ON THE EQUIDISTRIBUTION OF TOTALLY GEODESIC SUBMANIFOLDS IN COMPACT LOCALLY SYMMETRIC SPACES AND APPLICATION TO BOUNDEDNESS RESULTS FOR NEGATIVE CURVES

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Abstract. We prove an equidistribution result for totally geodesic submanifolds in a compact locally symmetric space. In the case of Hermitian locally symmetric spaces, this gives a convergence theorem for currents of integration along totally geodesic subvarieties. As a corollary, we obtain that on a complex surface which is a compact quotient of the bidisc or of the 2-ball, there is at most a finite number of totally geodesic curves with negative self intersection.

1. Introduction

Let $G$ be a connected semisimple real Lie group without compact factor and with finite center, $K < G$ a maximal compact subgroup and $\Gamma < G$ a torsion free cocompact lattice. We will denote by $X$ the symmetric space $G/K$, by $X$ the compact manifold $\Gamma \backslash G/K$ and by $\pi : \Gamma \backslash G \rightarrow X$ the natural projection.

We will be interested in compact totally geodesic (possibly singular) submanifolds of $X$. To simplify things, we will assume that all these submanifolds arise from (right) orbits in $\Gamma \backslash G$ of the same Lie subgroup $H$ of $G$.

More precisely, let $H < G$ be a connected semisimple Lie subgroup of $G$ without compact factor such that $H \cap K$ is a maximal compact subgroup of $H$. Then $Y := HK/K \subset G/K$ is a totally geodesic submanifold in $X$. Let $S$ be its stabilizer in $G$ (in particular $H \subset S$).

A subset $Y$ of $X$ will be called a closed totally geodesic submanifold of type $H$ if it is of the form $\Gamma \backslash gH K/K (= \Gamma \backslash gSK/K)$ for some $g \in G$ such that $g^{-1} \Gamma g \cap S$ is a lattice in $S$. Such a $Y$ is indeed a closed totally geodesic submanifold of $X$, which might be singular, and it supports a natural probability measure $\mu_Y$ which can be defined as follows. By assumption, the $S$-orbit $\Gamma g \cdot S \subset \Gamma \backslash G$ is closed and supports a unique $S$-invariant probability measure. We will denote by $\mu_Y$ the probability measure on $X$ whose support is $Y$ and which is defined as the push forward of the previous measure by the projection $\pi$. In the special case when $H = G$, we obtain the natural probability measure $\mu_X$ on $X$.

Our main result is the following

Theorem 1.1. Let $G$ be a connected semisimple real Lie group without compact factor and with finite center, $K < G$ a maximal compact subgroup and $\Gamma < G$ a torsion free cocompact lattice. Let $H < G$ be a connected semisimple Lie subgroup of $G$ without compact factor such that $Y := HK/K$ is totally geodesic in the symmetric space $X = G/K$ and let $S$ be the stabilizer of $Y$ in $G$.

Assume that

(1) the centralizer of $H$ in $G$ is contained in $K$. In particular, $Y$ is not a factor, i.e. there is no totally geodesic subspace of $X$ isometric to a non trivial Riemannian product $\mathcal{Y} \times Z$.

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Suppose that there exists a sequence \((Y_j)_{j \in \mathbb{N}}\) of closed totally geodesic submanifolds of type \(H\) in the manifold \(X = \Gamma \backslash G/K\), satisfying the following properties:

(2) the \(Y_j\)’s are pairwise distinct;
(3) no subsequence of \((Y_j)_{j \in \mathbb{N}}\) is contained in a closed totally geodesic proper submanifold of \(X\).

Then the sequence of probability measures \((\mu_{Y_j})_{j \in \mathbb{N}}\) converges to the probability measure \(\mu_X\).

Remark 1.2. Assumption (1) seems to be strong. Nevertheless, the conclusion of Theorem 1.1 is false if one only assumes (2) and (3) to be true as the following simple example shows: let \(X = \Sigma_1 \times \Sigma_2\) be the product of two Riemann surfaces of genus at least 2. If \((z_j)_{j \in \mathbb{N}}\) is a sequence of distinct points in \(\Sigma_1\) such that no subsequence is contained in a proper geodesic of \(\Sigma_1\), then the subvarieties \(Y_j = \{z_j\} \times \Sigma_2\) satisfy all the assumptions of Theorem 1.1 except (1), but for any subsequence of \((z_j)\) converging to some \(z \in \Sigma_1\), the corresponding subsequence of measures \(\mu_{Y_j}\) converges to \(\mu_{\{z\} \times \Sigma_2}\).

However, in the case of irreducible lattices of the bidisc, Assumption (1) is automatically satisfied (see the proof of Corollary 1.4 below).

Observe also that Assumption (1) is vacuous for rank 1 locally symmetric spaces.

In the case of Hermitian locally symmetric spaces, Theorem 1.1 gives a convergence result for currents of integration along closed complex totally geodesic subvarieties, suitably renormalized.

Corollary 1.3. Let \(G, K, \Gamma, X, H, \mathcal{Y}, S, (Y_j)_{j \in \mathbb{N}}\) be as in Theorem 1.1 and assume moreover that \(X\) is a Hermitian symmetric space and \(Y\) a Hermitian symmetric subspace of \(X\), so that the \(Y_j\)’s are complex subvarieties of the Kähler manifold \(X\). Let \(\omega_X\) be the Kähler form on \(X\) induced by the \(G\)-invariant Kähler form on \(X\). Then, for any \(k\)-form \(\eta\) on \(X\),

\[
\lim_{j \to +\infty} \frac{1}{k! \operatorname{vol}(Y_j)} \int_{Y_j} \eta = \frac{1}{n! \operatorname{vol}(X)} \int_X \eta \wedge \omega_X^{n-k}.
\]

Since our main applications, see below, will be when \(\operatorname{rk} X = 1\) and since it takes a simpler form in this case, we restate Corollary 1.3 for complex hyperbolic manifolds:

Corollary 1.4. Let \(X\) be a closed \(n\)-dimensional complex hyperbolic manifold. Let \(\omega_X\) be the Kähler form on \(X\) induced by the \(\operatorname{SU}(n,1)\)-invariant Kähler form on complex hyperbolic \(n\)-space \(\mathbb{C}H^n\). Assume that there exists a sequence \((Y_j)_{j \in \mathbb{N}}\) of pairwise distinct complex totally geodesic closed \(k\)-dimensional subvarieties in \(X\) and that no subsequence of \((Y_j)\) is contained in a closed totally geodesic proper subvariety of \(X\). Then, for any \(k\)-form \(\eta\) on \(X\),

\[
\lim_{j \to +\infty} \frac{1}{k! \operatorname{vol}(Y_j)} \int_{Y_j} \eta = \frac{1}{n! \operatorname{vol}(X)} \int_X \eta \wedge \omega_X^{n-k}.
\]

Complex hyperbolic manifolds satisfying the assumptions of the corollary exist: the arithmetic manifolds whose fundamental groups are the so-called uniform lattices of type \(I\) in the isometry group \(\operatorname{SU}(n,1)\) of complex hyperbolic \(n\)-space \(\mathbb{C}H^n\) are examples of manifolds supporting infinitely many totally geodesic subvarieties of dimension \(k\) for each \(1 \leq k < n\), and they are not all contained in a proper totally geodesic subvariety.

Our motivation for writing this note was twofold.

We were first interested in representations of cocompact complex hyperbolic lattices in isometry groups of Hermitian symmetric spaces. A complex hyperbolic lattice is a lattice \(\Gamma\) in the Lie group \(\operatorname{SU}(n,1)\). There is a number naturally associated to such a representation, called the Toledo invariant, which can be taken as a measure of the “complex size” of the
representation. It is defined as follows. Assume that the lattice $\Gamma$ is uniform and torsion free, so that $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ is a closed complex hyperbolic manifold of dimension $n$. Let $\mathcal{Y}$ be a Hermitian symmetric space of noncompact type, $\text{Isom}(\mathcal{Y})$ be the connected component of its isometry group, and $\rho : \Gamma \rightarrow \text{Isom}(\mathcal{Y})$ a representation, i.e. a group homomorphism. The space $\mathcal{Y}$ supports a $\text{Isom}(\mathcal{Y})$-invariant Kähler form $\omega_{\mathcal{Y}}$. Pulling-back this Kähler form by any $\Gamma$-equivariant map $f : \mathbb{H}^n_{\mathbb{C}} \rightarrow \mathcal{Y}$, and then, by $\Gamma$-invariance, integrating against the Kähler form $\omega_X$ of $X$ raised to the power $n-1$, gives a number which is independent of the choice of $f$ and hence only depends on the representation $\rho$. This is the Toledo invariant:

$$\tau(\rho) = \frac{1}{n!} \int_X f^* \omega_{\mathcal{Y}} \wedge \omega_X^{n-1}.$$  

Using bounded cohomology, M. Burger and A. Iozzi proved in [BI] that the Toledo invariant satisfies the following Milnor-Wood type inequality

$$|\tau(\rho)| \leq \text{rk}(\mathcal{Y}) \text{ vol}(X),$$

if the Kähler classes are normalized so that the minimum of the holomorphic sectional curvatures of $X$ and $\mathcal{Y}$ is $-1$.

In the very particular case where the complex hyperbolic manifold $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ has many totally geodesic (complex) curves, Corollary 1.4 gives directly an alternative proof if we take the inequality for (Riemann) surface groups for granted. Note that the inequality is easy to prove for surface groups using complex geometric methods (more precisely using the Higgs bundles associated to the representation, see e.g. [BGG1] and [BGG2]).

**Corollary 1.5.** Let $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ be a closed complex hyperbolic manifold. Assume that $X$ contains a sequence $(Y_j)_{n \in \mathbb{N}}$ of pairwise distinct totally geodesic closed curves in $X$ and that no subsequence of $(Y_j)_{n \in \mathbb{N}}$ is contained in a proper totally geodesic subvariety of $X$. Then any representation of $\Gamma$ in the isometry group of an irreducible Hermitian symmetric space of noncompact type satisfies the Milnor-Wood inequality.

**Remark 1.6.** A proof of the Milnor-Wood inequality using Higgs bundles, and without assuming that $\Gamma$ is a lattice of type I, is now available (see [KM]). It involves the tautological foliation by complex curves of the projectivized holomorphic tangent bundle of a complex hyperbolic manifold and, though quite different from the approach we propose here, the method shares the same philosophy consisting in using totally geodesic complex curves (which are not closed in the general case).

Our second motivation grew up from conversations in Oberwolfach, Germany, about the bounded negativity conjecture for Shimura surfaces. In February 2014, the authors were attending the mini-workshop “Kähler Groups” ([http://www.mfo.de/occasion/1409a/www_view](http://www.mfo.de/occasion/1409a/www_view)) organized by D. Kotschick and D. Toledo, and during the same week, there was another mini-workshop: “Negative Curves on Algebraic Surfaces” ([http://www.mfo.de/occasion/1409b/www_view](http://www.mfo.de/occasion/1409b/www_view)). As we learned from discussions with D. Toledo and participants of the latter workshop, there was an interest in boundedness results for negative curves on quotients of the 2-ball (see the report [DKMS]). Corollary 1.4 implies (we also include the already known case of the bidisc since it also follows from Theorem [LJ]):

**Corollary 1.7.** Let $X$ be closed complex surface whose universal cover is biholomorphic to either the 2-ball or the bidisc. Then $X$ only supports a finite number of totally geodesic curves with negative self intersection.
Apparently the interest in this result was indeed strong: it has been also obtained, very recently, by M. Möller and D. Toledo [MT]. We refer to their paper for background on Shimura surfaces and Shimura curves, and in particular for a discussion of the arithmetic quotients of SU(2,1) for the 2-ball and of SU(1,1) × SU(1,1) for the bidisc which admit infinite families of pairwise distinct totally geodesic curves. Their proof is also based on an equidistribution theorem, but only for curves in 2-dimensional Hermitian locally symmetric spaces (i.e. quotients of the 2-ball or the bidisc). We learned from their paper that L. Clozel and E. Ullmo ([CU, U]) proved several results about equidistribution of special varieties in Shimura varieties which seem to be very similar to Theorem 1.1.

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2. **Proofs**

Theorem 1.1 follows from considerations originating in the celebrated results of M. Ratner on unipotent flows, see e.g. [Rat] for a survey. The key result we are going to use is Y. Benoist Theorem 1.5].

Let us begin by giving the definitions needed to quote a downgraded version of [BQIII Theorem 1.5]. Our notation are a bit different from those of [BQIII]. Let $G$ be a real Lie group, $\Gamma$ a uniform lattice in $G$, and $H$ a Lie subgroup of $G$ such that $\text{Ad}(H)$ is a semisimple subgroup of $\text{GL}(g)$ with no compact factors.

A closed subset $Z$ of $\Gamma \backslash G$ is called a finite volume homogeneous subspace if the stabilizer $G_Z$ of $Z$ in $G$ acts transitively on $Z$ and preserves a Borel probability measure $\mu_Z$ on $Z$. If moreover $G_Z$ contains $H$, $Z$ is said $H$-ergodic if $H$ acts ergodically on $(Z, \mu_Z)$.

Let $S_{\Gamma \backslash G}(H)$ be the set of $H$-invariant and $H$-ergodic finite volume homogeneous subspaces $Z$ of $\Gamma \backslash G$. According to the main result of [BQII], every $H$-invariant $H$-ergodic probability measure on $\Gamma \backslash G$ is equal to $\mu_Z$ for some $Z$ in $S_{\Gamma \backslash G}(H)$. We may identify $S_{\Gamma \backslash G}(H)$ with a set of Borel probability measures on $\Gamma \backslash G$ through the map $Z \mapsto \mu_Z$. In particular $S_{\Gamma \backslash G}(H)$ is endowed with the topology of weak convergence, so that a sequence $(Z_n)$ in $S_{\Gamma \backslash G}(H)$ converges toward $Z \in S_{\Gamma \backslash G}(H)$ if and only if $\mu_{Z_n}$ converges toward $\mu_Z$.

Then [BQIII Theorem 1.5] implies the following:

**Theorem 2.1.** Let $G$ be a real Lie group, $\Gamma$ a uniform lattice in $G$, and $H$ a Lie subgroup of $G$ such that $\text{Ad}(H)$ is a semisimple subgroup of $\text{GL}(g)$ with no compact factors. Then

1. the space $S_{\Gamma \backslash G}(H)$ is compact;
2. if $(Z_n)$ is a sequence of $S_{\Gamma \backslash G}(H)$ converging to $Z \in S_{\Gamma \backslash G}(H)$, there exists a sequence $(\ell_n)$ of elements of the centralizer of $H$ in $G$ such that $Z_n \subset Z \cdot \ell_n$ for $n$ large.

Let us recall the notation from the introduction. Let $G$ be connected semisimple real Lie group without compact factor and with finite center, $K < G$ a maximal compact subgroup of $G$ and let us denote by $\mathcal{X}$ the symmetric space $G/K$ associated to $G$. Let $H < G$ be a semisimple Lie subgroup of $G$ without compact factor such that $H \cap K$ is a maximal compact subgroup of $H$. Then $\mathcal{Y} = H/(H \cap K) \subset G/K$ is a totally geodesic subspace of $\mathcal{X}$ and we denote by $S < G$ the stabilizer of $\mathcal{Y}$. If we set $V = S \cap K$, we also have $\mathcal{Y} \simeq S/V \simeq SK/K$.

**2.1. Proof of Theorem 1.1.** The closed manifold $X$ is the quotient $\Gamma \backslash \mathcal{X}$, where $\Gamma$ is a torsion free uniform lattice in $G$.

By assumption, for any $j$ there exists $g_j \in G$ such that $\Gamma_j = \Gamma \cap g_j S g_j^{-1}$ is a lattice in $g_j S g_j^{-1}$ and $Y_j = \Gamma \backslash g_j S K / K = \Gamma \backslash \Gamma g_j H K / K$ is the image of $\tilde{Y}_j = \Gamma_j \backslash g_j S V$ in $\Gamma \backslash G / K$. 
Remark that although the $\tilde{Y}_j$’s are smooth, the $Y_j$’s are in general singular since they might have self-intersection.

Consider the (right) $S$-invariant subsets $\Gamma\backslash\Gamma g_j S$ in $\Gamma\backslash G$. They support natural $S$-invariant probability measures, but these measures might be non ergodic with respect to the action of $H \subset S$. To get rid of this problem, we need to consider the action of $H$ on the orbit of a smaller subgroup than $S$. The stabilizer $S$ is a reductive group. There exists a subgroup $U < K$ centralizing $H$ such that $S = HU$. The intersection $H \cap U$ is the center of $H$ and hence is finite. Let $M_j$ be the “projection” of $g_j^{-1}\Gamma_j g_j$ to $U$, namely the group $\{u \in U$ such that $u = \gamma h$ for some $\gamma \in g_j^{-1}\Gamma_j g_j$ and $h \in H\}$, and let $M_j$ be the closure of $M_j$. Then $M_j$ is a compact subgroup of $U$ and this time, the right action of $H$ on the $M_j H$-invariant probability measure $\mu_j$ supported on $Z_j := \Gamma\backslash\Gamma g_j(M_j H)$ is ergodic. Indeed, a $H$-orbit in $Z_j$ is the same as a (left) $M_j H$-orbit in $M_j H$ (because $H$ is obviously normal in $M_j H$) and the group $M_j H$ is dense in $M_j H$ by construction (see [Ma, Prop. I.(4.5.1)]). Notice however that the push forward of the measure $\mu_j$ by the projection $\pi : \Gamma\backslash G \longrightarrow X = \Gamma\backslash G/K$ is the same as the push forward of the $S$-invariant probability measure on $\Gamma\backslash g_j S$ by $\pi$.

The measures $\mu_j$ belong to the set $S_{\Gamma\backslash G}(H)$ of $H$-invariant and $H$-ergodic finite volume homogeneous subspaces of $\Gamma\backslash G$ which is compact by Theorem [2,1]. Hence, after extraction of a subsequence, the sequence $(\mu_j)$ converges weakly to a $H$-ergodic probability measure $\mu$ whose support $\text{supp} \mu$ is $G^\mu$-homogeneous where $G^\mu := \{g \in G : \mu g = \mu\}$ is a (closed) Lie subgroup of $G$. Moreover, by the second part of the theorem, there exists a sequence $(\ell_j)$ of elements of the centralizer of $H$ in $G$ such that for $j$ large enough, $\text{supp} \mu_j \subset (\text{supp} \mu) \cdot \ell_j$.

We are going to prove that $G^\mu = G$ and Assumption (1) of Theorem [1,1] will come into play in order to neutralize the effect of the $\ell_j$’s. Before that, we will need two general results. The first one is very classical.

**Proposition-definition 2.2.** Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is reductive in $\mathfrak{g}$ if the two following equivalent properties are satisfied:

1. the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is semisimple;
2. $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ and $\mathfrak{a}$ are ideals of $\mathfrak{h}$, $\mathfrak{s}$ is semisimple, $\mathfrak{a}$ is abelian, and all the elements of $\mathfrak{a}$ are $\mathfrak{h}$-semisimple for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$.

**Proof.** It is clear that (2) implies (1) so that we only need to prove the reverse implication.

Since $\mathfrak{h}$ is a sub $\text{ad}(h)$-module of $\mathfrak{g}$, the adjoint action of $\mathfrak{h}$ on $\mathfrak{h}$ is also semisimple. Let $\tau$ be the radical ideal of $\mathfrak{h}$ and $\mathfrak{s}$ be a sub $\text{ad}(\mathfrak{h})$-module of $\mathfrak{h}$ which is complementary to $\tau$. Then $\mathfrak{s} \simeq \mathfrak{h}/\tau$ is a semisimple subalgebra of $\mathfrak{h}$. Since $\tau$ and $\mathfrak{s}$ are both ideals of $\mathfrak{h}$, $[\tau, \mathfrak{s}] = 0$ and hence $\mathfrak{h}$ only acts on $\tau$ by $\tau$. As a consequence, by semisimplicity, we have a decomposition $\tau = \tau' \oplus \mathfrak{v}$ where $\tau' = [\tau, \tau]$ and $\mathfrak{v}$ is an $\text{ad}(\tau)$-submodule of $\tau$, in particular $\mathfrak{v}$ is an abelian Lie subalgebra. But since $[\tau', \mathfrak{v}] = 0$, we have $\tau' = [\tau, \tau] = [\tau', \tau']$. As $\tau$ is solvable, this means that $\tau' = 0$ i.e. $\tau$ is abelian and also $[\tau, \mathfrak{h}] = 0$.

Let now $r \in \tau \subset \mathfrak{g}$ and $r = r_s + r_n$ be its Jordan-Chevalley decomposition where $r_s$ is semisimple, $r_n$ is nilpotent and $[r_s, r_n] = 0$. As $[r, \mathfrak{h}] = 0$, $[r_n, \mathfrak{h}] = 0$ hence the kernel of $\text{ad}(r_n)$ is $\text{ad}(\mathfrak{h})$-invariant and as such admits an $\text{ad}(\mathfrak{h})$-invariant complement. So, this complement is in particular $\text{ad}(r)$-invariant and then $\text{ad}(r_n)$-invariant. Since $r_n$ is nilpotent, $\text{Ker ad}(r_n)$ admits an $\text{ad}(r_n)$-invariant complement only if $r_n = 0$. \hfill $\Box$

**Proposition 2.3.** Let $G$ be a connected semisimple real Lie group and $\Gamma < G$ a cocompact lattice. Let $H$ be a closed connected subgroup of $G$ and assume that $\Gamma \cap H$ is a lattice of $H$. Then $\mathfrak{h}$ is reductive in $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the respective Lie algebras of $G$ and $H$. 


Proof. Let $R$ be the connected Lie subgroup of $H$ corresponding to the radical ideal $\mathfrak{r}$ of $\mathfrak{h}$ and $\pi : H \to H/R$ be the projection. Let $\Lambda := \Gamma \cap H$, let us denote by $\Delta$ the closure $\pi(\Gamma)$ of $\pi(\Gamma)$ in the group $H/R$, and its identity component by $\Delta^0$. Then, $\Delta^0$ is normalized by $\Delta$ and by a theorem of Auslander (see [Rag, Theorem 8.24]), $\Delta^0$ is solvable. Thus, $R_1 := \pi^{-1}(\Delta^0)$ is a solvable connected subgroup of $H$ containing $R$ and it is normalized by $\Lambda$. In particular, $\Lambda R_1$ is a subgroup of $G$ and moreover, $\Lambda R_1/R = \Delta$. Indeed, $R_1/R = \Delta^0$ is open in $\Delta$, so $\Lambda R_1/R$ is open, hence closed, in $\Delta$ and finally it is dense by construction. Therefore, $\Lambda R_1 = \pi^{-1}(\Delta)$ is a closed subgroup of $G$ and $\Lambda$ is a lattice in $\Lambda R_1$ thus $\Pi := \Gamma \cap R_1 = \Lambda \cap R_1$ is a lattice in $R_1$ by [Rag, Lemma 1.7].

We now use the cocompactness assumption on $\Gamma$ to prove that the elements of $\Pi$ are semisimple. It is well-known that in a semisimple Lie group, an element is semisimple if and only if its orbit under inner conjugation is closed (see [Rag] p. 5 for instance). Let $F \subset G$ be a compact fundamental domain for the action of $\Gamma$ and let $(g_0)$ be a sequence of elements of $G$ such that $g_0 \gamma g_0^{-1} \to g' \in G$. For any $n$, there exists $\gamma_n \in \Gamma$ and $f_n \in F$ such that $g_0 = f_n \gamma_n$. Taking a subsequence if necessary, we can assume that $f_n \to f \in F$ hence $\gamma_n \gamma_n^{-1} \to f^{-1} g' f$ and since $\Gamma$ is discrete, $\gamma_n \gamma_n^{-1} = f^{-1} g' f$ if $n$ is large enough, so that $g'$ is in the orbit of $\gamma$. As a consequence, the orbit of $\gamma$ is closed in $G$.

In particular, the adjoint action of any element of $\Pi = \Gamma \cap R_1$ on $\mathfrak{g}$ is semisimple. Also, recall that $R_1$ is solvable so that, by Lie’s theorem, the matrices of elements of $\text{ad}(t_f^\mathfrak{g}) \subset \text{gl}(t_f^\mathfrak{g})$ relative to a suitable basis of $t_f^\mathfrak{g}$ are upper triangular, where $t_f$ is the Lie algebra of $R_1$, $t_f^\mathfrak{g} = t_f \otimes \mathbb{C}$ and $t^\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}$. This implies that $[\text{ad}(t_f^\mathfrak{g}), \text{ad}(t_{f'}^\mathfrak{g})]$ is a nilpotent subalgebra of $\text{gl}(t_f^\mathfrak{g})$ and in particular, the elements of the derived group $[\text{Ad}(\Pi), \text{Ad}(\Pi)]$ are nilpotent. Hence, the elements of $\Pi$ are semisimple, hence trivial, and then $\Pi$ is abelian (since the kernel of the adjoint representation is the center of $G$). As a consequence, the adjoint action of $\Pi$ on $\mathfrak{g}$ is semisimple and since $\Pi$ is a lattice in $R_1$, it follows that the adjoint action of $R_1$ itself on $\mathfrak{g}$ is also semisimple. Indeed, let $W \subset \mathfrak{g}$ be an invariant vector subspace of $\mathfrak{g}$ under the adjoint action of $R_1$. Then, it is $\Pi$-invariant and since the action of $\Pi$ is semisimple, there exists a projection $p : \mathfrak{g} \to W$ which is $\Pi$-equivariant. Then we can set $q := \int_{\Pi \cap R_1} \text{Ad}(g^{-1}) \circ p \circ \text{Ad}(g) \, dg$ where $dg$ is the Haar measure on $R_1$ normalized in such a way that $\int_{\Pi \cap R_1} \, dg = 1$. The map $q : \mathfrak{g} \to W$ is again a projection and $R_1$-equivariant, so that $\text{Ker} \, q$ is an $R_1$-invariant complement subspace of $W$ in $\mathfrak{g}$.

The adjoint action of $t_f$ on $\mathfrak{g}$ being semisimple, $t_f$ is reductive in $\mathfrak{g}$ by definition. As it is solvable, its semisimple part is trivial and $t_f$ (as well as $\mathfrak{r} \subset t_f$) is abelian and only contains semisimple elements. Let $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{r}$ be a Levi decomposition of $\mathfrak{h}$ (which means that $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{h}$). We are left to show that $[\mathfrak{s}, \mathfrak{r}] = 0$.

Since $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}$, we have a linear representation $\Phi : \mathfrak{s} \to \text{gl}(\mathfrak{r})$ that we consider as a representation into $\text{gl}(t_f^\mathfrak{r})$ where $t_f^\mathfrak{r} = \mathfrak{r} \otimes \mathbb{C}$ is the complexification of $\mathfrak{r}$. Assume that $\text{Im} \, \Phi$ is not trivial. As $\mathfrak{s}$ is semisimple, $\Phi(\mathfrak{s})$ is also semisimple hence it contains elements which are not nilpotent. Let us choose $\zeta \in \mathfrak{s}$ such that $\Phi(\zeta) \in \text{gl}(t_f^\mathfrak{r})$ has a non zero eigenvalue $\lambda \in \mathbb{C}$ and let $\xi \in t_f^\mathfrak{r}$ ($\xi \neq 0$) such that $\text{ad}(\zeta)\xi = \lambda \xi$. Then, for any $t \in \mathbb{C}$, $\text{Ad}(\exp(t\zeta))\xi = \exp(t\lambda \xi)$ so that taking $t = \lambda s$ and letting $s$ go to $-\infty$ we see that $0$ is in the closure of the conjugacy class of $\xi$, i.e., $\xi$ is nilpotent. Now, we use that $t_f^\mathfrak{g}$ is solvable therefore by Lie’s theorem, the matrices of $t_f^\mathfrak{s}$ relative to a suitable basis of $t_f^\mathfrak{s}$ are upper triangular. Then clearly, $\xi + \xi^\mathfrak{s}$ and $\sqrt{-1}(\xi - \xi^\mathfrak{s})$ are both nilpotent elements of $\mathfrak{r}$ and (at least) one of them is non zero, which contradicts the fact that $\mathfrak{r}$ only contains semisimple elements. \hfill $\square$

Consider now the identity component $G_0^\mu$ of $G^\mu$ and its Lie algebra $\mathfrak{g}_0^\mu$. By Proposition 3.3, $\mathfrak{g}_0^\mu = \mathfrak{s} \oplus \tilde{\mathfrak{a}}$ is reductive. Let $\tilde{\mathfrak{a}}$ be the Lie subalgebra of $\mathfrak{g}$ associated to the Zariski closure in $G$ of the Lie group corresponding to $\mathfrak{a}$. Then, $\mathfrak{s}$ and $\tilde{\mathfrak{a}}$ still are in direct sum and the Lie subalgebra
s ⊕ a of g is reductive and algebraic. Moreover, as a is included in a Cartan subalgebra of g, this is also the case of a hence the elements of a are semisimple. So, by \[BH\] Lemma 1.5, there exists a Cartan involution on g stabilizing s ⊕ a.

Hence \(G_0^d\) has a totally geodesic orbit \(G_0^dK/K\) in \(X = G/K\) for some \(g' \in G\). Moreover, we know that \(\text{supp} \mu = \Gamma' \backslash \Gamma \Gamma gG^{n\mu}\) for some \(g \in G\) and we have seen above that if \(j\) is large, then \(Z_j \gamma_j = \Gamma' \backslash \Gamma \Gamma gG^{n\mu}\) for some \(\ell_j\) in the centralizer of \(H\). Hence \(\Gamma' \backslash \Gamma \Gamma gG^{n\mu} K/K \subset \Gamma' \backslash \Gamma gG^{n\mu} K/K\), because \(\ell_j \in K\) by Assumption (1). Therefore, there exists \(\gamma_j\) in \(\Gamma\) such that \(\gamma_j g_j HK/K \subset gG_0^dK/K \subset X\).

This actually implies that the totally geodesic submanifolds \(\gamma_j g_j HK/K\) are all included in the totally geodesic orbit \(gG_0^dK/K\) and thus that \(gG_0^d K = gG_0^d g' K/K\).

Let indeed \(d\) be the distance in the symmetric space \(X\). For all \(a \in G_0^d\), \(d(gaK, gag'K) = d(K, g'K)\) so that the set \(gG_0^dK\) is at bounded distance from the totally geodesic submanifold \(gG_0^d g' K/K\) of \(X\). The function

\[
\begin{align*}
\mathcal{X} &\longrightarrow \mathbb{R} \\
x &\longmapsto d(x, gG_0^d g' K/K)
\end{align*}
\]

is convex because \(gG_0^d g' K/K\) is totally geodesic. Therefore its restriction to \(\gamma_j g_j HK/K\), which is also totally geodesic, is both convex and bounded, hence constant equal to some \(d_j \in \mathbb{R}\). Then, if \(d_j \neq 0\), there exists an isometric embedding from the product \((\gamma_j g_j HK/K) \times \mathbb{R}\) to \(X\) which maps \((\gamma_j g_j HK/K) \times \{0\}\) to \(\gamma_j g_j HK/K\) and \((\gamma_j g_j HK/K) \times \{d_j\}\) to a totally geodesic subspace of \(gG_0^d g' K/K\), see e.g. \[BH\]. This is a contradiction with Assumption (1), hence \(d_j = 0\) for all \(j\) as claimed.

As a consequence, \(\Gamma' \backslash \Gamma gG_0^d K/K\) is a closed totally geodesic submanifold in \(X\) which contains the submanifolds \(Y_j\) (for \(j\) large enough). Because of Assumption (2), \(\Gamma' \backslash \Gamma gG_0^d K/K = \Gamma \backslash G/K\) which implies \(G_0^d = G\) since \(G_0^d\) is reductive.

In conclusion, \((\mu_j)\) is a sequence of elements of \(S_{\Gamma \backslash G}(SU(k,1))\) which is compact, and its sole limit point is the unique \(G\)-invariant probability measure \(\mu\) it converges to \(\mu\). Hence the sequence \((\mu_j)\) converges to \(\mu_X\).

### 2.2. Proof of Corollaries 1.3 and 1.4

In the setting of Corollary 1.3, the submanifolds \(Y_j\) are complex subvarieties of the Kähler manifold \(X\). Therefore the measures \(\mu_{Y_j}\) can be described in the following alternative way: the \(G\)-invariant Kähler form \(\omega\) on \(X\) descends to a Kähler form \(\omega_X\) on \(X\). Then, \(\mu_{Y_j}\) is the probability measure with support \(Y_j\) and density \(1/\text{vol}(Y_j) \omega^k\), where \(\text{vol}(Y_j) = \frac{1}{n!} \int_{Y_j} \omega^n X\) and \(\int_{Y_j}\) means integration over the smooth part of \(Y_j\).

Hence, if we apply Theorem 1.3, we obtain

\[
\lim_{j \to +\infty} \frac{1}{k! \text{vol}(Y_j)} \int_{Y_j} \eta = \frac{1}{|\omega^k|^2} \lim_{j \to +\infty} \frac{1}{k! \text{vol}(Y_j)} \int_{Y_j} \langle \eta, \omega^k \rangle \omega^k
\]

\[
= \frac{1}{|\omega^k|^2} \frac{1}{n! \text{vol}(X)} \int_X \langle \eta, \omega^k \rangle \omega^n
\]

\[
= \frac{1}{n! \text{vol}(X)} \int_X \eta \wedge \omega^{n-k}.
\]

In the case of \(n\)-dimensional complex hyperbolic manifolds, the group \(G\) is \(SU(n,1)\) and its maximal compact subgroup \(K\) is \(U(n)\). The corresponding Hermitian symmetric space is the complex hyperbolic space \(\mathbb{H}^n_{\mathbb{C}}\). There are only two types of totally geodesic submanifolds in \(\mathbb{H}^n_{\mathbb{C}}\): complex ones, which are isometric to lower dimensional complex hyperbolic spaces \(\mathbb{H}^k_{\mathbb{C}}\), and totally real ones, which are isometric to real hyperbolic space \(\mathbb{H}^k_{\mathbb{R}}\). Since we are only interested in totally complex geodesic submanifolds, fixing the \(H\)-type is the same as fixing
the dimension of the submanifold: for $k$-dimensional complex totally geodesic subvarieties, we may take $H = SU(k, 1)$ and $S = S(U(n - k) \times U(k, 1)) \simeq U(n - k) \times SU(k, 1)$ as the stabilizer of $Y \simeq \mathbb{H}_k^n$. As we already mentioned, Assumption (1) is vacuous for rank 1 locally symmetric spaces, and Corollary 1.4 follows immediately from Corollary 1.3.

2.3. Proof of Corollary 1.5 The representation $\rho$ of $\Gamma$ induces representations $\rho_j$ of the subgroups $\Gamma_j = \Gamma \cap g_j S_{\gamma j} = \pi_1(\tilde{Y}_j)$ of $\Gamma$. These subgroups are (up to conjugacy) lattices in $SU(1, 1)$. The known proofs of the Milnor-Wood inequality in dimension 1 give

$$\tau(\rho_j) = \int_{\tilde{Y}_j} f^*\omega_Y \leq \text{rk}(\gamma) \text{vol}(\tilde{Y}_j) = \text{rk}(\gamma) \text{vol}(Y_j).$$

Hence $\left(\frac{\tau(\rho_j)}{\text{vol}(Y_j)}\right)_{j \in \mathbb{N}}$ is bounded by $\text{rk}(\gamma)$, and converges to $\frac{\tau(\rho)}{\text{vol}(X)}$ by Corollary 1.3.

2.4. Proof of Corollary 1.7 In the case of the bidisc, i.e. $X = \Gamma \backslash (\Delta \times \Delta)$, if the lattice $\Gamma$ is reducible (i.e. if, up to a finite covering, $X$ is a product of Riemann surfaces) then the totally geodesic complex curves corresponding to the subgroups $H = SU(1, 1) \times \{e\}$ and $H = \{e\} \times SU(1, 1)$ of $G = SU(1, 1) \times SU(1, 1)$ all have vanishing self-intersection, and if the lattice is irreducible, such curves do not exist. As a consequence, we are only interested in the curves associated to the group $H \simeq SU(1, 1)$ which is diagonally embedded in $G$ and hence satisfies Assumption (1) of Theorem 1.1 so that we can apply Corollary 1.3.

In the case of the 2-ball, we can directly apply Corollary 1.4.

Assume now that there exist infinitely many totally geodesic curves $(C_j)_{j \in \mathbb{N}}$ with negative self intersection. Each of them defines an integral class $[C_j] \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$. We normalize the Kähler class $[\omega]$ of $X$ in such a way that $\int_X \omega^2 = 1$, so that $\text{vol}(X) = \frac{1}{2}$. We also set $A_j = [C_j] \cdot [\omega] = \int_{C_j} \omega = \text{vol}(C_j)$.

By the Hodge index theorem, the intersection form is negative definite on $[\omega]$, the orthogonal complement in $H^{1,1}(X, \mathbb{R})$ of the line generated by $[\omega]$. Let $\eta_1, \ldots, \eta_{h^{1,1} - 1}$ be an orthonormal basis of $[\omega]$ (i.e. $\eta_i \cdot \eta_j = -\delta_{ij}$) and for any $j$, write

$$\frac{1}{A_j}[C_j] = [\omega] + \sum_k \lambda^j_k \eta_k$$

where $\lambda^j_k \in \mathbb{Q}$. By Theorem 1.1 (in the formulation of Corollary 1.3), we know that for any closed $(1, 1)$-form $\varphi$ on $X$

$$\frac{1}{A_j}[C_j] \cdot [\varphi] = \frac{1}{A_j} \int_{C_j} \varphi \to +\infty \int_X \varphi \wedge \omega$$

hence

$$\lambda^j_k = -\frac{1}{A_j}[C_j] \cdot \eta_k \to +\infty 0,$$

and in particular the $|\lambda^j_k|$ are bounded by a uniform constant $A$.

Let us consider now the intersection numbers

$$I_j := \left([\omega] - \frac{1}{A_j}[C_j]\right) \cdot \frac{1}{A_j}[C_j].$$

On the one hand, for any $\varepsilon$, $|\lambda^j_k| < \varepsilon$ for any $k$, whenever $j$ is taken large enough. As a consequence,

$$|I_j| < h^{1,1} \varepsilon A.$$  

On the other hand, since $C_j^2 < 0$, we have $I_j \geq [\omega] \cdot \frac{1}{A_j}[C_j] = 1$ for any $j$, a contradiction.
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