GENERALISATIONS OF THE EINSTEIN–STRAUS MODEL TO CYLINDRICALLY SYMMETRIC SETTINGS

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We study generalisations of the Einstein–Straus model in cylindrically symmetric settings by considering the matching of a static space-time to a non-static spatially homogeneous space-time, preserving the symmetry. We find that such models possess severe restrictions, such as constancy of one of the metric coefficients in the non-static part. A consequence of this is that it is impossible to embed static locally cylindrically symmetric objects in reasonable spatially homogeneous cosmologies.

1. Introduction

Possible influences of the large scale evolution of the universe on local dynamics has been the subject of intense study in cosmology for a long time. As an attempt in answering this question, Einstein and Straus\textsuperscript{1} considered a model consisting of the matching of an inner Schwarzschild metric, approximating the local neighbourhood in the solar system, and an outer dust Friedmann-Lemaitre-Robertson-Walker (FLRW), representing the expanding universe. They showed that such matching was possible across any comoving 3-sphere, provided the total mass inside the 3-sphere was equal to the Schwarzschild mass contained in it. Thus according to this model the global expansion of the universe exerts no influence on the vacuum region surrounding the Schwarzschild mass.

Given the potential importance of this result, it is important to ask whether it still holds if some of the inevitable idealisations involved in the Einstein–Straus model are relaxed. A number of attempts have been made to generalise the original setting, by including other source fields and FLRW geometries\textsuperscript{2}. For example, it has been shown that in order to match a static spacetime to an expanding FLRW the configuration has to be "almost spherical"\textsuperscript{2}. This indicates that the original Einstein-Straus model is unstable. Realistic cosmological models, however, cannot be expected to be exactly homogeneous and isotropic, so it would be interesting to study modified Einstein-Straus models which include anisotropic cosmologies.

Here we summarize and extend recent results which allow anisotropies in the Einstein-Straus model\textsuperscript{2}. This is done by studying embeddings of static cylindrically symmetric cavities in expanding homogeneous and anisotropic universes.
2. Matching Conditions

We consider the matching between two locally cylindrically symmetric spacetimes, preserving the symmetry 6. The matching conditions between two spacetimes \((g^+, M^+)\) and \((g^-, M^-)\) across a 3–hypersurface \(\sigma\) are twofold3: firstly the equality of the first fundamental forms at \(\sigma\)

\[ g_{ab}^\pm = g_{\alpha\beta}^\pm e_a^\pm e_b^\pm |\sigma \]  

and secondly the equality of the generalised second fundamental forms at \(\sigma\)

\[ H_{ab}^\pm = -\ell_{\alpha}^\pm \epsilon^\pm_{\alpha} \nabla_{\beta}^\pm e_b^\pm |\sigma \],

where \(\ell_{\alpha}^\pm\) are the rigging forms4. These conditions imply, in turn, the Israel conditions which are the equality at \(\sigma\) of

\[ S_{\beta}^\pm = n^+\alpha T_{\alpha\beta}^\pm, \]  

where \(n^\pm\) are the normal vectors to \(\sigma\).

Here we take \((g^-, M^-)\) to be the most general cylindrically symmetric static metric

\[ ds^2^- = -A^2 dT^2 + B^2 d\rho^2 + C^2 d\phi^2 + D^2 d\bar{z}^2 + 2Ed\phi d\bar{z}, \]  

where \(A, B, C, D, E\) are functions of \(\rho\).

For \((g^+, M^+)\) we take the most general spatially homogeneous spacetimes which, in order to have a globally defined axis, are required to have a \(G_4\) on \(S_3\). These can be cast in a metric form adapted to the killing vectors \(\partial_{\phi}\) and \(\partial_{\bar{z}}:\)

\[ ds^2^+ = -\hat{A}^2 dt^2 + \hat{B}^2 dr^2 - 2\epsilon r \hat{B}^2 dr d\bar{z} + \hat{C}^2 d\phi^2 + 2\hat{E} d\phi d\bar{z} + \hat{D}^2 d\bar{z}^2, \]  

with

\[ \hat{A}^2 = 1; \quad \hat{B}^2 = b^2(t); \quad \hat{C}^2 = b^2(t)\Sigma^2(r) + na^2(t)(F(r)+k)^2, \]

\[ \hat{D}^2 = a^2(t) + \epsilon r b^2(t); \quad \hat{E} = na^2(t)(F(r)+k), \]

where \(\epsilon = 0, 1\) \(n = 0, 1\) \(en = ek = 0\) and \(\Sigma(r)\) and \(F(r)\) depend on \(k = 0, -1, 1\). Note that these metrics include all Bianchi types.

The matching conditions \(\text{(1)}\) then imply\(\text{(1)}\)

\[ \dot{\hat{D}} = \hat{D}; \quad \dot{\hat{C}} = \hat{C}; \quad \dot{\hat{E}} = \hat{E}. \]  

From \(\text{(2)}\) we obtain, in particular,

\[ \dot{\hat{D}}_t \hat{C}_r - \dot{\hat{D}}_r \hat{C}_t \equiv 0, \quad \dot{\hat{E}}_t \dot{\hat{D}}_r - \dot{\hat{E}}_r \dot{\hat{D}}_t \equiv 0, \quad \dot{\hat{E}}_t \hat{C}_r - \dot{\hat{E}}_r \hat{C}_t \equiv 0. \]  

from which we get \(n = 0\).
3. Results

Assuming that the locally cylindrically symmetric (LSC) static regions are spatially compact and simply connected, and using \( n = 0 \) together with the matching conditions, we obtain the following results:

**Proposition:** A non-static Bianchi II, VIII or IX spacetime cannot be matched to a LCS static region across a non-spacelike hypersurface preserving the symmetry.

**Theorem:** The only possible non-static spatially homogeneous spacetimes that can be matched to a LCS static region across a non-space-like hypersurface preserving the symmetry are given by

\[
ds^2 = -dt^2 + \beta^2 dz^2 + b^2(t) \left[ (dr - \epsilon r dz)^2 + \Sigma^2(r,k) d\phi^2 \right], \tag{8}
\]

where \( \beta \) is a constant and \( \epsilon = 0, 1 \) is such that \( \epsilon k = 0 \).

As a consequence of this theorem the only non-zero fluid expansion components are

\[
\theta_{11} = \theta_{22} = \frac{b_t}{b}. \tag{9}
\]

The vanishing of \( \theta_{33} \) is a severe constraint as far as cosmologically interesting models are concerned since there is no expansion along one spatial direction (spanned by \( \partial_z + \epsilon r \partial_r \)). Using the Israel conditions \( \text{[5]} \) we also prove the following corollaries:

**Corollary 1 (perfect fluid):** If the non-static spacetime has a perfect-fluid with \( \rho + p > 0 \) then it must have metric \( \text{[5]} \) with \( \epsilon = 0 \), \( b(t) = \sqrt{\alpha t - kt^2}, \alpha > 0 \) and

\[
\rho = p = \frac{\alpha^2}{4t^2(\alpha - kt)^2}. \tag{10}
\]

**Corollary 2 (vacuum):** If the static spacetime is vacuum then the non-static spacetime is also vacuum.

The above results show that there are no evolving perfect fluid Bianchi spacetimes with \( \rho \neq p \) satisfying the dominant energy condition and containing a locally cylindrically symmetric static cavity. They therefore demonstrate that the Einstein–Straus result cannot be generalised in this way to cylindrical symmetry.

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