Minimum degree condition for spanning generalized Halin graphs

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Abstract. A spanning tree with no vertices of degree 2 is called a Homeomorphically irreducible spanning tree (HIST). Based on a HIST embedded in the plane, a Halin graph is formed by connecting the leaves of the tree into a cycle following the cyclic order determined by the embedding. Both of the determination problems of whether a graph contains a HIST or whether a graph contains a spanning Halin graph are shown to be NP-complete. It was conjectured by Albertson, Berman, Hutchinson, and Thomassen in 1990 that every surface triangulation of at least four vertices contains a HIST (confirmed). And it was conjectured by Lovász and Plummer that every 4-connected plane triangulation contains a spanning Halin graph (disproved). Balancing the above two facts, in this paper, we consider generalized Halin graphs, a family of graph structures which are “stronger” than HISTs but “weaker” than Halin graphs in the sense of their construction constraints. To be exact, a generalized Halin graph is formed from a HIST by connecting its leaves into a cycle. Since a generalized Halin graph needs not to be planar, we investigate the minimum degree condition for a graph to contain it as a spanning subgraph. We show that there exists a positive integer \( n_0 \) such that any 3-connected graph with \( n \geq n_0 \) vertices and minimum degree at least \( \frac{2n + 3}{5} \) contains a spanning generalized Halin graph. As an application, the result implies that under the same condition, the graph \( G \) contains a wheel-minor of order at least \( \frac{n}{2} \). The minimum degree condition in the result is best possible.

Keywords. Homeomorphically irreducible spanning tree; Halin graph; Hamiltonian cycle

1 Introduction

A tree with no vertex of degree 2 is called a homeomorphically irreducible tree (HIT), and a spanning tree with no vertex of degree 2 is a homeomorphically irreducible spanning tree (HIST). A Halin graph, constructed by Halin in 1971 [9], is a graph formed from a plane embedding of a HIST by connecting its leaves into a cycle following the cyclic order determined by the embedding. In 1990, Albertson, Berman, Hutchinson, and Thomassen [1] showed that it is NP-complete to determine whether a graph contains a HIST. However, for special graph classes such as triangulations of surfaces, they conjectured that every triangulation of a surface with at least 4 vertices contains a HIST. The conjecture was confirmed in [6]. It was shown by Horton, Parker, and Borie [10] that it
is NP-complete to determine whether a graph contains a (spanning) Halin graph. Again, restricted to triangulations, Lovász and Plummer [14] conjectured that every 4-connected plane triangulation contains a spanning Halin graph. But the conjecture was disproved recently [5]. Since a Halin graph possesses many hamiltonian properties (e.g., see [3, 7, 4]), it seems that a graph has to have very “good properties” in order to contain a Halin graph as a spanning subgraph. For this reason, by relaxing on the planarity requirement, we define a \textit{generalized Halin graph} as a graph formed from a HIST by connecting its leaves into a cycle, and we study sufficient conditions for implying the containment of a spanning generalized Halin graph in a given graph.

Compared to Halin graphs, generalized Halin graphs are less studied. Kaiser et al. in [11] showed that a generalized Halin graph is \textit{prism Hamiltonian}; that is, the Cartesian product of a generalized Halin graph and $K_2$ is hamiltonian. Since a tree with no degree 2 vertices has more leaves than the non-leaves, a generalized Halin graph contains a cycle of length at least half of its order. Also, one can notice that by contracting the non-leaves of the underlying tree of a generalized Halin graph into a single vertex, a wheel graph is resulted with the contracted vertex as the hub, where a minor of a graph is obtained from the graph by deleting edges/contracting edges, or deleting vertices. Therefore, a generalized Halin graph contains a wheel-minor of order at least half of its order. The investigation on the properties of generalized Halin graphs is not of the interest of this paper. Instead, in this paper, we show the following two results.

**Theorem 1.1.** It is NP-complete to determine whether a graph contains a spanning generalized Halin graph.

**Theorem 1.2.** There exists a positive integer $n_0$ such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n + 3)/5$ contains a spanning generalized Halin graph. The result is best possible in the sense of the connectivity and minimum degree constraints.

Since a generalized Halin graph of order $n$ contains a wheel-minor of order at least $n/2$, we get the following corollary.

**Corollary 1.1.** There exists a positive integer $n_0$ such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n + 3)/5$ contains a wheel-minor of order at least $n/2$.

For notational convenience, for a graph $T$, we denote by $L(T)$ the set of degree 1 vertices of $T$ and $S(T) = V(T) - L(T)$. Also we abbreviate \textit{spanning generalized Halin graph} as \textit{SGHG} in what follows, and denote a generalized Halin graph as $H = T \cup C$, where $T$ is the underlying HIST of $H$ and $C$ is the cycle spanning on $L(T)$. The remaining of the paper is organized as follows. In Section 2, we prove Theorem 1.1 and show the sharpness of Theorem 1.2. In Section 3, we introduce some notations and lemmas, which are used in the proof of Theorem 1.2. We then proof Theorem 1.2 in Section 4.
2 Proof of Theorem 1.1 and the sharpness of Theorem 1.2

Proof of Theorem 1.1. It was shown by Albertson et al. [1] that it is NP-complete to decide whether a graph contains a HIST, and by the definition, a generalized Halin graph contains a HIST. Hence, we see that the problem of deciding whether an arbitrary graph contains an SGHG is in NP. To show the problem is NP-complete we assume the existence of a polynomial algorithm to test for an SGHG and use it to create a polynomial algorithm to test for a hamiltonian path between two vertices in an arbitrary graph. The decision problem for such hamiltonian paths is a classic NP-complete problem [8].

Let G be a graph and x, y ∈ V(G). We want to determine whether there exists a hamiltonian path connecting x and y. We first construct a new graph G′ and show that G contains a hamiltonian path between x and y if and only if G′ contains a HIST (the proof of this part is the same as the proof of Albertson et al. in [1]). Then based on G′, we construct a graph G″ and show that G′ contains a HIST if and only if G″ contains an SGHG.

Let \( \{z_1, z_2, \ldots, z_t\} = V(G) - \{x, y\} \). Then G′ is formed by adding new vertices \( \{z'_1, z'_2, \ldots, z'_t\} \) and new edges \( z_i z'_i : 1 ≤ i ≤ t \). It is clear that if P is a hamiltonian path between x and y, then \( P \cup \{z_i z'_i : 1 ≤ i ≤ t\} \) is a HIST of G′. Conversely, let T be a HIST of G′. Since \( 1 ≤ d_T(z'_i) ≤ d_G(z'_i) = 1 \), we get \( d_T(z'_i) = 1 \) for each i. Since \( N_G(z'_i) = \{z_i\} \) and T is a HIST, we have \( d_T(z_i) ≥ 3 \). Hence \( T - \{z'_1, z'_2, \ldots, z'_t\} \) is a tree with leaves possibly in \( \{x, y\} \). Since each tree has at least 2 leaves and a tree with exactly two leaves is a path, we conclude that \( T - \{z'_1, z'_2, \ldots, z'_t\} \) is a path between x and y.

Then based on G′, we construct a graph G″. First we add new vertices \( \{z''_1, z''_2, z''_3 : 1 ≤ i ≤ t\} \). Then we add edges \( \{z'_i z''_1, z'_i z''_2, z'_i z''_3, z'_1 z'_2, z'_1 z'_3 : 1 ≤ i ≤ t\} \). Finally, we connect all vertices in \( \{x, y\} \cup \{z'_1, z'_2, z'_3 : 1 ≤ i ≤ t\} \) into a cycle C″ such that \( \{z''_1 z''_2, z''_2 z''_3, z''_1 z''_3 : 1 ≤ i ≤ t\} \subseteq E(C″) \). If T′ is a HIST of G′, then T″ := T′ ∪ \( \{z''_1 z''_1, z''_2 z''_2, z''_3 z''_3 : 1 ≤ i ≤ t\} \) is a HIST of G″ and T″ ∪ C″ is an SGHG of G″. Conversely, suppose H = T ∪ C is an SGHG of G″. We claim that C = C″. This in turn gives that T = T″ and therefore T″ - \( \{z''_1, z''_2, z''_3 : 1 ≤ i ≤ t\} \) is a HIST of G′. To show that C = C″, we first show that \( z''_i ∈ L(T) \) for each i. Suppose on the contrary and assume, without loss of generality, that \( z''_i ∈ S(T) \). Then as \( N_{G''}(z''_i) = \{z''_1, z''_1, z''_1\} \), we get \( \{z''_2 z''_1, z''_2 z''_1, z''_2 z''_3\} \subseteq E(T) \). Since T is acyclic, \( z''_1 z''_1, z''_1 z''_1 \notin E(T) \). This in turn shows that \( \{z''_1, z''_1, z''_1\} \subseteq L(T) \). However, \( \{z''_1 z''_1, z''_1 z''_1, z''_1 z''_1\} \) forms a component of T, showing a contradiction. Then we show that \( z''_i, z''_i ∈ L(T) \) for each i. Suppose on the contrary and assume, without loss of generality, that \( z''_i ∈ S(T) \). By the previous argument, we have \( z''_i ∈ L(T) \). Then \( z''_i, z''_i ∈ L(T) \) as \( z''_i \) is on C and \( z''_i \) and \( z''_i \) are the only two neighbors of \( z''_i \) which can be on the cycle C. As \( d_{G''}(z''_1) = 3 \) and \( \{z''_1 z''_1\} \subseteq N_{G''}(z''_1) \), \( z''_1 z''_1, z''_1 z''_1 \subseteq E(T) \). Since \( z''_1 ∈ L(T) \) and \( z''_1, z''_1 ∈ L(T) \), we get \( z''_1 z''_1, z''_1 z''_1, z''_1 z''_1 \notin E(T) \). Since \( d_{G''}(z''_1) = d_{G''}(z''_1) = 3 \), we have \( z''_1 z''_1, z''_1 z''_1, z''_1 z''_1 ∈ E(C) \). However, \( z''_1 z''_1, z''_1 z''_1, z''_1 z''_1 \) forms a triangle but \( |V(C)| ≥ 4 \), showing a contradiction. So we have shown that \( \{z''_1, z''_2, z''_3 : 1 ≤ i ≤ t\} \subseteq L(T) \). This indicates that in the tree \( T - \{z''_1, z''_2, z''_3 : 1 ≤ i ≤ t\} \), each vertex \( z''_i \) has degree 1 and no vertices of degree 2. Hence
$T - \{z_{i1}', z_{i2}', z_{i3}' : 1 \leq i \leq t\}$ is a HIST of $G'$. 

Combining the arguments in the two paragraphs above, we see that $G$ has a hamiltonian path between $x$ and $y$ if and only if $G''$ has an SGHG. Hence a polynomial SGHG-tester becomes a polynomial path-tester.

Since a generalized Halin graph is 3-connected, the connectivity requirement in Theorem 1.2 is necessary. To show that the minimum degree requirement is best possible, we show the following proposition.

**Proposition 1.** Let $G(A, B) = K_{a,b}$ be a complete bipartite graph with $|A| = a$ and $|B| = b$. Then $G(A, B)$ has no HIST $T$ with $|L(T) \cap A| = |L(T) \cap B|$ if $b > \frac{3(a-1)}{2}$.

If a bipartite graph $G(A, B)$ contains an SGHG $H = T \cup C$, then $|L(T) \cap A| = |L(T) \cap B|$. Thus, by Proposition 1, it is easy to see that the complete bipartite graphs $K_{a,b}$ with $b = \frac{3a-1}{2}$ when $a$ is odd and $b = \frac{3a-2}{2}$ when $a$ is even does not have an SGHG. Let $n = a + b$. By direct computation, we get $\delta(K_{a,b}) = \frac{2n+1}{5}$ when $b = \frac{3a-1}{2}$ and $\delta(K_{a,b}) = \frac{2n+2}{5}$ when $b = \frac{3a-2}{2}$. We now prove Proposition 1.

**Proof of Proposition 1.** Suppose on the contrary that $G(A, B)$ contains a HIST $T$ such that $|L(T) \cap A| = |L(T) \cap B|$. Then

$$|S(T) \cap B| - |S(T) \cap A| = |B| - |L(T) \cap B| - (|A| - |L(T) \cap A|)$$

$$= |B| - |A| > \frac{3(a-1)}{2} - a = \frac{a-3}{2}.$$

Since $G(A, B)$ is bipartite and $T$ is a HIST of $G(A, B)$, we have $|S(T) \cap A| \geq 1$. Thus, from the inequalities above, we obtain $|S(T) \cap B| > (a-1)/2$. Since $T$ is a HIST, we have $d_T(y) \geq 3$ for each $y \in S(T) \cap B$. Let $E_B = \{e \in E(T) : e$ is incident to a vertex in $S(T) \cap B\}$. Denote by $T'$ the subgraph of $T$ induced on $E_B$. Notice that $T'$ is a forest of at least $3|S(T) \cap B|$ edges. Hence $T'$ has at least $3|S(T) \cap B| + 1$ vertices. As $T'$ is a bipartite graph with one partite set as $S(T) \cap B$, and another as a subset of $A$, we conclude that $|V(T) \cap A| = |V(T)| - |S(T) \cap B| \geq 2|S(T) \cap B| + 1$. Since $|S(T) \cap B| > (a-1)/2$, we then have $|V(T) \cap A| > a$. This gives a contradiction to the assumption $|A| = a$.

### 3 Notations and Lemmas

We consider in this paper simple and finite graphs only. Given a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively, and by $e(G)$ the size of $G$. Let $S \subseteq V(G)$ and $v \in V(G)$. Denote by $G[S]$ the subgraph of $G$ induced on $S$, and denote by $\Gamma_G(v, S)$ the set of neighbors of $v$ in $S$, and $deg_G(v, S) = |\Gamma_G(v, S)|$. When $S = V(G)$, we only write $\Gamma_G(v)$ and $deg_G(v)$. For two subsets $U_1, U_2 \subseteq V(G)$, let $\delta_G(U_1, U_2) = \min\{deg_G(u_1, U_2) : u_1 \in U_1\}$ and
\[ \Delta_G(U_1, U_2) = \max \{ \deg_G(u_1, U_2) : u_1 \in U_1 \} \]. Denote by \( E_G(U_1, U_2) \) the set of edges with one end in \( U_1 \) and the other in \( U_2 \), the cardinality of \( E_G(U_1, U_2) \) is denoted by \( e_G(U_1, U_2) \). Let \( u, v \in V(G) \) be two vertices. We write \( u \sim v \) if \( u \) and \( v \) are adjacent. A path connecting \( u \) and \( v \) is called a \((u, v)\)-path. If \( G \) is a bipartite graph with partite sets \( A \) and \( B \), we denote \( G \) by \( G(A, B) \) for specifying the two partite sets. A matching in \( G \) is a set of independent edges; a \( \wedge \)-matching is a set of vertex-disjoint copies of \( K_{1,2} \); and a claw-matching is a set of vertex-disjoint copies of \( K_{1,3} \). The set of degree 2 vertices in a \( \wedge \)-matching is called the center of the \( \wedge \)-matching; and the set of degree 3 vertices in a claw-matching is called the center of the claw-matching. A cycle \( C \) in a graph \( G \) is dominating if \( G - V(C) \) is an edgeless graph.

The Regularity Lemma of Szemerédi [18] and Blow-up lemma of Komlós et al. [12] are main tools in our proof of Theorem 1.2. For any two disjoint non-empty vertex-sets \( A \) and \( B \) of a graph \( G \), the density of \( A \) and \( B \) is the ratio \( d(A, B) := \frac{e(A, B)}{|A||B|} \). Let \( \varepsilon \) and \( \delta \) be two positive real numbers. The pair \((A, B)\) is called \( \varepsilon \)-regular if for every \( X \subseteq A \) and \( Y \subseteq B \) with \(|X| > \varepsilon|A| \) and \(|Y| > \varepsilon|B|\), \(|d(X, Y) - d(A, B)| < \varepsilon \) holds. In addition, if \( \deg(a, B) > \delta|B| \) for each \( a \in A \) and \( \deg(b, A) > \delta|A| \) for each \( b \in B \), we say \((A, B)\) an \((\varepsilon, \delta)\)-super regular pair.

**Lemma 3.1 (Regularity lemma-Degree form [18]).** For every \( \varepsilon > 0 \) there is an \( M = M(\varepsilon) \) such that if \( G \) is any graph with \( n \) vertices and \( d \in [0,1] \) is any real number, then there is a partition of the vertex set \( V(G) \) into \( l + 1 \) clusters \( V_0, V_1, \ldots, V_l \), and there is a spanning subgraph \( G' \subseteq G \) with the following properties.

- \( l \leq M \);
- \( |V_0| \leq \varepsilon n \); all clusters \( |V_i| = |V_j| \leq \lceil \varepsilon n \rceil \) for all \( 1 \leq i \neq j \leq l \);
- \( \deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n \) for all \( v \in V(G) \);
- \( e(G'[V_i]) = 0 \) for all \( i \geq 1 \);
- all pairs \((V_i, V_j) \) \((1 \leq i < j \leq l)\) are \( \varepsilon \)-regular, each with a density either 0 or greater than \( d \).

**Lemma 3.2 (Blow-up lemma-weak version [12]).** Given a graph \( R \) of order \( r \) and positive parameters \( \delta, \Delta \), there exists a positive \( \varepsilon = \varepsilon(\delta, \Delta, r) \) such that the following holds. Let \( n_1, n_2, \ldots, n_r \) be arbitrary positive integers and let us replace the vertices \( v_1, v_2, \ldots, v_r \) with pairwise disjoint sets \( V_1, V_2, \ldots, V_r \) of sizes \( n_1, n_2, \ldots, n_r \) (blowing up). We construct two graphs on the same vertex set \( V = \bigcup V_i \). The first graph \( K \) is obtained by replacing each edge \( v_i v_j \) of \( R \) with the complete bipartite graph between the corresponding vertex sets \( V_i \) and \( V_j \). A sparser graph \( G \) is constructed by replacing each edge \( v_i v_j \) arbitrarily with an \((\varepsilon, \delta)\)-super regular pair between \( V_i \) and \( V_j \). If a graph \( H \) with \( \Delta(H) \leq \Delta \) is embeddable into \( K \) then it is already embeddable into \( G \).

**Lemma 3.3 (Blow-up lemma-strengthened version [12]).** Given \( c > 0 \), there are positive numbers \( \varepsilon = \varepsilon(\delta, \Delta, r, c) \) and \( \gamma = \gamma(\delta, \Delta, r, c) \) such that the Blow-up lemma in the equal size case (all \(|V_i| \) are the same) remains true if for every \( i \) there are certain vertices \( x \) to be embedded into \( V_i \) whose images are a priori restricted to certain sets \( C_x \subseteq V_i \) provided that

(i) each \( C_x \) within a \( V_i \) is of size at least \( c|V_i| \);
The number of such restrictions within a $V_i$ is not more than $\gamma|V_i|$. 

We will use both the weak and strengthened versions of Blow-up lemma in our proof.

Besides the above two lemmas, we also need the two lemmas below regarding regular pairs.

Lemma 3.4. If $(A, B)$ is an $\varepsilon$-regular pair with density $d$, then for any $A' \subseteq A$ with $|A'| > \varepsilon|A|$, there are at most $\varepsilon|B|$ vertices $b \in B$ such that $\deg(b, A') \leq (d - \varepsilon)|A'|$.

Lemma 3.5 (Slicing lemma). Let $(A, B)$ be an $\varepsilon$-regular pair with density $d$, and for some $\nu > \varepsilon$, let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \nu|A|$, $|B'| \geq \nu|B|$. Then $(A', B')$ is an $\varepsilon'$-regular pair of density $d'$, where $\varepsilon' = \max \{\varepsilon/\nu, 2\varepsilon\}$ and $d' > d - \varepsilon$.

The following two results on hamiltonicity are used in finding cycles in the proofs.

Lemma 3.6 ([17]). If $G$ is a graph of order $n$ satisfying $d(x) + d(y) \geq n + 1$ for every pair of nonadjacent vertices $x, y \in V(G)$, then $G$ is hamiltonian-connected.

Lemma 3.7 ([15]). Let $G$ be a balanced bipartite graph with $2n$ vertices. If $d(x) + d(v) \geq n + 1$ for any two non-adjacent vertices $x, y \in V(G)$, then $G$ is hamiltonian.

4 Proof of Theorem 1.2

Given $0 \leq \beta < \alpha < 1$, we define the two extremal cases with parameters $\alpha$ and $\beta$ as follows.

Extremal Case 1. There exists a partition of $V(G)$ into $V_1$ and $V_2$ such that $|V_1| \geq (2/5 - 4\beta)n$ and $d(V_1, V_2) < \alpha$. Furthermore, $\deg(v_1, V_2) \leq 2\beta n$ for each $v_1 \in V_1$.

Extremal Case 2. There exists a partition of $V(G)$ into $V_1$ and $V_2$ such that $|V_1| > (3/5 - \alpha)n$ and $d(V_1, V_2) \geq 1 - 3\alpha$. Furthermore, $\deg(v_1, V_2) \geq (2n + 3)/5 - 2\beta n$ for each $v_1 \in V_1$.

Then Theorem 1.2 is shown through the following three theorems.

Theorem 4.1 (Non-extremal Case). For every $\alpha > 0$, there exists $\beta > 0$ and a positive integer $n_0$ such that if $G$ is a 3-connected graph with $n \geq n_0$ vertices and $\delta(G) \geq (2n + 3)/5 - \beta n$, then $G$ contains an SGHG or $G$ is in one of the two extremal cases.

Theorem 4.2 (Extremal Case 1). Suppose that $0 < \beta < \alpha < 1$ and $n$ is a sufficiently large integer. Let $G$ be a 3-connected graph on $n$ vertices with $\delta(G) \geq (2n + 3)/5$. If $G$ is in Extremal Case 1, then $G$ contains an SGHG.

Theorem 4.3 (Extremal Case 2). Suppose that $0 < \beta < \alpha < 1$ and $n$ is a sufficiently large integer. Let $G$ be a 3-connected graph on $n$ vertices with $\delta(G) \geq (2n + 3)/5$. If $G$ is in Extremal Case 2, then $G$ contains an SGHG.

We show Theorems 4.1-4.3 separately in the following three subsections.
4.1 Proof of Theorem 4.1

We fix the following sequence of parameters,

\[ 0 < \varepsilon \ll d \ll \beta \ll \alpha < 1, \]  \hspace{1cm} (1)

and specify their dependence as the proof proceeds. We let \( \beta \ll \alpha \) be the same \( \alpha \) and \( \beta \) as defined in the two extremal cases. Then we choose \( d \ll \beta \). Finally we choose

\[ \varepsilon = \min \left\{ \frac{1}{4} \varepsilon \left( \frac{d}{2} + \left\lceil \frac{2}{d^3} \right\rceil, 2, \frac{d}{2} \right), \frac{1}{9} \varepsilon \left( \frac{d}{2} \left\lceil \frac{3}{d^3} \right\rceil, 3 \right), \frac{1}{4} \varepsilon \left( \frac{d}{2} \left( 2, \frac{d}{2} \right) \right) \right\}, \]  \hspace{1cm} (2)

where \( \varepsilon \left( \frac{d}{2} + \left\lceil \frac{2}{d^3} \right\rceil, 3 \right) \) follows from the definition of the \( \varepsilon \) in the weak version of the Blow-up lemma and \( \varepsilon \left( \frac{d}{2}, \left\lceil \frac{3}{d^3} \right\rceil, 3 \right) \) and \( \varepsilon \left( \frac{d}{2}, 2, \frac{d}{2} \right) \) follow from the definition of the \( \varepsilon \) in the strengthened version of the Blow-up lemma. Choose \( n \) to be sufficiently large. In the proof, we omit non-necessary ceiling and floor functions.

Let \( G \) be a graph of order \( n \) such that \( \delta(G) \geq (2n + 3)/5 - \beta n \) and suppose that \( G \) is not in any of the two extremal cases. Applying the regularity lemma to \( G \) with parameters \( \varepsilon \) and \( d \), we obtain a partition of \( V(G) \) into \( l + 1 \) clusters \( V_0, V_1, \ldots, V_l \) for some \( l \leq M = M(\varepsilon) \), and a spanning subgraph \( G' \) of \( G \) with all described properties in Lemma 3.1 (the Regularity lemma). In particular, for all \( v \in V \),

\[ \deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n \geq (2/5 - \beta - d - \varepsilon)n \]

\[ \geq (2/5 - 2\beta)n \]  \hspace{1cm} (3)

and

\[ e(G') \geq e(G) - \frac{(d + \varepsilon)}{2} n^2 \geq e(G) - dn^2, \]

by using \( \varepsilon < d \).

We further assume that \( l = 2k \) is even; otherwise, we eliminate the last cluster \( V_l \) by removing all the vertices in this cluster to \( V_0 \). As a result, \( |V_0| \leq 2\varepsilon n \) and

\[ (1 - 2\varepsilon)n \leq lN = 2kN \leq n, \]  \hspace{1cm} (4)

here we assume that \( |V_i| = N \) for \( i \geq 1 \).

For each pair \( i \) and \( j \) with \( 1 \leq i < j \leq l \), we write \( V_i \sim V_j \) if \( d(V_i, V_j) \geq d \). We now consider the reduced graph \( G_r \), whose vertex set is \( \{1, 2, \ldots, l\} \), and two vertices \( i \) and \( j \) are adjacent if and only if \( V_i \sim V_j \). We claim that \( \delta(G_r) \geq (2/5 - 2\beta)l \). Suppose not, and let \( i_0 \in V(G_r) \) such that \( \deg(i_0, V(G_r)) < (2/5 - 2\beta)l \). Then, for the corresponding cluster \( V_{i_0} \) we have \( e_{G'}(V_{i_0}, V(G') - V_{i_0}) < |V_{i_0}|(2/5 - 2\beta)lN \). On the other hand, by using (3), we have \( e_{G'}(V_{i_0}, V(G') - V_{i_0}) \geq |V_{i_0}|(2/5 - 2\beta)n \). As \( lN \leq n \) from (4), we obtain a contradiction. The rest of the proof consists of the following steps.
Step 1. Show that $G_r$ contains a dominating cycle $C$ and there is a $\land$-matching in $G_r$ with all vertices in $V(G_r) - V(C)$ as its center. We distinguish two cases in Step 1, and each of the other steps will be separated into two cases correspondingly.

Case A. $C = X_1Y_1X_2Y_2 \cdots X_tY_t$ is an even cycle for some $t \leq k$.

Case B. $C = X_0X_1Y_1X_2Y_2 \cdots X_tY_t$ is an odd cycle for some $t < k$.

Notice that in Case B there is at least one vertex in $V(G_r) - V(C)$ by the assumption that $|V(G_r)| = l$ is even. In what follows, if we denote a vertex of $G_r$ by a capital letter, it means either a vertex of $G_r$ or the corresponding cluster in $G$, but the exact meaning will be clear from the context. For $1 \leq i \leq t$, we call $X_i$ and $Y_i$ the partners of each other, and write as $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

Since $C$ is not necessarily hamiltonian in $G_r$, we need to take care of the clusters of $G$ which are not represented on $C$. For each vertex $F \in V(G_r) - V(C)$, we partition the corresponding cluster $F$ into two small clusters $F_1$ and $F_2$ such that $-1 \leq |F_1| - |F_2| \leq 1$. We call each $F_1$ and $F_2$ a half-cluster. Then we group all the original clusters and the partitioned clusters into pairs $(A, B)$ and triples $(C, D, F)$ with $F$ as a half-cluster such that each pair $(A, B)$ and $(C, D)$ is still $\varepsilon$-regular with density $d$ and the pair $(D, F)$ is $2.1\varepsilon$-regular with density $d - \varepsilon$. Having the cluster groups like this, in the end, we will find “small” HITs within each pair $(A, B)$ or among each triple $(C, D, F)$.

Step 2. For each $1 \leq i \leq t - 1$, initiate two independent edges connecting $Y_i$ and $X_{i+1}$. In Case A, also initiate two independent edges connecting $X_1$ and $Y_t$; and in Case B, initiate two independent edges connecting the clusters in each pair of $X_0$ and $X_1$, and $X_0$ and $Y_t$.

Step 3. Make each regular pair in the new grouped pairs and triples given in Step 1 super-regular.

Step 4. Construct HITs covering all vertices in $V_0$ using vertices from the super-regular pairs obtained from Step 3, and obtain new super-regular pairs.

Step 5. Apply the Blow-up lemma to find a HIT between a super-regular pair resulted from Step 4 or among a triple $(A, B, F)$, where both $(A, F)$ and $(A, B)$ are super-regular pairs resulted from Step 4, and $F$ is a half cluster. In addition, in the construction, for each triple $(A, B, F)$, we require the HIT to use as many vertices as possible from $F$ as non-leaves.

Step 6. Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two disjoint paths covering all the leaves. Then connect all the HITs into a HIST of $G$ using edges guaranteed by the regularity and connect the disjoint paths into a cycle using the edges initiated in Step 2. The union of the HIST and the cycle gives an SGHG of $G$.

We now give details of each step. The assumption that $G$ is not in any of the two extremal cases leads to the following claim, which will be used in Step 1.

Claim 4.1. Each of the following holds for $G_r$. 

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(a) $G_r$ contains no cut-vertex set of size at most $3\beta l$;
(b) $G_r$ contains no independent set of size more than $(3/5 - \alpha/2)l$.

**Proof.** (a) Suppose instead that $G_r$ contains a vertex-cut $W$ of size at most $3\beta l$. As $\delta(G_r) \geq (2/5 - 2\beta)l$, then each component of $G_r - W$ has at least $(2/5 - 3\beta)l$ vertices. Let $U$ be the vertex set of one of the components of $G_r - W$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. We see that $|A|, |B| \geq (2/5 - 3\beta)lN \geq (2/5 - 4\beta)n$, and since $e(G) \leq e(G') + dn^2$, we have

$$e_G(A, B) \leq e_{G'}(A, B) + dn^2 \leq |W||A| + dn^2 \leq \beta lN(3/5 + 3\beta)lN + dn^2 \leq (3\beta/5 + 3\beta^2 + d)n^2 \quad \text{ (as } |A| \leq (3/5 + 3\beta)lN \text{ and } ln \leq n)$$

$$\leq \frac{25}{3}(3\beta/5 + 3\beta^2 + d)|A||B| \quad \text{ (since } |A||B| \geq 3n^2/25)$$

$$\leq \alpha|A||B| \quad \text{ (provided that } \frac{25}{3}(3\beta/5 + 3\beta^2 + d) < \alpha).$$

This shows that $d(A, B) < \alpha$. Since $deg_G(u, V(G_r) - U) = deg_G(u, W) \leq 3\beta l$ for each $u \in U$, we see that $deg_G(a, B) \leq 3\beta lN + (d + \varepsilon)n \leq 2\beta n$ for each $a \in A$ provided that $d + \varepsilon \leq \beta$. However, the above argument shows that $G$ is in Extremal Case 1, showing a contradiction.

(b) Suppose instead that $G_r$ contains an independent set $U$ of size larger than $(3/5 - \alpha/2)l$. Let $U' = V(G_r) - U$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. Then $|A| \geq (3/5 - \alpha/2)lN \geq (3/5 - \alpha)n$. For each vertex $v \in A$, since $deg_G(v, A) \leq deg_G(v, A) + (d + \varepsilon)n \leq \beta n$, we have $deg_G(v, B) \geq (2n + 3)/5 - \beta n - \beta n \geq (2n + 3)/5 - 2\beta n$. This gives that

$$d(A, B) \geq \frac{(2/5 - 2\beta)n}{|B|} \geq \frac{(2/5 - 2\beta)n}{(2/5 + \alpha)n} \geq 1 - 3\alpha,$$

provided that $\beta \leq \alpha/10 + 3\alpha^2/2$. We see that $G$ is in Extremal Case 2. \qed

**Step 1.** Show that $G_r$ contains a dominating cycle $C$, and there is a $\wedge$-matching in $G_r$ with all vertices in $V(G_r) - V(C)$ as its center.

We need some results on longest cycles and paths as follows.

**Lemma 4.1** ([16]). Let $G$ be a 2-connected graph on $n$ vertices with $\delta(G) \geq (n + 2)/3$. Then every longest cycle in $G$ is a dominating cycle.

**Lemma 4.2** ([2]). Let $G$ be a 2-connected graph on $n$ vertices with $\delta(G) \geq (n + 2)/3$. Then $G$ contains a cycle of length at least $\min\{n, n + (\delta(G) - \alpha(G))\}$, where $\alpha(G)$ is the size of a largest independent set in $G$.

**Lemma 4.3** ([13]). If $G$ is a 3-connected graph of order $n$ such that the degree sum of any four independent vertices is at least $3n/2 + 1$, then the number of vertices on a longest path and that on a longest cycle differs at most by 1.

By (a) of Claim 4.1, $G_r$ is $3\beta l$-connected. Since $n = Nl + |V_0| \leq (l + 2)\varepsilon n$, we get $l \geq 1/\varepsilon - 2$. Since $1/\varepsilon - 2 \geq 3/\beta$ (provided that $\beta \geq 3\varepsilon/(1 - 2\varepsilon)$), we then have $3\beta l \geq 3$. So $G_r$ is 3-connected.
By Claim 4.1 (b), $G_r$ has no independent set of size more than $(3/5 - \alpha/2)l$. Notice that $\delta(G_r) \geq (2/5 - 2\beta)l > (l + 2)/3$. Applying Lemma 4.1 and Lemma 4.2 on $G_r$, we see that there is a cycle $C$ in $G_r$ which is longest, dominating, and has length at least $(4/5 + \alpha/2 - 2\beta)l$. Let $W = V(G_r) - V(C)$. In Case B, we order and label the vertices of $C$ such that $X_0$ is adjacent to a vertex, say $Y_0 \in W$ (recall that $W \neq \emptyset$ in this case). We fix $(X_0, Y_0)$ as a pair at the first place $(X_0Y_0 \in E(G_r))$, as cluster in $G$, $(X_0, Y_0)$ is an $\varepsilon$-regular pair with density $d$). Let

$$W' = \begin{cases} W, & \text{if in Case A;} \\ W - \{Y_0\}, & \text{if in Case B.} \end{cases}$$

We have $|W'| \leq (1/5 - \alpha/2 + 2\beta)l$ if in Case A and $|W'| \leq (1/5 - \alpha/2 + 2\beta)l - 1$ if in Case B. So $2|W'| \leq (2/5 - \alpha + 4\beta)l < (2/5 - 2\beta)l$ (provided that $\beta < \alpha/6$) if in Case A and $2|W'| \leq (2/5 - \alpha + 4\beta)l - 2 < (2/5 - 2\beta)l - 1$ (provided that $\beta < \alpha/6$) if in Case B. Thus there is a $\wedge$-matching centered in all vertices in $W'$; furthermore, if in Case B, we can choose the matching such that $X_0$ is not covered by it. Let $M_\wedge$ be such a matching. For a vertex $X \in W'$, denote by $M_\wedge(X)$ the two vertices from $V(C)$ to which $X$ is adjacent in $M_\wedge$. Then we have two facts about vertices in $M_\wedge(X)$.

**Fact 1.** Let $X \in W'$. Then the two vertices in $M_\wedge(X)$ are non-consecutive on $C$. (By the assumption that $C$ is longest.)

**Fact 2.** Let $X \neq Y \in W'$. Then no two vertices from $M_\wedge(X) \cup M_\wedge(Y)$ are adjacent on $C$. (By applying Lemma 4.3.)

For a complete bipartite graph, if it contains an SGHG, then the ratio of the cardinalities of the two partite sets should be greater than $2/3$ as shown in Proposition 1. Since a longest dominating cycle in $G_r$ is not necessarily hamiltonian, we need to take care of the clusters of $G$ which are not represented by the vertices on $C$. One possible consideration is that for each $F \in V(G_r) - V(C)$, suppose $F$ is adjacent to $A \in V(C)$, recall $P(A)$ is the partner of $A$. Then as clusters, we consider the bipartite graph of $G$ with partite sets $A$ and $P(A) \cup F$. However, $|A|/|P(A) \cup F|$ is about $1/2$, which is less than $2/3$. For this reason, we partition $F \in V(G_r) - V(C)$ into two parts to attain the right ratio in the corresponding bipartite graphs. Suppose $M_\wedge(F) = \{D_1, D_2\} \subseteq V(C)$. As a cluster of $G$, we partition $F$ into $F_1$ and $F_2$ arbitrarily such that

$$|F_1| = \left\lfloor \frac{|F|}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad |F_2| = \left\lceil \frac{|F|}{2} \right\rceil = \left\lceil \frac{N}{2} \right\rceil.$$

We call each $F_i$ a half-cluster of $G$. Then we create two pairs $(D_i, F_i)$, and call $D_i$ the dominator of $F_i$, and $F_i$ the follower of $D_i$, and $(D_i, F_i)$ a DF-pair, for $i = 1, 2$. We have the following fact about a DF-pair.

**Fact 3.** Each DF-pair $(D, F)$ is $2.1\varepsilon$-regular with density at least $d - \varepsilon$. (By Slicing lemma.)

Also, by Fact 1 and Fact 2, if $D \in V(C)$ is a dominator, then $P(D)$, the partner of $D$, is not a dominator for any followers. As $X_0 \notin V(W')$, we know that $X_0$ is not a dominator for any half-clusters. We group the clusters and half-clusters of $G$ into $H$-pairs and $H$-triples in a way below.
For each pair \((X_i, Y_i)\) on \(C\), if \(\{X_i, Y_i\} \cap V(M_\lambda) = \emptyset\), we take \((X_i, Y_i)\) as an H-pair. Otherwise, \(\{|X_i, Y_i\} \cap V(M_\lambda)| = 1\) by Fact 1 and Fact 2. Since there is no difference for the proof for the case that \(X_i \in V(M_\lambda)\) or the case that \(Y_i \in V(M_\lambda)\), throughout the remaining proof, we always assume that \(Y_i \in V(M_\lambda)\) if \(\{X_i, Y_i\} \cap V(M_\lambda) \neq \emptyset\). In this case, there is a unique half-cluster \(F\) with \(Y_i\) as its dominator. Then we take \((X_i, Y_i, F)\) as an H-triple. We assign \((X_0, Y_0)\) as an H-pair.

**Step 2.** Initiating connecting edges.

Given an \(\varepsilon\)-regular pair \((A, B)\) of density \(d\) and a subset \(B' \subseteq B\), we say a vertex \(a \in A\) typical to \(B'\) if \(\deg(a, B') \geq (d - \varepsilon)|B'|\). Then by the regularity of \((A, B)\), the fact below holds.

**Fact 4.** If \((A, B)\) is an \(\varepsilon\)-regular pair, then at most \(\varepsilon|A|\) vertices of \(A\) are not typical to \(B' \subseteq B\) whenever \(|B'| > \varepsilon|B|\).

For each \(1 \leq i \leq t - 1\), choose \(y_i^* \in Y_i\) typical to both \(X_i\) and \(X_{i+1}\), and \(y_i^{**} \in Y_i\) typical to each of \(X_i\), \(X_{i+1}\), and \(\Gamma(y_i^*, X_i)\). Correspondingly, choose \(x_{i+1}^* \in \Gamma(y_i^*, X_{i+1})\) typical to \(Y_{i+1}\), and \(x_{i+1}^{**} \in \Gamma(y_i^{**}, X_{i+1})\) typical to both \(Y_{i+1}\) and \(\Gamma(x_{i+1}^*, Y_{i+1})\). For \(i = t\), we choose \(y_t^*\) and \(y_t^{**}\) the same way as for \(i < t\), but if in Case A, choose \(x_t^* \in \Gamma(y_t^*, X_1)\) typical to \(Y_1\), and \(x_t^{**} \in \Gamma(y_t^{**}, X_1)\) typical to both \(Y_1\) and \(\Gamma(x_t^*, Y_1)\); and if in Case B, choose \(x_0^* \in \Gamma(y_t^*, X_0)\) typical to \(X_1\), and \(x_0^{**} \in \Gamma(y_t^{**}, X_0)\) typical to both \(X_1\) and \(\Gamma(x_0^*, X_1)\). Furthermore, in Case B, we choose \(y_{t+1}^* \in X_0\) typical to both \(Y_0\) and \(X_1\), and \(y_{t+1}^{**} \in X_0\) typical to each of \(Y_0\), \(X_1\), and \(\Gamma(y_{t+1}^*, Y_0)\). Correspondingly, choose \(x_1^* \in \Gamma(y_{t+1}^*, X_1)\) typical to \(Y_1\) and \(x_1^{**} \in \Gamma(y_{t+1}^{**}, X_1)\) typical to both \(Y_1\) and \(\Gamma(x_1^*, Y_1)\). Additionally, we choose \(y_0^* \in \Gamma(y_{t+1}^*, Y_0)\) such that \(y_0^*\) is typical to \(X_0\), and choose \(y_0^{**} \in \Gamma(y_{t+1}^{**}, Y_0)\) such that \(y_0^{**}\) is typical to \(X_0\). Notice that by the choice of these vertices above, we have the following.

\[
\begin{aligned}
&\left\{\begin{array}{l}
y_i^* x_{i+1}^*, y_i^* x_{i+1}^{**} \in E(G), \\
x_1 y_i^*, x_1 x_i^* y_i^* \in E(G), \\
x_0 y_t^*, x_0^* x_t^*, x_1 y_{t+1}^*, x_1 x_{t+1}^*, y_0 y_{t+1}^*, y_0^* y_{t+1}^{**} \in E(G),
\end{array}\right. \\
&\text{for } 1 \leq i \leq t - 1; \\
&\text{in Case A;}
\end{aligned}
\]

By Fact 4, for each \(0 \leq i \leq t\), we have \(|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)|, |\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2 N\), and \(|\Gamma(y_{t+1}^*, Y_0) \cap \Gamma(y_{t+1}^{**}, Y_0)| \geq (d - \varepsilon)^2 N\).

**Step 3.** Super-regularizing the regular pairs in each H-pair and H-triple given in Step 1.

For each \(0 \leq i \leq t\), if \((X_i, Y_i)\) is an H-pair, let

\[X_i' = \{x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N\} \quad \text{and} \quad Y_i' = \{y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N\}.\]

By Fact 4, we have \(|X_i'|, |Y_i'| \geq (1 - \varepsilon)N\). Recall that \(x_i^*, x_i^{**} \in X_i\) and \(y_i^*, y_i^{**} \in Y_i\) are the initiated vertices in Step 2. For \(1 \leq i \leq t\), if \(|X_i' - \{x_i^*, x_i^{**}\}| \neq |Y_i' - \{y_i^*, y_i^{**}\}|\), say \(|X_i' - \{x_i^*, x_i^{**}\}| > |Y_i' - \{y_i^*, y_i^{**}\}|\), we then remove \(|X_i' - \{x_i^*, x_i^{**}\} - |Y_i' - \{y_i^*, y_i^{**}\}|\) vertices out from \(X_i' - \{x_i^*, x_i^{**}\}\), and denote the remaining set still as \(X_i'\). Denote \(Y_i' - \{y_i^*, y_i^{**}\}\) still as \(Y_i'\). We see that \(|X_i'| = |Y_i'|\).

As \(|Y_i'| \geq (1 - \varepsilon)N\) (to be precise, the lower bound should be \((1 - \varepsilon)N - 2\), however, the constant 2 can be made vanished by adjusting the \(\varepsilon\) factor, we ignore the slight different of the \(\varepsilon\)-factor here),
we have that $|X_i \cup Y_i - (X'_i \cup Y'_i)| \leq 2\varepsilon N$. For $i = 0$, if $|X'_i - \{x^*_i, x^{**}_i, y^{\ast}_t, y^{**}_t\} \neq |Y'_i - \{y^*_i, y^{**}_i\}|$, say $|X'_i - \{x^*_i, x^{**}_i, y^{\ast}_t, y^{**}_t\}| > |Y'_i - \{y^*_i, y^{**}_i\}|$, then we remove $|X'_i - \{x^*_i, x^{**}_i, y^{\ast}_t, y^{**}_t\}| - |Y'_i - \{y^*_i, y^{**}_i\}|$ vertices from $X'_i = \{x^*_i, x^{**}_i, y^{\ast}_t, y^{**}_t\}$ and denote the remaining set still as $X'_i$. Denote $Y'_i = \{y^*_i, y^{**}_i\}$ still as $Y'_i$. We see that $|X'_i| = |Y'_i|$. We call the resulting H-pairs super-regularized H-pairs. By Slicing lemma (Lemma 3.5) and the definitions of $X'_i, Y'_i$, we see that

**Fact 5.** Each super-regularized H-pair $(X'_i, Y'_i)$ is a $(2\varepsilon, d - 2\varepsilon)$-super-regular pair.

For each H-triple $(X_i, Y_i, F)$, by Fact 3, $(Y_i, F)$ is $2.1\varepsilon$-regular with density at least $d - \varepsilon$. Let

$$X'_i = \{x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N\},$$

$$Y'_i = \{y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N, \deg(y, F) \geq (d - 3.1\varepsilon)|F|\},$$

$$F' = \{f \in F : \deg(f, Y_i) \geq (d - 3.1\varepsilon)N\}.$$

Recall that $x^*_i, x^{**}_i \in X_i$ and $y^*_i, y^{**}_i \in Y_i$ are the initiated vertices in Step 2. We remove $x^*_i, x^{**}_i$ out from $X'_i$, and remove $y^*_i, y^{**}_i$ out from $Y'_i$. Still denote the resulted clusters as $X'_i$ and $Y'_i$, respectively. Remove $\lfloor d^3 N \rfloor$ vertices out from $F$, which consists of all vertices in $F - F'$ and any $\lfloor d^3 N \rfloor - |F - F'|$ vertices from $F'$ (we need to increase the ratio $|Y'_i|/|X'_i \cup F'|$ a little as later on we may use vertices in $Y'_i$ in constructing HITs covering vertices in $V_0$). Denote the resulting set still by $F'$. Then we see that $|X'_i| \geq (1 - \varepsilon)N$, $|Y'_i| \geq (1 - 3.1\varepsilon)N$, and $|F'| \geq (1 - 2.1\varepsilon)|F| - d^3 N \geq (1 - 2.1\varepsilon - 2d^3)N$. We call the resulted H-triples super-regularized H-triples. By the Slicing Lemma and the definitions above, the following is true.

**Fact 6.** For each super-regularized H-triple $(X'_i, Y'_i, F')$, $(X'_i, Y'_i)$ is a $(2\varepsilon, d - 3.1\varepsilon)$-super-regular, and $(Y'_i, F')$ is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$-super-regular.

Let $V^1_0$ be the union of the set of vertices from each $(X_i \cup Y_i - (X'_i \cup Y'_i)) - \{x^*_i, x^{**}_i, y^*_i, y^{**}_i\}$ (the set of vertices exists only if in Case B), where $(X_i, Y_i)$ is an H-pair, and let $V^2_0$ be the union of the set of vertices from each $(X_i \cup Y_i - (X'_i \cup Y'_i)) - \{x^*_i, x^{**}_i, y^*_i, y^{**}_i\}$, where $(X_i, Y_i, F)$ is an H-triple. Notice that for each H-pair $(X_i, Y_i)$, we have $|X_i \cup Y_i - (X'_i \cup Y'_i)| \leq 2\varepsilon N$; and for each H-triple $(X_i, Y_i, F)$, we have $|X_i - X'_i| \leq \varepsilon N$, $|Y_i - Y'_i| \leq (\varepsilon + 2.1\varepsilon)N$, and $|F - F'| \leq d^3 N$. Hence by using the facts that $|W'| \leq (1/5 - \alpha/2 + 2\beta)t, t = 1/2$, and $Nl \leq n$ from inequality (4), we get

$$|V^1_0| + |V^2_0| \leq 2\varepsilon Nl/2 + 2(1/5 - \alpha + 2\beta)(d^3 N + 2.1\varepsilon N) \leq 2d^3 Nl/5 + 2\varepsilon Nl \leq 2d^3 n/5 + 2\varepsilon n.$$

Let $V'_0 = V_0 \cup V^1_0 \cup V^2_0$. Then

$$|V'_0| \leq 2\varepsilon n + 2d^3 n/5 + 2\varepsilon n \leq d^3 n/2$$ (provided that $\varepsilon \leq d^3/40$).

**Step 4.** Construct small HITs covering all vertices in $V'_0$.

Consider a vertex $x \in V'_0$ and a cluster or a half-cluster $A$, we say that $x$ is adjacent to $A$, denoted by $x \sim A$, if $\deg(x, A) \geq (d - \varepsilon)|A|$. We call $A$ the partner of $x$. 

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Claim 4.2. For each vertex \( x \in V'_0 \), there is a cluster or a half-cluster \( A \) such that \( x \sim A \), where \( A \) is not a dominator, and we can assign all vertices in \( V'_0 \) to their partners which are not dominators such that each of the cluster or half-cluster is used by at most \( \frac{d^2N}{20} \) vertices from \( V'_0 \).

**Proof.** Suppose we have found partners for the first \( n < d^3n/2 \) (recall that \( |V'_0| \leq d^3n/2 \)) vertices of \( V'_0 \) such that no cluster or half-cluster is used by at most \( \frac{d^2N}{20} \) vertices. Let \( \Omega \) be the set of all clusters and half-clusters that are used exactly by \( d \) vertices. Then

\[
\frac{d^2N}{20} |\Omega| \leq m < d^3n/2 \leq d^3(2kN + 2\varepsilon n)/2 \\
\leq d^3kN + d^3 \frac{2kN}{1 - 2\varepsilon},
\]

by inequality (4). Therefore,

\[
|\Omega| \leq \frac{20d^3k}{d^2} + \frac{20d^3l}{d^2(1 - 2\varepsilon)} \\
\leq 10dl + 40dl \quad \text{(provided that } 1 - 2\varepsilon \geq 1/2 \text{)} \\
\leq \beta l \quad \text{(provided that } 50d \leq \beta \text{)}.
\]

Consider now a vertex \( v \in V'_0 \) not having a partner found so far. Let \( \mathcal{U} \) be the set of all non-dominator clusters and half-clusters adjacent to \( v \) not contained in \( \Omega \). We claim that \( |\mathcal{U}| \geq (\alpha - 7\beta)l \).

To see this, we first observe that any vertex \( v \in V'_0 \) is adjacent to at least \( (\alpha - 6\beta)l \) non-dominator clusters and half-clusters. For instead, as \( v \) may adjacent to \( 2|\mathcal{W}'| \) dominators, vertices in \( V'_0 \), or clusters \( A \) with \( deg(v, A) < (d - \varepsilon)|A| \), we have

\[
(2/5 - \beta)n \leq \deg_G(v) < (\alpha - 6\beta)lN + (2/5 + 4\beta - \alpha)lN + d^3n/2 + (d - \varepsilon)lN \\
\leq (2/5 - 2\beta + d^3/2 + d - \varepsilon)n \\
< (2/5 - 3\beta/2)n \quad \text{(provided that } d - \varepsilon + d^3/2 < \beta/2 \text{)},
\]

showing a contradiction. Since \( |\Omega| \leq \beta l \), we have \( |\mathcal{U}| \geq (2\alpha - 7\beta)l \). □

Now for each non-dominator cluster \( A \) (\( A \) is either a cluster \( X'_i, Y'_i \), or a half cluster \( F'_i \)), let \( I(A) \) be the set of vertices from \( V'_0 \) such that each of them has \( A \) as its partner. By Claim 4.2, we have \( |I(A)| \leq \frac{d^2N}{20} \).

We need three operations below for constructing small HITs covering vertices in \( V'_0 \).

**Operation I** Let \( (A, B) \) be an \((\varepsilon', \delta)\)-super-regular pair, and \( I \) a set of vertices disjoint from \( A \cup B \). Suppose that (i) \( \deg(x, B) \geq d'|B| > \varepsilon'|B| \) and \( \deg(x, B) \geq d'|B| \geq 3|I| \) for any \( x \in I \); (ii) \( (\delta - \varepsilon')d'|B| \geq 3|I| \); (iii) \( (\delta - \varepsilon')|A| > |I| \); and (iv) \( \delta|A| > 4|I| \). Then we can do the following operations on \((A, B)\) and \( I \).

Let \( I = \{x_1, x_2, \ldots, x_{|I|}\} \). We first assume that \( |I| \geq 2 \).

Since \((A, B)\) is \((\varepsilon', \delta)\)-super-regular, for each \( v \in \Gamma(x_i, B) \), \( |\Gamma(v, A)| \geq \delta |A| \). By condition (i), we have \( |\Gamma(x_i, B)| > \varepsilon'|B| \) for each \( i \). Applying Fact 4, we then know that there are at least
\((\delta - \varepsilon')|A| > |I|\) vertices from \(\Gamma(v, A)\) typical to \(\Gamma(x_{i+1}, B)\) for each \(1 \leq i \leq |I| - 1\). That is, there exists \(A_1 \subseteq \Gamma(v, A)\) with \(|A_1| \geq (\delta - \varepsilon')|A| > |I|\) such that for each \(a_1 \in A_1\), \(|\Gamma(a_1, \Gamma(x_{i+1}, B))| \geq (\delta - \varepsilon')d'|B| \geq 3|I|\). As \(\deg(x, B) \geq d'|B| \geq 3|I|\) for any \(x \in I\) and \((\delta - \varepsilon')d'|B| \geq 3|I|\), combining the above argument, we know there is a claw-matching \(M_I\) from \(I\) to \(B\) centered in \(I\) such that one vertex from \(\Gamma(x_i, V(M_I))\) and one vertex from \(\Gamma(x_{i+1}, V(M_I))\) have at least \((\delta - \varepsilon')|A| > |I|\) common neighbors in \(A\). Let \(x_{i1}, x_{i2}, x_{i3}\) be the three neighbors of \(x_i\) in \(M_I\) (in fact in \(B\)) and suppose that \(|\Gamma(x_{i3}, A) \cap \Gamma(x_{i+1}, A)| \geq |I|\). For \(1 \leq i \leq |I| - 1\), we then choose distinct vertices \(y_i \in \Gamma(x_{i3}, A) \cap \Gamma(x_{i+1}, A)\). By condition (iv), there is a \(\land\)-matching \(M_2\) between the vertex set \(\{x_{i3} : 1 \leq i \leq |I| - 1\}\) and the vertex set \(A - \{y_i : 1 \leq i \leq |I| - 1\}\) centered in the first set, a matching \(M_3\) between \(\{x_{i+1,1} : 1 \leq i \leq |I| - 1\}\) and \(A - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2)\) covering the first set, and a matching \(M_4\) between the vertex set \(\{y_i : 1 \leq i \leq |I| - 1\}\) and \(B - V(M_I)\) covering the first set. Finally, by using (iv) again, we can find three distinct vertices \(y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A) - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2) - V(M_3)\). Let \(T_B\) be the graph with \(V(T_B) = V(M_I) \cup \{y_i : 1 \leq i \leq |I| - 1\} \cup V(M_2) \cap V(M_3) \cap V(M_4) \cup \{y_{31}, y_{32}, y_{33}\}\) and \(E(T_B) = M_I \cup \{y_i x_{i3}, y_i x_{i+1,1} : 1 \leq i \leq |I| - 1\} \cup M_2 \cup M_3 \cup M_4 \cup \{x_{13} y_{31}, x_{13} y_{32}, x_{13} y_{33}\}\). If \(|I| = 1\), we choose \(x_{11}, x_{12}, x_{13} \in \Gamma(x_1, B)\) and \(y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A)\). Then let \(T_B\) be the graph with \(V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}, y_{33}\}\) and \(E(T_B) = \{x_1 x_{11}, x_1 x_{12}, x_1 x_{13}, x_{13} y_{31}, x_{13} y_{32}, x_{13} y_{33}\}\). In any case, we see that \(T_B\) is a HIT satisfying \(|V(T_B) \cap B| = |V(T_B) \cap A| = 4|I| - 1,\) \(|L(T_B) \cap B| = \min\{2|I| + 1, 3|I| - 1\}, |L(T_B) \cap A| = 3|I|\). \hspace{1cm} (6) We call \(T_B\) the insertion HIT associated with \(B\). Figure 1 gives a depiction of \(T_B\) for \(|I| = 1, 3\), respectively. 

![Figure 1: The HIT \(T_B\)](image)

**Operation II** Let \((A, B)\) be an \((\varepsilon', \delta)\)-super-regular pair, and \(I\) a set of vertices disjoint from \(A \cup B\). Suppose that (i) \(\deg(x, A) \geq d'|A| > \varepsilon'|A|\) and \(\deg(x, A) \geq d'|A| \geq 3|I|\) for any \(x \in I\); (ii)
$(\delta - \varepsilon')d'|A| \geq 3|I|$; (iii) $(\delta - 2\varepsilon')|B| > |I|$; and (iv) $\delta|B| > 3|I|$. Then we can do the following operations on $(A, B)$ and $I$.

Let $I = \{x_1, x_2, \cdots , x_{|I|}\}$. We first assume that $|I| \geq 3$.

Since $(A, B)$ is $(\varepsilon', \delta)$-super-regular, for each $v \in \Gamma(x_i, A)$, $|\Gamma(v, B)| \geq \delta|B|$. By condition (i), we have $|\Gamma(x_i, A)| > \varepsilon'|A|$ for each $i$. Applying Fact 4, we then know that there are at least $(\delta - 2\varepsilon')|B| > |I|$ vertices from $\Gamma(v, B)$ typical to both $\Gamma(x_{i+1}, A)$ and $\Gamma(x_{i+2}, A)$ for each $1 \leq i \leq |I| - 2$. That is, there exists $B_1 \subseteq \Gamma(v, B)$ with $|B_1| \geq (\delta - 2\varepsilon')|B| > |I|$ such that for each $b_1 \in B_1$, $|\Gamma(b_1, \Gamma(x_{i+1}, A))|, |\Gamma(b_1, \Gamma(x_{i+2}, A))| \geq (\delta - \varepsilon')d'|A| \geq 3|I|$. As $\deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|A| \geq 3|I|$, combining the above argument, we know there is a claw-matching $M_I$ from $I$ to $A$ centered in $I$ such that any one vertex from $\Gamma(x_i, V(M_I))$, any one vertex from $\Gamma(x_{i+1}, V(M_I))$, and any one vertex from $\Gamma(x_{i+2}, V(M_I))$ have at least $|I|$ common neighbors in $B$. Let $x_{i_1}, x_{i_2}, x_{i_3}$ be the three neighbors of $x_i$ in $M_I$ (in fact in $A$). For $i = 1$, choose $y_0 \in \Gamma(x_{13}, A) \cap \Gamma(x_{23}, A) \cap \Gamma(x_{33}, A)$. Let $h = \lceil(|I| - 3)/2 \rceil$. For $1 \leq k \leq h$, we then choose distinct vertices $y_k \in \Gamma(x_{1+2k, 2}, A) \cap \Gamma(x_{2+2k, 3}, A) \cap \Gamma(x_{3+2k, 3}, A)$ (if $|I| = 2k + 1$, let $\Gamma(x_{3+2k, 3}, A) = A$). By condition (iv), there is a matching $M$ between the vertex set $\{x_{13}, x_{1+2k, 2} : 1 \leq i \leq |I|, 1 \leq k \leq h\}$ and the vertex set $B - \{y_0, y_k : 1 \leq k \leq h\}$ covering the first set. If $|I|$ is even, choose $y_{31}, y_{32} \in \Gamma(x_{13}, B)$ such that they have not been chosen before; if $|I|$ is odd, choose $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, B)$ such that they have not been chosen before. Let $T_A$ be the graph with

$$V(T_A) = \begin{cases} V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd;} \end{cases}$$

and $E(T_A)$ containing all edges in $M_I \cup M \cup \{y_0x_{13}, y_0x_{23}, y_0x_{33}\}$ and all edges in

$$\begin{cases} \{x_{1+2k, 2}y_k, x_{2+2k, 2}y_k, x_{3+2k, 2}y_k, x_{1+2h, 2}y_h, x_{2+2h, 2}y_h : 1 \leq k \leq h - 1\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ \{x_{1+2k, 2}y_k, x_{2+2k, 2}y_k, x_{3+2k, 2}y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd.} \end{cases}$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$ and $y_{31}, y_{32} \in \Gamma(x_{13}, B)$, and then let $T_A$ be the graph with

$$V(T_B) = \{x_1, x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}\} \text{ and } E(T_B) = \{x_{11}x_{11}, x_{11}x_{12}, x_{11}x_{13}, y_{31}x_{13}, y_{32}x_{13}\}.$$  

If $|I| = 2$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A), x_{11}, x_{12}, x_{13} \in \Gamma(x_2, A), y \in \Gamma(x_{13}, B) \cap \Gamma(x_{21}, B), y_{11}, y_{12} \in \Gamma(x_{13}, B), \text{ and } y_{21}, y_{22} \in \Gamma(x_{21}, B)$ such that they are all distinct, then let $T_A$ be the graph with

$$V(T_B) = \{x_i, x_{i1}, x_{i2}, x_{i3}, y_{i1}, y_{i2} : i = 1, 2\} \quad \text{and} \quad E(T_B) = \{x_{i1}x_{i1}, x_{i1}x_{i2}, x_{i2}x_{i3}, x_{i3}y_{i}, x_{i2}y_{i1}, x_{i3}y_{11}, x_{i2}y_{12}, x_{i1}y_{21}, x_{i2}y_{22}\}.$$  

We see that $T_A$ is a tree which has a degree 2 vertex $y$ only if $|I| = 2$ and a degree 2 vertex $y_h$ only.
if \(|I| > 2\) and \(|I|\) is even. In addition, \(T_A\) satisfies the following.

\[
|V(T_A) \cap A| = 3|I| \quad \text{and} \quad |L(T_A) \cap A| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3; \end{cases}
\]

\[
|V(T_A) \cap B| = \begin{cases} 2, & \text{if } |I| = 1; \\ 2|I| + 1, & \text{if } |I| \geq 2; \end{cases} \quad \text{and}
\]

\[
|L(T_A) \cap B| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3. \end{cases}
\] (7)

In this case, we call \(T_A\) the insertion tree associated with \(A\). Notice that \(|L(T_A) \cap A| = |L(T_A) \cap B|\) always holds. Figure 1 gives a depiction of \(T_A\) for \(|I| = 1, 2, 5, 6\), respectively.

**Operation III** Let \((B, F)\) be an \((\varepsilon', \delta)\)-super-regular pair, and \(I\) a set of vertices disjoint from \(B \cup F\). Suppose that \(\text{deg}(x, F) \geq d'|F| \geq 3|I|\) for any \(x \in I\) and \(\delta|B| \geq 6|I|\). Then we can do the following operations on \((A, B)\) and \(I\).

Let \(I = \{x_1, x_2, \ldots, x_{|I|}\}\). Since \(\text{deg}(x, B) \geq d'|B| \geq 3|I|\) for any \(x \in I\), there is a claw-matching \(M_I\) from \(I\) to \(F\) centered in \(I\). Then as \(\delta|B| \geq 6|I|\), there is a \(\wedge\)-matching \(M_A\) from \(V(M_I) \cap F\) to \(B\) centered in \(V(M_I) \cap F\). Let \(T_F\) be the graph with

\[ V(T_B) = V(M_I) \cup V(M_A) \quad \text{and} \quad E(T_B) = M_I \cup M_A. \]

We see that \(T_F\) is a forest with no vertex of degree 2 satisfying

\[ |V(T_F) \cap F| = |S(T_F) \cap F| = 3|I| \quad \text{and} \quad |V(T_F) \cap B| = |L(T_F) \cap B| = 6|I|. \] (8)

We call \(T_F\) the insertion forest associated with \(F\).
Now for each H-pair \((X'_i, Y'_i)\), we may assume that \(I(X'_i) \neq \emptyset\) and \(I(Y'_i) \neq \emptyset\) for a uniform discussion, as the consequent argument is independent of the assumptions. Recall that \((X'_i, Y'_i)\) is \((2\varepsilon, d - 2\varepsilon)\)-super-regular by Fact 5. Notice that \(\deg(x, X'_i) \geq (d - \varepsilon)|X'_i|\) for each \(x \in I(X'_i)\), \(|I(X'_i)| \leq \frac{d^2N}{20}\), and \(|X'_i||Y'_i| \geq (1 - \varepsilon)N\). By simple calculations, we see that (i) \(\deg(x, X'_i) \geq (d - \varepsilon)|X'_i| > 2\varepsilon|X'_i|\) and \((d - \varepsilon)|X'_i| \geq 3d^2N/20\) for each \(x \in I(X'_i);\) (ii) \((d - 2\varepsilon - 2\varepsilon)(d - \varepsilon)|X'_i| > 3d^2N/20;\) (iii) \((d - 4\varepsilon)|Y'_i| > d^2N/20;\) and (iv) \((d - 2\varepsilon)|Y'_i| > d^2N/5 \geq 4I(X'_i)\). Thus all the conditions in Operation I are satisfied. So we can find a HIT \(T_{X'_i}\) associated with \(X'_i\).

As \(|V(T_{X'_i}) \cap X'_i| = |V(T_{X'_i}) \cap Y'_i| \leq 4|I(X'_i)| \leq \frac{d^2N}{5}\), we know that \((X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))\) is \((4\varepsilon, d - 2\varepsilon - d^2N/5)\)-super-regular. Since \(\deg(y, Y'_i) \geq (d - \varepsilon)|Y'_i|\) for each \(y \in I(Y'_i)\), we get \(\deg(y, Y'_i - V(T_{X'_i})) \geq (d - \varepsilon - d^2/5)|Y'_i|\) for each \(y \in I(Y'_i)\). By direct checking, conditions (i) \(\sim\) (iv) of Operation I are satisfied by the pair \((X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))\) and \(I(Y'_i)\). Then we use Operation I on \((X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))\) and \(I(Y'_i)\) to get a HIT \(T_{Y'_i}\) associated with \(Y'_i - V(T_{X'_i})\). Denote

\[
X'_i = X'_i - V(T_{X'_i}) - V(T_{Y'_i}) \quad \text{and} \quad Y'_i = Y'_i - V(T_{X'_i}) - V(T_{Y'_i}).
\]

By using (6) in Operation I, we have \(|X'_i| = |Y'_i| \geq (1 - 2d^2/5 - \varepsilon)N \geq N/2\). By Slicing lemma (Lemma 3.5) and Fact 5, we have the following.

**Fact 7.** For each H-pair \((X_i, Y_i)\), \((X'_i, Y'_i)\) is \((4\varepsilon, d - 2\varepsilon - d^2/5)\)-super-regular with \(|X'_i| = |Y'_i|\). We call \((X'_i, Y'_i)\) a ready H-pair.

Then for each H-triple \((X'_i, Y'_i, F')\), we may assume that \(I(X'_i) \neq \emptyset\) and \(I(F') \neq \emptyset\) (recall that \(Y_i\) is assumed to be the dominator of \(F\), so \(I(Y'_i) = \emptyset\) by the distribution principle of vertices in \(V'_0\) from Claim 4.2). By Fact 6, we know that \((X'_i, Y'_i)\) is \((2\varepsilon, d - 3.1\varepsilon)\)- super-regular and \((Y'_i, F')\) is \((4.2\varepsilon, d - 3.1\varepsilon - 2d^3)\)-super-regular. Notice also that \(|X'_i| \geq (1 - \varepsilon)N, |Y'_i| \geq (1 - 3\varepsilon)N, |F'| \geq (1 - 2.1\varepsilon - 2d^3)N/2\), and \(\deg(x, X'_i) \geq (d - \varepsilon)|X'_i|\) and \(\deg(y, F') \geq (d - \varepsilon)|F'|\) for each \(x \in I(X'_i)\) and each \(y \in I(F')\). Since \(|I(X'_i)|, |I(F')| \leq \frac{d^2N}{20}\) and \(\varepsilon \ll d \ll 1\), the conditions of Operation III are satisfied by \((Y'_i, F')\) and \(I(F')\) by direct calculations. Let \(T_{F'}\) be the insertion forest associated with \(F'\). Then we use Operation II on \((X'_i, Y'_i - V(T_{F'}))\) and \(I(X'_i)\) to get a tree \(T_{X'_i}\) associated with \(X'_i\). Denote

\[
X'_i = X'_i - V(T_{X'_i}), \quad Y'_i = Y'_i - V(T_{F'}) - V(T_{X'_i}), \quad \text{and} \quad F' = F' - V(T_{F'}).\]

By using (7) and (8) in Operation II and Operation III, respectively, we have \(|X'_i|, |Y'_i| \geq (1 - 3.1\varepsilon - 9d^2/20)N \geq N/2\) and \(|F'| \geq (1 - 2.1\varepsilon - 2d^3)N/2 - 3d^2N/20 \geq (1 - 2.1\varepsilon - 2d^3 - 3d^2/10)N/2\). By Slicing lemma and Fact 6, we have the following.

**Fact 8.** For each H-triple \((X_i, Y_i, F)\), \((X'_i, Y'_i)\) is \((4\varepsilon, d - 3.1\varepsilon - 9d^2/20)\)-super-regular and \((Y'_i, F')\) is \((8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)\)-super-regular. We call \((X'_i, Y'_i, F')\) a ready H-triple.

**Step 5.** Apply the Blow-up lemma to find a HIT within each ready H-pair and among each ready H-triple.
In order to apply the Blow-up Lemma, we first give two lemmas which assure the existence of a given subgraph in a complete bipartite graph.

**Lemma 4.4.** Suppose $0 < \varepsilon \ll d \ll 1$ and $N$ is a large integer. If $G(A, B)$ is a balanced complete bipartite graph with $(1 - \varepsilon - d^2/2)N \leq |A| = |B| \leq N$, then $G(A, B)$ contains a HIST $T_{\text{pair}}$ with $\Delta(T_{\text{pair}}) \leq [2/d^3]$ and $|L(T_{\text{pair}}) \cap A| - |L(T_{\text{pair}}) \cap B| = \ell$ for any given non-negative integer $\ell$ with $\ell \leq d^2N$.

**Proof.** By the symmetry, we only show that we can construct a HIST $T$ such that $|L(T) \cap A| - |L(T) \cap B| = \ell$. Let $\Delta' = [d^3N]$. We choose distinct $a_1, a_2, \ldots, a_N \in A$ and distinct $b_1, b_2, \ldots, b_{\Delta'-1} \in B$. Then we decompose all vertices in $B$ into $B_1, B_2, \ldots, B_{\Delta'}$ such that $3 \leq |B_i| \leq 1/d^3$, $B_i \cap B_{i+1} = \{b_i\}$ for $1 \leq i \leq \Delta' - 1$, and $B_i \cap B_j = \emptyset$ for $|i-j| > 1$. Now we choose $\ell + 1$ distinct vertices $b_{\Delta'}, b_{\Delta'+1}, \ldots, b_{\Delta'+\ell}$ from $B - \{b_i : 1 \leq i \leq \Delta' - 1\}$. As $\Delta' = [d^3N]$, $\ell + \Delta' \leq (d^2 + d^3)N + 1$, and thus

$$2(\ell + \Delta') \leq (2d^2 + 2d^3)N + 2 \leq (1 - d^2/2 - \varepsilon)N - [d^3N] \leq |A| - [d^3N].$$

Thus we can use all of the vertices in $\{b_i : 1 \leq i \leq \Delta' + \ell\}$ to cover all vertices in $A - \{a_i : 1 \leq i \leq \Delta' - 1\}$ such that each $b_i$ be adjacent to at least two distinct vertices. We partition $A - \{a_i : 1 \leq i \leq \Delta' - 1\}$ arbitrarily into $A_1, A_2, \ldots, A_{\ell + \Delta'}$ such that $2 \leq |A_i| \leq 1/d^3$. Now let $T$ be a spanning subgraph of $G(A, B)$ such that

$$E(T) = \{a_i b_j : b_i \in B_i, 1 \leq i \leq \Delta'\} \cup \{b_j a_j : a \in A_j, 1 \leq j \leq \Delta' + \ell\}.$$

Clearly, $\Delta(T) \leq [2/d^3]$. As $|A| = |B|$, $|S(T) \cap A| = \Delta'$, and $|S(T) \cap B| = \Delta' + \ell$, we then have that $|L(T) \cap A| - |L(T) \cap B| = \ell$. We denote $T$ as $T_{\text{pair}}$. 

**Lemma 4.5.** Suppose $0 < \varepsilon \ll d \ll 1$ and $N$ is a large integer. Let $G = G(A, B, F)$ be a tripartite graph with $V(G)$ partitioned into $A \cup B \cup F$ such that both $G[A \cup B]$ and $G[B \cup F]$ are complete bipartite graphs. If (i) $(1 - 4\varepsilon - d^2/2)N \leq |A|, |B| \leq N$, (ii) $(1/2 - 21\varepsilon - 3d^2/20 - d^3)N \leq |F| \leq (1/2 - d^3)N$, and (iii) for any given non-negative integer $l \leq 3d^3N/10$, we have $|B| - 2(|A \cup F| - |B| - l) \geq 3d^3N/2$ holds, then $G$ contains a HIST $T_{\text{triple}}$ and a path $P_{\text{triple}}$ spanning on a subset of $L(T_{\text{triple}})$ such that

(a) $T_{\text{triple}}$ is a HIST of $G$ with $\Delta(T_{\text{triple}}) \leq [3/d^3]$;

(b) $|L(T_{\text{triple}}) \cap B| = |L(T_{\text{triple}}) \cap (A \cup F)| - l$.

(c) $P_{\text{triple}}$ is a $(b, f)$-path on $L(T_{\text{triple}}) \cap F$ and any $|L(T_{\text{triple}}) \cap F|$ vertices from $L(T_{\text{triple}}) \cap B$, and $|V(P_{\text{triple}}) \cap F| \leq 5d^2N/6$.

**Proof.** Let $\Delta' = [d^3N/2]$. We choose distinct $b_1, b_2, \ldots, b_{\Delta'} \in B$ and partition all vertices in $F$ into $F_1, F_2, \ldots, F_{\Delta'}$ such that $3 \leq |F_i| \leq 1/d^3$. Then we choose distinct $a_1, a_2, \ldots, a_{\Delta'-1} \in A$ and decompose all vertices in $A$ into $A_1, A_2, \ldots, A_{\Delta'}$ such that $3 \leq |A_i| \leq 2/d^3$, $A_i \cap A_{i+1} = \{a_i\}$ for $1 \leq i \leq \Delta' - 1$, and $A_i \cap A_j = \emptyset$ for $|i-j| > 1$. Choose one more vertex, say $a_{\Delta'} \in A - \{a_i : 1 \leq i \leq \Delta' - 1\}$, and partition all vertices in $B$ into $B_1, B_2, \ldots, B_{\Delta'}$ such that $3 \leq |B_i| \leq 1/d^3$, $B_i \cap B_{i+1} = \{b_i\}$ for $1 \leq i \leq \Delta' - 1$, and $B_i \cap B_j = \emptyset$ for $|i-j| > 1$. Now we choose $\ell + 1$ distinct vertices $b_{\Delta'}, b_{\Delta'+1}, \ldots, b_{\Delta'+\ell}$ from $B - \{b_i : 1 \leq i \leq \Delta' - 1\}$. As $\Delta' = [d^3N/2]$, $\ell + \Delta' \leq (d^2 + d^3)N + 1$, and thus

$$2(\ell + \Delta') \leq (2d^2 + 2d^3)N + 2 \leq (1 - d^2/2 - \varepsilon)N - [d^3N] \leq |A| - [d^3N].$$

Thus we can use all of the vertices in $\{b_i : 1 \leq i \leq \Delta' + \ell\}$ to cover all vertices in $A - \{a_i : 1 \leq i \leq \Delta' - 1\}$ such that each $b_i$ be adjacent to at least two distinct vertices. We partition $A - \{a_i : 1 \leq i \leq \Delta' - 1\}$ arbitrarily into $A_1, A_2, \ldots, A_{\ell + \Delta'}$ such that $2 \leq |A_i| \leq 1/d^3$. Now let $T$ be a spanning subgraph of $G(A, B)$ such that

$$E(T) = \{a_i b_j : b_i \in B_i, 1 \leq i \leq \Delta'\} \cup \{b_j a_j : a \in A_j, 1 \leq j \leq \Delta' + \ell\}.$$
$i \leq \Delta' - 1$. Let $l' = |A \cup F| - |B| - l$. Notice that $l' > 0$. Now we choose $l'$ distinct vertices $f_1, f_2, \cdots, f_{l'}$ from $A - \{a_i : 1 \leq i \leq \Delta'\} \cup F$ (choose as many as possible from $F$ first) and partition any $2l'$ vertices of $B - \{b_i : 1 \leq i \leq \Delta'\}$ into $B_1, B_2, \cdots, B_{l'}$ such that $|B_i| = 2$. By (iii), we see that there are at least $[d^3N]$ vertices left in $B' = B - \{b_i : 1 \leq i \leq \Delta'\} - \bigcup_{i=1}^{l'} \{B_i\}$. Hence we can partition $B' = B'_1 \cup B'_2 \cup \cdots \cup B'_\Delta$, such that $|B'_{\Delta'}| \geq 2$ and $|B'_j| \geq 1$ for $j \neq \Delta'$. We let $T$ be a subgraph of $G$ on $A \cup B \cup F$ with

$$E(T) = \{b_if, b_ia, b_ab' : f \in F_i, a \in A_i, b' \in B'_i, 1 \leq i \leq \Delta'\} \cup \{f_ib : b \in B_i, 1 \leq i \leq l'\}.$$ 

By the construction, $T$ is a HIST of $G$, which clearly satisfies (a). Since $|S(T) \cap B| = \Delta'$ and $|S(T) \cap (A \cup F)| = \Delta' + l' = \Delta' + |A \cup F| - |B| - l$, we then see that $T$ satisfies (b). If $L(T) \cap F = \emptyset$, let $f \in L(T) \cap F$ and $b \in L(T) \cap B$, we can then take a $(b, f)$-path $P$ with $V(P) \cap F = L(T) \cup F$ and $|V(P)| = 2|L(T) \cap F|$. By (i) and (ii), we see that $l' = |A \cup F| - |B| - l \geq (1/2 - 6.1\varepsilon - 4d^2/5 - d^3)N$. Hence $|V(P) \cap F| = |F| - l' \leq 5d^2N/6$. Denote $T$ as $T_{\text{triple}}$ and $P$ as $P_{\text{triple}}$. $lacksquare$

Now for $1 \leq i \leq t$ and for each ready H-pair $(X_i^t, Y_i^s)$, suppose, without of loss generality, that $|(L(T_{X_i^t}) \cap Y_i^t)| = (2\varepsilon, d - 2\varepsilon)$-super-regular by Fact 5 and $|Y_i^t - Y_i^s| \leq 2d^2N/5$, we have $\text{deg}(x_a, Y_i^s) \geq (d - 2\varepsilon - d^2/2)N \geq dN/2$. Similarly, $\text{deg}(y_b, X_i^t) \geq (d - 2\varepsilon - d^2/2)N \geq dN/2$. Also, from Step 2, we have $\Gamma(x_i^s, Y_i^t), \Gamma(x_i^s, Y_i^t) \geq (d - 3\varepsilon)N$. So, $\Gamma(x_i^s, Y_i^t), \Gamma(x_i^s, Y_i^t) \geq (d - 3\varepsilon - d^2/2)N \geq dN/2$. Similarly, we have $\Gamma(y_i^s, X_i^t), \Gamma(y_i^s, X_i^t) \geq (d - 3\varepsilon - d^2/2)N \geq dN/2$. Recall that $(X_i^t, Y_i^t)$ is $(4\varepsilon, d - 2\varepsilon - 8d^2/20)$-super-regular by Fact 7, and therefore $(X_i^t, Y_i^t)$ is $(4\varepsilon, d/2)$-super-regular. By the strengthened version of the Blow-up lemma and Lemma 4.4 (the conditions are certainly satisfied by $X_i^t$ and $Y_i^s$), we can find a HIST $T_{i}^0 \cong T_{\text{pair}}$ on $X_i^t \cup Y_i^t$ such that there exist $y_a \in S(T_i^0) \cap \Gamma(x_a, Y_i^s), x_b \in S(T_i^0) \cap \Gamma(y_b, X_i^t), y_i' \in S(T_i^0) \cap \Gamma(x_i^s, Y_i^t), y_i'' \in S(T_i^0) \cap \Gamma(x_i^s, Y_i^t)$, and $x_i'' \in S(T_i^0) \cap \Gamma(y_i^s, X_i^t)$ such that $|L(T_i^0)| \cap X_i^t \cap X_i^t = |L(T_i^0)| \cap Y_i^t \cap Y_i^t = l'$. Hence

$$|L(T_i^0)| \cap X_i^t \cap X_i^t = |L(T_i^0)| \cap Y_i^t \cap Y_i^t \leq 2d^2N/5.$$ 

For the ready H-pair $(X_0^t, Y_0^s)$, let $x_a \in S(T_{X_0^t}) \cap X_0^t$ be a non-leaf of $T_{X_0^t}$ and $y_b \in S(T_{Y_0^s}) \cap Y_0^s$ a non-leaf of $T_{Y_0^s}$. By the strengthened version of the Blow-up lemma and Lemma 4.4 (the conditions are certainly satisfied by $X_0^t$ and $Y_0^s$), we can find a HIST $T_{i}^0 \cong T_{\text{pair}}$ on $X_0^t \cup Y_0^s$ such that there exist $y_0 \in S(T_0^0) \cap \Gamma(x_0^s, Y_0), y_0'' \in S(T_0^0) \cap \Gamma(x_0^s, X_0), x_0' \in S(T_0^0) \cap \Gamma(y_0^s, X_0), x_0'' \in S(T_0^0) \cap \Gamma(y_0^s, X_0)$ such that $|L(T_0^0)| \cap X_0^t \cap X_0^t = |L(T_0^0)| \cap Y_0^s \cap Y_0^s + 2$. Let

$$T^0 = T_{i}^0 \cup T_{X_0^t} \cup T_{Y_0^s} \cup \{x_{a}y_{a}, y_{b}x_{b}: x_0''y_0, y_0''x_0', x_0^s, y_0^s, x_0^t, y_0^t\} \leq L(T_0^0) \cap X_0^t \cap X_0^t \cap Y_0^s \cap Y_0^s \cap X_0^t \cap Y_0^s + 2.$$
For each ready triple \((X^*_i, Y^*_i, F^*)\), we know that \((X^*_i, Y^*_i)\) is \((4\varepsilon, d - 3.1\varepsilon - 9d^2/20)\)-super-regular and \((Y^*_i, F^*)\) is \((8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)\)-super-regular by Fact 8. Notice that \((1 - 4\varepsilon - 9d^2/20)N \leq |X^*_i|, Y^*_i| \leq N\) and \((1/2 - 2.1\varepsilon - 3d^2/30 - d^3)N \leq |F^*| \leq (1/2 - d^3)N\). Let \(|I(X^*_i)| = l'\) and \(|I(F^*)| = l/6\) for some integer \(l\). By Operation II we have \(|V(T_{X^*_i}) \cap X^*_i| \leq 3l'\) and \(|V(T_{X^*_i}) \cap Y^*_i| \leq 2l' + 1\). By Operation III we have \(|V(T_F^*) \cap F^*| = l/2\) and \(|V(T_F^*) \cap F^*| = l\). Notice that \(|L(T_{X^*_i}) \cap X^*_i| = |I(T_{X^*_i}) \cap Y^*_i|\). Hence,

\[
|Y^*_i| - 2(|X^*_i| \cup F^*| - |Y^*_i| - l) \geq 3(|Y^*_i| - 2l' - l - 1) - 2(|X^*_i| - 3l') - 2(|F^*| - l/2) + 2l
\]

\[= 3(|Y^*_i| - 2|X^*_i| - 2|F^*| - 3
\]

\[\geq 3(1 - 3.1\varepsilon)N - 2N - N + 2d^3N - 3 > 3d^3N/2.
\]

By the weak version of the Blow-up lemma (Lemma 3.2) and Lemma 4.5, we then can find a HIT \(T_1^i \cong T_{\text{triple}}\) on \(X^*_i \cup Y^*_i \cup F^*\) and a path \(P_i \cong P_{\text{triple}}\) spanning on \(L(T^*_i) \cap F^*\) and other \(|L(T^*_i) \cap F^*|\) vertices from \(Y^*_i\). Let \(y_a \in S(T_{X^*_i}) \cap Y^*_i\) be a non-leaf of \(T_{X^*_i}\) (take \(y_a\) as the degree 2 vertex if \(T_{X^*_i}\) has one) and \(y_a' \in S(T_{F^*}) \cap Y^*_i\) a non-leaf of \(T_{F^*}\). Then as \((Y^*_i, F^*)\) is \((4.1\varepsilon, d - 2.1\varepsilon - 2d^3)\)-super-regular, we have \(|\Gamma(y_a, F^*)'|, |\Gamma(y_a', F^*)| \geq (d - 2.1\varepsilon - 2d^3)N/2\). Since \(|F^*| \leq 3d^2N/20\), we then know that \(|\Gamma(y_a, F^*)'|, |\Gamma(y_a', F^*)| \geq (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2\). Since \(|F^*| \leq \frac{5d^2N/6}{(d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2}\), there exist \(f_a \in (S(T^*_i) \cap F^*) \cap \Gamma(y_a, F^*)\) and \(f_a' \in (S(T^*_i) \cap F^*) \cap \Gamma(y_a', F^*)\). For each \(x \in I(F^*),\) since \(\deg(x, F^*) \geq (d - \varepsilon)|F^*| \geq (d - \varepsilon)(1 - 2.1\varepsilon - d^3)N/2,\) we know there exists \(f' \in (S(T^*_i) \cap F^*) \cap \Gamma(x, F^*)\). From Step 2, we have \(|\Gamma(x^*_i, Y^*_i) \cap \Gamma(y^*_i, X_i)| \geq (d - \varepsilon)^2N\) and \(|\Gamma(y^*_i, X_i) \cap \Gamma(y^*_i, X_i)| \geq (d - \varepsilon)^2N\). Hence \(|\Gamma(x^*_i, Y^*_i) \cap \Gamma(y^*_i, X_i)| \geq \frac{(d - \varepsilon)^2 - 3.1\varepsilon}N\). Since \(|\Gamma(x^*_i, Y^*_i) \cap \Gamma(y^*_i, X_i)| \leq d^3N/2\), we see that there exists \(y' \in \Gamma(x^*_i, Y^*_i) \cap \Gamma(y^*_i, X_i) \cap L(T^*_i \cup T_{X^*_i} \cup T_{F^*})\). Similarly, there exists \(x' \in \Gamma(y^*_i, X_i) \cap \Gamma(x^*_i, Y^*_i) \cap L(T^*_i \cup T_{X^*_i} \cup T_{F^*})\). Let \(T^i = T^*_i \cup T_{X^*_i} \cup T_{F^*} \cup \{x f' : x \in I(F^*), f' \in (S(T^*_i) \cap F^*) \cap \Gamma(x, F^*)\} \cup \{y_a f_a, y_a' f_a'\} \cup \{y' x^*_i, y' x' y^*_i, y' x' y^*_i\}.\) It is clear that \(T^i\) is a HIT of \(X^*_i \cup Y^*_i \cup F^* \cup I(X^*_i) \cup I(F^*)\) such that

\[
\{x^*_i, x^*_i, y^*_i, y^*_i\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X^*_i| = |L(T^i) \cap Y^*_i|.
\]

Let \(H^i = T^i \cup P_i\). We call \(P_i\) the accompanying path of \(T^i\).

**Step 6.** Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two vertex-disjoint paths covering all the leaves. Then connect all the HITs into a HIT of \(G\) and connect the disjoint paths into a cycle using the edges initiated in Step 2.

Suppose \(1 \leq i \leq t.\) For each H-pair \((X_i, Y_i)\), let \(X^L_i = X^*_i \cap L(T^i) - \{x^*_i, x^*_i\} \) and \(Y^L_i = Y^*_i \cap L(T^i) - \{y^*_i, y^*_i\},\) and for each H-triple \((X_i, Y_i, F)\), let \(X^L_i = X^*_i \cap L(T^i \cup P_i) - \{x^*_i, x^*_i\} \) and \(Y^L_i = Y^*_i \cap L(T^i \cup P_i) - \{y^*_i, y^*_i\},\) where \(T^i\) is the HIT found in Step 5, and \(P_i\) is the accompanying path of \(T^i\). By Operations I, II and III, and the proofs of the Lemmas 4.4 and 4.5, we have \(I(X^*_i) \cup I(Y^*_i) \subseteq S(T_i)\) and \(F^i \cup I(F^*) \subseteq S(T_i \cup P_i)\). Thus, \(X^L_i \cup Y^L_i = L(T^i) - \{x^*_i, x^*_i, y^*_i, y^*_i\}\) for each H-pair and \(X^L_i \cup Y^L_i = L(T^i \cup P_i) - \{x^*_i, x^*_i, y^*_i, y^*_i\}\) for each H-triple. Furthermore, we have \(|X^L_i| = |Y^L_i|\). For the H-pair \((X_0, Y_0)\), let \(X^L_0 = X^*_0 \cap L(T^0) - \{x^*_0, x^*_0, y^*_0, y^*_0, y^*_0\}\) and \(Y^L_0 = Y^*_0 \cap L(T^0) - \{y^*_0, y^*_0\}\). We have \(X^L_0 \cup Y^L_0 = L(T^0) - \{x^*_0, x^*_0, y^*_0, y^*_0\} \) and \(|X^L_i| = |Y^L_i|\)
since from Step 5 we have \( |L(T^0) \cap X_0^i| = |L(T^0) \cap Y_0^i| + 2 \). By the construction of \( T_{pair} \) and \( H_{triple} \), we see that \( |S(T_i) \cap X_i^i|, |S(T_i) \cap Y_i^i| \leq d^2 N \). Since each H-pair \((X_i^i, Y_i^i)\) is \((2\varepsilon, d - 2\varepsilon)\)-super-regular, and each pair \((X_i^L, Y_i^L)\) from an H-triple \((X_i^i, Y_i^i, F^i)\) is \((2\varepsilon, d - 3.1\varepsilon)\)-super-regular, by Slicing Lemma, we then know that \((X_i^L, Y_i^L)\) is \((4\varepsilon, d - 3.1\varepsilon - d^2)\)-super-regular and hence is \((4\varepsilon, d/2)\)-super-regular.

For each \( 1 \leq i \leq t \), by the choice of \( x_i^*, y_i^*, y_i^{**} \), we have \( |\Gamma(x_i^*, Y_i)|, |\Gamma(x_i^*, Y_i^i)| \geq (d - \varepsilon)N \) and \( |\Gamma(y_i^*, X_i)|, |\Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)N \). Hence, \( |\Gamma(x_i^*, Y_i^L)|, |\Gamma(x_i^{**}, Y_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2 \) and \( |\Gamma(y_i^*, X_i^L)|, |\Gamma(y_i^{**}, X_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2 \). Similar results hold for the vertices \( x_0^*, x_{i+1}^*, y_{i+1}^* \). For each \( 0 \leq i \leq t \), we choose distinct vertices \( y_i^* \in \Gamma(x_i^*, Y_i^L), y_i^{**} \in \Gamma(x_i^{**}, Y_i^L) \) and \( x_i^* \in \Gamma(y_i^*, X_i^L), x_i^{**} \in \Gamma(y_i^{**}, X_i^L) \). If \( T_i \) does not have the accompany path, then by the strengthened version of the Blow-up lemma, we can find an \((x_i', y_i')\)-path \( P_{i1}^i \) and an \((x_i'', y_i'')\)-path \( P_{i2}^i \) such that \( P_{i1}^i \cup P_{i2}^i \) is spanning on \( X_i^L \cup Y_i^L \). If \( T_i \) has the accompany \((b, f)\)-path \( P_i \), we see that \( \text{deg}(b, X_i^L), \text{deg}(f, Y_i^L) \geq dN/2 \) as \((X_i^i, Y_i^i)\) is \((2\varepsilon, d - 3.1\varepsilon)\)-super-regular, and \((Y_i^i, F^i)\) is \((4\varepsilon, d - 3.1\varepsilon - d^2)\)-super-regular. Applying the strengthened version of the Blow-up lemma, we can find an \((x_i', y_i')\)-path \( P_{i1}^i \) and an \((x_i'', y_i'')\)-path \( P_{i2}^i \) such that \( P_{i1}^i \cup P_{i2}^i \) is spanning on \( X_i^L \cup Y_i^L \), and two consecutive internal vertices \( a', b' \in V(P_{i1}^i) \) with \( b' \in \Gamma(f, Y_i^L) \), and \( a' \in \Gamma(b, X_i^L) \). Let \( P_i^i = P_{i1}^i \cup P_i \cup \{bf', ba'\} - \{a'b'\} \). Notice that for the H-pair \((X_0, Y_0)\), the two vertices \( y_{i+1}^*, y_{i+1}^{**} \) are not used in this step, but we will connect them to \( y_0^* \) and \( y_0^{**} \), respectively, in next step.

We now connect the small HITs and paths together to find an SGHG of \( G \). In Case A, for \( 1 \leq i \leq t - 1 \), we have \( |S(T_i) \cap Y_i| \geq d^2 N/2 > \varepsilon N \) and \( |S(T_i^i) \cap X_i+1| \geq d^3 N/2 > \varepsilon N \). Since \((Y_i, X_i+1)\) is an \( \varepsilon \)-regular pair with density \( d \), we see that there is an edge \( e_i \) connecting \( S(T_i^i) \cap X_i+1 \) and \( S(T_i^i) \cap X_i+1 \). Let

\[
T = \bigcup_{i=1}^{t} T_i \cup \{e_i \mid 1 \leq i \leq t - 1\}.
\]

Then \( T \) is a HIST of \( G \). Let \( C \) be the cycle formed by all the paths in \( \bigcup_{i=1}^{t} (P_{i1}^i \cup P_{i2}^i) \) and all edges in the following set

\[
\{x_i^*, y_i^*, x_i^{**}, y_i^{**}, x_i^*, y_i^{**}, x_i^*: 1 \leq i \leq t\} \cup \{y_i^*, x_i^{**}, y_i^*, x_i^*: 1 \leq i \leq t - 1\} \cup \{y_i^*, x_i^*, y_i^*, x_i^*: 1 \leq i \leq t - 1\}.
\]

notices that the edges in \( \{y_i^*, x_i+1, y_i^{**}, x_i+1: 1 \leq i \leq t - 1\} \cup \{y_i^*, x_i^*, y_i^{**}, x_i: 1 \leq i \leq t - 1\} \) above are guaranteed in Step 2. It is easy to see that \( C \) is a cycle on \( L(T) \). Hence \( H = T \cup C \) is an SGHG of \( G \).

In Case B, for \( 1 \leq i \leq t - 1 \), we have \( |S(T_i) \cap Y_i| \geq d^3 N/2 > \varepsilon N \) and \( |S(T_i+1) \cap X_i+1| \geq d^3 N/2 > \varepsilon N \). Since \((Y_i, X_i+1)\) is an \( \varepsilon \)-regular pair with density \( d \), we see that there is an edge \( e_i \) connecting \( S(T_i+1) \cap X_i+1 \) and \( S(T_i+1) \cap X_i+1 \). Similarly, there is an edge \( e_0 \) connecting \( S(T_0) \cap X_0 \) and \( S(T_1) \cap X_1 \). Let

\[
T = \bigcup_{i=1}^{t} T_i \cup \{e_i \mid 0 \leq i \leq t - 1\}.
\]

Then \( T \) is a HIST of \( G \). Let \( C \) be the cycle formed by all paths in \( \bigcup_{i=1}^{t} (P_{i1}^i \cup P_{i2}^i) \) and all edges in
the set \( \{ y_0^t y_{t+1}^*, y_0^t y_{t+1}^*, y_i^t x_i^t, y_{t+1}^t x_i^t, x_0^t x_i^t, x_0^t y_i^t \} \) and in the following set
\[ \{ x_i^t y_i^t, x_i^t y_i^t, y_i^t x_i^t, y_i^t x_i^t : 0 \leq i \leq t \} \cup \{ y_i^t x_{i+1}^t, y_i^t x_{i+1}^t : 1 \leq i \leq t-1 \}. \]

It is easy to see that \( C \) is a cycle on \( L(T) \). Hence \( H = T \cup C \) is an SGHG of \( G \).

The proof of Theorem 4.1 is now finished. ■

4.2 Proof of Theorem 4.2

By the assumption that \( \text{deg}(v_1, V_2) \leq 2\beta n \) for each \( v_1 \in V_1 \) and the assumption that \( \delta(G) \geq (2n + 3)/5 \) in Extremal Case 1, we see that
\[ \delta(G[V_1]) \geq (2n + 3)/5 - 2\beta n. \] (9)

Then (9) implies that
\[ |V_1| \geq (2n + 3)/5 - 2\beta n \quad \text{and} \quad |V_2| \leq 3n/5 + 2\beta n. \] (10)

Also, by \( |V_2| \geq (2/5 - 4\beta)n \) in the assumption,
\[ |V_1| \leq (3/5 + 4\beta)n. \] (11)

We will construct an SGHG of \( G \) following several steps below.

**Step 1. Repartitioning**

Set \( \alpha_1 = \alpha^{1/3} \) and \( \alpha_2 = \alpha^{2/3} \). Let
\[ V'_1 = V_1 \quad \text{and} \quad V'_2 = \{ v \in V_2 \mid \text{deg}(v, V_1) \leq \alpha_1|V_1| \}. \]

Then by \( d(V_1, V_2) \leq \alpha \), we have
\[ \alpha_1|V_1||V_2 - V'_2| \leq e(V_1, V'_2) + e(V_1, V_2 - V'_2) = e(V_1, V_2) \leq \alpha|V_1||V_2|. \]

This gives that
\[ |V_2 - V'_2| \leq \alpha_2|V_2|. \] (12)

Denote \( V_{12}^0 = V_2 - V'_2 \). Then by the definition of \( V'_2 \), we have
\[ \delta(V_{12}^0, V'_1) > \alpha_1|V'_1| \quad \text{and} \quad \delta(G[V'_2]) \geq (2n + 3)/5 - \alpha_1|V'_1| \geq (2/5 - \alpha_1(3/5 + 4\beta))n, \] (13)

where the last inequality follows from (11).

Let \( n_i = |V'_i| \) for \( i = 1, 2 \). Then by (9) and (11),
\[ \delta(G[V'_1]) \geq (2n + 3)/5 - 2\beta n \geq \frac{2/5 - 2\beta}{3/5 + 4\beta} n_1 \geq (2/3 - 8\beta)n_1. \] (14)
and by (10) and the second inequality in (13),
\[
\delta(G[V_2]) \geq (2/5 - \alpha_1(3/5 + 4\beta))n \geq \frac{(2/5 - \alpha_1(3/5 + 4\beta))}{3/5 + 2\beta}n_2 \geq (2/3 - 1.1\alpha_1)n_2,
\]
provided that \( \beta \leq \frac{0.3\alpha_1}{9\alpha_1 + 20/3} \).

**Step 2. Finding three connecting edges**

AS \( G \) is 3-connected, there are 3 independent edges \( x_L^1y_L^1, x_L^2y_L^2 \) and \( x_Ny_N \) connecting \( V'_1 \cup V'_0 \) and \( V'_2 \) such that \( x_L^1, x_L^2, x_N \in V'_1 \cup V'_0 \) and \( y_L^1, y_L^2, y_N \in V'_2 \). In the remaining steps, we will find a HIST \( T_1 \) in \( G[V'_1 \cup V'_0] \) with \( x_N \) as a non-leaf and \( x_L^1, x_L^2 \) as leaves, and a HIST \( T_2 \) of \( G[V'_2] \) with \( y_N \) as a non-leaf and \( y_L^1, y_L^2 \) as leaves. Then \( T = T_1 \cup T_2 \cup \{x_Ny_N\} \) is a HIST of \( G \). By finding a hamiltonian \((x_L^1, x_L^2)\)-path \( P_1 \) on \( L(T_1) \), and a hamiltonian \((y_L^1, y_L^2)\)-path on \( L(T_2) \), we see that
\[
C := P_1 \cup P_2 \cup \{x_L^1y_L^1, x_L^2y_L^2\}
\]
forms a cycle on \( L(T) \). Hence \( H := T \cup C \) is an SGHG of \( G \).

**Step 3. Initiating two HITs**

In this step, we first initiate a HIT in \( G[V'_1 \cup V'_0] \) containing \( X_N \) as a non-leaf and \( x_L^1 \) and \( x_L^2 \) as leaves. Then, we initiate a HIT in \( G[V'_2] \) containing \( y_N \) as a non-leaf and \( y_L^1 \) and \( y_L^2 \) as leaves.

For \( x_L^1, x_L^2, x_N \in V'_1 \cup V'_0 \), by (9) and (13), each of them has at least \( \alpha_1|V'_1| \geq 9 \) neighbors in \( V'_1 \). Thus, we choose distinct \( z_L^1, z_1, z_L^2, z_2, z_N^1, z_N^2, z_N^3 \in V'_1 \) such that
\[
x_L^1 \sim z_L^1, z_1, \quad x_L^2 \sim z_L^2, z_2, \quad x_N \sim z_N^1, z_N^2, z_N^3.
\]
(Note that \( x_L^1 \) and \( x_L^2 \) may be from \( V'_0 \), and therefore they may not have too many neighbors in \( V'_1 \), we then choose \( z_L^1 \) and \( z_L^2 \) from \( V'_1 \) as their neighbors, respectively.)

By (14), we see that any two vertices in \( G[V'_1] \) have at least \((1/3 - 16\beta)n_1 \geq 14 \) neighbors in common. Thus, we can choose distinct vertices \( z_L^{11}, z_L^{22}, z_L^{12}, v_1^R \in V'_1 - \{x_L^1, x_L^2, x_N, z_L^1, z_1, z_L^2, z_N^1, z_N^2, z_N^3\} \) such that
\[
z_L^{11} \sim z_L^1, z_1, \quad z_L^{22} \sim z_L^2, z_2, \quad z_L^{12} \sim z_{11}, z_{22}, \quad v_1^R \sim z_{12}, z_L^1.
\]
Furthermore, by (14) again, we have \( \delta(G[V'_1]) \geq (2/3 - 8\beta)n_1 \geq 17 \). Choose \( z_1^1, z_2^2, z_N^1 \in V'_1 \) not chosen above such that
\[
z_1^1 \sim z_1^1, z_2^2 \sim z_2^2, z_N^1 \sim z_N^1.
\]
Let \( T_{11} \) be the graph with
\[
V(T_{11}) = \{x_L^1, x_L^2, x_N, z_L^1, z_1, z_L^2, z_2, z_N^1, z_N^2, z_N^3, v_1^R, z_1^2, z_2^2, z_N^1\}
\]
and with edges indicated above except the edges \( x_L^1z_L^1 \) and \( x_L^2z_L^2 \). We see that \( T_{11} \) is a tree with \( v_1^R \) as the only degree 2 vertex, and \( |V(T_{11})| = 17 \) and \( |L(T_{11})| = 9 \). Notice that in \( T_{11} \), \( z_L^1, x_L^1 \) and \( z_L^2, x_L^2 \) are leaves, and \( x_N \) is a non-leaf. Figure 3 gives a depiction of \( T_{11} \).
Notice that the edges $x_L^1z_L^1$ and $x_L^2z_L^2$ are not used in $T_{11}$. We will first construct a HIST $T_1$ in $G[V_1^1 \cup V_0^0]$ containing $T_{11}$ as a subgraph, then find a hamiltonian $(z_L^1, z_L^2)$-path on $L(T_1) - \{x_L^1, x_L^2\}$ by Lemma 3.6, finally by adding $x_L^1z_L^1$ and $x_L^2z_L^2$ to the path, we get a hamiltonian $(x_L^1, x_L^2)$-path on $L(T_1)$. The reason that we avoid using $x_L^1$ and $x_L^2$ is that when $x_L^1, x_L^2 \in V_1^0$, we may not be able to have the condition of Lemma 3.6 on $G[L(T_1)]$ in our final construction.

Then we initiate a HIT in $G[V_2^2]$ containing $y_L^1, y_L^2$ as leaves, and $y_N$ as a non-leaf.

As $y_L^1, y_L^2, y_N \in V_2'$, by (15) and the fact that each two vertices from $V_2'$ have at least $(1/3 - 2.2 \alpha_1)n_2 \geq 7$ common neighbors implied from (15), we can choose distinct vertices

$$y^{12}, y_N^1, y_N^2, y_N^3, v_2^R \in V_2' - \{y_L^1, y_L^2, y_N\}$$

such that

$$y^{12} \sim y_L^1, y_L^2, \quad y_N \sim y_N^1, y_N^2, y_N^3, \quad v_2^R \sim y^{12}, y_N. \quad (15)$$

Let $T_{21}$ be the graph with $V(T_{21}) = \{y_L^1, y_L^2, y_N, y^{12}, y_N^1, y_N^2, y_N^3, v_2^R\}$ and with $E(T_{21})$ described as in (15).

We see that $T_{21}$ is a tree with $v_2^R$ the only degree 2 vertex and $y_L^1, y_L^2 \in L(T_{21})$, $y_N \in S(T_{21})$ and

$$|V(T_{21}) \cap V_2'| = 8, \quad |L(T_{21}) \cap V_2'| = 5. \quad (16)$$

Denote

$$U_1 = V_1' - V(T_{11}), \quad U_2 = V_2' - V(T_{21}), \quad \text{and} \quad V_{12} = V_1^0 - V(T_{11}).$$

**Step 4. Absorbing vertices in $V_{12}^0$**

We may assume that $V_{12}^0 \neq \emptyset$. For otherwise, we skip this step. Let $|V_{12}| = n_{12}$ and $V_{12}^0 = \{x_1, x_2, \ldots, x_{n_{12}}\}$.

Since $|V(T_{11})| = 17$, by (13), we get

$$\delta(V_{12}^0, U_1) > \alpha_1|V_1'| - 17 \geq 3\alpha_2|V_2| \geq 3|V_2' - V_2| \geq 3|V_{12}^0|.$$
Thus, there is a claw-matching $M_c$ from $V_{12}^0$ to $U_1$ centered in $V_{12}^0$. For $i = 1, 2, \cdots, n_{12}$, let $x_{i1}, x_{i2}$ and $x_{i3}$ be the three neighbors of $x_i$ in $M_c$. If $n_{12} = 1$, let $T_a = M_c$, and we finish this step. Thus we assume $n_{12} \geq 2$.

By (14), each two vertices in in $V'_1$ have at least

$$(1/3 - 16\beta)n_1 \geq 6\alpha_2|V_{12}^0| + 17$$

(17)

neighbors in common. The above inequality holds as $n_1 \geq 2n/5 - 2\beta n$, $|V_2| \leq 3n/5 + 2\beta n$ by (10), and we can assume that $18\alpha_2/5 + 106\beta/15 + 12\alpha_2 \beta + 18/n - 32\beta^2 \leq 2/15$.

Thus, for each $i = 1, 2, \cdots, n_{12} - 1$, we can find distinct vertices $x_{13}^i, x_{23}^i, x_{33}^i, x_{i1+1,1}^i$ in $U_1 - V(M_c)$ such that

$$x_{13}^i \sim x_{i3}, x_{i+1,1}, \quad x_{23}^i \sim x_{13}^i, \quad x_{33}^i \sim x_{i3}, \quad x_{i1+1,1}^i \sim x_{i+1,1}. \quad (18)$$

Let $T_a$ be the graph with $V(T_a) = V(M_c) \cup \{x_{13}^i, x_{23}^i, x_{33}^i, x_{i1+1,1}^i : 1 \leq i \leq n_{12} - 1\}$, and $E(T_a)$ including all edges indicated in (18) for all $i$ and all edges in $M_c$. It is easy to see, by the construction, that $T_a$ is a HIT with

$$|V(T_a) \cap U_1| = 7n_{12} - 4 \quad \text{and} \quad |L(T_a) \cap U_1| = 4n_{12} - 1.$$ 

Using (17) again, we can find $x_{11}^{1} \in U_1 - V(T_a)$ such that $x_{N}^{11} \sim v_{1}^{R}, x_{11}$, where $v_{1}^{R} \in V(T_{11})$ and $x_{11} \in V(T_a)$. By (14),

$$\delta[G[V'_1]] \geq (2n + 3)/5 - 2\beta n \geq 6\alpha_2|V_{12}^0| + 20,$$

since $|V_2| \leq 2n/5 + 2\beta n$, and we can assume that $2\beta - 12\alpha_2 \beta - 18\alpha_2/5 - 21/n \leq 2/5$. So we can find distinct vertices $x_{N}^{12}, x_{11}^{1} \in U_1 - V(T_a) - \{x_{11}^{1}\}$ such that $x_{N}^{12} \sim x_{N}^{11}, x_{11}^{1} \sim x_{11}$.

Let $T_1^1$ be the graph with

$$V(T_1^1) = V(T_{11}) \cup V(T_a) \cup \{x_{N}^{11}, x_{N}^{12}, x_{11}^{1}\} \quad \text{and} \quad E(T_1^1) = E(T_{11}) \cup E(T_a) \cup \{x_{N}^{11}v_{R}, x_{N}^{11}x_{11}, x_{N}^{12}x_{N}^{11}, x_{11}^{1}x_{11}\}.$$ 

Then $T_1^1$ is a HIT such that

$$|V(T_1^1) \cap U_1| = 7n_{12} + 16 \quad \text{and} \quad |S(T_1^1) \cap U_1| = 3n_{12} + 7. \quad (19)$$

Denote $U_1' = U_1 - V(T_1^2)$ and $U_2' = U_2 - V(T_1^2)$.

**Step 5. Completion of HITs $T_1$ and $T_2$**

In this step, we construct a HIST $T_i$ in $G[V'_i]$ $(i = 1, 2)$ containing $T_i^1$ as an induced subgraph.

The following lemma guarantees the existence of a specified HIST in a graph with $n$ vertices and minimum degree at least $(2/3 - \alpha')n$ for some $0 < \alpha' \ll 1$.

**Lemma 4.6.** Let $H$ be an $n$-vertex graph with $\delta(H) \geq (2/3 - \alpha')n$ for some constant $0 < \alpha' \ll 1$. Then $H$ has a HIST $T_H$ satisfying

25
(i) $T_H$ has a vertex $v_R$ of degree at least $(2/3 - \alpha')n - 1$, and $v_R$ can be chosen arbitrarily from $V(H)$;

(ii) $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$.

**Proof.** Let $v_R \in V(H)$ be an arbitrary vertex. If $n(mod\ 2) \equiv deg(v_R) + 1(mod\ 2)$, then we let $N_R = N_H(v_R)$. For otherwise, let $N_R$ be a subset of $N(v_R)$ with $|N_H(v_R)| - 1$ elements. Let $T_{v_R}$ be the star with $V(T_{v_R}) = \{v_R\} \cup N_R$ and $E(T_{v_R}) = E(\{v_R\}, N_R)$. Let $V_0 = V(H) - V(T_{v_R})$. By $\delta(H) \geq (2/3 - \alpha')n$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. By the choice of $N_R$, we have $|V_0| \equiv 0(mod\ 2)$. If $V_0 = \emptyset$, then let $T_H = T_{v_R}$. For otherwise, we claim as follows.

**Claim 4.3.** Let $V_1 \subseteq V(H)$ be a subset with $|V_1| \geq (2/3 - \alpha')n - 1$ and $|V_1|(mod\ 2) \equiv n(mod\ 2)$. Then there exist two vertices from $V_0 = V(H) - V_1$ such that they have a common neighbor in $V_1$.

*Proof of Claim 4.3.* We assume that $|V_1| \leq (2/3 + 2\alpha')n$. For otherwise, $|V_0| < (1/3 - 2\alpha')n$. Since $\delta(H) \geq (2/3 - \alpha')n$, any two vertices of $H$ have at least $(1/3 - 2\alpha')n$ neighbors in common. By $|V_0| < (1/3 - 2\alpha')n$, any two vertices from $V_0$ have a common neighbor from $V_1$. We are done. Thus $|V_1| \leq (2/3 + 2\alpha')n$, and hence $|V_0| \geq (1/3 - 2\alpha')n \geq 3$. By the assumption that $|V_1| \geq (2/3 - \alpha')n - 1$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. This implies that $deg(v_0, V_1) \geq (1/3 - 2\alpha')n - 2$ for each $v_0 \in V_0$. As $|V_0| \geq 3$ and $3((1/3 - 2\alpha')n - 2) > (2/3 + 2\alpha')n \geq |V_1|$ (provided that $8\alpha' + 6/n < 1/3$), we see that there must be two vertices from $V_0$ such that they have a neighbor in common in $V_1$. □

By Claim 4.3, there exist two vertices $v_0^{11}, v_0^{12} \in V_0$ such that they have a common neighbor in $T_{v_R}$. Adding $v_0^{11}$ and $v_0^{12}$ to $T_{v_R}$ and two edges connecting them to one of their common neighbor in $V(T_{v_R})$. Let $T_{v_R}'$ be the resulting graph. Then we see that $T_{v_R}'$ is a HIT with $|V(T_{v_R}')| = |V(T_{v_R})| + 2$, and hence $|V(T_{v_R}')| + 2(mod\ 2) \equiv n(mod\ 2)$. Also $|V(T_{v_R}')| \geq |V(T_{v_R})| \geq (2/3 - \alpha')n - 1$. So we can use Claim 4.3 again to find another pair of vertices from $V_0 - \{v_0^{11}, v_0^{12}\}$ such that they have a common neighbor in $V(T_{v_R}' \cup \{v_0^{11}, v_0^{12}\})$. Adding the new pair of vertices and two edges connecting them to one of their common neighbor in $V(T_{v_R}' \cup \{v_0^{11}, v_0^{12}\})$ into $T_{v_R}'$, we get a new HIT $T_{v_R}'$. By repeating the above process another $l_0 = (|V_0| - 4)/2$ times, we get a HIT $T_{v_R}'$. Let $T_H = T_{v_R}'$. We claim that $T_H$ has the required properties in Lemma 4.6. Notice first that $d_{T_H}(v_R) \geq (2/3 - \alpha')n - 1$. Then since $T_H$ has $v_R$ and at most $|V_0|/2$ distinct vertices as non-leaves and $|V_0| \leq (1/3 + \alpha')n + 1$, we see that $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$. □

Let $H_1 = G[U_1' \cup \{v_R^1\}]$. Recall that $v_R^1$ is a non-leaf in $T_1$. By (14) and (19), and by noticing that $n_{12} \leq |V_2 - V_2^2| \leq \alpha_2|V_2| \leq 3\alpha_2n_1/2$ (by (10)), we see that

$$
\begin{align*}
\delta(H_1) & \geq (2/3 - 8\beta)n_1 - (7n_{12} + 19) \\
& \geq (2/3 - 8\beta)n_1 - 21\alpha_2n_1/2 - 19 \\
& \geq (2/3 - 11\alpha_2)|V(H_1)|.
\end{align*}
$$

Let $\alpha' = 11\alpha_2 \ll 1$ (by assuming $\alpha \ll 1$). By Lemma 4.6, we can find a HIT $T_1^f$ in $H_1$ with $v_1^R$ as the prescribed vertex in condition (i). It is easy to see that $T_1 := T_1^f \cup T_1^f$ is a HIT of $G[U_1' \cup V_0^{12}]$.
and
\[
    s_1 = |S(T_1) \cap V'_1| = |S(T_1^1) \cap V'_1| + |S(T_1^2) \cap V'_1| \\
    \leq 3n_{12} + 7 + (1/6 + 5.5\alpha_2)|V(H_1)| + 2 \text{ (by (19) and Lemma 4.6)} \\
    \leq 3n_{12} + 9 + (1/6 + 5.5\alpha_2)n_1 \\
    \leq (1/6 + 10.5\alpha_2)n_1 \text{ (by } n_{12} \leq 3\alpha_2n_1/2) .
\] (21)

Let \( H_2 = G[U'_2 \cup \{v^2_R\}] \). By (15) and (16), we see that
\[
    \delta(H_2) \geq (2/3 - 1.1\alpha_1)n_2 - 8 \geq (2/3 - 1.2\alpha_1)|V(H_2)| .
\]
By letting \( \alpha' = 1.2\alpha_1 \), we can find a HIT \( T'_2 \) in \( H_2 \) with \( v^2_R \) as the prescribed vertex in condition (i) of Lemma 4.6. Then \( T_2 := T'^2_1 \cup T'_2 \) is a HIST of \( G[V'_2] \). Also, notice that
\[
    s_2 = |S(T_2) \cap V'_2| = |S(T_2^1) \cap V'_2| + |S(T_2^2) \cap V'_2| \\
    \leq 3 + (1/6 + 0.6\alpha_1)|V(H_2)| + 2 \\
    \leq (1/6 + 0.7\alpha_2)n_2 ,
\] (22)
where the last inequality holds by assuming \( 5/n_2 \leq 0.1\alpha_2 \).

**Step 6. Finding two long paths.**

In this step, we first find a hamiltonian \((z^1_L, z^2_L)\)-path \( P_1^1 \) in \( G[L(T_1) - \{x^1_L, x^2_L\}] \); then find a hamiltonian \((y^1_L, y^2_L)\)-path \( P_2 \) in \( G[L(T_2)] \). Let \( G_{11} = G[L(T_1) - \{x^1_L, x^2_L\}] \) and \( n_{11} = |V(G_{11})| \). We will show that \( \delta(G_{11}) > \frac{1}{2}n_{11} \). We may assume \( s_1 \geq (1/6 - 8\beta)n_1 - 2 \). For otherwise, if \( s_1 < (1/6 - 8\beta)n_1 - 2 \), then by (14), we get
\[
    \delta(G_{11}) \geq \delta(G[V'_1]) - s_1 - 2 \\
    \geq (2/3 - 8\beta)n_1 - ((1/6 - 8\beta)n_1 - 1 - 2) - 2 \\
    \geq \frac{1}{2}n_1 + 1 \geq \frac{1}{2}n_{11} + 1 .
\]
Hence, \( s_1 \geq (1/6 - 8\beta)n_1 - 2 \), implying that
\[
    n_{11} \leq (5/6 + 8\beta)n_1 + 2 \quad \text{and thus} \quad n_1 \geq \frac{n_{11} - 2}{5/6 + 8\beta} .
\] (23)
Hence, by (21)
\[
    \delta(G_{11}) \geq \delta(G[V'_1]) - s_1 - 2 \geq (2/3 - 8\beta)n_1 - (1/6 + 10.5\alpha_2)n_1 - 2 \\
    \geq (1/2 - 8\beta - 11\alpha_2)n_1 \geq \frac{1/2 - 2\beta - 11\alpha_2}{5/6 + 2\beta}(n_{11} - 2) > n_{11}/2 ,
\]
the last inequality holds by assuming \( 3\beta + 11\alpha_2 + 2/n_{11} < 1/12 \). By applying Lemma 4.6 on \( G_{11} \), we find a hamiltonian \((z^1_L, z^2_L)\)-path \( P_1^1 \) in \( G_{11} \). Let \( P_1 = P_1^1 \cup \{z^1_L x^1_L, z^2_L x^2_L\} \). We see that \( P_1 \) is a hamiltonian \((x^1_L, x^2_L)\)-path on \( L(T_1) \).
Let $G_{22} = G[L(T_2)]$ and $n_{22} = |V(G_{22})|$. We will show that $\delta(G_{22}) > n_{22}/2$. We may assume that $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$. For otherwise, if $s_2 < (1/6 - 1.1\alpha_1)n_2 - 2$, then by (15), we see that

$$
\delta(G_{22}) \geq \delta(G[V'_2]) - s_2 - 2 \\
> (2/3 - 1.1\alpha_1)n_2 - ((1/6 - 1.1\alpha_1)n_2 - 2) - 2 \\
> n_2/2 \geq n_{22}/2.
$$

Thus, $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$, implying that

$$n_{22} \leq n_1 - s_2 \leq (5/6 + 1.1\alpha_1)n_2 + 2 \quad \text{gives that} \quad n_2 \geq \frac{n_{22} - 2}{5/6 + 1.1\alpha_1}.
$$

By (15) and (22),

$$
\delta(G_{22}) \geq \delta(G[V'_2]) - s_2 - 2 \\
\geq (2/3 - 1.1\alpha_1)n_2 - (1/6 + 0.7\alpha_1)n_2 - 2 \\
\geq (1/2 - 1.9\alpha_1)n_2 \geq \frac{1/2 - 1.9\alpha_2}{5/6 + 1.1\alpha_2}(n_{22} - 2) \\
> n_{22}/2.
$$

The last inequality follows by assuming that $2.45\alpha_1 + 2/n_{11} < 1/12$. Hence, by Lemma 4.6, there is a hamiltonian $(y_L^1, y_L^2)$-path $P_2$ in $G_{22}$.

**Step 7. Forming an SGHG**

Let $T = T_1 \cup T_2 \cup \{x_{NYN}\}$ and $C = P_1 \cup P_2 \cup \{x_L^1y_L^1, x_L^2y_L^2\}$. We see that $T$ is a HIST of $G$ with $L(T) = V(P_1) \cup V(P_2)$ and $C$ is a cycle spanning on $L(T)$. Hence $H = T \cup C$ is an SGHG of $G$.

**4.3 Proof of Theorem 4.3**

Notice that the assumption of Extremal Case 2 implies that

$$|V_1| > (3/5 - \alpha)n \quad \text{and} \quad |V_2| \geq (2/5 - 2\beta)n.
$$

We may assume that the graph $G$ is minimal with respect to the number of edges. This implies that no two adjacent vertices both have degree larger than $(2n + 3)/5$. (For otherwise, we could delete any edges incident to two vertices both with degree larger than $(2n + 3)/5$.) We construct an SGHG in $G$ step by step.

**Step 1. Repartitioning**

Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let

$$
V_2' = \{v \in V_2 \mid \deg(v, V_1) \geq (1 - 3\alpha_1)|V_1|\}, \\
V_0' = \{v \in V_2 - V_2' \mid \deg(v, V_1) \leq \alpha_1|V_2|/6\}, \\
V_1' = V_1 \cup V_0', \quad V_{12}' = V_2 - V_2' - V_0'.
$$
As \( d(V_1, V_2) \geq 1 - 3\alpha \), the following holds,

\[
(1 - 3\alpha)|V_1||V_2| \leq e_G(V_1, V_2) = e_G(V_1, V_2') + e_G(V_1, V_2 - V_2') \\
\leq |V_1||V_2'| + (1 - 3\alpha)|V_1||V_2 - V_2'|.
\]

The inequality implies that

\[
|V_2 - V_2'| \leq \alpha_2|V_2|.
\] (24)

As a consequence of moving vertices in \( V_2 - V_2' \) out from \( V_2 \), by (24) we get

\[
\delta(V_1, V_2') \geq (2n + 3)/5 - 6\beta|V_2| - \alpha_2|V_2| \\
\geq (2n + 3)/5 - 2\alpha_2|V_2|,
\] (25)

provided that \( 6\beta \leq \alpha_2 \). And as a consequence of moving vertices in \( V_0' \) to \( V_1 \),

\[
\delta(V_0', V_2') \geq \delta(G' - \Delta(V_0', V_1) - \Delta(V_0', V_2 - V_2')) \\
\geq (2n + 3)/5 - \alpha_1|V_2|/6 - \alpha_2|V_2| \\
\geq (2n + 3)/5 - \alpha_1|V_2|/3 \text{ (provided that } \alpha_2 \leq \alpha_1/6),
\] (26)

and

\[
\alpha_1|V_2|/6 < \delta(V_0', V_1') < (1 - 3\alpha_1)|V_1|.
\] (27)

From (25) and (26), we have

\[
\delta(V_0', V_2) \geq (2n + 3)/5 - \alpha_1|V_2|/3.
\] (28)

As

\[
\delta(V_2', V_1') \geq (1 - 3\alpha_1)|V_1| \geq (1 - 3\alpha_1)(3/5 - \alpha)n > \lceil (2n + 3)/5 \rceil,
\]

we get that

\[
deg(v_1') = \lceil (2n + 3)/5 \rceil
\] (30)

for each \( v_1' \in V_1' \), by the minimality assumption of \( e(G) \). Hence (28) and (30) give that

\[
\Delta(G|V_1'|) \leq \alpha_1|V_2|/3.
\] (31)

Step 2. Finding a vertex \( v_2' \) from \( V_2' \) with large degree in \( V_1' \)

Let

\[
e_{in} = e(G|V_1'|)
\]

be the number of edges within \( V_1' \), notice that \( e_{in} \) maybe 0. Then

\[
e_G(V_1', V_2' \cup V_0') = |V_1'|\lceil (2n + 3)/5 \rceil - 2e_{in}.
\] (33)
Let
\[ d_{in} = e_{in}/|V'_1| \quad \text{and} \quad |n_0| = |V'_2 \cup V^0_{12}| − \lfloor (2n + 3)/5 \rfloor. \] (34)

By (31) and the definition of \( d_{in} \) in (34), we have
\[ |d_{in}| \leq |V_2|/6. \]

In fact, since \( \Delta(V_1, V'_1) \leq \Delta(V_1, V_1) + \Delta(V_1, V'_0) \leq 2\beta n + |V'_0| \leq 2\beta n + \alpha_2|V_2| \), and \( \Delta(V'_0, V'_1) \leq \alpha_1|V_2|/6 + \alpha_2|V_2| \), more precisely, we have
\[
2d_{in} = 2e_{in}/|V'_1| \leq (2\beta n + \alpha_2|V_2|)|V_1|/|V'_1| + (\alpha_1|V_2|/6 + \alpha_2|V_2|)|V'_0|/|V'_1|
\leq (2\beta n + \alpha_2|V_2|) + \alpha_2(\alpha_1|V_2|/6 + \alpha_2|V_2|) \quad \text{(as } |V'_0| \leq \alpha_2|V_2| \text{ and } |V_1|, |V_2| \leq |V'_1|)\]
\leq 2\alpha_2|V_2| \quad \text{(provided that } 6\beta + \alpha/6 + \alpha^2_2 \leq \alpha_2). \quad (35)

Set

**Case A.** \( [(2n + 3)/5] - |V'_2 \cup V^0_{12}| = n_0 \geq 0; \)

**Case B.** \( |V'_2 \cup V^0_{12}| - [(2n + 3)/5] = n_0 \geq 1. \)

We have
\[
n_0 = \begin{cases} 
[(2n + 3)/5] - |V'_2 \cup V^0_{12}| \leq 2\beta n + \alpha_2|V_2| \leq (6\beta + \alpha_2)|V_2| \leq 2\alpha_2|V_2|, & \text{Case A,} \\
|V'_2 \cup V^0_{12}| - [(2n + 3)/5] \leq (2\beta + \alpha)n - [(2n + 3)/5] \leq \alpha n, & \text{Case B.} 
\end{cases} \quad (36)
\]

Then in case A,
\[
e_G(V'_1, V'_2 \cup V^0_{12}) = |V'_1|[(2n + 3)/5] - 2e_{in} \quad \text{(by (30))}
= |V'_1|(|V'_2 \cup V^0_{12}| + n_0 - 2d_{in})
\geq |V'_2 \cup V^0_{12}|(|V'_1| + 1.4n_0 - 3.2d_{in}),
\]
as \( 1.4|V'_2 \cup V^0_{12}| \leq 1.4((2n + 3)/5 + \alpha n) \leq (3/5 - \alpha)n < |V'_1| \) and \( 1.6|V'_2 \cup V^0_{12}| \geq 1.6((2n + 3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1| \) provided that \( 2.4\alpha < 1/25 \) and \( 5.2\beta + 2.6\alpha_2 \leq 1/25 \) respectively. Since \( e_G(V'_1, V'_2 \cup V^0_{12}) \leq |V'_2 \cup V^0_{12}| |V'_1| \), we have \( |V'_1| + 1.4n_0 - 3.2d_{in} \leq |V'_1| \), and thus \( 1.4n_0 \leq 3.2d_{in} \).

In Case B,
\[
e_G(V'_1, V'_2 \cup V^0_{12}) = |V'_1|[(2n + 3)/5] - 2e_{in} \quad \text{(by (30))}
= |V'_1|(|V'_2 \cup V^0_{12}| - n_0 - 2d_{in})
\geq |V'_2 \cup V^0_{12}|(|V'_1| - 1.6n_0 - 3.2d_{in}),
\]
as \( 1.6|V'_2 \cup V^0_{12}| \geq 1.6((2n + 3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1| \) provided that \( 5.2\beta + 2.6\alpha_2 \leq 1/25. \)
Let 
\[ d_t = \begin{cases} 
3.2d_{in} - 1.4n_0, & \text{if Case A,} \\
1.6n_0 + 3.2d_{in}, & \text{if Case B.} 
\end{cases} \] (37)

By (35) and (36), we see that 
\[ d_t \leq \begin{cases} 
3.2\alpha_2|V_2|, & \text{if Case A,} \\
6.4\alpha_2|V_2|, & \text{if Case B.} 
\end{cases} \] (38)

Then there is a vertex \( v_*^* \) in \( V'_2 \cup V^0_{12} \) of degree at least \( |V'_1| - d_t \). We will fix this vertex in what follows. In fact, such a vertex \( v_*^* \) is in \( V'_2 \) by the facts that 
\[ \delta(V^0_{12}, V'_1) < (1 - 3\alpha_1)|V_1| \quad \text{and} \quad |V'_1| - d_t \geq (1 - 3\alpha_1)|V_1|, \] (39)

where \( |V'_1| - d_t \geq (1 - 3\alpha_1)|V_1| \) holds because of (38).

**Step 3. Finding a matching \( M \) within \( G[\Gamma(v_*^*, V'_1)] \)**

In this step, if \( e_{in} \geq 1 \), we first find a matching within \( G[V'_1] \) of size at least \( e_{in} / (2 \Delta (G[V'_1])) \). We assume this by giving the following lemma.

**Lemma 4.7.** If \( G \) is a graph with maximum degree \( \Delta \), then \( G \) contains a matching of size at least \( \frac{|E(G)|}{2\Delta} \).

**Proof.** We use induction on \( |V(G)| \). We may assume that the graph is connected. For otherwise, we are done by the induction hypothesis. Let \( e = xy \in E(G) \) be an edge and \( G' = G - \{x, y\} \). Since \( |N_G(x) \cup N_G(y)| - |\{x, y\}| \leq 2(\Delta - 1) \), we have 
\[ e(G') \geq e(G) - 2(\Delta - 1) - 1 \geq e(G) - 2\Delta. \]

Hence, by the induction hypothesis, \( G' \) has a matching of size at least \( \frac{e(G) - 2\Delta}{2\Delta} = \frac{e(G)}{2\Delta} - 1 \). Adding \( e \) to the matching obtained in \( G' \) gives a matching of size at least \( \frac{e(G)}{2\Delta} \) in \( G \). \( \square \)

In case A, we take a matching in \( G[V'_1] \) of size at least \( \max\{|11d_{in}|, 11n_0\} \). This is possible because 
\[ \frac{e_{in}}{2 \Delta (G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} = \frac{3d_{in}}{2\alpha_1} \geq 11d_{in} \]
provided that \( \alpha \leq (\frac{3}{22})^3 \), and 
\[
2e_{in} \geq |V'_1|([2n + 3]/5) - |V'_1||V'_2| - (1 - 3\alpha_1)|V_1||V^0_{12}|
\geq |V'_1|([2n + 3]/5) - |V'_1|([2n + 3]/5) - n_0 - |V^0_{12}| - |V_1||V^0_{12}| + 3\alpha_1|V_1||V^0_{12}|
\geq |V'_1|n_0 + 3\alpha_1|V_1||V^0_{12}| \]
(40)

implying that 
\[ \frac{e_{in}}{2 \Delta (G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} \geq \frac{|V'_1|n_0/2}{2\alpha_1|V'_1|/3} \geq \frac{3n_0}{4\alpha_1} \geq 11n_0 \]

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provided that $\alpha \leq \left(\frac{3}{14}\right)^3$.

By (37), $|V'_1| - \Gamma(v^*_2, V'_1) \leq d_l \leq |3.2d_{in}|$, we can then choose a matching $M$ from $\Gamma(v^*_2, V'_1)$ such that

$$|M| = \max\{|7d_{in}|, 7n_0\}. \quad (41)$$

In case B, we take a matching in $G[V'_1]$ of size at least $|8d_{in}|$. This is possible as

$$\frac{e_{in}}{\Delta(G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} = \frac{3d_{in}}{2\alpha_1} \geq |8d_{in}|$$

provided that $\alpha \leq \left(\frac{3}{14}\right)^3$.

By the second equality of (37), $|V'_1| - \Gamma(v^*_2, V'_1) \leq |3.2d_{in} + 1.6n_0|$. If $n_0 < 2d_{in}$, then $|3.2d_{in} + 1.6n_0| \leq |7d_{in}|$. Thus, there is a matching $M$ within $\Gamma(v^*_2, V'_1)$ such that

$$|M| = \begin{cases} 
    \lfloor d_{in} \rfloor & \text{if } n_0 < 2d_{in}, \\
    0 & \text{if } n_0 \geq 2d_{in}. 
\end{cases} \quad (42)$$

We fix $M$ for denoting the matching we defined in this step hereafter.

**Step 4. Insertion**

In this step, we insert vertices in $V^0_{12}$ into $V'_1 - V(M)$. Let $I = V^0_{12} = \{x_1, x_2, \ldots, x_I\}$, $U_1 = \Gamma(v^*_2, V'_1) - V(M)$, and $U_2 = V'_2$. Then (i)

$$\delta(I, U_1) \geq \delta(I, V'_1) - |V(M)| - |V'_1 - \Gamma(v^*_2, V'_1)|$$
$$\geq \alpha_1|V_2|/6 - \max\{|7d_{in}|, 7n_0\} - \lfloor 1.6n_0 + 3.2d_{in} \rfloor,$$
$$\geq 3\alpha_2|V_2| \geq 3|I| \quad (\text{provided that } 23.4\alpha_2 \leq \alpha_1/6),$$

and from (28), we have (ii)

$$\delta(U_1, U_2 - \{v^*_1\}) \geq \lceil (2n + 3)/5 \rceil - \alpha_1|V_2|/3 - 1 > \alpha_2|V_2| \geq |I|.$$

By condition (i), there is a claw-matching $M_1$ between $I$ and $U_1$ centered in $I$. Suppose that $\Gamma(x_i, M_1) = \{x_{i0}, x_{i1}, x_{i2}\}$. We denote by $P_{x_i}$ the path $x_{i1}x_i x_{i2}$. By (ii), there is a matching $M_2$ between $\{x_{i0} \mid 1 \leq i \leq |I|\}$ and $U_2 - \{v^*_2\}$ covering $\{x_{i0} \mid 1 \leq i \leq |I|\}$. So far, we get two matchings $M_1$ and $M_2$.

Next we delete three types of edges not contained in

$$|I| \cup \bigcup_{i=1}^{\lceil |I|/5 \rceil - 1} E(P_{x_i}) \cup \{x_i x_{i0} : 1 \leq i \leq |I|\}.$$
Those edges include edges incident to a vertex in $I$, edges incident to a vertex in
$$
\bigcup_{i=1}^{\lvert I \rvert} \left( (\Gamma(x_{i1}) - \Gamma(x_{i2})) \cup (\Gamma(x_{i2}) - \Gamma(x_{i1})) \right),
$$
and one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both $x_{i1}$ and $x_{i2}$, for each $i = 1, 2, \cdots, \lvert I \rvert$.

For the resulting graph after the deletion of edges above, we contract each path $P_{x_i}$ ($1 \leq i \leq \lvert I \rvert$) into a single vertex $v_{x_i}$. We call each $v_{x_i}$ a wrapped vertex and call $P_{x_i}$ the preimage of $v_{x_i}$. Denote by $G^*$ the graph obtained by deleting and contracting the same edges as above, and let $U_2^* = V_2'$ and $U_1^* = V(G^*) - U_2^*$. (We will need the following degree condition in the end of this proof.) Since $|U_2^*| = |V_2'| \leq (2/5 + \alpha)n$, combining with (28), we have
$$
deg(v_{x_i}, U_2^*) \geq |\Gamma(x_{i1}, U_2^*) \cap \Gamma(x_{i2}, U_2^*)| - 1 \geq 2n/5 - \alpha_1|V_2|.
$$

By the above inequality and (28), we get the first inequality below in (43). Since one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both $x_{i1}$ and $x_{i2}$ is deleted in $G^*$ for each $i = 1, 2, \cdots, \lvert I \rvert$, combining with (29), we have the second inequality as follows.

$$
\begin{align*}
\delta(U_1^*, U_2^*) & \geq 2n/5 - \alpha_1|V_2|, \\
\delta(U_2^*, U_1^*) & \geq \delta(V_2', V_1') - 1 \geq (1 - 3\alpha_1)|V_1| - 1. 
\end{align*}
$$

Let $U_1'$ and $U_2'$ be the corresponding sets of $U_1$ and $U_2$, respectively, after the contraction. Let $T_W$ be the graph with

$$
V(T_W) = \{x_{i0}, v_{x_i} : 1 \leq i \leq \lvert I \rvert\} \cup (V(M_2) \cap U_2) \quad \text{and} \quad E(T_W) = \{x_{i0}v_{x_i} : 1 \leq i \leq \lvert I \rvert\} \cup E(M_2).
$$

By the construction,

$$
\lvert V(T_W) \cap U_1' \rvert = \lvert \{x_{i0}, v_{x_i} : 1 \leq i \leq \lvert I \rvert\} \rvert = 2\lvert I \rvert, \quad \lvert L(T_W) \cap U_1' \rvert = \lvert \{v_{x_i} : 1 \leq i \leq \lvert I \rvert\} \rvert = \lvert I \rvert, \quad \text{and} \quad \lvert V(T_W) \cap U_2' \rvert = \lvert L(T_I) \cap U_2' \rvert = \lvert V(M_2) \cap U_2' \rvert = \lvert I \rvert.
$$

Notice that $T_W$ is a forest with $\lvert I \rvert$ components and each vertex $x_{i0}$ ($1 \leq i \leq \lvert I \rvert$) has degree 2 in $T_W$. (We will make $T_W$ connected in the end by connecting each $x_{i0}$ to $v_2^*$.) See a depiction of this operation with $\lvert I \rvert = 1$ in Figure 4 below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{$T_W$ with $\lvert I \rvert = 1$}
\end{figure}
Let $U'_I = (V'_I - U_1) \cup U'_I - V(T_W)$, $U''_I = U'_2 - V(T_W)$, and $G_I$ the resulting graph with $V(G_I) = U'_I \cup U''_I$. We have that

$$|U'_I| = |V'_I| - 3|I| = |V'_I| - 3n_{12}, \quad |U''_I| = |V''_I| - |I| = |V'_I| - n_{12},$$
$$\delta(U'_I, U''_I) = \delta(V'_I, V''_I) - n_{12} \geq \left[(2n + 3)/5 \right] - \alpha_1|V_2|/3 - n_{12},$$
$$\delta(U''_I, U'_I) \geq \delta(V'_I, V''_I) - 3n_{12} \geq (1 - 3\alpha_1)|V_1| - 3n_{12}. \quad (44)$$

**Step 5. Matching Extension**

In this step, in the graph $G_I$, we do some operation on the matching $M$ found in Step 3. Notice that the vertices in $M$ are unused in Step 4. Recall that $|M| \leq \max\{7n_0, \left\lfloor 7d_{in} \right\rfloor \}$. By $|d_{in}| \leq \alpha_2|V_2|$ from (35) and $n_0 \leq 2\alpha_2|V_2|$ from (36), we get

$$|M| \leq 14\alpha_2|V_2|. \quad (45)$$

Hence, $\delta(U'_I, U''_I - \{v'_2\}) \geq \left[(2n + 3)/5 \right] - \alpha_1|V_2|/3 - n_{12} - 1 \geq |M|$. Let $V_M$ be the set of vertices containing exactly one end of each edge in $M$. Then there is a matching $M'$ between $V_M$ and $U_2 - \{v'_2\}$ covering $V_M$. Let $F_M$ be a forest with

$$V(F_M) = V(M) \cup (V(M') \cap U_2) \quad \text{and} \quad E(F_M) = E(M) \cup E(M').$$

Notice that

$$|V(F_M) \cap U_1| = 2|M|, \quad |L(F_M) \cap U_1| = |V(M) - V_M| = |M|,$$
$$|V(F_M) \cap U_2| = |L(F_M) \cap U_2| = |M|.$$

Notice that $F_M$ has $|M|$ components, and all vertices in $V_M$ has degree 2. (We will make $F_M$ a HIT later on by connecting each vertex in $V_M$ to the vertex $v'_2 \in U_2$) See Figure 5 for a depiction of $F_M$ with $|M| = 3$.

![Figure 5: F_M with |M| = 3](image)

Let

$$U^1_M = U^1_I - V(F_M) \quad \text{and} \quad U^2_M = U^2_I - V(F_M).$$

Notice that

$$|U^1_M| = |U^1_I| - 2|M| = |V'_I| - 3n_{12} - 2|M|,$$
$$|U^2_M| = |U^2_I| - |M| = |V'_I| - n_{12} - |M|, \quad (46)$$

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Suppose now the graph with vertices. Then by the conditions (i) and (ii), for each 1

\[ \delta(U^1_M, U^2_M) = \left\lceil \frac{2n + 3}{5} \right\rceil - \alpha_1|V_2|/3 - n^0_{12} - |M|, \]
\[ \delta(U^2_M, U^1_M) \geq (1 - 3\alpha_1)|V_1| - 3n^0_{12} - 2|M|. \] (47)

**Step 6. Distribute Remaining vertices in** \( U^1_M - \Gamma(v^*_2, V)' \)

Let

We may assume \( n_l \geq 1 \). For otherwise, we skip this step. By (38), we have

\[ n_l \leq \begin{cases} 3.2\alpha_2|V_2|, & \text{Case A,} \\ 6.4\alpha_2|V_2|, & \text{Case B.} \end{cases} \] (48)

By \( n^0_{12} \leq \alpha_2|V_2| \) from (24) and \( |M| \leq 14\alpha_2|V_2| \) from (45), we have (i)

\[ \delta(U^1_M, U^2_M) \geq \left\lceil \frac{2n + 3}{5} \right\rceil - \alpha_1|V_2|/3 - n^0_{12} - |M| \geq \left\lceil \frac{2n + 3}{5} \right\rceil - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \]
\[ \geq (1 - 3\alpha)|V_2| - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \text{ (as } \left\lceil \frac{2n + 3}{5} \right\rceil \geq (1 - 3\alpha)(2/5 + \alpha)n \text{)} \]
\[ \geq (1 - 3\alpha - \alpha_1/3 - 15\alpha_2)|V_2| \geq (1 - \alpha_1)|V_2| \text{ (provided } 3\alpha + 15\alpha_2 \leq 2\alpha_1/3 \text{)} \]
\[ \geq (1 - \alpha_1)|U^2_M|. \] (49)

By (46) and (48), we have (ii)

\[ |U^2_M| - 10\alpha_1|V_2| - \left\lceil n_l/10 \right\rceil - 1 \geq |V^*_2| - n^0_{12} - |M| - 16\alpha_1|V_2| - 0.64\alpha_2|V_2| - 2 \]
\[ \geq (1 - \alpha_2 - 14\alpha_2 - 10\alpha_1 - 0.64\alpha_2 - |V_2|/2)|V_2| \]
\[ \geq (1 - 11\alpha_1)|V_2| \text{ (provided } 15.64\alpha_2 + |V_2|/2 \leq \alpha_1 \text{)} \]
\[ \geq 0 \text{ (provided } 11\alpha_1 < 1 \text{).} \]

Let \( U_R = U^1_M - \Gamma(v^*_2, V)' \) and denote \( \left\lceil \frac{|U_R|}{10} \right\rceil = l \). Suppose first that \( |U_R| \geq 2 \). We partition \( U_R = U_{R_1} \cup U_{R_2} \cup \cdots \cup U_{R_l} \) arbitrary such that each set contains at least 2 and at most \( |U_R|/10 \) vertices. Then by the conditions (i) and (ii), for each 1 \( i \leq l \), there is a vertex \( y_i \in U_2 - \{v^*_2\} \) which is common to all vertices in \( U_{R_i} \), and is not used by any other \( U_{R_j} \) with \( j \neq i \). Let \( T_R \) be the graph with

\[ V(T_R) = U_R \cup \{y_i : 1 \leq i \leq l\} \text{ and } E(T_R) = \{xy_i : x \in U_{R_i}, 1 \leq i \leq l\}. \]

Suppose now \( |U_R| = 1 \), let \( U_R = \{x_R\} \). Choose \( x'_R \in U^1_M - U_R \) and \( y_R \in U^2_M - \{v^*_2\} \) be a vertex common to \( x_R \) and \( x'_R \). Let \( T_R \) be a tree with

\[ V(T_R) = \{x_R, x'_R, y_R\} \text{ and } E(T_R) = \{x_Ry_R, x'_Ry_R\}. \]

By the construction,

\[ |V(T_R) \cap U^1_M| = |L(T_R \cap U^1_M| = \max\{|U_R|, 2\}, \quad |V(T_R) \cap U^2_M| = l, \quad \text{and} \quad |L(T_R \cap U^2_M| = 0. \]
Then we have the idea, we skip the details.

Let $G$ be the graph with vertices in $U \cup \{v_1, v_2, \ldots, v_n\}$. Notice that $T_R$ is not connected when $|U_R| \geq 17$ and that $T_R$ may have degree 2 vertices in $V(T_R) \cap U^2_M$. Later on, by joining each vertex in $T_R \cap U^2_M$ to a vertex of a tree, we will make the resulting graph connected, and thereby eliminating the possible degree 2 vertices in $T_R$. Let

$$U^1_R = U^1_M - V(T_R) \quad \text{and} \quad U^2_R = U^2_M - V(T_R).$$

Then we have

$$|U^1_R| = |U^1_M| - n_l = |V^t_1| - 3n^0_{12} - 2|M| - \max\{2, n_l\},$$

$$|U^2_R| = |U^2_M| - \lceil n_l/10 \rceil = |V^t_2| - n^0_{12} - |M| - \lceil n_l/10 \rceil,$$

and

$$\delta(U^1_R, U^2_R) \geq \max \{ (2n + 3)/5 \} - \alpha_1|V_2|/3 - n^0_{12} - |M| - \lceil n_l/10 \rceil,$$

$$\delta(U^2_R, U^1_R) = (1 - 3\alpha_1)|V_1| - 3n^0_{12} - 2|M| - \max\{2, n_l\}.$$ (51)

Let $G_R$ be the subgraph of $G$ induced on $U^1_R \cup U^2_R$.

**Step 7. Completion of a HIST in $G_R$**

In this step, we find a HIST $T_{main}$ in $G_R$ such that

$$|L(T_{main}) \cap U^1_R| = |L(T_W)|/2 + |L(F_M) \cap U^1_I| + |L(T_R) \cap U^1_M| =$$

$$|L(T_{main}) \cap U^2_R| = |L(T_W)|/2 + |L(F_M) \cap U^2_I| + |L(T_R) \cap U^2_M|.$$ (52)

By the construction of $F_M$ and $T_R$, we have $|L(F_M) \cap U^1_I| = |L(F_M) \cap U^2_I|$ and $|L(T_R) \cap U^1_M| - |L(T_R) \cap U^2_M| = \max\{2, n_l\}$, respectively. So

$$|L(T_{main}) \cap U^2_R| - |L(T_{main}) \cap U^1_R| = \max\{2, n_l\}.$$ (53)

Notice that $v^*_2 \in U^2_R$, $v^*_2$ is adjacent to each vertex in $U^1_R$, and $V^t_1 - \Gamma(v^*_2, V^t_1) \subseteq V(U^1_R)$. We now construct $T_{main}$ step by step.

**Step 7.1**

Let $T^1_{main}$ be the graph with

$$V(T^1_{main}) = \{v^*_2\} \cup U^1_R \quad \text{and} \quad E(T^1_{main}) = \{v^*_2x \mid x \in U^1_R\}.$$ (54)

To make the requirement of (52) possible, we need to make at least

$$d_3 = |U^1_R| - |U^2_R| + \max\{2, n_l\},$$

$$= |V^t_1| - |V^t_2| - 2n^0_{12} - |M| + \lceil n_l/10 \rceil$$ (53)

vertices in $U^1_R$ with degree at least 3 in $T_{main}$, where the last inequality above follows from (50). Hereinafter, we assume that $\max\{2, n_l\} = n_l$. Since the proof for $\max\{2, n_l\} = 2$ follows the same idea, we skip the details.
Since all vertices in $U_R^1$ are included in $T_{main}^1$ and $T_{main}^1$ is connected, each vertex in $T_{main}^1$ needs to join to at least two distinct vertices from $U_R^2 - \{v^*_2\}$ to have degree no less than 3. Hence, to make a desired HIST $T_{main}$, it is necessary that

$$d_{fs} = |U_R^2| - 1 - 2d_3 = |V'_2| - n_{12}^0 - e_M - [n_l/10] - 1 - 2d_3 = 3|V'_2| - 2|V'_1| + 3n_{12}^0 + |M| - 3[n_l/10] - 1 \geq 0.$$  

We show (54) is true, separately, for each of Case A and Case B. For Case A, notice that

$$|V'_1| = n - [(2n + 3)/5] + n_0 \quad \text{and} \quad |V'_2| = [(2n + 3)/5] - n_0 - n_{12}^0.$$  

Hence,

$$3|V'_2| = 3[(2n + 3)/5] - 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V'_1| = 2n - 2[(2n + 3)/5] + 2n_0.$$  

Thus,

$$d_{fs} = 5[(2n + 3)/5] - 2n - 5n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3[n_l/10] - 1 \geq 2 - 5n_0 + |M| - 3[3.2d_{in}]/10 \quad \text{(by $n_l \leq d_l[3.2d_{in} - 1.4n_0]$ from (37))}$$

$$= 2 - 5n_0 + \max\{7n_0, |7d_{in}|\} - 3[3.2d_{in}]/10 \geq 0.$$  

Now we show (54) is true for case B. Notice that

$$|V'_1| = n - [(2n + 3)/5] + n_0 \quad \text{and} \quad |V'_2| = [(2n + 3)/5] + n_0 - n_{12}^0.$$  

So

$$3|V'_2| = 3[(2n + 3)/5] + 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V'_1| = 2n - 2[(2n + 3)/5] + 2n_0.$$  

Recall that $n_0 \geq 1$ in this case. We have

$$d_{fs} = 5[(2n + 3)/5] - 2n + n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3[n_l/10] - 1 \geq 2 + n_0 + |M| - 3[n_l/10]$$

$$= 2 + n_0 + |M| - 3[3.2d_{in} + 1.6n_0]/10 \quad \text{(by $n_l \leq [3.2d_{in} + 1.6n_0]$ from (37))}$$

$$\geq \begin{cases} 
2 + n_0 + [d_{in}] - [9.2d_{in}/10] - [4.8n_0/10] - 1 \geq 0, & \text{if } n_0 < 2d_{in}; \\
2 + n_0 - [9.2d_{in}/10] - [4.8n_0/10] - 1 \geq 0, & \text{if } n_0 \geq 2d_{in}.
\end{cases}$$  

We now in Step 2 below show that there is a way to make exactly $d_{fs}$ vertices in $T_{main}^1$ with degree 3 by joining each to two distinct vertices from $U_R^2 - \{v^*_2\}$.

**Step 7.2**
We first take $2d_3$ vertices from $U^2_R - \{v_i^*\}$. For those $2d_3$ vertices, pair them up into $d_3$ pairs. We show that for each pair of vertices, they have at least $d_3$ common neighbors in $U^1_R$. Using (51), $|M| \leq 14\alpha_2|V_2|$ from (45), $n_t \leq d_t \leq 6.4\alpha_2|V_2|$ from (38), we have
\[
\delta(U^2_R, U^1_R) \geq (1 - 3\alpha_1)|V_1| - 3n_t^0 - 2|M| - \max\{2, n_t\}
\geq |V_1| - 3\alpha_1|V_1| - 3\alpha_2|V_2| - 28\alpha_2|V_2| - 6.4\alpha_2|V_2|
\geq |V'_1| - |V_1 - V'_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|.
\] (55)

Since $|U^1_R| \leq |V'_1|$, we know that any two vertices in $U^2_R$ have at least
\[n_c = |V'_1| - |V'_2| - 2n_t^0 - |M| - [n_t/10]\]
\[\geq (3/5 - \alpha)n - 76.8\alpha_2|V_2| - 6\alpha_1|V_1| \quad \text{(by $|V'_1 - V'_1| = |V'_0| \leq |V'_2 - V'_2| \leq \alpha_2|V_2|$)}
\[\geq 3n/5 - 10\alpha_1|V_1| \quad \text{(provided that 76.8\alpha_2 + 3\alpha \leq 4\alpha_1)}\]

common neighbors in $U^1_R$. On the other hand,
\[d_3 = |V'_1| - |V'_2| - 2n_t^0 - |M| - [n_t/10]\]
\[\leq (3/5 - \alpha)n - (2n/5 - 2\beta n - |V_2 - V'_2|) + (1.6n_0 + 3.2|d_m|)/10 + 1
\[= n/5 - \alpha n + 2\beta n + |V_2 - V'_2| + (3.2\alpha_2|V_2| + 3.2\alpha_2|V_2|)/10 \quad \text{(by (35) and (36))}
\[\leq n/5 - \alpha n + 2\beta n + \alpha_2|V_2| + 0.64\alpha_2|V_2|
\[\leq n/5 + 2\alpha_1|V_2| < n_c \quad \text{(provided 12\alpha_1 < 2/5)}.
\]

Denote by \{u^1_1, u^2_1\}, \{u^1_2, u^2_2\}, \cdots, \{u^1_{d_3}, u^2_{d_3}\} the $d_3$ pairs of vertices from $U^2_R - \{v_i^*\}$. Then by the above argument, we can choose $d_3$ distinct vertices say $v_1, v_2, \cdots, v_{d_3}$ from $L(T^1_{\text{main}})$ such that $v_i \sim u^1_i, u^2_i$ for all $1 \leq i \leq d_3$.

Let $T^2_{\text{main}}$ be the graph with
\[V(T^2_{\text{main}}) = V(T^1_{\text{main}}) \cup \{u^1_i, u^2_i : 1 \leq i \leq d_3\} \quad \text{and} \quad E(T^2_{\text{main}}) = E(T^1_{\text{main}}) \cup \{v_iu^1_i, v_iu^2_i : 1 \leq i \leq d_3\}.
\] If $V(G_R) - V(T^2_{\text{main}}) = \emptyset$, we let $T_{\text{main}} = T^2_{\text{main}}$. For otherwise, we need one more step to finish constructing $T_{\text{main}}$.

**Step 7.3**

For the remaining vertices in $U^2_R - V(T^2_{\text{main}})$, we show that each of them has a neighbor in $S(T^2_{\text{main}}) \cap U^1_R$: that is, a neighbor in $U^1_R$ of degree 3 in $V(T^2_{\text{main}})$. This is clear, as by (55), we have
\[
\delta(U^2_R, U^1_R) \geq |V'_1| - |V'_1 - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|
\geq |U^1_R| - 38.4\alpha_2|V_2| - 3\alpha_1|V_1| \quad \text{(by $|V'_1 - V_1| \leq |V'_2 - V_2| \leq \alpha_2|V_2|$)}.
\]

Since $|S(T^2_{\text{main}}) \cap U^1_R| = d_3$, and
\[d_3 = |V_1| - |V'_2| - 2n_t^0 - 2|M| + [n_t/10]\]
\[\geq (3/5 - \alpha)n - (2/5 + \alpha)n - 2\alpha_2|V_2| - 28\alpha_2|V_2| + 0.64\alpha_2|V_2|
\[\geq n/5 - 2\alpha n - 29.36\alpha_2|V_2|
\[> 38.4\alpha_2|V_2| + 3\alpha_1|V_1| \quad \text{(provided 2\alpha + 67.76\alpha_2 + 3\alpha_1 < 1/5)}.
\]
Now, we join an edge between each vertex in \( U_2' = V(T_{main}^0) \) and a neighbor of the vertex in \( S(T_{main}) \cap U_1^R \). Let \( T_{main} \) be the resulting tree. By the construction procedure, it is easy to verify that \( T_{main} \) is a HIST of \( G_R \).

**Step 8. Connecting \( T_W, F_M, T_R, \) and \( V(T_{main}) \) into a connected graph**

In this step, we connect \( T_W, F_M, T_R, \) and \( V(T_{main}) \) into a connected graph. Recall that each degree 2 vertex in \( T_W \) and \( F_M \) is a neighbor of \( v^*_2 \). We join an edge connecting \( v^*_2 \) in \( V(T_{main}) \) and each degree 2 vertex in \( T_W \) and \( F_M \). By the argument in step 7.3 above, we know each vertex in \( V(T_R) \cap U_2^R \) has a neighbor in \( S(T_{main}) \cap U_1^R \). Thus, we join an edge between each vertex in \( V(T_R) \cap U_2^R \) to exactly one of its neighbor in \( S(T_{main}) \cap U_1^R \). Let \( T^* \) be the final resulting graph. Notice that \( I = V_{12}^0 = \{x_1, x_2, \ldots, x_I\} \subseteq L(T^*) \) is the set of the wrapped vertices from step 4. Recall that \( G^* \) is the graph obtained from \( G \) be deleting and contracting edges from step 4. Then by the constructions of \( T_W, F_M, T_R, \) and \( T_{main} \), we see that \( T^* \) is a HIST of \( G^* \) with \( |L(T^*) \cap U_1^*| = |L(T^*) \cap U_2^*| \).

**Step 9. Finding a cycle on \( L(T^*) \)**

Denote
\[
U_1^L = L(T^*) \cap U_1^*, \quad U_2^L = L(T^*) \cap U_2^* \quad \text{and} \quad G_L = G[E_G(U_1^L, U_2^L)].
\]

Notice that \( G_L \) is a balanced bipartite graph. And
\[
|S(T^*) \cap U_1^*| = d_3 \leq n/5 + 2\alpha_1|V_2| \quad \text{(by (56))}
\]
\[
|S(T^*) \cap U_2^*| = 1 + \left\lceil n_1/10 \right\rceil \leq 2 + 0.64\alpha_2|V_2| \quad \text{(by \( n_1 \leq d_1 \leq 6.4\alpha_2|V_2| \) from (38)).}
\]

Thus by (43),
\[
\delta_{G^*}(U_1^L, U_2^L) \geq 2n/5 - \alpha_1|V_2| - (2 + 0.64\alpha_2|V_2|) > 3n/10 > |U_2^L|/2 + 1,
\]
\[
\delta_{G^*}(U_2^L, U_1^L) \geq (1 - 3\alpha_1)|V_1| - 1 - (n/5 + 2\alpha_1|V_2|) > n/3 > |U_1^L|/2 + 1.
\]

By Lemma 3.7, \( G_L \) contains a hamiltonian cycle \( C' \).

**Step 10. Unwrap vertices in \( V(C') \cap \{v_{x_1}, v_{x_2}, \ldots, v_{x_{|I|}}\} \)**

On \( C' \), replace each vertex \( v_{x_i} \) with its preimage \( P_{x_i} = x_{i1}; x_{i2} \) for each \( i = 1, 2, \ldots, |I| \). Denote the resulting cycle by \( C \). Recall that \( x_{i1}, x_{i2} \in \Gamma(v^*_2) \) by the choice of \( x_{i1} \) and \( x_{i2} \). Let \( T \) be the graph on \( V(G) \) with
\[
E(T) = E(T^*) \cup \{v^*_2x_{i1}, v^*_2x_{i2} : i = 1, 2, \ldots, |I|\}.
\]

Then \( T \) is a HIST of \( G \). Let \( H = T \cup C \). Then \( H \) is an SGHG of \( G \).

The proof of Extremal Case 2 is finished.
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