Deterministic Bayesian information fusion
and the analysis of its performance

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Abstract

This paper develops a mathematical and computational framework for analyzing the expected performance of Bayesian data fusion, or joint statistical inference, within a sensor network. We use variational techniques to obtain the posterior expectation as the optimal fusion rule under a deterministic constraint and a quadratic cost, and study the smoothness and other properties of its classification performance. For a certain class of fusion problems, we prove that this fusion rule is also optimal in a much wider sense and satisfies strong asymptotic convergence results. We show how these results apply to a variety of examples with Gaussian, exponential and other statistics, and discuss computational methods for determining the fusion system’s performance in more general, large-scale problems. These results are motivated by studying the performance of fusing multi-modal radar and acoustic sensors for detecting explosive substances, but have broad applicability to other Bayesian decision problems.

Keywords: sensor data fusion, Bayes optimal decisions, computational statistics, machine learning, calculus of variations, probabilistic graphical models

AMS subject classification: 62C10, 49K30, 46N30

1 Introduction

Sensor networks are ubiquitous across many different domains, including wireless communications, temperature and process control, area surveillance, object tracking and numerous other fields [2, 6]. Large performance gains can be achieved in such networks by performing data fusion between the sensors, or combining information from the individual sensors to reach system-level decisions [9, 16, 24, 26]. The sensors are typically connected by wireless links to either a separate information collector (centralized fusion) or to each other (distributed fusion). Elementary fusion rules based on Boolean logic are used in many contexts due to their simplicity and ease of implementation. On the other hand, in most situations we have some knowledge of the statistical properties of the sensors’ outputs, and designing fusion rules that take this into account can provide much better performance [17, 24]. The fusion rule can be built to satisfy any of various statistical optimality criteria, such as achieving the maximum likelihood or the minimum Bayes risk, under any other constraints of the problem [17]. Sensor information fusion can also be understood as a special case of the more general problem of statistical data reduction, where the goal is to reduce information from

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a high-dimensional space into a low-dimensional one in some optimal manner.

In many sensor fusion applications, it is important for the fusion rule to be a deterministic function of the sensor outputs. Fusion techniques that incorporate randomness are widely used in the sensing literature and can be more easily optimized to achieve given performance targets, such as false or correct classification probabilities [19]. However, in certain applications such as the detection of explosive compounds, it is common for the number of positive targets to be several orders of magnitude smaller than the number of negative ones, which means that randomized fusion rules have a large variance and rarely achieve their expected theoretical performance on realistic sample sizes. For simple, binary decision-level fusion problems, deterministic rules are easy to find [24], but in general, requiring the fusion rule to be deterministic effectively introduces a nonconvex constraint that is difficult to incorporate into a numerical optimization framework. This type of constraint also complicates the calculation of a fusion rule’s expected classification performance, which is a key component of modeling and simulation efforts to design sensor layouts and to perform trade studies between different sensor configurations [23].

The goals of this paper are to study the fusion rule that is Bayes optimal among all deterministic fusion rules, to investigate its mathematical properties under different cost criteria and other problem constraints, and to develop a computational framework for finding its expected performance. These results are motivated by the standoff detection of threat substances using multi-modal sensors in a centralized fusion network. However, we formulate the problem in an abstract setting that makes minimal assumptions on the details of the sensors or the type of information they produce, in contrast to the relatively well-defined situations in much of the sensing literature (e.g., [5, 9, 21]). We first use variational techniques to derive the standard Bayesian posterior mean as the optimal deterministic fusion rule under a quadratic cost. We show that the resulting system’s classification performance is a smooth function under some regularity constraints on the problem, and describe extensions to more general settings where the posterior mean no longer applies. For a certain class of fusion scenarios with sub-Gaussian priors, we extend these results beyond the classical setting of a quadratic cost, showing that the same fusion rule remains optimal under higher order cost functions and that its classification performance exhibits stronger, pointwise-like asymptotic behavior. This mathematical theory is developed in Section 2 with the proofs of the theorems deferred to Appendix A. In Section 3 we apply these results to several illustrative examples with Gaussian, exponential and other statistics for the sensors and examine the different types of behavior that they exhibit. In Section 4 we finally discuss efficient computational methods for finding the performance of the fusion rule in practice, based on Monte Carlo integration techniques commonly used in machine learning and other fields.

2 Mathematical framework for data fusion

Suppose we have \( M \) sensors (random variables) \( \{A_m\}_{1 \leq m \leq M} \), each observing an object (hypothesis) \( H \) and producing outputs that are sent into a fusion center \( C \) (see Figure 1). The sensor outputs could represent simple true/false decisions, choices between several distinct classes, or collections of multiple continuous-valued physical features of an object. We want to design a fusion rule at \( C \) that uses the information from the \( A_m \) to minimize the Bayes risk, or in other words, the expected cost of making a wrong decision [24]. In real world problems, the sensor outputs \( A_m \) are typically deterministic functions of the random object \( H \) and any background noise. We thus make the standard assumption that the \( \{A_m\} \) are all conditionally independent given \( H \), which is valid as long as the noise is white and each sensor observes the object at a slightly different time.

In what follows, we will denote the density and distribution of any random variable \( X \) by \( d_X(x) \) and \( d_X(x)dx \).
respectively, and use the vector notation $A = \{A_m\}_{1 \leq m \leq M}$ (where each $A_m$ may itself be a vector). For any closed set $K \subset \mathbb{R}$, we let $C^0(K)$, $C^0_c(K)$ and $C^\infty(K)$ respectively be the space of continuous, bounded functions on $K$, the space of $C^0(K)$ functions with compact support, and the space of $C^0(K)$ functions that are smooth (infinitely differentiable with bounded derivatives on $K$). In order to incorporate deterministic fusion, we view a density as a first-order generalized function [12, 22] (i.e., a Schwartz distribution, but we use the former terminology to avoid confusion with the meaning of the term in probability theory). This formulation allows us to consider delta functions and other such function-like objects. It is equivalent to treating the distribution as a measure with singular components, but the generalized function framework is easier to work with in developing the theory below. The space of first-order generalized functions can be identified with the dual space of $C^0_c(K)$ and is denoted $D^0(K)$.

The fusion problem can now be described as follows. We are given the densities $d_{A_m|H}(a, h)$ for $1 \leq m \leq M$, the prior $d_H(h)$ and a cost function $\text{cost}(c, h)$. We want to minimize the Bayes risk, or the expected cost of making a decision $E(\text{cost}(C, H))$. The sensor densities $d_{A_m|H}$ represent either generic statistical models trained from experimental lab data or mathematical descriptions of the underlying sensor physics, and the prior $d_H$ comes from domain-specific operational knowledge. The fusion rule can be expressed as the density $d_{C|A}(c, a)$. We usually want the fusion center $C$ to produce the same type of information that $H$ represents, so it is reasonable to take the cost function to be a simple metric between $C$ and $H$ and motivates the choice $\text{cost}(c, h) = W(c - h)$ for an appropriate increasing function $W$.

We now proceed to establish several results on the optimal deterministic fusion rule and its performance characteristics. To maintain the clarity of exposition, we defer the proofs of these theorems to Appendix A. We first address the classical setting of a quadratic cost $W(x) = \frac{1}{2}x^2$. The optimal fusion rule in this case reduces to the usual posterior expectation $E(H|C)$ corresponding to a naive Bayes classifier, but we establish it under the abstract formulation discussed above and use a variational argument in the proof that will be the foundation for later results. For the rest of the section, we also assume that all the conditions of Proposition 1 are satisfied unless stated otherwise.
Proposition 1. Suppose we have a given object space $I \subseteq \mathbb{R}$, a collection of feature spaces $\{J_m\}_{1 \leq m \leq M}$ with $J_m \subseteq \mathbb{R}^{N_m}$, and a decision space $K$. Suppose $K$ is a single interval with $I \subseteq K$. Let the fusion problem be set up as described above, with the random variables $H$, $A_m$ and $C$ respectively taking on values in $I$, $J_m$ and $K$, and let $\text{cost}(c, h) = \frac{1}{2} (c - h)^2$. Assume that $d_H \in D^0(I)$, $E(H^2) < \infty$ and for every $m$, $d_{A_m|H} \in C^2(J_m \times I)$ and $d_{A_m|H} > 0$. Then there is an almost everywhere unique, deterministic, Bayes optimal fusion rule that combines the $\{A_m\}$ and produces outputs in $K$. It is given by the posterior expectation

$$
f(A) = \frac{\int_I \int_J \left( \prod_{m=1}^{M} d_{A_m|H}(A_m, h) \right) d_H(h) dh}{\int_I \left( \prod_{m=1}^{M} d_{A_m|H}(A_m, h) \right) d_H(h) dh}.
$$

The spaces $I$, $J$ and $K$ respectively represent the type of hidden information we want to estimate or classify between, the type of information we observe from the sensors and the type of decision we can make from those observations. We typically think of the objects and decisions as scalar quantities that may be discrete or continuous, while the feature spaces are potentially vector quantities (for example, a camera sensor that outputs images). Since we allow $d_H$ to be a generalized function, the results of Proposition 1 cover the case where the object space $I$ is discrete and finite by taking $d_H$ to be a weighted sum of delta functions. This is the case in most sensor fusion applications, where there are a finite and known number of anomaly classes.

Note that the deterministic constraint ((8) in the proof) is what allows the fusion rule to become the posterior expectation (1), and there can generally be other, random fusion rules (where $d_H$ is a density) that have better performance without this constraint imposed. Under some additional constraints on the problem, we can also study the properties of the classification performance of this fusion rule, given by the density $d_{C|H}$.

Theorem 2. Assume that $K$ and $J$ are compact, $d_{A_m|H}(\cdot, h) \in C^\infty(J_m)$ for all $m$, and for each point $a \in J$ there is at least one sensor $A_m$ and feature $n$, $1 \leq n \leq N_m$, such that

$$
\int_I \int_J \left( d_{A_m|H}(a_m, h) \frac{\partial d_{A_m|H}(a_m, h)}{\partial (a_m)_n} - d_{A_m|H}(a_m, h) \frac{\partial d_{A_m|H}(a_m, h)}{\partial (a_m')_n} \right) dh d_H(h) dh = 0.
$$

Then the classification performance of the fusion rule (1), given by $d_{C|H}(\cdot, h)$, is in $C^\infty(K)$ for each $h \in I$.

The performance function $d_{C|H}$ of the fusion rule is a generalization of the classical “confusion matrix” that describes the probabilities of correct and false classification in a binary, decision-level setting [24]. For every object $h \in I$, the values along the diagonal $d_{C|H}(h, h)$ are the likelihoods of the fusion rule giving the correct result, with a delta function $d_{C|H}(c, h) = \delta(c - h)$ corresponding to an ideal classifier (unattainable in a real situation). The condition (2) holds for simple examples such as those involving Gaussian statistics (see Section 3) but its exact form is not central to the result, as it can be replaced by a variety of weaker but more complicated conditions under which the stationary phase argument in the proof still holds. Under these conditions, Theorem 2 says that $d_{C|H}$ is actually a function, instead of merely a generalized function, and is meaningful at every point. The proof of Theorem 2 also provides a way to compute the performance function, by finding

$$
d_{C|H}(c, h) = \int_J \delta(c - f(a)) \left( \prod_{m=1}^{M} d_{A_m|H}(a_m, h) \right) da.
$$

This is effectively an integral over only the level sets of the fusion rule $\{a \in J : c = f(a)\}$ in the joint feature space. In practice, these level sets can be numerically approximated from the fusion rule as long as either the fusion rule does not have “flat regions” where the gradient $\nabla f$ is identically zero, or the decision space $K$ is discrete. This is usually the simplest and most efficient approach to computing $d_{C|H}$, although various “smoothed out” versions of (3) can be used instead, such as taking the Fourier transform of (3) and inverting it, as done in [13] in the proof.
The computation of the fusion rule \((1)\) itself is simple and involves an integration over only the (small) object space \(I\). On the other hand, increasing the number of sensors will rapidly increase the dimension of the integral \((3)\) for the performance function. In practice, each sensor generates only a moderate number of statistics and the individual feature spaces \(J_m\) are relatively small (with dimension \(N_m\) typically at most 10 or 20). This results in the joint density \(\mathbf{d}_{\mathcal{H}}(a, h)\) having a “block diagonal” dependence structure that allows the calculation of \(\mathbf{d}_{\mathcal{H}}\) to remain computationally tractable. We will describe a Monte Carlo-based approach to perform this calculation efficiently in Section 4.

We next extend Proposition 1 to more general decision spaces \(K\) that are not necessarily a single interval. This case is important for many applications in which \(I\) and \(K\) are both finite sets, the so-called binary decision and \(M\)-ary classification fusion problems. The case where \(I \not\subset K\) is also of interest in situations where there are several threat substances of interest, but we ultimately want to make a “true” or “false” decision at the fusion center. The posterior expectation \((1)\) is no longer a feasible solution and cannot be applied to this situation, but we still have the following result.

**Theorem 3.** If the decision space \(K\) is a closed set but otherwise unconstrained, then the Bayes optimal fusion rule is \(f^*(A) = Q(f(A))\), where \(Q\) is the quantization function defined for each \(x \in \mathbb{R}\) by choosing any \(x'\) from the set \(\{\arg\min_{x' \in K} |x-x'|\}\). This fusion rule is unique almost everywhere.

The modified fusion rule \(f^*\) is a generalization of well known formulas for binary (two element) spaces \(I, J_m\) and \(K\) \([15, 24]\), and can be computed easily in practice. For example, if \(I = K = \{0, 1\}\), then the fusion rule \(f\) given by \((1)\) may generally take on any value in \([0, 1]\) and is not feasible, but we can simply round it to the nearest integer to obtain the actual, optimal fusion rule \(f^*\) for our scenario. Note that in the proof of Theorem 3, the quadratic cost function is crucial and allows for the cancellation of the third term in \((17)\). The result of Theorem 4 will usually not hold for other costs.

We now consider cost functions more general than the quadratic one. Optimization problems in Bayesian statistics under arbitrary cost functions usually have no closed-form solutions, and the fusion rule would in most cases have to be determined numerically (except for some very specific densities, such as in \([27]\)). However, we identify a class of prior and sensor densities for which the same fusion rule \((1)\) turns out to be Bayes optimal for a much larger class of costs.

**Theorem 4.** Suppose that \(H\) is sub-Gaussian (i.e., \(E(e^{rH^2}) < \infty\) for some \(r > 0\)) and for each \(a \in I\), whenever \(E(e^{-2\pi i z H} | A = a) = 0\) for some \(z \in \mathbb{C}\), \(E(e^{2\pi i z H} | A = a) = 0\) as well. Let \(K\) be a single interval (possibly unbounded). Then the fusion rule \((1)\) is Bayes optimal for cost \((c, h) = (W(c-h), W)\), where \(W\) is any entire, even, nonnegative and convex function with \(\alpha_1(|x|^p - 1) \leq |W(x)| \leq \alpha_2(|x|^p + 1)\) for some constants \(\alpha_1, \alpha_2 > 0\) and \(p \geq 1\).

The conditions of Theorem 4 are satisfied, for example, when the object and sensor statistics are all Gaussian, a case that is discussed in more detail in Section 4. Many other scenarios where Theorem 4 holds can also be found, including cases with finitely supported priors \(H\) and other types of sensor distributions. For example, a prior of the form \(\mathbf{d}_H(h) = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} (\delta(h + m) + \delta(h - m))\), corresponding to \(I = \{m : m \in \mathbb{Z}, 1 \leq |m| \leq M\}\), together with Levy distributed sensor observations \(\mathbf{d}_{A_m H}(a_m, h) = \sqrt{\frac{|h|}{2\pi \sigma_m^2}} e^{-\frac{|h|^2}{2\sigma_m^2}}\) can be shown to result in \(\hat{G}(z)\) satisfying the “symmetric zeros” condition of Theorem 4.

The class of cost functions \(W\) addressed by Theorem 4 covers a wide variety of interesting cases. It includes functions of the form \(W(x) = \frac{1}{p} |x|^p\) with even exponents \(p\), as well as functions that asymptotically behave like \(|x|^p\) for any real \(p \geq 1\) (e.g., \(W(x) = \int_{-\infty}^{\infty} |x-y|^p e^{-y^2} dy\)). For such cost functions, larger values of \(p\) give
We finally examine the asymptotic properties of the fusion rule (1) as the number of sensors increases, with every $N_m = 1$ to simplify the notation. For fusion scenarios that are in the class covered by Theorem 4, we can prove much stronger asymptotic statements that cover not only the Bayes risk but also the fused classification performance for every possible object.

**Theorem 5.** Let each $J_m = \mathbb{R}$ and let $A_m$ satisfy $E(A_m|H) = H$ and $\text{var}(A_m|H) < R$ for some constant $R$. Suppose $d_H > 0$ on $I$ and $K$ is an interval such that $[\min h : h \in I] - \varepsilon, \max h : h \in I + \varepsilon] \subset K$ for some $\varepsilon > 0$. Then as $M \to \infty$, the Bayes risk of the fusion rule goes to 0. If in addition $I$ and $K$ are compact and for each $M$, $A$ and $H$ satisfy the conditions of Theorem 4 and $d_{CH}(., h)$ is a bounded, even function, then for every $h \in I$, $d_{CH}(., h) \to \delta(\cdot - h)$ in the weak-* sense as $M \to \infty$.

The proof of Theorem 5 shows that for a large number of sensors $M$, the Bayes risk of the optimal fusion rule is bounded above by the sample mean of the sensor outputs. The quadratic Bayes risk of the optimal fusion rule decays at least as fast as (and possibly faster than) the $O(\frac{1}{M})$ rate that the sample mean achieves, but it is usually quite different for fixed and realistic values of $M$. The conditions in Theorem 4 are the main things that enable the pointwise-like convergence result $d_{CH}(\cdot, h) \to \delta(\cdot - h)$, and the additional constraints of Theorem 5 (such as compact $I$ and $K$) simplify the proof but can be relaxed. If the object and feature distributions are not in the class covered by Theorem 4, the performance $d_{CH}$ of $f$ may not necessarily approach “perfect classification” (a delta function) and is only guaranteed to do so on average, in the sense of the Bayes risk under a quadratic cost going to zero.

### 3 Example data fusion scenarios

In this section, we consider a series of examples applying the theory from Section 2 to concrete data fusion and performance analysis problems. The simplest situation is when the sensor and object statistics are all Gaussian, and is one of only a few cases where the performance function of the fusion rule can be calculated symbolically. We study this case in detail.

**Proposition 6.** Let $I = K = \mathbb{R}$ and $J = J_m = \mathbb{R}^{M/2}$ for some even $M$. Suppose the object $H$ is Gaussian with mean 0 and variance 1. Suppose the sensors $A$ and $B$ observe $H$ and respectively output collections of features $\{A_m\}_{1 \leq m \leq M/2}$ and $\{B_m\}_{1 \leq m \leq M/2}$, where each $A_m|H$ and $B_m|H$ is Gaussian with mean $uH$ and variance $v$ for some fixed parameters $u$ and $v > 0$. Then the optimal fusion rule under a quadratic cost (1) is

$$f((A, B)) = \frac{u}{Mu^2 + v} \left( \sum_{m=1}^{M/2} A_m + \sum_{m=1}^{M/2} B_m \right),$$

and its performance and Bayes risk are given by

$$d_{CH}(c, h) = \frac{Mu^2 + v}{u\sqrt{2\pi}Mv} e^{-\frac{(Mu^2(c-h) + v)^2}{2Mu^2v}},$$

$$E((C - H)^2) = \frac{v}{Mu^2 + v}.$$
Furthermore, (4) is optimal for all costs $W$ satisfying the conditions of Theorem 4.

The situation described by Proposition 6 can be interpreted in the following way. The object space represents a degree of belief between the certain presence ($H = \infty$) and certain absence ($H = -\infty$) of an object of interest. Each of the features picked up by the sensors $A$ and $B$ provide some information on $H$, but with a known level of uncertainty $v$. The fusion rule (4) combines the features in a way that best matches the resulting (soft) decision with the original belief, using the available information from the sensors. The division of the $M$ features into two sensors $A$ and $B$ is arbitrary and they can equivalently be combined into a single sensor, but this formulation is useful in establishing Corollary 7 below.

Proposition 6 shows that the average performance of the fusion rule (as measured by the quadratic Bayes risk) is roughly inversely proportional to the number of features being fused. Note that the optimal fusion rule (4) is similar to but different from the sample mean of all the features $A_m$ and $B_m$, and that its performance is skewed by the prior on $H$ (see Figure 2). For a large number of features, it is easy to verify that as $M \to \infty$, the fusion rule (4) approaches the actual sample mean of the $A_m$ and $B_m$ and the performance satisfies $d_{C:H}(\cdot, h) \to \delta(\cdot - h)$ in the weak-$\star$ sense for each $h$, in line with Theorem 5.

![Figure 2: Fusion configuration from Proposition 6 with $u = v = 1$ and $M = 2$.](image)

Proposition 6 also allows us to compare the performance of directly fusing all features for the entire system, as opposed to having each sensor combine its own features internally according to (4) and then fusing the resulting sensor outputs using (4) again (see Figure 3 (a-b)). The latter corresponds to a typical approach taken in many real-world sensor fusion systems and is a version of distributed fusion with person-by-person optimization (PBPO), where the decision rule at each fusion center is chosen using only the properties of its own inputs and outputs, without regard to the rest of the graphical model (see [4, 9, 24, 25]). Distributed fusion networks are common in many applications with large numbers of sensors due to bandwidth or other communication constraints between the sensors. This typically incurs a loss in the overall system performance, but in the special case when the statistics are all Gaussian, the performance is unaffected.

**Corollary 7.** Let $I, J, K, H, A$ and $B$ be as given in Proposition 6 with $K^* = K$ and $u = v = 1$, so the Bayes risk of the fusion configuration in Proposition 6 is $\frac{1}{M+1}$. Now suppose $A$ and $B$ each have internal fusion centers that respectively reduce \{ $A_m$ \}$_{1 \leq m \leq M/2} \in J$ to a decision $A^* \in K^*$ and \{ $B_m$ \}$_{1 \leq m \leq M/2} \in J$ to a decision $B^* \in K^*$ using locally Bayes optimal decisions. Let $A^*$ and $B^*$ be combined at the system fusion center $C$ to
produce a locally optimal output in $K$, as in Figure 3(b). Then the Bayes risk of the entire fusion system is still \( \frac{1}{M+1} \).

The result of Corollary 7 is possible only because of the simple form of the fusion rule (4). The Bayes risk in a distributed fusion model like this will generally be lower in other cases due to a loss of information at $A^*$ and $B^*$. An example of this is discussed below.

Another example similar to Proposition 6 can be considered, using other types of sensor statistics.

**Proposition 8.** Let $I = K = J_m = [0, \infty)$. Suppose the object $H$ is exponentially distributed with rate parameter 1 (i.e., $E(H) = 1$), and there are $M$ sensors $A_m$ with exponentially distributed observations all having rate parameter $H$. Then the optimal fusion rule is $f(A) = \frac{M+1}{\sum_{m=1}^{M} A_m + 1}$. Its performance and Bayes risk are given by

\[
d_{C|H}(c, h) = \frac{(M+1)h^{M+1}}{M!c^{2}} \left( \frac{M+1}{c} - 1 \right)^{M} e^{-h(M+1)/c}, \quad c \in (0, M+1]
\]

\[
E((C - H)^2) = \frac{2}{M+2}.
\]

Proposition 8 illustrates a variety of different behavior from the Gaussian cases. The fusion rule is now quite different from the sample mean and only takes on values between 0 and $M+1$, even though the decision space is the entire positive real axis. This indicates that any fusion rule that produces decisions $f(A) > M + 1$ would compromise the good performance of the decisions for $0 < f(A) \leq M + 1$, to the extent that the overall Bayes risk of the system increases. The performance function is only meaningful for such values of $c$, but if we define it to be identically zero for $c \geq M + 1$, then it still converges to a delta function in the weak-* sense as $M \to \infty$.

![Figure 3](image-url:3) Graphical models of fusion configurations for (a) centralized fusion of features, and (b) distributed PBPO fusion of features, i.e., sensor-level fusion of features followed by system-level fusion of the resulting decisions.

We next look at a simple example with two sensors, each producing one feature, and a discrete, finite object space. It is generally no longer possible to study the fusion performance symbolically, but we can compute it numerically using (3). Let $I = \{0, 1, 2, 3\}$ and $J_1 = J_2 = \mathbb{R}$. For the sensors $A$ and $B$, let $A|H$ and $B|H$ be Gaussian variables for each $H \in I$, with $E(A|H) = E(B|H) = \{0, 1, 2, 3\}$, $\text{var}(A|H) = \{1.7, 0.4, 3, 1\}$ and $\text{var}(B|H) = \{0.5, 2, 0.7, 2\}$. We consider the cases $K = \{0, 1, 2, 3\}$ and $K = [0, 3]$, corresponding to hard or
soft decisions at the fusion center. The resulting fusion rules and their performance functions are shown in Figure 4. The hard decision fusion rule contains small “islands” surrounded by larger regions where different decisions are made, and this behavior is typical of problems where $I$ and $K$ are discrete but the $J_m$ are continuous. The soft decision scenario has an improved Bayes risk (0.35536) over the hard decision case (0.43775), reflecting the fact that more information is preserved about the object at the fusion center. However, some of the classification performance of the object $H = 1$ was traded off for improved performance with $H = 0$.

These scenarios can also be placed in the context of Corollary 7 and the distributed fusion model in Figure 3 (b). If we take $H, A$ and $B$ as above and set $K^* = \mathbb{R}$ and $K = [0, 3]$, then the fusion centers at $A^*$ and $B^*$ are simply identity mappings that pass the outputs of $A$ and $B$ into $C$, so the Bayes risk at $C$ is the same as the soft decision scenario above. On the other hand, if we take $K^* = \{0, 1, 2, 3\}$, so that the outputs of $A$ and $B$ are first reduced to hard decisions before the fusion at $C$, then the Bayes risk at $C$ turns out to be 0.57862, even though the final decision space $K$ is unchanged. This shows how distributed fusion can hurt performance when enough information is lost at $A^*$ and $B^*$, as opposed to fusing $A$ and $B$ directly at $C$.

Figure 4: Fusion rules and their performance with a discrete object space, Gaussian sensor outputs and a discrete (top) and continuous (bottom) decision space.
Another, similar type of scenario can be considered with discrete features and shows how random fusion rules can enter the picture. Let \( I = K = \{1, 2\} \) and \( J_1 = J_2 = \{0,1,2,3,\ldots\} \). Let \( P(H = 1) = P(H = 2) = \frac{1}{2} \), and \( A|H \) and \( B|H \) be Poisson variables with rate parameter \( H \). By Theorem 3, the deterministic fusion rule for a quadratic cost is given by taking \( f(A,B) = 1 \) when \( A + B \leq 2 \) and \( f(A,B) = 2 \) otherwise. In other words, there are only six \((A,B)\) pairs with \( f(A,B) = 1 \). The classification performance of this rule can be found explicitly, with the false positive rate \( P(H = 1|C = 2) = 1 - 5e^{-2} \) and the miss rate \( P(H = 2|C = 1) = 13e^{-4} \). Now the only way to improve the miss rate is by mapping one of the six \((A,B)\) values to \( f(A,B) = 2 \) instead, but there are a countable number of fusion rules that do this and their performances can only take on specific values. If we change the cost to be such that the false positive rate can be at most \( 1 - (5 - \varepsilon)e^{-2} \) for some small \( \varepsilon > 0 \), then there are random fusion rules with lower miss rates than any of the deterministic rules. For example, the random fusion rule \( g \) that takes \( g(1,1) = 2 \) with probability \( \varepsilon \) and 1 otherwise, and \( g(A,B) = f(A,B) \) for all other \((A,B)\), is a (non-unique) optimal choice with \( P(H = 1|C = 2) = 1 - (5 - \varepsilon)e^{-2} \) and \( P(H = 2|C = 1) = (13 - 4\varepsilon)e^{-4} \). This situation is typical when the feature and/or decision spaces are discrete, with a random fusion rule having more “wiggle room” to achieve specific classification probabilities, and is essentially a special case of the classical Neyman-Pearson lemma for likelihood ratio tests ([19], p. 23).

We finally consider one more numerical example with more complex features that have mixture distributions. Let \( I = K = [0,4] \) and \( J_1 = J_2 = [0,5] \). Suppose that \( H \) has a Gaussian prior with mean 0 and variance 1 as in Proposition 6 but \( A \) is an exponential-uniform mixture with the form \( d_{A|H}(a,h) = \frac{h}{2}e^{-ha} + \frac{1}{2\pi}\chi_{[0,\beta]}(a) \), where \( \chi \) is the indicator function, and \( B \) follows the Gaussian mixture distribution \( d_{B|H}(b,h) = \frac{1}{2} \left( \frac{\beta}{\pi} \right)^{-1/2}e^{-50(b-h)^2} + \frac{1}{2} \sqrt{\frac{4}{\pi}}e^{-h^2(b-0.7)^2} \). The optimal fusion rule and its performance function are shown in Figure 5. Note that the optimal fusion rule never outputs decisions greater than about 2.5, which is reflected in its performance and is similar to the scenario in Proposition 8.

Figure 5: Fusion configuration with mixture distributions.

### 4 Computational methods

We discuss some approaches to efficiently compute the performance \( d_{C|H} \) and Bayes risk \( E(\text{cost}(C,H)) \) of the fusion rule ([12]) in realistic scenarios with a large number of sensors or features. We assume that the densities \( d_{A|H} \) are available in a symbolic (but not necessarily closed) form, coming from either a physics-based
model or a kernel density estimate or other statistical model fitted to experimental data. The \( A_m | H \) can be discrete or continuous variables and have different dimensions \( N_m \), depending on what kinds of information each sensor puts out, but they are assumed to be well localized in their respective feature spaces. Having a symbolic expression for \( d_{A_m | H} \) as opposed to a tabulation on a discrete grid (for continuous variables) allows us to sample at points anywhere in \( J_m \), which is essential for the use of randomized integration methods that scale efficiently in the total number of dimensions \( N = \sum_{m=1}^{M} N_m \). For sensors that produce several different features \( (N_m \geq 1) \), \( d_{A_m | H} \) is often not specified explicitly but is given in terms of a probabilistic graphical model \( [14] \) that describes the dependencies among the individual features \( A_m \) and any intermediate (nuisance) variables.

In typical fusion scenarios in practice, it is common to combine several hundred features at once, and determining the performance \( d_{C | H} \) leads to a high dimensional integral in \( \{1\} \) that is intractable by conventional lattice-based approaches. However, one key property of statistical inference problems such as this is that the sensor densities \( d_{A_m | H} \) are all nonnegative, which means that the integral \( \{1\} \) involves no cancellation and the largest contribution comes from around the local maxima of the \( d_{A_m | H} \). We outline a Monte Carlo importance sampling approach that is motivated by these observations. We collect samples \( \{(a_i, h_i)\}_{1 \leq i \leq L} \) drawn from the proposal distribution \( (\prod_{m=1}^{M} d_{A_m | H}(a_m, h)d_{I_m}) d_{H}(h)dh \), which prioritizes points around the maxima of \( \{1\} \) and reduces the variance in the resulting estimates \( \{11\} \). \( H' \) is a variable either with the same distribution as \( H \) or the uniform distribution on \( I \). This reflects an accuracy tradeoff between finding the Bayes risk and the performance, with the former choice more efficient for calculating the Bayes risk and the latter preferable for finding the performance. For a given number of samples, the former choice will pick the points in \( H \) that contribute the most to the Bayes risk, but may leave a large portion of the domain \( I \times K \) uncovered by the samples and result in an inaccurate performance function. On the other hand, sampling \( H \) according to a uniform distribution on \( I \) will distribute the points evenly across \( I \times K \), even though only a few may add significantly to the Bayes risk. Note that finding the performance \( d_{C | H} \) corresponds to a “downstream” calculation in the probabilistic graphical model in Figure \( \{1\} \) where samples are generated at \( H \) and propagate downward through the sensors \( A_m \) into \( C \). This is in contrast to the more conventional task of doing posterior inference on data, which amounts to finding \( d_{H | A} \) and is an “upstream” calculation that involves the Bayes formula.

The sensor densities \( d_{A_m | H} \) can be sampled from using a variety of approaches, depending on how each one is specified. For graphical models, standard Markov Chain Monte Carlo (MCMC) methods such as Gibbs sampling and its variants (see \( \{3\} \) for details) allow us to obtain samples from a high-dimensional joint density that would be impossible to sample from directly, unless it has some special structure. However, one such case arises frequently in practice, where each \( A_m | H \) is jointly Gaussian. Many standard types of radar and acoustic sensors for explosive detection collect measurements such as the signal’s return time or the power at specific frequencies. These measurements are typically formed by averaging a large number of consecutive “looks” to smooth out the effects of noise, which results in each \( A_m | H \) being approximately an \( \mathbb{R}^{N_m} \)-valued Gaussian variable, and the resulting joint distribution is easy to sample from directly. The individual components of \( A_m | H \) are usually not independent, and may represent measurements such as the signal intensity at different frequencies, which are all influenced by an explosive substance with a given spectral profile. However, we can simply take \( N_m \) independent Gaussian samples \( G = \{G_n\}_{1 \leq n \leq N_m} \) using the Gaussian quantile function with mean 0 and variance 1, and “color” them appropriately by taking \( \mu + VG \), where \( \mu = \text{mean}(A_m | H) \) and \( VV^T = \text{cov}(A_m | H) \) is the Cholesky decomposition.

Once a sequence of samples \( \{(a_i, h_i)\} \) has been obtained, we can use the fact that for each \( a \) there is exactly one \( c \in K \) with \( c = f(a) \), so the delta function in the performance integral \( \{3\} \) never has to be computed or approximated explicitly. Instead, we discretize the decision space \( K \), and for each sample
we find the closest \(c\) in the discretized space and add \(\prod_{m=1}^{M} d_{A_m|H}(a_m, h)\) to the sum corresponding to that \((c, h)\) pair, effectively producing a weighted histogram of \(\{f(a|h)\}\) to determine \(d_{C|H}\). The Bayes risk may be found from the same samples \(\{(a, h)\}\) directly, or from the performance by computing \(\int_{X} K \cdot \text{cost}(c, h) d_{C|H}(c, h) d_{H}(h) dc dh\). Standard confidence bounds on the estimated Bayes risk and performance function can be found from the central limit theorem, which also holds for dependent variables with sufficiently good ergodicity or mixing properties (as is the case with some MCMC sampling patterns [13]).

For example, let \(B_L(W)\) be the Monte Carlo estimate of the Bayes risk \(B(W)\) under the cost function \(W\) with \(L\) samples taken from the proposal distribution with \(H' = H\). Then for any confidence level \(0 < R < 1\), as \(L \to \infty\),

\[
P \left( \left| B_L(W) - B(W) \right| < \left( \frac{2}{L} \int_{X} (W(f(a) - h) - B(W))^2 G(a, h) da dh \right)^{1/2} \text{erf}^{-1}(R) \right) \to R,
\]

and for fusion problems and costs covered by Theorem 4, this implies that for sufficiently large \(L\),

\[
P \left( \left| B_L(W) - B(W) \right| < \sqrt{\frac{2}{L} \left( B_L(W^2)^{1/2} + B_L(W) \right)} \text{erf}^{-1}(R) \right) \geq R.
\]

We use the approach discussed here to study a larger version of the fusion scenario considered in Proposition 6. In Figure 6, we take \(M = 300\) total features in Proposition 6 with 60000 i.i.d. points sampled from the proposal distribution with a uniform \(H'\). The performance can be compared with Figure 2 and, as expected, is much more concentrated along the main diagonal, with a correspondingly lower Bayes risk. Note that as we increase the number of features \(M\), the quadratic Bayes risk decays at least as fast as \(O(\frac{1}{M})\) by Theorem 5 so we need roughly \(L = O(M)\) sampling points to achieve a fixed relative error in the calculation.

![Figure 6: Fusion configuration from Proposition 6 with \(u = v = 1\) and \(M = 300\).](image)

5 Conclusion

We have described a mathematical and computational framework for analyzing the expected performance of deterministically combining statistical information under a specified optimality criterion. These results can be applied to many diverse situations, both in the sensors field as well as other domains, and can also
be extended in a number of other directions. In particular, many applications involve online formulations of
this problem where the sensor statistics are not known in advance and need to be estimated from real-time
data streams, and will be explored in future work.

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Appendix A: Proofs of Theorems

Proof of Proposition[7] Let $J = \prod_{m=1}^M J_m$ be the joint feature space with dimension $N = \sum_{m=1}^M N_m$. The minimum Bayes risk is

$$
\inf_{\{C: C(\omega) \in K\}} E(\text{cost}(C, H)) = \inf_{d_{C|A}} \int_{J \times K} \text{cost}(c, h) d_{C|A}(c, \cdot) \left( \prod_{m=1}^M d_{A_m|H}(a_{m, h}) \right) d_H(h) dc da dh
$$

subject to the constraint that for every $a \in J$, $d_{C|A}(\cdot, a)$ is a probability density, or in other words,

$$
\int_K d_{C|A}(c, a) dc = 1, \quad d_{C|A}(c, a) \geq 0.
$$

In addition, we want the fusion rule to be deterministic, so for all $a \in J$, there is a single finite point $f = f(a)$ in $K$ such that

$$
\text{supp}(d_{C|A}(\cdot, a)) = \{f(a)\}.
$$

The condition (8) makes the problem nonconvex, but it can be simplified as follows. Let $a$ be fixed. We want to show that $d_{C|A}(c, a) = \delta(c - f(a))$. The condition (8) is saying that there is a point $f$ such that $\int_K d_{C|A}(c, a) \phi(c) dc = 0$ for all test functions $\phi \in C^0_c(K)$ such that $\phi(f) = 0$. Since $d_{C|A}$ is compactly supported, this also holds for the larger class $\phi \in C^0(K)$ with $\phi(f) = 0$. Any $\tilde{\phi} \in C^0(K)$ can be put into this form by writing $\phi = \tilde{\phi} - \tilde{\phi}(f)$, and using (6) along with linearity implies that $\int_K d_{C|A}(c, a) \tilde{\phi}(c) dc = \tilde{\phi}(f)$. Since any generalized function in $D^0(K)$ is uniquely determined by its action on $C^0_c(K)$[22], it follows
that \( d_{C|A}(c, a) = \delta(c - f(a)) \), which implies (7) as well. This means that the minimization problem (5) is equivalent to

\[
\inf_{\{f : f(J) \leq K\}} \int_{J} \text{cost}(f(a), h) \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da \, dh,
\]

where there are no additional constraints.

We first consider the optimization problem (9) over the larger space defined by

\[
L^{2,H}(J) = \{ f : \| f \|_{L^{2,H}(J)} = \left( \int_{J} f(a)^{2} \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da \, dh \right)^{1/2} < \infty \}
\]

and without any other constraint. In this case, (9) becomes a standard problem of minimizing a positive quadratic form over a Hilbert space, with at least one feasible solution since \( E(H^2) < \infty \). There exists a unique solution to this problem in \( L^{2,H}(J) \) [18, 22], which is thus unique pointwise up to sets with zero \( N \)-dimensional Lebesgue measure. To find this solution, we set up the Euler-Lagrange equation [10]

\[
\frac{\partial}{\partial f} \int_{J} \frac{1}{2} (f - h)^{2} \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da \, dh = 0,
\]

which can be solved to obtain

\[
f(a) = \frac{\int_{J} h \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da}{\int_{J} \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da}.
\]

This is the only stationary point of the functional in (9) and the only candidate for a minimum. The denominator in (12) is simply the joint density \( d_{A}(a) \), and since \( d_{A_{m}|H}(a_{m}, h) > 0 \), it follows that \( d_{A}(a) > 0 \) as well. If either \( \sup(h : h \in K) \) or \( \inf(h : h \in K) \) are finite, we can check that

\[
\sup(f(J) : a \in J) \leq \sup_{a \in J} \frac{\int_{J} \sup(h : h \in I) \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da}{\int_{J} \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) d_{H}(h) \, da}
\]

\[
= \sup(h : h \in I)
\]

\[
\leq \sup(h : h \in K),
\]

and in the same manner, \( \inf f(J) \geq \inf(h : h \in K) \), so \( f \) is a feasible solution for (9) and is thus in fact the optimal fusion rule.

\[ \square \]

**Proof of Theorem 2.** First, note that if for all \( c \in K \), the level sets \( \{ a \in J : c = f(a) \} \) have zero \( N \)-dimensional measure, then the performance is given by

\[
d_{C|H}(c, h) = \int_{J} \delta(c - f(a)) \left( \prod_{m=1}^{M} d_{A_{m}|H}(a_{m}, h) \right) da.
\]

The integral of (14) over \( c \in K \) is always 1 since \( d_{C|H} \) is a probability density. If the level sets have positive measure, (14) is no longer well defined, but we can still examine a “smoothed out” version of (14).
Proof of Theorem 3. To simplify the notation, we let 

\[ \text{by considering the Fourier transform (or characteristic function) of } d_{c|H}, \text{ given by } E(e^{-2\pi iCz}|H). \]

Since \( \int_J d_{A_m|H}(a_m, h) da_m = 1, \)

\[
\sup_{z \in \mathbb{R}} \left| \frac{\partial^l}{\partial z^l} E(e^{-2\pi iCz}|H) \right| = \sup_{z \in \mathbb{R}} \left| E((-2\pi i)^l e^{-2\pi iCz}|H) \right|
\]

\[
= \sup_{z \in \mathbb{R}} \left| \int_J (-2\pi i f(a))^l e^{-2\pi iCz} \left( \prod_{m=1}^M d_{A_m|H}(a_m, H) \right) da \right|
\]

\[
\leq \left( 2\pi \|f\|_{L^\infty(J)} \right)^l
\]

\[
\leq (2\pi \max(|c| : c \in K))^l
\]

for every \( l \geq 0. \)

Now for the \( m' \)th sensor and \( n \)th feature that satisfies \( J \), the fact that \( d_{A_m|H} \) are all positive on \( I \) implies

\[
\left| \frac{\partial f(a)}{\partial (a_{m'})_n} \right| = \frac{1}{|d_{A}(a)|^2} \left| \int_I \int_I \left( d_{A_m|H}(a_m, \bar{h}) \frac{\partial d_{A_{m'}|H}(a_{m'}, h)}{\partial (a_{m'})_n} - d_{A_{m'}|H}(a_{m'}, h) \frac{\partial d_{A_m|H}(a_m, \bar{h})}{\partial (a_{m'})_n} \right) \times \left( \prod_{m \neq m'} d_{A_{m'}|H}(a_{m'}, h) d_{A_m|H}(a_m, \bar{h}) \right) dh d\bar{h} \right| > 0.
\]

This means that \( \nabla f(a) \) has a nonzero component for every \( a \in J \). We form a stationary phase approximation of the Fourier transform (see \[10,12\]) by integrating by parts \( l \) times,

\[
E(e^{-2\pi iCz}|H) = \left| \int_I \int_J \left( T_f^l \right) e^{-2\pi iCz} \left( \prod_{m=1}^M d_{A_m|H}(a_m, H) \right) da \right|
\]

\[
= \frac{1}{(2\pi)^l} \left| \int_J e^{-2\pi iCz} \left( T_f^l \right) \left( \prod_{m=1}^M d_{A_m|H}(a_m, H) \right) da \right|
\]

\[
\leq \frac{1}{(2\pi)^l} \left| \int_J \left( T_f^l \right) \left( \prod_{m=1}^M d_{A_m|H}(a_m, H) \right) da \right|
\]

where \( T_f \) is the linear differential operator \( T_f g(a) = \frac{\nabla g(a) \cdot \nabla f(a)}{|\nabla f(a)|^2} \) and \( T_f^* g(a) = -\text{div} \left( \frac{g(a) \nabla f(a)}{|\nabla f(a)|^3} \right) \) is its adjoint.

The integral in \( 16 \) is finite for each \( l \) due to the conditions on \( J \) and \( d_{A_m|H}. \) This means that for each \( h, \)

\( E(e^{-2\pi iCz}|H = h) \)

is a smooth function that decays faster than any polynomial. The inverse Fourier transform \( d_{c|H}(., h) \) is thus also a smooth function with the same decay \( 22. \)

\[ \square \]

Proof of Theorem 3. To simplify the notation, we let

\[
G(a, h) = \left( \prod_{m=1}^M d_{A_m|H}(a_m, h) \right) d_H(h).
\]

For any fusion rule \( g, \) the quadratic Bayes risk can be expanded by writing

\[
\frac{1}{2} \int_{I \times J} (g(a) - h)^2 G(a, h) da dh,
\]

\[
= \frac{1}{2} \int_{I \times J} \left( (g(a) - f^*(a))^2 + (f^*(a) - h)^2 + 2(g(a) - f^*(a))(f^*(a) - h) \right) G(a, h) da dh.
\]

(17)
For bounded $K$, the third term in (17) can be calculated using the fact that the densities $d_{A_m | H}$ and $d_H$ are all nonnegative,

$$\left| \int_{I \times J} 2(g(a) - f(a))(f(a) - h)G(a, h) da dh \right| \leq 2\|g - f\|_{L^2(J)} \int_J \left| \int_I (f(a) - h)G(a, h) dh \right| da$$

$$\leq 2|K| \int_J \left( \int_I \hat{h}G(a, \hat{h}) dh \int_I G(a, h) dh - \int_I hG(a, h) dh \right) | da$$

$$= 0. \tag{18}$$

Since (18) is independent of $K$, this in fact holds for unbounded $K$ as well. The second term in (17) is simply the Bayes risk of $f$ itself, so we only need to show that the first term (the Bayes risk of the additional contribution from the approximation error) is minimized by the feasible solution $\hat{z}$ for all $a$.

The sub-Gaussian condition on $H$ and the continuity of $d_{A_m | H}$ imply that $\int_{-\infty}^{\infty} e^{r \hat{h}^2}G(h)dh < RE(e^{r H^2})$ for some constant $R$ depending only on $a$. The Fourier transform $\hat{G}$ satisfies

$$\left| \hat{G}(z) \right| \leq \int_{-\infty}^{\infty} e^{2\pi \hat{h}|\text{Im}(z)|} e^{-r \hat{h}^2} e^{r \hat{h}^2} G(h) dh$$

$$\leq RE(e^{r H^2}) \max_{\hat{h} \in \mathbb{R}} \left( e^{2\pi \hat{h}|\text{Im}(z)| - r \hat{h}^2} \right)$$

$$= RE(e^{r H^2}) e^{\frac{-2\pi^2}{r} |\text{Im}(z)|^2} \tag{19}$$

for all $z \in \mathbb{C}$. This means that the integral defining $\hat{G}$ converges uniformly in $z$ on any compact subset of $\mathbb{C}$, so $\hat{G}$ is an entire function, and the bound (19) also shows that it has order at most 2 (see [1] for a definition).

Now since $W$ is even and entire, its Taylor series around the origin converges everywhere and contains only even powers $z^{2n}$, which means that the series for $W'$ has the form

$$W'(z) = \sum_{n=1}^{\infty} w_n z^{2n-1}.$$
The Euler-Lagrange equation for the problem (9) with the cost $W(c - h)$ is given by

$$0 = \frac{\partial}{\partial f} \int_I W(f - h) \left( \prod_{m=1}^M d_{a_m|H}(a_m, h) \right) d_H(h) dh$$

$$= \sum_{n=1}^{\infty} w_n(E \ast G)(f)$$

$$= \sum_{n=1}^{\infty} w_n \int_{-\infty}^{\infty} e^{2\pi if \hat{z}^n} \hat{G}(z) dz$$

$$= \sum_{n=1}^{\infty} w_n (-2\pi i)^{2n-1} \left( \frac{d}{dz} \right)^{2n-1} \left( e^{2\pi if \hat{z}^n} \hat{G}(z) \right)_{|z=0},$$

where (21) is justified because the tempered generalized function $E$ has the (compactly supported) Fourier transform $(-2\pi i)^{2n-1} \delta^{2n-1}$, while $\hat{G}$ is a smooth function. We want to show that every term in the sum (22) is zero. Proposition 1 says that it is zero if $w_n = 0$ for $n \geq 2$, and solving (22) for $f$ in this case gives

$$f = -\frac{\hat{G}'(0)}{2\pi i \hat{G}(0)},$$

where $\hat{G}(0) = d_A(a) > 0$ as in Proposition 1. Now let $\{\lambda_k\}_{k \in \mathbb{Z}}$ be the zeros of $\hat{G}$ in the right half of the complex plane, indexed in order of increasing absolute value. Since $E(e^{-2\pi i z} |A = a) = \hat{G}(z)$, our assumptions say that for each $\lambda_k$, $\hat{G}$ has another zero $-\lambda_k$ in the left half plane. We expand $\hat{G}$ using the Hadamard factorization theorem (1) for functions of order 2,

$$e^{2\pi if z} \hat{G}(z) = e^{2\pi if z} \hat{G}(0) e^{\frac{\partial(\omega)}{\partial(\omega^2)} z^2 + \frac{\partial(\omega^2)}{\partial(\omega^2)} z^4} \prod_k \left( 1 - \frac{z^2}{\lambda_k^2} \right) e^{\frac{z^2}{\lambda_k^2}} \left( 1 + \frac{z^2}{\lambda_k^2} \right) e^{-\frac{z^2}{\lambda_k^2}}.$$

This formula shows that the mapping $z^2 \rightarrow e^{2\pi if z} \hat{G}(z)$ has an analytic branch, or in other words, there is an entire function $T$ such that $T(z^2) = e^{2\pi if z} \hat{G}(z)$. This means that the Taylor series expansion of $e^{2\pi if z} \hat{G}(z)$ around $z = 0$ can only contain even powers $z^{2n}$, $n > 0$, with the odd terms all being zero. Therefore, each term in (22) is zero and $f$ satisfies the Euler-Lagrange equation (20) for any $W$. We only need to show that $f$ is in fact a minimum of (9), which can be done using the standard “direct method” from the calculus of variations (see [10] p. 443-453) and (3) for details). By the conditions on $W$, the functional in (9) is nonnegative and satisfies the bounds

$$\alpha_1 \|f\|_{L^p, H(J)} - \alpha_1 \int_{I \times J} |h|^p G(a, h) \, da \, dh \leq \int_{I \times J} W(f(a) - h) G(a, h) \, da \, dh$$

$$\leq \alpha_2 \|f\|_{L^p, H(J)} + \alpha_2 \int_{I \times J} |h|^p G(a, h) \, da \, dh,$$

where $L^p, H(J)$ is defined in the same manner as (10). The lower bound (23) together with the convexity of $W$ imply that the functional (9) is weakly lower semicontinuous and has a minimum in $L^p, H(J)$ [10] p. 448]. On the other hand, the upper bound (24) and the convexity of $W$ imply that any feasible solution of (20) is in fact a minimum of (9) [10] p. 452].
Proof of Theorem 5. Consider the fusion rule $g$ given by the sample mean $g(a) = \frac{1}{M} \sum_{m=1}^{M} a_m$. Since $\text{var}(A_m|H)$ is uniformly bounded, one form of the law of large numbers \cite{7} shows that $g(A|H)$ converges almost surely (and in distribution) to $H$. Since $H \in I$, this also means that $g(A|H)$ converges almost surely for sufficiently large $M$, so $g$ is in the feasible set of $\mathcal{F}$. For any test function $\phi \in C^0_c(I \times K)$, 

$$
\int I \int K \phi(c,h) d\mathcal{C}_H(c,h) d\mathcal{H}(h) dc \, dh = E(\phi(g(A),H)) \nonumber
$$

$$
= E(E(\phi(g(A|H),H))) \nonumber
$$

$$
\rightarrow E(\phi(H,H)) \nonumber
$$

$$
= \int I \phi(h,h) d\mathcal{H}(h) dh, \nonumber
$$

Since $d_H > 0$, this is equivalent to saying that

$$
\int I \int K \phi(c,h) d\mathcal{C}_H(c,h) dc \, dh \rightarrow \int I \phi(h,h) dh,
$$

which is the statement that $d_{\mathcal{C}_H}(c,h) \rightarrow \delta(c-h)$ in the weak-$\ast$ sense. Under any cost function for which $f$ given by \cite{1} is optimal and cost$(h,h) = 0$, we end up with

$$
E(\text{cost}(f(A),H)) \leq E(\text{cost}(g(A),H)) \rightarrow 0. \tag{25}
$$

Now if $A$ and $H$ satisfy the conditions of Theorem 4 then the limit \cite{4} holds for all appropriate costs $W(c-h)$ with $W(0) = 0$. We want to show that the costs $(c-h)^p$ for even $p$ are numerous enough to approximate anything in a space of test functions that $d_{\mathcal{C}_H}$ acts on. Since $d_{\mathcal{C}_H}$ is a bounded and nonnegative function, \cite{4} implies that for each $h \in I$,

$$
\int K W(c-h) d\mathcal{C}_H(c,h) dc \rightarrow W(0). \tag{26}
$$

This obviously holds for constant functions $W$ as well, where the Bayes risk is independent of $f$ or $M$. This means that \cite{6} also holds for functions in the linear span

$$
\mathcal{S} = \{ W(x) = \sum_{l=0}^{L} d_l x^{2l} : x = c-h \in \Lambda, d_l \in \mathbb{R} \},
$$

where $\Lambda = \{ c-h : c \in K, h \in I, c-h \geq 0 \}$. Since

$$
\int K W(c-h) d\mathcal{C}_H(c,h) dc \leq \|W\|_{L^\infty(\Lambda)},
$$

the limit \cite{6} holds uniformly over all $W \in \mathcal{S} \cap \mathcal{B}$, where $\mathcal{B}$ is the closed unit ball $\mathcal{B} = \{ W : \|W\|_{L^\infty(\Lambda)} \leq 1 \}$. It is easy to check that $\mathcal{S}$ is a vector space and an associative algebra (i.e., the product of two functions in $\mathcal{S}$ is also in $\mathcal{S}$), and for any two distinct points $x_1$ and $x_2$ in $\Lambda$, we can find $W \in \mathcal{S}$ such that $W(x_1) \neq W(x_2)$ by taking $W(x) = x^2$. Therefore, the Stone-Weierstrass theorem \cite{20} shows that $\mathcal{S}$ is dense in $C^0(\Lambda)$, and consequently that $\mathcal{S} \cap \mathcal{B}$ is dense in $C^0(\Lambda) \cap \mathcal{B}$. From \cite{26} and the fact that $d_{\mathcal{C}_H}(\cdot,h)$ is even, we conclude that $d_{\mathcal{C}_H}(\cdot,h) \rightarrow \delta(\cdot-h)$ for every $h \in I$. \hfill \Box
Proof of Proposition 6. To simplify the notation, we define \( A_{m+M/2} = B_m \) for \( 1 \leq m \leq M/2 \). This just reflects the fact that two sensors with \( M/2 \) independent (given \( H \)) features each are equivalent to \( M \) sensors with one such feature each, or to a single sensor with \( M \) such features. We have

\[
 f(A) = \frac{\int_{-\infty}^{\infty} h \prod_{m=1}^{M} \left( (2\pi v)^{-1/2} e^{-(A_m - uh)\sigma^2/(2v)} \right) \left( (2\pi)^{-1/2} e^{-h^2/2} \right) dh}{\int_{-\infty}^{\infty} \prod_{m=1}^{M} \left( (2\pi v)^{-1/2} e^{-(A_m - uh)\sigma^2/(2v)} \right) \left( (2\pi)^{-1/2} e^{-h^2/2} \right) dh} 
\]

\[
 = \frac{\int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-\left( \frac{M^2}{2\pi} + \frac{1}{2} \right) h^2 + \frac{M}{2} \sum_{m=1}^{M} A_m h} dh}{\int_{-\infty}^{\infty} \prod_{m=1}^{M} \left( (2\pi v)^{-1/2} e^{-(A_m - uh)\sigma^2/(2v)} \right) \left( (2\pi)^{-1/2} e^{-h^2/2} \right) dh} 
\]

\[
 = \frac{\left( \sum_{m=1}^{M} A_m \right) \exp \left( \frac{\sum_{m=1}^{M} A_m \sigma^2}{2(Mu^2 + v^2)} \right) \exp \left( \frac{(Mu^2 + v) v^{1/2}}{2} \right)}{\exp \left( \frac{M}{(Mu^2 + v)^2} - \frac{M^2}{2(Mu^2 + v)^2} \right) 2\pi u^2 v^2 + 2\pi u^2 v^2 H^2) \right) .} \]

Some straightforward (although technical) calculations give

\[
 d_{C|H}(c, h) = \int_{-\infty}^{\infty} E(e^{-2\pi i c H} | H = h) e^{2\pi i c} dz 
\]

\[
 = \frac{Mu^2 + v}{u \sqrt{2\pi Mu}} \exp \left( -\frac{(Mu^2(c - h) + cv)^2}{2Mu^2 v} \right) 
\]

and

\[
 E((C - H)^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (c - h)^2 d_{C|H}(c, h) (2\pi)^{-1/2} e^{-H^2/2} dcdh 
\]

\[
 = \frac{v}{u \sqrt{2\pi M(Mu^2 + v)}} \int_{-\infty}^{\infty} e^{-\frac{M^2 + v^2}{2Mv^2} c} dc 
\]

\[
 = \frac{v}{Mu^2 + v} 
\]

Finally, a similar calculation gives

\[
 E(e^{-2\pi i c H} | A) = \sqrt{\frac{v}{(Mu^2 + v)(2\pi v)^M}} \exp \left( \frac{(2\pi i v z - u \sum_{m=1}^{M} A_m)^2}{2v(Mu^2 + v)} - \frac{\left( \sum_{m=1}^{M} A_m \right)^2}{2v} \right) 
\]

so \( E(e^{-2\pi i c H} | A) \) has no zeros in the complex plane for any \( A \), and the other conditions of Theorem 4 obviously hold.
Proof of Corollary\[7\] By Proposition\[6\],\[4\], the locally optimal fusion rules at $A^*$ and $B^*$ are given by $A^* = \frac{\sum_{m=1}^{M/2} A_m}{M/2 + 1}$ and $B^* = \frac{\sum_{m=1}^{M/2} B_m}{M/2 + 1}$. These are Gaussian variables with means $\frac{M}{M/2}h$ and variances $\frac{2M}{(M+2)^2}$. Using these inputs, we apply (4) again to determine the locally optimal system fusion rule $C$. The fusion rule is $C = \frac{M^2}{2M^2} (A^* + B^*)$ and its Bayes risk is $\frac{M+2}{2M+2} \frac{\sum_{m=1}^{M/2} A_m}{M/2 + 1} = \frac{1}{M+1}$.

Proof of Proposition\[8\] From the standard integral definition of the gamma function, the fusion rule is

$$f(A) = \frac{\int_0^M h \prod_{m=1}^M (h^{-A_m}) e^{-h} dh}{\int_0^M \prod_{m=1}^M (h^{-A_m}) e^{-h} dh} = \frac{(\sum_{m=1}^M A_m + 1)^{-(M+1)!} (M+1)!}{(\sum_{m=1}^M A_m + 1)^{-(M+1)} M!} = \frac{M+1}{\sum_{m=1}^M A_m + 1}.$$

Note that we always have $C = f(A) \in (0, M+1]$. To determine its performance, the formula\[14\] is easier to apply directly instead of taking the Fourier transform.

$$d_{CH(c, h)} = \int \delta(c - f(a)) h^M e^{-h} \prod_{m=1}^M \frac{d}{dc} e^{-h} dh = \int \sum_{\sum_{m=1}^M a_m = M+1} \frac{d}{dc} e^{-h} dh |da| = (M+1)h^{M+1} c^{-2} e^{-h} \prod_{m=1}^M \frac{d}{dc} e^{-h} dh = (M+1)h^{M+1} c^{-2} e^{-h} \prod_{m=1}^M \frac{d}{dc} e^{-h} dh = \frac{(M+1)h^{M+1} c^{-2}}{M!} \left( \frac{M+1}{c} - 1 \right)^M e^{-h} \prod_{m=1}^M \frac{d}{dc} e^{-h} dh.$$

The quadratic Bayes risk can be evaluated in a similar way,

$$E((C - H)^2) = \int_0^{M+1} \frac{M+1}{M!} \left( \frac{M+1}{c} - 1 \right)^M \int_0^M (c^{-2} h^{M+1} - 2 c^{-1} h^{M+2} + h^{M+3}) e^{-h} dh dc = \int_0^{M+1} \frac{M+1}{M!} \left( \frac{M+1}{c} - 1 \right)^M \left( \frac{c^{M+2} (M+1)!}{(M+1)^{M+2}} - \frac{2 c^{M+3} (M+2)!}{(M+1)^{M+3}} + \frac{c^{M+4} (M+3)!}{(M+1)^{M+4}} \right) dc = \frac{M+3}{(M+1)^{M+2}} \int_0^{M+1} c^2 (M+1-c) M dc = \frac{2}{M+1}.$$

\[\Box\]