Internal space renormalization group for the Luttinger model

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The absence of fermionic, asymptotical one-particle states in the Luttinger model raises the suspicion that the interactions are actually strong at the vicinity of the Fermi points. The functional internal space renormalization group method, a systematic scheme for the computation of effective coupling strengths off the Fermi points, is applied to shed some light on this issue. A simple truncation of the evolution equation shows that the theory is indeed strongly coupled at the Fermi points for arbitrarily small value of the bare coupling constant and the removal of the cutoff is blocked by Landau poles. A peculiar feature of the normal ordering scheme is pointed out to explain the absence of these effects in the bosonized solution.

I. INTRODUCTION

One dimensional massless fermions display rather peculiar features. Owing to another specialty of one dimension, to the equivalence of the helicity and the chirality, the left and right moving fermion numbers are individually conserved in chiral invariant theories, like in the massless Thirring [1] or Gross-Neveu [2] models. The physics of electrons moving in a one-dimensional wire can effectively be described by linearizing the dispersion relation around the Fermi points. The resulting Luttinger model [3], a theory of relativistic, massless fermions, is the massless Thirring model except that the dispersion relations of the left and right moving fermions are displaced horizontally with respect to each other in order to accommodate the chemical potential. This model provides a rich structure to compare with experimentally well controlled situations [4].

One dimensional models can elegantly be renormalized by means of normal ordering and the works aiming at the Luttinger model are almost exclusively based on this method. Most widespread use of normal ordering is the bosonization. The bosonization of the excitations within a given fermion number sector of the Fock space is in principle possible in any dimensions. But, another interesting low dimensional “accident”, the bosonized effective theory [5–7] is local in dimension. The bosonized Luttinger model consists of free bosonic excitations which stand for the asymptotical, multi particle-hole states of the fermionic representation [5, 6, 8]. The most striking phenomenon of this chiral model is the loss of Fermi-liquid properties, the continuity of momentum distributions at the Fermi points at vanishing temperature. In the absence of a transverse direction to the motion the slightest interactions destabilize the particle-like asymptotical states which always propagate with the same speed. The asymptotical states of the theory, if they exist, must correspond either to massive particles or to non-interacting quasi-particles.

The non-interacting nature of the asymptotical sector seems to be consistent with the vanishing of the beta functions, confirmed in a number ways, independently of the bosonization. One chain of arguments is based on the application of Ward identities which are used to show a perfect cancellation between the self energy and the vertex corrections and to render the simple RPA exact [9–11]. The Wilsonian blocking employed in momentum space for the coupling constant of the bare action produces vanishing beta function [12, 13], too.

It is worthwhile mentioning a superficial analogy with the confinement of color charges in QCD. In fact, the non-propagating nature of the elementary fermionic excitations in one dimension is equivalent to the cancellation of the residue of the particle pole in the Green functions and the scattering amplitude. These are in turn the hallmark of confinement in hadron physics, the absence of colored asymptotical states. But the closer look betrays that the two phenomena are actually different. In fact, the bare particles of the one-dimensional world remain non-propagating for either sign of the coupling constant, i.e. both for attractive and for repulsive interactions, as a result of the kinematical origin of the effect. On the contrary, the attractive color forces are crucial in removing the colored asymptotical states in QCD.

Nevertheless, the superficial analogy between these cases raises other puzzles. The quark confinement is already a rather spectacular phenomenon because it changes the degrees of freedom with the scale of observation. But the
strong correlations between far separated quarks is reflected in the strong interactions among the quasiparticles, the hadrons. What happens in the bosonized model is even more surprising: How could the complete screening between "hadrons" take place in contrast to the strong, long-range correlations between the elementary constituents? Another question comes from the fact that classically scale invariant, renormalizable models display dimensional transmutation [14]. It arises when a (partial)resummation of the perturbation expansion produces a combination of the cutoff Λ and the dimensionless bare couplings in observables which remains finite in the limit Λ → ∞. This must happen unless the IR scaling laws remain the same as in the UV regime. The non-interacting, massless particle-hole quasiparticles posses no scale, any scale generated would indicate finite interaction strength. Why there is no dimensional transmutation, or different scaling regimes in the Luttinger model? No other classically scale invariant, renormalizable model shows such a simple structure.

These questions lead to a one-loop renormalization group study of the Luttinger model [15] which gave non-trivial beta functions, strongly coupled dynamics and singular momentum dependence in the effective interaction strength around the Fermi point. Furthermore, there were UV Landau poles in certain kinematical regimes of the effective coupling strengths, suggesting problems with renormalizability. This introduces the question whether the Luttinger model is indeed renormalizable.

We argue below that renormalization schemes based on normal ordering are rather fragile in the sense that cutoff independent, finite physical effects, such as dimensional transmutation and Landau poles, may arise from parts of the renormalized action which are vanishing when the cutoff is removed. In order to see if such phenomena are indeed present in the Luttinger model we turn to another powerful method, the functional renormalization group in the internal space [16].

In the usual renormalization group strategy the UV cutoff is lowered and the dynamics of the degrees of freedom, eliminated in this manner, is taken into account by adjusting the parameters of an effective bare action. On contrary, the fluctuations are turned on gradually in the internal space scheme and their effects are accumulated by the evolution equation for the effective action [17]. The advantage of this method is that it induces weaker non-local artifacts as blocking in momentum space and it gives access to effective interaction strengths off the Fermi points. These features are essential for the Luttinger model where eventual strong interactions appear as a singularity of the effective interaction strengths in momentum space in the vicinity of the Fermi points.

The only approximation needed in the functional renormalization group method is the restriction of the evolution equation onto a functional space with manageable size. The perturbation expansion and the functional renormalization group based solutions of a model can be contrasted by stating that one resums infinitely many orders generated by few relevant vertices in the former and uses one-loop expressions but infinitely many effective vertices in the latter procedure. We made a modest step in this work in extending the computation towards more coupling constants and considered the Luttinger model with two running coupling constants whose flow is obtained in the one-loop level.

Our results confirm the absence of evolution of the coupling constant when it is considered just at the Fermi points. But as soon as one comes off the Fermi points even by an infinitesimal amount our renormalization group scheme produces a non-trivial evolution which renders the theory strongly coupled in the vicinity of the Fermi point for arbitrary bare parameters. The cutoff generates Landau poles which seems to block the way of the removal of the cutoff. These results contradict the picture emerging from the normal ordering solutions and suggest the survival of cutoff effects.

The plan of the paper is the following. The model and its generating functionals are introduced in Section II. A peculiar feature of the normal ordering schemes is discussed in Section III. The internal space renormalization group is briefly described and applied to the Luttinger model in Section IV. The projection of the evolution of the equation into a suitable chosen functional subspace is discussed in Section V. The projected evolution equation for the effective action is given in Section VI. The simplest application of the evolution equation is presented in Section VII where the effective coupling strength is chosen at the Fermi points and the vanishing beta function is recovered. Section VIII contains the main results of this work, the integration of the beta functions for coupling strength off the Fermi points. Our results are summarized in Section IX. Two appendices facilitate the reading of the paper. The first sums up the notation and conventions used in the paper and the second contains few details of the computation of the one-loop integrals of the evolution equation.
II. LUTTINGER MODEL

The Luttinger model and its generating functional for the connected and the one-particle irreducible (1PI) Green functions are introduced in this section. The model is defined by the action \( S[\psi^\dagger, \psi] = S_0[\psi^\dagger, \psi] + S_1[\psi^\dagger, \psi] \), with

\[
S_0[\psi^\dagger, \psi] = \sum_{\sigma} \int_x \psi^\dagger_{\sigma;x} [\partial_t + \epsilon_\sigma (-i\partial_x)] \psi_{\sigma;x}
\]

\[
S_1[\psi^\dagger, \psi] = \sum_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} \int_{x_1, y_1, x_2, y_2} g^\sigma_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} \psi^\dagger_{\sigma_1, x_1} \psi_{\sigma_2, y_1} \psi^\dagger_{\sigma_3, x_2} \psi_{\sigma_4, y_2},
\]

where the field operator \( \psi_{\sigma;x} \) stands for left or right moving fermions with \( \sigma = -1 \) or \( +1 \), respectively and

\[
\epsilon_\sigma(p) = \sigma p - k_F.
\]

See Appendix A for notations and conventions. The separate conservation of number of left and right movers is achieved by allowing the interactions \( g^+_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} \) only to be non-vanishing.

Symmetry under space inversion requires \( g^+_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = g^+_{\sigma_4, \sigma_3, \sigma_2, \sigma_1} \) and \( g^-_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = g^-_{\sigma_4, \sigma_3, \sigma_2, \sigma_1} \).

In order to determine the space-dependence of the coupling constants one introduces the Fourier transforms

\[
\psi_{\sigma;x} = \int_p \Theta(\sigma p) e^{ipx} \psi_{\sigma;p}, \quad \psi^\dagger_{\sigma;x} = \int_p \Theta(\sigma p) e^{-ipx} \psi^\dagger_{\sigma;p}
\]

and

\[
g^\sigma_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = \int_{p_1, p_2, p_3, p_4} e^{i[p_1 x_1 - ip_2 y_1 + ip_3 x_2 - ip_4 y_2]} \delta_{p_1 - p_2 + p_3 - p_4} g^\sigma_{\sigma_1, \sigma_2, \sigma_3, \sigma_4}.
\]

The coupling constants which order the Grassmanián fields in the path integral according to Eq. (1) are given by

\[
\hat g^0_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = e^{i\eta(2p_1^0 - p_2^0 - p_3^0 + 2p_4^0)} g^0 \times
\]

\[
\hat g^+_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = e^{i\eta(2p_1^0 - p_2^0 - p_3^0 + 2p_4^0)} g^+.
\]

The generating functional \( W[j, j^\dagger] \) for the connected Green functions is given by

\[
e^{W[j, j^\dagger]} = \int D[\psi] D[\psi^\dagger] e^{-S[\psi^\dagger, \psi] + \sum_\sigma (\psi_{\dagger \sigma} j_\sigma + j^\dagger_\sigma \psi_{\sigma})}.
\]

The effective action, defined by the Legendre transformation,

\[
\Gamma[\psi^\dagger, \psi] = -W[j, j^\dagger] + \sum_\sigma (\Psi^\dagger_{\sigma} j_\sigma + j^\dagger_\sigma \Psi_{\sigma})
\]

with

\[
\Psi^\dagger_{\sigma;x} = -\frac{\delta W}{\delta j_{\sigma;x}} \quad \Psi_{\sigma;x} = \frac{\delta W}{\delta j^\dagger_{\sigma;x}}
\]

generates the 1PI vertex functions by its functional Taylor expansion.

The generating functional (6) needs regulator. For the sake of a better insight into the cutoff effects let us mention four easily implementable regulators. (i) The simplest regulator is a sharp cutoff in momentum space, the restriction of the Fourier modes of the field configuration to \( |p - \sigma k_F| < \Lambda \). (ii) We can construct a slightly different sharp cutoff regulators by placing the system onto space-time lattice. The most important change is the bending of the dispersion relations into smooth, periodic functions. (iii) We shall need smooth cutoff which is easiest to reach by smearing the local terms in the interaction Lagrangian. One possibility is to introduce the smeared field

\[
\psi_B_{\sigma;x} = \int_y \rho_{x-y} \psi_{\sigma;y}
\]
and use the action $S[\psi^\dagger, \psi] = S_0[\psi^\dagger, \psi] + S_i[\psi_B^\dagger, \psi_B]$ in Eq. (6). A natural choice for smearing is
\[ \rho_x = \frac{\Lambda^2}{\pi} e^{-x^2/\Lambda^2} \] (10)
which amounts to the use of
\[ \psi_B^{\sigma,p} = \rho_p \psi_{\sigma,p} \] (11)
with
\[ \rho_p = e^{-p^2/4\Lambda^2}. \] (12)
We note that the smearing function $\rho_p = \rho_\Lambda(p)$, with
\[ \rho_\Lambda(p) = \begin{cases} 1 & |p| \leq \Lambda \\ 0 & |p| > \Lambda \end{cases} \] (13)
brings us back to sharp momentum cutoff. (iv) Another smooth cutoff which preserves the local form of the current in the interactions and splits the two currents only is obtained by using the interaction
\[ S_B[\psi^\dagger, \psi] = \sum_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} \int x_1, y_1, x_2, y_2, z \rho_\Lambda g_{\sigma_1, \sigma_2, y_1, x_2, y_2} \psi_{\sigma_1, x_1 + z}^{\dagger} \psi_{\sigma_2, y_1 + z} \psi_{\sigma_3, x_2} \psi_{\sigma_4, y_2}. \] (14)
in Eq. (6). Each regularization admits bare Ward-identities.

III. BPHZ SCHEME AND BOSONIZATION

Most of the studies of the Luttinger model was carried out in an elegant subtraction scheme in the Hamiltonian formalism which is based on normal ordering. We highlight in this section the peculiarity of the results by showing that the normal ordering is actually very similar to the BPHZ subtraction scheme.

The BPHZ subtraction scheme is based on the observation [18, 19] that the primitive divergences of an irreducible Green function appear in the first few order of the Taylor expansion in the external momenta around zero. These divergences are subtracted by the introduction of local counterterms, given just by these few divergent terms. This procedure can be organized in such a manner that the subtraction is carried out on the level of the integrand of the loop integrals [20]. The resulting scheme is remarkable because it has no divergences whatsoever, the loop integrals are manifestly finite [21].

The bosonization in 1+1 dimensions has been thoroughly discussed in a number of papers [3, 5–7] and we are content to make few remarks about the cutoff effects by making the regularization explicit. The bosonization starts with the explicit construction of the correspondence between the fermionic and the bosonic field operators and continues with the demonstration that the Hamiltonian of the Luttinger model is of massless, noninteracting boson fields [3, 5]. We shall consider the the verification of the bosonic commutators in details.

The field operator corresponding to a chiral fermion in a box of size $L$ and periodic boundary conditions is given by
\[ \psi(x) = \frac{1}{L} \sum_k e^{ikx} c_k, \] (15)
\[ k = 2\pi n_k/L, \] where $n_k$ is integer. These operators obey the set of canonical anticommutation relation whose only non-vanishing member in momentum space is \{ $c_k, c_k^{\dagger}$ \} = $\delta_{k,k'}$. The boson operators are introduced by the equation
\[ b_q = -\frac{i}{\sqrt{4\Lambda}} \sum_k e_{k-q}^{\dagger} c_k \] (16)
for $q > 0$ and
\[ \phi(x) = \frac{1}{L} \sum_{q>0} e^{iqx} b_k. \] (17)
The canonical bosonic commutators for $b_q$ and $b_q^\dagger$ follow in a trivial manner except the relation for

$$[b_q, b_q^\dagger] = \frac{1}{n_q} \sum_k c_k^\dagger c_k^0 - \frac{1}{n_q} \sum_k c_k^\dagger c_{k-q}^0.$$  \hfill (18)

The diagonal matrix elements of this equation contains the difference of two divergent quantities and needs a cutoff. The regulator plays two roles simultaneously: It (i) makes the expressions finite by suppressing the short distance modes and, and (ii) defines an order of the summation/integration. The latter is rarely needed because the counterterms, pulled into the loop summation/integration render the convergence absolute and hence independent of the order. But there might observables with sums/integrals which are finite but non-absolutely convergent. These have no counterterms but still need a definite and consistent prescription in order to render the theory well defined. These observables are called anomalous.

In lattice regularization the Brillouin zone is periodic and the shift $k \rightarrow k + q$ in the first term is allowed and renders the commutator vanishing. The same argument applies for sharp momentum cutoff, as well [3].

But the regularization scheme where smooth cutoff is combined by normal ordering gives different result [5]. We shall use the regulator (iii) introduced in Section II which yields

$$b_B = - \frac{i}{\sqrt{n_q}} \sum_k \rho_k^* \rho_k c_k^\dagger q c_k$$

as bare operators for the interaction Lagrangian. We define the fermionic vacuum by $c_k|0\rangle = c_k^\dagger|0\rangle = 0$ for $k, -k' > 0$ and the corresponding normal ordering of an operator $O$ is

$$:O := O - \langle 0|O|0\rangle.$$  \hfill (20)

The regulated Eq. (18) now reads as

$$[b_B, b_B^\dagger] = \frac{1}{n_q} \sum_k |\rho_k - q|^2 c_k^\dagger c_k - \frac{1}{n_q} \sum_k |\rho_k|^2 c_k^\dagger c_{k-q}$$

$$= \frac{1}{n_q} \sum_k |\rho_k - q|^2 : c_k^\dagger c_k : - \frac{1}{n_q} \sum_k |\rho_k|^2 : c_k^\dagger c_{k-q} : + \frac{1}{n_q} \sum_k |\rho_k|^2 \langle 0|c_k^\dagger c_{k-q} |0\rangle - \frac{1}{n_q} \sum_k |\rho_k - q|^2 \langle 0|c_k^\dagger c_k |0\rangle.$$  \hfill (21)

One can show that any matrix element of the sums is absolute convergent and the reordering $k \rightarrow k + q$ is allowed, giving

$$[b_B, b_B^\dagger] = \mathbb{1}(1 + C_q) + D_q$$  \hfill (22)

where

$$C_q = \frac{1}{n_q} \sum_{k<0} (|\rho_k|^2 - |\rho_{k-q}|^2) + \frac{1}{n_q} \sum_{k=0}^q (|\rho_k|^2 - 1)$$

$$D_q = \frac{1}{n_q} \sum_k (|\rho_{k-q}|^2 - |\rho_{k+q}|^2) : c_k^\dagger c_k :$$  \hfill (23)

In the works using bosonization the usual regulator is (iv) of Section II which leads to Eq. (21) with $\rho_k = 1$. The sum with the normal ordered product gives absolute convergent matrix elements but the last two sums diverge and renders this cutoff unacceptable.

In order to understand better the problem of the regularization (iv) let us point out the similarity of the normal ordered scheme and the BPHZ method. One subtraction is sufficient in 1+1 dimensions and only its value differs in the normal ordered and the BPHZ schemes. The normal ordering shares the advantages of the BPHZ scheme, namely simplicity, the disappearance of divergences. But this does not mean that these schemes need no regulator whatsoever. The following two circumstances make the regulator unavoidable.

If there are non-absolutely convergent but finite loop summations/integration, anomalous observables, in the theory then regulator is needed in order to define their value in a consistent manner. The consistency is important, eg. one could have chosen any order in the second summation of the last line in Eq. (21) with $\rho_k = 1$ which yields finite result. But it is not obvious how to define sums which appear in other loops. One needs a consistent regulator which settle this issue on the level of the bare Hamiltonian and not of particular expressions.
The other instance which requires regularization is when the cutoff manages to generate a finite scale. An example is the dimensional transmutation. The only way the dimensional transmutation can appear in the BPHZ scheme is through the finite cutoff-dependence which is vanishing in the limit $\Lambda \to \infty$ because all divergent cutoff dependence which generates the new scale in other schemes are now subtracted. Therefore the finite expressions of the cutoff which tend to zero in the limit $\Lambda \to \infty$ must be retained before resumming the perturbation expansion in order no tot miss dimensional transmutation. There are simpler or more complicate schemes, where the same renormalized physics is recovered in a straightforward or more involved manner. The BPHZ scheme which is very powerful for massive models without dimensional transmutation but becomes highly nontrivial for massless models. This happens because its renormalized parameters are not closely related to the true effective coupling constants and finite, physical effects (eg. dimensional transmutation) arise from infinitesimal small modifications of the BPHZ renormalized dynamics. As far as the BPHZ is concerned, one can save it with the expense of some complications, by expanding the irreducible Green functions around non-vanishing energy-momenta, but the normal ordering can not be rescued in the absence of any scale to rely upon.

Another case when the regulator is needed is when the cutoff can not be moved beyond a certain threshold. This can easily happen in a perturbatively renormalizable theory, where the divergences are eliminated order by order in the perturbation expansion. In fact, the renormalization condition which fixes the theory is given as a set of equations, containing certain experimentally observed values on one side and the corresponding expressions on the other side. These expressions contain the cutoff and the bare parameters in a highly non-linear manner and it may happen that they have no solution with finite values of the bare parameters beyond certain scale, usually called Landau pole scale, for the cutoff.

After these general remark we now return to the discussion of Eq. (22). The commutation relation of Ref. [5] is obtained from Eq. (22) by setting $C_q = D_q = 0$, quantities of $O(q^2/\Lambda^2)$. After this simplification the bosonization is quickly completed by expressing the normal ordered fermionic Hamiltonian by means of the boson field and by performing a Bogolubov transformation. The result is a normal ordered Hamiltonian for a massless, non-interactive boson field.

Our point is that the regulator (iv) regulates all Feynman graphs contributing to Green functions because it suppresses interactions beyond the cutoff scale. But it does not remove divergences, such as in Eq. (22), from the unperturbed dynamics even if the regulator is amended by normal ordering. Normal ordering is a reordering of the loop contributions only and not regulator. The control of all divergences actually leads to the $O(q^2/\Lambda^2)$ terms in Eq. (22). They are certainly negligible as far as the canonical commutation relations are concerned at face value in the renormalized theory but they modify the bosonic Hamiltonian which is based on these relations and generate some $O(\Lambda^{-2})$ interactions. These interactions which are dropped by the formal limit $\Lambda \to \infty$ in the normal ordered scheme may generate finite effects via dimensional transmutations or Landau poles.

We shall, in this work, look for cutoff effects in the theory defined by sharp momentum space cutoff in order to see if the omission of the $O(q^2/\Lambda^2)$ terms in Eq. (22) is indeed justified. This choice of regulator brings us back to the discussion of different regularizations after Eq. (18). Have we not already lost bosonization with all of its consequences when sharp momentum cutoff is used?

The fact that different, but acceptable regularizations produce different result for a finite value of the cutoff is not surprising. All what one expects from renormalizable theories is that different cutoff theories converge to the same renormalized one when the cutoffs are removed. The usual argument based on universality assures that there is a unique Luttinger model, as long as its theory is renormalizable, whatever acceptable regulator is used. It might naturally happen that certain effects are more or less easily identifiable when a given regulator is used but the numerical value of the observables must agree. For example, there is a well known no-go theorem [22] that chiral fermions, as in Eq. (15), can not be accommodated within the framework of the usual quantum field theories when lattice regulator is used. But rather involved constructions which can be viewed as particular regularization schemes and violate the conditions of the no-go theorem can realize chiral lattice fermions in particular limits. In a similar manner, we may regard the fragility of bosonization as an indication that this scheme concentrates a large amount of informations in the $O(\Lambda^{-2})$ part of the renormalized action. In other schemes these informations spread over the finite part of the action. If they play appreciable role in the dynamics then bosonized models require an unusually careful treatment.

IV. RENORMALIZATION GROUP IN THE INTERNAL SPACE

The traditional renormalization group method [23] consists of the successive elimination of the modes. There are two possible strategies in determining the order of the elimination. The conventional way is to eliminate modes at the running UV cutoff $\Lambda$ which moves in the IR direction. One can imagine two different ways to justify this procedure. (i) One starts with the unavoidable non-local effects of quantum field theories. It is well known that a theory is
always non-local at distances which are shorter than the UV cutoff scale. These involved and dangerous correlations are turned on gradually during the blocking if the UV cutoff moves from the UV to the IR direction. (ii) Another strategy to fix the order of the modes to be eliminated is that the more perturbative modes should be eliminated by means of a simpler effective theory. Hence one should tackle with the more nonperturbative modes by means of a more sophisticated, "dressed" effective theory which is obtained after having already eliminated a large number of modes. The kinetic energy usually renders the short distance modes perturbative therefore the modes are eliminated in decreasing order of their momentum, their scale parameter in the space-time, called external space. This latter corresponds to the convention of considering the field variables \( \phi(x) \) as a mapping from the external into the internal space.

The caveat in the reasoning (i) is that the non-local effects usually reach well beyond the naive length scales \( 1/\Lambda \). In fact, let us consider a smooth momentum space cutoff which leaves the modes with momentum \( p \to \infty \) are non-physical, too. In particular, sharp momentum cutoff always generates infinitely long range correlations. Their distance scales \[1/\Lambda \] or fermion bilinears \[25-28\].

The Callan-Symanzik scheme in which the momentum independent, relativistic mass term is used to control the elimination of the modes \[16\] appears to be a scheme with a minimal amount of non-locality. When the mass term is forbidden, as in the Luttinger model, then we may use the coupling constant as a control parameter. More precisely, we rescale the coupling constant by a factor \( \lambda^2 \) which will be evolved from 0 to 1. This scheme will yield a gradual turning on the "dangerous" part of the quantum fluctuations which are not factorizable according to the Wick-theorem. The only non-local feature of this suppression scheme comes from the energy-momentum dependence of the interactions, needed for the separation of the left and right modes. But this is a \( \lambda \)-independent real physical effect and is a necessary ingredient.

Another advantageous feature of this scheme when applied for fermionic systems can be seen in the following manner. In fermionic models at finite density the dangerous, soft modes are around the Fermi surface and the blocking should zoom in this area at the final stage of the evolution. This usually requires a rather involved geometric construction which can easily be avoided by controlling the interaction strength. A rescaling of the fermionic fields by \( \lambda^{-1/2} \) removes the factor \( \lambda \) from the usual quartic interaction term and generates the coefficient \( 1/\lambda \) in front of the quadratic part of the action, i.e. multiplies the propagators by \( \lambda \). One can see again that the momentum dependence brought by this control parameter is dictated by the true, \( \lambda \)-independent dynamics, the kinetic energy, instead of an artificial device and it may build up the eventually strongly coupled dynamics around the Fermi surface.

Beyond the issue of preserving locality better, the most important difference between the external and internal space schemes from the point of view of fermionic system is the possibility in the latter case to obtain effective interaction strengths off the Fermi surface. In fact, the external space blocking methods zoom into the Fermi surface at the end point of the evolution only. As a result, any off-Fermi surface region is handled without the dynamical contributions of modes between the region in question and the Fermi surface. This happens because these modes appear at a later stage of the evolution only when the effective coupling strengths off the Fermi surface have already been fixed. On
the contrary, the internal space scheme blocking has no impact on the external space and leaves the possibility of tracing the evolution of the effective coupling strengths at in an arbitrary kinematical region.

The realization of the internal space blocking scheme starts with the extension

$$S_{\lambda}^{\Phi}[\psi^{\dagger}, \psi] = \sum_{\sigma} \int x [\psi^{\dagger}_{\sigma,x} (-\partial_{t} + \epsilon_{\sigma}(-i\partial_{x})) \psi_{\sigma,x} + \lambda^{2} \sum_{\sigma_{1,2,3,4}} \sum_{x_{1,2,3,4}} g_{\sigma_{1,2,3,4}}^{\Phi} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger}]$$

(24)

of the action (1) which can be written after the rescaling $\psi \rightarrow \lambda^{1/2} \psi$, $\psi^{\dagger} \rightarrow \lambda^{1/2} \psi^{\dagger}$ as

$$S_{\lambda}^{\Phi}[\psi^{\dagger}, \psi] = \frac{1}{\lambda} \sum_{\sigma} \int x [\psi^{\dagger}_{\sigma,x} (-\partial_{t} + \epsilon_{\sigma}(-i\partial_{x})) \psi_{\sigma,x} + \lambda^{2} \sum_{\sigma_{1,2,3,4}} \sum_{x_{1,2,3,4}} g_{\sigma_{1,2,3,4}}^{\Phi} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger} \psi_{\sigma_{1,2,3,4}}^{\dagger}]$$

(25)

The evolution equation for the generating functional is obtained by simply applying the derivative with respect to the control parameter $\lambda$ on Eq. (6) where the action is given by Eq. (25),

$$\partial_{\lambda} W[j_{\sigma}, j^{\dagger}_{\sigma}] = \frac{1}{\lambda^{2}} e^{-W[j^{\dagger}, j]} \int \mathcal{D}[\psi] \mathcal{D}[\psi^{\dagger}] e^{-S_{\lambda}[\psi^{\dagger}, \psi] + \sum_{\sigma} \left( \psi_{\sigma,x}^{\dagger} \partial_{t} \psi_{\sigma,x} + j_{\sigma} \psi_{\sigma,x}^{\dagger} \right)}$$

$$= \frac{1}{\lambda^{2}} \sum_{\sigma} \text{Tr} \left[ \left( W^{(2)}_{j_{\sigma}, j^{\dagger}_{\sigma}} + W^{(2)}_{j^{\dagger}_{\sigma}, j_{\sigma}} \right) \cdot \left[ -\partial_{t} + \epsilon_{\sigma}(-i\partial_{x}) \right] \right].$$

(26)

The truncation of this functional differential equation is rather difficult because the functional $W[j_{\sigma}, j^{\dagger}_{\sigma}]$ is too complicated due to strong non-local features. The 1PI vertex functions are supposed to be more local than the connected Green functions therefore we express this equation in terms of the effective action. We introduce the notation

$$F_{\Phi, \chi}^{(2)} = \frac{\delta^{2} F[\phi, \chi]}{\delta \phi \delta \chi},$$

(27)

use the relation

$$\int \mathcal{D}[\psi^{\dagger}] \mathcal{D}[\psi] e^{-S_{\lambda}[\psi^{\dagger}, \psi] + \sum_{\sigma} \left( \psi_{\sigma,x}^{\dagger} \partial_{t} \psi_{\sigma,x} + j_{\sigma} \psi_{\sigma,x}^{\dagger} \right)}$$

$$= -\delta_{z,x} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(28)

and find

$$\partial_{\lambda} \Gamma[\Psi^{\dagger}, \Psi] = -\partial_{t} W[j_{\sigma}, j^{\dagger}_{\sigma}]$$

$$= \frac{1}{\lambda^{2}} \sum_{\sigma} \text{Tr} \left[ \left( \Gamma^{(2)}_{\Psi^{\dagger}_{\sigma}, \Psi_{\sigma}} - \Psi_{\sigma} \Psi_{\sigma}^{\dagger} \right) \cdot \left[ -\partial_{t} + \epsilon_{\sigma}(-i\partial_{x}) \right] \right].$$

(29)

The solution will be sought by means of the functional form

$$\Gamma[\Psi^{\dagger}, \Psi] = \tilde{\Gamma}[\Psi^{\dagger}, \Psi] + \frac{1}{\lambda} \sum_{\sigma = 1} \Psi_{\sigma} \cdot \left[ -\partial_{t} + \epsilon_{\sigma}(-i\partial_{x}) \right] \cdot \Psi_{\sigma},$$

(30)

for which the evolution equation reads as

$$\partial_{\lambda} \tilde{\Gamma}[\Psi^{\dagger}, \Psi] = -\frac{1}{\lambda^{2}} \sum_{\sigma} \text{Tr} \left( \Gamma^{(2)}_{\Psi^{\dagger}_{\sigma}, \Psi_{\sigma}} - \Psi_{\sigma} \Psi_{\sigma}^{\dagger} \right) \cdot \left[ -\partial_{t} + \epsilon_{\sigma}(-i\partial_{x}) \right] \right).$$

(31)
V. ANSATZ

The general solution of the functional differential equation (31) is beyond our capacities and we have arrived at the
only approximation one has to rely upon in this scheme, the truncation of the evolution equation. By assuming an
analytical dependence in the field variables we write \( \tilde{\Gamma}[\Psi^\dagger, \Psi] \) in the form of a functional Taylor series where arbitrary
high powers of the fields appear and the coefficient functions, the 1PI vertices, are arbitrary functions of their space-
time or energy-momentum variables. The \( n \)-th order terms in this series represent \( n \)-body correlations and can be shown to be suppressed in the framework of the Wilsonian the external space renormalization group method in the
limit when the sharp momentum space cutoff used approaches zero \([11]\). Though this result seems to be somehow
fragile, the nonlocal interactions generated by the sharp cutoff imposed in the IR regime may substantially change the kinetic energy and thus invalidate the argument, we shall ignore the the terms \( n \geq 6 \) for the sake of simplicity. But the collinear divergences will be taken more seriously and an arbitrary space-time dependence will be retained for the vertex functions. Our choice for the "potential" \( \tilde{\Gamma}[\Psi^\dagger, \Psi] \) is

\[
\tilde{\Gamma}[\Psi^\dagger, \Psi] = \Gamma_0 - \sum_{\sigma} \tilde{\Psi}^\dagger_{\sigma} \cdot \Sigma_{\sigma} \cdot \tilde{\Psi}_{\sigma} - \int_{x_1,y_1,x_2,y_2} \gamma_{x_1,y_1,x_2,y_2} \tilde{\Psi}^\dagger_{-x_1} \tilde{\Psi}^{-y_1} \tilde{\Psi}^\dagger_{+x_2} \tilde{\Psi}^{-y_2} \\
- \int_{x_1,y_1,x_2,y_2} \gamma_{x_1,y_1,x_2,y_2} \tilde{\Psi}^\dagger_{+x_1} \tilde{\Psi}^{+y_1} \tilde{\Psi}^\dagger_{+x_2} \tilde{\Psi}^{+y_2} + \tilde{\Psi}^\dagger_{-x_1} \tilde{\Psi}^{-y_1} \tilde{\Psi}^\dagger_{-x_2} \tilde{\Psi}^{-y_2},
\]

where the effective vertices obey the symmetries \( \gamma_{x_1,y_1,x_2,y_2} = \gamma_{x_2,y_2,x_1,y_1} = -\gamma_{x_2,y_2,x_1,y_1} \).

We shall be interested in the question if the theory is strongly coupled around the Fermi points. It should be kept
in mind that the eventual divergence of the vertex functions at some points does not necessarily mean that the model
is strongly coupled in that kinematical regime. In fact, any physically acceptable definition of the strength of the two-
body interaction involves a finite resolution in the energy-momentum space and seeks an average, a representative value of the four-point vertex function in a small but finite area. Such a construction can be approximated by the introduction of an appropriately chosen normalized, smooth smearing function \( \rho_{p,q,r,s} \) in the definition of the dimensionless running coupling constant,

\[
\gamma = \int_{p,q,r,s} \delta_{p-q+r-s,0} \rho_{p,q,r,s} \hat{\gamma}_{p,q,r,s}
\]

where \( \hat{\gamma}_{p,q,r,s} \) is obtained from the vertex function by separating off the trivial energy-momentum conservation,

\[
\gamma_{x_1,y_1,x_2,y_2} = \int_{p,q,r,s} e^{ipx_1-iqy_1+irx_2-isy_2} \delta_{p-q+r-s,0} \hat{\gamma}_{p,q,r,s}.
\]

Guided by such a construction we shall call the theory strongly coupled at a certain kinematical point if the four point
function \( \hat{\gamma}_{p,q,r,s} \) either takes a uniformly large value around or displays a non-integrable singularity at that point.

One can explore the weakly coupled regime of the theory in a consistent manner with our truncation, (32). But
whenever the dimensionless running coupling constants assume values which is not anymore small with respect to one
then our computation must stop.

VI. EVOLUTION EQUATION

In order to arrive at the evolution equation for our ansatz we need the inverse of the second functional derivative,

\[
\Gamma^{(2)} = \begin{pmatrix}
\Gamma^{(2)}_{\Psi_{\sigma}, \Psi^{\dagger}_{\sigma}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi^{\dagger}, \Psi} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} \\
\Gamma^{(2)}_{\Psi^{\dagger}, \Psi} & \Gamma^{(2)}_{\Psi^{\dagger}, \Psi} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} \\
\Gamma^{(2)}_{\Psi^{\dagger}, \Psi} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} \\
\Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}} & \Gamma^{(2)}_{\Psi, \Psi^{\dagger}}
\end{pmatrix},
\]

computed in a manner which is consistent with the truncation. For this end we write it in the form

\[
\Gamma^{(2)} = G^{-1}_0 + \Delta^x + \Delta^u
\]
by means of the diagonal block matrix

$$G_0^{-1,x,y} = \begin{pmatrix} G_{+;x,y}^{-1} & 0 & 0 \\ 0 & G_{-;x,y}^{-1} & 0 \\ 0 & 0 & -G_{+;y,x}^{-1} & 0 \\ 0 & 0 & 0 & -G_{-;y,x}^{-1} \end{pmatrix}$$  \tag{37}$$

composed of the free propagator

$$G_{\sigma}^{-1} = \frac{1}{\lambda} [-\partial_t + \epsilon_{\sigma}(-i\partial_x)] + \Sigma_{\sigma},$$  \tag{38}$$

the self energies

$$\Delta^x_{x,y} = \int_{z_1, z_2} \left( \begin{array}{cccc} \gamma^x_{1, y, z_1, y} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & \gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} \\ -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & \gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} \\ 0 & -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & \gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 \\ -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & -\gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & \gamma^x_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} \end{array} \right),$$  \tag{39}$$

and

$$\Delta^u_{x,y} = \int_{z_1, z_2} \left( \begin{array}{cccc} 4 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & 2 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 \\ 0 & 4 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & 2 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} \\ 2 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & -4 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 \\ 0 & 2 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} & 0 & -4 \gamma^u_{1, y, z_1, z_2} \Psi_{+; z_1, z_2} \Psi_{+; z_2} \end{array} \right).$$  \tag{40}$$

The matrix elements of the second functional derivative in the desired approximation are finally obtained by means of the Neumann expansion,

$$\left(\Gamma^{(2)}\right)^{-1} = G_0 - G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0 + G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0 + \cdots$$  \tag{41}$$

which yields the evolution equation

$$\partial_\lambda \tilde{\Gamma} = -\frac{1}{\lambda^2} \text{Tr} \left\{ [-\partial_t + \epsilon_{\sigma}(-i\partial_x)] \cdot [G_0 - G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0 + G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0 \cdot (\Delta^x + \Delta^u) \cdot G_0] \right\}$$  \tag{42}$$

for our ansatz. The first contribution arising from \(G_0\) in the second parenthesis on the right hand side is field independent and can be ignored. The quadratic and quartic terms generate the evolution for the self energy and the effective coupling constants, respectively, to be considered below.

### A. Quadratic part

The quadratic order of the left and the right hand sides of the evolution equation are

$$\int_{z_1, z_2} \langle \Psi_{+; z_2} \partial_\lambda \Sigma_{+, z_1, z_2} \Psi_{+; z_2} + \Psi_{+; z_1} \partial_\lambda \Sigma_{-, z_1, z_2} \Psi_{-, z_2} \rangle$$  \tag{43}$$

and

$$-\frac{1}{\lambda^2} \int_{z_1, z_2, z_3, z_4} \left[ G_{+, z_4, z_1} (\Delta^x_{1,1} + \Delta^u_{1,1}) \right]_{z_1, z_2} G_{+, z_2, z_3} \left[ -\partial_t + \epsilon_+ (-i\partial_x) \right]_{z_3, z_4},$$

$$-\frac{1}{\lambda^2} \int_{z_1, z_2, z_3, z_4} \left[ G_{-, z_4, z_1} (\Delta^x_{2,2} + \Delta^u_{2,2}) \right]_{z_1, z_2} G_{-, z_2, z_3} \left[ -\partial_t + \epsilon_- (-i\partial_x) \right]_{z_3, z_4}$$  \tag{44}$$

respectively. By equating these expressions one finds the evolution equations for the self energy,

$$\partial_\lambda \Sigma_{-, x, y} = -\frac{1}{\lambda^2} \int_{z_1, z_2, z_3, z_4} \left\{ G_{+, z_4, z_1} \gamma^x_{y, z_1, z_2, z_3} \left[ -\partial_t + \epsilon_+ (-i\partial_x) \right]_{z_3, z_4} + 4G_{-, z_4, z_1} \gamma^u_{y, z_1, z_2, z_3} \left[ -\partial_t + \epsilon_- (-i\partial_x) \right]_{z_3, z_4} \right\},$$

$$\partial_\lambda \Sigma_{+, x, y} = -\frac{1}{\lambda^2} \int_{z_1, z_2, z_3, z_4} \left\{ G_{-, z_4, z_1} \gamma^x_{y, z_1, z_2, z_3} \left[ -\partial_t + \epsilon_- (-i\partial_x) \right]_{z_3, z_4} + 4G_{+, z_4, z_1} \gamma^u_{y, z_1, z_2, z_3} \left[ -\partial_t + \epsilon_+ (-i\partial_x) \right]_{z_3, z_4} \right\}.$$  \tag{45}$$
The structure of these equations reflects the Callan-Symanzik strategy in the framework of evolution in $\lambda$. In fact, the relation
\[
\partial_\lambda G_\sigma = \partial_\lambda \left( \frac{1}{x} \left[ -\partial_t + \epsilon_\sigma (-i\partial_x) \right] + \Sigma_\sigma \right) = -\frac{1}{\lambda^2} G_\sigma \cdot \left[ -\partial_t + \epsilon_\sigma (-i\partial_x) \right] \cdot G_\sigma
\]  
(46)
where $\partial_\lambda$ acts on the explicit, driving $\lambda$-dependence produces
\[
\partial_\lambda \Sigma; x, y = \partial_\lambda \int_{z_1, z_2} \left( \gamma^x_{x,y,z_1,z_2} G^{z_2,z_1} + 4\gamma'^x_{z_1,z_2,x,y} G^{z_2,z_1,}, \right)
\]
\[
\partial_\lambda \Sigma; x, y = \partial_\lambda \int_{z_1, z_2} \left( \gamma^x_{x,y,z_1,z_2} G^{z_2,z_1} + 4\gamma'^x_{z_1,z_2,x,y} G^{z_2,z_1,}, \right)
\]
the perturbative, one-loop Callan-Symanzik beta function for the propagator. The Fourier transform of the self energy
\[
\Sigma_{\sigma;x,y} = \int_{p,q} e^{ipx+ipy} \delta_{p+q,0} \Sigma_{\sigma,p,q},
\]
allows us to write the evolution equation in a simpler form,
\[
\partial_\lambda \Sigma; p = \partial_\lambda \int \frac{\gamma^x_{x,y,z_1,z_2}}{x} \left[ -\partial_t + \epsilon_\sigma (-i\partial_x) \right] + \Sigma_\sigma
\]
(49)
It is worthwhile noting the appearance of different derivatives on the two sides of these equations. In fact, had we found $\partial_\lambda$ on both sides the solution would have produced the standard one-loop results for the self energy. The derivative $\partial_\lambda$ on the right hand side makes sure that the effective vertices $\hat{\gamma}^x, \hat{\gamma}'^x$ which in principle depend on $\lambda$ are nevertheless kept intact. This procedure shows the essence of the infinitesimal change of the control parameter can safely be computed perturbatively by means of the actual effective vertices and the nonperturbative resummation arises from the repetition, i.e. the integration of this procedure.

**B. Quartic part**

The evolution equation gives in the quartic order
\[
\lambda^2 \partial_\lambda \gamma^x_{1,y_1,2,y_2} = -\int_{z_1, z_2, z_3} \left\{ G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1} \} + G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1}
\]
\[
\lambda^2 \partial_\lambda \gamma^x_{1,y_1,2,y_2} = -\int_{z_1, z_2, z_3} \left\{ G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1} \} + G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1}
\]
(50)
The Callan-Symanzik strategy can be used again to simplify the right hand sides as
\[
\partial_\lambda \gamma^x_{1,y_1,2,y_2} = \partial_\lambda \int_{z_1, z_2, z_3} \left\{ G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1} \} + G^{z_3,z_1} \left[ \delta_{z_1, z_2} \left[ -4\gamma^x_{1,y_1,2,y_2} G^{z_3,z_1} \right] - 4\gamma'^x_{1,y_1,2,y_2} \right] G^{z_3,z_1}
\]
(51)
where the one-loop beta functions appear on the right hand side. These relations assume the form

\[
\partial_\lambda \gamma^X_{p,q,r,s} = \partial_\lambda \int_{u,v} [G_{+;u}(-4\gamma^X_{p,q,u,v}G_{+;v}^\gamma_{u,r,s} + \gamma^\gamma_{q,u,s}G_{-;v}^\gamma_{p,v,r,u}) + G_{-;u}(\gamma^\gamma_{u,q,v}G_{+;v}^\gamma_{p,v,u,v} - 4\gamma^\gamma_{u,v,r,s}G_{-;v}^\gamma_{u,u,p,q})] \\
\partial_\lambda \gamma^\gamma_{p,q,r,s} = \partial_\lambda \int_{u,v} [G_{+;u}(-\gamma^\gamma_{r,s,u,v}G_{+;v}^\gamma_{p,q,u} + \gamma^\gamma_{r,q,u,u}G_{+;v}^\gamma_{p,s,v,u}) + G_{-;u}(-16\gamma^\gamma_{u,v,p,q}G_{-;v}^\gamma_{u,r,s} + 16\gamma^\gamma_{u,v,p,s}G_{-;v}^\gamma_{u,r,q} - 4\gamma^\gamma_{u,q,u,s}G_{-;v}^\gamma_{p,v,u,v})].
\]

in momentum space.

VII. RUNNING COUPLING CONSTANT AT THE FERMI POINT

We first confront our evolution equation with the standard one-loop Wilsonian renormalization group approach to the Luttinger model [12, 13] where the effective bare coupling constant is supposed to be energy-momentum independent and its value is given by the coefficient of the quartic term in the fermion fields taken at the one-shell Fermi-point. Furthermore we set \( \gamma^\gamma = 0 \) at the initial conditions and at any subsequent stage of the evolution. The evolution of the self-energy, Eq. (49) is now

\[
\partial_\lambda \Sigma_{\pm;p} = -\partial_\lambda \int_q \frac{\hat{\gamma}^X(\lambda)e^{iq_0}}{2(piq_0 - e_{\pm};q) - \Sigma_{\pm}}.
\]

Because the self-energy corrections are \( \mathcal{O}(\gamma) \), \( \Sigma_{\pm} \) can safely be ignored in the right hand side in the asymptotically weakly coupled initial conditions, imposed at \( \lambda_0 \approx 0 \). The right hand side starts with a \( p \)-independent value at the beginning and obviously keeps this simple form during the rest of the evolution,

\[
\partial_\lambda \Sigma_{\pm;p} = -\hat{\gamma}^0(\lambda) \int_q \frac{e^{iq_0}}{2(piq_0 - e_{\pm};q)}.
\]

The self energy remains real and it can be absorbed in the renormalization of the chemical potential. Let us continue with the evolution of \( \gamma^X(\lambda) \) which, according to Eq. (52) is governed by the equation

\[
\partial_\lambda \hat{\gamma}^X(\lambda) = \partial_\lambda \int_{u,v} \gamma^X(\lambda)[G_{+;u}G_{-;v} + (\lambda)G_{-;u}G_{+;v+2k_F}].
\]

The more detailed form of the two contributions on the right hand side,

\[
\int G_{+;u}G_{-;v} = -\int_{-\Lambda}^\Lambda \frac{du}{2\pi} \int_{-\infty}^{+\infty} \frac{du_0}{2\pi} \frac{1}{(iu_0 - u)(iu_0 + u)} \]

\[
\int G_{-;u}G_{+;v+2k_F} = \int_{-\Lambda}^\Lambda \frac{du}{2\pi} \int_{-\infty}^{+\infty} \frac{du_0}{2\pi} \frac{1}{(iu_0 + u)(iu_0 - u)}
\]

shows the vanishing of the beta function,

\[
\partial_\lambda \gamma^X_0 = 0,
\]

the absence of dressing in agreement in Refs. [12, 13].

Notice that the crucial step of this argument, the formal cancellation of the two integrals in Eqs. (56), takes place at the Fermi points only. A more careful study of the evolution equation (52) presented in the next Section reveals that the incomplete cancellation between the contributions on the right hand side of Eq. (55) generates strongly coupled dynamics in the vicinity of the Fermi points.

VIII. \( \beta \) FUNCTIONS OFF THE FERMI POINT

Let us consider now the complete evolution. One side of the problem is the evolution of the propagator, driven by the quadratic part, Eq. (49). The other one is the evolution of the two-body interactions, given by the
quartic pieces, Eqs. (52). This solution of this set of coupled integro-differential equations represents an involved numerical problem, owing to the fact that the evolution of momentum dependent functions actually involves infinitely many parameters. We reduce the complexity in this work in order to arrive at an analytically treatable problem by suppressing the evolution of the quadratic part in our ansatz (32), i.e. we impose \( \Sigma = 0 \), and leave the building up of the true interactions for the quartic effective vertices.

The quartic evolution, shown by Eqs. (52), is still a formidable numerical problem due to the singular structures generated in the integrand and will be simplified in the following manner. We assume that the naive ideas about the dominance of the interaction vertices by momentum independent parts is valid and introduce the usual running coupling constants, given by the strength of effective interactions at a certain kinematical point. The only generalization with respect to the previous Section is that the running coupling constant can be defined off the Fermi points. We shall then check the consistency of this picture by looking into the magnitude and the momentum-dependence of the effective two-body vertices obtained in this approximation.

The \( \beta \)-functions for the coupling constants \( \gamma_{\hat{p},\hat{q},\hat{r},\hat{s}}^{\kappa} = \gamma^\kappa \) and \( \gamma_{\hat{p},\hat{q},\hat{r},\hat{s}}^{\kappa'} = \gamma'^{\kappa'} \) at a given point \((\hat{p}, \hat{q}, \hat{r}, \hat{s})\) are introduced as

\[
\beta^\kappa = \beta^\kappa_{\hat{p},\hat{q},\hat{r},\hat{s}} = \lambda \partial_\lambda \gamma^\kappa_{\hat{p},\hat{q},\hat{r},\hat{s}}, \\
\beta'^{\kappa'} = \beta'^{\kappa'}_{\hat{p},\hat{q},\hat{r},\hat{s}} = \lambda \partial_\lambda \gamma'^{\kappa'}_{\hat{p},\hat{q},\hat{r},\hat{s}}.
\]

(58)

Let us assume that the region around the point \((\hat{p}, \hat{q}, \hat{r}, \hat{s})\) dominates the dynamics. Then we can approximate the solution of the integro-differential equations (52) by replacing the vertex function \( \gamma_{\hat{p},\hat{q},\hat{r},\hat{s}} \) by the momentum-independent \( \gamma = \gamma_{\hat{p},\hat{q},\hat{r},\hat{s}} \) in the loop integrals. The approximation is better if the point \((\hat{p}, \hat{q}, \hat{r}, \hat{s})\) is closer to the Fermi point.

We shall parametrize the energy-momentum as

\[
(-i\omega, \textbf{p}) = (\pm \textbf{p} + \Delta_p, \pm \textbf{k}_F)
\]

(59)

where the plus and minus sign stand for the right and left moving particles, respectively. The parameters \( \Delta_p, \Delta_q, \Delta_r, \) and \( \Delta_s \) measure the distance from the mass shell. The momentum conservation yields \( \hat{s} - \hat{r} = \hat{p} - \hat{q} \) and the energy conservation leads to the condition \( 2(\hat{p} - \hat{q}) = \Delta_p - \Delta_q + \Delta_r - \Delta_s \) and \( 0 = \Delta_p - \Delta_q + \Delta_r - \Delta_s \) in equations for \( \gamma^\kappa \) and \( \gamma'^{\kappa'} \), respectively.

The beta functions in Eqs. (52) can then be written as

\[
\beta^\kappa = -\lambda^2(\beta^{\kappa\kappa}_{\kappa} \gamma^\kappa + \beta^{\kappa\kappa'}_{\kappa} \gamma'^{\kappa'}), \\
\beta'^{\kappa'} = -\lambda^2(\beta^{\kappa\kappa'}_{\kappa} \gamma^\kappa + \beta^{\kappa\kappa}_{\kappa} \gamma'^{\kappa'}),
\]

(60)

with

\[
\beta^{\kappa\kappa}_{\kappa} = -2\pi \ln \left| \frac{(\hat{q} - \Delta_q + \Delta_r)(2\hat{r} - \Delta_q + \Delta_r)(2\hat{r} + \Delta_p + \Delta_r)^2 - 4\Delta^2}{(2\hat{r} + \Delta_p + \Delta_r)(2\hat{r} - \Delta_p - \Delta_r)(2\hat{q} + \Delta_q - \Delta_r)^2 - 4\Delta^2} \right|
\]

\[
\beta^{\kappa\kappa'}_{\kappa} = -16\pi \frac{\hat{p} - \hat{q}}{\Delta_p - \Delta_q + 2(\hat{p} - \hat{q})} + \frac{\hat{p} - \hat{q}}{\Delta_p - \Delta_q}
= -8\pi \left( 1 + \frac{\Delta_r - \Delta_s}{\Delta_p - \Delta_q} \right) \left( \frac{1}{2 + \frac{\Delta_r - \Delta_s}{\Delta_p - \Delta_q}} + 1 \right)
\]

\[
\beta^{\kappa\kappa}_{\kappa'} = -4\pi \left( \frac{\hat{q} - \hat{p}}{\Delta_q - \Delta_p + 2(\hat{q} - \hat{p})} + \frac{\hat{r} - \hat{q} + 2\textbf{k}_F}{\Delta_r - \Delta_q + 2\hat{q} - 2\textbf{k}_F} \right)
\]

\[
\beta^{\kappa\kappa'}_{\kappa'} = -4\pi \left( \frac{\hat{q} - \hat{p}}{\Delta_q - \Delta_p + 2(\hat{q} - \hat{p})} + \frac{\hat{r} - \hat{q} + 2\textbf{k}_F}{\Delta_r - \Delta_q + 2\hat{q} - 2\textbf{k}_F} - \frac{\hat{p} + \hat{r} + 2\textbf{k}_F}{\Delta_p + \Delta_r + 2\hat{r} + 2\textbf{k}_F} \right).
\]

(61)

See Appendix B for details.

The beta functions display singularities arbitrarily close to the Fermi points which lead to singularities in the effective coupling constants [29]. We shall integrate out the beta functions in order to see the singularity structure in a bit more detailed manner. Two kinematical regions will be considered, (a): \( \hat{q} = \hat{p}, \Delta = 0, \hat{r}, \hat{p} \ll k_F, \Lambda \), and (b): \( \hat{q} = \hat{p}, 2\hat{r} + \Delta_p + \Delta_r = 2\hat{p} + \Delta_q - \Delta_r, \hat{r}, \hat{p} \ll k_F, \) where \( \beta^{\kappa\kappa}_{\kappa} = 0, \beta^{\kappa\kappa'}_{\kappa} \approx 4\pi, \beta^{\kappa\kappa}_{\kappa'} \approx 36\pi \) by ignoring \( O(\hat{p}/k_F), O(\hat{r}/k_F) \) direction dependent terms and

\[
\beta^{\kappa\kappa'}_{\kappa} \approx \begin{cases} 
2\pi \frac{\frac{\hat{p}^2}{\hat{q}^2}}{\ln \left( \frac{2\hat{p} + \Delta_q - \Delta_r}{2\hat{p} - \Delta_q + \Delta_r} \right)} & (a), \\
\end{cases}
\]

(62)
The evolution of $\gamma^x$ is autonomous,

$$\lambda \frac{d\gamma^x}{d\lambda} = -\lambda^2 \beta^x \gamma^x \lambda^2,$$

with the solution,

$$\gamma^x(\lambda) = \frac{\gamma_0^x}{1 + \gamma_0^x \beta_x \lambda^2},$$

where $\gamma^x(0) = \gamma_0^x$ denotes the bare coupling constant which is supposed to be small, $|\gamma_0^x| \ll 1$. The solution (64) shows that our truncation resums the geometrical series of the diagonal part of the vertex function. The truncation of the effective action is consistent and the system remains weakly coupled for $\gamma_0^x \beta_x^x > 0$, in which case $|\gamma^x(\lambda)|$ decreases during the evolution. On the contrary, for $\gamma_0^x \beta_x^x < -1$ a Landau-pole is predicted at

$$\lambda_L = \sqrt{-\frac{1}{\gamma_0^0 \beta_x^x}} < 1,$$

before the desired bare coupling strength $\gamma_0$ is reached. The Landau-pole appears in region (a) due to the cutoff effects when the inequality

$$\frac{1}{2\pi} < \gamma_0^x \frac{\hat{p}^2 - \hat{r}^2}{\Lambda^2}$$

is satisfied by the bare theory. This pole would be absent in a typical high energy application when the cutoff is large but may appear as a non-universal effect when the model is considered as an effective theory with low cutoff, $\Lambda \ll k_F$. The Landau-pole is always present in region (b), in the vicinity of the mass shell and the Fermi point, where the inequality

$$\frac{1}{2\pi} < \gamma_0^x \ln \left| \frac{1 - \Delta_q - \Delta_r}{1 + \Delta_q - \Delta_r} \right|$$

can always be satisfied for an arbitrary value of $\gamma_0^x$.

The effective coupling constants $\gamma''$ is obtained by integrating the beta function

$$\beta'' = -4\lambda^2 \pi (\gamma^x + 9\gamma''^2).$$

The evolution is autonomous for $\gamma^x(\lambda) \approx 0$ and gives

$$\gamma''(\lambda) = \frac{\gamma_0''}{1 + 36\pi \gamma_0'' \lambda^2}$$

with another Landau-pole when

$$\gamma_0'' < -\frac{1}{36\pi}.$$ 

When $\gamma^x(\lambda)$ approaches a Landau-pole then $\gamma'' \to -\infty$ because the singularity is non-integrable.

It is worthwhile noting that $\beta''$ and together with it the coupling strengths $\gamma^x(\lambda)$ and $\gamma''(\lambda)$ are scale invariant in regime (b), i.e. the external scale parameters $k_F$ and $\Lambda$ decouple from the dynamics. As a result, the beta functions, obtained by means of the usual renormalization group schemes where the energy-momentum components of subtraction point scale with $\lambda$ in a homogeneous manner, are vanishing. Furthermore, the beta functions (61) and the effective coupling strengths become scale invariant in the limit $\Lambda \to \infty$ at the subtraction point, $\omega_p = \omega_q = \omega_r = \omega_s = 0$, $\hat{p}/3 = \hat{q} = -\hat{r} = 2$, used in Ref. [10].

**IX. SUMMARY**

The transmutation of quasiparticles between the short and the long distance scales is a challenging problem from particle to condensed matter physics. We considered one dimensional massless fermions, the Luttinger model in this
work and applied the functional form of the internal space renormalization group with sharp momentum cutoff which yields a functional differential equation for the effective action. The initial conditions for the differential equation are known. This equation was truncated in such a manner that the weakly coupled nature of the interactions can be checked. The solution shows strong interactions and singular momentum dependence in the vicinity of the Fermi points in the two-body channels and Landau poles for arbitrary bare parameters. But our results are rather preliminary, one should systematically enlarge the ansatz for the effective action and study the stability of Landau poles in the result.

We argued that cutoff effects, in particular Landau poles, are generated by the $O(\Lambda^{-2})$ part of the renormalized action in the normal ordered schemes, such as bosonization. We believe that more work is needed on the one hand, to increase the reliability of the results obtained by completely regulated bare theories and on the other hand, to clarify the role the $O(\Lambda^{-2})$ interactions may play in the bosonized theory.

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APPENDIX A: NOTATION, CONVENTIONS

The space-time and Fourier-space integrals are defined as

$$\int_x = \int d^2 x = \int d x \, d t = a_s a_t \sum_x, \quad \int_p = \int \frac{d^2 p}{(2\pi)^2} = \sum_p, \quad f_x = \int_p f e^{ipx},$$

(A1)

where $x = (t, x)$, $p = (\omega, p)$. In Euclidean space-time one has $px = p^0 x^0 + px$. The letters $a, b, \ldots$ and $p, q, \ldots$ denote space-time or energy-momentum variables, respectively. The integration is sometime shown as scalar product, $f \cdot g = \int_x f_x g_x$.

The functional derivative is defined in $d$-dimensional space-time lattice as

$$\frac{\delta}{\delta \Psi_x} = \frac{1}{a_s a_t} \frac{\partial}{\partial \Psi^L_x}$$

(A2)

where $\Psi^L$ is the lattice field variable whose dimension is removed by the lattice spacing and the factor $1/a_s a_t$ is needed in order to satisfy the relation

$$\frac{\delta}{\delta \Psi_y} \int_x f(\phi_x) = f'(\Psi_y).$$

(A3)

In an analogous manner we have

$$\frac{\delta}{\delta \Psi_p} = V \frac{\partial}{\partial \Psi^L_p}.$$

(A4)

APPENDIX B: ONE-LOOP INTEGRALS

This Appendix contains some details concerning the computation of the one-loop integrals of Eqs. (52) by means of sharp momentum space cutoff. In order to simplify the results we assume that the absolute value of the momenta, counted from the Fermi point does not exceed $\Lambda/4$. 
1. $\gamma^\times$

For the first equation we need the following integrals,

\[
I_1 = \frac{1}{\lambda^2} \int_{t,\omega} G_{\lambda,t} G_{\lambda,p-q+t},
\]

\[
I_2 = \frac{1}{\lambda^2} \int_{t,\omega} G_{\lambda,t} G_{\lambda,p+r-t},
\]

\[
I_3 = \frac{1}{\lambda^2} \int_{t,\omega} G_{\lambda,t} G_{\lambda,t-q+r},
\]

\[
I_4 = \frac{1}{\lambda^2} \int_{t,\omega} G_{\lambda,t} G_{\lambda,t+r-s},
\]

(B1)

where

\[
G_\sigma = \frac{\lambda}{-\partial_t + \epsilon_\sigma (-i\partial_x)}.
\]

(B2)

One uses the residuum theorem in integrating over the frequency in the integral

\[
I_1 = \int_{-\infty}^{\infty} du \int_{-\infty}^{+\infty} d\omega \frac{\rho_\lambda(u) \rho_\lambda(u + p-q)}{(\omega + iu)(\omega + \omega_q - \omega_r + i(u + p-q))}
\]

and finds

\[
I_1 = 2i\pi \int_{-\infty}^{\infty} du \rho_\lambda(u) \rho_\lambda(u + p-q) \frac{\Theta(-u) - \Theta(-u + q - p)}{\omega_p - \omega_q + i(p-q)}
\]

(B3)

(B4)

which results

\[
I_1 = \frac{2\pi(\hat{p} - \hat{q})}{\Delta_p - \Delta_q + 2(\hat{p} - \hat{q})}.
\]

(B5)

The other three integrals can be computed in a similar manner. For

\[
I_2 = \int_{-\infty}^{\infty} du \int_{-\infty}^{+\infty} d\omega \frac{\rho_\lambda(u) \rho_\lambda(u - p-r)}{(\omega + iu)(\omega - \omega_p - \omega_r - i(u - p-r))}
\]

we find

\[
I_2 = F_2(0) - F_2(-\Lambda) - [F_2(\Lambda) - F_2(\hat{p} + \hat{r})]
\]

(B6)

(B7)

for \(|\hat{p}|, |\hat{r}| < \Lambda/4\) where

\[
F_2(u) = -\pi \ln \left| u - \frac{\hat{p} + \hat{r}}{2} - i\frac{\omega_p + \omega_r}{2} \right| = -\pi \ln \left| u - \hat{r} - \frac{\Delta_p + \Delta_r}{2} \right|.
\]

(B8)

The final result is

\[
I_2 = -\pi \ln \left| \frac{\hat{r} + \frac{\Delta_p + \Delta_r}{2}}{\Lambda - \hat{r} - \frac{\Delta_p + \Delta_r}{2}} \right|
\]

(B9)

The integral

\[
I_3 = -\int_{-\infty}^{\infty} du \int_{-\infty}^{+\infty} d\omega \frac{\rho_\lambda(u) \rho_\lambda(u - q+r)}{(\omega - iu)(\omega - \omega_q + \omega_r + i(u - q + r - 2k_F))}
\]

becomes

\[
I_3 = -F_3(\Lambda) - F_3(-\Lambda) + F_3(0) + F_3(\hat{q} - \hat{r})
\]

(B10)

(B11)
where
\[ F_{3}(u) = \pi \ln \left| u + \frac{\hat{r} - \hat{q}}{2} + i \frac{\omega_{q} - \omega_{r}}{2} \right| \] (B12)

after the integration over the frequency. Trivial steps give
\[ I_{3} = \pi \ln \left| \frac{\left( \frac{\Delta_{s} - \Delta_{r}}{2} - q \right) \left( \frac{\Delta_{s} - \Delta_{r}}{2} - \hat{r} \right)}{q + \frac{\Delta_{s} - \Delta_{r}}{2} - \Lambda^{2}} \right| \] (B13)

Finally, the last integral,
\[ I_{4} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\omega \frac{\rho_{\Lambda}(u)\rho_{\Lambda}(u - s + r)}{(\omega - iu)(\omega + \omega_{r} - \omega_{s} + i(-u - r + s))} \] (B14)
gives
\[ I_{4} = 2i\pi \int_{-\Lambda}^{+\Lambda} du \left[ \frac{\Theta(u)}{\omega_{r} - \omega_{s} + i(s - r)} + \frac{\Theta(u + r - s)}{\omega_{s} - \omega_{r} - i(s - r)} \right] \]
\[ = \frac{2\pi(\hat{r} - \hat{q})}{\Delta_{s} - \Delta_{r} + 2(\hat{r} - \hat{q})}. \] (B15)

2. \( \gamma'' \)

For the second equation in Eqs. (52) we need the following five integrals,
\[ J_{1} = \frac{1}{\Lambda^{2}} \int_{t,\omega} G_{+;t}(G_{+;t-s+t}, \] \( J_{2} = \frac{1}{\Lambda^{2}} \int_{t,\omega} G_{+;t}(G_{+;t-q+t}, \] \( J_{3} = \frac{1}{\Lambda^{2}} \int_{t,\omega} G_{-;t}(G_{-;t-r-t}, \] \( J_{4} = \frac{1}{\Lambda^{2}} \int_{t,\omega} G_{-;t}(G_{-;t-s}, \] \( J_{5} = \frac{1}{\Lambda^{2}} \int_{t,\omega} G_{-;t}(G_{-;u-r+q}. \] (B16)

Due to the left and right symmetry properties \( J_{4} = J_{1} \) and \( J_{5} = J_{2} \). The same procedure than in the case of the previous integrals gives
\[ J_{1} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\omega \frac{\rho_{\Lambda}(u)\rho_{\Lambda}(u - s + r)}{(\omega + iu)(\omega + \omega_{r} - \omega_{s} + i(u + r - s))} \]
\[ = 2i\pi \int_{-\infty}^{\infty} du \rho_{\Lambda}(u)\rho_{\Lambda}(u - s + r) \left[ \frac{\Theta(-u)}{\omega_{r} - \omega_{s} + i(r - s)} + \frac{\Theta(-u + s - r)}{\omega_{s} - \omega_{r} - i(r - s)} \right] \]
\[ = \frac{2\pi(\hat{r} - \hat{q})}{\Delta_{r} - \Delta_{s} + 2(\hat{r} - \hat{q})}. \] (B17)

and
\[ J_{2} = -\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\omega \frac{\rho_{\Lambda}(u)\rho_{\Lambda}(u - q + r)}{(\omega + iu)(\omega + \omega_{r} - \omega_{q} + i(u + r - q))} \]
\[ = -2i\pi \int_{-\infty}^{\infty} du \rho_{\Lambda}(u)\rho_{\Lambda}(u - s + r) \left[ \frac{\Theta(-u)}{\omega_{r} - \omega_{q} + i(r - q)} + \frac{\Theta(-u - r + q)}{\omega_{q} - \omega_{r} - i(r - q)} \right] \]
\[ = 2\pi \frac{\hat{r} - \hat{q} + 2k_{F}}{\Delta_{r} - \Delta_{q} + 2q - 2k_{F}}. \] (B18)
Finally, the last independent integral is

\[
J_3 = -\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} d\omega \left( \frac{\rho_\Lambda(u)\rho_\Lambda(u - p - r)}{(\omega - iu)(\omega - \omega_p - \omega_r - i(u - p - r - 2k_F))} \right)
= -2i\pi \int_{-\infty}^{+\infty} du \rho_\Lambda(u) \rho_\Lambda(u - p - r) \left[ \frac{\Theta(u)}{-\omega_p - \omega_r - i(-p - r - 2k_F)} + \frac{\Theta(u - p - r - 2k_F)}{\omega_p + \omega_r + i(-p - r - 2k_F)} \right]
= -2\pi \frac{\hat{p} + \hat{r} + 2k_F}{\Delta_p + \Delta_r + 2\hat{r} + 2k_F}
\]

(B19)

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