Some Rational Approximations and Bounds for Bateman’s G-Function

Omelsaad Ahfaf 1, Mansour Mahmoud 2,* and Ahmed Talat 3

Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; omahfaf76@gmail.com or omahfaf@uoa.edu.ly
2 Mathematics Department, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
3 Mathematics and Computer Sciences Department, Faculty of Science, Port Said University, Port Said 42526, Egypt; a_t_amer@yahoo.com or ahmed.talat@uoa.edu.ly
* Correspondence: mansour@mans.edu.eg

Abstract: Symmetrical patterns exist in the nature of inequalities, which play a basic role in theoretical and applied mathematics. In several studies, inequalities present accurate approximations of functions based on their symmetry properties. In this paper, we present the following rational approximations for Bateman’s G-function $G(w) = \frac{1}{w} + \left[2w^2 + \sum_{j=1}^{n} 4w^{2j-2} \right]^{-1} + O\left(\frac{1}{w^{n+1}}\right)$, where $a_1 = \frac{1}{4}$, and $a_j = \frac{1}{j+1} \frac{(1-2^{2j-1})B_{2j-2}}{2} + \sum_{v=1}^{j-1} \frac{(1-2^{2j-2v-1})B_{2j-2v-2}}{2} \nu$, $j > 1$. As a consequence, we introduced some new bounds of $G(w)$ and a completely monotonic function involving it.

Keywords: psi function; Padé approximant; Bateman’s G-function; bounds; completely monotonic; applications

MSC: 33B15; 41A21; 26A48; 26D15

1. Introduction

Bateman’s G-function is defined by Harry Bateman (1882–1946) ([1], p. 20) as

$$G(w) = \psi((1 + w)/2) - \psi(w/2), \quad w \in \mathbb{R} - \{0, -1, -2, \ldots\}$$

(1)

where the Psi function $\psi(w) = \frac{d}{dw} \ln \Gamma(w)$ and $\Gamma$ is the classical Euler Gamma function ([2], p. 3). The function $G(w)$ satisfies

$$G(w) = 2 \int_{0}^{\infty} \frac{e^{-wv}}{1 + e^{-v}} dv, \quad w > 0$$

(2)

and

$$G(w) + G(w + 1) = 2w^{-1}.$$  

(3)

For more details about the applications and inequalities of the function $G(w)$, please see [1,3–11] and the references therein.

A function $C(w)$ defined for $w > 0$ is called completely monotonic if $C^{(r)}(w)$ exists $\forall r \in \mathbb{N}$ such that

$$(-1)^r C^{(r)}(w) \geq 0, \quad w > 0; \quad r \in \mathbb{N}.$$  

There are several remarkable applications of completely monotonic functions in many scientific branches; for more information about this topic, we refer to [12–14]. The convergence of the integral ([15], p. 160):

$$C(w) = \int_{0}^{\infty} e^{-vw} dK(v), \quad w \geq 0$$

(4)
where $\kappa(w)$ is bounded and non-decreasing for $w \geq 0$, gives the necessary and sufficient condition for $G(w)$ to be completely monotonic on $w \geq 0$.

In [11], Qiu and Vuorinen deduced the double inequality:

$$\frac{4(1.5 - \ln 4)}{w^2} < G(w) - w^{-1} < \frac{1}{2w^2}, \quad w > 0.5$$

(5)

and Mortici [10] improve it by

$$\psi(w) < \psi(w + r) < \psi(w) + \psi(r) + \gamma - r + r^{-1}, \quad w \geq 1; \quad 0 < r < 1$$

(6)

where $\gamma$ is the Euler constant. In [5], Mahmoud and Agarwal improved the lower bound of Inequality (5) for $w > \left(\frac{4 - 12 \ln 2}{16 \ln 2 - 11}\right)^{1/2} \approx 2.74$ by

$$G(w) > \frac{1}{2w^2 + \frac{1}{4}}, \quad w > 0.$$  

(7)

Furthermore, they presented the asymptotic formula:

$$G(w) - w^{-1} \sim \sum_{n=1}^{\infty} \frac{(2n - 1)B_{2n}}{nw^{2n}}, \quad w \to \infty$$

(8)

where the $B_n$s are the Bernoulli numbers.

In [7], Mahmoud, Talat, and Moustafa studied the following approximation family:

$$\lambda(\xi, w) = \ln \left(\frac{w + \xi + 1}{w + \xi}\right) + \frac{2}{w(w + 1)}, \quad \xi \in [1, 2]; \quad w > 0$$

which is asymptotically equivalent to $G(w)$ for $w \to \infty$, and they proved the inequality:

$$\ln \left(\frac{w + a + 1}{w + a}\right) - \frac{w - 1}{w(w + 1)} < G(w) - w^{-1} < \ln \left(\frac{w + b + 1}{w + b}\right) - \frac{w - 1}{w(w + 1)};$$

(9)

where $a = \frac{4}{e^2 - 4}$ and $b = 1$ are the best possible. In [3] Hegazi, Mahmoud, Talat, and Moustafa deduced that:

$$G(w) - \frac{1}{w} > \ln \left(1 + \frac{1}{w + \sqrt{\frac{2}{3}}}\right) + \frac{\sqrt{6} - 3w}{3w(w + 1)} + \frac{1}{6\sqrt{6}w^4}, \quad w \geq 2$$

(10)

$$G(w) - \frac{1}{w} < \ln \left(1 + \frac{1}{w - \sqrt{\frac{2}{3}}}\right) - \frac{\sqrt{6} + 3w}{3w(w + 1)} - \frac{1}{6\sqrt{6}w^4}, \quad w \geq \sqrt{\frac{2}{3}}$$

(11)

$$G(w) - \frac{1}{w} < \frac{1}{2} \ln \left(1 + \frac{2w + 3}{w^2 + 2w + \frac{48}{16}}\right) - \frac{w - 1}{w(w + 1)}, \quad w \geq \frac{576}{1000}$$

(12)

$$G(w) - \frac{1}{w} > \frac{1}{2} \ln \left(1 + \frac{2w + c}{w^2 + 2w + \frac{4}{3}}\right) - \frac{w - 1}{w(w + 1)}, \quad w > 0$$

(13)

and

$$G(w) - \frac{1}{w} < \frac{1}{2} \ln \left(1 + \frac{2w + d}{w^2 + 2w + \frac{4}{3}}\right) - \frac{w - 1}{w(w + 1)}, \quad w > 0$$

(14)
where \( c = 3 \) and \( d = \frac{c^4 - 16}{12} \) are the best possible.

In [9], Mahmoud, Talat, Moustafa, and Agarwal proved

\[
\frac{1}{2w^2 + 1} < G(w) - \frac{1}{w} < \frac{1}{2w^2}, \quad w > 0 \tag{15}
\]

which presents improvements of the lower bounds of (7) for \( w > 0 \), Inequality (9) for \( w > 2.48 \), Inequality (10) for \( w > 1.96 \), and Inequality (13) for \( w > 4.13 \). Furthermore, it improves the upper bounds of (9) for \( w > 2.54 \), Inequality (11) for \( w > \sqrt{2} \), Inequality (12) for \( w > 5.72 \), and Inequality (14) for \( w > 2.53 \). They showed that

\[
\sum_{n=1}^{m} \left( 2^{2n} - 1 \right) B_{2n} \frac{1}{n!w^{2n}} < G(w) - \frac{1}{w} < \sum_{n=1}^{l} \left( 2^{2n} - 1 \right) B_{2n} \frac{1}{n!w^{2n}}, \quad m, l = 1, 2, ...
\tag{16}
\]

and deduced that

\[
\frac{1}{2w^2 + l} < G(w) - \frac{1}{w} < \frac{1}{2w^2 + k}, \quad w > 0
\]

with the best possible constants \( l = 1 \) and \( k = 0 \), which is a refinement of the lower bound of Inequality (7).

Recently, Mahmoud and Almuashi [8] studied the generalized Bateman’s G-function \( G_\mu(w) \) defined by

\[
G_\mu(w) = \psi\left( \frac{w + \mu}{2} \right) - \psi\left( \frac{w}{2} \right), \quad w \neq -2r, -2r - \mu; \quad 0 < \mu < 2; \quad r = 0, 1, 2, ...
\]

and presented some of its properties. Furthermore, they presented the following inequality:

\[
\ln\left( 1 + \frac{\mu}{w + \rho} \right) < G_\mu(w) - \frac{2\mu}{(w + \mu)w} < \ln\left( 1 + \frac{\mu}{w + \iota} \right), \quad w > 0; \quad \mu \in (0, 2)
\]

where \( \rho = \frac{\mu}{\sqrt{1 + 4\mu^2}} + \frac{\mu}{2} \) and \( \iota = 1 \) are the best possible.

The outline of the paper is as follows. Section 1 provides the definition, some relations, asymptotic expansions, and some inequalities of the Bateman G-function. The Padé approximant is defined in Section 2, and some rational approximations of \( G(w) \) are calculated. In Section 3, some new bounds of \( G(w) \) are presented based on Padé approximants, and we show that our new inequalities improve some recently published ones. We prove the complete monotonicity property of a function involving \( G(w) \) in Section 4.

2. Some Padé Approximants of Bateman’s G-Function

In this section, we present some Padé approximants of the function \( G(w) \), which present the best rational approximations with the given order of a function.

Consider the formal power series:

\[
h(w) = c_0 + c_1 w + c_2 w^2 + ..., \]

then the rational function:

\[
[r, s]_h(w) = \frac{\sum_{i=0}^{r} a_i w^i}{1 + \sum_{i=1}^{s} b_i w^i}, \quad r \geq 0; \quad s \geq 1 \tag{17}
\]
is called the Padé approximant of order \((r, s)\) of the function \(h(w)\) ([16], Chapter 1; [17], p. 96) and [18] where

\[
[r,s]_h(w) - h(w) = O(w^{r+s+1}), \quad w \rightarrow \infty
\]  

(18)

and the coefficients \(b_i's\) are the solution of the system:

\[
\begin{align*}
0 &= c_{r+1} + c_{r}b_1 + \cdots + c_{r-s+1}b_s \\
0 &= c_{r+2} + c_{r+1}b_1 + \cdots + c_{r-s+2}b_s \\
&\quad \vdots \\
0 &= c_{r+s} + c_{r+s-1}b_1 + \cdots + c_rb_s,
\end{align*}
\]

(19)

with \(c_i = 0\) for \(i < 0\), and the coefficients \(a_i's\) are given by

\[
\begin{align*}
a_0 &= c_0 \\
a_1 &= c_1 + c_0 b_1 \\
&\quad \vdots \\
a_r &= c_r + c_{r-1} b_1 + \cdots + c_{r-s} b_s.
\end{align*}
\]

(20)

Theorem 1. The Padé approximant of order \((2, 2n)\) of the function \(f(w) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)B_{2k}}{kw^k}\) is given by

\[
[2, 2n]_f(w) = \frac{a_0 + a_1 w^{-2}}{1 + \sum_{i=1}^{n} b_i w^{-2i}},
\]

(21)

where \(a_0 = 0\), \(a_1 = \frac{1}{2}\), \(b_1 = \frac{1}{2}\), and

\[
b_j = -2 \left( \frac{(2^{2j+2}-1)B_{2j+2}}{j+1} + \sum_{v=1}^{j-1} \frac{(2^{2j-2v+2}-1)B_{2j-2v+2}}{j-v+1} b_v \right), \quad j > 1.
\]

Proof. For the function:

\[
f(\sqrt{w}) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)B_{2k}}{kw^k}
\]

the Padé approximant of order \((1, n)\) is given by

\[
[1, n]_f(\sqrt{w}) = \frac{\sum_{v=0}^{n} a_v w^{-v}}{1 + \sum_{v=1}^{n} b_v w^{-v}}
\]

where the coefficients \(b_i's\) are the solution of the system

\[
\begin{align*}
0 &= c_1 + \frac{1}{2} b_1 \\
0 &= \frac{1}{2} + \frac{1}{4} b_1 + \frac{1}{2} b_2 \\
&\quad \vdots \\
0 &= \frac{(2^{2n+2}-1)B_{2n+2}}{2n+2} + \frac{(2^{2n} - 1)B_{2n}}{n} b_1 + \cdots + \frac{15 B_4}{2} b_{n-1} + \frac{1}{2} b_n,
\end{align*}
\]

and

\[
\begin{align*}
a_0 &= 0 \\
a_1 &= \frac{1}{2}.
\end{align*}
\]
Then,

$$[2, 2n]_f(w) = \frac{w^{-2}}{2(1 + \sum_{i=1}^{n} b_i w^{-2i})} \tag{22}$$

with $b_1 = \frac{1}{2}$, and $b_j = 2 \left( \frac{(1-2^{j+1})b_{2j+2}}{j+1} + \sum_{i=1}^{j-1} \frac{(1-2^{j-2i+1})b_{2j-2i+2}}{j-i+1} b_i \right)$ for $j = 2, 3, ..., n$.

To obtain the Padé approximant of order (2, 16) of the function $f(w)$, we consider the system:

\[
\begin{align*}
    b_2 &= -2 \left( \frac{1}{2} - \frac{1}{4} b_1 \right) \\
    b_3 &= -2 \left( -\frac{17}{8} + \frac{1}{2} b_1 + \frac{1}{4} b_2 \right) \\
    b_4 &= -2 \left( \frac{31}{2} + \frac{17}{8} b_1 + \frac{1}{2} b_2 - \frac{1}{4} b_3 \right) \\
    b_5 &= -2 \left( \frac{5461}{2} - \frac{691}{4} b_1 - \frac{31}{2} b_2 + \frac{17}{8} b_3 - \frac{1}{2} b_4 - \frac{1}{4} b_5 \right) \\
    b_6 &= -2 \left( \frac{929,569}{16} + \frac{31}{2} b_1 - \frac{17}{8} b_2 + \frac{1}{2} b_3 - \frac{1}{4} b_4 \right) \\
    b_7 &= -2 \left( \frac{3,202,291}{2} - \frac{929,569}{16} b_1 + \frac{31}{2} b_2 - \frac{17}{8} b_3 + \frac{1}{2} b_4 - \frac{1}{4} b_5 \right),
\end{align*}
\]

and hence, we obtain $b_2 = -\frac{3}{4}$, $b_3 = \frac{27}{8}$, $b_4 = -\frac{423}{16}$, $b_5 = \frac{9927}{32}$, $b_6 = -\frac{324,423}{64}$, $b_7 = \frac{14,098,527}{128}$, and $b_8 = -\frac{787,622,823}{256}$. Then,

\[
\begin{align*}
    [2, 2]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2}} \\
    [2, 4]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4}} \\
    [2, 6]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6}} \\
    [2, 8]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6} - \frac{423}{16} w^{-8}} \\
    [2, 10]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6} - \frac{423}{16} w^{-8} + \frac{9927}{32} w^{-10}} \\
    [2, 12]_f(w) &= \frac{w^{-2} / 2}{1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6} - \frac{423}{16} w^{-8} + \frac{9927}{32} w^{-10} - \frac{324,423}{64} w^{-12}} \\
    [2, 14]_f(w) &= \frac{w^{-2}}{l_1(w)} \\
    [2, 16]_f(w) &= \frac{w^{-2}}{l_2(w)},
\end{align*}
\]

where

\[
\begin{align*}
l_1(w) &= 2(1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6} - \frac{423}{16} w^{-8} + \frac{9927}{32} w^{-10} - \frac{324,423}{64} w^{-12} + \frac{14,098,527}{128} w^{-14}) \\
l_2(w) &= 2(1 + \frac{1}{4} w^{-2} - \frac{3}{4} w^{-4} + \frac{27}{8} w^{-6} - \frac{423}{16} w^{-8} + \frac{9927}{32} w^{-10} - \frac{324,423}{64} w^{-12} + \frac{14,098,527}{128} w^{-14} - \frac{787,622,823}{256} w^{-16}).
\end{align*}
\]
3. Some Rational Bounds of Bateman’s G-Function

In this section, we use Padé approximants to formulate new bounds of the function $G(w)$.

Recall that, if the real-valued function $\xi(t)$ defined for $t > 0$, $\lim_{t \to \infty} \xi(t) = 0$ and $\xi(t + r) < \xi(t)$, $r \in \mathbb{N}$, then $\xi(t) > 0$ for $t > 0$ (see [19]).

**Theorem 2.** The following inequality holds:

$$|2, 6|_f(w) < G(w) - w^{-1} < |2, 4|_f(w),$$

where the lower bound and the upper bound hold for $w > 0$ and $w > \frac{\sqrt{13} - 1}{2} \simeq 0.80709$, respectively.

**Proof.** Let $g_4(w) = G(w) - \frac{1}{w} - |2, 4|_f(w)$. Then, by using Relation (3), we have

$$g_4(w + 2) - g_4(w) = \frac{18(24w^2 + 48w + 23)}{w(w + 1)(w + 2)(4w^4 + 2w^2 - 3)(4w^4 + 32w^3 + 98w^2 + 136w + 69)}.$$  \(\ldots(23)\)

Since the equation $4w^4 + 2w^2 - 3 = 0$ has only one positive root at $w_1 = \frac{\sqrt{13} - 1}{2}$, then $g_4(w + 2) > g_4(w), \forall w > w_1$ with $\lim_{w \to \infty} g_4(w) = 0$. Then, we obtain $g_4(w) < 0, \forall w > w_1$, which implies $G(w) - \frac{1}{w} < |2, 4|_f(w), \forall w > w_1$. Let $g_6(w) = G(w) - \frac{1}{w} - |2, 6|_f(w)$, then by using Relation (3), we have

$$g_6(w + 2) - g_6(w) = -\frac{18(940w^4 + 3760w^3 + 6316w^2 + 5112w + 1737)}{l_3(w)},$$

where $l_3(w) = w(w + 1)(w + 2)(8w^6 + 4w^4 - 6w^2 + 27)(8w^6 + 96w^5 + 484w^4 + 1312w^3 + 2010w^2 + 1640w + 579)$. Since the equation $8w^3 + 4w^2 - 6w + 27 = 0$ has no positive real root, then $g_6(w + 2) < g_6(w), \forall w > 0$ with $\lim_{w \to \infty} g_6(w) = 0$. Then, we obtain $g_6(w) > 0, \forall w > 0$, which implies $|2, 6|_f(w) < G(w) - \frac{1}{w}, \forall w > 0$. \(\square\)

**Theorem 3.** The following inequality holds:

$$|2, 14|_f(w) < G(w) - \frac{1}{w} < |2, 16|_f(w),$$

where the lower bound and the upper bound hold for $w > 0$ and $w > \frac{13}{2}, \text{respectively.}$

**Proof.** Let $g_{14}(w) = G(w) - \frac{1}{w} - |2, 14|_f(w)$, then by using Relation (3):

$$g_{14}(w + 2) - g_{14}(w) = -\frac{18r(w)}{w(w + 1)(w + 2)s(w)} < 0,$$

where

$$r(w) = 576, 752w^{13} + 17, 879, 312w^{12} + 247, 006, 376w^{11} + 1, 951, 966, 872w^{10} + 9, 023, 932, 048w^9 + 29, 349, 843, 952w^8 + 69, 495, 292, 288w^7 + 123, 190, 424, 736w^6 + 164, 419, 139, 520w^5 + 163, 742, 349, 820w^4 + 118, 336, 385, 968w^3 + 57, 293, 959, 996w^2 + 15, 984, 084, 792w + 157, 65, 707, 337 > 0, \quad w > 0$$

and
\[ s(w) = 4096w^{24} + 114,688w^{23} + 1,495,040w^{22} + 12,034,048w^{21} + 66,882,560w^{20} \\
+ 271,851,520w^{19} + 834,863,104w^{18} + 1,972,912,128w^{17} + 3,617,878,272w^{16} \\
+ 5,148,431,360w^{15} + 5,640,385,280w^{14} + 4,677,807,616w^{13} + 2,830,319,744w^{12} \\
+ 1,716,108,544w^{11} + 8,253,448,064w^{10} + 46,222,128,512w^9 + 176,800,158,576w^8 \\
+ 648,131,097,984w^7 + 899,802,659,232w^6 + 526,187,276,736w^5 \\
+ 1,331,567,184,348w^4 + 997,589,356,848w^3 + 496,128,363,804w^2 \\
+ 141,254,459,640w + 141,891,366,033 > 0, \quad w > 0. \\
\]

Furthermore, \( \lim_{w \to \infty} g_{14}(w) = 0 \) and hence, \( g_{14}(w) > 0, \forall w > 0 \), which gives \( [2, 14] f(w) < G(w) - \frac{1}{w}, \forall w > 0 \). Let \( g_{16}(w) = G(w) - \frac{1}{w} - [2, 16] f(w) \), then by using Relation (3), we have

\[ g_{16}(w + 2) - g_{16}(w) = \frac{18r_1(w)}{w(w + 1)(w + 2)s_1(w)s_2(w)}, \]

where

\[ r_1(w + 1) = 7,832,993,923,840w^{14} + 219,323,829,867,520w^{13} \\
+ 2,951,459,430,912,880w^{12} + 25,215,669,729,930,752w^{11} \\
+ 152,317,731,513,578,304w^{10} + 686,224,197,180,084,480w^9 \\
+ 2,371,783,010,371,491,504w^8 + 6,374,300,275,957,107,456w^7 \\
+ 13,359,225,218,824,632,416w^6 + 21,688,648,087,649,664,128w^5 \\
+ 26,807,655,086,077,277,460w^4 + 24,427,148,696,743,456,928w^3 \\
+ 15,494,208,668,209,128,860w^2 + 6,126,480,357,977,215,344w \\
+ 1,081,166,851,750,059,759 > 0, \quad w > 0, \]

\[ s_1(w + 1) = 256w^{16} + 12,288w^{15} + 276,608w^{14} + 3,876,096w^{13} + 37,844,160w^{12} \\
+ 272,975,616w^{11} + 1,504,750,176w^{10} + 6,466,091,328w^9 \\
+ 21,889,518,768w^8 + 58,570,846,848w^7 + 123,458,967,864w^6 \\
+ 202,842,619,632w^5 + 254,654,382,220w^4 + 236,146,329,936w^3 \\
+ 152,546,318,910w^2 + 61,418,471,508w + 10,955,689,299 > 0, \quad w > 0. \]
and
\[
\begin{align*}
s_2 \left( w + \frac{13}{5} \right) & = \\
& = 256w^{16} + \frac{94,208}{5}w^{15} + \frac{3,250,816}{5}w^{14} + \frac{349,058,304}{25}w^{13} \\
& + \frac{26,107,488,704}{125}w^{12} + \frac{7,211,309,234,944}{3125}w^{11} + \frac{30,437,288,868,672}{15,625}w^{10} \\
& + \frac{2,002,484,736,219,456}{390,625}w^9 + \frac{51,883,806,254,460,144}{7,812,500}w^8 \\
& + \frac{1,062,345,679,901,482,624}{390,625}w^7 + \frac{85,663,214,450,859,557,752}{9,765,625}w^6 \\
& + \frac{1,076,675,005,717,334,916,016}{48,828,125}w^5 + \frac{2,067,776,254,875,249,912,424}{48,828,125}w^4 \\
& + \frac{14,665,131,542,761,625,296,849}{3,440,625}w^3 + \frac{72,444,882,017,256,139,899,286}{1,220,703,125}w^2 \\
& + \frac{1,113,540,298,780,701,469,947,172}{30,517,578,125}w + \frac{1,604,764,820,494,673,221,424,691}{305,175,781,250}w^2 \\
& > 0 \quad w > 0.
\end{align*}
\]

Then, \( g_{16}(w + 2) > g_{16}(w) \), \( \forall w > \frac{13}{5} \) with \( \lim_{w \to \infty} g_{16}(w) = 0 \). Then, we have \( g_{16}(w) < 0 \), \( \forall w > \frac{13}{5} \), which means \( [2,16]_f(w) > G(w) - \frac{1}{w}, \forall w > \frac{13}{5} \). \( \Box \)

**Remark 1.** The lower bound of (23) improves the lower bound of (15) for \( w > \sqrt{\frac{\sqrt{13} - 1}{2}} \).

**Remark 2.** The upper bound of (23) improves the upper bound of (15) for \( w \in \mathbb{R}^+ - \left( \sqrt{\frac{\sqrt{13} - 1}{2}}, \frac{3}{5} \right) \).

**Remark 3.** Let
\[
b_{16}(w) = \frac{1}{2w^2} - \frac{1}{4w^4} + \frac{1}{2w^6} - \frac{17}{8w^8} + \frac{31}{2w^{10}} - \frac{691}{4w^{12}} + \frac{5461}{2w^{14}} - \frac{929,569}{16w^{16}},
\]
and
\[
b_{18}(w) = \frac{1}{2w^2} - \frac{1}{4w^4} + \frac{1}{2w^6} - \frac{17}{8w^8} + \frac{31}{2w^{10}} - \frac{691}{4w^{12}} + \frac{5461}{2w^{14}} - \frac{929,569}{16w^{16}} + \frac{3,202,291}{2w^{18}}.
\]
Consider the difference:
\[
[2,14]_f(w) - b_{16}(w) = \frac{r_3(w)}{16w^{16}s_3(w)},
\]
where
\[
r_3(w) = 128,654,692w^{12} - 262,161,780w^{10} + 1,299,430,638w^8 - 10,170,269,640w^6 + 104,226,438,528w^4
\]
\[- 1,219,083,574,950w^2 + 13,105,533,644,863,
\]
and
\[
s_3(w) = 128w^{14} + 64w^{12} - 96w^{10} + 432w^8 - 3384w^6 + 39,708w^4 - 648,846w^2 + 14,098,527.
\]
Since the equations:
\[
128,654,692w^2 - 262,161,780w + 299,430,638 = 0,
\]
1,000,000,000w^4 - 10,170,269,640w^3 + 104,226,438,528w^2 - 1,219,083,574,950w + 13,105,535,644,863 = 0,
200w^4 - 3384w^3 + 39,708w^2 - 648,846w + 14,098,527 = 0,

and

128w^3 + 64w^2 - 96w + 232 = 0

have no positive real roots, then \( r_3(w) \) and \( s_3(w) \) are positive \( \forall \ w > 0 \), which shows that the lower bound of (24) improves the lower bound of (15) for \( \ w > 0 \). Furthermore, consider the difference:

\[
b_{18}(w) - [2,16]/(w) = r_4(w)/16w^{18}s_4(w),
\]

where

\[
r_4\left(w + \frac{26}{9}\right) =
\frac{6,953,960,836w^{14}}{44,139,882,177,895,712w^{11}} + \frac{2,531,241,744,304w^{13}}{59049} + \frac{426,660,551,197,660w^{12}}{207,444,862,008,627,300,848w^6}
+ \frac{1,313,486,526,410,399,774w^{10}}{177,147} + \frac{161,200,595,978,180,233,592w^9}{1,594,323}
+ \frac{60,682,986,467,807,767,571,456w^7}{387,420,489} + \frac{1,359,824,925,032,346,851,475,104w^6}{1,348,907}
+ \frac{99190,199,144,227,314,857,593,450w^5}{3,486,784,401} + \frac{52,622,035,382,970,646,596,271,591w^2}{282,429,536,481}
+ \frac{125,974,211,992,242,994,684,174,061,164w}{2,541,865,828,329} + \frac{96,838,271,852,334,590,005,044}{22,876,792,454,961},
\]

\[w > 0.\]

and

\[
s_4\left(w + \frac{26}{9}\right) =
\frac{106,496w^{15}}{9} + \frac{6,925,696w^{14}}{27} + \frac{2,523,469,312w^{13}}{729} + \frac{213,550,796,096w^{12}}{531,441}
+ \frac{6,450,595,881,472w^{11}}{19,683} + \frac{637,979,089,706,848w^{10}}{531,441} + \frac{23,764,147,568,110,976w^9}{478,969}
+ \frac{4,450,595,881,472w^{11}}{387,420,489} + \frac{637,979,089,706,848w^{10}}{531,441} + \frac{23,764,147,568,110,976w^9}{478,969}
+ \frac{7,748,950,743,106,704w^8}{3,486,784,401} + \frac{14,03,341,767,918,187,371,872w^5}{10,460,353,203}
+ \frac{295,580,430,758,925,054,136w^6}{282,429,536,481} + \frac{1,403,341,767,918,187,371,872w^5}{10,460,353,203}
+ \frac{45,820,036,824,602,356,573,316w^4}{7,625,597,484,987} + \frac{368,352,992,789,069,056,571,296w^3}{2,541,865,828,329}
+ \frac{687,632,404,365,282,872,950,378w^2}{205,891,132,094,649} + \frac{7,208,442,438,570,664,907,699,096w}{9,099,132,094,649},
\]

\[w > 0.\]

Then, \( b_{18}(w) - [2,16]/(w) > 0 \) for \( \ w > \frac{26}{9} \), which shows that the upper bound of (24) improves the upper bound of (15) for \( \ w > \frac{26}{9} \).
4. A Completely Monotonic Function Involving $G(w)$

In this section, we present another advantage of Padé approximants in formulating new completely monotonic functions involving the function $G(w)$.

**Theorem 4.** The function:

$$M(w) = \frac{(w + 3)(4w^5 + 34w^4 + 122w^3 + 228w^2 + 225w + 92)}{w(w + 1)(w + 2)(4w^4 + 32w^3 + 98w^2 + 136w + 69)} - G(w),$$

is completely monotonic for $w > 0$, that is

$$(-1)^r M^{(r)}(w) \geq 0 \quad w > 0; \ r \in \mathbb{N}.$$

**Proof.** The function $M$ satisfies

$$M(w) = -\int_0^\infty m(t) \frac{\sqrt{13} \cosh(\frac{t}{2}) e^{-\frac{3t^2}{2}}}{\sqrt{13}(e^{t} + 1)} e^{-wt} dt,$$

where

$$m(t) = 2 \tanh\left(\frac{t}{2}\right) - \left(\frac{\sqrt{13} \sin\left(\frac{1}{2} \sqrt{13} t\right)}{\sqrt{13}} + \frac{\sqrt{13} - 1 \sinh\left(\frac{1}{2} \sqrt{13} - 1 t\right)}{\sqrt{13}} \right).$$

Using the relation:

$$2 \tanh\left(\frac{t}{2}\right) < \left(\frac{31^9}{362/880} - \frac{17}{20/160} + \frac{t^5}{120} - \frac{t^3}{12} + t\right) = -\sum_{n=11}^{\infty} a_n \frac{\mu^n}{362/880 n!}$$

where

$$a_n = (n - 2)(31n^4 - 186n^3 + 500n^2 - 150n + 189)(n - 10)(n - 8)(n - 6)(n - 4)$$

we obtain

$$2 \tanh\left(\frac{t}{2}\right) < \left(\frac{31^9}{362/880} - \frac{17}{20/160} + \frac{t^5}{120} - \frac{t^3}{12} + t\right), \quad t > 0. \quad (25)$$

Taylor’s formula with the remainder of the function $\sin t$ gives the following double inequality ([20], p. 284) for $n \in \mathbb{N}$:

$$\sin t = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1} t^{2\nu-1}}{(2\nu - 1)!} + E_{2\nu}(t), \quad E_{2\nu}(t) \leq \frac{|t|^{2\nu+1}}{(2\nu + 1)!}, \quad -\infty < t < \infty \quad (26)$$

where $|\sin^{(n+1)} t| \leq 1$. Furthermore, using the expansion:

$$\sinh t = \sum_{\nu=0}^{\infty} \frac{t^{2\nu+1}}{(2\nu + 1)!}, \quad -\infty < t < \infty$$

we obtain $s \in \mathbb{N}$

$$\sinh\left(\frac{1}{2} \sqrt{13} - 1 t\right) > \sum_{\nu=0}^{s} \frac{1}{(2\nu + 1)!} \left(\frac{1}{2} \sqrt{13} - 1 t\right)^{2\nu+1}, \quad t > 0 \quad (27)$$
From the inequalities (26) with \( r = 3 \) and (27) with \( s = 3 \), we obtain

\[
\sqrt{1 + \sqrt{13}} \sin \left( \frac{1}{2} \sqrt{1 + \sqrt{13}} t \right) + \frac{\sqrt{\sqrt{13} - 1} \sin \left( \frac{1}{2} \sqrt{\sqrt{13} - 1} t \right)}{\sqrt{13}} > - \left( \frac{t^7}{5760} + \frac{t^5}{120} - \frac{t^3}{12} + t \right), \quad t > 0.
\]

(28)

Hence, from the inequalities (25) and (28), we obtain

\[
m(t) < 0, \quad 0 < t < 9 \sqrt{\frac{3}{31}}.
\]

(29)

Now, the function \( 2 \tanh \left( \frac{t}{2} \right) \) is increasing and concave on \( t \in (0, \infty) \) with a limit equal to 2 as \( t \to \infty \). However, the line \( \left( \frac{1}{2} - \frac{1}{2\sqrt{13}} \right) t > 1 \) for \( t \geq 9 \sqrt{\frac{3}{31}} \), then we obtain

\[
2 \tanh \left( \frac{t}{2} \right) < \left( \frac{1}{2} - \frac{1}{2\sqrt{13}} \right) t + 1, \quad t \geq 9 \sqrt{\frac{3}{31}}.
\]

(30)

Using Inequality (28), we obtain

\[
\sqrt{1 + \sqrt{13}} \sin \left( \frac{1}{2} \sqrt{1 + \sqrt{13}} t \right) + \frac{\sqrt{\sqrt{13} - 1} \sin \left( \frac{1}{2} \sqrt{\sqrt{13} - 1} t \right)}{\sqrt{13}} > f_1(t), \quad t > 0
\]

where

\[
f_1(t) = - \frac{t^7}{5760} + \frac{t^5}{120} - \frac{t^3}{12} + \left( \frac{1}{2} + \frac{1}{2\sqrt{13}} \right) t - 1, \quad t > 0.
\]

The function \( f_1(t) \) is convex on \((t_1, t_2)\), where \( t_i = 2\sqrt{\frac{1}{2} \left( 20 + (-1)^i \sqrt{190} \right)}; \ i = 1, 2 \) with the two conditions \( f_1 \left( 9 \sqrt{\frac{3}{31}} \right) > 0 \) and \( f_1' \left( 9 \sqrt{\frac{3}{31}} \right) > 0 \). Then, \( f_1(t) \) is a positive increasing function on \( \left[ 9 \sqrt{\frac{3}{31}}, t_2 \right) \). Furthermore, \( f_1(t) \) is a concave function on \( (t_2, \infty) \) with \( f_1(6) > 0 \), and hence, \( f_1(t) \) is positive on \( [t_2, 6] \). Then, \( f_1(t) > 0 \) on \( 9 \sqrt{\frac{3}{31}}, 6 \). From the inequalities (26) with \( n = 1 \) and (27) with \( s = 3 \), we obtain

\[
\sqrt{1 + \sqrt{13}} \sin \left( \frac{1}{2} \sqrt{1 + \sqrt{13}} t \right) + \frac{\sqrt{\sqrt{13} - 1} \sin \left( \frac{1}{2} \sqrt{\sqrt{13} - 1} t \right)}{\sqrt{13}} > f_2(t), \quad t > 0
\]

where

\[
f_2(t) = \frac{(31\sqrt{13} - 91) t^7}{1,048,320} + \frac{(26 - 5\sqrt{13}) t^5}{6240} - \frac{t^3}{12} + \frac{1}{26} \left( 13 + \sqrt{13} \right) t - 1, \quad t > 0.
\]
$f_2(t)$ is a convex function on $\left(\frac{1}{3}\sqrt{2\left(\sqrt{210} + 11\sqrt{13}\right) - 15\left(1 + \sqrt{13}\right)}}, \infty\right)$ with $f_2(6) > 0$ and $f_2'(6) > 0$. Hence, $f_2(t) > 0$ for $t \geq 6$. Then, we have

$$\frac{\sqrt{1 + \sqrt{13}} \sin\left(\frac{1}{2}\sqrt{1 + \sqrt{13}} t\right)}{\sqrt{13}} + \frac{\sqrt{\sqrt{13} - 1} \sinh\left(\frac{1}{2}\sqrt{\sqrt{13} - 1} t\right)}{\sqrt{13}} > \left(\frac{1}{2} - \frac{1}{2\sqrt{13}}\right)t + 1, \quad t \geq 9\sqrt{\frac{3}{31}}.$$  \hfill (31)

Hence, from Inequalities (30) and (31), we have

$$m(t) < 0, \quad t \geq 9\sqrt{\frac{3}{31}}.$$  \hfill (32)

The two inequalities (29) and (32) complete the proof. \hfill \Box

5. Conclusions

The Padé approximant method presents some rational approximations for Bateman’s $G$-function $G(w)$. These approximations provided us with new inequalities of the function $G(w)$ with completely monotonic functions involving it. We presented proofs to clarify the novelty of our results, which could be of interest to a large part of the readers. This method is considered a powerful tool in deducing estimates and inequalities for several other special functions.

**Author Contributions:** Writing to Original draft, O.A., M.M. and A.T. All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Erdelyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; California Institute of Technology-Bateman Manuscript Project, 1953–1955; McGraw-Hill Inc.: New York, NY, USA, 1981; Volume I–III.
2. Andrews, G.E.; Askey, R.A.; Roy, R. *Special Functions*; Encyclopedia of Mathematics and Its Applications 71; Cambridge University Press: Cambridge, UK, 1999.
3. Hegazi, A.; Mahmoud, M.; Talat, A.; Moustafa, H. Some best approximation formulas and inequalities for the Bateman’s $G$-function. *J. Comput. Anal. Appl.* 2019, 27, 118–135.
4. Kiryakova, V. A guide to special functions in fractional calculus. *Mathematics* 2021, 9, 106. [CrossRef]
5. Mahmoud, M.; Agarwal, R.P. Bounds for Bateman’s $G$-function and its applications. *Georgian Math. J.* 2016, 23, 579–586. [CrossRef]
6. Mahmoud, M.; Almuashi, H. On some inequalities of the Bateman’s $G$-function. *J. Comput. Anal. Appl.* 2017, 22, 672–683.
7. Mahmoud, M.; Talat, A.; Moustafa, H. Some approximations of the Bateman’s $G$–function. *J. Comput. Anal. Appl.* 2017, 23, 1165–1178.
8. Mahmoud, M.; Almuashi, H. Generalized Bateman’s $G$-function and its bounds. *J. Comput. Anal. Appl.* 2018, 24, 23–40.
9. Mahmoud, M.; Talat, A.; Moustafa, H.; Agarwal, R.P. Completely monotonic functions involving Bateman’s $G$-function. *J. Comput. Anal. Appl.* 2021, 29, 970–986.
10. Mortici, C. A sharp inequality involving the psi function. *Acta Univ. Apulensis* 2010, 22, 41–45.
11. Qiu, S.-L.; Vuorinen, M. Some properties of the gamma and psi functions with applications. *Math. Comp.* 2004, 74, 723–742. [CrossRef]
12. Alzer, H.; Berg, C. Some classes of completely monotonic functions. *Ann. Acad. Sci. Fenn.* 2002, 27, 445–460. [CrossRef]
13. Haeringen, H.V. Completely monotonic and related functions. *J. Math. Anal. Appl.* 1996, 204, 389–408. [CrossRef]
14. Wang, X.F.; Ismail, M.E.H.; Batir, N.; Guo, S. A necessary and sufficient condition for sequences to be minimal completely monotonic. *Adv. Differ. Equ.* 2020, 2020, 665. [CrossRef]
15. Widder, D.V. *The Laplace Transform*; Princeton University Press: Princeton, NJ, USA, 1946.
16. Baker, G.A., Jr.; Graves–Morris, P.R. *Padé Approximants*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1996.
17. Brezinski, C. *Computational Aspects of Linear Control*; Kluwer: Dordrecht, The Netherland, 2002.
18. Brezinski, C.; Redivo-Zaglia, M. New representations of Padé, Padé-type, and partial Padé approximants. *J. Comput. Appl. Math.* 2015, 284, 69–77. [CrossRef]
19. Qi, F.; Guo, S.-L.; Guo, B.-N. Completely monotonicity of some functions involving polygamma functions. *J. Comput. Appl. Math.* 2010, 233, 2149–2160. [CrossRef]
20. Apostol, T.M. *Calculus, Volume I, One-Variable Calculus, with an Introduction to Linear Algebra*, 2nd ed.; John Wiley & Sons: Hoboken, NJ, USA, 1967.