Further study on the conformable fractional Gauss hypergeometric function

Mahmoud Abul-Ez\textsuperscript{a,}\textsuperscript{*}, Mohra Zayed\textsuperscript{b}\textsuperscript{†} and Ali Youssef\textsuperscript{c,}\textsuperscript{‡}

\textsuperscript{a,c} Mathematics Department, Faculty of Science, Sohag University, Sohag 82524, Egypt.
\textsuperscript{b} Mathematics Department, College of Science, King Khalid University, Abha, Saudi Arabia.

Abstract

This paper presents a somewhat exhaustive study on the conformable fractional Gauss hypergeometric function (CFGHF). We start by solving the conformable fractional Gauss hypergeometric equation (CFGHE) about the fractional regular singular points $x = 1$ and $x = \infty$. Next, various generating functions of the CFGHF are established. We also develop some differential forms for the CFGHF. Subsequently, differential operators and the contiguous relations are reported. Furthermore, we introduce the conformable fractional integral representation and the fractional Laplace transform of CFGHF. As an application, and after making a suitable change of the independent variable, we provide general solutions of some known conformable fractional differential equations, which could be written by means of the CFGHF.

Keywords: Special functions; Gauss hypergeometric functions; Conformable fractional calculus; Fractional differential equations.

1 Introduction

We believe that the old topics in classical analysis of special functions had been generalized to either the fractional calculus or the higher dimensional setting. Fractional calculus has recently attracted considerable attention. It is defined as a generalization of differentiation and integration to an arbitrary order. It has become a fascinating branch of applied mathematics, which has recently stimulated mathematicians and physicists. Indeed, it represents a powerful tool to study a myriad of problems from different fields of science, such as statistical mechanics, control theory, signal and image processing, thermodynamics and quantum mechanics (see [1–13]).
Over the last four decades, several interesting and useful extensions of many of the familiar special functions, such as the Gamma, Beta, and Gauss hypergeometric functions were considered by various authors of whom we may mention \[14–18\]. Functions of hypergeometric type constitute an important class of special functions. The hypergeometric function \(2F_1(\mu, \nu; c; x)\) plays a significant role in mathematical analysis and its applications. This function allows one to solve many interesting mathematical and physical problems, such as conformal mapping of triangular domains bounded by line segments or circular arcs and various problems of quantum mechanics. Most of the functions that occur in the analysis are classified as special cases of the hypergeometric functions. Gauss first introduced and studied hypergeometric series, paying particular attention to the cases when a series converges to an elementary function which leads to study the hypergeometric series. Eventually, elementary functions and several other important functions in mathematics can be expressed in terms of hypergeometric functions. Hypergeometric functions can also be described as the solutions of special second-order linear differential equations, which are the hypergeometric differential equations. Riemann was the first to exploit this idea and introduced a special symbol to classify hypergeometric functions by singularities and exponents of differential equations. The hypergeometric function is a solution of the following Euler’s hypergeometric differential equation

\[
x(1-x)\frac{d^2y}{dx^2} + [c - (\mu + \nu + 1)x] \frac{dy}{dx} - \mu\nu y = 0, \tag{1.1}
\]

which has three regular singular points 0, 1, and \(\infty\). The generalization of this equation to three arbitrary regular singular points is given by Riemann’s differential equation. Any second order differential equation with three regular singular points can be converted to the hypergeometric differential equation by changing of variables.

Recently, as a conformable fractional derivative introduced of \[21\], the authors in \[19\] used the new concept of fractional regular singular points with the technique of fractional power series to solve the CFGHDE about \(x = 0\). They also introduced the form of the conformable fractional derivative and the integral representation of the fractional Gaussian function. Besides, the solution of the fractional \(k\)–hypergeometric differential equation was introduced in \[20\]. As the Gauss hypergeometric differential equation appears in many problems of physics, engineering, applied science, as well as finance and many other important problems, it largely motivates us to conduct the present study.

In the present paper, we intend to continue the work of Abu Hammad et al. \[19\] by finding the solutions of CFGHE about the fractional regular singular points \(x = 1\) and \(x = \infty\). Afterward, we give a wide study on the CFGHF as follows. First, various generating functions of the CFGHF are established. Besides, some differential forms are developed for the CFGHF. Then, differential operators and contiguous relations are derived. Furthermore, we introduce the conformable fractional integral representation and the fractional Laplace transform of the CFGHF. As an application, and after making a suitable change to the independent variable, we derive the general solutions of some conformable fractional differential equations (CFDE), which could be written in terms of the CFGHF.
2 Preliminaries and basic concepts

Many definitions of fractional derivatives are obtained and compared. The most popular definitions of fractional derivatives are due to Riemann-Liouville and Caputo. It is pointed out that these definitions used an integral form and lacked some basic properties, such as product rule, quotient rule, and chain rule.

In 2014, Khalil et al. [21] introduced a surprising and satisfying definition of the fractional derivative that is analog to the classical derivative definition called conformable fractional derivative (CFD). Their definition runs, as follows:

**Definition 2.1.** Let $f : \Omega \subseteq (0, \infty) \to \mathbb{R}$ and $x \in \Omega$. The conformable fractional derivative of order $\alpha \in (0, 1]$ for $f$ at $x$ is defined as

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h},$$

whenever the limit exists. The function $f$ is called $\alpha-$ conformable fractional differentiable at $x$. For $x = 0$, $D^{\alpha} f(0) = \lim_{h \to 0^+} D^{\alpha} f(x)$ if such a limit exists.

This definition carries very important and natural properties. Let $D^{\alpha}$ denote the conformable fractional derivative (CFD) operator of order $\alpha$. We recall from [21–23] some of its general properties as follows.

- **Linearity:** $D^{\alpha} (af + bg)(t) = aD^{\alpha} f(t) + bD^{\alpha} g(t)$, for all $a, b \in \mathbb{R}$.
- **Product rule:** $D^{\alpha} (fg)(t) = f(t)D^{\alpha} g(t) + g(t)D^{\alpha} f(t)$.
- **Quotient rule:** $D^{\alpha} \left( \frac{f}{g} \right)(t) = \frac{g(t)D^{\alpha} f(t) - f(t)D^{\alpha} g(t)}{g^2(t)}$, where $g(t) \neq 0$.
- **Chain rule:** $D^{\alpha} (f \circ g)(t) = D^{\alpha} f(g(t)) . D^{\alpha} g(t) . g(t)^{\alpha - 1}$.

Notice that for $\alpha = 1$ in the $\alpha-$ conformable fractional derivative, we get the corresponding classical limit definition of the derivative. Also, a function could be $\alpha-$ conformable differentiable at a point but not differentiable in the ordinary sense. For more details, we refer to [21–22, 24].

Any linear homogenous differential equations of order two with three regular singularities can be reduced to (1.1). The hypergeometric function is known as a solution to the hypergeometric equation (1.1). One of the solutions of the hypergeometric equation is given by the following Gauss hypergeometric series in the form

$$2F_1(\mu, \nu; c; x) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} x^n \quad (|x| < 1) \quad (2.1)$$

where $(b)_n$ stands for the usual Pochhammer symbol defined by

$$(b)_n = b (b + 1) (b + 2) \ldots (b + n - 1) = \frac{\Gamma(b + n)}{\Gamma(b)}, \quad n \in \mathbb{N} \text{ and } (b)_0 = 1.$$
Choosing the values of the parameters $\mu, \nu$, and $c$ in an appropriately, one can obtain many elementary and special functions as particular cases of the Gauss hypergeometric series. For instance, the complete elliptic integrals of the first and the second kinds, the Legendre associated functions, ultra-spherical polynomials, and many others are special cases of the function $\text{}_2F_1(\mu, \nu; c; x)$.

**Definition 2.2.** Two hypergeometric functions are said to be contiguous if their parameters $\mu, \nu$, and $c$ differ by integers. The relations made by contiguous functions are said to be contiguous function relations.

**Definition 2.3.** The point $x = a$ is called an $\alpha$–regular singular point for the equation

$$D^\alpha D^\alpha y + P(x) D^\alpha y + Q(x) y = 0,$$

(2.2)

if $\lim_{x \to a^+} (x^\alpha - a) P(x)$ and $\lim_{x \to a^+} (x^\alpha - a)^2 Q(x)$ exist.

**Definition 2.4.** A series $\sum_{n=0}^{\infty} a_n x^{\alpha n}$ is called a fractional Maclaurin power series.

**Remark 2.1.** We will use $D^{n\alpha}$ to denote $D^\alpha D^\alpha \ldots D^\alpha$ $n$–times. If $D^{n\alpha} f$ exists for all $n$ in some interval $[0, \lambda]$ then one can write $f$ in the form of a fractional power series

**Definition 2.5.** Suppose that $f : (0, \infty) \to \mathbb{R}$ is $\alpha$–differentiable, $\alpha \in (0, 1]$, then the $\alpha$–fractional integral of $f$ is defined by

$$I^\alpha_1 f(t) = I^\alpha_1 (t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx, \; t \geq 0.$$

For the infinite double series, we have the following useful Lemma (see [26]), which will be used in the sequel.

**Lemma 2.1.**

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{k,n} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} a_{j,m-j} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k,n-k}$$

(2.3)

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{k,n+k}$$

(2.4)

**3 Solutions of the conformable fractional Gauss hypergeometric differential equation**

In our current study, we are interested to consider a generalization of the differential equation (1.1) to fractional Gauss hypergeometric differential equation, where the involving derivative is CFD. More precisely, we study the equation in the form

$$x^\alpha (1 - x^\alpha) D^\alpha D^\alpha y + \alpha [c - (\mu + \nu + 1) x^\alpha] D^\alpha y - \alpha^2 \mu \nu y = 0,$$

(3.1)
where $\alpha \in (0, 1]$ and $\mu, \nu$ and $c$ are reals.

The new concept of fractional regular singular point together with the technique of fractional power series are used to solve the CFGHDE \((3.1)\).

Dividing \((3.1)\) by $x^\alpha (1 - x^\alpha)$, we get

\[
D^\alpha D^\alpha y + \frac{\alpha \left\{ c - (\mu + \nu + 1) x^\alpha \right\}}{x^\alpha (1 - x^\alpha)} D^\alpha y - \frac{\alpha^2 \mu \nu}{x^\alpha (1 - x^\alpha)} y = 0.
\]  

(3.2)

Comparing \((3.2)\) with \((2.2)\), we have

\[
P(x) = \frac{\alpha \left\{ c - (\mu + \nu + 1) x^\alpha \right\}}{x^\alpha (1 - x^\alpha)} \quad \text{and} \quad Q(x) = \frac{-\alpha^2 \mu \nu}{x^\alpha (1 - x^\alpha)}.
\]

Clearly $x = 0$, $x = 1$ and $x = \infty$ are $\alpha$–regular singular points for \((3.1)\).

In 2020, the authors in \([19]\) used the technique of fractional power series to obtain the general solution of \((3.1)\) about $x = 0$ as

\[
y = A \, _2F_1(\mu, \nu; c; x^\alpha) + B \, x^\alpha (1 - c) \, _2F_1(1 - c + \mu, 1 - c + \nu; 2 - c; x^\alpha),
\]

where $A$ and $B$ are arbitrary constants and $\, _2F_1(\mu, \nu; c; x^\alpha)$ is CFGHF defined by

\[
_2F_1(\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} x^\alpha; \quad |x^\alpha| < 1.
\]  

(3.3)

In fact, we will use a similar technique of \([19]\) to solve the equation \((3.1)\) about the two $\alpha$–regular singular points $x = 1$ and $x = \infty$.

### 3.1 Solution of the CFGHE about $x = 1$

As $x = 1$ is an $\alpha$–regular singular point of \((3.1)\), therefore, the solution of \((3.1)\) can be obtained in a series of powers of $(x^\alpha - 1)$ as follows:

Taking $x^\alpha = 1 - t^\alpha$, this transfers the point $x = 1$ to the point $t = 0$ and therefore, we obtain the series solution of the following transformed CFDE in terms of the series of powers of $t^\alpha$:

\[
t^\alpha (1 - t^\alpha) \, D^\alpha D^\alpha y(t) + \alpha \left\{ [\mu + \nu + 1 - c] - (\mu + \nu + 1) t^\alpha \right\} D^\alpha y(t) - \alpha^2 \mu \nu y(t) = 0.
\]  

(3.4)

Putting $c' = \mu + \nu + 1 - c$, in \((3.4)\), we get

\[
t^\alpha (1 - t^\alpha) \, D^\alpha D^\alpha y(t) + \alpha \left\{ c' - (\mu + \nu + 1) t^\alpha \right\} D^\alpha y(t) - \alpha^2 \mu \nu y(t) = 0.
\]  

(3.5)

This conformable fractional differential equation is similar to CFGHE \((3.1)\). So, the two linearly independent solutions of \((3.5)\) can be stated in the form

\[
y_1 = \, _2F_1(\mu, \nu; c'; t^\alpha) \quad \text{and} \quad y_2 = t^\alpha (1 - c') \, _2F_1(1 - c' + \mu, 1 - c' + \nu; 2 - c'; t^\alpha)
\]  

(3.6)
Now, replacing $c'$ by $(\mu + \nu + 1 - c)$ and $t^\alpha$ by $(1 - x^\alpha)$ in (3.6), we get
\[ y_1 = _2F_1 (\mu; \nu; \mu + \nu + 1 - c; t^\alpha) \]
and
\[ y_2 = (1 - x^\alpha)^{(c-\mu-\nu)} _2F_1 (c - \nu, c - \mu; c - \mu - \nu + 1; 1 - x^\alpha) \]
Thus the general solution of equation (3.1) about $x = 1$ is given by
\[ y = A _2F_1 (\mu; \nu; \mu + \nu + 1 - c; t^\alpha) + B (1 - x^\alpha)^{(c-\mu-\nu)} _2F_1 (c - \nu, c - \mu; c - \mu - \nu + 1; 1 - x^\alpha), \] (3.7)
where $A$ and $B$ are arbitrary constants.

### 3.2 Solution of the CFGHE about $x = \infty$

As $x = \infty$ is an $\alpha-$regular singular point of (3.1), thus, the solution of (3.1) can be obtained in a series about $x = \infty$ by putting $x^\alpha = \frac{1}{\zeta^\alpha}$ in (3.1). Therefore,
\[ D_x^\alpha y = -\zeta^2 D_\zeta^\alpha y \text{ and } D_x^\alpha D_x^\alpha y = \left[ 2\alpha \zeta^3 D_\zeta^\alpha y + \zeta^{4\alpha} D_\zeta^\alpha D_\zeta^\alpha y \right]. \] (3.8)

In view of (3.1), we get
\[ \zeta^{2\alpha} (1 - \zeta^\alpha) D_\zeta^\alpha D_\zeta^\alpha y + \alpha \{ 2\zeta^\alpha (1 - \zeta^\alpha) + c \zeta^{2\alpha} - \zeta^\alpha (\mu + \nu + 1) \} D_\zeta^\alpha y + \alpha^2 \mu \nu y = 0 \] (3.9)

Now, to find the solution, we proceed as follows. Let $y = \sum_{n=0}^{\infty} a_n \zeta^{\alpha(n+s)}; a_0 \neq 0$ be the series solution of equation (3.9) about $\zeta = 0$. Then from the basic properties of the CFD we get
\[ D_\zeta^\alpha y = \sum_{n=0}^{\infty} a_n \alpha (s + n) \zeta^{\alpha(s+n-1)} \text{ and } D_\zeta^\alpha D_\zeta^\alpha y = \sum_{n=0}^{\infty} a_n \alpha^2 (s + n) (s + n - 1) \zeta^{\alpha(s+n-2)} \]
Thus, owing to (3.9), we have
\[ \zeta^{2\alpha} (1 - \zeta^\alpha) \sum_{n=0}^{\infty} a_n \alpha^2 (s + n) (s + n - 1) \zeta^{\alpha(s+n-2)} + \alpha \{ 2\zeta^\alpha (1 - \zeta^\alpha) + c \zeta^{2\alpha} - \zeta^\alpha (\mu + \nu + 1) \} \]
\[ \times \sum_{n=0}^{\infty} a_n \alpha (s + n) \zeta^{\alpha(s+n-1)} + \alpha^2 \mu \nu \sum_{n=0}^{\infty} a_n \zeta^{\alpha(s+n)} = 0, \]
therefore,
\[ \sum_{n=0}^{\infty} \alpha^2 a_n [(s + n) (s + n - 1) + 2 (s + n) - (\mu + \nu + 1) (s + n) + \mu \nu] \zeta^{\alpha(s+n)} \]
\[ - \sum_{n=0}^{\infty} \alpha^2 a_n [(s + n) (s + n - 1) + 2 (s + n) - c (s + n)] \zeta^{\alpha(s+n+1)} = 0. \]
Then we have
\[
\begin{align*}
\alpha^2 a_0 & [s (s - 1) + 2 s - s (\mu + \nu + 1) + \mu \nu] \zeta_s \\
& + \sum_{n=1}^{\infty} \alpha^2 a_n [(s + n) (s + n - 1) + 2 (s + n) - (\mu + \nu + 1) (s + n) + \mu \nu] \zeta_{s+n} \\
& - \sum_{n=0}^{\infty} \alpha^2 a_n [(s + n) (s + n - 1) + 2 (s + n) - c (s + n)] \zeta_{s+n+1} = 0.
\end{align*}
\]

A shift of index yields
\[
\begin{align*}
\alpha^2 a_0 & [s (s - 1) + 2 s - s (\mu + \nu + 1) + \mu \nu] \zeta_s \\
& + \alpha^2 \sum_{n=0}^{\infty} a_{n+1} [(s + n + 1) (s + n) + 2 (s + n + 1) - (\mu + \nu + 1) (s + n + 1) + \mu \nu] \zeta_{s+n+1} = 0
\end{align*}
\] (3.10)

Equating the coefficients of \( \zeta_s \) to zero in (3.10), we get the following indicial equation
\[
s^2 - s (\mu + \nu) + \mu \nu = 0
\] (3.11)

This equation (3.11) has two indicial roots \( s = s_1 = \mu \) and \( s = s_2 = \nu \).

Again, equating to zero the coefficient of \( \zeta_{s+n+1} \) in (3.10), yields the recursion relation for \( a_n \)
\[
a_{n+1} = \frac{[(s + n) (s + n - 1) + 2 (s + n) - c (s + n)]}{[(s + n + 1) (s + n) + 2 (s + n + 1) - (\mu + \nu + 1) (s + n + 1) + \mu \nu]} a_n,
\] (3.12)

or
\[
a_{n+1} = \frac{(s + n) (s + n + 1 - c)}{(s + n + 1) [(s + n + 1) - \mu - \nu + \mu \nu]} a_n
\] (3.13)

To find the first solution of (3.9), putting \( s = \mu \) in (3.13), we get
\[
a_{n+1} = \frac{(\mu + n) (\mu + n + 1 - c)}{(\mu + n + 1) [(n + 1) - \nu + \mu \nu]} a_n
\]

Note that if \( n = 0 \), one can see
\[
a_1 = \frac{(\mu) (\mu - c + 1)}{[\mu - \nu + 1]} a_0
\]

and for \( n = 1 \), we obtain
\[
a_2 = \frac{(\mu + 1) (\mu - c + 2)}{[\mu - \nu + 2]} a_1 = \frac{(\mu) (\mu + 1) (\mu - c + 1) (\mu - c + 2)}{2 [\mu - \nu + 1] [\mu - \nu + 2]} a_0
\]
Using the Pochhammer symbol we have

\[ a_2 = \frac{(\mu)_2 (\mu - c + 1)_2}{2! (\mu - \nu + 1)_2} a_0. \]

In general, we may write

\[ a_n = \frac{(\mu)_n (\mu - c + 1)_n}{n! (\mu - \nu + 1)_n} a_0. \]  \hspace{1cm} \text{(3.14)}

Letting \( a_0 = A \), the first solution \( y_1 \) is given by

\[ y_1 = A \sum_{n=0}^{\infty} \frac{(\mu)_n (\mu - c + 1)_n}{(\mu - \nu + 1)_n} \frac{\zeta \alpha (\mu + n)}{n!} = A \zeta \alpha \ \frac{2F_1 (\mu, \mu - c + 1; \mu - \nu + 1; \zeta \alpha)}{n!} \]

\[ = A x^{-\alpha \mu} \ 2F_1 \left( \mu, \mu - c + 1; \mu - \nu + 1; \frac{1}{x^\alpha} \right) \]

To find the second solution of (3.9), putting \( s = \nu \) in (3.13), we have

\[ a_{n+1} = \frac{(\nu + n) (\nu + n + 1 - c)}{(\nu + n + 1) [ (n + 1) - \mu + \nu \mu]} a_n = \frac{(\nu + n) (\nu + n + 1 - c)}{(n + 1) [ (n + 1) + \nu - \mu]} a_n, \]

from which we get

\[ a_1 = \frac{(\nu) (\nu - c + 1)}{[\nu - \mu + 1]} a_0. \]

Thus

\[ a_2 = \frac{(\nu + 1) (\nu - c + 2)}{[\nu - \mu + 2]} \frac{a_1}{a_0} = \frac{(\nu + 1) (\nu - c + 1) (\nu - c + 2)}{2 [\nu - \mu + 1] [\nu - \mu + 2]} a_0. \]

Again by Pochhammer symbol yields

\[ a_2 = \frac{(\nu)_2 (\nu - c + 1)_2}{2! (\nu - \mu + 1)_2} a_0 \]

and in general

\[ a_n = \frac{(\nu)_n (\nu - c + 1)_n}{n! (\nu - \mu + 1)_n} a_0 \]  \hspace{1cm} \text{(3.15)}

Putting \( a_0 = B \), the second solution \( y_2 \) is given by

\[ y_2 = B \sum_{n=0}^{\infty} \frac{(\nu)_n (\nu - c + 1)_n}{(\nu - \mu + 1)_n} \frac{\zeta \alpha (\nu + n)}{n!} = B \zeta \alpha \ \frac{2F_1 (\nu, \nu - c + 1; \nu - \mu + 1; \zeta \alpha)}{n!} \]

\[ = B x^{-\alpha \nu} \ 2F_1 \left( \nu, \nu - c + 1; \nu - \mu + 1; \frac{1}{x^\alpha} \right) \]

Therefore, the general solution of (3.9) about \( x = \infty \) is

\[ y = A x^{-\alpha \mu} \ 2F_1 \left( \mu, \mu - c + 1; \mu - \nu + 1; \frac{1}{x^\alpha} \right) + B x^{-\alpha \nu} \ 2F_1 \left( \nu, \nu - c + 1; \nu - \mu + 1; \frac{1}{x^\alpha} \right) \]

where \( A \) and \( B \) are arbitrary constants.
Remark 3.1. It is worthy to mention that the presented CFDE (3.9) is distinct to that one which was treated in \cite{19, 20}. In fact (3.9) extended the Gauss hypergeometric differential equation given in \cite{31} to the conformable fractional context.

4 Generating functions

Generating functions are important way to transform formal power series into functions and to analyze asymptotic properties of sequences. In what follows we characterize the CFGHF by means of various generating functions.

Theorem 4.1. For $\alpha \in (0, 1]$, the following generating function holds true

$$\sum_{m=0}^{\infty} (\mu)_m \ 2F1 (\mu + m, \nu; x^\alpha) \cdot \frac{n^\alpha}{m!} = (1 - t^\alpha)^{-\mu} \ 2F1 \left( \mu, \nu; c; \frac{x^\alpha}{1 - t^\alpha} \right),$$  \quad (4.1)

where $|x^\alpha| < 1$, and $|t^\alpha| < 1$.

Proof. For convenience, let $\mathcal{Z}$ denote the left-hand side of (4.1). In view of (3.3), we have

$$\mathcal{Z} = \sum_{m=0}^{\infty} (\mu)_m \left\{ \sum_{n=0}^{\infty} \frac{(\mu + m)_n (\nu)_n x^\alpha}{n!} \right\} \cdot \frac{n^\alpha}{m!}. \quad (4.2)$$

Changing the order of summations in (4.2) and make use of identity $(\mu)_m (\mu + m)_n = (\mu)_{m+n} = (\mu)_n (\mu + n)_m$, yields

$$\mathcal{Z} = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n x^\alpha}{n!} \cdot \sum_{m=0}^{\infty} \frac{(\mu + n)_m}{m!} \ t^\alpha m. \quad (4.3)$$

Using the equality $\sum_{m=0}^{\infty} \frac{(\mu+n)_m}{m!} \ t^\alpha m = (1 - t^\alpha)^{-(\mu+n)}$; $|t^\alpha| < 1$ and the definition (3.3) immediately leads to the required result. \hfill \square

Theorem 4.2. For $\alpha \in (0, 1]$, we have the following relation

$$\sum_{m=0}^{\infty} (\mu)_m \ 2F1 (-m, \nu; x^\alpha) \cdot \frac{n^\alpha}{m!} = (1 - t^\alpha)^{-\mu} \ 2F1 \left( \mu, \nu; c; \frac{-x^\alpha t^\alpha}{1 - t^\alpha} \right),$$  \quad (4.3)

where $|x^\alpha| < 1$, and $|t^\alpha| < 1$

Proof. For short, set $\mathcal{Z}$ to denote the left-hand side of (4.3). Using (3.3), one gets

$$\mathcal{Z} = \sum_{m=0}^{\infty} (\mu)_m \left\{ \sum_{n=0}^{\infty} \frac{(-m)_n (\nu)_n x^\alpha}{n!} \right\} \cdot \frac{n^\alpha}{m!}. \quad (4.4)$$
Since \((-m)_n = 0\) if \(n > m\), then we may write

\[
\zeta = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(\mu)_m}{m!} \frac{(-m)_n}{(c)_n} \frac{(\nu)_n}{n!} x^\alpha t^{\alpha m}
\]

Using the congruence relation \((-m)_n = \frac{(-1)^n m!}{(m-n)!}\), we get

\[
\zeta = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^n (\mu)_m}{(c)_n} \frac{(\nu)_n}{n!} x^\alpha t^{\alpha m} \tag{4.5}
\]

Using lemma 2.1 equation (4.5) becomes

\[
\zeta = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (\mu + m)_m}{(c)_m} \frac{(\nu)_n}{n!} x^\alpha t^{\alpha (n+m)} \tag{4.6}
\]

Changing the order of summations in (4.6) and make use of identity \((\mu + m)_m = (\mu)_n (\mu + n)_m\), we get

\[
\zeta = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n} \frac{(-x^\alpha t^\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(\mu + n)_m}{m!} t^{\alpha m}
\]

Using the binomial relation \(\sum_{m=0}^{\infty} \frac{(\mu + n)_m}{m!} t^{\alpha m} = (1 - t^\alpha)^{-(\mu + n)}\), \((|t^\alpha| < 1)\), it follows that

\[
\zeta = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n} \frac{(-x^\alpha t^\alpha)_n}{n!} (1 - t^\alpha)^{-(\mu + n)} = (1 - t^\alpha)^{-\mu} \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n} \frac{(-x^\alpha t^\alpha)_n}{n!} \left(\frac{1}{1 - t^\alpha}\right)
\]

\[
= (1 - t^\alpha)^{-\mu} \, _2F_1\left(\mu, \nu; c; \frac{-x^\alpha t^\alpha}{1 - t^\alpha}\right). \tag{4.7}
\]

\[\square\]

**Theorem 4.3.** For \(\alpha \in (0, 1]\), the following generating relation is valid

\[
\sum_{m=0}^{\infty} \frac{(\mu)_m (\nu)_m}{(c)_m} \frac{x^{\alpha m}}{m!} = _2F_1\left(\mu + m, \nu + m; c + m; x^\alpha\right) \cdot \frac{t^{\alpha m}}{m!} = _2F_1\left(\mu, \nu; c; x^\alpha + t^\alpha\right), \tag{4.7}
\]

where \(|x^\alpha| < 1\), \(|t^\alpha| < 1\), and \(|x^\alpha + t^\alpha| < 1\).

**Proof.** Let \(\zeta\) denote the left-hand side of (4.7), then using (3.3), we obtain

\[
\zeta = \sum_{m=0}^{\infty} \left(\frac{(\mu)_m (\nu)_m}{(c)_m} \left\{ \sum_{n=0}^{\infty} \frac{(\mu + m)_n (\nu + m)_n}{(c + m)_n} \frac{x^{\alpha n}}{n!} \right\} \right) \cdot \frac{t^{\alpha m}}{m!}
\]

With the help of \((\mu + m)_n = (\mu)_n (\mu + n)_n\), we get

\[
\zeta = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu + n)_n (\nu + n)_n}{(c)_n + n!} x^{\alpha n} t^{\alpha m}.
\]
Using lemma 2.1 we have
\[ \Im = \sum_{m=0}^{\infty} \sum_{n=0}^{m} (\mu)_{m} \frac{(\nu)_{m}}{(c)_{m} n! (m-n)!} x^{\alpha} t^{\alpha (m-n)} \]
\[ = \sum_{m=0}^{\infty} (\mu)_{m} (\nu)_{m} \sum_{n=0}^{m} \frac{m!}{n! (m-n)!} x^{\alpha} t^{\alpha (m-n)}. \]

The binomial theorem immediately gives
\[ \Im = \sum_{m=0}^{\infty} (\mu)_{m} (\nu)_{m} (x^{\alpha} + t^{\alpha})^{m} \]
\[ = \text{2F1} (\mu, \nu; c; x^{\alpha} + t^{\alpha}) \]
as required. \( \square \)

5 Transmutation formulas and differential forms

5.1 Transmutation formulas

Theorem 5.1. For \(|x^{\alpha}| < 1\) and \(|\frac{x^{\alpha}}{1-x^{\alpha}}| < 1\), the following identity is satisfied
\[ \text{2F1} (\mu, \nu; c; x^{\alpha}) = (1-x^{\alpha})^{-\mu} \text{2F1} \left( \mu, c-\nu; c; \frac{-x^{\alpha}}{1-x^{\alpha}} \right). \quad \text{(5.1)} \]

Proof. Consider
\[ (1-x^{\alpha})^{-\mu} \text{2F1} \left( \mu, c-\nu; c; \frac{-x^{\alpha}}{1-x^{\alpha}} \right) = \sum_{k=0}^{\infty} \frac{(-1)^{k} (\mu)_{k} (c-\nu)_{k}}{(c)_{k} k!} x^{\alpha k} (1-x^{\alpha})^{-(k+\mu)}. \]

In view of the expansion \((1-x^{\alpha})^{-\mu} = \sum_{n=0}^{\infty} \frac{(\mu)_{n}}{n!} x^{\alpha n}; \ |x^{\alpha}| < 1\), we may write
\[ (1-x^{\alpha})^{-\mu} \text{2F1} \left( \mu, c-\nu; c; \frac{-x^{\alpha}}{1-x^{\alpha}} \right) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k} (\mu)_{k} (c-\nu)_{k} (\mu+k)_{n}}{(c)_{k} k! n!} x^{\alpha (k+n)}. \]

Using the identity \((\mu+k)_{n} = (\mu)_{k+n}\), it is obvious
\[ (1-x^{\alpha})^{-\mu} \text{2F1} \left( \mu, c-\nu; c; \frac{-x^{\alpha}}{1-x^{\alpha}} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k} (\mu)_{k+n} (c-\nu)_{k} x^{\alpha (k+n)}}{(c)_{k} k! n!} \quad \text{(5.2)} \]

In virtue of lemma 2.1 and the fact \((-n)_{k} = \frac{(-1)^{k} n!}{(n-k)!}\) one easily gets
\[ (1-x^{\alpha})^{-\mu} \text{2F1} \left( \mu, c-\nu; c; \frac{-x^{\alpha}}{1-x^{\alpha}} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_{k} (c-\nu)_{k} (\mu)_{n} x^{\alpha n}}{(c)_{k} k! n!}. \quad \text{(5.3)} \]
Since \((-n)_k = 0\) if \(k > n\), then (5.3) becomes

\[
(1 - x^\alpha)^{-\mu} \, _2F_1 \left( \mu, c - \nu; c; \frac{-x^\alpha}{1 - x^\alpha} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-n)_k (c - \nu)_k (\mu)_n}{(c)_k k!} \frac{x^{\alpha n}}{n!}.
\]  

(5.4)

Since the inner sum on the right of (5.4) is a terminating hypergeometric series, then

\[
(1 - x^\alpha)^{-\mu} \, _2F_1 \left( \mu, c - \nu; c; \frac{-x^\alpha}{1 - x^\alpha} \right) = \sum_{n=0}^{\infty} 2F_1 (-n, c - \nu; 1) \frac{\mu^{\alpha n}}{n!}.
\]  

(5.5)

Due to \(2F_1 (-n, c - \nu; 1) = \frac{(\nu)_n}{(c)_n}\), the proof is therefore completed.

\[\square\]

**Theorem 5.2.** For \(|x^\alpha| < 1\), the following identity is true

\[
2F_1 (\mu, \nu; c; x^\alpha) = (1 - x^\alpha)^{\mu - \nu - c} \, 2F_1 (c - \mu, c - \nu; c; x^\alpha)
\]  

(5.6)

**Proof.** By using assertion of theorem 5.1 and assuming that \(y^\alpha = \frac{-x^\alpha}{1 - x^\alpha}\), it follows that

\[
2F_1 (\mu, c - \nu; c; y^\alpha) = (1 - y^\alpha)^{-(c - \nu)} \, 2F_1 \left( c - \mu, c - \nu; c; \frac{-y^\alpha}{1 - y^\alpha} \right).
\]  

(5.7)

From the assumption, we have \(x^\alpha = \frac{-y^\alpha}{1 - y^\alpha}\) which gives

\[
2F_1 \left( \mu, c - \nu; c; \frac{-x^\alpha}{1 - x^\alpha} \right) = (1 - x^\alpha)^{(c - \nu)} \, 2F_1 (c - \mu, c - \nu; c; x^\alpha).
\]  

(5.8)

A combinations of (5.8) and (5.1), the result follows.

\[\square\]

5.2 Some differential forms

Now, according to the notation \(D^{\alpha n}\), and due to the fact \(D^{\alpha} x^p = px^{\alpha - p}\) with \(\alpha \in (0, 1]\) and \(|x^\alpha| < 1\), we state some interesting conformable fractional differential formulas for \(2F_1 (\mu, \nu; c; x^\alpha)\) as follows

\[
D^{\alpha} \, _2F_1 (\mu, \nu; c; x^\alpha) = \frac{\alpha \mu \nu}{c} \, _2F_1 (\mu + 1, \nu + 1; c + 1; x^\alpha)
\]  

(5.9)

\[
D^{\alpha n} \, _2F_1 (\mu, \nu; c; x^\alpha) = \frac{\alpha^n (\mu)_n (\nu)_n}{(c)_n} \, _2F_1 (\mu + n, \nu + n; c + n; x^\alpha)
\]  

(5.10)

\[
D^{\alpha} \left\{ x^{\alpha (\mu + n - 1)} \, _2F_1 (\mu, \nu; c; x^\alpha) \right\} = \alpha^n (\mu)_n \, x^{\alpha (\mu - 1)} \, _2F_1 (\mu + n, \nu; c; x^\alpha)
\]  

(5.11)

\[
D^{\alpha} \left\{ x^{\alpha (c - 1)} \, _2F_1 (\mu, \nu; c; x^\alpha) \right\} = \alpha^n (c - n)_n \, x^{\alpha (c - n - 1)} \, _2F_1 (\mu, \nu; c - n; x^\alpha)
\]  

(5.12)

\[
D^{\alpha} \left\{ x^{\alpha (c - \mu + n - 1)} (1 - x^\alpha)^{\mu + \nu - c} \, _2F_1 (\mu, \nu; c; x^\alpha) \right\} = \alpha^n (c - \mu)_n \, x^{\alpha (c - \mu - 1)} (1 - x^\alpha)^{\mu + \nu - c - n} \, _2F_1 (\mu - n, \nu; c; x^\alpha)
\]  

(5.13)
\( D^{n\alpha} \{ (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \frac{\alpha^n (c - \mu)_n (c - \gamma)_n}{(c)_n} (1 - x^\alpha)^{\mu + \nu - c - n} \, 2F_1 (\mu, \nu; c + n; x^\alpha) \) \tag{5.14}

\[ D^{n\alpha} \{ x^{(\alpha - 1)} (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \alpha^n (c - n)_n \, x^{(\alpha - n - 1)} (1 - x^\alpha)^{\mu + \nu - c - n} \, 2F_1 (\mu - n, \nu - n; c - n; x^\alpha) \] \tag{5.15}

\[ D^{n\alpha} \{ x^{(n + \alpha - 1)} (1 - x^\alpha)^{n + \mu + \nu - c} \, 2F_1 (\mu + n, \nu + n; c + n; x^\alpha) \} = \alpha^n (c)_n \, x^{(\alpha - c - 1)} (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \] \tag{5.16}

Such formulas can be proved using the series expansions of \( 2F_1 (\mu, \nu; c; x^\alpha) \) as given in (5.3). However, we are going to prove the validity of (5.11) and (5.14), while the other formulas can be proved similarly. First, note that

\[ D^{n\alpha} \{ x^{\alpha (\mu + n - 1)} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \sum_{k=0}^{\infty} \frac{(\mu)_k (\nu)_k}{(c)_k k!} D^{n\alpha} \{ x^{\alpha (\mu + n + k - 1)} \} \]

The action of the conformable derivative gives

\[ D^{n\alpha} \{ x^{\alpha (\mu + n - 1)} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \sum_{k=0}^{\infty} \frac{(\mu)_k (\nu)_k}{(c)_k k!} \alpha^n \frac{\Gamma (\mu + n + k)}{\Gamma (\mu + k)} \, x^{\alpha (\mu + k - 1)} \]

\[ = \alpha^n \sum_{k=0}^{\infty} \frac{(\mu + n)_k (\nu)_k}{(c)_k k!} \, x^{\alpha (\mu + k - 1)} \]

Knowing that \((\mu)_n = (\mu)_n (\mu + n)_k\), it can be seen

\[ D^{n\alpha} \{ x^{\alpha (\mu + n - 1)} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \alpha^n (\mu)_n \, x^{\alpha (\mu - 1)} \sum_{k=0}^{\infty} \frac{(\mu + n)_k (\nu)_k}{(c)_k k!} \, x^{\alpha k} \]

\[ = \alpha^n (\mu)_n \, x^{\alpha (\mu - 1)} \, 2F_1 (\mu + n, \nu; c; x^\alpha) \]

as required.

In view of (5.6), we have

\[ D^{n\alpha} \{ (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = D^{n\alpha} \{ 2F_1 (\mu, \nu; c; x^\alpha) \} \] \tag{5.17}

Using (5.10), we get

\[ D^{n\alpha} \{ (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \frac{\alpha^n (c - \mu)_n (c - \nu)_n}{(c)_n} \, 2F_1 (\mu + n, c - \nu + n; c + n; x^\alpha) \] \tag{5.18}

Return to (5.6), we obtain

\[ D^{n\alpha} \{ (1 - x^\alpha)^{\mu + \nu - c} \, 2F_1 (\mu, \nu; c; x^\alpha) \} = \frac{\alpha^n (c - \mu)_n (c - \gamma)_n}{(c)_n} \, (1 - x^\alpha)^{\mu + \nu - c - n} \times \, 2F_1 (\mu, \nu; c + n; x^\alpha) \] \tag{5.19}

Remark 5.1. In case of \( \mu = -n \) in (5.16), we obtain

\[ D^{n\alpha} \{ x^{\alpha (n + c - 1)} \, (1 - x^\alpha)^{\nu - c} \} = \alpha^n (c)_n \, x^{\alpha (c - 1)} (1 - x^\alpha)^{\nu - c - n} \, 2F_1 (-n, \nu; c; x^\alpha) \] \tag{5.20}
6 Differential operator and contiguous relations

Following [26], define the conformable fractional operator $\theta^\alpha$ in the form

$$\theta^\alpha = \frac{1}{\alpha} x^\alpha D^\alpha. \quad (6.1)$$

This operator has the particularly pleasant property that $\theta^\alpha x^{n\alpha} = n x^{n\alpha}$, which makes it handy to be used on power series. In this section, relying on definition 2.2, we establish several results concerning contiguous relations for the CFGHF. To achieve that, we have to prove the following lemma.

**Lemma 6.1.** Let $\alpha \in (0, 1]$, then the CFGHF $2 \text{F}_1 (\mu, \nu; c; x^\alpha)$ satisfies the following

$$(\theta^\alpha + \mu)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \mu_2 \text{F}_1 (\mu + 1, \nu; c; x^\alpha) \quad (6.2)$$

$$(\theta^\alpha + \nu)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \nu_2 \text{F}_1 (\mu, \nu + 1; c; x^\alpha) \quad (6.3)$$

$$(\theta^\alpha + c - 1)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = (c - 1)_2 \text{F}_1 (\mu, \nu; c - 1; x^\alpha) \quad (6.4)$$

**Proof.** Using (3.3) and (6.1) it follows that

$$(\theta^\alpha + \mu)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} (\theta^\alpha + \mu) x^{\alpha n}$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} (n + \mu) x^{\alpha n} = \sum_{n=0}^{\infty} \frac{(\mu+n+1) (\nu)_n}{(c)_n n!} x^{\alpha n}$$

$$= \sum_{n=0}^{\infty} \frac{\mu (\mu+1)_n (\nu)_n}{(c)_n n!} x^{\alpha n} = \mu_2 \text{F}_1 (\mu + 1, \nu; c; x^\alpha).$$

Similarly, we have

$$(\theta^\alpha + \nu)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \nu_2 \text{F}_1 (\mu, \nu + 1; c; x^\alpha).$$

Along the same way

$$(\theta^\alpha + c - 1)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} (\theta^\alpha + c - 1) x^{\alpha n} = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} (n + c - 1) x^{\alpha n}.$$ 

Therefore,

$$(\theta^\alpha + c - 1)_2 \text{F}_1 (\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} x^{\alpha n}$$

$$= \sum_{n=0}^{\infty} \frac{(c-1)_n (\nu)_n}{(c-1)_n n!} x^{\alpha n} = (c - 1)_2 \text{F}_1 (\mu, \nu; c - 1; x^\alpha).$$

□
The following result is a consequence of Lemma 6.1.

**Theorem 6.1.** Let $\alpha \in (0, 1]$, then the CFGHF $\, _2F_1 (\mu, \nu; c; x^\alpha) \,$ satisfies the following contiguous relations

\[
(\mu - \nu) \, _2F_1 (\mu, \nu; c; x^\alpha) = \mu \, _2F_1 (\mu + 1, \nu; c; x^\alpha) - \nu \, _2F_1 (\mu, \nu + 1; c; x^\alpha)
\]  

(6.5)

and

\[
(\mu + c - 1) \, _2F_1 (\mu, \nu; c; x^\alpha) = \mu \, _2F_1 (\mu + 1, \nu; c; x^\alpha) - (c - 1) \, _2F_1 (\mu, \nu - 1; x^\alpha)
\]  

(6.6)

**Proof.** Using (6.2) and (6.3) immediately give (6.5) and similarly (6.2) and (6.4) assert (6.6). \[\square\]

**Theorem 6.2.** Let $\alpha \in (0, 1]$, then the CFGHF $\, _2F_1 (\mu, \nu; c; x^\alpha) \,$ satisfies the following contiguous relation

\[
[\mu + (\nu - c) \, x^\alpha] \, _2F_1 (\mu, \nu; c; x^\alpha) = \mu \, (1 - x^\alpha) \, _2F_1 (\mu + 1, \nu; c; x^\alpha)
\]  

(6.7)

\[-c^{-1} (c - \mu) (c - \nu) \, x^\alpha \, _2F_1 (\mu, \nu + 1; c; x^\alpha)\]

**Proof.** Consider

\[
\theta^\alpha \, _2F_1 (\mu, \nu; c; x^\alpha) = \sum_{n=1}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} \, n^\alpha x^n
\]

A shift of index gives

\[
\theta^\alpha \, _2F_1 (\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu)_{n+1} (\nu)_{n+1}}{(c)_{n+1} (n+1)!} \, x^{n+1} = x^\alpha \sum_{n=0}^{\infty} \frac{(\mu + n) (\nu + n)}{(c + n) n!} \, (\mu)_n (\nu)_n x^n
\]  

(6.8)

Since,

\[
\frac{(\mu + n) (\nu + n)}{(c + n)} = n + (\mu + \nu - c) + \frac{(c - \mu) (c - \nu)}{c + n},
\]

then equation (6.8) yields

\[
\theta^\alpha \, _2F_1 (\mu, \nu; c; x^\alpha) = x^\alpha \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} n^\alpha x^n + (\mu + \nu - c) x^\alpha \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} x^n
\]

\[+ x^\alpha \frac{(c - \mu) (c - \nu)}{c} \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c + n) (c)_n n!} x^n
\]

\[= x^\alpha \theta^\alpha \, _2F_1 (\mu, \nu; c; x^\alpha) + (\mu + \nu - c) x^\alpha \, _2F_1 (\mu, \nu; c; x^\alpha)
\]

\[+ \frac{(c - \mu) (c - \nu)}{c} \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c + 1)_n n!} x^n
\]
By operating \( \theta \alpha \) we obtain
\[
(1 - x^\alpha) \theta^\alpha_2 F_1 (\mu, \nu; c; x^\alpha) = -\mu (1 - x^\alpha) \ 2F_1 (\mu, \nu; c; x^\alpha) + \mu (1 - x^\alpha) \ 2F_1 (\mu + 1, \nu; c; x^\alpha)
\]
which implies together with (6.9) the required relation. \( \square \)

**Theorem 6.3.** For \( \alpha \in (0, 1] \), then the CFGHF, \( 2F_1 (\mu, \nu; c; x^\alpha) \) satisfies the following contiguous relation
\[
(1 - x^\alpha) \ 2F_1 (\mu, \nu; c; x^\alpha) = \ 2F_1 (\mu - 1, \nu; c; x^\alpha) - c^{-1} (c - \nu) x^\alpha \ 2F_1 (\mu, \nu; c + 1; x^\alpha)
\]
(6.10)
\[
(1 - x^\alpha) \ 2F_1 (\mu, \nu; c; x^\alpha) = \ 2F_1 (\mu, \nu - 1; c; x^\alpha) - c^{-1} (c - \mu) x^\alpha \ 2F_1 (\mu, \nu; c + 1; x^\alpha)
\]
(6.11)

**Proof.** By operating \( \theta^\alpha \) we obtain
\[
\theta^\alpha_2 F_1 (\mu - 1, \nu; c; x^\alpha) = \theta^\alpha \sum_{n=0}^{\infty} \frac{(\mu - 1)_n (\nu)_n x^{\alpha n}}{(c)_n n!} = \sum_{n=1}^{\infty} \frac{(\mu - 1)_n (\nu)_n x^{\alpha n}}{(c)_n n!}.
\]

A shift of index yields
\[
\theta^\alpha_2 F_1 (\mu - 1, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{(\mu - 1)_{n+1} (\nu)_{n+1} x^{\alpha (n+1)}}{(c)_{n+1} (n+1)!}
\]
\[
= (\mu - 1) x^\alpha \sum_{n=0}^{\infty} \frac{(\nu + n) (\mu)_n (\nu)_n x^{\alpha n}}{(c + n) (c)_n n!}
\]
(6.12)

But \( \frac{(\nu + n)}{(c + n)} = 1 - \frac{c - \nu}{c + n} \), thus (6.12) becomes
\[
\theta^\alpha_2 F_1 (\mu - 1, \nu; c; x^\alpha) = (\mu - 1) x^\alpha \left[ \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n x^{\alpha n}}{(c)_n n!} + \frac{c - \nu}{c} \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n x^{\alpha n}}{(c + n) (c)_n n!} \right]
\]
\[
= (\mu - 1) x^\alpha \left[ \ 2F_1 (\mu, \nu; c; x^\alpha) - \frac{c - \nu}{c} \ 2F_1 (\mu, \nu; c + 1; x^\alpha) \right]
\]
which yield
\[
\theta^\alpha_2 F_1 (\mu - 1, \nu; c; x^\alpha) = (\mu - 1) x^\alpha \ 2F_1 (\mu, \nu; c; x^\alpha) - c^{-1} (c - \nu) (\mu - 1) x^\alpha \ 2F_1 (\mu, \nu; c + 1; x^\alpha).
\]
(6.13)

Now, replacing \( \mu \) by \( (\mu - 1) \) in (6.2) implies that
\[
\theta^\alpha_2 F_1 (\mu - 1, \nu; c; x^\alpha) = - (\mu - 1) \ 2F_1 (\mu - 1, \nu; c; x^\alpha) + (\mu - 1) \ 2F_1 (\mu, \nu; c; x^\alpha).
\]
(6.14)

From (6.13) and (6.14), the relation (6.10) is verified. Similarly, since \( \mu \) and \( \nu \) can be interchanged without affecting the hypergeometric series, (6.11) yields. \( \square \)
Observe that from the contiguous relations we just derived in Theorems 6.1, 6.2, and 6.3 we can obtain further relations by performing some suitable eliminations as follows.

From (6.7) and (6.10), we get

\[
[2\mu - c + (\nu - \mu)x^\alpha] \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = \mu (1 - x^\alpha) \, \binom{2F_1 (\mu + 1, \nu; c; x^\alpha)}{\mu + 1} - (c - \mu) \, \binom{2F_1 (\mu - 1, \nu; c; x^\alpha)}{\mu - 1}.
\]  

(6.15)

A combination of (6.7) and (6.11) gives

\[
[\mu + \gamma - c] \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = \mu (1 - x^\alpha) \, \binom{2F_1 (\mu + 1, \nu; c; x^\alpha)}{\mu + 1} - (c - \gamma) \, \binom{2F_1 (\mu, \nu - 1; c; x^\alpha)}{\mu}.
\]  

(6.16)

Inserting (6.5) in (6.15), satisfies

\[
[c - \mu - \nu] \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = \mu (1 - x^\alpha) \, \binom{2F_1 (\mu - 1, \nu; c; x^\alpha)}{\mu - 1} - (1 - x^\alpha) \, \binom{2F_1 (\mu, \nu + 1; c; x^\alpha)}{\mu}.
\]  

(6.17)

Also, from (6.15) and (6.16), we get

\[
(\nu - \mu) \, (1 - x^\alpha) \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = \mu (1 - x^\alpha) \, \binom{2F_1 (\mu - 1, \nu; c; x^\alpha)}{\mu - 1} - (c - \nu) \, \binom{2F_1 (\mu, \nu - 1; c; x^\alpha)}{\mu}.
\]  

(6.18)

Use (6.6) and (6.16) to obtain

\[
[1 - \mu + (c - \nu - 1)x^\alpha] \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = (c - \mu) \, \binom{2F_1 (\mu - 1, \nu; c; x^\alpha)}{\mu - 1} - (c - 1) \, \binom{2F_1 (\mu, \nu + 1; c; x^\alpha)}{\mu}.
\]  

(6.19)

By interchanging \( \mu \) and \( \nu \) in (6.15), we have

\[
[2\nu - c + (\mu - \nu)x^\alpha] \, \binom{2F_1 (\mu, \nu; c; x^\alpha)}{\mu} = \nu (1 - x^\alpha) \, \binom{2F_1 (\mu, \nu + 1; c; x^\alpha)}{\mu} - (c - \nu) \, \binom{2F_1 (\mu, \nu - 1; c; x^\alpha)}{\mu}.
\]  

(6.20)

We append this section by driving the CFGHE. The conformable fractional operator (3.3) can be employed to derive a conformable fractional differential equation characterized by (3.3).

Relation (3.3) with the operator \( \theta^\alpha \) defined by (6.1) gives

\[
\theta^\alpha (\theta^\alpha + c - 1) y = \theta^\alpha \sum_{n=0}^\infty \binom{(\mu)_n (\nu)_n}{(c)_n n!} (n + c - 1) x^{\alpha(n)} = \sum_{n=1}^\infty \binom{(\mu)_n (\nu)_n}{(c)_n n!} (n + c - 1) x^{\alpha(n)}
\]

A shift of index yields

\[
\theta^\alpha (\theta^\alpha + c - 1) y = \sum_{n=0}^\infty \binom{(\mu)_{n+1} (\nu)_{n+1}}{(c)_{n+1} n!} (n + c) x^{\alpha(n+1)} = x^\alpha \sum_{n=0}^\infty \binom{(\mu)_n (\nu)_n}{(c)_n n!} (n + \mu) (n + \nu) x^{\alpha(n)} = x^\alpha (\theta^\alpha + \mu) (\theta^\alpha + \nu) y
\]

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This shows $y = _2F_1(\mu, \nu; c; x^\alpha)$ is a solution of the following CFDE

$$[\theta^\alpha (\theta^\alpha + c - 1) - x^\alpha (\theta^\alpha + \mu) (\theta^\alpha + \nu)] y = 0, \quad \theta^\alpha = \frac{1}{\alpha} x^\alpha D^\alpha \quad (6.21)$$

Owing to $\theta^\alpha y = \frac{1}{\alpha} x^\alpha D^\alpha y$ and $\theta^\alpha \theta^\alpha y = \frac{1}{\alpha^2} x^{2\alpha} D^\alpha y + \frac{1}{\alpha} x^\alpha D^\alpha$, then equation (6.21) can be written in the form

$$x^\alpha (1 - x^\alpha) D^\alpha D^\alpha y + \alpha [c - (\mu + \nu + 1) x^\alpha] D^\alpha y - \alpha^2 \mu \nu y = 0,$$

which coincide with (3.1)

### 7 Conformable fractional integral of CFGHF

Taking into account the $\alpha$-integral given in Definition 2.5, we provide some forms of fractional integral related to the $\alpha$-Gauss hypergeometric function. Thus according to Definition 2.5, it follows that

$$I_\alpha f(x) = \int_0^x t^{\alpha-1} f(t) \, dt. \quad (7.1)$$

In this regard, we state the following important result given in [21].

**Lemma 7.1.** Suppose that $f : [0, \infty) \to \mathbb{R}$ is $\alpha$-differentiable for $\alpha \in (0, 1]$, then for all $x > 0$ one can write:

$$I_\alpha D^\alpha (f(x)) = f(x) - f(0) \quad (7.2)$$

With the aid of (7.1) and (7.2), the following result can be deduced.

**Theorem 7.1.** For $\alpha \in (0, 1]$, then the conformable fractional integral $I_\alpha$ of CFGHF, $_2F_1(\mu, \nu; c; x^\alpha)$ can be written as

$$I_\alpha _2F_1(\mu, \nu; c; x^\alpha) = \frac{(c - 1)}{\alpha (\mu - 1) (\nu - 1)} [2F_1(\mu - 1, \nu - 1; c - 1; x^\alpha) - 1] \quad (7.3)$$

**Proof.** Relation (5.9), gives

$$D^\alpha _2F_1(\mu - 1, \nu - 1; c - 1; x^\alpha) = \frac{\alpha (\mu - 1) (\nu - 1)}{(c - 1)} _2F_1(\mu, \nu; c; x^\alpha)$$

Acting by the conformable fractional integral on both sides we obtain

$$I_\alpha D^\alpha _2F_1(\mu - 1, \nu - 1; c - 1; x^\alpha) = \frac{\alpha (\mu - 1) (\nu - 1)}{(c - 1)} I_\alpha _2F_1(\mu, \nu; c; x^\alpha)$$

Using (7.2), we have

$$_2F_1(\mu - 1, \nu - 1; c - 1; x^\alpha) - 1 = \frac{\alpha (\mu - 1) (\nu - 1)}{(c - 1)} I_\alpha _2F_1(\mu, \nu; c; x^\alpha)$$
Therefore
\[ I_\alpha \, _2F_1 (\mu, \nu; c; x^\alpha) = \frac{(c - 1)}{\alpha (\mu - 1) (\nu - 1)} [2F_1 (\mu - 1, \nu - 1; c - 1; x^\alpha) - 1] \]
as required.

**Theorem 7.2.** For \( \alpha \in (0, 1) \), then the CFGHF, \(_2F_1 (\mu, \nu; c; x^\alpha)\) has a conformable fractional integral representation in the form
\[ _2F_1 (\mu, \nu; c; x^\alpha) = 1 + \frac{\alpha \mu \nu}{c} \int_0^x _2F_1 (\mu + 1, \nu + 1; c + 1; t^\alpha) \, d_\alpha t \]
where \( d_\alpha t = t^{\alpha - 1} dt \)

**Proof.** In view of theorem 7.1, we obtain
\[ I_\alpha \, _2F_1 (\mu + 1, \nu + 1; c + 1; x^\alpha) = \frac{c}{\alpha \mu \nu} [2F_1 (\mu, \nu; c; x^\alpha) - 1] \]

Hence,
\[ _2F_1 (\mu, \nu; c; x^\alpha) = 1 + \frac{\alpha \mu \nu}{c} I_\alpha [2F_1 (\mu + 1, \nu + 1; c + 1; x^\alpha)] \]
\[ = 1 + \frac{\alpha \mu \nu}{c} \int_0^x _2F_1 (\mu + 1, \nu + 1; c + 1; t^\alpha) \, d_\alpha t \]
\[ = 1 + \frac{\alpha \mu \nu}{c} \int_0^x \beta (c, \nu, c + 1; t^\alpha) \, t^{\alpha - 1} dt \]
as required.

Now, following [20], we state the following result

**Theorem 7.3.** For \( \alpha \in (0, 1) \) and \( c > \nu > 0 \), the CFGHF, \(_2F_1 (\mu, \nu; c; x^\alpha)\) has an integral representation
\[ _2F_1 (\mu, \nu; c; x^\alpha) = \frac{\Gamma (c)}{\Gamma (\nu) \Gamma (c - \nu)} \int_0^1 \tau^{\nu - 1} (1 - \tau)^{c - \nu - 1} (1 - x^\alpha \tau)^{-\mu} \, d\tau \quad (7.4) \]

**Proof.** From the definition of CFGHF (3.3), we have
\[ _2F_1 (\mu, \nu; c; x^\alpha) = \sum_{n=0}^{\infty} \frac{\Gamma (\nu + n) \Gamma (c)}{\Gamma (\nu) \Gamma (c + n) n!} x^{\alpha n} \]
\[ = \frac{\Gamma (c)}{\Gamma (\nu) \Gamma (c - \nu)} \sum_{n=0}^{\infty} \frac{\Gamma (\nu + n) \Gamma (c - \nu)}{\Gamma (c + n) n!} x^{\alpha n} \]
\[ = \frac{\Gamma (c)}{\Gamma (\nu) \Gamma (c - \nu)} \sum_{n=0}^{\infty} \beta (c - \nu, \nu + n) \frac{(\mu)_n}{n!} x^{\alpha n}. \]
Using the integral form of beta function, we get

\[
2F_1(\mu, \nu; c; x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c-\nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} (x^\alpha \tau)^n \, d\tau.
\]

By using the identity \(\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} t^n = (1 - t)^{-(\mu)}; \quad |t| < 1\), we have

\[
2F_1(\mu, \nu; c; x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c-\nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu} \, d\tau
\]
as required. \(\Box\)

### 8 Recursion formulas for \(2F_1(\mu, \nu; c; x^\alpha)\)

Employing the assertion in theorem 7.3, and owing to the results given in [30], we state the following recursion formulas.

**Theorem 8.1.** The following recursion formulas hold for the CFGHF

\[
2F_1(\mu + n, \nu; c; x^\alpha) = 2F_1(\mu, \nu; c; x^\alpha) + \frac{\nu x^\alpha}{c} \sum_{k=1}^{n} 2F_1(\mu + n - k + 1, \nu + 1; c + 1; x^\alpha) \quad (8.1)
\]

\[
2F_1(\mu - n, \nu; c; x^\alpha) = 2F_1(\mu, \nu; c; x^\alpha) - \frac{\nu x^\alpha}{c} \sum_{k=1}^{n} 2F_1(\mu - k + 1, \nu + 1; c + 1; x^\alpha), \quad (8.2)
\]

where \(|x^\alpha| < 1\), \(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

**Proof.** By means of (7.4), we have

\[
2F_1(\mu + n, \nu; c; x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c-\nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu-n-1} \, d\tau
\]

\[
= \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c-\nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu-n-1} \, d\tau
\]

\[
- \frac{x^\alpha}{\alpha (\mu + n)} \left\{ \frac{\alpha (\mu + n) \Gamma(c)}{\Gamma(\nu) \Gamma(c-\nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu-n-1} \, d\tau \right\}
\]

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In virtue of conformable derivative, we may write

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_{0}^{1} \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu-n-1} d\tau \]

\[ - \frac{x^\alpha}{\alpha (\mu + n)} D^\alpha \left\{ \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_{0}^{1} \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{-\mu-n} d\tau \right\}. \]

Again using (7.4), we have

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = 2F_1 (\mu + n + 1, \nu; c; x^\alpha) - \frac{x^\alpha}{\alpha (\mu + n)} D^\alpha \left\{ 2F_1 (\mu + n, \nu; c; x^\alpha) \right\}. \]

Thus,

\[ 2F_1 (\mu + n - 1, \nu; c; x^\alpha) = 2F_1 (\mu + n, \nu; c; x^\alpha) - \frac{x^\alpha}{\alpha (\mu + n - 1)} D^\alpha \left\{ 2F_1 (\mu + n - 1, \nu; c; x^\alpha) \right\}, \]

or

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = 2F_1 (\mu + n - 1, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha (\mu + n - 1)} D^\alpha \left\{ 2F_1 (\mu + n - 1, \nu; c; x^\alpha) \right\} \]

Applying this last identity (8.3), we get

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = 2F_1 (\mu + n - 2, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha (\mu + n - 2)} D^\alpha \left\{ 2F_1 (\mu + n - 2, \nu; c; x^\alpha) \right\} \]

\[ + \frac{x^\alpha}{\alpha (\mu + n - 1)} D^\alpha \left\{ 2F_1 (\mu + n - 1, \nu; c; x^\alpha) \right\} \]

\[ = 2F_1 (\mu + n - 2, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha} \sum_{k=1}^{2} \frac{1}{(\mu + n - k)} D^\alpha \left\{ 2F_1 (\mu + n - k, \nu; c; x^\alpha) \right\}. \]

Again apply (8.3) recursively n-times, we obtain

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = 2F_1 (\mu, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha} \sum_{k=1}^{n} \frac{1}{(\mu + n - k)} D^\alpha \left\{ 2F_1 (\mu + n - k, \nu; c; x^\alpha) \right\}. \]

Using (5.9), we have

\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = 2F_1 (\mu, \nu; c; x^\alpha) + \frac{x^\alpha \nu}{c} \sum_{k=1}^{n} \left\{ 2F_1 (\mu + n - k + 1, \nu + 1; c + 1; x^\alpha) \right\}. \]
Furthermore, the assertion of theorem 7.3 gives

\[
2F_1(\mu - n, \nu; c; x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu-1} d\tau
\]

\[
- \frac{\Gamma(c) x^\alpha}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu-1} d\tau
\]

\[
= \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu-1} d\tau
\]

\[
- \frac{x^\alpha}{\alpha (\mu - n)} \left\{ \frac{\alpha (\mu - n) \Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu-1} d\tau \right\}
\]

\[
= \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu-1} d\tau
\]

\[
- \frac{x^\alpha}{\alpha (\mu - n)} D^\alpha \left\{ \frac{\Gamma(c)}{\Gamma(\nu) \Gamma(c - \nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau)^{n-\mu} d\tau \right\}
\]

Relying on the integral representation (7.4), we have

\[
2F_1(\mu - n, \nu; c; x^\alpha) = 2F_1(\mu - n + 1, \nu; c; x^\alpha) - \frac{x^\alpha}{\alpha (\mu - n)} D^\alpha \{2F_1(\mu - n, \nu; c; x^\alpha)\}.
\]

Therefore,

\[
2F_1(\mu - n - 1, \nu; c; x^\alpha) = 2F_1(\mu - n, \nu; c; x^\alpha) - \frac{x^\alpha}{\alpha (\mu - n - 1)} D^\alpha \{2F_1(\mu - n - 1, \nu; c; x^\alpha)\},
\]

or

\[
2F_1(\mu - n, \nu; c; x^\alpha) = 2F_1(\mu - n - 1, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha (\mu - n - 1)} D^\alpha \{2F_1(\mu - n - 1, \nu; c; x^\alpha)\}.
\]  \hspace{1cm} (8.5)

Applying relation (8.5) recursively, we obtain

\[
2F_1(\mu - n, \nu; c; x^\alpha) = 2F_1(\mu - n - 2, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha (\mu - n - 2)} D^\alpha \{2F_1(\mu - n - 2, \nu; c; x^\alpha)\}
\]

\[
+ \frac{x^\alpha}{\alpha (\mu - n - 1)} D^\alpha \{2F_1(\mu - n - 1, \nu; c; x^\alpha)\}
\]

\[
= 2F_1(\mu - n - 2, \nu; c; x^\alpha) + \frac{x^\alpha}{\alpha} \sum_{k=1}^{2} \frac{1}{(\mu - n - k)} D^\alpha \{2F_1(\mu - n - k, \nu; c; x^\alpha)\}.
\]
Repeating the recurrence relation (8.5) \(n\)-times and dappling the derivative formula (5.9), we have
\[
2F_1 (\mu - n, \nu; c; x^{\alpha}) = 2F_1 (\mu - 2n, \nu; c; x^{\alpha}) + \frac{x^{\alpha}}{c} \sum_{k=1}^{n} \{2F_1 (\mu - n - k + 1, \nu + 1; c + 1; x^{\alpha})\}. 
\]

The relation (8.2) follows directly from (8.6) upon replacing \(\mu\) by \((\mu + n)\) where \(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\). \(\square\)

**Theorem 8.2.** For \(\alpha \in (0, 1]\), the following recursion formulas hold true for the CFGHF, \(2F_1 (\mu, \nu; c; x^{\alpha})\)
\[
2F_1 (\mu + n, \nu; c; x^{\alpha}) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\nu)_k}{(c)_k} x^{\alpha k} 2F_1 (\mu + k, \nu + k; c + k; x^{\alpha}), \quad (8.7)
\]
and
\[
2F_1 (\mu - n, \nu; c; x^{\alpha}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\nu)_k}{(c)_k} x^{\alpha k} 2F_1 (\mu, \nu + k; c + k; x^{\alpha}), \quad (8.8)
\]
where \(|x^{\alpha}| < 1\), \(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

**Proof.** From (8.1) of theorem 8.1 with \(n = 1\), we see
\[
2F_1 (\mu + 1, \nu; c; x^{\alpha}) = 2F_1 (\mu, \nu; c; x^{\alpha}) + \frac{\nu x^{\alpha}}{c} 2F_1 (\mu + 1, \nu + 1; c + 1; x^{\alpha}), \quad (8.9)
\]
with \(n = 2\), we have
\[
2F_1 (\mu + 2, \nu; c; x^{\alpha}) = 2F_1 (\mu, \nu; c; x^{\alpha}) + \frac{\nu x^{\alpha}}{c} 2F_1 (\mu + 1, \nu + 1; c + 1; x^{\alpha}) + \frac{\nu x^{\alpha}}{c} 2F_1 (\mu + 2, \nu + 1; c + 1; x^{\alpha}), \quad (8.10)
\]
Making use of (8.9) and (8.10), we obtain
\[
2F_1 (\mu + 2, \nu; c; x^{\alpha}) = 2F_1 (\mu, \nu; c; x^{\alpha}) + \frac{2\nu x^{\alpha}}{c} 2F_1 (\mu + 1, \nu + 1; c + 1; x^{\alpha}) + \frac{\nu (\nu + 1) x^{2\alpha}}{c (c + 1)} 2F_1 (\mu + 2, \nu + 2; c + 2; x^{\alpha}). \quad (8.11)
\]
Using (8.9) and (8.11) with \(n = 3\), it follows that
\[
2F_1 (\mu + 3, \nu; c; x^{\alpha}) = 2F_1 (\mu, \nu; c; x^{\alpha}) + \frac{3\nu x^{\alpha}}{c} 2F_1 (\mu + 1, \nu + 1; c + 1; x^{\alpha}) + \frac{3\nu (\nu + 1) x^{2\alpha}}{c (c + 1)} 2F_1 (\mu + 2, \nu + 2; c + 2; x^{\alpha}) + \frac{\nu (\nu + 1) (\nu + 2) x^{3\alpha}}{c (c + 1) (c + 2)} 2F_1 (\mu + 3, \nu + 3; c + 3; x^{\alpha}). \quad (8.12)
\]
Relation (8.12) can be written in the form
\[ 2F_1 (\mu + 3, \nu; c; x^\alpha) = \sum_{k=0}^{3} \binom{3}{k} \frac{\nu}{(c)^k} x^{\alpha k} 2F_1 (\mu + k, \nu + k; c + k; x^\alpha). \]

In general, we may write that
\[ 2F_1 (\mu + n, \nu; c; x^\alpha) = \sum_{k=0}^{n} \binom{n}{k} \frac{\nu}{(c)^k} x^{\alpha k} 2F_1 (\mu + k, \nu + k; c + k; x^\alpha). \]

In order to prove (8.8), we note from (8.2) of theorem 8.1 (with \( n = 1 \)) that
\[ 2F_1 (\mu - 1, \nu; c; x^\alpha) = 2F_1 (\mu, \nu; c; x^\alpha) - \frac{\nu x^\alpha}{c} 2F_1 (\mu, \nu + 1; c + 1; x^\alpha). \]

Similarly, (with \( n = 2 \)) yields
\[ 2F_1 (\mu - 2, \nu; c; x^\alpha) = 2F_1 (\mu, \nu; c; x^\alpha) - \frac{\nu x^\alpha}{c} 2F_1 (\mu, \nu + 1; c + 1; x^\alpha) - \frac{\nu (\nu + 1) x^{2\alpha}}{c (c + 1)} 2F_1 (\mu, \nu + 2; c + 2; x^\alpha). \]

Inserting (8.13) in (8.14), we get
\[ 2F_1 (\mu - 2, \nu; c; x^\alpha) = 2F_1 (\mu, \nu; c; x^\alpha) - \frac{2\nu x^\alpha}{c} 2F_1 (\mu, \nu + 1; c + 1; x^\alpha) + \frac{\nu (\nu + 1) x^{2\alpha}}{c (c + 1)} 2F_1 (\mu, \nu + 2; c + 2; x^\alpha). \]

Using the Pochhammer symbol we may write (8.15) as
\[ 2F_1 (\mu - 2, \nu; c; x^\alpha) = \sum_{k=0}^{2} (-1)^k \binom{2}{k} \frac{\nu}{(c)^k} x^{\alpha k} 2F_1 (\mu, \nu + k; c + k; x^\alpha). \]

Thus, in general, we may write
\[ 2F_1 (\mu - n, \nu; c; x^\alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\nu}{(c)^k} x^{\alpha k} 2F_1 (\mu, \nu + k; c + k; x^\alpha), \]

just as required in (8.8).

\[ \square \]

**Theorem 8.3.** For \( \alpha \in (0, 1] \), the following recursion formulas hold true for the CFGHF,
\[ 2F_1 (\mu, \nu; c; x^\alpha) \]
\[ 2F_1 (\mu, \nu; c + n; x^\alpha) = \frac{(c)^n}{(c - \nu)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\nu}{(c)^k} x^{\alpha k} 2F_1 (\mu, \nu + k; c + k; x^\alpha), \]
\[ (|x^\alpha| < 1, c + n \notin \mathbb{Z}_0^-, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \]

(8.16)
Proof. In view of (7.4), we have
\[ 2F_1 (\mu, \nu; c + n; x^\alpha) = \frac{\Gamma (c + n)}{\Gamma (\nu) \Gamma (c + n - \nu)} \int_0^1 \tau^{\nu-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau) d\tau. \]

Using the binomial theorem, we obtain
\[ 2F_1 (\mu, \nu; c + n; x^\alpha) = \frac{\Gamma (c + n)}{\Gamma (\nu) \Gamma (c + n - \nu)} \int_0^1 \sum_{k=0}^n (-1)^k \binom{n}{k} \tau^{\nu+k-1} (1 - \tau)^{c-\nu-1} (1 - x^\alpha \tau) d\tau. \]  

Using the definition of the Pochhammer symbol, we may write (8.17) as
\[ 2F_1 (\mu, \nu; c + n; x^\alpha) = \frac{(-1)^n}{(c - \nu)_n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma (\nu)_k}{(c)_k} \frac{\Gamma (c + k)}{\Gamma (\nu + k) \Gamma (c - \nu)} \int_0^1 \tau^{\nu+k-1} (1 - \tau)^{c-k-\nu-1} (1 - x^\alpha \tau) d\tau. \]

Applying (7.4), we obtain
\[ 2F_1 (\mu, \nu; c + n; x^\alpha) = \frac{(-1)^n}{(c - \nu)_n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma (\nu)_k}{(c)_k} \frac{\Gamma (c + k)}{\Gamma (\nu + k) \Gamma (c - \nu)} 2F_1 (\mu, \nu + k; c + k; x^\alpha), \]

just as required in theorem 8.3.

\[ \Box \]

9 Fractional Laplace transform of the CFGHF

In [22], Abdeljawad defined the fractional Laplace transform in the conformable sense as follows:

Definition 9.1. [22] Let \( \alpha \in (0, 1] \) and \( f : [0, \infty) \to \mathbb{R} \) be real valued function. Then the fractional Laplace transform of order \( \alpha \) is defined by
\[ L_\alpha [f(t)] = F_\alpha (s) = \int_0^\infty e^{-s \frac{t^\alpha}{\alpha}} f(t) \, dt = \int_0^\infty e^{-s \frac{t^\alpha}{\alpha}} f(t) \, t^{\alpha-1} \, dt. \]  

Remark 9.1. If \( \alpha = 1 \), then (9.1) is the classical definition of the Laplace transform of integer order.

Also, the author in [22] gave the following interesting results.

Lemma 9.1. [22] Let \( \alpha \in (0, 1] \) and \( f : [0, \infty) \to \mathbb{R} \) be real valued function such that \( L_\alpha [f(t)] = F_\alpha (s) \) exist. Then \( F_\alpha (s) = L \left[ f(\alpha t)^{\frac{1}{\alpha}} \right] \), where \( L [f(t)] = \int_0^\infty e^{-st} f(t) \, dt \).

Lemma 9.2. [22] The following the conformable fractional Laplace transform of certain functions:
(1) \( L_\alpha[1] = \frac{1}{s}; \ s > 0 \)

(2) \( L_\alpha[t^p] = \alpha^p \frac{\Gamma(1+p)}{s^{1+p}}; \ s > 0 \)

(3) \( L_\alpha[e^{kx_\alpha}] = \frac{1}{s-k} \)

Owing to the definition of CFGHF and applying the conformable fractional Laplace
transform operator of an arbitrary order \( \gamma \in (0, 1] \), we have

\[
L_\gamma \left[ 2F_1(\mu, \nu; c; x^\alpha) \right] = L_\gamma \left[ \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} x^{\alpha n} \right] = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} L_\gamma \{ x^{\alpha n} \} \tag{9.2}
\]

Using (2) of lemma 9.2, we obtain

\[
L_\gamma \left[ 2F_1(\mu, \nu; c; x^\alpha) \right] = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} \gamma^{\alpha n} \frac{\Gamma(1+n)}{s^{1+n}} \tag{9.3}
\]

Remark 9.2. If \( \gamma = \alpha \) in (9.3) we have

\[
L_\alpha \left[ 2F_1(\mu, \nu; c; x^\alpha) \right] = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n n!} \alpha^n \Gamma(1+n) \frac{1}{s^{1+n}} = \sum_{n=0}^{\infty} \frac{\alpha^n (\mu)_n (\nu)_n}{(c)_n n!} \tag{9.4}
\]

Theorem 9.1. Let \( \alpha \in (0, 1] \) and \( 2F_1(\mu, \nu; c; x^\alpha) \) be a conformable fractional hypergeometric
function, then we have

\[
L_\alpha \left[ 2F_1(\mu, \nu; 1; x^\alpha \left( 1 - e^{-\frac{\nu}{\alpha}} \right) ) \right] = \frac{1}{s} 2F_1(\mu, \nu; s+1; x^\alpha) \tag{9.5}
\]

Proof. Using (3.3) and (9.1), one can see

\[
L_\alpha \left[ 2F_1(\mu, \nu; 1; x^\alpha \left( 1 - e^{-\frac{\nu}{\alpha}} \right) ) \right] = L_\alpha \left[ \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(1)_n n!} x^{\alpha n} \left( 1 - e^{-\frac{\nu}{\alpha}} \right)^n \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{n!} x^{\alpha n} L_\alpha \left[ \frac{1}{n!} \left( 1 - e^{-\frac{\nu}{\alpha}} \right)^n \right] \tag{9.6}
\]

But

\[
L_\alpha \left[ \frac{1}{n!} \left( 1 - e^{-\frac{\nu}{\alpha}} \right)^n \right] = L_\alpha \left[ \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} e^{-k\frac{\nu}{\alpha}} \right] = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} L_\alpha \left\{ e^{-k\frac{\nu}{\alpha}} \right\}
\]

Using (3) of lemma 9.2, we have

\[
L_\alpha \left[ \frac{1}{n!} \left( 1 - e^{-\frac{\nu}{\alpha}} \right)^n \right] = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{1}{s+k}
\]
Since \((-n)_k = 0\) if \(k > n\), then we can write

\[
L_\alpha \left[ \frac{1}{n!} \left( 1 - e^{-\frac{\alpha}{a}} \right)^n \right] = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k}{k! (s + k)}
\]

Using \(\frac{(s)_k}{s(s+1)_k} = \frac{1}{s+n}\), (9.7) becomes

\[
L_\alpha \left[ \frac{1}{n!} \left( 1 - e^{-\frac{\alpha}{a}} \right)^n \right] = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (s)_k}{s (s+1)_k k!} = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{(s)_k}{(s+1)_k} \left( 1 - e^{-\frac{\alpha}{a}} \right)^n = \frac{1}{s.n!} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{(s)_k}{(s+1)_k} \left( 1 - e^{-\frac{\alpha}{a}} \right)^n
\]

Substituting (9.8) into (9.6), we have

\[
L_\alpha \left[ 2F1 \left( \mu, \nu; 1; x^\alpha (1 - e^{-\frac{\alpha}{a}}) \right) \right] = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{n!} \left[ \frac{1}{s (s+1)_n} x^{\alpha n} \right] = \frac{1}{s} 2F1 \left( \mu, \nu; s+1; x^\alpha \right)
\]

as required. \(\square\)

**Theorem 9.2.** Let \(\alpha \in (0, 1]\) and \(2F1 (\mu, \nu; c; x^\alpha)\) be a conformable fractional hypergeometric function, then

\[
L_\alpha \left[ t^{\alpha n} \sin (at^\alpha) \right] = \frac{a^{n+1} \Gamma (n + 2)}{s^{n+2}} 2F1 \left( \frac{n + 2}{2}, \frac{n + 3}{2}; \frac{3}{2}; -\left( \frac{\alpha a}{s} \right)^2 \right)
\]

**Proof.** First, we see that

\[
L_\alpha \left[ t^{\alpha n} \sin (at^\alpha) \right] = L_\alpha \left[ t^{\alpha n} \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} t^{\alpha (2k+1)} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} L_\alpha \left\{ t^{\alpha (n+2k+1)} \right\}
\]

Using (2) of lemma 9.2, it follows that

\[
L_\alpha \left[ t^{\alpha n} \sin (at^\alpha) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} \alpha^{n+2k+1} \frac{\Gamma (n + 2k + 2)}{s^{n+2k+2}} = \frac{a^{n+1} \Gamma (n + 2)}{s^{n+2}} \sum_{k=0}^{\infty} \frac{\Gamma (n + 2k + 2)}{(n + 2)(2k+1)!} \left( \frac{-\alpha^2 a^2}{s^2} \right)^k
\]

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But \((n + 2)_{2k} = \left(\frac{n+2}{2}\right)_k \cdot \left(\frac{n+3}{2}\right)_k\) and \((2)_{2k} = (1)_{k} \cdot \left(\frac{3}{2}\right)_k = \left(\frac{3}{2}\right)_k k!\). Therefore

\[
L_{\alpha} \left[ t^{\alpha n} \sin (at^\alpha) \right] = \frac{a\alpha^{n+1} \Gamma(n + 2)}{s^{n+2}} \sum_{k=0}^{\infty} \left(\frac{n+2}{2}\right)_k \cdot \left(\frac{n+3}{2}\right)_k \frac{(-\alpha^2 a^2)^k}{(2)_k k!} = \frac{a\alpha^{n+1} \Gamma(n + 2)}{s^{n+2}} 2F_1 \left(\frac{n+2}{2}, \frac{n+3}{2}; \frac{3}{2}; -\left(\frac{\alpha a}{s}\right)^2\right).
\]

\[
10 \text{ Applications}
\]

The general solution of a wide class of conformable fractional differential equations of mathematical physics can be written in terms of the CFGHF after using a suitable change of independent variable. This phenomenon will be illustrated through the following interesting discussion.

Abul-Ez et al. \[28\] gave the hypergeometric representation of the conformable fractional Legendre polynomials \(P_{\alpha n}(x)\), as

\[
P_{\alpha n}(x) = 2F_1 \left(\frac{-n}{2}, \frac{n+1}{2}; 1; \frac{1-x^\alpha}{2}\right).
\]

This formula can be easily obtained through the CFGHE as follows.

Note that, the conformable fractional Legendre polynomials \(P_{\alpha n}(x)\) satisfy the conformable fractional differential equation

\[
(1 - x^{2\alpha}) D_x^{\alpha} D_x^{\alpha} P_{\alpha n}(x) - 2\alpha x^{\alpha} D_x^{\alpha} P_{\alpha n}(x) + \alpha^2 n (n + 1) P_{\alpha n}(x) = 0.
\]

(10.1)

With the help of \(t^\alpha = \frac{1-x^\alpha}{2}\), we get

\[
D_x^{\alpha} P_{\alpha n} = \left(\frac{-1}{2}\right) D_t^{\alpha} P_{\alpha n}, \text{ and } D_x^{\alpha} D_x^{\alpha} P_{\alpha n} = \frac{1}{4} D_t^{\alpha} D_t^{\alpha} P_{\alpha n}.
\]

Using (10.1), we obtain

\[
t^\alpha (1 - t^\alpha) D_t^{\alpha} D_t^{\alpha} P_{\alpha n} + \alpha \{1 - 2t^\alpha\} D_t^{\alpha} P_{\alpha n} + \alpha^2 n (n + 1) P_{\alpha n} = 0.
\]

(10.2)

Comparing the last equation (10.2) with the CFGHE (3.1), we obtain the parameters \(\mu\), \(\nu\) and \(c\), such that

\[
\mu = -n, \quad \nu = n + 1 \quad \text{and} \quad c = 1,
\]

Hence, we may write the conformable fractional Legendre polynomials as

\[
P_{\alpha n}(x) = 2F_1 \left(\frac{-n}{2}, \frac{n+1}{2}; 1; t^\alpha\right)
= 2F_1 \left(\frac{-n}{2}, \frac{n+1}{2}; 1; \frac{1-x^\alpha}{2}\right).
\]
Example 10.1. Consider the following conformable fractional differential equation

\[(1 - e^{x^\alpha} ) D_x^\alpha D_x^\alpha y + \frac{\alpha}{2} D_x^\alpha y + \alpha^2 e^{x^\alpha} y = 0 \]  \hspace{1cm} (10.3)

Then the general solution of (10.3) can be easily deduced as follows.

Let \( t^\alpha = (1 - e^{x^\alpha} ) \), then we have

\[ D_x^\alpha y = -e^{x^\alpha} D_t^\alpha y = - (1 - t^\alpha ) D_t^\alpha y \]

and

\[ D_x^\alpha D_x^\alpha y = -\alpha e^{x^\alpha} D_t^\alpha y + e^{2x^\alpha} D_t^\alpha D_t^\alpha y = (1 - t^\alpha )^2 D_t^\alpha D_t^\alpha y - \alpha (1 - t^\alpha ) D_t^\alpha y \]

Now, in view of (10.3), it can be easily seen that,

\[ t^\alpha [ (1 - t^\alpha )^2 D_t^\alpha D_t^\alpha y - \alpha (1 - t^\alpha ) D_t^\alpha y ] - \frac{\alpha}{2} (1 - t^\alpha ) D_t^\alpha y + \alpha^2 (1 - t^\alpha ) y = 0 \]  \hspace{1cm} (10.4)

Simplifying (10.4), we get

\[ t^\alpha (1 - t^\alpha ) D_t^\alpha D_t^\alpha y + \alpha \left\{ \frac{-1}{2} - t^\alpha \right\} D_t^\alpha y + \alpha^2 y = 0 \]  \hspace{1cm} (10.5)

Comparing (10.5) with the CFGHE (3.1), we obtain \( \mu + \nu = 0 \), \( \mu \nu = -1 \) and \( c = -\frac{1}{2} \). Thus, \( \mu = 1 \) and \( \gamma = -1 \). Therefore, the general solution of the CFDE (10.3) can be given in the form

\[ y = A \, _2F_1(\mu, \nu; c; t^\alpha) + B \, t^{\alpha(1-c)} \, _2F_1(\mu - c + 1, \nu - c + 1; 2 - c; t^\alpha) \]

\[ = A \, _2F_1 \left( 1, -1; \frac{-1}{2}; 1 - e^{x^\alpha} \right) + B \, \left[ 1 - e^{x^\alpha} \right]^{\frac{3}{2}} \, _2F_1 \left( \frac{5}{2}, \frac{1}{2}; \frac{5}{2}; 1 - e^{x^\alpha} \right) \]

where \( A \) and \( B \) are arbitrary constants.

The strategy used in the preceding example can be easily applied to solve some famous differential equations such as, Chebyshev, Fibonacci, and Lucas differential equations in the framework of fractional calculus. Handled by Chebyshev, Fibonacci, and Lucas differential equations have advantages due to their own importance in applications. Thus, we may mention that the properties of Chebyshev polynomials are used to give a numerical solution of the conformable space-time fractional wave equation, see [37]. The Fibonacci polynomial is a polynomial sequence, which can be considered as a generalization circular for the Fibonacci numbers. It is used in many applications, e.g., biology, statistics, physics, and computer science [38]. The Fibonacci and Lucas sequences of both polynomials and numbers are of great importance in a variety of topics, such as number theory, combinatorics, and numerical analysis. For these studies, we refer to [38–41]. Table 1 provides the general solutions of such famous differential equations briefly.
Conformable fractional Chebyshev differential equation

| CF Chebyshev DE | \((1 - x^{2\alpha}) D_x^\alpha D_x^\alpha y - \alpha x^{\alpha} D_x^\alpha y + \alpha^2 n^2 y = 0\) |
|-----------------|----------------------------------------------------------------|
| Suitable tranformation | \(t^\alpha = \frac{1 - x^{\alpha}}{2}\) |
| Transformed equation | \(t^\alpha (1 - t^\alpha) D_t^\alpha D_t^\alpha y + \alpha \left\{ \frac{1}{2} - t^\alpha \right\} D_t^\alpha y + \alpha^2 n^2 y = 0\) |
| Parameters | \(\mu = -n, \quad \nu = n\) and \(c = \frac{1}{2}\). |
| General solution | \(y = A\binom{\nu}{\mu} F\left(-n, \frac{n}{2}; \frac{1}{2}; \frac{1 - x^{\alpha}}{2}\right) + B\binom{-\nu}{\mu} F\left(-n, \frac{n}{2}; \frac{1}{2}; \frac{1 - x^{\alpha}}{2}\right)\) |

Conformable fractional Fibonacci differential equation

| CF Fibonacci DE | \((x^{2\alpha} + 4) D_x^\alpha D_x^\alpha y + 3\alpha x^{\alpha} D_x^\alpha y - \alpha^2 (n^2 - 1) y = 0\) |
|-----------------|----------------------------------------------------------------|
| Suitable tranformation | \(t^\alpha = \left(1 + \frac{x^{2\alpha}}{4}\right)\) |
| Transformed equation | \(t^\alpha (1 - t^\alpha) D_t^\alpha D_t^\alpha y + \alpha \left\{ \frac{3}{2} - 2t^\alpha \right\} D_t^\alpha y - \alpha^2 (1 - n^2) y = 0\) |
| Parameters | \(\mu = \frac{1 - n}{2}, \quad \nu = \frac{1 + n}{2}\) and \(c = \frac{3}{2}\) |
| General solution | \(y = A\binom{\nu}{\mu} F\left(-\frac{n}{2}, \frac{n}{2}; \frac{1}{2}; 1 + \frac{x^{2\alpha}}{4}\right) + B\binom{\nu}{\mu} F\left(-\frac{n}{2}, \frac{n}{2}; \frac{1}{2}; 1 + \frac{x^{2\alpha}}{4}\right)\) |

Conformable fractional Lucas differential equation

| CF Lucas DE | \((x^{2\alpha} + 4) D_x^\alpha D_x^\alpha y + \alpha x^{\alpha} D_x^\alpha y - \alpha^2 n^2 y = 0\) |
|-------------|----------------------------------------------------------------|
| Suitable tranformation | \(t^\alpha = \left(1 + \frac{x^{2\alpha}}{4}\right)\) |
| Transformed equation | \(t^\alpha (1 - t^\alpha) D_t^\alpha D_t^\alpha y + \alpha \left\{ \frac{1}{2} - t^\alpha \right\} D_t^\alpha y + \alpha^2 n^2 y = 0\) |
| Parameters | \(\mu = \frac{n}{2}, \quad \nu = \frac{-n}{2}\) and \(c = \frac{1}{2}\) |
| General solution | \(y = A\binom{\nu}{\mu} F\left(\frac{n}{2}, \frac{-n}{2}; \frac{1}{2}; 1 + \frac{x^{2\alpha}}{4}\right) + B\binom{\nu}{\mu} F\left(\frac{n}{2}, \frac{-n}{2}; \frac{1}{2}; 1 + \frac{x^{2\alpha}}{4}\right)\) |

Table 1: General Solutions of some famous CDEs

### 11 Conclusion

The Gaussian hypergeometric function \(\binom{\nu}{\mu} F\left(\chi; c; x\right)\) has been studied extensively from its mathematical point of view [32]. This occurs probably, in part, due to its many applications on a large variety of physical and mathematical problems. In quantum mechanics, the solution of the Schrödinger equation for some systems is expressed in terms of \(\binom{\nu}{\mu} F\left(\chi; c; x\right)\) functions, as observed in solving the Pöschl-Teller, Wood-Saxon, or Hulthén en potentials [33]. Another very important case is related to the angular momentum theory since the eigenfunctions of the angular momentum operators are written in terms of \(\binom{\nu}{\mu} F\left(\chi; c; x\right)\) functions [34]. One important tool related to such problems is then provided by the derivatives of the \(\binom{\nu}{\mu} F\left(\chi; c; x\right)\) function with respect to the parameters \(\mu, \nu,\) and \(c\) since they allow one, for example, to write a Taylor expansion around given values \(\mu_0, \nu_0,\) or \(c_0\). Therefore, the importance of the Gaussian hypergeometric differential equation motivates one to give a detailed study on the CFGHF.
The solutions of the CFGHE are given to improve and generalize those given in [19]. Besides, many interesting properties, and useful formulas of CFGHF are presented. Finally, supported examples, showing that a class of conformable fractional differential equations of mathematical physics can be solved by means of the CFGHF.

It is interesting to mention that the obtained results of the current work has treated various famous aspects such as, generating functions, differential forms, contiguous relations, and recursion formulas, for which they have been generalized and developed in the context of the fractional setting. These aspects play important roles in themselves and their diverse applications. In fact, most of the special functions of mathematical physics and engineering, for instance, the Jacobi and Laguerre polynomials can be expressed in terms of the Gauss hypergeometric function and other related hypergeometric functions. Therefore, the numerous generating functions involving extensions and generalizations of the Gauss hypergeometric function are capable of playing important roles in the theory of special functions of applied mathematics and mathematical physics, see [35].

The derivatives of any order of the GHF $\text{$_2F_1$} (\mu, \nu; c; x)$ with respect to the parameters $\mu, \nu$, and $c$, which can be expressed in terms of generalizations of multivariable Kampe de Fériet functions, have many applications (see the work of [36]). We may recall that applications of the contiguous function relation range from the evaluation of hypergeometric series to the derivation of the summation and transformation formulas for such series; these can be used to evaluate the contiguous functions to a hypergeometric function, see [27]. Furthermore, using some contiguous function relations for the classical Gauss hypergeometric series $\text{$_2F_1$}$, several new recursion formulas for the Appell functions $F_2$ with important applications have been the subject of some research work, see for example [30] and reference therein. In conclusion, it is rather interesting to consider a wide generalization of the Gaussian hypergeometric function in the forthcoming work either in the framework of fractional calculus or in a higher dimensional setting. Our concluded results can be used for a wide variety of cases.

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