Asymptotic properties of neutral type linear systems

Leonid Berezansky\textsuperscript{a}, Elena Braverman\textsuperscript{b}

\textsuperscript{a}Dept. of Math., Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
\textsuperscript{b}Dept. of Math. and Stats., University of Calgary, Calgary, AB T2N 1N4, Canada

Abstract
Exponential stability and solution estimates are investigated for a delay system
\[
\dot{x}(t) - A(t)\dot{x}(g(t)) = \sum_{k=1}^{m} B_k(t)x(h_k(t))
\]
of a neutral type, where $A$ and $B_k$ are $n \times n$ bounded matrix functions, and $g, h_k$ are delayed arguments. Stability tests are applicable to a wide class of linear neutral systems with time-varying coefficients and delays. In addition, explicit exponential estimates for solutions of both homogeneous and non-homogeneous neutral systems are obtained for the first time. These inequalities are not just asymptotic estimates, they are valid on every finite segment and evaluate both short- and long-term behaviour of solutions.

Keywords: linear neutral delay system, exponential stability, non-autonomous system, exponential estimates of solutions, matrix measure

AMS subject classification: 34K20, 34K40, 34K25, 34K06

1. Introduction
There are many papers and monographs on stability of scalar neutral differential equations, see a review of explicit stability tests for this class of equations in \cite{3,4}. For linear vector neutral delay equations, explicit exponential stability results can be found in the monographs \cite{1,11,19,20,27} and the papers \cite{2,9,12,17,23,24,25,26}.

A most popular approach to study stability for all classes of functional differential equations, including neutral, is the application of Lyapunov-Krasovskii functionals \cite{1,19,20,27}. LMI (linear matrix inequality) method is also based on the method of Lyapunov-Krasovskii functionals \cite{10,22,26}. Some stability results were obtained by using the Bohl-Perron theorem \cite{1,3,4,5,6,7,9} and application of special properties of matrices, including fundamental matrix functions for differential equations \cite{11,14,15,24,25}. Several tests for linear and nonlinear scalar neutral equations were obtained by the fixed point method \cite{18,30}. Stability conditions based on the application of fixed point theorems usually include the first and the second derivatives of delay functions. A more detailed discussion of these methods and comparison of stability results of the present paper to some earlier obtained is postponed to the end of the paper.

Recently in \cite{4}, exponential estimates were derived for a neutral delay equation
\[
\dot{x}(t) - a(t)\dot{x}(g(t)) = \sum_{k=1}^{m} b_k(t)x(h_k(t)), \quad t \geq t_0 \geq 0
\]
in the scalar case. In the present paper we extend the results obtained in [4] to a system
\[\dot{x}(t) - A(t)x(g(t)) = \sum_{k=1}^{m} B_k(t)x(h_k(t)), \ t \geq t_0 \geq 0, \] (1.2)
where some of the stability tests are new also for scalar equation (1.1) (we will discuss this question in the last section) and for a vector linear delay differential equations without the neutral part when \(A(t) \equiv 0\).

We study non-autonomous system (1.2) and get stability tests where not only the maximal of all the delays but each \(h_k\) is taken into account.

These tests are explicit, we compare them to some earlier results. Our second goal is to develop exponential estimates for solutions of a non-homogeneous version of (1.2), with coefficient of the exponential part described by a maximum of the initial functions, while the maximum of the right-hand side leads to a constant bound, not an exponential estimate of a solution. We evaluate short-term, as well as long-term (asymptotic) behaviour of a solution. To the best of our knowledge, for neutral differential systems such exponential estimates have not been known, and the present paper fills the gap.

Equation (1.2) includes equations without the neutral part
\[\dot{x}(t) = \sum_{k=1}^{m} B_k(t)x(h_k(t)), \ t \geq t_0 \geq 0, \] an equation with one delay \((m = 1)\) and autonomous equations.

In the monograph [19], the method of Lyapunov-Krasovskii functionals was applied to local asymptotic stability for vector nonlinear delay differential equations. Relevant Theorem 3.1 in [19, p. 286] was actually obtained for nonlinear equations, and we reformulate it for the linear case
\[\dot{x}(t) = B(t)x(t - h), \ t \geq 0, \] (1.3)
where \(B\) is an \(n \times n\) matrix with locally essentially bounded entries, and \(h > 0\) is a constant delay.

Below, \(\| \cdot \|\) is a norm in \(\mathbb{R}^n\), and \(\mu(A)\) is a matrix measure which is defined in the next section.

**Proposition 1.** Let \(t_0 \geq 0\), \(\mu(B(t)) \leq -\beta < 0\) and \(h \sup_{t \geq t_0} \|B(t)\|^2 < \inf_{t \geq t_0} |\mu(B(t))|\).

Then equation (1.3) is uniformly asymptotically stable.

A significant part of known asymptotic stability tests for neutral equations was obtained for autonomous systems with a non-delay term, such as
\[\dot{x}(t) - A_1\dot{x}(t - h_1) = A_0x(t) + A_2x(t - h_2), \ h_1, h_2 > 0, \ t \geq t_0. \] (1.4)

The following stability tests were obtained using Lyapunov-Krasovskii functionals.

**Proposition 2.** [19, P. 367, 22] Let \(\|A_1\| < 1\) and \(\mu(A_0) + \|A_0\| + \|A_1\||A_2\| < 1 - \|A_1\| < 0\).

Then system (1.4) is asymptotically stable.

**Proposition 3.** [19, P. 369] If \(\|A_1\| + h_2\|A_2\| < 1\) and
\[\mu(A_0 + A_2) + \|A_0 + A_2\|((\|A_1\| + h_2\|A_2\|) < 0\]
then system (1.4) is asymptotically stable.
The structure of the paper is the following. Section 2 presents preliminaries, while main results on solution estimates and exponential stability are obtained in Section 3. We get solution estimates for system (1.2) or its non-homogeneous counterpart and derive stability tests for (1.2). Section 4 illustrates novelty of the results with examples and indicates possible directions of further research. We also discuss Propositions 1-3 and compare them to the results obtained in the paper in Section 4.

2. Preliminaries

There exists a well developed theory of neutral linear differential equations including existence, uniqueness, representation of solutions, see e.g. [1, 11, 16, 19, 20]. We use some results and definitions from these monographs, including definitions of asymptotic and uniform exponential stability.

Let \( \|x\| \) be an arbitrary vector norm of \( x \in \mathbb{R}^n \), the same notation is used for the induced matrix norm. The matrix measure (often referred to as the logarithmic matrix norm) of an \( n \times n \) matrix \( C \) is defined as

\[
\mu(C) = \lim_{\nu \to 0^+} \nu^{-1} \left( \|E + \nu C\| - 1 \right),
\]

where \( E \) is the \( n \times n \) identity matrix. Some of its properties are useful in calculations:

\[
|\mu(C)| \leq \|C\|, \quad \mu(C_1 + C_2) \leq \mu(C_1) + \mu(C_2), \quad \mu(\lambda C) = \lambda \mu(C), \quad \lambda > 0.
\]

For details, we refer readers to [28, 29] and [19, Table 3.1, p. 286].

In particular, if we use the maximum norm \( \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \) then

\[
\|C\| = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |c_{ij}| \right\}, \quad \mu(C) = \max_{1 \leq i \leq n} \left\{ c_{ii} + \sum_{j \neq i} |c_{ij}| \right\}.
\]

For either a fixed interval \( J = [t_0, t_1] \) or \( J = [t_0, \infty) \), consider the space \( L_\infty(J) \) with an essential supremum norm \( \|x\|_J = \text{ess sup}_{t \in J} \|x(t)\| \). Further, by sup of any Lebesgue measurable function we mean an essential supremum without mentioning it.

For system (1.2), the functions \( A, B_k, g, h_k \) are assumed to be Lebesgue measurable on \([t_0, \infty)\), where \( t_0 \geq 0 \) is an arbitrary initial point, \( n \times n \) matrix functions \( A \) and \( B_k \) are essentially bounded on \([t_0, \infty)\), delays in \( g \) and \( h_k \) are bounded: \( 0 \leq t - g(t) \leq \sigma, 0 \leq t - h_k(t) \leq \tau_k \) for \( \sigma > 0, \tau_k > 0 \), \( k = 1, \ldots, m \). The coefficient \( A \) satisfies \( \|A(t)\| \leq a_0 < 1 \) for some \( a_0 < 1 \) for \( t \geq t_0 \).

The condition on the neutral delay that, once the Lebesgue measure of the set \( U \) is zero, the set \( g^{-1}(U) \) has also the zero measure, ensures that \( x(g(t)) \) is properly defined and measurable, once \( x \) is measurable.

In addition to (1.2), we introduce a system

\[
\dot{x}(t) - A(t)\dot{x}(g(t)) = \sum_{k=1}^{m} B_k(t)x(h_k(t)) + f(t), \quad t \geq t_0 \tag{2.1}
\]

with initial conditions

\[
x(t) = \Phi(t) \text{ for } t \leq t_0, \quad \dot{x}(t) = \Psi(t) \text{ for } t < t_0. \tag{2.2}
\]
Here \( f \in L_\infty([t_0, c]) \) for any \( c > t_0 \), while \( \Phi : [t_0 - \max_k \tau_k, t_0] \to \mathbb{R}^n \) and \( \Psi : [t_0 - \sigma, t_0) \to \mathbb{R}^n \), in addition to boundedness, are supposed to be Borel measurable.

Everywhere below we let these assumptions be satisfied, for example, considering (2.1), (2.2) or any delay system, without repeating them.

By a solution of problem (2.1), (2.2) we mean a locally absolutely continuous on \( [t_0, \infty) \) function \( x : \mathbb{R} \to \mathbb{R}^n \) satisfying system (2.1) for almost all \( t \in [t_0, \infty) \) and initial conditions (2.2) when \( t \leq t_0 \).

The above conditions on the parameters of the equations and initial conditions guarantee existence and uniqueness of a solution of initial value problem (2.1), (2.2) [1], and a solution representation which will later be applied.

For each \( s \geq t_0 \) we introduce the fundamental matrix as a solution \( X(t, s) \) of

\[
\dot{x}(t) - A(t)x(t) = \sum_{k=1}^m B_k(t)x(h_k(t)), \quad x(t) = 0, \quad \dot{x}(t) = 0, \quad t < s, \quad x(s) = E,
\]

where \( E \) is the identity matrix. By definition, \( X(t, s) \) vanishes whenever \( t < s \).

If there exist positive numbers \( M \) and \( \gamma \) for which any solution of (1.2), (2.2) satisfies

\[
\|x(t)\| \leq Me^{-\gamma(t-t_0)} \left[ \sup_{t \in [t_0 - \max_{k \leq m} \tau_k, t_0]} \|\Phi(t)\| + \sup_{t \in [t_0 - \sigma, t_0]} \|\Psi(t)\| \right], \quad t \geq t_0,
\]

where neither \( M \) nor \( \gamma \) depends on \( t_0 \geq 0 \) and the initial functions \( \Phi \) and \( \Psi \), system (1.2) is called uniformly exponentially stable. Also, exponential estimates of a fundamental matrix \( X(t, s) \) will be considered \( \|X(t, s)\| \leq M_0 e^{-\gamma_0(t-s)} \) for \( t \geq s \geq t_0 \) and some \( M_0 > 0 \) and \( \gamma_0 > 0 \) which hold with \( M_0 = M, \gamma_0 = \gamma \) if (1.2) is uniformly exponentially stable.

Define a linear bounded operator on the space \( L_\infty([t_0, t_1]) \) as

\[
(Sy)(t) = \begin{cases} 
A(t)y(g(t)), & g(t) \geq t_0, \\
0, & g(t) < t_0.
\end{cases}
\]

Note that there exists a unique solution (see, for example, [1]) of problem (2.1), (2.2). This solution has the representation

\[
x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s) \left[ (I - S)^{-1} f \right](s)ds \\
+ \int_{t_0}^{t_0 + \sigma} X(t, s) \left[ (I - S)^{-1}(A(\cdot)\Psi(g(\cdot))) \right](s)ds \\
+ \sum_{k=1}^{m} \int_{t_0}^{t_0 + \tau_k} X(t, s) \left[ (I - S)^{-1}(B_k(\cdot)\Phi(h_k(\cdot))) \right](s)ds,
\]

where \( I : L_\infty(J) \to L_\infty(J) \) is the identity operator, and we set \( \Psi(g(t)) = 0 \) for \( g(t) \geq t_0 \) and \( \Phi(h_k(t)) = 0 \) whenever \( h_k(t) \geq t_0 \). For any \( t_1 > t_0 \), the norm of \( (I - S)^{-1} \) satisfies [1]

\[
\|(I - S)^{-1}\|_{L_\infty([t_0, t_1]) \to L_\infty([t_0, t_1])} \leq \frac{1}{1 - \|A\|_{[t_0, \infty)}}.\]
We use an auxiliary system with a term not involving any delay

$$
\dot{z}(t) - A_0(t)\dot{z}(g(t)) = C(t)z(t) + \sum_{k=0}^{m} D_k(t)z(h_k(t)), \ t \geq t_0. \tag{2.5}
$$

First, let us evaluate the fundamental matrix of system (2.5). Denote

$$
D(t) := \sum_{k=0}^{m} D_k(t). \tag{2.6}
$$

**Lemma 1.** Let \(\|A_0\|_{[t_0, \infty)} < 1\), there exist an \(\alpha_0 < 0\) such that \(\mu(C(t) + D(t)) \leq \alpha_0\), \(Z\) be a fundamental matrix of system (2.5), and

$$
K_0 := \frac{\|C\|_{[t_0, \infty)} + \sum_{k=0}^{m} \|D_k\|_{[t_0, \infty)}}{1 - \|A_0\|_{[t_0, \infty)}} \left(\|A_0\|_{[t_0, \infty)} + \sum_{k=0}^{m} \frac{\tau_k}{\mu(C + D)}\right) < 1. \tag{2.7}
$$

Then, \(Z(t, s)\) satisfies

$$
\|Z(t, s)\| \leq K, \ t \geq s \geq t_0, \tag{2.8}
$$

where the bound \(K\) is \(K = (1 - K_0)^{-1}\).

**Proof.** We denote \(z(t) = Z(t, t_0)\) to make expressions shorter. By definition, \(z\) satisfies (2.5), where the initial matrix is \(z(t_0) = E\), and the initial functions are identically equal to the zero matrix.

For an arbitrary \(t_1 > t_0\), denote \(J = [t_0, t_1]\). By (2.4) and (2.5), we get the inequality

$$
\|\dot{z}\| \leq \frac{\|C\|_{[t_0, \infty)} + \sum_{k=0}^{m} \|D_k\|_{[t_0, \infty)}}{1 - \|A_0\|_{[t_0, \infty)}} \|z\|. \tag{2.9}
$$

A fundamental matrix \(Y(t, s)\) of the ordinary differential equation

$$
\dot{y}(t) = (C(t) + D(t))y(t)
$$

has an exponential estimate \([3, \text{Page 9}]\)

$$
\|Y(t, s)\| \leq e^{\int_{s}^{t} \mu(C(\xi) + D(\xi))d\xi}. \tag{2.10}
$$

Further, we rewrite (2.5) with a different non-delay term

$$
\dot{z}(t) = [C(t) + D(t)]z(t) + A_0(t)\dot{z}(g(t)) - \sum_{k=0}^{m} D_k(t) \int_{h_k(t)}^{t} \dot{z}(\xi)d\xi. \tag{2.11}
$$

Integrating from \(t_0\) to \(t \leq t_1\), we get:

$$
z(t) = Y(t, t_0) + \int_{t_0}^{t} Y(t, s)\mu(C(s) + D(s)) \left[ \frac{A_0(s)}{\mu(C(s) + D(s))}\dot{z}(g(s)) - \sum_{k=0}^{m} \frac{D_k(s)}{\mu(C(s) + D(s))} \int_{h_k(s)}^{s} \dot{z}(\xi)d\xi \right] ds.
$$
Therefore, using (2.9), (2.10), inequality \( \mu(C(t) + D(t)) \leq \alpha_0 < 0 \) leads to
\[
\left\| \int_{t_0}^{t} Y(t, s) \mu(C(s) + D(s)) \, ds \right\| \leq 1
\]
and also recalling \( K_0 \) from (2.7),
\[
\|z\| \leq 1 + \left( \left\| \frac{A_0}{\mu(C + D)} \right\|_{[t_0, \infty)} + \sum_{k=0}^{m} \tau_k \left\| \frac{D_k}{\mu(C + D)} \right\|_{[t_0, \infty)} \right) \|z\| \leq 1 + K_0 \|z\|.
\]
We get \( \|Z(t, t_0)\|_{[t_0, t_1]} \leq (1 - K_0)^{-1} \), and the expression in the right-hand side does not depend on \( t_1 \), which implies \( \|Z(t, t_0)\|_{[t_0, \infty)} \leq (1 - K_0)^{-1} \). This inequality is also valid if \( t_0 \) is replaced with \( s > t_0 \). Hence estimate (2.8) is valid, which concludes the proof.

**Lemma 2.** Let \( \|A_0\|_{[t_0, \infty)} + \sum_{k=0}^{m} \tau_k \|D_k\|_{[t_0, \infty)} < 1 \), \( \mu(C(t) + D(t)) \leq \alpha_0 \) for some \( \alpha_0 < 0 \), where \( D \) from (2.6) is used, \( Z \) be a fundamental matrix of (2.3), and the constant
\[
L_0 := \frac{\|C + D\|_{[t_0, \infty)}}{1 - \|A_0\|_{[t_0, \infty)} - \sum_{k=0}^{m} \tau_k \|D_k\|_{[t_0, \infty)}} \left( \left\| \frac{A_0}{\mu(C + D)} \right\|_{[t_0, \infty)} + \sum_{k=0}^{m} \tau_k \left\| \frac{D_k}{\mu(C + D)} \right\|_{[t_0, \infty)} \right) < 1.
\]
Then, \( Z(t, s) \) satisfies (2.8), where \( L = (1 - L_0)^{-1} \) is used instead of \( K \).

**Proof.** Again, \( z(t) = Z(t, t_0) \) satisfies (2.5) with the zero initial matrix-functions and \( z(t_0) = E \). Let \( J = [t_0, t_1] \) for any \( t_1 > t_0 \). Equality (2.11) implies
\[
\|\dot{z}\| \leq \|C + D\|_{[t_0, \infty)} \|z\| + \left( \|A_0\|_{[t_0, \infty)} + \sum_{k=0}^{m} \tau_k \|D_k\|_{[t_0, \infty)} \right) \|\dot{z}\|,
\]
therefore
\[
\|\dot{z}\| \leq \frac{\|C + D\|_{[t_0, \infty)}}{1 - \|A_0\|_{[t_0, \infty)} - \sum_{k=0}^{m} \tau_k \|D_k\|_{[t_0, \infty)}} \|z\|.
\]
The rest of the proof repeats the scheme for Lemma 1 and thus is omitted.

**3. Main Results**

3.1. **Boundedness and solution estimates**

For any \( \lambda \in \mathbb{R} \), denote the matrix function
\[
P(t) := \sum_{k=1}^{m} e^{\lambda(t-h_k(t))} B_k(t) - \lambda e^{\lambda(t-g(t))} A(t) + \lambda E. \tag{3.1}
\]
We will show that, once the matrix measure of \( P(t) \) in (3.1) is less than a constant negative number, for a positive \( \lambda \) and under some other natural assumptions, (1.2) is globally exponentially stable, and a solution estimate can be derived.
Theorem 1. Let $\lambda$ and $\beta$ be positive constants for which

$$\mu(P(t)) \leq -\beta, \quad t \geq t_0, \quad e^{\lambda t} \|A\|_{[t_0,\infty)} < 1, \quad (3.2)$$

$$M_1 := \frac{\lambda + \sum_{k=1}^{m} e^{\lambda t_k} \|B_k\|_{[t_0,\infty)} + \lambda e^{\lambda t} \|A\|_{[t_0,\infty)}}{1 - e^{\lambda t} \|A\|_{[t_0,\infty)}} \times \left(\frac{\|A\|_{[t_0,\infty)}}{\mu(P)} \right) \left((1 + \lambda \sigma) e^{\lambda t} + \sum_{k=1}^{m} \left(\frac{B_k}{\mu(P)} \right) e^{\lambda t_k} \right) < 1, \quad (3.3)$$

where $P$ is defined in (3.1). Then, for $M_0 := (1 - M_1)^{-1}$, the solution of (2.1), (2.2) satisfies

$$\|x(t)\| \leq M_0 e^{-\lambda(t-t_0)} \left[\|x(t_0)\| + \frac{e^{\lambda t} - 1}{\lambda(1 - \|A\|_{[t_0,\infty)})} \|A\|_{[t_0,\infty)} \|\Psi\|_{[t_0-\sigma,t_0)} \right]$$

$$+ \sum_{k=1}^{m} \frac{e^{\lambda t_k} - 1}{\lambda(1 - \|A\|_{[t_0,\infty)})} \|B_k\|_{[t_0,\infty)} \|\Phi\|_{[t_0-\tau_k, t_0)}$$

$$+ M_0 \frac{\|\Psi\|_{[0,t_0]} + \|\Phi\|_{[0,t_0]}}{\lambda(1 - \|A\|_{[t_0,\infty)})} \|f\|_{[t_0,t_0]} \quad (3.4)$$

Proof. We start with homogeneous system (1.2). Substituting $x(t) = e^{-\lambda(t-t_0)} y(t)$ into (1.2), we obtain

$$\dot{y}(t) - A(t) e^{\lambda(t-g(t))} \dot{y}(g(t)) = \lambda y(t) - \lambda e^{\lambda(t-g(t))} A(t) y(g(t)) + \sum_{k=1}^{m} e^{\lambda(t-h_k(t))} B_k(t) y(h_k(t)) \quad (3.5)$$

Equation (3.3) has the form of (2.5) with

$$A_0(t) = e^{\lambda(t-g(t))} A(t), \quad C(t) = \lambda E, \quad D_0(t) = -\lambda e^{\lambda(t-g(t))} A(t), \quad h_0(t) = g(t),$$

$$D_k(t) = e^{\lambda(t-h_k(t))} B_k(t), \quad k = 1, \ldots, m, \quad D(t) = \sum_{k=0}^{m} D_k(t).$$

Therefore $P(t) = C(t) + D(t),$

$$\left\|\frac{A_0}{\mu(C + D)}\right\|_{[t_0,\infty)} \leq e^{\lambda t} \left\|\frac{A}{\mu(P)}\right\|_{[t_0,\infty)} \left\|\frac{D_0}{\mu(C + D)}\right\|_{[t_0,\infty)} \leq \lambda e^{\lambda t} \left\|\frac{A}{\mu(P)}\right\|_{[t_0,\infty)} ,$$

$$\left\|\frac{D_k}{\mu(C + D)}\right\|_{[t_0,\infty)} \leq e^{\lambda t_k} \left\|\frac{B_k}{\mu(P)}\right\|_{[t_0,\infty)}, \quad k = 1, \ldots, m,$$

$$\left\|C\right\|_{[t_0,\infty)} + \sum_{k=0}^{m} \left\|D_k\right\|_{[t_0,\infty)} \leq \frac{\lambda + \sum_{k=1}^{m} e^{\lambda t_k} \|B_k\|_{[t_0,\infty)} + \lambda e^{\lambda t} \|A\|_{[t_0,\infty)}}{1 - e^{\lambda t} \|A\|_{[t_0,\infty)}} .$$

Inequalities (3.2) and (3.3) imply the assumptions of Lemma 1 in particular, (2.7). Let $Y(t,s)$ be a fundamental matrix of (3.5), then we can apply Lemma 1 to deduce $\|Y(t,s)\| \leq M_0$. Once $X(t,s)$ is a fundamental matrix of system (1.2), it satisfies $X(t,s) = e^{-\lambda(t-s)} Y(t,s)$. This implies
the exponential estimate \(\|X(t, s)\| \leq M_0 e^{-\lambda(t-s)}\). Let \(x\) be a solution of problem (1.2), (2.2). We use solution representation (2.3) and inequality (2.4)

\[
\|x(t)\| \leq \|X(t, t_0)\|\|x_0\| + \int_{t_0}^{t_0 + \sigma} \|X(t, s)\| (I - S)^{-1} L_{\infty}[t_0, t_1] \|A(s)\|\|\Psi(g(s))\| ds \\
+ \sum_{k=1}^{m} \int_{t_0}^{t_0 + \tau_k} \|X(t, s)\| (I - S)^{-1} L_{\infty}[t_0, t_1] \|B_k(s)\|\|\Phi(h_k(s))\| ds \\
\leq M_0 e^{-\lambda(t-t_0)} \|x(t_0)\| + \frac{M_0}{\lambda (1 - \|A\|_{[t_0, \infty])}} \|A\|_{[t_0, \infty]} \left( e^{-\lambda(t-t_0-\sigma)} - e^{-\lambda(t-t_0)} \right) \|\Psi\|_{[t_0-\sigma, t_0]} \\
+ \frac{M_0}{\lambda (1 - \|A\|_{[t_0, \infty])}} \sum_{k=1}^{m} \|B_k\|_{[t_0, \infty]} \left( e^{-\lambda(t-t_0-\tau_k)} - e^{-\lambda(t-t_0)} \right) \|\Phi\|_{[t_0-\tau_k, t_0]},
\]

which immediately yields (3.4) when \(f \equiv 0\).

For any \(f\) in (2.1), we employ the above inequality for \(\|X(t, s)\|\) and

\[
\left\| \int_{t_0}^{t} X(t, s) [(I - S)^{-1} f](s) ds \right\| \leq \frac{M_0}{\lambda (1 - \|A\|_{[t_0, \infty])}} \|f\|_{[t_0, t]}.
\]

\[\square\]

**Theorem 2.** Let \(\lambda\) and \(\beta\) be positive constants for which

\[
\mu(P(t)) \leq -\beta, \quad t \geq t_0, \quad (1 + \lambda \sigma) e^{\lambda \sigma} \|A\|_{[t_0, \infty]} + \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \|B_k\|_{[t_0, \infty]} < 1,
\]

\[
M_2 := \frac{\lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} \|B_k\|_{[t_0, \infty]} + \lambda e^{\lambda \sigma} \|A\|_{[t_0, \infty]}}{1 - (1 + \lambda \sigma) e^{\lambda \sigma} \|A\|_{[t_0, \infty]} - \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \|B_k\|_{[t_0, \infty]}} \\
\times \left( \left\| \frac{A}{\mu(P)} \right\|_{[t_0, \infty]} (1 + \lambda \sigma) e^{\lambda \sigma} + \sum_{k=1}^{m} \left\| \frac{B_k}{\mu(P)} \right\|_{[t_0, \infty]} e^{\lambda \tau_k \tau_k} \right) < 1,
\]

where \(P\) is defined in (3.1). Then the solution \(x\) of (2.1), (2.2) satisfies (3.4) with \(M_0 := (1 - M_2)^{-1}\).

**Proof.** The proof of the theorem is similar to the proof of Theorem 1 if we use Lemma 2 rather than Lemma 1 (3.5) and the same notation. \[\square\]

Let \(m = 1\), consider

\[
\dot{x}(t) - A(t) \dot{x}(g(t)) = B(t) x(h(t)) + f(t)
\]

with initial conditions (2.2). Denote

\[
P_1(t) := e^{(t-h(t))} B(t) - \lambda e^{(t-g(t))} A(t) + \lambda E.
\]

**Corollary 1.** Let \(\lambda\) and \(\beta\) be positive constants for which

\[
\mu(P_1(t)) \leq -\beta, \quad t \geq t_0, \quad e^{\lambda \sigma} \|A\|_{[t_0, \infty]} < 1,
\]

\[\square\]
Theorem 4. Then equation (1.2) is uniformly exponentially stable.

Let

\[ \sum \] and

\[ \sum \]

Theorem 3.

Then, for \( M \)

which is an initial value problem for a particular case of (1. 2) without the neutral part.

Then for the solution

Theorems 1 and 2 immediately yield exponential stability conditions.

Denote

Theorem 2 leads to exponential estimates similar to those in Corollaries 1 and 2.

Consider a delay system

(3.6), (2.2) and

of problem (3.6), (2.2) and

\[ (3.10) \]

\[ (3.9) \]

\[ (3.11) \]

\[ (3.8) \]

\[ (3.10) \]

which is an initial value problem for a particular case of (1.2) without the neutral part.

Denote

\[ P_2(t) := \sum_{k=1}^{m} e^{\lambda(t-h_k(t))} B_k(t) + \lambda E. \]

Corollary 2. Let \( \lambda \) and \( \beta \) be positive constants for which \( \mu(P_2(t)) \leq -\beta \) for \( t \geq t_0 \) and

\[ M_4 := \left( \lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} \| B_k \|_{[t_0,\infty)} \right) \sum_{k=1}^{m} \| B_k \|_{[t_0,\infty)} \| e^{\lambda \tau_k \tau_k} < 1. \]

Then, for \( M_0 := (1 - M_4)^{-1} \), the solution \( x \) of problem \( (3.10) \) satisfies

\[ \| x(t) \| \leq M_0 e^{-\lambda(t-t_0)} \left[ \| x(t_0) \| + \sum_{k=1}^{m} e^{\lambda \tau_k} \| B_k \|_{[t_0,\infty)} \| \Phi \|_{[t_0-\tau_k,t_0]} \right] + \frac{M_0}{\lambda} \| f \|_{[t_0,t]}. \]

Theorem 2 leads to exponential estimates similar to those in Corollaries 1 and 2.

3.2. Exponential stability

Theorems 1 and 2 immediately yield exponential stability conditions.

Theorem 3. Let \( \beta \) be a positive constant for which \( B(t) := \sum_{k=1}^{m} B_k(t) \), \( \mu(B(t)) \leq -\beta \), \( \| A \|_{[t_0,\infty)} < 1 \) and

\[ \sum_{k=1}^{m} \| B_k \|_{[t_0,\infty)} \left( \| A \|_{[t_0,\infty)} + \sum_{k=1}^{m} \| B_k \|_{[t_0,\infty)} \| \right) < 1. \]

Then equation (1.2) is uniformly exponentially stable.

Theorem 4. Let \( \beta \) be a positive constant for which \( \mu(B(t)) \leq -\beta \), where

\[ B(t) := \sum_{k=1}^{m} B_k(t), \quad \| A \|_{[t_0,\infty)} + \sum_{k=1}^{m} \tau_k \| B_k \|_{[t_0,\infty)} < 1 \]

\[ 9 \]
and
\[
\left( 1 - \|A\|_{[t_0,\infty)} - \sum_{k=1}^{m} \tau_k \|B_k\|_{[t_0,\infty)} \right) \left( \frac{\|A\|_{\mu(B)}{[t_0,\infty)}}{\|B\|_{[t_0,\infty)}} + \sum_{k=1}^{m} \tau_k \frac{\|B_k\|_{[t_0,\infty)}}{\|\mu(B)\|_{[t_0,\infty)}} \right) < 1.
\]

Then equation (1.2) is uniformly exponentially stable.

For \( C = \{c_{ij}\}_{i,j=1}^{n} \), denote the matrix \( |C| := \{c_{ij}\}_{i,j=1}^{n} \). If \( |B(t)| \leq \overline{B} \) for any \( t \in [t_0, \infty) \), where \( \overline{B} \) is a constant matrix, then \( \|B\|_{[t_0,\infty)} \leq \|\overline{B}\| \).

**Corollary 3.** Let \( \beta \) be a positive constant for which \( \mu(B(t)) \leq -\beta \), where \( B(t) := \sum_{k=1}^{m} B_k(t) \), \( |A| \leq \overline{A} \), \( |B_k(t)| \leq \overline{B}_k \), \( |B(t)| \leq \overline{B} \), \( |A| \leq 1 \), and at least one of the following conditions holds:
\[
\sum_{k=1}^{m} \|B_k\| \left( \|A\| + \sum_{k=1}^{m} \tau_k \|B_k\| \right) < \beta \left(1 - \|A\|\right);
\]
\[
\|\overline{B}\| \left( \|A\| + \sum_{k=1}^{m} \tau_k \|B_k\| \right) < \beta \left(1 - \|A\| - \sum_{k=1}^{m} \tau_k \|B_k\| \right).
\]

Then equation (1.2) is uniformly exponentially stable.

**Corollary 4.** Let \( \beta \) be a positive constant for which \( \mu(B(t)) \leq -\beta \) for any \( t \geq t_0 \), \( \|A\|_{[t_0,\infty)} < 1 \),
\[
\left( \frac{\|A\|_{\mu(B)}{[t_0,\infty)}}{\|B\|_{[t_0,\infty)}} + \tau \frac{\|B\|_{\mu(B)}{[t_0,\infty)}}{\|B\|_{[t_0,\infty)}} \right) \|B\|_{[t_0,\infty)} < 1 - \|A\|_{[t_0,\infty)}.
\]

Then equation (3.10) with \( f \equiv 0 \) is uniformly exponentially stable.

**Corollary 5.** Let \( \beta \) be a positive constant for which \( \mu(B(t)) \leq -\beta \) for any \( t \geq t_0 \), where \( B(t) := \sum_{k=1}^{m} B_k(t) \), and either
\[
1) \sum_{k=1}^{m} \|B_k\|_{[t_0,\infty)} \sum_{k=1}^{m} \tau_k \frac{\|B_k\|_{\mu(B)}{[t_0,\infty)}}{\|\mu(B)\|_{[t_0,\infty)}} < 1
\]
or
\[
2) \|B\|_{[t_0,\infty)} \sum_{k=1}^{m} \tau_k \frac{\|B_k\|_{\mu(B)}{[t_0,\infty)}}{\|\mu(B)\|_{[t_0,\infty)}} < 1 - \sum_{k=1}^{m} \tau_k \|B_k\|_{[t_0,\infty)}
\]
is satisfied. Then, system (3.10) with \( f \equiv 0 \) is uniformly exponentially stable.

Consider the system
\[
\dot{x}(t) = B(t)x(t(h(t))).
\]

**Corollary 6.** Let \( \beta \) be a positive constant for which \( \mu(B(t)) \leq -\beta \) for any \( t \geq t_0 \), and
\[
\tau \frac{\|B\|_{\mu(B)}{[t_0,\infty)}}{\|B\|_{[t_0,\infty)}} < 1.
\]

Then system (3.12) is uniformly exponentially stable.
4. Examples and Discussion

First, let us notice that Corollary 6 extends the stability test of Proposition 1 to a system with a variable delay.

Next, to compare stability tests of the present paper with known ones, consider a linear scalar neutral differential equation

\[ \dot{x}(t) - a(t)\dot{x}(g(t)) = \sum_{k=1}^{m} b_k(t)x(h_k(t)), \quad t \geq t_0 \geq 0, \quad (4.1) \]

where \(|a(t)| \leq a_0 < 1, t - g(t) \leq \sigma, t - h_k(t) \leq \tau_k\). Theorems 3 and 4 imply the following stability tests for scalar equation (4.1).

**Corollary 7.** Assume that for some \( \beta > 0 \), \( b(t) := \sum_{k=1}^{m} b_k(t) \leq -\beta < 0, \quad t \geq t_0 \) and either

\[
\sum_{k=1}^{m} \|b_k\|_{[t_0, \infty)} \left( \left\| \frac{a}{b} \right\| + \sum_{k=1}^{m} \tau_k \left\| \frac{b_k}{b} \right\| \right) < 1 - \|a\|_{[t_0, \infty)} \quad (4.2)
\]

or

\[
\|b\|_{[t_0, \infty)} \left( \left\| \frac{a}{b} \right\| + \sum_{k=1}^{m} \tau_k \left\| \frac{b_k}{b} \right\| \right) < 1 - \|a\|_{[t_0, \infty)} - \sum_{k=1}^{m} \tau_k \|b_k\|. \quad (4.3)
\]

Then equation (4.1) is uniformly exponentially stable.

Condition (4.2) is known and coincides with \([3, \text{Corollary 5.4}]\). However, the stability test in (4.3) is new, to the best of our knowledge, and independent of (4.2) which is illustrated in the following example.

**Example 1.** Consider a scalar equation with variable oscillating coefficients

\[
\dot{x}(t) - \nu [0.1 \sin t \dot{x}(g(t)) = -(1 - 3 \cos t)x(t) - (1 + 3 \cos t)x(h(t))], \quad (4.4)
\]

where \( \nu > 0, t - g(t) \leq \sigma, t - h(t) \leq 1 = \tau \). Here \( m = 2 \) and

\[
\|a\|_{[t_0, \infty)} = 0.1 \nu, \quad \|b_1\|_{[t_0, \infty)} = \|b_2\|_{[t_0, \infty)} = 4 \nu, \quad b = b_1 + b_2 = -2 \nu < 0, \quad \tau = 1.
\]

By (4.2) equation (4.4) is uniformly exponentially stable if \( \nu < \frac{1}{16.5} \), by (4.3) the stability condition is \( \nu < \frac{1}{72} \). Hence for this equation condition (4.3) is better than (4.2).

In equation (4.4), the value of \( \|b_1 + b_2\|_{[t_0, \infty)} \) is significantly less than \( \|b_1\|_{[t_0, \infty)} + \|b_2\|_{[t_0, \infty)} \). If they are close, condition (4.2) is better than (4.3). For example, if the term of \( 3 \cos t \) is omitted in the coefficients in (4.4), estimate in (4.2) becomes sharper than in (4.3) \( \nu < \frac{1}{12} \) compared to \( \nu < \frac{1}{72} \), respectively.

Note that all coefficients in (4.4) of Example 1 are oscillating, while most known stability tests even for equations without the neutral part deal with positive coefficients.

To compare the results of the present paper with known stability tests, we adapt Theorems 3 and 4 to a non-autonomous equation generalizing (4.4)

\[
\dot{x}(t) - A_1(t)\dot{x}(H_1(t)) = A_0(t)x(t) + A_2(t)x(H_2(t)), \quad t \geq t_0, \quad (4.5)
\]

where \( 0 \leq t - H_1(t) \leq h_1, 0 \leq t - H_2(t) \leq h_2 \) for some \( h_1, h_2 > 0 \).
Corollary 8. Assume that for some $\beta > 0$, $\mu(A_0(t) + A_2(t)) \leq -\beta < 0$, $t \geq t_0$ and at least one of the following conditions holds:

$$\|A_1\|_{[t_0,\infty)} < 1, \quad -\beta + \frac{(\|A_0\|_{[t_0,\infty)} + \|A_2\|_{[t_0,\infty)})(\|A_1\|_{[t_0,\infty)} + h_2\|A_2\|_{[t_0,\infty)})}{1 - \|A_1\|_{[t_0,\infty)}} < 0;\quad (4.6)$$

Then equation (4.5) is uniformly exponentially stable.

Proof. Assume first that (4.6) holds. We will apply Theorem 3 for Proof. Then equation (4.5) initially stable.

Inequality (4.8) is equivalent to the second inequality in (4.6).

Proposition 3 in the case of constant coefficients and delays.

Corollary 8. Assume that for some $\mu(A_0(t) + A_2(t)) \leq -\beta < 0$, $t \geq t_0$ and at least one of the following conditions holds:

$$\|A_1\|_{[t_0,\infty)} < 1, \quad -\beta + \frac{(\|A_0\|_{[t_0,\infty)} + \|A_2\|_{[t_0,\infty)})(\|A_1\|_{[t_0,\infty)} + h_2\|A_2\|_{[t_0,\infty)})}{1 - \|A_1\|_{[t_0,\infty)} - h_2\|A_2\|_{[t_0,\infty)}} < 0.\quad (4.7)$$

Inequality (4.8) is equivalent to the second inequality in (4.6). All conditions of Theorem 3 hold for equation (4.5), hence this equation is uniformly exponentially stable.

The second part of the corollary follows from Theorem 4 and is justified in a similar way. □

Proposition 2 is independent of Proposition 3 and of Corollary 8. Condition (4.6) in Corollary 8 is independent of both propositions and of the stability test in (4.7), while (4.7) coincides with Proposition 3 in the case of constant coefficients and delays.

Several stability results for linear neutral systems were obtained by the LMI method based on Lyapunov-Krasovskii functionals. Usually neutral systems studied by the LMI method have constant matrix coefficients and bounded derivatives of delay functions.

Consider now the following example of a non-autonomous system of neutral type with variable matrix coefficients $n \times n$. Note that with the growth of dimension the calculations in LMI increase dramatically.

Example 2. Consider system (1.2) for $m = 2$

$$\dot{x}(t) - A(t)x(t) = B_1(t)x(h_1(t)) + B_2(t)x(h_2(t)), \quad t \geq t_0 \geq 0,\quad (4.9)$$

where $B_1(t) = \sin^2 t$ $C$, $B_2(t) = \cos^2 t$ $C$, $C$ is a tri-diagonal and $A$ is a variable matrix

$$C = \begin{pmatrix} -\alpha & \beta & 0 & 0 & \ldots & 0 & 0 \\ \beta/2 & -\alpha & \beta/2 & 0 & \ldots & 0 & 0 \\ 0 & \beta/2 & -\alpha & \beta/2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \beta & -\alpha \end{pmatrix}, \quad A = \gamma \begin{pmatrix} \cos t & \cos 2t & \cos 3t & \ldots & \cos nt \\ \cos^2 t & \cos 2t & \cos^3 2t & \ldots & \cos^2 nt \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos^n t & \cos^n 2t & \cos^n 3t & \ldots & \cos^n nt \end{pmatrix},$$

12
\( \alpha > 0, h_1(t) = t - 0.1 \sin t, h_2(t) = t - 0.1 \cos t, t - g(t) \leq \sigma \). We have for the norm \( \| \cdot \|_{[0, \infty)} \):

\[
\| A \|_{(0, \infty)} = n |\gamma|, \quad \tau_i = \sup_{t \geq 0} (t - h_i(t)) = 0.1, \quad \| B_i \|_{(0, \infty)} = \alpha + |\beta|, \quad i = 1, 2,
\]

\( B = B_1 + B_2, \mu(B) = -\alpha + |\beta| \). Let

\[
|\beta| < \alpha, \quad n |\gamma| < 1, \quad 2(\alpha + |\beta|)[n |\gamma| + 0.2(\alpha + |\beta|)] < (1 - n |\gamma|)(\alpha - |\beta|).
\]

(4.10)

Then, by Theorem \( \square \) equation (4.9) is uniformly exponentially stable.

In particular, conditions (1.10) hold for \( \alpha = 0.4, \quad |\beta| = 0.1, \quad n |\gamma| = 0.01 \).

Assume in addition \( \| A \| = n |\gamma| = 0.01, \lambda = 0.06, \sigma = 0.1 \). Hence \( \| B_i \|_{(0, \infty)} = 0.5, \mu(C) = -0.3 \)
and calculate by Theorem \( \square \) an estimate for solution \( x \) of the equation

\[
\dot{x}(t) - A(t)\dot{x}(g(t)) = B_1(t)\dot{x}(h_1(t)) + B_2(t)\dot{x}(h_2(t)) + f(t), \quad t \geq t_0 \geq 0,
\]

(4.11)

with initial condition \( (2.2) \). We have

\[
\mu(P(t)) = \mu \left( e^{\lambda(t-h_1(t))}B_1(t) + e^{\lambda(t-h_2(t))}B_2(t) - \lambda e^{\lambda(t-g(t))}A(t) + \lambda E \right)
\]

\[
\leq \mu \left( e^{\lambda(t-h_1(t))}B_1(t) + e^{\lambda(t-h_2(t))}B_2(t) \right) + \mu \left( -\lambda e^{\lambda(t-g(t))}A(t) \right) + \mu(\lambda E).
\]

Then

\[
\mu \left( e^{\lambda(t-h_1(t))}B_1(t) + e^{\lambda(t-h_2(t))}B_2(t) \right) = \left( e^{\lambda(t-h_1(t))} \sin^2 t + e^{\lambda(t-h_2(t))} \cos^2 t \right) \mu(C).
\]

Since \( \mu(C) = -0.3 < 0 \) and

\[
e^{\lambda(t-h_1(t))} \sin^2 t + e^{\lambda(t-h_2(t))} \cos^2 t \geq \sin^2 t + \cos^2 t = 1,
\]

we get

\[
\mu \left( e^{\lambda(t-h_1(t))}B_1(t) + e^{\lambda(t-h_2(t))}B_2(t) \right) \leq \mu(C) = -0.3.
\]

Next,

\[
\mu \left( -\lambda e^{\lambda(t-g(t))}A(t) \right) \leq \lambda e^{\lambda(t-g(t))}\|A\|_{[0, \infty)} \leq 0.0006e^{0.006}, \quad \mu(\lambda E) = \lambda = 0.06.
\]

Then \( \mu(P(t)) \leq -0.23939 < 0 \) and

\[
M_1 := \frac{\lambda + \sum_{k=1}^{2} e^{\lambda \tau_k} B_k(\|t_0, \infty) + \lambda e^{\lambda \sigma} A(\|t_0, \infty)}{1 - e^{\lambda \sigma} A(\|t_0, \infty)} \left( \| A \|_{\mu(P)(\|t_0, \infty)} + \sum_{k=1}^{2} \left( B_k \| \mu(P) \|_{t_0, \infty} e^{\lambda \tau_k} \right) \right).
\]

By numerical calculations, \( M_1 \leq 0.5 \) and \( M_0 = (1 - M_1)^{-1} \leq 2 \). For the solution \( x \) of problem (4.11), (2.2) we have (here we omitted the norm indices)

\[
\| x(t) \| \leq 2e^{-0.06(t-t_0)} \left[ \| x(t_0) \| + 0.00102\| \Psi \| + 0.102\| \Phi \| \right] + 33.6\| f \|, \quad t \geq t_0 \geq 0.
\]
Let us conclude with some additional comments. Several explicit stability conditions for delay and neutral systems were obtained by application of the Bohl-Perron theorem \([1, 5, 6, 7, 9, 11]\). In general, these tests do not coincide with results of this paper even in the scalar case, see discussion in \([4]\). However, some conditions can be the same, for example, Corollary 5 of the present paper and \([7, \text{Corollary 1}]\) coincide.

In interesting papers \([24, 25]\), the authors study neutral systems in the Hale form. They obtained explicit exponential stability conditions for rather general linear neutral systems. The method applied in \([24, 25]\) is based on some specific spectral matrix properties, such as a Metzler matrix with non-negative off-diagonal entries.

Fixed point methods also lead to explicit stability condition, but so far they were applied only to scalar neutral equations, and used the first and the second derivatives of delay functions. The advantage of fixed point approach is that it can also be applied to nonlinear equations. The method described here is only considered for linear systems.

Finally, we suggest some relevant topics for future research.

1. Here we used the essential supremum norm which led to exponential stability tests including essential supremum estimates. It would be interesting to get integral stability conditions for neutral systems considered in the paper, for the scalar case see \([3, \text{Theorem 4.7}]\).
2. Obtain stability conditions assuming that either delays or coefficients, or both, can be unbounded.
3. Investigate exponential stability for neutral systems of a higher order.
4. Are uniform asymptotic stability and uniform exponential stability equivalent for linear neutral systems with bounded delays? For equations without a neutral term this is known \([16, 21]\).

Acknowledgment

E. Braverman was partially supported by NSERC, the grant RGPIN-2020-03934. The authors are very grateful to the anonymous referees whose thoughtful comments significantly contributed to the quality of presentation.

References

[1] N. V. Azbelev and P. M. Simonov, Stability of Differential Equations with Aftereffect. Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003.

[2] M. V. Barbarossa and H.-O. Walther, Linearized stability for a new class of neutral equations with state-dependent delay, Differ. Equ. Dyn. Syst. 24 (2016), 63–79.

[3] L. Berezansky and E. Braverman, On stability of linear neutral differential equations with variable delays, Czechoslovak Math. J. 69(144) (2019), 863–891.

[4] L. Berezansky and E. Braverman, A new stability test for linear neutral differential equations, Appl. Math. Lett. 108 (2020), 106515, doi https://doi.org/10.1016/j.aml.2020.106515.

[5] L. Berezansky, J. Diblik, Z. Svoboda and Z. Šmarda, Exponential stability of linear delayed differential systems, Appl. Math. Comput. 320 (2018), 474–484.
[6] L. Berezansky, J. Diblík, Z. Svoboda and Z. Šmarda, Exponential stability tests for linear delayed differential systems depending on all delays, *J. Dynam. Diff. Equat.* **31** (2019), 2095–2108.

[7] L. Berezansky, J. Diblík, Z. Svoboda and Z. Šmarda, Simple uniform exponential stability tests for linear delayed vector differential equation, submitted.

[8] W. A. Coppel, Dichotomies in stability theory. Lecture Notes in Mathematics, **629** Springer-Verlag, Berlin-New York, 1978.

[9] A. Domoshnitsky, M. Gitman and R. Shklyar, Stability and estimate of solution to uncertain neutral delay systems, *Bound. Value Probl.* 2014, 2014:55, 14 pp.

[10] E. Fridman, New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems, *Systems Control Lett.* **43** (2001), 309–319.

[11] M. I. Gil’, Stability of Neutral Functional Differential Equations, *Atlantis Studies in Differential Equations*, **3**. Atlantis Press, Paris, 2014.

[12] K. Gopalsamy, A simple stability criterion for linear neutral differential systems, *Funkcialaj Ekvacioj*, **28** (1985), 33–38.

[13] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics. Mathematics and its Applications, **74**. Kluwer Academic Publishers Group, Dordrecht, 1992.

[14] I. Győri and M. Pituk, Special solutions of neutral functional differential equations, *J. Inequal. Appl.* **6** (2001), 99–117.

[15] I. Győri and F. Hartung, Preservation of stability in a linear neutral differential equation under delay perturbations, *Dynam. Systems Appl.* **10** (2001), 225–242.

[16] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential equations. *Applied Mathematical Sciences*, **99**. Springer-Verlag, New York, 1993.

[17] Q.-L. Han, Stability criteria for a class of linear neutral systems with time-varying discrete and distributed delays, *IMA J. Math. Control Inform.* **20** (2003), 371–386.

[18] C. Jin and J. Luo, Fixed points and stability in neutral differential equations with variable delays, *Proc. Amer. Math. Soc.* **136** (2008), 909–918.

[19] V. Kolmanovskii and A. Myshkis, Introduction to the Theory and Applications of Functional-Differential Equations, *Mathematics and its Applications* **463**, Kluwer Academic Publishers, Dordrecht, 1999.

[20] V. B. Kolmanovskiĭ and V. R. Nosov, Stability of Functional-Differential Equations, *Mathematics in Science and Engineering* **180**, Academic Press, London, 1986.

[21] A. Kulikov and V. Malygina, On relation between uniform asymptotic stability and exponential stability of linear differential equations, *Electron. J. Qual. Theory Differ. Equ.* 2015, No. 65, 8 pp.
[22] M. Liu, Global exponential stability analysis for neutral delay-differential systems: an LMI approach, Internat. J. Systems Sci. 37 (2006), 777–783.

[23] L. M. Li, Stability of linear neutral delay-differential systems, Bull. Austral. Math. Soc. 38 (1988), 339–344.

[24] P. H. A. Ngoc and Q. Ha, On exponential stability of linear non-autonomous functional differential equations of neutral type, Internat. J. Control 90 (2017), 454–462.

[25] P. H. A. Ngoc and N. Trinh, Novel criteria for exponential stability of linear neutral time-varying differential systems, IEEE Trans. Automat. Control 61 (2016), 1590–1594.

[26] J. H. Park and S. Won, Stability analysis for neutral delay-differential systems, J. Franklin Inst. 337 (2000), 1–9.

[27] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Functional Differential Equations, Springer, Dordrecht, Heidelberg, New York, London, 2013.

[28] G. Söderlind, The logarithmic norm. History and modern theory, BIT 46 (2006), 631–652.

[29] Z. Zahreddine, Matrix measure and application to stability of matrices and interval dynamical systems, Int. J. Math. Math. Sci. 2003:2 (2003), 75–85.

[30] D. Zhao, New criteria for stability of neutral differential equations with variable delays by fixed points method, Adv. Difference Equ. 2011:48 (2011), 11 pp.