On well-posedness of incompressible two-phase flows with phase transitions: the case of non-equal densities

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Dedicated to Yoshihiro Shibata on the occasion of his 60th anniversary.

Abstract. The basic model for incompressible two-phase flows with phase transitions consistent with thermodynamics is studied. The latter means that the total energy is conserved and the total entropy is nondecreasing. We consider the case of constant but non-equal densities of the phases, complementing our previous paper (Prüss et al. in Evol Equ Control Theory 1:171–194, 2012) where the case of equal densities is analyzed. The local well-posedness of such problems is proved by means of the technique of maximal $L_p$-regularity, in a configuration where the interface is nearly flat and initial data are small.

1. Introduction

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain of class $C^3$, $n \geq 1$. $\Omega$ contains two phases: at time $t$, phase $k$, $k = 1, 2$, occupies subdomain $\Omega_k(t)$ of $\Omega$. Assume $\partial \Omega_1(t) \cap \partial \Omega = \emptyset$; this means no boundary intersection and no contact angles. The closed compact hypersurface $\Gamma(t) := \partial \Omega_1(t) \subset \Omega$ forms the interface between the phases.

Let $u$ denote the velocity field, $\pi$ the pressure field, $T(u, \pi, \theta)$ the stress tensor, $D(u) = (\nabla u + [\nabla u]^T)/2$ the rate of deformation tensor, $\theta$ the (absolute) temperature field, $\nu_{\Gamma}$ the outer normal of $\partial \Omega_1$, $u_{\Gamma}$ the interface velocity, $V_{\Gamma} = u_{\Gamma} \cdot \nu_{\Gamma}$ the normal velocity of $\Gamma(t)$, $H_{\Gamma} = H(\Gamma(t)) = -\text{div}_{\Gamma} v_{\Gamma}$ the curvature of $\Gamma(t)$, $j$ the phase flux, and

$$[[v]] = (v|_{\Omega_2(t)} - v|_{\Omega_1(t)})|_{\Gamma(t)}$$

the jump of a quantity $v$ across $\Gamma(t)$.

Mathematics Subject Classification (2010): Primary 35R35; Secondary 35Q30, 76D45, 76T10.

Keywords: Two-phase Navier–Stokes equations, Surface tension, Phase transitions, Well-posedness, Maximal $L_p$-regularity.

S.S. expresses her thanks for hospitality to the Institute of Mathematics, Martin-Luther-Universität Halle-Wittenberg, where important parts of this work originated. The research of S.S was partially supported by JSPS Grant-in-Aid for Scientific Research (B)—24340025 and Challenging Exploratory Research—23654048, MEXT.
Let \( \rho_k > 0 \) denote the densities of \( \Omega_k(t) \). In this paper, we consider the case of non-equal densities \( \rho_1 \neq \rho_2 \). In order to economize our notation, we set
\[
\rho = \rho_1 \chi_{\Omega_1(t)} + \rho_2 \chi_{\Omega_2(t)},
\]
where \( \chi_D \) denotes the indicator function of a set \( D \), and this notation is employed for \( \mu, \kappa, d \), etc. as well. We just keep in mind that the coefficients depend on the phases.

By an Incompressible Two-Phase Flow with Phase Transition we mean the following problem: Find a family of closed compact hypersurfaces \( \{ \Gamma(t) \}_{t \geq 0} \) contained in \( \Omega \) and appropriately smooth functions \( u : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}^{n+1} \), and \( \pi, \theta : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R} \) such that
\[
\rho (\partial_t u + u \cdot \nabla u) - \text{div} T(u, \pi, \theta) = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t), \quad t > 0,
\]
\[
T(u, \pi, \theta) = 2\mu(\theta)D(u) - \pi I, \quad \text{div} u = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t), \quad t > 0,
\]
\[
\left[ \frac{1}{\rho} \right] j^2 v_\Gamma - \| T(u, \pi, \theta) v_\Gamma \| - \sigma H v_\Gamma = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
\left[ u \right] - \left[ \frac{1}{\rho} \right] j v_\Gamma = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
u_\Gamma - u \cdot v_\Gamma + \frac{1}{\rho} j = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
\rho \kappa(\theta)(\partial_t \theta + u \cdot \nabla \theta) - \text{div}(d(\theta) \nabla \theta) - 2\mu(\theta)|D(u)|^2 = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t), \quad t > 0,
\]
\[
l(\theta) j + \| d(\theta) \partial_v \theta \| = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
\| \theta \| = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
\partial_v \theta = 0 \quad \text{on} \quad \partial \Omega, \quad t > 0,
\]
\[
\theta(0) = \theta_0 \quad \text{in} \quad \Omega,
\]
\[
(1.1)
\]
\[
\psi(\theta) + \left[ \frac{1}{2\rho^2} \right] j^2 - \left[ \frac{T(u, \pi, \theta) v_\Gamma \cdot v_\Gamma}{\rho} \right] = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
V_\Gamma - u \cdot v_\Gamma + \frac{1}{\rho} j = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0,
\]
\[
\Gamma(0) = \Gamma_0.
\]
\[
(1.2)
\]
Several quantities are derived from the specific free energy \( \psi_k(\theta) \) in phase \( k \) as follows.
\begin{itemize}
\item \( \epsilon_k(\theta) := \psi_k(\theta) + \theta \eta_k(\theta) \) the internal energy,
\item \( \eta_k(\theta) := -\psi'_k(\theta) \) the entropy,
\item \( \kappa_k(\theta) := \epsilon'_k(\theta) = -\theta \psi''_k(\theta) > 0 \) the heat capacity,
\item \( l(\theta) := \theta \| \psi'(\theta) \| = -\theta \| \eta(\theta) \| \) the latent heat.
\end{itemize}

Further, \( d_k(\theta) > 0 \) denotes the coefficient of heat conduction in Fourier’s law, \( \mu_k(\theta) > 0 \) the viscosity in Newton’s law, and \( \sigma > 0 \) the constant coefficient of surface tension.
Concerning the second equation of (1.3), we remind that balance of mass across \( \Gamma(t) \) requires \([\rho(u - u_\Gamma)] \cdot v_\Gamma = 0\), which implies

\[
\begin{align*}
  j &= \rho_1(u_1 - u_\Gamma) \cdot v_\Gamma = \rho_2(u_2 - u_\Gamma) \cdot v_\Gamma,
  
  \text{and so}
  
  u_1 \cdot v_\Gamma - \frac{1}{\rho_1} j &= u_2 \cdot v_\Gamma - \frac{1}{\rho_2} j.
\end{align*}
\]

Therefore, this equation is well-defined on \( \Gamma(t) \).

This model is explained in more detail in our previous paper [19], where we consider the case of equal densities. It has been recently proposed by Anderson et al. [1], see also the monographs by Ishii [12] and Ishii and Takashi [13], and it is thermodynamically consistent in the sense that in the absence of exterior forces and heat sources, the total energy is preserved and the total entropy is nondecreasing, see [19]. It is in some sense the simplest sharp interface model for incompressible Newtonian two-phase flows taking into account phase transitions driven by temperature.

Note that in the case of equal densities, the phase flux \( j \) does not enter (1.1), and so in this case, we obtain essentially a Stefan problem with surface tension, which is only weakly coupled to the standard two-phase Navier–Stokes problem via temperature-dependent viscosities. We call this case *temperature dominated*, and it has been studied in [19]. But in the case of different densities, the phase flux \( j \) causes a jump in the velocity field on the interface, which leads to so-called *Stefan currents* that are convections driven by phase transitions. In this situation, it turns out that the heat problem (1.2) is only weakly coupled to (1.1) and (1.3), we call this case *velocity dominated*. The resulting two-phase Navier–Stokes problem is non-standard, and therefore, it requires a new analysis.

The analytical properties of the problem appear to be different in these two cases. The spaces for well-posedness are not the same, and in the velocity-dominated case, the pressure is uniquely determined, while in the temperature-dominated case, it is only unique up to a constant. In the temperature-dominated case \([\rho] = 0\), the phase flux \( j \) can be eliminated by solving the second equation in (1.2) for \( j \). This yields

\[
  j = -\|[d(\theta) \partial_\nu \theta] / l(\theta)\|,
\]

as long as \( l(\theta) \neq 0\); this is the essential well-posedness condition in this case. Then, the equation describing the evolution of the interface becomes

\[
  V_\Gamma = u_\Gamma \cdot v_\Gamma + [d(\theta) \partial_\nu \theta] / \rho l(\theta).
\]

On the other hand, in the velocity-determined case \([\rho] \neq 0\), we can eliminate \( j \) by taking the inner product of the fourth equation in (1.1) with \( v_\Gamma \) to the result

\[
  j = \|[u \cdot v_\Gamma] / [1/\rho]\|.
\]
In this case, the equation for $V_\Gamma$ becomes

$$V_\Gamma = \|\rho u \cdot v_\Gamma\| / \|\rho\|,$$

which does not contain temperature, in contrast to the first case. Therefore, the analysis for these two cases necessarily is different, too.

There is a large literature on isothermal incompressible Newtonian two-phase flows without phase transitions [2,15,22,23,25,26], and also on the two-phase Stefan problem with surface tension modeling temperature driven phase transitions [3,8,18,21,24]. On the other hand, mathematical work on two-phase flow problems including phase transitions is rare. In this direction, we only know the papers by Hoffmann and Starovoitov [10,11] dealing with a simplified two-phase flow model, and Kusaka and Tani [16,17] which is two-phase for temperature but only one phase is moving. The papers of Di Benedetto and Friedman [4] and Di Benedetto and O’Leary [5] deal with weak solutions of conduction–convection problems with phase change. However, none of these papers considers models which are consistent with thermodynamics.

It is the purpose of this paper to present a rigorous analysis of problem (1.1), (1.2), (1.3) in the framework of $L_p$-theory in the case of non-equal densities and an initial interface which is nearly flat. We consider the nonlinear problem (1.1)–(1.3) for $\Omega = \mathbb{R}^{n+1}$ and a nearly flat interface represented as a graph over $\mathbb{R}^n$, namely in the regions

$$\Omega_k(t) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^k(y - h(t, x)) > 0, \ t \geq 0\}, \ k = 1, 2,$$

with interface

$$\Gamma(t) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y - h(t, x) = 0, \ t \geq 0\}.$$

We let $\Omega_0 = \Omega_1(0) \cup \Omega_2(0)$ and $v_0$ be the outer normal of $\Omega_1(0)$.

The main result of this paper is the local well-posedness of (1.1)–(1.3).

THEOREM 1.1. Let $p > n + 3$, $\rho_1, \rho_2, \sigma > 0$ be constant, $\rho_1 \neq \rho_2$, and suppose $\psi_k \in C^3(0, \infty), \mu_k, d_k \in C^2(0, \infty)$ are such that

$$\kappa_k(s) = -s\psi_k''(s) > 0, \ \mu_k(s) > 0, \ d_k(s) > 0 \ s \in (0, \infty), \ k = 1, 2.$$

Let the initial interface $\Gamma_0$ be given by a graph $x \mapsto (x, h_0(x)), x \in \mathbb{R}^n$, and let $\theta_\infty > 0$ be the constant temperature at infinity.

Then, given any finite interval $J = [0, a]$, there exists $\eta > 0$ such that (1.1)–(1.3) admits a unique $L_p$-solution on $J$ provided the smallness conditions

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_\infty\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{2-2/p}(\mathbb{R}^n)} \leq \eta,$$

and the compatibility conditions

$$\text{div}u_0 = 0 \ \text{in} \ \Omega_0,$$

$$P_{\Gamma_0}\|\mu(\theta_0) D(u_0)v_0\| = 0, \ P_{\Gamma_0}\|u_0\| = 0 \ \text{on} \ \Gamma_0,$$

$$\|\theta_0\| = 0, \ (l(\theta_0)/\|1/\rho\|)[u_0 \cdot v_0] + \|d(\theta_0)\partial_{v_0}\theta_0\| = 0 \ \text{on} \ \Gamma_0,$$
are satisfied. Here \( D(u_0) = (\nabla u_0 + [\nabla u_0]) / 2 \), and \( P_{\Gamma_0} = I - \nu_{\Gamma_0} \otimes \nu_{\Gamma_0} \) denotes the projection onto the tangent bundle of \( \Gamma_0 \).

The notion \( L_p \)-solution is explained in more detail in Section 5. For a proof of this result, we perform a detailed analysis of the linearized problem in an \( L_p \)-setting, following the approach in [22] for the standard two-phase Navier–Stokes problem without phase transitions. This requires the detection and analysis of the underlying boundary symbol. We then show maximal regularity for the linear part of the problem and finally employ the contraction mapping principle to solve the nonlinear problem. In a forthcoming paper, we will consider problem (1.1), (1.2), (1.3) in general geometries without smallness assumptions.

The plan for this paper is as follows. In Sect. 2, we transform the problem to the configuration of a fixed flat interface. The principal part of the linearization is studied in Sect. 3, and the property of maximal \( L_p \)-regularity is proved in Sect. 4. The last section contains the proof of well-posedness for the nonlinear problem.

2. Transformation to a flat interface

In the situation of a nearly flat interface, the nonlinear problem (1.1)–(1.3) can be transformed to a problem on \( \dot{R}^{n+1} := R^{n+1} \setminus [R^n \times \{0\}] \) by means of the transformations

\[
\begin{align*}
v(t, x, y) &:= (u_1, \ldots, u_n)^T(t, x, y + h(t, x)), \\
w(t, x, y) &:= u_{n+1}(t, x, y + h(t, x)), \\
\bar{\theta}(t, x, y) &:= \theta(t, x, y + h(t, x)) - \theta_\infty, \\
\bar{\pi}(t, x, y) &:= \pi(t, x, y + h(t, x)) - \pi_\infty,
\end{align*}
\]

where \( t \in J = [0, a], \ x \in \mathbb{R}^{n-1}, \ y \in \mathbb{R}, \ y \neq 0 \). Here, \( \theta_\infty > 0 \) denotes the (equilibrium) temperature at infinity and \( \pi_\infty \) the corresponding (equilibrium) pressure at infinity defined by the relations

\[
\begin{align*}
\| \psi(\theta_\infty) \| + \| \pi_\infty / \rho \| &= 0, \\
\| \pi_\infty \| &= 0.
\end{align*}
\]

With a slight abuse of notation, we will denote in the sequel the transformed velocity again by \( u \), that is, \( u = (v, w)^T \), the transformed temperature by \( \theta \), and the transformed pressure by \( \pi \). For given initial data \( u_0(x) \) and \( \theta_0(x) \), we set again \( u_0(x, y) := u_0(x', y + h_0(x')) \) and \( \theta_0(x, y) := \theta_0(x', y + h_0(x')) - \theta_\infty \) and define

\[
\mu_0 = \mu(\theta_\infty), \quad \kappa_0 = \kappa(\theta_\infty), \quad d_0 = d(\theta_\infty), \quad l_0 = l(\theta_\infty).
\]

With this notation, we have the transformed problem

\[
\begin{align*}
\rho \partial_t u - \mu_0 \Delta u + \nabla \pi &= F_u(u, \pi, \theta, h) & \text{in} & \ R^{n+1}, & t > 0, \\
\text{div} u &= F_d(u, h) & \text{in} & \ R^{n+1}, & t > 0,
\end{align*}
\]
where the phase flux \( j \) already has been eliminated, according to Sect. 1. Here, it reads

\[
j = \frac{\| w \| - \| v \| \cdot \nabla_x h}{\sqrt{1 + |\nabla_x h|^2\|1/\rho\|}} = \| w \| \sqrt{1 + |\nabla_x h|^2} \left( \|1/\rho\| \right).
\]

The nonlinear right-hand sides are defined by

\[
F_u(u, \pi, \theta, h) = (F_v(u, \pi, \theta, h), F_w(u, \theta, h))^T,
\]

\[
F_v(u, \pi, \theta, h) = (\mu(\theta) - \mu_0) \Delta v
\]

\[
+ \mu(\theta) \left(-\Delta_x h \partial_y v - 2\nabla_x h \cdot \nabla_x \partial_y v + |\nabla_x h|^2 \partial^2_y v\right)
\]

\[
- \rho(\nu \cdot \nabla_x v + w \partial_y v - \nabla_x h \partial_y v) + \rho \partial_t h \partial_y v + \nabla_x h \partial_y \pi
\]

\[
+ (\nabla_x v + [\nabla_x v]^T) - (\nabla_x h \otimes \partial_y v + \partial_y v \otimes \nabla_x h)\mu'(\theta)\nabla_x \theta
\]

\[
+ (\partial_y v + \nabla_x w - \nabla_x h \partial_y w) \mu'(\theta) \partial_y \theta,
\]

\[
F_w(u, \theta, h) = (\mu(\theta) - \mu_0) \Delta w
\]

\[
+ \mu(\theta) \left(-\Delta_x h \partial_y w - 2\nabla_x h \cdot \nabla_x \partial_y w + |\nabla_x h|^2 \partial^2_y w\right)
\]

\[
- \rho(\nu \cdot \nabla_x w + w \partial_y w - \nabla_x h \partial_y w) + \rho \partial_t h \partial_y w
\]

\[
+ ([\nabla_x v]^T + [\nabla_x w]^T - \partial_y w[\nabla_x h]^T) \mu'(\theta) \nabla_x \theta + 2\partial_y w \mu'(\theta) \partial_y \theta,
\]

\[
F_d(u, h) = \nabla_x h \cdot \partial_y v = \partial_y (\nabla_x h \cdot v),
\]

\[
G_v(u, \theta, h) = \| (\mu(\theta) - \mu_0)(\partial_y v + \nabla_x h) \| - \| \mu(\theta)(\nabla v + [\nabla v]^T) \| \nabla_x h
\]

\[
+ \| (\mu(\theta)(\nabla_x h(\partial_y v \cdot \nabla_x h) + \partial_y v|\nabla_x h|^2 - \nabla_x h \partial_y w) \| 
\]

\[
+ \| (\mu(\theta)(-\partial_y v + \nabla_x w) \cdot \nabla_x h + 2\partial_y w + \partial_y w|\nabla_x h|^2) \| \nabla_x h
\]

\[
+ \| \rho^{-1} (1 + |\nabla_x h|^2) \| w^2 |\nabla_x h|,
\]

\[
G_w(u, \theta, h) = \| (\mu(\theta) - \mu_0) 2\partial_y w \| - \| \mu(\theta)(\partial_y v + \nabla_x w) \cdot \nabla_x h \| 
\]

\[
+ \| (\mu(\theta) \partial_y w) \| \nabla_x h |^2 - \| \rho^{-1} (1 + |\nabla_x h|^2) \| w^2 - \sigma J(h),
\]

\[
G_j(u, h) = -\| w \| \nabla_x h,
\]

\[
F_\theta(u, \theta, h) = \rho(\kappa_0 - \kappa(\theta)) \partial_t \theta + (d(\theta) - d_0) \Delta \theta
\]
\[ + \rho \kappa(\theta) \{ \partial_t h \partial_y \theta - v \cdot \nabla \theta + (v \cdot \nabla h) \partial_y \theta - w \partial_y \theta \} \]
\[ + d'(\theta) \{ |\nabla \nabla h| - \nabla h \partial_y \theta \}^2 + (\partial_y \theta)^2 \]
\[ + (\mu(\theta)/2) |\nabla \nabla v + [\nabla \nabla v] \nabla h - \nabla h \otimes \partial_y v - \partial_y v \otimes \nabla h|^2 \]
\[ + (\mu(\theta)/2) |\partial_y v + \nabla x w - \partial_y w \nabla \nabla h|^2 + 2 |\partial_y w|^2 \],

\[ G_\theta(u, \theta, h) = \left\{ \left[ (d(\theta) - d_0) \partial_y \theta \right] - \left[ d(\theta) \nabla \nabla \theta \cdot \nabla h \right] \right\} \]
\[ + (l(\theta)/\|1/\rho\|)(1 + |\nabla \nabla h|^2)\|w\|, \]
\[ G_\pi(u, \theta, h) = - \left[ \left[ \psi(\theta + \theta_\infty) - \psi(\theta_\infty) \right] + 2 \left( (\mu(\theta) - \mu_0) \partial_y w / \rho \right) \right] \]
\[ - \left[ \frac{1}{2 \rho^2} \right] (1 + |\nabla \nabla h|^2) \left[ \frac{1}{\rho} \right]^{-2} \|w\|^2 - 2 \left[ \frac{\mu(\theta)}{\rho} \partial_y v \cdot \nabla h \right] \]
\[ + \frac{2}{1 + |\nabla \nabla h|^2} \left[ \frac{\mu(\theta)}{\rho} \right] \left[ (\nabla v \nabla h) \cdot \nabla h - \nabla x w \cdot \nabla h \right] \}
\[ G_h(u, h) = - \left[ \frac{\| \rho v \cdot \nabla h \|}{\| \rho \|} \right]. \]

The curvature of \( \Gamma(t) \) is given by

\[ H(\Gamma(t)) = \text{div}_x \left( \frac{\nabla \nabla h(t, x)}{\sqrt{1 + |\nabla \nabla h(t, x)|^2}} \right) = \Delta_x h - J(h), \]

with

\[ J(h) = \frac{|\nabla \nabla h|^2 \Delta_x h}{(1 + \sqrt{1 + |\nabla \nabla h|^2})^{3/2}} \]
\[ + \frac{\nabla \nabla h \cdot (\nabla^2 h \cdot \nabla \nabla h)}{(1 + |\nabla \nabla h|^2)^{3/2}}, \]

where \( \nabla^2 h \) denotes the Hessian of \( h \).

Concerning the boundary condition

\[ \left[ \| \rho^{-1} \| j^2 v \right]_\Gamma - \left[ \| \mu(\theta) \| (\nabla u + [\nabla u]^T) \right]_\Gamma = (\sigma H_\Gamma - \| \pi \|)_\Gamma \] (2.2)

in (1.1), multiplying (2.2) by \( \sqrt{1 + |\nabla \nabla h|^2} v, v = e_{n+1} \), we obtain

\[ \sigma H_\Gamma - \| \pi \| = - \left[ \| \mu(\theta) \| (\nabla u + [\nabla u]^T) \right] \sqrt{1 + |\nabla \nabla h|^2} v_\Gamma \cdot v + \| \rho^{-1} \| j^2. \]

Inserting this relation into (2.2), we obtain the nonlinear term \( G_v(u, \theta, h) \) which neither contains the curvature nor the pressure jump \( \| \pi \| \) (cf. [19, Section 4]).

### 3. The linear problem

The principal part of the linearized problem in the case of a nearly flat initial interface reads as follows

\[ \rho \partial_t u - \mu \Delta u + \nabla \pi = f_u \quad \text{in } \mathbb{R}^{n+1}, \quad t > 0, \]
\[ \text{div} u = f_d \quad \text{in } \mathbb{R}^{n+1}, \quad t > 0, \]
\[-2\left[\mu D(u)v + \|\pi\|v - \sigma(\Delta_x h)v = g_u \quad \text{on } \mathbb{R}^n, \quad t > 0, \right.\]
\[
\left.\|v\| = g_j \quad \text{on } \mathbb{R}^n, \quad t > 0, \quad \right.\]
\[
\left. u(0) = u_0 \quad \text{in } \mathbb{R}^{n+1}, \right.\]
\[
\rho \kappa \partial_t \theta - d \Delta \theta = f_\theta \quad \text{in } \mathbb{R}^{n+1}, \quad t > 0, \]
\[
-\|d \partial_{t} \theta\| = g_\theta \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\|\theta\| = 0 \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\theta(0) = \theta_0 \quad \text{in } \mathbb{R}^{n+1}, \]
\[
-2\left[\frac{\|\mu D(u)v \cdot v\|}{\rho} + \|\pi\|\right] = g_\pi \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\partial_t h - \|\rho u \cdot v\|/\|\rho\| = g_h \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\left. h(0) = h_0 \quad \text{on } \mathbb{R}^n, \right.\]

where $\mu_k, \kappa_k, d_k, \rho_k, k = 1, 2$, are constants, $v = e_{n+1}$, and $u = (v, w)^T$. We assume as always in this paper $\|\rho\| = \rho_2 - \rho_1 \neq 0$. Apparently, (3.2) decouples from the remaining problem. Since it is well-known that this problem has maximal $L_p$-regularity, we concentrate on the remaining one. It reduces to two separate problems. With $u = (v, w)^T$, the first one is the following non-standard Stokes problem.

\[
\rho \partial_t u - \mu \Delta u + \nabla \pi = f_u \quad \text{in } \mathbb{R}^{n+1}, \quad t > 0, \]
\[
\text{div } u = f_d \quad \text{in } \mathbb{R}^{n+1}, \quad t > 0, \]
\[
-2\|\mu D(u)v\| + \|\pi\|v = g_u = (g_v, g_w)^T \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\|v\| = g_j \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
-2\|\mu D(u)v \cdot v/\rho\| + \|\pi/\rho\| = g_\pi \quad \text{on } \mathbb{R}^n, \quad t > 0, \]
\[
\left. u(0) = u_0 \quad \text{in } \mathbb{R}^{n+1}. \right.\]

Having solved the first one, the second one results from replacing $g_w$ by $g_w + \sigma \Delta_x h$ and solving

\[
\partial_t h - \|\rho w\|/\|\rho\| = g_h, \quad t > 0, \quad h(0) = 0. \]

Before stating maximal regularity results of linear problems, let us introduce the relevant function spaces. Let $\Omega \subset \mathbb{R}^m$ be open and $X$ be an arbitrary Banach space. By $L_p(\Omega; X)$ and $H^s_p(\Omega; X)$, for $1 \leq p \leq \infty, s \in \mathbb{R}$, we denote the $X$-valued Lebesgue and the $X$-valued Bessel potential spaces of order $s$, respectively. We will also make use of the fractional Sobolev-Slobodeckij spaces $W^s_p(\Omega; X), 1 \leq p < \infty, s > 0, s \not\in \mathbb{N}$ with norm

\[
\|g\|_{W^s_p(\Omega; X)} = \|g\|_{W^{|s|}_p(\Omega; X)} + \sum_{|\alpha| = |s|} \left( \int_\Omega \int_\Omega \frac{\|\partial^\alpha g(x) - \partial^\alpha g(y)\|_X^p}{|x - y|^{m+(s-|\alpha|)p}} \, dx \, dy \right)^{1/p}, \]
where \([s]\) denotes the largest integer smaller than \(s\). Let \(J = [0, a]\). We set

\[
0 W_p^s(J, X) := \begin{cases} g \in W_p^s(J, X) : g(0) = g'(0) = \cdots = g^{(k)}(0) = 0, \\
W_p^s(J, X), & \text{if } s < 1/p.
\end{cases}
\]

The spaces \(0 H_p^s(J; X)\) are defined analogously. We remind that \(H_p^k = W_p^k\) for \(k \in \mathbb{N}\) and \(1 < p < \infty\) and that \(W_p^s = B_{pp}^s\) for \(s > 0\), \(s \notin \mathbb{N}\).

For \(s \in \mathbb{R}\) and \(1 < p < \infty\), we consider the homogeneous Bessel-potential space \(\hat{H}_p^s(\mathbb{R}^n)\) of order \(s\), defined by

\[
\hat{H}_p^s(\mathbb{R}^n) := \{ g \in S'(\mathbb{R}^n) : \hat{i}^s g \in L_p(\mathbb{R}^n) \},
\]

\[
\|g\|_{\hat{H}_p^s(\mathbb{R}^n)} = \|\hat{i}^s g\|_{L_p(\mathbb{R}^n)},
\]

where \(S'(\mathbb{R}^n)\) denotes the space of all tempered distributions, and \(\hat{i}^s\) is the Riesz potential given by

\[
\hat{i}^s g := (-\Delta)^{s/2} g = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} g), \quad g \in S'(\mathbb{R}^n).
\]

For \(s \in \mathbb{R} \setminus \mathbb{Z}\), the homogeneous Sobolev-Slobodeckij spaces \(\hat{W}_p^s(\mathbb{R}^n)\) of fractional order can be obtained by real interpolation as

\[
\hat{W}_p^s(\mathbb{R}^n) := \left( \hat{H}_p^k(\mathbb{R}^n), \hat{H}_p^{k+1}(\mathbb{R}^n) \right)_{s-k,p}, \quad k < s < k + 1,
\]

where \((\cdot, \cdot)_{\theta, p}\) is the real interpolation functor.

For problem (3.4), we have the following maximal regularity result.

**THEOREM 3.1.** Let \(1 < p < \infty\) be fixed, \(p \neq 3/2, 3\), and assume that \(\rho_k\) and \(\mu_k\) are positive constants for \(k = 1, 2\), \(\rho_2 \neq \rho_1\), and set \(J = [0, a]\). Then, the Stokes problem (3.4) admits a unique solution \((u, \pi)\), \(u = (v, w)\), with regularity

\[
u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p \left( J; H^2_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \right),
\]

\[

\|w\| \in H^1_p \left( J; \hat{W}_p^{-1/p}(\mathbb{R}^n) \right), \pi \in L_p \left( J; \hat{H}^1_p(\mathbb{R}^{n+1}) \right),
\]

if and only if the data \((f_u, f_d, g_v, g_w, g_j, g_\pi, u_0)\) satisfy the following regularity and compatibility conditions:

(a) \(f_u \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))\),

(b) \(f_d \in H^1_p \left( J; \hat{H}^{-1}_p(\mathbb{R}^{n+1}) \right) \cap L_p \left( J; H^1_p(\mathbb{R}^{n+1}) \right)\).

(c) \(g_v \in W_p^{1/2-1/p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p \left( J; W_p^{-1/p}(\mathbb{R}^n, \mathbb{R}^n) \right)\),

(d) \(g_w, g_\pi \in L_p \left( J; \hat{W}_p^{-1/p}(\mathbb{R}^n) \right)\),

(e) \(g_j \in W_p^{1/2-1/p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p \left( J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^n) \right)\).
(f) \( u_0 \in W^{2-2/p}_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \),

(g) \( \text{div}\, u_0 = f_d(0) \) in \( \mathbb{R}^{n+1} \) and \( \| v_0 \| = g_j(0) \) on \( \mathbb{R}^n \) if \( p > 3/2 \),

(h) \(-\| \mu \partial_y v_0 \| - \| \mu \nabla_x w_0 \| = g_v(0) \) on \( \mathbb{R}^n \) if \( p > 3 \).

In addition, for the pressure traces \( \pi_k \) on the interface we have

\[
\pi_k \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p \left( J; W^{1-1/p}_p(\mathbb{R}^n) \right)
\]

if and only if

\[
g_w, g_\pi \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p \left( J; W^{1-1/p}_p(\mathbb{R}^n) \right).
\]

The solution map \( [(f_u, f_d, g_v, g_w, g_j, g_\pi, u_0) \mapsto (u, \pi)] \) is continuous between the corresponding spaces.

For the problem (3.1) and (3.3), we also have maximal regularity result in the \( L_p \)-setting.

**THEOREM 3.2.** Let \( 1 < p < \infty \) be fixed, \( p \neq 3/2, 3 \), and assume that \( \rho_k \) and \( \mu_k \) are positive constants for \( k = 1, 2 \), \( \rho_2 \neq \rho_1 \), and set \( J = [0, a] \). Then, the Stokes problem with free boundary (3.1) and (3.3) admits a unique solution \((u, \pi, h), u = (v, w)\), with regularity

\[
u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p \left( J; H^2_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \right),
\]

\[
\| w \| \in H^1_p \left( J; W^{1-1/p}_p(\mathbb{R}^n) \right), \quad \pi \in L_p \left( J; H^1_p(\mathbb{R}^{n+1}) \right),
\]

\[
\pi_1, \pi_2 \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p \left( J; W^{1-1/p}_p(\mathbb{R}^n) \right),
\]

\[
h \in W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p \left( J; W^{2-1/p}_p(\mathbb{R}^n) \right) \cap L_p \left( J; W^{3-1/p}_p(\mathbb{R}^n) \right),
\]

if and only if the data \((f_u, f_d, g_v, g_j, g_\pi, g_h, u_0, h_0)\) satisfy the following regularity and compatibility conditions:

(a) \( f_u \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \),

(b) \( f_d \in H^1_p \left( J; H^{-1}_p(\mathbb{R}^{n+1}) \right) \cap L_p \left( J; H^1_p(\mathbb{R}^{n+1}) \right) \),

(c) \( (g_u, g_\pi) \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+2})) \cap L_p \left( J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+2}) \right) \),

(d) \( (g_j, g_h) \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p \left( J; W^{2-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1}) \right) \),

(e) \( u_0 \in W^{2-2/p}_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}), h_0 \in W^{3-2/p}_p(\mathbb{R}^n) \),

(f) \( \text{div}\, u_0 = f_d(0) \) in \( \mathbb{R}^{n+1} \) and \( \| v_0 \| = g_j(0) \) on \( \mathbb{R}^n \) if \( p > 3/2 \),

(g) \(-\| \mu \partial_y v_0 \| - \| \mu \nabla_x w_0 \| = g_v(0) \) on \( \mathbb{R}^n \) if \( p > 3 \).

The solution map \( [(f_u, f_d, g_v, g_j, g_\pi, g_h, u_0, h_0) \mapsto (u, \pi, h)] \) is continuous between the corresponding spaces.
Equation (3.2) is a two-phase heat problem with Neumann condition, which has the property of the maximal $L_p$-regularity (cf. [8]). Therefore, the linearized problem (3.1)–(3.3) has the property of maximal $L_p$-regularity as well. To state this result, we introduce appropriate function spaces. We set
\[ E_u(a) := H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \]

and define the regularity of the data space for (3.1)–(3.3) as
\[ F(\gamma) := E_u(a) \times E_\pi(a) \times E_{\gamma\pi}(a) \times E_\theta(a) \times E_h(a). \]

We denote by $\gamma\pi$ the two one-sided traces of $\pi$ on $\mathbb{R}^n$. The generic elements of $E(a)$ are functions $(u, \pi, \gamma\pi, \theta, h). E(a)$ is a Banach space with norm
\[ \| (u, \pi, \gamma\pi, \theta, h) \|_{E(a)} = \| u \|_{E_u(a)} + \| \pi \|_{E_\pi(a)} + \| \gamma\pi \|_{E_{\gamma\pi}(a)} + \| \theta \|_{E_\theta(a)} + \| h \|_{E_h(a)}. \]

Moreover we set
\[ F_u(a) := L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})), \]
\[ F_d(a) := H_p^1(J; \mathbb{R}^{n+1}) \cap L_p(J; H_p^2(\mathbb{R}^{n+1})), \]
\[ G_u(a) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1})), \]
\[ G_j(a) := W_p^{1-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^n)), \]
\[ F_\theta(a) := L_p(J; L_p(\mathbb{R}^{n+1})), \]
\[ G_\theta(a) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)), \]
\[ G_\pi(a) := G_\theta(a), \]
\[ G_h(a) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)) \]

and define the regularity of the data space for (3.1)–(3.3) as
\[ F(a) := F_u(a) \times F_d(a) \times G_u(a) \times G_j(a) \times F_\theta(a) \times G_\theta(a) \times G_\pi(a) \times G_h(a). \]

$F(a)$ is a Banach space with its natural norm, and the generic elements of $F(a)$ are functions $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$. Finally, we define the time trace space $X_\gamma$ of $E(a)$ as
\[ X_\gamma := W_p^{2-2/p}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \times W_p^{2-2/p}(\mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n). \]
The main result on the linearized problem (3.1)–(3.3) now can be stated as

**THEOREM 3.3.** Let $1 < p < \infty$ be fixed, $p \neq 3/2, 3$, and assume that $\rho_k$ and $\mu_k$ are positive constants for $k = 1, 2, \rho_2 \neq \rho_1, \kappa, d > 0$.

Then, the linear problem (3.1)–(3.3) admits a unique solution $(u, \pi, \gamma \pi, \theta, h) \in E(a)$ if and only if the data $(u_0, \theta_0, h_0)$ and $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$ satisfy the regularity conditions

$$(u_0, \theta_0, h_0) \in X_\gamma, \quad (f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h) \in F(a),$$

and the compatibility conditions

$$\text{div } u_0 = f_d(0) \quad \text{in } \dot{R}^{n+1},$$
$$-\|\mu(\partial_y v_0 + \nabla_x w_0)\| = g_v(0), \quad \|v_0\| = g_j(0) \quad \text{on } \mathbb{R}^n,$$
$$\|\theta_0\| = 0, \quad -d\partial_y \theta_0 = g_\theta(0) \quad \text{on } \mathbb{R}^n.$$

The solution map $[(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h, u_0, \theta_0, h_0) \mapsto (u, \pi, \gamma \pi, \theta, h)]$ is continuous between the corresponding spaces.

4. Proofs for Theorems 3.1 and 3.2

For necessity, we employ trace arguments as in [15,22]. The more difficult part of sufficiency also follows the lines of these papers, but has to be modified as it is more involved.

1. In order to remove $u_0$ which has a jump at the interface, we first solve the parabolic problem

$$\rho \partial_t u_1 - \mu \Delta u_1 = f_u \quad \text{in } \dot{R}^{n+1}, \quad t > 0,$$
$$-2 P_{\dot{R}^{n+1}}\|\mu D(u_1)\| = g_v \quad \text{on } \mathbb{R}^n, \quad t > 0,$$
$$-2\|\mu \partial_y w_1\| = \tilde{g}_w \quad \text{on } \mathbb{R}^n, \quad t > 0,$$
$$\|u_1\| = \tilde{g}_j \quad \text{on } \mathbb{R}^n, \quad t > 0,$$
$$u_1(0) = u_0 \quad \text{in } \dot{R}^{n+1},$$

with

$$\tilde{g}_w = -2e^{\Delta_x t}\|\mu \partial_y w_0\|, \quad \tilde{g}_j = E_{\dot{R}^{n+1}g_j} + (e^{\Delta_x t}\|w_0\|) v,$$

where $E_{\dot{R}^{n+1}g_j} := (g_j, 0)^T$, to meet the necessary compatibility conditions, and set $\pi_1 = 0$. Then,

$$u_1 \in H^1_p(J; L_p(\dot{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p\left(J; H^2_p(\dot{R}^{n+1}, \mathbb{R}^{n+1})\right),$$
$$\|w_1\| \in H^1_p\left(J; \dot{W}^{-1/p}_{p} (\mathbb{R}^n)\right).$$
Next, to remove the divergence data we solve the problem

$$\rho \partial_t u_2 - \mu \Delta u_2 + \nabla \pi_2 = 0 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$\text{div } u_2 = f_d - \text{div } u_1 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$-2[\mu D(u_2)v] + [\pi_2]v = (0, \bar{g}_w)^T \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$[u_2] = 0 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$u_2(0) = 0 \quad \text{in } \mathbb{R}^{n+1}, \quad (4.2)$$

with

$$\bar{g}_w = g_w - \bar{g}_w.$$

According to [22], this problem has a unique solution in the maximal $L_p$-regularity class. Note that the compatibility conditions do not involve the normal part of the stress boundary condition. Therefore, it remains to study the problem

$$\rho \partial_t u - \mu \Delta u + \nabla \pi = 0 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$\text{div } u = 0 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$-2[\mu D(u)v] + [\pi]v = \sigma \Delta_x h \nu \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$[v] = 0 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$u(0) = 0 \quad \text{in } \mathbb{R}^{n+1}, \quad (4.3)$$

$$-2[\mu D(u)v \cdot \nu/\rho] + [\pi/\rho] = g_2 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$\partial_t h - [\rho w]/[\rho] = g_3 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$h(0) = 0 \quad \text{on } \mathbb{R}^n. \quad (4.4)$$

This way the problem for $h$ is decoupled from the Stokes problem

$$\rho \partial_t u - \mu \Delta u + \nabla \pi = 0 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$\text{div } u = 0 \quad \text{in } \mathbb{R}^{n+1}, \ t > 0,$$

$$-2[\mu D(u)v] + [\pi]v = (0, g_1)^T \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$[v] = 0 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$-2[\mu D(u)v \cdot \nu/\rho] + [\pi/\rho] = g_2 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$u(0) = 0 \quad \text{in } \mathbb{R}^{n+1}, \quad (4.5)$$

where we have set $g_1 = \sigma \Delta_x h$ and $u = (v, w)$. Having solved this problem for given $h$ we insert into and solve the remaining equation for $h$, which reads.

$$\partial_t h - [\rho w]/[\rho] = g_3 \quad \text{on } \mathbb{R}^n, \ t > 0,$$

$$h(0) = 0 \quad \text{on } \mathbb{R}^n. \quad (4.6)$$
Here, the remaining data satisfy

\[ g_1, g_2 \in L^p(J; \dot{W}^{1-1/p}_p(\mathbb{R}^n)), \]
\[ g_3 \in 0W^{1-1/2p}_p(J; L^p(\mathbb{R}^n)) \cap L^p(J; W^{2-1/p}_p(\mathbb{R}^n)). \]

2. Assume for a moment that we have a solution of (4.5) in the proper regularity class even on the half-line \( J = \mathbb{R}_+ \). Then, we may employ the Laplace transform in \( t \) and the Fourier transform in the tangential variables \( x \in \mathbb{R}^n \), to obtain the following boundary value problem for a system of ordinary differential equations on \( \mathbb{R}^a \):

\[
\begin{aligned}
\omega^2 \hat{\nu} - \mu_0 \partial_y^2 \hat{\nu} + i \xi \hat{\pi} &= 0, \quad y \neq 0, \\
\omega^2 \hat{\nu} - \mu_0 \partial_y^2 \hat{\nu} + \partial_y \hat{\pi} &= 0, \quad y \neq 0, \\
(i \xi | \hat{\nu}) + \partial_y \hat{\nu} &= 0, \quad y \neq 0.
\end{aligned}
\]

Here we have set \( \omega^2 = \rho_k \lambda + \mu_k |\xi|^2, \) \( k = 1, 2, \) and

\[
\hat{v}_k(\lambda, \xi, y) = (2\pi)^{-n/2} \int_0^\infty \int_{\mathbb{R}^a} e^{-\lambda t} e^{-i(x|\xi)} v(t, x, y) \, dx \, dt, \quad (-1)^k y > 0.
\]

This system of equations is easily solved to the result

\[
\begin{bmatrix}
\hat{\nu}_2 \\
\hat{\nu}_2 \\
\pi_2
\end{bmatrix} = e^{-\omega y / \sqrt{\mu_2}} \begin{bmatrix}
a_2 \\
\sqrt{\mu_2} (i \xi |a_2) \\
0
\end{bmatrix} + a_2 e^{-|\xi| y} \begin{bmatrix}
-i \xi \\
|\xi| \\
\rho_2 \lambda
\end{bmatrix}, \quad (4.7)
\]

for \( y > 0 \), and

\[
\begin{bmatrix}
\hat{\nu}_1 \\
\hat{\nu}_1 \\
\pi_1
\end{bmatrix} = e^{\omega y / \sqrt{\mu_1}} \begin{bmatrix}
a_1 \\
-\sqrt{\mu_1} (i \xi |a_1) \\
0
\end{bmatrix} + a_1 e^{\xi y} \begin{bmatrix}
-i \xi \\
-|\xi| \\
\rho_1 \lambda
\end{bmatrix}, \quad (4.8)
\]

for \( y < 0 \). Here \( a_k \in \mathbb{C}^n \) and \( \alpha_k \in \mathbb{C} \) have to be determined by the interface conditions which in frequency domain read

\[
\hat{v}_1(0) - \hat{v}_2(0) = 0,
\]
\[
\mu_2 (\partial_y \hat{v}_2(0) + i \xi \hat{w}_2(0)) - \mu_1 (\partial_y \hat{v}_1(0) + i \xi \hat{w}_1(0)) = 0,
\]
\[
-2\mu_2 \partial_y \hat{w}_2(0) + \pi_2(0) + 2\mu_1 \partial_y \hat{w}_1(0) - \hat{\pi}(0) = \hat{g}_1,
\]
\[
-2(\mu_2 / \rho_2) \partial_y \hat{w}_2(0) + \pi_2(0) / \rho_2 + 2(\mu_1 / \rho_2) \partial_y \hat{w}_1(0) - \hat{\pi}(0) / \rho_2 = \hat{g}_2.
\]

Inserting the representation of the transformed solution into the first two of these equations, we obtain the following system.

\[
a_2 - a_1 = i \xi (\alpha_2 - \alpha_1),
\]
\[
\sqrt{\mu_2 \omega a_2} + \sqrt{\mu_1 \omega a_1} = i \xi \left\{ 2 |\xi| (\mu_1 a_1 + \mu_2 a_2) + (\mu_1 \sqrt{\mu_1 / \omega_1}) \beta_1 + (\mu_2 \sqrt{\mu_2 / \omega_2}) \beta_2 \right\},
\]

where we have set \( \beta_k = i \xi \cdot a_k, k = 1, 2 \). This system can be solved for \( a_k \) to the result
\[
a_1 = i \xi \frac{2 |\xi| (\mu_1 a_1 + \mu_2 a_2) + (\mu_1 \sqrt{\mu_1 / \omega_1}) \beta_1 + (\mu_2 \sqrt{\mu_2 / \omega_2}) \beta_2 - \sqrt{\mu_2} \omega_2 (\alpha_2 - \alpha_1)}{\sqrt{\mu_1} \omega_1 + \sqrt{\mu_2} \omega_2},
\]
and
\[
a_2 = i \xi \frac{2 |\xi| (\mu_1 a_1 + \mu_2 a_2) + (\mu_1 \sqrt{\mu_1 / \omega_1}) \beta_1 + (\mu_2 \sqrt{\mu_2 / \omega_2}) \beta_2 + \sqrt{\mu_1} \omega_1 (\alpha_2 - \alpha_1)}{\sqrt{\mu_1} \omega_1 + \sqrt{\mu_2} \omega_2}.
\]

Multiplying former equations with \( i \xi \) we obtain
\[
\beta_2 - \beta_1 = |\xi|^2 (\alpha_1 - \alpha_2),
\]
\[
\gamma_1 \beta_1 + \gamma_2 \beta_2 = -2 |\xi|^3 (\mu_1 a_1 + \mu_2 a_2),
\]
where \( \gamma_k = \sqrt{\mu_k \omega_k + \mu_k |\xi|^2 \sqrt{\mu_k} / \omega_k}, k = 1, 2 \). We solve this system for \( \beta_k \), concluding
\[
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} = \frac{|\xi|^2}{\gamma_1 + \gamma_2} \begin{bmatrix}
2 \mu_1 |\xi| + \gamma_2 2 \mu_2 |\xi| - \gamma_1 2 \mu_2 |\xi| + \gamma_1 \\
\gamma_1 + \gamma_2
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}.
\]

Inserting the transformed solution into the two remaining stress conditions leads to
\[
2 (\mu_2 \beta_2 + \mu_2 |\xi|^2 \alpha_2) + \rho_2 \lambda \alpha_2 - 2 (\mu_1 \beta_1 + \mu_1 |\xi|^2 \alpha_1) - \rho_1 \lambda \alpha_1 = \hat{g}_1,
\]
\[
2 (\mu_2 \beta_2 + \mu_2 |\xi|^2 \alpha_2) / \rho_2 + \lambda \alpha_2 - 2 (\mu_1 \beta_1 + \mu_1 |\xi|^2 \alpha_1) / \rho_1 - \lambda \alpha_1 = \hat{g}_2.
\]
Using the formulas for \( \beta_k \) and solving the resulting system in terms of \( \alpha_k \), we arrive after some elementary algebra at the expressions
\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix} = \frac{1}{\| \rho \| \lambda \epsilon} \begin{bmatrix}
1 + 2 \rho_2 \| \mu / \rho \| \epsilon_2 - \rho_2 (1 + 2 \| \mu \| \epsilon_2) \\
1 - 2 \rho_1 \| \mu / \rho \| \epsilon_1 - \rho_1 (1 - 2 \| \mu \| \epsilon_1)
\end{bmatrix} \begin{bmatrix}
\hat{g}_1 \\
\hat{g}_2
\end{bmatrix},
\]
where we have set \( \epsilon_k = \rho_k \lambda \sqrt{\mu_k} |\xi|^2 / (\omega_k (\gamma_1 + \gamma_2) (\omega_k + \sqrt{\mu_k} |\xi|)^2), k = 1, 2 \), and
\[
\epsilon = 1 + 2 (\mu_1 \epsilon_1 + \mu_2 \epsilon_2).
\]
Here, we observe that the surface pressure \( \pi_k \) have transforms \( \lambda \alpha_k \). Since the entries in the matrix defining \( \lambda \alpha_k \) are bounded and holomorphic, we may conclude that \( \pi_k \) have the same regularity as \( g_k \), and that the pressure \( \pi \) belongs to \( L_p (J; \hat{H}_p^1 (\mathbb{R}^{n+1})) \).

Next, let us compute the boundary velocities \( v^b_1 = v^b_k \) and \( w^b_k \), where \( v^b_k (x) = v_k (x, 0) \) and \( w^b_k (x) = w_k (x, 0) \). Their transforms are given by
\[
\hat{v}^b_k = a_k - i \xi \alpha_k, \quad \hat{w}^b_k = (-1)^k \left( \frac{\sqrt{\mu_k}}{\omega_k} \beta_k + |\xi| \alpha_k \right).
\]
Some algebra yield for $w^b_k$ the representation
\[
\begin{bmatrix}
\hat{w}_k^b \\
\hat{w}_2^b
\end{bmatrix} = \frac{|\xi|}{\omega_1\omega_2} \begin{bmatrix}
\frac{-\rho_1\omega_2}{\gamma_1+\gamma_2} (\sqrt{H_1} + \frac{\gamma_2}{\omega_1+\sqrt{\mu_1}\xi}) \\
\frac{-\rho_2\sqrt{\mu_1}\mu_2}{\gamma_1+\gamma_2} (\omega_2-\sqrt{\mu_2}\xi) \\
\frac{-\rho_2\omega_1}{\gamma_1+\gamma_2} (\sqrt{H_2} + \frac{\gamma_1}{\omega_2+\sqrt{\mu_2}\xi}) \\
\end{bmatrix} \begin{bmatrix}
\lambda\alpha_1 \\
\lambda\alpha_2
\end{bmatrix}.
\tag{4.11}
\]

This representation shows that $\hat{w}_k^b$ is bounded by $|\xi|/\omega_1\omega_2$. As in [22], Section 4, we obtain that the operator with symbol $|\xi|/\omega_1\omega_2$ maps $L_p(J; \hat{W}^{1-1/p}_p(\mathbb{R}^n))$ into the right space for the boundary values of $w$, i.e., we have
\[w^b_k \in 0H^1_p(J; \hat{W}^{-1/p}_{p-1}(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_{p-1}(\mathbb{R}^n)).\]

To keep this paper self-contained, we prove the mapping properties stated above. We set $G := \partial_t$ in $X := L_p(J; L_p(\mathbb{R}^n))$ with domain $D(G) = 0H^1_p(J; L_p(\mathbb{R}^n))$. It is well-known that $G$ is closed, invertible and sectorial with angle $\pi/2$, and $-G$ is the generator of a $C_0$-semigroup of contractions in $L_p(\mathbb{R}^n)$. Moreover, $G$ admits an $H^\infty$-calculus in $X$ with $H^\infty$-angle $\pi/2$ as well; see e.g. [9]. The symbol of $G$ is $\lambda$, the time covariable.

Next, we set $D_n := -\Delta$, the Laplacian in $L_p(\mathbb{R}^n)$ with domain $D(D_n) = H^2_p(\mathbb{R}^n)$. It is well-known that $D_n$ is closed and sectorial with angle 0, and it admits a bounded $H^\infty$-calculus which is even $\mathcal{R}$-bounded with $\mathcal{R}H^\infty$-angle 0; see e.g. [6]. These results also hold for the canonical extension of $D_n$ to $X$, and also for the fractional power $D_n^{1/2}$ of $D_n$, its domain is $D(D_n^{1/2}) = L_p(J; H^1_p(\mathbb{R}^n))$. The symbol of $D_n$ is $|\xi|^2$ and that of $D_n^{1/2}$ is $|\xi|$, where $\xi$ is the covariable of $x$. By the Dore-Venni theorem for sums of commuting sectorial operators (cf. [7]), we see that $L_k := \rho_k G + \mu_k D_n$ with natural domain
\[D(L_k) = D(G) \cap D(D_n) = 0H^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n))\]
are closed, invertible and sectorial with angle $\pi/2$. $L_k$ also admits a bounded $H^\infty$-calculus in $X$ with $H^\infty$-angle $\pi/2$ (cf. [20]). The same results are valid for operators $F_k = L_k^{1/2}$, their $H^\infty$-angle is $\pi/4$, and their domains are
\[D(F_k) = D(G^{1/2}) \cap D(D_n^{1/2}) = 0H^{1/2}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n)).\]

The symbol of $L_k$ is $\rho_k \lambda + \mu_k |\xi|^2$ and that of $F_k$ is $\sqrt{\rho_k \lambda + \mu_k |\xi|^2}$. $F_k^{-1}$ have the following mapping properties:
\[F_k^{-1} : L_p(J; L_p(\mathbb{R}^n)) \to 0H^{1/2}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n)),\]
\[F_k^{-1} : L_p(J; H^1_p(\mathbb{R}^n)) \to 0H^{1/2}_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n)).\]
Now we consider the operator \( N := D_n^{1/2} F_1^{-1} F_1^{-1} \) with domain
\[
\mathcal{D}(N) = \left\{ h \in \mathcal{R}(F_1) : F_1^{-1} h \in \mathcal{R}(F_2), F_2^{-1} F_1^{-1} h \in \mathcal{D}\left(D_n^{1/2}\right) \right\}.
\]

\( N \) has the following mapping properties:
\[
N : L_p(J; L_p(\mathbb{R}^n)) \to 0 H_p^1 \left( J; \dot{H}_p^{-1}(\mathbb{R}^n) \right) \cap L_p \left( J; H_p^1(\mathbb{R}^n) \right),
\]
\[
N : L_p \left( J; \dot{H}_p^1(\mathbb{R}^n) \right) \to 0 H_p^1 \left( J; L_p(\mathbb{R}^n) \right) \cap L_p \left( J; \dot{H}_p^2(\mathbb{R}^n) \right)
\]
\[
\to 0 H_p^1 \left( J; L_p(\mathbb{R}^n) \right) \cap L_p \left( J; H_p^2(\mathbb{R}^n) \right).
\]

Therefore, by threefold real interpolation \((\cdot, \cdot)_{1-1/p, p}\), we obtain
\[
N : L_p \left( J; \dot{W}_p^{-1/p}(\mathbb{R}^n) \right) \to 0 H_p^1 \left( J; \dot{W}_p^{-1/p}(\mathbb{R}^n) \right) \cap L_p \left( J; W_p^{2-1/p}(\mathbb{R}^n) \right),
\]
which is the desired property.

To obtain the regularity of the boundary values \( \nu_k^b \) of \( v \), we write
\[
\hat{\nu}_k^b = \hat{\nu}_2^b = a_2 - i \xi \alpha_2
\]
\[
= \frac{i \xi}{\sqrt{\mu_1 \omega_1} + \sqrt{\mu_2 \omega_2}} \left\{ \mu_1 \left( \frac{\sqrt{\mu_1}}{\omega_1} \beta_1 + \alpha_1 |\xi| \right) + \mu_2 \left( \frac{\sqrt{\mu_2}}{\omega_2} \beta_2 + \alpha_2 |\xi| \right) \right\}
\]
\[
- \frac{i \xi}{\sqrt{\mu_1 \omega_1} + \sqrt{\mu_2 \omega_2}} \left\{ \frac{\rho_1 \sqrt{\mu_1}}{\omega_1 + \sqrt{\mu_1} |\xi|} \lambda \alpha_1 + \frac{\rho_1 \sqrt{\mu_1}}{\omega_1 + \sqrt{\mu_1} |\xi|} \lambda \alpha_2 \right\}.
\]

This representation shows that also \( \hat{\nu}_k^b \) is bounded by \( |\xi| \hat{g}_1 / \omega_1 \omega_2 \), and the same argument as above yields
\[
\nu_k^b \in 0 H_p^1 \left( J; \dot{W}_p^{-1/p}(\mathbb{R}^n; \mathbb{R}^n) \right) \cap L_p \left( J; W_p^{2-1/p}(\mathbb{R}^n; \mathbb{R}^n) \right)
\]
\[
\to 0 W_p^{1-1/2p} \left( J; L_p(\mathbb{R}^n; \mathbb{R}^n) \right).
\]

This follows from the fact that \( \dot{W}_p^{-1/p}(\mathbb{R}^n) \leftrightarrow W_p^{-1/p}(\mathbb{R}^n) \) by a similar argument to the proof of [18, Lemma 6.3] where we set \( Au := (1 - \Delta)u \).

Therefore, the boundary values of \( u \) from above and below have the required regularity; hence, solving the Stokes problem with these boundary values separately in the upper and the lower half-space, we obtain as in [22], Section 4, that \( u \) has maximal \( L_p \)-regularity. This completes the proof of Theorem 3.1.

3. To extract the boundary symbol, we set \( \hat{g}_1 = -\sigma |\xi|^2 \hat{h}, \hat{g}_2 = 0 \) and observe that the transformed equation for \( \hat{h} \) reads
\[
\lambda \hat{h} - \| \rho \hat{w}(0) \| / \| \rho \| = \hat{g}_3.
\]

We have
\[
\| \rho \hat{w}(0) \| = \rho_2(\sqrt{\mu_2} \beta_2 / \omega_2 + |\xi| \alpha_2) - \rho_1(-\sqrt{\mu_1} \beta_1 / \omega_1 - |\xi| \alpha_1); \quad (4.12)
\]
hence, inserting the expressions for $\alpha_k$ and $\beta_k$, we obtain after some more algebra

$$s(\lambda, |\xi|)\hat{h} = \hat{g}_4.$$  

(4.13)

Here $g_4$ is determined by the data alone and has the same regularity as $g_3$. We set $\tau = |\xi|$. The boundary symbol $s(\lambda, \tau)$ is defined by

$$s(\lambda, \tau) = \lambda + \frac{\sigma \tau}{\|\rho\|^2} m(z),$$

(4.14)

where we employed the scaling $z = \lambda/\tau^2$. The holomorphic function $m(z)$ in turn is given by

$$(\mu_1 \varphi_1(z) + \mu_2 \varphi_2(z))m(z) = 2\frac{\rho_1 \rho_2}{\omega_1(z)} - \frac{1}{\omega_1(z)\omega_2(z)} - \frac{1}{\omega_1(z) + 1 \omega_2(z) + 1}$$

$$+ \frac{\rho_1^2 \mu_2 \varphi_2(z)}{\mu_1 \omega_1(z) (\omega_1(z) + 1)} + \frac{\rho_2^2 \mu_1 \varphi_1(z)}{\mu_2 \omega_2(z) (\omega_2(z) + 1)} + \frac{\rho_1^2}{\omega_1(z)} + \frac{\rho_2^2}{\omega_2(z)},$$

(4.15)

with the abbreviations

$$\omega_k(z) = \sqrt{1 + \rho_k z/\mu_k}, \quad \varphi_k(z) = \omega_k(z) + \frac{3}{\omega_k(z) + 1} - \frac{1}{\omega_k(z)(\omega_k(z) + 1)}.$$

We derive the formula in the “Appendix”. Note that $\omega_k(z)$ is holomorphic in the sliced plane $\mathbb{C} \setminus (-\infty, -\mu_k/\rho_k]$; hence, the function $\varphi_k(z)$ has this property as well. This function has exactly one zero $z_k$ in this set, it is real and satisfies $-\mu_k/\rho_k < z_k < -8\mu_k/9\rho_k < 0$. It is easy to see that $\varphi_k$ maps $\mathbb{C}_+$ into $\mathbb{C}_+$, and as $\varphi_k(0) = 2$ and $\varphi_k(z) \sim \sqrt{\rho_k z/\mu_k}$ as $z \to \infty$, we see that $\varphi_k(\mathbb{C}_+) \subset \Sigma_{\varphi_k}$, for some angle $\varphi_k < \pi/2$. By continuity of the argument function, this implies that $\varphi_k(\Sigma_{\pi/2+\eta}) \subset \mathbb{C}_+$, for some $\eta > 0$. Therefore, $\varphi(z) := \mu_1 \varphi_1(z) + \mu_2 \varphi_2(z)$ also has this property, in particular $\varphi(z)$ cannot vanish in $\Sigma_{\pi/2+\eta}$. This implies that $m(z)$ is holomorphic in this sector and in a ball $B_{r_0}(0)$ for some $r_0 > 0$. We obtain for the asymptotics of $m(z)$

$$m(0) = \frac{1}{2} \left( \frac{\rho_1^2}{\mu_1} + \frac{\rho_2^2}{\mu_2} \right) > 0, \quad \lim_{|z| \to \infty} z m(z) = \rho_1 + \rho_2 \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{R}_-.$$

Thus, there is a constant $M = M(r, \phi) > 0$ such that

$$|m(z)| \leq \frac{M}{1 + |z|}, \quad z \in \Sigma_\phi \cup B_r(0)$$

for each $\phi < \pi/2 + \eta$ and $r < r_0$. From this estimate, it is easy to conclude

$$|s(\lambda, \tau)| \leq C_\eta(|\lambda| + |\tau|), \quad \lambda \in \Sigma_{\pi/2+\eta}, \quad \tau \in \Sigma_\eta,$$

whenever $\eta > 0$ is small enough. Conversely, since $m(0) > 0$, given a small $\eta > 0$ we find $r_\eta \in (0, r_0)$ such that $m(z) \in \Sigma_{\pi/2-3\eta}$ and $|m(z)| \geq m(0)/2$, for all $z \in B_{r_\eta}(0)$. This implies that there is a constant $c_\eta > 0$ such that

$$|s(\lambda, \tau)| \geq c_\eta(|\lambda| + |\tau|), \quad \lambda \in \Sigma_{\pi/2+\eta}, \quad |\lambda| \leq r_\eta |\tau|^2.$$
On the other hand, choosing $|\lambda| \geq C|\tau|$ we obtain
\[
|s(\lambda, \tau)| \geq |\lambda| - \sigma \|\rho\|^{-2} M |\tau|
\geq \frac{1}{2}|\lambda| + \left( \frac{C}{2} - \sigma \|\rho\|^{-2} M \right) |\tau| \geq c_\eta(|\lambda| + |\tau|),
\]
for all $\lambda \in \Sigma_{\pi/2+\eta}$, $\tau \in \Sigma_{\eta}$ such that $|\lambda| \geq C|\tau|$ with $C > 2\sigma \|\rho\|^{-2} M$. If $\lambda_0$ is chosen large enough, this implies the lower bound
\[
|s(\lambda, \tau)| \geq c(|\lambda| + |\tau|), \quad \lambda \in \Sigma_{\pi/2+\eta}, \quad \tau \in \Sigma_{\eta}, \quad |\lambda| \geq \lambda_0.
\]
Thus, the boundary symbol for the Stokes problem with phase transition has the same mapping behavior for $|\lambda| > \lambda_0$ as that without phase transition.

We prove this assertion following the arguments in [22, Section 5]. By means of the $\mathcal{R}$-boundedness of the functional calculus for $D_n$ in $K_p^s(\mathbb{R}^n)$, where $K = \{H, W\}$ (cf. [6]), we see that
\[
\left( \lambda + D_n^{1/2} \right) s^{-1} \left( \lambda, D_n^{1/2} \right)
\]
is of class $H^\infty$ and $\mathcal{R}$-bounded on $\Sigma_{\pi/2+\eta}\setminus B_{\lambda_0}(0)$. The operator-valued $H^\infty$-calculus for $G = \partial_\eta$ on $0H^r_p(J; K_p^s(\mathbb{R}^n))$ with $H^\infty$-angle $\pi/2$ (cf. [9]) allows for an application of the Kalton and Weis theorem [14, Theorem 4.4] for the operator-valued function $(\lambda + D_n^{1/2})s^{-1}(\lambda, D_n^{1/2})$, which implies boundedness of
\[
\left( G + D_n^{1/2} \right) s^{-1} \left( G, D_n^{1/2} \right) \quad \text{in} \quad 0H^r_p \left( J; K_p^s(\mathbb{R}^n) \right).
\]
This shows that $s^{-1}(G, D_n)$ has the following mapping properties:
\[
s^{-1} \left( G, D_n^{1/2} \right) : 0H^r_p \left( J; K_p^s(\mathbb{R}^n) \right) \to 0H^{r+1}_p \left( J; K_p^{s+1}(\mathbb{R}^n) \right)
\]
\[
\cap 0H^r_p \left( J; K_p^{s+1}(\mathbb{R}^n) \right).
\]
(4.16)

We conclude that $S$ is invertible and that $S^{-1} = s^{-1}(G, D_n^{1/2})$. Choosing $r = 0$ and $s = 2 - 1/p$ and $K = W$ in (4.16) yields
\[
S^{-1} : L_p \left( J; W_p^{2-1/p}(\mathbb{R}^n) \right) \to 0H^1_p \left( J; W_p^{2-1/p}(\mathbb{R}^n) \right) \cap L_p \left( J; W_p^{3-1/p}(\mathbb{R}^n) \right).
\]
(4.17)

Moreover, we deduce from (4.16) that
\[
S^{-1} : L_p(J; L_p(\mathbb{R}^n)) \to 0H^1_p(J; L_p(\mathbb{R}^n)),
\]
\[
S^{-1} : 0H^1_p(J; L_p(\mathbb{R}^n)) \to 0H^2_p(J; L_p(\mathbb{R}^n)).
\]

Interpolating with the real method $(\cdot, \cdot)_{1-1/p, p}$ yields
\[
S^{-1} : 0W_p^{1-1/p}(J; L_p(\mathbb{R}^n)) \to 0W_p^{2-1/p}(J; L_p(\mathbb{R}^n)).
\]

As a consequence, the equation $Sh = g_4$ has a unique solution in the right regularity class, for each $g_4 \in 0W_p^{1-1/p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$. This completes the proof of Theorem 3.2.
5. Local well-posedness of the nonlinear problem

In this section, we prove Theorem 1.1. Given \( h_0 \in W^{3-2/p}_p(\mathbb{R}^n) \), we define
\[
\Xi_{h_0}(x, y) := (x, y + h_0(x)), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}.
\]
Letting \( \Omega_{h_0,k} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^k(y - h_0(x)) > 0\} \) and \( \Omega_{h_0} := \Omega_{h_0,1} \cup \Omega_{h_0,2} \), by the assumption \( p > n + 3 \) we obtain from Sobolev’s embedding theorem that
\[
\Xi_{h_0} \in \text{Diff}^2(\mathbb{R}^n, \Omega_{h_0}) \cap \text{Diff}^2(\mathbb{R}^n, \Omega_{h_0,1}) \cap \text{Diff}^2(\Omega^+_n, \Omega_{h_0,2}),
\]
namely \( \Xi_{h_0} \) yields a \( C^2 \)-diffeomorphism between the indicated domains. The inverse transform is given by \( \Xi_{h_0}^{-1}(x, y) = (x, y - h_0(x)) \). It then follows from the chain rule and transformation rule for integrals that
\[
\Xi_{h_0}^* \in \text{Isom} \left( H^m_p(\mathbb{R}^n, \Omega_{h_0}), H^m_p(\Omega_{h_0}) \right), \quad [\Xi_{h_0}^*]^{-1} = \Xi_{h_0}^m \quad m = 0, 1, 2,
\]
where we use the notation
\[
\Xi_{h_0}^* f = f \circ \Xi_{h_0} : \Omega_{h_0} \to \mathbb{R}^m, \\
\Xi_{h_0}^* g = g \circ \Xi_{h_0}^{-1} : \mathbb{R}^{n+1} \to \mathbb{R}^m,
\]
for the pull-back and push-forward operators, where \( m \) is non-negative integer.

Therefore, to prove Theorem 1.1, it is enough to prove the following result.

THEOREM 5.1. (Unique existence of solutions for the nonlinear problem (2.1))
Let \( p > n + 3 \) and \( \rho_1, \rho_2, \sigma > 0 \) be constant, \( \rho_1 \neq \rho_2 \), and let \( \psi_k \in C^3(0, \infty) \), \( \mu_k, d_k \in C^2(0, \infty) \) be such that
\[
k_k(s) = -s \psi''(s) > 0, \quad \mu_k(s) > 0, \quad d_k(s) > 0 \quad s \in (0, \infty), \quad k = 1, 2.
\]
Suppose the initial data
\[
(u_0, \theta_0, h_0) \in X_p := W^{2-2/p}_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \times W^{2-2/p}_p(\mathbb{R}^{n+1}) \times W^{3-2/p}_p(\mathbb{R}^n)
\]
are given, and assume that the compatibility conditions
\[
\text{div} (\Xi_{h_0}^* u_0) = 0 \quad \text{in} \ \Omega_0 / \Gamma_0, \\
\|\mu P_{\Gamma_0} D(\Xi_{h_0}^* u_0) v_0\| = 0, \quad \|P_{\Gamma_0} \Xi_{h_0}^* u_0\| = 0 \quad \text{on} \ \Gamma_0, \\
\|\Xi_{h_0}^* \theta_0\| = 0, \quad \|d_0 \partial_v \Xi_{h_0}^* \theta_0\| + \ell(\psi_{h_0} \theta_0 + \theta_\infty)\|\rho^{-1}\|^{-1}\|\Xi_{h_0}^* u_0 \cdot v_0\| = 0 \quad \text{on} \ \Gamma_0
\]
are satisfied. Here \( D(u) = (\nabla u + [\nabla u]^T)/2 \) and \( P_{\Gamma_0} = I - v_\Gamma_0 \otimes v_\Gamma_0 \). Then, for any time \( a > 0 \) there exists \( \eta > 0 \) such that for
\[
\|u_0\|_{W^{2-2/p}_p(\mathbb{R}^{n+1})} + \|\theta_0\|_{W^{2-2/p}_p(\mathbb{R}^{n+1})} + \|h_0\|_{W^{3-2/p}_p(\mathbb{R}^n)} \leq \eta,
\]
the nonlinear problem (2.1) admits a unique solution \((u, \pi, \gamma \pi, \theta, h) \in E(a)\). It depends continuously on the data.
In order to economize our notation, we set $z = (u, \pi, \gamma \pi, \theta, h)$ for $(u, \pi, \gamma \pi, \theta, h) \in \mathbb{E}(a)$. and set $z_0 = (u_0, \theta_0, h_0)$ for $(u_0, \theta_0, h_0) \in X_\gamma$. With this notation, the nonlinear problem (2.1) can be recast as

$$Lz = N(z), \quad z(0) = z_0. \quad (5.2)$$

where $L$ denotes the linear operator on the left hand side of (2.1), and $N$ denotes the nonlinear mapping on its right-hand side.

In the following, we say that a function space is a multiplication algebra if it is a Banach algebra under the operator of multiplication.

**Lemma 5.2.** (Lemma 6.1 in [22]) Suppose $p > n + 3$. Then,

(a) $E_{\gamma \pi}(a), \mathbb{G}_j(a), \mathbb{G}_\pi(a), \mathbb{G}_\theta(a), \mathbb{G}_\pi(a),$ and $\mathbb{G}_h(a)$ are multiplication algebras.

(b) $E_n(a) \hookrightarrow BUC(J; BUC^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$

(c) $E_\theta(a) \hookrightarrow BUC(J; BUC^1(\mathbb{R}^{n+1})) \cap BUC(J; BUC(\mathbb{R}^n))$

(d) $E_{\pi, \gamma}(a) \hookrightarrow BUC(J; BUC(\mathbb{R}^n))^2$

(e) $E_\theta(a) \hookrightarrow BUC^1(J; BC^1(\mathbb{R}^n)) \cap BUC(J; BC^2(\mathbb{R}^n))$

Concerning the nonlinearity $N$, we have the following result.

**Proposition 5.3.** Let $p > n + 3$ and $\rho_1, \rho_2, \sigma > 0$ be constant, $\rho_1 \neq \rho_2$, assume $\psi_k \in C^3(0, \infty), \mu_k, d_k \in C^2(0, \infty)$, and let $a > 0$ be fixed. Then, the nonlinear map $N$ satisfies

$$N \in C^1(\mathbb{E}(a), \mathbb{F}(a)) \text{ and } N(0) = 0, \quad DN(0) = 0,$$

where $DN$ denotes the Fréchet derivative of $N$.

**Proof.** This result is proved in a similar way as Proposition 6.2 in [22].

Now we prove Theorem 5.1, where $a > 0$ is a fixed life time.

**Step 1.** First, we reduce the problem to initial values $0$ and resolve the compatibility conditions. Thanks to Theorem 6.3 in [22], we find an extension $f^*_d \in \mathbb{F}_d(a)$ which satisfies $f^*_d(0) = \text{div} u_0$. We define

$$g^*_v = e^{\Delta_t} G_v(u_0, \theta_0, h_0), \quad g^*_w = e^{\Delta_t} G_w(u_0, \theta_0, h_0),$$

$$g^*_j = e^{\Delta_t} G_j(u_0, h_0), \quad g^*_\theta = e^{\Delta_t} G_\theta(u_0, \theta_0, h_0),$$

$$g^*_\pi = e^{\Delta_t} G_\pi(u_0, \theta_0, h_0), \quad g^*_h = e^{\Delta_t} G_h(u_0, h_0), \quad (5.3)$$

and set $f^*_u = f^*_\theta = 0$. With these extensions, by Theorem 3.3, we may solve the linear problem (3.1)–(3.3) with initial data $(u_0, \theta_0, h_0)$ and inhomogeneities $(f^*_u, f^*_d, g^*_u, g^*_j, f^*_\theta, g^*_\theta, g^*_\pi, g^*_h)$ which satisfy the required regularity conditions, and by construction, the required compatibility conditions, to obtain a unique solution

$$z^* = (u^*, \pi^*, \gamma \pi^*, \theta^*, h^*) \in \mathbb{E}(a)$$

with $u^*(0) = u_0, \theta^*(0) = \theta_0$, and $h^*(0) = h_0$. As the solution map of Theorem 3.3 is continuous, we know that for any $r > 0$, there exists $\eta > 0$ such that

$$\|z_0\|_{X_\gamma} \leq \eta \Rightarrow \|z^*\|_{\mathbb{E}(a)} \leq r. \quad (5.4)$$
Step 2. We rewrite problem (5.2) as

\[ Lz = N(z + z^*) - Lz^* =: K(z), \quad z \in \mathbb{E}(a), \]

where \( \mathbb{E}(a) := \{ z \in \mathbb{E}(a) : z(0) = 0 \} \). The space \( \mathbb{E}(a) \) is defined analogously. The solution \( z + z^* \) of (2.1) is obtained from the fixed point problem \( z = L^{-1} K(z) \), since Theorem 3.3 implies that \( L : \mathbb{E}(a) \to \mathbb{E}(a) \) is an isomorphism with

\[ \| L^{-1} \|_{\mathbb{E}(a) \to \mathbb{E}(a)} \leq M. \]

We may assume that \( M \geq 1 \). Thanks to Proposition 5.3 and due to \( K(0) = N(z^*) - Lz^* \), we may choose first \( r > 0 \) and then \( \eta > 0 \) sufficiently small such that

\[ \| K(0) \|_{\mathbb{E}(a)} \leq \frac{r}{2M}, \quad \| DK(z) \|_{\mathbb{E}(a) \to \mathbb{E}(a)} \leq \frac{1}{2M} \]

for all \( z \in \mathbb{E}(a) \) with \( \| z \|_{\mathbb{E}(a)} \leq r \), hence

\[ \| K(z) \|_{\mathbb{E}(a)} \leq \frac{r}{M} \quad \text{for} \quad \| z \|_{\mathbb{E}(a)} \leq r, \]

which ensures that \( L^{-1} K : \mathbb{B}_{\mathbb{E}(a)}(0, r) \to \mathbb{B}_{\mathbb{E}(a)}(0, r) \) is a contraction. Thus, we may employ the contraction mapping principle to obtain a unique solution on the fixed time interval \([0, a]\). As the map \( z_0 \mapsto z_* \) is continuous, the solution map \( z_0 \mapsto z \) is continuous as well. This completes the proof of Theorem 5.1.

6. Appendix

In this “Appendix”, we prove formula (4.14) for the boundary symbol, namely we compute the function \( m(\xi) \). By (4.12), (4.9), and (4.10) with \( \hat{g}_1 = -\sigma |\xi|^2 \hat{h} \) and \( \hat{g}_2 = 0 \), we obtain with \( \tau = |\xi| \)

\[
\begin{align*}
-\| \rho \hat{w}(0) \|_{\| \rho \|} & = -\| \rho \|^{-1} \left( \tau \alpha_1 \rho_1 + \tau \alpha_2 \rho_2 + \rho_1 \sqrt{\mu_1} \beta_1 / \omega_1 + \rho_2 \sqrt{\mu_2} \beta_2 / \omega_2 \right) \\
& = -\| \rho \|^{-1} \left[ \alpha_1 \left\{ \tau \rho_1 - \tau^2 / \gamma (\rho_1 \sqrt{\mu_1} / \omega_1 (2 \mu_1 \tau + \gamma_2) + \rho_2 \sqrt{\mu_2} / \omega_2 (2 \mu_1 \tau - \gamma_1)) \right\} \\
& \quad + \alpha_2 \left\{ \tau \rho_2 - \tau^2 / \gamma (\rho_1 \sqrt{\mu_1} / \omega_1 (2 \mu_2 \tau - \gamma_2) + \rho_2 \sqrt{\mu_2} / \omega_2 (2 \mu_2 \tau + \gamma_1)) \right\} \right] \\
& = \frac{\sigma \tau^3 \hat{h}}{\gamma \| \rho \|^2 \lambda \varepsilon} \left[ (1 + 2 \rho_2 \varepsilon_2 \| \mu / \rho \|) \\
& \times \left\{ \gamma_1 \rho_1 - \left( \rho_1 \tau \sqrt{\mu_1} / \omega_1 (2 \mu_1 \tau + \gamma_2) + \rho_2 \tau \sqrt{\mu_2} / \omega_2 (2 \mu_1 \tau - \gamma_1) \right) \right\} \\
& \quad + (1 - 2 \rho_1 \varepsilon_1 \| \mu / \rho \|) \\
& \times \left\{ \gamma_2 \rho_2 - \left( \rho_1 \tau \sqrt{\mu_1} / \omega_1 (2 \mu_2 \tau - \gamma_2) + \rho_2 \tau \sqrt{\mu_2} / \omega_2 (2 \mu_2 \tau + \gamma_1) \right) \right\} \right],
\end{align*}
\] (6.1)

where we have set \( \gamma := \gamma_1 + \gamma_2 \). The scaling \( z := \lambda / \tau^2, \tau := |\xi| \) yields

\[
\omega_k = \sqrt{\rho_k \lambda + \mu_k \tau^2} = \sqrt{\mu_k \tau \omega_k(z)}, \quad \omega_k(z) = \sqrt{1 + \rho_k z / \mu_k},
\]
\[ \gamma_k = \sqrt{\mu_k + \mu k \tau^2} \sqrt{\frac{\mu k}{\omega k}} = \mu k \gamma (\omega k(z) + 1/\omega k(z)), \]
\[ \varepsilon_k = \rho k \lambda \sqrt{\mu k \tau^2} / (\omega k \gamma (\omega k + \sqrt{\mu k \tau^2}) = (\gamma \omega k(z))^{-1}(\omega k(z) - 1)/(\omega k(z) + 1), \]
hence
\[ 2\mu k \tau - \gamma_k = -\frac{\mu k \tau}{\omega k(z)} (\omega k(z) - 1)^2 = -\frac{\rho k z}{\omega k(z)} \frac{\omega k(z) - 1}{\omega k(z) + 1}, \]
\[ \rho_1 \gamma_2 - \rho_1 \tau \sqrt{\mu_1 \gamma_2} / \omega_1 = \rho_1^2 \mu_2 \mu_1 (\omega_2(z) + \frac{1}{\omega_2(z)}) \frac{z}{\omega_1(z)(\omega_1(z) + 1)}, \]
\[ \rho_1 \gamma_1 - \rho_1 \tau \sqrt{\mu_1 \mu_2} / \omega_1 = \rho_1^2 \tau z / \omega_1(z), \]
\[ \gamma \varepsilon = \gamma + 2 \mu_1 \gamma \varepsilon_1 + 2 \mu_2 \gamma \varepsilon_2 = \mu_1 \varphi_1(z) + \mu_2 \varphi_2(z), \]
where
\[ \varphi_k(z) = \omega_k(z) + \frac{1}{\omega_k(z)} + \frac{2}{\omega_k(z)} \frac{\omega_k(z) - 1}{\omega_k(z) + 1} = \omega_k(z) + \frac{3}{\omega_k(z) + 1} - \frac{1}{\omega_k(z)(\omega_k(z) + 1)}. \]
Substituting these expressions into (6.1), we obtain
\[ \frac{\| \rho \dot{\varrho} (0) \|}{\| \rho \|} = \frac{\sigma \tau \hat{h}}{\| \rho \|^2 (\mu_1 \varphi_1 + \mu_2 \varphi_2)} \times \left[ (1 + 2 \rho_2 \varepsilon_2 \| \mu / \rho \|) \left( \frac{\rho_1^2}{\omega_1} + \frac{\rho_2^2 \mu_2}{\mu_1} (\omega_2 + \frac{1}{\omega_2}) \frac{1}{\omega_1(\omega_1 + 1)} + \frac{\rho_1 \rho_2}{\omega_1 \omega_2 (\omega_1 + 1)} \right) \right. \]
\[ \left. + (1 - 2 \rho_1 \varepsilon_1 \| \mu / \rho \|) \left( \frac{\rho_2^2}{\omega_2} + \frac{\rho_2^2 \mu_1}{\mu_2} (\omega_1 + \frac{1}{\omega_1}) \frac{1}{\omega_2(\omega_2 + 1)} + \frac{\rho_1 \rho_2}{\omega_1 \omega_2 (\omega_2 + 1)} \right) \right]. \]
Expanding and collecting terms, we see that the coefficients of \( \rho_1^2, \rho_1 \rho_2, \rho_2^2 \) in the square brackets on the right-hand side eventually become
\[ \frac{1}{\omega_1} + \frac{\mu_2 \varphi_2}{\mu_1 \omega_1 (\omega_1 + 1)}, \]
\[ \frac{2}{\omega_1 \omega_2 (\omega_1 + 1)} \frac{\omega_1 - 1}{\omega_2 - 1}, \]
\[ \frac{1}{\omega_2} + \frac{\mu_1 \varphi_1}{\mu_2 \omega_2 (\omega_2 + 1)}. \]
Note that \( \varepsilon_k \) contains \( \gamma \) in the denominator; in it is important to recognize that it factors. Finally, we obtain
\[ \frac{\| \rho \dot{\varrho} (0) \|}{\| \rho \|} = \frac{\sigma \tau \hat{h}}{\| \rho \|^2 (\mu_1 \varphi_1(z) + \mu_2 \varphi_2(z))} \left[ \frac{\rho_1 \rho_2}{\omega_1(\omega_1(z) - 1)} + \frac{\rho_1 \rho_2}{\omega_2(\omega_2(z) - 1)} \right. \]
\[ \left. + \frac{\rho_1^2 \mu_2 \varphi_2(z)}{\mu_1 \omega_1 (\omega_1(z) + 1)} + \frac{\rho_2^2 \mu_1 \varphi_1(z)}{\mu_2 \omega_2 (\omega_2(z) + 1)} + \frac{\rho_1^2}{\omega_1(z)} + \frac{\rho_2^2}{\omega_2(z)} \right], \]
which proves (4.15).

**Acknowledgments**

We would like to thank the referee for his thoughtful comments which helped to improve this article.
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