Topical Review

Theoretical cosmology

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Abstract
We review current theoretical cosmology, including fundamental and mathematical cosmology and physical cosmology (as well as cosmology in the quantum realm), with an emphasis on open questions.

Keywords: cosmology, theoretical, review

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1. Introduction

Cosmology concerns the study of the large scale behavior of the Universe within a theory of
gravity, which is usually assumed to be general relativity (GR)\(^3\). It has a unique nature that
makes it a distinctive science in terms of its relation to both scientific explanation and testing.

The uniqueness of the Universe

There is only one Universe which we effectively see from one spacetime point (because it
is so large) [1]. This is the foundational constraint in terms of both scientific theory (how do
we distinguish laws from initial conditions?) and observational testing of our models:

- We can only observe our Universe on one past light cone.
- We have to deduce four dimensional (4D) spacetime structure from a 2D image; distance
  estimations are therefore key.
- We cannot see many copies of the universe to deduce laws governing how universes operate.

Therefore, we have to compare the one universe with simulations of what might have been.

This consequently leads to the important question of what variations from our model need
explaining and what are statistical anomalies that do not need any explanation (i.e. cosmic
variance). This question arises, for example, regarding some cosmic microwave background
(CMB) anomalies.

The background model

Cosmology is the study of the behaviour of the Universe when small-scale structures (such
as, for example, stars and galaxies) can be neglected. The ‘Cosmological Principle’, which
can be regarded as a generalization of the Copernican principle, is often assumed to be valid.
This principle asserts that: \textit{On large scales the Universe can be well-modeled by a solution
to Einstein’s equations which is spatially homogeneous and isotropic.} This implies that a
preferred notion of cosmological time exists\(^4\) such that at each instant of time, space appears
the same in all directions (isotropy) and at all places (spatial homogeneity) on the largest
scales. This is, of course, certainly not true on smaller scales such as the astrophysical scales
of galaxies, and it would thus be better if the cosmological principle could be deduced rather
than assumed \textit{a priori} (i.e. could late time spatial homogeneity and isotropy be derived as a
dynamical consequence of the Einstein field equations (EFE) under suitable physical conditions
and for appropriate initial data). This has been addressed, in part, within the inflationary
paradigm, when scalar fields are dynamically important in the early Universe.

The cosmological principle leads to a background Friedmann–Lemaître–Robertson–
Walker (FLRW) model, and the EFE determine its dynamics. The concordance spatially
homogeneous and isotropic FLRW model (with a three-dimensional comoving spatial sec-
tion of constant curvature which is assumed simply connected) with a cosmological constant,
\(\Lambda\), representing the cosmological constant as an interpretation of dark energy, and CDM is
the acronym for cold dark matter (or so-called \(\Lambda\)CDM cosmology or standard cosmology for
short\(^5\), has been very successful in describing current observations. Early universe inflation

\(^3\)Commonly used acronyms used in this paper include: Cold dark matter (CDM). Cosmic microwave background
(CMB). Einstein field equations (EFE). Friedmann–Lemaître–Robertson–Walker (FLRW). Gravitational waves
(GW). General relativity (GR). Large scale structure (LSS). Linear perturbation theory (LPT). Loop quantum cos-
mology (LQC). Loop quantum gravity (LQG). Primordial gravitational waves (PGW). Primordial non-Gaussianities
(PNG). Quantum gravity (QG).

\(^4\)Except in the degenerate cases of spacetimes of constant curvature (de Sitter, anti-de Sitter and Minkowski space-
times). Such universe models do not correspond to the real Universe, which has preferred world lines everywhere [1].

\(^5\)In this review, by standard cosmology we shall mean an FLRW \(\Lambda\)CDM cosmological model that includes inflation
and linear perturbations but does not include a non-zero spatial curvature (see the text for details).
is often regarded as a part of the concordance model. The background spatial curvature of the universe, often characterized by the normalized curvature parameter ($\Omega_k$), is predicted to be negligible by most of simple inflationary models. Regardless of whether inflation is regarded as part of the standard model, spatial curvature is often assumed zero.

**Inhomogeneous models**

One of the greatest challenges in cosmology is understanding the origin of the structure of the Universe. An essential feature in structure formation is the study of inhomogeneities and anisotropies in cosmology. There are three approaches:

- Using exact solutions and properties where possible [2]. In particular, the Lemaître–Tolman–Bondi (‘LTB’) spherically symmetric dust model has been widely used, while the ‘Strenge Losungen’ approach of Ehlers, Kundt, Sachs, and Trümper at Hamburg provides a powerful method of examining generic properties of fluid solutions.
- Perturbed models where structure formation can be investigated, as pioneered by Lifshitz, Peebles, Sachs and Wolfe, Bardeen (see below).
- Numerical simulations, mainly Newtonian, but now being extended to the GR by various groups.

In particular, this enables an investigation of the scalar-tensor ratio and CMB polarization, and redshift space distortions and non-Gaussianities, which are key to testing inflationary universe models.

It is also important to consider the averaging, backreaction, and fitting problems relating the perturbed and background models. The main point here is that the same spacetime domain can be modeled at different averaging scales to obtain, for example, models representing galactic scales $L_1$, galaxy cluster scales $L_2$, large scale structure scales $L_3$, and cosmological scales $L_4$, with corresponding metrics, Ricci tensors, and matter tensors; the issue then is, firstly, how the FE at different scales are related [3] and, secondly, how observations at these different scales are related [4].

**Perturbed models**

In particular, the structure of the Universe can be investigated in cosmology via perturbed FLRW models. A technical issue that arises is the gauge issue: how do we map the background model (smooth) to a more realistic (lumpy) model? One must either handle gauge freedom by very carefully delineating what freedom remains at each stage of coordinate specialisation, or use gauge covariant variables (see later).

Cosmic inflation provides a causal mechanism for the generation of primordial cosmological perturbations in our Universe, through the generation of quantum fluctuations in the inflaton field which act as seeds for the observed anisotropies in the CMB and large scale structure (LSS) of our Universe. Although inflationary cosmology is not the only game in town, it is the simplest and perhaps the only scenario which is currently self-consistent from the point of view of low energy effective field theory. The recent Planck observations confirm that the primordial curvature perturbations are almost scale-invariant and Gaussian. In the standard cosmology, the primordial perturbations, corresponding to the seeds for the LSS, are chosen from a Gaussian distribution with random phases. This assumption is justified based on experimental evidence, regardless of whether or not inflation is assumed.

Predictions arising for matter power spectra and CMB anisotropy power spectra can then be compared with observations; this is a central feature of cosmology today. Together with comparisons of element abundance observations with primordial nucleosynthesis predictions, this has turned cosmology from philosophy to a solid physical theory. Finally, quantum fluctuations of the metric during inflation, imprinted in primordial B-mode perturbations of the
CMB, are perhaps the most vivid evidence conceivable for the reality of quantum gravity (QG) in the early history of our Universe. Indeed, any direct detection of primordial gravitational waves (PGW) and primordial non-Gaussianities (PNG) with the specific features predicted by inflation would provide strong independent support to this framework.

In this article we review the philosophical, mathematical, theoretical, physical (and quantum) challenges to the standard cosmology. For the most part, important non-theoretical issues (such as, for example, experiments and data analysis) are not discussed.

1.1. Fundamental issues

Cosmology is a strange beast. On the one hand it has evolved into a mature science, complete with observations, data analysis and numerical methods. On the other hand it contains philosophical assumptions that are not always scientific; this includes, e.g. the assumption of spatial homogeneity and isotropy at large scales outside our particle horizon and issues regarding inflation and the multiverse. As well as philosophical questions, there are fundamental physical problems (e.g. what is the appropriate model for matter, and what is the applicability of coarse graining) as well as mathematical issues (e.g. the gauge invariance problem in cosmology). Indeed, many of the open problems in theoretical cosmology involve the nature of the origin and details of cosmic inflation.

1.1.1. Open problems and GR. Noted problems have always been of importance and part of the culture in mathematics [5]. The twenty-three problems by Hilbert [6] are perhaps the most well known problems in mathematics. In addition, the set of fifteen problems presented in [7] nicely illustrate a number of open problems in mathematical physics. The most important and interesting unsolved problems in fundamental theoretical physics include foundational problems of quantum mechanics and the unification of particles and forces and the fine tuning problem in the quantum regime, the problem of quantum gravity and, of course, the problem of ‘cosmological mysteries’ [8]. However, it should be noted that some of them are in fact philosophical problems, in that they are not dealing with any conflict with observations.

We are primarily interested here in problems which we shall refer to as problems in theoretical cosmology, and particularly those that are susceptible to a rigorous treatment within mathematical cosmology. Problems in GR have been discussed elsewhere [9]. There are some problems in GR that are relevant in cosmology, and theorems can be extended into the cosmological regime by including models with matter. For example, generic spacelike singularities, traditionally regarded as being cosmological singularities, have been studied in detail [10]. It is also of interest to extend mathematical stability results to the case of a non-zero cosmological constant [11].

1.1.2. Philosophical issues. Philosophical problems have always played an important role in cosmology [12]; e.g. are we situated at the center of the Universe or not and, even in the earliest days of Einstein, is the Universe static or evolving. In addition, in cosmology the dynamical laws governing the evolution of the universe, the classical EFE, require boundary conditions to yield solutions. But in cosmology, by definition, there is no rest of the Universe to pass their specification off to. The cosmological boundary conditions must be one of the fundamental laws of physics.

There are a number of important philosophical issues that include the following: There is only one Universe. Consistency of one model does not rule out alternative models. What can a statistical analysis with only one data point tell us? What is observable? Due to the existence
of horizons, the Universe is only observed on or within our past light cone. A typical question in cosmology is: Why is the Universe so smooth. Must a suitable explanation be in terms of ‘genericity’ (of possible initial conditions), or can specialness lead to a possible explanation. There is no physical law that is violated by fine tuning. Indeed, perhaps the Universe is fine-tuned due to anthropic reasons. However, there are many caveats in describing physical processes (e.g. inflation) in terms of naturalness. Indeed, in cosmology the whole concept of ‘naturalness’ is suspect. Let us discuss some of these issues in a little more detail.

In observational cosmology, the amount of information that can be expected to be collected via astronomical observations is limited since we occupy a particular vantage point in the Universe; we are limited in what we can observe by visual and causal horizons (see discussion below). It can be argued that the observational limit may be approached in the foreseeable future, at least regarding some specific scientific hypotheses [12]. There is no certainty that the amount and types of information that can be collected will be sufficient to test all reasonable postulated hypotheses statistically. There is under-determination both in principle and in practice [12, 13]. This consequently leads to a natural view of model inference as inference to the best explanation/model, since some degree of explanatory ambiguity appears unavoidable in principle; inference in cosmology is based on a Bayesian interpretation of probability which includes a priori assumptions explicitly.

In physical cosmology, we are gravely compromised because we can only test physics directly up to the highest energies attainable by collisions at facilities such as the LHC, or from what we can deduce indirectly by cosmic ray observations. Hence we have to guess what extrapolation from known physics into the unknown is most likely to be correct; different extrapolations (e.g. string theory or loop quantum gravity) give different outcomes. As we cannot test directly the physics of inflation or of dark energy, theorists in fact rely mainly on Synge’s g-method discussed below: we conclude matter has the properties we would like it to have, in order to fit with astronomical observations.

1.1.3. Underlying theory. It has been argued [14] that the measure problem, and hence model inference, is ill defined due to ambiguity in the concepts of probability, in the situation where additional empirical observations cannot add any significant new information. However, inference in cosmological models can be made conceptually well-defined by extending the concept of probability to general valuations (using principles of uniformity and consistency) [14].

For example, an important area is empirical tests of the inflationary paradigm which necessitates, in principle, the specification or derivation of an a priori probability of inflation occurring (‘the measure problem’). The weakness of all models of inflation is consequently in the initial conditions [15]. To assert that the flatness of the Universe or the expected value for $\Lambda$ is predicted by inflation is absolutely meaningless without such an appropriate measure (this is particularly true in the case of the multiverse [16]).

The fundamental problem is that the theory of inflation cannot be proven to be correct. Falsifying a ‘bad theory’ (such as the the multiverse solution to the cosmological constant problem [17]) may be impossible [16], since parameters can be added without limit. But it should be possible to falsify a ‘good theory’, like inflation [18]. Perhaps the best way to make progress may be to probe the falsification of inflation, for which there is a robust predicted CMB polarization signal (induced by GW at the onset of inflation) [16].

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6 The commonly accepted solution to the mass hierarchy problem at the Planck scale necessitates an anti-de Sitter space-time and a negative $\Lambda$. However, if the sign of $\Lambda$ is allowed to have anthropic freedom, the concept of using Bayesian constraints to yield a non-zero value for $\Lambda$ from below must be discarded [16].
1.1.4. Assumptions. It is necessary to make assumptions to derive models to be used for cosmological predictions and check with observational data. But what precisely are these assumptions and how do they affect the results that come out; e.g. is the reason that small backreaction effects are obtained in computations because of the assumptions that are put in by hand at the beginning? We can only confirm the consistency of assumptions; we cannot rule out alternative explanations. The assumption of a FLRW background (cosmological principle) on cosmological scales presents a number of problems. There is no solid way to test spatial homogeneity, even in principle, by direct tests such as (redshift, distance observations), because we cannot control the possible time evolution of sources and so cannot be confident they are good standard candles (we do not, for example, have a solid understanding of supernova explosions and how they might depend on metallicity, or of radio source evolution).

However, observations of structure growth on the one hand and matter-light interactions via the kinematic Sunyaev–Zeldovich effect on the other do indeed give rise to solid constraints on inhomogeneity [19, 20], and indicate that approximate spatial homogeneity does indeed hold within our past light cone. Due to the existence of horizons, we can only observe the Universe on or within our past light cone (on cosmological scales). Assumptions beyond the horizon (Hubble scales) are impossible to test and so are, in effect, unscientific.

1.1.5. Homogeneity scale. The homogeneity scale is not actually theoretically determined, even in principle, in the standard cosmological model. It is just ‘pasted in’ to the standard model a postieri to help fit observations. Even then, what is the derived homogeneity scale implied (from the statistics observed). This question is important in the backreaction question.

There are a number of different approaches to the definition of a scale of statistical homogeneity. Even if we consider the standard model setting, then the homogeneity scale depends on the statistical measure used. But there are arguments that such a definition is not met and will never be met observationally [21]. Perhaps there is a different notion (e.g. using ergodicity) of statistical homogeneity in terms of an average positive density. But, any practical measure of statistical homogeneity is not directly based on a fundamental relation, but rather on the scale dependence of galaxy-galaxy correlation functions in observations [22].

Observationally, and based on the two-point correlation function, the smallest scale at which any measure of statistical homogeneity can emerge by the current epoch is in the range $70–120h^{-1}$ Mpc. Indeed, if all N-point correlations of the galaxy distribution are considered, then the homogeneity scale can only be reached, if at all, on scales above $700h^{-1}$ Mpc [21] (also see [23]).

1.1.6. Local and global coordinates. Perhaps, most importantly, what are the assumptions that underscore the use of an inertial coordinate system over a Hubble scale ‘background’ patch in which to do perturbation computations or in specifying initial conditions for numerical GR evolution. In particular, what are the assumptions necessary for the existence of Gaussian normal coordinates and hence a ‘global’ time and a ‘global’ inertial (Cartesian and orthogonal) spatial coordinate system (and thus a $1 + 3$ split) on the ‘background’ patch. This necessitates an irrotational congruence of fluid-comoving observers and, of course, is related to a choice of lapse and shift and hence a well defined gauge. And it essentially amounts to assuming that fluctuations propagate on a fixed absolute Newtonian background (with post-Newtonian corrections). Global inertial coordinates on a dynamical FLRW background in a GR framework are not conceptually possible [24]. A collection of spatially contiguous but causally disconnected regions which evolve according to GR on small scales do not generally evolve as a single collective background solution of GR on large cosmological scales.
1.1.7 Periodic boundary conditions in structure formation studies. In addition, what are the assumptions necessary for periodic boundary conditions (appropriate on scales comparable to the homogeneity scale) used in structure formation studies and numerical simulations? In particular, periodic boundary conditions impose a constraint on the global spatial curvature and force it to vanish [25, 26]. Strictly speaking, the (average) spatial curvature is only zero in the standard cosmology in which the FLRW universe possesses the space-time structure $R \times M^3$, in which $M^3$ is a three-dimensional spatial comoving simply connected infinite Euclidean three-space of constant curvature. The EFE govern the local properties of space-time but not the global geometry or the topology of the Universe at large. Nonstandard models with a compact spatial topology (or small universes) which are periodic due to topological identifications (and are hence not necessarily spatially flat) are also of interest and have observational consequences [27]. In particular, it has been shown that CMB data are compatible with the possibility that we live in a small Universe having the shape of a flat three-torus with a sufficiently large volume [28].

1.1.8. Weak field approach. Finally, what are the assumptions behind the weak field approach, the applicability of perturbation theory (and use of Fourier analysis), Gaussian initial conditions, averaging and the neglect of backreaction? To different degrees they all assume a small (or zero) spatial curvature. In particular, all global averages of spatial curvature are expected to coincide with that in the corresponding exact FLRW model to a high degree of accuracy when averaging linear Gaussian perturbations. In addition, in cosmology we can observe directions, redshifts, fluxes, but not distances. To infer a distance from observations in cosmology, we always use a model. Hence the real space correlation function and its Fourier transform, the power spectrum, are model dependent.

Essentially we conclude that within standard cosmology the spatial curvature is assumed to be zero (or at least very small and below the order of other approximations) in order for the analysis to be valid. In any case, the standard model cannot be used to predict a small spatial curvature. We will revisit the issue of spatial curvature later.

1.1.9. Quantum realm and multiverse. Are there possible differences from GR at very small scales that result from a theory of QG? In particular, do they have any relevance in the cosmological realm, and conversely what is the impact of cosmology on quantum mechanics [29]. For example, are there any fundamental particles that have yet to be observed and, if so, what are their properties? Do they (or the recently observed Higgs boson) have any relevance for cosmology. There is also the issue of whether singularities can be resolved in GR by quantum effects and whether singularity theorems are possible in higher dimensions, that are relevant in cosmology.

Both QG and inflation motivate the idea of a multiverse, in which there exists a wide range of fundamental theories (or, at least, different versions of the same fundamental theory with different physical parameters) and our own Universe is but one possibility [30]. In this scenario the question then arises as to why our own particular Universe has such finely tuned properties that allow for the existence of life. This has led to an explanation in terms of the so-called anthropic principle, which asserts that our Universe must have the properties it does because otherwise we would not be here to ask such a question. The cosmology of a multiverse leads to a number of philosophical questions. For example, is the multiverse even a scientific theory.

1.2. Definition of a cosmological model

A cosmological model has the following components [1].

*Spacetime geometry:* The spacetime geometry $(M, g)$ is defined by a smooth Lorentzian metric $g$ (characterizing the macroscopic gravitational field) defined on a smooth
differentiable manifold $M$ [31]. The scale over which the cosmological model is valid should be specified.

Field equations and equations of motion: To complete the definition of a cosmological model, we must specify the physical relationship (interaction) between the macroscopic geometry and the matter fields, including how the matter responds to the macroscopic geometry. We also need to know the trajectories along which the cosmological matter and light moves. In standard theory, the space-time metric, $g$, is determined by the matter present via the EFE:

\[ G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab} - \Lambda g_{ab} \]  

(1)

where the total energy momentum tensor, $T_{ab}$, is the sum of the stress tensors of all matter components present: $T_{ab} = \sum_i T_{ab}^{(i)}$, $\kappa$ is essentially the gravitational constant, and $\Lambda$ is the cosmological constant. In colloquial terms: \textit{Matter curves spacetime}. Because of the Bianchi identities, $R_{ab[cd]} = 0$, the definition on the left of (1) implies the identity $G_{ab}^{;b} = 0$ and hence, provided $\Lambda$ is indeed constant, that:

\[ G_{ab}^{;b} = 0 \Rightarrow T_{ab}^{;b} = 0; \]  

(2)

that is, energy-momentum conservation follows identically from the FE (1). The covariant derivatives in (2) depend on the space-time geometry, so in colloquial terms: \textit{Space-time tells matter how to move}. The key non-linearity of GR follows from the combination of these two statements, and the fact that $R_{ab}$ is a highly non-linear function of $g_{ab}(x)$. In GR a test particle follows a timelike or null geodesic. But a system that behaves as point particles on small scales may not necessarily do so on larger scales. That is, if the particles traverse timelike geodesics in the microgeometry, in principle, the macroscopic (averaged) matter need not follow timelike geodesics of the macrogeometry. However, the fundamental congruence is, in essence, the average of the timelike congruences along which particles move in the microgeometry, and defining the effective conserved energy-momentum tensor $T_{ab}^e$ through the EFE ensures timelike geodesic motion. In addition, the (average) motion of a photon is not necessarily on a null geodesic in the averaged macrogeometry, which will affect observations.

\textbf{Matter:} We require a consistent model for the matter on the characteristic cosmological (e.g. averaging) scale, and its appropriate (averaged) physical properties. Differentiation between the gravitational field and the matter fields is known not to be scale invariant and, in particular, a perfect fluid is not a scale invariant phenomenon [32]; averaging in the ‘mean field theory’ in the presence of gravity changes the equation of state of the matter [33]. In this framework all of the qualitative effects of averaging are absorbed into the redefined effective energy-momentum tensor $T_{ab}^e$ and the redefined effective equation state of the macro-matter, where $T_{ab}^e$ is conserved relative to the macrogeometery. The definition of the Landau frame for any combination of matter fields and radiation is invariant when matter and matter-radiation interactions take place due to local momentum conservation.

\textbf{Timelike congruence:} There is a preferred unit timelike congruence $u (u^a u_a = -1)$, defined locally at each event, associated with a family of fundamental observers (at late times) or the average motion of energy (at earlier times). In the case that there is more than one matter component, implying the existence of more than one fundamental macroscopic timelike congruence, we can always identify a fundamental macroscopic timelike
congruence represented by the four-velocity of the averaged matter in the model; i.e. the matter fields admit a formulation in terms of an averaged matter content which defines an average (macroscopic) timelike congruence. This then leads to a covariant 1 + 3 split of spacetime [1]. Mathematically this implies that the spacetime is topologically restricted and is \( I^-\)-non-degenerate, and consequently the spacetime is uniquely characterized by its scalar curvature invariants [34]. For example, for a perfect fluid \( \mathbf{u} \) is the timelike eigen-function of the Ricci tensor.

Observationally, this cosmological rest frame is determined as the frame wherein the CBR dipole is eliminated (the Solar System is moving at about 370 km s\(^{-1}\) relative to this rest frame). Note that the existence of this preferred rest frame is an important case of a broken symmetry: while the underlying theory is Lorentz invariant, it’s cosmologically relevant solutions are not (in particular, at no point in the history of the universe is it actually de-Sitter—much less anti-de Sitter).

**A note on modified theories of gravity:** Let us make a brief comment here. A key issue is whether GR is, in fact, the correct theory of gravity, especially on galactic and cosmological scales. Recent developments in testing GR on cosmological scales within modified theories of gravity were reviewed in [35, 36]. In particular, modified gravity theories have played an important role in the dark energy problem. Many questions can be posed in the context of modified gravity theories which include, for example, the general applicability of the BKL behaviour in the neighborhood of a cosmological singularity. We will not discuss such questions here, except for the particular question of whether isotropic singularities are typical in modified gravity theories.

### 1.3. Problems in mathematical cosmology

In GR, a sufficiently differentiable 4-dimensional Lorentzian manifold is assumed [31]. The Lorentz metric, \( g \), which characterizes the causal structure of \( M \), is assumed to obey the EFE, which constitute a hyperbolic system of quasi-linear partial differential equations which are coupled to additional partial differential equations describing the matter content [37]. The Cauchy problem is of particular interest, in which the unknown variables in the constraint equations of the governing EFE, consisting of a three-dimensional Riemannian metric and a symmetric tensor (in addition to initial data for any matter fields present), constitute the initial data for the remaining EFEs. Primarily the vacuum case is considered in attempting to prove theorems in GR, but this is not the case of relevance in cosmology. Viable cosmological models contain both matter and radiation, which in physically realistic cases then define a geometrically preferred timelike four-velocity field [1] which, because of (1), is related to an eigenvector of the matter stress tensor (which is unique if we assume realistic energy conditions [31]).

The EFE are invariant under an arbitrary change of coordinates (general covariance), which complicates the way they should be formulated in order for global properties to be investigated [38]. The vacuum EFEs are not hyperbolic in the normal sense. But utilizing general covariance, in harmonic coordinates the vacuum EFEs do represent a quasilinear hyperbolic system and thus the Cauchy problem is indeed well posed and local existence is guaranteed by standard results [39]. It can also be shown that if the constraints (and any gauge conditions) are satisfied initially, they are preserved by the evolution. Many analogues of the results in the vacuum case are known for the EFE coupled to different kinds of matter, including perfect fluids, gases governed by kinetic theory, scalar fields, Maxwell fields, Yang–Mills fields, and various combinations of these. Any results obtained for (perfect) fluids are generally only
applicable in restricted circumstances such as, for example, when the energy density is uniformly bounded away from zero (in the region of interest) [37]. The existence of global solutions for models with more exotic matter, such as stringy matter, has also been studied [40].

1.3.1. Singularity theorems. The concepts of geodesic incompleteness (to characterize singularities) and closed trapped surfaces [41] were first introduced in the singularity theorem due to Penrose [42]. Hawking and Ellis [43] then proved that closed trapped surfaces will indeed exist in the reversed direction of time in cosmology, due to the gravitational effect of the CMB. Hawking subsequently realized that closed trapped surfaces will also be present in any expanding Universe in its past, which would then inevitability lead to an initial singularity under reasonable conditions within GR [44]. This led to the famous Hawking and Penrose singularity theorem [45].

The singularity theorems prove the inevitability of spacetime singularities in GR under rather general conditions [42, 45], but they say very little about the actual nature of generic singularities. We should note that there are generic spacetimes which do not have singularities [46]. In particular, the proof of the Penrose singularity theorem does not guarantee that a trapped surface will occur in the evolution. It was proven [47] that for vacuum spacetimes a trapped surface can, indeed, form dynamically from regular initial data free of any trapped surfaces. This result was subsequently generalized in [48, 49]. A number of questions still exist, which include proving more general singularity theorems with weaker energy conditions and with weaker differentiability, and determining any relationship between geodesic incompleteness and the divergence of curvature [46]. But perhaps the most important open problem within GR is cosmic censorship [9].

1.3.2. Bouncing models. Using exotic matter, or alternative modified theories of gravity, can classically lead to the initial cosmological (or big bang) singularity being replaced by a big bounce, a smooth transition from contraction to an expanding universe [50], which may help to resolve some fundamental problems in cosmology. Bounce models have utilized ideas like branes and extra dimensions [51], Penrose’s conformal cyclic cosmology [52] (which leads to an interest in an isotropic singularity), string gas [53], and others [50, 54].

The matter bounce scenario faces significant problems. In particular, the contracting phase is unstable against anisotropies [55] and inhomogeneities [56]. In addition, there is no suppression of GW compared to cosmological perturbations, and hence the amplitude of GW (as well as possible induced non-Gaussianities) may be in excess of the observational bounds. In a computational study of the evolution of adiabatic perturbations in a nonsingular bounce within the ekpyrotic cosmological scenario [57], it was shown that the bounce is disrupted in regions with significant spatial inhomogeneity and anisotropy compared with the background energy density, but is achieved in regions that are relatively spatially homogeneous and isotropic.

The specially fine-tuned and simple examples studied to date, particularly those based on three spatial dimensions, scalar fields and, most importantly, a non-singular bounce that occurs at densities well below the Planck scale where QG effects are small [58], are arguably instructive in pointing to more physical bouncing cosmological models, and may present realistic alternatives to inflation to obtain successful structure formation (which we will discuss below).

The precise properties of a cosmic bounce depend upon the way in which it is generated, and many mechanisms have been proposed for this both classically and non-classically. Bounces can occur due to QG effects associated with string theory [59] and loop quantum gravity [60, 61]. In particular, in loop quantum cosmology there is a bounce when the energy density reaches a maximum value of approximately one half of the Planck density (although it
is also possible that bounces occur without a QG regime ever occurring [62], because if inflation occurs, the inflaton field violates the energy conditions needed for the classical singularity theorems to be applicable). We will discuss this in more detail later.

1.3.3. Mathematical results. Some applications in GR can be studied via Einstein–Yang–Mills (EYM) theory (which is relevant to cosmological models containing Maxwell fields and form fields and is perhaps a prototype to studying fields in, for example, string theory). Mathematical results when generalized to Maxwell and YM matter in 4D [63] are known (and have been studied in two dimensions less by wave maps with values on spheres [64, 65]).

Global existence in Minkowski spacetime, assuming initial data of sufficiently high differentiability, was first investigated in [66]. The uniqueness theorem for the 4D Schwarzschild spacetime was presented in [67]. The uniqueness theorem for the Kerr spacetime was proven in [68]. In the non-vacuum case the uniqueness of the rotating electrically charged black hole solution of Kerr–Newman has not yet been generally proven [69]. Once uniqueness has been established, the next step is to prove stability under perturbations. Minkowski spacetime has been shown to be globally stable [70, 71].

1.3.4. Extension to cosmology. Many of these problems in GR can be extended to the cosmological realm [9]. The uniqueness and stability of solutions to the EFE in GR are important7, and can be generalized to cosmological spacetimes (with a cosmological constant). Generic spacelike singularities are traditionally referred to as being cosmological singularities [10]. In particular, the stability of de Sitter spacetime will be discussed later. There are also a number of questions in the quantum realm [5], such as singularity resolution in GR by quantum effects and higher dimensional models, which are of interest in cosmology.

In essence the perturbation studies leading to theories of structure formation are stability studies of FLRW models. With ordinary equations of state, initial instabilities will grow but with a rate that depends on the background model expansion. Thus if there is no expansion, inhomogeneity will grow exponentially with time; with power law expansion, they will grow as a fractional power of time; and with exponential expansion, they will tend to freeze out. However, the way this happens depends on the comoving wavelength of the perturbation relative to the scale set by the Hubble expansion rate at that time8. These studies hold while the perturbation is linear, and have been heroically extended to the non-linear case (see later). However numerical simulations are required for the strongly non-linear case [25].

1.3.5. Computational cosmology. Numerical calculations have always played a central role in GR. Indeed, numerical computations support many of the conjectures in GR and their counterparts in cosmology and have led to a number of very important theoretical advances [9]. For example, the investigation of the mathematical stability of AdS spacetime includes fundamental numerical work and cosmic censorship is supported by numerical computations. In addition, the role of numerics in the understanding of the BKL dynamics, and in various other problems in cosmology and higher dimensional gravity, has been important. In fact,

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7 A full proof of the linear stability of Schwarzschild spacetime has recently been established [72]. The non-linear stability of the Schwarzschild spacetime is still elusive [73] (however, see [74]). Proving the non-linear stability of Kerr has become one of the primary areas of mathematical work in GR [71, 75]). All numerical results, and current observational data, provide evidence that the Kerr (and Kerr–Newman) black holes are non-linearly stable [76].

8 Often erroneously called the ‘Horizon’. It has nothing to do with causality, i.e. with effects related to the speed of light.
numerical computations are now commonly used to address fundamental issues within full GR cosmology [77–80].

1.4. Cosmological observations

What turns cosmology from a mathematical endeavour to a scientific theory is its ability to produce observational predictions that can be tested. Since the initiation of cosmology as a science by Lemaître in [81], telescopes of ever increasing power, covering all wavelengths and both Earth-based and in satellites, have led to a plethora of detailed tests of the models leading to the era of ‘precision cosmology’. The tests are essentially of two kinds: direct tests of the background models based on some kind of ‘standard candle’ or ‘standard ruler’, and indirect tests based on studying the statistics both of structures (inhomogeneities) on the one hand, and their effects on the CMB on the other. Both kinds of results produce broadly concordant results, but the latter give tighter restrictions on the background model than the former, because what kinds of structures can form depends on the dynamics of the background model. The basic restriction: The basic observational restriction in cosmology is that given the scales involved, we can only observe the Universe from one space-time event (‘here and now’) [1]. This would not be the case if the Universe were say the size of the Solar System, but that is not the case: a key discovery has been the immense size of the Universe, dwarfing the scales of galaxies which themselves dwarf the scale of the Solar System. This leads to major limits on what is observable, because of visual horizons for each kind of radiation or particle: for example, the CMB is observed on single surface (two-sphere) of last scattering. The furthest matter we can observe can be influenced by matter even further out, but such indirect effects are limited by the particle horizon: the furthest matter that can have had causal influence on us by influences travelling to us at speeds limited by the speed of light since the start of the Universe.

1.4.1. Anomalies. Within theoretical cosmology there needs to be an adequate explanation of observational anomalies, which are bound to occur as we make ever more detailed models of the structures and their effects on the CMB. Geometric optics must be utilized and model independent observations are sought. In general, data analysis and statistical methods are not discussed here. However, observations do, of course, lead to theoretical questions. Are there important neglected selection/detection effects [82]; i.e. what else can exist that we have not yet seen or detected? Observations sometimes lead to ridiculous predictions (e.g. \( w < -1 \); phantom matter); care must be taken not to be led into unphysical parameter space. Appropriate explanations of observational anomalies may well lead to new fundamental physics and questions.

The standard cosmology has been extremely successful in describing current observations, up to various possible anomalies and tensions [83], and particularly some statistical features in the CMB [84] and the existence of structures on gigaparsec scales such as the cold spot and some super-voids [85].

Although primordial nucleosynthesis has been very successful in accounting for the abundances of helium and deuterium, lithium has been found to be overpredicted by a factor of about three [86]. Lithium, along with deuterium, is destroyed in stars, and consequently it’s observation constitutes evidence (and a measure) of the primordial abundance after any appropriate corrections. To date there has been some claims of relief in this tension, but there is no satisfactory resolution of the lithium problem.

A seldom asked question is whether the CMB and matter dipoles are in agreement [87]. Tests of differential cosmic expansion on such scales rely on very large distance and redshift
catalogues, which are noisy and are subject to numerous observational biases which must be accounted for. In addition, ideally any test should be performed in a model independent manner, which requires removing the FLRW assumptions that are often taken for granted in many investigations. To date, such a model independent test has been performed for full sky spherical averages of local expansion [88], using the COMPOSITE and Cosmicflows-II catalogues; it was found with very strong Bayesian evidence that the spherically averaged expansion is significantly more uniform in the rest frame of the local group (LG) of galaxies than in the standard CMB rest frame. It was subsequently shown by that this result is consistent with Newtonian N-body simulations in the standard cosmology framework [89]. The future of such tests is discussed in [90], concluding that the amplitude of the matter dipole can be significantly larger than that of the CMB dipole. Its redshift dependence encodes information on the evolution of the Universe and on the tracers.

Perhaps more controversially, it has also been suggested that a ‘dark flow’ may be responsible for part of the motion of large objects that has been observed. An analysis of the local bulk flow of galaxies indicates a lack of convergence to the CMB frame beyond 100 Mpc [91], which contradicts standard cosmological expectations. Indeed, there is an anomalously high and approximately constant bulk flow of roughly 250 km s$^{-1}$ extending all the way out to the Shapley supercluster at approximately 260 Mpc, as indicated by low redshift supernova data. Furthermore, there is a discrepancy which has been confirmed by 6dF galaxy redshift data [92].

1.4.2. Tension in the Hubble constant. The recent determination of the local value of the Hubble constant based on direct measurements of supernovae made with the Hubble space telescope [93] is now 3.3 standard deviations higher than the value derived from the most recent data on the power spectrum temperature features in the CMB provided by the Planck satellite in a $\Lambda$CDM model. Although it is unlikely that there are no systematic errors (since the value of the Hubble constant has historically been a source of controversy), the difference might be a pointer towards new physics [94]. So this is perhaps the most important anomaly that needs to be addressed.

Although a large number of authors have proposed several different mechanisms to explain this tension, after three years of improved analyses and data sets, the tension in the Hubble constant between the various cosmological datasets not only persists but is even more statistically significant. The recent analysis of [93] found no compelling arguments to question the validity of the dataset used. Indeed, the recent determination of the local value of the Hubble constant by Riess et al in [93] of $H_0 = 73.24 \pm 1.74$ km s$^{-1}$ Mpc$^{-1}$ at 68% confidence level is now about 3 standard deviations higher than the (global) value derived from the earlier 2015 CMB anisotropy data provided by the Planck satellite assuming a $\Lambda$CDM model [95]. This tension only gets worse when we compare the Riess et al 2018 value of $H_0 = 73.52 \pm 1.62$ km s$^{-1}$ Mpc$^{-1}$ [96] to the Planck 2018 value of $H_0 = 67.27 \pm 0.60$ km s$^{-1}$ Mpc$^{-1}$ [84].

In order to investigate possible solutions to the Hubble constant tension a number of proposals have been made [97]. For example, in [98] it was shown that the best-fit to current experimental results includes an additional fourth, sterile, neutrino family with a mass of an eV order suggested by flavour oscillations. This would imply an additional relativistic degree of freedom ($N_{\text{eff}} = 4$) in the standard model, which may alleviate the $H_0$ tension. Recently it was argued that GW could represent a new kind of standard ‘sirens’ that will allow for $H_0$ to be constrained in a model independent way [99]. It is unlikely that inhomogeneities and cosmic variance can resolve the tension [26]. However, there are suggestions that the emergence of spatial curvature may alleviate the tension [26, 100–103]. Any definitive measurement of a non-zero spatial curvature would be crucial in cosmology. We will revisit this later.
2. Problems in theoretical cosmology

2.1. Acceleration: dark energy

The most fundamental questions in cosmology, perhaps, concern dark matter and dark energy, both of which are ‘detected’ by their gravitational interactions but can not be directly observed [104].

Indeed, the dark energy problem is believed to be one of the major obstacles to progress in theoretical physics [105, 106]. Weinberg discussed the *cosmological constant problem* in detail [107]. Conventional quantum field theory (QFT) predicts an enormous energy density for the vacuum. However, the GR equivalence principle asserts that all forms of mass and energy gravitate in an identical manner, which then implies that the vacuum energy is gravitationally equivalent to a cosmological constant and would consequently have a huge effect on the spacetime curvature. But the observed value for the effective cosmological constant is so very tiny (in comparison to the predictions of QFT) that a ‘bare’ cosmological constant, whose origin is currently not known, is necessary to cancel out this enormous vacuum energy to at least $10^{-120}$. This impossibly difficult fine-tuning problem becomes even worse if we include higher order LQG corrections [108].

A number of authors, including Weinberg, have offered the opinion that of all of the possible solutions to the dark energy problem, perhaps the most reasonable is the anthropic bound, which is itself very controversial [17]. However, another possibility is that the quantum vacuum does not gravitate. This will be true if the real gravitational theory is unimodular gravity, leading to the trace-free EFE [109].

Furthermore, the expansion of the Universe has been increasing for the last few billion years [110, 111]. Within the paradigm of standard cosmology, it is usually proposed that this acceleration is caused by a so-called dark energy, which effectively has the same properties as a very small cosmological constant (which is a repulsive gravitational force in GR). This *cosmological coincidence problem*, which necessitates a possible explanation for why the particular small observed valued of the cosmological constant currently is of a similar magnitude to that of the matter density in the Universe, is an additional problem. In particular, it is often postulated that dark energy is not due to a pure cosmological constant but that dynamical models such as, for example, quintessence and phantom energy scalar field models, are more reasonable. Alternative explanations for these gravitational effects have been proposed within theories with modified gravity on large scales, which consequently do not necessitate new forms of matter. The possibility of an effective acceleration of the Universe due to backreaction has also been discussed.

2.2. Acceleration: inflation

Inflation is a central part of modern theoretical cosmology. The assumption of zero spatial curvature ($k = 0$) is certainly well motivated in the standard model by inflation.

Before the development of inflation, it was already known that a scale invariant (Harrison-Zeldovich) power spectrum is a good fit to the data. But its origin was mysterious and there was no convincing physical mechanism to explain it. However, inflation naturally implies this property as a result of cosmological perturbations of quantum mechanical origin. Moreover, it allows a bridge to be built between theoretical considerations and actual astrophysical measurements. One fundamental assumption of inflation is that, initially, the quantum perturbations are placed in the vacuum state [112].
As noted earlier, models with a positive cosmological constant are asymptotic at late times to the inflationary de Sitter spacetime [113, 114]. Scalar field models with an increasing rate of (volume) expansion are also future inflationary. For models with an exponential potential, global asymptotic results can be obtained [115, 116]. Inflationary behavior is also possible in scalar field models with a power law potential, but typically occurs during an intermediate epoch rather than asymptotically to the future. Local results in this case are possible, but they are difficult to obtain and this problem is usually studied numerically.

There are a number of fundamental questions, which include the following. What exactly is the conjectured inflaton? What is the precise physical details of cosmic inflation? If inflation is self-sustaining due to the amplification of fluctuations in the quantum regime, is it still taking place in some (distant) regions of the Universe? And, if so, does inflation consequently give rise to an infinite number of ‘bubble universes’? In this case, under what (initial) conditions can such a multiverse exist? An investigation of ‘bubble universes’, in which our own Universe is but one of many that nucleate and grow within an ever-expanding false vacuum, has been undertaken (primarily computationally). For example, the interactions between such bubbles were investigated in [117].

Cosmological inflation is usually taken as a reasonable explanation for the fact that the Universe is apparently more uniform on larger scales than is anticipated within the standard cosmology (the horizon problem). However, there are other possible explanations. But how does inflation start? And, perhaps most importantly, what is the generality for the onset of inflation for generic spatially inhomogeneous initial data? We note that a rigorous formulation of this question is problematic due to the fact that there are so many different inflationary theories and since there are no ‘natural’ conditions for the initial data. However, any such natural initial conditions are expected to contain some degree of inhomogeneity9. Unfortunately, such initial data does not necessarily lead to inflation. Although it is known that large field inflation can occur for simple inhomogeneous initial data (at least for energies with substantial initial gradients and when the inflaton field is on the inflation supporting portion of the potential to begin with), it has also been shown that small field inflation is significantly less robust in the presence of inhomogeneities [119] (also see [117] and [120]).

2.2.1. Alternatives to inflation. Although inflation is the most widely acceptable mechanism for the generation of almost scale invariant (and nearly Gaussian adiabatic density) fluctuations to explain the origin of structure on large scales, possible alternatives include GR spikes [121], conformal cyclic cosmology [52] and QG fluctuations [122]. In particular, Penrose has argued that since inflation fails to take fully into account the huge gravitational entropy that would be associated with black holes in a generic spacetime, inflation is incredibly unlikely to start, and smooth out the universe, if its initial state is generic [52]. In addition, in the approach of [122] results from non-perturbative studies of QG regarding the large distance behavior of gravitational and matter two-point functions are utilized; non-trivial scaling dimensions exist due to a nontrivial ultraviolet renormalization group fixed point in 4D, motivating an explanation for the galaxy power spectrum based on the non-perturbative quantum field-theoretical treatment of GR. Perhaps the most widely accepted alternative to inflation to obtain successful structure formation and which is consistent with current observations [123] is the matter bounce scenario, in which new physics resolves the cosmological singularity.

9 Note that preliminary calculations in quantum field theory suggest that vacuum fluctuations could induce an enormous cosmological constant [118].
2.2.2. Bouncing models revisited. Bouncing models include the ekpyrotic and emergent string gas scenarios [123]. The ekpyrotic scenarios [51] are bouncing cosmologies which avoid the problems of the anisotropy and overproduction of GW in the matter bounce scenario, since the dynamics of the contracting phase is governed by a matter field (e.g. a scalar field with negative exponential potential) whose energy density increases faster than the contribution of anisotropies. In ekpyrotic scenarios, in which the bounce is not necessarily symmetric, fluctuations on all currently observable scales start inside the Hubble radius at earlier times, leading to structure that is formed causally and hence a solution of the horizon problem in the same way as in standard big bang cosmology and as in the usual matter bounce. But, contrary to the matter bounce scenario, during contraction the growth of fluctuations on super-Hubble scales is too weak to produce a scale-invariant spectrum from an initial vacuum state, leading to a subsequent blue spectrum of curvature fluctuations and GW [123]. Therefore, initial vacuum perturbations cannot describe the observed structures in the Universe. In addition, a negligible amplitude of GW is predicted on cosmological scales. However, a scale invariant spectrum of curvature fluctuations can be obtained by using primordial vacuum fluctuations in a second scalar field in the ekpyrotic scenario [124].

Another alternative to cosmological inflation is the emergent string gas scenario [53], based on a possible extended quasi-static period in the very early Universe dynamically dominated by a thermal gas of fundamental strings, after which there is a transition to the expanding radiation phase of standard cosmology. The thermal fluctuations of a gas of closed strings on a compact space with toroidal topology, which do not originate quantum mechanically (unlike in most models of inflation), then produce a scale-invariant spectrum of curvature fluctuations and GW. The tilt of the spectrum of curvature fluctuations is predicted to be red as in inflation, but that of the GW is slightly blue, in contrast to what is obtained in inflation.

We should note that although some of the alternatives to inflation are suggested by ideas motivated by QG, it is also of interest to know whether inflation occurs naturally within QG. We will discuss this later.

2.3. The physics horizon and Synge’s g-method

The physics horizon: The basic problem as regards inflation and any attempts to model what happened at earlier times in the history of the Universe is that we run into the physics horizon: we simply do not know what the relevant physics was at those early times. The reason is that we cannot construct particle colliders that extend to such high energies. Thus we are forced either to extrapolate tested physics at lower energies to these higher energies, with the outcome depending on what aspect of lower energy physics we decide to extrapolate (because we believe it is more fundamental than other aspects), or to make a phenomenological model of the relevant physics.

Synge’s g-method: A very common phenomenological method used is Synge’s g-method: running the EFE backwards [20]. That is, in eqn. (1), instead of trying to solve it from right to left (given a matter source, find a metric \(g\) that corresponds to that matter source), rather choose the metric and then find the matter source that fits. That is, select a metric \(g\) with some desirable properties, calculate the corresponding Ricci tensor \(R_{ab}\) and Einstein tensor \(G_{ab}\) and then use (1) to find the matter tensor \(T_{ab}\) so that (1) is identically satisfied, and voila! we have an exact solution of the EFE that has the desired geometric properties. No differential equations have to be solved. The logic is: via (1),

\[
\{g_{ab}\} \Rightarrow \{R_{ab}\} \Rightarrow \{G_{ab}\} \Rightarrow \{T_{ab}\}.
\]  (3)
One classic example is choosing an inflationary scale factor $a(t)$ that leads to structure formation in the early Universe that agrees with observations. We can then run the EFE backward as in (3) to find a potential $V(\phi)$ for an effective scalar field $\phi$ that will give the desired evolution $a(t)$. It is a theorem that one almost always can find such a potential [125], essentially because the energy momentum conservation equations are in that case equivalent to the Klein–Gordon equation for the field $\phi$; but there is no real physics behind claims of the existence of such a scalar field. It has not been related to any matter or field that has been demonstrated to exist in any other context.

2.4. Dynamical behaviour of cosmological solutions

The dynamical laws governing the evolution of the universe are the classical EFEs. It is of interest to study exact cosmological solutions and especially spatially inhomogeneous cosmologies [2], and their qualitative and numerical behaviour. Dynamical systems representations of the evolution of cosmological solutions are very useful [126, 127]. In particular, it is of interest to extend stability results to the study of cosmological models with matter and in the case of a non-zero cosmological constant [11].

2.4.1. Stability of cosmological solutions. This concerns the question of whether the evolution of the EFE under small perturbations is qualitatively similar to the evolution of the underlying exact cosmological solution (e.g. by including small-scale fluctuations). This problem involves the investigation of the (late time) behavior of a complex set of partial differential equations about a specific cosmological solution [128]. The asymptotic behaviour of solutions in cosmology was reviewed in [127].

We note that for a vanishing cosmological constant and matter that satisfies the usual energy conditions, spatially homogeneous spacetimes of (general) Bianchi type IX recollapse and consequently do not expand for ever. This result is formalized in the so-called closed universe recollapse conjecture [129], which was proven in [130]. However, Bianchi type IX spacetimes need not recollapse in the case that a positive cosmological constant is present. The study of the stability of de Sitter spacetime for generic initial data is very important, particularly within the context of inflation (although, as noted earlier, precise statements concerning the generality of inflation are problematic).

2.4.2. Stability of de Sitter spacetime. A stability result for de Sitter spacetime (vacuum and a positive cosmological constant) for small generic initial data was proven in [113]. Therefore, de Sitter spacetime is a local attractor for expanding cosmologies containing a positive cosmological constant. In addition, it was proven that any expanding spatially homogeneous model (in which the matter obeys the strong and dominant energy conditions) that does not recollapse is future asymptotic to an isotropic de Sitter spacetime [114]. This so-called ‘cosmic no–hair’ theorem is independent of the particular matter fields present. The remaining question is whether general, initially expanding, cosmological solutions corresponding to initial data for the EFE with a positive cosmological constant and physical matter exist globally in time. It is known that this is indeed the case for a variety of matter models (utilizing the methods of [131]). Global stability results have also been proven for inflationary models with a scalar field with an exponential potential [115, 116]. It is, of course, of considerable interest to investigate the cosmic no–hair theorem in the inhomogeneous case. A number of partial results are known in the case of a positive cosmological constant [132].
The possible quantum instability of de Sitter spacetime has also been investigated. In a semi-classical analysis of backreaction in an expanding universe with a conformally coupled scalar field and a cosmological constant, it was advocated that de Sitter spacetime is unstable to quantum corrections and might, in fact, decay. In principle, this could consequently provide a mechanism that might alleviate the cosmological constant problem and also, perhaps, the fine-tuning problems that occur for the very flat inflationary potentials that are necessitated by observations.

In particular, it has been suggested that de Sitter spacetime is unstable due to infrared effects, in that the backreaction of super-Hubble scale GW could contribute negatively to the effective cosmological constant and thereby cause the latter to diminish. Indeed, from an investigation of the backreaction effect of long wavelength cosmological perturbations it was found that at one LQG order super-Hubble cosmological perturbations do give rise to a negative contribution to the cosmological constant [133]. It has consequently been proposed that this backreaction could then lead to a late time scaling solution for which the contribution of the cosmological constant tracks the contribution of the matter to the total energy density; that is, the cosmological constant obtains a negative contribution from infrared fluctuations whose magnitude increases with time [101].

2.4.3. The nature of cosmological singularities. Although the singularity theorems imply that singularities occur generally in GR, they say very little about their nature [46]. For example, singularities can occur in tilted Bianchi cosmologies in which all of the scalar quantities remain finite [134]. However, such cosmological models are likely not generic. Belinski, Khalatnikov and Lifshitz (BKL) [135] have conjectured that within GR, and for a generic inhomogeneous cosmology, the approach to the spacelike singularity into the past is vacuum dominated, local and oscillatory, obeying the the so-called BKL or mixmaster dynamics. In particular, due to the non-linearity of the EFE, if the matter is not an effective massless scalar field, then sufficiently close to the singularity all matter terms can be neglected in the FE relative to the dynamical anisotropy. BKL have confirmed that the assumptions they utilized are consistent with the EFE. However, that does not imply that their assumptions are always valid in general situations of physical interest. Numerical simulations have recently been used to verify the BKL dynamics in special classes of spacetimes [136, 137]. Rigorous mathematical results on the dynamical behaviour of Bianchi type VIII and IX cosmological models have also been presented [138].

Up to now there have essentially been three main approaches to investigate the structure of generic singularities, including the original heuristic BKL metric approach and the so-called Hamiltonian approach. The dynamical systems approach [127], in which the EFE are reformulated as a scale invariant asymptotically regularized dynamical system (i.e. a first order system of autonomous ordinary or partial differential equations) in the approach towards a generic spacelike singularity, allows for a more mathematically rigorous study of cosmological singularities. A dynamical systems formulation of the EFE (in which no symmetries were assumed a priori) was presented in [139], which resulted in a detailed description of the generic attractor, precisely formulated conjectures concerning the asymptotic dynamical behavior toward a generic spacelike singularity, and a well-defined framework for the numerical study of cosmological singularities [140]. It should be noted that these studies assume that the singularity is spacelike, but there is no reason that this has to be so (this is not, in fact, generic). The effect of GR spikes on the BKL dynamics and on the initial cosmological singularity was reviewed in [9].
2.4.4. Isotropic singularity. Penrose [52] has utilized entropy considerations to motivate the ‘Weyl curvature hypothesis’ that asserts that on approach to an initial cosmological singularity the Weyl curvature tensor should tend to zero or at least remain bounded (this conjecture subsequently led to the conformal cyclic cosmology proposal). It is difficult to represent this proposal mathematically but the clearly formulated geometric condition presented in [141], that the conformal structure should remain regular at the singularity, is closely related to the original Penrose proposal. Such singularities are called isotropic or conformal singularities. It is known [142] that solutions of the EFE for a radiation perfect fluid that admit an isotropic singularity are uniquely characterized by particular free data specified at the singularity. The required data is essentially half as much as the data necessary in the case of a regular Cauchy hypersurface. This result was generalized to the case of a perfect fluid with a linear equation of state [143], and can be further extended to more general matter models (e.g. more general fluids and a collisionless gas of massless particles) [37].

As noted earlier, we do not aim to discuss alternative theories of gravity in this review. However, it is of cosmological interest to determine whether isotropic singularities are typical in any modified theories of gravity. For example, the past stability of the isotropic FLRW vacuum solution, on approach to an initial cosmological singularity, in the class of theories of gravity containing higher-order curvature terms in the GR Lagrangian, has been investigated [144]. In particular, a special isotropic vacuum solution was found to exist, which behaves like a radiative FLRW model, that is past stable to small anisotropies and inhomogeneities (which is not the case in GR). Exact solutions with an isotropic singularity for specific values of the perfect fluid equation of state parameter have also been obtained in a higher dimensional flat anisotropic Universe in Gauss–Bonnet gravity [145]. A number of simplistic cosmological solutions of theories of gravity containing a quadratic Ricci curvature term in the Einstein–Hilbert Lagrangian have also been investigated [146].

3. Problems in physical cosmology

The predicted distribution of dark matter in the Universe is based on observations of galaxy rotation curves, nucleosynthesis estimates and computations of structure formation [147]. The nature of the missing dark matter is not yet known (e.g. whether it is due to a particle or whether the dark matter phenomena is not characterized by any type of matter but rather by a modification of GR). But it is, in general, anticipated that this particular problem will be explained within conventional physics. More recently primordial black holes have been invoked to explain the missing dark matter and to alleviate some of the problems associated with the CDM scenario (see later) [148].

3.1. Origin of structure

The CMB anisotropies and structure observed on large angular scales are computed using linear perturbations about the standard background cosmological model. However, such large scale structure could never have been in causal contact within conventional cosmology and hence its origin cannot be explained by it without invoking inflation. In general, the testable predictions of inflationary models are scale-invariant and nearly Gaussian adiabatic density fluctuations and almost, but not exactly, a scale-invariant stochastic background of relic GW. However, and as noted earlier, possible alternatives to inflation to obtain successful structure formation consistent with current observations [123] exist, including the popular matter bounce cosmologies.
3.1.1. Large scale structure of the universe. In the standard cosmology it is assumed that
cosmic structure at sufficiently large scales grew out of small initial fluctuations at early times,
and we can study their evolution within (cosmological) linear perturbation theory (LPT)
[149]. We assume that on large scales there is a well defined mean density and on intermedi-
ate scales, the density differs little from it. This is a highly non-trivial assumption, which is
perhaps justified by the isotropy of the CMB. It is usual to use a fluid model for matter and a
kinetic theory model for radiation.

At late times and sufficiently small scales fluctuations of the cosmic density are not small.
The density inside a galaxy is about two orders of magnitude greater than the mean density
of the Universe, and LPT is then not adequate to study structure formation on galaxy-cluster
scales of a few Mpc and less. It is necessary to treat clustering non-linearly using N-body
simulations. Since this is mainly relevant on scales much smaller than the Hubble scale, it has
usually been studied in the past with non-relativistic N-body simulations. On intermediate to
small scales, density perturbations can become large. Inside a galaxy they are small, and even
inside a galaxy cluster the motion of galaxies is essentially decoupled from the Hubble flow
(i.e. clusters do not expand). Therefore, the gravitational potential of a galaxy remains small,
and in the Newtonian (longitudinal) gauge, metric perturbations remain small. In the past,
this together with the smallness of peculiar velocities has been used to argue that Newtonian
N-body simulations are sufficient.

In the adiabatic case, the last scattering surface is a surface of constant baryon density, so
the observed CMB fluctuations do not represent density fluctuations, as is often stated [150].
Thus, in standard perturbation theory language, this shows that in the uniform density gauge
(which for adiabatic perturbation is the same as the uniform temperature gauge) the density
fluctuations are given exactly by the redshift fluctuations. In the non-adiabatic case this will
no longer be true. The main shortcoming of the conventional analysis is, of course, the instan-
taneous recombination approximation (accurate to a few percent only for multipoles with
$\ell < 100$); to go beyond this one has to use a Boltzmann approach [150] (although nothing
changes conceptually). Also, in principle we cannot neglect radiation or neutrino (even mas-
vive) velocities. In addition, Newtonian simulations only consider 1 (of in general 6) degrees
of freedom, and observations are made on the relativistic, perturbed light cone. Hence relativ-
istic calculations are needed.

3.1.2. Perturbation theory. The complexity of the distribution of the actual matter and energy
in our observed Universe, consisting of stars and galaxies that form clusters and superclusters
of galaxies across a broad range of scales, cannot be described within the standard spatially
homogeneous model. To do this we must to be able to describe spatial inhomogeneity and anisotropy using a perturbative approach starting from the uniform FLRW model as a back-
ground solution [151]. The perturbations live on the four-dimensional background spacetime,
which is split into three-dimensional spatial hypersurfaces utilizing a ($1 + 3$) decomposition.
Within the standard cosmology a flat background spatial metric ($k = 0$) in LPT is assumed,
which is consistent with current observations. For generalisations to spatially hyperbolic or
spherical FLRW models see, e.g. [152].

The introduction of a spatially homogeneous background spacetime to describe the inho-
mogeneous Universe leads to an ambiguity in the choice of coordinates. Selecting a set of
coordinates in the (real) inhomogeneous Universe, which will then be described by an FLRW
model plus perturbations, essentially amounts to the assignation of a mapping between spa-
cetime points in the inhomogeneous Universe and the spatially homogeneous background
model. The freedom in this selection is the gauge freedom, or gauge problem, in GR pertur-
bation theory. Either the gauge freedom must be handled very carefully by delineating what
freedom remains at each stage of coordinate specialisation [153], by using gauge covariant variables [154], or utilizing 1 + 3 gauge invariant and covariant variables [155].

Indeed, gauge-invariant variables are widely utilized since they constitute a theoretically effective way to extract predictions from a gravitational field theory applied to the Universe for large-scale linear evolution [152]. In addition, by using gauge-invariant variables the analysis is reduced to the study of only three decoupled second order ordinary differential equations, and they represent physical quantities that can be immediately connected to observations. In the review [151] the focus was on how to construct a variety of gauge invariant variables to deal with perturbations in different cosmological models at first order and beyond. Most work to date has been done only to linear order where the perturbations obey linear FE.

As a theoretical application the origin of primordial curvature and isocurvature perturbations from field perturbations during inflation in the very early Universe can be considered. LPT allows the primordial spectra to be related to quantum fluctuations in the metric and matter fields at considerably higher energies. In the most simple single field inflationary models it is, in fact, possible to equate the primordial density perturbation with the curvature perturbation during inflation, which essentially remains constant on very large scales for adiabatic density perturbations. The observed power spectrum of primordial perturbations revealed by the CMB and LSS is thus a powerful probe of inflationary models of the very early Universe.

The outstanding problems within LPT are mostly technical issues and, in particular, include the important questions of the physical cut off to the short and long wavelength modes and the convergence of the perturbations (and hence the validity of the perturbative approach itself).

3.1.3. Non-linear perturbations. The new frontier in cosmological perturbation theory is the investigation of non-linear primordial perturbations, at second-order and beyond. Although the simple evolution equations obtained at linear order can be extended to non-linear order [151], the non-linearity of the EFE becomes evident and consequently the resulting definitions of gauge invariant quantities at second order clearly become more complicated than those at first order. Recently, perturbations at second order [156] and, more generally, non perturbative effects have been studied, where there are certainly more foundational problems.

Perturbative methods allow quantitative statements but have limited domains of validity. Recently, several groups have started to develop relativistic simulations [77]. Numerical relativistic N-body simulations are a unique tool to study more realistic scenarios, and appear to compare well to numerical relativity fluid simulations [157]. However, assumptions are still made that need to verified. In particular, care must be taken in applying Newtonian intuition to GR. For example [25], do not solve the full EFE and use the fact that the gravitational potential is very small, but spatial derivatives, and second derivatives, are not small. Therefore, when computing the Einstein tensor they go only to first order in the gravitational potentials and their time derivatives (but also include quadratic terms of first spatial derivatives and all orders for second spatial derivatives).

New qualitatively effects occur beyond linear order. The non-linearity of the FE inevitably leads to mixing between scalar, vector and tensor modes and the existence of primordial density perturbations consequently generate vector and tensor modes. Non-linearities then permit additional information to be determined from the primordial perturbations. A lot of effort is currently being devoted to the investigation of higher order correlations (and issues of gauge dependence). Non-Gaussianity in the primordial density perturbation distribution would uncover interactions beyond the linear theory. Such interactions are minimal (suppressed by slow-roll parameters) in the simplest single field inflation models, so any detection of primordial non-Gaussianity would cause a major reassessment about our knowledge of the very early Universe. In principle, this approach can be easily extended to higher-orders,
although large primordial non-Gaussianity is expected to dominate over non-linearity in the transfer functions.

However, cosmological perturbation theory based on a cosmological $1 + 3$ split is ill-suited to address important questions concerning non-linear dynamics or to evaluate the viability of scenarios based on classical modifications of GR. A new formulation of a fully non-perturbative approach has been advocated [158], along with a gauge fixing protocol that enables the study of these issues (and especially the linear mode stability in spatially homogeneous and nearly homogeneous backgrounds) in a wide range of cosmological scenarios, based on a method that has been successful in analyzing dynamical systems in mathematical and numerical GR based on the generalized harmonic formulation of the EFE.

3.1.4. Non-linear regime. At the non-linear order a variety of different effects come into play, including gravitational lensing of the source by the intervening matter and the fact that redshift is affected by peculiar motion, both of which have relatively simple Newtonian counterparts. But there are a host of complicated relativistic corrections once light propagation is worked out in more detail. There are selection effects too: we are much more likely to observe sources in halos, some objects are obscured from view by bright clusters, and so on.

Within the context of perturbation theory it is relatively easy to predict the expectation value of the bias in the Hubble diagram for a random direction [159]. The full second-order correction to the distance-redshift relation has been calculated within cosmological perturbation theory, yielding the observed redshift and the lensing magnification to second order appropriate for most investigations of dark energy models [160]. These results were used in [161] to calculate the impact of second-order perturbations on the measurement of the distance to the last-scattering surface, where relativistic effects can lead to significantly biased measurements of the cosmological parameters at the sub-percent to percent level if they are neglected.

The somewhat unexpected percent level amplitude of this correction was discussed in [162], but the focus therein was on the effect of gravitational lensing only and thus did not consider the perturbations of the observed redshift, notably due to peculiar velocities, which can lead to a further bias in parameter estimation. In addition, [163] noted that the notion of average is adapted to the observation of the Hubble diagram and may differ from the most common angular or ensemble averages, and suggested a possible non-perturbative way for computing the effects of inhomogeneities on observations based on light-like signals using the geodesic light-cone gauge to explicitly solve the geodetic-deviation equation.

In order to comprehensively address the issue of the bias of the distance-redshift relation, previous work was improved upon by fully evaluating the effect of second-order perturbations on the Hubble diagram [159]. In particular, the notion of average which affects bias in observations of the Hubble diagram for inhomogeneity of the Universe was carefully derived, and its bias at second-order in cosmological perturbations was calculated. It was found that this bias considerably affects direct estimations of the evolution of the cosmological parameters, and particularly the equation of state of dark-energy. Despite the fact that the bias effects can reach the percent level on some parameters, errors in the standard inference of cosmological parameters remain less than the uncertainties in observations [159].

In further work [80], a non-perturbative and fully relativistic numerical calculation of the observed luminosity distance and redshift for a realistic cosmological source catalog in a standard cosmology was undertaken to investigate the bias and scatter, mainly due to gravitational lensing and peculiar velocities, in the presence of cosmic structures. The numerical experiments provide conclusive evidence that the non-linear relativistic evolution of inhomogeneities, once consistently combined with the kinematics of light propagation on
the inhomogeneous spacetime geometry, does not lead to an unexpectedly large bias on the distance-redshift correlation in an ensemble of cosmological sources. However, inhomogeneities introduce a significant non-Gaussian scatter that can give a large standard error on the mean when only a small sample of sources is available. But even for large, high-quality supernovae samples this scatter can bias the inferred cosmological parameters at the percent level [80].

It was argued in [164], using a fully relativistic treatment, that cosmic variance (i.e. the effects of the local structures such as galaxy clusters and voids relative to statistical predictions from a family of models) is of a similar order of magnitude to current observational errors and consequently needs to be taken into consideration in local measurements of the Hubble expansion rate within the standard cosmology. In addition, the constraint equation relating metric and density perturbations in GR is inherently non-linear, and leads to an effective and intrinsic non-Gaussianity in the large-scale dark matter density field on large scales (even when the primordial metric perturbation is itself Gaussian) [165].

3.1.5. Non-Gaussianities. In standard cosmology, the primordial perturbations corresponding to the seeds for the LSS are selected from a Gaussian distribution with random phases, justified primarily from the fact that primordial non-Gaussianity (PNG) has not yet been observed and also theoretically (e.g. the central limit theorem); thus a Gaussian random field constitutes a satisfactory representation of the properties of density fluctuations. However, any deviation from perfect Gaussianity will, in principle, reveal important information on the early Universe, and an investigation of PNG is especially relevant if these initial conditions were generated by some dynamical process such as, for example, inflation. In particular, a direct measurement of non-Gaussianity would permit us to move beyond the free-field limit, yielding important information about the degrees of freedom, the possible symmetries and the interactions characterizing the inflationary action. The current status of the modelling of, and the searching for, PNG of cosmological perturbations was reviewed in [166].

In order to evaluate PNG from the early Universe to the present time, it is necessary to self-consistently calculate non-Gaussianity during inflation. We must then evolve scalar and tensor perturbations to second order outside the horizon, matching conserved second-order gauge-invariant variables to their values at the end of inflation (appropriately taking into account reheating). Finally, we need to investigate the evolution of the perturbations after they re-entered the Hubble radius, by computing the second-order radiation transfer function and matter transfer function for the CMB and LSS, respectively. Although these calculations are very complicated, PNG represents an important tool to probe fundamental physics during inflation at energies from the grand unified scale, since different inflationary models predict different amplitudes and shapes of the bispectrum, which complements the search for primordial gravitational-waves (PGW) (via a stochastic GW background).

The Planck satellite has produced good measurements of higher-order CMB correlations, resulting in considerable stringent constraints on PNG. The latest data regarding non-Gaussianity tested the local, equilateral, orthogonal (and various other) shapes for the bispectrum and led to new constraints on the primordial trispectrum parameter [84]. The most extreme possibilities have been excluded by CMB and LSS observations, and now primarily the detection of (or constraints from) mild or weak deviations from primordial Gaussian initial conditions are sought, characterized by a small parameter, $f_{\text{NL}}$, compatible with observations. Even though the sensitivity is not comparable to CMB data [84], the bispectra for redshift catalogues can be determined (e.g. the three-point correlation functions for the WiggleZ and BAO spectroscopic surveys) [167], and interesting observational bounds on the local $f_{\text{NL}}$ from
current constraints on the power spectrum can be obtained (see [166] and references within). Neglecting complications arising from the breaking of statistical isotropy (such as sky-cut, noise, etc) the procedure is, in general, to fit the theoretical bispectrum template, and $f_{NL}$ is found to be approximately 0.01 in generic inflation [168].

PNG is certainly the best way of practically investigating the only guaranteed prediction of inflation [16]. Indeed, even though standard models of slow-roll inflation only predict tiny deviations from Gaussianity (consistent with the Planck results), specific oscillatory PNG features can be indicative of particular string-theory models. Therefore, the search for PNG is of interest for theoretically well-motivated models of inflation and the Planck results can potentially severely constrain a variety of classes of inflationary models beyond the simplest paradigm. However, only the failure to find any such evidence for PNG can falsify inflation.

There are some outstanding issues regarding non-Gaussianity [166]. First, it has been argued that the consistency relation is certainly not observable for single field inflation since, in the strictly squeezed limit, this term can be gauged away by an appropriate coordinate transformation (so that the only residual term is proportional to the same order of the amplitude of tensor modes). Second, in the non-linear evolution of the matter perturbations in GR the second order dark matter dynamics leads to post-Newtonian-like contributions which mimic local PNG. A recent estimate of the effective non-Gaussianity due to GR light cone effects comparable to a PNG signal were discussed in [166], which would correspond in the comoving gauge to an $f_{NL}$ in the pure squeezed limit. Therefore, such a GR PNG signature may not be detectable via any cosmological observables.

3.1.6. Simulations and post-Newtonian cosmological perturbations. There has been a lot of recent interest in testing the validity of GR using cosmological observables related to structure formation. Since the physics involved in horizon-sized cosmological perturbations is quite different to that which occurs on smaller scales, where galaxies and clusters of galaxies are present, this is challenging. LPT [151] is not suitable for investigating gravitational fields associated with structures that have highly non-linear density contrasts (which necessarily have to be small in order for the perturbative expansion to be well defined). GR numerical simulations using, for example, the gevolution code developed by Adamek, Durrer and co-workers [25], have proven to be an important new tool for studying structure formation. Targeted fully relativistic non-linear simulations with an evolving non-zero spatial curvature have also been developed [100].

Alternatively, two-parameter post-Newtonian cosmological perturbation schemes have been proposed [169]. Indeed, recent progress [170] has been made in applying the techniques from post-Newtonian expansions of the gravitational FE into cosmology in the presence of highly non-linear structures to relate the functions that parameterize gravity on non-linear scales to those that parameterize it on very large scales. This so-called parameterized post-Newtonian cosmology (PPNC) has been used to analyse alternative theories of gravity [170]. This was achieved by simultaneously expanding all of the relevant equations in terms of two parameters; the first associated with the expansion parameter of LPT, and the second characterizing the order-of-smallness from post-Newtonian theory [169]. An alternative Lagrangian-coordinates based approximation scheme to provide a unified treatment for the two leading-order regimes was presented in [171].
3.2. Black holes and gravitational waves

3.2.1. Gravitational waves. Recent progress in numerical GR has allowed for a detailed investigation of the collision of two compact objects (such as, e.g. black holes and neutron stars). In such a violent inspiral an enormous amount of gravitational radiation is emitted. The detection and subsequent analysis of the gravitational wave (GW) signals produced by black hole mergers necessitate extremely accurate theoretical predictions that can be utilized as template waveforms that can then be used to cross-correlate with the output of GW detectors. This is, of course, of fundamental import in view of the recent LIGO observations [172]. Indeed, such an analysis led to the direct detection of GW by the LIGO-Virgo collaboration [173]. To a large extent the numerical problem has been solved in the case of a black-hole merger, although the relatively simple properties of the two-body non-linear gravity waveforms [174] have not been fully understood mathematically. There is also the recent binary neutron star merger event, which is much more difficult to model within GR. There are a number of open problems, particularly concerning the physical nature of the recently observed merger events [175]. GW astronomy will potentially play an increasingly important role within cosmology [176]. For example, there is a promise that they will allow very good direct estimates of the distance of colliding black holes, avoiding the need for the usual cosmic distance ladder.

3.2.2. Primordial gravitational waves. Primordial GW (PGW) add to the relativistic degrees of freedom of the cosmological fluid. Any change in the particle physics content, perhaps due to a change of phase or freeze-out of a species, will leave a characteristic imprint on an otherwise featureless spectrum of PGW. The existence of a stochastic PGW background at a detectable level would then probe new physics beyond the standard cosmological model, and this may be possible with the Laser Interferometer Space Antenna (LISA) [177].

Recently, a class of early-Universe scenarios has been theoretically identified which produce a strongly amplified, blue-tilted spectrum of GW [178]. Detection of GW over a broad range of frequencies can provide important information concerning the underlying source [178], and also may well be of relevance for the spectrum of GW emitted by other scaling sources. In addition, a population of massive primordial black holes (PBHs) would be anticipated to generate a stochastic background of GW [179], regardless of whether they form binaries or not. The focus is usually on the GW generated by either stellar black holes (observable by LIGO) or supermassive black holes (observable by LISA). However, with an extended PBH mass function, the GW background ought to encompass both of these limits and also every intermediate frequency. Many supermassive black holes are in binary pairs that orbit together and eventually merge, emitting GW in the process. The LISA detection window includes mergers of black holes in the mass range of \(10^4\)–\(10^7\) solar masses [180]. Due to the possibility that the coalescing black holes observed by LIGO [173] could be of primordial origin, black holes in the intermediate mass range of \(10\)–\(10^3\) solar masses are of particular interest since such PBHs might contribute to the dark matter (see below).

The primary goal of CMB observations is the polarization signal induced by GW at the start of inflation. There is a considerable effort underway to obtain stricter limits on the tensor-to-scalar ratio, \(r\), the quantitative measure of the ratio of the primordial amplitude of the B-mode (or shearing) polarization component due to GW to the scalar (or compressive) mode of CMB temperature fluctuations associated with the density fluctuations that seeded structure formation. While PGW have not yet been detected, the upper limit on \(r\) from the BICEP2/Keck CMB polarization experiments [84] (in conjunction with Planck temperature measurements and other data) is less than or equal to approximately 0.07 at the 95% confidence level. However,
the tensor amplitude predicted depends on the (fourth power of the) energy scale of inflation, and so the primordial polarization signal could, in principle, be unmeasurably small [16].

3.2.3. Primordial black holes. The possibility of $10^{-10}$–$10^3$ solar mass objects is of particular interest in view of the recent detection of black-hole mergers by LIGO which has, in particular, revitalized the interest in stellar mass black holes of around thirty solar masses (which are larger than initially expected) [173], and especially non-evaporating primordial black holes (PBHs). In particular, it has been suggested that massive PBHs could provide the dark matter [181] or the supermassive black holes which reside in galactic nuclei and power quasars [182].

The most natural mechanism for PBH formation involves the collapse of primordial inhomogeneities, such as might arise from inflation (or spontaneously at some kind of phase transition). Interest in PBH increased due to the discovery that black holes radiate [183], since only PBH could be small enough for this to be relevant cosmologically. Indeed, evaporating PBHs have been invoked to explain several cosmological features [148]. Since it was initially believed that PBHs would grow as fast as the Universe during the radiation-dominated era and consequently attain a huge mass by the present time, it was thought that PBH never formed and could thus be excluded. However, such an argument is essentially Newtonian and neglects the cosmological expansion, and in [184] it was shown that there is no self-similar solution in which a black hole can grow as fast as the Universe. Therefore, once formed, their contribution to the dark matter of the Universe grows with time (the mass of non-evaporating PBH is unchanged after formation and can only grow if they accrete matter) [185].

PBH would have the particle horizon mass at formation and could form as early as the Planck epoch, when QG forces are comparable to gravitational forces that at later epochs are far too weak on particle scales. However, as the Universe expands and cools, tiny black holes of Planck mass all quickly disappear. More massive black holes live longer and should survive until today as early Universe relics [184]. Attention has consequently shifted to larger PBHs, which are unaffected by Hawking radiation. Such PBHs might have important cosmological consequences.

Perhaps the most exciting possibility is that PBH larger than $10^3$ solar masses could provide the dark matter which comprises 25% of the critical density [148]. Since PBHs formed in the radiation-dominated era, they are not subject to the well-known cosmological nucleosynthesis constraint that baryons can contribute at most 5% to the critical density. PBH should thus be classified as non-baryonic and behave like any other form of cold dark matter (CDM). The subject has consequently become very popular and non-evaporating PBHs may turn out to play a more important cosmological role than evaporating ones.

PBHs could provide the dark matter but a number of constraints restrict their possible mass ranges [181], including those arising from gravitational microlensing, but PBHs at a level of 10% of the dark matter are still possible over a wide range of masses. The PBH density might be much less than the dark matter density, but the PBHs are not necessarily required to provide all of the dark matter [148]. For intermediate mass black holes of $10^3$ solar masses a dark matter mass fraction of only 0.1% still allows for important consequences for structure formation. Cosmological structures could be generated either individually through a ‘seed’ effect or collectively through the ‘Poisson’ effect (fluctuations in the black hole population generates an initial density perturbation for PBH dark matter), consequently alleviating some of the possible problems associated with the standard CDM scenario (even when they may only contribute a small portion of the dark matter density). Both mechanisms for generating

\[10\] An alternative to PBHs includes persistent (or ‘pre-big-bang’) black holes occurring in bouncing cosmologies [186].
fluctuations then amplify through gravitational instability to bind massive regions [182] and have been considered as either alternatives or in conjunction with other CDM scenarios.

3.3. Effects of structure on observations: gravitational lensing

A particularly important cosmological question is whether gravitational lensing significantly alters the distance-redshift relation \( D(z) \) to the CMB last scattering surface or the mean flux density of sources. Any such \( D(z) \) bias could change CMB cosmology, and the corresponding bias in the mean flux density could alter supernova cosmology.

In spatially homogeneous and isotropic cosmologies the ratio between the proper size of a source and the angular diameter distance is a function of redshift only. In an inhomogeneous Universe, lensing by intervening metric fluctuations can cause magnification of the angular size, with a corresponding change of flux density, since surface brightness is not affected by gravitational lensing. Therefore, the apparent distance to objects at a given redshift can effectively become a randomly fluctuating quantity.

Using conservation of photons (i.e. flux conservation), Weinberg [187] argued that in the case of transparent lenses there is no mean flux density amplification, so that the uniform universe formula for \( D(z) \) remains unchanged (where the averaging is over sources, and the result relies on the implicit assumption that the area of a constant-\( z \) surface is unaffected by gravitational lensing). This issue has recently been revisited [188], and it was argued that in an ensemble averaged (and more appropriate cosmological) sense, the perturbation to the area of a surface of constant redshift is in reality a very small (approximately one part in one million) effect, supporting Weinberg’s argument and validating the usual treatment of gravitational lensing in the analysis of CMB anisotropies.

However, Weinberg’s argument regarding the mean flux density appears to contradict well-known theorems of gravitational lensing, such as the focusing theorem. Non-linear relativistic perturbation theory to second order indicates that there is bias in the area of a surface of constant redshift and in the mean distance to the CMB last scattering surface. Indeed, a lot of investigations of gravitational lensing continue to advocate significant effects in the mean. Bolejko [189] (also see references in [188]) has provided a comprehensive review of such studies, some of which claim large effects, some of which obtain effects at the level of a few percent (which would still be important), while others argue that the effects are exceedingly small. A non-vanishing perturbation to the mean flux densities of distant sources caused by intervening structures, at least for sources that are viewed along lines of sight that avoid mass concentrations, effectively contradict Weinberg’s result. Recent non-linear analysis does suggest that non-linear effects have not been proven to be negligible [150, 159, 80].

3.4. Backreaction and averaging

Averaging in GR is a fundamental problem within mathematical cosmology [3]. The cosmological FE on the largest scales are derived by averaging or coarse graining the EFE of GR. A solution of this problem is critical for the correct interpretation of cosmological data [24] (on the largest scales the dynamical behavior can be significantly different from the dynamics in the standard cosmology; e.g. the expansion rate can be greatly affected [190]).

First, it is of great importance to provide a rigorous mathematical definition for averaging (tensors on a differential manifold) in GR. A spacetime or space volume averaging approach must be well defined and generally covariant [191, 192], and produce the structure equations for the averaged macroscopic geometry (and give a prescription for the correlation
functions in the macroscopic FE which emerge in the averaging of the non-linear FE), which do not necessarily take on precisely the same mathematical form as the original FE [192]. It is straightforward to average scalar quantities and since, in general, a spacetime is determined entirely by its scalar curvature invariants, a specific spacetime averaging scheme based on scalar invariants only has been proposed [193]. In addition, only scalar quantities are (space volume) averaged within the \((1 + 3)\) cosmological spacetime splitting approach of Buchert [190].

Although the standard FLRW \(\Lambda CDM\) cosmology has, to date, been very successful in explaining all of the observational data (up to a number of potential anomalies and tensions [83]) it does require, as yet undetected, sources of dark energy density that currently dominate the dynamics of the Universe. More importantly, the actual Universe is neither isotropic nor spatially homogeneous on small scales. Indeed, observations of the current late epoch Universe uncovers a very complicated picture in which the largest gravitationally bound structures, consisting of clusters of galaxies of different sizes, form, in turn, ‘knots, filaments and sheets that thread and surround very underdense voids’ [194]. An enormous fraction of the volume of the current Universe is, in fact, contained within voids of a single characteristic size of about 30 megaparsecs [195] with an almost ‘empty’ density contrast [196].

In principle, a number of coarse grainings over different scales is required to reasonably model the observed complicated gravitationally bound large scale structures [22]. In standard cosmology it is implicitly taken that the matter distribution on the largest scale can be modeled by an ‘effective averaged out’ stress-energy tensor, regardless of the physical details of the actual coarse graining at each scale. However, based on the two-point galaxy correlation function, the very smallest scale on which there can be a reasonable definition of statistical homogeneity is 70–120 megaparsecs [197], and even then variations for the number density of galaxies on the order of several percent still arise in the largest possible survey volumes [21, 198]. It is fair to say that it is not at all clear what the largest scale is that matter and geometry on smaller scales can be coarse-grained such that the average evolution is still an exact solution of the EFE.

A smooth macroscopic geometry (with macroscopic matter fields), applicable on cosmological scales, is obtained after an appropriate averaging. The coarse graining of the EFE for local inhomogeneities on small scales can generally lead to important backreaction effects (consisting of not just the mean cosmic variance) [199] on the average dynamics of the Universe [190]. In addition, all cosmological observations are deduced from null geodesics (the paths of photons) which travel enormous distances, preferentially traversing the underdense voids of the actual Universe. But inhomogeneities perturb curved null geodesics, so that observed luminosity distances can be significantly affected.

A consistent approach to cosmology is consequently to treat GR as a mesoscopic theory, which is applicable only on the mesoscopic scales for which it has actually been verified, containing a mesoscopic metric field and a mesoscopic geometry. The effective macroscopic dynamical equations on cosmological scales are then obtained by averaging. It had originally been hoped that such a backreaction approach might help resolve the dark energy and dark matter problems. However, it now seems unlikely that backreaction can replace dark energy (although large effects are theoretically possible from inhomogeneities and averaging [24]). But it can certainly affect precision cosmology at the level of 1% [26] and may offer a better understanding of some issues in cosmology (such as the emergence of a homogeneity scale and non-zero spatial curvature due to non-linear evolution of cosmic structure).

### 3.4.1. Backreaction magnitude

This last point is very important, and since it has been a source of some controversy let us summarize briefly here. The Universe is very inhomogeneous on small scales at the present time but smooth on large scales. It must be remembered
that density perturbations \( \frac{\delta \rho}{\rho} \simeq 10^{38} \) on Earth, but the metric is very close to Minkowski. To establish the backreaction effects we need approximation methods to deal with metric perturbations \( \frac{\delta h}{h} \simeq 10^{-5} \) but second derivatives \( \simeq 10^{28} \). Various approaches have been tried:

- Zalaletdinov [192] developed a very complex bimetric averaging formalism that can, in principle, be applied in general; the effect of such averaging on cosmological observations was estimated to be of the order of about 1% [200]. A global Ricci deformation flow for the metric, which is generically applicable in cosmology, was introduced by Carfora.
- Buchert [190] developed an explicit \((1+3)\) spatial averaging scheme, although the scheme is not fully deterministic and depends on some ad hoc phenomenological assumptions. Models based on this scheme, and particularly the ‘timescape cosmology’ of Wiltshire which utilizes time-dilation effects in voids, can predict very large effects, and there have been claims that the results are sufficient to explain dark energy [24].
- There have been various approximation schemes that have claimed that the backreaction effects are negligible \( \simeq 10^{-5} \), including a scheme by Green and Wald [199, 201] which uses distributional methods that does not involve explicit averaging.
- Durrer and collaborators have developed detailed second order calculations that predict percent level changes (i.e. that are sufficient to be of significance in GR precision cosmology studies), and they (using the ‘gevolution’ numerical code) and others have subsequently confirmed this with N-body simulations.

The most reasonable outcome of this debate, at least in our view and particularly in light of the latter results, is that observable differences caused by backreaction effects will be of the order of 1%.

3.5. Spatial curvature

Current constraints on the background spatial curvature, characterized by \( \Omega_k \), within the standard cosmology are often used to ‘demonstrate’ that it is dynamically negligible: \( \Omega_k \sim 5 \times 10^{-3} \) (95% confidence level) [95]. However, in standard cosmology the spatial curvature is assumed to be zero (or at least very small and below the order of other approximations) for the analysis to be valid. Therefore, strictly speaking, the standard model cannot be used to predict a small spatial curvature.

In general, \( \Omega_k \) is assumed to be constrained to be very small primarily based on CMB data. However, the recently measured temperature and polarization power spectra of the CMB provides a 99% confidence level detection of a negative \( \Omega_k = -0.044 \) \((\pm 0.018, -0.015)\), which corresponds to a positive spatial curvature [84]. Direct measurements of the spatial curvature \( \Omega_k \) using low-redshift data such as supernovae, baryon acoustic oscillations (BAO) and Hubble constant observations (as opposed to fitting the FLRW model to the data) do not place tight constraints on the spatial curvature and allow for a large range of possible values (but do include spatial flatness). Low-redshift observations often rely on some CMB priors [202] and, in addition, are sensitive to the assumptions about the nature of dark energy\(^{11}\).

Attempts at a consistent analysis of CMB anisotropy data in the non-flat case suggest a closed model with \( \Omega_k \sim 1\% \) [103, 204]. Including low redshift data, \( \Omega_k = -0.086 \pm 0.078 \) was obtained [204], which provides weak evidence in favor of a closed spatial geometry (at

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\(^{11}\) For late Universe observables there is significant degeneracy between \( \Omega_k \) and dark energy parameters; the standard approach is to treat \( \Omega_k \) and these parameters as independent quantities, and to marginalize over the dark energy parameters [203].
the level of $1.1\sigma$), with stronger evidence for closed spatial hypersurfaces (at a significantly higher $\sigma$ level) coming from dynamical dark energy models [103] (see also [205]). The inclusion of CMB lensing reconstruction and low redshift observations, and especially BAO data, gives a model dependent constraint of $\Omega_k = -0.0007 \pm 0.0019$ [84].

As an illustration, constraints on the phenomenological two curvature model (which has a simple parametrized backreaction contribution [200] leading to decoupled spatial curvature parameters $\Omega_{k_g}, \Omega_{k_d}$ in the metric and the Friedmann equation, respectively, and which reduces to the standard cosmology when $\Omega_{k_g} = \Omega_{k_d}$), were investigated in [102]. It was found that the constraints on the two spatial curvature parameters are significantly weaker than in the standard model, with constraints on $\Omega_{k_g}$ an order of magnitude tighter than those on $\Omega_{k_d}$, and there are tantalizing hints from Bayesian model selection statistics that the data favor $\Omega_{k_d} \neq \Omega_{k_g}$ at a high level of confidence.

Observations on recently emerged, present-day (large-scale mean) average negative curvature are weak and not easy to measure [206]. Local inhomogeneities and perturbations to the distance-redshift relation at second-order contribute a monopole at the sub-percent level, leading to a shift in the apparent value of the spatial curvature (as do other GR curvature effects in inhomogeneous spacetimes). Indeed, in an investigation of how future measurements of $\Omega_k$ are affected by GR effects, it was shown that constraints on the curvature parameter may be strongly biased if cosmic magnification is not included in the analysis [207].

Given that current curvature upper limits are at least one order of magnitude away from the level required to probe most of these effects, there is an imperative to continue pushing the curvature parameter, $\Omega_k$, constraints to greater precision (i.e. to about the 0.01\% level). These will become increasingly measurable in future surveys such as the Euclid satellite. In addition, the current curvature parameter estimations are not yet at the cosmic variance limit (beyond which constraints cannot be meaningfully improved due to the cosmic variance of horizon scale perturbations); indeed, the current measurements are more than one order of magnitude away from the limiting threshold [207]. The prospects for further improving measurements of spatial curvature are discussed in [208]. Most importantly, we are interested in model independent [209] and explicitly CMB-independent [203] checks of the cosmic flatness.

However, currently there is no fully independent constraint with an appropriate accuracy for a value of $\Omega_k$ of approximately less than 0.01 on the cosmic flatness from cosmological probes. In principle, a small non-zero measurement of $\Omega_k$ perhaps indicates that the assumptions in the standard model are not met, thereby motivating models with curvature at the level of a few percent. Such models are certainly not consistent with simple inflationary models in which $\Omega_k$ is expected to be negligible [104]. We remark that an observation of non-zero spatial curvature, even at the level of a percent or so, could be the result of backreaction effects and be a signal of non-trivial averaging effects [192]. Note that calculations imply a small positive spatial curvature [200] (although backreaction estimates have tended to give a negative mean curvature [206]).

If the geometry of the universe does indeed deviate slightly from the standard FLRW geometry (for example, due to the evolution of cosmic structures), then the spatial curvature will no longer necessarily be constrained to be constant and any effective spatial flatness may not be preserved. An investigation of a small emerging spatial curvature can be undertaken by relativistic cosmological simulations [77, 25]. However, such simulations need to include all relativistic corrections and can suffer from gauge issues [25, 210]. In particular, using a fully inhomogeneous, anisotropic cosmological numerical simulation, it was shown that [26]: (i) On small scales, below the measured homogeneity scale of the standard cosmology, deviations in cosmological parameters of 6\%–31\% were found (in general agreement with LPT
and with deviations depending on an observer’s physical location). (ii) On the approximate homogeneity scale of the Universe mean cosmological parameters consistent to about 1% with the corresponding standard cosmology were found (although the parameters can deviate from these mean values by 4%–9% again depending on the physical location in the simulation domain). (iii) Above the homogeneity scale of the Universe, 2%–3% variations in mean spatial curvature and backreaction were found.

As noted above, attempts to study relativistic models of inhomogeneities rely upon metric forms that are designed to be ‘close to’ the spatially homogeneous and isotropic metric form. However, these can not also be used to address the cosmological backreaction problem; backreaction can only be present if the structure–emerging average spatial curvature, and hence the large-scale average of cosmological variables, are allowed to evolve [211]. A dynamical coupling of matter and geometry on small scales which allows spatial curvature to vary is a natural feature of GR. Indeed, the requirement that spatial curvature remains constant as in an FLRW model on arbitrarily large scales of cosmological averaging is not a natural consequence of any principles of GR. Schemes that suppress average curvature evolution (e.g. by employing periodic boundary conditions as in Newtonian models and neglecting global curvature evolution) can not describe global backreaction but only cosmic variance [24]. Moreover, within standard cosmology, spatial fluctuations are conceived to evolve on an assumed background FLRW geometry, but this description only makes sense with respect to their spatial average distribution and its evolution. We note that even small fluctuations within averaging schemes are also subject to gauge issues [212]. In principle, large effects are possible from inhomogeneities and averaging [22, 24].

Recently, a relativistic (Simsilun) simulation based on the approximation of a ‘silent universe’ was presented [100]. The simulation begins with perturbations around a (flat) standard model (with initial conditions set up using the Planck data). The perturbations are allowed to have non-zero spatial curvature. Initially, the negative curvature of underdense regions is compensated by the positive curvature of overdense regions [200, 213]. But once the evolution enters the non-linear regime, this symmetry is broken and the mean spatial curvature of the universe slowly drifts from zero towards negative curvature induced by cosmic voids (which occupy more volume than other regions). The results of the Simsilun simulation indicate that the present-day curvature of our Universe is $\Omega_k \sim 0.1$, as compared to the spatial flatness of the early universe.

It should be emphasised that the fact that structure formation implies that the present-day Universe is volume-dominated by voids and is characterized by on average negative curvature is a subtle issue that follows from the result that intrinsic curvature does not obey a conservation law [24, 23]. Indeed, it dispels the naive expectation that on large scales the distribution of positive spatial curvature for high-density regions and negative spatial curvature for the voids, averages out to the almost or exactly zero spatial curvature assumed.

4. Problems from the quantum realm

There are a number of very fundamental problems in the quantum regime, culminating in the question of whether there is a single unified theory of quantum gravity (QG). And, in particular, is this ‘theory of everything’ string theory? Some problems in the quantum realm are relevant for cosmology. For example, do there exist any fundamental particles that are predicted by QG that have not yet been observed and, if so, what are their properties and are they of importance in cosmology? In particular, the detection of the Higgs boson seems to complete the standard model, but with additional new physics that is needed to protect the
particle mass from quantum corrections (that could increase it by 14 orders of magnitude). It is believed that supersymmetry is the most reasonable solution to this naturalness problem, but the most simple supersymmetric models have not proved successful and, to date, there is no convincing mechanism to break supersymmetry nor to determine the multiple parameters of the supersymmetric theory. In addition, does a theory of QG lead to a multiverse in cosmology? And, perhaps most importantly, do theories of QG naturally lead to inflation?

4.1. The problem of quantum gravity

The standard model of particle physics concerns only three forces: namely, electromagnetism and the strong and weak nuclear forces. A primary goal of theoretical physics is to derive a theory of QG in which all four forces, including that of gravitation, are unified within a single field theory. Up to now, no attempt at such a unification has been successful. In particular, it is of interest to know whether QG can be formulated for cosmology and whether there is any extension of quantum mechanics required for QG, and especially quantum cosmology?

Quantum cosmology gives rise to a number questions concerning a possible theory for the initial cosmological state [29], which include: what laws or principles might characterize the initial conditions of the Universe and what are the subsequent predictions of the initial conditions for the Universe on macro-, meso- and micro-scopic scales?

Let us first briefly discuss two cosmological problems that originate in QG and have a very distinct mathematical formulation of particular interest here.

4.1.1. Higher dimensions. Ordinary spacetime is 4D, but additional dimensions are possible in, for example, string theory [214]. At the classical level, gravity has a much richer structure in higher dimensions than in 4D. In particular, there has been a lot of work done on the uniqueness and stability of black holes in arbitrary dimensions [215]. For example, closed trapped surfaces and singularity theorems in higher dimensions have been discussed [216] and the positive mass theorem has been proven in arbitrary dimension [217]. However, the problem of stability in higher dimensions is much more difficult. Indeed, there is evidence from numerical simulations to indicate that there are higher dimensional black holes that are not stable [215]. In addition, the question of cosmic censorship in higher dimensions is extremely difficult and is perhaps not even well posed. In fact, there is numerical evidence that suggests that cosmic censorship does not hold [218] and that black holes are not necessarily stable to gravitational perturbations in higher dimensions [219]. Indeed, black holes become highly deformed at very large angular momenta and resemble black branes, and in spacetime dimensions greater than six exhibit an ‘ultraspinning instability’ [220].

Higher dimensional spacetime manifolds are also considered in a number of cosmological scenarios. For example, in the cosmological context all known mathematical results can be investigated in models with a non-zero cosmological constant. In addition, theoretical results, such as the dynamical stability of higher dimensional cosmological models, are of interest. In particular, spatially homogeneous cosmologies in higher dimensions, and especially extensions of the BKL analysis, have been investigated [221].

4.1.2. AdS/CFT correspondence. Anti de Sitter (AdS) spacetimes are of interest in QG theories formulated in terms of string theory due to the AdS/CFT (or Maldacena gauge/gravity duality) correspondence, in which string theory on an asymptotically AdS spacetime is conjectured to be equivalent to a conformally invariant quantum field theory (CFT) on its boundary [222, 223]. This holographic paradigm leads to a number of cosmological questions. In
particular, the AdS/CFT conjecture strongly motivates the dynamical investigation of asymptotically AdS spacetimes. But, of course, such a spacetime is clearly not this Universe. In addition, recently it has been conjectured that AdS spacetimes are unstable to arbitrarily small perturbations [224].

The global non-linear stability of AdS has been investigated in spherically symmetric massless scalar field models within GR [225]. Numerical evidence seems to indicate that AdS spacetimes are non-linearly unstable to a ‘weakly turbulent mechanism’ in which an arbitrarily small black hole is formed whose mass is determined by the initial energy. Such a non-linear instability appears to happen for various typical perturbations. However, there are also many perturbations that do not lead to an instability, which consequently implies the existence of ‘islands of stability’ [226, 227]. It is of great interest to study the non-linear stability of AdS with no assumptions on symmetry; however, such a study is currently intractable both analytically and numerically. But the general gravitational case is clearly richer than the case of spherical symmetry analysed to date [226]. Therefore, it is of great significance to determine if the conjectured non-linear instability in AdS spacetime in more general models behaves in a similar or a different way to that in spherically symmetric scalar field collapse [224].

4.2. Singularity resolution

4.2.1. Singularity resolution and a quantum singularity theorem. The existence of singularities indicates a failure of GR when the classical spacetime curvature is sufficiently large. This is exactly when QG effects are anticipated to be important. Therefore, the problem of if, and when, QG can extend solutions of classical GR beyond the singularities is crucial [228]. It is, of course, pertinent to determine whether all singularities can be removed in QG. However, it is certainly not true that all singularities can be resolved within string theory; for example, it is known that the string in an exact plane wave background does not propagate through the curvature singularity in a well-behaved manner [229].

Gauge/gravity duality, which can be regarded as providing an indirect formulation of string theory [230], has been utilized to study singularities in the quantum realm and investigate cosmic censorship with asymptotically AdS initial data. The existence of a quantum version of cosmic censorship was suggested from holographic QG [231]. It has been deduced that a large class of bounces through cosmological singularities are forbidden. Consequently, although some singularities can indeed be resolved, a novel singularity theorem is possible. Therefore, it is important to determine whether a quantum mechanical generalization of any of the singularity theorems exists, which would subsequently imply that singularities are inevitable even in quantum settings. In particular, it has been shown that a fine-grained generalized second law of horizon thermodynamics can be used to prove the inevitability of singularities [232], thereby extending the classical singularity theorem of Penrose [41] to the semi-classical regime. It is plausible that this result, which was constructed in the context of semiclassical gravity, will still hold in a complete theory of QG [232]. Therefore, not all singularities can be resolved within QG.

4.2.2. Cosmological singularity resolution. Cosmological singularity resolution can be investigated within loop quantum gravity (LQG) and string theory. (Black hole singularity resolution was reviewed in [5].) LQG is a non-perturbative canonical quantization of gravity. It has been suggested that singularities may be generically resolved within LQG as a result of QG effects [233]. In particular, the classical big bang singularity is replaced by a symmetric quantum big bounce when the energy density is of the order of the Planck density, which
occurs without any violation of the energy conditions or fine tuning. The resulting quantum big bounce connects the currently expanding universe to a pre-bounce contracting classical universe.

The application of LQG in the context of cosmology is referred to as loop quantum cosmology (LQC). In LQC the infinite degrees of freedom reduce to a finite number due to spatial homogeneity. A variety of spatially homogeneous cosmologies have been investigated [234]. In particular, solutions to the effective equations for the general class of Bianchi type IX cosmological spacetimes has been investigated within LQC computationally, wherein the big bang singularity was shown to be resolved [235]. The reduction of symmetries within LQC involves a very considerable simplification, and consequently crucial aspects of the dynamics may be neglected. However, partly due to evidence supporting the BKL conjecture, it is believed that the singularity resolution in spatially homogeneous cosmologies does capture important features of singularity resolution in more general spatially inhomogeneous cosmological models [234, 236]. There are ongoing attempts to include spatial inhomogeneities in the analysis [237].

Various singularities have been investigated within standard LQC. It has been conjectured that all curvature singularities which result in geodesic incompleteness are so-called strong singularities (such as the big bang in GR). In recent years a number of other types of cosmological singularities have been obtained, which include the big rip and the big freeze, and sudden and generalized sudden singularities. Of these, the big rip and big freeze are strong singularities within GR, whereas sudden and generalized sudden singularities are weak singularities. Using a phenomenological matter model in GR, it has been established that strong singularities are, in general, resolved in LQC, whereas quantum geometry does not usually affect weak singularities [238]. A comprehensive investigation of the resolution of a variety of singularities within modified LQC models, in which the bounce can be asymmetric and the bounce density can be affected, was performed using an effective spacetime description and compared with the analysis in standard LQC [238].

4.3. Quantum gravity and inflation

Although some of the alternatives to inflation alluded to earlier are suggested by ideas motivated by QG, it is also of interest to determine whether inflation occurs naturally within QG. For example, it appears to be difficult to get inflation within string theory [239]. In particular, so-called swampland criteria constrain inflationary models and there are no-go theorems for the existence of de-Sitter vacua in critical string theory. The fact that exact de Sitter solutions with a positive cosmological constant cannot describe the late-time behaviour of the Universe [239] is often interpreted as ‘bad news’ for string theory.

The observations of Planck 2018 (of the almost scale-invariant and Gaussian primordial curvature perturbations) [84] are compatible with the predictions of simple single scalar field inflation models with a canonical kinetic term and an appropriately flat self-interaction potential minimally coupled to gravity. However, despite the success of the single-field slow-roll inflation model, it is not straightforward to embed such a model within a fundamental theory [240].

However, the so-called $\alpha$-attractor models and, in particular, the KKLMMT model [241], have been actively studied. The most attractive theoretical properties of these models is their conformal symmetry and their successful embedding into supergravity via hyperbolic geometry. The KKLMMT model is often acknowledged as the first to discuss the origin of D-brane inflation within string theory [241], and provides the motivation for more general string inspired cosmological models. These models predict values for the spectral index and the
tensor-to-scalar ratio which match observational data well. Thus, phenomenological D-brane inflation has attained renewed importance, independent of its string theory origin, since Planck 2018 [84]. Indeed, it has been shown [242] that further phenomenological models of D-brane inflation can be derived within the string theory approach (see also [240]). Because scalar fields (such as, for example, moduli fields) occur ubiquitously in fundamental theories such as supergravity and string/M theory, multi-field generalizations of the $\alpha$-attractor models have also been considered [243].

A number of inflationary cosmologies have been suggested within the context of string/M-theory [239, 241]. However, very few models exist that can be embedded within LQC [244]. In particular, there are a number of approaches to QG which include bouncing regimes. In resolving the initial singularity, it is of interest to determine whether slow-roll inflation is subsequently allowed (or is even natural). Inflation within the context of LQC, and how the bounce affects the evolution of the inflaton (as compared to the normal scenario with no bounce), was investigated in [245]. The evolution of the inflaton from the initial bounce was studied analytically for a number of important potentials in the case that the inflaton is taken to be the same scalar field that gives rise to the LQC bounce. It was found [245] that LQC, or any bouncing model in which the total energy density of the inflaton field is bounded at the transition, does provide a viable description of the pre-inflationary epoch and the subsequent smooth evolution to the standard inflationary era. The results were particularly encouraging in that the bounds obtained theoretically (on the critical bounce value for the inflaton field in order for there to subsequently be an appropriate slow roll inflationary regime) match (where appropriate) the known results from the numerical dynamics of the fully non-linear LQC.

4.3.1. String inflation. Cosmological inflation and its realization within QG and, in particular, in string theory, was reviewed in [240]. Examples of string inflation include brane and axion inflation. There are also string inspired effective field theories. Since string theory is considerably more constrained, some effective field theories that are apparently consistent at low energies do not, in fact, admit ultraviolet QG completions (leading to improved predictivity). However, there are indications that it might not be possible to embed simple inflationary models in string theory [239, 240]. One problem is that in order to obtain a period of slow-roll inflation from simple scalar field potentials, field values in excess of the Planck mass are required. But for sufficiently large values of the fields string effects on the shape of the potential must be included, which tend to destroy its required flatness except perhaps in the case of special field symmetries [123] (however, even then string theoretical arguments such as the so-called ‘Weak Gravity Conjecture’ [246] can lead to the effective field theory analysis being invalidated).

In addition, generating effective theories from string theory can also lead to different ideas as to what a natural (or a minimal) inflationary model might be. Indeed, a comprehensive understanding of naturalness within string theory is elusive. However, a general feature of all stringy constructions is the existence of a number of light scalar fields, so while multiple ‘unnecessary’ fields might be considered non-minimal in many field theory models, they are ubiquitous within string theory. Time-dependent solutions with string scale curvatures are crucial for any further comprehension, especially if we hope to progress from the paradigm of an effective theory for the massless modes.

To date, it is fair to say that there have not been any convincing realizations of inflation in the context of superstring theory. Making predictions in string theory is made exceedingly difficult by the landscape problem that string theory has an enormous number of vacua. Despite the fact that dynamics within the landscape is not well understood, it appears that false vacuum eternal inflation is an unavoidable consequence. In addition, all 4D de Sitter vacua in
supersymmetric string theories are metastable, since 10D supersymmetric Minkowski spacetime has zero energy, but de Sitter spacetime has positive vacuum energy. In particular, there are well-known no-go theorems for the existence of stable de-Sitter vacua in critical string theory [239]. This is a real problem for inflation should string theory be the final theory of QG.

The so-called string swampland criteria constrain inflationary models [247]. In addition, the second of the swampland conjectures implies, as noted above, that exact de Sitter solutions with a positive cosmological constant cannot describe the fate of the Universe at late times within string theory [239]. Dynamical dark energy scalar field models must also satisfy particular criteria so as to avoid the swampland. The observational implications of such string-theory criteria on quintessence models and the accompanying constraints on the dark energy were studied in [248]. However, since string theory does not naturally lead to scalar fields with an appropriate energy scale to be a reasonable candidate for quintessence, novel physics from string theory must be introduced to explain dark energy. In some very special models it is possible to characterize the Planck-suppressed corrections to the string theory inflatonary action, leading to the first indications for inflation within string theory [240]. But many critical challenges still remain. Indeed, the ‘simple’ cosmological observations (of the almost scale-invariant and Gaussian primordial curvature perturbations measured by Planck) to date are often interpreted as an argument against complex models of inflation in string theory (however, see [240]).

5. Concluding remarks

We have reviewed recent developments and described a number of open questions in the field of theoretical cosmology. We described the concordance cosmological model and the standard paradigms of modern cosmology, and then discussed a number of fundamental issues and open theoretical questions, emphasizing the various assumptions made and identifying which results are independent of these assumptions. Indeed, standard cosmology contains a number of philosophical assumptions that are not always scientific, including the assumption of spatial homogeneity and isotropy at large scales outside our particle horizon. Perhaps a more tangible fundamental issue concerns the measure problem and the issue of initial conditions in inflation. Many of the fundamental problems arise due to the inhomogeneities in the Universe. However, this is also one of the great strengths of present day cosmology: our models predict what structure will occur, and consequently the astounding development of observational projects determining in great detail the characteristics of such structure that serve to give strong limits on cosmological parameters.

Cosmology is not only a mathematical endeavour, but it is a testable scientific theory due to its ability to produce observational predictions. In recent times there has been a plethora of such detailed tests, leading to the so-called era of precision cosmology. Perhaps fundamental questions are less relevant for current working cosmologists, who are more concerned with physical cosmology and data and statistical analysis. But as the modern emphasis changes to more physical and observational issues, theoretical cosmology is still important and fundamental questions persist. In some sense, we hope to record here the state of the art as it now exists.

A qualitative analysis of the properties of cosmological models and the problems of the stability of cosmological solutions and of singularities is important in mathematical cosmology. A number of open problems in theoretical cosmology involve the nature of the origin and details of cosmic inflation, and its relation to fundamental physics. Perhaps the most urgent open problems of theoretical cosmology include the early and late time accelerated expansion
of the universe and the role of the cosmological constant $\Lambda$. As we have emphasized, computational cosmology is becoming an increasingly important tool in the investigation of theoretical and physical cosmology.

We then reviewed a number of open problems in physical cosmology, with particular focus on perturbation theory (and gauge issues) and the formation and distribution of large scale structure in the Universe at present times (and especially in the non-linear regime). Backreaction is still an important issue, although perhaps the more formal mathematical averaging problem is currently more relevant. Finally, gravitational wave astronomy will potentially play an increasingly important role within cosmology. Indeed, there is a robust prediction within inflation for a gravitational wave induced CMB polarization signal.

We also discussed cosmological problems in quantum gravity, including the possible resolution of cosmological singularities and the crucial issue of the role of inflation within quantum gravity.

Finally we have emphasized that, given the uniqueness of the Universe and the limitations on the domain we can explore by any conceivable observations, it is key to carry out all possible consistency tests of our models. For example, the first and foremost is the age of the Universe: is the Universe older than its stellar and galactic content? If not, cosmology is in deep trouble. Fortunately this consistency test seems to be satisfied at present (thanks to the cosmological constant). Another consistency test is that all number counts must display a dipole aligned with the CMB dipole; this is presently being contested.

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