ON DISPLACEABILITY OF PRE-LAGRANGIAN FIBERS IN TORIC CONTACT MANIFOLDS

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Abstract. In this note we explore displaceability of generic toric fibers (pre-Lagrangian toric fibers) in toric contact manifolds. We obtain some results complementary to the symplectic rigidity case. In particular, we show that every generic orbit in the toric contact sphere $S^{2d-1}$, $d \geq 2$, is displaceable while every generic orbit in $T^d \times S^{d-1}$, $d \geq 2$, is non-displaceable. The first result is related to the non-orderability of the sphere, while the second one seems to be related to the existence of a free toric action. We discuss possible generalizations of these examples.

1. Introduction

A contact toric manifold is a co-oriented contact manifold $(V^{2d-1}, \xi)$ with an effective action of the torus $T^d$, that preserves the contact structure $\xi$. As $T^d$ is compact one can always choose a $T^d$-invariant contact form $\alpha$. For a $T^d$-invariant contact form $\alpha$ we define an $\alpha$-moment map $\mu_{\alpha}: V \to (t^d)^*$ by

$$\langle \mu_{\alpha}(p), X \rangle = \alpha_{\eta}(X_p),$$

where $X$ is the vector field on $V$ generated by $X \in t^d$ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $t^d$ and $(t^d)^*$. We identify the Lie algebra $t^d$ of $T^d$ with $\mathbb{R}^d$ via exponential map, using a convention $S^1 \simeq \mathbb{R}/\mathbb{Z}$, and view the moment map $\mu_{\alpha}$ as a map to $(\mathbb{R}^d)^*$. Fibers of $\mu_{\alpha}$ are $T^d$-orbits (see [L1, Lemma 3.16]).

Recall that the symplectization of $(V^{2d-1}, \xi)$ is the symplectic manifold $SV = \{(p, \eta_p) \in T^*V \mid \ker \eta_p = \xi_p \text{ and } \eta_p \text{ agrees with coorientation of } \xi_p\}$ with symplectic form $d\lambda_{SV}$, where $\lambda$ is the canonical Liouville 1-form on $T^*V$. The standard $\mathbb{R}_+$-action on $SV$ defined by $t \cdot (p, \eta_p) \mapsto (p, t\eta_p)$ makes $SV$ a principal $\mathbb{R}_+$-bundle over $V$. We denote by $\pi : SV \to V$ the projection $\pi(p, \eta_p) = p$. Any contact form for $\xi$ is a section of this bundle. Furthermore, if $V$ is equipped with a contact action of $T^d$, the lift of this action to $T^*V$ is symplectic and keeps $SV$ invariant, making $SV$ a symplectic toric manifold. The corresponding moment map $\Phi : SV \to (t^d)^*$, called contact moment map, is given by

$$\langle \Phi(p, \eta_p), X \rangle = \eta_p(X_p)$$

and does not depend on a choice of a contact form. Moreover, for any $T^d$-invariant contact form $\alpha$ holds $\Phi \circ \alpha = \mu_{\alpha}$ (see [L1] Proposition 2.8). Since the $T^d$-action on $SV$ is by
bundle transformations, the projection $\pi$ maps diffeomorphically generic $T^d$-orbits in $SV$ to generic $T^d$-orbits in $V$. Thus, if $\alpha$ is an invariant contact form, $\pi|_{\Phi^{-1}(c)} : \Phi^{-1}(c) \to \mu^{-1}_\alpha(c)$ is a diffeomorphism, for any $c \in \mu_\alpha(V) \cap \text{Int} SV$.

A submanifold $L \subset V$ is a **pre-Lagrangian** if it is a diffeomorphic image under projection $\pi$ of some Lagrangian submanifold $L' \subset SV$. Any generic (full dimensional) orbit in a toric symplectic manifold is a Lagrangian submanifold. Therefore any generic $T^d$-orbit in a toric contact manifold is a pre-Lagrangian. If $\alpha$ is a $T^d$-invariant contact form then all pre-Lagrangian toric fibers are of the form $\mu^{-1}_\alpha(c)$, for some $c \in \mu_\alpha(V) \cap \text{Int} SV$ and therefore are completely determined by a moment map $\mu_\alpha$. By Theorem 4.2 and in [L1], if $\dim V > 3$ then all toric fibers are connected.

Note that if $\pi : (V, \xi = \ker \alpha) \to (M, \omega)$ is a prequantization map, with $(M, \omega)$ a symplectic toric manifold, then $(V, \xi)$ is a toric contact manifold and for each Lagrangian toric fiber $L' \subset M$ its pre image $\pi^{-1}(L') \subset V$ is a pre-Lagrangian toric fiber in $V$ (See Section 2).

In symplectic geometry a time dependent Hamiltonian function induces a Hamiltonian isotopy. Lagrangian toric fiber $L \subset M$ in a toric symplectic manifold $(M, \omega)$ is non-displaceable if $\varphi_1(L) \cap L \neq \emptyset$ for every Hamiltonian isotopy $\varphi_t$ on $M$.

In contact setting, a time dependent Hamiltonian function $h : [0, 1] \times V \to \mathbb{R}$ on a contact manifold $V$ also induces a Hamiltonian contact isotopy $\varphi_t$. This isotopy is obtained by integrating a vector field $X_{ht}$ defined uniquely by

$$\alpha(X_{ht}) = h_t \text{ and } X_{ht} \alpha = dh_t(R_\alpha)\alpha - dh_t, \quad (1.1)$$

where $R_\alpha$ is the Reeb vector field. Note that $\varphi_t$ depends on the choice of a contact form $\alpha$.

Contrary to symplectic case where Hamiltonian diffeomorphisms form a non-trivial subset of symplectic isotopies, in contact setting every contact isotopy on $V$ starting at identity is a Hamiltonian isotopy. Precisely, any such isotopy $\varphi_t$ defines uniquely vector field $X_t$ by $X_t \circ \varphi_t = \frac{d\varphi_t}{dt}$, and one obtains the corresponding Hamiltonian by putting $h_t = \alpha(X_t)$. Therefore analyzing displaceability under Hamiltonian contact isotopies is equivalent to analyzing displaceability under contact isotopies. Translating the notion of non-displaceable Lagrangian fiber in symplectic manifold to the contact setting we obtain the following definition. A pre-Lagrangian toric fiber $L \subset V$ is called **non-displaceable** if $\varphi_1(L) \cap L \neq \emptyset$ for every contact isotopy $\varphi_t$ on $V$ starting at identity. Otherwise, it is called displaceable. Therefore, in contact setting we have in some sense “more” candidates to displace sets than in symplectic setting. Contrary to the symplectic case, where every compact symplectic toric manifold contains a non-displaceable fiber (see [EPP], [W], [MP]), contact toric manifolds may have all pre-Lagrangian fibers displaceable.

**Theorem 1.1.** Every pre-Lagrangian toric fiber in the standard contact toric sphere $S^{2d-1}$, $d \geq 2$, is displaceable.

The proof of the above Theorem is deeply related to the non-orderability of the sphere. Eliashberg and Polterovich in [EPP] defined a relation $\preceq$ on the universal cover of the identity component of the contactomorphism group, $\tilde{\text{Cont}}_0(V, \xi)$: two contact isotopies satisfy $\{\phi\} \preceq \{\psi\}$ if and only if $\{\phi^{-1} \circ \psi\}$ is generated by a positive Hamiltonian function. This relation is always reflexive and transitive. If it is also anti-symmetric then it defines
a bi-invariant order on $\widetilde{\text{Cont}}_0(V, \xi)$ and the contact manifold $(V, \xi)$ is called **orderable**. Equivalently, according to [ElP Proposition 2.1.A], a contact manifold is orderable if there are no contractible loops of contactomorphisms generated by a strictly positive contact Hamiltonian. The first known family of non-orderable contact manifolds was given by Eliashberg, Kim and Polterovich [EKP Theorem 1.16]. They showed that an ideal contact boundary of a product $M \times \mathbb{C}^d$ of a Liouville manifold $M$ and $(\mathbb{C}^d, \omega = \frac{i}{2} \sum dz \wedge d\bar{z})$ is not orderable for $d \geq 2$ (see [EKP Section 1.5]). When $M$ is a point, ideal contact boundary of $\mathbb{C}^d$ is the standard contact sphere $S^{2d-1}$. Hence, the sphere $S^{2d-1}$ is non-orderable if $d \geq 2$. In the proof of their non-orderability result, Eliashberg, Kim and Polterovich build a positive contractible loop as a composition of certain contactomorphisms. One of these building blocks is, what they call, a distinguished contact isotopy. In the case of $S^{2d-1}$ they give explicit formula for this isotopy. Moreover Giroux in [Gi] gives another formula for a contact isotopy of $S^{2d-1}$ playing the same role in the construction of positive contractible loop. This contact isotopy displaces all pre-Lagrangian toric fibers in $S^{2d-1}$, $d \geq 1$. In Section 4 we examine displaceability of generic fibers of $S^1 \times S^{2d}$, with $d \geq 2$. These manifolds are ideal contact boundaries of $T^*S^1 \times \mathbb{C}^d$ hence they are non-orderable, when $d \geq 2$. We believe that also in this case all generic fibers are displaceable. We prove displaceability for a large collection of them.

A connection between orderability and non-displaceability of certain subsets of a contact manifold has already been studied. It was proved by Eliashberg and Polterovich that a contact manifold is orderable if it contains a pair with the stable intersection property ([ElP Theorem 2.3.A]), that is a pair $(L, K)$, where $L$ is pre-Lagrangian and $K$ is either pre-Lagrangian or Legendrian, such that stabilizations of $K$ and $L$ cannot be displaced from each other in $V \times T^*S^1$, the stabilization of $V$. For precise definitions see [ElP Section 2.2]. If a pre-Lagrangian $L$ paired with itself has stable intersection property, i.e. if it is stably non-displaceable, then it is non-displaceable. To the best of our knowledge, there are no known examples of non-displaceable compact pre-Lagrangians without stable self-intersection property. If these notions turn out to coincide, the theorem of Eliashberg and Polterovich would say that the existence of non-displaceable pre-Lagrangian implies orderability. Moreover, any compact toric contact manifold admitting a monotone quasi-morphism with a vanishing property is orderable (see [BZ Theorem 1.26]), and contains a non-displaceable pre-Lagrangian toric fiber (what can be proved using the same technique as Entov-Polterovich [EnP Theorem 2.1]). Based on our work and the above results, we conjecture

**Conjecture 1.2.** If a compact toric contact manifold is not orderable then all its pre-Lagrangian toric fibers are displaceable.

We point out that non-compact contact toric manifolds $\mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n}$ are orderable (see [Bl], [S]), even though all their pre-Lagrangian toric fibers are displaceable (this can be seen with similar methods as in Section 2). The above conjecture implies that no such compact example could exist.

**Remark 1.3.** We define a **stem** in a contact toric manifold analogously as Entov-Polterovich [EnP] in symplectic case: a pre-Lagrangian toric fiber is a stem if every other pre-Lagrangian
toric fiber is displaceable. In symplectic setting a stem, if exists, is unique and non-displaceable ([EnP]), whereas, in contact setting this is not necessarily true. In $S^{2d-1}, d \geq 2$, every pre-Lagrangian toric fiber is a stem and none of them is non-displaceable. Moreover, according to Conjecture 1.2, the same should be true for every non-orderable toric contact manifold. However, in the family of contact toric manifolds admitting a monotone quasi-morphism with the vanishing property, stems behave as in symplectic case - if a stem exists then it is unique and non-displaceable (see Corollary 1.19 and Corollary 1.11 in [BZ]).

We observe another difference in rigidity properties of symplectic and contact manifolds. Using a method of probes it can be shown that any compact toric symplectic manifold contains uncountably many displaceable Lagrangian toric fibers (see [McD],[ABM]). In contrast, a contact toric manifold does not need to contain any displaceable pre-Lagrangian toric fiber.

**Theorem 1.4.** Every pre-Lagrangian toric fiber in the toric contact manifold $T^d \times S^{d-1}, d \geq 2$, (co-sphere bundle of $T^d$), is non-displaceable.

In [EHS] Eliashberg, Hofer and Salamon, using Floer theory for contact manifolds, show that the graph of any non-vanishing closed 1-form on a manifold is a non-displaceable pre-Lagrangian submanifold in a cosphere bundle over that manifold. Theorem 1.4 is proved by observing that every pre-Lagrangian toric fiber in $T^d \times S^{d-1}$ is a graph of such a form on $T^d$.

**Remark 1.5.** A stronger result holds: any pre-Lagrangian toric fiber in $T^d \times S^{d-1}, d \geq 2$, has a stable intersection property (see [EHS Example 2.4.A]), that is, it is stably non-displaceable. Due to [EHS Theorem 2.3.A] it follows that $T^d \times S^{d-1}, d \geq 2$, is orderable.

Note that the toric action on $T^d \times S^{d-1}$ is a free action. We believe that the non-existence of displaceable toric fibers is related to the existence of a free action. Toric action on a compact symplectic toric manifold is never free. It is free only on the pre-image of the interior of the moment map image. Moreover one can easily see that even a Hamiltonian circle action on a compact symplectic toric manifold is never free. A Hamiltonian moment map, as any smooth function on a compact manifold, must attain its extrema. Non-degeneracy of symplectic form implies that Hamiltonian vector field vanishes at these points, thus they must be fixed under the circle action. In the contact setting, Hamiltonian vector field would only need to be in the kernel of $d\alpha$ at these points. All toric contact manifolds admitting a free toric action are listed in Lerman’s Classification Theorem [L1 Theorem 2.18]. Precisely, if toric action on a contact toric manifold $V$ is free and dim $V = 3$, then $V \cong T^2 \times S^1$ and a contact form at a point $(\theta_1, \theta_2, \theta) \in T^2 \times S^1 \cong (\mathbb{R}/\mathbb{Z})^3$ is $\cos(2\pi k\theta) d\theta_1 + \sin(2\pi k\theta) d\theta_2$, for some $k \in \mathbb{Z}_{\geq 1}$. When $k = 1$ we obtain a co-sphere bundle of $T^2$ and thus by Theorem 1.4 all pre-Lagrangian toric fibers are non-displaceable. When $k > 1$ pre-Lagrangian toric fibers are not connected - they have exactly $k$ components: $T^2 \times \{ e^{2\pi i (\theta + \frac{j}{k})} \}, j = 0, \ldots, k - 1$. If dim $V = 2d - 1 > 3$ then $V$ is a principal $T^d$-bundle over the sphere $S^{d-1}$ with the unique toric contact structure. Since principal $T^d$-bundles over a manifold are in one-to-one correspondence with the second cohomology class of the manifold with coefficients in $\mathbb{Z}^d$ and since $H^2(S^{d-1}, \mathbb{Z}^d) = 0$ for $d-1 \neq 2$, it follows that when $d > 3$ this bundle is the trivial bundle $T^d \times S^{d-1}$. Thus, by Theorem 1.4 all pre-Lagrangian toric fibers are non-displaceable.
Hence the only toric contact manifolds with a free toric action that are not included in the statement of Theorem 1.4 are \((T^3, \ker(\cos(2\pi k\theta)\,d\theta_1 + \sin(2\pi k\theta)\,d\theta_2)), k > 1\) and non-trivial \(T^3\)-bundles over \(S^2\). It would be interesting to see if in these cases all pre-Lagrangian toric fibers are non-displaceable.

**Remark 1.6.** In Section 2 we observe that most of the contact toric manifolds whose toric action is not free have displaceable pre-Lagrangian fibers. Here by “most” we mean any toric contact manifold of Reeb type (see Corollary 2.13) and any toric contact manifold not of Reeb type whose moment cone does not contain any linear subspace of dimension one less than the dimension of the torus (see Proposition 2.14). The only contact toric manifolds with non free toric action which are not included in the above list are \(T^{d-1} \times S^d, d \geq 2\).

**Organization.** In Section 2 we present some methods for proving displaceability of pre-Lagrangian fibers by analyzing prequantization map and contact reduction. Theorem 1.1 is proved in Section 3. In Section 4 we use a squeezing method of Eliashberg, Kim and Polterovich to displace certain pre-Lagrangians in \(S^1 \times S^{2d}, d \geq 2\). Section 5 is devoted to the proof of Theorem 1.4.

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2. Methods for displacing.

2.1. Prequantization of a toric symplectic manifold and displaceability. Let \((M, \omega)\) be a symplectic manifold with the integral cohomology class \([\omega]\). A prequantization of \((M, \omega)\) is a principal \(S^1\)-bundle \(\pi: V \to M\) with the first Chern class \([\omega]\). There is a connection 1-form \(\alpha\) on \(V\) such that \(\pi^*\omega = d\alpha\) and \((V, \xi = \ker\alpha)\) is a contact manifold (see [BW]). Moreover, the Reeb vector field \(R_\alpha\) for the contact form \(\alpha\) generates the \(S^1\)-action on \(V\) that makes \(V\) a principle bundle over \(M\). The orbits of the Reeb vector field \(R_\alpha\) are the fibers of the bundle and \(d\pi(R_\alpha) = 0\). If \(V\) is a prequantization of \((M, \omega)\) then for every finite subgroup \(\mathbb{Z}_k \subset S^1\) the quotient submanifold \(V/\mathbb{Z}_k\) is a prequantization of \((M, k\omega)\).

If \((M^{2n}, \omega)\) is a compact toric symplectic manifold then the \(T^n\)-action lifts to a contact \(T^n\)-action on \(V\). Together with the \(S^1\)-action given by the Reeb flow this gives a \(T^{n+1}\)-action on the prequantization \((V^{2n+1}, \xi)\) making it a contact toric manifold (see [L2]). Preimages, under \(\pi\), of \(T^n\)-orbits in \(M\) are \(T^{n+1}\)-orbits in \(V\). Thus for any Lagrangian toric fiber \(L' \subset M\), \(\pi^{-1}(L') \subset V\) is a pre-Lagrangian toric fiber. Moreover, the image of the contact moment map \(\Phi: SV \to (\mathbb{R}^n)^* \times (\mathbb{R})^*\) is a cone over the moment map image \(\Delta\) of \(M\), namely it is

\[
C = \{a(x, 1) \in (\mathbb{R}^n)^* \times (\mathbb{R})^* | x \in \Delta, a \in \mathbb{R}_{>0}\}.
\]

Symplectic reduction of \(SV\) with respect to the circle \(\{1\} \times S^1 \subset T^{n+1}\) taken at level 1 gives back the symplectic toric manifold \((M, \omega)\) (see [L2] Lemma 3.7).

**Example 2.1.** \((S^{2d-1}, \alpha_{st} = \frac{i}{2} \sum_{i=1}^{d} (z_id\bar{z}_i - \bar{z}_idz_i))\) is a prequantization of the smooth complex projective space \((\mathbb{C}P^{d-1}, \frac{1}{\pi} \omega_{FS})\) as the first Chern class of the Hopf fibration \(S^{2d-1} \to \mathbb{C}P^{d-1}\) is \(\frac{1}{\pi} \omega_{FS}\). Performing the above construction for the standard \(T^{d-1}\)-action on \((\mathbb{C}P^{d-1}, \omega_{FS})\), we equip the prequantization space \(S^{2d-1}\) with a toric \(T^d\)-action which turns
out to be the same action as the standard action of $T^d$ on $S^{2d-1}$ coming from the standard $T^d$-action on $\mathbb{C}^d$. For more details on the toric contact sphere see Section 3. Furthermore, the real projective space $\mathbb{R}P^{2d-1} = S^{2d-1}/\mathbb{Z}_2$ and the lens spaces $L_p^{2d-1} = S^{2d-1}/\mathbb{Z}_p, p \in \mathbb{N}$, with contact forms induced by $\alpha_{st}$ are prequantizations of $(\mathbb{C}P^{d-1}, \frac{1}{2} \omega_{FS})$ and $(\mathbb{C}P^{d-1}, \frac{1}{2} \omega_{FS})$ respectively. Hence, they are toric contact manifolds.

Now we examine a connection between non-displaceability in a toric symplectic manifold $(M, \omega)$ and in its prequantization $(V, \xi)$. By $\text{Ham}(M, \omega)$ we denote Hamiltonian isotopies on $(M, \omega)$ and by $\text{Cont}_0(V, \xi)$ contact isotopies on $V$ starting at identity.

**Lemma 2.2. (Lifting property for prequantization)** Let $\varphi \in \text{Ham}(M, \omega)$. Then, there is $\tilde{\varphi} \in \text{Cont}_0(V, \xi)$ such that $\pi \circ \tilde{\varphi} = \varphi \circ \pi$.

**Proof.** Let $\varphi_t$ be a Hamiltonian isotopy such that $\varphi = \varphi_1$. Let $h_t : M \to \mathbb{R}$ be the corresponding time dependent Hamiltonian. That is

$$X_{h_t, \omega} = -dh_t,$$

where $X_{h_t}$ is a Hamiltonian vector field on $M$ generated by contact isotopy $\varphi_t$. Let us define $\tilde{h}_t : V \to \mathbb{R}$ by $\tilde{h}_t = h_t \circ \pi$. Note that $d\tilde{h}_t(R_{\alpha}) = 0$. Let $\tilde{\varphi}_t$ be a contact isotopy generated by $\tilde{h}_t$. $\tilde{\varphi} = \tilde{\varphi}_1$ has desired properties. Indeed, let $X_{\tilde{h}_t}$ be a vector field on $V$ whose flow is $\tilde{\varphi}_t$. We prove

$$d\pi \circ X_{\tilde{h}_t} = X_{h_t} \circ \pi$$

by showing that $d\pi \circ X_{\tilde{h}_t}$ defines a Hamiltonian vector field on $\pi(V) = M$ corresponding to $h_t$. Let $u \in TM$ be any vector. Since $d\pi : TV \to TM$ is surjective map there is a vector $\tilde{u} \in TV$ such that $d\pi(\tilde{u}) = u$. It follows that

$$\omega(d\pi(X_{\tilde{h}_t}), u) = \omega(d\pi(X_{\tilde{h}_t}), d\pi(\tilde{u})) = d\alpha(X_{\tilde{h}_t}, \tilde{u})$$

$$= dh_t(R_{\alpha})\alpha(\tilde{u}) - d\tilde{h}_t(\tilde{u}) = -d\tilde{h}_t(\tilde{u}) = -dh_t(u).$$

Since Hamiltonian vector field corresponding to $h_t$ is unique it follows that $d\pi \circ X_{\tilde{h}_t} = X_{h_t} \circ \pi$. By uniqueness of the flow it follows that $\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi$. \hfill $\square$

What follows is an analogue of the result of Abreu and Macarini on preserving non-displaceability under symplectic reduction. For the proof see [AM1, Proposition 3.2].

**Proposition 2.3.** Let $\varphi \in \text{Ham}(M, \omega)$ and let $\tilde{\varphi} \in \text{Cont}_0(V, \xi)$ be the lift of $\varphi$ given by the previous lemma. Let $L \subset M$ be any Lagrangian toric fiber and $\tilde{L} = \pi^{-1}(L) \subset V$ corresponding pre-Lagrangian toric fiber. If $\varphi(L) \cap \tilde{L} = \emptyset$ then $\tilde{\varphi} (\tilde{L}) \cap \tilde{L} = \emptyset$. Equivalently, if $\tilde{L} \subset V$ is non-displaceable then $L = \pi(\tilde{L}) \subset M$ is non-displaceable.

**Remark 2.4.** Using McDuff’s method of probes ([McD]) one can show that any compact toric symplectic manifold contains uncountably many displaceable Lagrangian toric fibers. Therefore a contact manifold whose all pre-Lagrangian toric fibers are non-displaceable (like $T^n \times S^{n-1}$) cannot be a prequantization of any compact toric symplectic manifold.

**Corollary 2.5.** If $L \subset M$ is a stem, then $\pi^{-1}(L) \subset V$ is a (contact) stem.
Remark 2.6. In [BZ] Borman and Zapolsky showed that a prequantization of any even monotone toric symplectic manifold (with the symplectic form scaled appropriately) admits a monotone quasimorphism with the vanishing property. Therefore, by Remark 1.3 the stem in such prequantization is unique and non-displaceable.

Example 2.7. The central fiber in $\mathbb{CP}^{d-1}$, Clifford torus

$$T_{\mathbb{CP}} = \{[z_1, \ldots, z_d] \in \mathbb{CP}^{d-1} | |z_1|^2 = \cdots = |z_d|^2\}$$

is proved to be non-displaceable (Cho-Poddar [CP]). On the other hand, using method of probes, introduced by McDuff [McD], it can be shown that all other Lagrangian toric fibers in $\mathbb{CP}^{d-1}$ are displaceable. Therefore the Clifford torus is a stem. By Corollary 2.5 the pre-images of the Clifford torus in $S^{2d-1}$, $\mathbb{RP}^{2d-1}$ and $I^{2d-1}_p = \mathbb{RP}^{2d-1}/\mathbb{Z}_p$ (under prequantization map) are also stems. However their displaceability properties are very different. The real projective space admits a monotone quasimorphism with the vanishing property and therefore its stem is non-displaceable. It is expected, though not yet proved, that all lens spaces also admit such quasimorphism. That would imply that preimage of Clifford torus in any $I^{2d-1}_p$ is non-displaceable. In the case of $S^{2d-1}$ preimage of Clifford torus is displaceable as shown in Theorem 1.4 (even though $S^{2d-1}$ does not admit a monotone quasimorphism with the vanishing property).

2.2. Contact reduction of a toric contact manifold and displaceability. We first recall a notion of a contact reduction. Suppose a Lie group $G$ acts on a contact manifold $(\tilde{V}, \tilde{\xi}, \tilde{\alpha})$ preserving $\tilde{\alpha}$. Let $\mu_G : \tilde{V} \to g^*$ denote the corresponding moment map. Suppose that $0 \in g^*$ is a regular value and that $G$ acts freely and properly on the level $\mu_G^{-1}(0)$. Then $V = \mu_G^{-1}(0)/G$ is a contact manifold. Precisely, if $\rho : \mu_G^{-1}(0) \to \mu_G^{-1}(0)/G$ is the projection, then there is a contact form $\alpha$ on $V$ such that $\rho^*\alpha = \tilde{\alpha}$ on $\mu_G^{-1}(0)$.

Moreover, if $V^{2d+1}$ is a toric contact manifold with a toric $T^{d+1}$-action and a moment map $\mu_{\tilde{\alpha}}$ and if an action of $G \cong T^k, k < d+1$ is a subaction of the toric action, then $V$ is also a toric contact manifold, with the toric action of $T^{d+1-k} \cong T^{d+1}/T^k$ and with a moment map $\mu_{\alpha}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mu_G^{-1}(0) & \xrightarrow{\mu_{\tilde{\alpha}}} & \Delta \subset (\mathbb{R}^{d+1})^* \\
\rho \downarrow & & \downarrow \iota^* \\
V & \xrightarrow{\mu_{\alpha}} & \Delta \subset (\mathbb{R}^{d+1-k})^*
\end{array}$$

where the map $\iota^*$ on the right is the dual of the map $\mathbb{R}^{d+1-k} \hookrightarrow \mathbb{R}^{d+1}$ induced by inclusion $T^k \hookrightarrow T^{d+1}$. From the commutativity of the diagram we see that a $\rho$-pre image of any pre-Lagrangian toric fiber in $V$ is a pre-Lagrangian toric fiber in $\tilde{V}$.

Example 2.8. Consider the toric contact manifold $(S^1 \times S^2, \beta), d \geq 1$ where

$$S^1 \times S^2 = \{(e^{2\pi i \theta}, z_1, \ldots, z_d, h) \in S^1 \times \mathbb{C}^d \times \mathbb{R} | \sum_{j=1}^d |z_j|^2 + h^2 = 1\},$$
\[ \beta = \hbar d\theta + \alpha_{st}, \quad \alpha_{st} = \frac{1}{4} \sum_{j=1}^{d} \left( z_j d\overline{z}_j - \overline{z}_j dz_j \right), \]
and the toric \( T^{d+1} \)-action is given by
\[ (t_0, t_1, \ldots, t_d) \ast \left( e^{2\pi i\theta}, z_1, \ldots, z_d, h \right) \mapsto (t_0 e^{2\pi i\theta}, t_1 z_1, \ldots, t_d z_d, h). \]

This toric contact manifold is not of Reeb type (see Section 4). The standard toric contact sphere \((\mathbb{S}^{2d-1}, \alpha_{st})\) is the \( \mathbb{S}^1 \)-contact reduction of the toric contact manifold \((\mathbb{S}^1 \times \mathbb{S}^{2d}, \beta)\) where the circle \( \mathbb{S}^1 \) acts on \((\mathbb{S}^1 \times \mathbb{S}^{2d}, \beta)\) by
\[ t_0 \ast \left( e^{2\pi i\theta}, z_1, \ldots, z_d, h \right) \mapsto (t_0 e^{2\pi i\theta}, z_1, \ldots, z_d, h). \]

We now establish a connection in rigidity of fibers in a toric contact manifold \((\tilde{V}, \tilde{\xi}, \tilde{\alpha})\) and in its contact reduction \((V, \xi, \alpha)\). Since the proofs are analogue to the symplectic case we refer to [AMI, Section 3].

**Lemma 2.9.** (Lifting property for contact reduction) Let \( \varphi \in \text{Cont}_0(V, \xi) \). Then, there is \( \tilde{\varphi} \in \text{Cont}_0(\tilde{V}, \tilde{\xi}) \) such that \( \rho \circ \tilde{\varphi} = \varphi \circ \rho \).

**Proposition 2.10.** If \( L \subset V \) is a displaceable pre-Lagrangian toric fiber, then \( \tilde{L} = \rho^{-1}(L) \subset \tilde{V} \) is a displaceable pre-Lagrangian toric fiber.

**Example 2.11.** Example 2.8 shows that a sphere \( \mathbb{S}^{2d-1} \) is a contact reduction of \( \mathbb{S}^1 \times \mathbb{S}^{2d} \). Thus by Theorem 1.1 and the previous Proposition we get that any pre-Lagrangian toric fiber
\[ \tilde{L} = \{(e^{2\pi i\theta}, z_1, \ldots, z_d, 0) \in \mathbb{S}^1 \times \mathbb{S}^{2d} \mid |z_j| = c_j, \ j = 1, \ldots, d \} \]
is displaceable.

### 2.3. Toric contact manifolds of Reeb type.

Let \( V \) be a toric contact manifold. Denote by \( R_\alpha \) the Reeb vector field corresponding to \( T^d \)-invariant contact form \( \alpha \), that is the unique vector field such that
\[ R_\alpha \ast d\alpha = 0 \quad \text{and} \quad \alpha(R_\alpha) = 1. \]

Denote by \( R^t_\alpha \) its flow. For any point \( p \in V \) and any vector \( X \in \mathbb{R}^d \) we have
\[ \left. \frac{d}{dt} \right|_{t=0} \mu_\alpha(R^t_\alpha(p)), X \right) = \langle d\mu_\alpha(R_\alpha(p)), X \rangle = d\alpha(R_\alpha(p), X) = 0. \]

Thus \( \left. \frac{d}{dt} \right|_{t=0} \mu_\alpha(R^t_\alpha(p)) = 0 \) and \( \mu_\alpha(R^t_\alpha(p)) = \mu_\alpha(p) \), i.e the Reeb orbit is contained in the \( T^d \)-orbit. If in addition the Reeb vector field corresponds to some \( \eta \in t^d \) in the Lie algebra of \( T^d \) then we call \( V \) a toric contact manifold of Reeb type. For these manifolds one can always perturb contact form (keeping same contact structure) so that an integral multiple of the associated Reeb vector field corresponds to a lattice element \( \eta \in t^d_\mathbb{Z} \) and thus the Reeb flow generates a \( \mathbb{S}^1 \)-action which is subaction of the action of \( T^d \) ([BG]). Lerman proved that toric contact manifolds of Reeb type are classified by strictly convex good cones ([L1, Definition 2.17] and [L1, Theorem 2.18]). Note that any toric contact manifold that is a prequantization of a toric symplectic orbifold is of Reeb type. Moreover, every contact toric manifold of Reeb type can be presented (not uniquely) as a prequantization of some toric symplectic orbifold, as we explain below. Recall that toric symplectic orbifolds are classified by rational and simple polytopes with positive integral labels attached to each facet ([L1]).

The points in the preimage of facet with label \( m \) have neighborhood modeled on \( \mathbb{C}^d / \mathbb{Z}_m \).
Let $V$ be any toric contact manifold and $C$ the corresponding good strictly convex cone. Denote by $v_1, \ldots, v_N \in \mathbb{Z}^d$ the primitive inward vectors normal to the facets of the cone $C$. That is

$$C = \bigcap_{i=1}^{N} \{x = (x_1, \ldots, x_d) \in (\mathbb{R}^d)^* | \langle x, v_i \rangle \geq 0 \}.$$ 

For any $R = \sum_{i=1}^{N} a_i v_i$ with $a_1, \ldots, a_N \in \mathbb{R}_{>0}$ there exists a $T^d$ invariant contact form $\alpha$ such that $R = R_\alpha$ is the Reeb vector field corresponding to it (see [AM2] Proposition 2.19). Take any such $R$ in the lattice of the Lie algebra of $T$, i.e. $R = \sum_{i=1}^{N} a_i v_i \in \mathbb{Z}^d$, $a_1, \ldots, a_N \in \mathbb{R}_{>0}$. The corresponding Reeb flow generates $S^1$-action on $V$ and one can perform symplectic reduction of the symplectization $SV$ with respect to the lift of that $S^1$-action. Result of this reduction is a symplectic orbifold $M$ and $V$ is a prequantization of $M$ ([Bo, Theorem 2.7]; see also [L2], Lemma 3.7.) The polytope corresponding to $M$ is the intersection of cone $C$ with a hyperplane perpendicular to $R$. Note that this intersection is always a compact polytope. Indeed, if the hyperplane is given by $h := \{x \in (\mathbb{R}^d)^* | \langle x, R \rangle = c \}$ then any $x \in C \cap h$ must satisfy

$$a_1 \langle x, v_1 \rangle + \cdots + a_N \langle x, v_N \rangle = c, \langle x, v_i \rangle \geq 0, \ldots, \langle x, v_N \rangle \geq 0.$$ 

As $a_j > 0$ and $v_1, \ldots, v_N$ generate the whole $\mathbb{R}^d$, the above condition implies that $0 \leq \langle x, v_j \rangle \leq \frac{c}{a_j}$ and $C \cap h$ is compact (it is empty if $c < 0$). The facets of the polytope of $M$ are intersections of the facets of $C$ with the hyperplane $h$. The label of a facet corresponding to the facet of $C$ with inward normal $v_j$ is the index of lattice generated by $v_j$ and $R$ inside $\text{span}_\mathbb{R}(v_j, R) \cap \mathbb{Z}^d$, where $\mathbb{Z}^d$ is the lattice of $\mathbb{R}^d = t^d$. If the collection $\{v_j, R\}$ can be completed to a $\mathbb{Z}$ basis of $\mathbb{Z}^d$, the label on the corresponding facet is 1.

**Lemma 2.12.** If $V$ is a toric contact manifold of Reeb type then $V$ can be presented as a prequantization of some symplectic toric orbifold having at least one label equal to 1.

**Proof.** Let $C$ be a good, strictly convex cone corresponding to $V$, with inward normals $v_1, \ldots, v_N \in \mathbb{R}^d$. Suppose the facets corresponding to normals $v_1, \ldots, v_{d-1}$ intersect in an edge. The assumption that the cone is good implies that the vectors $v_1, \ldots, v_{d-1}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^{d-1} = \text{span}_\mathbb{R}(v_1, \ldots, v_{d-1}) \cap \mathbb{Z}$. Without loss of generality we can assume that $v_1 = (1, 0, \ldots, 0)$, $\ldots$, $v_{d-1} = (0, \ldots, 0, 1, 0)$. We show that one can choose $a_j > 0$, $j = 1, \ldots, N$, so that $R = \sum_{j=1}^{N} a_j v_j = (c_1, \ldots, c_{d-1}, 1)$ or $R = (c_1, \ldots, c_{d-1}, -1)$ with each $c_j \in \mathbb{Z}$. In both cases the collection $\{v_1, \ldots, v_{d-1}, R\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. This implies that $V$ is a prequantization of a symplectic toric orbifold obtained as a reduction of $SV$ by the circle action generated by $R$, and that facets of this orbifold corresponding to normals $v_1, \ldots, v_{d-1}$ have labels 1.

For each $v_j$ let $(v_{j,1}, \ldots, v_{j,d})$ denote its coordinates. Suppose that at least one of $v_{d,d}, \ldots, v_{N,d}$ is positive. We now find constants $a_j > 0$ such that $R = \sum_{j=1}^{N} a_j v_j = (c_1, \ldots, c_{d-1}, 1)$. If all $v_{d,d}, \ldots, v_{N,d}$ were non positive we would choose $a_j > 0$, so that $R = (c_1, \ldots, c_{d-1}, -1) \in \mathbb{Z}^d$. Note that at least one of $v_{d,d}, \ldots, v_{N,d}$ is non-zero as $v_1, \ldots, v_N$ form a basis of $\mathbb{R}^d$. Take any $a_d, \ldots, a_N \in \mathbb{R}_+$ such that $\sum_{j=d}^{N} a_j v_{j,d} = 1$ (so also $\sum_{j=1}^{N} a_j v_{j,d} = 1$). For $a_j$, $j = 1, \ldots, d-1$, take any positive real number such that $a_j + \sum_{l=d}^{N} a_l v_{l,j}$ is an
integer. For example one can take \( a_j = -\sum_{l=1}^{N} a_l v_{l,j} \) if this sum was non-positive, and \( a_j = \left[ \sum_{l=1}^{N} a_l v_{l,j} \right] - \sum_{l=1}^{N} a_l v_{l,j} \) if this sum is positive. In both cases \( \sum_{l=1}^{N} v_{l,j} = a_j + \sum_{l=1}^{N} a_l v_{l,j} \) is an integer.

\( \square \)

**Corollary 2.13.** Any toric contact manifold \( V \) of Reeb type contains at least one displaceable pre-Lagrangian toric fiber.

**Proof.** McDuff’s method of probes for displacing Lagrangian fibers in symplectic toric manifolds (McD; see also ABM), though developed for toric manifolds, can be generalized to symplectic toric orbifolds as long as the probe starts at a facet with label 1. Using Lemma 2.12 method of probes and Proposition 2.3 we can displace (infinitely) many pre-Lagrangian toric fibers.

\( \square \)

2.4. **Toric contact manifolds not of Reeb type.** Toric contact manifolds not of Reeb type are the ones whose moment cones are convex but not strictly convex. According to Lerman’s Classification Theorem ([L1, Theorem 2.18]), if the toric action is free then these manifolds are diffeomorphic to \( T^3 \) (contact structure may vary) or to principal \( T^d \)-bundle over \( S^{d-1} \). Each such bundle has a unique invariant contact structure. If \( d > 3 \) the bundle must be trivial, so isomorphic to the co-sphere bundle of the torus and all the pre-Lagrangian toric fibers are non-displaceable by Theorem [L4]. Below we concentrate on the contact toric manifolds not of Reeb type whose action is not free and whose dimension is greater than 3. (Dimension 3 is exceptional. For example, the usual convexity and connectedness properties fail in dimension 3.) These are determined by their (not strictly convex) good moment cones. Let \( k \) be the dimension of the maximal linear subspace contained in the good moment cone, and \( (d+k), d \geq 0, \) be the dimension of the torus acting. Then such a cone is isomorphic to the cone of \( T^k \times S^{2d+k-1} \) and therefore the manifold must be equivariantly contactomorphic to \( T^k \times S^{2d+k-1} \) ([L1, Theorem 2.18]; see also the proof of Theorem 1.3 in Section 7).

**Proposition 2.14.** Contact toric manifolds \( T^k \times S^{2d+k-1} \), with \( d > 1 \), contain displaceable pre-Lagrangian toric fibers.

**Proof.** Write \( k = 2n + \delta, n \in \mathbb{N} \cup \{0\}, \delta = 0, 1 \) to keep track of parity of \( k \). Using the coordinates

\[
T^k \times S^{2d+2n+\delta-1} = \{(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_k}, z_1, \ldots, z_{d+n}, \delta \hat{h}) \in T^k \times \mathbb{C}^{d+n} \mid \sum_{j=1}^{d+n} |z_j|^2 + \delta \hat{h}^2 = 1 \}
\]

the contact structure is given as a kernel of the following contact form

\[
\beta_{2n+\delta} = \sum_{l=1}^{n} (x_l d\theta_{2l-1} + y_l d\theta_{2l}) + \delta \hat{h} d\theta_{2n+1} + \frac{i}{4} \sum_{j=1}^{d} (z_j d\overline{z}_j - \overline{z}_j dz_j),
\]

for \( z_{d+l} = x_l + iy_l, l = 1, \ldots, n \). We write \( (x, y) \) as an abbreviation for \( (x_1, y_1, \ldots, x_n, y_n) \). The toric \( T^{d+k} \)-action on \( T^k \times S^{2d+k-1} \) is given by

\[
(s_1, \ldots, s_k, t_1, \ldots, t_d) \ast (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_k}, z_1, \ldots, z_{d+n}, x, y, \delta \hat{h}) \mapsto (s_1 \cdot e^{-i\theta_1}, \ldots, s_k \cdot e^{-i\theta_k}, z_1 s_1^{-1}, \ldots, z_{d+n} s_k^{-1}, t_1 x s_1^{-1}, \ldots, t_d x s_k^{-1}, y, \delta \hat{h})
\]
A moment map with respect to the contact form $\beta_k$ is
\[
\mu_{\beta_k}(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_k}, z_1, \ldots, z_d, x, y, \delta h) = \pi(|z_1|^2, \ldots, |z_d|^2, x, y, \delta h).
\]
A pre-Lagrangian toric fiber in $T^k \times S^{2d+k-1}$ with respect to the toric action is any submanifold
\[
L = \{(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_k}, z_1, \ldots, z_d, p, q, \delta c) \in T^k \times S^{2d+k-1} | \sum_{j=1}^{d} c_j^2 + \sum_{l=1}^{n}(p_{d+l}^2 + q_{d+l}^2) + \delta c^2 = 1 \}\]
where $(p, q) = (p_{d+1}, q_{d+1}, \ldots, p_{d+n}, q_{d+n}) \in \mathbb{R}^n$, $c \in \mathbb{R}^n \cup \{0\}$ and $c_j \in \mathbb{R}^+, j = 1, \ldots, d$ are constants such that $\sum_{j=1}^{d} c_j^2 + \sum_{l=1}^{n}(p_{d+l}^2 + q_{d+l}^2) + \delta c^2 = 1$.

Reducing $(T^k \times S^{2d+k-1}, \beta_k)$ with respect to the first $k$ coordinates of $T^{d+k}$ we obtain the standard toric contact sphere $(S^{2d-1}, \alpha_{st})$. It follows that any fiber $L \subset T^k \times S^{2d+k-1}$ with $p = q = c = 0$ is displaceable. □

In fact, we show that more fibers in $(T^k \times S^{2d+k-1}, \beta_k)$ are displaceable. See Remark 4.8.

3. Proof of Theorem 1.1

We consider the standard contact sphere $(S^{2d-1}, \xi_{st}), d \geq 2$ where
\[
S^{2d-1} = \{(z_1, \ldots, z_d) \in \mathbb{C}^d | \sum_{j=1}^{d} |z_j|^2 = 1 \} \text{ and } \xi_{st} = T^d \cap JS^{2d-1}.
\]
The standard contact form is $\alpha_{st} = \frac{i}{4} \sum_{j=1}^{d} (z_j d\overline{z}_j - \overline{z}_j dz_j)|_{S^{2d-1}}$. The standard $T^d$-action on $S^{2d-1}$ defined by
\[
(t_1, \ldots, t_d) \ast (z_1, \ldots, z_d) \mapsto (t_1 z_1, \ldots, t_d z_d)
\]
makes the sphere $(S^{2d-1}, \xi_{st})$ a toric contact manifold. A moment map with respect to $\alpha_{st}$ is
\[
\mu_{\alpha_{st}}(z_1, \ldots, z_d) = \pi(|z_1|^2, \ldots, |z_d|^2).
\]
Hence, a pre-Lagrangian toric fiber with respect to this action is any submanifold
\[
L = \{(z_1, \ldots, z_d) \in S^{2d-1} | |z_j| = c_j, j = 1, \ldots, d \}\]
where $c_j \in \mathbb{R}^+_+ \text{ are constants such that } 0 < c_j < 1 \text{ and } \sum_{j=1}^{d} c_j^2 = 1$.

As explained in Section 2 and, in particular, in Example 2.1 and Example 2.7 all fibers other than the central fiber $c_1 = \cdots = c_d$ we can displace using the Lifting property for a prequantization. In this Section we show that all fibers, including the central one, can be displaced using only one contact isotopy. The isotopy we are using originally appeared in Giroux’s paper [Gi], where he proved non-orderability of the sphere.

Consider the map $\tau_t : B^{2d} \rightarrow \mathbb{C}^d, t \geq 0$ defined by
\[
\tau_t(z_1, \ldots, z_d) = \frac{1}{\cosh t + z_1 \sinh t}(\sinh t + z_1 \cosh t, z_2, \ldots, z_d),
\]
For every \( t > 0 \) whose domain is
\[
B^{2d} = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid \sum_{j=1}^d |z_j|^2 = 1\},
\]
\[
cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}.
\]
Let us check that \( \tau_t \) is well defined, i.e. that \( \cosh t + z_1 \sinh t \not= 0 \) for every \( |z_1| \leq 1 \) and every \( t \geq 0 \). Indeed, if \( t = 0 \) the expression is equal to 1, so assume there is some \( z_1 = x + iy \) and \( t > 0 \) such that \( \cosh t + z_1 \sinh t = 0 \). That means that \( \cosh t + x \sinh t = 0 \) and \( y \sinh t = 0 \). Consider the function \( f(x) = \cosh t + x \sinh t \) on the domain \( |x| \leq 1 \), for a fixed constant \( t > 0 \). We have \( f(-1) = e^{-t} > 0 \) and \( f'(x) = \cosh t > 0 \), hence \( f(x) > 0 \) for all \( |x| \leq 1 \).

By straightforward calculation we check that \( \tau_t \) is a complex automorphism of the unit ball \( (B^{2d}, \sum \frac{dz_j \wedge d\overline{z}_j}{2}) \). Since every complex automorphism of the unit ball \( B^{2d} \subset \mathbb{C}^d \) restricts to a contactomorphism of its boundary, i.e. the standard contact sphere \( (S^{2d-1}, \alpha_{st}) \), it follows that \( \tau_t \) is a contactomorphism on \( S^{2d-1} \) for every \( t \geq 0 \). Since \( \tau_0 \) is identity map, \( \tau_t \) is a contact isotopy starting at identity.

Now take any pre-Lagrangian toric fiber \( L \) given by \( \alpha \). We claim that for \( t > 0 \) large enough
\[
L \cap \tau_t(L) = \emptyset.
\]
It is enough to show that \( |z_1| \neq 1 \) for every \( z_1 \in \mathbb{C} \) such that \( |z_1| = c_1 \) (recall that the parameter \( c_1 \in (0,1) \)). Write \( z_1 = x + iy \). Hence \( x^2 + y^2 = c_1^2 \) and
\[
|z_1|^2 = |\cosh t + (x + iy) \sinh t|^2 = (\cosh t + x \sinh t)^2 + (y \sinh t)^2 = \cosh^2 t + 2x \cosh t \sinh t + c_1^2 \sinh^2 t.
\]

For every \( t > 0 \) we consider the following linear function
\[
f_t(x) = \cosh^2 t + 2x \cosh t \sinh t + c_1^2 \sinh^2 t - 1
\]
whose domain is \( |x| \leq c_1 \). It follows that
\[
f_t'(x) = 2 \cosh t \sinh t = \frac{e^{2t} - e^{-2t}}{2} > 0, \quad \text{since} \quad t > 0.
\]
That is, \( f_t \) is a monotone function and \( f(x) > f(-c_1) \), for every \( x > -c_1 \). Next,
\[
f_t(-c_1) = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - e^{-2t}}{2} c_1 + \frac{e^{2t} - 2 + e^{-2t}}{4} c_1^2 - 1
\]
\[
= e^{2t} \left( \frac{1}{4} - \frac{1}{2} c_1 + \frac{1}{4} c_1^2 \right) + e^{-2t} \left( \frac{1}{4} + \frac{1}{2} c_1 + \frac{1}{4} c_1^2 \right) - \left( \frac{1}{2} + \frac{1}{2} c_1^2 \right)
\]
\[
= e^{2t} \left( \frac{c_1 - 1}{4} \right)^2 + e^{-2t} \left( \frac{c_1 + 1}{4} \right)^2 - \frac{1}{2} - \frac{1}{2} c_1^2.
\]

As \( c_1 \neq 1 \) one can take \( t > \frac{1}{2} (\ln 2 + \ln \left( \frac{c_1^2+1}{(c_1-1)^2} \right) \). For this choice of \( t \) we have that \( f_t(-c_1) > 0 \), hence \( f_t(x) > 0 \) for every \( |x| \leq c_1 < 1 \). We conclude that \( |\cosh t + z_1 \sinh t| > 1 \).

**Remark 3.1.** Note that while proving \( L \cap \tau_t(L) = \emptyset \) we used the fact that \( d > 1 \). In the case \( d = 1 \) the map \( \tau \) is a well defined contactomorphism of a circle. As the only pre-Lagrangian of \( S^1 \) is the whole \( S^1 \), it is trivially non-displaceable.
4. Displaceability in $S^1 \times S^{2d}$ and the Squeezing Method

This Section is another step toward proving our conjecture that non-orderable toric contact manifold have all pre-Lagrangian toric fibers displaceable. In what follows we analyze a contact toric manifold $(S^1 \times S^{2d}, \ker \beta)$, $d \geq 2$,

$$S^1 \times S^{2d} = \{(e^{2\pi i \theta}, z_1, \ldots, z_d, h) \in S^1 \times \mathbb{C}^d \times \mathbb{R} \mid \sum_{i=1}^d |z_i|^2 + h^2 = 1\},$$

$$\beta = h d\theta + \frac{i}{4} \sum_{j=1}^d (z_j d\bar{z}_j - \bar{z}_j dz_j) = h d\theta + \frac{1}{2} \sum_{j=1}^d (x_j dy_j - y_j dx_j),$$

where $z_j = x_j + iy_j$ (see Example 2.8). To simplify the notation we often write $x_j dy_j - y_j dx_j$, i.e. $\beta = h d\theta + \frac{1}{2}(x dy - y dx)$. The above manifold is non-orderable as it is an ideal contact boundary of $(T^*S^1 \times \mathbb{C}^d, dh \wedge d\theta + \sum_{j=1}^d dx_j \wedge dy_j)$ with $d \geq 2$ ([EKP], Theorem 1.16). We discuss displaceability of generic fibers. It is not known if the manifold $S^1 \times S^{2d}$, obtained from the above family when $d = 1$, is orderable.

A moment map with respect to the contact form $\beta$ is

$$\mu_{\beta}(\theta, z_1, \ldots, z_d, h) = \pi(|z_1|^2, \ldots, |z_d|^2, h).$$

The cone over the image of $\mu_{\beta}$ is $\mathbb{R} \times (\mathbb{R}_{\geq 0})^d$, thus it is a good cone according to definition in [L1, Definition 2.17] (but not strictly convex). Note that this definition of a good cone does not require strict convexity. Later the adjective "good" started to be used meaning good and strictly convex. Recall that contact toric manifolds with good and strictly convex moment cones are of Reeb type. Hence, $(S^1 \times S^{2d}, \ker \beta)$ is not of Reeb type.

Generic orbits of $T^{d+1}$ action are pre-Lagrangian submanifolds of the form

$$\{(e^{2\pi i \theta}, z_1, \ldots, z_d, c) \in S^1 \times S^{2d} \mid |z_j| = c_j, \ j = 1, \ldots, d\} \tag{4.1}$$

for some constants $c \in [0, 1], c_j \in (0, 1)$ such that $\sum_{j=1}^d c_j^2 + c^2 = 1$.

Propositions 4.1 and 4.2 below prove that fibers contained in $S^1 \times \{h = 0\}$ and the fibers in certain neighborhood of $S^1 \times \{h^2 = 1\}$ are displaceable if $d \geq 2$. If true, Conjecture 1.2 would imply that every pre-Lagrangian toric fiber in $(S^1 \times S^{2d}, \beta), d \geq 2$ is displaceable.

**Proposition 4.1.** There exists an open neighborhood of $S^1 \times \{h = 0\}$ such that all pre-Lagrangian toric fibers contained in that neighborhood are displaceable.

**Proof.** Recall that a contact reduction of $S^1 \times S^{2d}$ at level $h = 0$ is $S^{2d-1}$ (see Example 2.8). Displaceability of fibers in $S^1 \times \{h = 0\}$ is explained in Example 2.11. Proposition follows by observing that displaceability is an open property. □

**Proposition 4.2.** There exists a certain open neighborhood of $S^1 \times \{h^2 = 1\}$ such that all pre-Lagrangian toric fibers contained in that neighborhood are displaceable.

We prove this Proposition using the squeezing method. In [EKP] Eliashberg, Kim and Polterovich analyzed the contact manifold $(S^1 \times \mathbb{R}^{2d}, \ker(d\theta - \frac{1}{2}(x dy - y dx)))$, $d > 1$, where $\theta$ is the coordinate of $S^1 = \mathbb{R}/\mathbb{Z}$ and $(x_1, y_1, \ldots, x_d, y_d)$ are coordinates on $\mathbb{R}^{2d}$ and proved the following “squeezing” result.
Theorem 4.3. [EKP] [Theorem 1.3] There exists a compactly supported contact isotopy $\Phi_t$ of $(S^1 \times \mathbb{R}^d, \ker(d\theta - \frac{1}{2} (x dy - y dx)))$, $d > 1$, starting at identity, that maps the closure of the product of $S^1$ and an open ball $B(r_1) = \{(x,y) \in \mathbb{R}^d, \ |(x,y)|^2 < r_1^2 \}$ into $S^1 \times B(r_2)$ for any $r_1, r_2 < \sqrt{1/\pi}$.

Note the difference in notation: in [EKP] they label balls with their capacities, while we label them with their radii. This squeezing isotopy requires some extra room, for example, $S^1 \times B(1/\sqrt{k\pi})$ can be squeezed into itself by an isotopy with support in $S^1 \times B(\sqrt{k/\pi})$ for any $\rho > 1/(k-1)$, but not by an isotopy with support only in $S^1 \times B(1/\sqrt{(k-1)\pi})$, see [EKP] Theorem 1.5 and Remark 1.23.

Remark 4.4. Conjugating with a diffeomorphism of $S^1 \times \mathbb{R}^d$ changing orientation on the $S^1$ factor one can show that Theorem 4.3 holds also for the contact manifold $(S^1 \times \mathbb{R}^d, \ker(d\theta + \frac{1}{2} x dy - y dx))$.

Below we recall equivariant version of Gray Stability Theorem and Isotopy Extension Theorem that will be needed to prove Proposition 4.2.

Theorem 4.5. [LW] Theorem 5.1 (Equivariant Gray Stability Theorem.) Suppose a Lie group $G$ acts on a contact manifold $V$ and $\{\alpha_t\}, t \in [0,1]$ is a smoothly varying family of invariant contact forms. Suppose that $N \subset V$ is an invariant submanifold and that
\[
\frac{d}{dt}\alpha_t(x) = \frac{d}{dt}d\alpha_t(x) = 0
\]
for all $x \in N$. Then there is a family of equivariant diffeomorphisms $\{\psi_t\}$ of $V$ with $\psi_t|_N = Id$ and $\psi_t^*(\alpha_t) = f_t\alpha_0$ for all $t$ and for some family of positive functions $\{f_t\}$.

Theorem 4.6. [Ge] Theorem 2.6.12 (Isotopy Extension Theorem) Suppose that $f_t: (P, \xi_P) \hookrightarrow (V, \xi_V)$ is an isotopy of contact embeddings, i.e. $(f_t)_*\xi_P = \xi_V$ of a closed contact manifold $(P, \xi_P)$. Then, there is Hamiltonian isotopy $\tilde{\varphi}_t$ on $V$ such that $\tilde{\varphi}_t \circ f_0 = f_t$ on $P$.

To prove Proposition 4.2 we work in an isomorphic contact toric manifolds. We start with a lemma showing how we can change the 1-form $\beta$ to obtain a 1-form which is still contact and $T^d+1$-invariant.

Lemma 4.7. Consider a manifold $S^1 \times S^d$ with a 1-form $\beta_g$
\[
\beta_g = g(h) d\theta + \frac{1}{2} (x dy - y dx)
\]
where $g: [-1,1] \rightarrow [-1,1]$ is some smooth function. Then $(S^1 \times S^d, \ker \beta_g)$ is a contact manifold if and only if for all $h \in [-1,1]$
\[
2h g(h) + g'(h)(1 - h^2) \neq 0.
\]

Proof. $(S^1 \times S^d, \ker \beta_g)$ is a contact manifold if and only if $\beta_g \wedge (d\beta_g)^d$ is nowhere vanishing on $S^1 \times S^d$. Note that $d\beta_g = g' dh \wedge d\theta + \sum_{j=1}^d dx_j \wedge dy_j$, thus

\[
\beta_g \wedge (d\beta_g)^d = d! g(h) d\theta \wedge dx \wedge dy + \frac{1}{2} d! g'(h) \sum_{j=1}^d ((x_j dy_j - y_j dx_j) \wedge dx_k \wedge dy_k) dh \wedge d\theta,
\]
where $dx \wedge dy = \bigwedge_{j=1}^{d} dx_j \wedge dy_j$. As $h^2 + \sum_j (x_j^2 + y_j^2) = 1$ we have $hdh + \sum_j (x_j dx_j + y_j dy_j) = 0$. Therefore at points with $h \neq 0$ we have $dh = -\sum_j (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j})$ and

$$\beta_h \wedge (d\beta_h)^d = d! g(h) d\theta \wedge dx \wedge dy + \frac{1}{2} d! g'(h) \frac{1-h^2}{h} d\theta \wedge dx \wedge dy.$$ 

For these points $\beta_h \wedge (d\beta_h)^d \neq 0$ if and only if $2h g(h) + g'(h)(1-h^2) \neq 0$. At points where $x_j \neq 0$ one can substitute $dx_j = -\frac{h}{x_j} dh - \frac{y_j}{x_j} dy_j - \sum_{k \neq j} (\frac{x_k}{x_j} dx_k + \frac{y_k}{x_j} dy_k)$ and get that

$$\beta_h \wedge (d\beta_h)^d = (d! g(h) \frac{h}{x_j} + \frac{1}{2} d! g'(h) \frac{1-h^2}{x_j}) dy_j \sum_{k \neq j} dx_k \wedge dy_k \wedge dh \wedge d\theta,$$

thus for these points we also get that $\beta_h \wedge (d\beta_h)^d \neq 0$ if and only if $2h g(h) + g'(h)(1-h^2) \neq 0$. The same condition is obtained if $y_j \neq 0$.

**Proof of Proposition 4.2.** First we concentrate on the pre-Lagrangians in a neighborhood of $S^1 \times \{ h = -1 \}$. Fix $1 > \varepsilon > 0$ and a smooth, non-decreasing function $\delta = \delta_\varepsilon : [-1,1] \to [-1,1]$ such that

$$\delta(h) = \begin{cases} -1 & \text{if } -1 \leq h \leq -\varepsilon, \\ h & \text{if } 0 \leq h \leq 1, \end{cases}$$

and strictly increasing on $(-\varepsilon,0)$. Consider the smooth family of 1-forms

$$\beta_t := ((1-t)h + t\delta(h)) d\theta + \frac{1}{2} (x dy - y dx), \quad t \in [0,1].$$

All the forms $\beta_t$ are $T^{d+1}$-invariant and $\beta_0 = \beta$ on the set $N = S^1 \times \{ h = -1 \} \cup \{ h \geq 0 \}$.

By Lemma 4.7 they are contact if and only if

$$2h((1-t)h + t\delta(h)) + ((1-t) + t\delta'(h)) (1-h^2) \neq 0.$$ 

Note that $2h((1-t)h + t\delta(h)) \geq 0$ and is equal to 0 if and only if $h = 0$, while $((1-t) + t\delta'(h))(1-h^2) \geq 0$ and is equal to 0 if and only if $h = \pm 1$ or $h \leq -\varepsilon$ and $t = 1$. This shows that $\beta_t$ are contact forms. Let $\xi_t = \ker \beta_t$ denote the induced contact structure on $S^1 \times S^{2d}$.

Notice that the moment map for our $T^{d+1}$-action, with respect to the contact form $\beta_t$ is given by

$$\mu_t(\theta, z_1, \ldots, z_d, h) = \pi(|z_1|^2, \ldots, |z_d|^2, (1-t)h + t\delta(h)).$$

Applying Equivariant Gray Stability Theorem (Theorem 4.5) we obtain an isotopy of equivariant diffeomorphisms $\{ \psi_t \}$ of $S^1 \times S^{2d}$ with $\psi_t|_N = Id$ and $\psi_t^*(\beta_t) = f_t \beta$ for all $t$ and for some family of positive functions $\{ f_t \}$, so $(\psi_t)_*(\xi) = \xi_t$. The fact that $\psi_t$ is equivariant implies that the generic toric fibers are mapped to generic toric fibers.

Note that the contact manifold $(S^1 \times S^{2d}, \ker \beta_1)$ in the neighborhood of $S^1 \times \{ h = -1 \}$ “looks like” a neighborhood of $S^1 \times \{ 0 \}$ in the contact manifold $(S^1 \times \mathbb{R}^{2d}, \ker (d\theta - \frac{1}{2} (x dy - y dx)))$ considered by Eliashberg, Kim and Polterovich in [EKP] (see Theorem 4.3 above). Precisely, the natural embedding of the closure of $S^1 \times B(\sqrt{1-\varepsilon^2})$ into $(S^1 \times S^{2d}, \ker \beta_1)$ given by $(\theta, x, y) \mapsto (\theta, x, y, -\sqrt{1-\left(|(x, y)| \right)^2})$ is a contact embedding.

Let $\eta = \eta(\varepsilon), \eta > 0$, be such that $S^1 \times B(\sqrt{1-\eta^2})$ can be squeezed into itself inside $S^1 \times B(\sqrt{1-\varepsilon^2})$ by an isotopy $\Phi_\eta = \Phi_\eta^0$ which exists by the result of Eliashberg-Kim-Polterovich quoted here as Theorem 4.3. Theorem 4.6 implies that there exist an isotopy of
the whole \((S^1 \times S^{2d}, \ker \beta_1)\) restricting to \(\Phi_t\) on the image of the closure of \(S^1 \times B(\sqrt{1 - \epsilon^2})\). By a slight abuse of notation we still call this isotopy \(\Phi_t\).

The above isotopy displaces pre-Lagrangian toric fibers of \((S^1 \times S^{2d}, \ker \beta_1)\) which are of the form \(L^\eta = S^1 \times \{(z_1, \ldots, z_d, \eta) \in S^{2d} | |z_j| = c_j, j = 1, \ldots, d\}\) for some \(0 < c_j < 1\) with \(c_1^2 + \cdots + c_d^2 = 1 - \eta^2\). The isotopy \(\Psi_t \circ \Phi_t \circ \Psi_1\) displaces these pre-Lagrangian fibers of our original contact manifold \((S^1 \times S^{2d}, \ker \beta_0)\) that map under \(\Psi_1\) to \(L^\eta\) of above form. Varying \(\epsilon\) in \((0, 1)\) and \(\eta\) in appropriate range, we displace all pre-Lagrangians of \((S^1 \times S^{2d}, \ker \beta_0)\) in some open neighborhood of \(S^1 \times \{h = -1\}\). Moreover, in view of Remark 4.8, analogous construction to the one presented above performed in the neighborhood of \(S^1 \times \{h = 1\}\) allows to obtain Hamiltonian isotopy displacing pre-Lagrangian toric fibers in some open neighborhood of \(S^1 \times \{h = 1\}\). \(\square\)

**Remark 4.8.** Consider a toric contact manifold \((T^k \times S^{2d+k-1}, \beta_k)\) from Proposition 2.14. It is an ideal contact boundary of \((T^*S^1)^k \times \mathbb{C}^d\), hence non-orderable, when \(d \geq 2\). We expect all pre-Lagrangian toric fibers to be displaceable. Many of them can be displaced using contact reduction method. \((T^{k-1} \times S^{2d+k-2}, \beta_{k-1})\) is the \(S^1\)-contact reduction of \((T^k \times S^{2d+k-1}, \beta_k)\) where \(S^1\) acts on the last coordinate of \(T^k\). Every displaceable fiber in \(S^1 \times S^{2d}\) lifts to a displaceable fiber in \(T^2 \times S^{2d+1}\), which in turn lifts to a displaceable fiber in \(T^3 \times S^{2d+2}\). Continuing this process we obtain some displaceable fibers in \((T^k \times S^{2d+k-1}, \beta_k)\).

## 5. Proof of Theorem 1.4

**Cosphere bundle.** Let \(N\) be any smooth manifold. Take a cosphere bundle \(P_+ T^*N\), that is, a fiber bundle over \(N\) with a fiber over a point \(p \in N\) given by

\[
P_+(T^*_pN) = (T^*_pN \setminus \{0\})/_{\eta \sim c_0, c > 0},
\]

and with the structure group \(O(2d, \mathbb{R})\), where \(d = \dim N\). The Liouville form on \(T^*N\) descends to a contact form on \(P_+ T^*N\) making it a contact manifold. Its symplectization is \(T^*N\) without the zero section, \(S(P_+ T^*N) = T^*N \setminus \{0\}\). The graph of any closed 1-form on \(N\) is a Lagrangian submanifold of \(T^*N\). Take any nowhere zero closed 1-form \(\alpha\) on \(N\). Then, the graph of \(\alpha\) is a Lagrangian submanifold in \(T^*N \setminus \{0\}\). Denote it by \(\tilde{L}_\alpha\). If \(\pi: S(P_+ T^*N) \to P_+ T^*N\) is the projection from symplectization then \(L_\alpha = \pi(\tilde{L}_\alpha)\) is a pre-Lagrangian submanifold in \(P_+ T^*N\). Thus we conclude: to every nowhere zero closed 1-form \(\alpha\) on \(N\) corresponds a pre-Lagrangian submanifold in \(P_+ T^*N\).

Now let \(N\) be the torus \(T^d\) and take a cosphere bundle over \(T^d\) (see [EH, 2.4.1]). We have that \((P_+ T^*_p T^d \cong T^d \times S^{d-1}, \xi = \ker(\sum_{i=1}^{d} x_i d\theta_i))\) where

\[
T^d \times S^{d-1} = \{(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d}, x_1, \ldots, x_d) \in T^d \times \mathbb{R}^d | \sum_{i=1}^{d} x_i^2 = 1\}.
\]

A \(T^d\)-action on \((T^d \times S^{d-1}, \sum_{i=1}^{d} x_i d\theta_i)\) given by

\[
(t_1, \ldots, t_d) * (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d}, x) \to (t_1 e^{2\pi i \theta_1}, \ldots, t_d e^{2\pi i \theta_d}, x),
\]
where $x = (x_1, \ldots, x_d)$, is an effective action by contactomorphisms so it is a toric action. A moment map $\mu : T^d \times S^{d-1} \to (\mathbb{R}^d)^*$ is given by
\[
\mu(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d}, x_1, \ldots, x_d) = \pi(x_1, \ldots, x_d).
\]
Hence, every pre-Lagrangian toric fiber in $T^d \times S^{d-1}$ is given by $T^d \times \{p\}$, for some point $p \in S^{d-1}$.

Now, for any point $p = (p_1, \ldots, p_d) \in S^{d-1}$ take $\alpha = \sum_{i=1}^d p_i d\theta_i$. Being a nowhere zero closed 1-form on $T^d$ to $\alpha$ corresponds pre-Lagrangian $L_\alpha$ in $T^d \times S^{d-1}$, as explained above. Note that $L_\alpha = T^d \times \{p\}$. We conclude that every pre-Lagrangian toric fiber in $T^d \times S^{d-1}$ is an image under $\pi$ of a graph of a nowhere zero closed 1-form on $T^d$. From \[EH{}S\] Corollary 2.5.2 we see that every such a pre-Lagrangian is non-displaceable. This proves Theorem 1.4.

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