Heavy quarkonium decays and transitions in the language of effective field theories

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Abstract. Heavy quarkonium decays and transitions are discussed in the framework of non-relativistic effective field theories. Emphasis is put on the matching procedure in the non-perturbative regime. Some exact results valid for the magnetic dipole couplings are discussed.

INTRODUCTION

In the last years the B factories, CLEO and BES have produced a large amount of new data for heavy quarkonium observables [1]. These data are not only interesting because they may signal new states or new decay or production mechanisms, but also because heavy quarkonium is a system that to a large extent can rigorously be studied in QCD. Therefore, any new understanding of it may potentially provide new insight on the non-perturbative dynamics of QCD.

Heavy quarkonium, as a non-relativistic bound state, is characterized by a hierarchy of energy scales: \( m, mv \) and \( mv^2 \), where \( m \) is the heavy-quark mass and \( v \ll 1 \) the heavy-quark relative velocity. Whenever a system is described by a hierarchy of scales, observables may be calculated by expanding one scale with respect to the other. An effective field theory (EFT) is a field theory that makes this expansion explicit at the Lagrangian level. To be more precise, let us call \( H \) a system described by a fundamental Lagrangian \( \mathcal{L} \) and suppose it characterized by 2 scales: \( \Lambda \gg \lambda \). The EFT Lagrangian, \( \mathcal{L}_{\text{EFT}} \), suitable to describe \( H \) at scales lower than \( \Lambda \), is characterized by (1) a cut off \( \Lambda \gg \mu \gg \lambda \); (2) some degrees of freedom that exist at scales lower than \( \mu \). The Lagrangian \( \mathcal{L}_{\text{EFT}} \) is then made of all operators \( O_n \) that may be built from the effective degrees of freedom and are consistent with the symmetries of the original Lagrangian \( \mathcal{L} \):

\[
\mathcal{L}_{\text{EFT}} = \sum_n c_n(\Lambda/\mu) \frac{O_n(\mu, \lambda)}{\Lambda^n}.
\]

The advantage is that, once the scale \( \mu \) has been run down to \( \lambda \), the power counting is homogeneous \( \langle O_n \rangle \sim \lambda^n \), so that the EFT is, indeed, organized as an expansion in \( \lambda/\Lambda \). Despite the EFT not being renormalizable in the traditional sense, it is renormalizable order by order in \( \lambda/\Lambda \). The matching coefficients \( c_n(\Lambda/\mu) \) encode the non-analytic behaviour in \( \Lambda \). They are calculated by imposing that \( \mathcal{L}_{\text{EFT}} \) and \( \mathcal{L} \) describe the same physics at any finite order in the expansion. The procedure is called matching. Finally, we note that if \( \Lambda \gg \Lambda_{\text{QCD}} \), \( c_n(\Lambda/\mu) \) may be calculated in perturbation theory, if \( \Lambda \sim \Lambda_{\text{QCD}} \), the matching must rely on non-perturbative methods.
Several effective field theories for heavy quarkonium that take full advantage of the non-relativistic hierarchy of scales have been developed and used over the last decade. For a recent review we refer to [2]. NRQCD is the EFT that exploits the hierarchy $\Lambda = m \gg \lambda = mv$ [3, 4]. Since $m \gg \Lambda_{QCD}$, the matching coefficients of NRQCD may be calculated in perturbation theory. pNRQCD is the EFT that exploits the hierarchy $\Lambda = mv \gg \lambda = mv^2$ [5, 6]. If $\Lambda_{QCD} \sim mv^2$, then the matching to pNRQCD may be still done in perturbation theory. We call weak coupling this regime, which may be suited to describe ground-state quarkonium. If $\Lambda_{QCD} \sim mv$, then the matching to pNRQCD is non perturbative. We call strong coupling this regime, which may be suited to describe excited quarkonium states.

The fact that EFTs may be built to describe heavy quarkonium in the strong-coupling regime follows from the observation that the non-relativistic hierarchy of scales survives also below $\Lambda_{QCD}$ [7, 8]. The complication of the strong-coupling regime comes from the non-perturbative matching and from new scales that may arise in loops sensitive to $\Lambda_{QCD}$. An example is the three-momentum scale $\sqrt{m \Lambda_{QCD}}$ discussed in [9]. Nevertheless, many advantages remain in treating even strongly-coupled heavy quarkonium in an EFT framework.

In the following we shall sketch a unified framework for the description of inclusive and electromagnetic decays, and radiative transitions in the framework of strongly coupled pNRQCD. For a treatment of inclusive and electromagnetic decay widths in the weak-coupling regime we refer to [2] and references therein. For a treatment of magnetic dipole transitions in the weak-coupling regime we refer to [10].

**NRQCD**

NRQCD is the EFT that follows from QCD when modes of energy or momentum $m$ are integrated out. The structure of the EFT Lagrangian is like Eq. (1) with $\Lambda = m$ and $\lambda = mv \sim \Lambda_{QCD}$. The scale $mv$ is sometimes called soft. The degrees of freedom of the EFT Lagrangian are quarks, antiquarks and gluons with energy and momentum lower than $m$ (we neglect light quarks).

The NRQCD Lagrangian may be written as

$$\mathcal{L}_{NRQCD} = \mathcal{L}_{2-f} + \mathcal{L}_{4-f} + \mathcal{L}_{light},$$

where

$$\mathcal{L}_{2-f} = \psi^\dagger \left( iD_0 + \frac{D^2}{2m} \right) \psi + \frac{c_F}{2m} \psi^\dagger \sigma \cdot gB \psi - \frac{2c_F - 1}{8m^2} \psi^\dagger \sigma \cdot [-iD \times, gE] \psi + \ldots + \left[ \psi \rightarrow i\sigma^2 \chi^\dagger \right],$$

$$\mathcal{L}_{light} = -\frac{1}{4} F^{\mu \nu a} F_{\mu \nu}^a.$$
The term $\mathcal{L}_{4-f}$ stands for the 4-fermion part of the NRQCD Lagrangian (for an explicit expression see, for instance, [4]).

The coefficient $c_F$ is a matching coefficient of the EFT. In Eq. (5), we have made use of reparameterization invariance to reduce the other matching coefficients to this one. It is known at two loops and may be found in [11]. The 4-fermion matching coefficients encode the contribution of the annihilation graphs. As a consequence they develop an imaginary part. We refer to [12] for an updated list of them and for references.

Let us give some definitions concerning the Fock space of NRQCD. If $H_{\text{NRQCD}}$ is the Hamiltonian of NRQCD, we call $|n; r, R\rangle$ the eigenstates of $H_{\text{NRQCD}}$, and $E_n$ the corresponding eigenvalues. $r$ stands for the relative distance of the two heavy quarks and $R$ for their centre-of-mass coordinate. Both are good quantum numbers in the static limit. With $n$ we indicate a generic set of conserved quantum numbers. $|n; r, R\rangle$ and $E_n(r, R; \nabla_r, \nabla_R)$ satisfy the system of equations:

$$H_{\text{NRQCD}}|n; r, R\rangle = \int d^4r' d^4R'|n; r', R'\rangle E_n(r', R'; \nabla_{r'}, \nabla_{R'}) \delta^3(r' - r) \delta^3(R' - R),$$  \hspace{1cm} (5)

$$\langle m; r, R|n; r', R'\rangle = \delta_{nm} \delta^3(r - r') \delta^3(R - R').$$  \hspace{1cm} (6)

**INCLUSIVE DECAYS IN PNRQCD**

pNRQCD is the EFT that follows from NRQCD when gluons of energy or momentum and quarks of energy larger than $mv^2$ and quarks of momentum larger than $mv$ are integrated out. The structure of the EFT Lagrangian is like Eq. (1) with $\Lambda = mv \sim \Lambda_{\text{QCD}}$ and $\lambda = mv^2$. The scale $mv^2$ is sometimes called ultrasoft. In the strong-coupling regime, if the gluonic excitations between the two heavy quarks develop a mass gap of order $\Lambda_{\text{QCD}}$, then they are all integrated out from the theory. Therefore, the degrees of freedom of the EFT Lagrangian are only singlet quarkonium fields. The Lagrangian of pNRQCD is very simple:

$$\mathcal{L}_{\text{pNRQCD}} = \int d^3r \text{Tr} \left\{ S^3 \left( i\partial_0 + \frac{\nabla_R^2}{4m} + \frac{\nabla_r^2}{m} - V_S \right) S \right\}. $$  \hspace{1cm} (7)

$S$ is a non-local field, function of $r$, $R$ and $t$, $2 \otimes 2$ in spin space and a $3 \otimes 3$ singlet in colour space. The trace is taken over colour and spin indices.

All complications go into the potential $V_S$, which is a non-perturbative function of $r$ to be determined by a non-perturbative matching procedure. In general $V_S$ contains also an imaginary part inherited from the matching coefficients of the 4-fermion operators of NRQCD. The matching condition reads

$$\langle q; r', R'|H_{\text{NRQCD}}|q; r, R\rangle = \left( -\frac{\nabla_R^2}{4m} - \frac{\nabla_r^2}{m} + V_S \right) \delta^3(r' - r) \delta^3(R' - R).$$  \hspace{1cm} (8)

The matching condition determines $V_S$ as a function of quantities defined in NRQCD (the left-hand side of Eq. (8)). Once $V_S$ has been determined, one may calculate the
solutions $\Phi_H(r)$ and $E_H$ of the Schrödinger equation

$$\left(-\frac{\nabla^2}{m} + V_S\right) \Phi_H(r) = E_H \Phi_H(r).$$  \hfill (9)

Using the optical theorem, the inclusive decay width to light particles (l.p.) is given by

$$\Gamma_{H\rightarrow l.p.} = -2\text{Im} \langle H(0)| - \mathcal{L}_{\text{PNRQCD}}|H(0)\rangle.$$  \hfill (10)

$|H(0)\rangle$ stands for a quarkonium state in the rest-frame ($P = 0$):

$$|H(0)\rangle = \int d^3r \int d^3R \text{Tr} \left\{ \Phi_H(r) S^\dagger(r, R) \right\} |0\rangle,$$  \hfill (11)

where $|0\rangle$ is the Fock subspace containing no heavy quarks but an arbitrary number of ultrasoft particles. Note that, since $\Phi_H$ and $L_{\text{PNRQCD}}$ have been calculated through the matching procedure, Eq. (10) provides, indeed, a practical tool to calculate the inclusive decay width. Explicit applications of Eq. (10) have been worked out in [13, 14, 15].

**RADIATIVE TRANSITIONS IN PNRQCD**

Radiative transitions may be described in the same EFT framework that we have discussed so far by enlarging the gauge group to $SU_c(3) \times U_{\text{em}}(1)$. This means that more degrees of freedom have to be taken into account (photons) and more operators added to the EFT Lagrangians. We will concentrate in the following on magnetic dipole transitions [10].

At the level of the NRQCD Lagrangian, magnetic transitions are accounted for by replacing $iD_0 \rightarrow iD_0 - ee_Q A_0^{\text{em}}$ and $iD \rightarrow iD + ee_Q A^{\text{em}}$ in Eq. (3),

$$\mathcal{L}_{2-f} \rightarrow \mathcal{L}_{2-f} + \frac{c_{F}^{\text{em}}}{2m} \psi^\dagger \sigma \cdot ee_Q B^{\text{em}} \psi - \frac{2c_{F}^{\text{em}} - 1}{8m^2} \psi^\dagger \sigma \cdot [-iD \times, ee_Q E^{\text{em}}] \psi$$

$$+ \frac{c_{W}^{\text{em}}}{8m^3} \psi^\dagger \{ D^2, \sigma \cdot ee_Q B^{\text{em}} \} \psi - \frac{c_{W}^{\text{em}} - 1}{4m^3} \psi^\dagger D^i \sigma \cdot ee_Q B^{\text{em}} D^i \psi$$

$$+ \frac{c_F^{\text{em}}}{8m^3} \psi^\dagger \left( \sigma \cdot D ee_Q B^{\text{em}} \cdot D + D \cdot ee_Q B^{\text{em}} \sigma \cdot D \right) \psi + \cdots + [\psi \rightarrow i\sigma^2 \chi^*],$$  \hfill (12)

and $\mathcal{L}_{\text{light}} \rightarrow \mathcal{L}_{\text{light}} - \frac{1}{4} F_{\mu
u}^{\text{em}} F^{\mu\nu}_{\text{em}}$, where the gauge fields with upperscript “em” are electromagnetic fields and $ee_Q$ stands for the charge of the quark of flavour $Q$.

The coefficients $c_{F}^{\text{em}}$ and $c_{W}^{\text{em}}$ are new matching coefficients of the EFT associated with the electromagnetic couplings. Again, we have made use of reparameterization invariance to reduce their number. All coefficients are known at least at one-loop level.
In particular, we have\(^1\)

\[
\begin{align*}
    c_F^{\text{em}} &\equiv 1 + \kappa_Q^{\text{em}} = 1 + \frac{4}{3} \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2), \\
    c_W^{\text{em}} &\equiv 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left( \frac{1}{12} + \frac{4}{3} \ln \frac{m}{\mu} \right) + \mathcal{O}(\alpha_s^2),
\end{align*}
\]  

(13) (14)

\(\kappa_Q^{\text{em}}\) is usually identified with the anomalous magnetic moment of the heavy quark.

At the level of pNRQCD, magnetic transitions involving ultrasoft photons are described by adding to the Lagrangian (7) the electromagnetic Lagrangian \(-F^{\mu\nu} F_{\mu\nu}\) and a term \(\mathcal{L}_{\gamma p\text{NRQCD}}\) responsible for the coupling of the quarkonium to the electromagnetic field:

\[
\mathcal{L}_{\gamma p\text{NRQCD}} = \int d^3 r \text{Tr} \left\{ V_A^{\text{em}} S^\dagger r \cdot e e_Q E^{\text{em}} S \right. \\
+ \frac{V_S^{\text{em}}}{2m} \left\{ S^\dagger, \sigma \cdot e e_Q B^{\text{em}} \right\} S + \frac{V_S^{\text{em}}}{16m} \left\{ S^\dagger, \sigma \cdot [\hat{r} \times (\hat{r} \times e e_Q B^{\text{em}})] \right\} S \\
+ \frac{V_S^{\text{em}}}{4m^2 r} \left\{ S^\dagger, \sigma \cdot [i \nabla_R \times, e e_Q E^{\text{em}}] \right\} S - \frac{V_S^{\text{em}}}{16m^2 r} \left\{ S^\dagger, \sigma \cdot [-i \nabla_r \times, \hat{r} (\nabla_R e e_Q E^{\text{em}})] \right\} S \\
+ \frac{V_S^{\text{em}}}{4m^3} \left\{ S^\dagger, \sigma \cdot e e_Q B^{\text{em}} \right\} \nabla^2 S + \frac{V_S^{\text{em}}}{4m^3} \left\{ S^\dagger, \sigma [e e_Q B^{\text{em}}, ] \right\} \nabla^2 S \\
+ \frac{V_S^{\text{em}}}{4m^3 r^2} \left\{ S^\dagger, \sigma \cdot [\hat{r} \times (\hat{r} \times e e_Q B^{\text{em}})] \right\} S + \frac{V_S^{\text{em}}}{4m^3 r^2} \left\{ S^\dagger, \sigma \cdot e e_Q B^{\text{em}} \right\} S + \cdots \right\}.
\]  

(15)

All gauge fields are calculated in the centre-of-mass coordinate \(R\). The field \(S\) is understood as a singlet also under \(U_{\text{em}}(1)\) gauge transformations.

In the centre-of-mass of the initial quarkonium state, the power counting goes as follows: \(\nabla_r \sim m v, r \sim 1/mv\), the electromagnetic fields associated to the external photons go like \(E^{\text{em}}, B^{\text{em}} \sim k_f^2\). The centre-of-mass derivative \(\nabla\) acting on the recoiling final quarkonium state or emitted photon is of order \(k_f\), where \(k_f\) is the energy and momentum of the emitted photon.

The coefficients \(V\) in Eq. (15) are the matching coefficients of pNRQCD. They encode high-energy contributions to the electromagnetic couplings and are of the same nature as \(V_S\) in Eq. (7). In the strong-coupling regime they are determined by nonperturbative matching of 5-points Green functions involving two external quarks, two external antiquarks and an external photon. Let us consider the matching condition for

\(^1\) The coefficients get also QED corrections, but they are numerically negligible.
the $1/m$ operators, it reads

$$
\langle 0; \mathbf{r}', \mathbf{R}' \rangle \otimes \langle \gamma | \left( \frac{e_F^\text{em}}{2m} \int d^3 x \, \psi^\dagger \sigma \cdot ee_Q \mathbf{B}^\text{em} \psi + [\psi \rightarrow i \sigma^2 \chi^*] \right) | 0 \rangle \otimes | 0; \mathbf{r}, \mathbf{R} \rangle =
$$

$$
\left( \frac{V_S^\sigma B}{2m} + \frac{V_S^2 (\mathbf{r} \cdot \nabla)^2 \sigma B}{16m} \right) \left( \sigma^{(1)} + \sigma^{(2)} \right) \cdot \langle \gamma | ee_Q \mathbf{B}^\text{em} | 0 \rangle \delta^3 (\mathbf{r}' - \mathbf{r}) \delta^3 (\mathbf{R}' - \mathbf{R}) .
$$

Since corrections to the state $| 0; \mathbf{r}, \mathbf{R} \rangle$ involving derivatives or spins are $1/m$ suppressed (see Eq. (3)), $\sigma \cdot ee_Q \mathbf{B}^\text{em}$ effectively behaves as the identity operator. As a consequence, the electromagnetic matrix element decouples in the left-hand side. From the normalization condition (6) it follows that

$$
V_S^\sigma B = V_S^\sigma c_F^\text{em} .
$$

This is a rather remarkable result that holds to all orders in the strong-coupling constant and non-perturbatively. It excludes that the $1/m$ magnetic coupling of the quarkonium field is affected by any soft contribution. A fortiori, it excludes large anomalous non-perturbative corrections to this coupling. Similar arguments lead to the following exact results at order $1/m^2$:

$$
V_S^\sigma r \times \sigma B = \frac{r^2}{2} V_S^{(0)} , \quad V_S^\sigma r = 0 , \quad V_S^\sigma \nabla \times \sigma B = 0 , \quad V_S^\sigma \nabla \cdot \sigma B = 2 c_F^\text{em} - 1 ,
$$

(18)

where $V_S^{(0)}$ is the static part of the $V_S$ potential. The first equality follows from the fact that Poincaré invariance protects the spin-orbit coupling [17, 18]. The second one remarkably states that to all orders in the strong-coupling constant and non-perturbatively the existence of an effective scalar interaction, which has been often advocated in phenomenological models, is excluded. The third one that those matching coefficients, like the one in Eq. (17), get only hard contributions.

The matching of the $1/m^3$ terms is more complicated. One reason is that at this order kinetic energy and spin-dependent corrections affect the state $| 0; \mathbf{r}, \mathbf{R} \rangle$ and $\sigma \cdot ee_Q \mathbf{B}^\text{em}$ does not behave anymore like the identity operator.

Once the matching has been completed, the transition width is given by:

$$
\Gamma_{H \rightarrow H'} = \int \frac{d^3 P'}{(2\pi)^3} \frac{d^3 P}{(2\pi)^3} (2\pi)^4 \delta^4 (P_H - k - P') | \mathcal{A} [H(0) \rightarrow H'(P')] \gamma(k) |^2 ,
$$

(19)

where

$$
\mathcal{A} [H(0) \rightarrow H'(-k)] \delta^3 (P' + k) = \langle H'(P') \gamma(k) \rangle - \int d^3 R \mathcal{L}_{pNRQCD} | H(0) \rangle .
$$

(20)

The overline stands for the sum over the final-state polarizations and the average over the initial state ones. $P_H = (M_H, 0)$ stands for the four-momentum of the initial-state quarkonium of mass $M_H$. The state $| H(P) \rangle$ is the state (11) boosted by $-P/M_H$. The Lorentz-boost transformations may be read from [17, 18].
CONCLUSIONS

We have discussed in an unified framework inclusive and electromagnetic decays, and radiative transitions of heavy quarkonium in a regime where the typical momentum transfer is of order $\Lambda_{\text{QCD}}$. Noteworthy, also in this situation suitable effective field theories may be constructed, systematic expansions exploited and exact results derived.

It seems rather unlikely that the non-perturbative matching, once completed at order $1/m^3$, will support the formulas traditionally and universally used so far to describe radiative transitions at relative order $v^2$ and derived from phenomenological assumptions \[\text{[19,20]}\]. This may possibly shade some light, for instance, on the radiative transition data for the $\Upsilon$ system recently collected at CLEO \[\text{[21]}\], whose understanding is problematic in many phenomenological models.

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REFERENCES

1. N. Brambilla et al., CERN-2005-005, (CERN, Geneva, 2005) [arXiv:hep-ph/0412158].
2. N. Brambilla, A. Pineda, J. Soto and A. Vairo, arXiv:hep-ph/0410047.
3. W. E. Caswell and G. P. Lepage, Phys. Lett. B 167, 437 (1986).
4. G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D 51, 1125 (1995) [Erratum-ibid. D 55, 5853 (1997)] [hep-ph/9407339].
5. A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. 64, 428 (1998) [arXiv:hep-ph/9707481].
6. N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B 566, 275 (2000) [arXiv:hep-ph/9907240].
7. N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D 63, 014023 (2001) [arXiv:hep-ph/0002250].
8. A. Pineda and A. Vairo, Phys. Rev. D 63, 054007 (2001) [Erratum-ibid. D 64, 039902 (2001)] [arXiv:hep-ph/0009145].
9. N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Lett. B 580, 60 (2004) [arXiv:hep-ph/0307159].
10. N. Brambilla, Y. Jia and A. Vairo, arXiv:hep-ph/0512369.
11. A. Czarnecki and A. G. Grozin, Phys. Lett. B 405, 142 (1997) [arXiv:hep-ph/9701415].
12. A. Vairo, Mod. Phys. Lett. A 19, 253 (2004) [arXiv:hep-ph/0311303].
13. N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, Phys. Rev. Lett. 88, 012003 (2002) [arXiv:hep-ph/0109130].
14. N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D 67, 034018 (2003) [arXiv:hep-ph/0208019].
15. A. Vairo, Nucl. Phys. Proc. Suppl. 115, 166 (2003) [arXiv:hep-ph/0205128].
16. A. V. Manohar, Phys. Rev. D 56, 230 (1997) [hep-ph/9701294].
17. N. Brambilla, D. Gromes and A. Vairo, Phys. Lett. B 576, 314 (2003) [hep-ph/0306107].
18. N. Brambilla, D. Gromes and A. Vairo, Phys. Rev. D 64, 076010 (2001) [arXiv:hep-ph/0104068].
19. H. Grotch and K. J. Sebastian, Phys. Rev. D 25, 2944 (1982).
20. H. Grotch, D. A. Owen and K. J. Sebastian, Phys. Rev. D 30, 1924 (1984).
21. M. Artuso et al. [CLEO Collaboration], Phys. Rev. Lett. 94, 032001 (2005) [hep-ex/0411068].