1. Introduction

Abelian varieties over finitely generated fields are amongst the most intensively and frequently studied classical issues in algebraic geometry. Let $\overline{F}$ be a separable closure of some finitely generated extension $F$ of a prime field of any characteristic, and let $\ell$ be a prime which is invertible in $F$. Recall that for an Abelian $g$-fold $Y$ over $F$ its $\ell$-adic Tate module $T_\ell Y := \lim_{\leftarrow} Y(F)[\ell^n]$ (resp. $V_\ell Y := \mathbb{Q} \otimes T_\ell Y$) is a free $\mathbb{Z}_\ell$-module (resp. $\mathbb{Q}_\ell$-vector space) of rank $2g$. Its significance stems from an extremely interesting $\text{Gal}(\overline{F}/F)$-action thereon, and one defines the $\ell$-adic arithmetic monodromy group of $Y/F$ to be the smallest $\mathbb{Q}_\ell$-algebraic subgroup $G_\ell \subset \text{GL}(V_\ell/Y_\ell)$ containing the image of the monodromy representation:

$$\rho_\ell : \text{Gal}(\overline{F}/F) \to \text{GL}(V_\ell Y_\ell/\mathbb{Q}_\ell)$$

\[\text{References}\]

\[\text{1. Introduction}\]

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The following classical results are indispensable for further insights into $G_\ell$ and its neutral component $G_\ell^0$:

(i) $G_\ell$ is reductive and $\mathbb{Q}_\ell \otimes \text{End}(Y) \cong \text{End}_{\mathbb{Q}_\ell}(V_\ell^Y_{\overline{\mathbb{F}}})$, given that $\mathbb{Z}_\ell \otimes \text{End}(Y)$ agrees with $\text{End}_{\mathbb{Z}_\ell(\text{Gal}(\overline{\mathbb{F}}/F))}(T_\ell Y_{\overline{\mathbb{F}}})$ (cf. [17]).

(ii) The quotient $G_\ell/G_\ell^0$ is independent of $\ell$ in the following strong sense: There exists a Galois extension $F^\circ$ of $F$ such that $\text{Gal}(\overline{F}/F^\circ) = \rho_\ell^{-1}(G_\ell^0(\mathbb{Q}_\ell))$ holds for all $\ell$, so that $\text{Gal}(F^\circ/F) \cong G_\ell/G_\ell^0(\mathbb{Q}_\ell)$, notice that $G_\ell(\mathbb{Q}_\ell)/G_\ell^0(\mathbb{Q}_\ell)$ is Zariski dense in $G_\ell/G_\ell^0$ (cf. [18 Theorem 3.6]).

(iii) The image $\rho_\ell(\text{Gal}(\overline{F}/F))$ is a compact subgroup of $G_\ell(\mathbb{Q}_\ell)$ and its intersection with the $\mathbb{Q}_\ell$-valued elements of the commutator subgroup $G_\ell^{\text{der}} := [G_\ell^0, G_\ell^0]$ is open therein (cf. [18] and notice $G_\ell^0 \neq G_\ell^{\text{der}}$ because $G_\ell^0 \rightarrow \mathbb{G}_{m,\mathbb{Q}_\ell}$). In the special case $\text{char}(F) = 0$ we may choose an inclusion $\iota$ of $\overline{F}$ into an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, giving rise to the Hodge-Tate decomposition

$$\mathbb{C}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell Y_{\overline{\mathbb{F}}^\iota} \cong \mathbb{C}_\ell(1) \otimes_{\mathbb{A}_F} \text{Lie} Y \oplus \mathbb{C}_\ell \otimes_{\mathbb{A}_F} \text{Lie} Y^t,$$

where $\mathbb{C}_\ell$ is the $\ell$-adic completion of $\overline{\mathbb{Q}}_\ell$ and $Y^t$ is the dual Abelian variety. The resulting cocharacter

$$\mu : \mathbb{G}_{m,\mathbb{C}_\ell} \rightarrow G_\ell^{\text{der}}_{\mathbb{C}_\ell}$$

possesses two weights in the $\mathbb{C}_\ell$-vector space [2], namely 0 and 1. Together with certain group theoretical observations this implies severe restrictions for the structure of $G_\ell^0$ and its tautological representation: In fact, it is known that the simple components of $G_\ell^{\text{der}}$ are of type $A_n$, $B_n$, $C_n$ or $D_n$ while the irreducible summands of $\mathbb{C}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell Y_{\overline{\mathbb{F}}^\iota}$ are (tensor products of) minuscule representations (cf. [18 Corollary 5.11]). A natural conjecture, considered and studied by Yuri Zarhin, says that this microweight-phenomenon should pertain if $\text{char}(F) = p > 0$ (cf. [27 Subsection 0.4]). The focus of this note does lie on the special case $\text{char}(F) = p > 2$ and its purpose is to construct and to study certain examples for which $G_\ell^{\text{der}}$ is a non-trivial group of adjoint type. This strongly contradicts Zarhin’s conjecture, because the only minuscule representation of such groups is the trivial one, given that minuscule representations are already completely determined by their restriction to the center (cf. [11 Chapter III, Section 13, Exercise 13]).

More recently Anna Cadoret and Akio Tamagawa introduced the ghost of $Y$: This is an Abelian variety $\mathbb{V}$ over an unspecified finite field $\mathbb{F}_q \supset \mathbb{F}_p$ with the nice property that the $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$-representations $V_\ell^\mathbb{V}_{\overline{\mathbb{F}}_p}$ agree with the $\mathbb{Q}_\ell$-spaces of $T_\ell$-invariants of the Tate vector spaces $V_\ell Y^\mathbb{V}_{\overline{\mathbb{F}}_p}$ for choices of $\ell \notin \{0, p\}$ and maximal tori $T_\ell \subset G_\ell^{\text{der}}$, please see to [5 Section 6] for details. If $Y$ happens to satisfy Zarhin’s conjecture (e.g. in the ordinary case, according to [27 Corollary 4.3.1] or by [19 Corollary 6.2]), then the Galois representations $V_\ell^\mathbb{V}_{\overline{\mathbb{F}}_p}$ agree with the $\mathbb{Q}_\ell$-spaces of $G_\ell^{\text{der}}$-invariants of the Tate vector spaces $V_\ell Y^\mathbb{V}_{\overline{\mathbb{F}}_p}$ for every $\ell$. Therefore, the ghost of $Y$ is a good measure for the failure of Zarhin’s conjecture.

I thank Claudia Glanemann for encouragement, Prof. Ikeda and Prof. Goldring for many conversations on the conjectures [1 and 2] of section 4 and Prof. Cadoret for pointing out the reference [18] and for setting the stimulating problem of characterizing the ghosts of the said counterexamples. Eventually we determine their dimension and their formal isogeny types. We hope to rekindle interest and foster awareness about further peculiarities of the case char$(F) > 0$, as pointed out in [5] and [23 Question 2A].
2. First Example

In this section we prove the existence of Abelian 6-folds whose \( \ell \)-adic arithmetic monodromy groups are certain \( \mathbb{Q}_\ell \)-forms of the group \( \mathbb{G}_m \times \text{SO}(3)^2 \). Using Zarhin’s theorem one can deduce this property from the structure of their endomorphism algebras, which we manufacture by deforming a supersingular Abelian 6-fold over some algebraically closed field \( k \) of characteristic \( p > 2 \), followed by a descent to an unspecified finitely generated ground ring contained in the \( k \)-algebra of power series \( k[[t]] \). More specifically, we fix a real quadratic number field \( K \) and a quaternion algebra \( D \) over \( K \) such that \( p \) is inert and unramified in \( K \) and \( D \) splits at all but the two archimedean places of \( K \). Eventually, we use the Serre-Tate theorem to construct polarized Abelian 6-folds \( Y \) over \( k[[t]] \) such that the endomorphism algebra of the geometric fiber of \( Y \) over the generic point of \( \text{Spec} k[[t]] \) is isomorphic to \( D \), i.e. of type \( \text{III}(2) \), when using the notation of [22] Paragraph 7.2.

2.1. Some Morita equivalences. Let \( k \) and \( D/K \) be as above. Let \( \mathcal{O}_K \) (resp. \( \mathcal{O}_D \)) be the ring of integers of \( K \) (resp. a maximal order in \( D \)). Fix an embedding \( \mathbb{Z}_p \otimes \mathcal{O}_K \hookrightarrow W(k) \) and an \( \mathcal{O}_K \)-linear isomorphism \( \mathbb{Z}_p \otimes \mathcal{O}_D \cong \text{Mat}(2 \times 2, W(F_p^2)) \), where \( F_{p^2} = \{ a \in k \mid \alpha^{p^2} = a \} \cong \mathcal{O}_K/p\mathcal{O}_K \). Let \( e_1, e_2 \) and \( s \) be the elements of \( \mathbb{Z}_p \otimes \mathcal{O}_D \) corresponding to the 2 \( \times \) 2-matrices \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Now let \( Y \) be an Abelian \( g \)-fold over \( K \), let \( \lambda : Y \to Y^t \) be a polarization and assume that some action \( \iota : \mathcal{O}_D \to \text{End}(Y) \) preserves the Rosati involution \( * \). The covariant Dieudonné module \( N = \mathbb{D}(Y[p^\infty]) \) inherits a non-degenerate and skew-symmetric Weil pairing \( \psi : N \times N \to W(k) \) from the polarization \( \lambda \). The \( \text{Mat}(2 \times 2, \mathbb{Z}_p) \)-action determines a decomposition \( N = M^{\oplus 2} \), as \( s \) swaps the eigenspaces \( N[e_2] \) and \( N[e_1] \). Moreover, \( * \) must agree with the main involution on \( D \) being non-split at the archimedean places. In view of \( e_{3-i} = e_i^* \) and \( -s = s^* \) there exists a unique symmetric pairing \( \phi : M \times M \to W(k) \) such that \( \psi(x, y) \) can be written as \( \phi(x_1, y_1) - \phi(x_1, y_2), \phi(x_1, y_2) = (y_1, y_2) = y \) and \( x_1, x_2, y_1, y_2 \in M \). It goes without saying that \( \phi \) satisfies the usual relation

\[
F \phi(x, V(y)) = \phi(F(x), y),
\]

as does \( \psi \), simply because it comes from a Cartier self-duality on \( Y[p^\infty] \). At last, observe that the \( \mathcal{O}_K \)-action on \( M \) induces a further decomposition \( M = M_0 \oplus M_1 \) according to \( W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_K = W(k)^{\oplus 2} \). It is easy to see that \( F(M_0) + V(M_1) \subset M_{1-i} \) and that \( \phi \) vanishes on \( M_0 \times M_1 \).

In short: Every triple \((Y, \lambda, \iota)\) as above gives rise to a \( \mathbb{Z}/2\mathbb{Z} \)-graded Dieudonné module \( M_0 \oplus M_1 \) which is equipped with a non-degenerate symmetric pairing \( \phi \).

The \( W(k) \)-rank of each of the two eigenspaces \( M_0 \) and \( M_1 \) is just \( \frac{g}{2} \), so that \( g \) is necessarily even.

2.2. On \( \mathbb{Z}/2\mathbb{Z} \)-graded symmetric Dieudonné-modules of height 6. In the sequel we need a classification of the formal isogeny types of \( \mathbb{Z}/2\mathbb{Z} \)-graded symmetric Dieudonné modules of height 6, and it turns out that there are just two of them, namely \( G^2_{1,1} \) and \( G^0_{0,1} \oplus G^1_{1,1} \). Observe that there must be at least one occurrence of \( G^1_{1,1} \), because otherwise there would be constituents of the form \( G^3_{0,1} \) or \( G_{1,2} \), none of which allow an action of a quadratic local field. So it remains to classify formal isogeny types of \( \mathbb{Z}/2\mathbb{Z} \)-graded symmetric Dieudonné modules of height 4, which are easily seen to be \( G^2_{1,1} \) or \( G^0_{0,1} \oplus G^2_{1,0} \).
Moreover, the supersingular skeleton of a lattice with a polarized Abelian variety is a polarized Abelian variety and consider our prime example of a supersingular variety $\phi$ is anisotropic.

is supersingular and gives rise to a supersingular skeleton $I$ on which the form $\phi$ is anisotropic.

Our prime example of a supersingular $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné module of height 6 is the following: Consider $M_i = W(k)x_i \oplus W(k)y_i \oplus W(k)z_i$ for $i \in \{0, 1\}$ and take $F(x_0) = x_1, F(y_0) = y_1, z_0 = V(z_1), F(x_1) = z_0, y_1 = V(y_0), z_1 = V(x_0)$ and $(y_0, y_0) = (x_0, z_0) = (x_1, z_1) = 1$ and $(y_1, y_1) = p$ and $(z_1, z_1) = (z_1, y_1) = (x_1, y_1) = (x_i, x_i) = 0$ for $i \in \{0, 1\}$. One readily checks that its skeleton can be written as

$$I = \{p\alpha x_0 + \beta y_0 + F^2 \alpha z_0 \mid \alpha \in W(F_p^2)[\frac{1}{p}], \beta \in W(F_p^2)[\frac{1}{p}]\},$$

on which $\phi$ restricts to the quadratic form $\beta^2 + 2p\mathbb{N}(\alpha)$, so that this specimen is anisotropic.

2.3. Deformations. We proceed to the construction of a deformation, which is based on the Serre-Tate theorem [12].

**Lemma 2.1.** For any $k$ and $D/K$ as above, there exists a polarized Abelian 6-fold $(Y_0, \lambda_0)$ with Rosati invariant $\mathcal{O}_D$-action $i_0 : \mathcal{O}_D \to \text{End}(Y_0)$ whose associated $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné module $\mathcal{D}(Y_0[p^{\infty}])[e_2]$ is anisotropic.

**Proof.** Let us write $(\ldots)$ for the Hilbert symbol of the local field $\mathbb{Z}_p \otimes K = W(F_p^2)[\frac{1}{p}]$, and choose totally positive elements $a, b, c \in \mathcal{O}_K$ with $(-1, -abc) = (b, c)(a, c)(a, b)$, so that the quadratic form $ax^2 + by^2 + cz^2$ on $\mathcal{O}_K^{\oplus 3}$ is anisotropic over $\mathbb{Z}_p \otimes K$ and positive definite over $\mathbb{R} \otimes K$. Observe that the tensor product of a lattice with a polarized Abelian variety is a polarized Abelian variety and consider $(Y_0, \lambda_0) := \mathcal{O}_K^{\oplus 3} \otimes (E, \lambda_E)$, where $(E, \lambda_E)$ is a canonically principally polarized supersingular elliptic curve over $k$. Once we choose an isomorphism between $D$ and $K \otimes \text{End}(E)$ we obtain canonical actions $D \to \text{End}(Y_0)$ and $D \to \text{End}(\mathcal{O}_K \otimes E)$. Moreover, the supersingular skeleton of $\mathcal{D}(Y_0[p^{\infty}])[e_2]$ can be described as the $K$-linear tensor product of $K^{\oplus 3}$ with the supersingular skeleton of the $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné module of $\mathcal{O}_K \otimes (E, \lambda_E)$, which is of height 2. This proves our lemma, because multiplication with a non-zero element of $W(F_p^2)$ does not change the property of being anisotropic.

Since $k[[t]]$ is a complete noetherian local ring with perfect residue field of characteristic $p > 2$ there exists a convenient description of the category of $p$-divisible groups of some finite height $h$ and of some dimension $d \in \{0, \ldots, h\}$ over $k[[t]]$ in terms of so-called Dieudonné displays in the sense of [28]. To this end one must introduced a certain subring $\hat{W}(k[[t]]) = W(k) \oplus \hat{W}(tk[[t]]) \subset W(k[[t]])$ and

$$I = \{x \in \mathbb{Q} \otimes M_0 \mid F^2(x) = px\},$$

which are $\frac{2}{2}$-dimensional vector spaces over the unramified quadratic local field $W(F_p^2)[\frac{1}{p}]$. The form $\phi$ restricts to a quadratic form on $I$, in view of $F^2(x, y) = (F^2(x), F^2(y)) = (x, y)$, which follows from (3). By slight abuse of language we say that a $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné module $M$ is anisotropic if and only if $M$ is supersingular and gives rise to a supersingular skeleton $I$ on which the form $\phi$ is anisotropic. 

Since $k[[t]]$ is a complete noetherian local ring with perfect residue field of characteristic $p > 2$ there exists a convenient description of the category of $p$-divisible groups of some finite height $h$ and of some dimension $d \in \{0, \ldots, h\}$ over $k[[t]]$ in terms of so-called Dieudonné displays in the sense of [28]. To this end one must introduced a certain subring $\hat{W}(k[[t]]) = W(k) \oplus \hat{W}(tk[[t]]) \subset W(k[[t]])$ and
consider the category of quadruples $(P, Q, F, V^{-1})$ where $P$ is a free $\hat{W}(k[[t]])$-module of rank $h$, $Q \subset P$ is a submodule such that $P/Q$ is a free $k[[t]]$-module of rank $d$, $V^{-1} : P \to P$ is an $F$-linear homomorphism whose image generates $P$ as a $\hat{W}(k[[t]])$-module, and $F : P \to P$ is an $F$-linear homomorphism satisfying $V^{-1}(V a \cdot x) = aF(x)$ for all $a \in \hat{W}(k[[t]])$ and $x \in P$. Zink proves that these two categories are equivalent, moreover the Dieudonné display of the dual $p$-divisible group can be described in the usual way. At last every Dieudonné module $M$ gives rise to a Dieudonné display by putting $P := \hat{W}(k[[t]]) \otimes_{W(k)} M$ while $Q$ is the kernel of $P \to k[[t]] \otimes_k M/V M$.

**Proposition 2.2.** For any $k$ and $D/K$ as above, there exists a polarized Abelian 6-fold with Rosati invariant $\mathcal{O}_D$-action $(Y, \lambda, \iota)$ over $k[[t]]$, such that the formal isogeny type of the special (resp. generic) fiber of the $p$-divisible group $\hat{Y}[p^\infty]$ is $G_{1,1}^6$ (resp. $G_{0,1}^4 \oplus G_{1,0}^4 \oplus G_{1,1}^4$).

**Proof.** By lemma 2.1 there exists a polarized Abelian 6-fold $(Y_0, \lambda_0)$ with Rosati invariant $\mathcal{O}_D$-action $\iota_0 : \mathcal{O}_D \to \text{End}(Y_0)$ whose associated $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné module is isomorphic to our key example $M = M_0 \oplus M_1$, as given at the end of subsection 2.2. We start out from the trivial deformation, which is described by the $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné display $(P_0 \oplus P_1, Q_0 \oplus Q_1, F, V^{-1})$, essentially obtained by change of base from $W(k)$ to $\hat{W}(k[[t]])$. Now let us write $[t] \in \hat{W}(tk[[t]]) \subset \hat{W}(k[[t]])$ for the Teichmüller lift of the element $t \in k[[t]]$ and consider the $\mathbb{Z}/2\mathbb{Z}$-graded $\hat{W}(k[[t]])$-linear automorphism $U$ on $P := P_0 \oplus P_1$ which is given by:

\[
\begin{align*}
x_0 & \mapsto x_0 \\
y_0 & \mapsto y_0 + [t]x_0 \\
z_0 & \mapsto z_0 - [t]y_0 - \frac{[t]^2}{2}x_0 \\
x_1 & \mapsto x_1 \\
y_1 & \mapsto y_1 \\
z_1 & \mapsto z_1
\end{align*}
\]

Observe that $U$ respects the $\mathbb{Z}/2\mathbb{Z}$-gradation and (the prolongation to $P$ of) the form $\phi$. The deformation we wish to study arises from $(P_0 \oplus P_1, Q_0 \oplus Q_1, F, V^{-1})$ by precomposing the maps $F$ and $V^{-1}$ with $U$ (N.B.: The composition of a linear map with a $F$-linear one is a $F$-linear map). Studying the Hasse-Witt matrix, as in [20], shows that the $p$-rank of the generic fiber of our deformation is non-zero. □

**Lemma 2.3.** Let $k$ and let $D/K$ be as above. For every polarized Abelian 6-fold $(Y, \lambda)$ with Rosati invariant action $\iota : D \to \text{End}^0(Y)$ one of the following assertions holds:

1. $Y$ has complex multiplication.
2. $\text{End}^0(Y)$ is isomorphic to $D$.

Moreover, there do exist triples $(Y, \lambda, \iota)$ for which the latter case occurs (by our proposition 2.2).

**Proof.** Recall that over an algebraically closed field of positive characteristic the property of having complex multiplication is equivalent to being isogenous to an Abelian variety definable over a finite field. So let us consider a triple $(Y, \lambda, \iota)$ which
satisfies none of the two assertions above. Since supersingular Abelian varieties do have complex multiplication, we know that the formal isogeny type of \( Y[p^\infty] \) must be \( G_{2,1}^1 \oplus G_{1,0}^1 \). We claim that \( Y \) is (absolutely) simple. The occurrence of two different isogeny factors would lead straightforwardly to a decomposition \( X \times_k Z \) each of whose factors is acted on by certain orders of \( D \), where \( \dim_k X = 2 \) and \( \dim_k Z = 4 \). The formal isogeny type of \( X \) (resp. \( Z \)) must be \( G_{1,1}^1 \) (resp. \( G_{0,1}^1 \oplus G_{1,0}^1 \)). We deduce that \( X \) has complex multiplication. However the ordinary Abelian 4-fold \( Z \) with \( D \)-action has complex multiplication too, as one can see from the theory of canonical lifts and the analogous fact in characteristic 0. Having ruled out the \( X \times_k Z \)-case we proceed to whether or not \( Y \) could be a power of a single isogeny factor \( Z \), and looking at the formal isogeny type leaves no possibility but \( Y \) being isogenous to \( Z \times_k Z \). As observed in [22, Paragraph 7.2], the endomorphism algebra of an (absolutely) simple Abelian solid cannot be a definite quaternion algebra (i.e. of type III(1), when using the notation of loc.cit.). In fact all possible endomorphism types can be read off from the classification which is given there: so \( \text{End}^0(Z) \) is either equal to \( \mathbb{Q} \) or isomorphic to a totally real cubic field or a (skew) field extension of degree 2, 6 or 18 over \( \mathbb{Q} \), provided that it possesses a positive involution of the second kind (i.e. of type I(1), I(3), IV(1,1), IV(3,1) or IV(1,3)). Again we would obtain a contradiction, since \( D \) cannot be accommodated in \( \text{Mat}(2 \times 2, \text{End}^0(Z)) \) in the first three cases, while \( Z \) is of CM-type in the last two cases.

Since we checked the simplicity of \( Y \), we know that \( B := \text{End}^0(Y) \) is a skew-field. Let \( H \) be a maximal commutative sub-algebra of \( D \), which is a quartic field containing \( K \). Extending to a maximal commutative sub-algebra of \( B \) yields a field of degree 4 or 12, of which the latter is ruled out by our assumption that \( Y \) was not of CM type. We deduce that \( H \) remains maximal commutative in \( B \), so that the center of \( B \) is contained in \( K \). If \( B \) was strictly bigger than \( D \) its center would be nothing but \( \mathbb{Q} \), so that the former is just a form of \( \text{Mat}(4 \times 4, \mathbb{Q}) \). Its invariant is contained in the 2-torsion of the Brauer group of \( \mathbb{Q} \), given that the Rosati involution is an isomorphism between \( B \) and \( B^{\text{opp}} \). It follows that \( B \) has the shape \( \text{Mat}(2 \times 2, B^0) \), which contradicts with \( B \) being a skew-field. \( \square \)

24. Conclusions. Fix \( \overline{F} \supset F \supset \mathbb{F}_{p^2} \) as in the introduction and let \( D/K \) be as above and let \((Y, \lambda)\) be a polarized Abelian 6-fold over \( F \) which is equipped with a Rosati invariant action \( \iota : D \to \text{End}^0(Y) \). Let \( H_\ell \subset \text{GL}(V_{Y_{\overline{F}}} / \mathbb{Q}_\ell) \) be the \( \mathbb{Q}_\ell \)-subgroup consisting of elements which commute with the \( \mathbb{Q}_\ell \otimes D \)-action on \( V_{Y_{\overline{F}}} \) and preserve the Weil-pairing \( V_{Y_{\overline{F}}} \times V_{Y_{\overline{F}}} \to \mathbb{Q}_\ell(1) \) up to a scalar in \( \mathbb{Q}_\ell \). Observe that \( (1) \) factors canonically through a homomorphism

\[
\text{Gal}(\overline{F}/F) \to H_\ell(\mathbb{Q}_\ell)
\]

so that we may regard the \( \ell \)-adic arithmetic monodromy group of \( Y/F \) as a subgroup \( G_\ell \subset H_\ell \). Let \( H_\ell^0 \) be the neutral component of \( H_\ell \). According to subsection 2.1 we know that \( H_\ell^0 \) is isomorphic to \( \mathbb{G}_m \times \text{SO}(3)^2 \), in particular the commutator subgroup of \( H_\ell^0 \) is of adjoint type, since it is a form of \( \text{SO}(3)^2 \).

Lemma 2.4. Let \( D/K \) be as above and consider a polarized Abelian 6-fold with Rosati invariant \( \mathcal{O}_D \)-action \((Y, \lambda, \iota)\) over some finitely generated extension \( F \) of \( \mathbb{F}_{p^2} \). Then, one of the following assertions holds:
The neutral component of the $\ell$-adic arithmetic monodromy group of $Y/F$ is a torus.

Moreover, the latter case does occur.

Proof. In view of [17] we have:

\[(5) \quad \mathbb{Q}_\ell \otimes \text{End}(Y_{\overline{F}}) \cong \text{End}_{G_\ell}(V_Y)\]

In order to determine $G^0_{\ell}$ in the non-CM case, we have to establish the surjectivity of the two projections $G^0_{\ell,\overline{K}} \to \mathbb{G}_m$ and $G^0_{\ell,\overline{K}} \to \text{SO}(3)^2$. The former is clear from $\mathbb{Q}_\ell(1) \not\cong \mathbb{Q}_\ell$ and to do the latter we may assume that $G^0_{\ell,\overline{K}}$ was conjugated to a subgroup of $\mathbb{G}_m \times \text{SO}(3)$ or $\text{SO}(3) \times \mathbb{G}_m$ or the diagonal $\text{SO}(3) \subset \text{SO}(3)^2$. However, $\mathbb{Q}_\ell \otimes_{\mathbb{Q}_p} \text{End}_{G_\ell}(V_Y)_{\overline{F}}$ would be isomorphic to $\text{Mat}(2 \times 2, \mathbb{Q}_\ell)^4$ or $\text{Mat}(4 \times 4, \mathbb{Q}_\ell)$, in these three cases, thus contradicting [3] as $\mathbb{Q}_\ell \otimes \text{End}(Y_{\overline{F}}) \cong \text{Mat}(2 \times 2, \mathbb{Q}_\ell)^2$. Now let $\mathcal{Y} \to \text{Spec} \mathbb{F}_q$ be (a model of) the ghost of $Y$ in the sense of [3] Section 6, where $q$ is a sufficiently big power of $p$. We deduce a contradiction as $\text{End}^0(\mathcal{Y}_{\mathbb{F}_p}) = \text{End}^0(\mathcal{Y}) = B$.

Proceeding to the structure of $\mathcal{Y}$ we choose maximal tori $T_\ell \subset G^\text{der}_{\ell}$, for each prime $\ell \neq p$. Our Morita equivalence shows that the $G_\ell$-representation $V_Y$ gives rise to a four-dimensional space of $T_\ell$-invariants, because any maximal torus of $\text{SO}(3)$ does fix a one-dimensional subspace in its standard representation. The theorem of Tate and

\[(6) \quad V_Y \otimes_{\mathbb{F}_p} \mathbb{Q}_p \cong V_Y^{T_\ell} \]

yields homomorphisms $\mathbb{Q}_\ell \otimes \text{Mat}(2 \times 2, K) \cong \mathbb{Q}_\ell \otimes D = \mathbb{Q}_\ell \otimes B$ and proves $\text{dim}_q \mathcal{Y} = 2$. Choosing $\ell$ to be inert in $K$ demonstrates $\text{dim}_q B \geq 8$, in fact we may assume equality, because the only other possibility is $\text{dim}_q B = 16$, which would already imply the supersingularity of $\mathcal{Y}$. Known results on endomorphism algebras of Abelian surfaces imply that $\mathcal{Y}$ is the square of an elliptic curve with complex multiplication by an imaginary quadratic number field $C$, the case of a simple $\mathcal{Y}$ can be ruled out with [22] Proposition(6.1) for instance. Whence it follows $B \cong \text{Mat}(2 \times 2, C)$ and continuing to stick to the case of primes that are inert in $K$ reveals that all of them are also inert in $C$, given that $\mathbb{Q}_\ell \otimes \text{Mat}(2 \times 2, K) \cong \mathbb{Q}_\ell \otimes \text{Mat}(2 \times 2, C)$. We deduce a contradiction, namely that a real quadratic number field would agree with an imaginary one, so that we do obtain $\text{dim}_q B = 16$ together with the supersingularity of $\mathcal{Y}$. \hfill \Box

Remark 2.5. The earliest counterexamples to Zarhin’s conjecture could be extracted from [21] Example(1.5.1), which uses the Mumford-Faltings-Chai construction to produce principally polarized Abelian 5-folds $Y \to \text{Spec} \mathbb{F}_p((t))$ such that $\text{End}(Y_{\mathbb{F}_p((t))})$ is a maximal order in a quaternion algebra $D$ with $\mathbb{Q}_v \otimes D \cong \text{Mat}(2 \times 2, \mathbb{Q}_v)$ iff $v \notin \{p, \infty\}$. Upon descending to an unspecified finitely generated subfield $F \subset \mathbb{F}_p((t))$, we deduce that the space of $G^\text{der}_{\ell}$-invariants of $V_Y$ is zero whereas the space of $T_\ell$-invariants of $V_Y$ is non-zero for any maximal torus $T_\ell$ of $G^\text{der}_{\ell}$, which is the neutral component of the $\ell$-adic geometric monodromy group (of an arbitrary model over $F$) of $Y$. Indeed, if the PEL-envelope $H_\ell \subset \text{GL}(V_Y_{\overline{F}}/\mathbb{Q}_\ell)$ is defined as in the beginning of this subsection, then its neutral component would
agree with \( \mathbb{G}_m \times \text{SO}(5) \) over \( \mathbb{Q}_\ell \) and any maximal torus of \( \text{SO}(5) \) does fix a one-dimensional subspace in the standard representation. Just as in the proof of our lemma 2.4, we infer the existence of a non-trivial ghost \( Y \), so that Oort and van der Put’s example cannot satisfy Zarhin’s conjecture (cf. [5 Subsubsection 6.2.1]). Some of the results in this section were announced in my talk [2], which was inspired by a problem of Frans Oort on whether every positive rational number can be written in the form

\[
\frac{2 \dim Y}{[\text{End}^0(Y) : \mathbb{Q}]} = \text{integer},
\]

where \( Y \) runs through all simple Abelian varieties over algebraically closed extensions of \( \mathbb{F}_p \). (N.B.: If the characteristic was zero, then \( 7 \) would be a natural number, namely the dimension of the rational period lattice \( H_1(Y(\mathbb{C}), \mathbb{Q}) \) as a vector space over the skew-field \( \text{End}^0(Y) \), cf. [23 Question 2A]). The lemma 2.3 of this section gives a solution for the number \( \frac{3}{2} \) whereas [21] gives solutions for any \( \frac{q}{g} \) with \( 5 \leq g \in \mathbb{N} \).

3. Second example

In this section we obtain the existence of non-CM Abelian \( 7 \cdot 8 = 56 \)-folds whose \( \ell \)-adic geometric monodromy groups are certain \( \mathbb{Q}_\ell \)-forms of a certain number of copies of groups of type \( G_2 \). Our construction hinges on a choice of a CM field of degree \( 2 \cdot 8 = 16 \), in fact an elaboration of the method of [4] yields the following more specific result:

**Theorem 3.1.** Suppose that \( L^+ \) is a totally real number field of degree \( r > 7 \). Assume that some odd rational prime \( p \) is inert and unramified in \( L^+ \), so that \( \mathbb{Q}_p \otimes L^+ 
\cong L_\rho^+ \cong W(\mathbb{F}_{p^r})[\frac{1}{p}] \), where \( q^+ \) is the sole prime of \( L^+ \) over \( p \). Moreover, let \( L \) be a totally imaginary quadratic extension of \( L^+ \) and assume that \( q^+ \) splits in \( L \), so that \( q^+ = qq^* \), where \( * \) denotes the non-trivial element of \( \text{Gal}(L/L^+) \) and \( q \) is one of the two primes of \( L \) over \( q^+ \). Then there exists a polarized Abelian \( 7r = g \)-fold with Rosati invariant \( \mathcal{O}_L \)-action \((Y, \lambda, \iota)\) over some finitely generated extension \( F \) of \( \mathbb{F}_p \) such that:

- For every prime \( \mathfrak{p} \) of \( L \), the smallest \( \mathfrak{p} \)-algebraic subgroup of \( \text{GL}(V_\mathfrak{p}/L_\mathfrak{p}) \) containing \( \rho_\mathfrak{p}(\text{Gal}(\mathbb{F}_p/F)) \) agrees with the product of the homotheties with a simple group of type \( G_2 \) over \( L_\mathfrak{p} \), where \( V_\mathfrak{p} = \mathbb{Q} \otimes \varprojlim \ Y(F)[\mathfrak{p}^n] \).
- The formal isogeny type of \( Y[q^{\mathfrak{p}_\infty}] \) is \( G_{2r} \oplus G_{1,r-1} \oplus G_{2,r-2} \).
- The ghost (in the sense of [4] Section 6) of \( Y \) is an Abelian \( r \)-fold allowing complex multiplication by \( L \) and the formal isogeny type of its \( p \)-divisible group is \( G_{1,r-1} \oplus G_{r-1,1} \).

The proof of theorem 3.1 is explained in the subsection 3.2. Our assumption ”\( r > 7 \)” enters into a construction aiming at a description of the \( p \)-divisible group \( Y[q^{\mathfrak{p}_\infty}] \) with \( W(\mathbb{F}_{p^r}) \)-action over \( k[[t]] \). The idea is to choose \( Y[q^{\mathfrak{p}_\infty}] \) in the isogeny class of a direct sum of a constant \( p \)-divisible group of height \( 3r \) with two copies of a non-constant \( p \)-divisible group of height \( 2r \) while the dimensions of the \( \mathbb{F}_{p^r} \)-eigenspaces of \( \text{Lie} Y[q^{\mathfrak{p}_\infty}] \) are as big as possible, so that the Newton slopes of the generic fiber are the ones given in theorem 3.1. This construction is explained in the next subsection, which is an elaboration of [4 Subsection 2.2]. With a little bit of extra work theorem 3.1 can probably be proved for any \( r \geq 4 \), possibly by using
the improved method of [3]. It is tempting to speculate on the cases \( r \in \{2, 3\} \),
which could be consequences of Emerton’s \( p \)-adic variational Hodge conjecture, cf. [4, Conjecture(2.2)].
For the case \( r = 8 \), our construction was announced in the introduction of [3].

3.1. On \( \mathbb{Z}/r\mathbb{Z} \)-graded Frobenius modules with \( \text{SL}(2)_{\mathbb{K}(\mathfrak{f}, \rho)} \)-structure. We
need to introduce Zink’s windows in the generality which we are going to use,
namely over \( k[[t]] \), where \( k \) is an algebraically closed field of characteristic \( p \). Let us
write \( \mathfrak{r} \) for the Frobenius lift on \( W(k)[[t]] \) with \( \mathfrak{r}(t) = t^p \).
In this setting a Dieudonné \( W(k)[[t]] \)-window is a quadruple \( (M, M_1, \phi) \),
where \( M \) is a finitely generated free\( W(k)[[t]] \)-module, \( M_1 \subset M \) is a \( W(k)[[t]] \)-submodule such that \( M/M_1 \) is a free \( k[[t]] \)-module and \( \phi : M \to M \) is a \( \mathfrak{r} \)-linear homomorphism such that \( \phi(M_1) \) generates\( W(k)[[t]] \)-submodule \( pM \). Zink’s nilpotence condition [29, Definition 3] defines
his full subcategory of \( W(k)[[t]] \)-windows, which turns out to be equivalent to the
category of formal \( p \)-divisible groups over \( k[[t]] \), according to [29, Theorem 4]. We
will write \( BT \) for the equivalence from the former to the latter. By a \( \mathbb{Z}/r\mathbb{Z} \)-gradation
on a Dieudonné \( W(k)[[t]] \)-window \( (M, M_1, \phi) \) we mean compatible \( \mathbb{Z}/r\mathbb{Z} \)-gradations
on \( M \) and \( M_1 \) such that \( \phi \) is homogeneous of degree 1.

Lemma 3.2. Fix \( r > 7 \) and an auxiliary \( \mathbb{Z}/r\mathbb{Z} \)-graded Dieudonné module\( H = \bigoplus_{\sigma = 1}^r H_\sigma \)
of formal isogeny type \( G_{r-1,1} \). Then there exists a \( \mathbb{Z}/r\mathbb{Z} \)-graded \( W(k)[[t]] \)-
window \( \tilde{K} = \bigoplus_{\sigma = 1}^r \tilde{K}_\sigma \) whose special (resp. generic) fibre is of formal isogeny type\( G_{r-1,1}^2 \) (resp. \( G_{r-2,2} \oplus G_{1,0} \)) and a \( \mathbb{Z}/r\mathbb{Z} \)-graded \( W(k)[[t]] \)-window \( \tilde{M} = \bigoplus_{\sigma = 1}^r M_\sigma \)
which is isogenous to
\[
\tilde{K} \oplus W(k)[[t]] \trianglelefteq W(k) H^{\otimes 3}
\]
and satisfies \( \text{rk}_{k[[t]]} \tilde{M}_\sigma/\tilde{M}_{\sigma,1} \geq 6 \) for every \( \sigma \).

Proof. The assumption on the formal isogeny type of \( H = \bigoplus_{\sigma = 1}^r H_\sigma \) implies that
\( \text{rk}_{W(k)} H_\sigma = 1 \) (resp. \( \dim_k H_\sigma/\text{H}_{\sigma,1} = 1 \)) holds for every (resp. for all but one)
element \( \sigma \) of \( \mathbb{Z}/r\mathbb{Z} \), i.e.
\[
\dim_k H_\sigma/\text{H}_{\sigma,1} = \begin{cases} 
0 & \sigma \equiv \sigma_1 \pmod{r} \\
1 & \text{otherwise}
\end{cases}
\]
\[
\text{rk}_{W(k)} H_\sigma = 1
\]
for some \( \sigma_1 \in \mathbb{Z} \). Notice that the requested properties of \( \tilde{K} \) (resp. \( \tilde{M} \)) force its
special fiber to lie in the isogeny class of \( H^{\otimes 2} \) (resp. \( H^{\otimes 7} \)). Fix \( \sigma_2 \in \mathbb{Z} \) satisfying
\( 5 \leq \sigma_2 - \sigma_1 \leq r - 2 \), along with a \( \mathbb{Z}/r\mathbb{Z} \)-graded Dieudonné module \( K = \bigoplus_{\sigma = 1}^r K_\sigma \)
of formal isogeny type \( G_{r-1,1}^2 \) and satisfying
\[
\dim_k K_\sigma/K_{\sigma,1} = \begin{cases} 
1 & \sigma \equiv \sigma_1 \pmod{r} \\
2 & \text{otherwise}
\end{cases}
\]
\[
\text{rk}_{W(k)} K_\sigma = 2
\]
for every \( \sigma \). Working in the category of windows, we describe an equicharacteristic
deformation of \( K \) whose generic fiber has the formal isogeny type \( G_{r-2,2} \oplus G_{1,0} \).
We start out from \( K_\sigma := W(k)[[t]] \otimes_{W(k)} K_\sigma \) and define a new Frobenius thereon
by precomposition (of the $\tau$-linear extension to $\tilde{K}_{\sigma-1}$) of $\phi : K_{\sigma-1} \to K_{\sigma}$ with

$$U_{\sigma} := \begin{cases} 
\text{id}_{K_{\sigma}} + t \otimes N_1 & \sigma \equiv \sigma_1 \pmod{r} \\
\text{id}_{K_{\sigma}} + t \otimes N_2 & \sigma \equiv \sigma_2 \pmod{r} \\
\text{id}_{K_{\sigma}} & \text{otherwise}
\end{cases}$$

where $N_j$ denotes endomorphisms of $K_{\sigma_j}$ satisfying $\ker(N_j) = N_jK_{\sigma_j} \not\subseteq K_{\sigma_j-1}$ for $j \in \{1,2\}$. Indeed, it is known that $W(k((t))) \otimes_{W(k)}[t]$ has non-zero $p$-rank, at least for good choices of $N_1$ and $N_2$ according to [20, Proposition 4.1.4].

The definition of the window $\tilde{K}$ is not completed before one has decreed $\tilde{K}_{\sigma,1} := p\tilde{K}_{\sigma} + W(k)[[t]] \otimes_{W(k)} K_{\sigma,1}$. Our prime interest lies in $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné sublattices:

$$(8) \quad M_{\sigma} \subset K_{\sigma}^2 \oplus H_{\sigma}^2$$

We require that $M_{\sigma}$ satisfies $\dim_k M_{\sigma}/M_{\sigma,1} \geq 6$ for every $\sigma$ and that (8) is an equality for $\sigma \in \{\sigma_1, \sigma_2\}$. Let us check that lattices with these properties exist: Starting out from $M_j := K_{\sigma_j}^2 \oplus H_{\sigma_j}^2$, we observe that the $W(k)$-length of $M_j/\phi^{r_1-\sigma_j}(M_j)$ (resp. $M_j/\phi^{r_2-\sigma_j}(M_j)$) is equal to 5 (resp. equal to 2). So let us pick flags of $W(k)$-modules

$$pM_j \subseteq \phi^{r_1-\sigma_j}(M_j) = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq F_4 \subsetneq M_j = F_5 = \cdots$$

(resp. $pM_j \subseteq \phi^{r_2-\sigma_j}(M_j) = E_0 \subsetneq E_1 \subsetneq M_j = E_2 = \cdots$) and define $M_{\sigma} = \phi^{r_1-\sigma_j}(M_j)$ provided that $\sigma_1 \leq \sigma \leq \sigma_2$ (resp. $M_{\sigma} = \phi^{r_2-\sigma_j}(M_j)$), whenever $\sigma_2 - r \leq \sigma \leq \sigma_1$. We still have to construct our $\mathbb{Z}/r\mathbb{Z}$-graded $W(k)[[t]]$-window $\bigoplus_{\sigma=1}^2 M_{\sigma}$. Again, we start out from $M_{\sigma} := W(k)[[t]] \otimes_{W(k)} M_{\sigma}$ and define a new Frobenius thereon by precomposition (of the $\tau$-linear extension to $M_{\sigma-1}$) of $\phi : M_{\sigma-1} \to M_{\sigma}$ with

$$U_{\sigma} := \begin{cases} 
\text{id}_{M_{\sigma}} + t \otimes N'_1 & \sigma \equiv \sigma_1 \pmod{r} \\
\text{id}_{M_{\sigma}} + t \otimes N'_2 & \sigma \equiv \sigma_2 \pmod{r} \\
\text{id}_{M_{\sigma}} & \text{otherwise}
\end{cases}$$

where $N'_j$ denotes the endomorphism of $M_{\sigma}$ which agrees with $N_j$ on the two copies of $K_{\sigma_j}$ and vanishes on the three copies of $H_{\sigma_j}$.

In this note the terminology ”Frobenius-module” is used for pairs $(M, \phi)$ consisting of a finitely generated free $W(k)[[t]][\frac{1}{p}]$-module $M$ and an isomorphism:

$$^\tau M := W(k)[[t]][\frac{1}{p}] \otimes_{W(k)[[t]][\frac{1}{p}]} M \xrightarrow{\phi} M$$

By a $\mathbb{Z}/r\mathbb{Z}$-gradation on a Frobenius-module $(M, \phi)$ we mean a decomposition $M = \bigoplus_{\sigma=1}^2 M_{\sigma}$ satisfying $\phi(M_{\sigma}) \subset M_{\sigma+1}$. The category of $\mathbb{Z}/r\mathbb{Z}$-graded Frobenius-modules forms a $K(\mathbb{F}_p)$-linear rigid $\otimes$-category in the usual way, where $K(\mathbb{F}_p)$ denotes the field $W(k)[[t]][\frac{1}{p}]$. In the following result $\operatorname{Rep}_0(\operatorname{SL}(2)_{K(\mathbb{F}_p)})$ stands for the $K(\mathbb{F}_p)$-linear tannakian category of finite dimensional representations of the group $\operatorname{SL}(2)$ over the ground field $K(\mathbb{F}_p)$.

**Lemma 3.3.** Consider a $\mathbb{Z}/r\mathbb{Z}$-graded Frobenius-module $K = \bigoplus_{\sigma=1}^2 K_{\sigma}$. Assume that all Newton slopes of its special (resp. generic) fiber are zero (resp. non-zero) and that each $K_{\sigma}$ is free of rank 2. Then there exists a fully faithful $K(\mathbb{F}_p)$-linear
using the methods of [15, Proposition 29] there is a canonical isomorphism

\[ \Theta : K_0 \otimes_{W(k)[[t]]} K(t) \{ \{ t \} \} \xrightarrow{\cong} J \otimes_{K(F_{p^r})} K(k) \{ \{ t \} \} \]

where \( K(k) \{ \{ t \} \} \subset K(k)[[t]] \) denotes the subring of power series that converge on the open unit disc. Let \( G_0 \) be the smallest \( K(F_{p^r}) \)-subgroup of \( \text{GL}(J/K(F_{p^r})) \) containing the element:

\[ \theta := (\Theta \otimes 1) \circ (1 \otimes \Theta)^{-1} \in \text{GL}(J/K(F_{p^r}))(K(k) \{ \{ t \} \}) \otimes_{W(k)[[t]]} K(k) \{ \{ t \} \} \]

Since \( W(k)[[t]] \) is faithfully flat we can use descent theory to construct a faithful functor \( M \) from \( \text{Rep}_0(G_0) \) to the \( K(F_{p^r}) \)-linear rigid \( \otimes \)-category of \( \mathbb{Z}/r\mathbb{Z} \)-graded Frobenius-modules. Notice \( G_0 \subset \text{SL}(J/K(F_{p^r})) \), because \( \bigoplus_{\sigma=1}^r \Lambda^2 W(k)[[t]][[t]]_p^* K_\sigma \) must be constant. It remains to prove that \( G_0 \subset \text{SL}(J/K(F_{p^r})) \). Due to slope reasons we must have \( 0 < \text{dim}_{K(F_{p^r})} G_0 \), just look at the generic fiber, which is an isocrystal over the perfection of \( k((t)) \). However, the only maximal proper subgroups of \( \text{SL}(J/K(F_{p^r})) \) of positive dimension are the Borel group and (four different forms of) \( G_m \times \{ \pm 1 \} \). We leave to the reader to check that in each of these five cases the special and generic Newton-polygons of any Frobenius module with \( G_0 \)-structure would agree. \[ \square \]

3.2. **Proof of theorem 3.1** Consider the \( \mathbb{Q} \)-group \( G := \text{Aut}(C) \), where \( C \) stands for the 8-dimensional division algebra of octonions over \( \mathbb{Q} \). Notice that \( G \) splits over all non-archimedean local fields and that \( G(\mathbb{R}) \) is compact. Observe that the Lie-algebra \( g \) of \( G \) can be identified with the space of derivations of \( C \) and that the actions of both \( G \) and \( g \) preserve the positive definite symmetric norm form \( C \times C \to \mathbb{Q}; (x,y) \mapsto xy+x\bar{y} \) and the 7-dimensional subspace \( C_0 = \{ x \in C \mid \bar{x} = -x \} \) of purely imaginary octonions. The next step consists of choosing a homomorphism

\[ \pi : \text{SL}(2)_{K(F_{p^r})} \to G_{K(F_{p^r})} \]

satisfying the following:

(i) No proper \( L^+ \)-subgroup of \( G_{L^+} \) contains the image of \( \pi \).

(ii) There exists an isomorphism \( K(F_{p^r}) \otimes C_0 \cong \text{std}^3_{K(F_{p^r})} \oplus K(F_{p^r})^{\otimes 3} \) in the category of \( \text{SL}(2)_{K(F_{p^r})} \)-representations.

The irrationality property (i) may be enforced by a conjugation with a suitable element of \( G(K(F_{p^r})) \) while the conjugacy class of \( \pi \) is induced from the long simple root of some based root system of the split group \( G_{Q_2} \), alternatively one could think of our embedded \( \text{SL}(2) \) as the commutator subgroup of the Levi factor of the maximal proper standard parabolic subgroup arising from the removal of the short simple root. Let \( \overline{X} \) be a polarized Abelian variety with complex multiplication.
by $\mathcal{O}_L$ such that the formal isogeny type of $X[q^{±∞}]$ is $G_{r-1,1}$. Consider the $\mathbb{Z}/r\mathbb{Z}$-graded windows $\bar{K}$ and $\bar{M}$, as provided by lemma 3.2 when applied to the $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné module $H$ with $BT(\bigoplus_{\sigma=1}^r H_\sigma) \cong X[q^{±∞}]$. Moreover, let us write $M$ for the fully faithful $\mathbb{Z}/r\mathbb{Z}$-graded Frobenius module with $SL(2)_K(r)\sigma^r$-structure resulting from applying lemma 3.3 to $\mathbb{Q} \otimes \bigoplus_{\sigma=1}^r H_\sigma \otimes W(k) \bar{K}_\sigma$. Observe that the special fiber $BT(W(k) \otimes W(k)[t]) M$ of $BT(M)$ is canonically isogenous to $C_0 \otimes X[q^{±∞}]$, simply because part (ii) of lemma 3.3 tells us that $\mathbb{Q} \otimes M$ agrees with $M(\pi) \otimes H$ (by slight abuse of notation we may regard $\pi$ as a representation of $SL(2)_K(r)\sigma^r$) on $L_{q^r} \otimes C_0 \cong \text{std}_{k^2}^{E_2} \oplus L_{q^r}^{E_2}$. This puts us into a position allowing the use of the Serre-Tate theorem: Over $k[[t]]$ there exists a canonical $p$-principally polarized Abelian scheme $Y^{(1)}$ with Rosati-invariant $O_L$-action such that its special fiber lies in the isogeny class $C_0 \otimes X$ while:

$$Y^{(1)}[q^{±∞}] \cong BT\left(\bigoplus_{\sigma=1}^r \tilde{M}_\sigma\right)$$

The crux of our argument is the 2nd exterior power Abelian scheme of $Y^{(1)}$, which was discovered somewhat implicitly in [25, Chapter IV, Paragraph 5, Exercise 1] over $\mathbb{C}$ and was generalized and reconsidered over arbitrary $W(\mathbb{F}_{p^r})$-schemes in [3, Subsection 4.3]. Just like $Y^{(1)}$ itself, its 2nd exterior power is equipped with a $p$-principal polarization and a Rosati-invariant $O_L$-action, in fact its formation depends on another auxiliary $p$-principally polarized Abelian variety $Y^{(2)}$ over $W(k)$, such that the formal isogeny type of $Y^{(2)}[q^{±∞}]$ is $G_{l,1}^{2,0}$. If one writes $N$ for the $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné module with $BT(\bigoplus_{\sigma=1}^r N_\sigma) \cong Y^{(0)}[q^{±∞}]$, then the 2nd exterior power $Y^{(2)}$ satisfies:

$$Y^{(2)}[q^{±∞}] \cong BT\left(\bigoplus_{\sigma=1}^r (\tilde{N}_\sigma \otimes W(k) \bigwedge^2 \tilde{M}_\sigma)\right)$$

Moreover, according to [3, Proposition 5.1] we have a commutative diagram:

$$\begin{array}{ccc}
\text{sym}^2_L \text{End}_L^{(1)}(Y^{(1)} \times_{k[[t]]} k) & \longrightarrow & \text{sym}^2_{K(\mathbb{F}_{p^2})} \text{End}(W(k) \otimes W(k)[t]) \bigoplus_{\sigma=1}^r \tilde{M}_\sigma \\
\downarrow & & \downarrow \\
\text{End}_L^{(2)}(Y^{(2)} \times_{k[[t]]} k) & \longrightarrow & \text{End}(W(k) \otimes W(k)[t]) \bigoplus_{\sigma=1}^r \bigwedge^2 W(k) \tilde{M}_\sigma
\end{array}$$

where the horizontal arrows are induced from the isomorphisms (10) and (11). We have a decomposition $\bigwedge^2 C_0 \cong \mathfrak{g} \oplus C_0$, of which the projection to the first summand is sketched in [11, Chapter V, Section 19, Exercise 5] while its projection to the second summand results from the commutator of octonions. The full faithfulness of $M$ implies:

$$\begin{align*}
\text{End}_L^{(1)}(Y^{(1)}) &= \{ \alpha \in L \otimes \text{End}(C_0) \mid \{ \alpha_{q^r}, \alpha_{q^r}^* \} \subset \text{End}_{SL(2)}(L_{q^r} \otimes C_0) \} = L \otimes \text{End}_G(C_0) = L \\
\text{End}_L^{(2)}(Y^{(2)}) &= \{ \alpha \in L \otimes \text{End}(\bigwedge^2 C_0) \mid \{ \alpha_{q^r}, \alpha_{q^r}^* \} \subset \text{End}_{SL(2)}(L_{q^r} \otimes \bigwedge^2 C_0) \} = L \otimes \text{End}_G(\bigwedge^2 C_0) \cong L \oplus L,
\end{align*}$$
where \(\alpha \) (resp. \(\alpha^*\)) denotes the image of \(\alpha\) (resp. \(\alpha^*\)) in \(L_t \otimes \text{End}(C_0)\) or \(\bigwedge^2 C_0\).

The final step of the proof consists of choosing a model \(Y\) of the generic fiber of \(\mathcal{Y}^{(1)}\) over some finitely generated subfield \(F \subset k((t))\). It does no harm to assume that all endomorphisms of \(Y\) and its 2nd exterior power are defined over \(F\), and we also decree \(F\) to contain \(\mathbb{F}_{p^r}\). The result follows from applying Zarhin’s theorem to \(Y\) and its 2nd exterior power, combined with some multilinear bookkeeping of Tate modules involving \(\ell\)-adic analogs of \((11)\) and \((12)\).

The ghost \(\mathcal{Y}\) of \(Y\) must be a power of \(\mathcal{X}\), since \(\mathcal{X}^{\otimes 2}\) is isogenous to a specialization of \(Y\) while \(\mathcal{X}\) is simple, because the formal isogeny type \(G_{1,r-1} \oplus G_{r-1,1}\) cannot be written as a sum of two self-dual ones. In order to show the last assertion of theorem \([5,4]\) it remains to show that \([L : \mathbb{Q}] = \dim_{\mathbb{Q}_l} \mathcal{V}_l Y_{\mathcal{F}}^{T_l}\) for some maximal torus \(T_l \subset G^{\text{der}}\). Indeed, observe that \(G_l\) commutes with \(L\) so that:

\[
\mathcal{V}_l Y_{\mathcal{F}}^{T_l} \cong \bigoplus_{v \mid l} \mathcal{V}_l Y_{\mathcal{F}}^{T_l}
\]

Moreover, the description of the Zariski closure of \(\rho_v(\text{Gal}(\overline{F}/F))\) shows that \(\mathcal{V}_l Y_{\mathcal{F}}^{T_l}\) is a one-dimensional vector space over \(L_t\) for each \(t\) and \([L : \mathbb{Q}] = \sum_{v \mid l}[L_t : \mathbb{Q}_l]\).

**Remark 3.4.** Please see to \([9]\) for an explanation of exterior powers of one-dimensional \(p\)-divisible groups by means of a multilinear Dieudonné theory, as suggested by Richard Pink and Hadi Hedayatzadeh. Eventually, this theory has lead to a \(\left(\lim_{n \to \infty} Y^{(0)}[q^n]\right) \otimes_{\mathcal{O}_{L_q}} Y^{(2)}[q^{\infty}]\)-valued alternating pairing on \(Y^{(1)}[q^{\infty}]\), please see to \([10]\) Construction 2.5 for more general assertions.

### 4. On Two Moduli Spaces

Our two examples arose from \(\mathbb{F}_p[[t]]\)-sections in moduli spaces of Abelian varieties with a certain kind of additional structure. We round off the treatment with soberly introducing these moduli spaces, whose ties to the theory of Shimura varieties deserve further study, as initiated in \([3]\).

#### 4.1. First moduli space.**

Recall that over an arbitrary number field, isometry classes of three-dimensional quadratic spaces with discriminant 1 are classified by the sets of their anisotropic places, which are arbitrary finite sets of even cardinality. Specializing to our totally real quadratic field \(K\), we fix an embedding \(v : K \hookrightarrow \mathbb{R}\) and a quadratic space \(V\) which is anisotropic at \(v\) and isotropic at the other real embedding of \(K\). Consider an odd rational prime \(p\) which is inert and unramified in \(K\) and such that \(V\) is isotropic at the unique prime above \(p\). Furthermore, let us fix a polarized Abelian surface \(\mathcal{Y}\) over \(\mathcal{O}_K/p\mathcal{O}_K\) with a Rosati invariant \(\mathcal{O}_D\)-action, where \(\mathcal{O}_D \supset \mathcal{O}_K\) is as in subsection 2.1. The kernel of the diagonal \(Z(p) \otimes \mathcal{O}_K \otimes \mathcal{O}_K \to Z(p) \otimes \mathcal{O}_K\) is generated by a unique idempotent, which we denote by \(e\). Mimicking the formalism of \([14]\) Section 5 we introduce the locally compact rings of adeles \(\mathbb{A} := \mathbb{R} \times \mathbb{Q} \otimes \mathbb{Z}\) and \(\mathbb{A}^{\infty,p} := \mathbb{Q} \otimes \prod_{\ell \neq p} \mathbb{Z}_\ell\) and consider the \(\mathcal{O}_K/p\mathcal{O}_K\)-functor \(\mathcal{M}\) whose value on some connected \(\mathcal{O}_K/p\mathcal{O}_K\)-scheme \(S\) is given by the set of quadruples \((Y, \lambda, \iota, \eta)\) with the following properties:

- \(Y \to S\) is an Abelian 6-fold, equipped with an action \(\iota : \mathcal{O}_D \to Z(p) \otimes \text{End}(Y)\),
up to $\mathbb{Z}_p$-isogeny. Moreover, we require that $\text{Lie} \, Y[1 - e]$ (resp. $\text{Lie} \, Y[e]$) is a projective $\mathcal{O}_S$-module of rank $4$ (resp. $2$), here notice that $e$ gives rise to an idempotent in $\mathcal{O}_K \otimes \mathcal{O}_S$, so that $\text{Lie} \, Y = \text{Lie} \, Y[1 - e] \oplus \text{Lie} \, Y[e]$.

- $\lambda : Y \rightarrow Y^t$ is a $p$-integral quasipolarization (coming from a positive element in the Neron-Severi group tensorized with $\mathbb{Z}_p$) which satisfies $\lambda \circ \iota(\alpha^*) = \iota(\alpha)^t \circ \lambda$ for any $\alpha \in D$. Moreover, we require that the $p$-adic valuation of $\text{deg}(\lambda)$ is $2$, thus inducing an isogeny $Y[p^\infty] \rightarrow Y^t[p^\infty]$ of degree $p^2$.

- $\eta : V \otimes_K H^1_\ell(\mathbb{Q}_s, \mathbb{A}^\infty \otimes) \cong H^1_\ell(Y_s, \mathbb{A}^\infty \otimes)$ is a $\pi_1^\ell(S, s)$-invariant $\mathbb{A}^\infty \otimes \mathcal{O}$-linear isometry, where $s$ is an arbitrary geometric point on $S$ (N.B.: Both sides possess natural $\mathbb{A}^\infty \otimes(1)$-valued pairings).

Let $H$ be the $K$-group of $K$-linear isometries of $V$. Observe that every $g \in H(\mathbb{A}^\infty \otimes K)$ gives rise to an automorphism of $\overline{\mathcal{M}}$, as $(Y, \lambda, t, \eta)$ can be sent to $(Y, \lambda, t, \eta \circ g)$. One may speculate on whether or not $\overline{\mathcal{M}}$ possesses analogues in the theory of Rapoport-Zink spaces of PEL type in the sense of [24, Definition 3.18], but it seems hard to apply loc.cit. directly. This is due to condition (iii) of [24, Definition 3.18], which requires our associated reductive $\mathbb{Q}_p$-group to possess a cocharacter with weights $0$ and $1$ in its standard representation, thus ruling out the orthogonal group in three variables. Nevertheless, it seems worthwhile to try to adapt [24] to the case at hand and similar ones, so that one can study $\overline{\mathcal{M}}$ with an applicable notion of local model in the sense of [24, Definition 3.27]. At the face of these methods $\overline{\mathcal{M}}$ might well be formally smooth of relative dimension one over $\mathcal{O}_K/p\mathcal{O}_K$, but I conjecture that the following even nicer description is valid:

**Conjecture 1.** Let $H/K$ be as above and let us write $K_p \subset H(\mathbb{Q}_p \otimes K)$ for the stabilizer of some self-dual $\mathbb{Z}_p \otimes \mathcal{O}_K$-lattice in $\mathbb{Q}_p \otimes V$ and let $K_\infty \subset H(\mathbb{R} \otimes K)$ be the product of the neutral component of some maximal compact subgroup with the center, so that $K_\infty \cong O(3, \mathbb{R}) \times SO(2, \mathbb{R}) \times O(1, \mathbb{R})$. There exists a flat, formally smooth and universally closed $\mathbb{Z}_p \otimes \mathcal{O}_K$-scheme $\overline{\mathcal{M}}$ with $H(\mathbb{A}^\infty \otimes K)$-action such that

$$H(K) \backslash H(\mathbb{A} \otimes K) / (K_\infty \times K_p) \cong \mathcal{M}(\mathbb{C}[v])$$

$$\overline{\mathcal{M}} \cong \mathcal{M} \times_{\mathbb{Z}_p \otimes \mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$$

holds $H(\mathbb{A}^\infty \otimes K)$-equivariantly.

The complex structure on the left hand side of (13) needs an isomorphism $h$ between $\mathbb{C}^\times / \mathbb{R}^\times$ and the subgroup $SO(2, \mathbb{R})$ of $K_\infty$, so that $(\text{Res}_{K/\mathbb{Q}} H^\circ, h)$ becomes a Shimura datum with trivial weight homomorphism. In fact it turns out to be of Abelian type, so that the methods of [13] clearly yield a scheme $\mathcal{M}$ satisfying (13). The conjectured characterization of its special fiber in terms of polarized Abelian $6$-folds with $O_D$-action, i.e. (14), is inspired by the following naive heuristic:

For any perfect field $k \supset \mathcal{O}_K/p\mathcal{O}_K$ one expects the $W(k)$-points on $\mathcal{M}$ to be given by a $\mathbb{Z}(p)$-motive $M$ of rank $6$ over $\text{Spec} \, W(k)$ which is equipped with a $\mathbb{Z}(p) \otimes \mathcal{O}_K$-action, a perfect symmetric polarization $M \times M \rightarrow \mathcal{Z}(p)(0)$, and a $\mathbb{A}^\infty \otimes K$-linear isometry $\eta$ between $\mathbb{A}^\infty \otimes V$ and the étale realization $M_{\text{ét}}$ of $M$, subject to the conditions

$$\text{Fil}^1 M_{\text{ét}} = M_{\text{ét}} \neq \text{Fil}^0 M_{\text{ét}} \supset M_{\text{ét}}[e] =: M_1$$

$$\text{Fil}^2 M_{\text{ét}} = 0 \neq \text{Fil}^1 M_{\text{ét}} \subset M_{\text{ét}}[1 - e] =: M_0$$
where $M_{dR}$ stands for the de Rham realization, which is equipped with a Hodge-filtration with respect to which the eigenspace $M_1$ (resp. $M_0$) has three steps (resp. one single step). Whence it follows $p^{-1}M_0/FM_1 \cong k \oplus W(k)/p^2W(k) \cong FM_1/pM_0$ and $FM_0 = M_1$, where $F : \mathbb{Q} \otimes M_1 \to \mathbb{Q} \otimes M_{1-1}$ stands for the natural Frobenius with respect to which the $W(k)$-lattice $M_{dR}$ is strongly divisible. Observe that $M_0 \cap FM_1 \subset FM_1 \subset M_0 + FM_1$ is a self-dual chain of $W(k)$-lattices in $\mathbb{Q} \otimes M_0$ and let $N_1 \subset M_1 \subset N_1$ be its preimage under $F : \mathbb{Q} \otimes M_1 \to \mathbb{Q} \otimes M_0$, notice that $N_1/FM_0 \cong k \cong FN_1/M_0$. It does no harm to assume that $\mathfrak{g}$ (resp. the degree of its fixed polarization) satisfies $\operatorname{Lie} \mathfrak{g}[e] = 0$ (resp. has $p$-adic valuation equal to 2), so that the $\mathbb{Z}/2\mathbb{Z}$-graded symmetric Dieudonné-module $L_0 \oplus L_1 = H^1_{cris}(\mathbb{Q}/W(k))[e_2]$ has height two and fulfills the properties $FL_0 = VL_0 = L_1$ and $FL_1 = VL_1 = pL_0$ while its $W(k)(1)$-valued Weil-pairing restricts to a perfect pairing on $L_0$. By slight abuse of notation, we denote the $\mathcal{O}_K$-linear tensor product of the special fiber of $M$ with the "motive" of $\mathfrak{g}$ by $M \otimes_{\mathcal{O}_K} \mathfrak{g}$. This object is a "$\mathbb{Z}(p)$-motive" of rank 12 over $\operatorname{Spec} k$ which is equipped with a $\mathcal{O}_d \otimes \mathcal{O}_D$-action and a natural skew-symmetric polarization, and its $\mathbb{Z}/2\mathbb{Z}$-graded crystalline realization is $M_0 \otimes L_0 \oplus M_1 \otimes L_1$. Inspired by [14] Definition 5.1, one could imagine that every Dieudonné lattice in the crystalline realization of a polarized $\mathbb{Z}(p)$-motive gives rise to a $\mathbb{Z}(p)$-isogeny class of polarized Abelian varieties. Applying this principle to the Dieudonné lattice $M_0 \otimes L_0 \oplus N_1 \otimes L_1$ yields a $\mathbb{Z}(p)$-isogeny class of polarized Abelian 6-folds $(Y, \lambda)$ with an $\mathbb{Z}(p)$-isogeny $M \otimes_{\mathcal{O}_K} \mathfrak{g} \to Y$ inducing isomorphisms $M_{ct} \otimes_{\mathcal{O}_K^{\infty}} H^1_{cris}(\mathfrak{g}, \mathbb{A}^{\infty}) \cong H^1_{ct}(Y, \mathbb{A}^{\infty})$ and $M_0 \otimes L_0 \oplus N_1 \otimes L_1 \cong H^1_{cris}(Y/W(k))[e_2]$. Notice that the latter implies $\dim_k \operatorname{Lie} Y[1-\tau] = 4$ and $\dim_k \operatorname{Lie} Y[e] = 2$. 

**Remark 4.1.** It would be nice to adapt the theory of local models of [24] to investigate the moduli space of principally polarized Abelian 5-folds with an action of a maximal order in the quaternion algebra which is non-split at $p$ and $\infty$ (i.e. the example pointed out by Oort and van der Put).

### 4.2 Second moduli space

Now let $L \supset L^+ \supset \mathbb{Q}$ and $q | q^+ | p$ be as in theorem [3.1]. Since $q^*$ and $q$ are the only divisors of $q^+$, the subset of embeddings $L \hookrightarrow \overline{\mathcal{L}}_p$ which factor through a continuous $L_q^* \hookrightarrow \overline{\mathcal{L}}_p$ one forms a CM-type $\Phi(0)$ i.e. $\operatorname{Hom}(L, \overline{\mathcal{L}}_p) \setminus \Phi(0) = \Phi(0) \circ s$. Pick another CM-type $\Phi \subset \operatorname{Hom}(L, \overline{\mathcal{L}}_p)$ such that $\Phi \setminus \Phi(0)$ consists of a single embedding, say $\iota_p$. We want to fix one more auxiliary embedding, namely from $\overline{\mathcal{L}}_p$ into $\mathbb{C}$, and choose an element $-\nu^* = \nu \in L \setminus \{0\}$ such that the image of $\phi(\nu)$ in $\mathbb{C}$ lies in the lower half-plane for every $\phi \in \Phi$. Our $G_2$-examples of section 3 are harder to analyze from a moduli theoretic perspective, which is primarily due to the detour via $\operatorname{SL}(2)$ and some other choices. Still one could happily pick a finitely generated $\ast$-invariant $\mathbb{Z}(p) \otimes \mathcal{O}_L$-algebra $\mathcal{R} \subset \mathbb{Z}(p) \otimes \mathcal{O}_L^{\infty}$ satisfying $\mathcal{R}[\frac{1}{p}] = \mathbb{L}^{\otimes 2}$, along with a subset $\Omega \subseteq \mathbb{Z}/r\mathbb{Z}$ of cardinality seven and consider the $W(\mathcal{F}_p)$-scheme $\mathbb{M}_{\mathcal{R}, \Omega}$ parameterizing quintuples $(Y^{(1)}, \lambda^{(1)}, \iota^{(1)}, \iota, \eta^{(1)})$ over connected $W(\mathcal{F}_p)$-schemes $S$ with the following properties:

- $Y^{(1)} \to S$ is an Abelian $7r$-fold up to $\mathbb{Z}(p)$-isogeny and $\lambda^{(1)}$ is a homogeneous class of polarizations on $Y^{(1)}$ allowing a $p$-principal representative (thus inducing a skew-symmetric self-duality $(Y^{(1)}[p^{\infty}])^* \cong Y^{(1)}[p^{\infty}])$.
- $\iota^{(1)} : \mathcal{O}_k \to \mathbb{Z}(p) \otimes \operatorname{End}(Y^{(1)})$ is a Rosati-invariant action such that $\iota_p \sigma^*$-eigenspace of Lie $Y$ is an invertible $\mathcal{O}_S$-module (resp. vanishes) whenever $\sigma \in \Omega$ (resp. $\sigma \notin \Omega$), where $\tau$ denotes the absolute Frobenius on $L_q$. 

such that the formal isogeny types of perfect field are one of

\[ H^1_t(Y_s^{(1)}, \mathbb{A}^\infty) \] is a \( \mathbb{A}^\infty \)-linear and \( \pi^t(S, s) \)-invariant similitude, where the skew-Hermitian pairing on \( L \) is defined by \( \psi(x, y) = tr_{L/Q}(xy^*) \), the euclidean \( \mathbb{Q} \)-space of purely imaginary octonions is denoted by \( C_0 \) and \( s \) is an arbitrary geometric point on \( S \).

- \( \iota : \mathcal{R} \to \mathbb{Z} \) is a Rosati-invariant \( \mathcal{O}_L \)-linear action, where \((Y^{(2)}, \lambda^{(2)}, \iota^{(2)})\) denotes the (canonically homogeneously \( p \)-principally polarized) 2nd exterior power of \((Y^{(1)}, \lambda^{(1)}, \iota^{(1)})\), which is formed with the help of the catalyst \( Y^{(0)} \) (cf. \cite[Theorem 4.8]{Bue2005}). Moreover, we also request the \( \mathcal{R} \)-linearity of the level structure

\[
\mathbb{A}^\infty \otimes \left( \bigwedge^2 C_0 \otimes L \right) \xrightarrow{2} H^1_t(Y_s^{(0)}, \mathbb{A}^\infty) \otimes \mathbb{A}^\infty \otimes L \to H^1_t(Y_s^{(2)}, \mathbb{A}^\infty),
\]

which is naturally inherited from \( \eta^t \).

Just as in subsection 3.2 we write \( G/\mathbb{Q} \) for the \( \mathbb{R} \)-compact \( \mathbb{Q} \)-form of the simple algebraic group of type \( G_2 \) and we let \( Z \) be the \( \mathbb{Q} \)-torus arising from the kernel of

\[
G_m \times \text{Res}_{L'/\mathbb{Q}} G_m \to \text{Res}_{L^+/\mathbb{Q}}, \quad (t, a) \mapsto taa^*,
\]

notice that \( Z(\mathbb{R}) \cong C^* \times SO(2, \mathbb{R})^{-1} \). The generic fiber of \( \mathfrak{M}_{\mathcal{R}, \Omega} \) is empty, but what can be said about the \( Z(\mathbb{A}^\infty) \times G(\mathbb{A}^\infty \otimes L^+) \times \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \)-set \( \mathfrak{M}_{\mathcal{R}, \Omega}(\overline{\mathbb{F}}_p) \) or the \( Z(\mathbb{A}^\infty) \times G(\mathbb{A}^\infty \otimes L^+) \times \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \)-representations \( H^1_t(\mathfrak{M}_{\mathcal{R}, \Omega} \times \text{W}(\mathbb{F}_p)) \)?

In this direction it seems reasonable to try to replace \( \mathfrak{M}_{\mathcal{R}, \Omega} \times \text{W}(\mathbb{F}_p) \) by a smaller and more canonical variety, in the spirit of the following:

**Conjecture 2.** If \( \mathcal{R} \) is sufficiently small, then the special fiber of \( \mathfrak{M}_{\mathcal{R}, \Omega} \) contains a \( Z(\mathbb{A}^\infty) \times G(\mathbb{A}^\infty \otimes L^+) \)-invariant subvariety \( \emptyset \neq \mathfrak{m} \) of dimension 6 over \( \mathbb{F}_p \) such that the formal isogeny types of \( Y^{(1)}[\mathbb{Q}^\infty] \) at any points on \( \mathfrak{m} \) with values in a perfect field are one of

- \( G_{0,1}^2 \oplus G_{1,r-1}^2 \oplus G_{2,r-2}^2 \)
- \( G_{0,1}^2 \oplus G_{1,2r-1} \oplus G_{1,r-1} \oplus G_{3,2r-3} \oplus G_{2,r-2} \)
- \( G_{1,2r-1} \oplus G_{1,r-1} \oplus G_{3,2r-3} \)
- \( G_{1,r-1}^2 \)

giving rise to locally closed subvarieties of dimension 6, 5, 4 and 3. Furthermore, \( G(\mathbb{A}^\infty \otimes L^+) \) acts trivially on the set of irreducible components of \( \mathfrak{m} \setminus \mathbb{F}_p \).

Our guesses on the dimensions of the Newton strata stand in line with their (by \cite{Bue2005}) known purity and with known properties of affine Deligne-Lusztig varieties (cf. \cite[Conjecture 1.0.1]{Bue2005}). The expected occurrence of four Newton strata stems from their Newton cocharacters factoring through \( Z \times \text{Res}_{L^+/\mathbb{Q}} G \).

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