§0. Summary of results

0.1. Introduction. In this paper we address the following three problems.

A. Calculate the Betti numbers and Euler characteristics of moduli spaces \( \overline{M}_{0,n} \) of stable \( n \)-pointed curves of genus zero (see e.g. [Ke]), or rather an appropriate generating function for these numbers.

B. The same for the space \( X[n] \), a natural compactification of the space of \( n \) pairwise distinct labelled points on a non–singular compact algebraic variety \( X \) constructed for \( \dim X = 1 \) in [BG] and in general in [FMPh]. (Beilinson and Ginzburg called this space “Resolution of Diagonals”, Fulton and MacPherson use the term “Configuration Spaces”).

C. Calculate the contribution of multiple coverings in the problem of counting rational curves on Calabi–Yau threefolds (see [AM], [Ko], and more detailed explanations below).

All these problems are united by the fact that available algebro–geometric information allows us to represent the corresponding numbers as a sum over trees with markings. M. Kontsevich in [Ko] invoked a general formula of perturbation theory in order to reduce the calculation of the relevant generating functions to the problem of finding the critical value of an appropriate formal potential. We solve problems A and B by applying this formalism in a simpler geometric context than that of [Ko]. Problem C is taken from [Ko]; we were able to directly complete Kontsevich’s calculation in this case and obtain a simple closed answer.

We will now describe our results (0.3—0.5) and technique (0.6) in some detail.

0.2. General setup. Let \( Y \) be an algebraic variety over \( \mathbb{C} \), possibly non–smooth and non–compact. Following [FMPh] we denote by \( P_Y(q) \) the virtual Poincaré polynomial of \( Y \) which is uniquely defined by the following properties.

a). If \( Y \) is smooth and compact, then

\[
P_Y(q) = \sum_j \dim H^j(Y)q^j. \tag{0.1}
\]

In particular

\[
\chi(Y) = P_Y(-1). \tag{0.2}
\]

b). If \( X = \bigsqcup_i X_i \) is a finite union of pairwise disjoint locally closed strata, then

\[
P_Y(q) = \sum_i P_{Y_i}(q). \tag{0.3}
\]

c). \( P_{Y \times Z}(q) = P_Y(q)P_Z(q) \). It follows that if \( Y \) is a fibration over base \( B \) with fiber \( F \) locally trivial in Zariski topology, then \( P_Y(c) = P_B(c)P_F(c) \).
A definition of \( P_Y(q) \) can be given using the weight filtration on the cohomology with compact support:
\[
P_Y(q) = \sum_{i,j} (-1)^{i+j} \dim (\text{gr}_{W}^i H^j_c(Y, \mathbb{Q})) q^j.
\] (0.4)

We apply the additivity formula (0.3) to the strata of the natural stratifications of \( \overline{M}_{0,n} \) and \( X[n] \) in Problems A, B. These strata can be indexed by marked trees describing various coalescing patterns of \( n \)-point configurations.

In [Ko], the role of \( Y \) is played by a compactification \( M(W) \) of the space of parametrised rational curves in some manifold \( W \). The relevant trees describe Gromov type degenerations of these curves. Kontsevich calculates certain Chern numbers of vector bundles over \( M(W) \) and uses Bott’s fixed point formula instead of (0.3) in order to represent them as a sum of local contributions. To make Bott’s formula applicable, Kontsevich assumes that \( W \) is endowed with a torus action and lifts this action to \( M(W) \). (Actually, his \( M(W) \) is not a manifold but a smooth stack).

0.3. Moduli spaces. We put
\[
\varphi(q,t) := t + \sum_{n=2}^{\infty} \frac{P_{M_{0,n+1}}(q) t^n}{n!} \in \mathbb{Q}[q][[t]],
\] (0.5)
\[
\chi(t) := \varphi(-1,t) = t + \sum_{n=2}^{\infty} \chi(\overline{M}_{0,n+1}) t^n \frac{t^n}{n!} \in \mathbb{Q}[[t]].
\] (0.6)

0.3.1. Theorem. a) \( \varphi(q,t) \) is the unique root in \( t + t^2 \mathbb{Q}[q][[t]] \) of any one of the following functional/differential equations in \( t \) with parameter \( q \) :
\[
(1 + \varphi) q^2 = q^4 \varphi - q^2 (q^2 - 1) t + 1,
\] (0.7)
\[
(1 + q^2 t - q^2 \varphi) \varphi_t = 1 + \varphi.
\] (0.8)

b) \( \chi \) is the unique root in \( t + t^2 \mathbb{Q}[[t]] \) of any one of the similar equations
\[
(1 + \chi) \log(1 + \chi) = 2\chi - t,
\] (0.9)
\[
(1 + t - \chi)\chi_t = 1 + \chi.
\] (0.10)

Equations (0.8) and (0.10) are equivalent to the following recursive formulas for the Poincaré polynomials. Put \( p_n = p_n(q) = P_{M_{0,n+1}} / n! \).

0.3.2. Corollary. We have for \( n \geq 1 \):
\[
(n+1)p_{n+1} = p_n + q^2 \sum_{i+j=n+1 \atop i \geq 2} j p_i p_j,
\] (0.11)
\[
P_{\overline{M}_{0,n+2}}(q) = P_{\overline{M}_{0,n+1}}(q) + q^2 \sum_{i+j=n+1} \binom{n}{i} P_{\overline{M}_{0,i+1}}(q) P_{\overline{M}_{0,j+1}}(q).
\] (0.12)
One can compare (0.11) with recursive formulas in [Ke], p. 550.

From (0.10) one sees that the function inverse to $\chi$ has a critical point at $t = e - 2$. Don Zagier has shown me how to derive from this the following asymptotical formula:

$$\chi(M_{0,n+1}) \cong \frac{1}{\sqrt{n}} \left( \frac{n}{e^2 - 2e} \right)^{n - \frac{1}{2}}.$$

We will prove Theorem 0.3.1 in §1. We will also discuss the ramification properties of $\varphi$ as a function of $t$ for $q^2 \neq 1$.

0.4. Configuration spaces. For a compact smooth algebraic manifold $X$ of dimension $m$, set

$$\psi_X(q,t) = 1 + \sum_{n \geq 1} P_X[n](q) \frac{t^n}{n!} \in \mathbb{Q}[q][[t]], \quad (0.13)$$

$$\chi_X(t) = \psi_X(-1,t) = 1 + \sum_{n \geq 1} \chi(X[n]) \frac{t^n}{n!} \in \mathbb{Q}[[t]]. \quad (0.14)$$

Put also

$$\kappa_m = \frac{q^{2m} - 1}{q^2 - 1} = P_{m-1}(q).$$

0.4.1. Theorem. Denote by $y^0 = y^0(q,t)$ the unique root in $t + t^2 \mathbb{Q}[q^2][[t]]$ of any one of the following equations:

$$\kappa_m (1 + y^0) q^{2m} = q^{2m} (q^{2m} + \kappa_m - 1) y^0 - q^{2m} (q^{2m} - 1) t + \kappa_m, \quad (0.15)$$

$$[q^{2m} t + 1 - (q^{2m} - 1 + \kappa_m) y^0] y^0_t = 1 + y^0. \quad (0.16)$$

Then we have in $\mathbb{Q}[[t]]$:

$$\psi_X(q,t) = (1 + y^0)^{P_X(q)}. \quad (0.17)$$

0.4.2. Theorem. Denote by $\eta = \eta(t)$ the unique root in $t + t^2 \mathbb{Q}[[t]]$ of any one of the following equations:

$$m (1 + \eta) \log(1 + \eta) = (m + 1) \eta - t, \quad (0.18)$$

$$(t + 1 - m \eta) \eta_t = 1 + \eta. \quad (0.19)$$

Then we have in $\mathbb{Q}[[t]]$:

$$\chi_X(t) = (1 + \eta)^{\chi(X)}. \quad (0.20)$$

Theorems 0.4.1 and 0.4.2 are proved in §2.

I am grateful to C. Soulé who remarked that (0.17) follows from a less neat identity which I deduced initially. He has also informed me that he and H. Gillet constructed a map $X \mapsto \left[ h^*(X) \right]$ from varieties to the $K_0$-ring of Grothendieck’s motives having all the formal properties of the virtual Poincaré polynomial. We can more or less mechanically use it in all our constructions; in particular, $q^2$ will be replaced by Tate’s motive $[h^2(\mathbb{P}^1)]$. 
For the reader’s convenience, we list the first terms of the generating series we have considered:

\[
\varphi(q,t) = t + \frac{t^2}{2!} + \frac{t^3}{3!}(q^2 + 1) + \frac{t^4}{4!}(q^4 + 5q^2 + 1) + \frac{t^5}{5!}(q^6 + 16q^4 + 16q^2 + 1) +
\]

\[
\frac{t^6}{6!}(q^8 + 42q^6 + 127q^4 + 42q^2 + 1) + \ldots,
\]

\[
P^{-1}\varphi_X(q,t) = t(\kappa_m + P - 1) +
\]

\[
\frac{t^2}{2!}[(P - 1)(P - 2) + \kappa_m(q^{2m} - 2) + 3(P - 1)\kappa_m + 3\kappa_m^2] +
\]

\[
\frac{t^3}{3!}[P^3 - 6P^2 + 11P - 6 + \kappa_m(6P^2 - 26P + 26 + 4Pq^{2m} - 9q^{2m} + q^{4m}) +
\]

\[
\kappa_m^2(15P + 10q^{2m} - 35) + 15\kappa_m^3] + \ldots
\]

where we put \( P = P_X(q) \).

0.5. Multiple coverings. Consider the following general problem of enumerative geometry.

**Problem** \( P_{g,k}(X, \beta, \mathcal{I}) \). Given a projective algebraic manifold \( X \), find the number of parametrised algebraic curves of genus \( g \) in \( X \), in the homology class \( \beta \), with \( k \) marked points, satisfying some incidence conditions \( \mathcal{I} \).

Notice that in this vaguely stated problem we implicitly assume that the number of solutions is only “virtually” finite, and look for the number of virtual solutions.

In [Ko], Maxim Kontsevich suggested a general scheme allowing him to simultaneously define this number for a wide class of problems and to calculate it in many cases using Bott’s residue formula. In the three examples he considered in full detail we have \( X = \mathbb{P}^n \) for some \( n, g = 0 \), and \( \beta = d[\mathbb{P}^1] \) for some \( d \geq 1 \). The remaining data is as follows.

(i) \( n = 2 \): \( X = \mathbb{P}^2 \), \( k = 3d - 1 \). The problem is to find the number of rational curves of degree \( d \) in \( \mathbb{P}^2 \) passing through \( 3d - 1 \) points in general positions.

(ii) \( n = 4 \): \( X = \mathbb{P}^4 \), \( k = 0 \). The problem is to find the number of rational curves of degree \( d \) lying in a quintic hypersurface \( V \).

(iii) \( n = 1 \): \( X = \mathbb{P}^1 \), \( k = 0 \). Here we additionally assume that \( X \) is a rational curve embedded in the quintic threefold (or a more general Calabi–Yau threefold) with normal sheaf \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \), and the problem is to calculate the contribution of maps of degree \( d \), \( \mathbb{P}^1 \rightarrow X \), to the number of solutions of problem (ii).

Using a different definition of the last contribution which we denote \( m_d \) P. Aspinwall and D. Morrison [AM] calculated it and confirmed an earlier prediction by P. Candelas et al.

In this note we show that Kontsevich’s formula gives the same answer:

0.5.1. Theorem. \( m_d = d^{-3} \).

0.6. Summation over trees. A tree \( \tau \) here is a finite connected simply connected CW–complex. We denote by \( V_\tau \) the set of its vertices, \( E_\tau \) the set of its
edges. Valency $|v|$ of a vertex $v \in V_\tau$ is the number of edges adjoining $v$. A flag of a vertex $v \in V_\tau$ is a pair $(v, e)$ where $v$ is a vertex, and $e$ is an adjoining edge.

A marking of a tree $\tau$ is a vaguely defined notion. It may consist of a family of marks of given type(s) put onto vertices, edges, flags, and satisfying certain restrictions. Below we will describe precisely a family of markings which we will call standard ones.

The generating functions $\varphi$ studied above and in [Ko] are calculated in three steps.

**STEP 1.** Represent $\varphi$ as an (infinite) sum of certain weights $w_\varphi(\tau, \mu)$ taken over isomorphism classes of marked trees $(\tau, \mu)$:

$$\varphi = \sum_{(\tau, \mu)/(\text{iso})} w_\varphi(\tau, \mu). \quad (0.21)$$

This stage involves a combinatorial encoding of the raw algebro–geometric data, determining type of marking and weights.

**STEP 2.** Try to rewrite (0.21) in a standard form of the following type. Choose a set $A$ (finite or countable) and a family of symmetric tensors indexed by $A$: $g^{ab}, a, b \in A; C_{a_1 \ldots a_k}, a_i \in A, k \geq 1$. The coordinates $g^{ab}, C_{a_1 \ldots a_k}$ must be elements of a topological commutative ring.

The standard marking corresponding to this data is a map $f : F_\tau \to A$.

The standard weight of a marked tree $(\tau, f)$ corresponding to this data is

$$w(\tau, f) := \frac{1}{|\text{Aut } \tau|} \prod_{\alpha \in E_\tau} f(\partial \alpha) \prod_{v \in V_\tau} C_{f(\sigma v)}. \quad (0.22)$$

Here we use the following notation. For an edge $\alpha$, $\partial \alpha$ denotes the set of two flags of this edge, and $f(\partial \alpha)$ is the set of two marks $(a, b)$ put on these flags by $f$. Similarly, for a vertex $v$, $\sigma v$ denotes the set of all flags containing $v$, and $f(\sigma v)$ is the respective family of marks.

Finally, the standard sum over trees, or in physics speak, a partition function is

$$Z := \sum_{\tau/(\text{iso})} \sum_{f : F_\tau \to A} w(\tau, f). \quad (0.23)$$

The passage from (0.21) to (0.23) is not completely automatic and indeed not always possible. Luckily, in can be made for all the problems discussed in [Ko] and here. I cannot explain conceptually why this is so. In particular, the factor $1/|\text{Aut } \tau|$ in the Problems A, B, resp. C, occurs for different geometric reasons.

If we managed to represent (0.21) in the form (0.23), then we can try to complete the calculation of $\varphi = Z$ with the help of the following identity.

Assume that the matrix $(g^{ab})$ has an inverse matrix $(g_{ab})$.

**STEP 3.** Consider an auxiliary family of independent variables (fields) $\varphi = \{\varphi_a \mid a \in A\}$. Construct the formal function (potential)

$$S(\varphi) = -\sum_{a, b \in A} g_{ab} \frac{\varphi_a \varphi_b}{2} + \sum_{k \geq 1, a_i \in A} \frac{1}{k!} C_{a_1 \ldots a_k} \varphi_{a_1} \cdots \varphi_{a_k}. \quad (0.24)$$

Denote by $\varphi^0 = \{\varphi^0_a \mid a \in A\}$ an appropriate critical point of $S(\varphi)$ that is, a solution of equations $\frac{\partial S}{\partial \varphi} |_{\varphi = \varphi^0} = 0, a \in A$. 

0.6.1. Claim.

\[ Z = S^{\text{crit}} = S(\varphi^0). \]  

(0.25)

This remains a “physical” statement until we specify the relevant topological ring containing \( g \) and \( C \), prove the existence and uniqueness of \( \varphi^0 \), and the convergence of \( S(\varphi^0) \). (See [Ko] for the standard physical argument “proving” 0.6.1). For example, considering \( (g^{ab}, C_{a_1, \ldots, a_k}) \) as independent formal variables, one can treat (0.25) as a formal series in these variables, and prove (0.6.1) as an identity in a localization of this ring.

Anyway, STEP 3 involves three calculational difficulties.

a). We must be able to sum \( S(\varphi) \). In our problems A,B this is easy. In [Ko], a partial success is achieved, reducing \( S(\varphi) \) to a new potential which is quadratic in \( \varphi_a \) but highly non-linear in a finite set of new auxiliary variables.

b). We must be able to solve \( dS = 0 \) and to find \( \varphi^0 \).

c). We must be able to calculate \( S(\varphi^0) \).

The following trick, also well known to physicists, will allow us in certain cases to avoid the last unpleasant calculation.

We will deform the data \( (g^{ab}, C_{a_1, \ldots, a_k}) \) by introducing independent parameters \( t = \{ t_a \mid a \in A \} \) and replacing \( C_a \) by \( t_a C_a \). The rest of the data \( A, g^{ab}, C_{a_1, \ldots, a_k} \) for \( k \geq 2 \) remains unchanged. Let \( Z^t, S^t, \varphi_0^t \) be respectively the deformed partition function, potential, and the critical point. Then we have

0.6.2. Claim. For all \( a \in A \), we have

\[ \frac{\partial Z^t}{\partial t_a} = C_a \varphi_0^t. \]  

(0.26)

From the view point of generating functions, we lose no information replacing (0.25) by (0.26).

To deduce (0.26) from (0.25), one applies Claim 0.6.1 to \( Z^t \) and differentiates in \( t \):

\[ \frac{\partial Z^t}{\partial t_a} = \frac{\partial}{\partial t_a} (S^t(\varphi)|_{\varphi = \varphi^0}) = \]

\[ \sum_b \frac{\partial S^t(\varphi)}{\partial \varphi_b} |_{\varphi = \varphi^0} \frac{\partial \varphi_b^0}{\partial t_a} + \frac{\partial S^t}{\partial t_a} |_{\varphi = \varphi^0} = C_a \varphi_0^t \]

because \( S^t \) depends on \( t \) only via linear terms \( \sum t_a C_a \varphi_a \).

On the other hand, to prove (0.26) in a formal context, one can totally bypass Claim 0.6.1 and simply apply a universal inversion formula to the formal map \( (\varphi_a) \mapsto (\partial S^t/\partial \varphi_a) \) giving simultaneously existence, uniqueness, and expression for \( \varphi_0^t \) as a sum over trees. Such inversion formulas are classical. The version closest to our needs is given in [GK]; the only difference is that \( \partial S^t/\partial \varphi_a \) at 0 does not vanish. We leave details to the reader.

Functional equations (0.7), (0.9), (0.15), (0.18) are essentially relations for coordinates of the critical point. Differential equations are obtained from them by differentiating in \( t \).
Acknowledgements. I am grateful to M. Kontsevich for many enlightening explanations, and to Don Zagier for teaching me PARI. After this work was written, I learned that E. Getzler proved (0.7) and (0.9) by essentially the same method.

§1. Moduli spaces

In this section, we prove the Theorem 0.3.1 following the three step procedure described in 0.6.

1.1. Marked trees and strata. A tree is called stable if $|v| \neq 2$ for all vertices $v$. If $|v| = 1$ we call $v$ end vertex. Let $V^1_\tau$ be the set of end vertices. An $n$–marking of $\tau$ is a bijection $\mu : V^1_\tau \to \{1, \ldots, n\}$. We also put $V^0_\tau = V \setminus V^1_\tau$ and refer to it as the set of interior vertices.

Let now $(C; x_1, \ldots, x_n)$ be a compact connected curve of arithmetical genus zero with $n \geq 3$ labelled non–singular points. The combinatorial structure of this curve is described by the following stable tree with $n$–marking $(\tau, \mu)$:

$V^0_\tau = \{\text{irreducible components of } C\}$,

$V^1_\tau = \{x_1, \ldots, x_n\}$;

$\mu : x_i \mapsto i$; an edge connects two interior vertices if the respective components of $C$ have non–empty intersection; an edge connects an interior vertex to an end vertex if the respective point belongs to the respective component.

Denote now by $M(\tau, \mu) \subset \overline{M}_{0,n}$ the set of points parametrising stable curves of the type $(\tau, \mu)$. If $\tau$ has only one interior vertex, $M(\tau, \mu) := M_{0,n}$ is the big cell. The following statement summarises the main properties of these sets; for a proof, see [Ke].

1.1.1. Proposition. a). $M(\tau, \mu)$ is a locally closed subset of $\overline{M}_{0,n}$ depending only on (the isomorphism class of) $(\tau, \mu)$.

b). $\overline{M}_{0,n}$ is the union of pairwise disjoint strata $M(\tau, \mu)$ for all marked stable $n$–trees $(\tau, \mu)$.

c). For any $(\tau, \mu)$,

$$M(\tau, \mu) \simeq \prod_{v \in V^0_\tau} M_{0,|v|}.$$  

Notice that there exists exactly one stable tree $\bullet \longrightarrow \bullet$ which does not correspond to any stable curve.

We can now calculate Poincaré polynomials.

1.1.2. Proposition. We have

$$P_{M(\tau, \mu)}(q) = \prod_{v \in V^0_\tau} P_{M_{0,|v|}}(q),$$  

$$P_{M_{0,n}}(q) = \left(\frac{q^2 - 2}{k - 3}\right)(k - 3)!.  \tag{1.2}$$  

Proof. (1.1) follows from the Proposition 1.1.1 and the multiplicativity of Poincaré polynomials.

To prove (1.2), one can use the following geometric facts. First, the morphism $\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ forgetting the last marked point is (canonically isomorphic to) the universal curve. Second, the infinity of the source consists of structure sections.
and fibers at infinity of the target. Therefore, over the big cell $M_{0,n}$ this morphism is a Zariski locally trivial fibration with fiber $\mathbb{P}^1$, and $M_{0,n+1} = \pi^{-1}(M_{0,n}) \setminus \{\text{union of structure sections}\}$.

From the additivity of Poincaré polynomials it follows that

$$P_{M_{0,n+1}}(q) = P_{M_{0,n}}(q)P_{\mathbb{P}^1}(q) - nP_{M_{0,n}}(q) = (q^2 + 1 - n)P_{M_{0,n}}(q).$$

Since $P_{M_{0,3}}(q) = 1$, we get (1.2).

Summarizing, we have for $n \geq 3$:

$$P_{M_{0,n}}(q)t^n = \sum_{(\tau,\mu)/(\text{iso})} \prod_{v \in V^{\tau}_0} \left(\frac{q^2 - 2}{|v| - 3}\right)(|v| - 3)! \prod_{v \in V^{\tau}_1} t,$$

where $t$ is a new formal variable, and the sum is taken over $n$-marked stable trees.

### 1.2. Passage to the standard marking.

Comparing (1.3) to (0.22) and (0.23) we are more or less compelled to choose $A = \{\ast\}$ (one element set), $g^{**} = 1$, $C_* = t$, $C^{**} = 0$ (this gives weight zero to non–stable trees), and finally, denoting by $C_k$ the component with $k \geq 3$ indices,

$$C_k = \left(\frac{q^2 - 2}{k - 3}\right)(k - 3)!.$$

In particular, we can forget about $f : F_\tau \to \{\ast\}$.

This makes the weight of $(\tau, \mu)$ depend only on $\tau/(\text{iso})$, but not $\mu$. Now, if $|V^1_\tau| = n$, the set of all $n$–markings of $\tau$ consists of $n!$ elements and is effectively acted upon by the group $\text{Aut} \tau$. Therefore,

$$\text{card } \{(\tau, \mu)/(\text{iso})\} = \frac{n!}{|\text{Aut} \tau|} \text{card } \{\tau\}/(\text{iso}).$$

Putting together (1.3), (0.22), and (0.23), we see finally that $\Phi(q,t) = Z^t$ where

$$\Phi(q,t) := \frac{t^2}{2!} + \sum_{n \geq 3} \frac{t^n}{n!}P_{M_{0,n}}(q),$$

$$Z^t := \sum_{\tau/(\text{iso})} \frac{1}{|\text{Aut} \tau|} \prod_{v \in V_\tau} C_{|v|}.$$ (1.6)

The summation in (1.6) is now taken over all trees, the term $t^2/2$ in (1.5) comes from the two–vertex tree, and the generating function argument $t$ in (1.5) corresponds precisely to the deformation parameter $t$ introduced at the end of the subsection 0.6.

We will now use (0.26) in order to calculate

$$\frac{\partial Z^t}{\partial t} = \frac{\partial \Phi(q,t)}{\partial t} := \varphi(q,t).$$

### 1.3. Potential.

From (0.24) and (1.4) one sees that

$$S^t(\varphi) = -\frac{\varphi^2}{2} + t\varphi + \sum_{k \geq 3} C_k\varphi^k =$$

$$-\frac{\varphi^2}{2} + t\varphi + \sum_{k \geq 3} \left(\frac{q^2 - 3}{k - 3}\right) \frac{\varphi^k}{k(k-1)(k-2)}.$$ This can easily be summed. We need only the derivative.
1.3.1. Proposition. For generic $q$ we have

$$\frac{\partial}{\partial \varphi} S^t(\varphi) = \frac{(1 + \varphi)q^2 - 1 - q^4 \varphi}{q^2(q^2 - 1)} + t,$$

(1.7)

and for $q = -1$,

$$\frac{\partial}{\partial \varphi} S^t(\varphi) = (1 + \varphi) \log(1 + \varphi) - 2 \varphi + t.$$

(1.8)

1.4. End of the proof. We see now that (0.7), resp. (0.9), are equations for the critical point $d_\varphi S^t = 0$. Differentiating them in $t$ and eliminating $(1 + \varphi)q^2$, resp. $\log(1 + \varphi)$, we get (0.8), resp. (0.10).

1.5. Ramification of $\varphi(q, t)$ as a function of $t$. If $q^2$ is rational but $\neq 1$, we see from (0.7) that $\varphi$ is an algebraic function of $t$ of genus 0. Otherwise it is transcendental and infinitely valued. In order to understand its topology, we can use the following classical trick.

Consider the differential equation for a function $y = y(x)$:

$$yy_x = ax + by; \ a, b \in \mathbb{C}. \quad (1.9)$$

Let $w_{1,2}$ be roots of its characteristic equation

$$w^2 - bw - a = 0. \quad (1.10)$$

Assume that $w_1 \neq w_2$ and put

$$A_1 = \frac{w_1}{w_2 - w_1}, \ A_2 = \frac{w_2}{w_1 - w_2} \quad (1.11)$$

so that $A_1 + A_2 = -1$. A direct calculation shows:

**Proposition 1.5.1.** Put $w(x) = y(x)/x$. Then the general solution of (1.9) is given by the implicit equation

$$Cx = (w - w_1)^{A_1}(w - w_2)^{A_2}, \quad (1.12)$$

where $C$ is an arbitrary constant.

We can apply this to (0.8) putting

$$y = 1 + q^2 t - q^2 \varphi, \ x = q^2 t + q^2 + 1.$$ 

Then we find

$$w_1 = 1, \ w_2 = q^{-2}, \ A_1 = \frac{q^2}{1 - q^2}, \ A_2 = \frac{1}{q^2 - 1}.$$ 

One can calculate $C$ evaluating (1.12) at the point $t = 0$ where we have $x = q^2 + 1, \ y = 1, \ w = (q^2 + 1)^{-1}$.

§2. Configuration spaces
In this section, we prove Theorems 0.4.1 and 0.4.2.

2.1. Nests and strata. Let $X$ be a smooth compact algebraic variety. The configuration space $X[n]$, $n \geq 2$, is defined in [FMPh] as the closure of its big cell $X^n \setminus \bigcup_{i<j} \Delta_{ij}$ (where $\Delta_{ij}$ is the diagonal $x_i = x_j$) in $X^n \times \prod S \tilde{X}^S$, where $S$ runs over subsets $S \subset \{1, \ldots, n\}$, $|S| \geq 2$; $X^S$ denotes the respective partial product of $X$'s, and $\tilde{X}^S$ is the blow up of the small diagonal $\Delta_S$ in $X^S$.

Every $S$ determines a divisor at infinity $D(S) \subset X[n]$. Namely, let $\pi_S : X[n] \to X^S$ be the canonical projection. Then $\pi_S^{-1}(\Delta_S) = \cup_{T \supseteq S} D(T)$.

The natural stratification of $X[n]$ described in [FMPh] consists of (open subsets of) intersections $X(S) = \cap_{i=1}^r D(S_i)$ corresponding to sets $S = \{S_1, \ldots, S_r\}$ of subsets in $\{1, \ldots, n\}$ called nests.

2.1.1. Definition. a). $S = \{S_1, \ldots, S_r\}$ is a nest (or $n$-nest) if $|S_i| \geq 2$ for all $i$, and either $S_i \subset S_j$ or $S_j \subset S_i$ for all $i, j$ such that $S_i \cap S_j \neq \emptyset$.

In particular, $S = \emptyset$ is a nest, and $S = \{S\}$ is a nest, if $|S| \geq 2$.

b). A nest $S$ is called whole (resp. broken) if $\{1, \ldots, n\} \in S$ (resp. $\{1, \ldots, n\} \notin S$).

Denote by $X(S) \subset \overline{X(S)} = \cap_{S \in S} D(S)$ the subset of points not belonging to smaller closed strata. The following facts are proved in [FMPh].

2.1.2. Proposition. a). For any $n \geq 2$ and $n$-nest $S$, $X(S)$ is a locally closed subset of $X[n]$.

b). $X[n]$ is the union of pairwise disjoint strata $X(S)$ for all $n$-nests $S$.

2.2. From nests to marked trees. As in 1.1 we consider a bijection $\mu : V^1_r \to \{1, \ldots, n\}$ as a part of the appropriate marking for our problem. The remaining data is supplied by choosing orientation of all edges.

2.2.1. Definition. A tree $\tau$ marked in this way is called admissible iff:

a). Every vertex of $\tau$ except of one has exactly one incoming edge.

b). The exceptional vertex has only outgoing edges, and their number is $\geq 2$.

This vertex is called source.

c). All interior vertices with possible exception of source have valency $\geq 3$.

2.2.2. Proposition. The following maps are (1,1):

\{broken n-nests\} $\to \{whole n-nests\} \to \{admissible marked n-trees\}$/iso,

$$\mathcal{S} \mapsto \mathcal{S} \cup \{\{1, \ldots, n\}\} \mapsto \tau(S) = \tau(S \cup \{\{1, \ldots, n\}\}).$$

Here $\tau$ is defined by its sets of vertices and edges: if $S = \{S_1, \ldots, S_r\}$, then

$$V_\tau = \{\tilde{S}_1, \ldots, \tilde{S}_{n+r}\} := \{S_1, \ldots, S_r, \{1\}, \ldots, \{n\}\},$$

and an edge oriented from $\tilde{S}_i$ to $\tilde{S}_j$ connects these two vertices iff $\tilde{S}_j \subset \tilde{S}_i$ and no $\tilde{S}_k$ lies strictly in between these two subsets.

This is proved by direct observation. The following facts are worth mentioning.
a). \(\{1, \ldots, n\}\) is the source of \(\tau(S)\) for any \(S\).

b). \(\{1\}, \ldots, \{n\}\) are all end vertices.

c). \(i \in S_j\) iff one can pass from \(S_j \in V_\tau\) to \(\{i\} \in V_\tau\) in \(\tau\) by going always in positive direction.

A reader is advised to convince him–or herself that the source has valency \(\geq 2\) and all other interior vertices have valency \(\geq 3\).

Denote the source by \(s\) and the set of the remaining interior vertices \(V_\tau^0\).

2.2.3. Proposition ([FMPh]). The virtual Poincaré polynomials of strata \(X(S)\) are given by the following formulas (we add a formal variable \(t\)).

If \(S\) is a broken \(n\)-nest, \(s \in V_\tau(S)\):

\[
t^n P_X(S)(q) = \left(\frac{P_X(q)}{|s|}\right)|s|! \times \prod_{v \in V_\tau^0(S)} \kappa_m \left(q^{2m} - 2\right)(|v| - 3)! \times \prod_{v \in V_\tau^1(S)} t. \tag{2.1}
\]

If \(S\) is a whole \(n\)-nest:

\[
t^n P_X(S)(q) = P_X(q)\kappa_m \left(q^{2m} - 2\right)(|s| - 2)! \times \prod_{v \in V_\tau^0(S)} \kappa_m \left(q^{2m} - 2\right)(|v| - 3)! \times \prod_{v \in V_\tau^1(S)} t. \tag{2.2}
\]

Comparing (2.1) and (2.2) one sees that one can express the joint contribution of two nests corresponding to an admissible marked tree \(\tau\) as a product of local weights corresponding to all vertices of \(\tau\). The local weight of the source will be

\[
\left(\frac{P_X(q)}{|s|}\right)|s|! + P_X(q)\kappa_m \left(q^{2m} - 2\right)(|s| - 2)!
\]

and the remaining local weights in (2.1) and (2.2) coincide and depend only on the valency.

2.3. Passage to the standard marking. We make the following choices.

Put \(A = \{+, -\}\). Interpret a mark + (resp. −) on a flag as incoming (resp. outgoing) orientation of this flag. Thus, \(f : F_\tau \to A\) is a choice of orientation of all flags.

Put \(g^{++} = g^{--} = 1, g^{+-} = g^{-+} = 0\). This makes the standard weight of \((\tau, f)\) vanish unless all edges are unambiguously oriented by \(f\).

Put \(C_+ = t\) (see (2.1) and (2.2)) and \(C_- = 0\). The last choice makes the standard weight vanish unless all end edges are oriented outwards.

Put \(C_{+-} = C_{-+} = 0\). This excludes vertices of the type \(\rightarrow \bullet \rightarrow\).

Put also \(C_{a_1, \ldots, a_k} = 0\) if \(\{+, +\} \subset \{a_1, \ldots, a_k\}\). This eliminates vertices with \(\geq 2\) incoming edges.

For tensors with \(k \geq 2\) minuses among the indices we put

\[
C_{-\ldots} = \left(\frac{P_X(q)}{t}\right)^k + \kappa_m P_X(q) \left(q^{2m} - 2\right)(k - 2)! \tag{2.3}
\]
(because only the source has all outgoing edges), and

\[ C_{+-\ldots} = \kappa_m \left( \frac{q^{2m} - 2}{k - 2} \right)(k - 2)! \]

(2.4)

(cf. (2.1) and (2.2)).

The standard weight of a marked tree defined by this data again is independent on the part \( \mu : V_1^1 \to \{1, \ldots, n\} \) of the initial marking which accounts for the factor \( \frac{n!}{|\text{Aut} \tau|} \) below.

Summarizing, we put

\[ \Phi_X(q, t) := \sum_{n \geq 2} t^n \frac{n!}{n!} P_X[n](q), \]

(2.5)

and get from the previous discussion

\[ Z_t := \sum_{\tau / (iso)} \frac{1}{|\text{Aut} \tau|} \sum_{f:F_\tau \to \{+,-\}} \prod_{\alpha \in E_\tau} g^{f(\partial \alpha)} \prod_{v \in V_\tau} C_{f(\sigma v)}, \]

(2.6)

(we have two arguments \( x, y \) but only one \( t = t_+ \) because \( C_- = 0 \)).

2.4. Potential. We change notation: \( \varphi_+ = x, \varphi_- = y \). From 2.3 we see that (already \( t \)-deformed) potential is

\[ S^t(x, y) = -xy + tx + \kappa_m \sum_{k=2}^{\infty} \left( \frac{q^{2m} - 2}{k - 2} \right) \frac{xy^k}{k(k - 1)} + \]

\[ \sum_{k=2}^{\infty} \left( \frac{\underline{P_X(q)}}{k} \right) y^k + \kappa_m \underline{P_X(q)} \sum_{k=2}^{\infty} \left( \frac{q^{2m} - 2}{k - 2} \right) \frac{y^k}{k(k - 1)} \]

(2.8)

(we have two arguments \( x, y \) but only one \( t = t_+ \) because \( C_- = 0 \)).

We must solve the system

\[ \frac{\partial S^t}{\partial x} |_{x^0, y^0} = \frac{\partial S^t}{\partial y} |_{x^0, y^0} = 0, \]

(2.9)

and (0.26) then tells us that

\[ \frac{\partial}{\partial t} Z^t = \varphi_X(q, t) = x^0. \]

(2.10)

Again, \( S^t(x, y) \) can be easily summed. To write down the functional equation, we need only \( x \)-derivative which for general \( q \) is

\[ \frac{\partial S^t}{\partial x} = -y + t + \kappa_m \frac{(1 + y)q^{2m} - 1 - q^{2m}y}{q^{2m} - 3m - 1}. \]

(2.11)
For \( q = -1 \):
\[
\frac{\partial S^t}{\partial x} = -y + t + m[(1 + y) \log(1 + y) - y].
\]  

(2.12)

2.5. End of the proof. We now see that (0.15), resp (0.18), are the equations defining \( y^0 \). Differentiating in \( t \) we get (0.16) and (0.19). And since \( S^t(x,y) \) is linear in \( x \), the vanishing of the \( y \)-derivative gives an explicit expression of \( x_0 \) via \( y_0^0 \):
\[
\varphi_X(q,t) = \frac{P_X(q)(1 + y^0)P_X(q) + (q^{2m} + \kappa_m - 1)y^0 - q^{2m}t - 1}{1 + (1 - q^{2m} - \kappa_m)y^0 + q^{2m}t}.
\]

To see that this is equivalent to (0.17) one can differentiate (0.17) in \( t \) and use (0.16).

2.6. Ramification of \( y^0 \). Replaying the game of 1.5, we put (changing the meaning of \( x,y \) in favor of those in 1.5):
\[
y = y(q,t) := q^{2m}t + 1 - (q^{2m} + \kappa_m - 1)y^0(q,t),
\]
\[
x := t + \frac{q^{2m} + \kappa_m}{q^{2m}}, \quad w(q,t) = y/x.
\]

Then (0.16) becomes
\[
yy_x = -q^{2m}x + (q^{2m} + 1)y
\]
so that in the notation of 1.5
\[
w_1 = 1, \quad w_2 = q^{2m}, \quad A_1 = \frac{1}{q^{2m} - 1}, \quad A_2 = \frac{q^{2m}}{1 - q^{2m}},
\]
and finally
\[
Cx = (w - w_1)^{A_1}(w - w_2)^{A_2}
\]
for some \( C \).

§3. Multiple coverings

3.1. Kontsevich’s formula for Problem C. Kontsevich represents \( m_d \) as a rational function of two variables \( \lambda_1, \lambda_2 \) which is formally homogeneous of degree zero and actually is expected to be a constant.

Geometrically, this statement must be a corollary of Bott’s fixed point formula for smooth stacks. The \( \lambda \)-variables in this context are coordinates of a toric vector field on the target \( \mathbb{P}^1 \). Until this has been worked out, we simply go ahead with Kontsevich and take this independence for granted.

The function in question is a sum of contributions indexed by isomorphism classes of connected trees \( \tau \) endowed with markings: each vertex \( v \) is marked by \( f_v = 1 \) or 2 so that no neighbors have the same mark; each edge \( \alpha \) is marked by a positive integer \( d_\alpha \). Only those marked trees contribute to \( m_d \) for which \( \deg \tau := \sum_\alpha d_\alpha = d \).

We introduce the following notation for a marked tree \( \tau \): \( F= \) the number of vertices marked by 2; \( \sigma_j = \sum_{v \in v_j} d_v \); \( \sigma_j = \sum_{v \in v_j} (|v| - 1) \); \( j = 1,2 \).
Then we have

\[
m_d = (\lambda_1 - \lambda_2)^{2d} \sum_{\tau : \deg \tau = d} \frac{1}{|\text{Aut} \tau|} (-1)^{d + F} \lambda_1^{2w_1} \lambda_2^{2w_2} V(\tau) E(\tau),
\]

where

\[
V(\tau) = \prod_v \sigma_v^{[v]-3}, \quad E(\tau) = \prod_{\alpha} \frac{d_{\alpha}^3}{d_{\alpha}!} \prod_{a+b=d, a, b \geq 1} (a\lambda_1 + b\lambda_2)^2.
\]

3.2. Theorem. \(m_d = d^{-3}\).

Proof. We will calculate the value of \(m_d\) at \(\lambda_1 = 1, \lambda_2 = 0\). The drastic simplification results from the fact that the factor \(\lambda_2^{2w_2}\) vanishes unless \(w_2 = 0\). Now, \(w_2 = 0\) implies that \(\tau\) has no vertices of multiplicity \(\geq 2\) marked by 2. Hence \(\tau\) either has only one edge, or is a star with central vertex marked 1, and end vertices marked by 2. We will consider the first case as one ray star as well.

Now, let \(\tau\) be such a star of degree \(d\). The set \(\{d_\alpha\}\) forms a partition of \(d\) into positive summands which uniquely defines the isomorphism class of \(\tau\). It is convenient to write this partition as the set of multiplicities \(R = \{r_1, r_2, \ldots\}\), where \(r_i\) is the number of edges marked by \(i\) so that \(\sum_i i r_i = d\). Obviously, \(|\text{Aut} \tau| = \prod_i r_i!\).

After some reshuffling, our assertion thus reduces to the following identity:

\[
(?) \sum_R \frac{1}{\prod_i r_i!} \prod_i (-\frac{d}{i})^{r_i} = (-1)^d.
\]

Now, the left hand side of (\?) can be obtained in the following way. Consider the formal series \(e^{\sum_{i \geq 1} y_i t^i}\), take its terms of degree \(d\) in \(t\) and put in them \(y_i = -d/i\). But we can clearly proceed in reverse order first making the substitution \(y_i = -d/i\). Then the series in the exponent becomes \(\sum_i (-d/i) t^i = d \log(1 - t)\), so that finally we get the coefficient of \(t^d\) in \((1 - t)^d\). QED

Remark. One can observe that \((-1)^d\) coincides with the contribution of just one trivial partition: \(r_d = 1\). The remaining terms cancel. Geometrically, this means that degenerating configurations do not contribute with this choice of vector field. Algebraically, this can be rewritten as an equality of two sums, one over proper partitions with odd, another with even number of summands.

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