LL'M'M'} (-i)^{L-L'} \Delta_L(k, \eta_0) \Delta_{L'}^*(k, \eta_0) Y_{LM}^*(\hat{k}) Y_{L'M'}(\hat{k}) \times \int d\Omega_\theta d\Omega_\phi' Y_{\ell m}^*(\hat{\mathbf{p}}) Y_{\ell' m'}^*(\hat{\mathbf{p}}') Y_{LM}(\hat{\mathbf{p}}) Y_{L'M'}^*(\hat{\mathbf{p}}'). \] (A.4)

It is then straightforward to compute the angular integrals, given the orthonormality of the spherical harmonics. This gives
\[ C_{\ell' m m'} = (-i)^{\ell-\ell'} \int \frac{d^3k}{(2\pi)^3} \frac{(4\pi)^2}{k^3} P(k) \Delta_\ell(k, \eta_0) \Delta_{\ell'}^*(k, \eta_0) Y_{\ell m}^*(\hat{k}) Y_{\ell' m'}(\hat{k}). \] (A.5)

Note that this expression is valid for any form of anisotropic initial curvature power spectrum if the universe is isotropized by the time of reheating.

By assumption, the power spectrum in the model considered here is axially symmetric around the anisotropic \( z \)-axis. We therefore use polar coordinates in momentum space, where \( \theta \) is the angle between \( \mathbf{k} \) and the \( z \)-axis. Due to the residual symmetry, the integral over the second angular coordinate \( \phi \) is trivial. More specifically, we expand
\[ Y_{\ell m}(\theta, \phi) = e^{im\phi} \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!}} P_{\ell m}(\xi), \] (A.6)
where \( \xi = \cos \theta \), and we use the fact that
\[ \int_0^{2\pi} d\phi Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) = 2\pi \delta_{mm'} \frac{1}{4\pi} \sqrt{\frac{(2\ell + 1)(2\ell' + 1)(\ell - m)!(\ell' - m)!}{(\ell + m)!((\ell' + m)!}}} \times P_{\ell}^m(\cos \theta) P_{\ell'}^m(\cos \theta), \] (A.7)

Appendix B. Explicit computation of the perturbations

In this appendix we perform the explicit computation of the perturbations about the background geometry (21). Since this is a linearized computation, we ignore the nonlinear coupling between different modes. We can therefore fix a comoving momentum \( \mathbf{k} \), and study the evolutions of the modes having that momentum. The computation is exhaustive as long as we can solve the problem for any arbitrary value of \( \mathbf{k} \). Due to the residual symmetry in the \( y-z \) plane, we can fix \( k_z = 0 \) without any loss of generality. More appropriately, we denote by \( k_1 \) the component of the momentum along the anisotropic \( x \) direction, and by \( k_2 \) the component in the orthogonal plane. We then choose the \( y-z \) coordinates such that the orthogonal momentum lies on the \( y \) direction. In coordinate...
space, this amounts to considering modes which depend on $x$ and $y$ only\textsuperscript{13}. The most general perturbations of the metric can be then written as

\[
g_{\mu\nu} = \begin{pmatrix} -a^2(1 + 2\Phi) & a\partial_1\chi & a\partial_2 B & b^2 B_3 \\ a^2(1 - 2\Psi) & b^2\partial_1 B & b^2\partial_1\tilde{B}_3 & b^2\partial_2 E_3 \\ & b^2(1 - 2\Sigma + 2\partial_2^2 E) & b^2\partial_2 E_3 & b^2(1 - 2\Sigma) \end{pmatrix}. \tag{B.1} \]

In addition, there is the perturbation of the inflaton field $\phi = \phi + \delta\phi$.

In the isotropic computation, the modes are classified into tensor, vector, or scalar modes, according to how they transform under 3D spatial rotations. This is particularly useful, since, due to the isotropy of the background, perturbations belonging to different groups are decoupled at the linearized level [31, 34]. A 3D scalar has one degree of freedom. A 3D vector has one spatial index; however, due to the transversality condition ($\partial_i V_i = 0$), it has two degrees of freedom. A 3D tensor has two spatial indices, and it is required to be symmetric, transverse ($\partial_i h_{ij} = 0$), and traceless. Therefore, it also has 2 degrees of freedom. Altogether, the 10 + 1 initial perturbations (of the metric and of the inflaton field), are classified into 5 scalar, 2 vector, and 1 tensor modes\textsuperscript{14}.

Here we do the same, but we classify the perturbations according to how they transform under 2D rotations (since modes will be decoupled only if they transform differently under 2D rotations in the $y$–$z$ plane, due to the restricted isotropy of the background). A 2D scalar has one degree of freedom. A 2D vector has 2 spatial indices, and is required to be transverse; therefore, it has one degree of freedom. Contrary to the previous case, 2D tensors instead do not exist, since there are no degrees of freedom left, once we impose the mode to be symmetric, transverse, and traceless. Therefore, the 10 + 1 initial perturbations are classified into 8 scalar and 3 vector modes.

We conclude this discussion with a note on our notation. A 2D vector in the $y$–$z$ plane is in general indicated with $V_i$, where $i = 2, 3$. The transversality condition is then $\partial_2 V_2 + \partial_3 V_3 = 0$. However, as mentioned before equation (B.1), we can impose that the modes do not depend on the third coordinate, without any loss of generality. Therefore, we can set $V_3 = 0$, and the degree of freedom of the 2D vector is explicitly encoded into its component $V_3$. This justifies the choice of the 2D vector modes $B_3, \tilde{B}_3, E_3$ appearing in the metric (B.1). The parametrization of the 2D scalars entering in (B.1) has been chosen for computational convenience.

### B.1. Gauge choice

We can then proceed by either constructing gauge invariant perturbations, or by choosing a gauge which completely removes the gauge freedom (in the standard case, this is typically done by the choice of the longitudinal gauge). The two procedures are completely

\textsuperscript{13} In our computation we disregard the cases where the $x$ and $y$ dependence are trivial. Namely, we always assume that both $k_1$ and $k_2$ are different from zero. These two cases are of zero measure in the integral (7) and can therefore be disregarded.

\textsuperscript{14} We stress that this is only the first step in the computation, which is dictated by mathematical convenience. Not all of these modes are physical; altogether, 4 degrees of freedom are gauge modes, and 4 other ones are non-dynamical. See the discussion in section 4.1.
equivalent, due to gauge invariance. We choose the second option, by setting
\[ \delta g_{12} = \delta g_{23} = \delta g_{22} = \delta g_{33} = 0. \]  
(B.2)

Unlike the more conventional gauge choices, we chose not to remove any perturbation entering in the \( g_{0\mu} \) metric elements. As we mentioned in section 4.1, these modes are non-dynamical [36]. Therefore, with this gauge choice, we know from the start that the non-dynamical modes of (B.1) are \( \Phi, \chi, B, \) and \( B_3. \) This modes will be then integrated out from the action of the perturbations (see below).

It is straightforward, although tedious, to show that this choice (i) can be made, and (ii) completely fixes the gauge freedom. This is true both for the isotropic (\( b = a \)) and anisotropic background. Under the general coordinate transformation \( x^\mu \rightarrow x^\mu + \xi^\mu \), the metric transforms as
\[ \delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - (0) g_{\mu\nu,\alpha} \xi^\alpha - (0) g_{\mu\alpha,\nu} \xi^\alpha - (0) g_{\alpha\nu,\mu} \xi^\alpha. \]  
(B.3)

We parametrize \( \xi^\mu = (\xi^0, \partial_1 \xi^1, \partial_2 \xi, \xi^3) \), so that \( \xi_3 \) is the 2D vector degree of freedom, while the other three degrees of freedom are 2D scalars. Equation (B.3) then explicitly gives
\[ \Phi \rightarrow \Phi - \frac{a'}{a} \xi^0 - \xi^0', \]
\[ \chi \rightarrow \chi + a \left( \xi^0 - \xi^1' \right), \]
\[ B \rightarrow B + a \xi^0 - \frac{b^2}{a} \xi^1', \]
\[ \Psi \rightarrow \Psi + \frac{d'}{a} \xi^0 + \partial_1^2 \xi^1, \]
\[ \tilde{B} \rightarrow \tilde{B} - \frac{a^2}{b^2} \xi^1 - \xi, \]
\[ \Sigma \rightarrow \Sigma + \frac{b'}{b} \xi^0, \]
\[ E \rightarrow E - \xi, \]
\[ B_3 \rightarrow B_3 - b \xi_3, \]
\[ \tilde{B}_3 \rightarrow \tilde{B}_3 - \xi_3, \]
\[ E_3 \rightarrow E_3 - \xi_3, \]
\[ \delta \phi \rightarrow \delta \phi - \phi' \xi^0 \]  
(B.4)

(for completeness, in the last line we have indicated how the perturbation of the inflaton transforms). Our gauge choice corresponds to \( \tilde{B} = \Sigma = E = E_3 = 0 \). It is easy to see that this is possible for one and only one choice of the parameters \( \xi^\mu \).

**B.2. Comparison with the standard modes in the isotropic limit (\( b = a \))**

It is instructive to see how the conventional modes (tensor and scalar modes of [34]) rewrite in this gauge, once the background becomes isotropic. This will help us interpreting our results. To do so, we start from the conventional mode parametrization, without 3D vectors (which are not supported during inflation) and with the scalar modes written in
the longitudinal gauge. In the momentum space (where the comparison is easier) we have:

\[
g_{\mu \nu} = a^2 \begin{pmatrix}
-1 - 2\Phi & 0 & 0 & 0 \\
1 - 2\bar{\Phi} + \frac{k^2}{k^2 + 1} \bar{h}_+ & 0 & -\frac{k^2}{k^2 + 1} \bar{h}_+ & \frac{k}{k^2 + 1} \bar{h}_x \\
1 - 2\bar{\Phi} + \frac{k^2}{k^2 + 1} \bar{h}_+ & 0 & -\frac{k^2}{k^2 + 1} \bar{h}_+ & \frac{k}{k^2 + 1} \bar{h}_x \\
1 - 2\bar{\Phi} + \frac{k^2}{k^2 + 1} \bar{h}_+ & 0 & -\frac{k^2}{k^2 + 1} \bar{h}_+ & \frac{k}{k^2 + 1} \bar{h}_x \\
1 - 2\bar{\Phi} + \frac{k^2}{k^2 + 1} \bar{h}_+ & 0 & -\frac{k^2}{k^2 + 1} \bar{h}_+ & \frac{k}{k^2 + 1} \bar{h}_x
\end{pmatrix}, \tag{B.5}
\]

where \( k^2 = k_1^2 + k_2^2 \).

In terms of the general decomposition (B.1) (written in the isotropic limit \( b = a \)) the expression (B.5) rewrites

\[
\Phi = \bar{\Phi}, \quad \chi = B = B_3 = 0, \\
\Psi = \bar{\Psi} - \frac{k_2^2}{2k^2} \bar{h}_+, \quad \tilde{B} = \frac{\bar{h}_+}{k^2}, \quad E = -\frac{2k^2 + k_3^2}{2k^2 k_2^2} \bar{h}_+, \quad \Sigma = \bar{\Psi} + \frac{\bar{h}_+}{2}, \\
\tilde{B}_3 = i \frac{k_2}{k_1 k} \bar{h}_x, \quad E_3 = -i \frac{k_1}{k k_2} \bar{h}_x.
\]

It is straightforward to write down how the final (and properly normalized) gauge invariant perturbations \( v, h_+, \) and \( h_\times \) appear in the longitudinal gauge [34]:

\[
v = a \left[ \delta \bar{\phi} + \frac{a' \phi'}{a} \bar{\Psi} \right], \quad h_+ = \frac{M_p}{\sqrt{2}} a \bar{h}_+, \quad h_\times = \frac{M_p}{\sqrt{2}} a \bar{h}_\times, \tag{B.7}
\]

where \( \delta \bar{\phi} \) is the perturbation of the inflaton in the longitudinal gauge.

We then transform from the longitudinal gauge to our gauge (B.2). Namely, we start from the perturbations (B.1) in the longitudinal gauge (B.6), and perform an infinitesimal change of coordinates \( x^\mu \to x^\mu + \xi^\mu \), such that the transformed modes satisfy the conditions that specify our gauge, \( \tilde{B} = \Sigma = E = E_3 = 0 \) (the explicit transformations of the modes are given by equations (B.4) in the isotropic limit \( b = a, b' = a' \)). This can be achieved for a unique choice of the infinitesimal transformation parameters \( \xi^\mu \):

\[
\xi^0 = -\frac{a}{a'} \left( \bar{\Psi} + \frac{\bar{h}_+}{2} \right), \quad \xi^1 = \frac{2k_1^2 + 3k_2^2}{2k^2 k_2^2} \bar{h}_+, \\
\xi = -\frac{2k_1^2 + k_2^2}{2k^2 k_2^2} \bar{h}_+, \quad \xi_3 = -i \frac{k_1}{k k_2} \bar{h}_\times. \tag{B.8}
\]

Starting from the longitudinal gauge (B.6), and performing the infinitesimal transformations (B.4) with the parameters (B.8), the dynamical modes \( \delta \bar{\phi}, \Psi, \tilde{B}_3 \) in our gauge become

\[
\delta \bar{\phi} = \delta \bar{\phi} + \frac{\phi'}{a'} \left( \bar{\Psi} + \frac{\bar{h}_+}{2} \right), \quad \Psi = -\frac{k_2^2}{k_1 k} \bar{h}_+, \quad \tilde{B}_3 = i \frac{k}{k_1 k_2} \bar{h}_\times. \tag{B.9}
\]

We invert these relations to express the modes in the longitudinal gauge in terms of our modes. We then insert the resulting expressions into equations (B.7); in this way, we find how our modes combine into the gauge invariant combinations \( v, h_+, h_\times \):

\[
a \left[ \delta \bar{\phi} + \frac{k_2^2}{2k^2} \frac{\phi'}{a'} \bar{\Psi} \right] = v, \quad -a \frac{M_p}{\sqrt{2}} \frac{k_2^2}{k^2} \Psi = h_+, \quad -ia \frac{M_p}{\sqrt{2}} \frac{k_1 k_2}{k} \tilde{B}_3 = h_\times. \tag{B.10}
\]

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Journal of Cosmology and Astroparticle Physics 11 (2007) 005 (stacks.iop.org/JCAP/2007/i=11/a=005) 25
B.3. 2D vectors

We now compute the quadratic action for the 2D vectors. We insert the decomposition (B.1) into the action (12), ignoring the 2D scalars. We also set $E_3 = 0$, according to the gauge choice (B.2), and we expand the action up to second order in the remaining 2D modes $B_3$ and $\tilde{B}_3$. This gives

$$S_{(2)}^{2Dvec} = \frac{M_p^2}{4} \int d\eta d^3x \ b^2 \left[ \frac{b^2}{a^2} \left( \tilde{B}_{3,1}^2 + B_{3,1}^2 - 2 B_{3,1} \tilde{B}_{3,1}' \right) + B_{3,2}^2 - \tilde{B}_{3,1,2}^2 \right] + B_1^{vec}, \quad (B.11)$$

where the boundary terms are

$$B_1^{vec} = M_p^2 \int d\eta d^3x \left[ \mathcal{F}_y' + \partial_1 \mathcal{F}_x + \partial_2 \mathcal{F}_y \right],$$

$$\mathcal{F}_y = \frac{b^4}{2a} \left[ - (H_a + 2 H_b) B_3^2 - \frac{1}{b^2} \left( \frac{b^2}{a} \tilde{B}_{3,1}^2 \right)' \right],$$

$$\mathcal{F}_x = \frac{1}{a} \left( \frac{b^4}{a} B_3 \tilde{B}_{3,1} \right)' - \frac{b^4}{2a^2} (B_3^2)'_1,$$

$$\mathcal{F}_y = \frac{b^2}{2} \left( \tilde{B}_{3,1}^2 - B_3^2 \right)'_2$$

($H_a = \dot{a}/a$ and $H_b = \dot{b}/b$ are the Hubble rates with respect to physical time, see equations (11)). Next, we Fourier transform along the spatial coordinates. We can see from the action (B.11) that the mode $B_3$ is non-dynamical, and it can be integrated out (as we discussed in 4.1, the modes $\delta g_{0\mu}$ are non-dynamical). Extremizing the Fourier transformed action with respect to $B_3$ gives

$$B_3 = \frac{p_1^2}{p^2} \tilde{B}_3,$$  \quad (B.13)

where we have used for shortness the components of the physical momentum $p_1 = k_1/a$ and $p_2 = k_2/b$, and the total magnitude $p = \sqrt{p_1^2 + p_2^2}$. We insert this value for $B_3$ back in the Fourier transformed action. The resulting action depends only on $\tilde{B}_3$ and its first derivative. We can cast it in the form

$$S_{(2)}^{2Dvec} = \frac{1}{2} \int d\eta d^3k \left[ |H_x'|^2 - \omega_x^2 |H_x|^2 \right] + B_1^{vec} + B_2^{vec}, \quad (B.14)$$

where

$$H_x \equiv \frac{M_p}{\sqrt{2}} \frac{b k_1 k_2}{\sqrt{k_1^2 + (a^2/b^2) k_2^2}} \tilde{B}_3,$$

$$\omega_x^2 \equiv k_1^2 + a^2 \left[ \frac{k_1^2}{b^2} - H_a^2 - H_b^2 + \frac{\dot{\phi}^2}{2M_p^2} + (H_a - H_b)^2 \frac{k_1^2 (k_1^2 + 4(a^2/b^2)k_2^2)}{(k_1^2 + (a^2/b^2)k_2^2)^2} \right], \quad (B.15)$$

and where the newly introduced boundary term is

$$B_2^{vec} = \frac{1}{2} \int d\eta d^3k \left\{ - H_b + (H_a - H_b) \frac{p_2^2}{p^2} \right\} a |H_x|^2 \right\}'.$$  \quad (B.16)
Namely, the action (B.14), disregarding the boundary terms, has been written in terms of the canonically normalized combination $H_\times$. This mode coincides with the standard mode $h_\times$ in the isotropic limit, as can be seen by comparing its definition with the third of (B.10) when $b = a$ (the modes actually differ by an unphysical phase). Its equation of motion, appearing in equation (22) of the main text, is immediately obtained from (B.14).

**B.4. 2D scalars**

We now compute the quadratic action for the 2D scalars. The computation proceeds analogously to the one of previous subsection, although it is more involved due to the presence of a larger number of modes. We start from the metric (B.1), in the gauge (B.2), and we now ignore the 2D vectors. We also include the perturbation of the inflaton field in the present computation. We insert this in the action (12) and we expand it to quadratic order in the perturbations. We find

\[
S_{(2)}^{2D \text{ sca}} = \frac{M_p^2}{4} \int d\eta d^3x a^2 b^2 \left\{ \frac{2}{M_p^2} \left( \frac{1}{a^2} \delta \phi'^2 - \frac{1}{a^2} \delta \phi_1^2 - \frac{1}{b^2} \delta \phi_{2}^2 \right) \right. \\
+ \frac{4}{b^2} \left[ - \frac{\dot{\phi}}{M_p^2} \delta \phi + (H_a + H_b) \Phi + (H_a - H_b) \Psi + \frac{1}{a} \Psi' \right] B_{1,2} \\
+ \frac{1}{a^2 b^2} (B_{1,2} - \chi_{1,2})^2 - \frac{4 \ddot{\phi}}{M_p^2} a^{2} (\Psi + \Phi) \delta \phi' + \frac{4 V'}{M_p^2} (\Psi - \Phi) \delta \phi \\
+ \frac{4}{\alpha^2} \left( 2H_b \Phi - \frac{\ddot{\phi}}{M_p^2} \delta \phi \right) \chi_{1,2} - \frac{4}{\alpha^2} B_{2} \Psi_{2} - \frac{8}{\alpha} H_b \Phi \Psi' \\
- \frac{2 V''}{M_p^2} \delta \phi'^2 + 2 \left[ \frac{\phi^2}{M_p^2} - 2 B_{a} (2 H_a + H_b) \right] \Phi^2 \right\} + \mathcal{B}_{1}^{\text{sc}}, \tag{B.17}
\]

where we stress that $\dot{\phi}$ is the derivative of the background inflaton with respect to physical time, and where the boundary terms are

\[
\mathcal{B}_{1}^{\text{sc}} = \frac{M_p^2}{2} \int d\eta \, d^3x \left[ \mathcal{F}_{\eta}' + \partial_1 \mathcal{F}_x + \partial_2 \mathcal{F}_y \right],
\]

\[
\mathcal{F}_\eta = a b^2 \left[ (H_a + 2 H_b) \left( 3 \Phi^2 + 2 \Phi \Psi - \frac{1}{a^2} \chi_{1,2} - \frac{1}{b^2} B_{2}^2 \right) \\
- H_a \Psi^2 + \frac{2}{a} (\Phi - \Psi) \Psi' - \frac{2}{a^2} \chi_{1,2} \Phi_{1,2} - \frac{2}{b^2} B_{2} \Phi_{2,2} \right],
\]

\[
\mathcal{F}_x = b^2 \left[ 2 \left( 2 H_b \chi_{1,2} + \Phi_{1,2} + \frac{1}{a} \chi_{1} \right) (\Phi - \Psi) - \frac{2}{b^2} B_{2} \Phi_{2,2} + \frac{2}{a} \chi_{1,2} \Phi' \right],
\]

\[
\mathcal{F}_y = 2 a^2 \left\{ \left[ (H_a + H_b) B_{2} + \Phi_{2} + \Psi_{2} + \frac{1}{a} B_{2} \right] (\Phi + \Psi) \\
+ \frac{1}{a^2} \left[ (B - \chi)_{1,2} \chi_{1,1} + \left( \chi_{1,1} + a (\Phi + \Psi') \right) B_{2} \right] \right\}.
\]

We now find that the modes $\Phi$, $\chi$, and $B$ are non-dynamical (they are the $\delta g_{0\mu}$ modes),
while the two remaining modes, $\Psi$ and $\delta \phi$ are dynamical. We can then integrate out the non-dynamical perturbations analogously to what we did for the 2D vectors, and obtain an action in terms of the two dynamical modes only.

After Fourier transforming the spatial component, extremizing the action wrt $\chi^*$ gives

$$\chi = B + \frac{2 b^2}{k_B^2} \left( \frac{\dot{\phi}}{M_p^2} \delta \phi - 2 H_b \Phi \right).$$

(B.18)

Inserting this back into the Fourier transformed action gives

$$S^{2D \text{ sca}} = M_p^2 \int d\eta d^3k a^2 b^2 \left\{ \frac{1}{2} \frac{1}{a^2 M_p^2} |\delta \phi'|^2 - \frac{1}{2M_p^2} \left[ p_1^2 + p_2^2 + V'' + \frac{2 p_1^2 \dot{\phi}^2}{p_2^2 M_p^2} \right] |\delta \phi|^2 \right. $$

$$+ \left. \left[ \frac{\dot{\phi}^2}{2 M_p^2} - H_b \left( 2 H_a + H_b \left( 1 + \frac{4 p_1^2}{p_2^2} \right) \right) \right] |\Phi|^2 \right. $$

$$+ \left. \frac{1}{2} \left[ \left( 2 H_0 p_1^2 + (H_a + H_b) p_2^2 \right) \Phi - \frac{\dot{\phi}}{M_p^2} \left[ p_1^2 + p_2^2 \right] \delta \phi + p_2^2 \frac{b}{a^2} \left( a \Psi \right) \right] B^* \right. $$

$$+ \left. \left[ - p_2^2 \Psi + 4 H_0 \frac{\dot{\phi}^2}{p_2^2 M_p^2} \delta \phi - \frac{\dot{\phi}}{M_p^2} \delta \phi' - \frac{2 H_b}{a} \Psi - \frac{V'}{M_p^2} \delta \phi \right] \Phi^* \right. $$

$$- \left. \frac{1}{M_p^2} \left( \frac{\dot{\phi}}{a} \delta \phi' - V' \delta \phi \right) \Psi^* + \text{h.c.} \right\} + \mathcal{B}_1^{\text{sca}}. \quad \text{(B.19)}$$

We extremize this action wrt $B^*$. The resulting equation of motion can be written as an equation for $\Phi$

$$\Phi = \frac{p_2^2}{H_a p_2^2 + H_b (2 p_1^2 + p_2^2)} \left[ \frac{\dot{\phi}}{M_p^2} \left( 1 + \frac{p_1^2}{p_2^2} \right) \delta \phi - \frac{b}{a^2} \left( a \Psi \right) \right].$$

(B.20)

When we insert this back into (B.20), also $B$ disappears from the action. The resulting action can be finally cast in the form

$$S = \frac{1}{2} \int d\eta d^3k \left[ |V'|^2 + |H_+|^2 - (V^*, H_+) \right] \Omega^2 \left( \frac{V}{H_+} \right) + \mathcal{B}_1^{\text{sca}} + \mathcal{B}_2^{\text{sca}}, \quad \text{(B.21)}$$

where the canonically normalized combinations are

$$V \equiv b \left[ \delta \phi + \frac{p_2^2 \dot{\phi}}{H_a p_2^2 + H_b (2 p_1^2 + p_2^2)} \right], \quad H_+ \equiv \frac{\sqrt{2} b M_p^2 p_2^2 H_b}{H_a p_2^2 + H_b (2 p_1^2 + p_2^2)} \Psi. \quad \text{(B.22)}$$

We see by comparing with the first two of (B.10) that these modes reduce to the analogous (lower case) ones in the isotropic limit (up to an unphysical phase for $H_+$).
The term $\Omega^2$ entering in the action is the square frequency matrix

$$\Omega^2 = \begin{pmatrix} \omega_{11}^2 & \omega_{12}^2 \\ \omega_{12}^2 & \omega_{22}^2 \end{pmatrix},$$

$$(\omega_{11}^2)^2 = \left(p_1^2 + p_2^2 - 2 H_a H_b + \frac{\dot{\phi}^2}{2 M_p^2} + \frac{2 H_a \dot{\phi}^2}{H_b M_p^2} + \frac{1}{2} \frac{\dot{\phi}^4}{H_b M_p^2} + 2 \frac{\dot{\phi} V'}{H_b M_p^2} + V'' \right)$$

$$+ \frac{p_2^2 (H_a - H_b)^2}{2 H_b p_1^2 + (H_a + H_b) p_2^2} \frac{\dot{\phi}}{M_p} \left[ 4 H_b - \frac{p_2^2}{[2 H_b p_1^2 + (H_a + H_b) p_2^2]} \right],$$

$$(\omega_{12}^2)^2 = \left(p_1^2 + p_2^2 - 2 H_a H_b + \frac{\dot{\phi}^2}{2 M_p^2} \right)$$

$$+ \frac{p_2^2 (H_a - H_b)^2}{2 H_b p_1^2 + (H_a + H_b) p_2^2} \frac{\dot{\phi}}{M_p} \left[ -3 H_b \frac{\dot{\phi}}{M_p} + \frac{1}{2} \frac{\dot{\phi}^3}{H_b M_p^2} - V' \right]$$

$$- \frac{p_2^2 (H_a - H_b)}{2 H_b p_1^2 + (H_a + H_b) p_2^2} \frac{\dot{\phi}}{M_p} \left( H_b + \frac{1}{2} \frac{\dot{\phi}^2}{H_b M_p^2} \right).$$

The equation of motion for the 2D scalars, appearing in equation (22) of the main text, is immediately obtained from (B.22).

The second boundary term is

$$B_2^{\text{scal}} = \int d\eta d^3k \left[ (V^*, H^*_+) \left( \begin{array}{cc} F_{VV} & F_{VH} \\ F_{VH} & F_{HH} \end{array} \right) \left( \begin{array}{c} V \\ H_+ \end{array} \right) \right]^\prime,$$

$$F_{VV} = -\frac{a}{2} \left( H_b + \frac{(p_1^2 + p_2^2) \dot{\phi}^2}{(H_a p_1^2 + H_b (2p_1^2 + p_2^2)) M_p^2} \right),$$

$$F_{VH} = \frac{a}{2 \sqrt{2} M_p} \left[ \frac{V'}{H_b} - 2 \dot{\phi} \frac{(2 H_b p_1^2 + H_a p_2 (p_1^2 - p_2^2))}{p_2^2 (H_a p_1^2 + H_b (2p_1^2 + p_2^2))} \right],$$

$$F_{HH} = a \left\{ -\frac{\dot{\phi} V'}{4 H_b^2 M_p^2} + \frac{4 H_b p_1^2 + H_a p_2^2}{4 M_p^2 H_b^2} \dot{\phi}^2 + \frac{2 H_b p_1^2 + (H_a + H_b) p_2^2}{4 H_b^2} \right.$$

$$+ \frac{1}{p_2^2 (H_a p_2^2 + H_b (2p_1^2 + p_2^2))} \left[ -2 H_b (H_a - H_b) p_1^4 \right.$$

$$\left. - (2 H_a^2 + H_a H_b - 2 H_b^2) p_1^2 p_2^2 - \frac{H_a}{2 H_b} (H_a^2 + 2 H_a H_b - H_b^2) p_2^4 \right\}.$$
Appendix C. Dependence of the frequency, and of the adiabaticity condition on the choice of conformal time

In appendix B, we compute the frequencies of the perturbations in conformal time $\eta$, defined in equation (21). Here, we study how the action of the modes changes when we use another conformal time $\tau$.

Rather than studying the 2D vector and scalar sector separately, we can give a more compact presentation by considering $N$ canonically normalized fields $X_i$ (after Fourier transforming the spatial coordinates), with the action

$$S = \frac{1}{2} \int d\eta d^3k \left[ \frac{dX^\dagger}{d\eta} \frac{dX}{d\eta} - X^+ \Omega^2_\eta X \right], \quad X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix},$$

where $\Omega^2_\eta$ is a Hermitian matrix (in the present application, $N = 1$ and 2 for the 2D vector and the 2D scalar sector, respectively).

Under the time redefinition

$$d\eta \equiv f^{-2}(\tau) \, d\tau$$

this action becomes

$$S = \frac{1}{2} \int d\tau d^3k \left[ f^2 \frac{dX^\dagger}{d\tau} \frac{dX}{d\tau} - f^{-2} X^+ \Omega^2_\tau X \right].$$

With this new time variable, the fields $X_i$ are no longer canonical. We can however introduce new canonical fields

$$Y_i \equiv f X_i,$$

In terms of these fields, the action reads

$$S = \frac{1}{2} \int d\tau d^3k \left[ \frac{dY^\dagger}{d\tau} \frac{dY}{d\tau} - Y^+ \Omega^2_\tau Y \right] + B,$$

where the new square frequency matrix is

$$\Omega^2_\tau \equiv \frac{1}{f^4} \Omega^2_\eta - \frac{1}{f} \frac{d^2 f}{d\tau^2} \mathbb{I},$$

while the boundary term is

$$B = -\frac{1}{2} \int d\tau d^3k \frac{d}{d\tau} \left( \frac{1}{f} \frac{dY}{d\tau} Y^\dagger Y \right).$$

In principle, the two actions (C.1) and (C.5) lead to two different quantization procedures for the fields, and to two different initial conditions for the cosmological application. We want to determine under which conditions these two procedures lead to the same physical result\textsuperscript{15}. Let us now assume that the off-diagonal terms of both

\textsuperscript{15} The physical perturbations will be linear combinations of $X_i$ or, equivalently, $Y_i$. We can equivalently compute the evolution of these physical modes by writing and solving the Einstein equations in terms of the fields $X_i$ or $Y_i$; the only difference that may arise from using one or the other set of fields is in the initial conditions that the two quantizations (either in terms of the time variable $\eta$ or $\tau$) impose.
\( \Omega_\eta \) and \( \Omega_\tau \) can be neglected at asymptotically early times (as this is the case for the modes studied in the present work, see section 4.2). As discussed after equation (25), if the frequency \( \Omega_\eta \) is adiabatically evolving, we can set the initial conditions (up to an irrelevant phase)

\[
(X_i)_{in} = \frac{1}{\sqrt{2\Omega_\eta}}, \quad \left(\frac{dX_i}{d\eta}\right)_{in} = -i \sqrt{\frac{\Omega_\eta}{2}}, \quad \text{if } \frac{1}{\Omega_\eta^2} \frac{d\Omega_\eta}{d\eta} \ll 1, \quad (C.8)
\]

where \( \Omega_\eta \) here refers to the diagonal entry corresponding to the mode \( X_i \).

On the other hand, if \( \Omega_\tau \) is also adiabatically evolving, we can also impose

\[
(Y_i)_{in} = \frac{1}{\sqrt{2\Omega_\tau}}, \quad \left(\frac{dY_i}{d\tau}\right)_{in} = -i \sqrt{\frac{\Omega_\tau}{2}}, \quad \text{if } \frac{1}{\Omega_\tau^2} \frac{d\Omega_\tau}{d\tau} \ll 1. \quad (C.9)
\]

The initial conditions (C.9) can be used to set the initial conditions for \( X_i \) and \( dX_i/d\eta \) through the relations (C.2) and (C.4). Proceeding in this way, we find

\[
\{ (Y_i)_{in}, \left(\frac{dY_i}{d\tau}\right)_{in} \} \rightarrow \left\{ (X_i)_{in} = \frac{1}{\sqrt{2\Omega_\eta}} f = \frac{1}{\sqrt{2\Omega_\eta}} \left[ 1 - f^3 \frac{d^2 f}{\Omega_\eta^2 d\tau^2} \right]^{-1/4}, \quad (C.10) \right. \]

\[
\left. \left( \frac{dX_i}{d\eta} \right)_{in} = -i \sqrt{\frac{\Omega_\eta}{2}} \left[ 1 - f^3 \frac{d^2 f}{\Omega_\eta^2 d\tau^2} \right]^{-1/4} \frac{f df}{\sqrt{2\Omega_\eta}} \left[ 1 - f^3 \frac{d^2 f}{\Omega_\eta^2 d\tau^2} \right]^{-1/4} \right. \]

If these conditions agree with those obtained by using the fields \( X_i \) and the time \( \eta \) directly, namely equations (C.8), then the two quantizations are equivalent. This happens only if

\[
\frac{df}{d\tau} \ll \Omega_\eta, \quad f^3 \left| \frac{d^2 f}{d\tau^2} \right| \ll \Omega_\eta^2 \quad (C.11)
\]

at asymptotically early times. One can verify that these conditions are equivalent to

\[
\frac{1}{f} \left| \frac{df}{d\eta} \right| \ll \Omega_\eta, \quad \frac{1}{f} \left| \frac{d^2 f}{d\eta^2} \right| \ll \Omega_\eta^2. \quad (C.12)
\]

Let us now discuss under which circumstances these conditions are met. As we mentioned, we can consistently set both (C.8) and (C.9) only if both \( \Omega_\eta \) and \( \Omega_\tau \) are adiabatically evolving (each with respect to its own time). Equation (C.6) can be differentiated to give

\[
\frac{1}{\Omega_\tau^2} \frac{d\Omega_\tau}{d\tau} = \frac{1}{2} \left[ \frac{d\Omega_\eta^2}{d\tau} \right]^{3/2} = \frac{1}{\Omega_\eta^2} \frac{d\Omega_\eta}{d\eta} - \frac{1}{16} \frac{1}{f} \frac{df}{d\eta} + \frac{1}{2\Omega_\eta^2} \left[ -12 \left( \frac{1}{f} \frac{df}{d\eta} \right)^3 + 9 \frac{df}{d\eta} \frac{1}{f} \frac{d^2 f}{d\eta^2} - \frac{1}{f} \frac{d^3 f}{d\eta^3} \right]. \quad (C.13)
\]

Barring accidental cancellations between the different terms, both frequencies are adiabatically evolving only if

\[
\frac{1}{f} \left| \frac{d^\alpha f}{d\eta^\alpha} \right| \ll \Omega_\eta^\alpha, \quad \alpha = 1, 2, 3. \quad (C.14)
\]
These conditions also imply that the two quantization procedures are equivalent, cf equations \( (C.12) \).

To summarize, we can consistently use the two different time variables \( \eta \) and \( \tau \), provided the relation among them varies sufficiently slowly. If conditions \( (C.14) \) are met, and if the frequency \( \Omega_\eta \) is adiabatically evolving, then also the frequency \( \Omega_\tau \) is adiabatically evolving in its own time. These two procedures lead to equivalent initial conditions for the physical modes. On the other hand, we see that not all choices for conformal time are appropriate for the quantization.

We proceed by discussing what these findings imply for the Bianchi I backgrounds studied in the main text. For sake of clarity, we study the positive and negative branches separately in the two following sections.

C.1. Positive branch

We show in section 4.2 that, for the positive branch, one can quantize the perturbations in the conformal time \( \eta \), defined in equation \( (21) \). Here, we want to investigate whether some other time variable \( \tau \) can be used. For definiteness, let us restrict our attention to time redefinitions of the form

\[
d\eta = f^{-2}(\tau)\,d\tau = a^{-2\alpha}b^{-2\beta}\,d\tau, \tag{C.15}
\]

where \( a \) and \( b \) are the two scale factors, and \( \alpha \) and \( \beta \) two constant parameters (this is the most immediate generalization of what is done in the isotropic case). Equations \( (C.14) \) give the three conditions that \( f \) needs to satisfy for the two choices of time to provide the same initial conditions. We perform the computation using the time \( \eta \), and we then present the early time expansion in physical time:

\[
\frac{1}{f} \left| \frac{df}{d\eta} \right| = |\alpha| a(t)H_a(t) \left[ 1 + O \left( m^2 t^2 \right) \right] \ll \Omega_{\eta},
\]

\[
\frac{1}{f} \left| \frac{d^2f}{d\eta^2} \right| = \alpha^2 a^2(t)H_a^2(t) \left[ 1 + O \left( m^2 t^2 \right) \right] \ll \Omega_{\eta}^2, \tag{C.16}
\]

\[
\frac{1}{f} \left| \frac{d^3f}{d\eta^3} \right| = |\alpha^3| a^3(t)H_a^3(t) \left[ 1 + O \left( m^2 t^2 \right) \right] \ll \Omega_{\eta}^3.
\]

Namely, these three conditions coincide\(^{16} \). From equation \( (29) \) we have \( \Omega_{\eta}^2 = k_1^2 + O(m^2 t^2) \) for all the three modes, where \( k_1 \) is the longitudinal (comoving) momentum of the mode in the anisotropic \( x \)-direction. Therefore, we must have \( |\alpha| a(t)H_a(t) \ll k_1 \) at asymptotically early times. The product \( aH_a \) is nearly constant all throughout the anisotropic sage, and very close to the value \( k_{\text{iso}} \) defined in section 4.3. Therefore, the three conditions \( (C.16) \) become

\[
k_1 \gg |\alpha| k_{\text{iso}}. \tag{C.17}
\]

The modes with \( k_1 \gg k_{\text{iso}} \) are those leaving the horizon after the universe has become isotropic, and we recover the standard result for them (see section 4.3). The modes with

\(^{16} \)To be rigorous, we showed that the three conditions \( (C.14) \) are sufficient to ensure that the two quantizations are equivalent, but not necessary, since there may be cancellations in equation \( (C.13) \); however, the specific form of \( (C.16) \) shows that these conditions are also necessary, once we restrict our attention to time redefinitions of the type \( (C.15) \).
smaller longitudinal momentum are those sensitive to the evolution of the universe during the anisotropic stage, and therefore are those of real interest for the present analysis. We can only quantize them using a time variable for which $\alpha = 0$ in equation (C.15). We remark that $b$ is constant at asymptotically early times. Therefore, if we use the scale factors in the definition of the conformal time, the time $\eta$ used in the main text is the only possible one, up to a trivial constant rescaling.

C.2. Negative branch

Let us start from the early time frequencies for the modes canonically normalized with respect to conformal time $\eta$. As shown in equation (32) we find

$$\Omega_{\eta}^2 = a^2 \left[ -\frac{5}{9t^2} + \frac{k^2}{b^2} + O(1) \right], \quad 2 \text{ d vector},$$

$$\Omega_{\eta}^2 = a^2 \left[ \frac{4}{9t^2} + \frac{k^2}{b^2} + O(1) \right], \quad 2 \text{ d scalar},$$

where, in the 2D scalar case, $\Omega_{\eta}^2$ refers to the dominant diagonal entries of the frequency matrix (which are identical). We recall that $a \propto t^{-1/3}$, $b \propto t^{2/3}$ at asymptotically early times (as always, although the frequency refers to the conformal time $\eta$, we show the expansion in terms of physical time $t$). As discussed in the main text, these frequencies are not adiabatically evolving at early times; moreover the 2D vector mode is tachyonic.

Let us discuss whether the situation improves if we use a different time variable $\tau$; as for the positive branch, we consider time redefinitions of the form (C.15).

Let us discuss the 2D vector mode first. Evaluating the two equations (C.6) and (C.13) we find

$$\Omega_{\tau}^2 = \frac{a^{-4\alpha+2} b^{-4\beta}}{9t^2} \left[ (\alpha - 2\beta + 1)(\alpha - 2\beta - 5) \right] \left[ 1 + O\left(m^{2/3}t^{2/3}\right) \right],$$

$$\frac{1}{\Omega_{\tau}^2} \frac{d\Omega_{\tau}}{d\tau} = \frac{[\alpha - 2(1 + \beta)]}{\sqrt{(\alpha - 2\beta + 1)(\alpha - 2\beta - 5)}} \left[ 1 + O\left(m^{2/3}t^{2/3}\right) \right].$$

We can obtain an adiabatic initial evolution of the frequency if we choose the parameters of the transformation such that $\alpha = 2(1 + \beta)$. However, in this case, the frequency $\Omega_{\tau}$ becomes

$$\Omega_{\tau}^2 = -\frac{a^{-8\beta-6} b^{-4\beta}}{t^2} \left[ 1 + O\left(m^{2/3}t^{2/3}\right) \right].$$

By using the early time asymptotics for the scale factors, we can see that the frequency (C.20) approaches a constant negative value close to the initial singularity. This leads to a tachyonic growth of the mode, and to the breakdown of the linearized computation (see the main text).

Let us finally discuss the 2D scalar sector. For generic values of $\alpha$ and $\beta$ we now find

$$\Omega_{\tau}^2 = \frac{a^{-4\alpha+2} b^{-4\beta}}{9t^2} \left\{ [\alpha - 2(1 + \beta)]^2 + 9t^2 k^2_2/b^2 \right\} + O(1),$$

$$\frac{1}{\Omega_{\tau}^2} \frac{d\Omega_{\tau}}{d\tau} = \frac{2[\alpha - 2(1 + \beta)]^3 + 9t^2 (3 + 2\alpha - 4\beta) k^2_2/b^2 + O(t^2)}{\left[ [\alpha - 2(1 + \beta)]^2 + 9t^2 k^2_2/b^2 + O(t^3) \right]^{3/2}}.$$
Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB

For \( \alpha \neq 2(1 + \beta) \), the second expression evaluates to 2 at asymptotically early time. For \( \alpha = 2(1 + \beta) \), the dominant term vanishes both at the numerator and at the denominator, and the adiabaticity condition approaches \( 1/ (3 t p_2) \). This quantity diverges as \( t^{-1/3} \) as the time approaches the initial singularity at \( t = 0 \). Therefore, no time variable \( \tau \) chosen as in (C.15) leads to an initial adiabatic evolution for the 2D scalars.

Note added. While this work was being completed, we became aware of the work [41], which also develops a formalism for the computation of anisotropies in Bianchi I models. Although the approach is rather different, and a different conformal time is used in [41], we have verified that the evolution for the canonical modes obtained in [41] agrees with ours in the limit in which the background has a 2D residual isotropy. The work [41] does not study the background solutions for the model to the extent done here, and it therefore does not discuss the initial conditions for the perturbations, nor the resulting power spectrum and CMB temperature anisotropies. We also note that some of the results for the perturbations in the anisotropic model discussed here had also been summarized in [42].

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