Tests of the Porter-Thomas Distribution for Reduced Partial Neutron Widths

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Given \( N \) data points drawn from a \( \chi^2 \)-distribution, we use Bayesian inference to determine most likely values and \( N \)-dependent confidence intervals for the width \( \sigma \) and the number \( k \) of degrees of freedom of that distribution. Using reduced partial neutron widths measured in a number of nuclei, a guessed value of \( \sigma \), and a maximum-likelihood approach (different from Bayesian inference), Koehler et al. \textsuperscript{[1]} and Koehler \textsuperscript{[2]} have determined the most likely \( k \)-values of \( \chi^2 \)-distributions that fit the data. In all cases they find values for \( k \) that differ substantially from \( k = 1 \) (the value characterizing the Porter-Thomas distribution (PTD) predicted by random-matrix theory). The authors conclude that the validity of the PTD must be rejected with considerable statistical significance. We show that the value of \( \sigma \) guessed in Refs. \textsuperscript{[1, 2]} lies far outside the Bayesian confidence interval for \( \sigma \), casting serious doubt on the results of and the conclusions drawn in Refs. \textsuperscript{[1, 2]}. We also show that \( \sigma \) and \( k \) must both be determined from the data. Comparison of the results with the Bayesian confidence intervals would then decide on acceptance or rejection of the PTD.

I. INTRODUCTION

In 2010 and 2011, Koehler et al. \textsuperscript{[1]} and Koehler \textsuperscript{[2]} have tested the Porter-Thomas distribution predicted by random-matrix theory \textsuperscript{[3]} against the distribution of reduced partial neutron widths measured in a number of nuclei. Because of possible \( p \)-wave admixtures and because of the difficulty to identify narrow \( s \)-wave resonances unambiguously, the authors have used a cutoff, considering only neutron resonances with reduced partial widths larger than some energy-dependent value \( y_0 \). They have applied a maximum likelihood approach to the resonance widths so obtained and have determined the most likely value of the parameter \( k \) of a \( \chi^2 \)-distribution with \( k \) degrees of freedom (subject to the same cutoff) \textsuperscript{[1]}. Using 411 (158) measured partial widths in \( ^{194}\text{Pt} \) (in \( ^{192}\text{Pt} \), respectively), the most numerous sets of resonances in single nuclei, Koehler et al. obtain a value of \( k \) close to 0.5. They conclude that the validity of the Porter-Thomas distribution (a \( \chi^2 \)-distribution with \( k = 1 \) degrees of freedom) must be rejected with a statistical significance of at least 99.997 per cent. For the nuclear data ensemble, Koehler finds \( k = 1.217 \pm 0.092 \) and a statistical significance for rejection of 98.17 per cent \textsuperscript{[2]}.

Random-matrix theory forms the basis of the statistical theory of nuclear reactions \textsuperscript{[4]}. Nuclear reaction cross sections that either cannot be measured with sufficient accuracy or cannot be measured altogether are calculated using the statistical theory. The results are widely used in astrophysics, reactor shielding, material science, medicine, and biology. The disagreement found in Refs. \textsuperscript{[1, 2]} is important for fundamental and applied science.

The results of Refs. \textsuperscript{[1, 2]} have, therefore, caused quite a stir in the theoretical nuclear-physics community \textsuperscript{[5–11]}. The Porter-Thomas distribution follows from orthogonal invariance, one of the pillars of random-matrix theory for time-reversal invariant systems. The terms that couple an orthogonally invariant Hamiltonian to the neutron channel and the numerous gamma channels break orthogonal invariance and may cause deviations of the distribution of reduced partial neutron widths from the Porter-Thomas distribution. That possibility has been investigated in a number of papers \textsuperscript{[6–11]}. It is now firmly established, however, that breaking of orthogonal invariance due to coupling to the channels is too weak to account for the discrepancy found in Refs. \textsuperscript{[1, 2]}, and the problem persists.

Here we address the discrepancy from a point of view that seems to have escaped attention so far. We ask: Are the significant deviations of the results of Refs. \textsuperscript{[1, 2]} from the Porter-Thomas distribution caused by the limited number of partial resonance widths available? In other words, must such deviations be expected (and not taken as evidence against the Porter-Thomas distribution)? To answer the question, we use the fact that each \( \chi^2 \)-distribution is characterized by two parameters, the number \( k \) of degrees of freedom and the width \( \sigma \). We determine the minimum number of data points (i.e., reduced partial neutron widths) needed to determine \( k \) and \( \sigma \) with sufficient statistical accuracy in a sufficiently narrow interval of values. We do so using Bayesian inference \textsuperscript{[12]}.

The overall Bayesian approach to the \( \chi^2 \)-distributions is described in Section \textsuperscript{III}. In Section \textsuperscript{IVII} it is applied to estimating \( \sigma \), in Section \textsuperscript{VII} it is applied to estimating \( k \). In Section \textsuperscript{VI} we combine both results and determine the optimum value for the pair \(( \sigma, k )\). Section \textsuperscript{VII} contains a summary and a discussion of the implications of our work for the data analysis of Refs. \textsuperscript{[1, 2]}. Some auxiliary calculations are deferred to the Appendix.
II. BAYESIAN INFERENCE

In this Section we define the $\chi^2$-distributions without and with cutoff and the Bayesian likelihood functions for these distributions. Given $k = 1,2,\ldots$ independent zero-centered Gaussian-distributed real random variables $x_l$, $l = 1,\ldots,k$, with equal variance $\sigma^2 \neq 0$, the random variable $y$ is defined by $y = \sum_{l=1}^{k} x_l^2$. The normalized probability distribution of $y$ is a $\chi^2$-distribution with $k$ degrees of freedom and width $\sigma$,

$$P_k,\sigma(y) = \frac{1}{\Gamma(k/2)(2\sigma^2)^{k/2}} y^{(k/2)-1} \exp\{-y/(2\sigma^2)\} . \tag{1}$$

Here $\Gamma(x)$ is the Gamma function [13]. The normalized $\chi^2$-distribution with cutoff at $y = y_0$ is

$$\tilde{P}_k,\sigma(y) = \Theta(y - y_0) \frac{1}{\Gamma(k/2, y_0/(2\sigma^2))(2\sigma^2)^{k/2}} y^{(k/2)-1} \times \exp\{-y/(2\sigma^2)\} . \tag{2}$$

Here $\Theta$ is the Heaviside function and $\Gamma(x, s)$ is the upper incomplete Gamma function [13].

In general terms, our problem can be stated as follows. Given the normalized probability distribution $P(y|\xi)$ of a variable $y$ in terms of a parameter $\xi$, and given a number $N$ of measured values $(y_1, y_2,\ldots, y_N)$ of $y$ (the “data points”), jointly referred to as $y$, how to ascertain that $\xi$ lies with a given probability within a given range of values? In our case, $P(y|\xi)$ stands either for the distribution [14] or for the distribution [12], $\xi$ stands either for $\sigma$ or for $k$, and the data points are the measured reduced partial neutron widths. Bayes’ theorem [12] gives the answer in terms of the “likelihood function”, i.e., the normalized posterior probability distribution $\Pi(\xi|y)$ for $\xi$, a function of the $N$ data points. We first display the answer for the case $N = 1$ because it is simple and intuitively convincing. The theorem states that

$$\Pi(\xi|y_1) = \frac{P(y_1|\xi)\mu(\xi)}{\int d\xi \mu(\xi)P(y_1|\xi)} . \tag{3}$$

Given a data point $y_1$, the probability $\Pi(\xi|y_1)$ to find the value $\xi$ equals the probability $P(y_1|\xi)$ to find, given the parameter $\xi$, the value $y_1$ modulo two factors, the normalization factor in the denominator of Eq. (3), and $\mu(\xi)$. The latter factor (the “prior”) accounts for whatever $a$ priori knowledge we may have about $\xi$ before any data are taken. It is possible, for instance, that $\xi$ is confined to some interval. That knowledge, irrelevant for $P(y|\xi)$, is embodied in $\mu(\xi)$. The generalization of Eq. (3) to $N$ data points is

$$\Pi(\xi|y) = \frac{\mu(\xi) \prod_{n=1}^{N} P(y_n|\xi)}{\int d\xi \mu(\xi) \prod_{n=1}^{N} P(y_n|\xi)} . \tag{4}$$

The data points are independent, and $\Pi(\xi|y)$ is the product of their probabilities. Little is usually known about the prior distribution $\mu(\xi)$ of $\xi$, and arguments of invariance and symmetry are, therefore, used to determine that quantity [12]. For $\xi = \sigma$, possible choices of $\mu(\xi)$ are discussed in Section III. We show that for $N \gg 1$ these choices do not influence our results.

For $N \gg 1$ the function $\Pi(\xi|y)$ tends toward a Gaussian. That is a consequence of the central limit theorem. The maximum yields the most likely value of $\xi$, the $N$-dependent width of the Gaussian is used to determine the width of the interval within which $\xi$ is found with a predetermined probability (the “confidence interval”). The approach of $\Pi(\xi|y)$ toward the Gaussian is not uniform in $\xi$, and corrections to the Gaussian have to be taken into account.

In implementing that approach, we use Eq. (4) to determine in Section III for fixed $k$ a range of most likely values of $\sigma$. That range depends upon $N$. We do so for both distributions [14] and [12], without and with cutoff. In Section IV we proceed analogously for $k$.

III. BAYESIAN ESTIMATION OF $\sigma$

For $N$ reduced neutron widths $(y_1, y_2,\ldots, y_N)$ written jointly as $y$, we apply, for fixed $k$, Bayes’ theorem [14] to the determination of $\sigma$. For the full distribution [14], the normalized posterior distribution $\Pi_k(\sigma|y)$ of $\sigma$ is given by

$$\Pi_k(\sigma|y) = \frac{\mu(\sigma) \prod_{n=1}^{N} P_k,\sigma(y_n)}{\int d\sigma \mu(\sigma) \prod_{n=1}^{N} P_k,\sigma(y_n)} . \tag{5}$$

For the distribution [12] with cutoff, the normalized posterior distribution is

$$\tilde{\Pi}_k(\sigma|y) = \frac{\mu(\sigma) \prod_{n=1}^{N} \tilde{P}_k,\sigma(y_n)}{\int d\sigma \mu(\sigma) \prod_{n=1}^{N} \tilde{P}_k,\sigma(y_n)} . \tag{6}$$

Here the cutoff may be different for each data point $y_n$. For the prior distribution $\mu(\sigma)$ we consider two options, both formulated as invariance requirements on the integration measure $\mu(\sigma) d\sigma$. (i) The measure is invariant under translations $\sigma \rightarrow \sigma + \rho$ for all real $\rho$. That yields $\mu(\sigma) = \mu_0$ with $\mu_0$ some real constant. (ii) The measure is invariant under scale transformations $\sigma \rightarrow \rho \sigma$ for all positive $\rho$. That yields $\mu(\sigma) = \mu_0/\sigma$ with $\mu_0$ some positive constant. We write $\mu(\sigma) = \mu_0/\sigma^\alpha$ with $\alpha = 0$ ($\alpha = 1$) for case (i) (case (ii), respectively). We show that for $N \gg 1$ the two options yield results that differ only to order $1/N$. To leading order our results are, thus, independent of the choice of the prior. In both cases the integration over $\sigma^*$ in Eqs. (6) extends from $-\infty$ to $+\infty$, and the factor $\mu_0$ cancels in numerator and denominator of the right-hand side of these equations. To

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2 The functions [4], [5] have an essential singularity at $\sigma = 0$. Translational invariance can, nevertheless, be imposed if we replace $\sigma^* \rightarrow \sigma^* + \delta$ with positive infinitesimal $\delta$ and let $\delta \rightarrow 0$ after the integration over $\sigma$ is done.
define $\Pi_k$ and $\tilde{\Pi}_k$ unambiguously we replace in Eqs. (1) and (2) $\sigma^k \to |\sigma|^k$. Then $\Pi_k$ and $\tilde{\Pi}_k$ are symmetric in $\sigma$ about the point $\sigma = 0$. The normalized distributions (4, 5) are referred to as the likelihood functions for $\sigma$ for a given data set $(y_1, y_2, \ldots, y_N)$. They are given an additional index $\alpha = 1, 2$ to distinguish the two choices of the invariant measure.

A. Full Distribution

From Eqs. (11), (5) and for $\alpha = 0, 1$ we have

$$\Pi_{k,\alpha}(\sigma|y) = \frac{\sigma^{-\alpha}}{N_{\alpha}} \prod_{n=1}^{N} \left[ \frac{1}{(2\sigma^2)^{k/2}} y_n^{(k/2)-1} \times \exp\{-y_n/(2\sigma^2)\} \right]$$

$$= \frac{1}{N_{\alpha}} \frac{1}{\sigma^\alpha} \exp \left\{ -(Nk/2) \ln(2\sigma^2) + \left[ (k/2) - 1 \right] \sum_{n=1}^{N} \ln y_n - \sum_{n=1}^{N} y_n/(2\sigma^2) \right\}.$$  (7)

Here $N_{\alpha}$ is the normalization factor. The exponential of $[(k/2) - 1] \sum_n \ln y_n$ does not depend on $\sigma$ and cancels in numerator and denominator. It follows that the likelihood function (4) for $\sigma$ depends on the $N$ data points $(y_1, y_2, \ldots, y_N)$ only via the arithmetic mean value

$$m = \frac{1}{N} \sum_{n=1}^{N} y_n.$$  (8)

That is expected. We define

$$R = 2m/k, \quad N_0 = Nk^2/2.$$  (9)

The likelihood function takes the universal form

$$\Pi_{\alpha}(\sigma|y) = \frac{\sigma^{-\alpha}}{f_{N_0}} \exp\{-N_0[\ln(2\sigma^2) + R/(2\sigma^2)]\}.$$  (10)

The function $\Pi_{\alpha}(\sigma|y)$ depends on the input parameters $(N, k, m)$ only via the parameters $N_0$ and $R$ defined in Eqs. (9). It is convenient to introduce the dimensionless variable

$$z = \frac{2\sigma^2}{R} = \frac{\sigma^2}{m/k}, \quad d\sigma = \frac{\sqrt{R}}{\sqrt{8\pi}} dz.$$  (11)

We absorb the factor $\sqrt{R}/\sqrt{8\pi}$ into the likelihood function. In the range $0 \leq z < \infty$ the integration measure for $z$ is then equal to unity, and

$$\Pi_{\alpha}(z) = \frac{1}{N_0 z^{(1+\alpha)/2}} \exp\{-N_0[\ln z + 1/z]\}$$

$$= \frac{1}{N_0} \exp\{-(N0/2)[(1+\alpha)/2] \ln z\}.$$  (12)

As a consequence of the scaling of $\sigma$ in Eq. (11), $\Pi_{\alpha}(z)$ depends on the data points only via the number $N_0$. That is why $y$ does not appear as argument of $\Pi_{\alpha}(z)$. The normalization integrals $N_{\alpha}$ are calculated in the Appendix and given by

$$N_0 = \frac{c(2N_0 - 3)}{c(N_0 - 2)} \sqrt{\frac{2\pi}{N_0}} \exp\{N_0\},$$

$$N_1 = \frac{c(N_0 - 1)}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{N_0 - 1}} \exp\{N_0\}.$$  (13)

The coefficients $c(n)$ arise in the context of Stirling's formula and for integer $n$ obey

$$\sqrt{2\pi} \leq c(n) \leq e.$$  (14)

Combining Eqs. (12) and (13) we have

$$\Pi_{\alpha}(z) = \frac{1}{C_{\alpha}} \sqrt{\frac{N_0 - \alpha}{2\pi}} \frac{1}{z^{(1+\alpha)/2}} \times \exp\{-N_0[\ln z + 1/z - 1]\}.$$  (15)

The coefficients $C_{\alpha}$ are given by the first factors on the right-hand sides of Eqs. (13). The maximum of the term in the exponent is at $z = 1$ or, from Eq. (11), at

$$m = \frac{ka^2}{R}.$$  (16)

That shows that for $N \to \infty$, $m$ approaches asymptotically the mean value $(y) = ka^2$ of the distribution (11), as is to be expected. Writing $z = 1 + \epsilon$ we expand $\Pi_{\alpha}(z)$ in powers of $\epsilon$, keeping only the $\epsilon$-dependent factors and of these, only terms up to third order. We define $\gamma_\alpha = (1+\alpha)/(2N_0)$. In the exponent that gives

$$-N_0 \left[ -N_0 \gamma_\alpha + \frac{1}{2} \left( 1 - \gamma_\alpha \right) \epsilon^2 - \frac{2}{3} \epsilon^3 \left( 1 - \gamma_\alpha/2 \right) \right].$$  (17)

We use that in Eq. (15) and integrate the resulting expression over $\epsilon$ from $-q$ to $+q$. With $\zeta = \epsilon \sqrt{N_0 - \alpha}$ we obtain

$$\frac{1}{C_{\alpha}} \sqrt{\frac{N_0 - \alpha}{2\pi}} \int_{-q}^{+q} \sqrt{N_0 - \alpha} d\zeta \exp\{-N_0[\ln z + 1/z - 1]\}$$

$$\times \exp\{-(1+\alpha)/2 \ln z\}$$

$$\times \exp\{\frac{1}{2} \left( 1 - \gamma_\alpha \right) N_0 - \alpha \epsilon^2\}.$$  (18)

The expansion in expression (18) effectively proceeds in powers of $\zeta/\sqrt{N_0}$. For $|\zeta| \leq 3$ and $N_0 = 900$ that quantity is in magnitude smaller than or equal to $1/10$. In relation to the term of second order, the third-order term in expression (18) is also of order $1/10$. Terms of higher order need be taken into account only for smaller values of $N_0$. The inequalities (14) allow for a range of values of the coefficients $C_{\alpha}$. The least stringent bounds on $N_0$ are obtained by using the upper bounds on $1/C_{\alpha}$. These are

$$1/C_1 \leq 1 \quad \text{and} \quad 1/C_0 \leq e/\sqrt{2\pi}.$$  (19)

We use the first of these, putting $C_1 = 1$. The second actually overestimates $1/C_0$. 20
By definition, the integral $\int \sigma$ is bounded from above by unity. Numerical calculation shows that this condition is met only for $1/C_0 = 1$. In effect, we replace both coefficients $C_0$ by unity.

The integral $\int \sigma$ yields the probability $p$ to find $z$ in the interval $(1-q, 1+q)$ as a function of the parameters $q$ and $N_0$. Conversely, choosing $p$ and the said interval we may use the integral to find $q$ and a minimum value of $N_0$. For instance, for $\alpha = 0$, $N_0 = 900$ and $\sqrt{N_0} = 3$, the interval is $0.9 \leq z \leq 1.1$ and $p = 0.99923$. For $N_0 = 900$ and $\sqrt{N_0} = 2.9$, we get $0.903 \leq z \leq 1.097$ and $p = 0.9980$. Very similar results are obtained for $\alpha = 1$, confirming our claim that for $N_0 \gg 1$ the Bayesian prior has little influence on our results. It may be desirable to reduce the size of the interval so that $z$ has little influence on our results. It may be desirable to reduce the size of the interval so that $z$ differs from unity (and $\sigma^2$ from $m/k$) by less than 5 per cent, but to keep the probability $p$ fixed. That is the case for $\sqrt{N_0} = 2.9$ and $N_0 = 3600$. The interval for $z$ can also be reduced by reducing $q$. Doing so and keeping $N_0$ fixed reduces $p$. One quickly reaches a value for $p < 1$ for which the probability $(1-p)$ for $z$ to lie outside the interval $(1-q, 1+q)$ is unacceptably large.

We use these figures for $k = 1$, $N = 2N_0$, the case of central interest in Refs. [1, 2]. The maximum of the distribution is at $\sigma^2 = m$. For $N = 1800$, $\sigma^2$ is found with probability 0.99923 in the interval $0.9 \leq \sigma^2/m \leq 1.1$ and with probability 0.9980 in the interval $0.903 \leq \sigma^2/m \leq 1.097$. For $N = 400$, close to the case of $^{194}$Pt in Ref. [2], $\sigma^2$ is found with probability 0.99923 in the interval $0.785 \leq \sigma^2/m \leq 1.214$.

### B. Distribution with Cutoff

In view of the close similarity of the results obtained for the two choices of the invariant measure in Section IIIA we confine ourselves here to the case of translational invariance, put $\alpha = 0$, and drop the index $\alpha$ in what follows. We assume that the cutoff $y_0$ is fixed, i.e., is the same for all data points $(y_1, y_2, \ldots, y_N)$. In principle, it is possible to allow for different cutoff values $y_n$, one for each data point. We have not found a way, however, to handle that case analytically and completely. At the end of Section IIIA we demonstrate, however, that the results obtained for fixed cutoff are actually representative of the more general case with different cutoff values.

We define the mean value $M$ of the data points as in Eq. (8) but display explicitly the fact that by definition all data points obey $y_n > y_0$. Thus,

$$M = \frac{1}{N} \sum_{n=1}^{N} y_n (y_n - y_0) .$$

The normalized Bayesian likelihood function (9) is given by

$$\tilde{\Pi}_k(\sigma|y, y_0) = \exp \left\{ - \frac{(Nk/2) \ln(2\sigma^2)}{2} - \frac{(NM/(2\sigma^2))}{N} \right\} .$$

The upper incomplete Gamma function in Eq. (20), defined in Sec. 8.350 no. 2 of Ref. [12], depends upon $\sigma$. That changes the form of the probability distribution (10) for $\sigma$ in a non-universal way. The maximum is obtained by putting the derivative with respect to $2\sigma^2$ equal to zero. That condition can be written as (see Sect. 8.358 no. 4 of Ref. [12])

$$M = k\sigma^2 + (2\sigma^2) \frac{[y_0/(2\sigma^2)]}{\Gamma(k/2, y_0/(2\sigma^2))} ,$$

or

$$\Gamma(k/2, y_0/(2\sigma^2)) = \exp\left\{ -y_0/(2\sigma^2) \right\} \mathcal{P}_{(k/2) - 1} ,$$

and Eq. (20) becomes

$$\tilde{\Pi}_k(\sigma|y, y_0) = \exp \left\{ - \frac{(Nk/2) \ln(2\sigma^2)}{2} - \frac{NM}{2}\sigma^2 \right\} .$$

Here $\mathcal{P}_{(k/2) - 1}$ is a polynomial of order $(k/2) - 1$ in $y_0/(2\sigma^2)$, with $P_0 = 1$ for $k = 2$ and $P_1 = 1 + y_0/(2\sigma^2)$ for $k = 4$.

Eq. (20) differs from Eq. (10) in two respects. First, $M$ is replaced by $(M-y_0)$, i.e., each of the $y_n$’s is replaced by $y_n - y_0$. Second, the non-universal term $N \ln P$ affects the location of the maximum (see Eq. (21)) and, in particular, the expansion around that maximum. For $k = 2$ we have $P_0 = 1$. Except for the replacement $m \to (M-y_0)$, the distribution (20) takes the universal form (12) for $\sigma = (M-y_0)/2$. All other conclusions of Section IIIA remain unaltered. For $k = 4$ we have $P_1 = 1 + y_0/(2\sigma^2)$, and the exponent in Eq. (22) becomes

$$-N \left\{ \ln(2\sigma^2) + \ln(2\sigma^2 + y_0) + \frac{(M - y_0)}{2\sigma^2} \right\} .$$
In analogy to Eq. (11) and since \( k = 4 \) we define

\[
2\sigma^2 = \frac{1}{2}(M - y_0)z, \quad \beta = \frac{2y_0}{m - y_0}.
\]

In expression (24) we omit terms independent of \( z \) generated by the variable transformation (25). Expression (24) becomes

\[
-N\left\{ \ln z + \ln(z + \beta) + \frac{2}{z} \right\}.
\]

(26)

Putting the derivative equal to zero yields a quadratic equation. The solution that tends to unity for \( \beta \to 0 \) is

\[
z_0 = \frac{1}{2}(1 - \frac{1}{\beta}) + \frac{1}{\sqrt{2}}\sqrt{(1 - \frac{1}{\beta})^2 + 4\beta}
\]

(27)

and yields \( 4\sigma_0^2 = (M - y_0)z_0 \) at the maximum of expression (26). We write \( z = z_0(1 + \epsilon) \) and expand expression (26) in powers of \( \epsilon \), omitting the constant term and keeping terms up to third order. We obtain

\[
-N\frac{1}{2}\epsilon^2\left( \frac{4}{z_0} - 1 - \frac{\epsilon^2}{z_0^2} + \frac{2\epsilon}{z_0(z + \beta)} \right)
+ N\frac{1}{3}\epsilon^3\left( \frac{6}{z_0} - \frac{z_0^3}{(z_0 + \beta)^3} \right)
\]

(28)

For \( \beta = 0, z_0 = 1 \) the big round brackets are equal to 2 and 4, respectively, and expression (28) is equal to expression (17) for \( k = 4 \), \( y_0 = 0 \). We estimate the range of \( \beta \) as function of the cutoff using Eq. (25). By construction, \( M > y_0 \). For the data of Ref. 1 (see Figure 3) we have \( M \approx 2y_0 \). Thus, a reasonable estimate for the range of \( \beta \) is \( 0 \leq \beta \leq 2 \). In that range \( z_0 \) ranges from 1 to \( \sqrt{2} \), the first big round bracket in expression (28) ranges from 2 to \( \approx 1.66 \), and the second big round bracket from 4 to \( \approx 3.17 \). At the upper end of the ranges these values are equivalent to the replacement \( N \to 0.8N \). That increases the lower bound on \( N \) by a factor 2. Thus, for \( k = 2 \) the results of Section III A on the required minimum size of the data set are not affected by the cutoff while for \( k = 4 \) the cutoff marginally increases the lower bound on the required size of the data set.

For odd \( k \) the analogue of Eq. (22) involves error functions which can only be handled numerically. However, the \( \chi^2 \)-distribution (1) depends on \( k \) analytically and smoothly. Hence, there is no reason to doubt that the results obtained for \( k = 2 \) and \( k = 4 \) hold similarly also for \( k = 1 \) and \( k = 3 \). We conclude that the bounds on \( p \) and \( N \) derived in Section III A apply similarly also for the distribution with cutoff. The main difference is the value of \( \sigma \) for large \( N \). We have shown in Section III A that without cutoff we have \( \sigma^2 \to m/k \) for large \( N \). In the present Section, we have shown that with a uniform cutoff at \( y_0 \) and for \( k = 2 \), we have \( \sigma^2 \to (M - y_0)/2 \), a significant change compared to \( \sigma^2 \to m/2 \) without cutoff. Although the case \( k = 1 \) with cutoff is not accessible analytically, the case \( k = 2 \) makes us expect that here, too, the limiting value of \( \sigma^2 \) differs markedly from \( m \).

C. Discussion

The application of Bayes’ theorem to the determination of the parameter \( \sigma \) in the \( \chi^2 \)-distribution (1) for fixed \( k \) is, in principle, simple and straightforward. The likelihood function for \( \sigma \) possesses a single maximum. As the number \( N \) of data points becomes large, the location of the maximum becomes independent of \( N \) and of the Bayesian prior. It then defines the most likely value of \( \sigma^2 \) in terms of the expectation value \( \langle y \rangle \). For \( N \gg 1 \), the likelihood function assumes the form of a Gaussian. The width of the Gaussian is proportional to \( 1/\sqrt{N} \) and yields an estimate of the probability to find \( \sigma^2 \) within some narrow interval centered at the maximum. For realistic values of \( N \) (i.e., for a few hundred data points), the Gaussian approximation is replaced by the analytical form of the normalized likelihood function. Integration of that function over an interval of \( \sigma \)-values centered at the maximum yields the probability \( p \) to find \( \sigma \) within that interval. For fixed \( p \) the size of the interval shrinks with increasing \( N \). Fixing \( p \) and the size of the interval determines the minimum number \( N \) of data points.

In Section III A we have implemented that program for the \( \chi^2 \)-distribution (1) without cutoff. The most likely value of \( \sigma \) is given by \( \sigma^2 = m/k \). That result is expected because for \( y = \sum_{i=1}^k x_i^2 \) defined as in Section II in terms of a sum of real Gaussian-distributed zero-centered random variables \( x_i \) with equal variance \( \sigma^2 \), we obviously have \( \langle y \rangle = k\sigma^2 \). The Bayesian approach defines a confidence interval for \( \sigma^2 \) in terms of \( N \), or vice versa. For useful values of \( p \) and acceptable values of the size of the interval, we find \( N_0 \geq 900 \). Eq. (3) yields \( N = 2N_0/k \). For fixed \( N_0 \), the bound on \( N \) decreases with increasing \( k \). That is because the \( \chi^2 \)-distribution (1) is most asymmetric for \( k = 1 \). For the Porter-Thomas distribution \( (k = 1) \), we obtain \( N \geq 1800 \).

The implementation of that program for the \( \chi^2 \)-distribution (2) with cutoff in Section III B encounters an additional problem. The relation \( \sigma^2 = m/k \) is useless because \( m \) as defined in Eq. 8 cannot be determined from the data. The mean value \( M \) defined in Eq. (19) in terms of the data points that are actually available is surely bigger than \( m \) so that \( M = c m \) with \( c \geq 1 \). The relation \( \sigma^2 = m/k \) remains intact but is now written as \( \sigma^2 = M/(ck) \). Here \( c \) depends upon \( k \) and the cutoff \( y_0 \). Qualitatively, \( c \) depends on these parameters as follows. For \( k \gg 1 \) the maximum of the \( \chi^2 \)-distribution (1) shifts to ever larger \( k \)-values, and the cutoff becomes increasingly irrelevant. Therefore we have \( c \to 1 \) for \( k \to \infty \). The converse is true for \( k \to 0^+ \). The bulk of the \( \chi^2 \)-distribution becomes ever more narrowly concentrated near zero. In fact, the relation \( \sigma^2 = m/k \) shows that \( m \to 0 \) for \( k \to 0^+ \). On the other hand, \( M \) attains by definition a finite value \( \geq y_0 \) in that limit. Hence \( c \to \infty \) for \( k \to 0 \). We conclude that \( c \geq 1 \) diverges for \( k \to 0 \) and decreases monotonically with increasing \( k \). For fixed \( k \), \( c \) is expected to grow monotonically with \( y_0 \). The most likely value for \( \sigma^2 \) of the Bayesian likelihood func-
tion in Eq. (21) takes these facts into account. Because of the occurrence of the incomplete Gamma function, that equation cannot be solved analytically and in general for \( \sigma^2 \). We have confined ourselves to determining the most likely value of \( \sigma^2 \) for \( k = 2 \) and \( k = 4 \) and for a fixed value of \( y_0 \) as only these cases can be worked out analytically. \( M/\kappa \) and \( \sigma^2 = M/\kappa \) without cutoff. For \( k = 4 \), the most likely value is \( \sigma^2 = 4 \) (see Eq. (27)) and obtain \( \sigma^2 = M\sqrt{2}/8 \). That value differs by the factor \( \sqrt{2}/2 \approx 0.7 \) from the value \( \sigma^2 = m/4 \) without cutoff. For \( k = 1 \) we expect an even smaller value \( \sigma^2 \approx 0.4M \). For the bounds on \( N \), the modifications due to the cutoff are less drastic and amount to an increase of 20 per cent or so.

When a separate cutoff \( y_{0,n} \) is used for each data point \( y_n \), Eq. (21) is replaced by

\[
M = k\sigma^2 \quad (29)
\]

\[
+ (2\sigma^2 \sum_{n=1}^{N} \left[ \frac{y_{0,n}/(2\sigma^2)^{k/2}}{\Gamma(k/2, y_{0,n}/(2\sigma^2))} \exp\left(-y_{0,n}/(2\sigma^2)^{k/2}\right) \right] .
\]

Comparison of Eqs. (29) and (21) suggests defining \( y_0 \) by putting

\[
(2\sigma^2 \sum_{n=1}^{N} \left[ \frac{y_{0,n}/(2\sigma^2)^{k/2}}{\Gamma(k/2, y_{0,n}/(2\sigma^2))} \exp\left(-y_{0,n}/(2\sigma^2)^{k/2}\right) \right] ) = (2\sigma^2 \sum_{n=1}^{N} \left[ \frac{y_{0,n}/(2\sigma^2)^{k/2}}{\Gamma(k/2, y_{0,n}/(2\sigma^2))} \exp\left(-y_{0,n}/(2\sigma^2)^{k/2}\right) \right] )
\]

The second line in that equation can be read as an average over the cutoff parameters \( y_{0,n} \). It comprises only positive contributions. Moreover, each term in the sum increases monotonically with increasing \( y_{0,n} \). Therefore, Eq. (30) possesses a unique solution \( y_0 \). That solution determines the factor \( c \) reducing \( M/k \). Thus, the conclusions drawn above on the dependence of \( c \) on the cutoff parameter \( y_0 \) remain valid. The confidence interval for \( \sigma \) cannot be determined by the solution \( y_0 \) of Eq. (30) but must be determined from all the \( y_{0,n} \). It is reasonable to expect, however, that the estimates in Section III.B for that interval remain valid.

IV. BAYESIAN ESTIMATION OF \( k \)

In Section III we have defined the minimum number of data points needed to determine for fixed \( k \) and required accuracy the parameter \( \sigma \) in the \( \chi^2 \)-distributions \( P_{k,\sigma}(y) \) of Eq. (1) and \( P_{k,\sigma}(y) \) of Eq. (2). With the help of Bayes’ theorem, we now proceed analogously for the parameter \( k \). For \( N \) data points \( (y_1, y_2, \ldots, y_N) \) (written below as \( y \)) and for a fixed value of \( \sigma \) we determine the most likely value of \( k \) and a confidence interval for that parameter. At the end of the Section we combine both approaches to determine the optimum value for the pair \((k, \sigma)\) of parameters that characterize the \( \chi^2 \)-distributions in Eqs. (1) and (2) and the accuracy with which these can be determined. As in Section III we address first the full \( \chi^2 \) distribution (1) and then the distribution (2) with cutoff.

For the \( \chi^2 \)-distribution of Eq. (1), the normalized Bayesian likelihood function \( \Pi(k|y) \) is

\[
\Pi(k|y) = \frac{\mu(k) \prod_{n=1}^{N} P_{k,\sigma}(y_n)}{\int dk \mu(k) \prod_{n=1}^{N} P_{k,\sigma}(y_n)} .
\]

We consider \( k \) as a continuous variable in the range \( 0 < k < \infty \). A meaningful invariance requirement on the prior distribution \( \mu(k) \) is scale invariance so that \( \mu(k) = \mu_0/k \). Then the integral in the denominator extends from 0 to \( \infty \), and \( \mu_0 \) cancels in numerator and denominator.

We use Eq. (1) and write Eq. (31) as

\[
\frac{k^{-1}}{N} \left[(2\sigma^2)^{1/2}\Gamma(k/2)\right]^{-N} \exp\left\{-N\ln\left[(2\sigma^2)^{1/2}\Gamma(k/2)\right]\right\} \left(\frac{\ln y_n}{(2\sigma^2)^{1/2}}\right)^{N} \exp\left\{-\frac{N\ln y_n}{(2\sigma^2)^{1/2}}\right\} .
\]

Here \( N \) is the normalization factor. That factor is well defined because the Gamma function diverges sufficiently rapidly for both \( k \to \infty \) and \( k \to 0^+ \). The factors \((2\sigma^2)^{-N/2} \) and \( \exp\{-N\ln(2\sigma^2)\} \) are independent of \( k \), cancel in numerator and denominator, and are omitted in what follows. The likelihood function \( \Pi(k|y) \) depends on the \( N \) data points only via the parameter

\[
\mathcal{L} = \frac{1}{N} \sum_{n} \ln\left[\frac{\ln y_n}{(2\sigma^2)^{1/2}}\right] .
\]

For fixed \( \sigma^2 \), \( \mathcal{L} \) is determined by the geometric mean value of the data points. The \( N \)-dependent terms in the numerator of \( \Pi(k|y) \) are

\[
\exp\{-N \ln \Gamma(k/2) + N\ln(k/2) - 1\mathcal{L}\} .
\]

The function of \( k \) in the exponent has a maximum at \( k = k_0 \) where

\[
\mathcal{L} = \frac{d}{dx} \ln \Gamma(x) \bigg|_{x=(k_0/2)} .
\]

For \( x \to 0^+ \) (for \( x \to +\infty \), the logarithmic derivative of the Gamma function, written as \( f'(x) \), approaches \(-\infty \) \((+\infty \), respectively). In the interval \( 0^+ \leq x < \infty \), \( f'(x) \) increases monotonically with \( x \). Therefore, Eq. (35) always has a uniquely defined solution \( k_0 \). The accuracy with which \( k_0 \) is determined depends on the probability to find \( k = k_0(1 + \varepsilon) \) within some interval centered at \( k_0 \). That interval is obtained by expanding the exponent in expression (34) in powers of \( \varepsilon \). We write \( f''(k/2), f'''(k/2) \) for the higher-order logarithmic derivatives of the Gamma function. Keeping only terms of second and third order in \( \varepsilon \) we find

\[
\frac{-Nk_0}{2} \left( \frac{1}{2} f''(k_0/2) \right)^{\varepsilon^2} + \frac{1}{4} f'''(k_0/2) k_0^2 \varepsilon^3
\]

(36)
To evaluate expression (35) we use Sect. 8.362, no. 1 of Ref. [13],
\[
f'(x) = -\frac{1}{x} - \gamma - \sum_{n=1}^{\infty} \left( \frac{1}{n} + x \right)^{-n} \tag{37}
\]
Here \(\gamma \approx 0.577216\) is Euler’s constant. That gives
\[
f''(x) = \frac{1}{x^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},
\]
\[
f'''(x) = -\frac{2}{x^3} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^3}. \tag{38}
\]
We evaluate expression (39) for two representative values \(k_0 = 1\) and \(k_0 = 2\) of \(k_0\). With \(x = (k_0/2)\) Eqs. (38) yield
\[
f''(1/2) = 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \approx 4.9300,
\]
\[
f'''(1/2) = -16 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \approx -16.8288,
\]
\[
f''(1) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} = 1.6400,
\]
\[
f'''(1) = -2 \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \approx -2.4042. \tag{39}
\]
Expression (39) becomes
\[
\text{For } k = 1: \quad -\frac{N}{2} (1.23\varepsilon^2 - 0.70\varepsilon^3),
\]
\[
\text{For } k = 2: \quad -\frac{N}{2} (1.64\varepsilon^2 - 0.80\varepsilon^3). \tag{40}
\]
These figures are quite close to each other indicating that the constraints on \(N\) derived from Eqs. (10) hold similarly for all values of \(k\) in the interval \(1 \leq k \leq 2\) and beyond. In Section IIIA we have used the full analytical knowledge of the likelihood function in combination with an expansion as in expression (39) to determine the interval confining \(\sigma^2\). That cannot be done here because the normalization integral involves the Gamma function and is not available analytically. We resort to the rule (12) usually applied in that case. We request that the term of third order in expression (39) be less than or equal in magnitude to 1/10 times the term of second order everywhere in the interval \((-3, +3)\). When used for the figures in Section IIIA that rule yields \(N \geq 1070\), close to the value of \(N_0 = 900\) obtained there. For \(k = 1\) and expression (10) that gives \(N \geq 180\) and \(|\varepsilon| \leq 0.17\). For \(N = 400\) (representative for Ref. [1]) we find \(|\varepsilon| \leq 0.11\), for \(N = 50\) (representative for Ref. [2]) we find \(|\varepsilon| \leq 0.31\). In all cases, the upper bound on \(\varepsilon\) defines the interval \((1 - \varepsilon) \leq 1 \leq (1 + \varepsilon)\) confining \(k\). The expansion used in expression (39) is determined entirely by the logarithmic derivative of the Gamma function and independent of the actual value of \(\sigma^2\) and so are, therefore, the bounds on \(k\) just obtained.

With cutoff and for fixed \(\sigma\), the Bayesian likelihood function for the distribution (2) as function of \(k\) is
\[
\tilde{\mathcal{L}}(k|y) = \frac{\mu(k) \prod_{n=1}^{N} \tilde{P}_{k,\sigma}(y_n)}{\prod_{n=1}^{N} P_{k,\sigma}(y_n)}. \tag{41}
\]
We again assume that the cutoff \(y_0\) is the same for all data points \((y_1, y_2, \ldots, y_N)\) all of which obey \(y_n > y_0\). Then the numerator in Eq. (41) is
\[
\mu(k)[(2\sigma^2)^{1/2} \Gamma(k/2, y_0/(2\sigma^2))]^{-N} \exp\{-N m/(2\sigma^2)\} \times \exp\{N((k/2) - 1) \mathcal{L}\}. \tag{42}
\]
The incomplete upper Gamma function disappears rapidly for \(k \to \infty\) but takes a finite value for \(k \to 0^+\). Therefore, the normalization integral in Eq. (41) exists only if we choose \(\mu(k) = \mu_0\). We do so even though the problem lacks translational invariance. The analogue of Eq. (35) is
\[
\mathcal{L}(x) = \frac{d}{dx} \ln \Gamma(x, y_0/(2\sigma^2)) \bigg|_{x=(k_0/2)}. \tag{43}
\]
The logarithmic derivative of the incomplete upper Gamma function has a large negative value at \(k = 0\) and diverges for \(k \to \infty\). Therefore, a solution \(k_0\) of Eq. (43) always exists. The accuracy with which \(k_0\) is determined is defined in the same manner as in Eqs. (39) to (40). Details depend upon the actual choice of \(y_0\) and are not investigated here. We only remark that to account for the cutoff, it is probably convenient to write \(\Gamma(k/2, y_0/(2\sigma^2)) = \Gamma(k/2) - \gamma(k/2, y_0/(2\sigma^2))\) and to use Eqs. (39) for \(\Gamma(x)\), see Sect. 8.350, no. 1, of Ref. [12]. For the incomplete lower Gamma function \(\gamma(x, s)\) powerful series expansions are available. In any case there is no reason to assume that the results will differ significantly from those of the case without cutoff.

Combining the result of Section III with the present one is straightforward without cutoff. In Section IIIA the most likely value for \(\sigma\) was found to be \(\sigma = m/k\). That yields \(N \tilde{\mathcal{L}} = \sum_n \ln(y_n/2m)\), and Eq. (35) becomes
\[
\frac{1}{N} \sum_n \ln(y_n/m) + \ln(k_0/2) = f'(k_0/2). \tag{44}
\]
Eq. (44) allows for a direct determination of \(k_0\) from the input parameters. The uncertainty of \(\sigma^2\) values due to the finite number of data points causes via Eq. (33) the same uncertainty (percentagewise) of \(\mathcal{L}\). We estimate the resulting uncertainty in \(k_0\) using Eq. (35). The derivative of the right-hand side of that equation equals \(f''(k_0/2)/2\). At \(k = 1\), that has the value 4.93. Thus, the uncertainty of \(k_0\) is about 1/5 of that of \(\sigma^2\). That must be combined with the uncertainty in \(k_0\) estimated below expressions (40). For the data in Ref. [1], the uncertainty in \(\sigma^2\) (in \(k_0\)) amounts to 20 percent (10 percent, respectively), giving a total uncertainty for \(k_0\) of about 11 percent.

With cutoff, the situation is more complex. For small \(k\) we lack an analytical expression for the most likely value of \(\sigma^2\). For \(k \gg 1\) we may use \(\sigma^2 = m/k\) as before.
But for \( k \to 0 \), \( \langle y \rangle \) tends toward a nonzero value (and not to zero, as the relation \( k\sigma^2 = m \) would suggest). For purposes of orientation we may put \( M = (k + a)\sigma^2 \) and fit \( a \) to the result for \( k = 2 \) in Section IIIB. That gives \( M = k\sigma^2 + 2y_0 \). Here \( M \) is given by Eq. (19). We may use that expression for \( \sigma \) in Eq. (43). That yields a first guess for \( k_0 \). For higher accuracy, it is necessary to solve Eq. (21) numerically. We have shown, in any case, that for \( k = 1 \), \( \sigma^2 \) is equal to or smaller than about 0.4 \( M \). The influence of the cutoff on the uncertainty of \( \sigma^2 \) and \( k_0 \) is probably less strong. For \( \sigma^2 \) that was shown in Section III For \( k_0 \) it follows because in the interval \( 0 \leq k \leq \infty \), the derivatives of the Gamma function and of the upper incomplete Gamma function are, for cutoff values \( y_0/(2\sigma^2) \approx 1/2 \), quite close to each other except at the lower end points.

V. SUMMARY AND CONCLUSIONS

The \( \chi^2 \)-distributions in Eqs. 1 and 2 depend on the number \( k \) of degrees of freedom and on the parameter \( \sigma \) characterizing the width of the distributions. We have used Bayes’ theorem to determine these parameters from a set of \( N \) data points \((y_1, y_2, \ldots, y_N)\). The maxima of the two Bayesian likelihood functions, probability distributions for \( k \) and \( \sigma \), respectively, define the most probable values of these parameters. They yield \( k \) in terms of the logarithm of the geometric mean value of the data points and as a function of \( \sigma \), and \( \sigma \) as the arithmetic mean value \( m \) of the data points and as a function of \( k \). The combination of both yields unique values for \( k \) and \( \sigma \). The width of each of the two Bayesian likelihood functions around their maxima depends on the number \( N \) of data points. Requesting that \( k \) lies with a given probability within a given interval centered at the maximum of its Bayesian likelihood function, fixes the minimum number \( N \) of data points needed for \( k \), and correspondingly for \( \sigma \). Conversely, the number \( N \) of data points determines the confidence intervals for \( k \) and \( \sigma \). As a rule of thumb we have shown that for realistic values of \( N \), the confidence interval for \( \sigma^2(k) \) has a width of about \( \pm 20 \) percent (\( \pm 10 \) percent, respectively) around the maximum.

The results for \( k \) obtained in Refs. 1, 2 lie outside the range for \( k \) determined above for the Porter-Thomas distribution, roughly given by \( 0.89 \leq k \leq 1.11 \). That seems to confirm the rejection of the Porter-Thomas distribution in Refs. 1,2. However, in Refs. 1, 2 the parameter \( \sigma^2 \) in the \( \chi^2 \)-distribution with cutoff is replaced by the value \( \sigma^2 = M/k \), and the result is used in a maximum-likelihood analysis with the aim to find the value of \( k \) that best fits the data. Here \( M \) is the arithmetic average of the data points all of which are bigger than the cutoff. That replacement is incorrect because \( M \) is bigger by a factor \( c > 1 \) than \( m \), the arithmetic mean without cutoff. The correct relation is \( \sigma^2 = m/k = M/(ck) \). That reduces by \( 1/c \) the value of \( \sigma^2 \) used in Refs. 1, 2. For \( k = 4 \) and \( k = 2 \) the factor \( c \) was found to be about 1.3 and 2.0, respectively. For \( k = 1 \) we expect \( c \approx 2.5 \). That is a major change which is likely to affect the analysis of Refs. 1, 2 in a substantial manner and casts doubt on the results and the conclusions.

Why did Koehler et al. and Koehler choose a value for \( \sigma^2 \) that is so far outside the Bayesian confidence interval? Refs. 1 and 2 use the \( \chi^2 \)-distribution for \( k \) degrees of freedom written as in Eq. (1) of Ref. 2. In our notation and for s-wave resonances that equation reads

\[
P_k = \frac{k}{2^{k/2}(2\sigma^2)^{k/2}} \exp \left\{ -\frac{ky}{2\sigma^2} \right\}. \quad (45)
\]

Eq. (45) is obtained from Eq. (1) upon replacing \( \sigma^2 \to \langle y \rangle/k \). While Eq. (1) holds generally, such replacement and the resulting Eq. (45) hold only under two restrictive conditions. (i) The number \( N \) of data points is large. The resulting Bayesian confidence interval for \( \sigma^2 \) is so narrow that \( \sigma^2 \) may be replaced by the most likely value. (ii) The most likely value of \( \sigma^2 \) is given by \( \langle y \rangle/k \). That condition (i) is not met by the data sets in Refs. 1 and 2. That is addressed in the paragraph that follows. More importantly, condition (ii) holds for the case without cutoff (see Eq. (45)) but only for that case. With cutoff, \( \sigma^2 \) is determined by the implicit Eq. (21). We have shown that the result differs markedly from \( \langle y \rangle/k \). Using Eq. (45) as the starting point of a maximum-likelihood analysis as done in Refs. 1 and 2 for the case with cutoff is flawed from the beginning. Combined with the replacement \( \langle y \rangle \to M \) it implies the guess \( \sigma^2 = M/k \) which is incorrect.

Replacing \( \sigma^2 \) in the \( \chi^2 \)-distribution by some fixed value entails a second problem. Such replacement is correct only in the limit \( N \to \infty \) where the maximum of the Bayesian likelihood function is completely sharp. For finite \( N \) the Bayesian likelihood function defines an \( N \)-dependent interval of acceptable values for \( \sigma^2 \). That interval may be fairly wide. For \(^{194}\text{Pt}\) we have \( N = 411 \). For \( N = 400 \) and \( k = 1 \) Eq. (19) yields \( N_0 = 200 \). Putting \( \sqrt{N}/N_0 = 2.9 \) as at the end of Section IIIA we obtain a probability \( p = 0.9980 \) for \( \sigma^2 \) to differ by less than \( \pm 20 \) percent from its most likely value. For the data in Ref. 2 the size of the \( \sigma^2 \)-interval increases by factors that lie between 4.5 and 2. A set of \( N \) data points corresponds to a pair \( (\sigma^2, k) \) of values that lie somewhere in these intervals. Different sets of \( N \) data points determine different pairs \( (\sigma^2, k) \) of values. It is, therefore, incorrect to determine \( k \) by choosing arbitrarily some value of \( \sigma^2 \) within the given interval. Both \( \sigma^2 \) and \( k \) must be determined from the given data set. That can be done, for instance, with the help of the maximum-likelihood analysis of Refs. 1, 2.

Within the intervals defined by \( N \), that determines to which \( \chi^2 \)-distribution the data belong.

Using the method of Ref. 1 for a simultaneous determination of \( \sigma^2 \) and of \( k \) may be realistic for the Pt isotopes 1 where the number of data points is large. The Nuclear Data Ensemble 2 comprises 24 nuclides. In all but two of these, the number \( N \) of resonances is between 17 and 109. For each nuclide, \( \sigma^2 \) and \( k \) must be determined independently. That is necessary because
the mean value of the input parameters $y_a$ (and, therefore, $\sigma^2$) depends upon the neutron strength function which varies with mass number. Thus, application of the method of Ref. [1] to the Nuclear Data Ensemble poses a challenge.

In summary, the analysis of Refs. [2][3][4] is unsatisfactory as it stands. The results are inconclusive. A reevaluation is needed that determines both \(\sigma^2\) and \(k\) from the data and compares the result with confidence intervals derived from the Bayesian analysis. Only on that basis will it be possible to decide whether the data support the Porter-Thomas distribution, or whether that distribution must be rejected.

Acknowledgement. We acknowledge useful correspondence by P. Koehler.

APPENDIX: NORMALIZATION INTEGRALS

The normalization integral \(N_0\) in Eq. [11] is calculated separately for \(\alpha = 0\) and \(\alpha = 1\). For \(\alpha = 0\) we substitute \(z = N_0 x^{-2}\), \((1/z)dz = -(2\sqrt{N_0}/x^2)dx\). That yields

\[
N_0 = \int_0^\infty \frac{dz}{\sqrt{\pi}} z^{-N_0} \exp\{-N_0/z\}
= N_0^{(1/2)-N_0} \int_{-\infty}^{+\infty} dx \ (x^2)^{-N_0-1} \exp\{-x^2\}
= N_0^{(1/2)-N_0} \left(\frac{\partial N_0^{-1}}{\partial \beta}\right)_{\beta=1}
\times \frac{\partial N_0^{-1}}{\partial \beta}\int_{-\infty}^{+\infty} dx \ \exp\{-\beta x^2\}
= \sqrt{\pi} N_0^{(1/2)-N_0} (-N_0-1) \left(\frac{\partial N_0^{-1}}{\partial \beta}\right)_{\beta=1}
= 2\sqrt{\pi} N_0 (2N_0)^{-N_0} (2N_0-3)! \ .
\]

(46)

We use Stirling’s formula,

\[
n! = c(n) \ n^{n+(1/2)} \exp\{-n\} \ , \ \text{with} \ \sqrt{2\pi} \leq c(n) \leq e \ \text{for all} \ n .
\]

That gives

\[
(2N_0-3)! = \frac{(2N_0-3)!}{2^{N_0-2}(N_0-2)!}
= \frac{c(2N_0-3)}{c(N_0-2)} \frac{(2N_0-3)^{2N_0-3+(1/2)}}{2^{N_0-2}(N_0-2)^N_0-3}/2) \exp\{1-N_0\}
\approx \frac{c(2N_0-3)}{c(N_0-2)} \exp\{1-N_0\}
\times \exp\{(N_0-1) \ln N_0 + (N_0-1/2) \ln 2 - 1\} .
\]

The normalization integral becomes

\[
N_0 = \frac{c(2N_0-3)}{c(N_0-2)} \sqrt{\frac{2\pi}{N_0}} \exp\{-N_0\} .
\]

(49)

For \(\alpha = 1\) we substitute \(z = N_0 x/x\), \(dz = -(N_0/x^2)dx\) and obtain, using Stirling’s formula [17],

\[
N_1 = N_0^{-N_0} \int_0^\infty dx \ x^{N_0-1} \exp\{-x\}
= N_0^{-N_0} (N_0-1)! \approx \frac{c(N_0-1)}{\sqrt{N_0-1}} \exp\{-N_0\} .
\]

(50)

For \(k\) even and \(s = (k/2) - 1 = 0, 1, 2, \ldots\) the upper incomplete Gamma function is given by

\[
\Gamma(k/2, y_0/(2\sigma^2)) = \int_{y_0/(2\sigma^2)}^\infty t^{s} \exp\{-t\} \ dt
= (-)^s \frac{\partial^s}{\partial \beta^s} \exp\{-\beta y_0/(2\sigma^2)\}\bigg|_{\beta=1}
= \beta_s \exp\{-y_0/(2\sigma^2)\} .
\]

(51)

Here \(\beta_s\) is a polynomial in \(y_0/(2\sigma^2)\) of order \(s\), with

\[
\beta_0 = 1 , \ \beta_1 = 1 + y_0/(2\sigma^2) .
\]

(52)

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