Ostrowski-Sugeno fuzzy inequalities

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ABSTRACT

We present Ostrowski-Sugeno fuzzy type inequalities. These are Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties are investigated. Tight upper bounds to the deviation of a function from its Sugeno-fuzzy averages are given. This work is greatly inspired by [3] and [1].

RESUMEN

Presentamos desigualdades de Ostrowski-Sugeno de tipo fuzzy. Estas son desigualdades de tipo Ostrowski en el contexto de integrales fuzzy de Sugeno y se investigan sus propiedades especiales. Se entregan cotas superiores ajustadas para la desviación de una función de sus promedios fuzzy de Sugeno. Este trabajo está inspirado principalmente por [3] y [1].

Keywords and Phrases: Sugeno fuzzy, integral, function fuzzy average, deviation from fuzzy mean, fuzzy Ostrowski inequality.

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1 Introduction

The famous Ostrowski inequality motivates this work and has as follows:

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(y) \, dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f\|_{\infty},
\]

where \( f \in C'([a, b]) \), \( x \in [a, b] \), and it is a sharp inequality. One can easily notice that

\[
\left( \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) = \frac{(x-a)^2 + (b-x)^2}{2(b-a)}.
\]

Another motivation is author’s article [1].

First we give a survey about Sugeno fuzzy integral and its basic properties. Then we derive a series of Ostrowski-like inequalities to all directions in the context of Sugeno integral and its basic important particular properties. We also give applications to special cases of our problem we deal with.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented.

**Definition 2.1.** (Fuzzy measure [3, 7]) Let \( \Sigma \) be a \( \sigma \)-algebra of subsets of \( X \), and let \( \mu : \Sigma \rightarrow [0, +\infty] \) be a non-negative extended real-valued set function. We say that \( \mu \) is a fuzzy measure iff:

1. \( \mu(\emptyset) = 0 \),
2. \( E, F \in \Sigma : E \subseteq F \) imply \( \mu(E) \leq \mu(F) \) (monotonicity),
3. \( E_n \in \Sigma \ (n \in \mathbb{N}) \), \( E_1 \supset E_2 \supset \ldots \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \) (continuity from below);
4. \( E_n \in \Sigma \ (n \in \mathbb{N}) \), \( E_1 \supset E_2 \supset \ldots \), \( \mu(E_1) < \infty \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) \) (continuity from above).

Let \((X, \Sigma, \mu)\) be a fuzzy measure space and \( f \) be a non-negative real-valued function on \( X \). We denote by \( F_+ \) the set of all non-negative real valued measurable functions, and by \( L_\alpha f \) the set: \( L_\alpha f := \{ x \in X : f(x) \geq \alpha \} \), the \( \alpha \)-level of \( f \) for \( \alpha \geq 0 \).

**Definition 2.2.** Let \((X, \Sigma, \mu)\) be a fuzzy measure space. If \( f \in F_+ \) and \( A \in \Sigma \), then the Sugeno integral (fuzzy integral) \([4] \) of \( f \) on \( A \) with respect to the fuzzy measure \( \mu \) is defined by

\[
(S) \int_{A} f \, d\mu := \bigvee_{\alpha \geq 0} (\alpha \land \mu(A \cap L_\alpha f)),
\]

where \( \bigvee \) and \( \land \) denote the sup and inf on \([0, \infty]\), respectively.
The basic properties of Sugeno integral follow:

**Theorem 2.3.** ([7, 8]) Let \((X, \Sigma, \mu)\) be a fuzzy measure space with \(A, B \in \Sigma\) and \(f, g \in \mathcal{F}_+\). Then

1) \((S) \int_A f \, d\mu \leq \mu (A)\);
2) \((S) \int_A k d\mu = k \cap \mu (A)\) for a non-negative constant \(k\);
3) if \(f \leq g\) on \(A\), then \((S) \int_A f \, d\mu \leq (S) \int_A g \, d\mu\);
4) if \(A \subseteq B\), then \((S) \int_A f \, d\mu \leq (S) \int_B f \, d\mu\);
5) \(\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S) \int_A f \, d\mu \leq \alpha\);
6) if \(\mu(A) < \infty\), then \(\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S) \int_A f \, d\mu \geq \alpha\);
7) when \(A = X\), \((S) \int_A f \, d\mu = \vee_{\alpha \geq 0} (\alpha \cap \mu(L_\alpha f))\);
8) if \(\alpha \leq \beta\), then \(L_\alpha f \leq L_\beta f\);
9) \((S) \int_A f \, d\mu \geq 0\).

**Theorem 2.4.** ([7, p. 135]) Let \(f \in \mathcal{F}_+\), the class of all finite nonnegative measurable functions on \((X, \Sigma, \mu)\). Then

1) if \(\mu(A) = 0\), then \((S) \int_A f \, d\mu = 0\), for any \(f \in \mathcal{F}_+\);
2) if \((S) \int_A f \, d\mu = 0\), then \(\mu(A \cap \{x| f(x) > 0\}) = 0\);
3) \((S) \int_A f \, d\mu = (S) \int_A f \cdot \chi_A \, d\mu\), where \(\chi_A\) is the characteristic function of \(A\);
4) \((S) \int_A (f + a) \, d\mu \leq (S) \int_A f \, d\mu + (S) \int_A ad\mu\), for any constant \(a \in (0, \infty)\).

**Corollary 2.5.** ([7, p. 136]) Let \(f, f_1, f_2 \in \mathcal{F}_+\). Then

1) \((S) \int_A (f_1 \lor f_2) \, d\mu \geq (S) \int_A f_1 \, d\mu \lor (S) \int_A f_2 \, d\mu\);
2) \((S) \int_A (f_1 \land f_2) \, d\mu \leq (S) \int_A f_1 \, d\mu \land (S) \int_A f_2 \, d\mu\);
3) \((S) \int_{A \cup B} f \, d\mu \geq (S) \int_A f \, d\mu \lor (S) \int_B f \, d\mu\);
4) \((S) \int_{A \cap B} f \, d\mu \leq (S) \int_A f \, d\mu \land (S) \int_B f \, d\mu\).

In general we have

\[ (S) \int_A (f_1 + f_2) \, d\mu \neq (S) \int_A f_1 \, d\mu + (S) \int_A f_2 \, d\mu, \]

and

\[ (S) \int_A af \, d\mu \neq a(S) \int_A f \, d\mu, \text{ where } a \in \mathbb{R}, \]

see [7, p. 137].

**Lemma 2.6.** ([7, p. 138]) \((S) \int_A f \, d\mu = \infty\) if and only if \(\mu(A \cap L_\alpha f) = \infty\) for any \(\alpha \in [0, \infty)\).

We need
Definition 2.7. (2) A fuzzy measure \( \mu \) is subadditive iff
\[
\mu(A \cup B) \leq \mu(A) + \mu(B), \quad \text{for all } A, B \in \Sigma.
\]

We mention the following result

Theorem 2.8. (2) If \( \mu \) is subadditive, then
\[
(\mathcal{S}) \int_X (f + g) \, d\mu \leq (\mathcal{S}) \int_X f \, d\mu + (\mathcal{S}) \int_X g \, d\mu,
\]
for all measurable functions \( f, g : X \to [0, \infty) \).

Moreover, if (2) holds for all measurable functions \( f, g : X \to [0, \infty) \) and \( \mu(X) < \infty \), then \( \mu \) is subadditive.

Notice here in (7) we have that \( \alpha \in [0, \infty) \).

We have the following corollary.

Corollary 2.9. If \( \mu \) is subadditive, \( n \in \mathbb{N} \), and \( f : X \to [0, \infty) \) is a measurable function, then
\[
(\mathcal{S}) \int_X nf \, d\mu \leq n (\mathcal{S}) \int_X f \, d\mu,
\]
in particular it holds
\[
(\mathcal{S}) \int_A nf \, d\mu \leq n (\mathcal{S}) \int_A f \, d\mu,
\]
for any \( A \in \Sigma \).

Proof. By inequality (2). \hfill \Box

A very important property of Sugeno integral follows.

Theorem 2.10. If \( \mu \) is subadditive measure, and \( f : X \to [0, \infty) \) is a measurable function, and \( c > 0 \), then
\[
(\mathcal{S}) \int_A cf \, d\mu \leq (c + 1)(\mathcal{S}) \int_A f \, d\mu,
\]
for any \( A \in \Sigma \).

Proof. Let the ceiling \( \lceil c \rceil = m \in \mathbb{N} \), then by Theorem 2.8 (3) and (4) we get
\[
(\mathcal{S}) \int_A cf \, d\mu \leq (\mathcal{S}) \int_A mf \, d\mu \leq m(\mathcal{S}) \int_A f \, d\mu \leq (c + 1)(\mathcal{S}) \int_A f \, d\mu,
\]
proving (5). \hfill \Box
3 Main Results

From now on in this article we work on the fuzzy measure space \([a, b], \mathcal{B}, \mu\), where \([a, b] \subseteq \mathbb{R}, \mathcal{B}\) is the Borel \(\sigma\)-algebra on \([a, b]\), and \(\mu\) is a finite fuzzy measure on \(\mathcal{B}\). Typically we take it to be subadditive.

The functions \(f\) we deal with here are continuous from \([a, b]\) into \(\mathbb{R}_+\).

We make the following remark

**Remark 3.1.** Let \(f \in C^1([a, b], \mathbb{R}_+),\) and \(\mu\) is a subadditive fuzzy measure such that \(\mu([a, b]) > 0\), \(x \in [a, b]\). We will estimate

\[
E := \left| \int_{[a,b]} f(x) \, d\mu(t) - \mu([a, b]) \wedge f(x) \right| \quad (6)
\]

(by Theorem 2.3 (2))

\[
= \left| \int_{[a,b]} f(t) \, d\mu(t) - \int_{[a,b]} f(x) \, d\mu(t) \right|.
\]

We notice that

\[
f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),
\]

then (by Theorem 2.3 (3) and Theorem 2.4 (4))

\[
\int_{[a,b]} f(t) \, d\mu(t) \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t) + \int_{[a,b]} f(x) \, d\mu(t), \quad (7)
\]

that is

\[
\int_{[a,b]} f(t) \, d\mu(t) - \int_{[a,b]} f(x) \, d\mu(t) \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t). \quad (8)
\]

Similarly, we have

\[
f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),
\]

then (by Theorem 2.3 (3) and Theorem 2.4)

\[
\int_{[a,b]} f(x) \, d\mu(t) \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t) + \int_{[a,b]} f(t) \, d\mu(t),
\]

that is

\[
\int_{[a,b]} f(x) \, d\mu(t) - \int_{[a,b]} f(t) \, d\mu(t) \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t). \quad (9)
\]

By (8) and (9) we derive that

\[
\left| \int_{[a,b]} f(t) \, d\mu(t) - \int_{[a,b]} f(x) \, d\mu(t) \right| \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t). \quad (10)
\]
Consequently it holds
\[ E \left( \frac{\|\mathcal{D}_f\|}{\mu([a,b])} \right) \int_{[a,b]} |f(t) - f(x)| \, d\mu(t) \]

(and by $|f(t) - f(x)| \leq \|f\|_{\infty} |t - x|$)
\[ \leq (S) \int_{[a,b]} \|f\|_{\infty} |t - x| \, d\mu(t) \leq (\|f\|_{\infty} + 1) (S) \int_{[a,b]} |t - x| \, d\mu(t). \]  \tag{11}

We have proved the following Ostrowski-like inequality
\[ \left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \frac{\mu([a,b] \land f(x))}{\mu([a,b])} \right| \leq \frac{\|f\|_{\infty} + 1}{\mu([a,b])} (S) \int_{[a,b]} |t - x| \, d\mu(t). \]  \tag{12}

The last inequality can be better written as follows:
\[ \left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \land \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{\|f\|_{\infty} + 1}{\mu([a,b])} (S) \int_{[a,b]} |t - x| \, d\mu(t). \]  \tag{13}

Notice here that \( 1 \land \frac{f(x)}{\mu([a,b])} \leq 1 \), and
\[ \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) \leq \frac{\mu([a,b])}{\mu([a,b])} = 1, \quad \text{where} \quad (S) \int_{[a,b]} f(t) \, d\mu(t) \geq 0. \]

I.e. If \( f : [a,b] \to \mathbb{R}_+ \) is a Lipschitz function of order \( 0 < \alpha \leq 1 \), i.e. \( |f(x) - f(y)| \leq K|x - y|^\alpha \), \( \forall x, y \in [a,b] \), where \( K > 0 \), denoted by \( f \in \text{Lip}_a,K ([a,b], \mathbb{R}_+) \), then we get similarly the following Ostrowski-like inequality:
\[ \left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \land \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{(K + 1)}{\mu([a,b])} (S) \int_{[a,b]} |t - x|^\alpha \, d\mu(t). \]  \tag{14}

We have proved the following Ostrowski-Sugeno inequalities:

**Theorem 3.2.** Suppose that \( \mu \) is a fuzzy subadditive measure with \( \mu([a,b]) > 0 \), \( x \in [a,b] \).

1) Let \( f \in C^1 ([a,b], \mathbb{R}_+) \), then
\[ \left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \land \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{\|f\|_{\infty} + 1}{\mu([a,b])} (S) \int_{[a,b]} |t - x| \, d\mu(t). \]  \tag{15}
2) Let \( f \in \text{Lip}_{\alpha,K} ([a, b], \mathbb{R}_+) \), \( 0 < \alpha \leq 1 \), then
\[
\left| \frac{1}{\mu([a, b])} \right| (S) \int_{[a, b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu([a, b])} \right) \leq \frac{(K + 1)}{\mu([a, b])} (S) \int_{[a, b]} |t - x|^\alpha \, d\mu(t).
\]
(16)

We make the following remark

**Remark 3.3.** Let \( f \in C^1 ([a, b], \mathbb{R}_+) \) and \( g \in C^1 ([a, b]) \), by Cauchy’s mean value theorem we get that
\[
(f(t) - f(x))' = \left( \frac{f(x)}{g'(c)} \right) (g(t) - g(x)),
\]
for some \( c \) between \( t \) and \( x \); for any \( t, x \in [a, b] \).

If \( g'(c) \neq 0 \), we have
\[
(f(t) - f(x)) = \left( \frac{f'(c)}{g'(c)} \right) (g(t) - g(x)).
\]

Here we assume that \( g'(t) \neq 0 \), \( \forall \, t \in [a, b] \). Hence it holds
\[
|f(t) - f(x)| \leq \left\| \frac{f'}{g'} \right\|_\infty |g(t) - g(x)|,
\]
(17)
for all \( t, x \in [a, b] \).

We have again as before (see (11))
\[
E \leq (S) \int_{[a, b]} |f(t) - f(x)| \, d\mu(t) \overset{(\text{by } (11))}{\leq} \left( \left\| \frac{f'}{g'} \right\|_\infty + 1 \right) (S) \int_{[a, b]} |g(t) - g(x)| \, d\mu(t).
\]
(18)

We have established the following general Ostrowski-Sugeno inequality:

**Theorem 3.4.** Suppose that \( \mu \) is a fuzzy subadditive measure with \( \mu([a, b]) > 0 \), \( x \in [a, b] \). Let \( f \in C^1 ([a, b], \mathbb{R}_+) \) and \( g \in C^1 ([a, b]) \) with \( g'(t) \neq 0 \), \( \forall \, t \in [a, b] \). Then
\[
\left| \frac{1}{\mu([a, b])} \right| (S) \int_{[a, b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu([a, b])} \right) \leq \left( \left\| \frac{f'}{g'} \right\|_\infty + 1 \right) (S) \int_{[a, b]} |g(t) - g(x)| \, d\mu(t).
\]
(19)
We give for \( g(t) = e^t \) the next result

**Corollary 3.5.** Suppose that \( \mu \) is a fuzzy subadditive measure with \( \mu([a, b]) > 0 \), \( x \in [a, b] \). Let 
\[ f \in C^1([a, b], \mathbb{R}_+) \], then
\[
\left| \frac{1}{\mu([a, b])} \left[ \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right] \right| \leq \left( \frac{\|f'\|_\infty + 1}{\mu([a, b])} \right) \left( S \int_{[a,b]} |e^t - e^x| \, d\mu(t) \right). \tag{20}
\]

When \( g(t) = \ln t \) we get the following corollary.

**Corollary 3.6.** Suppose that \( \mu \) is a fuzzy subadditive measure with \( \mu([a, b]) > 0 \), \( x \in [a, b] \) and \( a > 0 \). Let 
\[ f \in C^1([a, b], \mathbb{R}_+) \]. Then
\[
\left| \frac{1}{\mu([a, b])} \left[ \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right] \right| \leq \left( \frac{\|tf'(t)\|_\infty + 1}{\mu([a, b])} \right) \left( S \int_{[a,b]} \left| \ln \frac{t}{x} \right| \, d\mu(t) \right). \tag{21}
\]

Many other applications of Theorem 3.4 could follow but we stop it here.

We make the following remark.

**Remark 3.7.** Let \( f \in C([a, b], \mathbb{R}_+) \cap C^{n+1}([a, b]) \), \( n \in \mathbb{N} \), \( x \in [a, b] \). Then by Taylor’s theorem we get
\[
f(y) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} (y - x)^k + R_n(x, y), \tag{22}\]
where the remainder
\[
R_n(x, y) := \int_{x}^{y} \left( f^{(n)}(t) - f^{(n)}(x) \right) \frac{(y - t)^{n-1}}{(n-1)!} \, dt; \tag{23}\]
here \( y \) can be \( \geq x \) or \( \leq x \).

By [1] we get that
\[
|R_n(x, y)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n + 1)!} |y - x|^{n+1}, \quad \text{for all } x, y \in [a, b]. \tag{24}\]

Here we assume \( f^{(k)}(x) = 0 \), for all \( k = 1, \ldots, n \).

Therefore it holds
\[
|f(t) - f(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n + 1)!} |t - x|^{n+1}, \quad \text{for all } t, x \in [a, b]. \tag{25}\]
Here we have again

\[
E \leq \int_{[a,b]} |f(t) - f(x)| \, d\mu(t) \quad (\text{by Theorem } 2.3 \text{ and } 25)
\]

\[
(S) \int_{[a,b]} \frac{||f^{(n+1)}||_\infty}{(n+1)!} |t-x|^{n+1} \, d\mu(t) \quad (\text{by } 24)
\]

\[
\left( \frac{||f^{(n+1)}||_\infty}{(n+1)!} + 1 \right) (S) \int_{[a,b]} |t-x|^{n+1} \, d\mu(t).
\]  \hspace{1cm} (26)

We have derived the following high order Ostrowski-Sugeno inequality:

**Theorem 3.8.** Let \( f \in [C([a,b], \mathbb{R}_+)] \cap \mathcal{C}([a,b]), n \in \mathbb{N}, x \in [a,b] \). We assume that \( f^{(k)}(x) = 0 \), all \( k = 1, \ldots, n \). Here \( \mu \) is subadditive with \( \mu([a,b]) > 0 \). Then

\[
\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(x)}{\mu([a,b])} \right) \right| \leq
\]

\[
\left( \frac{||f^{(n+1)}||_\infty}{(n+1)!} + 1 \right) \frac{\mu([a,b])}{\mu([a,b])} (S) \int_{[a,b]} |t-x|^{n+1} \, d\mu(t),
\]  \hspace{1cm} (27)

which generalizes (15).

When \( x = \frac{a+b}{2} \) we get the following corollary

**Corollary 3.9.** Let \( f \in [C([a,b], \mathbb{R}_+)] \cap \mathcal{C}([a,b]), n \in \mathbb{N} \). Assume that \( f^{(k)}(\frac{a+b}{2}) = 0 \), \( k = 1, \ldots, n \). Here \( \mu \) is subadditive with \( \mu([a,b]) > 0 \). Then

\[
\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left( 1 \wedge \frac{f(\frac{a+b}{2})}{\mu([a,b])} \right) \right| \leq
\]

\[
\left( \frac{||f^{(n+1)}||_\infty}{(n+1)!} + 1 \right) \frac{\mu([a,b])}{\mu([a,b])} (S) \int_{[a,b]} \left| t - \frac{a+b}{2} \right|^{n+1} \, d\mu(t).
\]  \hspace{1cm} (28)
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