INEQUALITIES FOR PRODUCTS OF POLYNOMIALS I

I. E. PRITSKER AND S. RUSCHEWEYH

Abstract. We study inequalities connecting the product of uniform norms of polynomials with the norm of their product. This circle of problems include the Gelfond-Mahler inequality for the unit disk and the Kneser-Borwein inequality for the segment $[-1,1]$. Furthermore, the asymptotically sharp constants are known for such inequalities over arbitrary compact sets in the complex plane. It is shown here that this best constant is smallest (namely: 2) for a disk. We also conjecture that it takes its largest value for a segment, among all compact connected sets in the plane.

1. The problem and its history

Let $E$ be a compact set in the complex plane $\mathbb{C}$. For a function $f : E \to \mathbb{C}$ define the uniform (sup) norm as follows:

$$\|f\|_E = \sup_{z \in E} |f(z)|.$$  

Clearly $\|f_1 f_2\|_E \leq \|f_1\|_E \|f_2\|_E$, but this inequality is not reversible, in general, not even with a constant factor in front of the right hand side. Indeed, $\|f_1\|_E \|f_2\|_E \leq C \|f_1 f_2\|_E$ does not hold for functions with disjoint supports in $E$, for example. However, the situation is quite different for algebraic polynomials $\{p_k(z)\}_{k=1}^m$ and their product $p(z) := \prod_{k=1}^m p_k(z)$. Polynomial inequalities of the form

$$\prod_{k=1}^m \|p_k\|_E \leq C \|p\|_E,$$  

(1.1)

exist and are readily available. One of the first results in this direction is due to Kneser [19], for $E = [-1,1]$ and $m = 2$ (see also Aumann [1]), who proved that

$$\|p_1\|_{[-1,1]} \|p_2\|_{[-1,1]} \leq K_{\ell,n} \|p_1p_2\|_{[-1,1]}, \quad \deg p_1 = \ell, \quad \deg p_2 = n - \ell, \quad$$  

(1.2)

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where

\[ K_{\ell,n} := 2^{n-1} \prod_{k=1}^{\ell} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \prod_{k=1}^{n-\ell} \left( 1 + \cos \frac{2k-1}{2n} \pi \right). \]

Note that equality holds in (1.2) for the Chebyshev polynomial
\[ t(z) = \cos n \arccos z = p_1(z)p_2(z), \]
with a proper choice of the factors \( p_1(z) \) and \( p_2(z) \). P. B. Borwein [7] generalized this to the multifactor inequality

\[ \prod_{k=1}^{m} \|p_k\|_{[-1,1]} \leq 2^{n-1} \prod_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)^2 \|p\|_{[-1,1]}. \]

He also showed that

\[ 2^{n-1} \prod_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)^2 \sim (3.20991 \ldots)^n \text{ as } n \to \infty. \]

A different version of inequality (1.1) for \( E = D \), where \( D := \{ w : |w| \leq 1 \} \) is the closed unit disk, was considered by Gelfond [15, p. 135] in connection with the theory of transcendental numbers:

\[ \prod_{k=1}^{m} \|p_k\|_D \leq e^n \|p\|_D. \]

The latter inequality was improved by Mahler [23], who replaced \( e \) by 2:

\[ \prod_{k=1}^{m} \|p_k\|_D \leq 2^n \|p\|_D. \]

It is easy to see that the base 2 cannot be decreased, if \( m = n \) and \( n \to \infty \). However, (1.7) has recently been further improved in two directions. D. W. Boyd [9] [10] showed that, given the number of factors \( m \) in (1.7), one has

\[ \prod_{k=1}^{m} \|p_k\|_D \leq (C_m)^n \|p\|_D, \]

where

\[ C_m := \exp \left( \frac{m}{\pi} \int_0^{\pi/m} \log \left( 2 \cos \frac{t}{2} \right) dt \right) \]

is asymptotically best possible for each fixed \( m \), as \( n \to \infty \). Kroó and Pritsker [20] showed that, for any \( m \leq n \),

\[ \prod_{k=1}^{m} \|p_k\|_D \leq 2^{n-1} \|p\|_D, \]

where equality holds in (1.10) for each \( n \in \mathbb{N} \), with \( m = n \) and \( p(z) = z^n - 1 \).

Inequalities (1.2)-(1.10) clearly indicate that the constant \( C \) in (1.1) grows exponentially fast with \( n \), with the base for the exponential depending on the
set $E$. A natural general problem arising here is to find the smallest constant $M_E > 0$, such that

$$
(1.11) \quad \prod_{k=1}^{m} \|p_k\|_E \leq M_E^n \|p\|_E
$$

for arbitrary algebraic polynomials $\{p_k(z)\}_{k=1}^{m}$ with complex coefficients, where $p(z) = \prod_{k=1}^{m} p_k(z)$ and $n = \deg p$. The solution of this problem is based on the logarithmic potential theory (cf. [36] and [35]). Let $\text{cap}(E)$ be the logarithmic capacity of a compact set $E \subset \mathbb{C}$. For $E$ with $\text{cap}(E) > 0$, denote the equilibrium measure of $E$ by $\mu_E$. We remark that $\mu_E$ is a positive unit Borel measure supported on $\partial E$ (see [36, p. 55]). Define

$$
(1.12) \quad d_E(z) := \max_{t \in E} |z - t|, \quad z \in \mathbb{C},
$$

which is clearly a positive and continuous function in $\mathbb{C}$. It is easy to see that the logarithm of this distance function is subharmonic in $\mathbb{C}$. Furthermore, it has the following integral representation

$$
\log d_E(z) = \int \log |z - t|d\sigma_E(t), \quad z \in \mathbb{C},
$$

where $\sigma_E$ is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, see Lemma 5.1 of [31] and [22]. For further in-depth analysis of the representing measure $\sigma_E$, we refer to the recent paper of Gardiner and Netuka [14]. This integral representation is the key fact used by the first author to prove the following result [31].

**Theorem 1.1.** Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$. Then the best constant $M_E$ in (1.11) is given by

$$
(1.13) \quad M_E = \exp \left( \frac{\int \log d_E(z)d\mu_E(z)}{\text{cap}(E)} \right).
$$

Theorem 1.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [36, p. 56]). In particular, if $E$ is a continuum, i.e., a connected set, then we obtain a simple universal bound for $M_E$ [31]:

**Corollary 1.2.** Let $E \subset \mathbb{C}$ be a bounded continuum (not a single point). Then we have

$$
(1.14) \quad M_E \leq \frac{\text{diam}(E)}{\text{cap}(E)} \leq 4,
$$

where $\text{diam}(E)$ is the Euclidean diameter of the set $E$.

On the other hand, for non-connected sets $E$ the constants $M_E$ can be arbitrarily large. For example, consider $E_k = [-\sqrt{k} + 4, -\sqrt{k}] \cup [\sqrt{k}, \sqrt{k} + 4]$,
so that \( \text{cap}(E_k) = 1 \) \[35\] and

\[
M_E = \exp \left( \int \log d_{E_k}(z) \, d\mu_{E_k}(z) \right) \geq e^{\log(2\sqrt{\pi})} \to \infty \quad \text{as} \quad k \to \infty.
\]

For the closed unit disk \( D \), we have that \( \text{cap}(D) = 1 \) \[36, \text{p. 84}\] and that

(1.15) \[d\mu_D = \frac{d\theta}{2\pi}, \]

where \( d\theta \) is the arclength on \( \partial D \). Thus Theorem 1.1 yields

(1.16) \[M_D = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log d_D(e^{i\theta}) \, d\theta \right) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log 2 \, d\theta \right) = 2,
\]

so that we immediately obtain Mahler’s inequality (1.7).

If \( E = [-1, 1] \) then \( \text{cap}([-1, 1]) = 1/2 \) and

(1.17) \[d\mu_{[-1,1]} = \frac{dx}{\pi \sqrt{1-x^2}}, \quad x \in [-1, 1],
\]

which is the Chebyshev (or arcsin) distribution (see \[36, \text{p. 84}\]). Using Theorem 1.1, we obtain

\[
M_{[-1,1]} = 2 \exp \left( \frac{1}{\pi} \int_{-1}^1 \frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^2}} \, dx \right) = 2 \exp \left( \frac{2}{\pi} \int_0^1 \frac{\log(1+x)}{\sqrt{1-x^2}} \, dx \right)
\]

(1.18) \[= 2 \exp \left( \frac{2}{\pi} \int_0^{\pi/2} \log(1 + \sin t) \, dt \right) \approx 3.2099123,
\]

which gives the asymptotic version of Borwein’s inequality (1.4)-(1.5).

Considering the above analysis of Theorem 1.1, it is natural to conjecture that the sharp universal bounds for \( M_E \) are given by

(1.19) \[2 = M_D \leq M_E \leq M_{[-1,1]} \approx 3.2099123,
\]

for any bounded non-degenerate continuum \( E \), see \[33\].

It follows directly from the definition that \( M_E \) is invariant with respect to the similarity transformations of the plane. Thus we can normalize the problem by setting \( \text{cap}(E) = 1 \). Thus, equivalently, we want to find the maximum and the minimum of the functional

(1.20) \[\tau(E) := \int \log d_E(z) \, d\mu_E(z)
\]

over all compact connected sets \( E \) in the plane satisfying the above normalization. These questions are addressed in Section 2 of the paper. Section 3 discusses a more refined version of our problem on the best constant in (1.1). All proofs are given in Section 4.

In the forthcoming paper \[34\], we consider various improved bounds of the constant \( M_E \), e.g., bounds for rotationally symmetric sets. From a different perspective, the results of Boyd (1.8)-(1.9) suggest that for some sets the
constant \( M_E \) can be replaced by a smaller one, if the number of factors is fixed. We characterize such sets in [34], and find the improved constant.

The problems considered in this paper have many applications in analysis, number theory and computational mathematics. We mention specifically applications in transcendence theory (see Gelfond [15]), and in designing algorithms for factoring polynomials (see Boyd [11] and Landau [21]). A survey of the results involving norms different from the sup norm (e.g., Bombieri norms) can be found in [11]. For polynomials in several variables, see the results of Mahler [24] for the polydisk, of Avanissian and Mignotte [2] for the unit ball in \( \mathbb{C}^k \). Also, see Beauzamy and Enflo [5], and Beauzamy, Bombieri, Enflo and Montgomery [4] for multivariate polynomials in different norms.

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2. Sharp bounds for the constant \( M_E \)

We study bounds for the constant \( M_E \) in this section, where \( E \subset \mathbb{C} \) is a compact set satisfying \( \text{cap}(E) > 0 \). Our main goal here is to prove (1.19). It is convenient to first give some general observations on the properties of \( M_E \).

**Theorem 2.1.** Let \( I \subset E \) be compact sets in \( \mathbb{C} \), \( \text{cap}(I) > 0 \). Denote the unbounded components of \( \mathbb{C} \setminus E \) and \( \mathbb{C} \setminus I \) by \( \Omega_E \) and \( \Omega_I \). If \( d_E(z) = d_I(z) \) for all \( z \in \partial \Omega_I \) then \( M_E \leq M_I \), with equality holding only when \( \text{cap}(\Omega_I \setminus \Omega_E) = 0 \).

This theorem gives several interesting consequences. In particular, we show that if the set \( E \) is contained in a disk whose diameter coincides with the diameter of \( E \) then its constant \( M_E \) does not exceed that of a segment. Thus segments indeed maximize \( M_E \) among such sets. Denote the closed disk of radius \( r \) centered at \( z \) by \( D(z, r) \).

**Corollary 2.2.** Let \( z, w \in E \) satisfy \( \text{diam} E = |z - w| \) and \( [z, w] \subset E \). If \( E \subset D\left(\frac{z + w}{2}, \frac{\text{diam} E}{2}\right) \) then \( M_E \leq M_{[z, w]} = M_{[-2, 2]} \).

The next results shows that the constant decreases when the set is enlarged in a certain way.

**Corollary 2.3.** Let \( E^* := \cap_{z \in \partial \Omega_E} D(z, d_E(z)) \), where \( E \subset \mathbb{C} \) is compact, \( \text{cap}(E) > 0 \). If \( H \) is a compact set such that \( E \subset H \subset E^* \), then \( M_H \leq M_E \). Equality holds if and only if \( \text{cap}(\Omega_E \setminus \Omega_H) = 0 \).
Let \( \text{conv}(H) \) be the convex hull of \( H \). The operation of taking the convex hull of a set satisfies the assumption of Corollary 2.3 (or Theorem 2.1), which gives

**Corollary 2.4.** Let \( V \subset \mathbb{C} \) be a compact set, \( \text{cap}(V) > 0 \). If \( H := \overline{\mathbb{C}} \setminus \Omega_V \) is not convex, then \( M_{\text{conv}(H)} < M_H \).

The above results help us to show that the minimum of \( M_E \) is attained for the closed unit disk \( D \), among all sets of positive capacity (connected or otherwise).

**Theorem 2.5.** Let \( E \subset \mathbb{C} \) be an arbitrary compact set, \( \text{cap}(E) > 0 \). Then \( M_E \geq 2 \), where equality holds if and only if \( \overline{\mathbb{C}} \setminus \Omega_E \) is a closed disk.

In other words, \( M_E = 2 \) only for sets whose polynomial convex hull is a disk. This may also be described by saying that \( M_E = 2 \) if and only if \( \partial U \subset E \subset U \), where \( U \) is a closed disk.

Proving that the maximum of \( M_E \) for arbitrary continua is attained for a segment is a more difficult problem. In fact, it is related to some old open problems on the moments of the equilibrium measure (or circular means of conformal maps), see Pólya and Schiffer [27], and Pommerenke [28]. In particular, we use the results of [27] and [28] to show that

**Theorem 2.6.** Let \( E \subset \mathbb{C} \) be a connected compact set, \( \text{cap}(E) > 0 \).

(i) If the center of mass \( c := \int z \, d\mu_E(z) \) for \( \mu_E \) belongs to \( E \), then

\[
(2.1) \quad M_E < 2 + 4.02/\pi \approx 3.279606.
\]

(ii) If \( E \) is convex then

\[
(2.2) \quad M_E < 2 + 4/\pi \approx 3.27324.
\]

This should be compared with \( M_{[-2,2]} = M_{[-1,1]} \approx 3.2099123 \).

After this paper had been written, a new related manuscript [3] appeared. That manuscript contains a proof of our conjecture \( M_E \leq M_{[-2,2]} \) for centrally symmetric continua, as well as another quite general conjecture (if true) implying \( M_E \leq M_{[-2,2]} \) holds for all continua.

3. **Refined problem**

The constant \( M_E \) represents the base of rather crude exponential asymptotic for the constant in inequality (1.1). A more refined question is to find the sharp constant attained with equality. Such constants are known in the case of a segment, see (1.4) and [7]; and in the case of a disk, see (1.10) and [20]. Let \( E \) be any compact set in the plane, and let \( \prod_{k=1}^m p_k(z) = \prod_{j=1}^n (z - z_j) \), where \( p_k(z) \) are arbitrary monic polynomials with complex coefficients. Define the
constant

\[ C_E(n) := \sup_{p_k} \prod_{k=1}^{m} \|p_k\|_E = \sup_{z_j \in \mathbb{C}} \prod_{j=1}^{n} \|z - z_j\|_E. \]

If \( \text{cap}(E) > 0 \) then it follows from Theorem 1.1 that \( 1 \leq C_E(n) \leq M^n_E \). The refined version of our conjecture in (1.19) is as follows:

\[ 2^{n-1} = C_D(n) \leq C_E(n) \leq C_{[-2,2]}(n) = 2^{n-1} \prod_{k=1}^{[n/2]} \left( 1 + \cos \frac{2k - 1}{2n \pi} \right)^2 \]

for any connected compact set \( E \) of positive capacity.

4. Proofs

Proof of Theorem 2.1 Since \( I \subset E \), we have that \( \text{cap}(E) \geq \text{cap}(I) > 0 \). Let \( g_E(z, \infty) \) and \( g_I(z, \infty) \) be the Green’s functions for \( \Omega_E \) and \( \Omega_I \), with poles in infinity. We follow the standard convention by setting \( g_E(z, \infty) = 0 \), \( z \notin \overline{\Omega_E} \) and \( g_I(z, \infty) = 0 \), \( z \notin \overline{\Omega_I} \). It follows from the maximum principle that \( g_E(z, \infty) \leq g_I(z, \infty) \) for all \( z \in \mathbb{C} \). Furthermore, this inequality is strict in \( \Omega_E \), unless \( \text{cap}(\Omega_I \setminus \Omega_E) = 0 \).

Using the integral representation for \( d_E(z) \) from Lemma 5.1 of [31] (see also [22] and [14]) and the Fubini theorem, we obtain that

\[
\log M_E = \int \log d_E(z) \, d\mu_E(z) - \log \text{cap}(E) = \int \left( \int \log |z - t| \, d\sigma_E(t) \right) \, d\mu_E(z) - \log \text{cap}(E) = \int g_E(t, \infty) \, d\sigma_E(t),
\]

where the last equality follows from the well known identity \( g_E(t, \infty) = \int \log |z - t| \, d\mu_E(z) - \log \text{cap}(E) \) [35]. It is clear that

\[
\int g_E(t, \infty) \, d\sigma_E(t) \leq \int g_I(t, \infty) \, d\sigma_E(t),
\]

with equality possible if and only if \( \text{cap}(\Omega_I \setminus \Omega_E) = 0 \). Indeed, if we have equality in the above inequality, then \( g_E(z, \infty) = g_I(z, \infty) \) for all \( z \in \text{supp} \sigma_E \). But \( \text{supp} \sigma_E \) is unbounded, so that \( g_E(z, \infty) = g_I(z, \infty) \) in \( \Omega_E \) by the maximum
principle. Hence we obtain that
\[
\log M_E \leq \int g_I(t, \infty) \, d\sigma_E(t) = \int \left( \int \log |z - t| \, d\mu_I(z) - \log \text{cap}(I) \right) \, d\sigma_E(t)
\]
\[
= \int \log d_E(z) \, d\mu_I(z) - \log \text{cap}(I) = \int \log d_I(z) \, d\mu_I(z) - \log \text{cap}(I)
\]
\[
= \log M_I,
\]
with equality if and only if \( \text{cap}(\Omega_I \setminus \Omega_E) = 0 \).

Note that we used \( \text{supp} \mu_I \subset \partial \Omega_I \), so that \( d_E(z) = d_I(z) \) for \( z \in \text{supp} \mu_I \).

\[ \square \]

Proof of Corollary 2.2. Let \( I = [z, w] \) be the segment connecting the points \( z \) and \( w \), i.e., the common diameter of \( E \) and the disk containing it. Observe that we have \( d_E(t) = d_I(t) \) for all \( t \in \partial \Omega_I = I \) under the stated geometric conditions. Since all assumptions of Theorem 2.1 are satisfied, we obtain that
\[
M_E \leq M_{[z,w]} = M_{[z,2,2]},
\]
where the last equality follows from the invariance with respect to the similarity transformations of the plane.

\[ \square \]

Proof of Corollary 2.3. Observe that \( E \subset D(z, d_E(z)) \) for any \( z \in \mathbb{C} \). Hence \( E \subset E^* \). Since \( E \subset H \subset E^* \), we immediately obtain that \( d_E(z) \leq d_H(z) \leq d_{E^*}(z) \), \( z \in \mathbb{C} \). On the other hand, the definition of \( E^* \) gives that \( d_E(z) = d_{E^*}(z) \) for all \( z \in \partial \Omega_E \). Therefore \( d_E(z) = d_H(z) \) for all \( z \in \partial \Omega_E \), and the result follows from Theorem 2.1.

\[ \square \]

Proof of Corollary 2.4. We apply Theorem 2.1 again, with \( I = H \) and \( E = \text{conv}(H) \). It was shown in \[22\] that \( d_H(z) = d_{\text{conv}(H)}(z) \) for all \( z \in \mathbb{C} \), where \( H \) is an arbitrary compact set. Since \( H \) is not convex in our case, we obtain that \( \text{cap}(\Omega_I \setminus \Omega_E) > 0 \) and \( M_E < M_I \).

\[ \square \]

For the proof of Theorem 2.5 we need a special case of the following lemma, which may be of some independent interest. Let \( \Delta := \{ w : |w| > 1 \} \), and \( \mathbb{D} := \{ z : |z| < 1 \} \) the unit disk.

Lemma 4.1. Let \( \Gamma \) be a Jordan domain and let \( \Psi(z) := cw + \sum_{k=0}^{\infty} a_k w^{-k} \) be a conformal map of \( \Delta \) onto \( \Omega_F \). Furthermore assume that
\[
(4.1) \quad \forall x, z \in \partial \Delta : \quad |\Psi(z) - \Psi(x)| \leq |\Psi(z) - \Psi(-z)|.
\]
Then \( \Gamma \) is a disk.

Proof. First note that by Carathéodory’s theorem [30, p. 18] \( \Psi \) extends to a homeomorphism of \( \overline{\Delta} \), so that (4.1) makes sense. Also there is no loss of generality in assuming \( 0 \in \Gamma \), so that \( \Psi(z) \neq 0 \) in \( \overline{\Delta} \). Let
\[
g(z) := \frac{1}{\Psi(1/z)}, \quad z \in \overline{\mathbb{D}}.
\]
Then \( g(z) = z/c + \sum_{k=2}^{\infty} b_k z^k \) is a homeomorphism of \( \overline{D} \) onto the closure of the Jordan domain \( \Gamma^* \), the interior domain of the Jordan curve \( 1/\partial \Gamma \). Note that \( g(0) = 0, g'(0) = 1/c \neq 0 \).

Let \( 1/z \in \partial D \), and in (4.1) we replace \( 1/x \in \partial D \) by \(-1/xz\) which is also in \( \partial D \). Condition (4.1) then becomes

\[
1 \geq \left| \frac{g(z) - g(-z)}{g(z) - g(-z)} \right| = \left| \frac{xg(-z) g(-xz) - g(z)}{g(-xz) g(-z) - g(z)} \right|, \quad x, z \in \partial D.
\]

Note that the function

\[
F(x, z) := \frac{xg(-z) g(-xz) - g(z)}{g(-xz) g(-z) - g(z)}
\]

is analytic in \((x, z) \in \mathbb{D}^2\), and by the maximum principle, applied to both variables separately, we find that

\[
|F(x, z)| \leq 1, \quad x, z \in \mathbb{D}.
\]

Now fix \( z_0 \) with \( 0 < |z_0| < 1 \). Then \( x \mapsto F(x, z_0) \) is analytic in \( \overline{D} \), satisfies \( |F(x, z_0)| \leq 1 \) for \( x \in \overline{D} \), and, in addition, \( F(1, z_0) = 1 \). The Julia-Wolf Lemma [30, p. 82] then says that \( F'(1, z_0) > 0 \), or

\[
1 + \frac{-z_0g'(-z_0)}{g(-z_0)} \frac{g(z_0)}{g(-z_0) - g(z_0)} > 0.
\]

Obviously this must be true for any \( z_0 \), and so, by the identity principle, we are left with the relation

\[
\frac{-zg'(-z)}{g(-z)} \frac{g(z)}{g(-z) - g(z)} \equiv \alpha, \quad z \in \mathbb{D},
\]

where \( \alpha > -1 \) is some real constant. Letting \( z \to 0 \), we find \( \alpha = -\frac{1}{2} \). Hence

\[
\frac{zg'(z)}{g(z)} \frac{g(-z)}{g(-z) - g(z)} = \frac{1}{2}, \quad z \in \mathbb{D}.
\]

In terms of \( \Psi \) this reads

\[
2w\Psi'(w) = \Psi(w) - \Psi(-w), \quad w \in \Omega_\Gamma.
\]

From this we conclude that \( w\Psi'(w) \) is an odd function, which, in turn, implies that \( \Phi(w) := \Psi(w) - a_0 \) is odd as well. For \( \Phi \) we then get the equation \( w\Phi'(w) = \Phi(w) \), or \( \Phi(w) = cw \). This implies \( \Psi(w) = cw + a_0 \) and therefore that \( \Gamma \) is a disk.

\[\Box\]

\textbf{Proof of Theorem 2.3}. Note that for any compact set \( E \), we have \( M_E = M_W \), where \( W := \overline{C} \setminus \Omega_E \). This follows because \( \mu_E = \mu_W \) [35] and \( d_E(z) = d_W(z), \quad z \in \mathbb{C} \). Corollary 2.4 now implies that

\[
\inf\{M_E : E \text{ is compact}\} = \inf\{M_H : H \text{ is convex and compact}\}.
\]
Hence we can assume that $E$ is convex from the start. We also set $\operatorname{cap}(E) = 1$, because $M_E$ is invariant under similarity transforms. Thus $\partial E$ is a rectifiable Jordan curve (or a segment when $E = \partial E$). The following argument that shows $M_E \geq 2$ for all connected sets is due to A. Solynin. Let $\Psi : \Delta \to \Omega_E$ be the standard conformal map:

$$
\Psi(w) = w + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \quad w \in \Delta.
$$

Recall that $\Psi$ can be extended as a homeomorphism of $\Delta$ onto $\Omega_E$, with $\Psi(T) = \partial E$, $T := \partial \Delta$. It is clear that $d_E(\Psi(e^{it})) \geq |\Psi(e^{it}) - \Psi(-e^{it})|$, $t \in [0, 2\pi)$.

Since $\Psi(w)$ is univalent in $\Delta$, the function

$$
H(w) := \frac{\Psi(w) - \Psi(-w)}{w}
$$

is analytic and non-vanishing in $\Delta$, including $w = \infty$. Furthermore, $H(\infty) := \lim_{w \to \infty} H(w) = 2$. It follows that $h(w) := \log |H(w)|$ is harmonic in $\Delta$. Recall that the equilibrium measure $\mu_E$ is the harmonic measure of $\Omega_E$ at $\infty$, which is invariant under the conformal transformation $\Psi$, see [35]. Hence

$$
\log M_E = \int \log d_E(z) d\mu_E(z) = \frac{1}{2\pi} \int_0^{2\pi} \log d_E(\Psi(e^{it})) dt \\
\geq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{\Psi(e^{it}) - \Psi(-e^{it})}{e^{it}} \right| dt = \log 2,
$$

where we used the Mean Value Theorem for $h(w)$ on the last step. Thus we conclude that $M_E \geq 2 = M_D$ holds for all compact sets $E$.

Recall that $M_E = M_W$, where $W = \mathbb{C} \setminus \Omega_E$. If $M_E = 2$ then $M_W = 2$, so that $W$ must be convex by Corollary [2.4]. Since $M_W > 3.2$ for any segment, we have that $W$ is the closure of a convex domain. We can assume that $\operatorname{cap}(W) = 1$ after a dilation. Repeating the above argument for $W$ instead of $E$, we obtain that

$$
\log 2 = \log M_W = \frac{1}{2\pi} \int_0^{2\pi} \log d_W(\Psi(e^{it})) dt \\
\geq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \Psi(e^{it}) - \Psi(-e^{it}) \right| dt = \log 2.
$$

It follows that

$$
\int_0^{2\pi} \left( \log d_W(\Psi(e^{it})) - \log \left| \Psi(e^{it}) - \Psi(-e^{it}) \right| \right) dt = 0,
$$

and that $d_W(\Psi(e^{it})) = |\Psi(e^{it}) - \Psi(-e^{it})|$ a.e. on $[0, 2\pi)$. But these functions are clearly continuous, so that

$$
d_W(\Psi(e^{it})) = |\Psi(e^{it}) - \Psi(-e^{it})| \quad \forall t \in \mathbb{R}.
$$
An application of Lemma 4.1 with \( \Gamma \) the interior domain of \( W \) shows that \( W \) must be a disk. We would also like to mention that A. Solynin obtained a different proof of the fact that \( M_E = 2 \) for a connected set \( E \) implies \( W \) is a disk. \( \Box \)

**Proof of Theorem 2.6.** Recall that \( M_E \) is invariant under similarity transformations. Hence we can assume again that \( \text{cap}(E) = 1 \) and \( \int z d\mu_E(z) = 0 \).

The latter condition means that the center of mass for the equilibrium measure is at the origin. If we introduce the conformal map \( \Psi : \Delta \to \Omega_E \), as in the previous proof, then this condition translates into \( a_0 = 0 \), i.e.,

\[
\Psi(w) = w + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \quad w \in \Delta.
\]

Theorem 1.4 of [29, p. 19] gives that \( E \subset D(0, 2) \), so that \( d_E(z) \leq 2 + |z|, \ z \in E \), by the triangle inequality. Note that this is sharp for \( E = [-2, 2] \). Applying Jensen’s inequality, we have

\[
\log M_E = \int \log d_E(z) d\mu_E(z) \leq \int \log(2 + |z|) d\mu_E(z) < \log \left( 2 + \int |z| d\mu_E(z) \right).
\]

 Estimates (2.1) and (2.2) now follow from the results of Pommerenke [28], and of Pólya and Schiffer [27], who estimated the integral

\[
\int |z| d\mu_E(z) = \frac{1}{2\pi} \int_0^{2\pi} |\Psi(e^{it})| dt < \frac{4.02}{\pi} \ (\text{or} \ \leq \frac{4}{\pi}),
\]

under the corresponding assumptions. \( \Box \)

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DEPARTMENT OF MATHEMATICS, 401 MATHEMATICAL SCIENCES, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078-1058, U.S.A.
E-mail address: igor@math.okstate.edu

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WÜRZBURG, AM HUBLAND, 97074 WÜRZBURG, GERMANY
E-mail address: ruscheweyh@mathematik.uni-wuerzburg.de