ON THE FINITE GENERATION OF FIBER RING OF INVARIANT JET DIFFERENTIALS

MOHAMMAD REZA RAHMATI

Abstract. We consider invariant jet differentials, in complex hyperbolicity. The main result of this paper concerns the finite generation of the ring of invariant k-jets as a generalization of a conjecture which was first mentioned in [5] and later studied in [1, 11, 12, 21]. The conjecture in [5] states that the fiber ring of the space of invariant jet differentials of a projective manifold is finitely generated on the regular locus, i.e. the normalized ring of invariant k-jets is finitely generated. Berczi-Kirwan has a partial attempt toward the question in [1] (2012), but the conjecture remains open. We prove this conjecture independently. Our method is different and quite new to this question. The analytic automorphism group of regular k-jets of holomorphic curves on a projective variety X is a non-reductive subgroup of the general linear group GL_kC. In this case, the proof of the Chevalley theorem on the invariant polynomials in the fiber rings falls into difficulties. Different methods are required for the proof of the finite generation of the ring of invariants. We employ some techniques of algebraic Lie groups (not necessarily reductive) together with primary results obtained by Berczi and Kirwan (2012) to prove the finite generation of the ring of invariant jets. We study two actions of the algebraic group G_k, the first its adjoint action on its Lie algebra and the second the above action on the space of k-jets. We prove both of these two actions have generic stabilizers and obtain various structure theorems and results on the associated quotient maps. Finally, we present a result on the structure of the representation ring of G_k.

1. Introduction

In this text, we initiate a general method to study the (non-reductive) invariant theory of the transformation group of k-jets of entire holomorphic curves on a projective variety X, under the reparametrizations of the domain (C, 0), [1, 11, 12, 15]. Our method uses the theory of generic stabilizers of Lie group actions on manifolds, [2, 5, 6, 31, 24, 27, 30, 32, 34, 35]. Specifically, we prove several structure theorems for the Lie group G_k related to its adjoint action on g_k = LieG_k and also its action on the space of k-jets of complex curves on X. We study both of these actions and show both actions have generic stabilizers. Several immediate consequences of this property have been presented in the paper as new results. From the well-known facts it is not hard to see that the G_k-action on V = J_kC^n is stable. We use a theorem of Vust [see Theorem 2.13 on the module of coinvariants Mor(g_k, V^*) and a result of Igusa [17, 24, 30] [see Theorem 2.13] to prove the finite generation of the ring of invariant k-jets. The idea is that by the Theorem of Vust the module of coinvariants has a finite basis. One can try to apply

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The moduli of $k$-jets of holomorphic curves (denoted by $J_kX$) on a projective variety $X$ (maybe singular) is one of the interesting geometric constructions that reflects some of the topological properties of the variety. As a classifying space, it can be embedded in a Grassmann variety and, consequently in a projective space. Its compactification and boundary points have been one of the subjects of study for mathematicians in the last decades [1, 13, 20]. For geometric reasons, one should look for a coordinate-free definition of the $k$-jets on a projective variety. One considers moduli of invariant $k$-jets of complex curves independent of the parametrizing variable in $\mathbb{C}$. The latter moduli space is a quotient space of the ordinary moduli of $k$-jets by a non-reductive subgroup of the Lie group $GL_k\mathbb{C}$. The ring of regular functions on the space of invariant $k$-jets and the stalk of this ring at a regular jet is meaningful to consider, [1, 11, 12, 13]. We prove that the stalk of this local ring is finitely generated at a regular point.

1.1. Motivation and related works. Since the work of Green and Griffiths [15], developed by Demailly [11, 12], Diverio, Merker, and Rousseau in [13], the action of the local automorphism group $G_k$ of $(\mathbb{C}, 0)$ on the bundle $J_kTX$ of $k$-jets of holomorphic curves $f: \mathbb{C} \to X$ in a complex projective manifold $X$ has been a focus of the investigation. The transformation group of $k$-jets under reparametrization of the curve is a non-reductive subgroup $G_k = \mathbb{C}^* \ltimes U_k$ of $GL_k(\mathbb{C})$, where $U_k$ is the unipotent radical consisting of upper-triangular $k \times k$ matrices of a certain type. Unlike the reductive case, here, one can not deduce the Chevalley kind of theorems. Besides, it is still unknown if the fiber ring of regular invariant functions under the action of $G_k$ or the action of its unipotent part is finitely generated. This conjecture has been attempted in [1] but is still open. Theorem 4.7 proves this well-known conjecture mentioned in [13]. We prove a more general statement for the whole coordinate ring of $J_k\mathbb{C}^n/G_k$. In general, there are many relations in the ring $\mathbb{C}[J_k]^{G_k}$. A major reference exploring many of these relations is the original work of J. Merker [21] following the works of Demailly and others. We fundamentally use the notation and techniques in [23, 24, 45] and employ various results from the theory of algebraic group (not necessarily reductive) actions on algebraic varieties, [30].

1.2. The problem and contribution. Assume $X$ is a projective manifold. The Lie group $G_k$ of reparametrizations of a holomorphic curve $f: \mathbb{C} \to X$ is the group of matrices of the form,

\[
G_k := \begin{bmatrix}
    a_1 & a_2 & a_3 & \ldots & a_k \\
    0 & a_1^2 & 2a_1a_2 & \ldots & a_1a_{k-1} + \ldots + a_{k-1}a_1 \\
    0 & 0 & a_1^3 & \ldots & 3a_1^2a_{k-2} + \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & a_1^k,
\end{bmatrix}
\]

where $0 \neq a_1 \in \mathbb{C}^*$ and all others $a_i \in \mathbb{C}$. The formula for the entries of the matrix is

\[
[G_k]_{ij} = \sum_{s_1 + s_2 + \ldots + s_i = j} a_{s_1} \ldots a_{s_i}
\]

where $s_i \geq 1$. The Lie group $G_k$ is a subgroup of $GL_k\mathbb{C}$ and can be written as $G_k = \mathbb{C}^* \ltimes U_k$, where $U_k$ is the unipotent radical and sits inside $SL_k$. The group
there are 5 independent variables, \( f \)

the algebra of invariants is just \( Demailly \) \([11]\) to prove the Kobayashi hyperbolicity conjecture \([18]\).

invariant jets play an essential role in the strategy introduced by Green-Griffiths \((1.7)\)

Then the ring of invariant polynomials \( \text{Conjecture } 1.1. \)

In this text, we attend to prove the following open question.

By replacing \((1.4)\) in \((1.3)\) we have:

\[
(1.5) \quad f(t) = (a_1 t + a_2 t^2 + \ldots + a_k t^k) f'(0) + \frac{(a_1 t + a_2 t^2 + \ldots + a_k t^k)^2}{2!} f''(0) \\
+ \ldots + \frac{(a_1 t + a_2 t^2 + \ldots + a_k t^k)^k}{k!} f^{(k)}(0), \quad \text{mod } t^{k+1}.
\]

Regrouping the terms shows that the following matrix multiplication defines the action of \( G_k \) on the \( k \)-jets,

\[
(1.6) \quad [f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!].
\]

The unipotent elements correspond to the matrices with \( a_1 = 1 \). The exponents along the diagonal elements arise for weights under the reparametrization \( \phi(t) : t \mapsto \lambda t \) in the Taylor series of \( f \). The action of \( G_k \) on the generic vectors \( f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k! \) induces an action of \( G_k \) on the polynomial ring \( \mathbb{C}[f'(0), f''(0), \ldots, f^{(k)}(0)]^{G_k} \), where we have considered \( f^{(k)}(0) \) as a germ of variable.

In this text, we attend to prove the following open question.

**Conjecture 1.1.** Assume \( f = (f_1, \ldots, f_n) \) and consider \( f^{(k)} \) as a formal vector. Then the ring of invariant polynomials

\[
\mathcal{O}(J_k \mathbb{C}^n)^{G_k} = \mathbb{C}[f'(0), f''(0), \ldots, f^{(k)}(0)]^{G_k}
\]

is finitely generated.

The problem of determining the finite generation of the localized complex, graded algebras \([\mathcal{O}(J_k)]^G_x\) where \( x \) is a regular jet, has been a long-time open problem. The above conjecture is somewhat more general and stronger conjecture. The invariant jets play an essential role in the strategy introduced by Green-Griffiths \([13]\) and Demailly \([11]\) to prove the Kobayashi hyperbolicity conjecture \([18]\).

**Example 1.2.** In general, there are many relations in the ring \( \mathbb{C}[J_k]^{G_k} \). For \( k = 1 \) the algebra of invariants is just \( \mathbb{C}[f'_1, f'_2, \ldots, f'_n] \). For \( k = 2 \) the algebra \( E_{2,n} \) is generated by \( f'_1 \) and the order=2 Wronskians \( f'_1 f''_j - f'_j f''_1 \). For \( k = 3 \) and \( n = 2 \) there are 5 independent variables,

\[
f'_1, f'_2, \quad f'_1 f''_2 - f'_2 f''_1, \quad [f'_1 f''_j - f'_j f''_1, f'_1], \quad [f'_1 f''_j - f'_j f''_1, f'_2],
\]

where the bracket operation is defined formally by \([p, q] = nq \cdot \partial p - mp \cdot \partial q\), for \( p \) and \( q \) polynomials of order \( n \) and \( m \) respectively, (see section \([3]\) for the order of jet variables). In the next dimension \( n = 3 \) for jets of the same order \( k = 3 \), the Demailly–Semple algebra \( E_{3,3} \) is generated by 16 mutually independent invariants, Rousseau, \([32]\). A major reference exploring the basic invariants and their relations,
is the original work of J. Merker following the works of Demailly, Rousseau, and others.

We initiate a new method to study the non-reductive invariant theory of the Lie group $G_k$. We stress several results on the structure theory of $G_k$ and its Lie algebra. Our method employs the theory of generic stabilizers in the action of Lie groups on algebraic varieties. In particular, we prove that the adjoint action of $G_k$ on its Lie algebra and the action $G_k$ on $J_k\mathbb{C}^n$ have generic stabilizers [see Theorems 4.1 and 4.4]. Several immediate consequences of these results have been proved in Section 4: Theorems 4.2 and 4.5 stress these results. We employ a fundamental theorem of Igusa [see [24, 30]] and some techniques of the module of coinvariants to answer the Conjecture 1.1 resolved in Theorem 4.7 in the paper. All the Theorems in section 4 are our contributions and new results.

1.3. Content. The remainder of this paper is organized as follows. Section 2 presents some fundamental aspects of the invariant theory of algebraic groups (not necessarily reductive Lie groups) and their Lie algebras. Section 3 contains basic definitions of the jet bundles on projective manifolds and the Lie group of transformations of $k$-jets of holomorphic curves under reparametrizations of their domains, namely $G_k$. Section 4 consists of the paper’s main results on the (non-reductive) invariant theory of the Lie group $G_k$; all the material of this section is the paper’s new results and contributions. Moreover, a proof of the well-known conjecture on finite generation of the total fiber ring of invariant $k$-jets of entire curves on a projective manifold is given in such a section.

2. Preliminaries on non-reductive Lie groups and algebras

This section explains some general facts on the invariant theory of algebraic groups. We do not stress that the Lie group $G$ be reductive. Where ever this assumption is necessary, we have specifically mentioned it. The primary purpose of this section is to provide some preliminary materials that are used in Section 4.

2.1. Invariant theory of non-reductive Lie groups. Let $G$ be an algebraic group that acts on an affine variety $X$. Denote by $\mathbb{C}[X]^G$ the algebra of $G$-regular invariant functions on $X$. If $\mathbb{C}[X]^G$ is finitely generated, then we set

\[(2.1) \quad X//G := \text{Spec}(\mathbb{C}[X]^G),\]

and we have the quotient morphism

\[(2.2) \quad \pi : X \longrightarrow X//G\]

which corresponds to the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. Now, denote by $G_x$ the stabilizer of the point $x \in X$. Set $\mathfrak{g} = \text{Lie}(G)$ and

\[(2.3) \quad \mathfrak{g}_x = \text{Lie}(G_x) \subset \mathfrak{g}.\]

Assume $G$ is an affine algebraic group acting on $X$, then by a theorem of Rosenlicht (see Theorem 2.1 below [24, 34, 35]), there is a dense $G$-stable subset $U \subset X$ such that the functions in $\mathbb{C}(X)^G$ separate the $G$-orbits in $U$. In this case, one has

\[(2.4) \quad \text{trdegree}(\mathbb{C}[X]^G) = \dim X - \text{Max}_x \dim(G_x)\]

We formalize this in the following theorem.
Theorem 2.1. (M. Rosenlicht) [33, 34, 24 theorem 1.1] Let $K \subset \mathbb{C}(X)^G$ be a subfield. Then, $K = \mathbb{C}(X)^G$ if $K$ separates the $G$-orbits in a dense open subset of $X$.

Theorem 2.1 motivates the following definition, which plays a crucial role in this text.

Definition 2.2. If in an open dense subset $U$ of $X$, all the stabilizers $g_x$ are $G$-conjugate, we say the action of $G$ on $X$ has a generic stabilizer and call the points in $U$ to be generic. Similarly, one defines by $G_x$ the generic isotropy subgroups.

The existence of a generic isotropy group implies the existence of a generic stabilizer. The converse is also true [31, 32, 30]. Let us define the set of regular points by,

$$X^{\text{reg}} = \{ x \in X \mid \dim G_x = \max_x \dim G.z \}.$$  

The definition (2.5) depends on the algebraic group $G$. Sometimes we denote this by $G$-regular points. The generic stabilizers always exist if $G$ is reductive and $X$ is a smooth variety. We list some results on the existence of generic stabilizers, in case the algebraic group $G$ is reductive, [see [30] section 7],

- If $G$ is reductive and $X$ is an irreducible affine variety on which $G$ acts, then the generic stabilizers exist [Richardson-Luna [31, 32]].
- If $G$ is reductive, $X$ is affine, and the action is stable, then the generic stabilizers exist.
- If $G$ is reductive and $X$ is irreducible smooth projective variety, and $X^G \neq \emptyset$, then the generic stabilizers exist.
- If $G$ is reductive and $X$ irreducible smooth projective variety with $\text{Pic}(X) = \mathbb{Z}$, then the generic stabilizers exist.
- If $G$ is connected and $X$ is an arbitrary irreducible variety, then there are generic stabilizers of the action of a Borel and a unipotent subgroup of $G$, [Brion-Luna-Vust [5, 45]].

We wish to consider the generic stabilizer property in a non-reductive case. In general, we have the following theorem due to Richardson.

Theorem 2.3. (R. W. Richardson) [34 page 229] Let $G$ be an algebraic group acting on an irreducible algebraic variety $X$. For $x \in X$, let $U_x, L_x$ be the unipotent radical and Levi subgroup of $G_x$. There are non-singular subvarieties $X_1, ..., X_n$ of $X$ such that

- $X = \bigcup_j X_j$.
- $X_j$ is open in $X \setminus \bigcup_{i=1}^{j-1} X_i$.
- For any $j$ and any $x, y \in X_j$ the groups $L_x, L_y$ are conjugate in $G$.
- The projection of the set $\{(x, u) \mid x \in X_j, u \in U_x\} \subset X_j \times G$ on $X_j$ is smooth for any $j$ (has surjective differential).

In the Theorem 2.3 the strata $X_j$ coincides with the set of all $G$-orbits of dimension $j$. Thus the $X_j$ are invariant under the $G$ action, cf. [31] page 228. The theorem states that for a general action of an algebraic group $G$ on an algebraic variety $X$, we have a stratification of the variety into disjoint smooth subvarieties $X_j$, where on each $X_j$ the restriction of quotient mapping has generic stabilizers.
With only these assumptions, one still can not deduce if \( C \) is finitely generated. However, if each \( X_j \) may satisfy a codimension two condition (C.2.C), one can proceed more for this result.

Assume \( G \) is a complex algebraic group (not necessarily reductive) and,\[
\rho : G \rightarrow GL(V)
\]
is a finite dimensional representation of \( G \), and \( \varrho : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \) is the corresponding representation of \( \mathfrak{g} \). If \( v \in V \) let define\[
V^{\varrho_v} = \{ x \in V \mid \varrho_v.x = 0 \}.
\]
If \( \Omega \subset V \), then set\[
N(\Omega) = \{ g \in G \mid g.\Omega \subset \Omega \}, \quad Z(\Omega) = \{ g \in G \mid g.x = x, \quad \forall x \in \Omega \}.
\]

**Theorem 2.4** ([24] Lemma 1.2, Lemma 1.3, prop. 1.4). Assume \( \rho : G \rightarrow GL(V) \) is a representation and \( \Omega = V^{\varrho_v} \). We have the following:

- \( \text{Lie } Z(\Omega) = \mathfrak{g}_x \).
- \( N(\Omega) = N_G(Z(\Omega)) = N_G(\mathfrak{g}_x) \).
- \( Z(\Omega) \leq N(\Omega) \).
- If \( x \in \Omega^{\text{reg}} \), then \( G.x \cap \Omega = N(\Omega).x \).
- \( \mathfrak{g}.x \cap \Omega = \mathfrak{n}_G(\mathfrak{g}_v).x \), holds for \( x \in \Omega^{\text{reg}} \).
- \( Y = G.\Omega \) is a \( G \)-stable irreducible subvariety of \( V \).
- The restriction map yields an isomorphism
  \[
  \mathbb{C}(Y)^G \xrightarrow{\cong} \mathbb{C}(\Omega)^{N(\Omega)/Z(\Omega)}.
  \]
- The induced map on algebras gives an embedding,
  \[
  \mathbb{C}[Y]^G \longrightarrow \mathbb{C}[\Omega]^{N(\Omega)/Z(\Omega)}.
  \]
  which in general is not onto.

The group \( W = N(\Omega)/Z(\Omega) \) is called the Weyl group of the action of \( G \) on \( V \). It can be an infinite group. We want to characterize the property of having a generic stabilizer for the action of \( G \) on a vector space \( V \) and propose to state Chevalley-type theorems.

**Proposition 2.5** ((Elashvili) [14], [24] lemma 1.6, see also [30] sec. 7). If \( v \in V \), then \( G.V^{\varrho_v} \) is dense in \( V \) iff

\[
V = \mathfrak{g}.v + V^{\varrho_v}.
\]

In order that the group \( G_v \) be a generic isotropy, it is necessary and sufficient that

- the set \( \{ u \in V^{G_v} \mid G_v = G_u \} \) be dense in \( V^{G_v} \).
- \( V = \mathfrak{g}.v + V^{G_v} \).

The last criterion in Proposition 2.5 follows by Definition 2.2 and the map \( G \times V^{G_v} \rightarrow V \) has constant rank on the set \( G \times \{ u \in V^{G_v} \mid G_v = G_u \} \). Differentiating this argument yields that, in order for the tangent algebra \( \mathfrak{g}_v \) to be a generic stabilizer is necessary and sufficient that:

- the set \( \{ u \in V^{\varrho_v} \mid \mathfrak{g}_v = \mathfrak{g}_u \} \) be dense in \( V^{\varrho_v} \).
- \( V = \mathfrak{g}.v + V^{\varrho_v} \).
where the argument follows by analyzing the map $G \times V^\mathfrak{v} \rightarrow V$. The existence of a non-trivial generic stabilizer yields a Chevalley-type theorem for the field of invariants. If $v \in V$ is a generic point and $\Omega = V^\mathfrak{v}$ then the last item in Theorem 2.4 gives the desired claim.

**Theorem 2.6.** Consider the adjoint representation,

\begin{equation}
\text{Ad} : G \rightarrow GL(\mathfrak{g}), \quad g \mapsto \text{Ad}(g) : (x \mapsto g.x.g^{-1}).
\end{equation}

If $G$ is reductive, then the adjoint representation has the following properties:

- It is self dual with $\mathfrak{h}$ as a generic stabilizer.
- The algebra $\mathbb{C}[X]^G$ is a polynomial ring.
- The restriction homomorphism induces an isomorphism [Chevalley Theorem, see [24, 30]].

\begin{equation}
\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W.
\end{equation}

- The quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/G$ is equi-dimensional.
- The fiber $\pi^{-1}(0)$ is the union of finitely many orbits.

These properties may fail if $G$ is replaced with an arbitrary (possibly non-reductive) Lie group. Specifically, the first item is refused to be true. The coadjoint representation of a non-reductive group shows different properties than the adjoint one. Let

\begin{equation}
ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})
\end{equation}

be the derivative of Ad at zero. In this case the stabilizer $\mathfrak{g}_x = \mathfrak{z}_\mathfrak{g}(x)$, is the centralizer of $x$. A point $x \in \mathfrak{g}$ is generic iff [cf. [24]],

\begin{equation}
[\mathfrak{g}, x] + \mathfrak{g}_{\mathfrak{z}_\mathfrak{g}(x)} = \mathfrak{g}.
\end{equation}

Recall that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a nilpotent subalgebra such that $\mathfrak{n}_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$. Cartan subalgebras always exist, and all are conjugate under $G$. [38].

**Proposition 2.7** ([24] props. 2.1 and 2.5). The adjoint action of the Lie group $G$ on its Lie algebra $\mathfrak{g}$ has a generic stabilizer if and only if either of the following equivalent conditions holds:

- There exists $x \in \mathfrak{g}$ such that the centralizer $\mathfrak{z}_\mathfrak{g}(x)$ is commutative and

\begin{equation}
[\mathfrak{g}, x] \oplus \mathfrak{z}_\mathfrak{g}(x) = \mathfrak{g}
\end{equation}

holds.

- The Cartan subalgebras of $\mathfrak{g}$ are commutative.

If $x \in \mathfrak{g}$ is generic, then $\mathfrak{n}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(x)) = \mathfrak{z}_\mathfrak{g}(x)$, and the commutativity of $\mathfrak{h}$ implies $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(x)$. We shall assume that a generic point has a generic isotropy group (always).

**Theorem 2.8** ([24] theorem 2.6). Assume $\mathfrak{g}$ has a generic stabilizer (centralizer) under the adjoint action, and $x \in \mathfrak{g}$ be a generic point where $Z_G(x)$ is generic. Then,

- $Z(\mathfrak{z}_\mathfrak{g}(x)) = Z_G(x)$,
- $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}(\mathfrak{z}_\mathfrak{g}(x))^W$,
- The Weyl group $W = N_G(\mathfrak{z}_\mathfrak{g}(x))/Z_G(x)$ is finite.

Theorem 2.8 provides a way to establish the finite generation of $\mathbb{C}[\mathfrak{g}]^G$ when the Weyl group $W$ is finite.
Remark 2.9. The last property may fail for the coadjoint representation, That is, the $\text{Ad}^* : G \to \text{Aut}(\mathfrak{g}^*)$ is defined by,

\[(2.17) \quad (\text{Ad}^*_g \mu, Y) = (\mu, \text{Ad}_{g^{-1}}Y) \quad g \in G, Y \in \mathfrak{g}, \mu \in \mathfrak{g}^*,\]

where $\langle \mu, Y \rangle$ denotes the value of the linear functional $\mu$ on the vector $Y$. The Weyl group of the coadjoint representation may not be finite, [see [24] for details].

Assume that the algebraic group $G$ acts on a variety $X$, and let $V$ be a $G$-module. Consider the module of covariants $(\mathbb{C}[X] \otimes V)^G$ and define,

\[(2.18) \quad \text{ev}_x : (\mathbb{C}[X] \otimes V)^G \to V, \quad F \otimes v \mapsto F(x)v.\]

A natural question is if the image of the map $\text{ev}_x$ captures the information on $V^{G_x}$. The following theorem answers this question under some codimension 2-condition for the complement of the $G$-orbit of $x$ in its closure. The map $\text{ev}_x$ plays a crucial role in the study of the basic invariants of the ring $\mathbb{C}[V]^{G_x}$.

\textbf{Theorem 2.10} ([27] theorem 1). Let $G$ be a reductive algebraic group acting on an affine variety $X$. If for some $x \in X$, the closure orbit $G.x$ is a normal subvariety of $X$, and $\text{codim}_{\mathbb{C}[X]}(G.x \setminus G.x) \geq 2$, then image$(\text{ev}_x) = V^{G_x}$.

We denote by $\mathfrak{g}^{\text{reg}}$ the regular elements of $\mathfrak{g}$ under the adjoint action of $G$ (cf. [2.3]). Let $T$ be a maximal torus of $G$. Assume $G$ and $V$ be as in Theorem 2.10. Let denote by $\text{Mor}_G(X, V)$ the set of all $G$-equivariant maps from $X$ to $V$. $\text{Mor}_G(X, V)$, has a natural structure of $\mathbb{C}[X]^G$-module. This $\mathbb{C}[X]^G$-module is said to be the module of covariants (of type $V$). It is easily seen that $\text{Mor}_G(X, V)$ can be identified with $(\mathbb{C}[X] \otimes V)^G$. For $F \in \text{Mor}_G(X, V)$, define the polynomial $f \in \mathbb{C}[X \times V]^G$ by the rule $f(x, \xi) = \langle F(x), \xi \rangle$, where $\langle ., . \rangle : V \times V^* \to \mathbb{C}$ is the natural pairing. Then $f \in \mathbb{C}[X \times V]^G$, cf. [24, 25]. With the above nomenclature, we have the following result due to Kostant [18].

\textbf{Theorem 2.11.} (Kostant) [18] page 385] Assume $G$ is a reductive algebraic group acting on the vector space $V$ ($V$ is a $G$-module). Then,

1. $\dim V^{G_x} = \dim V^T$ for any $x \in \mathfrak{g}^{\text{reg}}$.
2. The module $\text{Mor}_G(\mathfrak{g}, V)$ is freely generated over $\mathbb{C}[\mathfrak{g}]^G$ and has rank equal to $\dim V^T$.

We need a version of the Theorems 2.10 and 2.11 in a non-reductive case. In fact, under some mild conditions, these theorems can be extended to the non-reductive case. We make the following definition.

\textbf{Definition 2.12} ([30] page 156, [24]). The action of the algebraic group $G$ on a variety $X$ is said to be stable if there exists in $X$ a non-empty open subset such that the orbits of its points are closed. An action of $G$ on the vector space $V$ is said to be stable if the union of the closed orbits is dense in $V$.

If the action of $G$ on $V$ is stable, a generic stabilizer is reductive, and $\mathbb{C}(V)^G$ is the quotient field of $\mathbb{C}[V]^G$, [42, 30]. We have the following theorem due to Vust, see [43] Chap. 3.

\textbf{Theorem 2.13} (T. Vust). ([43] Chap. 3, [24] page 6] Assume the action of the algebraic group $G$ on the vector space $V$ is stable, and has a generic stabilizer $H = g.i.g(G : V)$ (generic isotropy group of the action of $G$ on $V$). Then $H$ is reductive and the rank of the $\mathbb{C}[\mathfrak{g}]^G$-module $\text{Mor}_G(\mathfrak{g}, V)$ is $\dim V^H$. 
Remark 2.11. If $G$ is reductive the claim of Theorem 2.13 will trivially hold. Thus the special value of the Vust Theorem appears when $G$ is non-reductive, but has generic stabilizers, 24, 27. For the reader’s convenience, we outline a proof: first, rank $\text{Mor}_G(g, V) = \dim V^H$, because, if $F \in \text{Mor}_G(g, V)$ is $G$-equivariant, then $F(v) \in V^{G_v}$ for any $v \in V$. Applying this to the open set of $G$-generic elements in $V$, we obtain rank $\text{Mor}_G(g, V) \leq \dim V^H$. Second, the "evaluation" map $ev : \text{Mor}_G(g, V) \to V^{G_v}, F \to F(v)$, is onto whenever $Gv = \overline{Gv}$, see 27, Theorem 1. Hence if generic $G$-orbits in $V$ are closed (and isomorphic to $G/H$), then the upper bound $\dim V^H$ is attained, see 25 for a more detailed discussion. Theorem 2.14 is crucial in this paper and plays an important role in the proof of our main result, namely Theorem 4.5.

If $f \in \mathbb{C}[V]$, then, the differential of $f$ defines a map $v \mapsto df(v)$ in $\text{Hom}(V, V^*)$. If the function $f$ belong to $\mathbb{C}[V]^G$, then the resulting map lies in $\text{Mor}_G(V, V^*)$, i.e., $G$-equivariant morphisms $V \to V^*$, [sec. 4].

Theorem 2.15 (T. Vust 45, 24 theorem 4.5). Assume $G$ acts on $V$ such that $\mathbb{C}[V]^G$ is a polynomial algebra, and the quotient map $\pi : V \to V/G$ is equidimensional. Let $H$ be a generic isotropy group for the action, and suppose $N_G(H)/H$ is finite. Let $f_1, ..., f_r$ is a set of basic invariants generating $\mathbb{C}[V]^G$. Then, $\text{Mor}_G(V, V^*)$ is freely generated by $df_1, ..., df_r$.

Remark 2.16. In 24 the properties of the invariant ring $\mathbb{C}[V]^G$ have been studied by considering the module of covariants $[\mathbb{C}[X] \otimes V]^G$. There have been determined the exact conditions under which it is finitely generated over $\mathbb{C}[g]^G$. The basic invariants of $\mathbb{C}[g]$ is determined via Theorem 2.15.

We need the following definition, from 24, to state the Igusa theorem.

Definition 2.17. 24, 17 Assume $X$ is an algebraic variety. We say that an open subset $\Omega \subset X$ is big if $X \setminus \Omega$ contains no divisor.

Theorem 2.18. [Igusa 14, 24 lemma 6.1, 20] Theorem 4.12 page 190] Let $G$ be an algebraic group that regularly acts on an algebraic affine variety $X$. Suppose $S$ is an integrally closed finitely generated subalgebra of $\mathbb{C}[X]^G$ such that the morphism $\pi : X \to \text{Spec}(S)$ satisfies:

- The fibers of $\pi$ over a dense open subset of $\text{Spec}(S)$ contains a dense $G$-orbit.
- $\text{im}(\pi)$ contains a big open subset of $\text{Spec}(S)$.

Then, $S = \mathbb{C}[X]^G$. In particular, the algebra $\mathbb{C}[X]^G$ is finitely generated.

The first condition in the Igusa Theorem applies to the Rosenlicht Theorem and implies $\mathbb{C}[x]^G = \mathbb{C}[\text{Spec}(S)]$. The first condition implies that $\pi$ is the dominant morphism; hence the induced map on the structure sheaves is a monomorphism (denoted by $\pi^*$). The second condition employs the Richardson lemma in birational geometry, 3 3.2 Lemme 1. The condition implies that if $f \in \mathbb{C}[X]^G$ is constant on the fibers of $\pi$ in general position, it has the form $f = \pi^*g$ for $g \in \mathbb{C}(Y)$. In a normal variety, the set that a rational function is not defined is either empty or has codimension 1. The second condition can be replaced by "For any point $s$ of some non-empty open subset of $S$, the fiber $\pi^{-1}(s)$ contains exactly one closed orbit", cf. 24, 70 loc. cit.
2.2. Representation ring of algebraic groups. The material of this subsection is to introduce a Chevalley-type theorem for the representation ring of compact connected semisimple algebraic groups. We generalize these results for the group of reparametrization of $k$-jets of analytic curves in Theorem 4.11. We recall some basic facts on the Grothendieck ring of $G$-modules for general connected complex Lie group $G$. Let $G$ be a nontrivial connected complex Lie group, and $T \subset G$ be a maximal torus. We denote the representation ring of $G$ by $R(G) = \bigoplus \mathbb{Z}[\rho]$, the free abelian group generated by the irreducible representations $\rho$ of $G$. It has a ring structure by tensor product. The ring $R(G)$ is a subring of the ring of class functions on $G$ by the formation of characters. The Weyl group $W = W(G, T) = N_G(T)/T$ acts naturally on $R(T)$. The restriction of class functions to $T$ makes $R(G)$ a subring of $R(T)^W$. Let $r = \dim(T)$ and $\Phi = \Phi(G, T)$ be the associated root system with $B = \{\alpha_1, \ldots, \alpha_r\}$ a basis of $\Phi$. Assume $\varpi_1, \ldots, \varpi_r$ are the fundamental weights with respect to $(G, T, B)$, and $V_{\varpi_1}, \ldots, V_{\varpi_r}$ are the fundamental representations of $G$. For any dominant weight $\lambda = \sum_j n_j \varpi_j$, the irreducible representation $V_{\lambda}$ of $G$, with highest weight $\lambda$, occurs exactly once in $V_{\varpi_1} \otimes \cdots \otimes V_{\varpi_r}$ due to the theorem of the highest weight. The following theorem is well known.

**Theorem 2.19** (see [8] page 291). If $G$ is semisimple and simply connected, then $R(G)$ is a polynomial ring on the fundamental representations. More specifically, the map $\mathbb{Z}[X_1, \ldots, X_r] \to R(G)$ defined by $X_j \mapsto [V_{\varpi_j}]$ is an isomorphism.

The representation ring of a compact connected Lie group satisfies the following finite-type theorem.

**Theorem 2.20** ([8] page 293). If $G$ is a compact connected Lie group, then $R(G) = R(T)^W$.

In the case of semisimple and connected group $G$, one notes that the character group of $T$ is $X(T) = P = \bigoplus \mathbb{Z} \varpi_j$. Thus, the representation ring of $T$ satisfies $R(T) = \mathbb{Z}[X(T)] = \mathbb{Z}[P]$. Then, $R(T)^W$ is a free $\mathbb{Z}$-module with a basis given by the sums along the $W$-orbit in $P$. Because of the simple transitive action of the Weyl group $W$ on the Weyl chambers in $X(T)_R$, we have $\overline{C(B)} \cap P = \mathbb{Z}_{\geq 0} \varpi_j$, where $C(B)$ is the Weyl chamber corresponding to the basis $B$. In other words, any $W$-orbit in $P$ meets a dominant weight. Besides, in the composition,

$$Z[X_1, \ldots, X_r] \to R(G) \hookrightarrow R(T)^W,$$

and by considering the above observation, one deduces that the map $Z[X_1, \ldots, X_r] \to R(T)^W$ is surjective, cf. [8].

3. The bundle of $k$-jets of holomorphic curves

This section reviews basic definitions and facts about the bundle of $k$-jets of holomorphic curves on a projective variety. In addition, this section introduces the basic notations in use in the following sections.

The references for this section are [11], [12], [13]. Let $X$ be a complex projective manifold of dimension $n$. The $k$-jet bundle,

$$J_k \rightarrow X$$

is the bundle of germs of curves in $X$ whose fiber at $x \in X$ is the set of equivalence classes of germs of holomorphic curves.

$$f : (\mathbb{C}, 0) \rightarrow (X, x)$$
with equivalence relation,

\[(3.3) \quad f \equiv_k g \iff f^{(j)}(0) = g^{(j)}(0), \quad 0 \leq j \leq k.\]

By choosing local holomorphic coordinates around \(x\), the fiber \(J_{k,x}\) elements can be represented by the Taylor expansion:

\[(3.4) \quad f(t) = tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1}).\]

Setting \(f = (f_1, \ldots, f_n)\) on an open neighborhood of \(0 \in \mathbb{C}\), the fiber can be given as

\[(3.5) \quad J_{k,x} = \{(f'(0), \ldots, f^{(k)}(0))\} = \mathbb{C}^n.\]

We can regard \(J_{k}X\) obtained from a sequence of vector bundles \(X_k \to X_{k-1} \to \ldots \to X\), parametrized by the vectors of the components in (3.3). The projective bundle \(J_{k}(X)/\mathbb{C}^*\) is called the Green-Griffiths bundle; [see [11, 12, 15] for details]. Let’s \(G_k\) be the group of local analytic reparametrizations of \((\mathbb{C}, 0)\) of the form:

\[(3.6) \quad t \mapsto \phi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*.\]

Applying the change of coordinates, \(\phi\) on a fiber element produces action on the \(k\)-jets given as the following matrix multiplication,

\[(3.7) \quad \left[ f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k! \right] = \left[ \begin{array}{cccc} a_1 & a_2 & a_3 & \ldots & a_k \\ 0 & a_2^2 & 2a_1a_2 & \ldots & a_1a_{k-1} + \ldots + a_{k-1}a_1 \\ 0 & 0 & a_3^3 & \ldots & 3a_1^2a_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & a_k^k \end{array} \right].\]

Then, \(G_k\) is the subgroup of \(GL_k \mathbb{C}\) of transformation \(k \times k\)-matrices of the form appearing in (3.7). Now, denote the unipotent elements corresponding to the matrices with \(\alpha_1 = 1\) by \(U_k\). There is a short exact sequence,

\[(3.8) \quad 1 \to U_k \to G_k \to \mathbb{C}^* \to 0.\]

The subgroup \(U_k\) is the unipotent radical of \(G_k\), see [1]. The action of \(\mathbb{C}^*\) on \(k\)-jets is

\[(3.9) \quad \lambda \times \left( f'(0), \ldots, f^{(k)}(0) \right) \mapsto \left( \lambda f'(0), \ldots, \lambda^k f^{(k)}(0) \right).\]

An alternative way to define these bundles is through their ring of sections. In a more general setup, we may consider the jets tangent to \(V\), a holomorphic subbundle of the tangent bundle \(TX\) of \(X\). Green and Griffiths [15] introduce the vector bundle:

\[(3.10) \quad E_{k,m}^{GG}V^* \to X\]

whose fibers are polynomials,

\[(3.11) \quad Q(z, \xi) = \sum_{\alpha} A_{\alpha}(z) \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}\]

of weighted degree \(m\), where \(z\) is a local variable on \(X\) and \(\xi_i\) is the variable on the fibers of

\[(3.12) \quad X_i \to X_{i-1} \quad (1 \leq i \leq n).\]

One can identify \(E_{k,m}^{GG}V^* = \bigoplus_{m} E_{k,m}^{GG}V^*\) with the projective bundle of germs of \(k\)-jets of the analytic maps \(f : (\mathbb{C}, 0) \to (X, x)\) tangent to \(V\).
attached to $\xi_1, \ldots, \xi_k, \ldots$ are 1, 2, ..., $k$, ..., respectively, and the total weighted degree of a homogeneous germ is defined as

$$|\alpha| = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k.$$  

The $\mathbb{C}^*$-action on the space of germs of holomorphic maps $f : \mathbb{C} \to X$ is defined as in (3.9). Then, we have

$$(3.13) E_{k}^{GG}V^* = \bigoplus_m E_{k,m}^{GG}V^* = J_kV/\mathbb{C}^*.$$  

**Remark 3.1.** The symbol $V^*$ used in $E_{k,m}^{GG}V^*$ is only a notation reminding the duality appearing in the fact that $S^*V = \text{Sym}^V V$ can be understood as polynomial algebra on the basis of $V^*$. The analytic structure on $E_{k,m}^{GG}V^*$ comes from the fact that one can arrange the germs to depend holomorphically on their initial values at 0.

One can consider the group of all analytic reparametrization automorphisms of $J_kV$ and the invariant bundle $(J_kV)^{G_k}$. We denote by $E_{k,m}V^*$ the Demailly-Semple bundle whose fiber at $x$ consists of $U_k$-invariant polynomials on the fiber coordinates of $J_kV$ at $x$ of weighted degree $m$. Also, set $E_k = \bigoplus_m E_{k,m}$ to be the Demailly-Semple bundle of graded algebras of invariant jets, [14, 19, 18, 16].

Recall that, a $k$-jet $f : \mathbb{C} \to X$ is regular if $f'(0) \neq 0$. We call such a jet non-degenerate when it has maximal rank in local coordinates.

**Proposition 3.2.** [7] There is a $G_k$-invariant algebraic morphism

$$\phi : J_k^{\text{reg}} \longrightarrow \text{Hom}(\mathbb{C}^k, \text{Sym}^{\leq k}\mathbb{C}^n)$$

that induces an injective map,

$$\Phi^{\text{Gr}} : J_k^{\text{reg}}/G_k \hookrightarrow \text{Grass}(k, \text{Sym}^{\leq k}\mathbb{C}^n)$$

given explicitly by

$$(3.15) (f', f'', \ldots, f^{(k)}) \longmapsto \left( f', f'' + (f')^2, \ldots, \sum_{i_1 + \ldots + i_s = d} \frac{1}{i_1! \ldots i_s!} f^{(i_1)} \ldots f^{(i_s)}, \ldots \right),$$

where we have identified $J_k\mathbb{C}^n$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by putting the coordinates of $f^{(j)}$ in the $j$-th column of a $n \times k$ matrix. The map $\phi$ also gives an injective map

$$\phi^{\text{Flag}} : J_k^{\text{reg}}/G_k \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k}\mathbb{C}^n)$$

defined explicitly by

$$(3.16) (f^{(1)}, f^{(2)}, \ldots, f^{(k)}) \longmapsto \left( \text{im}(\phi(f^{(1)})) \subset \text{im}(\phi(f^{(2)})) \subset \ldots \right),$$

where we have considered $f^{(j)} \subset \mathbb{C}^n$ and

$$(3.17) \text{Flag}_k(\mathbb{C}^n) = \{ F_0 \subset F_1 \subset \ldots \subset F_k \subset \mathbb{C}^n \}$$

of flags of length $k$. We also have

$$(3.18) \phi^{\text{Gr}} = \phi^{\text{Flag}} \circ \pi_k$$

where

$$(3.19) \pi_k : \text{Flag}_k(\text{Sym}^{\leq k}\mathbb{C}^n)) \longrightarrow \text{Grass}(k, \text{Sym}^{\leq k})$$
is the projection to the $k$-dimensional subspace. Besides, the composition of $\phi^Gr$ with the Plucker embedding,

$$\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n),$$

gives an embedding

$$\phi^\text{proj} : J^\text{reg}_k \hookrightarrow \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n).$$

**Definition 3.3.** [1] Let $e_1, ..., e_n$ be the standard basis of $\mathbb{C}^n$. Then, a basis of $\text{Sym}^{\leq k} \mathbb{C}^n$ consists of

$$\{e_{i_1} \cdots e_i \mid e_j \in \Pi_{\leq k}, \Pi_{\leq k} = \{ \{i_1 \leq i_2 \leq \cdots \leq i_s\} \} \}$$

and a basis of $\mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n)$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_i, i_1 \leq i_2 \leq \cdots \leq i_s, s \leq k\}.$$

Then, the set

$$\{e_{i_1} \wedge \cdots \wedge e_i \mid e_j \in \Pi_{\leq k}, \Pi_{\leq k} = \{i_1 \leq i_2 \leq \cdots \leq i_s\} \}$$

represents a basis of $\mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n)$. The corresponding coordinates of $x \in \text{Sym}^{\leq k} \mathbb{C}^n$ will be denoted by $x_{e_{i_1} \cdots e_i}$. Also, we denote by $A_{n,k} \subset \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n)$ the points whose projection on $\wedge^k \mathbb{C}^n$ is non-zero, i.e., at least one $x_{i_1 \cdots i_k} \neq 0$.

**Proposition 3.4.** [1] Assume $k \leq n$. Let

$$z_k = \phi^\text{proj}(e_1, ..., e_k) = [e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge (\sum e_{i_1} \cdots e_{i_k})].$$

The group $\text{GL}(n)$ acts on $z$ as follows. If $g \in \text{GL}(n)$ has the column vectors $v_1, ..., v_n$, then

$$g.z_k = \phi^\text{proj}(v_1, ..., v_n) = [v_1 \wedge (v_2 + v_1^2) \wedge \cdots \wedge (\sum v_{i_1} \cdots v_{i_k})].$$

Besides,

- If $n \geq k$, then the image of non-degenerate $k$-jets is the $\text{GL}(n)$ orbit of $z$, and it is denoted by $X_{n,k}$.
- If $n \geq k$, then the image of regular $k$-jets is a finite union of $\text{GL}(n)$-orbits, and it is denoted by $Y_{n,k}$.
- If $k > n$, then $X_{n,k}$ and $Y_{n,k}$ are $\text{GL}(n)$-invariant quasi-projective varieties, with no dense $\text{GL}(n)$-orbits.

Regular jets are an open subset $J_k \mathbb{C}^n$, as the condition defining the singularity is an algebraic equation. If $n > 1$, then one has

$$\text{codim}_{J_k \mathbb{C}^n}(J_k \mathbb{C}^n \setminus J^\text{reg}_k \mathbb{C}^n) \geq 2$$

cf. [1] loc. cit. We use this inequality to apply the Igusa theorem for the action of $G_k$ on $J_k \mathbb{C}^n$ in the next section.

**Remark 3.5** ([1], Theorems 6.1 and 6.2). One of the main results of [1] is the following. For $N$ large enough the stabilizer of $p_k = z_k \otimes e_1^\otimes N \in \wedge^k \text{Sym}^{\leq k} \mathbb{C}^k \otimes (\mathbb{C}^k)^\otimes N$ for the $SL_k \mathbb{C}$-action is $U_k$. Furthermore, the complement of the $SL_k$-orbit of $p_k$ in its closure has a codimension of at least two. Although this result supports the proof of our results better, we do not need to stand on that for the applications of the Igusa theorem, and the inequality (3.30) is enough in this case.
4. Main results: The invariant theory of $G_k$

In this section, we present the main contributions of this paper. All the theorems and their proof appear to be new in the literature. We consider the two actions of $G_k$ on $\mathfrak{g}_k$ (adjoint action), and also on $J_k C^n$, separately. We shall work with the notations introduced in Sections 2 and 3.

4.1. The adjoint action of $G_k$ on $\mathfrak{g}_k$. Let $\mathfrak{g}_k = \text{Lie}(G_k)$, and

\[ \text{Ad}_k : G_k \to GL(\mathfrak{g}_k) \]

be the adjoint representation. Also, set $\text{ad}_k : \mathfrak{g}_k \to \mathfrak{gl}(\mathfrak{g}_k)$ be its derivative, i.e., the corresponding representation of its Lie algebra (bracket). The decomposition $G_k = C^* \ltimes U_k$ implies:

\[ \mathfrak{g}_k = C \oplus \mathfrak{u}_k, \]

where $\mathfrak{u}_k = \text{Lie}(U_k)$. Our first theorem expresses the generic stabilizer property for the action (4.1).

**Theorem 4.1.** (Main result.) The adjoint action of $G_k$ on the Lie algebra $\mathfrak{g}_k$ has a generic stabilizer. In particular, the Cartan subalgebras of $\mathfrak{g}_k$ are commutative.

**Proof.** We first check that the Elashvili criterion holds for the adjoint action $\text{Ad}_k : G_k \to GL(\mathfrak{g}_k)$, that is,

\[ [\mathfrak{g}_k, x] + \mathfrak{j}_{\mathfrak{g}_k}(x) = \mathfrak{g}_k, \]

for some $x \in \mathfrak{g}_k$. One can check this in a way to find an $x \in \mathfrak{g}_k$, such that $\mathfrak{j}_{\mathfrak{g}_k}(x)$ is commutative and $[\mathfrak{g}_k, x] + \mathfrak{j}_{\mathfrak{g}_k}(x) = \mathfrak{g}_k$. The equation implies $\text{image}(\text{ad}(x)) = \text{image}(\text{ad}^2(x))$; this will hold as long as we choose $x$ to be a semisimple element. In this case, $\text{image}(\text{ad}(x))$ generates $\mathfrak{u}_k$. This is due to choose $u \in \mathfrak{u}_k$ in a suitable basis, then, $\text{ad}(x)(u) = \alpha_u u$ for a complex number $\alpha_u$. On the other hand, if $x$ is semisimple, $\mathfrak{j}_{\mathfrak{g}_k}(x)$ consists of diagonal elements. In the latter case, it follows that $\mathfrak{j}_{\mathfrak{g}_k}(x)$ is commutative. Theorem 4.1 and Proposition 2.1 complete the proof of having generic stabilizers, see [24] page 9. The proof of the last claim is also a consequence of Proposition 2.1 and is equivalent to the generic stabilizer property of adjoint action of $G_k$. In other words, $\mathfrak{j}_{\mathfrak{g}_k}(x) = H$ for $x \in \mathfrak{g}_k^{\text{reg}}$ is a Cartan subalgebra of $\mathfrak{g}_k$.

Using Theorem 4.1, we can prove the following Chevalley-type theorem for the adjoint action of $G_k$ on $\mathfrak{g}_k$, which appears as our following main result.

**Theorem 4.2.** (Main result.) Let $H$ be a generic stabilizer of the adjoint action of $G_k$ on $\mathfrak{g}_k$, (called generic centralizer in this case), and $\mathfrak{h} = \text{Lie}(H)$. Then, we have:

1. $\mathbb{C}(\mathfrak{g}_k)^{G_k} = \mathbb{C}(\mathfrak{h})^W$.
2. The Weyl group $W = N_{G_k}(H)/H$ (of the action) is finite.
3. $\mathbb{C}[\mathfrak{g}_k]^{G_k} = \mathbb{C}[\mathfrak{h}]^W$.
4. The invariant ring $\mathbb{C}[\mathfrak{g}_k]^{G_k}$ is finitely generated.

**Proof.** The first and the second items are direct consequences of Theorem 2.1 and the fact that $\mathfrak{j}_{\mathfrak{g}_k}(x) = H$ for $x \in \mathfrak{g}_k^{\text{reg}}$ is a Cartan subalgebra. For the third item,
we have the inclusion $\mathbb{C}[g_k]^{G_k} \subset \mathbb{C}[h]^W$. This is because for $U = g^x = g h = h$ the restriction map easily gives an inclusion, 

\[(4.4) \quad \mathbb{C}[g_k]^{G_k} \subset \mathbb{C}[U]^{N(U)/Z(U)}.\]

Now, the equality follows from the finiteness of the Weyl group $W = N(U)/Z(U)$. It also follows that $\mathbb{C}[g_k]^{G_k}$ is finitely generated, which is the last item. \hfill \Box

Below, we give an alternative and direct proof of the Theorem 4.2.

**Proof. (Alternative proof of Theorem 4.2)** Recall that $G_k$ is the semi-direct product of its unipotent radical $U_k$ with the multiplicative group $\mathbb{C}^*$ of scalar reparametrizations. Moreover, $\mathbb{C}^*$ acts linearly on the Lie algebra of $U_k$ with positive weights. So $\mathbb{C}^*$ is its own normalizer in $G_k$, and hence is a Cartan subgroup; the corresponding Weyl group is trivial. Likewise, $Lie(\mathbb{C}^*) = \mathbb{C}$ is a Cartan subalgebra of $Lie(G_k)$, with trivial Weyl group. Also, the algebra of $\mathbb{C}^*$ invariant polynomial functions on $Lie(G_k)$ is a polynomial ring in one variable generated by the projection $Lie(G_k) \rightarrow \mathbb{C}$ (the differential at the neutral element of the homomorphism $G_k \rightarrow \mathbb{C}^*$). Since this projection is $G_k$ invariant, it generates the algebra of $G_k$ invariants; in other terms, it is the quotient morphism. This morphism is smooth, with fiber at 0 (the null cone) being $Lie(U_k)$. \hfill \Box

Theorems 4.1 and 4.2 imply several important properties for the quotient space $g_k/G_k$ that we list in the next theorem.

**Theorem 4.3.** (Main result.) Under the adjoint action of $G_k$ on $g_k$, the following holds:

1. $\max_{x \in g_k} \dim G_k.x = \dim g_k - \dim g//G_k$.
2. The quotient map $\pi : g_k \rightarrow g_k//G_k$ is equi-dimensional.
3. Let $\Omega = \{x \in g_k \mid d_x \pi \text{ is onto}\}$, then, $g_k \setminus \Omega$ contains no divisor.
4. The null cone $\mathcal{R}_{G_k}(g_k) = \pi^{-1}(0)$ is an irreducible complete intersection of dimension $\dim g_k - \dim g_k//G_k$.

**Proof.** For any $d$ set $g_k^{(d)} = \{x \in g_k \mid \dim G_k.x \leq d\}$. Then $\emptyset \subset g_k^{(1)} \subset g_k^{(0)} \subset \ldots$, and each set $g_k^{(i)}$ is closed in $g_k$. For $m = \max_x \dim G_k.x$ the set $g_k^{(m)}$ coincides with $g_k$, and the set $g_k \setminus g_k^{(m-1)}$ is nonempty and open in $g_k$. Then the dimension of the orbit of a point in $g_k$ in general position is equal to $m$. Because $\dim G_k = \dim G_k.x + \dim G_{k.x}$, it follows that the stabilizer of a point of $g_k$ in general position has dimension $\dim G_{k.x} = \dim G_k - m$. When $G_{k.x}$ is generic the stabilizers of all the points in general positions are conjugate to $G_{k.x}$. This conjugacy holds on the open subset $g_k \setminus X_{m-1}$. By the Rosenlicht Theorem and identity 2.3 the transcendence degree of the field $\mathbb{C}(g_k)^{G_k}$ is equal to $\dim G_k - m$. This implies equality in item (1) because this transcendence degree is equal to $\dim g_k//G_k$.

The second item (2) is equivalent to the first item. Because the maximal dimension of the fibers $G_k.x$ of $\pi$ is equal to the dimension of the null cone $\mathcal{R}_{G_k}(g_k)$ i.e. the fiber that contains zero, and the minimal dimension is equal to the dimension of a typical fiber i.e. $\dim g_k - \dim g//G_k$. This shows that item (2) is equivalent to $\dim \mathcal{R}_{G_k}(g_k) = \dim g_k - \dim g_k//G_k$. Thus (1) and (2) are equivalent.

By what we said above $\Omega \supset g_k^{(d)}$. We prove item (3) using an argument of [30] in Theorem 3.11 at page 172. For a generic $x \in g_k$, the stabilizer satisfies $G_{k,x} = H$, where $H$ is a Cartan subgroup of $G_k$ and thus is a reductive subgroup.
This implies that $\mathbb{C}[G_k/G_k,\mathfrak{a}]$ is finitely generated. Then, the above-mentioned theorem of Grosshans implies that codim$(G_k,\mathfrak{a}) \geq 2$. Therefore, the regular points of $\mathfrak{g}_k$ have the property that codim$(\mathfrak{g}_k \setminus \mathfrak{g}_k^{\text{reg}}) \geq 2$. Because $\Omega \supset \mathfrak{g}_k^{\text{reg}}$. Thus, codim$(\mathfrak{g}_k \setminus \Omega) \geq 2$.

The null cone i.e. the fiber that contains zero is the set of common zero’s of all homogeneous invariants of positive degree. By Hilbert Nullstellensatz, algebraically independent homogeneous invariants $f_1, \ldots, f_s$ constitute a system of parameters of the algebra $\mathbb{C}[\mathfrak{g}_k]^{G_k}$ if and only if their set of common zero’s coincide the null cone. The ideal of the null cone is generated by the basic invariants of $\mathbb{C}[\mathfrak{g}_k]^{G_k}$, cf. [24], see also [30] page 196. The last item follows.

4.2. The action of $G_k$ on $J_k\mathbb{C}^n$. We want to prove similar properties for the action of $G_k$ on the space of $k$-jets. That is, to put $V = J_k\mathbb{C}^n$, $G = G_k$, $\mathfrak{g} = \mathfrak{g}_k = \text{Lie}(G_k)$. Let us consider the representation

\begin{equation}
\rho : G_k \rightarrow \text{GL}(J_k\mathbb{C}^n)
\end{equation}

be the natural action. Again, the first result is the generic stabilizer property for the action of $\rho$.

**Theorem 4.4.** (Main result.) The action of $G_k$ on $J_k\mathbb{C}^n$ has generic stabilizer.

**Proof.** We successively put the vectors’ entries in a $k$-jet i.e. the vectors $\xi_1 = f'(0), \xi_2 = f''(0), \ldots, \xi_k = f^{(k)}(0)$ in the matrix columns. Thus, a $k$-jet on $\mathbb{C}^n$ identifies with an element of $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$. Then stabilizing a $k$-jet by an element of $G_k$ can be shown as the following,

\begin{equation}
\begin{bmatrix}
\xi_{11} & \cdots & \xi_{1n} \\
\xi_{21} & \cdots & \xi_{2n} \\
\vdots & \ddots & \vdots \\
\xi_{k1} & \cdots & \xi_{kn}
\end{bmatrix}
\begin{bmatrix}
a_1 & a_2 & \cdots & a_k \\
0 & a_1^2 & \cdots & a_1a_{k-1} + \cdots + a_{k-1}a_1 \\
0 & 0 & \cdots & 3a_1^2a_{k-2} + \cdots \\
0 & 0 & \cdots & a_k
\end{bmatrix}
= 
\begin{bmatrix}
\xi_{11} & \cdots & \xi_{1n} \\
\xi_{21} & \cdots & \xi_{2n} \\
\vdots & \ddots & \vdots \\
\xi_{k1} & \cdots & \xi_{kn}
\end{bmatrix}
\end{equation}

where $\xi_i = (\xi_{i1}, \ldots, \xi_{in})$. On the open subset of regular jets where all $\xi_j \neq 0$, $j = 1, \ldots, k$ one can easily understand the conjugacy condition for the existence of a generic stabilizer. Let’s explain this in a basic manner. It is possible to take (transform) any regular $k$-jet $x$ to the one where $\xi_1 = [1,0,\ldots,0]$ by applying an element $g \in G_k$. Continuing this way one can apply successively elements of $G_k$ to make all $\xi_j = [1,0,\ldots,0]$. It follows that on the open set $\Omega = \{\xi_j \neq 0, \forall j = 1,2,\ldots,k\}$ the stabilizers of all elements are conjugate to the $k$-jet $x_0 = \{\xi_j = [1,0,\ldots,0], j = 1,2,\ldots,k\}$. This shows the existence of generic stabilizers.

One can repeat this proof on each open stratum defined as the various intersections on the complements of the set where ”exactly” one of the vectors $\xi_j$ is identically zero. □

**Corollary 4.5.** (Main result.) We have:

- The set $U = G_k[J_k\mathbb{C}^n]^{\mathfrak{g}_k}$ is dense in $J_k\mathbb{C}^n$.
- $\mathbb{C}(J_k\mathbb{C}^n/G_k) = \mathbb{C}((J_k\mathbb{C}^n)^{\mathfrak{g}_k})^W$, where $W = N(U)/Z(U)$ is the Weyl group of the action.

**Proof.** The proof is a consequence of Theorems [2.8] and [4.4] □
Remark 4.6. The above corollary does not assert anything on the Weyl group of the action of $G_k$ on $J_k \mathbb{C}^n$ to be finite. Thus we may not proceed the same as the adjoint action of $G_k$ in the previous subsection. However, we still have other tools that can be worked out to prove the finite generation of rings of invariants.

We now come to the main result of this paper. We want to use the literature above to prove the finite generation of the fiber ring of the moduli of $k$-jets. Our method uses the Igusa lemma and the Theorem 2.13 by Vust.

**Theorem 4.7.** [Main result. Finite generation of the coordinate ring of invariant jets] The algebra $\mathbb{C}[(J_k \mathbb{C}^n)^{G_k}]$ is finitely generated.

**Proof.** Consider the space $\mathfrak{g}_k \times J_k \mathbb{C}^n$ with the diagonal action of $G_k$, and $V = J_k \mathbb{C}^n = \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ considered as a vector space. First, the action of $G_k$ on $V$ is regular in the sense that it is defined by algebraic maps. From Theorem 4.3 the action of $G_k$ on $V$ has generic stabilizers. By the explanation of the proof of Theorem 4.3 or from [1], section 4 the orbits of the regular jets are closed algebraic sets [where the algebraic equations are given op. cit.]. This shows that the action of $G_k$ on $V$ is also stable [in the sense of Definition 2.1.2]. By Theorem 2.1.3 $\text{Mor}_{G_k}(\mathfrak{g}_k, V^*)$ is a free $\mathbb{C}[\mathfrak{g}_k]^{G_k}$-module of finite rank, say $m$. Let $f_1, \ldots, f_m$ be a basis for this module and $f_1, \ldots, f_m$ the corresponding $G_k$-invariants on $\mathfrak{g}_k \times J_k \mathbb{C}^n$, i.e.,

$$f_i(x, \xi) := \langle F_i(x), \xi \rangle, \quad \xi \in J_k \mathbb{C}^n.$$  

That is $f_i \in \mathbb{C}[\mathfrak{g}_k \times V]^{G_k}$. We prove that $\mathbb{C}[\mathfrak{g}_k \times J_k \mathbb{C}^n]^{G_k}$ is finitely generated by the coordinate functions on $\mathfrak{g}_k$ and the polynomials $f_i$, $i = 1, \ldots, m$.

We wish to apply Theorem 2.1.3 (Igusa lemma). Set

$$X_m = \{x \in \mathfrak{g}_k \mid \text{dim span}\{F_1(x), \ldots, F_m(x)\} = m\}.$$  

That is, $X_m$ is the set of those $x$, where the vectors $F_i(x) \in V^* = (J_k \mathbb{C}^n)^*$, $i = 1, \ldots, m$, are linearly independent. Then, $X_m \supset \mathfrak{g}_k^{reg}$ and is a big open subset of $\mathfrak{g}_k$. More precisely, codim $\mathfrak{g}_k(\mathfrak{g}_k \setminus X_m) \geq 2$ (see the argument in the proof of Theorems 4.3 and 4.10 item (3)). Let $x_1, \ldots, x_r$ be the coordinates on $\mathfrak{g}_k$, where $r = \text{dim } \mathfrak{g}_k$. Then $x_1, \ldots, x_r, f_1, \ldots, f_m$ are algebraically independent because their differentials are linearly independent on $X_m \times J_k \mathbb{C}^n$. Consider the mapping

$$\pi : \mathfrak{g}_k \times J_k \mathbb{C}^n \to \text{Spec } \mathbb{C}[x_1, \ldots, x_r, f_1, \ldots, f_m] = \mathbb{C}^{r+m},$$

$$\pi(x, \xi) = (x, f_1(x, \xi), \ldots, f_m(x, \xi))$$

(4.7)

We identify $\mathbb{C}^{r+m}$ with $\mathfrak{g}_k \times \mathbb{C}^m$. If $x = (x_1, \ldots, x_r) \in X_m$, then the $F_i(x)$'s are linearly independent in $(J_k \mathbb{C})^*$, so that the system

$$f_i(x, \xi) = \langle F_i(x), \xi \rangle = \beta_i, \quad i = 1, \ldots, m$$

has a solution $\xi \in J_k \mathbb{C}^n$ for any $m$-tuple $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$. Hence $\text{Image}(\pi) \supset X_m \times \mathbb{C}^m$, which means that $\text{Image}(\pi)$ contains a big open subset of $\mathbb{C}^{r+m}$. It follows from the above that

$$\mathfrak{g}_k^{reg} = X_m \cap \{y \in \mathfrak{g}_k \mid \text{dim } G_k \cdot y = r - m\}.$$  

Take $x \in \mathfrak{g}_k^{reg}$, and let $\xi \beta$ be a solution to the system $f_i(x, \xi) = \beta_i$. Then $\pi^{-1}(x, \beta) \ni (x, \xi \beta)$ and

$$\pi^{-1}(x, \beta) \ni G_k \cdot (x, \xi \beta), \quad x \in \mathfrak{g}_k^{reg}.$$
Since \( x \in X_m \), we have \( \dim \pi^{-1}(x, \beta) = r - m \). On the other hand, \( \dim [g_k, x] = r - m \), by the definition of \( g^{reg} \), cf. [24] page 9. Hence
\[
\pi^{-1}(x, \beta) = G_k \cdot (x, \xi_\beta), \quad x \in g_k^{reg}
\]
for dimension reason. Thus, a generic fiber of \( \pi \) is a \( G_k \)-orbit, and the Theorem 2.18 applies here. It follows that \( \mathbb{C}[g_k \times J_k \mathbb{C}^n]^{G_k} \) is finitely generated.

On the other hand because \( f_1, \ldots, f_m \) were already \( G_k \)-invariant we have,
\[
\mathbb{C}[g_k \times J_k \mathbb{C}^n]^{G_k} = \mathbb{C}[g_k]^{G_k}[f_1, \ldots, f_m]
\]
It follows that \( \mathbb{C}[J_k \mathbb{C}^n]^{G_k} = \mathbb{C}[\overline{f_1}, \ldots, \overline{f_m}] \), where \( \overline{f_i} \) is the projected image of \( f_i \) in the coordinate ring of \( \mathbb{C}[J_k \mathbb{C}^n]^{G_k} \), is finitely generated. \( \square \)

Remark 4.8. The technique applied in the above for the proof of Theorem 4.7 was frequently in use in the texts [25, 27, 24] on finite generation of invariants of \( g \rtimes V \). For instance, it is similarly applied in the proof of theorem 6.2 in [24]. But there \( G \) is reductive and a theorem of Kostant has been applied instead of the Vust Theorem 2.13. The codimension 2 argument also was supported by a theorem of Kostant. The argument with the Vust Theorem 2.13 appears in [25] with \( G \) to be an algebraic group and not necessarily reductive. To apply the Vust Theorem one needs also the stability condition for the action. In the above references the object \( g \rtimes V \) has also the structure of a Lie algebra. In our proof above, we do not have a semi-direct product structure and we just replaced the product space \( g_k \times J_k \mathbb{C}^n \) with diagonal \( G_k \)-action. By the way, an interesting related space is probably \( g_k \times J_k \mathbb{C}^n \) which has a Lie algebra structure, see [27].

Remark 4.9. We have proved the well-known conjecture mentioned in [13], on finite generation of fiber ring of regular invariant jets. The conjecture asserts that; The algebra \( \mathbb{C}[(J_k \mathbb{C}^n)^{G_k}] \) is finitely generated, where \( J_k \mathbb{C}^n/G_k \rightarrow J_k \mathbb{C}^n/G_k \) is the normalization. In particular, the stalk or the fiber ring of \( J_k \mathbb{C}^n/G_k \) at a regular point is finitely generated, which worked out in [1] but the proof is incomplete and the gap is still there, see also [11, 12, 13]. Theorem 4.7 proves a more general statement for the whole coordinate ring of \( J_k \mathbb{C}^n/G_k \).

We have the following analog of Theorem 4.3 for the \( \rho \)-action.

**Theorem 4.10.** (Main result) Under the action of \( \rho \) on \( V = J_k \mathbb{C}^n \), the following holds:

1. \( \max_{x \in J_k \mathbb{C}^n} \dim G_k \cdot x = \dim J_k \mathbb{C}^n - \dim J_k \mathbb{C}^n/G_k \).
2. The quotient map \( \pi : J_k \mathbb{C}^n \rightarrow J_k \mathbb{C}^n/G_k \) is equidimensional over the normal locus.
3. Let \( \Omega = \{ x \in J_k \mid d_x \pi \text{ is onto} \} \), then, \( J_k \mathbb{C}^n \setminus \Omega \) contains no divisors.
4. The null cone \( \mathfrak{R}_{G_k}(J_k \mathbb{C}^n) = \pi^{-1}(\pi(0)) \) is an irreducible complete intersection of dimension \( \dim J_k \mathbb{C}^n - \dim J_k \mathbb{C}^n/G_k \).

**Proof.** The proof is analogous to the proof of Theorem 4.3 with the action replaced by the representation \( \rho \), where Theorem 4.3 applies.

Items 1 and 2 are equivalent to each other and both follow from the fact that the action of \( G_k \) on \( V = J_k \mathbb{C}^n \) has generic stabilizers, i.e. Theorem 4.3 [compare with the proof of Theorem 4.3]. Item 3 is followed by a similar argument in the proof of Theorem 4.3 of the similar item: If we let \( f_1, \ldots, f_m \) be the basic invariants
for the action of $G_k$ on $\mathbb{C}[J_k \mathbb{C}^n]$, and set
\begin{equation}
(4.8) \quad X_r = \{ x \in J_k \mathbb{C}^n | \dim \langle df_1(x), \ldots, df_m(x) \rangle = r \},
\end{equation}
then, $X_n \supset J^\text{reg}_k \mathbb{C}^n$ and because we already had $\text{codim}_{J_k \mathbb{C}^n}(J_k \mathbb{C}^n \setminus J^\text{reg}_k \mathbb{C}^n) \geq 2$ by (3.30), the item (3) follows. The proof and explanation of item (4) is identical to that of Theorem 4.9 for the last item. \hfill \square

Our last result concerns a Chevalley-type theorem for the representation ring of the Lie group $G_k$. We prove a finiteness result for the Grothendieck ring of finitely generated $G_k$-modules.

**Theorem 4.11.** (Main result.) Assume $G_k$ is the group of reparametrizations of $k$-jets of holomorphic curves. Let $T_k$ be a maximal torus of $G_k$ and $W_k$ the associated Weyl group. Then, the representation ring of $G_k$ satisfies $R(G_k) = R(T_k)^{W_k}$.

**Proof.** We have $G_k = \mathbb{C}^* \times U_k$. Let $G' = [G_k, G_k] = U_k$ be the derived group of $G_k$, and $H \rightarrow G'$ be its universal (finite) cover. Then, we have $G_k = Z \times G'/\pi_1$, where $Z$ is a maximal central torus and $\pi_1 \subset Z \times G'$ is a finite central and closed subgroup.

On the other hand, as we also mentioned in the proof of the Elashvili condition in Theorem 4.11 by letting $x \in \mathfrak{g}_k \subset \mathfrak{gl}_k$ to be a generic semisimple element implies that $\text{ad}(x)(\mathfrak{g}_k) = \mathfrak{u}_k$. This is because the ad function is the same as the one in $\mathfrak{gl}_k$, and we may choose a complete set of root vectors that are eigenvectors of $\text{ad}(x)$. This proves that $[\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k$. In other words, $\mathfrak{g}_k$ is semisimple. Therefore, $G_k$ is semisimple.

By Theorem 4.2, the Weyl group $W_k$ is finite. Summing up the above two paragraphs, $H$ is semisimple and connected. Now, the Theorem claim holds for $Z$ and $H$. Therefore, it holds for $Z \times H$. By the first paragraph in the proof, it remains to show that the same claim descends to central finite quotients. This has been done in \[8\] page 294, but we brief the argument below. First, we note that if $G_1 = G_2/\pi_1$ is such a quotient, then we have $T_1 = T_2/\pi_1$, which implies $W(G_1, T_1) = N_{G_1}(T_1)/T_1 = N_{G_2}(T_2)/T_2 = W(G_2, T_2)$. Therefore, we have the following diagram:

\begin{equation}
\begin{array}{ccc}
R(G_1) & \longrightarrow & R(T_1)^{W_1} \\
\downarrow & & \downarrow \\
R(G_2) & \longrightarrow & R(T_2)^{W_2}
\end{array}
\end{equation}

where all the arrows are inclusions; considering class functions, $R(T_i)$ is the $Z$-span of irreducible characters trivial on $Z$. The same holds for $R(G_i)$ by the Schur lemma. Passing to $W$-invariants, the class functions in $R(T_i)^{W_i}$ are invariant under $Z$-translation. But a class function on $G_i$ is determined by its restriction on $T_i$. Thus, the $Z$ invariance of the class functions can be checked on the torus. The claim follows. \hfill \square

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Perception and Robotics Laboratory [LAPyR], Center for Research in Optics
Email address: rahmati@cio.mx