NOTE ON BOUNDS FOR EIGENVALUES USING TRACES

R. Sharma, R. Kumar and R. Saini
Department of Mathematics, Himachal Pradesh University, Shimla 171 005, India
email: rajesh_hpu_math@yahoo.co.in

Abstract. We show that various old and new bounds involving eigenvalues of a complex $n \times n$ matrix are immediate consequences of the inequalities involving variance of real and complex numbers.

MSC 2010: 15A18, 15A45, 65F35

Keywords: Variance, Samuelson’s inequality, trace, eigenvalue.
1 Introduction

It is useful to have bounds for eigenvalues and spread in terms of the functions of entries of the given matrix. Such bounds have been studied extensively in literature, see [1-18]. Bounds on eigenvalues and spread of a matrix $A$ in terms of the traces of $A$ and $A^2$ are of special interests. These bounds are in fact the immediate consequences of inequalities for real or complex numbers. In this note we point out some more such bounds related to the variance of real and complex numbers.

Let $x_1, x_2, \ldots, x_n$ denote $n$ real numbers. Their arithmetic mean is the number

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad (1.1)$$

and the variance is

$$S_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2. \quad (1.2)$$

Samuelson’s inequality [12] says that

$$S_x^2 \geq \frac{1}{n-1} (x_j - \bar{x})^2, \quad (1.3)$$

for all $j = 1, 2, \ldots, n$.

The Nagy inequality [11] gives a lower bound for the variance,

$$S_x^2 \geq \frac{1}{2n} \max_{i,j} (x_i - x_j)^2, \quad (1.4)$$

A more general inequality due to Fahmy and Prochan [7] says that for $x_1 \leq x_2 \leq \ldots \leq x_n$, we have

$$S_x^2 \geq \frac{l(n-k+1)}{n(n+l-k+1)} (x_k - x_l)^2, \quad (1.5)$$

where $1 \leq l < k \leq n$.

We here study some extensions of these inequalities for the complex numbers and discuss their applications. Let $z_1, z_2, \ldots, z_n$ denote $n$ complex numbers. Their arithmetic mean is the number

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i, \quad (1.6)$$

Let

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (z_i - \bar{z})^2, \quad (1.7)$$
and
\[ S_z^2 = \frac{1}{n} \sum_{i=1}^{n} |z_i - \bar{z}|^2. \]  \hfill (1.8)

For \( z_i = x_i + iy_i \), we have \( S_z^2 = S_x^2 + S_y^2 \). On substituting \( 2x_i = z_i + \bar{z}_i \) in (1.2), we get that
\[ 2S_x^2 = S_z^2 + \text{Re} S^2 \quad \text{and} \quad 2S_y^2 = S_z^2 - \text{Re} S^2. \]  \hfill (1.9)

We first prove some basic inequalities involving real and complex numbers in the following lemmas, and use these inequalities to derive several bounds for the eigenvalues in Section 2.

**Lemma 1.1.** Let \( z_i = x_i + iy_i, \ i = 1, 2, \ldots, n \). For \( x_1 \leq x_2 \leq \ldots \leq x_n \),
\[ |x_k - x_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} \left( S_z^2 + \text{Re} S^2 \right), \]  \hfill (1.10)

For \( y_1 \leq y_2 \leq \ldots \leq y_n \),
\[ |y_k - y_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} \left( S_z^2 - \text{Re} S^2 \right). \]  \hfill (1.11)

Also, the inequalities
\[ |z_k - z_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} \left( S_z^2 + |S^2| \right), \]  \hfill (1.12)

hold for some permutation of complex numbers \( z_i, i = 1, 2, \ldots, n \), and \( 1 \leq l < k \leq n \).

**Proof.** For any complex number \( z \) there is a complex number \( \alpha \) with \( |\alpha| = 1 \) such that \( \text{Re}(\alpha z) = |z| \). Therefore, for \( x_i = \text{Re}(\alpha z_i) \) we can choose \( \alpha \) such that \( |\alpha| = 1 \) and
\[ |x_k - x_l| = |\text{Re}(\alpha z_k) - \text{Re}(\alpha z_l)| = |z_k - z_l|. \]  \hfill (1.13)

Without restricting generality, assume that \( z_1, z_2, \ldots, z_n \) is a permutation of complex numbers \( z_i \) such that \( \text{Re}(\alpha z_1) \leq \text{Re}(\alpha z_2) \leq \ldots \leq \text{Re}(\alpha z_n) \). Put \( x_i = \text{Re}(\alpha z_i) \) in (1.2), we find that
\[ 2S_x^2 = S_z^2 + \frac{1}{n} \text{Re} \left( \alpha^2 \sum_{i=1}^{n} (z_i - \bar{z})^2 \right). \]  \hfill (1.14)

For \( |\alpha| = 1 \), we have
\[ \left| \frac{1}{n} \text{Re} \left( \alpha^2 \sum_{i=1}^{n} (z_i - \bar{z})^2 \right) \right| \leq |S^2|. \]  \hfill (1.15)
It follows from (1.14) and (1.15) that for any complex number $\alpha$ with $|\alpha| = 1$ and $x_i = \text{Re}(\alpha z_i)$, we have
\[
2S_x^2 \leq S_z^2 + |S|^2.
\]
(1.16)
Substituting (1.13) in (1.5) and use (1.16), we immediately get (1.12).
The inequalities (1.10) and (1.11) follow immediately on using (1.9) in (1.5).

**Lemma 1.2.** Let
\[
\mu_2 = \sum_{i=1}^{n} p_i \left( x_i - \mu_1' \right)^2,
\]
(1.17)
where $\mu_1' = \sum_{i=1}^{n} p_i x_i$ and $p_i$ are non-negative real numbers such that $\sum_{i=1}^{n} p_i = 1$ and $p_j \neq 1$. Then
\[
\mu_2 \geq \frac{p_j}{1 - p_j} \left( \mu_1' - x_j \right)^2,
\]
(1.18)
for all $j = 1, 2, \ldots, n$.

**Proof.** From (1.17), we have
\[
\mu_2 = p_j \left( x_j - \mu_1' \right)^2 + (1 - p_j) \frac{1}{1 - p_j} \sum_{i=1, i \neq j}^{n} p_i \left( x_i - \mu_1' \right)^2.
\]
(1.19)
It follows from the Cauchy-Schwarz inequality that
\[
\frac{1}{1 - p_j} \sum_{i=1, i \neq j}^{n} p_i \left( x_i - \mu_1' \right)^2 \geq \left( \frac{1}{1 - p_j} \sum_{i=1, i \neq j}^{n} p_i \left( x_i - \mu_1' \right) \right)^2.
\]
(1.20)
On the other hand, the sum of all the deviations from the mean is zero, therefore
\[
\sum_{i=1}^{n} p_i \left( x_i - \mu_1' \right) = 0,
\]
and we get that
\[
\sum_{i=1, i \neq j}^{n} p_i \left( x_i - \mu_1' \right) = p_j \left( x_j - \mu_1' \right).
\]
(1.21)
Combining (1.20) and (1.21), we find that
\[
\frac{1}{1 - p_j} \sum_{i=1, i \neq j}^{n} p_i \left( x_i - \mu_1' \right)^2 \geq \left( \frac{p_j \left( x_j - \mu_1' \right)}{1 - p_j} \right)^2.
\]
(1.22)
Insert (1.22) in (1.19), a little computation leads to (1.18).
Lemma 1.3. For \( x_1 \leq x_2 \leq \ldots \leq x_n \), the inequality
\[
S_x^2 \geq \frac{k}{n-k} (\bar{x} - x_j)^2
\]
holds for \( j = k \) and \( j = n - k + 1 \) with \( k \leq n - k + 1 \).

Proof. For \( \bar{x} \geq x_k \), we have
\[
S_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \geq \frac{k}{n} (x_k - \bar{x})^2 + \frac{1}{n} \sum_{i=k+1}^{n} (x_i - \bar{x})^2.
\]
Let
\[
\bar{y} = \frac{1}{n} \left( k x_k + \sum_{i=k+1}^{n} x_i \right),
\]
and
\[
S_y^2 = \frac{1}{n} \left( k (x_k - \bar{y})^2 + \sum_{i=k+1}^{n} (x_i - \bar{y})^2 \right).
\]
It is clear that \( \bar{y} \geq \bar{x} \) and
\[
S_x^2 \geq \frac{k}{n} (x_k - \bar{x})^2 + \frac{1}{n} \sum_{i=k+1}^{n} (x_i - \bar{x})^2 \geq S_y^2.
\]
Using Lemma 1.2, we have
\[
S_x^2 \geq S_y^2 \geq \frac{k}{n-k} (\bar{y} - x_k)^2 \geq \frac{k}{n-k} (\bar{x} - x_k)^2,
\]
for \( k = 1, 2, \ldots, n - 1 \).

On using similar argument we find that for \( \bar{x} \leq x_k \), we have
\[
S_x^2 \geq \frac{n-k+1}{k-1} (x_k - \bar{x})^2,
\]
k = 2, 3, .., n. Thus,
\[
S_x^2 \geq \min \left\{ \frac{k}{n-k} (\bar{x} - x_k)^2, \frac{n-k+1}{k-1} (x_k - \bar{x})^2 \right\},
\]
for \( k = 2, 3, \ldots, n - 1 \). The assertions of lemma now follow from (1.25) and the fact that
\((n-k+1)(n-k) \geq k(k-1)\) if and only if \( k \leq n - k + 1 \). ■

Lemma 1.4. Let \( z_i = x_i + iy_i \), \( i = 1, 2, \ldots n \). For \( x_1 \leq x_2 \leq \ldots \leq x_n \), the inequalities
\[
S_z^2 + \text{Re} S^2 \geq \frac{2k}{n-k} \left| \frac{1}{n} \sum_{i=1}^{n} x_i - x_j \right|^2,
\]
(1.26)
hold for \( j = k \) and \( j = n - k + 1 \) with \( k \leq n - k + 1 \). Likewise, for \( y_1 \leq y_2 \leq \ldots \leq y_n \), we have

\[
S_z^2 - \text{Re} S^2 \geq \frac{2k}{n-k} \left| \frac{1}{n} \sum_{i=1}^{n} y_i - y_j \right|^2.
\]

Also,

\[
S_z^2 + |S^2| \geq \frac{2k}{n-k} \left| \frac{1}{n} \sum_{i=1}^{n} z_i - z_j \right|.
\]

**Proof.** Use (1.6)-(1.9), (1.16) and Lemma 1.3; we immediately get the inequalities (1.26)-(1.28).

**Lemma 1.5.** For \( n \) complex numbers \( z_i \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} |z_i - \tilde{z}|^{2r} \geq \frac{1 + (n-1)^{2r-1}}{n(n-1)^{2r-1}} |z_j - \tilde{z}|^{2r},
\]

where \( \tilde{z} \) is given in (1.6).

**Proof.** We write

\[
\frac{1}{n} \sum_{i=1}^{n} |z_i - \tilde{z}|^{2r} = \frac{|z_j - \tilde{z}|^{2r}}{n} + \frac{n-1}{n} \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}|^{2r}.
\]

For \( m \) positive real numbers \( y_i, i = 1, 2, \ldots, m \),

\[
\frac{1}{m} \sum_{i=1}^{m} y_i^k \geq \left( \frac{1}{m} \sum_{i=1}^{m} y_i \right)^k, \quad k = 1, 2, \ldots.
\]

Applying (1.31) to \( n - 1 \) positive real numbers \( |z_i - \tilde{z}|^{2r}, i = 1, 2, \ldots, n \) and \( i \neq j \), we get

\[
\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (|z_i - \tilde{z}|^{2r})^r \geq \left( \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}|^2 \right)^r.
\]

Also, we have

\[
\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}|^2 \geq \left( \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}| \right)^2.
\]

On the other hand the sum of all the deviations from the mean is zero, therefore

\[
\sum_{i=1}^{n} (z_i - \bar{z}) = 0,
\]
and we get that
\[ |\tilde{z} - z_j| \leq \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}|. \]  
(1.34)

Combining (1.31)-(1.34), we find that
\[ \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} |z_i - \tilde{z}|^2 \geq \left( \frac{|z_j - \tilde{z}|}{n-1} \right)^2. \]  
(1.35)

Insert (1.35) in (1.30), a little computation leads to (1.29). ■

2 Bounds on eigenvalues using traces

Let \( A = (a_{ij}) \) be an \( n \times n \) complex matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Let \( \lambda_i = \alpha_i + i\beta_i, \ i = 1, 2, \ldots, n \). Then
\[ \tilde{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{\text{tr}A}{n}, \]  
(2.1)
\[ S^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \lambda_i - \tilde{\lambda} \right)^2 = \frac{\text{tr}A^2}{n} - \left( \frac{\text{tr}A}{n} \right)^2, \]  
(2.2)
and
\[ S^2_{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \left| \lambda_i - \tilde{\lambda} \right|^2 = \frac{1}{n} \sum_{i=1}^{n} |\lambda_i|^2 - \frac{\left| \text{tr}A \right|^2}{n}, \]  
(2.3)
where \( \text{tr}A \) denotes the trace of \( A \). Likewise, we have
\[ S^2_{\alpha} = \frac{1}{n} \sum_{i=1}^{n} (\alpha_i - \tilde{\alpha})^2 = \frac{S^2_{\lambda} + \text{Re} S^2}{2}, \]  
(2.4)
and
\[ S^2_{\beta} = \frac{1}{n} \sum_{i=1}^{n} (\beta_i - \tilde{\beta})^2 = \frac{S^2_{\lambda} - \text{Re} S^2}{2}. \]  
(2.5)

Theorem 2.1. Let \( A \) be an \( n \times n \) complex matrix. Then, the disk
\[ \left| z - \frac{\text{tr}A}{n} \right| \leq \sqrt{\frac{n-k}{2k} \left( S^2_{\lambda} + |S|^2 \right)}, \]  
(2.6)
contains at least \( n - 2k + 2 \) eigenvalues of \( A \) where \( k \) is a positive integer less than or equal to \( \frac{n+1}{2} \). Also, the disks
\[ \left| x - \frac{\text{Re} \text{tr}A}{n} \right| \leq \sqrt{\frac{n-k}{2k} \left( S^2_{\lambda} + \text{Re} S^2 \right)} \]  
(2.7)
and

\[ |y - \frac{\text{Im tr} A}{n}| \leq \sqrt{\frac{n - k}{2k}} (S_\lambda^2 - \text{Re} S^2), \tag{2.8} \]

contains real and imaginary parts of eigenvalues, respectively.

**Proof.** Let \( x_i = \text{Re} (\alpha \lambda_i) \). Then, there is a complex number \( \alpha \) with \( |\alpha| = 1 \) such that

\[ |x_i - \frac{1}{n} \sum_{i=1}^{n} x_i| = \left| \text{Re} \left( \lambda_i - \frac{\text{tr} A}{n} \right) \right| = \left| \lambda_i - \frac{\text{tr} A}{n} \right|. \]

It follows from Lemma 1.4 that

\[ \left| \lambda_i - \frac{\text{tr} A}{n} \right| \leq \sqrt{\frac{n - k}{2k}} \left( S_\lambda^2 + |S|^2 \right), \tag{2.9} \]

for \( i = k \) and \( i = n - k + 1 \) with \( k \leq n - k + 1 \). Let \( D_k \) denote the \( k^{th} \) disk in (2.9).

It is easy to see that \( D_{\frac{n+1}{2}} \subseteq D_{\frac{n-1}{2}} \subseteq \ldots \subseteq D_2 \subseteq D_1 \) when \( n \) is odd. By Lemma 1.4, the eigenvalue \( \lambda_{\frac{n+1}{2}} \) lies in the disk \( D_{\frac{n+1}{2}} \), while \( \lambda_{\frac{n-1}{2}} \) and \( \lambda_{\frac{n+3}{2}} \) lie in \( D_{\frac{n-1}{2}} \). So the disk \( D_{\frac{n-1}{2}} \) contains at least three eigenvalues. Repeating the above process, we can easily see that the disk \( D_k \) contains \( n - 2k + 2 \) eigenvalues. Likewise, the inequalities (2.7) and (2.8) follow from (1.26) and (1.27), respectively. \( \blacksquare \)

For a normal matrix, we have

\[ S_\lambda^2 = \frac{\text{tr} A A^*}{n} - \left| \frac{\text{tr} A}{n} \right|^2. \tag{2.10} \]

Using this in (2.6)-(2.8), we can calculate the corresponding upper bounds and hence the regions containing eigenvalues. For arbitrary matrices, we can use the various upper bounds for \( \sum_{i=1}^{n} |\lambda_i|^2 \), see [17]. For special case \( k = 1 \), we get Theorem 2.7 of Huang and Wang [8]. It is clear that if rank of matrix is \( m \); we can replace \( n \) by \( m \).

**Theorem 2.2.** Let \( A \) be an \( n \times n \) complex matrix with eigenvalues \( \lambda_i = \lambda_i = \alpha_i + i\beta_i \), \( i = 1, 2, \ldots n \). For \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \), the inequalities

\[ |\alpha_k - \alpha_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} (S_\lambda^2 + \text{Re} S^2), \tag{2.11} \]

hold for \( 1 \leq l < k \leq n \), and for \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \)

\[ |\beta_k - \beta_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} (S_\lambda^2 - \text{Re} S^2). \tag{2.12} \]
Also, the inequalities
\[ |\lambda_k - \lambda_l|^2 \leq \frac{n(n + l - k + 1)}{2l(n - k + 1)} \left( S^2_\lambda + |S^2| \right), \]  
hold for some permutation of complex numbers \( z_i, i = 1, 2, ..., n \) and \( 1 \leq l < k \leq n \).

**Proof.** The assertions of the theorem follow easily from Lemma 1.1. ■

It may be noted here that for \( l = 1 \) and \( k = n \), Theorem 2.2 provides bounds for the spread, \( \text{Spd}(A) = \max_{i,j} |\lambda_i - \lambda_j| \).

**Corollary 2.3.** Let \( A \) be an \( n \times n \) complex matrix with at least two distinct eigenvalues. Let \( \lambda_k = \alpha_k + i\beta_k, k = 1, 2, ... n \) be any eigenvalue of \( A \). For \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \), the disk
\[ |x - \alpha_l| \leq \frac{n}{\sqrt{2(n - 1)}} \sqrt{S^2_\alpha + \text{Re} S^2}, \]  
contains real part of one more eigenvalue. For \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \)
\[ |y - \beta_1| \leq \frac{n}{\sqrt{2(n - 1)}} \sqrt{S^2_\beta - \text{Re} S^2}, \]  
contains imaginary part of one more eigenvalue. Also, the disk
\[ |z - \lambda_l| \leq \frac{n}{\sqrt{2(n - 1)}} \sqrt{S^2_\lambda + |S^2|}, \]  
contains one more eigenvalue of \( A \).

**Proof.** It follows from Theorem 2.2 that for \( k = l + 1 \),
\[ 2 |\alpha_{l+1} - \alpha_l|^2 \leq \frac{n^2}{2l(n - l)} \left( S^2_\lambda + \text{Re} S^2 \right), \]  
\[ 2 |\beta_{l+1} - \beta_l|^2 \leq \frac{n^2}{2l(n - l)} \left( S^2_\beta - \text{Re} S^2 \right), \]  
and
\[ 2 |\lambda_{l+1} - \lambda_l|^2 \leq \frac{n^2}{l(n - l)} \left( S^2_\lambda + |S^2| \right), \]  
\( l = 1, 2, ..., n - 1 \). The inequalities (2.14)-(2.16) respectively follow from (2.17)-(2.19), and the fact that \( l(n - l) \geq n - 1 \) for \( l = 1, 2, ..., n - 1 \). ■

The above corollary is useful when one of the eigenvalues is known. For example, for a singular matrix one eigenvalue is zero.
Theorem 2.4. Let $A$ be an $n \times n$ complex matrix. Then all the eigenvalues of $A$ are contained in the disk

$$|z - \frac{\text{tr}A}{n}| \leq \left( \frac{(n-1)^{2r-1}}{1+(n-1)^{2r-1}} \sum_{i=1}^{n} |\lambda_i(B)|^{2r} \right)^{\frac{1}{2r}},$$

(2.20)

where $B = A - \frac{\text{tr}A}{n} I$ and $r = 1, 2, \ldots$.

Proof. Let $\lambda_i$ be eigenvalues of $A$, then the eigenvalues of $B^r$ are $(\lambda_i - \frac{\text{tr}A}{n})^r$, $i = 1, 2, \ldots, n$. The assertions of the theorem now follow on using Lemma 1.5. $\blacksquare$

Note that

$$\sum_{i=1}^{n} |\lambda_i(B)|^{2r} = \sum_{i=1}^{n} |\lambda_i(B)^r|^2 \leq \sqrt{\left(\|B^r\|_2^2 - \frac{1}{n} |\text{tr}B^r|^2\right)^2 - \frac{1}{2} \|B^r(B^r)^* - (B^r)^*B^r\|_2^2 + \frac{1}{n} |\text{tr}B^r|^2}.$$ 

Using this and other relation in (2.20), we can obtain various bounds for eigenvalues.

Theorem 2.5. Let $A$ be an $n \times n$ complex normal matrix. Then, one eigenvalue of $A$ lies on or outside the circle

$$|z - \frac{\text{tr}A}{n}| = \left(\text{tr}B^r(B^r)^*\right)^{\frac{1}{2r}}.$$ 

(2.21)

Proof. The proof of theorem follows from the fact that

$$\frac{\text{tr}B^r(B^r)^*}{n} = \frac{1}{n} \sum_{i=1}^{n} \left|\lambda_i - \frac{\text{tr}A}{n}\right|^{2r} \leq \left|\lambda_j - \frac{\text{tr}A}{n}\right|^{2r}.$$ 

$\blacksquare$

Let the eigenvalues of a complex $n \times n$ matrix $A$ are all real, as in case of a Hermitian matrix. Wolkowicz and Styan [18] have shown that

$$\lambda_{\text{max}} \geq \frac{\text{tr}A}{n} + \sqrt{\frac{\text{tr}B^2}{n(n-1)}}$$

and

$$\lambda_{\text{min}} \leq \frac{\text{tr}A}{n} - \sqrt{\frac{\text{tr}B^2}{n(n-1)}}.$$ 

We prove extensions of these inequalities in the following theorem.
Theorem 2.6. Let $A$ be an $n \times n$ complex matrix with real eigenvalues $\lambda_i, i = 1, 2, \ldots, n$. Then
\[
\lambda_{\text{max}} \geq \frac{\text{tr}A}{n} + \frac{\text{tr}B^2}{n} \left( \frac{1 + (n - 1)^{2r-1}}{(n - 1)^{2r-1} \text{tr}B^{2r}} \right)^{\frac{1}{2r}} \tag{2.22}
\]
and
\[
\lambda_{\text{min}} \leq \frac{\text{tr}A}{n} - \frac{\text{tr}B^2}{n} \left( \frac{1 + (n - 1)^{2r-1}}{(n - 1)^{2r-1} \text{tr}B^{2r}} \right)^{\frac{1}{2r}}, \tag{2.23}
\]
where $B = A - \frac{\text{tr}A}{n} I$.

Proof. We have [3]
\[
\frac{1}{n} \text{tr}B^2 \leq \left( \lambda_{\text{max}} - \frac{\text{tr}A}{n} \right) \left( \frac{\text{tr}A}{n} - \lambda_{\text{min}} \right).
\]
Therefore,
\[
\lambda_{\text{max}} \geq \frac{\text{tr}A}{n} + \frac{1}{n} \frac{\text{tr}B^2}{\text{tr}A/n - \lambda_{\text{min}}}. \tag{2.24}
\]
It follows from Theorem 2.4 that
\[
\lambda_{\text{max}} \leq \frac{\text{tr}A}{n} + \left( \frac{(n - 1)^{2r-1}}{1 + (n - 1)^{2r-1} \text{tr}B^{2r}} \right)^{\frac{1}{2r}}. \tag{2.25}
\]
Combining (2.24) and (2.25), we get (2.23). The inequality (2.22) follows on using similar arguments. ■

Example 1. Let
\[
A = \begin{bmatrix}
4 & 0 & 2 & 3 \\
0 & 5 & 0 & 1 \\
2 & 0 & 6 & 0 \\
3 & 1 & 0 & 7
\end{bmatrix}.
\]
The estimates of Wolkowicz and Styan [18] give $\lambda_{\text{max}} \geq 7.1583$ and $\lambda_{\text{min}} \leq 3.841$ while our estimates (2.22) and (2.23) for $r = 2$ give $\lambda_{\text{max}} \geq 7.2586$ and $\lambda_{\text{min}} \leq 3.7414$, respectively.

Acknowledgements. The authors are grateful to Prof. Rajendra Bhatia for the useful discussions and suggestions. The first two authors thank I.S.I. Delhi for a visit in January 2014 when this work had begun.
References

[1] E.R. Barnes, A.J. Hoffman, Bounds for the spectrum of normal matrices, *Linear Algebra Appl.*, 201:79-90, 1994.

[2] R. Bhatia, Matrix Analysis, *Springer Verlag New York*, 2000.

[3] R. Bhatia, C. Davis, A better bound on the variance, *Amer. Math. Monthly*, 107:353-357, 2000.

[4] R. Bhatia, R. Sharma, Some inequalities for positive linear maps, *Linear Algebra Appl.*, 436:1562-1571, 2012.

[5] R. Bhatia, R. Sharma, Positive linear maps and spreads of matrices, *Amer. Math. Monthly*, 121:619-624, 2014.

[6] R. Bhatia, R. Sharma, Positive linear maps and spreads of matrices-II, Preprint.

[7] S. Fahmy, F. Proschan, Bounds on differences of order statistics, *J. Amer. Statist. Assoc.*, 35:46-47, 1981.

[8] T.Z. Huang, L. Wang, Improving upper bounds for eigenvalues of complex matrices using traces, *Linear Algebra Appl.*, 426:841-854, 2007.

[9] E. Jiang, X. Zhan, Lower bounds for the spread of a Hermitian matrices, *Linear Algebra Appl.*, 256:153-163, 1997.

[10] C.R. Johnson, R. Kumar, H. Wolkowicz, Lower bounds for the spread of a matrix, *Linear Algebra Appl.*, 71:161-173, 1985.

[11] J.V.S. Nagy, Uber algebraische Gleichungen mit lauter reellen Wurzeln [in German], *Jahresber. Deutsch. Math.-Verein.*, 27:37-43, 1918.

[12] P.A. Samuelson, How deviant can you be?, *J. Amer. Statist. Assoc.*, 63:1522-1525, 1968.

[13] P. Tarazarga, Eigenvalue estimates for symmetric matrices, *Linear Algebra Appl.*, 135:171-179, 1990.

[14] K. J. Merikoski, R. Kumar, Characterization and lower bounds for the spread of a normal matrix, *Linear Algebra Appl.* 346:13-31, 2003.
[15] L. Mirsky, The spread of a matrix, *Mathematika* 3:127-130, 1956.

[16] L. Mirsky, Inequalities for normal and Hermitian matrices, *Duke Math. J.* 24:591-598, 1957.

[17] R. Sharma, R. Kumar, Remark on upper bounds for the spread of a matrix, *Linear Algebra Appl.*, 438:4359-4362, 2013.

[18] H. Wolkowicz, G.P.H. Styan, Bounds for eigenvalues using traces, *Linear Algebra Appl.*, 29:471-506, 1980.