

TORSION IN CHOW GROUPS OF ZERO CYCLES OF HOMOGENEOUS PROJECTIVE VARIETIES

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ABSTRACT. We give bounds on the order of torsion in the Chow group of zero dimensional cycles for isotropic Grassmannians and Brauer-Severi flag varieties. To do this, we introduce tools to understand the behavior of torsion in Chow groups with coefficients under morphisms of proper varieties.

1. Introduction

The problem of understanding torsion in the Chow group of projective homogeneous varieties has had rich and surprising applications. A topic of much investigation, it has given important insight into the study of central simple algebras and quadratic forms (see [BZZ13, Kar95, Kar96, Kar91, KM90, Mer95]). The Chow group of zero dimensional cycles plays an important role both due to its connection with rational points, and because of its connections to the structure of Chow groups in other dimensions (for example, see [KM13]).

In the last few years significant progress has been made by showing that for certain classes of projective homogeneous varieties, the Chow groups of zero dimensional cycles are torsion free [Kra10, PSZ, CM]. Despite these results, many gaps in our understanding still remain.

In this paper, we introduce an alternative approach, relying on Rost’s theory of Chow groups with coefficients as developed in [Ros96], in order to obtain results for new classes of projective homogeneous varieties. Instead of proving the non-existence of torsion for zero dimensional cycles, we give bounds on the order of torsion.

Given a geometrically rational variety $X$ defined over a field $k$, the Chow group of zero dimensional cycles of degree 0 is always a torsion group. More precisely, it is not hard to see by a standard restriction-corestriction argument, that if $X_L$ is a rational variety for $L/k$ a finite field extension, then the order of every element of this group divides $[L:k]$. On the other hand, in many cases, we know that these bounds on the torsion are not sharp. In fact, for many classes of projective homogeneous varieties, it is shown in [CM, Kra10, PSZ], that this group is trivial in many cases despite the varieties themselves being far from rational. It is natural to ask, therefore, in which cases does this group possess nontrivial torsion, and how can one give bounds on its order?

In this paper, we show for certain classes of varieties, namely isotropic Grassmannians of quadratic forms, and Brauer-Severi flag varieties for algebras of period 2, one may consistently achieve better bounds on torsion than would be implied by the above restriction-corestriction argument.
It is also an interesting and open question to understand to what extent torsion in Chow groups of zero dimensional cycles on one homogeneous variety contributes to the torsion in the Chow groups of higher dimensional cycles of other homogeneous varieties via natural Chow correspondences.

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2. Notation and Preliminaries

Let $k$ be a field. We let $K^M_\bullet(k)$ denote the Milnor $K$-theory ring of $k$. We will make frequent use of the machinery of Rost’s “Chow groups with coefficients,” as defined in [Ros96]. We recall in particular, that a cycle module $M$ (see [Ros96, Def.2.1]), defines for every field extension $L/k$, a graded Abelian group $M(L) = M_\bullet(L)$, which is a graded module for the Milnor $K$-theory ring $K^M_\bullet(k)$. These are also equipped with, restriction maps $r_{L/k} : M(k) \to M(L)$ for every field extension $L/k$ and for finite field extensions $L/k$, corestriction maps $c_{L/k} : M(L) \to M(k)$ which are compatible with the corresponding maps in Milnor $K$-theory (see [Ros96, Def. 1.1]).

For a cycle module $M$ and a variety $X$, setting

$$C_p(X; M, q) = \coprod_{x \in X_{(q)}} M_q(k(x)),$$

we obtain a complex

$$\xymatrix{ & C_{p+1}(X; M, q + 1) \ar[rr]^{d_X} & & C_p(X; M, q) \ar[rr]^{d_X} & & C_{p-1}(X; M, q - 1) \ar[rr] & & }$$

Following [Ros96, (3.2)], we denote the kernel of the map $C_p(X; M, q) \to C_{p-1}(X; M, q - 1)$ by $Z_p(X; M, q)$, and the homology at the middle term by $A_p(X; M, q)$. For a class $\alpha \in Z_p(X; M, q)$, we let $[\alpha]$ denote its class in $A_p(X; M, q)$.

We refer to a pair $(z, m)$ where $z \in X_{(q)}$ and $m \in M_q(z)$ as a prime chain, and identify it with $m \in M_q(z) \subset \coprod_{x \in X_{(q)}} M_q(k(x)) = C_p(X; M, q)$. If $d_X(z, m) = 0$, we refer to it also as a prime cycle. We will occasionally need to use the notation $(z, m)_X$ if it is necessary to make the variety $X$ explicit.

We recall that the functors $C_p(\_ ; M, q)$, $Z_p(\_ ; M, q)$, and $A_p(\_ ; M, q)$ are covariant with respect to proper maps. If $\pi : X \to Y$ is proper, then we denote all the corresponding pushforward maps by $\pi_*$. For a prime chain $(x, m) \in C_p(X; M, q)$, these are defined by $\pi_*(x, m) = 0$ if $k(x)/k(\pi(x))$ is not a finite field extension, and otherwise, we set $\pi_*(x, m) = (\pi(x), c_{k(x)/k(\pi(x))}(m))$.

Definition 2.1. Let $X$ be a proper variety. We let $A_0(X; M, q)$ denote the kernel of the map $A_0(X; M, q) \to A_0(Spec k; M, q) = M_q(k)$ induced by the pushforward for the structure map $X \to \text{Spec } k$. 2
Definition 2.2. Let $X$ be a variety over $k$. We write $A_p(X, \mathbb{Z})$ for $A_p(X; K^*_M, 0) = CH_p(X)$, and $A_0(X, \mathbb{Z})$ for $A_0(X; K^*_M, 0)$.

We note that this notation differs somewhat from that used in [Kra10], but is more consistent with the notation of [Ros96].

Definition 2.3. Let $M$ be a cycle module over $k$, let $F/k$ be an arbitrary field extension, and let $X/F$ be a variety. Let $M_F$ denote the cycle module restricted to the extension fields of $F$. We define $t(X; M, q)$ to be elements of $K^*_M(k)$ which annihilate $A_0(X; M_F, q)$. That is:

$$t(X; M, q) = \text{ann}_{K^*_M(k)} A_0(X; M_F, q).$$

We set $t_0(X; M, q) = t(X; M, q) \cap K^*_0 = t(X; M, q) \cap \mathbb{Z}$.

For a morphism $\pi : X \to Y$, and $W \to Y$ a morphism, we let $X_W$ denote the fiber product $X \times_Y W$.

Definition 2.4. Let $\pi : X \to Y$ be a proper morphism. We set $t(\pi_0(M), M)$ (respectively $t_0(\pi_0(M), M)$) to be the intersection of the ideals $t(X_y, M_{k(y)})$ (resp. $t_0(X_y, M_{k(y)})$) as $y$ ranges over all the points of $Y$ of dimension $p$.

Definition 2.5. Let $X$ be a variety over $k$. We define $\text{ind}(X)$ to be the greatest common divisor of the degrees of all finite field extensions $L/k$ such that $X(L) \neq \emptyset$.

Definition 2.6. Let $\pi : X \to Y$ be a morphism. We define $\text{ind}_p(\pi)$ to be the least common multiple of all integers of the form $\text{ind}(X_y)$ over $y \in Y(\mathbb{p})$.

3. Behavior of torsion in Chow groups under morphisms

Lemma 3.1. Let $\pi : X \to Y$ be a morphism. Then

$$(\text{ind}_p(\pi)) C_p(Y; M, q) \subset \pi_* C_p(X; M, q).$$

Proof. To prove the statement, it is enough to show that for every prime chain (i.e. each chain consisting of a single term) $(y, m)$, $y \in Y(\mathbb{p})$ and $m \in M_y(k(y))$, that $\text{ind}_p(\pi)(y, m) \in \pi_* C_p(X; M, q)$. By definition of the index, we may find a collection of points $x_1, \ldots, x_r \in X(\mathbb{p})$ lying over $y$ such that $x_i \equiv \text{Spec}(L_i)$ and $\gcd([L_i : k(y)]) | \text{ind}_p(\pi)(y, m)$. We will show that $[L_i : k(y)](y, m) \in \pi_* C_p(X; M, q)$, which will complete the proof.

To do this, we simply consider $(x_i, r_{L_i/k(y)}(m)) \in C_p(X; M, q)$. By definition of the map $\pi_*$, we find $\pi_*(x_i, r_{L_i/k(y)}(m)) = (y, [L_i : k(y)]m) = [L_i : k(y)](y, m)$ as desired.

Lemma 3.2. If $\alpha \in \ker \left( \pi_* : A_p(X; M, q) \to A_p(Y; M, q) \right)$, then there exists $\tilde{\alpha} \in Z_p(X; M, q)$ such that $[\tilde{\alpha}] = (\text{ind}_{p+1}(\pi)) \alpha$ and

$\quad \tilde{\alpha} \in \ker \left( \pi_* : Z_p(X; M, q) \to Z_p(Y; M, q) \right).$

Proof. Choose a representative $\tilde{\beta}$ for $\alpha$ in $Z_p(X; M, q)$. By definition, since the class of $\pi_\beta$ is 0, we may find $\gamma \in C_{p+1}(Y; M, q + 1)$ such that $d_\gamma \gamma = \pi_* \beta$. By Lemma 3.1, we may find a cycle $\gamma' \in C_{p+1}(X; M, q + 1)$ such that $\pi_* \gamma' = (\text{ind}_{p+1}) \gamma'$. Defining $\tilde{\alpha} = (\text{ind}_{p+1}) \tilde{\beta} - d_\gamma \gamma'$, we then have that $[\tilde{\alpha}] = [\tilde{\beta}] = \alpha$, and by [Ros96, Prop. 4.6(1)], we have

$$\pi_* \tilde{\alpha} = (\text{ind}_{p+1}) \pi_* \tilde{\beta} - \pi_* d_\gamma \gamma' = (\text{ind}_{p+1}) \pi_* \tilde{\beta} - (\text{ind}_{p+1}) d_\gamma \gamma' = 0,$$

as desired.
**Lemma 3.3.** Suppose we have a morphism of proper varieties \( \pi : X \to Y \), and \( \alpha \in Z_0(X; M, q) \) such that \( \pi_*\alpha = 0 \) in \( Z_0(X; M, q) \). Then we may find a reduced scheme \( W \), finite over \( k \), and an inclusion \( j : W \to Y \), such that if we consider the fiber product diagram:

\[
\begin{array}{ccc}
X_W & \xrightarrow{i} & X \\
\downarrow{\pi_W} & & \downarrow{\pi} \\
W & \xrightarrow{j} & Y
\end{array}
\]

then we have \( \alpha = i_*\beta \) for some \( \beta \in \ker((\pi_W)_*: Z_0(X_W; M, q) \to Z_0(W; M, q)) \).

**Proof.** We note that by the definition of a cycle complex [Ros96, Def. 3.2], it follows immediately that the pushforward is injective on the level of complexes for closed immersions. In fact, we may naturally identify the cycle complex of a closed subscheme with a subcomplex of the cycle complex of the overscheme. Consequently, it suffices to show that we may find \( \beta \in C_0(X_W; M, q) \) with \( i_*\beta = \alpha \), and the final conclusion will follow automatically. But this now follows by setting \( W \) to be the image of the support of the 0-dimensional chain \( \alpha \).

**Lemma 3.4.** Let \( \pi : X \to Y \) be a morphism of proper \( k \)-varieties. Then we have

\[
\ker((\pi_* : A_0(X; M, q) \to A_0(Y; M, q)))(\ind_1(\pi)t(\pi_{(0)}; M, q)) = 0.
\]

**Proof.** Let \( \alpha \in \ker((\pi_* : A_0(X; M, q) \to A_0(Y; M, q)) \). By Lemma 3.2, we may find \( \tilde{\alpha} \in Z_0(X; M, q) \) with \( [\tilde{\alpha}] = (\ind_1(\pi)\alpha) \) and \( \pi_*\tilde{\alpha} = 0 \). By Lemma 3.3, we may find a reduced scheme \( W \), finite over \( k \), and an inclusion \( j : W \to Y \), such that if we set \( i : X_W \to X \) to be the natural fiber product map, we have \( \tilde{\alpha} = i_*\beta \) for some \( \beta \in Z_0(X_W; M, q) \). From the definition of \( t \), it follows that we have \( t(\pi_{(0)}; M)[\tilde{\alpha}] = 0 \). All together this tells us:

\[
t(\pi_{(0)}; M)(\ind_1(\pi)\alpha) = t(\pi_{(0)}; M)[\tilde{\alpha}] = 0,
\]

as desired.

**Proposition 3.5.** Let \( \pi : X \to Y \) be a morphism of proper varieties over \( k \). Then

\[
t(Y; M, q)t(\pi_{(0)}; M, q) \ind_1(\pi) \subset t(X; M, q), \text{ and } \ind_0(\pi)t(X; M, q) \subset t(Y; M, q)
\]

**Proof.** Let \( \alpha \in \overline{A}_0(X; M, q) \). We need to show that

\[
t(Y; M, q)t(\pi_{(0)}; M, q) \ind_1(\pi)\alpha = 0.
\]

Since \( \pi_*\alpha \in \overline{A}_0(Y; M, q) \), it follows that \( t(Y; M, q)\pi_*\alpha = \pi_*(t(Y; M, q)\alpha) = 0 \). But now by Lemma 3.4, we conclude that \( t_0(\pi, M)(\ind_1(\pi)t(Y; M)\alpha = 0 \) as desired.

Next, suppose that \( \beta \in \overline{A}_0(Y; M, q) \). We need to show that

\[
t(X; M, q) \ind_0(\pi)\beta = 0.
\]

By Lemma 3.1, we may find \( \gamma \in \overline{A}_0(X; M, q) \) with \( \pi_*\gamma = (\ind_0(\pi))\beta \). Since \( t(X; M, q)\gamma = 0 \), the desired conclusion follows.

**Remark 3.6.** Let \( X \) be a variety and \( x \in X \) a closed point with residue field \( L = k(x) \). Let \( m \in M(L) \). Then the prime chain \( (x, m)_X \) may be identified with the pushforward of the corresponding prime chain \( (x, m)_X \).
We can see this as follows. We may view $x$ as both a morphism $\text{Spec}(L) \to X$ or $\text{Spec}(L) \to X_L$. To remove ambiguity, we will temporarily denote the first by $x$ and the second by $x'$. We have a commutative diagram:

$$
\begin{array}{ccc}
\text{Spec}(L) & \xrightarrow{x'} & X_L \\
& \searrow & \downarrow p \\
& x & X
\end{array}
$$

Identifying $A_0(\text{Spec}(L), M) = M(L)$, we have $x_*(m) = (x, m)_X$ and $x'_*(m) = (x', m)_{X_L}$. It therefore follows from the commutativity of the diagram that $p_*(x', m)_{X_L} = (x, m)_X$ as claimed.

**Lemma 3.7.** Suppose that $X$ and $Y$ are proper varieties over $k$ such that $X_{k(Y)}$ and $Y_{k(X)}$ are both rational varieties, then $A_0(X; M) \cong A_0(Y; M)$.

**Proof.** In this case, if we consider $Z = X \times Y$, we see that $Z$ is birationally isomorphic to $X \times \mathbb{P}^{\dim Y}$ and to $Y \times \mathbb{P}^{\dim X}$. The result therefore follows from [KM13, Cor. RC.13] and [Ful98, Thm. 3.3(b)].

**Corollary 3.8.** Suppose that $X$ is geometrically rational. Then $\overline{A}_0(X; M)$ is torsion.

**Proof.** This follows from the standard restriction-corestriction argument and the fact that $\overline{A}_0(X_T, M_T)$ is trivial by Lemma 3.7.

Following [Kra10, Def. 3.15], we say that a morphism $\pi : X \to Y$ of $k$-varieties has rational fibers if for every field extension $L/k$ and every $L$-point $y \in Y(L)$, the scheme theoretic fiber $X_y$ is rational.

**Corollary 3.9.** Suppose that $\pi : X \to Y$ is morphism of proper $k$-varieties with rational fibers. Then for any cycle module $M$, we have $A_0(X; M, q) = A_0(Y; M, q)$.

**Proof.** This follows from the fact that since the generic fiber of $\pi$ is rational, we have $k(X)$ is totally transcendental over $k(Y)$. This implies that $X$ is birationally isomorphic to $Y \times \mathbb{P}^{\dim X}$, and the result then follows from [KM13, Cor. RC.13] and [Ful98, Thm. 3.3(b)].

Let $G$ be a $k$-linear algebraic group acting on a $k$-variety $X$. Following [HHK09, p. 243], we say that $G$ acts transitively on the points of $X$ if, for every field extension $L/k$, $G(L)$ acts transitively on $X(L)$.

**Proposition 3.10.** Let $G$ be a $k$-rational linear algebraic group acting on proper $k$-varieties $X$ and $Y$, and suppose $\pi : X \to Y$ is a $G$-equivariant morphism. Suppose that $G$ acts transitively on the points of $Y$. Then for $y \in Y(k)$, the inclusion:

$$i : X_y \to X$$

induces an surjective map $i_* : A_0(X_y, M, q) \to A_0(X, M, q)$.

**Proof.** Suppose that we have a prime cycle of the form $(z, m)$ where $z \in X_{(0)}$ is a closed point with residue field $L = k(z)$ and $m \in M_q(L)$. It will suffice to show that $(z, m)$ is equivalent to a prime cycle of the form $(x, n)$ for some $x$ with $\pi(x) = y$. As in Remark 3.6, if we set $p_X : X_L \to X$ (respectively $p_Y : Y_L \to Y$) to be the natural projection maps, then regarding $z$ also as a point of $X_L$, we have $(z, m)_X = (p_X)_*(z, m)_{X_L}$.

Since $G$ acts transitively on the points of $Y$, we may find a $g \in G(L)$ such that $g(\pi(z)) = y_L$. Since $G$ is $R$-trivial, we may find a rational curve $\varphi : \mathbb{P}^1_L \dashrightarrow G_L$ such that $\varphi(0) = \text{id}_G, \varphi(1) = g$. 

\[5\]
We may then define a new rational curve \( \psi : \mathbb{P}^1_L \to X_L \) in \( X_L \) by \( \psi(t) = \varphi(t)(z) \), and by the properness of \( X_L \), \( \psi \) extends to a morphism.

Consider the exact sequence obtained using [Ros96, Prop. 2.2]:

\[
C_1(\mathbb{P}^1_L; M_L, q) \xrightarrow{d} C_0(\mathbb{P}^1_L; M_L, q) \xrightarrow{r} M(L),
\]

where \( r \) is the structure morphism for the \( L \)-scheme \( \mathbb{P}^1_L \). Since the cycle \( (0, m)p_i - (1, m)p_i \) is in the kernel of \( r \), we may choose \( \alpha \in C_1(\mathbb{P}^1_L; M, q) \) with \( d_{p_i} \alpha = (0, m)p_i - (1, m)p_i \). Applying \( \psi \), we find

\[
(z, m)_{X_L} = \psi_*(0, m) \sim \psi_*(0, m) - d_{\alpha} \psi_\alpha \sim \psi_*(0, m) - \psi_*(d_{p_i} \alpha) \sim \psi_*(0, m - d_{p_i} \alpha) \sim \psi_*(1, m) = (g(z), m)_{X_L}.
\]

Now, applying \( \langle p_X \rangle \), to each side, we find

\[
(z, m)_X \sim (g(z), c_{L/k} m)
\]

But now, we note that we have \( \pi(g(z)) = g(\pi(z)) = y \) which implies that \( (g(z), c_{L/k} m) \in i_* (Z_0(X; M, q)) \). as desired. \( \square \)

4. ISOTROPIC GRASSMANNIANS

Throughout the section, we fix a field \( k \) of characteristic not 2. Suppose we have a regular quadratic form \( \phi \) over \( k \). We recall that the splitting pattern of \( \phi \), introduced and studied in [Kne76, Kne77, HR93], is defined to be the collection of all possible Witt indices which \( \phi \) attains upon field extensions of \( k \). We write these as

\[
i(\phi) = i_0 < i_1 < \cdots < i_h = \left\lfloor \frac{\dim \phi}{2} \right\rfloor,
\]

and we call \( h = h(\phi) \) the height if the form \( \phi \).

**Lemma 4.1.** Let \( (V, \phi) \) be a quadratic space and let \( W < V \) be a totally isotropic subspace. Choose a hyperbolic space \( H \) of dimension \( 2 \dim W \) containing \( W \), and write \( V = H \perp K \). Then we have a natural bijection of sets

\[
\{ W < U < V \mid U \text{ totally isotropic} \} \leftrightarrow \{ P < K \mid P \text{ totally isotropic} \}
\]

given by \( U \mapsto U \cap K \) and \( P \mapsto P + W \).

**Proof.** Clearly both maps are well defined and the composition

\[
P \mapsto P + W \mapsto (P + W) \cap K = P
\]

shows that the map \( P \mapsto P + W \) is injective with the other map as a one sided inverse. It therefore suffices to show that the map \( P \mapsto P + W \) is surjective.

Suppose that \( W < U < V \) with \( U \) totally isotropic, and let \( u \in U \). It suffices to show that we may write \( u = u' + w \) where \( w \in W \) and \( u' \in U \cap K \).

Since \( U \) is totally isotropic and \( W \subset U \), it follows that \( U \subset W^\perp = W \perp K \). We may therefore write \( u = u' + w \) where \( w \in W \) and \( u' \in K \). But since \( w \in W \subset U \) and \( u \in U \), it follows that \( u' \in U \) and hence \( u' \in K \cap U \) as desired. \( \square \)

**Proposition 4.2.** Let \( \phi \) be a regular quadratic form, and suppose that \( 2i(\phi) + 2 \leq \dim \phi \). Then \( \overline{\Lambda}_0(X_{i(\phi)+1}; \mathbb{Z}) = 0 \).
Proof. Set $X = X_{i(\phi),i+1}, Y = X_{i(\phi)}$ and let $\mathcal{X} = X_{i(\phi),i(\phi)+1}$ be the variety of isotropic flags of dimension $i(\phi), i(\phi) + 1$. Note that he statement of the Proposition is equivalent to the statement that $t_0(X, \mathbb{Z}) = \mathbb{Z}$. We have natural projection maps:

$$
\mathcal{X} \xrightarrow{\pi_1} Y \\
\pi_2 \downarrow \\
X.
$$

We may identify the fibers of $\pi_2$ over an isotropic plane $W < V$ with the Grassmannian of $i(\phi)$-dimensional subspaces of $W$. In particular, each of the fibers of $\pi_2$ is $R$-trivial. We therefore have $t_0(\pi_2, \mathbb{Z}) = \mathbb{Z}$ and $\text{ind}(\pi_2) = 1$. By Proposition 3.5 we may conclude that $t_0(\mathcal{X}, \mathbb{Z}) = t_0(X, \mathbb{Z})$.

By the definition if $i(\phi)$, $Y$ has a rational point $y \in Y(k)$. By Lemma 4.1, we may identify the fiber $\mathcal{X}_y$ with a quadric hypersurface. By [Swa89] or [Kra10] Thm 8.8, we have $\overline{A}_0(\mathcal{X}_y, \mathbb{Z}) = 0$ and therefore by Proposition 3.10 using $G = \text{SO}(\phi)$, we conclude $\overline{A}_0(\mathcal{X}, \mathbb{Z}) = 0$, completing the proof.

Lemma 4.3. Let $\phi$ be a regular quadratic form over $k$, and suppose that $m$ is a positive integer with $i_j < m \leq i_{j+1}$ for some $0 < j < h(\phi)$. Then for any cycle module $\Lambda$, we have $t_0(X_\Lambda(\phi); M, q) = t_0(X_{i_{j+1}}(\phi); M, q)$.

Proof. By Lemma 4.1 we may identify the fibers of the morphism $X_{m,i_{j+1}}(\phi) \to X_m$ with an isotropic Grassmannian of a quadratic form Witt equivalent to $\phi$, and by the definition of the splitting pattern, each of these fibers is nonempty and hence rational. We also note that each of the fibers of the map $X_{m,i_{j+1}}(\phi) \to X_{i_{j+1}}(\phi)$ can be identified with a Grassmannian $G(m, i_{j+1})$, and hence are also rational. The result therefore follows from Corollary 3.9.

Theorem 4.4. Let $\phi$ be a regular quadratic form over $k$, and suppose that $m$ is a positive integer with $i_j < m \leq i_{j+1}$ for some $0 < j < h(\phi)$. Then $\overline{A}_0(X_\Lambda(\phi), \mathbb{Z})$ is $2^j$-torsion.

Proof. We prove this by induction on $j$, starting with $j = 0$. In this case, we have $i_0(\phi) = i(\phi) = i$, and by Proposition 4.2 we have $\overline{A}_0(X_{i+1}, \mathbb{Z}) = 0$. Consequently, $t(X_{i+1}, \mathbb{Z}) = (1)$. By Lemma 4.3 $t(X_{i+1}, \mathbb{Z}) = t(X_i, \mathbb{Z}) = t(X_m, \mathbb{Z})$, which implies $t(X_m, \mathbb{Z}) = (1)$, and $\overline{A}_0(X_m, \mathbb{Z}) = 0$ as claimed.

For the induction step, suppose that $\overline{A}_0(X_{i_j}(\phi), \mathbb{Z})$ is $2^{j-1}$-torsion. Considering the morphism $\pi : X_{i_{j+1}} \to X_{i_j}$, we find that the fibers are quadric hypersurfaces by Lemma 4.1. In particular, $t(\pi, \mathbb{Z}) = (1)$, and $\text{ind}(\pi)|2$. We therefore have $2^j \in 2t(X_i) \subset t(X_{i_{j+1}})$. But since $X_{i_{j+1}} \to X_{i_{j+1}}$ has fibers isomorphic to Grassmannians $G(i_j, i_j + 1)$ which are therefore rational, it follows from Corollary 3.9 that $t(X_{i_{j+1}}, \mathbb{Z}) = t(X_{i_{j+1}})$, showing that $2^j \in t(X_{i_{j+1}})$ as claimed.

5. Generalized Severi-Brauer varieties

Let $A$ be a central simple algebra of degree $n$. Given integers $1 \leq n_1 < n_2 < \cdots < n_k < n$, we set $X_{n_1,\ldots,n_k}(A)$ to be the Brauer-Severi flag variety parametrizing flags of right ideals $I_1 < \cdots < I_k \subset A$, where $\dim I_i = n_i$ (see [Kra10] Def. 4.1 for a functorial description of this variety). For a right ideal $I < A$, and for a positive integer $m$ with $mn < \dim I$, we set $X_m(I)$
to be the variety of right ideals of $A$ of dimension $mn$ contained in $I$. We recall from [Kra10, Thm. 4.8] that $X^m(I) \cong X^m(D)$ for some central simple algebra $D$ of degree $\dim I$, Brauer equivalent to $A$. It follows from Lemma 5.7 that $A_0(X_{n_1,\ldots,n_k}(A);M,q) = A_0(X_d(A);M,q)$ for any cycle module $M$, where $d$ is the gcd of the integers $n_i$ and the index of $A$. In particular, in understanding $t(X_{n_1,\ldots,n_k}(A))$ it suffices to consider Brauer-Severi flag varieties of the form $X_d(A)$ with $d \mid \text{ind}(A)$.

**Theorem 5.1.** Let $A$ be a central simple algebra of period 2 and index $2m$. Then for $d \mid m$, we have $\overline{A}_0(X_{2d}(A),\mathbb{Z})$ is $m$-torsion.

We note that since $\text{ind}(A)$ and $\text{per}(A)$ always have the same prime factors, it follows that $m$ is a power of 2.

**Proof.** Consider the natural projection morphisms

$$
\xymatrix{ & X_{2,2d}(A) \ar[dl]_{\pi_2} \ar[dr]^{\pi_1} & \\
X_{2d}(A) & & X_2(A).
}
$$

To examine the fibers of $\pi_1$, we may switch to the opposite algebra $A^{op}$ and consider the isomorphisms $X_{2,2d}(A) = X_{2m-2d,2m-2}(A^{op})$, $X_2(A) = X_{2m-2}(A^{op})$, given by taking right ideals to their left annihilators. In particular, we find that we may identify the fibers of $\pi_1$ with varieties of the form $X_{2m-2d}(I) = X_{2m-2d}(D)$ for an algebra $D$ of degree $2m - 2$ and index 2. In particular, since they are projective homogeneous varieties with rational points, the fibers of $\pi_1$ are rational trivial, and hence $t(X_2(A),\mathbb{Z}) = t(X_{2d,2},\mathbb{Z})$ by Corollary 5.9.

On the other hand, we may identify the fibers of $\pi_2$ with varieties of the form $X_{2}(D)$ where $D$ is a central simple algebra of degree $2m$. In particular, $t(X_{2}(D),\mathbb{Z}) = (1)$ by [Kra10 Thm. 7.3] and $\text{ind}(X_{2}(D),\mathbb{Z}) = m$ by [Kra10 Lem. 7.1]. It therefore follows from Proposition 3.5 that $\overline{A}_0(X_{2,2m})(A)$ and hence $\overline{A}_0(X_{2d}(A))$ is $2^m$-torsion as desired. \hfill $\Box$

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