Formulation of an electrostatic field with a charge density in the presence of a minimal length based on the Kempf algebra

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Abstract – In a series of papers, Kempf and co-workers (J. Phys. A: Math. Gen., 30 (1997) 2093; Phys. Rev. D, 52 (1995) 1108; Phys. Rev. D, 55 (1997) 7909) introduced a D-dimensional (β, β')-two-parameter deformed Heisenberg algebra which leads to a nonzero minimal observable length. In this work, the Lagrangian formulation of an electrostatic field in three spatial dimensions described by Kempf algebra is studied in the case in which β' = 2β up to first order over the deformation parameter β. It is shown that there is a similarity between electrostatics in the presence of a minimal length (modified electrostatics) and higher-derivative Podolsky’s electrostatics. The important property of this modified electrostatics is that the classical self-energy of a point charge becomes a finite value. Two different upper bounds on the isotropic minimal length of this modified electrostatics are estimated. The first upper bound will be found by treating the modified electrostatics as a classical electromagnetic system, while the second one will be estimated by considering the modified electrostatics as a quantum field-theoretic model. It should be noted that the quantum upper bound on the isotropic minimal length in this paper is near to the electroweak length scale (ℓ̲_{electroweak} ∼ 10^{-18} m).

Introduction. – One of the fundamental problems in theoretical physics is the unification between Einstein’s general relativity and quantum mechanics. The most interesting consequence of this unification is the appearance of a minimal observable length on the order of the Planck length. Today’s theoretical physicists know that the existence of a minimal observable length leads to a generalized Heisenberg uncertainty principle. This generalized or gravitational uncertainty principle (GUP) can be written as

\[ \Delta X \Delta P \geq \frac{\hbar}{2} \left[ 1 + \alpha^2 \ell_P^2 \left( \frac{\Delta P}{\hbar} \right)^2 \right], \]  

(1)

where ℓ_P is the Planck length and α is a positive numerical constant [1]. At low energies, the second term on the right-hand side of eq. (1) may be neglected (α^2 ℓ_P^2 (∆P/ℏ)^2 ≪ 1), and we have ΔXΔP ≫ ℏ/2. The high-energy limit is obtained when α^2 ℓ_P^2 (∆P/ℏ)^2 ∼ 1. In this limit eq. (1) leads to a minimal observable length (ΔX)_{min} = αℓ_P. Many physicists believe that introducing a minimal observable length into a quantum field theory can eliminate the divergences [2–5]. In the past few years, formulation of quantum field theory and gravity in the presence of a minimal observable length have been studied extensively [6–19].

This paper is organized as follows. In the second section, the D-dimensional (β, β')-two-parameter deformed Heisenberg algebra introduced by Kempf is reviewed and it is shown that the above algebra leads to a minimal observable distance [6–8]. In the third section, the Lagrangian formulation of an electrostatic field in three spatial dimensions described by Kempf algebra is presented in the case in which β' = 2β up to first order over the deformation parameter β. We show that there is a similarity between electrostatics in the presence of a minimal length and higher-derivative Podolsky’s electrostatics in three spatial dimensions. The important property of electrostatics in the presence of a minimal length is that the self-energy of a point charge becomes a finite value.

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In the fourth section, two upper bounds on the minimal length of these modified electrodynamics are obtained. The first upper bound will be found by treating the modified electrodynamics as a classical electromagnetic system, while the second one will be obtained by considering the modified electrodynamics as a quantum field-theoretic model. Finally, the numerical values for the classical and quantum upper bounds on the isotropic minimal length in this work are compared with the results in previous investigations. SI units are used throughout this paper.

A brief review of the Kempf algebra. – Let us begin with a brief review of the Kempf algebra, which is a modified Heisenberg algebra that describes a D-dimensional quantized space [6–8]. The D-dimensional Kempf algebra is specified by the following modified commutation relations:

\[
[X_i, P_j] = i\hbar \left[ \delta_{ij} (1 + \beta \mathbf{P}^2) + \beta' P_i P_j \right],
\]

\[
[X_i, X_j] = i\hbar \left( 2\beta - \beta' \right) + \beta(2\beta + \beta')\mathbf{P}^2 \left( P_i X_j - P_j X_i \right),
\]

\[
[P_i, P_j] = 0,
\]

where \(i, j = 1, 2, \ldots, D\) and \(\beta, \beta'\) are two deformation parameters which are non-negative \((\beta, \beta' \geq 0)\) and have the same dimensions, i.e., \([\beta] = [\beta'] = (\text{momentum})^{-2}\). Also, \(X_i\) and \(P_i\) are position and momentum operators in the deformed space. Using (2) and the Schwarz inequality for a quantum state, the uncertainty relation for position and momentum by assuming that \(\Delta P_i\) is isotropic \((\Delta P_i = \Delta P_j = \ldots = \Delta P_D)\) becomes [12]

\[
(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2} \left[ 1 + (D\beta + \beta') (\Delta P_i)^2 + \gamma \right],
\]

where

\[
\gamma = \beta \sum_{k=1}^{D} (P_k)^2 + \beta' (P_i)^2.
\]

The relation (5) leads to an isotropic minimal length which is given by

\[
(\Delta X_i)_{\text{min}} = \hbar \sqrt{D\beta + \beta'}, \quad \forall i \in \{1, 2, \ldots, D\}.
\]

In this work, we only consider the special case \(\beta' = 2\beta\), in which the position operators commute to first order in \(\beta\), i.e., \([X_i, X_j] = 0\).

In such a linear approximation, the Kempf algebra reads

\[
[X_i, P_j] = i\hbar \left[ \delta_{ij} (1 + \beta \mathbf{P}^2) + 2\beta P_i P_j \right],
\]

\[
[X_i, X_j] = 0,
\]

\[
[P_i, P_j] = 0,
\]

In ref. [15], Braun showed that the following representations satisfy (9)–(11), at the first order in \(\beta\):

\[
X_i = x_i,
\]

\[
P_i = p_i(1 + \beta p_i^2).
\]

Note that the representations (7), (8) and (12), (13) coincide when \(\beta' = 2\beta\).

Lagrangian formulation of an electrostatic field with a charge density based on the Kempf algebra. – The Lagrangian density for an electrostatic field with a charge density \(\rho(x)\) in three spatial dimensions \((D = 3)\) is [20]

\[
\mathcal{L}(\phi, \partial_\phi, \partial_\phi^2) = \frac{1}{2} \varepsilon_0 (\partial_\phi \phi)(\partial_\phi \phi) - \rho \phi,
\]

where \(\phi(x)\) is the electrostatic potential. The Euler-Lagrange equation for the electrostatic potential is

\[
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\phi \left( \frac{\partial \mathcal{L}}{\partial (\partial_\phi \phi)} \right) = 0.
\]

If we substitute the Lagrangian density (14) in the Euler-Lagrange equation (15), we will obtain the Poisson equation as follows:

\[
\nabla^2 \phi(x) = -\frac{\rho(x)}{\varepsilon_0},
\]

where \(\nabla^2 := \partial_\phi \partial_\phi\) is the Laplace operator. The Poisson equation (16) is equivalent to the following two equations:

\[
\nabla \cdot \mathbf{E}(x) = \frac{\rho(x)}{\varepsilon_0},
\]

\[
\nabla \times \mathbf{E}(x) = 0,
\]

where \(\mathbf{E}(x) = -\nabla \phi(x)\) is the electrostatic field.

Now, let us obtain the Lagrangian density for an electrostatic field in the presence of a minimal length based on the Kempf algebra. For such a purpose, we must replace the usual position and derivative operators \(x_i\) and \(\partial_\phi\) with the modified position and derivative operators \(X_i = x_i\) and \(\nabla_i := (1 - \beta \hbar^2 \nabla^2) \partial_\phi\) according to (12) and (13) in the Lagrangian density (14), i.e.,

\[
\mathcal{L} = \frac{1}{\varepsilon_0} (\nabla_i \phi)(\nabla_i \phi) - \rho \phi
\]

\[
= \frac{1}{\varepsilon_0} [(1 - \beta \hbar^2 \nabla^2) \partial_\phi \phi] (\nabla_i \phi)(\nabla_i \phi) - \rho \phi
\]

\[
= \frac{1}{\varepsilon_0} (\partial_\phi \phi)(\partial_\phi \phi) + \varepsilon_0 (h \sqrt{\beta})^2 (\nabla_i \phi)(\nabla_i \phi)
\]

\[
-\rho \phi + \mathcal{O} \left( (h \sqrt{\beta})^4 \right),
\]

where \(\mathcal{O} \left( (h \sqrt{\beta})^4 \right)\) is a higher order term in \(\hbar\).
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After neglecting terms of order \((\hbar \sqrt{\beta})^4\) and dropping out the total derivative term \(-\varepsilon_0(\hbar \sqrt{\beta})^2 \partial_j [(\partial_i \phi)(\partial_i \partial_j \phi)]\), the Lagrangian density (19) will be equivalent to the following Lagrangian density:

\[
\mathcal{L}(\phi, \partial_i \phi, \partial_i \partial_j \phi) = \frac{1}{2} \varepsilon_0 (\partial_i \phi)(\partial_i \phi) + \varepsilon_0(\hbar \sqrt{\beta})^2 (\partial_i \partial_j \phi)(\partial_i \partial_j \phi) - \rho \phi. \tag{20}
\]

The term \(\varepsilon_0(\hbar \sqrt{\beta})^2 (\partial_i \partial_j \phi)(\partial_i \partial_j \phi)\) in (20) can be considered as a minimal length effect. The Euler-Lagrange equation for the generalized Lagrangian density (20) is \([21,22]\]

\[
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_i \left( \frac{\partial \mathcal{L}}{\partial (\partial_i \phi)} \right) + \partial_i \partial_j \left( \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_j \phi)} \right) = 0. \tag{21}
\]

If we substitute (20) into (21), we will obtain the Poisson equation in the presence of a minimal length as follows:

\[
(1 - 2(\hbar \sqrt{\beta})^2 \nabla^2) \nabla^2 \phi(x) = -\frac{\rho(x)}{\varepsilon_0}. \tag{22}
\]

The modified Poisson equation (22) has been introduced earlier by Tkachuk in ref. [23] with a different approach. The modified Poisson equation (22) is equivalent to the following two equations:

\[
(1 - a^2 \nabla^2) \nabla \cdot E(x) = \frac{\rho(x)}{\varepsilon_0}, \tag{23}
\]

\[
\nabla \times E(x) = 0, \tag{24}
\]

where \(a := \hbar \sqrt{\beta}\). Equations (23) and (24) are fundamental equations of Podolsky’s electrostatics [24,25], and \(a\) is called Podolsky’s characteristic length [26–30].

Now, we want to obtain the total potential energy of an electrostatic field in the presence of a minimal length. Using the linear higher-order equations (22) and (23) together with eq. (24), the general expression for the total potential energy becomes

\[
U = \frac{1}{2} \int \rho(x) \phi(x) d^3 x
- \frac{1}{8} \varepsilon_0 \int \phi(x) [\nabla^2 \phi(x) - \hbar \sqrt{2\beta} [\nabla \cdot E(x)]^2 + 2 E \cdot \nabla E] d^3 x
- \frac{1}{2} \varepsilon_0 \int [E^2 - (\hbar \sqrt{\beta})^2 (\nabla \cdot E)^2] d^3 x
= \frac{1}{2} \varepsilon_0 \int [E^2 + (\hbar \sqrt{\beta})^2 (\nabla \cdot E)^2] d^3 x, \tag{25}
\]

where we assumed that \(E \cdot V \cdot E\) falls off faster than \(|x|^{-2}\) as \(|x| \to \infty\). According to eq. (25) the energy density of an electrostatic field in the presence of a minimal length is given by

\[
u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \varepsilon_0 (\hbar \sqrt{\beta})^2 (\nabla \cdot E)^2. \tag{26}
\]

The term \(\frac{1}{2} \varepsilon_0 (\hbar \sqrt{\beta})^2 (\nabla \cdot E)^2\) in eq. (26) shows the effect of minimal length corrections. Using the similarity between Podolsky’s electrostatics and our modified electrostatics the electrostatic potential \(\phi(x)\) for a point charge \(q\) with charge density \(\rho(x) = q \delta(x)\) can be written as

\[
\phi(x) = \frac{q}{4 \pi \varepsilon_0 |x|} \left(1 - e^{-\frac{|x|}{\hbar \sqrt{\beta}}}\right). \tag{27}
\]

In contrast with the usual Maxwell’s electrostatics the electrostatic potential (27) at the origin has a finite value \(\phi(0) = \frac{q}{4 \pi \varepsilon_0 \hbar \sqrt{\beta}}\). Now, using eq. (27) together with \(E(x) = -\nabla \phi(x)\) the electric field due to a point charge \(q\) is given by

\[
E(x) = \frac{q}{4 \pi \varepsilon_0 |x|^2} \left[1 - \left(1 + \frac{|x|}{\hbar \sqrt{\beta}}\right) e^{-\frac{|x|}{\hbar \sqrt{\beta}}}\right] \frac{x}{|x|}. \tag{28}
\]

If we substitute (28) into (25) and performing the integration, we will obtain the total potential energy of a point charge as \(U = \frac{q^2}{8 \pi \varepsilon_0 \hbar \sqrt{\beta}}\). By using eq. (28), we obtain the modified Coulomb’s law for the electrostatic interaction between a test charge \(q_0\) and the point charge \(q\) as follows:

\[
F(x) = \frac{q_0 q}{4 \pi \varepsilon_0 |x|^2} \left[1 - \left(1 + \frac{|x|}{\hbar \sqrt{\beta}}\right) e^{-\frac{|x|}{\hbar \sqrt{\beta}}}\right] \frac{x}{|x|}. \tag{29}
\]

Finding the upper bound on the isotropic minimal length in modified electrostatics. – If we substitute \(\beta' = 2 \beta\) into eq. (6), we will obtain the isotropic minimal length in three spatial dimensions as follows:

\[
(\Delta X_i)_{\text{min}} = \hbar \sqrt{5\beta}, \quad \forall i \in \{1, 2, 3\}. \tag{31}
\]

According to eq. (31) the isotropic minimal length can be expressed in terms of Podolsky’s characteristic length \(a = \hbar \sqrt{\beta}\) as

\[
(\Delta X_i)_{\text{min}} = \sqrt{\frac{10}{2}} a, \quad \forall i \in \{1, 2, 3\}. \tag{32}
\]

Now we are ready to estimate the classical and quantum upper bounds on the isotropic minimal length in modified electrostatics.

A classical upper bound on the isotropic minimal length. In ref. [28], Accioly and Scatena have obtained a classical bound on the Podolsky’s characteristic length \(a\). This classical bound on \(a\) was found using the data from a very accurate experiment carried out by Pilmore and Lawton [31] to test the Coulomb’s law of force between charges. According to refs. [28] and [30] the classical upper limit on \(a\) is

\[
a \lesssim 5.1 \times 10^{-10} \text{ m}. \tag{33}
\]

Using (33) in (32), we obtain the following classical upper bound for the isotropic minimal length:

\[
(\Delta X_i)_{\text{min}} \lesssim 8.06 \times 10^{-10} \text{ m}. \tag{34}
\]

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A quantum upper bound on the isotropic minimal length.

In a series of papers, Accioly et al. [26,28,30] obtained a quantum bound on Podolsky’s characteristic length $a$ by computing the anomalous magnetic moment of the electron in the framework of Podolsky’s theory. The quantum upper limit on $a$ is [26,28,30]

$$a \leq 4.7 \times 10^{-18} \text{ m.}$$

Using (35) in (32), we obtain the following quantum upper bound for the isotropic minimal length:

$$(\Delta X_i)_{\text{min}} \leq 7.4 \times 10^{-18} \text{ m.}$$

Note that the classical upper bound for the isotropic minimal length in our modified electrostatics is about eight orders of magnitude larger than the quantum upper bound, i.e.,

$$(\Delta X_i)_{\text{min-classical}} \sim 10^8(\Delta X_i)_{\text{min-quantum}}.$$  

Conclusions. – After 1930, many theoretical physicists have attempted to introduce a minimal observable length into quantum field theory [32,33]. The idea of a minimal observable distance received a great attention in the physics community when it was developed in details by Heisenberg and March [32,33].

Nowadays, we know that the existence of a minimal observable length leads to a generalized uncertainty principle. An immediate consequence of the generalized uncertainty principle is a modification of position and momentum operators according to eqs. (12) and (13) for $\beta = 2\beta$. We have shown that the Lagrangian formulation of an electrostatic field with a charge density in the presence of a minimal observable length leads to a fourth-order Poisson equation. Also, we proved that there is a similarity between electrostatics in the presence of a minimal length and Podolsky’s electrostatics. Another interesting property of electrostatics in the presence of a minimal length is that the classical self-energy of a point charge becomes a finite value.

Now, let us compare the classical and quantum upper bounds on the isotropic minimal length in this work with the results of refs. [34–36]. In ref. [34] the motion of neutrons in a gravitational quantum well has been studied by Brau and Buisseret and it was shown that $(\Delta X_i)_{\text{min}} \leq 2.41 \times 10^{-9} \text{ m.}$ In ref. [35] Nouicer has studied the Casimir effect in the presence of a minimal length and obtains $(\Delta X_i)_{\text{min}} \leq 15 \times 10^{-9} \text{ m.}$ The classical upper limit in eq. (34) is compatible with the results of refs. [34] and [35]. In ref. [36] it was deduced that the upper bound of the minimal length ranges from $2.4 \times 10^{-17} \text{ m}$ to $3.3 \times 10^{-18} \text{ m.}$ The quantum upper limit on the isotropic minimal length in eq. (36) is near to the results of ref. [36]. It is necessary to note that the quantum upper bound on the isotropic minimal length in this paper, i.e., $7.4 \times 10^{-18} \text{ m}$ is also near to the electroweak length scale ($\ell_{\text{electroweak}} \sim 10^{-18} \text{ m}$). Recently, Smailagic and Spallucci have proposed a novel way to formulate quantum field theory in the presence of a minimal length [37]. Using Smailagic-Spallucci approach, Gaete and Spallucci introduced a $U(1)$ gauge field with a non-local kinetic term as follows:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu \nu} \varepsilon^{\alpha \beta \mu \nu} F^{\mu \nu} - J^\alpha A_\alpha,$$  

where $\sqrt{\beta}$ is the minimal length of model [38]. The authors of ref. [38] have shown in appendix A of their paper that in the electrostatic case and to first order in $\theta$, the non-local Lagrangian (38) leads to the following modified Gauss’s law:

$$(1 - \theta \nabla^2) \cdot E(x) = \frac{\rho(x)}{\varepsilon_0}.$$  

A comparison between eqs. (23) and (39) shows that there is an equivalence between the Gaete-Spallucci electrostatics to first order in $\theta$ and the electrostatics with a minimal observable length. It should be noted that the electrostatics with a minimal observable length in this study is only correct to the first order in the deformation parameter $\beta$, while the Gaete-Spallucci electrostatics is valid to all orders in $\theta$.

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