FREE INFINITE DIVISIBILITY FOR ULTRASPERICALS

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Abstract. We prove that the integral powers of the semicircular distribution
are freely infinitely divisible. As a byproduct we get another proof of the free
infinite divisibility of the classical Gaussian distribution.

1. Introduction

The semicircular distribution, with density
\[ d\gamma_1(t) = \frac{1}{2\pi} \sqrt{4 - t^2} 1_{[-2,2]} dt \]
is a very important probability measure, appearing as the distributional limit of
the eigenvalues of large selfadjoint random matrices with independent entries. More
important to us, this measure plays a central role in free probability theory since
it arises from the free version of the central limit theorem, in particular, it is freely
infinitely divisible (f. i. d. for short).

In a different context, the semicircular distribution belongs to a family
of measures called ultraspherical (or hyperspherical) distributions, with density
\[ d\gamma_n(t) = c_n(4 - t^2)^n - \frac{1}{2} dt \]
Here, the normalizing constant \( c_n \) is simply the reciprocal of \( \int_{-2}^{2} (4 - t^2)^{n-1/2} dt \),
which, as a direct computation shows, equals \( 4^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n-2)^2 \cdot (2n)^2} \pi \).

The importance of this family comes, on one hand, from a geometrical
point of view in the following way. Let \( x = (x_1, x_2, \ldots, x_{2n+2}) \) be a random vector in \( \mathbb{R}^{2n+2} \) with spherical symmetry and let \( v = (v_1, \ldots, v_{2n+2}) \) be any
fixed unit vector in \( \mathbb{R}^{2n+2} \). If we denote by \( \mathbb{P} := \frac{x}{\|x\|} \) and by \( \lambda \) the dot product \( \lambda := \mathbb{P} \cdot v \), then \( 2\lambda \) has
a distribution which is ultraspherical \( \gamma_n \) \cite{14,16}. Moreover, properly normalized,
\( \gamma_n \) converges to the Normal (or classical Gaussian) distribution when \( n \) tends to
infinity, a result known as Poincaré’s theorem; \cite{22}. For this reason, as explained
by McKean \cite{18}, one can think of the Wiener measure (all whose marginals are
Gaussian) as the uniform distribution on an infinite dimensional sphere.

On the other hand, as observed in Arizmendi and Perez-Abreu \cite{9}, the
ultraspherical family contains all the Gaussian distributions with respect to the 5
fundamental independences in non-commutative probability, as classified by Muraki
\cite{20}. Specifically, the left-boundary case \( n = -1 \) is the symmetric Bernoulli distribution appearing in the central limit theorem with respect to Boolean convolution
(Speicher and Woroudi \cite{23}). Similarly, for \( n = 0 \) we obtain the arcsine distribution, which plays the same role in monotone and antimonotone convolutions as the Normal distribution does in classical probability (Muraki \cite{19}). As mentioned
before, the case \( n = 1 \), which is the semicircle distribution, and the right-boundary

\textit{Date:} May 2, 2014.

\textit{Work of S. Belinschi was supported by a Discovery Grant from NSERC.}

\textit{Octavio Arizmendi was supported by DFG-Deutsche Forschungsgemeinschaft Project SP419/8-1.}
case $n = \infty$, which is the Normal distribution, correspond to the free and classical central limits theorems, respectively.

From this observations, the free infinite divisibility of the ultraspherical distributions was considered in [3] as a mean to prove that the Normal distribution $\gamma_\infty$ is f. i. d. The authors proved using kurtosis arguments that $\gamma_n$ is not f. i. d. for $n < 1$ and conjectured that it is f. i. d. for all $n \in [1, +\infty)$. The free infinite divisibility of the Normal distribution was proved in [7]. However the conjecture for the ultraspherical distributions remained open.

In this paper we prove that $\gamma_n$ is f. i. d. for $n \in \mathbb{N}$. The method we employ in our proof is similar to the method used in [1, 7]. We will construct an inverse to the Cauchy transform $G_{\gamma_n}$ defined on the whole lower half-plane. Then the Voiculescu transform $\phi_{\gamma_n}(1/z) = G^{-1}_{\gamma_n}(z) - (1/z)$ has an extension to the whole complex upper half-plane $\mathbb{C}^+$.

The following theorem from [10] allows us to conclude:

**Theorem 1.** A Borel probability measure $\mu$ on the real line is $\boxplus$-infinitely divisible if and only if its Voiculescu transform $\phi_\mu(z)$ extends to an analytic function $\phi_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^-$. 

As a direct consequence, we get another proof of the free infinite divisibility of the Normal distribution.

The free infinite divisibility of other families including the semicircle and Normal distributions have been considered recently. A particularly important one are the so-called $q$-Gaussian distributions introduced by Bożejko and Speicher in [12, 11]. This family was proved to be f. i. d. for all $q \in [0, 1]$ in [1].

Finally, let us mention that, at this point, all the proofs of the free infinite divisibility of the Normal distribution are purely analytical and, even though some combinatorial considerations were noted in [7], a more conceptual explanation of this fact is still not known. However, connections between ultraspherical distributions with random matrices are known, see for example [2]. Also, ultraspherical distributions appear in connection to quantum groups [4, 5, 6]. Moreover, the free analogue of ultraspherical laws was found in [6]. So, we think that studying this family in more detail may lead to a better understanding of the free infinite divisibility of the Normal distribution.

2. **Properties of $\gamma_n$**

We shall first collect some data about our objects. The Cauchy (or Cauchy-Stieltjes) transform of a finite Borel measure $\mu$ on the real line is defined by

$$G_\mu(z) = \int_\mathbb{R} \frac{1}{z-t} d\mu(t), \quad z \notin \text{supp}(\mu),$$

where $\text{supp}(\mu)$ denotes the topological support of $\mu$. This function is analytic on its domain, and whenever $\mu$ is positive, it maps $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$. The Cauchy transform of the semicircular distribution is known to have a nice expression:

$$G_{\gamma_1}(z) = \frac{z - \sqrt{z^2 - 4}}{2}, \quad z \in \mathbb{C}^+, \quad (1)$$

where the square root branch is chosen so that $\lim_{z \rightarrow +\infty} zG_{\gamma_1}(z) = 1$. One can easily verify that
(1) \( G_{\gamma_1}(\mathbb{C}^+)^{-} = \mathbb{D} \cap \mathbb{C}^{-} \), and the correspondence is bijective. Moreover, 
(2) \( G_{\gamma_1}([-2, 2]) = \{ z \in \mathbb{C}^{-} : |z| = 1 \} \), 
(3) \( G_{\gamma_1}((\infty, -2]) = [-1, 0], G_{\gamma_1}((2, \infty)) = (0, 1) \).

It is remarkable that this function has an analytic extension through \( \mathbb{R} \setminus [-2, 2] \) which satisfies \( G_{\gamma_1}(\overline{z}) = \overline{G_{\gamma_1}(z)} \), and a different extension through \((-2, 2)\) which satisfies the more convenient condition that \( G_{\gamma_1}(z) = \frac{\overline{z} \pm \sqrt{z^2 - 4}}{2} \), where the square root is chosen with the same condition as before. From now on whenever we write \( G_{\gamma_1} \) we shall refer to this extension: \( G_{\gamma_1} : \mathbb{C} \setminus \{ (\infty, -2] \cup [2, \infty) \} \to \mathbb{C}^{-} \). It satisfies

(1) \( G_{\gamma_1}(\mathbb{C}^{-}) = \mathbb{D} \setminus \mathbb{D} \), 
(2) \( G_{\gamma_1}|_{\mathbb{C}^{-}}((\infty, -2]) = (-\infty, -1], G_{\gamma_1}|_{\mathbb{C}^{-}}((2, +\infty)) = [1, +\infty), \) and 
(3) \( G_{\gamma_1}([-2, 2]) = \{ z \in \mathbb{C}^{-} : |z| = 1 \} \).

(The reader should note that the restriction of \( G_{\gamma_1} \) to the upper half-plane has a different extension to the complement of \([-2, 2] \) than the restriction of \( G_{\gamma_1} \) to the lower half-plane. On the lower half-plane, \( G_{\gamma_1} \) behaves like the extension through \( \mathbb{R} \setminus [-2, 2] \) of the function \( \overline{G_{\gamma_1}|_{\mathbb{C}^{+}}(z)} \).

It has been shown in [15] Proposition 3.1 that for any \( \lambda > 0 \) there exists a positive constant \( d_{\lambda} \) so that

\[
(2) \quad \int_{-2}^{2} \frac{1}{(z-t)^{\lambda}}(4-t^2)^{\lambda - 1/2} dt = d_{\lambda} \left( \int_{-2}^{2} \frac{1}{2\pi(z-t)} \sqrt{4-t^2} \right)^\lambda, \quad z \in \mathbb{C}^{+}.
\]

This will allow us to establish some useful functional and differential equations.

Note that \( G^{(k)}_{\mu}(z) = k! \int \frac{(-1)^k}{(z-t)^{k+1}} \, d\mu(t), \ k \in \mathbb{N}, z \in \mathbb{C}^{+} \).

**Lemma 2.** For any \( z \in \mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty)\} \), we have

\[
G_{\gamma_n}(z) = Q_n(z^{2})G_{\gamma_{1}}(z) + zP_n(z^{2}),
\]

where

\[
Q_n(X) = 2\pi c_n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (-1)^j 4^{n-1-j} X^j = 2\pi c_n (4 - X)^{n-1},
\]

\[
P_n(X) = 2\pi c_n \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \frac{(-1)^j k 4^{n-j+k}}{(j+k-1)! (n-j-k)!} C_{j-1} X^{k-1},
\]

and \( c_n = \frac{n!}{2\pi 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)(2n-1)} \). In particular, the Cauchy transform of \( \gamma_n \) extends to an analytic function on \( \mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty)\} \),
Proof. The proof is a simple direct computation:

\[
G_{\gamma_n}(z) = 2\pi c_n \int_{-2}^{2} \frac{(4 - t^2)^{n-1}}{2\pi (z-t)} \sqrt{4 - t^2} \, dt
\]

\[
= 2\pi c_n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (-1)^j 4^{n-1-j} \int_{-2}^{2} \frac{t^{2j}}{2\pi (z-t)} \sqrt{4 - t^2} \, dt
\]

\[
= 2\pi c_n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (-1)^j 4^{n-1-j} \left[ z^{2j} G_{\gamma_1}(z) - \sum_{k=0}^{j-1} C_k z^{2(j-k)-1} \right]
\]

\[
= \left[ 2\pi c_n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (-1)^j 4^{n-1-j} z^{2j} \right] G_{\gamma_1}(z)
\]

\[
- 2\pi c_n \sum_{j=1}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (-1)^j 4^{n-1-j} \sum_{k=0}^{j-1} C_k z^{2(j-k)-1}
\]

\[
+ 2\pi c_n \sum_{k=1}^{n-1} \left( \sum_{j=1}^{n-k} \frac{(-1)^{j+k} 4^{n-j+k} (n-1)!}{(j+k-1)!(n-j-k)!} C_{j-1} \right) z^{2k-1}.
\]

\[
\square
\]

Lemma 3. For any \( z \in \mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty]\} \), we have

\[
(-1)^n G_{\gamma_n}^{(n-1)}(z) = \frac{d^n}{dz^n} G_{\gamma_1}(z)^n.
\]

Proof. The statement is a direct consequence of formula (2) and analytic continuation. Indeed, for \( \lambda = n \), the left hand side of (2) is, as noted above, the \((n-1)\)th derivative of \( G_{\gamma_n} \), up to a multiplicative constant depending only in \( n \). Since \( n \in \mathbb{N}, n > 0 \), it follows that

\[
(-1)^n n! \int_{-2}^{2} \frac{1}{(z-t)^n} (4 - t^2)^{n-1/2} \, dt = \frac{d^n}{dz^n} \left( \int_{-2}^{2} \frac{1}{z-t} (4 - t^2)^{n-1} \sqrt{4 - t^2} \, dt \right).
\]

On the other hand, for any \( k \in \mathbb{N} \), we have

\[
\frac{1}{2\pi} \int_{-2}^{2} \frac{t^{2k}}{z-t} \sqrt{4 - t^2} \, dt = \frac{1}{2\pi} \int_{-2}^{2} \frac{(t-z) \left( \sum_{j=0}^{2k-1} t^{2k-j-1} z^j \right)}{z-t} \sqrt{4 - t^2} \, dt
\]

\[+ z^{2k} \frac{1}{2\pi} \int_{-2}^{2} \frac{1}{z-t} \sqrt{4 - t^2} \, dt
\]

\[= z^{2k} G_{\gamma_1}(z) - \sum_{j=0}^{k-1} C_j z^{2(k-j)-1},
\]
where $C_j$ is the $j$th Catalan number, the $2j$th moment of the standard semicircular distribution $\gamma_1$. We shall record this equality for future reference:

$$
\frac{1}{2\pi} \int_{-2}^{2} z^2 \frac{1}{z-t} \sqrt{4-t^2} \, dt = z^{2k} G_{\gamma_1}(z) - \sum_{j=0}^{k-1} C_j z^{2(k-j)-1}, \quad z \notin (-\infty, -2] \cup [2, +\infty).
$$

Since $n \in \mathbb{N}$, $n > 0$, it follows that

$$
\frac{d^n}{dz^n} \left( \int_{-2}^{2} \frac{1}{z-t} (4 - t^2)^{n-1} \sqrt{4-t^2} \, dt \right) = \frac{d^n}{dz^n} \left( Q_n(z^2) G_{\gamma_1}(z) + z P_n(z^2) \right).
$$

Here $P_n$ and $Q_n$ are polynomials obtained in the previous lemma. Since the right hand term has a continuation from $\mathbb{C}^+$ to $\mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty)\}$, we have proved our statement.

It might be useful to see the Cauchy transforms of the first two ultrasphericals:

$$
G_{\gamma_2}(z) = 2\pi c_2 (-z^2 + 4) G_{\gamma_1}(z) + 2\pi c_2 z,
$$

and

$$
G_{\gamma_3}(z) = 2\pi c_3 (z^4 - 8z^2 + 16) G_{\gamma_1}(z) + 2\pi c_3 z (-z^2 + 7).
$$

We shall need also the following recurrence relation:

$$
G_{\gamma_{n+1}}(z) = \frac{n+1}{2(n+1)} (4-z^2) G_{\gamma_n}(z) + \frac{n+1}{2(n+1)} z.
$$

**Remark 4.** The above lemma has an interesting consequence for the behaviour of $G_{\gamma_n}$ along the imaginary axis. It follows trivially from the symmetry of $\gamma_n$ that $G_{\gamma_n}(iy) \subseteq i\mathbb{R}$. This implies that $G_{\gamma_n}'(iy) \in \mathbb{R}$ for all $y \in \mathbb{R}$. Moreover, as it will be shown below (independent from this remark), $G_{\gamma_n}'(iy) \neq 0$ whenever $G_{\gamma_n}(iy) \in \mathbb{C}^-$. Since this is true for all $y > 0$, it follows that $G_{\gamma_n}'(iy)$ has constant sign along the positive half of the imaginary axis. It can be found through direct computation that for large $y$ we have $G_{\gamma_n}'(iy) > 0$, so it follows that $G_{\gamma_n}'(iy)$ remains positive for all $y > 0$. We claim that in fact this must hold for all $y \in \mathbb{R}$. Indeed, otherwise it would be necessary that $G_{\gamma_n}'(iy_0) = 0$ for some $y_0 \leq 0$. However, since $\Re G_{\gamma_n}(iy) < 0$, this would contradict the lemma below. In particular, this allows us to conclude that $G_{\gamma_n}$ maps bijectively $i\mathbb{R}$ onto $i(-\infty, 0)$ (with $\lim_{y \to -\infty} G_{\gamma_n}(iy) = -i\infty$, $\lim_{y \to +\infty} G_{\gamma_n}(iy) = 0$).

3. Main Results

**Lemma 5.** If $G_{\gamma_n}(z) \in \mathbb{C}^-$, then $G_{\gamma_n}'(z) \neq 0$. Moreover, the extension of $G_{\gamma_n}|_{\mathbb{C}^+}$ to the real line is continuous, bounded and injective.

**Proof.** For $n = 1$, the statement is trivial. For $n = 2$, it follows directly from Lemma 4 that $G_{\gamma_2}'(z) \neq 0$ for all $z \in \mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty)\}$. For general $n$, we
note that
\[
G'_{\gamma_{n+1}}(z) = -c_{n+1} \int_{-2}^{2} \frac{(4-t^2)^{n+\frac{1}{2}}}{(z-t)^2} \, dt \\
= -c_{n+1} \frac{(4-t^2)^{n+\frac{1}{2}}}{z-t} + c_{n+1} \left( n + \frac{1}{2} \right) \int_{-2}^{2} \frac{-2t(4-t^2)^{n-\frac{1}{2}}}{z-t} \, dt \\
= \frac{c_{n+1}}{c_n} \frac{(2n+1)c_n}{c_n} \int_{-2}^{2} \frac{(z-t)(4-t^2)^{n-\frac{1}{2}}}{z-t} \, dt \\
= \frac{c_{n+1}}{c_n} (2n+1)[1-zG_{\gamma_n}(z)] \\
= \frac{n+1}{2} [1-zG_{\gamma_n}(z)], \quad z \in \mathbb{C}^+.
\]

Analytic continuation guarantees that this relation holds on the whole domain \( \mathbb{C} \setminus \{(-\infty,-2) \cup [2,\infty)\} \). Thus, the equality \( G'_{\gamma_{n+1}}(z) = 0 \) implies necessarily \( G_{\gamma_n}(z) = \frac{1}{z} \). If \( z \in \mathbb{C}^+ \cup [-2,2] \), this equality implies that \( \gamma_n = \delta_0 \), an obvious contradiction. If \( z \in \mathbb{C}^- \), then we must recall from (4) that

\[
z \left[ \frac{2(2n+1)}{n+1} G_{\gamma_{n+1}}(z) - z \right] \frac{1}{4-z^2} = 1.
\]

This is equivalent to

\[
\frac{2(2n+1)}{n+1} G_{\gamma_{n+1}}(z) = \frac{4}{z} \in \mathbb{C}^+.
\]

This contradicts the hypothesis of our lemma.

We shall now prove that the continuous extension of \( G_{\gamma_n} \) from the upper half-plane to the real line is bounded and continuous, and that \( G_{\gamma_n} |_{\mathbb{R} \cup \{\infty\}} \) is injective. Continuity of \( G_{\gamma_n} \) follows trivially from the continuity of \( G_{\gamma_1} \) and Lemma 2. Recall that \( G_{\gamma_1}(\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}) = \overline{D \cap C} \), so boundedness is obvious for \( n = 1 \). This together with Lemma 2 guarantees that for any fixed \( n \in \mathbb{N}, R > 0 \), the set \( G_{\gamma_n}((\mathbb{C} \cup \{\infty\}) \cap \{ z \in \mathbb{C} : |z| \le R \}) \) is bounded. On the other hand, for \( R > 0 \) large enough, we know that \( G_{\gamma_n}((\mathbb{C} \cup \{\infty\}) \setminus \{ z \in \mathbb{C} : |z| \le R \}) \) is a bounded neighbourhood of zero. This proves that \( G_{\gamma_n} \) is bounded on \( \mathbb{C}^+ \cup \mathbb{R} \) for all \( n \in \mathbb{N} \).

As seen above, \( G'_{\gamma_n}(z) \neq 0 \) for all \( z \in \mathbb{C}^+ \). The inequality \( G'_{\gamma_n}(x) < 0 \) for all \( x \in \mathbb{R} \setminus [-2,2] \) is a trivial consequence of the definition of the Cauchy-Stieltjes transform and the fact that \( \gamma_n \) is supported by \([-2,2] \). Equation (4) guarantees that \( G_{\gamma_n}(\pm 2) = \frac{n}{2n-2} \). Employing again Lemma 2 we obtain for \( x \in (-2,2) \) that \( \Re G_{\gamma_n}(x) = \pi c_n x (4-x^2)^{n-1} + P_n(x^2) \) and \( \Im G_{\gamma_n}(x) = -\pi c_n (4-x^2)^{n-\frac{1}{2}} \). The derivative of the imaginary part is, by direct computation, strictly positive on \((0,2)\), strictly negative on \((-2,0)\) and equal to zero in \(0,\pm 2\) (as real function in \(\pm 2\)). This already guarantees the injectivity of \( G_{\gamma_n} \) on the intervals \((-\infty,0] \) and \([0,\infty) \). We still need to show that \( G_{\gamma_n}([-2,0)) \cap G_{\gamma_n}((0,2]) = \emptyset \). The identity principle together with the analyticity of \( G_{\gamma_n} \) on \((-2,2)\) guarantee that, if this intersection is nonempty, then it can only be a discrete set. Recalling that \( \gamma_n \) is a symmetric measure, it follows that a point \( w \in G_{\gamma_n}([-2,0)) \cap G_{\gamma_n}((0,2]) \) must be of the form \( w = G_{\gamma_n}(r) = G_{\gamma_n}(-r) \) for some \( r \in (0,2) \). In particular, \( w \in -i(0,\infty) \). Using equation (4), we find that if \( \Re G_{\gamma_n}(r) = 0 \), then \( \Re [(4-r^2) G_{\gamma_{n-1}}(r + r)] = 0 \), so that \( \Re G_{\gamma_{n-1}}(r) = -\frac{4}{4-r^2} < 0 \). Since \( G_{\gamma_{n-1}}(2) = \frac{n-1}{2n-3} > 0 \), for \( G_{\gamma_{n-1}} \) we can find a new \( r \in (0,2) \) so that \( G_{\gamma_{n-1}}(r) \in i\mathbb{R} \). Repeating, we find that there exists
satisfies the condition $i$ is symmetric with respect to the imaginary axis and we must have $G_{\gamma_n}$ is injective on $\mathbb{R}$, as claimed in our lemma. □

A very important consequence of the above lemma is that $G_{\gamma_n}$ is injective on the whole upper half-plane (see [21]).

**Theorem 6.** For each $n \in \mathbb{N}, n > 0$, there exists a simply connected domain $D_n$ so that $\mathbb{C}^+ \cup (-2, 2) \subset D_n \subseteq \mathbb{C} \setminus \{(-\infty, -2] \cup [2, +\infty)\}$ and $G_{\gamma_n}: D_n \to \mathbb{C}^-$ is a conformal map. In particular, $\gamma_n$ is freely infinitely divisible for each $n \geq 1$.

Remark 4 gives an indication of how $D_n$ must look like: it must be symmetric with respect to the imaginary axis and we must have $i \mathbb{R} \subset D_n$ for all $n$. We shall make all this precise in our proof below.

**Proof.** We observe first that, according to Lemma 5, $G_{\gamma_n}$ is injective on $\mathbb{C}^+ \cup (-2, 2)$, and according to Remark 4 it is injective on the imaginary line. Moreover, the continuous extension of $G_{\gamma_n}|_{\mathbb{C}^+}$ to $\mathbb{R} \cup \{\infty\}$ is a simple closed curve which cuts the imaginary axis exactly twice, namely in $G_{\gamma_n}(0)$ and $0 = G_{\gamma_n}(+\infty)$. Thus, $G_{\gamma_n}$ is injective on a small enough complex neighbourhood of $\mathbb{C}^+ \cup (-2, 2) \cup i \mathbb{R}$.

Recall that the extension of $G_{\gamma_n}$ to the lower half-plane through $(-2, 2)$ satisfies the condition

$$\lim_{z \to \infty} \frac{G_{\gamma_n}(z)}{z} = 1.$$  

Using Lemma 2 we immediately observe a similar behaviour of $G_{\gamma_n}$. Indeed, $Q_n(z^2)$ has degree $2n - 2$ and $zP_n(z^2)$ has degree $2n - 3$. In particular, the asymptotics at infinity of $G_{\gamma_n}$ in the lower half-plane is of order $z^{2n-1}$. Considering the extension of $G_{\gamma_n}|_{\mathbb{C}^-}$ to the upper half-plane through the complement of $[-2, 2]$ we obtain a map which is $2n - 1$-to-$1$ on a neighbourhood of infinity and which preserves the real and imaginary lines close to infinity.

For each $t > 0$, define $\eta_t$ to be the open segment that unites $-it \in i(-\infty, 0)$ and $\frac{1}{2n - 1} + t \in (\frac{1}{2n - 1}, +\infty)$. It is clear that $\bigcup_{t > 0} \eta_t = \{z \in \mathbb{C}^-: \mathbb{R}z > 0\}$. For each fixed $t > 0$ we shall prove that there exists a unique simple path $a_t$ uniting $G_{\gamma_n}^{-1}(-it)$ to a point in $[2, +\infty)$ so that $G_{\gamma_n}(a_t) = \eta_t$.

The above remarks regarding the injectivity of $G_{\gamma_n}$ on a neighbourhood of the imaginary line guarantee that $G_{\gamma_n}^{-1}$ is well-defined on $i(-\infty, 0)$, and so $G_{\gamma_n}^{-1}(-it)$ is the unique choice for starting $a_t$. This choice imposes that $a_t$ moves into the right half of the complex plane (recall that $G_{\gamma_n}'$ is positive on the imaginary line). Let us now extend $a_t$ under the condition that $G_{\gamma_n}(a_t) \subseteq \eta_t$. We claim that we can extend $a_t$ uniquely all the way to having $G_{\gamma_n}(a_t) = \eta_t$. Moreover, $a_t$ will eventually enter the lower half-plane and approach $(2, +\infty)$ from the lower right quadrant. Indeed, assume towards contradiction that this is not the case. There are two possible obstacles to this extension: first obstacle is a critical point for $G_{\gamma_n}$ along $a_t$, and the second is the possibility that the extension of $a_t$ leaves $\{z \in \mathbb{C}: \mathbb{R}z > 0\} \setminus [2, +\infty)$. Let us discuss the first obstacle first: if there is a point $c_0$ to which $a_t$ has been extended by continuity so that $G_{\gamma_n}'(c_0) = 0$, then, as long as $a_t$ has not left $\{z \in \mathbb{C}: \mathbb{R}z > 0\} \setminus [2, +\infty)$, we have a direct contradiction to Lemma 5 since $G_{\gamma_n}(c_0) \in \mathbb{C}^-$ by construction. Thus, we need next to discard the case when $a_t$ leaves $\{z \in \mathbb{C}: \mathbb{R}z > 0\} \setminus [2, +\infty)$. It is obvious that it cannot leave it through $i\mathbb{R}$. Since both $G_{\gamma_n}|_{\mathbb{C}^+}$ and $G_{\gamma_n}|_{\mathbb{C}^-}$ map $\mathbb{R} \setminus [-2, 2]$ to $\mathbb{R}$, it is equally
obvious that $a_t$ cannot leave through $[2, +\infty)$, either from above (i.e. $\mathbb{C}^+$) or below (i.e. $\mathbb{C}^-$). Since $\lim_{z \to \infty} G_{\gamma_n}(z) = \infty$ when the limit is taken with values of $z$ in the lower half-plane, and $\lim_{z \to \infty} G_{\gamma_n}(z) = 0$ when the limit is taken with values of $z$ in the upper half-plane, $a_t$ cannot leave $\{z \in \mathbb{C} : \Re z > 0\} \setminus [2, +\infty)$ through infinity either. So $a_t$ cannot leave this set at all. We conclude that $a_t$ can indeed be extended so that $G_{\gamma_n} : a_t \mapsto \eta_t$ bijectively.

Recalling that there is a simply connected neighbourhood $V$ of $\mathbb{C}^+ \cup (-2, 2) \cup i\mathbb{R}$ on which $G_{\gamma_n}$ is injective, we find the simply connected open set $G_{\gamma_n}(V) \subset \mathbb{C}^-$, which contains $i(-\infty, 0)$, on which we can uniquely define an analytic map $G_{\gamma_n}^{-1}$ which satisfies $G_{\gamma_n} \circ G_{\gamma_n}^{-1} = \text{Id}_{G_{\gamma_n}(V)}$ and $G_{\gamma_n}^{-1} \circ G_{\gamma_n} = \text{Id}_V$. Symmetry of $\gamma_n$ implies that, first, we may assume $V$ to be symmetric with respect to $i\mathbb{R}$ and, second, that $G_{\gamma_n}^{-1}$ must also satisfy $G_{\gamma_n}^{-1}(u + iv) = G_{\gamma_n}^{-1}(-u + iv)$. From Lemma 2 it follows that $z = Q_n(G_{\gamma_n}^{-1}(z))^2 G_{\gamma_n}(G_{\gamma_n}^{-1}(z)) + G_{\gamma_n}^{-1}(z) P_n(G_{\gamma_n}^{-1}(z))$; since $P_n, Q_n$ are polynomials and $G_{\gamma_n}$ is defined on $\mathbb{C} \setminus \{(-\infty, -2) \cup [2, +\infty)\}$, this relation holds for all $z \in G_{\gamma_n}(V)$. This equation will allow us to extend $G_{\gamma_n}^{-1}$ to all of $\mathbb{C}^-$. Indeed, for any point $w \in \mathbb{C}^-$ there exists a unique path $\eta_t$ so that $w \in \eta_t$. We claim that $G_{\gamma_n}^{-1}$ extends uniquely along $\eta_t$: since we have shown the existence and uniqueness of a path $a_t \subset \{z \in \mathbb{C} : \Re z > 0\} \setminus [2, +\infty)$ so that $G_{\gamma_n}$ maps $a_t$ bijectively onto $\eta_t$, it is clear that we can define $G_{\gamma_n}^{-1}$ on $\eta_t$ and with values in $a_t$. In addition, we recall that, $\eta_t$ being in the lower half-plane and $a_t$ in $\{z \in \mathbb{C} : \Re z > 0\} \setminus [2, +\infty)$, Lemma 3 guarantees that $G_{\gamma_n}(w) = Q_n(w^2) G_{\gamma_n}(w) + w P_n(w^2)$ has a nonzero derivative at each $w \in a_t$, and thus, by the analytic inverse function theorem, $G_{\gamma_n}^{-1}$ can be defined as an analytic map on a neighbourhood of $\eta_t$. Thus, since $\bigcup_{t>0} \eta_t = \{z \in \mathbb{C}^- : \Re z > 0\}$, we have extended our map $G_{\gamma_n}^{-1}$ to $V \cup \{z \in \mathbb{C}^- : \Re z > 0\}$. We extend it to $\mathbb{C}^-$ by the formula $G_{\gamma_n}^{-1}(u + iv) = G_{\gamma_n}^{-1}(-u + iv)$. This completes our proof, with $D_n = \bigcup_{t>0} a_t \cup i\mathbb{R} \cup \bigcup_{t>0} (-a_t)$.

$\square$

Remark 7 (free divisibility indicator). In terms of the free divisibility indicator $\phi(\mu)$ defined in [8], the last theorem says that $\phi(\gamma_n) \geq 1$, for $n \in \mathbb{N}$. Using results in [3] it is easily shown that $\phi(\gamma_n) \leq \frac{2n+1}{n+2}$.

The free infinite divisibility of the Gaussian distribution then follows.

Corollary 8. The classical Gaussian distribution $d\gamma_{\infty}(t) := (2\pi)^{-1} e^{-t^2/2} dt$ is freely infinitely divisible.

Proof. The class of f. i. d. ’s is closed with respect to the weak convergence. Since, by Poincaré’s theorem [22], the distributions $\gamma_n$, properly normalized, converge weakly to a Gaussian distribution, we get the result by Theorem 6. $\square$

Finally, from our main result we see that some beta distributions are freely infinitely divisible. Recall that a beta distribution is given by its density function

$$d_{\text{Beta}(\alpha, \beta)}(t) = \frac{1}{B(\alpha, \beta)} (t)^{\alpha-1} (1-t)^{\beta-1} 1_{[0,1]} dt.$$ 

Corollary 9. For $n \in \mathbb{N}$ the following families are freely infinitely divisible:

1) The beta distributions $\text{Beta}(n + 1/2, n + 1/2)$.

2) The beta distributions $\text{Beta}(1/2, n + 1/2)$ and $\text{Beta}(n + 1/2, 1/2)$.
Proof. 1) If $X \sim \gamma_n$ then $1/4(1 - X)$ has a distribution $\text{Beta}(n + 1/2, n + 1/2)$. Since free infinite divisibility is preserved under affine transformations we see that $\text{Beta}(n + 1/2, n + 1/2)$ is f. i. d.

2) It was proved recently in [2] that the square of any symmetric f. i. d. is also f. i. d. Now if $X \sim \gamma_n$ then $Y = 1/4X^2$ has a distribution $\text{Beta}(1/2, n + 1/2)$ which then is f. i. d. Finally, for the same reason as in (1), the distribution of $1 - Y$ shall be f. i. d. which is a $\text{Beta}(n + 1/2, 1/2)$.

□

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