Solvability for Stokes system in Hölder spaces in bounded domains and its applications

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Abstract

We consider Stokes system in bounded domains and we present conditions of given data, in particular, boundary data, which ensure Hölder continuity of solutions. For Hölder continuous solutions for the Stokes system the normal component of boundary data requires a bit more regular than boundary data of Hölder continuous solutions for the heat equation. We also construct an example, which shows that Hölder continuity is no longer valid, unless the proposed condition of boundary data is fulfilled. As an application, we consider a certain general types of nonlinear systems coupled to fluid equations and local well-posedness is established in Hölder spaces.

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1 Introduction

In this paper, we study the initial and boundary value problem of non-stationary Stokes system in bounded domains with $C^2$ boundary in $\mathbb{R}^n$, $n \geq 2$. To be more precise, we consider

$$\partial_t u - \Delta u + \nabla p = \nabla \cdot F + f, \quad \text{div} \ u = 0 \quad \text{in} \quad Q_T := \Omega \times [0, T]$$

with initial condition and boundary condition

$$u(x, 0) = u_0(x), \quad u(x, t) = \phi(x, t) \quad \text{on} \quad \partial \Omega \times [0, T],$$

where vector field $f$ and tensor $F = (F_{ij})$ are given external forces. Here we assume that the compatibility conditions hold

$$\text{div} \ u_0 = 0, \quad u_0(x) = \phi(x, 0) \quad \text{on} \quad \partial \Omega, \quad \int_{\partial \Omega} \phi(x, t) \cdot n = 0 \quad \text{for all} \quad t \in [0, T],$$

where $n$ is the outward unit normal vector on $\partial \Omega$.

Our main objective of this paper is to establish well-posedness of the Stokes system (1.1)-(1.2) in the Hölder spaces, $C^{\alpha, \frac{1}{2}}_{\Omega, \overline{Q}_T}$.

We recall some known results related to our concerns. In case $u_0 \in C^s(\mathbb{R}^3_+), f \in C^{s-2, \frac{1}{s}-1}(\mathbb{R}^3_+ \times (0, T)), F = 0$ and $\phi \in C^{s, \frac{1}{2}}(\mathbb{R}^2 \times (0, T))$ for $2 < s < 3$, Solonnikov showed in [23] that a unique solution of the Stokes system (1.1)-(1.2) exists so that

$$\|u\|_{C^{s, \frac{1}{2}}(\mathbb{R}^3_+ \times (0, T))} \leq C \left( \|u_0\|_{C^s(\mathbb{R}^3_+)} + \|\phi\|_{C^{s, \frac{1}{2}}(\mathbb{R}^2 \times (0, T))} + \|R'(D_t \phi_0)\|_{L^\infty(\mathbb{R}^2; C^\frac{1}{2}_T)} + \|f\|_{C^{s-2, \frac{1}{s}-1}((\mathbb{R}^3_+ \times (0, T))} \right).$$
When \( \Omega \) is a bounded domain with \( 0 < \alpha < 1 \), the first author and Jin [6] proved that in case that \( f = 0 \), the following estimate holds: for \( 0 < \alpha < 1 \)

\[
\|u\|_{C^\alpha, \frac{\alpha}{2}(\mathbb{R}^n \times (0,T))} \leq c\left(\|u_0\|_{C^\alpha(\mathbb{R}^n_+)} + \|R'u_0\|_{C^\alpha(\mathbb{R}^n_+)} + \|\phi\|_{C^\alpha, \frac{\alpha}{2}(\mathbb{R}^n_+ \times (0,T))} + \|R'\phi\|_{C^\alpha, \frac{\alpha}{2}(\mathbb{R}^n_+ \times (0,T))} + \max\{ T^\frac{\alpha}{2}, T^\frac{\alpha}{2} + \frac{\alpha}{2} \}\right). 
\]

When \( \Omega \) is a bounded domain with \( C^{2+\alpha} > 0 \), Solonnikov [22] showed that if \( f = 0 \), \( \mathcal{F} = 0 \) and \( \phi \in C(\partial \Omega \times (0, T)) \) with \( \phi \cdot n = 0 \), then the solution \( u \) of (1.1)-(1.2) is continuous in \( \Omega \times (0, T) \) such that

\[
\|u\|_{L^\infty(\Omega \times (0,T))} \leq c\|\phi\|_{L^\infty(\partial \Omega \times (0,T))}. \tag{1.4} \]

The estimate (1.4) was improved by the first author and Choe as following inequality (see [5])

\[
\|u\|_{L^\infty(\Omega \times (0,T))} \leq c\left(\|\phi\|_{L^\infty(\partial \Omega \times (0,T))} + \max_{t \in (0,T)} \|\phi \cdot n(t)\|_{Dini, \partial \Omega}\right),
\]

where for an \( r_0 > 0 \)

\[
\|f\|_{Dini, \partial \Omega} = \sup_{P \in \partial \Omega} \int_{0}^{r_0} \omega(f)(r, P) \frac{dr}{r}, \quad \omega(f)(r, P) = \sup_{Q \in B_r(P) \cap \Omega} |f(Q) - f(P)|.
\]

There are various literatures for the solvability of the Stokes system (1.1)-(1.2) with homogeneous boundary data, that is, with \( \phi = 0 \) (see e.g. [5], [15], [18]-[19], and references therein). In particular, the following estimate is derived in [22]:

\[
\|u\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))} \leq c\left(\|u_0\|_{L^\infty(\mathbb{R}^n_+)} + \|R'u_0\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))}\right),
\]

where \( u_0 \in C(\mathbb{R}^n_+) \) and \( f = 0, \mathcal{F} = (F_{ij})_{i,j=1}^n \in C(\mathbb{R}^n_+ \times (0, T)) \) with \( \text{div} u_0 = 0 \), \( u_0|_{x_n=0} = 0 \), \( F_{nj}|_{x_n=0} = 0 \) for \( j = 1, 2, \ldots, n \) (see also [15]).

We compare the system (1.1)-(1.3) to similar situation of the heat equation

\[
\partial_t v - \Delta v = \nabla \cdot f + f \quad \text{in} \quad Q_T := \Omega \times [0, T] \tag{1.5}
\]

with initial condition and boundary conditions

\[
v(x, 0) = v_0(x) \quad \text{in} \quad \Omega, \quad v(x, t) = \phi(x, t) \quad \text{on} \quad \partial \Omega \times [0, T]. \tag{1.6}
\]

If we assume that

\[
v_0 \in C^{\alpha+1}(\Omega), \quad \phi \in C^{\alpha, \frac{\alpha}{2}}(\partial \Omega \times [0, T]), \quad f \in L^\infty(\Omega \times (0, T)), \quad \mathcal{F} \in C^{\alpha, \frac{\alpha}{2}}(\Omega \times [0, T]), \tag{1.7}
\]

we then obtain the following estimate:

\[
\|v\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega \times [0, T])} \leq c\left(\|v_0\|_{C^\alpha(\Omega)} + \|\phi\|_{C^{\alpha, \frac{\alpha}{2}}(\partial \Omega \times [0, T])} + T^{1-\frac{\alpha}{2}}\|f\|_{L^\infty(\Omega \times (0,T))} + T^\frac{\alpha}{2}\|\mathcal{F}\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega \times [0, T])}\right). \tag{1.8}
\]

The estimate (1.8) is probably known to experts, but we show it for clarity in section 2. In fact, we prove more than the above estimate (see Theorem 5 for the details). Definition of Hölder spaces \( C^{\alpha+1}(\Omega), C^{\alpha, \frac{\alpha}{2}}(\Omega \times [0, T]) \) and \( C^{\alpha, \frac{\alpha}{2}}(\partial \Omega \times [0, T]) \) are given in section 2.
Corollary 1 In case that the Dirichlet boundary condition in \((1.6)\), \(v = \phi\) on \(\partial \Omega \times (0, T)\), is replaced by the Neumann condition \(\frac{\partial v}{\partial n} = \psi\) on \(\partial \Omega\), if \(\psi \in C^{1+\alpha, \frac{1}{2}+\frac{3}{2}\alpha}(\partial \Omega \times [0, T])\) is assumed, then the same estimate as \((1.8)\) can be valid.

Because of non-local effect for the Stokes system, the estimate \((1.8)\) is not clear. If we assume, however, further additional assumptions for \(u_0\) and \(\phi\), then the Hölder estimate is available. To be more precisely, if we assume, instead of \((1.7)\), that

\[
\begin{align*}
    u_0 &\in C_{\partial \Omega}^{\alpha}(\Omega), & \phi &\in C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T]), & \phi \cdot n &\in C_{\partial \Omega}^{\alpha}(\partial \Omega; C^{\frac{1}{2}\beta}[0, T]), \\
    f &\in L^\infty(0, T; C_{\partial \Omega}^{\alpha}(\Omega)), & F &\in C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T]), \quad (1.9)
\end{align*}
\]

then the similar estimate as \((1.8)\) can be obtained. The details of function spaces \(C_{\partial \Omega}^{\alpha}(\Omega)\), \(C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])\) and \(C_{\partial \Omega}^{\alpha}(\partial \Omega; C^{\frac{1}{2}\beta}[0, T])\) are also given in section 2.

Our first main result reads as follows:

Theorem 1 Let \(0 < \alpha < 1\). Let \(u_0, \phi, f, F\) satisfy the conditions \((1.9)\) - \((1.10)\). Furthermore, \(u_0\) and \(\phi\) satisfy the compatibility condition \((1.3)\). Then, there exists a unique weak solution \(u\) of the Stokes equations of \((1.1)\) - \((1.2)\) in the class \(C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])\) such that

\[
\begin{align*}
    \|u\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])} &\leq C(\|\phi\|_{C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])} + \|\phi \cdot n\|_{C_{\partial \Omega}^{\alpha}(\partial \Omega; C^{\frac{1}{2}\beta}[0, T])} + \|u_0\|_{C_{\partial \Omega}^{\alpha}(\Omega)}) \\
    &\quad + \max(T, T^{2-\frac{1}{2}\beta}) \|F\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])} + \max(T, T^{2-\frac{1}{2}\beta}) \|f\|_{L^\infty(0, T; C_{\partial \Omega}^{\alpha}(\Omega))}.
\end{align*}
\]

The notion of weak solution of the Stokes system in the class \(C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])\) is given in section 3 (see Definition 1).

We note that \(\phi \in C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])\) implies the condition \((1.9)\) with replacement of \(\alpha\) by \(\beta\) for any \(0 < \beta < \alpha\), since

\[
\begin{align*}
    \|\phi\|_{C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])} + \|\phi \cdot n\|_{C_{\partial \Omega}^{\alpha}(\partial \Omega; C^{\frac{1}{2}\beta}[0, T])} &\leq c(\|\phi\|_{C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])}.
\end{align*}
\]

Similarly, we also note that

\[
\begin{align*}
    \|u_0\|_{C_{\partial \Omega}^{\beta}(\Omega)} &\leq \|u_0\|_{C^{\alpha}(\Omega)}; & \|F\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])} &\leq \|F\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])};
\end{align*}
\]

\[
\begin{align*}
    \|f\|_{L^\infty(0, T; C_{\partial \Omega}^{\beta}(\Omega))} &\leq c \|f\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])}.
\end{align*}
\]

Therefore, a direct consequence of Theorem 1 is the following:

Corollary 1 Let \(\alpha \in (0, 1)\). Suppose that \(u_0 \in C^{\alpha}(\Omega)\), \(\phi \in C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])\), and \(u_0\) and \(\phi\) satisfy the compatibility condition \((1.3)\). Assume further that \(f, F \in C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])\). Then, there exists unique weak solution \(u\) of \((1.1)\) - \((1.2)\) in the class \(C^{\beta, \frac{1}{2}\beta}(\Omega \times [0, T])\) for any \(\beta < \alpha\). Furthermore, \(u\) satisfies

\[
\begin{align*}
    \|u\|_{C^{\beta, \frac{1}{2}\beta}(\Omega \times [0, T])} &\leq c(\|u_0\|_{C^{\alpha}(\Omega)} + \|\phi\|_{C^{\alpha, \frac{1}{2}\beta}(\partial \Omega \times [0, T])} + \max(T, T^{2-\frac{1}{2}\beta}) \|F\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])} \\
    &\quad + \max(T, T^{2-\frac{1}{2}\beta}) \|f\|_{C^{\alpha, \frac{1}{2}\beta}(\Omega \times [0, T])}).
\end{align*}
\]
Next, we show that \( \beta \) cannot be extended to \( \alpha \) in Corollary \ref{cor1}. To be more precise, there exists a boundary data \( \phi \in C^{\alpha,\frac{\alpha}{2}}(\partial \Omega \times [0,T]) \) such that \( u \notin C^{\alpha,\frac{\alpha}{2}}(\Omega \times [0,T]) \), even if \( u_0, f \) and \( F \) are smooth. This implies that the result in Theorem \ref{thm1} seems optimal. Our second result is to construct a solution \( u \notin C^{\alpha,\frac{\alpha}{2}}(\Omega \times [0,T]) \) of \((1.1)-(1.3)\) when \( \phi \in C^{\alpha,\frac{\alpha}{2}}(\partial \Omega \times [0,T]) \).

**Theorem 2** Theorem \ref{thm1} is not true, if \( \phi \) is assumed to belong to \( C^{\alpha,\frac{\alpha}{2}}(\partial \Omega \times [0,T]) \) only.

As an application, we consider nonlinear types of drift equations coupled fluid equations. Let \( \rho : \Omega \times [0,T] \to \mathbb{R}, \theta : \Omega \times [0,T] \to \mathbb{R} \) and \( u : \Omega \times [0,T] \to \mathbb{R}^n \) satisfy

\[
\partial_t \rho + u \cdot \nabla \rho - \Delta \rho = \nabla \cdot F(\rho, \theta, \nabla \theta, u), \tag{1.11} \]
\[
\partial_t \theta + u \cdot \nabla \theta - \Delta \theta = f(\rho, \theta, \nabla \theta, u), \tag{1.12} \]
\[
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = G(\rho, \theta, \nabla \theta, u), \quad \text{div} \, u = 0 \tag{1.13} \]

with initial data \( \rho_0, \theta_0 \) and \( u_0 \). Here \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( F, G : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) are \( C^1 \) scalar and vector valued functions with polynomial growth conditions. To be more precise, we assume that for \( (x,y,z,w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \) there exists an integer \( l \) with \( 1 \leq l < \infty \) such that \( F, G, \) and \( \phi \) satisfy

\[
|f(x,y,z,w)| + |F(x,y,z,w)| + |G(x,y,z,w)| \leq C (1 + |x| + |y| + |z| + |w|)^l, \tag{1.14} \]
\[
|\nabla f(x,y,z,w)| + |\nabla F(x,y,z,w)| + |\nabla G(x,y,z,w)| \leq C (1 + |x| + |y| + |z| + |w|)^{l-1}. \tag{1.15} \]

Under our consideration, no-flux boundary conditions are assigned for \( \rho \) and \( \theta \) and no-slip boundary condition of \( u \) is assumed, namely

\[
\frac{\partial \rho}{\partial n} = 0, \quad \frac{\partial \theta}{\partial n} = 0, \quad u = 0 \quad \text{on} \quad \partial \Omega. \tag{1.16} \]

For nonlinear system \((1.11)-(1.16)\), with the aid of results in Theorem \ref{thm1}, we can also establish local well-posedness in the Hölder spaces. Our last main result reads as follows:

**Theorem 3** Let the initial data \( (\rho_0, \theta_0, u_0) \) be given in \( C^\alpha(\Omega) \times C^{\alpha+1}(\Omega) \times C_0^\alpha(\Omega) \) for \( \alpha \in (0,1) \). Assume that \( F, G \) and \( f \) satisfy the assumption \((1.11)-(1.13)\). There exists \( T_1 > 0 \) such that a pair of unique solution \( (\rho, \theta, u) \) for \((1.11)-(1.13)\) with \((1.16)\) can be constructed in the class \( C^{\alpha,\frac{\alpha}{2}}(\Omega \times [0,T_1]) \times C^{\alpha+1,\frac{\alpha+1}{2}}(\Omega \times [0,T_1]) \times C^{\alpha,\frac{\alpha}{2}}(\partial \Omega \times [0,T_1]) \).

**Remark 2** The result of Theorem \ref{thm3} could be applicable to various types of concrete equations involving fluid motions. For an specific example, the Keller-Segel-Navier-Stokes equations, a mathematical model describing the dynamics of a certain bacteria living in fluid and consuming oxygen, can be considered. For such model we can establish local well-posedness in the Hölder space as a consequence of Theorem \ref{thm3} (see section 4 for more details).

This paper is organized as follows. In Section 2, Hölder estimates of solutions for the heat equations are computed. Section 3, 4 and 5 are devoted to providing the proofs of Theorem \ref{thm1}, Theorem \ref{thm2} and Theorem \ref{thm3} respectively. Some technical lemmas are proved in Appendix.
2 Preliminaries

We first introduce the notation and present preparatory results that are useful to our analysis. We start with the notation. Let $\Omega$ be an open domain in $\mathbb{R}^n$. The letter $c$ is used to represent a generic constant, which may change from line to line, and $c(\ast, \cdots, \ast)$ is considered a positive constant depending on $\ast, \cdots, \ast$. We introduce a homogeneous Hölder space in $\Omega$ with exponent $\alpha \in (0, 1)$, denoted by $\dot{C}^\alpha(\Omega)$, defined by

$$\dot{C}^\alpha(\Omega) := \{ f \in L^1(\Omega) : \| f \|_{\dot{C}^\alpha(\Omega)} := \sup_{x, y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \}.$$  

Usual Hölder space with exponent $\alpha \in (0, 1)$, denoted by $C^\alpha(\Omega)$, is specified as

$$C^\alpha(\Omega) := \{ f \in L^1(\Omega) : \| f \|_{C^\alpha(\Omega)} := \| f \|_{L^\infty(\Omega)} + \| f \|_{\dot{C}^\alpha(\Omega)} < \infty \}.$$  

Furthermore, we introduce following function classes

$$\dot{C}^\alpha_{D_{\eta}}(\Omega) = \{ f \in \dot{C}^\alpha(\Omega) : \| f \|_{\dot{C}^\alpha_{D_{\eta}}(\Omega)} := \sup_{x, y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha \eta(|x - y|)} < \infty \},$$

where $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is increasing Dini continuous, namely $\int_0^1 \frac{\eta(r)}{r} \, dr < \infty$. Similarly, we define $\dot{C}^\alpha_{D_{\eta}}(\Omega)$ by

$$\dot{C}^\alpha_{D_{\eta}}(\Omega) = \{ f \in \dot{C}^\alpha(\Omega) : \| f \|_{\dot{C}^\alpha_{D_{\eta}}(\Omega)} := \| f \|_{\dot{C}^\alpha(\Omega)} + \| f \|_{\dot{C}^\alpha_{D_{\eta}}(\Omega)} < \infty \}.$$  

In case of non-stationary function $f \in L^1(\Omega \times (0, T))$, we recall a seminorm of $f$, which is Hölder continuous with exponent $\alpha \in (0, 1)$ in spatial and temporal variable, denoted by $\| f \|_{\dot{C}^{\frac{1}{2} \alpha}(\Omega \times [0, T])}$, indicated as follows:

$$\| f \|_{\dot{C}^{\frac{1}{2} \alpha}(\Omega \times [0, T])} := \| f \|_{L^\infty(0, T) ; \dot{C}^\alpha(\Omega))} + \| f \|_{L^\infty(\Omega ; \dot{C}^{\frac{1}{2} \alpha}(0, T))} = \sup_t \sup_{x, y} \frac{|f(x, t) - f(y, t)|}{|x - y|^\alpha} + \sup_t \sup_{x, s} \frac{|f(x, t) - f(x, s)|}{|t - s|^\frac{1}{2} \alpha}.$$  

We also remind an Hölder space with exponent $\alpha \in (0, 1)$ in $\Omega \times (0, T)$, written as $C^{\frac{1}{2} \alpha}(\Omega \times [0, T])$, which is given by

$$C^{\frac{1}{2} \alpha}(\Omega \times [0, T]) := \{ f \in L^1 : \| f \|_{C^{\frac{1}{2} \alpha}(\Omega \times [0, T])} := \| f \|_{L^\infty(\Omega \times [0, T])} + \| f \|_{\dot{C}^{\frac{1}{2} \alpha}(\Omega \times [0, T])} < \infty \}.$$  

Let $\eta$ be a increasing Dini-function defined above. To treat non-zero boundary data under our considerations, we also introduce a function class $\dot{C}^{\frac{1}{2} \alpha}_{D_{\eta}}(\partial \Omega ; \dot{C}^{\frac{1}{2} \alpha}(0, T))$ defined by

$$\dot{C}^{\frac{1}{2} \alpha}_{D_{\eta}}(\partial \Omega ; \dot{C}^{\frac{1}{2} \alpha}(0, T)) := \left\{ f \mid \sup_{P, Q \in \partial \Omega} \frac{\| f(P, \cdot) - f(Q, \cdot) \|_{\dot{C}^{\frac{1}{2} \alpha}(0, T)}}{\eta(|P - Q|)} < \infty \right\}, \quad \alpha \in (0, 1),$$

equipped with the norm $\| f \|_{\dot{C}^{\frac{1}{2} \alpha}_{D_{\eta}}(\partial \Omega ; \dot{C}^{\frac{1}{2} \alpha}(0, T))} := \sup_{P, Q \in \partial \Omega} \frac{\| f(P, \cdot) - f(Q, \cdot) \|_{\dot{C}^{\frac{1}{2} \alpha}(0, T)}}{\eta(|P - Q|)}$, which is equivalently as

$$\| f \|_{\dot{C}^{\frac{1}{2} \alpha}_{D_{\eta}}(\partial \Omega ; \dot{C}^{\frac{1}{2} \alpha}(0, T))} = \sup_{P, Q \in \partial \Omega} \sup_{s, t \in [0, T]} \frac{|f(P, t) - f(P, s) - f(Q, t) + f(Q, s)|}{|t - s|^\frac{1}{2} \alpha \eta(|P - Q|)}.$$  

\[5\]
For our purpose, as a limiting case of $\alpha = 0$, we introduce
\[
L^\infty(0, T; \hat{C}_{D_\eta}(\overline{\Omega})) := \{f \mid \sup_t \sup_{x,y \in \Omega} \frac{|f(x,t) - f(y,t)|}{\eta(|x-y|)} < \infty\}.
\]

We recall some estimates of heat equations in following lemmas.

**Lemma 4** Let $\alpha \in (0, \infty)$, $0 < T < \infty$ and $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, which belongs to $C_{D_\eta}^\alpha(\mathbb{R}^n)$. We set $W(x,t) := \int_{\mathbb{R}^n} \Gamma(x-z,t)u_0(z)dz$, where $\Gamma$ is the heat kernel. Then, $W \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])$ and $W$ satisfies
\[
\|W\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])} \leq c\|u_0\|_{C^{\alpha}(\mathbb{R}^n)}. \tag{2.1} \]

Furthermore, if $\alpha \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, then
\[
\|W \cdot n\|_{C_{D_\eta}^\alpha(\partial \Omega; C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T]))} \leq c\|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)}. \tag{2.2} \]

**Proof.** Since the estimates in (2.1) are well-known, we omit its details (see e.g. [15]) and we just show the estimate (2.2). Indeed, using $u_0 \in C_{D_\eta}^\alpha(\mathbb{R}^n)$, we compute for $P, Q \in \partial \Omega$
\[
W(P,t) \cdot n(P) - W(P,s) \cdot n(P) - W(Q,t) \cdot n(Q) + W(Q,s) \cdot n(Q)
= \int_{\mathbb{R}^n} \left(\Gamma(z,t) - \Gamma(z,s)\right) u_0(P - z) \cdot n(P) - u_0(Q - z) \cdot n(Q) \, dz
= \int_{\mathbb{R}^n} \left(\Gamma(z,t) - \Gamma(z,s)\right) u_0(P - z) \cdot n(P) - u_0(P) \cdot n(P) - u_0(Q - z) \cdot n(Q) + u_0(Q) \cdot n(Q) \, dz.
\]

For the second equality, we used $\int_{\mathbb{R}^n} (\Gamma(z,t) - \Gamma(z,s)) \, dz = 0$ for all $0 < s, t$. We note that
\[
|u_0(P - z) \cdot n(P) - u_0(P) \cdot n(P) - u_0(Q - z) \cdot n(Q) + u_0(Q) \cdot n(Q)|
\leq c\left(\|u_0\|_{C^\alpha(\mathbb{R}^n)}|P - Q| + \|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)} \eta(|P - Q|)\right) |z|^{\alpha}.
\]

Hence, for $s < t$, we have
\[
|W(P,t) \cdot n(P) - W(P,s) \cdot n(P) - W(Q,t) \cdot n(Q) + W(Q,s) \cdot n(Q)|
\leq \|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\Gamma(z,t) - \Gamma(z,s)||z|^\alpha dz(\eta(|P - Q|) + |P - Q|)
\leq \|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)} \int_s^t \int_{\mathbb{R}^n} |D_\tau \Gamma(z,\tau)||z|^\alpha dz(\eta(|P - Q|) + |P - Q|)
\leq c\|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)} \int_s^t \frac{1}{\tau^{\frac{\alpha}{2}}} d\tau(\eta(|P - Q|) + |P - Q|)
\leq c\|u_0\|_{C_{D_\eta}^\alpha(\mathbb{R}^n)} (t-s)^{\frac{1}{2}} \eta(|P - Q|) + |P - Q|).
\]

This completes the proof.\[\square\]
For notational convention, we denote for a measurable function $f$ in $\mathbb{R}^n \times \mathbb{R}$
\[
\Lambda_0(f)(x,t) := \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)f(y,s)dyds,
\]
(2.3)
where $\Gamma$ is the heat kernel.

Next, we also present estimates of heat equation with external force with zero initial. It may be probably well-known to experts, we present its details in the Appendix for reader’s convenience.

\textbf{Lemma 5} Let $T > 0$, $0 < \alpha < 1$, $f \in C^{\alpha,\frac{1}{2}\alpha}(\mathbb{R}^n \times [0,T])$ and $\Lambda_0(f)$ be defined in (2.3). Then,
\[
\|\Lambda_0(f)\|_{C^{\alpha,\frac{1}{2}\alpha}(\mathbb{R}^n \times [0,T])} \leq cT^{1-\frac{1}{2}\alpha}\|f\|_{L^{\infty}(\mathbb{R}^n \times (0,T))},
\]
(2.4)
\[
\|\Lambda_0(f)\|_{C^{\alpha+1,\frac{1}{2}\alpha+\frac{1}{2}}(\mathbb{R}^n \times [0,T])} \leq c\max \left( T^{\frac{1}{2}}, T^{\frac{1}{2} - \frac{1}{2}\alpha} \right) \|f\|_{L^{\infty}(0,T;C^{\alpha}(\mathbb{R}^n))},
\]
(2.5)
\[
\|\nabla \Lambda_0(f)\|_{C^{\alpha,\frac{1}{2}\alpha+\frac{1}{2}}(\mathbb{R}^n \times [0,T])} \leq c\|f\|_{L^{\infty}(0,T;C^{\alpha}(\mathbb{R}^n))},
\]
(2.6)
\[
\|\nabla \Lambda_0(f)\|_{C^{\alpha,\frac{1}{2}\alpha}(\mathbb{R}^n \times [0,T])} \leq cT^{\frac{1}{2}\alpha}\|f\|_{L^{\infty}(0,T;C^{\alpha}(\mathbb{R}^n))},
\]
(2.7)
\[
\|\nabla \Lambda_0(f)\|_{C^{\alpha+\epsilon,\frac{1}{2}\alpha+\frac{1}{2}(\alpha+\epsilon)}(\mathbb{R}^n \times [0,T])} \leq cT^{\frac{1}{2}\alpha-\frac{\epsilon}{2}}\|f\|_{L^{\infty}(0,T;C^{\alpha}(\mathbb{R}^n))}, \quad 0 < \epsilon < 1.
\]
(2.8)

Lastly, we consider the initial-boundary value problem of heat equation (1.5)-(1.6). Here we assume that $v_0 \in C^{\alpha+k}(\Omega)$, $\psi \in C^{\alpha+k,\frac{1}{2}(\alpha+k)}(\partial \Omega \times [0,T])$, $f \in C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])$ and $F \in C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])$, where $k$ is either 0 or 1. We let $\tilde{f}, \tilde{F} \in C^{\alpha,\frac{1}{2}\alpha}(\mathbb{R}^n \times [0,T])$ an extension of $f, F$, respectively, such that $\|\tilde{f}\|_{C^{\alpha,\frac{1}{2}\alpha}(\mathbb{R}^n \times [0,T])} \leq c\|f\|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])}$. Similarly, we denote by $\tilde{v}_0$ the extension of $v_0$ such that $\|\tilde{v}_0\|_{C^{\alpha+k,\frac{1}{2}(\alpha+k)}(\Omega \times [0,T])} \leq c\|v_0\|_{C^{\alpha+k,\frac{1}{2}(\alpha+k)}(\Omega \times [0,T])}$.

\textbf{Theorem 6} Let $\Omega$ be an bounded domain with $C^2$ boundary. Suppose that $f, F \in C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])$, $\psi \in C^{\alpha+k,\frac{1}{2}(\alpha+k)}(\partial \Omega \times [0,T])$ and $v_0 \in C^{\alpha+k}(\Omega)$ with $\psi|_{t=0} = v_0|_{\partial \Omega}$, where $k = 0$ or $k = 1$. Then, there exists a unique solution $v \in C^{\alpha+k,\frac{1}{2}(\alpha+k)}(\Omega \times [0,T])$ of (1.5)-(1.6) and $v$ satisfies
\[
\|v\|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])} \leq c\|v_0\|_{C^{\alpha}(\Omega)} + \|\psi\|_{C^{\alpha,\frac{1}{2}\alpha}(\partial \Omega \times [0,T])} + T^{1-\frac{1}{2}\alpha}\|f\|_{L^{\infty}(\Omega \times (0,T))} + T^{\frac{1}{2}}\|F\|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])},
\]
(2.9)
\[
\|v\|_{C^{\alpha+1,\frac{1}{2}(\alpha+1)}(\Omega \times [0,T])} \leq c\|v_0\|_{C^{\alpha+1}(\Omega)} + \|\psi\|_{C^{\alpha+1,\frac{1}{2}(\alpha+1)}(\partial \Omega \times [0,T])} + \max \left( T^{\frac{1}{2}}, T^{(\frac{1}{2} - \frac{\alpha}{2})} \right) \|f\|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])} + \|F\|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])},
\]
(2.10)
Remark 3 In case that the boundary condition in \((1.5)\), \(v = \psi\) on \(\partial \Omega\), is replaced by the Neumann condition \(\frac{\partial v}{\partial n} = \psi\) on \(\partial \Omega\), if \(\psi \in C^{1+k+\alpha, \frac{1}{2}(1+k+\alpha)}(\partial \Omega \times [0, T])\) is assumed, then the same result of Theorem 4 can be obtained.

3 Proof of Theorem 1

In this section, we consider the boundary value problem of the following Stokes system \((1.1)-(1.2)\). Let \(\Omega\) be a \(C^2\) bounded domain in \(\mathbb{R}^n\). First we introduce the notion of weak solutions for the Stokes system.

**Definition 7** Suppose that \(\mathcal{F} = \{F_j\}_{j=1}^n \in C^{\alpha, \frac{1}{2}}(\bar{\Omega} \times [0, T]), f \in L^\infty(\Omega \times (0, T)), g \in C^{\alpha, \frac{1}{2}}(\partial \Omega \times [0, T])\) and \(v_0 \in C^\alpha(\bar{\Omega}).\) We say that a vector field \(u\) is a weak solution in the class \(C^{\alpha, \frac{1}{2}}(\bar{\Omega} \times [0, T])\) for the Stokes system \((1.1)-(1.2)\) if the following conditions are satisfied:
(i) $u \in C^{\alpha, \frac{1}{2}}(\Omega \times [0, T])$ and $\nabla u \in L^\infty_{\text{loc}}(\Omega \times (0, T))$.

(ii) For each $\Phi \in C_0^\infty(\Omega \times (0, T))$ with $\text{div}_x \Phi = 0$

$$\int_0^T \int_\Omega \nabla u : \nabla \Phi \, dx \, dt = \int_0^T \int_\Omega u \cdot \Phi_t + f \cdot \Phi - F : \nabla \Phi \, dx \, dt$$

(iii) $u(x, 0) = u_0(x)$ in $\Omega$.

(iv) $u(P, t) = g(P, t)$ in $P \in \partial \Omega \times (0, T)$.

For $f$ and $F$ given in Theorem 1, we denote by $\tilde{f}$ and $\tilde{F}$ the extension of $f$ and $F$, respectively, to $\mathbb{R}^n \times (0, T)$ such that $\tilde{f}$ and $\tilde{F}$ have compact supports. Let $\mathbb{P}$ be the Helmholtz projection operator on $\mathbb{R}^n$ such that

$$[\mathbb{P} \tilde{f}]_j(x, t) = \delta_{ij} \tilde{f}_i + \int_{\mathbb{R}^n} D_x D_x N(x - y) \tilde{f}_i(y, t) \, dy = \delta_{ij} \tilde{f}_i + R_i R_j \tilde{f}_i,$$

$$[\mathbb{P} \text{div} \tilde{F}]_j = D_x \left( \delta_{ij} \tilde{F}_{ki} + R_i R_j \tilde{F}_{ki} \right),$$

where $R_i$ is Riesz transform in $\mathbb{R}$.

We define $V^1$ and $V^2$ by

$$V^1_1(x, t) := \Lambda_0([\mathbb{P} \tilde{f}]_j)(x, t) = \Lambda_0[\delta_{ij} \tilde{f}_i + R_i R_j \tilde{f}_i](x, t),$$

$$V^2_1(x, t) := \Lambda_0([\mathbb{P} \text{div} \tilde{F}]_j)(x, t) = -D_x \Lambda_0(\delta_{ij} \tilde{F}_{ki} + R_i R_j \tilde{F}_{ki})(x, t).$$

We observe that $V^1$ and $V^2$ satisfy the equations

$$V^1_1 - \Delta V^1 = \mathbb{P} \tilde{f}, \quad \text{div} V^1 = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T),$$

$$V^1_{1|t=0} = 0 \quad \text{on} \quad \mathbb{R}^n.$$

$$V^2_1 - \Delta V^2 = \mathbb{P} \text{div} \tilde{F}, \quad \text{div} V^2 = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T),$$

$$V^2_{1|t=0} = 0 \quad \text{on} \quad \mathbb{R}^n.$$

Since support of $\tilde{f}$ is bounded, we obtain $\|R_i R_j \tilde{f}\|_{L^\infty(\mathbb{R}^n)} \leq c\|\tilde{f}\|_{L^\infty(0, T; \mathfrak{C}_{D_0}(\mathbb{R}^n))}$. By (2.4), we have

$$\|V^1\|_{C^{\alpha, \frac{1}{2}}(\mathbb{R}^n \times [0, T])} \leq cT^{1-\frac{1}{2\alpha}} \|\tilde{f} + R_i R_j \tilde{f}\|_{L^\infty(\mathbb{R}^n \times (0, T))}$$

$$\leq cT^{1-\frac{1}{2\alpha}} \left( \|\tilde{f}\|_{L^\infty(\mathbb{R}^n \times (0, T))} + \|\tilde{F}\|_{L^\infty(0, T; \mathfrak{C}_{D_0}(\mathbb{R}^n))} \right)$$

$$\leq cT^{1-\frac{1}{2\alpha}} \|\tilde{F}\|_{L^\infty(0, T; \mathfrak{C}_{D_0}(\mathbb{R}^n))},$$

$$\|V^1\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq cT \|\tilde{f} + R_i R_j \tilde{f}\|_{L^\infty(\mathbb{R}^n \times (0, T))}$$

$$\leq cT \|\tilde{F}\|_{L^\infty(0, T; \mathfrak{C}_{D_0}(\mathbb{R}^n))}.$$
Moreover, we note that

\[
V^1(P, t) \cdot n(P) - V^1(P, s) \cdot n(P) - V^1(Q, t) \cdot n(Q) + V^1(Q, s) \cdot n(Q) = \left( V^1(P, t) - V^1(P, s) \right) \cdot \left( n(P) - n(Q) \right) + \left( V^1(P, t) - V^1(Q, t) - V^1(P, s) + V^1(Q, s) \right) \cdot n(Q)
\]

\[:= I_1 + I_2. \quad (3.2) \]

From (2.4), we get

\[
|I_1| \leq c \|V^1\|_{L^\infty(\mathbb{R}^n; C^{\frac{1}{2}}(\gamma_0, T))} |t - s|^{\frac{1}{2}} |P - Q| \leq c T^{1-\frac{1}{2}} \|\tilde{f}\|_{L^\infty(0, T; C_{D_0}(\mathbb{R}^n))} |t - s|^{\frac{1}{2}} |P - Q|.
\]

(3.3)

And

\[
I_2 = \int_s^t \int_{\mathbb{R}^n} \left( \Gamma(P - z, t - \tau) - \Gamma(Q - z, t - \tau) \right) \mathbb{P}[\tilde{f}](z, \tau) d\tau d\tau
\]

\[
+ \int_0^s \int_{\mathbb{R}^n} \left( \Gamma(P - z, t - \tau) - \Gamma(P - z, s - \tau) - \Gamma(Q - z, t - \tau) + \Gamma(Q - z, s - \tau) \right) \mathbb{P}[\tilde{f}](z, \tau) d\tau d\tau.
\]

The first term is

\[
|I_1| \leq c \|V^1\|_{L^\infty(\mathbb{R}^n; C^{\frac{1}{2}}(\gamma_0, T))} |t - s|^{\frac{1}{2}} |P - Q| \leq c T^{1-\frac{1}{2}} \|\tilde{f}\|_{L^\infty(0, T; C_{D_0}(\mathbb{R}^n))} |t - s|^{\frac{1}{2}} |P - Q|.
\]

(3.3)

The second term is

\[
|I_2| \leq c \|\tilde{f}\|_{L^\infty(0, T; C_{D_0}(\mathbb{R}^n))} |P - Q| (t - s)^{\frac{1}{2}}.
\]

(3.4)

Hence, we have

\[
I_2 \leq \|\tilde{f}\|_{L^\infty(0, T; C_{D_0}(\mathbb{R}^n))} |P - Q| (t - s)^{\frac{1}{2}}.
\]

(3.4)

By (2.2) - (3.4), we have

\[
\|V^1 \cdot n\|_{C_{D_0}(\partial \Omega; C^{\frac{1}{2}}(0, T))} \leq c \max(T^{1-\frac{1}{2}}, T^{\frac{1}{2}}) \|\tilde{f}\|_{L^\infty(0, T; C_{D_0}(\mathbb{R}))}.
\]

(3.5)
Next, we estimate $V^2$. Since $R_i : \dot{\mathcal{C}}^\alpha(\mathbb{R}^n) \rightarrow \dot{\mathcal{C}}^\alpha(\mathbb{R}^n)$ is bounded, by (2.3), we have

$$||V^2||_{\dot{\mathcal{C}}^\alpha(\mathbb{R}^n \times [0,T])} \leq c \max(T^{\frac{1}{2}}, T^{\frac{1}{2} - \frac{\alpha}{2}}) ||\mathcal{P}[\mathcal{F}]||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} \leq c \max(T^{\frac{1}{2}}, T^{\frac{1}{2} - \frac{\alpha}{2}}) ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))}.$$ 

We estimate $||V^2||_{L^\infty(\mathbb{R}^n \times (0,T))}$. It is well known that $D_x \Gamma_t \in \mathcal{H}^1(\mathbb{R}^n)$ with $D_x \Gamma_t \mathcal{H}^1(\mathbb{R}^n) \leq ct^{-\frac{1}{2}}$, where $\mathcal{H}^1(\mathbb{R}^n)$ denotes Hardy space. Since $R_i : BMO(\mathbb{R}^n) \rightarrow BMO(\mathbb{R}^n)$ is bounded, we have

$$||V^2||_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (||\mathcal{F}||_{L^\infty(\mathbb{R}^n \times (0,T))} \int_0^t \int_\mathbb{R}^n D_x \Gamma(x - y, t) dy + \int_0^t \int_\mathbb{R}^n \int_\mathbb{R}^n \int_0^t |R_i R_j \mathcal{F}(t)||_{BMO(\mathbb{R}^n)} |D_x \Gamma_\mathcal{H}^2(\mathbb{R}^n)|)$$

$$\leq c t^{\frac{1}{2}} ||\mathcal{F}||_{L^\infty(\mathbb{R}^n \times (0,T))}.$$ 

Hence, we have

$$||V^2||_{\dot{\mathcal{C}}^\alpha(\mathbb{R}^n \times [0,T])} \leq c \max(T^{\frac{1}{2}}, T^{\frac{1}{2} - \frac{\alpha}{2}}) ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))}.$$ 

Moreover,

$$V^2(P,t) \cdot n(P) - V^2(P,s) \cdot n(P) - V^2(Q,t) \cdot n(Q) + V^2(Q,s) \cdot n(Q)$$

$$= (V^2(P,t) - V^2(P,s)) \cdot (n(P) - n(Q)) + (V^2(P,t) - V^2(P,s) - V^2(Q,t) + V^2(Q,s)) \cdot n(Q)$$

$$:= I_1 + I_2.$$ 

By (2.7), we get

$$I_1 \leq c ||V^2||_{L^\infty(\mathbb{R}^n \times [0,T])} |t - s|^{\frac{1}{2} + \alpha} |P - Q| \leq c T^{\frac{1}{2}} ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} |t - s|^{\frac{1}{2} + \alpha} |P - Q|$$

and

$$\int_0^t \int_\mathbb{R}^n \nabla G(z,t - \tau)(\mathcal{P}[\mathcal{F}](P - z, \tau) - \mathcal{P}[\mathcal{F}](Q - z, \tau)) dzd\tau$$

$$\quad + \int_0^s \int_\mathbb{R}^n \nabla G(z,t - \tau) - \nabla G(z,s - \tau)(\mathcal{P}[\mathcal{F}](P - z, \tau) - \mathcal{P}[\mathcal{F}](Q - z, \tau)) dzd\tau.$$ 

The first term is dominated by

$$|P - Q|^{\alpha} ||\mathcal{P}[\mathcal{F}]||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} \int_s^t (t - \tau)^{\frac{1}{2} - \alpha} d\tau \leq c ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} |P - Q|^{\alpha} (t - s)^{\frac{1}{2}}.$$ 

The second term is

$$\int_0^s \int_\mathbb{R}^n \int_0^1 \int_0^1 D_p \Gamma(z, \lambda t + (1 - \lambda)s - \tau)(\mathcal{P}[\mathcal{F}](P - z, \tau) - \mathcal{P}[\mathcal{F}](Q - z, \tau)) d\lambda dzd\tau$$

$$\leq c ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} |P - Q|^{\alpha} \int_0^1 \int_0^s (\lambda t + (1 - \lambda)s - \tau)^{-\frac{1}{2} + \alpha} d\tau d\lambda$$

$$\leq c ||\mathcal{F}||_{L^\infty(0,T;\dot{\mathcal{C}}^\alpha(\mathbb{R}^n))} |P - Q|^{\alpha} (t - s)^{\frac{1}{2}}.$$
Hence, we have
\[ |II_2| \leq c \| F \|_{L^\infty(0,T;C^1_2[\mathbb{R}^n])} |P - Q|^\frac{1}{2}. \]  
\[ (3.9) \]

From (3.7)-(3.9), we have
\[ \| V^2 \cdot n \|_{C_D(\partial \Omega; C^2_2[0,T])} \leq c \max \left( T^\frac{1}{2}, T^\frac{1}{2} - \frac{1}{4} \right) \| F \|_{L^\infty(0,T;C^1_2(\Omega))}. \]  
\[ (3.10) \]

Next we treat initial data \( u_0 \). Let \( \tilde{u}_0 \) be an extension of \( u_0 \) satisfying that \( \text{div} \tilde{u}_0 = 0 \) in \( \mathbb{R}^n \). Letting \( v \) by
\[ v(x,t) := \int_{\mathbb{R}^n} \Gamma(x - y,t)\tilde{u}_0(y)dy. \]
We observe that \( v \) satisfies the equations
\[ v_t - \Delta v = 0, \quad \text{div} v = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \]
\[ v|_{t=0} = \tilde{u}_0 \quad \text{on} \quad \mathbb{R}^n. \]

By Lemma [4], we have
\[ \| v \|_{C^\alpha, \frac{1}{2}(\mathbb{R}^n \times [0,T])} \leq c \| \tilde{u}_0 \|_{C^\alpha(\mathbb{R}^n)} \leq c \| u_0 \|_{C^\alpha(\overline{\Omega})}, \]
\[ \| v \cdot n \|_{C_D(\partial \Omega; C^2_2[0,T])} \leq c \| \tilde{u}_0 \|_{C^2_2[0,T]} \leq c \| u_0 \|_{C^2_2[0,T)}. \]

We denote \( G \) as
\[ G = \phi - V^1|_{\partial \Omega \times (0,T)} - V^2|_{\partial \Omega \times (0,T)} - v|_{\partial \Omega \times (0,T)} \]
We note that \( G|_{t=0} = 0 \) if \( \phi|_{t=0} = u_0 \) on \( \partial \Omega \) and also observe that \( G \) satisfies
\[ \| G \|_{C^\alpha, \frac{1}{2}(\partial \Omega \times (0,T))} \leq c \left( \| \phi \|_{C^\alpha, \frac{1}{2}(\partial \Omega \times (0,T))} + \max \left( T^\frac{1}{2}, T^\frac{1}{2} - \frac{1}{4} \right) \| f \|_{L^\infty(0,T;C_D(\overline{\Omega}))} \right) \]
\[ + \max \left( T^\frac{1}{2}, T^\frac{1}{2} - \frac{1}{4} \right) \| F \|_{L^\infty(0,T;C^1_2(\Omega))} + \| u_0 \|_{C^\alpha(\overline{\Omega})} \right), \]
\[ \| G \cdot n \|_{C_D(\partial \Omega; C^2_2[0,T])} \leq c \left( \| \phi \cdot n \|_{C_D(\partial \Omega; C^2_2[0,T])} + \| u_0 \|_{C^2_2[0,T]} \right) \max \left( T^\frac{1}{2} - \frac{1}{4}, T^\frac{1}{2} - \frac{1}{4} \right) \| f \|_{L^\infty(0,T;C_D(\overline{\Omega}))}. \]

We decompose the solution \( u \) in (1.1)-(1.2) as the form of \( u = V^1 + V^2 + v + w \), where \( w \) solves the following equations:
\[ \begin{cases} 
\begin{align*}
 w_t - \Delta w + \nabla \pi &= 0, & \Omega \times (0,T), \\
 \text{div} w &= 0, & \Omega \times (0,T), \\
 w|_{\partial \Omega \times (0,T)} &= G, & w|_{t=0} = 0.
\end{align*}
\end{cases} \]  
\[ (3.11) \]

Hence, solving the equations (1.1)-(1.2) is reduced to treat the equations (3.13). For the estimate in Theorem [11] it suffices to obtain that
\[ \| w \|_{C^\alpha, \frac{1}{2}(\Omega \times [0,T])} \leq c \left( \| G \cdot n \|_{C_D(\partial \Omega; C^2_2[0,T])} + \| G \|_{C^\alpha, \frac{1}{2}(\partial \Omega \times [0,T])} \right). \]  
\[ (3.12) \]
3.1 Invertibility of boundary integral operators

In this subsection, we will provide the estimate (3.12). Denoting $w$, $\pi$ and $G$ in (3.11) by $u$, $q$ and $g$, respectively, we consider

\[
\begin{align*}
&\left\{\begin{array}{l}
u_t - \Delta u + \nabla q = 0, \quad \Omega \times (0, T), \\
\text{div } u = 0, \quad \Omega \times (0, T), \\
w|_{\partial \Omega \times (0, T)} = g, \quad u|_{t=0} = 0.
\end{array}\right.
\end{align*}
\]

Due the result of Solonnikov [20], the solution of (3.13) can be written in the form

\[
u(x, t) = U[\Phi](x, t) + \nabla V[\Psi](x, t),
\]

where $\nu$ is electrostatic potential of a single layer, i.e.,

\[
V[\Psi](x, t) = \int_{\partial \Omega} N(x - Q) \Psi(Q, t)dQ,
\]

where $N$ is fundamental solution of Laplace equation. On the other hand, $U$ is referred as the hydrodynamical potential, which is defined by

\[
U[\Phi](x, t) = \int_0^t \int_{\partial \Omega} G(x, Q, t - s) \Phi(Q, s)dQds.
\]

Here $G$ is the tensor given by

\[
G(x, Q, t) = -2 \frac{\partial \Gamma(x - Q, t)}{\partial n_Q} \left(I - n(Q) \otimes n(Q)\right) + 4 \left(\nabla x - n(Q) \frac{\partial}{\partial n}\right) q(x, Q, t),
\]

where $n(Q)$ is unit outer normal vector at $Q \in \partial \Omega$. The corresponding pressure tensor is given as

\[
q(x, Q, t) = \int_{\Pi(x, Q)} \frac{\partial \Gamma(Z - Q, t)}{\partial n} \nabla N(x - Z)dZ,
\]

where $\Pi(x, Q)$ is the layer between the tangent plane at $Q \in \partial \Omega$ and the parallel plane passing through the point $x$ (see [20] pp 115-117). We recall some estimates of $G = (G_{ij})_{i,j=1,2,\ldots,n}$ (see section 3 in [14]). Let $P, Q, Z \in \partial \Omega$. We then have for all $0 < \lambda < 1$

\[
|G_{ij}(P, Q, t)| \leq c_\lambda \frac{1}{t^{1+\lambda/2} \left(|P - Q|^2 + t\right)^{n-2\lambda/2}},
\]

\[
|G_{ij}(P, Z, t) - G_{ij}(Q, Z, t)| \leq c_\lambda \frac{|P - Q|^\lambda}{t^{1+\lambda/2} \left(|P - Z|^2 + t\right)^{n-2\lambda/2}}, \quad \text{if } |P - Q| \leq \frac{1}{2}|P - Z|.
\]

Let $\Phi$ and $\Psi$ satisfy the following condition:

\[
\Phi(P, t) \cdot n(P) = 0, \quad \int_{\partial \Omega} \Psi(Q, t)dQ = 0.
\]
For convenience, for any vector field $h$ defined on $\partial \Omega$ we denote by $h_{\text{tan}}$ the tangential componential of $h$, i.e. $h_{\text{tan}} = h - n(h \cdot n)$. By the (3.14), we solve the following equations

$$\Phi + U_{\text{tan}}[\Phi] + \nabla_S V[\Psi] = g_{\text{tan}},$$

$$\Psi + K^*[\Psi] + n \cdot U[\Phi] = g \cdot n$$

(3.21) boundary system

(see [20] pp 120, (2.26)). Here, $V[\Psi]$ and $U[\Phi]$ are the direct values of (3.15) and (3.16) on $\partial \Omega$, respectively, i.e.

$$U[\Phi](P,t) = \int_0^t \int_{\partial \Omega} G(P,Q,t-s) \Phi(Q,s) dQ ds,$$  
$$V[\Psi](P,t) = \int_{\partial \Omega} N(P - Q) \Psi(Q,t) dQ, \quad P \in \partial \Omega$$  

(3.22) CK1107-2

and

$$K^* \Psi(P,t) = \text{p.v.} c_n \int_{\partial \Omega} \frac{(P - Q) \cdot n(P)}{|P - Q|^n} \Psi(Q,t) dQ.$$  

(3.24) double-layer

In addition, $\nabla_S V$ indicates the tangential gradient of $V$ on $\partial \Omega$, namely $\nabla_S V = \nabla V - n \frac{\partial V}{\partial n}$.

**Lemma 8** Let $P, Q \in \partial \Omega$ and $0 < \alpha < 1$. There is $\delta = \delta(\alpha)$ with $0 < \delta$ such that the tensor $G$ given in (3.17) satisfies

$$\int_0^t \int_{\partial \Omega} |G(P,Z,\tau) - G(Q,Z,\tau)| dZ d\tau \leq c t^\delta |P - Q|\alpha,$$  

(3.25) CK-August

$$\int_s^t \int_{\partial \Omega} |G(P,Z,\tau)| dZ d\tau \leq c (t - s)\delta, \quad 0 \leq s < t, \quad P \in \partial \Omega.$$  

(3.26) CK-June20-200

**Proof.** First, we prove (3.25). Let $r = |P - Q|$. Then, we have

$$\int_0^t \int_{\partial \Omega} |G(P,Z,\tau) - G(Q,Z,\tau)| dZ d\tau = \int_0^t \int_{|Z| < 2r} \cdots dZ d\tau + \int_0^t \int_{|Z| > 2r} \cdots dZ d\tau := I_1 + I_2.$$

Via the inequality (3.18), for $0 < \lambda < 1$ we have

$$I_1 \leq c \int_0^t \int_{|Z| < 2r} \frac{1}{|Z|^2 + \tau} \frac{n - 2\lambda}{n - 2\lambda} \frac{n - 2\lambda}{n - 2\lambda} dZ d\tau \leq c \int_0^t \tau^{-1 + \frac{\lambda}{2}} \int_{|Z| < 2r} \frac{1}{|Z|^2 + 1} dZ d\tau.$$

In case that $\alpha < \frac{1}{2}$, we take $\lambda$ with $0 < \lambda < \frac{1}{2}$. If $r^2 \leq t$, then

$$I_1 \leq c \int_0^{r^2} \tau^{-1 + \frac{\lambda}{2}} d\tau + c \int_{r^2}^t \tau^{-1 + \frac{\lambda}{2}} (\frac{r}{\sqrt{\tau}})^{n-1} d\tau \leq cr^\lambda \leq c t^{\frac{1}{2} - \frac{\lambda}{2}} r^\alpha.$$

On the other hand, if $r^2 > t$, then we have

$$I_1 \leq c \int_0^t \tau^{-1 + \frac{\lambda}{2}} d\tau \leq c \frac{1}{2} t^{\frac{1}{2} - \frac{\lambda}{2}} r^\alpha \leq c t^{\frac{1}{2} - \frac{\lambda}{2}} r^\alpha.$$
In case that $\alpha \geq \frac{1}{2}$, we take $\lambda$ with $\alpha < \lambda < \frac{1}{2} + \frac{1}{2}\alpha$. If $r^2 \leq t$, then

$$I_1 \leq c \int_0^t \tau^{-1+\frac{\lambda}{2}} (\frac{r}{\sqrt{\tau}})^{-1+2\lambda} d\tau + c \int_{r^2}^t \tau^{-1+\frac{\lambda}{2}} (\frac{r}{\sqrt{\tau}})^{n-1} d\tau \leq cr^\lambda \leq ct^{\frac{1}{2}\lambda - \frac{1}{2}\alpha} r^\alpha.$$  

In case that $r^2 > t$, we get

$$I_1 \leq c \int_0^t \tau^{-1+\frac{\lambda}{2}} (\frac{r}{\sqrt{\tau}})^{-1+2\lambda} d\tau \leq r^{-1+2\lambda} t^{\frac{1}{2} - \frac{\lambda}{2}} \leq ct^{\frac{1}{2}\lambda - \frac{1}{2}\alpha} r^\alpha.$$  

For $I_2$, we take $\lambda > \alpha$. Using the estimate (3.19), we have

$$I_2 \leq cr^\lambda \int_0^t \int_{|Z|>2r} \frac{1}{\tau^{-1+\frac{\lambda}{2}} (|Z|^2 + \tau)^{\frac{n-\lambda}{2}}} dZd\tau \leq cr^\lambda \int_0^t \tau^{-1} \int_{|Z|>2r} \frac{1}{\tau^{-1+\frac{\lambda}{2}} (|Z|^2 + 1)^{\frac{n-\lambda}{2}}} dZd\tau.$$  

If $r^2 \leq t$, then

$$I_2 \leq cr^\lambda \int_0^r \tau^{-1+\frac{1}{2}} (\frac{r}{\sqrt{\tau}})^{-1+\lambda} d\tau + cr^\lambda \int_0^t \tau^{-1} d\tau \leq cr^\lambda + cr^\lambda \ln \frac{t}{r^2} \leq ct^{\frac{1}{2}\lambda - \frac{1}{2}\alpha} r^\alpha.$$  

For the last inequality, we used the fact of $|\ln \frac{t}{r^2}| \leq c(\frac{1}{r^2})^{\frac{1}{2} - \frac{\alpha}{2}}$ for $\frac{1}{r^2} \geq 1$. On the other hand, if $r^2 > t$, we have

$$I_2 \leq cr^\lambda \int_0^t \tau^{-1} \int_{|Z|>2r} \frac{1}{\tau^{-1+\frac{1}{2}} (|Z|^2 + \tau)^{\frac{n-\lambda}{2}}} dZd\tau.$$  

It remains to prove (3.26). Due the inequality (3.18), we have

$$\int s \int_{\partial \Omega} |G(P, Z, \tau)| dZd\tau \leq c \int s \int_{\partial \Omega} \frac{1}{\tau^{-1+\frac{1}{2}} (|P - Z|^2 + \tau)^{\frac{n-\lambda}{2}}} dZd\tau.$$  

Since $\Omega$ is a bounded domain, taking $\frac{1}{2} < \lambda < 1$, we can see that $\int_{\partial \Omega} \frac{1}{(|P - Z|^2 + \tau)^{\frac{n-\lambda}{2}}} dZ < c\lambda$, and thus, the estimate (3.26) is immediate.

**Lemma 9** Let $U$ be the hydrodynamical potential in (3.22). Suppose that $\Phi \in L^\infty(\partial \Omega; C^\frac{\alpha}{2} [0, T])$ and $\Phi|_{t=0} = 0$. Then, there is $c > 0$ such that

$$\|U[\Phi]\|_{C^\alpha; C^\frac{\alpha}{2} (\partial \Omega \times [0, T])} \leq cT^\delta \|\Phi\|_{L^\infty(\partial \Omega; C^\frac{\alpha}{2} [0, T])},$$  

(3.27) **inequality508-1**

$$\|U[\Phi]\|_{C^\alpha; C^\frac{\alpha}{2} (\partial \Omega \times [0, T])} \leq cT^\delta \|\Phi\|_{L^\infty(\partial \Omega; C^\frac{\alpha}{2} [0, T])},$$  

(3.28) **inequality508-2**

where $\delta > 0$ is the number in Lemma 8.

**Proof.** We first note, due to (3.26) in Lemma 8 that

$$\|U[\Phi]\|_{L^\infty(\partial \Omega \times (0, T))} \leq \|\Phi\|_{L^\infty(\partial \Omega \times (0, T))} \int_0^T |G(\cdot, Q, t)| dQ dt \leq cT^\delta \|\Phi\|_{L^\infty(\partial \Omega \times (0, T))}. \quad (3.29)$$  

1209linfty
We first note, due to (3.25) in Lemma 8, that
\[
|U[\Phi](P,t) - U[\Phi](Q,t)| \leq c \int_0^t \int_{\partial \Omega} |G(P,Z,t - \tau) - G(Q,Z,t - \tau)||\Phi(Z,\tau)| dZ d\tau
\]
\[
\leq c \|\Phi\|_{L^\infty(\partial \Omega \times [0,T])} \int_0^t \int_{\partial \Omega} |G(P,Z,\tau) - G(Q,Z,\tau)| dZ d\tau
\]
\[
\leq c \|\Phi\|_{L^\infty(\partial \Omega \times [0,T])} |P-Q|^\alpha t^\delta.
\]
Therefore, we obtain
\[
\|U[\Phi]\|_{L^\infty(0,T;\dot{C}^\alpha(\partial \Omega))} \leq cT^\delta \|\Phi\|_{L^\infty(\partial \Omega \times [0,T])}.
\] (3.30)

We also obtain for \( s<t \)
\[
U[\Phi](P,t) - U[\Phi](P,s) = \int_s^t \int_{\partial \Omega} G(P,Z,\tau)\Phi(Z,t-\tau) dZ d\tau
\]
\[
+ \int_0^s \int_{\partial \Omega} G(P,Z,\tau)(\Phi(Z,t-\tau) - \Phi(Z,s-\tau)) dZ d\tau := I_1 + I_2.
\]
Since \( \Phi(Z,0) = 0 \), we have
\[
|I_1| \leq \int_s^t \int_{\partial \Omega} |G(P,Z,\tau)||\Phi(Z,t-\tau) - \Phi(Z,\tau)| dZ d\tau
\]
\[
\leq \|\Phi\|_{L^\infty(\partial \Omega;\dot{C}^\alpha([0,T]))} |t-s|^\frac{\alpha}{2} \int_s^t \int_{\partial \Omega} |G(P,Z,\tau)| dZ d\tau,
\]
\[
|I_2| \leq \int_0^s \int_{\partial \Omega} |G(P,Z,\tau)||\Phi(Z,t-\tau) - \Phi(Z,s-\tau)| dZ d\tau
\]
\[
\leq \|\Phi\|_{L^\infty(\partial \Omega;\dot{C}^\alpha([0,T]))} |t-s|^\frac{\alpha}{2} \int_0^s \int_{\partial \Omega} |G(P,Z,\tau)| dZ d\tau.
\]
Hence, via (3.26) in Lemma 8, we obtain
\[
\|U[\Phi]\|_{L^\infty(\partial \Omega;\dot{C}^\alpha([0,T]))} \leq cT^\delta \|\Phi\|_{L^\infty(\partial \Omega;\dot{C}^\alpha([0,T]))}.
\] (3.31)

By (3.29), (3.30) and (3.31), we completes the proof of (3.27).

For \( s<t \), we note that
\[
U[\Phi](P,t) - U[\Phi](P,s) - U[\Phi](Q,t) + U[\Phi](Q,s)
\]
\[
= \int_s^t \int_{\partial \Omega} (G(P,Z,\tau) - G(Q,Z,\tau))\Phi(Z,t-\tau) dZ d\tau
\]
\[
+ \int_0^s \int_{\partial \Omega} (G(P,Z,\tau) - G(Q,Z,\tau))(\Phi(Z,t-\tau) - \Phi(Z,s-\tau)) dZ d\tau = I_1 + I_2.
\]
Lemma 10

Let $V$ be the electrostatic potential of a single layer given in (3.23). Suppose that $\Psi \in C^{\alpha,\frac{1}{2}}(\partial \Omega \times [0,T]) \cap C_2^\alpha(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])$. Then, there is $c > 0$ such that

\[
\|\nabla SV[\Psi]\|_{C^{\alpha,\frac{1}{2}}(\partial \Omega \times [0,T])} \leq c\left(\|\Psi\|_{C^{\alpha,\frac{1}{2}}(\partial \Omega \times [0,T])} + \|\Psi\|_{C_2^\alpha(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}\right),
\]

Furthermore, if $\int_{\partial \Omega} \Psi(Z,t)\,dQ = 0$ for all $t \in [0,T]$, then

\[
\|\Psi\|_{C^{\alpha,\frac{1}{2}}(\partial \Omega \times [0,T])} \leq c\|(I + K^*)\Psi\|_{C^{\alpha,\frac{1}{2}}(\partial \Omega \times [0,T])},
\]

\[
\|\Psi\|_{C_2^\alpha(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])} \leq c\|(I + K^*)\Psi\|_{C_2^\alpha(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}.
\]

Proof. Since $\nabla SV[\Psi] : \hat{c}_{2\alpha}^0(\partial \Omega) \to \hat{c}_{2\alpha}^0(\partial \Omega)$ is bounded, it follows that

\[
\|\nabla SV[\Psi]\|_{L^\infty([0,T]; C^{\alpha,\frac{1}{2}}(\partial \Omega))} \leq c\|\Psi\|_{L^\infty(0,T; C^{\alpha,\frac{1}{2}}(\partial \Omega))},
\]

Next, we will show that

\[
\|\nabla SV[\Psi]\|_{L^\infty(\partial \Omega; C_2^\alpha[0,T])} \leq c\left(\|\Psi\|_{L^\infty(\partial \Omega; C_2^\alpha[0,T])} + \|\Psi\|_{C_2^\alpha(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}\right).
\]

Indeed, we compute

\[
\nabla SV[\Psi](P,t) - \nabla SV[\Psi](P,s) = \sum_{l=1}^{n-1} T_l(P) \int_{\partial \Omega} \frac{(P - Z) \cdot (T_l(P) - T_l(Z))}{|P - Z|^n} (\Psi(Z,t) - \Psi(Z,s)) \,dZ
\]

\[
+ \sum_{l=1}^{n-1} T_l(P) \int_{\partial \Omega} \frac{(P - Z) \cdot T_l(Z)}{|P - Z|^n} (\Psi(Z,t) - \Psi(Z,s)) \,dZ := K_1 + K_2,
\]

where $T_l(P), 1 \leq l \leq n - 1$ are tangential vector on $P \in \partial \Omega$. Since $|T_l(P) - T_l(Q)| \leq c|P - Q|$, one can easily see the first term $K_1$ is estimated as

\[
|K_1| \leq c\|\Psi\|_{L^\infty(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}|t - s|^{\frac{\alpha}{2}}.
\]

Since $\int_{\partial \Omega} \frac{(P - Z) \cdot T_l(Z)}{|P - Z|^n} \,dZ = 0$, the second term $K_2$ can be estimated as follows:

\[
|K_2| = \left| \int_{\partial \Omega} \frac{(P - Z) \cdot T_l(Z)}{|P - Z|^n} (\Psi(Z,t) - \Psi(Z,s) - \Psi(P,t) + \Psi(P,s)) \,dZ \right|
\]

Again using $\Phi(Z,0) = 0$, we obtain

\[
|I_1| \leq \int_s^t \int_{\partial \Omega} |G(P, Z, \tau) - G(Q, Z, \tau)| |\Phi(Z, t - \tau) - \Phi(Z, 0)| \,dZ \,d\tau
\]

\[
\leq \|\Phi\|_{L^\infty(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}|t - s|^{\frac{\alpha}{2}} \int_s^t \int_{\partial \Omega} |G(P, Z, \tau) - G(Q, Z, \tau)| \,dZ \,d\tau,
\]

\[
|I_2| \leq \int_0^s \int_{\partial \Omega} |G(P, Z, \tau) - G(Q, Z, \tau)| |\Phi(Z, t - \tau) - \Phi(Z, s - \tau)| \,dZ \,d\tau
\]

\[
\leq \|\Phi\|_{L^\infty(\partial \Omega; \hat{c}_{2\alpha}^0[0,T])}|t - s|^{\frac{\alpha}{2}} \int_s^t \int_{\partial \Omega} |G(P, Z, \tau) - G(Q, Z, \tau)| \,dZ \,d\tau.
\]

By (3.23) in Lemma 8, we obtain (3.28). This completes the proof.
Adding up (3.37) and (3.38), we deduce (3.36).

Using the same argument, we get

$$\| \nabla SV[\Psi] \|_{L^\infty(\Omega \times (0,T))} \leq c(\| \Psi \|_{L^\infty(\Omega \times (0,T))} + \| \Psi \|_{C_{\sigma,D}(\partial \Omega \times \{0,T\})}).$$  \hspace{1cm} (3.39) \text{CK-June21-100}

Hence, from (3.33), (3.35) and (3.39), we complete the proof of (3.32).

It remains to show the estimates (3.33)-(3.34). Since $\Omega$ is a $C^2$ domain, $K^* : C^\alpha_{\sigma,D_0}(\partial \Omega) \to C^\alpha_{\sigma,D_0}(\partial \Omega)$ is compact operator, where

$$C^\alpha_{\sigma,D_0}(\partial \Omega) := \{ \Psi \in C^\alpha_{D_0}(\partial \Omega) \mid \int_{\partial \Omega} \Psi = 0 \}.$$

Since $I + K^* : C^\alpha_{\sigma,D_0}(\partial \Omega) \to C^\alpha_{\sigma,D_0}(\partial \Omega)$ is injective, by Fredholm operator theory, $I + K^* : C^\alpha_{\sigma,D_0}(\partial \Omega) \to C^\alpha_{\sigma,D_0}(\partial \Omega)$ is bijective operator. Using the same argument, we note that $I + K^* : C^\alpha_{\sigma}(\partial \Omega) \to C^\alpha_{\sigma}(\partial \Omega)$ and $I + K^* : L^\infty_{\sigma}(\partial \Omega) \to L^\infty_{\sigma}(\partial \Omega)$ are bijective operators. Hence, for $\Psi$ satisfying $\int_{\partial \Omega} \Psi = 0$, we have

$$\| \Psi \|_{C^\alpha(\partial \Omega)} \leq c(\| I + K^* \| \Psi \|_{C^\alpha(\partial \Omega)}),$$

$$\| \Psi \|_{C^\alpha_{D_0}(\partial \Omega)} \leq c(\| I + K^* \| \Psi \|_{C^\alpha_{D_0}(\partial \Omega)}),$$

$$\| \Psi \|_{L^\infty(\partial \Omega)} \leq c(\| I + K^* \| \Psi \|_{L^\infty(\partial \Omega)}).$$

In particular, for $s, t \in [0, T]$, we obtain

$$\| \Psi(t) - \Psi(s) \|_{L^\infty(\partial \Omega)} \leq c(\| I + K^* \| \| \Psi(t) - \Psi(s) \|_{L^\infty(\partial \Omega)}),$$

$$\| \Psi(t) - \Psi(s) \|_{C^\alpha_{D_0}(\partial \Omega)} \leq c(\| I + K^* \| \| \Psi(t) - \Psi(s) \|_{C^\alpha_{D_0}(\partial \Omega)}).$$

The above estimates immediately imply (3.33)-(3.34). This completes the proof. \hfill \square

By Lemma 3, we have

$$\| U \|_{C_0^{\alpha,1/2}(\partial \Omega \times [0,T]) \to C_0^{\alpha,1/2}(\partial \Omega \times [0,T])}, \quad \| U \|_{C_D^{\alpha,1/2}(\partial \Omega \times [0,T]) \to C_D^{\alpha,1/2}(\partial \Omega \times [0,T])} \leq cT^\delta,$$  \hspace{1cm} (3.40) \text{1124-1}

where

$$C_0^{\alpha,1/2}(\partial \Omega \times [0,T]) = \{ f \in C^{\alpha,1/2}(\partial \Omega \times [0,T]) \mid f|_{t=0} = 0 \},$$

$$C_D^{\alpha,1/2}(\partial \Omega \times [0,T]) = \{ f \in C_D^{\alpha,1/2}(\partial \Omega \times [0,T]) \mid f|_{t=0} = 0 \}.$$

Hence, for $cT^\delta < 1$, the operators $I + U_{tan} : C_0^{\alpha,1/2}(\partial \Omega \times [0,T]) \to C_0^{\alpha,1/2}(\partial \Omega \times [0,T])$ and $I + U_{tan} : C_D^{\alpha,1/2}(\partial \Omega \times [0,T]) \to C_D^{\alpha,1/2}(\partial \Omega \times [0,T])$ are bijective. Therefore, we have that there is $T_0 > 0$ such that for $T \leq T_0$ and $\Psi$ satisfying $\Psi|_{t=0} = 0$,

$$\| \Phi \|_{C_0^{\alpha,1/2}(\Omega \times [0,T])} \leq c(\| \Phi + U_{tan}[\Phi] \|_{C_0^{\alpha,1/2}(\partial \Omega \times [0,T])}),$$

$$\| \Phi \|_{C_D^{\alpha,1/2}(\Omega \times [0,T])} \leq c(\| \Phi + U_{tan}[\Phi] \|_{C_D^{\alpha,1/2}(\partial \Omega \times [0,T])}).$$  \hspace{1cm} (3.41) \text{lemma3}
Proposition 1 Let $T < \infty$. Suppose that $g \in C^0_{0}((\partial\Omega \times [0, T])$ with $g \cdot n \in \dot{C}_{D_{n}}(\partial\Omega; \dot{C}^{1, \alpha}_{\Omega})(0, T)$, and satisfying the condition $\int_{\partial\Omega} g(Q, t) \cdot n(Q) dQ = 0, \quad \forall t \in (0, T)$, then, the system (3.21) has a unique solution $\Phi, \Psi \in C^0_{0}((\partial\Omega \times [0, T])$, $\Psi \in \dot{C}^{1, \alpha}_{\Omega}(0, T]$) with the conditions (3.20). Furthermore, $(\Phi, \Psi)$ satisfies the following inequality:

$$
\|\Phi\|_{C^0_{0}((\partial\Omega \times [0, T])} + \|\Psi\|_{C^0_{0}((\partial\Omega \times [0, T])} \leq c \left( \|g\|_{C^0_{0}((\partial\Omega \times [0, T])} + \|g \cdot n\|_{C^0_{0}((\partial\Omega \times [0, T])} \right),
$$

where $c = c(T)$.

Proof. Let $T \leq T_0$, where $T_0$ is a constant defined (3.41). By (3.41), we solve the following equation:

$$
\Phi_1 + U_{tan}[\Phi_1] = g_{tan}
$$

and $\Phi_1$ satisfies

$$
\Phi_1 \in C^0_{0}((\partial\Omega \times [0, T]), \quad \|\Phi_1\|_{C^0_{0}((\partial\Omega \times [0, T])} \leq c \|g_{tan}\|_{C^0_{0}((\partial\Omega \times [0, T])},
$$

and $\Phi_1 \cdot n = 0$. Note that since $U[\Phi_1]$ is divergence-free, $\int_{\partial\Omega} n \cdot U[\Phi_1] = 0$. In the proof of Lemma 10 there is $\Psi_1$ we solve

$$
\Psi_1 + K^*[\Psi_1] = g \cdot n - n \cdot U[\Phi_1]
$$

and by (3.41) and Lemma 10 $\Psi_1 \in C^0_{0}((\partial\Omega \times [0, T])$ satisfies

$$
\|\Psi_1\|_{C^0_{0}((\partial\Omega \times [0, T])} \leq c \left( \|g \cdot n\|_{C^0_{0}((\partial\Omega \times [0, T])} + T^\delta \|\Phi_1\|_{C^0_{0}((\partial\Omega \times [0, T])} \right),
$$

Iteratively, we define $(\Phi_{m+1}, \Psi_{m+1})$ for any $m = 1, 2, \cdots$ as follows:

$$
\Phi_{m+1} + U_{tan}[\Phi_{m+1}] = g_{tan} - \nabla_S V[\Psi_m],
\Psi_{m+1} + K^*[\Psi_{m+1}] = g \cdot n - n \cdot U[\Phi_{m+1}],
$$

We then note that $\Phi_{m+1}$ satisfies

$$
\|\Phi_{m+1}\|_{C^0_{0}((\partial\Omega \times [0, T])} \leq c \left( \|g_{tan}\|_{C^0_{0}((\partial\Omega \times [0, T])} + \|\nabla_S V[\Psi_m]\|_{C^0_{0}((\partial\Omega \times [0, T])} \right),
$$

where we used Lemma 10. On the other hand, for $\Psi_{m+1}$ we observe that

$$
\|\Psi_{m+1}\|_{C^0_{0}((\partial\Omega \times [0, T])} \leq c \left( \|g \cdot n\|_{C^0_{0}((\partial\Omega \times [0, T])} + T^\delta \|\Phi_{m+1}\|_{C^0_{0}((\partial\Omega \times [0, T])} \right),
$$

$$
\|\Psi_{m+1}\|_{C^0_{0}((0, T; \dot{C}_{D_n}(\partial\Omega))} \leq c \left( \|g \cdot n\|_{C^0_{0}((0, T; \dot{C}_{D_n}(\partial\Omega))} + T^\delta \|\Phi_{m+1}\|_{C^0_{0}((\partial\Omega \times [0, T])} \right),
$$

where we used (3.41) and Lemma 10.

For uniformly convergence, we denote $\phi_m = \Phi_{m+1} - \Phi_m$ and $\psi_m = \Psi_{m+1} - \Psi_m$. Then, $(\phi_m, \psi_m)$ solves

$$
\phi_{m+1} + U_{tan}[\phi_{m+1}] = -\nabla_S V[\psi_m],
$$

iterate
\[ \psi_{m+1} + K^*[\psi_{m+1}] = -n \cdot U[\psi_{m+1}] \]

and it satisfies
\[
\begin{align*}
\|\phi_{m+1}\|_{c^a, \Omega \times [0,T]} &\leq c(\|\psi_m\|_{c^a, \Omega \times [0,T]} + \|\psi_m\|_{c^a, \Omega \times [0,T]}), \\
\|\psi_{m+1}\|_{c^a, \Omega \times [0,T]} &\leq cT^\delta \|\phi_{m+1}\|_{c^a, \Omega \times [0,T]}, \\
\|\psi_{m+1}\|_{c^a, \Omegad_0(\Omega)} &\leq cT^\delta \|\phi_{m+1}\|_{c^a, \Omega \times [0,T]}.
\end{align*}
\]

Hence, we obtain
\[
\begin{align*}
\|\phi_{m+1}\|_{c^a, \Omega \times [0,T]} &\leq cT^\delta \|\phi_m\|_{c^a, \Omega \times [0,T]}, \\
\|\psi_{m+1}\|_{c^a, \Omega \times [0,T]} &\leq cT^\delta \|\psi_m\|_{c^a, \Omega \times [0,T]}.
\end{align*}
\]

This implies that there is \( T^* > 0 \) with \( T^* \leq T_0 \) such that \( \{\Phi_m, \Psi_m\} \) converges for some \((\Phi^1, \Psi^1) \in C^\alpha_0(\partial \Omega \times [0,T^*]) \times C^\alpha_0(\partial \Omega \times [0,T^*])\). From (3.32), (3.33) and (3.41), \((\Psi^1, \Psi^1)\) satisfy
\[
\begin{align*}
\Phi^1 + U_{tan}[\Phi^1] + \nabla SV[\Psi^1] &= g_{tan}, \\
\Psi^1 + K^*[\Psi^1] + n \cdot U[\Phi^1] &= g \cdot n \quad \text{in} \quad \Omega \times (0,T^*)
\end{align*}
\]

and
\[
\begin{align*}
\|\Phi^1\|_{c^a, \Omega \times [0,T^*]} + \|\Psi^1\|_{c^a, \Omega \times [0,T^*]} &
\leq c(T^*)^2 \left( \|\phi_m\|_{c^a, \Omega \times [0,T^*]} + \|\psi_m\|_{c^a, \Omega \times [0,T^*]} \right).
\end{align*}
\]

To construct \((\Phi, \Psi)\) up to any time \( T \), we introduce \( h \), which is given as
\[
h(P, t) = \Phi^1(P, T^*) + \Psi^1(P, T^*)n(P) + \int_0^T \int_{\partial \Omega} G(t + T^* - \tau) \Phi^1(Q, \tau) dQ d\tau + \nabla V[\Psi^1(T^*)](P)
\]
for \( t \in [0, T^*] \) such that \( h(P, 0) = g(P, T^*) \). For \( t \in [0, T^*] \), let us \( g^1(P, t) = g(P, T^* + t) - h(P, t) \) such that \( g^1 \in C^\alpha_0(\partial \Omega \times [0,T^*]) \) and \( g^1 \cdot n \in \hat{C}_d(\partial \Omega; \hat{C}^{\frac{1}{2}a}([0,T^*])) \). By above argument, there is \((\Phi^2, \Psi^2) \in C^\alpha_0(\partial \Omega \times [0,T^*]) \times C^\alpha_0(\partial \Omega \times [0,T^*])\) such that
\[
\begin{align*}
\Phi^2 + U_{tan}[\Phi^2] + \nabla SV[\Psi^2] &= h_{tan}, \\
\Psi^2 + K^*[\Psi^2] + U[\Phi^2] \cdot n &= h^1 \cdot n.
\end{align*}
\]

We define \((\Phi, \Psi)\) by
\[
\begin{align*}
\Phi(P, t) &= \begin{cases} \Phi^1(P, t), & 0 \leq t \leq T^*, \\
\Phi^2(P, t - T^*) + \Phi^1(P, T^*), & T^* \leq t \leq 2T^*.
\end{cases}
\end{align*}
\]
\[
\Psi(P, t) = \begin{cases} \Psi^1(P, t), & 0 \leq t \leq T^*, \\
\Psi^2(P, t - T^*) + \Psi^1(P, T^*), & T^* \leq t \leq 2T^*.
\end{cases}
\]

Then, we obtain \( \Phi \in C^\alpha_0(\partial \Omega \times [0,2T^*]) \), \( \Psi \in C^\alpha_0(\partial \Omega \times [0,2T^*]) \cap C^\alpha_d(\partial \Omega \times [0,2T^*]) \) and
\[
\begin{align*}
\Phi + U_{tan}[\Phi] + \nabla SV[\Psi] &= g_{tan}, \quad \text{in} \quad \partial \Omega \times [0,2T^*], \\
\Psi + K^*[\Psi] + U[\Phi] \cdot n &= g \cdot n
\end{align*}
\]

We repeat the above procedure until we reach any given time \( T \). This completes the proof. \( \square \)
3.2 Global estimates

Next, we estimate the global estimates of solution of Stokes equations.

Proposition 2 Let \( \Phi \in C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T]) \) and \( \Psi \in C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T]) \cap \hat{C}_{0}^{\alpha}(0, T; \hat{C}_{D, \alpha}(\partial\Omega)) \) satisfying (3.20). Suppose that \( U(\Phi) \) and \( V(\Psi) \) are defined in (3.16) and (3.15). Then,

\[
||U(\Phi)||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])} \leq c ||\Phi||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])},
\]

(3.45) CK-June21-8000

\[
||\nabla V(\Psi)||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])} \leq c \left( ||\Psi||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])} + ||\Psi||_{C_{0}^{\alpha, \frac{1}{2}}(0, T; \hat{C}_{D, \alpha}(\partial\Omega))} \right).
\]

(3.46) CK-June21-820

Proof. Let \( x \in \Omega \). Choose \( P_{x} \in \partial\Omega \) satisfying \( \delta(x) = |x - P_{x}| \). Using the rotation and translation, we may assume that \( x = (0, x_{n}) \), \( P_{x} = 0 \) and \( \delta(x) = x_{n} \). We recall that

\[
|G(x, Q, t)| \leq c_{\lambda} \frac{\delta(x)^{\lambda}}{t^{\lambda+\frac{1}{2}}(|x - Q|^{2} + t)^{\frac{\lambda}{2}}}, \quad 0 < \lambda < 1
\]

\[
|D_{x}G(x, Q, t)| \leq \frac{c}{t^{\lambda}(x_{n}^{2} + t)^{\frac{\lambda}{2}}(|x - Q|^{2} + t)^{\frac{\lambda}{2}}}
\]

(3.47) 609inequality-2

(see [13] and [20]). From the first inequality of (3.47), we have

\[
|U(\Phi)(x, t)| \leq c ||\Phi||_{L^{\infty}(\Omega \times [0, T])} \int_{0}^{t} \int_{\partial\Omega} \frac{\delta(x)^{\lambda}}{t^{\lambda+\frac{1}{2}}(|x - Q|^{2} + \tau)^{\frac{\lambda}{2}}} dQ d\tau
\]

\[
\leq c ||\Phi||_{L^{\infty}(\Omega \times [0, T])}.
\]

To complete the proof of (3.45), we first show that

\[
\sup_{(x, t) \in \Omega \times [0, T]} \delta^{1-\alpha}(x) |D_{x}U(x, t)| \leq c ||\Phi||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])}.
\]

(3.48) weight

It is known (see e.g. [12] Theorem 4.1 and [9] Theorem 1.4) that the estimate (3.48) implies

\[
||U(\Phi)||_{L^{\infty}(0, T; \hat{C}_{D, \alpha}(\partial\Omega))} \leq c ||\Phi||_{C_{0}^{\alpha, \frac{1}{2}}(\partial\Omega \times [0, T])},
\]

(3.49) CK-June21-800

Since \( \Phi(0, 0) = 0 \), we compute

\[
D_{x}U(\Phi)(x, t) = \int_{0}^{t} \int_{\partial\Omega} \nabla_{x}G(x, Q, t - s)(\Phi(Q, \tau) - \Phi(P_{x}, \tau)) dQ d\tau + \int_{-\infty}^{t} \int_{\partial\Omega} \nabla_{x}G(x, Q, t - s)(\Phi(P_{x}, \tau) - \Phi(P_{x}, t)) dQ d\tau + \Phi(P_{x}, t) \int_{-\infty}^{t} \int_{\partial\Omega} \nabla_{x}G(x, Q, t - s) dQ d\tau := I_{1} + I_{2} + I_{3}.
\]

Referring to [14] (4.5), we note that there exists a small \( \epsilon > 0 \) such that

\[
|I_{3}| \leq c_{\epsilon} x_{n}^{-\epsilon} ||\Phi||_{L^{\infty}(0, T; \partial\Omega)}.
\]
Noting that for $Q \in \partial \Omega$, $|x - Q|^2 \approx x_n^2 + |P_x - Q|^2$ and using (3.47), we estimate $I_1$ and $I_2$ as

$$
|I_1| \leq c\|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \int_0^t \frac{1}{(t - \tau)^{\frac{3}{2}}(x_n^2 + t - \tau)^{\frac{1}{2}}} \int_{\partial \Omega} \frac{|P_x - Q|}{(|P_x - Q|^2 + x_n^2 + t - \tau)^{\frac{3}{2}}} dQd\tau
\leq c\|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \int_0^t \frac{1}{(t - \tau)^{\frac{3}{2}}(x_n^2 + t - \tau)^{\frac{1}{2}}} \int_{\partial \Omega} \frac{|P_x - Q|}{(|P_x - Q|^2 + x_n^2 + t - \tau)^{\frac{3}{2}}} dQd\tau
\leq c\|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} x_n^{-1+\alpha}.
$$

$$
|I_2| \leq c\|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \int_0^t \frac{1}{(t - \tau)^{\frac{3}{2}}(x_n^2 + t - \tau)^{\frac{1}{2}}} \int_{\partial \Omega} \frac{(t - \tau)^{\frac{3}{2}}}{(|P_x - Q|^2 + x_n^2 + t - \tau)^{\frac{3}{2}}} dQd\tau
\leq c\|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} x_n^{-1+\alpha}.
$$

Summing up above estimates, we obtain (3.48).

Let $h > 0$ and we compute

$$
\mathcal{U}[\Phi](x, t + h) - \mathcal{U}[\Phi](x, t) = \int_0^{t+h} \int_{\partial \Omega} G(x, Q, t + h - \tau) \Phi(Q, \tau) dQ d\tau - \int_0^t \int_{\partial \Omega} G(x, Q, t - \tau) \Phi(Q, \tau) dQ d\tau
= \int_0^t \int_{\partial \Omega} G(x, Q, \tau) \Phi(Q, t + h - \tau) dQ d\tau + \int_0^t \int_{\partial \Omega} G(x, Q, \tau) \Phi(Q, t - \tau) dQ d\tau + \int_t^{t+h} \int_{\partial \Omega} G(x, Q, \tau) \Phi(Q, t + h - \tau) dQ d\tau
:= I_1 + I_2.
$$

By (3.26), we have

$$
|I_1| \leq h^{\frac{1}{2}} \alpha \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \int_0^t \int_{\partial \Omega} |G(x, Q, \tau)| dQ d\tau
\leq c h^{\frac{1}{2}} \alpha t^{\delta} \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))},
$$

$$
|I_2| \leq \int_0^{t+h} \int_{\partial \Omega} G(x, Q, \tau) \left( \Phi(Q, t + h - \tau) - \Phi(Q, 0) \right) dQ d\tau
\leq h^{\frac{1}{2}} \alpha \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \int_0^{t+h} \int_{\partial \Omega} |G(x, Q, \tau)| dQ d\tau
\leq c h^{\frac{1}{2}} \alpha (t + h)^{\delta} \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))}.
$$

Hence, we have

$$
|\mathcal{U}[\Phi](x, t + h) - \mathcal{U}[\Phi](x, t)| \leq c \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))} h^{\frac{1}{2}}.
$$

Therefore, we obtain

$$
\|\mathcal{U}[\Phi]\|_{L^\infty(0,T;C^\omega(\partial \Omega))} \leq c \|\Phi\|_{L^\infty(0,T;C^\omega(\partial \Omega))}.
$$

By (3.49) and (3.50), we obtain (3.45).
It remains to show (3.46). By the well known result of the harmonic function, boundedness of \( K^* : L^\infty(0, T; C^\alpha(\partial \Omega)) \to L^\infty(0, T; C^\alpha(\partial \Omega)) \) and (3.32), we have

\[
\| \nabla V[\Psi] \|_{L^\infty(0, T; C^\alpha(\partial \Omega))} \leq c \| \nabla V[\Psi] \|_{L^\infty(0, T; C^\alpha(\partial \Omega))} \leq c \left( \| \nabla V[\Psi] \|_{L^\infty(0, T; C^\alpha(\partial \Omega))} + \| (I + K^*)[\Psi] \|_{L^\infty(0, T; C^\alpha(\partial \Omega))} \right).
\]

For \( t \neq s \), we note that

\[
\nabla_x V[\Psi](x, t) - \nabla_x V[\Psi](x, s) = \nabla_x \int_{\partial \Omega} N(x - Q)(\Psi(Q, t) - \Psi(Q, s))dQ.
\]

By maximum principle of the harmonic function, we have

\[
|\nabla_x V[\Psi](x, t) - \nabla_x V[\Psi](x, s)| \leq \sup_{Q \in \partial \Omega} |\nabla_x V[\Psi](Q, t) - \nabla_x V[\Psi](Q, s)|.
\]

We note that

\[
\nabla_x V[\Psi](Q, t) = \sum_{i=1}^{n-1} (\nabla_x V[\Psi](Q, t) \cdot T_i(Q)) T_i(Q) + (\nabla_x V[\Psi](Q, t) \cdot n(Q)) n(Q).
\]

Here,

\[
\nabla_x V[\Psi](Q, t) \cdot T_i(Q) = p.v \int_{\partial \Omega} \frac{(Q - Z) \cdot T_i(Q)}{|Q - Z|^n} \Psi(Z, t)dZ,
\]

\[
\nabla_x V[\Psi](Q, t) \cdot n(Q) = (I + K^*)[\Psi](Q, t).
\]

Here \( K^*[\Psi](Q, t) \) is defined in (3.24). Since \( \Omega \) is a smooth domain, it is known that

\[
|K^*[\Psi(\cdot, t) - \Psi(\cdot, s)](Q)| \leq c \| \Psi(\cdot, t) - \Psi(\cdot, s) \|_{L^\infty(\partial \Omega)}.
\]

Since \( \int_{\partial \Omega} \frac{(Q - Z) \cdot T_i(Q)}{|Q - Z|^n}dZ = 0 \), we have

\[
(\nabla_x V[\Psi](Q, t) - \nabla_x V[\Psi](Q, s)) \cdot T_i(Q)
\]

\[
= \int_{\partial \Omega} \frac{(Q - Z) \cdot T_i(Q) - (Q - Z_i)}{|Q - Z|^n} (\Psi(Z, t) - \Psi(Z, s))dZ
\]

\[
+ \int_{\partial \Omega} \frac{(Q - Z) \cdot T_i(Z)}{|Q - Z|^n} (\Psi(Z, t) - \Psi(Q, t) - \Psi(Q, s) + \Psi(Z, s))dZ
\]

\[
\leq c |t - s|^{\frac{1}{n}} \| \Psi \|_{L^\infty(\partial \Omega; C^{\frac{1}{n}}(\partial \Omega))} + |t - s|^{\frac{1}{n}} \| \Psi \|_{C^{\frac{1}{n}}(0, T; C^\alpha(\partial \Omega))} \int_{\partial \Omega} \frac{\eta(|Q - Z|)}{|Q - Z|^n - 1}dZ
\]

\[
\leq c |t - s|^{\frac{1}{n}} \left( \| \Psi \|_{L^\infty(\partial \Omega; C^{\frac{1}{n}}(\partial \Omega))} + \| \Psi \|_{C^{\frac{1}{n}}(0, T; C^\alpha(\partial \Omega))} \right).
\]

Therefore, we obtain

\[
\| \nabla_x V[\Psi] \|_{L^\infty(\Omega; C^{\frac{1}{n}}(0, T])} = \sup_{t, s \in [0, T]} \frac{\| \nabla_x \psi(\cdot, t) - \nabla_x \psi(\cdot, s) \|_{L^\infty(\partial \Omega)}}{|t - s|^{\frac{1}{n}} \leq \| \Psi \|_{C^{\frac{1}{n}}(0, T; C^\alpha(\partial \Omega))}.
\]

This completes the proof. \( \square \)
Remark 4 Let $\Omega_\delta = \{ x \in \Omega \mid \text{dist} (x, \text{supp } \Psi) \geq \delta \}$ for $\delta > 0$. Then, from (3.51), we can obtain

$$\| \nabla x V[\Psi] \|_{L^\infty(\Omega_\delta; \mathcal{C}^{1,\alpha}_0[0,T])} \leq c_\delta \| \Psi \|_{L^\infty(\partial \Omega \cap \mathcal{C}^{1,\alpha}_0[0,T])}.$$  

This implies

$$\| u \|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega_\delta \times [0,T])} \leq c_\delta \left( \| \Phi \|_{C^{\alpha,\frac{1}{2}\alpha}(\partial \Omega \times [0,T])} + \| \Psi \|_{C^{\alpha,\frac{1}{2}\alpha}(\partial \Omega \times [0,T])} \right).$$

Summarizing the above results, we obtain the following.

Theorem 11 Let $0 < \alpha < 1$. Let $g \in \mathcal{C}^{\alpha,\frac{1}{2}\alpha}(\partial \Omega \times (0,T))$ such that $g \cdot n \in \hat{\mathcal{C}}^\alpha(0,T; \hat{\mathcal{C}}_{Dn}(\partial \Omega))$ and $\int_{\partial \Omega} g(Q,t) n(Q)dQ = 0$. Then, there exists a unique solution $u \in \mathcal{C}^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])$ of the Stokes system

$$u_t - \Delta u + \nabla P = 0, \quad \text{div } u = 0, \quad \text{in } \Omega \times (0,T)$$

$$u|_{\partial \Omega \times (0,T)} = g, \quad u(x,0) = 0.$$  

Furthermore, $u$ satisfies

$$\| u \|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])} \leq c\left( \| u_0 \|_{C^{\alpha}_0(\Omega)} + \| u_0 \|_{\mathcal{C}_{Dn}(\Omega)} + \| g \|_{C^{\alpha,\frac{1}{2}\alpha}(\Omega \times [0,T])} + \| g \cdot n \|_{\mathcal{C}^\alpha(0,T; \mathcal{C}_{Dn}(\partial \Omega))} \right). \quad (3.52)$$

Proof. From Proposition [1] there is $(\Phi, \Psi) \in \mathcal{C}^{\alpha,\frac{1}{2}\alpha}(\partial \Omega) \times \mathcal{C}^{\alpha}_0(0,T; \mathcal{C}_{Dn}(\partial \Omega))$ such that

$$\Phi + U_{tan}[\Psi] + \nabla S V[\Psi] = g_{tan},$$

$$\Psi + K^*[\Psi] + n \cdot U[\Phi] = g \cdot n.$$  

Let $u(x,t) = U[\Phi](x,t) + V[\Psi](x,t)$. Then, the estimate (3.52) of $u$ is consequences of Proposition [2]. This completes the proof.

As mentioned earlier, results of Theorem [1] is direct due to Theorem [11]. Since its verification is direct, we omit its details.

4 Construction of an example in Theorem [2]

In this section, we construct an example, which shows that the condition of boundary data in Theorem [1] is crucial.

Proof of Theorem [2]

We consider the Stokes system (1.1) in two dimensions. Suppose that $\Omega \subset \mathbb{R}^2_+$ and a part of boundary is flat and it contains, via translation, the open unit interval, e.g. $\{ x_1 \in \mathbb{R} : |x_1| < 2 \}$. We let $g = (g_1, g_2) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ such that $g_1$ is identically zero, that is $g_1 = 0$, and $g_2$ is defined by

$$g_2(x_1, t) = (|x_1|^2 + |t|)^{\frac{1}{2}\alpha} \left( \arctan \frac{x_1}{t} \right)^{\frac{1}{2}\alpha} \left( \arctan \frac{t}{x_1} \right)^{\frac{1}{2}\alpha} \chi_{\{ x_1 > 0 \}}(x_1) \chi_{\{ t > 0 \}}(t) \phi(x_1),$$

where $\phi \in C^\infty_c(-1,1)$ with $\phi \equiv 1$ in $(-\frac{1}{2}, \frac{1}{2})$. Clearly, $g$ is supported in $B_1 \times \mathbb{R}_+$ and $g \in \mathcal{C}^{\alpha,\frac{1}{2}\alpha}(\mathbb{R} \times \mathbb{R})$ (See Theorem 1.4 in [9]). However, we can see that $g \notin \hat{\mathcal{C}}_{Dn}(\mathbb{R} \times \mathbb{R}).$
Indeed, suppose that \( g \in \mathcal{C}^1_{D_0}(\mathbb{R} \times \mathbb{R}) \) for some Dini-continuous function \( \eta_0 \). Note that \( \liminf_{r \to 0} \eta_0(r) = 0 \). Taking \( x_1 = t^2 \), we have

\[
\|g\|_{\mathcal{C}^1_{D_0}(\mathbb{R} \times \mathbb{R})} \geq \frac{|g(0,0) - g(0,s) - g(x_1,0) + g(x_1,t)|}{t^{\frac{\alpha}{2}} \eta_0(x_1)}
\]

\[
= \frac{g(t^2,t)}{t^{\frac{\alpha}{2}} \eta_0(t^2)} = \frac{2^{\frac{\alpha}{2}} t^{\frac{\alpha}{2}}}{t^{\frac{\alpha}{2}} \eta_0(t^2)} = \frac{2^\alpha}{\eta_0(t^2)} \to \infty \quad \text{as} \ t \to 0.
\]

We consider the Stokes system in a half-space with boundary data \( g \) and the solution \( u = (u^1, u^2) \) is represented by (see \[13\] and \[20\])

\[
u^i(x,t) = \sum_{j=1}^2 \int_0^t \int_\mathbb{R} K_{ij}(x_1 - y_1, x_2, t - s) g_j(y_1, s) dy_1 ds, \quad i = 1, 2,
\]

where

\[
K_{ij}(x_1 - y_1, x_2, t) = -2\delta_{ij} D_{x_1} \Gamma(x_1 - y_1, x_2, t) - L_{ij}(x_1 - y_1, x_2, t)
\]

\[
+ \delta_{j2} \delta(t) D_{x_1} N(x_1 - y_1, x_2), \quad i, j = 1, 2
\]

with

\[
L_{ij}(x,t) = D_{x_j} \int_0^t \int_\mathbb{R} D_{x_i} \Gamma(z,t) D_{x_i} N(x - z) dz, \quad i, j = 1, 2.
\]

From Remark 4, we obtain that \( u \in C^{\alpha, \frac{\alpha}{2}}(\partial \Omega \times [0,1]) \). This completes the proof.

Here the tangential component of \( u \), i.e. \( u^1 \), is given by

\[
u^1(x,t) = -\int_0^t \int_\mathbb{R} L_{12}(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds + \int_\mathbb{R} D_{y_1} N(x_1 - y_1, x_2) g_2(y_1, t) dy_1
\]

\[
= -\int_0^t \int_\mathbb{R} L_{21}(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds - \int_0^t \int_\mathbb{R} \int_\mathbb{R} D_{x_2} \Gamma(x_1 - y_1, x_2, t - s) H g_2(y_1, s) dy_1 ds
\]

\[
+ \int_\mathbb{R} D_{y_1} N(x_1 - y_1, x_2) g_2(y_1, t) dy_1 := I_1(x,t) + I_2(x,t) + I_3(x,t),
\]

where \( H \) is a Hilbert transform defined as

\[
H g(y_1) = \text{p.v.} \frac{1}{\pi} \int \frac{1}{y_1 - z_1} g(z_1) dz_1 = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y_1 - z_1| > \epsilon} \frac{1}{y_1 - z_1} g(z_1) dz_1.
\]

It can be checked that \( I_1 \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R} \times \mathbb{R}) \) (see e.g. \[6\]) and so we obtain \( |I_1(0, x_2, t) - I_1(0,0,0)| \leq c (x_2^2 + t)^{\frac{\alpha}{2}} \). Hence, we have

\[
|u^1(0, x_2, t) - u^1(0,0)| = |u^1(0, x_2, t)| \geq |I_2(0, x_2, t) + I_3(0, x_2, t)| - c (x_2^2 + t)^{\frac{\alpha}{2}}.
\]

Now, we estimate \( |I_2(0, x_2, t)| \). Since \( g_2 \) is a function in Holder continuous and has compact support in \( x_1 \in (0,1) \), \( H g_2 \) is bounded in \( \mathbb{R} \) and \( |H g_2(y_1)| \leq c |y_1|^{-1} \) for \( |y_1| \geq 1/2 \). For \( |y_1| \leq \frac{1}{2} \), we have

\[
H g_2(y_1, s) = \int_\mathbb{R} \frac{1}{y_1 - z_1} g_2(z_1, \tau) dz_1 = \int_0^{|y_1|} \cdots dz_1 + \int_{|y_1|}^{2|y_1|} \cdots dz_1 + \int_{2|y_1|}^1 \cdots dz_1.
\]
Here, using change of variable, the first and second terms are estimated as follows:

\[
\int_0^{\left| y_1 \right|} \cdots dz_1 + \int_0^{\left| y_1 \right|} \cdots dz_1 = \int_0^{\left| y_1 \right|} \frac{1}{z_1} (g_2(y_1 - z_1, \tau) - g_2(y_1 + z_1, \tau)) dz_1 \leq c \int_0^{\left| y_1 \right|} \frac{1}{z_1^\alpha} dz_1 \approx \left| y_1 \right|^\alpha.
\]

It remains to estimate the third term in (4.1). Firstly, in case that \( \tau > (2\left| y_1 \right|)^2 \), we obtain

\[
\int_{2\left| y_1 \right|}^{1} \frac{1}{y_1 - z_1} g_2(z_1, \tau) dz_1 \approx \int_{2\left| y_1 \right|}^{1} \frac{1}{y_1 - z_1} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1 \\
\approx -\int_{2\left| y_1 \right|}^{1} \frac{1}{\left| z_1 \right|^\alpha} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1 = -\left( \frac{1}{2\alpha} (\tau^{\frac{1}{2}\alpha} - \tau^{-\frac{1}{2}\alpha} |y_1|^2) - \frac{1}{\alpha} |y_1|^\alpha - \tau^{\frac{1}{2}\alpha} \ln \tau^{\frac{1}{2}} \right). 
\]

On the other hand, if \( \tau > (2\left| y_1 \right|)^2 \), then we have

\[
\int_{2\left| y_1 \right|}^{1} \frac{1}{y_1 - z_1} g_2(z_1, \tau) dz_1 = \int_{2\left| y_1 \right|}^{1} \frac{1}{y_1 - z_1} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1 \\
+ \int_{\tau^{\frac{1}{2}}}^{1} \frac{1}{y_1 - z_1} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1
\]

\[
\approx -\int_{2\left| y_1 \right|}^{1} \frac{1}{\left| z_1 \right|^\alpha} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1 - \int_{\tau^{\frac{1}{2}}}^{1} \frac{1}{\left| z_1 \right|^\alpha} (\left| z_1 \right|^2 + \tau)^{\frac{1}{2}\alpha} \left( \arctan \frac{z_1^\alpha}{\tau^{\frac{1}{2}\alpha}} \right) \left( \arctan \frac{\tau^{\frac{1}{2}\alpha}}{z_1^\alpha} \right) dz_1
\]

\[
= \left( \frac{1}{2\alpha} (\tau^{\frac{1}{2}\alpha} - \tau^{-\frac{1}{2}\alpha} |y_1|^2) - \frac{1}{\alpha} |y_1|^\alpha - \tau^{\frac{1}{2}\alpha} \ln \tau^{\frac{1}{2}} \right). 
\]

For \( x_1 = 0 \) and \( x_2 > 0 \), we get

\[
I_2(0, x_2, t) = \int_0^t \int_{\mathbb{R}} D_2 \Gamma(-y_1, x_2, t - s) H g_2(y_1, s) dy_1 ds = J_1 + J_2,
\]

where

\[
J_1 := \int_0^t \int_{\left| y_1 \right| > \frac{1}{2}} D_2 \Gamma(-y_1, x_2, t - s) H g_2(y_1, s) dy_1 ds,
\]

\[
J_2 := \int_0^t \int_{\left| y_1 \right| \leq \frac{1}{2}} D_2 \Gamma(-y_1, x_2, t - s) H g_2(y_1, s) dy_1 ds.
\]

Noting that \( \int_{\frac{-\tau}{\sqrt{s}}}^{\frac{\tau}{\sqrt{s}}} e^{-y_1^2} \frac{1}{y_1} dy_1 ds \leq ce^{-\frac{4}{\tau^2}} \) for \( s \leq 1 \) and \( e^{-a} \leq c_k a^{-k} \) for \( a, k > 0 \), we have

\[
J_1 \leq 2 \int_0^t \int_{\frac{1}{2}}^{\infty} D_2 \Gamma(y_1, x_2, t - s) \frac{1}{y_1} dy_1 ds \leq c \int_0^t \int_{\frac{1}{2}}^{\infty} e^{-\frac{x_2^2}{s^2}} \left( \int_{\frac{-\tau}{\sqrt{s}}}^{\frac{\tau}{\sqrt{s}}} e^{-y_1^2} \frac{1}{y_1} dy_1 ds \right) ds \leq c \int_0^t \frac{x_2}{s^2} e^{-\frac{x_2^2}{s^2}} ds \int_{\frac{-\tau}{\sqrt{s}}}^{\frac{\tau}{\sqrt{s}}} e^{-y_1^2} \frac{1}{y_1} dy_1 ds.
\]

\[
\leq c \int_0^t \frac{x_2}{s^2} e^{-\frac{x_2^2}{s^2}} ds \leq c \int_0^t \frac{x_2}{s^2} e^{-\frac{x_2^2}{s^2}} ds.
\]
Using the change of variables ($\eta := x^2/s$) and taking $k = \frac{\alpha+1}{2} < 1$, we have

$$J_1 \leq c \int_0^\infty x^2 e^{-k \eta} d\eta \leq c x^\alpha_2.$$ 

Next, from (4.2) and (4.3), we have

$$J_2 = \int_0^t \int_{|y_1| \leq \frac{1}{2}} D_2 \Gamma(y_1, x_2, s) H g_2(y_1, t-s) dy_1 ds \approx - \int_0^t \int_{s^2} x_2 e^{-\frac{y_1^2}{2}} \left( \int_0^{s^2} e^{-\frac{y_1^2}{2}} (s^{\frac{1}{2} \alpha} - y_1^{\alpha}) + (s^{\frac{1}{2} \alpha} - y_1^{\alpha}) - s^{\frac{1}{2} \alpha} \ln s + \frac{s^{\frac{1}{2} \alpha}}{2} \right) dy_1 ds$$

$$- \int_0^1 \int_{s^\frac{1}{2}} x_2 e^{-\frac{y_1^2}{2}} \left( \ln y + s^{\alpha} y_1^{-\alpha} \right) dy_1 ds := J_2^1 + J_2^2.$$ 

$J_2^1$ is computed as follows:

$$J_2^1 \leq c \int_0^t \int_{s^\frac{1}{2}} x_2 e^{-\frac{y_1^2}{2}} s^{\alpha} \ln s \int_0^{s^\frac{1}{2}} e^{-\frac{y_1^2}{2}} dy_1 ds$$

$$= c \int_0^t \int_{s^\frac{1}{2}} x_2 e^{-\frac{y_1^2}{2}} s^{\alpha} \ln s \int_0^{s^\frac{1}{2}} e^{-y_1^2} dy_1 ds$$

$$= c \int_0^t x_2 e^{-\frac{x^2}{2}} s^{\alpha} \left( \int_0^{s^2} e^{-\frac{y_1^2}{2}} dy_1 \right) ds$$

$$= cx^\alpha_2 \int_{s^\frac{1}{2}} x_2 e^{-\frac{x^2}{2}} s^{\alpha} \ln s dx_1 ds$$

$$\leq cx^\alpha_2 \left( \ln x_2 \right) \int_0^\infty s^{\frac{1}{2} - \frac{1}{2} \alpha} e^{-s} ds + \int_{s^\frac{1}{2}}^\infty s^{\frac{1}{2} - \frac{1}{2} \alpha} \ln s e^{-s} ds.$$ 

Similarly, $J_2^2$ is estimated by

$$J_2^2 \leq c \int_0^t \int_{s^\frac{1}{2}} x_2 e^{-\frac{y_1^2}{2}} s^{\alpha} \left( \int_0^{s^\frac{1}{2}} e^{-\frac{y_1^2}{2}} dy_1 \right) ds$$

$$= c \int_0^t x_2 e^{-\frac{x^2}{2}} s^{\alpha} \left( \int_0^{s^\frac{1}{2}} e^{-y_1^2} dy_1 \right) ds$$

$$\leq c \int_0^t x_2 e^{-\frac{x^2}{2}} s^{\alpha} \ln s dx_1 ds$$

$$= cx^\alpha_2 \int_{s^\frac{1}{2}} x_2 e^{-\frac{x^2}{2}} s^{\alpha} \ln s dx_1 ds$$

$$\leq cx^\alpha_2 \left( \ln x_2 \right) \int_0^\infty s^{\frac{1}{2} - \frac{1}{2} \alpha} e^{-s} ds + \int_{s^\frac{1}{2}}^\infty s^{\frac{1}{2} - \frac{1}{2} \alpha} \ln s e^{-s} ds.$$ 

Hence, for $x^2 \leq t$, we have

$$|I_2| \leq cx^\alpha_2 \ln x_2.$$ 

Now, we estimate $I_3$. 

27
Proof of Theorem 3

In this section, we present the proof of Theorem 3.

Proof of Theorem 3

\[ I_3(0, x_2, t) = \int_{\mathbb{R}} \frac{z_1}{z_1^2 + x_2^2} g_2(z_1, t) dz_1 = \int_0^{x_2} \cdots dz_1 + \int_{x_2}^{2x_2} \cdots dz_1 + \int_{2x_2}^1 \cdots dz_1. \quad (4.4) \]

Again, due to the change of variable, the first and second terms are estimated by

\[ \int_{x_2}^{2x_2} \cdots dz_1 + \int_{2x_2}^1 \cdots dz_1 = \int_0^{x_2} \frac{z_1}{z_1^2 + x_2^2} (g(y_1 - z_1, \tau) - g(y_1 + z_1, \tau)) dz_1 \leq c \int_0^{x_2} \frac{1}{z_1} z_1^\alpha dz_1 \approx x_2^\alpha. \]

In case that \( t > (2x_2)^2 \), the last term in (4.4) is computed as follows:

\[
\begin{aligned}
\int_{2x_2}^1 \frac{z_1}{z_1^2 + x_2^2} g(z_1, t) dz_1 &= \int_{2x_2}^{\frac{t^2}{2}} \frac{z_1}{z_1^2 + x_2^2} (|z_1|^2 + t) \frac{1}{z_1^\alpha} \arctan(\frac{z_1}{t}) \frac{1}{z_1^\alpha} dz_1 \\
&\quad + \int_{\frac{t^2}{2}}^1 \frac{z_1}{z_1^2 + x_2^2} (|z_1|^2 + t) \frac{1}{z_1^\alpha} \arctan(\frac{z_1}{t}) \frac{1}{z_1^\alpha} dz_1 \\
&\approx - \int_{2x_2}^{\frac{t^2}{2}} \frac{1}{z_1^\alpha} (|z_1|^2 + t) \frac{1}{z_1^\alpha} dz_1 - \int_{\frac{t^2}{2}}^1 \frac{1}{z_1^\alpha} (|z_1|^2 + t) \frac{1}{z_1^\alpha} dz_1 \\
&\geq c \left( \frac{1}{t^\alpha} |\ln t| - x_2^\alpha - t \frac{1}{t^\alpha} \right).
\end{aligned}
\]

Hence, for \( x_2^3 \leq t \), we have

\[ |I_3(0, x_2, t)| \geq c \left( \frac{1}{t^\alpha} |\ln t| - x_2^\alpha - t \frac{1}{t^\alpha} \right). \]

Summing up above estimates, for \( x_2 \leq t^\frac{2}{3} \ll 1 \), we obtain

\[ |I_2(0, x_2, t) + I_3(0, x_2, t)| \geq c \left( \frac{1}{t^\alpha} |\ln t| - x_2^\alpha |\ln x_2| - x_2^\alpha - t \frac{1}{t^\alpha} \right). \]

Therefore, we conclude that \( u_1 \notin C^{\alpha, \frac{3}{2} \alpha}(Q_1^+), \) since

\[ \|u_1\|_{C^{\alpha, \frac{3}{2} \alpha}(Q_1^+ (0,0))} = \sup_{x,t} \frac{|u(x, t) - u(y, s)|}{(|x - y|^2 + (t - s)\frac{1}{t^\alpha})^{\frac{1}{t^\alpha}}} \geq c \sup_{x_1=0,x_2=t<\frac{1}{2}} \frac{|u(0, x_2, t) - u(0, 0)|}{(x_2^3 + t) \frac{1}{t^\alpha}} \geq c \sup_{t<\frac{1}{2}} \frac{t \frac{1}{t^\alpha} |\ln t| - t \frac{1}{t^\alpha} |\ln t| - t \frac{2}{t^\alpha}}{t \frac{1}{t^\alpha}} = \infty. \]
We introduce function classes $X(\Omega)$ and $X(Q_T)$ defined as follows:

$$X^\alpha(\Omega) := C^\alpha(\Omega) \times C^{\alpha+1}(\Omega) \times C^\alpha_\partial(\Omega), \quad X^\alpha(Q_T) := C^{\alpha,\frac{\alpha}{2}}(\{Q_T\}) \times C^{\alpha+1,\frac{\alpha+1}{2}}(\{Q_T\}) \times C^{\alpha,\frac{\alpha}{2}}(\{Q_T\})$$

with norms

$$\|(\rho, \theta, u)\|_{X^\alpha(\Omega)} := \|\rho\|_{C^\alpha(\Omega)} + \|\theta\|_{C^{\alpha+1}(\Omega)} + \|u\|_{C^\alpha_\partial(\Omega)},$$

$$\|(\rho, \theta, u)\|_{X^\alpha(Q_T)} := \|\rho\|_{C^{\alpha,\frac{\alpha}{2}}(\{Q_T\})} + \|\theta\|_{C^{\alpha+1,\frac{\alpha+1}{2}}(\{Q_T\})} + \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\{Q_T\})}.$$  

Let $(\rho_0, \theta_0, u_0) \in X^\alpha(\Omega)$. We consider

$$\rho_1^t - \Delta \rho_1 = 0, \quad \theta_1^t - \Delta \theta_1 = 0, \quad u_1^t - \Delta u_1 + \nabla p_1 = 0, \quad \text{div } u_1 = 0 \quad \text{in } Q_T$$

with initial and boundary conditions

$$\rho_1(x, 0) = \rho_0(x), \quad \theta_1(x, 0) = \theta_0(x), \quad u_1(x, 0) = u_0(x),$$

$$\frac{\partial \rho_1}{\partial n} = 0, \quad \frac{\partial \theta_1}{\partial n} = 0, \quad u_1 = 0 \quad \text{on } \partial \Omega \times (0, T).$$

(5.1)  

By Theorem 1, Theorem 6 and Remark 8, we get

$$\|(\rho_1, \theta_1, u_1)\|_{X^\alpha(Q_T)} \leq c \|(\rho_0, \theta_0, u_0)\|_{X^\alpha(\Omega)}.$$

For $m = 1, 2, \cdots$ we define iteratively by $(\rho^{m+1}, \theta^{m+1}, u^{m+1})$ a solution of the following equations in $Q_T$:

$$\rho^{m+1}_t - \Delta \rho^{m+1} = -\text{div}(u^m \rho^m) + \nabla \cdot F^m,$$

$$\theta^{m+1}_t - \Delta \theta^{m+1} = -u^m \cdot \nabla \theta^m + f^m,$$

$$u^{m+1}_t - \Delta u^{m+1} + \nabla p^{m+1} = -\nabla \cdot (u^m \otimes u^m) + G^m, \quad \text{div } u^{m+1} = 0$$

with the same boundary and initial conditions as (5.1) and (5.2). Here $F^m, f^m$ and $G^m$ denote

$$F^m := F(\rho^m, \theta^m, \nabla \theta^m, u^m), \quad f^m := f(\rho^m, \theta^m, \nabla \theta^m, u^m), \quad G^m := G(\rho^m, \theta^m, \nabla \theta^m, u^m).$$

We fix $T > 0$, which will be specified later. Then, we have

$$\|(\rho^{m+1}, \theta^{m+1}, u^{m+1})\|_{X^\alpha(Q_T)} \leq c \left( \|(\rho_0, \theta_0, u_0)\|_{X^\alpha(\Omega)} + T^\frac{1}{2} \|u^m \rho^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + T^\frac{1}{2} \|F^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} 
+ \max(T^\frac{1}{2}, T^\frac{1}{2}) \|u^m \cdot \nabla \theta^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \max(T^\frac{1}{2}, T^\frac{1}{2}) \|f^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} 
+ \max(T^\frac{1}{2}, T^\frac{1}{2}) \|u^m \otimes u^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \max(T, T^\frac{1}{2}) \|G^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \right).$$

(5.3)

Let $M_0 = \frac{c}{2} \|(\rho_0, \theta_0, u_0)\|_{X^\alpha(\Omega)}$, where $c$ is the constant in (5.3). Suppose that $M$ is a number with $M > M_0$ such that

$$\|(\rho^m, \theta^m, u^m)\|_{X^\alpha(Q_T)} < M.$$  

We then see that

$$\|u^m \rho^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|u^m \cdot \nabla \theta^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|u^m \otimes u^m\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$$

29
Choosing sufficiently small $X$, Via the argument of contraction mapping, the constructed sequence is indeed convergent in $\mathbb{R}^d$.

We then see that $(\rho^m, \theta^m, u^m)$ solve the following system:

$$
\begin{align*}
\partial_t \rho^m + \nabla \cdot (\rho^m u^m) &= \nabla \cdot (F^m - F^{m-1}) , \\
\partial_t \theta^m + \nabla \cdot (\theta^m u^m) &= U^m \nabla \theta^m + u^{m-1} \nabla \theta^m + f^m - f^{m-1}, \\
U_t^{m+1} - \Delta U_t^{m+1} &= \nabla P^m = -\nabla (u^m \otimes u^m + U^m \otimes u^{m-1}) + G^m - G^{m-1},
\end{align*}
$$

with following initial and boundary conditions

$$
\frac{\partial \rho^{m+1}}{\partial n} = 0, \quad \frac{\partial \theta^{m+1}}{\partial n} = 0, \quad U^{m+1} = 0 \quad \text{on} \quad \partial \Omega \times (0, T),
$$

and initial zero conditions, namely $\rho^{m+1}(x, 0) = \theta^{m+1}(x, 0) = U^{m+1}(x, 0) = 0$. Then, we have

$$
\|(\rho^{m+1}, \theta^{m+1}, U^{m+1})\|_{X^\alpha(Q_T)} \leq c \max(T, T^{\frac{1}{2} - \frac{\alpha}{2}}) \left(\|(\rho^m, \theta^m, U^m)\|_{X^\alpha(Q_T)}\right)^{\nu_1} \left(\|(\rho^m, \theta^m, u^m)\|_{X^\alpha(Q_T)} + \|(\rho^m, \theta^m, u^{m-1})\|_{X^\alpha(Q_T)}^{l-1} \|u^m\|_{X^\alpha(Q_T)}\right).
$$

Choosing sufficiently small $T$, we obtain

$$
\|(\rho^{m+1}, \theta^{m+1}, U^{m+1})\|_{X^\alpha(Q_T)} \leq \frac{1}{2} \|(\rho^m, \theta^m, U^m)\|_{X^\alpha(Q_T)}.
$$

Via the argument of contraction mapping, the constructed sequence is indeed convergent in $X^\alpha(Q_T)$, provided that $T$ is sufficiently small. Let $(\rho, \theta, u)$ be the limit of $(\rho^m, \theta^m, u^m)$. Then, it is direct that $(\rho, \theta, u)$ is the unique solution of $(\ref{eq:1.1.1})-(\ref{eq:1.1.2})$. Since its verification is rather standard, we skip its details.

As an application of Theorem 3, we establish local well-posedness in Hölder spaces for a mathematical model describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in $\mathbb{R}^2$. Such a model was proposed by Tuval et al.\cite{24}, formulating the dynamics of swimming bacteria, $Bacillus subtilis$, which is given as

$$
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (\chi(c)n \nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -k(c)n, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -n \nabla \phi, \quad \nabla \cdot u = 0.
\end{align*}
$$
where \( c(t, x) : Q_T \rightarrow \mathbb{R}^+, n(t, x) : Q_T \rightarrow \mathbb{R}^+, u(t, x) : Q_T \rightarrow \mathbb{R}^d \) and \( p(t, x) : Q_T \rightarrow \mathbb{R} \) denote the oxygen concentration, cell concentration, fluid velocity, and scalar pressure, respectively. Here \( \mathbb{R}^+ \) indicates the set of non-negative real numbers.

The nonnegative functions \( k(c) \) and \( \chi(c) \) denote the oxygen consumption rate and the aerobic sensitivity, respectively. i.e. \( k, \chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( k(c) = k(c(x, t)) \) and \( \chi(c) = \chi(c(x, t)) \). Initial data are given by \((n_0(x), c_0(x), u_0(x))\) with \( n_0(x), c_0(x) \geq 0 \) and \( \nabla \cdot u_0 = 0 \).

There are many known results for the system (5.7) regarding existence, regularity and asymptotics. We do not recall previous results but give some list of reference (see e.g. [3], [4], [7], [8], [10], [16], [24], [25], [26]). As far as the authors’ concerned, local-wellposedness is not known for the system (5.7) and it is, however, a direct consequence [4], [7], [8], [10], [16], [24], [25], [26].

**Theorem 12** Let the initial data \((n_0, c_0, u_0)\) be given in \( C^\alpha(\overline{\Omega}) \times C^{\alpha+1}(\overline{\Omega}) \times C^\alpha_{\partial \Omega}(\overline{\Omega}) \) for \( \alpha \in (0, 1) \) with \( n_0 \geq 0 \) and \( c_0 \geq 0 \). Assume that \( \chi, k, \chi', k' \) are all non-negative and \( \chi, k \in C^m(\mathbb{R}^+) \) and \( k(0) = 0 \). \( \|\nabla^l \phi\|_{L^\infty} < \infty \) for \( 1 \leq |l| \leq m \). There exists \( T > 0 \) such that unique solutions \((n, c, u)\) of (5.7) exist in the class \( C^{\alpha, \frac{1}{2}}(\overline{Q}_T) \times C^{\alpha+1, \frac{1}{2}}(\overline{Q}_T) \times C^{\alpha, \frac{1}{2}}(\overline{Q}_T) \) for any \( t < T \).

The result of Theorem 12 is a direct consequence of Theorem 3 and we skip its details.

**Appendix**

In this Appendix, we present the proof of Lemma 5.

**Proof of Lemma 5**

First, we prove the estimate (2.5). Direct computations show that

\[
|D_x \Lambda_0(f)(x, t) - D_y \Lambda_0(f)(y, t)| = \left| \int_0^t \int_{\mathbb{R}^n} D_z \Gamma(z, t - \tau)(f(x - z, \tau) - f(y - z, t - \tau))dzd\tau \right|
\]

\[
\leq c\|f\|_{L^\infty(0, T; C^\alpha(\mathbb{R}^n))}|x - y|^{\alpha} \int_0^t (t - \tau)^{-\frac{1}{2}}d\tau
\]

\[
\leq cT^{\frac{1}{2}}|x - y|^{\alpha}\|f\|_{L^\infty(0, T; C^\alpha(\mathbb{R}^n))}. \tag{5.8}
\]

Hence, we have

\[
\|\Lambda_0(f)\|_{L^\infty(0, T; C^{\alpha, \frac{1}{2}}(\mathbb{R}^n))} \leq cT^{\frac{1}{2}}\|f\|_{L^\infty(0, T; C^\alpha(\mathbb{R}^n))}. \tag{5.9}
\]

On the other hand, for \( s < t \), we obtain

\[
|\Lambda_0(f)(x, t) - \Lambda_0(f)(x, s)| \leq \left| \int_s^t \int_{\mathbb{R}^n} \Gamma(x - z, t - \tau)f(z, \tau)dzd\tau \right|
\]

\[
+ \left| \int_0^s \int_{\mathbb{R}^n} (\Gamma(x - z, t - \tau) - \Gamma(x - z, s - \tau))f(z, \tau)dzd\tau \right| = I_1 + I_2.
\]
We first estimate $I_1$.

$$I_1 \leq \|f\|_{L^\infty(\mathbb{R}^n \times (0,T))} \int_s^t \int_{\mathbb{R}^n} \Gamma(z,t-\tau)dzd\tau$$

$$= \|f\|_{L^\infty(\mathbb{R}^n \times (0,T))} (t-s) \leq T^{\frac{1}{2}-\frac{1}{2}\alpha} \|f\|_{L^\infty(\mathbb{R}^n \times (0,T))} (t-s)^{\frac{\alpha}{2}+\frac{1}{2}}.$$

For $I_2$, since $\int_{\mathbb{R}^n} (\Gamma(x-z,t-\tau) - \Gamma(x-z,s-\tau)) dz = 0$, we have

$$I_2 = \left| \int_0^s \int_{\mathbb{R}^n} (\Gamma(x-z,t-\tau) - \Gamma(x-z,s-\tau))(f(z,\tau) - f(x,\tau)) dzd\tau \right|$$

$$\leq c \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))} \int_0^s \int_{\mathbb{R}^n} |x-z|^\alpha |\Gamma(x-z,t-\tau) - \Gamma(x-z,s-\tau)| dzd\tau$$

$$\leq c \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))} \int_0^s \int_{\mathbb{R}^n} |x-z|^\alpha \int_s^t |D_\theta \Gamma(x-z,\theta-\tau)| d\theta dzd\tau$$

$$\leq c \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))} \int_0^s \int_s^t (\theta-\tau)^{\frac{\alpha}{2}-1} d\theta d\tau \leq c \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))} (t^{\frac{\alpha}{2}+1} - s^{\frac{\alpha}{2}+1}).$$

By mean-value theorem, there is $\xi \in (s,t)$ such that

$$s^{\frac{\alpha}{2}+1} - t^{\frac{\alpha}{2}+1} = (\frac{\alpha}{2} + 1)\xi^{\frac{\alpha}{2}}(t-s) \leq cT^{\frac{1}{2}}(t-s)^{\frac{\alpha}{2}+\frac{1}{2}}.$$

Hence, we have

$$I_2 \leq cT^{\frac{1}{2}} \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))}.$$

Combining above estimates, we obtain

$$\|\Lambda_0(f)\|_{L^\infty(\mathbb{R}^n;\mathcal{C}_0^{\frac{\alpha}{2}+\frac{1}{2}}[0,T])} \leq \max(T^{\frac{1}{2}}, T^{\frac{1}{2}-\frac{\alpha}{2}}) \|f\|_{L^\infty(\mathbb{R}^n;\mathcal{C}_0^{\frac{\alpha}{2}}[0,T])}.$$  \(\text{(5.10)}\)

Via (5.9) and (5.10), we obtain (2.5).

We prove the estimate (2.7). Due to (5.8), it is direct that

$$\|\nabla \Lambda_0(f)\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))} \leq T^{\frac{1}{2}} \|f\|_{L^\infty(0,T;\mathcal{C}_0^{\alpha}(\mathbb{R}^n))}.$$  \(\text{(5.11)}\)

Therefore, it suffices to show

$$\|\nabla \Lambda_0(f)\|_{L^\infty(\mathbb{R}^n;\mathcal{C}_0^{\frac{\alpha}{2}}[0,T])} \leq T^{\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^n;\mathcal{C}_0^{\frac{\alpha}{2}}[0,T])}.$$  \(\text{(5.12)}\)

Indeed, we note that for $s < t$,

$$|\nabla \Lambda_0(f)(x,t) - \nabla \Lambda_0(f)(x,s)| \leq \left| \int_s^t \int_{\mathbb{R}^n} \nabla_x \Gamma(x-z,t-\tau) f(z,\tau) dzd\tau \right|$$

$$+ \left| \int_0^s \int_{\mathbb{R}^n} (\nabla_x \Gamma(x-z,t-\tau) - D_x \Gamma(x-z,s-\tau)) f(z,\tau) dzd\tau \right| = J_1 + J_2.$$
We first estimate $J_1$. Since $\int_{\mathbb{R}^n} D_x \Gamma(z, \tau) dz = 0$ for $\tau > 0$, using change of variables, we have

\[
J_1 \leq \int_s^t \int_{\mathbb{R}^n} D_x \Gamma(z, t - \tau) (f(x, z, \tau) - f(x, \tau)) dz d\tau
\]

\[
\leq \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t \int_{\mathbb{R}^n} |D_x \Gamma(z, t - \tau)| |z|^a dz d\tau
\]

\[
\leq \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t (t - \tau)^{-\frac{1}{2} + \frac{a}{2}} d\tau
\]

\[
\leq T^\frac{1}{2} \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))}(t - s)^\frac{a}{2}.
\]

Using $\int_{\mathbb{R}^n} (D_x \Gamma(x, z, t - \tau) - D_x \Gamma(x, z, s - \tau)) dz = 0$, we estimate $J_2$ as follows:

\[
J_2 \leq \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t \int_{\mathbb{R}^n} |x - z|^a |D_x \Gamma(x, z, t - \tau) - D_x \Gamma(x, z, s - \tau)| dz d\tau
\]

\[
\leq \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t \int_{\mathbb{R}^n} |x - z|^a \int_s^t |D_\theta D_x \Gamma(x, z, \theta - \tau)| d\theta dz d\tau
\]

\[
\leq \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t \int_s^t (t - \tau)^{\frac{3}{2} - \frac{a}{2}} d\tau
\]

\[
= \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))} \int_s^t ((s - \tau)^{\frac{3}{2} - \frac{a}{2}} - (t - \tau)^{\frac{3}{2} - \frac{a}{2}}) d\tau
\]

\[
\leq c T^\frac{1}{2} \|f\|_{L^\infty(0, T; \mathcal{C}^a(\mathbb{R}^n))}(t - s)^\frac{a}{2}.
\]

This implies the estimate (5.12), and therefore, together with (5.11), we obtain (2.17).

Next, we prove the estimate (2.6). Let $\hat{f}(\xi), \xi \in \mathbb{R}^n$ denote the Fourier transform of $f(x), x \in \mathbb{R}^n$. Let $S(\mathbb{R}^n)$ denote the Schwartz space on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be the dual space of the Schwartz space $S(\mathbb{R}^n)$. Fix a Schwartz function $\psi \in S(\mathbb{R}^n)$ satisfying $\hat{\psi}(\xi) = 0$ on $\frac{1}{2} < |\xi| < 2$, $\hat{\psi}(\xi) = 0$ elsewhere, and $\sum_{j=-\infty}^{\infty} \hat{\psi}(2^{-j} \xi) = 1$ for $\xi \neq 0$. Let

\[
\hat{\psi}_j(\xi) := \hat{\psi}(2^{-j} \xi), \quad (j = 0, \pm 1, \pm 2, \ldots).
\]

Note that

\[
\hat{\mathcal{C}}^a(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) \mid \sup_{-\infty < j < \infty} 2^{-aj} \|f \ast \psi_j\|_{L^\infty(\mathbb{R}^n)} < \infty\},
\]

where $\ast$ is convolution in $\mathbb{R}^n$. Let $\Psi = \psi_{-1} + \psi_0 + \psi_1$ and $\Psi_j(\xi) = \Psi(2^{-j} \xi)$ such that $\text{supp} \Psi_j \subset \{2^{-j-2} < |\xi| < 2^{-j+2}\}$ and $\Psi \equiv 1$ in $2^{j-1} < |\xi| < 2^{j+1}$. We observe that

\[
\nabla \Lambda_0 f \ast \psi_j(\xi, t) = \int_0^t \xi e^{-(t-\tau)|\xi|^2} \hat{f}(\xi, \tau) \hat{\psi}_j(\xi) d\tau
\]

\[
= \int_0^t \xi \Phi_j(\xi) e^{-(t-\tau)|\xi|^2} \hat{f}(\xi, \tau) \hat{\psi}_j(\xi) d\tau.
\]
Note that the $L^\infty$-multiplier norms of $\xi \Phi_j(\xi)e^{-(t-\tau)|\xi|^2}$ and $2^j\xi \Phi(\xi)e^{-(t-\tau)2^j|\xi|^2}$ are same (see Theorem 6.1.3 in [1]). By Lemma 13, the $L^\infty$-multiplier norm of $\xi \Phi(\xi)e^{-(t-\tau)2^j|\xi|^2}$ is dominated by $e^{-\frac{1}{8}(t-\tau)2^j}$. Hence, we have

\[
\|\nabla \Lambda_0 f \ast \psi_j(t)\|_{L^\infty(\mathbb{R}^n)} \leq \int_0^t \| \mathcal{F}^{-1}(\xi \Phi_j(\xi)e^{-(t-\tau)|\xi|^2} \hat{\psi}_j(\tau)) \|_{L^\infty(\mathbb{R}^n)} d\tau
\]
\[
\leq \int_0^t 2^j e^{-\frac{1}{8}(t-\tau)2^j} \| (f \ast \psi_j)(\tau) \|_{L^\infty(\mathbb{R}^n)} d\tau
\]
\[
\leq 2^{(1-\alpha)j} \int_0^t e^{-\frac{1}{8}(t-\tau)2^j} \| f(\tau) \|_{L^\infty(\mathbb{R}^n)} d\tau
\]
\[
\leq 2^{-(1+\alpha)j} \| f \|_{L^\infty(0,T;\mathcal{C}^{\alpha}(\mathbb{R}^n))} \int_0^{2^{2j}t} e^{-\frac{1}{8}\tau} d\tau. \tag{5.13}\]

Hence, we have

\[
\|\nabla \Lambda_0 f\|_{L^\infty(0,T;\mathcal{C}^{\alpha+\epsilon}(\mathbb{R}^n))} \leq c\|f\|_{L^\infty(0,T;\mathcal{C}^{\alpha}(\mathbb{R}^n))}. \tag{5.14}\]

By the same argument of (5.12), we have

\[
\|\nabla \Lambda_0(f)\|_{L^\infty(\mathbb{R}^n;\mathcal{C}^{\alpha+\frac{1}{2}}[0,T])} \leq c\|f\|_{L^\infty(\mathbb{R}^n \times C^{\frac{1}{2}}[0,T])}. \tag{5.15}\]

By (5.14) and (5.15), we obtain (2.6).

For (2.8), note that for $\epsilon < 1$, we have

\[
\int_0^{2^{2j}t} e^{-\frac{1}{8}\tau} d\tau \leq c \int_0^{2^{2j}t} \tau^{-\frac{1}{2}+\frac{1}{8}\epsilon} d\tau = c(2^{2j}t)^{\frac{1}{2}-\frac{1}{8}\epsilon}.
\]

Hence, from (5.13), we have

\[
\|\nabla \Lambda_0 f\|_{L^\infty(0,T;\mathcal{C}^{\alpha+\epsilon}(\mathbb{R}^n))} \leq cT^{\frac{1}{2}-\frac{1}{8}\epsilon} \| f \|_{L^\infty(0,T;\mathcal{C}^{\alpha}(\mathbb{R}^n))}. \tag{5.16}\]

By the same argument of (5.12), we have

\[
\|\nabla \Lambda_0(f)\|_{L^\infty(\mathbb{R}^n;\mathcal{C}^{\alpha+\frac{1}{2}}[0,T])} \leq cT^{\frac{1}{2}-\frac{1}{8}\epsilon} \| f \|_{L^\infty(0,T;\mathcal{C}^{\alpha}(\mathbb{R}^n))}. \tag{5.17}\]

By (5.16) and (5.17), we obtain (2.8). This completes the proof.

Lemma 13 Let $\rho_{ij}(\xi) = \xi \Psi_j(\xi)e^{-t|\xi|^2}$ for each integer $j$. Then $\rho_{ij}(\xi)$ is a $L^\infty(\mathbb{R})$-multiplier with the finite norm $M(t,j)$. Moreover for $t > 0$

\[
M(t,j) \leq c e^{-\frac{1}{4}t2^j} \sum_{0 \leq i \leq n} t^{i2^j} \leq c e^{-\frac{1}{8}(t2^j)}. \tag{5.18}\]

Proof. The $L^\infty(\mathbb{R}^n)$-multiplier norms $M(t,j)$ of $\rho_{ij}(\xi)$ is equal to the $L^\infty(\mathbb{R}^n)$-multiplier norm of $\rho_{ij}'(\xi) := 2^j \xi \Psi_j(\xi)e^{-t2^j|\xi|^2}$, $\Psi_j'(\xi) = \xi \Psi_j(\xi)$(See Theorem 6.1.3 in [1]). Now, we make use of
the Lemma 6.1.5 of [1]. Let \( \beta = (\beta_1, \beta_2, \cdots, \beta_n) \in (\mathbb{N} \cup \{0\})^n \). Since \( \text{supp} (\Psi^t) \subset \{ \xi \in \mathbb{R}^n | \frac{1}{4} < |\xi| < 4 \} \), we have

\[
|D_{\xi}^{\beta} \rho_{ij}^t(\xi)| \leq c \sum_{0 \leq i \leq |\beta|} t^{i} 2^{2ij} e^{-\frac{1}{4} t^{22j}} \chi_{\frac{1}{4} < |\xi| < 4}(\xi)
\]

\[
\leq c e^{-\frac{1}{8} t^{22j}} \chi_{\frac{1}{4} < |\xi| < 4}(\xi),
\]

where \( \chi_A \) is the characteristic function on a set \( A \). Hence, applying Lemma 6.1.5 of [1], we completes the proof.

\[\Box\]

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