AFFINE $\widehat{sl}_p$ CONTROLS THE MODULAR REPRESENTATION THEORY OF THE SYMMETRIC GROUP AND RELATED HECKE ALGEBRAS

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Abstract. In this paper we prove theorems that describe how the representation theory of the affine Hecke algebra of type $A$ and of related algebras such as the group algebra $\mathbb{F} S_n$ of the symmetric group are controlled by integrable highest weight representations of the characteristic zero affine Lie algebra $\widehat{sl}_\ell$. In particular we parameterise the representations of these algebras by the nodes of the crystal graph, and give various Hecke theoretic descriptions of the edges.

As a consequence we find for each prime $p$ a basis of the integrable representations of $\widehat{sl}_\ell$ which shares many of the remarkable properties, such as positivity, of the global crystal basis/canonical basis of Lusztig and Kashiwara. This $p$-canonical basis is the usual one when $p = 0$, and the crystal of the $p$-canonical basis is always the usual one.

The paper is self-contained, and our techniques are elementary (no perverse sheaves or algebraic geometry is invoked).

1. Introduction

This paper is concerned with the representation theory of the affine Hecke algebra of type $A$ and of the cyclotomic Hecke algebras.

We parameterise the finite dimensional representations of these algebras over any field $\mathbb{F}$ and prove various results about the behaviour of irreducible modules under restriction and induction. Each of these algebras contains a large commutative subalgebra, and we also describe how the failure of this algebra to act semisimply controls the combinatorics of the representation theory.

In contrast to the existing literature on these algebras, we prove all our results without the use of sophisticated machinery or explicit combinatorics—perverse sheaves and the geometry of the graded nilpotent cones are notably absent from this work, as is the study of partitions.

In their place we prove the following theorem, which is a remarkable rigidity property of the representation theory: there is an action of the characteristic zero affine Lie algebra $\widehat{sl}_\ell$ on the Grothendieck group of Hecke algebra representations. Furthermore, the irreducible Hecke representations define the natural crystal structure on the $\widehat{sl}_\ell$ representation.

As an immediate consequence we recover the explicit combinatorics of the Hecke algebra representation theory. For example, the simplest case of these theorems identifies the Grothendieck group of symmetric group representations in characteristic $p$ with an integral form of the basic representation of $\widehat{sl}_p$. This representation has a construction as a Fock space (the “principal realisation”). The well known natural parametrisation by $p$-regular partitions of the irreducible characteristic $p$ representations of the symmetric group follows immediately. In particular, this
explains why the generating function for the number of irreducible mod-$p$ representations of $S_n$ is just the character of the basic representation of $\hat{sl}_p$.

To describe the results in more detail, consider the direct sum over all $n$ of the Grothendieck group of representations of the affine Hecke algebra of type $A_n$. (This, and all other terms, are carefully defined in the body of the paper).

Similarly, consider the direct sum over all $n$ of the Grothendieck group of representations of the symmetric group $S_n$. Then it is a classical observation that these are cocommutative Hopf algebras; for the symmetric group this has been rediscovered many times (see [Mc]), but for the affine Hecke algebra this is due to [BZ]. In theorem 14.1 we identify this algebra—it is just the dual to the enveloping algebra $\tilde{U}_{\eta}$ of the upper triangular part of the affine algebra $\hat{sl}_l$; here $l$ is the order of the parameter $q \in \mathbb{R}^\times$ which enters in the definition of the affine Hecke algebra.

More generally, consider the cyclotomic Hecke algebra defined by Ariki and Koike [AK]. This is a deformation of the group algebra of the wreath product of the symmetric group $S_n$ with a cyclic group of order $r$. The deformation depends on an $r$-tuple of elements of $\mathbb{R}^\times$, $\lambda = (q_1, \ldots, q_r)$, and we denote the corresponding algebra $H^\lambda_n$. When $r = 1$ and $\lambda = (1)$ this is just the finite Hecke algebra.

If we now sum the Grothendieck groups of representations of $H^\lambda_n$ for fixed $\lambda$ it is no longer true that this is a Hopf algebra. However it is obviously a comodule for the Hopf algebra (dual to $\tilde{U}_{\eta}$) built out of the affine Hecke algebra. The first of our main theorems, theorem 14.2, says that it has many more symmetries—that it is in fact dual to a module for the entire affine algebra $\hat{sl}_l$, and moreover that this module is an irreducible integrable highest weight module with highest weight determined by $\lambda$.

Even in the classical case of the symmetric group and its deformations ($r = 1$) this is new information: it identifies the Hopf algebra built out of $S_n$ with the principal realisation of the basic representation of $\hat{sl}_l$, and it identifies the action of $\hat{sl}_l$ by vertex operators on this representation. For other cyclotomic Hecke algebras it extends the results of [A].

To prove this theorem we must introduce several new ingredients. The first is the action of the Chevalley generators $f_i$ of the lower triangular part $\eta^\circ$ of $\hat{sl}_l$. Unlike the operators $e_i$ of $\eta^\circ$ which have been known since the 1950s, and which arise in an obvious way from the affine Hecke algebra, the definition of the $f_i$ is new to this paper. After a variety of preliminary results on the affine Hecke algebra, we begin the study of these operators in section 8.

The operators $e_i$ and $f_i$ are defined directly on the module category, but will not satisfy the defining relations of $\hat{sl}_l$ before we pass to the Grothendieck group. However, by considering the cosocle filtration of these operators (i.e. before passing to the Grothendieck group), we can define the leading term of the operators $e_i$ and $f_i$. These leading terms have a beautiful interpretation as “crystal operators” [Ka], and allow us to define the crystal graph structure of the representations. This structure generalises the classical “branching laws” for representations of the symmetric group.

The crystal graph is a graph with nodes given by irreducible representations, and with an edge between irreducible representation $M$ of $H^\lambda_n$ and $N$ of $H^\lambda_{n-1}$ if $N$ occurs in the cosocle of the restriction of $M$. The operators $e^+_i$ which are dual to $e_i$ refine restriction, and we can label this edge with an $i$ if $N$ occurs in $e^+_i M$. 
Our second main theorem (theorem 14.3) is that this graph is precisely the usual crystal graph of the representation determined by \( \lambda \). This gives a combinatorial parametrisation of irreducible representations, and shows this parametrisation depends only on \( l \)—the order of \( q \) in \( R^\times \). (When \( R = \mathbb{C} \) the modules for \( H_{\text{aff}}^n \) were first parameterised in [BZ] when \( l = \infty \), and in [G] for arbitrary \( l \). Subsequently Vigneras conjectured [V] that the parametrisation depends only on \( l \), and not on \( R \).)

To prove this we must engage in a detailed study of the modules for the affine Hecke algebra. We begin by showing that \( e_i^* M \) has a simple cosocle. This result, which is the main result of [GV], generalises the classical “multiplicity one” properties of restriction for complex representations of the symmetric group, and the corresponding property for characteristic \( p \) representations [Kv].

We then study the relationship between the crystal operators and the failure of semisimplicity of both the Hecke algebra and a large commutative subalgebra of it (the functions \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) on the maximal torus of \( GL_n \)). We find that though \( e_i^* M \) is a complicated module which is far from being semisimple, it has a uniserial part to its composition series which admits a clean description in various ways.

In particular, the cosocle of \( e_i^* M \) occurs in a uniserial chain inside \( e_i^* M \), and the length of the chain in which it occurs is precisely the maximal size of a Jordan block for \( X_n \) on \( M \). We prove that this length can also be read off the image of \( M \) inside the Grothendieck group of \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \)-modules.

These results, and the analogous (but harder) results for induction, form the bulk of section 9.

Finally, we can reinterpret the theorems 14.3, 14.2 as defining a family of interesting bases in the representations of \( \widehat{sl}_\ell \). Each of these bases has the same crystal graph, but these \( p \)-canonical bases are all different. In the case \( R = \mathbb{C} \), the 0-canonical basis coincides with the basis defined by Lusztig and Kashiwara. Each \( p \)-canonical basis shares all of the remarkable properties of the 0-canonical basis—for example, the structure constants of \( e_i \) and \( f_i \) are non-negative integers, and the basis of the Verma descends to a basis of the integrable representations. These bases are just the dual to the irreducible representations, and \( p \) is the characteristic of the field \( R \). (The dual to the \( p \)-canonical basis also has a Hecke theoretic interpretation—they are dual to the projective representations).

To summarise, this paper “explains” all of the combinatorics of the representation theory of the symmetric group and Hecke algebras—it is just the combinatorics of the crystal graph of \( \widehat{sl}_\ell \). (Our proofs are free of such combinatorics.)

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2. Notation

Throughout the paper, we fix a field \( R \), and an invertible element \( q \in R \), i.e., a homomorphism \( \mathbb{Z}[q, q^{-1}] \to R \). We write \( \mu_q = \{ q^i \mid i \in \mathbb{Z} \} \) for the powers of \( q \) in \( R \), and \( \ell = |\mu_q| \), so \( \ell \in \mathbb{N} \cup \{ \infty \} \), and \( \ell \) is the order of \( q \in R^\times \).

We further assume that \( q \neq 1 \). This is for clarity of exposition; the changes in the statements of definitions and theorems that must be made when \( q = 1 \) are sketched.
in section II. The most interesting case when $q = 1$ is the modular representation theory of the symmetric group. Then $R = \mathbb{F}_p$ and $q = 1$, but $\ell = p$; see section II.

Define the Dynkin diagram of $\mu_q$ to be the directed graph with vertices the elements of $\mu_q$, and an edge $q^i \to q^j$ if $q^{i-j} = 1$. (A slightly classier notation, which we often use, is to write $i, j \in \mu_q$ instead of $q^i, q^j \in \mu_q$, and then $i \to j$ if $ij^{-1} = q$.) The isomorphism type of this graph depends only on $\ell$, and not on more general properties of $q$ or $R$. This feature will be mirrored by the properties of the representation theory we study. (Conjecturally [3], the representation theory depends only on the characteristic of $R$ and the Dynkin diagram $\mu_q$.)

The Dynkin diagram of $\mu_q$ defines an affine Lie algebra $\widehat{\mathfrak{sl}}_\ell$. The theorems of this paper are a description of how this Lie algebra controls the representation theory of the affine and cyclotomic Hecke algebras. We recall the definition and basic properties of $\widehat{\mathfrak{sl}}_\ell$ and its representation theory (as found in [Kac]) in section 3.

Most of the results about Hecke algebras hold for arbitrary rings $R$ when appropriately formulated. This is an easy exercise, but for clarity we have phrased results only for fields.

2.1. Some common notation. If $A$ is an $R$-algebra, we write $A$-mod for the category of all left $A$-modules, and $\text{Rep } A$ for the category of left $A$-modules which are finite dimensional as $R$-modules. Also write $\text{Proj } A$ for the subcategory of $\text{Rep } A$ consisting of finite dimensional projective $A$-modules. (If $R$ is an arbitrary ring, we would also need to define various subcategories, such as the category of $A$-modules which are projective as $R$-modules, and so on.)

We recall that the socle of a module $M$, denoted $\text{soc}(M)$, is the largest semisimple submodule of $M$, and that the cosocle of $M$, denoted $\text{cosoc}(M)$, is its largest semisimple quotient.

We write $S_n$ for the symmetric group on $n$ letters, $s_i = (i \ i + 1)$ for the simple transpositions, $\ell : S_n \to \mathbb{N}$ for the length function.

2.2. Grothendieck Group. If $\mathfrak{C}$ is an abelian category, we write $K(\mathfrak{C})$ for the Grothendieck group of $\mathfrak{C}$. This is the quotient of the free abelian group with generators the objects $M \in \mathfrak{C}$ by the ideal generated by the elements $M_1 - M_2 + M_3$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathfrak{C}$. If objects in $\mathfrak{C}$ have finite length and unique composition factors, we write $K(\mathfrak{C})^*$ for the topological dual of $K(\mathfrak{C})$; i.e. for the linear functions $f : K(\mathfrak{C}) \to \mathbb{Z}$ such that $f(M) = 0$ for all but finitely many isomorphism classes of irreducible objects $M \in \mathfrak{C}$.

If $M \in \mathfrak{C}$, let’s write $[M]$ for its image in $K(\mathfrak{C})$. Then as $M$ runs through the irreducible objects in $\mathfrak{C}$, the elements $[M]$ form a basis of $K(\mathfrak{C})$, and the functions

$$\delta_M : K(\mathfrak{C}) \to \mathbb{Z}, \quad \delta_M(N) = \begin{cases} 0 & \text{if } M \ncong N, \text{ } N \text{ irreducible} \\ 1 & \text{if } M \cong N \end{cases}$$

form a basis of $K(\mathfrak{C})^*$. More generally, if $N \in \mathfrak{C}$ and $M$ is an irreducible object in $\mathfrak{C}$, write $[M : N]$ for the multiplicity of $M$ in a Jordan-Hölder series of $N$, and extend this to $[\quad] : K(\mathfrak{C}) \times K(\mathfrak{C}) \to \mathbb{Z}$ by bilinearity. Then write, for any $M \in K(\mathfrak{C})$, $\delta_M : K(\mathfrak{C}) \to \mathbb{Z}$ for the function $N \mapsto [M : N]$. 

Now, if \( F: \mathcal{C} \to \mathcal{C}' \) is an exact functor of abelian categories, we get an induced \( \mathbb{Z} \)-linear map \( F: K(\mathcal{C}) \to K(\mathcal{C}') \), and we can define its transpose \( F^*: K(\mathcal{C}')^* \to K(\mathcal{C})^* \) by \( F^*f = fF \).

We will apply all this to the category \( \text{Rep} \ A \) of finite dimensional representations of an algebra \( A \) (over \( R \)). Suppose that \( A \) and \( A' \) are two such algebras, and that the cosocle of \( A \) and \( A' \) are direct sums of matrix algebras over \( R \); i.e. that they are separable algebras. Then the irreducible representations of \( A \otimes A' \) are of the form \( M \boxtimes M' \), where \( M \) is an irreducible \( A \)-module, \( M' \) is an irreducible \( A' \)-module. More generally, we recall that under such assumptions

**Lemma 2.1.** \( K(A \otimes A'\text{-mod}) = K(A\text{-mod}) \otimes K(A'\text{-mod}) \)

which is certainly not true before passing to the Grothendieck group. (Here, the tensor product is algebraic—elements consist of finite linear combinations of the elements \([M] \otimes [N] \) as \([M], [N] \) run through a basis of \( K(A\text{-mod}) \) and \( K(A'\text{-mod}) \) respectively.)

Write \( K(\mathcal{C})_\mathbb{Q} = K(\mathcal{C}) \otimes_\mathbb{Z} \mathbb{Q} \). As \( K(\mathcal{C}) \) is a torsion free \( \mathbb{Z} \)-module, \( K(\mathcal{C})_\mathbb{Q} \) is a \( \mathbb{Q} \)-vector space with distinguished sublattice \( K(\mathcal{C}) \subset K(\mathcal{C})_\mathbb{Q} \).

3. **Summary of properties of the affine Lie algebra \( \widehat{\mathfrak{sl}}_\ell \)**

First suppose \( \ell \in \mathbb{N} \). If \( A \) is a ring, we write \( \mathfrak{sl}_\ell(A) \) for the Lie algebra of trace zero \( \ell \times \ell \) matrices over \( A \) and \( \widehat{\mathfrak{sl}}_\ell(A) \) for the central extension of \( \mathfrak{sl}_\ell(A[t, t^{-1}]) \) by \( A \)

\[
0 \to A \cdot c \to \widehat{\mathfrak{sl}}_\ell(A) \to \mathfrak{sl}_\ell(A[t, t^{-1}]) \to 0.
\]

This has Lie bracket \([f, g] = (fg - gf) + \text{tr} \text{Res}(\frac{df}{dt} \cdot g)c\), where \( f, g \in \mathfrak{sl}_\ell(A[t, t^{-1}]) \) and \( \text{Res} \) denotes the coefficient of \( t^{-1} \).

If \( \ell = \infty \), let us abuse notation and write \( \widehat{\mathfrak{sl}}_\ell(A) \) for the Lie algebra \( \mathfrak{gl}_\infty(A) \)

of infinite matrices in which only finitely many entries in any row or column are non-zero. With this convention, the rest of this section is valid for \( \ell \in \mathbb{N} \cup \{ \infty \} \).

Write \( U_\mathbb{Q}\widehat{\mathfrak{sl}}_\ell \) for the enveloping algebra of \( \widehat{\mathfrak{sl}}_\ell(\mathbb{Q}) \). This has generators \( e_i, f_i, h_i \) for \( i \in \mu_\ell \)

and relations

\[
[e_i, f_j] = \delta_{ij}h_i \quad [h_i, e_j] = c_{ij}e_j \quad [h_i, f_j] = -c_{ij}f_j
\]

\[
(ad e_i)^{1-c_{ij}} e_j = 0 \quad (ad f_i)^{1-c_{ij}} f_j = 0
\]

where \( c_{ij} = 2\delta_{ij} - (\delta_{ij-1, q} + \delta_{ji-1, q}) \) is the Cartan matrix of \( \widehat{\mathfrak{sl}}_\ell \).

Write \( U_\mathbb{Z}\widehat{\mathfrak{sl}}_\ell \) for Kostant’s integral form of \( U_\mathbb{Q}\widehat{\mathfrak{sl}}_\ell \). This is a Hopf algebra over \( \mathbb{Z} \), contained in \( U_\mathbb{Q}\widehat{\mathfrak{sl}}_\ell \) as a lattice. It is generated as an algebra by the elements

\[
e_i^{(n)} = \frac{e_i^n}{n!} \quad f_i^{(n)} = \frac{f_i^n}{n!} \left( \frac{h_i}{n} \right) = \frac{h_i(h_i - 1) \cdots (h_i - n + 1)}{n!}, \quad i \in \mu_\ell, n \in \mathbb{N}
\]

with relations induced from the relations for \( e_i, f_i, \) and \( h_i \). (It is possible to write the relations explicitly; but we will not need this.)

We write \( U_\mathbb{Z}\eta_\ell \) for the Hopf subalgebra of \( U_\mathbb{Z}\widehat{\mathfrak{sl}}_\ell \) generated by \( e_i^{(n)}, i \in \mu_\ell, n \in \mathbb{N} \), and \( U_\mathbb{Z}\mu_\ell \) for the Hopf subalgebra of \( U_\mathbb{Z}\widehat{\mathfrak{sl}}_\ell \) generated by \( f_i^{(n)}, i \in \mu_\ell, n \in \mathbb{N} \).

Recall that a representation \( V \) of \( U_\mathbb{Q}\widehat{\mathfrak{sl}}_\ell \) is called *integrable of lowest weight* if...
(i) Each $h_i$ acts semisimply
(ii) Each $e_i$ and $f_i$ acts locally nilpotently
(iii) There exists a finite dimensional subspace $W \subseteq V$ such that $U\eta \cdot W = V$ and that the category of such representations is semisimple.

Given an integrable lowest weight module $V$, the space of invariants for $U\eta$ (lowest weight vectors) is preserved by each $h_i$. If this space is one dimensional $V$ is irreducible, and if $1^\lambda$ is a lowest weight vector with $h_i 1^\lambda = \lambda_i \cdot 1^\lambda \quad \forall i$, then $\lambda_i \in \mathbb{N}$ and $\sum \lambda_i < \infty$.

This sets up a correspondence between

\[(\text{functions } \lambda: \mu \to \mathbb{N}, \sum \lambda_i < \infty) \quad \text{and} \quad (\text{irreducible integrable lowest weight modules})\]

Given $\lambda$, write $L_\lambda$ for the corresponding irreducible module.

As a $U\eta$-module, $L_\lambda$ is generated by a lowest weight vector $1^\lambda$, and $L_\lambda = U\eta / U\eta \langle e_i^{\lambda_i + 1} \mid i \in \mu \rangle$.

In particular, as the intersection of the left ideals $\bigcap_{\lambda: \mu \to \mathbb{N}} U\eta \langle e_i^{\lambda_i + 1} \mid i \in \mu \rangle$ is trivial, it follows that if $x \in U\eta$ acts as zero on every integrable lowest weight representation, then $x = 0$.

We write $\Lambda_0: \mu \to \mathbb{N}$ for the function $\Lambda_0(i) = \delta_{i,1}$. The corresponding representation is called the basic representation of $\mathfrak{sl}_\ell$. More generally, define the fundamental weights $\Lambda_j: \mu \to \mathbb{N}$ by $\Lambda_j(i) = \delta_{i,j}$ and define the roots $\alpha_i: \mu \to \mathbb{Z}$ by $\alpha_i = 2\Lambda_i - \Lambda_{qi} - \Lambda_{q^{-1}i}$.

Each $L_\lambda$ carries a non-degenerate symmetric bilinear form, the Shapovalov form $(\ , \ )$, which is determined by requiring

\[(1^\lambda, 1^\lambda) = 1\]

and

\[(e_ix, y) = (x, f_iy) \quad \text{for all } i \in \mu.\]

4. Definitions and first properties of Hecke algebras

4.1. The finite Hecke algebra, $H_n$ is the $R$-algebra with generators

$T_1, \ldots, T_{n-1}$

and relations

1. **braid relations** $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $T_iT_j = T_jT_i$, $|i - j| > 1$
2. **quadratic relations** $(T_i + 1)(T_i - q) = 0$.

The braid relations imply that if $w = s_{i_1} \cdots s_{i_r}$ and $\ell(w) = r$, then $T_{i_1} \cdots T_{i_r}$ depends only on $w \in S_n$. It is denoted $T_w$, and the $T_w$, $w \in S_n$ form a basis of $H_n$ over $R$.

The **affine Hecke algebra** $H_n^{\text{aff}}$ is the $R$-algebra, which as an $R$-module is isomorphic to

$H_n \otimes_R R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

The algebra structure is given by requiring that $H_n$ and $R[X_i^{\pm 1}]$ are subalgebras, and that

\[T_iX_iT_i = qX_i+1.\]
Equivalently, if \( f \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \), then
\[
(4) \quad T_i f - s_i f T_i = (q - 1) \frac{f - s_i f}{1 - X_i X_{i+1}}
\]
where \( s_i \in S_n \) acts on \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) by permuting \( X_i \) and \( X_{i+1} \).

The affine Hecke algebra so defined is an extension by a Laurent polynomial algebra of the Hecke algebra associated to the Coxeter group with Dynkin diagram \( \mu_q \). This definition and the isomorphism is due to Bernstein.

**Proposition 4.1.** (Bernstein) The center of \( H_n^{\text{aff}} \), \( Z(H_n^{\text{aff}}) \), is isomorphic to symmetric Laurent polynomials.

\[
Z(H_n^{\text{aff}}) \simeq R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}
\]

**Proof.** The relation \([4]\) makes it clear that \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n} \) is contained in the center. Conversely, suppose \( h = \sum T_w f_w \in Z(H_n^{\text{aff}}) \) where \( f_w \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). Let \( u \in S_n \) be maximal with respect to Bruhat order such that \( f_u \neq 0 \). If \( u \neq 1 \) then there exists some \( i \) such that \( u(i) \neq i \). As \( X_i T_w = T_w X_{w^{-1}(i)} + \sum w' < w X_{w'} g_{w'} \) for some \( g_{w'} \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \), the coefficient of \( T_u \) is different in \( X_i h \) and \( h X_i \).

Hence \( h = f_1 \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). But then \([4]\) implies \( h \) is \( S_n \)-invariant.

**Corollary 4.2.** If \( M \in H_n^{\text{aff}}\)-mod is absolutely irreducible, then \( M \) is finite dimensional, and in fact \( \dim_R M \leq n! \).

**Proof.** As \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) is a free \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n} \)-module of rank \( n! \), \( H_n^{\text{aff}} \) is a free module over \( Z(H_n^{\text{aff}}) \) of rank \((n!)^2\). Dixmier’s version of Schur’s lemma implies that the center of \( H_n^{\text{aff}} \) acts by scalars on absolutely irreducible modules, and hence \( M \) is an irreducible module for a finite dimensional algebra of dimension \((n!)^2\).

Suppose \( R \) is algebraically closed. Then the characters (i.e. one dimensional representations) of the center \( Z(H_n^{\text{aff}}) \) are the orbits of \( S_n \) on \((R^*)^n = \text{Spec}R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). Given any finite dimensional module \( M \in \text{Rep} H_n^{\text{aff}} \), we can write \( M \) as a direct sum of \textit{generalized} eigenspaces for the commutative subalgebra \( R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \), say

\[
M = \bigoplus_{s \in (R^*)^n} M[s]
\]

where \( s = (s_1, \ldots, s_n) \) and \( M[s] = \{ m \in M \mid (X_i - s_i)^{\text{dim} M} m = 0, 1 \leq i \leq n \} \).

If we also write

\[
M_s = \{ m \in M \mid X_i m = s_i m, 1 \leq i \leq n \}
\]

for the \textit{actual} simultaneous eigenspace of the elements \( X_i \), we have

\[
M_s \neq 0 \iff M[s] \neq 0.
\]

Further, if \( M \) is irreducible, then \( Z(H_n^{\text{aff}}) \) acts by the central character \( S_n \cdot s \in (R^*)^n / S_n \) if and only if there exists some \( w \in S_n \) such that \( M_{ws} \neq 0 \).

To generalize this to all of \( \text{Rep} H_n^{\text{aff}} \), recall the following general property of algebras.
Lemma 4.3. Let $A$ be an $R$-algebra, and $Z \subset A$ a central subalgebra. If $M, N$ are two $A$-modules on which $Z$ acts by different one dimensional characters, then

$$\text{Ext}^i_A(M, N) = 0, \quad \text{for all } i.$$ 

Now, let $M$ be an indecomposable finite dimensional $H_n^{\text{aff}}$-module. This has a finite filtration by irreducible $H_n^{\text{aff}}$-modules, and the lemma implies $Z(H_n^{\text{aff}})$ acts by the same character on each subquotient. It follows that

Proposition 4.4. $\text{Rep}H_n^{\text{aff}} \simeq \bigoplus_{s \in (R^\times)^n/S_n} \text{Rep}_s H_n^{\text{aff}}$ (direct sum of categories). A module $M$ is in $\text{Rep}_s H_n^{\text{aff}}$ if and only if the support of $M$ as a $Z(H_n^{\text{aff}})$-module is $s \in (R^\times)^n/S_n$, i.e. if and only if (i) there exists an $s' \in (R^\times)^n$ with $M_{s'} \neq 0$, and $S_n \cdot s' = s$, and (ii) if $M_{s'} = 0$, then $S_n \cdot s' = s$.

The summands above are called blocks of the category $\text{Rep}H_n^{\text{aff}}$. If $s \in (R^\times)^n$, $\text{Rep}_s H_n^{\text{aff}} = \lim_k \text{Rep}(H_n^{\text{aff}}/Z^k H_n^{\text{aff}})$, where $Z_s = \{f \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^S_n \mid f(s) = 0\}$. If $N \in \text{Rep}_s H_n^{\text{aff}}$ we say that $N$ has central character $s$.

4.2. Fix a function $\lambda: \mu_q \to \mathbb{Z}_+$ such that $\sum_{i \geq 0} \lambda_i = r < \infty$.

The Ariki-Koike algebra, or cyclotomic Hecke algebra, is the $R$-algebra with generators $T_1, \ldots, T_{n-1}$ and $T_0$

and relations

$$\prod_{q \in \mu_q} (T_0 - q^i)^{\lambda_i} = 0$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$$

as well as the braid relations (1) and quadratic relations (2) in the definition of the finite Hecke algebra above for $i \geq 1$.

In particular, if $r = 1$ then this is just the finite Hecke algebra, and if $r = 2$ this is the Hecke algebra of type $B_n$ or $C_n$ with possibly unequal parameters. The finite Hecke algebra is always a subalgebra of the cyclotomic Hecke algebra.

There is a surjective algebra homomorphism, first defined by Cherednik

$$\text{ev} = \text{ev}_\lambda: H_n^{\text{aff}} \to H_n^\lambda$$

defined on the generators by $T_i \mapsto T_i, \quad 1 \leq i \leq n - 1$

$$X_1 \mapsto T_0, \quad X_i \mapsto q^{1-i} T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1}$$

Write $\text{ev}^*: \text{Rep} H_n^\lambda \to \text{Rep} H_n^{\text{aff}}$ for the induced map of modules.

The image $\text{ev}(R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}])$ form a commutative subalgebra of $H_n^\lambda$, (“Murphy operators”) and as $\text{ev}$ is surjective, the image $\text{ev}(Z(H_n^{\text{aff}}))$ is contained in the center of $H_n^\lambda$. This implies the category of $H_n^{\text{aff}}$-modules splits up into a direct sum, indexed by characters of $Z(H_n^{\text{aff}})$,

$$\text{Rep} H_n^\lambda = \bigoplus \text{Rep}_s H_n^\lambda.$$ 

The key result about these algebras is:

Proposition 4.5 (Ariki-Koike). The algebra $H_n^\lambda$ is finite dimensional, of dimension $r^n n!$, where $r = \sum_{i \in \mu_q} \lambda_i$. 
The image under $ev$ of the elements
\[ X_1^{a_1} \cdots X_n^{a_n} T_w, \quad 0 \leq a_i < r, w \in S_n \]
form a basis of $H_n^\lambda$.

Consider the modules $M$ for $H_n^{\text{aff}}$ such that the only eigenvalues of $X_1$ on $M$ are powers of $q$. Such modules form a full subcategory of $\text{Rep} H_n^{\text{aff}}$ which we denote $\text{Rep}_q H_n^{\text{aff}}$.

If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact in $\text{Rep} H_n^{\text{aff}}$, and any two of the modules $M_i$ are in $\text{Rep}_q H_n^{\text{aff}}$, then so is the third, i.e. $\text{Rep}_q H_n^{\text{aff}}$ is closed under subquotients and extensions.

Further, $\text{Rep}_q H_n^{\text{aff}}$ is a direct sum of blocks of the category $\text{Rep} H_n^{\text{aff}}$; i.e.

**Lemma 4.6.** If $M \in \text{Rep}_q H_n^{\text{aff}}$, and $N \in \text{Rep} H_n^{\text{aff}}$ with $\text{Ext}^*(M, N) \neq 0$ or $\text{Ext}^*(N, M) \neq 0$, then $N \in \text{Rep}_q H_n^{\text{aff}}$ also. More precisely,

\[ \text{Rep}_q H_n^{\text{aff}} = \bigoplus_{s \in \mu_q^n / S_n} \text{Rep}_s H_n^{\text{aff}}. \]

This is immediate from the following lemma, and the above description of the center of $H_n^{\text{aff}}$.

**Lemma 4.7.** If $M$ is a module for $H_n^{\text{aff}}$ and there is some $i$ such that the only eigenvalues for $X_i$ on $M$ are powers of $q$, then for all $j$, the only eigenvalues of $X_j$ on $M$ are powers of $q$.

**Proof.** Write $X = X_{-1}, T = T_{-1}$. Let $v$ be an eigenvector for $X$ and for $TXT = qX$. It is enough to show that the eigenvalues of $X$ and $TXT$ on $v$ differ by a power of $q$. Consider the space spanned by $v$ and $Tv$. This is $X$-stable. If it is two dimensional, then with respect to the basis $v, Tv$ we have $T$ has matrix\[
\begin{pmatrix}
0 & q \\
1 & q - 1
\end{pmatrix}
\]
and $X$ has matrix\[
\begin{pmatrix}
\mu & a \\
0 & \mu'
\end{pmatrix}.
\]
It follows that $TXT$ has matrix\[
\begin{pmatrix}
qu' & q\mu'(q - 1) \\
\delta & q\mu + (q - 1)\delta
\end{pmatrix}
\]
where $\delta = a + \mu'(q - 1)$. By assumption, $\delta = 0$, so $TXT$ has eigenvalues $q\mu, q\mu'$ and we are done. If $Tv$ is a multiple of $v$, then either $Tv = -v$ or $Tv = qv$. In the first case $TXT$ has eigenvalue $\mu$, in the second case $q^2\mu$, and we’re done. \qed

Given $\lambda; \mu \to \mathbb{Z}_+; \sum \lambda_i < \infty$, it is immediate from the definition of $H_n^\lambda$ that if $M \in \text{Rep} H_n^\lambda$, $ev^* M \in \text{Rep}_q H_n^{\text{aff}}$. Define $\text{Rep}_q H_n^{\text{aff}}$ to be the full subcategory of $\text{Rep}_q H_n^{\text{aff}}$ whose objects are the modules $M$ such that the Jordan blocks of $X_1$ on $M$ with eigenvalue $q^i$ have size less than or equal to $\lambda_i$ (and there are no other eigenvalues), i.e. $\text{Rep}_q H_n^{\text{aff}}$ consists of modules annihilated by $\prod (X_1 - q^i)^{\lambda_i}$.

Then if $M \in \text{Rep}_q H_n^{\text{aff}}$, and $N$ is a subquotient of $M$, then $N \in \text{Rep}_q H_n^{\text{aff}}$ also. However, extensions of modules in $\text{Rep}_q H_n^{\text{aff}}$ need not be in $\text{Rep}_q H_n^{\text{aff}}$.

We clearly have

\[ ev^*: \text{Rep} H_n^\lambda \to \text{Rep}_q H_n^{\text{aff}} \]
is an equivalence of categories.

Given any $M \in \text{Rep}_q H_n^{\text{aff}}$, there are infinitely many $\lambda$ such that $M \in \text{Rep}_q H_n^{\text{aff}}$. More precisely, define a partial order on $\lambda$ by $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i$. Then

\[ \mu \leq \lambda \implies \text{Rep}_q^\mu H_n^{\text{aff}} \subseteq \text{Rep}_q^\lambda H_n^{\text{aff}} \]
and
\[ \text{Rep}_{q} H_{n}^{\text{aff}} = \lim_{\rightarrow} \lambda \text{Rep}_{q} H_{n}^{\text{aff}}. \]

The exact functor \( \text{ev}^{*} = \text{ev}_{\lambda}^{*} : \text{Rep} H_{n}^{\lambda} \to \text{Rep} H_{n}^{\text{aff}} \) has a left adjoint \( \text{pr}_{\lambda} \), and a right adjoint \( \tilde{\text{pr}}_{\lambda} \). To define them, write
\[ I_{\lambda} = \ker(\text{ev} : H_{n}^{\text{aff}} \to H_{n}^{\lambda}). \]

As \( \text{ev} \) is an algebra homomorphism, \( I_{\lambda} \) is a two sided ideal, and as \( \text{ev} \) is surjective \( H_{n}^{\lambda} = H_{n}^{\text{aff}} / I_{\lambda} \).

Now if \( N \in \text{Rep} H_{n}^{\text{aff}} \), we have \( N / I_{\lambda} N \) is an \( H_{n}^{\lambda} \)-module on which \( I_{\lambda} \) acts trivially and so \( N / I_{\lambda} N \) is an \( H_{n}^{\lambda} \)-module. If \( M \in \text{Rep} H_{n}^{\lambda} \) we have
\[ \text{Hom}_{H_{n}^{\lambda}}(N / I_{\lambda} N, M) = \text{Hom}_{H_{n}^{\text{aff}}}(N / I_{\lambda} N, \text{ev}^{*} M) = \text{Hom}_{H_{n}^{\text{aff}}}(N, \text{ev}^{*} M) \]
and so if we define
\[ \text{pr}_{\lambda}(N) = N / I_{\lambda} N \]
and
\[ \tilde{\text{pr}}_{\lambda}(N) = N I_{\lambda} \]
we have proved that \( \text{pr}_{\lambda} \) is the left adjoint to \( \text{ev}^{*} \). Similarly, \( \tilde{\text{pr}}_{\lambda} \) is the right adjoint, and neither functor is exact.

Define
\[ \text{Rep}_{q}^{\text{aff}} = \bigoplus_{n \geq 0} \text{Rep}_{q} H_{n}^{\text{aff}} \]
\[ \text{Rep}_{q}^{\lambda} = \bigoplus_{n \geq 0} \text{Rep} H_{n}^{\lambda} \overset{\text{ev}^{*}}{\longrightarrow} \bigoplus_{n \geq 0} \text{Rep}_{q} H_{n}^{\text{aff}} \]
\[ \text{Rep}_{q}^{\text{fin}} = \bigoplus_{n \geq 0} \text{Rep}_{q}^{\text{fin}} \overset{\text{ev}^{*}}{\longrightarrow} \bigoplus_{n \geq 0} \text{Rep}_{q}^{\lambda} H_{n}^{\text{aff}}. \]

We will investigate some rigid structures on these categories.

5. Generalities on Induction and Restriction

The results in this section are easy and are mostly well known (with the possible exception of 5.6). We omit proofs which are trivial variants of what is already in the literature.

5.1. Recall that if \( A \subset B \) are \( R \)-algebras, the exact functor of restriction
\[ \text{Res}_{A}^{B} : \text{B-mod} \to \text{A-mod} \]
has left and right adjoints, \( \text{Ind} \) and \( \tilde{\text{Ind}} \) defined by
\[ \text{Ind}_{A}^{B} : \text{A-mod} \to \text{B-mod} \quad M \mapsto B \otimes_{A} M \]
\[ \tilde{\text{Ind}}_{A}^{B} : \text{A-mod} \to \text{B-mod} \quad M \mapsto \text{Hom}_{A}(B, M); \]
i.e. we have the Frobenius reciprocity
\[ \text{Hom}_{B}(\text{Ind}_{A}^{B} M, N) = \text{Hom}_{A}(M, \text{Res} N) \quad \text{Hom}_{B}(N, \tilde{\text{Ind}} M) = \text{Hom}_{A}(\text{Res} N, M). \]
If \( B \) is a free \( A \)-module, then \( \text{Ind} \) and \( \tilde{\text{Ind}} \) are exact functors also. Further, if \( A \subset B \subset C \) are inclusions of \( R \)-algebras, we have transitivity of induction and restriction:
\[ \text{Res}_{A}^{B} \text{Res}_{B}^{C} = \text{Res}_{A}^{C}, \quad \text{Ind}_{B}^{C} \text{Ind}_{A}^{B} = \text{Ind}_{A}^{C}, \quad \tilde{\text{Ind}}_{B}^{C} \tilde{\text{Ind}}_{A}^{B} = \tilde{\text{Ind}}_{A}^{C}. \]
Now apply these remarks to affine and cyclotomic Hecke algebras. Given a sequence \( P = (a_1, \ldots, a_k) \) of non-negative integers, we have an obvious embedding
\[
H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}} \hookrightarrow H_{a_1 + \cdots + a_k}^{\text{aff}}
\]
which makes \( H_{a_1 + \cdots + a_k}^{\text{aff}} \) a free \( H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}} \)-module. Applying the previous remarks we get exact functors \( \text{Res}, \text{Ind}, \text{Ind}^{\text{aff}} \). These functors depend on the order \( (a_1, \ldots, a_k) \) and not just on the underlying set!

Write \( \text{Rep}_q(H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}}) \) for the full subcategory of modules for \( H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}} \) on which each \( X_j \) acts with eigenvalues in \( \mu_q \). Then the following is evident.

**Lemma 5.1.** (i) \( \text{Res} \) and \( \text{Ind} \) define functors \( \text{Rep}_q(H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}}) = \text{Rep}_q H_{a_1 + \cdots + a_k}^{\text{aff}}. \)

(ii) \( K(\text{Rep}_q(H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}})) = K(\text{Rep}_q H_{a_1}^{\text{aff}}) \otimes \cdots \otimes K(\text{Rep}_q H_{a_k}^{\text{aff}}). \)

Similarly, we may define induction and restriction for finite Hecke algebras, i.e. functors
\[
\text{Rep}_P^{\text{Ind}} \xrightarrow{\text{Res}} \text{Rep}_H.
\]

It is clear that

**Lemma 5.2.** \( \text{ev}^* \text{Res}^{H_{n}^{\text{aff}}}_{H_{P}} = \text{Res}^{H_{n}^{\text{aff}}}_{H_{P}^{\text{aff}}} \text{ev}^*. \)

But note that though \( \text{Ind}^{H_{n}^{\text{aff}}}_{H_{P}^{\text{aff}}} \text{ev}^* \) and \( \text{ev}^* \text{Ind}^{H_{n}}_{H_{P}} \) seem quite different, they are related. The results of section 5.3 describe the relation.

5.2. **Mackey formula.** We now state the Mackey formula. Unfortunately, it requires some notation.

Given a sequence \( P = (a_1, \ldots, a_k) \) of positive integers, as above, with \( \sum a_i = n \), write \( H_P^{\text{aff}} = H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}} \hookrightarrow H_n^{\text{aff}}, S_P = S_{a_1} \times \cdots \times S_{a_k} \hookrightarrow S_n, \) and \( H_P = H_{a_1} \otimes \cdots \otimes H_{a_k}. \) Then if \( P' \) is another such sequence, we define a partial order on \( S_P \setminus S_n / S_{P'} \) by setting \( x < y \) if the maximal length coset representative of \( x \) is less than that of \( y \) in Bruhat order.

Now if \( x \) is a minimal length coset representative of an element \( x \in S_P \setminus S_n / S_{P'} \), write \( P \cap \pi P' \) for the new sequence of positive integers defined by listing the parts of the ordered partition of \( 1, \ldots, n \) defined by \( P \cap \pi P' \). Note \( H_{P' \cap \pi P'}^{\text{aff}} \) is isomorphic to \( H_{P' \cap \pi P'}^{\text{aff}} \) by the isomorphism “conjugation by \( x^{\pi} \)” which sends
\[
T_w \mapsto T_{xwx^{-1}}, \quad X_i \mapsto X_{x^{-1}i}.
\]

**Lemma 5.3.** (i) \( H_n^{\text{aff}} \) is a free right \( H_P^{\text{aff}} \)-module, and
\[
H_n^{\text{aff}} = \bigoplus_{w \in S_P \setminus S_n} H_P^{\text{aff}} \cdot T_w.
\]

Similarly for \( H_n \).

(ii) If \( x \in S_P \setminus S_n / S_{P'}, \) then
\[
H_P^{\text{aff}} \cdot T_x \cdot H_P^{\text{aff}} = \sum_{y \leq x} H_P^{\text{aff}} \cdot T_y \cdot H_P^{\text{aff}} = \bigoplus_{a \leq x} H_P^{\text{aff}} \cdot T_a = \bigoplus_{b \leq x} T_b \cdot H_P^{\text{aff}}.
\]
Corollary 5.4. Res_{H_n^{\text{aff}}} H_{P^{\text{aff}}} M admits a filtration with subquotients isomorphic to
\[
\begin{align*}
\text{Ind}_{P^{\text{aff}} \cap xP^{\text{aff}}}^{H_{P^{\text{aff}}}} w^{-1} \text{Res}_{P^{\text{aff}} \cap x^{-1} P^{\text{aff}}}^{H_{P^{\text{aff}}}} M.
\end{align*}
\]

In the above filtration, Ind_{P^{\text{aff}} \cap xP^{\text{aff}}}^{H_{P^{\text{aff}}}} Res_{P^{\text{aff}} \cap x^{-1} P^{\text{aff}}}^{H_{P^{\text{aff}}}} M always sits as a subobject of Res Ind M.

If we apply the above corollary to \( P = (1, \ldots, 1) \), we get a particularly nice consequence. Write
\[
\text{ch}(M) = \sum_{s \in (R^\times)^n} \dim M[s] \cdot [s] \in K(\text{Rep} R[X_1^{\pm1}, \ldots, X_n^{\pm1}])
\]
for the character of \( M \) as a module for \( R[X_1^{\pm1}, \ldots, X_n^{\pm1}] \).

Lemma 5.5 (shuffle lemma). If \( M \in \text{Rep}_q H_n^{\text{aff}}, N \in \text{Rep}_q H_m^{\text{aff}}, \) then
\[
\text{ch} \text{Ind}(M \boxtimes N) = \sum_{s'} (\dim M[s'] \cdot \dim N[s''])[s]
\]
where if \( s = (s_1, \ldots, s_{n+m}), s' \) is a subsequence \( (s_{i_1}, \ldots, s_{i_n}) \) where \( i_1 < \cdots < i_n \), and \( s'' \) is the sequence obtained from \( s \) by removing \( s' \). In other words, the spectrum of \( \text{Ind}(M \boxtimes N) \) is obtained by shuffling the spectrum of \( M \) and \( N \).

5.3. Boring central characters. Let \( P = (a_1, \ldots, a_k) \) be a sequence of positive integers, \( \sum a_i = n \), and write \( H_{P^{\text{aff}}} = H_{a_1}^{\text{aff}} \otimes \cdots \otimes H_{a_k}^{\text{aff}} \hookrightarrow H_n^{\text{aff}} \) as before. Write \( S = R[X_1^{\pm1}, \ldots, X_n^{\pm1}] \), so that \( Z(H_{P^{\text{aff}}}) = S^{S_P} \), and form the enhanced Hecke algebra
\[
\widetilde{H}_{P^{\text{aff}}} = H_{P^{\text{aff}}} \otimes_{Z(H_{P^{\text{aff}}})} S.
\]
Clearly \( Z(\widetilde{H}_{P^{\text{aff}}}) = S \). Define the \( q \)-discriminant, \( \Delta_q^P : (R^\times)^n \to R \) by
\[
\Delta_q^P(s_1, \ldots, s_n) = \prod (s_i - qs_j)
\]
where the product runs over all pairs \((i, j)\) such that \( 1 \leq i, j \leq n \), and it is not the case that both \( i \) and \( j \) lie in an interval \( a_1 + \cdots + a_r + 1, \ldots, a_1 + \cdots + a_r + a_{r+1} \). Also write
\[
\Delta(s_1, \ldots, s_n) = \Delta_1^{(1,1,\ldots,1)}(s_1, \ldots, s_n) = \prod_{i \neq j} (s_i - s_j)
\]
for the discriminant. So if \( \Delta(s) \neq 0 \), there are \( n! \) points in the \( S_n \)-orbit of \( s \). Now, if \( s \in (R^\times)^n \), let’s write \( \text{Rep}_q H_{P^{\text{aff}}} \) for the category of finite dimensional \( H_{P^{\text{aff}}} \)-modules \( M \) such that the support of \( M \) as an \( Z(\widetilde{H}_{P^{\text{aff}}}) \)-module is \( s \in (R^\times)^n \).
Theorem 5.6. (i) If \( \Delta_q(s) \neq 0 \), induction defines an equivalence of categories
\[
\text{Rep}_s H_n^\text{aff} \to \text{Rep}_s H_n^\text{aff}, \quad M \mapsto H_n^\text{aff} \otimes_{H_n^\text{aff}}^\text{Ind} M.
\]
(ii) If \( \Delta(s) \neq 0 \), there is an equivalence of categories
\[
\text{Rep}_s H_n^\text{aff} \cong \text{Rep}_s H_n^\text{aff}.
\]
(iii) Regardless, there is always an isomorphism
\[
\text{K}(\text{Rep}_s H_n^\text{aff}) \cong \text{K}(\text{Rep}_s H_n^\text{aff}).
\]

In particular, if \( M \) is an irreducible \( H_n^\text{aff} \)-module with central character \( S_n \cdot s \), then by defining that \( S \) acts via evaluation at \( s \), we get an irreducible \( H_n^\text{aff} \)-module, and conversely every irreducible module in \( \text{Rep}_s H_n^\text{aff} \) is of such a form.

The following particular case will be of great importance to us.

Corollary 5.7. \([K]\) The \( H_n^\text{aff} \)-module \( \text{Ind} H_n^\text{aff}(q^i J_1 \boxtimes \cdots \boxtimes q^i J_1) \) is irreducible. It is the unique irreducible \( H_n^\text{aff} \)-module with central character \( (q^i \cdots q^i) \).

Notice that the theorem tells us completely how to reconstruct all of \( \text{Rep}_q H_n^\text{aff} \) once we understand \( \text{Rep}_q H_n^\text{aff} \). (This is one good reason for concentrating on \( \text{Rep}_q H_n^\text{aff} \) for the rest of the paper!)

5.4. A useful criterion for irreducibility.

Proposition 5.8. \( \text{Ind}_{H_n^\text{aff} \otimes H_n^\text{aff}} H_n^\text{aff}(M \boxtimes N) \simeq \text{Ind}_{H_n^\text{aff} \otimes H_n^\text{aff}}^H H_n^\text{aff}(N \boxtimes M) \).

The preceding proposition, when combined with the following useful observation, allows us to detect whether certain induced modules are irreducible.

Lemma 5.9. If \( \text{Ind}(M \boxtimes N) \simeq \text{Ind}(M \boxtimes N) \), and \( M \boxtimes N \) is an irreducible module which occurs with multiplicity one in a composition series for \( \text{Res Ind}(M \boxtimes N) \), then \( \text{Ind}(M \boxtimes N) \) is irreducible.

Proof. Let \( 0 \to K \to \text{Ind}(M \boxtimes N) \to Q \to 0 \) be an exact sequence. Then \( M \boxtimes N \) is a submodule of \( Q \), if \( Q \neq 0 \), by Frobenius reciprocity, and as \( \text{Ind}(M \boxtimes N) \simeq \text{Ind}(M \boxtimes N) \), \( M \boxtimes N \) is also a submodule of \( K \) if \( K \neq 0 \). As \( M \boxtimes N \) occurs with multiplicity one, either \( K \) or \( Q \) is zero, and so \( \text{Ind}(M \boxtimes N) \) is irreducible. \( \square \)

6. Examples

The computations in this section will be used in sections 10 and 12.

6.1. \( H_1^\text{aff} \). Recall \( H_1^\text{aff} = R[x, x^{-1}] \).

For \( q^i \in \mu_q \), and \( n \geq 1 \), let’s write \( q^i J_n \) for the rank \( n \) Jordan block with eigenvalue \( q^i \), i.e.

\[ q^i J_n := R[x]/(x - q^i)^n. \]

This is an indecomposable \( H_1^\text{aff} \)-module, and conversely every indecomposable module in \( \text{Rep}_q H_1^\text{aff} \) is of the form \( q^i J_n \) for a unique \( q^i \in \mu_q \) and \( n \geq 1 \). Further

\[
\text{Hom}(q^i J_n, q^j J_{n'}) = \text{Ext}^1(q^i J_n, q^j J_{n'}) = \begin{cases} 0 & \text{if } q^i \neq q^j \\ R^\text{min}(n, n') & \text{if } q^i = q^j \end{cases}
\]
and
\[ \text{Ext}^k(q^i J_n, q^i J_m) = 0 \quad \text{if} \ k \neq 0, 1. \]

Hence \( \text{Rep}_H^\text{aff} \) is the direct sum of \( \mu_q \)-copies of the same category, the category of finite Jordan blocks (with a fixed eigenvalue).

6.2. \( H^\text{aff} \)-modules. We list all the indecomposable \( H^\text{aff} \)-modules. First, an easy computation. Write \( S = R[X_1^{-1}, X_2^\pm 1] \).

**Lemma 6.1.** Set \( p = (1 - X_1 X_2^{-1}) T_1 - (q - 1) \). Then

(i) \( X_1 p = p X_2, \ X_2 p = p X_1 \),

(ii) \( p^2 = (q - X_1 X_2^{-1})(q - X_2 X_1^{-1}) \) is central.

We describe \( \text{Rep}_{s_1, s_2} H^\text{aff} \). Choose \( s = (s_1, s_2) \) in the orbit \( \{(s_1, s_2), (s_2, s_1)\} \), and write \( s' = (s_2, s_1) \). Let \( M \) be a module, and write \( M = M_s + M_{s'} \) for its decomposition into generalized eigenspaces for \( S \).

Clearly \( p M_s \subseteq M_{s'} \) and \( p M_{s'} \subseteq M_s \) by (6.1).

Case 1. \( s_1 \neq s_2 \), and \( s_1 s_2^{-1} \neq q^{\pm 1} \). Then \( p^2 \) is an invertible semisimple automorphism, and \( M = M_s \oplus p M_s \). We have inverse equivalences of categories

\[ \text{Rep}_s S \cong \text{Rep}_s H^\text{aff}, \quad A \mapsto H^\text{aff} \otimes_S A, \quad M_s \mapsto M. \]

Case 2. \( s_2 = q s_1 \), \( s \neq 1 \)

In this case, \( p^2 \) is nilpotent, and \( 1 - X_1 X_2^{-1} \) is invertible. Now, \( \text{ker} \ p \) is \( S \)-stable, and on \( \text{ker} \ p \), \( T_1 \) is \( \frac{q^{-1}}{1 - X_1 X_2^{-1}} \). It follows that \( \text{ker} \ p \) is an \( H^\text{aff} \)-submodule, and that \( \text{ker} \ p = (\text{ker} \ p)_s \oplus (\text{ker} \ p)_{s'} \) is a decomposition as \( H^\text{aff} \)-modules. Further, the indecomposable summands of \( \text{ker} \ p \) as an \( H^\text{fin} \)-module are the indecomposable summands as an \( H^\text{aff} \)-module. First suppose \( q \neq -1 \). Then \( T_1 \) acts semisimply on \( \text{ker} \ p \), hence so does \( X_1 X_2^{-1} \). We suppose for simplicity that the central element \( X_1 X_2 \) acts semisimply.

Then \( \text{ker} \ p \) is a direct sum of one dimensional \( H^\text{aff} \)-modules. On the subspace on which \( T_1 \) acts as \( -1 \), we have \( X_1 = q X_2 \); on the subspace on which \( T_1 \) acts as \( q \), we have \( X_2 = q X_1 \). Now suppose \( q = -1 \). Then \( (T_1 + 1)^2 = 0 = (1 - X_1 X_2^{-1})^2 \), and again there are two isomorphism classes of indecomposable \( H^\text{fin} \)-modules: namely \( T_1 \) acts as \(-1\) or as \( \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \).

Let \( A \in \text{Rep}_{(s_1, q s_1)} \), and set \( A' \in \text{Rep}_{(qs_1, s_1)} \) to be \( (X_1 \leftrightarrow X_2)^* A \). Put \( A_+ = \text{Im}(\text{Ind} A \to \text{Ind} A') = p \text{Ind} A, \quad A_- = \text{Im}(\text{Ind} A' \to \text{Ind} A) = p \text{Ind} A', \) and \( A_0 = \text{Ind} A \). Then if \( A \) is indecomposable, so are \( A_0, A_+, A_- \); and if \( q = -1 \) these give a complete list of (non-isomorphic) indecomposables. For all \( q \), the modules \( (J_1 \boxtimes q J_1)_+ \) and \( (J_1 \boxtimes q J_1)_- \) are the distinct irreducibles.

Case 3. \( s = (s_1, s_1) \).

We have nothing general to add to the description of these modules already given by theorem 6.6; however we record the following, which is proved by direct computation.

**Lemma 6.2.** \( \text{Res}_{H^3}^{H^2} \text{Ind}_{H^2}^{H^\text{aff} \otimes H^2} (q^i J_n \boxtimes q^i J_n) = (a - 1) \cdot q^i J_n + (b - 1) \cdot q^i J_n + q^i J_{n+b} \).

6.3. \( H^3 \). We list the number of the irreducible \( H^3 \)-modules for which \( \prod (s_i - q s_i) = 0 \) and some of the characters of these modules. Note that as \( \text{Rep}_s \), \( H^3 \cong \text{Rep}_{s'} H^\text{s} \), where \( s' = (\alpha s_1, \ldots, \alpha s_n) \) for some \( \alpha \in R^\times \), we may assume \( s_1 = 1 \).
q = -1:
Rep_{(1,1,q)}: There are 3 irreducible modules with central character (11q); two are 2 dimensional, and one is 1 dimensional. Their characters are 2(11q), 2(q11) and (1q1).
q^3 = 1:
There are 6 irreducible representations in Rep_{(1,q,q^3)}; 2 in Rep_{(1,1,q)}, 2 in Rep_{(1,q,q^3)}.
The modules in Rep_{(1,1,q)} have characters 2(11q) + (1q1) and 2(q11) + (1q1).

6.4. \( H_q^{aff} \) when \( q = -1 \).
Rep_{(1,1,1,q)} has three irreducibles: 6(111q) + 2(11q1) = Ind2(11q) \boxtimes 1, 2(11q1) + 2(q11) = Ind((1q1) \boxtimes a) = (aq1) + (qa1) + (q1a) are irreducible. Except for Rep_{(1,q,q^2)} these modules have the same character as when \( q^3 = 1 \).

7. Bernstein-MacDonald Hopf algebra
Induction and restriction give \( K(Rep_q^{aff}) \) and \( K(Rep_q^{fin}) \) the structure of bialgebras. Precisely, if \( M \in Rep_q H_{a_1}^{aff} \), \( N \in Rep_q H_{a_2}^{aff} \), define multiplication
\[ M \cdot N = \text{Ind}_{H_{a_1}^{aff} \otimes H_{a_2}^{aff}}^{H^{aff}} M \boxtimes N \]
and comultiplication \( \Delta M = \bigoplus_{a_1 + a_2 = n} \Delta_{a_1,a_2} M \), where \( \Delta_{a_1,a_2} M = \text{Res}_{H_{a_1}^{aff} \otimes H_{a_2}^{aff}}^{H^{aff}} M \) (and similarly for \( Rep_q^{fin} \)). Then as induction and restriction are exact functors, these descend to give functors
\[ K(Rep_q H_{a_1}^{aff}) \otimes K(Rep_q H_{a_2}^{aff}) = K(Rep_q (H_{a_1}^{aff} \otimes H_{a_2}^{aff})) \xrightarrow{\Delta_{a_1,a_2}} K(Rep_q H_{a_1+a_2}^{aff}) \]
and the properties of Ind and Res translate into the axioms of a bialgebra; viz:
transitivity of induction becomes associativity of multiplication; transitivity of restriction becomes coassociativity of comultiplication; the trivial representation of the trivial algebra \( H_0^{aff} \) is the unit, ditto for the counit, and the Mackey formula is the statement that \( \Delta \) is an algebra homomorphism.

Similarly, define multiplication and comultiplication on \( K(Rep_q^{fin}) \).

Remark 1. The comultiplication on \( K(Rep_q^{aff}) \) is not cocommutative. However, as we shall see, multiplication on \( K(Rep_q^{aff}) \) is commutative. Hence the bialgebra \( K(Rep_q^{aff}) \) is the dual to the enveloping algebra of a Lie algebra; and Theorem 4.1 identifies this algebra. (The structure of the categories \( Rep_q^{fin} \) will be essential for this identification.) Note that \( K(Rep_q^{fin}) \) is both commutative and cocommutative, but because of the previous remark, and because of the structure of this as an algebra over \( \mathbb{Z} \), it is more natural to think of this as the dual to an enveloping algebra also. (That way it is a polynomial algebra; if we don’t take the dual it is a divided power algebra.)
Remark 2. Structure on $\bigoplus \text{Rep} S_n$. For $r > 1$, we can form embeddings of the wreath product
\[
S_n \wr \mathbb{Z}_r \times S_m \wr \mathbb{Z}_r \hookrightarrow S_{n+m} \wr \mathbb{Z}_r
\]
making $\bigoplus_{n\geq 0} \text{Rep}(S_n \wr \mathbb{Z}_r)$ into a Hopf algebra which again classically is known to be the $r^{th}$ tensor product of the Fock space $\bigoplus \text{Rep} S_n$ (see [Mc] for example).

The functors $\text{Rep} \lambda$ are a deformation of the wreath product $S_n \wr \mathbb{Z}_r$, and one can ask if the embedding (5) exists for them. It is a consequence of theorem 14.3 that it does not. However, we always have an action of $K(\text{Rep}^\text{fin}_q)$ on $K(\text{Rep}^\lambda_q)$, which is the Fock space structure when $\lambda = \Lambda_0$.

This action of $K(\text{Rep}^\text{fin}_q)$ on $K(\text{Rep}^\lambda_q)$, even when $\lambda = \Lambda_0$ is only part of the story—the coaction of $K(\text{Rep}^\text{aff}_q)$ provides much more structure. Eventually we will see that the affine algebra $\hat{\mathfrak{sl}}_\ell$ acts on $K(\text{Rep}^\lambda_q)$, and the Bernstein-MacDonald Hopf algebra structure on $K(\text{Rep}^\text{fin}_q)$ is just the principal realization of $\hat{\mathfrak{sl}}_\ell$. But first:

As $\text{ev}^* \circ \text{Res} = \text{Res} \circ \text{ev}^*$, we have

**Lemma 7.1.** The functor $K(\text{Rep}^\text{fin}_q) \xrightarrow{\text{ev}^*} K(\text{Rep}^\text{aff}_q)$ is a homomorphism of coalgebras. In particular, $K(\text{Rep}^\text{fin}_q)$ is a comodule for $K(\text{Rep}^\text{aff}_q)$.

**Warning 1.** $\text{ev}^*$ is not a homomorphism of algebras.

More generally, if $M \in \text{Rep}^\lambda_q$, then $\text{Res}^\text{aff}_q H^\lambda_n \otimes H^\lambda_b \text{ev}^*(M)$ is an element of $\text{Rep}^\lambda_q (H^\lambda_n \otimes H^\lambda_b)$ and

**Lemma 7.2.** $K(\text{Rep}^\lambda_q)$ is a comodule for $K(\text{Rep}^\text{aff}_q)$.

This is saying there are significantly more operations of $K(\text{Rep}^\lambda_q)$ than appear at first sight. In particular, any exact functor $\text{Rep}^\text{aff}_q \xrightarrow{F} R$-mod gives a functor $K(\text{Rep}^\lambda_q) \rightarrow K(\text{Rep}^\lambda_q)$ by composing the comodule structure with $\otimes_R F$

$K(\text{Rep}^\lambda_q) \rightarrow K(\text{Rep}^\lambda_q) \otimes K(\text{Rep}^\text{aff}_q) \xrightarrow{1d \otimes F} K(\text{Rep}^\lambda_q)$.

If $\otimes_R F$ satisfies some finiteness conditions, we obtain left and right adjoint functors $\text{Rep}^\lambda_q \rightarrow \text{Rep}^\lambda_q$ (which in favorable cases agree and are exact). We will now compute some examples of this.

8. THE FUNCTORS $e^*_i$ AND $f^*_i$

In this section we define an action of the generators of $\mathfrak{sl}_\ell$ on $K(\text{Rep}^\lambda_q) = \bigoplus_{n\geq 0} K(\text{Rep} H^\lambda_n)$.

Define functors $e^*_i$ for $q^i \in \mu_q$,
\[
e^*_i : \text{Rep}^\lambda_q H^\lambda_n \rightarrow \text{Rep}^\lambda_q H^\lambda_{n-1} \quad e^*_i : \text{Rep}^\lambda_q H^\lambda_n \rightarrow \text{Rep}^\lambda_q
\]
as follows. If $M \in \text{Rep} H^\lambda_n$, $e^*_i M$ is the generalized eigenspace of $X_n$ with eigenvalue $q^i$. As $X_n$ commutes with $H^\lambda_{n-1}$, $e^*_i M$ is an $H^\lambda_{n-1}$-module. Clearly $X_1$ acts in the same way on $M$ and $e^*_i M$, so if we define $e^*_i M = e^*_i (\text{ev}^*(M))$ for $M \in \text{Rep} H^\lambda_n$, then we have $e^*_i M \in \text{Rep}^\lambda_q$ also.

**Lemma 8.1.** The functors $e^*_i : \text{Rep}^\text{aff}_q \rightarrow \text{Rep}^\text{aff}_q$, $e^*_i : \text{Rep}^\lambda_q \rightarrow \text{Rep}^\lambda_q$ are exact.
Proof. $e_i^*$ is the composite of the exact functors of restriction and the functor of generalized eigenspace, which is exact on torsion (and in particular finite dimensional) $R[x]$-modules.

Remark 3. In the abstract language favored in the previous section, we can write

$$e_i^*M = \lim_m \ker((X_n - q^i)^m; \text{Res}^{H^n_{n-1} \otimes H^n_{n}} M)$$

where we have identified $\text{Rep}^\lambda H^n_n$ with $\text{Rep}^\lambda H^n_n$ via $\text{ev}^\ast$. Note that as $M$ is finite dimensional the direct limit stabilizes.

As the functors $e_i^*$ are exact, we may try and find their adjoints. For formal reasons, adjoints exist in $H^n_n\text{-mod}$ and $H^n_n\text{-mod}$, but that is no guarantee they exist in $\text{Rep} H^n_n$ or $\text{Rep} H^n_n$ (i.e. that they are finite dimensional). Nonetheless, define functors $f_i^*$, $f_i^*$ from $H^n_{n-1}\text{-mod}$ to $H^n_n\text{-mod}$ by setting, for $M \in H^n_{n-1}\text{-mod}$,

$$f_i^*(M) = \lim_m \text{pr}_\lambda(\text{Ind}^{H^n_{n-1} \otimes H^n_{n}}(M \boxtimes q^i J_m))$$

and

$$f_i^*(M) = \lim_m \text{pr} \lambda(\text{Ind}^{H^n_{n-1} \otimes H^n_{n}}(M \boxtimes q^i J_m))$$

where $q^i J_m$ is the Jordan block of size $m$ and eigenvalue $q^i$, $\text{pr}$ and $\text{pr}$ are the left and right adjoints to $\text{ev}^*$: $\text{Rep} H^n_n \rightarrow \text{Rep} H^n_n$, and the direct and inverse limits are taken with respect to the systems

$$q^i J_0 \hookrightarrow q^i J_1 \hookrightarrow \cdots \hookrightarrow q^i J_m \hookrightarrow \cdots, \quad q^i J_0 \leftarrow \cdots \leftarrow q^i J_m \leftarrow \cdots$$

given by multiplication by $(x - q^i):R[x]/(x - q^i)^m \rightarrow R[x]/(x - q^i)^m$, and $1 \hookrightarrow 1$, respectively.

Let us abbreviate $\text{Res}^{H^n_{n-1} \otimes H^n_{n}}$ by $\text{Res}$, and similarly abbreviate $\text{Ind}^{H^n_{n-1} \otimes H^n_{n}}$ by $\text{Ind}$.

Proposition 8.2. If $N \in H^n_{n}\text{-mod}$, the inverse system

$$\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m)$$

stabilizes after finitely many terms.

Proof. If $M$ is an $H^n_{n+1}\text{-module}$ generated by an $R$-subspace $W \subset M$, then $\text{pr}_\lambda(M)$ is an $H^n_{n+1}\text{-module}$ generated by the image of $W$. In particular, if $M$ is finitely generated, then $\text{pr}_\lambda(M)$ is finite dimensional (it is a quotient of the finite dimensional vector space $H^n_{n+1} \otimes W$). Now if $N \in H^n_{n}\text{-mod}$ is generated by a subspace $W'$, $\text{Ind}(N \boxtimes q^i J_m)$ is generated by $W' \boxtimes q^i J_1 \simeq W'$ and the system $\text{pr}(\text{Ind}(N \boxtimes q^i J_m))$ are all quotients of a fixed $H^n_{n+1}\text{-module}$ $H^n_{n+1} \otimes W'$, which if $N$ is finitely generated is finite dimensional. Hence the inverse system stabilizes in this case.

A more careful analysis shows that the system stabilizes for $m$ greater than a fixed constant which depends on $n, \lambda$ and not on $W'$. This gives the proposition in general, but as we will not use it for non-finitely generated modules, we will omit further details.

Corollary 8.3. If $M \in \text{Rep} H^n_{n-1}$, then $f_i^* M$ and $f_i^* M$ are finite dimensional, i.e. $f_i^*, \hat{f}_i^*$ restrict to functors $\text{Rep} H^n_{n-1} \rightarrow \text{Rep} H^n_{n}$.
We have defined \( f_i^* \), \( \hat{f}_i^* \) so that

**Proposition 8.4.** The functor \( f_i^*: \text{Rep} H_{n-1}^\lambda \to \text{Rep} H_n^\lambda \) is left adjoint to \( e_i^*: \text{Rep} H_n^\lambda \to \text{Rep} H_{n-1}^\lambda \). Similarly, \( \hat{f}_i^* \) is right adjoint to \( e_i^* \).

\[
\text{Proof.} \quad \text{We prove } f_i^* \text{ is left adjoint; the proof for } \hat{f}_i^* \text{ is similar. This is almost, but not quite, a formal result. To see that, observe that if } M \in H_n^\lambda, \\
e_i^* M = \lim_{m} \ker((X_n - q^i)^m, \text{Res } M) \\
= \lim_{m} \text{Hom}_{R[X_n^{\pm 1}]}(q^i J_m, \text{Res } M)
\]

where the second limit is taken over the system \( q^i J_0 \leftarrow q^i J_1 \leftarrow \cdots \) used in the definition of \( f_i^* \). This equals

\[
\text{Hom}_{R[X_n^{\pm 1}]}(\lim_{m} q^i J_m, \text{Res } M).
\]

if \( M \) is finite dimensional, or more generally \( R[X_n^{\pm 1}]/(X_n - q^i) \)-torsion, but not in general. (For example, if \( n = 1 \) and \( M = R[[X_n - q^i]] \) they clearly differ, and indeed \( e_i^* \) is not exact on the category of all \( H_n^\lambda \)-modules.)

However, if \( M \in H_n^\lambda \)-mod, then as \( H_n^\lambda \) is finite dimensional, these are equal, and the direct and inverse limits above stabilize after finitely many terms (for a suitable \( \lambda > \dim H_n^\lambda \). Hence if \( M \in H_n^\lambda \)-mod, and \( N \in H_{n-1}^\lambda \)-mod, then

\[
\text{Hom}_{H_n^\lambda}(N, \text{ev}^*(e_i^* M)) = \text{Hom}_{H_{n-1}^\lambda}(N, \lim_{m} \text{Hom}_{R[X_n]}(q^i J_m, \text{Res } \text{ev}^* M)) \\
= \lim_{m} \text{Hom}_{H_{n-1}^\lambda}(N, \text{Hom}_{R[X_n]}(q^i J_m, \text{Res } \text{ev}^* M))
\]

as the direct limit stabilizes after finitely many terms, and this equals

\[
= \lim_{m} \text{Hom}_{H_{n-1}^\lambda} \otimes H_n^\lambda(N \boxtimes q^i J_m, \text{Res } \text{ev}^* M) \\
= \lim_{m} \text{Hom}_{H_n^\lambda}(\text{Ind}(N \boxtimes q^i J_m), \text{ev}^* M),
\]

as \( \text{Ind} \) is left adjoint to \( \text{Res} \). As \( \text{pr}_\lambda = \text{pr} \) is left adjoint to \( \text{ev}^* \), we get

\[
= \lim_{m} \text{Hom}_{H_n^\lambda}(\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m), M) \\
= \text{Hom}_{H_n^\lambda}(\lim_{m} \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m), M) \\
= \text{Hom}_{H_n^\lambda}(f_i^* N, M)
\]

where we have used, once again, the fact that this limit of Hom’s stabilizes after finitely many terms.

In particular, restricting to finite dimensional modules \( \text{Rep}^\lambda_q \), we get

**Corollary 8.5.** \( f_i^*: \text{Rep} H_{n-1}^\lambda \to \text{Rep} H_n^\lambda \) is left adjoint to \( e_i^*: \text{Rep} H_n^\lambda \to \text{Rep} H_{n-1}^\lambda \), and \( \hat{f}_i^*: \text{Rep} H_{n-1}^\lambda \to \text{Rep} H_n^\lambda \) is right adjoint.

\[
\text{Note that the proof above shows for any } N \in H_{n-1}^\lambda \text{-mod, and any } M \in \text{Rep} H_n^\lambda, \\
\text{that } \text{Hom}(f_i^* N, M) = \text{Hom}(\text{pr}_\lambda \text{Ind}(N \boxtimes \lim_{m} q^i J_m), M)
\]
so by uniqueness of adjoints
\[ f_i^* N = \text{pr}_\lambda \text{Ind}(N \boxtimes \lim_m q^i J_m). \]

(But it is certainly not true that \( f_i^* N = \text{pr}_\lambda \text{Ind}(N \boxtimes \lim_m q^i J_m) \) for arbitrary \( N \in H^{\text{aff}}_{n-1} \)-mod, or even for arbitrary \( N \in \text{Rep} H^\lambda_{n-1} \). Obviously.)

Recall that \( \text{Rep} H^\lambda_\text{aff} = \bigoplus \text{Rep}_s H^\lambda_\text{aff} \), where the sum is over \((R^\times)^n/S_n\), the orbits of the symmetric group on \((R^\times)^n\). A module \( M \) is in \( \text{Rep}_s H^\lambda_\text{aff} \) if \( M \), when considered as a module for \( Z(H^\lambda_\text{aff}) \), has support a single orbit of \( S_n \) on \((R^\times)^n\). Further, if \( M \) is indecomposable and \( s = (s_1, \ldots, s_n) \in (R^\times)^n \) is such that the weight space \( M_\mu = \{m \in M \mid X_i \cdot m = s_i m, 1 \leq i \leq n \} \) is non-zero, then \( M \in \text{Rep}_s H^\lambda_\text{aff} \). (Conversely, if \( M \) is indecomposable, there is always some \( s \in (R^\times)^n \) such that \( M_\mu \neq 0 \).

If \( s = (s_1, \ldots, s_n) \), and \( q^i \in \{s_1, \ldots, s_n\} \), say \( s_n = q^i \), let’s write \( s \setminus q^i \) for the unique orbit in \((R^\times)^n/S_n \) obtained by deleting \( q^i \) from the list. Then lemma \[ \text{Lemma 8.6.} \] has a refinement (which needs no proof).

**Lemma 8.6.** \( e_i^* \) is a functor \( \text{Rep}_s H^\lambda_\text{aff} \to \text{Rep}_{s \setminus \{q^i\}} H^\lambda_{n-1} \), and \( e_i^* M = 0 \) if \( q^i \notin \{s_1, \ldots, s_n\} \).

Dually, write \( s + q^i \) for the orbit of \( s_{n+1} \cdot (s_1, \ldots, s_n, q^i) \in (R^\times)^{n+1}/S_{n+1} \). We can now refine the previous proposition to:

**Proposition 8.7.** \( f_i^*: \text{Rep}_s H^\lambda_{n-1} \to \text{Rep}_{s+q^i} H^\lambda_n \) is left adjoint to \( e_i^*: \text{Rep}_{s+q^i} H^\lambda_n \to \text{Rep}_s H^\lambda_{n-1} \), and \( f_i^* \) is right adjoint.

**Proof.** Let \( M \) be an indecomposable \( H^\lambda_{n-1} \)-module, and \( v \) a non-zero element in the weight space \( M_\mu \) for some \( s \in (R^\times)^n \). Then all vectors in \( Rv \boxtimes q^i J_m \) are in \( M_\mu \) in the generalized eigenspace with eigenvalue \( s + q^i \), and hence all subquotients of \( \text{Ind}(ev^* M \boxtimes q^i J_m) \) are in \( \text{Rep}_{s+q^i} H^\lambda_n \). But \( N \in \text{Rep}_s H^\lambda_\text{aff} \), then \( \text{pr}_\lambda N \in \text{Rep}_s H^\lambda_n \), and so \( f_i^* M \in \text{Rep}_{s+q^i} H^\lambda_n \). (The proof for \( f_i^* \) is similar.)

**Lemma 8.8.** If \( M \in \text{Rep} H^\lambda_n \), then

(i) \( \text{Res}^{H^\lambda_n}_{H^\lambda_{n-1}} M \simeq \bigoplus_{i \in \mu_q} e_i^* M \)

(ii) \( \text{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n} M \simeq \bigoplus_{i \in \mu_q} f_i^* M \simeq \bigoplus_{i \in \mu_q} \bar{f}^*_i M \)

**Proof.** For \( M \in \text{Rep} H^\lambda_n \), \( X_n \) acts on \( M \) with eigenvalues in \( \mu_q \), so \((8.8)\) is immediate from the definition of \( e_i^* \). But then \((8.8)\) follows as both \( \sum f_i^* \) and \( \text{Ind}^{H^\lambda_{n+1}}_{H^\lambda_{n-1}} \) are left adjoint to \( \text{Res}^{H^\lambda_{n+1}}_{H^\lambda_{n-1}} : \text{Rep} H^\lambda_n \to \text{Rep} H^\lambda_{n-1} \), and left adjoints are unique when they exist.

**Corollary 8.9.** As functors \( \text{Rep}^\lambda_q \to \text{Rep}^\lambda_q \), the functors \( f_i^*, \bar{f}^*_i \) satisfy

(i) \( f_i^* \simeq \bar{f}^*_i \)

(ii) \( f_i^* \) is exact.

**Proof.** As \( f_i^*, \bar{f}^*_i \) are direct summands of the exact functor \( \text{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n} \), they are exact. Proposition \[ \text{Proposition 8.7} \] identifies them as the same direct summands.
Remark 4. We could also use this as a definition of \( f_i^* \), but then we would not see the uniform dependence on \( \lambda \), or be able to prove the theorems of the next section.

Remark 5. Note that for \( N \in \text{Rep} H^\lambda_{n-1} \),

\[
\text{Ind}(ev^* N \boxtimes q^i J_m) \neq \text{Ind}(ev^* N \boxtimes q^i J_m).
\]

Because \( f_i^* \) is left adjoint to an exact functor, or because restriction and induction take free modules to free modules, we observe

**Lemma 8.10.** If \( M \in \text{Rep}_q^\lambda \) is projective, then so are \( e_i^* M \) and \( f_i^* M \).

### 8.1. Divided Powers

We can do slightly better. Fix \( i \in \mu_q \). For each \( n \geq 1 \) we define the divided powers of \( e_i^* \) and \( f_i^* \) as exact functors \( \text{Rep}_q^\lambda \to \text{Rep}_q^\lambda \).

First, define \( e_i^{(n)*}: \text{Rep}_q^\lambda_{a+n} \to \text{Rep}_q^\lambda_a \) as follows

\[
e_i^{(n)*} (M) = \lim_{\rightarrow} \text{Hom}_{H^\lambda_q \otimes H^\lambda_n} (H^\lambda_a \boxtimes P, \text{Res}_{H^\lambda_q \otimes H^\lambda_n}^\lambda M)
\]

where the limit is taken over the small category whose objects are the finite dimensional approximations \( P \) to a projective module for \( H^\lambda_n \) which surjects onto \( H^\lambda_{a+n} \otimes H^\lambda_n q^i J_1 \otimes \cdots \otimes q^i J_1 \). More precisely, the objects of this category are the diagrams

\[
\mathcal{P} \to P \to \text{Ind}_{H^\lambda_q \otimes H^\lambda_n} (q^i J_1 \otimes \cdots \otimes q^i J_1)
\]

where \( P \) is a finite dimensional \( H^\lambda_n \)-module and \( \mathcal{P} \) is a projective module for \( H^\lambda_n \). Morphisms from \( P \) to \( P' \) are commutative diagrams such that all maps are surjective.

Then define \( e_i^{(n)*}: \text{Rep}_{q}^\lambda_{a+n} \to \text{Rep}_q^\lambda_a \) as \( e_i^{(n)*} \circ ev^* \), and define \( f_i^{(n)*}: \text{Rep}_q^\lambda_a \to \text{Rep}_q^\lambda_{a+n} \) by

\[
f_i^{(n)*} (M) = \lim_{\leftarrow} \text{pr}_1 \text{Ind}_{H^\lambda_q \otimes H^\lambda_n} (M \boxtimes P).
\]

We omit the proof that these are reasonable definitions—that they are exact functors, that \( f_i^{(n)*} \) is both left and right adjoint to \( e_i^{(n)*} \), that they preserve the properties of being finite dimensional, etc.

More generally given an irreducible module \( L \in \text{Rep}_q^\lambda \) we get functors

\[
\Delta_L: \text{Rep}_q^\lambda_{a+n} \to \text{Rep}_q^\lambda_a, \quad m_L: \text{Rep}_q^\lambda_a \to \text{Rep}_q^\lambda_{a+n}
\]

by mimicking the above construction—just replace \( \text{Ind}(q^i J_1 \otimes \cdots \otimes q^i J_1) \) above with \( L \). If \( L \) is the module \( q^i J_1 \) for \( H^\lambda_q \), \( \Delta_L = e_i^* = e_i^{(1)*} \) and \( m_L = f_i^* = f_i^{(1)*} \).

In general (and even for \( L = q^i J_1 \)), these functors only have good properties in the Grothendieck group. It is clear \( \Delta_L \) is the composite of the comodule action of \( K(\text{Rep}_q^\lambda) \) on \( K(\text{Rep}_q^\lambda) \) with the function \( \delta_L \) of “multiplicity of \( L \) in the Jordan-Holder series”

\[
\Delta_L: K(\text{Rep}_q^\lambda) \xrightarrow{\Delta} K(\text{Rep}_q^\lambda) \otimes K(\text{Rep}_q^\lambda) \xrightarrow{1 \otimes [L: -]} K(\text{Rep}_q^\lambda) \otimes_{\mathbb{Z}} K(\text{Rep}_q^\lambda)
\]

(see section 2.2 for the notation). In order to describe \( m_L \) on the level of the Grothendieck group we must introduce the Shapovalov inner product in \( K(\text{Rep}_q^\lambda) \); then \( m_L \) will be the adjoint of \( \Delta_L \) with respect to this inner product. This is done in section 1.
9. CRYSTAL GRAPH

In this section we will study certain “highest order” approximations to \(e_i^*\) and \(f_i^*\). These approximations make sense for both cyclotomic and affine Hecke algebras, unlike \(f_i^*\) itself, which is not defined for \(\text{Rep}^\text{aff}_q\). We summarise the properties of these operators in a combinatorial structure, the crystal graph.

There are three main results in this section. The first is theorem 9.4, which is a strong multiplicity one theorem for restriction and induction. This tells us that these operators in a combinatorial structure, the crystal graph.

The other two results are theorems 9.13 and theorem 9.15. These describe the close connection between the crystal graph structure and the representation theory of the Hecke algebra.

Theorem 9.13 shows that the integers \(\varepsilon_i(M)\) have various interpretations. By definition, if \(M\) is irreducible \(\varepsilon_i(M)\) is the length of the longest chain of \(q\)'s that end the spectrum of \(M\). But we show that it is also the maximum size of a Jordan block of \(X_n\) with eigenvalue \(q^i\) on the module \(M\), i.e. that it measures the failure of semisimplicity of the action of \(X_n\). It is also the dimension of \(\text{Hom}(e_i^* M, e_i^* M)\) — another subtle measure of lack of semisimplicity, as well as the multiplicity of the cosocle of \(e_i^* M\) in \(e_i^* M\). These last two statements show that the cosocle of \(e_i^* (M)\) fits into \(e_i^* (M)\) in a uniserial chain of length \(\varepsilon_i (M)\) in as simple a way as possible.

The analogous results for \(f_i^*\) is theorem 9.15. This is much harder, and the particular proof we give requires the results of the next few sections. The asymmetry is a nice shadow of the fact that whereas the dual of a finite dimensional lowest weight \(\mathfrak{sl}_2\) module is again a lowest weight module, the dual of an integrable lowest weight \(\mathfrak{sl}_\ell\)-module is a highest weight module. Nonetheless, the \(\mathfrak{sl}_2\) structure is the relevant one, and we eventually show \(\varphi_i (M)\) may also be read off the spectrum of \(M\).

The results of section 11 show that \(\varepsilon_i, \varphi_i\) also admit an interpretation in terms of the structure of projective modules.

9.1. First properties. To save repetition in notation, we will allow \(H_n^\lambda\) to denote the affine Hecke algebra as well as the cyclotomic Hecke algebra, i.e. we allow \(\lambda\) to be either the symbol “aff”, or a function \(\lambda: J_q \to \mathbb{Z}_+\) with \(\sum \lambda_i < \infty\). We say \(\lambda\) is affine or cyclotomic, as appropriate. For consistency in notation, define \(\text{pr}^\text{aff}, \text{ev}^\text{aff}\) to be the identity functors from \(\text{Rep}_q H_n^\text{aff} \to \text{Rep}_q H_n^\text{aff}\).

With this understood we write \(B_\lambda\) for the disjoint union of the set of isomorphism classes of irreducible representations on \(H_n^\lambda\), for \(n \geq 0\). Let \(\text{ZB}_\lambda\) be the free abelian group on the set \(B_\lambda\), and define a non-degenerate symmetric pairing on \(\text{ZB}_\lambda\) by making the elements of \(B_\lambda\) an orthonormal basis. We will identify \(\mathbb{Z}_+ B_\lambda\) with isomorphism classes of semisimple \(H_n^\lambda\)-modules, and the symmetric pairing with \(\dim_R \text{Hom}(-, -)\). (There is a canonical isomorphism of abelian groups—without the symmetric pairing—from the Grothendieck group \(K(\text{Rep}_q^\lambda)\) to \(\text{ZB}_\lambda\), but this will not be so useful for what follows.)

Now, define for \(M \in \text{Rep} H_n^\lambda\)

\[
\tilde{f}_i(M) = \text{pr}_\lambda \text{cosoc} \text{Ind}_{H_{n-1}^\text{aff} \otimes H_1^\text{aff}}^{H_n^\text{aff} \otimes H_1^\text{aff}}(\text{ev}_\lambda^* M \boxtimes q^i J_1)
\]

\[
\tilde{e}_i(M) = \text{soc} \text{Hom}_{R[\mathbb{X}_n^{\pm 1}]}(q^i J_1, \text{Res}_{H_{n-1}^\text{aff} \otimes H_1^\text{aff}} H_n^\text{aff} \otimes H_1^\text{aff} \text{ev}_\lambda^* M).
\]
As \( \bar{e}_i(M) \) is semisimple and \( \bar{e}_i(M \oplus M') = \bar{e}_i(M) \oplus \bar{e}_i(M') \), \( \bar{e}_i \) defines a \( \mathbb{Z}_+ \)-linear operator on isomorphism classes of semisimple \( H^\lambda_n \)-modules, and hence an operator \( \bar{e}_i : \mathbb{Z}B_\lambda \to \mathbb{Z}B_\lambda \). Similarly, \( \bar{f}_i \) defines an operator \( \mathbb{Z}B_\lambda \to \mathbb{Z}B_\lambda \), as \( \text{pr}_\lambda \) of a semisimple module is still semisimple. These are adjoint operators with respect to the inner product on \( \mathbb{Z}B_\lambda \), i.e.:

**Lemma 9.1.** If \( N \) is a semisimple \( H^\lambda_{n-1} \)-module, and \( M \) a semisimple \( H^\lambda_n \)-module,

\[
\text{Hom}_{H^\lambda_{n-1}}(N, \bar{e}_i M) = \text{Hom}_{H^\lambda_n}(\bar{f}_i N, M).
\]

**Proof.** As \( N \) is semisimple,

\[
\text{Hom}_{H^\text{aff}_{n-1}}(N, \text{soc Hom}_{R[X, \frac{\lambda}{n}]}(q^i J_1, \text{Res ev}^* M)) = \text{Hom}_{H^\text{aff}_{n-1}}(N, \text{Hom}_{H^\text{aff}_{n-1}}(\text{soc Hom}_{R[X, \frac{\lambda}{n}]}(q^i J_1, \text{Res ev}^* M))) = \text{Hom}_{H^\text{aff}_{n-1}}(N \otimes q^i J_1, \text{ev}^* M) = \text{Hom}_{H^\lambda_n}(\text{Ind}(N \otimes q^i J_1), \text{ev}^* M)
\]

by adjointness. Now, as \( \text{ev}^* M \) is semisimple, this is

\[
\text{Hom}_{H^\lambda_n}(\text{cosoc Ind}(N \otimes q^i J_1), \text{ev}^* M) = \text{Hom}_{H^\lambda_n}(\text{pr}_\lambda \text{cosoc Ind}(N \otimes q^i J_1), M).
\]

As with the operators \( f_i^* \) the order of the operations defining \( \bar{f}_i \) is not crucial. We have:

**Lemma 9.2.** \( \bar{f}_i(M) = \text{cosoc}(\text{pr}_\lambda \text{Ind}_{H^\lambda_{n-1} \otimes H^\lambda_1}(\text{ev}^* M \otimes q^i J_1)) \).

**Proof.** We may finish the above proof differently, noticing \( \text{ev}^* M \) is still semisimple, to get

\[
\text{Hom}_{H^\lambda_{n-1}}(N, \bar{e}_i M) = \text{Hom}_{H^\lambda_n}(\text{pr}_\lambda \text{Ind}(N \otimes q^i J_1), M)
\]

\[
= \text{Hom}_{H^\lambda_n}(\text{cosoc pr}_\lambda \text{Ind}(N \otimes q^i J_1), M).
\]

It follows that both \( \bar{f}_i \) and this new operator define adjoints to \( \bar{e}_i : \mathbb{Z}B_\lambda \to \mathbb{Z}B_\lambda \). But the symmetric pairing \( \text{Hom}(\ , \ ) \) is non-degenerate, and so these two operators must be equal.

In fact, we will show in theorem 9.9 that we may even replace \( J_1 \) with \( J_m \) in the definitions of \( \bar{e}_i \) and \( \bar{f}_i \).

We have implicitly used the map

\[
\text{ev}^* : \mathbb{Z}B_\lambda \to \mathbb{Z}B^\text{aff}
\]

induced by \( \text{ev}^* : \text{Rep } H^\lambda_n \to \text{Rep } H^\lambda_n \). The definitions of \( e_i^* \) and \( f_i^* \) make the following lemma obvious.

**Lemma 9.3.**

(i) \( \bar{e}_i \text{ev}^* M = \text{ev}^* \bar{e}_i M, \) for \( M \in B_\lambda \).

(ii) If \( M \in B_\lambda \), \( \text{ev}^* \bar{f}_i M \) is a direct summand of \( \bar{f}_i(\text{ev}^* M) \).
Theorem 9.4. If $M$ is an irreducible $H_n^\lambda$-module, then $\bar{e}_i M$ and $\bar{f}_i M$ are either irreducible or zero. Further, if $N \neq 0$, then $\bar{e}_i M = N$ if and only if $M = \bar{f}_i N$.

The theorem will be proved after proposition 9.8 below.

As a result, we can summarize the operators $\bar{e}_i$ and $\bar{f}_i$ in the datum of an oriented graph, with edges labelled by the elements of $\mu_q$. The vertices of the graph are the elements of $B_\lambda$, and $M$ is joined to $N$ by an arrow colored by $i \in \mu_q$

$$M \rightarrow N$$

if $\bar{e}_i M = N$; equivalently if $N = \bar{f}_i M$. This datum $(B_\lambda, \bar{e}_i, \bar{f}_i)$ is called a crystal graph, after Kashiwara. Note that as a consequence of the lemma 9.3, ev* induces an inclusion of crystal graphs, $B_\lambda \rightarrow B_{\text{aff}}$. We record one immediate formal consequence of this fact:

Corollary 9.5. If $M \in B_{\text{aff}}$, $(\text{pr}_\lambda \bar{f}_i)^k M = \text{pr}_\lambda (\bar{f}_i^k M)$.

Proof. By induction on $k$, $(\text{pr}_\lambda \bar{f}_i)^k M = \text{pr}_\lambda \bar{f}_i \text{pr}_\lambda \bar{f}_i^{k-1} M$, and $\text{pr}_\lambda \bar{f}_i^{k-1} M$ is either 0 or $\bar{f}_i^{-1} M$, so $(\text{pr}_\lambda \bar{f}_i)^k M$ is either 0 or $\text{pr}_\lambda \bar{f}_i^k M$. So if $\text{pr}_\lambda \bar{f}_i^k M = 0$, then $(\text{pr}_\lambda \bar{f}_i)^k M = 0$ also, and if $\text{pr}_\lambda \bar{f}_i^k M \neq 0$, it equals $\bar{f}_i^k M$ (by the theorem), so $\bar{e}_i (\text{pr}_\lambda \bar{f}_i^k M) = \bar{f}_i^{-1} M \in \text{Rep} H_n^\lambda$, and so $\bar{f}_i^{-1} M = \text{pr}_\lambda \bar{f}_i^{k-1} M$. Hence $(\text{pr}_\lambda \bar{f}_i)^k M = \text{pr}_\lambda \bar{f}_i^k M$ here also. $\square$

Define, for $M \in B_\lambda$

$$\varepsilon_i (M) = \max \{ n \geq 0 \mid \bar{e}_i^n M \neq 0 \}$$

$$\varphi_i (M) = \max \{ n \geq 0 \mid \bar{f}_i^n M \neq 0 \}$$

so that $\varepsilon_i (M) \in \mathbb{Z}_+$ (and indeed if $M \in \text{Rep} H_n^\lambda$, $\varepsilon_i (M) \leq n$), and $\varphi_i (M) \in \mathbb{Z}_+ \cup \{ \infty \}$. Note that if $\lambda = \text{aff}$ then $\varphi_i (M) = \infty$ always. We will see in theorem 9.15 that if $\lambda$ is cyclotomic then $\varphi_i (M)$ is always finite.

9.2. Detailed study of the crystal graph. We now start to seriously study the crystal graph. In order to lighten notation, let’s agree to write $\text{Ind} = \text{Ind}_{a_1 \ldots a_k} = \text{Ind}_{H_{n+1}^\lambda \otimes \ldots \otimes H_{n+k}^\lambda}$ for the induction functor between modules for affine Hecke algebras, and to omit indices when this causes no confusion; and similarly for restriction.

Let us also write

$$q^i K_n = \text{Ind}_{H_{n+1}^\lambda \otimes \ldots \otimes H_{1}^\lambda} (q^i J_1 \boxtimes \ldots \boxtimes q^i J_1)$$

for the unique irreducible $H_{n+1}^\lambda$-module with central character $(q^i \cdot q^i)$, and, if $M \in \text{Rep} H_{n+1}^\lambda$ and $N \in \text{Rep} H_n^\lambda$

$$\text{Hom}_{H_n^\lambda} (M, N)$$

for the $H_n^\lambda$-module $\text{Hom}_{H_{n+1}^\lambda \otimes H_n^\lambda} (H_n^\lambda \boxtimes M, \text{Res}_{H_{n+1}^\lambda \otimes H_n^\lambda}^{H_n^\lambda} N)$. So in our lighter notation, $\bar{e}_i M = \text{soc} \text{Hom}_{H_1^\lambda} (q^i J_1, M)$.

Lemma 9.6. Let $M \in \text{Rep} H_{n+1}^\lambda$. The following are equivalent

(i) $\varepsilon_i (M) \geq n$

(ii) $\bar{e}_i^n M \neq 0$

(iii) $\text{Hom}_{H_n^\lambda} (q^i K_n, M) \neq 0$. 
Proof. Clearly, (i) and (ii) are equivalent to the existence of a nonzero map of $R[X_{a,\pm 1}, \ldots, X_{a+n,\pm 1}]$-modules

$$q^i J_1 \boxtimes \cdots \boxtimes q^i J_1 \to \text{Res}_{H_a^\text{aff}}^{H_a^\text{aff}} \otimes \cdots \otimes H_a^\text{aff}^\text{aff} \otimes M.$$ 

By Frobenius reciprocity, this is equivalent to the existence of a nonzero map of $H_a^\text{aff}$-modules

$$q^i K_n \to \text{Res}_{H_a^\text{aff}}^{H_a^\text{aff}} \otimes H_a^\text{aff}^\text{aff} \otimes M.$$

The crucial technical result we will need to prove the theorem is:

**Proposition 9.7.** Let $M \in \text{Rep} H_a^\text{aff}$, and suppose $\varepsilon_i(M) = 0$. Then the exact sequence of $H_a^\text{aff} \otimes H_a^\text{aff}$-modules

$$0 \to M \boxtimes q^i K_n \to \text{Res}_{a,n}^{a+n} (M \boxtimes q^i K_n) \to \mathfrak{A} \to 0$$

splits. Moreover, for every subquotient $A$ of $\mathfrak{A}$,

$$\text{Ext}_{H_a^\text{aff} \otimes H_a^\text{aff}}(A, H_a^\text{aff} \boxtimes q^i K_n) = 0.$$

Proof. By the Mackey formula, $\mathfrak{A}$ admits a filtration whose graded pieces are

$$\Gamma_w = \text{Ind}_{\{(a,n)\}}^{(a,n)} w(a,n) w^{-1} \text{Res}_{\{(a,n)\}}^{(a,n)} (M \boxtimes q^i K_n)$$

where $w$ runs over representatives for all the cosets $S_{(a,n)} \backslash S_{a+n}/S_{(a,n)}$ except for the coset $S_{(a,n)}$.

Now consider $\Gamma_w$ as a module for $R[X_{a,\pm 1}, \ldots, X_{a+n,\pm 1}]$, and suppose $s = (s_{a+1}, \ldots, s_{a+n}) \in (R^X)^n$ is in its support; i.e. $\Gamma_w[s] \neq 0$. Fix $w \in S_{(a,n)} \backslash S_{a+n}/S_{(a,n)}$, not the identity double coset $S_{(a,n)}$. By the shuffle lemma [5.5], there must be some $s_\gamma$, $a < \gamma \leq a+n$, such that the $s_\gamma$-weight space of $X_a$ on $M$ is nonzero, i.e. $\{ m \in M \mid (X_a - s_\gamma)^\text{dim} M \cdot m = 0 \} \neq 0$. As $\Gamma_w N = 0$, $s_\gamma \neq q^i$. So we have shown there is some $\gamma$, $a < \gamma \leq a+n$, with $s_\gamma \neq q^i$.

It follows that $\Gamma_w$, when considered as a module for $Z(H_a^\text{aff})$ has support disjoint from $(q^i \cdots q^i)$. As this is the support of $q^i K_n$, we see that all subquotients of $\Gamma_w$, considered as an $H_a^\text{aff}$-module, are in different blocks from $q^i K_n$. The proposition follows. 

Write $\tilde{f}_i^{(n)} M = \text{cosoc} \text{Ind}(M \boxtimes q^i K_n)$, for $M \in \text{Rep} H_a^\text{aff}$.

**Proposition 9.8.** Suppose $M \in \text{Rep} H_a^\text{aff}$ is irreducible, and $\varepsilon_i(M) = 0$. Then

(i) $\tilde{f}_i^{(n)} M$ is irreducible.

(ii) If $A$ is an irreducible subquotient of the kernel of the map

$$\text{Ind}(M \boxtimes q^i K_n) \to \text{cosoc} \text{Ind}(M \boxtimes q^i K_n)$$

then $\varepsilon_i(A) < n$. Further, $\varepsilon_i(\tilde{f}_i^{(n)} M) = n$.

(iii) If $M, N \in \text{Rep} H_a^\text{aff}$ are irreducible, and $\varepsilon_i(M) = 0$ then $\tilde{f}_i^{(n)} M = \tilde{f}_i^{(n)} N$ implies $M = N$.

(iv) $\tilde{f}_i^{(n)} \tilde{f}_i^{(m)} M = \tilde{f}_i^{(n+m)} M$. In particular, $\tilde{f}_i^{(n)} M = \tilde{f}_i^{(n)} M$.

Proof. (i) As $\tilde{f}_i^{(n)} M$ is semisimple, we will know it is irreducible if we show $d = \dim \text{Hom}(\tilde{f}_i^{(n)} M, \tilde{f}_i^{(n)} M)$ equals 1. Note that $\text{cosoc}(X) = 0$ only if $X = 0$, so $\tilde{f}_i^{(n)} M$ is nonzero, and we must show $d \leq 1$. 


Write $S$ for the kernel
\[(6) \quad 0 \to S \to \text{Ind}(M \boxtimes q^iK_n) \to \bar{f}_i^{(n)}M = \text{cosoc} \text{Ind}(M \boxtimes q^iK_n) \to 0.\]

As $\text{Res} = \text{Res}_{n,a}^{n+a}$ is exact, we have an exact sequence
\[0 \to \text{Res} S \to \text{Res} \text{Ind}(M \boxtimes q^iK_n) \to \text{Res} \bar{f}_i^{(n)}M \to 0,
\]
and by the technical proposition 9.7, $\text{Res} \text{Ind}(M \boxtimes q^iK_n) = M \boxtimes q^iK_n \oplus \mathfrak{A}$. So $\text{Res} \bar{f}_i^{(n)}M = \frac{M \boxtimes q^iK_n}{M \boxtimes q^iK_n} \oplus \mathfrak{A}$, where $M \boxtimes q^iK_n$ is a quotient of $M \boxtimes q^iK_n$, and $\mathfrak{A}$ is a quotient of $\mathfrak{A}$. Now,
\[\text{Hom}(\bar{f}_i^{(n)}M, \bar{f}_i^{(n)}M) = \text{Hom}(\text{Ind}(M \boxtimes q^iK_n), \bar{f}_i^{(n)}M) = \text{Hom}(M \boxtimes q^iK_n, \text{Res} \bar{f}_i^{(n)}M)\]
as $\bar{f}_i^{(n)}M$ is semisimple, and $\text{Ind}$ adjoint to $\text{Res}$. But this equals
\[\text{Hom}(M \boxtimes q^iK_n, \frac{M \boxtimes q^iK_n}{M \boxtimes q^iK_n})\]
by the technical proposition 9.7. As $M \boxtimes q^iK_n$ is irreducible, either $\frac{M \boxtimes q^iK_n}{M \boxtimes q^iK_n}$ is zero, and $d = 0$, or $M \boxtimes q^iK_n = M \boxtimes q^iK_n$, and $d = 1$. We have already observed $d \neq 0$, so $\bar{f}_i^{(n)}M$ is irreducible, nonzero.

(ii) As the sequence \[(6)\] is exact, this also shows that $\text{Res} S$ embeds into $\text{Res} \mathfrak{A}$. Now let $A$ be an irreducible subquotient of $S$. Then to show $\varepsilon_i(A) < n$, it suffices to show $\text{Hom}_{H_n^{\text{aff}}}(q^iK_n, \text{Res}^{n+a}_a A)$ is zero. But $\text{Res} A$ is a subquotient of $\text{Res} \mathfrak{A}$, and the technical proposition 9.7 gives the result. As $\text{Res} \bar{f}_i^{(n)}M = M \boxtimes q^iK_n \oplus \mathfrak{A}$, it is clear that $\varepsilon_i(\bar{f}_i^{(n)}M) = n$.

(iii) Suppose $N \in \text{Rep} H_n^{\text{aff}}$ is irreducible. Then by semisimplicity of $\bar{f}_i^{(n)}M$ and adjunction, we get
\[\text{Hom}(\bar{f}_i^{(n)}N, \bar{f}_i^{(n)}M) = \text{Hom}(N \boxtimes q^iK_n, \text{Res} \bar{f}_i^{(n)}M) = \text{Hom}(N \boxtimes q^iK_n, M \boxtimes q^iK_n), \text{ as before} = \text{Hom}(N, M), \text{ as } N, M \text{ semisimple}.\]
So $\bar{f}_i^{(n)}N = \bar{f}_i^{(n)}M \iff N = M$.

(iv) To show (iv), observe that as we have a surjection
\[\text{Ind}(M \boxtimes q^iK_m) \boxtimes q^iK_n \twoheadrightarrow \text{cosoc} \text{Ind}(M \boxtimes q^iK_m) \boxtimes q^iK_n = \bar{f}_i^{(m)}M \boxtimes q^iK_n\]
we also get a surjection
\[\text{Ind}(M \boxtimes q^iK_{m+n}) = \text{Ind}(M \boxtimes q^iK_m) \boxtimes q^iK_n) \to \text{Ind}(\bar{f}_i^{(m)}M \boxtimes q^iK_n).\]
Given a surjective map $X \to Y$, we get an induced surjection on cosocles, $\text{cosoc} X \twoheadrightarrow \text{cosoc} Y$, so a surjective map
\[\bar{f}_i^{(m+n)}M \twoheadrightarrow \bar{f}_i^{(n)}(\bar{f}_i^{(m)}M).\]
As $\bar{f}_i^{(m+n)}M$ is irreducible, this is an isomorphism. \(\square\)

We now prove the theorem 9.4.
Proof. It is clearly enough to prove it for $H_n^{\text{aff}}$-modules, as lemma 9.8 implies the result for cyclotomic Hecke algebras.

Suppose $\overline{M} \in B_{\text{aff}}$ is an irreducible $H_m^{\text{aff}}$-module, and write $n = \varepsilon_i(\overline{M})$. Then if $M$ is semisimple,

$$\text{Hom}(M, \overline{\varepsilon_i^n(M)}) = \text{Hom}(\overline{f_i^n M}, \overline{M}).$$

So if $M$ is a simple summand of $\overline{\varepsilon_i^n(M)}$, then $\overline{\varepsilon_i} M = 0$ and the proposition 9.8 shows $\overline{f_i^n M} = \overline{M}$. If $M'$ is another simple summand of $\overline{\varepsilon_i^n(M)}$, then $\overline{f_i^n M'} = \overline{f_i^n M}$, so $M' = M$. It follows that $M = \overline{\varepsilon_i^n(M)}$.

We have just seen that every irreducible module is of the form $\overline{f_i^n M}$, where $\varepsilon_i(M) = 0$. It follows that $\overline{f_i} N$ is irreducible for all irreducible modules $N$. Finally, let $X$ be a summand of $\overline{\varepsilon_i} f_i^n M$, where $\varepsilon_i(M) = 0$, $M$ is irreducible. Then $X = \overline{f_i} n N$, for some $N$ with $\varepsilon_i(N) = 0$, as $\text{Hom}(X, \overline{\varepsilon_i} f_i^n M) = \text{Hom}(\overline{f_i} X, \overline{f_i} n M)$. It follows that $\overline{f_i} X = \overline{f_i} n M$, so $\overline{f_i} n+1 N = \overline{f_i} n M$, whence $m = n - 1$ and $N = M$. So $\overline{\varepsilon_i} f_i^n M = X = \overline{f_i} n M$, as desired. 

The previous theorem used the result that $q^i K_n$ is an irreducible in $\text{Rep}_s H_n^{\text{aff}}$, where $s = (q^1 \cdots q^i)$. The next theorem uses the fact that this is the only irreducible in this block.

**Theorem 9.9.** (i) For any irreducible $M \in \text{Rep}_s H_n^{\text{aff}}$, and $m > 0$

$$\overline{f_i}(M) = \text{pr}_\lambda \text{cosoc} \text{Ind}_{H_n^{\text{aff}} \otimes H_1^{\text{aff}}} (\text{ev}_s^* M \boxtimes q^i J_m).$$

In particular, if $\lambda$ is cyclotomic, then $\overline{f_i}(M) = \text{cosoc} f_i^*(M)$.

(ii) For any irreducible $M \in \text{Rep}_s H_n^{\text{aff}}$, and $m > 0$

$$\overline{e_i}(M) = \text{soc} \text{Hom}_{R(X_n \oplus 1)} (q^i J_m, \text{Res}_{H_n^{\text{aff}} \otimes H_1^{\text{aff}}} \text{ev}_s^* M).$$

In particular, $\overline{e_i}(M) = \text{soc} e_i^*(M)$.

**Proof.** It is enough to prove (i). For, if $M$ and $N$ are semisimple $H_n^{\lambda}$-modules, then by (i) and adjointness $\text{Hom}(\overline{f_i} M, N) = \text{Hom} (\text{cosoc} f_i^* M, N) = \text{Hom} (f_i^* M, N) = \text{Hom}(M, \overline{e_i} N) = \text{Hom}(M, \text{soc} e_i^* N)$ and so both $\overline{e_i}$ and $\text{soc} e_i^*$ are adjoint to $\overline{f_i} : \mathbb{Z} B\lambda \to \mathbb{Z} B\lambda$. As the symmetric pairing $\text{Hom}(\ ,\ )$ is non-degenerate, they must be equal.

So we prove (i). In fact, we prove a stronger statement. Let $P \in \text{Rep}_s H_n^{\text{aff}}$ be any indecomposable module with cosocle $q^i K_n$, so $P \rightarrow q^i K_n$. Define $f_P(M) = \text{cosoc} \text{Ind}(M \boxtimes P)$, for any $M \in \text{Rep}_q H_n^{\text{aff}}$. Clearly $f_P(M) \rightarrow \overline{f_i}(M)$. We will show that if $\varepsilon_i(M) = 0$, $f_P(M) = \overline{f_i}(M) = \overline{f_i}(M)$ is irreducible. Now, the socle filtration of $P$ has subquotients isomorphic to $q^i K_n$. It follows that we can filter $\text{Ind}(M \boxtimes P)$ with subquotients isomorphic to $\text{Ind}(M \boxtimes q^i K_n)$. Hence $f_P(M) = \text{cosoc} \text{Ind}(M \boxtimes P)$ is some summand of some copies of $\text{cosoc} \text{Ind}(M \boxtimes q^i K_n) = \overline{f}_i(M)$ (for any filtered module $Z = \cup Z_j$, cosocle(Z) is a summand of $\oplus \text{cosoc}(Z_j/Z_{j-1})$).

But

$$\text{Hom}(f_P M, \overline{f}_i(M)) = \text{Hom} (\text{Ind}(M \boxtimes P), \overline{f}_i(M)) = \text{Hom}(M \boxtimes P, \text{Res}(\overline{f}_i(M)) = \text{Hom}(M \boxtimes P, M \boxtimes q^i K_n) = R.$$

It follows that $f_P M = \overline{f}_i(M)$.
Note that this implies if cosoc($P$) = $q^iK_n$, and cosoc($P'$) = $q^iK_n'$, then $f_P f'_P = \tilde{f}_i^n \tilde{f}'_1^{n} = \tilde{f}_1^{n+n} = f_{\text{Ind}(P \boxtimes P')}$. Now, if $\mathcal{M}$ is any simple $H_n^\lambda$-module, $\mathcal{M} = \tilde{f}_i^{(n)}\mathcal{M}$ for some $M$ with $\varepsilon_i(\mathcal{M}) = 0$. Take $P = \text{Ind}_{n+1}^n(q^iK_n \boxtimes q^i J_{n+1})$. Then $f_{\tilde{q}^i J_{n+1}}(\mathcal{M}) = f_P(\mathcal{M}) = \tilde{f}_1^{n+i}M = f_i \mathcal{M}$ is irreducible, as desired.

This has the following simple consequence:

**Corollary 9.10.** [GV]

(i) Let $M$ be a simple $H_n^\lambda$-module, $\lambda$ cyclotomic. Then the cosocle of $\text{Ind}_{H_n^\lambda}^{H_n^\lambda+1} M$ and the socle of $\text{Res}_{H_n^\lambda}^{H_n^\lambda+1} M$ are multiplicity free.

(ii) If $M$ is a simple $H_n^{\text{aff}}$-module, $\text{soc Res}_{H_n^{\text{aff}}}^{H_n^{\text{aff}}+1} M$ is multiplicity free.

**Corollary 9.11.** If $M$ is irreducible, $e_i^*M$ and $f_i^*M$ are indecomposable.

**Proposition 9.12.** (i) $\text{soc}(e_i^*M) \simeq \text{soc}(e_i M)$, for $M \in \text{Rep} H_n^\lambda$.

(ii) $\text{soc}(f_i^*M) \simeq \text{soc}(f_i M)$, for $M \in \text{Rep} H_n^\lambda$, $\lambda$ cyclotomic.

**Warning 2.** However, $\text{soc Ind}_{H_n^{\text{aff}}}^{H_n^{\text{aff}}+1}(M \boxtimes q^i J_1) \neq \text{soc Ind}_{H_n^{\text{aff}}}^{H_n^{\text{aff}}+1}(M \boxtimes q^i J_1)$.

For example, if $n = 2$, and $M = q^{i-1} J_1$, they are clearly different.

We finish the section with some alternate characterizations of $\varepsilon_i$ and $\varphi_i$.

**Theorem 9.13.** Let $N$ be an irreducible module in $\text{Rep}^\lambda_{H_n}$. Then

(i) $\varepsilon_i(N)$ is the maximal size of a Jordan block for $R[[X_n-q^i]]$ on $N$.

(ii) In $K(\text{Rep}^\lambda_n)$, we have $e_i^*N = \varepsilon_i(N) \cdot \tilde{e}_i(N) + \sum a_\alpha M_\alpha$, where $M_\alpha$ are irreducible modules with $\varepsilon_i(M_\alpha) < \varepsilon_i(N) - 1$.

(iii) $\varepsilon_i(N) = \dim \text{Hom}(e_i^*N, e_i^*N)$.

**Proof.** We may suppose that $N = \tilde{f}_i^n M = \text{cosoc Ind}(M \boxtimes q^i K_n)$, where $M$ is an irreducible $H_n^\lambda$-module and $\varepsilon_i(M) = 0$. By the Mackey formula and “shuffle lemma,” if $P \rightarrow q^i K_n$ is some module covering $q^i K_n$ we have an exact sequence

$$0 \rightarrow \text{Ind}_{n,n-1,1}^{n,n+1,1}(M \boxtimes P) \rightarrow \text{Res}_{n,n-1,1}^{n,n+1}(M \boxtimes P) \rightarrow Q \rightarrow 0$$

where $X_n$ has no $q^i$-eigenspace on $Q$, i.e.

$$\text{Hom}_{H_n^{\text{aff}}}^{H_n^{\text{aff}}+1 \boxtimes H_n^{\text{aff}}(H_{n+1}^{\text{aff}} \boxtimes q^i J_1), Q} = 0.$$ 

Hence taking the $q^i$-eigenspace of $X_n$, we get

$$e_i^* \text{Ind}_{n,n-1}^{n,n+1}(M \boxtimes P) = \text{Ind}_{n,n-1}^{n,n+1}(M \boxtimes e_i^* P)$$

and that $\tilde{\varepsilon}_i(\text{Ind}(M \boxtimes P)) = \tilde{\varepsilon}_i(P)$, where we denote the maximal size of a Jordan block of $R[[X_n-q^i]]$ on a module $N$ by $\tilde{\varepsilon}_i(N)$. We now apply this for $P = q^i K_n$. Write, for $m \leq n$

$$0 \rightarrow S_m \rightarrow \text{Ind}(M \boxtimes q^i K_m) \rightarrow \tilde{f}_i^m M \rightarrow 0$$

for the exact sequence defining $\tilde{f}_i^m M$. We prove (ii). Apply $e_i^*$ to this exact sequence, to get

$$0 \rightarrow e_i^* S_n \rightarrow \text{Ind}(M \boxtimes e_i^*(q^i K_n)) \rightarrow e_i^* \tilde{f}_i^n M \rightarrow 0.$$
Now, in $K(\text{Rep}_n^\lambda)$, $e_i^*(q^i K_n)$ is a multiple of $q^i K_{n-1}$, as this is the unique irreducible module with this central character. Comparing dimensions, we see that $e_i^*(q^i K_n) = n \cdot q^i K_{n-1}$, and so in $K(\text{Rep}_{n+\mu}^\lambda)$ we have
\[ e_i^*(\tilde{f}_i^n M) = n \cdot \text{Ind}(M \otimes q^n K_{n-1}) - e_i^*(S_n) = n \tilde{f}_i^{n-1} M + nS_{n-1} - e_i^*(S_n). \]
By proposition 9.8, (ii) and its proof, the left hand side of the proposition is of the desired form. Finally, as $\tilde{f}_i^n M$ is in $K(\text{Rep}_H^\lambda)$, it follows the non-zero terms in this expression are also (after cancelling).

We now prove (i). As $\text{Ind}(M \otimes q^n K_n) \rightarrow \tilde{f}_i^n M$, (*) implies
\[ \tilde{e}_i(\tilde{f}_i^n M) \leq \tilde{e}_i(q^n K_n). \]
But by proposition 9.8, (ii) and its proof, the $H_n^\lambda \otimes H_n^\text{aff}$-module $M \otimes q^n K_n$ does occur as a submodule of $\text{Res}(\tilde{f}_i^n M)$ (in fact, with multiplicity one). It follows that
\[ \tilde{e}_i(\tilde{f}_i^n M) \geq \tilde{e}_i(q^n K_n). \]

It remains to show $\tilde{e}_i(q^n K_n) = n = \varepsilon_i(q^n K_n)$.

We show inductively that if $\alpha_m \neq 0$ in the exact sequence
\[ 0 \rightarrow \text{Hom}(q^n J_{m-1}, q^n K_n) \rightarrow \text{Hom}(q^n J_m, q^n K_n) \xrightarrow{\alpha_m} \text{Hom}(q^n J_1, q^n K_n) \]
then $\text{Hom}(q^n J_m, q^n K_n) = m \cdot q^n K_{n-1}$. in the Grothendieck group of $H_n^{\text{aff}}$ modules. As $\tilde{e}_i(q^n K_n)$ is the smallest integer $m$ for which $\alpha_{m+1} = 0$, and $\text{Hom}(q^n J_{\tilde{e}_i(q^n K_n)}, q^n K_n) = e_i^*(q^n K_n)$, this will finish us up. But $\text{Hom}(q^n J_1, q^n K_n)$ is $\tilde{e}_i(q^n K_n)$, which is $q^n K_{n-1}$ as it is irreducible and non-zero. Our induction starts with $m = 0$, where it is clear by these remarks. Likewise, if $m > 0$, then if $\alpha_m$ is non-zero it must be a surjection onto $q^n K_{n-1}$, hence the result.

We now show that (i) and (ii) imply (iii). By (i), the operators of multiplication by $(X_n - q^i)^k$ are non-zero for $0 \leq k < \varepsilon_i(N)$, and zero for $k = \varepsilon_i(N)$. Hence they are linearly independent, and $\dim \text{Hom}(e_i^* N, e_i^* N) \geq \varepsilon_i(N)$. For the reverse inequality, observe that for any module $X$ with cosocle $\tilde{e}_i N$
\[ \dim \text{Hom}(X, e_i^* N) \leq \varepsilon_i(N), \]
as one sees by filtering $e_i^* N$ so it has semisimple quotients and applying (ii). Now take $X = e_i^* N$. \hfill \Box

Let us write, for an irreducible $H_n^{\text{aff}}$-module $N$
\[ \varepsilon_i^*(N) = \varepsilon_i(\sigma^s N) \]
where $\sigma: H_n^{\text{aff}} \rightarrow H_n^{\text{aff}}$, $T_i \mapsto -(T_{n-i} + 1 - q)$, $X_i \mapsto X_{n+1-i}$ is the diagram automorphism of $H_n^{\text{aff}}$. Then $\varepsilon_i^*(N)$ is the maximum, as $s \in (R^\times)^n$ varies with $N_s \neq 0$, of the length of the chain of $q$’s beginning $s$. (This is immediate as $\varepsilon_i(N)$ is the maximal length of the chain of $q$’s ending $s$ with $N_s \neq 0$.) The previous theorem tells us $\varepsilon_i^*(N)$ is also the maximal size of a Jordan block for $R[[X_1 - q^i]]$ on $N$; hence

**Corollary 9.14.** If $N$ is an irreducible $H_n^{\text{aff}}$-module, then
\[ \text{pr}_\lambda N \neq 0 \iff \varepsilon_i^*(N) \leq \lambda_i, \quad \text{for all } i \in \mu_q. \]

We now give descriptions of $\varphi_i$ parallel to that of $\varepsilon_i$. The proof of the following theorem will not be completed until section 12.
Theorem 9.15. Let $N$ be an irreducible module in $\text{Rep}_q^\lambda$, where $\lambda$ is cyclotomic. Then

(i) $\varphi_i(N)$ is the smallest integer $m$ for which $f_i^* N = \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m)$.

(ii) In the Grothendieck group $K(\text{Rep}_q^\lambda)$,

$$f_i^* N = \varphi_i(N) \cdot \bar{f}_i(N) + \sum a'_\alpha M'_\alpha$$

where $M'_\alpha$ are irreducible modules with $\varepsilon_i(M'_\alpha) < \varepsilon_i(N) + 1$.

(iii) $\varphi_i(N) = \dim \text{Hom}(f_i^* N, f_i^* N)$.

Proof. We first show (i) and (ii) are equivalent. Suppose the surjective map

$$\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m) \twoheadrightarrow \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_{m-1})$$

is not an isomorphism. We claim that it follows that

$$\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m) = m \cdot \bar{f}_i(N) + \text{smaller terms},$$

(where we mean equality in the Grothendieck group, and smaller terms means a sum of modules $A$ for which $\varepsilon_i(A) < \varepsilon_i(\bar{f}_i(N))$). The proof is by induction. We have an exact sequence

$$\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m) \xrightarrow{\alpha_m} \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m) \xrightarrow{\gamma_m} \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_{m-1}) \rightarrow 0.$$ 

By assumption, $\alpha_m \neq 0$, so the image of $\alpha_m$ contains a copy of the irreducible module $\text{cosoc} \text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_1) = \bar{f}_i(N)$. As $\text{pr}_\lambda \text{Ind}(N \boxtimes q^i J_m) = \bar{f}_i(N) + \text{smaller terms}$, the induction is complete if we show that $\gamma_m$ is an isomorphism implies $\gamma_m + k$ is also, for $k \geq 0$. But this is clear—factor $\alpha_m + k = \theta_m^k \circ \alpha_m$, where $\theta_m^k$ is the map induced from $x^k : J_m \rightarrow J_{m+k}$. But now $\alpha_m = 0$ and $\alpha_m + k = 0$ and hence $\gamma_m + k$ is an isomorphism.

Write $\tilde{\varphi}_i(N)$ for the smallest integer $m$ for which $f_i^* N = \text{pr} \text{Ind}(N \boxtimes q^i J_m)$. We have just shown that for $m > \tilde{\varphi}_i(N)$, $N \boxtimes q^i \check{J}_{\tilde{\varphi}_i,N}$ is a submodule of $\text{Res} \text{pr} \text{Ind}(N \boxtimes q^i J_m)$. It follows that for such $m$, $R^\tilde{\varphi}_i N = \text{Hom}(N \boxtimes q^i J_m, N \boxtimes q^i \check{J}_{\tilde{\varphi}_i,N})$ is a submodule of

$$\text{Hom}(\text{Ind}(N \boxtimes q^i J_m), \text{pr} \text{Ind}(N \boxtimes q^i J_m)) = \text{Hom}(f_i^* N, f_i^* N),$$

and so $\dim \text{Hom}(f_i^* N, f_i^* N) \geq \tilde{\varphi}_i(N)$. On the other hand, as in the proof of Theorem 9.13, for any module $X$ with cosocle $f_i N$, we have

$$\dim \text{Hom}(X, f_i^* N) \leq \varphi_i(N).$$

Taking $X = f_i^* N$ we see (iii) is equivalent to (i) and (ii).

Now let us show $\tilde{\varphi}_i(N) \geq \varphi_i(N)$. But this is immediate from Frobenius reciprocity: any embedding $R[X]/(X_1 - q_i)^m \rightarrow q^i K_m$ induces a map $\text{Ind}(N \boxtimes q^i J_{\varphi_i(N)}) \twoheadrightarrow \text{Res} \text{pr} \text{Ind}(N \boxtimes q^i K_{\varphi_i(N)})$, which when restricted to $H^a \otimes H^b_{\text{aff}}$ induces an injection $N \boxtimes q^i J_{\varphi_i(N)} \hookrightarrow \text{Res} f_i^{\varphi_i(N)} N$. It follows that $\tilde{\varphi}_i(N) \geq \varphi_i(N)$. The reverse inequality will be proved in section 12.

10. Serre relations

In this section we study the relation of $\varepsilon_i(f_{iq+1} N)$ to $\varepsilon_i(N)$. To simplify notation we will write $i = 1$, so $iq = q$; this has no effect other than to make the notation readable. We will also write $K_m$ rather than $q^0 K_m$, and similarly for $J_m$, and we shall write $q^i$ rather than $q^i J_1$ when this causes no confusion.
The case $q = -1$ is different from the case $q \neq -1$; for most of the section we will assume $q \neq -1$. The main results of the section are proposition 10.4 which holds for $q \neq -1$, and its counterpart proposition 10.3, which covers the case $q = -1$.

$q \neq -1$ First, suppose $q \neq -1$. Then we have a short exact sequence of $H^\text{aff}_n^\text{aff}$-modules

$$0 \to 1_{(1q)} \to \text{Ind}(q \boxtimes 1) \to \text{St}_{(q1)} \to 0$$

which does not split; here $1_{(1q)}$ and $\text{St}_{(q1)}$ are the one dimensional modules with spectra $(1q)$ and $(q1)$ respectively.

Hence for $m \geq 1$ we have a short exact sequence (**)

$$0 \to \text{Ind}(N \boxtimes 1_{(1q)} \boxtimes K_{m-1}) \to \text{Ind}(N \boxtimes q \boxtimes K_m) \to \text{Ind}(N \boxtimes \text{St}_{(q1)} \boxtimes K_{m-1}) \to 0.$$

**Proposition 10.1.** Suppose $q \neq -1$. Then for all irreducible $N \in \text{Rep } H^\lambda_n$, and all $m \geq 1$

(i) $\text{Ind}(1_{(1q)} \boxtimes K_m) \cong \text{Ind}(K_m \boxtimes 1_{(1q)})$ is irreducible.

(ii) $\text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)})$ is indecomposable, with simple cosocle.

(iii) $\text{Ind}((\text{St}_{(q1)} \boxtimes K_m) \cong \text{Ind}(K_m \boxtimes \text{St}_{(q1)})$ is irreducible.

(iv) $\text{Ind}(N \boxtimes K_m \boxtimes \text{St}_{(q1)})$ is indecomposable, with simple cosocle.

**Proof.** We prove (i) and (ii); as (iii) and (iv) are similar (or indeed are formal consequences, upon applying the obvious automorphism of $H_n^\text{aff}$). By the classification of $H_n^\text{aff}$-modules, when $q \neq -1$,

$$\text{Ind}(1_{(1q)} \boxtimes 1) \cong \text{Ind}(1 \boxtimes 1_{(1q)})$$

is irreducible; hence by transitivity of induction

$$\text{Ind}(1_{(1q)} \boxtimes 1 \boxtimes \cdots \boxtimes 1) = \text{Ind}(1 \boxtimes 1_{(1q)} \boxtimes \cdots \boxtimes 1) = \cdots = \text{Ind}(1 \boxtimes \cdots \boxtimes 1 \boxtimes 1_{(1q)}).$$

But the Mackey formula implies that $1_{(1q)} \boxtimes K_m$ occurs with multiplicity one in $\text{Res} \text{Ind}(1_{(1q)} \boxtimes K_m)$, hence the useful observation (lemma 9.3) shows $\text{Ind}(K_m \boxtimes 1_{(1q)})$ is irreducible.

To prove (ii), observe that if $X \to Y$ is a surjection, then $\text{cosoc}(X) \to \text{cosoc}(Y)$, so as $N$ is a quotient of $\text{Ind}(\tilde{N} \boxtimes K_m)$ for some $\tilde{N}$ with $\varepsilon_1(\tilde{N}) = 0$, it is enough to show (ii) when $\varepsilon_1(N) = 0$. We play with the Mackey formula again as in theorem 9.13, so we will be terse. Write $\text{Res} = \text{Res}_{a+m+2}^{a+m+1}$. Then the sequence

$$0 \to N \boxtimes \text{Ind}(K_m \boxtimes 1_{(1q)}) \to \text{Res} \text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)}) \to U \to 0$$

splits, as the shuffle lemma shows the central character of $U$ differs from that of

$$\text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)}).$$

It follows, as in 9.8, that

$$\text{Res} \text{cosoc} \text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)}) = N \boxtimes \text{Ind}(K_m \boxtimes 1_{(1q)}) \oplus \overline{U},$$

where $\overline{U}$ is some quotient of $U$, and that

$$\text{Hom}((\text{cosoc} \text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)}), \text{cosoc} \text{Ind}(N \boxtimes K_m \boxtimes 1_{(1q)})) = \text{Hom}(N \boxtimes \text{Ind}(K_m \boxtimes 1_{(1q)}), N \boxtimes \text{Ind}(K_m \boxtimes 1_{(1q)})) = R,$$

the last equality as $\text{Ind}(K_m \boxtimes 1_{(1q)})$ is irreducible by (i).

For $N \in B_\lambda$, define $\tilde{f}_{q \lambda}(N) = \text{pr}_1 \text{cosoc} \text{Ind}(N \boxtimes 1_{(1q)})$, and $\tilde{f}_{q \lambda}(N) = \text{pr}_1 \text{cosoc} \text{Ind}(N \boxtimes \text{St}_{(q1)})$. As a corollary of the proposition, $f_{q \lambda} N$ and $\tilde{f}_{q \lambda} N$ are irreducible modules, if $q \neq -1$. (If $q = -1$ this is no longer true.)
Lemma 10.2. (i) If $A$ is an irreducible constituent of $\text{Ind}(N \boxtimes 1_{(q)} \boxtimes K_{m-1})$, then $\varepsilon_1(A) \leq \varepsilon_1(N) + m - 1$.
(ii) If $A$ is an irreducible constituent of $\text{Ind}(N \boxtimes K_{m-1} \boxtimes \text{St}_{(q_1)})$, then $\varepsilon_1(A) \leq \varepsilon_1(N) + m$. Moreover, if $\varepsilon_1(A) = \varepsilon_1(N) + m$, then $A$ is an irreducible constituent of $\text{Ind}(\tilde{f}_1^{m-1}N \boxtimes \text{St}_{(q_1)})$.
(iii) If $Q$ is an irreducible quotient of $\text{Ind}(N \boxtimes K_{m-1} \boxtimes \text{St}_{(q_1)})$, then $\varepsilon_1(Q) = \varepsilon_1(N) + m$.

Proof. (i) and the first part of (ii) are immediate from the shuffle lemma. As $\text{Ind}(N \boxtimes K_{m-1}) = \tilde{f}_1^{m-1}N + \sum a_\alpha M_\alpha$, where $\varepsilon_1(M_\alpha) < \varepsilon_1(N) + m - 1$ by proposition, the last part of (ii) follows by another application of the shuffle lemma.

Finally, to prove (iii), it is enough to show $\varepsilon_1(Q) \geq \varepsilon_1(N) + m$. Further, we may again assume $\varepsilon_1(N) = 0$, as the cosocle of $\text{Ind}(\varepsilon_1(N)N \boxtimes K_{\varepsilon_1(N)+m-1} \boxtimes \text{St}_{(q_1)})$ surjects onto the cosocle of $\text{Ind}(N \boxtimes K_{m-1} \boxtimes \text{St}_{(q_1)})$. Now, if $Q$ is an irreducible quotient as in (iii), Frobenius reciprocity gives a non-zero homomorphism $\varepsilon_1(N) \boxtimes \text{Ind}(K_{m-1} \boxtimes \text{St}_{(q_1)}) \rightarrow Q$; hence $\varepsilon_1(Q) \geq \varepsilon_1(N) \boxtimes \text{Ind}(K_{m-1} \boxtimes \text{St}_{(q_1)})$. But $\text{Ind}(K_{m-1} \boxtimes \text{St}_{(q_1)}) = \text{Ind}(\text{St}_{(q_1)} \boxtimes K_{m-1})$ by the previous proposition, and so $\varepsilon_1(Q) \geq m$.

We have shown that
$$\text{Ind}(N \boxtimes 1_{(q)} \boxtimes K_{m-1}) = \tilde{f}_1 q \tilde{f}_1^{m-1}N + \text{smaller terms},$$
and
$$\text{Ind}(N \boxtimes \text{St}_{(q_1)} \boxtimes K_{m-1}) = \text{cosoc} + \text{smaller terms},$$
where all the terms in the cosocle have $\varepsilon_1 = \varepsilon_1(N) + m$. As $\text{Ind}(N \boxtimes q \boxtimes K_m)$ surjects onto $\tilde{f}_1 q \tilde{f}_1^{m-1}N$ for $m \geq 0$, and we have a filtration of $\text{Ind}(N \boxtimes q \boxtimes K_m)$ as in (**), we have proved most of

Proposition 10.3. Precisely one of the following alternatives hold

(i) For all $m \geq 1$, $\tilde{f}_1 q \tilde{f}_1^{m-1}N$, and for all $m \geq 0$
\[ \varepsilon_1(\tilde{f}_1 q \tilde{f}_1^{m-1}N) = m - 1 + \varepsilon_1(N), \quad \text{or} \]

(ii) For all $m \geq 1$, $\tilde{f}_1 q \tilde{f}_1^{m-1}N$ is a summand of cosoc $\text{Ind}(\tilde{f}_1^{m-1}N \boxtimes \text{St}_{(q_1)})$, and for all $m \geq 0$,
\[ \varepsilon_1(\tilde{f}_1 q \tilde{f}_1^{m-1}N) = m + \varepsilon_1(N). \]

Proof. We have proved everything for $m \geq 1$; to finish we must only observe that $\varepsilon_1(\tilde{f}_1 q \tilde{f}_1^{m-1}N) = \varepsilon_1(\tilde{f}_1 q \tilde{f}_1^{m-1}N) + 1$, and so we have the assertions for $m = 0$ also.

Now let $N$ be an irreducible $H^\lambda_n$-module, with $\lambda$ cyclotomic. Let us agree to write $f_i$ for the affine crystal operator $\text{cosoc} \text{Ind}(\bullet \boxtimes q^i)$, and write $pr \tilde{f}_i$ for the cyclotomic crystal operator.

Proposition 10.4. Let $N$ be an irreducible $H^\lambda_n$-module, and suppose $pr \lambda \tilde{f}_q N \neq 0$. Then precisely one of the following holds:

(i) $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N) - 1$, and $\varphi_1(\tilde{f}_q N) = \varphi_1(N)$, \quad or \quad $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N) - 1$,
(ii) $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N)$, and $\varphi_1(\tilde{f}_q N) = \varphi_1(N) + 1$. 

The precise meaning of $\lambda$ is explained in the context.
Proof. Write $\varphi = \varphi_1(N)$. First suppose $\varepsilon_1(\tilde{f}_qN) = \varepsilon_1(N) - 1$. Then the surjection $\text{Ind}(1_q) \to 1_{(1q)}$ induces a surjection, hence isomorphism

$$\tilde{f}_q f_1^m N = \cosoc \text{Ind}(N \boxtimes K_m \boxtimes q) \to \cosoc \text{Ind}(N \boxtimes K_{m-1} \boxtimes 1_{(1q)})$$

$$= \tilde{f}_q f_1^{m-1} N = \tilde{f}_q f_1 N$$

for all $m \geq 1$. It follows that $\text{pr}_\lambda \tilde{f}_q f_1^{\varepsilon} \tilde{f}_q N = \text{pr}_\lambda \tilde{f}_q f_1^{\varepsilon+1} N = 0$. To show $\text{pr}_\lambda \tilde{f}_q f_1^{\varepsilon} f_q N \neq 0$, we compute $\varepsilon_\alpha(\tilde{f}_q f_1^m f_q N)$ for all $\alpha \in \mu_q$ and invoke corollary 9.14.

The shuffle lemma shows $\varepsilon_1^\wedge(\text{Ind}(f_1^m N \boxtimes q)) \leq \varepsilon_1^\wedge(\tilde{f}_q N)$, hence the definition of $\varphi$ and corollary 9.14 shows $\varepsilon_1^\wedge(\tilde{f}_q f_q N) \leq \lambda_1$.

Similarly, the shuffle lemma shows that for all $m$, $\varepsilon_1^\wedge \text{Ind}(\tilde{f}_q N \boxtimes K_m) \leq \varepsilon_1^\wedge(\tilde{f}_q N)$ and $\varepsilon_0^\wedge(\tilde{f}_q N) \leq \lambda_q$ by corollary 9.14 and the assumption $\text{pr}_\lambda \tilde{f}_q N \neq 0$. We clearly have $\varepsilon_\alpha^\wedge(\tilde{f}_q f_q N) \leq \varepsilon_\alpha^\wedge(N) \leq \lambda_\alpha$ for $\alpha \neq \{1, q\}$, and all $m \geq 0$. Thus another application of corollary 9.14 shows $\text{pr}_\lambda (\tilde{f}_q f_q N) \neq 0$, so $\varphi_1(\tilde{f}_q N) = \varphi$.

Now suppose $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N)$. The argument of the last paragraph applies equally well here, so to show $\text{pr}_\lambda (\tilde{f}_q f_1^{\varepsilon} \tilde{f}_q N) \neq 0$ we must only show $\varepsilon_1^\wedge(\tilde{f}_q f_q N) \leq \varepsilon_1^\wedge(\tilde{f}_q f_1^{m-1} N)$ for all $m \geq 1$. But this is immediate from the shuffle lemma, as in this case $\tilde{f}_q f_1 \tilde{f}_q N$ is a quotient of $\text{Ind}(\tilde{f}_q f_1^{m-1} N \boxtimes \mathbf{St}_{(q1)})$.

Finally, observe that Frobenius reciprocity implies that $\tilde{f}_q f_1^{m-1} N \boxtimes \mathbf{St}_{(q1)}$ is contained in $\text{Res}(\tilde{f}_q f_1^{m-1} N)$, so if $\varepsilon_1^\wedge(\tilde{f}_q f_1^{m-1} N) > \lambda_1$, then $\varepsilon_1^\wedge(\tilde{f}_q f_q N) > \lambda_1$ also. It follows that $\text{pr}_\lambda \tilde{f}_q f_1^{m-1} f_q N = 0$, and so $\varphi_1(\tilde{f}_q N) = \varphi + 1$ in this case. \[q = -1\]

We now suppose $q = -1$. The analogue of proposition 10.4 is

**Proposition 10.5.** Let $N$ be an irreducible $H_n^\lambda$-module, and suppose $\text{pr}_\lambda \tilde{f}_q N \neq 0$. Then precisely one of the following holds:

(i) $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N) - 2$, and $\varphi_1(\tilde{f}_q N) = \varphi_1(N)$, or

(ii) $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N) - 1$, and $\varphi_1(\tilde{f}_q N) = \varphi_1(N) + 1$, or

(iii) $\varepsilon_1(\tilde{f}_q N) = \varepsilon_1(N)$, and $\varphi_1(\tilde{f}_q N) = \varphi_1(N) + 2$.

**Proof.** We merely sketch the necessary modifications in the definitions needed to prove this. Recall that there are $3$ irreducible representations of $H_n^\alpha$ with central character $\mathbf{S}_3 \cdot (11q)$. Denote them $\alpha, \overline{\alpha}, \gamma$ with $\text{ch} \alpha = 2(11q)$, $\text{ch} \overline{\alpha} = 2(q11)$, $\text{ch} \gamma = (1q1)$. The representations $\text{Ind}(1 \boxtimes \alpha)$ and $\text{Ind}(1 \boxtimes \gamma)$ are irreducible, so if we define

$$\tilde{f}_{11q}(M) = \text{pr}_\lambda \cosoc \text{Ind}(M \boxtimes \alpha)$$
$$\tilde{f}_{1q1}(M) = \text{pr}_\lambda \cosoc \text{Ind}(M \boxtimes \gamma)$$
$$\tilde{f}_{q11}(M) = \text{pr}_\lambda \cosoc \text{Ind}(M \boxtimes \overline{\alpha})$$

then the analogue of proposition 10.4 is that each of these operators takes irreducible modules to irreducible modules (or zero). Using the exact sequence $0 \to \gamma \to \text{Ind}(1 \boxtimes 1_{(1q)}) \to \alpha \to 0$, we see that for $m \geq 2$ there is a $4$ step filtration of $\text{Ind}(N \boxtimes q \boxtimes K_m)$

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \text{Ind}(N \boxtimes q \boxtimes K_m)$$
where
\[ F_1 = \text{Ind}(N \boxtimes \text{Ind} (\lambda \boxtimes K_{m-2})), \quad F_2/F_1 = F_3/F_2 = \text{Ind}(N \boxtimes \text{Ind} (\gamma \boxtimes K_{m-2})), \]
\[ F_4/F_3 = \text{Ind}(N \boxtimes \text{Ind} (\pi \boxtimes K_{m-2})). \]

Arguing as before, one sees that the cosocle of each of these subquotients consists of a sum of terms with fixed \( \varepsilon_1 \): respectively \( \varepsilon_1 - m \) is \( \varepsilon_1(N) - 2, \varepsilon_1(N) - 1, \) and \( \varepsilon_1(N) \). Hence \( f_1^m f_q N \), which is a quotient of \( \text{Ind}(N \boxtimes q \boxtimes K_m) \), is a quotient of precisely one of those three subquotients (for \( m \geq 2 \)). The rest of the proof is as before.

**11. Shapovalov Form**

Let \( A \) be a finite dimensional algebra over \( R \), and \( \text{Proj}(A) \) be the category of finite dimensional projective \( A \)-modules. As \( \text{Hom}_A(P, -) \) is an exact functor on \( \text{Rep} A \) if \( P \) is projective, there is a well defined bilinear form
\[ (\quad , \quad) : K(\text{Proj} A) \otimes \mathbb{Z} K(\text{Rep} A) \to \mathbb{Z} \]
given by \( ([P], [M]) = \dim \text{Hom}_A(P, M) \).

If \( M \) is a simple \( A \)-module, we write \( \mathcal{P}_M \) for the projective cover of \( M \), so \( \mathcal{P}_M \) is an indecomposable projective with \( \text{cosoc} (\mathcal{P}_M) = M \). Then \( (\mathcal{P}_M, -) \) is the multiplicity of \( M \) in the composition series of \( N \), i.e. \( (\mathcal{P}_M, -) = \delta_M : K(\text{mod} A) \to \mathbb{Z} \), and \( (\quad , \quad) \) defines an isomorphism \( K(\text{Proj} A) \to K(\text{Rep} A)^* \).

Now consider the natural map
\[ K(\text{Proj} A) \to K(\text{Rep} A). \]

This is an injection if and only if it becomes an isomorphism after tensoring \( \otimes \mathbb{Z} \mathbb{Q} \), and this happens precisely when the bilinear form in non-degenerate when restricted to
\[ K(\text{Proj} A) \otimes \mathbb{Z} K(\text{Proj} A) \to \mathbb{Z}. \]

We now show this happens for the cyclotomic Hecke algebra. In this section \( H_n^\lambda \)
denotes the cyclotomic Hecke algebra only, i.e. \( \lambda : \mu_q \to \mathbb{Z}_+ \) is a function with \( \sum \lambda_i < \infty \).

**Theorem 11.1.** The above pairing \((P, P') \mapsto \dim \text{Hom}(P, P')\)
\[ (\quad , \quad) : K(\text{Rep} H_n^\lambda)^* \otimes \mathbb{Z} K(\text{Rep} H_n^\lambda)^* \to \mathbb{Z} \]
is a non-degenerate symmetric form.

We call this form, or the induced form
\[ K(\text{Rep} H_n^\lambda) \otimes \mathbb{Z} K(\text{Rep} H_n^\lambda) \to \mathbb{Q} \]
the Shapovalov form.

Observe that as \( e_i^* \) and \( f_i^* \) carry projective modules to projective modules, they act on \( K(\text{Proj} H_n^\lambda) \). Clearly the action is just the transpose of the action of \( e_i^* \) and \( f_i^* \) on \( K(\text{Rep} H_n^\lambda) \). Hence the following lemma is an immediate consequence of the results of section 13. (We reassure the reader that the results of this section are not used in the proof of theorem 11.13).

**Lemma 11.2.** Let \( N \) be an irreducible \( H_n^\lambda \)-module, and set \( \varepsilon = \varepsilon_i(N), \varphi = \varphi_i(N). \)
Then
\[(i) \ e_i^*(\varepsilon)P_N = (\varepsilon + \varphi)P_{\varepsilon_i^*N} + \sum_{Q, \varphi_i(Q) > \varepsilon} a_Q P_{\varepsilon_i^*Q}, \text{ for some } a_Q \in \mathbb{N}. \]
Proof. If $S$ is an irreducible $H_{a-\varepsilon}^\lambda$-module, then $\dim \text{Hom}(e_i^\varepsilon \mathcal{P}_N, S) = 0$. Let $\varphi_i(S) \geq \varepsilon$. So we may assume $S = \overline{\varepsilon}_i Q$, where $Q$ is an irreducible $H_{a-\varepsilon}^\lambda$-module, and then

$$a_Q = \dim \text{Hom}(e_i^\varepsilon \mathcal{P}_N, \overline{\varepsilon}_i Q) = \dim \text{Hom}(\mathcal{P}_N, f_i^\varepsilon \overline{\varepsilon}_i Q).$$

But $f_i^\varepsilon \overline{\varepsilon}_i Q = (\varepsilon_i^{(Q)} + \varepsilon)Q + \sum_{\varepsilon_i(M) < \varepsilon_i(Q)} \alpha_M M$, by proposition 11.8.

Now suppose that $\varepsilon_i(Q) = \varepsilon = \varepsilon_i(N)$. Then the terms $M$ in the above sum have $\varepsilon_i(M) < \varepsilon$, so none are isomorphic to $N$. It follows that if $\varepsilon_i(Q) = \varepsilon$, $a_Q = 0$ unless $Q = N$, and $a_N = (\varepsilon + \varepsilon_i^\varepsilon)$. The lemma is immediate.

We now prove the theorem.

Proof. We may suppose that projective covers of irreducible modules in $K(\text{Rep } H_{a-\varepsilon}^\lambda)$ are linearly independent for $a' < a$, and that $a > 0$. Suppose we have a relation

$$\sum c_M \mathcal{P}_M = 0$$

in $K(\text{Rep } H_{a-\varepsilon}^\lambda)$ with not all the $c_M$ equal to zero. Choose an $i \in \mu_q$ and a simple module $N$ such that $c_N \neq 0$, and $\varepsilon = \varepsilon_i(N)$ is maximal among terms in this sum. We may choose $i \in \mu_q$ so that $\varepsilon > 0$, as for a module $N$, $\varepsilon_i(N) = 0$ for all $i \in \mu_q$ implies that $\text{Res}_{H_{a-\varepsilon}^\lambda} \text{Proj}_{H_{a-\varepsilon}^\lambda} H_{a}^\lambda N = 0$, which is absurd for $a > 0$.

Now apply $e_i^\varepsilon$ to this sum. By the lemma, we get an equality

$$\sum_{N : \varepsilon_i(N) = \varepsilon} \left( \varepsilon + \varepsilon_i(N) \right) c_N \mathcal{P}_{\overline{\varepsilon}_i N} + X = 0$$

in $K(\text{Rep } H_{a-\varepsilon}^\lambda)$, where $X$ is a sum of terms of the form $\mathcal{P}_{\overline{\varepsilon}_i Q}$, with $Q \in \text{Rep}_q H_{a-\varepsilon}^\lambda$ and $\varepsilon_i(Q) > \varepsilon$. In particular, all the terms in the sum are distinct projective modules in $\text{Rep } H_{a-\varepsilon}^\lambda$. By our inductive assumption these terms are linearly independent, hence $X = c_N = 0$. This contradicts our choice of $c_N$, and shows the Shapovalov form is non-degenerate.

It remains to show it is symmetric. Again, induct on $a$. Clearly the form is symmetric on $K(\text{Rep } H_{a}^\lambda)$, and $(f_i^\varepsilon x, y) = (x, e_i^\varepsilon y) = (e_i^\varepsilon y, x) = (y, f_i^\varepsilon x)$ where we have used adjunction twice, and the inductive hypothesis. So the form is symmetric on the image of $1_{\lambda}$ under the operators $e_i^\varepsilon$ and $f_i^\varepsilon$. We must merely show that this is everything. This follows from the following two lemmas.

Lemma 11.3. Let $N$ be an irreducible $H_a^\lambda$-module, and set $\varepsilon = \varepsilon_i(N), \varphi = \varphi_i(N)$. Then

1. $f_i^{\varepsilon+\varphi} \mathcal{P}_N = (\varepsilon + \varphi) f_i^\varepsilon \mathcal{P}_N + \sum_{Q : \varphi_i(Q) > \varphi} a_Q \mathcal{P}_{f_i^\varphi Q}$ for some coefficients $a_Q \in \mathbb{N}$.
2. If $\varepsilon > \varphi$, then $f_i^{\varepsilon+\varphi} \mathcal{P}_N = \sum_{Q : \varphi_i(Q) \geq \varepsilon, \varphi} a_Q \mathcal{P}_{f_i^\varepsilon Q}$ for some coefficients $a_Q \in \mathbb{N}$.

We omit the proof.

Lemma 11.4. Every $\mathcal{P}_N \in K(\text{Proj } H_a^\lambda)$ can be written as a sum of monomial words in $f_i^{\varepsilon}$ with integer coefficients.
Proof. Again, we assume the result for \(a' < a\), and suppose \(a > 0\). Fix \(i \in \mu_q\), and \(r > 0\), and suppose the result is true for all irreducible \(N\) with \(\varphi_i(N) > r > 0\). As there are finitely many irreducible modules in \(\text{Rep}H_n^\lambda\), our induction on \(r\) starts successfully, somewhere. Let \(M\) be irreducible, and \(\varepsilon_i(M) = r\). Apply the above lemma to \(N = \tilde{e}_i^r M\), to get

\[
P_M = f_i^{(r)} \mathcal{P}_{\tilde{e}_i^r M} - \sum_{Q; \varphi_i(Q) > r} a_Q \mathcal{P}_{\tilde{f}_i^r Q}
\]

which by induction is of the desired form. Arguing in this way for each \(i \in \mu_q\), we see the result is true except perhaps for modules \(N\) for which \(\varphi_i(N) = 0\) for all \(i \in \mu_q\). Such a module would have \(\text{Ind}_{H_n^{\lambda+1}}^{H_n^\lambda} N = 0\), which is absurd. Hence the result is true for all modules.

12. RELATIONS IN THE GROTHENDIECK GROUP

We show the operators \(e_i^*\), \(f_i^*\), \(h_i^*\) acting on the Grothendieck group of modules for the cyclotomic and affine Hecke algebras satisfy the defining relations of \(\hat{\mathfrak{s}l}_k\). This is not true before passage to the Grothendieck group. We know of several proofs of this result. Aside from the one given here, another would be to simply compute explicitly the action of \(e_i^*\) and \(f_i^*\) on a suitable basis of \(\text{Rep}K(\lambda)\). Such a basis is given by the Specht modules, first defined by Ariki and Koike.

In order to do this, one must (i) define these modules, (ii) show they span the Grothendieck group \(\text{Rep}K(\lambda)\), and (iii) explicitly compute \(e_i^*\) and \(f_i^*\) on these modules. This requires quite some work (of a combinatorial nature) which can, however, be found in the literature.

In the approach below, we derive the defining relations for \(\hat{\mathfrak{s}l}_k\) from our general theory (which is built from an explicit study of the representations of \(H_n^{\text{aff}}, n \leq 4\)). As a consequence we rederive and explain these properties of Specht modules, obtaining a conceptual explanation for their combinatorics: it is just a realization of the crystal graph.

**Proposition 12.1.** The operators \(e_i^*: \text{Rep}K(\lambda) \rightarrow \text{Rep}K(\lambda), \quad e_i^*: \text{Rep}K^{\text{aff}}(\lambda) \rightarrow \text{Rep}K^{\text{aff}}(\lambda)\) satisfy the Serre relations; i.e. if \(i, j \in \mu_q\), \(ij^{-1} \neq q^{\pm 1}\), then as operators on the Grothendieck group

\[
e_i^* e_j^* = e_j^* e_i^*: K(\lambda) \rightarrow K(\lambda),
\]

If \(ij^{-1} = q^{\pm 1}\), and \(q \neq q^{-1}\), then

\[
e_i^* e_j^* + e_j^* e_i^* = 2e_i^* e_j^* e_i^*: K(\lambda) \rightarrow K(\lambda),
\]

If \(ij^{-1} = q\) and \(q = q^{-1}\), then

\[
e_i^* e_j^* + 3e_j^* e_i^* e_j^* = 3e_i^* e_j^* e_i^*: K(\lambda) \rightarrow K(\lambda).
\]

**Proof.** This reduces to checking the result on irreducible modules for \(H_2^{\text{aff}}\) (if \(ij^{-1} \neq q^{\pm 1}\), \(H_3^{\text{aff}}\) (if \(ij^{-1} = q^{\pm 1}\), and \(q \neq q^{-1}\)), and \(H_4^{\text{aff}}\) (in the remaining case). To see this, observe that as \(e_i^*\) commutes with \(\text{ev}^*: K(\lambda) \rightarrow K^{\text{aff}}(\lambda)\), and \(\text{ev}^*\) is injective, it is enough to check that the \(e_i^*\) satisfy the Serre relations on \(K(\lambda)\).

But, as observed in [4], \(e_i^*\) is just the component \(\text{Id} \otimes \delta_{ij}\Delta_k\) of \(\Delta_{n,1}\) and \(\Delta\) is coassociative. So it is enough to check the relations involving a word of length \(k\) in the \(e_i^*\)’s on \(H_k^{\text{aff}}\).
We must now check the Serre relations for the irreducible $H_2^{\text{aff}}$-modules, $n \leq 4$. These modules were listed in section 3, and the relations follow by examining each module individually.

Remark 6. The argument reducing the proposition to a case by case check sounds simpler if phrased in terms of $K(\text{Rep}_{q}^\lambda)^*$—algebras are easier to think with than coalgebras. It becomes even clearer when stated explicitly: Let $M \in \text{Rep } H_n^{\text{aff}}$.

Consider $e_i^* e_j^* (M)$. This is $\lim_{m, m'} \text{Hom}(H_n^{\text{aff}} \otimes q^i J_m \otimes q^j J_{m'}, \text{Res}_{H_n^{\text{aff}} \otimes H_n^{\text{aff}} \otimes H_n^{\text{aff}}}^n M)$.

Factor $\text{Res}_{n, 1, 1}^n M = \text{Res}_{n, 1, 1}^n \text{Res}_{n, 2}^n M$. $\text{Res}_{n, 2}^n M$ has a filtration with graded pieces simple modules $N_\alpha \boxtimes \Gamma_\alpha$, with $N_\alpha$ a simple $H_n^{\text{aff}}$-module, and $\Gamma_\alpha$ a simple $H_2^{\text{aff}}$-module. The image on $e_i^* e_j^* (M)$ in the Grothendieck group depends only on the value of the exact functor

$$\lim_{m, m'} \text{Hom}(H_n^{\text{aff}} \otimes q^i J_m \otimes q^j J_{m'}, \text{Res}_{n, 1, 1}^n ( )) = \text{Id} \otimes \lim_{m, m'} \text{Hom}_{K(\mathbb{A}^{n+1, n+2})}(q^i J_m \otimes q^j J_{m'}, \text{Res}_{1, 1}^n ( ))$$

on the pieces $N_\alpha \boxtimes \Gamma_\alpha$, and this functor depends only on $\Gamma_\alpha$. As $e_i^* e_j^*$ has a similar description, acting on the pieces $N_\alpha \boxtimes \Gamma_\alpha$ by some other functor depending only on $\Gamma_\alpha$, it suffices to check equality on $H_2^{\text{aff}}$-modules. And again, as these functors on $H_2^{\text{aff}}$ are exact, it suffices to check equality on simple modules.

Remark 7. In fact, the Serre relations are a formal consequence of the following (see section 3.3.3 of [Kac]): (i) The relations $[e_i^*, f_j^*] = \delta_{ij} h_i^*$ and $[h_i^*, e_j^*] = c_{ij} e_j^*$, proved below, (ii) the Shapovalov form of section 1 is non-degenerate, and (iii) the union $\bigcup \text{Rep}^\lambda = \text{Rep}_{q}^\lambda$. (This last condition ensures that the Serre relations hold not just as operators on the irreducible module $K(\text{Rep}_{q}^\lambda)$, but also on the module $K(\text{Rep}_{q}^\lambda)$.) This is no simplification, as the proof we have chosen to give of $[h_i^*, e_j^*] = c_{ij} e_j^*$ essentially consists of verifying a more precise form of the Serre relations.

Corollary 12.2. The operators $f_i^*: K(\text{Rep}_{q}^\lambda) \rightarrow K(\text{Rep}_{q}^\lambda)$ satisfy the Serre relations.

Proof. It is enough to show they satisfy the Serre relations as maps from $K(\text{Rep}_{q}^\lambda)_\mathbb{Q} \rightarrow K(\text{Rep}_{q}^\lambda)_\mathbb{Q}$. But the Shapovalov form of section 1 is non-degenerate, and $f_i^*$ is adjoint to $e_i^*$ with respect to this form. As the $e_i^*$ satisfy the Serre relations, so do the $f_i^*$.

Proposition 12.3. $n! e_i^{(n)*} = e_i^{*n}$, $n! f_i^{(n)*} = f_i^{*n}: K(\text{Rep}_{q}^\lambda) \rightarrow K(\text{Rep}_{q}^\lambda)$

Proof. As before, to check $n! e_i^{(n)*} = e_i^{*n}$, it suffices to check on simple $H_n^{\text{aff}}$-modules. Both left and right hand sides are zero on all simple $H_n^{\text{aff}}$-modules except $q^i K_\alpha$, where both are $n!$. This implies $n! f_i^{(n)*} = f_i^{*n}$, by an argument similar to the above one.
We must now determine the relations between \(e_i^*\) and \(f_j^*\). The Mackey formula for the cyclotomic Hecke algebra implies

\[
(7) \quad [\text{Res}_{H_n^\lambda}^{H_{n-1}^\lambda}, \text{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda}](M) = \sum \lambda_i M.
\]

As \(e_i^*\) is a refined version of restriction, and \(f_j^*\) is a refined variant of induction, the next theorem should be regarded as a sharpening of \(\text{(6)}\). The proof will occupy the rest of this section.

Define for \(N\) an irreducible \(H_n^\lambda\)-module

\[
h_i^*(N) = (\varphi_i(N) - \varepsilon_i(N))N
\]

and more generally define \((h_i^*)(N) = (\varphi_i(N) - \varepsilon_i(N))\cdot N\), so that \((h_i^*)K(\text{Rep}_q^\lambda) \to K(\text{Rep}_q^\lambda)\).

**Theorem 12.4.** \([e_i^*, f_j^*] = \delta_{ij} h_i^* : K(\text{Rep}_q^\lambda) \to K(\text{Rep}_q^\lambda)\).

We begin by showing

**Proposition 12.5.** Suppose \(M\) is an irreducible \(H_n^\lambda\)-module. Then \([e_i^*, f_j^*](M)\) is a multiple of \(M\), and \([e_i^*, f_j^*](M) = 0\) if \(i \neq j\).

**Proof.** For \(m \gg 0\) we have a surjection

\[
\text{Ind}(M \boxtimes q^i J_m) \twoheadrightarrow f_j^* M \to 0.
\]

Apply \(\text{pr}\) \(e_i^*\). As \(e_i^*\) is exact, and \(\text{pr}\) is right exact, we still get a surjection

\[
\text{pr} e_i^* \text{Ind}(M \boxtimes q^i J_m) \twoheadrightarrow e_i^* f_j^* M \to 0.
\]

But by the Mackey formula, we have an exact sequence

\[
0 \to \delta_{ij} m M \to e_i^* \text{Ind}(M \boxtimes q^i J_m) \to \text{Ind}(e_i^* M \boxtimes q^i J_m) \to 0,
\]

and hence, as \(M\) is irreducible, an exact sequence

\[
0 \to \delta_{ij} m' M \to \text{pr} e_i^* \text{Ind}(M \boxtimes q^i J_m) \to f_j^* e_i^* M \to 0
\]

for some \(m' \leq m\), if \(m \gg 0\). Hence

\[
\delta_{ij} m' M + f_j^* e_i^* M \geq e_i^* f_j^* M \geq 0
\]

where we write \(A \geq B\) if for each irreducible \(N\), the multiplicity \([N : A]\) of \(N\) in \(A\) is not less than the multiplicity \([N : B]\) of \(N\) in \(B\). Sum \(\text{(8)}\) over \(i, j \in \mu_q\), we get

\[
(9) \quad aM + \text{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} \text{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} M \geq \text{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} \text{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} M
\]

for some \(a \geq 0\).

Next we claim that in \(K(\text{Rep}_q^\lambda)\)

\[
(10) \quad \left(\sum \lambda_i\right) M + \text{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} \text{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} M = \text{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} \text{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} M.
\]

Granting this for the moment, let \(N\) be an irreducible \(H_n^\lambda\)-module with \(N \neq M\). Then comparing the multiplicity of \(N\) in \(\text{(10)}\) and in \(\text{(9)}\), we see that all inequalities in \(\text{(6)}\) must be equalities, and so the multiplicity of \(N\) in \(f_j^* e_i^* M\) equals the multiplicity of \(N\) in \(e_i^* f_j^* M\); so \([e_i^*, f_j^*](M) = 0\) if \(i \neq j\).
So to prove the proposition it remains to show \( \text{[10]} \). This is immediate from the Mackey formula for \( H_n^\lambda \). A weak form of this may be immediately deduced from the Mackey formula for \( H_n^\text{aff} \); we omit further details.

For \( N \) an irreducible \( H_n^\lambda \)-module, write \( \text{wt}_i(N) = \varphi_i(N) - \varepsilon_i(N) \), and

\[
\text{wt}(N) = \sum \text{wt}_i(N) \lambda_i: \mu_q \to \mathbb{Z}.
\]

Define a function \( \delta_i: R^n \to \mathbb{N} \) by

\[
\delta_i(s_1, \ldots, s_n) = \# \{ a | 1 \leq a \leq n, s_a = i \},
\]

and set \( \text{wt}_i(s) = -2\delta_i(s) + \delta_q(s) + \delta_{q^{-1}}(s) \). Recall \( \alpha_i = 2\lambda_i - \Lambda_q - \Lambda_{q^{-1}} \).

The following theorem which is a summary of the results of section 10, tells us that both \( \varepsilon_i(N) \) and \( \varphi_i(N) \) may be read off the spectrum of \( N \), and that their difference \( \text{wt}_i(N) \) depends only on the central character of \( N \).

**Theorem 12.6.** Let \( N \) be an irreducible \( H_n^\lambda \)-module with central character \( s \). Then

(i) \( \text{wt}(f_i N) = \text{wt}(N) - \alpha_i \), if \( f_i N \neq 0 \).

(ii) \( \text{wt}_i(1_\lambda) = \lambda_i \), where \( 1_\lambda \) is the irreducible \( H_0^\lambda \)-module.

(iii) \( \text{wt}_i(N) = \varphi_i(N) - \varepsilon_i(N) = \lambda_i + \text{wt}_i(s) \).

**Proof.** As every irreducible module is obtained from \( 1_\lambda \) by a sequence of raising operators \( f_i \), it is clear that (i) and (ii) are equivalent to (iii). (ii) is immediate from the definition of \( f_i \) and the description \( H_0^\lambda = R[x]/ \prod(x - q^i) \lambda_i \). So we prove (i).

We first observe that as \( N = \bar{e}_i f_i N \) if \( f_i N \neq 0 \), we have \( \text{wt}_i(f_i N) = \text{wt}_i(N) - 2 \). Further, as \( [e_j^*, f_i^*] = 0 = [f_j^*, f_i^*] \) if \( j \notin \{ q_i, i, q^{-1} i \} \), for such \( j \) \( \text{wt}_i(f_j N) = \text{wt}_i(N) \), in agreement with (i). Hence for \( q \neq -1 \), the content of the theorem is the assertion

\[
\text{wt}_i(f_{qi} N) = \text{wt}_i(N) + 1, \quad \text{if} \quad f_{qi} N \neq 0.
\]

This is immediate from proposition \( \text{[10]} \). Likewise, if \( q = -1 \), proposition \( \text{[10]} \) is equivalent to the assertion of the theorem.

We are now in a position to finish the proofs of theorem 9.15 and theorem 12.4. For \( M \) an irreducible module, let \( \bar{\varphi}_i(M) \) be the integer defined in theorem 9.15. Then by theorem 9.15 (iii),

\[
\sum_{i \in \mu_q} \bar{\varphi}_i(M) = \sum_{i,j \in \mu_q} \dim \text{Hom}(f_i^* M, f_j^* M)
\]

\[
= \sum_{i,j \in \mu_q} \dim \text{Hom}(M, e_j^* f_i^* M)
\]

\[
= \dim \text{Hom}(M, \text{Res}_H^{H_n^\lambda} \text{Ind}_H^{H_n^\lambda+1} M)
\]

and by \( \text{[10]} \), this is

\[
\leq \sum \lambda_i + \sum \varepsilon_i(M).
\]

But \( \bar{\varphi}_i(M) \geq \varphi_i(M) \) for all \( i \in \mu_q \), by the last paragraph of theorem 9.15, and by theorem 12.6 (iii)

\[
\sum \lambda_i = \sum (\bar{\varphi}_i(M) - \varepsilon_i(M)).
\]

It follows that

\[
\sum \lambda_i = \sum (\varphi_i(M) - \varepsilon_i(M)) \leq \sum (\bar{\varphi}(M) - \varepsilon(M)) \leq \sum \lambda_i,
\]
and hence all the inequalities above are equalities. In particular \( \tilde{\varphi}_i(M) = \varphi_i(M) \), for all \( i \), completing the proof of theorem 0.13.

Finally, theorem 0.13 (ii) and theorem 0.13 (ii) now give

\[
[e_i^* f_i^* M - f_i^* e_i^* M : M] = (\varepsilon_i(M) + 1)\varphi_i(M) - (\varphi_i(M) + 1)\varepsilon_i(M)
= \varphi_i(M) - \varepsilon_i(M)
\]

completing the proof of theorem 12.4.

To finish verifying the defining relations of \( \tilde{\mathfrak{sl}}_\ell \), we need only observe that theorem 12.4 immediately implies

**Lemma 12.7.** \([h_i^*, e_i^*] = c_{ij} e_j^* \), \([h_i^*, f_i^*] = -c_{ij} f_j^* : K(\text{Rep}_q^\text{aff}) \to K(\text{Rep}_q^\text{aff}) \), where \( c_{ij} = 2\delta_{ij} - \delta_{i,j} - \delta_{q_{ij}} \) is the Cartan matrix of \( \tilde{\mathfrak{sl}}_\ell \).

**Remark.** M. Vazirani has recently improved on theorem 12.4 by determining the relation between \( e_i^* f_i^* M \) and \( f_i^* e_i^* M \) for \( M \) irreducible, before passage to the Grothendieck group. Her results extend and clarify theorems 0.15(iii) and 0.13(iii).

13. **Uniqueness of the Crystal**

In this section we determine the crystal graph of \( K(\text{Rep}_q^\text{aff}) \). This admits many different combinatorial descriptions, each of which it is possible to interpret Hecke-theoretically. Rather than do this, we prove one more property of the crystal \( B_\text{aff} \) of \( K(\text{Rep}_q^\text{aff}) \). This property (proposition 13.1), together with what we have proved earlier, is already sufficiently strong to show combinatorially the uniqueness of the crystal. In fact other than 13.1 all we need is that the crystals \( B_\lambda \) admit a description purely in terms of the crystal \( B_\text{aff} \) and the involution on \( B_\text{aff} \) induced by the anti-automorphism \( \sigma^* \).

Recall that we write \( \tilde{e}_i^\wedge = \sigma^* \tilde{e}_i \sigma^* \), and also define \( e_i^* \wedge = \sigma^* e_i^* \sigma^* \) and \( f_i^\wedge = \sigma^* f_i \sigma^* \).

**Proposition 13.1.** Let \( M \) be an irreducible \( H_n^\text{aff} \)-module, and write \( c = \varepsilon_i^\wedge(M) \)

(i) Suppose \( \varepsilon_i^\wedge(\tilde{f}_i M) = \varepsilon_i^\wedge(M) \). Then

\[
(\tilde{e}_i^\wedge)^c(\tilde{f}_i M) = \tilde{f}_i(\tilde{e}_i^\wedge c M)
\]

(ii) If \( \varepsilon_i^\wedge(\tilde{f}_i M) = \varepsilon_i^\wedge(M) + 1 \), then \( \tilde{e}_i^\wedge \tilde{f}_i M = M \).

**Proof.** (i) We have \( M = (\tilde{f}_i^\wedge)^c N = \cosoc \text{Ind}(q^i \text{Rep}_q \boxtimes N) \), where \( N \) is an irreducible \( H_n^\text{aff} \) module with \( \varepsilon_i^\wedge(N) = 0 \).

Set \( Q_a = (e_i^* \wedge)^{-a} \tilde{f}_i M \), so that in the Grothendieck group \( Q_a \) is some number of copies of \((\tilde{e}_i^\wedge)^c \tilde{f}_i M \) plus terms with strictly smaller \( \varepsilon_i^\wedge \). In particular we have that \( \varepsilon_i^\wedge(A) \leq a \) for all \( A \) that occur in \( Q_a \), and \( Q_0 \) is just some copies of \((\tilde{e}_i^\wedge)^c \tilde{f}_i M \).

We will show by decreasing induction on \( a \) that there is a non-zero map

\[
\gamma_a : \text{Ind}(q^i \text{Rep}_q \boxtimes N \boxtimes q^i J_1) \to Q_a.
\]

If \( a = c \), \( Q_a = \tilde{f}_i M = \cosoc \text{Ind}(M \boxtimes q^i J_1) \) is a quotient of \( \text{Ind}(q^i \text{Rep}_q \boxtimes q^i J_1) \) so our induction starts. Now suppose \( \gamma_a \) exists, and \( a \geq 1 \). Consider \( \text{Res}_{q^{i+n-1}}^{q^n \text{Rep}_q} \text{Ind}(q^i \text{Rep}_q \boxtimes N \boxtimes q^i J_1) \). By the Mackey formula, this has a three step filtration \( 0 \subset \tilde{F}_1 \subset \tilde{F}_2 \subset \tilde{F}_3 \)
with successive quotients
\[
F_1 = \text{Ind}_{1,n}^{1,a+n} \text{Res}_{1,n}^{a,n} (q^i K_a \boxtimes N) \boxtimes q^i J_1),
\]
\[
F_2 = \text{Ind}_{1,n}^{1,a+n} \text{Res}_{1,n}^{a,n} (q^i K_a \boxtimes N) \boxtimes q^i J_1),
\]
\[
F_3 = \text{Ind}_{1,n}^{1,a+n} (q^i J_1 \boxtimes q^i K_a \boxtimes N),
\]

where \(w\) is the obvious permutation.

As \(\gamma_a \neq 0\), Frobenius reciprocity gives a copy of \(q^i K_a \boxtimes N \boxtimes q^i J_1\) in the image of \(\gamma_a\), and so
\[
\tilde{\gamma}_a := \epsilon_i^\gamma : \epsilon_i^\gamma \text{Ind}(q^i K_a \boxtimes N \boxtimes q^i J_1) \to \epsilon_i^\gamma Q_a = Q_a^{-1}
\]
is non-zero. Suppose that \(\tilde{\gamma}_a\) is zero when restricted to the \(q^i\)-eigenspace of \(X_1\) on \(F_1\). As there is no \(q^i\)-eigenspace of \(X_1\) on \(F_2/F_1\), we must have a non-zero homomorphism from \(F_3/F_2\) to \(Q_a^{-1}\), i.e. a non-zero homomorphism
\[
\text{Ind}(q^i K_a \boxtimes N) \to Q_a^{-1}.
\]

But \(\epsilon_i^\gamma(\text{cosoc Ind}(q^i K_a \boxtimes N)) = a > \epsilon_i^\gamma(A)\), for any constituent \(A\) of \(Q_a^{-1}\). So this is not possible, and it must be that \(\tilde{\gamma}_a\) restricts to a non-zero homomorphism on the \(q^i\)-eigenspace of \(X_1\) on \(F_1\). As \(\epsilon_i^\gamma(q^i K_a)\) has a filtration with subquotients \(q^i K_a^{-1}\), there must be a non-zero map \(\gamma_a^{-1} : \text{Ind}(q^i K_a^{-1} \boxtimes N \boxtimes q^i J_1) \to Q_a^{-1}\).

We now take \(a = 0\) and conclude there is a non-zero homomorphism
\[
\gamma_0 : \text{Ind}(N \boxtimes q^i J_1) \to Q_0,
\]

hence \(\tilde{f}_i(\overline{\epsilon_i^\gamma})^c M = \tilde{f}_i N = \text{cosoc Ind}(N \boxtimes q^i J_1)\) is a subquotient of \(Q_0\). But \(Q_0\) is a multiple of \((\overline{\epsilon_i^\gamma})^c \tilde{f}_i M\), so we have indeed shown that \(\tilde{f}_i(\overline{\epsilon_i^\gamma})^c M = (\overline{\epsilon_i^\gamma})^c \tilde{f}_i M\).

(ii) Again write \(N = (\overline{\epsilon_i^\gamma})^c M\). As multiplication is commutative in the bialgebra \(K(\text{Rep}_{\text{aff}} q^\gamma)\), \(\text{Ind}(q^i K_a \boxtimes N \boxtimes q^i J_1)\) equals \(\text{Ind}(q^i K_a \boxtimes q^i J_1 \boxtimes N)\) in the Grothendieck group. Hence \(\text{Ind}(q^i K_a \boxtimes N \boxtimes q^i J_1)\) is \((\overline{\epsilon_i^\gamma})^{c+1} N = \overline{\epsilon_i^\gamma} M\) plus terms \(A\) with \(\epsilon_i^\gamma(A) \leq c\). As \(\text{Ind}(q^i K_a \boxtimes N \boxtimes q^i J_1)\) surjects onto \(\tilde{f}_i M\), if \(\epsilon_i^\gamma(\tilde{f}_i M) = c + 1\) it must be that \(\tilde{f}_i M = \tilde{f}_i M\).

This is the last property we will need of the representations of the affine Hecke algebra. The next proposition is a formal consequence of what we have already proved.

For \(M\) an irreducible \(H_n^{\text{aff}}\)-module with central character \(s\), define
\[
\text{wt}_s^i(M) = \text{wt}_s(s).
\]

Also define \((\overline{\epsilon_i^\gamma})^{\max}(M) = (\overline{\epsilon_i^\gamma})^{\gamma}(M)\), where \(c = \epsilon_i^\gamma(M)\).

**Proposition 13.2.** Let \(M\) be an irreducible \(H_n^{\text{aff}}\)-module, and write \(c = \epsilon_i^\gamma(M)\), \(M = (\overline{\epsilon_i^\gamma})^{\max}(M)\).

(i) \(\epsilon_i(M) = \text{max}(\epsilon_i(M), c - \text{wt}_s^i(M))\).

(ii) Suppose \(\epsilon_i(M) > 0\). Then
\[
\epsilon_i^\gamma(\overline{\epsilon_i}(M)) = \begin{cases} c, & \text{if } \epsilon_i(\overline{M}) \geq c - \text{wt}_s^i(\overline{M}) \\ c - 1, & \text{if } \epsilon_i(\overline{M}) < c - \text{wt}_s^i(\overline{M}) \end{cases}
\]

(iii) Suppose \(\epsilon_i(M) > 0\). Then
\[
(\overline{\epsilon_i^\gamma})^{\max}(\overline{\epsilon_i}(M)) = \begin{cases} \overline{\epsilon_i}(\overline{M}), & \text{if } \epsilon_i(\overline{M}) \geq c - \text{wt}_s^i(\overline{M}) \\ \overline{M}, & \text{if } \epsilon_i(\overline{M}) < c - \text{wt}_s^i(\overline{M}) \end{cases}
\]
Proof. Let \( \lambda : \mu_q \to \mathbb{N} \) be such that \( \sum \lambda_i < \infty \), and suppose \( N \in \text{Rep} H_\lambda^s \). Then corollary 9.14 tells us that \( k = \varphi_i(N) \) means

\[
\varepsilon_i^\lambda(N) \leq \varepsilon_i^\lambda(\bar{f}_i^k N) = \lambda_i, \quad \text{and} \quad \varepsilon_i^\lambda(\bar{f}_i^{k+1} N) = \lambda_i + 1,
\]

and that theorem 12.6 shows that \( \varphi_i(N) = \lambda_i + \varepsilon_i(N) + \text{wt}'_i(N) \).

Take \( N = \bar{e}_i^m M \), where \( m = \varepsilon_i(M) \), and define \( \lambda(y) : \mu_q \to \mathbb{N} \) by setting \( \lambda_i(y) = \varepsilon_i^\lambda(N) + y \), and setting \( \lambda_j(a) \) to be any integer much greater than \( m + y + a \), when \( j \neq i \). Then it follows from the previous paragraph applied to \( \lambda(0), \lambda(1), \ldots \) that for all \( s \)

\[
\varepsilon_i^\lambda(\bar{f}_i^s N) = \begin{cases} 
\varepsilon_i^\lambda(N), & s \leq \varepsilon_i^\lambda(N) + \varepsilon_i(N) + \text{wt}'_i(N) \\
 s - \varepsilon_i(N) - \text{wt}'_i(N), & s \geq \varepsilon_i^\lambda(N) + \varepsilon_i(N) + \text{wt}'_i(N)
\end{cases}
\]

and as \( \varepsilon_i(N) = 0 \) it follows that

\[
\varepsilon_i^\lambda(\bar{f}_i^m N) = \max(\varepsilon_i^\lambda(N), m - \text{wt}'_i(N)),
\]

i.e. that \( \varepsilon_i^\lambda(M) = \max(\varepsilon_i^\lambda(\bar{e}_i^\sigma(M) M), \varepsilon_i(M) - \text{wt}'(\bar{e}_i^\sigma(M) M)) \). Applying this to \( \sigma^* M \) we get (i).

To see (ii), observe that \( \varepsilon_i^\lambda(\bar{e}_i M) = \varepsilon_i^\lambda(M) - 1 \) precisely when

\[
m = \varepsilon_i(M) > \text{wt}'_i(N) + \varepsilon_i^\lambda(N)
\]

and that otherwise \( \varepsilon_i^\lambda(\bar{e}_i M) = \varepsilon_i^\lambda(M) \). But \( \text{wt}'_i(N) = \text{wt}'_i(M) + 2m \), and so (ii) follows if we show that \( \text{wt}'_i(M) + \varepsilon_i^\lambda(N) + m < 0 \) if and only if \( \text{wt}'_i(M) + \varepsilon_i(\bar{M}) + c < 0 \). But \( \text{wt}'_i(M) + \varepsilon_i^\lambda(N) + m = \max(\text{wt}'_i(M) + \varepsilon_i^\lambda(N) + \varepsilon_i(M), \varepsilon_i^\lambda(N) - c) \), by (i), and \( \varepsilon_i^\lambda(N) - c \leq 0 \) always. Similarly \( \text{wt}'_i(M) + \varepsilon_i(M) + c = \max(\text{wt}'_i(M) + \varepsilon_i^\lambda(N) + \varepsilon_i(M), \varepsilon_i(M) + m) \), and \( \varepsilon_i(M) - m \leq 0 \) always. Finally, proposition 13.1 shows that \( \varepsilon_i(M) = m \) if and only if \( \varepsilon_i^\lambda(N) = c \); hence (ii).

Now (iii) is immediate from (ii) and proposition 13.1.

\[\square\]

13.1. Combinatorial consequences. We now show that we have enough properties to completely describe the tensor category of “integrable lowest weight crystals”, and hence to describe the crystals themselves. We follow [Ka], especially 8.2. (This is a kind of purely combinatorial Tannakian property.)

So we recall some ideas of Kashiwara [Ka]. Recall that if \( B_1 \) and \( B_2 \) are two crystals, their tensor product \( B_1 \otimes B_2 \) is the crystal whose underlying set is \( B_1 \times B_2 \) and with

\[
\bar{e}_i(b_1 \otimes b_2) = \begin{cases} 
\bar{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\
b_1 \otimes \bar{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2)
\end{cases}
\]

\[
\bar{f}_i(b_1 \otimes b_2) = \begin{cases} 
\bar{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\
b_1 \otimes \bar{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2)
\end{cases}
\]

Define the crystal \( B_i \), for \( i \in \mu_q \), to have underlying set \( \{b_i(n) \mid n \in \mathbb{Z}\} \) and set

\[
\varepsilon_j(b_i(n)) = \begin{cases} 
-n, & j = i \\
-\infty, & j \neq i 
\end{cases}, \quad \varphi_j(b_i(n)) = \begin{cases} 
n, & j = i \\
-\infty, & j \neq i 
\end{cases}
\]

\[
\bar{e}_j(b_i(n)) = \begin{cases} 
b_i(n + 1), & j = i \\
0, & j \neq i 
\end{cases}, \quad \bar{f}_j(b_i(n)) = \begin{cases} 
b_i(n - 1), & j = i \\
0, & j \neq i 
\end{cases}
\]

and write \( b_i = b_i(0) \).
Recall that a strict embedding of crystals is an injective map \( \psi : B_1 \rightarrow B_2 \) such that \( \psi \) commutes with \( e_i \) and \( f_i \) for all \( i \in \mu_q \). Define \( B_\infty \) to be the same crystal as \( B_{aff} \), except that we set \( \varphi_s(M) := e_i(M) + \text{wt}_s \) if \( M \) has central character \( s \).

We can now rephrase proposition 13.2 as

**Proposition 13.3.** For each \( i \in \mu_q \), define a map \( \Psi_i : B_\infty \rightarrow B_{aff} \otimes B_i \) by sending \( M \) to \( (\epsilon_i)^{(c)}(M) \otimes f^{\mu_0}b_1 \), where \( c = \ell_i(M) \). Then \( \Psi_i \) is a strict embedding of crystals.

It is a result of Kashiwara that this determines the crystal \( B_\infty \) (see [KS] and proposition 3.2.3 of [KS]). For the convenience of the reader we reproduce the argument. Choose a sequence \( (i_1, i_2, \ldots) \) in \( \mu_q \) so that each \( i \in \mu_q \) appears infinitely often. Define a sequence \( \Phi_n : B_\infty \rightarrow B_{aff} \otimes B_{i_1} \otimes \cdots \otimes B_{i_n} \) by \( \Phi_n = \Psi_{i_n} \circ \cdots \circ \Psi_{i_1} \). Then for any \( b \in B_\infty \) there exists an \( n \) such that \( \Phi_n(b) = b_0 \otimes f_{i_1}^{\mu_0}b_{i_2} \otimes \cdots \otimes f_{i_n}^{\mu_0}b_1 \). The sequence \( (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \) does not depend on \( n \). This embeds \( B_\infty \) as the smallest subcrystal of \( B_{Kas} \) containing \( (0, 0, \ldots) \), where we define \( B_{Kas} \) to be the crystal whose underlying set is the set of sequences \( \{(a_i) \in Z \mid a_i = 0 \text{ for } i \gg \} \), and whose crystal structure is defined by sending \( (a_1, \ldots, a_n, 0, \ldots) \) to \( \cdots \otimes b_{i_2}^{-1}a_{i_2} \otimes b_{i_1}(a_{i_1}) \). Hence \( B_\infty \) is completely determined by the Dynkin diagram \( \mu_q \), as \( B_{Kas} \) is.

14. REAPING THE HARVEST

In this section we summarise the theorems of the previous sections and identify the Hopf algebra \( K(\text{Rep}^\text{aff}_q) \) and its comodules \( K(\text{Rep}_q^\text{aff}) \). As a consequence of this rigid structure, we obtain a parameterization of irreducible modules for \( H_n^\text{aff} \) and \( H_n^\text{aff} \) over any field \( R \) that depends only on \( n, \lambda \), and \( \ell = |\mu_q| \) (and not on \( R \), or even the characteristic of \( R \)). We observe that the crystal graph of \( B_\lambda \) we have defined coincides with the crystal graph defined by Kashiwara and Lusztig. We emphasise that there is no further Hecke theoretic content in the parameterisation of representations—any of the many combinatorial descriptions of the crystal basis gives a combinatorial parameterisation of \( B_\lambda \). Thus we can label modules by tuples of partitions, or Littelmann paths, or paths in a perfect crystal, for example. The identification of this with Deligne-Langlands parameters \( K[1, 3] \) is a pleasant exercise.

For convenience we dualize \( K(\text{Rep}^\text{aff}_q) \), and denote the adjoints to \( e_i^{(n)*}, f_i^{(n)*}, (h_i^\ell) \) by \( e_i^{(n)*}, f_i^{(n)*}, (h_i^\ell) \). Also write \( 1_\lambda \) for the generator of \( K(\text{Rep} H_0^\lambda) = Z \) dual to the trivial representation of \( H_0^\lambda \).

**Theorem 14.1.** The map \( U_{\mathfrak{g}^\text{aff}} \rightarrow K(\text{Rep}^\text{aff}_q)^* \) which sends the generators \( e_i^{(n)} \) to the elements \( e_i^{(n)}1_{aff} = \delta_q\ell_iK_\lambda \) is an isomorphism of bialgebras.

**Theorem 14.2.** The operators \( e_i^{(n)}, f_i^{(n)}, (h_i^\ell) \) define a structure of a \( U_{\mathfrak{g}^\text{aff}} \)-module on \( K(\text{Rep}^\text{aff}_q)^* \). The module \( K(\text{Rep}^\text{aff}_q)^* \) is a \( \mathbb{Z} \)-form of the irreducible integrable lowest weight module for \( \mathfrak{g}^\text{aff} \) with lowest weight \( \lambda \) (and lowest weight vector \( 1_\lambda \)). Under this identification, the Shapovalov form on the module becomes the form of section 14.

Remark 9. The bialgebra structure on \( K(\text{Rep}^\text{fin}_q)^* \) is that given by the principal realisation of the basic representation, i.e. the identification of this with the Hopf algebra \( Z[x_i \mid \ell \neq i] \).
Theorem 14.3. The crystal graph of the $\hat{\mathfrak{sl}}_\ell$-module $K(\text{Rep}_q^\lambda)^*$ is the graph $(B_\lambda, e_i, f_i)$ defined in section 13.

We prove theorems 14.1 and 14.2 simultaneously.

Step 1: As $K(\text{Rep}_q^\lambda)^*$ is a torsion-free $\mathbb{Z}$-module, and $e_i^{(n)}$, $f_i^{(n)}$, $(h_i^n)$ are well defined operators which are determined by the actions of $e_i$, $f_i$ and $h_i$, it is enough to check that the defining relations on $e_i$, $f_i$, $h_i$ are satisfied. This was done in the last section, and so we have a well defined action of $U_q\hat{\mathfrak{sl}}_\ell$ on $K(\text{Rep}_q^\lambda)^*$. Similarly, we have a well defined bialgebra morphism $U_{\mathbb{Z}^+}\hat{\mathfrak{sl}}_\ell \to K(\text{Rep}_q^\lambda)^*$.

Step 2: The map $U_{\mathbb{Z}^+}\eta \to K(\text{Rep}_q^\lambda)^*$ is surjective. This follows from lemma 1.4, which is precisely the statement that the map $U_{\mathbb{Z}^+}\eta \to K(\text{Rep}_q^\lambda)^*$ is surjective, and the fact that every irreducible $\mathfrak{h}^\text{aff}_n$-module is a $H^\text{aff}_n$ module for some $\lambda$.

Step 3: We show $K(\text{Rep}_q^\lambda)^*_{\mathbb{Q}}$ is the irreducible $\hat{\mathfrak{sl}}_\ell$-module with lowest weight $\lambda$. But $K(\text{Rep}_q^\lambda)^*_{\mathbb{Q}}$ is a $U_q\hat{\mathfrak{sl}}_\ell$-module on which $e_i$ and $f_i$ act locally nilpotently (theorems 9.13 and 9.15), and on which $h_i$ acts semisimply. By step 2 it is generated by $1_\lambda$ as a $U_{\mathbb{Z}^+}\eta$-module. Hence it is an irreducible integrable lowest weight module. By theorem 12.6 (ii) this weight is $\lambda$.

Step 4: Finally we show the map $U_{\mathbb{Z}^+}\eta \to K(\text{Rep}_q^\lambda)^*$ is injective. Let $x$ be in its kernel. As the action of $U_{\mathbb{Z}^+}\eta$ on $K(\text{Rep}_q^\lambda)^*$ factors through $K(\text{Rep}_q^\lambda)^*$, $x$ acts as zero on every $K(\text{Rep}_q^\lambda)^*$ hence on every integrable lowest weight module. It follows that $x = 0$.

This concludes the proof of theorems 14.1 and 14.2. Theorem 14.3 follows from the description of the crystal $B^\text{aff}_n$ in section 13 and the fact that the crystal of $U_{\eta}$ has the same description $[K]_n$.

14.1. The $p$-canonical basis. The above theorems prompt the following definition. Fix a prime $p \geq 0$, and a positive integer $\ell$.

Definition 1. The $p$-canonical basis of $\hat{\mathfrak{sl}}_\ell$ is the basis of the module $U_{\mathbb{Z}^+}\eta$ given by the dual of the irreducible $\mathfrak{h}^\text{aff}_n$-modules, where $\mathfrak{h}^\text{aff}_n$ is the affine Hecke algebra over $R = \mathbb{F}_p$, an algebraically closed field of characteristic $p$, and $q \in R$ is a primitive $\ell^{th}$ root of unity. (If $p = 0$, take $R = \mathbb{C}$.)

(If $(p, \ell) \neq 1$, we can still define such a basis. The case $p = \ell$ is explained in the next section.) This basis has the following pleasant properties, among others.

(i) The basis of $U_{\mathbb{Z}^+}\eta$ descends to give a basis for all integrable lowest weight modules.

(ii) The structure constants of $e_i$ and $f_i$ on this basis are non-negative integers.

(iii) The 0-canonical basis is a non-negative integral combination of the $p$-canonical basis, for each prime $p$.

(iv) The 0-canonical basis is the canonical basis (= global crystal basis) of Lusztig and Kashiwara.

Note that we have given (elementary!) proofs of (i)–(iii) in this paper; property (iv) is immediate from the Kazhdan-Lusztig description of the affine Hecke algebra in geometric terms and Lusztig’s definition of the canonical basis in terms of perverse sheaves on quivers; this is explained (tersely) in $[K]$. 
15. Modifications when $q = 1$.

If $q = 1$ all of the above theorems and constructions go through, without change, once we make the appropriate definitions.

Define $\mu_q$ to be the image of $\mathbb{Z} \to R$, $1 \mapsto 1$, so $\ell$ is the characteristic of $R$, $\ell \in \mathbb{N} \cup \{\infty\}$. Instead of the affine Hecke algebra, we work with the degenerate or graded affine Hecke algebra $\overline{H}_n^{gr}$, defined by Drinfeld [D] and Lusztig [L]. This is isomorphic as an $R$-module to

$$R[S_n] \otimes R[X_1, \ldots, X_n]$$

with algebra structure defined by requiring that $R[S_n]$ and $R[X_i]$ are subalgebras, and that

$$s_i \cdot f - s_i f \cdot s_i = \frac{f - s_i f}{X_i - X_{i+1}}.$$

Given a function $\lambda : \mu_q \to \mathbb{Z}_+$, such that $\sum \lambda_i < \infty$, define the degenerate cyclotomic algebra $\overline{T}_n = \overline{H}_n^{gr}/I_\lambda$, where $I_\lambda = \overline{H}_n^{gr}/\prod_{i \in \mu_q} (X_1 - i)^{\lambda_i} \cdot \overline{H}_n^{gr}$. If $\lambda = \Lambda_0$, $\overline{T}_n^{\Lambda_0} = R[S_n]$, in general one shows that $\dim R \overline{T}_n^{\mu_q} = r^n n!$, where $r = \sum \lambda_i$. We define $\text{Rep}_q \overline{T}_n^{\mu_q}$ to be the subcategory of $\overline{H}_n^{gr}$-modules on which $X_1$ acts with eigenvalues in $\mu_q$. All other definitions and theorems are as before, once we agree to write the group law in $\hat{\mathbb{L}}_p = \mu_q$ multiplicatively, so $q_i$ denotes what is usually written $i + 1$. No changes are necessary in the proofs.

In particular, take $R = \mathbb{F}_p$, to see that $\hat{\mathfrak{sl}}_p$ controls the representation theory of the symmetric group in characteristic $p$.

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