Calculation of depressurization coordinate in underground and offshore gas pipelines

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Abstract. The paper introduces a method for calculating the depressurization coordinate of an extended ultra-high pressure gas pipeline from the data measurements of pressure and flow rate at the outlet. The method is based on a mathematical model of a non-isothermal steady single-phase gas mixture flow through a pipeline of constant section on the quasi-linearization method. It allows to calculate the depressurization coordinate for extended underground and offshore ultra-high pressure gas pipelines with high accuracy, while considering varying leakage size and location. Examples of solving problems of practical interest are given. The influence of the flow rate and pressure settings error on the coordinate of depressurization calculating error has been demonstrated.

1. Introduction
An analysis of known accidents in surface and underground gas pipelines leads to the conclusion that most of them are associated with the depressurization of the pipeline. Timely detection and elimination of gas leaks are important tasks. It is especially difficult to locate leaks in underground and offshore pipelines, for which visual observation of the state and the environment is difficult. To date, many approaches have been proposed to solve gas pipeline depressurization site detecting problem. Leak detection methods are usually divided into external and internal [1, 2]. External methods include pipeline inspection, acoustic and optical methods. External methods also include monitoring of the soil adjacent to the gas pipeline, pipeline route satellite surveying, acoustic and optical tests of the gas pipeline state.

Acoustic methods [1, 3, 4] are based on the use of sensors installed along the pipeline. However, the recorded data includes not only the signal of the leak, but also the background signals of the gas pipeline normal operation. One of the problems with acoustic leak detection is separating the signal generated by the leak itself from the background signals. In [4], a method is proposed that allows you to eliminate signal parts that belong to the background noise. The filtered data is processed, and as a result, location of the leak site is determined with high accuracy. The authors of [4] note that this method allows to detect small gas leaks quickly. The effectiveness of the acoustic method depends on the distance between the sensors. For long pipelines, installing and maintaining a large number of sensors can be very costly. Leak detection technique based on acoustic methods alone may be of little value due to the potential for false alarms.

Optical methods [1, 5, 6, 7] are based on fiber optic sensors that can be used to measure various physical and chemical parameters. The optical fiber is placed as close to the pipeline as possible so that it can detect vibrations near the pipeline and other possible changes. When a
disturbance occurs, the signal passing through the optical fiber undergoes a change. This change
is detected and interpreted in order to determine the type of event that caused the change in the
signal. Optical methods can quickly locate and assess the leak, but these methods are expensive.

Thus, using acoustic and optical methods, it is possible to detect the location and size of
the gas leak quickly and accurately. However, these methods require expensive equipment to
be installed and maintained, resulting in high operating costs. A substantial review of acoustic
and optical methods is given, for example, in [1, 8].

Internal (model) methods [9, 10, 11, 12, 13, 14] are based on the gas transportation process
computer modeling, i.e., on transportation mathematical models and on computing systems
based on these models. Computer modeling has now become a powerful tool in solving
many engineering problems in the design and operation of gas pipelines. Computing systems
are installed on computers in the dispatch center. They allow real-time monitoring of gas
transportation. The disadvantage of model methods is the impossibility of using them to
determine the cause of the leak, in addition, model methods are difficult to use when there
is more than one leak.

Model methods based on computer simulation are economical and easy to implement. When
using the simulation method, the fact that the change in outlet pressure due to the occurrence
of a leak does not occur immediately must be considered. Therefore, in contrast to acoustic
and optical methods, the simulation method allows detecting a leak only sometime after its
occurrence.

Numerous publications have been dedicated to the gas leakage site location calculation by
the modeling method, starting from [11] in the 70s of the last century, and up to the present
time, for example, [12].

In this work [12], an extremely simplified mathematical model of gas transportation is used.
The model assumes a constant cross-section of the gas pipeline, one-dimensionality, stationary
and isothermal processes, constancy of the gas compressibility coefficient, and a negligible value
of inertia forces compared to pressure. The system of equations of such a model is easily
integrated, which leads to an uncomplicated explicit analytical dependence of the pressure on
the coordinate along the pipeline. This model allowed the authors to obtain a simple formula
for calculating the coordinates of the gas leakage location based on the values of pressure and
flow rate at the inlet and outlet of the gas pipeline. Unfortunately, in most gas pipelines,
especially underground and offshore, the processes are essentially non-isothermal and the gas
compressibility factor depends both on the changing temperature and on the pressure; therefore,
it is possible to directly use the results of [12] in rare cases. However, in terms of methodology,
work [12] is of undoubted interest.

A promising direction in the effectiveness study of the various methods for detecting leaks in
pipes is presented in the recently published interesting work [15]. It describes in detail a pipeline
leak simulator and its study of the effectiveness of acoustic, optical and model methods. The
authors conclude that acoustic and optical techniques, combined with simulation techniques,
can not only detect, locate and quantify a leak, but also prevent it.

This report [8] made in 2019 at the SPE conference on gas and oil technologies also needs
to be noted. In it, along with an overview of papers on gas leak detection using mathematical
modeling of processes in pipelines, an overview of papers on new trends associated with the use
of measuring probes is given.

In this article, we propose an effective method for calculating the coordinate of the
depressurization from the data of pressure and flow rate at the outlet measurements in an
extended underground or offshore gas pipeline. This method relates to internal (model) methods.
It is based on the Bellman quasi-linearization method [16] and on a mathematical model of a
non-isothermal steady natural gas flow through a pipeline, which was thoroughly researched in
this book [17]. It possesses several features linked to the choice of the initial approximation
and the algorithm for calculating a new iteration in the problem of identifying the coordinate of depressurization which provided its efficiency. The calculation of one problem variant takes no more than a few minutes on an average computer. The proposed method is an effective tool for detecting gas leaks in extended underground and offshore gas pipelines.

Calculation of the gas pipeline depressurization coordinate can be formulated as an inverse problem. A reliable solution of the inverse problem, as it is known, is possible only under the condition that the corresponding direct problem is well studied, namely, an adequate mathematical model of the process has been created and an effective algorithm for calculating the process parameters has been proposed. The model and algorithm for solving the direct problem of the gas mixture flow through an extended offshore gas pipeline at high pressures are given in our book [17]. There is also a solution given to the problem of identifying hard-to-determine model parameters from experimental data, such as the coefficient of hydraulic resistance and the total coefficient of heat transfer. The solution of this problem makes it possible to ensure the adequacy of the gas flow mathematical model while dealing with real problems.

2. Mathematical model of steady gas mixture flow through a gas pipeline of constant circular cross section

**Continuity equation:**

\[ \rho u S = Q^0, \]

**Momentum balance:**

\[ \frac{d}{dz} \left( p + \rho u^2 \right) = -\frac{\lambda \rho |u|}{4R} + \rho g \cos \alpha(z), \]

**Heat equation:**

\[ \rho u c_p \frac{dT}{dz} = p u T \left( \frac{\partial V}{\partial T} \right)_p \frac{dp}{dz} + \frac{\lambda \rho u^2 |u|}{4R} + 2\beta \frac{T^* - T}{R}, \]

**Equation of state:**

\[ pV = R_g T Z(p, T), \quad V = 1/\rho. \]

In the system of equations (1)–(4) \( Z(p, T) \) is the compressibility factor; \( V \) — specific gas volume; \( R_g \) — gas constant that depends on the composition of the gas mixture; \( z, L \) — coordinate along the axis of the gas pipeline and its length; \( \rho, p, T, u \) — density, pressure, temperature and gas velocity, respectively; \( R, S \) — inner radius and cross-sectional area of the gas pipeline; \( g \) — acceleration of gravity; \( \alpha(z) \) — the angle between the direction of gravity and the gas pipeline axis in the \( z \)th section; \( \lambda \) — hydraulic resistance coefficient; \( \beta \) — total heat transfer coefficient; \( c_p \) — specific heat of the gas mixture at constant pressure coefficient of; \( Q^0 \) — gas mass flow; \( T^* \) — environment temperature.

In the general case, \( \lambda, c_p, \beta \) and \( T^* \) can be \( z \) coordinate functions. Adequacy of the mathematical model (1)–(4) is ensured by the correct choice of following functions, and the compressibility coefficient \( Z(p, T) \) dependence from pressure and temperature in the investigated range of pressure, temperature, and gas flow rate changes.

This book [17] describes in detail calculation methods of \( \lambda \) and \( \beta \), there are also estimates of the influence of their variability on calculated flow characteristics. The specific heat coefficient \( c_p \) (4) can be calculated using one of the known thermodynamic formulas [18] or determined from the corresponding experimental data at the selected equation of state. The system of equations (1)–(4) is supplemented by the boundary condition at the gas pipeline inlet:

\[ z = 0, \quad p = p_0, \quad T = T_0. \]

The proposed model covers a wide range of practical tasks. It allows one to take into account the non-isothermal nature of the processes, the complex thermodynamics of the gas mixture.
at ultra-high pressures, the relief of the route, and the conditions of heat exchange with the environment.

This is how it differs from other works on calculating leak location coordinates based on modeling, for example, from this work [12].

The direct task is to calculate the pressure distributions \( p(z) \), temperature \( T(z) \) and gas density \( \rho(z) \) by the system of equations (1)–(4) under the boundary condition (5). A solution of the system of equations (1)–(4) in a wide range of changes \( p_0, T_0, Q^0, R, L \) exists, is unique, and can be obtained numerically, for example, by the Runge – Kutta method.

3. The gas mixture equation of state

It is convenient to use pressure and temperature as independent thermodynamic variables in tasks of gas leak locating. This determines the choice of the equation of state in the form of equation (4). There is some difficulty in the fact that, in a wide range of temperature changes at high pressures, it is not easy to set a single form of the dependence of the compressibility coefficient on \( p \) and \( T \) for gas mixtures of different compositions. As is known [19, 20], analytical equations of state, such as the Redlich – Kwong, Peng – Robinson, Benedict – Webb – Rubin equations, are more accurate. However, the analytical equations of state are cubic relatively to the specific volume \( V \) and to the compressibility factor \( Z(p, T) \). This complicates their direct use in model (1)–(3).

When solving real problems, it is most reliable to determine the form of the compressibility coefficient dependence \( Z(p, T) \) via experimental data \( p - V - T \) obtained for the investigated gas mixture in the investigated range of temperature and pressure changes.

A lot of approximate dependences of the compressibility coefficient \( Z(p, T) \) in high-pressure regions have been proposed. For example, in this book [21] the dependence of V.V. Latonov – G.R. Gurevich is proposed, which approximates the well-known graphs of Brown et al. It has the following form:

\[
Z_{LG}(p_b, T_b) = (0.4 \lg T_b + 0.73)p_b + 0.1p_b. \tag{6}
\]

Dimensionless temperature \( T_b \) and pressure \( p_b \) in (6) are equal: \( T_b = T/T_r, \ p_b = p/p_r, \ p \ — \) pressure in MPa, \( T \ — \) gas temperature in K. For pressures not exceeding 30 MPa, authors recommend calculating the pseudocritical parameters \( p_r, T_r \) of gas mixtures according to the expressions of Hankinson, Thomas, and Phillips:

\[
p_r = 0.006894 \left( 709.604 - 58.718 \frac{M}{28.96} \right) \text{ (MPa)},
\]

\[
T_r = \left( 170.491 + 307.44 \frac{M}{28.96} \right)/1.8 \text{ (K)},
\]

\( M \) (g/mol) — molar mass of a gas mixture.

In this paper, as experimental data for \( Z_{exp} - p - T \), we used calculation data of according to the Redlich – Kwong equation of state for a gas mixture with a predominance of methane, which is typical for many problems of natural gas transportation. (The composition of this mixture of 12 components, critical pressures \( p_c \) and mixture temperature \( T_c \), as well as the gas constant \( R_g \) are taken from this book [17], their values are given below in the set of parameters (35).) The dependency building algorithm \( Z(p, T) \) by experimental data is as follows. To visually determine the nature of \( Z_{exp} \) dependence on pressure and temperature, surface graphs \( Z_{exp}(p, T) \) with a different number of experimental points (from several thousand to a million) are plotted. For the investigated gas mixture in the investigated range of pressure and temperature changes, the analysis of the plotted graphs led to the conclusion that this surface should be approximated by a second-order surface satisfactorily, i. e., the objective function should be of the form:

\[
Z(p, T) = a_1T^2 + a_2T + a_3p^2 + a_4p + a_5 + a_6Tp.
\]
Optimal values of $a_1, . . . , a_6$ coefficients were calculated by the least squares method [22] with a given objective function. There is free source code software for this approach, namely, Gnuplot [23] and LAPACK library [24, 25].

The dependency $Z(\tilde{p}, \tilde{T})$ found via the least squares method algorithm is as follows:

$$Z(\tilde{p}, \tilde{T}) = n_1 \tilde{T}^2 + n_2 \tilde{T} + n_3 \tilde{p}^2 + n_4 \tilde{p} + n_5 + n_6 \tilde{T} \tilde{p}. \tag{7}$$

Here $\tilde{p}$ and $\tilde{T}$ — dimensionless pressure and temperature related to their critical values $p_c$ and $T_c$ for the given gas mixture, $n_1, . . . , n_6$ — dimensionless constants, the values of which depend on the composition of the gas mixture.

As is known, the Redlich – Kwong equation in terms of the compressibility factor $Z_{RK}$ [19] is written as:

$$Z_{RK}^3 - Z_{RK}^2 + (A - B^2 - B)Z_{RK} - AB = 0,$$

$$A = \frac{\Omega_a \tilde{p}}{T}, \quad B = \frac{\Omega_b \tilde{p}}{T}, \quad \tilde{p} = p/p_c, \quad \tilde{T} = T/T_c. \tag{8}$$

In Table 1 there is an example of calculating the compressibility coefficients $Z (7), Z_{LG} (6)$, and $Z_{RK} (8)$. The values are present for the given gas mixture at 19 MPa pressure for temperatures of 10, 15, 20, 25, 30 degrees Celsius.

| $T$, ℃ | 10   | 15   | 20   | 25   | 30   |
|--------|------|------|------|------|------|
| $Z$    | 0.78969 | 0.80177 | 0.81343 | 0.82466 | 0.83546 |
| $Z_{RK}$ | 0.78888 | 0.80121 | 0.81309 | 0.82448 | 0.83538 |
| $Z_{LG}$ | 0.80146 | 0.80763 | 0.81377 | 0.81987 | 0.82594 |

As shown by the calculations, the maximum deviation of compressibility factor $Z$ modulus, calculated by approximation (7), from its exact value $Z_{RK} (8)$ does not exceed 0.00081 in the range of pressure change from 15 to 21 MPa and temperature change from 5 to 35 degrees Celsius.

In additional calculations, the found approximation (7) of the compressibility coefficient was used.

4. The task of locating a gas leak

Let us denote by $z_a$ the dimensional section coordinate of the gas pipeline in which the leakage occurs. In the absence of leakage at the outlet from the gas pipeline in the steady state mode, the flow rate is constant and equal to the flow rate $Q^0$ at the inlet.

If a leak occurs due to the depressurization of the pipeline, after a while, a new stationary mode is established, in which the pressure and flow rate at the outlet take on new values $P(L)$, $Q(L)$ ($P(L) > P^0(L)$, $Q(L) < Q^0$), in this case, the equalities are:

$$Q = \begin{cases} 
Q^0, & 0 \leq z < z_a, \\
Q(L), & z_a \leq z \leq L.
\end{cases} \tag{9}$$

The difference $(Q^0 - Q(L))$ determines the size of the gas leak. (The values $P(L), P^0(L), Q(L), Q^0$ are dimensional.)
Let us demonstrate a method for solving the inverse problem of calculating the coordinate $z_0$ for inlet pressures of about 20 MPa, provided that the gas pipeline route is horizontal and values $\lambda$, $c_p$, $\beta$ and $T^*$ are constant.

Consideration of values $\lambda$, $c_p$, $\beta$ and $T^*$ dependency from pressure, temperature, and conditions along the route, as well as taking into account the surface pattern of the route, do not introduce any fundamental difficulties. Let us reduce the system of equations (1)–(4), (7) with boundary condition (5) to dimensionless form, setting:

$$\tilde{z} = \frac{z}{l_x}, \quad \tilde{T} = \frac{T}{T_x}, \quad \tilde{\rho} = \frac{p}{p_x}, \quad \tilde{\rho}_x = \frac{p}{R_g T_x Z(p_x, T_x)}$$

Let us introduce the dimensionless coordinate with boundary condition (5) to dimensionless form, setting:

$$u_x = \frac{Q^0}{\rho_x S}, \quad \tilde{\rho} = \frac{\rho}{\rho_x}, \quad \tilde{u} = \frac{u}{u_x}, \quad \tilde{T}^* = \frac{T^*}{T_x}.$$

Let us take independent characteristic values $l_x$, $T_x$, $p_x$ as equal: $l_x = L$, $T_x = T_c$, $p_x = p_c$. The resulting dimensionless system of equations is reduced in a known way to a system consisting of two ordinary differential equations, resolved with respect to the derivatives $d\tilde{\rho}/d\tilde{z}$, $d\tilde{T}/d\tilde{z}$ and to the dimensionless equation of state.

Let us omit the wavy lines at the dimensionless values wherever this does not lead to ambiguity and write the dimensionless model of these processes in the absence of gas leakage.

\[
\begin{align*}
\rho u &= 1, \\
\frac{dp}{dz} &= \frac{f_5 - f_2 f_6}{f_1 - f_2 f_3} = F_1(p, T), \\
\frac{dT}{dz} &= \frac{f_1 f_6 - f_3 f_5}{f_1 - f_2 f_3} = F_2(p, T), \\
\rho &= \frac{pZ(1, 1)}{TZ(p, T)}
\end{align*}
\]

\[z = 0, \quad p(0) = \frac{p_0}{p_x}, \quad T(0) = \frac{T_0}{T_x}\]

Dimensionless functions $f_1, \ldots, f_6$ in system (11) are in the form:

\[
\begin{align*}
f_1(p, T) &= 1 + m_1 \frac{T}{p} \left( -Z + \frac{\partial Z}{\partial p} \right), \\
f_2(p, T) &= \frac{m_1}{p} \left( Z + T \frac{\partial Z}{\partial T} \right), \\
f_3(p, T) &= -m_3 \frac{T}{p} \left( Z + T \frac{\partial Z}{\partial T} \right), \\
f_5(p, T) &= \frac{TZ}{p}, \\
f_6(p, T) &= m_5 \left( \frac{TZ}{p} \right)^2 + m_6 (T^* - T).
\end{align*}
\]

Dimensionless complexes $m_1, \ldots, m_6$ for constant values $\lambda$, $\beta$, $c_p$, $T^*$ are equal:

\[
\begin{align*}
m_1 &= \frac{(Q^0)^2 R_g T_x}{S^2 p_x^2}, \quad m_2 = -m_1 \frac{\lambda L}{4R}, \quad m_3 = \frac{R_g}{c_p}, \\
m_5 &= \frac{\lambda L (Q^0)^2 R_x^2 T_x}{4RS^2 c_p p_x^2}, \quad m_6 = \frac{2\beta \pi RL}{Q^0 c_p}.
\end{align*}
\]

The function $Z(p, T)$ of the dimensionless arguments $p$ and $T$ is written in the form of equation (7). Let us introduce the dimensionless coordinate $l$ of the gas leakage site: $l = z_0/L$. 

\[
doi:10.1088/1742-6596/2162/1/012023
\]
5. Direct problem in the presence of a gas leak

Let the dimensionless coordinate \( l \) of the gas leakage location be known. The gas pipeline is divided into two sections: \( z \in [0, l] \) and \( z \in [l, 1] \).

In the first section, (11) the system is solved in the region \( z \in [0, l] \) with dimensionless complexes (14), calculated for \( Q^0 \) with the boundary condition (12). In the second section (11), the system is solved in the region \( z \in [l, 1] \), with dimensionless complexes (14), calculated for \( Q^0 = Q(L) \), with the boundary condition:

\[
  z = l : \quad p = p_1(l), \quad T = T_1(l),
\]

in which the values \( p_1(l), T_1(l) \) are calculated in the first section.

As a result of solving the direct problem for given values \( l, p_0, T_0, Q_0 \) and \( Q(L) \), distributions of dimensionless pressure, temperature, and density in the first and second sections are determined.

6. Inverse problem of calculating the dimensionless coordinate \( l \) of a gas leak

The inverse problem is solved by the parameter identification method. In this method, it is assumed that the desired parameter is included in the right-hand sides of the differential equations of the system (11). According to [11], let us introduce new dimensionless coordinates \( y \) and \( x \), determining them as follows (15):

\[
\begin{align*}
  \text{in the first section} & \\
  y &= \frac{z}{2l}, \quad z \in [0, l] \rightarrow y \in [0, 1/2],
\end{align*}
\]

\[
\begin{align*}
  \text{in the second section} & \\
  x &= 1 - \frac{1 - z}{2(1 - l)}, \quad z \in [l, 1] \rightarrow x \in [1/2, 1].
\end{align*}
\]

Let us write the differential equations of the dimensionless system (11) in terms of the dimensionless coordinate \( y \) for the first section, supplementing them with the condition of the \( l \) parameter invariability:

\[
\begin{align*}
\frac{dp_1}{dy} &= 2lF_1(p_1, T_1) = \Phi_1(p_1, T_1, l), \\
\frac{dT_1}{dy} &= 2lF_2(p_1, T_1) = \Phi_2(p_1, T_1, l), \\
\frac{dl}{dy} &= 0.
\end{align*}
\]

Boundary condition:

\[
  y = 0 : \quad p_1 = p_0/p_x, \quad T_1 = T_0/T_x.
\]

For the second section, the differential equations of the dimensionless system (11) in terms of the dimensionless coordinate \( x \), supplemented by the condition of the invariability of the parameter \( l \), are in form:

\[
\begin{align*}
\frac{dp_2}{dx} &= 2(1 - l)\tilde{F}_1(p_2, T_2) = \Psi_1(p_2, T_2, l), \\
\frac{dT_2}{dx} &= 2(1 - l)\tilde{F}_2(p_2, T_2) = \Psi_2(p_2, T_2, l), \\
\frac{dl}{dx} &= 0.
\end{align*}
\]
Boundary condition:

\[ x = \frac{1}{2} : \quad p_2 = p_1(y) \bigg|_{y=\frac{1}{2}}, \quad T_2 = T_1(y) \bigg|_{y=\frac{1}{2}}. \]  

(19)

In formulas (18) \( \tilde{F}_1(p_2, T_2) \) and \( \tilde{F}_2(p_2, T_2) \) functions are calculated for dimensionless complexes (14), in which the flow rate \( Q^0 \) equals \( Q(L) \).

In the first and second sections, the \( l \) parameter identification problem is solved by the Bellman quasilinearization method [16]. This method in this book [17] solved the task of identifying the \( \lambda \) and \( \beta \) parameters in model (1)–(3) for the given range of problems.

In accordance with Bellman’s method, the solution of both system (16) and system (18) is sought iteratively; at each iteration, the solution of systems of nonlinear equations is replaced by the solution of systems of corresponding linearized equations with respect to the previous iteration.

Let us demonstrate the algorithm of the method by the example of solving the system (16) for the first section.

7. Calculation of the dimensionless leak coordinate \( l \) at the \((s + 1)^{th}\) iteration

Let us represent system (16) in vector form:

\[
\frac{d\mathbf{u}}{dy} = \Phi(\mathbf{u}), \quad \mathbf{u} = \begin{pmatrix} p_1 \\ T_1 \\ l \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ 0 \end{pmatrix}.
\]  

(20)

Let the vector of unknowns be found at the \( s \)th iteration. The vector of unknowns \( \mathbf{u}^{s+1} \) at the \((s + 1)^{th}\) iteration is determined from the following linearized equation:

\[
\frac{d\mathbf{u}^{s+1}}{dy} = \Phi(\mathbf{u}^s) + J^s(\mathbf{u}^{s+1} - \mathbf{u}^s).
\]  

(21)

Jacobi matrix \( J^s \), which equals:

\[
J^s = \begin{pmatrix} \Phi_{1p} & \Phi_{1T} & \Phi_{1l} \\ \Phi_{2p} & \Phi_{2T} & \Phi_{2l} \\ 0 & 0 & 0 \end{pmatrix},
\]  

(22)

at each point \((y)\) calculated at the values of pressure \( p_1^s(y) \), temperature \( T_1^s(y) \) and \( l^s \), found at the \( s \)th iteration. Elements of the Jacobi matrix are partial derivatives of the functions \( \Phi_1, \Phi_2 \) for \( p_1, T_1 \) and \( l \), for example:

\[ \Phi_{2T}(y) = \Phi_{2T}(p_1^s(y), T_1^s(y), l^s) = \frac{\partial \Phi_2}{\partial T_1}. \]

The research has confirmed the validity of this hypothesis for both the first and second sections of the gas pipeline. As an illustration of the validity of this hypothesis, Fig. 1 shows
the dependence of the pressure in the second section at different points \((x)\) for different values of the parameter \(l\) and different flow rates at the outlet and inlet of the gas pipeline (i.e., at different gas leak rates) for the set of parameters presented below (35).

In vector form, linear representations (23) are written in the form:

\[
\mathbf{u}^{s+1} = C\mathbf{u}^{s+1} + \mathbf{g}, \quad C = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} a_3 \\ a_4 \\ 0 \end{pmatrix}.
\]  

(24)

![Figure 1. Dependence of pressure on dimensionless \(l\) and \(x\) at different \(Q(L)\); \(---\) \(Q(L) = 400 \text{ kg/s}\), \(-----\) \(Q(L) = 430 \text{ kg/s}\).](image)

Linearized equation (21) considering representation (24) of the vector \(\mathbf{u}^{s+1}\) leads to the following equations for the matrix \(C\) and the vector \(\mathbf{g}\):

\[
\frac{dC}{dy} = J^s C, \quad \frac{d\mathbf{g}}{dy} = \Phi (\mathbf{u}^s) + J^s (\mathbf{g} - \mathbf{u}^s).
\]  

(25) \hspace{1cm} (26)

Matrix equation (25) is equivalent to the system of two ordinary differential equations for the functions \(a_1(y)\) and \(a_2(y)\). Vector equation (26) is equivalent to the system of two ordinary differential equations for the functions \(a_3(y)\) and \(a_4(y)\). Boundary conditions for functions \(a_1(y)\), \(a_2(y)\), \(a_3(y)\), \(a_4(y)\) follow from the boundary condition (17), which takes place at any iteration, and from the linear representation (23). These boundary conditions are as follows:

\[
a_1(0) = 0, \quad a_2(0) = 0, \quad a_3(0) = \frac{p_0}{p_x}, \quad a_4(0) = \frac{T_0}{T_x}.
\]  

(27)

As a result of the numerical solution of the resulting two systems of ordinary differential equations, the functions are calculated \(a_1(y)\), \(a_2(y)\), \(a_3(y)\), \(a_4(y)\) in particular, their values at \(y = 1/2\).

In the second section, system (18) is solved in a similar way with the boundary condition (19). Linear representation of \(p_2^{s+1}(x)\) and \(T_2^{s+1}(x)\) functions is used from \(l^{s+1}\) in every point \((x)\) of the second section:

\[
p_2^{s+1}(x) = b_1(x) l^{s+1} + b_3(x), \\
T_2^{s+1}(x) = b_2(x) l^{s+1} + b_4(x).
\]  

(28)

Functions \(b_1(y)\), \(b_2(y)\), \(b_3(y)\), \(b_4(y)\) are calculated under boundary conditions:

\[
x = \frac{1}{2}: \quad b_1 = a_1 \left( \frac{1}{2} \right), \quad b_2 = a_2 \left( \frac{1}{2} \right), \quad b_3 = a_3 \left( \frac{1}{2} \right), \quad b_4 = a_4 \left( \frac{1}{2} \right).
\]  

(29)
which follow from conditions (19) valid at any iteration. By calculated \( b_1(x) \) and \( b_3(x) \) functions the values \( b_1(1) \) and \( b_3(1) \) at the gas pipeline outlet are defined. From the linear representation (28) for \( x = 1 \) at the equality at the outlet for the pressure follows:

\[
p^{s+1}_2(1) = b_1(1) l^{s+1} + b_3(1),
\]

which allows to find the \( l^{s+1} \) value of the leak site dimensionless coordinate at the \((s + 1)\)th iteration:

\[
l^{s+1} = \frac{p^{s+1}_2(1) - b_3(1)}{b_1(1)}. \tag{30}
\]

Pressure \( p^{s+1}_2(1) \) at every iteration is known and equals: \( p^{s+1}_2 = P(L)/p_x \), here \( P(L) \) — measured pressure at the gas pipeline outlet (dimensional).

Thus, the solution to the problem of calculating the dimensionless coordinate of the gas leak at the \((s + 1)\)th iteration is given by this formula (30). The condition for the iterative process conclusion is the fulfillment of the inequality:

\[
|l^{s+1} - l^s| \leq \varepsilon, \tag{31}
\]

in which the value \( \varepsilon \) is determined in result of a computational experiment. To establish the practical convergence of the iterative process, it is sufficient to make sure that with an increase in the iteration number, the value \( \delta l^{s+1} \) decreases, which equals:

\[
\delta l^{s+1} = \left| \frac{l^{s+1} - l^s}{l^s} \right|.
\]

In contrast to [11], to calculate the pressure and temperature distributions at the \((s + 1)\)th iteration, the proposed method uses the solution of the direct problem with the found coordinate of the leak site \( l^{s+1} \). The solution of the identification problem for the parameter \( l \) at the next iterations is carried out similarly to the presented algorithm.

8. Initial approximation

As is known, when solving problems by the iterative method, the choice of the initial approximation is of great importance. In present range of gas leak detection problems, the numerical solution of the direct problem (paragraph 5) at a given location of the gas leak is not difficult, the calculation of one variant takes several seconds on a medium-power personal computer (for example, computers with a 4th Gen Intel Core i5 processor on Haswell architecture).

This allows to numerically form a dependence when making calculations for a specific gas pipeline in the investigated range of pressure and flow rate changes at the outlet:

\[
p_2 < p_L < p_1, \quad q_2 < q_L < 1, \tag{32}
\]

\( p_2 \) — minimum dimensionless outlet pressure at a given dimensionless outlet flow \( q_L \), which is realized at \( l \to 1 \), \( p_1 \) — maximum dimensionless outlet pressure, which is realized at \( l \to 0 \) (\( l \) — dimensionless coordinate of the gas leak, \( q_L = Q(L)/Q_0 \), \( q_2 \) — minimum dimensionless outlet flow rate, which is realized at maximum gas leakage).

For many problems in the investigated range (32) of flow rate changes at the output, the dependence \( l(p_L) \) is practically linear. This is the case for the set of parameters (35) of the example given below. Linearity of dependence \( l(p_L) \) allows us to propose the following algorithm for calculating the gas leak coordinate \( l^0 \) in the initial approximation.
From the solution of the direct problem (paragraph 5) for a given flow rate at the outlet, the values of dimensionless pressures \( p_1 \) and \( p_2 \) are calculated at \( l = 0.05 \) and \( l = 0.95 \) respectively.

If the specified value of the dimensionless pressure \( p_L \) at the gas pipeline outlet satisfies the inequality \( p_L \leq p_2 \), the value \( l^0 \) is assigned the value: \( l^0 := 0.95 \), if \( p_L \) satisfies inequality \( p_L \geq p_1 \), \( l^0 \) is assigned the value: \( l^0 := 0.05 \). For values \( p_L \), which satisfy inequality (32), the value \( l^0 \) is calculated by the formula:

\[
l^0 = l_1 + \frac{p_1 - p_L}{p_1 - p_2}(l_2 - l_1),
\]

in which the values \( l_1, l_2, p_1, p_2 \) are:

\[
l_1 = 0.05, \quad l_2 = 0.95, \quad p_1 = p_L(0.05), \quad p_2 = p_L(0.95).
\]

In equalities (34), all values are dimensionless, value \( p_L \) is also dimensionless, equal to the ratio of the dimensional pressure \( P_L \) at the outlet to the dimensional characteristic pressure \( p_x \).

To calculate the distributions of dimensionless pressure and temperature in the initial approximation, the direct problem is solved at \( l = l^0 \) by the system (16) with boundary conditions (17) and by the system (18) with boundary conditions (19).

9. Examples of solving test problems
The following (typical for real natural gas transportation problems) parameters of a steady flow along an extended gas pipeline were set:

\[
R = 0.5 \text{ m}, \quad L = 300 \text{ km}, \quad Q^0 = 450 \text{ kg/s}, \quad \lambda = 0.0087, \quad |
\]

\[
\beta = 11.0 \text{ W/(m}^2\text{K}), \quad p_0 = 200 \text{ atm} \quad T_0 = 308.15 \text{ K}, \quad p_c = 4.59776 \text{ MPa}, \quad T_c = 193.69853 \text{ K}, \quad T^* = 278.15 \text{ K}, \quad R_g = 493.58891 \text{ m}^2/(\text{s}^2\text{K}).
\]

(The critical gas mixture parameter values are given in (35) with an accuracy of five digits after the decimal point; in all calculations, a significantly higher accuracy was used.) Parameter values \( p_c, T_c, R_g \) in (35) correspond to a gas mixture with a predominance of methane; the composition of the mixture is given in the book [17].

The calculations used the average value of the heat capacity coefficient in the investigated range of pressure and temperature changes. In the examples below, the “experimental” new flow rate value at the outlet \( Q(L) \) was set as well as the exact leak site coordinate \( z_a \) (that was be further defined). From this data, for this set of parameters (35) and from the solution of the direct problem, the dimensional pressure \( P(L) \) atm at the outlet was calculated, which was then taken as an “experimental”. The flow rate at the outlet value of the and the leak coordinate of \( z_a \) varied in given examples.

For a given flow rate at the outlet, the dimensional pressure values at the gas pipeline outlet at \( l_1 = 0.05 \) and \( l_2 = 0.95 \) were calculated from the solution of the direct problem (paragraph 5). At \( Q(L) = 400 \text{ kg/s} \) these pressures are:

\[
P_1 = 180.684 \text{ atm}, \quad P_2 = 175.865 \text{ atm}.
\]

The following designations are adopted:

\( z_a^{(s)} \), \( m \) — dimensional coordinate of the leak site calculated at the \( s \)th iteration; \( \delta l_s = |l^s - l^{s-1}| \) — the absolute value of difference between the dimensionless leak coordinates at the \( s \)th and \((s-1)\)th iterations; \( \delta_z = |z_a^{(s)} - z_a| \) — the absolute value of the difference between the dimensional leak coordinate at the \( s \)th iteration and the given value \( z_a \) (m).

**Example 1.** \( Q(L) = 400 \text{ kg/s}, \text{ t} = 0.01, \text{ z}_a = 3 \text{ km}, P(L) = 180.937 \text{ atm}, P(L) > P_1 \rightarrow l^0 := 0.05; \) the coordinate of the leak site \( z_a^{(3)} = 2986.0 \text{ m}, \) the coordinate of the leak site calculating accuracy \( \delta_z = 14.0 \text{ m} \) (table 2).
Example 2. \( Q(L) = 400 \text{ kg/s}, l = 0.35, z_a = 105 \text{ km}, P(L) = 179.001 \text{ atm}, \) calculating \( t^0 \) by formula (33); the coordinate of the leak site \( z_a^{(2)} = 104.995 \text{ km}, \) the coordinate of the leak site calculating accuracy \( \delta_z = 5.2 \text{ m} \) (table 3).

Example 3. \( Q(L) = 400 \text{ kg/s}, l = 0.55, z_a = 165 \text{ km}, P(L) = 177.951 \text{ atm}, \) calculating \( t^0 \) by formula (33); the coordinate of the leak site \( z_a^{(2)} = 164.993 \text{ km}, \) the coordinate of the leak site calculating accuracy \( \delta_z = 6.6 \text{ m} \) (table 4).

Example 4. \( Q(L) = 400 \text{ kg/s}, l = 0.985, z_a = 295.50 \text{ km}, P(L) = 175.682 \text{ atm}, \) \( P(L) < P_2 \rightarrow t^0 := 0.95; \) the coordinate of the leak site \( z_a^{(2)} = 295.5 \text{ km}, \) the coordinate of the leak site calculating accuracy \( \delta_z = 8.5 \text{ m} \) (table 5).

| \( s \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) |
|---|---|---|---|---|
| \( l_s \) | 0.05 | 0.009 | 0.010 | 0.010 |
| \( \delta l_s \) | – | 0.041 | 0.0009 | 0.6 \cdot 10^{-6} |

Table 2. Calculation for \( l = 0.01 \).

| \( s \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|---|
| \( l_s \) | 0.3643 | 0.34998 | 0.35000 |
| \( \delta l_s \) | – | 0.01431 | 0.347 \cdot 10^{-6} |

Table 3. Calculation for \( l = 0.35 \).

| \( s \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|---|
| \( l_s \) | 0.560 | 0.54998 | 0.55000 |
| \( \delta l_s \) | – | 0.01040 | 0.434 \cdot 10^{-6} |

Table 4. Calculation for \( l = 0.55 \).

| \( s \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|---|
| \( l_s \) | 0.950 | 0.98503 | 0.98502 |
| \( \delta l_s \) | – | 0.03503 | 0.428 \cdot 10^{-7} |

Table 5. Calculation for \( l = 0.985 \).

The given examples, as well as numerous calculations for other values of the outlet flow rate and other leak site variants, led to the conclusion that there is a practical convergence of the iterative method. For example, from the calculation data of example 1 it follows that with increasing \( s \) value \( \delta^{(s)} = \left| \frac{P_s^{(s)} - P_s^{(s-1)}}{P_s^{(s-1)}} \right| \) decreases as follows:

\[ \delta^{(1)} = 0.819, \quad \delta^{(2)} = 0.098, \quad \delta^{(3)} = 0.000061. \]

Influence of the error in setting the pressure \( P(L) \) at the gas pipeline outlet. Let us demonstrate the influence of this error on the coordinate of the leak site calculating accuracy. Assuming that: \( Q(L) = 400 \text{ kg/s}, l = 0.35. \) From the solution of the direct problem (example 2) it follows: \( P(L) = 179.001 \text{ atm}. \)

Let the measured pressure \( \tilde{P}(L) \) differ from \( P(L) \) by 0.005 atm, namely, \( \tilde{P}(L) = 179.006 \text{ atm}. \) Calculation by formula (33) shows that the initial approximation \( t^0 \) at this pressure \( \tilde{P}(L) \) equals: \( t^0 = 0.36336. \) The iterative process in this example converges at the third iteration, while \( t^3 = 0.34904, \delta_z = 278.6 \text{ m} \).

Influence of the error in setting the flow rate \( Q(L) \) at the gas pipeline outlet. Assuming that flow rate is set with accuracy 1\%, let the measured flow rate be: \( Q(L) = 396 \text{ kg/s}, \) the dimensionless leak coordinate is: \( l = 0.35. \) When the flow rate changes, the values \( P_1 \) and \( P_2 \) (36), namely, \( P_1 = 183.458 \text{ atm}, P_2 = 178.215 \text{ atm}. \) Following the above algorithm for the initial approximation choosing, we find that: \( t^0 = 0.36406. \) The iterative process in this example converges at the third iteration, while \( t^3 = 0.35004, \delta_z = 13.7 \text{ m}. \)
**Gas leakage reduction.** The numerical experiments have shown that with constant parameters of the numerical solution, a decrease in the leakage value increases the leak site calculating error. For example, with an increase in the gas flow rate at the outlet from 400 kg/s to 425 kg/s, the amount of gas leakage \( (Q^0 - Q(L)) \) decreases twice at \( Q^0 = 450 \) kg/s. The leak site coordinates calculating error at \( Q(L) = 425 \) kg/s equals: \( \delta_z = 10.5 \) m, which is about twice as much as \( \delta_z = 5.2 \) m shown in Example 2. This leads to the following conclusion: To ensure constant accuracy of a gas leak coordinates calculation, it is necessary to use the step along the \( z \) coordinate in the numerical solution of the identification problem, which is reduced by about the same factor as the amount of leakage decreases.

In each real problem the necessary step size \( \Delta z \) in the numerical solution and the accuracy value \( \varepsilon \) in the iterative process end condition are determined in advance using computational experiments. That ensures the required accuracy of calculating the gas pipeline depressurization coordinate.

The proposed leak detection technique has limitations that are typical for all model methods: it allows calculating the leak site location site only some time after its occurrence, and it does not allow to determine the cause of the leak.

Generalization of the proposed technique for the presence of more than one leak requires additional research.

10. Conclusion

The given examples, as well as solutions of numerous variants of the gas pipeline depressurization coordinate calculation problem for other parameter values, led to the following conclusions.

The proposed technique allows to calculate promptly the depressurization coordinate for extended underground and offshore \((L > 500 \) km) ultra-high pressure \((p > 20 \) MPa) gas pipelines with an accuracy of tens of meters in a wide leakage size and location range.

The proposed algorithm for choosing the initial approximation ensures a high convergence rate of the iterative process, which allows calculating leakage site location in real time.

The greatest influence on the calculation accuracy is caused by the pressure measurement accuracy.

As shown in this paper, the proposed technique allows calculating the depressurization coordinate even with a decrease in leakage.

Studies have shown that in the case of small leaks, approximately 0.4%, the leak site location can be calculated with an accuracy of up to tens of meters, if the equipment provides high accuracy in measuring the pressure and flow rate at the outlet.

The created calculation programs set is simple to implement, the proposed technique does not require an installation of expensive equipment along the gas pipeline and allows to calculate a leakage site location in extended underground and offshore gas pipelines with high accuracy.

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