Proportionate growth in patterns formed in the rotor-router model

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Received 2 January 2014
Accepted for publication 24 October 2014
Published 25 November 2014

Abstract. We study the growing patterns formed in the rotor-router model by adding $N$ walkers at the centre of an $L \times L$ two-dimensional square lattice, starting with a periodic background of arrows, and relaxing to a stable configuration. The pattern is made of a large number of triangular and quadrilateral regions, where in each region all arrows point in the same direction. We show that the pattern formed by arrows which have been rotated at least one full circle may be described in terms of a tiling of the plane by squares of different sizes. The sizes of these squares, and the size of the pattern, grow linearly with $N$ for $1 \ll N < 2L$. We use the Brooks–Smith–Stone–Tutte theorem relating tilings of squares by smaller squares to resistor networks, to determine the exact relative sizes of these tiles for large $N$. The scaling limit of the number of visits $\phi(\xi, \eta)$ as a function of the scaled position $(\xi, \eta)$ is also determined. We also present numerical evidence that the deviations of the sizes of the different squares in the tiling from the asymptotic linear growth law are always $O(1)$, and are quasiperiodic functions of $N$.

Keywords: cellular automata, self-organized criticality (theory), patterns, pattern formation (theory)

ArXiv ePrint: 1312.6888
1. Introduction

As baby animals grow from birth to adulthood, the different parts of the body grow roughly proportionately to each other. This is called proportionate growth. While proportionate growth is quite typical in the animal kingdom, examples of proportionate growth outside biology are hard to find. Proportionate growth is qualitatively different from the growth phenomena that have usually been studied in physics, e.g. diffusion limited aggregation, invasion percolation, formation of snow-flakes, none of which show this property. Recently, we have shown that growing patterns formed in the abelian sandpile model (ASM) show proportionate growth [1]. Here, the patterns are formed by depositing particles one by one at the origin of a finite square lattice and letting the configuration relax using the ASM relaxation rules, until it becomes stable. The growth rates and internal structures of patterns formed starting from various periodic backgrounds were also characterized [2, 3]. The effect of various forms of noise on the patterns has also been studied [4]. The existence of the asymptotic limit was rigorously...
proved for the special case of a pattern formed on an empty background, in [19]. A review of these results may be found in [5].

In this paper, we extend these results for pattern formation showing proportionate growth to a different model—the rotor-router model. This model shares the abelian property with the sandpile model, and in fact, the algebra of the particle addition operators is the same [6]. We characterize the structure of these patterns. The characterization here turns out to be simpler than for the sandpile model patterns. This is due to the fact that here, one deals only with piece-wise linear functions, while in the sandpile problem these functions are in general piece-wise quadratic [5]. We show how the relative sizes of different elements in the asymptotic patterns so formed can be calculated exactly from knowledge of the adjacency structure of the pattern. We also present numerical evidence that the sizes of different features in the pattern show interesting quasiperiodic properties as a function of \( N \), and in fact the deviations of these from the linear growth law in \( N \) seem to remain of magnitude \( \leq 1 \), so that if the size of feature \( P \) increases on the average as \( \alpha_P N \) for large \( N \), where \( \alpha_P \) is some \( P \)-dependent constant, the difference between the actual value for any finite \( N \) and the integer \( \lfloor \alpha_P N \rfloor \) is \( O(1) \) for all \( N \).

The rotor-router model is a simple model of a deterministic walk, in which the walker locally modifies the medium it moves in, affecting its subsequent motion when it returns to the same site. The model was originally introduced in the context of self-organized criticality, and called the Eulerian Walker model [7,8]. The name comes from the fact that on a finite undirected graph, the walk eventually settles into a Euler cycle, in which each edge of the graph is visited exactly once in each direction. It was independently proposed by Jim Propp as a derandomized version of the random walk, and called the rotor-router model [9]. The latter nomenclature seems to describe the model more succinctly, and will be used in this paper. Arising from the general interest in derandomized algorithms in computer science, and the fact that algorithms using derandomization with rotor-router rules seem to perform better than randomized algorithms, the differences between the fluctuation properties of the rotor-router walk and the random walk have been the subject of much interest [10]. Reviews of earlier work on the rotor-router model may be found in [11] and [12]. An introductory discussion of derandomization techniques in computer science may be found in [13].

If we allow walkers to leave the lattice at the boundaries, and new walkers are added at randomly selected sites, the rotor-router model reaches a critical steady-state which shows self-organized criticality [7]. Furthermore, like the abelian sandpile model, if walkers are always added at the same site, the pattern of arrows obtained after relaxation shows proportionate growth. The diameter of these patterns and the sizes of their internal elements grow linearly with the number of walkers added. The simplest pattern of this type (figure 1) was in fact first observed by Holroyd and Propp in [10], but these authors were mainly interested in the relation of this problem to random walks. The escape rates of walkers and the behaviour of the pattern diameter as a function of the number of walkers were studied by Florescu et al [14]. They showed that, in \( d \geq 2 \) dimensions, the diameter of the patterns by depositing \( N \) particles on a periodic transient background grows with \( N \) as \( \beta N \) where \( \beta \) is a finite constant depending on the background. The characterization of the patterns itself has not been undertaken so far. We do this here.
Figure 1. Pattern formed by depositing rotor-router walkers at the origin of a square lattice, starting from an initial configuration of all arrows pointing East. (a) the pattern after 200 walkers and (b) after 800 walkers. Note that (b) is scaled down by a factor of four to make it equal in size to (a). Colour code: dark blue—→, light blue—↑, yellow—←, red—↓.

We briefly mention other related works. A version of the rotor-router model where the walker stops walking when it visits a previously unvisited site has been studied by Levine and Peres [15, 16]. They found that the aggregate formed by these walkers around the origin on a 2D lattice has a circular shape, similar to that in Internal Diffusion Limited Aggregation, and derived sharp bounds on the fluctuations in the radius of this ‘Propp circle’. The Propp circle also shows an internal structure similar to the patterns we study here, but this structure has not been studied in detail. Some nice pictures of the Propp circle, and related patterns in the sandpile model may be found in [17]. Kapri and Dhar studied the pattern produced by a single walker walking on a random background in 2D [18], and found numerical evidence that the asymptotic shape of the pattern is a circle. They also argued that the average number of visits to a site \( \vec{r} \) up to time \( t \) tends to a scaling function of the form \( t^{1/3} g(\vec{r}t^{-1/3}) \).

The outline of this paper is as follows: We define the model precisely in section 2, and recapitulate some known results. In section 3, we study the pattern obtained from the initial configuration with all arrows parallel. We define the asymptotic pattern, and the asymptotic visit function for the pattern. In section 4, using the Brooks–Smith–Stone–Tutte (BSST) theorem, we construct the resistor network corresponding to the tiling seen in the pattern and determine the sizes of the tiles. We obtain an exact characterization of the asymptotic pattern, and determine the relative sizes of different elements. Thus we also determine the visit function for the pattern in the large \( N \) limit. In section 5, we extend the analysis to some other patterns obtained starting from different periodic initial configurations. In section 6, we provide evidence that the deviations in the diameter and sizes of various elements in the pattern from simple linear dependence on \( N \) are bounded and show quasiperiodicity. Section 7 contains a summary of our results, and some concluding remarks. In the appendix, we re-derive the results of section 4.3 without invoking the BSST theorem.

doi:10.1088/1742-5468/2014/11/P11030
2. Definition of the model

We will consider the rotor-router model on a two-dimensional square lattice. There is an arrow attached to every lattice site, which points in the direction of one of its four neighbours. When a walker reaches a site, it rotates the arrow attached to that site by 90° counterclockwise and takes a step in the new direction of the arrow (figure 2). For sites that have been visited by the walker at least once, the current direction of the arrow shows the direction of the last exit of the walker from that site. Since the update procedure is deterministic, the configuration of arrows and the position of the walker after \( n \) steps is fully determined by the initial configuration.

The rotor-router as defined above can be generalized to an arbitrary graph by introducing the notion of a ‘stack’ on each site: a stack is an infinite sequence of instructions of the type ‘go to \( X \)’, where \( X \) is one of the neighbours of the site. On arrival at any site, the walker follows the instruction of the topmost entry of the stack at that site. Once it is obeyed, this instruction is deleted (or ‘popped’) from the list, so that the next time the walker arrives at the site, it finds a new instruction. If stack is an infinite repetition of ‘North–West–South–East’, we get the model we study in this paper. If the entries are chosen randomly, we get a random stack. A random stack on every site leads to a random walk on the graph.

We consider a finite square lattice, with sites labelled as \((x, y)\), where \( x \) and \( y \) are integers, with \(|x| \leq L/2\), and \(|y| \leq L/2\). The first walker starts at the origin. The arrows attached to the boundary sites can point out of the lattice. The walker then ‘falls off’ the lattice when it follows such an arrow. When this happens, a new walker is introduced at the origin.

We study the configuration of arrows when the \( N \)-th walker has just left the system, and the \((N + 1)\)-th walker is not yet introduced. First consider the case where the initial configuration is such that all arrows are parallel, pointing due East. Clearly, the first walker put at the origin would move along the positive \( y \)-axis, rotating the arrows it encounters to point North. The second walker rotates the arrow at the origin to point West, and then walks North along the vertical line \( x = -1 \). The third walker returns to the origin once, before leaving along the vertical line \( x = +1 \). And so on.

Figure 3 shows the configurations left behind by the walker after 2, 5 and 20 walkers. We denote sites having different directions of arrows by small filled squares of different colours. Then the region of the lattice visited at least once by a walker is seen to consist of four distinct regions: (i) a long vertical stripe, with an extra triangular region at one end, in which all arrows are pointed north, showing the walkers’ paths before they leave from the top boundary, (ii) a dart-shaped region with all arrows pointing west, (iii) a
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Figure 3. The arrangement of the arrows starting from the initial background with all spins pointing East after (a) two (b) five and (c) twenty walkers. The lattice depicted is 50×50. Colour code: dark blue—→, light blue—↑, yellow—←, red—↓.

triangular region where sites have been visited thrice and are hence pointing south, and (iv) a growing octagonal region around the origin. The octagonal region is made of sites where the arrows have undergone at least one complete rotation, while sites on the rest of the lattice have been visited less than four times. We call this octagonal region the ‘pattern’. We first note that the internal structure of the pattern grows proportionately with the pattern itself. This is seen by comparing figures 1(a) and (b), which show the pattern after the number of walkers \( N = 400 \) and 1600 respectively. The size of the pattern has grown four times, but the internal structure has not changed much, except for near the origin, where the smaller patches become better-resolved, and some new small patches can be seen.

The diameter \( D_N \) of the pattern is defined as the extent of the pattern along the \( x \)-axis, after \( N \) walkers have exited from the lattice. From the simulations we find that, asymptotically, \( D_N = N/2 + O(1) \). We will take this as an observation based on numerical studies. A rigorous proof is not yet available. The behaviour of \( D_N \) as a function of \( N \) is discussed in detail in section 6.

We can see that the octagon in the figure is made of triangular or dart-shaped quadrilateral regions in which all arrows point in the same direction (shown with the same colour). We call these three- or four-sided polygonal regions ‘patches of constant orientation’. As \( N \) is increased, the size of these patches increases, but their shapes and relative sizes remain nearly unchanged. The linear growth cannot be exact, as the coefficient of the linear growth rate is typically irrational, but the diameters of all the patches are necessarily integers.

We now describe how a walker explores a single patch after entering it (figure 4). From the evolution rules of the walker, it is easy to check that when a walker enters a patch, it visits all the sites in the patch four times, and the sites on the boundaries one, two or three times, before leaving the patch. Thus, the orientation of arrows in the patch does not change with \( N \), except at the boundaries of the patch, which may be shifted by one lattice spacing. A new walker, introduced at the origin, moves from one patch to another until it exits the octagonal region, after which it follows a non-self-intersecting path to the sink. As a result of this motion, the boundaries of visited patches shift, and the overall pattern grows in size.
Figure 4. (a) A schematic representation of the trail of the walker in a single red patch: When going to the right the path is a straight line, while going to the left, it moves between two adjacent horizontal rows, as indicated in the topmost part of the path. The colour code: dark blue→, light blue↑, yellow←, red↓. (b) Successive arrow configurations encountered during the walker’s overall leftward motion. The position of the walker is indicated by an unfilled circle.

3. The visit function

In this paper, we will study the structure of the asymptotic pattern in the limit of large $N$. The asymptotic pattern is defined as follows.

Assign the numerical values 0, 1, 2 and 3 to arrows pointing in the directions East, North, West and South respectively. Call $\rho_N(x,y)$ the numerical value assigned to the direction of the arrow at the site $(x,y)$ after $N$ walkers have exited the lattice. We define the scaled coordinates $\xi = x/D_N$, $\eta = y/D_N$. The asymptotic pattern is defined by the discrete function $\rho(\xi, \eta)$ in the unit square centred at the origin, given by the limit

$$\rho(\xi, \eta) = \lim_{N \to \infty} \rho_N([D_N \xi], [D_N \eta])$$

(1)

The function $\rho(\xi, \eta)$ is a piece-wise-constant integer function of its arguments, and gives the orientation of the arrow at point $(\xi, \eta)$ in the scaled pattern. We will defer a formal proof of the existence of this limit to future studies. For the sandpile patterns, the existence of the asymptotic pattern has been proved for the initially empty background by Pegden and Smart in [19].

With our normalization convention and the observation that $D_N = N/2 + O(1)$, and using the fact that the pattern is symmetric about the $y$-axis, the left and right boundaries of the scaled pattern are at $|\xi| = 1/2$. From the observation that the pattern is in the shape of an octagon, and has equal extent in horizontal and vertical directions, we infer that the upper and lower boundaries of the scaled pattern lie along the lines $|\eta| = 1/2$. 

doi:10.1088/1742-5468/2014/11/P11030
From figure 1, one can see that the outer patches are bigger and hence are well-formed even for small \( N \), while one has to increase \( N \) in order to distinguish patches with smaller diameters. The asymptotic pattern as defined above has well-resolved triangular or dart-shaped patches at all levels, with sharp boundaries. The boundaries between well-resolved patches seem to be of width 2 at most in the original unscaled units (This may be checked by zooming into figure 1 in the electronic version of this paper.).

The exact characterization of the sandpile patterns was given in terms of the toppling function \( \Phi_N(x, y) \) which gives the number of topplings at any lattice point \((x, y)\), when \( N \) particles are added at the origin and the configuration is relaxed [5]. For the rotor-router model, the corresponding function, which we also denote \( \Phi_N(x, y) \) is the so-called visit function, which counts the total number of full rotations undergone by the arrow attached to the site \((x, y)\) due to the first \( N \) walkers. The region where \( \Phi_N \) is non-zero is just the octagonal region which we have earlier defined as our pattern. Define the scaled visit function \( \phi(\xi, \eta) \) by

\[
\phi(\xi, \eta) = \lim_{N \to \infty} \frac{\Phi_N(\lfloor D_N \xi \rfloor, \lfloor D_N \eta \rfloor)}{D_N}
\]

Note that the total number of visits to the site \((x, y)\) is given by \( 4\Phi_N(x, y) + \rho_N(x, y) \).

Now we argue that \( \Phi_N \) in every patch is the sum of a linear function of \( x \) and \( y \) and a periodic part, with the periodicity the same as that of the initial background (For examples of patterns on backgrounds where the unit cell is \( 2 \times 2 \), see section 5, and figures 10, 11 and 13.) Let us take a patch of constant orientation, where the basis vectors of the periodic arrow configuration within the patch are \( \hat{e}_1 \) and \( \hat{e}_2 \). Define \( \Delta_j \Phi(\bar{R}) \) for \( j = 1, 2 \) as the finite difference \( \Phi_N(\bar{R} + \hat{e}_j) - \Phi_N(\bar{R}) \). The key observation that enables full characterization of the pattern is that whenever \( \bar{R} \) and \( \bar{R} + \hat{e}_j \) are both within a given patch of constant orientation, \( \Delta_j \Phi(\bar{R}) \) is independent of \( \bar{R} \) and \( N \). The non-dependence on \( N \) comes from the Eulerian property of the walkers illustrated in figure 4: whenever a walker comes to a patch, it visits all sites in the patch the same number of times. The translational invariance within a patch, namely that \( \Delta_j \Phi(\bar{R}) \) does not depend on \( \bar{R} \), comes from the fact that the boundary between two different patches of constant orientation shifts by the same amount each time a new walker visits the patch. Since all sites along the boundary are visited the same number of times whenever the boundary is shifted, this gives rise to translational invariance in the direction along the boundary. Since the shift of the boundary is always the same, this gives rise to an invariance of finite differences in the direction normal to the boundary. Hence \( \Delta_j \Phi(\bar{R}) \) must be the same for all sites \( \bar{R} \) in the same patch, equivalent under translations by \( \hat{e}_1 \) and \( \hat{e}_2 \).

The argument above does not apply for two sites \( \bar{R}_1 \) and \( \bar{R}_2 \) within the same unit cell. We now argue that \( \Delta_j \Phi(\bar{R}) \) cannot be different for two such sites either. Consider two adjacent sites \( \bar{A} \) and \( \bar{B} \), within the same unit cell. Now, for \( s \) such that sites \( \bar{A} + s\hat{e}_1 \) and \( \bar{B} + s\hat{e}_1 \) are still within the patch, \( \Phi(\bar{A} + s\hat{e}_1) - \Phi(\bar{A}) = s\delta_A \) and \( \Phi(\bar{B} + s\hat{e}_1) - \Phi(\bar{B}) = s\delta_B \), which gives \( \Phi(\bar{A} + s\hat{e}_1) - \Phi(\bar{B} + s\hat{e}_1) = s(\delta_A - \delta_B) \). This difference grows with \( s \), but since \( \bar{A} + s\hat{e}_1 \) and \( \bar{B} + s\hat{e}_1 \) are adjacent sites, the difference between the number of visits at both sites cannot be arbitrarily large. This is possible only if \( \delta_A = \delta_B \). Hence we must have \( \Delta_j \Phi(\bar{R}) \) the same for all sites \( \bar{R} \) within a single patch of constant orientation.

We note that for a linearly growing pattern, this result also follows from the scaling property of proportionate growth. This argument is an adaptation of the one for linearly
growing sandpiles in [5]. Consider the scaled toppling function \( \phi(\xi, \eta) \) within a patch of constant orientation. We expand \( \phi(\xi, \eta) \) about a point \((\xi_0, \eta_0)\) within the patch (not at the boundary), assuming that the scaled visit function is a twice-differentiable function within a given patch. We also assume that the vector of finite differences \( \Delta \phi \rightarrow D_N \nabla \phi(\eta, \xi) \) point-wise, as \( N \rightarrow \infty \), where both sides are expressed in the basis formed by the basis vectors of the unit cell in the patch. One gets \( \phi(\xi_0 + \Delta \xi, \eta_0 + \Delta \eta) = \phi(\xi_0, \eta_0) + v_1 \Delta \eta + v_2 \Delta \xi + v_3 (\Delta \eta)^2 + v_4 (\Delta \xi)^2 + \ldots \), where \( v_1, v_2, v_3, v_4 \) are finite constants. Rescaling back to the visit function \( \Phi_N(x, y) = D_N \phi(\eta, \xi) \), and using \( \Delta \xi = \Delta x/D_N, \Delta \eta = \Delta y/D_N \), we find that near the point \((x, y)\), \( \Phi_N(x_0 + \Delta x, y_0 + \Delta y) = \Phi_N(x_0, y_0) + v_1 \Delta x + v_2 \Delta y + (v_3 \Delta x)^2/D_N + v_4 (\Delta y)^2/D_N + \ldots \) In this expansion, the second order terms scale as \( 1/D_N \), and can be made arbitrarily small for large \( D_N \). But, \( \Phi_N \) has to be an integer valued function for integer values of \( \Delta x \) and \( \Delta y \) for arbitrary \( D_N \) however large and this implies that \( v_3, v_4 \) and all higher coefficients have to be exactly zero. Therefore, within each patch, \( \phi(\xi, \eta) \) is a linear function of \( \xi \) and \( \eta \).

The above discussion implies that within a single patch

\[
\Phi_N(x, y) = f_{\text{per}}(x, y) + F_N + v_1 x + v_2 y
\]

where \( f_{\text{per}} \) is a periodic function on the lattice with zero mean, \( v_1 \) and \( v_2 \) are constants which depend on the patch. The constant \( F_N \) is an \( N \)-dependent integer, which is the same for all sites within a patch, as long as they stay in the patch. The periodicity of \( f_{\text{per}} \) would be expected to be the same as the periodicity of the starting background configuration. To get the continuum limit function \( \phi(\xi, \eta) \), we have to average \( \Phi_N(x, y) \) over a ‘coarse-graining scale’ much larger than 1. Then, since the mean value of \( f_{\text{per}} \) may be assumed to be zero without any loss of generality, it does not contribute to \( \phi(\xi, \eta) \), and we get

\[
\phi(\xi, \eta) = f + v_1 \xi + v_2 \eta,
\]

where \( f = \lim_{N \rightarrow \infty} F_N/D_N \). Thus \( f \) measures the mean increase in the number of rotations at a site \((x, y)\) in the patch, per unit increase of diameter, as long as it stays within the patch.

Let \( \delta \vec{x} \) be a basis vector of the superlattice of the background pattern. Within a single patch, this difference \( \Phi_N(\vec{x} + \delta \vec{x}) - \Phi_N(\vec{x}) = (v_1, v_2) \cdot \delta \vec{x} \) has to be an integer. For the present pattern, the periodicity in both directions is 1, and \( \delta \vec{x} \) equals \((1, 0)\) or \((0, 1)\). Hence \( v_1, v_2 \) can only take integral values, and \( f_{\text{per}}(x, y) \) may be taken to be 0. Thus, within a patch, \( \Phi_N(\xi, \eta) \) is of the form

\[
\Phi_N(x, y) = F_N(m, n) + mx + ny, \quad \text{and}
\]

\[
\phi(\xi, \eta) = f_{m, n} + m \xi + n \eta
\]

where \( m \) and \( n \) are integers depending on the patch, and \( F_N(m, n) \) is an integer. The coefficients of \( x \) and \( y \) in equation (5) do not change as \( N \) is changed, as long as the point \((x, y)\) stays within a given patch.

For the pattern in figure 1, within a given patch, \( \nabla \phi(\xi, \eta) = (m, n) \) and hence is constant. However, its value might change discontinuously across patch boundaries (but it does not always, as is shown in figure 5). Since \( \nabla \phi \) does not always change across boundaries between patches of constant orientation, we define patches of constant gradient of the visit function, or more briefly ‘visit-function-patches’ as the regions across which \( \nabla \phi \) is the same constant. By the continuity of the scaled visit function \( \phi(\xi, \eta) \), the value of
Figure 5. (a) The first quadrant of the pattern in figure 1 showing patches of constant orientation, and (b) the same area showing the visit-function-patches. Within a single visit-function-patch, $\nabla \Phi$ is constant. We plot the two components of $\nabla \Phi$ using a two-colour-checkerboard code, with the colour on odd pixels showing $-\nabla_x \Phi$, and the even pixels showing $-\nabla_y \Phi$. The values of $\nabla \Phi$ are also directly displayed in the patch. Note that there are four patches of different arrow orientation within the circle shown in (a), whereas there are only two regions with different values of $\nabla \Phi_N$ in the corresponding circle in (b). The colour code in (a) is: dark blue—→, light blue—↑, yellow—←, red—↓.

the constant part $f_{m,n}$ also does not change across boundaries when $m$ and $n$ don’t change. It is convenient to label a visit-function-patch using the value of $\nabla \phi$ in the patch. Then the patch $(m, n)$ refers to the visit-function patch for which $\nabla \phi = (m, n)$. To compare visit-function-patches with patches of constant orientation, compare figures 5(a) and (b).

Consider two adjacent patches with scaled visit functions $\phi_1 = a_1 + m\xi + n\eta$ and $\phi_2 = a_2 + m'\xi + n'\eta$. Then from the continuity of $\phi$ it implies that the equation of the boundary is $(m-m')\xi + (n-n')\eta = a_2 - a_1$. Since the slopes of the boundaries in figure 1 are either 0, $\pm 1$ or $\infty$, the vector $(m-m', n-n')$ has to be an integer multiple of $(0, 1)$, $(1, 0)$ or $(1, 1)$. The results of our numerical study show that the multiplicative constant is unity, that is, $(m, n)$ only changes by $(0, \pm 1)$, $(\pm 1, 0)$ or $(\pm 1, \pm 1)$ across a patch-boundary in this pattern. Since we do not observe patches with only diagonal boundaries, we conclude that all values of $(m, n)$ are present. Figure 6 shows the arrangement of these patches in the actual figure.

We note that there is a crowding of different patches near the origin in the pattern. One way to avoid this, and be able to see details of the pattern near the origin, is to take a $1/r$ transform of the picture. Such a transformation was first used in [20] for sandpile patterns. Under this transform, a point with polar coordinates $(r, \theta)$ is mapped to the point $(1/r, \theta)$ in the transformed picture, and the colour in the transformed picture is given the colour of the corresponding point in the original pattern. Taking a transform of a pattern produced on the $1000 \times 1000$ lattice, we get figure 7(a). The colours in the picture are chosen to show $\nabla \phi(\xi, \eta)$, of which figure 5(b) shows the first quadrant. Under this mapping, straight line boundaries map to non-straight arcs. However, the
Figure 6. (a) A zoomed-in figure of some patches in the first quadrant in figure 5(b), illustrating the arrangement of visit-function-patches, labelled by their \((m, n)\) values. Note that the vector \(\nabla \Phi_N\) changes by \((0, 1)\) across a horizontal boundary, by \((1, 0)\) across a vertical one, and by \((1, 1)\) across a diagonal boundary. (b) The patch \((m, n)\) and its neighbours. The points A and B lie on a line of slope \(\pm 1\). In section 4.3, we assign the vertex \((m, n)\) in the resistor network to the lower boundary of the patch \((m, n)\), to make a correspondence between visit-function-patches and a tiling of the pattern by squares.

Figure 7. (a) The \(1/r\) transform of the plot of \(|\nabla \Phi_N|\), of which figure 5(b) is the first quadrant. As in that figure, the odd and even sites give \(|\nabla_x \Phi_N|\) and \(|\nabla_y \Phi_N|\) respectively. The red spots in the patches next to the central patch are artifacts caused by the \(1/r\) transform of the diagonal lines within patches of the visit function in figure 5(b). (b) Central part of the adjacency graph of the visit function, the first quadrant of which corresponds to the patches shown in figure 5(b). The central node is the vertex \((0, 0)\) corresponding to the outermost patch where \(\phi(\xi, \eta) = 0\).
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transformation preserves the adjacency structure of different patches (i.e. which patches share a boundary). In the transformed picture, we can notice more easily a rectangular grid-like structure, such that the visit-function-patch with label \((m, n)\) is found in the neighborhood of the point with Cartesian coordinates \((m, n)\) in the transformed picture. This is shown in figure 7(b), where we have shown the adjacency graph of visit-function-patches in the pattern. Here a node in the graph denotes a patch, and we draw an edge between two nodes only if they share a common boundary.

Thus, we see that the adjacency graph for patches of the visit function is an infinite graph which takes the form of a square lattice with extra diagonal bonds, of which figure 7(b) shows the neighborhood of the node corresponding to the patch at infinity.

4. Characterizing the pattern as a tiling

In this section we will determine the sizes of the different patches in the pattern, using as input the observed arrangement of these elements in the pattern. We do this by noting that the observed pattern may be thought of as a tiling of a square by smaller squares, where each square tile is made of two triangles of different colours (compare figures 1 and 9(a)) . We will ignore the colours, and focus on the sizes and the adjacency relations between different tiles.

We consider the asymptotic pattern defined by equation (1) (figure 1). This rescaled asymptotic pattern has a diameter \(D = 1\), and has features at arbitrarily small scales. From this pattern one constructs a tiling of the unit square as follows: we take each tilted boundary between patches, and construct its bounding square. The bounding square of a tilted line is the square for which this line is the diagonal. Drawing these bounding squares for each diagonal line, and then erasing these diagonal lines, one gets from figure 1 to figure 9(a). If one assumes that the asymptotic pattern is fully tiled by patches of constant orientation, it follows that the squares tiles so constructed also tile the pattern without overlaps or voids, as each patch of constant orientation is fully covered by one square (when the patch is a triangle), or by two squares (when it is a dart). Each square tile thus overlaps two patches of constant orientation. Alternatively, one could start from the six-sided visit-function-patches, and construct squares as above from the two diagonal boundaries. Then the remaining area of the patch forms a single square. That is, a patch of the visit function is made of three squares, two shared with other patches, and one lying entirely within the patch. See figure 6(b).

4.1. The Brooks–Smith–Stone–Tutte mapping

The Brooks–Smith–Stone–Tutte (BSST) mapping [21] between square tilings and resistor networks allows one to find the sizes of all the squares in a square tiling knowing only their arrangement in the tiling. We first recapitulate the BSST mapping and then use it to calculate the sizes of patches in figure 1.

Consider the tiling of a rectangle by squares shown in figure 8. We refer to the outer rectangle as the ‘bounding’ rectangle. We associate with a given tiling a resistor network as follows: With each horizontal line, we associate a node of a resistor network. Two nodes are joined by a \((1 \, \Omega)\) resistor if parts of the corresponding segments form opposite edges.
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Figure 8. An application of the BSST theorem. (a) A tiling of a rectangle by squares. (b) The resistor network constructed from the tiling. All the edges of the graph have unit resistance. We put a voltage source across the graph equal to the height of the rectangle and determine the currents produced in each edge. (c) The lengths of the squares determined from the values of the currents in (b), converted to integers by multiplying by 32 (The height of the big square becomes 32. Its width can be calculated from the currents to be 33.). Note that the arrows only denote the direction of the current (from top to bottom, in the tiling).

of a tiling square. The resistors (that is, the edges) in the network thus correspond to squares. We associate a voltage $V(v)$ with the nodes $v$ of the network, where $V(v)$ is the vertical height of the horizontal edge corresponding to the node $v$ in the tiling. Assigning unit resistance to all resistors, the current between nodes $v$ and $u$ is $V(v) - V(u)$, and this equals the length of the side of the square joining the corresponding horizontal lines in the tiling. Then the Kirchoff current balance conditions of the resistor network corresponds to the condition that the total horizontal length $L(v)$ of the segment $v$ is the same whether calculated using its set of upper neighbours $\{u\}$ or the set of lower neighbours $\{\ell\}$

$$\sum_{\{u\}} [V(u) - V(v)] = \sum_{\{\ell\}} [V(v) - V(\ell)] = L(v), \text{ or}$$

$$\sum_{n} [V(n) - V(m)] = 0 \quad (8)$$

where the summation in the second line is over all neighbours $n$ of the node $m$ in the graph. Denoting by $L$ and $U$ the vertices corresponding to the lower and upper edges of the bounding square, we define $V(U) = 1$ and $V(L) = 0$. Then, the unique solution for the voltage developed on the nodes of the resistor network constructed in the above way gives the heights (in the original tiling) of all the horizontal segments.

Note that when four squares meet at a point, the horizontal segment going through the point can be considered as one segment or two. An example is the meeting point of segments G and B in figure 9(a). In such cases, we will always choose the latter, more general, option (The former case corresponds to the degenerate case when the voltages at B and G are equal.).

doi:10.1088/1742-5468/2014/11/P11030
Figure 9. (a) The pattern in figure 1 as a square tiling. The dashed lines indicate segments not present in the pattern in figure 1, but drawn to help visualisation. The first few levels of the corresponding network are shown. (b) Part of the resistor network corresponding to the tiling in (a). (c) The graph in part (b), showing the different labelled nodes, redrawn as a square lattice. The edges between nodes labelled in (a) by capital letters are in black, with arrows indicating the direction of the current flow. Note that some nodes (for example B and G) correspond to horizontal segments which lie at the same height, but have not been grouped together as a single segment. The arrows only denote the direction of the current.

4.2. The tiling as a resistor network on a square grid

The resistor network corresponding to this tiling, to a depth of three layers starting from the outer boundary, is shown in figure 9(b). Drawing more layers of the resistor network, one will get an infinite graph with an infinite number of nodes. Most of these nodes correspond to the edges of very tiny square tiles near the centre of the pattern.

The main simplification that allows us to analyse this problem exactly is the fact that the resistor network graph has a very simple structure: it is equivalent to the resistor-network formed by connecting unit resistors to make a square grid, with only one missing bond between nodes A and K (figure 9(c)). One can easily verify that the networks in figures 9(b) and (c) are equivalent, being different pictorial representations of the same graph. For the present, we will take this crucial observation as an induction from the features of the observed pattern.

For the resistor network shown in figure 9(c), it is convenient to label nodes by their integer coordinates \((m, n)\). We will choose the coordinates so that the node A has the Euclidean coordinate \((0,0)\) and the node K has the coordinate \((0, -1)\). The boundary conditions for the voltage on this resistor network are given by the positions of the upper and lower horizontal segments of the bounding square. Since the pattern has been rescaled to be of unit diameter centred at the origin, this gives \(V(0, 0) = 1/2\), and \(V(0, -1) = -1/2\).

Then equation (8) in our case becomes

\[
V(m, n) = \frac{1}{4}[V(m + 1, n) + V(m - 1, n) + V(m, n + 1) + V(m, n - 1)]
\]  

(9)

doi:10.1088/1742-5468/2014/11/P11030
The solution of the resistor network problem for this graph is well known, $V(m, n)$ being given by the following superposition of Green’s functions for an infinite square lattice $G_{sq}(m, n)$

\[
V(m, n) = 2(G_{sq}(m, n + 1) - G_{sq}(m, n))
\]

where

\[
G_{sq}(m, n) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \frac{1 - \cos(k_1 m + k_2 n)}{2 - \cos k_1 - \cos k_2}
\]

It can be verified that indeed $V(0, 0) = 1/2$. For finite values of $m$ and $n$, this integral can be evaluated in closed form, and the result is of the form $a_{m,n} + b_{m,n}/\pi$, where $a_{m,n}$ and $b_{m,n}$ are rational numbers [22]. A computationally efficient formula for $G_{sq}(m, n)$, which can be used in Mathematica™ to get exact expressions for the first few $V(m, n)$, is given in [23],

\[
G_{sq}(m, n) = \int_{0}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{\{m,s\} \cos(ns)}}{\sinh(s)}
\]

with $\cosh(s) = 2 - \cos(y)$.

From this solution we get the sizes of various elements in the pattern. For example, the sizes of the big squares at the four corners of figure 9(a) are given by the difference in the vertical co-ordinates of lines A and B. From figure 9(c) this is equal to $V(A) - V(B) = V(0,0) - V(1,0)$. Using the values $V(1,0) = \frac{\pi}{2} - 1/2$ and $V(0,0) = 1/2$, the size of these largest squares in the pattern relative to the size of the pattern is $1 - \frac{2}{\pi}$.

For $m, n \gg 1$, the corresponding horizontal lines get closer and closer to the origin of the original pattern. On the resistor network for large $m, n$, $V(m, n)$ looks like the electric potential due to a dipole at the origin, and tends to zero for large $m, n$. $V(m,n) \sim \frac{\cos\theta}{(m^2 + n^2)^{1/2}} \sim \frac{n}{(m^2 + n^2)}$ in this region. Consider moving along the $y$-axis of the resistor network. Then $V(m=0,n) \sim 1/n$. In the tiling, this gives the distance from the origin of the $n$th ring of squares (counted from the outside inwards) as $r_n \sim 1/n$. Then the size of the squares in this ring varies as $\frac{\partial r_n}{\partial n} \sim \frac{1}{n^2}$.

### 4.3. Determining the visit function

In the previous section, we were able to determine the sizes of patches by solving the Kirchoff equations for a resistor network that was a square lattice. In figure 7, we also encountered the square lattice as the adjacency graph for the patches. These are somewhat different constructions, in that the nodes in the adjacency graph are patches, but nodes in the BSST construction are horizontal line segments in the pattern. This difference is resolved by noting that we can associate a unique horizontal segment to each visit-function patch, which we choose to be its lower horizontal boundary. Label this horizontal boundary by the same label as the patch, that is, $(m, n)$. The choice of the lower boundary for this association ensures that the labels on the adjacency graph and the resistor network are the same (For instance, the lower boundary of the bounding square is the lower horizontal boundary of the patch $(0, -1)$, and also the node $(0, -1)$ in the resistor network in figures 9(b) and (c).). 

To determine the visit function for the scaled pattern, we have to determine $f_{m,n}$ for all $(m,n)$ in equation (6). The visit function in the patch with label $(m,n)$ is

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\[ \phi_1(\xi, \eta) = m\xi + n\eta + f_{m,n}. \]

For two adjacent patches with labels \((m, n)\) and \((m, n + 1)\), continuity of the visit function along the horizontal boundary between them gives the position of the boundary as

\[ y = f_{m,n} - f_{m,n+1} \]  

Since this is the lower boundary of the patch \((m, n)\), we can make the identification \(y = V(m, n)\) in equation (10), giving

\[ f_{m,n} = -2G_{sq}(m, n) \]  

This gives \(f_{0,0} = 0\) and \(f_{0,1} = -1/2\), which is consistent with the normalisation that the upper boundary of the pattern (the boundary between patches \((0, 1)\) and \((0, 0)\)) lies along the line \(x = 1/2\). The \(f\)'s are negative because patches with positive \(m, n\) lie in the upper right quadrant of the pattern. For large \(m\) and \(n\), \(f_{m,n} \sim -\log (m^2 + n^2)\).

5. Other starting backgrounds

In this section, we study the patterns formed by walkers deposited on the origin of backgrounds formed by periodic repetition of unit cells of size 2 \(\times\) 2. Using the symmetries of rotation, reflection and translation to classify all the 2 \(\times\) 2 unit cells, we get 18 inequivalent classes of unit cells. However, the number of different asymptotic patterns (classified according to the adjacency structure of their patches) is only 4. The reason for this is not clear. For example, the unit cell \(\to\uparrow\to\to\to\) clearly belongs to a different symmetry class than the unit cell \(\to\uparrow\to\to\), which forms the background with all arrows parallel, and forms the pattern in figure 1. Yet the pattern formed on the former background has the same structure. We call this the type I pattern. This pattern has already been discussed in the previous two sections. We will discuss the remaining three inequivalent pattern types in this section.

5.1. Type II

We first consider the background formed, starting from the unit cell in figure 10(a). The pattern generated on this background is shown in figure 10(b). We see from the picture generated that this pattern can be seen as a tiling of the unit square by rectangles with an aspect ratio 1:2. The BSST mapping is easily generalised to a tiling by rectangles: the resistor in the network corresponding to a rectangle with vertical to horizontal side ratio \(p/q\) has resistance \(R = p/q\). The corresponding resistor network is drawn in figure 10(c). Interestingly, this network has the same structure as the one for the pattern studied in section 4, with different resistances along the bonds depending on the aspect ratio of the tile. The vertical bonds have \(R_1 = 2\Omega\) and the horizontal bonds have \(R_2 = \frac{1}{2}\Omega\). The solution for the potential on such a lattice is given by the so-called rectangular lattice Green’s function \(G_{rect}(R; m, n)\), which is the generalization of the square lattice Green’s function to a lattice with resistance \(R\ \Omega\) along the vertical bonds and \(1\Omega\) along the horizontal bonds. This lattice is equivalent to the present lattice provided all resistances are scaled down by a factor 2; here this is built into the scale factor \(Q\) in equation (16).

doi:10.1088/1742-5468/2014/11/P11030

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Figure 10. Pattern formed on the initial background generated by the 2 × 2 unit cell given in (a) is shown in (b) after 700 walkers put at the origin have left the lattice. Colour code: dark blue—→, light blue—↑, yellow←, red—↓. (c) The resistor network for the pattern, where black bonds denote a resistance of 1 Ω, red, a resistance of 0.5 Ω and green, a resistance of 2 Ω.

The computationally efficient formula for \( G_{\text{rect}}(m, n) \) is [23]

\[
G_{\text{rect}}(R; m, n) = \int_0^\pi dy \frac{1 - e^{|m|s \cos(ny)}}{2\pi \sinh(s)}
\]

where \( s \) is defined implicitly \( \cosh(s) = 1 + \frac{1}{R} - \frac{1}{4\pi^2} \cos y \).

The solution of \( V(m, n) \) with \( V(0, 0) = \frac{1}{2} \) and \( V(0, -1) = -\frac{1}{2} \) is given by the superposition of two Green’s functions:

\[
V(m, n) = Q\left[ G_{\text{rect}}(R = 4; m, n + 1) - G_{\text{rect}}(R = 4; m, n) \right]
\]

The charge \( Q \) is determined by the condition that \( V(0, 0) = \frac{1}{2} \), which gives \( Q = 1/[4 - \frac{2}{\pi} \tan^{-1}(2)] \). The height of the big rectangles at the corners relative to the height of the figure is hence given by \( l = V(0, 0) - V(0, 1) = \frac{1}{2} - \frac{Q}{2} (1 - \frac{2}{\pi} \tan^{-1} \frac{1}{2}) \approx 0.259286 \).

We can also determine the aspect ratio of the figure, by also using the BSST mapping for the pattern rotated by 90°. The structure of the resistor network is again the same (figure 10(c)), except that the 0.5 Ω and 2 Ω bonds are interchanged. Hence, for the new network, \( V'(m, n) = V(n, m) \frac{Q'}{Q} \), where \( Q' \) is determined by the condition that \( V(0, 0) = \frac{1}{2} \). This gives \( Q' = (1 - \frac{2}{\pi} \tan^{-1} \frac{1}{2})^{-1} \). One can then calculate the breadth of the big rectangle at the corner relative to the width of the figure as \( b' = \frac{1}{2} - \frac{Q'}{2} (2 - \frac{4}{\pi} \tan^{-1}(2)) \approx 0.43433 \). One then uses the condition that \( 2lH = b'W \) to determine the aspect ratio of the figure to be \( H/W \approx 0.83755 \).

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Figure 11. Pattern formed by the initial background generated by the $2 \times 2$ unit cell given in (a) is shown in (b) after 900 walkers put at the origin have left the lattice. Colour code: dark blue→, light blue↑, yellow←, red↓.

Figure 12. (a) The resistor network for the pattern in figure 11(b), where black bonds denote a resistance of $1 \Omega$, red, a resistance of $0.5 \Omega$ and green, a resistance of $2 \Omega$. (b) The resulting network, when we replace the $2 \Omega$ resistors by a series combination of two $1 \Omega$ resistors, and connect the new added neighbouring nodes by $1 \Omega$ resistors (shown as pink bonds in figure). Since in our problem, the end points of the pink bonds are at the same potential, this does not change the currents. In this new network, all resistors have equal resistance.

5.2. Type III

Now consider the initial periodic structure having the unit cell in figure 11(a). The resulting pattern after 900 walkers is shown in figure 11(b). The characterization and analysis of this pattern is very similar to previous cases. The corresponding resistor network is given in figure 12(a). It is also a square lattice, but the bonds connecting sites $(i, 0)$ and $(i, -1)$ for all integers $i$ have $R' = 2 \Omega$ whereas the rest of the bonds have $R = 1 \Omega$. Figure 12(b) gives an equivalent square lattice structure for this network.

Using the BSST mapping, the vertical positions of horizontal lines with respect to the centre of the figure in the upper left half of figure 11(b) are given by the voltages in the first quadrant in the square lattice in figure 12(b), taking the point with $+\frac{1}{2}$ as the origin.
In terms of the square lattice Green’s function $G_{\text{sq}}(m, n)$, equation (12),

$$V_1(m, n) = Q_1(G_{\text{sq}}(m, n + 2) - G_{\text{sq}}(m, n))$$

(17)

The condition $V(0, 0) = \frac{1}{2}$ then gives $Q_1 = (2 - \frac{4}{\pi})^{-1}$. To determine the aspect ratio of the figure, one determines the positions of the horizontal lines with labels $(m, n) = (0, 1)$ and $(1, 0)$, call them $l_1$ and $l_2$. The size of the big squares at the boundary is then $(\frac{1}{2} - l_2)$ and that of the square between then is $(\frac{1}{2} - l_1)$. The horizontal extent of the pattern is then $(\frac{1}{2} - l_1) + 2(\frac{1}{2} - l_2) = (\pi - 2)^{-1}$. Since the voltages are normalized such that the vertical extent of the pattern is 1, this gives the aspect ratio as $H/W = (\pi - 2)^{-1}$.

### 5.3. Type IV

Holroyd and Propp [10] classify initial backgrounds on infinite lattices into two types. ‘Transient’ backgrounds are those where the walker visits all sites only finitely many times before reaching infinity, while ‘recurrent’ backgrounds are those where it visits each site infinitely many times before reaching infinity. For periodic backgrounds on a lattice of size $L \times L$, we will call a background transient, if the first walker exits in a time of order $O(L)$, and recurrent if the time is $O(L^3)$. The backgrounds we have studied so far in this paper were all transient. For the backgrounds studied so far, after $N$ walkers have been put on an $L \times L$ lattice, the total time taken for all the walkers to reach the sink is, for $N \gg 1$, $AN^3 + BNL + O(N^2)$. For $1 \ll N \ll L$, this is linear in $L$. When the size of the pattern becomes comparable to the lattice, this becomes $O(L^3)$. $A$ and $B$ are constants for a given background.

Now we consider the initial periodic structure having the unit cell in figure 13(a). Here, on an $L \times L$ lattice, the first walker takes $O(L^3)$ steps to get to the sink, and hence this background is recurrent. Figure 13(b) shows the pattern left behind by the first walker. This resulting background is no longer recurrent, and the next walker takes just $L/4$ steps to reach the boundary. Putting more walkers at the origin after the first walker has exited, one gets the pattern observed in figure 13(c).

The resistor network for this tiling is shown in figure 14(a), and is again a square lattice with not all resistances equal to $1\,\Omega$. Thus, at least starting from $2 \times 2$ unit cells,
we find that the adjacency graph of patches is the same for all types of patterns formed, and the corresponding resistor networks differ not in their structure, but only in the values of resistance. Figure 14(b) gives an equivalent square lattice structure for this network.

The voltages in the first quadrant of figure 14(a) are given by

\[ V_2(m, n) = Q_2(G_{sq}(m, n - 2) + G_{sq}(m + 1, n - 2) - G_{sq}(m, n) - G_{sq}(m + 1, n)) \] (18)

where \(Q_2\) is determined by the condition that \(V(1, 0) = \frac{1}{2}\). The aspect ratio for this pattern can be observed from the symmetry of the pattern itself to be unity.

6. Bounded fluctuations and quasiperiodicity

In this section we examine the growth with \(N\) of the horizontal and vertical extent of the full pattern in figure 1, and the sizes of the different patches which comprise it. As the pattern shows proportionate growth, each of these would grow as \(\alpha_P N\) for large \(N\), where \(\alpha_P\) is in general an irrational number, which depends on the patch. We would like to study how much the actual size at a given value of \(N\) differs from the linear growth expectation. Our surprising observation is that for the patterns studied here, for all patches that we looked at, the actual value of the diameter of the patch stays as close to the linear growth as is possible, differing from \(\lfloor \alpha_P N \rfloor\) by not more than \(O(1)\).

As the first example, consider the horizontal extent (distance of the boundary from the origin) of the pattern along the positive \(x\)-axis. Denote this extent after \(N\) walkers have left the lattice, by \(H(N)\). Similarly, let \(H_1(N)\) be the distance from the origin of the vertical boundary of the patch of constant orientation \((1, 0)\) (the largest dark blue triangle in figure 5(a)). In section 4, we showed that, for large \(N\),

\[ \lim_{N \to \infty} H(N)/N = 1/4, \]

\[ \lim_{N \to \infty} H_1(N)/N = \left(\frac{3}{4} - \frac{2}{\pi}\right) N. \] (20)
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Figure 15. For the pattern in figure 1, (in red) the deviation of $H_1(N)$, the horizontal extent of the second-farthest vertical line along the positive $x$-axis measured every fourth step from the asymptotic linear value $(3/4 - 2/\pi)N$, and deviation from $\text{Nint}[(3/4 - 2/\pi)(N-3)]$ (in blue). On the $x$-axis is $\lfloor N/4 \rfloor$ ($H_1(N)$ only changes every fourth step).

We find that, after an initial transient, $H(N)$ increases precisely periodically, increasing by 1 every fourth step. In fact, we can write down an exact formula for $H(N)$:

$$H(N) = \lfloor (N + 1)/4 \rfloor$$

for $N \geq 3$.

Now consider the behaviour of $H_1(N)$ with $N$. We find that $H_1(N)$ increases only when $H(N)$ increases, but, clearly, as the average rate of increase is lower, sometimes $H(N)$ will increase, but $H_1(N)$ will not. In figure 15, we have plotted $H_1(N) - (3/4 - 2/\pi)N$ every fourth step versus $\lfloor N/4 \rfloor$. For all $N$ in the range plotted, these deviations from the linear growth are always of magnitude strictly less than 1, but have a complicated dependence on $N$.

Thus, we have seen that the exact value of $H_1(N)$ is very well approximated by $(3/4 - 2/\pi)N$, and the actual value is either the integer just above or just below this real approximant. As a simple exercise, we considered the sequence $g_N = \text{Nint}(f(N-a))$, where $\text{Nint}()$ is the nearest-integer function, $f = (3/4 - 2/\pi)$, and $a$ is an adjustable constant. Figure 15 shows that the difference between the actual sequence and this sequence, in the range plotted, is nonzero only about 1% of the time, for the choice $a = 3$. Also, when it is not zero, it takes the value $-1$. The Fourier transform for $g_N$ is shown in figure 16 along with the Fourier transform of the actual sequence. It is seen that the peaks (when the Brillouin zone of frequencies is scaled to go from 0 to 1) for the sequence are at multiples of $f = 3 - 8/\pi$ as close an accuracy as the finite length of the sequence allows. While most of the peaks for the real sequence fall at the same frequencies, their heights are slightly smaller. There are also additional peaks which are not there in this approximating sequence. In the next few paragraphs we examine the origin of these extra peaks by taking a closer look at the difference between the actual sequence and this simple approximant.

The growth of $H_1(N)$ may be encoded in a binary sequence $\{h_i\}$ defined as follows: Let $h'_j$ denote the increase in $H_1(N)$ when $H(N)$ increases from $j-1$ to $j$. Clearly,
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Figure 16. The power spectra (on a log scale and arbitrary units) of (in red) the actual sequence $h'_{N/4}$ for the pattern and (in blue) the Fourier transform of the sequence Nint$[(3 - \frac{8}{\pi})i]$. 

$$H_1(N) = \sum_{j=0}^{N/4} h'_j.$$ Also, asymptotically, $\sum_{j=0}^{N} h'_j = (3 - 8/\pi)N$. The power spectrum for the sequence $\{h'_j\}$ obtained from the pattern is plotted in figure 16. There are a number of peaks of various magnitudes, which is the typical behaviour of quasiperiodic sequences.

We now construct a sequence of 1’s and 0’ which has a growth rate $3 - 8/\pi$, using a different deterministic rule: the best rational approximant to $(3/4 - 2/\pi)$ using the nearest-integer continued fraction representation. The sequence of continued fractions obtained by truncating the expansion at different orders is easily seen to be

$$1/2, 5/11, 39/86, \ldots \tag{21}$$

Denote the string ‘01’ by A. For $N < 11$, the best sequence of 0s (no growth during the time-step) and 1s (denoting growth by one unit), which best approximates the irrational growth rate $(3/4 - 2/\pi)$ would be the sequence AAA..., that is, 010101..., such that the total length is less than $N$. For $N > 11$, one would need to have five 1’s for every six 0’s, and hence the best sequence (for $N < 86$) would be $(A^50)(A^50)(A^50)\ldots$. Compare this with the actual sequence $h'_{N/4}$ from $N/4 = 242$ to $N/4 = 511$: 

$$(A^50)(A^50)(A^50)(A^50)(A^50)(A^40)(A^50)(A^40)(A^60)(A^40)(A^50)(A^60)(A^40)(A^60)(A^40)$$

$$\ldots \tag{22}$$

$$\ldots (A^40)(A^40)(A^50)(A^50)(A^50)(A^50)(A^50)(A^50)(A^50)(A^40)(A^60)(A^40)$$

It is seen that sometimes the string $(A^50)(A^50)$, in the simplest rational approximant at this order, is replaced by the string $(A^40)(A^60)$. We have no good explanation of why this should occur, except to say that this is probably caused by steric effects of nearby patches.

doi:10.1088/1742-5468/2014/11/P11030
This effect of nearby patches is felt even at the boundary of the pattern. Let $V(N)$ denote the vertical extent of the pattern along the positive $y$-axis after $N$ walkers. Figure 17(a) shows $V(N) - \frac{N}{3} + \frac{1}{4}$ versus $N$ and we plot its power spectrum in figure 17(b). This sequence is not periodic, unlike the horizontal extent, although the associated growth rate is the same rational number $\frac{1}{4}$. Again, we note that the magnitude of the difference is always $O(1)$, in fact being $\leq 1$ in the range plotted.

Our numerical studies suggest that this behaviour, where deviations from the exact linear growth are bounded and quasiperiodic, is a general property of the positions of all patch boundaries in the pattern.

7. Summary and concluding remarks

In this paper, we studied the growing patterns formed by depositing rotor-router walkers on the origin of a finite two-dimensional lattice, starting with periodic background configurations of arrows. We find that the patterns show rich internal structure which scales proportionately with the size of the pattern. We characterized this structure by mapping it to a resistor network and solving the Kirchoff equations on the network. We make some reasonable assumptions that the regularities seen in the observed patterns continue to hold for larger $N$, and this allows us determine the exact asymptotic scaling limit of the visit function $\phi(\xi, \eta)$. We find that the function $\phi(\xi, \eta)$ is a piece-wise linear function of its arguments, with integer slopes. We presented numerical evidence that the magnitude of deviation of the pattern size from the linear growth estimate remains less than 1, and is a quasiperiodic function of the number of walkers.

It would be interesting to explore other patterns where the basic tiles are not squares. For example, in the sandpile model, patterns on a triangular lattice were found to be hexagonal in shape and composed of triangular tiles [3]. For such tiles, there is no equivalent of the Brooks–Smith–Stone–Tutte theorem. Also, the quasiperiodicity and reason why the deviations in the diameters of patches from linear dependence on $N$ in these rotor-router patterns are as small as possible (the deviation never exceeds 1 in magnitude) remains to be understood. It would certainly be desirable to have a more rigorous derivation of the presumably exact results presented in this paper.
Acknowledgments

DD would like to acknowledge partial support by the Indian DST via the grant DST-SR/S2/JCB-24/2005.

Appendix. Derivation of equation (9) from matching of boundary conditions

Consider the arrangement of six-sided visit-function-patches shown in figure 6 (b). We note that the point A lies on the boundary of patches labeled \((m,n)\) and \((m-1,n)\). Continuity of \(\phi\) across the boundary implies that

\[
\phi(A) = f_{m-1,n} + (m-1) \xi_A + n \eta_A = f_{m,n} + m \xi_A + n \eta_A,
\]

where \(A \equiv (\xi_A, \eta_A)\). Thus

\[
\xi_A = f_{m-1,n} - f_{m,n},
\]

(A2)

Similar arguments give

\[
\eta_A = f_{m,n} - f_{m,n+1},
\]

(A3)

\[
\xi_B = f_{m,n} - f_{m+1,n},
\]

(A4)

\[
\eta_B = f_{m,n-1} - f_{m,n}.
\]

Now, we note that the line AB was a boundary of a patch of constant orientation, and has a slope that can only take one of the values 0, ±1, ∞. In this case, it must be 1. Hence we get

\[
f_{m+1,n} + f_{m-1,n} + f_{m,n-1} + f_{m,n+1} - 4f_{m,n} = 0
\]

(A5)

which is equation (9), obtained earlier using the BSST theorem.

References

[1] Dhar D, Sadhu T and Chandra S 2009 Pattern formation in growing sandpiles Europhys. Lett. 85 48002
[2] Sadhu T and Dhar D 2010 Pattern formation in growing sandpiles with multiple sources or sinks J. Stat. Phys. 138 815
[3] Sadhu T and Dhar D 2012 Pattern formation in fast-growing sandpiles Phys. Rev. E 85 021107
[4] Dhar D and Sadhu T 2011 The effect of noise on patterns formed by growing sandpiles J. Stat. Mech. P03001
[5] Dhar D and Sadhu T 2013 A sandpile model for proportionate growth J. Stat. Mech. P11006
[6] Dhar D 2006 Theoretical studies of self-organised criticality Physica A 369 29
[7] Priezzhev V B, Dhar D, Dhar A and Krishnamurthy S 1996 Eulerian Walkers as a model of self-organised criticality Phys. Rev. Lett. 77 5079
[8] Povolotsky A M, Priezzhev V B and Scherbakov R R 1998 Dynamics of Eulerian Walkers Phys. Rev. E 58 5449
[9] Propp J 2010 Discrete analog computing with rotor-routers Chaos 20 037110
[10] Holroyd A E and Propp J 2010 Rotor Walks and Markov chains Contemp. Math. 520 105
[11] Holroyd A E, Levine L, Meszaros K, Peres Y, Propp J and Wilson D B 2008 Chip-firing and rotor-routing on directed graphs Prog. Probab. 60 331
[12] Levine L 2002 The rotor-router model Thesis Harvard University Senior
[13] Vadhan S P Pseudorandomness preprint available from http://people.seas.harvard.edu/~salil/pseudorandomness/

doi:10.1088/1742-5468/2014/11/P11030
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[14] Florescu L, Ganguly S, Levine L and Peres Y 2014 Escape rates for rotor walk in $\mathbb{Z}^d$ SIAM J. Discrete Math. 28 323
[15] Levine L and Peres Y 2008 Spherical asymptotics for the rotor-router model in $\mathbb{Z}^d$ Indiana Univ. Math. J. 57 431
[16] Levine L and Peres Y 2009 Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile Potential Anal. 30 1
[17] Kleber M 2005 Goldbug variations Math. Intelligencer 27 55
[18] Kapri R and Dhar D 2009 Asymptotic shape of the region visited by an Eulerian Walker Phys. Rev. E 80 051118
[19] Pegden W and Smart C K 2013 Convergence of the Abelian sandpile Duke Math. J. 162 627
[20] Ostojic S 2003 Patterns formed by addition of grains to only one site of an abelian sandpile Physica A 318 187
[21] Brooks R L, Smith C A B, Stone A H and Tutte W T 1940 The dissection of rectangles into squares Duke Math. J. 7 312
[22] Spitzer F 1964 Principles of Random Walk (Princeton: Van Nostrand) pp 124, 148–51
[23] Cserti J 2000 Application of the lattice Green’s function for calculating the resistance of an infinite networks of resistors Am. J. Phys. 68 896

doi:10.1088/1742-5468/2014/11/P11030