Abstract: In this short note, exploits of constructions of $F$-structures coupled with technology developed by Cheeger-Gromov and Paternain-Petean are seen to yield a procedure to compute minimal entropy, minimal volume, Yamabe invariant and to study collapsing with bounded sectional curvature on inequivalent smooth structures and inequivalent PL-structures within a fixed homeomorphism class. We compute these fundamental Riemannian invariants on all exotic and all fake tori of dimension greater than four as an example. We observe that the minimal volume is not an invariant of the smooth structures on the homeomorphism type of the $n$-torus, yet the Yamabe invariant does discern the standard smooth structure from all the others. We also observe that the fundamental group places no restriction on the vanishing of the minimal volume and collapse with bounded sectional curvature in odd-dimensional manifolds.

1. Introduction and main results

Within a fixed homeomorphism type, we refer as inequivalent smoothings to different smooth structures up to diffeomorphism and as inequivalent PL-structures to different PL-structures up to PL-homeomorphism. The first examples of inequivalent smoothings were provided by Milnor [M56] on the homeomorphism type of the 7-sphere $S^7$. A classical success of surgery theory established the existence of both inequivalent smoothings and inequivalent PL-structures on high-dimensional manifolds [W69]. Inequivalent smoothings and PL-structures on the $n$-torus for were shown to exist by Casson, Wall, and Hsiang-Shaneson [HS69, HSW69], provided $n \geq 5$. They are respectively known as exotic and fake tori. The torus $T^n = S^1 \times \cdots \times S^1$ is called the standard $n$-torus for $n \in \mathbb{N}$. We refer as homotopy $n$-sphere and homotopy $n$-torus to smooth manifolds that are homeomorphic to $S^n$ and $T^n$, respectively.

It is known that inequivalent smoothings need not share certain basic geometric properties with the standard smooth structure. Hitchin has shown in [H74] that certain homotopy spheres do not admit a Riemannian metric of positive scalar curvature, while the round metric on the standard $S^n$ has positive sectional curvature. When it comes to computations of fundamental invariants in Riemannian geometry, Euclidean geometry canonically serves as a basic model. For example, among the many interesting geometric traits of the $n$-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ are the existence of Riemannian metrics of zero sectional curvature, hence the vanishing and realization of its Yamabe invariant (see Section 2.3 for the definition). Gromov’s minimal volume is zero for the torus, and so is its minimal entropy (see Section 2.2 and Section 2.3 for definitions). Moreover, Riemannian manifolds with zero sectional
curvature make prototypes to study collapsing since they observe that collapse by rescaling preserves bounded sectional curvature (see Section 2.2 for definitions). Although these invariants play a fundamental role in Riemannian geometry, their computations are as challenging as they are scarce.

We observe in this note that exploits of \( F \)-structures and results of Gromov [G82], Cheeger-Gromov [CG80], and Paternain-Petean [PP1], among others, yield a procedure to calculate the aforementioned invariants on inequivalent smoothings and PL-structures for manifolds of dimension at least five. This procedure enlarges considerably the set of examples for which the values of the invariants are known. We sample the procedure on every exotic and fake torus of dimension at least five, and on exotic spheres. We show that the minimal volume does not distinguish different smooth nor PL-structures on a homotopy n-torus, while the Yamabe invariant does discern the standard smooth structure. The precise statement is as follows.

**Theorem 1.** Every homotopy n-torus \( M \) with \( n \geq 5 \) admits a polarized \( F \)-structure. Consequently,

\[
\text{MinVol}(M) = 0,
\]

and \( M \) collapses with bounded sectional curvature.

The Yamabe invariant satisfies

\[
\mathcal{Y}(M) = 0,
\]

and it is realized, i.e., there exists a scalar-flat Riemannian metric on \( M \) if and only if \( M \) is diffeomorphic or PL-homeomorphic to the n-torus \( S^1 \times \cdots \times S^1 \).

Theorem 1 can be compared to results in [Be98] and [BaT12]. Bessières has shown that the minimal volume is sensitive to inequivalent smoothings on homeomorphism types of hyperbolic manifolds [Be98]. Baues-Tuschmann have shown that the vanishing of the more restrictive invariant \( D\text{-MinVol} \) does discern the standard smooth structure on the n-torus within a set of inequivalent ones (see Section 2.2 for the definition of the invariant). Theorem 1 says this is not the case for MinVol of any homotopy n-torus. Moreover, we observe the following corollary of Baues-Tuschmann’s work.

**Proposition 1.** Within the homeomorphism type of the n-torus for \( n \geq 5 \) there are smoothings and PL-structures for which \( D\text{-MinVol} \) decreases under finite coverings.

On the other hand, for every choice of smoothing or PL-structure, MinVol is invariant under finite coverings.

Moreover, Theorem 1 answers a question of Baues-Tuschmann [BaT12]; see Remark [3].

Theorem 1 and the results in this note can be generalized to many more inequivalent smoothings and PL-structures on a broader range of homeomorphism types (see Lemma 1, Theorem 6, and Remark 1). Results of Montgomery-Yang [MoY68] and Cheeger-Gromov [CG80] are coupled in Theorem 7 to observe that every smoothing of the 7-sphere has vanishing minimal volume and collapses with bounded sectional curvature. Similar results on other homotopy spheres regarding minimal entropy and collapsing with sectional curvature bounded from below are
given in Section 2.5. Similar computations on homotopy real projective n-spaces are considered in Lemma 1. We point out in Theorem 6 that results of Hitchin [H74] and Petean [P00] imply that every homotopy sphere that does not admit a Riemannian metric of positive scalar curvature has vanishing and unrealized Yamabe invariant. Proposition 4 samples another extension on a smoothing on the complex projective 5-space.

We also show that the fundamental group places no restriction on the vanishing of the minimal volume and collapse with bounded sectional curvature for odd-dimensional manifolds. We also observe a similar statement on even-dimensional manifolds regarding entropy, and collapse with sectional curvature bounded from below.

**Theorem 2.** Let $G$ be a finitely presented group, and let $k$ be an integer greater than one. There exist a closed smooth $(2k+1)$-manifold $X(G)$ and a closed smooth $(2k+2)$-manifold $Y(G)$ with fundamental groups $\pi_1(X(G)) \cong G \cong \pi_1(Y(G))$, and such that both manifolds admit a $T$-structure.

The $T$-structure on $X(G)$ is polarized. Consequently,

$$\text{MinVol}(X(G)) = 0,$$

and $X(G)$ collapses with bounded sectional curvature.

Similarly,

$$h(Y(G)) = 0 = \text{Vol}_K(Y(G)),$$

and $Y(G)$ collapses with sectional curvature bounded from below.

For values $k \geq 4$, there is a $(2k+2)$-manifold with fundamental group $G$ that admits a polarized $T$-structure.

The note is structured as follows. The definitions, machinery and main results that we build upon are recollected in Section 2. Section 2.1 contains the definition of $F$-structures, the principal tool of this paper, as well as several useful constructions. The result used to conclude on the vanishing of the minimal volume and on the collapse, along with their definitions is recalled in Section 2.2. The result that guarantees the vanishing of the minimal entropy is stated in Section 2.3. The vanishing and non-realization of the Yamabe invariant follows from the main results of Section 2.4 and Section 2.5. Existence of $F$-structures on exotic spheres, and on exotic and fake tori are considered in Section 2.6 and Section 2.7. The proofs of Theorem 1 and Theorem 2 are given in Section 3.1.

## 2. Definitions and background results

We collect in this section the definitions and main results used in this manuscript in order to make it as self-contained as possible.

### 2.1. $F$-structures and constructions

The definition of an $F$-structure below was given in [PP09, Section 2] and it is equivalent to the original one introduced by Cheeger-Gromov [GS82, CG86].

**Definition 1.** An $F$-structure on a smooth closed manifold $M$ is given by

1. a finite open cover $\{U_1, \ldots, U_N\}$ of $M$;
2. a finite Galois covering $\pi_i : \tilde{U}_i \to U_i$ with $\Gamma_i$ a group of deck transformations for $1 \leq i \leq N$;
(3) a smooth effective torus action with finite kernel of a $k_i$-dimensional torus
\[ \phi_i : T^{k_i} \to Diff(\tilde{U}_i) \]
for $1 \leq i \leq N$;

(4) a representation $\Phi_i : \Gamma_i \to Aut(T^{k_i})$ such that
\[ \gamma(\phi_i(t)(x)) = \phi_i(\Gamma_i(\gamma)(t))(\gamma x) \]
for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$, and $x \in \tilde{U}_i$;

(5) for any subcollection $\{U_{i_1}, \ldots, U_{i_l}\}$ that satisfies
\[ \tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_l} = \emptyset, \]
the following compatibility condition holds: let $\tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_l}$ be the set of all points $(x_{i_1}, \ldots, x_{i_l}) \in \tilde{U}_{i_1} \times \cdots \times \tilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \cdots = \pi_{i_l}(x_{i_l})$. The set $\tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_l}$ covers $\pi^{-1}(U_{\pi_{i_1}^{-1}(x_{i_1})}) \subset \tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_l}$ for all $1 \leq j \leq l$. It is required that $\phi_i$ leaves $\pi^{-1}(U_{\pi_{i_1}^{-1}(x_{i_1})})$ invariant, and it lifts to an action on $\tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_l}$ such that all lifted actions commute.

(6) An $\mathcal{F}$-structure is called a $T$-structure if the Galois coverings $\pi_i : \tilde{U}_i \to U_i$ in Item (1) can be taken to be trivial for every $i$.

Locally-free torus actions on smooth closed manifolds yield toy-examples of polarized $T$-structures.

**Example 1.** Pseudofree circle actions give rise to polarized $T$-structures. A pseudofree circle action
\[ \Psi : S^1 \times M \to M \]
is said to be pseudofree if it is not free, every orbit is one-dimensional, and if the isotropy group of $\Psi$ is the identity except for isolated exceptional orbits, where it is a finite cyclic group \[ \text{MoY68}. \]
Since all isotropy groups of such an action are discrete, the action is locally-free.

Soma \[ \text{So81}. \]
showed that existence of polarized $T$-structures on 3-manifolds is closed under connected sums, and Gromov \[ \text{G82} \]
(Appendix 2) (cf. \[ \text{CG86} \]
Example A.1) remarked that the result holds in all odd dimensions. With broader generality, Paternain-Petean have shown that the existence of $T$-structures is closed under connected sums of closed manifolds of dimensions greater than two \[ \text{PP03, Theorem 5.9}. \]

**Proposition 2.** Soma, Gromov. Let $X$ and $Y$ be closed smooth $(2k+1)$-manifolds with $k \in \mathbb{N}$ that admit a polarized $T$-structure. Their connected sum $X \# Y$ admits a polarized $T$-structure.

The following proof of Proposition 2 is due to to Paternain-Petean \[ \text{PP03} \]
Proof of Theorem 5.9]; it is a word by word repetition of their argument modulo a small tweak to justify the addition of the adjective 'polarized'. The proof is included for the sake of completeness.

**Proof.** Paternain-Petean’s construction of the polarized $T$-structure on $X \# Y$ is as follows. The connected sum $X \# Y$ is deconstructed into a union of three pieces. Each piece is equipped with a polarized $T$-structure such that the $T$-structures induced on adjacent boundary components are compatible with each other (see Definition 1). The three polarized $T$-structures then glue together to a (global) polarized $T$-structure on $X \# Y$.

Begin by finding embedded solid tori
where $D_x$ and $D_y$ are small $2k$-balls centered at points $x \in X$ and $y \in Y$ respectively. We describe this step of the procedure just for $X$ given that it is the very same for $Y$. If needed, modify the $\mathcal{T}$-structure on $X$ and take a point $x \in X$ contained in a single open set $U_j$ (for a fixed $j$) of the polarized $\mathcal{T}$-structure. Without loss of generality, we can assume that the corresponding torus $T^k_1$ acting on $U_j$ is a circle, and that the point $x$ is contained in a regular orbit. The small $2k$-ball $D_x$ centered at $x$ is taken to be transverse to this circle action. The embedded solid torus $S^1 \times D_x$ is obtained as the union of orbits through $D_x$. At this point of the proof we have located embedded solid tori inside $X$ and $Y$.

The connected sum is to be performed inside these tori. We use the existence of a diffeomorphism between $S^1 \times D_x \# S^1 \times D_y$ and $(S^1 \times D_x \setminus S^{2k-1} \times D^2)$, where the boundary component coming from the $S^{2k-1} \times D^2$ piece that was carved out is identified with the boundary $\partial(S^1 \times D_y) = S^1 \times S^{2k-1}$. To see the decomposition, split the $2k$-disk around $x$ as

\[
D_x = D_{\epsilon_1} \cup (S^{2k-1} \times [\epsilon_1, \epsilon_2]),
\]

where $D_{\epsilon_1}$ for small constants $\epsilon_1, \epsilon_2$. The $2$-disk factor $\{pt\} \times D^2 \subset S^{2k-1} \times D^2$ is a small $2$-disk around a point contained in the middle of $S^{2k-1} \times [\epsilon_1, \epsilon_2]$ and transverse to $\{pt\} \times S^{2k-1} \subset S^1 \times S^{2k-1} \times [\epsilon_1, \epsilon_2]$.

The manifold $X \# Y$ is constructed as the union of the pieces

\[
(X \setminus S^1 \times D_x) \cup S^1 \times D_{\epsilon_1} \cup \cdots \cup S^1 \times D_{\epsilon_{m-1}}
\]

and

\[
Y \setminus S^1 \times D_y
\]

The choices of $\mathcal{T}$-structures are the following. Equip pieces $\mathcal{V}$ and $\mathcal{W}$ with the initial $\mathcal{T}$-structure, which is polarized by hypothesis. Take a free circle action on the $S^{2k-1}$ factor of $S^1 \times S^{2k-1} \times [\epsilon_1, \epsilon_2] \setminus (S^{2k-1} \times D^2)$. This is the point of the proof where the hypothesis on the dimension of the manifolds to be $2k + 1 \geq 3$ is used. This equips piece $\mathcal{X}$ with a polarized $\mathcal{T}$-structure. To see that these structures are compatible on the boundary components, we argue as follows. The action induced on each boundary component of pieces $\mathcal{V}, \mathcal{W}$ and $\mathcal{X}$ by our choices of $\mathcal{T}$-structures pastes together to the canonical action on the circle factor into a $T^2$-action. Hence, the conditions of Definition $\mathcal{I}$ are satisfied, and the connected sum $X \# Y$ has a polarized $\mathcal{T}$-structure. $\square$

We finish the section by pointing out a construction of $\mathcal{S}$-structures on orbit spaces that is useful for the results in Section 2.5 and the proof of Theorem 1.

**Proposition 3.** Let $\tilde{M}$ and $M$ be closed smooth $n$-manifolds and let

\[
\tilde{M} \xrightarrow{pr} M
\]

be a finite covering. Suppose $\tilde{M}$ admits a free transitive torus action

\[
\tilde{\phi} : T^k \to \text{Diff}(\tilde{M})
\]

There is a polarized $\mathcal{S}$-structure on $M$.  

---

\[
S^1 \times D_x \hookrightarrow X \quad \text{and} \quad S^1 \times D_y \hookrightarrow Y,
\]
Proof. The free transitive torus action $\tilde{\phi}$ yields a pure polarized $T$-structure on $\tilde{M}$ with a single element of its covering $V := V(\tilde{y})$ for every $y \in \tilde{M}$. We use this structure to construct a pure polarized $F$-structure on $M$. The covering $\{U_1, \ldots, U_N\}$ in Item (1) of Definition \[ is to be chosen as follows. For a point $x_i \in M$, the set $pr^{-1}(x)$ is discrete. Let $q$ be its cardinality, i.e., the degree of the covering $\tilde{M} \to M$. Denote by $\tilde{x}_{i,l}$ the lifts of $x_i$ to $\tilde{M}$. Take a ball $U_i := B_i(x_i, \epsilon)$ about $x_i$ for an arbitrarily small $\epsilon > 0$. Pick this constant such that the set

$$ pr^{-1}(B_i(x_i, \epsilon)) = \bigcup_{1 \leq l \leq q} \tilde{U}_{i,l} $$

consists of the disjoint union of $q$ open sets, the restriction of the covering map $pr(\tilde{U}_{i,l}) \to U_i$ is a diffeomorphism for an open subset $U_i \subset M$, and such that the overlap

$$ pr^{-1}(B_i(x_i, \epsilon)) \cap pr^{-1}(B_j(x_j, \epsilon)) $$

either empty or path-connected for $x_i, x_j \in M$. The set $\{V \cap \tilde{U}_{i,l}\}$ is discrete and consists of a disjoint union of $q$ elements that we continue to denote by $\tilde{U}_{i,l}$, and that are contained in $\tilde{M}$. Fix a point $x \in M$ and take $x_i \in U_i$ for every open set $\{U_1, \ldots, U_N\}$. By hypothesis, $pr^{-1}(x)$ is a discrete subset. Fix an element $y \in pr^{-1}(x) \subset \tilde{M}$. Equip $M$ with a Riemannian metric $g$, and let $\tilde{g}$ be the metric on $\tilde{M}$ that makes $(\tilde{M}, \tilde{g}) \to (M, g)$ a Riemannian submersion. Let $\alpha_i : [0, 1] \to M$ be a geodesic with initial and end points $\alpha_i(0) = x$ and $\alpha_i(1) = x_i$. There is a geodesic $\tilde{\alpha}_i : [0, 1] \to \tilde{M}$ with $\tilde{\alpha}_i(0) = y$ and $\tilde{\alpha}_i(1) = \tilde{x}_i \in pr^{-1}(x_i)$ that is obtained by lifting $\alpha_i$. In particular $\tilde{x}_i \in \tilde{U}_m$ for some open subset $\tilde{U}_m \subset \tilde{M}$. Pick an element $\tilde{U}_i$ with $pr(\tilde{U}_i) = U_i$ and set $\pi_i := pr|_{\tilde{U}_i}$. In particular, $\pi_i : \tilde{U}_i \to U_i$ is a diffeomorphism. The overlaps $U_i \cap U_j$ are path-connected or empty for arbitrary points $x_i, x_j \in M$. We can define a smooth torus action

$$ \phi_i : T^{k_i} \to Diff(\tilde{U}_i) $$

with finite kernel as in Item (3) of Definition \[ as the restriction of the torus action $\tilde{\phi} : T^k \to Diff(V)$ to the subset $\tilde{U}_i \subset \tilde{M}$ for every $1 \leq i \leq N$. If the intersections $\tilde{U}_i \cap \tilde{U}_j$ and $\tilde{U}_i \cap \tilde{U}_j$ are both nonempty, we have a trivial covering. If the intersection $\tilde{U}_i \cap \tilde{U}_j$ is not empty, yet $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, then there is a Deck transformation $g : M \to M$ such that $g\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$. The corresponding group of Deck transformations on the open set is finite by our assumption. The commutativity condition in Item (4) and invariance of Item (5) of Definition \[ are immediately satisfied. Since the torus action $\phi$ is assumed to be transitive, we have $k_i = k$ for every $i$, and the $F$-structure will be pure. Since its isotropy groups are trivial at every point, the $F$-structure on $M$ will be polarized. Hence, we have constructed a pure polarized $F$-structure on $M$. □

2.2. Collapsing, and minimal volume. Cheeger-Gromov used $F$-structures to study the collapsing of Riemannian manifolds with bounded sectional curvature \[. Before stating their result at the end of this section, we first recall concepts involved. A smooth manifold $M$ collapses with bounded sectional/Ricci/scalar curvature if and only there exists a sequence of Riemannian metrics $\{g_j\}$ for
which the sectional/Ricci/scalar curvature is uniformly bounded, but their volumes \(\{Vol(M, g_j)\}\) approach zero as \(j \to \infty\). Of course, collapse with bounded sectional curvature implies collapse with bounded Ricci curvature, which in turn implies collapse with bounded scalar curvature.

In the same vein, the following two fundamental Riemannian invariants are defined. We denote by \(K_g\) the sectional curvature of a Riemannian metric \(g\), and assume the normalization \(Vol(M, g) = 1\). Define

\[
MinVol(M) := \inf_g \{Vol(M, g) : |K_g| \leq 1\}
\]

and

\[
Vol_K(M) := \inf_g \{Vol(M, g) : K_g \geq -1\}.
\]

The invariant \(MinVol(M)\) is known as the minimal volume and was introduced by Gromov [G82]. A profound utility of \(F\)-structures is observed in the following result.

**Theorem 3.** Cheeger-Gromov [CG86]. The minimal volume of a manifold \(M\) that admits a polarized \(F\)-structure vanishes; so does \(Vol_K(M)\) and \(M\) collapses with bounded sectional curvature.

For a real number \(D > 0\), the more restrictive invariant \(D\)-MinVol\((M)\) of a closed smooth Riemannian manifold \((M, g)\) can be defined as follows. Much like \(MinVol(M)\), it is required that the infimum of \(vol(M, g)\) is to be taken over all metrics \(g\) and additionally requiring that the diameter remains bounded from above by \(D\). As mentioned in the introduction, Baues-Tuschmann have shown that for certain exotic tori, the vanishing of \(D\)-MinVol discerns the standard smooth structure [BaT12].

### 2.3. Minimal entropy, and Yamabe invariant

We now recall the definitions of the remaining Riemannian invariants that we are investigating. The minimal entropy \(h(M)\) is the infimum of the topological entropy of the geodesic flow of a smooth metric \(g\) on \(M\) such that \(Vol(M, g) = 1\). The inequality

\[
[h(M)]^n \leq (n - 1)^n MinVol(M)
\]

is known to hold [PP03 page 417]. It is unknown if the minimal entropy depends on the choice of smooth structure. In this paper, we provide a myriad of examples in terms of homeomorphism types where it does not.

Let us now recall the definition of the Yamabe invariant [B87 S87]. Let

\[
\gamma := [g] = \{ug : M \to \mathbb{R}^+\}
\]

be a conformal class of Riemannian metrics on \((M, g)\). The Yamabe constant of \((M, \gamma)\) is

\[
\mathcal{Y}(M, \gamma) := \inf_{g \in \gamma} \frac{\int_M \text{scal}_g \text{dvol}_g}{(Vol(M, g))^{\frac{n}{n-2}}}.
\]

The Yamabe invariant of \(M\) is defined as

\[
\mathcal{Y}(M) := \sup_{\gamma} \mathcal{Y}(M, \gamma).
\]
The following result allows us to use the existence of a $\mathcal{T}$-structure in order to compute these invariants.

**Theorem 4.** Paternain-Petean [PP03]. The minimal entropy of a closed smooth manifold that admits a $\mathcal{T}$-structure vanishes. If the dimension of the manifold is at least three, then it collapses with bounded scalar curvature, and its Yamabe invariant is nonnegative.

Collapse with bounded scalar curvature and collapse with scalar curvature bounded from below are equivalent for manifolds of dimension at least three [PP03, Proposition 7.1].

### 2.4. Non-realization of Yamabe invariant zero in our examples

Recall that we say that the Yamabe invariant is realized if there exists a Riemannian metric on $M$ with vanishing scalar curvature. The non-realization of the invariant for our examples is a corollary of the following result.

**Theorem 5.** Bieberbach, Schoen-Yau [SY79], Gromov-Lawson [GL80]. There are no Riemannian metrics of positive scalar curvature on a homotopy $n$-torus. Any Riemannian metric of non-negative scalar curvature on a homotopy $n$-torus is flat. In such a case, $M$ is diffeomorphic to $T^n$.

We finish this brief section with a pair of sample results.

**Theorem 6.** Hitchin [H74], Petean [P00]. Let $\Sigma$ be a homotopy $n$-sphere such that $\alpha(\Sigma) \neq 0$. Its Yamabe invariant is

\[(21) \quad \mathcal{Y}(\Sigma) = 0\]

is not realized, i.e., there are no scalar-flat Riemannian metrics on $\Sigma$.

**Proof.** Petean has shown that the Yamabe invariant of a closed simply connected $n$-manifold with $n \geq 5$ is nonnegative [P00]. Hitchin has shown that $\alpha(\Sigma) \neq 0$ implies that $\Sigma$ does not admit a Riemannian metric of positive scalar curvature [H74]. Since the Yamabe invariant of a closed manifold is positive if and only if the manifold admits a Riemannian metric of positive scalar curvature, these results imply the vanishing $\mathcal{Y}(\Sigma) = 0$. To argue that the Yamabe invariant is unrealized, we proceed by contradiction. Suppose there exists a scalar-flat Riemannian metric $g$ on $\Sigma$. Since $\alpha(\Sigma) \neq 0$ by assumption, the homotopy sphere has a non-trivial parallel spinor, and the Riemannian manifold $(\Sigma, g)$ thus has special holonomy [H74]. This is a contradiction, since homotopy spheres have generic holonomy. Hence, $\Sigma$ does not admit a scalar-flat Riemannian metric, and its Yamabe invariant is not realized. \qed

**Proposition 4.** There exists a smoothing $M$ of $\mathbb{CP}^5$ such that

\[(22) \quad h(M) = 0 = \text{Vol}_K(M),\]

and $M$ collapses with sectional curvature bounded from below.

Its Yamabe invariant is

\[(23) \quad \mathcal{Y}(M) = 0,\]

and it is not realized, i.e., there are no scalar-flat Riemannian metrics on $M$. 

Proof. Let $\Sigma$ be a homotopy 10-sphere such that $\alpha(\Sigma) \neq 0$; this sphere was proven to exist in [M65, Ad66]. The smooth manifold $M := \mathbb{C}P^5 \# \Sigma$ is homeomorphic to the complex projective 5-space. There is a nontrivial circle action on $\mathbb{C}P^5$ and on $\Sigma$ (see Proposition 5), and [PP03, Theorem 5.9] implies that there is a $T$-structure on $M$. Theorem 4 implies that the minimal entropy of $M$ is zero, and that it collapses with sectional curvature bounded from below (see [PP03, Theorem B]). In particular, $\mathcal{Y}(M) \geq 0$. Since $\alpha(M) = \alpha(\mathbb{C}P^5 \# \Sigma) = \alpha(\mathbb{C}P^5) + \alpha(\Sigma)$ [H74], it follows that $\mathcal{Y}(M) = 0$. The claim about the nonexistence of scalar-flat Riemannian metrics on $M$ follows from the argument given in the proof of Theorem 6. \qed

2.5. Homotopy $n$-spheres and projective $n$-spaces, and (polarized) $T$-structures. We now collect existence results of $T$-structures on homotopy $n$-spheres, and indicate further constructions of these structures. Existence questions of circle actions on homotopy spheres has been a classical area of research since the late 1960’s see [HsWhs64, MoY68, Sc72] and references therein. A sample result is the following proposition [Sc72].

Proposition 5. Let $\Sigma$ be a homotopy $n$-sphere for $5 \leq n \leq 13$ that that bounds a spin manifold. There is a $T$-structure on $\Sigma$. Consequently, 

$$h(\Sigma) = 0 = Vol_K(\Sigma),$$

and $\Sigma$ collapses with sectional curvature bounded from below.

Since the characteristic numbers of a smooth manifold that admits a polarized $T$-structure vanish, the $T$-structures of Proposition 5 on the even-dimensional homotopy spheres are not polarized. Regarding existence of these structures, we have the following. As it was mentioned in Example 4, a pseudofree circle action on a smooth manifold $M$ gives rise to a $T$-structure that is polarized. Montgomery-Yang [MoY68] have shown that all 28 smoothing on $S^7$ admit such an action. Coupling their work with Theorem 3 yields the following result.

Theorem 7. Montgomery-Yang, Gromov-Cheeger. Every smooth manifold that is homeomorphic to the 7-sphere admits a polarized $T$-structure.

Consequently, every smoothing of $S^7$ has zero minimal volume, and collapses with bounded sectional curvature.

Examples in dimension eleven were constructed by W. C. Hsiang-W. Y. Hsiang [HsWhs64, Theorem 2]. They showed the existence of an exotic 11-sphere on which the circle acts freely. To construct more (polarized) $T$-structures on homotopy spheres consider the following. The group of homotopy spheres that bound parallelizable manifolds are generated by the Kervaire spheres $K^{2n-1}$, which are exotic spheres provided $n + 1$ is not a power of 2. All Kervaire spheres arise as Brieskorn manifolds in the following manner [Br66]. Suppose $n \geq 3$ is an odd integer number, and $d = \pm 3 \mod 8$. Then a homotopy sphere $\Sigma^{2n-1}$ is a real algebraic submanifold of $\mathbb{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0$$

and

$$|z_0|^2 + |z_1|^2 + \cdots |z_n|^2 = 1.$$
A myriad of examples with nontrivial fundamental group arise verbatim. As mentioned in Tuschmann [Tu12], the Brieskorn manifolds that correspond to an even value of $n$ and that have the rational homology of a sphere have a polarized $T$-structure.

We point out that the results in this section apply to more inequivalent smoothings. For example, inequivalent smoothings on high-dimensional spheres form a group under the operation of connected sum of manifolds [KeM63]. Hence, it suffices to equip a homotopy sphere that generates this group with a (polarized) $T$-structure to conclude that all smoothings of such a homotopy sphere have a (polarized) $T$-structure. As it was discussed in Section 2.1, Gromov, and Paternain-Petean have shown that $T$-structures behave well under connected sums. We sample an application to other homeomorphism types in the following lemma.

**Lemma 1.** There are 28 inequivalent smoothings of $\mathbb{R}P^7$ that admit a polarized $F$-structure.

Consequently, the smoothings have zero minimal volume, and they collapses with bounded sectional curvature.

**Proof.** There is a polarized $T$-structure on the real projective 7-space. Consider the connected sum $\mathbb{R}P^7 \# \Sigma^7$, which is homeomorphic to the projective 7-space. Theorems[7] and Proposition[2] imply that any such connected sum admits a polarized $T$-structure. Cheeger-Gromov result now says that its minimal volume is zero and that it collapses with bounded sectional curvature. We can choose homotopy 7-spheres $\Sigma_i^7$ and $\Sigma_j^7$ such that looking at universal covers $S^4 \# \Sigma_i \# \Sigma_i$ and $S^4 \# \Sigma_j \# \Sigma_j$ of two homeomorphic manifolds $\mathbb{R}P^7 \# \Sigma_i^7$ and $\mathbb{R}P^7 \# \Sigma_j^7$, we conclude that they are diffeomorphic if and only if $i = j$. The lemma now follows since there are 28 smoothings on $S^7$.

**Remark 1.** Similar results apply to more homeomorphism types. The inertia group of the complex projective $n$-space is not trivial for certain values of $n$ [Ka68]. Vanishing of the minimal entropy and collapse with sectional curvature bounded from below can be obtained for inequivalent smoothings on the homeomorphism type of $\mathbb{C}P^n$.

2.6. Homotopy $n$-tori and polarized $T$-structures. Hsiang-Wall [HsW69] have shown that a closed $n$-manifold that is homotopy equivalent to the $n$-torus $T^n$ is homeomorphic to it provided $n \geq 5$. A homotopy $n$-torus with a PL-structure is parallelizable, hence smoothable [W99, Chapter 15A]. As it was mentioned in the introduction of this note, Casson, Wall and Hsiang-Shaneson have shown existence of inequivalent smoothings (up to diffeomorphism) and inequivalent PL-structures (up to PL-homeomorphism) on homotopy $n$-tori.

The set $S^{PL}(T^n)$ of equivalence classes up to PL-homeomorphism of closed $n$-dimensional PL-manifolds that are homotopy equivalent to the $n$-torus is in bijective correspondence with $H^3(T^n; \mathbb{Z}/2)$ [W69, HsS69, W99]. There is an isomorphism

$$[T^n, G/PL] \cong \bigoplus_{0 \leq i \leq n} H^i(T^n; \pi_i(G/PL)).$$
where the groups $\pi_i(G/P_L)$ are finite. According to Wall [W69, Corollary] and Hsiang-Shaneson [HsS69, Theorem B], a closed n-dimensional PL-manifold that is homotopy equivalent to the n-torus with $n \geq 5$ is finitely covered by the standard $T^n$. The normal PL-invariants are natural for coverings and any element of the group $H^r(T^n; A)$ with $A$ finite is killed by passing to a finite cover. Coverings corresponding to subgroups of $\pi_1 = \mathbb{Z}^n$ of odd index are fake tori; those that correspond to subgroups of even index are standard. The set of smoothings on a PL-structure class are in bijective correspondence with elements of

$$[T^n, PL/O] \cong \bigoplus_{i \leq n} H^i(T^n; \pi_i(PL/O))$$

and the normal invariants, as in the PL-structures case, are natural for coverings [W99]; indeed, the homotopy groups of the loop space $PL/O$ are finite and any element of the cohomology group of a homotopy n-torus with coefficients in the finite group $\pi_i(PL/O)$ can be killed by passing to a finite cover. We are indebted to Terry Wall for explaining this to us. The key structural result of homotopy tori that we use to prove Theorem 1 and Proposition 1 is the following theorem.

**Theorem 8.** Hsiang-Shaneson [HsS69], Wall [W69, W99]. For $n \geq 5$, every homotopy n-torus is finitely covered by the standard $T^n$.

Before showing that every manifold homeomorphic to the n-torus of dimension at least five admits a polarized $T$-structure, we mention the following two constructions of polarized $T$-structures on exotic tori.

**Example 2.** Let $\Sigma^n$ be a homotopy n-sphere. The connected sum

$$T^n \# \Sigma^n$$

is homeomorphic to the n-torus, and it is diffeomorphic to $T^n$ if and only if $\Sigma^n$ is diffeomorphic to $S^n$, since the inertia group of the n-torus $I(T^n)$ is trivial [Sc71, Section 3]. Recall that the group $I(T^n)$ consists of all exotic n-spheres $\Sigma^n$ such that $T^n \# \Sigma^n$ is orientation-preservingly diffeomorphic to $T^n$. In particular, different smoothings on the n-sphere yield exotic n-tori.

Proposition 2 and Proposition 3 yield two methods to see that any such exotic torus admits a polarized $F$-structure; in several cases a polarized $T$-structure. If the homotopy sphere is known to admit a polarized $F$-structure, Proposition 6 implies that the exotic torus $T^n \# \Sigma^n$ admits a polarized $F$-structure since $T^n$ admits polarized $T$-structures that arise as free torus actions. Theorem 7 yields seven-dimensional examples.

For any homotopy sphere, a finite cover $T^n \# \Sigma^n \# \cdots \# \Sigma^n$ of $T^n \# \Sigma^n$ is diffeomorphic to the standard n-torus [KcM63]. Proposition 8 implies that any such exotic torus admits a polarized $F$-structure, and consequently has zero minimal volume and collapses with bounded sectional curvature by Theorem 8.

**Remark 2.** The previous connected sum construction of manifolds homeomorphic to the n-torus produces inequivalent smoothings but not inequivalent PL-structures. For any homotopy sphere $\Sigma^n$, the manifolds $T^n \# \Sigma^n$ and $T^n$ are PL-homeomorphic by the Alexander trick.

Fake n-tori are constructed as mapping tori

$$M_h := [0, 1] \times T^n/(1, x)(0, h(x))$$
for a diffeomorphism $h : T^n \rightarrow T^n$, as described in Farrell-Gogolev [FG14]. If the diffeomorphism $h$ is not PL pseudo-isotopic to the identity map $id_{T^n}$, then the mapping torus $M_h$ is a fake torus [FG14 Corollary 2.3]. The fake torus $M_h$ is the total space of a $T^n$-bundle over the circle. There is a bijection between the smooth pseudo-isotopy classes of diffeomorphisms $h$ whose induced homomorphisms on fundamental groups satisfy $h_* = id_{\pi_1(T^n)}$, and the smooth structures $\theta$ on a homotopy torus $M$ such that the inclusion map $\sigma : T^n \times \{0\} \hookrightarrow (M \times S^1, \theta)$ is a smooth embedding. The smooth structures correspond to the subgroup of elements $\varphi \in [T^{n+1}, \text{TOP}/\text{O}]$ such that $\varphi \circ \sigma$ is null-homotopic [W99, FG14]. It is straightforward to construct a polarized $T$-structure on $M_h$.

We now proceed to prove that any smooth manifold homeomorphic to a high-dimensional torus can be equipped with a pure polarized $T$-structure.

**Proposition 6.** Every smoothing and every PL-structure on the homeomorphism type of $T^n$ with $n \geq 5$ admits a pure polarized $T$-structure.

**Proof.** The $n$-torus acts on itself by translations transitively and freely. Since every closed smooth manifold homeomorphic to $T^n$ has a finite covering diffeomorphic to the standard torus and $T^n$, the claim follows from Proposition 3. □

3. Proofs

3.1. **Proof of Theorem 1.** Homotopy $n$-tori admit a polarized $\mathcal{F}$-structure by Proposition 6. Their minimal volume is zero and they collapse with bounded sectional curvature by Theorem 3. Their minimal entropy vanishes and their Yamabe invariant is nonnegative by Theorem 4. Any manifold homeomorphic to the $n$-torus is enlargeable in the sense of Gromov-Lawson, and hence does not admit a Riemannian metric of positive scalar curvature [GL80], as stated in Theorem 5 (cf. [SY79]). The Yamabe invariant of a smooth closed manifold $M$ is positive if and only if $M$ admits a Riemannian metric of positive scalar curvature. Hence, we conclude that a homotopy $n$-torus has zero Yamabe invariant. Other than the standard torus, no homotopy $n$-torus admits a scalar-flat Riemannian metric by Theorem 5.

3.2. **Proof of Proposition 1.** This result is a produce of the work of Baues-Tuschmann [BaT12], Theorem 8 and Theorem 1. Indeed, let $M$ be an exotic/fake torus that is not a product of a standard torus and a lower-dimensional exotic/fake tori; exotic tori of this kind are considered in Example 2. Baues-Tuschmann have shown that

\[(31) \quad D - \text{MinVol}(M) > 0.\]

On the other hand,

\[(32) \quad D - \text{MinVol}(T^n) = 0.\]

Since any homotopy $n$-tori is finitely covered by the standard $n$-torus, the corollary follows.

**Remark 3.** On a question of Baues-Tuschmann. It was asked in [BaT12 Question 1.7] if an exotic/fake torus that is not a product of a standard torus and a lower-dimensional exotic/fake torus always has non-vanishing minimal volume. Theorem 1 answers their question in the negative. As we have mentioned in They have shown that the finer invariant $D$-MinVol does discern the standard smooth structure within...
a subset of homotopy tori with inequivalent smooth structures and inequivalent PL-structures. In Theorem 2, it is the realization of the Yamabe invariant that tells the standard smooth structure apart from any other.

3.3. Proof of Theorem 2. The manifolds $X(G)$ and $Y(G)$ are obtained as the union of two pieces glued along a common boundary. Each piece is equipped with a circle action that is compatible with the other along the common boundary, thus they paste together to a $T$-structure. Recall $G$ is an arbitrary finitely presented group. A classical result states the existence of a smooth closed orientable $n$-manifold $M(G)$ whose fundamental group is isomorphic to $G$ for every integer $n \geq 4$ (cf. [D12]).

The choices of pieces and circle actions are the following. The first piece is the product manifold $M(G) \times S^1$ with fundamental group $G \times \mathbb{Z}$, and consider a free circle action $\sigma_1$ on its $S^1$ factor. This gives a polarized $T$-structure on the $(n+1)$-dimensional manifold $M(G) \times S^1$. The second piece is $S^{n-1} \times D^2$ and consider a nontrivial circle action $\sigma_2$ on the sphere $S^{n-1} \times \{d\} \subset S^{n-1} \times D^2$ for $d \in D^2$. Here is the point of the proof where the odd-dimensional case is discerned from the even-dimensional: for even values of $n$, we can pick a free circle action on $S^{n-1}$.

Carve out a tubular neighborhood of $\{x\} \times S^1 \subset M(G) \times S^1$ for a point $x \in M(G)$. This neighborhood is diffeomorphic to $D^n \times S^1$. We construct a closed smooth $(n+1)$-manifold

$$W(G, n+1) := (M(G) \times S^1 \setminus D^n \times S^1) \cup (S^{n-1} \times D^2),$$

where the pieces are identified along their $S^{n-1} \times S^1$ boundaries. The Seifert-van Kampen theorem implies that the fundamental group of $W(G, n)$ is isomorphic to $G$. Indeed, the isotopy class of the loop $\{x\} \times S^1 \subset M(G) \times S^1$ is the representative for the homotopy class of the generator of the infinite cyclic factor in the group of $\pi_1(M(G) \times S^1) = G \times \mathbb{Z}t$. The induced identification from (33) sets $t = 0$ in $\pi_1(W(G, n+1))$ and this group is isomorphic to $G$.

Due to our choices of circle actions, there is a $T^2$-action $(\sigma_2, \sigma_1)$ on the $S^{n-1} \times S^1$ boundary of each piece; each circle action acts on a factor. Again, for $n$ even, the action is free. Therefore, we have constructed a $T$-structure on $W(G, n+1)$ for every value of $n$ and choice of $G$. If the dimension $n+1$ of $W(G, n+1)$ is odd, the structure is polarized.

We now synchronize the notation notation used with the one in the statement of Theorem 2. Recall $k \geq 2$. For even values $n = 2k$, we express the odd-dimension of the manifold as $(n+1) = (2k+1)$ and set $X(G) := W(G, n+1)$. For odd values $n = (2k+1)$, we express $(n+1) = (2k+2)$, and set $Y(G) := W(G, n+1)$.

Finally, for $k \geq 4$, the $(2k+2)$-manifold $M(G) \times (S^3 \times S^3) \times S^2 \times \cdots \times S^2$ admits a polarized $T$-structure that arises from a free circle action on an 3-sphere factor.

4. Acknowledgements

We thank Frank Quinn, Andrew Ranicki, and Terry Wall for helpful e-mail correspondence. We thank Oliver Baues, Fernando Galaz-García, Marco Radeschi, and Wilderlich Tuschmann for interesting conversations that helped us improve this manuscript. We gratefully acknowledge support from the University of Fribourg and the organizers of its Riemannian Topology Seminar 2015 for a very pleasant and productive meeting during which part of this paper was written.
References

[Ad66] J. F. Adams, *On the groups \( J(X) \). IV*, Topology 5 (1966), 21 - 71.

[BaT12] O. Baues and W. Tuschmann, *Seifert fiberings and collapsing infrasolv spaces*, arXiv:1209.2450v1 (2012).

[B87] A. L. Besse, *Einstein manifolds*, Ergeb. der Math. und ihrer Grenz. Springer-Verlag, Berlin - Heidelberg - New York, 1987.

[Be98] A. L. Bessières, *Un théorème de rigidité différentielle*, Comment. Math. Helv. 73 (1998), 443 - 479.

[Br66] E. Brieskorn, *Beispiele zur differentialtopologie von singularitäten*, Invent. Math 2 (1966), 1 - 14.

[CG86] J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded I*, J. Diff. Geom. 23 (1986), 309 - 346.

[DI2] M. Dehn, *Über unendliche diskontinuierliche Gruppen*, Math. Annalen 71 (1912), 16 - 144.

[FG14] F. T. Farrell and A Gogolev, *Examples of expanding endomorphisms on fake tori*, J. Topol. 7 (2014), 805 - 816.

[G82] M. Gromov, *Volume and bounded cohomology*, Publ. Math. Inst. Hautes Etud. Sci. 56 (1982), 1 - 99.

[GL80] M. Gromov and H. B. Lawson, *Spin and scalar curvature in the presence of a fundamental group*, I, Ann. of Math. Vol. 111 (1980), 209 - 230.

[H74] N. Hitchin, *Harmonic spinors*, Advances in Math. 14 (1974), 1 - 55.

[HsWa64] W. C. Hsiang and W. Y. Hsiang, *Some free differentiable actions of \( S^1 \) and \( S^3 \) on 11-spheres*, Q. J. Math 15 (1964), 371 - 374.

[HsSh69] W. C. Hsiang and J. L. Shaneson, *Fake tori, the Annulus conjecture, and the Conjectures of Kirby*, Proc. Nat. Acad. Sci. U. S. A Vol. 62 (1969), 687 - 691.

[HsWa69] W. C. Hsiang and C. T. C. Wall, *On homotopy tori II*, Bull. London Math. Soc. 1 (1969), 341 - 342.

[Ka68] K. Kawakubo, *Inertia groups of low dimensional complex projective spaces and some free differentiable actions on spheres. I*, Proc. Japan Acad. 44 (1968), 873 - 875.

[KeM63] M. A. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. of Math. 77 (1963), 504 - 537.

[M56] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. 64 (1956), 399 - 405.

[M65] J. Milnor, *Remarks concerning spin manifolds*, Differential and Combinatorial Topology, 55 - 62, Princeton Univ. Press, Princeton, NJ, 1968.

[MoYo68] D. Montgomery and C. T. Yang, *Differentiable actions on homotopy seven spheres II*, Proc. Conf. on Transformation Groups (1967), 125 - 134. Springer-Verlag, New York, 1968.

[PP03] G. P. Paternain and J. Petean, *Minimal entropy and collapsing with curvature bounded from below*, Invent. Math. 151 (2003), 415 - 450.

[PP09] G. P. Paternain and J. Petean, *Collapsing manifolds obtained by Kummer-type constructions*, Trans. Amer. Math. Soc. 361 No. 8 (2009), 4077 - 4090.

[P00] J. Petean, *The Yamabe invariant of simply connected manifolds*, J. Reine Angew. Math. 523 (2000), 225 - 231.

[S87] R. Schoen, *Variational theory for the total scalar functional for Riemannian metrics and related topics*, Springer Lect. Notes Math. 1365 (1987), 120 - 154.

[SY79] R. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), 159 - 183.

[Sc71] R. Schultz, *On the inertia group of a product of spheres*, Trans. Amer. Math. Soc. 156 (1971), 137 - 153.

[Sc72] R. Schultz, *Circle actions on homotopy spheres bounding plumbing manifolds*, 36 (1972), 297 - 300.

[Sm62] S. Smale, *On the structure of 5-manifolds*, Ann. of Math. 75 Vol. 2 (1962), 38 - 46.

[So81] T. Soma, *The Gromov invariant of links*, Invent. Math. 64 (1981), 445 - 454.

[Tu12] W. Tuschmann, *Collapsing and almost nonnegative curvature in Global Differential Geometry*, C. Bär et al (eds.) pp. 93 - 106, Springer Proceedings in Mathematics 17, Springer Verlag 2012.

[W69] C. T. C. Wall, *On homotopy tori and the annulus theorem*, Bull. London Math. Soc. 1 (1969), 95 - 97.
[W99] C. T. C. Wall, *Surgery on compact manifolds* Second Ed. Edited and with a foreword by A. A. Ranicki. Mathematical Surveys and Monographs, 69. Amer. Math. Soc., Providence, RI, 1999. xvi + 203 pp.

*Scuola Internazionale Superiore di Studi Avanzati, Via Bonomea 265, 34136, Trieste, Italy*

*E-mail address: rtorres@sissa.it*