BERGMAN KERNEL ALONG THE KÄHLER RICCI FLOW AND TIAN’S CONJECTURE

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ABSTRACT. In this paper, we study the behavior of Bergman kernels along the Kähler Ricci flow on Fano manifolds. We show that the Bergman kernels are equivalent along the Kähler Ricci flow under certain condition on the Ricci curvature of the initial metric. Then, using a recent work of Tian and Zhang, we can solve a conjecture of Tian for Fano manifolds of complex dimension ≤ 3.

1. INTRODUCTION

A Fano manifold is a compact Kähler manifold with positive first Chern class. It has been one of main problems in Kähler geometry to study if a Fano manifold admits a Kähler-Einstein metric since the Calabi-Yau theorem on Ricci-flat Kähler metrics in 70’s and the Aubin-Yau theorem on Kähler-Einstein metrics with negative scalar curvature. This problem is more difficult because there are new obstructions to the existence. It was conjectured that the existence of Kähler-Einstein metrics on M is equivalent to the K-stability, the Yau-Tian-Donaldson conjecture in the case of Fano manifolds.

Theorem 1.1. Let M be a Fano manifold without non-zero holomorphic vector fields, then M admits a Kähler-Einstein metric if and only if it is K-stable.

The necessary part of this theorem is proved by Tian in [29]. Last Fall, Tian gave a proof for the sufficient part (see [31]) by establishing the partial $C^0$-estimate for conic Kähler-Einstein metrics. Another proof for the sufficient part was given in [9, 10, 11].

An older approach for solving the conjecture is to solve the following complex Monge-Amperé equations by the continuity method:

\begin{equation}
(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m = e^{h - t\varphi} \omega^m, \quad \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0,
\end{equation}

where $\omega$ is a given Kähler metric with its Kähler class $[\omega] = 2\pi c_1(M)$ and $h$ is uniquely determined by

\[
\text{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h, \quad \int_M (e^h - 1)\omega^m = 0.
\]

Let $I$ be the set of $t$ for which (1.1) is solvable. Then we have known: (1) By the well-known Calabi-Yau theorem, $I$ is non-empty; (2) In 1983, Aubin proved that $I$ is open; (3) If we can have an a prior $C^0$-estimate for the solutions of (1.1), then $I$ is closed and consequently, there is a Kähler-Einstein metric on $M$.

Key words and phrases. Kähler Ricci flow, scalar curvature, Bergman kernel, partial $C^0$-estimate.
However, the $C^0$-estimate does not hold in general since there are many Fano manifolds which do not admit any Kähler-Einstein metrics. The existence of Kähler-Einstein metrics required certain geometric stability on the underlying Fano manifolds. In 90’s, Tian proposed a program towards establishing the existence of Kähler-Einstein metrics. The key technical ingredient of this program is a partial $C^0$-estimate conjecture (Conjecture 1.3). Tian affirmed that if one can prove this conjecture (Conjecture 1.3) for the solutions of (1.1), then one can use the K-stability to derive the a priori $C^0$-estimate for the solutions of (1.1), and consequently, the existence of Kähler-Einstein metrics. In this paper, we will solve Tian’s partial $C^0$-estimate conjecture for complex dimension $\leq 3$.

Let $(M, \omega)$ be a Fano manifold and $K^{-1}_M$ be its anti-canonical bundle. Choose a hermitian metric $H_\omega$ with $\omega$ as its curvature form and any orthonormal basis $\{S_i\}_{1 \leq i \leq N}$ of $H^0(M, K^{-l}_M)$, with respect to the induced inner product induced by $H_\omega \otimes l\omega$ and $\omega$, where $N = \dim H^0(M, K^{-l}_M)$. Then, following [31], we define the Bergman kernel by

$$\rho_{\omega,l}(x) = \sum_{i=1}^{N} ||S_i||^2_{H^0_\omega}(x).$$

This is independent of the choice of $H_\omega$ and the orthonormal basis $\{S_i\}$.

**Remark 1.2.** Denote

$$||S||^2_{H^0_\omega} = \int_M ||S||^2_{H^0_\omega}(x)d\mu(x)$$

and

$$\eta_\omega(x) = \sup_{||S||^2_{H^0_\omega} = 1} ||S||^2_{H^0_\omega}(x),$$

then we have the following inequalities:

$$\frac{1}{N}\rho_{\omega,l}(x) \leq \eta_\omega(x) \leq \rho_{\omega,l}(x).$$

Denote by $\mathcal{K}(R_0, V_0, m)$ the set of all compact Kähler manifolds $(M, \omega)$ of complex dimension $m$ such that $|\omega| = 2\pi c_1(M)$, $\text{Ric}(\omega) \geq R_0 > 0$ and $\text{Vol}(M, \omega) \geq V_0 > 0$. In 1990, Tian proposed the following conjecture:

**Conjecture 1.3** (Tian [28, 30]). For each $(M, \omega) \in \mathcal{K}(R_0, V_0, m)$, there are uniform constants $c_k = c(m, k, R_0, V_0) > 0$ $(k \geq 1)$ and $l_i \to \infty$ with $i \geq 0$ and $l_0 = l_0(m)$, such that for all $l = l_i$, we have

$$\rho_{\omega,l} > c_l > 0.$$

**Remark 1.4.** Tian also mentioned a stronger version of Conjecture 1.3: There are uniform constants $c_k = c(m, k, R_0, V_0) > 0$ for $k \geq 0$ and $l_0 = l_0(m)$, such that for all $l \geq l_0$, we have $\rho_{\omega,l} > c_l > 0$.

In this paper, using recent regularity theory developed by Tian and Zhang [33] for Kähler Ricci flow, we solve Tian’s conjecture for complex dimensions 2 and 3:

**Theorem 1.5.** Let $(M, \omega)$ be a compact Kähler manifold of complex dimension 2 or 3 and. We further assume a positive Ricci curvature lower bound $\text{Ric}(\omega) \geq R_0 > 0$ and a volume lower bound $V \geq V_0 > 0$. Then there are uniform constants
\( c_k = c(m, k, R_0, V_0) > 0 \) for all \( k \geq 1 \) and \( l_i \to \infty \) with \( i \geq 0 \) and \( l_0 = l_0(m, R_0, V_0) \), such that for all \( l = l_i \), we have

\[ \rho_{\omega, l} > c_l > 0. \]

Following Tian’s approach, we can prove (see also [36])

**Corollary 1.6.** The Yau-Tian-Donaldson conjecture holds for complex dimension \( \leq 3 \).

We can also prove a stronger version of Tian’s Conjecture in complex dimension 1,

**Theorem 1.7.** Let \((M, \omega)\) be a Kähler manifold of complex dimension 1 and with positive first Chern class. Then for all \( l \in \mathbb{N}_+ \), the Bergman kernel \( \rho_{\omega, l} \) has a uniform positive lower bound \( c_l > 0 \), depending only on positive curvature lower bound, volume lower bound and \( l \):

\[ \rho_{\omega, l} > c_l > 0. \]

In order to study the Bergman kernel, we consider the Kähler Ricci flow

\[ \partial_t g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} = u_{i\bar{j}}, \quad t > 0 \]

on a compact Kähler manifold \( M \) of complex dimension \( m \) and with \( c_1(M) > 0 \). We can show that the Bergman kernels are equivalent along the Kähler Ricci flow (1.5), see Theorem 6.10.

Given any initial Kähler metric \( g(0) \), Cao [3] proved that (1.5) has a solution for all time \( t \geq 0 \). Moreover, Perelman (see [25]) proved that the scalar curvature \( R \) is uniformly bounded, and the Ricci potential \( u \) is uniformly bounded in \( C^1 \) norm, with respect to \( g(t) \). One can easily check that these uniform bounds depend on the Sobolev constant \( C_s \) of \( g(0) \), the volume \( V \) of \( g(0) \), the scalar curvature upper and lower bound of \( g(0) \), the upper bound of \( |\nabla u| \). The following theorem is essentially due to Perelman,

**Theorem 1.8.** (see [25]) Let \( g(t) \) be a Kähler Ricci flow (1.5) on a Fano manifold \( M \) of complex dimension \( m \). There exists a uniform constant \( C \) so that

\[ |R(g(t))| \leq C, \quad \text{diam}(M, g(t)) \leq C, \quad ||u||_{C^1} \leq C, \]

where the constant \( C \) depends only on dimension \( m \), \( \text{Vol}(M, g(0)) \), the \( L^2 \)-Sobolev constant \( C_s \) of \( g(0) \), bounds of \( |R(g(0))| \) and \( |\nabla u(0)| \).

In this paper, we prove that in all complex dimensions, the scalar curvature and the gradient of Ricci potential \( |\nabla u|^2 \) for \((M, g(t))\) have a bound \( Ct^{-\frac{m}{2m}} \) in small time. Here \( C \) depends only on a lower bound of Ricci curvature, the volume bound of \( g(0) \) and an upper bound of diameter of \( g(0) \). The key result of this paper is the following:

**Theorem 1.9.** Let \( g(t) \) be a Kähler Ricci flow (1.5) on a Fano manifold \( M \) of real dimension \( n = 2m \). Then there exists a uniform constant \( C \), which depends only on constant \( n_0 \) (\( n_0 = n \) if \( n \geq 3 \), \( n_0 > 2 \) if \( n = 2 \)), the lower bound of \( \text{Ric}(g(0)) \), the volume \( \text{Vol}(g(0)) \) and the upper bound of diameter of \( g(0) \). Such that for all \( 0 < t < 1 \), we have

\[ \sup_{x \in M} |R(x, t)| \leq C t^{-\frac{n_0 + 2}{2m}}. \]
and

$$(1.7) \quad \sup_{x \in M} |\nabla u(x,t)|^2 \leq \frac{C}{t^{-n_0/2}}.$$ 

Remark 1.10. Note that, one can deduce a uniform lower bound for $u$ by Moser’s iteration using the $L^2$-Sobolev inequality along the Kähler Ricci flow (Lemma 2.4), so the gradient bound of $u$ implies the $C^1$-norm of $u$. Applying the results of Perelman (Theorem 1.8) and Theorem 1.9, we can estimate the diameter, the $C^1$ norm of Ricci potential, scalar curvature upper bound for all time along the Kähler Ricci flow.

Theorem 1.9 is a corollary of the following theorem. Its proof replies on Moser’s iteration combined with the $L^2$-Sobolev inequality along the Ricci flow (see [20]). Note that the term involving the scalar curvature will be a good term when applying Moser’s iteration.

**Theorem 1.11.** Let $g(t)$ be a Kähler Ricci flow on a Fano manifold $M$ of real dimension $n$.

$$\partial_t g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} = u_{i\bar{j}}, \quad t > 0$$

Let $f$ be a nonnegative Lipschitz continuous function on $M \times [0, \infty)$ satisfying

$$(1.8) \quad \frac{\partial f}{\partial t} \leq \Delta f + af$$

on $M \times [0, \infty)$ in the weak sense, where $a \geq 0$, then for any $0 < t < 1$, $p > 0$, we have

$$\sup_{x \in M} |f(x,t)| \leq \frac{C}{t^{-n_0/2}} \left( \int_0^1 \int_M f(x,t)^p d\mu(t) dt \right)^{\frac{1}{p}},$$

where $C$ is a positive constant depending only on $a$, $p$, constant $n_0$ ($n_0 = n$, if $n \geq 3$, $n_0 > 2$, if $n = 2$), Sobolev constant $C_s$ of initial metric, volume $V$ of initial metric and negative lower bound of $R(0)$.

Notations: Let $R = R(x,t)$ be the scalar curvature at time $t$, $V$ be the volume of $g(0)$, $d$ be the diameter upper bound of $g(0)$, $d\mu(t)$ be the volume element of $g(t)$. Denote $\text{diam}(M, g)$ and $\text{Vol}(M, g)$ to be the diameter and volume of $(M, g)$.

The organization of this paper is as follows: In the next section, we give a proof for Theorem 1.8, we will consider carefully how the quantities rely on the initial metric. In section 3, we prove Theorem 1.11. The main idea is Moser’s iteration for parabolic equation (see [20]). In section 4, applying Theorem 1.11, we give the proof of theorem 1.9. In section 5, we consider the complex 1 case and prove Theorem 1.7. We will divide the proof into several lemmas. In section 6, we consider high dimension case, we show that Bergman kernels are equivalent along the Kähler Ricci flow and complete the proof of Theorem 1.5.

2. Perelman’s Scalar Curvature Estimate

In this section, we will give a proof of Theorem 1.8. The method is mainly similar to [25] and [4]. We will consider carefully all the quantities how to rely on the initial metric. We only prove complex dimension $\geq 2$ case. For complex dimension 1 Fano manifold, the proof is similar by noticing that Proposition 2.3 can be replaced with Lemma 3.2. First of all, we will show that the Ricci potential has a uniform lower bound, and then using maximum principle we can control the
gradient of Ricci potential and scalar curvature upper bound. At last, a diameter upper bound estimate will conclude the proof.

Now we will prove a uniform Ricci potential lower bound. Firstly, we need to show the scalar curvature has a uniform lower bound.

**Lemma 2.1.** There exists a constant $C > 0$ such that the scalar curvature $R$ of $g(t)$ satisfies the estimate

$$R(x,t) \geq -C$$

for all $t \geq 0$ and all $x \in M$. Here constant $C$ depends only on the lower bound of $R(g(0))$.

**Proof.** By directly computing, we have the evolution of $R$,

$$\frac{\partial}{\partial t} R = \Delta R + |\text{Ric}|^2 - R.$$

Let $R_{\min}(0)$ be the minimum of $R(x,0)$ on $M$. If $R_{\min} \geq 0$, then by maximum principle, we have $R(x,t) \geq 0$ for all $t > 0$ and all $x \in M$.

Now suppose $R_{\min}(0) < 0$. Set $F(x,t) = R(x,t) - R_{\min}(0)$. Then, $F(x,0) \geq 0$ and $F$ satisfies

$$\frac{\partial}{\partial t} F = \Delta F + |\text{Ric}|^2 - F - R_{\min}(0) > \Delta + |\text{Ric}|^2 - F.$$

Hence it follows again from the maximum principle that $F \geq 0$ on $M \times [0, \infty]$, i.e.,

$$R(x,t) \geq R_{\min}(0)$$

for all $t > 0$ and all $x \in M$. □

Next, we will show Perelman’s $\kappa$ non-collapsing theorem, we need the following:

**Lemma 2.2.** Let $\hat{g}_{ij}(s)$, $0 \leq s < 1$ and $g_{ij}(t)$, $0 \leq t < \infty$, be solutions to the Kähler Ricci flow (2.1) and (2.2) respectively,

$$\partial_s \hat{g}_{ij}(s) = -R_{ij}(s), \ 0 \leq s < 1, \ \hat{g}_{ij}(0) = g_{ij}$$

and

$$\partial_t g_{ij}(t) = g_{ij}(t) - R_{ij}(t), \ t > 0, \ g_{ij}(0) = g_{ij}.$$

Then $\hat{g}_{ij}(s)$ and $g_{ij}(t)$ are related by

$$\hat{g}_{ij}(s) = (1-s)g_{ij}(t(s)), \ t = -\log(1-s)$$

and

$$g_{ij}(t) = e^t \hat{g}_{ij}(s(t)), \ s = 1 - e^{-t}.$$ □

In order to show Perelman’s $\kappa$-noncollapsed theorem, we only need to prove the following:

**Proposition 2.3.** (Ye [34]) Consider the Kähler Ricci flow (2.2) on Fano manifold. Then there are positive constants $A$ and $B$ depending only on the dimension $m$, a non-positive lower bound for $R_g(0)$, a positive lower bound for $\text{Vol}(M,g(0))$, an upper bound for Sobolev constant $C_s(M,g(0))$. Such that, for each $t > 0$ and all $f \in W^{1,2}(M)$ there holds

$$\left( \int_M f^{\frac{2m}{m-1}} d\mu(t) \right)^{\frac{m-1}{m}} \leq A \int_M (|\nabla f|^2 + \frac{R}{4} f^2) d\mu(t) + B \int_M f^2 d\mu(t).$$
Consequently, let $L > 0$ and assume $R \leq \frac{1}{r}$ on a geodesic ball $B(x, r)$ with $0 < r \leq L$. Then there holds

\begin{equation}
\text{Vol}(B(x, r)) \geq \left(\frac{1}{2^{2m+3}A + 2BL^2}\right)^m r^{2m}.
\end{equation}

Proof. By the monotonicity of $W$-entropy functional and the flow (2.1) in Lemma 2.2, one can show that

\begin{equation}
\left(\int_M f^{2m} d\tilde{\mu}(t)\right)^{\frac{m-1}{m}} \leq A \int_M (|\nabla f|^2 + \frac{R}{4} f^2) d\tilde{\mu}(s) + B \int_M f^2 d\tilde{\mu}(s)
\end{equation}

where constants $A$ and $B$ depend only on the quantities stated in the proposition. By the relation between (2.1) and (2.2), we have

\begin{equation}
\left(\int_M f^{2m} d\mu(t)\right)^{\frac{m-1}{m}} \leq A \int_M (|\nabla f|^2 + \frac{R}{4} f^2) d\mu(t) + B e^{-t} \int_M f^2 d\mu(t)
\end{equation}

\begin{equation}
\leq A \int_M (|\nabla f|^2 + \frac{R}{4} f^2) d\mu(t) + B \int_M f^2 d\mu(t).
\end{equation}

At last, to prove the $\kappa$-noncollapsed, we can assume $r = 1$ since we can scale the metric with factor $\frac{1}{r}$. That is $\tilde{g} = \frac{1}{r} g$. Thus we have

\begin{equation}
\left(\int_M f^{2m} d\tilde{\mu}(t)\right)^{\frac{m-1}{m}} \leq A \int_M (|\nabla f|^2 + \frac{\tilde{R}}{4} f^2) d\tilde{\mu}(t) + B \int_M f^2 d\tilde{\mu}(t)
\end{equation}

\begin{equation}
\leq A \int_M (|\nabla f|^2 + \frac{\tilde{R}}{4} f^2) d\mu(t) + B \int_M f^2 d\mu(t)
\end{equation}

and $\tilde{R} \leq 1$ on geodesic ball $B_{\tilde{g}}(x, 1)$. Then a standard argument implies the lower bound of $\text{Vol}_{\tilde{g}}(B_{\tilde{g}}(x, 1))$. One can find more details in Ye [34].

By taking the trace of (1.5), we get $\Delta u = m - R$. Normalize $u$ so that

\begin{equation}
\int_M e^{-u} d\mu(t) = (2\pi)^m.
\end{equation}

Then we have

Lemma 2.4. Function $u(t)$ is uniformly bounded from below. That is

\begin{equation}
u(t) \geq -C,
\end{equation}

where constant $C$ depends only on the constants in Proposition 2.3.

Proof. Since $\Delta u = m - R$, we have

\begin{equation}
\Delta e^{-u} = -\Delta e^{-u} + |\nabla u|^2 e^{-u} \geq (R - m) e^{-u}.
\end{equation}

Denote $f = e^{-u}$. We have

\begin{equation}
-\Delta f + Rf \leq mf.
\end{equation}

Multiplying $f^p$ to both sides of (2.5) and integrating by parts, we have

\begin{equation}
\frac{4p}{(p+1)^2} \int_M |\nabla f|^2 d\mu(t) + \int_M R f^{p+1} d\mu(t) \leq m \int_M f^{p+1} d\mu(t).
\end{equation}

That is

\begin{equation}
\int_M |\nabla f|^2 d\mu(t) + \frac{(p+1)^2}{p} \int_M \frac{R}{4} f^{p+1} d\mu(t) \leq \frac{m(p+1)^2}{4p} \int_M f^{p+1} d\mu(t).
\end{equation}
Since $R$ has a uniform lower bound, so
\[ \int_M |\nabla f|^2 \frac{e^{\frac{R}{2}}}{\pi} d\mu(t) + \int_M \frac{R}{4} f^{p+1} d\mu(t) \leq C p \int_M f^{p+1} d\mu(t). \]
Then by Proposition 2.3 and Moser’s iteration, we deduce
\[ \sup_{x \in M} f(x) \leq C \int_M f d\mu(t) = C \int_M e^{-u} d\mu(t) = C(2\pi)^m. \]
This provides a pointwise lower bound of $u$. \hfill \Box

Define
\[ W(g, f, \tau) = (4\pi \tau)^{-m} \int_M e^{-f} \{2\tau(R + |\nabla f|^2) + f - 2m\} d\mu, \int_M e^{-f} d\mu = (4\pi \tau)^m \]
to be Perelman’s $W$-entropy functional for $g$ as in [21]. By directly computing, we have
\[ \frac{\partial}{\partial t} W(g(t), f(t), \frac{1}{2}) = (2\pi)^{-m} \int_M e^{-f} \left( |\nabla \nabla f|^2 - R + m \right) d\mu(t) \geq 0 \]
along the following
\[ \partial_t f = -\Delta f + |\nabla f|^2 - R + m, \int_M e^{-f} d\mu(t) = (2\pi)^m, \]
\[ \partial_t g_{ij}(t) = g_{ij}(t) - R_{ij}(t). \]
Define
\[ \mu(g, \tau) = \inf_{\{f| \int_M e^{-f} d\mu = (4\pi \tau)^m\}} W(g, f, \tau). \]
Then
\[ A_0 = \mu(g(0), \frac{1}{2}) \leq \mu(g(t), \frac{1}{2}) \]
\[ \leq \int_M (2\pi)^{-m} e^{-u}(R + |\nabla u|^2 + u - 2m) d\mu(t) \]
\[ = \int_M (2\pi)^{-m} e^{-u}(-\Delta u + |\nabla u|^2 + u - m) d\mu(t) \]
\[ = -m + (2\pi)^{-m} \int_M e^{-u} d\mu(t). \]
On the other hand, let $F = e^{-\frac{u}{2}}(2\pi)^{-\frac{m}{2}}$. Then $\int_M F^2 d\mu(t) = 1$.
\[ W(g, f, \frac{1}{2}) = \int_M \left( RF^2 + 4|\nabla F|^2 - F^2 \log F^2 \right) d\mu - 2m - m \log(2\pi) \geq -C. \]
Here we have used the $L^2$-Sobolev inequality along the Kähler Ricci flow and the uniform lower bound of scalar curvature. The constant depends only on the constants in Proposition 2.3. Particularly, we have $A_0 \geq -C$. Moreover, the function $xe^{-x}$ is bounded from above. Thus we have
\[ \text{Lemma 2.5. Denote } a = -(2\pi)^{-m} \int_M e^{-u} d\mu(t), \text{ then} \]
\[ |a(t)| \leq C \]
where the constant $C$ depends only on the volume of $g(0)$, a lower bound of $R(g(0))$ and an upper bound of the Sobolev constant $C_s = C_s(M, g(0))$. \hfill \Box
The Ricci potential $u(x,t)$ satisfies
\[ \partial_t \partial_j u = g_{ij} - R_{ij}. \]
Differentiating this, we have
\[ \partial_t \partial_j u_t = g_{ij} - R_{ij} + \frac{\partial}{\partial t} \partial_i \partial_j \log \det(g_{ij}) \]
which implies
\[ \frac{\partial}{\partial t} u = \Delta u + u + \varphi(t). \]
However,
\[ 0 = \frac{\partial}{\partial t} \int_M e^{-u} d\mu(t) = \int_M e^{-u} (- \partial_t u + \Delta u) d\mu(t) = \int_M e^{-u} (-u - \varphi(t)) d\mu(t). \]
Thus
\[ \varphi(t) = -(2\pi)^{-m} \int_M e^{-u} u d\mu(t) = a. \]
By maximum principle, one can easily prove the following:

**Lemma 2.6.** There is a uniform constant $C$, so that
\[ |\nabla u|^2(x,t) \leq C(u+C), \]
where the constant depends only on $\text{Vol}(M,g(0))$, the $L^2$-Sobolev constant $C_s$ of $g(0)$ and upper bounds of $\|R(g(0))\|$ and $|\nabla u|$.\]

**Proof.** This is essentially a parabolic version of Yau’s gradient estimate in [24]. By Lemma 2.4, we have $u(x,t) \geq -C$. Choosing $B = C + 1$, then $u(x,t) + B \geq 1$. Let $H = \frac{\nabla u^2}{u+B}$. In order to show (2.7), we only need to estimate an upper bound for $H$. By directly computing, we have
\[ (\partial_t - \Delta)H = (B-a)\frac{\nabla u^2}{(u+B)^2} - \frac{\nabla^2 \nabla u^2 + |\nabla \nabla u|^2}{u+B} + \frac{2(\nabla \nabla u^2, \nabla u)}{(u+B)^2} - \frac{2|\nabla u|^4}{(u+B)^3}. \]
For each $T > 0$, suppose $H$ attains its maximum at $(x_0, t_0)$ on $M \times [0, T]$. If $t_0 = 0$, the upper bound of $H$ follows easily by the bound for $|\nabla u|(g(0))$. Assume $t_0 > 0$. Then at $(x_0, t_0)$ we have
\[ \partial_t H(x_0, t_0) \geq 0, \ \Delta H(x_0, t_0) = 0, \ \Delta H(x_0, t_0) \leq 0. \]
Substituting these into (2.9), we obtain
\[ (B-a)\frac{\nabla u^2}{u+B}(x_0, t_0) \geq |\nabla \nabla u^2(x_0, t_0) + |\nabla \nabla u|^2(x_0, t_0). \]
On the other hand, since $\nabla H(x_0, t_0) = 0$, we have
\[ \nabla |\nabla u|^2(x_0, t_0) = \frac{|\nabla u|^2 \nabla u}{u+B}(x_0, t_0). \]
Thus at \((x_0, t_0)\),

\[
\frac{|\nabla u|^3}{u + B} = \frac{|\nabla |\nabla u|^2|}{u + B} \leq \frac{|\nabla u||\nabla u| + |\nabla u|}{u + B} \\
\leq \sqrt{2}|\nabla u|(\frac{|\nabla u|^2 + |\nabla u|^2}{u + B}^\frac{1}{2}).
\]

Combining with (2.10), we have

\[
2(B - a)H(x_0, t_0) \geq H^2(x_0, t_0).
\]

Hence \(H(x_0, t_0) \leq 2(B - a)\). Let \(T \to \infty\) and note that \(a\) is bounded. We complete the proof of (2.7).

Now we turn to the proof of (2.8). Our goal is to prove that \(-\Delta u\) is bounded by \(C(u + C)\), which yields (2.8), since \(\Delta u = n - R\). Let \(K = \frac{\Delta u}{u + B}\), where \(B\) is a uniform constant as above. Similar computation as before gives that

\[
(\partial_t - \Delta)K = \frac{|\nabla u|^2}{u + B} + \frac{(-\Delta u)(B - a)}{(u + B)^2} + 2\frac{\langle \nabla K, \nabla u \rangle}{u + B}.
\]

Combining this with (2.9), we have

\[
(\partial_t - \Delta)(K + 2H) = \frac{-2|\nabla u|^2}{u + B} + \frac{(-\Delta u + 2|\nabla u|^2)(B - a)}{(u + B)^2} + 2\frac{\langle \nabla (K + 2H), \nabla u \rangle}{u + B}.
\]

For each \(T > 0\), suppose \(2H + K\) attains its maximum at \((x_0, t_0)\) on \(M \times [0, T]\). If \(t_0 = 0\), the upper bound of \(2H + K\) follows easily by the bound for \(|\nabla u|(0)\) and \(|R|(0)\). Assume \(t_0 > 0\). Then at \((x_0, t_0)\) we have

\[
\frac{(-\Delta u + 2|\nabla u|^2)(B - a)}{(u + B)^2} \leq \frac{|\nabla u|^2 + 2|\nabla u|^2}{u + B} \geq \frac{|\nabla u|^2}{u + B} \geq \frac{(\Delta u)^2}{m(u + B)}.
\]

Thus

\[
\frac{1}{m} \left(\frac{\Delta u}{u + B}\right)^2 + (B - a)\frac{\Delta u}{u + B} \leq \frac{2(B - a)|\nabla u|^2}{u + B} \leq C.
\]

Here we have used the fact that \(u + B \geq 1\). Hence \(|\Delta u|_u + B|(x_0, t_0) \leq C\). Now for each \((x, t) \in M \times [0, T]\),

\[
\frac{-\Delta u}{u + B}(x, t) \leq \frac{-\Delta u + 2|\nabla u|^2}{u + B}(x, t) \leq \frac{-\Delta u + 2|\nabla u|^2}{u + B}(x_0, t_0) = \frac{-\Delta u}{u + B}(x_0, t_0) + \frac{2|\nabla u|^2}{u + B}(x_0, t_0) \leq C.
\]

Let \(T \to \infty\). We finish the proof. \(\square\)

**Corollary 2.7.** There exits a constant \(C\) depending only on the constant of Lemma 2.6, such that,

\[
\begin{align*}
\frac{u(y, t)}{(x_0, t_0)} &\leq C(\text{dist}_t^2(\hat{x}, y) + 1), \\
R(y, t) &\leq C(\text{dist}_t^2(\hat{x}, y) + 1), \\
|\nabla u|^2(y, t) &\leq C(\text{dist}_t^2(\hat{x}, y) + 1)
\end{align*}
\]

for all \(t > 0\) and \(y \in M\), where \(u(\hat{x}, t) = \min_{x \in M} u(x, t)\).
Proof. By Lemma 2.6, we only need to estimate \( u(x, t) \). Actually, by (2.7), we have
\[
|\nabla \sqrt{u + C}| \leq C.
\]
Hence we have
\[
\sqrt{u + C}(x, t) \leq \sqrt{u + C}(\hat{x}, t) + C \text{dist}(\hat{x}, x),
\]
where \( u(\hat{x}, t) = \min_{x \in M} u(x, t) \). On the other hand, since \( \int_M e^{-u} d\mu(t) = (2\pi)^m \), we have
\[
u(\hat{x}, t) \leq \log \left( \frac{V}{(2\pi)^m} \right).
\]
So
\[
u(y, t) \leq C(\text{dist}^2(\hat{x}, y) + 1).
\]
The other two inequalities follow from this and Lemma 2.6. \( \square \)

Notice the results in Corollary 2.7. To prove Theorem 1.8, it suffices to estimate the diameter upper bound. Let \( B(k_1, k_2) = \{ z : 2^{k_1} \leq \text{dist}(\hat{x}, z) \leq 2^{k_2} \} \). Consider an annular \( B(k, k + 1) \). By Corollary 2.7 we have that \( R \leq C2^{k} \) on \( B(k, k + 1) \) and note that \( B(k, k + 1) \) contains at least \( 2^{k_1 - 1} \) balls of radii \( \frac{1}{2} \). By Proposition 2.3 we have
\[
\text{Vol}(B(k, k + 1)) \geq \sum_i \text{Vol}(B(x_i, 2^{-k})) \geq C2^{2k - 2km},
\]
where the constant \( C \) depends only on the constant in Corollary 2.7 and constants in Proposition 2.3.

**Lemma 2.8.** For each \( \epsilon > 0 \), if \( \text{diam}(M, g(t)) \geq C \epsilon \), we can find \( B(k_1, k_2) \) such that
\[
\text{Vol}(B(k_1, k_2)) < \epsilon,
\]
\[
\text{Vol}(B(k_1, k_2)) \leq 2^{10m} \text{Vol}(B(k_1 + 2, k_2 - 2)).
\]
Here we can choose \( C_\epsilon = 2^{\log V/C} + 2^{\log (V/C)} + 2 + 1 \) and \( C \) is the constant in (2.11)

**Proof.** Denote \( k_0 = \frac{\log V/C}{2(2m + 8)\log 2} + 2 \) and assume \( \text{diam}(M, g) \geq 2k_0 4^{k_0+1} \). Then we will show that for each \( \frac{k_0}{2} \leq k \leq \frac{k_0}{2} 4^{k_0} \), there exits \( B(k_1, k_2) \) so that \( 2k \leq k_1 < k_2 \leq 2k + 1 \) and
\[
\text{Vol}(B(k_1, k_2)) \leq 2^{10m} \text{Vol}(B(k_1 + 2, k_2 - 2)).
\]
Otherwise, by (2.11)
\[
V \geq \text{Vol}(B(2k, 6k + 1)) \geq 2^{10m} \text{Vol}(B(2k + 2, 6k - 1)) > 2^{10m} k \text{Vol}(B(4k + 4k + 1)) \geq C2^{110m} \text{Vol}(B(4k + 8k - 8km) = C2^{22km + 8k}).
\]
Thus
\[
k \leq \frac{\log V/C}{2(2m + 8)\log 2}.
\]
On the other hand, there must be some \( 0 \leq l \leq \frac{V}{C} \) such that
\[
\text{Vol}(B(k_0 4^l, k_0 4^{l+1})) < \epsilon.
\]
Otherwise,

\[ V \geq \sum_{i=0}^{\lfloor \frac{2}{\epsilon} \rfloor} \text{Vol}(B(k_0^4i, k_0^4i+1)) \geq \left( \left\lfloor \frac{V}{\epsilon} \right\rfloor + 1 \right) \epsilon > V. \]

Getting together all the above arguments will imply the lemma. \( \square \)

**Lemma 2.9.** For each \( 0 < k_1 < k_2 < \infty \), there exist \( r_1, r_2 \) and a uniform constant \( C \) such that \( 2^{k_1} \leq r_1 \leq 2^{k_1+1} \), \( 2^{k_2-1} \leq r_2 \leq 2^{k_2} \) and

\[ \int_{B(r_1, r_2)} R d\mu(t) \leq C \text{Vol}(B(k_1, k_2)), \]

where \( B(r_1, r_2) = \{ z \in M : r_1 \leq \text{dist}(z, \hat{x}) \leq r_2 \} \) and the constant \( C \) depends only on the constant in Corollary 2.7.

**Proof.** First of all, since

\[ \frac{d}{dr} \text{Vol}(B(r)) = \text{Vol}(S(r)), \]

we have

\[ \text{Vol}(B(k_1, k_1+1)) = \int_{2^{k_1}}^{2^{k_1+1}} \text{Vol}(S(r)) dr. \]

Here \( S(r) \) denotes the geodesic sphere of radius \( r \) centered at \( \hat{x} \) with respect to \( g(t) \). Hence, we can choose \( r_1 \in [2^{k_1}, 2^{k_1+1}] \) such that

\[ \text{Vol}(S(r_1)) \leq \frac{\text{Vol}(B(k_1, k_1+1))}{2^{k_1}} \leq \frac{\text{Vol}(B(k_1, k_2))}{2^{k_1}}. \]

Similarly, there exists \( r_2 \in [2^{k_2-1}, 2^{k_2}] \) such that

\[ \text{Vol}(S(r_2)) \leq \frac{\text{Vol}(B(k_2-1, k_2))}{2^{k_1}} \leq \frac{\text{Vol}(B(k_1, k_2))}{2^{k_2}}. \]

Next, by integration by parts and Corollary 2.7,

\[ \left| \int_{B(r_1, r_2)} \Delta u d\mu(t) \right| \leq \int_{S(r_1)} |\nabla u| d\sigma(t) + \int_{S(r_2)} |\nabla u| d\sigma(t) \]

\[ \leq \frac{\text{Vol}(B(k_1, k_2))}{2^{k_1}} C 2^{k_1+1} + \frac{\text{Vol}(B(k_1, k_2))}{2^{k_2}} C 2^{k_2+1} \]

\[ \leq 4C\text{Vol}(B(k_1, k_2)). \]

Therefore, since \( R = -\Delta u + m \), it follows that

\[ \int_{B(r_1, r_2)} R d\mu(t) \leq (m+4)C \text{Vol}(B(k_1, k_2)) \]

proving Lemma 2.9. Here \( C \) is the constant in Corollary 2.7. \( \square \)

In order to control the diameter of \( M \), we only need to show the following:

**Lemma 2.10.** There exists a constant \( \epsilon_0 > 0 \). If \( 0 < \epsilon < \epsilon_0 \), we can’t find \( B(k_1, k_2) \) such that

\begin{align*}
\text{Vol}(B(k_1, k_2)) &< \epsilon, \\
\text{Vol}(B(k_1, k_2)) &\leq 2^{10m} \text{Vol}(B(k_1 + 2, k_2 - 2)).
\end{align*}

Here we can choose \( \epsilon_0 = (2\pi)^m e^{6C2^{10m} + A_0 + 2m} \) and constant \( C \) is the constant in Lemma 2.9 and \( A_0 = \mu(g(0), \frac{1}{2}) \).
Proof. Actually, if we can find $B(k_1, k_2)$ such that (2.12) holds. Then we choose $r_1, r_2$ as in Lemma 2.9. Define a cut off function $0 \leq \phi \leq 1$,

$$\phi(s) = \begin{cases} 
1, & 2^{k_1+2} \leq s \leq 2^{k_2-2},
0, & \text{outside } [r_1, r_2].
\end{cases}$$

Then $|\phi'| \leq 1$ everywhere. Let

$$F(y) = e^L \phi(\text{dist}_4(\hat{x}, y)),$$

where the constant $L$ is chosen so that

$$(2\pi)^m = \int_M F^2 d\mu(t) = e^{2L} \int_{B(r_1, r_2)} \phi^2 d\mu(t).$$

Since $\text{Vol}(B(r_1, r_2)) \leq \text{Vol}(B(k_1, k_2)) < \epsilon$, thus $L \geq \frac{1}{2} \log \left(\frac{2\pi}{\epsilon}\right)$.

By monotonicity of $\mathcal{W}$-entropy functional, we have

$$A_0 = \mu(g(0), \frac{1}{2}) \leq \mu(g(t), \frac{1}{2})$$

$$\leq (2\pi)^{-m} \int_M (RF^2 + 4|\nabla F|^2 - F^2 \log F^2) \, d\mu(t) - 2m$$

$$= (2\pi)^{-m} e^{2L} \int_{B(r_1, r_2)} \left( R\phi^2 + 4|\phi'|^2 - \phi^2 \log \phi^2 \right) \, d\mu(t) - 2L - 2m.$$

By Lemma 2.9 we have

$$e^{2L} \int_{B(r_1, r_2)} R\phi^2 \, d\mu(t) \leq C e^{2L} \text{Vol}(B(k_1, k_2))$$

$$\leq C e^{2L} 2^{10m} \text{Vol}(B(k_1 + 2, k_2 - 2))$$

$$\leq C 2^{10m} \int_{B(r_1, r_2)} F^2 \, d\mu(t) = C 2^{10m} (2\pi)^m.$$

On the other hand, using $|\phi'| \leq 1$ and $-s \log s \leq e^{-s}$, we have

$$e^{2L} \int_{B(r_1, r_2)} (4|\phi'|^2 - \phi^2 \log \phi^2) \, d\mu(t) \leq 5e^{2L} \text{Vol}(B(k_1, k_2))$$

$$\leq 5Ce^{2L} 2^{10m} \text{Vol}(B(k_1 + 2, k_2 - 2))$$

$$\leq 5C 2^{10m} \int_{B(r_1, r_2)} F^2 \, d\mu(t) = 5C 2^{10m} (2\pi)^m$$

for $0 \leq s \leq 1$. The above constant $C$ is the uniform constant in Lemma 2.9.

Therefore,

$$A_0 \leq -2(L + m) + 6C 2^{10m}.$$

Hence we have

$$\log \left(\frac{(2\pi)^m}{\epsilon}\right) \leq 2L \leq 6C 2^{10m} - A_0 - 2m.$$

Thus it provides

$$\epsilon \geq (2\pi)^m e^{6C 2^{10m} + A_0 + 2m}$$

Combining Lemma 2.8 and Lemma 2.10 will finish the proof of Theorem 1.8. □
3. A Linear Parabolic Estimate

The main purpose of this section is to prove Theorem 1.11. We need two lemmas. The following lemma is due to Rugang Ye [34] and Qi S. Zhang [35], for \( n \geq 3 \), see also Proposition 2.3.

**Lemma 3.1.** Let \((M, g(t))\) be a Kähler Ricci flow with real dimension \( n \), \( C_1(M) > 0 \). At time \( t = 0 \), the following \( L^2 \) Sobolev inequality holds

\[
\left( \int_M f(x) \frac{2^n}{n} d\mu(0) \right)^{\frac{n-2}{n}} \leq C_s \left( \int_M |\nabla f(x)|^2 d\mu(0) + \int_M f^2(x) d\mu(0) \right),
\]

Then along the Kähler Ricci flow we have

\[
\left( \int_M f(x) \frac{2^n}{n} d\mu(t) \right)^{\frac{n-2}{n}} \leq A \left( \int_M |\nabla f(x)|^2 + (R(x,t) + C_0) f^2(x) d\mu(t) \right)
\]

for all \( 0 \leq t \leq 1 \), where the constants \( A, C_0 \) depend only on dimension \( n \), Sobolev constant \( C_s \), volume \( V \) of \( g(0) \) and a lower bound of \( R(g(0)) \).

For \( n = 2 \), the following lemma follows from Theorem 1 and Theorem 2 of [17], Lemma 4.1 of [34] and a standard argument of [1].

**Lemma 3.2.** Let \((M, g(t))\) be a Kähler Ricci flow with real dimension \( 2 \), \( C_1(M) > 0 \). At time \( t = 0 \), the following \( L^1 \) Sobolev inequality holds

\[
\left( \int_M f(x)^2 d\mu(0) \right)^{\frac{1}{2}} \leq C_s \left( \int_M |\nabla f(x)| d\mu(0) + \int_M |f(x)| d\mu(0) \right).
\]

Then along the Kähler Ricci flow holds

\[
\left( \int_M f(x) \frac{2^{n_0}}{n_0} d\mu(t) \right)^{\frac{n_0-2}{n_0}} \leq A \left( \int_M |\nabla f(x)|^2 + (R(x,t) + C_0) f^2(x) d\mu(t) \right)
\]

for all \( 0 \leq t \leq 1 \) and \( n_0 > 2 \), where the constants \( A, C_0 \) depend only on constant \( n_0 \), Sobolev constant \( C_s \), volume \( V \) of \( g(0) \) and a lower bound of \( R(g(0)) \).

Outline of the proof. Consider the flow (2.1). By Theorem 1 and Theorem 2 of [17], for all \( 0 < s \leq 1 \), \( \sigma > 0 \), \( f \in C^\infty(M, \tilde{g}(s)) \) and \( \|f\|_{L^2(M, \tilde{g}(s))} = 1 \), we obtain

\[
\int_M f^2 \log f^2 d\tilde{\mu}(s) \leq \sigma \int_M \left( 4|\nabla f(x)|^2 + \tilde{R} f^2(x) \right) d\tilde{\mu}(s) - \log \sigma + C.
\]

Taking the minimum of the right hand side respect to \( \sigma \), then for all \( n_0 > 2 \), we have

\[
\int_M f^2 \log f^2 d\tilde{\mu}(s) \leq \log \left( C \int_M \left( 4|\nabla f(x)|^2 + (\tilde{R} + C_0) f^2(x) \right) d\tilde{\mu}(s) \right)
\]

\[
\leq \frac{n_0}{2} \log \left( C \int_M \left( 4|\nabla f(x)|^2 + (\tilde{R} + C_0) f^2(x) \right) d\tilde{\mu}(s) \right).
\]

The following argument is standard, see section 10.2 of [1] and Proposition 2.3. □

**Remark 3.3.** One can easily check that the constants \( A \) in Lemma 3.1 and 3.2 do not depend on the time \( t \), since the constant \( C \) in (3.2) doesn’t depend on \( t \). However, here we only need \( t \leq 1 \).
Now we are ready to prove theorem 1.11:

**Proof of Theorem 1.11.**

Case $n \geq 1$: Let $f^p$, $p \geq 1$ be a test function in (1.8). We have

$$\int_M f^p f^t d\mu(t) - \int_M f^p \Delta f d\mu(t) \leq a \int_M f^{p+1} d\mu(t).$$

Integration by parts, we have

$$\frac{1}{p+1} \int_M (f^{p+1}) d\mu(t) + \frac{4p}{(p+1)^2} \int_M |\nabla f^\frac{p+1}{2}|^2 d\mu(t) \leq a \int_M f^{p+1} d\mu(t).$$

Moreover, since $\partial_t d\mu(t) = \Delta u d\mu(t) = \left(\frac{n}{2} - R\right) d\mu(t)$,

$$\frac{1}{p+1} \partial_t \int_M f^{p+1} d\mu(t) + \frac{1}{p+1} \int_M R f^{p+1} d\mu(t) + \frac{4p}{(p+1)^2} \int_M |\nabla f^\frac{p+1}{2}|^2 d\mu(t) \leq (a + \frac{n}{p+1}) \int_M f^{p+1} d\mu(t).$$

Multiplying both sides $p + 1$, since $4^p \geq 2(p + 1)$ for all $p \geq 1$, we get

$$\partial_t \int_M f^{p+1} d\mu(t) + \int_M R f^{p+1} d\mu(t) + 2 \int_M |\nabla f^\frac{p+1}{2}|^2 d\mu(t) \leq (a(p+1) + n) \int_M f^{p+1} d\mu(t).$$

Since scalar curvature has a lower bound $-C_0$, then

$$\partial_t \int_M f^{p+1} d\mu(t) + \frac{1}{2} \left( \int_M (R + C_0) f^{p+1} d\mu(t) + 4|\nabla f^\frac{p+1}{2}|^2 d\mu(t) \right) \leq \left( a(p+1) + n + C_0 \right) \int_M f^{p+1} d\mu(t).$$

(3.3)

For any $1 > \sigma > \tau > 0$, let

$$\psi(t) = \begin{cases} 0, & 0 \leq t \leq \tau \\ \frac{t-\tau}{\sigma-\tau}, & \tau \leq t \leq \sigma \\ 1, & \sigma \leq t \leq 1 \end{cases}$$

Multiplying (3.3) by $\psi$, we obtain

$$\partial_t \left( \psi \int_M f^{p+1} d\mu(t) \right) + \frac{1}{2} \psi \left( \int_M (R + C_0) f^{p+1} d\mu(t) + 4|\nabla f^\frac{p+1}{2}|^2 d\mu(t) \right) \leq \left[ (a(p+1) + n + C_0) \psi + \psi' \right] \int_M f^{p+1} d\mu(t).$$

Integrating this with respect to $t$ we get

$$\sup_{\sigma \leq t \leq 1} \int_M f^{p+1} d\mu(t) + \frac{1}{2} \left( \int_\sigma^1 \int_M [(R + C_0) f^{p+1} d\mu(t) + 4|\nabla f^\frac{p+1}{2}|^2 d\mu(t)] dt \right) \leq \left[ (a(p+1) + n + C_0) + \frac{1}{\sigma - \tau} \right] \int_\tau^1 \int_M f^{p+1} d\mu(t) dt.$$
Applying the $L^2$ Sobolev inequality along the Kähler Ricci flow (Lemma 3.1), we deduce
\[
\int_{\sigma}^{1} \int_{M} f^{(p+1)(1+\frac{\sigma}{2})} d\mu(t) dt \\
\leq \int_{\sigma}^{1} \left( \int_{M} f^{p+1} d\mu(t) \right)^{\frac{2}{p}} \left( \int_{M} f^{(p+1)\frac{\sigma}{2}} d\mu(t) \right)^{\frac{p}{2}} dt \\
\leq \sup_{\sigma \leq t \leq 1} \left( \int_{M} f^{p+1} d\mu(t) \right)^{\frac{2}{p}} \int_{\sigma}^{1} A \left( \int_{M} [(R + C_0)f^{p+1} d\mu(t)] + 4|\nabla f^{\frac{2(p+1)}{p}}|^2 d\mu(t) \right) dt \\
\leq 2A[(p + 1)a + n + C_0 + \frac{1}{\sigma - \tau}]^{1 + \frac{2}{p}} \left( \int_{\tau}^{1} \int_{M} f^{p+1} d\mu(t) dt \right)^{1 + \frac{2}{p}}.
\]

We put
\[
H(p, \tau) = \left( \int_{\tau}^{1} \int_{M} f^p d\mu(t) dt \right)^{\frac{1}{p}}, \text{ for any } p \geq 2, 0 < \tau < 1.
\]

Thus
\[
H(p(1 + \frac{2}{n}), \sigma) \leq (2A)^{\frac{1}{p(1 + \frac{2}{n})}} [p a + n + C_0 + \frac{1}{\sigma - \tau}] \frac{1}{p} H(p, \tau).
\]

Fix $0 < t_0 < t_1 < 1$ and set $\chi = 1 + \frac{2}{n}$, $p_k = p_0 \chi^k$, $\tau_k = t_0 + (1 - \frac{1}{\chi^k})(t_1 - t_0)$. Then we have
\[
H(p_{k+1}, \tau_{k+1}) \leq (2A)^{\frac{1}{\tau_{k+1}}} [p_k a + n + C_0 + \frac{1}{t_1 - t_0} \chi \chi^k]^{\frac{1}{p_k}} H(p_k, \tau_k).
\]

By iteration, we have
\[
H(p_{m+1}, \tau_{m+1}) \leq (2A)^{\sum_{k=0}^{m} \frac{1}{\tau_{k+1}}} [p_0 a + n + C_0 + \frac{1}{t_1 - t_0} \chi^{\sum_{k=0}^{m} 1 + \frac{1}{p_k}} \chi^{\sum_{k=0}^{m} \frac{1}{p_k}} H(p_0, \tau_0).
\]

Letting $m \to \infty$, we obtain
\[
H(p_\infty, \tau_\infty) \leq C_1 [p_0 a + n + C_0 + \frac{n + 2}{2(t_1 - t_0)}]^{\frac{n + 2}{2p_0}} H(p_0, \tau_0)
\]
for all $p_0 \geq 2$. That is
\[
\sup_{(x,t) \in M \times [t_1, 1]} |f(x, t)| \leq C_1 [p_0 a + n + C_0 + \frac{n + 2}{2(t_1 - t_0)}]^{\frac{n + 2}{2p_0}} \left( \int_{t_0}^{1} \int_{M} f^{p_0} d\mu(t) dt \right)^{\frac{1}{p_0}}.
\]

Since $0 < t_0 < t_1 < 1$, we have
\[
\sup_{(x,t) \in M \times [t_1, 1]} |f(x, t)| \leq \frac{C_2}{(t_1 - t_0)^{\frac{n + 2}{2p_0}}} \left( \int_{t_0}^{1} \int_{M} f^{p_0} d\mu(t) dt \right)^{\frac{1}{p_0}}, \text{ for all } p_0 \geq 2,
\]
where $C_2$ depending only on $p_0, a$, dimension $n$, Sobolev constant $C_s$, volume $V$ and a lower bound of $R(g(0))$. For $0 < p < 2$, we set
\[
h(s) = \sup_{(x,t) \in M \times [s, 1]} |f(x, t)|.
\]
So

\[ h(t_1) \leq \frac{C_2}{(t_1 - t_0)^{\frac{n+2}{2p}}} \left( \int_0^1 \int_M f^2 d\mu(t) dt \right)^{\frac{1}{p}} \]

\[ \leq h(t_0)^\frac{1}{2} \left( \frac{C_2}{(t_1 - t_0)^{\frac{n+2}{2p}}} \right)^{\frac{1}{2}} \left( \int_0^1 \int_M f^p d\mu(t) dt \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} h(t_0) + \frac{C_3}{(t_1 - t_0)^{\frac{n+2}{2p}}} \left( \int_0^1 \int_M f^p d\mu(t) dt \right)^{\frac{1}{p}}. \]

By an iteration lemma (Lemma 4.3 of [15]), we get

\[ h(t_1) \leq \frac{C_4}{(t_1 - t_0)^{\frac{n+2}{2p}}} \left( \int_0^1 \int_M f^p d\mu(t) dt \right)^{\frac{1}{p}}, \]

for all \( 0 < t_0 < t_1 < 1, p > 0. \)

Letting \( t_0 \to 0 \) we have

\[ h(t_1) \leq \frac{C_4}{(t_1 - t_0)^{\frac{n+2}{2p}}} \left( \int_0^1 \int_M f^p d\mu(t) dt \right)^{\frac{1}{p}},\]

where constant \( C_4 \) depends only on \( p, a, \) dimension \( n, \) Sobolev constant \( C_s, \) volume \( V \) and a lower bound of \( R(0). \)

Case \( n = 2: \) we only need to replace Lemma 3.1 with Lemma 3.2. \( \square \)

Remark 3.4. From the above proof, one can find that it also holds for Ricci flow on real Riemannian manifold.

4. Proof of Theorem 1.9

The purpose of this section is to prove Theorem 1.9. This mainly bases on the linear parabolic estimate in Theorem 1.11. Since along the Kähler Ricci flow (1.5) we have the evolution equations, see [25]

\[ (\partial_t - \Delta) R = |\nabla \bar{\nabla} u|^2 + R - \frac{n}{2}, \]

\[ (\partial_t - \Delta) |\nabla u|^2 = -|\nabla \nabla u|^2 - |\nabla u|^2 + |\nabla u|^2. \]

Hence

\[ (\partial_t - \Delta)(R + |\nabla u|^2) = -|\nabla \nabla u|^2 + R + |\nabla u|^2 - \frac{n}{2} \leq R + |\nabla u|^2. \]

Applying Theorem 1.11, we have

\[ (4.1) \sup_{x \in M} |R + |\nabla u|^2|(x, t) \leq \frac{C}{t^{\frac{n+2}{2p}}} \int_0^1 \int_M |R + |\nabla u|^2| d\mu(t) dt. \]

for all \( 0 < t < 1, \) where constant \( C \) depends only on dimension \( n, \) Sobolev constant \( C_s \) of \( g(0) \) and a lower bound of \( R(g(0)). \) Since Ricci curvature lower bound \(-K, \)
volume lower bound \( V_0 \) and diameter upper bound \( d \) can deduce the \( L^2 \) Sobolev inequality (see Theorem 3.2 of [16] or [18]). So \( C \) depends only on \( n, K, V_0 \) and \( d. \)
In order to get an upper bound of $R$, it suffices to estimate \( \int_0^1 \int_M |R + |\nabla u|^2|d\mu(t)dt \). Since \( \Delta u = \frac{n}{2} - R \leq \frac{n}{2} + C_0 \), then

\[
\int_M |\Delta u|d\mu(t) = \int_M |\Delta u - (\frac{n}{2} + C_0) + (\frac{n}{2} + C_0)|d\mu(t)
\leq \int_M (\frac{n}{2} + C_0) - \Delta u + (\frac{n}{2} + C_0)d\mu(t)
= (n + 2C_0)V.
\]

Hence \( \int_M |R|d\mu(t) \leq C_1 V \), and then

\[
(4.2) \quad \int_0^1 \int_M |R|d\mu(t)dt \leq C_1 V.
\]

On the other hand, normalize \( u \) by \( \int_M e^{-u}d\mu(t) = (2\pi)^{\frac{n}{2}} \). The evolution equation of \( u \) is given by (2.6),

\[
\partial_t u = \Delta u + u + a, \quad a = -\int_M ue^{-u}(2\pi)^{\frac{n}{2}} \leq C'.
\]

where \( C' \) depends only on dimension \( n \), volume of \( g(0) \), Sobolev constant \( C_s \) of \( g(0) \) and a lower bound of \( R(g(0)) \). Thus

\[
(4.3) \quad \int_M ud\mu(t) = \int_M ud\mu(t) + \int_M ud\mu(t)
= \int_M ud\mu(t) + aV + \int_M u\Delta u\mu(t)
= \int_M ud\mu(t) + aV - \int_M |\nabla u|^2d\mu(t)
\leq \int_M ud\mu(t) + C'V
\]

and

\[
(4.4) \quad \int_M d\mu(t) \leq \left( \int_M ud\mu(0) + C'V \right) e^t - CV \leq \left( \int_M ud\mu(0) + C'V \right) e^t.
\]

Integrating (4.3) over \([0, 1]\), we have

\[
\int_0^1 \partial_t \int_M ud\mu(t)dt = \int_0^1 \int_M ud\mu(t)dt + aV - \int_0^1 \int_M |\nabla u|^2d\mu(t)dt.
\]

By (4.4) we have

\[
\int_0^1 \int_M |\nabla u|^2d\mu(t)dt
\leq \int_0^1 \int_M ud\mu(t)dt + aV + \int_M ud\mu(0) - \int_M ud\mu(1)
\leq C'V + (e - 1) \left( \int_M ud\mu(0) + C'V \right) + \int_M ud\mu(0) - \int_M ud\mu(1).
\]

Since \( \int_M e^{-u}d\mu(1) = (2\pi)^{\frac{n}{2}} \), by Jensen’s inequality

\[
\ln \left( \frac{(2\pi)^{\frac{n}{2}}}{V} \right) = \ln \left( \int_M e^{-u}\frac{d\mu(1)}{V} \right) \geq \frac{1}{V} \int_M -ud\mu(1).
\]
Hence we get
\[\int_0^1 \int_M |\nabla u|^2 d\mu(t)dt \leq C'V + (e - 1) \left( \int_M u d\mu(0) + C'V \right) + \int_M u d\mu(0) + V \ln \left( \frac{(2\pi)^{\frac{n}{2}}}{V} \right).\]
Combining (4.2) with (4.5), we arrive at
\[\int_0^1 \int_M |R + |\nabla u|^2| d\mu(t)dt \leq C_1 V + e \left( \int_M u d\mu(0) + C'V \right) + V \ln \left( \frac{(2\pi)^{\frac{n}{2}}}{V} \right).\]
In order to deduce an upper bound of scalar curvature \(R\) and gradient bound of Ricci potential \(u\), it suffices to estimate the upper bound of \(\int_M u d\mu(0)\).

Actually, since we have a lower bound of Ricci curvature \(\text{Ric}(\omega(0)) \geq -K\omega(0)\), volume lower bound \(V_0 > 0\) and diameter upper bound \(d\) at time \(t = 0\), we can get a lower bound of the Green function \(\Gamma(x, y) \geq -B\) at time \(t = 0\) (see [2] and [12]), where \(B\) depends only on \(K, V_0, d\) and \(n\). At time \(t = 0\), since \(\int_M e^{-u} d\mu(0) = (2\pi)^{\frac{n}{2}}\), there must be a point \(x_0\) such that \(u(x_0) \leq -\ln \left( \frac{(2\pi)^{\frac{n}{2}}}{V_0} \right)\). Then by Green’s formula
\[u(x_0) = \frac{1}{V} \int_M u d\mu(0) + \int_M \Gamma(x_0, y)(-\Delta u(y))d\mu(0),\]
that is
\[\frac{1}{V} \int_M u d\mu(0) = u(x_0) + \int_M \Gamma(x_0, y)\Delta u(y) d\mu(0) = u(x_0) + \int_M (\Gamma(x_0, y) + B)\Delta u(y) d\mu(0) \leq u(x_0) + \left( \frac{n}{2} + C_0 \right) \int_M \Gamma(x_0, y) + Bd\mu(0) \leq -\ln \left( \frac{(2\pi)^{\frac{n}{2}}}{V} \right) + \left( \frac{n}{2} + C_0 \right) BV.\]
Substituting (4.7) into (4.6), we have
\[\int_0^1 \int_M |R + |\nabla u|^2| d\mu(t)dt \leq C_2\]
where constant \(C_2\) depends only on \(K, n, V_0\) and \(d\). Hence we finish the proof. \(\square\)

5. Partial \(C^0\) estimate on \(S^2\)

In this section, suppose \((M, \omega)\) is a complex dimension 1 Fano manifold with \(R(\omega) \geq R_0 > 0\), \(\text{Vol}(M, \omega) \geq V_0 > 0\). Consider the heat flow (see [29])
\[\frac{\partial f}{\partial s} = \log \left( \frac{(\omega + \partial_\overline{\partial} f)}{\omega} \right) + f - h_\omega, \quad f|_{s=0} = 0.\]
This is in fact the Kähler Ricci flow. Here \(h_\omega\) is Ricci potential of \(\omega\). We will denote by \(f_s, \omega_s\) and \(R_s\), the function \(f(s, \cdot)\), the Kähler form \(\omega + \partial_\overline{\partial} f_s\) and scalar curvature \(R(\omega_s)\).
Remark 5.1. In this section, the following constants $C$ depend only on $R_0$ and $V_0$. It may change line by line. Moreover, constants $C_p, c_s$ will depend on $p$ or $s$.

Differentiating (5.1), we obtain
\[ \partial \overline{\partial} \left( \frac{\partial f_s}{\partial s} \right) = -\text{Ric}(\omega_s) + \omega_s \]
This implies that the Ricci potential $h_{\omega_s} = -\frac{\partial f_s}{\partial s} + c_s$ where $c_s$ is constant. Since $f_0 = 0$, we have $c_0 = 0$.

Lemma 5.2. There exist constants $V'_0, d$ depending only on $R_0, V_0$ such that, for all $s > 0$, holds
\[ R(\omega_s) \geq 0, \quad V_0 \leq \text{Vol}(M, \omega_s) \leq V'_0, \quad \text{diam}(M, \omega_s) \leq d. \]
What’s more, for all $s \geq R_0 > 0$, $R(\omega_s)$ has a uniform upper bound.

Proof. We have the evolution equation along the Kähler Ricci flow,
\[ \frac{\partial R_s}{\partial s} = \Delta_s R_s + R_s^2 - R_s, \]
Since $R(\omega) \geq R_0 > 0$, by maximum principle, $R_s \geq 0$. On the other hand, $R_s^2 - R_s \geq -\frac{1}{4}$, then
\[ \frac{\partial R_s}{\partial s} \geq \Delta_s R_s - \frac{1}{4} \]
Applying maximum principle again, we obtain
\[ R(\omega_s)_{\text{min}} \geq R_0 - \frac{1}{4} s \]
for time $0 \leq s \leq R_0$. Then we have $R(\omega_s) \geq \frac{3}{4} R_0 > 0$ for time $0 \leq s \leq R_0$. By Meyer’s diameter theorem, we obtain the upper bound of diameter. By volume comparison theorem and the upper bound of diameter, we deduce the upper bound of volume.

For $s \geq R_0$, applying Theorem 1.9, we have an upper bound of $R(\omega R_0)$ and an upper bound of the $C^1$-norm for $h_{\omega R_0}$. Using Perelman’s theorem, i.e., Theorem 1.8, we get the upper bound of $R(\omega_s)$ and diam$(M, \omega_s)$ for all $s \geq R_0$. \qed

Normalize $h_{\omega_s}$ by $\int_M e^{h_{\omega_s} \omega_s} = 2\pi$. Then along the Kähler Ricci flow, we have the evolution equation of $h_{\omega_s}$, see [25]
\[ (5.2) \quad \frac{\partial h_{\omega_s}}{\partial s} = \Delta_s h_{\omega_s} + h_{\omega_s} - a, \]
where $a = \frac{1}{2\pi} \int_M h_{\omega_s} e^{h_{\omega_s} \omega_s}$ is uniformly bounded.

Lemma 5.3. $\forall p \geq 1, \forall s \geq 0$, there exists constant $C_p$ depending only on $p, R_0$ and $V_0$ such that
\[ ||h_{\omega_s}||_{L^p(M, \omega_s)} \leq C_p. \]

Proof. Since $-\overline{\partial} \partial h_{\omega_s} = -\text{Ric}(\omega_s) + \omega_s$, by taking the trace, we get
\[ (5.3) \quad -\Delta_s h_{\omega_s} = 1 - R_s. \]
By Lemma 5.2, we can estimate the Green function $\Gamma_s(x, y)$ of $\Delta_s$ at all time $s$ (see also [2],[12],[13]),
\[ (5.4) \quad -C < \Gamma_s(x, y) \leq C|\log(d(x, y))| + C \]
where constant $C$ depends only on $R_0$, $V_0$.

Since $\int_M e^{h_\omega - \omega_s} = 2\pi$, by Jensen’s inequality, we have $\int_M h_\omega - \omega_s \leq C$. On the other hand, by the lower bound estimate of $\Gamma_\omega(x,y)$, we can estimate the lower bound of $\int_M h_\omega - \omega_s$. Actually, since $\int_M e^{h_\omega - \omega_s} = 2\pi$, there must be a point $x_0$ such that $h_\omega(x_0) = \log(\frac{2\pi}{V})$. Then by Green’s formula

$$\frac{1}{V} \int_M h_\omega - \omega_s = h_\omega(x_0) + \int_M \Gamma_\omega(x,y) \Delta_s h_\omega - \omega_s$$

$$= h_\omega(x_0) + \int_M (\Gamma_\omega(x,y) + C)(R_\omega - 1)\omega_s$$

$$\geq h_\omega(x_0) - \int_M (\Gamma_\omega(x,y) + C)\omega_s$$

$$= \log(\frac{2\pi}{V}) - CV.$$

Hence, there exists a uniform constant $C$ such that the following holds

$$\frac{1}{V} \int_M h_\omega - \omega_s \leq C.$$

Using Green’s formula again, we have

(5.5) \hspace{1cm} h_\omega(x) = \frac{1}{V} \int_M h_\omega - \omega_s + \int_M \Gamma_\omega(x,y)(-\Delta_s h_\omega - \omega_s)\omega_s(y).

Moreover, $||\Delta_\omega h_\omega||_{L^1(M,\omega_s)}$ is uniformly bounded. By (5.3) and Gauss-Bonnet theorem,

$$||\Delta_\omega h_\omega||_{L^1(M,\omega_s)} \leq V_0 + \int_M R_\omega \omega_s \leq C.$$

Applying Young’s inequality to (5.5), we arrive at

$$||h_\omega||_{L^p(M,\omega_s)} \leq C + \sup_{x \in M} \left( \int_M |\Gamma_\omega(x,y)|^p \omega_s(y) \right)^{\frac{1}{p}} ||\Delta_\omega h_\omega||_{L^1(M,\omega_s)}.$$

Claim: $\int_M |\Gamma_\omega(x,y)|^p \omega_s(y) \leq C_p$, for some constant $C_p$ depending only on $p$, $R_0$, $V_0$.

Actually, by inequality (5.4), we only need to show $\int_M |\log(d(x,y))|^p \omega_s(y)$ is uniformly bounded.

$$\int_M |\log(d(x,y))|^p \omega_s(y) \leq \int_{B(x,1)} |\log(d(x,y))|^p \omega_s(y) + \int_{M \setminus B(x,1)} |\log(d(x,y))|^p \omega_s(y)$$

$$\leq \sum_{k=0}^{\infty} \int_{B(x,\frac{1}{2k}) \setminus B(x,\frac{1}{2k+1})} |\log(d(x,y))|^p \omega_s(y) + V_0^\prime \log(d)^p$$

$$\leq \sum_{k=0}^{\infty} |\log(\frac{1}{2k+1})|^p \int_{B(x,\frac{1}{2k})} \omega_s(y) + V_0^\prime \log(d)^p$$

$$\leq \sum_{k=0}^{\infty} |\log(2)(k+1)|^p \text{vol}(B(x,\frac{1}{2k})) + V_0^\prime \log(d)^p$$

$$\leq \sum_{k=0}^{\infty} |\log(2)(k+1)|^p \frac{e}{4k} + V_0^\prime \log(d)^p$$

$$\leq C_p.$$
where we have used the volume comparison, so the Claim holds. Hence the $L^p$-norm of $h_{\omega_s}$ is uniformly bounded. □

**Lemma 5.4.** There exists a constant $C$ depending only on $R_0, V_0$ such that

\begin{equation}
|c_s| \leq C(e^s - 1), \text{ for all } s > 0
\end{equation}

where $c_s = \frac{\partial f_s}{\partial s} + h_{\omega_s}$.

**Proof.** Differentiating (5.1) we have

\begin{equation}
\frac{\partial}{\partial s} \left( \frac{\partial f_s}{\partial s} \right) = \Delta_s \left( \frac{\partial f_s}{\partial s} \right) + \frac{\partial f_s}{\partial s}
\end{equation}

and combine this with (5.2). We have

\[
\frac{\partial}{\partial s} \left( \frac{\partial f_s}{\partial s} + h_{\omega_s} \right) = \Delta_s \left( \frac{\partial f_s}{\partial s} + h_{\omega_s} \right) + \left( \frac{\partial f_s}{\partial s} + h_{\omega_s} \right) - a.
\]

That is

\[
\frac{\partial}{\partial s} c_s = c_s - a.
\]

Since $c_0 = 0$, $a$ is bounded, we can deduce the bound for $c_s$ easily,

\[|c_s| \leq C(e^s - 1).\]

□

Now we can estimate the Kähler potential,

**Lemma 5.5.** There exists a constant $C$ depending only on $R_0, V_0$ such that

\[|f_s(x)| \leq C\sqrt{s}\]

for all $s \leq 1$.

**Proof.** By Lemma 5.3 and Lemma 5.4, we have

\[\left| \frac{\partial f_s}{\partial s} \right|_{L^p} \leq C_p, \text{ for all } s \leq 1.\]

Combining Theorem 1.11 with equation (5.7), we obtain

\[\left| \frac{\partial f_s}{\partial s} \right| \leq \frac{C_p n_0}{s^{\frac{n_0}{2p}}}, \text{ for all } s \leq 1.\]

Choose $n_0 = 3$, $p = 5$. We have $|\frac{\partial f_s}{\partial s}| \leq \frac{C}{\sqrt{s}}$. Then

\[|f_t(x) - f_0(x)| \leq \int_0^t \left| \frac{\partial f_s}{\partial s} \right| ds \leq C\sqrt{t}\]

for all $t \leq 1$. Noting that $f_0 = 0$, then we have $|f_s(x)| \leq C\sqrt{s}$, for all $s \leq 1$. □

Since

\[\text{Ric}(\omega_s) - \overline{\partial} \overline{\partial} h_{\omega_s} = \omega_s,
\]

we can choose $\omega_s e^{h_{\omega_s}}$ as a Hermitian metric of anti-canonical line bundle $K_{M}^{-1}$ with curvature form $\omega_s$, denoting by $H_{\omega_s}$.

**Lemma 5.6.** For all $s \leq 1, H_\omega$ and $H_{\omega_s}$ are equivalent. i.e., there exists a constant $C$ depending only on $R_0, V_0$ such that

\[\frac{1}{C} H_\omega \leq H_{\omega_s} \leq C H_\omega.
\]
Proof. By equation (5.1), we have
\[ e^{\frac{\partial f_s}{\partial s}} - f_s + h_\omega = \frac{\omega_s}{\omega} . \]
Thus
\[ e^{f_s} - f_s = \frac{\omega_s e^{-\frac{\partial f_s}{\partial s}} + c_s}{\omega e^{h_\omega}} = \frac{H_{\omega_s}}{H_\omega} . \]
By Lemma 5.4 and Lemma 5.5, we conclude
\[ \frac{1}{C} \leq \frac{H_{\omega_s}}{H_\omega} \leq C \]
for all \( s \leq 1 \).

Now we turn to prove Theorem 1.7.

Proof of Theorem 1.7. We will argue by contradiction. Suppose there exists a family Fano manifolds \((M^t, \omega^t)\) satisfying \( R(\omega^t) \geq R_0 \), \( \text{Vol}(M^t, \omega^t) \geq V_0 \), but there exist \( \{x_i \in M^t\} \) such that
\[ \rho_{\omega^t, t}(x_i) \to 0, \text{ when } i \to \infty. \]
Assume \( S^t \in H^0(M^t, K_{M^t}^{-1}) \) satisfying \( ||S^t||_{H^0_{\omega^t}}^2 = 1 \),
\[ ||S^t||_{H^0_{\omega^t}}^2 (x_i) = \sup_{||S||_{H^0_{\omega^t}}^2 = 1} ||S||_{H^0_{\omega^t}}^2 (x_i) = \eta_{\omega^t}(x_i). \]
Here \( \omega^t \) is the Kähler form at time \( t = 1 \) along the Kähler Ricci flow with initial metric \( \omega^t \) on Fano manifold \( M^t \). Then by Lemma 5.6 and Remark 1.2,
\[ \eta_{\omega^t}(x_i) = \frac{||S^t||_{H^0_{\omega^t}}^2 (x_i)}{||S||_{H^0_{\omega^t}}^2 (x_i)} \cdot ||S||_{H^0_{\omega^t}}^2 (x_i) \cdot \eta_{\omega^t}(x_i) \leq C ||S||_{H^0_{\omega^t}}^2 \cdot \rho_{\omega^t, t}(x_i) . \]
Using Remark 1.2 again, we obtain
\[ \rho_{\omega^t, t}(x_i) \leq NC ||S||_{H^0_{\omega^t}}^2 \cdot \rho_{\omega^t, t}(x_i) . \]
By Lemma 5.2, we have \( 0 \leq R(\omega^t) \leq C \), \( V_0 \leq \text{Vol}(M^t, \omega^t) \leq V_0' \). Then by Moser’s iteration we have \( ||S^t||_{H^0_{\omega^t}}^2 (x) \leq C \), for all \( x \in M^t \) (see[27]). Since \( H_{\omega^t}^{\nabla t} \) and \( H_{\omega^t}^{\nabla t} \) are equivalent, so
\[ ||S^t||_{H^0_{\omega^t}}^2 (x) \leq C \]
Thus
\[ ||S||_{H^0_{\omega^t}}^2 \leq CV_0' \]
Hence
\[ \rho_{\omega^t, t}(x_i) \leq NC \rho_{\omega^t, t}(x_i) \to 0, \text{ when } i \to \infty. \]
But the family \((M_i, \omega_i)\) have bounded curvature, bounded volume, bounded diameter. By Hamilton’s compactness theorem for Ricci flow([14]), \(\rho_{\omega_i, t}(x)\) have a uniform lower bound. This is a contradiction. Hence we complete the proof. □

**Remark 5.7.** Under the above estimates, we can also prove that the Bergman kernel is uniformly continuous along the Kähler Ricci flow by showing the measure is uniformly continuous.

### 6. Partial \(C^0\) estimate for complex dimension \(\geq 2\)

Suppose \((M, \omega)\) is a complex dimension \(m = \frac{n}{2} \geq 2\) Fano manifold, with \(\text{Ric}(\omega) \geq R_0\), \(\text{Vol}(M, \omega) \geq V_0\), \(\text{diam}(M, \omega) \leq d\). Consider the heat flow (see [29])

\[
\frac{\partial f}{\partial s} = \log \left( \frac{(\omega + \partial f)^m}{\omega^m} \right) + f - h_\omega, \quad f|_{s=0} = 0.
\]

This is in fact the Kähler Ricci flow. Here \(h_\omega\) is Ricci potential of \(\omega\). We will denote by \(f_s\) and \(\omega_s\) the function \(f(s, \cdot)\) and the Kähler form \(\omega + \partial \bar{\partial} f_s\).

**Remark 6.1.** In this section, the following constants \(C\) depend only on \(R_0, V_0, d, n\). It maybe change line by line.

**Remark 6.2.** In order to estimate the Bergman kernel for high dimension, we only need to estimate the Kähler potential like the case of complex dimension 1. However, the estimates of dimension 1 mostly depend on the property of dimension 1 which we can’t extend to high dimension. Hence we must need another approach.

First, we can estimate the upper bound of \(h_\omega\). The proof is similar to Lemma 2.4.

**Lemma 6.3.** Suppose \(h_\omega\) is the Ricci potential of \(\omega\). Normalize it by \(\int_M e^{h_\omega} \omega^m = (2\pi)^m\). Then there is a uniform constant \(C\) depending only on \(R_0, V_0, d\) and \(m\) such that

\[
(6.2) \quad h_\omega \leq C.
\]

**Proof.** Since \(\text{Ric}(\omega) - \partial \bar{\partial} h_\omega = \omega\), taking the trace respect to \(\omega\), we obtain

\[
\Delta h_\omega = R - m.
\]

Thus

\[
\Delta e^{h_\omega} = \Delta h_\omega e^{h_\omega} + |\nabla h_\omega|^2 e^{h_\omega}
\]

\[
\geq (R - m)e^{h_\omega}
\]

\[
\geq -Ce^{h_\omega}.
\]

Moreover, since \(\text{Ric}(\omega) \geq R_0\), \(\text{Vol}(M, \omega) \geq V_0\), \(\text{diam}(M, \omega) \leq d\), then the \(L^2\)-Sobolev inequality holds,

\[
\left( \int_M \phi^\frac{2m}{m-1} \omega^m \right)^{\frac{m-1}{m}} \leq C \left( \int_M |\nabla \phi|^2 \omega^m + \int_M \phi^2 \omega^m \right).
\]

The Moser’s iteration and inequality (6.3) imply

\[
e^{h_\omega} \leq C \int_M e^{h_\omega} \omega^m = C(2\pi)^m.
\]

Hence the upper bound of \(h_\omega\) follows. □
Now we can estimate the Kähler potential $f_s$.

**Lemma 6.4.** Let $f_s$ be the Kähler potential of $\omega_s$, satisfying equation (6.1). Then for all $s \geq 0$, we have a lower bound estimate for $f_s$,

\begin{equation}
(6.4) \quad f_s \geq C(1 - e^s).
\end{equation}

**Proof.** By Lemma 6.3,

\[
\frac{\partial f_s}{\partial s} = \log \left( \frac{\omega_s^m}{\omega_m} \right) + f_s - h_\omega \geq \log \left( \frac{\omega_s^m}{\omega_m} \right) + f_s - C.
\]

Using maximum principle, we have

\[
\frac{\partial f_s}{\partial s} \geq f_s - C.
\]

Then

\[
\partial_s (f_se^{-s}) \geq -Ce^{-s}.
\]

Since $f_0 = 0$, we deduce

\[
f_s \geq C(1 - e^s).
\]

\[\square\]

**Lemma 6.5.** Let $f_s$ be the Kähler potential of $\omega_s$, satisfying equation (6.1). Then for all $s \geq 0$, we have an upper bound estimate for $f_s$,

\begin{equation}
(6.5) \quad f_s \leq Ce^s.
\end{equation}

**Proof.** Since $\omega_s = \omega + \partial T f_s > 0$, taking trace respect to $\omega$, we have

\[
m + \Delta f_s > 0
\]

By the assumption of initial metric $\omega$, we can control the Green function lower bound $\Gamma(x, y) \geq -C$. Applying Green’s formula, we have

\[
f_s(x) = \frac{1}{V} \int_M f_s \omega^m + \int_M \Gamma(x, y)(-\Delta f_s)\omega^m
\]

\[
= \frac{1}{V} \int_M f_s \omega^m + \int_M (\Gamma(x, y) + C)(-\Delta f_s)\omega^m
\]

\[
\leq \frac{1}{V} \int_M f_s \omega^m + m \int_M (\Gamma(x, y) + C)\omega^m
\]

\[
\leq mCV + \frac{1}{V} \int_M f_s \omega^m.
\]

(6.6)
In order to get an upper bound of $f_s$, it suffices to estimate $\frac{1}{V} \int_M f_s \omega^m$. Actually, by (6.1) and Jensen’s inequality,

$$\frac{\partial}{\partial s} \left( \frac{1}{V} \int_M f_s \omega^m \right) = \frac{1}{V} \int_M \frac{\partial f_s}{\partial s} \omega^m$$

$$= \frac{1}{V} \int_M \log \left( \frac{\omega^m}{\omega^m} \right) \omega^m + \frac{1}{V} \int_M f_s \omega^m - \frac{1}{V} \int_M h \omega^m$$

$$\leq \log \left( \frac{\omega^m}{\omega^m} \right) + \frac{1}{V} \int_M f_s \omega^m - \frac{1}{V} \int_M h \omega^m$$

$$\leq \frac{1}{V} \int_M f_s \omega^m - \frac{1}{V} \int_M h \omega^m.$$  

(6.7)

On the other hand, we have a lower bound estimate of $\frac{1}{V} \int_M h \omega^m \geq -C$, see (4.7), noting that $-u = h \omega^m$. Hence

$$\frac{\partial}{\partial s} \left( \frac{1}{V} \int_M f_s \omega^m \right) \leq C + \frac{1}{V} \int_M f_s \omega^m.$$ 

Since $f_0 = 0$, then we get the upper bound of $\frac{1}{V} \int_M f_s \omega^m$,

$$\frac{1}{V} \int_M f_s \omega^m \leq C(e^s - 1).$$

Substituting into (6.6), the lemma follows. □

Remark 6.6. Since we can estimate the $L^1$-norm of Kähler potential $f$, one can use Moser’s iteration to get the upper bound of $f_s$. Furthermore the upper bound will go to zero when time go to zero.

Remark 6.7. If we have a $C^0$ bound of the Ricci potential $h \omega$, by using Maximum principal theorem, one can easily deduce all the above bound(see [29]). However, we do not have the $C^0$ bound for the Ricci potential $h \omega$ here.

We can choose $\omega^m e^{h \omega}$ as a Hermitian metric of anti-canonical line bundle $K^{-1}_M$ with curvature form $\omega_s$, since

$$\text{Ric}(\omega_s) - \overline{\partial} \overline{\partial} h \omega_s = \omega_s.$$ 

Denote $\omega^m e^{h \omega}$ by $H_\omega$. Noticing that the constant $c_s = h \omega + \frac{\partial f_s}{\partial s}$ have a uniform bound in Lemma 5.4 and combining Lemma 6.4 with Lemma 6.5, we have

**Lemma 6.8.** For all $s \leq 1$, $H_\omega$ and $H_\omega_s$ are equivalent, i.e.,

$$\frac{1}{C} H_\omega \leq H_\omega_s \leq C H_\omega.$$ 

where the constant $C$ depends only on $R_0$, $V_0$, $m$ and $d$.

**Proof.** By equation (6.1), we have

$$e^{\frac{\partial f_s}{\partial s} + h - f_s} = \frac{\omega^m}{\omega^m}.$$ 

Thus

$$e^{c_s - f_s} = \frac{\omega^m e^{\frac{\partial f_s}{\partial s} + c_s}}{\omega^m e^{h \omega}} = \frac{H_\omega_s}{H_\omega}.$$ 

By Lemma 5.4, Lemma 6.4, and Lemma 6.5, we deduce

$$\frac{1}{C} \leq \frac{H_\omega_s}{H_\omega} \leq C.$$
for all $s \leq 1$.

The diameter upper bound is also under control.

**Lemma 6.9.** For $\frac{1}{2} \leq s \leq 1$, there exits a uniform constant $D > 0$ depending only on $R_0$, $V_0$, $d$ and $m$ such that

\begin{equation}
\text{diam}(M, \omega_s) \leq D.
\end{equation}

**Proof.** Since along the Kähler Ricci flow, the following $L^2$-Sobolev inequality holds (see Ye [34] or Zhang [35]),

\[
\left( \int_M \phi(x) \frac{2m}{m-1} \omega_s^m \right)^{\frac{m-1}{m}} \leq A \left( \int_M [4|\nabla \phi(x)|^2 + (R_s + C_0) \phi^2(x)] \omega_s^m \right)
\]

for all $0 \leq s \leq 1$ and $\phi \in W^{1,2}(M, \omega_s)$. For $\frac{1}{2} \leq s \leq 1$, by Theorem 1.9, we have $|R_s| \leq C$. Hence

\[
\left( \int_M \phi(x) \frac{2m}{m-1} \omega_s^m \right)^{\frac{m-1}{m}} \leq C \left( \int_M [4|\nabla \phi(x)|^2 + \phi^2(x)] \omega_s^m \right).
\]

Since the $L^2$-Sobolev inequality implies the non-collapsing of volume (see Lemma 2.2 of [16]) and the volume is preserved along the Kähler Ricci flow, thus there must be a uniform upper bound for diameter.

**Theorem 6.10.** For all $\frac{1}{2} \leq s \leq 1$, all $l \geq 1$, the Bergman kernels $\rho_{\omega, l}$ and $\rho_{\omega_s, l}$ are equivalent. i.e., there exists a constant $C_l$ depending only on $R_0$, $V_0$, $d$, $l$ and $m$, such that

\begin{equation}
\frac{1}{C_l} \rho_{\omega, l} \leq \rho_{\omega_s, l} \leq C_l \rho_{\omega, l}.
\end{equation}

**Proof.** Assume $S_s \in H^0(M, K_M^{-l})$, $x \in M$, satisfying $||S_s||^2_{L^2_{\omega_s}} = 1$,

\[
||S_s||^2_{L^2_{\omega_s}}(x) = \sup_{||S||^2_{L^2_{\omega_s}} = 1} ||S||^2_{L^2_{\omega_s}}(x) = \eta_{\omega_s}(x).
\]

Then, by Lemma 6.8 and Remark 1.2

\[
\eta_{\omega_s}(x) = \frac{||S_s||^2_{L^2_{\omega_s}}(x)}{||S_s||^2_{L^2_{\omega_s}}(x)} \cdot ||S_s||^2_{L^2_{\omega_s}} \cdot ||S_s||^2_{L^2_{\omega_s}}(x) \cdot ||S_s||^2_{L^2_{\omega_s}}(x)
\]

\[
\leq ||S_s||^2_{L^2_{\omega_s}}(x) \cdot ||S_s||^2_{L^2_{\omega_s}}(x) \cdot \eta_{\omega_s}(x)
\]

\[
\leq C ||S_s||^2_{L^2_{\omega_s}}(x) \cdot \rho_{\omega, l}(x).
\]

Using Remark 1.2 again, we obtain

\[
\rho_{\omega, l}(x) \leq NC ||S_s||^2_{L^2_{\omega_s}}(x).
\]

Due to Zhang [35] or Ye [34], and the upper bound estimate of scalar curvature in Theorem 1.9, we have the $L^2$-Sobolev inequality holds, for all $\frac{1}{2} \leq s \leq 1$,

\[
\left( \int_M \phi(x) \frac{2m}{m-1} \omega_s^m \right)^{\frac{m-1}{m}} \leq C \left( \int_M [4|\nabla \phi(x)|^2 + \phi^2(x)] \omega_s^m \right), \forall \phi \in W^{1,2}(M, \omega_s).
\]
Moreover, for section $S_\omega$, we have equation (see Tian [27])
\[ \Delta_\omega |S_\omega|^2 \leq ||\nabla S_\omega||^2_{H^{2,1}_\omega} - ml||S_\omega||^2_{H^{2,1}_\omega} \geq -ml||S_\omega||^2_{H^{2,1}_\omega}. \]
Applying Moser’s iteration, for all $y \in M$, we deduce
\[ ||S_\omega||^2_{H^{2,1}_\omega}(y) \leq C||S_\omega||^2_{H^{2,1}_\omega} = C. \]
By the equivalence of $H^{2,1}_\omega$ and $H^{2,1}_\omega$, we have
\[ ||S_\omega||^2_{H^{2,1}_\omega}(y) \leq C. \]
This implies
\[ ||S_\omega||^2_{H^{2,1}_\omega} \leq CV. \]
Hence
\[ \rho_{\omega,t}(x) \leq NC\rho_{\omega,t}(x). \]
The proof of the other part is similar.

In order to prove Theorem 1.5, we only need to prove the following:

**Theorem 6.11.** For any family of Fano manifolds $\{(M, \omega^i)\}$ with complex dimension $m = 2, 3$ and $\text{Ric}(\omega^i) \geq R_0$, $\text{Vol}(M, \omega^i) \geq V_0$, $\text{diam}(M, \omega^i) \leq d$, there exists a subsequence $\{(M, \omega^{i_k})\}$ and sequence $l_k \to \infty$, such that for all $l = l_k$
\[ \inf_{l_k} \inf_{x \in M} \rho_{\omega^{i_k},l}(x) > 0. \]

**Proof.** Let $(M, \omega^t)$ be the manifolds at time $t = s$ along the Kähler Ricci flow with initial metric $\omega^s$. By the above theorem 6.10 (the equivalence of Bergman kernels), if we can show a uniform lower bound for the Bergman kernel $\rho_{\omega^t,l}$ of $(M, \omega^t)$ with a sequence of $l \to \infty$, then the Theorem follows. We need the following lemmas developed in Tian and Zhang’s paper [33].

First of all, the $L^4$-estimate of Ricci curvature is the key step to prove the theorem.

**Lemma 6.12.** ([33]) Let $(M, \omega)$ be a Fano manifold with complex dimension $m$, $\text{Ric}(\omega) \geq R_0$, $\text{Vol}(M, \omega) \geq V_0 > 0$, $\text{diam}(M, \omega) \leq d$. Then along the Kähler Ricci flow
\[ \partial_t g_{ij} = g_{ij} - R_{ij} = u_{ij}, \omega(0) = \omega, \]
there exists $C = C(R_0, V_0, d, m)$ such that
\[ \int_M |\text{Ric}(\omega_t)|^4 \omega_t^m \leq C, \forall t \in \left[ \frac{1}{2}, 1 \right]. \]

**Proof.** The proof is similar to [33] by noticing our estimates in Theorem 1.9,
\[ \|u(t)\|_{C^0} + \|\nabla u(t)\|_{C^0} + \|\Delta u(t)\|_{C^0} \leq C, \forall t \in \left[ \frac{1}{2}, 1 \right] \]
where $C = C(R_0, V_0, d, m)$. Combining (6.12) with the Chern-Weil theory, we have the following $L^2$-estimate
\[ \int_M (|\nabla u|^2 + |\nabla \nabla u|^2 + |Rm|^2) \omega_t^m \leq C, \forall t \in \left[ \frac{1}{2}, 1 \right]. \]
Then by Bochner formula and integration by parts, we have $L^4$-estimate
\[ \int_M |\nabla \nabla u|^4 \omega_t^m \leq C \int_M (|\nabla \nabla \nabla u|^2 + |\nabla \nabla \nabla u|^2) \omega_t^m, \]
and
\begin{equation}
\int_M |\nabla u|^4 \omega^m_t \leq C \int_M (|\nabla \nabla \nabla u|^2 + |\nabla \nabla \nabla u|^2 + |\nabla \nabla \nabla u|^2) \omega^m_t
\end{equation}
for all $t \in [\frac{1}{2}, 1]$. In order to prove the $L^4$-bound of Ricci curvature, it suffices to estimate the $L^2$-bound of the third derivatives of $u$. Actually, by integration by parts and (6.12), we have
\begin{equation}
\int_M (|\nabla \nabla \nabla u|^2 + |\nabla \nabla \nabla u|^2) \omega^m_t \leq C \int_M (|\nabla \nabla u|^2 + |Rm|^2 + |\nabla u|^2) \omega^m_t, \forall t \in [\frac{1}{2}, 1].
\end{equation}
In [33], by using the evolution equation of $|\nabla \nabla u|^2$ and (6.16) the authors can estimate the upper bound of $\int_M |\nabla \nabla \nabla u|^2 \omega^m_t$. Thus we finish the proof where the constants $C$ above depending only on $R_0$, $V_0$, $d$ and $m$. One can find more details in [33].

By using the methods developed by Cheeger-Colding [5, 6, 7] and Cheeger-Colding-Tian [8] and Petersen-Wei [22, 23], the authors in [33] can prove

Lemma 6.13. ([33]) Let $\{ (M_i, \omega^i) \}$ be a family of Fano manifolds with complex dimension $m$ and
\begin{equation}
\int_M |\text{Ric}|^p \leq \Lambda.
\end{equation}
We further assume the non-collapsing, $\text{Vol}(B_r(x)) \geq kr^{2m}$, for all $x \in M_i$, $r \leq 1$, where $p > m$ and $\kappa, \Gamma$ are uniform positive constants. Then there exists a subsequence such that $(M_i, \omega^i)$ is convergent in the pointed Gromov-Hausdorff topology
\begin{equation}
(M_i, \omega^i) \overset{d\text{-GH}}{\longrightarrow} (M_\infty, d),
\end{equation}
and the followings hold,
(i) $M_\infty = S \cup R$ such that the singular set $S$ is a closed set of codimension $\geq 4$ and $R$ is convex in $M_\infty$;
(ii) There exists a $C^\alpha$, $\forall \alpha < 2 - \frac{m}{p}$, metric $g_\infty$ on $R$ which induces $d$;
(iii) $\omega^i$ converges to $g_\infty$ in the $C^\alpha$ topology on $R$. \hfill \Box

Combining the $L^4$-bound estimate of Ricci curvature along the Kähler Ricci flow (Lemma 6.12) and applying Perelman’s pseudolocality theorem [21] of Ricci flow and Shi’s higher derivative estimate to curvature [26], with the same argument in section 3.3 of [33] we can show that the convergence is smooth on the regular set along the Kähler Ricci flow:

Lemma 6.14. ([33]) With the same assumptions as Theorem 6.11, denote by $\omega^i_1$, the 1 time slice of the Kähler Ricci flow starting from $\omega^i$. Then up to a subsequence we have
\begin{equation}
(M, \omega^i_1) \overset{d\text{-GH}}{\longrightarrow} (M_\infty, d),
\end{equation}
and the limit $M_\infty$ is smooth outside a closed subset $S$ of real codimension $\geq 4$ and $d$ is induced by a smooth Kähler metric $g_\infty$ on $M_\infty \setminus S$. Moreover, $\omega^i_1$ converge to $g_\infty$ in $C^\infty$-topology outside $S$. \hfill \Box
Now by noticing the estimate of Ricci potential,
\[
\|u(t)\|_{C^0} + \|\nabla u(t)\|_{C^0} + \|\Delta u(t)\|_{C^0} \leq C, \quad \forall t \in \left[\frac{1}{2}, 1\right]
\]
and the smooth convergence of \((M, \omega^t_1)\), we can finish the proof of Theorem 6.11 with the similar arguments as in [31, 32], see also the proof of Theorem 5.1 in [33].

We remark that the \(C^1\) estimate to \(u(t)\) will be used in the iteration arguments to cancel the bad terms containing \(\nabla\nabla u(t)\).

\[\square\]

Since the nonnegative holomorphic bisectional curvature is preserved along the Kähler Ricci flow (see [19]), then combining with the upper bound estimate of scalar curvature (Theorem 1.9), the diameter estimate (Lemma 6.9) and Hamilton’s compactness theorem([14]), we have

**Theorem 6.15.** Let \((M, \omega)\) be a Fano manifold of complex dimension \(m \geq 1\), with nonnegative holomorphic bisectional curvature, volume lower bound \(V_0 > 0\) and diameter upper bound \(d\). Then for all \(l \in \mathbb{N}_+\), the Bergman kernel \(\rho_{\omega, l}\) has a uniform positive lower bound
\[
(6.21) \quad \rho_{\omega, l} > c_l > 0
\]
where the constant \(c_l\) depends only on \(m\), \(V_0\), \(d\) and \(l\).

\[\square\]

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