Block-Coordinate Minimization for Large SDPs with Block-Diagonal Constraints

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Abstract. The so-called Burer-Monteiro method is a well-studied technique for solving large-scale semidefinite programs (SDPs) via low-rank factorization. The main idea is to solve rank-restricted, albeit non-convex, surrogates instead of the original SDP. Recent works have shown that, in an important class of SDPs with an elegant geometric structure, one can find globally optimal solutions to the SDP by finding rank-deficient second-order critical points of an unconstrained Riemannian optimization problem. Hence, in such problems, the Burer-Monteiro approach can provide a scalable and reliable alternative to interior-point methods that scale poorly. Among various Riemannian optimization methods proposed, block-coordinate minimization (BCM) is of particular interest due to its simplicity. Erdogdu et al. [7] in their recent work proposed BCM for problems over the Cartesian product of unit spheres and provided global convergence rate estimates for the algorithm. This report extends the BCM algorithm and the global convergence rate analysis of Erdogdu et al. [7] from problems over the Cartesian product of unit spheres to the Cartesian product of Stiefel manifolds. The latter more general setting has important applications such as synchronization over the special orthogonal (SO) and special Euclidean (SE) groups.

Notations and Preliminaries
For a $dn \times dn$ matrix $X$ formed by blocks of size $d \times d$, $X_{[i,j]} \in \mathbb{R}^{d \times d}$ refers to its $(i,j)$-th block. $\| \cdot \|_*$ is the matrix nuclear norm. $\langle \cdot , \cdot \rangle$ is the Frobenius inner product. The Stiefel manifold is defined as $\text{St}(d, r) \triangleq \{ Y \in \mathbb{R}^{r \times d} : Y^TY = I_d \}$ and is equipped with the Riemannian metric induced by its $\mathbb{R}^{r \times d}$ embedding. Define the product manifold $\text{St}(d, r)^n \triangleq \{ Y = [Y_1 \ Y_2 \ldots Y_n] \in \mathbb{R}^{r \times dn} : Y_i \in \text{St}(d, r) \}$. Given a function $F : \text{St}(d, r)^n \rightarrow \mathbb{R}$, $\nabla F$ and $\text{grad} F$ are the ambient Euclidean and Riemannian gradient of $F$, respectively. We have that $\text{grad} F(Y) = \text{Proj}_Y(\nabla F(Y))$, where $\text{Proj}$ is the orthogonal projection operator onto the tangent space of the product manifold at $Y \in \text{St}(d, r)^n$ [2].

1 Problem Formulation
We are interested in solving large-scale SDPs with the following structure.
Problem 1 (SDP). Let $Q \in \mathbb{R}^{dn \times dn}$.

\[
\begin{align*}
\text{minimize} & \quad \trace(QX), \\
\text{subject to} & \quad X_{[i,i]} = I_d, \forall i \in [n], \\
& \quad X \succeq 0.
\end{align*}
\]

Problem 1 arises frequently as SDP relaxations of important non-convex or combinatorial optimization problems. A comprehensive list of applications can be found in [5]. In particular, for $d = 1$, notable examples include Max-Cut [9], MAP inference on graphical models [8], and community detection [3]. For $d > 1$, examples include pose-graph optimization (synchronization over the special Euclidean group) [13], rotation synchronization (synchronization over the special orthogonal group) [15], phase synchronization [14], and spherical embedding [17].

In many of these applications, the problem size (i.e., $d \times n$) can be quite large. This is the case, e.g., in pose-graph optimization where $d \times n$ typically well exceeds $10^4$. Interior-point methods are often impractical for solving such large-scale instances of Problem 1. To address this issue, Burer and Monteiro [6] propose to impose a low-rank factorization on the decision variable $X$, and solve the resulting non-convex problem; see Problem 2 below. Note that Problem 1 is equivalent to Problem 2 after introducing an additional non-convex constraint $\text{rank}(X) \leq r$.

Problem 2 (rank-restricted SDP, Riemannian optimization form).

\[
\begin{align*}
\text{minimize} & \quad \trace(QY^TY), \\
\text{subject to} & \quad Y \in \text{St}(d, r)^n.
\end{align*}
\]

Remark 1. Following [7, 16], we assume without loss of generality that $Q$ is symmetric and $Q_{[i,i]} = 0_d, \forall i \in [n]$. If $Q$ is not symmetric, replacing it with its symmetric part $\frac{1}{2}(Q + Q^\top)$ does not change the objective value, since $X = Y^TY$ is symmetric. In addition, setting each $Q_{[i,i]} = 0_d$ only decreases the objective by a constant value $\trace(Q_{[i,i]})$.

It has been established that if $Y$ is a rank-deficient second-order critical point of Problem 2, then it is a global minimizer of Problem 2, and furthermore $X = Y^TY$ will be a globally optimal solution for Problem 1; see [4] and references therein. This has motivated the use of Riemannian optimization methods for finding local solutions to the unconstrained Riemannian optimization problem (Problem 2). Block-coordinate minimization (BCM) methods, among others, have been proposed for solving Problem 2 on the Cartesian product of unit spheres $\text{St}(1, r)^n$ [7, 11, 16]. This paper closely follows the recent work of Erdoğdu et al. [7] and extends the BCM algorithm and its global convergence rate analysis to cover the more general case of Cartesian product of Stiefel manifolds $\text{St}(d, r)^n$ with arbitrary $d \leq r$. 
2 Algorithm

In this section, we present the generalized BCM algorithm for the product manifold \( St(d, r)^n \). Note that in Problem 2, the overall cost function can be decomposed as \( F(Y) = \text{tr}(QY^T Y) = \sum_{i=1}^{n} F_i(Y_i) \), where the contribution of a single variable \( Y_i \) is, \( F_i(Y_i) = \langle G_i, Y_i \rangle \), \( G_i \triangleq \sum_{j \in [n] \setminus i} Y_j Q_{[j,i]} \).

We make the crucial observation that after fixing all other variables \( Y_j \) where \( j \neq i \), the problem of minimizing \( F_i(Y_i) \) subject to \( Y_i \in St(d, r) \) admits a closed-form solution. Let \( U_i \Sigma_i V_{i}^T \) be the singular value decomposition of \( -G_i \). Then the optimal \( Y_i \) is given by \( Y_i^\star = U_i I_{r \times d} V_{i}^T \) [12, Theorem 2.1]. This motivates the following block-coordinate method, outlined in Algorithm 1, for solving Problem 2. Following [7], we consider two sampling schemes related to the choice of \( p_i \):

- Uniform sampling: \( p_i = 1/n, \forall i \in [n] \).
- Importance sampling: \( p_i = \|G_i\|_\star / \sum_{j=1}^{n} \|G_j\|_\star, \forall i \in [n] \).

**Algorithm 1** BLOCK-COORDINATE MINIMIZATION (BCM) FOR PROBLEM 2

1: Initialize \( Y_i^0 \in St(r, d), \forall i \in [n] \). Compute \( G_i^0 = \sum_{j \in [n] \setminus i} Y_j Q_{[j,i]}, \forall i \in [n] \).
2: for \( k = 0, 1, \ldots \) do
3: Randomly select \( i_k = i \) with probability \( p_i, \forall i \in [n] \).
4: \( Y_{ik}^{k+1} \leftarrow U_{ik} I_{r \times d} V_{ik}^T \), where \( U_{ik} \Sigma_{ik} V_{ik}^T \) is the SVD of \( -G_{ik} \).
5: \( G_{ik}^{k+1} \leftarrow G_{ik}^k - Y_{ik}^k Q_{[ik,i]} + Y_{ik}^{k+1} Q_{[ik,i]}, \forall i \neq i_k \).
6: end for

3 Global Convergence Analysis

In this section, we extend the global convergence rate analysis provided by Erdogdu et al. [7]. Specifically, we show that the established global convergence rate estimates for BCM and the associated proof techniques can be generalized from the Cartesian product of spheres \( St(1, r)^n \) to the Cartesian product of Stiefel manifolds \( St(d, r)^n \), for any \( d \leq r \). Interestingly, our results reduce exactly to the corresponding technical statements in [7] after setting \( d = 1 \).

Recall that in Algorithm 1, each iteration minimizes the contribution of a single variable block \( Y_{ik} \) to the cost function while keeping the other blocks fixed. Thus the sequence of iterates generated by BCM will yield nonincreasing cost values. The following lemma confirms that this is indeed the case, and furthermore, quantifies the cost decrease in terms of \( Y_i \)'s and \( G_i \)'s.
Lemma 1. Define $F(Y) = \text{tr}(QY^TY) = \sum_{i=1}^n F_i(Y_i) = \sum_{i=1}^n \langle G_i, Y_i \rangle$. Let $Y^k$ denote the value of $Y$ at the $k$th iteration of Algorithm 1. Each iteration of BCM yields a descent on the cost function:

$$F(Y^{k+1}) - F(Y^k) = -2 \left( \|G^k_{i_k}\|_* + \langle G^k_{i_k}, Y^k_{i_k} \rangle \right) \leq 0. \quad (4)$$

In addition to the cost decrease, another key quantity we must investigate is the Frobenius norm of the Riemannian gradient $\|\text{grad} F(Y)\|_F$.

Lemma 2. For $i \in [n]$, define $A_i \triangleq \frac{1}{2}(Y_i^T G_i + G_i^T Y_i)$. Then,

$$\|\text{grad} F(Y)\|_F^2 = 4 \sum_{i=1}^n (\|G_i\|_F^2 - \|A_i\|_F^2). \quad (5)$$

We are now ready to give the main theoretical results of this section. Theorem 1 and Corollary 1 below establish the global sublinear convergence rate of BCM with uniform sampling and importance sampling, respectively. In particular, after sufficient number of iterations, BCM with either sampling strategy will produce a solution with arbitrarily small gradient norm in expectation. In other words, Algorithm 1 is guaranteed to converge to a first-order critical point of Problem 2 in expectation. We note that these results generalize the global convergence proof and rate estimates given in Theorem 3.2 and Corollary 3.3 in [7].

Theorem 1. Let $F^*$ be the optimal value of Problem 2. Define $C_1(Q) \triangleq \max_i \sum_{j \neq i} \|Q_{[j,i]}\|_*$. Then, for any $K \geq \lceil 2dnC_1(Q)(F(Y^0) - F^*)/\epsilon \rceil$ iterations, BCM with uniform sampling is guaranteed to return a solution $Y^k$, for some $k \in [K-1]$, such that $\mathbb{E} \|\text{grad} F(Y^k)\|^2_F \leq \epsilon$.

Corollary 1. Let $F^*$ be the optimal value of Problem 2. Define $C_2(Q) \triangleq \sum_{i \neq j} \|Q_{[i,j]}\|_*$. Then, for any $K \geq \lceil 2dC_2(Q)(F(Y^0) - F^*)/\epsilon \rceil$ iterations, BCM with importance sampling is guaranteed to return a solution $Y^k$, for some $k \in [K-1]$, such that $\mathbb{E} \|\text{grad} F(Y^k)\|^2_F \leq \epsilon$.

Remark 2. We note that the BCM algorithm and the analysis presented in this section readily extend to Cartesian product of Stiefel manifolds with different number of orthonormal frames, i.e., $\text{St}(d_1, r) \times \text{St}(d_2, r) \times \ldots \times \text{St}(d_n, r)$, where $d_i \leq r, \forall i \in [n]$. A similar global sublinear convergence rate can be proved, with small changes in the constants (for example, $d$ will be replaced with $\max_{i \in [n]} d_i$).
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Appendix A  Proofs

Proof (Lemma 1). Starting from the definition of $F(Y^{k+1})$,

$$F(Y^{k+1}) = \sum_{i=1}^{n} \langle Y_{ik}^{k+1}, G_{ik}^{k+1} \rangle$$

$$= \langle Y_{ik}^{k+1}, G_{ik}^{k+1} \rangle + \sum_{i \neq ik} \langle Y_{ik}^{k+1}, G_{ik}^{k+1} \rangle$$

$$= \langle Y_{ik}^{k+1}, G_{ik}^{k} \rangle + \sum_{i \neq ik} \langle Y_{ik}^{k}, G_{ik}^{k+1} \rangle$$

$$= -\|G_{ik}^{k}\|_{*} + \sum_{i \neq ik} \langle Y_{ik}^{k}, G_{ik}^{k} - Y_{ik}^{k}Q_{[i,k]_i} + Y_{ik}^{k+1}Q_{[i,k]_i} \rangle \quad (6)$$

$$= -\|G_{ik}^{k}\|_{*} + \sum_{i \neq ik} \langle Y_{ik}^{k}, G_{ik}^{k} \rangle + \sum_{i \neq ik} \langle Y_{ik}^{k}, (Y_{ik}^{k+1} - Y_{ik}^{k})Q_{[i,k]_i} \rangle$$

$$= -\|G_{ik}^{k}\|_{*} + \sum_{i \neq ik} \langle Y_{ik}^{k}, G_{ik}^{k} \rangle + \sum_{i \neq ik} \langle Y_{ik}^{k}Q_{[i,k]_i}, Y_{ik}^{k+1} - Y_{ik}^{k} \rangle$$

$$= -\|G_{ik}^{k}\|_{*} + \sum_{i \neq ik} \langle Y_{ik}^{k}, G_{ik}^{k} \rangle + (G_{ik}^{k}, Y_{ik}^{k+1} - Y_{ik}^{k})$$

$$= -2\|G_{ik}^{k}\|_{*} - 2\langle G_{ik}^{k}, Y_{ik}^{k} \rangle + \sum_{i} \langle Y_{ik}^{k}, G_{ik}^{k} \rangle$$

$$= -2\|G_{ik}^{k}\|_{*} - 2\langle G_{ik}^{k}, Y_{ik}^{k} \rangle + F(Y^{k}).$$

Moreover, using the general von Neumann trace theorem, we can upper bound the absolute value of the second term in the above expression:

$$|\langle G_{ik}^{k}, Y_{ik}^{k} \rangle| \leq \sum_{i=1}^{d} \sigma_{i}(G_{ik}^{k}) \sigma_{i}(Y_{ik}^{k})$$

$$= \sum_{i=1}^{d} \sigma_{i}(G_{ik}^{k}) = \|G_{ik}^{k}\|_{*} \quad (7)$$

This ensures that each iteration of the BCM algorithm yields a descent on the objective value:

$$F(Y^{k+1}) - F(Y^{k}) = -2(\|G_{ik}^{k}\|_{*} + \langle G_{ik}^{k}, Y_{ik}^{k} \rangle) \leq 0.$$
Proof (Lemma 2). Starting from the definition of the Riemannian gradient,
\[
\text{grad} \ F(Y) = \text{Proj}_Y(\nabla F(Y)) \\
= 2 \text{Proj}_Y(YQ) \\
= 2 \left\{ YQ - Y \frac{1}{2} \text{BlockDiag}_d(Y^T YQ + QY^T Y) \right\}
\] 
(8)

Expand the terms in \( \frac{1}{2} \text{BlockDiag}_d(Y^T YQ + QY^T Y) \). The \( i \)-th diagonal block is given by,
\[
\frac{1}{2} \text{BlockDiag}_d(Y^T YQ + QY^T Y)_{[i,i]} = \frac{1}{2} \sum_{j=1}^{n} (Y_i^T Y_j Q_{[j,i]} + Q_{[i,j]} Y_j^T Y_i) \\
= \frac{1}{2} Y_i^T (\sum_{j \neq i} Y_j Q_{[j,i]}) + \frac{1}{2} (\sum_{j \neq i} Q_{[i,j]} Y_j^T) Y_i \\
= \frac{1}{2} Y_i^T G_i + \frac{1}{2} G_i^T Y_i \\
= A_i
\] 
(9)

Above, we have used the assumption that \( Q_{[i,i]} = 0_d \). Using (9) we can now simplify (8),
\[
\text{grad} \ F(Y) = 2Y \begin{bmatrix} -A_1 & Q_{[1,2]} & \ldots & Q_{[1,n]} \\ Q_{[2,1]} & -A_2 & \ldots & Q_{[2,n]} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{[n,1]} & Q_{[n,2]} & \ldots & -A_n \end{bmatrix}
\] 
(10)

View \( \text{grad} \ F(Y) \in \mathbb{R}^{r \times dn} \) as a row block matrix with blocks of size \( r \times d \). Then, the \([1,i]-\)th block can be expressed as,
\[
\text{grad} \ F(Y)_{[1,i]} = 2(-Y_i A_i + G_i)
\] 
(11)

The squared Frobenius norm is given by,
\[
\| \text{grad} \ F(Y)_{[1,i]} \|_F^2 = 4 \text{tr}((-Y_i A_i + G_i)^T (-Y_i A_i + G_i)) \\
= 4 \text{tr}(A_i^T Y_i^T Y_i A_i + G_i^T G_i - G_i^T Y_i A_i - A_i^T Y_i^T G_i) \\
= 4(\|A_i\|_F^2 + \|G_i\|_F^2 - \text{tr}((G_i^T Y_i + Y_i^T G_i) A_i)) \\
= 4(\|A_i\|_F^2 + \|G_i\|_F^2 - 2 \|A_i\|_F^2) \\
= 4(\|G_i\|_F^2 - \|A_i\|_F^2).
\] 
(12)

Finally, the squared Frobenius norm of the entire gradient is simply the sum over the individual blocks,
\[
\| \text{grad} \ F(Y) \|_F^2 = 4 \sum_{i=1}^{n} (\|G_i\|_F^2 - \|A_i\|_F^2).
\] 
(13)
Proof (Theorem 1). We first note that, since $F$ is a continuous function and St$(d, r)^n$ is a compact manifold, $F^*$ must be bounded from below. From Lemma 1,

$$F(Y^k) - F(Y^{k+1}) = 2\left(\|G^k_i\|_* + \langle G^k_i, Y^k_i \rangle\right)$$

$$= 2\left(\|G^k_i\|^2_F + \|G^k_i\|_* \langle G^k_i, Y^k_i \rangle\right)$$

$$\geq 2\left(\|G^k_i\|^2_F - \langle G^k_i, Y^k_i \rangle^2\right)$$

$$= 2\left(\|G^k_i\|^2_F - \mathrm{tr}(A^k_i)^2\right)$$

$$\geq 2\left(\|G^k_i\|^2_F - \|A^k_i\|^2_F\right)$$

(14)

where the first inequality follows from the general von Neumann trace theorem, and the second inequality follows from Lemma 5. Given $Y^k$, take the expectation over the next iteration of BCM,

$$F(Y^k) - E_k F(Y^{k+1}) \geq 2 \sum_{i=1}^n P_i \frac{\|G^k_i\|^2_F - \|A^k_i\|^2_{F'}}{\|G^k_i\|_*}$$

$$\geq \frac{2}{n \max_i \|G^k_i\|_*} \sum_{i=1}^n (\|G^k_i\|^2_F - \|A^k_i\|^2_{F'})$$

$$= \frac{1}{2n \max_i \|G^k_i\|_*} \|\mathrm{grad} F(Y^k)\|^2_{F'}$$

(15)

where the second equality follows from Lemma 2. The last inequality holds because each $\|G^k_i\|_*$ can be upper bounded as,

$$\|G^k_i\|_* = \|\sum_{j \neq i} Y^k_j Q_{[j,i]}\|_*$$

$$\leq \sum_{j \neq i} \|Y^k_j Q_{[j,i]}\|_*$$

$$\leq \sum_{j \neq i} \|Y^k_j\|_* \|Q_{[j,i]}\|_*$$

$$= d \sum_{j \neq i} \|Q_{[j,i]}\|_*$$

$$\leq d C_1(Q)$$

(16)
To prove the theorem, suppose that \( \mathbb{E} \| \nabla F(Y^k) \|_F^2 > \epsilon, \forall k \in [K - 1] \) for some integer \( K \). Then we have,

\[
F(Y^0) - F^* \geq F(Y^0) - \mathbb{E} F(Y^K) \\
= \sum_{k=0}^{K-1} \mathbb{E} [F(Y^k) - F(Y^{k+1})] \\
= \sum_{k=0}^{K-1} \mathbb{E} [F(Y^k) - \mathbb{E}_k F(Y^{k+1})] \\
\geq \frac{1}{2dnC_1(Q)} \sum_{k=0}^{K-1} \mathbb{E} \| \nabla F(Y^k) \|_F^2 \\
> \frac{K \epsilon}{2dnC_1(Q)} \tag{17}
\]

By contradiction, since \( F^* \) is bounded from below, the algorithm returns a solution with \( \| \nabla F(Y^k) \|_F^2 \leq \epsilon \), for some \( k \in [K - 1] \), provided that

\[
K > \frac{2dnC_1(Q)(F(Y^0) - F^*)}{\epsilon} \tag{18}
\]
Appendix B  Miscellaneous Lemmas

Lemma 3. Let $M \in \mathbb{R}^{n \times n}$ have ordered singular values $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ and eigenvalues $\lambda_1, \ldots, \lambda_n$.

(a) For any $p > 0$,
\[ \sum_{i=1}^{n} |\lambda_i|^p \leq \sum_{i=1}^{n} \sigma_i^p \tag{19} \]

(b) \[ \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \leq \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j \tag{20} \]

Proof.
(a) This is a special case of \cite[Theorem 3.3.13(b)]{10} with $k = n$.
(b) From \cite[Theorem 2.16]{18}, the compound matrix $\wedge^2 M$ has eigenvalues $\lambda_i \lambda_j$ and singular values $\sigma_i \sigma_j$. Applying (19) on $\wedge^2 M$ with $p = 1$ gives the desired inequality.\footnote{The authors thank Darij Grinberg \cite{1} for his help with the proof.}

Lemma 4. Let $G \in \mathbb{R}^{r \times d}, Y \in \text{St}(d, r)$. Let $\sigma_i(\cdot)$, $\lambda_i(\cdot)$ return the $i$th algebraically largest singular value and eigenvalue, respectively. For $i \in [d]$,
\[ \sigma_i(Y^T G) \leq \sigma_i(G) \tag{21} \]

Proof. To show (21), we can equivalently show that,
\[ \lambda_i(G^T Y Y^T G) \leq \lambda_i(G^T G) \tag{22} \]

We make use of the min-max characterization of eigenvalues,
\[ \lambda_i(G^T Y Y^T G) = \max_{\dim(S)=i} \min_{0 \neq v \in S} \frac{v^T G^T Y Y^T G v}{v^T v} \]
\[ = \max_{\dim(S)=i} \min_{0 \neq v \in S} \frac{v^T G^T Y Y^T G v}{v^T G^T G v} \cdot \frac{v^T G^T G v}{v^T v} \]
\[ \leq \max_{\dim(S)=i} \min_{0 \neq v \in S} \|Y^T\| \frac{v^T G^T G v}{v^T v} \]
\[ = \max_{\dim(S)=i} \min_{0 \neq v \in S} \frac{v^T G^T G v}{v^T v} \]
\[ = \lambda_i(G^T G) \tag{23} \]

where the operator norm satisfies $\|Y^T\| = 1$.\footnote{The authors thank Darij Grinberg \cite{1} for his help with the proof.}
Lemma 5. Let $G \in \mathbb{R}^{r \times d}, Y \in \text{St}(d, r)$, and define $A = \frac{1}{2}(G^T Y + Y^T G)$.

$$\text{tr}(A)^2 - \|A\|_F^2 \leq \|G\|_+^2 - \|G\|_F^2$$  \hspace{1cm} (24)

Proof. First, note that

$$\|A\|_F^2 = \frac{1}{2} \text{tr}(G^T Y G^T Y) + \frac{1}{2} \|G^T Y\|_F^2$$

$$= \frac{1}{2} \sum_{i=1}^{d} \lambda_i^2(G^T Y) + \frac{1}{2} \sum_{i=1}^{d} \sigma_i^2(G^T Y)$$

$$\geq \text{tr}(G^T Y G^T Y)$$  \hspace{1cm} (25)

where the inequality holds by invoking (19) with $p=2$. Starting from (25), the left hand side of (24) can be upper bounded by,

$$\text{tr}(A)^2 - \|A\|_F^2 \leq \text{tr}(G^T Y)^2 - \text{tr}(G^T Y G^T Y)$$

$$= 2 \sum_{1 \leq i < j \leq d} \lambda_i(G^T Y)\lambda_j(G^T Y)$$

$$\leq 2 \sum_{1 \leq i < j \leq d} |\lambda_i(G^T Y)\lambda_j(G^T Y)|$$

$$\leq 2 \sum_{1 \leq i < j \leq d} \sigma_i(G^T Y)\sigma_j(G^T Y)$$

$$\leq 2 \sum_{1 \leq i < j \leq d} \sigma_i(G)\sigma_j(G)$$

$$= \|G\|_+^2 - \|G\|_F^2$$  \hspace{1cm} (26)

Above, the third inequality uses part (b) of Lemma 3, and the fourth inequality uses Lemma 4.