HARNACK TYPE INEQUALITY FOR AN ELLIPTIC EQUATION.

SAMY SKANDER BAHOURA

Abstract. We give a sup x inf inequality for an elliptic equation.

1. Introduction and Main Results

We are on Riemannian manifold $(M, g)$ of dimension $n \geq 3$. In this paper we denote $\Delta_g = -\nabla^j (\nabla_j)$ the Laplace-Beltrami operator and $N = \frac{2n}{n-2}$.

We consider the following equation

$$\Delta_g u = V u^{N-1} + u^\alpha, \quad u > 0.$$  \hspace{1cm} (1)

Where $V$ is a function and $\alpha \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right]$.

For $a, b, A > 0$, we consider a sequence $(u_i, V_i)_i$ of solutions of the previous equation with the following conditions:

$$0 < a \leq V_i \leq b < +\infty,$$

$$||\nabla V_i||_\infty \leq A.$$

Here we study some properties of this nonlinear elliptic equation. We try to find some estimates of type sup x inf. We denote by $S_g$ the scalar curvature.

There are many existence and compactness results which concern this type of equations, see for example [1-21]. In particular in [1], we can find some results about the Yamabe equation and the Prescribed scalar curvature equation. Many methods where used to solve these problems, as a variational approach and some other topological methods. Note that the problems come from the nonlinearity of the critical Sobolev exponent. We can find in [1] some uniform estimates for various equations on the unit sphere or for the Monge-Ampere equation. Note that Tian and Siu proved uniform upper and lower bounds for the sup + inf for the Monge-Ampere equation under some condition on the Chern class, see [1]. In the case of the Scalar curvature equation and in dimension 2 Shafrir used the isoperimetric inequality of Alexandrov to prove an inequality of type sup + inf with only $L^\infty$ assumption on the prescribed curvature, see [21]. The result of Shafrir is an extention of a result of Brezis and Merle, see [4] and later, Brezis-Li-Shafrir proved a sharp sup + inf inequality for the same equation with Lipschitzian assumption on the prescribed scalar curvature, see [3]. Li in[17] extend the previous last result to compact Riemanian surfaces. In the higher dimensional case, we can find in [15] a proof of the sup x inf inequality in the constant case for the scalar curvature equation on open set of $\mathbb{R}^n$. We have various estimates in [2] when we consider the nonconstant case. To prove our result, we use a blow-up analysis and the moving-plane method, based on the maximum principle and the Hopf Lemma as showed in [2, 3, 15, 17], and a condition on the scalar curvature is sufficient to prove the estimate.

Our main result is:

**Theorem 1.1.** Assume $S_g > 0$ on $M$, then, for every compact $K$ of $M$, there exist a positive constant $c = c(\alpha, a, b, A, K, M, n, g)$ such that:

$$\sup_K u_i \times \inf_M u_i \leq c.$$  \hspace{1cm} (2)

**Remark:** in the case where $(M, g) = (\Omega \subset \mathbb{R}^n, \delta)$ an open set of the euclidean space with the flat metric, we have the same inequality on compact sets of $\Omega$ in this case the scalar curvature $S_\delta \equiv 0$, see [2].
If we consider the Green function $G$ of the Laplacian with Dirichlet condition on small balls of $M$, we can have a positive lower bound for $G$ and we have the following corollary:

**Corollary 1.2.** Assume $S_g > 0$ on $M$, then, for every compact $K$ of $M$, there exist a positive constant $c' = c'(\alpha, a, b, A, K, M, n, g)$ such that:

$$\int_K u_i^{2n} dv_g \leq c'.$$

2. **Proof of the theorem.**

Let us consider $x_0 \in M$, by a conformal change of the metric $\tilde{g} = \varphi^{4/(n-2)}g$ with $\varphi > 0$ we can consider an equation of type:

$$\Delta \tilde{g} u + R_{\tilde{g}} u = Vu^{N-1} + \varphi^{\alpha+1-N}u^\alpha + R_g \varphi^{2-N} u, \ u > 0,$$

with,

$$Ricci_{\tilde{g}}(x_0) = 0.$$

Here; $R_g = \frac{n-2}{4(n-1)}S_g$ and $R_{\tilde{g}} = \frac{n-2}{4(n-1)}S_{\tilde{g}}$

It is clear see the computations in a previous paper [2], it is sufficient to consider an equation of type:

$$\Delta \tilde{g} u = Vu^{N-1} + u^\alpha + \mu u, \ u > 0,$$

with,

$$Ricci \equiv 0 \text{ and } \mu > 0.$$

**Part I: The metric in polar coordinates.**

Let $(M, g)$ a Riemannian manifold. We note $g_{x,ij}$ the local expression of the metric $g$ in the exponential map centered in $x$.

We are concerning by the polar coordinates expression of the metric. Using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{\theta i}(r, \theta)d\theta^i d\theta^j = dt^2 + r^2 g_{ij}(r, \theta)d\theta^id\theta^j = g_{x,ij}dx^i dx^j,$$

in a polar chart with origin $x^*$, $[0, \epsilon_0[ \times U_k$, with $(U^k, \psi)$ a chart of $S_{n-1}$. We can write the element volume:

$$dV_g = r^{n-1}\sqrt{|\tilde{g}|}dr d\theta^1 \ldots d\theta^{n-1} = \sqrt{|det(g_{x,ij})|}dx^1 \ldots dx^n,$$

then,

$$dV_{\tilde{g}} = r^{n-1}\sqrt{|\tilde{g}|}dr d\theta^1 \ldots d\theta^{n-1} = \sqrt{|det(g_{x,ij})|}dx^1 \ldots dx^n,$$

where, $\alpha^k$ is such that, $d\alpha_{n-1} = \alpha^k(\theta)d\theta^1 \ldots d\theta^{n-1}$. (Riemannian volume element of the sphere in the chart $(U^l, \psi)$). Then,

$$\sqrt{|\tilde{g}|} = \alpha^k(\theta)\sqrt{|det(g_{x,ij})|}.$$

Clearly, we have the following proposition:
Proposition 2.1. Let \( x_0 \in M \), there exist \( \epsilon_1 > 0 \) and if we reduce \( U^k \), we have:

\[
|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_\theta \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \forall x \in B(x_0, \epsilon_1) \\forall r \in [0, \epsilon_1], \forall \theta \in U^k.
\]

and,

\[
|\partial_r |\tilde{g}^k(x, r, \theta)| + |\partial_r \partial_\theta |\tilde{g}^k(x, r, \theta)| \leq Cr, \forall x \in B(x_0, \epsilon_1) \\forall r \in [0, \epsilon_1], \forall \theta \in U^k.
\]

Remark:

\( \partial_r [\log \sqrt{|\tilde{g}^k|}] \) is a local function of \( \theta \), and the restriction of the global function on the sphere \( \mathbb{S}_{n-1} \), \( \partial_r [\log \sqrt{\det(g_{x,ij})}] \). We will note, \( J(x, r, \theta) = \sqrt{\det(g_{x,ij})} \).

Part II: The laplacian in polar coordinates

Let’s write the laplacian in \( [0, \epsilon_1] \times U^k \),

\[
-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log \sqrt{|\tilde{g}^k|}] \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_\theta (\tilde{g}^{\alpha \beta} \partial_\alpha \sqrt{|\tilde{g}^k|} \partial_\beta).
\]

We have,

\[
-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r \log J(x, r, \theta) \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_\theta (\tilde{g}^{\alpha \beta} \partial_\alpha \sqrt{|\tilde{g}^k|} \partial_\beta).
\]

We write the laplacian (radial and angular decomposition),

\[
-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log J(x, r, \theta)] \partial_r - \Delta_{S_r(x)},
\]

where \( \Delta_{S_r(x)} \) is the laplacian on the sphere \( S_r(x) \).

We set \( L_\theta(x, r)(...) = r^2 \Delta_{S_r(x)}(...)[\exp_\theta(r\theta)] \), clearly, this operator is a laplacian on \( \mathbb{S}_{n-1} \) for particular metric. We write,

\[
L_\theta(x, r) = \Delta_{g_{x,r},\mathbb{S}_{n-1}},
\]

and,

\[
\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log J(x, r, \theta)] \partial_r - \frac{1}{r^2} L_\theta(x, r).
\]

If, \( u \) is function on \( M \), then, \( \tilde{u}(r, \theta) = u[\exp_\theta(r\theta)] \) is the corresponding function in polar coordinates centered in \( x \). We have,

\[
(4) \quad -\Delta u = \partial_{rr} \tilde{u} + \frac{n-1}{r} \partial_r \tilde{u} + \partial_r [\log J(x, r, \theta)] \partial_r \tilde{u} - \frac{1}{r^2} L_\theta(x, r) \tilde{u}.
\]

Part III: "Blow-up" and "Moving-plane" methods

The "blow-up" analysis

Let, \( (u_i) \), a sequence of functions on \( M \) such that,

\[
(5) \quad \Delta \tilde{g}_{ij} + R_{ij} u_i = V_i u_i^{N-1} + \varphi^{\alpha - N} u_i^\alpha + R_{ij} \varphi^{2-N} u_i, \quad u_i > 0, \quad N = \frac{2n}{n-2},
\]

It is sufficient to consider an equation of type:

\[
(6) \quad \Delta \tilde{g} u = Vu^{N-1} + \mu u, \quad u > 0.
\]

with \( \text{Ricci} \equiv 0 \) and \( \mu > 0 \).

We argue by contradiction and we suppose that \( \sup \times \inf \) is not bounded.

We assume that:

\[
\forall \ c, R > 0 \ \exists \ u_{c,R} \ \text{solution of} \ (E) \ \text{such that}:
\]

\[
(3) \quad \Delta \tilde{g} u = Vu^{N-1} + \mu u, \quad u > 0.
\]
\[ R^{n-2} \sup_{B(x_0,R)} u_i \cdot R(R) \geq c. \quad (H) \]

**Proposition 2.2.** There exist a sequence of points \( (y_i) \), \( y_i \to x_0 \) and two sequences of positive real number \( (l_i) \), \( (l_i) \), \( l_i \to 0 \), \( L_i \to +\infty \), such that if we consider \( v_i(y) = \frac{u_i[\exp_{y_i}(y)]}{u_i(y_i)} \), we have:

i) \( 0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \beta_i \to 1. \)

ii) \( v_i(y) \to \left( \frac{1}{1 + |y|^2} \right)^{(n-2)/2}, \) uniformly on every compact set of \( \mathbb{R}^n \).

iii) \( l_i^{(n-2)/2}[u_i(y_i)] \times \inf_M u_i \to +\infty \)

**Proof:**

Without loss of generality, we can assume that:

\[ V(x_0) = n(n - 2). \]

We use the hypothesis \((H)\). We can take two sequences \( R_i > 0, R_i \to 0 \) and \( c_i \to +\infty \), such that,

\[ R_i^{1/2} \sup_{B(x_0,R_i)} u_i \cdot R(R) \geq c_i \to +\infty. \]

Let, \( x_i \in B(x_0, R_i) \), such that \( \sup_{B(x_0,R_i)} u_i = u_i(x_i) \) and \( s_i(x) = [R_i-d(x,x_i)]^{(n-2)/2} u_i(x), x \in B(x_i, R_i) \). Then, \( x_i \to x_0 \).

We have,

\[ \max_{B(s_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{1/2} u_i(x) \geq \sqrt{c_i} \to +\infty. \]

Set:

\[ l_i = R_i - d(y_i, x_i), \quad u_i(y) = u_i[\exp_{y_i}(y)], \quad v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)]^{2/((n-2))})]}{u_i(y_i)} \]

Clearly, \( y_i \to x_0 \). We obtain:

\[ L_i = \frac{l_i}{(c_i)^{1/2(n-2)}}[u_i(y_i)]^{2/((n-2))} = \frac{[s_i(y_i)]^{2/((n-2))}}{c_i^{1/2(n-2)}}, \quad c_i^{1/2(n-2)} = c_i \to +\infty. \]

If \( |z| \leq L_i \), then \( y = \exp_{y_i}(z/[u_i(y_i)]^{2/((n-2))}) \in B(y_i, \delta_i, l_i) \) with \( \delta_i = \frac{1}{(c_i)^{1/2(n-2)}} \) and \( d(y, y_i) < R_i - d(y, x_i) \), thus, \( d(y, x_i) < R_i \) and, \( s_i(y) \leq s_i(y_i) \), we can write,

\[ u_i(y)[R_i - d(y, y_i)]^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2} \]

But, \( d(y, y_i) \leq \delta_i, l_i, R_i > l_i \) and \( R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i) \), we obtain,

\[ 0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \frac{l_i}{l_i(1 - \delta_i)} \leq 2^{(n-2)/2}. \]

We set, \( \beta_i = \left( \frac{1}{1 - \delta_i} \right)^{(n-2)/2} \), clearly \( \beta_i \to 1. \)

The function \( v_i \) is solution of:
By elliptic estimates and Ascoli, Ladyzenskaya theorems, \((v_i)_i\) converge uniformly on each compact to the function \(v\) solution on \(\mathbb{R}^n\) of,

\[
\Delta v = n(n-2)v^{n-1}, \quad v(0) = 1, \quad 0 \leq v \leq 2^{(n-2)/n},
\]

By using maximum principle, we have \(v > 0\) on \(\mathbb{R}^n\), the result of Caffarelli-Gidas-Spruck (see [6]) give, \(v(y) = \left(\frac{1}{1 + |y|^2}\right)^{(n-2)/2}\). We have the same properties for \(v_i\) in the previous paper [2].

**Polar coordinates and “moving-plane” method**

Let,

\[
w_i(t, \theta) = e^{(n-2)/2} \tilde{u}_i(e^t, \theta) = e^{(n-2)t/2} u_i \exp_i(e^t \theta), \text{ et } a(y_i, t, \theta) = \log \, J(y_i, e^t, \theta).
\]

We set \(\delta = \frac{(n + 2) - (n - 2)\alpha}{2}\).

**Lemma 2.3.** The function \(w_i\) is solution of:

\[
-\partial_{tt} w_i - \partial_t a \partial_t w_i - L_{\theta}(y_i, e^t) + cw_i = V_i w_i^{N-1} + e^{\delta t} w_i^\alpha + \mu e^{2t} w_i,
\]

with,

\[
c = c(y_i, t, \theta) = \left(\frac{n-2}{2}\right)^2 + \frac{n-2}{2} \partial_t a.
\]

**Proof:**

We write:

\[
\partial_t w_i = e^{nt/2} \partial_r \tilde{u}_i + \frac{n-2}{2} w_i, \quad \partial_{tt} w_i = e^{(n+2)t/2} \left[ \partial_r \tilde{u}_i + \frac{n-1}{e^t} \partial_r \tilde{u}_i \right] + \left(\frac{n-2}{2}\right)^2 w_i.
\]

\[
\partial_t a = e^t \partial_r \log \, J(y_i, e^t, \theta), \quad \partial_t a \partial_t w_i = e^{(n+2)t/2} \left[ \partial_r \log \, J \partial_r \tilde{u}_i \right] \partial_t a w_i + \frac{n-2}{2} \partial_t a w_i.
\]

the lemma is proved.

Now we have, \(\partial_t a = \frac{\partial_t b_1}{b_1}\), \(b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0\),

We can write,

\[
-\frac{1}{\sqrt{b_1}} \partial_{tt}(\sqrt{b_1} w_i) - L_{\theta}(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = V_i w_i^{N-1} + e^{\delta t} w_i^\alpha + \mu e^{2t} w_i,
\]

where, \(b_2(t, \theta) = \partial_t (\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_t b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2\).

Let,

\[
\tilde{w}_i = \sqrt{b_1} w_i,
\]
Lemma 2.4. The function \( \tilde{w}_i \) is solution of:

\[
-\partial_t \tilde{w}_i + \Delta_{g_{y_i, \xi}} \tilde{w}_i + 2 \nabla \phi(\tilde{w}_i) \nabla \phi \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)\tilde{w}_i =
\]

\[
= V_i \left( \frac{1}{b_1} \right)^{(N-2)/2} \tilde{w}_i^{N-1} + e^{\delta t} \left( \frac{1}{b_1} \right)^{(n-1)/2} \tilde{w}_i^\alpha + \mu e^{2t} \tilde{w}_i,
\]

where, \( c_2 = \frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, \xi}} \tilde{w}_i + |\nabla \phi \log(\sqrt{b_1})|^2 \).

Proof:

We have:

\[
-\partial_t \tilde{w}_i = \sqrt{b_1} \Delta_{g_{y_i, \xi}} \tilde{w}_i + (c + b_2)\tilde{w}_i = V_i \left( \frac{1}{b_1} \right)^{(N-2)/2} \tilde{w}_i^{N-1} + e^{\delta t} \left( \frac{1}{b_1} \right)^{(n-1)/2} \tilde{w}_i^\alpha + \mu e^{2t} \tilde{w}_i,
\]

But,

\[
\Delta_{g_{y_i, \xi}} \tilde{w}_i = \sqrt{b_1} \Delta_{g_{y_i, \xi}} \tilde{w}_i - 2 \nabla \phi \nabla \phi \log(\sqrt{b_1}) - c_2 \tilde{w}_i,
\]

we deduce than,

\[
\sqrt{b_1} \Delta_{g_{y_i, \xi}} \tilde{w}_i = \Delta_{g_{y_i, \xi}} \tilde{w}_i + 2 \nabla \phi \nabla \phi \log(\sqrt{b_1}) - c_2 \tilde{w}_i,
\]

with \( c_2 = \frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, \xi}} \tilde{w}_i + |\nabla \phi \log(\sqrt{b_1})|^2 \). The lemma is proved.

The "moving-plane" method:

Let \( \xi \), a real number, and suppose \( \xi_i \leq t \). We set \( t^{\xi_i} = 2 \xi_i - t \) and \( \tilde{w}_i^{\xi_i} (t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta) \).

We have,

\[
-\partial_t \tilde{w}_i^{\xi_i} + \Delta_{g_{y_i, \xi}} \tilde{w}_i^{\xi_i} + 2 \nabla \phi(\tilde{w}_i^{\xi_i}) \nabla \phi \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}), b_2(t^{\xi_i}) - c_2] \tilde{w}_i^{\xi_i} =
\]

\[
= V_i \left( \frac{1}{b_1} \right)^{(N-2)/2} \tilde{w}_i^{\xi_i}^{N-1} +
+ e^{\delta t} \left( \frac{1}{b_1} \right)^{(n-1)/2} \tilde{w}_i^{\xi_i}^\alpha + \mu e^{2t} \tilde{w}_i,
\]

By using the same arguments than in [2], we have:

**Proposition 2.5.** We have:

1) \( \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \forall \theta \in S_{n-1} \).

For all \( \beta > 0 \), there exist \( c_\beta > 0 \) such that:

2) \( \frac{1}{c_\beta} e^{(n-2)t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{(n-2)t/2}, \forall t \leq \beta, \forall \theta \in S_{n-1} \).
We set,
\[
\bar{Z}_t = -\partial_t (\ldots) + \Delta_{g_{b_t, e^{\xi_t}} - g_{b_{t-1}, e^{\xi_{t-1}}}} (\ldots) + 2 \nabla \log(\sqrt{b_t}) + (c + b_t^{-1/2}b_2 - c_2)(\ldots)
\]

**Remark:** In the operator \(\bar{Z}_t\), by using the proposition 3, the coefficient \(c + b_t^{-1/2}b_2 - c_2\) satisfies:
\[
c + b_t^{-1/2}b_2 - c_2 \geq k' > 0, \quad \text{for } t << 0,
\]
It is fundamental if we want to apply Hopf maximum principle.

We set, \(\delta = \frac{(n + 2) - (n - 2)\alpha}{2}\).

**Goal:**

Like in [2], we have elliptic second order operator. Here it is \(\bar{Z}_i\), the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:
\[
\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0, \text{ if } \bar{w}_i^{\xi_i} - \bar{w}_i \leq 0.
\]

We write:
\[
\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) = (\Delta_{g_{b_t, e^{\xi_t}} - g_{b_{t-1}, e^{\xi_{t-1}}}})(\bar{w}_i^{\xi_i}) +
+ 2(\nabla_{\theta, e^{\xi_i}} - \nabla_{\theta, e^{\xi_i}})(\bar{w}_i^{\xi_i}) \log(\sqrt{b_t}) + 2 \nabla \nabla_{\theta, e^{\xi_i}}(\bar{w}_i^{\xi_i}) \log(\sqrt{b_t}) - \log \sqrt{b_t} +
+ 2 \nabla_{\theta, e^{\xi_i}}(\bar{w}_i^{\xi_i}) \cdot (\nabla_{\theta, e^{\xi_i}} - \nabla_{\theta, e^{\xi_i}}) \log \sqrt{b_t} - [(c + b_t^{-1/2}b_2 - c_2)^i - (c + b_t^{-1/2}b_2 - c_2)]\bar{w}_i^{\xi_i} +
+ \nu_i^{\xi_i} \left( \frac{1}{b_t} \right)^{(N-2)/2}(\bar{w}_i^{\xi_i})^{N-1} - V_i \left( \frac{1}{b_t} \right)^{(N-2)/2} \bar{w}_i^{N-1} +
+ e^{\xi_i} b_t^{-1/2} (\bar{w}_i^{\xi_i})^{\alpha - e^{\xi_i} b_t^{1/2}(\bar{w}_i^{\xi_i})^{\alpha} + \mu e^{2\xi_i} \bar{w}_i^{\xi_i} - e^{2t} \bar{w}_i^{\xi_i} \right) \quad (** 1)
\]
Clearly, we have:

**Lemma 2.6.**
\[
b_1(y_t, t, \theta) = 1 - \frac{1}{3} \text{Ricci}_{g_t}(\theta, \theta) e^{2t} + \ldots,
\]
\[
R_g(e^t \theta) = R_g(y_t) + <\nabla R_g(y_t) \theta > e^t + \ldots.
\]

According to proposition 1 and lemma 3,

**Proposition 2.7.**
\[
\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq \tilde{A}(e^t - e^{\xi_i})(\bar{w}_i^{\xi_i})^{N-1} + (1/2)(e^{4\xi_i} - e^{4t})(\bar{w}_i^{\xi_i})^\alpha +
+ C|e^{2t} - e^{2\xi_i}| \left[ |\nabla_{g_{b_t}^{\xi_i}}| + |\nabla^2_{g_{b_t}^{\xi_i}}| + o(1)(\bar{w}_i^{\xi_i})^{N-1} + (\bar{w}_i^{\xi_i})^\alpha + \mu \bar{w}_i^{\xi_i} \right].
\]

**Proof:**

We use proposition 1, we have:
\[
a(y_t, t, \theta) = \log J(y_t, e^t, \theta) = \log b_1, \quad |\partial_t b_1(t)| + |\partial_{t-a} b_1(t)| + |\partial_{t-a} a(t)| \leq C e^{2t},
\]
and,
\[
|\partial_{x_{a}} b_1(t)| + |\partial_{x_{a}} y_{a} b_1(t)| + |\partial_{x_{a}} a_{a} b_1(t)| \leq C e^{2t},
\]
then,
We have:

\[ i \cdot \theta \cdot \xi \]

\[ j \cdot t \]

\[ (i) = e^{(n-2)\frac{(\lambda - \lambda_1) + (\xi_1 - t)}{2\nu_1} + \frac{\xi_1 - t}{2\nu_1}(\lambda_1 + \lambda - t)\theta} \leq C_i, \]

\[
\frac{\partial_{\theta_j} w_k(t, \theta)}{w_k(t)} = e^{(n-2)\frac{(\lambda - \lambda_1) + (\xi_1 - t)}{2\nu_1} + \frac{\xi_1 - t}{2\nu_1}(\lambda_1 + \lambda - t)\theta} \leq C_i.
\]

\[ \Delta_{\theta_i, \theta_j} \leq C_\lambda e^{2\epsilon t} - e^{2\epsilon \tilde{\lambda}_1} \left\| \nabla \theta \tilde{w}_i + \nabla^2_\theta (\tilde{w}_i) \right\|,
\]

We take \( C = \max \{C_i, 1 \leq i \leq q \} \) and if we use \((**+1)\), we obtain proposition 4.

We have:

\[
\frac{\partial_{\theta_j} w_k(t, \theta)}{w_k(t)} = e^{(n-2)\frac{(\lambda - \lambda_1) + (\xi_1 - t)}{2\nu_1} + \frac{\xi_1 - t}{2\nu_1}(\lambda_1 + \lambda - t)\theta} \leq C_i.
\]

Also,

\[ c_2 = \frac{1}{\sqrt{b_1}} \Delta_{\theta_i, \theta_j} \nabla \theta \left( \sqrt{b_1} \right) + \left| \nabla \theta \log (\sqrt{b_1}) \right|^2. \]

Then,

\[ \partial_t c(y, t, \theta) = \frac{\left( n - 2 \right)}{2} \partial_t a, \]

by proposition 1,

\[ |\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K e^{2t}. \]

We have:

\[ w_i(2\xi_1 - t, \theta) = e^{(n-2)\frac{(\lambda - \lambda_1) + (\xi_1 - t)}{2\nu_1} + \frac{\xi_1 - t}{2\nu_1}(\lambda_1 + \lambda - t)\theta} \leq 2^{(n-2)/2} e^{n-2} = \tilde{c}. \]
We set $\delta = \frac{(n+2) - (n-2)\alpha}{2}$.

The left right side are denoted $Z_1$ et $Z_2$, we can write:

$$Z_1 = (\bar{V}_i - \bar{V}_i)(\bar{w}_i^{\xi_i})^{N-1} + \bar{V}_i[(\bar{w}_i^{\xi_i})^{N-1} - \bar{w}_i^{N-1}],$$

and,

$$Z_2 = e^\delta t [(\bar{w}_i^{\xi_i})^\alpha - (\bar{w}_i)^\alpha] + (\bar{w}_i^{\xi_i})^\alpha (e^{\delta t \xi_i} - e^{\delta t}).$$

We can write the part with nonlinear terms as:

$$(\bar{w}_i^{\xi_i})^\alpha [(A w_i^{\xi_i,N-1-\alpha} + B) (e^t - e^{t \xi_i}) + c (e^{\delta t \xi_i} - e^{\delta t})].$$

Because $w_i^{\xi_i} \leq \bar{c}$, we have:

$$-\bar{Z}_1(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq (\bar{w}_i^{\xi_i})^\alpha [A e^{(N-1-\alpha)} + B] (e^t - e^{t \xi_i}) + e^{\delta t \xi_i} + (\bar{w}_i^{\xi_i})^\alpha (\mu/2)(e^{2t \xi_i} - e^{2t}).$$

But $\alpha \in ]\frac{n}{n-2}, \frac{n+2}{n-2}[, \quad \delta = \frac{n+2 - (n-2)\alpha}{2} \in ]0, 1[.$

We obtain for $t \leq t_0 < 0$:

$$e^t \leq e^{(1-\delta)t} e^{\delta t}, \quad \text{pour tout } t \leq t_0.$$ 

and, $t^{\xi_i} \leq t (\xi_i \leq t)$, we integrate:

$$(e^{\delta t \xi_i} - e^{\delta t}) \leq \frac{\delta}{e^{(1-\delta)t_0}} (e^{\xi_i} - e^t).$$

Finally:

$$-\bar{Z}_1(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq (\bar{w}_i^{\xi_i})^\alpha [- \frac{\delta e}{e^{(1-\delta)t_0}} + A \bar{e}^{N-1-\alpha} + B] (e^t - e^{t \xi_i}) + (\bar{w}_i^{\xi_i})^\alpha (\mu/2)(e^{2t \xi_i} - e^{2t}).$$

We apply proposition 3. We take $t_i = \log \sqrt{t_i}$ with $t_i$ like in proposition 2. The fact

$\sqrt{t_i} u_i(y_i)^{(n-2)/(n-2)} \to +\infty$ (see proposition 2), implies $t_i = \log \sqrt{t_i} > \frac{2}{n-2} \log u_i(y_i) + 2 = \lambda_i + 2$. Finally, we can work on $]-\infty, t_i[.$

We define $\xi_i$ by:

$$\xi_i = \sup \{ \lambda \leq \lambda_i + 2, \bar{w}_i(2\lambda - t, \theta) - \bar{w}_i(t, \theta) \leq 0 \text{ on } [\lambda, t_i] \times S_{n-1} \}.$$

If we use proposition 4 and the similar technics that in [B2] we can deduce by Hopf maximum principle,

$$\min_{S_{n-1}} \bar{w}_i(t_i, \theta) \leq \max_{S_{n-1}} \bar{w}_i(2\xi_i - t_i, \theta),$$

which implies,

$$t_i^{(n-2)/2} u_i(y_i) \times \min_M \bar{u} \leq c.$$

It is in contradiction with proposition 2.

Then we have,

$$\sup_{K} u \times \inf_{M} u \leq c = c(\alpha, a, A, K, M, g, n).$$
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Department of Mathematics, Pierre et Marie Curie University, 75005 Paris France.
E-mail address: samybahoura@yahoo.fr, bahoura@ccr.jussieu.fr