Submanifold Differential Operators in \( \mathcal{D} \)-Module Theory I: Schrödinger Operators

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Abstract.
For this quarter of century, differential operators in a lower dimensional submanifold embedded or immersed in real \( n \)-dimensional euclidean space \( \mathbb{E}^n \) have been studied as quantum mechanical models, which are realized as restriction of the operators in \( \mathbb{E}^n \) to the submanifold. For this decade, the Dirac operators in the submanifold have been investigated in such a scheme, which are identified with operators of the Frenet-Serret relation for a space curve case and of the generalized Weierstrass relation for a conformal surface case. These Dirac operators are concerned well in the differential geometry, since they completely represent the submanifolds. In this and a future series of articles, we will give mathematical construction of the differential operators on a submanifold in \( \mathbb{E}^n \) in terms of \( \mathcal{D} \)-module theory and rewrite recent results of the Dirac operators mathematically. In this article, we will formulate Schrödinger operators in a low-dimensional submanifold in \( \mathbb{E}^n \).

MCS Codes: 32C38, 34L40, 35Q40

Key Words: Laplacian, Schrödinger operator, Submanifold, \( \mathcal{D} \)-Module

§1. Introduction
Recently it becomes recognized that the Dirac operators play important roles in geometry e.g., differential, algebraic, arithmetic geometry and so on. Pinkall gave an invited talk in the international congress of mathematicians in 1998 on the relation between immersed surfaces in three and/or four dimensional euclidean space \( \mathbb{E}^n \), \( (n = 3, 4) \) and Dirac operators, which was worked with Pedit [PP]. They constructed quaternion differential geometry and reduced the Dirac operators, which exhibit the geometrical properties of the surface. The Dirac operators of \( \mathbb{E}^n \) \( (n = 3, 4) \) also had been discovered by Konopelchenko in studies on geometrical interpretation of soliton theory [Ko1, 2, KT] and by Burgess and Jensen [BJ] and me [Mat3, Mat4] in the framework of the quantum physics. Further on case of \( \mathbb{E}^3 \), Friedrich obtained it by investigation of spin bundle [Fr].

Our Dirac operator is purely constructed in analytic category as we will show and is directly related to index theorems [Mat2, Mat3, TM]. Thus I believe that it is important to reformulate our works in the framework of pure mathematics and to translate them for mathematicians. In this and a future series of articles [II], I will mathematically formulate the canonical Schrödinger operator and Dirac operator on a submanifold in \( \mathbb{E}^n \). Indeed, there have appeared similar studies [DES, FH, RB] only on the Schrödinger operator case but it does not look enough to overcome several obstacle between physics and mathematics.

The submanifold quantum mechanics, which I called, was opened by Jensen and Koppe in 1971 [JK] and rediscovered by da Costa in 1982 [dC]. They considered a quantum particle confined in a subspace in our three dimensional euclidean space \( \mathbb{E}^3 \) which can be regarded as a low dimensional submanifold by taking a certain limit. (Since confinement of quantum particle into a subspace is realized in a certain case [DWH], their investigation is not so fictitious.) They found a canonical Laplacian by constructing the Schrödinger equation in the submanifold, which differs from the ordinary Beltrami-Laplace operator [dC, JK]: For a surface embedded in \( \mathbb{E}^3 \) case, the submanifold Laplacian \( \Delta_{S \hookrightarrow \mathbb{E}^3} \) is expressed by

\[-\Delta_{S \hookrightarrow \mathbb{E}^3} := -\Delta_S - (K - H),\]
where ΔS is a Beltrami-Laplace operator, K is Gauss curvature and H is the mean curvature of S. For a curve C in E^3 case, we have

\[-\Delta_{C\rightarrow E^3} := -\Delta_C - \frac{1}{4}k^2,
\]

where \(\Delta_C\) is a Beltrami-Laplace operator on C and \(k\) is curvature of the curve.

However submanifold quantum mechanics needs very subtle treatments. In fact, there are several different types of theories of quantum mechanics for submanifolds. For example, it is well-known that restriction of quantum particle can be performed using Dirac constraint quantization scheme [Dir2]. Let an equation \(f = 0\) represent a hypersurface in n-dimensional \(\mathbb{E}^n\). We can apply the Dirac constraint scheme [Dir2] to this system with a constraint \(f = 0\). Alternatively we can also deal with \(\dot{f} = 0\) constraint, where dot means derivative in time \([\text{INTT}]\). These results differ; \(\dot{f} = 0\) case agrees with the results of Jensen and Koppe [JK] and da Costa [dC] whereas \(f = 0\) case is very natural but \(\dot{f} = 0\) case is not. Since the results of Jensen and Koppe [JK] and da Costa [dC] connect with fruitful results such as the generalized Weierstrass equations as we will show in the introduction in [II], \(\dot{f} = 0\) case is very natural but \(f = 0\) case is not. In fact as a physical problem should be determined by local information, the constraint \(\dot{f} = 0\) consists only of local data whereas \(f = 0\) contains global information and is a fancy constraint from physical viewpoint [INTT, Mat1].

Accordingly when we make a theory of a submanifold quantum mechanics, we must pay many attentions on its treatment.

The self-adjoint operator is one of objects to need attentions. In order to show importance of self-adjoint operator rather than canonical commutation relation (or generating relation of Weyl algebra), let us consider a radial differential operator of polar coordinate in S. For a curve C in E^3 case, we have

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Even though, $\sqrt{-\nabla^2}$ is self-adjoint, $\sqrt{-\nabla^2}$ is not observable in general; the ordinary Schrödinger operator can not be expressed by the spectral decomposition using eigen vectors of operator $\sqrt{-\nabla^2}$ [Dirl]. (For example, radial momentum in hydrogen atom can not observed.) However by an appropriate confine potential with infinite height, we can confine a quantum particle in a thin ring or thin surface; mathematically speaking, we can impose a Dirichlet boundary condition and restrict support of functions or the domain of the Schrödinger operator [dC, DES, JK, SM]. Then normal mode for the codimensional direction in the ordinary Schrödinger equation, which is expressed by (bilinear of) $\sqrt{-\nabla^2}$ is well-defined and $\sqrt{-\nabla^2}$ behaves as momentum operator. (In mesoscopic quantum mechanical system, normal mode can be observed as subband state [DWH].)

Let us take a squeezing limit so that thickness of subspace or support of functions is negligible and the subspace can be regarded as a lower dimensional submanifold $S$ itself. Then we can argue a system of differential operators defined over the subspace; a self-adjoint differential operator along the normal direction can be constructed similar to $\sqrt{-\nabla^2}$. By integrating the hamiltonian over the normal mode, we obtain a submanifold quantum mechanics. This procedure is similar to the techniques to get the Thom isomorphism using Berezin integration method [BGV, Y].

However it is not easy to justify such squeezing limit including Dirac operators using concept of function space or the domain of the Schrödinger operator [dC, DES, JK, SM]. Then normal mode, we obtain a submanifold quantum mechanics. This procedure is similar to the techniques to get the Thom isomorphism using Berezin integration method [BGV, Y].

In squeezing limit, we must evaluate divergence of eigenvalue of normal direction.

Thus in this article, I will make an attempt to reformulate it using $\mathcal{D}$-module theory. In fact, Sato said that to study noncommutative system sometimes needs an appropriate topology instead of ordinary topologies which are used in the operator algebra, e.g., weak topology in Hilbert space [S]. Further in the submanifold quantum mechanics, we need a restriction of the differential operator while the restriction is the most natural concept in the sheaf theory and $\mathcal{D}$-module theory is based upon the sheaf theory. Thus I believe that my attempt is more natural than others approaches to the submanifold quantum mechanics.

The $\mathcal{D}$-module theory is based on Hilbert space even though Duclos, Exner and Štovíček attempted for Schrödinger operators [DES]: In squeezing limit, we must evaluate divergence of eigenvalue of normal direction.

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The $\mathcal{D}$-module theory was began by Sato as a algebraic analysis [Bjo, K]. For a noncommutative ring $\mathcal{D}_{\mathbb{E}^n}$ of differential operators over $n$-dimensional parameter spaces and a given differential equation $Pu = 0$ for $P \in \mathcal{D}_{\mathbb{E}^n}$, let us consider a left $\mathcal{D}_{\mathbb{E}^n}$-module

$$\mathcal{M}_{\mathbb{E}^n} = \mathcal{D}_{\mathbb{E}^n}/\mathcal{D}_{\mathbb{E}^n} P.$$ 

Then the ring homomorphism of $\mathcal{M}_{\mathbb{E}^n}$ to a function space such as analytic function space $C^\infty_{\mathbb{E}^n}$ is ring isomorphic to solution space of the equation $Pu = 0$ [Bjo, Cou, HT]. For a more general case, above quotient space is replaced with a coherent module. In the $\mathcal{D}$-module theory, the differential operators on a submanifold have been studied in detail. However in these studies, our differential operators in submanifold quantum mechanics have not ever appeared as long as I know.

The algorithm to construct the submanifold quantum mechanics in $\mathcal{D}$-module theory for a hypersurface $S$ in $n$-dimensional euclidean space $\mathbb{E}^n$ is as follows.

1. We construct a quantum equation and its related $\mathcal{D}_{\mathbb{E}^n}$-module $\mathcal{M}_{\mathbb{E}^n}$ in $\mathbb{E}^n$ with the natural metric, e.g., for the free Schrödinger equation, $-\Delta_{\mathbb{E}^n} \psi = 0$, $\mathcal{M}_{\mathbb{E}^n} = \mathcal{D}_{\mathbb{E}^n}/(\mathcal{D}_{\mathbb{E}^n} (-\Delta_{\mathbb{E}^n}))$; Here $\mathcal{D}_{\mathbb{E}^n} := C^\infty_{\mathbb{E}^n}[\partial_1, \ldots, \partial_n], C^\infty_{\mathbb{E}^n}$ is complex valued analytic functions over $\mathbb{E}^n$ and $\Delta_{\mathbb{E}^n}$ is the Beltrami-Laplace operator in $\mathbb{E}^n$.

2. We embed (or immerse) a real analytic hypersurface $S$ in $\mathbb{E}^n$, which is given by an equation $f = 0$.

3. We find a local system along a tubular neighborhood $T_S$ of the submanifold $S$ and calculate the inverse image $\mathcal{M}_{T_S}$ of $\mathcal{M}_{\mathbb{E}^n}$ to $T_S$ and $\Delta_{T_S} := \Delta_{\mathbb{E}^n}|_{T_S}$ for $T_S \hookrightarrow \mathbb{E}^n$.

4. We find a self-adjoint operator $\sqrt{-\nabla_{\mathbb{E}^n}^S}$ along the normal direction of $S$ over an open set $U$ of $S$. 

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(5) We define a quotient module $\mathcal{M}_{S\to T_S}$ by an exact sequence,

$$0 \rightarrow \mathcal{M}_{T_S}|_S \xrightarrow{\nabla^S_A \perp} \mathcal{M}_{T_S}|_S S \rightarrow \mathcal{M}_{S\to E^n} \rightarrow 0.$$ 

(6) $\mathcal{M}_{S\to E^n}$ is a coherent $\mathcal{D}_S$-module related to submanifold quantum mechanics over the submanifold $S$. We will define a submanifold quantum mechanical operator $\Delta_{S\to E^n}$ by the exact sequence,

$$0 \rightarrow \mathcal{D}_S \xrightarrow{\Delta_{S\to T_S}} \mathcal{D}_S \rightarrow \mathcal{M}_{S\to E^n} \rightarrow 0.$$ 

This scheme does not need any limit procedure and avoid the disease of divergence. (5) and (6) can be rewritten as follows if you choose a local coordinate system,

$$(5')$$ We will define $\Delta_{S\to T_S} := \Delta_{T_S}|_S$ and $\Delta_{S\to E^n} := \Delta_{S\to T_S}|_{\sqrt{-1} \nabla^S_A \perp = 0}$ for a standard form of $\Delta_{S\to T_S}$, where all $\nabla^S_A \perp$ are put right side of each terms of $\Delta_{S\to T_S}$.

Physically speaking, these processes are naturally performed when we introduce the confinement potential along the submanifold with the same thin thickness [dC, JK].

Contents are as follows. In §2, we will quickly review $\mathcal{D}$-module theory and sheaf theory. In §3, we will define the adjoint operator using Hodge $\ast$ product, though it can be defined using extension in the cohomology theory. We will introduce the half-form and the self-adjoint momentum operator.

In §4, we will define the Schrödinger operator in a lower dimensional submanifold in $E^n$ and give theorems.

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§2. Foliation and $\mathcal{D}_M$-Module

Although we will not construct a theory in the category of differential geometry whose morphism is $(C^\infty)$-diffeomorphism, we are concerned with differential geometry and physical system rather than merely algebraic structure. Accordingly we will treat real analytic objects in this article.
Let \( M \) be a real \( n \)-dimensional analytic manifold without singularity and \( C^\omega_M \) is a structure sheaf of real analytic functions over \( M \); \((M, C^\omega_M)\) is an algebraized analytic manifold. A sheaf \( S \) (of sets) over a topological space \( X \) is characterized by a triple \((S, \pi, X)\) because it is defined so that there is a local homeomorphism \( \pi : S \to X \). In this article, we choose \( M \) as such a topological space \( X \). Economy of notations make us to denote \( S := (S, \pi, M) \) for abbreviation. Further we will write a set of (local) sections of \( S \) over an open set \( U \) of \( M \) as \( S(U) \) or \( \Gamma(S, U) \). Using the category equivalence between category of sheaves and category of complete presheaves (due to theorem 13.1 in [Mal1]), we will mix them here.

Notations 2.1 (Sheaves). [Mal1, Mal2, Bjo]

1. Complexification of \( C^\omega_M \) is denoted by \( C^\omega M \).
2. Let us denote a sheaf \( C_M \) consisting of a locally constant functions \( \mathbb{C} \) over \( M \). Similarly \( \mathbb{R}_M, \mathbb{Z}_M, \mathbb{Z}_2 M \) and so on.
3. \( 1_M \) means a unit element sheaf of multiplicative group sheaf \( C^\times_M \) of \( C_M \).
4. If \( E \) is a finite rank of locally free \( C^\omega_M \)-module, we call it a vector sheaf.
5. The tangent sheaf over \( M \), which is a vector sheaf, is written as \( \Theta^\omega_M := \text{Der}_M(C^\omega_M) \) := \( \Theta_M \otimes \mathbb{C} \).
6. The complex valued tangent sheaf is \( \Theta^\omega_M = \text{Der}_M(C^\omega_M) := \Theta_M \otimes \mathbb{C} \).
7. Let us denote the sheafification of homomorphism of \( A \)-module sheaves \( M \) and \( M' \) over \( M \) by \( \text{Hom}_A(M, M') \). Similarly the sheafification of endomorphism of \( A \)-module sheaves \( M \) over \( M \) by \( \text{End}_A(M) \).
8. A locally constant sheaf \( L_M \) over \( M \), called local system, is defined so that its stalks are finite dimensional real vector spaces \( \mathbb{R}^m \).
9. A group sheaf generated by the presheaf,

\[
U \to \text{GL}(n, C^\omega_M)(U),
\]

is denoted by general linear group sheaf \( \mathcal{GL}(n, C^\omega_M) \), where \( U \) is an open set of \( M \) and \( \text{GL}(n, C^\omega_M) \) is general linear group for \( C^\omega_M(U) \)-valued \( n \)-matrix. (p. 285 in [Mal1])

10. A sheaf homomorphism as group from \( \mathcal{GL}(n, C^\omega_M) \) to \( C^\times_M \) is denoted by \( \text{det} \). (p.294-p.295 in [Mal1])

Proposition 2.2.

For each point \( p \in M \), there exists an open neighborhood \( U_p \) around \( m \) with a local coordinate system \( \{x_i, \partial_i\}_{1 \leq i \leq n} \) satisfied with,

\[
x_i \in \Gamma(U, C^\omega_M), \quad \Theta_M(U) = \bigoplus_{i=1}^n \Gamma(U, C^\omega_M) \partial_i, \quad [\partial_i, x_j] = \partial_i x_j - x_j \partial_i = \delta_{ij}.
\]

Proof. see [Bjo] p.11 proposition 1.1.18 and p.17 remark under definition 1.2.2. \( \square \)
Definition 2.3. (1.2.8 in [Bjo])
In a chart, we write sections $\delta$ and $\delta'$ in $\Theta_M(U)$, $\delta = \sum_i f_i(x)\partial_i$, $\delta' = \sum_i g_i(x)\partial_i$, where $U$ is an open set of $M$, $f$'s and $g$'s are in $C^\omega_M(U)$. The commutator is defined by
$$[\delta, \delta'] = \sum_{u,v} (f_v\partial_v(g_u)\partial_u - g_u\partial_u(f_v)\partial_v), \quad [\delta, g] = \sum_{u,v} f_v\partial_v g.$$

Definition 2.4 (Differential Ring Sheaf). [Bjo, TH]
We will denote the subring sheaf $\mathcal{D}_M$ of $\text{End}_{\mathbb{R}_M}(C^\omega_M)$, which is generated by the complete presheaves $\Gamma(C^\omega_M, U)$ and $\Gamma(\Theta_M, U)$ and has local expression,
$$\Gamma(U, \mathcal{D}_M) = \bigoplus_{\alpha \in \mathbb{N}^n} \Gamma(U, C^\omega_M)\partial^\alpha,$$
where $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \in \Gamma(\Theta_M, U)$. We will define $\mathcal{D}_M^C$ as $\mathcal{D}_M \otimes \mathbb{C}$.

Remark 2.5 (Standard Form Representation). [Bjo]
The local expression is based upon the standard form representation, which is, for any $P \in \Gamma(U, \mathcal{D}_M)$, given as,
$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha, \quad a_\alpha \in C^\omega_M(U).$$

Definition 2.6 (Filter). [Bjo, TH]
Let us define the filter $F$ associated with rank as a sheaf morphism by complete presheaf generated by the sections for an open set $U$ of $M$,
$$(F_l\mathcal{D}_M)(U) = \sum_{|\alpha| \leq l, \alpha \in \mathbb{N}^n} \Gamma(U, C^\omega_M)\partial^\alpha,$$
where $|\alpha| = \sum_i \alpha_i$. Further for an open set $V$ of $M$, the filter $F_l \mathcal{D}_M$ is defined as
$$(F_l\mathcal{D}_M)(V) := \{ P \in \mathcal{D}_M(V) \mid \rho_{UV} P \in F_l(\mathcal{D}_M(U)), \ (U \subset V) \},$$
where $\rho_{UV}$ is a restriction associated with the complete presheaf $\mathcal{D}_M$.

Proposition 2.7.
(1) $F_l$ is increasing filter and defined over $M$;
$$\mathcal{D}_M = \bigcup_{m \in \mathbb{N}} F_l\mathcal{D}_M, \quad F_l\mathcal{D}_M \subset F_m\mathcal{D}_M, \text{ for } l < m.$$
(2) $F_0\mathcal{D}_M = C^\omega_M$, $(F_l\mathcal{D}_M)(F_m\mathcal{D}_M) = F_{l+m}\mathcal{D}_M$ as $C^\omega_M$-module.
(3) For $P \in F_l\mathcal{D}_M$ and $Q \in F_m\mathcal{D}_M$, $[P, Q] \in F_{l+m-1}\mathcal{D}_M$, where $F_l\mathcal{D}_M = 0$ for $l < 0$; symbolically
$$[(F_l\mathcal{D}_M), (F_m\mathcal{D}_M)] \subset F_{l+m-1}\mathcal{D}_M.$$
(4) We will define a symbol, $\text{gr}_l \mathcal{D}_M := \bigoplus_{m=0}^{l}(F_l\mathcal{D}_M)/(F_{l-1}\mathcal{D}_M)$. Then $\text{gr}_l \mathcal{D}_M$ is a commutative ring and is regarded as a commutative $C^*$ algebra.

Proof. [Bjo] p.17-p.18 and [TH] □
Definition 2.8 (Differential Form). (p.240 in [Mal1])
(1) We will define the duality \( \Omega^1_M := \mathcal{H}om(\Theta_M, \mathcal{C}^\omega_M) \) generated by \( \omega(\theta) \in \mathcal{C}^\omega_M(U) \) for \( \omega \in \Gamma(U, \Omega_M) \) and \( \theta \in \Gamma(U, \Theta_M) \) where \( U \) is an open set of \( M \).
(2) Let \( \Omega_M \) be a sheaf of differential form as a graded commutative ring generated by \( \Omega^1_M \),
\[
\Omega_M = \bigcup_{m \in \mathbb{Z}_{\geq 0}} \Omega^p_M,
\]
where \( \Omega^p_M \) is set of p-forms and \( \Omega^q_M = 0 \) (\( q > n \)) vanishes: \( \mathbb{Z}_{\geq 0} := \{ n \in \mathbb{Z} \mid n \geq 0 \} \).
(3) The exterior derivative is expressed by \( d : \Omega^p_M \to \Omega^{p+1}_M \).

Definition 2.9 (Lie Derivative). [TH,W]
For an open set \( U \) of \( M \) and \( \theta \in \Theta_M(U) \), the Lie derivative \( \text{Lie}_\theta \) is defined as follows:
(1) for \( f \in \Gamma(U, \mathcal{C}^\omega_M) \), \( \text{Lie}_{\theta f} = \theta f \).
(2) For \( \theta_1 \in \Theta_M(U) \) \( (a = 1, 2) \), \( \text{Lie}_{\theta_1} = \theta_1 \).
(3) For \( \theta_1 \in \Theta_M(U) \) \( (a = 1, 2) \) and \( \omega \in \Omega^1_M(U) \), \( \text{Lie}_{\theta_1}(\omega(\theta_2)) = \omega(\text{Lie}_{\theta_1}(\theta_2)) = \theta_1(\omega(\theta_2)) - \omega([\theta_1, \theta_2]) \).
(4) For \( \theta_1, \theta_2, \ldots, \theta_n \in \Theta_M(U) \), \( \text{Lie}_\theta(\omega(\theta_1, \theta_2, \ldots, \theta_n)) = \theta(\omega(\theta_1, \theta_2, \ldots, \theta_n)) = \sum_{i=1}^n \omega(\theta_1, \ldots, [\theta, \theta_i], \ldots, \theta_n) \).

Proposition 2.10.
For an open set \( U \) of \( M \), \( f \in \mathcal{C}^\omega_M(U) \) and \( \omega \in \Omega^1_M(U) \), \( \text{Lie}_\theta f = \text{Lie}_\theta f \omega = \text{Lie}_\theta f \).

Proof. Direct computation shows it [TH]. \( \square \)

Definition 2.11 (Right Action). ([TH], [Bjo])
The left action of \( \theta \) section of \( \Theta(U) \) for an open sets \( U \) of \( M \) is defined by
\[
\omega \cdot \theta := -\text{Lie}_\theta \omega,
\]
where \( \omega \in \Omega^1_M(U) \).

Definition 2.12 (Integrable Connection). (1.2.10 in [Bjo])
Let \( \mathcal{F} \) be \( \mathcal{C}^\omega_M \)-module. Put \( \mathcal{I}(\mathcal{F}) := \mathcal{H}om_{\mathcal{C}^\omega_M}(\mathcal{F}, \mathcal{F}) \). A global section \( \nabla \) of \( \mathcal{H}om_{\mathcal{C}^\omega_M}(\Theta_X, \mathcal{I}(\mathcal{F})) \) is called an integrable connection on \( \mathcal{F} \) if the following holds for any open set \( U \) of \( M \):
(1) For \( \alpha \in \mathcal{C}^\omega_M(U) \) and \( f \in \mathcal{F}(U) \), \( \nabla_\delta(f) = \delta \alpha \cdot f + \alpha \nabla_\delta(f) \).
(2) \( \nabla_{[\delta, \delta']} = [\nabla_\delta, \nabla_{\delta'}] \).
(3) \( \nabla \) is involutive, i.e., \( [\nabla_\delta, \nabla_{\delta'}] \) is a commutator in \( \mathcal{I}(\mathcal{F}) \).

Definition 2.13 (Category). (1.1.24 and 1.2.11 in [Bjo])
(1) Denote by \( \text{Mod}^L(\mathcal{D}_M) \) the category of left \( \mathcal{D}_M \)-module.
(2) Denote by \( \text{Mod}(\mathcal{C}^\omega_M) \) the category of \( \mathcal{C}^\omega_M \)-module.
(3) For is the forgetful functor from \( \text{Mod}^L(\mathcal{D}_M) \) to \( \text{Mod}(\mathcal{C}^\omega_M) \).
Proposition 2.14. (1.2.11 in [Bjo])
Consider the category $\mathcal{A}$ whose objects consist of pair $(N, \nabla)$, where $N \in \text{Mod}(\mathcal{C}_M^\omega)$ and $\nabla$ is an integral connection on $N$. Morphisms are defined as follows:

$$\text{Hom}_A((N, \nabla), (N', \nabla')) = \{ \varphi \in \text{Hom}_{\mathcal{C}_M^\omega}(N, N') \mid \nabla' \circ \varphi = \varphi \circ \nabla \}.$$ 

Then $\text{Mod}^L(D_M)$ and $\mathcal{A}$ are category equivalent by the functor $\mu : \text{Mod}^L(D_M) \to \mathcal{A}$ for which $\mu(M) = (\text{for}(M), \nabla)$ where $\nabla_\delta(m) = \delta(m)$ for every $\delta \in \Theta_M$ and $m \in M \in \text{Mod}^L(D_M)$.

Proof. [Bjo] p.19 Theorem 1.2.12. □

This means that we can find an object in $\text{Mod}^L(D_M)$ for any an integrable connection set $(\nabla, N)$.

Definition 2.15 (Horizontal Section). (1.3.7 in [Bjo])
The left annihilator of $1_M$ be denoted by.

$$\text{Ann}_{D_M}(1_M) := \{ P \in D_M \mid P(1_M) = 0 \}.$$ 

For every $M \in \text{Mod}^L(D_M)$, we introduce $\mathbb{R}_M$-module, called the sheaf of horizontal sections of $M$,

$$\text{hor}(M) := \text{Hom}_{D_M}(D_M/\text{Ann}_{D_M}(1_M), M).$$

Then we have

$$M \approx \mathcal{C}_M^\omega \otimes_{\mathbb{R}_M} \text{hor}(M), \quad D_M/\text{Ann}_{D_M}(1_M) \approx F_0D_M \approx \mathcal{C}_M^\omega.$$ 

Proposition 2.16 (Connections). (1.3.9 in [Bjo])

1. For a local system $\mathcal{L}_M$, the left $D_M$-module $\mathcal{C}_M^\omega \otimes_{\mathbb{R}_M} \mathcal{L}_M$ is denote by $\text{Con}(\mathcal{L}_M)$. Let a set of local systems $\mathcal{L}_M$ be $\mathcal{L}_M$ and $\text{Con}(D_M) := \{ \text{Con}(\mathcal{L}_M) \mid \mathcal{L}_M \in \mathcal{L}_M \}$. $\mathcal{L}_M$ and $\text{Con}(D_M)$ are category equivalent due to the relation; for $\mathcal{L}_M, \mathcal{L}_M' \in \mathcal{L}_M$

$$\text{Hom}_{D_M}(\text{Con}(\mathcal{L}_M), \text{Con}(\mathcal{L}_M')) = \text{Hom}_{\mathbb{R}_M}(\mathcal{L}_M, \mathcal{L}_M').$$

2. For a local system $\mathcal{L}_M$, there exists a correspondence,

$$\text{Hom}_{D_M}(D_M/\text{Ann}_{D_M}(\mathcal{L}_M), \mathcal{C}_M^\omega) = \mathcal{L}_M,$$

where

$$\text{Ann}_{D_M}(\mathcal{L}_M) := \{ P \in D_M \mid P(\mathcal{L}_M) = 0 \}.$$ 

Proof. [Bjo] p.22-p.23. □

If $\mathcal{L}_M$ is a subset of $\Theta_M$, this theorem is essentially the same as the Frobenius integrable theorem [Mal2]. The local system can be regarded as distribution in the terminology of differential geometry [W].
Let $S$ be a real analytic submanifold without singularity of the manifold $M$. For a point $s \in S$, there is an open neighborhood of $s$, there exist real analytic functions $Q_1, \ldots, Q_d \in \mathcal{C}_M^\omega$ such that

$$S \cap U = \{ s \subset U \mid Q_1(s) = \cdots = Q_d(s) = 0 \},$$

where $d = n - k$.

Let the natural embedding be expressed by $\iota_S : S \hookrightarrow M$. Then for a sheaf $\mathcal{F}$ over $M$, we will define $\mathcal{F}|_S := \iota_S^{-1}\mathcal{F}$.

**Definition 2.17 (Inverse Image of $\mathcal{D}_M$-module).**

The sheaf $\mathcal{D}_{S \to M}$ over $S$ is defined as $\mathcal{D}_{S \to M} = \mathcal{C}_S^\omega \otimes_{\mathcal{C}_M^\omega|S} (\mathcal{D}_M|_S)$.

If we will assign the local coordinate of open neighborhood of a point $s \in S \subset M$, $q^1 = q^2 = \cdots = q^d = 0$. Locally $\mathcal{D}_{S \to M}$ is expressed as,

$$\mathcal{D}_{S \to M} = \mathcal{D}_M|_S/(q^1(\mathcal{D}_M|_S) + q^2(\mathcal{D}_M|_S) + \cdots + q^d(\mathcal{D}_M|_S)).$$

Next we will introduce the Riemannian metric in sheaf theory along the line of the arguments of Mallios [Mal1,2].

**Definition 2.18 (Ordered Algebraized Space).** (Definition 8.1 in [Mal1])

Let $\mathcal{A}$ be a real ring sheaf whose local sections are real valued. Suppose the $(M, \mathcal{A})$ is algebraized space and $\mathcal{A}^+$ is a subsheaf of the real ring sheaf $\mathcal{A}$, whose local sections are positive real valued. $(M, \mathcal{A}, \mathcal{A}^+)$ is an ordered algebraized space, viz, for any local section $\lambda \in \mathbb{R}^0_M(U)$, $\lambda \mathcal{A}^+(U) \subset \mathcal{A}^+(U)$, $\mathcal{A}^+(U) + \mathcal{A}^+(U) \subset \mathcal{A}^+(U)$ and $\mathcal{A}^+(U)\mathcal{A}^+(U) \subset \mathcal{A}^+(U)$.

Then it is obvious that there exists an ordered algebraized space $(M, \mathcal{C}_M^\omega, \mathcal{C}_M^\omega +)$.

**Definition 2.19 (Inner Product Module).** (p.318 in [Mal1])

Suppose that $(M, \mathcal{A}, \mathcal{A}^+)$ is an order algebraized space and $\mathcal{E}$ is $\mathcal{A}$-module. We say that $(\mathcal{E}, g_\mathcal{E})$ has an $\mathcal{A}$-valued inner product and $(\mathcal{E}, g_\mathcal{E})$ is an inner product $\mathcal{A}$-module on $M$, if a sheaf morphism $g_\mathcal{E} : \mathcal{E} \oplus \mathcal{E} \to \mathcal{A}$ is satisfied following conditions:

1. $g_\mathcal{E}$ is a $\mathcal{A}$ bilinear morphism.
2. $g_\mathcal{E}$ is positive definite; for any local section $s \in \mathcal{E}(U)$ over an open set $U$, $g_\mathcal{E}(s, s) \in \mathcal{A}^+(U)$ such that $g_\mathcal{E}(s, s) = 0$ if and only if $s = 0$ in $\mathcal{E}(U)$.
3. $g_\mathcal{E}$ is symmetric; for any local two sections $s$ and $t$ of $\mathcal{E}(U)$ over an open set $U$, $g_\mathcal{E}(s, t) = g_\mathcal{E}(t, s)$.

**Definition 2.20 (Riemannian Metric).** (p.320 in [Mal1])

We say that an order algebraized space $(M, \mathcal{C}_M^\omega, \mathcal{C}_M^\omega +)$ has a Riemannian metric $(\Theta_M, g_M)$ as a sheaf morphism $g_M : \Theta_M \to \Omega^1_M$ if it is satisfied with following conditions:

1. The sheaf morphism $g_M : \Theta_M \to \Omega^1_M$ is $\mathcal{C}_M^\omega$-isomorphism.
Lemma 2.22. and the integrable connection of \( \Theta_M \) is denoted by

\[
\theta g_M(\theta_1, \theta_2) = g_M(\theta \theta_1, \theta_2) + g_M(\theta_1, \theta \theta_2).
\]

Here we will note strictly fine sheaf, which is used in the definition of the Riemannian module in [Mal1] in the category for sheaves of smooth functions \( C^\infty \). By using forgetful functor from the category of \( C^\infty \) to that of \( C^\infty_M \), it turns out that image of the functor is a subset of strictly fine sheaves [Mal1]. Hence above definition is not contradict with that in [Mal1].

Further we will define a morphism

\[
\tilde{g}_M : \Omega^1_M \to \Theta_M,
\]

as \( \tilde{g}_M \circ g_M = id\Theta_M \) and \( g_M \circ \tilde{g}_M = id\Omega^1_M \). Then due to their duality, \( \tilde{g}_M : \Omega^1_M \oplus \Omega^1_M \to C^\infty_M \).

Since analytic manifold \( M \) is given by a solution of a certain differential equations and we assume that our considered manifold \( M \) has no singular, from proposition 2.16, it has a local coordinate system. Using local frame \((x^a)_{a=1, \ldots, n}\), as the duality is expressed by \( < dx^a, \partial_x > = \delta^a_x \) and \( \Theta_x \) is decomposed as \( \Theta_x = \sum_i C^a_i \partial_i \) where germs of \( \Theta_M \), \( C^a_i \) at \( x \), we can express it

\[
g_{Mi,j} \equiv g_M(i, j) := g_M(\partial_i, \partial_j),
\]

and thus

\[
g_M \equiv g_{Mi,j} dx^i \otimes C^a_i dx^j.
\]

Since \( g_M \) can be realized as a section of \( \mathcal{GL}(n, C^\omega_M) \), there is a map \( \det: \mathcal{GL}(n, C^\omega_M) \to C^\omega_M \), we will denote \( g_M := \det(g_M(i, j)) \).

Definition 2.21 (Normal Sheaf).

Let us define the normal sheaf \( \Theta^\perp_S \) as \( C^\omega_S \)-module sheaf by the exact sequence,

\[
0 \to \Theta_S \to i^{-1}_S \Theta_M \to \Theta^\perp_S \to 0,
\]

and the integrable connection of \( \Theta^\perp_S \) as \( \nabla^\perp_S \in \text{Hom}_{C^\omega_S}(\Theta_S, \mathcal{I}(\Theta^\perp_S)) \).

Lemma 2.22.

Let \( \Omega^1_S := \text{Ann}_{i^{-1}_S \Omega^1_M}(\Theta_S) = \{ \omega \in i^{-1}_S \Omega^1_M \mid \text{for} \forall \theta \in \Theta_S, \omega(\theta) = 0 \} \). Then we have a natural correspondence

\[
\tilde{g}_M(\Omega^1_S) = \Theta^\perp_S.
\]

By applying the proposition 2.14 and 2.16, we obtain the theorem.

Theorem 2.23 (Existence of Tubular neighborhood).

Suppose \( (S, C^\omega_S) \) is a \( k \)-dimensional non-singular real analytic algebraized submanifold of \( (M, C^\omega_M) \):

\[
i_S : S \hookrightarrow M.
\]

We have a tubular neighborhood \( (T_S, \mathcal{L}^\parallel_{T_S}) \) of \( S \) satisfied with following conditions,

1. \( T_S \) is an open set of \( M \), whose dimension is the same as \( M \) as a manifold, such that there are natural real analytic inclusions, \( i_S : S \hookrightarrow T_S \), and \( i_{T_S} : T_S \hookrightarrow M \), where \( i_S \equiv i_{T_S} \circ i_S \).
2. There is a real analytic projection from \( T_S \) to \( S \), \( \pi_{T_S} : T_S \to S \) such that \( \pi_{T_S} \circ i_S = id_S \).
(3) The tangent sheaf of $\Theta_{T_S}$ is $\Theta_{T_S} = i_{T_S}^{-1}\Theta_M$ and has a direct decomposition as a $\mathcal{C}_T^{\omega}$-module,

$$\Theta_{T_S} = \Theta_{T_S}^\parallel \oplus \Theta_{T_S}^\perp,$$

where $i_{T_S}^{-1}\Theta_{T_S}^\parallel = \Theta_S^\parallel$ and $i_{T_S}^{-1}\Theta_{T_S}^\perp = \Theta_S^\perp$.

(4) $\mathcal{L}_{T_S}^\parallel$ is a local system over $T_S$ such that $\mathcal{D}_M$-module $\mathcal{M}_{T_S}^\parallel := \text{Ann}_{\mathcal{D}_T S} (\mathcal{L}_{T_S}^\parallel)$ is generated by $\Theta_{T_S}^\parallel$ and, whose local sections are given as $\Gamma(U, \mathcal{L}_{T_S}^\parallel) = \Gamma(U, \mathbb{R}_M^{n-k})$ for any open set $U$ in $T_S$.

**Proof.** Using proposition 2.16 and the fact that $M$ has no singular, we can prove them. However they are also proved in concepts in Frobenius integrability theorem [AM, Mal2, W] and complete parallelism [Mal2] p.136. Later is familiar for differential geometers. □

**Remark 2.24 (Tubular neighborhood).**

1. $T_S$ has a foliation structure and $(\mathcal{L}_{T_S}^\parallel, \pi_{T_S}, S)$ is a sheaf over $S$.
2. For the metric $g_M$ of $M$, $T_S$ and $S$ have the induced metric $i_{T_S}^{-1}g$ and $i_{S}^{-1}g$ respectively.
3. We can define a local coordinate system of an open set $U$ of $T_S$: $p \in U$, $p$ is expressed by $(s^1, \ldots, s^k, q^{k+1}, \ldots, q^n)$, where $(s^1, \ldots, s^k)$ is a local coordinate system of $S$ by $\pi_{T_S} U$.
   Then $\mathcal{L}_{T_S}^\parallel(U)$ is characterized by a constant section $(q_k)_{a=1, \ldots, n-k} \in \mathbb{R}_M^{n-k}(U)$ and exhibits a leaf of foliation.

§3. Self Adjoint Operator

Let $\text{Mod}^L\mathcal{D}_M$ and $\text{Mod}^R\mathcal{D}_M$ be abelian category of left and right $\mathcal{D}_M$-modules.

**Proposition 3.1.**

1. For $\mathcal{M} \in \text{Mod}^L\mathcal{D}_M$, if we define the action of tangent sheaf $\Theta_M$ to $\mathcal{M} \otimes \mathcal{C}_M^\omega \omega_M^n$ as

$$\theta m \otimes \omega := -(\theta m) \otimes \omega + m \otimes \omega \theta,$$

where $m \otimes \omega \in \mathcal{M} \otimes \mathcal{C}_M^\omega \omega_M^n$, $\theta \in \Theta_M$, then $\mathcal{M} \otimes \mathcal{C}_M^\omega \omega_M^n$ can be regarded as the right $\mathcal{D}_M$-module.

2. $\omega_M^n$ can be regarded as the isomorphism as abelian category, which maps

$$\omega_M^n : \text{Mod}^L\mathcal{D}_M \to \text{Mod}^R\mathcal{D}_M.$$

According to the arguments of Mallios [Mal2, p. 343], we will introduce the Hodge *-operator, volume element and Beltrami-Laplace operator. Hereafter we will sometimes use Einstein convention; we will sum over an index if it appear twice in a term.
Definition 3.2. [AM, Mall1, N, W]

Let $(M, C^1_M, C^{r+}_M)$ be a non-singular analytic manifold endowed with the a Riemannian $C^r_M$-module $(\Theta_M, g_M)$. We have the Hodge operator as an element of automorphism of $\Omega_M$,

$$\ast : \Omega^n_M \to \Omega^{n-p}_M.$$  

By local chart $U$, the metric is expressed by $g_M = g_{Mi,j} dx^i \otimes dx^j$ over $U$, there $\tilde{g}^{ij}_M$ is the inverse matrix of $g_{Mi,j}$, and its determinant is expressed by $\det g_M := \det g_{Mi,j}$, then the Hodge operator is represented by,

$$\ast : \omega = \omega|_{i_1,i_2,\ldots,i_p} dx^{i_1} dx^{i_2} \cdots dx^{i_p}$$

$$\mapsto \sum_j \frac{\sqrt{g_M}}{(n-q)!} \omega|_{i_1,i_2,\ldots,i_p} \epsilon_{i_1,i_2,\ldots,i_p}^{j_{p+1},j_{p+2},\ldots,j_n} dx^{j_{p+1}} dx^{j_{p+2}} \cdots dx^{j_n},$$

where $\epsilon_{i_1,i_2,\ldots,i_n}$ is a section of $\mathbb{Z}_{2M}(U)$

$$\epsilon_{i_1,i_2,\ldots,i_n} = \begin{cases} 1 & \text{if } (i_1,i_2,\ldots,i_n) \text{ is an even permutation of } (1,2,\ldots,n) \\ -1 & \text{if } (i_1,i_2,\ldots,i_n) \text{ is an odd permutation of } (1,2,\ldots,n) \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{i_1,i_2,\ldots,i_p}^{j_{p+1},j_{p+2},\ldots,j_n} := \frac{\tilde{g}^{i_1,j_{p+1}} \tilde{g}^{i_2,j_{p+2}} \cdots \tilde{g}^{i_p,j_n}}{\sqrt{g_M}}.$$

Here we will express $dx^{i_1} dx^{i_2} \cdots dx^{i_p}$ by $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$ for abbreviation.

If necessary, we can introduce an orthonormal system [Mal1] and then the definition of the Hodge star operator becomes simpler.

Proposition 3.3.

$$\epsilon_{i_1,i_2,\ldots,i_n}^{-1} = \tilde{g}_M^{-1} \epsilon_{i_1,i_2,\ldots,i_n}^{-1}.$$

Proof. Direct computation gives the result. □

Proposition 3.4.

1. Let $w_M$ is the volume form of $(\Omega_M, g_M)$ and then

$$\ast 1_M = w_M \in \Omega^n(M), \quad \ast w_M = 1_M.$$

2. For $P \in D_M$, there exists $f \in C^\infty_M$ such that $F_0(w_M \cdot P) \equiv fw_M$.

Proof. From definition, (1) is obvious. Direct computations give (2). □

Definition 3.5 (Adjoint Operator).

For an open set $U$ of $M$ and $P \in D_M^C(U)$, we define the adjoint operator $P^\dagger$ as

$$P^\dagger := \ast (\ast 1_M \cdot P) \equiv \ast (\overline{w_M \cdot P}),$$

where $\overline{\cdot}$ means the complex conjugate. If $P^\dagger = P$, we will call it self-adjoint operator.
Remark 3.6.
We note that this adjoint operator can be expressed by the extension operation [Bjo, TH].

Definition 3.7.
The Beltrami-Laplace operator $\Delta$ of the Riemannian module $(M, g_M)$ is defined by
$$\Delta_M = \ast d \ast d.$$

Proposition 3.8.
There is a local system $\mathcal{L}^n_M$ such that $\Theta_M(U) = \Gamma(U, C_M^w \otimes_{\mathbb{R}} \mathcal{L}^n_M)$ or there is a basis $\{\theta_i\}_{i=1,\ldots,n}$ of $\mathcal{L}^n_M(U)$, $\Theta_M(U) = \oplus_{i=1}^n C_M^w(U) \theta_i$ for an open set $U$ in $M$.

Proof. From proposition 2.2, we can suppose that $\{x^i, \partial_i\}_{1 \leq i \leq n}$ is a local coordinate system for an open set $U$ of $M$ such that $[\partial_i, x^j] = \delta^j_i$. Then $c_i \partial_i$ $(1 \leq i \leq n, c_i \in \mathbb{R}(U))$ is an element of $\mathcal{L}^n_M(U)$ and $\xi = \sum_i a_i \partial_i \in \Theta_M(U)$ $a_i \in C_M^w(U)$. □

Inversely, for $\xi = \sum_i a_i \partial_i \in \Theta_M(U)$, $a_i \in C_M^w(U)$ and $\partial_i \in \mathcal{L}^n_M$, we can find a local coordinate system $y^i$ for $V \subset (\cup_i \text{supp}(a_i)) \cap U$, such that $\xi = \partial/\partial y^i$, where supp means the support of $a_i$. This can be proved by solving the differential equation $a_i = \partial y^i/\partial x^i$.

Proposition 3.9.
For a local coordinate $(x_1, \cdots, x_n)$ and inverse matrix $g^{ij}_M$ of its metric $g_{Mij}$ and $\tilde{g}^{ij}_M \equiv \tilde{g}_M(dx^i, dx^j)$, the Beltrami-Laplace operator is expressed by $\Delta_M$ as $\Delta_M = g_M^{-1/2} \partial_i g_M^{1/2} g^{ij}_M \partial_j$.

Proof. For a local coordinate system, the volume form $w_M$ is expressed by $w_M = g_M^{1/2} dx^1 \cdots dx^n$. □

Here we will define the Schrödinger system as a $D$-module related to the free Schrödinger equation $-\Delta_M \psi = 0$.

Definition 3.10 (Schrödinger System).
We will define the Schrödinger system $S_M$ by the exact sequence,
$$0 \longrightarrow D_M \xrightarrow{-\Delta_M} D_M \longrightarrow S_M \longrightarrow 0.$$

As mentioned in the §1 introduction, we will give a lemma on the Weyl algebra and thus introduce half form, wave system and their related anti-self-adjoint differential operators as follows.

Lemma 3.11.
There exist elements $\{\partial_i\}_{1 \leq i \leq n} \in \mathcal{C}_M^w$ such that they are of generators $\{x^i, \partial_i\}_{1 \leq i \leq n}$ of Weyl algebra, $[x^i, \partial_j] = \delta^j_i$ and $(\sqrt{-1}\partial_i)^\dagger \neq \sqrt{-1}\partial_i$.

Definition 3.12 (Half Form).
We will define a sheaf of half form $\sqrt{\mathcal{O}_M^2}$ as $\mathcal{C}_M^w$-module whose section over an open set $U$ of $M$ is given by $f \otimes_{\mathcal{R}_M} \sqrt{w_M}$, where $f \in \mathcal{C}_M^w(U)$ and $\sqrt{w_M}$ is defined as follows
\begin{enumerate}
\item $w_M := (\sqrt{w_M})^2 = w_M \sqrt{w_M}$,
\item $\ast \sqrt{w_M} = \sqrt{w_M}$,
\item For $\theta \in \Theta_M(U)$, the left handed action is given by $\theta(\sqrt{w_M}) := F_0(\theta \sqrt{w_M}) = \frac{1}{2} f \sqrt{w_M}$, if $F_0(\text{Lie}_\theta(w_M)) \equiv f w_M$.
\item The right handed action is $\sqrt{w_M} \cdot \theta = -\theta \sqrt{w_M}$.
\item For $\alpha \in \mathcal{C}_M^w$, $\theta \alpha \sqrt{w_M} = (\theta(\alpha)) \sqrt{w_M} + \alpha \theta \sqrt{w_M}$.
\end{enumerate}
We will denote $C\otimes R_M \sqrt{\Omega_M^n}$ by $\sqrt{\Omega_M^n}$.

If $M$ is a Riemannian surface with complex dimension one, $\sqrt{\Omega_M^n}$ is essentially the same as the prime form. Further $\sqrt{w_M}$ essentially appears in calculation of the gravitational or lorentzian anomaly in the elementary particle physics. Accordingly it is a natural variable.

**Definition 3.13 (Wave System $\sqrt{C_M^n}$).**

A $\mathcal{C}_M^n$-module $\sqrt{\mathcal{C}_M^n}$, called wave system, is defined by a closed presheaf of the bilinear morphism; for $\psi_a \in \sqrt{\mathcal{C}_M^n}(U)$ ($a = 1, 2$), $*\psi_1 \cdot \psi_2 \in \Omega_M^n(U)$.

We have the relation,

$$*\sqrt{\mathcal{C}_M^n}(U) \oplus \sqrt{\mathcal{C}_M^n}(U) \approx \Omega_M^n(U) \approx \mathcal{C}_M^n \cdot w_M(U).$$

We note that $\sqrt{\mathcal{C}_M^n}$ is isomorphic to $\mathcal{C}_M^n$ itself because basic field of $\mathcal{C}_M^n$ is a complete field $\mathbb{C}$.

**Definition 3.14 (Anti-Self-Adjoint Connection $\nabla_{M\theta}^{\text{SA}}$ of Wave System).**

Let us define an anti-self-adjoint connection $\nabla_{M\theta}^{\text{SA}} \in \text{Hom}_{\mathcal{C}_M}(\mathcal{C}_M^n, \mathcal{I}(\sqrt{\mathcal{C}_M^n}))$ by local relations for $\theta \in \mathcal{L}_n^\text{M}(U)$ and $\sqrt{w_M} \in \sqrt{\Omega_M^n}(U)$,

$$\nabla_{M\theta}^{\text{SA}} := *(\sqrt{w_M} \theta \sqrt{w_M}).$$

Here we will denote a sheaf of set of $\nabla_{M\theta}^{\text{SA}}$ by $\Xi_M^{\text{SA}}$.

**Proposition 3.15 (Local Expression of $\nabla_{M\theta}^{\text{SA}}$).**

For a local coordinate system $\{x^i, \partial_i\}_{1 \leq i \leq n}$ such that $[\partial_i, x^j] = \delta_i^j$, $\partial_i$ ($1 \leq i \leq n$) is an element of $\mathcal{L}_M^n$, and $w_M$ is expressed by

$$w_M = \sqrt{g_M} dx_1 dx_2 \cdots dx_n,$$

we have the local expression of $\nabla_{M\theta}^{\text{SA}}$ of $\theta = \partial_i$ case,

$$\nabla_{M\partial_i}^{\text{SA}} = \partial_i + \frac{1}{4} \partial_i \log g_M = \sqrt{g_M^{-1}} \partial_i \sqrt{g_M}.$$
Proposition 3.17 (Anti-Self-Adjoint Connection $\nabla^{SA}_{M \theta}$).

For an open set $U$ of $M$, $\alpha \in \mathcal{C}_\omega^\infty(M(U))$ and $\psi \in \sqrt{\mathcal{O}^\omega_{\text{Ad}}(U)}$, following holds.

1. $\nabla^{SA}_{M \theta} \alpha = \alpha \nabla^{SA}_{M \theta}$.
2. $\nabla^{SA}_{M \theta}(\alpha \psi) = \theta(\alpha) \psi + \alpha \nabla^{SA}_{M \theta}(\psi)$.
3. $[\nabla^{SA}_{M \theta}, \nabla^{SA}_{M \theta}'] = \nabla^{SA}_{M [\theta, \theta']}$. 

$\nabla^{SA}_{M \theta}$ is an integrable connection if it is a global section.

Proof. Hence, (1) and (2) are obvious. For $[\partial_i, \partial_j] = 0$, $[\partial_i, \partial_j]g_M = 0$. We obtain (4). □

Definition 3.18 (Momentum Operator $p_\theta$).

Let $U$ is an open set of $M$.

1. Let us define a momentum operator $p_\theta$ of $\theta \in \mathcal{L}_n^\infty(M(U))$, which consists of an integrable connection $\nabla^{SA}_{M \theta} \in \Xi^{SA}_{M}(\mathcal{L}_n^\infty(M(U)))$ for $\theta \in \mathcal{L}_n^\infty(M(U))$,

$$p_\theta := \sqrt{-1} \nabla^{SA}_{M \theta}.$$ 

2. $\mathcal{P}_M$ is $\mathbb{C}_M$-module generated by $p_\theta$ for $\theta \in \mathcal{L}_n^\infty(M(U))$, i.e., $\mathcal{P}_M \equiv \sqrt{-1} \Xi^{SA}_{M}$.

Proposition 3.19 (Momentum Operator $p_\theta$).

1. $p_\theta$ of $\theta \in \mathcal{L}_n^\infty(M)$ is self-adjoint, i.e., $p_\theta^\dagger = p_\theta$.
2. The Beltrami-Laplace operator $\Delta_M$ is locally expressed by

$$\Delta_M = \nabla^{SA}_{M i} g_M \nabla^{SA}_{M j} + \frac{1}{4}(\partial_j \log g_M)(\partial_i g_M)$$

$$+ \frac{1}{4}g_M^{ij}(\partial_i \partial_j \log g_M) + \frac{1}{16}g_M^{ij}(\partial_i \log g_M)(\partial_j \log g_M).$$

and $\nabla^{SA}_{M i} g_M \nabla^{SA}_{M j} = -p_i g_M^{ij} p_j$. Here we have used notations $\nabla^{SA}_{M i} := \nabla^{SA}_{M \partial_i}$ and $p_i := p_{\partial_i}$.

Proof. $w_M \cdot \nabla^{SA}_{M \partial_i} w_M = -\text{Lie}_{\partial_i} w_M + \frac{1}{4} \partial_i \log g_M w_M = w_M(-\theta - \frac{1}{4} \partial_i \log g_M)$. From 3.15, (1) is obvious. By direct computation, we obtain $\nabla^{SA}_{M i} g_M \nabla^{SA}_{M j}$. □

Corollary 3.20.

$\Delta_M \equiv p_i g_M^{ij} p_j$ modulo $F_0(\mathcal{P}_M)$.

Definition 3.21 (Momentum Operator $q_\theta$).

Let $U$ is an open set of $M$.

1. Let us define a momentum operator $q_\theta$ of $\theta \in \mathcal{L}_n^\infty(M(U))$, which consists of an integrable connection $\theta$ of $\Gamma \left( U, \mathcal{H}om_{\mathbb{R}_M}(\mathcal{L}_n^\infty(M), \mathcal{I}(\sqrt{\mathcal{O}^\omega_{\text{Ad}}(n)})) \right)$,

$$q_\theta := \sqrt{-1} \partial_\theta,$$
In other words, there is an equivalent functor, \( g \) of \( \theta \) for \( \psi \) Let the proposition 3.15, for sections of \( \Omega^C_M \) and morphisms are elements of a differential ring sheaf \( D^C_M \) generated by \( \mathcal{P}_M \) with coefficient \( \sqrt{C^*_M} \approx C^*_M \).

Let \( \mathcal{B} \) be a category whose objects are \( \mathcal{C}_M \)-modules \( \sqrt{\Omega^C_M} \) and morphisms are elements of a differential ring sheaf \( D^C_M \) generated by \( \mathcal{Q}_M \) with coefficients \( C^*_M \).

In other words, there is an equivalent functor, \( \xi_M : (\sqrt{C^*_M}, D^C_M) \to (\sqrt{\Omega^C_M}, D^C_M) \).

**Remark 3.22 (Momentum Operator \( p_\theta \)).**

Noting \( \ast \) for \( \sqrt{\Omega^C_M} \) is identity operator and we have the relation,
\[ \ast \sqrt{\Omega^C_M} \oplus \sqrt{\Omega^C_M} \approx \Omega^C_M \approx C^*_M \cdot w_M. \]

We should modify the definition of \( \dagger \) and let \( q_\theta \) be self-adjoint; \( q_\theta \dagger = q_\theta \).

**Proposition 3.23.**

Following categories are isomorphic.

1. Let \( \mathcal{A} \) be a category whose objects are \( \mathcal{C}_M \)-modules \( \sqrt{C^*_M} \) and morphisms are elements of a differential ring sheaf \( D^C_M \) generated by \( \mathcal{P}_M \) with coefficient \( \sqrt{C^*_M} \approx C^*_M \).

2. Let \( \mathcal{B} \) be a category whose objects are \( \mathcal{C}_M \)-modules \( \sqrt{\Omega^C_M} \) and morphisms are elements of a differential ring sheaf \( D^C_M \) generated by \( \mathcal{Q}_M \) with coefficients \( C^*_M \).

**Lemma 3.24.**

Let \( U \) be an open set of \( M \).

1. There is an equivalent functor \( \mathcal{A}_M \) from \( D^C_M \) to \( D^C_{\Omega_M} \). For \( P \in D^C_M \), \( g_M^{1/4} P \, q_M^{-1/4} \in D^C_{\Omega_M} \).

2. There is an equivalent functor \( \mathcal{B}_M \) from \( D^C_M \) to \( D^C_{\mathcal{P}_M} \); \( \mathcal{B}_M := \xi_M^{-1} \circ \mathcal{A}_M \). For an element of \( P \in D^C_M(U), \mathcal{B}(P) = P \).

3. \( P' \in D^C_{\Omega_M}(U) \) locally has a standard form representation,
\[ P' = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \theta^\alpha. \]

4. \( P' \in D^C_{\mathcal{P}_M}(U) \) locally has a standard form representation,
\[ P' = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \nabla^\alpha_M. \]

5. The functors \( \mathcal{A}_M \) and \( \mathcal{B}_M \) are extended to functors of the category of left \( D^C_M \)-modules and that of left \( D^C_{\Omega_M} \)-modules or left \( D^C_{\mathcal{P}_M} \)-modules.
Proof. From the propositions 3.15 and 3.23 give the correspondences (1) and (2), which are essentially the same as the hermitianization known in the Sturm-Liouville operator [Arf] as showed in the introduction. For an element $\partial_i$ of $\Theta_M(U)$, we have a local expression, $\overline{\sigma}_M(\partial_i) = \nabla_M^{S \alpha} - (\partial_i \log g_M)/4 = g_M^{-1/4} \partial_j g_M^{1/4} g_M^{-1/4} = \partial_i$. The expression of (3) can be obtained by expanding $g_M^{-1/4} \partial_j g_M^{1/4} = \partial^a + \ldots$. (4) is guaranteed from (3) and $\xi^{-1}$. Next we will show (5). For a quotient space $M = D_M^C/D_M^C R$ of $R \in D_M^C$ and $P_1, P_2, Q \in D_M^C$ such that $P_1 - P_2 = Q R$, $\overline{\sigma}_M(P_1 - P_2) = g_M^{-1/4} Q g_M^{-1/4} R g_M^{-1/4} = \overline{\sigma}_M(Q) \overline{\sigma}_M(R)$. Thus $\overline{\sigma}_M(M)$ can be defined as $\overline{\sigma}_M(M) := D_M^C / D_M^C \overline{\sigma}_M(R)$. Similarly we can naturally define $\overline{\sigma}_M$ for general coherent module. \[ \square \]

Due to the properties of $\overline{\sigma}_M$, we can mix $\overline{\sigma}_M(D_M^C)$ and $D_M^C$. In fact we did not discriminate them in the introduction and will not in [II]. However in this article, in order to see the action of $\overline{\sigma}_M$, we will explicitly express it.

Corollary 3.25.

(1) In the category $B$, we have local expression of $\overline{\sigma}_M(\Delta_M) = g_M^{1/4} \Delta_M g_M^{-1/4}$,

$$\overline{\sigma}_M(\Delta_M) = \partial_i \overline{\sigma}_M^i \partial_i + \frac{1}{4} (\partial_i \log g_M) (\partial_i \overline{\sigma}_M^i) + \frac{1}{16} \overline{\sigma}_M^i (\partial_i \partial_j g_M) + \frac{1}{16} \overline{\sigma}_M^i (\partial_i \log g_M) (\partial_j \log g_M).$$

(2) $\overline{\sigma}_M(\Delta_M)$ agrees with proposition 3.19 (2), $\Delta_M = \overline{\sigma}_M(\Delta_M)$.

§4. Schrödinger Operators in a submanifold $S \hookrightarrow \mathbb{E}^n$

Let $S$ be a $k$-dimensional real analytic compact submanifold of $\mathbb{E}^n$, $i_S : S \hookrightarrow \mathbb{E}^n$ and $T_S$ be its associated tubular neighborhood; $i_S : S \hookrightarrow T_S$ and $i_T : T_S \hookrightarrow \mathbb{E}^n$ such that $i_S \equiv i_T \circ i_S$. $T_S$ has a projection $\pi_{T_S} : T_S \rightarrow S$. Since $\mathbb{E}^n$ has a natural metric $g_{\mathbb{E}^n}$, $T_S$ and $S$ have Riemannian modules $(i_S^{-1} \Theta_{\mathbb{E}^n}, i_T^{-1} g_{\mathbb{E}^n})$ and $(\Theta_S, i_S^{-1} g_{\mathbb{E}^n})$. Further we put $i_T(S \Theta_{T_S}(\Theta_{T_S}^1)) := C_S^{-1} \otimes i_S^{-1} C_{T_S}^1 i_T^{-1} \Xi_{T_S}(\Theta_{T_S}^1)$.

The Schrödinger system $S_{T_S}$ of $T_S$ is given as $S_{T_S \rightarrow \mathbb{E}^n} := i_T^{-1} S_{\mathbb{E}^n} := D_{T_S \rightarrow \mathbb{E}^n} \otimes i_T^{-1} D_{\mathbb{E}^n} i_T^{-1} S_{\mathbb{E}^n}$ and $\overline{S}_{S \rightarrow T_S} := i_T^{-1} S_{\mathbb{E}^n}(S_{T_S}) := \overline{S}(S) \otimes i_T^{-1} \overline{S}_{T_S}(D_{T_S}) i_T^{-1} \overline{S}_{T_S}(S_{T_S})$.

Further we will use the notations in §2 and §3 and will not neglect the action of $\overline{\sigma}$’s.

Proposition 4.1.

(1) For $i_S^* S_{T_S} := D_S \otimes i_T^{-1} D_{T_S} i_T^{-1} S_{T_S}$,

$$i_S^* S_{T_S} = i_T^* S_{\mathbb{E}^n} := D_S \otimes i_T^{-1} D_{\mathbb{E}^n} i_T^{-1} S_{\mathbb{E}^n}.$$  

(2) Let the anti-self-adjoint connection $\nabla_S^{\mathbb{E}^n} \chi \in \Gamma(U, i_S^* \Xi_{T_S}(\Theta_{T_S}^1))$ for an open set $U$ in $S$. There is an injective endomorphism of the Schrödinger system $\overline{S}_{S \rightarrow T_S}$,

$$\eta_{\chi}^{\text{conf}} : \overline{S}_{S \rightarrow T_S} \rightarrow \overline{S}_{S \rightarrow T_S},$$
for $P \in \Gamma(U, S_{S \to T_S})$, $\eta^\text{conf}_\alpha(P) = P \nabla^{S_A}_\alpha \in \Gamma(U, S_{S \to T_S})$. Then we have a submodule of $S_{S \to T_S}$, $\eta^\text{conf} : (S_{S \to T_S})^{n-k} \to \sum_{\alpha=k+1}^{n} S_{S \to T_S} \nabla^{S_A}_\alpha \subset S_{S \to T_S}$.

**Proof.** Due to the relation $\iota_S \equiv i_T \circ i_S$, (1) is obvious. From lemma 3.24, we choose elements $P_1$, $P_2$ and $Q$ in $(i^*_S \sigma_{T_S} (D_T))$ such that $P_1 \equiv P_2 \in S_{S \to T_S}$ i.e., $P_1 - P_2 = Q[i^*_S \sigma_{T_S} (\Delta_{T_S})]$. Noting the relation $[Q[i^*_S \sigma_{T_S} (\Delta_{T_S})] \nabla^{S_A}_\alpha] = 0$, $Q[i^*_S \sigma_{T_S} (\Delta_{T_S})] \nabla^{S_A}_\alpha = Q \nabla^{S_A}_\alpha [i^*_S \sigma_{T_S} (\Delta_{T_S})] \equiv 0$ in $S_{S \to T_S}$. Thus it is injective. □

**Definition 4.2.**

(1) We will define a coherent $i^*_S \sigma_{T_S} (D_T)$-module $\overline{S}$ in $T_S$ by the exact sequence, $(S_{S \to T_S})^{n-k} \eta\rightarrow S_{S \to T_S} \rightarrow \overline{S} \rightarrow 0$.

(2) We will define a coherent $D_S$-module by $S_{S \to \mathbb{E}^n} := \sigma^{-1} S_{S \to T_S}$.

Let us call it submanifold Schrödinger system.

(3) When the submanifold Schrödinger system $S_{S \to \mathbb{E}^n}$ is decomposed by the exact sequence, 

$0 \to D_S -\Delta_{S \to \mathbb{E}^n} D_S \to S_{S \to \mathbb{E}^n} \to 0$,

where $\Delta_{S \to \mathbb{E}^n} - \Delta_S \in F_0 D_S$, we will call $\Delta_{S \to \mathbb{E}^n}$ the submanifold Schrödinger operator.

**Proposition 4.3.**

(1) $\overline{S}_{S \to \mathbb{E}^n}$ is uniquely determined.

(2) The definitions in 4.2 are naturally extended to a submanifold immersed in $\mathbb{E}^n$.

**Proof.** From the definition which does not depend upon the coordinate system, (1) is obvious. When we construct $\Delta_{S \to \mathbb{E}^n}$ and others, we used only local data. Hence (2) is also obvious. □

Now we will state our main theorem in this article.

**Theorem 4.4.**

(1) $k = 1$ and $n = 3$ case, $S$ is curve $C$,

$$-\Delta_{C \to \mathbb{E}^3} = -\partial_s^2 - \frac{1}{4}|\kappa_C|^2,$$

where $s$ is the arclength of the curve $C$, $\kappa_C$ is the complex curvature of $C$, defined by $\kappa_C = \kappa(s) \exp \left( \sqrt{-1} \int_s^0 \tau ds \right)$ using the Frenet-Serret curvature $\kappa$ and torsion $\tau$.

(2) $k = 2$ and $n = 3$ case, $S$ is a conformal surface,

$$-\Delta_{S \to \mathbb{E}^2} = -\Delta_S - (H^2 - K),$$

where $H$ and $K$ are the mean and Gauss curvatures.
These operators agree with the operators obtained by Jensen and Koppe [JK] and da Costa [dC]. Investigation of these equations might mean the properties of these submanifolds. Similar attempt was done [HL] for $-\Delta_S - 2H^2$ but these operators are more natural because they are related to Frenet-Serret and generalized Weierstrass relations [II].

In order to prove theorem 4.4, we will set up the language to express the submanifold system. We will note that these concepts of differential geometry, such as the Christoffel symbol, curvature and so on, are translated to language in sheaf theory by Mallios [Mal1,2]. Thus although we will use them in classical ways, they could be written more abstractly if one prefers.

An affine vector in $\mathbb{E}^n$ is given by $(Y^1, Y^2, \cdots, Y^n)$ as the Cartesian coordinate system and in its tangent space $T_p \mathbb{E}^n$, the bases are $\partial_i := \partial/\partial Y^i$, $i = 1, 2, \cdots, n$, $< \partial_i, dY^j >= \delta^j_i$. Here Latin indices $i, j, k$, are for the Cartesian coordinate of $\mathbb{E}^n$. $\Theta_{\mathbb{E}^n} = \mathbb{C}_{\mathbb{E}^n}^\omega \otimes_{\mathbb{E}^n} \mathcal{L}_{\mathbb{E}^n}^\omega$. The Riemannian metric $g_{\mathbb{E}^n}$ in the euclidean space is given as

$$g_{\mathbb{E}^n} = \delta_{i,j} dY^i \otimes dY^j.$$ 

Let the equations $Q^a(Y^1, Y^2, \cdots, Y^n) = 0$, $(a = 1, \cdots, d \equiv n - k$ express surface of $S$; the Pfaffian is expressed by $dQ^a = 0$’s and by Frobenius integrable theorem, there are vector fields given by the bases $\partial_\alpha := \partial/\partial s^\alpha$ which are satisfied with

$$< \partial_\alpha, dQ^\beta >= 0, \quad \text{for } \forall \alpha \forall \beta.$$ 

Hence the local coordinate of the submanifold is given as $(s^1, s^2, \cdots, s^k)$ or $(s^\alpha)$. Let us employ the conventions that the beginning of the Greek ($\alpha, \beta, \gamma, \cdots$) runs from 1 to $k$ and it with dot ($\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$) runs form $k + 1$ to $n$.

**Notation 4.5.**

1. A point $p$ in $T_S$ is expressed by the local coordinate $(u^\mu) := (s^1, s^2, \cdots, s^k, q^{k+1}, \cdots, q^n)$, $\mu = 1, 2, \cdots, n$ where $(s^1, \cdots, s^k)$ is a local coordinate of $\pi_{T_S} p$; We assume that the beginning of the Greek ($\alpha, \beta, \gamma, \cdots$) runs from 1 to $k$ and they with dot ($\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$) runs from $k + 1$ to $n$.

2. Let $(u^\mu) = (s^\alpha, q^\beta)$, where the middle part of the Greek $(\mu, \nu, \lambda, \cdots)$ run from 1 to $n$.

3. $e_\alpha := \partial_\alpha := \partial/\partial s^\alpha$ is a base of $\Theta_S(U)$. For $e_\alpha \in \iota_S^{-1}\Theta_{\mathbb{E}^n}(U)$, $e_\alpha$ is expressed by $e_\alpha = e^\mu\partial_{\mu}$.

4. Using $< e_\alpha, e^\beta >= \delta^\beta_\alpha$, $e^\beta = ds^\beta \in \Omega_S^1(U)$.

5. $e_\dot{\alpha} := \partial_\dot{\alpha} := \partial/\partial q^\beta$ is a base of $\Theta_{\mathbb{E}^n}(U)$. For $e_\dot{\alpha} \in \iota_{S^{-1}}\Theta_{\mathbb{E}^n}(U)$, $e_\dot{\alpha}$ is expressed by $e_\dot{\alpha} = e^j\partial_j$.

6. Let $(\iota_S^{-1}g_{\mathbb{E}^n})(e_\mu, e_\nu) = \delta_{i,j}e^\mu e^j\nu = g_{\mathbb{E}^n}\mu\nu$. Then we have induced metric $g_{T_S} := \iota_{T_S}g_{\mathbb{E}^n}$ and $g_S := \iota_S^{-1}g_{\mathbb{E}^n}$. Then we will express $g_S$ by the relation,

$$g_S = \iota_S^{-1}(\delta_{i,j}d(x^i) \otimes d(x^j)) = g_{S,\alpha\beta}ds^\alpha \otimes ds^\beta,$$

where

$$g_{S,\alpha\beta} := g_S(\partial_\alpha, \partial_\beta) := \iota_S^{-1}g_{\mathbb{E}^n}(e_\alpha, e_\beta).$$
Proposition 4.6.
Let $\mathcal{D}_{S \rightarrow T_S} := C^2 \otimes i_{S^{-1}} C^2 (i_{S^{-1}} T_S (D_{T_S}))$.

1. For an element $P$ of left $\mathcal{D}_{S \rightarrow T_S}$-module $S \rightarrow T_S$, there exists $Q \in \varphi_{T_S} S_T$ such that $P = Q |_{q^\alpha = 0, \alpha = k + 1, \ldots, n}$ for a normal coordinate $(q^\alpha, \alpha = k + 1, \ldots, n)$.
2. For an element $P$ of left $\mathcal{D}_S$-module $S \rightarrow T_S$, there exists $Q \in \varphi_{T_S} S_T$ such that $P = Q |_{\nabla^S A_{TS} = 0, \alpha = k + 1, \ldots, n}$.
3. $\Delta_{S \rightarrow E^n}$ is uniquely determined.

Proof. (1) and (2) are obvious from the definition 4.2. Since we tuned $\Delta_{S \rightarrow E^n}$ using $\Delta_S$, there is no multiplicative freedom. From proposition 4.3 (1), (3) is obvious. □

Proposition 4.7.
An affine vector (coordinate) $Y \equiv (Y^i)$ in $T_S \subset E^n$ is expressed by,
$$Y = X + e_{q^\alpha} q^\alpha,$$
for a certain affine vector $X$ of $S$.

Proof. By setting $X = \pi T_S Y$, it is obvious. □

For a case of immersion, this expression is not unique but locally unique.

Proposition 4.8.
For $U \subset T_S$, the induced metric of $T_S$ from $E^n$ has a direct sum form,
$$g_{T_S} := i_{T_S} g_{E^n} = g_{T_S^\parallel} \oplus g_{T_S^\perp},$$
where $g_{T_S^\perp}$ is trivial structure. In local coordinate,
$$g_{T_S^\parallel} = \delta_{\alpha\beta} dq^\alpha \otimes dq^\beta, \quad g_{T_S^\perp} = g_{T_S^\alpha\beta} ds^\alpha \otimes ds^\beta,$$
or for $g_{T_S^\mu,\nu} := g_{T_S}(\partial_{\mu}, \partial_{\nu})$
$$g_{T_S^\alpha\beta} = \delta_{\alpha\beta}, \quad g_{T_S^\alpha\beta} = g_{T_S^\alpha\beta} = 0,$$
where $\partial_{\mu} := \partial / \partial u^\mu$.

In order to prove this proposition and to give a concrete expression of $g_{S^\parallel}$ using $g_S$ and $q^\alpha$, we will consider the intrinsic and the extrinsic properties of $S \subset E^n$ e.g., the Weingarten map $[E,G]$.

Proposition 4.9 (intrinsic properties).
We can define the Riemannian connection consisting with this metric $g_S$ for $\theta, \xi \in \Theta_S$,
$$D_\theta \xi := \theta \xi - g_S(\theta \xi, e_{q^\beta}) e_{q^\alpha} g_{T_S}^{\alpha\beta},$$
where $g_{T_S}^{\alpha\beta}$ is the inverse matrix of $g_{T_S^\alpha\beta}$.

Proof. See chapter 12 and 13 in [G]. □
Proposition 4.10 (Weingarten map).
For a base $e_\alpha$ of $\Theta_S$ and $e_\beta \in T_{S^{-1}} E^n$, $\partial_\alpha e_\beta$ is an element of $T_{S^{-1}} E^n$ and is expressed by

$$\partial_\alpha e_\beta = \beta_\alpha e_\alpha + \gamma_\alpha e_\beta, \quad \gamma_\alpha := g_S(\beta_\alpha e_\alpha, e_\beta).$$

$-\gamma_\alpha : \Theta_S \rightarrow T_{S^{-1}} E^n$ is called as the Weingarten map.

Proof. See chapter 12 and 13 in [G] and p.162-164 in [E].  □

Proposition 4.11 (Second fundamental Form).

The second fundamental $\gamma^\alpha \beta_\alpha := g_S(\partial_\alpha e_\beta, e_\beta)$ is connected with the Weingarten map,

$$\gamma^\alpha \beta_\alpha = -g_S(e_\beta, \gamma^\gamma e_\gamma), \quad \gamma^\alpha \beta_\alpha = -g_S, g_S, g_S, g_S.$$

Proof. [E] Due to $g_S(e_\alpha, e_\beta) = 0$ and $\partial_\beta g_S(e_\alpha, e_\beta) = 0$, we prove it.  □

Lemma 4.12.

There exist the normal vectors $e_\alpha \in \Theta_S$ satisfied with,

$$\partial_\alpha e_\alpha = \gamma^\beta_{\alpha \beta} e_\beta.$$

Proof. Let $(\gamma, e)$ in proposition 4.11 be rewrite $(\gamma, \tilde{e})$. From the proposition 4.11, the derivative of a general normal orthonormal base $e_\alpha$ is given as $\partial_\alpha e_\alpha = \tilde{\gamma}^\alpha_{\beta \alpha} e_\beta + \tilde{\gamma}^\alpha_{\alpha \beta} e_\beta$.

From $g_{E^n}(e_\alpha, e_\beta) = \delta_{\alpha, \beta}$, for $\theta \in \Theta_S(U)$, i.e., $\theta = f^\alpha \partial_\alpha$ at $U \subset S$, $g_{E^n}(\partial_\alpha e_\beta, e_\alpha) = -g_{E^n}(e_\beta, \partial_\alpha e_\alpha)$, we have $\tilde{\gamma}^\alpha_{\beta \alpha} = -\tilde{\gamma}^\alpha_{\beta \alpha}$, and $\tilde{\gamma}^\alpha_{\alpha \beta} \equiv 0$ (not summed over $\alpha$). In other words, there are $k(n-k)(n-k-1)/2$ degrees of freedom; $\tilde{\gamma}^\alpha_{\beta \alpha}$ for $\alpha = 1, \cdots, k$. Thus we will employ an element $G$ of $SO(n-k)$ transformation so that

$$(\partial_\alpha + \tilde{\gamma}^\alpha_{\beta \alpha}) = G^{-1}(\partial_\alpha)G.$$

It is obvious that the solution of this differential equation locally exists, e.g.,

$$\begin{pmatrix} e_\alpha \\ e_\beta \end{pmatrix} = G_{\alpha \beta} \begin{pmatrix} \tilde{e}_\alpha \\ \tilde{e}_\beta \end{pmatrix},$$

$$G_{\alpha \beta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta := \int^1_s \frac{ds}{s} \tilde{\gamma}^\alpha_{\beta 1} + \int^2 s^2 \frac{ds}{s} \tilde{\gamma}^\alpha_{\beta 2} + \cdots + \int^k s^k \tilde{\gamma}^\alpha_{\beta k}.$$

The topological structure of $E^n$ is simple and the normal bundle of $S$ exists if the submanifold $S$ is not wild. From the assumptions on $S$, there is no singularity in $S$. Thus these solutions globally exist. This transformation is sometimes called as Hashimoto transformation.  □
Lemma 4.13.
The moving frame of $T_S$, $E^i = \pi_{T_S}^{-1}(\partial_i) \in \Theta_{T_S}$ ($\mu = 1, \cdots, n$), is expressed by $E^i \mu \partial_i \in i_{T_S}^{-1}\Theta_{E^\alpha}$ and

$$E^i_\alpha = e^i_\alpha + q^\alpha \gamma^\beta \alpha \beta \varepsilon^{i}_\beta, \quad E^i_\dot{\alpha} = e^i_{\dot{\alpha}}.$$

Proof. Using proposition 4.7, direct computation leads this result. \qed

Proof of Proposition 4.8.

Lemma 4.12 and $g_{TS} = E^i_{\mu}E^i_{\mu}dx^i \otimes dx^i$ lead the result in proposition 4.8. \qed

Its inverse matrix is denoted by $(E^\mu_i)$.

Corollary 4.14.

1. The metric in $T_S$ is expressed as

$$g_{TS} = g_S + (g_{TS}^{(2)}(q^\beta)\big)^2,$$

$$g_{T_S}(\partial_\alpha, \partial_\beta) = g_{S\alpha\beta} + [\gamma^\alpha_{\alpha\alpha}g_{S\beta} + g_{S\alpha\gamma}g_{S\beta\gamma}][\gamma^\alpha - [\gamma^\alpha_{\alpha\alpha}g_{S\delta\gamma}g_{S\beta\gamma}]q^\alpha q^\beta].$$

2. $g_{TS} := \det_{n \times n}(g_{TS}.\mu, \nu)$ is $g_{TS} = \det_{k \times k}(g_{T_S}(\alpha, \beta))$ and

$$g_{TS} = g_S\left\{1 + 2tr_{k \times k}(\gamma^\alpha_{\alpha\beta})q^\alpha + \left[2tr_{k \times k}(\gamma^\alpha_{\alpha\beta})tr_{k \times k}(\gamma^\alpha_{\beta\gamma}) - tr_{k \times k}(\gamma^\alpha_{\alpha\beta}\gamma^\alpha_{\gamma\beta})\right]q^\alpha q^\beta + O(q^\alpha q^\beta q^\gamma) + \cdots\right\}.$$

Example 4.15.

1. In coordinate, for the case of $n = 3, k = 1$;

$$g_{T_S} = g_S(1 - |\kappa_C|^3 + |\kappa_C|^2(q^3)^2)^2,$$

where $\kappa_C := \gamma^3_{12} + \sqrt{\gamma^2_{13}}$ is the complex curvature of $C$. Here $\kappa_C = \kappa(s) \exp\left(\sqrt{-1}\int^s \tau ds\right)$ is also given by the Frenet-Serret curvature $\kappa$ and torsion $\tau$.

2. In coordinate, for the case of $n = 3, k = 2$;

$$g_{T_S} = g_S(1 - 2Hq^3 + K(q^3)^2)^2,$$

where $H := \text{tr}(\gamma^3_{33})/2$ is the mean curvature and $K := \det(-\gamma^3_{33})$ is the Gauss curvature.

3. In coordinate, for the case of $n = 3$ and $k = 2$;

$$g_{TS} = g_S(1 + \text{tr}_2(\gamma^3_{33})q^3 + \text{tr}_2(\gamma^4_{43})q^4 + K(q^3, q^4))^2.$$

We will denote

$$H_{\alpha-2} := -\frac{1}{2}\text{tr}_2(\gamma^\alpha_{\alpha\beta}).$$

We can introduce the "complex mean curvature" $H_{\alpha} = H_1 + iH_2$.  

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Proof. Since $g_T^\parallel = (\det((\partial_\alpha x^i))^2$, we obtain (1) after calculation of $\det(\partial_\alpha x^i)$ using the fact that for a $2 \times 2$ matrix $A$, $\det(1 + A) = 1 + \text{tr}A + \det A$. Similarly we have (2) and (3). \qed

Lemma 4.16.

$S_{s \to T_S}$ can be expressed by

$$\text{Hom}_{D_T S}(S_{s \to T_S}, C_T S) = \{ \psi \in C_T S \mid \Delta_{s \to T_S} \psi = 0 \},$$

where

$$\Delta_{s \to T_S} = \Delta_S - \frac{1}{4} g_{T_S}^\beta (\text{tr}_{k \times k}(\gamma^\alpha_\beta)) \left( \text{tr}_{k \times k}(\gamma^\alpha_\beta) \right) + \frac{1}{2} \left( \frac{g_{T_S}^\beta}{g_S} \text{tr}_{k \times k}(\gamma^\delta_\alpha_\beta \gamma^\alpha_\beta \delta) \right).$$

Proof. From the proposition 4.6, we have $\Delta_{s \to T_S} \equiv \Delta_S|_{q^\alpha = 0, \alpha = k+1, \cdots, n}$. Proposition 3.19 gives

$$\Delta_T S = \nabla_{T_S \mu} (g_{T_S}^\mu)\nabla_S^{\alpha \beta} + \frac{1}{4} (\partial_\mu \log g_{T_S}) (\partial_\mu g_{T_S}^\mu)$$

$$+ \frac{1}{4} g_{T_S}^{\mu \nu} (\partial_\mu \log g_{T_S}) + \frac{1}{16} g_{T_S}^{\mu \nu} (\partial_\mu \log g_{T_S}) (\partial_\nu \log g_{T_S}).$$

Since for $n, m \in \mathbb{Z}$, we have the relation $\partial_\alpha^\alpha \partial_\beta^m g_{T_S} \equiv \partial_\alpha^\alpha \partial_\beta^m g_S \mod q^\alpha$, $\pi_T S \Delta_S$ is given by,

$$\pi_T S \Delta_S \equiv \nabla_{T_S \alpha} (g_{T_S}^\alpha)\nabla_{T_S}^{\alpha \beta} + \frac{1}{4} (\partial_\alpha \log g_{T_S}) (\partial_\alpha g_{T_S}^\alpha) + \frac{1}{4} g_{T_S}^{\alpha \beta} (\partial_\beta \log g_{T_S})$$

$$+ \frac{1}{16} g_{T_S}^{\alpha \beta} (\partial_\beta \log g_{T_S}) \mod q^\alpha.$$ 

By noting $\pi_T S \Delta_S \equiv \Delta_S$ and proposition 4.8 and by computing the remainder of $\Delta_T S - \pi_T S \Delta_S$, we obtain

$$\Delta_{s \to T_S} = \Delta_S + \nabla_{T_S \alpha} (g_{T_S}^\alpha)\nabla_{T_S}^{\alpha \beta} - \frac{1}{4} g_{T_S}^\beta \text{tr}_{k \times k}(\gamma^\alpha_\beta \delta) \left( \text{tr}_{k \times k}(\gamma^\alpha_\beta) \right)$$

$$+ \frac{1}{2} \left( \frac{g_{T_S}^\beta}{g_S} \text{tr}_{k \times k}(\gamma^\delta_\alpha_\beta \gamma^\alpha_\beta \delta) \right).$$

From the proposition 4.6 again, we have $\Delta_{s \to T_S} \equiv \Delta_S|_{q^\alpha = 0, \alpha = k+1, \cdots, n}$. \qed

Proof of Theorem 4.4.

Direct computations of the operator in 4.16 leads us to obtain the theorem 4.4 noting the examples 4.14. \qed
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