Aspects of Kink-Like Structures in 2D Dilaton Gravity

Francisco Cleiton E. Lima* and Carlos Alberto S. Almeida*

The topological structures that arise from two-dimensional (2D) models are relevant physically and the first step toward understanding more complex systems. In this work, one studies the kink-like solutions of the matter field that emerge in a 2D dilaton gravity scenario. Considering this scenario, the linear stability of the matter field and the translational mode is examined. Due to the specific profile of the solutions found, a differential configurational complexity (DCC) analysis of the system is necessary to confirm the nature of the structures. Furthermore, the interface and scattering process is studied, considering a pair of kink-antikink-like solutions in a fixed time \( t = 0 \).

1. Introduction

It is notorious that high-dimensional cosmological models suggest some explanations for open-ended problems in high-energy physics.\(^{11}\) These models can provide us with interpretations of open-ended issues in physics as the problem related to the hierarchy,\(^{2,3}\) the origin of dark matter,\(^{4}\) the cosmological constant,\(^{5,6}\) and cosmic acceleration.\(^{6,7}\) However, a simple alternative to understanding these cosmological models is to analyze the two-dimensional (2D) gravitational models.\(^{8}\) In the last decades, the study of 2D models has shown promise and has contributed to the return of the interest on these structures.\(^{9-12}\) Furthermore, the interforce and scattering process is studied, considering a pair of kink-antikink-like solutions in a fixed time \( t = 0 \).

The MMSS action is interpreted as a limit when \( D \to 2 \) for general relativity. The fact is that models governed by an MMSS-like action have been shown effective for studies of black hole chemistry and entropic gravity. On the other hand, the study of the dynamics of the matter field is still a little-frequented area. Motivated by these discussions, we will, throughout this work, study the dynamics of self-gravitating topological structures in 2D dilaton gravity.

The action of MMSS is interpreted as a limit when \( D \to 2 \) for general relativity. The fact is that models governed by an MMSS-like action have been shown effective for studies of black hole chemistry and entropic gravity. On the other hand, the study of the dynamics of the matter field is still a little-frequented area. Motivated by these discussions, we will, throughout this work, study the dynamics of self-gravitating topological structures in 2D dilaton gravity.

The main feature that motivates the studies of 2D structures is that, in low dimensions, we obtain the field by elementary techniques. Moreover, one can describe several physical systems approximately or effectively assuming one-dimensional field configurations.\(^{24}\) Additionally, to reach the 2D field configurations, the presence of domain walls in the theory is necessary. On the other hand, to confirm the existence of the domain wall, we can use arguments emerging from the information theory, i.e., the configurational entropy (CE) and its variants. Several works, in various dimensions, have used fundamentals from the CE to predict and complement the understanding of topological structures.\(^{25,26}\) This interpretation is possible since these formalisms can provide criteria to control the stability of field configurations based only on the informative content. For a review of the relevant features and applications of the CE and their properties to predict possible phase transitions, see the papers of Gleiser et al.\(^{27-30}\)

Recently, have been performed several studies on 2D structures.\(^{31-35}\) This interest is due to the diversified results that arise in the inquiry of the kink structures. Some impressive and promising results about 2D configurations appear in studies of the long-range interactions between kink and antikink.\(^{36-38}\)

---

F. C. E. Lima
Programa de Pós-graduação em Física
Universidade Federal do Maranhão
Campus Universitário do Bacanga
São Luís, Maranhão 65080-805, Brazil
E-mail: cleiton.estevao@fisica.ufc.br

C. A. S. Almeida
Departamento de Física
Universidade Federal do Ceará (UFC)
Fortaleza, Ceará 60455-760, Brazil
E-mail: carlos@fisica.ufc.br

C. A. S. Almeida
Institute of Cosmology
Department of Physics and Astronomy
Tufts University
Medford, Massachusetts 02155, USA

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/prop.202300051

DOI: 10.1002/prop.202300051
Other attractive results involving these topological structures also appear in the study of point particles and zero-branes,[39,40] in domain walls,[41] and investigations on the emergence of the universe.[42] Thus, one can conclude that these researches on 2D configurations are of great interest and help us understand more complex problems in theoretical physics.[43,44]

Our central purpose in this paper is the study of the emergence of self-gravitating topological structures in 2D dilaton gravity. Then, one seeks to understand the influence of the dilaton field and the spacetime curvature on the matter field. Furthermore, we investigate which internal force, or long-range interference, causes this interaction. The scattering process produced by this interference also is examined. Finally, adopting arguments from CE, the phase transitions and the class of solutions of the structure are investigated.

We organize our work as follows: We discuss the model and topological solutions of the matter field and its stability in Section 2. In Section 3, one calculates the inter-kink force and investigates the structure scattering. Posteriorly, we study the phase transitions of the matter field in Section 4. Finally, in Section 5, our findings are announced.

2. The Model and its Stability

We know from the literature that the Einstein tensor in 2D is null. Thus, to describe a model in 2D gravity, it is necessary to extend the Einstein-Hilbert action. An immediate way to extend Einstein’s gravity occurs when introducing a dilaton field coupled non-minimally with the metric. Given this consideration, let us consider the MMSS-type action[21]:

\[ S = \frac{1}{\kappa} \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \phi R + \kappa \left( -\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi - V(\psi) \right) \right] \]  (2)

where \( R \) is the Ricci scalar, \( \kappa \) is the gravitational coupling, \( \phi \) is the dilaton field, and \( \psi \) is a real scalar field.

The variation of action leads to equations of motion, i.e., the Einstein equation

\[ \kappa T_{\mu \nu} = -\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} g_{\mu \nu} (\nabla^\alpha \phi \nabla_\alpha \phi + 4 \nabla_\alpha \nabla^\alpha \phi) - \nabla_\mu \nabla_\nu \phi \]  (3)

the dilaton field equation

\[ \nabla_\mu \nabla_\nu \phi + R = 0 \]  (4)

and the scalar field equation

\[ \nabla_\mu \nabla^\mu \psi + \frac{dV}{d\psi} = 0 \]  (5)

The stress-energy tensor is

\[ T_{\mu \nu} = \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu \nu} (\nabla^\alpha \psi \nabla_\alpha \psi + 2V) \]  (6)

For our study of the 2D structures, let us consider the metric given by

\[ ds^2 = e^{2A}(\bar{dt}^2 + dr^2) \]  (7)

The metric (7) is widely used in the study of topological structures in extra dimensions.[43,45] The term \( A(r) \) is the warp function.

Substituting the metric (7) in the static Einstein equation (3), one obtains

\[ 2\phi''(r) + \phi'(r)^2 - 2A'(r)\phi'(r) = \kappa \psi'(r)^2 \]  (8)

and

\[ \phi''(r) + A'(r)\phi'(r) = -\kappa V(\psi) \]  (9)

Analyzing the Equations (4), (5), (8), and (9), one can conclude that only three of them are linearly independent. Indeed, we can derive the Equation (9) using the Equations (4), (5), and (8). Besides, one way to solve the above equation system is to use the first-order formalism, see ref. [46, 47]. In first-order formalism, it assumes a superpotential related to the interaction of the matter field to reduce the order of the equations of motion. In this work, we will not use first-order formalism, i.e., we will solve the second-order equations of the system numerically.

For simplicity, it is convenient to write the action (2) in terms of the fields \( \phi(r), \psi(r) \), and \( A(r) \). For this, we note that the Ricci scalar is

\[ R = -2e^{-2A(r)}A''(r) \]  (10)

Therefore, the action is

\[ S = \frac{1}{\kappa} \int d^2x \left[ -\frac{1}{2} \phi'(r)^2 - 2A'(r)\phi'(r) - \frac{1}{2} \kappa \psi'(r)^2 - \kappa e^{2A(r)}V(\psi) \right] \]  (11)

where the prime notation is the derivative regarding the position variable.

For the topological structures to arise is necessary to preserve the spontaneous symmetry breaking. Thus, we adopt a \( \varphi^4 \)-like interaction, i.e.,

\[ V(\psi) = \frac{\lambda}{2} (\psi^2 - \bar{\psi}^2)^2 \]  (12)

Allow us to start from the assumption that we know the warp function. Consequently, action happens to have two fields with the independent variable \( r \). These fields will describe the matter and dilaton fields. The equations of motion of scalar and dilaton fields are, respectively, given by

\[ \psi''(r) + 2\lambda e^{2A(r)}\psi(r)(\psi^2 - \bar{\psi}^2) = 0 \]  (13)

and

\[ \phi''(r) - 2A''(r) = 0 \]  (14)

From Equation (14), we obtain a relation between the dilaton field and the warp function, namely,

\[ \phi(r) = 2A(r) + \beta r + \gamma \]  (15)

To preserve the translational invariance of the \( \psi(r) \) field is convenient to assume the condition \( \beta = \gamma = 0 \). This condition is ap-
propriate because it also allows describing a topological matter field\(^1\). For more details, see refs. [8, 41]. Moreover, due to the choice of potential in Equation (12), the warp factor assumes the form

\[
e^{2A(r)} = \cosh^{-2\nu}(\lambda_0 r),
\]

where the \(\lambda_0\)-parameter makes the argument of hyperbolic dimensionless. Indeed, one can assume \(\lambda_0 = 1\). In 5D brane worlds, this profile of the warp function is widely used.\([43,44]\) In refs. [43, 44], using this profile of the warp function, the authors demonstrated the symmetrical and asymmetrical solutions for a thick brane in 5D. In our 2D case, the warp factor (16) leads us to the following equation for the kink structures

\[
\psi''(r) + 2\lambda \cosh^{-2\nu}(r)\psi(r)(\nu^2 - \psi(r)^2) = 0
\]

For the metric (7) and the warp factor (16), the dilaton field profile is given by

\[
\phi(r) = \ln(\cosh^{-2\nu}(r))
\]

We displayed the profile of the dilaton field in Figure 1. Note that our results for the dilaton field behavior are similar to those found in ref. [48].

Equation (17) is easily solved using numerical techniques. For simulation, we use topological boundary conditions, i.e.,

\[
\psi(r \to \pm \infty) = \pm \nu.
\]

where, for convenience, one adopts \(\nu = 1\).

As a result, the configuration of the \(\psi(r)\) field found has a double-kink-like profile. We expose the double-kink-like solutions in Figure 2a. Meanwhile, the double-antikink-like configurations are displayed in Figure 2b.

\[1\] Topological matter fields are fields in which the topological boundary conditions are valid. In this case, these conditions are \(\psi(r \to \pm \infty) \to \pm \nu\).

As a result, the configuration of the \(\psi(r)\) field found has a double-kink-like profile. We expose the double-kink-like solutions in Figure 2a. Meanwhile, the double-antikink-like configurations are displayed in Figure 2b.

2.1. The Stability of the Solutions

Let us analyze the linear stability of the kink-antikink-like self-gravitating solutions performing small perturbations in the field solutions (Figure 2). Allow us to consider this small perturbation around arbitrary static solutions. This perturbation of the metric is given by

\[
\delta g_{\mu\nu} = e^{2A(r)} \left( h_{\mu\nu}(r, t) \Phi(r, t) + \Phi(r, t) h_{\mu\nu}(r, t) \right)
\]

and for the first order, the perturbation of the inverse metric is

\[
\delta g^{\mu\nu} = -e^{-2A(r)} h^{\mu\nu}
\]

As assumed in refs. [8, 49], it is convenient to consider a new parameter makes the argument of hyperbolic differential equation

\[
A_i \ddot{\delta \phi} - 2A_i' \dot{\delta \phi} + \delta \phi + \delta \phi' + 2h_r \dot{\phi}'' + \frac{1}{2} h_r \dot{\phi}'' + \frac{1}{2} \dot{\phi}'' = \kappa \left( \psi' \Psi' + \psi'' \Psi - \frac{1}{2} \Psi \dot{\Psi} h_r \right)
\]

From the linearization of the scalar field (5), we found a new perturbation equation of the matter field, namely,

\[
\delta \psi'' + 2A_i \frac{\psi'''}{\psi} \delta \psi - \frac{\psi'''}{\psi} \Psi' \delta \psi - \frac{1}{2} \psi \dot{h}' \dot{\psi}'' + \frac{1}{2} h_r \dot{\phi}'' + \frac{1}{2} \dot{\phi}'' = 0
\]

But note that we have three independent perturbation equations. Nonetheless, we should notice that the perturbation variables are not all independent. By analyzing the invariance of the perturbed equations, it comes to

\[
\delta \psi'' - V_{\psi}(r) \delta \psi = 0
\]

For more details on perturbative calculations, see refs. [8, 49].

The effective potential is given by

\[
V_{\psi}(r) = 4A_i \psi'' - 2A_i \frac{\psi'''}{\psi} \psi'' + 2 \left( \frac{\psi'''}{\psi'} \right)^2 - 2 \frac{\psi'''}{\psi'} + \psi'''
\]

This effective potential is illustrated in Figure 3a.
Figure 2. Matter field when $p = 1.00, 1.05,$ and $1.15$.

Figure 3. (a) Effective potential. (b) Translational mode.

Let us now, as usual, assume that $\delta \psi = \xi(r)e^{i\omega t}$. This consideration leads us to

$$-\xi''(r) + V_{\text{eff}}(r)\xi(r) = \omega^2 \xi(r)$$  \hspace{1cm} (28)

By arguments similar to those of supersymmetric quantum mechanics (see refs. [8, 36, 49]), one obtains that the translational mode of the kink-like structure is

$$\xi_0(r) = C_0 \frac{\psi'(r)}{2A'(r)}$$  \hspace{1cm} (29)

where $C_0$ is a normalization constant. The translational mode of the kink-like structure was obtained and exposed in Figure 3b.

It is interesting to highlight some similarities of our model with results of works in extra-dimensions. For example, in braneworlds, the effective potentials have singularities. These singularities suggest that the scalar modes are non-normalizable. Even as models in 5D, our theory appears to be free of long-range scalar force problems and indicates the nonexistence of resonance. For details on this discussion, please see refs. [45, 50, 51]. We will confirm this scattering result in Section 4. A peculiar feature arises in the study of the translational mode. In truth, the zero modes behave as if the topological structure were described by a double-kink-like configuration, thus admit-
ting multiple-phase transitions. We will explore this result when discussing the DCC of the matter field.

3. Inter-kink Force

In this section, we are interested in studying the force which acts between structure pairs with opposite topological charge. Our interest in the interface of this system is because this investigation can give us information about the process of structure scattering.\(^2\)[41]

The inter-kink force or interface is an internal force between topological structures as they approach each other. Thus, this force can induce a scattering process (or not) between the topological structures. If the interforce between configurations is attractive, a scattering process will occur in the dynamic case.\(^2\)[41] Particularly, the scattering phenomenon plays a relevant role in topological field theories because, from this process, it is possible to state whether a structure is a true soliton.\(^2\)[41]

To study the interface, let us start by assuming that our solutions (Figure 2a and b) can form a pair of kink-antikink-like structures widely separated. The spacing between these structures is \(2a\). For convenience, the combination of the solutions of Equation (17) is exhibited in Figure (4).

The force exerted on a topological structure must obey Newton’s second law. To calculate this force, we must analyze the momentum of the field. The integral of the momentum density \(T^0 = \epsilon^{AB} T_{0} \) will give us the momentum of the structure in a large region around the kink. For the configuration located at \(x = -a\), we will choose to look at the localized field momentum at \((-a - R, -a + R)\), i.e.,

\[
P = \int_{-a-R}^{-a+R} dr \cosh^{-2p}(r) \psi(r, t) \psi'(r, t)\]

where the dot notation denotes the time derivative.

For convenience, the combination of the solutions of Equation (4) is initially static. This consideration will tell us that

\[
\dot{\psi}(x,t)|_{x=0} = 0
\]

what gives us

\[
F = \int_{-a-R}^{-a+R} \cosh^{-2p}(r) \left[ \frac{1}{2} \frac{d\psi'^2(r,t)}{dr} + \cosh^{-2p}(r) \frac{dV}{dr} + \psi(r, t) \psi'(r, t) \right] dr
\]

To finish our considerations, we will assume that the configuration in Figure (4) is initially static. This consideration will tell us that

\[
\dot{\psi}(x,t)|_{x=0} = 0
\]

where \(V\) is the interaction.

We can numerically calculate the structure interface. To reach our purpose, i.e., for simulation issues, we will consider \(|a| = 5\) and \(|R| = 4\). To calculate the force, we substitute the numerical solutions of the matter field \(\psi(r)\) (solutions shown in Figure 2) in Equation (35).

The numerical result of the magnitude of the interface generated by the kink-antikink-like interaction shown in Figure 4, is displayed in the Table 1 for various values of the \(p\)-parameter.

The plot of the interface density (35) is shown in Figure 5. Note that in the boundary region, where kink-antikink-like structures interact, the force density profile becomes “oscillating” with a sharp change in the amplitude of the interface density. Note that the interface density in the center of the structures (i.e., at \(r = 0\)) is null. In addition, note that the smaller the \(p\)-parameter, the smaller the contribution of the warp factor, and the smaller the interface intensity.

3.1. Structure Scattering Phenomenon

Allow us to study the scattering process of the kink-antikink solutions that our model support. In order, we investigate the equa-

\[
\text{Figure 4. Interaction of the Kink-antikink-like widely separated in fixed time.}
\]

Table 1. Numerical results of the interface between kink-antikink-like structures.

| \(p\)-parameter | Interface |
|-----------------|-----------|
| 1.00            | \(-0.1023 \cdot 10^6\) |
| 1.05            | \(-3.6197 \cdot 10^{-9}\) |
| 1.10            | \(-3.2558 \cdot 10^{-11}\) |

\(2\) We will study the scattering of this structure Section 3.1.
tions of motion considering the dynamic case. In this scenario, one finds two solutions again, i.e., kink-like and antikink-like configurations. Numerically, we assume that the kink structures are initially at $r = +a$. Meanwhile, the antikink-like solutions are at $r = -a$. Furthermore, one considers, initially, that these structures move in opposite directions with initial velocity $v_{in}$. Doing this procedure, we study the scattering of structures. We present the process result in Figure 6.

For simulation purposes, one adopts steps of $10^{-4}$ in the numerical computation. Analyzing the numerical results, we found the critical values for the initial velocity ($v_{in}$), i.e., $v_{in} \approx 0.965$. The $v_{in}$ separates the scattering process into two regimes. If $0.965 < v_{in} < 0.153$, the structures collide, annihilating each other and radiating their energy. Meanwhile, for $v_{in} > 0.153$, one notes a structure scattering (see Figure 7). The scattering process occurs for speed $0.153 < v_{in} < 0.965$. In this case, after the collision, one perceives that each structure carries an opposite topological charge, and the interface is attractive when the structures are far apart (see Table 1). However, when they are closer together, they repel, reflect, and move away, giving rise to the scattering of an elastic collision. Mathematically, this event class is given by

$$\psi_{[-1,+1]} \cup \psi_{[+1,-1]} \rightarrow \psi_{[+1,-1]} \cup \psi_{[-1,+1]}.$$  

(36)

We expose the illustration of this effect in Figures 7 and 8.

4. Phase Transitions of the Matter Field

A natural inquiry is: does the matter field assume other field configurations, such as multi-kink solutions? To answer this question, we will use arguments from information theory, namely, configurational entropy (CE). Indeed, we will use the discussions of a variant from the CE, known as differential configurational complexity (DCC). Let us start our analysis by pointing out that CE emerged from the theory proposed in 1948 by Claude E. Shannon. The communication theory proposed by Shannon tells us that

$$S = -\sum \rho_j \ln \rho_j$$  

(37)

where $\rho_j$ represents the probability $j$-th given the corpus of every possible message. Shannon’s information arises pursuing to quantify the maximum rate of information transmission between an addressee and an addressee. In other words, Shannon’s entropy gives us the best way for information to propagate in a medium. For more details, see refs. [52–54].

Based on Shannon’s concept, the CE concept arises. CE appears as an informational complexity measuring applied the localized field configurations. The message contents of the CE are components of the power spectrum. Thus, to accomplish our purpose, let us use the concept of a CE variant, i.e., the DCC, namely,

$$S = \int \rho(k) \ln \rho(k) \, dk$$  

(38)

As mentioned in ref. [54], DCC contemplates measuring the informational complexity of a localized field configuration. For a field configuration with energy $E(r)$, one describes the 2D wave modes decomposition by Fourier’s transform:

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int E(r) e^{ikr} \, dr$$  

(39)

A device sensitive to the full-wave spectrum will detect a wave mode within a volume $dk$ centered on $k$ with the probability proportional to the mode power. The 2D probability is

$$p(k|dk) \propto |\mathcal{F}(k)|^2 \, dk$$  

(40)

This probability allows us to write the modal fraction as

$$f(k) = \frac{|\mathcal{F}(k)|}{|\mathcal{F}(k)|}$$  

(41)

so that the DCC is

$$S_c = -\int f(k) \ln[f(k)] \, dk$$  

(42)
Figure 7. Scattering of the structures.

The integrand of Equation (42) is called entropic density.

It is necessary to investigate the energy density of the matter field to calculate the DCC. In this case, the energy density is

\[
\mathcal{E}(r) = \cosh^{-2p}(r) \left[ \cosh^{-2p}(r) \psi'(r) + \frac{1}{2} \left( r^2 - \psi(r)^2 \right)^2 \right] \quad (43)
\]

The matter field has the localized energy profile shown in Figure 9.

To achieve our aspiration of analyzing the phase transitions, let us assume the numerical solutions of the matter field (Figure 2).

With numerical result (43), we calculate the Fourier transform (39). Applying the modal fraction (41) and the energy density (43), we numerically calculate the entropic density (42) and the DCC (Table 2).

The numerical result of the entropic density of the DCC shows an absolute minimum at the origin from the power space, i.e., \( k = 0 \). This minimum occurs in the range of values \(|k| < 0.69\). By simulation, it is clear that this point describes an absolute minimum, which indicates the presence of a domain wall around \( r = 0 \). The existence of the domain wall suggests the presence of multiple walls (or at least two) very close together and suggests...
4. Double-Phase Transition

Furthermore, the curved geometry seems to influence the matter field phase transition producing double-kink-like and double-antikink-like solutions shown in Figures 2a and b. Our calculations of the modal fraction and DCC (Figure 10a and b) guarantee that the matter field should have a profile similar to the class of kinks solutions. Finally, the DCC indicates that the structures are more likely to occur when $p = 1$.

5. Conclusion

In this work, we study self-gravitating 2D solutions in dilaton gravity. To investigate the solution stability, we use some arguments by Zhong et al.\cite{8,49} in dilaton gauge ($\delta \phi = 0$). Furthermore, we use a similar approach to reference\cite{24} to investigate the zero-mode (translational) of the structure. We also calculate the interference between configurations with opposite topological charges and analyze the scattering phenomenon. The structure phase transitions were analyzed using DCC arguments.

Adopting a canonical matter field in dilaton gravity arises, naturally, the following question: What is the influence of dilaton gravity on the matter field? The answer to this question appears in the results found in the second section. Indeed, the 2D dilaton gravity with the metric (7) induces deformation in the matter field. In this way, the solutions that describe this field are double-kink-like profiles. This fascinating result is analogous to the results found in the brane-splitting process in cosmological scenarios.\cite{12,24,44} This similarity in the results leads us to believe that the deformation of the kink structures to double-kink is related to the warp factor profile chosen.

Considering the formation of a pair of structures (widely spaced) with opposite topological charges, we observe that the field configurations suffer the action of an interforce. This interforce is attractive, inducing the scattering process. The structure scattering is particularly interesting because the process will depend on the initial velocity. In our model, the scattering process has two possible results. For initial speed in the range $0.965 < v_{in} < 1.053$, the structures will collide and annihilate each other, radiating their energy. On the other hand, for initial velocity in the range $0.153 < v_{in} < 0.965$, we have an elastic scattering process.

The peculiar profile of the matter field appears numerically to resemble a double-kink configuration. Although, indications of the numerical solution and stability are not enough affirmations to guarantee that the topological solutions are true double-kinks. For this analysis, we study the energy profile of the matter field. Furthermore, the arguments from the configurational entropy (precisely, the DCC) help us confirm the previous result. The calculation of energy and DCC suggests that multiple phase transitions (or at least two) occur around the neighborhood of the origin. The domain walls are in a region around the range of values in the power spectrum, namely, $-0.69 < k < 0.69$. This result allows us to interpret that there is a double-phase transition. DCC data permit us to affirm that our structures are not true double-kink.

It is important to observe that when the $p$-parameter increases, the warp factor $e^{2A}$ becomes more significant. Also, while the $p$-parameter increases, the field configurations go from a more compact field configuration to a smoother profile. This result implies that the interforce is smaller for $p$ smaller.

The results obtained throughout this work open new doors for a better understanding of the aspects of the topological structures. That is because they will help us to understand the more complex models in high dimensions. Furthermore, this discus-
tion could be extended to models with 2D modified gravity.[56]

We hope to bring these discussions in some future works.

Acknowledgements

F. C. E. Lima (FCEL) expresses gratitude to the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) for the doctoral scholarship number 88887.372425/2019-00 received from August 2019 to July 2023. Furthermore, F. C. E. Lima acknowledges the Department of Physics from the Universidade Federal do Ceará (UFC) for hospitality. C. A. S. Almeida (CASA) is thankful to the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), grant no. 309553/2021-0, for financial support. Additionally, CASA would like to thank the Print-UFC CAPES program, project number 88887.837980/2023-0, for funding. C. A. S. Almeida acknowledges the Department of Physics and Astronomy at Tufts University for its warm hospitality. The authors thank the anonymous referee for their criticisms, comments, and suggestions.

Corrections added on October 4, 2023 after first online publication: The Author name and affiliations have been corrected.

Conflict of Interest

The authors declare no conflict of interest.

Keywords

2D Dilaton gravity, kink-like solutions, topological structures

Received: February 26, 2023
Revised: June 15, 2023

[1] K. A. Bronnikov, S. A. Kononogov, V. N. Melnikov, Gen. Relativ. Gravit. 2006, 38, 1215.
[2] L. Randall, R. Sundrum, Phys. Rev. Lett. 1999, 83, 3370.
[3] N. Arkani-Hamed, S. Dimopoulos, J. March-Russel, Phys. Rev. D 2001, 63, 064020.
[4] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, N. Kaloper, J. High Energ. Phys. 2000, 012, 010.
[5] J. W. Chen, M. A. Luty, E. Ponton, J. High Energ. Phys. 2000, 09, 012.
[6] J. Khoury, B. A. Ovrut, P. J. Steinhardt, N. Turok, Phys. Rev. D 2001, 64, 123522.
[7] S. Räsänen, Nucl. Phys. B 2002, 626, 183.
[8] Y. Zhong, F.-Y. Li, X.-D. Liu, Phys. Lett. B 2001, 522, 136716.
[9] M. Henneaux, Phys. Rev. Lett. 1985, 54, 959.
[10] S. P. de Alwis, Phys. Lett. B 1992, 289, 278.
[11] C. Vaz, L. Witten, Phys. Lett. B 1994, 325, 27.
[12] C. Vaz, L. Witten, Class. Quantum Gravity 1996, 13, L59.
[13] C. G. Callan Jr, S. B. Giddings, J. A. Harvey, A. Strominger, Phys. Rev. D 1992, 45, R1005.
[14] A. Bilal, J. Callan, G. Curtis, Nucl. Phys. B 1993, 394, 73.
[15] J. G. Russo, L. Susskind, L. Thorlacius, Phys. Lett. B 1992, 292, 13.
[16] D. Grumiller, W. Kummer, D. V. Vassilevich, Phys. Lett. B 2002, 369, 327.
[17] T. Ishii, S. Okumura, J.-i. Sakamoto, K. Yoshida, Nucl. Phys. B 2020, 951, 114901.
[18] T. Hartman, E. Shaghoulian, A. Strominger, J. High Energ. Phys. 2020, 07, 022.
[19] C. Teitelboim, Phys. Lett. B 1983, 126, 41.
[20] R. Jackiw, Nucl. Phys. B 1985, 252, 343.
[21] R. B. Mann, S. Morssink, A. Sikkema, T. Steele, Phys. Rev. D 1991, 43, 3948.
[22] A. M. Frassino, R. B. Mann, J. R. Moreira, Phys. Rev. D 2015, 92, 124069.
[23] R. Mann, J. Mureika, Phys. Lett. B 2011, 703, 167.
[24] E. Belendryasova, V. A. Gani, K. G. Zloshchastiev, Phys. Lett. B 2021, 823, 136776.
[25] R. A. C. Correa, R. da Rocha, Eur. Phys. J. C 2015, 75, 522.
[26] N. R. F. Braga, R. da Rocha, Phys. Lett. B 2017, 767, 386.
[27] M. Gleiser, N. Stamatopoulos, Phys. Lett. B 2012, 713, 304.
[28] M. Gleiser, N. Stamatopoulos, Phys. Rev. D 2012, 86, 045004.
[29] M. Gleiser, D. Sowinski, Phys. Lett. B 2013, 727, 272.
[30] M. Gleiser, N. Graham, Phys. Rev. D 2014, 89, 083502.
[31] N. S. Manton, K. Oles, T. Romaničzukiewicz, A. Wereszczyński, Phys. Rev. D 2021, 103, 025024.
[32] K. G. Zloshchastiev, Europhys. Lett. 2018, 122, 39001.
[33] K. G. Zloshchastiev, Acta Phys. Pol. B 2011, 42, 261.
[34] T. Sugiyama, Prog. Theor. Phys. 1979, 61, 1550.
[35] D. K. Campbell, J. F. Schonfeld, C. A. Wingate, Physica 1983, 90, 1.
[36] V. A. Gani, V. Lensky, M. A. Lizunova, J. High Energ. Phys. 2015, 08, 1.
[37] E. Belendryasova, V. A. Gani, Commun. Nonlinear Sci. Numer. Simul. 2019, 67, 414.
[38] N. S. Manton, J. Phys. A: Math. Theor. 2019, 52, 065401.
[39] K. G. Zloshchastiev, Mod. Phys. Lett. A 2000, 15, 67.
[40] K. G. Zloshchastiev, Phys. Lett. B 2001, 519, 111.
[41] T. Vachaspati, Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons, Cambridge University Press, Cambridge, UK 2006.
[42] V. A. Gani, A. A. Kirillov, S. G. Rubin, J. Cosmol. Astropart. Phys. 2018, 04, 042.
[43] A. R. P. Moreira, J. E. G. Silva, F. C. E. Lima, C. A. S. Almeida, Phys. Rev. D 2021, 103, 064046.
[44] A. R. P. Moreira, F. C. E. Lima, C. A. S. Almeida, Int. J. Mod. Phys. D 2022, 31, 2250080.
[45] J. Yang, Y.-L. Li, Y. Zhong, Y. Li, Phys. Rev. D 2012, 85, 084033.
[46] A. R. P. Moreira, F. C. E. Lima, J. E. G. Silva, C. A. S. Lima, Eur. Phys. J. C 2021, 81, 1081.
[47] D. Bazeia, C. B. Gomes, L. Losano, R. Menezes, Phys. Lett. B 2006, 633, 415.
[48] B. Stoetzel, Phys. Rev. D 1995, 52, 2192.
[49] Y. Zhong, J. High Energ. Phys. 2021, 04, 118.
[50] Y.-X. Liu, Y. Zhong, K. Yang, Europhys. Lett. 2010, 90, 51001.
[51] Y. Zhong, Y.-X. Liu, Z.-H. Zhao, Phys. Rev. D 2014, 89, 104034.
[52] M. Gleiser, M. Stephens, D. Sowinski, Phys Rev. D 2018, 97, 096007.
[53] C. E. Shannon, Bell Syst. Tech. J. 1948, 27, 379.
[54] F. C. E. Lima, C. A. S. Almeida, Eur. Phys. J. C 2021, 81, 1044.
[55] M. Gleiser, D. Sowinski, Phys. Rev. D 2018, 98, 056026.
[56] H.-J. Schmidt, Gen. Relativ. Gravit. 1999, 31, 1187.