RIGIDITY OF MANIFOLDS WITH BOUNDARY UNDER A LOWER RICCI CURVATURE BOUND

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Abstract. We study Riemannian manifolds with boundary under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary. We prove a volume comparison theorem of Bishop-Gromov type concerning the volumes of the metric neighborhoods of the boundaries. We conclude several rigidity theorems. As one of them, we obtain a volume growth rigidity theorem. We also show a splitting theorem of Cheeger-Gromoll type under the assumption of the existence of a single ray.

1. Introduction

In this paper, we study Riemannian manifolds with boundary under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary. Heintze and Karcher in [HK], and Kasue in [K2] ([K1]), have proved several comparison theorems for such manifolds with boundary. Furthermore, Kasue has proved rigidity theorems in [K3], [K4] for such manifolds with boundary (see also [K5], [H]). These rigidity theorems state that if such manifolds satisfy suitable rigid conditions, then there exist diffeomorphisms preserving the Riemannian metrics between the manifolds and the model spaces. Other rigidity results have been also studied in [dCX] and [X], and so on.

In order to develop the geometry of such manifolds with boundary, we prove a volume comparison theorem of Bishop-Gromov type concerning the metric neighborhoods of the boundaries, and produce a volume growth rigidity theorem. We also prove a splitting theorem of Cheeger-Gromoll type under the assumption of the existence of a single ray emanating from the boundary. We obtain a lower bound for the smallest Dirichlet eigenvalues for the $p$-Laplacians. We also add a
rigidity result to the list of the rigidity results obtained in [K4] on the smallest Dirichlet eigenvalues for the Laplacians.

The preceding rigidity results mentioned above have stated the existence of Riemannian isometries between manifolds with boundary and the model spaces. On the other hand, our rigidity results discussed below states the existence of isometries as metric spaces from a viewpoint of metric geometry. These notions are equivalent to each other (see Subsection 2.3).

1.1. Main results. For $\kappa \in \mathbb{R}$, we denote by $M^n_\kappa$ the $n$-dimensional space form with constant curvature $\kappa$, and by $g^n_\kappa$ the standard Riemannian metric on $M^n_\kappa$.

We say that $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition if there exists a closed geodesic ball $B^n_{\kappa,\lambda}$ in $M^n_\kappa$ with non-empty boundary $\partial B^n_{\kappa,\lambda}$ such that $\partial B^n_{\kappa,\lambda}$ has a constant mean curvature $\lambda$. We denote by $C_{\kappa,\lambda}$ the radius of $B^n_{\kappa,\lambda}$. We see that $\kappa$ and $\lambda$ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. Let $s_{\kappa,\lambda}(t)$ be a unique solution of the so-called Jacobi-equation

$$f''(t) + \kappa f(t) = 0$$

with initial conditions $f(0) = 1$ and $f'(0) = -\lambda$. We see that $\kappa$ and $\lambda$ satisfy the ball-condition if and only if the equation $s_{\kappa,\lambda}(t) = 0$ has a positive solution; in particular, $C_{\kappa,\lambda}$ is the minimal positive solution of the equation $s_{\kappa,\lambda}(t) = 0$.

We denote by $S^{n-1}$ the $(n-1)$-dimensional standard unit sphere. Let $ds^{2}_{n-1}$ be the canonical metric on $S^{n-1}$. For an arbitrary pair of $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define an $n$-dimensional model space $M^n_{\kappa,\lambda}$ with constant mean curvature boundary with Riemannian metric $g^n_{\kappa,\lambda}$ as follows: If $\kappa > 0$, then we put $(M^n_{\kappa,\lambda}, g^n_{\kappa,\lambda}) := (B^n_{\kappa,\lambda}, g^n_\kappa|_{B^n_{\kappa,\lambda}})$. If $\kappa \leq 0$, then

$$(M^n_{\kappa,\lambda}, g^n_{\kappa,\lambda}) := \begin{cases} (B^n_{\kappa,\lambda}, g^n_\kappa|_{B^n_{\kappa,\lambda}}) & \text{if } \lambda > \sqrt{|\kappa|}, \\ (M^n_\kappa \setminus \text{Int } B^n_{\kappa,-\lambda}, g^n_\kappa|_{M^n_\kappa \setminus \text{Int } B^n_{\kappa,-\lambda}}) & \text{if } \lambda < -\sqrt{|\kappa|}, \\ ([0,\infty) \times S^{n-1}, dt^2 + s^2_{\kappa,\lambda}(t)ds^2_{n-1}) & \text{if } |\lambda| = \sqrt{|\kappa|}, \\ ([t_{\kappa,\lambda}, \infty) \times S^{n-1}, dt^2 + s^2_{\kappa,0}(t)ds^2_{n-1}) & \text{if } |\lambda| < \sqrt{|\kappa|}, \end{cases}$$

where $t_{\kappa,\lambda}$ is the unique solution of the equation $s'_{\kappa,0}(t)/s_{\kappa,0}(t) = -\lambda$ under the assumptions $\kappa < 0$ and $|\lambda| < \sqrt{|\kappa|}$. We denote by $h^n_{\kappa,\lambda}$ the induced Riemannian metric on $\partial M^n_{\kappa,\lambda}$.

For $n \geq 2$, let $M$ be an $n$-dimensional, connected Riemannian manifold with boundary with Riemannian metric $g$. The boundary $\partial M$ is assumed to be smooth. We say that $M$ is complete if for the Riemannian distance $d_M$ on $M$ induced from the length structure determined by
Let \( M \) be an \( n \)-dimensional, connected, complete Riemannian manifold with boundary \( \partial M \) with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \lambda \). Let \( \rho_{\partial M} : M \to \mathbb{R} \) be the distance function from \( \partial M \) defined as
\[
\rho_{\partial M}(p) := \text{dist}(p, \partial M).
\]
The inscribed radius of \( M \) is defined as
\[
D(M, \partial M) := \sup_{p \in M} \rho_{\partial M}(p).
\]

Let \( h > 0 \), we put \( B_r(\partial M) := \{ p \in M \mid \rho_{\partial M}(p) \leq r \} \). We denote by \( \text{vol}_g \) the Riemannian volume on \( M \) induced from \( g \).

One of the main results is the following volume comparison theorem:

**Theorem 1.1.** For \( \kappa \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \), and for \( n \geq 2 \), let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \lambda \). Let \( \partial M \) be compact. Then for all \( r, R \in (0, \infty) \) with \( r \leq R \), we have
\[
\frac{\text{vol}_g B_R(\partial M)}{\text{vol}_g B_r(\partial M)} \leq \frac{\text{vol}_{g^{\kappa,\lambda}} B_R(\partial M^{\kappa,\lambda}_n)}{\text{vol}_{g^{\kappa,\lambda}} B_r(\partial M^{\kappa,\lambda}_n)}.
\]

Theorem 1.1 is an analogue to the Bishop-Gromov volume comparison theorem ([Gr1], [Gr2]). What happens in the equality case can be described by using the Jacobi fields along the geodesics perpendicular to the boundary (see Remark 4.3 and Proposition 5.3).

**Remark 1.1.** Theorem 1.1 is a relative volume comparison theorem. Under the same setting as in Theorem 1.1, it has been proved in [HK] that the absolute volume comparison inequality
\[
\frac{\text{vol}_g B_r(\partial M)}{\text{vol}_h \partial M} \leq \frac{\text{vol}_{g^{\kappa,\lambda}} B_r(\partial M^{\kappa,\lambda}_n)}{\text{vol}_{h^{\kappa-1,\lambda}} \partial M^{\kappa,\lambda}_n}
\]
holds for every \( r > 0 \). Similar volume comparison inequalities for submanifolds have been studied in [HK].

**Remark 1.2.** It has been shown in [K3] that if \( \kappa \) and \( \lambda \) satisfy the ball-condition, then \( D(M, \partial M) \leq C_{\kappa,\lambda} \) (see Lemma 4.4); moreover, if we have a point \( p_0 \in M \) such that \( \rho_{\partial M}(p_0) = C_{\kappa,\lambda} \), then \( M \) is isometric to \( B_{C_{\kappa,\lambda}} \) (see Theorem 4.5).

**Remark 1.3.** It has been recently shown in [L] that if \( M \) is an \( n \)-dimensional, connected complete Riemannian manifold with boundary...
such that $\text{Ric}_M \geq 0$ and $H_{\partial M} \geq \lambda > 0$, then $D(M, \partial M) \leq C_{0, \lambda}$; moreover, if $\partial M$ is compact, then $M$ is compact, and the equality holds if and only if $M$ is isometric to $B^n_{0, \lambda}$. It has been recently proved in [LW] that for $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$, if $M$ is an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$, then $D(M, \partial M) \leq C_{\kappa, \lambda}$; moreover, if $\partial M$ is compact, then the equality holds if and only if $M$ is isometric to $B^n_{\kappa, \lambda}$. A similar result has been proved in [LW] for manifolds with boundary under a lower Bakry-Émery Ricci curvature bound. It has been also recently stated in [G] that if $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition, and if $M$ is an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$, then $D(M, \partial M) \leq C_{\kappa, \lambda}$; moreover, if $\partial M$ is compact, then the equality holds if and only if $M$ is isometric to $B^n_{\kappa, \lambda}$.

For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define $[0, \infty) \times_{\kappa, \lambda} \partial M$ as the warped product $([0, \infty) \times \partial M, dt^2 + s_{\kappa, \lambda}^2(t)h)$ with Riemannian metric $dt^2 + s_{\kappa, \lambda}^2(t)h$, and we put $d_{\kappa, \lambda} := d_{[0, \infty) \times_{\kappa, \lambda} \partial M}$. We put $\bar{C}_{\kappa, \lambda} := C_{\kappa, \lambda}$ if $\kappa$ and $\lambda$ satisfy the ball-condition; otherwise, $\bar{C}_{\kappa, \lambda} := \infty$. We define a function $\bar{s}_{\kappa, \lambda} : [0, \infty) \to \mathbb{R}$ by

$$\bar{s}_{\kappa, \lambda}(t) := \begin{cases} s_{\kappa, \lambda}(t) & \text{if } t \leq \bar{C}_{\kappa, \lambda}, \\ 0 & \text{if } t > \bar{C}_{\kappa, \lambda}, \end{cases}$$

and define a function $f_{n, \kappa, \lambda} : [0, \infty) \to \mathbb{R}$ by

$$f_{n, \kappa, \lambda}(t) := \int_0^t \bar{s}_{\kappa, \lambda}^{n-1}(u) \, du.$$

Theorem 1.1 yields the following volume growth rigidity theorem:

**Theorem 1.2.** For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, and for $n \geq 2$, let $M$ be an $n$-dimensional Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Let $h$ denote the induced Riemannian metric on $\partial M$. If

$$\liminf_{r \to \infty} \frac{\text{vol}_g B_r(\partial M)}{f_{n, \kappa, \lambda}(r)} \geq \text{vol}_h \partial M,$$

then the metric space $(M, d_M)$ is isometric to $([0, \bar{C}_{\kappa, \lambda}) \times_{\kappa, \lambda} \partial M, d_{\kappa, \lambda})$. Moreover, if $\kappa$ and $\lambda$ satisfy the ball-condition, then $(M, d_M)$ is isometric to $(B^n_{\kappa, \lambda}, d_{B^n_{\kappa, \lambda}})$. 

Remark 1.4. Under the same setting as in Theorem 1.2 by Theorem 1.1, we always have the following (see Proposition 5.1):
\[ \limsup_{r \to \infty} \frac{\text{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \leq \text{vol}_h \partial M. \]

Theorem 1.2 is certainly concerned with a rigidity phenomenon.

1.2. Splitting theorems. In [K3], Kasue has proved the following splitting theorem. For \( \kappa \leq 0 \), let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \sqrt{|\kappa|} \). If \( M \) is non-compact and \( \partial M \) is compact, then \((M, d_M)\) is isometric to \((\mathbb{R}_+, \kappa, \sqrt{|\kappa|} \partial M, d_{\kappa, \sqrt{|\kappa|}})\). The same result has been proved in [CK].

In [K3], the proof of the splitting theorem is based on the original proof of the Cheeger-Gromoll splitting theorem in [CG]. For a ray \( \gamma \) on \( M \), let \( b_\gamma \) be the busemann function on \( M \) for \( \gamma \). The key points in [K3] are to show the existence of a ray \( \gamma \) on \( M \) such that for all \( t \geq 0 \) we have \( \rho_{\partial M}(\gamma(t)) = t \), and the subharmonicity of the function \( b_\gamma - \rho_{\partial M} \) in a distribution sense, and to apply an analytic maximal principle (see [GT]). In [CK], the splitting theorem has been proved by using the Calabi maximal principle ([C]) similarly to the elementary proof of the Cheeger-Gromoll splitting theorem developed in [EH]. It seems that the proof in [CK] relies on the compactness of \( \partial M \).

Let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary. For \( x \in \partial M \), we denote by \( u_x \) the unit inner normal vector at \( x \). Let \( \gamma_x : [0, T) \to M \) be the geodesic with initial conditions \( \gamma_x(0) = x \) and \( \gamma_x'(0) = u_x \). We define a function \( \tau : \partial M \to \mathbb{R} \cup \{\infty\} \) by
\[ \tau(x) := \sup\{ t > 0 \mid \rho_{\partial M}(\gamma_x(t)) = t \}. \]

We point out that the following splitting theorem holds for the case where the boundary is not necessarily compact.

**Theorem 1.3.** For \( n \geq 2 \) and \( \kappa \leq 0 \), let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \sqrt{|\kappa|} \). Assume that for some \( x \in \partial M \), we have \( \tau(x) = \infty \). Then \((M, d_M)\) is isometric to \((\mathbb{R}_+, \kappa, \sqrt{|\kappa|} \partial M, d_{\kappa, \sqrt{|\kappa|}})\).

Theorem 1.3 can be proved by a similar way to that of the proof of the splitting theorem in [K3]. We give a proof of Theorem 1.3 in which we use the Calabi maximal principle. Our proof can be regarded as an elementary proof of the splitting theorem in [K3].

**Remark 1.5.** In Theorem 1.3 if \( \partial M \) is non-compact, then we can not replace the assumption of \( \tau \) with that of the existence of a single ray.
orthogonally emanating from the boundary. For instance, we put

\[ M := \{(p,q) \in \mathbb{R}^2 \mid p < 0, p^2 + q^2 \leq 1\} \cup \{(p,q) \in \mathbb{R}^2 \mid p \geq 0, |q| \leq 1\}. \]

Observe that \( M \) is a 2-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M = 0 \) and \( H_{\partial M} \geq 0 \). For all \( x \in \partial M \), we have \( \tau(x) = 1 \). The geodesic \( \gamma_{(-1,0)} \) is a ray in \( M \). On the other hand, \( M \) is not isometric to the standard product \([0,\infty) \times \partial M\).

1.3. Eigenvalues. Let \( M \) be a Riemannian manifold with boundary with Riemannian metric \( g \). For \( p \in [1,\infty) \), the \((1,p)\)-Sobolev space \( W^{1,p}_0(M) \) on \( M \) with compact support is defined as the completion of the set of all smooth functions on \( M \) whose support is compact and contained in \( \text{Int} M \) with respect to the standard \((1,p)\)-Sobolev norm. Let \( \| \cdot \| \) denote the standard norm induced from \( g \), and \( \text{div} \) the divergence with respect to \( g \). For \( p \in [1,\infty) \), the \( p\)-Laplacian \( \Delta_p f \) for \( f \in W^{1,p}_0(M) \) is defined as

\[ \Delta_p f := -\text{div} \left( \|\nabla f\|^{p-2} \nabla f \right), \]

where the equality holds in a weak sense on \( W^{1,p}_0(M) \). A real number \( \lambda \) is said to be a \( p\)-Dirichlet eigenvalue for \( \Delta_p \) on \( M \) if we have a non-zero function \( f \) in \( W^{1,p}_0(M) \) such that \( \Delta_p f = \lambda |f|^{p-2} f \) holds on \( \text{Int} M \) in a weak sense on \( W^{1,p}_0(M) \), and \( f|_{\partial M} = 0 \). For \( p \in [1,\infty) \), the Rayleigh quotient \( R_p(f) \) for \( f \in W^{1,p}_0(M) \) is defined as

\[ R_p(f) := \frac{\int_M \|\nabla f\|^p \text{d} \text{vol}_g}{\int_M |f|^p \text{d} \text{vol}_g}. \]

We put \( \mu_{1,p}(M) := \inf_f R_p(f) \), where the infimum is taken over all non-zero functions in \( W^{1,p}_0(M) \). The value \( \mu_{1,2}(M) \) is equal to the infimum of the spectrum of \( \Delta_2 \) on \( M \). If \( M \) is compact, and if \( p \in (1,\infty) \), then \( \mu_{1,p}(M) \) is equal to the infimum of the set of all \( p\)-Dirichlet eigenvalues for \( \Delta_p \) on \( M \).

Due to the volume estimate obtained in \[ K5 \], we obtain the following:

**Theorem 1.4.** For \( \kappa \in \mathbb{R} \), \( \lambda \in \mathbb{R} \) and \( D \in (0,\bar{C}_{\kappa,\lambda}] \), and for \( n \geq 2 \), let \( M \) be an \( n\)-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M \geq (n-1)\kappa \), \( H_{\partial M} \geq \lambda \) and \( D(M,\partial M) \leq D \). Let \( \partial M \) be compact. Then for all \( p \in (1,\infty) \), we have

\[ \mu_{1,p}(M) \geq \left( pC(n,\kappa,\lambda,D) \right)^{-p}, \]

where \( C(n,\kappa,\lambda,D) \) is a positive constant defined by

\[ C(n,\kappa,\lambda,D) := \sup_{t \in [0,D]} \frac{\int_t^D s_{n,\kappa,\lambda}^{-1}(s) \text{d}s}{s_{n,\kappa,\lambda}^{-1}(t)}. \]
Remark 1.6. In Theorem 1.4 since \( \partial M \) is compact, \( D(M, \partial M) \) is finite if and only if \( M \) is compact (see Lemma 3.3). We see that \( C(n, \kappa, \lambda, \infty) \) is finite if and only if \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \); in this case, we have \( C(n, \kappa, \lambda, D) = ((n-1)\lambda)^{-1} \left( 1 - e^{-(n-1)\lambda D} \right) \); in particular, \( (2 C(n, \kappa, \lambda, \infty))^2 = ((n-1)\lambda/2)^2 \).

Remark 1.7. For compact manifolds with boundary of non-negative Ricci curvature, similar lower bounds for \( \mu_{1,p} \) to that in Theorem 1.4 have been obtained in \([K4]\), in \([Z1]\) and in \([Z2]\).

We recall the works in \([K4]\) for compact manifolds with boundary. Let \( n \geq 2, \kappa, \lambda \in \mathbb{R} \) and \( D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\} \). Kasue has proved in \([K4]\) that there exists a positive constant \( \mu_{n,k,\lambda,D} \) such that for every \( n \)-dimensional, connected compact Riemannian manifold \( M \) with boundary such that \( \text{Ric}_M \geq (n-1)\kappa, H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \), we have \( \mu_{1,2}(M) \geq \mu_{n,k,\lambda,D} \); moreover, in some extremal case, the equality holds if and only if \( M \) is isometric to some model space. The extremal case happens only if \( \kappa \) and \( \lambda \) satisfy the ball-condition or the condition that the equation \( s_{\kappa,\lambda}'(t) = 0 \) has a positive solution. Note that the equation \( s_{\kappa,\lambda}'(t) = 0 \) has a positive solution if and only if either (1) \( \kappa = 0 \) and \( \lambda = 0 \); (2) \( \kappa < 0 \) and \( \lambda \in (0, \sqrt{|\kappa|}) \); or (3) \( \kappa > 0 \) and \( \lambda \in (-\infty, 0) \). Let

\[
\bar{\mu}_{n,k,\lambda,D} := \left( 4 \sup_{t \in (0, D)} \int_0^D s_{\kappa,\lambda}^{n-1}(s) \, ds \int_0^t \int_0^s \int_0^u \int_0^v \int_0^w ds \, du \, dv \, dw \right)^{-1}.
\]

It has been shown in \([K4]\) that \( \mu_{n,k,\lambda,D} > \bar{\mu}_{n,k,\lambda,D} \). Therefore, for every \( n \)-dimensional, connected compact Riemannian manifold \( M \) with boundary such that \( \text{Ric}_M \geq (n-1)\kappa, H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \), we have \( \mu_{1,2}(M) > \bar{\mu}_{n,k,\lambda,D} \). This estimate for \( \mu_{1,2} \) is better than that in Theorem 1.4.

Let \( n \geq 2, \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \). The model space \( M^n_{\kappa,\lambda} \) is non-compact. For \( t \in [0, \infty) \), we put \( \phi_{n,k,\lambda}(t) := t e^{(n-1)\lambda t} \). The smooth function \( \phi_{n,k,\lambda} \circ \rho_{\partial M^n_{\kappa,\lambda}} \) on \( M^n_{\kappa,\lambda} \) satisfies \( R_2(\phi_{n,k,\lambda} \circ \rho_{\partial M^n_{\kappa,\lambda}}) = ((n-1)\lambda/2)^2 \); hence, \( \mu_{1,2}(M^n_{\kappa,\lambda}) \leq ((n-1)\lambda/2)^2 \). Notice that the value \( (2 C(n, \kappa, \lambda, \infty))^2 \) in Theorem 1.4 is equal to \( ((n-1)\lambda/2)^2 \) (see Remark 1.6). Theorem 1.4 implies \( \mu_{1,2}(M^n_{\kappa,\lambda}) = ((n-1)\lambda/2)^2 \). Let \( D \in (0, \infty) \). As mentioned above, we have already known in \([K4]\) that for every \( n \)-dimensional, connected compact Riemannian manifold \( M \) with boundary such that \( \text{Ric}_M \geq (n-1)\kappa, H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \), we have \( \mu_{1,2}(M) > \bar{\mu}_{n,k,\lambda,D} \). The value \( \bar{\mu}_{n,k,\lambda,D} \) is equal to \( ((n-1)\lambda/2)^2 \left( 1 - e^{-(n-1)\lambda D/2} \right)^{-2} \), and tends to \( \mu_{1,2}(M^n_{\kappa,\lambda}) \) as \( D \to \infty \).
By using Theorem 1.4 and the splitting theorem in [K3], we add the following result for not necessarily compact manifolds with boundary to the list of the rigidity results obtained in [K4].

**Theorem 1.5.** Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $n \geq 2$, let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n - 1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Then for all $p \in (1, \infty)$, we have

$$\mu_{1,p}(M) \geq \left(\frac{(n - 1)\lambda}{p}\right)^p;$$

if the equality holds, then $(M, d_M)$ is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$; moreover, if $p = 2$, then the equality holds if and only if $(M, d_M)$ is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$.

**Remark 1.8.** In Theorem 1.5, the author does not know whether in the case of $p \neq 2$ the value $\mu_{1,p}([0, \infty) \times_{\kappa,\lambda} \partial M)$ is equal to $((n - 1)\lambda/p)^p$.

In [CC], Cheeger and Colding have proved the segment inequality for complete Riemannian manifolds under a lower Ricci curvature bound. They have mentioned that their segment inequality gives a lower bound for the smallest Dirichlet eigenvalue for the Laplacian on a closed ball.

Based on the proof of Theorem 1.1, we prove a segment inequality of Cheeger-Colding type for manifolds with boundary (see Proposition 7.2). Using our segment inequality, we obtain a lower bound for $\mu_{1,p}$ smaller than the lower bound in Theorem 1.4 (see Proposition 7.4).

**1.4. Organization.** In Section 2, we prepare some notations and recall the basic facts on Riemannian manifolds with boundary.

In Section 3, for a connected complete Riemannian manifold with boundary, we study the basic properties of the cut locus for the boundary. The basic properties seem to be well-known, however, they have not been summarized in any literature. For the sake of the readers’, we discuss them in order to prove our results.

In Section 4, by using the study of the cut locus for the boundary in Section 3, we prove Theorem 1.1.

In Section 5, we prove Theorem 1.2. The rigidity follows from the study in the equality case in Theorem 1.1.

In Section 6, we prove Theorem 1.3.

In Section 7, we prove Theorems 1.4 and 1.5. We also prove a segment inequality (see Proposition 7.2). After that, we show the Poincaré inequality (see Lemma 7.3), and we conclude Proposition 7.4.
Addendum. After completing the first draft of this paper, the author has been informed by Sormani of the paper [P]. Let $M$ be a connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq 0$ and $H_{\partial M} \geq \lambda$. The paper [P] contains a Laplacian comparison theorem for $\rho_{\partial M}$ everywhere in a barrier sense, a theorem of volume estimates of the metric neighborhoods of $\partial M$, and applications to studies of convergences of such manifolds with boundary. In a former draft of this paper, the author has obtained lower bounds for the smallest Dirichlet eigenvalue for the Laplacian. The author has found that some better estimates have been already obtained in [K4].

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2. Preliminaries

We refer to [BBI] for the basics of metric geometry, and to [S] for the basics of Riemannian manifolds with boundary.

2.1. Metric spaces. Let $(X, d_X)$ be a metric space. For $r > 0$ and $A \subset X$, we denote by $U_r(A)$ the open $r$-neighborhood of $A$ in $X$, and by $B_r(A)$ the closed one.

For a metric space $(X, d_X)$, the length metric $\bar{d}_X$ is defined as follows: For two points $x_1, x_2 \in X$, we put $\bar{d}_X(x_1, x_2)$ to the infimum of the length of curves connecting $x_1$ and $x_2$ with respect to $d_X$. A metric space $(X, d_X)$ is said to be a length space if $d_X = \bar{d}_X$.

Let $(X, d_X)$ be a metric space. For an interval $I \subset \mathbb{R}$, let $\gamma : I \to X$ be a curve. We say that $\gamma$ is a normal minimal geodesic if for all $s, t \in I$, we have $d_X(\gamma(s), \gamma(t)) = |s - t|$, and $\gamma$ is a normal geodesic if for each $t \in I$, we have an interval $J \subset I$ with $t \in J$ such that $\gamma|_J$ is a normal minimal geodesic. A metric space $(X, d_X)$ is said to be a geodesic space if for every two points in $X$, there exists a normal minimal geodesic connecting them. A metric space is proper if all closed bounded subsets of the space are compact. The Hopf-Rinow theorem for length spaces states that if a length space $(X, d_X)$ is complete and locally compact, and if $d_X < \infty$, then $(X, d_X)$ is a proper geodesic space.
2.2. Riemannian manifolds with boundary. For $n \geq 2$, let $M$ be an $n$-dimensional, connected Riemannian manifold with (smooth) boundary with Riemannian metric $g$. For $p \in \text{Int } M$, let $T_p M$ be the tangent space at $p$ on $M$, and let $U_p M$ be the unit tangent sphere at $p$ on $M$. We denote by $\| \cdot \|$ the standard norm induced from $g$. If $v_1, \ldots, v_k \in T_p M$ are linearly independent, then we see $\|v_1 \wedge \cdots \wedge v_k\| = \sqrt{\det(g(v_i, v_j))}$. Let $d_M$ be the length metric induced from $g$. If $M$ is complete with respect to $d_M$, then the Hopf-Rinow theorem for length spaces tells us that the metric space $(M, d_M)$ is a proper geodesic space.

For $x \in \partial M$, and the tangent space $T_x \partial M$ at $x$ on $\partial M$, let $T_x^\perp \partial M$ be the orthogonal complement of $T_x \partial M$ in the tangent space at $x$ on $M$. Take $u \in T_x^\perp \partial M$. For the second fundamental form $S$ of $\partial M$, let $A_u : T_x \partial M \to T_x \partial M$ be the shape operator for $u$ defined as

$$g(A_u v, w) := g(S(v, w), u).$$

Let $u_x \in T_x^\perp \partial M$ denote the unit inner normal vector at $x$. The mean curvature $H_x$ at $x$ is defined by

$$H_x := \frac{1}{n - 1} \text{trace } A_{u_x}.$$

For the normal tangent bundle $T^\perp \partial M := \bigcup_{x \in \partial M} T_x^\perp \partial M$ of $\partial M$, let

$$0(T^\perp \partial M)$$

be the zero-section $\bigcup_{x \in \partial M} \{0_x \in T_x^\perp \partial M\}$ of $T^\perp \partial M$. For $r > 0$, we put

$$U_r(0(T^\perp \partial M)) := \bigcup_{x \in \partial M} \{t u_x \in T_x^\perp \partial M \mid t \in [0, r)\}.$$ 

For $x \in \partial M$, we denote by $\gamma_x : [0, T) \to M$ the normal geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. On an open neighborhood of $0(T^\perp \partial M)$ in $T^\perp \partial M$, the normal exponential map $\exp^\perp$ of $\partial M$ is defined as follows: For $x \in \partial M$ and $u \in T_x^\perp \partial M$, put

$$\exp^\perp(x, u) := \gamma_u(\|u\|).$$

Since the boundary $\partial M$ is smooth, there exists an open neighborhood $U$ of $\partial M$ such that $\exp^\perp|_{(\exp^\perp)^{-1}(U \setminus \partial M)}$ is a diffeomorphism onto $U \setminus \partial M$, and for every $p \in U$, we have a unique point $x \in \partial M$ such that $d_M(p, x) = d_M(p, \partial M)$, and then $\gamma_x|_{[0, d_M(p, \partial M)]}$ is a unique normal minimal geodesic in $M$ from $x$ to $p$. We call such an open set $U$ a normal neighborhood of $\partial M$. If $\partial M$ is compact, then for some $r > 0$, the set $U_r(\partial M)$ is a normal neighborhood of $\partial M$.

We say that a Jacobi field $Y$ along $\gamma_x$ is a $\partial M$-Jacobi field if $Y$ satisfies the following initial conditions:

$$Y(0) \in T_x \partial M, \quad Y'(0) + A_{u_x}Y(0) \in T_x^\perp \partial M.$$

We say that $\gamma_x(t_0)$ is a conjugate point of $\partial M$ along $\gamma_x$ if there exists a non-zero $\partial M$-Jacobi field $Y$ along $\gamma_x$ with $Y(t_0) = 0$. Let $\tau_1(x)$ denote
the first conjugate value for \( \partial M \) along \( \gamma_x \). It is well-known that for all \( x \in \partial M \) and \( t > \tau_1(x) \), we have \( t > d_M(\gamma_x(t), \partial M) \).

For all \( x \in \partial M \) and \( t \in [0, \tau_1(x)] \), we denote by \( \theta(t, x) \) the absolute value of the Jacobian of \( \exp_x^\perp \) at \( (x, tu_x) \in T^\perp_x \partial M \). For each \( x \in \partial M \), we choose an orthonormal basis \( \{e_{x,i}\}_{i=1}^{n-1} \) of \( T_x \partial M \). For each \( i = 1, \ldots, n-1 \), let \( Y_{x,i} \) be the \( \partial M \)-Jacobi field along \( \gamma_x \) with initial conditions \( Y_{x,i}(0) = e_{x,i} \) and \( Y'_{x,i}(0) = -A_{u,i} e_{x,i} \). Note that for all \( x \in \partial M \) and \( t \in [0, \tau_1(x)] \), we have \( \theta(t, x) = \|Y_{x,1}(t) \wedge \cdots \wedge Y_{x,n-1}(t)\| \).

This does not depend on the choice of the orthonormal basis.

2.3. Distance rigidity and metric rigidity. For \( i = 1, 2 \), let \( M_i \) be \( n \)-dimensional, connected complete Riemannian manifolds with boundary with Riemannian metric \( g_i \). For each \( i \), the boundary \( \partial M_i \) carries the induced Riemannian metric \( h_i \).

**Definition 2.1.** We say that a homeomorphism \( \Phi : M_1 \to M_2 \) is a *Riemannian isometry with boundary* from \( M_1 \) to \( M_2 \) if \( \Phi \) satisfies the following conditions:

1. \( \Phi|_{\text{Int} M_1} : \text{Int} M_1 \to \text{Int} M_2 \) is smooth, and \( (\Phi|_{\text{Int} M_1})^*(g_2) = g_1 \);
2. \( \Phi|_{\partial M_1} : \partial M_1 \to \partial M_2 \) is smooth, and \( (\Phi|_{\partial M_1})^*(h_2) = h_1 \).

If we have a Riemannian isometry \( \Phi : M_1 \to M_2 \) with boundary, then the inverse \( \Phi^{-1} \) is also a Riemannian isometry with boundary.

It seems that the following is well-known.

**Lemma 2.1.** For \( i = 1, 2 \), let \( M_i \) be \( n \)-dimensional, connected complete Riemannian manifolds with boundary with Riemannian metric \( g_i \). Then we have a Riemannian isometry with boundary from \( M_1 \) to \( M_2 \) if and only if the metric space \( (M_1, d_{M_1}) \) is isometric to \( (M_2, d_{M_2}) \).

2.4. Comparison theorem. For \( \kappa \in \mathbb{R} \), let \( s_\kappa(t) \) be a unique solution of the so-called Jacobi-equation \( f''(t) + \kappa f(t) = 0 \) with initial conditions \( f(0) = 0 \) and \( f'(0) = 1 \).

The Laplacian \( \Delta \) of a smooth function on a Riemannian manifold is defined by the minus the trace of its Hessian. For manifolds without boundary under a lower Ricci curvature bound, we have the Laplacian comparison theorem for the distance function from a single point. For manifolds with boundary, we have:

**Lemma 2.2.** Let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M \geq (n-1)\kappa \). Take \( p \in \text{Int} M \) and \( u \in U_p M \). Let \( \rho_p : M \to \mathbb{R} \) be the function defined as \( \rho_p(q) := d_M(p, q) \), and let \( \gamma_u : [0, t_0) \to M \) be the normal minimal geodesic with initial conditions \( \gamma_u(0) = p \) and \( \gamma'_u(0) = u \) such that \( \gamma_u \)
lies in \(\text{Int} \, M\). Then for all \(t \in (0, t_0)\), we have
\[
\Delta \rho_p(\gamma_u(t)) \geq -(n-1) \frac{s'_{\kappa}(t)}{s_{\kappa}(t)}.
\]

3. Cut locus for the boundary

Let \(M\) be an \(n\)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \(g\).

3.1. Foot points. For a point \(p \in M\), we call \(x \in \partial M\) a foot point on \(\partial M\) of \(p\) if \(d_M(p,x) = d_M(p,\partial M\)). Since \((M,d_M)\) is proper, every point in \(M\) has at least one foot point on \(\partial M\).

**Lemma 3.1.** For \(p \in \text{Int} \, M\), let \(x \in \partial M\) be a foot point on \(\partial M\) of \(p\). Then we have a unique normal minimal geodesic \(\gamma : [0,l] \to M\) from \(x\) to \(p\) such that \(\gamma = \gamma|_{[0,l]}\), where \(l = \rho_{\partial M}(p)\). In particular, \(\gamma'(0) = u_x\) and \(\gamma|_{[0,l]}\) lies in \(\text{Int} \, M\).

**Proof.** Since \((M, d_M)\) is geodesic, we have a normal minimal geodesic \(\gamma : [0,l] \to M\) from \(x\) to \(p\). Since \(x\) is a foot point on \(\partial M\) of \(p\), we see that \(\gamma|_{[0,l]}\) lies in \(\text{Int} \, M\). We take a normal neighborhood \(U\) of \(\partial M\). If \(p \in U \setminus \partial M\), then \(x\) is a unique foot point on \(\partial M\) of \(p\), and \(\gamma = \gamma|_{[0,l]}\); in particular, we have \(\gamma'(0) = u_x\). Even if \(p \notin U \setminus \partial M\), then for every sufficiently small \(t > 0\), we see that \(x\) is the foot point on \(\partial M\) of \(\gamma(t)\). Hence, \(\gamma'(0) = u_x\). This implies \(\gamma = \gamma|_{[0,l]}\).

3.2. Cut locus. Let \(\tau : \partial M \to \mathbb{R} \cup \{\infty\}\) be the function defined as
\[
\tau(x) := \sup \{ t > 0 \mid \rho_{\partial M}(\gamma_x(t)) = t \}.
\]
We see that \(\tau\) is continuous on \(\partial M\). By Lemma 3.1 and the property of \(\tau_1\), for all \(x \in \partial M\), we have \(0 < \tau(x) \leq \tau_1(x)\).

By Lemma 3.1, we have the following:

**Lemma 3.2.** For all \(r > 0\), we have
\[
B_r(\partial M) = \exp\left( \bigcup_{x \in \partial M} \{ tu_x \mid t \in [0, \min\{r, \tau(x)\}] \} \right).
\]

**Proof.** Take \(p \in B_r(\partial M)\), and let \(x\) be a foot point on \(\partial M\) of \(p\). By Lemma 3.1, we have a unique normal minimal geodesic \(\gamma : [0,l] \to M\) from \(x\) to \(p\) such that \(\gamma = \gamma|_{[0,l]}\), where \(l = \rho_{\partial M}(p)\). Since \(x\) is a foot point on \(\partial M\) of \(p\), we have \(l \leq r\), and \(l \leq \tau(x)\). Hence,
\[
B_r(\partial M) \subset \exp\left( \bigcup_{x \in \partial M} \{ tu_x \mid t \in [0, \min\{r, \tau(x)\}] \} \right).
\]
On the other hand, take \( x \in \partial M \) and \( t \in [0, \min \{ r, \tau(x) \}] \). By the definition of \( \tau \), the point \( x \) is a foot point on \( \partial M \) of \( \gamma_x(t) \). Therefore, \( \rho_{\partial M}(\gamma_x(t)) = t \leq r \). This implies the opposite inclusion. \( \square \)

For the inscribed radius \( D(M, \partial M) \) of \( M \), from the definition of \( \tau \), it follows that \( \sup_{x \in \partial M} \tau(x) \leq D(M, \partial M) \). Lemma 3.1 implies the opposite. Hence, we have \( D(M, \partial M) = \sup_{x \in \partial M} \tau(x) \). We put \( TD_{\partial M} := \bigcup_{x \in \partial M} \{ t u_x \in T_x^\perp \partial M \mid t \in [0, \tau(x)] \} \), \( T\mathrm{Cut} \partial M := \bigcup_{x \in \partial M} \{ \tau(x) u_x \in T_x^\perp \partial M \} \), and define \( D_{\partial M} := \exp^\perp (TD_{\partial M}) \) and \( \mathrm{Cut} \partial M := \exp^\perp (T\mathrm{Cut} \partial M) \). We call \( \mathrm{Cut} \partial M \) the cut locus for the boundary \( \partial M \). By Lemma 3.1, we have \( \text{Int} M = (D_{\partial M} \setminus \partial M) \sqcup \mathrm{Cut} \partial M \) and \( M = D_{\partial M} \sqcup \mathrm{Cut} \partial M \).

The continuity of \( \tau \) tells us the following:

**Lemma 3.3.** Suppose that \( \partial M \) is compact. Then \( D(M, \partial M) < \infty \) if and only if \( M \) is compact.

**Proof.** If \( D(M, \partial M) < \infty \), then \( \sup_{x \in \partial M} \tau(x) < \infty \). By the continuity of \( \tau \), the set \( TD_{\partial M} \sqcup T\mathrm{Cut} \partial M \) is closed in \( T^\perp \partial M \). Since \( \partial M \) is compact, the set is compact in \( T^\perp \partial M \). The set \( D_{\partial M} \sqcup \mathrm{Cut} \partial M \) coincides with \( M \). The continuity of \( \exp^\perp \) implies that \( M \) is compact. On the other hand, if \( M \) is compact, then the function \( \rho_{\partial M} \) is finite on \( M \); in particular, \( D(M, \partial M) < \infty \). \( \square \)

Furthermore, we have:

**Proposition 3.4.** \( \text{vol}_g \mathrm{Cut} \partial M = 0 \).

**Proof.** By the continuity of \( \tau \), the graph \( \{(x, \tau(x)) \mid x \in \partial M \} \) of \( \tau \) is a null set of \( \partial M \times \mathbb{R} \), and \( T\mathrm{Cut} \partial M \) is also a null set of \( T^\perp \partial M \). Since \( \exp^\perp \) is smooth, we obtain \( \text{vol}_g \mathrm{Cut} \partial M = 0 \). \( \square \)

We have the following basic characterization of \( \tau \). The proof is left to the readers.

**Lemma 3.5.** Take \( x \in \partial M \) with \( \tau(x) < \infty \). Then \( T = \tau(x) \) if and only if \( T = \rho_{\partial M}(\gamma_x(T)) \), and at least one of the following holds:

1. \( \gamma_x(T) \) is the first conjugate point of \( \partial M \) along \( \gamma_x \);  
2. there exists a foot point \( y \in \partial M \setminus \{x\} \) on \( \partial M \) of \( \gamma_x(T) \).

From Lemma 3.5, we derive the following:

**Lemma 3.6.** \( \mathrm{Cut} \partial M \cap D_{\partial M} = \emptyset \). In particular, 
\[ \text{Int} M = (D_{\partial M} \setminus \partial M) \sqcup \mathrm{Cut} \partial M, \quad M = D_{\partial M} \sqcup \mathrm{Cut} \partial M. \]
Proof. Suppose that we have \( p \in \text{Cut } \partial M \cap D_{\partial M} \). Then we have \( x, y \in \partial M \) and \( l \in (0, \tau(y)) \) such that \( p = \gamma_x(\tau(x)) = \gamma_y(l) \). By the definition of \( \tau \), we have \( l = \tau(x) \); in particular, \( x \neq y \). Furthermore, by the definition of \( \tau \), we see that \( x \) and \( y \) are foot points on \( \partial M \) of \( p \). By Lemma 3.9, we have \( l = \tau(y) \). This is a contradiction. Therefore, we have \( \text{Cut } \partial M \cap D_{\partial M} = \emptyset \). Since \( \text{Int } M = (D_{\partial M} \setminus \partial M) \cup \text{Cut } \partial M \) and \( M = D_{\partial M} \cup \text{Cut } \partial M \), we prove the lemma. \( \square \)

For the connectedness of the boundary, we show:

Lemma 3.7. If \( \text{Cut } \partial M = \emptyset \), then \( \partial M \) is connected.

Proof. Suppose that \( \partial M \) is not connected. Let \( \partial M_i, i \geq 2 \), be the connected components of \( \partial M \). By Lemma 3.5 for every \( p \in D_{\partial M} \setminus \partial M \), we have a unique foot point on \( \partial M \) of \( p \). For each \( i \), we denote by \( D_{\partial M_i} \) the set of all points in \( D_{\partial M} \setminus \partial M \) whose foot points are contained in \( \partial M_i \). By the continuity of \( \tau \), the sets \( D_{\partial M_i} \setminus \partial M, i \geq 2 \), are mutually disjoint domains in \( \text{Int } M \). Lemma 3.6 implies that \( \text{Int } M \) coincides with \( (\bigcup_{i \geq 2} D_{\partial M_i}) \sqcup \text{Cut } \partial M \). Since \( \text{Cut } \partial M = \emptyset \), the set \( \text{Int } M \) is not connected. This is a contradiction. \( \square \)

By the continuity of \( \tau \), the set \( TD_{\partial M} \setminus 0(T^\perp \partial M) \) is a domain in \( T^\perp \partial M \). Using Lemma 3.5 we see the following:

Lemma 3.8. \( TD_{\partial M} \setminus 0(T^\perp \partial M) \) is a maximal domain in \( T^\perp \partial M \) on which \( \text{exp}^\perp \) is regular and injective.

We show the smoothness of \( \rho_{\partial M} \) on the set \( \text{Int } M \setminus \text{Cut } \partial M \).

Proposition 3.9. The function \( \rho_{\partial M} \) is smooth on \( \text{Int } M \setminus \text{Cut } \partial M \). Moreover, for each \( p \in \text{Int } M \setminus \text{Cut } \partial M \), the gradient vector \( \nabla \rho_{\partial M}(p) \) of \( \rho_{\partial M} \) at \( p \) is given by \( \nabla \rho_{\partial M}(p) = \gamma'(l) \), where \( \gamma : [0, l] \to M \) is the normal minimal geodesic from the foot point on \( \partial M \) of \( p \) to \( p \).

Proof. By Lemma 3.8 the map \( \text{exp}^\perp |_{TD_{\partial M}\setminus 0(T^\perp \partial M)} \) is a diffeomorphism onto \( D_{\partial M} \setminus \partial M \). Lemma 3.6 implies \( \text{Int } M \setminus \text{Cut } \partial M = D_{\partial M} \setminus \partial M \). For all \( q \in \text{Int } M \setminus \text{Cut } \partial M \), we have \( \rho_{\partial M}(q) = \| (\text{exp}^\perp)^{-1}(q) \|. \) Hence, \( \rho_{\partial M} \) is smooth on \( \text{Int } M \setminus \text{Cut } \partial M \).

For any vector \( v \in T_p M \), we take a smooth curve \( c : (-\epsilon, \epsilon) \to \text{Int } M \) tangent to \( v \) at \( p = c(0) \). We may assume \( c(s) \in \text{Int } M \setminus \text{Cut } \partial M \) when \( |s| \) is sufficiently small. By Lemma 3.5 we have a unique foot point \( \bar{c}(s) \) on \( \partial M \) of \( c(s) \). By Lemma 3.1 we obtain a smooth variation of \( \gamma \) by taking normal minimal geodesics in \( M \) from \( \bar{c}(s) \) to \( c(s) \). The first variation formula for the variation implies \( (\rho_{\partial M} \circ c)'(0) = g(v, \gamma'(l)) \). Therefore, we have \( \nabla \rho_{\partial M}(p) = \gamma'(l) \). \( \square \)
4. Comparison theorems

4.1. Basic comparison. We refer to the following absolute comparison inequality that has been shown in [HK].

**Lemma 4.1 ([HK]).** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$. Take a point $x \in \partial M$. Suppose that for all $t \in (0, \min\{\tau_1(x), \tilde{C}_{\kappa,\lambda}\})$, we have $\text{Ric}_g(\gamma_x'(t)) \geq (n-1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $t \in (0, \min\{\tau_1(x), \tilde{C}_{\kappa,\lambda}\})$, we have

$$\frac{\theta'(t, x)}{\theta(t, x)} \leq (n-1)\frac{s_{\kappa,\lambda}'(t)}{s_{\kappa,\lambda}(t)}.$$

**Remark 4.1.** In the case in Lemma 4.1, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_x$ with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{ux}e_{x,i}$. Then there exists $t_0 \in (0, \min\{\tau_1(x), \tilde{C}_{\kappa,\lambda}\})$ such that

$$\frac{\theta'(t_0, x)}{\theta(t_0, x)} = (n-1)\frac{s_{\kappa,\lambda}'(t_0)}{s_{\kappa,\lambda}(t_0)},$$

if and only if for all $i = 1, \ldots, n-1$ and $t \in [0, t_0]$, we have $Y_{x,i}(t) = s_{\kappa,\lambda}(t)E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along $\gamma_x$ with initial condition $E_{x,i}(0) = e_{x,i}$ (see [HK]).

The following Laplacian comparison theorem has been stated in [K2].

**Theorem 4.2 ([K2]).** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$. Take $x \in \partial M$. Suppose that for all $t \in (0, \tau(x))$, we have $\text{Ric}_g(\gamma_x'(t)) \geq (n-1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $t \in (0, \tau(x))$, we have

$$\Delta \rho_{\partial M}(\gamma_x(t)) \geq -(n-1)\frac{s_{\kappa,\lambda}'(t)}{s_{\kappa,\lambda}(t)}.$$

**Remark 4.2.** In the case in Theorem 4.2, for all $t \in (0, \tau(x))$, we have $\Delta \rho_{\partial M}(\gamma_x(t)) = -\theta'(t, x)/\theta(t, x)$. Therefore, the equality case in Theorem 4.2 results into that in Lemma 4.1 (see Remark 4.1).

By Lemma 4.1, we have the following relative comparison inequality.

**Lemma 4.3.** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$. Take a point $x \in \partial M$. Suppose that for all $t \in (0, \min\{\tau_1(x), \tilde{C}_{\kappa,\lambda}\})$, we
have $\text{Ric}_g(\gamma'_x(t)) \geq (n - 1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $s, t \in [0, \min\{\tau_1(x), \bar{C}_{\kappa, \lambda}\}]$ with $s \leq t$,
\[
\frac{\theta(t, x)}{\theta(s, x)} \leq \frac{s^{n-1}_{\kappa, \lambda}(t)}{s^{n-1}_{\kappa, \lambda}(s)}.
\]

In particular, if $\kappa$ and $\lambda$ satisfy the ball-condition, then $\tau_1(x) \leq C_{\kappa, \lambda}$.

Proof. Take $\bar{x} \in \partial M^n_{\kappa, \lambda}$. By Lemma 4.1, for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa, \lambda}\})$
\[
\frac{d}{dt} \log \frac{\theta(t, \bar{x})}{\theta(t, x)} = \frac{\theta'(t, \bar{x})}{\theta(t, \bar{x})} - \frac{\theta'(t, x)}{\theta(t, x)} \geq 0.
\]
Hence, for all $s, t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa, \lambda}\})$ with $s \leq t$, we have
\[
\frac{\theta(t, x)}{\theta(s, x)} \leq \frac{\theta(t, \bar{x})}{\theta(s, \bar{x})}.
\]
In the inequality, by letting $s \to 0$, we have $\theta(t, x) \leq \theta(t, \bar{x})$. Hence, for all $s, t \in [0, \min\{\tau_1(x), \bar{C}_{\kappa, \lambda}\}]$ with $s \leq t$, we have the desired inequality.

Let $\kappa$ and $\lambda$ satisfy the ball-condition. We suppose $C_{\kappa, \lambda} < \tau_1(x)$. For all $t \in [0, C_{\kappa, \lambda})$, we have $\theta(t, x) \leq s^{n-1}_{\kappa, \lambda}(t)$. By letting $t \to C_{\kappa, \lambda}$, we have $\theta(C_{\kappa, \lambda}, x) = 0$. Since $C_{\kappa, \lambda} < \tau_1(x)$, the point $\gamma_x(C_{\kappa, \lambda})$ is not a conjugate point of $\partial M$ along $\gamma_x$. Hence, there exists a $\partial M$-Jacobi field $Y$ along $\gamma_x$ such that $Y(C_{\kappa, \lambda}) = 0$; in particular, $\gamma_x(C_{\kappa, \lambda})$ is a conjugate point of $\partial M$ along $\gamma_x$. This is a contradiction. Therefore, we have $\tau_1(x) \leq C_{\kappa, \lambda}$. \hfill $\Box$

4.2. Inscribed radius comparison. Using Lemma 4.3 we will give a proof of the following lemma that has been already proved in [K3].

Lemma 4.4 ([K3]). Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n - 1)\kappa$ and $H_{\partial M} \geq \lambda$. Then for all $x \in \partial M$, we have $\tau(x) \leq C_{\kappa, \lambda}$; in particular, $D(M, \partial M) \leq C_{\kappa, \lambda}$.

Proof. Take $x \in \partial M$. By the definition of $\tau$, the geodesic $\gamma_x|_{[0, \tau(x)])}$ lies in $\text{Int} M$. If $C_{\kappa, \lambda} < \tau(x)$, then by Lemma 4.3 we see that $\gamma_x(C_{\kappa, \lambda})$ is a conjugate point of $\partial M$ along $\gamma_x$. We obtain $\tau_1(x) < \tau(x)$. This contradicts the relation between $\tau$ and $\tau_1$. Hence, $\tau(x) \leq C_{\kappa, \lambda}$. \hfill $\Box$

The following rigidity theorem has been proved in [K3].

Theorem 4.5 ([K3]). Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n - 1)\kappa$ and $H_{\partial M} \geq \lambda$. If we have a point $p \in M$ such that $\rho_{\partial M}(p) = C_{\kappa, \lambda}$, then the metric space $(M, d_M)$ is isometric to $(B^n_{\kappa, \lambda}, d_{B^n_{\kappa, \lambda}})$. 

4.3. **Volume comparison.** For the proof of Theorem 1.1 we need the following basic lemma. We omit the proof.

**Lemma 4.6.** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$. Let $\partial M$ be compact. Suppose that we have $r > 0$ such that $U_r(\partial M)$ is a normal neighborhood of $\partial M$. Then we have

$$\text{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^r \theta(t, x) \, dt \, d\text{vol}_h,$$

where $h$ is the induced Riemannian metric on $\partial M$.

From Lemma 4.6, we derive the following:

**Lemma 4.7.** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$. Let $\partial M$ be compact. Then for all $r > 0$, we have

$$\text{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^{\min\{r, \tau(x)\}} \theta(t, x) \, dt \, d\text{vol}_h,$$

where $h$ is the induced Riemannian metric on $\partial M$.

**Proof.** Take $r > 0$. By Lemma 3.2 we have

$$B_r(\partial M) = \exp^\perp \left( \bigcup_{x \in \partial M} \{tu_x \mid t \in [0, \min\{r, \tau(x)\}]\} \right).$$

From Lemma 3.8 it follows that the map $\exp^\perp$ is diffeomorphic on $\bigcup_{x \in \partial M} \{tu_x \mid t \in (0, \min\{r, \tau(x)\})\}$. Therefore, by Proposition 3.4 and Lemma 4.6 we have the desired equality. \(\square\)

We prove Theorem 1.1

**Proof.** We define a function $\bar{\theta} : [0, \infty) \times \partial M \to \mathbb{R}$ by

$$\bar{\theta}(t, x) := \begin{cases} \theta(t, x) & \text{if } t \leq \tau(x), \\ 0 & \text{if } t > \tau(x). \end{cases}$$

By Lemma 4.7 we have

$$\text{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^r \bar{\theta}(t, x) \, dt \, d\text{vol}_h.$$

Lemma 4.4 implies that for each $x \in \partial M$, we have $\tau(x) \leq \tilde{C}_{\kappa, \lambda}$. Therefore, from Lemma 4.3 it follows that for all $s, t \in [0, \infty)$ with $s \leq t$,

$$\bar{\theta}(t, x) \tilde{s}_{\kappa, \lambda}^{n-1}(s) \leq \bar{\theta}(s, x) \tilde{s}_{\kappa, \lambda}^{n-1}(t).$$
Integrating the both sides of the above inequality over $[0, r]$ with respect to $s$, and then doing that over $[r, R]$ with respect to $t$, we see
\[
\frac{\int_r^R \tilde{\theta}(t, x) \, dt}{\int_0^r \theta(s, x) \, ds} \leq \frac{\int_r^R s_{\kappa, \lambda}^{n-1}(t) \, dt}{\int_0^r s_{\kappa, \lambda}^{n-1}(s) \, ds}.
\]
Hence, we have
\[
\frac{\text{vol}_g B_R(\partial M)}{\text{vol}_g B_r(\partial M)} = 1 + \frac{\int_{\partial M} \int_0^r \tilde{\theta}(t, x) \, dt \, d\text{vol}_h}{\int_{\partial M} \int_0^r \theta(s, x) \, ds \, d\text{vol}_h} \leq 1 + \frac{\int_r^R s_{\kappa, \lambda}^{n-1}(t) \, dt}{\int_0^r s_{\kappa, \lambda}^{n-1}(s) \, ds} = \frac{\text{vol}_h B_R(\partial M_{n, \kappa, \lambda})}{\text{vol}_h B_r(\partial M_{n, \kappa, \lambda})}.
\]
This completes the proof of Theorem 1.1.

Remark 4.3. In the case in Theorem 1.1, we suppose that there exists $R > 0$ such that for all $r \in (0, R]$, we have
\[
\frac{\text{vol}_g B_R(\partial M)}{\text{vol}_g B_r(\partial M)} = \frac{\text{vol}_h B_R(\partial M_{n, \kappa, \lambda})}{\text{vol}_h B_r(\partial M_{n, \kappa, \lambda})}.
\]
In this case, for all $t \in (0, R]$ and $x \in \partial M$, we have $\tilde{\theta}(t, x) = s_{\kappa, \lambda}^{n-1}(t)$. We choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $Y_{x,i}$ be the $\partial M$-Jacobi field along $\gamma_x$ with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. For all $i = 1, \ldots, n-1$, and for all $t \in [0, \min\{R, C_{\kappa, \lambda}\}]$ and $x \in \partial M$, we have $Y_{x,i}(t) = s_{\kappa, \lambda}(t) E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along $\gamma_x$ with initial condition $E_{x,i}(0) = e_{x,i}$.

5. Volume growth rigidity

5.1. Volume growth. By Theorem 1.1, we have the following:

**Proposition 5.1.** Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Let $h$ denote the induced Riemannian metric on $\partial M$. Then
\[
\limsup_{r \to \infty} \frac{\text{vol}_g B_r(\partial M)}{\text{vol}_h B_r(\partial M)} \leq \text{vol}_h \partial M.
\]

**Proof.** Take $r > 0$. By Lemma 4.7, we have
\[
\text{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^{\min\{r, \tau(x)\}} \theta(t, x) \, dt \, d\text{vol}_h.
\]
By Lemma 4.3, for all $x \in \partial M$ and $t \in (0, \min\{r, \tau(x)\})$, we have $\theta(t, x) \leq s_{\kappa, \lambda}^{n-1}(t)$. Integrating the both sides of the inequality over $(0, \min\{r, \tau(x)\})$ with respect to $t$, and then doing that over $\partial M$ with
respect to $x$, we see $\text{vol}_g B_r(\partial M)/f_{n,\kappa,\lambda}(r) \leq \text{vol}_h \partial M$. Letting $r \to \infty$, we obtain the desired inequality. 

\section{Volume Growth Rigidity}

In the equality case in Theorem 1.1, $\tau$ satisfies the following property:

\textbf{Lemma 5.2.} Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Assume that there exists $R > 0$ such that for all $r \in (0, R]$, we have

$$\frac{\text{vol}_g B_R(\partial M)}{\text{vol}_g B_r(\partial M)} = \frac{\text{vol}_B(\partial M^n_{\kappa,\lambda})}{\text{vol}_B(\partial M^n_{\kappa,\lambda})}.$$ 

Then for all $x \in \partial M$, we have $\tau(x) \geq R$.

\textbf{Proof.} Suppose that for some $x_0 \in \partial M$, we have $\tau(x_0) < R$. Put $t_0 := \tau(x_0)$. Take $\epsilon > 0$ with $t_0 + \epsilon < R$. By the continuity of $\tau$, there exists a closed geodesic ball $B$ in $\partial M$ centered at $x_0$ such that for all $x \in B$, we have $\tau(x) \leq t_0 + \epsilon$. By Lemma 4.3, we see that $\text{vol}_g B_R(\partial M)$ is not larger than

$$\int_{\partial M \setminus B} \int_0^{\min\{R, \tau(x)\}} s_{n-1}^n(t) \, dt \, dh + \int_B \int_0^{t_0 + \epsilon} s_{n-1}^n(t) \, dt \, dh.$$ 

This is smaller than $(\text{vol}_h \partial M) f_{n,\kappa,\lambda}(R)$. On the other hand, by the assumption, we see that $f_{n,\kappa,\lambda}(R)$ is equal to $\text{vol}_g B_R(\partial M)/\text{vol}_h \partial M$. This is a contradiction. 

In the case in Lemma 5.2, for every $r \in (0, R)$, the level set $\rho_{\partial M}^{-1}(r)$ is an $(n-1)$-dimensional submanifold of $M$. In particular, $(B_r(\partial M), g)$ is an $n$-dimensional (not necessarily, connected) complete Riemannian manifold with boundary. We denote by $d_{B_r(\partial M)}$ and by $d_{\kappa,\lambda,r}$ the Riemannian distances on $(B_r(\partial M), g)$ and on $[0, r] \times_{\kappa,\lambda} \partial M$, respectively.

\textbf{Proposition 5.3.} Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Assume that there exists $R > 0$ such that for all $r \in (0, R]$, we have

$$\frac{\text{vol}_g B_R(\partial M)}{\text{vol}_g B_r(\partial M)} = \frac{\text{vol}_B(\partial M^n_{\kappa,\lambda})}{\text{vol}_B(\partial M^n_{\kappa,\lambda})}.$$ 

Then for every $r \in (0, R)$, the metric space $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda,r})$. 
Proof. Take \( r \in (0, R) \). By Lemma 5.2 for all \( x \in \partial M \), we have \( \tau(x) > r \); in particular, \( B_r(\partial M) \cap \text{Cut} \partial M = \emptyset \). Each connected component of \( \partial M \) one-to-one corresponds to the connected component of \( B_r(\partial M) \). Therefore, we may assume that \( B_r(\partial M) \) is connected.

By Theorem 1.1 for all \( t \in (0, R) \) and \( x \in \partial M \), we have \( \theta(t, x) = s_n(t) \). Choose an orthonormal basis \( \{e_{x,i}\}_{i=1}^{n-1} \) of \( T_x \partial M \). For each \( i = 1, \ldots, n-1 \), let \( Y_{x,i} \) be the \( \partial M \)-Jacobi field along \( \gamma_x \) with initial conditions \( Y_{x,i}(0) = e_{x,i} \) and \( Y'_{x,i}(0) = -\Lambda_{x,i} e_{x,i} \). For all \( t \in [0, \min\{R, \rho_{\partial M}\}] \) and \( x \in \partial M \), we have \( Y_{x,i}(t) = s_{\kappa,\lambda}(t) E_{x,i}(t) \), where \( E_{x,i} \) are the parallel vector fields along \( \gamma_x \) with initial condition \( E_{x,i}(0) = e_{x,i} \) (see Remark 4.3). Define a map \( \Phi : [0, r] \times \partial M \to B_r(\partial M) \) by \( \Phi(t, x) := \gamma_x(t) \).

For every \( p \in (0, r) \times \partial M \), the map \( D(\Phi|_{[0,r] \times \partial M})_p \) sends an orthonormal basis of \( T_p([0,r] \times \partial M) \) to that of \( T_{\Phi(p)} B_r(\partial M) \), and for every \( x \in \{0, r\} \times \partial M \), the map \( D(\Phi|_{[0,r] \times \partial M})_x \) sends an orthonormal basis of \( T_x ([0, r] \times \partial M) \) to that of \( T_{\Phi(x)} B_r(\partial M) \). Hence, \( \Phi \) is a Riemannian isometry with boundary from \([0, r] \times \partial M \) to \( B_r(\partial M) \).

5.3. Proof of Theorem 1.2. Let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \lambda \). Let \( \partial M \) be compact. We assume

\[
\liminf_{r \to \infty} \frac{\text{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \geq \text{vol}_h \partial M.
\]

By Proposition 5.1, for all \( r, R \in (0, \infty) \) with \( r \leq R \), we have

\[
\frac{\text{vol}_g B_R(\partial M)}{f_{n,\kappa,\lambda}(R)} = \frac{\text{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} = \text{vol}_h \partial M.
\]

From Lemma 5.2 it follows that for all \( x \in \partial M \), we have \( \tau(x) = C_{\kappa,\lambda} \).

If \( \kappa \) and \( \lambda \) satisfy the ball-condition, then Lemmas 3.3 and 4.4 imply that \( M \) is compact; in particular, we have a point \( p \in M \) such that \( \rho_{\partial M}(p) = D(M, \partial M) = C_{\kappa,\lambda} \). Hence, from Theorem 4.5, it follows that \((M, d_M)\) is isometric to \((B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})\).

If \( \kappa \) and \( \lambda \) do not satisfy the ball-condition, then \( \text{Cut} \partial M = \emptyset \). From Lemma 3.7 it follows that \( \partial M \) is connected. Take a sequence \( \{r_i\} \) with \( r_i \to \infty \). By Proposition 5.3 for each \( r_i \), we obtain a Riemannian isometry \( \Phi_i : [0, r_i] \times \kappa,\lambda \partial M \to B_{r_i}(\partial M) \) with boundary from \([0, r_i] \times \kappa,\lambda \partial M \) to \( B_{r_i}(\partial M) \) defined by \( \Phi_i(t, x) := \gamma_x(t) \). Since for all \( x \in \partial M \) it holds that \( \tau(x) = \infty \), we have a Riemannian isometry \( \Phi : [0, \infty) \times \kappa,\lambda \partial M \to M \) with boundary from \([0, \infty) \times \kappa,\lambda \partial M \) to \( M \) defined by \( \Phi(t, x) := \gamma_x(t) \) satisfying \( \Phi([0,r_i] \times \kappa,\lambda \partial M) = \Phi_i \). Hence, \((M, d_M)\) is isometric to \((0, \infty) \times \kappa,\lambda \partial M, d_{\kappa,\lambda}\). We complete the proof. \( \square \)
5.4. Curvature of the boundary. For the Ricci curvature on \( \partial M \), we have the following:

**Lemma 5.4.** Let \( M \) be an \( n \)-dimensional Riemannian manifold with boundary with Riemannian metric \( g \). Let \( h \) denote the induced Riemannian metric on \( \partial M \). Take a point \( x \in \partial M \), and choose an orthonormal basis \( \{ e_{x,i} \}_{i=1}^{n-1} \) of \( T_x \partial M \). Put \( u := e_{x,1} \). Then

\[
\text{Ric}_h(u) = \text{Ric}_g(u) - K_g(u_x, u) + \text{trace}\ A_S(u,u) - \sum_{i=1}^{n-1} \| S(u, e_{x,i}) \|^2,
\]

where \( K_g(u_x, u) \) is the sectional curvature at \( x \) in \((M, g)\) determined by \( u_x \) and \( u \).

**Proof.** Note that \( \text{Ric}_h(u) = \sum_{i=2}^{n-1} K_h(u, e_{x,i}) \). By the Gauss formula,

\[
\text{Ric}_h(u) = \sum_{i=2}^{n-1} K_g(u, e_{x,i}) + g(S(u, u), S(e_{x,i}, e_{x,i})) - \| S(u, e_{x,i}) \|^2.
\]

Since \( u, e_{x,2}, \ldots, e_{x,n-1}, u_x \) are orthogonal to each other, we have

\[
\text{Ric}_g(u) = \sum_{i=2}^{n-1} K_g(u, e_{x,i}) + K_g(u, u_x).
\]

On the other hand, we see

\[
\sum_{i=1}^{n-1} g(S(u, u), S(e_{x,i}, e_{x,i})) = \sum_{i=1}^{n-1} g(A_S(u,u))e_{x,i} = \text{trace} A_S(u,u).
\]

Combining these equalities, we have the formula. \qed

To study our rigidity cases, we need the following:

**Lemma 5.5.** Let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n - 1)\kappa \). If \((M, d_M)\) is isometric to \((\mathbf{R}, \kappa) \times \kappa \lambda \partial M, d_{\kappa, \lambda})\), then we have \( \text{Ric}_{\partial M} \geq (n - 2)(\kappa + \lambda^2) \).

**Proof.** We have a Riemannian isometry with boundary from \( M \) to \([0, \infty) \times_{\kappa, \lambda} \partial M\). For each \( x \in \partial M \), choose an orthonormal basis \( \{ e_{x,i} \}_{i=1}^{n-1} \) of \( T_x \partial M \). For each \( i = 1, \ldots, n - 1 \), let \( Y_{x,i} \) be the \( \partial M \)-Jacobi field along \( \gamma_x \) with initial conditions \( Y_{x,i}(0) = e_{x,i} \) and \( Y'_{x,i}(0) = -A_{u_x} e_{x,i} \). We have \( Y_{x,i}(t) = s_{\kappa, \lambda}(t) E_{x,i}(t) \), where \( E_{x,i} \) are the parallel vector fields along \( \gamma_x \) with initial condition \( E_{x,i}(0) = e_{x,i} \). Then \( A_{u_x} e_{x,i} = -Y'_{x,i}(0) = \lambda e_{x,i} \) and \( Y''_{x,i}(0) = \kappa e_{x,1} \). Hence, \( \text{trace} A_{u_x} = (n - 1)\kappa \) and \( K_g(u_x, e_{x,1}) = \kappa \). For all \( i \) we have \( S(e_{x,i}, e_{x,i}) = \lambda u_x \), and for all \( i \neq j \) we have \( S(e_{x,i}, e_{x,j}) = 0 \). By Lemma 5.4 and \( \text{Ric}_M \geq (n - 1)\kappa \), we have \( \text{Ric}_{\partial M} \geq (n - 2)(\kappa + \lambda^2) \). \qed
5.5. **Complement rigidity.** For $\kappa > 0$, let $M$ be an $n$-dimensional, connected complete Riemannian manifold (without boundary) with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$. By the Bishop volume comparison theorem ([BC]), $\text{vol}_g M \leq \text{vol} M^n_\kappa$; the equality holds if and only if $M$ is isometric to $M^n_\kappa$.

The following is concerned with the complements of metric balls.

**Corollary 5.6.** Let $\kappa \in \mathbb{R}$ and $-\lambda \in \mathbb{R}$ satisfy the ball-condition. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. Let $h$ denote the induced Riemannian metric on $\partial M$. If

$$\liminf_{r \to \infty} \frac{\text{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \geq \text{vol}_h \partial M, \quad \text{vol}_h \partial M \geq \text{vol}_{h_{n-1}} \partial B^n_{\kappa,\lambda},$$

then $(M, d_M)$ is isometric to $(M^n_\kappa \setminus \text{Int} B^n_{\kappa,\lambda}, d_{M^n_\kappa \setminus \text{Int} B^n_{\kappa,\lambda}})$.

**Proof.** By Theorem [1.2], $(M, d_M)$ is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Lemma [5.3] implies $\text{Ric}_{\partial M} \geq (n-2)(\kappa + \lambda^2)$. Since $\kappa$ and $-\lambda$ satisfy the ball-condition, $(\partial M, h)$ is a connected complete Riemannian manifold of positive Ricci curvature. By the assumption $\text{vol}_h \partial M \geq \text{vol}_{h_{n-1}} \partial B^n_{\kappa,\lambda}$, and by the Bishop volume comparison theorem, $(\partial M, h)$ is isometric to $(\partial B^n_{\kappa,\lambda}, h_{n-1}^{n-1})$. It turns out that $M$ and $M^n_\kappa \setminus \text{Int} B^n_{\kappa,\lambda}$ are isometric to each other as metric spaces. \qed

6. **Splitting theorems**

Let $M$ be a connected complete Riemannian manifold with boundary. A normal geodesic $\gamma : [0, \infty) \to M$ is said to be a ray if for all $s, t \in [0, \infty)$, we have $d_M(\gamma(s), \gamma(t)) = |s-t|$. For a ray $\gamma : [0, \infty) \to M$, the function $b_\gamma : M \to \mathbb{R}$ defined as

$$b_\gamma(p) := \lim_{t \to \infty} (t - d_M(p, \gamma(t)))$$

is called the busemann function of $\gamma$.

**Lemma 6.1.** Let $M$ be a connected complete Riemannian manifold with boundary. Suppose that for some $x_0 \in \partial M$, we have $\tau(x_0) = \infty$. Take a point $p \in \text{Int} M$. If $b_{\gamma_{x_0}}(p) = \rho_{\partial M}(p)$, then $p \notin \text{Cut} \partial M$. Moreover, for the unique foot point $x$ on $\partial M$ of $p$, we have $\tau(x) = \infty$.

**Proof.** Since $\tau(x_0) = \infty$, the normal geodesic $\gamma_{x_0} : [0, \infty) \to M$ is a ray. Since $\rho_{\partial M}$ is 1-Lipschitz, for all $q \in M$, we have $b_{\gamma_{x_0}}(q) \leq \rho_{\partial M}(q)$.

Take a foot point $x$ on $\partial M$ of $p$. Suppose $p \in \text{Cut} \partial M$. We have $\tau(x) < \infty$ and $p = \gamma_x(\tau(x))$. Take $\epsilon > 0$ with $B_\epsilon(p) \subseteq \text{Int} M$, and
a sequence \( \{t_i\} \) with \( t_i \to \infty \). For each \( i \), we take a normal minimal geodesic \( \gamma_i : [0, l_i] \to M \) from \( p \) to \( \gamma_{x_0}(t_i) \). Then \( \gamma_i|_{[0, \epsilon]} \) lies in \( \text{Int} \ M \).

Put \( u_i := \gamma'_i(0) \in U_pM \). By taking a subsequence, for some \( u \in U_pM \), we have \( u_i \to u \) in \( U_pM \). We denote by \( \gamma_u : [0, T) \to M \) the normal geodesic with initial conditions \( \gamma_u(0) = p \) and \( \gamma'_u(0) = u \). We have

\[
t_i - d_M(p, \gamma_{x_0}(t_i)) = -\epsilon + (t_i - d_M(\gamma_i(\epsilon), \gamma_{x_0}(t_i))).
\]

By letting \( i \to \infty \), we have \( b_{\gamma_{x_0}}(p) = -\epsilon + b_{\gamma_{x_0}}(\gamma_u(\epsilon)) \). From the assumption \( b_{\gamma_{x_0}}(p) = \rho_{\partial M}(p) \), it follows that \( \rho_{\partial M}(p) \leq -\epsilon + \rho_{\partial M}(\gamma_u(\epsilon)) \).

On the other hand, since \( \rho_{\partial M} \) is 1-Lipschitz, we have the opposite. Therefore, \( d_M(x, \gamma_u(\epsilon)) \) is equal to \( d_M(x, p) + d_M(p, \gamma_u(\epsilon)) \); in particular, we see \( u = \gamma'_x(\tau(x)) \).

Furthermore, \( \rho_{\partial M}(\gamma_x(\tau(x) + \epsilon)) = \tau(x) + \epsilon \). This contradicts the definition of \( \tau \). Hence, \( p \notin \text{Cut} \partial M \), and \( x \) is the unique foot point on \( \partial M \) of \( p \).

Put \( l := \rho_{\partial M}(p) \). We see that for every sufficiently small \( \epsilon > 0 \), we have \( b_{\gamma_{x_0}}(\gamma_x(l + \epsilon)) = \rho_{\partial M}(\gamma_x(l + \epsilon)) \). In particular, for all \( t \in [l, \infty) \), we have \( b_{\gamma_{x_0}}(\gamma_x(t)) = \rho_{\partial M}(\gamma_x(t)) \). It follows that \( \tau(x) = \infty \). \( \square \)

Let \( M \) be a connected complete Riemannian manifold with boundary, and let \( \gamma : [0, \infty) \to M \) be a ray. Take \( p \in \text{Int} \ M \), and a sequence \( \{t_i\} \) with \( t_i \to \infty \). For each \( i \), let \( \gamma_i : [0, l_i] \to M \) be a normal minimal geodesic from \( p \) to \( \gamma(t_i) \). Since \( \gamma \) is a ray, we have \( l_i \to \infty \). Take a sequence \( \{T_j\} \) with \( T_j \to \infty \). Since \( M \) is proper, we have a subsequence \( \{\gamma_{1,i}\} \) of \( \{\gamma_i\} \), and a normal minimal geodesic \( \gamma_{p,1} : [0, T_1] \to M \) from \( p \) to \( \gamma_{p,1}(T_1) \) such that \( \gamma_{1,i}|_{[0, T_1]} \) uniformly converges to \( \gamma_{p,1} \).

Furthermore, we have a subsequence \( \{\gamma_{2,i}\} \) of \( \{\gamma_{1,i}\} \), and a normal minimal geodesic \( \gamma_{p,2} : [0, T_2] \to M \) from \( p \) to \( \gamma_{p,2}(T_2) \) such that \( \gamma_{2,i}|_{[0, T_2]} \) uniformly converges to \( \gamma_{p,2} \), where \( \gamma_{p,2}|_{[0, T_1]} = \gamma_{p,1} \).

By a diagonal argument, we obtain a subsequence \( \{\gamma_k\} \) of \( \{\gamma_i\} \), and a ray \( \gamma_p : [0, \infty) \to M \) such that for every \( t \in (0, \infty) \), we have \( \gamma_k(t) \to \gamma_p(t) \) as \( k \to \infty \). We call such a ray \( \gamma_p \) an asymptote for \( \gamma \) from \( p \).

**Lemma 6.2.** Let \( M \) be a connected complete Riemannian manifold with boundary. Suppose that for some \( x_0 \in \partial M \), we have \( \tau(x_0) = \infty \). Take \( l > 0 \), and put \( p := \gamma_{x_0}(l) \). Then there exists \( \epsilon > 0 \) such that for all \( q \in B_r(p) \), all asymptotes for the ray \( \gamma_{x_0} \) from \( q \) lie in \( \text{Int} \ M \).

**Proof.** The proof is by contradiction. Suppose that there exists a sequence \( \{q_i\} \) in \( \text{Int} \ M \) with \( q_i \to p \) such that for each \( i \), we have an asymptote \( \gamma_i \) for \( \gamma_{x_0} \) from \( q_i \) such that \( \gamma_i \) does not lie in \( \text{Int} \ M \).

Now, \( M \) is proper. Therefore, by taking a subsequence of \( \{\gamma_i\} \), we may assume that there exists a ray \( \gamma_p : [0, \infty) \to M \) such that for every \( t \in [0, \infty) \), we have \( \gamma_i(t) \to \gamma_p(t) \) as \( i \to \infty \). Since all \( \gamma_i \) are...
asymptotes for $\gamma_{x_0}$, for all $t \in [0, \infty)$, we have $b_{\gamma_{x_0}}(q_i) = -t + b_{\gamma_{x_0}}(\gamma_i(t))$. By letting $i \to \infty$, we obtain $b_{\gamma_{x_0}}(p) = -t + b_{\gamma_{x_0}}(\gamma_p(t))$. Note that $b_{\gamma_{x_0}}(p) = \rho_{\partial M}(p)$. Since $\rho_{\partial M}$ is 1-Lipschitz, $d_M(\gamma_p(t), x_0)$ is equal to $d_M(\gamma_p(t), p) + d_M(p, x_0)$. In particular, $\gamma_p|_{(0, \infty)}$ coincides with $\gamma_{x_0}|_{(t, \infty)}$. Since $q_i \in \operatorname{Int} M$ for each $i$, we have $u_i := \gamma_i'(0) \in U_{q_i} M$. We have $q_i \to p$ in $M$. Therefore, by taking a subsequence of $\{u_i\}$, for some $u \in U_p M$ we have $u_i \to u$ in the unit tangent bundle on $\operatorname{Int} M$. Since $\gamma_p|_{(0, \infty)}$ coincides with $\gamma_{x_0}|_{(t, \infty)}$, we have $u = \gamma'_x(l)$. Put
\[ t_i := \sup\{t > 0 \mid \gamma_i([0, t)) \subset \operatorname{Int} M\} \]
and $x_i := \gamma_i(t_i) \in \partial M$. Since all $\gamma_i$ are asymptotes for $\gamma_{x_0}$, and since $\rho_{\partial M}(x_i) = 0$ for all $i$, we have
\[ b_{\gamma_{x_0}}(q_i) = -t_i + b_{\gamma_{x_0}}(x_i) \leq -t_i. \]
We see $b_{\gamma_{x_0}}(q_i) \to l$ as $i \to \infty$. Therefore, the sequence $\{t_i\}$ does not diverge. We may assume that for some $x \in \partial M$, the sequence $\{x_i\}$ converges to $x$ in $\partial M$. Since $u = \gamma'_x(l)$, the ray $\gamma_{x_0}$ passes through $x$. This contradicts that $\gamma_{x_0}|_{(0, \infty)}$ lies in $\operatorname{Int} M$. \hfill \Box

Let $M$ be a connected complete Riemannian manifold with boundary. Take a point $p \in \operatorname{Int} M$, and a continuous function $f : M \to \mathbb{R}$. We say that a function $\bar{f} : M \to \mathbb{R}$ is a support function of $f$ at $p$ if we have $\bar{f}(p) = f(p)$, and for all $q \in M$, we have $\bar{f}(q) \leq f(q)$.

Take a domain $U$ in $\operatorname{Int} M$. We say that $f$ is subharmonic on $U$ if for each $\epsilon > 0$, and for each $p \in U$, there exists a support function $f_{p, \epsilon} : M \to \mathbb{R}$ of $f$ at $p$ such that $f_{p, \epsilon}$ is smooth on an open neighborhood of $p$, and $\Delta f_{p, \epsilon}(p) \leq \epsilon$. The maximal principle in [C] tells us that if a subharmonic function on $U$ takes the maximal value at a point in $U$, then the function must be constant.

We prove Theorem 1.3 by using the maximal principle in [C].

Proof. For $\kappa \leq 0$, let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\operatorname{Ric}_M \geq (n - 1)\kappa$ and $H_{\partial M} \geq \sqrt{|\kappa|}$. Assume that for $x \in \partial M$, we have $\tau(x) = \infty$. Let $\partial M_0$ be the connected component of $\partial M$ containing $x$. Put
\[ \Omega := \{y \in \partial M_0 \mid \tau(y) = \infty\}. \]
The assumption implies $\Omega \neq \emptyset$. By the continuity of the function $\tau$, we see that $\Omega$ is closed in $\partial M_0$.

We show the openness of $\Omega$ in $\partial M_0$. Let $x_0 \in \Omega$. Take $l > 0$, and put $p_0 := \gamma_{x_0}(l)$. There exists an open neighborhood $U$ of $p_0$ in $\operatorname{Int} M$ such that $U \subset D_{\partial M}$. Taking $U$ smaller, we may assume that for each point $q \in U$, the unique foot point on $\partial M$ of $q$ is contained in $\partial M_0$. 

[References]
By Lemma \ref{lem:6.2}, we have \( \epsilon > 0 \) such that for all \( q \in B_\epsilon(p_0) \), asymptotes for \( \gamma_{x_0} \) from \( q \) lie in \( \text{Int} \, M \). We may assume \( U \subset B_\epsilon(p_0) \).

We prove that the function \( b_{\gamma_{x_0}} - \rho_{\partial M} \) is subharmonic on \( U \). By Proposition \ref{prop:3.9}, \( \rho_{\partial M} \) is smooth on \( U \). Fix a point \( q_0 \in U \), and take an asymptote \( \gamma_{q_0} : [0, \infty) \to M \) for \( \gamma_{x_0} \) from \( q_0 \). For \( t > 0 \), define a function \( b_{\gamma_{x_0},t} : M \to \mathbb{R} \) by

\[
b_{\gamma_{x_0},t}(p) := b_{x_0}(q_0) + t - d_M(p, \gamma_{q_0}(t)).
\]

We see that \( b_{\gamma_{x_0},t} - \rho_{\partial M} \) is a support function of \( b_{\gamma_{x_0}} - \rho_{\partial M} \) at \( q_0 \).

Since \( \gamma_{q_0} \) is a ray contained in \( \text{Int} \, M \), for every \( t \in (0, \infty) \), the function \( b_{\gamma_{x_0},t} \) is smooth on a neighborhood of \( q_0 \) in \( \text{Int} \, M \). By Lemma \ref{lem:2.2} we have \( \Delta b_{\gamma_{x_0},t}(q_0) \leq (n-1)(s'_\kappa(t)/s_\kappa(t)) \). Note that \( s'_\kappa(t)/s_\kappa(t) \to \sqrt{\kappa} \) as \( t \to \infty \). On the other hand, by Theorem \ref{thm:4.2} for all \( q \in U \), we have \( \Delta \rho_{\partial M}(q) \geq (n-1)\sqrt{\kappa} \). Hence, \( b_{\gamma_{x_0}} - \rho_{\partial M} \) is subharmonic on \( U \). The function \( b_{\gamma_{x_0}} - \rho_{\partial M} \) takes the maximal value 0 at \( p_0 \). The maximal principle in \cite{C} implies that \( b_{\gamma_{x_0}} \) coincides with \( \rho_{\partial M} \) on \( U \).

From Lemma \ref{lem:6.1}, it follows that \( \Omega \) is open in \( \partial M \).

For all \( x \in \partial M_0 \), we have \( \tau(x) = \infty \). We put

\[
TD_{\partial M_0} := \bigcup_{x \in \partial M_0} \{ t u_x \mid t \in (0, \infty) \}.
\]

By Lemma \ref{lem:3.8}, \( \exp^\perp |_{TD_{\partial M_0}} : TD_{\partial M_0} \to \exp^\perp(TD_{\partial M_0}) \) is a diffeomorphism. Therefore, the set \( \exp^\perp(TD_{\partial M_0}) \) is open and closed in \( \text{Int} \, M \).

Since \( \text{Int} \, M \) is connected, \( \exp^\perp(TD_{\partial M_0}) \) coincides with \( \text{Int} \, M \). In particular, \( \partial M \) is connected and \( \text{Cut} \, \partial M = \emptyset \). For all \( p \in \text{Int} \, M \), the equality in Theorem \ref{thm:4.2} holds at \( p \). For each \( x \in \partial M \), choose an orthonormal basis \( \{ e_{x,i} \}_{i=1}^{n-1} \) of \( T_x \partial M \). For each \( i = 1, \ldots, n-1 \), let \( Y_{x,i} \) be the \( \partial M \)-Jacobi field along \( \gamma_x \) with initial conditions \( Y_{x,i}(0) = e_{x,i} \) and \( Y'_{x,i}(0) = -A_{u_x} e_{x,i} \). Then we have \( Y_{x,i}(t) = s_{\kappa, \sqrt{\kappa}}(t) E_{x,i}(t) \),

where \( E_{x,i} \) is the parallel vector fields along \( \gamma_x \) with initial condition \( E_{x,i}(0) = e_{x,i} \) (see Remark \ref{rem:4.2}). Define a map \( \Phi : [0, \infty) \times \partial M \to M \) by \( \Phi(t, x) := \gamma_x(t) \).

For every \( p \in (0, \infty) \times \partial M \), the map \( D(\Phi)|_{(0, \infty) \times \partial M} \) sends an orthonormal basis of \( T_p((0, \infty) \times \partial M) \) to that of \( T_{\Phi(p)} M \), and for every \( x \in \{ 0 \} \times \partial M \), the map \( D(\Phi)|_{\{ 0 \} \times \partial M} \) sends an orthonormal basis of \( T_x((0, \infty) \times \partial M) \) to that of \( T_{\Phi(x)} \partial M \). Therefore, \( \Phi \) is a Riemannian isometry with boundary from \( [0, \infty) \times \kappa, \sqrt{\kappa} \partial M \) to \( M \). We complete the proof of Theorem \ref{thm:1.3}.

The Cheeger-Gromoll splitting theorem \cite{CG1} states that if \( M \) is an \( n \)-dimensional, connected complete Riemannian manifold of non-negative Ricci curvature, and if \( M \) contains a line, then there exists
an \((n - 1)\)-dimensional Riemannian manifold \(N\) of non-negative Ricci curvature such that \(M\) is isometric to the standard product \(\mathbb{R} \times N\).

**Corollary 6.3.** For \(\kappa \leq 0\), let \(M\) be an \(n\)-dimensional, connected complete Riemannian manifold with boundary such that \(\text{Ric}_M \geq (n - 1)\kappa\) and \(H_{\partial M} \geq \sqrt{|\kappa|}\). Suppose that for some \(x \in \partial M\), we have \(\tau(x) = \infty\). Then there exist \(k \in \{0, \ldots, n - 1\}\), and an \((n - 1 - k)\)-dimensional, connected complete Riemannian manifold \(N\) of non-negative Ricci curvature containing no line such that \((\partial M, d_{\partial M})\) is isometric to the standard product metric space \((\mathbb{R}^k \times N, d_{\mathbb{R}^k \times N})\). In particular, \((M, d_M)\) is isometric to \(([0, \infty) \times \kappa, \sqrt{|\kappa|}(\mathbb{R}^k \times N), d_{\kappa, \sqrt{|\kappa|}})\).

**Proof.** From Theorem 1.3, it follows that the metric space \((M, d_M)\) is isometric to \(([0, \infty) \times \kappa, \sqrt{|\kappa|}(\mathbb{R}^k \times N), d_{\kappa, \sqrt{|\kappa|}})\). Lemma 5.5 implies \(\text{Ric}_{\partial M} \geq 0\). Applying the Cheeger-Gromoll splitting theorem to \(\partial M\) inductively, we see that \((\partial M, d_{\partial M})\) is isometric to \((\mathbb{R}^k \times N, d_{\mathbb{R}^k \times N})\) for some \(k\). \(\square\)

### 7. The First Eigenvalues

#### 7.1. Lower bounds

Let \(M\) be a connected, complete Riemannian manifold with boundary with Riemannian metric \(g\). For a relatively compact domain \(\Omega\) in \(M\) such that \(\partial \Omega\) is a smooth hypersurface in \(M\), we denote by \(\text{vol}_{\partial \Omega}\) the Riemannian volume measure on \(\partial \Omega\) induced from the induced Riemannian metric on \(\partial \Omega\). For \(\alpha \in (0, \infty)\), the **Dirichlet \(\alpha\)-isoperimetric constant** \(ID_\alpha(M)\) of \(M\) is defined as

\[
ID_\alpha(M) := \inf_{\Omega} \frac{\text{vol}_{\partial \Omega} \partial \Omega}{(\text{vol}_g \Omega)^{1/\alpha}},
\]

where the infimum is taken over all relatively compact domains \(\Omega\) in \(M\) such that \(\partial \Omega\) is a smooth hypersurface in \(M\) and \(\partial \Omega \cap \partial M = \emptyset\). The **Dirichlet \(\alpha\)-Sobolev constant** \(SD_\alpha(M)\) of \(M\) is defined as

\[
SD_\alpha(M) := \inf_{f \in W^{1,1}_0(M)} \frac{\int_M \|\nabla f\| d\text{vol}_g}{\left(\int_M |f|^{\alpha} d\text{vol}_g\right)^{1/\alpha}}.
\]

For all \(\alpha \in (0, \infty)\), we have \(ID_\alpha(M) = SD_\alpha(M)\). This relationship between the isoperimetric constant and the Sobolev constant has been formally established in [FF] (see e.g., [Ch], [Li]), and later used in [Che] for the estimate of the first Dirichlet eigenvalue of the Laplacian.

The following volume estimate has been proved in [K5].

**Proposition 7.1 ([K5]).** Let \(M\) be an \(n\)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \(g\)
such that \( \text{Ric}_M \geq (n-1)\kappa \) and \( H_{\partial M} \geq \lambda \). Let \( \Omega \) be a relatively compact domain in \( M \) such that \( \partial \Omega \) is a smooth hypersurface in \( M \). Then

\[
\text{vol}_g \Omega \leq \text{vol}_{\partial \Omega} \sup_{t \in (\delta_1(\Omega), \delta_2(\Omega))} \int_t^{\delta_2(\Omega)} s_{n-1}^{n-1}(s) ds \frac{s_{n-1}^{n-1}(t)}{s_{n-1}^{n-1}(t)},
\]

where \( \delta_1(\Omega) := \inf_{p \in \Omega} \rho_{\partial M}(p) \) and \( \delta_2(\Omega) := \sup_{p \in \Omega} \rho_{\partial M}(p) \).

The equality case in Proposition 7.1 has been also studied in [K5]. We prove Theorem 1.4.

\[\text{Proof.}\] Let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n-1)\kappa \), \( H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \). Let \( \partial M \) be compact. Recall that the positive constant \( C(n, \kappa, \lambda, D) \) is defined as

\[
C(n, \kappa, \lambda, D) := \sup_{t \in [0, D]} \int_t^D s_{n-1}^{n-1}(s) ds \frac{s_{n-1}^{n-1}(t)}{s_{n-1}^{n-1}(t)}.
\]

Let \( \Omega \) be a relatively compact domain in \( M \) such that \( \partial \Omega \) is a smooth hypersurface in \( M \) and \( \partial \Omega \cap \partial M = \emptyset \). By Proposition 7.1

\[
\text{vol}_g \Omega \leq \text{vol}_{\partial \Omega} \sup_{t \in (0, D)} \int_t^D s_{n-1}^{n-1}(s) ds \frac{s_{n-1}^{n-1}(t)}{s_{n-1}^{n-1}(t)} = C(n, \kappa, \lambda, D) \text{vol}_{\partial \Omega} \partial \Omega.
\]

From the relationship \( ID_1(M) = SD_1(M) \), it follows that \( SD_1(M) \geq C(n, \kappa, \lambda, D)^{-1} \). Therefore, for all \( \phi \in W_0^{1,1}(M) \), we have the following Poincaré inequality:

\[
\int_M |\phi|^p d \text{vol}_g \leq C(n, \kappa, \lambda, D) \int_M \|\nabla \phi\| d \text{vol}_g.
\]

For a fixed \( p \in (1, \infty) \), let \( \psi \) be a non-zero function in \( W_0^{1,p}(M) \). Put \( q := p \left(1 - \frac{1}{p}\right)^{-1} \). In the Poincaré inequality, by replacing \( \phi \) with \( \psi^p \), and by the Hölder inequality, we see

\[
\int_M |\psi|^p d \text{vol}_g \leq p C(n, \kappa, \lambda, D) \int_M |\psi|^{p-1} \|\nabla \psi\| d \text{vol}_g \leq p C(n, \kappa, \lambda, D) \left( \int_M |\psi|^p d \text{vol}_g \right)^{1/p} \left( \int_M \|\nabla \psi\|^p d \text{vol}_g \right)^{1/p}.
\]

Considering the Rayleigh quotient \( R_p(\psi) \), we obtain the inequality \( \mu_{1,p}(M) \geq (p C(n, \kappa, \lambda, D))^{-p} \). This proves Theorem 1.4.

We next prove Theorem 1.5.
Proof. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary such that $\text{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Let $\partial M$ be compact. We put $D := D(M, \partial M) \in (0, \infty]$. We have

$$C(n, \kappa, \lambda, D) = ((n-1)\lambda)^{-1} \left( 1 - e^{-(n-1)\lambda D} \right).$$

The right hand side is monotone increasing as $D \to \infty$. By Theorem 1.4, for all $p \in (1, \infty)$ we have $\mu_{1,p}(M) \geq ((n-1)\lambda/p)^p$. We assume $\mu_{1,p}(M) = ((n-1)\lambda/p)^p$. By Theorem 1.4 we have $D = \infty$. Since $\partial M$ is compact, Lemma 3.3 and $D = \infty$ imply that $M$ is non-compact. It has been proved in [K3] as a splitting theorem (see Subsection 1.2) that if $M$ is non-compact and $\partial M$ is compact, then $(M,d_M)$ is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Therefore, $(M,d_M)$ is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$.

Let $p = 2$, and let $(M,d_M)$ be isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Let $\phi_{n,\kappa,\lambda} : [0,\infty) \to [0,\infty)$ be a smooth function defined by

$$\phi_{n,\kappa,\lambda}(t) := t e^{-\frac{(n-1)\lambda t}{2}}.$$

Then the smooth function $\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}$ on $M$ satisfies

$$\Delta_2(\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}) = \left( \frac{(n-1)\lambda}{2} \right)^2 (\phi_{n,\kappa,\lambda} \circ \rho_{\partial M})$$

on $M$; in particular,

$$\mu_{1,2}(M) \leq R_2(\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}) = \left( \frac{(n-1)\lambda}{2} \right)^2.$$

Therefore, $\mu_{1,2}(M) = ((n-1)\lambda/2)^2$. This proves Theorem 1.5. \qed

7.2. Segment inequality. For $n \geq 2$, $\kappa, \lambda \in \mathbb{R}$, and $D \in (0, \bar{C}_{\kappa,\lambda}]$, let $C_1(n, \kappa, \lambda, D)$ be the positive constant defined as

$$C_1(n, \kappa, \lambda, D) := \sup_{l \in (0,D)} \sup_{t \in (0,l)} \frac{s_{n-1}^{n-1}(l)}{s_{n-1}^{n-1}(t)}.$$

We prove the following segment inequality:

**Proposition 7.2.** For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric $g$ such that $\text{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M, \partial M) \leq D$. Let $f : M \to \mathbb{R}$ be a non-negative integrable function on $M$, and define a function $E_f : M \to \mathbb{R}$ by

$$E_f(p) := \inf_{x \in \partial M} \int_0^{\rho_{\partial M}(p)} f(\gamma_x(t)) \, dt,$$
where the infimum is taken over all foot points \( x \) on \( \partial M \) of \( p \). Then
\[
\int_M E_f \, d\text{vol}_g \leq C_1(n, \kappa, \lambda, D) D \int_M f \, d\text{vol}_g.
\]

Proof. Put \( C_1 := C_1(n, \kappa, \lambda, D) \). Fix \( x \in \partial M \) and \( l \in (0, \tau(x)) \). Observe that \( x \) is the unique foot point on \( \partial M \) of \( \gamma_x(l) \), and \( \gamma_x|_{0,l} \) lies in \( \text{Int} M \). By Lemma 4.3 for all \( t \in [0, l] \) we have
\[
E_f(\gamma_x(l)) \theta(l, x) \leq C_1 \int_0^l f(\gamma_x(t)) \theta(t, x) \, dt.
\]
Integrating the both sides, we see
\[
\int_{\tau(x)}^0 E_f(\gamma_x(l)) \theta(l, x) \, dl \leq C_1 D \int_{\tau(x)}^0 f(\gamma_x(t)) \theta(t, x) \, dt.
\]
Lemma 3.6 implies \( M = \exp^{\perp} (\bigcup_{x \in \partial M} \{ tu_x \mid t \in [0, \tau(x)] \}) \). From Lemma 3.8, it follows that \( \exp^{\perp} |_{T_{\partial M} \setminus 0} \) is a diffeomorphism onto \( D \setminus \partial M \). By Proposition 3.4, we have \( \text{vol}_g \text{Cut} \partial M = 0 \). Integrating the both sides of the above inequality over \( \partial M \) with respect to \( x \), we obtain the desired segment inequality. \( \square \)

From Proposition 7.2, we derive the following Poincaré inequality:

**Lemma 7.3.** For \( D \in (0, \tilde{C}_{\kappa, \lambda}] \setminus \{ \infty \} \), let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric \( g \) such that \( \text{Ric}_M \geq (n-1)\kappa, H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \). Let \( \psi : M \to \mathbb{R} \) be an integrable function on \( M \) such that \( \psi|_{\text{Int} M} \) is smooth on \( \text{Int} M \), and \( \psi|_{\partial M} = 0 \). Assume \( \int_M \|
abla \psi\| \, d\text{vol}_g < \infty \). Then
\[
\int_M |\psi| \, d\text{vol}_g \leq C_1(n, \kappa, \lambda, D) D \int_M \|
abla \psi\| \, d\text{vol}_g.
\]

Proof. Put \( f := \|\nabla \psi\| \), and let \( E_f \) be the function defined in Proposition 7.2. For each \( p \in D_{\partial M} \), let \( x \) be the foot point on \( \partial M \) of \( p \). By the Cauchy-Schwarz inequality, we have \( |\psi(p) - \psi(x)| \leq E_f(p) \).
Since \( \psi|_{\partial M} = 0 \), we have \( |\psi(p)| \leq E_f(p) \). Integrate the both sides of the inequality over \( D_{\partial M} \) with respect to \( p \). By Proposition 7.2 and \( \text{vol}_g \text{Cut} \partial M = 0 \), we arrived at the desired inequality. \( \square \)

We next prove the following weaker than Theorem 1.4.

**Proposition 7.4.** For \( D \in (0, \tilde{C}_{\kappa, \lambda}] \), let \( M \) be an \( n \)-dimensional, connected complete Riemannian manifold with boundary such that \( \text{Ric}_M \geq (n-1)\kappa, H_{\partial M} \geq \lambda \) and \( D(M, \partial M) \leq D \). Let \( M \) be compact. Then for all \( p \in (1, \infty) \), we have
\[
\mu_{1,p}(M) \geq (p C_1(n, \kappa, \lambda, D) D)^{-p}.
\]
Proof. For a fixed \( p \in (1, \infty) \), let \( \psi \) be a non-zero function in \( W^{1,p}_0(M) \). We may assume that \( \psi \) is smooth on \( M \). In Lemma [7.3] by replacing \( \psi \) with \( \psi^p \), we have

\[
\int_M |\psi|^p \, d\text{vol}_g \leq p C_1(n, \kappa, \lambda, D) D \int_M \|\nabla \psi\| \, d\text{vol}_g.
\]

From the Hölder inequality, we derive

\[
E_{p}(\psi) \geq (p C_1(n, \kappa, \lambda, D) D)^{-p}.
\]

This proves Proposition 7.4. \( \square \)

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