Genuine Network Multpartite Entanglement

Miguel Navascués,1 Elie Wolfe,2 Denis Rosset,2 and Alejandro Pozas-Kerstjens3,4

1Institute for Quantum Optics and Quantum Information (IQOQI) Vienna, Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria
2Perimeter Institute for Theoretical Physics, 31 Caroline St. N., Waterloo, Ontario, Canada, N2L 2Y5
3Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040 Madrid, Spain
4ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

The standard definition of genuine multipartite entanglement stems from the need to assess the quantum control over an ever-growing number of quantum systems. We argue that this notion is easy to hack: in fact, a source capable of distributing bipartite entanglement can, by itself, generate genuine k-partite entangled states for any k. We propose an alternative definition for genuine multipartite entanglement, whereby a quantum state is genuine network k-entangled if it cannot be produced by applying local trace-preserving maps over several (k-1)-partite states distributed among the parties, even with the aid of global shared randomness. We provide analytic and numerical witnesses of genuine network entanglement, and we reinterpret many past quantum experiments as demonstrations of this feature.

The existence of multipartite quantum states which cannot be prepared locally is at the heart of many communication protocols in quantum information science, such as quantum teleportation [1], dense coding [2], entanglement-based quantum key distribution [3] and the violation of Bell inequalities [4, 5]. Most importantly, for the last two decades, the ability to entangle an ever-growing number of photons or atoms has been regarded as a benchmark for the experimental quantum control of optical systems [6–9].

Since any multipartite quantum state where two parts share a singlet can be regarded as “entangled”, another, more demanding, notion of entanglement was required to assess the progress of quantum technologies. The accepted answer was genuine multipartite entanglement [10–12]. Genuine multipartite entanglement has since become a standard for quantum many-body experiments [6–9, 13]. But, is it a universal measure?

In this paper, we argue the opposite and present an alternative and stronger definition, genuine network multipartite entanglement, which we formulate in terms of quantum networks [14]. First, we define and compare the two approaches. Next, we present general criteria to detect genuine network entanglement, and discuss the tightness of the bounds so obtained. Finally, we single out past experiments in quantum optics that can single out past experiments in quantum optics that can

$$\hat{\rho} = |\phi^+\rangle\langle\phi^+| \otimes \rho(K_3,\ldots,K_n), \quad |\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

does not admit a decomposition of the form (1).

In order to address this issue, an extended definition of multipartite separability was proposed [10–12]. Intuitively, a state is k-partite entangled if, in order to produce it, one must create k-partite entangled states and distribute them among the n parties in such a way that no party receives less than one subsystem. More formally, we say that an n-partite state is separable with respect to a partition $S_1|\ldots|S_n$ of $\{K_1,\ldots,K_n\}$ if it can be expressed as

$$\rho = \sum_j w_j \rho^{(S_1)}_j \otimes \cdots \otimes \rho^{(S_n)}_j.$$  

An n-partite state is genuine k-partite entangled (or has entanglement depth k) if it cannot be expressed as a convex combination of quantum states, each of which is separable with respect to at least one partition $S_1|S_2|\ldots$ of $\{1,\ldots,n\}$ with $|S_\ell| \leq k-1,$ for all $\ell.$ Using this definition, the state $\hat{\rho}$ in Eq. (2) is certainly genuine 2-entangled. However, $\hat{\rho}$ is not genuine 3-entangled so long as its marginal $\rho(K_3,\ldots,K_n)$ is fully separable.

This notion of multipartite entanglement is easy to cheat, as we show next. For simplicity, we consider a tripartite scenario $(n=3)$ and rename the Hilbert spaces $A, B$ and $C$; we split $A$ into three local subspaces $A', A''$ and $A'''$, the same for $B$ and $C$. Now, let $\rho_{A'B'C'} = |\phi^+\rangle\langle\phi^+|_{A'B'} \otimes |0\rangle\langle0|_{C'},$ and similarly $\rho_{A'B'C''} = |\phi^+\rangle\langle\phi^+|_{B'C''} \otimes |0\rangle\langle0|_{A''},$ while
the deterministic operation at Alice's by a family of linear bounded operators on the Hilbert space defined formally. Let Λ be a classical random variable Alice, Bob and Charlie certify that the state produced acting on systems randomness Λ to jointly influence the local operations. In addition, we provide Eve with unlimited shared transformations known as Local Operations and Shared Randomness (LOSR) [15, 16].

Genuine network entanglement. We explain our definition using an adversarial approach. Eve is a vendor selling a source of tripartite quantum states to three honest scientists Alice, Bob and Charlie. Eve pretends that her device produces a valuable entangled tripartite state ρ_ABC. Unbeknown to the scientists, the source sold to them is actually composed of cheaper components: quantum sources that produce the bipartite entangled states σ_A'B'c, σ_C'A', σ_B'C', see Figure 1. Alice receives the A', A'' subsystems of the states σ_A'B', σ_C'A', σ_B'C'. Those can in principle interact within Alice’s experimental setup, giving rise to a new quantum system A: that is what Alice eventually probes. Similarly, Bob (Charlie) will have access to system B (C), whose state is the result of a deterministic interaction between systems B', B'' (C', C''). In addition, we provide Eve with unlimited shared randomness Λ to jointly influence the local operations acting on systems A', A'', B'B'' and C'C''.

By performing local tomography on the state ρ_ABC, can Alice, Bob and Charlie certify that the state produced by Eve’s network is indeed a valuable tripartite quantum state?

The family of states that they try to rule out can be defined formally. Let Λ be a classical random variable with distribution P_Λ(λ) sent at the three labs (for example through radio broadcast). Denoting by B(H) the set of bounded operators on the Hilbert space H, we describe the deterministic operation at Alice’s by a family of linear maps \{Ω_Λ^A\}_Λ, where each Ω_Λ^A has type

\[ \Omega_Λ^A : B(A' \otimes A'') \to B(A) \]

and each Ω_Λ^A is completely positive and trace preserving. For completeness, the other maps correspond to Ω_Λ^B : B(B' \otimes B'') \to B(B) and Ω_Λ^C : B(C' \otimes C'') \to B(C), so that the state ρ_ABC is

\[ \rho_{ABC} = \sum P_Λ(λ) [Ω_Λ^A \otimes Ω_Λ^B \otimes Ω_Λ^C] (σ), \]

where \[ σ = σ_A'B'c \otimes σ_C'A' \otimes σ_B'C'. \]

The valuable states, those genuine network 3-entangled, are those which cannot be written the way described by Eq. (4). It is easy to see that the set of states of the form (4) is closed under tensor products and LOSR transformations. That is, the set of network 2-entangled states is a self-contained class within the resource theory of LOSR entanglement [15, 16].

Note that, in the considered scenario, rather than the state σ_A'B'c \otimes σ_C'A' \otimes σ_B'C', Eve could also distribute Alice, Bob and Charlie arbitrary convex combinations of states of the form σ_A'B'c (i) \otimes σ_C'A' (i) \otimes σ_B'C' (i), for some values of i. Since the dimensionality of the primed spaces is unbounded, though, this strategy can be simulated with the operations allowed by Eq. (4). Indeed, it suffices to distribute the tensor product of the states σ_A'B'c (i) \otimes σ_C'A' (i) \otimes σ_B'C' (i) and embed the index i within the hidden variable Λ (whose dimension is also unbounded). The index i would then signal in which pair of Hilbert spaces at party Z’s the map Ω_Λ^C is to be applied.

The definition of genuine network entanglement can be straightforwardly extended to the n-partite case.

**Definition 1.** A multipartite quantum state is genuine network k-entangled if it cannot be generated by distributing entangled states among subsets of maximum k-1 parties, and letting the parties apply local trace-preserving maps, those maps being possibly correlated through global shared randomness.
Witnesses of genuine network entanglement. The certification of $\rho_{ABC}$ being genuine network 3-entangled is complicated, as the dimensions of the Hilbert spaces $\mathcal{A}', \ldots, \mathcal{C}''$ are in principle unbounded. To classify the degree of a state’s network multipartititeness we must somehow determine if the state can come about from a particular quantum causal process. The study of quantum causal processes has experienced great progress [14, 17–20], and many techniques have recently been developed [19, 21, 22]. Herein, we adapt the inflation technique for causal inference [19, 23] in order derive witnesses for genuine network entanglement.

As a starter, we consider a three qudit state $\rho_{ABC}$, and quantify its proximity to the Greenberger-Horne-Zeilinger (GHZ) state [24] via the fidelity

$$F_{\text{GHZ}} = \langle \text{GHZ}_d | \rho_{ABC} | \text{GHZ}_d \rangle,$$

where $| \text{GHZ}_d \rangle = \sum_{i=1}^{d} |i i i\rangle / \sqrt{d}$.

If $\rho_{ABC}$ is of the form (4), then there exists a random variable $\Lambda$, quantum states $\sigma_{A'B''}, \sigma_{B'C''}$, and families of CPTP maps $\{\Omega_A^\Lambda\}_\Lambda$, $\{\Omega_B^\Lambda\}_\Lambda$, and $\{\Omega_C^\Lambda\}_\Lambda$ that generate $\rho_{ABC}$. Now, by using these resources, one can consider inflation scenarios that involve multiple copies of them, wired in ways different to the original scenario but still respecting the structure of the Hilbert spaces on which the maps act. Quantum states compatible with Eq. (4) will satisfy constraints defined on these new scenarios, and a violation of such constraints will constitute a proof of genuine network 3-entanglement.

We next show how to analytically obtain genuine network entanglement witnesses using the ring inflation of Figure 2. The states produced in the inflation scenario are described by the density matrices $\tau_{A_1B_1C_1A_2B_2C_2}$ and $\gamma_{A_1B_1C_1A_2B_2C_2}$, which are essentially unknown to us, as we do not know how Eve’s devices act when they are wired differently.

However, the states $\tau$ and $\gamma$ are subject to several consistency constraints. To begin, with, $\tau$ is symmetric under the exchange of systems $A_1B_1C_1$ by systems $A_2B_2C_2$, and so is $\gamma$ under the exchange of $A_3B_3C_3$ by $A_4B_4C_4$. In addition, we observe that

$$\tau_{(A_1B_1C_1)} = \tau_{(A_2B_2C_2)} = \rho_{ABC}.$$  

Still, we cannot say that $\tau_{A_1B_1C_1A_2B_2C_2} = \rho_{ABC} \otimes \rho_{ABC}$ as the production of the two triangles could be classically correlated through the shared randomness $\Lambda$. However, the state $\tau$ is separable across the $A_1B_1C_1/A_2B_2C_2$ partition. Both $\tau$ and $\gamma$ are related to each other through the constraints

$$\gamma_{(A_1B_1A_2B_2)} = \tau_{(A_1B_1A_2B_2)}$$

and $\gamma_{(B_3C_3B_4C_4)} = \tau_{(B_3C_3B_4C_4)}$ and $\gamma_{(C_3A_4C_4)} = \tau_{(C_3A_4C_4)}$. Furthermore, $\tau$ and $\gamma$ have trace one and are semidefinite positive. Finally, the reduced state $\gamma_{(A_2B_3C_3B_4)}$ is separable across the $A_2B_3C_3/B_4$ partition; and additional constraints of that type follow from cyclic symmetry.

Let us now provide some intuition as to why any state $\rho_{ABC}$ admitting such extensions $\tau, \gamma$ cannot be arbitrarily close to the GHZ state. Suppose, indeed, that $F_{\text{GHZ}} = 1$, i.e., $\rho_{ABC} = |\text{GHZ}_d\rangle \langle \text{GHZ}_d|$ and that there exist extensions $\tau, \gamma$ satisfying the constraints above. A measurement in the computational basis of the sites $A_3B_3C_3$ of $\gamma$ will generate the random variables $a_3, b_3, c_3$. Since $\gamma_{(A_3B_3C_3)} = \rho_{(AB)} = \frac{1}{d} \sum_{i=1}^{d} |i i i\rangle \langle i i i|$, it must be the case that $a_3, b_3$ are perfectly correlated. The same considerations hold for $b_3$ and $c_3$. Since $a_3, b_3$ and $b_3, c_3$ are pair-wise perfectly correlated, so are $a_3, c_3$. Now, from the condition $\gamma_{(A_3A_4C_4)} = \gamma_{(C_4A_4C_4)}$, we have that the distribution of $c_3$ and $a_3$ must be the same as that of $c_2$ and $a_1$. Hence $c_2$ and $a_1$ must be perfectly correlated. However, $\tau_{(A_1B_1C_1)}$ is a pure state, since $\tau_{(A_1B_1C_1)} = |\text{GHZ}_d\rangle \langle \text{GHZ}_d|$, and hence it must be in a product state with respect to any other system, such as $C_2$. It follows that a measurement in the computational basis of the sites $A_1$ and $C_2$ will produce two uncorrelated random variables $a_1, c_2$. We thus reach a contradiction.

The previous argument just invalidates the case $F_{\text{GHZ}} = 1$. A more elaborate argument (see Appendix A for a proof) shows that, if $a, b, c$ are the random variables resulting from measuring $\rho_{ABC}$ locally, then any network 2-entangled state $\rho_{ABC}$ must satisfy

$$H(a:b) + H(b:c) - H(b) \leq S(\rho_{AB}) - S(\rho_{ABC}) + S(\rho_{BC}).$$

Here $H(x), H(x : y)$ and $S(\rho)$ respectively denote the Shannon entropy of variable $x$; the mutual information between the random variables $x, y$; and the von Neumann entropy of state $\rho$. Condition (8) is clearly violated if $\rho_{ABC} \approx |\text{GHZ}_d\rangle \langle \text{GHZ}_d|$ and the measurements are carried in the computational basis.

Another constraint satisfied by states satisfying eq. (4), expressed in terms of the GHZ fidelity, is

$$F_{\text{GHZ}} \leq \frac{2d(3d + \sqrt{2d^2 - 1})}{1 - 2d^2 + 9d^2}.$$  

Remarkably, in order to derive Eqs. (8) and (9), it is not necessary to invoke the existence of the six-partite states $\tau, \gamma$, but that of their reduced density matrices $\tau_{(A_1B_1C_1C_2)}, \gamma_{(A_1B_1C_1C_2)}$. As shown in Appendix B, both expressions (8) and (9) can be generalized to detect genuine network $k$-entanglement.

For $d=2$, Eq. (9) establishes that any tripartite state with $F_{\text{GHZ}} > \frac{1}{3} \left(6 + \sqrt{3}\right) \approx 0.9372$ is genuine network 3-entangled. As it turns out, this inequality is not tight: it can be improved to $F_{\text{GHZ}} > \frac{1}{3} \left(6 + \sqrt{3}\right) \approx 0.6803$ by means of semidefinite programming applied to the ring inflation.

The variables in the corresponding program are trace-one SDP matrices $\tau_{A_1B_1C_1A_2B_2C_2}$ and $\gamma_{A_1B_1C_1A_2B_2C_2}$ of size
FIG. 2. Ring inflation of the triangle scenario in Figure 1, containing copies of the state processing devices $\Omega_{A i B j C k}$; we label such copies according to their output Hilbert space $A_i B_j C_k$, where $i, j, k$ is the index of the copy. These devices process copies of the quantum resources $\sigma_{A'B'C'}$, $\sigma_{B'C'A'}$ and $\sigma_{C'A'B'}$. To simplify the drawing, we omitted the indices of these copies and only indicate their original type. Note that, despite the fact that the wirings between states and CPTP maps are different than in the original scenario, every copy of a CPTP map acts on copies of the states determined by the original scenario.

64 × 64, subject to linear constraints of the form (6) and (7), as well as to the permutational symmetry $1 \leftrightarrow 2, 3 \leftrightarrow 4$. For all states $\rho_{\mathcal{X}\mathcal{Y}}$ separable across a $\mathcal{X}/\mathcal{Y}$ partition, we add a Positivity under Partial Transposition (PPT) constraint $(\rho_{\mathcal{X}\mathcal{Y}})^{1 \gamma} \geq 0$ [25]. This applies to $\tau$ across the $A_1 B_1 C_1 / A_2 B_2 C_2$ partition, and to reduced states of $\gamma$ for the partitions $A_1 B_2 C_3 / B_3 C_1 A_2, B_3 C_3 A_1 / C_4, C_3 A_4 B_4 / A_5$.

The bound $F_{\text{GHZ}} > 1 + \mathcal{J}$ is obtained by maximizing $\langle \text{GHZ}_2 | \rho_{ABC} | \text{GHZ}_2 \rangle$ subject to the constraints above—a typical instance of a semidefinite program—using the optimization toolbox CVX [26] and the solver MOSEK [27].

We also employed the semidefinite optimization procedure using as reference the W state [28], $|W\rangle \equiv \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle)$, concluding that any 3-qubit state $\rho_{ABC}$ with $\langle W | \rho_{ABC} | W \rangle > 0.7602$ is genuine network 3-entangled.

Armed with these witnesses, we find that several past experiments in quantum optics can be interpreted as demonstrations of genuine network tripartite entanglement [29–32]. Indeed, in all those experiments, the fidelity of the prepared states with respect to GHZ or W states is greater than the bounds derived above for network-bipartite states. The prepared states are thus certified to contain genuine network tripartite entanglement. 

Robustness to detection inefficiency. In many experimental setups, due to low detector efficiencies, the carriers transmitting the quantum information are often unobserved. The standard prescription in such a scenario consists in discarding the experimental data gathered when not all detectors click. Coming back to our adversarial setup, this post-selection of measurement results opens a loophole that Eve can in principle exploit to fool Alice, Bob and Charlie. It is possible to contemplate this contingency in the calculations above, and thus bound the detection efficiency needed for certifying genuine network entanglement under post-selection.

Let $p$ indicate the fraction of experimental data preserved by postselection, i.e., the probability that all three detectors click. If $\rho_{ABC}^p$ is the state reconstructed after postselection, then all that can be said about the true tripartite quantum state $\rho_{ABC}$ before the postselection took place is that

$$\rho_{ABC} - p \times \rho_{ABC}^p \geq 0.$$  

As before, linear optimizations over the set of postselected states $\rho_{ABC}^p$ can be conducted via semidefinite programming. In such instances one continues to correlate the inflated states $\tau$ and $\gamma$ to the true (albeit unknown) tripartite state $\rho_{ABC}$, and Eq. (10) is merely added as an extra constraint. We find critical postselection probabilities beyond which one can still certify genuine network tripartite entanglement via GHZ fidelity ($p_c \approx 0.685$) or W fidelity ($p_c \approx 0.765$).

Conclusions. In this paper we have argued that the standard definition of genuine multipartite entanglement is not appropriate to assess the quantum control over an ever-growing number of quantum systems. We proposed an alternative definition, genuine network multipartite entanglement, that captures the potential of a source to distribute entanglement over a number of spatially separated parties. We provided analytic and numerical tools to detect genuine network tripartite entanglement, and also indicated how the definition can be adapted to situations where there may be local postselections on each party’s lab. Furthermore, the construction can be adapted to detect genuine network $n$-partite entanglement for any $n$.

While quite general, our numerical methods to detect genuine network entanglement demand considerable memory resources, to the point that we were not able to derive new entanglement witnesses for tripartite qutrit states in a normal computer. In addition, there exist significant gaps between the bounds we derived on GHZ and W state fidelities via SDP relaxations and the lower bounds obtained using standard variational techniques [33, 34]. Using such algorithms, we were not able to give lower bounds to the GHZ and W fidelities larger than 0.5170 and 2/3, respectively. A topic for future research is thus to develop better techniques for the characterization of genuine network multipartite entanglement.

Note added. After completing this manuscript, we became aware of the work of [35], whose authors consider a scenario very similar to that depicted in Figure 1. Crucially, they restrict the maps $\Omega_{A_i B_j C_k}$ to be unitary transformations, acting on convex combinations of bipartite states. The restriction to unitary maps not only
allows upper-bounding the dimensionality of the source states $\sigma_{A'B'C'},\sigma_{C'A'B'},\sigma_{B'C'A'}$, but it also severely constrains the resulting set of states $\Delta_C$: as shown in [35], tripartite qubit states in $\Delta_C$ cannot be genuinely tripartite entangled. This contrasts with the GHZ fidelity greater than $1/2$ reported above, achievable by states of the form (4).

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Appendix A: analytic witnesses for genuine network tripartite entanglement

The goal of this appendix is to prove the following result:

**Theorem 1.** Let \( \rho_{ABC} \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \) be a tripartite quantum state, and let \( a,b,c \) be the outcomes which result when we locally probe subsystems \( A,B,C \). If \( \rho_{ABC} \) is not a genuine network 3-entangled, then it must satisfy the relations:

\[
\langle \text{GHZ}_d | \rho | \text{GHZ}_d \rangle \leq \frac{2d(3d + \sqrt{2d^2-1})}{1-2d^2+9d^2}, \tag{11}
\]

\[
H(a:b) + H(b:c) - H(b) \leq S(\rho_A) + S(\rho_{AB}), \tag{12}
\]

where \( S(\rho_A) \) and \( S(\rho_{AB}) \) respectively denote the von Neumann entropy of \( \rho_A \) and the conditional entropy of system \( A \) with respect to \( BC \), i.e., \( S(\rho_{ABC}) - S(\rho_{BC}) \).

The intuition behind the proofs of both inequalities is the same. First, we assume that the state \( \rho_{ABC} \) admits the six-partite extensions \( \gamma_{A_1,B_1,C_1,A_2,B_2,C_2} \) and \( \gamma_{A_3,B_3,C_3,A_4,B_4,C_4} \) described in the main text. Then we prove that a strong correlation between the random variables \( a_3,b_3 \) and \( b_3,c_3 \) implies a strong correlation between the variables \( a_3,c_3 \), and hence a strong correlation between the variables \( a_1,c_2 \). Next, we show that the correlation between the variables \( a_1,c_2 \) is upper bounded in some way by the purity of the original state \( \rho_{ABC} \). To obtain one bound or another we rely on different measures of correlation and purity.

The following lemma will establish the transitivity of strongly correlated variables.

**Lemma 1.** Let \( x, y, z \) be jointly distributed random variables. Then, the following inequalities hold:

\[
P(x = z) \geq P(x = y) + P(y = z) - 1, \tag{13}
\]

\[
H(x : z) \geq H(x : y) + H(y : z) - H(y), \tag{14}
\]

**Proof.** Let \( P(x,y,z) \) be the joint probability distribution of the three variables. Then we have that

\[
P(x = y) + P(y = z) - P(x = z) = \sum_i P(i,i,i) + \sum_{j \neq i} P(i,i,j) + \sum_{j \neq i} P(j,i,i) - \sum_{j \neq i} P(i,j,i) \]

\[
\leq \sum_i P(i,i,i) + \sum_{j \neq i} P(i,i,j) + P(j,i,i). \tag{15}
\]

The right-hand side of the above equation contains the probabilities of a set of incompatible events. Its sum is thus bounded by 1, hence proving inequality (13).

To prove Eq. (14), we invoke strong subadditivity. Namely, for any three random variables \( x,y,z \), it holds that

\[
H(x,y,z) \leq H(x,y) + H(y,z) - H(y). \tag{16}
\]

The left-hand side of the equation above can be lower bounded by \( H(x,z) \). It follows that \( H(x : z) = H(x) + H(z) - H(x,z) \) is lower bounded by \( H(x) + H(z) - H(x,y) - H(y,z) + H(y) \). This, in turn, equals the right-hand side of Eq. (14). \( \square \)

The next lemma will relate the purity of a tripartite state with the correlations it can establish with other systems.

**Lemma 2.** Consider a four-partite quantum state \( \rho_{ABCY} \), with \( F = |\text{GHZ}_d \rangle \langle \text{GHZ}_d | \), and suppose that \( a,y \) are the result of measuring systems \( A,Y \) in the computational basis, then the inequality

\[
P(a = y) \leq 1 + \left( \frac{1}{d} - 1 \right) F + 2\sqrt{\frac{(1 - F)}{d}} \tag{17}
\]

holds. Moreover, independently of the nature of the measurements, the relation

\[
H(a:y) \leq S(A) + S(\rho_{ABC}). \tag{18}
\]

is satisfied.

**Proof.** Suppose that we measure systems \( A,Y \) in the computational basis, and define the operator

\[
E \equiv \sum_{i=1}^{d} |i\rangle \langle i| \otimes |0\rangle \langle 0| \otimes |i\rangle \langle i|. \tag{19}
\]

Then, \( P(a = y) = \text{tr}[E \rho] \). Furthermore, one can verify that

\[
(P_0 \otimes 1) E (P_0 \otimes 1) = \frac{1}{d} |\text{GHZ}_d \rangle \langle \text{GHZ}_d | \otimes 1, \tag{20}
\]

where \( P_0, P_1 \) are the projectors defined by \( P_0 = |\text{GHZ}_d \rangle \langle \text{GHZ}_d |, \ P_1 = 1 - P_0 \).

We have that

\[
\text{tr}[E \rho] = \sum_{i,j=0,1} \omega_{ij}, \tag{21}
\]
where $\omega$ is the $2 \times 2$ matrix defined by

$$\omega_{ij} = \text{tr}[\sigma(P_i \otimes \mathbb{1})E(P_j \otimes \mathbb{1})].$$

(22)

$\omega$ is positive semidefinite. Indeed, take an arbitrary vector $|c\rangle$. Then,

$$\langle c|\omega|c\rangle = \text{tr}\left[\sigma\left(\sum_i c_i^* P_i \otimes \mathbb{1}\right)E\left(\sum_j c_j P_j \otimes \mathbb{1}\right)\right] \geq 0,$$  

(23)

where the last inequality stems from the fact that both $\sigma$ and $E$ are positive semidefinite.

From the positive-semidefiniteness of $\omega$ it follows that $|\omega_{01}| \leq \sqrt{\omega_{00}\omega_{11}}$. On the other hand, by (20) we have that

$$\omega_{00} = \frac{1}{d} \text{tr}[\sigma(|\text{GHZ}_d \rangle \langle \text{GHZ}_d| \otimes \mathbb{1}_d)] = \frac{F}{d}. $$

(24)

In addition, $\omega_{11} = \text{tr}[\tilde{\sigma} E]$, where $\tilde{\sigma}$ is the positive semidefinite operator defined by

$$\tilde{\sigma} \equiv (P_1 \otimes \mathbb{1})\sigma(P_1 \otimes \mathbb{1}). $$

(25)

Note that $\text{tr}[\tilde{\sigma}] = \text{tr}[\sigma(\text{ABC})P_1] = 1 - F$. Since the operator $E$ has norm 1, it follows that $\omega_{11} = \text{tr}[\tilde{\sigma} E] \leq 1 - F$. Putting all together, we have that

$$P(a = y) \leq \frac{F}{d^2} + 1 - F + 2\sqrt{\frac{F(1 - F)}{d}}. $$

(26)

This proves Eq. (17).

Let $\alpha, \beta$ be two quantum systems. By $S(\alpha|\beta) = S(\alpha\beta) - S(\beta)$ we denote the conditional quantum information; by $S(\alpha : \beta) = S(\alpha) + S(\beta) - S(\alpha\beta)$, the quantum mutual information. To prove Eq. (18), we invoke weak monotonicity, namely, the fact that for any three quantum subsystems $\alpha, \beta, \gamma$, $S(\alpha|\beta) + S(\alpha|\gamma) \geq 0$. Taking $\alpha = A, \beta = BC, \gamma = Y$, we have that $-S(A|Y) \leq S(A|BC)$. By the data processing inequality it thus follows that

$$H(a : y) \leq S(A : Y) = S(A) - S(A|Y) \leq S(A) + S(A|BC).$$

(27)

Having reached this point, we are ready to prove part of Theorem 1. Choose Positive Operator Valued Measures (POVMs) $M_A, M_B, M_C$ and use them to probe the type-$A$, type-$B$ and type-$C$ subsystems of $\gamma$ and $\tau$, thus obtaining the random variables $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$. From the constraints $\rho(\text{ABC}) = \gamma(\text{A}, \text{B}, \text{C}), \rho(\text{BC}) = \gamma(\text{B}, \text{C})$ and Lemma 1, we arrive at the relations

$$P(a = b) + P(b = c) - 1 = P(a_3 = b_3) + P(b_3 = c_3) - 1 \leq P(a_3 = c_3), $$

(28)

and

$$H(a : b) + H(b : c) - H(b) = H(a_3 : b_3) + H(b_3 : c_3) - H(b_3) \leq H(a_3 : c_3), $$

(29)

where $a, b, c$ is the result of locally measuring $\rho_{\text{ABC}}$ according to the POVMs $M_A, M_B, M_C$.

On the other hand, $\gamma(\text{A}, \text{C}) = \tau(\text{A}, \text{C})$, and hence $H(a_3 : c_3) = H(a_1 : c_2)$ and $P(a_3 = c_3) = P(a_1 = c_2)$. Invoking Lemma 2, the right hand side of Eq. (29) is upper bounded by the right-hand side of Eq. (18). This proves Eq. (12).

If $M_A, M_B, M_C$ moreover correspond to measurements in the computational basis, then we can invoke again Lemma 2 to bound the right-hand side of Eq. (28) with the right-hand side of Eq. (17). This gives:

$$P(a = b) + P(b = c) - 1 \leq 1 + \left(\frac{1}{d} - 1\right)F + 2\sqrt{\frac{F(1 - F)}{d}}.$$ 

(30)

To prove Eq. (11), we need to lower bound $p(a = b)$ and $p(b = c)$ in terms of the GHZ fidelity. The necessary bound is provided by the next lemma.

**Lemma 3.** Let $\rho_{\text{ABC}} \in \mathcal{B}(\mathcal{C}^d \otimes \mathcal{C}^d \otimes \mathcal{C}^d)$ be a tripartite quantum state, and let $F = \langle \text{GHZ}_d | \rho | \text{GHZ}_d \rangle$ be its GHZ fidelity. If we measure any two systems $A, B, C$ in the computational basis, the corresponding random variables $a, b$ will satisfy:

$$P(a = b) \geq F.$$ 

(31)

**Proof.** Without loss of generality, suppose that we measure systems $A$ and $B$, obtaining the random variables $a$ and $b$, respectively. Notice that $P(a = b) = \text{tr}[S \rho]$, with $S$ defined by

$$S \equiv \sum_{i=1}^{d} |i\rangle \langle i| \otimes \mathbb{1}_d.$$ 

(32)

Then one can verify that

$$P(b) = \text{tr}[S \rho] = \sum_{i,j=0,1} \text{tr}[\rho P_i \rho S_j] = \text{tr}[|\text{GHZ}_d\rangle \langle \text{GHZ}_d| \rho] + \text{tr}[P_4 S_1 \rho] \geq F.$$ 

(34)

It follows that

$$P(a = b) = \text{tr}[S \rho] = \sum_{i,j=0,1} \text{tr}[\rho P_i \rho S_j] \geq F.$$ 

(30)

Now, use the previous lemma to lower bound the left-hand side of Eq. (30) by $2F - 1$. Solving the inequality for $F$, we arrive at Eq. (11).

**Appendix B: analytic witnesses for genuine network $k$-entanglement**

The arguments leading to Theorem 1 can be extended to detect network $k$-entanglement. Consider
this time a $k$-partite quantum state $\rho_{X^0...X^{k-1}}$, and suppose that it can be generated by applying correlated local maps to $(k-1)$-partite quantum states of the form $\{\sigma^{\otimes j},\cdots,\sigma^{\otimes (k-2)\otimes j}:j=0,\ldots,k-1\}$, where $\otimes$ indicates addition modulo $k$. As in the tripartite case, we consider a double network with nodes $X^0_j$, $X^{k-1}_j$, $X^0_j$, $X^1_j$, $X^2_{j-1}$, ..., $X^2_{k-1}$. For $j=0,\ldots,k-1$, we distribute two copies of the states $\rho_{\sigma^{\otimes j}}$ to systems $\lambda^0_j, \lambda^1_j, \lambda^{k-1}_j$; $\lambda^0_j, \lambda^1_j, \lambda^{k-1}_j$, $\lambda^0_j, \lambda^1_j, \lambda^{k-1}_j$, respectively. By applying the maps $\Omega^\lambda_{\lambda^0}$ over systems $\lambda^0_j, \lambda^1_j$ and averaging over $\lambda$, we obtain a $2k$-partite quantum state $\tilde{\tau}_{X^0...X^{k-1}}$ with $\tilde{\tau}_{X^0...X^{k-1}} = \tilde{\tau}(\rho_{X^0...X^{k-1}})$.

On the contrary, consider the systems $\lambda^0_j, \lambda^1_j, \lambda^{k-1}_j$, $\lambda^0_j, \lambda^1_j, \lambda^{k-1}_j$, order them as $0,1,\ldots,2k-1$, and distribute each state $\rho_{\sigma^{\otimes j}}$ to the systems $0+j$ (mod $2k$), $k-1+j$ (mod $2k$), for $j=0,\ldots,2k-1$. Applying the maps $\Omega^\lambda_{\lambda^0}$ and averaging over $\lambda$, we end up with a $2k$-partite state $\gamma_{0...2k-1}$ with the following properties:

1. $\gamma_{i+1} = \rho_{X^i,X^{i+1}}$. This is so because, in the previous construction, each node $i$ shares with node $i+1$ the $k-2$ states $\{\tau_{\sigma^{\otimes i+1}},\cdots,\tau_{\sigma^{\otimes i+1}}\}$. These are all the states which those two nodes would have shared had they been part of the network that built $\rho_{X^0...X^{k-1}}$. Hence, their joint state must correspond to the latter’s reduced state $\rho_{\sigma^{\otimes i+1}}$.

2. $\gamma_{X^0...X^{k-1}} = \tau_{\gamma_{X^0...X^{k-1}}}$. This follows from the fact that the states used to generate $\gamma$ were distributed in such a way that no states are shared by systems $i$ and $i+k-1$ (mod $2k$).

We are ready to derive new witnesses. Say that the original state $\rho_{X^0...X^{k-1}}$ has a high fidelity with the $k$-partite GHZ state, i.e.,

$$F_{GHZ_d^k} = \langle GHZ^0_d^k|\rho|GHZ^0_d^k \rangle,$$

where $|GHZ^0_d^k\rangle = \sum_{j=1}^{d^{k-1}} \frac{(j_{\otimes k})}{\sqrt{d}}$.

We locally measure $\rho$ in some basis, obtaining the random variables $x^0,\ldots,x^{k-1}$. If $F_{GHZ_d^k}$ is high enough and the measurement basis is close to the computational one, then one should expect to find a high correlation between $x^j$, $x^j$, for $i,j=0,\ldots,k-1$. This implies that a measurement of $\gamma$ in the computational basis will produce random variables $x^0,\ldots,x^{k-1} \chi_i$ with very high coincidence probability $P(x^i = x^{i+1})$ and mutual information $H(x^i : x^{i+1})$ between neighboring sites. Applying Lemma 1 recursively, and taking into account that the distributions of $x^0, x^{k-1}$ and $x^i, x^{i+1}$ are the same, we have that

$$\sum_{i=0}^{k-2} H(x^i : x^{i+1}) - \sum_{i=1}^{k-2} H(x^i) \leq H(x^0 : x^{k-1})$$

and

$$\sum_{i=0}^{k-2} P(x^i = x^{i+1}) - k-2 \leq P(x^0 = x^{k-1}).$$