Quantum phase space picture of Bose-Einstein Condensates in a double well

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We present a quantum phase space model of Bose-Einstein condensate (BEC) in a double well potential. In a quantum two-mode approximation we examine the eigenvectors and eigenvalues and find that the energy correlation diagram indicates a transition from a delocalized to a fragmented regime. Phase space information is extracted from the stationary quantum states using the Husimi distribution function. We show that the mean-field phase space characteristics of a nonrigid physical pendulum arises from the exact quantum states, and that only 4 to 8 particles per well are needed to reach the semiclassical limit. For a driven double well BEC, we show that the classical chaotic dynamics is manifest in the dynamics of the quantum states. Phase space analogy also suggests that a π phase displaced wavepacket put on the unstable fixed point on a separatrix bifurcates to create a superposition of two pendulum rotor states - a macroscopic superposition state of BEC. We show that the choice of initial barrier height and ramping, following a π phase imprinting on the condensate, can be used to generate controlled entangled number states with tunable extremity and sharpness.

I. INTRODUCTION

Although Bose-Einstein condensate (BEC) is well described by mean-field theory [1], it has many aspects that can only be described in a quantum picture containing a proper description of correlations. Examples include number squeezing [2] and the superfluid to Mott insulator transition [3] observed recently in optical lattices. The essential underlying physics can be understood with the study of a simpler double well BEC with a variable barrier height in the well known quantum two-mode approximation [6,7]. Quantum fluids in a double well potential exhibit many rich phenomena related to the coherence, e.g. the Josephson effect [8] and the deBroglie wave interference [9]. A mean-field description although appropriate in explaining these ‘Josephson related effects’ cannot describe the ‘number squeezing effects’ described earlier. In this paper we develop a quantum phase space picture of BEC in a double well and study the connection between the mean-field and quantum effects. As important applications of our model, we investigate dynamics in phase space, study quantum manifestations of classical chaos in a driven double well, and show dynamic generation of tunable entangled number states with well defined and controlled entanglement.

It was long ago noted by Anderson [11] that the Josephson effect, namely two quantum fluids connected by a tunnel junction [12], may be modeled as a physical pendulum. Similarly, Smerzi et al. in Refs. [13] showed that the semi-classical (large N) dynamics of two weakly linked BECs can be modeled as a classical nonrigid physical pendulum. We begin, here, with the full quantum mechanical description of a double well BEC in a two-mode approximation [6,7,14], and show that the mean-field semiclassical limit of a nonrigid physical pendulum emerges from the exact quantum treatment. By treating the phase and the number difference of the condensates in two wells as conjugate variables, phase space information is extracted from the exact (two mode) quantum wavefunction using the Husimi projection [15] of semi-classical quantum mechanics. We show that these phase space projections of exact quantum eigenstates are localized on the known classical energy contours of the nonrigid physical pendulum [13], and thus the mean-field classical phase space properties, such as libration and π states, are seen to be a property of the exact quantum eigenstates. We explore quantum classical correspondence for the stationary states in phase space as a function of particle number, and show that the semiclassical limit already emerges for particle numbers as small as 4 to 8 per well.

The quantum phase space model also reveals an underlying time dependent semi-classical dynamics in phase space. In a study of the dynamics of a displaced coherent state, we show a surprisingly close correspondence between classical whorls and quantum dynamics even for N as small as 4 per well. We further illustrate that a sinusoidally driven double well BEC (a driven physical pendulum) shows clear signatures of classical chaos in the quantum phase space. This can be contrasted with a different property of a chaotic system - the recently observed phenomenon of dynamical tunneling [17,18], which is a quantum motion between two resonance zones in phase space not allowed within the classical dynamics. We also discuss the dynamics of a coherent ground state after a sudden change of barrier height [2,10]. We show that the oscillations between a number squeezed and a phase squeezed state is a rotation of a pulsing ellipse in the phase space.

Due to the macroscopic nature of its wavefunction, BEC should be an ideal system for the generation of macroscopic quantum superposition states (Schrödinger cat states). The creation of macroscopic superposition states in various condensed matter systems has received attention [19]. In the context of BEC, several authors have suggested producing such states [20–23], although none have been demonstrated experimentally. We show how such macroscopic quantum superposition states are
generated in phase space with a single component BEC in a double well. Starting with a ground state centered at the origin and displacing it through a \( \pi \) phase imprinting to put it on the hyperbolic fixed point of the classical phase space, the autonomous dynamics splits the wavepacket along the separatrix to create entangled number states of the form

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|n_L, N-n_L\rangle + |N-n_L, n_L\rangle) \tag{1}
\]

where \(|n_L, n_R\rangle\) denotes a state with \(n_L\) particles in the left well, \(n_R\) in the right well, and \(N = n_L + n_R\). The idea of the exploitation of unstable fixed points to generate such entangled states with BEC in a double well and spinor condensates in a single trap has also been discussed in the works of Polkovnikov et al. [22] and Micheli et al. [23], a discussion of which is given in Sec. V. Unlike in other proposals [20,22,23], we use the barrier height to control the squeezing of the initial BEC ground state, followed by a continuous change of barrier height, to control both the extremity (the value of \(n_L\)) and the sharpness (the spread around \(n_L\)) of the final entangled state. A very simple particle loss scheme [21] is used here to test the robustness of the entangled states.

The article is organized as follows. In Sec. II we introduce the model Hamiltonian, and examine its ground and excited states. In Sec. III we find the Husimi probability distribution function for the quantum states, show that the quantum states are localized on the classical phase space orbits of a known nonrigid physical pendulum. In Sec. IV we analyze phase space dynamics for a displaced wavepacket, study chaotic dynamics of a driven double well and explain phase space rotation of a ground state. In Sec. V we provide a phase space analysis of the generation of tunable entangled states. Remarks and summary in Sec. VI conclude the paper.

II. QUANTUM TWO-STATE MODEL

A. Model Hamiltonian

The many-body Hamiltonian for a system of \(N\) weakly interacting bosons in an external potential \(V(r)\), in second quantization, is given by

\[
\hat{H} = \int dr \hat{\Psi}^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \hat{\Psi}(r) + \frac{g}{2} \int dr \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r) \hat{\Psi}(r) \hat{\Psi}(r) \tag{2}
\]

where \(\hat{\Psi}(r)\) and \(\hat{\Psi}^\dagger(r)\) are the bosonic annihilation and creation field operators, \(m\) is the particle mass, and \(g = \frac{4\pi a_s \hbar^2}{m}\) where \(a_s\) is the s-wave scattering length.

In studies of double-well BEC or two-component spinor condensates, the low-energy many-body Hamiltonian in Eq. (2) can be simplified in the well-known two-mode approximation [6,7]. Many authors have studied the double-well condensate using the two-mode approximation. We use the model introduced by Speckens and Sipe [7]. The exclusion of the nonlinear tunneling terms in this model gives rise to the Bose-Hubbard model [4]. The full two-mode Hamiltonian is

\[
\hat{H} = \epsilon_{LL} \hat{N}_L + \epsilon_{RR} \hat{N}_R + (\epsilon_{LR} + gT_1(\hat{N} - 1)) \times (a_L^\dagger a_R + a_R^\dagger a_L) + \frac{2\hbar^2}{m} (\hat{N}_L^2 + \hat{N}_R^2 - \hat{N}) \tag{3}
\]

where \(\hat{N}_L = a_L^\dagger a_L\), \(\hat{N}_R = a_R^\dagger a_R\), \(\hat{N} = \hat{N}_L + \hat{N}_R\) and

\[
\epsilon_{ij} = \int dr \phi_i(r) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \phi_j(r) \tag{4}
\]

where \(i, j = L, R\).

\[
T_0 = \int dr \phi_L^\dagger(r) \phi_L^\dagger(r) \phi_L(r) \phi_R(r); \quad T_1 = \int dr \phi_L^\dagger(r) \phi_R(r) \phi_L(r) \phi_R(r); \quad T_2 = \int dr \phi_L^\dagger(r) \phi_R(r) \phi_R^\dagger(r) \phi_L^\dagger(r) \tag{5}
\]

Here \(\phi_L\) and \(\phi_R\) are the left and right localized single particle Schrödinger wavefunctions, the \(\epsilon_{LL}\) and \(\epsilon_{RR}\) are the energies of a single particle in the left and right wells, \(\epsilon_{LR}\) is the single particle tunneling amplitude; \(T_0\) is the mean-field energy in each well and \(T_{1,2}\) are nonlinear tunneling matrix elements.

We make a one parameter approximation [14] of the single particle energies and the tunneling matrix elements:

\[
g = 1; \epsilon_{LL} = \epsilon_{RR} = T_0 = 1; \epsilon_{LR} = T_1 = -e^{-\alpha}; \quad T_2 = -e^{-2\alpha}. \tag{6}
\]

This parametrization allows a simple study of continuous change in the linear and non-linear tunneling through variation of a single parameter \(\alpha\). In our computations with this model we ignore the \(T_2\) term which scales as \(\exp(-2\alpha)\). The model Hamiltonian then reduces to

\[
\hat{H} = \epsilon_{LL} \hat{N}_L + \epsilon_{RR} \hat{N}_R + (\epsilon_{LR} + gT_1(\hat{N} - 1)) \times (a_L^\dagger a_R + a_R^\dagger a_L) + \frac{2\hbar^2}{m} (\hat{N}_L^2 + \hat{N}_R^2 - \hat{N}) \tag{7}
\]

B. Fock State Analysis

The most general state vector is a superposition of all the number states

\[
|\Psi\rangle = \sum_{n_L=0}^{N} c_{n_L}^{(i)} |n_L, N-n_L\rangle \tag{8}
\]

where
Finding the eigenvalues and eigenvectors of the model Hamiltonian in the Fock basis can be easily accomplished by diagonalizing a \((N + 1) \times (N + 1)\) tridiagonal matrix. Authors in Ref. [7] studied condensate fragmentation by looking at the ground state as the barrier is raised. We extend their analysis to look at the coefficients of the higher lying states and examine the energy correlation diagram. Fig. 1 shows all 21 eigenvalues for a system of 20 particles in a double well for \(\alpha\) ranging from 0 to 5. For this range of \(\alpha\), the tunneling parameters vary from 1 to 0.0067, going from a low barrier to a high barrier leading to a fragmented condensate with fixed number of particles in each well. The correlation diagram shows avoided crossings and energy level merging. As \(\alpha\) increases the levels start to get doubly degenerate; at a value of about \(\alpha = 1.8\) the highest levels are degenerate, and all but the ground state is degenerate for higher values of \(\alpha\).

Looking at the coefficients of eigenvectors reveals interesting characteristics of the ground and excited states. Fig. 2(a) and (b) shows the coefficients of the eigenvectors for the two lowest lying states for 40 particles. The lowest delocalized states appear to be like the co-ordinate space wave functions of a harmonic oscillator. These are the states that are below the crossover ridge in a correlation diagram as in Fig. 1. For states over the ridge, a similar list of coefficients for two higher lying states are shown in Figs. 2 (c) and (d). These do not look like the harmonic oscillator wave functions. These are examples of states that are superpositions of a macroscopic number of particles on left and right well. For these nearly degenerate Schrödinger cat-like even and odd states, a very high precision arithmetic is required to get the coefficients.

III. QUANTUM MECHANICAL PHASE SPACE ANALYSIS

A. Classical Hamiltonian

The classical Hamiltonian that describes the mean-field dynamics of BEC in a double well has been analyzed in several papers [13,24]. In a mean-field assumption [1] for the two-mode double well, and for large enough \(N\), the operators \(\hat{a}_j\) can be replaced by the c-numbers \(\sqrt{\langle n_j \rangle} e^{i\theta_j}\), where \(j = L, R\). With this assumption and defining \(n = n_L + n_R\), \(\theta = \theta_L - \theta_R\), and starting with our model Hamiltonian Eq. (7) gives the classical Hamiltonian

\[
H_{cl} = E_c n^2 - E_J \sqrt{1 - \left(\frac{2n}{N}\right)^2} \cos \theta + \frac{E_c}{4} N^2 - \frac{E_c}{2} N + \epsilon_{LL} N_L + \epsilon_{RR} N_R
\]  

(10)

where \(E_c = g T_0\) and \(E_J = -N (\epsilon_{LR} + g T_1 (N - 1))\). Here \(n\) and \(\theta\) are conjugate variables and the equations of motion are

\[
\dot{n} = -E_J \sqrt{1 - \left(\frac{2n}{N}\right)^2} \sin \theta
\]  

(11a)

\[
\dot{\theta} = 2 E_J n + \frac{4 E_J n}{\sqrt{1 - \left(\frac{2n}{N}\right)^2}} \cos \theta
\]  

(11b)

Eq. (10) is the Hamiltonian of a nonrigid physical pendulum where \(n\) and \(\theta\) are the angle and angular momentum of the pendulum. The phase space of a nonrigid physical pendulum allows novel dynamical regimes such as the macroscopic quantum self-trapping (MQST) and \(\pi\)-motions [13]. MQST refers to the incomplete oscillations of the populations between the two wells. \(\pi\)-motion refers to oscillations such that the average relative phase remains \(\pi\).

B. Husimi Distribution Function

Since the phase-space distribution function allows one to describe the quantum aspects of a system with as much classical language allowed, it is a popular tool to study semi-classical physics. Among the most popular distribution functions used are the Wigner distribution, Husimi distribution, and the Q-function [15,25]. They are all related - the Q-function is a special case of Husimi distribution function, and a smoothing of the Wigner function with a squeezed Gaussian gives the Husimi distribution [15].

Husimi distribution function can be used to project, in a squeezed coherent state representation, the classical \((q,p)\) phase space behavior from a stationary quantum wavefunction. Coherent state representation of the electromagnetic field, where \(n\) and \(\theta\) are conjugate variables corresponding to the number and phase of the electromagnetic fields were introduced by Glauber [26]. The \((q,p)\) coherent state [27] is defined as

\[
|\beta\rangle = e^{(-|\beta|^2/2)} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle
\]  

(12)

which is a superposition of the harmonic oscillator eigenstates \(|n\rangle\), here \(\beta = q + ip\). For BEC in a double well, the phase difference \(\theta = \theta_L - \theta_R\) and the number difference \(n = n_L - n_R\) are the conjugate variable analogous to \(q\) and \(p\) respectively. Therefore, in \((n,\theta)\) representations, the coordinate and momentum representations of a squeezed coherent state is

\[
\langle \theta' | \theta + in \rangle = \frac{1}{(\pi\kappa)^{1/4}} \exp[-i n \theta' - \frac{(\theta' - \theta)^2}{2\kappa}]
\]  

(13)

\[
\langle n' | \theta + in \rangle = \frac{1}{(\pi\kappa)^{1/4}} \exp[-i \theta n' - \frac{(n' - n)^2}{2\kappa}]
\]  

(14)
In this representation a probability distribution function can be defined as

\[ P_j(n, \theta) = |\langle \theta + in|\Psi_j \rangle|^2 \]

where

\[ \langle \theta + in|\Psi_j \rangle = \frac{1}{(\pi \kappa)^{1/4}} \sum_{n'=-N/2}^{N/2} c_{n'}^{\dagger} \exp\{i\theta n' - \frac{(n' - n)^2}{2\kappa}\} \]

Here \( n' = \frac{n_L - n_R}{2} \), rather than being the simpler left particle counter, and \( c_n \) is the corresponding Fock-state coefficient. Husimi function is defined for any value of the squeezing parameter \( \kappa \). The Q-function in quantum optics is a special case of Husimi distribution function whenever \( \kappa = \omega \), where \( \omega \) is the frequency of a coherent state Gaussian wavepacket [15]. The ‘coarse-graining’ parameter \( \kappa \) determines the relative resolution in phase space in the conjugate variables number and phase.

C. Quantum Classical Connection for the Eigenstates

It is natural to ask what aspects of the mean-field phase space properties of a nonrigid physical pendulum [13] are contained in the exact quantum treatment. We explore that question here by investigating the ground and excited states of the two-mode quantum Hamiltonian, and extracting phase space information through the use of the Husimi distribution function. Fig. 3(a) shows the classical energy contours for 40 particles for parameter values \( \alpha = 4, \ g = 1, \ T_0 = 1 \). For these same parameters, Fig. 3 panels (b), (c), (d) and (e) show the Husimi distributions for the ground state, 6th, 12th and 35th states respectively. The Husimi projections confirm the physical pendulum characteristics of the eigenstates. As is evident from the panels, the ground state is a minimum uncertainty wave-packet in both number and phase that is centered at the origin, the harmonic-oscillator-like low lying excited states are the analog of pendulum librations, and the higher lying cat-like states are the analog of pendulum rotor motions, with a clear signature of the quantum separatrix state where the libration and rotation states separate.

A systematic exploration is made of the quantum classical correspondence in phase space for different number of particles. Fig. 4 shows the Husimi distribution for \( N = 16, 8, 4, 2 \) in panels (a), (b), (c) and (d) respectively. For each of the particle numbers, it shows the ground state, a low lying oscillator state, a higher lying separatrix state, and a macroscopic superposition state. Although the classical energy contours (as shown in Fig. 3(a)) are the same for all different particle numbers, we see here that for \( N = 4 \) and \( N = 2 \) the minimum uncertainty spread of the eigenstates blur the clear signature of a pendulum phase space structure. It is interesting to note that only 4 particles per well particles are needed to reach the semi-classical limit where the classical phase space structure is evident. For a very large number of particles the Husimi distributions of the eigenstates become sharper approaching the classical limit of a line trajectory.

A fundamental difference between the classical trajectories and the quantum states is visible in the rotor state in Fig. 3(e) which is a superposition of most particles in the left and right wells. In the classical sense this corresponds to two different trajectories corresponding to rotor motions of a physical pendulum in two opposite directions. The quantum states always maintain the parity of the Hamiltonian and hence the combinations of two such classical trajectories make up a quantum state. The localized motion corresponding to one classical trajectory is known as macroscopic quantum self-trapping (MQST) [13]. Such parity violating states also appear as stationary solutions of the Gross-Pitaevskii equation in a double well [28].

In order for the quantum Hamiltonian to correspond to a momentum-shortened physical pendulum, there should exist \( \pi \) type motions [13] among the quantum states. A change in the parameters to \( \alpha = 4, \ g = 0.1 \) and \( T_0 = 0.1 \), puts us in a slightly different regime as shown in Fig. 5(a) showing dynamical regimes with an average phase difference of \( \pi \). The Husimi projections in panels (b), (c), (d) and (e) are respectively for the 12th, 30th, 34th and the 41st state. Here the higher-lying quantum states are the analog of \( \pi \)-motions of the mean-field classical Hamiltonian. Again only 4 to 8 particles per well are needed to reach the semiclassical limit.

IV. DYNAMICS IN PHASE SPACE

A. Comparison of classical and quantum dynamics

To illustrate the applications of the quantum phase space picture, here we make a comparison of the quantum and classical phase space dynamics. Investigation of the quantum classical correspondence in phase space by approximating a Gaussian wavepacket with a swarm of points in the classical phase space, although widespread in quantum chaos literature, has not been performed for BEC. This type of comparison between non-averaged quantities contains the maximum amount of information allowed. By approximating the quantum wavepacket with a swarm of points in the classical phase space, the mean-field and quantum dynamics is compared for 8 particles in Fig. 6. The first column shows the quantum dynamics in Husimi projection space for a \( N/4 \) displaced wavepacket, and the second column shows the corresponding classical points initially, after the first, second and fourth cycles respectively. The effects of dephasing is
apparent in the quantum phase space in panels (e) and (g). The classical trajectories develop a narrow whorl-type structure as shown in Fig. 6(f) and (h). Surprisingly even for such small number of particles the classical and quantum dynamics is comparable; the quantum states are localized in the region of the classical points with high phase space density. For a longer time scale the whorls become more convoluted and finer, and the quantum dynamics shows prominent interference effects such as recurrences as discussed next.

Schrödinger [29] first pointed out that quantum time evolution of a displaced harmonic oscillator ground state led to a minimum uncertainty wavepacket which evolves in time following its classical phase space trajectory without any spreading. In the nonlinear pendulum considered here, a ground state displaced by a small amount will evolve in phase space without much spreading. However a state which is farther from the origin will show the effects of nonlinearity and quantum interference a lot quicker. After the full delocalization occurs, the interference effects become pronounced for longer times. Localized peaks appear which again delocalizes with the appearance of new peaks. Fig. 7 shows such fractional revivals [30,31] in the Husimi projection space for N=40.

B. Quantum-Classical correspondence for classically chaotic dynamics

In the context of chaotic dynamics in BEC, dynamical tunnelling of untracold atoms from a BEC in a modulated periodic potential has been observed [17], and a theoretical study of a similar system has been done using the Floquet operator [18]. These authors showed that exact quantum dynamics of the system can exhibit classically forbidden tunnelling between two regular regions in the corresponding classical phase space, a phenomenon known as dynamical tunneling [16]. Here we study instead the similarities in the dynamics in the classical and quantum phase space. A driven pendulum is a well known example of a one and half degree of freedom classical system exhibiting chaos. For an analogous system of a driven double well BEC, we make a comparison of the quantum states at different times with the corresponding classical trajectories and illustrate signatures of quantum chaos. Such comparison is done in phase space most usefully between the Husimi projection of a quantum state and the corresponding classical band of points initially in the same region of phase space [33]. For a diagnostic to the classical phase space, Fig. 8 shows the Poincaré section for 200 particles and $\alpha = 2.5 + 2.5\cos(10t)$. As the amplitude of the driving force becomes larger the whole phase space becomes chaotic.

For comparisons in the chaotic region, the Husimi distribution of the superpositions of 128th and 129th eigenstate at different times are shown in Fig. 9 on the right panels. The classical trajectories of similar points are shown in the left panels. At shorter time $t=0.17$ as shown in Fig. 9(c) and (d), the quantum state very nicely follows the classical points. Panels (e) and (f) are a comparison for points showing a visibly chaotic yet localized pattern both in the classical and quantum phase space. The effect of chaotic dynamics fully takes effect at $t=5$ when the classical phase space points are diffused throughout the whole region as shown in (g). A comparison with the Husimi projections in (h) makes evident the manifestations of chaos in the quantum dynamics. A state initially localized in the regular regions of phase space does not give rise to such chaotic structures. In the limit when $\hbar \to 0$ or equivalently $1/N \to 0$, the discrete quantum energy spectrum becomes continuous and the quantum mechanics will more closely follow classical mechanics; any evidence of chaos in the quantum dynamics will be better represented in such comparisons.

C. Relative number and phase squeezing

Ground state number-squeezing with a variable barrier height in double and multi-well systems has been discussed and observed by many authors [2,3,7]. The case of a sudden change of barrier height on a coherent ground state, which we analyze here, has been discussed on a theoretical basis [2,10]. In Ref [10], the authors consider the evolution, in the space of number differences, of an initially perfect binomial number distribution state, and find that for an optimal value of parameters in the Hamiltonian, the initial state periodically evolves to a relatively number squeezed state.

We perform here a quantum phase space analysis of this phenomenon, and find this to be a propety of coherent ground state evolving under a Hamiltonian for which it is not an eigenstate. We show that the initial state rotates in the number-angle phase space and thus becomes elongated or well defined in number and phase periodically. We illusrate this with an example: the ground state for $\alpha = 0$ very closely approximates a state with a binomial distribution of Fock state coefficients. With a sudden raising of barrier to $\alpha = 3$, we follow the evolution of the state in phase space. The initial coherent ground state is not an eigenstate of the changed potential and hence will time evolve accordingly. As shown in the quantum phase space in Fig. 10 panels (a), the initial state is rather well defined in phase ($\theta$) and elongated in number difference ($n$). Further evolution in the new potential rotates the elongation in phase space such that after a certain period it becomes well defined in $n$ (as in panel (d)) or it is relatively number squeezed. A full cycle is shown in Fig. 10; in (f) the evolution brings it back to the initial coherent state.
V. GENERATING TUNABLE ENTANGLED STATES USING PHASE ENGINEERING

The quantum phase space model presented here points to a simple way that an entangled state can be generated with a single component BEC in a double well. A wave-packet π phase displaced to the unstable hyperbolic fixed point of a classical phase space bifurcates along the separatrix if allowed to time evolve. With the above motivation, here we provide a visual explanation in phase space of the creation of controlled entangled number states of a BEC in a double well via phase imprinting on the part of the condensate in one of the wells followed by a continuous change of barrier height. When properly implemented this results in a state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|n_L, N-n_L\rangle + |N-n_L, n_L\rangle)$$

(17)

where \(|n_L, n_R\rangle\) denotes a state with \(n_L\) particles in the left well, \(n_R\) in the right well, with total number of particles \(N = n_L + n_R\). Unlike in other proposals [20–23], we can use the barrier height to control the squeezing of the initial BEC ground state followed by a continuous change of barrier height to control both the extremity (the value of \(n_L\) \((n_L = 0, 1, 2...N)\)) and the sharpness (the spread around \(n_L\)) of the entangled state. An extreme entangled state would correspond to \(n_L = 0\) or \(N\).

Writing phases on part of a condensate is experimentally feasible via interaction with a far-off-resonance laser. This method has been used to generate dark solitons and measure their velocities due to a phase offset [34]. Mathematically, such a method corresponds to multiplying the coefficient of each of the Fock states in the expansion of an eigenstate by \(e^{in\ell \theta}\), where \(|n_L\rangle\) is the corresponding Fock state, and \(\theta\) is the phase offset for particles in the left well. By \(\pi\) phase imprinting the condensate in one well, the ground state centered at the origin \((0,0)\) in phase space is displaced to the unstable equilibrium point \((0,\pi)\) on the separatrix. Using exact quantum time evolution within the framework of the two mode model, the resulting quantum wave-packet bifurcates as expected. If the barrier is raised as discussed below, the wave-packet is permanantly split, resulting in a superposition of two classical rotor states.

A. Entangled state generation without decoherence

In the situation when there is no decoherence, well controlled entangled states can be generated within the two-mode quantum dynamics. As an example, Figs. 11 show how a number entangled state with 1000 particles is generated. Fig. 11 shows the evolution in phase space using Husimi projections - (a) the ground state, (b) a \(\pi\)-phase imprinted state, (c) and (d) show subsequent evolution in the process of bifurcating the state; further evolution along with a change of barrier totally splits and traps the state symmetrically above the separatrix, as shown in (e), finally giving rise to an entangled state in (f). Here the barrier height is ramped up in time as \(\alpha = 3 + 2t\). When an entangled state is reached the barrier is suddenly raised to essentially halt the evolution. With different initial barrier heights and the same ramping of the potential, the extremity of the entangled states can be tuned. Examples are shown in Fig. 12 where the different values of the barrier heights are \(\alpha = 1 + 2t\), \(\alpha = 3 + 2t\) and \(\alpha = 5 + 2t\) for rows (1), (2) and (3) respectively. The columns show: (a) the barrier height and the ramping, (b) the respective ground state, (c) the final entangled state at the end of the ramping, and (d) a close view of the coefficients for the final state shows that these are rather sharply peaked entangled states. As is evident from the pictures, the initial squeezing of the ground state determines the extremity of the final entangled state. The rate at which the barrier is ramped determines the sharpness.

B. Entangled state generation with loss

Macroscopic superposition states are not observed mainly due to interaction with the environment. In elastic collisions where the total number of atom is conserved, phase damping destroys the quantum coherence [35]. In the case where the number of particles are not conserved, the loss of even a single particle destroys an extreme entangled state [21], as can be seen with the operation of a destruction operator to such a state

$$\hat{a}_1(|N,0\rangle + |0,N\rangle)/\sqrt{2} = \sqrt{N/2}|N-1,0\rangle$$

(18)

The robustness of the entangled states is tested with such a loss scheme. It is likely that particles from the condensate will be lost during the evolution of the state when the barrier is raised. This is simulated by the operation of the destruction operator at different time intervals during the evolution and taking particles out randomly from either well at each time. Fig. 13 shows different realizations of loss of different number of particles from the least extreme entangled state example in Fig. 12, third row. Panels (a) and (b) are two different simulations for a loss of 10 particles during the evolution. Panels (c) and (d) show two different runs for a loss of 30 particles from the same entangled state. Results for extreme entangled states are not shown here as such states are totally destroyed, meaning all the particles are localized in one well. The simulations suggests that a less extreme entangled state is more robust, so it may be desirable to sacrifice the extremity of a cat state in order for it to survive in a realistic laboratory setting. To compare the effects of loss for sharpness, an entangled state which is not sharp and has a Gaussian spread has a better chance of having nonvanishing coefficients after the loss of particles. So the most robust state would be a
less extreme entangled state with a Gaussian width of coefficients around the two peaks. The coherence is not lost in destroying particles in the fashion done here - this is evident in the density matrix [35] for panel (a) as shown in panel (e). The off-diagonal peaks in the density matrix that quantifies the coherence remains a geometric mean of the diagonal elements since we have not introduced phase damping; coherence vanishes only when the final state is localized in one well.

C. Discussions

During our development of the quantum phase space picture for the double well BEC since 2002 [14,36], several other authors have also noted that metastable quantum states and dynamical instability can be exploited to produce entangled states in a double well [22] and in a spinor condensate [23]. All these findings are consistent with the phase space model introduced in this paper; our demonstration of the tunability and sharpening of the entangled states in a double well setting provides a useful improvement which may be important for experimental detection and other practical purposes. The Wigner distribution function, the Gaussian average of which is the Husimi distribution, has also emerged as a valuable tool to the description of entangled state generation in a spinor condensate [23].

VI. REMARKS AND SUMMARY

We have developed a quantum mechanical phase space picture of a double-well Bose-Einstein condensate in the two-mode approximation. In a mean-field approximation, the two-mode Hamiltonian reduces to the Hamiltonian of a nonrigid physical pendulum. Examination of the Husimi projections of the stationary quantum states reveals how the mean-field classical phase space follows directly from quantum mechanics. We have found eigenstate structures that are localized like classical oscillating states, free-rotor states and $\pi$ states.

The Husimi probability distribution turns out to be an extremely useful tool to study BECs in a double-well. Through its study we found unifying connections and new insights into the double well phase space and its dynamics. For a driven double well, quantum states are found to diffuse into the chaotic region of phase space analogous to classical chaos. A $\pi$ phase imprinted condensate put on an unstable fixed point of the classical phase space bifurcates along the separatrix if allowed to time evolve. The extremity and the sharpness of the entangled states produced in this scheme can be tuned with the initial barrier height and the appropriate ramping of the potential. The model developed here may find applications in the studies of other double well BEC dynamics, such as in a study of asymmetric wells, effects of change of scattering lengths, transitions connected to avoided crossings, topics in quantum chaos and studies of the effects of decoherence.

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FIG. 1. Energy correlation diagram for 20 particles showing the eigenvalues as a function of barrier height $\alpha$. Note the merging of energy levels as tunneling decreases.

FIG. 2. Fock state coefficients for N=40 for (a) the ground state, (b) the first excited state, (c) the 30th state and (d) the 31st state. Low lying states are similar to harmonic oscillator wavefunctions, whereas the higher lying states are macroscopic quantum superpositions of particles simultaneously in both wells.

FIG. 3. Comparison of the classical nonrigid physical pendulum phase space with the Husimi distributions for different energy eigenstates for 40 particles. Shown are (a) classical energy contour. Husimi projections for (b) ground state (c) 6th (d) 12th and (e) 35th state.
FIG. 4. Quantum-classical correspondence in phase space as functions of number of particles. Shown are the ground state, an oscillator state, a state near the separatrix and an entangled state for particle numbers (a) 16 (b) 8 (c) 4 and (d) 2. A clear signature of classical pendulum phase space is manifest for $N = 8$. 


FIG. 5. Comparison of the classical and quantum phase space for N=40 showing the analog of π states in the exact quantum treatment. Shown are (a) classical energy contour. Husimi projections are for (b) 12th (c) 30th (d) 34th and (e) 41st states. (d) and (e) are the analogs self-trapped π states of mean-field theory, the quantum states here preserve parity.

FIG. 6. A comparison of quantum and classical dynamics for N=8. We see that the classical points very closely follow the quantum phase space density. The panels are for (a) initially, and after (b) the first cycle, (c) second cycle and (d) fourth cycles. The quantum interference effects for shorter times seem to have localizing effects in the region with high density of classical whorls ((g) and (h)). For much longer times quantum dynamics shows recurrences as in the next figure.
FIG. 7. Husimi projections showing fractional revivals in the dynamics of a phase displaced ground state for N=40. The recurrence time here is T=12.375. The panels show (a) ground state, (b) ground state phase displaced by π/2, and revivals approximately at (c) T/4, (d) T/3, (e) T/2 and (f) T. (e) is an example of a macroscopic superposition of two coherent states, and (f) is approximately a full revival of (b).

FIG. 8. A composite Poincaré surface of section for 100 trajectories evenly spaced on θ = 0. This is for N=200, and for a sinusoidal barrier $\alpha = 2.5 + 2.5\cos(10t)$.

FIG. 9. Comparison of classical and quantum dynamics for points in the chaotic regions of phase space for N=200. Right panels show Husimi projections for the time evolution of the localized superposition of 128th and 129th eigenstate and the left panels show the time evolution of three bands of classical trajectories initially localized in the same region. (a),(b) at t=0 (c),(d) t=0.17 (e),(f) t=1.7 and (g),(h) t=5. Quantum states are visibly localized around the chaotic classical points.
FIG. 10. Husimi projections showing rotations in phase space for $N=40$. (a) The initial phase squeezed or coherent state at $t=0$, (b) slightly rotated state at $t=0.075$, (c) $t=0.09$, (d) at $t=0.125$, a number squeezed state, (e) $t=0.165$ (f) at $t=0.25$ the evolution brings the state back to the initial phase squeezed state.

FIG. 11. Shown is the evolution to an entangled state of $N=1000$ in Husimi projection space. (a) The ground state at $t=0$, (b) the $\pi$-phase imprinted ground state at the hyperbolic fixed point, (c) at $t=0.01$ the wave-packet is bifurcating along the separatrix, (d) at $t=0.016$ it continues to move along the separatrix, (e) at $t=0.4$ the states become trapped as we increase the barrier, and (f) at $t=2.3$ a sharply peaked entangled state is obtained.
FIG. 12. Shown are the entangled states for N=1000 with different initial heights of the barrier and therefore different initial squeezings of the BEC ground state, but the same ramping of the potential. Row (1) shows the states where $\alpha = 1 + 2t$: (a) the parameter $\alpha$ as a function of time, (b) the ground state, (c) the final entangled state, and (d) a magnified view of the Fock-state coefficients. Rows (2) and (3) show the results for $\alpha = 3 + 2t$ and $\alpha = 5 + 2t$ respectively. The initial barrier height controls the extremity of the entangled states. Note that for clarity the axes in the panels have different scalings.

FIG. 13. Effects of loss of particles on entangled states. (a) and (b) show the effect of loss of 10 particles on the less extreme entangled state example of the third row in the previous figure. (c) and (d) show the effects of loss of 30 particles. (e) shows density matrix for panel (a) denoting that the coherence is not lost.