(1,1)-knots via the mapping class group of the twice punctured torus

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Abstract

We develop an algebraic representation for (1,1)-knots using the mapping class group of the twice punctured torus $MCG_2(T)$. We prove that every (1,1)-knot in a lens space $L(p,q)$ can be represented by the composition of an element of a certain rank two free subgroup of $MCG_2(T)$ with a standard element only depending on the ambient space. As notable examples, we obtain a representation of this type for all torus knots and for all two-bridge knots. Moreover, we give explicit cyclic presentations for the fundamental groups of the cyclic branched coverings of torus knots of type $(k,ck+2)$.

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1 Introduction and preliminaries

The topological properties of (1,1)-knots, also called genus one 1-bridge knots, have recently been investigated in several papers (see [1, 5, 6, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 21, 24, 25, 26]). These knots are very important in the light of some results and conjectures involving Dehn surgery on knots (see in particular [9] and [25]). Moreover, the strict connection between cyclic branched coverings of (1,1)-knots and cyclic presentations of groups have been pointed out in [3, 12] and [21].

Roughly speaking, a (1,1)-knot is a knot which can be obtained by gluing along the boundary two solid tori with a trivial arc properly embedded. A
more formal definition follows. A set of mutually disjoint arcs \( \{t_1, \ldots, t_b\} \) properly embedded in a handlebody \( H \) is trivial if there exist \( b \) mutually disjoint discs \( D_1, \ldots, D_b \subset H \) such that \( t_i \cap D_i = t_i \cap \partial D_i = t_i \) and \( \partial D_i - t_i \subset \partial H \) for all \( i, j = 1, \ldots, b \) and \( i \neq j \). Let \( M = H \cup \varphi H' \) be a genus \( g \) Heegaard splitting of a closed orientable 3-manifold \( M \) and let \( F = \partial H = \partial H' \); a link \( L \subset M \) is said to be in \( b \)-bridge position with respect to \( F \) if: (i) \( L \) intersects \( F \) transversally and (ii) \( L \cap H \) and \( L \cap H' \) are both the union of \( b \) mutually disjoint properly embedded trivial arcs. The splitting is called a \((g,b)\)-decomposition of \( L \). A link \( L \) is called a \((g,b)\)-link if it admits a \((g,b)\)-decomposition. Note that a \((0,b)\)-link is a link in \( S^3 \) which admits a \( b \)-bridge presentation in the usual sense. So the notion of \((g,b)\)-decomposition of links in 3-manifolds generalizes the classical bridge (or plat) decomposition of links in \( S^3 \) (see [7]). Obviously, a \((g,1)\)-link is a knot, for every \( g \geq 0 \).

Therefore, a \((1,1)\)-knot \( K \) is a knot in a lens space \( L(p,q) \) (possibly in \( S^3 \)) which admits a \((1,1)\)-decomposition

\[
(L(p,q), K) = (H, A) \cup_{\varphi} (H', A'),
\]

where \( \varphi : (\partial H', \partial A') \to (\partial H, \partial A) \) is an (attaching) homeomorphism which reverses the standard orientation on the tori (see Figure 1). It is well known that the family of \((1,1)\)-knots contains all torus knots (trivially) and all two-bridge knots (see [16]) in \( S^3 \).

In this paper we develop an algebraic representation of \((1,1)\)-knots through elements of \( \text{MCG}_2(T) \), the mapping class group of the twice punctured torus. In Section 2 we establish the connection between the two objects. In Section 3 we prove that every \((1,1)\)-knot in a lens space \( L(p,q) \) can be represented by an element of \( \text{MCG}_2(T) \) which is the composition of an element of a certain rank two free subgroup and of a standard element only depending on the ambient space \( L(p,q) \). This representation will be called “standard”. As a notable application, in Sections 4 and 5 we obtain standard representations for the two most important classes of \((1,1)\)-knots in \( S^3 \): the torus knots and the two-bridge knots. Moreover, applying certain results obtained in [5], we give explicit cyclic presentations for the fundamental groups of all cyclic branched coverings of torus knots of type \((k,ck+2)\), with \( c, k > 0 \) and \( k \) odd.

In what follows, the symbol \( L(p,q) \) will denote any lens space, including \( S^3 = L(1,0) \) and \( S^1 \times S^2 = L(0,1) \). Moreover, homotopy and homology classes will be denoted with the same symbol of the representing loops.
Let $F_g$ be a closed orientable surface of genus $g$ and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a finite set of distinguished points of $F_g$, called punctures. We denote by $\mathcal{H}(F_g, \mathcal{P})$ the group of orientation-preserving homeomorphisms $h : F_g \to F_g$ such that $h(\mathcal{P}) = \mathcal{P}$. The punctured mapping class group of $F_g$ relative to $\mathcal{P}$ is the group of the isotopy classes of elements of $\mathcal{H}(F_g, \mathcal{P})$. Up to isomorphism, the punctured mapping class group of a fixed surface $F_g$ relative to $\mathcal{P}$ only depends on the cardinality $n$ of $\mathcal{P}$. Therefore, we can simply speak of the $n$-punctured mapping class group of $F_g$, denoting it by $MCG_n(F_g)$. Moreover, for isotopy classes we will use the same symbol of the representing homeomorphisms.

The $n$-punctured pure mapping class group of $F_g$ is the subgroup $PMCG_n(F_g)$ of $MCG_n(F_g)$ consisting of the elements pointwise fixing the punctures. There is a standard exact sequence

$$1 \to PMCG_n(F_g) \to MCG_n(F_g) \to \Sigma_n \to 1,$$

where $\Sigma_n$ is the symmetric group on $n$ elements. A presentation of all punctured mapping class groups can be found in [11] and in [17].
In this paper we are interested in the two-punctured mapping class group of the torus $MCG_2(T)$. According to previously cited papers, a set of generators for $MCG_2(T)$ is given by a rotation $\rho$ of $\pi$ radians which exchanges the punctures and the right-handed Dehn twists $t_\alpha, t_\beta, t_\gamma$ around the curves $\alpha, \beta, \gamma$ respectively, as depicted in Figure 2. Since $\rho$ commutes with the other generators, we have

$$MCG_2(T) \cong PMCG_2(T) \oplus \mathbb{Z}_2.$$  

The following presentation for $PMCG_2(T)$ has been obtained in [22]:

$$\langle t_\alpha, t_\beta, t_\gamma \mid t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta, t_\alpha t_\gamma t_\alpha = t_\gamma t_\alpha t_\gamma, t_\beta t_\gamma = t_\gamma t_\beta, (t_\alpha t_\beta t_\gamma)^4 = 1 \rangle. \tag{1}$$

The group $PMCG_2(T)$ (as well as $MCG_2(T)$) naturally maps by an epimorphism to the mapping class group of the torus $MCG(T) \cong SL(2, \mathbb{Z})$, which is generated by $t_\alpha$ and $t_\beta = t_\gamma$. So we have an epimorphism

$$\Omega : PMCG_2(T) \to SL(2, \mathbb{Z})$$

defined by $\Omega(t_\alpha) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\Omega(t_\beta) = \Omega(t_\gamma) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

The group $\ker \Omega$ will play a fundamental role in our discussion. In order to investigate its structure, let us consider the two elements $\tau_m = t_\beta t_\gamma^{-1}$ and $\tau_l = t_\eta t_\alpha^{-1}$, where $t_\eta$ is the right-handed Dehn twist around the curve $\eta$ depicted in Figure 3. The effect of $\tau_m$ and $\tau_l$ is to slide one puncture (say $P_2$) respectively along a meridian and along a longitude of the torus, as shown in Figure 3. Observe that, since $\eta = \tau_m^{-1}(\alpha)$, we have $t_\eta = \tau_m^{-1} t_\alpha \tau_m$.

The following result can be obtained from [31, Th. 1] and [22, Th. 5] by classical techniques.
Proposition 1 The group \( \ker \Omega \) is freely generated by \( \tau_m = t_\beta t_\gamma^{-1} \) and \( \tau_l = \gamma_\eta t_\alpha^{-1} \), where \( t_\eta = \tau_m^{-1} t_\alpha \tau_m \).

Now, let \( K \subset L(p,q) \) be a \((1,1)\)-knot with \((1,1)\)-decomposition \((L(p,q),K) = (H,A) \cup_\varphi (H',A')\) and let \( \mu : (H,A) \to (H',A') \) be a fixed orientation-reversing homeomorphism, then \( \psi = \varphi \mu |_{\partial H} \) is an orientation-preserving homeomorphism of \((\partial H, \partial A) = (T, \{P_1, P_2\})\). Moreover, since two isotopic attaching homeomorphisms produce equivalent \((1,1)\)-knots, we have a natural surjective map from the twice punctured mapping class group of the torus \( \text{MCG}_2(T) \) to the class \( \mathcal{K}_{1,1} \) of all \((1,1)\)-knots

\[ \Theta : \psi \in \text{MCG}_2(T) \mapsto K_\psi \in \mathcal{K}_{1,1}. \]

If \( \Omega(\psi) = \begin{pmatrix} q & s \\ p & r \end{pmatrix} \), then \( K_\psi \) is a \((1,1)\)-knot in the lens space \( L(|p|,|q|) \) if and only if \( p = \pm 1 \).

As will be proved in Section 3, we have the following “trivial” examples:

i) if either \( \psi = 1 \) or \( \psi = t_\beta \) or \( \psi = t_\gamma \), then \( K_\psi \) is the trivial knot in \( S^1 \times S^2 \).
ii) if $\psi = t_\alpha$, then $K_\psi$ is the trivial knot in $S^3$.

Moreover, it is possible to prove that if $\psi = t_\alpha t_\beta t_\alpha t_\alpha t_\gamma t_\alpha$, then $K_\psi$ is the knot $S^1 \times \{P\} \subset S^1 \times S^2$, where $P$ is any point of $S^2$. So, in this case, $K_\psi$ is a standard generator for the first homology group of $S^1 \times S^2$.

Every element $\psi$ of $\text{MCG}_2(T)$ can be written as $\psi = \psi' \rho^k$, $k \in \{0, 1\}$, where $\psi' \in \text{PMCG}_2(T)$. Since $\rho$ can be extended to a homeomorphism of the pair $(H, A)$, the $(1,1)$-knots $K_\psi$ and $K_{\psi'}$ are equivalent. So, for our discussion it is enough to consider the restriction

$$\Theta' = \Theta|_{\text{PMCG}_2(T)} : \psi \in \text{PMCG}_2(T) \mapsto K_\psi \in K_{1,1}.$$ 

### 3 Standard decomposition

In this section we show that every $(1,1)$-knot $K \subset L(p,q)$ admits a representation by the composition of an element in $\ker \Omega$ and an element which only depends on $L(p,q)$. A representation of this type will be called “standard”. Note that a similar result, using a rank three free subgroup of $\text{MCG}_2(T)$, has been obtained in [6, Theorem 3].

First of all, we deal with trivial knots in lens space. Let $T$ be the subgroup of $\text{PMCG}_2(T)$ generated by $t_\alpha$ and $t_\beta$. There exists a disk $D \subset H$, with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset T$, such that $D \cap \alpha = D \cap \beta = \emptyset$. So any element of $T$ produces a trivial knot in a certain lens space. On the other hand, any trivial knot in a lens space admits a representation through an element of $T$, as will be proved in Proposition 3.

We need a preparatory result.

**Lemma 2** Let $K$ be a $(1,1)$-knot in $L(p,q)$. Then, for each $r, s \in \mathbb{Z}$ such that $qr - ps = 1$ there exists $\psi \in \text{PMCG}_2(T)$, with $\Omega(\psi) = \begin{pmatrix} q & s \\ p & r \end{pmatrix}$, such that $K = K_\psi$.

**Proof.** Let $K = K_\psi$, with $\Omega(\tilde{\psi}) = \begin{pmatrix} q & s \\ p & \bar{r} \end{pmatrix}$. Since $q\bar{r} - p\bar{s} = 1$, there exist $c \in \mathbb{Z}$ such that $r = \bar{r} + cp$ and $s = \bar{s} + cq$. If $\psi = \tilde{\psi} t_{-\bar{c}}$, we have $K_\psi = K_{\psi'}$. 


since $t^{-c}$ can be extended to a homeomorphism of the pair $(H, A)$. Moreover
\[
\Omega(\psi) = \Omega(\overline{\psi})\Omega(t^{-c}) = \begin{pmatrix} q & s \\ p & r \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & s + cq \\ p & r + cp \end{pmatrix}.
\]

For integers $p, q$ such that $0 < q < p$ and $\gcd(p, q) = 1$ consider the sequence of equations of the Euclidean algorithm (with $r_0 = p$, $r_1 = q$):
\[
r_0 = a_1 r_1 + r_2 \\
r_1 = a_2 r_2 + r_3 \\
\vdots \\
r_{m-2} = a_{m-1} r_{m-1} + r_m \\
r_{m-1} = a_m r_m,
\]
with $r_1 > r_2 > \cdots > r_{m-1} > r_m = 1$.

The $a_i$’s are the coefficients of the continued fraction
\[
\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_m}}}.\]
In the following we will use the notation $p/q = [a_1, a_2, \ldots, a_m]$.

**Proposition 3**  
- The trivial knot in $S^3 = L(1, 0)$ is represented by $\psi_{1,0} = t_\beta t_\alpha t_\beta$.

- The trivial knot in $S^1 \times S^2 = L(0, 1)$ is represented by $\psi_{0,1} = 1$.

- Let $p, q$ be integers such that $0 < q < p$ and $\gcd(p, q) = 1$. If $\frac{p}{q} = [a_1, a_2, \ldots, a_m]$, then the trivial knot in the lens space $L(p, q)$ is represented by

\[
\psi_{p,q} = \begin{cases} t_\alpha t^{-a_2} \cdots t^{-a_m} & \text{if } m \text{ is odd} \\
t_\alpha t^{-a_2} \cdots t^{-a_m} t_\beta t_\alpha t_\beta & \text{if } m \text{ is even} \end{cases}
\]

**Proof.** Since all the involved homeomorphisms belong to $\mathcal{T}$, all the knots are trivial. It is easy to check (see also [4, p. 186]) that, for suitable $r, s \in \mathbb{Z}$, we have:
\[
\begin{pmatrix} q & s \\ p & r \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix} & \text{if } m \text{ is odd}, \\
\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{if } m \text{ is even}. \end{cases}
\]
Since $\Omega(t^0_\alpha) = \begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix}$, $\Omega(t^1_\beta) = \begin{pmatrix} 1 & -a_i \\ 0 & 1 \end{pmatrix}$, and $\Omega(t_\beta t_\alpha t_\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the statement is obtained. 

Now we can prove the result announced at the beginning of this section.

**Theorem 4** Let $K$ be a $(1, 1)$-knot in $L(p, q)$. Then there exist $\psi', \psi'' \in \ker \Omega$ such that $K = K_\psi$, with $\psi = \psi' \psi_{p,q} = \psi_{p,q} \psi''$.

**Proof.** By Lemma 2, there exists $\psi$, with $\Omega(\psi) = \Omega(\psi_{p,q})$, such that $K = K_\psi$. It suffices to define $\psi' = \psi \psi_{p,q}^{-1}$ and $\psi'' = \psi_{p,q}^{-1} \psi$. 

A representation $\psi \in PMCG_2(T)$ of a $(1, 1)$-knot will be called **standard** if $\psi$ is of the type described in the previous theorem.

We point out that $(1, 1)$-knots admit different (usually infinitely many) standard representations. For example $\tau_m^c$ represents the trivial knot in $S^1 \times S^2$, for all $c \in \mathbb{Z}$.

### 4 Representation of torus knots

In this section we give a standard representation for all torus knots in $S^3$. Let $K = t(k, h)$ be a torus knot of type $(k, h)$. Then $\gcd(k, h) = 1$, and we can assume that $K$ lies on the boundary $T = \partial H$ of a genus one handlebody $H$ canonically embedded in $S^3$. The homology class of $K$ is $hl + km$, where $l$ and $m$ respectively denote a longitude and a meridian of $T$. By slightly pushing (the interior of) an arc $A' \subset K$ outside $H$ and $K - A'$ inside $H$, we obtain an obvious $(1, 1)$-decomposition of $K$. Observe that $0 < |k| < h$ can be assumed without loss of generality (see [4, p. 45]).

In the next statement $\lfloor x \rfloor$ denotes the integral part of $x$.

**Theorem 5** The torus knot $t(k, h) \subset S^3$ is the $(1, 1)$-knot $K_\psi$ with:

$$\psi = \prod_{i=1}^{h} (\tau_{m}^{\lfloor (i-1)k/h \rfloor - \lfloor ik/h \rfloor} \tau_{l}^{-1}) t_\beta t_\alpha t_\beta,$$

where $\tau_m = t_\beta t_{\gamma}^{-1}$ and $\tau_l = \tau_{m}^{-1} t_\alpha \tau_{m} t_{\alpha}^{-1}$.

**Proof.** Up to isotopy, we can suppose that the arc $A = K_\psi - \text{int}(A')$ lies on $\partial H$, as in Figure 4. The arc $A$ can be transformed into an arc $\bar{A}$ in
such a way that $\tilde{A} \cup A'$ is a trivial knot in $S^3$, represented by the standard homeomorphism $\psi_{1,0} = t_\beta t_\alpha t_\beta$, via a suitable sequence of homeomorphisms $\tau_l$ and $\tau_m$, according to the following algorithm. Consider the sequence of equations:

$$
k = q_1 h + r_1,
2k = q_2 h + r_2,
\vdots
hk = q_h h + r_h,
$$

where $0 \leq r_i < h$, for $i = 1, \ldots, h$. Moreover, define $q_0 = 0$. So $q_i = \lfloor ik/h \rfloor$, for $i = 0, 1, \ldots, h$. Now define the homeomorphisms $\psi_i = \tau_l^{h(q_i-q_{i-1})}$, for $i = 1, \ldots, h$. Figure 5 depicts the effect of $\tau_l$ and $\tau_l \tau_m$ on $A$. As a consequence, the homeomorphism $\phi = \psi_h \psi_{h-1} \cdots \psi_1$ transforms the arc $A$ into the arc $\tilde{A}$ (Figure 7 shows the case $t(5,7)$), and therefore we have $\psi_{1,0} = \phi \psi$. So $\phi^{-1}\psi_{1,0}$ represents the torus knot $t(k, h)$. ■

For example, $t(5,7) = K_{\psi}$, with $\psi = \tau_l^{-1}(\tau_m^{-1})^2 \tau_l^{-1}(\tau_m^{-1})^3 t_\beta t_\alpha t_\beta$ (see Figure 6).

As a consequence, we obtain a cyclic presentation for the fundamental group for all cyclic branched coverings of a particular class of torus knots.

**Proposition 6** The fundamental group of the $n$-fold cyclic branched covering of the torus knot $t(k, ck+2)$, with $k > 1$ odd and $c > 0$, admits the cyclic
presentation $G_n(w)$, where $w$ is equal to

\[
\prod_{i=0}^{(k-3)/2} \prod_{j=0}^{c(k-1)/2} x_{1-i(ck+2)+jk} \prod_{l=0}^{c(k+1)/2} x_{ck(k-1)/2-\gamma(ck+2)-\alpha k} \prod_{m=0}^{c(k-1)/2} x_{1-(k-1)(ck+2)/2+mk}
\]

(subscripts are taken modulo $n$).

**Proof.** Let $r = (k - 1)/2$. From Theorem 5 we have $t(k, ck + 2) = K\psi$ with $\psi = (\tau_1^{c\tau_m^{-1}}\tau_1^{c\tau_m^{-1}}\tau_1^{c\tau_m^{-1}})^r\tau_1^{c\tau_m^{-1}}\tau_1^{c\tau_m^{-1}}t_\beta t_\beta$. Applying Proposition 1], we obtain

\[
\pi_1(S^3 - t(k, ck + 2)) = \langle \alpha, \gamma \mid r(\alpha, \gamma) \rangle,
\]

with $r(\alpha, \gamma) = (\gamma^{-1}(\alpha^{cr+1}\gamma^{-1}(\alpha^{-c(r+1)^{-1}}))\gamma^{-1}\alpha^{cr+1})$. Then $H_1(S^3 - t(k, ck + 2)) = \langle \alpha, \gamma \mid \alpha - k\gamma \rangle$. Since, up to equivalence, $\omega_f(\gamma) = 1$, we have $\omega_f(\alpha) = k$. We set $\alpha = x\gamma^k$, therefore

\[
\pi_1(S^3 - t(k, ck + 2)) = \langle x, \gamma \mid \bar{r}(x, \gamma) \rangle,
\]

with $\bar{r}(x, \gamma) = (\gamma^{-1}(x\gamma^k)^{1+c(k-1)/2}\gamma^{-1}(\gamma^{-k}x\gamma^{-c(k+1)/2})(k-1)^{1+ck+1}(c(k-1)/2)}$. The statement derives from a straightforward application of [5, Theorem 7],

For example, the fundamental group of the $n$-fold cyclic branched covering
Figure 6: Trivialization of $t(5, 7)$. 

11
of \( t(5, 7) \) admits the cyclic presentation \( G_n(w) \), where
\[
w = x_{15}x_{20}x_{25}x_{24}^{-1}x_{19}^{-1}x_{14}^{-1}x_9^{-1}x_8x_{13}x_{18}x_{17}^{-1}x_7^{-1}x_2^{-1}x_1x_6x_{11}.
\]

5 Representation of two-bridge knots

In this section we give a standard representation for all two-bridge knots in \( S^3 \). Let \( b(a/b) \) be a non-trivial two-bridge knot in \( S^3 \) of type \((a,b)\). Then we can assume \( \gcd(a, b) = 1 \), \( a \) odd, \( b \) even and \( 0 < |b| < a \), without loss of generality (see [4, Ch. 12B]). It is known that \( b(a/b) \) admits a Conway presentation with an even number of even parameters \([2a_1, 2b_1, \ldots, 2a_n, 2b_n] \) (see Figure 7), satisfying the following relation:

\[
a/b = 2a_1 + \frac{1}{2b_1 + \frac{1}{2a_2 + \cdots + \frac{1}{2a_n}}}.
\]

![Figure 7: Conway presentation for two-bridge knots.](image)

**Theorem 7** The two-bridge knot \( b(a/b) \subset S^3 \) having Conway parameters \([2a_1, 2b_1, \ldots, 2a_n, 2b_n] \) is the \((1,1)\)-knot \( K_\psi \) with:

\[
\psi = t_{1}\tau_{m}^{-1} t_{a}^{-1} t_{1}^{\alpha_{1}} \tau_{m}^{-1} t_{2}^{-1} \cdots \tau_{m}^{-1} t_{n}^{-1} \tau_{m}^{-1} t_{2}^{-1} \cdots t_{1}^{-1} \tau_{m}^{-1} t_{a}^{-1},
\]
where \( t_{\varepsilon} = \tau_{1}^{-1} \tau_{m} \tau_{1}^{-1} \) is the right-handed Dehn twist around the curve \( \varepsilon \) depicted in Figure 8.

**Proof.** Figure 8 shows the result of the application of \( \tau_{m}^{-b_{n}} t_{\varepsilon}^{a_{n}} \cdots \tau_{m}^{-b_{1}} t_{\varepsilon}^{a_{1}} \). By applying \( \psi_{1,0} = t_{1} t_{a} t_{1} \) we obtain the two-bridge knot with Conway parameters \([2a_1, 2b_1, \ldots, 2a_n, 2b_n] \).
Figure 8: Standard representation of two-bridge knots.
Now we show that \( t_{\varepsilon} = \tau_i^{-1} \tau_m \tau_m^{-1} \) (note that no disk bounded by \( \varepsilon \) and properly embedded in \( H \) is disjoint from \( A \)). Referring to Figure 9 the following “lantern” relation \( t_{\delta_1} t_{\delta_2} = t_{\tau} t_{\varepsilon} \) holds (see [23]). So we obtain \( \zeta = t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1}(\gamma) \) and therefore \( t_{\zeta} = t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1}. \) Since \( t_{\delta_1} = t_{\delta_2} = 1 \) we have \( t_{\varepsilon} = t_{\delta_1} t_{\zeta}^{-1} t_{\beta}^{-1} = t_{\delta_1} t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta^{-1} t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1}. \) Now, using the relations of \([11]\) we get

\[
 t_{\varepsilon} = t_{\delta_1} t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} = t_\gamma t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\tau t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1} = t_\gamma t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1} = t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1} = t_\gamma^{-1} t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1} = t_\gamma^{-1} t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_{\beta}^{-1} = \tau_m^{-1} t_\alpha^{-1} t_m t_\alpha t_\tau t_\alpha^{-1} t_m^{-1} = \tau_m^{-1} t_\alpha^{-1} t_m t_\alpha t_\tau t_\alpha^{-1} t_m^{-1} = t_\eta^{-1} t_\alpha t_\tau t_\eta^{-1} t_\alpha^{-1} t_m^{-1} = t_\eta^{-1} t_\alpha t_\tau t_\alpha^{-1} t_m^{-1}.
\]

For example, the figure-eight knot \( b(5/2) \), which has Conway parameters \([2,2]\), is the knot \( K_{\psi} \) with \( \psi = t_\beta t_\alpha t_\beta^{-1} t_\varepsilon \) (see Figure 10).
Figure 10: Standard representation of the figure-eight knot.
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