Polynomial approximations of a class of stochastic multiscale elasticity problems *

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Abstract

We consider a class of elasticity equations in $\mathbb{R}^d$ whose elastic moduli depend on $n$ separated microscopic scales. The moduli are random and expressed as a linear expansion of a countable sequence of random variables which are independently and identically uniformly distributed in a compact interval. The multiscale displacement problem, the multiscale Hellinger-Reissner mixed problem that allows for computing the stress directly, and the multiscale mixed problem with a penalty term for nearly incompressible isotropic materials are considered. The stochastic problems are studied via deterministic problems that depend on a countable number of real parameters which represent the probabilistic law of the stochastic equations. We study the multiscale homogenized problems that contain all the macroscopic and microscopic information. The solutions of these multiscale homogenized problems are written as generalized polynomial chaos (gpc) expansions. We approximate these solutions by semidiscrete Galerkin approximating problems that project into the spaces of functions with only a finite number of $N$ gpc modes. Assuming summability properties for the coefficients of the elastic moduli’s expansion, we deduce bounds and summability properties for the solutions’ gpc expansion coefficients. These bounds imply explicit rates of convergence in terms of $N$ when the gpc modes used for the Galerkin approximation are chosen to correspond to the best $N$ terms in the gpc expansion. For the mixed problem with a penalty term for nearly incompressible materials, we show that the rate of convergence for the best $N$ term approximation is independent of the Lamé constants’ ratio when it goes to $\infty$. Correctors for the homogenization problem are deduced. From these we establish correctors for the solutions of the parametric multiscale problems in terms of the semidiscrete Galerkin approximations. For two scale problems, an explicit homogenization rate which is uniform with respect to the parameters is deduced. Together with the best $N$ term rate, it provides an explicit convergence rate for the correctors of the parametric multiscale problems. For nearly incompressible materials, we obtain a homogenization rate that is independent of the ratio of the Lamé constants, so that the error for the corrector is also independent of this ratio.

1 Introduction

We consider a multiscale elasticity problem in $\mathbb{R}^d$ whose elastic tensor is random and is a linear combination of a sequence of random variables which are independently and uniformly distributed in a compact interval. The elastic tensor depends on $n$ separable microscopic scales and is periodic with respect to each of these scales. We consider the multiscale displacement problem, the multiscale Hellinger-Reissner mixed problem that allows for computing the stress tensor directly, and the multiscale mixed problem with a penalty term for nearly incompressible isotropic materials. We study the random problems via deterministic ones whose elastic tensor depends on an infinite sequence of real parameters. The space of parameter sequences is equipped with a probability measure that is the law of the sequence of random

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variables that the random elastic tensor depends on. Thus the deterministic parametric multiscale solution is the law of the solution of the stochastic multiscale problem. Solving this parametric equation, we obtain statistic properties of the stochastic multiscale solution.

For multiscale problems, a direct numerical procedure is prohibitively expensive. The problem is approximated by the homogenization limit when all the microscopic scales converge to 0 (3,14). To obtain the microscopic information, a part from the solution of the homogenized problem, we need also the scale interacting terms. We therefore apply the multiscale convergence (see 16, 1 and 2) to obtain the multiscale homogenized problem that contains all the macroscopic and microscopic information. The problem is posed in a high dimensional tensorized domain: if the original multiscale problem is posed in $\mathbb{R}^d$ and depends on $n$ microscopic scales and one macroscopic scale, this problem is posed in $\mathbb{R}^{n+1}d$. However, as demonstrated in Hoang and Schwab 10 for multiscale elliptic equations and in Xia and Hoang 13 for multiscale elasticity equations, the sparse tensor finite element approach is capable of solving these high dimensional multiscale homogenized problems with an essentially optimal complexity which is essentially equal to that for solving a problem in $\mathbb{R}^d$. Though we do not address finite element approximation in the paper, this is the motivation for us to consider polynomial approximations for the solutions of the stochastic/parametric multiscale homogenized problems. We write their solutions in terms of a generalized polynomial chaos (gpc) expansion with respect to a system of multivariate polynomials which forms an orthonormal basis for the $L^2$ Lebesgue space of the parameter sequences. Following Cohen et al. 17 and 18 and other related papers (12, 11, 15), we study the best $N$ term approximation for the gpc expansion of the high dimensional multiscale homogenized problems. When assuming summability for the coefficients of the elastic moduli’s expansion, we deduce summability properties for the coefficients of the gpc expansion. From this, an explicit error estimate for the semidiscrete Galerkin approximating problem which projects into the spaces of functions with only $N$ gpc modes can be deduced when these modes are chosen to correspond to the best $N$ terms in the gpc expansion. In many cases, this rate is superior over the Monte Carlo $N^{-1/2}$ rate.

To approximate the solutions of the multiscale problems, we derive correctors from the solution of the semidiscrete Galerkin problems. For two scales, an explicit homogenization error is available. To employ this error for the parametric problem, we prove that it is uniform with respect to the parameter sequences. From this a corrector for the multiscale problem in the mean square norm with respect to the parameter space is deduced. The rate of convergence is the sum of the best $N$ term semidiscrete Galerkin rate and the uniform homogenization rate. For more than two scale problems, an explicit homogenization rate is not available. However, we can construct a corrector for the parametric homogenization problem. From this, a corrector for the parametric multiscale problem in the mean square norm using the solution of the semidiscrete Galerkin problem is deduced, without an explicit error.

For multiscale elliptic problems, the framework has been applied in 12 where the complex method to deduce bounds for the gpc coefficients is employed. Here we employ the real method developed in 17. However, the main contributions of this paper are the studies of the stochastic/parametric high dimensional multiscale homogenized problems for the multiscale mixed Hellinger-Reissner formula and for the multiscale mixed problem with a penalty term for nearly incompressible materials. We adapt the approach in 17 for single scale macroscopic equations to deduce the best $N$ term approximations. For the mixed problems, the inverse of the elastic tensor depends on the random variables in the expansion nonlinearly. To employ the linear dependence, we formulate the equivalent mixed problems that use the elastic tensor instead of its inverse. For the Hellinger-Reissner problem, showing the well-poseness of the semidiscrete Galerkin problem of the equivalent form, and deducing bounds for the coefficients of the gpc expansion require careful manipulation of various inf-sup conditions. For the mixed problem with a penalty term, we obtain a best $N$ term approximation whose rate is independent of the ratio of the Lamé constants. We also obtain a homogenization rate of convergence for nearly incompressible materials that is independent of this ratio. To the best of our knowledge, this homogenization rate is new. Together with the best $N$ term approximation, we construct a corrector for the stochastic/parametric multiscale nearly incompressible problem with an error independent of the ratio of the Lamé constants.

The paper is organized as follows. In the next section, we set up the multiscale problem, define the multiscale and the random structures of the elastic tensors. We formulate the multiscale random displacement problem, the multiscale mixed Hellinger-Reissner problems and the multiscale mixed problem with a penalty term for nearly incompressible materials. Section 3 studies the deterministic parametric problems. We show that their solutions are measurable with respect to the $\sigma$-algebra of the parameter space.
Therefore the solution of the original stochastic problem can be deduced from the parametric solution by inserting the random sequences into the place of the parameters. We establish the multiscale homogenized problems in this section. The next three sections are devoted to approximating the deterministic parametric high dimensional multiscale homogenized problems. Approximations for the displacement problem is considered in Section 4. We first write the high dimensional solution as a gpc expansion. We then consider the semidiscrete Galerkin problem that projects into a subspace of functions with only $N$ fixed gpc modes. To get an explicit rate of convergence, we deduce bounds for the coefficient functions of the gpc expansion. From these bounds, we derive the rate of convergence when the gpc modes are chosen to correspond to the best $N$ terms in the gpc expansion. We consider approximation of the mixed high dimensional multiscale homogenized problem for the Hellinger-Reissner setting and its equivalent form using the elastic tensor (but not its inverse) in Section 5. Employing standard estimates for saddle point problems, we deduce bounds for the coefficient functions of the gpc expansions of the displacement vector and the stress tensor. We then deduce an explicit convergence rate for the semidiscrete Galerkin problem when the finite number of gpc modes is chosen corresponding to the best $N$ terms in the gpc expansion. Approximation for the nearly incompressible problem is studied in Section 6. We consider both the mixed problem with a penalty term and the equivalent one using the Lamé constant expansion. Approximation for the nearly incompressible problem is studied in Section 6. We consider both the mixed problem with a penalty term and the equivalent one using the Lamé constant $\lambda$ instead of $1/\lambda$. We deduce bounds for the coefficients of the gpc expansion, from which we show that the best $N$ term approximation achieves a rate of convergence that is independent of the ratio of the Lamé constants when $\lambda$ goes to $\infty$. In Section 7, we use the semidiscrete Galerkin approximations to deduce correctors for the parametric multiscale problems. For two-scale problems, we derive a homogenization rate which is uniform with respect to the parameters. From this a corrector for the parametric multiscale problem is deduced in the mean square norm with respect to the parameter space with an explicit error that is the sum of the uniform homogenization error and the best $N$ term approximation rate. For nearly incompressible materials, we prove a homogenization rate that is independent of the ratio of the Lamé constants. The error for the correctors of the stochastic/parametric two-scale problem is thus independent of this ratio. For problems that depend on more than one microscopic scales, we derive a corrector for the parametric homogenization problem, without a rate of convergence, which implies a corrector in the mean square norm with respect to the parameter space for the solution of the parametric multiscale problem, without an explicit rate of convergence.

Throughout this paper, repeated indices indicate summation. Notation $\nabla$ without an explicit variable denotes the gradient with respect to $x$. Similarly, $\epsilon$ without an explicit variable denotes the elastic strain tensor with respect to $x$. We denote by $\cdot$ the inner product in $R^{d \times d}_{\text{sym}}$. For a sequence of integers $\nu = (\nu_1, \nu_2, \ldots)$ with only a finite number of terms being non zero, we denote by $\nu! = \nu_1! \nu_2! \ldots$; and for a sequence of real numbers $d = (d_1, d_2, \ldots)$, we denote by $d^\nu = d_1^{\nu_1} d_2^{\nu_2} \ldots$. The notation $\#$ indicates spaces of periodic functions, in particular $H^k_\#(Y)$ denotes Sobolev spaces of periodic functions, and $C^k_\#(Y)$ denotes the space of $k$ time differentiable periodic functions.

## 2 Setting-up of the problems

Let $D \subset R^d$ be a bounded domain. Let $Y$ be the unit cube $(0,1)^d \subset R^d$ and $Y_1, \ldots, Y_n$ be $n$ copies of $Y$. We denote by $Y = Y_1 \times \ldots \times Y_n$ and $y = (y_1, \ldots, y_n) \in Y$. Let $(\Omega, \Sigma, P)$ be a probability space where $\Sigma$ is the sigma algebra and $P$ is the probability measure. The elastic tensor is a random function $a(\omega, x, y_1, \ldots, y_n) : \Omega \times D \times Y \rightarrow L^\infty(\Omega, C(D, C_\#(Y_1 \times \ldots \times Y_n)))^{d^2}$ where $C_\#(Y_1 \times \ldots \times Y_n) = C_\#(Y)$ denotes the space of continuous functions that are periodic with respect to $y_i$ with the period $Y_i$ for $i = 1, \ldots, n$. The tensor function $a$ is assumed to be symmetric: for $i, j, k, l = 1, \ldots, d$

$$a_{ijkl} = a_{ijlk} = a_{klij}.$$  

The random tensor $a$ satisfies the coerciveness and boundedness condition

$$a(\omega; x, y) \xi \cdot \xi \geq \alpha |\xi|^2, \quad a(\omega; x, y) \xi \cdot \zeta \leq \beta |\xi||\zeta| \quad \forall \xi, \zeta \in R^{d \times d}_{\text{sym}}$$  

(2.1)

where the constants $\alpha > 0, \beta > 0$ are independent of $\omega \in \Omega$, $x \in D$ and $y \in Y$. To define the $n$ microscopic scales that the multiscale elasticity problem depends on, let $\epsilon > 0$ be a small quantity and $\epsilon_i$ ($i = 1, \ldots, n$) be $n$ positive functions of $\epsilon$ which satisfy the scale separation assumption

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon_{i+1}(\epsilon)}{\epsilon_i(\epsilon)} = 0, \quad i = 1, \ldots, n - 1.$$
Without loss of generality, we assume that $\varepsilon_1(\varepsilon) = \varepsilon$. We define the random multiscale elastic moduli as

$$a^\varepsilon(\omega; x) = a \left( \omega; x; \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n} \right).$$

To define the probability distribution of the random elastic moduli $a$, we assume that there are independent random variables $z_m : \Omega \to [-1, 1]$ which are uniformly distributed in $[-1, 1]$, and there are fourth order symmetric tensor functions $\psi_m : D \times Y \to \mathbb{R}^d$ so that

$$a(\omega; x, y) = \bar{a}(x, y) + \sum_{m=1}^{\infty} z_m(\omega)\psi_m(x, y). \quad (2.2)$$

The fourth order tensor function $\bar{a}(x, y)$ is the mean value of $a$. It is symmetric and satisfies

$$\bar{a}(x, y)\xi : \xi \geq \alpha_0|\xi|^2, \quad |\bar{a}(x, y)\xi : \xi| \leq \beta_0|\xi||\xi| \quad (2.3)$$

for $\alpha_0 > 0, \beta_0 > 0$ and all $\xi, \zeta \in \mathbb{R}^{d\times d}_{sym}$. For the uniform coerciveness and boundedness (2.1), we assume further that there are positive constants $\beta_m$ such that for all $\xi, \zeta \in \mathbb{R}^{d\times d}_{sym}$

$$|\psi_m(x, y)\xi : \zeta| \leq \beta_m|\xi||\zeta| \quad (2.4)$$

and that

$$\sum_{m=1}^{\infty} \beta_m \leq \frac{\kappa}{1 + \kappa} \alpha_0 \quad (2.5)$$

for $\kappa > 0$. We can then take the constants $\alpha$ and $\beta$ in (2.1) as

$$\alpha = \frac{\alpha_0}{1 + \kappa}, \quad \beta = \beta_0 + \frac{\kappa}{1 + \kappa} \alpha_0. \quad (2.6)$$

Let $\Gamma$ be a subset of the boundary $\partial D$. Let $H_0^1(D)$ be the subspace of $H^1(D)$ of functions with zero trace on $\Gamma$. Let $V = H_0^1(D)^d$. We denote by $(\cdot, \cdot)$ the inner product in $L^2(D)^d$, extended to the dual pairing relation $(\cdot, \cdot)_{V', V}$, and also the inner product in $L^2(D)^{d\times d}$. Let $f \in V'$ be the forcing function which is assumed to be deterministic. We study the multiscale elasticity equation

$$-\frac{\partial}{\partial x_j} \left( \sigma_{ijk\ell}(\omega; x)\epsilon_{kl}(u^\varepsilon)(\omega; x) \right) = f_i(x),$$

for all $i = 1, \ldots, d$, with the zero boundary condition $u^\varepsilon = 0$ on $\Gamma$ and the traction free condition on $\partial D \setminus \Gamma$. Here $\epsilon$ denotes the elastic strain tensor

$$\epsilon_{ij}(w) = \frac{1}{2} \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right)$$

for functions $w \in H^1(D)^d$. In variational form this problem is written as: Find $u^\varepsilon \in V$ so that

$$\int_D \sigma^\varepsilon(\omega; x)\epsilon(v)(\omega; x) : \epsilon(v)(x)dx = \int_D f(x) \cdot v(x)dx \quad \forall v \in V. \quad (2.7)$$

From Korn’s inequality, there is a constant $C$ that only depends on $\alpha$ and $\beta$ in (2.1) and the domain $D$ so that $\|w\|_V \leq C\|f\|_{V'}$.

We study also the Hellinger-Reissner mixed problem that allows for computing the stress tensor $\sigma^\varepsilon(\omega; x) = \sigma^\varepsilon(\omega; x)\epsilon(u^\varepsilon(\omega; x))$ directly. Let $\mathcal{H} = L^2(D)^{d\times d}_{sym}$. The problem is: Find $(\sigma^\varepsilon(\omega; \cdot), u^\varepsilon(\omega; \cdot)) \in \mathcal{H} \times V$ so that

$$\left\{ \begin{array}{ll} (\alpha^\varepsilon)^{-1}(\omega; \cdot)\sigma^\varepsilon(\omega; \cdot, \tau) - (\tau, \epsilon(u^\varepsilon)(\omega; \cdot)) = 0, & \forall \tau \in \mathcal{H} \\ -\sigma^\varepsilon(\omega; \cdot, \epsilon(v)) = -(f, v), & \forall v \in V. \end{array} \right. \quad (2.8)$$

For each $\tau \in \mathcal{H}$, $\alpha^\varepsilon(\omega; \cdot)\tau$ also belongs to $\mathcal{H}$. Therefore the following mixed problem is equivalent to (2.8):

$$\left\{ \begin{array}{ll} (\sigma^\varepsilon(\omega; \cdot), \tau) - (\tau, \alpha^\varepsilon(\omega; \cdot)\epsilon(u^\varepsilon)(\omega; \cdot)) = 0, & \forall \tau \in \mathcal{H} \\ -\sigma^\varepsilon(\omega; \cdot, \epsilon(v)) = -(f, v), & \forall v \in V. \end{array} \right. \quad (2.9)$$
We assume a similar structure for the Lamé constants \( \mu \) and \( \varepsilon \inf-sup \) conditions for (2.8) hold uniformly with respect to \( \varepsilon \). Therefore \( \|\sigma^\varepsilon\|_H + \|u^\varepsilon\|_V \leq \epsilon\|f\|_V, \forall \varepsilon > 0 \).

For isotropic materials, the multiplying constant of the best \( \varepsilon \inf-sup \) is written as

\[
a_{ijkl}(\omega; x, y) = \mu(\omega; x, y)(\delta_{i4}\delta_{j4} + \delta_{i4}\delta_{j4}) + \lambda(\omega; x, y)\delta_{i4}\delta_{j4}.
\]

(2.10)

We assume a similar structure for the Lamé constants \( \mu(\omega; x, y) \) and \( \lambda(\omega; x, y) \), i.e.

\[
\mu(\omega; x, y) = \Omega(x, y) + \sum_{m=1}^{N} z_m(\omega)\mu_m(x, y), \quad \lambda(\omega; x, y) = \Lambda(x, y) + \sum_{m=1}^{N} z_m(\omega)\lambda_m(x, y).
\]

(2.11)

Let \( \gamma_m = \sup_{(x, y) \in D \times Y} |\mu_m(x, y)| \) and \( \delta_m = \sup_{(x, y) \in D \times Y} |\lambda_m(x, y)| \). We define \( \Omega_{\min} = \inf_{(x, y) \in D \times Y} \Omega(x, y) \) and \( \Lambda_{\min} = \inf_{(x, y) \in D \times Y} \Lambda(x, y) \). For the uniform coerciveness and boundedness of the tensor \( a_{ijkl} \), we assume: There exists a constant \( \kappa > 0 \) such that

\[
\sum_{m=1}^{\infty} \gamma_m \leq \frac{\kappa}{\kappa + 1} \Omega_{\min}, \quad \sum_{m=1}^{\infty} \delta_m \leq \frac{\kappa}{\kappa + 1} \Lambda_{\min}, \quad \text{where} \quad \Omega_{\min} > 0, \Lambda_{\min} > 0.
\]

(2.12)

For isotropic materials, the multiplying constant of the best \( N \) term convergence rates in the displacement and the Hellinger-Reissner settings depends on the ratio of the Lamé constants which is very large when the materials are nearly incompressible. We therefore consider the mixed problem with a penalty term and show that the best \( N \) term convergence rate can be established to be independent of this ratio. Let \( H = L^2(D) \). The mixed problem is: Find \((u^\varepsilon(\omega; \cdot), p^\varepsilon(\omega; \cdot)) \in V \times H \) such that

\[
2\mu^\varepsilon(\omega; \cdot)\epsilon(u^\varepsilon(\omega; \cdot), \epsilon(v(\cdot))) + (\text{div} v(\cdot), p^\varepsilon(\omega; \cdot)) = (f, v), \quad \forall v \in V,
\]

\[
(\text{div} u^\varepsilon(\omega; \cdot), q(\cdot)) - \left( \frac{1}{\lambda^\varepsilon(\omega; \cdot)} p^\varepsilon(\omega; \cdot), q(\cdot) \right) = 0, \quad \forall q \in H.
\]

(2.13)

To employ the linear dependence of \( \lambda(\omega; x, y) \) on \( z_m \), we will consider the equivalent mixed problem

\[
2\mu^\varepsilon(\omega; \cdot)\epsilon(u^\varepsilon(\omega; \cdot), \epsilon(v(\cdot))) + (\text{div} v(\cdot), p^\varepsilon(\omega; \cdot)) = (f, v), \quad \forall v \in V,
\]

\[
\left( \frac{1}{\Lambda^\varepsilon(\omega; \cdot)} \text{div} u^\varepsilon(\omega; \cdot), q(\cdot) \right) - \left( \frac{1}{\Lambda_{\min}} p^\varepsilon(\omega; \cdot), q(\cdot) \right) = 0, \quad \forall q \in H.
\]

(2.14)

We consider the case where \( \text{meas}(\partial D \setminus \Gamma) > 0 \). Problem (2.13) has a unique solution (see, e.g. [5]) and \( \|u^\varepsilon\|_V + \|p^\varepsilon\|_H + \|p^\varepsilon/(\lambda^\varepsilon)^{1/2}\|_H \leq c\|f\|_V \) where \( c \) is independent of \( \varepsilon \). From (2.12), \( \inf_{x \in D} \lambda^\varepsilon(\omega; x) \geq \Lambda_{\min}/(1 + \kappa) \) so \( \|u^\varepsilon\|_V + \|p^\varepsilon\|_H \leq c\|f\|_V \), \( \forall \varepsilon \).

**Remark 2.1** When \( \text{meas}(\partial D \setminus \Gamma) = 0 \), i.e. \( V = H^1_0(D) \), the solution of (2.13) is bounded with respect to the norm \( \|u^\varepsilon\|_V + \|p^\varepsilon\|_H + \|p^\varepsilon/(\lambda^\varepsilon)^{1/2}\|_H \leq c\|f\|_V \) when \( \Lambda_{\min} \) goes to \( \infty \). Therefore for nearly incompressible problems, we only consider the case where \( \text{meas}(\partial D \setminus \Gamma) > 0 \).

3 Deterministic parametric problems

To study the law of the solutions of stochastic problems we study parametric problems whose elastic moduli depend on parameter sequences in \( U = [-1, 1]^N \). We first define the probability space.

**3.1 Probability space**

For the space of parameter sequences \( U = [-1, 1]^N \), we introduce the \( \sigma \) algebra \( \Sigma_U = \mathcal{B}([-1, 1]^N) \) where \( \mathcal{B}([-1, 1]) \) is the Borel \( \sigma \)-algebra on \([-1, 1]\). We define the probability measure \( \rho \) on \((U, \Sigma_U)\) as

\[
d\rho(z) = \frac{\infty}{\sum_{m=1}^{2} dz_m}.
\]
As $d\omega /2$ is a probability measure on $[-1, 1]$, $d\rho$ is a probability measure on $U$ so $(U, \Sigma_U, \rho)$ is a probability space. As $z_m$ are independently distributed on $[-1, 1]$, for $S = \prod_{m=1}^{\infty} S_m \subset U$ where $S_m \subset [-1, 1]$, \[ \rho(S) = \prod_{m=1}^{\infty} \mathcal{P}[\omega : z_m(\omega) \in S_m]. \]

### 3.2 Parametric deterministic problems

For $\psi_m(x)$ in (2.8), we define the deterministic parametric elastic moduli for each $z \in U$ as \[ a(z, x, y) = \bar{a}(x, y) + \sum_{m=1}^{\infty} z_m \psi_m(x, y). \]

Conditions (2.8a) and (2.8b) guarantee that $a(z, x, y)$ is well defined for all $x \in D$ and $y \in Y$ and that \[ a(z, x, y)\xi : \xi \geq |\xi|^2, \quad |a(z, x, y)\xi : \zeta| \leq \beta |\xi||\zeta| \quad \forall \xi, \zeta \in \mathbb{R}^{d \times d}_{sym}. \] (3.1)

The multiscale parametric elastic moduli are defined as \[ a^\varepsilon(z, x) = a(z, x, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n). \]

We consider the parametric elasticity equation:

\[ -\frac{\partial}{\partial x_j} (a^\varepsilon_{ijkl}(u^\varepsilon))(z, x,j) = f_i, \quad i = 1, \ldots, d \]

with the Dirichlet boundary condition $u^\varepsilon = 0$ on $\Gamma$ and the traction free condition on $\partial D \setminus \Gamma$. In variational form, this problem becomes: Find $u^\varepsilon(z, \cdot) \in V$ such that \[ \int_D a^\varepsilon(z, x) \epsilon(u^\varepsilon)(z, x) : \epsilon(v)(x) dx = \int_D f(x) \cdot v(x) dx \forall v \in V. \] (3.2)

From Lax-Milgram lemma and Korn’s inequality, this problem has a unique solution $u^\varepsilon(z, x)$ that satisfies $\|u^\varepsilon(z, x)\|_V \leq c\|f\|_V$, where $c$ is independent of $\varepsilon$ and $z$.

We consider the parametric Hellinger-Reissner mixed problem: Find $(\sigma^\varepsilon(z, \cdot), u^\varepsilon(z, \cdot)) \in H \times V$ so that \[ \left\{ \begin{array}{l} ((a^\varepsilon)^{-1}(z, \cdot)\sigma^\varepsilon(z, \cdot), \tau) - (\tau, \epsilon(u^\varepsilon))(z, \cdot)) = 0, \quad \forall \tau \in H \\ - (\sigma^\varepsilon(z, \cdot), \epsilon(v)) = -(f, v), \quad \forall v \in V. \end{array} \right. \] (3.3)

This problem is equivalent to: Find $(\sigma^\varepsilon(z, \cdot), u^\varepsilon(z, \cdot)) \in H \times V$ such that \[ \left\{ \begin{array}{l} (\sigma^\varepsilon(z, \cdot), \tau) - (\tau, a^\varepsilon(z, \cdot)\epsilon(u^\varepsilon))(z, \cdot)) = 0, \quad \forall \tau \in H \\ - (\sigma^\varepsilon(z, \cdot), \epsilon(v)) = -(f, v), \quad \forall v \in V. \end{array} \right. \] (3.4)

As the bilinear form $((a^\varepsilon)^{-1}\sigma, \tau)$ on $H \times H$ is uniformly coercive with respect to $\varepsilon$, the Hellinger-Reissner mixed problem (3.3) has a unique solution $(\sigma^\varepsilon, u^\varepsilon)$ which satisfies \[ \|\sigma^\varepsilon\|_H + \|u^\varepsilon\|_V \leq c\|f\|_V. \] (3.5)

For nearly incompressible problems, we restrict our consideration to the case where the measure of $\partial D \setminus \Gamma$ is positive. For each $z \in U = [-1, 1]^N$, we consider the parametric Lamé constants \[ \mu(z, x, y) = \bar{\mu}(x, y) + \sum_{m=1}^{\infty} z_m \mu_m(x, y), \quad \text{and} \quad \lambda(z, x, y) = \bar{\lambda}(x, y) + \sum_{m=1}^{\infty} z_m \lambda_m(x, y). \]

From (2.8), for all $(x, y) \in D \times Y$ and $z \in U$ we have \[ \mu_{\min} := \frac{1}{1 + \kappa} \bar{\mu}_{\min} \leq \mu(z, x, y) \leq \sup_{(x, y) \in D \times Y} \bar{\mu}(x, y) + \frac{\kappa}{1 + \kappa} \bar{\mu}_{\min} := \mu_{\max}, \] (3.6) \[ \lambda_{\min} := \frac{1}{1 + \kappa} \bar{\lambda}_{\min} \leq \lambda(z, x, y) \leq \sup_{(x, y) \in D \times Y} \bar{\lambda}(x, y) + \frac{\kappa}{1 + \kappa} \bar{\lambda}_{\min} := \lambda_{\max}. \] (3.7)
For problems (2.13) and (2.14), we consider the parametric problems: Find \((u^\varepsilon(z;\cdot), p^\varepsilon(z;\cdot)) \in V \times H\) such that
\[
\begin{aligned}
2(\mu^\varepsilon(z;\cdot)e(u^\varepsilon(z;\cdot)), e(v(\cdot))) &+ (\text{div} \, v(\cdot), p^\varepsilon(z;\cdot)) = (f, v), \quad \forall v \in V, \\
(\text{div} \, u^\varepsilon(z;\cdot), q(\cdot)) &- \left(\frac{1}{\lambda^\varepsilon(z;\cdot)}p^\varepsilon(z;\cdot), q(\cdot)\right) = 0, \quad \forall q \in L^2(D)
\end{aligned}
\]
(3.8)
and: Find \((u^\varepsilon(z;\cdot), p^\varepsilon(z;\cdot)) \in V \times H\) such that
\[
\begin{aligned}
2(\mu^\varepsilon(z;\cdot)e(u^\varepsilon(z;\cdot)), e(v(\cdot))) &+ (\text{div} \, v(\cdot), p^\varepsilon(z;\cdot)) = (f, v), \quad \forall v \in V, \\
\left(\frac{\lambda^\varepsilon(z;\cdot)}{\lambda_{\min}} \text{div} \, u^\varepsilon(z;\cdot), q(\cdot)\right) &- \left(\frac{1}{\lambda_{\min}}p^\varepsilon(z;\cdot), q(\cdot)\right) = 0, \quad \forall q \in L^2(D).
\end{aligned}
\]
(3.9)
Problem (3.8) has a unique solution which satisfies \(\|u^\varepsilon\|_V + \|p^\varepsilon\|_H + \|p^\varepsilon/(\lambda^\varepsilon)^{1/2}\|_H \leq c\|f\|_V\). As \(\lambda^\varepsilon \geq \lambda_{\min}/(1 + \kappa)\), we deduce that
\[
\|u^\varepsilon\|_V + \|p^\varepsilon\|_H \leq c\|f\|_V.
\]
(3.10)
To connect the parametric problems to the stochastic problems we prove that with respect to the probability measure \((U, \Sigma_U, \rho)\) the solutions are measurable.

**Proposition 3.1** The solution \(u^\varepsilon(z;\cdot)\) of problem (3.2) as a map from \(U\) to \(V\) is measurable. The solution \((\sigma^\varepsilon(z;\cdot), u^\varepsilon(z;\cdot))\) of problems (3.3) and (3.4) as a map from \(U\) to \(H \times V\) is measurable. The solution \((u^\varepsilon(z;\cdot), p^\varepsilon(z;\cdot))\) of problems (3.8) and (3.9) as a map from \(U\) to \(V \times H\) is measurable.

**Proof** We present the proof for problem (3.2). The proofs for other problems are similar. Let \(z = (z_1, z_2, \ldots)\) and \(z' = (z'_1, z'_2, \ldots)\) in \(U\). Let \(w = u^\varepsilon(z;\cdot) - u^\varepsilon(z';\cdot)\). We then have
\[
\begin{aligned}
\frac{\partial}{\partial x_j}(a^\varepsilon_{ijkl}(z;\cdot)\epsilon_{kl}(w)) &= \frac{\partial}{\partial x_j}((a^\varepsilon_{ijkl}(z';\cdot) - a^\varepsilon_{ijkl}(z;\cdot))\epsilon_{kl}(u^\varepsilon)(z';\cdot)).
\end{aligned}
\]
From this,
\[
\int_D a^\varepsilon(z;\cdot)\epsilon(w)(x) : \epsilon(w)(x)dx = \int_D (a^\varepsilon(z';\cdot) - a^\varepsilon(z;\cdot))\epsilon(w)(z';\cdot) : \epsilon(w)(x)dx.
\]
As \(u^\varepsilon(z;\cdot)\) is uniformly bounded in \(V\) with respect to \(z \in U\), we deduce from (2.4) that
\[
\|w\|_V \leq c\|a^\varepsilon(z;\cdot) - a^\varepsilon(z';\cdot)\|_{L^\infty(D)} \leq c\|a(z;\cdot) - a(z';\cdot)\|_{L^\infty(D \times Y)}\epsilon^t.
\]
Thus
\[
\|u^\varepsilon(z;\cdot) - u^\varepsilon(z';\cdot)\|_V \leq c\sup_m |z_m - z'_m| \leq c\|z - z'\|_{\ell^\infty(\mathbb{N})}.
\]
The mapping \(u^\varepsilon : U \ni z \mapsto u^\varepsilon(z;\cdot) \in V\) is thus Lipschitz with respect to the \(\ell^\infty(\mathbb{N})\) norm, and is therefore measurable.

We therefore have:

**Proposition 3.2** For the displacement problem (2.7), almost surely, the random solution \(u^\varepsilon(\omega;\cdot)\) can be obtained from the parametric solution \(u^\varepsilon(z;\cdot)\) of (3.2) by
\[
u^\varepsilon(\omega;\cdot) = u^\varepsilon(z;\cdot)|_{z = z(\omega)}.
\]
For the stochastic mixed problems (2.9) and (2.10), almost surely,
\[
\sigma^\varepsilon(\omega;\cdot) = \sigma^\varepsilon(z;\cdot)|_{z = z(\omega)}, \quad \text{and} \quad u^\varepsilon(\omega;\cdot) = u^\varepsilon(z;\cdot)|_{z = z(\omega)}.
\]
For the stochastic mixed problems (2.13) and (2.14) almost surely
\[
u^\varepsilon(\omega;\cdot) = u^\varepsilon(z;\cdot)|_{z = z(\omega)}, \quad \text{and} \quad p^\varepsilon(\omega;\cdot) = p^\varepsilon(z;\cdot)|_{z = z(\omega)}.
\]
3.3 Multiscale homegenized problems

We study the multiscale problems via multiscale convergence. We first recall the concept of multiscale convergence developed byNguetseng [10], Allaire [1] and Allaire and Briane [2].

**Definition 3.3** A sequence \( \{w^\varepsilon\}_\varepsilon \subset L^2(D) \) n + 1-scale converges to a function \( u^0 \in L^2(D \times Y) \) if

\[
\lim_{\varepsilon \to 0} \int_D w^\varepsilon(x) \phi(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}) dx = \int_D \int_Y u^0(x, y) \phi(x, y) dy dx \quad \forall \phi \in L^2(D, C_0^\infty(Y)).
\]

We have the following results whose proofs can be found in [10], [1] and [2].

**Proposition 3.4** From a bounded sequence in \( L^2(D) \), there exists an n + 1-scale convergent subsequence.

**Proposition 3.5** From a bounded sequence \( \{w^\varepsilon\}_\varepsilon \in H^1(D) \), there exists a subsequence (not renumbered) so that \( w^\varepsilon \) converges weakly to \( u^0 \) in \( H^1(D) \), and n functions \( w^i \in L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_0(Y_i)/\mathbb{R}) \) \((i = 1, \ldots, n)\) such that \( \nabla w^\varepsilon \) n + 1-scale converges to \( \nabla_x u^0 + \nabla_{y_1} u^1 + \ldots + \nabla_{y_n} u^n \).

We denote by \( V_i = L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_0(Y_i)/\mathbb{R}) \), and \( V = V \times V_1 \times \ldots \times V_n \). The space \( V \) is equipped with the norm

\[
\|v\|_V = \|v^0\|_V + \sum_{i=1}^n \|v^i\|_{V_i}
\]

for \( v = (v^0, v^1, \ldots, v^n) \in V \). For \( v \in V \), for \( i, j = 1, \ldots, d \), we denote by

\[
\epsilon_{ij}(v) = \frac{1}{2} \left[ \frac{\partial v^0}{\partial x_i} \frac{\partial v^j}{\partial y_i} + \frac{\partial v^j}{\partial x_i} \frac{\partial v^0}{\partial y_i} + \frac{\partial v^i}{\partial y_i} \frac{\partial v^j}{\partial x_i} + \frac{\partial v^j}{\partial y_i} \frac{\partial v^i}{\partial x_i} \right].
\]

We have the following Korn type inequality.

**Lemma 3.6** There is a constant \( c \) such that for all \( v \in V \),

\[
\|\epsilon(v)\|_{L^2(D \times Y)^{d \times d}} \geq c \|v\|_V. \tag{3.11}
\]

**Proof** For \( v = (v^0, v^1, \ldots, v^n) \in V \), we have

\[
\|\epsilon(v)\|_{L^2(D \times Y)^{d \times d}} = \|\epsilon_{\varepsilon}(v^0)\|_H^2 + \sum_{i=1}^n \|\epsilon_{y_i}(v^i)\|_{L^2(D \times Y_1 \times \ldots \times Y_i)^{d \times d}}^2.
\]

We get the conclusion from Korn’s inequality for functions in \( V \) and for functions in \( (H^1(Y)/\mathbb{R})^d \). \( \square \)

For problem (3.2), we have the following result.

**Proposition 3.7** For each \( z \in U \), there is \( u(z; \cdot, \cdot) = (u^0, u^1, \ldots, u^n) \in V \) such that the parametric solution \( u^\varepsilon(z; \cdot) \) of problem (3.2) converges weakly to \( u^0(z; \cdot) \in V \) and \( \epsilon(u^\varepsilon) \) n + 1-scale converges to \( \epsilon(u(z; \cdot, \cdot)) \). The function \( u \) satisfies the problem

\[
b(z; u, v) := \int_D \int_Y a(z; x, y) \epsilon(u)(z; x, y) : \epsilon(v)(x, y) dy dx = \int_D f(x) \cdot v^0(x) dx \tag{3.12}
\]

for all \( v = (v^0, v^1, \ldots, v^n) \in V \). Problem (3.12) is well-posed. There is a constant \( c \) such that

\[
\|u(z; \cdot, \cdot)\|_V \leq c \|f\|_V, \quad \forall z \in U. \tag{3.13}
\]

**Proof** The proof of this proposition is standard, see, e.g., [1]. As \( u^\varepsilon(z; \cdot) \) is uniformly bounded in \( V \), there is a subsequence (not renumbered), a function \( u = (u^0, u^1, \ldots, u^n) \in V \) so that \( \epsilon_{\varepsilon ij}(u^\varepsilon) \) n + 1-scale converges to \( \epsilon_{ij}(u^0) + \epsilon_{y_i y_j}(u^1) + \ldots + \epsilon_{y_n y_j}(u^n) \). By selecting the test function in (3.2) as

\[
v(x) = v^0(x) + \sum_{i=1}^n \epsilon_i v^i(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_i})
\]

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for \(v^0 \in \mathcal{D}(D)^d\) and \(v^i(x,y_1,\ldots,y_i) \in \mathcal{D}(D,C(Y_1 \times \ldots \times Y_i))^d\) we get (3.12) from a density argument. The coercivity of \(b\) follows from \(b(z;u,u) \geq c_1\|\epsilon(u)\|^2_{L^2(D \times Y)}\) and (3.14). The boundedness of \(b\) follows from (2.11). Problem (3.13) thus has a unique solution that satisfies (3.15).

Let \(\mathcal{H} = L^2(D \times Y; \mathbb{R}^{d \times d})\). Let \(\mathcal{X} = \mathcal{H} \times \mathcal{V}\) with the norm \(\|(\tau,v)\|_{\mathcal{H}} = \|\tau\|_{\mathcal{H}} + \|v\|_{\mathcal{V}}\). We define the bilinear forms \(b_1 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}\) and \(b_2 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}\) as

\[
\begin{align*}
\quad & b_1((\sigma,u),(\tau,v)) = \int_D \int_Y [a(\xi(x,y))(\sigma(x,y) - \tau(x,y)) + (\sigma(x,y) - \tau(x,y))(\epsilon(u)(x,y))] \, dy \, dx, \\
\quad & b_2((\sigma,u),(\tau,v)) = \int_D \int_Y \epsilon(\tau,x,y) : \epsilon(u)(x,y) \, dy \, dx.
\end{align*}
\]

We define the linear form \(f : \mathcal{V} \rightarrow \mathbb{R}\) as

\[
f(v) = -\int_D f(x) \cdot v^0(x) \, dx
\]

for \(v = (v^0,v^1,\ldots,v^n) \in \mathcal{V}\). We have the following results.

**Proposition 3.8** For problems (2.13) and (2.14), the sequence \(\sigma^x_n + 1\)-scale converges to \(\sigma \in \mathcal{H}\) and there is a function \(u = (u^0,u^1,\ldots,u^n) \in \mathcal{V}\) such that for all \(i,j = 1,\ldots,d\), \(\epsilon_{ij}(u^n)\) \(n + 1\)-scale converges to \(\epsilon_{ij}(u)\). The functions \(\sigma\) and \(u\) satisfy

\[
b_1(z;\sigma,u),(\tau,v)) = F(v), \quad (\forall \tau,v) \in \mathcal{X}.
\]

and

\[
b_2(z;\sigma,u),(\tau,v)) = F(v), \quad (\forall \tau,v) \in \mathcal{X}.
\]

Problems (3.14) and (3.15) possess a unique solution. There are constants \(\chi_1\) and \(\chi_2\) such that

\[
\inf_{(\sigma,u)} \sup_{(\tau,v)} b_1(z;\sigma,u),(\tau,v)) \|z\|_{\mathcal{H}} \geq \chi_1, \quad \text{and} \quad \inf_{(\sigma,u)} \sup_{(\tau,v)} b_2(z;\sigma,u),(\tau,v)) \|z\|_{\mathcal{H}} \geq \chi_2 \quad \forall z \in U.
\]

**Proof** From (3.3), there is a \(z^0 \in \mathcal{H}\) and \(u(z^0) = (u^0,u^1,\ldots,u^n) \in \mathcal{V}\) and a subsequence (not renumbered) so that \(\sigma^x_n \rightarrow \sigma\) \(n + 1\)-scale converges to \(\sigma\), \(u^0 \rightarrow u^0\) in \(H^1(D)\) and \(\epsilon_{ij}(u^n)\) \(n + 1\)-scale converges to \(\epsilon_{ij}(u^n) + \epsilon_{ij}(u^0)\). Let \(\tau(x,y) \in C(D \times Y)^{d \times d}\) and \(v^0 \in D(D)^d\) and \(v^i \in \mathcal{D}(D,C(Y_1 \times \ldots \times Y_i))^d\). Let \(\tau(x,y) = \tau(x,y)^t - \frac{\tau(x,y)}{\xi} \frac{\tau(x,y)}{\xi}\) and \(v(x) = v^0(x) + \sum_{i=1}^n \epsilon_{ij}(x)\frac{\tau(x,y)}{\xi}\) be the test function in (3.14). We then get (3.14). We derive problem (3.15) similarly.

The mapping \(\mathcal{H} : \tau \mapsto a\tau \in \mathcal{H}\) is one-to-one so problems (3.14) and (3.15) are equivalent. The inf-sup condition for \(b_1\) follows from a standard procedure for mixed elasticity problem (see, e.g., [5]), using the uniform coerciveness of \(a^{-1}\) and Korn’s inequality (3.11).

For a function \(v = (v^0,v^1,\ldots,v^n) \in \mathcal{V}\), we define by

\[
\quad \text{div} v = \text{div}_x v^0 + \text{div}_y v^1 + \ldots + \text{div}_y v^n.
\]

Let \(\mathcal{H} = L^2(D \times Y)\). For mixed problems (2.13) and (2.14), we need the following result.

**Lemma 3.9** There is a constant \(c_0\) such that \(\forall q \in \mathcal{H}\), there is a function \(v \in \mathcal{V}\) so that

\[
\quad \text{div} v(x,y) = q(x,y) \quad \text{and} \quad \|v\|_{\mathcal{V}} \leq \frac{\|q\|_{\mathcal{H}}}{c_0}.
\]

**Proof** We denote by

\[
\begin{align*}
Q_0(x) &= \int_Y q(x,y) \, dy, \quad Q_n(x,y_1,\ldots,y_n) = q(x,y) - \int_{Y_n} q(x,y) \, dy_n, \\
Q_i(x,y_1,\ldots,y_i) &= \int_{Y_{i+1}} \ldots \int_{Y_n} q(x,y) \, dy_n \ldots dy_{i+1} - \int_{Y_{i+1}} \ldots \int_{Y_n} q(x,y) \, dy_n \ldots dy_i, \quad i = 1,\ldots,n-1.
\end{align*}
\]
Then \( q(x, y) = Q_0(x) + Q_1(x, y_1) + \ldots + Q_n(x, y_1, \ldots, y_n) \). Since \( \text{meas}(\partial D \setminus \Gamma) > 0 \), there is a function \( v^0 \in V \) and a constant \( c \) such that \( \text{div}_x v^0(x) = Q_0(x) \), and \( \|v^0\|_V \leq \|Q_0\|_{H/c} \) (see [12] Lemma 4.9 page 181). For \( i = 1, \ldots, n \), there is a function \( v_i \in V \) such that

\[
\text{div}_x v_i(x, y_1, \ldots, y_i) = Q_i(x, y_1, \ldots, y_i), \quad \text{and} \quad \|v_i(x, y_1, \ldots, y_{i-1}, \cdot)\|_{H^1(Y_i)} \leq \frac{\|Q_i(x, y_1, \ldots, y_{i-1}, \cdot)\|_{L^2(Y_i)}}{c_i}
\]

where the constant \( c_i \) is independent of \( q \). We note that

\[
\|Q_0\|_{L^2(D)} \leq \|q\|_{L^2(D \times Y)}, \quad \text{and} \quad \|Q_i\|_{L^2(D \times Y_1 \times \ldots \times Y_i)} \leq 2\|q\|_{L^2(D \times Y)}.
\]

From these we get the conclusion.

Let \( X = V \times H \) with the norm \( \|(u, p)\|_X = \|u\|_V + \|p\|_H \). We define the bilinear forms \( b_3, b_4 : X \times X \to \mathbb{R} \) as

\[
b_3(z; (u, p), (v, q)) = \int_D \int_Y \left[ 2\mu(z; x, y)e(u(z; x, y)) : e(v(x), y) + \text{div}(v(x, y)p(z; x, y)) \right] dydx
\]

\[
+ \int_D \int_Y \left[ \text{div}u(z; x, y)q(x, y) - \frac{1}{\lambda(z; x, y)}p(z; x, y)q(x, y) \right] dydx
\]

and

\[
b_4(z; (u, p), (v, q)) = \int_D \int_Y \left[ 2\mu(z; x, y)e(u(z; x, y)) : e(v(x), y) + \text{div}(v(x, y)p(z; x, y)) \right] dydx
\]

\[
+ \int_D \int_Y \left[ \frac{\lambda(z; x, y)}{\lambda_{\min}} \text{div}u(z; x, y)q(x, y) - \frac{1}{\lambda_{\min}}p(z; x, y)q(x, y) \right] dydx.
\]

We have the following results.

**Proposition 3.10** For problems (3.17) and (3.18), the sequence \( p^* \) \((n+1)\)-scale converges to \( p \in H \) and there is a function \( u = (u^0, u^1, \ldots, u^n) \in V \) such that \( \epsilon_{i,j}(u^*) \) \((n+1)\)-scale converges to \( \epsilon(u) \). The functions \( p \) and \( u \) satisfy:

\[
b_3(z; (u, p), (v, q)) = -f(v), \quad \forall (v, q) \in X,
\]

and:

\[
b_4(z; (u, p), (v, q)) = -f(v), \quad \forall (v, q) \in X.
\]

**Proof** The proof of the limiting problems (3.17) and (3.18) is standard. We use \( v(x) = v^0(x) + \sum_{i=1}^n \epsilon_i v_i(x, \frac{x_1}{\epsilon_1}, \ldots, \frac{x_n}{\epsilon_n}) \), where \( v^0 \in D(D)^d \) and \( v^i \in D(D, C^\#_i (Y_1 \times \ldots \times Y_i))^d \) and \( q(x, \frac{x_1}{\epsilon_1}, \ldots, \frac{x_n}{\epsilon_n}) \) where \( q(x, y) \in C(D, C^\#_i (Y)) \) as test functions.

Due to (3.17), \( \frac{x}{\epsilon} \) is uniformly bounded above and below with respect to \( x \) and \( y \) so for all \( q \in H \), \( \frac{x}{\epsilon} \in H \). Problems (3.17) and (3.18) are equivalent. The \( \text{sup} \) condition of \( b_3 \) follows the standard procedure for mixed problem with a penalty term (see [5]) using Lemma 3.9. We note that the norm \( \|v\|_V + \|q\|_H + \|q/\lambda^2\|_H \) is equivalent to \( \|(v, q)\|_X \) uniformly with respect to \( z \in U \). This norm equivalence is uniform with respect to \( \lambda \) if \( \lambda_{\min} > \vartheta \) and \( \kappa < \kappa_0 \) for fixed constants \( \vartheta \) and \( \kappa_0 \). For each \( (u, p), p) \in X \),

\[
\sup_{(v, q) \in X} \frac{b_3(z; (u, p), (v, q))}{\|(u, p)\|_X \|(v, q)\|_X} = \sup_{(v, q) \in X} \frac{b_3(z; (u, p), (v, \frac{1}{\lambda_{\min}}q))}{\|(u, p)\|_X \|(v, \frac{1}{\lambda_{\min}}q)\|_X \|(v, q)\|_X}
\]

(3.20)

The \( \text{sup} \) condition for \( b_4 \) follows from the inequality \( \frac{\lambda q}{\lambda_{\min}} \|q\|_H \geq \frac{1}{1 + \kappa} \|q\|_H \).

We have the following measurability results. \( \square \)
Proposition 3.11 Solution $u(z;\cdot,\cdot)$ of problem (3.12) as a map from $U$ to $V$ is measurable. Solution $(\sigma(z;\cdot,\cdot), u(z;\cdot,\cdot)))$ of problems (3.14) and (3.15) as a map from $U$ to $X$ is measurable. Solution $(u(z;\cdot,\cdot), p(z;\cdot,\cdot))$ of problems (3.17) and (3.18) as a map from $U$ to $X$ is measurable.

The proof is similar to that of Proposition 3.1 so we do not present it in details here.

4 Approximation of the displacement problem (3.12)

Let $\mathbf{V} = L^2(U, \rho; V)$. From (3.13) and Proposition 3.1, $u$ as a function of $z, x$ and $y$ belongs to $\mathbf{V}$. We define the bilinear form $B : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and the linear form $F : \mathbf{V} \to \mathbb{R}$ as

$$B(u, v) = \int_U b(z; u, v) d\rho(z), \quad F(v) = \int_D f(x) \cdot v^0(z; x) dx d\rho(z),$$

where $b$ is the bilinear form in (3.12). We consider the problem: Find $u \in \mathbf{V}$ so that

$$B(u, v) = F(v) \quad \forall v \in \mathbf{V}. \quad (4.1)$$

As $b$ is uniformly coercive and bounded with respect to $z$ so $B$ is bounded and coercive. Problem (4.1) has a unique solution. To approximate this solution, we identify a basis of $L^2(U, \rho)$.

4.1 Orthonormal basis of $L^2(U, \rho)$

Let $\mathcal{F}$ be the set of all sequences $\nu = (\nu_j)_{j \geq 1}$ of non-negative integers $\nu_j$ such that only finitely many $\nu_j$ are nonzero. We consider the Legendre polynomials $L_\nu(t)$ normalized so that

$$\int_{-1}^{1} \frac{1}{2} L_\nu(t)^2 dt = 1.$$

As $L_{\nu_j}(t) = 1$ when $j$ is sufficiently large, we can define the multivariate Legendre polynomials as

$$L_\nu(z) = \prod_{j \geq 1} L_{\nu_j}(z_j)$$

which form an orthonormal basis of $L^2(U, \rho)$. We can therefore write $u$ as

$$u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu(z) \quad \text{where} \quad u_\nu = \int_U u(z; \cdot, \cdot) L_\nu(z) d\rho(z). \quad (4.2)$$

4.2 Semidiscrete Galerkin approximation in $z$

Let $\Lambda$ be a finite subset of $\mathcal{F}$. We define by

$$\mathbf{V}_\Lambda = \{ v_\Lambda \in \mathbf{V} : v_\Lambda(z; x, y) = \sum_{\nu \in \Lambda} v_\nu(x, y) L_\nu(z), v_\nu \in \mathbf{V} \} \subset \mathbf{V}. $$

We consider the semidiscrete Galerkin approximation: Find $u_\Lambda \in \mathbf{V}_\Lambda$ such that

$$B(u_\Lambda, v_\Lambda) = F(v_\Lambda), \quad \forall v_\Lambda \in \mathbf{V}_\Lambda. \quad (4.3)$$

The following error estimate for the semidiscrete Galerkin problem (4.3) holds.

Lemma 4.1 The solution $u_\Lambda$ of problem (4.3) satisfies

$$\| u - u_\Lambda \|_\mathbf{V} \leq c \left( \sum_{\nu \in \Lambda} \| u_\nu \|_\mathbf{V}^2 \right)^{1/2}.$$

Proof From Cea’s Lemma we have $\| u - u_\Lambda \|_\mathbf{V} \leq c \inf_{v_\Lambda \in \mathbf{V}} \| u - v_\Lambda \|_\mathbf{V}$. Letting $v_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu$, we get the conclusion using the orthonormality of $L_\nu$. □
4.3 Bounds for $\|u_\nu\|_V$

To get the best $N$ term approximation rate for $u$, we deduce bounds for $\|u_\nu\|_V$. For $d_m = \beta_m / \alpha$ where $\beta_m$ and $\alpha$ are the constants in (4.3) and (4.4), we denote by $d = (d_1, d_2, \ldots) \in \mathbb{R}^N$. We have:

**Proposition 4.2** For the solution $u$ of (3.12), there is a constant $C$ independent of $\nu$ such that

$$\|u_\nu(z; \cdot, \cdot)\|_V \leq C|\nu|d^\nu.$$

**Proof** The proof follows the ideas of [7] which has been adapted for stochastic elasticity problems in [17]. We show that there is $C_0$ independent of $\nu$ such that

$$\|\epsilon(\partial_z^\nu u)\|_H \leq C_0|\nu|d^\nu \quad \forall \nu \in F. \quad \text{(4.4)}$$

From (3.13), we have $\|\epsilon(u)(z)\|_H^2 \leq \|v\|_V^2$, so $\|\epsilon(u)(z)\|_H \leq C_0 \forall z \in U$. Differentiating both sides of (3.12), we get

$$\int_D \int_Y a(z; x, y)\epsilon(\partial_z^\nu u)(z; x, y) : \epsilon(v)(x, y) dy dx = - \sum_{m, \nu_m \neq 0} \nu_m \int_D \int_Y \psi_m(x, y)\epsilon(\partial_z^{\nu_m - \nu} u)(z; x, y) : \epsilon(v)(x, y) dy dx$$

where $e_m$ is the $m$th unit vector in $\mathbb{N}^N$. Therefore

$$\|\epsilon(\partial_z^\nu u)(z; \cdot, \cdot)\|_H \leq \sum_m \nu_m \frac{\beta_m}{\alpha} \|\epsilon(\partial_z^{\nu_m - \nu} u)(z; \cdot, \cdot)\|_H.$$

Assuming that (4.4) holds for $\nu - e_m$, then

$$\|\epsilon(\partial_z^\nu u)(z; \cdot, \cdot)\|_H \leq C_0 \sum_m \nu_m d_m (|\nu| - 1)!d^{\nu - \nu_m} = C_0|\nu|d^\nu.$$

Using (3.11), we get $\|\partial_z^\nu u(z; \cdot, \cdot)\|_V \leq C|\nu|d^\nu$ where $C$ is independent of $\nu$. Let $d_m = d_m / \sqrt{\alpha}$, $d = (d_1, d_2, \ldots)$. We establish the following bounds.

**Proposition 4.3** For the expansion (4.2), for a constant $C$ independent of $\nu$

$$\|u_\nu(z; \cdot, \cdot)\|_V \leq \frac{C|\nu|!d^\nu}{\nu!d^\nu}.$$

The proof of this proposition uses formula (4.2) and an integration by parts argument following exactly the procedure in Section 6 of Cohen et al. [7], using the bound in Proposition 4.2.

4.4 Best $N$-term approximation for the solution $u$ of problem (3.12)

We deduce the rate of convergence for the best $N$-term approximation for the solution $u$. To do so we first establish the summability of $(\|u_\nu\|)_\nu$. We first assume the following.

**Assumption 4.4** The sequence $(\beta_m)_m$ belongs to $\ell^p(\mathbb{N})$ for a constant $0 < p < 1$.

For the summability of $(\|u_\nu\|)_\nu$ we employ the following result which is proved in [7].

**Lemma 4.5** The sequence $(\frac{\nu!d^\nu}{\nu!d^\nu})_\nu \in \ell^p(\mathcal{F})$ if and only if $\|d\|_{\ell^p(\mathbb{N})} < 1$ and $d \in \ell^p(\mathbb{N})$.

From (4.3) and (4.4) we have that $(1/\alpha) \sum_{m=1}^\infty \beta_m \leq \kappa$ so $\sum_{m=1}^\infty \bar{d}_m < 1$ when $\kappa < \sqrt{\alpha}$. This, together with Assumption 4.4 implies $(\frac{\nu!d^\nu}{\nu!d^\nu})_\nu \in \ell^p(\mathcal{F})$. We therefore have

**Proposition 4.6** Under Assumption 4.4 if $\kappa \leq \sqrt{\alpha}$, then $(\|u_\nu\|_V)_\nu \in \ell^p(\mathcal{F})$.

The rate of convergence of the best $N$ term approximation is deduced using the following result.
Lemma 4.7 (Stechkin) Let \((b_n)\) be a decreasing sequence of positive numbers, then for \(0 < p < q\)

\[
\left( \sum_{n>N} b_n^q \right)^{1/q} \leq N^{1/q-1/p} \left( \sum_{n\geq 1} b_n^p \right)^{1/p}.
\]

With these results, we then have.

Theorem 4.8 Under Assumption 4.4 with \(\kappa < \sqrt{3}\), for any \(N\) there is a set \(\Lambda \subset F\) of cardinality not larger than \(N\) such that the error of the semidiscrete Galerkin approximation problem (4.3) satisfies

\[
\|u - u_\Lambda\|_V \leq CN^{-s},
\]

where \(C\) is independent of \(N\) and \(s = 1/p - 1/2\).

Proof Let \(\Lambda\) be the set of index sequences \(\nu \in F\) corresponding to the \(N\) terms \(u_\nu\) in (4.2) with the largest norms \(\|u_\nu\|_V\). We then get the rate of convergence from Lemma 4.7. \(\blacksquare\)

5 Approximation of mixed problems (3.14) and (3.15)

We consider the polynomial chaos approximation for problems (3.14) and (3.15) in this section.

5.1 Deterministic parametric Hellinger-Reissner mixed problems

From Propositions 3.8, the solution of (3.14) and (3.15) satisfies \(\sigma \in L^2(U, \rho; H) := H\) and \(u \in L^2(U, \rho; V) := V\). Let \(X = H \times V\). We define the bilinear forms \(B_1, B_2 : X \times X \to \mathbb{R}\) and the linear form \(F : X \to \mathbb{R}\) as

\[
B_1((\sigma, u), (\tau, v)) = \int_U b_1(z; (\sigma, u), (\tau, v))d\rho(z), \quad B_2((\sigma, u), (\tau, v)) = \int_U b_2(z; (\sigma, u), (\tau, v))d\rho(z),
\]

\[
F((\tau, v)) = -\int_U \int_D f(x) \cdot v^0(z; x)dxd\rho(z), \quad v = (v^0, v^1, \ldots, v^n) \in V.
\]

We consider problems: Find \((\sigma, u) \in X\) such that

\[
B_1((\sigma, u), (\tau, v)) = F((\tau, v)) \quad \forall (\tau, v) \in V \tag{5.1}
\]

and

\[
B_2((\sigma, u), (\tau, v)) = F((\tau, v)) \quad \forall (\tau, v) \in V. \tag{5.2}
\]

Proposition 5.1 Problems (5.1) and (5.2) are equivalent and well-posed.

Proof The proof for the well-posedness of (5.1) is standard. As \(\tau \mapsto a\tau\) is a one-to-one map from \(H\) to \(H\); these two problems are equivalent. The inf-sup condition for \(B_1\) follows from the standard procedure. \(\blacksquare\)

5.2 Semidiscrete Galerkin approximation in \(z\) of (5.1) and (5.2)

We write \(\sigma\) and \(u\) in terms of the multivariate Legendre polynomials \(L_\nu\) as

\[
\sigma = \sum_{\nu \in F} \sigma_\nu L_\nu, \quad u = \sum_{\nu \in F} u_\nu L_\nu, \quad \sigma_\nu \in H, \quad u_\nu \in V.
\]

Let \(\Lambda\) be a subset of \(F\) of finite cardinality. Let

\[
V_\Lambda = \{v_\Lambda \in V : v_\Lambda = \sum_{\nu \in \Lambda} v_\nu L_\nu, \nu_\nu \in V\}, \quad H_\Lambda = \{\tau_\Lambda \in H : \tau_\Lambda = \sum_{\nu \in \Lambda} \tau_\nu L_\nu, \tau_\nu \in H\}
\]
and $X^f_A = H_A \times V_A$. We consider the semidiscrete problems: Find $(\sigma_A, u_A) \in X_A$ so that

$$B_1((\sigma_A, u_A), (\tau_A, v_A)) = F((\tau_A, v_A)) \quad \forall (\tau_A, v_A) \in X_A$$

and: Find $(\sigma_A, u_A) \in X_A$ so that

$$B_2((\sigma_A, u_A), (\tau_A, v_A)) = F((\tau_A, v_A)) \quad \forall (\tau_A, v_A) \in X_A.$$ 

These problems are not equivalent as generally for $\tau \in H_A$, $a_{\tau}$ is not in $H_A$. We therefore establish their well-posedness separately.

**Proposition 5.2** Problem $\mathcal{H}_A^f$ is well-posed.

**Proof** The proof of this proposition follows standard proof for saddle point problems. □

For problem $\mathcal{H}_A^f$, we assume that there is a positive constant $\tilde{\kappa}$ such that

$$\sum_{m=1}^{\infty} \beta_m \leq \frac{\tilde{\kappa} \alpha_0^2}{1 + \kappa \alpha_0 + \beta_0}$$

which is stronger than $\mathcal{H}_A^f$. With this assumption, we have.

**Proposition 5.3** If $\mathcal{H}_A^f$ holds, problem $\mathcal{H}_A^f$ is well-posed.

**Proof** We adapt the proof in Xia and Hoang [17] for parametric elasticity equations. We define

$$\mathcal{H}_A^f = \{ \zeta \in H^f : \zeta = \epsilon(v), v \in V \}.$$

From inequality $\mathcal{H}_A^f$, this is a closed subspace of $\mathcal{H}_A$. We define

$$\mathcal{H}_A^f = \{ \zeta_A \in H^f : \zeta_A = \sum_{\nu \in \Lambda} \epsilon(v_\nu)(x, y)L_\nu(z), v_\nu \in V \}.$$

We define the bilinear forms $\bar{a}_A : H_A \times H_A \to \mathbb{R}$ and $b_A : H_A \times H_A^f \to \mathbb{R}$ as

$$\bar{a}_A(\sigma_A, \tau_A) = \int_U \int_D \int_Y \bar{a}^{-1}(x, y)\sigma_A(z; x, y) : \tau_A(z; x, y) d\gamma(x) d\rho(y),$$

$$b_A(\tau_A, \zeta_A) = -\int_U \int_D \int_Y \tau_A \cdot \zeta_A d\gamma(x) d\rho(y).$$

Let $X^f_A = H_A \times H_A^f$. The bilinear form $B_A : X^f_A \times X^f_A$ is defined as

$$B_A((\sigma_A, \xi_A), (\tau_A, \zeta_A)) = \bar{a}_A(\sigma_A, \tau_A) + b_A(\tau_A, \xi_A) + b_A(\sigma_A, \zeta_A).$$

We define the bilinear from $B_{2A} : X_A^f \times X_A^f \to \mathbb{R}$ as

$$B_{2A}((\sigma_A, \xi_A), (\tau_A, \zeta_A)) = \bar{a}_A(\sigma_A, \tau_A) + b_A(\tau_A, \xi_A) + b_A(\sigma_A, \zeta_A) - \int_U \int_D \int_Y \tau_A(z; x, y) : \left( \sum_{m=1}^{\infty} z_m \bar{a}^{-1}(z; x, y) \psi_m(x, y) \xi_A(z; x, y) \right) d\gamma(x) d\rho(y).$$

Let $K$ be the kernel of the map $H_A \ni \tau_A \mapsto b_A(\tau_A, \cdot) \in (H_A^f)'$. From $\mathcal{H}_A$ we deduce that for all $z \in U$, $x \in D$ and $y \in Y$ all the eigenvalues of the map $\mathbb{R}_+ \ni a \mapsto \bar{a}^{-1} a \in \mathbb{R}_+$ are not larger than $\beta_0$ so all the eigenvalues of the map $\tau \mapsto \bar{a}^{-1} \tau$ are not smaller than $1/\beta_0$. Thus for all $\eta \in \mathbb{R}_+ \ni a^{-1} \eta : \eta \geq ||\eta||_{\mathbb{R}_+}^2/\beta_0$. Then for all $\sigma_A \in K$

$$\sup_{\tau_A \in K} \frac{\bar{a}_A(\sigma_A, \tau_A)}{||\tau_A||_2^{H_A}} \geq \frac{\bar{a}_A(\sigma_A, \sigma_A)}{||\sigma_A||^2_{\mathcal{H}_A}} \geq \frac{1}{\beta_0}.$$

For $\sigma_A \in H_A$, let $\bar{\sigma}_A$ be the orthogonal projection of $\sigma_A$ to $H_A$. For all $\zeta_A \in H_A$, we have

$$\sup_{\zeta_A \in H_A} \frac{B_A((\bar{\sigma}_A, \zeta_A), (\zeta_A, 0))}{||\zeta_A||_{\mathcal{H}_A}} \leq \sup_{\zeta_A \in H_A} \frac{b_A(\bar{\sigma}_A, \zeta_A)}{||\zeta_A||_{\mathcal{H}_A}} \geq \frac{b_A(\bar{\sigma}_A, \bar{\sigma}_A)}{||\bar{\sigma}_A||_{\mathcal{H}_A}} = ||\bar{\sigma}_A||_{\mathcal{H}_A}.$$
Since $\hat{\sigma}_\Lambda$ is the orthogonal projection of $\sigma_\Lambda$ to $H^c_\Lambda$, we have $b_\Lambda(\sigma_\Lambda, \zeta_\Lambda) = b_\Lambda(\hat{\sigma}_\Lambda, \zeta_\Lambda)$. Therefore

$$\|\hat{\sigma}_\Lambda\|_{H^c_\Lambda} \leq \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}}. \quad (5.6)$$

Since $\sigma_\Lambda - \hat{\sigma}_\Lambda \in K$, we have

$$\frac{1}{\beta_0} \|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{H^c_\Lambda} \leq \sup_{\tau_\Lambda \in K} \frac{\bar{a}_\Lambda(\sigma_\Lambda - \hat{\sigma}_\Lambda, \tau_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} = \sup_{\tau_\Lambda \in K} \frac{\bar{a}_\Lambda(\sigma_\Lambda - \hat{\sigma}_\Lambda, \tau_\Lambda) + b_\Lambda(\tau_\Lambda, \zeta_\Lambda) + b_\Lambda(\sigma_\Lambda, 0)}{\|\tau_\Lambda\|_{H^c_\Lambda}}$$

$$\leq \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \|\bar{a}_\Lambda\|_{H^c_\Lambda \times H^c_\Lambda \to \mathbb{R}} \|\hat{\sigma}_\Lambda\|_{H^c_\Lambda}$$

for all $\xi \in H^c_\Lambda$. Using

$$B_{\Lambda}(\sigma_\Lambda, \zeta_\Lambda), (\tau_\Lambda, \zeta_\Lambda) \leq B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda), (\tau_\Lambda, \zeta_\Lambda) + \frac{1}{\alpha_0} \left( \sum_{m=1}^{\infty} \beta_m \right) \|\tau_\Lambda\|_{H^c_\Lambda} \|\xi_\Lambda\|_{H^c_\Lambda},$$

we deduce that

$$\|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{H^c_\Lambda} \leq \frac{\beta_0}{\alpha_0} \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \frac{\beta_0}{\alpha_0} \left( \sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{H^c_\Lambda} + \frac{\beta_0}{\alpha_0} \|\hat{\sigma}_\Lambda\|_{H^c_\Lambda}. \quad (5.7)$$

Thus together with (5.6) we obtain

$$\|\sigma_\Lambda\|_{H^c_\Lambda} \leq \|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{H^c_\Lambda} + \|\hat{\sigma}_\Lambda\|_{H^c_\Lambda}$$

$$\leq \left( \frac{\beta_0}{\alpha_0} + 1 + \frac{\beta_0}{\alpha_0} \right) \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \frac{\beta_0}{\alpha_0} \left( \sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{H^c_\Lambda}.$$

For $\xi_\Lambda \in H^c_\Lambda$, we have

$$\|\xi_\Lambda\|_{H^c_\Lambda} \leq \sup_{\tau_\Lambda \in H^c_\Lambda} \frac{b_\Lambda(\tau_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}}$$

$$\leq \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda) + b_\Lambda(\tau_\Lambda, \zeta_\Lambda) + b_\Lambda(\sigma_\Lambda, 0)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \sup_{\tau_\Lambda \in H^c_\Lambda} \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}}$$

$$\leq \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \|\bar{a}_\Lambda\|_{H^c_\Lambda \times H^c_\Lambda \to \mathbb{R}} \|\sigma_\Lambda\|_{H^c_\Lambda}$$

$$\leq \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \frac{1}{\alpha_0} \left( \sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{H^c_\Lambda} + \frac{1}{\alpha_0} \left( \sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{H^c_\Lambda}$$

$$\leq \left( 1 + \frac{1}{\alpha_0} \beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}} + \left( 1 + \frac{\beta_0}{\alpha_0} \right) \left( \sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{H^c_\Lambda}. \quad (5.8)$$

From (5.8), we deduce that

$$\frac{1}{1 + \xi} \|\xi_\Lambda\|_{H^c_\Lambda} \leq \left( 1 + \frac{1}{\alpha_0} \beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\tau_\Lambda\|_{H^c_\Lambda}}.$$

From (5.7) and (5.8),

$$\inf_{(\sigma_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \sup_{(\tau_\Lambda, \zeta_\Lambda) \in \mathcal{X}_\Lambda} \frac{B_{2\Lambda}(\sigma_\Lambda, \zeta_\Lambda)}{\|\sigma_\Lambda\|_{H^c_\Lambda} \|\tau_\Lambda\|_{H^c_\Lambda}} \geq c$$
for a constant $c$ independent of $\Lambda$. For $B_2$, we have

$$
\frac{B_2([\sigma, u_A], (\tau_A, v_A))}{\| [\sigma, u_A] \|_{\mathcal{X}} \| (\tau_A, v_A) \|_{\mathcal{X}}} = \frac{B_{2A}([\sigma, e(u_A)], (\bar{\sigma}, e(v_A)))}{\| [\sigma, e(u_A)] \|_{\mathcal{X}} \| (\bar{\sigma}, e(v_A)) \|_{\mathcal{X}}},
$$

From (5.11), we get the uniform inf-sup condition with respect to $\Lambda$ for bilinear form $B_2$ in (5.4). □

We therefore have the following result on the error estimates.

**Proposition 5.4** There is a constant $C$ independent of $\Lambda$ such that the solution of problems (5.3) satisfies

$$
\| (\sigma - \sigma_A), (u - u_A) \|_{\mathcal{X}} \leq C \inf_{(\tau, v) \in \mathcal{X}} \| (\sigma - \tau), (u - v) \|_{\mathcal{X}} \leq C \left[ \left( \sum_{\nu \notin A} \| \sigma_{\nu} \|_{H} \right)^{1/2} + \left( \sum_{\nu \notin A} \| u_{\nu} \|_{V} \right)^{1/2} \right].
$$

If condition (5.5) holds, then the solution of problem (5.3) satisfies

$$
\| (\sigma - \sigma_A), (u - u_A) \|_{\mathcal{X}} \leq \inf_{(\tau, v) \in \mathcal{X}} \| (\sigma - \tau), (u - v) \|_{\mathcal{X}} \leq C \left[ \left( \sum_{\nu \notin A} \| \sigma_{\nu} \|_{H} \right)^{1/2} + \left( \sum_{\nu \notin A} \| u_{\nu} \|_{V} \right)^{1/2} \right].
$$

### 5.3 Bounds for $\| u_{\nu} \|_{V}$ and $\| \sigma_{\nu} \|_{H}$

We deduce in this section explicit bounds for $\| u_{\nu} \|_{V}$ and $\| \sigma_{\nu} \|_{H}$. We denote by $\delta = (\delta_1, \delta_2, \ldots)$ where

$$
\delta_m = \frac{(1/\alpha_0 + \beta_0/\alpha_1^2)\beta_m}{1 - (1/\alpha_0 + \beta_0/\alpha_1^2)(\sum_{m=1}^{\infty} \beta_m)}.
$$

**Proposition 5.5** If (5.5) holds, then there is a constant $C$ independent of $\nu$ such that

$$
\| \partial_{x_2}^\nu \sigma \|_{H} + \| \partial_{x_2}^\nu u \|_{V} \leq C \| \nu \|^{1/2}.
$$

**Proof** For each $z \in U$, the solution of (5.1) and (5.2) satisfies $\| (\sigma(z, \cdot), u(z, \cdot)) \|_{\mathcal{X}} \leq c \| f \|_{V}$. Let $\mathcal{X}' = H \times H'$ be equipped with the norm $\| (\tau, \zeta) \|_{\mathcal{X}'} = \| \tau \|_{H} + \| \zeta \|_{H}$. We have

$$
\| (\sigma(z, \cdot), e(u)(z, \cdot)) \|_{\mathcal{X}'} \leq c \| f \|_{V}.
$$

Differentiating (5.19), we have for all $(\tau, v) \in \mathcal{X}$:

$$
\int_{D} \int_{Y} \partial_{x_2}^\nu \sigma(z, x, y) : \tau(x, y) \, dxdy - \int_{D} \int_{Y} \sigma(z, x, y) : \tau(x, y) \, dxdy + \int_{D} \int_{Y} \psi_m(x, y) \epsilon(\partial_{x_2}^\nu u)(x, y) \, dxdy
$$

As $\bar{\Lambda} \ni \tau \mapsto a(z, \cdot, \cdot) \tau \in H$ is one to one, we can rewrite equations (5.9) as: $\forall (\tau, \zeta) \in \mathcal{X}^c$

$$
\int_{D} \int_{Y} \partial_{x_2}^\nu \sigma(z, x, y) : \tau(x, y) \, dxdy - \int_{D} \int_{Y} \tau(x, y) : \epsilon(\partial_{x_2}^\nu u)(z, x, y) \, dxdy
$$

$$
= \sum_{m=1}^{\infty} \nu_m \int_{D} \int_{Y} \tau(x, y) : \psi_m(x, y) \epsilon(\partial_{x_2}^{\nu-1} u)(z, x, y) \, dxdy
$$

$$
+ \sum_{m=1}^{\infty} \int_{D} \int_{Y} \tau(x, y) : \psi_m(x, y) \epsilon(\partial_{x_2}^{\nu} u)(z, x, y) \, dxdy + \sum_{m=1}^{\infty} \int_{D} \int_{Y} \tau(x, y) : \zeta(x, y) \, dxdy
$$

We have the following inf-sup conditions

$$
\inf_{\sigma \in \mathcal{H}} \sup_{\tau \in \mathcal{H}} \int_{D} \int_{Y} a^{-1}(x, y) \sigma(x, y) : \tau(x, y) \, dxdy \geq \frac{1}{\beta_0}, \quad \text{and} \quad \inf_{\zeta \in \mathcal{H}} \sup_{\tau \in \mathcal{H}} \int_{D} \int_{Y} \tau(x, y) : \zeta(x, y) \, dxdy \geq 1.
$$

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Further for all $\sigma$ and $\tau$ in $\mathcal{H}$,
\[
\left| \int_D \int_Y \tilde{a}^{-1}(x, y) \xi(x, y) : \zeta(x, y) dy dx \right| \leq \frac{1}{\alpha_0} \|\xi\|_{\mathcal{H}} \|\zeta\|_{\mathcal{H}}.
\]
Using standard estimates for solutions of saddle point problems (Theorem 2.31 of [9]), we have
\[
\|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq \left( 1 + \frac{\beta_0}{\alpha_0} \right) \left( \frac{1}{\alpha_0} \sum_{m=1}^\infty \nu_m \beta_m \|\varepsilon(\partial^{\nu\kappa}_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \right)
\]
\[
+ \frac{1}{\alpha_0} \left( \sum_{m=1}^\infty \beta_m \right) \|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}},
\]
which implies
\[
\left( 1 - \left( \frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0} \right) \sum_{m=1}^\infty \beta_m \right) \|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq \left( \frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0} \right) \sum_{m=1}^\infty \nu_m \beta_m \|\varepsilon(\partial^{\nu\kappa}_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}}.
\]
Therefore
\[
\|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq \sum_{m=1}^\infty \nu_m \delta_m \|\varepsilon(\partial^{\nu\kappa}_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}}.
\]
By induction, assuming that $\|\varepsilon(\partial^{\nu\kappa}_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq c(\nu| - 1)\delta^{\nu\kappa}$ for all $\zeta \in U$, then
\[
\|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq \sum_{m=1}^\infty \nu_m (\nu| - 1)\delta_m \delta^{\nu\kappa} = c|\nu|\delta^{\nu}\delta^\kappa.
\]
From inequality 3.3, we have $\|\partial^\nu_x u(z)\|_K \leq c|\nu|\delta^\nu$. From standard estimates for solutions of saddle point problems and 5.10, we get
\[
\|\partial^\nu_x \sigma(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \leq \beta_0 \left( \frac{1}{\alpha_0} \sum_{m=1}^\infty \nu_m \beta_m \|\varepsilon(\partial^{\nu\kappa}_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}} \right) + \frac{1}{\alpha_0} \left( \sum_{m=1}^\infty \beta_m \right) \|\varepsilon(\partial^\nu_x u)(\zeta; \cdot, \cdot)\|_{\mathcal{H}}
\]
\[
\leq \beta_0 \left( \frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0} \right) \sum_{m=1}^\infty \beta_m \sum_{m=1}^\infty \nu_m \delta_m (\nu| - 1)\delta^{\nu\kappa} + \frac{\beta_0}{\alpha_0} \left( \sum_{m=1}^\infty \beta_m \right) c|\nu|\delta^{\nu\kappa}
\]
\[
\leq C|\nu|\delta^{\nu\kappa}.
\]
We have the estimates:

Proposition 5.6 If condition (5.5) holds, then
\[
\|(\sigma_{\nu}, u_{\nu})\|_{K} \leq C|\nu|\delta^{\nu}.
\]

The proof of this proposition is similar to that of Proposition 4.3

5.4 Best $N$ term convergence rate

We deduce the rate of convergence for the semidiscrete Galerkin problems (5.1) and (5.4).

Proposition 5.7 Under Assumption 4.2 if (5.3) holds with $\tilde{\nu} < \sqrt{3}$ then $\|(\sigma_{\nu}, u_{\nu})\|_{K} \in \ell^p(F)$.

Proof For $\left( \frac{\nu!}{\nu^!} \delta^{\nu} \right)_{\nu \in F}$ to be in $\ell^p(F)$ we need $\sum_m \delta_m < 1$, i.e.
\[
\left( \frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0} \right) \sum_{m=1}^\infty \beta_m < \sqrt{3} - \sqrt{3} \left( \frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0} \right) \sum_{m=1}^\infty \beta_m
\]
\[
\sum_{m=1}^{\infty} \beta_m < \frac{\sqrt{3}}{1 + \sqrt{3} \alpha_0 + \beta_0}
\]
which holds when \( \kappa < \sqrt{3} \).

Let \( s = 1/p - 1/2 \). We then deduce

**Theorem 5.8** Under Assumption [4.4] if (5.5) holds with \( \kappa < \sqrt{3} \) then for any integer \( N \), there is a set \( \Lambda \subset \mathcal{F} \) of cardinality not larger than \( N \) such that the error of the semidiscrete Galerkin problems (5.3) and (5.4) satisfies

\[
\|\sigma - \sigma_{\Lambda}\|_{\mathcal{H}} + \|u - u_{\Lambda}\|_{\mathcal{V}} \leq cN^{-s},
\]
where \( c \) does not depend on \( N \).

The proof of this theorem uses Lemma 4.7 and Proposition 5.7.

### 6 Approximations for mixed problems (3.17) and (3.18)

We consider approximation for mixed problems for nearly incompressible materials in this section. We show that the best \( N \)-term convergence rate does not depend on the ratio of the Lamé constants when this ratio goes to \( \infty \). As stated in Remark 2.1, we restrict our consideration to the case where \( \text{meas}(\partial D \setminus \Gamma) > 0 \).

#### 6.1 Deterministic parametric mixed problems for nearly incompressible materials

From Proposition 3.11 and the uniform boundedness of \( \|u(z; \cdot, p(z; \cdot))\|_{\mathcal{X}} \), the solution of (3.17) and (3.18) satisfies \( u \in L^2(U, \rho; \mathcal{V}) := \mathcal{V} \) and \( p \in L^2(U, \rho; \mathcal{H}) := \mathcal{H} \). Let \( \mathcal{X} = \mathcal{V} \times \mathcal{H} \). We define the bilinear forms \( B_3, B_4 : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) as

\[
B_3((u, p), (v, q)) = \int_U b_3(z; (u, p), (v, q)) \, d\rho(z), \quad B_4((u, p), (v, q)) = \int_U b_4(z; (u, p), (v, q)) \, d\rho(z).
\]

The linear form \( F : \mathcal{X} \to \mathbb{R} \) is defined as

\[
F((v, q)) = \int_U \int_D f(x) \cdot v^0(z; x) \, dxd\rho(z).
\]

We consider problems:

Find \( (u, p) \in \mathcal{X} \) such that \( B_3((u, p), (v, q)) = F((v, q)), \quad \forall (v, q) \in \mathcal{X} \) \hspace{2cm} (6.1)

and:

Find \( (u, p) \in \mathcal{X} \) such that \( B_4((u, p), (v, q)) = F((v, q)), \quad \forall (v, q) \in \mathcal{X} \) \hspace{2cm} (6.2)

**Proposition 6.1** Problems (6.1) and (6.2) are equivalent and possess a unique solution.

The proof of this proposition follows standard procedure for saddle point problems with a penalty term.

#### 6.2 Semidiscrete Galerkin approximation in \( z \) of (6.1) and (6.2)

As \( u \in L^2(U, \rho; \mathcal{V}) := \mathcal{V} \) and \( p \in L^2(U, \rho; \mathcal{H}) := \mathcal{H} \), we can write them as

\[
u \in \mathcal{F}
\]

For a subset \( \Lambda \subset \mathcal{F} \) of finite cardinality, we denote by

\[
\mathcal{V}_{\Lambda} = \left\{ v_{\Lambda} \in \mathcal{V} : v_{\Lambda} = \sum_{\nu \in \Lambda} v_{\nu} L_{\nu} \right\}, \quad \mathcal{H}_{\Lambda} = \left\{ p_{\Lambda} \in \mathcal{H} : p_{\Lambda} = \sum_{\nu \in \Lambda} p_{\nu} L_{\nu} \right\},
\]

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and $\mathbf{X}_\Lambda = \mathbf{V}_\Lambda \times \mathbf{H}_\Lambda$. We consider the semidiscrete Galerkin approximation for (6.1) and (6.2) as follows:

Find $(u_\Lambda, p_\Lambda) \in \mathbf{X}_\Lambda$ so that $B_3((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda)) = F((v_\Lambda, q_\Lambda))$, $\forall (v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda$. (6.3) and:

Find $(u_\Lambda, p_\Lambda) \in \mathbf{X}_\Lambda$ so that $B_4((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda)) = F((v_\Lambda, q_\Lambda))$, $\forall (v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda$. (6.4)

We then have the following result.

Proposition 6.2  There exists a constant $\chi'_3 > 0$ independent of $\Lambda$ such that

$$\inf_{(u_\Lambda, p_\Lambda) \in \mathbf{X}_\Lambda} \sup_{(v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda} \frac{B_3((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda))}{\| (u_\Lambda, p_\Lambda) \|_\mathbf{X} \| (v_\Lambda, q_\Lambda) \|_\mathbf{X}} \geq \chi'_3. \quad (6.5)$$

The inf-sup condition (6.5) is the standard result for saddle point problems with a penalty term. For problem (6.1) we have the following result.

Proposition 6.3  Assume that (2.12) holds with $\kappa \leq \kappa_0$ where $\kappa_0 > 0$ is a fixed constant. There is a constant $\vartheta_1$ depending on $\kappa_0$, $\mu$ and the domain $D$ such that if $\Lambda_{\min} > \vartheta_1$ then

$$\inf_{(u_\Lambda, p_\Lambda) \in \mathbf{X}_\Lambda} \sup_{(v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda} \frac{B_4((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda))}{\| (u_\Lambda, p_\Lambda) \|_\mathbf{X} \| (v_\Lambda, q_\Lambda) \|_\mathbf{X}} \geq \chi'_4, \quad (6.6)$$

where $\chi'_4$ is independent of $\Lambda$. Problem (6.4) has a unique solution.

Proof  For $q_\Lambda = \sum_{v \in \Lambda} q_v L_v \in \mathbf{H}_\Lambda$, from Lemma 3.9 we can choose $v_\Lambda \in \mathbf{V}$ so that $\nabla v_\Lambda = q_\Lambda$ and $\| v_\Lambda \|_\mathbf{H} \leq \frac{1}{c_0} \| q_\Lambda \|_\mathbf{H}$. Then $v_\Lambda = \sum_{v \in \Lambda} v_v L_v$ satisfies $\| v_\Lambda \|_\mathbf{V} \leq \frac{1}{c_0} \| q_\Lambda \|_\mathbf{H}$. From this we deduce

$$\inf_{q_\Lambda \in \mathbf{H}_\Lambda} \sup_{v_\Lambda \in \mathbf{V}_\Lambda} \frac{\int_U \int_D \int_Y \nabla v_\Lambda(z, x, y)q_\Lambda(z, x, y) dy dx dp(z)}{\| v_\Lambda \|_\mathbf{V} \| q_\Lambda \|_\mathbf{H}} \geq c_0.$$

Let $\mu^*$ be a constant such that

$$\left| \int_U \int_D \int_Y \mu(z, x, y) e(u(z, x, y)) : e(v(z, x, y)) dy dx dp(z) \right| \leq \mu^* \| u \|_\mathbf{V} \| v \|_\mathbf{V} \quad \forall u, v \in \mathbf{V}. \quad (6.7)$$

Let $(u_\Lambda, p_\Lambda) \in \mathbf{X}_\Lambda$. Adapting the approach in [1], we first consider the case

$$\| u_\Lambda \|_\mathbf{V} \leq \frac{c_0 \| p_\Lambda \|_\mathbf{H}}{4 \mu^*}. \quad (6.8)$$

From (6.7), we get

$$c_0 \| p_\Lambda \|_\mathbf{H} \leq \sup_{v_\Lambda \in \mathbf{V}_\Lambda} \frac{\int_U \int_D \int_Y \nabla v_\Lambda(z, x, y) p_\Lambda(z, x, y) dy dx dp(z)}{\| v_\Lambda \|_\mathbf{V}}$$

$$= \sup_{v_\Lambda \in \mathbf{V}_\Lambda} \frac{B_4((u_\Lambda, p_\Lambda), (v_\Lambda, 0))}{\| (v_\Lambda, q_\Lambda) \|_\mathbf{X}} - 2 \int_U \int_D \int_Y \mu(z, x, y) e(u(z, x, y)) : e(v(z, x, y)) dy dx dp(z) \frac{\| v_\Lambda \|_\mathbf{V}}{\| v_\Lambda \|_\mathbf{V}}$$

$$\leq \sup_{(v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda} \frac{B_4((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda))}{\| (v_\Lambda, q_\Lambda) \|_\mathbf{X}} + 2 \mu^* \| u_\Lambda \|_\mathbf{V} + \frac{c_0}{2} \| p_\Lambda \|_\mathbf{V}. \quad (6.9)$$

Using (6.8), we obtain

$$\sup_{(v_\Lambda, q_\Lambda) \in \mathbf{X}_\Lambda} \frac{B_4((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda))}{\| (v_\Lambda, q_\Lambda) \|_\mathbf{X}} \geq \frac{c_0}{2} \| p_\Lambda \|_\mathbf{V} \geq \min \left\{ \frac{c_0}{4}, \mu^* \right\} \| (u_\Lambda, p_\Lambda) \|_\mathbf{X}.$$
We then consider the case
\[ \|u_A\|_X > \frac{\varepsilon_0 \|p_A\|_H}{4\mu^*}. \] (6.9)

Let \( c_1 > 0 \) be a constant such that
\[ 2 \int_U \int_D \mu(z; x, y) \epsilon(u_A(z; x, y)) : \epsilon(u_A(z; x, y)) \, dy \, dx \geq c_1 \|u_A\|_X^2. \]

Choosing \( q_A = t \div u_A \) we have
\[ B_4((u_A, p_A), (u_A, q_A)) \geq c_1 \|u_A\|_X^2 - \|p_A\|_H \|\div u_A\|_H + \frac{t}{1 + \kappa} \|\div u_A\|_H^2 - \frac{t}{\lambda_{\min}} \|p_A\|_H \|\div u_A\|_H. \]

When \( \lambda_{\min} \geq t \)
\[ B_4((u_A, p_A), (u_A, q_A)) \geq c_1 \|u_A\|_X^2 + \frac{t}{1 + \kappa} \|\div u_A\|_H^2 - 2\|p_A\|_H \|\div u_A\|_H. \]

Using (6.19), there is \( c_5 > 0 \) so that
\[ B_4((u_A, p_A), (u_A, q_A)) \geq c_5 \left( \|u_A\|_X + \|q_A\|_H \right)^2 \]
for all positive constants \( c_5 \). For \( c_4 = \frac{c_1}{2c_2^2} \) and \( \lambda_{\min} \geq \theta_1 = \frac{1}{c_3} \Lambda_0^2 (1 + \kappa) \), we get
\[ B_4((u_A, p_A), (u_A, q_A)) \geq c_5 \left( \|u_A\|_X + \|q_A\|_H \right)^2 \]
for \( c_5 = \frac{1}{2} \min \{ c_2, \frac{c_1}{2c_2^2} \} \). Therefore for a constant \( c_6 \) independent of \( \lambda \)
\[ \frac{B_4((u_A, p_A), (u_A, q_A))}{\|u_A\|_X + \|q_A\|_H} \geq c_6 \left( \|u_A\|_X + \|p_A\|_H \right). \]

### 6.3 Bounds for \( u_\nu \) and \( p_\nu \)

We now establish bounds for \( \|(u_\nu, p_\nu)\|_X \).

**Proposition 6.4** When \( \kappa < \kappa_0 \), there are positive constants \( \theta_2, C_i, i = 1, 2, \ldots, 5 \) depending on \( \kappa_0, \mu \) and the domain \( D \) such that if \( \lambda_{\min} > \theta_2 \), for \( 0 < \zeta < 1 \),
\[ \|\epsilon(\partial_\nu \nu)\|_H + \frac{1}{\lambda_{\min}^\zeta} \|\lambda \div \partial_\nu \nu\|_H \leq \left( 1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}} \right) \sum_{m=1}^\infty \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_\nu^{\nu-e_\nu} \nu)\|_H \]
\[ + \left( 1 + \frac{C_2}{\lambda_{\min}} + \frac{C_3}{\lambda_{\min}} \right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\min}} \frac{1}{\lambda_{min}} \|\lambda \div \partial_\nu^{\nu-e_\nu} \nu\|_H. \]

**Proof** Differentiating (3.18), we get
\[ 2 \int_D \int_Y \mu(z; x, y) \epsilon(\partial_\nu \nu(z; x, y)) : \epsilon(v(z; x, y)) \, dy \, dx + \int_D \int_Y \div v(z; x, y) \partial_\nu \nu(z; x, y) \, dy \, dx \]
\[ = \int_D \int_Y \frac{\lambda(z; x, y)}{\lambda_{\min}} \div \partial_\nu \nu(z; x, y) q(z; y) \, dy \, dx + \int_D \int_Y \frac{1}{\lambda_{\min}} \partial_\nu \nu(z; x, y) q(z; y) \, dy \, dx \]
\[ = -2 \sum_{m=1}^\infty \nu_m \int_D \int_Y \mu_m(z; x, y) \epsilon(\partial_\nu^{\nu-e_\nu} \nu(z; x, y)) : \epsilon(v(z; x, y)) \, dy \, dx \]
\[ - \sum_{m=1}^\infty \nu_m \int_D \int_Y \frac{\lambda_m}{\lambda_{\min}} \div \partial_\nu^{\nu-e_\nu} \nu(z; x, y) q(z; y) \, dy \, dx \]
for all \( (v, q) \in \mathbf{X} \). For each \( q \in H \), \( \lambda q / \lambda_{\text{min}} \in H \) so we can rewrite this equation as

\[
2 \int_D \int_Y \mu(z; x, y) e(\partial_z^p u(z; x, y)) \cdot e(v(x, y)) \, dy \, dx + \int_D \int_Y \nabla v(x, y) \partial_z^p u(z; x, y) \, dy \, dx
\]

\[
= -2 \sum_{m=1}^{\infty} \nu_m \int_D \int_Y \mu_m(x, y) e(\partial_z^{\nu_m} u(z; x, y)) \cdot e(v(x, y)) \, dy \, dx,
\]

\[
(6.10)
\]

\[
\int_D \int_Y \nabla \partial_z^p u(z; x, y) q(x, y) \, dy \, dx
\]

\[
= \int_D \int_Y \left( \frac{1}{\lambda(z; x, y)} \partial_z^p u(z; x, y) - \sum_{m=1}^{\infty} \nu_m \frac{\lambda_m(x, y)}{\lambda(z; x, y)} \nabla \partial_z^{\nu_m} u(z; x, y) \right) q(x, y) \, dy \, dx.
\]

\[
(6.11)
\]

We denote by

\[
g(z; x, y) = \frac{1}{\lambda(z; x, y)} \partial_z^p u(z; x, y) - \sum_{m=1}^{\infty} \nu_m \frac{\lambda_m(x, y)}{\lambda(z; x, y)} \nabla \partial_z^{\nu_m} u(z; x, y), \quad (z; x, y) \in H.
\]

From Lemma 3.3, there is a function \( u_0 \in V \) such that \( \nabla u_0 = g \) and \( c_0 \| u_0 \|_V \leq \| g \|_H \). For all \( v \in V \) with \( \nabla v = 0 \)

\[
2 \int_D \int_Y \mu(z; x, y) \left( e(\partial_z^p u(z; x, y)) - e(u_0(z; x, y)) \right) \cdot e(v(x, y)) \, dy \, dx
\]

\[
= -2 \sum_{m=1}^{\infty} \nu_m \int_D \int_Y \mu_m(x, y) e(\partial_z^{\nu_m} u(z; x, y)) \cdot e(v(x, y)) \, dy \, dx
\]

\[
- 2 \int_D \int_Y \mu(z; x, y) e(u_0(z; x, y)) \cdot e(v(x, y)) \, dy \, dx.
\]

Letting \( v = \partial_z^p u - u_0 \) we obtain

\[
\| e(\partial_z^p u) - e(u_0) \|_H \leq \sum_{m=1}^{\infty} \nu_m \gamma_m \mu_{\text{min}} \| e(\partial_z^{\nu_m} u) \|_H + \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \| e(u_0) \|_H,
\]

which implies

\[
\| e(\partial_z^p u) \|_H \leq \sum_{m=1}^{\infty} \nu_m \gamma_m \mu_{\text{min}} \| e(\partial_z^{\nu_m} u) \|_H + \left( 1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \right) \| e(u_0) \|_H.
\]

Let \( c_7(d) \) be a constant such that \( \| e(v) \|_H \leq c_7 \| v \|_V \) for all \( v \in V \). We have

\[
\| e(\partial_z^p u) \|_H \leq \sum_{m=1}^{\infty} \nu_m \gamma_m \mu_{\text{min}} \| e(\partial_z^{\nu_m} u) \|_H + \left( 1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \right) \frac{c_7}{c_0} \| g \|_H
\]

\[
\leq \sum_{m=1}^{\infty} \nu_m \gamma_m \mu_{\text{min}} \| e(\partial_z^{\nu_m} u) \|_H + \left( 1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \right) \frac{c_7}{c_0} \lambda_{\text{min}} \left( \| \partial^p p \|_H + \sum_{m=1}^{\infty} \nu_m \delta_m \| \nabla \partial_z^{\nu_m} u \|_H \right)
\]

\[
\leq \sum_{m=1}^{\infty} \nu_m \gamma_m \mu_{\text{min}} \| e(\partial_z^{\nu_m} u) \|_H + \left( 1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \right) \frac{c_7}{c_0} \lambda_{\text{min}} \left( \| \partial^p p \|_H + \sum_{m=1}^{\infty} \nu_m \delta_m \| \lambda \nabla \partial_z^{\nu_m} u \|_H \right).
\]

\[
(6.12)
\]

From Lemma 3.3, there is \( v \in V \) such that \( \nabla v = \partial_z^p \) and \( c_0 \| v \|_V \leq \| \partial_z^p \|_H \). We deduce from \( 6.10 \)

\[
\| \partial_z^p \|_H^2 \leq 2 \mu_{\text{max}} \| e(\partial_z^p u) \|_H \| e(v) \|_H + 2 \sum_{m=1}^{\infty} \nu_m \gamma_m \| e(\partial_z^{\nu_m} u) \|_H \| e(v) \|_H
\]

\[
\leq 2 \mu_{\text{max}} \frac{c_7}{c_0} \| e(\partial_z^p u) \|_H \| \partial_z^p \|_H + 2 \sum_{m=1}^{\infty} \nu_m \gamma_m \frac{c_7}{c_0} \| e(\partial_z^{\nu_m} u) \|_H \| \partial_z^p \|_H.
\]
Thus, using (6.12),
\[
\|\partial_z^\nu p\|_H \leq 2\mu_{\text{max}} \frac{c_7}{c_0} \|\epsilon(\partial_z^\nu u)\|_H + 2 \sum_{m=1}^\infty \nu_m \gamma_m \frac{c_7}{c_0} \|\epsilon(\partial_z^\nu - e_m u)\|_H
\]
\[
\leq 2 \frac{c_7}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H + 2 \frac{c_7^2}{c_0^2} \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \|\partial_z^\nu p\|_H \tag{6.13}
\]
\[
+ 2 \frac{c_7^2 \mu_{\text{max}}}{c_0^2 \mu_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H.
\]
When
\[
\lambda_{\text{min}} > \vartheta_2 := 4\mu_{\text{max}}(1 + \kappa_0) \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \frac{c_7^2}{c_0^2}
\]
we have
\[
\lambda_{\text{min}} = \frac{\lambda_{\text{min}}}{1 + \kappa} > 4\mu_{\text{max}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \frac{c_7^2}{c_0^2}
\]
Thus
\[
\|\partial_z^\nu p\|_H \leq 4 \frac{c_7}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H
\]
\[
+ 4\mu_{\text{max}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \frac{c_7^2}{c_0^2} \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H \tag{6.14}
\]
From (6.12) and (6.14), it follows that
\[
\|\epsilon(\partial_z^\nu u)\|_H \leq \left(1 + \frac{4 \mu_{\text{min}}}{\mu_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)^2\right) \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H
\]
\[
+ \frac{c_7}{c_0 \lambda_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \left(1 + 4 \frac{c_7^2}{c_0^2} \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)\right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H
\]
\[
= \left(1 + \frac{C_1}{\lambda_{\text{min}}} \right) \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H + \left(\frac{C_2}{\lambda_{\text{min}}} + \frac{C_3}{\lambda_{\text{min}}}^2\right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H \tag{6.15}
\]
the constants
\[
C_1 = \frac{4 \mu_{\text{min}}}{c_0^2} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)^2, \quad C_2 = \frac{c_7}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right), \quad C_3 = \frac{4 \mu_{\text{max}} c_7^2}{c_0^2} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)^2,
\]
do not depend on \(\lambda_{\text{min}}\). From (6.13), we have
\[
\|\lambda \text{div } \partial_z^\nu u\|_H \leq \|\partial_z^\nu p\|_H + \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H
\]
\[
\leq 4 \frac{c_7 \mu_{\text{min}}}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H
\]
\[
+ \left(1 + 4 \frac{c_7^2}{c_0^2} \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)\right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H \tag{6.16}
\]
\[
= C_4 \sum_{m=1}^\infty \nu_m \gamma_m \|\epsilon(\partial_z^\nu - e_m u)\|_H + \left(1 + \frac{C_5}{\lambda_{\text{min}}} \right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div } \partial_z^\nu - e_m u\|_H,
\]
where the constants
\[
C_4 = 4 \frac{c_7 \mu_{\text{min}}}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right), \quad C_5 = 4 \frac{\mu_{\text{max}} c_7^2}{c_0^2} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right)\]
do not depend on $\lambda_{\text{min}}$. Thus for $0 < \zeta < 1$

$$
\|\epsilon(\partial_u^\nu u)\|_\mathcal{H} + \frac{1}{\lambda_{\text{min}}} \|\lambda \text{div} \partial_u^\nu u\|_\mathcal{H} \leq \left(1 + \frac{C_1}{\lambda_{\text{min}}} + \frac{C_4}{\lambda_{\text{min}}^\zeta}\right) \sum_{m=1}^\infty \nu_m \frac{\gamma_m}{\mu_{\text{min}}} \epsilon(\partial_u^{\nu - c_m} u)\|_\mathcal{H} + \left(1 + \frac{C_2}{\lambda_{\text{min}}} + \frac{C_3}{\lambda_{\text{min}}^\zeta} + \frac{C_5}{\lambda_{\text{min}}^2}\right) \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} \|\lambda \text{div} \partial_u^{\nu - c_m} u\|_\mathcal{H}.
$$

(6.17)

Letting

$$
\hat{d}_m = \max \left\{ \frac{\gamma_m}{\mu_{\text{min}}} \left(1 + \frac{C_1}{\lambda_{\text{min}}} + \frac{C_4}{\lambda_{\text{min}}^\zeta}\right), \frac{\delta_m}{\lambda_{\text{min}}} \left(1 + \frac{C_2}{\lambda_{\text{min}}} + \frac{C_3}{\lambda_{\text{min}}^\zeta} + \frac{C_5}{\lambda_{\text{min}}^2}\right) \right\}
$$

and $\hat{d} = (\hat{d}_1, \hat{d}_2, \ldots)$ we have

$$
\|\epsilon(\partial_u^\nu u)\|_\mathcal{H} + \frac{1}{\lambda_{\text{min}}} \|\lambda \text{div} \partial_u^\nu u\|_\mathcal{H} \leq \sum_{m=1}^\infty \nu_m \hat{d}_m \left(\|\epsilon(\partial_u^{\nu - c_m} u)\|_\mathcal{H} + \frac{1}{\lambda_{\text{min}}} \|\lambda \text{div} \partial_u^{\nu - c_m} u\|_\mathcal{H}\right).
$$

(6.18)

We thus have the following result.

**Proposition 6.5** If $\kappa < \kappa_0$ and $\lambda_{\text{min}} > \vartheta_2$ for the constants $\kappa_0$ and $\vartheta_2$ in Proposition 4.4 there is a constant $C_0$ such that the solution $(u, p) \in \mathbb{X}$ of problems (3.14) and (3.18) satisfies

$$
\|\partial_u^\nu u\|_V + \|\partial_u^\nu p\|_H \leq C_0 |\nu| |\hat{d}^{\nu'} \forall \nu \in \mathcal{F}.
$$

**Proof** From (3.19), there is a constant $c$ independent of $\lambda$ such that $\|\epsilon(u)\|_\mathcal{H} + \|p\|_H \leq c \|f\|_V$. As $p = \lambda \text{div} u$ and $\lambda_{\text{min}} = \frac{\kappa_{\text{min}}}{1 + \kappa_{\text{min}}^\zeta} > \frac{\vartheta_2}{1 + \kappa_{\text{min}}}$, there is a constant $C$ independent of $\lambda$ such that

$$
\|\epsilon(u)\|_\mathcal{H} + \frac{1}{\lambda_{\text{min}}} \|\lambda \text{div} u\|_H \leq C.
$$

(6.19)

From (6.18) we can show by induction that $\|\epsilon(\partial_u^\nu u)\|_\mathcal{H} + \frac{1}{\lambda_{\text{min}}} \|\lambda \text{div} \partial_u^\nu u\|_H \leq C_0 |\nu| |\hat{d}^{\nu'}$. The bound for $\|\partial_u^\nu p\|_V$ follows from (3.11). From (6.14),

$$
\|\partial_u^\nu p\|_H \leq \frac{4C}{c_0} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \sum_{m=1}^\infty \nu_m \frac{\gamma_m}{\mu_{\text{min}}} (|\nu| - 1) |\hat{d}^{\nu - c_m}| + 4\mu_{\text{max}} \left(1 + \frac{\mu_{\text{max}}}{\mu_{\text{min}}}\right) \frac{c_0^2}{\lambda_{\text{min}}^\zeta} \sum_{m=1}^\infty \nu_m \frac{\delta_m}{\lambda_{\text{min}}} (|\nu| - 1) |\hat{d}^{\nu - c_m}|.
$$

We therefore get the bound for $\|\partial_u^\nu p\|_H$.

**Proof**

Let $\delta_m = \frac{\delta_m}{\lambda_{\text{min}}}$ and $\delta = (\delta_1, \delta_2, \ldots)$. We then have the following proposition.

**Proposition 6.6** The coefficients $u_\nu$ and $p_\nu$ of the gpc expansion for $u$ and $p$ satisfy

$$
\|u_\nu\|_V + \|p_\nu\|_H \leq c \frac{|\nu|!}{\nu!} \hat{d}^{\nu'}.
$$

(6.20)

**6.4 Best N term convergence rate**

To quantify the rate of convergence for the best $N$ term approximation for $u$ and $p$, we need to establish the summability property of $u_\nu$ and $p_\nu$. We first make the following assumption on the summability of the coefficients of the expansion (2.11).

**Assumption 6.7** The constants $\gamma_m$ and $\delta_m$ in (2.12) satisfy $\sum_{m=1}^\infty \gamma_m^p < \infty$ and $\sum_{m=1}^\infty \delta_m^p < \infty$. The sequence $\{\gamma_m\}$ and $\{\delta_m\}$ satisfy

$$
\sum_{m=1}^\infty \max \left\{ \frac{\gamma_m}{\lambda_{\text{min}}^\zeta}, \frac{\delta_m}{\lambda_{\text{min}}^2}\right\} \leq \frac{\kappa}{1 + \kappa}.
$$

(6.21)
Under this assumption, we have

**Proposition 6.8** Let $\theta > 0$. If Assumption $[\text{6.4}]$ holds with $\kappa < \frac{\sqrt{3}}{1+\theta}$ then there is a constant $\vartheta_3 > 0$ depending on $\theta$, $\mu$ and the domain $D$ such that when $\lambda_{\min} > \vartheta_3$, $\{\|u_\nu\|_V\}_\nu$ and $\{\|p_\nu\|_H\}_\nu$ are in $\ell^3(F)$.

**Proof** When $\sum_{m=1}^\infty \delta_m < 1$, i.e. $\sum_{m=1}^\infty \delta_m < \sqrt{3}$ the sequences $\{\|u_\nu\|_V\}_\nu$ and $\{\|p_\nu\|_H\}_\nu$ belong to $\ell^3(F)$. From (6.6) and (6.7) we have that

$$
\sum_{m=1}^\infty \delta_m = \sum_{m=1}^\infty \max \left\{ \frac{\gamma_m}{\mu_{\min}} \left( 1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\max}^\kappa} \right), \frac{\delta_m}{\lambda_{\min}} \left( 1 + \frac{C_2}{\lambda_{\min}^\kappa} + \frac{C_3}{\lambda_{\max}^\kappa} + \frac{C_5}{\lambda_{\min}^\kappa} \right) \right\} 
\leq \kappa \max \left\{ 1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\max}^\kappa}, 1 + \frac{C_2}{\lambda_{\min}^\kappa} + \frac{C_3}{\lambda_{\max}^\kappa} + \frac{C_5}{\lambda_{\min}^\kappa} \right\}.
$$

If $\lambda_{\min} > \vartheta_3(\theta)$ for a constant $\theta > 0$ then $\lambda_{\min} = \frac{\kappa}{1+\kappa}$ satisfies

$$
\max \left\{ \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\max}^\kappa}, \frac{C_2}{\lambda_{\min}^\kappa} + \frac{C_3}{\lambda_{\max}^\kappa} + \frac{C_5}{\lambda_{\min}^\kappa} \right\} < \theta.
$$

Thus $\sum_{m=1}^\infty \delta_m < \sqrt{3}$ due to $\kappa < \sqrt{3}/(1+\theta)$. From this, we get the conclusion. \[ \square \]

Let $s = \frac{\theta}{p} - 1$. Let $\lambda_{\max} = \sup_{x \in D} X'(x)$. We then deduce the best $N$-term convergence rate for the approximations (6.3) and (6.4).

**Theorem 6.9** Let $\theta > 0$. If Assumption $[\text{6.4}]$ holds with $\kappa < \sqrt{3}/(1+\theta)$, then there is a constant $\vartheta > 0$ depending on $\theta, \mu$ such that when $\lambda_{\min} > \vartheta$, for each $N$ there is a set $\Lambda \subset F$ with cardinality not more than $\lambda_{\min}^\vartheta$ and $\lambda_{\max}^\vartheta$ and the solution $(u_\nu, p_\nu)$ of problems (3.17) and (3.18) and their approximations $(u_\Lambda, p_\Lambda)$ in (6.3) and (6.4) respectively satisfy

$$
\|u - u_\Lambda\|_V + \|p - p_\Lambda\|_H \leq C N^{-s},
$$

where $C$ depends only on $\frac{\lambda_{\max}}{\lambda_{\min}}$, $\{\|u_\nu\|_V\}_\nu$, $\{\|p_\nu\|_H\}_\nu$, and $\{\|u_\nu\|_V\}_\nu$ and $\{\|p_\nu\|_H\}_\nu$, and in particular it does not depend on the ratio $\lambda_{\max}/\mu_{\min}$ when $\lambda_{\min} \to +\infty$.

**Proof** The approximation $(u_\Lambda, p_\Lambda)$ of (6.4) satisfies

$$
\|u - u_\Lambda\|_V + \|p - p_\Lambda\|_H \leq \left( 1 + \frac{B_4}{\lambda_4} \right) \left( \left\| \sum_{\nu \notin \Lambda} u_\nu \right\|_V + \left\| \sum_{\nu \notin \Lambda} p_\nu \right\|_H \right),
$$

where $\lambda_4$ is the constant in (6.0). The bilinear form $B_4$ satisfies

$$
|B_4((u_\Lambda, p_\Lambda), (v_\Lambda, q_\Lambda))| \leq \left( 2\mu^* + 1 + \frac{\lambda_{\max}}{\lambda_{\min}} + \frac{1}{\lambda_{\min}} \right) \|u_\Lambda, p_\Lambda\|_X \|v_\Lambda, q_\Lambda\|_X.
$$

Let $\vartheta = \max\{\vartheta_1, \vartheta_2, \vartheta_3\}$ where $\vartheta_1, \vartheta_2$ and $\vartheta_3$ are the constants defined above. We have

$$
\|u - u_\Lambda\|_V + \|p - p_\Lambda\|_H \leq \left( 1 + \frac{1}{\lambda_4} \left( 2\mu^* + 1 + \frac{\lambda_{\max}}{\lambda_{\min}} + \frac{1}{\lambda_{\min}} \right) \right) \left( \left\| \sum_{\nu \notin \Lambda} u_\nu \right\|_V + \left\| \sum_{\nu \notin \Lambda} p_\nu \right\|_H \right)
\leq C \left[ \left( \sum_{\nu \notin \Lambda} \|u_\nu\|_V \right)^{\frac{1}{2}} + \left( \sum_{\nu \notin \Lambda} \|p_\nu\|_H \right)^{\frac{1}{2}} \right],
$$

where $C$ only depends on $\frac{\lambda_{\max}}{\lambda_{\min}}$. Letting $\Lambda$ be the set corresponding to the $N$ largest bounds $C|\nu|\|\nu\|^\vartheta/\vartheta^\vartheta$, we get the conclusion. The proof for $B_3$ is similar. \[ \square \]
7 Correctors for the solutions of the multiscale problems

7.1 Two-scale problem

For two scale problems, we can deduce an explicit homogenization rate of convergence. For conciseness, we denote \( a(z; x, y) \) as \( a(z; x, y) \). For \( 1 \leq r, s \leq d \), we define the second order symmetric tensor \( \epsilon_{rs} \in \mathbb{R}^{d \times d} \) as \( \epsilon_{rs} = \frac{1}{2}(\delta_{rk}\delta_{sd} + \delta_{rs}\delta_{kd}) \). The homogenized elastic moduli is determined by

\[
a^0_{ijkl} = \int_Y a(z; x, y)(\epsilon_{ij} + \epsilon_g(N^{rs})) : (\epsilon_{kl} + \epsilon_g(N^{kl})) dy.
\]

(7.1)

where \( N^r \) is the solution of the cell problem

\[
\int_Y a(z; x, y)(\epsilon_{rs} + \epsilon_g(N^{rs})) : (\epsilon_{s\ell} + \epsilon_g(N^{\ell})) dy = 0, \quad \forall \phi \in H^1_0(Y)^d.
\]

(7.2)

We have the following homogenization rate of convergence.

**Proposition 7.1** If \( u^0 \in L^\infty(U; H^2(D)^d) \), \( N^r \in L^\infty(U; C^1(\bar{D}, C^1(\bar{Y})) \cap H^2(Y))^d \) and \( \partial D \) is Lipschitz, then

\[
\|u^r(z; \cdot) - u^0(z; \cdot) - \epsilon u^1 \left( \frac{z; \cdot}{\epsilon} \right) \|_{L^\infty(U; H^1(D)^d)} \leq c\epsilon^{1/2}.
\]

(7.3)

The proof of this proposition is similar to that for the non-parametric case which can be found in [18].

The uniform constant \( c \) in the homogenization rate with respect to the parameters is due to the uniform boundedness and coerciveness of the elasticity moduli \( a \), and the uniform regularity of \( u^0 \) and \( N^r \). To have the required regularity for \( N^r \), we make the following assumption.

**Assumption 7.2** The fourth order tensors \( \bar{a} \) and \( \psi_m \in [2,2] \) belongs to \( C^2(D,C^2_0(Y))^d \) such that

\[
\sum_{m=1}^\infty \|\psi_m\|_{C^2(D,C^2_0(Y))^d} < \infty.
\]

The elastic tensor \( a(z; \cdot, \cdot) \) is then uniformly bounded in \( C^2(D,C^2_0(Y))^d \). We then have:

**Lemma 7.3** Under Assumption 7.2, \( N^r \in L^\infty(U; C^1(\bar{D}, C^1(\bar{Y})))^d \).

For the homogenization error estimate (7.3) to hold, we need the following result.

**Lemma 7.4** Assume that \( \partial D \) belongs to the \( C^1 \) class and \( f \in L^2(D)^d \). Then \( u^0 \in L^\infty(U; H^2(D)^d) \).

The proofs of Lemmas 7.3 and 7.4 use elliptic regularity (Theorems 4.16 and 4.18 of [15]). We refer to [18] for details.

From this we deduce.

**Proposition 7.5** Assume that \( \partial D \) belongs to the \( C^1 \) class and \( f \in L^2(D)^d \). Under Assumption 7.2, then

\[
\left\| \nabla u^r(z; \cdot) - \left[ \nabla u^0(z; \cdot) + \nabla_y u^1 \left( \frac{z; \cdot}{\epsilon} \right) \right] \right\|_{L^2(U; \mathbb{R}^{d \times d})} \leq c\epsilon^{1/2}.
\]

To deduce an approximation for the solution \( u^r \) of the multiscale parametric problem [5,2], we introduce the operator \( U^r : L^1(D \times Y) \to L^1(D) \) which is defined as

\[
U^r(\Phi)(x) = \int_Y \Phi \left( \frac{x}{\epsilon} \right) dt
\]

where \([\cdot]\) denotes the integer part with respect to \( Y \) and \( \cdot = \cdot - [\cdot] \). Let \( D^\epsilon \) be a \( 2\epsilon \) neighbourhood of \( D \), we have:

**Lemma 7.6** For \( \Phi \in L^1(D \times Y) \),

\[
\int_{D^\epsilon} U^r(\Phi)(x) dx = \int_D \int_{Y} \Phi(x, y) dy dx.
\]
A proof of this Lemma can be found in [6]. We then have the following result.

**Lemma 7.7** If \( u^0 \in L^\infty(U; H^2(D))^d \) and \( N^{\tau^s} \in L^\infty(U; C^1(D, C^1(Y)))^d \), then
\[
\sup_{x \in U} \int_D \left| \nabla_y u^1(z; x, \frac{x}{\varepsilon}) - U^\varepsilon(\nabla_y u^1(z; \cdot)) (x) \right|^2 \, dx \leq c \varepsilon^2
\]
where the constant \( c \) is independent of \( \varepsilon \) and \( z \in U \).

**Proof** This Lemma is essentially Lemma 5.5 in Hoang and Schwab [12]. It relies on the fact that
\[
\int_D \int_D \left| \varepsilon(u^0)(z; x) - \varepsilon(u^0) \left( z; \frac{x}{\varepsilon} + \varepsilon t \right) \right|^2 \, dx \, dt \leq c \varepsilon^2
\]
as \( \varepsilon(u^0) \in H^1(D)^{d \times d} \); \( c \) only depends on \( \| \varepsilon(u^0) \|_{H^1(D)^{d \times d}} \). The proof for this is quite technical so we refer to [12] for details. Further as \( N^{\tau^s} \in L^\infty(U; C^1(D, C^1(Y)))^d \),
\[
\esssup_{z \in U} \sup_{y \in Y} \left| \nabla_y N^{\tau^s} \left( z; x, \frac{x}{\varepsilon} \right) - \nabla_y N^{\tau^s} \left( z; \frac{x}{\varepsilon} + \varepsilon t, \frac{x}{\varepsilon} \right) \right| \leq c \varepsilon.
\]
From these we get the conclusion.

For the mixed problem (3.3), we have the following approximations.

**Theorem 7.8** Assume that the boundary \( \partial D \) belongs to the class \( C^1 \), and \( f \in L^2(D)^d \). If condition (5.5) holds, then for each \( N \) there is a set \( \Lambda_N \subset \mathcal{F} \) of cardinality not more than \( N \) such that the solution of the approximating problems (5.3) and (5.4) satisfy
\[
\| \nabla u^\varepsilon - [\nabla u^0_\Lambda + U^\varepsilon(\nabla u^1_\Lambda)] \|_{L^2(U; L^2(D)^{d \times d})} + \| \sigma^\varepsilon - U^\varepsilon(\sigma_\Lambda) \|_{L^2(U; L^2(D)^{d \times d})} \leq c(\varepsilon^{-1/2} + N^{-s})
\]

**Proof** From Proposition 7.7, we have
\[
\| \sigma^\varepsilon - a^\varepsilon(\varepsilon(u^0) + \varepsilon u^1)(\cdot, \cdot) \|_{L^2(U; L^2(D)^{d \times d})} \leq c \varepsilon^{1/2}
\]
i.e. \( \| \sigma^\varepsilon - a(\cdot, \cdot) \|_{L^2(U; L^2(D)^{d \times d})} \leq c \varepsilon^{1/2} \).

From Lemma 7.7, we have \( \sup_{x \in U} \int_D | \varepsilon(u^0)(z; x) + \varepsilon u^1(\cdot, \cdot, \varepsilon) - U^\varepsilon(\varepsilon(u^0)(z; \cdot) + \varepsilon u^1)(z; \cdot)) (x) |^2 \, dx \leq c \varepsilon^2 \). From the proof of Lemma 7.7, as \( \varepsilon(u^0)(z; \cdot) \) is uniformly bounded in \( H^1(D)^{d \times d} \), we have
\[
\sup_{x \in U} \int_D \left| \varepsilon(u^0)(z; x) - U^\varepsilon(\varepsilon(u^0))(z; x) \right|^2 \, dx \leq c \varepsilon^2.
\]
These imply \( \sup_{x \in U} \int_D \left| \varepsilon(u^0)(z; x) + \varepsilon u^1(\cdot, \cdot, \varepsilon) - U^\varepsilon(\varepsilon(u^0)(z; \cdot) + \varepsilon u^1)(z; \cdot)(\cdot) (x) \right|^2 \, dx \leq c \varepsilon^2 \). Therefore
\[
\sup_{x \in U} \int_D \left| \sigma(z; x, \varepsilon) - a(z; x, \varepsilon) \| U^\varepsilon(\varepsilon(u^0)(z; \cdot) + \varepsilon u^1)(z; \cdot)(\cdot) (x) \right|^2 \, dx \leq c \varepsilon^2.
\]
Since \( a \in L^\infty(U; C^1(D, C^1(Y))) \), \( \sup_{x \in U} \int_D \left| a(z; x, \varepsilon) - U^\varepsilon(a)(z; x) \right| \leq c \varepsilon \). Using \( U^\varepsilon(a)U^\varepsilon(\varepsilon(u^0) + \varepsilon u^1) = U^\varepsilon(a(\varepsilon(u^0) + \varepsilon u^1)) = U^\varepsilon(a) \), we deduce that
\[
\sup_{x \in U} \int_D \left| \sigma(z; x, \varepsilon) - U^\varepsilon(\sigma)(x) \right|^2 \, dx \leq c \varepsilon^2.
\]
From Theorem 7.8, by choosing \( \Lambda_N \) as the set corresponding to the indices \( \nu \) with the largest \( \| u_\nu \| V + \| \sigma_\nu \| \mathcal{K} \) we have
\[
\| U^\varepsilon(\nabla_y u^1 - \nabla_y u^1_\Lambda) \|_{L^2(U; L^2(D)^{d \times d})} + \| \sigma^\varepsilon - \sigma_\Lambda \|_{L^2(U; L^2(D)^{d \times d})} \leq c N^{1/2}
\]
\[
\| \nabla_y u^1 - \nabla_y u^1_\Lambda \|_{L^2(U; L^2(D)^{d \times d})} + \| \sigma^\varepsilon - \sigma_\Lambda \|_{L^2(U; L^2(D)^{d \times d})} \leq c N^{-s}.
\]
Therefore
\[
\| U^\varepsilon(\nabla_y u^1 - \nabla_y u^1_\Lambda) \|_{L^2(U; L^2(D)^{d \times d})} + \| \sigma^\varepsilon - \sigma_\Lambda \|_{L^2(U; L^2(D)^{d \times d})} \leq c(\varepsilon^{1/2} + N^{-s}).
\]
7.2 Two-scale nearly incompressible problem

The constant $c$ in estimate (7.3) depends explicitly on the ratio $\sup_{x,y} \lambda / \inf_{x,y} \mu$ which is very large when the material is nearly incompressible. In this section, we deduce a homogenization error rate that does not depend explicitly on this ratio. Let

$$||\alpha|| = \max_{ijklr} ||\alpha_{ijklr}||_{L^\infty(U;C^1(D,C(Y)))}, \quad ||N|| = \max_{rs} ||N_{rs}||_{L^\infty(U;C^1(D,C(Y)\cap H^2(Y)))}, \quad \text{and} \quad ||u_0|| = ||u_0||_{H^2(D)}$$

where $\alpha$ is the tensor defined in (7.7) below. We then have:

**Proposition 7.9** If $u^0 \in L^\infty(U;H^2(D)^d)$ and $N_{rs} \in L^\infty(U;C^1(\bar{D},C^1(\bar{Y}) \cap H^2(Y))^d$, then there are constants $c_1 = c_1(||\alpha||, ||N||, ||u_0||)$ and $c_2 = c_2(||N||, ||u_0||)$ such that

$$\left|\left|u^\varepsilon(z;\cdot) - u^0(z;\cdot) - \varepsilon u_1(z;\cdot, \frac{\cdot}{\varepsilon})\right|\right|_{H^1(D)^d} \leq c_1\varepsilon + c_2\varepsilon^{1/2}. \quad (7.4)$$

**Proof** We consider the cell problem

$$\begin{cases}
\int_Y \left[2\mu(z;x,y)(\varepsilon^s + \varepsilon_y(N_{rs}(z;x,y))) : \varepsilon(\phi) + \text{div}_{x,y} \phi(y)p_{rs}(z;x,y)\right] \, dy = 0 \\
\int_Y \left[\varepsilon^s + \text{div}_{y} N_{rs}(z;x,y)q(x,y) - \frac{1}{\lambda(z;x,y)}p_{rs}(y)q(y)\right] \, dy = 0
\end{cases} \quad (7.5)$$

for all $\phi \in H^1_\mu(Y)^d$ and $q \in L^2(Y)$. We can then write

$$u_1(z;x,y) = N_{rs}(z;x,y)e_{rs}(u_0(z;x)), \quad p(z;x,y) = p_{rs}(z;x,y)e_{rs}(u_0(z;x)).$$

Let $\mu^0$ be the fourth order tensor and $\lambda^0$ be the second order tensor defined by

$$\mu^0_{ijrs}(z;x) = 2\int_Y \mu(z;x,y)(\varepsilon^s_{ij} + \varepsilon_{yij}(N_{rs}(z;x,y))) \, dy, \quad \lambda^0_{rs}(z;x) = \int_Y p_{rs}(z;x,y) \, dy.$$ 

The homogenized elastic tensor is $a^0_{ijrs}(z;x) = \mu^0_{ijrs}(z;x) + \delta_{ij} \lambda^0_{rs}(z;x)$ ($a^0$ may not be isotropic). The homogenized equation, in the variational form, is

$$\int_D \left[\mu^0_{ijrs}e_{rs}(u_0(z;x))\varepsilon_{ij}(\phi(x)) + \lambda^0_{rs}(z;x)e_{rs}(u_0(z;x))\delta_{ij}(\varepsilon(x))\right] \, dx = \int_D f(x) \cdot \phi(x) \, dx.$$ 

We note that as $p_{rs}(z;x,y) = \lambda(z;x,y)(\varepsilon^s + \text{div}_{y} N_{rs}(z;x,y))$, this formula of $a^0_{ijrs}$ is consistent with (7.1). Let

$$u^{\varepsilon}(z;x) = u^0(z;x) + \varepsilon N_{rs}(z;x, \frac{\cdot}{\varepsilon})e_{rs}(u_0)(z;x) \text{ and } p^{\varepsilon}(z;x) = p_{rs}(z;x, \frac{\cdot}{\varepsilon})e_{rs}(u_0(z;x)).$$

For each function $\phi \in V$, we have

$$\int_D \left[2\mu^0(z;x)\varepsilon(u^{\varepsilon}(z;x)) : \varepsilon(\phi) + \text{div}_{x,y} \phi(x)p^{\varepsilon}(z;x)\right] \, dx$$

$$= \int_D \left[2\mu^0(z;x) \left(\varepsilon_{ij}(u^0(z;x)) + \varepsilon_{yij}N_{rs}(z;x, \frac{x}{\varepsilon})e_{rs}(u_0(z;x)) + \varepsilon_{ij}(N_{rs}(z;x, \frac{x}{\varepsilon}))e_{rs}(u_0(z;x))\right) + \frac{1}{2}\varepsilon \left(N_{rs}^s(z;x,y)\varepsilon_{r}\varepsilon_{s}(u_0(z;x))\right) \varepsilon_{ij}(\phi) \right] \, dx$$

$$+ \delta_{ij}p^{\varepsilon}(z;x, \frac{x}{\varepsilon})e_{rs}(u_0(z;x))\varepsilon_{ij}(\phi) \, dx$$

$$= \int_D \left[\mu^0_{ijrs}(z;x) + \delta_{ij} \lambda^0_{rs}(z;x)\right]e_{rs}(u^0(z;x))\varepsilon_{ij}(\phi(x)) \, dx + \int_D g_{ijk}(z;x, \frac{x}{\varepsilon})\varepsilon_{kl}(u^0(x))\varepsilon_{ij}(\phi(x)) \, dx$$

$$+ \varepsilon I_{ij}\varepsilon_{ij}(\phi)$$

where

$$g_{ijk}(z;x,y) = \mu(z;x,y)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 2\mu(z;x,y)\varepsilon_{yij}(N^{kl}(z;x,y)) + \delta_{ij}p^{kl}(z;x,y) - \mu^{0}_{ijk}l(z;x) - \delta_{ij} \lambda^0_{kl}(z;x)$$

(7.6)
and

\[ I_{ij} = \epsilon_{ij}(N^{rs}(z; x, x^r, x^s))_{rs}u_0(z; x) + \frac{1}{2} \left( N^{rs}_{ij}(z; x, y) \frac{\partial \epsilon_{rs}(u_0(z; x))}{\partial x_j} + N^{rs}_{ij}(z; x, y) \frac{\partial \epsilon_{rs}(u_0(z; x))}{\partial x_i} \right) \]

From (7.4), we deduce that

\[ \int_Y g_{ijkl}(z; x, y) dy = 0, \quad \text{and} \quad \frac{\partial}{\partial y_j} g_{ijkl}(z; x, y) = 0. \]

From the result in [14] page 7, there are functions

\[ \alpha_{ij}^{kl}(z; x, y) \in H_0^1(Y) \] with \( \alpha_{ij}^{kl} = \alpha_{ij}^{kl} \) and \( g_{ijkl}(z; x, y) = \frac{\partial}{\partial y_r} \alpha_{ij}^{kl}(z; x, y). \) (7.7)

Since \( g_{ijkl} = g_{ijkl}, \)

\[ \int_D g_{ijkl}(z; x, x^r, x^s)_{rs}u_0(z; x) dx = -\epsilon \int_D \alpha_{ij}^{kl}(z; x, x^r, x^s)_{rs}u_0(z; x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx - \epsilon \int_D r^i_j(z; x) \epsilon_{ij} \phi(z) dx = \epsilon \int_D r^i_j(z; x) \epsilon_{ij} \phi(z) dx \]

where

\[ r^i_j(z; x) = -\frac{\partial \alpha_{ij}^{kl}}{\partial x_r}(z; x, x^r, x^s)_{rs}u_0(z; x) - \alpha_{ij}^{kl}(z; x, x^r, x^s)_{rs}u_0(z; x) \].

Thus there is a constant \( c = c(\alpha, ||N||, ||u_0||) \) so that

\[ \left| \int_D [2\mu^c(z; x) \epsilon(u^{1c}(z; x)) : \epsilon(\phi) + \epsilon(\phi) p^c(z; x)] dx \right| \leq c \varepsilon \|

As

\[ \int_D (\mu_0^{ij}(z; x) + \epsilon_{ij} \lambda^0_{rs}(z; x)_{rs}u_0(z; x)_{rs}u_0(z; x))_{rs}u_0(z; x) \epsilon_{ij} \phi(z) dx \]

we have

\[ \left| \int_D [2\mu^c(z; x) \epsilon(u^{1c}(z; x) - u^c(z; x)) : \epsilon(\phi) + \epsilon(\phi) (p^c(z; x) - p^c(z; x))] dx \right| \leq c \varepsilon \|

Let \( \tau^c \in \mathcal{D}(D) \) be such that \( \tau^c(x) = 1 \) outside an \( \varepsilon \) neighbourhood \( \tilde{D} \subset D \) of \( D \) and \( \sup_{x \in D} \varepsilon |\nabla \tau^c(x)| < c \) for all \( \varepsilon \). Let

\[ u^{1c}(z; x) = u_0(z; x) + \epsilon \tau^c(z; x) N^{rs}(z; x, x^r, x^s)_{rs}u_0(z; x). \]

We have

\[ \frac{\partial}{\partial x_j}(u^{1c} - u^{1c})_{ij}(z; x) = -\frac{\partial \tau^c}{\partial x_j}(z; x) N^{rs}(z; x, x^r, x^s)_{rs}u_0(z; x) + \epsilon (1 - \tau^c(x)) \frac{\partial}{\partial x_j}(z; x, x^r, x^s)_{rs}u_0(z; x) + \epsilon (1 - \tau^c(x)) \frac{\partial}{\partial y_j}(z; x, x^r, x^s)_{rs}u_0(z; x) \]

We have that \( \|\phi\|^2_{L^2(D, \mathcal{D})} \leq \varepsilon c^2 \|\phi\|^2_{H^1(D)} + c \|\phi\|_{L^2(\mathcal{D})} \|\phi\|^2_{H^1(D)} \|\| \mathcal{D} \|_{H^1(D)} \leq \varepsilon \|\phi\|^2_{H^1(D)} \) for all \( \phi \in C^\infty(D) \) and therefore for all \( \phi \in H^1(D) \). As \( u_0 \in L^\infty(U; H^2(D)) \) so \( \epsilon_{rs}(u_0) \) \( \in L^\infty(U; H^1(D)) \). Together with \( N^{rs} \in L^\infty(U; C^3(\tilde{D}, C^1(Y))) \), there is a constant \( c = c(||N||, ||u_0||) \) such that for all \( i = 1, \ldots, d \)

\[ \left| u^{1c}(z) - u^{1c}(z) \right|_{H^1(D)} \leq c(||N||, ||u_0||)^{1/2}, \quad \forall z \in U. \] (7.8)
Therefore

$$2\int_D \mu^\varepsilon(z;x)\varepsilon(u^\varepsilon(z;x) - u^\varepsilon(z;x)) : \varepsilon(\phi)dx \leq c(||N||, ||u^0||)\varepsilon^{1/2}\|\phi\|_V.$$  

Thus

$$\int_D [2\mu^\varepsilon(z;x)\varepsilon(u^\varepsilon(z;x) - u^\varepsilon(z;x)) : \varepsilon(\phi) + \text{div}\phi(x)(\mu^\varepsilon(z;x) - \mu^\varepsilon(z;x))dx] \leq$$

$$\leq (c(||\alpha||, ||N||, ||u^0||)\varepsilon + c(||N||, ||u^0||))\varepsilon^{1/2}\|\phi\|_V. \quad (7.9)$$

We note that

$$\text{div}u^\varepsilon(z;x) = \text{div}^0(z;x) + \text{div}_y N^\varepsilon_s(z;x, \frac{x}{\varepsilon})\varepsilon_{rs}(u^0(z;x))
+ \varepsilon N^\varepsilon_s(z;x, \frac{x}{\varepsilon}) \cdot \text{grade}_{rs}(u^0(z;x)) + \varepsilon N^\varepsilon_s(z;x, \frac{x}{\varepsilon}) \cdot \text{grade}_{rs}(u^0(z;x))$$

$$= (e^\varepsilon_{rs} + \varepsilon N^\varepsilon_s(z;x, \frac{x}{\varepsilon})\varepsilon_{rs}(u^0(z;x)) + \varepsilon N^\varepsilon_s(z;x, \frac{x}{\varepsilon}) \cdot \text{grade}_{rs}(u^0(z;x))$$

$$+ \varepsilon \text{div}_x N^\varepsilon_s(z;x, \frac{x}{\varepsilon})\varepsilon_{rs}(u^0(z;x)) = \frac{1}{\lambda^\varepsilon(z;x)} p^\varepsilon(x) + \varepsilon N^\varepsilon_s(z;x, \frac{x}{\varepsilon}) \cdot \text{grade}_{rs}(u^0(z;x)) + \varepsilon \text{div}_x N^\varepsilon_s(z;x, \frac{x}{\varepsilon})\varepsilon_{rs}(u^0(z;x)) \quad (7.10)$$

Therefore

$$\int_D [\text{div}u^\varepsilon(z;x)q(x) - \frac{1}{\lambda^\varepsilon(z;x)} p^\varepsilon(z;x)q(x)]dx \leq c(||N||, ||u^0||)\varepsilon\|q\|_H,$$

where the constant $c(||N||, ||u^0||)$ does not depend on $\lambda_{\text{min}}$ when it goes to $\infty$. From (7.8), we deduce

$$\int_D [\text{div}u^\varepsilon(z;x)q(x) - \frac{1}{\lambda^\varepsilon(z;x)} p^\varepsilon(z;x)q(x)]dx \leq c(||N||, ||u^0||)\varepsilon^{1/2}\|q\|_H,$$

so

$$\int_D \left[\text{div}(u^\varepsilon(z;x) - u^\varepsilon(z;x))q(x) - \frac{1}{\lambda^\varepsilon(z;x)} (p^\varepsilon(z;x) - p^\varepsilon(z;x))q(x)\right]dx \leq c(||N||, ||u^0||)\varepsilon^{1/2}\|q\|_H. \quad (7.10)$$

From (7.11) and (7.10),

$$\|u^\varepsilon(z;x) - u^\varepsilon(z;x)\| \leq c(||\alpha||, ||N||, ||u^0||)\varepsilon + c(||N||, ||u^0||)\varepsilon^{1/2}$$

where the constants do not depend on $\lambda(z;x,y)$ when $\lambda_{\text{min}}$ goes to $\infty$. From this we get the conclusion. □

**Remark 7.10** The constant $c$ in the homogenization error (7.3) depends also on $||\alpha||$, $||N||$ and $||u^0||$ (see the detailed proof in (7)). It also depends explicitly on $\sup_{x,y} \lambda/\inf_{x,y} \mu$. The constants in (7.4) does not depend on this ratio, so this error is better than (7.3).

For the required regularity of $u^0$ and $N^\varepsilon_s$, we make the following assumption

**Assumption 7.11** The functions $\mu_m$ and $\lambda_m$ in (7.11) belong to $C^2(D,C^2(\Omega)^d$ such that

$$\sum_{m=1}^\infty \|\mu_m\|_{C^2(D,C^2(\Omega)^d)\varepsilon^s} + \|\lambda_m\|_{C^2(D,C^2(\Omega)^d)\varepsilon^s} < \infty.$$
Proposition 7.12 Under Assumption 7.11 if the boundary ∂D belongs to the C^1 class and f ∈ L^2(D)^d, then there are constants c_1 = c_1(||u||, ||N||, ||w||) and c_2 = c_2(||v||, ||w||) such that

\[ \|\nabla u^\varepsilon(z) - [\nabla u^0(z) + \nabla_y u^1 \left( z; \cdot \varepsilon \right)] \|_{L^2(U, \|u^0\|)} \leq c_1 \varepsilon + c_2 \varepsilon^{1/2}. \]

From this, we have the following approximation result.

Theorem 7.13 Assume that the boundary ∂D belongs to the C^1 class and f ∈ L^2(D)^d. Let θ > 0 be a constant. If Assumptions 6.7 and 7.11 hold with κ < \sqrt{\frac{1}{1+\theta}} then there is η > 0 depending on θ, μ such that the solution u of 7.3 and 6.18 and the solution u_A of problem 6.3 and 6.4 satisfy

\[ \|\nabla u^\varepsilon - \|\nabla u^\varepsilon_A + \mathcal{U}^\varepsilon(\nabla_y u^\varepsilon_A)\|_{L^2(U, \|u^0\|)} \leq c_1 \varepsilon + c_2 \varepsilon^{1/2} + c_3 N^{-s} \]

where s = 1/p - 1/2. The constant c_1 depends on ||u||, ||N|| and ||w||, the constant c_2 depends on ||N|| and ||w||, the constant c_3 depends on \( \lambda_{\min} \), \[ \|\|u_0\|\|\|v\|\|_{\mathcal{P}(F)} \|\|p_0\|\|_{\mathcal{P}(F)}. \]

7.3 Multiscale problems

We summarize briefly the derivation of the homogenized equation for the multiscale case. Details can be found in [18]. Let \( a^m(x, y_m) = a(x, y) \) for \( m = 1, \ldots, n - 1 \), the \( m \)th level homogenized coefficient \( a^m(z; x, y_m) \) is defined recursively as follows. Let \( N_{m+1}^s(z; x, y_{m+1}) \in V_{m+1} \) be the solution of the cell problem

\[ \int_{D} \int_{Y_{m+1}} a^{m+1}(z; x, y_{m+1})(e^{ry} + e_{y_{m+1}}(N_{m+1})^s) : e_{y_{m+1}}(\phi) d y_{m+1} d x = 0 \]  \( (7.11) \)

for all \( \phi \in V_{m+1} \). The \( m \)th level homogenized elastic moduli \( a^m(z; x, y_m) \) is

\[ a_{ijkl}^m(z; x, y_m) = \int_{Y_{m+1}} a^{m+1}(z; x, y_{m+1})(e^{ry} + e_{y_{m+1}}(N_{m+1})^s) : (e^{ij} + e_{y_{m+1}}(N_{m+1})^s) d y_{m+1}; \]

\( a^0(z; x) \) is the homogenized coefficient. The homogenized equation is

\[ -\frac{\partial}{\partial x_j}(a^0_{ijkl} e_{kl}(u_0)) = f_i. \] \( (7.13) \)

We deduce the convergence for the multiscale solution \( u^\varepsilon \) of the parametric multiscale problem \( 3.22 \) in this section. For problems with more than two scales, a homogenization rate of convergence similar to that in \( 3.23 \) is not available. However, we can deduce a corrector for the case where \( \varepsilon_i/\varepsilon_{i+1} \) is an integer for \( i = 1, \ldots, n - 1 \). We first define the operator \( \mathcal{T}^\varepsilon_n : L^1(D) \to L^1(D \times Y) \) as

\[ \mathcal{T}^\varepsilon_n(\phi)(x, y) = \phi \left( \varepsilon_1 \left[ \frac{x}{\varepsilon_1} \right] + \varepsilon_2 \left[ \frac{y_1}{\varepsilon_2/\varepsilon_1} \right] + \ldots + \varepsilon_n \left[ \frac{y_n-1}{\varepsilon_n/\varepsilon_{n-1}} \right] + \varepsilon_n y_n \right) \]

where \( \phi \in L^1(D) \) is understood to be zero outside \( D \). For each \( z \in U \), when \( \varepsilon \to 0 \), the solution \( u^\varepsilon(z) \) and its \( n + 1 \)-scale convergence limit \( u = (u^0, \ldots, u^n) \) which is the solution of problem \( 3.12 \) satisfies

\[ \mathcal{T}^\varepsilon_n \left( \frac{\partial u_j^1}{\partial x_j} \right) = \frac{\partial u_j^0}{\partial y_{ij}} + \frac{\partial u_j^1}{\partial y_{ij}} + \ldots + \frac{\partial u_j^n}{\partial y_{ij}} \]

in \( L^2(D \times Y) \) so

\[ \mathcal{T}^\varepsilon_n(\epsilon(u^\varepsilon)) \to \epsilon(u^0) + \epsilon_{y_1}(u^1) + \ldots + \epsilon_{y_n}(u^n) \quad \text{in} \quad L^2(D \times Y)^d. \] \( (7.14) \)

Letting \( D^\varepsilon \) be the \( 2\varepsilon \) neighbourhood of \( D \). We have

\[ \int_D \phi dx = \int_{D^\varepsilon} \int_{Y_1} \ldots \int_{Y_n} \mathcal{T}^\varepsilon_n(\phi) d y_{n+1} \ldots d y_1 d x \quad \forall \phi \in L^1(D). \] \( (7.15) \)
The proofs of Theorems 7.14 and 7.15 can be found in [6]. To deduce an approximation of \( u^\varepsilon \) in \( H^1(D)^d \) we define the operator \( U_n^\varepsilon : L^2(D \times Y) \to L^2(D) \) as

\[
U_n^\varepsilon(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi(\varepsilon_1 \frac{x}{\varepsilon_1} + \varepsilon_1 t_1, \varepsilon_2 \frac{x}{\varepsilon_2} + \varepsilon_2 t_2, \cdots, \varepsilon_n \frac{x}{\varepsilon_n} + \varepsilon_n t_n) dt_n \cdots dt_1
\]

for all functions \( \Phi \in L^1(D \times Y_1 \times \cdots \times Y_n) \). We assume further regularity for the elastic moduli.

**Assumption 7.14** The fourth order tensors \( \psi_m \) in (7.12) belong to \( C^2(D, C^2_\#(Y_1, \ldots, C^2_\#(Y_n)))^d \), which we denote as \( C^2(D, C^2_\#(Y))^d \), such that \( \sum_{m=1}^\infty \| \psi_m \|_{C^2(D, C^2_\#(Y))^d} < \infty \) and \( \sigma \in C^2(D, C^2_\#(Y))^d \).

We then have the following regularity result for the solution \( N^{\varepsilon s} \) of the cell problem (7.11).

**Lemma 7.15** Under Assumption 7.14 \( N^{\varepsilon s} \in L^\infty(U; C^1(D, C^1_\#(Y))) \) \( \forall r, s = 1, \ldots, d \) and \( i = 1, \ldots, n \). The proof is a routine generalization of the proof of Lemma 7.3. To deduce the correctors for \( u^\varepsilon(z) \), we employ the following result which is established in Xia and Hoang [17] for non-parametric problems.

**Lemma 7.16** Assume that \( u^0 \in L^\infty(U; H^2(D)^d) \) and \( N^{\varepsilon r} \in L^\infty(U; C^1(D, C^1_\#(Y))^d) \). Then

\[
\lim_{\varepsilon \to 0} \sup_{x \in D} \int_D \left| \varepsilon y_i(u^\varepsilon(z; x, \varepsilon_0 \varepsilon_1, \ldots, x_\varepsilon_i)) - \varepsilon_0 U_n^\varepsilon(\varepsilon y_i(u^\varepsilon))(z; x) \right|^2 dx = 0
\]

and

\[
\lim_{\varepsilon \to 0} \sup_{x \in U} \int_D \left| \nabla y_i u^\varepsilon(z; x, \varepsilon_0 \varepsilon_1, \ldots, x_\varepsilon_i) - \varepsilon_0 \nabla y_i U_n^\varepsilon(u^\varepsilon)(z; x) \right|^2 dx = 0.
\]

The proof is similar to that for the non-parametric case presented in [18]; we refer to [18] for details. We then have the following convergence result.

**Theorem 7.17** Assume that \( D \) is a \( C^1 \) domain and \( f \in L^2(D)^d \). Under Assumption 7.14

\[
\lim_{N \to \infty} \| \nabla u^\varepsilon - U_n^\varepsilon(\nabla u^\varepsilon_1 + \varepsilon_1 t_1, \ldots, \nabla u^\varepsilon_n + \varepsilon_n t_n) \|_{L^2(U; L^2(D)^{d \times d})} = 0
\]

where \( u_\Lambda = (u^0, u^1, \ldots, u^n) \) is the solution of problem (7.3) corresponding to the best \( N \) term set \( \Lambda \).

**Proof** Following the proof of Lemma 6.6 of [18], we have

\[
\lim_{\varepsilon \to 0} \| \nabla u^\varepsilon - U_n^\varepsilon(\nabla u^\varepsilon_1 + \ldots + \nabla u^\varepsilon_n) \|_{L^2(U; L^2(D)^{d \times d})} = 0.
\]

We get the conclusion by using

\[
\| U_n^\varepsilon(\nabla y_i u^\varepsilon_1 - \nabla y_i U_n^\varepsilon) \|_{L^2(U; L^2(D)^{d \times d})} \leq \| \nabla y_i u^\varepsilon_1 - \nabla y_i U_n^\varepsilon \|_{L^2(U; L^2(D)^{d \times d})} \to 0 \text{ when } N \to \infty.
\]

For the mixed problems, we have the following result.

**Theorem 7.18** Assume that the boundary \( \partial D \) belongs to \( C^1 \) and \( f \in L^2(D)^d \). Under Assumption 7.14

\[
\lim_{N \to \infty} \| \nabla y_i u^\varepsilon - [\nabla y_i u^\varepsilon + U_n^\varepsilon(\nabla y_i u^\varepsilon_1 + \ldots + \nabla y_i u^n) \|_{L^2(U; L^2(D)^{d \times d})} = 0 + \| \sigma^\varepsilon - U_n^\varepsilon(\sigma) \|_{L^{2}(U; L^{2}(D)^{d \times d})} = 0.
\]

**Proof** From (7.13), we have \( \lim_{\varepsilon \to 0} \| \varepsilon(u^\varepsilon) - U_n^\varepsilon(\varepsilon(u)) \|_{L^2(U; L^2(D)^{d \times d})} = 0 \), so

\[
\lim_{\varepsilon \to 0} \| \sigma^\varepsilon - a^\varepsilon U_n^\varepsilon(\varepsilon(u)) \|_{L^2(U; L^2(D)^{d \times d})} = 0. \text{ Since } a \in L^\infty(U; C^2(D, C^2_\#(Y)))^d,
\]

\[
\lim_{\varepsilon \to 0} \sup_{x \in D} \left| a(x; x, \varepsilon_1, \ldots, x_\varepsilon_n) - U_n^\varepsilon(a)(x) \right| = 0.
\]

Therefore \( \lim_{\varepsilon \to 0} \| \sigma^\varepsilon - U_n^\varepsilon(a(u)) \|_{L^2(U; L^2(D)^{d \times d})} = 0 \), i.e. \( \lim_{\varepsilon \to 0} \| \sigma^\varepsilon - U_n^\varepsilon(\sigma) \|_{L^2(U; L^2(D)^{d \times d})} = 0 \). From Theorem 7.18, \( \lim_{N \to \infty} \| U_n^\varepsilon(\sigma) - U_n^\varepsilon(\sigma) \|_{L^2(U; L^2(D)^{d \times d})} = 0 \). Therefore

\[
\lim_{N \to \infty} \| U_n^\varepsilon(\sigma) - U_n^\varepsilon(\sigma) \|_{L^2(U; L^2(D)^{d \times d})} = 0.
\]
Remark 7.19 The purpose of considering mixed problems with a penalty term for nearly incompressible materials is to derive an approximation whose error is independent of the ratio of the Lamé constants. For the general multiscale problems, we do not have an explicit error for the corrector so we do not consider the nearly incompressible problems separately in this section.

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