Implications in Sectionally Pseudocomplemented Posets

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Abstract. A sectionally pseudocomplemented poset $P$ is one which has the top element and in which every principal order filter is a pseudocomplemented poset. The sectional pseudocomplements give rise to an implication-like operation on $P$ which coincides with the relative pseudocomplementation if $P$ is relatively pseudocomplemented. We characterise this operation and study some elementary properties of upper semilattices, lower semilattices and lattices equipped with this kind of implication. We deal also with a few weaker versions of implication. Sectionally pseudocomplemented lattices have already been studied in the literature.

1. Introduction

This study is roused by the paper [4] the subject of which is lattices with the largest element and pseudocomplemented upper sections (principal filters). Such a lattice $(L, \land, \lor, 1)$ admits a partial binary operation $*$ defined as follows:

\begin{enumerate}
  \item $x \ast y = z$ if and only if $y \leq x$ and $z$ is the pseudocomplement of $x$ in $[y]$.
  \item if $x \in [y]$, then, for all $u \in [y]$, $u \leq x \ast y$ if and only if $u \land x = y$.
\end{enumerate}

In particular, if $y \leq x$ and $z$ is the pseudocomplement of $x$ relatively to $y$, i.e., if for all $u$, $u \leq z$ if and only if $u \land x = y$.

then $z = x \ast y$. The algebra $(L, \land, \lor, *, 1)$ could be called a sectionally pseudocomplemented lattice. The total binary operation $\rightarrow$ defined on $L$ by the condition

\begin{enumerate}
  \item $x \rightarrow y := (x \lor y) \ast y$
\end{enumerate}

is, evidently, an extension of *. Sectionally pseudocomplemented lattices and their extensions are explored further in [6, 7]. As noted in [6, Remark 2.2], the extension $(L, \lor, \land, \rightarrow, 1)$ of a distributive sectionally pseudocomplemented lattice is a Brouwerian lattice (Heyting algebra).

Another type of extension of the operation $*$:

\begin{enumerate}
  \item $x \rightarrow y := x \ast (x \land y)$
\end{enumerate}

was investigated in [5, 12]. (It should also be noted that meet semilattices with pseudocomplemented lower sections (principal ideals) have been studied already in [17, 19, 21].)

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A natural way to extend the notion of a pseudocomplementation to arbitrary posets has been discovered by several authors — see \[18, 26, 11\]. Correspondingly, sectional pseudocomplements can also be considered in posets that are not meet semilattices. By understanding pseudocomplements in (1) in this wider sense, we obtain, instead of (2), the following characteristic condition for sectional pseudocomplementation \(*\) in a poset (we write \(u \perp_y x\) to mean that there is no lower bound of \(u\) and \(x\) in \([y]\) distinct from \(y\)):

\[(5) \quad \text{if } x \in [y], \text{ then, for all } u \in [y], u \leq x * y \text{ if and only if } u \perp_y x.\]

This condition reduces to (2) if the poset is a meet semilattice.

The concept of pseudocomplementation may be further weakened in various ways. In this paper, we consider posets (in particular, semilattices and lattices) with weakened sectional pseudocomplementation (5), as well as its extension allied to (3).

2. Extensions of sectionally pseudocomplemented posets

Now suppose that \((A, \lor, 1)\) is a semilattice with unit, and \(*\) is the operation defined by (5). The extension of \(*\) given by (3) can, actually, be defined without an explicit reference to \(*\). We first note that, in \(A\), the condition \(u \perp y x\) actually means that \(u \land x\) exists and equals to \(y\). Indeed, if \(y \leq u, x\), then

\[
\begin{align*}
\forall v \in [y] \, (\text{if } v \leq u, x, \text{ then } v = y) \\
\Rightarrow \forall w \, (\text{if } w \lor y \leq u, x, \text{ then } w \lor y = y) \\
\Rightarrow \forall w \, (\text{if } w \leq u, x, \text{ then } w \leq y) \\
\Rightarrow \forall w \, (w \leq u, x \iff w \leq y).
\end{align*}
\]

Furthermore,

\[
z \leq (x \lor y) * y \iff z \lor y \leq (x \lor y) * y \iff z \lor y \perp_y x \lor y
\]

for all \(z \in A\). These observations lead us to the following lemma.

**Lemma 1.** A binary operation \(\rightarrow\) on \(A\) satisfies (6) if and only if, for all \(x, y\) and \(z\) in \(A\),

\[
(6) \quad z \leq x \rightarrow y = (z \lor y) \land (x \lor y) \text{ exists and equals to } y.
\]

The next theorem shows that the operation \(\rightarrow\) can be characterised even without reference to join.

**Theorem 2.** A binary operation \(\rightarrow\) on \(A\) satisfies (3) if and only if it has the following properties:

\[
(\rightarrow_1) \quad \text{if } x \leq y \to z, \text{ then } y \leq x \to z,
\]

\[
(\rightarrow_2) \quad \text{if } x \leq x \to y, \text{ then } x \leq y,
\]

\[
(\rightarrow_3) \quad \text{if the meet of } x \text{ and } y \text{ exists, then } x \leq y \to (x \land y).
\]

**Proof.** It is easily seen that \((\rightarrow_1)\), \((\rightarrow_2)\) and \((\rightarrow_3)\) hold for the operation \(\rightarrow\) characterised by (3).

Conversely, if the operation \(\rightarrow\) satisfies the conditions \((\rightarrow_1) - (\rightarrow_3)\) and \(*\) is its restriction defined by

\[
(7) \quad x * y = v \text{ if and only if } y \leq x \text{ and } v = x \rightarrow y,
\]
then, obviously, (3) holds true. Let us see, why \( x \ast y \) is the pseudocomplementation of \( x \) in \([y]\). Suppose that \( u, x \in [y] \). If \( u \leq x \ast y \) and \( y \leq v \leq u, x \), then \( v \leq u \leq x \rightarrow y \) and, furthermore, \( v \leq x \leq v \rightarrow y \) by (11), wherefrom \( v \leq y \), i.e. \( v = y \) by (12).

If, conversely, \( u \perp y x \), then \( y \) is the greatest lower bound of \( x \) and \( y \) in \([y]) \) and (as noted at the beginning of the section) even in \( L \), and then \( z \leq x \rightarrow y = x \ast y \) by (13). So, (6) holds by Lemma 1.

Now let \((A, \rightarrow, 1)\) be any algebra in which \( A \) is a poset with 1 the greatest element and \( \rightarrow \) is a binary operation obeying the conditions (14)–(16). It is an implicative algebra in the sense of [23], for the relation

\[
(8) \quad x \leq y \text{ if and only if } x \rightarrow y = 1 \tag{8}
\]

is an easy consequence of these conditions. Indeed, if \( 1 \leq x \rightarrow y \), then \( x \leq x \rightarrow y \), and the inequality \( x \leq y \) follows by (12). Conversely, suppose that \( x \leq y \). By (14), \( 1 \leq y \rightarrow y \), and then \( x \leq 1 \rightarrow y \) in virtue of (11).

We know from the proof of the theorem that \( A \) is an extension of a sectionally pseudocomplemented poset. It follows from Corollary 6 below that this extension is completely determined by the underlying poset. To remind that the characterised properties of \( \rightarrow \) were based on (3) rather than on (4), this kind of extension could even be termed a \( j \)-extension (‘\( j \)’ for ‘join’).

These observations motivate the following definition.

**Definition 3.** An algebra \((A, \rightarrow, 1)\) satisfying (14)–(16) is said to be a sectionally \( j \)-pseudocomplemented poset. The operation \( \rightarrow \) itself is called \( j \)-sectional pseudocomplementation.

Sectionally \( j \)-pseudocomplemented semilattices and lattices are defined similarly.

### 3. Weak BCK*-algebras

It turns out that many important properties of sectionally \( j \)-pseudocomplementation actually do not depend of (14) and (15). See Remark 7 for the motivation of the term ‘\( w \)BCK*-algebra’ used in the subsequent definition.

**Definition 4.** A weak BCK*-algebra, or just \( w \)BCK*-algebra, is an implicative algebra \((A, \rightarrow, 1)\), where \( \rightarrow \) satisfies (11). A \( w \)BCK*-algebra is said to be weakly contractive if it satisfies (12).

**Lemma 5.** In every \( w \)BCK*-algebra,
\[
(\rightarrow_4): \quad x \leq (x \rightarrow y) \rightarrow y,
(\rightarrow_5): \quad \text{if } x \leq y, \text{ then } y \rightarrow z \leq x \rightarrow z.
(\rightarrow_6): \quad y \leq (x \rightarrow y) \rightarrow y,
(\rightarrow_7): \quad ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y,
(\rightarrow_8): \quad x \rightarrow x = 1,
(\rightarrow_9): \quad x \leq y \rightarrow x,
(\rightarrow_{10}): \quad 1 \rightarrow x = x,
(\rightarrow_{11}): \quad x \rightarrow 1 = 1.
\]

**Proof.** (\( \rightarrow_4 \)) Trivially.
(\( \rightarrow_5 \)) By (14) and (11).
(\( \rightarrow_6 \)) Similarly.
(\( \rightarrow_7 \)) By (14) and (11), \((x \leq y) \rightarrow y \leq x \rightarrow y \). The converse inequality is a particular case of (13).
Follows from (8).

By (11) and (8), as $y \leq 1 = x \rightarrow x$.

By (8), the inequality $1 \rightarrow x \leq x$ follows from (11). Its converse is a particular case of (9).

Follows from (9). \hfill \Box

Now it can be shown that the structure of every wBCK*-algebra is completely determined by the structure of its sections. In particular, a sectionally pseudocomplemented poset admits at most one wBCK*-algebra extension.

Lemma 6. Suppose that $(A, \rightarrow)$ is a wBCK*-algebra and that $\ast$ is the restriction of $\rightarrow$ determined by (7). Then

$$ x \rightarrow y = \max\{ z \ast y : x, y \leq z \}. $$

Proof. Let $z := (x \rightarrow y) \rightarrow y$. Then $x, y \leq z$ by (11) and (8), and further $x \rightarrow y = z \rightarrow y = z \ast y$ by (7) and (7). On the other hand, if $x, y \leq z$, then $z \ast y = z \rightarrow y \leq x \rightarrow y$ by (5). \hfill \Box

Remark 7. It follows from (11) and (10) that $y \leq (y \rightarrow z) \rightarrow z \leq x \rightarrow z$ whenever $x \leq y \rightarrow z$. Therefore, this pair of conditions is equivalent to the axiom $\rightarrow$ of wBCK*-algebras. The latter term is motivated by this observation: if the algebra $(A, \rightarrow, 1)$ satisfies (8), (11) and the following strengthening of (9)

$$ x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z), $$

then it is the dual algebra (w.r.t. the ordering $\leq$) of a BCK-algebra (see, e.g., [16]). We adopt the asterick notation $\text{BCK}^*$ for such duals from [24].

A Hilbert algebra can be characterised as a positive implicative BCK*-algebra, i.e., a BCK*-algebra in which

$$ x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z); $$

see [3, 25, 20]. The latter condition may be replaced by the identity

$$ x \rightarrow (x \rightarrow y) = x \rightarrow y $$

[16, Theorem 8]; see also [25, Theorem 1]. Since this identity covers (9), every Hilbert algebra is an example of a weakly contractive wBCK*-algebra. See Corollary 16 below for a stronger result.

A particular kind of Hilbert algebras are relatively pseudocomplemented posets [24]: algebras $(A, \rightarrow, 1)$, where $A$ is a poset with 1 the maximum element and the operation $\rightarrow$ that satisfies the condition

$$ \text{(9)} \quad \text{if } u \leq x \rightarrow y \text{ and } v \leq x, u, \text{ then } v \leq y $$

as well as its converse—the following strengthening of (8):

$$ \text{(10)} \quad \text{if } v \leq y \text{ whenever } v \leq x, u, \text{ then } u \leq x \rightarrow y. $$

In fact, (9) is equivalent to its particular case

$$ \text{(11)} \quad \text{if } z \leq x, z \leq x \rightarrow y, \text{ then } z \leq y. $$

The subsequent lemma characterises the relation between sectionally j-pseudocomplemented and relatively pseudocomplemented posets more exactly.
Theorem 8. Let $A$ be a poset with the greatest element $1$. A binary operation $\to$ on $A$ is relative pseudocomplementation if and only if it satisfies $(\text{-11})$, $(\text{-2})$ and $(\text{-12})$.

Proof. It was established in the proof of Lemma 1 that $(\text{-11})$ is a consequence of $(\text{-11})$ and $(\text{-2})$. On the other hand, the conditions $(\text{-11})$ and $(\text{-2})$ are fulfilled in every relatively pseudocomplemented poset. Indeed, $(\text{-2})$ is a particular case of $(\text{-11})$ with $u = v = x$. To prove $(\text{-11})$, assume that $x \leq y \to z$. By $(\text{-11})$, then $v \leq x, y$ implies that $v \leq z$. Therefore $y \leq x \to z$ by $(\text{-11})$.

We shall say that a wBCK*-algebra $A$ is an upper (lower) wBCK*-semilattice or a wBCK*-lattice, if $A$ happens to be an upper (lower) semilattice or a lattice, respectively. A Hilbert algebra with infimum $\text{11}$, i.e., a lower semilattice-ordered Hilbert algebra, is an example of a weakly contractive lower wBCK*-semilattice. BCK*-semilattices and lattices have been studied in $\text{14}$. Relatively pseudocomplemented semilattices, known also as Brouwerian or implicative semilattices, form a subclass of sectionally $j$-pseudocomplemented lower BCK*-semilattices. Likewise, relatively pseudocomplemented, or implicative, lattices (Heyting algebras) form a subclass of BCK*-lattices.

Theorem 9. A sectionally $j$-pseudocomplemented lower semilattice (lattice) is relatively pseudocomplemented if and only if it satisfies the condition

$(\text{-12})$: if $x \leq y$, then $z \to x \leq z \to y$.

Proof. Due to $\text{10}$ and $\text{11}$, the condition $(\text{-12})$ is fulfilled in every relatively pseudocomplemented poset. Now assume that a wBCK*-algebra $A$ is a lower semilattice satisfying $(\text{-12})$; by the previous theorem it suffices to prove only that $(\text{-11})$ holds. Let $v \leq x$, $u$ implies, for every $v$, that $v \leq y$. Then $v \land x \leq y$ and, by $(\text{-2})$ and $(\text{-12})$,

As $(\text{-12})$ holds in all BCK*-algebras (see $\text{16}$ Theorem 2), we conclude that a lower BCK*-semilattice is relatively pseudocomplemented (i.e., is an implicative semilattice) if and only if it is sectionally $j$-pseudocomplemented.

4. Weak BCK*-algebras with condition $S$

Adapting the definition known for BCK-algebras (see, e.g., $\text{15}$ $\text{16}$), we shall say that a wBCK*-algebra $A$ satisfies condition $S$ if, for all $x$ and $y$, the subset \{ $z$: $x \leq y \to z$ \} has the least element. We may denote it by $x \cdot y$; this way a binary operation $\cdot$ on $A$ may be introduced. The couple of operations $(\cdot, \to)$ illustrates the following definition.

Definition 10. By a (binary) adjunction on a poset $A$ we mean a pair $(\cdot, \to)$ of binary operations on $A$ satisfying the condition

$(\text{12})$ $x \leq y \to z$ if and only if $xy \leq z$.

Proposition 11. An adjunction $(\cdot, \to)$ can equivalently be characterised by four conditions

(a1) $x \leq y \to xy$,
(a2) $(x \to y)x \leq y$,
(a3) if $x \leq y$, then $z \to x \leq z \to y$,
(a4) if $x \leq y$ then $xz \leq yz$. 

Note that (a3) coincides with (12).

**Theorem 12.** Suppose that \((·, →)\) is an adjunction on a poset \(A\) and \(1 \in A\). Then the following statements are equivalent:

(a) \((A, →, 1)\) is a \(wBCK^*\),

(b) \((A, ·, 1)\) is a commutative groupoid with the neutral element 1 which is also the largest element in \(A\).

If it is the case, then the \(wBCK^*\)-algebra is weakly contractive if and only if the groupoid is idempotent.

**Proof.** (a) \(→\) (b). Commutativity of \(·\) follows from (12) in virtue of (1). By (11), 1 is the maximum element in \(A\). By (12) and (12), \(1 · x ≤ x\). At last, \(x ≤ 1 · x\) by (a1) and (a).

(b) \(→\) (a). The two assumptions on 1 provide (8):

\[ x → y = 1 ⇔ 1 ≤ x → y ⇔ 1 · x ≤ y ⇔ x ≤ y. \]

We know from Remark 7 that (1) is a consequence of (4) and (7). The property (a2) together with commutativity of \(·\) allows us to prove (9). To obtain (9), assume that \(x ≤ y\). As \(·\) is commutative, it follows from (a4) that \(ux ≤ uy\) for all \(u\). Then, for every \(z\), \(uy ≤ z\) implies that \(ux ≤ z\). Therefore, \(u ≤ y → z\) implies that \(u ≤ x → z\). Hence, \(y → z ≤ x → z\).

For the last assertion note that

\[ x ≤ x → y ⇔ x · x ≤ y ⇔ x ≤ y, \]

if \(·\) is idempotent, and that

\[ x · x ≤ y ⇔ x ≤ x → y ⇔ x ≤ y, \]

if \(→\) is weakly contractive (the condition “if \(x ≤ y\), then \(x ≤ x → y\)” inverse to (7) follows from (9)). Therefore, idempotency of \(·\) turns out to be equivalent to condition that \(→\) has to be weakly contractive. \(\square\)

In virtue of (a4), if the operation \(·\) in an adjunction is commutative, then it gives rise to a partially ordered groupoid (po-groupoid). A commutative po-groupoid is said to be integral, if it has the neutral element which is also the maximum element, and residuated if the multiplication \(·\) has the adjoint operation \(→\). We shall use the acronym pocrig for a partially ordered commutative residuated and integral groupoid. Therefore, a pocrig can be viewed as an algebra of type \((A, ·, →, 1)\). A pocrig with associative multiplication is known as a pocrim; see [2, 3].

**Corollary 13.** An algebra \((A, →, 1)\) is a \(wBCK^*\)-algebra with condition \(S\) if and only if it is a reduct of a pocrig.

A similar correspondence between \(BCK^*\)-algebras with condition \(S\) and pocrims has already been noticed in the literature; see, e.g. [3]. It should be noted that some authors include multiplication in the signature of \(BCK^*\)-algebras with condition \(S\); then the class of \(BCK^*\)-algebras with condition \(S\) coincides with the class of pocrims.

The last assertion of Theorem 12 suggests that idempotent pocrigs and regular \(wBCK^*\)-algebras with condition \(S\) should be related to each other in the same way. In fact, we can say more about this situation.
The next lemma (which slightly improves Lemma 4.1 in [22]) implies that such a wBCK*-algebra is even relatively pseudocomplemented (hence, a Hilbert algebra — see the preceding section).

**Lemma 14.** An idempotent pocrig \((A, \cdot, \rightarrow, 1)\) is an implicative semilattice, i.e., \((A, \cdot, 1)\) is a lower semilattice with unit, and \(\rightarrow\) is relative pseudocomplementation on \(A\).

**Proof.** As multiplication in a pocrig is, by definition, isotone, it follows from \(x, y \leq 1\) that \(x \cdot y\) is a lower bound of \(x\) and \(y\). Assume that \(z \leq x, y\); then \(x \cdot z \leq x \cdot y\) and \(z = z \cdot z \leq x \cdot z\). Therefore, \(z \leq x \cdot y\), and \(x \cdot y\) is actually the greatest lower bound of \(x\) and \(y\). Then the adjoint \(\rightarrow\) of \(\cdot\) becomes relative pseudocomplementation on \(A\). □

**Corollary 15.** An algebra \((A, \rightarrow, 1)\) is a weakly contractive wBCK*-algebra with condition \(S\) if and only if it is a reduct of an implicative semilattice.

Hilbert algebras are just subreducts of implicative semilattices [9, Theorem 12] (see also Theorem 8 of [13]). This gives us the following characteristic of those wBCK*-algebras that are Hilbert algebras.

**Corollary 16.** A wBCK*-algebra is a Hilbert algebra if and only if it is a subalgebra of a weakly contractive wBCK*-algebra with condition \(S\).

By a multiplicative semilattice we, following [1], shall mean an upper semilattice with multiplication which is both left and right distributive.

**Theorem 17.** Suppose that \((\cdot, \rightarrow)\) is an adjunction on a poset \(A\), \(1 \in A\), and \(\lor\) be a binary operation on \(A\). Then the following statements are equivalent:

(a) \((A, \lor, \rightarrow, 1)\) is a wBCK*-semilattice,

(b) \((A, \cdot, 1)\) is a semilattice ordered commutative integral groupoid,

(c) \((A, \lor, \cdot, 1)\) is an integral and commutative multiplicative semilattice.

**Proof.** In virtue of Theorem 12, it remains to show that a semilattice ordered pocrig is distributive. For all \(u \in A\),

\[
(x \lor y)z \leq u \iff x \lor y \leq z \rightarrow u
\]

\[
\iff x \leq z \rightarrow u \text{ and } y \leq z \rightarrow u
\]

\[
\iff xz \leq u \text{ and } yx \leq u
\]

\[
\iff xz \lor yz \leq u. \quad \Box
\]

**Corollary 18.** An algebra \((A, \lor, \rightarrow, 1)\) is an upper wBCK*-semilattice with condition \(S\) if and only if it is a reduct of a semilattice ordered pocrig or, equivalently, of a residuated integral multiplicative semilattice.

5. **Some equational classes of expanded wBCK*-algebras**

It is well-known [27] that the class of BCK*-algebras is not a variety. As the condition \((\rightarrow 7)\) is, due to [4], essentially an equation, this remains true also for wBCK*-algebras. The situation changes when join or meet operation is added.

**Theorem 19.** Let \((A, \lor, 1)\) be a semilattice with unit and the natural ordering \(\leq\), and let \(\rightarrow\) be a binary operation on \(A\). Then the following statements are equivalent:

(a) \((A, \rightarrow, 1)\) is a wBCK*-algebra,
(b) \( \rightarrow \) satisfies the conditions \((-4), (-10)\) and
(b1) \( x \rightarrow (x \lor y) = 1 \),
(b2) \( (x \lor y) \rightarrow z \leq y \rightarrow z \).

**Proof.** Evidently, every wBCK*-semilattice satisfies the conditions listed in (b) — see (8) and \((-4), (-10)\). If, conversely, the conditions are satisfied in \((A, \lor, \rightarrow, 1)\), then the order relation \(\leq\) satisfies (8) in virtue of \((4), (10)\) and (b1), and then \((-12)\) follows from (b2) by (8).

The next theorem is proved similarly.

**Theorem 20.** Let \((A, \wedge, 1)\) be a lower semilattice with unit and the natural ordering \(\leq\), and let \(\rightarrow\) be a binary operation on \(A\). Then the following statements are equivalent:

(a) \((A, \rightarrow, 1)\) is a wBCK*-algebra,
(b) \(\rightarrow\) satisfies the conditions \((-4), (-10)\) and
(b1) \(x \land (x \rightarrow y) = 1\),
(b2) \(x \rightarrow z \leq (x \land y) \rightarrow z\).

**Corollary 21.** The following classes of algebras are equationally definable:

(a) the class of all upper wBCK*-semilattices,
(b) the class of all lower wBCK*-semilattices, as well as its subclasses of weakly contractive and of sectionally \(j\)-pseudocomplemented semilattices,
(c) the class of all wBCK*-lattices, as well as its subclasses of weakly contractive and of sectionally \(j\)-pseudocomplemented lattices,
(d) the class of all upper semilattice-ordered pocrigs,
(e) the class of all lower semilattice-ordered pocrigs,
(f) the class of all lattice-ordered pocrigs.

**Proof.** (a) Follows from Theorem 19.
(b) Follows from Theorem 20. Note that the condition \((-2)\) can be rewritten in a form of an equation as follows:
\((\rightarrow 13): x \land (x \rightarrow y) \leq y\).
Indeed, \((-2)\) is an easy consequence of \((\rightarrow 13)\). On the other hand, \(x \land (x \rightarrow y) \leq x \rightarrow y\), and then \(x \leq x \land (x \rightarrow y) \rightarrow y\) by \((\rightarrow 11)\). As \(x \land (x \rightarrow y) \leq x\), it follows by \((-2)\) that \(x \land (x \rightarrow y) \leq y\).
(c) Follows from (a) and (b).
(d), (e) Follow from (a) and (b) respectively, as in semilattices the four conditions listed in Proposition 11 are captured by equations.
(f) Follows from (d) and (e). \(\square\)

For BCK*-semilattices and lattices this was proved by Idziak in [14, Theorem 1]. As noted in [14], the variety of upper BCK*-semilattices is neither congruence permutable nor congruence distributive. Clearly, this concerns also upper wBCK*-semilattices. In contrast, the class of lower wBCK*-semilattices is even arithmetical; and so is the class of wBCK*-lattices. Our next theorem together with its proof generalises Theorem 2 of [14].

**Theorem 22.** The variety of lower wBCK*-semilattices is arithmetical.

**Proof.** The variety is congruence distributive, for it has a majority term
\(m(x, y, z) := (x \rightarrow y. \rightarrow y) \land (y \rightarrow z. \rightarrow z) \land (z \rightarrow x. \rightarrow x)\)
and congruence permutable, for it has a corresponding Mal’cev term

\[ p(x, y, z) := (x \to y \to z) \land (z \to y \to x) \]

(see (−8), (−10), (−4), (−6)). Hence, it is arithmetical.

Of course, then all subvarieties varieties of lower wBCK*-semilattices and of wBCK*-lattices mentioned in the corollary are also arithmetical.

**Remark 23.** Sectionally j-pseudocomplemented lattices are just the j-extensions of sectionally pseudocomplemented lattices mentioned in Introduction. Another equational description of such extensions was presented in [1] Theorem 2. It was stated in Theorems 5.1 and 5.3 of [6] that this variety is arithmetical and 1-regular. The easy proof of the latter theorem goes even for any variety of upper wBCK*-semilattices.

The next theorem and its proof are suggested by the similar result [14] Theorem 3 for pocrims.

**Theorem 24.** The variety of uppersemilattice-ordered pocrigs is arithmetical.

**Proof.** The corresponding Mal’cev terms are

\[ m(x, y, z) := x(x \to y) \lor y(y \to z) \lor z(z \to x), \]

and

\[ p(x, y, z) := x(y \to z) \lor z(y \to x). \]

□

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