ON MATRIX QUANTUM GROUPS OF TYPE $A_n$

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Abstract. Given a Hecke symmetry $R$, one can define a matrix bialgebra $E_R$ and a matrix Hopf algebra $H_R$, which are called function rings on the matrix quantum semi-group and matrix quantum groups associated to $R$. We show that for an even Hecke symmetry, the rational representations of the corresponding quantum group are absolutely reducible and that the fusion coefficients of simple representations depend only on the rank of the Hecke symmetry. Further we compute the quantum rank of simple representations. We also show that the quantum semi-group is “Zariski” dense in the quantum group. Finally we give a formula for the integral.

1. Introduction

Matrix quantum groups of type $A_n$ generalize the standard deformations of the matrix group $GL(n)$ introduced by Faddeev, Reshetikhin, Takhtajian [5]. They are defined in terms of Hecke symmetries. To a Hecke symmetry $R$ there are associated a bialgebra $E_R$ and a Hopf algebra $H_R$, which are considered as the ring of regular functions on the corresponding matrix quantum semi-group and group of type $A$, respectively. If $R$ is the Drinfel’d-Jimbo solution to the Yang-Baxter equation of type $A_n$, the Hopf algebra $H_R$ reduces to the above mentioned deformations.

Quantum groups of type $A$ were firstly studied by Gurevich [8], who generalized Lyubashenko’s results on vector symmetry for the case of Hecke symmetry. Many interesting properties of quantum matrix (semi-) groups were found [9 21, 11, 26, 10]. These results show on one hand the similarity in the theory of quantum matrix groups of type $A_n$ and the classical groups $GL(n)$, and on the other hand the relations between quantum matrix groups of type $A_n$ and the Hecke algebras.

In this work we study rational representations of matrix quantum groups of type $A_n$, i.e., comodules of Hopf algebras associated to even Hecke symmetries. For an even Hecke symmetry, one has a notion of rank, which is, in the classical case, equal to the dimension of the vector space, on which the Hecke symmetry is defined. It was showd by Gurevich that, in general, the rank and the dimension of the vector space may differ. We show that the category of these comodules is, up to a braided abelian equivalence, determined only by the rank of the Hecke symmetry.

We show that this category is a ribbon category and compute the quantum dimension of simple comodules. Further, we give the explicit formula for the integral on $H_R$.
The work is organized as follows. In Section 2 we briefly recall some definitions of the bialgebra $E_R$ and its Hopf envelope $H_R$. We focus ourselves on the case of Hecke symmetries. We show that the canonical map $E_R \rightarrow H_R$ is injective, which means that the matrix quantum groups of type $A$ is dense in the corresponding matrix quantum semi-group. In Section 3 we study comodules of $E_R$ and $H_R$ for an even Hecke symmetry $R$. We show that the category $H_R$-Comod is determined by $q$ and the rank of $R$. Further we compute some quantum dimension of simple $H_R$ comodules. In Section 4 we compute the integral on $H_R$ and $SH_R$.

In the Appendix we briefly recall some results on Hecke algebras from [1, 2] and prove some lemmas needed in our context.

Notation. Let us fix a field $\mathbb{K}$ of characteristic zero. All objects of the paper are defined over $\mathbb{K}$. Throughout the paper we shall deal with an element $q$ from $\mathbb{K}^\times$ which we will assume not to be a root of unity, but may be the unity itself.

The symmetric groups are denoted by $S_n$, the corresponding Hecke algebras are denoted by $\mathcal{H}_n$.

A partition $\lambda$ of a non-negative integer $n$ is a sequence of non-increasing non-negative integers whose sum, denoted by $|\lambda|$, is qual to $n$. The length $l(\lambda)$ is the cardinal of its non-zero components. The set of all partitions is denoted by $\mathcal{P}$, the set of partitions to a given $n$ is denoted by $\mathcal{P}_n$. The set of partitions of length $r$ is denoted by $\mathcal{P}^r$. The conjugate partition $\lambda'$ to a given partition $\lambda$ is the one, whose $i^{th}$ component is $\#\{\lambda_j | \lambda_j \geq i\}$. Some time we shorten the notation of a partition with repeated indices, e.g., $(2, 1^3) := (2, 1, 1, 1)$.

Analogously, a $\mathbb{Z}$-partition is a (finite) sequence of decreasing integers. The set of all $\mathbb{Z}$-partitions is denoted by $\mathcal{P}^\mathbb{Z}$.

2. The Matrix Quantum Groups and Quantum Semi-groups

2.1. Yang-Baxter Operators and the Associated Matrix Quantum (Semi-) Groups.

Let $V$ be a finite dimensional vector space over $\mathbb{K}$ and $R : V \otimes V \rightarrow V \otimes V$ be an invertible $\mathbb{K}$-linear operator. Fix a basis $x_1, x_2, \ldots, x_d$ of $V$ and let $\xi^1, \xi^2, \ldots, \xi^d$ be its dual basis in the dual vector space $V^*$ to $V$, $<\xi^i|x_j> = \delta^i_j$. Denote $z^i_j := \xi^i \otimes x_j$, then $z^i_j, 1 \leq i, j \leq d$ form a $\mathbb{K}$-linear basis of $V^* \otimes V$. Let $R_{ij}^{kl}$ be the matrix of $R$ with respect to the basis $x_i \otimes x_j$: $x_i \otimes x_j R = x_k \otimes x_l R_{ij}^{kl}$. The algebra $E_R$ is defined to be the factor algebra of the tensor algebra on $V^* \otimes V$ by the relations:

$$R_{ij}^{nm} z^k_m z^l_n = z^p_i z^q_j R_{pq}^{kl}, 1 \leq i, j, k, l \leq d. \quad (1)$$

Remark. The definition of $E_R$ is, in fact, basis free.

$E_R$ is a bialgebra, the coproduct and counit are given by $\Delta(z^i_j) = z^k_i \otimes z^k_j$ and $\varepsilon(z^i_j) = \delta^i_j$, respectively (we shall frequently use the convention of summing up by the indices that appear both in lower and upper places). We shall be interested in right $E_R$-comodules, the coaction of $E_R$ will be denoted by $\delta$.

$E_R$ is called matrix quantum semi-group iff $R$ satisfies the Yang-Baxter equation:

$$(R \otimes \text{id}_V)(\text{id}_V \otimes R)(R \otimes \text{id}_V) = (\text{id}_V \otimes R)(R \otimes \text{id}_V)(\text{id}_V \otimes R).$$

In this case $R$ is called Yang-Baxter operator.
The matrix quantum group associated to $R$ is defined to be the Hopf envelope of $E_R$. By definition, the Hopf envelope of a bialgebra $E$ is a pair $(H, i)$ consisting of a Hopf algebra $H$ and a bialgebra homomorphism $i : E \rightarrow H$, satisfying the following universal property:

for any Hopf algebra $F$ and a bialgebra homomorphism $e : E \rightarrow F$, there exists uniquely a Hopf algebra homomorphism $h : H \rightarrow F$, such that $e = h \circ i$.

This definition is dual to the definition of the maximal subgroup of a semi-group. Because of that we use this definition as the standard way of finding a quantum group from a given quantum semi-group. The Hopf envelope always exists (cf. [24, 29]).

Let $H_R$ be the Hopf envelope of $E_R$, then $H_R$ is, in general, infinitely generated as an algebra. However, with some more condition on $R$, $H_R$ may turn out to be finitely generated.

Since $R$ satisfies the Yang-Baxter equation, $E_R$ is coquasitriangular, i.e., $E_R$-comod is a braided category (cf. [19]). It is natural to ask, whether one can extend the coquasitriangular structure onto $H_R$.

Let $M$ be an $E_R$-comodule, which is finite dimensional as a vector space. $M$ is naturally endowed with an $H_R$-comodule structure, hence, so is its dual vector space $M^*$. In this case, the braiding on $M \otimes M^*$ is uniquely determined by the one on $M \otimes M$. In fact, if, with respect to some basis of $M$, the braiding on $M \otimes M^*$ has a matrix $T$ then the braiding on $M \otimes M^*$ with respect to this basis and its dual should should have a matrix $S$ that satisfies

$$S_{im}^{kn}T_{mj}^{nl} = \delta_i^l \delta_j^k \quad \text{and} \quad S_{im}^{kn}T_{mj}^{nl} = \delta_i^l \delta_j^k.$$

In particular, this holds for $M = V$ – the fundamental comodule. The braiding on $V \otimes V^*$ is given by:

$$x_l \otimes \xi^k R_{V,V^*} = \xi^l \otimes x_j P_{li}^{kj},$$

with $P$ satisfying $R_{jk}^{il}P_{ln}^{km} = \delta_n^i \delta_j^m$ (and hence $P_{jk}^{il}R_{ln}^{-1} = \delta_n^i \delta_j^m$). It was shown by Lyubashenko [20] that in this case, there exists a matrix $Q$ that satisfies $R_{jk}^{-1}P_{ln}^{km} = \delta_n^i \delta_j^m$, and the operator twisting $V^*$ and $V$ is given by:

$$\xi^l \otimes x_j R_{V,V^*} = x_l \otimes P_{li}^{kj} \xi^k R_{jk}^{-1}.$$

**Theorem 2.1.1.** [3] If $R$ is a closed Yang-Baxter operator then $H_R$ can be characterized as the factor algebra of the tensor algebra on $z_i^j, t_i^j, 1 \leq i, j \leq d$, by the following relations:

\begin{align}
R_{ij}^{mn}z_m^k z_n^l & = z_l^p z_j^q R_{pq}^{kl}, \\
t_j^i z_i^k & = \delta_k^i, \\
z_j^i t_i^k & = \delta_i^k.
\end{align}

The coproduct is: $\Delta(z_i^j) = z_i^j \otimes z_i^k, \Delta(t_i^j) = t_i^k \otimes t_i^j$, and the antipode is determined by $S(x_i^j) = t_i^j$.

**Proof.** According to the general construction of $H_R$ [24], to show that $H_R$ is a Hopf algebra it is sufficient to show that the antipode on $t_i^j$ is representable in terms of $z_i^j$. 

Multiplying (2) with $t_k^i$ from the left and $t^j_l$ from the right and using (3), (4), we have
\[ R_{ij}^{mn} t_k^i = t_m^z P_{pq}^{kl} \]  
(5)
or equivalently
\[ P_{ik}^{lp} z_n^t q^j = t_m^z P_{nm}^{ji} \]  
(6)
Setting $n = j$ in (6) and summing up by this index we get
\[ B_p^k = t_m^z P_{lm}^{ji} B_i^m, \]  
(7)
where $B_i^m := P_{lm}^{ii}$. Analogously, setting $k = p$ in (6) and summing up by this index, we get:
\[ z_k^l C_q^t = C_j^l, \]  
(8)
where $C_j^l := P_{jl}^{ii}$.

The crucial step is to show that $B$ and $C$ are invertible. It turns out that
\[ C^{-1j} = Q^{lj}_i, \quad B^{-1j} = Q^{jl}_i. \]
The proof of these equalities is based on the Yang-Baxter equation for $R$ and the derived equations of $P$ and $Q$ (cf. Equation (18)). Hence we have
\[ S(t_k^i) = B_k^l z_p^t B^{-1j}_p = C^{-1i} C^j_l. \]  
(9)
Thus we showed that $H_R$ is a Hopf algebra. It this then easy to show that this is the Hopf envelope of $E_R$. \lbrack 8 \rbrack

The matrices $B$ and $C$ introduced in the above proof play an important role in the study of comodules of $H_R$. They were called parity operator by Gurevich. In our context, we think the name reflection is more appropriate, since $B$ and $C$ are matrices of the two canonical comodule morphisms $V \rightarrow V^{\ast\ast}$ and $V^{\ast\ast} \rightarrow V$.

Equation (9) shows that $V$ being a simple comodule on $E_R$ may turn to be reducible being comodule on $H_R$. In [5], there was introduced a condition ([loc.cit], Eq. (20)), which was equivalent to the following:
\[ BC = \text{const} \cdot I, \]
that makes the second equation in (9) trivial.

2.2. Matrix Quantum (Semi-) Groups Associated with Hecke Symmetries. A Yang-Baxter operator $R$ is called Hecke operator if it satisfies the following equation
\[(R + 1)(R - q) = 0, \text{ for some } q \in \mathbb{K}^\times.\]
We shall assume once for all that $q$ is not a root of unity except the unity itself. Further, if $R$ is closed then $R$ is called Hecke symmetry.

We define the algebras $\Lambda_R$ and $S_R$ as the factor algebras of the tensor algebra upon $V$ by the following relations:
\[-x_i \otimes x_j = x_k \otimes x_k R_{ij}^{kl}, 1 \leq i, j \leq d,\]
\[qx_i \otimes x_j = x_k \otimes x_k R_{ij}^{kl}, 1 \leq i, j \leq d,\]  
(10)
Theorem 2.3.1. The algebras $E_n$ and $\rho(\mathcal{H}_n)$ are centralizers of each other in $\text{End}_K(V^\otimes n)$. Using standard arguments we deduce form this theorem

Corollary 2.3.2. Every simple $E_n$-comodule has the form
$$L \cong \text{Im}(\rho(E)),$$
where $E$ is a primitive idempotent of $\mathcal{H}_n$. Conversely, every primitive idempotent $E$ of $\mathcal{H}_n$ induces a simple comodule of $E_n$ iff $\rho(E) \neq 0$. Further, the modules defined by $E$ and $E'$ are non-isomorphic iff $E$ and $E'$ belong to different minimal two-sided ideals.
Remark. Since $\mathcal{H}_n$ is the direct product of matrix rings over $\mathbb{K}$, so is $E_n^*$. Therefore Schur’s lemma holds for $E_R$ – the automorphism ring of a simple $E_R$-comodule is $\mathbb{K}$.

Lemma 2.3.3. Let $R$ be a Hecke operator. Let $r$ be the largest number, such that $\Lambda^r_R \neq 0$. Assume that $r < \infty$, then $\Lambda^r_R$ is a simple $E_R$-comodule and for every simple $E_R$-comodule $N$, $\Lambda^r_R \otimes N$ is simple too.

Proof. It is known (cf. [4]) that:

$$\Lambda^r_R = \text{Im}(\rho(Y_r)),$$

where $Y_r$ is the minimal central idempotent that induces the signature representation of $\mathcal{H}_r$, $Y_r = \frac{1}{|r|/q} \sum_{w \in \mathfrak{S}_r} (-q)^{-l(w)} T_w$ (see Appendix). Thus, $\Lambda^r_R$ is a simple $E_R$-comodule, since $Y_r$ is a primitive idempotent. By assumption we have:

$$\rho(Y_r) \neq 0 \text{ and } \rho(Y_s) = 0, \forall s \geq r + 1.$$

Let $N$ be a simple $E_R$-comodule of homogeneous degree $m \geq 1$. By Lemma 2.3.3 there exists a primitive idempotent $E$ in $\mathcal{H}_m$, such that $N \cong \text{Im}(\rho(E))$. $\Lambda^r_R \otimes N$ is a homogeneous comodule of degree $m + r$. Embedding $\mathcal{H}_r$ and $\mathcal{H}_m$ into $\mathcal{H}_{m+r}$ as follows:

$$\mathcal{H}_r \ni T_i \mapsto T_i \in \mathcal{H}_{m+r}, 1 \leq i \leq r - 1$$

$$\mathcal{H}_m \ni T_j \mapsto T_{r+j} \in \mathcal{H}_{m+r}, 1 \leq j \leq m - 1,$$

we have $\Lambda^r_R \otimes N \cong \text{Im}(\rho_{m+r}(Y_r) E)$. Thus, it is sufficient to show that $\rho_{m+r}(Y_r) E$ is a primitive idempotent in $\rho(H_{m+r})$. This fact will be proved in the Appendix [4].

A direct consequence of this lemma is that $\Lambda^r_R \otimes \Lambda^s_R$ is a simple comodule. Hence, the braiding on $\Lambda^r_R \otimes \Lambda^s_R$ should be scalar. Since the identity operator is not closed, except when dim$_K\Lambda^r_R = 1$, $R$ is not closed if dim$_K\Lambda^r_R \geq 2$ (cf. [4]).

Now assume that $R$ is an even Hecke symmetry, then dim$_K\Lambda^r_R = 1$, and hence induces the quantum determinant denoted by $D$ in $E_R$. It is shown in [4], that $E_R[D^{-1}]$ is a Hopf algebra, hence coincides with $H_R$ (cf. [13]). Let $\Lambda^r_R^*$ denote the left dual $H_R$-comodule to $\Lambda^r_R$. Then, by definition of dual comodules and since the antipode sends a group-like element to its inverse,

$$\Lambda^r_R \otimes \Lambda^s_R^* \cong \Lambda^s_R^* \otimes \Lambda^r_R \cong I$$

as $H_R$-comodules.

Theorem 2.3.4. Let $R$ be a Hecke symmetry. Then the following assertions hold:

(i) The canonical homomorphism $i : E_R \rightarrow H_R$ is injective,

(ii) $H_R$ is cosemi-simple,

(iii) Simple comodules of $H_R$ have the form $N \otimes \Lambda^r_R \otimes^n$, where $N$ is a simple $E_R$-comodule, $n \geq 0$.

Proof. According to Lemma 2.3.3, $N \otimes \Lambda^r_R$ is simple whenever $N$ is. Hence, $D$ is not a zero-divisor in $E_R$. Thus, the localizing homomorphism $E_R \rightarrow E_R[D^{-1}] = H_R$ is injective, (i) is proved.

Let $M$ be a finite dimensional comodule on $H_R \cong E_R[D^{-1}]$, and $x_i, i = 1, 2, \ldots, m$ be a $\mathbb{K}$-basis of $M$, then there exist elements $a^i_j, 1 \leq i, j \leq m$ in $H_R$, such that $\delta(x_i) = x_j a^i_j$. Let $n$ be such an integer, that $a^i_j D^n \in E_R, \forall i, j$. Then $M \otimes \Lambda^r_R \otimes^n$ is an $E_R$-comodule.
Let $M \subset N$ be finite dimension $H_R$-comodules. There exists $n \in \mathbb{N}$ such that $M \otimes \Lambda_R^r \otimes^n$ and $N \otimes \Lambda_R^r \otimes^n$ are $E_R$-comodules. Hence, there exists $E_R$-comodule $P$, such that:

$$N \otimes \Lambda_R^r \otimes^n = M \otimes \Lambda_R^r \otimes^n \oplus P.$$ 

Therefore

$$N = M \oplus P \otimes \Lambda_R^r \otimes^n.$$ 

Thus every finite dimension $H_R$-comodule is absolutely reducible. Since every $H_R$-comodule is an injective limit of a system of finite dimension comodules, (ii) is proved. (iii) also follows from this discussion. 

We now show that the canonical homomorphism $i : E_R \longrightarrow H_R$ is injective for arbitrary Hecke symmetry $R$.

Let $\mathcal{V}$ be the $\mathbb{K}$-additive braided category, generated by $V$ and $R$, that is, objects of $\mathcal{V}$ are $V \otimes^n, n = 0, 1, 2, \ldots$, and morphisms in $\mathcal{V}$ are obtained from $R$ and the identity morphisms by composing, taking tensor products and linear combinations with coefficient in $\mathbb{K}$. Analogously, let $\mathcal{W}$ be the $\mathbb{K}$-additive monoidal category, generated by $V, V^*$ and $R_{V,V}, R_{V^*,V}, R_{V,V^*}$ and $ev_V, db_V$. Then $\mathcal{W}$ is a rigid category. In fact, $(V, ev_V \circ R_{V,V^*})$ is the dual to $V^*$.

We shall use graphical notation (cf.[16, 17]). The identity morphism $id_V$ is depicted by an arrow, oriented downward, its dual $id_{V^*}$ is depicted by an arrow, oriented upward:

\[
\begin{array}{c}
V \\
\downarrow \\
V
\end{array}
\quad
\begin{array}{c}
V^* \\
\uparrow \\
V^*
\end{array}
\]

The morphism $R_{V,V}$ is depicted by a braid:

\[
R_{V,V} := \begin{array}{c}
V & V \\
\rightarrow & \\
V & V
\end{array}
\]

To depict composition of morphism we place their pictures in successive order downward, the first morphism to be applied will be placed first. To depict tensor product of morphism, we place them in successive order from left to right.

The morphisms $ev_V$ and $db_V$ are depicted by special picture, the tangles:

\[
ev_V := \begin{array}{c}
V^* & V \\
\rightarrow & \\
V & V^*
\end{array} 
\quad
db_V := \begin{array}{c}
V & V^* \\
\rightarrow & \\
V & V^*
\end{array}
\]

The oriented tangles

\[
\begin{array}{c}
\circ \\
\circ
\end{array}
\]

are not assumed to present any morphism.

Reidermeister’s moves are allowed, except the second one:

\[
\begin{array}{c}
\circ = \circ = \cup \\
\circ
\end{array}
\]

Thus the (oriented) circle does not represent any morphism. Instead, the twisted circles

\[
\begin{array}{c}
\circ \quad \circ
\end{array}
\]
represent (different) morphisms $K \rightarrow K$

As a consequence, we obtain the presentation of $R_{V^*,V}, R_{V,V^*}, R_{V^*,V^*}$,

$$R_{V,V^*} := \begin{array}{c}
\begin{array}{c}
V^* \quad V^*
\end{array}
\end{array}$$

$$R_{V^*,V} := \begin{array}{c}
\begin{array}{c}
V \quad V^*
\end{array}
\end{array}$$

$$R_{V,V^*} := \begin{array}{c}
\begin{array}{c}
V \quad V
\end{array}
\end{array}$$

Since $R$ obeys the Hecke equation, we have

$$V^* = q^{-1} V^* - (1 - q^{-1}) V^*$$

(14)

$$V = q^{-1} V - (1 - q^{-1}) V$$

(15)

Further we have:

$$V V^* = q^{-1} V V^* - (1 - q^{-1}) V^*$$

(16)

$$V^* V = q^{-1} V^* V - (1 - q^{-1}) V^*$$

(17)

These equations follow immediately from the equations below:

$$\begin{array}{c}
\begin{array}{c}
V V^*
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
V
\end{array}
\end{array}$$

(18)

**Theorem 2.3.5.** Let $R$ be a closed Hecke operator then the canonical homomorphism $i : E_R \rightarrow H_R$ is injective.

**Proof.** It is known that $E_R$ (resp. $H_R$) is isomorphic to the Coend of the forgetful functor $v : V \rightarrow \text{Vect}_K$ (resp. $w : W \rightarrow \text{Vect}_K$) (cf. [28]). Relations on $E_R$ (resp. $H_R$) are obtained from morphisms in $V$ (resp. $W$). In fact, the relations on $E_R$ are obtained from $\text{Hom}_V(V^\otimes n, V^\otimes m)$. To show that $i : E_R \rightarrow H_R$ is injective, it is sufficient to show that

$$\text{Hom}_V(V^\otimes n, V^\otimes m) \cong \text{Hom}_W(V^\otimes n, V^\otimes m).$$

(19)

If $m \neq n$, then both sides of (19) are empty, hence assume that $n = m$.

A tangle which represents a morphism in $\text{Hom}_W(V^\otimes n, V^\otimes n)$ may contain some strands which orientate themselves downward and some closed strands. Using (16) and (17) we can disjoint the closed strands, i.e., represent the tangle as a combination of tangle in
which the closed strands are isolated, that is, they do not intersect with other strands. For example, we have:

![Diagram](image)

An isolated closed strand represents a morphism of $\mathbb{K}$, i.e., a scalar. Further, using (14), (15) and (18) we can transform a non-closed strand into one, that does not have local extremals. This means, a tangle with isolated closed strands is equivalent to a braid up to a scalar. Thus, we see that a tangle which represents a morphism in $\text{Hom}_W(V^\otimes n, V^\otimes n)$ is a linear combination of braids, hence the following equation holds

$$\text{Hom}_W(V^\otimes n, V^\otimes n) \cong \text{Hom}_V(V^\otimes n, V^\otimes n),$$

from which the theorem follows immediately.

3. Structure of $H_R$-Comod for an Even Hecke Symmetry $R$

As it was remarked in the previous section, simple $E_R$-comodules are parameterized by a subset of $\mathcal{P}$ – the set of all partition. We show in this section that for an even Hecke symmetry $R$ of rank $r$, this set is precisely $\mathcal{P}^r = \{\lambda \in \mathcal{P} | \lambda_{r+1} = 0\}$. The tactic is to use the 8dim. We show first that an $H_R$-comodule has a non-zero 8dim. Further we show that the 8dim of a simple comodule $M_\lambda$ is non-zero iff $\lambda \in \mathcal{P}^r$.

3.1. The 8dim of Simple Comodules. Let $\tau$ denote the braiding on $H_R$-Comod. In particular, $\tau_{V,V} = R_\tau, \tau_{V^*,V} = R_{V^*,V}, \tau_{V,V^*} = R_{V,V^*}$. Let $M$ be an object in $H_R$-Comod, we define its 8dim to be \[23\]:

$$\text{8dim}(M) := \text{ev}_M \tau_{M,M^*} \text{db}_M(1_\mathbb{K}).$$

For example $\text{8dim}(V) = \text{tr}(C)$. A crucial point in our study is the fact, again, due to Gurevich [9], that

$$\text{tr}(C) = -[r]_q.$$  \[20\]

This quantity is called the quantum rank of $R$.

More general, for a morphism $f : M \rightarrow M$ we define its 8tr to be

$$\text{8tr}(f) := \text{ev}_M \tau_{M,M^*} (f \otimes M^*) \text{db}_M(1_\mathbb{K}).$$

Then we have

$$\text{8dim}(M) = \text{8tr}(\text{id}_M), \quad \text{(21)}$$

and for morphisms $f : M \rightarrow N$ and $g : N \rightarrow M$,

$$\text{8tr}(fg) = \text{8tr}(gf). \quad \text{(22)}$$
Let $M = M_1 \oplus M_2$ with morphisms $p_i : M \to M_i$, $j_i : M_i \to M$, $p_i \circ j_i = 1_{M_i}$, $j_1 p_1 + j_2 p_2 = 1_M$. Then

\[
8\dim(M) = \db_M \tau_{M^*,M} \cdot (1 \otimes (j_1 p_1 + j_2 p_2)) \ev_M \\
= \tr(j_1 p_1) + \tr(j_2 p_2) \\
= \tr(p_1 j_1) + \tr(p_2 j_2) \\
= 8\dim(M_1) + 8\dim(M_2).
\]

Thus the $8\dim$ is additive.

The idea of using $8\dim$ comes from the following lemma.

**Lemma 3.1.1.** Let $M$ be a simple $H_R$-comodule, then $8\dim(M) \neq 0$.

**Proof.** Since $M$ is simple, $\Hom(M^* \otimes M, K) \cong \Hom(M, M) = K$. \hfill (23)

$db_M : K \to M \otimes M^*$ is not trivial, hence there exists a morphism $p : M \otimes M^* \to K$, such that $p \circ db_M = 1_K$. Thus $p \neq 0$, hence $0 \neq p \circ \tau_{M,M^*}^{-1} : M^* \otimes M \to K$. By (23) there exists $c \in K^*$, such that $c \cdot p \circ \tau_{M,M^*}^{-1} = \ev_M$, hence $\ev_M \tau db = c \neq 0$. \hfill □

Let us fix the following standard rule of defining the dual to the tensor product. Let $V_i$, $i = 1, 2, \ldots, k$ be $H_R$-comodules, then

\[
(V_1 \otimes V_2 \otimes \cdots \otimes V_k)^* := V_k^* \otimes V_{k-1}^* \otimes \cdots \otimes V_1^*,
\]

with the evaluation map $\ev = \ev_k \ev_{k-1} \cdots \ev_1$.

Let $M = \text{Im}(\rho(E))$ be a simple $H_R$-comodules, where $E$ is a primitive idempotent in $H_n$. Then

\[
8\dim(M) = 8\tr(\rho(E)).
\]

According to (22), the map $\rho$ composed with $8\tr$ gives us a trace map $H_n \to K$, denoted by $\tr_r$. We show that this trace map depends actually only on $q$ and $r = \text{rank} R$ and not on the operator $R$ itself.

For a morphism $f \in \text{End}^E_R(V^{\otimes n})$ we define a morphism $8\tr^n(f) \in \text{End}^E_R(V^{\otimes n-1})$ to be:

\[
8\tr^n(f) := (\id_{V^{\otimes n-1}} \otimes \ev_v \circ \tau_{V,V^*})(f \otimes \id_{V})(\id_{V^{\otimes n-1}} \otimes db).
\]

Thus, we have a map $\text{End}^E_R(V^\otimes n) \to \text{End}^E_R(V^\otimes n-1)$.

**Lemma 3.1.2.** Let $f \in \text{End}^E_R(V^\otimes n)$. Then

\[
8\tr(f) = 8\tr^1 \circ 8\tr^2 \circ \cdots \circ 8\tr^n(\tau_{w_n}^{-2} \circ f),
\]

where $w_n$ is the longest element of $\mathfrak{S}_n$ – the one, that reverses the order of the elements $1, 2, \ldots, n$. 

Proof. The above equation above can be best explained using pictorial representation. Below is the picture for $n = 3$.

Let $w \in S_n$ be expressed in the form $w = v_k v_{k+1} \cdots v_{n-1} w_1$, $w_1 \in S_{n-1}$. Then it is easy to check

$$8\text{tr}^n(\rho(T_w)) = \begin{cases} 
\rho_{n-1}(T_{v_k \cdots v_{n-2} w_1}) & \text{if } k \leq n - 1 \\
\text{tr}(C) \rho_{n-1}(T_{w_1}) & \text{if } k = n.
\end{cases}$$

Thus we see that $8\text{tr}^n$ depends only on $\text{tr}(C) = -[-r]_q$. Let us define an analogous operator $\text{tr}^n : \mathcal{H}_n \rightarrow \mathcal{H}_{n-1}$, setting

$$\text{tr}^n_r(T_w) = \begin{cases} 
T_{v_k \cdots v_{n-2} w_1} & \text{if } k \leq n - 1 \\
-[-r]_q T_{w_1} & \text{if } k = n,
\end{cases}$$

where $w = v_k v_{k+1} \cdots v_{n-1} w_1$, $w_1 \in S_{n-1}$. Then we have

$$\rho_{n-1} \circ \text{tr}^n_r = 8\text{tr}^n \circ \rho_n.$$  

Recall that the trace map $\text{tr}_r$ is defined as the composition of $\rho_n$ by $8\text{tr}$.

**Corollary 3.1.3.** We have

$$\text{tr}_r(E) = \text{tr}_r^1 \circ \text{tr}_r^2 \circ \cdots \circ \text{tr}_r^n(T_{w^{-2}_n} E).$$

Hence $\text{tr}_r$ depends only on $r$.

**Proof.** For $n = 0$, we have $\rho_0 : \mathcal{H}_0 \cong \mathbb{K} \cong \text{End}^E_R(V^0)$. Using equation (29), we have

$$\text{tr}_r(W) = 8\text{tr}(\rho_n(W))$$

$$= 8\text{tr}^1 \circ 8\text{tr}^2 \circ \cdots \circ 8\text{tr}^n(\tau_{w_n}^{-2} \rho_n(W))$$

$$= \rho_0(\text{tr}_r^1 \circ \text{tr}_r^2 \circ \cdots \circ \text{tr}_r^n(T_{w^{-2}_n} W))$$

$$= \text{tr}_r^1 \circ \text{tr}_r^2 \circ \cdots \circ \text{tr}_r^n(T_{w^{-2}_n} W) \blacksquare$$

**Proposition 3.1.4.** Let $R$ be an even Hecke symmetry of rank $r$. Let $\lambda \vdash n$ and $M = \text{Im}\rho_n(E_\lambda)$. Then $M_\lambda \neq 0$ (and hence is a simple comodule of $H_R$) iff $\lambda_{r+1} = 0$. 


Proof. By virtue of Lemma 3.1.1, it is sufficient to show that
\[ \text{tr}_r(T_{w_n}^{-2}E) := \text{tr}_r^1(\text{tr}_r^2(\cdots (\text{tr}_r^n(E)) \cdots )) \neq 0 \iff \lambda_{r+1} = 0. \]

Let \( R_{DJ} \) be the Drinfel’d-Jimbo \( R \)-matrix of type \( A_{r-1} \), defined in (12), then \( R_{DJ} \) has rank \( r \). Let \( \rho_{n}^{DJ} \) denote the representation of \( \mathcal{H}_n \) induced by \( R_{DJ} \). If for a partition \( \lambda \), \( \lambda_{r+1} = 0 \), then \( V^{\otimes n} \), being considered as \( \mathcal{H}_n \)-module, contains direct summand isomorphic to the simple \( \mathcal{H}_n \)-module \( S_\lambda \) (see Appendix) ([3], Proposition 5.1). On this module \( E_\lambda \neq 0 \), hence \( \text{Im}(\rho_n^{DJ}(E_\lambda)) \neq 0 \). By Lemma 3.1.1, \( \text{tr}_r(T_{w_n}^{-2}E_\lambda) \neq 0 \).

If \( \lambda_{r+1} \neq 0 \), then \( V^{\otimes n} \), being considered as \( \mathcal{H}_n \)-module through \( \rho_n^{DJ} \), does not contain direct summand that is isomorphic to \( S_\lambda \) as \( \mathcal{H}_n \)-module, so that \( E_\lambda \) acts as zero on \( V^{\otimes n} \). Hence \( \text{tr}_r(T_{w_n}^{-2}E_\lambda) \) acts as zero on \( V^{\otimes 0} = \mathbb{K} \). On the other hand, \( \text{tr}_r(T_{w_n}^{-2}E_\lambda) \) is a constant, hence it is equal to zero.

3.2. The Grothendieck Ring of \( H_R \)-Comod. Let \( E_\lambda \) and \( E_\mu \) be primitive idempotents of \( \mathcal{H}_n \) and \( \mathcal{H}_m \) respectively, \( M_\lambda = \rho_n(E_\lambda) \), \( M_\mu = \rho_m(E_\mu) \). Embed \( \mathcal{H}_n \) into \( \mathcal{H}_{n+m} \), \( T_i \mapsto T_i \) and \( \mathcal{H}_n \) into \( \mathcal{H}_{n+m} \), \( T_j \mapsto T_j^{n+1} \). Then \( M_\lambda \otimes M_\mu \cong \text{Im}(\rho_{n+m}(E_\lambda E_\mu)) \), where \( E_\mu^n \) is the image of \( E_\mu \) under the defined above embedding. The decomposition of \( E_\lambda E_\mu^n \) into primitive idempotents depends only on \( \lambda \) and \( \mu \) themselves. Consequently, in the decomposition

\[ M_\lambda \otimes M_\mu = \bigoplus_{\gamma \vdash |\lambda|+|\mu|} c_{\lambda \mu}^\gamma M_\gamma \]  

(31)

the coefficients \( c_{\lambda \mu}^\gamma \) coincide with the one for (rational) representations of \( GL(r) \), i.e., the Littlewood-Richardson coefficients.

Hence we have the determinantal form for \( M_\lambda \) (cf. [14], Section 19):

\[ M_\lambda \cong \det \left( (M_{\lambda_i-i+j})_{1 \leq i,j \leq r} \right), \]  

(32)

where, in the right-hand side, the determinant should be understood to be taken in the Grothendieck ring of \( H_R \)-Comod and \( M_{(k)} := 0 \) whenever \( k < 0 \).

We now characterize the dual comodule to \( M_\lambda \), thus furnish all \( H_R \)-comodules. Call sequence of \( r \) integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \) a \( \mathbb{Z} \)-partition. Let \( \overline{\lambda}_i := \lambda_i - \lambda_r \). Define

\[ M_\lambda := M_{\overline{\lambda}} \otimes \Lambda^{\otimes \lambda_r} \]

where \( \overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_r) \), and

\[ \Lambda^{\otimes \lambda_r} := \begin{cases} \Lambda^{\otimes \lambda_r} & \text{if } \lambda_r \geq 0; \\ \Lambda^{\otimes |\lambda_r|} & \text{otherwise.} \end{cases} \]

For a \( \mathbb{Z} \)-partition \( \lambda \), define \( -\lambda := (-\lambda_r, -\lambda_{r-1}, \ldots, -\lambda_1) \). We show that \( M_\lambda^* \cong M_{-\lambda} \).

In fact, since \( (M_\lambda^*)^* \cong M_\lambda \), we have

\[ \text{Hom}(M_{\lambda}^*, M_{-\lambda}) \cong \text{Hom}(\mathbb{K}, M_{-\lambda} \otimes M_\lambda) \]

\[ \cong \text{Hom}(\Lambda^{\otimes \lambda_1}, M_{\overline{\lambda}} \otimes M_\lambda) \]

\[ \cong M(c_{\lambda \lambda}^{(\lambda_1)}) \cong \mathbb{K}, \]

where \( c_{\lambda \lambda}^{(\lambda_1)} \) is the Littlewood-Richardson coefficient.
where $M(\xi_{\lambda})$ denotes the matrix ring of rank $c_{\lambda,\lambda}(\xi)$.

The last equation above holds, since it does not depend on $R$ and it holds in the classical case of $\text{GL}(n)$. One can also use Littlewood-Richardson’s rule (cf. [22]) to show directly that $c_{\lambda,\lambda}(\xi) = 1$.

**Theorem 3.2.1.** Let $R$ be an even Hecke symmetry of rank $r$. Then simple $E_R$ comodules are parameterized by partitions of length $\leq r$ and simple $H_R$-comodules are parameterized by $\mathbb{Z}$-partitions of length $\leq r$. For $\lambda \vdash n$, $\mu \vdash m$, we have

$$M(\lambda) \otimes M(\mu) \cong \bigoplus_{\gamma \in \mathbb{P}^n+m} c^{\gamma}_{\lambda,\mu} M(\gamma),$$

where $c^{\gamma}_{\lambda,\mu}$ are the Littlewood-Richardson coefficients. The dual to $M(\lambda)$ comodule is $M(-\lambda)$.

**Proof.** It remains to prove the last statement. Remark that every $E_R$-comodule is a submodule of $V^\otimes_n$, $n = 1, 2, \ldots$ and $\text{End}_{E}(V^\otimes_n)$ is the factor algebra of $H_n$ by the ideal generalized by minimal central idempotents that correspond to partitions of length larger than $r$. Thus, the category $E_R$-comod, up to braided equivalence, does not depend on $R$ but only on its rank $r$.

Analogous discussion holds for $H_R$-comodules. ■

From the above theorem one derives immediately an analog of Peter-Weyl’s decomposition for $H_R$. We have an isomorphism

$$\bigoplus_{\lambda \in \mathbb{P}^Z} \text{End}_\text{E}(M(\lambda)) \cong H_R.$$  \hspace{1cm} (34)

### 3.3. The Computation of $8\text{dim}$ and $\text{rdim}$ of Simple Comodules

The $8\text{dim}$ is an invariant in $H_R$-Comod, however it is not tensor multiplicative and hence is difficult to compute for non-simple comodules. Even the formula for $8\text{dim}(V^\otimes_n)$ is very complicated. There is a so called ribbon-dimension – $\text{rdim}$ – which is multiplicative and, on simple comodules, it differs from $8\text{dim}$ only by a power of $q$. In fact, for a simple $E_R$-comodule $M$ of degree $n$, defined by a primitive idempotent $E \in H_n$, its ribbon-dimension is, by definition,

$$\text{rdim}(M) := q^{n(r+1)} \text{tr}_r(T_{w_n}^2 E) = q^{n(r+1)} \text{tr}_1 \circ \text{tr}_2 \circ \cdots \circ \text{tr}_n(E).$$  \hspace{1cm} (35)

By definition, a ribbon in a strict braided category is a natural isomorphism $\theta$, subject to the following equations:

$$\theta_M \otimes N = (\theta_M \otimes \theta_N) \tau_{N,M} \tau_{M,N}, \quad \theta_M^* = (\theta_M)^*,$$

where $\tau$ denotes the braiding in the category. This definition first appeared in the works of Joyal and Street [15] and Turaev [30]. A strict braided rigid category with a ribbon is called ribbon category. For more detail on ribbon categories, the reader is referred to a book of C. Kassel [17]. For an object $M$ in a ribbon category, the ribbon-dimension, $\text{rdim}(M)$, is by definition the morphism

$$\text{ev}_M \tau_{M,\text{M}^*} (\theta_M \otimes \text{M}^*) \text{db}_M : I \rightarrow I.$$  \hspace{1cm} (37)
By virtue of (30), rdim is multiplicative (cf. [17], Chapter XIV).

**Lemma 3.3.1.** $H_{R}$-Comod possesses a ribbon $\theta$, such that for an idempotent $E$ in $\mathcal{H}_n$, and $M = \text{Im}(\rho(E))$, the ribbon on $M$ is given by

$$\theta_M = q^{n(r+1)/2}T_{w_n}^2 E.$$  

Thus, $H_{R}$-Comod is a ribbon category.

**Proof.** Assume that $\theta$ is a ribbon in $H_{R}$-Comod, then (cf. [27])

$$\theta_M^{-2} = (ev_M \tau_{M,M'} \otimes \text{id}_M)(\text{id}_M \otimes \tau_{M,M'} \circ db_M),$$

for any comodule $M$. Thus, using Equation (34), we have $\theta_V = q^{(r+1)/2} \text{id}_V$. Hence, according to the first axiom for $\theta$, for an idempotent $E \in \mathcal{H}_n$ and $M = \text{Im}$, we have

$$\theta_M = q^{n(r+1)/2} \rho(T_{w_n}^2 E).$$

Remark that $T_{w_n}^2$ is central in $\mathcal{H}_n$. Further, we have $\theta_{\Lambda_n} = q^{-n(r+1)/2} \text{id}_{\Lambda_n}$. Hence we set $\theta_{\Lambda^*} = q^{-n(r+1)/2} \text{id}_{\Lambda^*}$ and extend it on the whole category using Theorem 2.3.4 (iii) and the first axiom in (39).

It remains to verify $\theta_{\Lambda^*} = (\theta_M)^*$. We can assume $M \subset V^\otimes n$ hence reduce it to checking $\theta_{V^\otimes n^*} = (\theta_{V^\otimes n})^*$, and then to checking $\theta_{V^*} = (\theta_V)^*$, by using the identity

$$\tau_{V^\otimes W^\otimes V} = (\tau_{V^\otimes V})^*(V \otimes (\tau_{W^\otimes V} \circ \tau_{U^\otimes W}))(\tau_{U^\otimes V} \otimes W).$$

We have $V^* \cong \Lambda^{r-1} \otimes \Lambda^{r*}$. The problem reduces to checking

$$(\theta_{\Lambda^{r-1}} \otimes \theta_{\Lambda^{r*}}) \circ \tau_{\Lambda^{r-1},\Lambda^{r-1}} \circ \tau_{\Lambda^{r-1},\Lambda^{r*}} = \text{id}_{V^*}.$$  

Or, equivalently

$$\tau_{\Lambda^{r-1},\Lambda^{r-1}} \circ \tau_{\Lambda^{r-1},\Lambda^{r*}} = q^{r(r+1)} \text{id}_{\Lambda^{r-1} \otimes \Lambda^{r}}.$$  

Again, this equation follows from

$$\tau_{V^\otimes n,\Lambda^r} \tau_{\Lambda^{r-1},V^\otimes n} = q^{(r+1)n} \text{id}_{\Lambda^{r} \otimes V^\otimes n}.$$  

By means of Identity (39), an induction reduces it to the check of the above equation for the case $n = 1$. We have

$$\tau^2_{V^\otimes n, V^\otimes n} = R_r R_{r-1} \cdots R_1 R_1 R_2 \cdots R_r = q^r + (q-1)(q^{r-1} R_r + q^{r-2} R_r R_{r-1} + \cdots + R_1 \cdots R_r \cdots R_1).$$

Hence

$$\tau_{V,\Lambda^r} \tau_{\Lambda^{r-1},V^*} = \rho(Y_r) \otimes \text{id}_V \tau^2_{V^\otimes n, V^\otimes n} = q^{r+1} \text{id}_{\Lambda^r \otimes V^*}.$$  

Here we use the fact that $\rho(Y_{r+1}) = 0$. 

Using the definition in (37), we can now define, for $M = \rho(E)$, where $E$ is an idempotent,

$$\text{rdim}(M) = q^{n(r+1)/2} \text{tr}_r(T_{w_n}^2 E) = q^{n(r+1)/2} \text{tr}_r \circ \text{tr}_r \circ \cdots \circ \text{tr}_n(E).$$

According to (32) and since rdim is tensor multiplicative, we have the formula for the rdim of $M_\lambda$:

$$\text{rdim}(M_\lambda) = \det \left( \text{rdim}(M_\lambda)_{i+j} \right)_{1 \leq i, j \leq r}.$$
Direct computation shows that
\[
\text{rdim}(M_{(k)}) = q^{k(r+1)/2} \text{tr}_{r}^{1,2,\ldots,k}(E_{(k)}) = q^{-k(r-1)/2} \frac{[k+r-1]_{q}}{[r-1]_{q}[k]_{q}}.
\] (43)

Using Ex. I.3.1 of [22] we deduce

\begin{equation}
\text{Theorem 3.3.2. Let } M_{\lambda} \text{ be a simple } E_{R}-\text{comodule corresponding to a partition } \lambda. \text{ Then}
\end{equation}
\[
\text{rdim}(M_{\lambda}) = q^{\frac{1}{2}\lambda(\lambda+1)} - \sum_{i=1}^{r} \lambda_{i} \frac{[\lambda_{i} - \lambda_{j} + j - i]_{q}}{[j - i]_{q}}.
\] (44)

Since \text{rdim}(M_{(1^{r})}) = 1 and since the \text{rdim} is tensor multiplicative, Formula (44) holds for any \( \mathbb{Z} \)-partition. As a direct corollary of this theorem we have

\begin{equation}
\text{Proposition 3.3.3. Let } \lambda \text{ be a } \mathbb{Z}-\text{partition. Then}
\end{equation}
\[
8 \text{dim}(M_{\lambda}) = q^{-(\lambda(|\lambda|+2r-2i+2))} \prod_{1 \leq i < j \leq r} \frac{[\lambda_{i} - \lambda_{j} + j - i]_{q}}{[j - i]_{q}}.
\] (45)

Proof. It is sufficient to show that, for \( n = |\lambda| \),
\[
T_{w_{n}}E_{\lambda} = q^{-\left(\sum_{i=1}^{r} \lambda_{i}(\lambda_{i}-2i+1)+n(n-1))\right)/2} E_{\lambda}.
\] (46)

This is the assertion of Corollary A4 in Appendix.

3.4. The Special Matrix Quantum Groups \( S_{H}R \). Assume now that the quantum determinant commutes with elements of \( H_{R} \) (see [9] for a necessary and sufficient condition). Then we can set it equal to 1 to get the special matrix quantum group \( S_{H}R \).

\begin{equation}
\text{Theorem 3.4.1. The Hopf algebra } S_{H}R \text{ is co-semi-simple and its simple comodule are parameterized by partitions of length } \leq \text{rank}(R) - 1.
\end{equation}

Proof. In the isomorphism (34), assume that \( H_{\lambda} \) is the image of \( \text{End}_{\mathbb{K}}(M_{\lambda}) \). Then \( H_{\lambda} \) is a simple subcoalgebra of \( H_{R} \). The multiplication on \( H_{R} \) maps
\[
H_{\lambda} \otimes H_{\mu} \longrightarrow \bigoplus_{\gamma \vdash |\lambda|+|\mu|, l(\gamma) \leq r} H_{\gamma}, \quad (r = \text{rank}(R)).
\]
In particular, it maps \( H_{\lambda} \otimes H_{(1^{r})} \longrightarrow H_{\lambda \cup (1^{r})} \), where \( \lambda \cup (1^{r}) \) is the partition \( (\lambda_{1} + 1, \lambda_{2} + 1, \ldots, \lambda_{r} + 1) \), which is an isomorphism.

Recall that \( H_{(1^{r})} \) is spanned by the quantum determinant. Thus, if we set the quantum determinant equal to 1, (34) reduces to the isomorphism
\[
\bigoplus_{l(\lambda) \leq r - 1} H_{\lambda} \cong S_{H}R.
\] (47)

To see this, we consider the composition
\[
\bigoplus_{l(\lambda) \leq r - 1} H_{\lambda} \longrightarrow H_{R} \longrightarrow S_{H}R
\]
which is obviously surjective. For a partition \( \lambda \), let \( \overline{\lambda} = (\lambda_{1} - \lambda_{r}, \lambda_{2} - \lambda_{r}, \ldots, 0) \). The discussion above shows that \( H_{\lambda} \) and \( H_{\overline{\lambda}} \) have the same image in \( S_{H}R \). Further, since the
The determinant is not a zero-divisor and lies in the center of $H_R$, the above map in injective when restricted on each $H$, Whence (17) is an isomorphism.

The comodule $\Lambda^r$, being considered as a $SH_R$-comodule, is isomorphic to the trivial module. Hence, it is expected that this module has an $\text{dim}$ equal to 1. This can be made by rescaling $R \mapsto q^{-\frac{r+1}{2r}} R$. We call the rescaled $\text{dim}$ the normalized $\text{dim}$ and denote it by $\text{dim}_n$.

The last equation shows particularly that the $\text{dim}$ of $M_\lambda$ and of $M_\lambda \otimes \Lambda^r$ are equal. As a corollary of the above discussion we have

**Proposition 3.4.2.** Let $R$ be such an even Hecke symmetry that the quantum determinant commutes with all elements of $H_R$, so that one can set it equal to 1 to get the corresponding special matrix quantum groups $SH_R$. Then the coquasitriangular structure $\langle \cdot | \cdot \rangle$ on $SH_R$ can be defined, setting

$$
\langle z^k_j | z^l_j \rangle = q^{-\frac{r+1}{2r}} R_{ij}^{kl}.
$$

The ribbon in $SH_R$-Comod is then given by

$$
\tau_{V \otimes n} = q^{\frac{r(r^2-1)}{2r}} \tau_{w_{1n}}^{\frac{r}{2r}}.
$$

$\tau := q^{-\frac{r+1}{2r}} R$. Direct computation shows that the $\text{dim}$ remains unchanged.

From all results obtained we can interpret the quantum group of type $A_n$ as follows.

**Definition 3.4.3.** The quantum group $\text{GL}_q(r)$ ($\text{SL}_q(r)$) is a $K$-linear braided rigid semi-simple abelian category, whose simple objects are parameterized by $\mathbb{Z}$-partitions $\{\lambda|\lambda_{r+1} = 0\}$ (partitions $\{\lambda|\lambda_i = 0\}$) and the tensor product satisfy Equation (33). The braiding on the simple object $V$ that corresponds to partition (1) satisfies the equation $(R - q)(R + 1) = 0$.

Each even Hecke symmetry of rank $r$ induces a (monoidal) functor from $\text{GL}_q(r)$ into $\text{Vect}_{K}$, called a realization of this quantum group. For a realization of $\text{SL}_q(r)$ we need certain extra condition on the Hecke symmetry.

**Remark.** According to a result of Kazhdan-Wenzl [18], each monoidal category with the Grothendieck ring isomorphic to the one of the category of rational representations of $\text{GL}(r)$ ($\text{SL}(r)$) should be equivalent to $\text{GL}_q(r)$ ($\text{SL}_q(r)$) for some $q$.

According to Theorem 3.3.2, we have

**Proposition 3.4.4.** For $q \neq q'$, $\text{GL}_q(r)$ and $\text{GL}_{q'}(r)$ (resp. $\text{SL}_q(r)$ and $\text{SL}_{q'}(r)$) are not equivalent as braided categories.

Indeed, if there were such an equivalent, the $\text{dim}$ of the corresponding objects would be equal, which is obviously not the case if $q \neq q'$. 
4. The Integral

By definition, a (left) integral on a Hopf algebra \( H \) is a linear functional \( \int : H \rightarrow k \) satisfying the following condition:

\[
\text{if } \Delta(x) = \sum_{(x)} x_1 \otimes x_2 \text{ then } \int(x) = \sum_{(x)} x_1 \int(x_2). \tag{48}
\]

Since \( H_R \) and \( SH_R \) are co-semi-simple, such an integral on them exists. If we rescale it, setting \( \int(1) = 1 \), then it exists uniquely. Our aim in this section is to compute explicitly the integral for \( H_R \) and \( SH_R \).

4.1. The Integral on \( H_R \). Let us make \( H_R \) into a \( \mathbb{Z} \)-graded algebra by setting \( \deg_Z(z_j^i) = 1 \), \( \deg_Z(t_j^i) = -1 \). For \( x \) homogeneous, we can choose an expression \( \Delta(x) = x_1 \otimes x_2 \), such that \( \deg_Z(x) = \deg_Z(x_1) = \deg_Z(x_2) \). Hence, \( \int \) is not a homogeneous relation unless \( \deg_Z(x) = 0 \), while \( H_R \) is homogeneous. Thus, we proved the following.

**Lemma 4.1.1.** The value of \( \int \) on a homogeneous element of \( H_R \) (with respect to the above grading) is equal to zero unless the degree of this element is zero.

The problem reduces to finding the values of \( \int \) on monomials in \( z_j^i \)'s and \( t_j^k \)'s, in which the numbers of \( z_j^i \)'s and \( t_j^k \)'s are equal. By virtue of the relation in \( [3] \), every monomial in \( z_j^i \)'s and \( t_j^k \)'s can be expressed as a linear combination of monomials in which \( z_j^i \)'s are on the right of \( t_j^k \)'s. Hence it reduces to computing:

\[
\int(z_{i_1}^1 z_{i_2}^2 \cdots z_{i_n}^n t_{k_1}^1 t_{k_2}^2 \cdots t_{k_n}^n).
\]

At first sight, it is not clear if a set of values of \( \int \) on monomials in \( z_j^i \)'s and \( t_j^k \)'s is invertible, \( \int \) being a linear combination of \( z_j^i \)'s and \( t_j^k \)'s, which ensures that different ways of expressing yield the same result.

We need some notations. The Murphy operators \( L_m \), \( 1 \leq m \leq n \), were introduced in \( [2] \) as follows: \( L_1 := (1) \).

\[
L_m := q^{-1}T_m -1 + q^{-2}T_{m-2}T_m -T_{m-2} + \cdots + q^{-m+1}T_1T_m \cdots T_{m-1}T_1, m \geq 2.
\]

For an element \( W \) of \( \mathcal{H}_n \), we shall denote \( \overline{W} \) it image under the map \( \rho : \mathcal{H}_n \rightarrow \text{End}^{E_R}(V^{\otimes n}) \), the image of \( \mathcal{H}_n \) itself will be denoted by \( \overline{\mathcal{H}}_n \). Lemma \( \Delta.6 \) in the Appendix shows that the operator \( (\overline{L}_n - [-r]_q) \) is invertible in \( \overline{\mathcal{H}}_n = \text{End}^{E_R}(V^{\otimes n}) \).

**Theorem 4.1.2.** Let \( R \) be an even Hecke symmetry of rank \( r \). Then the integral of \( H_R \) can be given by the following formula

\[
\int Z_j^i T_K^L = \sum_{w \in \mathfrak{S}_n} q^{-1(w)} \left( \prod_{k=1}^n (\overline{L}_k - [-r]_q)^{-1}R_{w^{-1}C^{\otimes n}} \right)_L^J R_{wK'}^{J'}. \tag{49}
\]

\(^1\)There was a misprint in \( [2] \) where \( L_1 := 1 \).
where $I, J, K, L$ are multi-indices, $Z^I_1 := z_1^{j_1^1} z_2^{j_2^1} \cdots z_n^{j_n^1}$, and for $K = (k_1, k_2, \ldots, k_n)$, $K' := (k_n, k_{n-1}, \ldots, k_1)$. The matrix $C$ was introduced in Equation (8).

Proof. We should check that $\int$ satisfies (18) and is well defined, i.e., is compatible with the defining relations. The verification of (48) is rather trivial. Let us check that the formula for $\int$ is compatible with the relations (3) and (4). For simplicity let us denote $P_n := \prod_{k=1}^n (L_k - [-r]_q)$. Thus $\overline{P}_n = \rho_n (P_n)$ is invertible in $\overline{H}_n$.

Remark that (3) together with (3) implies (3). Thus we have to check

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} (\overline{P}_n^{-1} R_{w^{m-1}} C^{\otimes n})_{I_1 m} R_w J = \sum_{w \in \mathcal{G}_{n-1}} q^{-l(w)} (\overline{P}_{n-1} R_{w^{m-1}} C^{\otimes n-1})_{I_1 m} R_{w^{l}} J \delta^j_{k_n}, \quad (50)$$

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} (\overline{P}_n^{-1} R_{w^{m-1}} C^{\otimes n-1})_{I_1 m} R_w J_{n} C_p = \sum_{w \in \mathcal{G}_{n-1}} q^{-l(w)} (\overline{P}_{n-1} R_{w^{m-1}} C^{\otimes n-1})_{I_1 m} R_{w^{l}} J \delta^j_{k_n}, \quad (51)$$

here, for a multi-index $K = k_1, k_2, \ldots, k_n$ we denote $K_1 := k_1, k_2, \ldots, k_{n-1}$.

First, we need a technical lemma.

Lemma 4.1.3. The following relation holds in $\mathcal{H}_n$:

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} T_{w^{m-1}} \otimes \text{tr}^n(T_w) = \sum_{w \in \mathcal{G}_{(n-1, 1)}} q^{-l(w)} (L_n - [-r]_q) T_{w^{m-1}} \otimes T_w. \quad (52)$$

The proof will be given in the Appendix [A.7].

We check (51). By definition of $8\text{tr}$,

$$R_{w^{l}} J_{n} C_p = 8\text{tr}(R_{w^{l}})^n J_{n}.$$ 

Thus, without indices, (51) has the form

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} (\overline{P}_n^{-1} R_{w^{m-1}} C^{\otimes n-1}) \otimes 8\text{tr}^n(R_w) = \sum_{w \in \mathcal{G}_{(n-1, 1)}} q^{-l(w)} (\overline{P}_{n-1} R_{w^{m-1}} C^{\otimes n-1}) \otimes R_w.$$ 

Remark that $P_n = P_{n-1} (L_n - [-r])$, and $\overline{P}_{n-1}$ acts only on the first $n-1$ components of $V^{\otimes n}$, we see that the above equation follows from (52).

We check (50). (50) can be rewritten as

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} ((\overline{L}_n - [-r])^{-1} R_{w^{m-1}} C^{\otimes n-1})_{I_1 m} R_w J = \sum_{w \in \mathcal{G}_{n-1}} q^{-l(w)} (R_{w^{m-1}} C^{\otimes n-1})_{I_1 m} R_{w^{l}} J \delta^j_{k_n}.$$ 

Omitting indices and canceling $C^{\otimes n-1}$, it has the from

$$\sum_{w \in \mathcal{G}_n} q^{-l(w)} 8\text{tr}^n((\overline{L}_n - [-r])^{-1} R_{w^{m-1}}) \otimes R_w = \sum_{w \in \mathcal{G}_{n-1, 1}} q^{-l(w)} R_{w^{m-1}} \otimes R_w.$$ 

The trick here is to take $(\overline{L}_n - [-r])^{-1}$ out of $8\text{tr}^n$. To do this we consider the left multiplication by this element as an endomorphism of $\overline{H}_n$.

Let $P \in \mathcal{H}_n$ such that $\overline{P} = (\overline{L}_n - [-r]_q)^{-1}$. Let

$$(L_n - [-r]_q)T_v = \sum_{u \in \mathcal{G}_n} c^v_u T_u \quad \text{and} \quad PT_v = \sum_{u \in \mathcal{G}_n} d^v_u T_u.$$
then
\[ \sum_{w \in \mathfrak{S}_n} c_v^w d^w_u R_u = \delta_{u,v} R_v. \]

Set \( \langle T_u, T_v \rangle = \delta_{u,v} q^{(v)} \) and extend it linearly on \( \mathcal{H}_n \), we get a non-degenerate symmetric scalar product on \( \mathcal{H}_n \) which satisfies \( \langle hk, g \rangle = \langle h, g k^* \rangle \) and \( \langle h, g k \rangle = \langle g^* h, k \rangle \), where \( \star \) is the linear extension of the map \( T_w \mapsto T_{w^{-1}}, w \in \mathfrak{S}_n \) \( [1] \), Lemma 2.2). Since \( (L_n - [-r]_q) \) is \( \star \)-invariant, we have:
\[
q^{(v)} c_v^w = \langle T_w, (L_n - [-r]_q) T_v \rangle = \langle (L_n - [-r]_q) T_w, T_v \rangle.
\]

Hence
\[
(L_n - [-r]_q) T_w = \sum_{v \in \mathfrak{S}_n} q^{(v) - l(v)} c_v^w T_v. \tag{53}
\]

Since \( T_w, w \in \mathfrak{S}_n \) form a basis for \( \mathcal{H}_n \), we can conclude that \( c_v^w = q^{(v) - l(v)} c_u^v \). Applying first operation \( \star \) on \( (53) \) and then \( \rho \), we get
\[
R_{w^{-1}} (L_n - [-r]_q) = \sum_{v \in \mathfrak{S}_n} q^{l(v) - l(v)} c_v^w R_{w^{-1}}.
\]

Now, we have
\[
\sum_{v \in \mathfrak{S}_n} q^{l(v) - l(u)} \text{Str}^n((L_n - [-r]_q) R_{w^{-1}} \otimes R_{w} (L_n - [-r]_q))
= \sum_{w,u,v \in \mathfrak{S}_n} q^{l(v) - l(u)} d_{w^{-1}}^{w} \text{Str}^n (R_{w^{-1}}) \otimes q^{l(v) - l(u)} c_{u^{-1}}^w R_u
= \sum_{w,u,v \in \mathfrak{S}_n} q^{l(u) - l(v)} \text{Str}^n (R_{u^{-1}}) \otimes R_u \text{ (by virtue of Lemma 4.1.3)}
= \sum_{w \in \mathfrak{S}_n} q^{l(v) - l(v)} (R_{w^{-1}}) \otimes R_{w} (L_n - [-r]_q).
\]

Since \( (L_n - [-r]_q) \) is invertible, \( (50) \) is proved. \( \Box \)

4.2. The Integral on \( SH_R \). The integral on \( SH_R \) is easier to compute. By its definition, \( SH_R \) is a quotion of the bialgebra \( E_R \), generated by \( \{ x^i_j \mid 1 \leq i, j \leq d \} \), by setting the quantum determinant equal to 1. Thus, we have to compute \( \int (Z_I^J) \), where \( I, J \) are multi-indices.

Introduce a \( \mathbb{Z}_r \)-grading on \( SH_R \), setting \( \deg(z^i_j) = 1 \mod r \). Then \( SH_R \) is defined by \( \mathbb{Z}_r \)-homogeneous relation. From its definition we see that
\[
\int (Z_I^J) = 0, \quad \text{unless } l(I) = l(J) = 0 \mod r.
\]

Theorem 4.2.1. Let \( l(I) = l(J) = n = kr \) and \( F_{(k^r)} \) be the minimal central idempotent, that corresponds to \( (k^r) \). Let \( \Phi_k = \rho_n (F_{(k^r)}) \). Then
\[
\int (Z_I^J) = \Phi_{kI}^J. \tag{54}
\]
Proof. It is known that the integral has the following property. For a comodule \( M \) on \( SH_R \), the map \( M \to M, m \mapsto m_0 \int (m_1) \) is the projection from \( M \) onto its subspace of \( SH_R \)-coinvariants, that are \( n \in M, \delta(n) = n \otimes 1 \). Moreover, the integral is determined by this property.

Since every simple \( SH_R \)-comodule is a submodule of \( V \otimes n \) for some \( n \), it is sufficient to check the above property for \( V \otimes n \). Notice that the subspace of \( SH_R \)-coinvariant in \( V \otimes n \) is the sum of all \( V \otimes n \)-subcomodules that are isomorphic to \( \mathbb{K} \). From the proof of Theorem 3.4.1, each submodule of \( V \otimes n \), isomorphic to \( \mathbb{K} \), is determined by a primitive idempotent of \( \mathcal{H}_n \), that corresponds to \((k^r)\). In other words, if \( V \otimes n \) contains a non-trivial subspace of \( SH_R \)-coinvariant, then \( n = kr \) for some \( k \) and for the image of primitive idempotent, corresponding to \((k^r)\), under \( \rho_n \) is non-zero (Corollary 2.3.2). As an easy consequence, we see that the subspace of \( SH_R \)-coinvariant is precisely \( \text{Im}(\rho_n(F(k^r))) \). \( \Box \)

In Theorem 4.2.3 a full system of mutually orthogonal primitive idempotent, corresponding to a partition \( \lambda \), is given. Consequently, their sum is the minimal central idempotent corresponding to \( \lambda \):

\[
F_\lambda = \sum_{i=1}^{d_\lambda} E_{i,\lambda}.
\]

We give below a simpler description of \( \Phi_k \). First we remark

Lemma 4.2.2. The minimal central idempotent \( F_\lambda \) can be obtained from a primitive central idempotent \( E_\lambda \) in the following way:

\[
F_\lambda = \text{tr}(E_\lambda) \sum_{w \in \mathcal{S}_n} q^{-l(w)}T_w E_\lambda T_{w^{-1}}
\]

where \( \text{tr}(h) := \langle h, 1 \rangle \), \( \langle , \rangle \) is the scalar product introduced in the previous subsection.

Proof. Let \( A_\lambda \) be the two-sided ideal generated by \( F_\lambda \). Thus, \( A_\lambda \) is a full matrix ring, of degree \( d_\lambda \). Let \( \{E_{\lambda}^{ij}\}^{1 \leq i, j \leq d_\lambda} \) be a basis of \( A_\lambda \) with the property

\[
E_{\lambda}^{ij} E_{\lambda}^{kl} = \delta_{ik}^{j} E_{\lambda}^{jl}.
\]

Thus, \( E_{\lambda}^{ij} \) are primitive idempotents and \( F_\lambda = \sum_i E_{\lambda}^{ii} \). We let \( \lambda \) run it the set of all partitions of \( n \), then \( \{E_{\lambda}^{ij}\}^{1 \leq i, j \leq d_\lambda} \) form a basis for \( \mathcal{H}_n \).

The map \( \text{tr} \) is by definition a trace map, which is faithful, i.e. \( \forall G, \exists H, \text{tr}(GH) \neq 0 \). By standard argument (see [3]) \( \text{tr}(E_{\lambda}^{ij}) = 0 \) unless \( i = j \) and \( \text{tr}(E_{\lambda}^{ij}) \) are equal for all \( i \) to a non-zero constant \( k_\lambda \). Hence \( \{\frac{1}{k_\lambda} E_{\lambda}^{ij} | 1 \leq i, j \leq d_\lambda \} \) form a basis, dual to the described above basis with respect to the scalar product \( \langle , \rangle \). On the other hand, the bases \( \{T_w | w \in \mathcal{S}_n \} \) and \( \{q^{-l(w)}T_w | w \in \mathcal{S}_n \} \) are dual w.r.t this scalar product, too. Therefore we have:

\[
\sum_{w \in \mathcal{S}_n} q^{-l(w)}T_w HT_{w^{-1}} = \sum_{i \leq k, j \leq d_\lambda} \frac{1}{k_\lambda} E_{\lambda}^{ij} H E_{\lambda}^{ji}, \quad \text{for any } H \in \mathcal{H}_n.
\]

In particular, for \( H = E_\lambda = E_{\lambda}^{kk} \), we have

\[
\sum_{w \in \mathcal{S}_n} q^{-l(w)}T_w E_{\lambda} T_{w^{-1}} = \frac{1}{k_\lambda} F_\lambda. \fp
The value of \( \text{tr}(E_\lambda) \) can be computed explicitly:

\[
\text{tr}(E_\lambda) = q^{\sum \lambda_i(i-1)} \prod_{k=1}^{n} \frac{1}{[r_\lambda(k) + r]_q} \prod_{j < i} \left[ \frac{[\lambda_i - \lambda_j + j - i]_q}{[j - i]_q} \right].
\]  

(55)

The proof of this formula will be given in a forthcoming paper \[12\].

In our case, the operator \( \overline{E}_{(k^r)} \), according to Lemma 2.3.3, is the product

\[
\overline{E}_{(k^r)} = \overline{\mathbf{Y}}_r \otimes \overline{\mathbf{Y}}_r \otimes \cdots \otimes \overline{\mathbf{Y}}_r.
\]

Thus, we have

**Corollary 4.2.3.** The operator \( \Phi_k \) can be given by

\[
\Phi_k = \text{tr}(E_{(k^r)}) \sum_{w \in \mathfrak{S}_k} q^{-l(w)} R_w \overline{\mathbf{Y}}_r \otimes \mathbf{Y}_r^{-1}
\]

\[
= q^{kr(r-1)/2} \frac{[0]_q! [1]_q! \cdots [k - 1]_q!}{[r]_q! [r + 1]_q! \cdots [r + k - 1]_q!} \sum_{w \in \mathfrak{S}_k} q^{-l(w)} R_w \overline{\mathbf{Y}}_r \otimes \mathbf{Y}_r^{-1}.
\]  

(56)

Comparing the two formulae for integral on \( H_R \) and \( SH_R \), we notice that the integral on the latter can be obtained from the integral on the other one in the following sense.

The quantum determinant can be represented in the form \( q^{(r+1)/2} B_{M} \mathbf{Y}_r^{M} Z_{N}^{r} \), and it inverse is \( q^{(r+1)/2} B_{M} \mathbf{Y}_r^{M} Z_{N}^{r} \), the matrix \( B \) was introduced in Equation (47). Therefore, if we apply the formula (19) for the element \( Z_{I} B_{M} \mathbf{Y}_r^{M} Z_{N}^{r} \), \( I, J, K, L \in \mathbb{N}^{kr} \) then we have

\[
\int (q^{(r+1)/2} Z_{I} B_{M} \mathbf{Y}_r^{M} Z_{N}^{r})
\]

\[
= q^{(r+1)/2} \sum_{w \in \mathfrak{S}_k} q^{-l(w)} \left( \prod_{m=1}^{kr} (R_{w^{-1} \mathcal{L} w} - [-r]_q)^{-1} C_{M} \otimes \mathbf{Y}_r^{-1} \right)^{J}
\]

\[
= q^{kr(r-1)/2} \frac{[0]_q! [1]_q! \cdots [k - 1]_q!}{[r]_q! [r + 1]_q! \cdots [r + k - 1]_q!} \sum_{w \in \mathfrak{S}_k} q^{-l(w)} (R_{w} \mathbf{Y}_r \otimes \mathbf{Y}_r^{-1})^{J}
\]

in the second equation we use Equations (28) and (29) and the fact that \( BC = q^{-r+1} I \) which follows from Equation (14).

**Appendix A. The Hecke Algebras**

We recall in this Appendix the definition and some important properties of the Hecke algebra. Then we derive from them some results that are needed in our context.

The symmetric groups \( \mathfrak{S}_n \) consists of permutations of the set \( \{1, 2, \ldots, n\} \). It is generated by the basic transpositions \( v_i = (i, i + 1), 1 \leq i \leq n - 1 \). The length of a permutation \( v \) is the number of pairs \( (i, j), 1 \leq i < j \leq n \), such that \( iv > jv \). This is equal to the minimal number of \( v_i \)’s needed to express \( v \). Irreducible representations of \( \mathfrak{S}_n \) are indexed by partitions of \( n \).

Let \( \lambda \) be a partition of \( n \), a \( \lambda \)-tableau \( a^\lambda \) is a matrix \( a^\lambda \), \( 1 \leq j \leq \lambda_i, i = 1, 2, \ldots \), where \( a^\lambda \) are different elements from the set \( \{1, 2, \ldots, n\} \). A tableau is said to be row- (column-).
standard if the numbers $1, 2, \ldots, n$ increase along the rows (columns) and standard if it is row- and column-standard.

**A.1. The Hecke Algebra $H_n$**

Let $\mathbb{K}$ be a field of characteristic zero, $0 \neq q \in \mathbb{K}$. As a $\mathbb{K}$-vector space, the Hecke algebra $H_{n, q}$ is spanned by $\{ T_v \mid v \in \mathfrak{S}_n \}$. The multiplication satisfies the following relations:

- $T_1 = 1_{H_{n, q}}$,
- $T_v T_w = T_{vw}$ iff $l(vw) = l(v) + l(w)$,
- $T_{v_i} = (q - 1)T_{v_i} + q 1$, $v_i = (i, i + 1)$.

Let us denote $T_i := T_{v_i}$, $1 \leq i \leq n - 1$. As is the case of $\mathfrak{S}_n$, $T_i$ generate $H_n$ and satisfy:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, 1 \leq i \leq n - 2.$$ 

If $q$ is not a root of unity then $H_n = H_{n, q}$ is semi-simple.

For $n \in \mathbb{Z}$, define $[n]_q := (q^n - 1)/(q - 1)$ and $[\pm n]_q! := [\pm 1]_q[\pm 2]_q \cdots [\pm n]_q$, $[0]_q! := 1$.

We denote,

$$X_n := \frac{1}{[n]_q!} \sum_{w \in \mathfrak{S}_n} T_w, \quad Y_n := \frac{1}{[n]_q!} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} T_w.$$ 

$X_n$ (resp. $Y_n$) is a minimal central primitive idempotent of $H_n$, which induces the trivial (resp. signature) representation of $H_n$

$$T_w X_n = X_n T_w = q^{l(w)} X_n, \quad T_w Y_n = Y_n T_w = (-1)^{l(w)} Y_n.$$ 

**A.2. Murphy Operators and Primitive Idempotents**

Primitive idempotents of $H_n$ can be expressed in terms of Murphy operators. The Murphy operators $L_m, 1 \leq m \leq n$, were introduced in [2] as follows: $L_1 := 0$,

$$L_m := q^{-1}T_{m-1} + q^{-2}T_{m-2}T_{m-1} + \cdots + q^{-m+1}T_1 T_2 \cdots T_{m-1} T_1, m \geq 2.$$ 

The following equality will be frequently used:

$$T_k T_{k+1} \cdots T_{m-1} T_m T_{m-1} \cdots T_{k+1} T_k = T_m T_{m-1} \cdots T_{k+1} T_k T_{k+1} \cdots T_{m-1} T_m,$$

$k < m \leq n - 1$.

We summarize some properties of Murphy operators obtained in [2] in a theorem.

**A.3. Theorem**

(i) $L_m, 1 \leq m \leq n$ commute and symmetric polynomials on them form the center of $H_n$ ([2], Theorem 2.14).

(ii) Simple $H_n$ modules are parameterized by partitions of $n$. The simple module $S_\lambda$, $\lambda \vdash n$, has dimension $d_\lambda$ equal to the number of standard $\lambda$-tableaux. There exists a basis (called Young’s semi-normal basis), indexed by standard $\lambda$-tableaux, $t_i = t_{i, \lambda}, i = 1, 2, \ldots, d_\lambda$, such that

$$l_i L_m = [r_{i, \lambda}(m)]_q t_i, \quad (57)$$

$r_{i, \lambda}(m) := k - j$, where $(j, k)$ is the coordinate of $m$ in the standard $\lambda$-tableau $a^\lambda_i$ ([2], Lemma 4.6).}

\[2\] In [2], $r_{i, \lambda}(m)$ is defined to be $[k - j]_q$. 

(iii) The set
\[
\left\{ E_{i,\lambda} := \prod_{|k| \leq n-1} \prod_{k \neq r_i,\lambda(m)} \frac{L_m - [k]_q}{[r_i,\lambda(m)]_q - [k]_q} \left| 1 \leq i \leq d, \lambda \vdash n \right. \right\}
\]
is a complete set of primitive idempotents of \( \mathcal{H}_n \) (\cite{2}, Theorem 5.2).

As a direct consequence of Theorem A.3 we have
\[
\prod_{|c| \leq m-1} (L_m - [c]_q) = 0, \quad (58)
\]
since \(|r_{i,\lambda}(m)| \leq m - 1\). Hence we have
\[
E_\lambda = \prod_{1 \leq m \leq n} \frac{L_m - [k]_q}{[r_{i,\lambda}(m)]_q - [k]_q}. \quad (59)
\]

**A.4. Corollary** For any primitive idempotent \( E \in \mathcal{H}_n \), we have
\[
T_{w_n}^2 E_\lambda = q^{-s(\lambda)-n(n-1)/2} E_\lambda.
\]

*Proof.* According to (14), \( T_{w_n}^2 E = \prod_{k=1}^n q^k (1 + (q-1)L_k) \). Equation (58) and Formula (53) imply the desired equation. \( \blacksquare \)

**A.5. The completion of Lemma 2.3.3**

Assume that \( n = m + r, \quad r > 1 \). For an element \( H \in \mathcal{H}_n \) denote \( \overline{H} := \rho(h) \) and for an element \( K \in \mathcal{H}_m \), \( K^r \) its image under the embedding \( \mathcal{H}_m \ni T_j \rightarrow T_{r+j} \in \mathcal{H}_n \). Assume that
\[
E = E_\lambda = \prod_{1 \leq k \leq m} \frac{T_k - [c]_q}{[r_{i,\lambda}(k)]_q - [c]_q},
\]
where \( \lambda \) is a partition of \( m \).

We have \( \overline{Y}_r (T - T_{r+1}) = [r+1]_q \overline{Y}_{r+1} = 0 \). On the other hand, for \( k \geq 1 \),
\[
L_{k+r} = L_k + q^{-k+1} T_{k+r-1} \cdots T_{r+1} L_{r+1} T_{r+1} \cdots T_{k+r-1}.
\]
Hence
\[
\overline{Y}_r T_{k+r} = \overline{Y}_r T_k + q^{-k+1} T_{k+r-1} \cdots T_{r+1} \overline{Y}_r T_{r+1} T_{r+1} \cdots T_{k+r-1} = \overline{Y}_r (1 + q T_k).
\]
Therefore
\[
\overline{Y}_r (T_{k+r} - [c]_q) = \overline{Y}_r (1 + q T_k - [c]_q) = q \overline{Y}_r (T_k - [c-1]_q). \quad (60)
\]
Remark that \( \overline{Y}_r \) commutes with all \( T_k \), thus
\[
\overline{Y}_r E^r = \overline{Y}_r \prod_{1 \leq k \leq m} \frac{q^{-1}(T_{k+r} - [c+1]_q)}{[r_{i,\lambda}(k)]_q - [c]_q} = \overline{Y}_r \prod_{1 \leq k \leq m} \frac{T_{k+r} - [c+1]_q}{[r_{i,\lambda}(k) + 1]_q - [c+1]_q}.
\]
Equations (60) and (58) imply
\[
\prod_{|c| \leq k-1} (T_{k+r} - [c + 1]q) = \prod_{|c| \leq k-1} (T_k - [c + 1]q) = 0,
\]
which means that
\[
\prod_{|c| \leq k-1} \frac{L_{k+r} - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q} \cdot L_{k+r} = \prod_{|c| \leq k-1} \frac{L_k - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q}.
\]
Hence, for \( r \geq 2, k = 1, 2, \ldots, m, \)
\[
\prod_{|c| \leq k-1} \frac{L_{k+r} - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q} = \prod_{|c| \leq k-1} \frac{L_k - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q}.
\]
Since \( L_{k+r} \) commute each other, \( Y_r \) is an idempotent, we have:
\[
\prod_{1 \leq k \leq m \atop |c| \leq k-1 \atop c \neq r_i,\lambda(k)} \frac{L_{k+r} - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q} = \prod_{1 \leq k \leq m \atop |c| \leq k-1} \frac{L_k - [c + 1]q}{[r_i,\lambda(k) + 1]q - [c + 1]q} = \prod_{1 \leq k \leq m \atop |c| \leq k-1 \atop c \neq r_i,\lambda(k)} Y_r E_r.
\]
The left-hand side term of this equality is a primitive idempotent, hence, so is the right-hand side term. \( \Box \)

**A.6. Lemma** Let \( R \) be an even Hecke symmetry of rank \( r \) and \( \rho_n : H_n \rightarrow \overline{H}_n \) be the corresponding representation of \( H_n \). Then \( \left( L_n \right) - [-r]q \) is invertible in \( \overline{H}_n \).

**Proof.** According to Proposition 3.1.4, \( \Psi_\lambda = 0 \) whenever \( \lambda_{r+1} \neq 0 \). Hence a simple module \( S_\lambda \) of \( H_n \) remains simple \( \overline{H}_n \)-module only if \( \lambda_{r+1} = 0 \). For these \( \lambda, r_i,\lambda(m) \geq -r + 1 \). Hence, according to Theorem 3.3, (ii), \( L_n - [-r]q \) is invertible on \( S_\lambda \). This holds for all \( \lambda \), thus, \( L_n - [-r]q \) is invertible on \( \overline{H}_n \). \( \Box \)

**A.7. The proof of Lemma 4.1.3**

the left-hand side of Equation (52) is equal to
\[
-[-r] \sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)} T_{w-1} \otimes T_w + \sum_{k=1}^{n-1} \sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)-k} T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_w T_{n-2} \cdots T_{n-k}.
\]
We have
\[
\sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)-k} T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_w T_{n-2} \cdots T_{n-k}
\]
\[
= \sum_{w \in \mathfrak{S}_{(n-1,1)}} \left[ q^{-l(w)-k} T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{wv_{n-2}} T_{n-2} \cdots T_{n-k} \right.
\]
\[
+ q^{-l(w)-k} T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{w} T_{n-2} \cdots T_{n-k} \right]
\]
\[
= \sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)-k} T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{wv_{n-2}} T_{n-2} \cdots T_{n-k}
\]
\[
+ qT_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{n-2} \cdots T_{n-k}
\]
\[
+ qT_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{w} T_{n-2} \cdots T_{n-k}
\]
\[
+ (q-1)T_{n-k} \cdots T_{n-1} T_{w-1} \otimes T_{w} T_{n-2} \cdots T_{n-k}
\]
\[
\begin{align*}
&= \sum_{w \in \mathfrak{S}_{n-1,1}} q^{-l(w)-k-1} \left[ T_{n-k} \cdots T_{n-1} (T_{n-2})^2 T_{w-1} \otimes T_{wv_{n-2}} T_{n-3} \cdots T_{n-2} \\
&\quad + qT_{n-1} T_{n-k} \cdots T_{n-2} T_{n-1} T_{w-1} \otimes T_w T_{n-3} \cdots T_{n-2} \right] \\
&= \sum_{w \in \mathfrak{S}_{n-1,1}} q^{-l(w)-k} T_{n-1} T_{n-k} \cdots T_{n-2} T_{n-1} T_{w-1} \otimes T_w T_{n-3} \cdots T_{n-2} \\
&= \cdots \text{ (repeating the above process } k - 2 \text{ times)} \\
&= \sum_{w \in \mathfrak{S}_{n-1,1}} q^{-l(w)-k} T_{n-1} T_{n-2} \cdots T_{n-k} T_{n-k+1} \cdots T_{n-2} T_{n-1} T_{w-1} \otimes T_w
\end{align*}
\]

Summing up the above equations for \( k = 1, 2, \ldots n - 1 \) we arrive at the desired equation.

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