ESTIMATES OF TRIPLE PRODUCTS OF AUTOMORPHIC FUNCTIONS II

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Abstract. We prove a sharp bound for the average value of the triple product of modular functions for the Hecke subgroup $\Gamma_0(N)$. Our result is an extension of the main result in [BR3] to a fixed cuspidal representation of the adele group $PGL_2(\mathbb{A})$.

1. Introduction

1.1. Maass forms. We recall the setup of [BR3] which should be read in conjunction with this appendix. Let $Y$ be a compact Riemann surface with a Riemannian metric of constant curvature $-1$ and the associated volume element $dv$. The corresponding Laplace-Beltrami operator is non-negative and has purely discrete spectrum on the space $L^2(Y, dv)$ of functions on $Y$. We will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$ its eigenvalues and by $\phi_i$ the corresponding eigenfunctions (normalized to have $L^2$ norm one). In the theory of automorphic forms the functions $\phi_i$ are called automorphic functions or Maass forms (after H. Maass, [M]). We write $\mu_i = (1 - \lambda_i^2)/2$ and $\phi_i = \phi_{\lambda_i}$ as is customary in representation theory of the group $PGL_2(\mathbb{R})$.

For any three Maass forms $\phi_i$, $\phi_j$, $\phi_k$ we define the following triple product or triple period:

$$c_{ijk} = \int_Y \phi_i \phi_j \phi_k dv.$$ (1)

One would like to bound the coefficient $c_{ijk}$ as a function of eigenvalues $\mu_i$, $\mu_j$, $\mu_k$. In particular, we would like to find bounds for these coefficients when one or more of these indices tend to infinity. The study of these triple coefficients goes back to pioneering works of Rankin and Selberg (see [Ra], [Se]), and reappeared in celebrated works of Waldspurger [W] and Jacquet [J] (also see [HK], [Wa], [Ich]). Recently, an interest in analytic questions related to triple products was initiated in the groundbreaking paper of Sarnak [Sa1] (see also [Go] for

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the first general result on exponential decay). This was motivated by
the widespread use of triple products in applications (e.g., [Sa2]).

In our paper [BR3] we considered the following problem. We fix two
Maass forms \( \phi = \phi_{\tau}, \phi = \phi_{\tau} \) as above and consider coefficients defined
by the triple period:

\[
 c_i = \int_Y \phi \phi' \phi_i dv
\]

(2)

as \( \{ \phi_i = \phi_{\lambda_i} \} \) run over the orthonormal basis of Maass forms. We
note first that one has exponential decay for the coefficients \( c_i \) in the
parameter \( |\lambda_i| \) as \( i \) goes to \( \infty \). For that reason, one renormalizes coeffi-
cients \( |c_i|^2 \) by an appropriate ratio of Gamma functions dictated by the
Watson formula [Wa] (see also Appendix in [BR3] where these factors
were computed from another point of view). Taking into account the
asymptotic behavior of these factors, we introduced normalized c oeffi-
cients

\[
|a_i|^2 = |\lambda_i|^2 \exp \left( \frac{\pi}{2} |\lambda_i| \right) \cdot |c_i|^2.
\]

(3)

Under such a normalization, we showed that

\[
\sum_{|\lambda_i| \leq T} |a_i|^2 \leq A \cdot T^2
\]

(4)

for some explicit constant \( A = A(\Gamma, \phi, \phi') \). According to the Weyl law,
there are approximately \( cT^2 \) terms in the above sum, and hence the
bound (4) consistent with the Lindelöf bound on average (in fact it is
not difficult to show that the bound (4) essentially is sharp, see [Re]).

There are various natural questions concerning the bound (4) which
were not discussed in [BR3]. These are mostly related to the depen-
dence of the constant \( A \) on various parameters (i.e., \( \Gamma, \phi, \phi' \)), and also
to the fact that we restricted the discussion to Maass forms, leaving
aside the case of holomorphic forms. Another restriction of the treat-
ment we presented was the fact that we used in an essential way the
compactness of \( Y \) (which was treated by us differently in [BR1]). All
these issues turn out to be important in applications. In this appendix,
we answer some of these questions for Hecke congruence subgroups.

1.1.1. Hecke subgroups. We consider Hecke subgroups \( \Gamma_0(N) \) of the
modular group. We normalize the scalar product on the quotient Rie-
mann surface \( Y_N = \Gamma_0(N) \setminus \mathbb{H} \) by \( \langle f, g \rangle_{Y_N} = \frac{1}{\text{vol}_{\mathbb{H}}(Y_N)} \int_{Y_N} f(z)\overline{g(z)}d\mu_{\mathbb{H}} \)
where \( d\mu_{\mathbb{H}} \) is the standard volume element on the upper half plane \( \mathbb{H} \)
(i.e., we normalize the volume element \( dv_{Y_N} \) on \( Y_N \) to have the total vol-
ume 1). Let \( \phi \) be a (primitive) Hecke-Maass form for the group \( \Gamma_0(N_0) \)
for some fixed level $N_0 \geq 1$. We assume that $\phi$ is normalized by the $L^2$-norm $||\phi||_{L^2(Y_{N_0})} = 1$. For an integer $\ell > 1$, denote by $\ell \phi(z) = \phi(\ell z)$ the corresponding old form for the Hecke subgroup $\Gamma_0(\ell N_0)$. The corresponding function $\ell \phi$ also turns out to be $L^2$-normalized on $Y_{\ell N_0}$ with respect to the normalization of measures we choose. This follows easily from the Rankin-Selberg method. For two such Maass forms $\phi$ and $\phi'$, we define triple products by

$$c_i(\ell) = \int_{Y_{\ell N_0}} \ell \phi \ell \phi' \phi_i \, dv_{Y_{\ell N_0}}$$

as $\{\phi_i = \phi_{\lambda_i}\}$ run over the orthonormal basis of Maass forms on $Y_{\ell N_0}$. We have the corresponding normalized coefficients

$$|a_i(\ell)|^2 = |\lambda_i|^2 \exp\left(\frac{\pi}{2} |\lambda_i|\right) |c_i(\ell)|^2 .$$

**Theorem A.** There exists an effectively computable constant $A$ such that the following bound holds for all $T \geq 1$,

$$\sum_{|\lambda_i| \leq T} |a_i(\ell)|^2 \leq A \cdot T^2 ,$$

where the summation is over an orthonormal basis of Maass forms for the group $\Gamma_0(\ell N_0)$. The constant $A$ depends on $N_0$, $\phi$ and $\phi'$, but not on $\ell$.

This could be viewed as the Lindelöf on the average bound in two parameters $\ell$ and $T$ (but only for old forms of course). There are $[\Gamma_0(\ell N_0) : \Gamma(1)] \cdot T^2$ terms in the sum according to the Weyl-Selberg law, and the resulting bound is consistent with this under our normalization of measures $\text{vol}(Y_{\ell N_0}) = 1$. In fact, there is also an analogous contribution from the Eisenstein series leading to the triple coefficients $a_{s,\kappa}(\ell)$ arising from triple products $\langle \ell \phi \ell \phi', E_{\kappa}(s) \rangle_{Y_{\ell N_0}}$ where $E_{\kappa}(s)$ is the Eisenstein series associated to a cusp $\kappa$ of $\Gamma_0(\ell N_0)$. We have then

$$\sum_{|\lambda_i| \leq T} |a_i(\ell)|^2 + \sum_{|\kappa|} \int_{-T}^T |a_{it,\kappa}(\ell)|^2 \, dt \leq A \cdot T^2 .$$

1.2. **The method.** The proof we presented in [BR3] was based on the uniqueness of the triple product in representation theory of the group $PGL_2(\mathbb{R})$. We review quickly the general ideas behind our proof. It is based on ideas from representation theory (see the seminal book [G6], and also [Bu], [L]). Namely, we use the fact that every automorphic form $\phi$ generates an automorphic representation of the group $G = PGL_2(\mathbb{R})$; this means that starting from $\phi$ we produce a smooth
irreducible representation of the group \( G \) in a space \( V \) and its realization \( \nu : V \to C^\infty(X) \) in the space of smooth functions on the automorphic space \( X = \Gamma \backslash \text{G} \) endowed with the invariant measure of the total mass one. We denote by \( \phi_v(x) = \nu(v)(x) \) the corresponding to \( v \in V \) automorphic function. The Maass form corresponds to a unit \( K \)-invariant vector \( e_0 \in V \), where \( K = \text{PSO}(2) \) is a maximal compact connected subgroup of \( G \). In our case, we have the family of spaces \( X_\ell = \Gamma_0(\ell N_0) \backslash \text{G} \) (we denote by \( X_0 = \Gamma_0(N_0) \backslash \text{G} \)), and the corresponding family of isometries \( \nu_\ell : V \to C^\infty(X_\ell) \) of the same abstract representation of \( G \). These are generated by old Maass forms \( \phi(\ell z) \) in the same adelic representation corresponding to the Hecke-Maass form \( \phi \).

The triple product \( c_i = \int_X \phi \phi' \phi_i dv \) extends to a \( G \)-equivariant trilinear form on the corresponding automorphic representations \( l^{aut} : V \otimes V' \otimes V_i \to \mathbb{C} \), where \( V = V_\tau, V' = V'_{\tau'}, V_i = V_{\lambda_i} \).

Then we use a general result from representation theory that such a \( G \)-equivariant trilinear form is unique up to a scalar, i.e., that the space \( \text{Hom}_G(V \otimes V' \otimes V_i, \mathbb{C}) \) is at most one-dimensional (see [O], [Mo], [Lo] and [P] for \( p \)-adic \( \text{GL}(2) \)). This implies that the automorphic form \( l^{aut} \) is proportional to an explicit “model” form \( l^{mod} \) which we describe using explicit realizations of representations of the group \( G \); it is important that this last form carries no arithmetic information.

Thus we can write \( l^{aut} = a_i \cdot l^{mod}_i \) for some constant \( a_i \) and hence \( c_i = l^{aut}_i(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) = a_i \cdot l^{mod}_i(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) \), where \( e_\tau, e_{\tau'}, e_{\lambda_i} \) are \( K \)-invariant unit vectors in the automorphic representations \( V, V', V_i \) corresponding to the automorphic forms \( \phi, \phi' \) and \( \phi_i \).

It turns out that the proportionality coefficient \( a_i \) in the last formula carries important “automorphic” information while the second factor carries no arithmetic information and can be computed in terms of Euler \( \Gamma \)-functions using explicit realizations of representations \( V_\tau, V_{\tau'} \) and \( V_{\lambda_i} \) (see Appendix to [BR3]). This second factor is responsible for the exponential decay, while the first factor \( a_i \) has a polynomial behavior in the parameter \( \lambda_i \).

In order to bound the quantities \( a_i \), we use the fact that they appear as coefficients in the spectral decomposition of the diagonal Hermitian form \( H_\Delta \) given by

\[
H_\Delta(v \otimes w) = \int_X |\phi_v(x)\phi_w(x)|^2 dx
\]
on the space \( E = V_\tau \otimes V_{\tau'} \). This gives an inequality \( \sum |a_i|^2 H_i \leq H_\Delta \) where \( H_i \) is a Hermitian form on \( E \) induced by the model trilinear form \( l^{mod}_i : V \otimes V' \otimes V_i \to \mathbb{C} \) as above.
Using the geometric properties of the diagonal form and simple explicit estimates of forms $H_i$, we establish the mean-value bound for the coefficients $|a_i|^2$. Here is where one obtains the dependence of the constant $A$ in (4) on parameters involved. In the method of [BR3], we used $L^2$ theory by averaging the form $H_\Delta$ and comparing the resulting form with the $L^2$-form. The coefficient $A$ that one obtains in such an argument depends in particular on the injectivity radius of $X$. While in certain cases it gives an optimal result, it obviously has two drawbacks. One is related to the possible non-compactness of $X$ since in the cusp the injectivity radius tends to zero. Another problem arises when one considers a sequence of subgroups with co-volume going to infinity along the sequence. In that case, the bound which the method of [BR3] provides for the constant (e.g., $A(\Gamma_0(p)) \leq \operatorname{vol}(Y_p) \approx p$, see [K]) is too weak for many applications. Both of these problems arise in the classical setup of Hecke subgroups $\Gamma_0(N)$. Here we obtain an optimal bound for old forms. We do not know how to obtain similar results for new forms. We will discuss improvements over a trivial bound for new forms elsewhere. Theorem 1.1.1 could be viewed as the exact analog of the result in [BR3] for a fixed adelic representation.

1.3. Proof of Theorem A. We have a family of various objects $(X_\ell, \nu_\ell, H^\ell_\Delta, a_i(\ell))$ parameterized by the level $\ell$. However the model Hermitian form $H_i$ is the same since the abstract representation $V$ of $PGL_2(\mathbb{R})$ does not change. The proof of the bound (4) given in [BR3] was based on the spectral decomposition $\sum |a_i(\ell)|^2 H_i \leq H^\ell_{\Delta}$ of the diagonal form and on the construction of the test vector $u = u_\ell \in E$ such that $H^\ell_{\Delta}(u) \leq aT^2$ and $H_i(u) \geq 1$ for $|\lambda_i| \leq T$. We now construct the test vector independently of $\ell$. The dependence on $\ell$ is hidden in the automorphic realization $\nu_\ell(u)$ as a function on $X_\ell$.

1.3.1. Construction of vector $u$. We slightly change the construction of the test vector $u_\ell$ given in Section 5.3.2 of [BR3]. Let us identify the space $E = V \otimes V'$ with a subspace of smooth functions $C^\infty(\mathbb{R} \times \mathbb{R})$. Choose a smooth non-negative function $\alpha \in C^\infty(\mathbb{R})$ with the support $\operatorname{supp}(\alpha) \subset [-0.1, 0.1]$ and $\int_{\mathbb{R}} \alpha(x) dx = 1$. Let $\|\alpha\|_{L^2(\mathbb{R})}^2 = c^2$ for some $c > 0$. Consider the diagonal element $a_\ell = \operatorname{diag}(T^{-\frac{1}{2}}, T^{\frac{1}{2}}) \in G$. We define vectors

\begin{equation}
\nu_\ell = T^{\frac{1}{2}+\tau} \cdot \pi_\ell(a_\ell) \alpha \quad \text{and} \quad \nu'_\ell = T^{\frac{1}{2}+\tau'} \cdot \pi_{\tau'}((1 - 1)) \nu_\ell.
\end{equation}

We set our test vector to $u_\ell(x, y) = \nu_\ell(x) \otimes \nu'_\ell(y)$.

Recall that the action is given by $\pi_\tau(\operatorname{diag}(a^{-1}, a)) v(x) = |a|^{1-\tau} v(a^2 x)$, and $\pi_{\tau'}((1 - 1)) v(x) = v(x+1)$. By an easy calculation we have $\|u_\ell\|_E^2 =$
\( c^2 T^2 \). We note that geometrically the vector \( u_T \) is a small non-negative bump function around the point \((0, 1) \in \mathbb{R}^2 \), with the support in the box of the size \( T^{-1} \), and satisfies \( \int_{\mathbb{R}^2} u_T(x, y) dx dy = 1 \). Computation identical to the one performed in Section 5.3.4 of [BR3] gives then \( H_i(u_T) \geq \beta \) for \(|\lambda_i| \leq T \) and some explicit \( \beta > 0 \) independent of \( \lambda_i \) (in fact for \(|\tau|, |\tau'| \leq T \), it does not depend on these either). We remark that the only difference with the construction of the test vector given in [BR3] is that here we constructed \( u_T \) with the help of the action of \( G \) on \( V \) (while in [BR3] we constructed essentially the same vector explicitly in the model). This will play a crucial role in our estimate of the corresponding automorphic function.

We now need to estimate \( H_{\Delta}^\ell(u_T) \). We claim that \( H_{\Delta}^\ell(u_T) \leq BT^2 \) for some explicit constant \( B \) independent of \( \ell \). Since we have

\[
H_{\Delta}^\ell(u_T) = \int_{X_\ell} |^\ell \phi_{v_T} |^2 dx \leq \frac{1}{2} \| ^\ell \phi_{v_T} \|_{L^2(X_\ell)}^2 + \frac{1}{2} \| ^\ell \phi_{v_T} \|_{L^2(X_\ell)}^2,
\]

it is enough to show that \( \| ^\ell \phi_{v_T} \|_{L^2(X_\ell)}^2 \leq \beta T^2 \). This would finish the proof of Theorem 1.1.1 following the argument in Section 4.7 [BR3]. Since \( \| ^\ell \phi_{v_T} \|_{L^2(X_\ell)}^2 = cT \), it is easy to see that such a bound is sharp. We claim that

\[
(10) \quad \sup_{x \in X_\ell} |^\ell \phi_{v_T}(x)| \leq \beta'' T^{\frac{1}{2}},
\]

for some \( \beta'' \) independent of \( \ell \). Here \( ^\ell \phi_{v_T}(x) = \nu_\ell(v_T)(x) \). Note that the bound provided by the Sobolev theorem [BR2] is too weak.

1.3.2. Supremum norm. Recall that we started with an \( L^2 \)-normalized Hecke-Maass form \( \phi \) on \( \Gamma_0(N_0) \), and the corresponding isometry of \( \nu : V \to C^\infty(X_0) \) of the principal series representation \( V \simeq V_\tau \). We then constructed another isometry \( \nu^\ell : V \to C^\infty(X_\ell) \) by using the map

\[
\nu(v) \left( \left( \begin{array}{c} \ell^{-\frac{1}{2}} \\ \ell^{\frac{1}{2}} \end{array} \right) x \right) = \phi_v \left( \left( \begin{array}{c} \ell^{-\frac{1}{2}} \\ \ell^{\frac{1}{2}} \end{array} \right) x \right),
\]

for any \( v \in V \). This relation might be viewed as the relation between functions on \( G \) invariant on the left for an appropriate \( \Gamma \) (e.g., for \( \Gamma_0(N_0) \) and for \( \Gamma_0(\ell N_0) \)). In particular, we see that the supremum of the function \( ^\ell \phi_v \) on \( X_\ell \) and that of the function \( \phi_v \) on \( X_0 \) are equal for the same vector \( v \in V \). Hence it is enough to show that \( \sup_{x \in X_0} |\phi_{v_T}(x)| \leq \beta'' T^{\frac{1}{4}} \). In fact, this is obvious since \( v_T = T^{\frac{1}{2}+\tau} \pi(a_T) \alpha \) is given by the
(scaled) action of $G$ on a fixed vector. We have

$$\sup_{x_0} |\phi_{vT}(x)| = T^{\frac{1}{2}} \sup_{x_0} |\phi_{\pi(aT)\alpha}(x)| =$$

$$T^{\frac{1}{2}} \sup_{x_0} |\phi_\alpha(x\cdot aT)| = T^{\frac{1}{2}} \sup_{x_0} |\phi_\alpha(x)| = \beta^\prime T^{\frac{1}{2}},$$

since $\alpha$ is a fixed vector in a fixed automorphic cuspidal representation $(\nu, V)$, and the action does not change the supremum norm. □

Remark. It is easy to see that the condition that forms $\ell \phi$ and $\ell \phi'$ be of the same level is not essential for the proof, as well as that the $s$ be Hecke forms. In particular, under our the normalization of measures on $X_\ell$, we see that for a vector $v \in V$, $L^2$-norms $||\phi_v||_{X_\ell} = ||\phi_v||_{X_{\ell'}}$ are equal if $\ell|\ell'$ (here we view the function $\ell \phi_v$ as both $\Gamma_0(\ell N_0)$-invariant function and as $\Gamma_0(\ell' N_0)$-invariant function). Hence for two Maass forms $\phi$ and $\phi'$ on $\Gamma_0(\ell N_0)$, we obtain the bound:

\begin{equation}
\sum_{|\lambda_i| \leq T} |(\ell_1 \phi \cdot \ell_2 \phi', \phi_i)_{Y_{\ell_1\ell_2(\ell N_0)}}|^2 \cdot |\lambda_i|^2 e^{\frac{\pi}{2}|\lambda_i|} \leq A \cdot T^2,
\end{equation}

where the summation is over an orthonormal basis of Maass forms for the subgroup $\Gamma_0(\ell_1\ell_2 N_0)$.

1.4. Holomorphic forms. The approach given above is applicable to holomorphic forms as well. In principle, there are no serious changes needed as compared to the Maass forms case. The main difficulty is that we have to fill up the gap left in [BR3] concerning the model trilinear functional for discrete series representations of $PGL_2(\mathbb{R})$.

Let $\phi^k, \phi'^k$ be (primitive) holomorphic forms of weight $k$ for the subgroup $\Gamma_0(N_0)$. We assume these are $L^2$-normalized. For $\ell > 1$, we consider (old) forms $\ell^k \phi^k(z) = \phi^k(\ell z)$ and $\ell^k \phi'^k = \phi'^k(\ell z)$ on $\Gamma_0(\ell N_0)$. Under our normalization of measures for $Y_{\ell N_0}$, we have $||\ell^k \phi^k||_{Y_{\ell N_0}} = ||\ell^k \phi'^k||_{Y_{\ell N_0}} = \ell^k \tau^k$. This follows from the Rankin-Selberg method. Hence it would have been more natural to consider normalized forms $\phi^k|_{[a_\ell]e} = \ell^k \phi^k$.

For a (norm one) Maass form $\phi_i$ on $\Gamma_0(\ell N_0)$, we define the corresponding triple coefficient by

\begin{equation}
\ell_i^k(\ell) = \int_{Y_{\ell N_0}} \ell^k \phi^k \overline{\phi'^k} \phi_i \ y^k \ d\nu_{Y_{\ell N_0}}.
\end{equation}
As with Maass forms, we renormalize these coefficients in accordance with the Watson formula by introducing normalized triple product coefficients
\[ |a^k_\ell| \leq |\lambda_i|^{2-2k} \exp \left( \frac{\pi}{2} |\lambda_i| \right) |c^k_\ell| \leq \lambda_i | \ell^k \exp \left( \frac{\pi}{2} \lambda_i \right) |c^k_\ell| \leq A \cdot T^2 , \]

**Theorem B.** There exists an effectively computable constant $A$ such that the following bound holds for all $T \geq 1$,
\[ \sum_{T \leq |\lambda_i| \leq T} |a^k_\ell|^2 \leq A \cdot T^2 , \]
where the summation is over an orthonormal basis of Maass forms for the group $\Gamma_0(N_0)$. The constant $A$ depends on $N_0$, $\phi$ and $\phi'$, but not on $\ell$.

**Remark.** The proof we give applies to a slightly more general setup of forms of different co-prime level $\ell_1$ and $\ell_2$. Namely, we have
\[ \sum_{T \leq |\lambda_i| \leq T} |\langle \ell_1 \phi^k \ell_2 \phi'^k, \phi_i \rangle_{Y_{\ell_1 \ell_2 N_0}}|^2 \leq (\ell_1 \ell_2)^k |\lambda_i|^{2-2k} e^{\frac{1}{2} \pi |\lambda_i|} \]
for two forms $\phi^k$ and $\phi'^k$ on $\Gamma_0(N_0)$. Breaking the interval $[1, T]$ into dyadic parts, we obtain for the full range,
\[ \sum_{|\lambda_i| \leq T} |\langle \ell_1 \phi^k \ell_2 \phi'^k, \phi_i \rangle_{Y_{\ell_1 \ell_2 N_0}}|^2 \leq (\ell_1 \ell_2)^k |\lambda_i|^{2-2k} e^{\frac{1}{2} \pi |\lambda_i|} \leq A \cdot T^2 \ln(T) . \]
This is slightly weaker than (7) for Maass forms.

1.5. **Proof of Theorem B.** As we seen in the case of Maass forms, the proof is based on the explicit form of the trilinear functional, its value on special vectors leading to the normalization (14), and the construction of test vectors for which we can estimate supremum norm effectively. We explain below changes and additions needed in order to carry out this scheme for discrete series.

1.5.1. **Discrete series.** Let $k \geq 2$ be an even integer, and $(D_k, \pi_{D_k})$ be the corresponding discrete series representation of $PGL_2(\mathbb{R})$. In particular, for $m \in 2\mathbb{Z}$, the space of $K$-types of weight $m$ is non-zero (and in this case is one-dimensional) if and only if $|m| \geq k$. This defines $\pi_k$ uniquely. Under the restriction to $PSL_2(\mathbb{R})$, the representation $\pi_k$ splits into two representations $(D^{\pm}_k, \pi^{\pm}_{D_k})$ of “holomorphic” and “anti-holomorphic” discrete series of $PSL_2(\mathbb{R})$, and the element $\delta = \text{diag}(1, -1)$ interchanges them.
We consider two realizations of discrete series as subrepresentations and as quotients of induced representations. Consider the space $H_{k-2}$ of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $k - 2$ (i.e., $f(tx) = t^{k-2}f(x)$ for any $t \in \mathbb{R}^\times$ and $0 \neq x \in \mathbb{R}^2$). We have the natural action of $GL_2(\mathbb{R})$ given by $\pi_{k-2}(g)f(x) = f(g^{-1}x) \cdot \det(g)^{(k-2)/2}$, which is trivial on the center and hence defines a representation $(H_{k-2}, \pi_{k-2})$ of $PGL_2(\mathbb{R})$. There exists a unique non-trivial invariant subspace $W_{k-2} \subset H_{k-2}$. The space $W_{k-2}$ is finite-dimensional, $\dim W_{k-2} = k - 1$, and is generated by monomials $x_1^m x_2^n$, $m + n = k - 2$. The quotient space $H_{k-2}/W_{k-2}$ is isomorphic to the space of smooth vectors of the discrete series representation $\pi_k$.

We also consider the dual situation. Let $\mathcal{H}_{-k}$ be the space of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $-k$. There is a natural $PGL_2(\mathbb{R})$-invariant pairing $\langle \cdot , \cdot \rangle : H_{k-2} \otimes \mathcal{H}_{-k} \to \mathbb{C}$ given by the integration over $S^1 \subset \mathbb{R}^2 \setminus 0$. Hence $\mathcal{H}_{-k}$ is the smooth dual of $H_{k-2}$, and vice versa. There exists a unique non-trivial invariant subspace $D_k^* \subset \mathcal{H}_{-k}$. The quotient $\mathcal{H}_{-k}/D_k^*$ is isomorphic to the finite-dimensional representation $W_{k-2}$.

Of course $D_k^*$ is isomorphic to $D_k$, but we will distinguish between two realizations of the same abstract representation as a subrepresentation $D_k^* \subset \mathcal{H}_{-k}$ and as a quotient $\mathcal{H}_{k-2} \to D_k$. We denote corresponding maps by $i_k : D_k^* \subset \mathcal{H}_{-k}$ and $q_k : \mathcal{H}_{k-2} \to D_k$.

1.5.2. Trilinear invariant functionals. Let $(V_{\lambda,\varepsilon}, \pi_{\lambda,\varepsilon})$ be a unitary representation of the principal series of $PGL_2(\mathbb{R})$. These are parameterized by $\lambda \in i\mathbb{R}$ and by $\varepsilon = 0, 1$ describing the action of the element $\delta$ (see [Bu]). The space $\text{Hom}_G(D_k \otimes D_k^* \otimes V_{\lambda,\varepsilon})$ is one-dimensional. We will work with the space of invariant trilinear functionals $\text{Hom}_G(D_k \otimes D_k^* \otimes V_{-\lambda,\varepsilon}, \mathbb{C})$ instead. We construct below a non-zero functional $l_{k,\lambda,\varepsilon}^{ind} \in \text{Hom}_G(H_{k-2} \otimes H_{-k} \otimes V_{-\lambda,\varepsilon}, \mathbb{C})$ for induced representations (in fact, this space is also one-dimensional) by means of (analytic continuation of) the explicit kernel. We use it to define a non-zero functional $l_{k,\lambda,\varepsilon}^{mod} \in \text{Hom}_G(D_k \otimes D_k^* \otimes V_{-\lambda,\varepsilon}, \mathbb{C})$. What is more important, we will use $l_{k,\lambda,\varepsilon}^{ind}$ in order to carry out our computations in a way similar to the principal series.

Let $l_{k,\lambda,\varepsilon}^{ind} \in \text{Hom}_G(H_{k-2} \otimes H_{-k} \otimes V_{-\lambda,\varepsilon}, \mathbb{C})$ be a non-zero invariant functional. Such a functional induces the corresponding functional on $H_{k-2} \otimes D_k^* \otimes V_{-\lambda,\varepsilon}$ since $D_k^* \subset H_{-k}$. Moreover, any such functional vanishes on the subspace $W_{k-2} \otimes D_k^* \otimes V_{-\lambda,\varepsilon}$ since there are no non-zero maps between $W_{k-2} \otimes D_k^*$ and $V_{\lambda,\varepsilon}$. Hence we obtain a functional $l_{k,\lambda,\varepsilon}^{mod} \in \text{Hom}_G(D_k \otimes D_k^* \otimes V_{-\lambda,\varepsilon}, \mathbb{C})$ on the corresponding quotient space. We denote by $T_{k,\lambda,\varepsilon}^{mod} : D_k \otimes D_k^* \to V_{\lambda,\varepsilon}$ the associated map, and by
\[ H_{k,\lambda,\epsilon}^{\text{mod}}(u) = \| T_{k,\lambda,\epsilon}^{\text{mod}}(u) \|_{V_{\lambda,\epsilon}}^2, \quad u \in D_k \otimes D_k^* \] is the corresponding Hermitian form.

1.5.3. Model functionals. We follow the construction from [BR3]. Denote

\[ K_{k,\lambda}(x, y, z) = |x - y|^{\frac{1}{k+1}} |x - z|^{-\frac{1}{k+1}} |y - z|^{-\frac{1}{k+1}}. \]

In order to construct \( l_{k,\lambda,\epsilon}^{\text{mod}} \in \text{Hom}_G(\mathcal{H}_{k-2} \otimes \mathcal{H}_{-k} \otimes V_{-\tau,\epsilon}, \mathbb{C}) \), we consider the following function in three variables \( x, y, z \in \mathbb{R}^3 \)

\[ K_{k-2,-k,\lambda,\epsilon}(x, y, z) = (\text{sgn}(x, y, z))^{\epsilon} \cdot K_{k,\lambda}(x, y, z), \]

where \( \text{sgn}(x_1, x_2, z_3) = \prod_{i \neq j} \text{sgn}(x_i - x_j) \) (this is an \( SL_2(\mathbb{R}) \)-invariant function on \( \mathbb{R}^3 \) distinguishing two open orbits). An analogous expression could be written in the circle model on the space \( C^\infty(S^1) \). Viewed as a kernel, \( K_{k-2,-k,\lambda,\epsilon} \) defines an invariant non-zero functional \( l_{k,\lambda,\epsilon}^{\text{ind}} \) on the (smooth part of) the representation \( \mathcal{H}_{k-2} \otimes \mathcal{H}_{-k} \otimes V_{-\tau,\epsilon} \subset C^\infty(\mathbb{R}^3) \).

Such a kernel should be understood in the regularized sense (e.g., analytically continued following [G1]). We are interested in \( \lambda \in i\mathbb{R}, \ |\lambda| \to \infty \), and hence all exponents in (19) are non-integer. This implies that the regularized kernel does not have a pole at relevant points.

We denoted by \( l_{k,\lambda,\epsilon}^{\text{mod}} \in \text{Hom}_G(D_k \otimes D_k^* \otimes V_{-\lambda,\epsilon}, \mathbb{C}) \) the corresponding model functional. The difference with principal series clearly lies in the fact that we only can compute the auxiliary functional \( l_{k,\lambda,\epsilon}^{\text{ind}} \). However, for \( k \) fixed, it turns out that necessary computations are essentially identical to the ones we performed for the principal series in [BR3].

1.5.4. Value on \( K \)-types. In order to obtain normalization (14) and to compare our model functional \( l_{k,\lambda,\epsilon}^{\text{mod}} \) to the automorphic triple product (13), we have to compute, or at least to bound, the value \( l_{k,\lambda,\epsilon}^{\text{mod}}(e_k \otimes e_{-k} \otimes e_0) \) where \( e_{\pm k} \in D_k \) are highest/lowest \( K \)-types of norm one, and \( e_0 \in V_{\lambda,\epsilon} \) is a \( K \)-fixed vector of norm one. For Maass forms, this is done in the Appendix of [BR3] by explicitly calculating this value in terms of \( \Gamma \)-functions. In fact, the relevant calculation is valid for \( K \)-fixed vectors for any three induced representations with generic values of parameters (i.e., those for which the final expression is well-defined). Using the action of Lie algebra of \( G \) (see [Lo] for the corresponding calculation where it is used to prove uniqueness), one obtains recurrence relations between values of the model functional on various weight vectors. For a generic value of \( \tau \), this allows one to reduce the computation of \( l_{\tau,\lambda,\epsilon}^{\text{mod}}(e_k \otimes e_{-k} \otimes e_0) \) to the value of \( l_{\tau,\lambda,\epsilon}^{\text{mod}}(e_0 \otimes e_0 \otimes e_0) \). By analytic continuation, this relation holds for our set of parameters corresponding
to discrete series. From this, one deduces the bound
\begin{equation}
|f_{k,a,e}^{\text{mod}}(e_k \otimes e_{-k} \otimes e_0)|^2 \leq a|\lambda_i|^{2k-2} \exp \left(-\frac{\pi}{2}|\lambda_i|\right),
\end{equation}
for some explicit constant $a > 0$. In fact, this is the actual order of the
magnitude for the above value.

**Remark.** There is a natural trilinear functional on Whittaker models of
representations of $G$. This is the model which appears in the Rankin-
Selberg method as a result of unfolding. The above computation (and
the similar one for Maass forms performed in [BR2]) shows that our
normalization of the trilinear functional and the one coming from the
Whittaker model coincide up to a constant of the absolute value one.

1.5.5. Test vectors. Our construction is very close to the construction
we made in Section 1.3.1 for principal series representations, with
appropriate modifications. We construct a test vector $w_T(x,y) = v_T(x) \otimes
v_T'(y) \in D_k \otimes D_k' \subset H_{k-2} \otimes H_{-k}$ satisfying $H^{\text{mod}}_{k,a,e}(w_T) \geq \beta > 0$ for
$T/2 \leq |\lambda| \leq T$ and some constant $\beta > 0$ independent of $\lambda$. Vec-
tors $v_T \in H_{k-2}$ and $v_T' \in H_{-k}$ are first constructed in the line model
of induced representations, and then we relate these to vectors in the
discrete series representation $D_k$.

Choose a smooth non-negative function $\alpha \in C^\infty(\mathbb{R})$ with the
support supp($\alpha$) $\subset [-0.1,0.1]$ and $\int_{\mathbb{R}} \alpha(x)dx = 1$. Consider the diagonal
element $a_T = \text{diag}(T^{-\frac{1}{2}}, T^{\frac{1}{2}}) \in G$. We define $w_T \in H_{k-2}$ by
\begin{equation}
w_T = T^{\frac{1}{2}} \cdot \pi_{k-2}(a_T)\alpha.
\end{equation}
Recall that the action is given by $\pi_{k-2}(\text{diag}(a^{-1}, a))v(x) = |a|^{2-k}v(a^2x)$. We note that geometrically the vector $w_T$ is a small non-negative bump
function around the point $0 \in \mathbb{R}$, with the support in the interval
$T^{-1} \cdot [-0.1,0.1]$, and satisfying $\int_{\mathbb{R}} w_T(x)dx = T^{\frac{3}{2}(1-k)}$. We now set
$v_T = q_k(w_T) \in D_k$. Note that $v_T = T^{\frac{3}{2}} \cdot \pi_{D_k}(a_T)\tilde{v}$ for some $\tilde{v} \in D_k$, i.e.,
the vector $v_T$ is obtained by the action of $G$ on some fixed vector in
$D_k$. This will be crucial in what follows since we will need to estimate
the supremum norm for the automorphic realization of the vector $v_T$.

The construction of the test vector $v_T' \in D_k'$ is slightly more com-
plicated since we can not simply project a vector to $D_k' \subset H_{-k}$ since
the value of the functional $f_{k,a,e}^{\text{ind}}$ might change significantly. Let $\alpha$ be as
above. We now view it as a vector in $H_{-k}$. We choose a smooth real
valued function $\alpha' \in C^\infty(\mathbb{R})$ satisfying the following properties:

1. supp($\alpha'$) $\subset [M-0.1, M+0.1]$, where the parameter $M$ is to be
chosen later,
2. $\int_{\mathbb{R}} x^m \alpha'(x)dx = -\int_{\mathbb{R}} x^m \alpha(x)dx$ for $0 \leq m \leq k-2$.
The last condition implies that the vector \( w = \alpha + \alpha' \in D_k^* \) since \( \int_{\mathbb{R}} x^m w(x)dx = 0 \) for \( 0 \leq m \leq k - 2 \). We define now the second test vector by

\[
(22) \quad v'_T = T^{\frac{1}{2}} \cdot \pi_{-k}(\begin{pmatrix} 1 & -1 \end{pmatrix} a_T)w.
\]

Clearly we have \( v'_T \in D_k^* \).

Recall that \( \pi_{-k}(\text{diag}(a^{-1}, a))v(x) = |a|^k v(a^2 x) \), and \( \pi_{-k}(\begin{pmatrix} 1 & -1 \end{pmatrix})v(x) = v(x+1) \). Hence geometrically the vector \( v'_T = \alpha_T + \alpha'_T \) is the sum of two bump functions \( \alpha_T \) and \( \alpha'_T \) with their supports satisfying \( \text{supp}(\alpha_T) \subset 1 + T^{-1}[-0.1, 0.1] \) and \( \text{supp}(\alpha'_T) \subset 1 + T^{-1}[M - 0.1, M + 0.1] \), both near the point \( 1 \in \mathbb{R} \). We also have \( \alpha_T \geq 0 \) and \( \int_{\mathbb{R}} \alpha_T(x)dx = T^{\frac{3}{2}}(k-1) \).

We now set our test vector to \( u_T = v_T \otimes v'_T \in D_k \otimes D_k^* \). We want to show that for \( T/2 \leq |\lambda| \leq T \),

\[
(23) \quad |\tilde{l}_{k,\lambda,\varepsilon}^{\text{mod}}(u_T \otimes u)| \geq c' > 0
\]

for some vector \( u \in V_{\lambda,\varepsilon} \) with \( ||u|| = 1 \), and with a constant \( c' > 0 \) independent of \( \lambda \).

As we explained before, \( \tilde{l}_{k,\lambda,\varepsilon}^{\text{mod}}(q_k(v) \otimes w \otimes u) = \tilde{l}_{k,\lambda,\varepsilon}^{\text{ind}}(v \otimes w \otimes u) \) for any triple \( v \otimes w \otimes u \in \mathcal{H}_{k-2} \otimes \mathcal{H}_{-k} \otimes V_{\lambda,\varepsilon} \). Hence we work with \( \tilde{l}_{k,\lambda,\varepsilon}^{\text{ind}}(w_T \otimes v'_T \otimes u) \) instead of \( \tilde{l}_{k,\lambda,\varepsilon}^{\text{mod}} \). This value is given by an explicit integral. Let \( K_{k,\lambda}(x, y, z) \) be as in \cite{[18]}. We have then

\[
(24) \quad \tilde{l}_{k,\lambda,\varepsilon}^{\text{ind}}(w_T \otimes v'_T \otimes u) = \\
\int K_{k,\lambda}(x, y, z)(\text{sgn}(x, y, z))^\varepsilon w_T(x)v'_T(y)u(z) \, dx \, dy \, dz.
\]

Hence it is enough to show that the absolute value of the integral

\[
(25) \quad I_{\lambda}(z) = \\
\int K_{k,\lambda}(x, y, z)w_T(x)v'_T(y) \, dx \, dy = \langle K_{k,\lambda}(x, y, z), w_T(x)v'_T(y) \rangle
\]

is not small for some interval of \( z \in \mathbb{R} \). We have \( I_{\lambda}(z) = K(z) + K'(z) \), where we denote by \( K(z) = \langle K_{k,\lambda}(x, y, z), w_T(x)\alpha_T(y) \rangle \) and by \( K'(z) = \langle K_{k,\lambda}(x, y, z), w_T(x)\alpha'_T(y) \rangle \).

Since the support of \( w_T \) and of \( \alpha_T \) is of the size \( T^{-1} \), it is easy to see that \( |K(z)| \geq c'' > 0 \) for some constant \( c'' > 0 \), for \( z \in [10, 20] \) and for \( |\lambda| \leq T \). This is because the function \( w_T(x)\alpha_T(y) \) is a non-negative function with the support in a small box of size \( T^{-1} \) around the point \((0, 1) \in \mathbb{R}^2 \), and that the gradient of the function \( K_{k,\lambda}(x, y, z) \) is bounded by \( |\lambda| \leq T \) in this box. Hence there are no significant cancellations in the integral \((25)\). We normalized our vectors so that \( \int_{\mathbb{R}^2} v_T(x)\alpha_T(y)dx \, dy = 1 \), and hence the integral \( K(z) \) is not small for \( z \).
not near singularities of the kernel $K_{k,\lambda}(x, y, z)$. This part is identical to our argument in Section 5.3.4 of [BR3].

We are left with the second term $K'(z)$. We want to show that there are no cancellations between two terms $K(z)$ and $K'(z)$ for $z \in [10, 20]$. The reason for this is that while these terms are of about the same size, their arguments are different, and not opposite if $\lambda$ is not too small (e.g., $T/2 \leq |\lambda| \leq T$). Namely, the argument of the kernel function $K_{k,\lambda}$ in [24] on the support of $u_T$ is given by

\[(26) \quad \frac{\lambda}{2T} (t_1(1 - z^{-1}) - t_2(1 - (z - 1)^{-1})) + \frac{\lambda}{2} \left(\ln(z) + \ln(z - 1)\right) + O(T^{-1}),\]

where $t_1 \in [-0.1, 0.1]$, $t_2 \in [0.9, 1.1]$ for $\alpha_T$, and $t_2 \in [M - 0.1, M + 0.1]$ for $\alpha'_T$. By choosing appropriate value of $M$, we can see that the difference of these arguments is not close to 0 and $\pi$ for any fixed $z \in [10, 20]$ and $T/2 \leq |\lambda| \leq T$, and hence integrals $K(z)$ and $K'(z)$ do not cancel each other since $u_T$ is real valued.

Hence we have shown that $H_{\lambda}(u_T) \geq c' > 0$ for $T/2 \leq |\lambda| \leq T$, and some explicit $c' > 0$ which is independent of $\lambda$.

1.5.6. **Raising the level.** We now discuss what happens when we change the level. Since our test vectors $v_T$ and $v'_T$ are not $K$-finite, we have to pass to automorphic functions on the space $X_\ell$. We use the standard notation $j(g, z) = \det(g)^{-\frac{k}{2}}(cz + d)$ for $g = (a \ b) \in G^+$ and $z \in \mathbb{H}$. Let $\phi^k$ be a primitive holomorphic form of weight $k$ on $\mathbb{H}$ for the subgroup $\Gamma_0(N_0)$. We normalize $\phi^k$ by its norm on $Y_{N_0}$. According to the well-known dictionary, we associate to $\phi^k$ the function $\phi_{e_k} \in C^\infty(X_0)$ given by

\[(27) \quad \phi_{e_k}(g) = \phi^k(g(i)) \cdot j(g, z)^{-k},\]

where $z = g(i)$. In the opposite direction, we have $\phi^k(g(i)) = \phi_{e_k}(g) \cdot j(g, z)^k$. We have the associated isometry $\nu_k = \nu_{\phi^k} : D_k \to C^\infty(X_{N_0})$ which gives $\nu_k(e_k) = \phi_{e_k}.$

Let $\ell > 1$ be an integer. We denote by $a_\ell = \text{diag}(\ell^\frac{1}{2}, \ell^{-\frac{1}{2}})$. For a given $\nu_k = \nu_{\phi^k} : D_k \to C^\infty(X_{N_0})$, we construct the corresponding isometry $\nu_k' : D_k \to C^\infty(X_\ell)$ as follows. For a vector $v \in D_k$, we consider the corresponding automorphic function

\[(28) \quad ^t \phi_{v}(x) = \nu_k'(v)(x) = \nu_k(v)(a_\ell x) = \phi_v(a_\ell x).\]

Obviously, we have $\sup_{X_\ell} |^t \phi_v| = \sup_{X_0} |\phi_v|$ for the same vector $v \in D_k$. We want to compare this to the classical normalization of old forms.
For the lowest weight vector $e_k \in D_k$, we have with $z = g(i)$

$$
\ell^k \phi_{e_k}(g) = \phi_{e_k}(a_\ell g) = \phi^k(\ell z) \cdot j(a_\ell g, i)^{-k} = \\
\ell^k \phi^k(\ell z) \cdot j(g, i)^{-k}.
$$

On the other hand, classically, old forms are given by $\ell^k \phi^k(z) = \phi^k(z)$. Hence we acquire the extra factor $\ell^k$.

Since, as we noted, test vectors $v_T \in D_k$ and $v_T' \in D_k^*$ are obtained by the (scaled) group action applied to fixed vectors, and since the operation of raising the level by $\ell$ does not change the supremum norm, we arrive at the following bound

$$
\sup_{X_\ell} |\ell^k \phi_{v_T}(x)| = \sup_{X_0} |\phi_{v_T}(x)| = \\
T^{\frac{1}{2}} \sup_{X_0} |\phi_{\pi(\ell x)}(x)| = T^{\frac{1}{2}} \sup_{X_0} |\phi(w)(x)| = \beta' T^{\frac{1}{2}},
$$

for some constant $\beta'$. The same holds for the automorphic function $\ell^k \phi_{v_T'}$. This implies that $||\ell^k \phi_{v_T} \ell^k \phi_{v_T'}||^2_{X_\ell} \leq \beta T^2$ for some $\beta > 0$ independent of $\ell$ and $T$.

To summarize, we have proved the bound

$$
\sum_{\frac{1}{2} \leq T \leq \lambda_i \leq T} |\langle \ell^k \phi^k \ell^k \phi, \phi, \phi \rangle_{Y_\ell}|^2 \cdot \ell^{2k} \lambda_i^{-2k} e^{\frac{1}{2} \pi |\lambda_i|} \leq A \cdot T^2,
$$

where the summation is over an orthonormal basis of Maass forms for the subgroup $\Gamma_0(\ell N_0)$.

The above argument also proves the case of forms with different level, i.e., the bound (16). Let $\ell_1$ and $\ell_2$ be two co-prime integers. Under our the normalization of measures on $X_{\ell_1}$, we see that for a vector $v \in V$, $L^2$-norms $||\ell^k \phi_v||_{X_{\ell}} = ||\ell^k \phi_v||_{X_{\ell'}}$ are equal if $\ell|\ell'$ (here we view the function $\ell^k \phi_v$ as both $\Gamma_0(\ell N_0)$-invariant function and as $\Gamma_0(\ell' N_0)$-invariant function). Obviously, the supremum norms of $\ell^k \phi_v$ on $X_{\ell}$ and on $X_{\ell'}$ are also coincide. Hence

$$
||\ell_1^k \phi_{v_T} \ell_2^k \phi_{v_T'}||^2_{X_{\ell_1 \ell_2}} \leq \frac{1}{2} ||\ell_1^k \phi_{v_T}||^2_{X_{\ell_1 \ell_2}} + \frac{1}{2} ||\ell_2^k \phi_{v_T'}||^2_{X_{\ell_1 \ell_2}} \leq \beta T^2.
$$

This implies (16).

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