Remarks on the spectral radius of $K_{r+1}$-saturated graphs

V. Nikiforov*

Abstract

Write $\rho(G)$ for the spectral radius of a graph $G$ and $S_{n,r}$ for the join $K_r \vee \overline{K}_{n-r}$.

Let $n > r \geq 2$ and $G$ be a $K_{r+1}$-saturated graph of order $n$.

Recently Kim, Kim, Kostochka, and O determined exactly the minimum value of $\rho(G)$ for $r = 2$, and found an asymptotically tight bound on $\rho(G)$ for $r \geq 3$. They also conjectured that

$$\rho(G) > \rho(S_{n,r-1}),$$

unless $G = S_{n,r-1}$.

In this note their conjecture is proved.

Keywords: $K_r$-saturated graph; spectral radius; sum of squares of degrees; extremal graph.

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1 Introduction

Given a graph $G$, we write $e(G)$ for the number of its edges and $\rho(G)$ for the spectral radius of its adjacency matrix. We write $S_{n,r}$ for the join $K_r \vee \overline{K}_{n-r}$.

A graph is called $K_r$-saturated if it is $K_r$-free, but the addition of any edge creates a copy of $K_r$ in it.

In [1], Erdős, Hajnal, and Moon established the following theorem:

**Theorem 1** If $n > r \geq 2$ and $G$ is a $K_{r+1}$-saturated graph of order $n$, then

$$e(G) > e(S_{n,r-1}),$$

unless $G = S_{n,r-1}$.

In a recent seminal paper Kim, Kim, Kostochka, and O [3] studied a spectral version of this result, in which $e(G)$ is replaced by $\rho(G)$.

For $r = 2$, they found the best possible result:

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*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA. Email: vnikifrv@memphis.edu
Theorem 2 If $n \geq 3$ and $G$ is a $K_3$-saturated graph of order $n$, then
\[ \rho(G) > \rho(S_{n,1}) = \sqrt{n - 1}, \]
unless $G$ is the star $S_{n,1}$ or a Moore graph of diameter 2.

For $r \geq 2$ these authors proved the following elegant combinatorial result ([3], Theorem 2.1).

Theorem 3 If $n > r \geq 2$ and $G$ is a $K_{r+1}$-saturated graph of order $n$, then
\[ \sum_{v \in V(G)} d^2(u) - 2d(u) \geq (r - 1)(n - r + 1) n. \tag{1} \]
If equality holds in (1), then the neighbors of every vertex of $G$ induce $S_{d(u),r-2}$.

Note that the necessary condition for equality given above is not part of the original statement, but can be readily extracted from its proof.

For $r \geq 3$, combining Theorems 1 and 3, the authors of [3] deduced an asymptotically tight lower bound for a $K_{r+1}$-saturated graph $G$ of order $n$:
\[ \rho(G) \geq \sqrt{\frac{(n-1)^2(r-1) + (r-1)^2(n-r+1)}{n}} = \rho(S_{n,r-1}) - \frac{r-2}{r} + \Theta \left( \frac{r^{1.5}}{\sqrt{n}} \right). \]

In addition, they made the following conjecture:

Conjecture 4 If $n > r \geq 3$ and $G$ is a $K_{r+1}$-saturated graph of order $n$, then
\[ \rho(G) > \rho(S_{n,r-1}), \]
unless $G = S_{n,r-1}$.

The purpose of this note is to prove Conjecture 4. Our approach relies on Theorem 3, but requires a few other ideas as well. In particular, we provide a more definite condition for equality in (1):

Lemma 5 If $n > r \geq 3$ and $G$ is a $K_{r+1}$-saturated graph of order $n$, then equality holds in (1) if and only if $G = S_{n,r-1}$.

Additionally, we show that Theorem 3 implies Theorem 1. The latter statement needs some clarification, for, in fact, Theorem 1 is used in the proof of Theorem 3; hence, the opposite implication seems to lack merit. However, Theorem 1 is used for a proof by induction on $r$, while the implication “Theorem 3 $\Rightarrow$ Theorem 1” is independent of $r$. Therefore, one can devise a general theorem that encompasses Theorems 1 and 2, and the statement of Conjecture 4.
2 Proof of Lemma 5

We shall prove that if equality holds in (1), then $G = S_{n,r-1}$. Our main tool is the condition for equality stated in Theorem 3.

Before going into details, let us give two definitions: first, write $K'_k$ for the graph obtained from $K_k$ by removing two incident edges; second, for any vertex $u \in V(G)$, write $G(u)$ for the subgraph induced by the neighbors of $u$.

Note that the condition for equality in Theorem 3 implies that for any $u \in V(G)$, the graph $G(u)$ is isomorphic to $S_{d(u),r-2}$.

Our main goal is to prove that $G$ does not contain an induced $K'_3$, and is therefore complete $r$-partite.

Assume for a contradiction that the vertices $u,v,w \in V(G)$ induce a copy of $K'_3$, and suppose that $\{u,v\} \in E(G)$. Hence, $\{w,u\} \notin E(G)$ and $\{w,v\} \notin E(G)$.

As $G + \{w,u\}$ contains a copy of $K_{r+1}$, it follows that $G$ has an $(r-1)$-clique $R_u$ whose vertices are joined to both $w$ and $u$. By symmetry, $G$ has an $(r-1)$-clique $R_v$ whose vertices are joined to both $w$ and $v$.

If $R_u$ and $R_v$ have a vertex $x$ in common, then $u,v,w$ induce a copy of $K'_3$ in $G(x)$, which is a contradiction, because $G(x) = S_{d(x),r-2}$.

If $R_u$ and $R_v$ have no vertices in common, then $R_u$ and $R_v$ are disjoint $(r-1)$-cliques in $G(w)$, which is a contradiction, because $G(w)$ is isomorphic to $S_{d(w),r-2}$ and $r \geq 3$.

Hence, $G$ contains no induced $K'_3$, and is therefore complete multipartite. As the clique number of $G$ is $r$, $G$ is complete $r$-partite.

Finally, suppose that $G$ has two vertex classes with two or more vertices and let $u$ be a vertex that does not belong to either of these classes. Clearly, $G(u)$ is not isomorphic to $S_{d(u),r-2}$, because $G(u)$ contains an induced 4-cycle. Hence, all vertex classes of $G$ except one consist of exactly one vertex, that is, $G = S_{n,r-1}$.

3 Proof of Conjecture 4

Our goal is to show that

$$\rho(G) \geq \rho(S_{n,r-1}).$$

It follows from our proof that if equality holds in (2), then equality holds in (1), and now Lemma 5 implies that $G = S_{n,r-1}$.

We prove inequality (2) as a sequence of simple propositions and observations.

Assume that $G$ is a $K_{r+1}$-saturated graph of order $n$ with minimum spectral radius $\rho(G)$ among all such graphs of order $n$. Obviously $G$ is connected.

Proposition 6 Every edge of $G$ belongs to a copy of $K_r$.

Proof Let $uv \in E(G)$. Note that $G - uv$ is not $K_{r+1}$-saturated, for if it were, $\rho(G - uv) < \rho(G)$, contrary to our choice of $G$.

That means that we can add an edge $xy \neq uv$ to $G - uv$ so that the graph $G - uv + xy$ is $K_{r+1}$-free.

But since $G + xy$ has a copy of $K_{r+1}$, say $Q$, it turns out that $uv$ belongs to $Q$.

Therefore, $uv$ belongs to $Q - xy$, and thus, $uv$ belongs to a copy of $K_r$. ☐
Let $A$ be the adjacency matrix of $G$ and set
\[ B := A^2 - (r - 2) A. \]

Clearly $B$ is a symmetric matrix.

In the subsequent arguments we use a few facts from matrix theory, which may be found in [4].

**Proposition 7** The matrix $B$ is nonnegative.

**Proof** Note that for any distinct vertices $u$ and $v$, the $(u, v)$th entry of $A^2$ is equal to the number of paths of length 2 joining $u$ to $v$.

Every edge of $G$ belongs to a copy of $K_r$; hence, for every edge $uv$, there are at least $r - 2$ paths of length 2 joining $u$ to $v$.

Therefore the $(u, v)$th entry of $A^2$ is at least $r - 2$, implying that $B$ is nonnegative. \(\square\)

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$, indexed in descending order, and suppose that $v_1, \ldots, v_n$ is an orthonormal basis of eigenvectors to $\lambda_1, \ldots, \lambda_n$. Additionally we suppose that $v_1$ is positive; we may, because $G$ is connected and thus $A$ is irreducible.

Since
\[ B v_i = \left( A^2 - (r - 2) A \right) v_i = \left( \lambda_i^2 - (r - 2) \lambda_i \right) v_i, \]
for every $i = 1, \ldots, n$, we see that the numbers
\[ \lambda_1^2 - (r - 2) \lambda_1, \ldots, \lambda_n^2 - (r - 2) \lambda_n \] (3)
are the eigenvalues of $B$ with corresponding eigenvectors $v_1, \ldots, v_n$.

Note, however, that the ordering of the eigenvalues (3) is not necessarily descending or ascending. Nevertheless, since $\rho(B)$ is a positive eigenvalue of $B$ and $x^2 - (r - 2) x$ is a convex function in $x$, the spectral radius of $B$ satisfies
\[ \rho(B) = \max \left\{ \lambda_1^2 - (r - 2) \lambda_1, \lambda_n^2 - (r - 2) \lambda_n \right\}. \]

In our case we can make a definite choice:

**Proposition 8** $\rho(B) = \lambda_1^2 - (r - 2) \lambda_1$.

**Proof** If $B$ is irreducible, then the Perron-Frobenius theory (see, e.g., [4], Theorems 4.1-4.4) implies that $\rho(B)$ has multiplicity one, and no other eigenvalue of $B$ can have a nonnegative eigenvector. Therefore,
\[ \rho(B) = \lambda_1^2 - (r - 2) \lambda_1, \]
proving the proposition if $B$ is irreducible.

If $B$ is reducible, then it is a direct sum of irreducible principal submatrices $B_1, \ldots, B_k$. Note that for every $i = 1, \ldots, k$, the number $\lambda_i^2 - (r - 2) \lambda_1$ is an eigenvalue to $B_i$ with a positive eigenvector, which is the restriction of $v_1$ to the index set of $B_i$. It follows that
\[ \rho(B_1) = \cdots = \rho(B_k) = \lambda_1^2 - (r - 2) \lambda_1, \]
and so,
\[ \rho(B) = \lambda_1^2 - (r - 2) \lambda_1, \]
completing the proof of the proposition.

We are now ready for the proof of (2).
Since \(B\) is a symmetric matrix, the Rayleigh principle implies that
\[ \rho(B) \geq \langle Bx, x \rangle \]
for any \(n\)-vector \(x\) of length one. Taking \(x := (n^{-1/2}, \ldots, n^{-1/2})\) and writing \(\Sigma C\) for the sum of all entries of a matrix \(C\), we see that
\[ \rho(B) \geq \langle Bx, x \rangle = \frac{1}{n} \sum_{v \in V(G)} d^2(u) - (r - 2) d(u). \]
Since
\[ \Sigma A^2 = \sum_{v \in V(G)} d^2(u), \]
we find that
\[ \Sigma (A^2 - (r - 2) A) = \frac{1}{n} \sum_{v \in V(G)} d^2(u) - (r - 2) d(u). \]
Hence,
\[ \lambda_1^2 - (r - 2) \lambda_1 \geq \frac{1}{n} \sum_{v \in V(G)} d^2(u) - (r - 2) d(u). \]

Now, Theorem 3 implies that
\[ \lambda_1^2 - (r - 2) \lambda_1 - (r - 1) (n - r + 1) \geq 0, \]
and hence
\[ \rho(G) = \lambda_1 \geq \frac{r - 2 + \sqrt{(r - 2)^2 + 4 (r - 1) (n - r + 1)}}{2} = \rho(S_{n,r-1}). \]
Inequality (2) is proved.

4 Theorem 3 implies Theorem 1

In [2] and independently in [5] the following bound has been proved:

**Theorem 9** If \(G\) is a graph of order \(n\), with \(m\) edges, and with minimum degree \(\delta\), then
\[
\rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - \delta n + \frac{(\delta + 1)^2}{4}}. \tag{4}
\]
Now, if \( n > r \geq 2 \) and \( G \) is a \( K_{r+1} \)-saturated graph of order \( n \), then \( \delta \geq r - 1 \), because every vertex belongs to a copy of \( K_r \). As the right side of (4) decreases in \( \delta \), we find that

\[
\rho(G) \leq \frac{r-2}{2} + \sqrt{2m - (r-1)n + \frac{r^2}{4}}.
\]

Combining this inequality with (2), after some algebra, we get Theorem 1:

\[
2m \geq 2(r-1)n - r(r-1) = 2e(S_{n,r-1}).
\]

The condition for equality follows immediately, except for \( r = 2 \), where the Moore graphs have to be discarded first.

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