Practical creation and detection of polarization Bell states using parametric down-conversion

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CANADA

Abstract

The generation and detection of maximally-entangled two-particle states, ‘Bell states,’ are crucial tasks in many quantum information protocols such as cryptography, teleportation, and dense coding. Unfortunately, they require strong inter-particle interactions lacking in optics. For this reason, it has not previously been possible to perform complete Bell state determination in optical systems. In this work, we show how a recently developed quantum interference technique for enhancing optical nonlinearities can make efficient Bell state measurement possible. We also discuss weaknesses of the scheme including why it cannot be used for unconditional quantum teleportation.
**INTRODUCTION**

The new science of quantum information builds on the recognition that entanglement, an essential but long underemphasized feature of quantum mechanics, can be a valuable resource. Many of the headline-grabbing quantum communication schemes (including quantum teleportation [1, 2, 3], dense coding [4, 5], and quantum cryptography [6, 7]) are based on the maximally-entangled two-particle quantum states called Bell states. Using the polarization states of a pair of photons in different spatial modes, the four Bell states are written as:

\[
|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|V\rangle_1|H\rangle_2 \pm |H\rangle_1|V\rangle_2)
\]

\[
|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2),
\]  

(1)

where \(|H\rangle\) and \(|V\rangle\) describe horizontal- and vertical-polarization states, and the subscripts 1 and 2 are spatial mode labels. These four states form a complete, orthonormal basis for the polarization states of a pair of photons. In each Bell state, a given photon is completely unpolarized but perfectly correlated with the polarization of the other photon. Photon Bell states were produced in atomic cascades for the first tests of the nonlocal predictions of quantum mechanics [8]. Since that time, parametric down-conversion sources [9, 10, 11, 12, 13] have replaced cascade sources due to their ease of use, high brightness, and the high-purity states they produce. However, down-conversion sources do not deterministically prepare photon Bell states, but rather states in which the Bell state component is in a coherent superposition with a dominant vacuum term; coincidence detection of photon pairs projects out only the two-photon component of the state.

While optical Bell state source technology has shown marked improvement, methods of distinguishing these states has proven a difficult challenge. Perhaps the most well-known example of why distinguishing Bell states is important comes from quantum teleportation. A general projective measurement is required for unconditional teleportation; experimental teleportation was originally limited to a maximum efficiency of 25% since only the singlet state, \(|\psi^-\rangle\), could be distinguished from the triplet states [2]. The challenge for measuring Bell states stems from the requirement for a strong inter-particle interaction, which is usually nonexistent for photons. Without such a nonlinearity, only two of the four states can be distinguished[14]. It was realized that a strong enough optical nonlinearity, typically
\(\chi^{(3)}\), could be used to mediate a photon-photon interaction. Unfortunately, even the nonlinearities of our best materials are far too weak. An experiment using standard nonlinear materials to demonstrate a scheme for unconditional teleportation was limited to extremely low efficiencies (on the order of \(10^{-10}\)) by the tiny nonlinearities involved \([13]\). Proposals for extending optical nonlinearities to the quantum level include schemes based on cavity QED \([16]\), electromagnetically-induced transparency \([17]\), photon-exchange interactions \([18]\), and quantum interference techniques \([19, 20]\). Using the latter, we have recently demonstrated a conditional-phase switch \([20]\) which is similar to the controlled-phase gate in quantum computation. In this work, we show how to apply the conditional-phase switch to the problem of Bell state detection. It should be noted that if recently published schemes for performing quantum computing with linear optics \([21, 22]\) could be experimentally realized, then the problem of distinguishing all four Bell states could be performed without the need for strong optical nonlinearities. Theoretical work has also shown that if the Bell state is embedded appropriately in a higher-dimensional Hilbert space, all of the Bell states can be distinguished \([23]\).

Strong optical nonlinearities are desired so that one can construct a controlled-\(\pi\), a specific case of the controlled-phase gate for photons. Such a gate and all one-qubit rotations form a universal set of gates for the more general problem of quantum computation – just as the NAND gate is universal for classical computation. The controlled-\(\pi\) transformation \([24]\) is described by:

\[
\begin{align*}
|0\rangle_1 |0\rangle_2 & \rightarrow |0\rangle_1 |0\rangle_2 \\
|0\rangle_1 |1\rangle_2 & \rightarrow |0\rangle_1 |1\rangle_2 \\
|1\rangle_1 |0\rangle_2 & \rightarrow |1\rangle_1 |0\rangle_2 \\
|1\rangle_1 |1\rangle_2 & \rightarrow -|1\rangle_1 |1\rangle_2 ,
\end{align*}
\]

in which the two qubit states are \(|0\rangle\) and \(|1\rangle\) and the subscript is the qubit label. This transformation does nothing to the input state unless both qubits have a value of \(|1\rangle\), in which case it applies a phase-shift of \(\pi\). On the surface this transformation appears to do nothing since an overall phase in quantum mechanics is meaningless. However, it is clearly nontrivial when applied to superpositions of states.

The polarization of the photon makes an ideal two-level system for encoding a qubit largely due to its relative immunity to environmental decoherence. A large enough \(\chi^{(3)}\)
nonlinearity could be used to effect the c-π transformation on a pair of photons. Given a polarization-dependent $\chi^{(3)}$, or through the use of polarizing beam-splitters, only photon pairs with, say, horizontal polarization would experience the nonlinear interaction and pick up the additional phase shift. Such a gate could then be incorporated into the optical implementation of the quantum circuits shown in Fig. 1a. and 2a. (similar circuits are discussed in [14, 25]). The circuit in Fig. 1a. converts, through unitary transformation, a state in the rectilinear product state basis (i.e. $|0\rangle_1|0\rangle_2, |0\rangle_1|1\rangle_2, |1\rangle_1|0\rangle_2,$ and $|1\rangle_1|1\rangle_2$) to the Bell basis. The circuit in Fig. 2a. performs the opposite function converting a Bell state via unitary transformation to the rectilinear basis. In essence, these circuits allow for the creation and removal of entanglement between pairs of qubits. If the qubit states $|0\rangle$ and $|1\rangle$ are encoded into the polarization states $|H\rangle$ and $|V\rangle$ in two different spatial modes 1 and 2, then an optical realization of the circuit in Fig. 2a. allows for the conversion of a photon pair in a Bell state to a rectilinear basis state. These four rectilinear basis states are easily distinguishable using the simple optical setup shown in Fig. 3. Thus, after passing the photon pair in a Bell state through the optical realization of the circuit in Fig. 2a., the subsequent detection of the rectilinear state is equivalent to determination of the Bell state.

The conditional-phase switch we propose is related to the controlled-phase gate of quantum computation and is described in the theory section of this work. The switching effect occurs in a $\chi^{(2)}$ nonlinear material that is pumped by a strong, classical beam. This pump beam is capable of creating pairs of down-converted photon pairs into a pair of output modes. Pairs of photons, in a coherent superposition with the vacuum, pass through the crystal into those same output modes. It is the interference between the amplitudes for multiple paths leading to a photon pair that greatly enhances the effective nonlinearity; since the down-converted light is only created in pairs, the interference only affects the amplitude for photon pairs. However, since the switching effect is based on an interference effect, it is intrinsically dependent on the phase and amplitude of the incoming beams. This has two consequences. First, the switch requires an input which is in a coherent superposition with the vacuum. In this way, the input has the required uncertain number of photons, since photon number and phase are conjugate quantities. And second, the switch works as described only for states in the correct superposition with the vacuum, not a general input state. As we will show, these conditions do allow for one to distinguish between the four Bell states provided they are in the correct superposition with the vacuum. Nonetheless,
the conditions are too stringent to allow for unconditional teleportation using this method.

First, we describe the effective nonlinearity. Then we show how the nonlinearity can be used to construct optical devices analogous to the quantum computation circuits shown in Fig. 1a. and Fig. 2a.

**THEORY**

**Effective Nonlinearity**

The general down-conversion state can be written as

\[
|\psi\rangle = |0\rangle + \varepsilon \begin{pmatrix}
|H\rangle_1 |H\rangle_2 & |H\rangle_1 |V\rangle_2 & |V\rangle_1 |H\rangle_2 & |V\rangle_1 |V\rangle_2
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix},
\]

where the part of the state describing photon pairs has been written as an inner product. The amplitudes for the polarization states \(|H\rangle_1 |H\rangle_2\), \(|H\rangle_1 |V\rangle_2\), \(|V\rangle_1 |H\rangle_2\), and \(|V\rangle_1 |V\rangle_2\) are \(\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma,\) and \(\varepsilon\delta\), respectively. Again, the subscripts 1 and 2 describe two different spatial modes. Throughout this theory section, we adopt a 4-dimensional vector representation to describe the polarization state of the photon pairs. In this more compact notation, the general state is written

\[
|\psi\rangle = |0\rangle + \varepsilon \begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix},
\]

In both cases, we have suppressed the normalization factor for clarity, and for the discussion here we will restrict ourselves to the case where the probability of having a photon pair at any given time is small, i.e. \(|\varepsilon|^2 \ll 1\) (as is always the case in real down-conversion experiments).

The effective nonlinearity \([20]\) can be described as follows. Modes 1 and 2 are of frequency \(\omega\) and pass through a \(\chi^{(2)}\) nonlinear crystal that is simultaneously pumped by a strong classical laser beam of frequency \(2\omega\) in mode p. The modes are so chosen such that the
nonlinear crystal can create degenerate horizontally-polarized photon pairs in spatial modes 1 and 2 via spontaneous parametric down-conversion, as shown in Fig. 4. The nonlinear process is mediated by the interaction Hamiltonian,

\[ H = g a_1^\dagger H a_2^\dagger H a_{p,V} + g^* a_1 H a_2 H a_{p,V}, \]  

(5)

where g is the coupling constant and \( a_i^{(f)} \) is the field annihilation (creation) operator for the \( i^{th} \) mode, and the subscripts \( H \) and \( V \) are the polarizations of the relevant modes for the type-I phase-matching. The pump laser is intense enough that we treat it classically by replacing its field operators with c-number amplitudes, \( \zeta \) and \( \zeta^* \):

\[ H = g \zeta a_1^\dagger H a_2^\dagger H + g^* \zeta^* a_1 H a_2 H. \]  

(6)

Due to phase-matching constraints, the nonlinear crystal can only produce horizontally-polarized photon pairs. In the weak coupling regime, we can use first-order perturbation theory to propagate our state under the interaction to,

\[ |\psi(t)\rangle = \left( 1 - \frac{it}{\hbar} H \right) |\psi\rangle \]  

(7)

\[ = |0\rangle + \varepsilon \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} - \frac{it}{\hbar} g \zeta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]  

(8)

\[ = |0\rangle + \varepsilon \begin{pmatrix} \alpha - \frac{it g \zeta}{\hbar} \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \]  

(9)

To first order, this Hamiltonian simply creates an amplitude for a horizontally-polarized pair of photons. This new down-conversion amplitude interferes with the preexisting amplitude for the \( HH \) term.

The transformation, as described here, does not appear unitary. This is due to a few approximations. We assume that the vacuum term in our state is unchanged, and neglect terms describing more than one pair of photons. These approximations are only valid in the relevant limit where \( |\varepsilon| \ll 1 \), where we can also suppress the normalization term for
clarity. However, the exact propagator follows from a hermitian Hamiltonian and is of course unitary.

As was shown in the “railcross experiment” [26] and in our subsequent work with photon pairs from coherent state inputs [19], interference between the amplitudes for existing pairs and for down-conversion can modulate the rate of pair production. Given the phase-matching scheme presented here, only the amplitude for \( HH \) pairs is affected. Accompanying this modulation of the photon pair production rate is a shift in the phase of the horizontally-polarized photon pair term. The down-conversion crystal impresses a \( \pi \) phase-shift on the \( HH \) term if the down-conversion amplitude, \(-igt\zeta/\hbar\) to be \(-2\varepsilon\alpha\). To implement a transformation analogous to the \( c-\pi \) (Eq. 2) in the coincidence basis, this is the only condition that must be enforced; the values for the coefficients \( \alpha \), \( \beta \), and \( \gamma \) are free. This condition takes the place of the more usual normalization condition on \( \alpha \), \( \beta \), \( \gamma \), and \( \delta \) to describe our state space. It can be enforced experimentally by controlling the amplitude and phase of the pump laser and/or the overall pair amplitude \( \varepsilon \). Unfortunately, this means that the gate cannot be utilized on arbitrary inputs without some prior information. Under these conditions, the crystal implements

\[
|0\rangle + \varepsilon \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \rightarrow |0\rangle + \varepsilon \begin{pmatrix} -\alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.
\]

If horizontal polarization is used to represent a logical ‘0’, this performs a transformation analogous to a \( c-\pi \) within the state space defined by our constraint on \( \alpha \). We do not use the conventional \( c-\pi \) so that we can use the common convention for the Hadamard gate later on without the need for additional quantum gates. We will now describe how this operation can be used to perform Bell state creation under certain conditions.

**Bell state creation**

The circuit in Fig. 1a. is capable of converting each rectilinear basis state to a different Bell state. To give a concrete example, we begin with the qubit pair in the state
$|0\rangle_1 |0\rangle_2$ represented as the 4-vector

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$  \hspace{1cm} (11)$$

where the rows now contain the amplitudes for the states $|0\rangle_1 |0\rangle_2$, $|0\rangle_1 |1\rangle_2$, $|1\rangle_1 |0\rangle_2$, and $|1\rangle_1 |1\rangle_2$. The circuit contains one-qubit Hadamard transformations which are defined by the $2 \times 2$ matrix,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$  \hspace{1cm} (12)$$

and the two-qubit $c-\pi$ gate whose operation has already been discussed. The circuit then takes the input state, $|\psi\rangle$, to the output state $|\psi'\rangle$ given by

$$|\psi'\rangle = (H_1 \otimes I_2) (c-\pi) (H_1 \otimes H_2) |\psi\rangle$$  \hspace{1cm} (13)$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (14)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (15)$$

This final state is the Bell state $|\phi^+\rangle$. Each different rectilinear state input will produce a different Bell state output through this circuit.

The conditional-phase operation can be incorporated into the optical device schematically represented in Fig. 1b that can perform a very similar transformation. Instead of using a state describing a pure photon pair as input, this device requires the input pair to be in a coherent superposition with the vacuum. As discussed previously, this is merely the output from a parametric down-conversion source (Eq. 3). Here we assume the coefficients are normalized according to $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$, such that $|\epsilon|^2$ is the probability of a
photon pair of any polarization being present. The photons have been created into spatial
modes 1 and 2 by an initial down-conversion crystal (not shown) to serve as input to the
optical device in Fig. 1b. Hadamard operations are accomplished via half-wave plates at
22.5 degrees, and the c-π has been replaced by the conditional-phase switch. The intial
state will evolve as follows through the device. The pair of Hadamard gates changes the
general state, $|\psi_1\rangle$, to $|\psi_2\rangle$,

$$|\psi_2\rangle = (H_1 \otimes H_2) |\psi_1\rangle$$  \hspace{1cm} (16)

$$= |0\rangle + \varepsilon \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$  \hspace{1cm} (17)

$$= |0\rangle + \varepsilon \frac{1}{2} \begin{pmatrix} \alpha + \beta + \gamma + \delta \\ \alpha - \beta + \gamma - \delta \\ \alpha + \beta - \gamma - \delta \\ \alpha - \beta - \gamma + \delta \end{pmatrix}$$  \hspace{1cm} (18)

This state passes through the conditional-phase shift, which is phase-matched to contribute
an amplitude of $-\varepsilon$ for horizontally-polarized photon pairs. It will evolve to $|\psi_3\rangle$,

$$|\psi_3\rangle = |0\rangle + \varepsilon \frac{1}{2} \begin{pmatrix} \alpha + \beta + \gamma + \delta \\ \alpha - \beta + \gamma - \delta \\ \alpha + \beta - \gamma - \delta \\ \alpha - \beta - \gamma + \delta \end{pmatrix} - \varepsilon \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (19)

$$= |0\rangle + \varepsilon \frac{1}{2} \begin{pmatrix} \alpha + \beta + \gamma + \delta - 2 \\ \alpha - \beta + \gamma - \delta \\ \alpha + \beta - \gamma - \delta \\ \alpha - \beta - \gamma + \delta \end{pmatrix}$$  \hspace{1cm} (20)

The final Hadamard gate acts only on mode 1, and converts $|\psi_3\rangle$ to the output state $|\psi'\rangle$, 

9
\[ |\psi'\rangle = (H_1 \otimes I_2) |\psi_3\rangle \]
\[ = |0\rangle + \frac{\varepsilon}{2\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} \alpha + \beta + \gamma + \delta - 2 \\ \alpha - \beta + \gamma - \delta \\ \alpha + \beta - \gamma - \delta \\ \alpha - \beta - \gamma + \delta \end{pmatrix} \]
\[ = |0\rangle + \frac{\varepsilon}{\sqrt{2}} \begin{pmatrix} \alpha + \beta - 1 \\ \alpha - \beta \\ \gamma + \delta - 1 \\ \gamma - \delta \end{pmatrix} \] (22)

If, for example, the input state to this device had only an amplitude for a horizontally-polarized photon pair (i.e. \(\alpha = 1\) and \(\beta, \gamma, \delta = 0\)), then the output state would be,
\[ |\psi'\rangle = |0\rangle + \varepsilon \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \]
\[ = |0\rangle - \varepsilon |\psi^-\rangle. \] (24)

The other 3 possible rectilinear basis inputs would each evolve to a different Bell state in a coherent superposition with the vacuum state. The resulting transformations on four possible rectilinear input states are
\[ |0\rangle + \varepsilon |H\rangle_1 |H\rangle_2 \rightarrow |0\rangle - \varepsilon |\psi^-\rangle \]
\[ |0\rangle + \varepsilon |H\rangle_1 |V\rangle_2 \rightarrow |0\rangle - \varepsilon |\psi^+\rangle \]
\[ |0\rangle + \varepsilon |V\rangle_1 |H\rangle_2 \rightarrow |0\rangle - \varepsilon |\phi^-\rangle \]
\[ |0\rangle + \varepsilon |V\rangle_1 |V\rangle_2 \rightarrow |0\rangle - \varepsilon |\phi^+\rangle. \] (26)

**Bell state detection**

The method just described for creating polarization Bell states is much more experimentally difficult than the elegant methods of doing so in a cleverly-oriented crystal or crystal
pair [11, 12]. What is unique about this method is that this device performs a one-to-one transformation between rectilinear basis states and Bell basis states. This device for creating the Bell states can, in fact, be run in reverse to distinguish between the four Bell states provided, again, that they are in a superposition with vacuum. Fig. 2a. shows a quantum circuit for transforming Bell states to the rectilinear basis, that is very similar in structure to the circuit shown in Fig. 1a. To give a concrete example, we can trace the evolution of the singlet state, $|\phi^\rightarrow\rangle$, through the device. The singlet state can be written in 4-vector notation as,

$$|\psi^\rightarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}. \quad (27)$$

The circuit transforms the input state to the output $|\psi'\rangle$ in the following way,

$$|\psi'\rangle = (H_1 \otimes H_2) (c-\pi) (H_1 \otimes I_2) |\phi^\rightarrow\rangle \quad (28)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad (29)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (30)$$

The output state is the product state $|1\rangle_1 |1\rangle_2$.

The optical device that performs the analogous transformation is shown in Fig. 2b. The device, again, uses half-wave plates to implement the Hadamard transformations, and the conditional-phase switch which is set to contribute an amplitude of $+\varepsilon$ for a horizontally-polarized photon pair. The input state to this device, $|\psi_1\rangle$, is again described by the general
down-conversion state,

\[ |\psi_1\rangle = |0\rangle + \varepsilon \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \]  

(31)

This state passes through the polarization rotator in mode 1 and will evolve to the state \( |\psi_2\rangle \),

\[ |\psi_2\rangle = (H_1 \otimes I_2) |\psi_1\rangle \]  

(32)

\[ = |0\rangle + \varepsilon \sqrt{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \]  

(33)

\[ = |0\rangle + \varepsilon \sqrt{2} \begin{pmatrix} \alpha + \gamma \\ \beta + \delta \\ \alpha - \gamma \\ \beta - \delta \end{pmatrix} \]  

(34)

This state is subsequently passed through the conditional-phase switch where the pump laser is set to the appropriate amplitude and phase to add an amplitude of \( +\varepsilon \) for a vertically-polarized photon pair. The state evolves to \( |\psi_3\rangle \) where

\[ |\psi_3\rangle = |0\rangle + \varepsilon \sqrt{2} \begin{pmatrix} \alpha + \gamma \\ \beta + \delta \\ \alpha - \gamma \\ \beta - \delta \end{pmatrix} + \varepsilon \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]  

(35)

\[ = |0\rangle + \varepsilon \sqrt{2} \begin{pmatrix} \alpha + \gamma + \sqrt{2} \\ \beta + \delta \\ \alpha - \gamma \\ \beta - \delta \end{pmatrix} \]  

(36)
Finally, this state passes through a pair of half-wave plates. The final state, $|\psi'\rangle$, is

$$|\psi'\rangle = |0\rangle + \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{\varepsilon}{\sqrt{2}} \begin{pmatrix} \alpha + \gamma + \sqrt{2} \\ \beta + \delta \\ \alpha - \gamma \\ \beta - \delta \end{pmatrix}$$

(37)

$$= |0\rangle + \sqrt{2}\varepsilon \begin{pmatrix} \alpha + \beta + \frac{1}{\sqrt{2}} \\ \alpha - \beta + \frac{1}{\sqrt{2}} \\ \gamma + \delta + \frac{1}{\sqrt{2}} \\ \gamma - \delta + \frac{1}{\sqrt{2}} \end{pmatrix}.$$  

(38)

If, for example our input state has $\alpha = \delta = -1/\sqrt{2}$ and $\beta = \gamma = 0$ (i.e. the input is $|0\rangle - \varepsilon |\phi^+\rangle$ – one of the outputs of the previous device), then the output state would be,

$$|\psi'\rangle = |0\rangle + \sqrt{2}\varepsilon \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

(39)

$$= |0\rangle + \varepsilon \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$  

(40)

That is, the output contains only an amplitude for a photon pair in the product state $|V\rangle_1 |V\rangle_2$. The results for all of the input states are simply stated:

$$|0\rangle - \varepsilon |\psi^-\rangle \rightarrow |0\rangle + \varepsilon |H\rangle_1 |H\rangle_2$$

$$|0\rangle - \varepsilon |\psi^+\rangle \rightarrow |0\rangle + \varepsilon |H\rangle_1 |V\rangle_2$$

$$|0\rangle - \varepsilon |\phi^-\rangle \rightarrow |0\rangle + \varepsilon |V\rangle_1 |H\rangle_2$$

$$|0\rangle - \varepsilon |\phi^+\rangle \rightarrow |0\rangle + \varepsilon |V\rangle_1 |V\rangle_2,$$

(41)

and are the inverse of the transformation the previous device performed.

In order to complete the measurement of the Bell state, the output of this device is passed through an optical device like the one in Fig. 3. The detection of a photon pair constitutes a successful measurement and will occur with probability $|\varepsilon|^2$ – the probability of having a Bell state in our input state. This probability ignores issues of detector and path efficiency.
DISCUSSION

We have proposed a way of implementing a transformation capable of converting the polarization state of a pair of photons from the rectilinear basis to the Bell state basis and vice versa provided the photon pairs are in a known coherent superposition with the vacuum. This transformation relies on a recently reported effective nonlinearity at the single-photon level [20]. Requiring the photon pair to be in a superposition with the vacuum seems unusual, but this type of superposition exists in all down-conversion sources of entangled photons. It is only upon performing a photon-counting coincidence measurement that the maximally-entangled behaviour is projected out. While these down-conversion sources of Bell states exist and are practical in the lab, the creation mechanism does not suggest how one might try to measure those Bell states. In the device discussed here, the Bell state creator and Bell state analyzer look very similar. The creator can essentially be run in reverse to make the analyzer.

This device cannot be used for performing unconditional quantum teleportation. The device is only capable of distinguishing the four Bell states; it is not capable of performing a general projective measurement in the Bell basis. This is due to the conditional-phase shifter’s dependence on the magnitude and phase of the amplitude for the Bell state component in the input state; the gate does not operate properly on arbitrary superpositions of Bell states. Nevertheless, the device discussed herein constitutes a novel way of manipulating the degree of entanglement between a pair of photons, and may find a use in other quantum optics applications, such as dense coding [1, 2]. The ability to entangle and disentangle photon pairs is a crucial step toward building scalable all-optical quantum computers.

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Figure Captions

Fig. 1. a) A quantum circuit and b) its optical analogue for the creation of Bell states from product states. a) The quantum circuit acts on a pair of input modes 1 and 2. The circuit uses one-qubit Hadamard gates, and a two-qubit controlled-$\pi$ gate. This circuit performs a unitary transformation on the inputs and takes each of the four possible qubit product states to a different Bell state. b) The optical analogue of the quantum circuit. In the diagram, $\lambda/2$ are half-wave plates oriented at 22.5 degrees and $\chi^{(2)}$ is a nonlinear material. The device is capable of converting the state of a photon pair in a product state of polarization to one of the Bell states, provided that the input is in the correct superposition with the vacuum.

Fig. 2. a) A quantum circuit and b) its optical analogue for the conversion of Bell states to product states. a) This quantum circuit takes a pair of qubits in input modes 1 and 2 and performs a unitary transformation that will convert a Bell state to a product state. b) The optical analogue of the quantum circuit takes a photon pair in a Bell state to a rectilinear product state, provided the photon pair is in the correct superposition with the vacuum.

Fig. 3. An optical device for distinguishing rectilinear basis states. This simple device can distinguish between the product states for the polarization of a pair of photons $|H\rangle_1|H\rangle_2$, $|H\rangle_1|V\rangle_2$, $|V\rangle_1|H\rangle_2$, and $|V\rangle_1|V\rangle_2$, where the subscripts 1 and 2 are mode labels. The device consists of a pair of polarizing beam-splitters (PBS) and 4 photon counting detectors monitoring their outputs. For example, the detection of a photon at detector 1 and detector 4 corresponds to the state $|H\rangle_1|V\rangle_2$.

Fig. 4. Schematic for the conditional-phase switch. A strong, classical, laser in mode $p$, of frequency $2\omega$, pumps a $\chi^{(2)}$ nonlinear material such that it can create down-conversion pairs in modes 1 and 2. A pair of input beams, of frequency $\omega$, pass through the nonlinear material into modes 1 and 2. Interference between the multiple paths leading to photon pairs at the output can be used to introduce a large phase shift on the amplitude for a photon pair.
(2)
