SUPERTROPICAL MATRIX ALGEBRA

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Abstract. The objective of this paper is to develop a general algebraic theory of supertropical matrix algebra, extending [11]. Our main results are as follows:
• The tropical determinant (i.e., permanent) is multiplicative when all the determinants involved are tangible.
• There exists an adjoint matrix \( \text{adj}(A) \) such that the matrix \( A \text{adj}(A) \) behaves much like the identity matrix (times \(|A|\)).
• Every matrix \( A \) is a supertropical root of its Hamilton-Cayley polynomial \( f_A \). If these roots are distinct, then \( A \) is conjugate (in a certain supertropical sense) to a diagonal matrix.
• The tropical determinant of a matrix \( A \) is a ghost iff the rows of \( A \) are tropically dependent, iff the columns of \( A \) are tropically dependent.
• Every root of \( f_A \) is a “supertropical” eigenvalue of \( A \) (appropriately defined), and has a tangible supertropical eigenvector.

1. Introduction

In [12], the abstract foundations of supertropical algebra were set forth, including the concept of a supertropical domain and supertropical semifield. The motivation was to overcome the difficulties inherent in studying polynomials over the max-plus algebra, by providing an algebraic structure that encompasses the max-plus algebra, thereby permitting a thorough study of polynomials and their roots and a direct algebraic-geometric development of tropical geometry.

Similarly, although there has been considerable interest recently in linear algebra over the max-plus algebra [1, 5, 17], the weakness of the inherent structure of the max-plus algebra has hampered a systematic development of the matrix theory. The object of this paper is to lay the groundwork for such a theory over a supertropical domain, which yields analogs of much the classical matrix theory for the max-plus algebra and also explains why other parts do not carry over.

The max-plus algebra is a special kind of idempotent semiring. In general, the matrix semiring over a semiring is also a semiring (to be described below in detail), but often loses some of its properties. So we also need to pinpoint some of those properties that are preserved in such matrix semirings. Our underlying structure is a semiring with ghosts, which we recall from [13] is a triplet \((R, \mathcal{G}_0, \nu)\), where \( R \) is a semiring with a unit element \( \mathbf{1}_R \) and with zero element \( \mathbf{0}_R \) (satisfying \( \mathbf{0}_R r = r \mathbf{0}_R = \mathbf{0}_R \) for every \( r \in R \), and often identified in the examples with \(-\infty\), as indicated below), \( \mathcal{G}_0 = \mathcal{G} \cup \{\mathbf{0}_R\} \) is a semiring ideal called the ghost ideal, and \( \nu : R \to \mathcal{G}_0 \), called the ghost map, is an idempotent semiring homomorphism (i.e., which preserves multiplication as well as addition).

We write \( a^\nu \) for \( \nu(a) \), called the \( \nu \)-value of \( a \). Thus, \( 1_R^\nu \) is multiplicatively idempotent, and serves as the unit element of \( \mathcal{G}_0 \). Two elements \( a \) and \( b \) in \( R \) are said to be matched if they have the same \( \nu \)-value; we say that \( a \) dominates \( b \) if \( a^\nu \geq b^\nu \).

For tropical applications, we focus on the tangible elements, which in this paper are defined as \( \mathcal{T} = R \setminus \mathcal{G}_0 \); they are defined more generally in [13] (cf. Remark [14] below). We write \( \mathcal{T}_0 \) for \( \mathcal{T} \cup \{0_R\} \). (Although \( 0_R \) is a ghost element, being part of the ghost ideal, it is useful to consider it together with the tangible elements when considering linear combinations.)

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Next, in [13 Definition 3.5], we defined a **supertropical semiring**, which is a commutative semiring with ghosts satisfying the extra properties:

- \( a + b = a'' \) if \( a'' = b'' \);
- \( a + b \in \{a, b\}, \forall a, b \in R \text{ s.t. } a'' \neq b'' \). (Equivalently, \( G_0 \) is ordered, via \( a'' \leq b'' \iff a'' + b'' = b'' \).)

Thus, \( a'' = 0_{R''} \iff a = a + 0_R = 0_{R''} = 0_R \).

It follows that \( a + b = 0_R \iff \max\{a'', b''\} = 0_R, \text{iff } a = b = 0_R \). Hence, no nonzero element has an additive inverse.

In studying supertropical semirings in [13], we defined a **supertropical domain** to be a supertropical semiring for which \( T \) is a monoid; we also assume here that the map \( \nu_T : T \to G \) (defined as the restriction from \( \nu \) to \( T \)) is onto. (See [13 Remark 3.11] for some immediate consequences of this definition, including a version of cancellation.) We also defined a **supertropical semifield** to be a supertropical domain \((R, G_0, \nu)\) for which \( T \) is a group.

Whereas the paper [13] focused on the theory of polynomials and their roots over supertropical semifields, in this paper we turn to the matrix theory of semirings with ghost ideals, and so bring in tropical determinants, i.e., permanents, and tropical linear algebra. We obtain a multiplicative rule for the tropical determinant (Theorem 6.5), a tropical theory of the adjoint matrix (Theorem 4.10), and the fact that a matrix is singular iff its rows, or its columns, are (tropically) dependent (Theorem 6.5). Some of our results follow [11], which handled the special case where \( R \) is the “extended tropical semiring” of the real numbers; the proofs here are somewhat more conceptual. Theorem 6.5 is extended in [14], which relies on this paper.

“Linear algebra over a semiring” is the title of a chapter in Golan’s book [8 Chapter 17], and there already exists a sizeable literature concerning linear algebra for the max-plus algebra, as summarized in [11]; there is the existence of eigenvectors for matrices over max-plus algebras. (See also [17] and [5] for results concerning tropical determinants and tropical rank.) Nevertheless, the ghost ideal here changes the flavor considerably, enabling us to define and utilize adjoint matrices and also obtain a supertropical version of the Hamilton-Cayley theorem, together with applications to obtain tangible supertropical eigenvectors for all roots of the characteristic polynomial.

To clarify our exposition for those versed in tropical mathematics, the examples in this paper are presented for the extended tropical semiring [10], the motivating example for supertropical semirings. For this semiring, denoted as \( D(R) \), we have \( T = G = R, 1_R = 0, \text{ and } 0_R = -\infty \), where its operations, \( + \) and \( \cdot \), are respective modifications of the standard max, + operations over the reals. In other words, we use **logarithmic notation** in all of our illustrations, whereas in the theorems, we use multiplicative notation which is more in accordance with the ring-theoretic structure, our source of intuition. We hope this does not cause undue confusion.

Throughout this paper, as in [13], we assume that \( \nu \) is given by

\[
\nu(a) = a + a. \tag{1.1}
\]

Although there are more general situations of interest in supertropical algebra, they can often be reduced to the setting here because of the following observation:

**Remark 1.1.** Suppose \((R, G_0, \nu)\), is a semiring with ghosts satisfying \((1_R + 1_R)' = 1_R''\), (but not necessarily satisfying \(1_R + 1_R = 1_R''\)), and \( T \subseteq R \setminus G_0 \) is any multiplicative monoid. Taking \( G_0' = R1_R'' \subseteq G_0 \), one can define a new semiring structure \( \bar{R} = T \cup G_0' \), as follows:

Multiplication is the restriction to \( R \) of multiplication in \( R \), so \( 0_R \) remains the zero element, \( 1_R \) remains the unit element, and \( 1_R'' \) still is multiplicatively idempotent.

The new addition in \( \bar{R} \) is given by \( 0_R + r = r + 0_R \) for all \( r \in R \); but now, the sum of two elements \( a \) and \( b \) in \( \bar{R} \) is defined to be their sum in \( R \) if it lies in \( T \), and is \((a + b)''\) otherwise. In particular, \( a + a = a'' \) in \( \bar{R} \), and \( \bar{R} \) is a supertropical semiring. The ghost ideal of \( \bar{R} \) is \( R1_R'' \), and the tangible part is \( T \).

Thus, we see that the “tangible” part of the algebraic structures of \( R \) and \( \bar{R} \) are the same, and in particular the theorems in this paper about \( M_n(R) \) also hold for \( M_n(\bar{R}) \).
We write “$a = b + \text{ghost}”$ to indicate that $a$ equals $b$ plus some undetermined ghost element. This can happen in two ways: Either $a \in \mathcal{T}$ (in which case $a = b$), or $a \in \mathcal{G}$ with $a^\nu \geq b^\nu$ (in which case $a = b + a$).

**Remark 1.2.** If $a = b + \text{ghost}$, then Equation (1.1) implies $a + b = b + b + \text{ghost} \in \mathcal{G}_0$, although the converse might fail.

One difference with [13] is that here we do not require our semirings to be commutative, since we must deal with semirings of matrices. Nevertheless, we do have the following important property:

**Remark 1.3 (The Frobenius property).** $(r + z)^m$ equals $r^m + z^m + \text{ghost}$, for all $m \in \mathbb{N}^+$, $r \in R$, and central $z \in R$. This is because

$$(r + z)^m = r^m + z^m + \sum_{1 \leq i < m} \binom{m}{i} r^i z^{m-i},$$

and each summand in the summation is ghost since $\binom{m}{i} > 1$ for $1 \leq i < m$. It follows that $(r + z)^m + (r^m + z^m)$ is ghost, whenever $z$ is central.

2. **Tropical modules and matrices**

Modules over semirings (called *semimodules* in [8]) are defined just as modules over rings, except that now the additive structure is that of a semigroup instead of a group. (Note that subtraction does not enter into the other axioms of a module over a ring.) Let us state this explicitly, for the reader's convenience.

**Definition 2.1.** Suppose $R$ is a semiring. An *$R$-module* $V$ is a semigroup $(V, +, 0_V)$ together with a scalar multiplication $R \times V \to V$ satisfying the following properties for all $r, v, w \in V$:

1. $r(v + w) = rv + rw$;
2. $(r_1 + r_2)v = r_1v + r_2v$;
3. $(r_1r_2)v = r_1(r_2v)$;
4. $1_Rv = v$;
5. $0_Rv = 0_V = r0_V$.

Note that this definition of module over a semiring $R$ coincides with the usual definition of module when $R$ is a ring, taking $-v = (-1_R)v$.

**Definition 2.2.** Suppose $(R, \mathcal{G}_0, \nu)$ is a semiring with ghosts. An *$R$-module with ghosts* $(V, \mathcal{H}_0, \mu)$ is an $R$-module $V$, together with a *ghost submodule* $\mathcal{H}_0$ and an $R$-module projection

$$\mu : V \to \mathcal{H}_0$$

satisfying the following axioms for all $r \in R$ and $v, w \in V$:

1. $\mu(rv) = r\mu(v) = r^\nu v$;
2. $\mu(v + w) = \mu(v) + \mu(w)$.

Note that (1) implies $\mathcal{G}_0 V \subseteq \mathcal{H}_0$.

Rather than developing the general module theory here, we content ourselves with the following example.

**Example 2.3.** The direct sum $V = \bigoplus_{j \in J} R$ of copies (indexed by $J$) of a supertropical semiring $R$ is denoted as $R^{(J)}$, with zero element $0_V = (0_R)$. The ghost submodule is $\mathcal{G}_0^{(J)}$. When $R$ is a supertropical semifield, $R^{(J)}$ is called a *tropical vector space* over $R$.

If we take $J = \{1, \ldots, n\}$, then the tropical module $R^{(n)}$ is denoted as $R^{(n)}$, which is the main example of tropical linear algebra. The *tangible vectors* of $R^{(n)}$ are defined as those $(a_1, \ldots, a_n)$ such that each $a_i \in \mathcal{H}_0$, but with some $a_i \neq 0_R$. (Note that there may be vectors that are neither tangible nor ghost, having some tangible components and some ghost components.)

**Definition 2.4.** The *standard base* of $R^{(n)}$ is defined as

$$e_1 = (1_R, 0_R, \ldots, 0_R), \quad e_2 = (0_R, 1_R, 0_R, \ldots, 0_R), \quad \ldots, \quad e_n = (0_R, 0_R, \ldots, 1_R).$$

Note that every element $(r_1, \ldots, r_n)$ of $R^{(n)}$ can be written (uniquely) in the form $\sum_{i=1}^n r_ie_i$. 

2.1. Matrices over semirings with ghosts. It is standard that for any semiring $R$, we have the semiring $M_n(R)$ of $n \times n$ matrices with entries in $R$, where addition and multiplication are induced from $R$ as in the familiar ring-theoretic matrix construction. The unit element of $M_n(R)$ is the identity matrix $1_R$ on the main diagonal and whose off-diagonal entries are $0_R$.

Given the designated ghost ideal $G_0$ of $R = (R, G_0, \nu)$, we define the ghost ideal $M_n(G_0)$ of $M_n(R)$ and thus we obtain the matrix semiring with ghosts $(M_n(R), M_n(G_0), \nu_s)$, where the ghost map $\nu_s$ on $M_n(R)$ is obtained by applying $\nu$ to each matrix entry.

Remark 2.5. The Frobenius property (Remark [3.3]) implies that for any matrix $A$ over a commutative semiring $R$ with ghosts and any $\alpha \in R$, the matrix $(A + \alpha I)^m$ equals $A^m + \alpha^m I + \text{ghost}$, in $M_n(R)$. Note that $(A + \alpha I)^m$ can differ from $A^m + \alpha^m I$; for example, take $A = \begin{pmatrix} 0_R & 1_R \\ 1_R & 0_R \end{pmatrix}$ with $m = 2$.

3. Tropical determinants

For the remainder of this paper, unless otherwise specified, we only consider matrices over supertropical domains $R = (R, G_0, \nu)$. A typical matrix is denoted as $A = (a_{i,j})$; for example, the zero matrix is $(0_R)$.

The tropical version of the determinant must be the permanent, since we do not have negation at our disposal. Nevertheless, its function in supertropical algebra is the analog of the familiar determinant. In [10], a counterexample was given to the proposed formula $|AB| = |A| \cdot |B|$. Let us see why such counterexamples exist, by providing a conceptual development of the tropical determinant that indicates when the formula does hold. As in classical algebra, when we study tropical determinants, we assume as a matter of course that the base semiring $R$ is commutative.

Theorem 3.1. Suppose $V = R^{(n)}$, taken with the standard base $(e_1, \ldots, e_n)$, over a supertropical (commutative) semiring $R = (R, T, G_0, \nu)$.

Define the function $\Phi_{\gamma} : V^{(n)} \to R$ by the following formula, where $v_i = (v_{1,i}, \ldots, v_{n,i})$:

$$\Phi_{\gamma}(v_1, \ldots, v_n) = \gamma \sum_{\pi \in S_n} v_{1,\pi(1)} \cdots v_{n,\pi(n)},$$

(3.1)

where $\gamma \in R$ is fixed. Then $\Phi_{\gamma}$ satisfies the following properties:

1. $\Phi_{\gamma}$ is linear in each component, in the sense that

$$\Phi_{\gamma}(v_1, \ldots, \alpha v_i + \alpha' v_i', \ldots, v_n) = \alpha \Phi_{\gamma}(v_1, \ldots, v_i, \ldots, v_n) + \alpha' \Phi_{\gamma}(v_1, \ldots, v_i', \ldots, v_n),$$

for all $\alpha, \alpha' \in R$ and $v_i, v_i' \in V$.

2. $\Phi_{\gamma}(v_1, \ldots, v_n) \in G_0$ if $v_i = v_j$ for some $i \neq j$.

3. $\Phi_{\gamma}(v_1, \ldots, v_n) = 0_R$ if $v_i = 0_V$ for some $i$.

4. $\Phi_{\gamma}(v_{\pi(1)}, \ldots, v_{\pi(n)}) = \Phi_{\gamma}(v_1, \ldots, v_n)$, for all $\pi \in S_n$.

5. $\Phi_{\gamma}(e_1, \ldots, e_n) = \gamma$.

Furthermore, $\Phi_{\gamma}$ is unique up to ghosts, in the sense that if $\Phi'_{\gamma}$ is another function satisfying the same properties (1)–(5), then either

$$\Phi'_{\gamma}(v_1, \ldots, v_n) = \Phi_{\gamma}(v_1, \ldots, v_n),$$

or $\Phi'_{\gamma}(v_1, \ldots, v_n) \in G_0$, with $(\Phi'_{\gamma}(v_1, \ldots, v_n))^\nu \geq (\Phi_{\gamma}(v_1, \ldots, v_n))^\nu$.

Proof. First of all, note that Formula (3.1) satisfies the conditions (1)–(5) of the assertion. Conversely, suppose $\Phi'_{\gamma}$ satisfies these conditions. Since $v_i = \sum_{j=1} v_{i,j} e_j$, we have (by linearity)

$$\Phi'_{\gamma}(v_1, \ldots, v_n) = \sum_{j_1, \ldots, j_n} v_{1,j_1} \cdots v_{n,j_n} \Phi'_{\gamma}(e_{j_1}, \ldots, e_{j_n}).$$

When any $j_s = j_t$, we get $\Phi'_{\gamma}(e_{j_1}, \ldots, e_{j_n}) \in G_0$ by property (2). If such ghost terms do not dominate all the $v_{1,\pi(1)} \cdots v_{n,\pi(n)} \Phi'_{\gamma}(e_{\pi(1)}, \ldots, e_{\pi(n)})$, $\pi \in S_n$, then

$$\Phi'_{\gamma}(v_1, \ldots, v_n) = \sum_{\pi \in S_n} v_{1,\pi(1)} \cdots v_{n,\pi(n)} \Phi'_{\gamma}(e_{\pi(1)}, \ldots, e_{\pi(n)}) = \gamma \sum_{\pi \in S_n} v_{1,\pi(1)} \cdots v_{n,\pi(n)}.$$
since, by conditions (4) and (5),

\[ \Phi'_\gamma(e_{\pi(1)}, \ldots, e_{\pi(n)}) = \Phi'(e_1, \ldots, e_n) = \gamma. \]

This proves the last assertion. \(\square\)

Remark 3.2. Condition (1) implies condition (3). Indeed,

\[ \Phi_\gamma(v_1, \ldots, 0_V, \ldots, v_n) = \Phi_\gamma(v_1, \ldots, 0_R v_i, \ldots, v_n) = 0_R \Phi_\gamma(v_1, \ldots, v_i, \ldots, v_n) = 0_R. \]

Remark 3.3. Actually, the same proof shows that \(\Phi_\gamma\) satisfies the following stronger property than (2):

- \(\Phi_\gamma(v_1, \ldots, v_n) \in G_0\) if \(v_i' = v_j'\) for some \(i \neq j\) (in other words, if the corresponding components have the same \(v\)-values).

Conversely, (1) and (4) imply that it is enough to verify (2) for the standard base \(e_1, \ldots, e_n\).

When \(\gamma = \mathbb{1}_R\), we denote \(\Phi_\gamma(v_1, \ldots, v_n)\) as \(|v_1, \ldots, v_n|\) and call this the normalized version of Formula (3.1). On the other hand, Theorem 3.1 points to a strange phenomenon: Ghosts produce “noise” which disrupts attempts to provide an analog to the classical determinantal theory, as we shall see.

We define the tropical determinant of a matrix \(A = (a_{i,j})\) as in Formula (3.1) (normalized) applied to the rows of \(A\):

\[ |(a_{i,j})| = \sum_{\pi \in S_n} a_{1, \pi(1)} \cdots a_{n, \pi(n)}, \tag{3.2} \]

which is the formula given in [11]. (Also see Remark 3.4.)

Remark 3.4. Defining the transpose \((a_{i,j})^t\) to be \((a_{j,i})\), we have

\[ |(a_{i,j})^t| = |(a_{i,j})|, \]

in view of Theorem 3.1 since

\[ \sum_{\pi \in S_n} a_{1, \pi(1)} \cdots a_{n, \pi(n)} = \sum_{\pi \in S_n} a_{\pi(1), 1} \cdots a_{\pi(n), n}. \]

As in classical linear algebra, we thus have analogous results if we use columns instead of rows.

Theorem 3.5. For any \(n \times n\) matrices over a supertropical semiring \(R\), we have

\[ |AB|^{\nu} \geq |A|^{\nu} |B|^{\nu}, \]

with \(|AB| = |A| |B|\) whenever \(|AB|\) is tangible. (In other words, \(|AB| = |A| |B| + \text{ghost.}\)

Proof. Define \(\Phi_{|B|}(A) = |AB|\). This satisfies all of the properties of Theorem 3.1, taking \(\gamma = |B|\), so must be \(\gamma |A| = |A| |B|\) except when \(|AB|\) is ghost and dominates \(|A| |B|\). \(\square\)

3.1. Tropically singular and nonsingular matrices. We start this subsection with the supertropical version of the terms “nonsingular” and “singular,” to be contrasted with the classical notion of invertibility:

Definition 3.6. A matrix \(A\) is nonsingular if \(|A| \in \mathcal{T}\); on the other hand, when \(|A| \in G_0\), we say that \(A\) is singular. When \(|A| = 0_R\), we say that \(A\) is strictly singular.

Note that if \(|A|\) is any ghost \(\neq 0_R\), then \(A\) is singular but not strictly singular. Although the two concepts of singular and strictly singular are analogous, the approach to their theories are quite different.

Remark 3.7. Let us study determinants via permutations, utilizing Formula (3.2) to analyze \(|A|\) where \(A = (a_{i,j})\). Clearly

\[ \nu(|A|) = \nu(a_{1, \sigma(1)} \cdots a_{n, \sigma(n)}) \]

if \(a_{1, \sigma(1)} \cdots a_{n, \sigma(n)}, \sigma \in S_n\), has the maximal \(\nu\)-value of all such products. We say a permutation \(\sigma \in S_n\) attains \(|A|\) if \(|A|^\nu = (a_{\sigma(1), 1} \cdots a_{\sigma(n), n})^{\nu}\).

- By definition, some permutation always attains \(|A|\).
- If there is a unique permutation \(\sigma\) which attains \(|A|\), then \(|A| = a_{1, \sigma(1)} \cdots a_{n, \sigma(n)}\). In this case, when \(|A|\) is ghost, then some \(a_{i, \sigma(i)}\) must be ghost.
• If at least two permutations attain $|A|$, then $A$ must be singular. Note in this case that if we replaced all nonzero entries of $A$ by tangible entries of the same $\nu$-value, then $A$ would still be singular.

• When $A$ is nonsingular, there is a unique permutation $\sigma$ which attains $|A|$; in this case each $a_{i,\sigma(i)}$ is tangible.

• When $|A| = 0_R$, then every permutation attains $|A|$, so we must have
  \[ a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = 0_R \]
  for each $\sigma \in S_n$. Accordingly, for each permutation $\sigma$, at least one of the $a_{i,\sigma(i)}$ is $0_R$ (where $i$ depends on $\sigma$).

Thus, $|A| = 0_R$ iff “enough” entries are $0_R$ to force each summand in Formula (3.2) to be $0_R$. This is a very strong property, which in classical matrix theory provides a description of singular subspaces. We elaborate this idea in Proposition 6.2.

We write $P_\sigma$ for the permutation matrix whose entry in the $(i, \sigma(i))$ position is $1_R$ (for each $1 \leq i \leq n$) and $0_R$ elsewhere; $P_\sigma$ is nonsingular for any $\sigma \in S_n$. Likewise, we write $\text{diag}\{a_1, \ldots, a_n\}$ for the diagonal matrix whose entry in the $(i, i)$ position is $a_i \in R$ and $0_R$ elsewhere.

**Example 3.8.** Any permutation matrix $P_\sigma$ is (classically) invertible; indeed, $P_\sigma^{-1} = P_{\sigma^{-1}}$. Also, the diagonal matrix $\text{diag}\{a_1, \ldots, a_n\}$ is invertible iff each $a_i$ is invertible in $R$, for then
  \[ \text{diag}\{a_1, \ldots, a_n\}^{-1} = \text{diag}\{a_1^{-1}, \ldots, a_n^{-1}\}. \]

The following easy result should be well known.

**Proposition 3.9.** Suppose $R$ is a supertropical semiring. A matrix $A \in M_n(R)$ is (multiplicatively) invertible, iff $A$ is a product of a permutation matrix with an invertible diagonal matrix.

**Proof.** Any invertible matrix $A$ is nonsingular, by Theorem 3.3 since $|AA^{-1}| = 1_R$. Thus, for the permutation $\sigma$ attaining $|A|$, we have $\{a_{\sigma(1),1}, \ldots, a_{\sigma(n),n}\} \subseteq T$. Replacing $A$ by $P_{\sigma^{-1}}A$, we may assume that the diagonal of $A$ is tangible; then, multiplying through by a suitable diagonal matrix, we may assume that the diagonal of $A$ is the identity matrix $I$. In other words, $A$ has the form $A = I + B$ for some matrix $B$ which is $0_R$ on the diagonal. Also, write $A^{-1} = D' + B'$ where $D'$ is diagonal and $B'$ is $0_R$ on the diagonal. But then, $I = AA^{-1} = D' + BD' + B' + BB'$, which can be $0_R$ off the diagonal only if $B = B' = (0_R)$. \[\square\]

**Remark 3.10.** The set
  \[ W = \{ Q_\sigma = P_\sigma D \mid D \text{ is invertible diagonal} \}, \]
which by Proposition 3.3A comprises the unique maximal subgroup of $M_n(R)$ (having the same identity element $I$), is in fact the (affine) Weyl group when $T = \mathbb{Z}$; cf. [9].

Thus, invertibility in supertropical matrices is a strong concept, and we want to consider the weaker notion of nonsingularity. We start by asking when the power of a nonsingular matrix is nonsingular.

**Example 3.11.** Let us compute $|A^2|$, for any $2 \times 2$ matrix
  \[
  A = \begin{pmatrix}
  a_{1,1} & a_{1,2} \\
  a_{2,1} & a_{2,2}
  \end{pmatrix},
  \]
and compare it to $|A|$. Clearly $A^2 = \begin{pmatrix}(a_{1,1})^2 + a_{1,2}a_{2,1} & a_{1,2}(a_{1,1} + a_{2,2}) \\
  a_{2,1}(a_{1,1} + a_{2,2}) & (a_{2,2})^2 + a_{1,2}a_{2,1}
  \end{pmatrix}$, so
  \[
  |A^2| = ((a_{1,1})^2 + a_{1,2}a_{2,1})((a_{2,2})^2 + a_{1,2}a_{2,1}) + (a_{1,1} + a_{2,2})^2 a_{1,2}a_{2,1}
  = \nu((a_{1,1})^2 + (a_{2,2})^2 a_{1,2}a_{2,1}) + (a_{1,1})^2 a_{2,2}a_{2,1} + (a_{2,2})^2 a_{1,1}a_{2,1} + \nu(a_{1,1}a_{2,2}a_{1,2}a_{2,1})
  \]
  \[
  = \nu((a_{1,1})^2 + (a_{2,2})^2 a_{1,2}a_{2,1}) + (a_{1,1}a_{2,2}a_{1,2}a_{2,1}) + (a_{1,1}a_{2,2} + a_{1,2}a_{2,1})^2.
  \]
The right side is ghost when
  \[
  \nu((a_{1,1})^2 + (a_{2,2})^2 a_{1,2}a_{2,1}) \geq \nu((a_{1,1}a_{2,2} + a_{1,2}a_{2,1})^2). \]
Assuming that $a_{1,1}^2 \geq a_{2,2}^2$, we get (3.3) iff $\nu((a_{1,1})^2) \geq \nu(a_{1,2}a_{2,1}) \geq \nu((a_{2,2})^2)$. (The situation for $a_{1,1}^2 \leq a_{2,2}^2$ is symmetric.) Let us examine the various cases in turn, where $A^2$ is nonsingular.
Case I: \( \nu((a_{1,1})^2) = \nu((a_{2,2})^2) > \nu(a_{1,2}a_{2,1}) \). Then
\[
A^2 = \begin{pmatrix} (a_{1,1})^2 & a_{1,2}a_{2,1} \\ a_{2,1}a_{1,1} & (a_{2,2})^2 \end{pmatrix},
\]
so the entries of \((a_{1,1})A\) and \(A^2\) are \(\nu\)-matched, and we see by iteration that \(A^{2^\nu}\) is nonsingular for every \(\nu\), and thus every power of \(A\) is nonsingular.

Case II: \( \nu((a_{2,2})^2) \leq \nu((a_{1,1})^2) < \nu(a_{1,2}a_{2,1}) \). Then
\[
A^2 = \begin{pmatrix} a_{1,2}a_{2,1} & a_{1,2}a_{1,1} \\ a_{2,1}a_{1,1} & a_{1,2}a_{2,1} \end{pmatrix},
\]
(where the off-diagonal terms are made ghost if \(a_{1,1} = a_{2,2}^2\), which has the form of Case I; hence, every power of \(A^2\), and thus of \(A\), is nonsingular.

Case III: \( \nu((a_{1,1})^2) > \nu((a_{2,2})^2) > \nu(a_{1,2}a_{2,1}) \). Then
\[
A^2 = \begin{pmatrix} (a_{1,1})^2 & a_{1,2}a_{1,1} \\ a_{2,1}a_{1,1} & (a_{2,2})^2 \end{pmatrix} = (a_{1,1}I)A',
\]
where \(A'\) differs from \(A\) only in the \((2,2)\)-entry, whose \(\nu\)-value has been reduced by a factor of \(\frac{1}{\sqrt{2}}\). Taking a high enough power of \(A\) will reduce \((a_{2,2})^2\) until it is dominated by \(a_{1,2}a_{2,1}\), and thus yield a singular matrix. Thus, some power of \(A\) will always be singular, even though \(A^2\) need not be singular.

Summarizing, \(A^2\) nonsingular implies every power of \(A\) is nonsingular except in Case III, which for any \(k\) provides an example where \(A^k\) is nonsingular but \(A^{k+1}\) is singular.

3.2. The digraph of a supertropical matrix. One major computational tool in tropical matrix theory is the weighted digraph \(G = (V, E)\) of an \(n \times n\) matrix \(A = (a_{i,j})\), which is defined to have vertex set \(V = \{1, \ldots, n\}\), and an edge \((i, j)\) from \(i\) to \(j\) of weight \(a_{i,j}\) whenever \(a_{i,j} \neq 0\).

We use \([\mathcal{G}]\) as a general reference for graphs. We always assume that \(V = \{1, \ldots, n\}\), for convenience of notation. The out-degree, \(d_{\text{out}}(i)\), of a vertex \(i\) is the number of edges emanating from \(i\), and the in-degree, \(d_{\text{in}}(j)\), is the number edges terminating at \(j\). A sink is a vertex \(j\) with \(d_{\text{out}}(j) = 0\), while a source is a vertex \(j\) with \(d_{\text{in}}(j) = 0\).

The length \(\ell(p)\) of a path \(p\) is the number of edges of the path. A path is simple if each vertex appears only once. A simple cycle is a simple path for which \(d_{\text{out}}(i) = d_{\text{in}}(i) = 1\) for every vertex \(i\) of the path; thus, the initial and terminal vertices are the same. A simple cycle of length 1 is then a loop. A simple cycle repeated several times is called a cycle; thus, for some \(m\), \(d_{\text{out}}(i) = d_{\text{in}}(i) = m\) for every vertex \(i\) of the cycle.

It turns out that the only edges of use to us are those that are parts of cycles. Accordingly, we define the reduced digraph \(G_A\) of \(A\) to be the graph obtained from the weighted digraph by erasing all edges that are not parts of cycles. Consequently, if there is a path from \(i\) to \(j\) in \(G_A\), there also is a path from \(j\) to \(i\). Hence, \(G_A\) can be written as a disjoint union of connected components, in each of which there is a path between any two vertices.

The weight \(w(p)\) of a path \(p\) is defined to be the tropical product of the weights of the edges comprising \(p\), counting multiplicity. The average weight of the path \(p\) is \(\sqrt{w(p)}\), where \(\ell = \ell(p)\) is the length of the path, i.e., the number of edges in the path. (As always, our product, being tropical, is really the sum, so we indeed are taking the average.) We order the weights according to their \(\nu\)-values. Then the \((i,j)\)-entry of \(A^k\), where \(A\) is a tangible matrix, corresponds to the highest weight of all the paths of length \(k\) from \(i\) to \(j\), and is a ghost whenever two distinct paths of length \(k\) have the same highest weight.

We define a \(k\)-multicycle \(C\) in a digraph to be the union of disjoint simple cycles, the sum of whose lengths is \(k\); its weight \(w(C)\) is the product of the weights of the component cycles. Thus, each \(n\)-multicycle passes through all the vertices; \(n\)-multicycles are also known in the literature as cyclic covers, or saturated matchings.

Remark 3.12. Writing a permutation \(\sigma\) as a product \(\sigma_1 \cdots \sigma_k\) of disjoint cyclic permutations, we see that each permutation corresponds to an \(n\)-multicycle. Conversely, any \(n\)-multicycle corresponds to a permutation, and their highest weight in \(G_A\) matches \(|A|\). In particular, when \(|A|\) is tangible, there is a unique \(n\)-multicycle having highest weight.
Let us review some well-known results about cycles and multicycles.

**Remark 3.13.** Given a graph $G = (V, E)$ where $d_{in}(i) \geq 1$ and $d_{out}(i) \geq 1$ for each $i \in V$, then $G$ contains a simple cycle. Indeed, otherwise $G$ must have a sink or source, $i \in V$, in contradiction to $d_{in}(i) \geq 1$ and $d_{out}(i) \geq 1$, respectively.

We also need a special case of the celebrated theorem of Birkhoff and Von Neumann [3], which states that every positive doubly stochastic $n \times n$ matrix is a convex combination of at most $n^2$ cyclic covers; more precisely, we quote the graph-theoretic version of Hall’s marriage theorem. Since Hall’s theorem is formulated for bipartite graphs, we note the following correspondence between digraphs having $n$ vertices and undirected bipartite graphs having $2n$ vertices.

**Remark 3.14.** Any digraph $G = (V, E)$ gives rise to a bipartite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ whose vertex set is $\tilde{V} = V \cup V'$, where $V'$ is a disjoint copy of $V$, and such that any edge $(i, j) \in E$ corresponds to an edge in $\tilde{E}$ from $i \in V$ to $j \in V'$. (Thus, the directed edges in $G$ correspond to undirected edges in $\tilde{G}$.)

**Theorem 3.15 (Hall’s marriage theorem).** Suppose $\tilde{G} = (\tilde{V}, \tilde{E})$ is an (undirected) bipartite graph, and for each $j \in \tilde{V}$ define
\[
N(j) = \{i \in \tilde{V} : \text{there is an edge in } \tilde{E} \text{ connecting } i \text{ and } j\}.
\]
For $S \subseteq \tilde{V}$, define $N(S) = \cup\{N(s) : s \in S\}$, and assume that $|N(S)| \geq |S|$ for every $S \subseteq \tilde{V}$. (Here $|S|$ denotes the order of the set $S$.) Then, for each $k \leq n$, $\tilde{G}$ contains a set of edges
\[
\{(\pi(1), 1), \ldots, (\pi(k), k)\}
\]
for some $\pi \in S_n$. (For $k = n$, this is called a matching).

A quick proof can be found in [6, Theorem 2.1.2] or [16]. This hypothesis provides the next lemma, motivated by an argument found in [2]:

**Lemma 3.16.** Assume that $G = (V, E)$ is a digraph, possibly with multiple edges. Then $G$ contains an $n$-multicycle, under any of the following conditions (for any $k \geq 1$ in (i) and $k > 1$ in the other parts):

(i) $d_{in}(j) = d_{out}(i) = k$ for all $i, j$.

(ii) $d_{in}(j) = k$ for all vertices $j$ except one (at most) with in-degree $k+1$ and one with in-degree $k-1$, and $d_{out}(i) = k$ for all vertices $i$.

(iii) $d_{out}(i) = k$ for all vertices $i$ except one (at most) with out-degree $k+1$ and one with out-degree $k-1$, and $d_{in}(j) = k$ for all vertices $j$.

(iv) $d_{out}(i) = k$ for all vertices $i$ except one (at most) with out-degree $k-1$, and $d_{in}(j) = k$ for all vertices $j$ except one (at most) with in-degree $k-1$.

**Proof.** We form a matrix $B$ whose $(i,j)$-entry is the number of (directed) edges from $i$ to $j$ in $G$, and a new bipartite graph $\tilde{G}$ obtained from the graph $G$ as in Remark 3.14. Thus, any nonzero entry $b_{i,j} \in B$ corresponds to an edge from $i \in V$ to $j \in V'$.

Note that any matching in $\tilde{G}$ corresponds to an $n$-multicycle of $G$. Thus, we need to verify the hypothesis of Hall’s marriage theorem on $\tilde{G}$. For any $S \subseteq \tilde{V} = V \cup V'$, write $U = N(S)$. We need to show that $|U| \geq |S|$. First of all, since by definition the neighbors of $V$ are in $V'$ and visa versa, it suffices to assume $S \subseteq V$ or $S \subseteq V'$.

(i) By symmetry, we assume that $S \subseteq V$. Then $U \subseteq V'$ and
\[
k|N(S)| = k|U| = \sum_{j \in U} \sum_{i \in N(j)} b_{i,j} \geq \sum_{j \in U} \sum_{i \in S} b_{i,j} = \sum_{i \in S} d_{out}(i) = k|S|,
\]
implying $|N(S)| \geq |S|$, as desired.

(ii) We modify the argument of (i), noting that if $a$ and $b$ are integers with $a > b - 1$ then $a \geq b$. First assume that $S \subseteq V'$. For any subset $U$ of $V'$, the number $t$ of edges (counting multiplicities) terminating in a vertex in $S$ is at least $(|S| - 1)k + 1$. But since any such edge starts at a vertex in $N(S)$, we see that $t \leq |N(S)|k$, so we conclude that $|N(S)| > |S| - 1$, and thus $|N(S)| \geq |S|$, as desired.
Now assume $S \subseteq V$. For any subset $S$ of $V$, the number $t$ of edges (counting multiplicities) starting in a vertex in $S$ is $|S|k$. But since any such edge starts at a vertex in $N(S)$, we see that $t \leq |N(S)|k + 1$, so again we conclude that $|N(S)| > |S| - 1$, and thus $|N(S)| \geq |S|$, as desired.

(iii) As in (ii).

(iv) Again the analogous argument holds. By symmetry, we assume that $S \subseteq V$. Now Equation \((3.6)\) becomes

\[ k|N(S)| = k|U| \geq \sum_{j \in U} d_{in}(j) = \sum_{j \in U \cap N(j)} b_{i,j} \geq \sum_{j \in U \cap i \in S} b_{i,j} = \sum_{i \in S} d_{out}(i) = k|S| - 1, \]

so again $|U| \geq |S|$. \(\square\)

**Proposition 3.17.** Assume that $G = (V, E)$ where each vertex $i \in V$ has $d_{in}(i) = d_{out}(i) = k$. Then $G$ is a union of $k$ distinct $n$-multicycles.

**Proof.** By the lemma, we have an $n$-multicycle which we may remove from $G$; we thereby obtain a graph where each vertex $i \in V$ has $d_{in}(i) = d_{out}(i) = k - 1$, and continue by induction on $k$. \(\square\)

### 4. Quasi-invertible Matrices and the Adjoint

**Definition 4.1.** A quasi-zero matrix $Z_G$ is a matrix equal to $0_R$ on the diagonal, and whose off-diagonal entries are ghosts or $0_R$. (Despite the notation, the quasi-zero matrix $Z_G$ is not unique, since the $\nu$-values of the ghost entries may vary.) A quasi-identity matrix $I_G$ is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_G$, where $Z_G$ is a quasi-zero matrix.

A matrix $B$ is a quasi-inverse for $A$ if $AB$ and $BA$ are quasi-identities. The matrix $A$ is quasi-inverse when $A$ has a quasi-inverse.

Thus, for any matrix $A$ and any quasi-identity, $I_G$, we have $AI_G = A + A_G$, where $A_G \in M_n(G_0)$. Also, $|I_G| = 1_R$ by the nonsingularity of $I_G$. Note that the identity matrix $I$ is itself a quasi-identity, and also is a quasi-inverse for any quasi-identity.

**Remark 4.2.**

(i) By definition, each quasi-identity $I_G$ is also quasi-inverse, since $I_G$ is a quasi-inverse of itself. Recall from semigroup theory that there is a one-to-one correspondence between (multiplicative) idempotent matrices in $M_n(R)$ and maximal (multiplicative) subgroups of $M_n(R)$; the idempotent matrix $I_G \in M_n(R)$ is the identity element of a unique maximal subgroup of $M_n(R)$, namely the group of units of $I_GM_n(R)I_G$; cf. \([15]\). Note that $M_n(R)$ has many other idempotents, nonsingular and singular.

(ii) Any quasi-identity matrix $I_G = (a_{i,j})$ must satisfy $a_{i,j}a_{j,i} < 1_R$ for $i \neq j$ and $a_{i,j}a_{j,k} \leq a_{i,k}$ for $i \neq k$, because $I_G$ is multiplicatively idempotent.

(iii) A slightly weaker notion, called pseudo-identity, is given in \([14]\). Note that a pseudo-identity need not be multiplicatively idempotent, as seen by considering upper triangular $3 \times 3$ matrices with ghost entries on the upper diagonal (cf. Example \([4.15]\) below); these do not necessarily satisfy the criterion $a_{i,j}a_{j,k} \leq a_{i,k}$ of (ii).

There is another formula to help us out.

**Definition 4.3.** The $(i,j)$-minor $A'_{i,j}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the $i$ row and $j$ column of $A$. The adjoint matrix $\text{adj}(A)$ of $A$ is defined as the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = |A'_{i,j}|$.

**Remark 4.4.** By definition, $a'_{i,j}$ can be computed as

\[ \sum_{\pi \in S_n, \ \pi(i)=j} a_{1,\pi(1)}a_{2,\pi(2)} \cdots a_{i-1,\pi(i-1)}a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)}. \]
Remark 4.5.

(i) Suppose \( A = (a_{i,j}) \). An easy calculation using Formula (3.2) yields
\[
|A| = \sum_{j=1}^{n} a_{i,j} a'_{i,j}, \quad \forall i. \tag{4.2}
\]

Consequently, \((a_{i,j} a'_{i,j})' | A |' \leq |A |' \) for each \( i, j \).

(ii) If we take \( k \neq i \), then replacing the \( i \) row by the \( k \) row in \( A \) yields a matrix with two identical rows; thus, its determinant is a ghost, and we thereby obtain
\[
\sum_{j=1}^{n} a_{i,j} a'_{k,j} \in G_0, \quad \forall k \neq i; \tag{4.3}
\]

Likewise
\[
\sum_{j=1}^{n} a_{j,i} b'_{j,k} \in G_0, \quad \forall k \neq i. \tag{4.4}
\]

This observation is significant since it is often useful to take \( b'_{i,j} \in T \). The same argument shows that if \( b_{i,j} \in R \) with the same \( \nu \)-value as \( a_{i,j} \), then
\[
\sum_{j=1}^{n} b_{i,j} b'_{k,j} \in G_0, \quad \forall k \neq i.
\]

Definition 4.6. For \( |A| \) is invertible, define
\[
I_A = \frac{\text{adj}(A)}{|A|}, \quad I'_A = \frac{\text{adj}(A)}{|A|} A.
\]

Putting together (i) and (ii) of Remark 4.5 shows that the matrices \( I_A \) and \( I'_A \) are the identity on the diagonal and ghost off the diagonal.

Example 4.7. Let us compute \( \text{adj}(AB) \), for any \( 2 \times 2 \) matrices
\[
A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix},
\]

and compare it to \( \text{adj}(B) \; \text{adj}(A) \). First, \( \text{adj}(A) = \begin{pmatrix} a_{2,2} & a_{1,2} \\ a_{2,1} & a_{1,1} \end{pmatrix}, \quad \text{adj}(B) = \begin{pmatrix} b_{2,2} & b_{1,2} \\ b_{2,1} & b_{1,1} \end{pmatrix} \), so
\[
\text{adj}(B) \; \text{adj}(A) = \begin{pmatrix} b_{2,2} a_{2,2} + b_{1,2} a_{2,1} \\ b_{2,1} a_{2,2} + b_{1,1} a_{2,1} \end{pmatrix} + \begin{pmatrix} b_{2,2} a_{1,2} + b_{1,2} a_{1,1} \\ b_{2,1} a_{1,2} + b_{1,1} a_{1,1} \end{pmatrix},
\]

which equals \( \text{adj}(AB) \)

However, for larger \( n \), this fails; for example, for the \( 3 \times 3 \) matrix
\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{we have} \quad A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \text{adj}(A^2) = (1^\nu),
\]

whereas

\[
\text{adj}(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\quad \text{and} \quad \text{adj}(A)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]

One does have the following fact, which illustrates the subtleties of the supertropical structure:

**Proposition 4.8.** \(\text{adj}(AB) = \text{adj}(B) \text{adj}(A) + \text{ghost} \).

**Proof.** Writing \(AB = (c_{i,j})\), we see that \(\text{adj}(AB) = (c'_{i,j})\) whereas the \((i,j)\)-entry of \(\text{adj}(B) \text{adj}(A)\) is \(\sum_k^n b'_{k,i} a'_{j,k}\). Since \(a'_{j,k} b'_{k,i}\) appears in \(c'_{j,i}\), we need only check that the other terms in \(c'_{j,i}\) occur in matching pairs that thus provide ghosts. These are sums of products the form

\[
d_{k_1,\pi(k_1)} d_{k_2,\pi(k_2)} \cdots d_{k_{n-1},\pi(k_{n-1})},
\]

where \(k_t \neq j\), \(\pi(k_t) \neq i\) for all \(1 \leq t \leq n-1\), and

\[
d_{k_t,\pi(k_t)} = a_{k_t,\ell_{t,\pi(k_t)}}.
\]

If the \(\ell\) do not repeat, we have a term from \(\text{adj}(B) \text{adj}(A)\). But if some \(\ell\) repeats, i.e., if we have

\[
d_{k_t,\pi(k_t)} = a_{k_t,\ell_{t,\pi(k_t)}}, \quad d_{k_u,\pi(k_u)} = a_{k_u,\ell_{t,\pi(k_u)}},
\]

then in computing \(c'_{j,i}\) we also have a contribution from \(\sigma\) where \(\sigma(k_t) = \pi(k_u)\) and \(\sigma(k_u) = \pi(k_t)\) (and otherwise \(\sigma = \pi\)), where we get

\[
a_{k_t,\ell_{t,\sigma(k_t)}} a_{k_u,\ell_{t,\pi(k_u)}} = a_{k_t,\ell_{t,\pi(k_u)}} a_{k_u,\ell_{t,\pi(k_t)}} = a_{k_t,\ell_{t,\pi(k_t)}} a_{k_u,\ell_{t,\pi(k_u)}},
\]

as desired. \(\square\)

We show below that the matrices \(I_A\) and \(I'_A\) of Definition 4.6 are quasi-identities. This requires some preparation. Our main technique of proof is to define a **string** (from the matrix \(A\)) to be a product \(\text{str} = a_{i_1,j_1} \cdots a_{i_k,j_k}\) of entries from \(A\) and, given such a string, to define the **digraph** \(G_{\text{str}}\) of the **string** to be the graph whose edges are \((i_1,j_1), \ldots, (i_k,j_k)\), counting multiplicities. For example, the digraph \(G_{\text{str}}\) of the string

\[
\text{str} = a_{1,2} a_{2,3} a_{3,1} a_{1,1} a_{2,3} a_{3,2}
\]

has edge set \(\{(1,1), (1,2), (2,3)\} \) (multiplicity 2), \((3,1), (3,2)\).

**Theorem 4.9.**

(i) \(\nu(|A|) = |A|^{n^2}\).

(ii) \(\nu(|A|) = |A|^{n^2-1}\).

**Proof.**

(i) First we claim that \(\nu(\text{adj}(A)) = \nu(|A|^{n^2})\). First note that the \((i,k)\)-entry of \(A \text{adj}(A)\) is \(\sum_{j=1}^n a_{i,j} a'_{k,j}\).

Hence, by definition of tropical determinant,

\[
|A| \text{adj}(A) = \sum_{\pi \in S_n} \sum_{j=1}^n \cdots \sum_{j_n=1}^n a_{1,j_1} a_{\pi(1),j_1} \cdots a_{n,j_n} a_{\pi(n),j_n}.
\]

Let \(\beta_1 = |A|^n\), and \(\beta_2\) denote the right side of (4.5). Clearly \(\beta_2^\nu \geq \beta_1^\nu\), seen by taking \(j_i = i\) and \(\pi = (1)\). (Noting that the diagonal entries of \(A \text{adj}(A)\) all are \(|A|\), we see that \(|A \text{ adj}(A)|\) has \(\nu\)-value at least that of \(|A|^n\).)

To prove the claim, it remains to show that \(\beta_2^\nu \leq \beta_1^\nu\). Viewing (4.5) as a sum of strings of entries of \(A\), consider a string of maximal \(\nu\)-value, and take its digraph (counting multiplicities). Any string occurs in some

\[
\sum_{j_1=1}^n \cdots \sum_{j_n=1}^n a_{1,j_1} a'_{\pi(1),j_1} \cdots a_{n,j_n} a'_{\pi(n),j_n},
\]

so we can subdivide our string into \(n\) substrings, each a summand of \(a_{i,j} a'_{\pi(i),j}\), as \(1 \leq i \leq n\). In each such substring we have \(n\) edges: The edge \((i,j_i)\) appears because of \(a_{i,j_i}\), and \(n - 1\) other edges appear in \(a'_{\pi(i),j_i}\), namely of the form

\[
a_{i_1,j_1} \cdots a_{i_{n-1},j'_{n-1}}
\]

where \(\{i_1, \ldots, i_{n-1}\} = \{1, \ldots, \pi(i)-1, \pi(i)+1, \ldots, n\}\) and \(\{j_1, \ldots, j_{n-1}\} = \{1, \ldots, j_i-1, j_i+1, \ldots, n\}\).
In each of these $n$ substrings, the in-degree of each vertex is exactly one (since $j_i$ appears in $a_{i,j_i}$, and all the other indices appear in the adjoint term $a'_{\pi(i),j_i}$); thus the total in-degree of each vertex in any string arising from (4.5) is $n$.

The total out-degree in any substring in (4.6) is:

$$d_{out}(i) = \begin{cases} 1 & \text{for each index when } \pi(i) = i; \\ 2 & \text{for } 0, \text{ if } \pi(i), 1 \text{ for all } i' \neq i, \pi(i) \text{ when } \pi(i) \neq i. \end{cases}$$

Since $\pi$ is a permutation, the total out-degree of each vertex in any string arising from (4.5) is

$$\left( \sum_i 1 \right) + 1 - 1 = n.$$}

Hence, by Proposition 3.17, the digraph of $A \text{adj}(A)$ is a union of $n$ $n$-multicycles, each of whose weights has $\nu$-value at most $|A|$, by Remark 3.12. Hence, the term (4.6) has $\nu$-value at most that of $|A|^n$, namely $\nu^n$, as desired.

When $|A|$ is tangential, there is a unique $n$-multicycle $C$ of highest weight, corresponding to some permutation $\sigma \in S_n$, and thus the term (4.6) is obtained precisely when $C$ is repeated $n$ times. This implies that $j$ must be $\sigma(i)$ in each leading term in (4.5), yielding a unique leading term, and $|A \text{adj}(A)| = |A|^n$.

When $|A|$ is not tangential, then either our $n$-multicycle of highest weight yields a ghost term, or we have several $n$-multicycles of highest weight, corresponding to permutations yielding equal contributions to $|A|$; hence $\beta_1$ and $\beta_2$ are ghosts, and again we have equality.

(ii) Recall the formula:

$$|\text{adj}(A)| = \sum_{\pi \in S_n} \prod_{i=1}^n a'_{i,\pi(i)}. \tag{4.7}$$

The digraph for each summand has in-degree and out-degree $(n-1)$ for each vertex (since $\pi$ is a permutation), so we can separate it into $(n-1)$ individual $n$-multicycles, each of which has weight of $\nu$-value at most $|A|^n$, proving

$$\nu(|\text{adj}(A)|) \leq \nu(|A|^{n-1}).$$

On the other hand, if we take a permutation $\pi \in S_n$ attaining $|A|$, then clearly, for each $i_0$, $\prod_{i \neq i_0} a_{i,\pi(i)} = a'_{i_0,\pi(i_0)}$, implying $a_{i,\pi(i)} a'_{i,\pi(i)} = |A|$, and thus

$$\nu(|\text{adj}(A)|) \geq \nu \left( \prod_{\pi \in S_n} \prod_{i=1}^n a'_{i,\pi(i)} \right) = \nu \left( \prod_{i=1}^n \frac{|A|}{a_{i,\pi(i)}} \right) = \nu \left( |A|^n / |A| \right) = \nu \left( |A|^{n-1} \right).$$

If $A$ is nonsingular, then $\text{adj}(A)$ is nonsingular, since we have only one term of maximal $\nu$-value in computing $|A|$ and thus $|\text{adj}(A)|$, yielding $|\text{adj}(A)| = |A|^{n-1}$.

If $A$ is singular, then so is $\text{adj}(A)$, concluding the proof.

(Note in the important case that $R$ is a supertropical domain and $A$ is nonsingular, the assertion of (ii) follows at once from (i), since Theorem 3.5 implies

$$|\text{adj}(A)| = \frac{|A \text{adj}(A)|}{|A|} = |A|^{n-1}.$$}

□

In case $|A|$ is invertible in $R$, we define the **canonical quasi-inverse** of $A$ to be

$$A^\nabla = \frac{1_R}{|A|} \text{adj}(A).$$

Thus $AA^\nabla = I_A$, and $A^\nabla A = I'_A$. Note that $I'_A$ and $I_A$ may differ off the diagonal, although

$$I_A A = AA^\nabla A = A I'_A.$$

For example, taking $A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$, we have $A^\nabla = \begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix}$; thus $AA^\nabla = \begin{pmatrix} 0 & (-2)^n \\ 1^n & 0 \end{pmatrix}$ whereas $A^\nabla A = \begin{pmatrix} 0 & 0^n \\ (-1)^n & 0 \end{pmatrix}$. The following result is given in [14], with different proof.
Corollary 4.10. When $|A|$ is invertible, $|I_A| = 1_R$.

Although $I_A$ is not the identity, we obtain other noteworthy properties from a closer examination of the reduced digraph $G_A$ of $A$, and of how it is used to compute $A \text{adj}(A)$. As before, we write $A = (a_{i,j})$ and $\text{adj}(A) = (a'_{i,j})$. Since the $(i,j)$ entry of $A \text{adj}(A)$ is $\sum a_{i,k} a'_{j,k}$, we examine the terms $a_{i,k} a'_{j,k}$ where $i \neq j$.

The digraph $G_{i,j,k}$ of $G_A$ corresponding to any string appearing in $a_{i,k} a'_{j,k}$ has in-degree 1 at each vertex (since $a'_{j,k}$ provides in-degree 1 at every vertex except $k$, and $a_{i,k}$ provides in-degree 1 at the vertex $k$); likewise $G_{i,j,k}$ has out-degree 2 at $i$, 0 at $j$, and 1 at each other vertex. Let us call such a subgraph an $n$-proto-multicycle.

Conversely, given an $n$-proto-multicycle $C$ having out-degree 2 at $i$ and 0 at $j$, we take $a_{i,k}$ corresponding to an edge of $C$, and note that the remaining edges correspond to some $(n-1)$-multicycle in the graph corresponding to $a'_{j,k}$; thus $C$ provides a term of $\nu$-value at most $a_{i,k} a'_{j,k}$. (Incidentally, since the out-degree at $i$ is 2, we have two possible choices of $k$ that provide the same $\nu$-value, thereby giving us an alternate proof that the off-diagonal entries of $A \text{adj}(A)$ are ghost.) Now we need another immediate consequence of Lemma 3.16.

Lemma 4.11. Assume that $G = (V,E)$, where each vertex $i \in V$ has $d_{\text{out}}(i) = k$, and all but two vertices have $d_{\text{in}}(i) = k$, and one vertex $i'$ has $d_{\text{in}}(i') = k + 1$ and one vertex $j'$ has $d_{\text{in}}(j') = k - 1$. Then $G$ is a union of $k - 1$ $n$-multicycles and an $n$-proto-multicycle.

Proof. By Lemma 3.16(iii), $G$ contains an $n$-multicycle, which we delete and then conclude by induction on $k$. □

Theorem 4.12. $(A \text{adj}(A))^2 = |A| A \text{adj}(A)$, for every matrix $A$.

Proof. We check that $(A \text{adj}(A))^2 = |A| A \text{adj}(A)$ at each entry. The $(i,j)$-entry $b_{i,j}$ of $(A \text{adj}(A))^2$ is

$$
\sum_{k,\ell,m=1}^{n} a_{i,k} a'_{\ell,k} a_{\ell,m} a'_{j,m}.
$$

Taking $\ell = j$ yields $\sum_{k,m} a_{i,k} a'_{j,k} a_{j,m} a'_{j,m} = |A| \sum_{k,m} a_{i,k} a'_{j,k}$, proving that $b_{i,j}$ has $\nu$-value at least that of the $(i,j)$-entry of $|A| A \text{adj}(A)$. The reverse inequality comes from Lemma 4.11, which enables us to extract an $n$-multicycle, whose $\nu$-value is at most $|A|$. Clearly the off-diagonal terms of $(A \text{adj}(A))^2$ are ghosts; the diagonal terms are all tangible if $A$ is nonsingular, for, in that case, the tropical determinant is tangible. □

Theorem 4.13. When $|A|$ is invertible, $A A^\nu$ and $A^\nu A$ are quasi-identities (not necessarily the same), and thus $A^\nu$ is a quasi-inverse for $A$.

Proof. This is Corollary 3.10 and Theorem 4.12 together. □

Remark 4.14. In case $R$ is a supertropical semifield, then $A^\nu$ has been defined whenever $|A| \in T$. We can also define $A^\nu$ for $|A| \neq 0_R$ ghost by dividing each entry of $\text{adj}(A)$ by some tangible element whose $\nu$-value is $|A|$. Then $A A^\nu = \tilde{I}_A$ where $\tilde{I}_A$ is $1_R$ on the diagonal and ghost off the diagonal, and Theorem 4.12 now implies that $(\tilde{I}_A)^2 = \tilde{I}_A$ since $(1_R)^2 = 1_R$. Likewise, we can write $A^\nu A = \tilde{I}_A$, where $\tilde{I}_A$ is $1_R$ on the diagonal and ghost off the diagonal, with $(\tilde{I}_A)^2 = \tilde{I}_A$. These observations enable us to treat singular matrices in an analogous manner to nonsingular ones, just as long as $|A| \neq 0_R$.

One might hope that the same proof of Theorem 4.12 would yield the better result that $A \text{adj}(A)A = |A| A$, (i.e., $A A^\nu A = A$ for $|A|$ invertible), which we call the “von Neumann regularity condition”, cf. [15]. Unfortunately, this is false in general! The difficulty is that one might not be able to extract an $n$-multicycle from

$$
a_{i,j} a'_{j,k} a_{k,\ell}.
$$

(4.8)

For example, when $n = 3$, we have the term

$$
a_{1,1}(a_{1,3} a_{3,2})a_{2,2} = a_{1,1} a'_{2,1} a_{2,2},
$$

which does not contain an $n$-multicycle. This is displayed explicitly in the following example (in logarithmic notation, as usual).
Example 4.15.

Let \( A = \begin{pmatrix} 10 & 0 & 10 \\ 0 & 10 & 0 \\ 0 & 10 & 1 \end{pmatrix} \). Then \( \text{adj}(A) = \begin{pmatrix} 11 & 20 & 20 \\ 1 & 11 & 10\nu \\ 20 & 20 & 20 \end{pmatrix} \).

\[ A \text{adj}(A) = \begin{pmatrix} 21 & 30\nu & 30\nu \\ 11\nu & 21 & 20\nu \\ 11\nu & 21\nu & 21\nu \end{pmatrix}, \quad \text{and} \quad A \text{adj}(A)A = \begin{pmatrix} 31 & 40\nu & 31\nu \\ 21\nu & 31 & 21\nu \\ 21\nu & 31\nu & 22 \end{pmatrix}. \]

As expected, the von Neumann regularity condition is ruined by the (1,2) position.

An even easier example of the same phenomenon can be seen via triangular matrices, again for \( n \geq 3 \).

Example 4.16.

Take \( A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \). Then \( \text{adj}(A) = \begin{pmatrix} 0 & a & b+ac \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \), and

\[ A \text{adj}(A)A = A \text{adj}(A) = \begin{pmatrix} 0 & a\nu & b\nu + (ac)\nu \\ 0 & 0 & c\nu \\ 0 & 0 & 0 \end{pmatrix} \neq A, \]

when \((ac)\nu > b\nu\).

From a positive perspective, if each digraph arising from (4.8) does contain an \( n \)-multicycle, then the matrix \( A \) satisfies the von Neumann regularity condition. In particular, this is true when \( n = 2 \).

Conversely to Theorem 4.13 we have

**Proposition 4.17.** Each quasi-identity \( I_G \) satisfies \( \text{adj}(I_G) = I_G^\nabla = I_G \), and thus \( I_{L_G} = I_G \).

Proof. Write \( I_G = (a_{i,j}) \). The \((i, j)\)-entry \( a'_{i,j} \) of \( \text{adj}(I_G) \) is the sum of those terms corresponding to \( a \)-paths in \( G_{I_G} \) having out-degree 0 at \( i \), in-degree 0 at \( j \), and otherwise out-degree and in-degree 1 at all vertices. When \( i = j \), then this is an \((n-1)\)-multicycle, which must have weight \( \leq 1_R \) since \( |I_G| = 1_R \), and we get \( 1_R \) from the string

\[ a_{1,1} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{n,n} = 1_R = 1_R. \]

Thus it remains to check those \( a'_{i,j} \) for \( i \neq j \). We need to show that \( a'_{i,j} = a_{j,i} \), which by hypothesis is ghost. In computing \( a'_{i,j} \), we have the term

\[ a_{j,i} \prod_{k \neq i,j} a_{k,k} = a_{i,j} 1_R \cdots 1_R = a_{j,i}, \]

implying \( a'_{i,j} \nu \geq a_{j,i} \nu \). But all strings in \( a'_{i,j} \) have \( \nu \)-value \( \leq a_{j,i} \), because they can be decomposed as the union of cycles and a path from \( j \) to \( i \); the weight of any cycle must have \( \nu \)-value at most \( 1_R \nu \) (since \( |I_G| = 1_R \)), and the weight of any path from \( j \) to \( i \) has \( \nu \)-value most \( a_{j,i} \) because \( I_G \) is idempotent. Thus, \( a'_{i,j} = a_{j,i}. \)

\[ \Box \]

We conclude that a necessary and sufficient condition for a matrix \( B \) to have the form \( AA^\nabla \) is for \( B \) to be a quasi-identity. By symmetry, this is also a necessary and sufficient condition for the matrix \( B \) to have the form \( A^\nabla A \) (but possibly with different \( A \)).

This leads to another positive result concerning von Neumann regularity. First we want to compare \( \text{adj}(A) \) and \( \text{adj}(AA^\nabla) \) for \( A \) nonsingular. One must be careful, since it is not necessarily the case that \( \text{adj}(AA^\nabla) = \text{adj}(A) \); for example, with \( A = \begin{pmatrix} 0 & 1 \\ -\infty & 0 \end{pmatrix} \), we have \( \text{adj}(A) = A \) but \( \text{adj}(A)A = \text{adj}(AA^\nabla) = \begin{pmatrix} 0 & 1^\nu \\ -\infty & 0 \end{pmatrix} \).
Lemma 4.18. The corresponding entries of $\text{adj}(AA^\nabla A)$ and $\text{adj}(A)$ have the same $\nu$-values.

Proof. Write $AA^\nabla A = (b_{i,j})$ and $\text{adj}(AA^\nabla A) = (b'_{i,j})$. Since $I_A = I + \text{ghost}$, clearly $b'_{i,j} \leq a'_{i,j}$. Thus, in computing any string for $b'_i$, which we recall is a product

$$b'_{i_1,j_1} \cdots b'_{i_{n-1},j_{n-1}}$$

where $\{i_1, \ldots, i_{n-1}\} = \{1, \ldots, n-1\}$ and $\{j_1, \ldots, j_{n-1}\} = \{1, \ldots, n-1\}$, we see that the out-degree is $n-1$ for $i$, and $n$ for all other vertices; likewise, the in-degree is $n-1$ for $j$, and $n$ for all other vertices. Hence, by Lemma 4.19 (iv) we can extract $n-1$ $n$-multicycles, each having value $\leq |A|$, and are left with a graph of out-degree 0 for $i$ and out-degree 1 for each other vertex, and in-degree 0 for $j$ and in-degree 1 for each other vertex; the product of the corresponding entries of $A$ is a summand of $a'_{i,j}$. In other words, $b'_i$ is a sum of terms, each of which is $1_R \leq \nu$ times $a'_{i,j}$, as desired.$\square$

Lemma 4.19. $|AA^\nabla A| = |A|^{\nu}$, for any matrix $A$ over a supertropical semifield.

Proof. Applying Theorem 4.19 to Lemma 4.18

$$\left(|AA^\nabla A|^{n-1}\right)^\nu = |\text{adj}(AA^\nabla A)|^{\nu} = |\text{adj}(A)|^{\nu} = \left(|A|^{n-1}\right)^\nu,$$

implying $|AA^\nabla A| = |A|^{\nu}$, since $G$ is an ordered group.$\square$

Proposition 4.20. $AA^\nabla A$ satisfies the von Neumann regularity property, for any nonsingular matrix $A$ over a supertropical semifield.

Proof. First we claim that $I_{AA^\nabla A} = I_A$. Indeed, since $I_{AA^\nabla A}$ and $I_A$ are both quasi-identities, it suffices to show that their respective off-diagonal entries have the same $\nu$-values (since they are ghost, by definition). But

$$I_{AA^\nabla A} = AA^\nabla A AA^\nabla A = \frac{1}{|A|} I_A A \text{ adj}(AA^\nabla A)$$

whereas

$$I_A = I_A^2 = \frac{1}{|A|} I_A A \text{ adj}(A).$$

The claim follows when we observe that the corresponding entries of $\text{adj}(AA^\nabla A)$ and $\text{adj}(A)$ have the same $\nu$-values, in view of Lemma 4.19.

But now, using the fact that $I_A$ is multiplicatively idempotent, we have

$$(AA^\nabla A)(AA^\nabla A)(AA^\nabla A) = I_{AA^\nabla A} AA^\nabla A = I_A AA^\nabla A = I_A^2 A = I_A A = AA^\nabla A.$$

Here is another application of the adjoint matrix, to be elaborated in a follow-up paper.

Remark 4.21. Suppose $|A|$ is invertible, and $v \in R^{n \times 1}$. Then the equation $Aw = v + \text{ghost}$ has the solution $w = A^\nabla v$. Indeed, writing $I_A = I + Z_G$ for some quasi-zero matrix $Z_G$, we have

$$Aw = AA^\nabla v = I_A v = (I + Z_G) v = v + \text{ghost}.$$

5. The Hamilton-Cayley theorem

Definition 5.1. Define the \textbf{characteristic polynomial} $f_A$ of the matrix $A$ to be

$$f_A = |\lambda I + A|,$$

the \textbf{essential characteristic polynomial} to be the essential part $f_A^{\text{ess}}$ of the characteristic polynomial $f_A$, cf. [13] Definition 4.9, and the \textbf{tangible characteristic polynomial} to be a tangible polynomial $f_A = \lambda^n + \sum_{i=1}^n \alpha_i \lambda^{n-i}$, where $\alpha_i \in R$ and $\alpha_i' = \alpha_i$, such that $f_A = \lambda^n + \sum_{i=1}^n \alpha_i \lambda^{n-i}$.

Under this notation, we see that $\alpha_k \in R$ is the highest weight of the $k$-multicycles in the reduced digraph $G_A$ of $A$.

Recall that the \textbf{roots} of a polynomial $f \in R[\lambda]$ are those elements $a \in R$ for which $f(a) \in G$. Thus, we say that a matrix $A$ satisfies a polynomial $f \in R[\lambda]$ if $f(A) \in M_n(G)$. Thus, we say that a matrix $A$ satisfies a polynomial $f \in R[\lambda]$ if $f(A) \in M_n(G)$. Thus, we say that a matrix $A$ satisfies a polynomial $f \in R[\lambda]$ if $f(A) \in M_n(G)$.
Theorem 5.2. *(Supertropical Hamilton-Cayley)* Any matrix $A$ satisfies both its characteristic polynomial $f_A$ and its tangible characteristic polynomial $\hat{f}_A$.

Proof. Let $B = \hat{f}_A(A) = A^n + \sum \hat{\alpha}_i A^{n-i}$. It suffices to prove that $B \in M_n(\mathbb{S}_0)$, i.e., that each entry $b_{u,v}$ is ghost. But $b_{u,v}$ is obtained as the maximum from the various contributions $\hat{\alpha}_i A^{n-i}$, each of which is the product of weights of disjoint simple cycles $C_1, \ldots, C_{t(u,v)}$ in the reduced digraph $G_A$ with each $C_j$ of length $n_j$, where $\sum_{j=1}^{t(u,v)} n_j = i$, multiplied by the weight of a path $p$ of $G_A$ of length $n - i$. If this last path $p$ intersects one of the cycles, say $C_1$, then we also have a path of length $n - i + n_1$ obtained by combining $p$ with $C_1$, in which case $b_{u,v}$ is matched by a term from $\hat{\alpha}_{i-n_1} A^{n-i+n_1}$, and thus is ghost. Thus, we may assume $p$ is disjoint from all the cycles. But this implies that the path $p$ traverses only $n - i$ vertices, which is the length of $p$, and thus $p$ must contain a cycle $C$ of some length $m \leq n - i$ (by the pigeonhole principle). But then $b_{u,v}$ is matched with a term from $\alpha_{i-m} A^{n-i-m}$, and thus is ghost. (When $m = n - i$, we have $u = v$, and $p$ itself is a cycle $C_{t(u,u)+1}$, so we match $b_{u,u}$ with a term from $|A|$.)

When all the $\hat{\alpha}_i$ contributing to $b_{u,v}$, and thus to $B$, are $0_R$, it means that the cycle of length $n$ is the unique cycle of minimal length. In this case, we have $\hat{f}_A(A) = A^n + |A| I$ is ghost. \[\]

We digress for a moment to improve Theorem 5.2 slightly, by looking closely at its proof. Given a polynomial $f = \alpha_n \lambda^n + \cdots + \alpha_1 \lambda + \alpha_0$, we define the polynomial $\hat{f}$ to be

$$\hat{f} = \hat{\alpha}_n \lambda^{n-1} + \cdots + \hat{\alpha}_2 \lambda + \hat{\alpha}_1,$$

where $\hat{\alpha}_i \in \mathcal{T}_0$ and $\hat{\alpha}_i' = \alpha_i'$. \[\]

Theorem 5.3. $\hat{f}_A(A) = \text{adj}(A) + \text{ghost}$, for any matrix $A$.

Proof. We first show that many entries of

$$B = \hat{f}_A(A) + \text{adj}(A) = A^{n-1} + \sum \hat{\alpha}_i A^{n-i-1} + \text{adj}(A)$$

are ghosts. The $(u,v)$-entry $b_{u,v}$ is obtained as having the largest $v$-value from the various $\alpha_i A^{n-i-1}$, which is the product of weights of disjoint simple cycles $C_1, \ldots, C_{t(u,v)}$, with each $C_j$ of length $n_j$, where $\sum_{j=1}^{t(u,v)} n_j = i$, together with the weight of a path $p$ of length $n - i - 1$. If this last path $p$ intersects one of the cycles, say $C_1$, then we also have a path of length $n - i + n_1 - 1$ obtained by combining $p$ with $C_1$, so we match $b_{u,v}$ with a term from $\hat{\alpha}_{i-n_1} A^{n-i+n_1-1}$. Thus, we have a ghost term unless $p$ is disjoint from all the cycles. But this implies that $p$ traverses only $n - i - 1$ vertices, which is its length. If $p$ contains a cycle $C$ of some length $m \leq n - i - 1$, then $b_{u,v}$ is matched by a term from $\alpha_{i-m} A^{n-i-1-m}$, and thus is ghost.

Thus, the only unmatched terms arise precisely when $p$ does not contain any cycle. In this case, $p$ must have the form

$$a_{k_1, \pi(k_1)} a_{k_2, \pi(k_2)} \cdots a_{k_m, \pi(k_m)},$$

where $k_t \neq u$ and $\pi(k_t) \neq v$ for all $1 \leq t \leq m$, and $\pi(k_t) = k_{t+1}$ for all $t < m$. But combining this with the cycles $C_1, \ldots, C_{t(u,v)}$ give us one of the summands in Equation (4.1) of Remark 4.4, and conversely any such summand can be matched with a disjoint union of simple cycles and some path of this form. Thus, we have decomposed $\hat{f}_A(A)$ as $\text{adj}(A)$ plus ghost terms. \[\]

Note 5.4. *Let us compare these two notions of characteristic polynomial. The tangible characteristic polynomial shows us that the powers of $A$ are tropically dependent (as defined in Definition 6.3 below). But, as we shall see, the characteristic polynomial is more appropriate when we work with eigenvalues, and its essential monomials play a special role.*

Note, however, that a monomial which is inessential with respect to substitutions in $R$, is not necessarily inessential with respect to matrix substitutions in $M_n(R)$. For example, consider the polynomial $f = \lambda^2 + \lambda + 2$; the term $\lambda$ is inessential for substitutions in $R$ but essential for matrix substitutions, seen by taking the matrix $A = \begin{pmatrix} -\infty & 1 \\ 1 & 0 \end{pmatrix}$ in logarithmic notation. In this case, $A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, so

$$f(A) = \begin{pmatrix} 2^\nu & 1^\nu \\ 1^\nu & 2^\nu \end{pmatrix}$$

is ghost, whereas $f^e(A) = \begin{pmatrix} 2^\nu & 1 \\ 1 & 2^\nu \end{pmatrix}$ is not ghost. The theory runs more smoothly when the characteristic polynomial is essential.
Note 5.5. We conclude from Theorem 5.2 that any \(2 \times 2\) matrix \(A\) satisfies
\[A^2 + \text{tr}(A)A + |A| I \in M_2(\mathbb{G}_0)\].

Here is an easy but important special case of Theorem 5.2.

Definition 5.6. A matrix \(A = (a_{i,j})\) is in lower ghost-triangular form if \(a_{i,j} \in \mathbb{G}_0\) for each \(i > j\).

Note that if \(A\) is nonsingular and is in lower ghost-triangular form, then its diagonal terms must all be tangible.

Example 5.7. Any matrix \(A = (a_{i,j})\) in lower ghost-triangular form satisfies the polynomial
\[
f = \prod_{i=1}^{n}(\lambda + a_{i,i}).
\]
One way of seeing this is to replace the \(a_{i,j}\) by 0 for all \(i > j\), and apply Theorem 5.2. Here is a direct verification. \(f(A) = (A + a_{1,1}I) \cdots (A + a_{n,n}I)\). In order to get a non-ghost entry in \(f(A)\), we need to multiply together \(n\) terms from the diagonal or above. However, the \((1,1)\) position in the first multiplicand starts with \(a_{1,1}e_{1,1}\) (where \(e_{i,j}\) denote the standard matrix units), so the first factor must be \(a_{i_1,j_1}e_{i_1,j_1}\) for \(j_1 \geq 2\). But the \((2,2)\) position in the second multiplicand starts with \(a_{2,2}e_{2,2}\), implying the second factor must be \(a_{i_2,j_2}e_{i_2,j_2}\) for \(j_2 \geq 3\). Continuing in this way, we see that the \((n-1)\)-factor must be \(a_{i_{n-1},j_{n-1}}e_{i_{n-1},j_{n-1}}\) for \(j_{n-1} \geq n\), in which case the last factor must be a ghost.

6. Applications to supertropical linear algebra

In this section, we see how tropical determinants apply to vectors over a supertropical domain \(R\). Our main objective is to characterize singularity of a matrix \(A\) in terms of tropical dependence of its rows.

First we start with a special case, where \(A\) is strictly singular, i.e., \(|A| = 0\). In view of Remark 5.2, the answer is a consequence of results in classical matrix theory, but anyway the statement and proof in this case are rather straightforward, so we present it here in full.

Definition 6.1. We say that a set \(v_1, \ldots, v_k\) of vectors has rank defect \(\ell\) if there are \(\ell\) columns, which we denote as \(j_1, \ldots, j_\ell\), such that \(v_{i,j_u}\) for all \(1 \leq i \leq k\) and \(1 \leq u \leq \ell\).

For example, the vectors \((2,0,R,2,0,R),(0,R,0,R,0,R,0,R)\) have rank defect 1, since they are all \(0\) in the second column.

Proposition 6.2. An \(n \times n\) matrix \(A\) has tropical determinant \(0\), iff, for some \(1 \leq k \leq n\), \(A\) has \(k\) rows having rank defect \(n + 1 - k\).

Proof. (⇒) If \(k = n\) then this is obvious, since some column is entirely \(0\). If \(n > k\), we take one of the columns \(j\) other than \(j_1, \ldots, j_k\) of Definition 6.1. Then for each \(i\), the \((i,j)\)-minor \(A_{i,j}\) has at least \(k - 1\) rows with rank defect \((n - 1) + 1 - k\), so has tropical determinant \(0\) by induction; hence \(|A| = 0\), by Formula (4.7).

(⇒) We are done if all entries of \(A\) are \(0\), so assume for convenience that \(a_{n,n} \neq 0\). Then the minor \(A_{n,n}\) has tropical determinant \(0\), so, by induction, \(A_{n,n}\) has \(k \geq 1\) rows of rank defect \((n - 1) + 1 - k = n - k\).

For notational convenience, we assume that \(a_{i,j} = 0\) for \(1 \leq i \leq k\) and \(1 \leq j \leq n - k\). Thus, we can partition \(A\) as the matrix
\[
A = \begin{pmatrix}
0 & B' \\
B'' & C
\end{pmatrix},
\]
where \(0\) denotes the \(k \times n-k\) zero matrix, \(B'\) is a \(k \times k\) matrix, \(B''\) is an \(n-k \times n-k\) matrix, and \(C\) is an \(n-k \times k\) matrix.

By inspection, \(|B'| |B''| = |A| = 0\); hence \(|B'| = 0_R\) OR \(|B''| = 0_R\). If \(|B'| = 0_R\), then, by induction, \(B'\) has \(k'\) rows of rank defect \(k+1-k', so\) altogether, the same \(k'\) rows in \(A\) have rank defect \((n-k)+1-k' = n+1-k', and we are done taking \(k'\) instead of \(k\).

If \(|B''| = 0_R\), then, by induction, \(B''\) has \(k''\) rows of rank defect \((n-k)+1-k'', so\) altogether, these \(k+k''\) rows in \(A\) have rank defect \(n+1-(k+k'')\), and we are done, taking \(k+k''\) instead of \(k\). \(\square\)
Now we turn to the supertropical version, whose statement has quite a different flavor of linear dependence.

**Definition 6.3.** Suppose \( V = (R(n), \mathcal{H}_0, \mu) \) is a module over a supertropical semiring \( R \). A subset \( W \subset V \) is **tropically dependent** if there is a finite sum \( \sum \alpha_i v_i \in \mathcal{H}_0 \), with each \( \alpha_i \in \mathcal{H}_0 \), but not all of them \( 0_R \); otherwise \( W \subset V \) is called **tropically independent**.

**Theorem 6.4.** (See [11] Corollary 3.3 and [14] Theorem 2.6) If vectors \( v_1, \ldots, v_n \in R(n) \) are tropically dependent, for \( R \) a supertropical domain, then \( |v_1, \ldots, v_n| \in \mathcal{G}_0 \).

**Proof.** Our proof follows the lines of [14] Theorem 2.6. Let \( A \) be the matrix whose \( i \)-th row is \( v_i \). Thus, writing \( v_i = (a_{i,1}, \ldots, a_{i,n}) \), we have \( A = (a_{i,j}) \). We need to prove that \( |A| \) is ghost, so for the remainder of the proof, we assume on the contrary that \( |A| \) is tangible, and aim for a contradiction.

Rearranging the rows and columns does not affect linear dependence of the rows, so we may assume that \( |A| \) is attained by the identity permutation, i.e., \( |A| = a_{1,1} \cdots a_{n,n} \), and is not attained by any other permutation.

We are given some dependence \( \sum \alpha_i v_i \in \mathcal{H}_0 \). First assume that \( \alpha_n = 0_R \); i.e., \( \sum_{i=1}^{n-1} \alpha_i v_i \in \mathcal{H}_0 \). If we erase the \( j \)-th column of the \( v_i \)'s, we are left with the minor \( A'_{n-1,j} \) whose rows clearly satisfy the same dependence. Then by induction, its tropical determinant \( A''_{n-1,j} \in \mathcal{G}_0 \), so

\[
|A| = \sum_{j=1}^n a_{n-1,j} A''_{n-1,j} \in \mathcal{G}_0,
\]

and we are done. Thus, we may assume that every \( \alpha_n \neq 0_R \).

Replacing \( v_i \) by \( \alpha_i v_i \) for \( 1 \leq i \leq n \), with \( \alpha_i \) tangible, we may assume that

\[
\sum v_i \in \mathcal{H}_0.
\]

We say \( a_{i,j} \in A \) is **critical** if \( a_{i',j} \geq a_{i,j} \) for each \( 1 \leq i' \leq n \); in other words, if \( a_{i,j} \) dominates all entries in the \( j \)-th column of \( A \). Note that for this particular matrix \( A \), any critical entry is either ghost, or is matched by another critical entry in the same column.

Let \( G_A \) denote reduced digraph of \( A \), let \( G' \) denote the sub-digraph of edges corresponding to critical entries, and let \( G'' \) denote the sub-digraph of \( G' \) after we erase all the loops of \( G' \). (The loops correspond to critical diagonal elements \( a_{i,i} \).)

Note that if some \( a_{i,i} \in \mathcal{G}_0 \) then \( |A| \in \mathcal{G}_0 \), and we are done. Thus, any critical diagonal entry must be tangible, and thus must be matched by another critical entry in the same column. It follows that \( G'' \) has in-degree \( \geq 1 \) in each vertex, so Remark 6.13 implies that \( G'' \) contains a cycle (which by definition of \( G'' \) is not a loop); this corresponds to

\[
a_{i_1,i_2} \cdots a_{i_{k-1},i_k} a_{i_k,i_1}
\]

where each entry is critical. Defining the permutation \( \pi \) by \( \pi(i_1) = i_2, \ldots, \pi(i_k) = i_1 \) and the identity elsewhere, it is clear that \( a_{i_1,i_2} \cdots a_{i_{k-1},i_k} a_{i_k,i_1} \) is dominated by \( a_{i_1,i_2} \cdots a_{i_{k-1},i_k} a_{i_k,i_1} \), and thus \( |A''| \) is also attained by \( \pi \), contrary to \( |A| \in \mathcal{T} \). \( \square \)

We look for the converse of Theorem 6.4. **Theorem 6.5.** (See [11] Corollary 3.3 and [14] Theorem 2.10) Suppose \( R \) is a supertropical domain. Vectors \( v_1, \ldots, v_n \in R(n) \) are tropically dependent, iff \( |A| \in \mathcal{G}_0 \), where \( A \) is the matrix whose rows are \( v_1, \ldots, v_n \). Furthermore, we explicitly display the tropical dependence in the proof.

**Proof.** \((\Rightarrow)\) By Theorem 6.4 \((\Leftarrow)\) Assuming that \( A \) is singular, we need to prove that the rows of \( A \) are tropically dependent. Arguing by induction \( n \), we assume that the theorem is true for \( (n-1) \), the case for \( n = 1 \) being obvious.

Rearranging the rows and columns of \( A \), we assume henceforth that the identity permutation \( \pi = (1) \) attains \( |A| \). Note that this hypothesis is not affected by multiplying through any row by a given tangible element, which we do repeatedly throughout the proof.

Let

\[
\gamma_\pi = v_{\pi(1), 1} \cdots v_{\pi(n), n}
\]

for each permutation \( \pi \) of \( \{1, \ldots, n\} \), and let

\[
\gamma = \gamma_{(1)} = v_{1,1} \cdots v_{n,n}.
\]
Thus \( \gamma^v = |A|^v = |A| \).

**Case I:** \( \gamma^v = \gamma^v_\pi \) for some permutation \( \pi \neq (1) \). Thus, \( \pi(i_0) \neq i_0 \) for some \( i_0 \); for notational convenience, we assume that \( \pi(1) \neq 1 \). Take \( \beta_\pi \in \mathcal{T}_0 \) of the same \( \nu \)-value as the tropical determinant \( |A_{i_1}| \) of the minor \( A_{i_1} \). Then \( \sum_{i=1}^n \beta a_{i_1,i} \) has the same \( \nu \)-value as \( \sum |A_{i_1}| a_{i_1,i} = |A| \), but is ghost since, by hypothesis, there are two leading summands in the determinant formula that match. Hence, \( \sum_{i=1}^n \beta a_{i_1,i} \in \mathcal{G}_0 \). On the other hand, for every \( j \neq 1 \), \( \sum_{i=1}^n \beta a_{i,i} \) is the tropical determinant of a matrix having two columns with the same \( \nu \)-values, so is in \( \mathcal{G}_0 \) by Equation (4.3). Thus, we are done unless all \( \beta_\pi = 0 \). In this case \( \gamma = 0_R \), so in view of Proposition 6.2 there is \( k \) for which \( A \) has \( k \) rows with rank defect \( n+1-k \). We need to conclude that these \( k \) rows are tropically dependent. By induction on \( n \), we may assume that \( n = k+1 \), and that the first entry of each row is \( 0_R \). If \( |A_{i_1}| \neq 0_R \), we are done by the above argument. If \( |A_{i_1}| = 0 \), we see by induction that \( v_2, \ldots, v_n \) are tropically dependent.

**Case II:** \( \gamma^v > \gamma^v_\pi \) for each permutation \( \pi \neq (1) \). Thus \( \gamma = |A| \in \mathcal{G}_0 \), so some \( a_{i_1} \in \mathcal{G}_0 \); renumbering the indices, we may assume that \( a_{i_1} \in \mathcal{G}_0 \). As in Case I, take \( \beta_i \in \mathcal{T}_0 \) of the same \( \nu \)-value as \( |A_{i_1}| \). Then \( \sum_{i=1}^n \beta a_{i_1,i} \) has the same \( \nu \)-value as \( \sum |A_{i_1}| a_{i_1,i} = |A| \), but is ghost since by hypothesis \( a_{i_1} \in \mathcal{G}_0 \). Again, by Equation (4.3), \( \sum_{i=1}^n \beta a_{i,i} \in \mathcal{G}_0 \), for all \( j \neq 1 \). Thus, \( \sum_{i=1}^n \beta v_i \in \mathcal{H}_0 \), as desired. \( \square \)

**Corollary 6.6.** (See [11] Corollary 3.3 and [13] Theorem 3.4.) The matrix \( A \in M_n(R) \) over a supertropical domain \( R \) is nonsingular iff the rows of \( A \) are tropically independent, iff the columns of \( A \) are tropically independent.

**Proof.** Apply the theorem to \( |A| \) and \( |A^t| \), which are the same. \( \square \)

**Corollary 6.7.** Any \( n+1 \) vectors in \( R^{(n)} \) are tropically dependent.

**Proof.** Expand their matrix to an \( (n+1) \times (n+1) \) matrix \( A \) by adding a column of zeroes at the beginning; obviously \( A \) is strictly singular, so its rows are tropically dependent. \( \square \)

As pointed out in [11] Observation 2.6], and as we have seen in Example [47] above, the square of a nonsingular matrix \( A \) need not be nonsingular.

### 6.1. The Vandermonde matrix.

One way of applying this method is by means of a version of the celebrated Vandermonde argument. Given \( a_1, \ldots, a_n \) in \( R \), define the Vandermonde matrix \( A \) to be the \( n \times n \) matrix \( (a_{i,j}) \), where \( a_{i,j} = a_i^{j-1} \) and \( a_0^0 = 1_R \). Recall from [13] Lemma 7.58 that its tropical determinant is

\[
|A| = \prod_{i \neq j} (a_i + a_j).
\]

(6.2)

**Remark 6.8.** Assume that \( A \) is a Vandermonde matrix \( (a_i^{j-1}) \) with respect to distinct \( a_1, \ldots, a_n \). By Formula (6.2), we see that if all the \( a_i \) are tangible, or if the only \( a_i \) which is ghost is the \( a_i \) of smallest \( \nu \)-value, then \( A \) is nonsingular; otherwise \( A \) is singular.

**Lemma 6.9.** If \( A \in M_n(R) \) and \( v \) is a tangible vector such that \( Av \) is a ghost vector, then the matrix \( A \) is singular.

**Proof.** The columns of \( A \) are tropically dependent, so \( A \) is singular by Corollary 6.6. \( \square \)

**Theorem 6.10.** Suppose \( v = (\gamma_1, \ldots, \gamma_n) \in R^{(n)} \) for \( R = (R, \mathcal{G}_0, \nu) \) a supertropical domain, and suppose \( \sum_{j=1}^n a_i^j \gamma_j \in \mathcal{G}_0^{(n)} \) for each \( i = 1, \ldots, n \), where \( a_1, \ldots, a_n \) are tangible. Then some \( \gamma_j \) is ghost.

**Proof.** Let \( A \) be the Vandermonde matrix \( (a_i^{j-1}) \). Then \( A \) is ghost, so we are done by the lemma. \( \square \)

**Example 6.11.** Despite these nice applications of the Vandermonde matrix, the Vandermonde matrix 

\[
A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\
\end{pmatrix}
\]

(over \( D(R) \)) has the poor behavior that \( A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\
\end{pmatrix} \), which is singular with tropical determinant \( 5^v \) whereas \( |A| = 2 \); cf. Example 4.7.

**Definition 6.12.** A matrix \( B_1 \) is (classically) **conjugate to** \( B \) if \( B_1 = A^\nu BA \) for some matrix \( A \) with \( |A| \) invertible in \( R \). More generally, a matrix \( B_1 \) is **tropically conjugate to** \( B \) if \( B_1 = A^\nu BA + \text{ghost} \) for some matrix \( A \) with \( |A| \) invertible.
Lemma 6.13. If $f \in R[\lambda]$ is a polynomial with constant term 0$_F$. Then for any nonsingular matrix $A$, 
\[ f(A^\vee BA) = A^\vee f(B)A + \text{ghost}. \]

Proof. It is enough to check the case that $f = \lambda^i$ for $i \geq 1$. Assume $B_1 = A^\vee BA$. Let $I_A = AA^\vee = (I + Z_g)$, where $Z_g$ is a quasi-zero matrix. For any $i > 0$, 
\[ (A^\vee BA)^i = A^\vee B(I + Z_g)B \cdots B(I + Z_g)BA = A^\vee B^iA + \text{ghost}. \]

\[ \square \]

Proposition 6.14. If $B$ satisfies a polynomial $f \in R[\lambda]$, $R$ is a supertropical domain, then every tropical conjugate of $B$ satisfies $f$.

Proof. It is enough to show that every conjugate of $B$ satisfies $f$, since the added ghost only yields extra ghost terms. Writing $f = g + \alpha_0$, where $g$ has constant term 0$_F$, we have 
\[ f(A^\vee BA) = A^\vee g(B)A + \text{ghost} + \alpha_0I, \]
whereas $A^\vee g(B)A + \alpha_0 A^\vee A = A^\vee f(B)A$ is ghost. Write $g(B) = (b_{i,j})$. The diagonal terms of $f(A^\vee BA)$ are ghost, since they are ghosts plus the diagonal terms of $f(B)$, which by hypothesis is ghost. Thus, we need only check the off-diagonal terms of $A^\vee g(B)A$, which when multiplied by $|A|$ have the form $\sum a_{j,k} a_{k,\ell}$, for $j \neq \ell$; we need to show that these are ghosts.

On the other hand, $f(B) = g(B) + \alpha_0 I$, so $f(B)$ and $f(A^\vee BA)$ agree off the diagonal. When $j \neq k$, $b_{j,k}$ is either ghost or is the same as the $(j,k)$-entry of $f(B)$, which is ghost by hypothesis, so we may assume that $j = k$. Now, when tangible, 
\[ \sum a_{j,k} a_{j,\ell} = \sum a_{j,k} \alpha_0 a_{j,\ell} = \alpha_0 \sum a_{j,k} a_{j,\ell}, \]
which is ghost by (4.8).

\[ \square \]

7. Supertropical eigenvectors

We work throughout with matrices over a supertropical semifield $F$.

Definition 7.1. A vector $v$ is an \textbf{eigenvector} of $A$, with \textbf{eigenvalue} $\beta$, if $Av = \beta v$. The eigenvalue $\beta$ with $\beta^\vee$ maximal is said to be of \textbf{highest weight}.

Definition [4] is standard (not requiring the language of ghosts), and indeed it is known [4] that any (tangible) matrix has an eigenvalue of highest weight. However, even counting multiplicities, the number of eigenvalues often is less than the size of the matrix, since certain roots of the characteristic polynomial may be “lost” as eigenvalues.

Example 7.2. The characteristic polynomial $f_A$ of 
\[ A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \]
over $F = D(\mathbb{R})$, is $(\lambda + 4)(\lambda + 1) + 0 = (\lambda + 4)(\lambda + 1)$, and indeed the vector $(4,0)$ is a eigenvector of $A$, with eigenvalue 4. However, there is no eigenvector having eigenvalue 1.

We rectify this deficiency by weakening Definition [4]. Actually, there are several possible definitions of supertropical eigenvalue. We present two of them; the second is stronger but suffices for our theory, so we call the first one “weak.”

Definition 7.3. A vector $v \neq 0$ is a \textbf{weak generalized supertropical eigenvector} of $A$, with (tangible) \textbf{weak generalized supertropical eigenvalue} $\beta \in T_0$, if $A^m v + \beta^m v$ is ghost for some $m$; the minimal such $m$ is called the \textbf{multiplicity} of the eigenvalue (and also of the eigenvector).

A tangible vector $v$ is a \textbf{generalized supertropical eigenvector} of $A$, with \textbf{generalized supertropical eigenvalue} $\beta \in T_0$, if 
\[ A^m v = \beta^m v + \text{ghost} \]
for some $m$; the minimal such $m$ is called the \textbf{multiplicity} of the eigenvalue (and also of the eigenvector).

A \textbf{supertropical eigenvalue} (resp. \textbf{supertropical eigenvector}) is a generalized supertropical eigenvalue (resp. generalized supertropical eigenvector) of multiplicity 1.
(Although weak generalized supertropical eigenvectors need not be tangible, generalized supertropical eigenvectors are required to be tangible, since we are about to prove that there are “enough” of them for a reasonable theory. Note that if we did not require \( \beta \) to be tangible, all vectors would be weak supertropical eigenvectors; indeed, for any given matrix \( A \) and vector \( v \), any large enough ghost element \( \beta \) would be a weak supertropical eigenvalue of \( A \) with respect to \( v \). On the other hand, this observation does not apply to the definition of supertropical eigenvectors.)

When \( \nu_T \) is 1:1 (which is the case in the applications to tropical geometry), tangible weak (generalized) supertropical eigenvectors are (generalized) supertropical eigenvectors, because of the following observation.

**Lemma 7.4.** Suppose \( \nu_T \) is 1:1. If \( v \) is tangible and \( A^m v + \beta^m v \) is ghost for \( \beta \in \mathcal{T} \), then \( A^m v = \beta^m v + \text{ghost} \).

**Proof.** Write \( v = (r_1, \ldots, r_n) \) where each \( r_i \in \mathcal{T}_0 \), and \( A^m v = (s_1, \ldots, s_n) \). But then \( \beta^m r_i \in \mathcal{T}_0 \), so the \( i \)-th component \( s_i + \beta^m r_i \) of \( A^m v + \beta^m v \) can be ghost only when \( s_i = \beta^m v_i \) or \( s_i \) is ghost dominating \( \beta^m v_i \), in which case

\[
s_i = s_i + \beta^m r_i = \beta^m v_i + \text{ghost}.
\]

\( \square \)

**Example 7.5.** The matrix \( A = \left( \begin{array}{cc} 4 & 0 \\ 0 & 1 \end{array} \right) \) of Example 7.4 also has the tangible supertropical eigenvector \( v = (0, 4) \), corresponding to the supertropical eigenvalue 1, since

\[
Av = (4^*, 5) = 1v + (4^*, 0^*).
\]

**Remark 7.6.** Let \( A_{\text{tan}} \) denote the matrix obtained by replacing each ghost entry of \( A \) by 0. Then \( A = A_{\text{tan}} + \text{ghost} \), so clearly every (generalized) supertropical eigenvalue of \( A_{\text{tan}} \) is a (generalized) supertropical eigenvalue of \( A \). This enables us to reduce many questions about supertropical eigenvalues to tangible matrices.

We also want to study supertropical eigenvalues in terms of other notions.

**Proposition 7.7.** The matrix \( A + \beta I \) is singular, iff \( \beta \) is a root of the characteristic polynomial \( f_A \) of \( A \).

**Proof.** The determinant of \( A + \beta I \) comes from \( n \)-multicycles of greatest weight. Since the contribution from \( \beta I \) comes from say \( n-k \) entries of \( \beta \) along the diagonal, the remaining \( k \) entries must come from a \( k \)-multicycle, in the graph of \( A \), which dominates the \( k \)-multicycles and has some total weight \( \alpha_k \). On the other hand, as already noted in the proof of Theorem 5.2, \( \alpha_k \) is precisely the coefficient of \( \lambda^{n-k} \) in \( f_A \). Thus, \( [A + \beta I] \in \mathcal{G} \) iff either \( \alpha_k \in \mathcal{G} \) or some other \( \alpha_k \beta^k \) matches \( \alpha_k \beta^k \) (and dominates all other \( \alpha_j \beta^j \)); but this is precisely the criterion for \( \beta \) to be a root of \( f_A \), proving the assertion. \( \square \)

**Proposition 7.8.** If \( v \) is a tangible supertropical eigenvector of \( A \) with supertropical eigenvalue \( \beta \), the matrix \( A + \beta I \) is singular (and thus \( \beta \) must be a (tropical) root of the characteristic polynomial \( f_A \)) of \( A \).

**Proof.** \( (A + \beta I)v \) is ghost, and thus so is \( \text{adj}(A + \beta I)(A + \beta I)v \). If \( A + \beta I \) were nonsingular this would be \( f_A(\beta)I_{A+\beta I}v \), implying \( I_{A+\beta I} \) is ghost, by Lemma 6.9, a contradiction. \( \square \)

Our goal is to prove the converse, that every tangible root of the characteristic polynomial of \( A \) is a supertropical eigenvalue (of a tangible supertropical eigenvector). First of all, let us reduce the theory to tangible matrices.

**Remark 7.9.** If \( \tilde{A} \) is a tangible matrix (i.e., all entries are in \( \mathcal{T}_0 \)), such that \( \tilde{A}^* = A^* \), then every tangible supertropical eigenvector \( v \) of \( \tilde{A} \) is a supertropical eigenvector of \( A \), with the same supertropical eigenvalue. (Indeed, let \( \beta \) be the eigenvalue of \( v \) for \( \tilde{A} \); obviously \( \tilde{A}v \) and \( Av \) are \( \nu \)-matched, with every tangible component of \( \tilde{A}v \) matched by a tangible component of \( Av \), so

\[
Av = \tilde{A}v + \text{ghost} = \beta v + \text{ghost}.
\]

**Theorem 7.10.** Assume that \( \nu|_T : T \to \mathcal{G} \) is 1:1. For any matrix \( A \), the dominant tangible root of the characteristic polynomial \( f_A^\nu = \lambda^\nu + \sum_{j=1}^t \alpha_j \lambda^{\nu-k_j} \) of \( A \) is an eigenvalue of \( A \), and has a tangible eigenvector. The matrix \( A \) has at least \( t \) supertropical tangible eigenvectors, whose respective tangible eigenvalues are precisely the tangible roots of \( f_A^\nu \).
Proof. Let $B = A + \beta I$. By Proposition 7.7, $B$ is singular, which implies by Corollary 6.6 that its columns $c_1, \ldots, c_n$ are tropically dependent. Taking tangible $\gamma_1, \ldots, \gamma_n$, not all of them $0_F$, such that $\sum \gamma_j c_j \in \mathcal{G}_0(n)$, and letting $v = (\gamma_1, \ldots, \gamma_n)$, we see that

$$Av + \beta v = Bv = \sum \gamma_j c_j \in \mathcal{G}_0(n),$$

implying by Lemma 7.3 that $Av = \beta v + \text{ghost}$, as desired. \qed

We have proved that the supertropical eigenvalues are precisely the roots of the characteristic polynomial. On the other hand, there may be extra cycles that also contribute weak supertropical eigenvectors, providing weak supertropical eigenvalues that are not roots of the characteristic polynomial. Let us illustrate this feature.

Example 7.11. Let $A$ be the $3 \times 3$ tropical matrix

$$
\begin{pmatrix}
-\infty & 14 & 8 \\
0 & -\infty & -\infty \\
0 & 1 & -\infty
\end{pmatrix},
$$
in logarithmic notation. The tangible characteristic polynomial is $\lambda^3 + 14\lambda + 9$ whose tangible roots are 7 and $-5$, and the supertropical tangible eigenvectors corresponding to $f_A$ are:

1. $(7, 0, 0)$ of eigenvalue 7, which arises from the cycle $(1, 2), (2, 1)$ of weight $\frac{14}{7} = 7$.
2. The tangible supertropical eigenvector $v = (0, 5, 11)$; here $Av = (19^\circ, 0, 6) = (-5)v + (19^\circ, -\infty, -\infty)$.

Note that the other cycles give rise to weak supertropical eigenvectors, although not tangible:

1. The cycle $(1, 3), (3, 1)$ yields the supertropical eigenvector $(10^\circ, 0, 6)$ of supertropical eigenvalue 4.
2. The cycle $(1, 3), (2, 1), (3, 2)$ of weight $\frac{2}{3} = 3$ yields the supertropical eigenvector $(6^\circ, 3^\circ, 0)$ of supertropical eigenvalue 3.

Example 7.12. Let $A$ be the $3 \times 3$ tropical matrix

$$
\begin{pmatrix}
-\infty & -\infty & 7 \\
4 & -\infty & -\infty \\
3 & 5 & -\infty
\end{pmatrix},
$$

over the extended max-plus semiring $D(\mathbb{R})$ (in logarithmic notation). We look for an eigenvector $(0, \gamma_2, \gamma_3)$, by means of rather crude computations. For any supertropical eigenvalue $\beta$, we have the three equations (in $\mathbb{R}$, with respect to the familiar addition and multiplication):

1. $7 + \gamma_3 = \beta$;
2. $4 = \gamma_2 + \beta$;
3. $\max\{3, 5 + \gamma_2\} = \gamma_3 + \beta$.

Adding the first two equations yields $\gamma_2 + \gamma_3 + 3 = 0$. Thus, plugging into (3) yields either $3 = \gamma_3 + \beta$ or $3\gamma_3 = -5$. In the former case, we get $v = (0, -1, -2)$, which is not an eigenvector since $Av = (5, 4, 4^\circ)$! (The reason is that reversing the steps in the proposed solution does not satisfy (3).)

On the other hand, $v = (0, -1^\circ, -2)$ is a weak supertropical eigenvector, since $Av = (5, 4, 4^\circ)$, and then

$$Av = 5v + (0^\circ, 0^\circ, 4^\circ),$$

thus 5 is a weak supertropical eigenvalue. Also $A^2v = (11^\circ, 9, 9)$, and $A^3v = (16, 15^\circ, 14^\circ)$, implying 5 is a supertropical eigenvalue of $A^2v$. But these weak supertropical eigenvectors are quite strange, since $A^3v = 16v + \text{ghost}$, whereby we see that $v$ is a generalized supertropical eigenvector for the generalized supertropical eigenvalue $\frac{16}{3}$.

In the latter case, we get $\gamma_3 = -\frac{5}{3}$, in which case $\gamma_2 = -\frac{4}{3}$, so $v = (0, -\frac{4}{3}, -\frac{5}{3})$, which is a supertropical eigenvector with supertropical eigenvalue $\frac{16}{3}$.

If one plays a bit more with the equations, one also gets the weak supertropical eigenvector $(0, -2, -1^\circ)$, with weak supertropical eigenvalue 6. But, again, $A^3v = 16v + \text{ghost}$.
The mystery can be cleared up by examining the characteristic polynomial \( \lambda^3 + 10\lambda + 16 \) of \( A \). The essential part of \( f_A \) is \( \lambda^3 + 16 \), whose only tangible root is \( \beta = \frac{16}{7} \), and indeed we get the supertropical eigenvector \((0, -\frac{4}{7}, -\frac{2}{7})\) by applying the proof to \( v_0 = (0, -\infty, -\infty) \) and \( \beta = \frac{16}{7} \).

Here is a surprising counterexample to a natural conjecture.

Example 7.13. Let \( A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \), of Example 6.11. Its characteristic polynomial is \( \lambda^2 + 2\lambda + 2 = (\lambda + 0)(\lambda + 2) \), whose roots are 2 and 0. The eigenvalue 2 has tangible eigenvector \( v = (0, 2) \) since \( Av = (2, 4) = 2v \), but there are no other tangible eigenvalues. \( A \) does have the tangible supertropical eigenvalue 0, with tangible supertropical eigenvector \( w = (2, 1) \), since

\[
Aw = (2, 3w^\nu) = 0w + (-\infty, 3w^\nu).
\]

Note that \( A + 0I = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \) is singular; i.e., \( |A + 0I| = 2^\nu \).

Furthermore, \( A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), which is singular, and

\[
A^4 = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = 4A,
\]

implying that \( A^2 \) is a root of \( \lambda^2 + 4A \), and thus \( A \) is a root of \( g = \lambda^4 + 4A^2 = (\lambda(\lambda + 2))^2 \), but 0 is not a root of \( g \) although it is a root of \( f_A \). This shows that the naive formulation of Frobenius’ theorem fails in the supertropical theory.

Let us say that a matrix \( A \) is separable if its characteristic polynomial \( f_A \) splits as the product of distinct monic linear tangible factors. (Equivalently, \( f_A = \sum_{i=0}^{n} \alpha_i \lambda^i \) is essential, with each \( \alpha_i \in F \) tangible.) Let \( U_A \) be the matrix whose columns are supertropical eigenvectors of \( A \). We conjecture that the matrix \( U_A \) is nonsingular. The argument seems to be rather intricate, involving a description of the multicycles of \( A \) in terms of its eigenvalues, so, for the time being, we insert this as a hypothesis.

Corollary 7.14. Every separable \( n \times n \) matrix \( A \) (for which \( U_A \) is nonsingular) is tropically conjugate to a diagonal matrix, in the sense that

\[
U_A^\top AU_A = D_A + \text{ghost},
\]

where \( D_A \) is the diagonal matrix whose entries \( \{\beta_1, \ldots, \beta_n\} \) are the supertropical eigenvalues of \( A \).

Proof. Suppose \( f = \prod_{i=1}^{n}(\lambda + \beta_i) \). Then taking supertropical eigenvectors \( v_i \) for which

\[
Av_i = \beta_i v_i + \text{ghost},
\]

we have \( AU_A = U_A D_A + \text{ghost} \), implying

\[
U_A^\top AU_A = U_A^\top U_A D_A + U_A^\top \text{ghost} = I_{U_A} D_A + \text{ghost} = (I + \text{ghost}) D_A + \text{ghost} = D_A + \text{ghost}.
\]

\( \square \)

References
[1] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebra, 2008. Preprint.
[2] Ambrosio, Andrea Proof of Birkhoff – von Neumann Theorem. PlanetMath.org, 2005.
[3] G. Birkhoff, Tres observaciones sobre el algebra lineal. Univ. Nac. Tucuman Rev, Ser. A, no. 5, pp. 147-151, (1946)
[4] R. A. Brualdi and H. J. Ryser. Combinatorial matrix theory. Cambridge University Press, 1991.
[5] M. Develin, F. Santos and B. Sturmfels, On the tropical rank of a matrix, Discrete and Computational Geometry, (eds. J.E. Goodman, J. Pach and E. Welzl), Mathematical Sciences Research Institute Publications, Volume 52 (2005), pp. 213-242, Cambridge University Press.
[6] R. Diestel, Graph theory, New York: Springer, New York, 1997.
[7] A. M. Gibbons. Algorithmic Graph Theory. Cambridge Univ. Press, Cambridge, UK, 1985.
[8] J. Golan. The theory of semirings with applications in mathematics and theoretical computer science, volume 54. Longman Sci & Tech., 1992.
[9] B. Hall. Lie Groups, Lie Algebras, and Representations: an Elementary Introduction. Springer, New York, 2003.
[10] Z. Izhakian. Tropical arithmetic and algebra of tropical matrices, Communications in Algebra, vol 37, pp 1445-1468, 2009. (preprint at arXiv:math.AG/0505458).
[11] Z. Izhakian. The tropical rank of a matrix. Preprint at arXiv:math.AC/0604208, 2005.
[12] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical semirings and supervaluations, Preprint, 2009.
[13] Z. Izhakian and L. Rowen. Supertropical algebra. Preprint at arXiv:0806.1175, 2007.
[14] Z. Izhakian and L. Rowen. The tropical rank of a matrix. Communications in Algebra, to appear.
[15] G. Lallement. Semigroups and Combinatorial Applications. John Wiley & Sons, Inc., New York, USA, 1979.
[16] M. Slone. Proof of Hall’s Marriage Theorem. PlanetMath. Org, 2002.
[17] J. Richter-Gebert, B. Sturmfels, and T. Theobald, First steps in tropical geometry, in "Idempotent Mathematics and Mathematical Physics", Proceedings Vienna 2003, (editors G.L. Litvinov and V.P. Maslov), Contemporary Mathematics 377 (2005) 289-317, American Mathematical Society,

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