Erice lectures on “The status of local supersymmetry”\textsuperscript{1}

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\textbf{Abstract}

In the first lecture we review the current status of local supersymmetry. In the second lecture we focus on D=11 supergravity as the low-energy limit of M-theory and pose the questions: (1) What are the D=11 symmetries? (2) How many supersymmetries can M-theory vacua preserve?

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1 Local supersymmetry in supergravity, superstrings and M-theory

Gravity exists, so if there is any truth to supersymmetry then any realistic supersymmetry theory must eventually be enlarged to a supersymmetric theory of matter and gravitation, known as supergravity. Supersymmetry without supergravity is not an option, though it may be a good approximation at energies below the Planck Scale.

Steven Weinberg, The Quantum Theory of Fields, Volume III, Supersymmetry

1.1 Supergravity

The organizers of the school requested that I review the status of “local supersymmetry”. Since local supersymmetry represents a large chunk of the last 25 years of research in theoretical high energy physics, I will necessarily be selective. Local supersymmetry appears in supergravity, superstrings, supermembranes and M-theory. A complete treatment of strings, branes and M-theory is beyond the scope of these lectures and they will deal mostly with supergravity. In my opinion there are currently four reasons why supergravity is interesting:

1) Ten dimensional and eleven dimensional supergravity respectively describe the low energy limits of string theory and M-theory, which represent our best hope for a unification of all fundamental phenomena: particle physics, black holes and cosmology. Supergravities in lower dimensions are also important for discussing compactifications. Pending such a final theory there are less sweeping but more tractable uses of supergravity such as:

2) The gauge-theory/supergravity correspondence allows us to use our knowledge of weakly coupled five-dimensional supergravity to probe strongly coupled four-dimensional gauge theories such as QCD.

3) Cosmological solutions of supergravity hold promise of explaining inflation and the current acceleration of the universe.

4) There is still no direct experimental evidence for supersymmetry but it might be the panacea for curing the ills of non-supersymmetric theories of particles and cosmology:

The gauge hierarchy problem
Electroweak symmetry breaking
Gauge coupling unification
Cold dark matter

Baryon asymmetry

Let us recall that global supersymmetry unifies bosons and fermions by requiring that our equations be invariant under a transformation involving a constant fermionic parameter $\epsilon$ which converts boson fields $B$ to fermion fields $F$ and vice versa. Symbolically

$$\delta F = \partial B \epsilon \quad \delta B = \bar{\epsilon} F$$

(1)

Here $B$ is commuting while $F$ and $\epsilon$ are anticommuting. There can be up to 4 such supersymmetries in four spacetime dimensions: simple $N = 1$ and extended $N = 2,4$. The maximum spin allowed is $s = 1$. The maximum spacetime dimension allowed is $D = 10$ corresponding to 16 spinor components.

Local supersymmetry means that we allow $\epsilon$ to be a function of the spacetime coordinates. The massless gauge field associated with local supersymmetry is a spin 3/2 fermion, the gravitino. Interestingly enough, local supersymmetry necessarily implies invariance under general coordinate transformations and so, as its name implies, the gravitino is the superpartner of the graviton. There can be up to 8 such supersymmetries in four spacetime dimensions: simple $N = 1$ and extended $N = 2,3,4,5,6,8$. The maximum spin allowed is $s = 2$. The maximum spacetime dimension allowed is $D = 11$ corresponding to 32 spinor components.

The status of local supersymmetry is largely the status of supergravity: the supersymmetric version of general relativity discovered in 1976. This is the original reason for the popularity of supergravity: it provides a natural framework in which to unify gravity with the strong, weak and electromagnetic forces. This is the top-down approach.

Local supersymmetry played a major part in many subsequent developments such as matter coupling to supergravities, the super Higgs mechanism, anti de Sitter supergravities, BPS black holes and supersymmetric sigma-models. Many of these contributed to the phenomenological application of supergravity-induced supersymmetry breaking in the physics beyond the standard model, as well as to the connection between Yang-Mills theories and supergravity via the AdS/CFT correspondence.

It is important not only as supersymmetric extension of gravity but has also had a significant impact on other fields. In standard general relativity it has given rise to positive energy theorems and to new results in the study of black holes, extra spacetime dimensions
and cosmology.

Since local supersymmetry places an upper limit on the dimension of spacetime, it naturally suggests that we incorporate the Kaluza-Klein idea that our universe may have hidden dimensions in addition to the familiar three space and one time.

Since my job is to evaluate the status of local supersymmetry, I shall not spend much time with introductions. Rather I wish in this first lecture to explain where it stands in the grand scheme of things and to what extent the top-down approaches enumerated in (1)-(3) above and bottom-up approaches of (4) are compatible. In this connection, we note that the criterion of chirality in four dimensions means that only simple $N=1$ supersymmetry could be directly relevant to observed particles. However, such models can emerge from both simple and extended theories in higher dimensions.

Early discussions of local supersymmetry may be found in the papers of Volkov and Soroka [1, 2]. Supergravity was introduced by Ferrara, Freedman and van Nieuwenhuizen [3] and by Deser and Zumino [4]. Introductions to supersymmetry and supergravity may be found in the books by Bagger and Wess [5], Gates, Grisaru, Rocek and Siegel [6], Srivastava [7], West [8], Freund [9], Bailin and Love [10] and Weinberg [11]. See also the Physics Reports of Sohnius [12], van Nieuwenhuizen [13] and Fayet and Ferrara [14] and the review by Lykken [15].

For phenomenological applications of local supersymmetry see the lecture of Ellis [19] and the Physics Reports by Nilles [17], Nanopoulos [16], Haber and Kane [18], and Chung, Everett, Kane, King, Lykken and Wang [20]. See also the TASI lectures of Dine [21], the Les Houches lectures of Ross [22] and the review by Raby [23].

For Kaluza-Klein theories and supergravity, see the Shelter Island lectures of Witten [28], the Physics Reports by Duff, Nilsson and Pope [29], the reprint volume by Appelquist, Chodos and Freund [30], the books by Castellani, D’Auria and Fre [31] and Salam and Sezgin [32] and the reviews by Duff [33] [34].

1.2 String theory

To paraphrase Weinberg:

Supergravity is itself only an effective nonrenormalizable theory which breaks down at the Planck energies. So if there is any truth to supersymmetry then any realistic theory must eventually be enlarged to superstrings which are ultraviolet finite. Supersymmetry without
superstrings is not an option.

Following the 1984 superstring revolution, the emphasis in the search for a final theory shifted away from the spacetime aspects of supergravity towards the two-dimensions of the string worldsheet. The five consistent superstrings: Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$ all feature spacetime local supersymmetry in ten dimensions. It plays a crucial part in discussions of superstring compactification from ten to four dimensions and, inter alia, has also stimulated research in pure mathematics, for example Calabi-Yau manifolds and manifolds of exceptional holonomy.

Introductions to string theory may be found in the books by Green, Schwarz and Witten [35] and Polchinski [36].

1.3 M-theory

To paraphrase Weinberg again:

_Superstring theory is itself only a perturbative theory which breaks down at strong coupling. So if there is any truth to supersymmetry then any realistic theory must eventually be enlarged to the non-perturbative M-theory, a theory involving higher dimensional extended objects: the super p-branes. Supersymmetry without M-theory is not an option._

In 1995 it was realized that a non-perturbative unification of the five consistent superstring theories is provided by M-theory, whose low-energy limit is eleven-dimensional supergravity. In addition to strings, M-theory involves p-dimensional extended objects, namely the p-branes which couple to the background fields of D=11 supergravity. This resolved the old mystery of why local supersymmetry allows a maximum of eleven dimensions while superstrings stop at ten. Indeed, many of the p-branes were first understood as classical solutions of the supergravity field equations. As a result, supergravity has returned to center stage.

M-theory is regarded by many as the dreamed-of final theory and has accordingly received an enormous amount of attention. It is curious, therefore, that two of the most basic questions of M-theory have until now remained unanswered:

i) **What are the D=11 symmetries?**

In the section we will argue that the equations of M-theory possess previously unidentified hidden spacetime (timelike and null) symmetries in addition to the well-known hidden internal (spacelike) symmetries. For $11 \geq d \geq 3$, these coincide with the general-
ized structure groups discussed below and take the form $\mathcal{G} = \text{SO}(d-1,1) \times G(\text{spacelike})$, $\mathcal{G} = \text{ISO}(d-1) \times G(\text{null})$ and $\mathcal{G} = \text{SO}(d) \times G(\text{timelike})$ with $1 \leq d < 11$. For example, $G(\text{spacelike}) = \text{SO}(16)$, $G(\text{null}) = [\text{SU}(8) \times \text{U}(1)] \ltimes \mathbb{R}^{56}$ and $G(\text{timelike}) = \text{SO}^*(16)$ when $d = 3$. The nomenclature derives from the fact that these symmetries also show up in the spacelike, null and timelike dimensional reductions of the theory. However, we emphasize that we are proposing them as background-independent symmetries of the full unreduced and untruncated $D = 11$ equations of motion, not merely their dimensional reduction. Although extending spacetime symmetries, there is no conflict with the Coleman-Mandula theorem. A more speculative idea is that there exists a yet-to-be-discovered version of $D = 11$ supergravity or $M$-theory that displays even bigger hidden symmetries corresponding to $\mathcal{G}$ with $d \leq 3$ which could be as large as $\text{SL}(32, \mathbb{R})$.

**ii) How many supersymmetries can vacua of $M$-theory preserve?**

The equations of $M$-theory display the maximum number of supersymmetries $N=32$, and so $n$, the number of supersymmetries preserved by a particular vacuum, must be some integer between 0 and 32. But are some values of $n$ forbidden and, if so, which ones? For quite some time it was widely believed that, aside from the maximal $n = 32$, $n$ is restricted to $0 \leq n \leq 16$ with $n = 16$ being realized by the fundamental BPS objects of $M$-theory: the M2-brane, the M5-brane, the M-wave and the M-monopole. The subsequent discovery of intersecting brane configurations with $n = 0, 1, 2, 3, 4, 5, 6, 8, 16$ lent credence to this argument. On the other hand, it has been shown that all values $0 \leq n \leq 32$ are in principle allowed by the $M$-theory algebra discussed in section 4.1, and examples of vacua with $16 < n < 32$ have indeed since been found. In fact, the values of $n$ that have been found “experimentally” to date are: $n = 0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32$.

In $M$-theory vacua with vanishing 4-form $F(4)$, one can invoke the ordinary Riemannian holonomy $H \subset \text{SO}(10,1)$ to account for unbroken supersymmetries $n = 1, 2, 3, 4, 6, 8, 16, 32$. To explain the more exotic fractions of supersymmetry, in particular $16 < n < 32$, we need to generalize the notion of holonomy to accommodate non-zero $F(4)$. In section 6 we show that the number of supersymmetries preserved by an $M$-theory vacuum is given by the number of singlets appearing in the decomposition of the 32-dimensional representation of $\mathcal{G}$ under $\mathcal{G} \supset \mathcal{H}$ where $\mathcal{G}$ are generalized structure groups that replace $\text{SO}(1,10)$ and $\mathcal{H}$ are generalized holonomy groups. In general we require the maximal $\mathcal{G}$, namely $\text{SL}(32, \mathbb{R})$, but smaller $\mathcal{G}$ appear in special cases such as product manifolds.
Reviews of $M$-theory may be found in the paper by Schwarz [38], the paper by Duff [39], the book by Duff [40], the lectures of Townsend [41] and the books by Kaku [42, 43]. Reviews on supermembranes are given in the Physics reports of Duff, Khuri and Lu [37], the TASI lectures by Duff [46] and the papers by Duff [44, 45] and Stelle [47], the books by Polchinski [36], Johnson [48] and Ortin [49].

2 Simple supersymmetry in four dimensions

2.1 The algebra

The $N = 1$ supersymmetry algebra takes the form

$$\{Q_\alpha, Q_\beta\} = 2(\gamma_a C)_{\alpha\beta} P^\mu$$

$$[Q_\alpha, P_\mu] = 0$$

$$[Q_\alpha, J_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\beta^\alpha Q_\beta$$

$$[Q_\alpha, R] = i(\gamma_5)_\beta^\alpha Q_\beta$$

(2)

together with the commutation relations of the Poincare group.

2.2 Wess-Zumino model

The simplest representation of this algebra is provided by the Wess-Zumino multiplet which consists of 2 scalars $A$ and $B$, a 4-component fermion $\chi$ and two auxiliary fields $F$ and $G$.

The free Wess-Zumino Lagrangian is given by

$$\mathcal{L}_{WZ} = -\frac{1}{2} \left[ (\partial_\mu A)^2 + (\partial_\mu B)^2 + \bar{\chi}\gamma^\mu \partial_\mu \chi - F^2 - G^2 \right]$$

The action is invariant under the supersymmetry transformations

$$\delta A = \frac{1}{2} \bar{\epsilon} \chi$$

$$\delta B = -\frac{1}{2} \bar{\epsilon} \gamma_5 \chi$$

$$\delta \chi = \frac{1}{2} \left[ \gamma^\mu \partial_\mu (A - i\gamma_5 B) + (F + i\gamma_5 G) \right] \epsilon$$

$$\delta F = \frac{1}{2} \bar{\epsilon} \gamma_5 \partial_\mu \chi$$
\[ \delta G = \frac{1}{2} \bar{\epsilon} \gamma^\mu \partial_\mu \chi \]  

(3)

It is now easy to see why supersymmetry is sometimes called “the square root of a translation”. For example

\[ [\delta_1, \delta_2] A = a^\mu \partial_\mu A \]  

(4)

where

\[ a^\mu = \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \]  

(5)

### 2.3 Super Yang-Mills

Another representation is provided by the vector multiplet which consists of a set of vectors \( A^i_\mu \), fermions \( \lambda^i \) and auxiliary fields \( D^i \). The Yang-Mills Lagrangian is given by

\[ L_{YM} = -\frac{1}{4} (F_{\mu\nu}^i)^2 - \frac{1}{2} \bar{\lambda}^i \partial \lambda^i + \frac{1}{2} (D^i)^2 \]  

(6)

The action is invariant under the supersymmetry transformations

\[ \delta A^i_\mu = \bar{\epsilon} \gamma_\mu \lambda^i \]

\[ \delta \lambda^i = \left(-\frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu}^i + i \gamma_5 D^i \right) \epsilon \]

\[ \delta D^i = i \bar{\epsilon} \gamma_5 \partial \lambda^i \]  

(7)

where

\[ F_{\mu\nu}^i = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g c^i_{jk} A^j_\mu A^k_\nu \]  

(8)

### 2.4 Simple supergravity

Finally we come to the tensor multiplet consisting of a vierbein \( e_a^\mu \), a gravitino \( \psi_\mu \) and auxiliary fields \( b_\mu \), \( M \) and \( N \). The supergravity lagrangian is

\[ L_{SUGRA} = \frac{1}{2\kappa^2} R - \frac{1}{2} \bar{\psi}_\mu R^\mu - \frac{1}{3} \epsilon (M^2 + N^2 - b_\mu b^\mu) \]  

(9)

where

\[ R = R_{\mu\nu} e_a^\mu e_b^\nu \]  

(10)

and

\[ \frac{1}{4} R_{\mu\nu}^{ab} \sigma_{ab} = [D_\mu, D_\nu] \]  

(11)
The transformations are now those of local supersymmetry where $\epsilon = \epsilon(x)$:

$$\delta e^a_\mu = \kappa \bar{\epsilon} \gamma^a \psi_\mu$$

$$\delta \psi_\mu = 2\kappa^{-1} D_\mu (w(e, \psi)) \bar{\epsilon} + i \gamma_5 \left( b_\mu - \frac{1}{3} \gamma_\mu \hat{b} \right) \bar{\epsilon} - \frac{1}{3} \gamma_\mu (M + i \gamma_5 N) \bar{\epsilon}$$

$$\delta M = -\frac{1}{2} e^{-1} \bar{\epsilon} \gamma_\mu R^\mu - \frac{\kappa}{2} i \bar{\epsilon} \gamma_5 b_\nu b^\nu - \kappa \bar{\epsilon} \gamma_\mu \psi_\nu M - \frac{\kappa}{2} \bar{\epsilon} (M + i \gamma_5 N) \gamma^\mu \psi_\mu$$

$$\delta N = -\frac{1}{2} e^{-1} \bar{\epsilon} \gamma_5 \gamma_\mu R^\mu + \frac{\kappa}{2} i \bar{\epsilon} \gamma_5 b_\nu b^\nu - \kappa \bar{\epsilon} \gamma_\mu \psi_\nu N - \frac{\kappa}{2} i \bar{\epsilon} \gamma_5 (M + i \gamma_5 N) \gamma^\mu \psi_\mu$$

$$\delta b_\mu = \frac{3i}{2} e^{-1} \bar{\epsilon} \gamma_5 \left( g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu \right) R^{\mu\nu} + \kappa \bar{\epsilon} \gamma_\mu \psi_\nu b_\mu - \frac{\kappa}{2} \bar{\epsilon} \gamma_\mu \psi_\nu b_\mu - \frac{\kappa}{2} i \bar{\epsilon} \gamma_5 (M + i \gamma_5 N) \bar{\epsilon} - \frac{ie}{4} \bar{\epsilon} \gamma_{\mu\nu\rho\sigma} b_{\mu} b_{\nu} \bar{\epsilon} \gamma_\rho \gamma_\sigma \psi_d$$

where

$$R^\mu = \epsilon^{\mu\nu\rho\kappa} i \gamma_5 \gamma_\nu D_\rho (w(e, \psi)) \gamma_\kappa$$

$$D_\mu (w(e, \psi)) = \partial_\mu + \frac{1}{4} w_{\mu ab} \sigma^{ab}$$

and

$$w_{\mu ab} = \frac{1}{2} \epsilon^a_\nu \left( \partial_\mu e_{b\nu} - \partial_\nu e_{a\mu} \right) - \frac{1}{2} \epsilon^a_\nu \left( \partial_\mu e_{b\nu} - \partial_\nu e_{a\mu} \right)$$

$$- \frac{1}{2} \epsilon^a_\rho \epsilon^\sigma_b \left( \partial_\mu e_{c\sigma} - \partial_\sigma e_{a \mu} \right) e^c_\mu$$

$$+ \frac{\kappa^2}{4} \left( \bar{\psi}_\mu \gamma_a \psi_b + \bar{\psi}_a \gamma_\mu \psi_b - \bar{\psi}_\mu \gamma_b \psi_a \right)$$

(15)

2.5 Off-shell versus on-shell

Since the auxiliary fields $F$, $G$, $D^i$, $b_\mu$, $M$ and $N$ enter only algebraically in the Lagrangians, they may be eliminated by their equations of motion, if so desired. With the auxiliary fields, however, the algebra closes off-shell whereas it closes only on-shell without them. It is useful to count the number of degrees of freedom in both cases.

Off-shell: For the Wess-Zumino multiplet, $A$ and $B$ each count 1, $\chi$ counts 4 and $F$ and $G$ each count 1, making 4 bose and 4 fermi in total. For the vector multiplet, $A_\mu$ counts 3, $\lambda$ counts 4 and $D$ counts 1, making 4 bose and 4 fermi in total. For the supergravity multiplet, $e^a_\mu$ counts 16 $- 10 = 6$, $\psi_\mu$ counts 16 $- 4 = 12$, $b_\mu$ counts 4 and $M$ and $N$ each count 1, making 12 bose plus 12 fermi in total.

On-shell: For the Wess-Zumino multiplet, $A$ and $B$ each count 1, $\chi$ counts 2 and $F$ and $G$ each count 0, making 2 bose and 2 fermi in total. For the vector multiplet, $A_\mu$ counts 2, $\lambda$
counts 2 and $D$ counts 0, making 2 bose and 2 fermi in total. For the supergravity multiplet, $e^\mu_a$ counts 2, $\psi_\mu$ counts 2, $b_\mu$ counts 0 and $M$ and $N$ each count 0, making 2 bose plus 2 fermi in total.

Note that supersymmetry always requires equal number of bose and fermi degrees of freedom both off-shell and on-shell.

### 2.6 Particle phenomenology

The requirement of chirality limits us to $N = 1$ and the most general such theory consists of $N = 1$ supergravity coupled to $N = 1$ Yang-Mills and $N = 1$ chiral multiplets. This theory is characterized by three functions of the chiral multiplets: the superpotential $W$, the Kahler potential $K$ and the gauge function $f$. The function $f$ is real while $W$ and $K$ are holomorphic.

Within this framework, one might wish to embed the standard model gauge groups $SU(3) \times SU(2) \times U(1)$ and three families of quarks and leptons. Of course this immediately doubles the number of elementary particles, since every particle we know of acquires a superpartner, none of which can be identified with a known particle. These have names like gauginos (winos, zinos, photinos and gluinos), higgsinos, squarks and sleptons. Moreover, unbroken supersymmetry implies that these superpartners are degenerate in mass with the known particles in obvious disagreement with experiment. In any realistic theory, therefore, supersymmetry must be broken. Since the equations of motion of the only known quantum consistent theories of gravity are supersymmetric, this breaking must be spontaneous. However, the resulting low-energy theory can be represented by a globally supersymmetric Lagrangian $L_{\text{soft}}$ with explicit but soft breaking terms. By soft we mean operators of dimensions 2 or 3. The bottom-up approach is thus to write down such a minimal supersymmetric standard model (MSSM) with mass parameters that are typically of the order of the electroweak to TeV scale. Counting the masses, coupling constants and phases, the most general such model has 124 parameters. Of course, experiment can provide constraints. Its claimed successes include resolutions of: the technical gauge hierarchy problem, the electroweak symmetry breaking problem, the gauge coupling unification problem, the cold dark matter problem and the baryon asymmetry problem.

In the literature, there is a plethora of different top-down proposals for how this spontaneous supersymmetry breaking may come about. The obvious tree-level TeV breaking in
which either the F or D auxiliary fields acquire vacuum expectation values seems to be ruled out by experiment. One alternative is the hidden sector framework where the theory can be split into two sectors with no renormalizable couplings between them: an observable sector containing the SM model particles and their superpartners, and hidden sector in which supersymmetry is broken by a dynamical mechanism such as gaugino condensation. The scale of supersymmetry breaking $M_S$ is hierarchically higher than a TeV.

There are various versions of these hidden sector models: gravity mediated models, gauge mediated models, bulk mediated models. In the latter scenario, the observable and hidden sectors reside on different branes embedded in a bulk spacetime of higher dimension.

Another alternative is D-term breaking which arises in extensions of the MSSM to GUTs or strings.

The hope, of course, is that the correct mechanism will be selected by the fundamental theory but owing to the vacuum degeneracy problem, there has been very little progress in this respect. In fact, neither string theory nor M-theory has yet been able to fix any of the 124 parameters.

## 3 Extended supersymmetry

### 3.1 The algebra

To discuss extended supersymmetry, it is more convenient to rewrite the (anti)commutation relations \[^2\] in terms of two-component Weyl spinors $Q_\alpha$ and $\bar{Q}_\dot{\alpha}$

\[
\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\dot{\alpha}, \bar{Q}_\dot{\beta}\} = 0 \\
\{Q_\alpha, \bar{Q}_\dot{\beta}\} = 2\sigma^{\mu}_{\alpha\dot{\beta}} P_\mu \\
[Q_\alpha, P_\mu] = [\bar{Q}_\dot{\alpha}, P_\mu] = 0
\]

(16)

in which dotted and undotted indices take the values $\alpha, \dot{\alpha} = 1, 2$.

We now allow for a set of $Q_\alpha$, labelled by an index $L$, which transform according to some representation of a compact Lie group $G$, and $\bar{Q}_{\dot{\alpha}}^L = (Q^L_\alpha)^*$ which transform according to the complex conjugate representation. The simple supersymmetry algebra (16) now generalizes to the extended supersymmetry algebra

\[
\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM}
\]
Table 1: Multiplicities for massless irreducible representations with maximal helicity 1 or less

\[
\begin{array}{|c|cccc|}
\hline
N & 1 & 1 & 2 & 2 & 4 \\
\hline
\text{Spin} & 1 & 1 & 2 & 2 & 4 \\
\text{Spin 1} & 1 & 1 & 2 & 2 & 4 \\
\text{Spin \( \frac{1}{2} \)} & 2 & - & 2 & 4 & 6 \\
\hline
\end{array}
\]

\[
\begin{align*}
\{\bar{Q}^\alpha_L, Q^\beta_M\} &= \epsilon_{\alpha\beta} Z^{LM} \\
\{Q^\alpha_L, Q^\beta_M\} &= 2\delta^{LM} \sigma^\mu_{\alpha\beta} P_\mu \\
[Q^L_\alpha, P_\mu] &= [\bar{Q}^L_\alpha, P_\mu] = 0 \\
[Q^L_\alpha, B_l] &= iS_l^{LM} Q^M_\alpha \\
[B_l, B_m] &= if_{lmk} B_k
\end{align*}
\]

where \( S_l^{LM} \) are the hermitian matrices of the representation containing the \( Q^L_\alpha \) and the \( B_k \) are the generators of the internal symmetry group \( G \). The \( Z^{LM} \) are central charges which commute with all the other generators.

### 3.2 Multiplets

We shall not detail the representation theory of the extended supersymmetry algebra (17) but simply quote some results. Massless irreducible representations with maximum helicity 1 and 2 are tabulated in Tables 1 and 2 respectively. Some massive representations with and without central charges are tabulated in Tables 3 and 4.

Discussions of representations of extended supersymmetry may be found in the Trieste Lectures of Ferrara and Savoy [51] and in the review of Strathdee [50].

### 3.3 Auxiliary fields?

When we come to extended supersymmetry and higher dimensions, the off-shell formalism is not always available. In \( D = 4 \), the finite set of auxiliary fields has been worked out only for \( N = 1 \) and \( N = 2 \) multiplets and some \( N = 4 \) supergravity/matter combinations. No
| N | Spin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|------|---|---|---|---|---|---|---|---|
| Spin 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Spin $\frac{3}{2}$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 |
| Spin 1 | 1 | 3 | 6 | 10 | 16 | 28 | 28 | 28 |
| Spin $\frac{1}{2}$ | 1 | 4 | 11 | 26 | 56 | 56 | 56 |
| Spin 0 | 2 | 10 | 30 | 70 | 70 | 70 |

Table 2: Multiplicity for massless on-shell representations with maximal helicity 2.

| N | Spin | 1 | 2 | 3 | 4 |
|---|------|---|---|---|---|
| Spin 2 | 1 | 1 | 1 | 1 | 1 |
| Spin $\frac{3}{2}$ | 1 | 2 | 1 | 4 | 5 + 1 |
| Spin 1 | 1 | 2 | 1 | 4 | 5 + 1 |
| Spin $\frac{1}{2}$ | 1 | 2 | 1 | 4 | 5 + 1 |
| Spin 0 | 2 | 1 | 5 | 4 | 1 |

Table 3: Some massive representations (without central charges) labelled in terms of the $USp(2N)$ representations.

theory beyond half-maximal has an off-shell formulation with a finite number of auxiliary fields. Harmonic superspace can extend the range but at the price of an infinite number. There is no known off-shell formulation for the maximally supersymmetric theories. This is a drawback since non-renormalization theorems are most transparent in the off-shell formalism. For example, the finiteness of the maximally supersymmetric N=4 Yang-Mills theory leads one to wonder whether the maximally supersymmetric N=8 supergravity might also have some peculiar ultraviolet properties.

The absence of a complete off-shell formalism also remains something of a mystery: is there some deeper meaning to all this?

Early discussions of ultraviolet divergences in extended supergravity may be found in the Trieste Lectures by Duff [24] and the paper by Howe and Lindstrom [25], and up-to-date
| \(N\) | 2 | 4 | 6 | 8 |
|-------|---|---|---|---|
| Spin 2 |   |   | 1 | 1 |
| Spin \(\frac{3}{2}\) |   | 1 | 1 | 6 |
| Spin 1 | 1 | 1 | 4 | 8 |
| Spin \(\frac{1}{2}\) | 1 | 2 | 4 | 14 | 14 | 14 | 14 | 6 | 27 |
| Spin 0 | 2 | 1 | 5 | 4 | 14 | 14 | 42 |

Table 4: Some massive representations with one central charge (\(|Z| = m\)). All states are complex.

ones in the review by Bern at al [26] and the paper by Howe and Stelle [27].

4 Eleven dimensions

4.1 The algebra

Eleven is the maximum spacetime dimension in which one can formulate a consistent supersymmetric theory, as was first recognized by Nahm in his classification of supersymmetry algebras. The easiest way to see this is to start in four dimensions and note that one supersymmetry relates states differing by one half unit of helicity. If we now make the reasonable assumption that there be no massless particles with spins greater than two, then we can allow up to a maximum of \(N = 8\) supersymmetries taking us from helicity \(-2\) through to helicity \(+2\). Since the minimal supersymmetry generator is a Majorana spinor with four off-shell components, this means a total of 32 spinor components. Now in a spacetime with \(D\) dimensions and signature \((1, D−1)\), the maximum value of \(D\) admitting a 32 component spinor is \(D = 11\). (Going to \(D = 12\), for example, would require 64 components.) See Table [7]. Furthermore, \(D = 11\) emerges naturally as the maximum dimension admitting supersymmetric extended objects.

3Conventions differ on how to count the supersymmetries and the more usual conventions are that \(N_{\text{max}} = 8\) in \(D = 5\) and \(N_{\text{max}} = 4\) in \(D = 7\)
| Dimension $(D \text{ or } d)$ | Minimal Spinor $(M \text{ or } m)$ | Supersymmetry $(N \text{ or } n)$ |
|-----------------------------|-----------------------------|-----------------------------|
| 11                          | 32                          | 1                           |
| 10                          | 16                          | 2, 1                        |
| 9                           | 16                          | 2, 1                        |
| 8                           | 16                          | 2, 1                        |
| 7                           | 16                          | 2, 1                        |
| 6                           | 8                           | 4, 3, 2, 1                  |
| 5                           | 8                           | 4, 3, 2, 1                  |
| 4                           | 4                           | 8, ..., 1                   |
| 3                           | 2                           | 16, ..., 1                  |
| 2                           | 1                           | 32, ..., 1                  |

Table 5: Minimal spinor components and supersymmetries.

The full D=11 supertranslation algebra is

$$\{Q_\alpha, Q_\beta\} = (CT^M)_{\alpha \beta} P_M + (CT_{MN})_{\alpha \beta} Z^{MN} + (CT_{MNPQR})_{\alpha \beta} Z^{MNPQR}.$$  \hspace{1cm} (18)

Note that the total number of algebraically independent charges that could appear on the right hand side is 528. The number actually appearing is

$$11 + 55 + 462 = 528$$  \hspace{1cm} (19)

so the algebra (18) is ‘maximally extended’. The three types of charge appearing on the right hand side are those associated with the supergraviton, the supermembrane and the superfivebrane, which are the three basic ingredients of M-theory. The time components $Z_0I$ and $Z_0IJKL$ are associated with the 8-brane and 6-brane of Type IIA theory that arise on compactification to D=10.

The M-theory algebra is treated in the papers by Townsend[53] and Gauntlett and Hull[52].

4.2 The multiplet

Not long after Nahm’s paper, Cremmer, Julia and Scherk realized that supergravity not only permits up to seven extra dimensions but in fact takes its simplest and most elegant
Table 6: On-shell degrees of freedom in $D$ dimensions. $\alpha = D/2$ if $D$ is even, $\alpha = (D-1)/2$ if $D$ is odd. We assume Majorana fermions and divide by two if the fermion is Majorana-Weyl. Similarly, we assume real bosons and divide by two if the tensor field strength is self-dual.

| Field          | Degree of Freedom |
|----------------|-------------------|
| $d$-bein      | $e^{A}_{M} / 4D(D-3)$ |
| gravitino     | $\Psi_{M} / 2(\alpha-1)(D-3)$ |
| $p$-form      | $A_{M_{1}M_{2}...M_{P}} / 2^{(\alpha-1)}$ |
| spinor        | $\chi / 2^{(\alpha-1)}$ |

form when written in its full eleven-dimensional glory. The unique $D=11, N=1$ supermultiplet is comprised of a graviton $g_{MN}$, a gravitino $\psi_{M}$ and 3-form gauge field $A_{MNP}$ with 44, 128 and 84 physical degrees of freedom, respectively. For a counting of on-shell degrees of freedom in higher dimensions, see Table 6. The theory may also be formulated in superspace. Ironically, however, these extra dimensions were not at first taken seriously but rather regarded merely as a useful device for deriving supergravities in four dimensions. Indeed $D=4, N=8$ supergravity was first obtained by Cremmer and Julia via the process of dimensional reduction i.e. by requiring that all the fields of $D=11, N=1$ supergravity be independent of the extra seven coordinates.

### 4.3 D=11 supergravity

For future reference we record the bosonic field equations

$$ R_{MN} = \frac{1}{12} \left( F_{MPQR} F_{N}^{PQR} - \frac{1}{12} g_{MN} F^{PQRS} F_{PQRS} \right) $$

and

$$ d * F_{(4)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0, $$

where $F_{(4)} = dA_{(3)}$. The supersymmetry transformation rule of the gravitino reduces in a purely bosonic background to

$$ \delta \Psi_{M} = \mathcal{D}_{M} \epsilon, $$

where the parameter $\epsilon$ is a 32-component anticommuting spinor, and where

$$ \mathcal{D}_{M} = D_{M} - \frac{1}{288} (\Gamma_{M}^{NPQR} - 8 \delta_{M}^{N} \Gamma^{PQR}) F_{NPQR}, $$

16
where $\Gamma^A$ are the $D = 11$ Dirac matrices and $\Gamma_{AB} = \Gamma_{[A} \Gamma_{B]}$. Here $D_M$ is the usual Riemannian covariant derivative involving the connection $\omega_M$ of the usual structure group $\text{Spin}(10,1)$, the double cover of $\text{SO}(10,1)$,

$$D_M = \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB}$$  \tag{24}$$

For many years the Kaluza-Klein idea of taking extra dimensions seriously was largely forgotten but the arrival of eleven-dimensional supergravity provided the missing impetus. The kind of four-dimensional world we end up with depends on how we compactify these extra dimensions: maybe seven of them would allow us to give a gravitational origin, a la Kaluza-Klein, to the strong and weak forces as well as the electromagnetic. In a very influential paper, Witten drew attention to the fact that in such a scheme the four-dimensional gauge group is determined by the isometry group of the compact manifold $\mathcal{K}$. Moreover, he proved (what to this day seems to be merely a gigantic coincidence) that seven is not only the maximum dimension of $\mathcal{K}$ permitted by supersymmetry but the minimum needed for the isometry group to coincide with the standard model gauge group $SU(3) \times SU(2) \times U(1)$.

In the early 80’s there was great interest in four-dimensional $N$-extended supergravities for which the global $SO(N)$ is promoted to a gauge symmetry. In these theories the underlying supersymmetry algebra is no longer Poincare but rather anti-de Sitter ($AdS_4$) and the Lagrangian has a non-vanishing cosmological constant $\Lambda$ proportional to the square of the gauge coupling constant $g$:

$$G\Lambda \sim -g^2$$ \tag{25}$$

where $G$ is Newton’s constant. The $N > 4$ gauged supergravities were particularly interesting since the cosmological constant $\Lambda$ does not get renormalized and hence the $SO(N)$ gauge symmetry has vanishing $\beta$-function\footnote{For $N \leq 4$, the beta function (which receives a contribution from the spin $3/2$ gravitinos) is positive and the pure supergravity theories are not asymptotically free. The addition of matter supermultiplets only makes the $\beta$ function more positive and hence gravitinos can never be confined.}. The relation (25) suggested that there might be a Kaluza-Klein interpretation since in such theories the coupling constant of the gauge group arising from the isometries of the extra dimensions is given by

$$g^2 \sim Gm^2$$ \tag{26}$$
where \( m^{-1} \) is the size of the compact space. Moreover, there is typically a negative cosmological constant

\[ \Lambda \sim -m^2 \]  

(27)

Combining (26) and (27), we recover (25). Indeed, the maximal \((D = 4, N = 8)\) gauged supergravity was seen to correspond to the massless sector of \((D = 11, N = 1)\) supergravity compactified on an \( S^7 \) whose metric admits an \( SO(8) \) isometry and 8 Killing spinors. An important ingredient in these developments that had been insufficiently emphasized in earlier work on Kaluza-Klein theory was that the \( AdS_4 \times S^7 \) geometry was not fed in by hand but resulted from a spontaneous compactification, i.e. the vacuum state was obtained by finding a stable solution of the higher-dimensional field equations. The mechanism of spontaneous compactification appropriate to the \( AdS_4 \times S^7 \) solution of eleven-dimensional supergravity was provided by the Freund-Rubin mechanism in which the 4-form field strength in spacetime \( F_{\mu \nu \rho \sigma} \) \((\mu = 0, 1, 2, 3)\) is proportional to the alternating symbol \( \epsilon_{\mu \nu \rho \sigma} \):

\[ F_{\mu \nu \rho \sigma} \sim \epsilon_{\mu \nu \rho \sigma} \]  

(28)

By applying a similar mechanism to the 7-form dual of this field strength one could also find compactifications on \( AdS_7 \times S^4 \) whose massless sector describes gauged maximal \( N = 4, SO(5) \) supergravity in \( D = 7 \). Type IIB supergravity in \( D = 10 \), with its self-dual 5-form field strength, also admits a Freund-Rubin compactification on \( AdS_5 \times S^5 \) whose massless sector describes gauged maximal \( N = 8 \) supergravity in \( D = 5 \).

In the three cases given above, the symmetry of the vacuum is described by the supergroups \( OSp(4|8) \), \( SU(2,2|4) \) and \( OSp(6,2|4) \) for the \( S^7 \), \( S^5 \) and \( S^4 \) compactifications respectively, as shown in Table 7. Each of these groups is known to admit the so-called singleton, doubleton or tripleton\(^5\) supermultiplets as shown in Table 8. We recall that sing-

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\(^5\)Our nomenclature is based on the \( AdS_4 \), \( AdS_5 \) and \( AdS_7 \) groups having ranks 2, 3 and 4, respectively, and differs from that of Gunaydin.
Table 8: Superconformal groups and their singleton, doubleton and tripleton representations.

gletons are those strange representations of $AdS$ first identified by Dirac which admit no analogue in flat spacetime. They have been much studied by Fronsdal and collaborators.

This Kaluza-Klein approach to $D = 11$ supergravity eventually fell out of favor for three reasons. First, in spite of its maximal supersymmetry and other intriguing features, eleven dimensional supergravity was, after all, still a field theory of gravity with all the attendant problems of non-renormalizability. The resolution of this problem had to await the dawn of $M$-theory, since we now regard $D = 11$ supergravity not as a fundamental theory in its own right but the effective low-energy Lagrangian of $M$-theory. Second, as emphasized by Witten, it is impossible to derive by the conventional Kaluza-Klein technique of compactifying on a manifold a chiral theory in four spacetime dimensions starting from a non-chiral theory such as eleven-dimensional supergravity. Ironically, Horava and Witten were to solve this problem years later by compactifying $M$-theory on something that is not a manifold, namely $S^1/Z_2$. Thirdly, these $AdS$ vacua necessarily have non-vanishing cosmological constant unless cancelled by fermion condensates and this was deemed unacceptable at the time. However, $AdS$ is currently undergoing a renaissance thanks to the $AdS/CFT$ correspondence.

A discussion of spinors and Dirac matrices in $D$ spacetime dimensions may be found in the reprint volume of Salam and Sezgin [32] and the book by West [8]. $D = 11$ supergravity is discussed in the paper of Cremmer, Julia and Scherk [64]. A summary of the $S^7$ and other $X_7$ compactifications of $D = 11$ supergravity down to $AdS_4$ may be found in the Physics Report of Duff, Nilsson and Pope [29].

Discussions of anti-de Sitter space and singletons in supergravity may be found in the Physics Reports by Duff, Nilsson and Pope [29], the review by Gunaydin in proceedings of the 1989 Trieste supermembrane conference [85], the book by Salam and Sezgin [32], and the TASI lectures by Duff [87].

A review of the AdS/CFT correspondence may be found in Physics Reports of Aharony,
5 Hidden spacetime symmetries in D=11

5.1 Spacelike, null and timelike reductions

Long ago, Cremmer and Julia pointed out that, when dimensionally reduced to \( d \) dimensions, \( D = 11 \) supergravity exhibits hidden symmetries. For example \( E_7(glocal) \times SU(8)(local) \) when \( d = 4 \) and \( E_8(glocal) \times SO(16)(local) \) when \( d = 3 \). Cremmer and Julia concentrated on the case where all extra dimensions are spacelike. Here we shall consider timelike and null reductions as well. The global symmetries remain the same but we shall focus on the local symmetries.

In fact, in anticipation of applications to vacuum supersymmetries in section 6, we shall focus particularly on the supercovariant derivative (23) as it appears in the gravitino variation of the dimensionally reduced theory. One finds that, after making a \( d/(11-d) \) split, the Lorentz subgroup \( G = SO(d-1,1) \times SO(11-d) \) can be enlarged to the generalized structure groups \( G = SO(d-1,1) \times G(spacelike) \), \( G = ISO(d-1) \times G(null) \) and \( G = SO(d) \times G(timelike) \) arising in the spacelike, null and timelike dimensional reduction, respectively. As we shall see, these generalized structure groups are the same as the hidden symmetries for \( d \geq 3 \) but differ for \( d < 3 \).

First we consider a spacelike dimensional reduction corresponding to a \( d/(11-d) \) split. Turning on only \( d \)-dimensional scalars, the reduction ansatz is particularly simple

\[
g^{(11)}_{MN} = \begin{pmatrix} \Delta^{-1/(d-2)} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}, \quad A^{(11)}_{ijk} = \phi_{ijk},
\]

where \( \Delta = \det g_{ij} \). For \( d \leq 5 \), we must also consider the possibility of dualizing either \( F(4) \) components or (for \( d = 3 \)) Kaluza-Klein vectors to scalars. We will return to such possibilities below. But for now we focus on \( d \geq 6 \). In this case, a standard dimensional reduction of the \( D = 11 \) gravitino transformation (22) yields the \( d \)-dimensional gravitino transformation

\[
\delta \psi_\mu = \hat{D}_\mu \epsilon
\]

where

\[
\hat{D}_\mu = \partial_\mu + \omega_\mu^{\alpha\beta} \gamma_{\alpha\beta} + Q_\mu^{ab} \Gamma_{ab} + \frac{1}{3!} e^{ia} e^{jb} e^{kc} \partial_\mu \phi_{ijk} \Gamma_{abc}.
\]
Here $\gamma_\alpha$ are SO($d - 1, 1$) Dirac matrices, while $\Gamma_\alpha$ are SO($11 - d$) Dirac matrices. For completeness, we also note that the $d$-dimensional dilatinos transform according to
\[
\delta \lambda_i = -\frac{1}{2} \gamma^\mu [P_{\mu ij} \Gamma^j - \frac{1}{36} (\Gamma_i^{jkl} - 6\delta_i^j \Gamma^{kl}) \partial_\mu \phi_{jkl}] \epsilon. \tag{32}
\]
In the above, the lower dimensional quantities are related to their $D = 11$ counterparts through
\[
\psi_\mu = \Delta^{\frac{1}{4(d-2)}} \left( \psi^{(11)}_\mu + \frac{1}{d-2} \gamma^i \Gamma^{(11)}_i \right), \quad \lambda_i = \Delta^{\frac{1}{4(d-2)}} \psi^{(11)}_i,
\]
\[
\epsilon = \Delta^{\frac{1}{4(d-2)}} \epsilon^{(11)},
\]
\[
Q^{ab}_\mu = e^{i[a} \partial_\mu e^{b]}_i, \quad P_{\mu ij} = e^{a}_{(i \partial_\mu e^{j})a}. \tag{33}
\]
This decomposition is suggestive of a generalized structure group with connection given by $\hat{D}_\mu$. However one additional requirement is necessary before declaring this an enlargement of SO($d - 1, 1$) $\times$ SO($11 - d$), and that is to ensure that the algebra generated by $\Gamma_{ab}$ and $\Gamma_{abc}$ closes within itself. Along this line, we note that the commutators of these internal Dirac matrices have the schematic structure
\[
[\Gamma^{(2)}, \Gamma^{(2)}] = \Gamma^{(2)}, \quad [\Gamma^{(2)}, \Gamma^{(3)}] = \Gamma^{(3)}, \quad [\Gamma^{(3)}, \Gamma^{(3)}] = \Gamma^{(6)} + \Gamma^{(2)}. \tag{34}
\]
Here the notation $\Gamma^{(n)}$ indicates the antisymmetric product of $n$ Dirac matrices, and the right hand sides of the commutators only indicate what possible terms may show up. The first commutator above merely indicates that the $\Gamma_{ab}$ matrices provide a representation of the Riemannian SO($11 - d$) structure group.

For $d \geq 6$, the internal space is restricted to five or fewer dimensions. In this case, the antisymmetric product $\Gamma^{(6)}$ cannot show up, and the algebra clearly closes on $\Gamma^{(2)}$ and $\Gamma^{(3)}$. Working out the extended structure groups for these cases results in the expected Cremmer and Julia groups listed in the first four lines in the second column of Table 9. A similar analysis follows for $d \leq 5$. However, in this case, we must also dualize an additional set of fields to see the hidden symmetries. For $d = 5$, an additional scalar arises from the dual of $F_{\mu\nu\rho\sigma}$; this yields an addition to (31) of the form $\hat{D}_\mu^{\text{additional}} = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma\lambda} F_{\nu\rho\sigma\lambda} \Gamma_{123456}$. This $\Gamma^{(6)}$ term is precisely what is necessary for the closure of the algebra of (34). Of course, in this case, we must also make note of the additional commutators
\[
[\Gamma^{(2)}, \Gamma^{(6)}] = \Gamma^{(6)}, \quad [\Gamma^{(3)}, \Gamma^{(6)}] = \Gamma^{(7)} + \Gamma^{(3)}, \quad [\Gamma^{(6)}, \Gamma^{(6)}] = \Gamma^{(10)} + \Gamma^{(6)} + \Gamma^{(2)}. \tag{35}
\]
However neither $\Gamma^{(7)}$ nor $\Gamma^{(10)}$ may show up in $d = 5$ for dimensional reasons.

The analysis for $d = 4$ is similar; however here $\hat{D}_\mu^{\text{additional}} = \frac{1}{3!} \epsilon_{\mu \nu \rho} e^{ia} F_{\nu \rho \alpha} \Gamma_a \Gamma_{1234567}$. Closure of the algebra on $\Gamma^{(2)}$, $\Gamma^{(3)}$ and $\Gamma^{(6)}$ then follows because, while $\Gamma^{(7)}$ may in principle arise in the middle commutator of (35), it turns out to be kinematically forbidden. For $d = 3$, on the other hand, in additional to a contribution $\hat{D}_\mu^{\text{additional}} = \frac{1}{2!} \epsilon_{\nu}^{\mu} e^{ia} e^{jb} F_{\nu \rho ij} \Gamma_{ab} \Gamma_{12345678}$, one must also dualize the Kaluza-Klein vectors $g_{\mu i}$. Doing so gives rise to a $\Gamma^{(7)}$ in the generalized connection which, in addition to the previously identified terms, completes the internal structure group to SO(16).

The remaining three cases, namely $d = 2$, $d = 1$ and $d = 0$ fall somewhat outside the framework presented above. This is because in these low dimensions the generalized connections $\hat{D}_\mu$ derived via reduction are partially incomplete. For $d = 2$, we find

$$\hat{D}_{\mu}^{(d=2)} = \partial_\mu + \omega_\mu^{\alpha \beta} \gamma_{\alpha \beta} + Q_\mu^{ab} \Gamma_{ab} + \frac{1}{9} (\delta^{\nu}_\mu - \frac{1}{2} \gamma^{\nu}_\mu) e^{ia} e^{jb} e^{kc} \partial_\nu \phi_{ijkl} \Gamma_{abc},$$

(36)

where $\gamma_{\mu \nu} = -\frac{1}{2} \epsilon_{\mu \nu} (\epsilon^{\alpha \beta} \gamma_{\alpha \beta})$ is necessarily proportional to the two-dimensional chirality matrix. Hence from a two-dimensional point of view, the scalars from the metric enter non-chirally, while the scalars from $F_{(4)}$ enter chirally. Taken together, the generalized connection (36) takes values in SO(16)$_+ \times$ SO(16)$_-$, which we regard as the enlarged structure group. However not all generators are present because of lack of chirality in the term proportional to $Q_\mu^{ab}$. Thus at this point the generalized structure group deviates from the hidden symmetry group, which would be an infinite dimensional subgroup of affine $E_8$. Similarly, for $d = 1$, closure of the derivative $\hat{D}_\mu^{(d=1)}$ results in an enlarged SO(32) structure group. However this is not obviously related to any actual hidden symmetry of the $1/10$ split. The $d = 0$ case is subject to the same caveats as the $d = 1$ and $d = 2$ cases: not all group generators are present in the covariant derivative. SL(32, R) requires $\{\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)}\}$ whereas only $\{\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)}\}$ appear in the covariant derivative.

Next we consider a timelike reduction for which we simply interchange a time and a space direction in the above analysis. This results in an internal Clifford algebra with signature $(10 - d, 1)$, and yields the extended symmetry groups indicated in the fourth column of Table 9. The same caveats concerning $d = 2, 1, 0$ apply in the timelike case.

Turning finally to the null case, we may replace one of the internal Dirac matrices with $\Gamma_{\pm}$ (where $+, -$ denote light-cone directions). Since $(\Gamma_{\pm})2 = 0$, this indicates that the extended structure groups for the null case are contractions of the corresponding spacelike (or timelike)
groups. In addition, by removing $\Gamma_+$ from the set of Dirac matrices, we essentially end up in the case of one fewer compactified dimensions. As a result, the $G(null)$ group in $d$-dimensions must have a semi-direct product structure involving the $G(spacelike)$ group in $(d + 1)$-dimensions. Of course, these groups also contain the original ISO$(10 - d)$ structure group as a subgroup. The resulting generalized structure groups are given in the third column of Table 9. Once again, the same caveats concerning $d = 2, 1, 0$ apply.

Spacelike reductions of $D=11$ supergravity may be found in the paper of Cremmer and Julia [65], null reductions in the paper of Duff and Liu [55] and timelike reductions in the paper of Hull and Julia [66]. Some of the noncompact groups appearing in the Table may be unfamiliar, but a nice discussion of their properties may be found in the book by Gilmore [67].

5.2 The complete uncompactified $D=11$ theory

Following Cremmer and Julia’s spacelike reduction, the question was then posed: do these symmetries appear magically only after dimensional reduction, or were they already present in the full uncompactified and untruncated $D = 11$ theory? The question was answered by de Wit and Nicolai who made a $d/(11 - d)$ split and fixed the gauge by setting to zero the off-diagonal components of the elfbein. They showed that in the resulting field equations the local symmetries are indeed already present, but the global symmetries are not. For example, after making the split $SO(10, 1) \supset SO(3, 1) \times SO(7)$, we find the enlarged symmetry $SO(3, 1) \times SU(8)$. There is no global $E_7$ invariance (although the 70 internal components of the metric and 3-form may nevertheless be assigned to an $E_7/SU(8)$ coset). Similar results were found for other values of $d$: in each case the internal subgroup $SO(11 - d)$ gets enlarged to some compact group $G(spacelike)$ while the spacetime subgroup $SO(d - 1, 1)$ remains intact\(^6\). Here we ask instead whether there are hidden spacetime symmetries. This is a question that could have been asked long ago, but we suspect that people may have been inhibited by the Coleman-Mandula theorem which forbids combining spacetime and internal symmetries. However, this is a statement about Poincare symmetries of the S-matrix and here we are concerned with Lorentz symmetries of the equations of motion, so there will be no conflict.

\(^6\)We keep the terminology “spacetime” and “internal” even though no compactification or dimensional reduction is implied.
Table 9: The generalized structure groups are given by $G = \text{SO}(d-1,1) \times G(\text{spacelike})$, $G = \text{ISO}(d-1) \times G(\text{null})$ and $G = \text{SO}(d) \times G(\text{timelike})$.

The explicit demonstration of $G(\text{spacelike})$ invariance by de Wit and Nicolai is very involved, to say the least. However, the result is quite simple: one finds the same $G(\text{spacelike})$ in the full uncompactified $D = 11$ theory as was already found in the spacelike dimensional reduction of Cremmer and Julia. Here we content ourselves with the educated guess that the same logic applies to $G(\text{timelike})$ and $G(\text{null})$: they are the same as what one finds by timelike and null reduction, respectively. The claim that the null and timelike symmetries are present in the full theory and not merely in its dimensional reductions might be proved by repeating the spacelike calculations of de Wit and Nicolai with the appropriate change of $\Gamma$ matrices. So we propose that, after making a $d/(11 - d)$ split, the Lorentz subgroup $G = \text{SO}(d-1,1) \times \text{SO}(11 - d)$ can be enlarged to the generalized structure groups $G = \text{SO}(d-1,1) \times G(\text{spacelike})$, $G = \text{ISO}(d-1) \times G(\text{null})$ and $G = \text{SO}(d) \times G(\text{timelike})$.

As we have seen, for $d > 2$ the groups $G(\text{spacelike})$, $G(\text{timelike})$ and $G(\text{null})$ are the same as those obtained from dimensional reductions. For the purposes of this section, however, their physical interpretation is very different. They are here proposed as symmetries of the full $D = 11$ equations of motion; there is no compactification involved, whether toroidal or
otherwise. (Note that by postulating that the generalized structure groups survive as hidden symmetries of the full uncompactified theory, we avoid the undesirable features associated with compactifications including a timelike direction such as closed timelike curves.)

For \( d \leq 2 \) it is less clear whether these generalized structure groups are actually hidden symmetries. Yet one might imagine that there exists a yet-to-be-discovered formulation of M-theory in which the \( d = 2 \) and \( d = 1 \) symmetries are realized. This would still be in keeping with the apparent need to make a non-covariant split and to make the corresponding gauge choice before the hidden symmetries emerge. A yet bolder conjecture, due to Hull, requiring no non-covariant split or gauge choice since \( d = 0 \) is that there exists a formulation of M-theory with the full \( \text{SL}(32, \mathbb{R}) \). This proposal is nevertheless very attractive since \( \text{SL}(32, \mathbb{R}) \) contains all the groups in Table 9 as subgroups and would thus answer the question of whether all these symmetries are present at the same time. This is an important issue deserving of further study.

We can apply similar logic to theories with fewer than 32 supersymmetries. Of course, if M-theory really underlies all supersymmetric theories then the corresponding vacua will all be special cases of the above. However, it is sometimes useful to focus on such a sub-theory, for example the Type I and heterotic strings with \( N = 16 \). Here \( G(\text{spacelike}) = \text{SO}(d) \times \text{SO}(d) \), \( G(\text{null}) = \text{ISO}(d-1) \times \text{ISO}(d-1) \) and \( G(\text{timelike}) = \text{SO}(d-1,1) \times \text{SO}(d-1,1) \).

Finally, we emphasize that despite the \( d/(11-d) \) split these symmetries refer to the full equations of motion and not to any particular background such as product manifolds. This issue of specific solutions of these equations is the subject of the next section.

Note that we have not considered the global symmetries such as \( E_7 \) for \( d=4 \), \( E_8 \) for \( d=3 \) and their infinite dimensional generalizations \( E_{11-d} \) for \( d \leq 2 \). These appear after dimensional reduction but, according to de Wit and Nicolai, not even the finite dimensional examples are symmetries of the full uncompactified theory. Discrete subgroups, known as U-dualities, do appear in M-theory, but so far only as symmetries of toroidally compactified vacua, not as background-independent symmetries of the equations of motion.

Hidden symmetries of the uncompactified \( D = 11 \) equations, as opposed to their dimensional reduction, are discussed in the papers by Duff [68], de Wit and Nicolai [69, 70], Duff and Liu [55], Hull [62] and Keurentjes [61, 62].

U-duality conjectures in membrane and M-theory may be found in the papers of Duff and Liu [98] and Hull and Townsend [99]. For a recent discussion of \( E_{11} \) see the paper by
6 Counting supersymmetries of D=11 vacua

6.1 Holonomy and supersymmetry

The equations of M-theory display the maximum number of supersymmetries \( N = 32 \), and so \( n \), the number of supersymmetries preserved by a particular vacuum, must be some integer \( 0 \leq n \leq 32 \). In vacua with vanishing 4-form \( F_{(4)} \), it is well known that \( n \) is given by the number of singlets appearing in the decomposition of the 32 of \( \text{SO}(1,10) \) under \( H \subset \text{SO}(1,10) \) where \( H \) is the holonomy group of the usual Riemannian connection \([24]\). This connection can account for vacua with \( n = 0, 1, 2, 3, 4, 6, 8, 16, 32 \).

Vacua with non-vanishing \( F_{(4)} \) allow more exotic fractions of supersymmetry, including \( 16 < n < 32 \). Here, however, it is necessary to generalize the notion of holonomy to accommodate the generalized connection \([23]\) that results from a non-vanishing \( F_{(4)} \). As discussed by Duff and Liu, the number of M-theory vacuum supersymmetries is now given by the number of singlets appearing in the decomposition of the 32 of \( G \) under \( H \subset G \) where \( H \) is the generalized holonomy group and \( G \) is the generalized structure group.

In subsequent papers by Hull and by Papadopoulos and Tsimpis it was shown that \( G \) may be as large as \( \text{SL}(32,R) \) and that an M-theory vacuum admits precisely \( n \) Killing spinors iff

\[
\text{SL}(31-n,R) \ltimes (n+1)R(31-n) \supseteq H \subseteq \text{SL}(32-n,R) \ltimes nR(32-n),
\]

i.e. the generalized holonomy is contained in \( \text{SL}(32-n,R) \ltimes nR(32-n) \) but is not contained in \( \text{SL}(31-n,R) \ltimes (n+1)R(31-n) \).

We recall that the number of supersymmetries preserved by an M-theory background depends on the number of covariantly constant spinors,

\[
D_M \epsilon = 0,
\]

called \textit{Killing} spinors. It is the presence of the terms involving the 4-form \( F_{(4)} \) in \([23]\) that makes this counting difficult. So let us first examine the simpler vacua for which \( F_{(4)} \) vanishes. Killing spinors then satisfy the integrability condition

\[
[D_M, D_N] \epsilon = \frac{1}{4} R_{MN}^{AB} \Gamma_{AB} \epsilon = 0,
\]
\[
\frac{d}{(11 - d)} \quad H \subset \text{SO}(11 - d) \subset \text{Spin}(10) \quad n
\]

| \( \frac{d}{11 - d} \) | \( H \subset \text{SO}(11 - d) \subset \text{Spin}(10) \) | \( n \) |
|-----------------|---------------------------------|-----|
| 7/4             | \( \text{SU}(2) \cong \text{Sp}(2) \) | 16  |
| 5/6             | \( \text{SU}(3) \)               | 8   |
| 4/7             | \( G_2 \)                        | 4   |
| 3/8             | \( \text{SU}(2) \times \text{SU}(2) \) | 8   |
|                 | \( \text{Sp}(4) \)              | 6   |
|                 | \( \text{SU}(4) \)              | 4   |
|                 | \( \text{Spin}(7) \)            | 2   |
| 1/10            | \( \text{SU}(2) \times \text{SU}(3) \) | 4   |
|                 | \( \text{SU}(5) \)              | 2   |

Table 10: Holonomy of static M-theory vacua with \( F^{(4)} = 0 \) and their supersymmetries.

where \( R^{AB \Gamma \epsilon}_{MN} \) is the Riemann tensor. The subgroup of \( \text{Spin}(10, 1) \) generated by this linear combination of \( \text{Spin}(10, 1) \) generators \( \Gamma_{AB} \) corresponds to the holonomy group \( H \) of the connection \( \omega_M \). We note that the same information is contained in the first order Killing spinor equation \( (38) \) and second-order integrability condition \( (39) \). One implies the other, at least locally. The number of supersymmetries, \( n \), is then given by the number of singlets appearing in the decomposition of the 32 of \( \text{Spin}(10, 1) \) under \( H \). In Euclidean signature, connections satisfying \( (39) \) are automatically Ricci-flat and hence solve the field equations when \( F^{(4)} = 0 \). In Lorentzian signature, however, they need only be Ricci-null so Ricci-flatness has to be imposed as an extra condition. In Euclidean signature, the holonomy groups have been classified. In Lorentzian signature, much less is known but the question of which subgroups \( H \) of \( \text{Spin}(10, 1) \) leave a spinor invariant has been answered by Bryant.

There are two sequences according as the Killing vector \( v_A = \epsilon \Gamma_A \epsilon \) is timelike or null. Since \( v^2 \leq 0 \), the spacelike \( v_A \) case does not arise. The timelike \( v_A \) case corresponds to static vacua, where \( H \subset \text{Spin}(10) \subset \text{Spin}(10, 1) \) while the null case to non-static vacua where \( H \subset \text{ISO}(9) \subset \text{Spin}(10, 1) \). It is then possible to determine the possible \( n \)-values and one finds \( n = 2, 4, 6, 8, 16, 32 \) for static vacua, and \( n = 1, 2, 3, 4, 8, 16, 32 \) for non-static vacua as shown in Table 10 and \( n = 1, 2, 3, 4, 8, 16, 32 \) for non-static vacua, as shown in Table 11.

The allowed \( n \) values for Riemannian connections may be found in the papers of Acharya et al \[74, 75\] and by Figueroa-O’Farrill \[71\].
Table 11: Holonomy of non-static M-theory vacua with $F_{(4)} = 0$ and their supersymmetries.

6.2 Generalized holonomy

In general we want to include vacua with $F_{(4)} \neq 0$. Such vacua are physically interesting for a variety of reasons. In particular, they typically have fewer moduli than their zero $F_{(4)}$ counterparts. Now, however, we face the problem that the connection in (23) is no longer the spin connection to which the bulk of the mathematical literature on holonomy groups is devoted. In addition to the Spin(10,1) generators $\Gamma_{AB}$, it is apparent from (23) that there are terms involving $\Gamma_{ABC}$ and $\Gamma_{ABCDE}$. In fact, the generalized connection takes its values in SL(32,R). Note, however, that some generators are missing from the covariant derivative. Denoting the antisymmetric product of $k$ Dirac matrices by $\Gamma^{(k)}$, the complete set of SL(32,R) generators include $\{\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)}\}$ whereas only $\{\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)}\}$ appear in the covariant derivative. Another way in which generalized holonomy differs from the Riemannian case is that, although the vanishing of the covariant derivative of the spinor implies the vanishing of the commutator, the converse is not true, as discussed below.

This generalized connection can preserve exotic fractions of supersymmetry forbidden by the Riemannian connection. For example, M-branes at angles include $n=5$, 11-dimensional pp-waves include $n=18, 20, 22, 24, 26$, squashed $N(1,1)$ spaces and M5-branes in a pp-wave background include $n=12$ and Godel universes include $n=14, 18, 20, 22, 24$. However, we can attempt to quantify this in terms of generalized holonomy groups $^7$.

$^7$In these lectures we focus on $D = 11$ but similar generalized holonomy can be invoked to count $n$ in Type IIB vacua, which include pp-waves with $n = 28$. 

\begin{center}
\begin{tabular}{c|cccc}
$d/(11-d)$ & $H \subset \text{ISO}(d-1) \times \text{ISO}(10-d) \subset \text{Spin}(10,1)$ & $n$ \\
\hline
10/1 & $R^9$ & 16 \\
6/5 & $R^5 \times (\text{SU}(2) \ltimes R^4)$ & 8 \\
4/7 & $R^3 \times (\text{SU}(3) \ltimes R^6)$ & 4 \\
3/8 & $R^2 \times (G_2 \ltimes R^7)$ & 2 \\
2/9 & $R \times (\text{SU}(2) \ltimes R^4) \times (\text{SU}(2) \ltimes R^4)$ & 4 \\
& $R \times (\text{Sp}(4) \ltimes R^8)$ & 3 \\
& $R \times (\text{SU}(4) \ltimes R^8)$ & 2 \\
& $R \times (\text{Spin}(7) \ltimes R^8)$ & 1 \\
\end{tabular}
\end{center}
Generalized holonomy means that one can assign a holonomy $H \subset G$ to the generalized connection appearing in the supercovariant derivative $\mathcal{D}$ where $G$ is the generalized structure group. The number of unbroken supersymmetries is then given by the number of $H$ singlets appearing in the decomposition of the 32 dimensional representation of $G$ under $H \subset G$.

For generic backgrounds we require that $G$ be the full $\text{SL}(32, \mathbb{R})$ while for special backgrounds smaller $G$ are sufficient. To see this, let us write the supercovariant derivative as

$$\mathcal{D}_M = \hat{D}_M + X_M,$$

for some other connection $\hat{D}_M$ and some covariant $32 \times 32$ matrix $X_M$. If we now specialize to backgrounds satisfying

$$X_M \epsilon = 0,$$

then the relevant structure group is $\hat{G} \subseteq G$.

Consider, for example, the connection $\hat{D}$ arising in dimensional reduction of $D = 11$ supergravity \[(31)\]. The condition \[(41)\] is just $\delta \lambda_i = 0$ where $\lambda_i$ are the dilatinos of the dimensionally reduced theory. In this case, the generalized holonomy is given by $\hat{H} \subseteq \hat{G}$ where the various $\hat{G}$ arising in spacelike, null and timelike compactifications are tabulated in Table 9 for different numbers of the compactified dimensions.

Another way in which generalized holonomy differs from Riemannian holonomy is that, although the vanishing of the covariant derivative implies the vanishing of the commutator, the converse is not true. Consequently, the second order integrability condition alone may be a misleading guide to the generalized holonomy group $H$.

To illustrate this, we consider Freund-Rubin vacua with $F_{(4)}$ given by

$$F_{\mu\nu\rho\sigma} = 3m \epsilon_{\mu\nu\rho\sigma},$$

where $\mu = 0, 1, 2, 3$ and $m$ is a constant with the dimensions of mass. This leads to an $AdS_4 \times X^7$ geometry. For such a product manifold, the supercovariant derivative splits as

$$\mathcal{D}_\mu = D_\mu + m \gamma_\mu \gamma_5,$$

and

$$\mathcal{D}_m = D_m - \frac{1}{2} m \Gamma_m,$$

and the Killing spinor equations reduce to

$$\mathcal{D}_\mu \epsilon(x) = 0$$

(45)
and
\[ \mathcal{D}_m \eta(y) = 0. \] (46)

Here \( \epsilon(x) \) is a 4-component spinor and \( \eta(y) \) is an 8-component spinor, transforming with Dirac matrices \( \gamma_\mu \) and \( \Gamma_m \) respectively. The first equation is satisfied automatically with our choice of \( AdS_4 \) spacetime and hence the number of \( D = 4 \) supersymmetries, \( 0 \leq N \leq 8 \), devolves upon the number of Killing spinors on \( X^7 \). They satisfy the integrability condition
\[ [\mathcal{D}_m, \mathcal{D}_n] \eta = -\frac{1}{4} C^{ab}_{mn} \Gamma_{ab} \eta = 0, \] (47)
where \( C^{ab}_{mn} \) is the Weyl tensor. Owing to this generalized connection, vacua with \( m \neq 0 \) present subtleties and novelties not present in the \( m = 0 \) case, for example the phenomenon of skew-whiffing. For each Freund-Rubin compactification, one may obtain another by reversing the orientation of \( X^7 \). The two may be distinguished by the labels \( \text{left} \) and \( \text{right} \). An equivalent way to obtain such vacua is to keep the orientation fixed but to make the replacement \( m \rightarrow -m \) thus reversing the sign of \( F_4 \). So the covariant derivative (44), and hence the condition for a Killing spinor, changes but the integrability condition (47) remains the same. With the exception of the round \( S^7 \), where both orientations give \( N = 8 \), at most one orientation can have \( N \geq 0 \). This is the skew-whiffing theorem.

The squashed \( S^7 \) provides a non-trivial example: the left squashed \( S^7 \) has \( N = 1 \) but the right squashed \( S^7 \) has \( N = 0 \). Other examples are provided by the left squashed \( N(1,1) \) spaces, one of which has \( N = 3 \) and the other \( N = 1 \), while the right squashed counterparts both have \( N = 0 \). (Note, incidentally, that \( N = 3 \) \( i.e. \), \( n = 12 \) can never arise in the Riemannian case.)

All this presents a dilemma. If the Killing spinor condition changes but the integrability condition does not, how does one give a holonomic interpretation to the different supersymmetries? We note that in (44), the \( SO(7) \) generators \( \Gamma_{ab} \), augmented by the presence of \( \Gamma_a \), together close on \( SO(8) \). Hence the generalized holonomy group satisfies \( \mathcal{H} \subset SO(8) \). We now ask how the 8 of \( SO(8) \) decomposes under \( \mathcal{H} \). In the case of the left squashed \( S^7 \), \( \mathcal{H} = SO(7)^- \), \( 8 \rightarrow 1 + 7 \) and \( N = 1 \), but for the right squashed \( S^7 \), \( \mathcal{H} = SO(7)^+ \), \( 8 \rightarrow 8 \) and \( N = 0 \). From the integrability condition alone, however, we would have concluded naively that \( \mathcal{H} = G_2 \) and that both orientations give \( N = 1 \).

Another context in which generalized holonomy may prove important is that of higher loop corrections to the M-theory Killing spinor equations with or without the presence of
non-vanishing $F_{(4)}$. Higher loops yield non-Riemannian corrections to the supercovariant derivative, even for vacua for which $F_{(4)} = 0$, thus rendering the Berger classification inapplicable. Although the Killing spinor equation receives higher order corrections, so does the metric, ensuring, for example, that $H = G_2$ Riemannian holonomy 7-manifolds still yield $N = 1$ in $D = 4$ when the non-Riemannian corrections are taken into account. This would require a generalized holonomy $\mathcal{H}$ for which the decomposition $8 \to 1 + 7$ continues to hold.

Generalized holonomy is discussed in the papers of Duff and Stelle [97], Duff [54], Duff and Liu [55], Hull [56], Papadopoulos and Tsimpis [57, 58], Batrachenko, Duff, Liu and Wen [59], Bandos, de Azcarraga, Izquierdo, Lukierski, Picon and Varela [60, 76] and Keurentjes [61, 62].

Skew-whiffing is discussed in the paper and Physics Report by Duff, Nilsson and Pope [91, 29] and the paper of van Nieuwenhuizen and Warner [90]. The squashed $S^7$ may be found in the papers of Awada, Duff and Pope [92] and Duff, Nilsson and Pope [91]. For the result that $SO(7)$ generators $\Gamma_{ab}$, augmented by presence of $\Gamma_a$, together close on $SO(8)$ see the paper by Castellani, D'Auria, Fre and van Nieuwenhuizen [93].

Higher loop corrections to the Killing spinor equation are treated in the paper by Lu, Pope, Stelle and Townsend [94].

### 6.3 Specific examples

In Table 12 we tabulate the results of computations of this generalized holonomy for the $n = 16$ examples of the M2-brane, the M5-brane, the M-wave (MW) and the M-monopole (MK), and for a variety of their $n = 8$ intersections: M5/MK, M2/MK/MK, M2/MK, M2/MW, M5/MW,MW/MK and M2/M5. As we can see, the generalized holonomy of M-theory solutions takes on a variety of guises. We make note of two features exhibited by these solutions. Firstly, it is clear that many generalized holonomy groups give rise to the same number $n$ of supersymmetries. This is a consequence of the fact that while $\mathcal{H}$ must satisfy the condition (37), there are nevertheless many possible subgroups of $\text{SL}(32 - n, \mathbb{R}) \ltimes n\mathbb{R}^{(32-n)}$ allowed by generalized holonomy. Secondly, as demonstrated by the plane wave solutions, knowledge of $\mathcal{H}$ by itself is insufficient for determining $n$; here $\mathcal{H} = R^9$, while $n$ may be any even integer between 16 and 26.

What this indicates is that, at least for counting supersymmetries, it is important to understand the embedding of $\mathcal{H}$ in $\mathcal{G}$. In contrast to the Riemannian case, different embed-
Table 12: Generalized holonomies of the objects investigated in the text. For $n = 16$, we have $\mathcal{H} \subseteq \text{SL}(16,\mathbb{R}) \times 16\mathbb{R}$, while for $n = 8$, it is instead $\mathcal{H} \subseteq \text{SL}(24,\mathbb{R}) \times 8\mathbb{R}$.

While the full generalized holonomy involves several factors, the transverse (or $\hat{D}$) holonomy is often simpler, e.g. $\text{SO}(5)$ for the M5 and $\text{SO}(8)$ for the M2. The results summarized in Table 12 are suggestive that the maximal compact subgroup of $\mathcal{H}$, which must be contained in $\text{SL}(32 - n,\mathbb{R})$, is often sufficient to determine the number of surviving supersymmetries. For example, the M2/MK/MK solution may be regarded as a 3/8 split, with a hyper-Kahler eight-dimensional transverse space. In this case, the $\hat{D}$ structure group is $\text{SO}(16)$, and the 32-component spinor decomposes under $\text{SO}(32) \supset \text{SO}(16) \supset \text{SO}(8) \times \text{SU}(2) \times \text{SU}(2)$ as $32 \rightarrow 2(16) \rightarrow 2(8,1,1) + 2(1,2,2) + 8(1,1,1)$ yielding eight singlets. Similarly, for the M5/MW intersection, we consider a 2/9 split, with the wave running along the two-dimensional longitudinal space. Since the $\hat{D}$ structure group is $\text{SO}(16) \times \text{SO}(16)$ and the maximal compact subgroup of $\text{SU}^*(8)$ is $\text{USp}(8)$, we obtain the decomposition $32 \rightarrow$
(16, 1) + (1, 16) → 4(4, 1) + (1, 8) + 8(1, 1) under $\text{SO}(32) \supset \text{SO}(16) \times \text{SO}(16) \supset \text{SO}(5) \times \text{USp}(8)$. This again yields $n = 8$. Note, however, that this analysis fails for the plane waves, as $\mathbb{R}^9$ has no compact subgroups.

Ultimately, one would hope to achieve a complete classification of M-theory vacua, either through generalized holonomy or other means. In this regard, one must also include the effects of higher order corrections and perhaps additional contributions beyond the supergravity itself.

### 6.4 The full M(onty)?

In sections 5 and 6 we have focused on the low energy limit of M-theory, but since the reasoning is based mainly on group theory, it seems reasonable to promote it to the full M-theory. Similar reasoning can be applied to M-theory in signatures (9,2) and (6,5), the so-called M$'$ and M$^*$ theories, but the groups will be different. When counting the $n$ value of a particular vacuum, however, we should be careful to note the phenomenon of *supersymmetry without supersymmetry*, where the supergravity approximation may fail to capture the full supersymmetry of an M-theory vacuum. For example, vacua related by T-duality and S-duality must, by definition, have the same $n$ values. Yet they can appear to be different in supergravity if one fails to take into account winding modes and non-perturbative solitons. So more work is needed to verify that the $n$ values found so far in $D = 11$ supergravity exhaust those of M-theory.

A different approach to supersymmetric vacua in M-theory is through the technique of $G$-structures. Hull has suggested that $G$-structures may be better suited to finding supersymmetric solutions whereas generalized holonomy may be better suited to classifying them. In any event, it would be useful to establish a dictionary for translating one technique into the other.

Ultimately, one would hope to achieve a complete classification of vacua for the full M-theory. In this regard, one must at least include the effects of M-theoretic corrections to the supergravity field equations and Killing spinor equations and perhaps even go beyond the geometric picture altogether. It seems likely, however, that counting supersymmetries by the number of singlets appearing in the decomposition 32 of $\text{SL}(32, \mathbb{R})$ under $\mathcal{H} \subset \text{SL}(32, \mathbb{R})$ will continue to be valid.

The various spacetime signatures in which M-theory can be formulated is discussed in the
paper by Blencowe and Duff [95]. M' and M* theories are treated in [96]. Supersymmetry without supersymmetry may be found in the papers of Duff, Lu and Pope [82, 83]. For G-structures, see the papers by Gauntlett, Martelli, Pakis, Sparks and Waldram [77, 78, 79, 80, 81] and by Hull [56]. Connections between generalized holonomy and G-structures in theories with 8 supercharges are discussed in the paper by Batrachenko and Wen [101].

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