Enumeration for the total number of all spanning forests of complete tripartite graph based on the combinatorial decomposition

Sung Sik U

Faculty of Mathematics, Kim Il Sung University, D.P.R Korea
e-mail address : usungsik@yahoo.com

Abstract

This paper discusses the enumeration for the total number of all rooted spanning forests of the labeled complete tripartite graph. We enumerate the total number by a combinatorial decomposition.

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1 Introduction

Y. Jin and C. Liu have enumerated the number of spanning forests of the labeled complete bipartite graph $K_{m,n}$ on $m$ and $n$ vertices by combinatorial method and by using the exponential generating function respectively ([2],[3]). And D. Stark [5] has found the asymptotic number of labeled spanning forests of the complete bipartite graph $K_{m,n}$ as $m \to \infty$ when $m \leq n$ and $n = o(m^{6/5})$. L. A. Szekely [6] gave a simple proof to the formula in [2] and a generalization for complete multipartite graphs. In [1], [4] a bijective proof of the enumeration of spanning trees of the complete tripartite graphs and the complete multipartite graphs has been given respectively.

Let $H_m, H_n, H_p$ denote the three disjoined vertex sets of the complete tripartite graph $K_{m,n,p}$, that is, $K_{m,n,p} = (H_m, H_n, H_p)$. Let $V(K_{m,n,p}) = H_m \cup H_n \cup H_p$ denote the vertex set of $K_{m,n,p}$. The out-degree of a vertex $z$ will be denoted by $d^+(z)$, while the in-degree of $z$ will be denoted by $d^-(z)$. Let $V(G)$ denote the vertex set of graph $G$. The goal of this paper is to give a closed formula of the enumeration for the total number of all spanning trees, forests of the labeled complete tripartite graph $K_{m,n,p}$ by the combinatorial method. Throughout this paper, we will consider only the labeled graphs.
2 Counting the number of spanning trees and forests of a labeled complete tripartite graph $K_{m,n,p}$

Let $T(m, n, p)$ denote the set of all labeled spanning trees of the complete tripartite graph $K_{m,n,p}$. Each tree $T$ in $K_{m,n,p}$ gives rise to labeled directed spanning tree $T'$ with $z$ as a root, and all edges are directed to towards $z$. Let $D(m, 0; n, 0; p, \{|z_1\}|)$ denote the set of all such directed trees with $z_1 \subset H_p$ as a root. Clearly, $|T_{m,n,p}| = |D(m, 0; n, 0; p, \{|z_1\}|)|.

For any $T \in D(m, 0; n, 0; p, \{|z_1\}|)$, $d^+(z_1) = 0, d^+(z) = 1, z \in V(K_{m,n,p})\{z_1\}$.

It is well known [1] that the number $f(m, l; n, k)$ of labeled spanning forests of $K_{m,n} = (H_m, H_n)$, where in the forest every tree is rooted, there are $l$ roots in $H_m$, $k$ roots in $H_n$, and the tree in the forest are not ordered, is equal to

$$f(m, l; n, k) = \frac{m^l}{l!} \frac{n^k}{k!} (m+n-l-k)^{m+n-l-k-1}.$$

Our proof is based on the following combinatorial decomposition. Given a rooted spanning tree of the complete tripartite graph $K_{m,n,p}$ where the root is in $H_p$, we remove the root vertex from the tree to obtain a spanning forest of the another tripartite graph. The roots of trees in this forest are in $H_m$ or $H_n$.

**Theorem 2.1.** The number $|T_{m,n,p}|$ of labeled spanning trees of the complete tripartite graph $K_{m,n,p}$ is as follows:

$$|T_{m,n,p}| = (m+n)^{p-1}(m+p)^{n-1}(n+p)^{m-1}(m+n+p).$$

**Proof.** We observe that a directed subgraph of $K_{m,n,p}$ belongs to $D(m, 0; n, 0; p, \{|z_1\}|)$ if and only if, in the subgraph,

$$d^+(z_1) = 0, d^+(z) = 1, z \in V(K_{m,n,p})\{z_1\}$$

and the subgraph is (weakly)connected. Let $D(m, l; n, k)$ denote the set of all spanning forests of complete bipartite graph $K_{m,n}$, with $l$ roots in $H_m$ and $k$ roots in $H_n$, that is,

$$f(m, l; n, k) = |D(m, l; n, k)|.$$

Let $F$ belongs to $D(m, l; n, k)$. From $F$, we will construct the rooted spanning forests of $K_{m,n,p}$ with root $z_1 \in H_p$ as follows. First, link an edge $(z, v)$ between every $z \in H_p\{z_1\}$ and some $v \in V(F)$ (where $V(F)$ denotes the vertex set of graph $F$).
There are \((m+n)^{p-1}\) ways. Notice that the obtained graph \(G\) has \(k+l\) (weakly) connected components each of which has a unique vertex in \(H_m \cup H_n\) of out-degree zero. Now, for any fixed integer \(t\), let \(G'\) denote a graph obtained by adding \(t\) edges consecutively to \(G\) as follows.

At each step we add an edge of the form \((a, b)\) where \(b\) is any vertex of \(H_p \setminus \{z_1\}\) and \(a \in H_m \cup H_n\) is a vertex of out-degree zero in any component not containing \(b\) in the graph already constructed.

The number of components decreases by one each time such an edge is added.

Since \(|H_p \setminus \{z_1\}| = p - 1\) and the number of components not containing \(b\) in the graph \(G\) already constructed is \(l + k - 1\), there are \((l-1)(l+k-1)\) choices for the first such edge. Similarly, there are \((l-1)(l+k-2)\) choices for the second edge and in general \((l-1)(l+k-t)\) choices for the \(t\)th edge, where, \(0 \leq t \leq l + k - 1\), because the number of components in the graph \(G\) is \(l + k\). The graph \(G'\) constructed like this has \(l + k - t\) components each of which has a unique vertex in \(H_m \cup H_n\) of out-degree zero and the remaining vertices all have out-degree; if we add edges from these vertices of out-degree zero to \(z_1\), we obtain a tree \(T'\) in \(D(m, 0; n, 0; p, \{\{z_1\}\})\) that contains \(G\) and in which \(d^-(z_1) = l + k - t\). The order in which the \(t\) edges are added to \(G\) to form \(G'\) is immaterial, so it follows that there are

\[
\frac{(p-1)(l+k-1)(p-1)(l+k-2) \cdots (p-1)(l+k-t)}{t!} = \binom{l+k-1}{t}(p-1)^t
\]

rooted spanning trees \(T'\) for fixed integer \(t\).

This implies that there are

\[
\sum_{t=0}^{l+k-1} \binom{l+k-1}{t}(p-1)^t = p^{l+k-1}
\]

spanning trees \(T\) in \(D(m, 0; n, 0; p, \{\{z_1\}\})\) that contain \(G\).

Hence

\[
|T_{m,n,p}| = |D(m, 0; n, 0; p, \{\{z_1\}\})| = \sum_{l=0}^{m} \sum_{k=0}^{n} f(m, l; n, k)(m+n)^{p-1}p^{l+k-1}
\]

\[
= \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{n}{k} m^{l-1} n^{k-1} (km + nl - kl)(m+n)^{p-1} p^{l+k-1}
\]

\[
= (m+n)^{p-1}(m+p)^{n-1}(n+p)^{m-1}(m+n+p).
\]

Therefore, we get the required result.

\[\square\]

**Corollary 2.2.** The number \(f(m, 0; n, 0; p, 1)\) of the labeled spanning trees of \(K_{m,n,p}\) with a root in \(H_p\) as follows:

\[
f(m, 0; n, 0; p, 1) = p(m+n)^{p-1}(m+p)^{n-1}(n+p)^{m-1}(m+n+p).
\]
Let $D(m, 0; n, 0; p, |\{z_{i_1}, z_{i_2}, \ldots, z_{i_r}\}|)$ be the set of the spanning forests of $K_{m,n,p}$ with roots $z_{i_1}, z_{i_2}, \ldots, z_{i_r}$ in $H_p$.

**Theorem 2.3.** The number $f(m, 0; n, 0; p, r)$ of the labeled spanning forests of the complete tripartite graph $K_{m,n,p}$ with $r$ roots in $H_p$ is as follows:

$$f(m, 0; n, 0; p, r) = \binom{p}{r} r(m + n)^{p-r}(m + p)^{n-1}(m + n + p).$$

**Proof.** Let $z_{i_1}, z_{i_2}, \cdots, z_{i_r}$ in $H_p$ be vertices given as roots, $Z' = H_p \setminus \{z_{i_1}, z_{i_2}, \cdots, z_{i_r}\}$ and $F$ belongs to $D(m, l; n, k)$. There are $\binom{p}{r}$ ways to choose the $r$ root in $H_p$. As in theorem 2.1, link an edge $(z, v)$ between every $z \in Z'$ and some $v \in V(F)$. There are $(m + n)^{p-r}$ ways.

Notice that the obtained graph $G$ has $k + l$ (weakly)connected components each of which has a unique vertex in $H_m \bigcup H_n$ of out-degree zero. As in the proof of theorem 2.1, for any fixed integer $t$ such that $0 \leq t \leq l + k - 1$, link an edge $(v, z)$ between any $z \in Z'$ and a vertex $v \in H_m \bigcup H_n$ of out-degree zero in any component not containing $z$ in the graph already constructed, we repeat this procedure $t$ times. There are

$$\frac{(p - 1)(l + k - 1)(p - 1)(l + k - 2) \cdots (p - 1)(l + k - t)}{t!} = \binom{l + k - 1}{t} (p - 1)^t$$

rooted spanning forests $F'$. The every forests $F'$ thus obtained has $l + k - t$ (weakly)connected components each of which has a unique vertex in $H_m \bigcup H_n$ of out-degree zero. The number of the ways linking edges from $l + k - t$ vertices of out-degree zero in these components to $r$ vertices $z_{i_1}, z_{i_2}, \ldots, z_{i_r}$ in $H_p \setminus Z'$ is equals to $r^{l+k-t}$.

The number of the spanning forests with $r$ roots $z_{i_1}, z_{i_2}, \cdots, z_{i_r}$ in $H_p$ of $K_{m,n,p}$ obtained from $F$ is as follows:

$$\sum_{t=0}^{l+k-1} \binom{l + k - 1}{t} (p - 1)^t r^{l+k-t} = p^{l+k-1} r.$$

Hence

$$|D(m, 0; n, 0; p, |\{z_{i_1}, z_{i_2}, \cdots, z_{i_r}\}|)| = \sum_{l=0}^{m} \sum_{k=0}^{n} f(m, l; n, k)(m + n)^{p-r} p^{l+k-1} r$$

$$= \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{n}{k} n^{m-l-1} m^{n-k-1} (km + nl - kl)(m + n)^{p-r} p^{l+k-1} r$$

$$= (m + n)^{1-r} r |D(m, 0; n, 0; p, |\{z_{i_1}\}|)|$$

$$= r(m + n)^{p-r} (m + p)^{n-1}(n + p)^{m-1}(m + n + p).$$
Therefore,

\[ f(m, 0; n, 0; p, r) = \binom{p}{r} |D(m, 0; n, 0; p, \{z_{i_1}, z_{i_2}, \ldots, z_{i_r}\})| \]
\[ = \binom{p}{r} r(m + n)^{p-r}(m + p)^{n-1}(n + p)^{m-1}(m + n + p). \]

\[ \square \]

3 Counting the total number of all spanning forests of \( K_{m,n,p} \)

**Theorem 3.1.** The total number \( S(m, n, p) \) of all spanning forests of \( K_{m,n,p} \) is as follows:

\[ S(m, n, p) = (m + n + 1)^{p-1}(m + p + 1)^{n-1}(n + p + 1)^{m-1}(m + n + p + 1)^2. \]

**Proof.** Let \( B(p, r) \) denote the set of spanning forests of the complete tripartite graph \( K_{m,n,p} \) in which \( r \) roots are in \( H_p \) and remain roots in \( H_m \) or \( H_n \). Let \( F \) belongs to \( D(n, l; n, k) \). From \( F \), we will construct the rooted spanning forests of \( K_{m,n,p} \) with \( r \) roots in \( H_p \) as follows. Let \( z_{i_1}, z_{i_2}, \ldots, z_{i_r} \in H_p \) be vertices given as roots. The number of ways which select \( r \) roots in \( H_n \) is equal to \( \binom{p}{r} \). Let \( Z' = H_p \setminus \{z_{i_1}, z_{i_2}, \ldots, z_{i_r}\} \).

First, link an edge \((z,v)\) between every \( z \in Z' \) and some \( v \in V(F) \) (i.e., vertex \( v \) of forest \( F \)). There are \((m + p)^{p-r}\) ways. Notice that the obtained graph \( G \) has \( k + l + r \) (including components consisting of \( z_{i_1}, z_{i_2}, \ldots, z_{i_r} \in H_p \)) weakly connected components.

Let \( t \) denote any fixed integer such that \( 0 \leq t \leq l + k - 1 \), \( H \) denote a graph obtained by adding \( t \) edges consecutively to \( G \) as follows. At each step we add an edge of the form \((a,b)\) where \( b \) is any vertex of \( Z' \) and \( a \in H_m \cup H_n \) is a root of any component not containing \( b \) in the graph already constructed. The number of components decreases by one each time an edge is added. Since \(|Z'| = p - r\) and the number of components not containing \( b \) in the graph \( G \) already constructed is \( l + k - 1 \), there are \((p-r)(l+k-1)\) choices for the first such edge, \((p-r)(l+k-2)\) choices for the second edge, \( \ldots \), and \((p-r)(l+k-t)\) choices for the \( t \)th edge. The order in which the \( t \) edges are added to \( G \) to form \( H \) is immaterial, so it follows that there are

\[ \frac{(p-r)(l+r-1)(p-r)(l+r-2)\cdots(p-r)(l+r-t)}{t!} = \binom{l+r-1}{t} \binom{p-r}{t} \]

ways.

The graph \( H \) constructed like this has \( l + k - t \) components(with the exception of components consisting of \( H_p \setminus Z' \)) each of which has a unique vertex in \( H_m \cup H_n \).
of out-degree zero and the remaining vertices all have out-degree one; if we add edges from some vertices of these vertices of out-degree zero to \( z_i, z_{i2}, \ldots, z_{ik} \), we obtain a forest in \( B(p, r) \) that contains \( G \). There are \((r + 1)^{l+k-t}\) ways. Therefore, this implies that there are

\[
\sum_{l=0}^{l+k-1} \binom{l + k - 1}{t} (p - r)^l (r + 1)^{l+k-t} = (r + 1)(p + 1)^{l+k-1}
\]

forests in \( B(n, r) \) that contain \( G \).

Hence,

\[
S(m, n, p) = \sum_{l=0}^{m} \sum_{k=0}^{n} \sum_{r=0}^{p} f(m, l; n, k)(m + n)^{p-r}(r + 1)(p + 1)^{l+k-1}
\]

\[= \sum_{l=0}^{m} \sum_{k=0}^{n} \sum_{r=0}^{p} \binom{m}{l} \binom{n}{k} \binom{p}{r} (m + n)^{m-l-1} n^{n-k-1} (km + ln - lk)(m + n)^{p-r}(r + 1)(p + 1)^{l+k-1}
\]

\[= (m + n + 1)^{p-1}(m + p + 1)^{n-1}(n + p + 1)^{m-1}(m + n + p + 1)^2.
\]

Therefore, we get the required result.

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