Research Article
On the Solution of Region-Based Straight-Line Mechanism Design Optimization

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Received 17 March 2022; Accepted 16 July 2022; Published 28 September 2022

A universal design method for synthesis problems of mechanisms that realize the approximate straight-line trajectory is presented in this paper. First, given the expected straight line and prescribed fixed pivots, a general mathematics model with angles as design parameters to determine the initial position of two side links is established, through which all infinite possible straight-line mechanisms are obtained. Then, kinematic constraints are imposed, including type, transmission angle, size, straightness, and defect. All feasible solution mechanisms that meet the constraints are calculated and can be expressed in solution regions. It is intuitive and comprehensive for designers to observe the distribution pattern of the solution. In the end, an optimal high-precision straight-line mechanism can be selected in the feasible solution regions by setting an optimization aim. The second-order osculating mechanism synthesis method can provide more solutions for designers, but designers can use the third-order osculating mechanism synthesis method when a higher straightness requirement is imposed. This method addresses the synthesis problem of this kind of mechanism for straight-line guidance and the problem of choosing an optimal solution from an infinite number of solutions.

1. Introduction

An approximate straight-line mechanism can replace guide rails to make a frictionless motion where trajectory is exceptionally similar to a straight line. It possesses practical value in engineering and is used in various machinery applications, such as industrial robots, harvesting machines, lifting machines, and intermittent machines. The theory and method of approximate straight-line mechanism synthesis have a history of nearly a hundred years and are still a popular point for scholars, with many related monographs and papers published. Sandor [1] compared the fourth form of the ESE with a double-valued solution with an extension of Bobillier’s construction and obtained a double-valued solution. Lyndon [2] introduced a new graphical method of the Euler–Savary equation typically encountered in synthesis straight-line mechanisms. Barker [3] described the classification system and the geometry of the solution space, laying the foundation for future work to study the properties of planar four-bar mechanisms. Shen [4] introduced Bobillier’s graphical method of the inflection circle and its characteristics and used it to synthesize a four-bar lifting mechanism that implements an approximate straight-line trajectory. Li and Wang [5] established a Ball point theory to solve the synthesis problem of approximate straight-line mechanism for giving point position and straight-line direction. The method can obtain multiple sets of crank-rocker mechanisms to meet the requirements. Yu [6] proposed a numerical comparison synthesis method for single and double approximate straight-line mechanisms, which can solve the synthesis problem of approximate straight-line mechanisms more effectively. Chiang and Bulatović [7–9] used the variable controlled deviation method and differential evolution algorithm (DE) to achieve high precision in many given points along a straight line. Still, the Burmester curve does not have a single and orderly expression. In addition, it is also complicated to find the required mechanism quickly from the infinite number of solutions.

After that, the theory made new progress in the four-bar mechanism. Hu [10] proposed a method for determining the
size of a four-bar mechanism by specifying three precise symmetry points to determine the instantaneous center, using the instantaneous center position method. Han [11] gave a modern synthesis approach to the classical mechanism synthesis problem. Wang [12] proposed a procedure to synthesize mechanisms for mixed motion and function generation, which has not been settled before. SM [13] has presented an algorithm for path generation synthesis of the four-bar linkage with three precision points. An optimal synthesis method of the four-bar path generator is introduced by Romero [14]. The method uses a robust mathematical formulation and an optimization algorithm to certify the robustness of the formulation. Wu [15] developed an analytical method to find accurate solutions of coupler curve equations for planar four-bar linkages based on a neoteric algebraic formulation of the coefficient equation system. Bai and Angeles [16] used a new approach of combining numbers and graphics, which enabled us to use a simple equation to solve the problem of mechanism synthesis. Brake [17] gave the dimension and degree of the whole issues in the Alt–Burmester family and provided more details concerning zero- and one-dimensional cases. Wang [18] proposed the geometric properties of spherical coupler curves and provided a sound theoretical basis for synthesizing the four-bar linkages.

At the same time, many design methods have been proposed. Liu [19] studied a novel design framework to complete the motion generation of the four-bar linkage. The aim is to improve the dynamic performance of the linkage. Kafash [20] designed an objective function for the optimization of path generation of four-bar linkages by differential evolution (DE). Sleesongsom [21] proposed a constraint handling technique for optimum path generation of four-bar linkages and presented a new technique to address the constraints of both the input crank rotation and the Grashof criterion. Wang [22] solved the synthesis problem of the approximate straight-line mechanism theoretically at this specific position when the instantaneous center is infinite and made some explorations to synthesize the second-order straight-line mechanism under particular conditions. Xu [23] proposed an approach to design large-displacement straight-line mechanisms with rotational flexural joints based on a viewpoint that the straight-line motion is regarded as a compromise of rigid and compliant parasitic motion of a rotational flexural joint. Wang [24] introduced the calculation of circle and center points by the Burmester theory and developed a program package to find a satisfying linkage. Pickard [25] proposed an appropriate synthesis method for uncertainties present in the geometric parameters of linkages during dimensional synthesis. Zimmerman [26] provided an accurate solution to satisfy any combination of these exact synthesis problems that was not over-constrained by using poles and rotation angles as constraints. Yin [27–29] deduced the formula of the synthetic crank-rocker straight-line mechanism and drew the crank-rocker mechanism solution region, and Yin proved that the straight-line accuracy of the mechanism with the Burmester points is generally better than that of the mechanism with Ball point. Then, a synthesis approach for selecting an optimal mechanism with a Ball-Burmester point is developed based on solution regions. Cui [30] extended a method for solving the synthesis problem of four-bar linkage with four specified positions. The method provides a reference for the synthesis of other multi-bar linkages.

The above research contributed to the development of classical approximate straight-line mechanism synthesis theory. However, more attention is paid to establishing and solving the method of straight-line mechanism synthesis models, with less focus on the performance analysis and optimization of infinite number mechanisms. Next, the design process and the parameter adjustment of these researches are not intuitive; with no prediction and no forward-looking aspect, designer cannot grasp the distribution patterns and trends of the mechanism solutions. Thus, the current design is more or less done manually by experience through repeated attempts, making the design inefficient. Overall, no unified theory or methodology for the mechanism synthesis realizes approximate straight-line trajectories.

With the rapid development and widespread use of computer technology, many mechanism synthesis problems are expected to make substantial progress. This paper proposes a unified design method in which computer programming can be quickly implemented to solve the synthesis problems of mechanisms that realize approximate straight-line trajectories. The second- or third-order oscillation mechanism can be in a general or special configuration. This method can effectively solve the problem that the optimization of an infinite number of approximate straight-line mechanisms is complicated, and the optimization process is not intuitive.

2. Inflection Circle Generation Method

For a linkage $A_0ABB_0$ as depicted in Figure 1, the instantaneous center $P$ can be obtained by extending $A_0A$ and $B_0B$ to intersect. Based on the Euler– Savary equation, the geometric relationships between the moving pivots $A$ and $B$, the curvature center $A_0$ and $B_0$, the inflection poles $J_A$ and $J_B$, and the instantaneous center $P$ are as follows:

$$AA_0 \times AJ_A = PA^2,\quad BB_0 \times BJ_B = PB^2.\quad (1)$$

The inflection poles $J_A$ and $J_B$ can be calculated by (1). Inflection circle is made by instantaneous center $P$, and inflection poles $J_AJ_B$. Any moving point $C$ on the inflection circle are satisfied:

$$PC = d \sin \alpha_1,\quad (2)$$

where $PC$ and the angle $\alpha_1$ represent the polar coordinates of the moving point $C$ and $d$ is the diameter of the inflection circle.

Any moving point $C$ located on the inflection circle, with the corresponding center of curvature tending to infinity, has a trajectory containing an approximately straight section, and the direction of the velocity $V_C$ at this point is perpendicular to $PC$. 
3. General Configuration Mechanism
Synthesis Model

When the mechanism is in the initial configuration, there are no specific positional relationships between the four bars; it is called general configuration. The design requirements are as follows.

This paper uses a point $C(x_C, y_C)$ and a direction angle $\beta$ to describe an expected straight line, where $\beta$ is specified to take a positive value when turned counterclockwise from the $x$-axis to the expected straight line. Conversely, the angle $\beta$ takes a negative value, as shown in Figure 2.

The frame is described by the coordinates of the two fixed pivots $A_0(x_{A0}, y_{A0})$ and $B_0(x_{B0}, y_{B0})$. It can also be characterized by a fixed pivot $A_0(x_{A0}, y_{A0})$ and a frame vector $r$. It is required that the moving pivots $A$ and $B$ are determined to guide point $C$ along the given direction to walk an approximate straight line.

3.1. Second-Order Osculation Straight-Line Mechanism.
The inflection circle is a set of points with the instantaneous radius of curvature that is infinite in the coupler plane, where the point is called the inflection point. The coupler curve has at least second-order osculation with the straight line at the inflection point, so point $C$ must be the inflection point. The second-order osculation straight-line mechanism synthesis method provides more design parameters compared with the third order in the design; as long as the approximate straight-line length and the straight-line deviation of the selected inflection point are within the allowable range, the point can be used as the working point on the coupler plane. Thus, the second-order method can provide designers with more options.

According to the inflection circle generation technology, the process of obtaining the inflection point for the four pivots of the known mechanism is shown in Figure 3.

The mechanism synthesis method proposed in this paper starts from the opposite of the problem using backward thinking to derive the straight-line mechanism under the condition that the straight-line point (inflection point) is known. The synthesis approach is as follows.

The first design variable is introduced to determine the instantaneous center $P$: direction angle $\theta$ of the side link $A_0A$. When $\theta$ is reversed from the $x$-axis to $A_0A$, it is taken as a positive value. Conversely, $\theta$ is taken as a negative value; $\theta \in (-\pi/2, \pi/2)$. Then, the intersection of $A_0A$ with normal of the expected straight line through point $C$ is instantaneous center $P$.

The coordinate of point $P$ is as follows:

$$x_P = \frac{\cot \beta \cdot x_C + y_C + \tan \theta \cdot x_{A0} - y_{A0}}{\cot \beta + \tan \theta},$$

$$y_P = \frac{\tan \theta \cdot y_C + \cot \beta \cdot \tan \theta x_C - x_{A0} + \cot \beta \cdot y_{A0}}{\cot \beta + \tan \theta}. \quad (3)$$

The points $P$ and $C$ are located on the inflection circle, and then the position angle $\gamma$ of the inflection circle is introduced as the second design variable. $\gamma$ is an angle between line $PC$ and line $PO$, its value interval is $(-\pi/2, \pi/2)$, and the angle $\gamma$ is positive when line $PC$ is turned counterclockwise to line $PO$. Then, the diameter $d$ of the inflection circle can be obtained:

$$d = \frac{PC}{\cos \gamma}. \quad (4)$$

The coordinate of the center point $O$ of the inflection circle is as follows:

$$x_O = x_P + d \cos \left(\frac{\alpha_{PC} + \gamma}{2}\right),$$

$$y_O = y_P + d \sin \left(\frac{\alpha_{PC} + \gamma}{2}\right), \quad (5)$$

where $\alpha_{PC}$ is the rotation angle of the vector $PC$ to the $x$-axis, $\alpha_{PC} \in (0, 2\pi)$, and its value is determined by $\tan \alpha_{PC} = y_P - y_C/(x_P - x_C).$
The inflection poles \( J_A \) and \( J_B \) are the intersections of lines \( PA_0 \) and \( PB_0 \) with the inflection circle, respectively. The coordinate of point \( J_A \) is as follows:
\[
\begin{align*}
x_{J_A} &= x_O + a_1 a_4 + a_5, \\
y_{J_A} &= y_O - a_4 + a_1 a_5,
\end{align*}
\]
where
\[
a_1 = \frac{y_P - y_{A0}}{x_P - x_{A0}},
\]
\[
a_2 = y_{A0} - a_1 x_{A0},
\]
\[
a_3 = a_1^2 + 1,
\]
\[
a_4 = \frac{a_1 x_O - y_O + a_5}{a_3},
\]
\[
a_5 = \pm \sqrt{\frac{d^2}{4a_3 - a_4^2}}.
\]

Two sets of coordinate values can be obtained by changing the sign of \( a_5 \), and \( J_A \) does not coincide with point \( P \). When calculating the point \( J_B \), we only need to replace \( x_{A0} \) and \( y_{A0} \) in (6) with \( x_{B0} \) and \( y_{B0} \) of the point \( B_0 \).

According to (1), the vector \( J_A A \) is calculated after the points \( A_0, P, \) and \( J_A \) are determined:
\[
J_A A = \frac{P J_A^2}{A_0 J_A + 2 J_A P}
\]

According to (8), we can determine the moving pivot \( A \) coordinate. In the same way, the moving pivot \( B \) can be determined by the points \( B_0, P, \) and \( J_B \).

Finally, the fixed pivots are connected to obtain the initial position \( A_0 B_0 B A C \) of the mechanism. An infinite number of second-order osculation straight-line mechanisms can be chosen by the designer through varying the values of the angles \( \theta \) and \( \gamma \).

3.2. Third-Order Osculation Straight-Line Mechanism. The intersection of the inflection circle and the curvature-stationary point curve beyond the polar point is Ball point. There are four infinitely close points between the coupler curve and the expected straight line at the Ball point. Consequently, the third-order osculation straight-line mechanism synthesis method can be used for straight-line mechanism design with high straightness requirements.

The coordinate of the instantaneous center is still calculated by (3). In particular, it is essential to note that the inflection circle should be calculated by the curvature-stationary point equation rather than chosen directly from the inflection circle bundle. The method to solve the inflection circle is as follows.

Because points \( A, B, \) and \( C \) are all moving points on the coupler plane, according to the Euler-Savary equation, we have the following:
\[
\begin{align*}
\frac{1}{PA} &= \frac{PA_0 + d \sin \alpha_a}{d \cdot PA_0 \sin \alpha_a}, \\
\frac{1}{PB} &= \frac{PB_0 + d \sin \alpha_b}{d \cdot PB_0 \sin \alpha_b}, \\
\frac{1}{PC} &= \frac{1}{d \sin \alpha_1},
\end{align*}
\]

where \( PA \) and \( \alpha_a, PB \) and \( \alpha_b, PC \) and \( \alpha_1 \) represent the polar diameter and polar angle of moving points \( A, B, \) and \( C \) in the polar coordinate system, respectively, as shown in Figure 4.

The curvature-stationary point curve equation is as follows:
\[
\frac{1}{R} = \frac{1}{(M \sin \alpha_a) + \frac{1}{(N \cos \alpha_a)}},
\]

where \( R \) and \( \alpha \) represent the polar position of an uncertain moving point, \( M \) and \( N \) are instantaneous invariants.

Point \( C \) is Ball point. Points \( A \) and \( B \) always make arc motions around their fixed pivots, so points \( A, B, \) and \( C \) are curvature invariant points and the curvature-stationary point curve equation is satisfied:
\[
\begin{align*}
\frac{1}{PA} &= \frac{1}{M \sin \alpha_a} + \frac{1}{N \cos \alpha_a}, \\
\frac{1}{PB} &= \frac{1}{M \sin \alpha_b} + \frac{1}{N \cos \alpha_b}, \\
\frac{1}{PC} &= \frac{1}{M \sin \alpha_1} + \frac{1}{N \cos \alpha_1},
\end{align*}
\]

Combining (9) and (11), we have the following:
\[
\begin{align*}
\frac{PA_0 + d \sin \alpha_a}{d \cdot PA_0 \sin \alpha_a} &= \frac{1}{M \sin \alpha_a} + \frac{1}{N \cos \alpha_a}, \\
\frac{PB_0 + d \sin \alpha_b}{d \cdot PB_0 \sin \alpha_b} &= \frac{1}{M \sin \alpha_b} + \frac{1}{N \cos \alpha_b}, \\
\frac{1}{d \sin \alpha_1} &= \frac{1}{M \sin \alpha_1} + \frac{1}{N \cos \alpha_1}.
\end{align*}
\]

Counteracting \( M \) and \( N \) in (12), we can obtain the following:
\[
\frac{PB_0 \sin(a_b - a_1) \sin \alpha_a \cos \alpha_a + PA_0 \sin(a_1 - a_2) \sin \alpha_b \cos \alpha_b}{PA_0 \cdot PB_0 \sin \alpha_a \cos \alpha_a \sin \alpha_b \cos \alpha_b \sin \alpha_1 \cos \alpha_1} = 0.
\]

The angles \( \alpha_a, \alpha_b, \) and \( \alpha_1 \) are defined as follows:
\( \alpha_n = \alpha_n' - \varphi, \)
\( \alpha_B = \alpha_B' - \varphi, \)
\( \alpha_1 = \alpha_1' - \varphi, \)  
\[
\text{(14)}
\]
where \( \alpha_n', \alpha_B', \alpha_1', \) and \( \varphi \) are angles between the rays \( PA_0, PB_0, \) \( PC, \) and the polar axis \( Pt \) to the \( x \)-axis, respectively.

It is easy to know that if we want to guarantee that (13) has solutions, the instantaneous center \( P \) must not coincide with the fixed pivots \( A_1 \) and \( B_0, \) the angles \( \alpha_n, \alpha_B, \) and \( \alpha_1 \) are not equal to \( 0 \) or \( \pi/2; \) the cubic curvature-stationary point curve does not degenerate; and the mechanism is in general configuration.

Substituting (14) into (13) and rearranging it, we have the following:
\[
\% k_1 \sin(\alpha_n' - \varphi) \cos(\alpha_B' - \varphi) - k_2 \sin(\alpha_1' - \varphi) \cos(\alpha_1' - \varphi) = 0,
\]
\[
\text{(15)}
\]
where
\[
k_1 = PB_0 \sin(\alpha_B - \alpha_1),
\]
\[
k_2 = PA_0 \sin(\alpha_N - \alpha_1),
\]
\[
\text{(16)}
\]
Only one unknown is direction angle \( \varphi \) of the polar axis \( Pt \) in (15). After rearranging (15), we can get
\[
\tan 2 \phi = \frac{(k_1 \sin 2\alpha_n' - k_2 \sin 2\alpha_1')}{(k_1 \cos 2\alpha_n' - k_2 \cos 2\alpha_1')},
\]
\[
\text{(17)}
\]
From (17), the direction angles \( \varphi \) and \( \varphi' \) of the polar axes \( P' \) lie in the interval \((0, \pi), \) and the difference between \( \varphi \) and \( \varphi' \) is \( \pi/2 \) (as shown in Figure 5); then, the polar angle \( \alpha_1 \) can be calculated from the third formula of (13) and substituted into the third formula of (8) to calculate the inflection circle diameter \( d. \) Position angle \( \gamma \) of the inflection circle is as follows:
\[
\gamma = \pi/2 - \alpha_1, \quad \gamma' = -\alpha_1.
\]
The center \( O \) of the inflection circle is calculated by (5). The following steps are the same as the second-order osculation mechanism and will not be repeated here.

### 4. Special Configuration Mechanism Synthesis Model

The curvature-stationary point curve will degenerate into the second-order curve (circle) and straight line from the third order if any of the angles \( \alpha_1, \alpha_n, \) and \( \alpha_B \) is equal to \( 0 \) or \( \pi/2. \) At the same time, the moving pivots \( A \) and \( B, \) the fixed pivots \( A_0 \) and \( B_0, \) the inflection poles \( J_A \) and \( J_B, \) the instantaneous center \( P, \) and the Ball point \( C \) are in a particular geometric position relationship.

When \( 1/M = 0, \) (10) can be simplified:
\[
R = N \cos \alpha,
\]
\[
\sin \alpha = 0.
\]
When \( 1/N = 0, \)
\[
R = M \sin \alpha,
\]
\[
\cos \alpha = 0.
\]
When \( 1/M = 0 \) and \( 1/N = 0, \)
\[
\cos \alpha = 0,
\]
\[
\sin \alpha = 0.
\]

Correspondences between the curvature-stationary curve degradation rules and mechanism configurations are shown in Table 1.

The knowns are point \( C(x_C, y_C) \) on the expected straight line, direction angle \( \theta \) of the expected straight line, and fixed pivot \( A_0 \) \((x_{A0}, y_{A0})\). It is essential to add the configuration with two side links \( (A_0A \) and \( B_0B) \) being perpendicular; only coordinate \( x \) of the fixed pivot \( A_0 \) needs to be given. The unknowns are fixed pivot \( B_0 \) \((x_{B0}, y_{B0})\) and the moving pivots \( A \) and \( B. \) The direction angle \( \theta \) of \( A_0A \) is the first design variable. The instantaneous center \( P \) is determined by (3) according to the direction angle \( \theta. \)

#### 4.1 Special Configuration \( 1/M = 0. \)

The mechanism is in the configuration in which a side link \( (A_0A) \) and a frame \( (A_0B_0) \) are collinear, \( 1/M = 0, \) and the cubic stationary point curve degenerates into a circle \( m_1 \) whose center is on the polar tangent \( Pt \) and a straight line \( m_2 \) that is coincident with the polar tangent \( Pt, \) as shown in Figure 6(a). Because a side link \( (A_0A) \) and a frame \( (A_0B_0) \) are collinear and two side links intersect in the instantaneous center \( P, \) the point \( P \) coincides with the point \( B_0. \) Then, according to (1), we can infer that the moving pivot \( B \) selected randomly on the polar tangent \( Pt \) can satisfy the Euler–Savary equation, and the points \( A \) and \( C \) are located on the circle \( m_1. \)

The inflection circle is obtained by solving for the angle \( \varphi \) of the polar axis \( Pt. \)

Points \( A \) and \( C \) satisfy the Euler–Savary equation. Thus, (9) is transformed into the following:
which satisfies the curvature-stationary point curve equation. From (18), we can obtain the following:

\[ PA = \frac{d \cdot PA_0 \sin \alpha_a}{PA_0 + d \sin \alpha_a} \]  

\[ (21) \]

At the same time, points \( A \) and \( C \) are located on the circle \( m_1 \), which satisfies the curvature-stationary point curve equation. From (18), we can obtain the following:

\[ PA = N \cos \alpha_a, \]

\[ PC = N \cos \alpha_1. \]  

We counteract \( N \) in (20) to get the following:

\[ PA = \frac{PC \cos \alpha_a}{\cos \alpha_1} \]  

\[ (23) \]

Equations (21) and (23) are combined:

\[ P_A = \frac{d \cdot PA_0 \sin \alpha_a}{PA_0 + d \sin \alpha_a} \]

\[ (21) \]

Table 1: Correspondences between the curvature-stationary curve degradation rules and mechanism configurations.

| Condition  | Equation | Curve form | Initial configuration | Position of moving pivots | Polar angle value | Relationship between points |
|------------|----------|------------|-----------------------|---------------------------|------------------|----------------------------|
| 1/M ≠ 0 and 1/N ≠ 0 | Equation (10) | Cubic curve | General configuration | General position | Equation (14) | \( P, A_0, B_0, A, B, J_A, J_B, \) and \( C \) do not coincide with each other |
| 1/M = 0 | Equation (18) | A straight line (line \( Pt \)) and a circle with its center on line \( Pt \) | A side link \((A_0A)\) and a frame \((A_0B_0)\) are collinear | Points \( A \) and \( C \) on a circle | \( \alpha_a = 0 \) | \( P, B_0, \) and \( J_B \) are coincident |
| 1/N = 0 | Equation (19) | A straight line (line \( Pn \)) and a circle with its center on line \( Pn \) | A side link \((A_0A)\) and a coupler \((AB)\) are collinear | Points \( A \) and \( C \) on a circle | \( \alpha_a = 0 \) | \( P, B, \) and \( J_B \) are coincident |
| 1/M = 0 and 1/N = 0 | Equation (20) | Two straight lines (line \( Pt \) and \( Pn \)) | A coupler \((AB)\) is parallel to a frame \((A_0B_0)\) | \( \alpha_1 = \pi/2 \) | \( C \) and \( J_B \) are coincident |
| 1/M = 0 and 1/N = 0 | Equation (21) | Two side links \((A_0A\) and \( B_0B\)) are perpendicular | \( \alpha_a = \pi/2, \alpha_b = 0 \) | \( C \) and \( J_A \) are coincident |

Figure 5: Mechanism synthesis model when \( 1/M = 0 \). (a) Special configuration with \( A_0A \) and \( A_0B_0 \) being collinear. (b) Special configuration with \( A_0A \) and \( AB \) being collinear.
Substituting the first formula of (14) into (24) and rearranging it, we get the following:

\[ \sin 2(\alpha_a' - \phi) = \frac{2k}{PC} \]  \hspace{1cm} (25)

where \( k = PA_0 \sin (\alpha_a - \alpha_1) \), \( k = PA_0 \sin (\alpha_a - \alpha_1) \); it represents the distance from the fixed pivot \( A_0 \) to the line \( PC \).

Two values of \( \phi \) in \( (0\sim\pi) \) are calculated by (23); the inflection circle is then determined. The process that follows is the same as that for the general configuration mechanism.

It is not hard to see that if we want (25) to have the solution, then it must satisfy the following:

\[ \left| \frac{2k}{PC} \right| \leq 1. \]  \hspace{1cm} (26)

The equation of the line \( PC \) is as follows:

\[ \frac{y - y_c}{x - x_c} = \tan \beta. \]  \hspace{1cm} (27)

Therefore, the following can be obtained:

\[ k = \frac{x_{A0} \tan \beta - y_{A0} - x_c \tan \beta + y_c}{\sqrt{\tan^2 \beta + 1}}. \]  \hspace{1cm} (28)

Meanwhile,

\[ PC = \sqrt{(x_p - x_c)^2 + (y_p - y_c)^2}. \]  \hspace{1cm} (29)

Get \( (x_p, y_p) \) from (3); the ranges of \( \theta \) are calculated by bringing (28) and (29) into (26).

When \( |2k| = |PC| \), \( \theta \) is the critical value. Suppose point \( P^* \) is located on \( PC \); the direct line of the link \( A_0A \) cannot fall
within the shaded area between $A_0P_1^*$ and $A_0P_2^*$, as shown in Figure 5.

The coordinate of the critical point $P^*$ is as follows:

\[
x_{P^*} = x_c + 2k \cos\left(\beta - \frac{\pi}{2}\right),
\]
\[
y_{P^*} = y_c + 2k \sin\left(\beta - \frac{\pi}{2}\right),
\]

where the two critical points $P_1^*$ and $P_2^*$ can be obtained by changing the plus and minus signs, and the direction angles of the corresponding link $A_0A$ are $\theta_1$ and $\theta_2$.

4.2. Special Configuration $1/N = 0$. When $1/N = 0$, the cubic stationary point curve degenerates into a circle $m_1$ whose center is on the polar normal $Pn$ and a line $m_2$ which is coincident with the polar normal $Pn$. The mechanism in three configurations can be designed according to this degradation law: the coupler is parallel to the frame $(A_0B_0)$, the Ball point is located on a side link $(B_0B)$, and the coupler is perpendicular to a side link $(A_0B_0)$.

For the three configuration mechanisms, the common characteristic is that Ball point $C$ is located on the polar normal line $Pn$ and the moving pivot $A$ is on the circle $m_1,m_2$; the difference is in the position of the moving pivot $B$, as shown in Figure 6.

The moving point $C$ must satisfy both the Euler–Savary equation and the curvature-stationary point curve equation; thus,

\[
d = \frac{PC}{\sin \alpha_1},
\]
\[
\cos \alpha_1 = 0.
\]

We can obtain the inflection circle diameter $d$ and direction angle $\varphi$ of the polar axis $Pt$:

\[
d = PC,
\varphi = \beta.
\]

After finding the inflection circle, the inflection pole $I_T$ and the corresponding moving pivot $A$ can be calculated according to (6) and (8). Then, the calculation steps of the moving pivot $B$ and the fixed pivot $B_0$ are as follows:

1. For the configuration with a coupler $(AB)$ parallel to a frame $(A_0B_0)$, moving pivots $A$ and $B$ are on the circle $m_1$ simultaneously. The second design variable is position angle $\gamma_B$ of the moving pivot $B$. $\gamma_B$ is defined as $\gamma_B = \angle CPB$, which takes on $(-\pi/2, \pi/2)$, and $\gamma_B$ is positive when line $PC$ is turned counterclockwise to line $PB$. We can obtain the following by (17):

\[
PA = M \sin \alpha_a,
PB = M \sin \alpha_b,
\]

where $\alpha_a = \gamma_B + \pi/2$.

So the solution for $PB$ is as follows:

\[
PB = \frac{PA \cos \gamma_B}{\sin \alpha_a}.
\]

Find the coordinate of the moving pivot $B$ by (35):

\[
x_B = x_p - PB \sin (\beta + \gamma_B),
\]
\[
y_B = y_p + PB \cos (\beta + \gamma_B).
\]

The coupler curve is symmetrical when the moving pivots $A$ and $B$ are axisymmetric about line $Pn$. Coupler $AB$ is parallel to the frame $A_0B_0$, thus,

\[
PB_0 = \frac{PA_0}{PA} PB.
\]

We can obtain the fixed pivot $B_0$ by (37).

2. For the configuration with Ball point on a side link $(B_0B)$, the moving pivot $B$ can be arbitrarily selected on the polar normal line $Pn$. Therefore, the reference point $T_B$ on the tangent line $Pt$ is introduced, and $PT_B$ is equal to the radius of the inflection circle.

The second design variable is position angle $\gamma_B$ of the moving pivot $B$. $\gamma_B$ is defined as $\gamma_B = \angle PT_B B$, which takes on $(-\pi/2, \pi/2)$, and $\gamma_B$ is positive when line $T_B P$ is turned counterclockwise to line $T_B B$; thus,

\[
PB = d \tan \frac{\gamma_B}{2}.
\]

Find the coordinate of the moving pivot $B$ by (38):

\[
x_B = x_p + PB \sin \beta,
\]
\[
y_B = y_p - PB \cos \beta.
\]

After the moving pivot $B$ and the inflection pole $I_B$ (coincident with Ball point $C$) are determined, the coordinate of the fixed pivot $B_0$ is calculated by the second formula of (1).

3. For the configuration with a coupler $(AB)$ perpendicular to a side link $(B_0B)$, the moving pivot $B$ is on the polar normal line $Pn$ and also on the circle $m_1$, which can be seen as a particular case of the configuration (1) and (2). At this moment, Ball point $C$ is on the side link $B_0B$. 

Since the moving pivots A and B are located on the circle $m_1$, rearranging (33), we get the following:

$$PB = \frac{PA \sin \alpha_b}{\sin \alpha_a}.$$  (40)

The moving pivot B is also on the polar normal line $Pn$; thus,

$$\cos \alpha_b = 0.$$  (41)

(36) and (41) are combined to solve $PB$, and the coordinate of the moving pivot $B$ is calculated by (39). Finally, the fixed pivot $B_0$ coordinates are calculated by the second formula of (1).

4.3. Special Configuration 1/M = 0, 1/N = 0. For the configuration with $A_0A \perp B_0B$, the stagnation point curve degenerates into a straight line $m_1$ coincident with the polar tangent $Pt$ and a straight line $m_2$ coincident with the polar normal line $Pn$ if the three points A, B, and C are all on the curvature-stagnation point curve. At this moment, point B must lie on the polar tangent $Pt$ if points A and C lie on the polar normal $Pn$, as shown in Figure 7.

In particular, the direction angle $\theta$ of the link $A_0A$ cannot be adjusted in the design; its relationship with the direction angle $\beta$ of the expected straight line is $\theta = \beta - \pi/2$. Link $A_0A$ coincides with the expected straight-line normal through point C, so the instant center $P$ can be selected randomly on line $Pn$. Introduce the position angle $\gamma_P$ of the instantaneous center $P$ as the first design variable; $\gamma_P$ is defined as $\gamma_P = \angle CT_P P$, which takes on ($-\pi/2, \pi/2$), where the point $T_P$ is a reference point on the expected straight line; let $CT_P = CA_0; CT_P = CA_0$; $\gamma_P$ is positive when line $T_P C$ is turned counterclockwise to line $T_P P$; thus,

$$CP = CA_0 \tan \gamma_P.$$  (42)

The instantaneous center $P$ (fixed pivot $B_0$) can be calculated by (42), and the moving pivot $A$ can be calculated by the first formula of (1).

The moving pivot $B$ can be selected randomly on line $Pt$. Introduce the position angle $\gamma_B$ as the second design variable. $\gamma_B$ is defined as $\gamma_B = \angle PCB$, which takes on ($-\pi/2, \pi/2$) and is positive when line $CP$ is turned counterclockwise to line $CB$; thus,

$$PB = CP \tan \gamma_B.$$  (43)

The moving pivot $B$ is determined by (43).

5. Visual Optimization Method

According to the synthesis model of the mechanism proposed in this paper, the design variables can be varied to obtain an infinite number of approximate straight-line mechanisms for the expected straight line and fixed pivots. The initial position of link $A_0A$ is determined by the first design variable. The link $B_0B$ is then determined for the third-order osculation of the general configuration and the configuration with a coupler (AB) perpendicular to a side link ($B_0B$). Nevertheless, link $B_0B$ needs to be determined by the second design variable for other configurations. Since both design variables take values in ($-\pi/2, \pi/2$), an infinite number of mechanisms can be expressed in a limited plane area and the area is defined mechanism solution region.

In practice, maybe, designers put forward some demands. The link length maximum and the range of pivot coordinates can restrict mechanism within specific workspaces. Transmission angle is used to measure the performance of a mechanism, and the designer should set its limit to promote sound force transmission. There are other constraints, such as straightness, types, and branch defect. Whether or not the constraints are considered and constraint scopes depend on the practical needs or designer's interests. A feasible solution region is defined as zones in which solutions satisfy all constraints above in the solution region.

5.1. Mechanism Optimization Steps. The main steps of the solution region optimization method can be summarized as below:

Step 1. Input initial conditions: the fixed pivot $A_0(xA_0, yA_0)$, point $C(x_C, y_C)$ on the expected straight line, and direction angle $\beta$ of the expected straight line.

Step 2. Select the range of design variables and the calculation step.

Step 3. Set geometric and kinematic constraints. Choose the optimization goal according to the design purpose, such as the longest approximate straight line and the minimum deviation of straight line.

Step 4. Mechanism synthesis: based on the above mathematical model, get all solutions to form solution regions and figure their attributes.

Step 5. Feasible solution regions are generated by screening out solutions which are satisfying all constraints.

Step 6. An optimum solution is obtained by comparison with the value of the optimization goal of the mechanism in feasible solution regions.
5.2. Example of Mechanism Synthesis. The expected straight-line and the fixed points are given in Table 2.

The set of kinematic constraints includes the following:

Frame: \(-100 \leq x_{A0}, y_{A0}, x_{B0}, y_{B0} \leq 100, \leq 100.\)

Types: crank-rocker linkage, double-rocker linkage, or triple-rocker linkage.

Transmission performance: minimum transmission angle \(y_{\text{min}}\) of crank-rocker mechanism \(\geq 30^\circ, \geq 30^\circ.\)

Link length: link length ratio \(l_R = l_{\text{min}}/l_{\text{max}} \geq 0.1,\) single link length \(l_I \leq 200,\) sum \(l_{\text{sum}}\) of link length \(\leq 600,\)

Straightness: deviation of straight line \(\Delta h \leq 0.5;\) length \(L\) of the approximate straight line \(\geq 60,\) as shown in Figure 8.

Defect: no branch defect for double-rocker linkage and triple-rocker linkage.

Coupler curve: width \(W \leq 500,\) height \(H \leq 200.

Optimization goal: It is desirable to have the most extended straight-line segment within the allowable deviation of the straight-line.

The optimal search algorithm is as follows:

1. Set the range of design variables and the step according to the working space and the demanded precision, such as the whole solution region \((-\pi/2, \pi/2)\) and step \(\Delta = 1^\circ.\)

2. The dimensional and performance parameters of the mechanism are calculated by changing the design variables according to the step. This paper gives the contour sheet of approximate straight-line length when the mechanism is in various configuration conditions, as shown in Figure 9.

The contour sheet of the third-order osculation straight-line mechanism and the unique configuration with a coupler (AB) perpendicular to a side link \((B_RB)\) are two-dimensional graphs because there is only one design variable. The ordinate represents the approximate straight-line length, as shown in Figures 9(b) and 9(c). The other configurations are three-dimensional graphs (the values on the line are the approximate straight-line length). For the third-order mechanism, two inflection circles can be solved according to (9), so there are two solution cases, as shown in Figure 9(b). For the particular configuration \(1/M = 0,\) two critical points \(P_1^*, P_2^*\) and two direction angles \(\theta_1^*, \theta_2^*\) of the corresponding side link \(A_0A\) can be obtained according to (24), so there are two solution cases and no solution regions as shown in Figures 9(d) and 9(e).

3. The mechanisms obtained by combining design variables in pairs are screened automatically. Check mechanism attributes item by item, as long as one does not conform to the constraints and the mechanism is unfeasible. The solution regions composed of feasible solutions that satisfy all constraints are drawn, as shown in Figure 9, in the red zone, and the optimal solution position is marked by a red dot.

4. The range of design variables and the step can be reduced to get the optimal solution for different configurations, as shown in Figure 10. Points \(D\) and \(D'\) are the junction of two branches in Figure 10, and it is easy to see that points \(D\) and \(D'\) are not on the approximate straight-line segment, which means that these mechanisms have no branch defect. The dimensional and performance parameters are given in Tables 3 and 4, where \(M1-M8\) correspond to the optimal solution in Figure 10.

From the example, the following can be seen:

1. Both the second-order and the third-order mechanism design methods can be used to solve the approximate straight-line mechanism synthesis problem.

2. The third-order mechanism solution is a subset of the synthesized second-order mechanism set which is characterized by the curvature of the trajectory at the point \(C\) as a stationary point. In the actual design, if the designer is very concerned about the straightness in the straight-line working segment, the location needs to be used as the link point \(C.\) Designers can consider choosing the third-order mechanism design method to converge quickly and

| Geometric features | \(x_{A0}\) | \(y_{A0}\) | \(x_{B0}\) | \(y_{B0}\) | \(x_C\) | \(y_C\) | \(\beta (^\circ)\) |
|--------------------|--------|--------|--------|--------|--------|--------|-----------|
| General            |        |        |        |        |        |        |           |
| Special            | 40     | —      | 80     | 0      | 20     | 50     | 140       |
| Others             | 10     | —      | —      | —      | —      | —      | —         |

![Figure 8: Straight-line performance parameters.](image-url)
Figure 9: Continued.
Figure 9: Contour sheets of the length. (a) Second-order osculation straight-line mechanism. (b) Third-order osculation straight-line mechanism. (c) Special configuration with $AB \perp B_0B$. (d) Special configuration with $A_0A$ and $A_0B_0$ being collinear. (e) Special configuration with $A_0A$ and $AB$ being collinear. (f) Special configuration with $AB \parallel A_0B_0$. (g) Special configuration with Ball point on $B_0B$. (h) Special configuration with $A_0A \perp B_0B$. 
Figure 10: Continued.
Table 3: Dimensional parameters.

| Mechanism | $\theta$    | $\gamma$ | $A_0$  | $B_0$  | $A$       | $B$       |
|-----------|-------------|----------|--------|--------|-----------|-----------|
| $M_1$     | 60.21       | 25.69    | (40, 10) | (80, 0) | (98.37, 111, 96) | (116.83, 103.54) |
| $M_2$     | 60.50       | —        | (40, 10) | (80, 0) | (98.66, 113.68) | (117.81, 109.87) |
| $M_3$     | 35.05       | -22.41   | (40, 10) | (-90.21, -81.34) | (-19.26, -31.57) | (-102.74, -48.29) |
| $M_4$     | 33.99       | -32.93   | (40, 10) | (-100.00, -23.83) | (-15.55, -27.46) | (-93.35, -73.17) |
| $M_5$     | 59.88       | -11.51   | (40, 10) | (-2617, 68.81) | (103.25, 119.02) | (71.95, 146.84) |
| $M_6$     | 59.41       | -48.40   | (40, 10) | (-22.72, -0.91) | (107.1, 123.5) | (84.53, 136.9) |
| $M_7$     | 60.11       | —        | (40, 10) | (3.86, 30.77) | (101.49, 116.99) | (84.43, 126.79) |
| $M_8$     | -78.15      | -8.2     | (40, 73.84) | (98.37, 111.96) | (-23.13, -1.4) | (-58.95, -77.33) |
find the mechanism with higher straight-line performance.

(3) If the third-order mechanism design method cannot get a satisfactory mechanism due to the motion space, transmission angle, branch defect, length, and so on, or the designer is more concerned about a longer straight-line working segment, the designer can consider using the second-order mechanism design method to obtain more mechanisms using the position angle of the inflection circle. Although the calculation and search time are extended, the optimal mechanism with good straight-line performance can be obtained more easily to satisfy the tolerance.

6. Conclusion

This paper presents a novel method to solve the synthesis problem of the approximate straight-line mechanism, providing a general and effective synthesis method for the mechanism that realizes an approximate straight-line trajectory. The method uses the analytical geometry method combined with computer graphics technology and visual optimization design. A unified mathematical model for the synthesis of this kind of mechanism is established, and the design variables are limited to \((-\pi/2, \pi/2)\) to generate a straight-line mechanism with at least second-order or third-order osculation. The designer can chose a general configuration or a certain specific configuration according to the design requirements. An infinite number of mechanisms will be obtained by changing the design variables. In addition to straightness, other motion conditions such as type, contour size of the coupler curve, and transmission performance must be considered for engineering purposes. To achieve efficient optimization, the kinematic properties of the mechanism that are of interest to the designer are calculated, and the mechanical properties are visualized graphically. By representing an infinite number of mechanisms in a finite solution region, the designer can get an intuitive and comprehensive understanding of the trend and distribution pattern of all mechanisms from the mechanism solution region with the design variables changing. Then, by applying kinematic constraints, the feasible mechanisms are automatically selected from the infinite number of mechanisms. The optimal high-precision straight-line mechanism satisfying linear performance and other kinematic requirements is solved according to the set optimization target, which provides a new technical means for designers to make fast and accurate decisions. The proposed method is an effective combination of traditional theory and modern design methods, which solves the synthesis problem of approximate straight-line mechanisms and the blindness of selecting from infinitely number mechanisms in the synthesis process.

Data Availability

The parameter data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was financially supported by both the National Natural Science Foundation of China (No. 51874235) and the Natural Science Foundation of Shaanxi Province (2021JLM-01).

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