Bounds of Dirichlet eigenvalues for Hardy-Leray operator

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Abstract
The purpose of this paper is to study the eigenvalues \( \{ \lambda_{\mu,i} \} \) for the Dirichlet Hardy-Leray operator, i.e.
\[
-\Delta u + \mu |x|^{-2} u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]
where \( -\Delta + \frac{\mu}{|x|^2} \) is the Hardy-Leray operator with \( \mu \geq -\frac{(N-2)^2}{4} \) and \( \Omega \) is a smooth bounded domain with \( 0 \in \Omega \). We provide lower bounds of \( \{ \lambda_{\mu,i} \} \), together with the Li-Yau’s one for \( \mu > -\frac{(N-2)^2}{4} \) and Karachalio’s one for \( \mu \in \left[ -\frac{(N-2)^2}{4}, 0 \right) \). Secondly, we obtain Cheng-Yang’s type upper bounds for \( \lambda_{\mu,k} \). Finally, we get the Weyl’s limit of eigenvalues which is independent of the potential’s parameter \( \mu \). This interesting phenomena indicates that the inverse-square potential does not play an essential role for the asymptotic behavior of the spectral of the problem considered.

Keywords: Dirichlet eigenvalues; Hardy-Leray operator.

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1 Introduction and main results
Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 2, 0 \in \Omega \) and \( \mu_0 := -\frac{(N-2)^2}{4} \) the best constant for the standard Hardy inequality. The purpose of this paper is to study the bounds of eigenvalues for the Dirichlet problem
\[
\begin{aligned}
\mathcal{L}_\mu u &= \lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2} \) for \( \mu \geq \mu_0 \) is the Hardy-Leray operator.

When \( \mu = 0 \), \( \mathcal{L}_\mu \) reduces to the Laplacian and in 1912, Weyl [11] showed that the \( k \)-th eigenvalue \( \lambda_k \) of the Dirichlet problem for any smooth bounded domain \( \Omega \) with the Laplacian operator has the asymptotic behavior
\[
\lim_{k \to +\infty} k^{-\frac{N}{N-2}} \lambda_k = c_N |\Omega|^{-\frac{N}{N-2}}, \quad \text{where } c_N = (2\pi)^2 |B_1|^{-\frac{N}{N-2}},
\]
where \( c_N \) is named as Weyl’s constant, \( |\Omega| \) is the volume of \( \Omega \) and \( B_1 \) is the unit ball centered at the origin. Later, Pólya [39] (in 1960) proved that
\[
\lambda_k \geq C |\Omega|^{-\frac{N}{N-2}} k^{-\frac{N}{N-2}},
\]
with \( C = c_N, (N = 2) \) and for any “plane-covering domain” in \( \mathbb{R}^2 \), (his proof also works in dimension \( N \geq 3 \)) and he also conjectured that (1.3) holds with \( C = c_N \) for any bounded domain.

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in $\mathbb{R}^N$. Later on, Lieb [34] proved (1.3) with some positive constant $C$ in a general bounded domain and then Li-Yau [33] improved this constant in (1.3) to
\begin{equation}
C_N := \frac{N}{N+2}c_N = \frac{N}{N+2}(2\pi)^2|B_1|^{-\frac{2}{N}}.
\end{equation}

The estimate (1.3) with $C = C_N$ is also called Berezin-Li-Yau’s inequality because the constant $C_N$ is achieved with the help of Legendre transform in an earlier result obtained by Berezin. Here we call $C_N$ the Li-Yau’s constant. The Berezin-Li-Yau inequality then is generalized in [14,17,29,31,36] for degenerate elliptic operators. Upper bounds for the first $k$-eigenvalues obtained in [20] is controlled by $CNk^{\frac{2}{N}}$ together with lower order terms; later on, Cheng-Yang in [13] developed a very interesting upper bound
\begin{equation}
\lambda_k \leq \left(1 + \frac{4}{N}\right)k^{\frac{2}{N}}\lambda_1,
\end{equation}
which is also named Cheng-Yang’s inequality. More results on minimizing the eigenvalues problems could be referred to [20,21,29], subject to the homogeneous Dirichlet boundary condition, to [11,13] with Neumann boundary condition. Our motivations of the Dirichlet eigenvalues for Hardy operators are twofold, the one is the Polya’s conjecture associating the zero of the Riemann Zeta function with the eigenvalue of a Hermitian operator and the latter is the role of the critical potential in the analysis of PDEs.

When $\mu \geq \mu_0$ and $\mu \neq 0$, the Hardy potential has homogeneity $-2$, which is critical from both mathematical and physical viewpoint. The Hardy-Leary problems have great applications to various fields as molecular physics [31], quantum cosmological models such as the Wheeler-de-Witt equation (see e.g. [2]) and combustion models [25]. The Hardy inequalities play a fundamental role in the study of Hardy-Leary problems. When the potential’s singularity $\{0\}$ is in $\Omega$, the Hardy inequality [18, (2)] (also see [4,5]) reads as
\begin{equation}
\int_{\Omega} |
abla u(x)|^2 dx + \mu_0 \int_{\Omega} \frac{u^2(x)}{|x|^2} dx \geq c_0 \int_{\Omega} u^2(x) dx, \quad \forall u \in H^1_0(\Omega)
\end{equation}
for some constant $c_0 > 0$. The operator $L_\mu$ is then positive definite for $\mu \in [\mu_0, +\infty)$. More properties of Hardy-Leary operator could be found in [7,37]. It is worth noting that the inverse square potential plays an essential role in isolated singularity of elliptic Hardy equations. Indeed, for a given ‘source’ $f$ defined in $\Omega$, the solutions of the Dirichlet problem
\begin{equation}
L_\mu u = f \quad \text{in } \Omega \setminus \{0\}, \quad u = 0 \quad \text{on } \partial \Omega
\end{equation}
with isolated singularities at $\{0\}$ are classified fully in [11], thanks to a new formulation of distributional identity associated to some specific weight. Extensive treatments of the associated semilinear problems are developed in [12,13] via an introduction of a notion of very weak solution.

Due to the inverse square potential, the spectral of the Hardy-Leary operator remains widely unexplored. In [19] the authors investigated the essential spectral for general Hardy-Leary potential, for problem (1.1), an increasing sequence of eigenvalues $\{\lambda_{\mu,k}(\Omega)\}_{k\in\mathbb{N}}$, simply denoted by $\{\lambda_{\mu,k}\}_{k\in\mathbb{N}}$ in the sequel, are obtained in [10]; [8] studied the optimal first eigenvalue with respect to the domain, and the lower bounds estimates of eigenvalues for (1.1) in [27] are derived by using the estimates of the related heat kernel and Sobolev inequalities. Precisely, the lower bounds state as following:

(i) when $N \geq 3$ and $\mu_0 < \mu < 0$, there holds that for $k \in \mathbb{N}$,
\begin{equation}
\lambda_{\mu,k} \geq \left(1 - \frac{\mu}{\mu_0}\right)\frac{N(N - 2)}{4e}\omega_{N-1}^{-\frac{2}{N}}|\Omega|^{-\frac{2}{N}}k^{\frac{2}{N}};
\end{equation}
(ii) when $N \geq 3$ and $\mu = \mu_0$, there holds that for $k \in \mathbb{N}$,
\begin{equation}
\lambda_{\mu_0,k} \geq e^{-1}S_N(N - 2)^{-\frac{2(N - 1)}{N}}\|X_1\|_{L^2(\Omega)}^2 \|X_1\|_{L^2(\Omega)}^2 k^{\frac{2}{N}},
\end{equation}
where $X_1(x) := (-\log(|x|))^{-1}$, $\forall x \in \Omega$, $D_0 := 2\max_{x \in \partial \Omega} |x|$, and $S_N := 2^{2/N}\pi^{1 + 1/N}\Gamma(\frac{N + 1}{2})$ is the best constant of Sobolev inequality in $\mathbb{R}^N$ and $\Gamma$ is the well-known Gamma function.
We note that the defect of the lower bound \((1.6)\) is the coefficient 
\[
\frac{N(N-2)}{4e}(1 - \frac{\mu}{\mu_0}) \to 0 \quad \text{as} \quad \mu \to \mu_0^+.
\]

It is natural to ask whether the Hardy-Leray potential plays an essential role in the estimates of the spectral or how it works on the related eigenvalues. Our main aim in this paper is to answer this question and establish the lower and upper bounds, and show the limit the eigenvalues as \(k \to +\infty\). Now we state our lower bounds as follows.

**Theorem 1.1.** Let \(D_0 = \max_{x \in \partial \Omega} |x| \) and \(\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}\) be the increasing sequence of eigenvalues of problem \((1.1)\).

(i) If \(\mu_0 \leq \mu < 0\) and \(N \geq 3\), then we have that for \(k \in \mathbb{N}\),

\[
\sum_{i=1}^{k} \lambda_{\mu,i} \geq \max \left\{ \left(1 - \frac{\mu}{\mu_0}\right) C_N |\Omega|^{-\frac{1}{2}} k^{1+\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1}, \, b_k e^{-1} \sigma_{\mu} k^{1+\frac{2}{N}} \right\},
\]

where \(C_N\) is the Li-Yau’s constant defined in \((1.4)\), \(b_k > \left(\frac{1}{2}\right)^{1+\frac{2}{N}}\), \(\lim_{k \to +\infty} b_k = \frac{N}{N+2}\) and

\[
\sigma_{\mu} = \max \left\{ \frac{1}{2} + \frac{N}{N+2}, \, (1 - \frac{\mu}{\mu_0}) \frac{N(N-2)}{4} \|\mathbf{x}_{1+\frac{1}{2}}\|_{L^2(\Omega)} \right\}.
\]

(ii) If \(\mu > 0\) and \(N \geq 2\), then we have that for \(k \in \mathbb{N}\),

\[
\sum_{i=1}^{k} \lambda_{\mu,i} \geq C_N |\Omega|^{-\frac{1}{2}} k^{1+\frac{2}{N}} + D_0^{-2} \mu k.
\]

We remark that the regularity of \(\Omega\) could be released to ‘bounded’ for the lower bounds and for \(\mu_0 \leq \mu < 0\), the lower bounds in part (i) consist of the Berezin-Li-Yau’s type bounds and Karachalio’s type bounds. In fact, the Li-Yau’s method can not be applied for \(\mu = \mu_0\) thanks to the factor \((1 - \frac{\mu}{\mu_0})\), which vanishes as \(\mu \to \mu_0^+\). This also occurs for the Karachalio’s estimate \((1.6)\). To overcome this decay, we develop the Karachalio’s method and our lower bound appears \(e^{-1} \sigma_{\mu} k^{1+\frac{2}{N}}\), where \(\sigma_{\mu}\) has the nondecreasing monotonicity for \(\mu \in [\mu_0, 0)\); Particularly

\[
\sigma_{\mu} \geq \sigma_{\mu_0} = S_N(N-2) \frac{2(N-1)}{N} \frac{1}{\|\mathbf{x}_{1+\frac{1}{2}}\|_{L^2(\Omega)}^{2(N-1)}} \geq 0.
\]

From the monotonicity of \(\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}\), we have the following corollary:

**Corollary 1.2.** Let \(\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}\) be the increasing sequence of eigenvalues of problem \((1.1)\).

(i) If \(\mu_0 \leq \mu < 0\) and \(N \geq 3\), then for \(k \in \mathbb{N}\),

\[
\lambda_{\mu,k} \geq \max \left\{ \left(1 - \frac{\mu}{\mu_0}\right) C_N |\Omega|^{-\frac{1}{2}} k^{1+\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1}, \, \sigma_{\mu} k^{1+\frac{2}{N}} \right\}.
\]

(ii) If \(\mu > 0\) and \(N \geq 2\), then for \(k \in \mathbb{N}\),

\[
\lambda_{\mu,k} \geq C_N |\Omega|^{-\frac{1}{2}} k^{\frac{2}{N}} + D_0^{-2} \mu k.
\]

In order to obtain upper bounds, we extend Cheng-Yang’s type inequality for \(\mu \geq \mu_0\) and our upper bounds state as follows:

**Theorem 1.3.** Let \(\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}\) be the increasing sequence of eigenvalues of problem \((1.1)\).

(i) If \(\mu \geq 0\) and \(N \geq 2\), then

\[
\lambda_{\mu,k} \leq \left(1 + \frac{4}{N}\right) k^{\frac{2}{N}} \lambda_{\mu,1}.
\]

(ii) If \(\mu \in [\mu_0, 0)\) and \(N \geq 3\), then

\[
\lambda_{\mu,k} \leq \left(1 + \frac{4}{N}\right) k^{\frac{2}{N}} \lambda_{\mu,1} + \mu D_0^{-2}.
\]
It is worth noting that the crucial inequality for obtaining the Cheng-Yang’s type inequality \( (1.8) \) is the following
\[
\sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \leq 2a_\mu \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \lambda_{\mu,i},
\]
which holds with \( a_\mu = \frac{2}{N} \) if \( \mu > 0 \). However, \( (1.10) \) has the constant \( a_\mu = \frac{2}{N} \frac{\mu - \mu_0}{\mu - \mu_0} \) if \( \mu \in (\mu_0, 0) \) for \( N \geq 3 \), then it will give an equality like \( \lambda_{\mu,k} \leq (1 + a_\mu) k^\mu \lambda_{\mu,1} \), which is too rough since \( a_\mu > \frac{2}{N} \) and \( \lim_{\mu \to \mu_0^+} a_\mu = +\infty \). So we adopt an approach of comparing with the Laplacian directly.

Combining the estimate of the first eigenvalue \( \lambda_{\mu,1} \), we have the following corollary:

**Corollary 1.4.** For \( N \geq 3 \) and \( \mu > 0 \), we have that
\[
\lambda_{\mu,k} \leq (1 + \frac{4}{N}) \left( \lambda_{0,1} + \mu ||\phi_{0,1}\||_L^2 \int_\Omega |x|^{-2}dx \right) k^{\frac{2}{N}}.
\]

For \( N \geq 2 \) and \( \mu > 0 \),
\[
\lambda_{\mu,k} \leq \left( 1 + \frac{4}{N} \right) \left( \lambda_{0,1} + c_0^{-2} \tau_+ (\mu) ||\phi_{0,1}\||_C^2 \int_\Omega \int_\Omega \int_\Omega \int \frac{|x|^{2\tau_+ (\mu)-2} dx}{\rho^2 (x) |x|^{2\tau_+ (\mu)} dx} \right) k^{\frac{2}{N}},
\]
where \( c_0 > 0 \), \( \tau_+ (\mu) = -\frac{N-2}{2} + \sqrt{\mu - \mu_0} \), \( (\lambda_{0,1}, \phi_{0,1}) \) is the first eigenvalue and associated eigenfunctions of \( (1.1) \) for \( \mu = 0 \) and \( \rho(x) = \text{dist}(x, \partial \Omega) \).

It is remarkable that the upper bound \( (1.11) \) is much simple, but it can’t be used in the case that \( N = 2 \) and the upper bound \( (1.12) \) is available for \( N \geq 2 \) with the help of the value \( \tau_+ (\mu) > 0 \) for \( \mu > 0 \).

We summarize the asymptotic of the sum of the first \( k \)-eigenvalues from Corollary \( (1.2) \) and Theorem \( (1.3) \) in the following table:

| \( \mu \) | \( (0, +\infty) \) | \( [\mu_0, 0) \) |
|---|---|---|
| LB of \( \lambda_{\mu,k} \) | \( C_N |\Omega|^{-\frac{2}{N}} k^{\frac{2}{N}} + \mu D_0^{-2} \) | \( \max \left\{ \left( 1 - \frac{\mu}{\mu_0} \right) C_N |\Omega|^{-\frac{2}{N}} k^{\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1}, \sigma_\mu k^{\frac{2}{N}} \right\} \) |
| UB of \( \lambda_{\mu,k} \) | \( \left( 1 + \frac{4}{N} \right) k^{\frac{2}{N}} \lambda_{0,1} \) | \( \left( 1 + \frac{4}{N} \right) k^{\frac{2}{N}} \lambda_{0,1} + \mu D_0^{-2} \) |

where LB and UB stand for Lower bound and Upper bound respectively.

Finally, we provide the Weyl’s limit of eigenvalues for Hardy-Leray operators.

**Theorem 1.5.** Assume that \( N \geq 2 \), \( \mu \geq \mu_0 \) and \( \{\lambda_{\mu,i}\}_{i \in \mathbb{N}} \) is the increasing sequence of eigenvalues of problem \( (1.7) \). Then there holds
\[
\lim_{k \to +\infty} \lambda_{\mu,k} k^{-\frac{2}{N}} = c_N |\Omega|^{-\frac{2}{N}},
\]
where \( c_N \) is the Weyl’s constant given in \( (1.2) \).

We notice that the limit of \( \{\lambda_{\mu,k} k^{-\frac{2}{N}} \}_{k \in \mathbb{N}} \) is independent of \( \mu \). Actually, this limit coincides the one for Laplacian, see \( (1.2) \). This answers our question that the inverse square potential term is a second role for the asymptotic eigenvalues.

The rest of this paper is organized as follows. In Section 2, we build the Berezin-Li-Yau’s type lower bounds and the Karachalio’s type lower bounds in Theorem \( (1.1) \). Section 3 is devoted to the Yang’s inequality and proof of Theorem \( (1.3) \). Finally, we prove the Weyl’s limit of eigenvalues in Theorem \( (1.5) \) in Section 4.
2 Lower bounds

2.1 Li-Yau’s type lower bounds

**Proposition 2.1.** Assume that \( N \geq 3, \mu_0 < \mu < 0 \) and \( \{\lambda_{\mu,i}\}_{i \in \mathbb{N}} \) is the increasing sequence of eigenvalues of problem (1.1). Then we have

(i) for \( \mu_0 < \mu < 0 \) and \( k \in \mathbb{N} \) there holds

\[
\sum_{i=1}^{k} \lambda_{\mu,i} \geq \left(1 - \frac{\mu}{\mu_0}\right) C_N |\Omega|^{-\frac{N}{2}} k^{1+\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1k};
\]

(ii) for \( \mu > 0 \) and \( k \in \mathbb{N} \) there holds

\[
\sum_{i=1}^{k} \lambda_{\mu,i} \geq C_N |\Omega|^{-\frac{N}{2}} k^{1+\frac{2}{N}} + D_0^{-2} \mu k,
\]

where we recall \( D_0 = \max_{x \in \partial \Omega} |x| \).

For \( N \geq 2, \mu \geq \mu_0 \), we denote \( H_\mu(\Omega) \) as the completion of \( C_0^\infty(\Omega) \) with the norm

\[
\|u\|_\mu = \sqrt{\int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} dx}
\]

and it is a Hilbert space with the inner product

\[
\langle u, v \rangle_\mu = \int_{\Omega} \nabla u \cdot \nabla v dx + \mu \int_{\Omega} \frac{uv}{|x|^2} dx.
\]

We remark that \( H_\mu(\Omega) = H_0^1(\Omega) \) for \( \mu > \mu_0 \) and if \( \mu_0 \neq 0 \), \( H_{\mu_0}(\Omega) \supseteq H_0^1(\Omega) \). The following lemma is crucial to get the Berezin-Li-Yau’s lower bound which is appeared in [33].

**Lemma 2.2.** [33, Lemma 1] If \( f \) is a real-valued function defined on \( \mathbb{R}^N \) with \( 0 \leq f \leq M_1 \) and

\[
\int_{\mathbb{R}^N} f(z) |z|^2 dz \leq M_2,
\]

then we have

\[
\int_{\mathbb{R}^N} f(z) dz \leq (M_1 |B_1|)^{\frac{2}{N+2}} M_2^{\frac{N}{N+2}} \left( \frac{N+2}{N} \right)^{\frac{N}{N+2}}.
\]

**Proof of Proposition 2.1.** Let \( (\lambda_{\mu,k}, \phi_k) \) be the eigenvalue and eigenfunction pair of (1.1) such that

\[
\|\phi_k\|_{L^2(\Omega)} = 1.
\]

Then

\[
\lambda_{\mu,k} = \min \left\{ \int_{\Omega} \left( |\nabla u|^2 + \frac{\mu}{|x|^2} u^2 \right) dx : u \in H_k(\Omega) \text{ with } \|u\|_{L^2(\Omega)} = 1 \right\},
\]

where

\[
H_1(\Omega) = H_\mu(\Omega) \quad \text{and} \quad H_k(\Omega) = \{ u \in H_\mu(\Omega) : \int_{\Omega} u \phi_j dx = 0 \text{ for } j = 1, \ldots, k-1 \} \quad \text{for } l > 1.
\]

Moreover, \( \{\phi_k \in H_\mu(\Omega) : k \in \mathbb{N}\} \) is an orthonormal basis of \( L^2(\Omega) \).

Denote

\[
\Phi_k(x, y) := \sum_{j=1}^{k} \phi_j(x) \phi_j(y)
\]
and its Fourier transform is then
\[ \hat{\Phi}_k(z, y) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \Phi_k(x, y)e^{iz \cdot x}dx. \]

Note that
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\hat{\Phi}_k(z, y)|^2 dz dy = \int_{\Omega} \int_{\mathbb{R}^N} |\Phi_k(x, y)|^2 dx dy = \sum_{j=1}^{k} \int_{\Omega} |\phi_j^2(x)| dx = k \]
and
\[ \int_{\mathbb{R}^N} |\hat{\Phi}_k(z, y)|^2 dy = (2\pi)^{-N} \int_{\Omega} \int_{\mathbb{R}^N} |\Phi_k(x, y)e^{iz \cdot x}|^2 dx dy = (2\pi)^{-N} \int_{\Omega} \sum_{j=1}^{k} \phi_j(x)e^{iz \cdot x} dx, \]
which implies by Bessel’s inequality (see [35, (1.2)]) that
\[ \int_{\mathbb{R}^N} |\hat{\Phi}_k(z, y)|^2 dy \leq (2\pi)^{-N} \int_{\Omega} |e^{iz \cdot x}|^2 dx = (2\pi)^{-N} |\Omega|. \]

Meanwhile, the Hardy inequality (1.4) implies that for \( \mu_0 < \mu < 0, \)
\[ \int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} dx = \frac{\mu}{\mu_0} \left( \int_{\Omega} |\nabla u|^2 dx + \mu_0 \int_{\Omega} \frac{u^2}{|x|^2} dx \right) \]
\[ + \left( 1 - \frac{\mu}{\mu_0} \right) \int_{\Omega} |\nabla u|^2 dx \]
\[ \geq (1 - \frac{\mu}{\mu_0}) \int_{\Omega} |\nabla u|^2 dx + \frac{\mu}{\mu_0} \lambda_{\mu,1} \int_{\Omega} u^2 dx, \]
thus, we have that
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\hat{\Phi}_k(z, y)|^2 |z|^2 dz dy = \int_{\mathbb{R}^N} \int_{\Omega} |\nabla \Phi_k(x, y)|^2 dx dy = \int_{\Omega} \left| \sum_{j=1}^{k} \nabla \phi_j(x) \right|^2 dx \]
\[ \leq (1 - \frac{\mu}{\mu_0})^{-1} \left( \int_{\Omega} \left| \sum_{j=1}^{k} \phi_j(x) \right|^2 dx + \mu \int_{\Omega} \frac{\sum_{j=1}^{k} \phi_j^2(x)}{|x|^2} dx - \frac{\mu}{\mu_0} \lambda_{\mu,1} \int_{\Omega} \sum_{j=1}^{k} \phi_j^2(x) dx \right) \]
\[ = (1 - \frac{\mu}{\mu_0})^{-1} \left( \sum_{j=1}^{k} \lambda_{\mu,j} - \frac{\mu}{\mu_0} \lambda_{\mu,1} k \right). \]

For \( \mu > 0, \) we have that
\[ \int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} dx \geq \int_{\Omega} |\nabla u|^2 dx + \mu D_0^{-2} \int_{\Omega} u^2 dx, \]
and then
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\hat{\Phi}_k(z, y)|^2 |z|^2 dz dy = \int_{\mathbb{R}^N} \int_{\Omega} |\nabla \Phi_k(x, y)|^2 dx dy \]
\[ \leq \int_{\Omega} \left| \sum_{j=1}^{k} \nabla \phi_j \right|^2 dx + \mu \int_{\Omega} \frac{\sum_{j=1}^{k} \phi_j^2(x)}{|x|^2} dx - D_0^{-2} \mu \int_{\Omega} \left( \sum_{j=1}^{k} \phi_j \right)^2 dx \]
\[ = \sum_{j=1}^{k} \lambda_{\mu,j} - \mu D_0^{-2} k. \]
Now for Case: $N \geq 3$, $\mu_0 < \mu < 0$. We apply Lemma 2.2 to the function $f(z) = \int_{\Omega} |\hat{\Phi}_k(z, y)|^2 dy$ with

$$M_1 = (2\pi)^{-N} |\Omega| \quad \text{and} \quad M_2 = (1 - \frac{\mu}{\mu_0})^{-1} \left( \sum_{j=1}^{k} \lambda_{\mu,j} - \frac{\mu}{\mu_0} \lambda_{\mu,1} k \right),$$

then we conclude that

$$k = \int_{\mathbb{R}^N} f(z) dz \leq (M_1 |B_1|)^{\frac{2}{N+2}} M_2^{\frac{N}{N+2}} \left( \frac{N + 2}{N} \right)^{\frac{N}{N+2}}$$

$$\leq \left( (2\pi)^{-N} |\Omega| |B_1| \right)^{\frac{2}{N+2}} \left( \sum_{j=1}^{k} \lambda_{\mu,j} - \frac{\mu}{\mu_0} \lambda_{\mu,1} k \right)^{\frac{N}{N+2}} \left( \frac{N + 2}{N} \right)^{\frac{N}{N+2}}$$

and

$$\sum_{j=1}^{k} \lambda_{\mu,j} \geq \left( 1 - \frac{\mu}{\mu_0} \right) C_N |\Omega|^{-\frac{2}{N+2}} |k^{1+\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1} k.$$

For Case: $\mu > 0$. We again apply Lemma 2.2 to the same function $f(z)$ and $M_1$, but

$$M_2 = \sum_{j=1}^{k} \lambda_{\mu,j} - \mu D_0^{-2} k,$$

then we conclude that

$$k = \int_{\mathbb{R}^N} f(z) dz \leq (M_1 |B_1|)^{\frac{2}{N+2}} M_2^{\frac{N}{N+2}} \left( \frac{N + 2}{N} \right)^{\frac{N}{N+2}}$$

$$\leq \left( (2\pi)^{-N} |\Omega| |B_1| \right)^{\frac{2}{N+2}} \left( \sum_{j=1}^{k} \lambda_{\mu,j} - \mu D_0^{-2} k \right)^{\frac{N}{N+2}} \left( \frac{N + 2}{N} \right)^{\frac{N}{N+2}}$$

and

$$\sum_{j=1}^{k} \lambda_{\mu,j} \geq C_N |\Omega|^{-\frac{2}{N+2}} |k^{1+\frac{2}{N}} + \mu D_0^{-2} k.$$

We complete the proof.

\[\Box\]

**Corollary 2.3.** Let $\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}$ be the increasing sequence of eigenvalues of problem (1.1).

(i) For $N \geq 3$, $\mu_0 < \mu < 0$ and $k \in \mathbb{N}$, we have that

$$\lambda_{\mu,k} \geq (1 - \frac{\mu}{\mu_0}) \left( C_N |\Omega|^{-\frac{2}{N+2}} |k^{\frac{2}{N}} + \frac{\mu}{\mu_0} \lambda_{\mu,1} \right).$$

(ii) For $N \geq 2$, $\mu > 0$ and $k \in \mathbb{N}$, we have that

$$\lambda_{\mu,k} \geq C_N |\Omega|^{-\frac{2}{N+2}} k^{\frac{2}{N}} + \mu D_0^{-2}.$$

**Proof.** From the increasing monotonicity of $k \mapsto \lambda_{\mu,k}$, we have that

$$\lambda_{\mu,k} \geq \frac{1}{k} \sum_{j=1}^{k} \lambda_{\mu,j},$$

which implies the lower bounds for $\lambda_{\mu,k}$ by Proposition 2.1.

\[\Box\]
2.2 Karachalio’s type lower bound

Recall that $S_N = 2^{2/N} \pi^{1+1/N} \Gamma(N+1)$ is the best constant of Sobolev inequality in $\mathbb{R}^N$ and $\Gamma$ is the Gamma function. Now we set

$$X_1(x) = (- \log \left( \frac{|x|}{D_0} \right))^{-1} \quad \text{and} \quad X_2(x) \equiv 1, \quad \forall x \in \Omega,$$

where we recall $D_0 = \max_{x \in \partial \Omega} |x|$. Note that

$$\|X_1^{N-1/2}\|_{L^2(\Omega)}^2 \leq \int_{B_{D_0}} X_1^{-N}(y)dy \leq \omega_{N-1} \int_0^{D_0} r^{N-1} (- \log \frac{r}{D_0})^{N-1} dr < +\infty$$

and $\|X_2^{N-1/2}\|_{L^2(\Omega)}^2 = |\Omega|$.

**Proposition 2.4.** Assume that $N \geq 3$, $\mu_0 \leq \mu < 0$ and $\{\lambda_{\mu,k}\}_{k \in \mathbb{N}}$ is the increasing sequence of eigenvalues of problem (1.1). Then for $k \in \mathbb{N}$ we have that

$$\lambda_{\mu,k} \geq e^{-1} \sigma_\mu k^{d_N},$$

where $\sigma_\mu$ is defined in [1.7].

**Proof.** When $\mu = \mu_0$, the proof is refereed for [28] and we focus on the case $\mu_0 < \mu < 0$.

Let $L_\mu$ be the Hardy-Leray operator with its domain defined as $D(L_\mu) = C_0^\infty(\Omega)$ and its Friedrich’s extension, still denoted by $L_\mu$, with its domain defined as

$$D(L_\mu) := \{ u \in H_\mu(\Omega) : L_\mu u \in L^2(\Omega) \},$$

which is a nonnegative self-adjoint operator on $L^2(\Omega)$ and the operator gives rise to the semigroup of operators $e^{-L_\mu t}$ for every $t > 0$, possessing an integral kernel $K(x,y,t) > 0$ for all $(x,y,t) \in \Omega \times \Omega \times (0, +\infty)$. Then $L_\mu$ has compact resolvent and $K$ can be represented as

$$K(x,y,t) = \sum_{i=1}^\infty e^{-\lambda_{\mu,i} t} \phi_i(x) \phi_i(y),$$

which solves the problem

$$\begin{cases}
\partial_t K + L_\mu K = 0 & \text{in } \Omega \times \Omega \times (0, +\infty), \\
K(x,y,t) > 0 & \text{in } \Omega \times \Omega \times (0, +\infty), \\
K(x,y,t) = 0 & \text{on } \partial \Omega \times \partial \Omega \times (0, +\infty).
\end{cases} \tag{2.1}$$

Since $\{\phi_i\}_{i \geq 1}$ is an orthonormal basis of $L^2(\Omega)$, it follows that

$$h(t) := \sum_{i=1}^\infty e^{-2\lambda_{\mu,i} t} = \int_{\Omega} \int_{\Omega} K^2(x,y,t) \, dx \, dy.$$

By the improved Hardy-Sobolev inequalities, [23] Theorem A] for $\mu \geq \mu_0$, (the following inequality is sharp for $\mu = \mu_0$)

$$\int_{\Omega} |\nabla u|^2 \, dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq \sigma_1 \left( \int_{\Omega} |u|^2 X_1(x) \frac{2(N-1)}{N-2} \, dx \right)^{\frac{2}{N}}, \quad \forall u \in C_0^\infty(\Omega)$$

and for $\mu > \mu_0$

$$\int_{\Omega} |\nabla u|^2 \, dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq \sigma_2 \left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{2}{N}}, \quad \forall u \in C_0^\infty(\Omega),$$
where $2^* = \frac{2N}{N-2}$,

$$\sigma_1 = SN(N-2)^{-\frac{2(N-1)}{N}} \quad \text{and} \quad \sigma_2 = \frac{N(N-2)}{4}(1 - \frac{\mu}{\mu_0})\omega_{N-1}.$$

By Hölder inequality, we get that for $j = 1, 2,$

$$h(t) \leq \int_{\Omega} \left(\int_{\Omega} X_j(y)^{\frac{2(N-1)}{N-2}} K^2(x, y, t) dy\right)^{\frac{N}{N-2}} \left(\int_{\Omega} X_j(y)^{-\frac{2(N-1)}{N-2}} K(x, y, t) dy\right)^{-\frac{2}{N-2}} dx$$

$$\leq \left[ \int_{\Omega} \left(\int_{\Omega} X_j(y)^{\frac{2(N-1)}{N-2}} K^2(x, y, t) dy\right)^{\frac{2}{N-2}} dx \right]^{\frac{N}{2(N-2)}} \left(\int_{\Omega} Q_j^2(x, t) dx\right)^{\frac{2}{2(N-2)}} \tag{2.2},$$

where $\frac{2(N-1)}{(N-2)(2^*-2)} = \frac{N-1}{2}$ and

$$Q_j(x, t) = \int_{\Omega} X_j(y)^{-\frac{N+1}{2}} K(x, y, t) dy.$$

We observe that $Q_j(x, t)$ is the solution of the Cauchy-Dirichlet problem:

$$
\begin{align*}
&\partial_t Q_j + \mathcal{L}_{\mu} Q_j = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
&Q_j(x, 0) = X_j^{\frac{1-N}{2}} \quad \text{in} \quad \Omega, \\
&Q_j(x, t) = 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty). 
\end{align*}
\tag{2.3}
$$

Multiplying the above equation (2.3) by $Q_j$, we get the energy equation

$$\frac{1}{2} \frac{d}{dt} \|Q_j(t)\|^2_{L^2(\Omega)} + \|Q_j(t)\|^2_{\mu} = 0$$

and then

$$\|Q_j(t)\|^2_{L^2(\Omega)} \leq \|Q_j(0)\|^2_{L^2(\Omega)} = \|X_j^{\frac{1-N}{2}}\|^2_{L^2(\Omega)}.$$

With the help of above inequalities, letting

$$C_j := \|X_j^{\frac{1-N}{2}}\|_{L^2(\Omega)}^{2(2^*-1)} \quad \text{for} \quad j = 1, 2,$$

for $\mu > \mu_0$, from (2.2) we obtain that

$$h^{\frac{2(2^*-1)}{2^*}}(t) \leq C_j \int_{\Omega} \left(\int_{\Omega} X_j(y)^{\frac{2(N-1)}{N-2}} K^2(x, y, t) dy\right)^{\frac{N}{N-2}} dx$$

$$\leq \frac{C_j}{\sigma_j} \int_{\Omega} \int_{\Omega} \left(\|\nabla_y K(x, y, t)\|^2 dx + \mu \frac{K^2(x, y, t)}{|y|^2} \right) dx dy$$

$$\leq \frac{C_j}{\sigma_j} \|Q_j(t)\|^2_{\mu} = -\frac{C_j}{\sigma_j} \frac{1}{2} \frac{d}{dt} \|Q_j(t)\|^2_{L^2(\Omega)}$$

$$\leq -\frac{C_j}{\sigma_j} \frac{1}{2} \frac{dh(t)}{dt},$$

which, using $h(0) = +\infty$, implies that

$$\sum_{i=1}^{\infty} e^{-2\lambda_{\mu,i}t} = h(t) \leq \left(\frac{2^*}{2\sigma_j(2^*-2)}\right)^{\frac{2^*}{2^*-2}} \|X_j^{\frac{1-N}{2}}\|^2_{L^2(\Omega)} t^{-\frac{2}{2^*-2}} \quad \text{for} \quad j = 1, 2.$$

Now we choose

$$t = \frac{2^*}{2(2^*-2)} \frac{1}{\lambda_{\mu,1}}$$
and 
\[ ke^{-\frac{\sigma^2}{2}} \leq \sum_{i=1}^{\infty} e^{-\frac{2^*\lambda_{\mu,i}}{\mu,k}} \leq \sigma_j^{-\frac{2^*}{\lambda_{\mu,k}}} \|X_j\|_{L^2(\Omega)}^2 \] for \( j = 1, 2 \).

Note that \( \frac{2^*}{\lambda_{\mu,k}} = \frac{N}{2} \) and we conclude that
\[ \lambda_{\mu,k} \geq e^{-1} \sigma_j^\frac{N}{2} \] for \( j = 1, 2 \).

As a consequence, we obtain that
\[ \lambda_{\mu,k} \geq e^{-1} \sigma_j^\frac{N}{2} \lambda_{\mu,k}, \]
where
\[ \sigma_j = \max \left\{ \sigma_1^\frac{1-N}{2} \|X_1\|_{L^2(\Omega)}^\frac{N}{2}, \sigma_2^\frac{\Omega}{2} \right\}. \]

We complete the proof. \( \square \)

**Proof of Theorem 1.1.** When \( N \geq 3 \) and \( \mu_0 \leq \mu < 0 \), the lower bounds
\[ \sum_{i=1}^{k} \lambda_{\mu,i} \geq \left( 1 - \frac{\mu}{\mu_0} \right) \left( C_N^\frac{\Omega}{2} + \frac{\mu}{\mu_0} \lambda_{\mu,1,k} \right) \]
follows from Proposition 2.1 part (i). Note that for \( \mu = \mu_0 \), the above bounds is true obviously since \( 1 - \frac{\mu}{\mu_0} = 0 \). Moreover, from Proposition 2.1, we have that
\[ \lambda_{\mu,k} \geq e^{-1} \sigma_k^\frac{N}{2}, \]
which implies that
\[ \sum_{i=1}^{k} \lambda_{\mu,i}(\Omega) \geq e^{-1} \sigma_k^\frac{N}{2} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^\frac{N}{2} \geq b_k e^{-1} \sigma_k^\frac{N}{2} + 1, \]
where
\[ b_k = \frac{1}{k} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^\frac{N}{2} \to \frac{N}{N+2} \] as \( k \to +\infty \)
and for any \( k \geq 2 \)
\[ b_k = \frac{1}{k} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^\frac{N}{2} \geq \frac{1}{k} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^\frac{N}{2} \geq \left( \frac{1}{2} \right)^\frac{N}{2} + 1. \]

When \( \mu > 0 \), it comes from Proposition 2.1 part (ii) directly and we complete the proof. \( \square \)

**Proof of Corollary 1.2.** It follows form Corollary 2.3 and Proposition 2.4 directly. \( \square \)

### 3 Upper bounds

Now we establish the upper bounds, more precisely we will use the following Cheng-Yang’s type inequality and extend it for \( \mu \geq \mu_0 \).

**Proposition 3.1.** Let \( 0 < \nu_1 \leq \nu_2 \leq \cdots \) be a sequence of numbers satisfying the following inequality
\[ \sum_{i=1}^{k} (\nu_{k+1} - \nu_i)^2 \leq 2 \varrho_0 \sum_{i=1}^{k} (\nu_{k+1} - \nu_i) \nu_i, \] (3.1)
where \( \varrho_0 \) is a positive constant. Then we have
\[ \nu_{k+1} \leq (1 + 2 \varrho_0) k^\varrho_0 \nu_1. \]
From Proposition (3.1), the key point is to obtain inequality (3.1) with explicit constant $\rho_0$. To this end, we have the following inequalities.

**Proposition 3.2.** Assume that $\mu > \mu_0$ and let $\{\lambda_{\mu,i}\}_{i \in \mathbb{N}}$ be the increasing sequence of eigenvalues of problem (1.1).

Then (i) for $N \geq 2$ and $\mu > 0$, one has

$$\sum_{i=1}^{k} (\lambda_{\mu,i} - \lambda_{\mu,i+1})^2 \leq \frac{4}{N} \sum_{i=1}^{k} (\lambda_{\mu,i} - \lambda_{\mu,i+1}) \lambda_{\mu,i};$$

(ii) for $N \geq 3$ and $\mu_0 < \mu < 0$, one has

$$\sum_{i=1}^{k} (\lambda_{\mu,i} - \lambda_{\mu,i+1})^2 \leq \frac{4}{N} \frac{-\mu_0}{\mu - \mu_0} \sum_{i=1}^{k} (\lambda_{\mu,i} - \lambda_{\mu,i+1}) \lambda_{\mu,i}.\quad (3.2)$$

**Proof.** The proof for $\mu = 0$ can be found in [14, Theorem 2.1]. Here the difference is to deal with the Hardy term and we give the proof for reader’s convenience.

Let $(\lambda_{\mu,i}, \phi_i)$ be the $i$-th eigenvalue and eigenfunctions of (1.1) with $\int_{\Omega} \phi_i \phi_j dx = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. Let $x = (x^1, \ldots, x^N)$ and $g = x^m$ with $m = 1, \ldots, N$ define a trial function $\varphi_i$ by

$$\varphi_i := g \phi_i - \sum_{j=1}^{k} a_{ij} \phi_j, \quad a_{ij} := \int_{\Omega} g \phi_i \phi_j dx = a_{ji}.$$ 

By the orthonormality of $\phi_i$ and the definition of $a_{ij}$, $\varphi_i$ is perpendicular to $\phi_j$, that is, for $i, j = 1, \ldots, k$, there holds $\int_{\Omega} \varphi_i \phi_j dx = 0$. Let

$$b_{ij} = \int_{\Omega} \phi_j \nabla g \cdot \nabla \phi_i dx$$

and then

$$\lambda_{\mu,j} a_{ij} = \int_{\Omega} g (L_{\mu} \phi_i) \phi_j dx = \int_{\Omega} \left( -2 \phi_j \nabla g \cdot \nabla \phi_i + g (L_{\mu} \phi_i) \phi_j dx \right) = -2b_{ij} + \lambda_{\mu,i} a_{ij},$$

i.e.

$$2b_{ij} = (\lambda_{\mu,i} - \lambda_{\mu,j}) a_{ij}.\quad (3.4)$$

Note that

$$L_{\mu} \varphi_i = \lambda_{\mu,i} g \phi_i - 2 \nabla g \cdot \nabla \phi_i - \sum_{j=1}^{k} a_{ij} \lambda_{\mu,j} \phi_j.$$ 

Hence, we infer that

$$\|\varphi_i\|_{\mu}^2 = \lambda_{\mu,i} \int_{\Omega} \varphi_i^2 dx - 2 \int_{\Omega} \varphi_i \nabla g \cdot \nabla \phi_i dx$$

$$= \lambda_{\mu,i} \int_{\Omega} \varphi_i^2 dx - 2 \int_{\Omega} (g \nabla g) \cdot (\phi_i \nabla \phi_i) dx + 2 \sum_{j=1}^{k} a_{ij} \int_{\Omega} \phi_j \nabla g \cdot \nabla \phi_i dx$$

$$= \lambda_{\mu,i} \int_{\Omega} \varphi_i^2 dx + 1 + \sum_{j=1}^{k} (\lambda_{\mu,i} - \lambda_{\mu,j}) a_{ij}^2,$$
where we note particularly

\[-2 \int_\Omega \varphi_i \nabla g \cdot \nabla \phi_i dx = 1 + \sum_{j=1}^k (\mu_{\mu,i} - \mu_{\mu,j}) a_{ij}^2. \tag{3.5}\]

From the Rayleigh-Ritz inequality, we have that

\[\lambda_{\mu,k+1} \int_\Omega \varphi_i^2 dx \leq \|\varphi_i\|_{\mu}^2 = \lambda_{\mu,i} \int_\Omega \varphi_i^2 dx + 1 + \sum_{j=1}^k (\mu_{\mu,i} - \mu_{\mu,j}) a_{ij}^2,\]

that is

\[(\lambda_{\mu,k+1} - \lambda_{\mu,i}) \int_\Omega \varphi_i^2 dx \leq 1 + \sum_{j=1}^k (\lambda_{\mu,i} - \lambda_{\mu,j}) a_{ij}^2. \tag{3.6}\]

Multiplying (3.5) by \((\lambda_{\mu,k+1} - \lambda_{\mu,i})^2\) and then taking sum on \(i\) from 1 through \(k\), we obtain that

\[-2 \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \int_\Omega \varphi_i \nabla g \cdot \nabla \phi_i dx = \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 + \sum_{i,j=1}^k (\lambda_{\mu,i} - \lambda_{\mu,j})(\lambda_{\mu,k+1} - \lambda_{\mu,i}) a_{ij}^2\]

\[= \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 - 4 \sum_{i,j=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i}) b_{ij}^2 := \theta\]

by the symmetry of \(a_{ij}\) and the anti-symmetry of \(b_{ij}\). Multiplying (3.6) by \((\lambda_{\mu,k+1} - \lambda_{\mu,i})^2\) and then taking sum on \(i\) from 1 through \(k\), we obtain that

\[\sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^3 \int_\Omega \varphi_i^2 dx \leq \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 - 4 \sum_{i,j=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i}) b_{ij}^2 = \theta.\]

Note that for arbitrary constant \(d_{ij}\) (to be determined later)

\[\theta^2 = \left( -2 \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \int_\Omega \varphi_i \nabla g \cdot \nabla \phi_i dx \right)^2\]

\[= 4 \left( \sum_{i=1}^k \int_\Omega (\mu_{\mu,k+1} - \lambda_{\mu,i})^2 \varphi_i \nabla g \cdot \nabla \phi_i dx - (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \int_\Omega \sum_{j=1}^k d_{ij} \varphi_j \phi_j dx \right)^2\]

\[\leq 4 \left( \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i})^3 \int_\Omega \varphi_i^2 dx \right) \left( \sum_{i=1}^k \int_\Omega \left( (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \nabla g \cdot \nabla \phi_i - \sum_{j=1}^k d_{ij} \phi_j \right)^2 dx \right)\]

\[\leq 4 \theta \int_\Omega \left( \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \left| \partial_m \phi_i \right|^2 \right) - 2 \sum_{i,j=1}^k d_{ij} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \phi_j \nabla g \cdot \nabla \phi_i + \left( \sum_{i,j=1}^k d_{ij} \phi_j \right)^2 dx,\]

then we have that

\[\theta \leq 4 \sum_{i=1}^k (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \int_\Omega \left| \partial_m \phi_i \right|^2 dx + 4 \left( -2 \sum_{i,j=1}^k d_{ij} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \phi_j + \sum_{i,j=1}^k d_{ij} \right),\]

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Taking \(d_{ij} = (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 b_{ij}\), we have that
\[
\theta = \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 - 4 \sum_{i,j=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) b_{ij}^2
\]
\[
\leq 4 \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \int_{\Omega} |\partial_m \phi_i|^2 dx - 4 \sum_{i,j=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) b_{ij}^2.
\]
Thus,
\[
\sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \leq 4 \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \int_{\Omega} |\partial_m \phi_i|^2 dx.
\]
Finally, we take sum on \(m\) from 1 through \(N\) and obtain that
\[
N \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \leq 4 \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \int_{\Omega} |\nabla \phi_i|^2 dx.
\]
Therefore we conclude that for \(\mu > 0\),
\[
N \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 < 4 \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \left( \int_{\Omega} |\nabla \phi_i|^2 dx + \mu \int_{\Omega} |\phi_i|^2 dx \right)
\]
\[
= 4 \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \lambda_{\mu,i}.
\]
While for \(\mu_0 < \mu < 0\), since we have that
\[
\int_{\Omega} |\nabla \phi_i|^2 dx \leq \frac{-\mu_0}{\mu - \mu_0} \left( \int_{\Omega} (|\nabla \phi_i|^2 + \mu |x|^2) dx \right) = \frac{-\mu_0}{\mu - \mu_0} \lambda_{\mu,i}
\]
and then
\[
N \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i})^2 \leq 4 \frac{-\mu_0}{\mu - \mu_0} \sum_{i=1}^{k} (\lambda_{\mu,k+1} - \lambda_{\mu,i}) \lambda_{\mu,i}.
\]
We complete the proof. \(\square\)

**Proof of Theorem 1.3**

Part (i): Case of \(\mu > 0\). Theorem 1.3 follows Proposition 3.1 and (3.2) with \(\vartheta_0 = \frac{2}{N}\). To be convenient, we sketch the proof here. Denote
\[
\Lambda_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_{\mu,i}, \quad T_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_{\mu,i}^2, \quad F_k = (1 + \frac{2}{N}) \Lambda_k^2 - T_k,
\]
then we have
\[
F_{k+1} \leq C(N,k) \left( \frac{k+1}{k} \right)^{\frac{4}{k}} F_k,
\]
where
\[
0 < C(N,k) := 1 - \frac{1}{6} \left( \frac{k}{k+1} \right)^{\frac{4}{k}} \frac{\left( 1 + \frac{2}{N} \right) \left( 1 + \frac{4}{N} \right)}{(k+1)^3} < 1,
\]
and then
\[
F_k \leq C(N,k-1) \left( \frac{k}{k-1} \right)^{\frac{4}{k}} F_{k-1} < \left( \frac{k}{k-1} \right)^{\frac{4}{k}} F_{k-1} \leq \cdots
\]
\[
< \left( \frac{k-1}{k-2} \right)^{\frac{4}{k}} \left( \frac{k-2}{k-3} \right)^{\frac{4}{k}} \cdots \left( \frac{2}{1} \right)^{\frac{4}{k}} F_1 = \frac{2}{N} \frac{4}{k} \lambda_{\mu,1}^2.
\]
On the other hand, (3.7) implies that

\[
\left( \lambda_{\mu,k+1} - \frac{2 + N}{N} \Lambda_k \right)^2 \leq \frac{4 + N}{N} F_k - \frac{4 + N}{N} \left( 2 + \frac{N}{N} \right)^2 \Lambda_k
\]

and then

\[
\frac{2}{4 + N} \lambda_{\mu,k+1}^2 + \frac{4 + N}{2 + N} \left( \lambda_{\mu,k+1} - \frac{4 + N}{N} \Lambda_k \right)^2 \leq \frac{4 + N}{N} F_k,
\]

that means

\[
\lambda_{\mu,k+1}^2 \leq \frac{N}{N + 2} \left( \frac{N + 4}{N} \right)^2 F_k \leq (1 + \frac{4}{N})^2 k^\frac{2}{N} \lambda_{\mu,1}.
\]

Thus, we have that

\[
\lambda_{\mu,k} \leq (1 + \frac{4}{N}) k^\frac{2}{N} \lambda_{\mu,1}.
\]  

(3.8)

**Part (ii): Case of \( \mu_0 \leq \mu < 0 \).** In this case, the constant \( \frac{4}{N} \frac{\mu_0}{\mu - \mu_0} > \frac{4}{N} \), which will produce a higher order for parameter \( k \) from Proposition 3.1. So we shall derive the upper bounds by comparing with the eigenvalues of Laplacian Dirichlet problem.

Let

\[
\tilde{L}_\mu = L_\mu - \mu D_0^{-2}
\]

and \( \lambda_{\tilde{L}_\mu,k} \) be the \( k \)-th Dirichlet eigenvalue of \( \tilde{L}_\mu \). Clearly we have

\[
\lambda_{\tilde{L}_\mu,k} = \lambda_{\mu,k} - \mu D_0^{-2}.
\]

Then for \( \mu \in [\mu_0, 0) \),

\[
\langle \tilde{L}_\mu u, u \rangle < \langle -\Delta u, u \rangle, \quad \forall u \in C_0^\infty(\Omega)
\]

It follows by [3, Theorem 10.2.2] that

\[
\lambda_{\mu,k} - \mu D_0^{-2} \leq \lambda_{0,k}, \quad k = 1, 2, \ldots
\]

together with the upper bound in [14] (2.10)

\[
\lambda_{0,k} \leq (1 + \frac{4}{N}) \lambda_{0,1} k^\frac{2}{N},
\]

we imply that

\[
\lambda_{\mu,k} \leq (1 + \frac{4}{N}) \lambda_{0,1} k^\frac{2}{N} + \mu D_0^{-2}, \quad k = 1, 2, \ldots.
\]

The proof is complete. \( \square \)

**Proof of Corollary 1.4** Recall that \( (\lambda_{0,1}, \phi_{0,1}) \) is the first Dirichlet eigenvalue and associated eigenfunctions of Laplacian, then for \( \mu > 0 \) and \( N \geq 3 \)

\[
\lambda_{\mu,1} \leq \int_{\Omega} |\nabla \phi_{0,1}|^2 dx + \mu \int_{\Omega} \frac{\phi_{0,1}^2}{|x|^2} dx \leq \lambda_{0,1} + \mu \| \phi_{0,1} \|^2_{L^\infty} \int_{\Omega} |x|^{-2} dx,
\]

which, together with (3.8), implies that

\[
\lambda_{\mu,k} \leq (1 + \frac{4}{N}) \lambda_{0,1} k^\frac{2}{N} + \mu D_0^{-2}, \quad k = 1, 2, \ldots.
\]

For \( \mu > 0 \) and \( N \geq 2 \), take test function \( u = \phi_{0,1} \Gamma_\mu \) where \( \Gamma_\mu(x) = |x|^{\tau_+}(\mu) \) with \( \tau_+(\mu) = -\frac{N - 2}{2} + \sqrt{\mu - \mu_0} \), which satisfying \( L_\mu \Gamma_\mu = 0 \) in \( \mathbb{R}^N \setminus \{0\} \). Then

\[
\lambda_{\mu,1} \leq \frac{\int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} \frac{u^2}{|x|^2} dx}{\| u \|^2_{L^2(\Omega)}}
\]
\[ \lambda_{0,1} + \frac{\int_\Omega \nabla \phi_{0,1}^2 \cdot \nabla \Gamma_\mu dx}{2 \int_\Omega \phi_{0,1}^2 \Gamma_\mu dx} \leq \lambda_{0,1} + c_1^{-2} \tau_+ (\mu) \| \phi_{0,1} \|_{C^1}^2 \frac{\int_\Omega \rho^2(x) |x|^{2\tau_+ (\mu)} dx}{\int_\Omega \rho^2(x) |x|^{2\tau_+ (\mu)} dx}, \]

where \( c_1 > 0 \) such that
\[ \phi_{0,1}(x) \geq c_1 \rho(x), \quad \forall x \in \Omega. \]

Together with (3.8), we obtain that
\[ \lambda_{\mu,k} \leq (1 + \frac{4}{N}) \left( \lambda_{0,1} + c_1^{-2} \tau_+ (\mu) \| \phi_{0,1} \|_{C^1}^2 \frac{\int_\Omega \rho^2(x) |x|^{2\tau_+ (\mu)} dx}{\int_\Omega \rho^2(x) |x|^{2\tau_+ (\mu)} dx} \right) k \frac{2}{N}. \]

The proof is complete. \( \square \)

4 Limit of Eigenvalues

Our proof of Weyl’s limit of eigenvalues relies on estimates of the partition function
\[ Z_\mu(t) := \sum_{i=1}^\infty e^{-\lambda_{\mu,i}t}, \quad t > 0, \]
which can be written as
\[ Z_\mu(t) = \int_0^\infty e^{-\beta t} dN_\mu(\beta), \quad (4.1) \]
where \( N_\mu(\beta) = \sum_{\lambda_{\mu,i} \leq \beta} 1 \) is the usual counting function. Additionally, it has another standard formula for the partition function
\[ Z_\mu(t) = \int p_{\mu,\Omega}(x,x,t) dx, \quad (4.2) \]
where \( p_{\mu,\Omega} \) is the heat kernel of Hardy operators \( L_\mu \) in domain \( \Omega \times \Omega \times (0, +\infty) \). More properties of heat kernel with Hardy potentials could be found in [22, 37]. In the whole space \( \Omega = \mathbb{R}^N \), we denote \( p_\mu = p_{\mu,\mathbb{R}^N} \). Particularly,
\[ p_0(x,y,t) = p_0(x-y,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}} \quad \text{for} \quad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty) \]
is the usual heat kernel of the Laplacian in \( \mathbb{R}^N \).

The following lemma concerns the estimate of heat kernel, which is the essential part for the Weyl’s limit.

Lemma 4.1. Let \( \mu \geq \mu_0 \) and
\[ \gamma_\mu(x,y,t) = p_0(x-y,t) - p_{\mu,\Omega}(x,y,t) \quad \text{for} \quad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty). \]

Then
(i) For \( \mu > 0 \), there exists \( c_2 > 0 \) independent of \( \mu \) such that
\[ 0 \leq \gamma_\mu(x,x,t) \leq c_2 \mu |x|^{-2} t^{-\frac{N+1}{2}}, \quad \forall x \in \Omega \setminus \{0\}, \forall t \in (0,1]. \]
(ii) For $\mu \in (\mu_0, 0)$, there exists $c_3 > 0$ such that
\[
|\gamma_{\mu}(x, x, t)| \leq c_3|x|^{2r(\mu)-2}t^{-\frac{N}{2}+1}, \quad \forall x \in \Omega \setminus \{0\}, \forall t \in (0, 1].
\]

(iii) For $\mu = \mu_0$, there exists $c_4 > 0$ such that
\[
|\gamma_{\mu}(x, x, t)| \leq c_4 \left( |x|^{-N+(\frac{N}{2}+2)}\left( t^{-\frac{N}{2}+1} + |x|^{-N+\frac{1}{2}}t^{-\frac{N}{2}} \right) \right), \quad \forall x \in \Omega \setminus \{0\}, \forall t \in (0, 1].
\]

**Proof.** Case of $\mu > 0$. Note that elementary properties infer directly that
\[
p_{\mu, \Omega}(x, y, t) \leq p_{0, \Omega}(x, y, t) \leq p_0(x - y, t) \quad \text{in} \quad \Omega \times \Omega \times (0, +\infty)
\]
and for fixed $y \in \Omega$, we have that
\[
\lambda_{\mu}(x, y, t) = \int_0^t \int_\Omega p_{\mu, \Omega}(x, y, t, t-s) \frac{\mu}{|z|^2} d\Omega d\tau,
\]
\[
\gamma_{\mu}(x, y, t) = \int_0^t \int_\Omega p_{\mu, \Omega}(x, y, t, t-s) \frac{\mu}{|z|^2} d\Omega d\tau.
\]

Note that $\gamma_{\mu}$ could be expressed by heat kernel, i.e. for $y \in \Omega \setminus \{0\}$,
\[
\gamma_{\mu}(x, y, t) = \int_0^t \int_\Omega p_{\mu, \Omega}(x, y, t, t-s) \frac{\mu}{|z|^2} d\Omega d\tau.
\]

For $N \geq 3$ and $x \in \Omega \setminus \{0\}$, we have that
\[
\gamma_{\mu}(x, x, t) \leq \frac{\mu}{(4\pi)^N} \left( \int_0^t \int_{\Omega \setminus B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{4(t-s)}} e^{-\frac{|x-y|^2}{4s}} dz ds \right)
\]
\[
+ \int_0^t \int_{B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{4(t-s)}} e^{-\frac{|x-y|^2}{4s}} dz ds
\]
\[
\leq \frac{\mu}{(4\pi)^N} \left( \int_0^t \int_{\Omega \setminus B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{4(t-s)}} e^{-\frac{|x-y|^2}{4s}} dz ds \right)
\]
\[
+ \int_0^t \int_{B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{16(t-s)}} \int_{\Omega \setminus B|z|} \frac{1}{|z|^2} e^{-\frac{|x-y|^2}{4s}} dz ds
\]
\[
\leq \frac{\mu}{(4\pi)^N} \left( \int_0^t \int_{\Omega \setminus B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{4(t-s)}} dz ds \right)
\]
\[
+ c_5 \omega_{\frac{N}{2}-1} \left( \frac{|x|}{2} \right)^{N-2} \int_0^t \int_{\Omega \setminus B|z|} \frac{1}{|z|^2} e^{-\frac{|x-z|^2}{16(t-s)}} dz ds
\]
\[
= \frac{\mu}{(4\pi)^N} \left( \frac{2}{N}+2 \right) \left( \frac{|x|}{2} \right)^{N-2} \int_0^t \int_{\Omega \setminus B|z|} e^{-\frac{|x-z|^2}{16(t-s)}} dz ds + c_5 \omega_{\frac{N}{2}-1} |x|^{N-2} + c_5 \omega_{\frac{N}{2}-1} \left( \frac{|x|}{2} \right)^{N-2},
\]
where $c_5 > 0$ such that
\[
e^{-a} \leq c_5 a^{-\frac{N}{2}} \quad \text{for all} \quad a \geq 1.
\]

Case of $\mu \in (\mu_0, 0)$. It is known that for $y \in \Omega \setminus \{0\}$,
\[
\gamma_{\mu}(x, y, t) = \int_0^t \int_\Omega p_{\mu, \Omega}(x, y, t-s) \frac{\mu}{|z|^2} d\Omega d\tau.
\]
It follows from [37, Theorem 3.10] that the corresponding heat kernel verifies that for $t \in (0, 1]$
\[
0 \leq p_{\mu, \Omega}(x, y, t) \leq p_{\mu, RN}(x, y, t) \leq c_6(|x||y|)^{\tau_+(\mu)} t^{-\frac{N}{2}} e^{-c_7|x-y|^2},
\]
where $c_6, c_7 > 0$ and $\tau_+(\mu) < 0$. We see that there is no order for $p_{\mu, \Omega}$ and $p_0$. Thus, for $x \in \Omega \setminus \{0\}$ and $t \in (0, 1]$
\[
|\gamma_{\mu}(x, x, t)| \leq |\mu| \int_0^t \int_{\Omega} p_{0, \Omega}(x, z, t-s) \frac{1}{|z|^2} p_{\mu, \Omega}(z, x, s) dz ds
\]
\[
\leq \frac{c_6|\mu|}{(4\pi)^{\frac{N}{2}}} |x|^{\tau_+(\mu)} \left( \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\mathbb{R}^N} |z|^{\tau_+(\mu)-2} e^{-\frac{|x-z|^2}{4(t-s)} - \frac{c_7|x-z|^2}{s}} dz ds \right)
\]
\[
+ \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\partial B_\mu(x)} |z|^{\tau_+(\mu)-2} e^{-\frac{|x-z|^2}{4(t-s)} - \frac{c_7|x-z|^2}{s}} dz ds
\]
\[
< \frac{c_6|\mu|}{(4\pi)^{\frac{N}{2}}} |x|^{\tau_+(\mu)} \left( \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-|z|^2} dz \int_0^t (c_7 t + (4 - c_7)s) \frac{N}{2} ds \right)
\]
\[
+ \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\partial B_\mu(x)} |z|^{\tau_+(\mu)-2} e^{-|z|^2} \frac{N}{2} ds
\]
\[
\leq \frac{c_6|\mu|}{(4\pi)^{\frac{N}{2}}} |x|^{\tau_+(\mu)} \left( \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-|z|^2} dz + c_9 \frac{\omega_{N-1}}{N-2} 2^{N-\tau_+(\mu) + 2} e^{-\frac{N}{2}} \right) |x|^{2\tau_+(\mu)-2} t^{-\frac{N}{2}+1},
\]
where $2\tau_+(\mu) - 2 > -N$ for $\mu \in (\mu_0, 0)$ and $c_8, c_9 > 0$.

Case of $\mu = \mu_0$. When $\mu = \mu_0$, the above inequality holds true, but the factor $|x|^{2\tau_+(\mu_0) - 2} = |x|^{-N}$ is non-integrable in $\Omega$. So we have to modify the above estimates.

From [22, Theorem 1.1] we have that $t \in (0, 1]$
\[
0 \leq p_{\mu, \Omega}(x, y, t) \leq c_{10}(|x||y|)^{\frac{2-N}{4}} t^{-\frac{N}{2}} e^{-c_{11}|x-y|^2}.
\]
We choose
\[
r = \frac{1}{4} |x|^{\frac{1}{N(N-2)}} \quad \text{and} \quad \theta = \frac{1}{4},
\]
then we have that
\[
|\gamma_{\mu}(x, x, t)| \leq \frac{c_{10}|\mu_0|}{(4\pi)^{\frac{N}{2}}} |x|^{\frac{2-N}{2}} \left( \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\mathbb{R}^N \setminus B_r(x)} |z|^{\frac{N+2}{2}} e^{-\frac{|x-z|^2}{4(t-s)} - \frac{c_{11}|x-z|^2}{s}} dz ds \right)
\]
\[
+ \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{B_r(x)} |z|^{\frac{N+2}{2}} e^{-\frac{|x-z|^2}{4(t-s)} - \frac{c_{11}|x-z|^2}{s}} dz ds
\]
\[
< \frac{c_{10}|\mu_0|}{(4\pi)^{\frac{N}{2}}} |x|^{\frac{2-N}{2}} \left( \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)} - \frac{c_{11}|x-y|^2}{s}} dz ds \right)
\]
\[
+ \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{B_r(0)} |z|^{\frac{N+2}{2}} e^{-\frac{|x|^2}{4(t-s)} - \frac{c_{11}|x|^2}{s}} dz ds
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then we obtain that
\[
\frac{c_{10}|p_0|}{(4\pi)^{\frac{N}{2}}} |x|^{\frac{2N}{2}} \left( r - \frac{N+2}{2} \right) \int_{\mathbb{R}^N} e^{-|z|^2} dz \int_0^t (c_{11} t + (4 - c_{11}) s)^{-\frac{N}{2}} ds \]
\[+ \frac{\omega_{N-1}}{N - 2} \frac{r + 2}{4} \int_{0}^{t} \frac{1}{t-s} \frac{c_{14}}{s} \left( \frac{|x|^2}{s} \right)^{-\frac{N+2}{2}} ds \]
\[\leq c_{12} |x|^{\frac{2N}{2}} r^{-\frac{N+2}{2}} t^{-\frac{N}{2} + 1} + c_{13} |x|^{1 + \theta - \frac{3}{2} N} r \frac{N+2}{2} t^{-\frac{N}{2} + \frac{7}{8}} \int_{0}^{t} (t-s)^{-\frac{N+2}{2}} ds \]
\[= c_{12} |x|^{-N + \frac{3}{2} N} r t^{-\frac{N}{2} + 1} + c_{13} |x|^{-N + \frac{7}{8}} t^{-\frac{N}{2} + \frac{7}{8}}, \]
where \( c_{12}, c_{13} > 0 \) and \( c_{14} > 0 \) verifies that
\[
e^{-a} \leq c_{14} a^{-\frac{N+2}{2}} \quad \text{for all} \ a \geq 1. \]

This completes the proof. \( \square \)

**Proof of Theorem 1.5.** From Lemma 4.1, we see that for \( t \in (0, 1) \) and \( \mu \geq \mu_0 \),
\[
t^{\frac{N}{2}} \left| \int_{\Omega} p_{\mu, \Omega}(x, x, t) dx - \int_{\Omega} p_0(0, t) dx \right| \leq \int_{\Omega} |r_{\mu}(x, x, t)| dx \]
\[\leq c_2 t^{\frac{N}{2}} \int_{\Omega} \max \{|x|^2, |x|^{2r_{\mu}(\mu)-2}\} dx \]
\[\to 0 \quad \text{as} \ t \to 0^+, \]
then we obtain that
\[
\lim_{t \to 0^+} t^{\frac{N}{2}} |Z_{\mu}(t) - Z_0(t)| = 0,
\]
where \( Z_0 \) is the partition function for Laplacian and direct computation shows that
\[
\lim_{t \to 0^+} t^{\frac{N}{2}} Z_0(t) = \frac{|\Omega|}{(4\pi)^{\frac{N}{2}}}. \]

So there holds
\[
\lim_{t \to 0^+} t^{\frac{N}{2}} Z_{\mu}(t) = \frac{|\Omega|}{(4\pi)^{\frac{N}{2}}}. \quad (4.4)
\]
From (4.4) and Karamata’s Tauberian theorem [38 Theorem 10.3], we get that
\[
N_{\mu}(\beta) \beta^{-\frac{N}{2}} \to \frac{|\Omega|}{\Gamma(\frac{N}{2} + 1)(4\pi)^{\frac{N}{2}}} \quad \text{as} \ \beta \to +\infty,
\]
which, taking \( \beta = \lambda_{\mu, k} \) and \( N(\lambda_{\mu, k}) = k \), implies that
\[
\lim_{k \to +\infty} \lambda_{\mu, k}^{-\frac{N}{2}} k = \frac{|\Omega|}{\Gamma(\frac{N}{2}) (4\pi)^{\frac{N}{2}}} = \frac{|\Omega| |B_1|}{(2\pi)^N}, \quad (4.5)
\]
where
\[
|B_1| = \frac{1}{N} \omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{N \Gamma(\frac{N}{2})}.
\]
Note that (4.5) is equivalent to (1.13) and the proof is completed. \( \square \)

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