Cosmological Spinning Multi-‘Black-Hole’ Solution in String Theory

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We give a cosmological spinning multi-‘black-hole’ solution in the Einstein-Maxwell-Dilaton-Axion theory with a positive cosmological constant. This solution is the cosmological dilatonic Israel-Wilson-Perjes solution and describes the collision of several spinning ‘black-holes’.

A cosmological multi-black hole solution has been discovered by Kastor and Traschen (KT solution) in the Einstein-Maxwell theory with a positive cosmological constant \( \Lambda \) (EM-\( \Lambda \)). The solution describes collisions of several black holes (BHs) in asymptotically deSitter space-times. It is worth noting that the solution is the first exact solution such that describes BH-BH collision. It also provides the test of the cosmic censorship\(^{2,3}\) and cosmic no-hair conjectures\(^{4}\) in the inflationary universe\(^{5}\).

One may be greatly interested in the spinning version of the KT solution because the rotating cases are generic. Apart from the cosmological cases, the spinning multi-soliton solution was found by Israel & Wilson\(^{6}\) and Perjes\(^{7}\) (IWP solution) in asymptotically flat space-times. Unfortunately, the solution has naked singularities due to the force balance condition \( Q = M \). On the other hand, we can expect that there are cases without singularities in asymptotically deSitter space-times even if \( Q = M \). This expectation relies on the causal structure of the single Kerr-Newman-deSitter space-time. So we can obtain the physical advantage of the cosmological version because of no singularity.

The existence of the spinning multi-soliton solution is deeply related to the force balance up to the gravitational spin-spin interaction. The force cancellation has been confirmed for the IWP solution\(^{8}\). The present author and Gen also showed that the force cancellation holds between a probe particle and the Kerr-Newman-deSitter space-time\(^{9}\). This observation strongly encourages us to find a new cosmological spinning multi-black-hole solution, that is, the spinning version of the KT solution.

In this Letter, however, we will give an exact solution in the four dimensional Einstein-Maxwell-Dilaton-Axion(EMDA) theory with a positive \( \Lambda \) (EMDA-\( \Lambda \)) which describes a low energy string theory except for a positive \( \Lambda \). The construction of
the solution is motivated by the chiral null model\textsuperscript{10} in five dimensions. In the four dimensional EMDA theory without the cosmological constant, there is the dilatonic IWP solution\textsuperscript{11} where the force cancellation holds\textsuperscript{12}. Via a certain dimensional reduction the solution turns out to be embedded into the chiral null model in five dimensions\textsuperscript{10}. Therefore we realise that it is better that we think of an exact solution in five dimensions and we are bearing the chiral null model spirit in mind. By virtue of the spirit, it is more easy to handle the EMDA-$\Lambda$ theory than the EM-$\Lambda$ theory.

In the Einstein frame the action of the four dimensional EMDA-$\Lambda$ theory is

$$S_4 = \int d^4 x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} e^{-4\phi} H_{\mu\nu\sigma} H^{\mu\nu\sigma} - e^{2\phi} A \right],$$

(1)

where $\phi$, $F_{\mu\nu}$ and $H_{\mu\nu\sigma}$ are the dilaton, field strength of the electromagnetic and three form tensor fields, respectively. In this paper, $A$ is assumed to be a positive constant.

Firstly, we give the expression of the exact solution without the explanation of the derivation. The metric $g_{\mu\nu}$, vector potential $A_\mu$ and the three form $H_{\mu\nu\sigma}$ are

$$g_{\mu\nu} dx^\mu dx^\nu = a^2 \left[ -F(d\tau + \omega_i dx^i)^2 + F^{-1} \delta_{ij} dx^i dx^j \right]$$

(2)

$$A = \frac{F}{\sqrt{2}} (d\tau + \omega_i dx^i)$$

(3)

and

$$H_{\mu\nu\sigma} = 24 A_{[\mu} \partial_{\sigma]} A_{\nu]},$$

(4)

where

$$F^{-1} = 1 + \text{Re} \left[ \sum_i \frac{M_i}{a^2 |\mathbf{x} - \mathbf{x}_i|} \right]$$

(5)

$$\omega = \text{Re} \left[ \sum_i \frac{N_i}{a^2 |\mathbf{x} - \mathbf{x}_i|} \right]$$

(6)

and the relation between $\omega_i$ and the above harmonic function $\omega$ is given by

$$\epsilon_{ijk} \partial_j \omega_k = \partial_i \omega.$$  

(7)

$M_i$ and $N_i$ are real parameters. $\mathbf{x}_i$ are complex constant vectors. The dilaton field is

$$e^{2\phi} = \frac{F}{a^2}.$$  

(8)

$a$ is the function of the time coordinate $\tau$ so that

$$a(\tau) = e^{H_0 \tau},$$

(9)
where \( H_0 := \pm \sqrt{\Lambda/2} \). This solution apparently induces the non-cosmological dilatonic IWP solution in the limit of \( H_0 = 0 \). It also does the non-rotating dilatonic KT solution in the limit of \( N_i = 0 \) and the real \( x_i \).

We can directly check that Eqs. (2)−(9) satisfy the field equations derived from the action of Eq. (1). However, as we said, our construction is inspired by the chiral null model in five dimensions. So we will guide our trace here. The action (Eq. (1)) can be obtained from the five dimensional theory as follows. The five dimensional low energy string action is given by

\[
S_5 = \int d^5x \sqrt{-G} \left[ R_G + 4 \nabla_M \phi \nabla^M \phi - \frac{1}{12} H^2 - \Lambda \right] e^{-2\phi},
\]

where \( M = y, 0, 1, 2, 3 \), \( H^2 = H_{IJK}H^{IJK} \) and \( H_{IJK} = 3 \partial_i B_{JK} \). In the Kaluza-Klein-like manner as

\[
G_{MN}dx^M dx^N = (dy + \sqrt{2}A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu,
\]

we obtain the four dimensional action

\[
S_4 = \int d^4x \sqrt{-\hat{g}} \left[ \hat{R} + 4 \hat{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{12} \hat{g}^{\alpha\beta} \hat{g}^{\rho\sigma} H_{\mu\alpha\rho} H_{\nu\beta\sigma}
\]

\[\]

\[
+ \hat{g}^{\mu\nu} \hat{g}^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \Lambda \right] e^{-2\phi},
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( H_{\mu\nu\sigma} = 3 \partial_{[\mu} B_{\nu\sigma]} + 24 A_{[\mu} \partial_{\nu]} A_{\sigma]} \) and we assumed that all of fields do not depend on the coordinate \( y \). Finally, rewriting the action in the Einstein frame where we can move to there by the conformal transformation, \( \hat{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu} \), we obtain the action of Eq. (1).

Inspired by the chiral null mode, we can set

\[
\hat{g}_{\mu\nu} dx^\mu dx^\nu = -F^2 (d\tau + \omega_i dx^i)^2 + \delta_{ij} dx^i dx^j
\]

\[
A = \frac{F}{\sqrt{2}} (d\tau + \omega_i dx^i)
\]

\[
B_{\mu y} = -\sqrt{2} A_\mu
\]

\[
B_{\mu\nu} = 0.
\]

In fact, in a certain case, the chiral null model form of the Lagrangian for the string theory becomes

\[
L = F \partial u [\partial_v + F^{-1} \partial_u + 2 \omega_i \partial x^i] + \partial x^i \partial x_i
\]

\[
= -F^2 (\partial \tau + \omega_i \partial x^i)(\partial \tau + \omega_i \partial x^i) + (\partial y + F \partial \tau + F \omega_i \partial x^i)(\partial y + F \partial \tau + F \omega_i \partial x^i)
\]

\[
+ F (\partial y \partial \tau + \partial \tau \partial y) + F \omega_i (\partial y \partial x^i - \partial x^i \partial y) + \partial x^i \partial x_i,
\]
where \( \partial = \partial_+ \) and \( \bar{\partial} = \partial_- \). From the first to the second and third line, we set \( u = y \) and \( v = 2\tau \). Here \( \omega_i \) is assumed to satisfy \( \varepsilon_{ijk} \partial_j \omega_k = \partial_i \omega \), where \( \omega \) a harmonic function. Since we are considering the system with a positive cosmological constant and this system is not exactly string theory, we remind readers that the chiral null model gives just reference to manage the problem. In the actual chiral null model \( F \) does not depend on the coordinates \( y \) and \( \tau \). On the other hand, \( F \) does on \( \tau \) in the present study. Comparing the second line of Eq. (17) with the string action, \( L = (G_{MN} + B_{MN}) \partial X^M \partial X^N \), we see that Eqs. (13), (14), (15) and (16) together with Eq. (11) are plausible arranging.

Using Eqs. (11) and (13), the five dimensional metric of Eq. (11) becomes

\[
G_{MN} dx^M dx^N = dy^2 + 2F dyd\tau + 2F \omega_i dy dx^i + \delta_{ij} dx^i dx^j. \tag{18}
\]

Equations which we should solve are derived by the variational principle of the action (10),

\[
\partial_I (\sqrt{-G} e^{-2\phi} H^{IJK} ) = 0 \tag{19}
\]

\[
\nabla_M \nabla^M \phi - \nabla_M \phi \nabla^M \phi + \frac{1}{4} \left( R_G - \frac{1}{12} H^2 - \Lambda \right) = 0 \tag{20}
\]

and

\[
R_{MN} - \frac{1}{2} G_{MN} R_G + 2 \nabla_M \nabla_N \phi - 2G_{MN} \nabla_K \nabla^K \phi + 2G_{MN} \nabla_K \phi \nabla^K \phi - \frac{1}{4} \left( H_M^{IJ} H_{NIJ} - \frac{1}{6} G_{MN} H^2 \right) + \frac{1}{2} G_{MN} \Lambda = 0. \tag{21}
\]

Substituting Eq. (20) into Eq. (21) the Einstein equation is simplified as

\[
R_{MN} + 2 \nabla_M \nabla_N \phi - \frac{1}{4} H_M^{IJ} H_{NIJ} = 0. \tag{22}
\]

First of all, we assume

\[
e^{2\phi} = \frac{F}{a^2}, \quad a(\tau) = e^{H_0 \tau} \tag{23}
\]

and

\[
\partial_\tau (a^2 F^{-1}) = 2H_0 a^2. \tag{24}
\]

thanks to the dilatonic KT solution \(^{13,14}\). Eq. (19) yields

\[
\partial_\tau (a^2 \omega_i) = 0 \tag{25}
\]

\[
\partial_i \partial^i F^{-1} = 0 \tag{26}
\]

and

\[
\partial_\tau (a^2 \partial_i F^{-1}) = 0 \tag{27}
\]
From Eq. (26) we can see that $F^{-1}$ is a harmonic function. Eqs. (25)~(27) have the solution with Eqs. (5) and (6). We can check in straightforward way that Eqs. (5) and (6)satisfy the rest equations (20) and (22). As a result, we obtain the solution of Eqs.(2)~(9) in the Einstein frame.

In the same way as the non-rotating dilatonic KT solution, we can see that the solution given by us here, the rotating dilatonic KT solution, has the timelike singularities which surrounds solitons. This is the reason why we say dashed ‘Black Hole’ in the title and the abstract of this paper. Rigorously speaking, the solution does not describe black holes. However, this type might be interpreted as a non-singular solution in a higher dimensions as Gibbons, Horowitz and Townsend have done.

In this Letter, for simplicity, we considered the EMDA-$\Lambda$ theory where we can be bearing the chiral null model spirit in mind. And now we could find the spinning version of the dilatonic KT solution. Thus, we surely expect that the spinning non-dilatonic solution exists as well as our present solution. We may think that the spinning version of the KT solution in the EM-$\Lambda$ theory is free from naked singularities because the KT solution in the pure Einstein-Maxwell theory is like that. Aided the study of the force balance up to the gravitational spin-spin interaction, we think that the present work can encourage us to find a spinning version of the KT solution in the EM-$\Lambda$ theory. Hopefully, we would like to discover the exact solution in near future. At the same time, the comprehensive features such as the global structure and the symmetry like supersymmetry should be reported together with the features of the present rotating dilatonic KT solution if there is.

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