Integrable motion of anisotropic space curves and surfaces induced by the Landau-Lifshitz equation

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Abstract

In this paper, we have studied the geometrical formulation of the Landau-Lifshitz equation (LLE) and established its geometrical equivalent counterpart as some generalized nonlinear Schrödinger equation. When the anisotropy vanishes, from this result we obtain the well-known results corresponding for the isotropic case, i.e. to the Heisenberg ferromagnet equation and the focusing nonlinear Schrödinger equation. The relations between the LLE and the differential geometry of space curves in the local and in nonlocal cases are studied. Using the well-known Sym-Tafel formula, the soliton surfaces induced by the LLE are briefly considered.

Key words: Integrable equations, Nonlinear Schrödinger equation, Heisenberg ferromagnet equation, Landau-Lifshitz equation, space curves, soliton solution, soliton surfaces, nonlocal integrable equations.

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Nonlinear dynamics of one-dimensional anisotropic ferromagnet can be described by the Landau-Lifshitz equation (LLE) \[1\]

\[ S_t = S \wedge S_{xx} + S \wedge JS, \]  

(1.1)

where \( J = \text{diag}(J_1, J_2, J_3) \) and \( J_1 \leq J_2 \leq J_3 \), \( S = (S_1, S_2, S_3) \) is the unit spin vector and \( S^2 = S_1^2 + \epsilon S_1^2 + \epsilon S_2^2 = 1 \). Here \( \epsilon = \pm 1 \), which gives rise to two different models corresponding to the \( su(2) \) \( (\epsilon = +1) \) and \( su(1,1) \) \( (\epsilon = -1) \) cases. The LLE is integrable \[2]-[3\]. It admits several integrable and non-integrable extensions and reductions in 1+1, 2+1 and 3+1 dimensions. The supersymmetric isotropic LLE is one of such generalizations in 1+1 dimensions \[5\]. In 2+1 dimensions, the LLE admits the following integrable spin systems: Ishimori equation \[6\], Myrzakulov-I (M-I) equation \[7\], Myrzakulov-IX (M-IX) equation \[8\], Myrzakulov-VIII (M-VIII) equation \[9\] and so on. The LLE has some interesting particular and limiting cases \[10]-[14\]. For example, in the case \( J_1 = J_2 = J_3 \) that is in the isotropic case, from (1.1) we obtain the following Heisenberg ferromagnet equation (HFE)

\[ S_t = S \wedge S_{xx}. \]  

(1.2)

At the same time, in the case \( J_1 = J_2 \) it is called the LLE with an easy plan. Also it is well-known that the nonlinear Schrödinger equation (NLSE) and sine-Gordon equation (SGE) are some limiting cases of LLE under the choice of the spectral parameter. In this sense, the LLE is a universal integrable model in 1+1 dimensions. The connection of the LLE (as well as the Bloch equation) with some types of two-level quantum systems was established in \[15\]-[16\].

The gauge equivalence between some integrable equations is an important tool in soliton theory \[17\]. Gauge equivalence of some integrable systems are well understood both in classical and discrete versions. In particular, it was shown that the LLE (1.1) is gauge equivalent to some nonlinear Schrödinger-like equation \[18\]-[19\]. Gauge equivalence of the LLE with an easy plan and the nonlinear Schrödinger equation was considered in \[20\]-[21\].

There are another type of equivalences between integrable equations which is called the geometrical equivalence or Lakshmanan equivalence \[22\] (see, also \[23\]-[33\] and references therein). The geometrical equivalence is related with the differential geometry of curves and surfaces. The relation between the differential geometry of curves and surfaces and integrable systems were investigated by many authors \[23\]-[59\]. These results give the geometrical formulation of integrable systems.

Recently, the integrable nonlocal soliton equations were introduced and studied \[60\]-[68\]. In particular, the nonlocal HFE were studied \[69\]-[70\]. Other interesting problems related with the subjects of this paper are integrable discrete HFE \[71\]-[72\].

In our previous works, we have studied some integrable and nonintegrable continuous classical spin chains. In this paper, we continue our investigations such types of spin chains, namely, we study the LLE. If exactly, in this paper we study the LLE (1.1) from the geometrical point of view. We construct the integrable motion of the anisotropic space curves and anisotropic surfaces induced by the LLE. Also we present the nonlocal versions of the LLE and its equivalent nonlinear Schrödinger-like equation. The integrable deformations of nonlocal space curves and the corresponding Serret-Frenet equation are discussed.
The paper is organized as follows. In Sec. 2, we briefly review some well-known results on the LLE. The geometrical formulation of the LLE in terms of anisotropic space curves is studied in Sec. 3. The nonlocal version of the LLE is presented in Sec. 4. Next, in Sec. 5 we discuss the nonlocal space curves formalism including the corresponding nonlocal Serret-Frenet equation. In Sec. 6, the gauge equivalence between the LLE and the nonlinear Schrödinger-like equation is discussed. In Sec. 7, we briefly consider the anisotropic soliton surfaces by using the famous Sym-Tafel formula. Some generalizations of the LLE are presented in Sec. 8. The hydrodynamical formulation of the LLE is given in Sec. 9. The quantum Heisenberg model is considered in Sec. 10. The dispersionless LLE is presented in Sec. 11 for some particular case. In Sec. 12, the self-similar solutions of the LLE is briefly discussed. The lattice (discrete) version of the LLE is considered in Sec. 13. We give conclusions in the last section.

2 Landau-Lifshitz equation

In this section we give some basic informations on the LLE like the equation of motion, three types of Lax pairs, Hamilton formulation, different parametrizations and limiting cases.

2.1 Equation

The LLE has been introduced in \[1\] and describes the dynamics of the orientation of the magnetization (or spin) in ferromagnetic materials. It constitutes nowadays a fundamental tool in the magnetic recording industry, due to its applications to ferromagnets. The LLE has the form \[\text{(10)}\]

\[ S_t = S \wedge S_{xx} + S \wedge JS, \tag{2.1} \]

where \( J = \text{diag}(J_1, J_2, J_3) \) and \( J_1 \leq J_2 \leq J_3 \). In components it takes the form

\[ S_{1t} = S_2 S_{3xx} - S_{2xx} S_3 + S_2 J_3 S_3 - J_2 S_2 S_3, \tag{2.2} \]

\[ S_{2t} = S_3 S_{1xx} - S_{3xx} S_1 + S_3 J_1 S_1 - J_3 S_3 S_1, \tag{2.3} \]

\[ S_{3t} = S_1 S_{2xx} - S_{1xx} S_2 + S_1 J_2 S_2 - J_1 S_1 S_2, \tag{2.4} \]

or

\[ S_{1t} = S_2 S_{3xx} - S_{2xx} S_3 - J_2 S_2 S_3, \tag{2.5} \]

\[ S_{2t} = S_3 S_{1xx} - S_{3xx} S_1 - J_3 S_3 S_1, \tag{2.6} \]

\[ S_{3t} = S_1 S_{2xx} - S_{1xx} S_2 - J_1 S_1 S_2, \tag{2.7} \]

where \( J_{ij} = J_i - J_j \), \( I_{ij} = J_i + J_j \). We also use the following form of the LLE

\[ i S^+_t = S^+ S_{3xx} - S^+ S_{2xx} S_3 + J_3 S_3 S^+ - \frac{1}{2} S_3 [J_1 (S^+ S^-) + J_2 (S^+ S^-)], \tag{2.8} \]

\[ i S^-_t = S^- S_{3xx} - S^- S_{2xx} S_3 - J_3 S_3 S^- + \frac{1}{2} S_3 [J_1 (S^+ S^-) - J_2 (S^+ S^-)], \tag{2.9} \]

\[ i S^+_{3t} = \frac{1}{2} (S^- S_{xx} - S^- S_{xx} S^+) - i (J_2 - J_1) S_1 S_2, \tag{2.10} \]

or

\[ i S^+_t = S^+ S_{3xx} - S^+ S_{2xx} S_3 + J_3 S_3 S^+ - \frac{1}{2} S_3 (I_{21} S^+ - J_2 S^-), \tag{2.11} \]

\[ i S^-_t = S^- S_{3xx} - S^- S_{2xx} S_3 - J_3 S_3 S^- - \frac{1}{2} S_3 (I_{21} S^+ - I_{21} S^-), \tag{2.12} \]

\[ i S^+_{3t} = \frac{1}{2} (S^- S_{xx} - S^- S_{xx} S^+) - \frac{1}{4} J_{12} (S^+ S^- - S^2). \tag{2.13} \]
The matrix LLE reads as
\[ iS_t = \frac{1}{2} [S, S_{xx} + S_J], \]  
where
\[ S = \sum_{k=1}^{3} S_k \sigma_k = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^2 = I, \quad S^\pm = S_1 \pm iS_2 \]  
\[ S_J = \sum_{k=1}^{3} J_k S_k \sigma_k = \begin{pmatrix} J_3 S_3 & J_1 S_1 - iJ_2 S_2 \\ J_1 S_1 + iJ_2 S_2 & -J_3 S_3 \end{pmatrix}. \]  
Here \( \sigma_j \) are Pauli matrices:
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
We note that the LLE is a nonlinear dispersive partial differential equation. The corresponding dispersion relation reads as
\[ \omega(k) = \pm \sqrt{|k|^4 + (J_{21} + J_{23})|k|^2 + J_{21}J_{23}}, \]  
what corresponds to the solutions of the form \( e^{i(kx - \omega t)} \) that is to linear sinusoidal waves of frequency \( \omega \) and wavenumber \( k \).

### 2.2 Lax representation

There are several forms of the Lax representations of the LLE: elliptic, \( 4 \times 4 \) polynomial and \( 6 \times 6 \) polynomial Lax pairs.

#### 2.2.1 Elliptic Lax pair

The elliptic Lax representation for the LLE is given by
\[ \Phi_x = U_1 \Phi, \]  
\[ \Phi_t = V_1 \Phi, \]  
where
\[ U_1 = -i \sum_{k=1}^{3} \frac{u_k S_k \sigma_k}{u_k}, \]  
\[ V_1 = 2i \sum_{k=1}^{3} \frac{u_1 u_2 u_3}{u_k} S_k \sigma_k - i \sum_{a,b,c=1}^{3} u_a \epsilon_{abc} S_b S_c \sigma_a. \]  
Here
\[ u_1 = \frac{\rho}{sn(\lambda, k)}, \quad u_2 = \frac{\rho dn(\lambda, k)}{sn(\lambda, k)}, \quad u_3 = \frac{\rho cn(\lambda, k)}{sn(\lambda, k)} \]  
and
\[ k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}, \quad \rho = \frac{1}{2} \sqrt{J_3 - J_1}, \quad 0 \leq k \leq 1, \quad \rho \geq 0. \]
Note that \( sn(\lambda, k) \), \( cn(\lambda, k) \), \( dn(\lambda, k) \) are Jacobi elliptic functions having the following limit cases
\[
\begin{align*}
  sn(\lambda, 0) &= \sin \lambda, & cn(\lambda, 0) &= \cos \lambda, & dn(\lambda, 0) &= 1, \\
  sn(\lambda, 1) &= \tanh \lambda, & cn(\lambda, 1) &= dn(\lambda, 1) = \frac{1}{\cosh \lambda}
\end{align*}
\] (2.25)
and
\[
  u_a^2 - u_b^2 = \frac{1}{4}(J_b - J_a).
\] (2.27)

### 2.2.2 Polynomial 4 × 4 Lax pair

The 4 × 4 polynomial Lax pair for the LLE has the form [73]–[75]

\[
U_1 = \frac{1}{2} A_1(\lambda + \tilde{J}),
\]
(2.28)
\[
V_1 = \frac{1}{2} (\frac{1}{2} A_1 - [A, A_{1x}] + \frac{1}{2} A_{2x})(\lambda + \tilde{J}),
\]
(2.29)
where
\[
A_1 = \frac{1}{2}
\begin{pmatrix}
  0 & S_1 & S_2 & S_3 \\
  -S_1 & 0 & S_3 & -S_2 \\
  -S_2 & -S_3 & 0 & S_1 \\
  -S_3 & S_2 & -S_1 & 0
\end{pmatrix},
\]
\[
A_{2x} = \frac{1}{2}
\begin{pmatrix}
  0 & J_1 S_1 & J_2 S_2 & -J_3 S_3 \\
  -J_1 S_1 & 0 & J_3 S_3 & -J_2 S_2 \\
  -J_2 S_2 & -J_3 S_3 & 0 & -J_1 S_1 \\
  J_3 S_3 & -J_2 S_2 & J_1 S_1 & 0
\end{pmatrix},
\]
\[
\tilde{J} =
\begin{pmatrix}
  -J_1 - J_2 + J_3 & 0 & 0 & 0 \\
  0 & -J_1 + J_2 - J_3 & 0 & 0 \\
  0 & 0 & J_1 - J_2 - J_3 & 0 \\
  0 & 0 & 0 & J_1 + J_2 + J_3
\end{pmatrix}.
\] (2.30)

### 2.2.3 Polynomial 6 × 6 Lax pair

The 6 × 6 polynomial Lax pair for the LLE has the form [73]–[75]

\[
U_1 = \frac{\lambda}{2} \left( M(S) 0 \right) + \frac{1}{2} \left( 0 0 \right) + \frac{1}{2} ad_{\tilde{J}} \left( M(S) 0 \right),
\]
(2.31)
\[
V_1 = \frac{\lambda^2}{4} \left( M(S) 0 \right) + \frac{\lambda}{2} \left( M(S \times S_x) 0 \right) + \frac{1}{2} ad_{\tilde{J}} \left( M(S) 0 \right) + K,
\]
(2.32)
where \( ad_{f} g = [f, g] \) and
\[
M(S) = \begin{pmatrix}
  0 & S_3 & -S_2 \\
  -S_3 & 0 & S_1 \\
  S_2 & -S_1 & 0
\end{pmatrix},
\]
\[
\tilde{J} = \begin{pmatrix}
  0 & J \\
  J & 0
\end{pmatrix},
\]
\[
[M(A), M(B)] = -M(A \times B),
\]
(2.33)
\[
K = -\frac{1}{4} \left( M(J^2 S) 0 \right) + \frac{1}{4} ad_{\tilde{J}} \left( M(S \times S_x) 0 \right),
\]
(2.34)

From the compatibility condition \( U_1 t - V_1 x + [U_1, V_1] = 0 \) we obtain
\[
M(S) t + [M(S), M(S)_{xx}] + [M(S, M(J S)) = 0,
\]
(2.35)
which is the LLE.
2.3 Hamiltonian structure

The LLE can be written in the Hamiltonian form as

\[ S_t = \{ H, S \}, \] (2.36)

where the Hamilton function is given by

\[ H = \frac{1}{2} \int (S_x^2 - J_1 S_1^2 - J_2 S_2^2 - J_3 S_3^2) \, dx. \] (2.37)

The Poisson bracket for the two functionals \( A, B \) has the form

\[ \{ A, B \} = - \sum_{a,b,c=1}^3 \epsilon_{abc} \int \frac{\delta A}{\delta S_a} \frac{\delta B}{\delta S_b} S_c \, dxdy. \] (2.38)

In particular, for the components of the spin vector the corresponding Poisson brackets read as

\[ \{ S_a(x), S_b(y) \} = -\epsilon_{abc} S_c(x) \delta(x - y). \] (2.39)

2.4 Different parametrizations

The LLE can be written in the different forms that follow from the different parametrizations of the spin vector \( S \). In this subsection we going to present some of these forms of the LLE.

2.4.1 \((\theta - \phi)\) - form

Consider the following angle parametrization of the LLE

\[ S^+ = e^{i\varphi} \sin \theta, \quad S^- = e^{-i\varphi} \sin \theta, \quad S_3 = \cos \theta, \] (2.40)

where \( \theta(x, t), \varphi(x, t) \) are some real functions. Then the LLE takes the form

\[ \theta_t + \varphi_{xx} \sin \theta + 2 \varphi_x \theta_x \cos \theta + J_2 \sin \theta \sin \varphi = 0, \] (2.41)
\[ \varphi_t \sin \theta - \theta_{xx} + (\varphi_x^2 + J_3 - J_1 \cos \varphi) \sin \theta \cos \theta = 0. \] (2.42)

2.4.2 \(w\) - form

Let us introduce the new complex function \( w \) as

\[ w = \frac{S^+}{1 + S_3} = \tan \frac{\theta}{2} e^{i\varphi}. \] (2.43)

Then in terms of this function, the LLE takes the form

\[ iw_t + w_{xx} - \frac{2\bar{w}w^2}{1 + |w|^2} - \frac{1}{2(1 + |w|^2)} \{ [(2J_3 - J_{21})(1 - |w|^2) + J_{12}(w^2 - \bar{w}^2)]w + J_{21}(1 - |w|^2)\bar{w} \} = 0. \] (2.44)
2.4.3 \((p - q)\) form

We now consider the following parametrization of the spin vector [76]

\[
S_1 = p(q^2 - 1) + q, \quad S_2 = ip(q^2 + 1) + iq, \quad S_3 = 2pq + 1.
\]  

(2.45)

In terms of these new variables \((p - q)\), the LLE reads as

\[
p_{t_1} = -\frac{\partial H}{\partial q} = p_{xx} - 2(p^2 q_x)_x,
\]

(2.46)

\[
q_{t_1} = \frac{\partial H}{\partial p} = -q_{xx} - 2pq^2_x,
\]

(2.47)

where

\[
H = p_x q_x - p^2(q_x^2 + r(q)) - \frac{1}{12} r''(q) - \frac{1}{2} r'(q), \quad r^V = 0, \quad r = -\frac{1}{4} \mathbf{S} \cdot \mathbf{J}, \quad t_1 = it.
\]

(2.48)

2.4.4 LLE as the Madelung equation

To study some properties of some nonlinear differential equations is very useful their hydrodynamical forms. The same conclusion is correct for the LLE. Let us find its hydrodynamical form. To derive the hydrodynamical form of the LLE, we consider the following Madelung-like transformation [14]

\[
S + \frac{1}{\sqrt{1 - S^2}} e^{-i(\phi - 0.5\pi)}.
\]

(2.49)

Then the LLE (1.1) takes the form

\[
\phi_t + \left(\frac{S_{3x}}{1 - S^2} + \frac{S_3 S_{3x}}{(1 - S^2)^2} + (\phi_x^2 - J_{23} + J_{21} \sin^2 \phi) S_3\right) = 0,
\]

(2.50)

\[
S_{3t} = \left(1 - S^2\right) \phi_x + 0.5 J_{21} (1 - S^2) \sin^2 2\phi = 0.
\]

(2.51)

This is the Madelung equation form of the LLE (1.1). Note that this Madelung form will be essential in the study of solutions of the LLE.

2.4.5 \(\tau_n\) - parametrization

Let us present one of very interesting parametrizations of the spin vector. Let the spin matrix \(S\) has the form

\[
S = g^{-1} \sigma_3 g,
\]

(2.52)

where \(g\) satisfies the following equations

\[
g_x = \left(\begin{array}{ccc}
0 & \tau_{n+1} & \tau_n \\
\tau_{n+1} & \tau_{n-1} & \tau_n^{-1} \\
0 & \tau_{n-1} & \tau_n^{-1}
\end{array}\right) g,
\]

(2.53)

\[
g_t = i \left(\begin{array}{ccc}
\tau_{n-1} & \tau_{n-1} & 0 \\
0 & \tau_{n+1} & \tau_{n-1}^{-1} \\
\tau_{n+1} & \tau_{n+1} & \tau_{n-1}^{-1}
\end{array}\right) g.
\]

(2.54)

The compatibility condition of this set of linear equations gives

\[
(D_t - D_x^2) \tau_{n+1} \cdot \tau_n = 0,
\]

(2.55)

\[
D_x^2 \tau_n \cdot \tau_n - 2 \tau_{n+1} \cdot \tau_{n-1} = 0.
\]

(2.56)
It is the second member of the TL hierarchy (see e.g. [77] and references therein). On the other hand, the spin matrix $S$ obeys the HFE (as $t \to it$)

$$2iS_t = [S, S_{xx}].$$  \hspace{1cm} (2.57)

At the same time the following functions

$$r = \tau_{n+1}\tau_n^{-1}, \quad q = \tau_{n-1}\tau_n$$  \hspace{1cm} (2.58)

solves the NLSE (2.61)-(2.62).

### 2.5 Limiting cases

In this subsection we want to present some well-known particular (limiting) cases of the LLE (see, e.g. Refs. [10],[14]).

#### 2.5.1 HFE

In the isotropic case $J_1 = J_2 = J_3$, the LLE turn to the HFE

$$S_t = S \wedge S_{xx}.$$  \hspace{1cm} (2.59)

The corresponding Lax pair of the HFE has the form

$$U_2 = -i\lambda S, \quad V_2 = -2i\lambda^2 S + \lambda SS_x.$$  \hspace{1cm} (2.60)

It is well-known that the HFE (1.2) or (2.59) is gauge [17] and geometrically [22] equivalent to the following NLSE

$$iq_t + q_{xx} - 2rq^2 = 0,$$

$$ir_t - r_{xx} + 2qr^2 = 0$$  \hspace{1cm} (2.61, 2.62)

with the Lax pair

$$U_1 = -i\lambda \sigma_3 + Q, \quad V_1 = -2i\lambda^2 \sigma_3 + \lambda V_1 + V_0.$$  \hspace{1cm} (2.63)

Here

$$r = -\epsilon \bar{q}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad V_1 = 2Q, \quad V_0 = i \begin{pmatrix} -rq & q_x \\ -r_x & rq \end{pmatrix},$$  \hspace{1cm} (2.64)

where bar stands for the complex conjugation and $\epsilon = \pm 1$ signals the focusing (+) and defocusing (−) nonlinearity. Let us introduce the new variables as

$$u = rq, \quad v = -\ln q.$$  \hspace{1cm} (2.65)

Then the NLSE takes the form

$$iu_t - u_{xx} - 2(uv_x)_x = 0,$$

$$iv_t + v_{xx} - v_r^2 + 2u = 0$$  \hspace{1cm} (2.66, 2.67)

which is the Toda lattice (see e.g. Refs. [76]-[77] and references therein).

#### 2.5.2 $J_1 = J_2 \neq J_3$ case

When $J_1 = J_2 \neq J_3$, the equation (1.1) is called the LLE for a spin chain with an easy plan.
2.5.3 Sine-Gordon equation

To get the sine-Gordon equation (SGE) from the LLE, we assume that

\[ S^2 = R^2, \quad S_1 = -\frac{\beta \pi}{2}, \quad S_2 = \sqrt{R^2 - \frac{\beta^2 \pi^2}{4} \sin \frac{\phi \pi}{2}}, \quad S_3 = \sqrt{R^2 - \frac{\beta^2 \pi^2}{4} \cos \frac{\phi \pi}{2}}, \]

\[ J_2 = J_1 + 1, \quad J_3 = J_2 + m^2 R^{-2} = 1 + J_1 + m^2 R^{-2}. \]  

(2.68)

(2.69)

Then the LLE takes the form

\[ S_t = R^{-2} S \wedge S_{xx} + S \wedge JS, \]  

(2.70)

so that in the limit \( R \to \infty \), it turns to the SGE

\[ \pi = \varphi_t, \]  

(2.71)

\[ \pi_t = \varphi_{xx} - \frac{m^2}{\beta \sin \beta \varphi} \]  

(2.72)

or

\[ \varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta} \sin \beta \varphi = 0. \]  

(2.73)

2.5.4 NLSE

Let us now we derive the NLSE from the LLE. To do that, we assume

\[ S^2 = 1, \quad S^+ = \sqrt{2q e^{2i\kappa \eta^{-1} t}}, \quad S_3 = \sqrt{1 - 2|q|^2}, \quad J_2 = J_1, \quad J_3 = J_1 - 2\kappa \eta^{-1}, \quad \kappa > 0. \]  

(2.74)

In this case, from the LLE (1.1) as \( \eta \to 0 \) we obtain the NLSE

\[ iq_t + q_{xx} + 2|q|^2 q = 0. \]  

(2.75)

3 Integrable deformations of anisotropic space curves

In this section, our goal is to present the geometrical formulation of the LLE in terms of space curves. To the end, we now consider the space curve in the Euclidean space \( R^3 \). Such space curves governed by the following Serret-Frenet equation (SFE) and its temporal counterpart equation (see e.g. [22], [55] and references therein)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_x = C
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix},
\]

(3.1)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_t = D
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix},
\]

(3.2)

where

\[
C = \begin{pmatrix}
  0 & \kappa & \sigma \\
-\kappa & 0 & \tau \\
-\sigma & -\tau & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
  0 & \omega_3 & \omega_2 \\
-\omega_3 & 0 & \omega_1 \\
-\omega_2 & -\omega_1 & 0
\end{pmatrix}.
\]

(3.3)

Here the functions \( \kappa \) is called normal curvature, \( \sigma \) is called geodesic curvature, \( \tau \) is called geodesic torsion and \( \omega_j \) are some real functions. The later functions must be expressed in terms of \( \kappa, \sigma, \tau \) and their derivatives. The compatibility conditions \( e_{jxt} = e_{jtx} \) of the equations (3.1)-(3.2) reads as

\[ C_x - D_x + [C, D] = 0. \]
For the elements this equation gives
\[ \kappa_t = \omega_{3x} - \tau \omega_2 + \sigma \omega_1, \]  
\[ \sigma_t = \omega_{2x} - \kappa \omega_1 + \tau \omega_3, \]  
\[ \tau_t = \omega_{1x} - \sigma \omega_3 + \kappa \omega_2. \]  
(3.4) (3.5) (3.6)

Now let us assume that
\[ \kappa = 0, \quad \sigma = r - q, \quad \tau = -i(r + q), \quad r = -\bar{q}, \]  
\[ \omega_1 = q_x - r_x - i(q_{21} + q_{12}), \quad \omega_2 = -i(r_x + q_x) + q_{21} - q_{12}, \quad \omega_3 = -2rq - 2iq_{11}. \]  
(3.7) (3.8)

Substituting these expressions of \( \kappa, \sigma, \tau, \omega \) into (3.4)-(3.6), we obtain the following nonlinear Schrödinger-like equation
\[ iq_t + q_{xx} - 2rq^2 - iR_{12} = 0, \]  
\[ ir_t - r_{xx} + 2qr^2 - iR_{21} = 0, \]  
\[ q_{11x} - (qq_{21} - rq_{12}) = 0, \]  
(3.9) (3.10) (3.11)

where
\[ R_{12} = q_{12x} + 2qq_{11}, \quad R_{21} = q_{21x} - 2rq_{11}. \]  
(3.12)

It is the case to mention that the system of equations (3.9)-(3.11) is integrable in the sense that it admits the following Lax representation \([18],[19]\)
\[ \Psi_t = U_2 \Psi, \]  
\[ \Psi_t = V_2 \Psi, \]  
(3.13) (3.14)

where
\[ U_2 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & -q_{11} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} = Q_x - [U_2, Q], \quad R_{21} = -R_{12}, \]  
(3.15)

\[ V_2 = -i(U_2^2 + U_{2x})\sigma_3 + Q = \begin{pmatrix} -irq + q_{11} -q x + q_{12} \\ -irx + q_{21} -iq - q_{11} \end{pmatrix}. \]  
(3.16)

Let us present here some useful formulas. Let \( \Psi \) has the form
\[ \Psi = \frac{g_1 - \bar{g}_2}{g_2 - \bar{g}_1}, \]  
(3.17)

where \( g_j \) are some complex functions. From (3.13) follows that
\[ \det(\Psi)|_x = (|g_1|^2 + |g_2|^2)_x = 0, \]  
(3.18)

that is \( |g_1|^2 + |g_2|^2 = c = const. \) Usually we assume \( c = 1, \) so that \( |g_1|^2 + |g_2|^2 = 1. \) Then the function \( R \) has the form \([18]\)
\[ R = \Psi[(\phi S)_x - \frac{i}{2}S_{Jx}]\Psi^{-1} \]  
(3.19)

or
\[ R = \Psi[(\phi S)_x - \frac{i}{2}S_{Jx}]\Psi^{-1} = iq\bigl[J_1(2Im(\bar{g}_2\bar{g}_1)) + J_2(2Re(\bar{g}_2\bar{g}_1)) + J_3(|\bar{g}_2|^2 - |g_1|^2)\bigr] - \]  
\[ i\bigl[J_1(\bar{g}_1^2 + \bar{g}_2^2)Im(\bar{g}_1^2\bar{g}_2 + \bar{g}_2^2\bar{g}_1) + J_2(\bar{g}_1^2 - \bar{g}_2^2)Re(\bar{g}_1^2\bar{g}_2 + \bar{g}_2^2\bar{g}_1) + 2J_3Re(\bar{g}_2\bar{g}_1)(\bar{g}_2\bar{g}_1 - \bar{g}_2\bar{g}_1), \]  
(3.20)

where \( \phi = J_1S_1^2 + J_2S_2^2 + J_3S_3^2 \) and we used the formula \( S_x = \Psi^{-1}[\sigma_3, U_2]\Psi. \) Now from (3.1)-(3.2), taking into account the last relation, we obtain the following equation for the unit vector \( e_3: \)
\[ e_{3t} = e_3 \times (e_{3xx} + J_e_3), \]  
(3.21)

which is nothing but the LLE (1.1) after the identification \( e_3 \equiv S. \)

**Theorem 1.** The Landau-Lifshitz equation (1.1) is geometrically equivalent to the nonlinear Schrödinger-like equation (3.9)-(3.11). In the isotropic case, this geometrical equivalence is reduced to the well-known geometric equivalence between the HFE and the NLSE \([22]\).
4 Nonlocal Landau-Lifshitz equation

In this section we want to present the nonlocal version of the LLE. It is well-known that in the nonlocal case, the unit spin vector $\mathbf{S} = (S_1, S_2, S_3)$ is the complex-valued vector \[69\]. The complex-valued spin vector $\mathbf{S}$ induced that the unit vectors $\mathbf{e}_j$ become also complex-valued. It means that the curvature $\kappa(t, x)$, the torsion $\tau(t, x)$ and $\omega_j$ are complex-valued functions \[68\]. Since in the nonlocal case, the spin vector $\mathbf{S}$ is the complex-valued vector function, so that we may decompose it as $\mathbf{S} = \mathbf{M} + i\mathbf{L}$, where $\mathbf{M}$ and $\mathbf{L}$ are already real valued vector functions \[69\]. These new real vectors satisfy the following relations

$$M^2 - L^2 = 1, \quad M \cdot L = 0. \quad (4.1)$$

From the LLE (1.1) follows that these real valued vector functions $\mathbf{M}$ and $\mathbf{L}$ satisfy the following set of coupled equations:

$$M_t = M \wedge M_{xx} - L \wedge L_{xx} + M \wedge J M - L \wedge J L, \quad (4.2)$$
$$L_t = M \wedge L_{xx} + L \wedge M_{xx} + L \wedge J M + M \wedge J L. \quad (4.3)$$

It is the desired nonlocal LLE. Note that in the isotropic case that is when $J_1 = J_2 = J_3$ this nonlocal LLE reduces to the following nonlocal HFE \[69\]

$$M_t = M \wedge M_{xx} - L \wedge L_{xx}, \quad (4.4)$$
$$L_t = M \wedge L_{xx} + L \wedge M_{xx}. \quad (4.5)$$

5 Nonlocal space curves and Nonlocal Serret-Frenet equation

As we mentioned in the previous section, in the nonlocal case, the unit vectors $\mathbf{e}_j$ and the functions $\kappa, \tau, \sigma, \omega_j$ are the complex-valued variables. As a consequence of these statements, the corresponding space curves describe by the nonlocal Serret-Frenet equation (SFE). To derive the nonlocal SFE, first we decompose the unit vectors $\mathbf{e}_j$ and functions $\kappa, \tau, \sigma, \omega_j$ into the real and image parts as

$$\mathbf{e}_1 = \mathbf{m}_1 + i\mathbf{l}_1, \quad \mathbf{e}_2 = \mathbf{m}_2 + i\mathbf{l}_2, \quad \mathbf{e}_3 = \mathbf{m}_3 + i\mathbf{l}_3, \quad (5.1)$$
$$\kappa = \kappa_1 + i\kappa_2, \quad \sigma = \sigma_1 + i\sigma_2, \quad \tau = \tau_1 + i\tau_2, \quad (5.2)$$
$$\omega_1 = \omega_{11} + i\omega_{12}, \quad \omega_2 = \omega_{21} + i\omega_{22}, \quad \omega_3 = \omega_{31} + i\omega_{32}, \quad (5.3)$$

where $\mathbf{m}_j, \mathbf{l}_j$ are the real valued vector functions and $\kappa_j, \sigma_j, \tau_j, \omega_{ij}$ are real functions. From

$$\mathbf{e}_j^2 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0 (i \neq j), \quad \mathbf{e}_i \wedge \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k \quad (5.4)$$

follows that the vectors $\mathbf{m}_j, \mathbf{l}_j$ satisfy the following relations

$$\mathbf{m}_j^2 - \mathbf{l}_j^2 = 1, \quad \mathbf{m}_j \cdot \mathbf{l}_j = 0, \quad \mathbf{m}_i \wedge \mathbf{m}_j - \mathbf{l}_i \wedge \mathbf{l}_j = \epsilon_{ijk} \mathbf{m}_k, \quad \mathbf{m}_i \wedge \mathbf{l}_j + \mathbf{l}_i \wedge \mathbf{m}_j = \epsilon_{ijk} \mathbf{l}_k \quad (5.5)$$

and for the case $i \neq j$ we have

$$\mathbf{m}_i \cdot \mathbf{m}_j - \mathbf{l}_i \cdot \mathbf{l}_j = 0, \quad \mathbf{m}_i \cdot \mathbf{l}_j + \mathbf{m}_j \cdot \mathbf{l}_i = 0. \quad (5.6)$$

We now ready to write the nonlocal SFE which follow from (3.1)-(3.2) and have the forms

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}_x = C_1 \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} - C_2 \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \quad \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_x = C_1 \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} + C_2 \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad (5.7)$$
The compatibility conditions $m_{jxt} = m_{jtx}$, $l_{jxt} = l_{jtx}$ of these equations read as

$$C_{1t} - D_{1x} + [C_1, D_1] - [C_2, D_2] = 0, \quad C_{2t} - D_{2x} + [C_1, D_2] + [C_2, D_1] = 0. \quad (5.10)$$

In components these equations take the following forms

$$\begin{align*}
\kappa_{1t} &= \omega_{31} - (\tau_1 \omega_{21} - \tau_2 \omega_{22}) + (\sigma_1 \omega_{11} - \sigma_2 \omega_{12}), \\
\kappa_{2t} &= \omega_{32} - (\tau_1 \omega_{22} - \tau_2 \omega_{21}) + (\sigma_2 \omega_{11} + \sigma_1 \omega_{12}), \\
\sigma_{1t} &= \omega_{11} - (\sigma_1 \omega_{31} - \sigma_2 \omega_{32}), \\
\sigma_{2t} &= \omega_{12} - (\sigma_1 \omega_{32} + \sigma_2 \omega_{31}).
\end{align*} \quad (5.11)$$

Now we set

$$\begin{align*}
\kappa &= 0, \quad \sigma = r - q, \quad \tau = -i(r + q), \quad r = -\bar{q}, \\
\omega_1 &= q_x - r_x - i(q_{21} + q_{12}), \quad \omega_2 = -i(r_x + q_x) + q_{21} - q_{12}, \quad \omega_3 = -2rq - 2iq_{11},
\end{align*} \quad (5.17)$$

where $r = \nu \bar{q}(\epsilon_1 x, \epsilon_2 t)$. As in the local case, we can show that the nonlinear LLE (4.2)-(4.3) is the gauge and geometrical equivalent to the following nonlinear nonlinear Schrödinger-like equation

$$\begin{align*}
iq_t + q_{xx} - 2rq^2 - iR_{12}(r, q) &= 0, \\
i\nu \bar{q}_t - r_{xx} + 2qr^2 - iR_{21}(r, q) &= 0, \\
q_{11x} - (q_{21} - q_{12}) &= 0,
\end{align*} \quad (5.19)$$

where

$$R_{12}(r, q) = q_{12x} + 2qq_{11}, \quad R_{21}(r, q) = q_{21x} - 2rq_{11}, \quad r = \nu \bar{q}(\epsilon_1 x, \epsilon_2 t), \quad \nu = \pm 1, \quad c^2 = 1. \quad (5.22)$$

### 6 Gauge equivalence between the LLE and NLS-like equation

The gauge equivalent counterpart of the LLE was constructed in [18]. For the noncompact case, the LLE (1.1) is called the modified LLE, the corresponding gauge equivalent was studied in [19]. Here let us consider the relation between the solutions of the LLE (1.1) and the nonlinear Schrödinger-like equation (3.9)-(3.11). Let $S$ is the solution of the LLE (1.1) and $g = \Psi$ is the solution linear equations (3.13)-(3.14). Then we have

$$S = g^{-1} \sigma_3 g, \quad g = A(S + \sigma_3) = \begin{pmatrix} f_1(S_3 + 1) & f_1 S^- \\ f_2 S^+ & -f_2(S_3 + 1) \end{pmatrix}, \quad A = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} = \text{diag}(e^{i\gamma}, e^{-i\gamma}), \quad (6.1)$$

where $\gamma$ is a real function, $\det(g) = 1$ and $\Delta = -2(S_3 + 1)$. Since

$$\begin{align*}
(S + \sigma_3)^{-1} &= -\frac{1}{\Delta} \begin{pmatrix} S_3 + 1 & -S^- \\ S^+ & -(S_3 + 1) \end{pmatrix}, \quad \Delta = \det(S + \sigma_3) = -2(S_3 + 1) \quad (6.2)
\end{align*}$$
we obtain

\[(S + \sigma_3)^{-1}(S + \sigma_3) = -\frac{1}{\Delta} \left( \begin{array}{cc} S_3 + 1 & S^- \\ S^+ & -(S_3 + 1) \end{array} \right) \left( \begin{array}{cc} S_3 + 1 & S^- \\ S^+ & -(S_3 + 1) \end{array} \right) = I. \quad (6.3)\]

Similarly, we calculate

\[g^{-1} = (S + \sigma_3)^{-1}A^{-1} = -\frac{1}{\Delta} \left( \begin{array}{cc} S_3 + 1 & S^- \\ S^+ & -(S_3 + 1) \end{array} \right) \left( \begin{array}{cc} e^{-i\gamma} & 0 \\ 0 & e^{i\gamma} \end{array} \right) = -\frac{1}{\Delta}(S + \sigma_3)A^{-1}. \quad (6.4)\]

To find the desired relation we recall that

\[S_x = g^{-1}[\sigma_3, U]g = 2g^{-1} \left( \begin{array}{cc} 0 & q \\ -r & 0 \end{array} \right) g. \quad (6.5)\]

Hence we get

\[\left( \begin{array}{cc} 0 & q \\ -r & 0 \end{array} \right) = \frac{1}{2}A(S + \sigma_3)S_x(S + \sigma_3)^{-1}A^{-1} = -\frac{1}{2\Delta}A(S + \sigma_3)S_x(S + \sigma_3)A^{-1} = -\frac{1}{2\Delta}A \cdot \mathbb{E}A^{-1} \quad (6.6)\]

and

\[\begin{aligned} (S + \sigma_3)S_x(S + \sigma_3) &= E = \left( \begin{array}{cc} 0 & e_{12} \\ e_{21} & 0 \end{array} \right) = 2 \left( \begin{array}{cc} 0 & S^-S_3x - (S_3 + 1)S_x^- \\ S^+S_3x - (S_3 + 1)S_x^+ & 0 \end{array} \right). \quad (6.7) \end{aligned}\]

Hence we obtain

\[\left( \begin{array}{cc} 0 & q \\ -r & 0 \end{array} \right) = \frac{1}{4(S_3 + 1)} \left( \begin{array}{cc} 0 & e_{12}e^{2i\gamma} \\ e_{21}e^{-2i\gamma} & 0 \end{array} \right) = \frac{1}{2(S_3 + 1)} \left( \begin{array}{cc} [S^+S_3x - (S_3 + 1)S_x^+]e^{-2i\gamma} \\ [S^-S_3x - (S_3 + 1)S_x^-]e^{2i\gamma} \end{array} \right), \quad (6.8)\]

or

\[\begin{aligned} q &= \frac{1}{2(S_3 + 1)}e_{12}e^{2i\gamma} = \frac{1}{2(S_3 + 1)}[S^-S_3x - (S_3 + 1)S_x^-]e^{2i\gamma}, \quad (6.9) \\ r &= \frac{1}{2(S_3 + 1)}e_{21}e^{-2i\gamma} = \frac{1}{2(S_3 + 1)}[S^+S_3x - (S_3 + 1)S_x^+]e^{-2i\gamma}, \quad (6.10) \end{aligned}\]

where \(r = -\bar{q}\). In terms of \(\theta - \varphi\) variables these expressions take the form

\[q = \frac{(i\varphi_x \sin \theta - \theta \varphi_x) e^{-i f \varphi_x \cos \theta dx}}{2}, \quad (6.11)\]

\[r = \frac{(i\varphi_x \sin \theta + \theta \varphi_x) e^{i f \varphi_x \cos \theta dx}}{2}, \quad (6.12)\]

where \(\epsilon = \pm 1\). Hence we obtain

\[rq = \frac{\varphi_x^2 \sin^2 \theta + \theta_x^2}{4} = \frac{1}{4}S_x^2. \quad (6.13)\]

### 7 Anisotropic soliton surfaces induced by the LLE

In this section, our aim is to present the anisotropic soliton surfaces induced by the LLE (1.1). To do that, let us recall that the position vector \(r = (r_1, r_2, r_3)\) of the soliton surface satisfies the certain two equations. Note that in this approach, the matrix form of the position vector defines by the Sym-Tafel formula

\[r = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 = \left( \begin{array}{cc} r_3 & r^- \\ r^+ & -r_3 \end{array} \right) = \Phi^{-1}\Phi_\lambda. \]
Hence we get the following two equations:

\[ r_x = Φ^{-1} U_λ Φ, \]
\[ r_t = Φ^{-1} V_λ Φ, \]

where

\[ U_λ = \frac{iρ}{sn^2(λ,k)} [cn(λ,k)dn(λ,k)S_1σ_1 + cn(λ,k)S_2σ_2 + dn(λ,k)S_3σ_3], \]

\[ V_λ = i [(u_2u_3)λS_1σ_1 + (u_1u_3)λS_2σ_2 + (u_1u_2)λS_3σ_3] - i \sum_{a,b,c=1}^3 u_{abc} S_a S_b S_c σ_a. \]

Here we used the following formulas

\[ [sn(λ)]_λ = cn(λ,k)dn(λ,k), \]
\[ [cn(λ)]_λ = -sn(λ,k)dn(λ,k), \]

\[ [dn(λ)]_λ = -k^2 sn(λ,k)cn(λ,k) \]

and

\[ sn^2(λ) + cn^2(λ) = 1, \quad dn^2(λ) + k^2 sn^2(λ) = 1, \quad dn^2(λ) - k^2 cn^2(λ) = k^2. \]

We now can write the first fundamental form of the desired anisotropic soliton surface

\[ I = r_x^2 dx^2 + 2r_x \cdot r_t dx dt + r_t^2 dt^2, \]

where

\[ r_x^2 = \frac{1}{2} tr(r_x^2), \quad r_x \cdot r_t = \frac{1}{2} tr(r_x r_t), \quad r_t^2 = \frac{1}{2} tr(r_t^2). \]

Similarly, we can construct the second fundamental form of the soliton surface corresponding to the LLE. In our case, the well-known Sym-Tafel formula has the form

\[ r = Φ^{-1} Φ_λ = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix}. \]

Using the following expressions

\[ Φ_λ = \begin{pmatrix} φ_{1λ} & -φ_{2λ} \\ φ_{2λ} & φ_{1λ} \end{pmatrix}, \quad \Phi^{-1} = \frac{1}{\det Φ} \begin{pmatrix} φ_1 & φ_2 \\ -φ_2 & φ_1 \end{pmatrix}, \quad \det Φ = |φ_1|^2 + |φ_2|^2, \]

we finally have

\[ r = \frac{1}{\det Φ} \begin{pmatrix} \tilde{φ}_1 φ_{1λ} + \tilde{φ}_2 φ_{2λ} & -\tilde{φ}_1 φ_{1λ} + \tilde{φ}_2 φ_{2λ} \\ -φ_2 φ_{1λ} + φ_1 φ_{2λ} & φ_2 φ_{1λ} + φ_1 φ_{2λ} \end{pmatrix}. \]

Hence for the components of the position vector \( r = (r_1, r_2, r_3) \) we obtain

\[ r^+ = r_1 + ir_2 = \frac{-φ_2 φ_{1λ} + φ_1 φ_{2λ}}{\det Φ}, \quad r^- = \frac{-\tilde{φ}_1 φ_{2λ} + \tilde{φ}_2 φ_{1λ}}{\det Φ}, \quad r_3 = \frac{\tilde{φ}_1 φ_{1λ} + \tilde{φ}_2 φ_{2λ}}{\det Φ} \]

or

\[ r_1 = \frac{-φ_2 φ_{1λ} + φ_1 φ_{2λ} - \tilde{φ}_1 φ_{2λ} + \tilde{φ}_2 φ_{1λ}}{2 \det Φ}, \]
\[ r_2 = \frac{-φ_2 φ_{1λ} + φ_1 φ_{2λ} + \tilde{φ}_1 φ_{2λ} - \tilde{φ}_2 φ_{1λ}}{2i \det Φ}, \]
\[ r_3 = \frac{\tilde{φ}_1 φ_{1λ} + \tilde{φ}_2 φ_{2λ}}{\det Φ}. \]
Let us now construct the soliton surface corresponding to the 1-soliton solution of the LLE (1.1) which we presented in the previous section. In this case, the components of the position vector are given by (7.13)-(7.15), where

$$\phi_1 = c_1 e^{-i(u_3x - 2u_1u_2t)}, \quad \phi_2 = c_2 e^{i(u_3x - 2u_1u_2t)}, \quad \bar{\phi}_1 = \bar{c}_1 e^{-\bar{\chi}}, \quad \bar{\phi}_2 = \bar{c}_2 e^{\bar{\chi}} \quad (7.16)$$

$$\phi_{1\lambda} = -i[u_3\lambda x - 2(u_1u_2)\lambda t]\phi_0, \quad \phi_{2\lambda} = i[u_3\lambda x - 2(u_1u_2)\lambda t]\phi_0. \quad (7.17)$$

Thus in this section we have presented the soliton surface given by the position vector $r$ corresponding to the 1-soliton solution of the LLE.

### 8 Some generalizations of the LLE

There are several integrable and nonintegrable generalizations of the LLE. In this section we will present some of them.

#### 8.1 Landau-Lifshitz-Schrödinger equation

One of generalizations is the Landau-Lifshitz-Schrödinger equation (LLSE) which has the form [78]

$$iS_t + \frac{1}{2}[S, S_{xx} + S_J] + 2Q_x + i[S, F_0] - [Q, SS_x] = 0, \quad Q_t - F_{0x} + [Q, F_0] = 0, \quad (8.1)$$

where

$$F_0 = V_0 + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad V_0 = i\begin{pmatrix} -rq & q_x \\ -r_x & rq \end{pmatrix}, \quad r = \epsilon q. \quad (8.3)$$

In the isotropic case $J_1 = J_2 = J_3$, this LLSE takes the form

$$iS_t + \frac{1}{2}[S, S_{xx}] + 2Q_x + i[S, F_0] - [Q, SS_x] = 0, \quad Q_t - F_{0x} + [Q, F_0] = 0. \quad (8.4)$$

This isotropic LLSE admits the following LR [78]

$$Z_x = U_3Z, \quad (8.6)$$

$$Z_t = V_3Z, \quad (8.7)$$

with the Lax pair

$$U_3 = -i\lambda S + Q, \quad (8.8)$$

$$V_3 = -2i\lambda^2 S + \lambda(SS_x + 2Q) + F_0. \quad (8.9)$$

#### 8.2 Myrzakulov-XCIX equation

Another integrable generalization of the LLE is the Myrzakulov-XCIX (M-XCIX) equation [24]-[25]. The anisotropic M-XCIX equation reads as

$$iS_t + \frac{1}{2}[S, S_{xx}] - 4i\lambda_0 S_x + \frac{1}{\lambda_0 + \omega}[S, W] + \frac{1}{2}[S, S_J] = 0, \quad (8.10)$$

$$iW_x + (\lambda_0 + \omega)[S, W] = 0. \quad (8.11)$$
where \( \omega = \text{const} \), \( S = \Sigma_{j=1}^{3} S_j(x, y, t) \sigma_j \) is a matrix analogue of the spin vector, \( W \) is the potential with the matrix form \( W = \Sigma_{j=1}^{3} W_j(x, y, t) \sigma_j \). In the isotropic case \( J_1 = J_2 = J_3 \) that is the isotropic M-XCIX equation

\[
i S_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{\omega} [S, W] = 0, \tag{8.12}
\]
\[
i W_x + \omega [S, W] = 0, \tag{8.13}
\]
is integrable by the IST. Its LR can be written in the form \([24]-[25]\)

\[
\Phi_x = U_4 \Phi, \tag{8.14}
\]
\[
\Phi_t = V_4 \Phi, \tag{8.15}
\]
where the Lax pair \( U_4 \) and \( V_4 \) have the form

\[
U_4 = -i \lambda S, \tag{8.16}
\]
\[
V_4 = \lambda^2 V_2 + \lambda V_1 + \left( \frac{i}{\lambda + \omega} - \frac{i}{\alpha} \right) W. \tag{8.17}
\]

Here

\[
V_2 = -2i S, \quad V_1 = SS_x. \tag{8.18}
\]

The M-XCIX equation (8.12)-(8.13) is gauge equivalent to the following SMBE \([24]-[25]\)

\[
i q_t + q_{xx} + 2\delta |q|^2 q - 2ip = 0, \tag{8.19}
\]
\[
p_x - 2i \omega p - 2\eta q = 0, \tag{8.20}
\]
\[
\eta_x + \delta (q^* p + p^* q) = 0, \tag{8.21}
\]

where \( q(x, t), p(x, t) \) are complex functions, \( \eta(x, t) \) is a real function, \( \omega \) is the real constant. This system is integrable by the IST. The corresponding LR is given by

\[
\Psi_x = U_5 \Psi, \tag{8.22}
\]
\[
\Psi_t = 2\lambda U_5 \Psi + B \Psi, \tag{8.23}
\]
where \( U_5 \) and \( B \) have the forms

\[
U_5 = -i \lambda \sigma_3 + U_0, \tag{8.24}
\]
\[
B = B_0 + \frac{i}{\lambda + \omega} B_{-1}. \tag{8.25}
\]

Here

\[
U_0 = \left( \begin{array}{cc} 0 & q \\ -q^* & 0 \end{array} \right), \quad B_0 = -i \delta |q|^2 \sigma_3 + i \left( \begin{array}{cc} 0 & q_x \\ \delta q_x^* & 0 \end{array} \right), \quad B_{-1} = \left( \begin{array}{cc} \eta & -p \\ -\delta p^* & -\eta \end{array} \right), \tag{8.26}
\]

where * means a complex conjugate and \( \delta = \pm 1 \), so that \( \delta = +1 \) corresponds to the attractive interaction and \( \delta = -1 \) to the repulsive interaction respectively.

### 8.3 LLE with the self-consistent variable anisotropy

Let us consider the variable anisotropy case that is \( J_j = \frac{1}{\lambda_0 + \omega} K_j(x, t) \). Then the LLE (1.1) takes the form

\[
S_t + S \wedge S_{xx} + S \wedge KS = 0, \tag{8.27}
\]
where \( K(x, t)S = (K_1(x, t)S_1 + K_2(x, t)S_2 + K_3(x, t)S_3) \). This equation is integrable if we additionally demand that the quantity \( KS \) to satisfy some additional equation, say, the following equation

\[
(KS)_x + \mu S \wedge KS = 0. \tag{8.28}
\]

Then the full set of equations has the form

\[
S_t + S \wedge S_{xx} + S \wedge KS = 0, \tag{8.29}
\]

\[
(KS)_x + \mu S \wedge KS = 0. \tag{8.30}
\]

8.4 Integrable LLE with the external magnetic field

The previous examples of the generalized LLE were in the absence of an external magnetic field. We now consider the LLE with the external magnetic field \( H \) which can be written as

\[
S_t + S \wedge S_{xx} + S \wedge JS + S \wedge H = 0, \tag{8.31}
\]

where \( H = (H_1, H_2, H_3) \) is the external magnetic field. We now demand that in this LLE with the external magnetic field, the vector field \( H \) is variable and satisfies the following additional equation

\[
H_x + \mu S \wedge H = 0, \tag{8.32}
\]

where \( \mu = const. \) Thus we obtain the following set of the closed equations

\[
S_t + S \wedge S_{xx} + S \wedge JS + S \wedge H = 0, \tag{8.33}
\]

\[
H_x + \mu S \wedge H = 0, \tag{8.34}
\]

which is, in fact, the LLE with the self-consistent external magnetic field \( H \). This LLE with the self-consistent external magnetic field in the isotropic case reads as

\[
S_t + S \wedge S_{xx} + S \wedge H = 0, \tag{8.35}
\]

\[
H_x + \mu S \wedge H = 0. \tag{8.36}
\]

This equation is integrable.

8.5 Ishimori equation

The famous anisotropic Ishimori equation (IE) has the form \[6\]

\[
S_t - S \wedge (S_{xx} + \alpha^2 S_{yy}) - u_y S_x - u_x S_y - S \wedge JS = 0, \tag{8.37}
\]

\[
u_{xx} - \alpha^2 u_{yy} + 2\alpha^2 S \cdot (S_x \wedge S_y) = 0. \tag{8.38}
\]

The isotropic IE is integrable and reads as

\[
S_t - S \wedge (S_{xx} + \alpha^2 S_{yy}) - u_y S_x - u_x S_y = 0, \tag{8.39}
\]

\[
u_{xx} - \alpha^2 u_{yy} + 2\alpha^2 S \cdot (S_x \wedge S_y) = 0, \tag{8.40}
\]

or

\[
vw_t + v_{xx} - \alpha w_{yy} - \frac{2\bar{w}(w_x^2 - w_y^2)}{1 + |w|^2} + i\beta(u_x w_y - u_y w_x) = 0, \tag{8.41}
\]

\[
u_{xx} + \nu u_{yy} - \frac{8Im(w_x w_y)}{1 + |w|^2} = 0. \tag{8.42}
\]
### 8.6 Myrzakulov-IX equation

The anisotropic Myrzakulov-IX (M-IX) equation is given by

\[
S_t - S \wedge M_1 S + iA_2 S_x + iA_1 S_y - S \wedge JS = 0, \quad (8.43)
\]

\[
M_2 u - 2a^2 S \cdot (S_x \wedge S_y) = 0, \quad (8.44)
\]

where \(M_j\) and \(A_j\) are some operators \[8\]-\[9\]. The isotropic M-IX equation is integrable. Note that the isotropic M-IX equation is equivalent to the following equation \[79\]-\[81\]

\[
iq_t + M_1 q + vq = 0, \quad (8.45)
\]

\[
 ip_t - M_1 p - vp = 0, \quad (8.46)
\]

\[
 M_2 v + 2M_1(pq) = 0. \quad (8.47)
\]

### 8.7 Myrzakulov-VIII equation

The M-IX equation admits some integrable reductions. For example, as \(a = b = -1\) it reduces to the anisotropic Myrzakulov-VIII (M-VIII) equation. The M-VIII reads as \[8\]-\[9\]

\[
iS_t + \frac{1}{2}[S, S_{xx}] + iuS_x + [S, S_J] = 0, \quad (8.48)
\]

\[
u_x + u_y + \frac{1}{4i} tr(S[S_x, S_y]) = 0. \quad (8.49)
\]

Note that the isotropic M-VIII equation is integrable. Its gauge equivalent counterpart has the form \[79\]-\[81\]

\[
iq_t + q_{xx} + vq = 0, \quad (8.50)
\]

\[
 ip_t - p_{xx} - vp = 0, \quad (8.51)
\]

\[
v_x + v_y + 2(pq)y = 0. \quad (8.52)
\]

### 8.8 Myrzakulov-XXXIV equation

The (1+1)-dimensional anisotropic Myrzakulov-XXXIV (M-XXXIV) equation is given by \[8\]-\[9\]

\[
S_t - S \wedge S_{xx} + uS_x - S \wedge JS = 0, \quad (8.53)
\]

\[
u_t + u_x - \beta(S^2_x) = 0. \quad (8.54)
\]

In the isotropic case, the M-XXXIV equation is integrable and is equivalent to the Yajima-Oikawa equation \[9\]

\[
iq_t + q_{xx} - vq = 0, \quad (8.55)
\]

\[
v_t + v_x + \delta(|q|^2)_x = 0. \quad (8.56)
\]

### 8.9 Myrzakulov-I equation

The anisotropic Myrzakulov-I (M-I) equation is given by \[7\]

\[
S_t - S \wedge S_{xy} - uS_x - S \wedge JS = 0, \quad (8.57)
\]

\[
u_x + S \cdot (S_x \wedge S_y) = 0. \quad (8.58)
\]
In the isotropic case, it takes the form
\[ S_t - S \wedge S_{xy} - uS_x = 0, \quad u_x + S \cdot (S_x \wedge S_y) = 0. \] (8.59)

or
\[ iw_t + w_{xy} - 2\bar{w}w_x w_y = 0, \quad u_x + \frac{2i(w_x w_y - w_x w_y)}{(1 + |w|^2)^2} = 0. \] (8.61)

The isotropic M-I equation is integrable by the IST method and equivalent to the following (2+1)-dimensional NLSE
\[ iq_t + q_{xy} - vq = 0, \quad v_x + 2(|q|^2)_y = 0. \] (8.63)

Note that in 1+1 dimension that is when \( y = x \), this NLSE reduces to the usual NLSE
\[ iq_t + q_{xx} + 2|q|^2q = 0. \] (8.65)

8.10 Landau-Lifshitz-Gilbert equation

From the physical point of view, one of very important generalizations of the LLE is the so-called Landau-Lifshitz-Gilbert equation (LLGE). It was proposed in 1955 by T. Gilbert as a modification of the LLE (1.1) to incorporate a damping term. This LLGE is given by [13]
\[ S_t + \beta S \wedge H_{eff} + \alpha S \wedge S \wedge H_{eff} = 0, \] (8.66)

where the vector function \( H_{eff} \) is the effective magnetic field as some kind derivative of the magnetic energy of the material. For more details, we refer to the survey [14] and references therein.

8.11 Landau-Lifshitz-Bloch equation

The Landau-Lifshitz-Bloch equation (LLBE) is one of important generalizations of the LLE. It describes the magnetization dynamics of magnetic particles at high temperatures without the restriction of a fixed magnetization length and thus allows for its longitudinal relaxation. The LLBE is given by (see i.g. [82])
\[ u_t = c_1 u \wedge H_{eff} + c_2 u \wedge (u \wedge H_{eff}) + c_3 \frac{1}{|u|^2}(u \cdot H_{eff})u, \] (8.67)

where \( u \) is the spin polarization, \( c_j \) are constants, \( H_{eff} \) is effective field.

9 LLE as the Madelung equation. Hydrodynamical formulations

To study some properties of the LLE, it is very useful its hydrodynamical form. To derive the hydrodynamical form of the LLE we consider the following Madelung like transformation [14]
\[ S^+ = \sqrt{1 - S^2} e^{-i(\phi - 0.5\pi)}. \] (9.1)
Then the LLE (1.1) takes the form

$$\begin{align*}
\phi_t + \left( \frac{S_{3x}}{1 - S_3^2} \right)_x - \frac{S_3 S_{3x}^2}{(1 - S_3^2)} + \left( \phi_x^2 - J_23 + J_12 \sin^2 \phi \right) S_3 &= 0, \\
S_3 x - (1 - S_3^2) \phi_x + 0.5 J_21 (1 - S_3^2) \sin^2 2\phi &= 0.
\end{align*}$$

(9.2, 9.3)

This is the Madelung equation form of the LLE (1.1). Note that this Madelung form will be essential in the study of solutions of the LLE. It is interesting to present the Madelung equation for the gauge and geometrical equivalent of the LLE, namely, for the NLS like equation (3.9)-(3.11). Let

$$q = \sqrt{\rho e^{i\phi}}.$$

For simplicity, the equation (3.9)-(3.11) we write in the form

$$i q_t + q_{xx} - v q = 0,$$

(9.4)

where

$$v = 2|q|^2 + iF_{12}, \quad F_{12} = -R_{12}/q, \quad F_{21} = -R_{21}.$$  

(9.5)

Then for the equation (3.9)-(3.11), the corresponding Madelung equations takes the form

$$\phi_t + \phi_x^2 + \left( \nu - \frac{\sqrt{\rho}}{2\sqrt{\rho}} \right) = 0, $$

$$\rho_t + (\nu \phi)_x = 0. $$

(9.6, 9.7)

10 Quantum Heisenberg model

It is well-known that the LLE (1.1) is the continuous classical limit of the quantum Heisenberg XYZ model:

$$\hat{H} = -\frac{1}{2} \sum_{j=1}^{N} \left( J_1 \hat{S}_{1j} \hat{S}_{1(j+1)} + J_2 \hat{S}_{2j} \hat{S}_{2(j+1)} + J_3 \hat{S}_{3j} \hat{S}_{3(j+1)} + h \hat{S}_{3j} \right)$$

(10.1)

where $h$ is the external magnetic field, $J_j$ are real-valued coupling constants. Three particular cases of this model: i) if $J_1 = J_2 = J_3$, the model is called quantum Heisenberg XXX model; ii) if $J_1 = J_2 \neq J_3$, it is the quantum Heisenberg XXZ model; iii) if $J_1 \neq J_2 \neq J_3$, this model is called the quantum Heisenberg XYZ model. Note that quantum Heisenberg XYZ model is one of examples of the so-called integrable quantum models. Such integrable quantum models can be solved by the quantum inverse scattering methods and/or by the Bethe ansatz. Finally we present the well-known result, namely, the spin operators $\hat{S}_j$ can be written in terms of auxiliary oscillators $a_j$ and their conjugates $a_j^+$ as

$$\hat{S}_j^+ = a_j, \quad \hat{S} = a_j^+(1 - a_j^+ a_j), \quad \hat{S}_{3j} = \frac{1}{2} - a_j^+ a_j.$$  

(10.2)

11 Dispersionless LLE

Let us present the dispersionless LLE. The vector LLE (1.1) is equivalent to the only one equation for the complex function $S^+(x, t)$. This equation has the form (2.11) that is

$$i S_t^+ = S^+ S_{3xx} - S_{xx}^+ S_3 + J_3 S_3 S^+ - \frac{1}{2} S_3 (I_21 S^+ - J_21 S^-), $$

(11.1)

where $S^- = \bar{S}^+$, $S_3 = \sqrt{1 - |S^+|^2}$. Let us consider the transformation

$$S^+ = \sqrt{\rho} e^{i\phi}.$$  

(11.2)
where \( u = 1 - S_3^2 \) and \( s \) are some real functions, \( \epsilon \) is some real parameter. In our case, the equation (11.1) takes the form

\[
 i\epsilon S_t^+ = \epsilon^2 (S^+ S_{3xx} - S^+_{xx} S_3) + J_3 S_3 S^+ - \frac{1}{2} S_3 (J_{21} S^+ - J_{21} S^-). 
\] (11.3)

For simplicity, let us consider the case \( J_1 = J_2 \) that is the LLE with an easy plan. In this case, the equation (11.3) takes the form

\[
 i\epsilon S_t^+ = \epsilon^2 (S^+ S_{3xx} - S^+_{xx} S_3) + J_3 S_3 S^+ - J_1 S_3 S^+. 
\] (11.4)

Hence and using the transformation (11.2) we obtain the following set of equations

\[
 s_t + \sqrt{1-u}(s_x^2 + J_{31}) = 0, 
\] (11.5)

\[
 u_t + 2\sqrt{1-u}(s_{xx} u + s_x u_x) = 0, 
\] (11.6)

or

\[
 v_t + [\sqrt{1-u}(v^2 + J_{31})]_x = 0, 
\] (11.7)

\[
 u_t + 2(uv)_x \sqrt{1-u} = 0, 
\] (11.8)

where \( v = s_x \). It is the desired dispersionless LLE (dLLE) for the easy plan case. Note that this dLLE is integrable. Finally, we note that the dispersionless isotropic LLE reads as

\[
 v_t + [v^2 \sqrt{1-u}]_x = 0, 
\] (11.9)

\[
 u_t + 2(uv)_x \sqrt{1-u} = 0. 
\] (11.10)

12 Self-similar solution of the LLE

In the theory of the nonlinear differential equations, their self-similar solutions play an important role [83]-[89]. For that reason, in this section we want just briefly mention some basic facts about the self-similar solutions, as example, for the LLGE (for details see e.g. [89]). Note that the LLGE contains the LLE as the particular case. In 1+1 dimension, the LLGE we write as [89]

\[
 S_t - \beta S \wedge S_{xxx} + \alpha S \wedge (S \wedge S_{xx}) = 0. 
\] (12.1)

Recall that the solution of the LLGE is called self-similar, if it satisfies the condition:

\[
 S(x, t) = S(\lambda x, \lambda^2 t), 
\] (12.2)

i.e the LLGE is invariant under this scaling. In [89] were studied the following two types self-similar solutions:

i) the expander self-similar solution

\[
 S(x, t) = f \left( \frac{x}{\sqrt{t-t_0}} \right), 
\] (12.3)

ii) the shrinker self-similar solution

\[
 S(x, t) = f \left( \frac{x}{\sqrt{T-t}} \right). 
\] (12.4)
13 Lattice LLE

Almost all main integrable systems have their mirror side in the form of the lattice or discrete integrable equations [90]-[92]. Here we want briefly present some basic facts of the lattice version of the LLE (LLLE). It has the form (see e.g. [10] and references therein)

\[
\frac{d\Gamma_{\alpha}^{(n)}}{dt} = \{H_{LLLE}, \Gamma_{\alpha}^{(n)}\},
\]

(13.1)

where \(\Gamma_{\alpha}^{(n)}\) are some dynamical variables and (for details see e.g. [10])

\[
H_{LLLE} = \sum_{n=1}^{N} \log h(\Gamma_{\alpha}^{(n)}, \Gamma_{\alpha}^{(n+1)}).
\]

(13.2)

The isotropic LLLE that is the discrete HFE is given by ([90]-[91])

\[
S_{jt} = 2 \left[ \frac{S_{j+1} \land S_{j}}{1 + S_{j+1} \cdot S_{j}} - \frac{S_{j} \land S_{j-1}}{1 + S_{j} \cdot S_{j-1}} \right].
\]

(13.3)

14 Conclusion

In this paper, we have established that the LLE (1.1) is geometrical equivalent to the generalized nonlinear Schrödinger like equation (3.9)-(3.11). When the anisotropy vanishes, from this result follows the well-known results corresponding for the continuous isotropic spin chain, i.e. the HFE (1.2). We have studied the relation between the LLE and the differential geometry of space curves in the local and nonlocal cases. Also the soliton surfaces induced by the LLE are briefly considered using the well-known Sym-Tafel formula.

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