ON VARIETIES OF GENERAL TYPE WITH MANY GLOBAL K-FORMS

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Abstract. We study projective $n$-folds of general type with many global $k$-forms ($1 \leq k \leq n$). We show that, for any nonsingular projective 3-fold $V$ of general type with either $h^{2,0}(V) \geq 108 \cdot 42^3 + 4$ or $\chi(\mathcal{O}_V) \geq 108 \cdot 42^3 + 5$, the 3-canonical map is stably birational. Similarly, for any nonsingular projective 4-fold $V$ of general type with sufficiently large $h^{2,0}(V)$, we prove that the 5-canonical map is stably birational. We also show that the canonical stability index of any smooth projective general type $n$-fold $X$ with $q(X) > n$ ($n \geq 4$) is less than or equal to the $(n - 1)$-th canonical stability index $r_{n-1}$.

1. Introduction

We work over an algebraically closed field of characteristic zero. Given a nonsingular projective variety $W$ of general type, it has been of great importance to calculate the canonical stability index $r_s(W)$. For any $n \in \mathbb{Z}_{>0}$, the $n$-th canonical stability index

$$r_n := \sup_W \{r_s(W) | W \text{ is a smooth projective } n\text{-fold of general type}\}$$

is a key global quantity of birational geometry. It is known that $r_1 = 3$ and, by Bombieri [3], $r_2 = 5$. Iano-Fletcher’s example in [26] shows $r_3 \geq 27$ and, by Chen-Chen [9, 10, 11] and Chen [15], $r_3 \leq 57$. For any $n \geq 4$, the theorem of Hacon-McKernan [25], Takayama [41] and Tsuji [46] tells $r_n < +\infty$. However, no effective upper bound is known aside the fact that $r_n > 2^{\frac{n^2}{2}}$ due to those interesting examples found by Esser-Totaro-Wang [20].

Restricting the interest to some more explicit cases, let us recall the following known results:

- When $\dim(W) = 3$ and $p_g(W) \geq 4$, $r_s(W) \leq 5 (= r_3^+)$ by Chen [13, Theorem 1.2]; when $\dim(W) = 3$ and $\text{vol}(W) > 12^3$, $r_s(W) \leq 6$ by Chen-Chen [9, 10, 11].

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\[r_s(W) \leq 5 (= r_3)\] by Todorov [15] Theorem 1.2 and Chen [14] Theorem 1.1;

- There are constants \(K(4), L(4)\) and \(L(5)\). When \(\dim(W) = 4\) and \(\vol(W) \geq K(4)\) (resp. \(p_g(W) \geq L(4)\)), \(r_s(W) \leq r_3\) (resp. \(\leq r_3^+\)) by Chen-Jiang [17] Theorem 1.4, Theorem 1.5]; When \(\dim(W) = 5\) and \(p_g(W) \geq L(5)\), \(r_s(W) \leq r_4^+\) by Chen-Jiang [17] Theorem 1.5]; (see [17] P. 2044 for the definition of \(r_3^+\))

- When \(\dim(W) = n \geq 4\), \(r_s(W) \leq \max\{r_{n-1}, (n-1)r_{n-2} + 2\}\) by Lacini [39] Theorem 1.3].

In this paper, we consider varieties with sufficiently many global \(k\)-forms. Our first main result is the following:

**Theorem 1.1.** (=Theorem [2,7]) Fix two integers \(n\) and \(k\) with \(n > 0\) and \(0 \leq k \leq n\). There exist positive numbers \(a_{n,k}\) and \(b_{n,k}\) such that the inequality

\[\vol(X) \geq a_{n,k}h^0(X, \Omega_X^k) - b_{n,k}\]

holds for every smooth projective \(n\)-fold \(X\) of general type.

The constants \(a_{n,k}\) and \(b_{n,k}\) are related to minimal volumes of varieties of general type of dimensions \(\leq n - 1\). When \(n\) is small, these numbers are explicit.

These Noether type inequalities also suggest that the pluricanonical systems on varieties with many global \(k\)-forms should behave well. We show that it is indeed the case by considering 3-folds and 4-folds with many global two forms and varieties in any dimension with many global 1-forms.

**Theorem 1.2.** Let \(V\) be any nonsingular projective 3-fold of general type with \(h^0(V, \Omega^2_V) \geq 108 \cdot 18^3 + 4\). Then the \(m\)-canonical map \(\varphi_{m,V}\) is birational for all \(m \geq 3\).

Note that \(\chi(\mathcal{O}_V) = 1 + h^0(V, \Omega^2_V) - q(V) - p_g(V)\). Hence we can also provide an alternative form of Theorem 1.2 as follows.

**Theorem 1.3.** Let \(V\) be any nonsingular projective 3-fold of general type with \(\chi(\mathcal{O}_V) \geq 108 \cdot 18^3 + 5\). Then \(\varphi_{m,V}\) is birational for all \(m \geq 3\).

A typical 3-fold \(V\) of general type with arbitrarily large \(\chi(\mathcal{O}_V)\) can be constructed as follows.

**Example 1.4.** Let \(C_i\) be a hyperelliptic curve of genus \(g_i > 1\) for \(i = 1, 2, 3\). We denote by \(\tau_i\) the hyperelliptic involution on \(C_i\) and let \(f_i: C_i \to \mathbb{P}^1\) be the hyperelliptic quotient for \(i = 1, 2, 3\). We then have

\[f_i^*\mathcal{O}_{C_i} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(- (g_i + 1)).\]

Let \(X := (C_1 \times C_2 \times C_3)/\langle \tau_1 \times \tau_2 \times \tau_3 \rangle\) be the diagonal quotient and let \(f: X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) be the natural morphism. Then \(X\) has finitely many singular points, which are isolated terminal quotient singularities. Let \(\tau: V \to X\) be a desingularization. Since \(X\) has
rational singularities, $R \tau_* O_V = O_X$. Considering the composition of morphisms 
\[ g : V \to X \to \mathbb{P}^1 \times \mathbb{P}^1, \]
we have
\[ R g_* O_V = R f_* R \tau_* O_V = f_* O_X \]
\[ = \left( (f_1, O_{C_1}) \boxtimes (f_2, O_{C_2}) \boxtimes (f_3, O_{C_3}) \right)^{(r_1 \times r_2 \times r_3)} \]
\[ = (O_{p_1} \boxtimes O_{p_1} \boxtimes O_{p_1}) \]
\[ \oplus (O_{p_1}(-g_1 + 1)) \boxtimes O_{p_1}(-g_2 + 1) \boxtimes O_{p_1}) \]
\[ \oplus (O_{p_1} \otimes O_{p_1}(-g_1 + 1)) \boxtimes O_{p_1}(-g_3 + 1)) \]
\[ \oplus (O_{p_1} \otimes O_{p_1} = (g_1 - 1)(g_2 - 1)(g_3 - 1)). \]

We then see that $h^1(V, O_V) = h^3(V, O_V) = 0$ and $h^2(V, O_V) = g_1 g_2 + g_1 g_3 + g_2 g_3$. Moreover, $\text{vol}(V) = \text{vol}(K_X) = \frac{1}{8} \text{vol}(C_1 \times C_2 \times C_3) = (g_1 - 1)(g_2 - 1)(g_3 - 1)$. If $g_1$ is large, so is $\text{vol}(V)$. Let $g_2 = 2$, then $V$ has a genus 2 fibration, $|2K_V|$ cannot be birational. Therefore, $r_s(V) = 3$ which means the statement in Theorem 1.2 is sharp.

We can extend Theorem 1.2 to dimension 4.

**Theorem 1.5.** There exists a constant $M(4)$ such that, for any non-singular projective 4-fold $V$ of general type with $h^0(V, O_V^2) \geq M(4)$, $\varphi_{m, V}$ is birational for all $m \geq 5$.

The next two examples show that the statement in Theorem 1.5 is sharp as well.

**Example 1.6.** Let $Y$ be a minimal irregular threefold of general type with $q(Y) = 1$ such that the general fiber of the Albanese morphism of $Y$ is an $(1, 2)$-surface (namely, the minimal model of this fiber has invariants $(K^2, p_g) = (1, 2)$). Let $V = Y \times C$, where $C$ is a smooth projective curve of any genus $g \geq 2$. Then, as $g$ is sufficiently large, $V$ is a fourfold with sufficiently large $h^{2,0}(V)$, but $|2K_V|$ cannot induce a birational map since $r_s(S) = 5$. Thus, when $g$ is large enough, $r_s(V) = 5$.

**Example 1.7.** We still denote by $S$ a minimal $(1, 2)$-surface. Let $S'$ be a smooth minimal surface of general type with $p_g(S') = g$. Let $V = S \times S'$. Then, when $g$ is large enough, so is $h^{2,0}(V)$, but $|4K_V|$ cannot induce a birational map. Hence $r_s(V) = 5$.

It is interesting to mention that, though constructions in Examples 1.6 and 1.7 are simple, those varieties with similar structures are exactly the most difficult cases while proving Theorem 1.5.

Observing that two digits “3” and “5” in the statements of Theorem 1.2 and Theorem 1.5 can be understood as $r_{n-2}$ where $n = \dim(V) = 4$. 
3, 4, one might put the higher dimensional analog as an interesting conjecture (see Section 7)! However the next example shows that, unlike in dimension 3, one cannot replace the condition “$h^2_0(V) \gg 0$” with the condition “$\chi(O_V) = \chi(\omega_V) \gg 0$” even in dimension 4. Thus the higher dimensional analog of Theorem 1.3 seems to be impossible. At least it fails in dimension 4.

Example 1.8. Let $C$ be a smooth projective curve and let $L$ be a very ample line bundle on $C$ with $d = \deg(L) \gg 1$. Let $\mu : Y \rightarrow \mathbb{P} := \mathbb{P}(1, 3, 4, 5, 14)$ be a resolution of the weighted projective space of dimension 4 such that $|\mu^*\mathcal{O}_\mathbb{P}(28)| = |M| + E$, where the effective $\mathbb{Q}$-divisor $E$ is the fixed part and the mobile part $|M|$ is base point free. Let $V \subset C \times Y$ be a general hypersurface of $|L \boxtimes M|$. Then $V$ is a smooth 4-fold and $\omega_V = ((K_C \otimes L) \boxtimes (K_Y \otimes M))_V$. Let $f : V \rightarrow C$ be the natural fibration. One sees that $R^i f_* \omega_V = 0$ for $i = 1, 2, 3$, since a general fiber of $f$ is birational to a hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 5, 14)$. Moreover, an easy computation shows that $f_* \omega_V = K_C \otimes L$ and $R^3 f_* \omega_V = K_C$. Hence, $\chi(\omega_V) = d \gg 0$, $h^{1,0}(V) = 0$ and that $|13K_V|$ cannot induce a birational map of $V$. In a word, $r_s(V) > 13 > r_2 = 5$.

Finally we study irregular varieties of general type and prove the following theorem:

Theorem 1.9. (=Theorem 7.2) Let $X$ be a smooth projective variety of general type of dimension $n \geq 4$. Assume that $q(X) > n$. Then $|mK_X|$ induces a birational map for all $m \geq r_{n-1}$.

A variety $X$ is an integral separated scheme of finite type over an algebraic closed field of characteristic 0. We will always work on normal projective varieties. Let $D_1$ and $D_2$ be two $\mathbb{Q}$-Weil divisors on a normal variety $X$. We say that $D_1 \geq D_2$ if $D_1 - D_2$ is an effective $\mathbb{Q}$-Weil divisor. We say that $D_1 \geq_\mathbb{Q} D_2$ if there exits a positive integer $M$ such that $M(D_1 - D_2)$ is a Cartier divisor and is linearly equivalent to an effective Cartier divisor. We write $D_1 \sim_\mathbb{Q} D_2$ if $M(D_1 - D_2)$ is a principal Cartier divisor for $M$ sufficiently large and divisible. We say a $\mathbb{Q}$-Cartier divisor $D$ on a normal projective variety $X$ is psuedo-effective if for any ample Cartier divisor $H$ on $X$ and any rational number $b > 0$, $D + bH$ is a big $\mathbb{Q}$-Cartier divisor.

Let $X$ be a normal variety and $D$ an effective $\mathbb{Q}$-Weil divisor on $X$. Let $\mu : X' \rightarrow X$ be a log resolution of $(X, D)$. We may write

$$K_{X'} = \mu^*(K_X + D) + \sum_E a(E; X, D)E,$$

where $E$ runs over all the distinct prime divisors of $X'$ and $a(E; X, D) \in \mathbb{Q}$. We call $a(E; X, D)$ the discrepancy of $E$ with respect to $(X, D)$. We say that $(X, D)$ is log canonical at $x \in X$ if $a(E; X, D) \geq -1$ for each $E$ such that $x \in \mu(E)$. Let $E \subset X'$ be a prime divisor with discrepancy
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−1. We say that $\mu(E)$ is a log canonical center or lc center of $(X, D)$ if $(X, D)$ is log canonical at a general point of $\mu(E)$. A log canonical center which is minimal with respect to the inclusion is called a minimal log canonical center. Assume $(X, D)$ is log canonical at $x \in X$ and let $C_1$ and $C_2$ be two lc centers of $(X, D)$ containing $x$. By $[30, \text{Proposition 1.5}]$, each irreducible component of $C_1 \cap C_2$ containing $x$ is also a lc center of $(X, D)$. In particular, the minimal lc center of $(X, D)$ at $x$ is well-defined. Let $E_1, \ldots, E_m$ be the divisors with discrepancy $\leq -1$ of $(X, D)$. Then $\mu(E_1 \cup \cdots \cup E_m)$ is called the non-klt locus of $(X, D)$, usually denoted by $N_{\text{klt}}(X, D)$.

When $X$ is smooth, we denote by $\mathcal{I}(D) = \mathcal{I}(X, D) = \mu^*O_X(\sum E \lceil -a(E; X, D)E \rceil)$ the multiplier ideal of $D$ (see $[38, \text{Section 9}]$). Then it is clear that $N_{\text{klt}}(X, D)$ is the support of the subscheme of $X$ defined by $\mathcal{I}(D)$.

Let $D$ be an effective $\mathbb{Q}$-divisor on a smooth variety $X$. We denote by $\text{lct}(X; D)$ the maximal positive rational number $t$ such that $(X, tD)$ is log canonical at each point of $X$. We call $\text{lct}(X; D)$ the log canonical threshold of $D$.

Let $F$ be a torsion-free coherent sheaf of rank $r$ on a smooth variety $X$. Let $j : U \subset X$ be the locus where $F$ is locally free. Then, $\text{codim}_X(X \setminus U) \geq 2$. We write $\det F$ to be the unique Cartier divisor on $X$, which extends $\wedge^r F$ on $U$. We also denote by $F^{**}$ the reflexive hull $j_*(j^*F)$ of $F$.

We say that a set $X$ of varieties is birationally bounded if there is a projective morphism between schemes, say $\tau : X \to T$, where $T$ is of finite type, such that for every element $X \in X$, there is a closed point $t \in T$ and a birational equivalence $X \dasharrow Z_t$.

We usually denote by $\epsilon$ a sufficiently small positive rational number.

2. Inequalities among birational invariants

2.1. General inequalities.

According to Hacon-McKernan $[25]$, Takayama $[44]$ and Tsuji $[46]$, given any positive rational number $M$, the set of smooth projective general type $n$-folds whose canonical volumes are upper bounded by $M$ is birationally bounded. Denote by $\nu_n$ the minimal volume among all smooth projective $n$-folds of general type. By the MMP and Birkar-Cascini-Hacon-McKernan $[2]$, any variety of general type has a minimal model.

Theorem 2.1. Fix two integers $n$ and $k$ with $n > 0$ and $0 \leq k \leq n$. There exist positive numbers $a_{n,k}$ and $b_{n,k}$ such that the inequality

$$\text{vol}(X) \geq a_{n,k} h^0(X, \Omega_X^k) - b_{n,k}$$

holds for every smooth projective $n$-fold $X$ of general type.

When $k = n$, we have $h^n(X, \mathcal{O}_X) = h^0(X, K_X) = p_g(X)$ and hence Theorem 2.1 is a generalization of the Noether type inequality (see
Chen-Jiang [17, Corollary 5.1]). When \( n = 2 \), we have Debarre’s inequality (see [18]): \( \text{vol}(X) \geq 2p_g + 2(q(X) - 4) \).

**Proof.** By Lemma 2.2 below, there exists a subsheaf \( \mathcal{F} \) of \( \Omega^k_X \) such that \( h^0(X, \det \mathcal{F}) \geq \binom{n}{k} \). We may replace \( \mathcal{F} \) by its saturation in \( \Omega^k_X \) and denote by \( \mathcal{Q} \) the corresponding quotient bundle. Set \( H := \det \mathcal{F} \) and \( L := \det \mathcal{Q} \). Then

\[
\left( \frac{n-1}{k-1} \right) K_X \sim \det(\Omega^k_X) \sim H + L.
\]

By Campana and Paun [7, Theorem 1.2], we know that \( L \) is pseudo-effective.

Modulo birational modifications, we may assume that \( |H| \) is base point free. We consider the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X_{\text{min}} \\
\phantom{X} \downarrow{\varphi_H} & & \downarrow{} \\
\mathbb{P}(H^0(X, H)) & & 
\end{array}
\]

where \( X_{\text{min}} \) is the minimal model of \( X \) and \( \varphi_H \) is the morphism induced by the linear system \( |H| \). Denote by \( \varphi_H : X \xrightarrow{\pi} \mathbb{P}(H^0(X, H)) \) the Stein factorization of \( \varphi_H \) and let \( d = \dim \Gamma \). Let \( F \) be a general fiber of \( f \).

Take \( d-1 \) general hyperplane sections \( H_1, \ldots, H_{d-1} \) of \( \mathbb{P}(H^0(X, H)) \). Let \( W = s^*(H_1) \cap \cdots \cap s^*(H_{d-1}) \) and \( X_W = f^{-1}(W) \). Then the induced morphism \( f_W := f|_{X_W} : X_W \rightarrow W \) is a fibration from a smooth projective variety \( X_W \) of dimension \( n - d + 1 \) to a smooth projective curve. Let \( a \geq 1 \) be the degree of \( s^*H_1 \) on \( W \). Note that \( a \geq h^0(X, H) - d \).

Then, by Kawamata’s restriction theorem (see [31]), for each \( m \geq 2 \),

\[
\left| am(K_{X_W} + \frac{1}{a}H|_{X_W}) \right|_f = \left| amK_F \right|.
\]

Repeatedly applying Kawamata’s restriction theorem, one gets, for \( m \geq 2 \),

\[
\begin{align*}
\left| ma(K_X + (d - 1 + \frac{1}{a})H) \right|_f &= \left| ma(K_X + \varphi^*_H H_1 + \cdots + \varphi^*_H H_{d-1} + \frac{1}{a}H) \right|_f \\
&= \left| ma(K_{X_W} + \frac{1}{a}H|_{X_W}) \right|_f \\
&= \left| maK_F \right|.
\end{align*}
\]

We take a rational number \( 0 < \epsilon \ll 1 \) and consider the \( \mathbb{Q} \)-divisor \((d - 1 + \frac{1}{a})L + \epsilon K_X \). Since \( L \) is pseudo-effective and \( K_X \) is big,

\[
M((d - 1 + \frac{1}{a})L + \epsilon K_X)
\]
is effective for sufficiently large and divisible integer $M$. Therefore,

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)K_X|$$

$$= |M(K_X + (d-1 + \frac{1}{a})H) + M((d-1 + \frac{1}{a})L + \epsilon K_X)|$$

$$\supset |M(K_X + (d-1 + \frac{1}{a})H)| + D,$$  

where $D \in |M((d-1 + \frac{1}{a})L + \epsilon K_X)|$ is an effective divisor. Restricting on $F$, by (2.1), we get

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)K_X|_F \supset |MK_F| + D|_F.$$ 

Modulo a further birational modification to $\pi$, we may assume that $\theta : F \to F_{\min}$ is a morphism onto one of its minimal model. Note that the free part of

$$|M\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)\pi^*K_{X_{\min}}|.$$ 

Thus

$$\left(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1 + \epsilon\right)\pi^*K_{X_{\min}}|_F \geq \theta^*K_{F_{\min}}.$$ 

We finally conclude that

$$\text{vol}(X) = (\pi^*K_{X_{\min}}^n) \geq \frac{1}{(n-1)^k} \cdot ((L + H) \cdot \pi^*K_{X_{\min}}^{n-1})$$

$$\geq \frac{1}{(n-1)^k} \cdot (H^d \cdot \pi^*K_{X_{\min}}^{n-d}) \geq \frac{a}{(n-1)^k} \cdot ((\pi^*K_{X_{\min}})^{n-d}_F)$$

$$\geq \frac{a}{(n-1)^k} \cdot \frac{\text{vol}(F)}{(\binom{n-1}{k-1}(d-1 + \frac{1}{a}) + 1)^{n-d}}$$

$$\geq \frac{h^0(X,H) - d}{(n-1)^d} \cdot \frac{\text{vol}(F)}{(\binom{n-1}{k-1}(d+1)^{n-d}}$$

$$\geq \frac{h^0(X,H)}{(n-1)^d} \cdot \frac{\text{vol}(F)}{(\binom{n-1}{k-1}(d+1)^{n-d}}$$

$$\geq \frac{\nu_{n-d}}{(n-1)^d} \cdot \frac{\text{vol}(F)}{(\binom{n-1}{k-1}(d+1)^{n-d}} \cdot (h^{k,0}(X) - d\binom{n}{k}). \ (2.2)$$

□
Lemma 2.2. Let \( \mathcal{E} \) be a torsion-free sheaf of rank \( r \) over a projective variety \( X \). Assume that \( h^0(X, \mathcal{E}) > 0 \). There exists a torsion-free subsheaf \( \mathcal{F} \subset \mathcal{E} \) such that \( h^0(X, \text{det} \mathcal{F}) \geq h^0(X, \mathcal{E}) r \).

Proof. We run induction on \( r \). When \( r = 1 \), it is a trivial statement. We now assume that \( \text{rank}(\mathcal{E}) = r > 1 \). We may assume that the evaluation \( \text{ev} : H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E} \) is generically surjective. Otherwise, the image \( \mathcal{E}' \) of the evaluation map is of rank \( \leq r - 1 \) and we replace \( \mathcal{E} \) by \( \mathcal{E}' \).

We then take \( s_1, \ldots, s_{r - 1} \in H^0(X, \mathcal{E}) \), which generate a subspace \( W \) of \( H^0(X, \mathcal{E}) \), and the evaluation map \( \text{ev}|_W : W \otimes \mathcal{O}_X \rightarrow \mathcal{E} \) is injective. We then take the wedge product:

\[
\phi_W : H^0(X, \mathcal{E}) \rightarrow H^0(X, \text{det} \mathcal{E})
\]

\[
s \rightarrow s_1 \wedge \cdots \wedge s_{r - 1} \wedge s.
\]

If \( h^0(X, \text{det} \mathcal{E}) \geq h^0(X, \mathcal{E}) r \), we are done. Otherwise,

\[
\dim \ker \phi_W > \frac{r - 1}{r} \cdot h^0(X, \mathcal{E}).
\]

Let \( W' := \ker \phi_W \). We now consider

\[
\text{ev}|_{W'} : W' \otimes \mathcal{O}_X \rightarrow \mathcal{E}.
\]

Since for each \( s \in W' \), \( s \) is linearly dependent with \( s_1, \ldots, s_{r - 1} \) at a general point of \( X \), the image of \( \text{ev}|_{W'} \) is a subsheaf \( \mathcal{E}'' \subset \mathcal{E} \) of rank \( r - 1 \). Note that \( H^0(X, \mathcal{E}'') \supset \mathcal{E}'' \). Thus, by induction, \( \mathcal{E}'' \) contain a subsheaf \( \mathcal{F} \) with \( h^0(X, \text{det} \mathcal{F}) \geq \frac{h^0(X, \mathcal{E})}{r - 1} > \frac{h^0(X, \mathcal{E})}{r} \).

The linear bound \( (2.2) \) is probably far from being optimal, but to the authors’ knowledge, this is the first explicit inequality between canonical volumes and intermediate Hodge numbers in high dimensions.

Corollary 2.3. Let \( X \) be a 3-fold of general type,

\[
\text{vol}(X) \geq \begin{cases} 
\frac{h^{2,0}(X) - 9}{24}, & h^{2,0}(X) \leq 13 \\
\frac{h^{2,0}(X) - 3}{54}, & h^{2,0}(X) \geq 14,
\end{cases}
\]

and

\[
\text{vol}(X) \geq \begin{cases} 
\frac{\chi(\mathcal{O}_X) - 10}{24}, & \chi(\mathcal{O}_X) \leq 14 \\
\frac{\chi(\mathcal{O}_X) - 1}{54}, & \chi(\mathcal{O}_X) \geq 15.
\end{cases}
\]

Proof. We note that \( \nu_1 = 2 \) and \( \nu_2 = 1 \). Hence, by \( (2.2) \), when \( n = 3 \) and \( k = 2 \), we have

\[
\text{vol}(X) \geq \min\left\{ \frac{h^{2,0}(X) - 9}{24}, \frac{h^{2,0}(X) - 6}{30}, \frac{h^{2,0}(X) - 3}{54} \right\}.
\]

Since \( \chi(\mathcal{O}_X) = 1 - q(X) + h^{2,0}(X) - p_g(X) \), we have \( h^{2,0}(X) \geq \chi(\mathcal{O}_X) - 1 \).
2.2. A stronger inequality between $\text{vol}$ and $q$.

In the rest of this section, we deduce a stronger inequality between the canonical volume and the irregularity via the Albanese morphism using generic vanishing theory. One may compare it with various Severi inequalities (see, for instance, [28]).

We first recall some results from generic vanishing. For a coherent sheaf $\mathcal{F}$ on an abelian variety $A$, we define the $i$-th cohomological support locus

$$V^i(\mathcal{F}) := \{ P \in \text{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes P) \neq 0 \}.$$ 

We say that $\mathcal{F}$ is a GV sheaf if $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) \geq i$ for each $i \geq 1$. Following [41], we define the generic vanishing index

$$gv(\mathcal{F}) := \min_{1 \leq i \leq \dim A} \{ \text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) - i \}.$$ 

for a GV sheaf $\mathcal{F}$. Note that, if $V^i(\mathcal{F}) = \emptyset$, we let $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) = \infty$. The main result of [41] states that, if $\mathcal{F}$ is a GV sheaf on $A$ with $gv(\mathcal{F}) < \infty$, $\chi(\mathcal{F}) \geq gv(\mathcal{F})$. Given a morphism $f : X \to A$ from a smooth projective variety $X$ to an abelian variety, the higher direct images $R^i f_* \omega_X$ are GV for each $i \geq 0$ (see [24]). Moreover, by Green-Lazarsfeld [23] and Simpson [43], $V^j(R^i f_* \omega_X)$ is a union of torsion translates of abelian subvarieties of $\text{Pic}^0(A)$ for each $i, j \geq 0$.

The following lemma is a kind of geometric version of Lemma 2.2 for $h^{1,0}(X)$.

**Lemma 2.4.** Let $f : Z \to A$ be a morphism from a smooth projective variety $Z$ to an abelian variety $A$. Assume that $f$ is generically finite onto its image and $f(Z) \subseteq A$ generates $A$. Then there exists a quotient between abelian varieties $q_B : A \to B$ with connected fibers such that, when taking the Stein factorization of $q_B \circ f : Z \to B$:

$$Z \xrightarrow{f} A \xrightarrow{q_B} B,$$

$f_B(Z_B) \subsetneq B$ generates $B$, any smooth model $Z_B$ of $Z_B$ is of general type, and

$$\chi(\omega_{Z_B}) \geq \frac{\dim A - \dim Z}{\dim Z}.$$ 

**Proof.** Let $n = \dim Z$ and $g = \dim A$. We run induction on $n$. When $n = 1$, the conclusion follows from the assumption that $f(Z)$ generates $A$. We then assume that $n \geq 2$.

Note that $f_* \omega_Z$ is a GV sheaf and, since $f$ is generically finite, $R^i f_* \omega_Z = 0$. We consider $gv(f_* \omega_Z)$. Since $H^n(A, f_* \omega_Z \otimes P) = H^n(Z, \omega_Z \otimes f^* P)$ for each $P \in \text{Pic}^0(A)$,

$$V^n(f_* \omega_Z) = \text{ker}(f^* : \text{Pic}^0(A) \to \text{Pic}^0(Z))$$

With the Stein factorization of $q_B \circ f : Z \to B$, $f_B(Z_B) \subsetneq B$ generates $B$, we then assume that $n \geq 2$.
consists of finitely many points. In particular, $gv(f_\ast \omega_Z) < \infty$. If $gv(f_\ast \omega_Z) \geq \frac{g-n}{n}$, we conclude from Pareschi-Popa [31] that $\chi(\omega_Z) \geq \frac{g-n}{n}$.

We then assume that $gv(f_\ast \omega_Z) = k < \frac{2}{n} - 1$ and

$$\text{codim}_{\text{Pic}^0(A)^{\text{V}^0}}(f_\ast \omega_Z) - i_0 = k$$

for some $1 \leq i_0 \leq n$. Since $n \geq 2$ and $\dim V^n(f_\ast \omega_Z) = 0$, we see that $1 \leq i_0 \leq n - 1$. Pick an irreducible component $W$ of $V^n(f_\ast \omega_Z)$ of codimension $i_0 + k$, then $W$ must be of the form $Q + \hat{C}$ where $\hat{C} \subset \text{Pic}^0(A)$ is an abelian subvariety and $Q \in \text{Pic}^0(A)$ is a torsion point. We then consider the dual quotient $q_C : A \to C := \text{Pic}^0(\hat{C})$. After taking further necessary birational modification to $q_C \circ f : Z \to C$, we obtain the Stein factorization: $Z \xrightarrow{q_C} Z_C \xrightarrow{f_C} C$, where $Z_C$ may be assumed smooth.

We claim that $\dim Z_C \leq n - i_0$. Indeed, since $Q + \hat{C} \subset V^n(f_\ast \omega_Z)$, for general $P \in \hat{C}$,

$$H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^* P)) = H^{i_0}(A, f_\ast \omega_Z \otimes Q \otimes q_C^* P) \neq 0.$$

On the other hand, by Kollár’s splitting (see [33, the main theorem]),

$$H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^* P)) \cong \bigoplus_{0 \leq j \leq i_0} H^j(Z_C, R^{i_0-j} q_{C*}(\omega_Z \otimes f^*Q) \otimes f_C^* P)$$

$$\cong \bigoplus_{0 \leq j \leq i_0} H^j(C, f_C* R^{i_0-j} q_{C*}(\omega_Z \otimes f^*Q) \otimes P).$$

By Hacon’s theorem (see [24]), all sheaves $f_C* R^{i_0-j} q_{C*}(\omega_Z \otimes f^*Q)$ are GV on $C$ for $0 \leq j \leq i_0$. Thus

$$H^{i_0}(Z, \omega_Z \otimes f^*(Q \otimes q_C^* P)) \simeq H^0(C, f_C* R^{i_0-j} q_{C*}(\omega_Z \otimes f^*Q) \otimes P) \neq 0.$$

This implies that $R^{i_0} q_{C*}(\omega_Z \otimes f^*Q) \neq 0$ and, by Kollár’s theorem [32, Theorem 2.1], $\dim Z - \dim Z_C \geq i_0$.

We then have

$$\frac{\dim C - \dim Z_C}{\dim Z_C} \geq \frac{g - i_0 - k}{n - i_0} - 1 \geq \frac{g - n - k}{n - i_0}$$

$$> \frac{g - n - \frac{2}{n} + 1}{n - 1} = \frac{g}{n} - 1.$$

Since $f_C(Z_C) \subseteq C$ generates $C$ and, by induction, there exists a further quotient $q_{CB} : C \to B$ with connected fibers between abelian varieties such that for the Stein factorization $Z_C \to Z_B \to B$ of $q_{CB} \circ f_C$, any smooth model of $Z_B$ or its image in $B$ is of general type, and

$$\chi(\omega_{Z_B}) \geq \frac{\dim C - \dim Z_C}{\dim Z_C} > \frac{g}{n} - 1.$$

□
Given any smooth projective $n$-fold $X$ of general type, for the case $k = 1$, Theorem 2.1 gives
\[ \text{vol}(X) \geq \min_{1 \leq d \leq n} \frac{\nu_{n-d}}{n(d+1)^{n-d}}(q(X) - dn), \]
which can be greatly improved as follows.

**Theorem 2.5.** Let $n > 0$ and $1 \leq d \leq n$. Set $\lambda_n := \min_{1 \leq d \leq n} \frac{\nu_{n-d}}{n(d+1)^{n-d}}$. The inequality
\[ \text{vol}(X) \geq 2(n-1)! \lambda_n(q(X) - n) \]
holds for any nonsingular projective $n$-fold $X$ of general type.

**Proof.** We may assume that $q(X) \geq n + 1$. Let $a_X : X \to A_X$ be the Albanese morphism of $X$. Taking the Stein factorization of $a_X$,
\[
\begin{array}{ccc}
    & & a_X \\
    & h & \\
 X & \downarrow & \downarrow a_X \\
 h_B & \downarrow & \downarrow h \\
 Z & f & \rightarrow A_X \\
 & \uparrow f & & \uparrow \\
 & Z_B & f_B & \rightarrow B,
\end{array}
\]
where $Z_B$ is smooth of general type, $\chi(\omega_{Z_B}) \geq \frac{q(X) - m}{m} \geq \frac{q(X) - n}{n}$, and $h_B$ is a fibration. Let $d = \dim Z_B \leq m$ and let $F$ be a general fiber of $h_B$.

By the Severi inequality (see [1] and [50]), $\text{vol}(Z_B) \geq 2d! \chi(\omega_{Z_B}) \geq 2d! \frac{q(X) - n}{n}$.

We have
\[ \text{vol}(X) \geq \frac{n!}{d!(n-d)!} \cdot \text{vol}(Z_B) \cdot \text{vol}(F) \cdot \frac{\nu_{n-d}}{(n-d)!}(q(X) - n). \]

Thus
\[ \frac{\nu_d}{d!} \leq \frac{2 \nu_{d-1}}{(d-1)!}. \]

It is natural to expect that $\nu_d < \nu_{d-1}$ when $d \geq 2$. If this is the case, we would have
\[ \text{vol}(X) \geq 2 \nu_{n-1}(q(X) - n) \]
for any smooth projective $n$-fold ($n \geq 3$) of general type.

**Remark 1.** By considering the product, we see that $\nu_d \leq 2d \nu_{d-1}$. Thus $\frac{\nu_d}{d!} \leq 2 \frac{\nu_{d-1}}{(d-1)!}$. It is natural to expect that $\nu_d < \nu_{d-1}$ when $d \geq 2$. If this is the case, we would have
\[ \text{vol}(X) \geq 2 \nu_{n-1}(q(X) - n) \]
3. Proof of Theorem 1.2

Let $V$ be a nonsingular projective 3-fold of general type. We show that $|3K_V|$ induces a birational map under the condition that $h^0(V, \mathcal{O}_V^2) \geq 108 \cdot 18^3 + 4$. The method naturally works for all $|mK_V|$ with $m \geq 4$.

By Corollary 2.3 we have $\text{vol}(V) > 2 \cdot 18^3$. Applying Fujita’s approximation (see [35, Subsection 11.4]), we write $K_V \sim_\mathbb{Q} A + E$, where $E$ is an effective $\mathbb{Q}$-divisor and $A$ is an ample $\mathbb{Q}$-divisor such that $0 < \text{vol}(V) - \text{vol}(A) \ll 1$.

We now apply the method of cutting non-klt locus in Hacon-McKernan [25], Takayama [44] and Tsuji [46], which is also exploited in Todorov [45] and in our previous work [17, Subsection 4.2].

Pick very general points $x, y \in V$. There exists an effective $\mathbb{Q}$-divisor $D_1 \sim_\mathbb{Q} t_1 K_V$ with $t_1 < 3 \sqrt{\frac{2}{\text{vol}(V)}} + \epsilon < \frac{1}{6}$, where $0 < \epsilon \ll 1$ such that $(V, D_1)$ is log canonical but not klt at $x$, and that $(V, D_1)$ is not klt at $y$. Modulo a small perturbation, we may also assume that the non-klt locus of $(V, D_1)$, passing through $x$, is the minimal log canonical center $V_1$.

3.1. The case with $\dim V_1 = 1$.

We apply Takayama’s induction to conclude that there exists a divisor $D_2 \sim_\mathbb{Q} t_2 K_V$ such that $t_2 \leq t_1 + \frac{2}{\text{vol}(V_1/K_V)} + \epsilon$, $(X, D_2)$ is log canonical at $x$, $\{x\}$ is an isolated component of $\text{Nklt}(X, D_2)$ at $x$, and $(X, D_2)$ is not klt at $y$. Moreover, by Takayama [44, Theorem 4.5], we know that $\text{vol}(V_1/K_V + D_1) \geq \text{vol}(V_1)$, where $V_1$ is the normalization of $V_1$. Thus

$$t_2 \leq t_1 + \frac{2(1 + t_1)}{2g(V_1) - 2} + \epsilon \leq 1 + 2t_1 + \epsilon.$$

Since $t_1 < \frac{1}{6}$, we can choose $t_2 < 2$.

We now conclude by using Nadel vanishing. Indeed, since $x$ and $y$ are very general, both $x$ and $y$ are not contained in the support of $E$. Thus we still have $x, y \in \text{Nklt}(V, D_2 + (2 - t_2)E)$ and $\{x\}$ is an isolated component of $\text{Nklt}(V, D_2 + (2 - t_2)E)$. Consider the short exact sequence

$$0 \to \mathcal{O}_V(3K_V) \otimes \mathcal{J}(D_2 + (2 - t_2)E) \to \mathcal{O}_V(3K_V) \to \mathcal{O}_V(3K_V) \otimes (\mathcal{O}_V/\mathcal{J}(D_2 + (2 - t_2)E)).$$

Since $2K_V - D_2 - (2 - t_2)E \sim_\mathbb{Q} (2 - t_2)A$ is ample, $H^1(V, \mathcal{O}_V(3K_V) \otimes \mathcal{J}(D_2 + (2 - t_2)E)) = 0$ by Nadel vanishing. Thus $|3K_V|$ separates $x$ and $y$.

3.2. The case with $\dim V_1 = 2$ and $\text{vol}(V_1) \geq 128$.

Similarly, there exists a divisor $D_2 \sim_\mathbb{Q} t_2 K_V$ such that $(X, D_2)$ is log canonical at $x$, $V_2 \subseteq V_1$ is the minimal log canonical center of $(X, D_2)$.
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at $x$ and $(X, D_2)$ is not klt at $y$, where

$$t_2 \leq t_1 + 2\sqrt{\frac{2}{\text{vol}_V(K_V)}} + \epsilon$$

$$\leq t_1 + 2(1 + t_1)\sqrt{\frac{2}{\text{vol}(V_1)}} + \epsilon.$$ 

Thus, if $\text{vol}(V_1) \geq 128$, since $t_1 < \frac{1}{6}$, $t_2 < \frac{1}{2}$, the statement follows from the argument in Subsection 3.1.

3.3. The case with $\dim V_1 = 2$ and $\text{vol}(V_1) \leq 127$. We apply a result of Todorov in [45, Lemma 3.2] to spread the minimal log canonical centers into a family. More precisely, there exists a smooth projective threefold $\tilde{V}$ with the following diagram:

$$\begin{array}{ccc}
\tilde{V} & \xrightarrow{\pi} & V \\
\downarrow{f} & & \downarrow{\pi} \\
C & & \\
\end{array}$$

where

(i) $f : \tilde{V} \to C$ is a surjective morphism to a smooth projective curve whose general fiber $F$ is a smooth projective surface of volume $\leq 127$;

(ii) $\pi$ is generically finite;

(iii) for $v \in \tilde{V}$ general, let $F_v$ be the fiber of $f$ passing through $v$ and $z = \pi(v)$, then $\pi|_{F_v} : F_v \to \pi(F_v)$ is birational onto its image and there exists an effective $\mathbb{Q}$-divisor $D_v \sim_\mathbb{Q} t_1 K_V$ such that $\pi(F_v)$ is the minimal log canonical center of $(V, D_v)$ at $z$.

3.3.1. The subcase with $\deg \pi = m \geq 2$. For a general point $z$ of $V$, the pre-image $\pi^{-1}(z)$ lies on $m$ distinct fibers of $f$ and we denote by $S_z$ the set of these fibers. We also observe that, for such a general $z \in V$ and for any $v \in \pi^{-1}(z)$, $z$ is a smooth point of $D_v$. In fact, locally $F_v$ maps onto $D_v$. The following argument is due to Todorov [45, Lemma 3.3].

If $S_x \neq S_y$, we may take $F_1 \in S_x$, $F_2 \in S_x \setminus S_y$, and $F_3 \in S_y \setminus S_x$. Let $D'_x, D''_x \sim_\mathbb{Q} t_1 K_V$ be the corresponding effective $\mathbb{Q}$-divisors such that $\pi(F_1)$ and $\pi(F_2)$ are respectively the minimal log canonical center of $(V, D'_x)$ and $(V, D''_x)$ at $x$. Let $D_y \sim_\mathbb{Q} t_1 K_V$ be the effective $\mathbb{Q}$-divisors such that $\pi(F_3)$ is the minimal log canonical center of $(V, D_y)$ at $y$. Note that $(V, D'_x + D''_x + D_y)$ is not klt at both $x$ and $y$. We set

$$c := \max\{ t \mid (V, t(D'_x + D''_x + D_y) \text{ is log canonical at } x) \}.$$ 

Then $c \in (0, 1]$ is a rational number. Moreover, since $x \notin D_y$ and $x$ is a smooth point of $D'_x$ and $D''_x$, the minimal log canonical center of
(V, c(D'_x + D''_x) + D_y) at x is contained in π(F_1) ∩ π(F_2) and, hence, is of dimension ≤ 1. It is also clear that (V, c(D'_x + D''_x) + D_y) is not klt at y.

If S_x = S_y, we take F_i ∈ S_x = S_y for i = 1, 2 and similarly denote by D' and D'' the corresponding effective Q-divisors. Both x and y are smooth point of D' and D''. Let

\[ c = \max\{\text{lct}_x(V, D' + D'') \cup \text{lct}_y(V, D' + D'')\}. \]

Similarly, 0 < c ≤ 1 is a rational number. After switching x and y, we may assume that (V, c(D' + D'')) is log canonical at x. Then the minimal log canonical center of (V, c(D' + D'')) is again contained in \( \pi(F_1) \cap \pi(F_2) \).

In conclusion, if deg \( \pi \geq 2 \), there exists an effective Q-divisor \( D_2 \sim Q \) such that \( (V, D_2) \) is log canonical at x, and \( D_2 \neq \emptyset \) is the minimal log canonical center of \( (V, D_2) \) at x and \( (V, D_2) \) is not klt at y, where 0 < t_2 ≤ 3t_1 < \( \frac{1}{2} \). The statement also follows from the argument in Subsection 3.3.2.

3.3.2. The subcase with \( \pi \) being birational. Since \( \pi \) is birational, we may simply assume that \( V = \tilde{V} \). Hence we have a fibration \( f : V \to C \) such that a general fiber \( F \) of \( f \) has its canonical volume \( \text{vol}(F) \leq 127 \).

We now apply the assumption that \( h^1(\omega_V) = h^0(\Omega_1^2) \geq 108 \cdot 18^3 + 4 \). We have

\[ h^1(V, \omega_V) = h^0(C, R^1f_*\omega_V) + h^1(C, f_*\omega_V) \]

by Leray’s spectral sequence. Note that \( h^1(C, f_*\omega_V) = h^0(C, (f_*\omega_{V/C})^*) \) by Serre duality. Moreover, \( f_*\omega_{V/C} \) is a locally free sheaf of rank \( p_g(F) \). Since \( \text{vol}(F) \leq 127 \), by Noether inequality \( p_g(F) \leq 250 \). Moreover, by Fujita’s theorem (see [21]), \( f_*\omega_{V/C} \) is nef. By considering the Harder-Narasimhan filtration of \( f_*\omega_{V/C} \), we see that

\[ h^0(C, (f_*\omega_{V/C})^*) \leq p_g(F) \leq 250. \]

Therefore \( h^0(C, R^1f_*\omega_V) \) is very large. In particular,

\[ q(F) = \text{rank}(R^1f_*\omega_V) > 0. \]

We may run the relative minimal model program for \( f : V \to C \). After resolving the finitely many terminal singularities of the relative minimal model, we may assume that a general fiber \( F \) of \( f \) is a minimal surface. Since \( F \) is irregular, \( p_g(F) \geq q(F) \geq 1 \). Hence the linear system \( |2K_F| \) is base point free (see [4] Chapter VII. Theorem 7.4) and \( |3K_F| \) induces a birational morphism of \( F \) (see [4] Proposition 7.3). By the main theorem of Kawamata [31], the restriction map

\[ |m(K_X + F)| \to |mK_F| \]

is surjective for \( m \geq 2 \). Fix a general divisor \( G \in |M(K_V + F)| \) for \( M \) sufficiently large. We also write \( K_V \sim Q A + E \), where \( A \) is an ample Q-divisor and \( E \) is an effective Q-divisor.
Since $\text{vol}(F) \leq 127$ and $\text{vol}(V) > 2 \cdot 18^3$, there exists an effective $\mathbb{Q}$-divisor $D$ such that $D \sim_{\mathbb{Q}} \lambda K_V$ and $D = F + D'$, where $D'$ is also an effective $\mathbb{Q}$-divisor and $\lambda^{-1} \approx \frac{\text{vol}(V)}{3 \cdot 127} > 2 \cdot 18^3 > 91$. (see, for instance, [17, the last paragraph of Page 2055]).

Fix two general fibers $F_1$ and $F_2$ of $f$, we introduce an effective $\mathbb{Q}$-divisor

$$Z := 4D + \frac{2 - 4\lambda - \epsilon}{M}G + (4\lambda - 4 + \epsilon)F + \epsilon E.$$ 

Note that $Z$ is $\mathbb{Q}$-effective. We denote by $\mathcal{J}(Z)$ the multiplier ideal of $Z$. Since $Z \sim_{\mathbb{Q}} (2 - \epsilon)K_V - 2F + \epsilon E$ and thus $2K_V - F_1 - F_2 - Z \sim_{\mathbb{Q}} \epsilon A$, by the Nadel vanishing theorem,

$$H^1(V, \mathcal{O}_V(3K_V - F_1 - F_2) \otimes \mathcal{J}(Z)) = 0.$$ 

Thus the restriction map

$$H^0(V, \mathcal{O}_V(3K_V) \otimes \mathcal{J}(Z))$$

$$\rightarrow H^0(F_1, \mathcal{O}_{F_1}(3K_{F_1}) \otimes \mathcal{J}(Z)|_{F_1}) \bigoplus H^0(F_2, \mathcal{O}_{F_2}(3K_{F_2}) \otimes \mathcal{J}(Z)|_{F_2})$$

is surjective.

By the restriction theorem ([38 Theorem 9.5.1]), $\mathcal{J}(F_1, Z|_{F_1}) \subset \mathcal{J}(Z)|_{F_1}$. Since $M$ can be sufficiently large, $|MK_{F_1}|$ is base point free, and $\epsilon$ can be sufficiently small, $\mathcal{J}(F_1, Z|_{F_1}) = \mathcal{J}(F_1, 4D|_{F_1})$.

When $\text{vol}(F) \geq 3$, since $D|_{F_1} \sim_{\mathbb{Q}} \lambda K_{F_1}$, we finish the proof by Lemma 3.1

When $\text{vol}(F) = 2$, we have $p_g(F) = q(F) = 1$. We may choose $\lambda$ such that

$$\lambda^{-1} \approx \frac{\text{vol}(V)}{3 \cdot 18^3} = 1944.$$ 

We finish the proof by Lemma 3.2. \qed

**Lemma 3.1.** Let $F$ be a minimal surface of general type with $K_F^2 \geq 3$. The linear system

$$|3K_F \otimes \mathcal{J}(\mu D_F)|$$

induces a birational map for any effective $\mathbb{Q}$-divisor $D_F \sim_{\mathbb{Q}} K_F$ and rational number $0 < \mu < 2 - \sqrt{3}$.

**Proof.** It is convenient to apply the $\mathbb{Q}$-divisor method on surfaces. Let $\sigma : \tilde{F} \rightarrow F$ be a log resolution of $(F, D_F)$. Then $\mathcal{J}(\mu D_F) = \sigma_* \mathcal{O}_{\tilde{F}}(K_{\tilde{F}}/\tilde{F} - [\mu \sigma^*(D_F)])$.

Hence $\sigma_*$ induces an isomorphism:

$$H^0(\tilde{F}, K_{\tilde{F}} + [2\sigma^*K_F - \sigma^*(\mu D_F)],) \cong H^0(F, 3K_F \otimes \mathcal{J}(\mu D_F)).$$

Recall the following theorem of Langer (see [37]), for a nef $\mathbb{Q}$-divisor $Q := 2\sigma^*K_F - \sigma^*(\mu D_F)$ on the surface $\tilde{F}$, if $Q^2 > 8$ and $(Q \cdot C) > \frac{4}{1 + \sqrt{1 - \frac{Q^2}{8}}} - 2$ for any curve $C$ passing through a very general point of $\tilde{F}$, then $|K_{\tilde{F}} + [Q]|$ induces a birational map of $\tilde{F}$. 

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Now we have $Q^2 = (2\sigma^* K_F - \sigma^*(\mu D_F))^2 > 3(K_F^2) \geq 9$. Thus it suffices to verify that

$$Q \cdot C > 3 = \frac{4}{1 + \sqrt{1 - \frac{8}{9}}} \geq \frac{4}{1 + \sqrt{1 - \frac{8}{9^2}}}$$

for any curve $C$ passing through a very general point of $\tilde{F}$. This is the case since, by Chen-Chen [11, Lemma 2.5], we always have $(\sigma^* K_F \cdot C) \geq 2$. □

**Lemma 3.2.** Assume that $F$ is a minimal surface of general type with $K_F^2 = 2$ and $p_g(F) = q(F) = 1$. Then $J(F, \mu D_F) = O_F$ for any effective $\mathbb{Q}$-divisor $D_F \sim Q K_F$ and any rational number $\mu: 0 < \mu < \frac{1}{26}$.

**Proof.** We have $h^0(F, 2K_F) = 3$.

Let $\sigma : F \to F_0$ be the contraction onto the canonical model of $F$. Then $K_F = \sigma^* K_{F_0}$. We denote by $H_0 \sim K_{F_0}$ the ample Cartier divisor on $F_0$. By the birational transformation rule (see [38, Theorem 9.2.33]), it suffices to show that $\lct(F_0; D_{F_0}) \geq \frac{1}{26}$ for any $D_{F_0} \sim Q H_0$.

We apply Kollár’s method (see the appendix of [16]). Since $H_0^2 = 2$ and $h^0(F_0, K_{F_0} + H_0) = 3$, we have

$$3 \geq \operatorname{mcd}(\lct(F_0; \frac{1}{2} D_{F_0})),$$

for any $D_{F_0} \sim Q H_0$ (see [16, Proposition A.3 and Remark A.7]). By [16, Proposition A.4], $\lct(F_0; \frac{1}{2} D_{F_0}) \geq \frac{1}{13}$. □

4. **Extension theorems**

We prove two extension theorems in this section. Both are crucial ingredients in the proof of Theorem 1.5. We hope that they are useful in other contexts.

4.1. **The first theorem.**

**Theorem 4.1.** Given a birationally bounded set $\mathfrak{X}$ of varieties of general type of dimension $n$. Let $f : V \to S$ be a fibration from a smooth projective variety $V$ onto a smooth projective surface $S$, whose general fiber $F$ is birationally equivalent to an element of $\mathfrak{X}$, and $h : S \to C$ a fibration from $S$ onto a smooth projective curve $C$. Let $g = f \circ h : V \to C$ be the composition of fibrations and denote by $Y$ a general fiber of $g$. For any integer $m \geq 2$, there exists a constant $M$ depending only on $m$ and $\mathfrak{X}$ such that either of the following statements holds for any $V$ with $K_V \geq Q MY$:

1. the restriction map $H^0(V, mK_V) \to H^0(Y, mK_Y)$ is surjective;
2. the restriction map $H^0(V, mK_V) \to H^0(F, mK_F)$ is surjective.
Proof. Let \( p \in C \) be a general point and let \( Y \) and \( Z \) be respectively the fiber of \( g \) and \( h \) over \( p \). We have the following commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{f_Y} & V \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{h} & S \\
\downarrow p & & \downarrow p \\
\end{array}
\]

Since \( K_Y \geq M_Y \), by the proof of [17, Theorem 3.7], the image of the restriction map

\[
H^0(V, mK_V) \to H^0(Y, mK_Y)
\]

contains

\[
H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}((m-1-\epsilon)K_Y + \epsilon D_Y)), \tag{4.1}
\]

where \( \epsilon = \frac{m}{M+1} \) and \( D_Y \sim_\mathbb{Q} K_Y \) is an effective \( \mathbb{Q} \)-divisor, and the definition of the multiplier ideal sheaf \( \mathcal{J}((m-1-\epsilon)K_Y + \epsilon D_Y) \) can be found in [17 Subsection 2.3].

Since \( F \) is birational to an element of \( \mathfrak{X} \), by Lemma [4.2] below, there exist constants \( M_1 \) and \( \epsilon_1 \), depending only on \( m \) and \( \mathfrak{X} \), such that whenever \( \operatorname{vol}(Y) > M_1 \) and \( \epsilon < \epsilon_1 \), the restriction map

\[
H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}((m-1-\epsilon)K_Y + \epsilon D_Y)) \to H^0(F, mK_F)
\]

is surjective.

Thus, when \( M > \frac{m}{\epsilon_1} \) and \( \operatorname{vol}(Y) > M_1 \), the restriction map

\[
H^0(V, mK_V) \to H^0(F, mK_F)
\]

is surjective.

If \( \operatorname{vol}(Y) \leq M_1 \), \( Y \) belongs to a birationally bounded family. By [17 Theorem 3.5], there exists a positive constant \( \epsilon_2 \), depending only on \( m \) and \( M_1 \), such that for any effective \( \mathbb{Q} \)-divisor \( D_Y \sim_\mathbb{Q} K_Y \) and any positive rational number \( \tau < \epsilon_2 \),

\[
H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}((m-1-\tau)K_Y + \tau D_Y)) \simeq H^0(Y, mK_Y).
\]

Thus, by [4.1], we see that when \( M > \frac{m}{\epsilon_2} \), the restriction map

\[
H^0(V, mK_V) \to H^0(Y, mK_Y)
\]

is surjective.

Hence we can take \( M = \max\{\frac{m}{\epsilon_1}, \frac{m}{\epsilon_2}\} \) and finish the proof of Theorem 4.1. \( \square \)

Lemma 4.2. Under the assumption of Theorem 4.1, let \( f_Y : Y \to Z \) be the induced morphism. There exist constants \( \epsilon_1 \) and \( M_1 \), depending
only on $m$ and $X$, such that whenever $\text{vol}(Y) > M_1$ and $0 < \eta < \epsilon_1$, for any effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_Y$, the restriction map $H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}((m-1-\eta)K_Y | + \eta D)) \to H^0(F, \mathcal{O}_F(mK_F))$ is surjective, where $F$ is a general fiber of $f_Y$.

Proof. After suitable birational modifications, we denote by $r : Y \to Y_0$ (resp. $r_F : F \to F_0$) the contraction morphism onto a minimal model $Y_0$ (resp. $F_0$). We write $K_Y = r^*K_{Y_0} + E_Y$ and $K_F = r_F^*K_{F_0} + E_F$. Note that $E_Y$ and $E_F$ are effective $\mathbb{Q}$-divisors and for $N$ sufficiently large and divisible, $|NK_{Y_0}|$ and $|NK_{F_0}|$ are base point free and we have

\[ |NK_Y| = r^*|NK_{Y_0}| + NE_Y, \]
\[ |NK_F| = r_F^*|NK_{F_0}| + NE_F. \]  

(4.2)

Let $\mu : \tilde{Y} \to Y$ be a log resolution of $(Y, E_Y + D)$, let $\tilde{F}$ be the strict transform of $F$, and let $\mu_F : \tilde{F} \to F$ be the induced morphism. We have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\mu_F} & F \\
\downarrow & & \downarrow r_F \\
\tilde{Y} & \xrightarrow{\mu} & Y \\
\downarrow f & & \downarrow r \\
Y_0 & & Y_0
\end{array}
\]

Define

\[ \mathcal{J} := \mathcal{J}((m-1-\eta)K_Y | + \eta D) \]
\[ = \mu_*\mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}/Y} - [(m-1-\eta)\mu^*E_Y + \eta \mu^*D]). \]

Then

\[ H^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}) \]
\[ \cong H^0(\tilde{Y}, K_{\tilde{Y}} + (m-1)\mu^*K_Y - [(m-1-\eta)\mu^*E_Y + \eta \mu^*D]). \]

We consider the restriction map:

\[ \psi : H^0(\tilde{F}, K_{\tilde{F}} + (m-1)\mu_F^*K_F - [(m-1-\eta)\mu^*E_Y + \eta \mu^*D] |_{\tilde{F}}) \]
\[ \subset H^0(F, mK_F). \]

Since $F$ is birationally equivalent to an element in $X$, $\text{vol}(F) \leq N_x$. Thus, by considering the asymptotic Riemann-Roch (see, for instance, [17 the proof of Theorem 3.7]), one has $\tau K_Y \geq_{\mathbb{Q}} F$ for any number $\tau > \frac{N_x \dim Y}{\text{vol}(Y)}$. In particular, by Kawamata’s extension theorem, for $N$ sufficiently large and divisible, there exists an effective divisor $V_N \sim N\tau K_F$ on $F$ such that

\[ |N(1+\tau)K_Y|_F \supset |NK_F| + V_N. \]  

(4.3)
Combining (4.2) and (4.3), we have
\begin{align}
(1 + \tau) r^* K_{Y_0}|_F & \sim_Q r^* K_{F_0} + Z_1, \\
(1 + \tau) E_Y|_F & \sim_Q E_F + Z_2, \\
\tau r^* K_{Y_0} & \sim_Q F + Z_3,
\end{align}
where $Z_1$ and $Z_2$ are effective $\mathbb{Q}$-divisors on $F$ such that $Z_1 + Z_2 \sim_Q \tau K_F$ and $Z_3$ is an effective $\mathbb{Q}$-divisor on $Y$. Note that
\[ Z_3|_F \sim_Q \tau r^* K_{Y_0}|_F = \frac{\tau}{1 + \tau} r^* K_{F_0} + \frac{\tau}{1 + \tau} Z_1. \]

We may and will choose $\epsilon_1$ sufficiently small and $M_1$ sufficiently large such that $\epsilon_1 + \frac{\text{dim} Y}{M_1} < 1$ and choose a rational number $\tau$ such that $0 < \tau - \frac{\text{dim} Y}{M_1} \ll 1$. Then
\begin{align*}
(m - 1)m^* K_Y - [(m - 1 - \eta)m^* E_Y + \eta m^* D] - \tilde{F} \\
\geq_Q (m - 1 - \eta)(r \circ \mu)^* K_{Y_0} - \mu^* F \\
\geq_Q (m - 1 - \eta - \tau)(r \circ \mu)^* K_{Y_0}
\end{align*}
is big. Define
\[ \mathcal{J}' := \mathcal{J}([((m - 1)m^* K_Y - [(m - 1 - \eta)m^* E_Y + \eta m^* D] - \tilde{F})|_{\tilde{F}}]). \]

By Nadel vanishing (see [38, Theorem 11.2.12]),
\[ H^1(\tilde{F}, \mathcal{O}_Y(K_Y + (m-1)m^* K_Y - [(m-1-\eta)m^* E_Y + \eta m^* D] - \tilde{F}) \otimes \mathcal{J}') = 0. \]

Thus the image of the restriction map $\Psi$ contains the subspace $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m-1)m^* K_F - [(m-1-\eta)m^* E_Y + \eta m^* D]|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}}$.

We want to show that
\begin{align*}
H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m-1)m^* K_F - [(m-1-\eta)m^* E_Y + \eta m^* D]|_{\tilde{F}}) \otimes \mathcal{J}'|_{\tilde{F}}) \\
= H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} + (m-1)m^* K_F)) \cong H^0(F, mK_F).
\end{align*}

We now study the sheaf $\mathcal{J}'|_{\tilde{F}}$. By (4.4), we have
\begin{align*}
(m - 1)m^* K_Y - [(m - 1 - \eta)m^* E_Y + \eta m^* D] - \tilde{F} \\
\sim_Q (m - 1 - \eta)(r \circ \mu)^* K_{Y_0} + V' - \tilde{F} \\
\sim_Q (m - 1 - \eta - \tau)(r \circ \mu)^* K_{Y_0} + V'',
\end{align*}
where $V' = \{(m - 1 - \eta)m^* E_Y + \eta m^* D\}$ and $V'' = V' + \mu^*(Z_3)$.

Since $K_{Y_0}$ is semiprimitive, $\mathcal{J}' \supset \mathcal{J}([V'']|_F) \supset \mathcal{J}(V'')$. By the restriction theorem of asymptotic multiplier ideals (see [38, Theorem 11.2.1]), we see that $\mathcal{J}'|_{\tilde{F}} \supset \mathcal{J}(\tilde{F}, V''|_{\tilde{F}})$.

We then have
\[ \mathcal{O}_{\tilde{F}}(-(m - 1 - \eta)m^* E_Y + \eta m^* D)|_{\tilde{F}} \otimes \mathcal{J}'|_{\tilde{F}} \supset \mathcal{J}((m - 1 - \eta)m^* E_Y + \eta m^* D)|_{\tilde{F}} + \mu^*(Z_3)|_{\tilde{F}}. \]
By \((1.14)\), we have
\[
((m-1-\eta)\mu^*E_Y + \eta \mu^*D)|_{\tilde{F}} + Z_3|_{\tilde{F}}
\]
\[\sim_{Q} \frac{m - 1 - \eta}{1 + \tau} \mu^*_{F}(E_F + Z_2) + \frac{\tau}{1 + \tau} \mu^*_{F}(r_{F}^*K_{F_0} + E_F) + \frac{\tau}{1 + \tau} \mu^*_{F}(Z_1 + Z_2) + \eta \mu^* D|_{\tilde{F}} \]
\[
\sim_{Q} \frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}E_F + \frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}(r_{F}^*K_{F_0} + E_F) + \frac{\tau}{1 + \tau} \mu^*_{F}(Z_1 + Z_2) + \eta \mu^* D|_{\tilde{F}} \]
\[
\leq_{Q} \frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}E_F + \frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}(r_{F}^*K_{F_0} + E_F) + \frac{\tau}{1 + \tau} \mu^*_{F}(Z_1 + Z_2) + \eta \mu^* D|_{\tilde{F}} = \frac{m\tau + \eta}{1 + \tau} \mu^*_{F}K_F,
\]
and
\[
\frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}E_F = (m - 1 - \frac{m\tau + \eta}{1 + \tau}) \mu^*_{F}E_F.
\]
Set \(\rho = \frac{m\tau + \eta}{1 + \tau}\) and choose an effective \(\mathbb{Q}\)-divisor \(D_F \sim_{Q} K_F\) so that
\[
\frac{m - 1 - \eta - \tau}{1 + \tau} \mu^*_{F}E_F + \frac{\tau}{1 + \tau} \mu^*_{F}(r_{F}^*K_{F_0} + E_F) + \frac{\tau}{1 + \tau} \mu^*_{F}(Z_1 + Z_2) + \eta \mu^* D|_{\tilde{F}} \leq_{Q} \rho \mu^*_{F}D_F.
\]
We then have
\[
O_{\tilde{F}}(-[(m - 1 - \eta)\mu^*E_Y + \eta \mu^*D]|_{\tilde{F}}) \otimes J'|_{\tilde{F}} \]
\[\supset \ J(\tilde{F}, (m - 1 - \rho)\mu^*E_F + \rho \mu^*D_F)
\]
\[= J(||(m - 1 - \rho)\mu^*K_F|| + \rho \mu^*D_F).
\]
Therefore we have
\[
H^0(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} + (m - 1)\mu^*K_F - [(m - 1 - \eta)\mu^*E_Y + \eta \mu^*D]|_{\tilde{F}}) \otimes J'|_{\tilde{F}}
\]
\[\supset H^0(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} + (m - 1)\mu^*K_F) \otimes J(||(m - 1 - \rho)\mu^*K_F|| + \rho \mu^*D_F)).
\]
By the birational transformation rule (see \[38\] Theorem 9.2.33),
\[
\mu_{F*}(O_{\tilde{F}}(K_{\tilde{F}}) \otimes J(||(m - 1 - \rho)\mu^*K_F|| + \rho \mu^*D_F)) = J(F, ||(m - 1 - \rho)K_F|| + \rho D_F).
\]
Therefore it follows that
\[
\mu_{F*}(\Psi) \supset H^0(F, O_F(mK_F) \otimes J(F, ||(m - 1 - \rho)K_F|| + \rho D_F)).
\]
By \[17\] Theorem 3.5, there exists a positive constant \(\epsilon_X\), depending only on \(m\) and \(X\), such that for each \(0 < \epsilon < \epsilon_X\) and \(D_F \sim_{Q} K_F\),
\[
H^0(F, O_F(mK_F) \otimes J(F, ||(m - 1 - \epsilon)K_F|| + \epsilon D_F))) = H^0(F, mK_F).
\]
Thus we may choose $M_1$ and $\epsilon_1$ such that $\epsilon_1 + \frac{m \cdot \dim Y \cdot N}{M_1} < \epsilon_X$. Then, for $0 < \tau - \frac{N \cdot \dim Y}{M_1} \ll 1$ and $0 < \eta < \epsilon_1$, we have $\rho = \frac{m \cdot \eta + \tau}{1 + \tau} < \epsilon_1 + \frac{m \cdot \dim Y \cdot N}{M_1} < \epsilon_X$ and hence the restriction map $\Psi$ is surjective. □

4.2. The second theorem.

**Theorem 4.3.** Let $f : V \to S$ be a fibration from a smooth projective variety of general type to a smooth projective variety $S$ and let $F$ be a general fiber of $f$. Assume that there exists an effective $\mathbb{Q}$-divisor $\Delta$ on $S$ such that $K_S + \Delta$ is big and nef, and

1. for any big divisor $H$ on $S$, $0 < \epsilon \ll 1$ and $N \in \mathbb{Z}_{> 0}$ sufficiently large and divisible, the restriction map
   \[ H^0(V, N(K_V - f^*(K_S + \Delta))) + N\epsilon f^*H) \to H^0(F,NK_F) \]
   is surjective;

2. there exists a rational number $a > 0$ such that, for a general point $s \in S$, there exists an effective $\mathbb{Q}$-divisor $D_s \sim_{\mathbb{Q}} a(K_S + \Delta)$ such that $(S, D_s)$ is log canonical at $s$ and $s$ is a lc center of $(S, D_s)$.

Then the restriction map
\[ H^0(V, mK_V) \to H^0(F, mK_F) \]
is surjective for $m \geq [a + \epsilon] + 1$.

**Proof.** Let $f_0 : V_0 \to S$ be a $f$-minimal model. After necessary birational modifications of $V$, we may and do assume that we have a birational morphism $\rho : V \to V_0$. Write $K_V = \rho^*K_{V_0} + \sum_{i=1}^{n} a_i E_i$.

We also fix $K_V \sim_{\mathbb{Q}} A + E$, where $E$ is an effective $\mathbb{Q}$-divisor and $A$ is an ample $\mathbb{Q}$-divisor. Modulo further birational modifications, we may assume that $\sum_{i=1}^{n} E_i + E$ has SNC support and its restriction on a general fiber $F$ of $f$ also has SNC support. We may assume that $E_i$ is $f$-horizontal for $1 \leq i \leq n_0$ and $E_i$ is $f$-vertical for $n_0 + 1 \leq i \leq n$. Since $V_0$ is a $f$-minimal model, for sufficiently large and divisible $N$,

\[ |NK_F| = |N\rho^*K_{V_0}|_F| + N(\sum_{i=1}^{n} a_i E_i)|_F \]

\[ = |N\rho^*K_{V_0}|_F| + N(\sum_{i=1}^{n_0} a_i E_i)|_F|, \quad (4.5) \]

where $N(\sum_{i=1}^{n} a_i E_i)|_F| = N(\sum_{i=1}^{n_0} a_i E_i)|_F|$ is the fixed part of $|NK_F|$ and the mobile part $|N\rho^*K_{V_0}|_F|$ is base point free.

For $s \in S$ general, by assumption, there exists an effective $\mathbb{Q}$-divisor $D_s \sim_{\mathbb{Q}} a(K_S + \Delta)$ such that $(S, D_s)$ is log canonical at $s$ and $s$ is a lc center. By Tie breaking (see, for instance, [35]), after a small perturbation of $D_s$ (thus $D_s \sim_{\mathbb{Q}} (a + \eta)(K_S + \Delta)$, where $0 < \eta \ll 1$), we may assume that $(S, D_s)$ is log canonical at $s$, $s$ is an isolated
component of \( \text{Nklt}(S, D_s) \), and there is a unique exceptional divisor with discrepancy \(-1\) over \( s \). Let \( F \) be the fiber of \( f \) over \( s \).

Let \( \rho_S : S' \rightarrow S \) be a log resolution of \((S, D_s)\). Then we have

\[
K_{S'} + E_{S'} = \rho_S^*(K_S + D_s) + E''_S,
\]

where \( E_S, E'_S \) and \( E''_S \) are effective \( \mathbb{Q} \)-divisors with no common component, \( E'_S \) is \( \rho_S \)-exceptional, \( E_S \) is the unique exceptional divisor with discrepancy \(-1\) over \( s \), and each component of \( E'_S \) meeting \( E_S \) has coefficient \(< 1\).

Since \((V, \text{Supp}(\sum_{i=1}^n E_i + E))\) is log smooth over an open neighborhood of \( s \in S \), we may take a log resolution of \((V, f^*D_s + \sum_{i=1}^n E_i + E)\), which is isomorphic to \( V \times_S S' \) over an open neighborhood of \( F = f^{-1}(s) \). Namely, there exists an open neighborhood \( U \) of \( s \) such that \((f \circ \rho_V)^{-1}(U) \simeq f^{-1}(U) \times_U \rho_S^{-1}(U)\). We have

\[
\begin{array}{ccc}
V' & \overset{\rho_V}{\longrightarrow} & V \\
\downarrow f' & & \downarrow f \\
S' & \overset{\rho_S}{\longrightarrow} & S,
\end{array}
\]

which summarizes all morphisms. Then we have

\[
K_{V'} + E_{V'} = \rho_V^*(K_V + f^*D_s) + E''_V,
\]

where \( E_V, E'_V, \) and \( E''_V \) are effective \( \mathbb{Q} \)-divisors without common component, \( E''_V \) is \( \rho_V \)-exceptional, \( E_V \) is the unique exceptional divisor with discrepancy \(-1\) over \( F \), each component of \( E'_V \) meeting \( E_V \) has coefficient \(< 1\), and \( E'_V \) is \( f' \)-vertical.

We then have

\[
K_{V'} + E_{V'} + E'_V + \epsilon \rho_V^*E = \rho_V^*(K_V + f^*D_s + \epsilon E) + E''_V.
\]

Since \( F \not\subset \text{Supp}(E) \), the coefficient of each component of \( E'_V + \epsilon \rho_V^*E \) meeting \( E_V \) is \(< 1\), while \( 0 < \epsilon \ll 1\).

For \( 0 < \epsilon \ll 1 \) and the sufficiently large and divisible \( M \), let \( b = m - 1 - \epsilon - (a + \eta) > 0 \) and let

\[
G \in |M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta))|
\]

be a general element. By the first assumption, for the sufficiently large and divisible \( M \), the restriction map

\[
H^0(V, M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta)) \rightarrow H^0(F, M(m - 1 - \epsilon)K_F)
\]

is surjective. Thus we may write

\[
|M((m - 1 - \epsilon)(K_V - f^*(K_S + \Delta)) + bf^*(K_S + \Delta))| = |L_M| + M(m - 1 - \epsilon) \sum_{i=1}^n a_i E_i,
\]
where $|L_M|$ is a linear system with $\text{Bs}|L_M| \cap F = \emptyset$ by (4.5) and (4.6).

After shrinking $U$, we may assume that the base locus on $|L_M|$ is contained in $f^{-1}(S \setminus U)$. Hence, we write $G = G_1 + M(m - 1 - \epsilon) \sum_{i=1}^{n} a_i E_i$, where $(V, \text{Supp}(\sum_{i=1}^{n} E_i + E + G_1))$ is log smooth over $U$ and $G_1|F$ is linearly equivalent to an irreducible smooth divisor which does not contain any component of $\rho_V(\text{Supp}(E'_V + \epsilon \rho_V^* E)) \cap F$. After blowing-up centers in $(f \circ \rho_V)^{-1}(S \setminus U)$, we may assume that $\rho_V$ is a log resolution of $(V, G + f^* D_s + E + \sum_{i=1}^{n} E_i)$.

We now study the multiplier ideal $J := J(\frac{1}{M} G + f^* D_s + \epsilon E)$. We have

$$K_V + E_V + E'_V + \epsilon \rho_V^* E + \frac{1}{M} \rho_V^* G_1 + (m - 1 - \epsilon) \sum_{i=1}^{m} a_i \rho_V^* E_i = \rho_V^*(K_V + \frac{1}{M} G + f^* D_s + \epsilon E) + E'_V.$$

Note that $\rho_V^* E_i = \tilde{E}_i + E'_i$, where $\tilde{E}_i = \rho_V^{-1}(E_i)$ is the strict transform of $E_i$ and $E'_i$ does not meet $E_V$ for $1 \leq i \leq n_0$ and $\rho_V^* E_i$ does not meet $E'_V$ for $n_0 + 1 \leq i \leq m$. Let $b_i$ be the coefficient of $\tilde{E}_i$ in $\rho_V^* E$ for $1 \leq i \leq m_0$.

Let $D_V = E'_V + \epsilon \rho_V^* E + \frac{1}{M} \rho_V^* G_1 + (m - 1 - \epsilon) \sum_{i=1}^{n} a_i \rho_V^* E_i$. We observe that the coefficient of $\tilde{E}_i$ in $D_V$ is $(m - 1 - \epsilon)a_i + \epsilon b_i$ for $1 \leq i \leq n_0$. The coefficient of other component of $D_V$, meeting $E'_V$, is $< 1$.

By definition,

$$J = \rho_V^* \mathcal{O}_V((E'_V - E_V - D_V)),$$

After shrinking $U$, we may assume that for any component of $D_V$, not meeting $E_V$, its image in $S$ is contained in the complement of $U$.

We claim that, for $0 < \epsilon \ll 1$,

$$J|_{f^{-1}(U)} = \mathcal{I}_F \cdot \mathcal{O}_{f^{-1}(U)}(-D),$$

where $0 \leq D \leq \sum_{i=1}^{n_0} [ma_i] E_i$ and $\mathcal{I}_F$ is the ideal sheaf of $F$.

Indeed,

$$[E'_V - E_V - D_V] = -E_V - \sum_{i=1}^{m_0} c_i \tilde{E}_i - Z_1 + Z_2,$$

where $Z_1$ and $Z_2$ are effective divisors without common components, $Z_2$ is $\rho_V$-exceptional, each component of $Z_1$ does not meet $E_V$.

Thus

$$J|_{f^{-1}(U)} = \rho_V^* \mathcal{O}_{\rho_V^{-1}(f^{-1}(U))}(-E_V - \sum_{i=1}^{m_0} c_i \tilde{E}_i)$$

and

$$J|_{f^{-1}(U)} = \mathcal{I}_F \cdot \mathcal{O}_{f^{-1}(U)}(-\sum_{i=1}^{m_0} c_i E_i)$$

(4.7)
where \( c_i = \lfloor (m - 1 - \epsilon)a_i + \epsilon b_i \rfloor \). Since \( 0 < \epsilon \ll 1 \), we have \( (m - 1 - \epsilon)a_i + \epsilon b_i < ma_i \) and, hence, \( c_i \leq \lfloor ma_i \rfloor \). Set \( D = \sum_{i=1}^{n_0} c_i E_i \). The claim is proved.

Note that \( (m-1)K_V - \left( \frac{1}{n}G + f^*D_s + \epsilon E \right) \sim_{\mathbb{Q}} \epsilon A \). By Nadel vanishing, we have \( H^1(V, \mathcal{O}_V(mK_V) \otimes \mathcal{J}) = 0 \).

Moreover, let \( Q \) be the quotient sheaf \( \mathcal{O}_V(-D) / \mathcal{J} \). By (4.7), we see that \( Q = \mathcal{O}_F(-D) \oplus Q^1 \) where \( \text{Supp}(Q^1) \subset V \setminus f^{-1}(U) \). By considering the short exact sequence

\[
0 \to \mathcal{J} \to \mathcal{O}_V(-D) \to Q \to 0,
\]

we have the surjective map:

\[
H^0(V, mK_V - D) \to H^0(F, mK_F - D|_F) \oplus H^0(Q^1(mK_V)).
\]

Since \( D \leq \sum_{i=1}^{n_0} \lfloor ma_i \rfloor E_i \) and \( \sum_{i=1}^{n_0} \lfloor ma_i \rfloor E_i |_F \) is contained in the fixed part of \( |mK_F| \), we conclude that the restriction map

\[
H^0(V, mK_V) \to H^0(F, mK_F)
\]

is surjective. \( \square \)

**Corollary 4.4.** Let \( f : V \to S \) be a fibration from a smooth projective variety of general type onto a smooth projective surface \( S \) and let \( F \) be a general fiber of \( f \). Assume that there exists an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( S \) such that \( K_S + \Delta \) is big and nef, and

1. for any big divisor \( H \) on \( S \), \( 0 < \epsilon \ll 1 \) and any sufficiently large and divisible number \( N \), the restriction map

\[
H^0(V, N(K_V - f^*(K_S + \Delta)) + N\epsilon f^*H) \to H^0(F, NK_F)
\]

is surjective;

2. \( (K_S + \Delta)^2 = \alpha > 0 \) and \( ((K_S + \Delta) \cdot C) = \beta > 0 \) for any curve \( C \subset S \) passing through very general points of \( S \).

Then the restriction map

\[
H^0(V, mK_V) \to H^0(F, mK_F)
\]

is surjective for \( m \geq \left\lceil \frac{2}{\alpha} + \frac{1}{\beta} + \epsilon \right\rceil + 1 \).

**Proof.** It is a direct application of Theorem 4.3. We just need to apply [34, Example-Theorem 6.3], which states that, for a very general point \( s \in S \), there exists an effective \( \mathbb{Q} \)-divisor \( D_s \sim_{\mathbb{Q}} (\frac{2}{\alpha} + \frac{1}{\beta})(K_S + \Delta) \) such that \( (S, D_s) \) is log canonical at \( s \) and \( s \) is a lc center of \( (S, D_s) \). \( \square \)

5. The proof of Theorem 1.5

The proof of Theorem 1.5 follows the same strategy as that of Theorem 1.2. A very difficult case appears, however, when the 4-fold \( V \) admits a fibration \( f : V \to S \) onto a surface, whose general fibers are surfaces with small birational invariants. The usual inductive argument is not sufficient to show the birationality of \( \Phi_{[5K_V]} \). We need to
apply Theorem 4.1, as well as a great deal of extra arguments, to deal with this difficult case.

First we remark that, by Theorem 2.1, \( \text{vol}(V) \gg 0 \) since \( h^0(\Omega^2_V) \gg 0 \).

Let \( x, y \in V \) be two very general points of \( V \).

We take an effective \( \mathbb{Q} \)-divisor \( D_1 \sim_{\mathbb{Q}} t_1 K_V \) such that \( t_1 < 4 \sqrt{2} \text{vol}(V) + \epsilon \), \( x, y \in \text{Nklt}(V, D_1) \), and after switching \( x \) and \( y \), we may assume that \( (V, D_1) \) is log canonical at \( x \). Let \( V_1 \) be the minimal log canonical center of \( (V, D_1) \) through \( x \).

5.1. The case with \( \dim V_1 = 3 \).

We have two ways, according to the value of \( \text{vol}(V_1) \), to cut down the log canonical center through \( x \).

If \( \text{vol}(V_1) \gg 0 \), by Takayama’s method, there exists an effective \( \mathbb{Q} \)-divisor \( D_2 \sim_{\mathbb{Q}} t_2 K_V \) such that

\[
0 < t_2 < 3t_1 + 3(1 + t_1) \sqrt{\frac{2}{\text{vol}(V_1)}} + \epsilon,
\]

\( x, y \in \text{Nklt}(V, D_2) \), and the minimal log canonical center of \( (V, D_2) \) through \( x \) is a closed subvariety \( V_2 \subset V_1 \). Note that \( 0 < t_1 < t_2 \ll 1 \) in this case. The problem is reduced to Subcase 5.2 and Subcase 5.3.

If \( \text{vol}(V_1) \) is upper bounded, we still apply Todorov’s inductions and spread the centers \( V_1 \) into a family in the similar way to that of Subsection 3.3. We have the following diagram:

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\pi} & V \\
\downarrow{\rho} & & \downarrow{f} \\
C & & 
\end{array}
\]

where \( \pi : \tilde{V} \to V \) is a generically finite and surjective morphism onto \( V \), \( \tilde{V} \) is nonsingular projective, \( f \) is a fibration onto a smooth curve \( C \) such that a general fiber \( F \) is birational onto \( V_1 \) via \( \pi \). There exists an effective \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} t_2 K_V \) such that \( D = V_1 + D' \), where \( D' \) is an effective \( \mathbb{Q} \)-divisor and \( V_1 \not\in \text{Supp}(D') \).

Assume that \( \deg \pi \geq 2 \). We conclude as in Subsection 3.3.1 that there exists an effective \( \mathbb{Q} \)-divisor \( D_2 \sim_{\mathbb{Q}} t_2 K_V \) such that \( t_2 \leq 3t_1 \), \( x, y \in \text{Nklt}(V, D_2) \), and that the minimal log canonical center of \( (V, D_2) \) through \( x \) is a closed subvariety \( V_2 \subset V_1 \). Note that \( 0 < t_2 < 3t_1 \ll 1 \) under the condition of Theorem 1.5. The problem is also reduced to Subcase 5.2 and Subcase 5.3.

Assume that \( \pi \) is birational. We simply identify \( \tilde{V} \) with \( V \) to save symbols. Note that

\[
h^2(V, \omega_V) = h^0(V, \Omega^2_V) \gg 0
\]

and

\[
h^2(V, \omega_V) = h^1(C, R^1 f_* \omega_V) + h^0(C, R^2 f_* \omega_V).
\]
Both $R^1f_*\omega_V$ and $R^2f_*\omega_V$ are torsion-free sheaves on $C$ by Kollár’s theorem \cite{[32]}. Moreover, $R^1f_*\omega_{V/C}$ is weakly positive (see, for instance, \cite{[17]} or \cite{[42] Theorem 1.4}) and, hence, nef on $C$.

Since $\text{vol}(F)$ is bounded, $\text{rank}(R^1f_*\omega_V) = h^{2,0}(F)$, hence by Serre duality, $h^1(C, R^1f_*\omega_V) = h^0(C, (R^1f_*\omega_{V/C})^*) \leq h^{2,0}(F)$. Hence $h^0(C, R^2f_*\omega_V)$ is sufficiently large. In particular, we have $R^2f_*\omega_V \neq 0$. Thus $F$ is an irregular 3-fold of general type. By the main theorem of \cite{[8]}, $|5K_F|$ induces a birational map of $F$.

Since $\text{vol}(V) \gg 0$ and fibers of $f$ are birationally bounded, by \cite{[17] Theorem 3.4], the restriction map

$$H^0(V, 5K_V) \to H^0(F_1, 5K_{F_1}) \oplus H^0(F_2, 5K_{F_2})$$

is surjective for any two different general fibers $F_1$ and $F_2$ of $f$. Thus $|5K_V|$ induces a birational map of $V$.

5.2. The case with $\dim V_2 = 2$ and $0 < t_2 \ll 1$.

After discussions in the previous subsection, we may assume that there exists an effective $\mathbb{Q}$-divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $0 < t_2 \ll 1$, $x, y \in \text{Nklt}(V, D_2)$, and that the minimal log canonical center $V_2$ of $(V, D_2)$ through $x$ is of dimension 2. Modulo a small perturbation, we may and do assume that $\text{Nklt}(V, D_2) = V_2$ locally around $x$.

Let $\tilde{V}_2$ be a smooth model of $V_2$. By Takayama’s induction, there exists an effective $\mathbb{Q}$-divisor $D_3 \sim_{\mathbb{Q}} t_3 K_V$ such that

$$0 < t_3 < t_2 + 2(1 + t_2)\sqrt{\frac{2}{\text{vol}(V_2)}} + \epsilon,$$

$x, y \in \text{Nklt}(V, D_2)$, and that the minimal log canonical center $V_3$ of $(V, D_3)$ through $x$ is a proper subset of $V_2$.

If $\text{vol}(V_2) \geq 4$, $t_3 < \sqrt{2} + (1 + \sqrt{2})t_2 + \epsilon$. The problem is reduced to Subcase 5.3 noting that $0 < t_3 - \sqrt{2} \ll 1$.

If $\text{vol}(V_2) \leq 3$, the bound for $t_3$ is not good enough for our purpose. We still apply Todorov’s method to discuss the following exclusive situations:

5.2.1. There exists an effective $\mathbb{Q}$-divisor $D_3 \sim_{\mathbb{Q}} t_3 K_V$ such that $0 < t_3 \leq 3t_2$, $x, y \in \text{Nklt}(V, D_2)$, and that the minimal log canonical center $V_3$ of $(V, D_3)$ through $x$ is a proper subset of $V_2$, for which the problem is reduced to Subcase 5.3 with $0 < t_3 \ll 1$;

5.2.2. (⋆) Modulo birational modifications, there exists a fibration $f : V \to S$ onto a smooth projective surface $S$ such that a general fiber $F$ of $f$ is a surface whose canonical volume is $\leq 3$, that there exists an effective $\mathbb{Q}$-divisor $D_2 \sim_{\mathbb{Q}} t_2 K_V$ such that $x, y \in \text{Nklt}(V, D_2)$, $(V, D_2)$ is log canonical at $x$ (after possibly switching $x$ and $y$), and that the minimal log canonical center at $x$ is the fiber $F_x$ of $f$ through $x$, $F_x$ is an irreducible component of $\text{Nklt}(V, D_2)$. The statement follows from Theorem 6.1.
5.3. The case with $\dim V_3 \leq 1$, $0 < t_2 \ll 1$ and $0 < t_3 - \sqrt{2} \ll 1$.

So far, except Situation $(\star)$ (i.e., 5.2.2.), we always have the following: we already have a rational number $0 < t_2 \ll 1$; there exists an effective $\mathbb{Q}$-divisor $D_3 \sim Q t_3 K_V$ such that

$$0 < t_3 \leq \max\{3t_2, \sqrt{2} + (1 + \sqrt{2})t_2 + \epsilon\}$$

($0 < \epsilon \ll 1$), $x, y \in \text{Nklt}(V, D_3)$, and that the minimal log canonical center $V_3$ of $(V, D_3)$ through $x$ is of dimension $\leq 1$.

By Takayama’s induction, there exists an effective $\mathbb{Q}$-divisor $D_4 \sim Q t_4 K_V$ such that

$$0 < t_4 \leq 2t_3 + 1 + \epsilon,$$

$x, y \in \text{Nklt}(V, D_4)$, and that $\{x\}$ is an isolated component of $\text{Nklt}(V, D_4)$. Since $0 < t_2 \ll 1$ and $2\sqrt{2} + 1 < 4$, we conclude by Nadel vanishing, as in Subsection 3.1, that $|5K_V|$ separates $x$ and $y$. Hence we have finished the proof of Theorem 1.5.

6. Proof for Subcase [5,2]2.

We deal with Subcase $(\star)$ in this section.

Theorem 6.1. Under Assumption $(\star)$ (i.e., Subcase [5,2]2), $|5K_V|$ induces a birational map of $V$.

The proof of Theorem 6.1 is elaborate. Recall that we have a fibration $f : V \rightarrow S$ onto a smooth projective surface $S$. We start with the following simple reduction.

Lemma 6.2. Under Assumption $(\star)$ (i.e., Subcase [5,2]2), if $x$ and $y$ are in different fibers of $f$, $|5K_V|$ separates $x$ and $y$.

Proof. By assumption, each component of $\text{Nklt}(V, D_2)$ containing $y$ does not contain $x$. It is then easier to cut down the log canonical centers.

Note that

$$\dim \text{Im}(H^0(V, mK_V) \rightarrow H^0(F_x, mK_{F_x})) \sim \frac{\text{vol}_{F_x}(K_X)}{2} m^2 + O(m)$$

for sufficiently large integers $m$. Hence we can choose an effective $\mathbb{Q}$-divisor $D \sim Q K_V$ such that $F_x$ is not contained in the support of $D$ and that, by Takayama [44, Theorem 4.5],

$$\text{mult}_x(D|_{F_x}) > \sqrt{\text{vol}_{F_x}(K_X)} - \epsilon \geq \sqrt{\frac{\text{vol}(K_F)}{1 + t_2}} - \epsilon.$$

Let $t'_3$ be the maximal rational number such that $(V, D_2 + t'_3 D)$ is log canonical at $x$. It is known that $t'_3 \leq \frac{2}{\text{mult}_x(D|_{F_x})} < \frac{2(1 + t_2)}{\sqrt{\text{vol}(K_F)}} + \epsilon.$

Let $D_3 = D_2 + t'_3 D \sim Q t_3 K_V$ and $t_3 = t_2 + t'_3$. Then the minimal log canonical center $V_3$ of $(V, D_3)$ is a proper subset of $F_x$ and there
exists an irreducible component of $\text{Nklt}(V, D_3)$ containing $y$ and not containing $x$.

Note that $t_3 < 2 + 3t_2 + \epsilon$. If $V_3 = \{x\}$, since there exists an irreducible component of $\text{Nklt}(V, D_3)$ containing $y$ and not containing $x$, we can take a small perturbation of $D_3$ such that $x$ is an isolated component of $\text{Nklt}(X, D_3)$ and $y \in \text{Nklt}(X, D_3)$. Note $t_3 < 4$. We can conclude the statement by Nadel vanishing.

If $V_3$ is a curve, we denote by $\overline{V_3}$ its normalization. Similarly, there exists an effective $\mathbb{Q}$-divisor $D_4 \sim_{\mathbb{Q}} t_4 K_V$ such that $t_4 < t_3 + \epsilon$.

\[ t_4 < t_3 + \frac{1 + t_3}{\text{vol}(V_3)} + \epsilon \leq \frac{3}{2} t_3 + \frac{1}{2} + \epsilon < \frac{7}{2} + \frac{9}{2} t_2 + \epsilon, \]

$\{x\}$ is a log canonical center of $(V, D_4)$, $(V, D_4)$ is not klt at $y$, and there exists an irreducible component of $\text{Nklt}(V, D_4)$ containing $y$, while not containing $x$. Since $t_2 \ll 1$ and then $t_4 < 4$, we conclude the statement as before. \hfill \Box

We then focus on proving that $|5K_V|_F$ induces a birational map for the general fiber $F$ of $f$.

6.1. The case with $\kappa(S) = 2$ or $\kappa(S) \leq 1$ and $p_g(S) \gg 0$.

**Lemma 6.3.** Under Assumption $(\star)$ (i.e., Subcase 5.2.2), if $S$ is of general type, $|5K_V|$ induces a birational map of $V$.

**Proof.** By Lemma 6.2, it suffices to show that $|5K_V|_F = |5K_F|$, since $|5K_F|$ induces a birational map of $F$. By simply blowing down $S$, we may assume that $S$ is minimal. Set $\Delta = 0$ and we will apply Corollary 4.4. Note that $\text{vol}(S) \geq 1$ and $(K_S \cdot C) \geq 1$ for each irreducible curve $C$ through a general point of $S$. Moreover, $f_*O_V(NK_V/S)$ is weakly positive by Viehweg’s theorem (see [47]) and hence condition (1) in Corollary 4.4 is also satisfied. \hfill \Box

Now we assume that $\kappa(S) \leq 1$. By Hodge symmetry, Serre duality and the assumption that $h^0(V, \Omega^2_V) \gg 0$, we see that $h^2(V, \omega_V) \gg 0$. Thus we have

\[ h^0(S, R^2 f_* \omega_V) + h^1(S, R^1 f_* \omega_V) + h^2(S, f_* \omega_V) = h^2(V, \omega_V) \gg 0. \]

We observe that $h^2(S, f_* \omega_V) \leq 3$. Indeed, since $\text{vol}(F) \leq 3$, we have $p_g(F) \leq 3$ by the Noether inequality. We take a general member $H$ in a very ample linear system of $S$. Let $f_H : V_H \to H$ be the induced morphism, where $V_H := f^{-1}(H)$. By Kollár [32 Theorem 2.1], we have the short exact sequence:

\[ 0 \to f_* \omega_V \to f_* \omega_V(H) \to f_{H_*} \omega_{V_H} \to 0 \]

and $H^2(S, f_* \omega_V(H)) = 0$. Thus we have the surjective map

\[ H^1(H, f_{H_*} \omega_{V_H}) \to H^2(V, f_* \omega_V). \]
We have already seen that \( h^1(H, f_{H^*} \omega_{V/H}) \leq p_g(F) \leq 3 \) using the semi-positivity of \( f_{H*} (\omega_{V/H}) \).

By Kollár [33], we know that \( R^2 f_* \omega_V = \omega_S \) and hence
\[
h^0(S, R^2 f_* \omega_V) = h^0(S, \omega_S) = p_g(S).
\]

**Lemma 6.4.** Under Assumption (⋆) (i.e., Subcase 5.2.2), if \( p_g(S) \gg 0 \), then \( |5K_V| \) induces a birational map of \( V \).

**Proof.** Since \( \kappa(S) \leq 1 \), we have \( \kappa(S) = 1 \). We may assume that \( S \) is minimal and denote by \( I_S : S \to C \) the Iitaka fibration of \( S \). By the canonical bundle formula, \( K_S = I_S^*(H + B) \), where \( H = I_S^* \omega_S \) and \( B \) is an effective \( \mathbb{Q} \)-divisor on \( C \) depending on the singular fibers of \( I_S \). We also have \( p_g(S) = h^0(S, K_S) = h^0(C, H) \). Hence \( \deg H \gg 0 \).

Let \( g : V \to S \xrightarrow{I_S} C \) be the composition of fibrations and let \( Y \) be a general fiber of \( g \). Since a general fiber of \( I_S \) is an elliptic curve, \( Y \) is an irregular threefold of general type.

By Viehweg’s weak positivity (see, for instance, [47]), we know that \( K_{V/S} + \epsilon K_V \) is big. Thus \( (1 + \epsilon)K_V \geq \mu f^* K_S \geq \mu g^* H \). By Theorem 1.1 one of the restriction maps:
\[
H^0(V, 5K_V) \to H^0(Y, 5K_Y),
\]
\[
H^0(V, 5K_V) \to H^0(F, 5K_F)
\]
is surjective. In the latter case, we are done since \( |5K_F| \) induces a birational map of \( F \). In the former case, we conclude by the main theorem of [8]. \( \square \)

6.2. **The case with \( \kappa(S) \leq 1 \) and \( p_g(S) \) being upper bounded.**

From now on within this section, we assume that \( \kappa(S) \leq 1 \) and \( p_g(S) \) is upper bounded. As discussed above, the condition of Theorem 1.5 forces \( h^1(S, R^1 f_* \omega_X) \gg 0 \). Note that \( R^1 f_* \omega_X \neq 0 \) implies that \( F \) is an irregular surface. Moreover, since \( \text{vol}(F) \leq 3 \), we have \( p_g(F) = q(F) = 1 \) and \( 2 \leq \text{vol}(F) \leq 3 \) by Debarre’s inequality that \( \text{vol}(F) \geq 2p_g(F) \) (see [13]). Let \( g : V \to X \) be the relative Albanese morphism of \( f \). Note that \( g \) exists by [5] Théorème 1 or [22] Theorem 2]. We have the commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{h} \\
S & &
\end{array}
\]

By the definition of the relative Albanese morphism, for \( s \in S \) general, let \( g_s : V_s \to X_s \) be the fibers of \( g : V \to X \) over \( s \), then \( g_s \) is the Albanese morphism of \( V_s \). Since \( q(V_s) = 1 \) and the Albanese morphism of \( V_s \) is a fibration. Thus a general fiber of \( h \) is a genus 1 curve and a general fiber of \( g \) is connected. Modulo birational modifications, we may also assume that the 3-fold \( X \) is nonsingular and projective.
By Kollár [33, Theorem 3.4], $R^1f_*\omega_V \cong h_*R^1g_*\omega_V \oplus R^1h_*g_*\omega_V$. Moreover, since $h^1(X, g_*\omega_V) = 0$ by the fact that $q(V) = 1$, we conclude that $R^1h_*g_*\omega_V = 0$. Thus

$$h^1(S, h_*\omega_X) = h^1(S, h_*R^1g_*\omega_V) = h^1(S, R^1f_*\omega_V) \gg 0.$$ 

Therefore $X$ is a 3-fold with many global two forms, i.e. $h^0(X, \Omega^2_X) \gg 0$. By the easy addition (see [40, Corollary 2.3]), we also have $\kappa(X) \leq \dim S = 2$.

Next we follow Campana and Peternell’s results in [6] to trace 2-forms on non-general type 3-folds to study the structure of $h: X \to S$ and to finish the proof of Theorem 1.5. We organize the argument according to the value of $\kappa(X)$.

6.2.1. The case with $\kappa(X) = -\infty$. The global holomorphic 2-forms on $X$ are induced from the maximal rationally connected quotient of $X$. More precisely, by Campana-Peternell [6] Theorem 3.1, there exists a morphism $h': X \to S'$ onto a minimal smooth projective surface $S'$ with $\kappa(S') \geq 0$ such that $H^0(X, \Omega^2_X) \cong H^0(S', K_{S'})$ has sufficiently large dimension. In particular, $h$ and $h'$ are not birational to each other. Since a general fiber $E$ of $h$ is an elliptic curve, the family of $h'(E)$ covers $S'$. Thus $\kappa(S') = 1$ and $E' \to h'(E')$ is an isogeny, since otherwise $S'$ couldn’t have non-negative Kodaira dimension. Denote by $l': S' \to C$ the Iitaka fibration of $S'$. Note that $l'$ is simply the morphism contracting the family of $h'(E)$. Thus we also have an induced morphism

$$I: S \to C \quad s \to l'(h^{-1}(s)),$$

so that we have the following commutative diagram:

$$\begin{array}{ccc}
V & \xrightarrow{g} & X \\
\downarrow f & & \downarrow h' \\
S & \xrightarrow{h} & S' \\
\downarrow l & & \downarrow l' \\
C & & C
\end{array}$$

Since $p_g(S') \gg 0$, there exists a line bundle $H$ on $C$ such that $\deg H \gg 0$ and $K_{S'} \geq Q l^*H$. We then again have

$$(1 + \epsilon)K_V \geq Q (l' \circ g)^*K_{S'} \geq (l' \circ h' \circ g)^*H.$$ 

Let $Y$ be a general fiber of $I \circ f$. Then, by Theorem 4.1, one of the restriction maps:

$$H^0(V, 5K_V) \to H^0(Y, 5K_Y),$$ 

$$H^0(V, 5K_V) \to H^0(F, 5K_F)$$
is surjective. Note that \( Y \) is again an irregular threefold of general type, because \( I' \) is an elliptic fibration. We then conclude as in the proof of Lemma 6.4.

6.2.2. The case with \( \kappa(X) = 0 \). By Campana-Peternell [6, Theorem 3.2], we see \( h^0(X, \Omega^2_X) \leq 3 \), contradicting to our assumption \( h^0(X, \Omega^2_X) \gg 0 \).

6.2.3. The case with \( \kappa(X) = 1 \). Let \( I_X : X \to C \) be the Iitaka fibration of \( X \).

A general fiber of \( I_X \) is birational either to an abelian surface, or to a hyperelliptic surface, or to a K3 surface, or to a Kummer surface. For a general point \( p \in C \), let \( X_p \) be the corresponding fiber of \( I_X \). We have exact sequences

\[
0 \to \Omega^2_X(-X_p) \to \Omega^2_X \to \Omega^2_X|_{X_p} \to 0
\]

and

\[
0 \to \Omega_{X_p} \to \Omega^2_X|_{X_p} \to \Omega^2_X|_{X_p} \to 0.
\]

In all cases, we have

\[
h^0(\Omega^2_X|_{X_p}) \leq h^0(\Omega_{X_p}) + h^0(\Omega^2_{X_p}) \leq 3.
\]

Let \( N = \lfloor \frac{h^0(\Omega^2_X)}{4} \rfloor \). For \( p_1, \ldots, p_N \) be general points of \( C \) and let \( X_{p_i} \) be the corresponding fiber over \( p_i \). Considering

\[
0 \to \Omega^2_X(-\sum_{i=1}^N X_{p_i}) \to \Omega^2_X \to \bigoplus_{i=1}^N \Omega^2_X|_{X_{p_i}} \to 0,
\]

we have \( h^0(\Omega^2_X(-\sum_{i=1}^N X_{p_i})) > 0 \).

Let \( \mathcal{L} = \mathcal{O}_X(L) \) be the saturation of \( I_X^*\Omega_C \subset \Omega_X \) and denote by \( \mathcal{F} \) the torsion-free quotient. Note that \( \mathcal{L} = I_X^*\omega_C \otimes \mathcal{O}_X(E) \), where \( E \) is an effective divisor supported on the singular fibers of \( I_X \). We have the exact sequence:

\[
0 \to \mathcal{L} \to \Omega_X \to \mathcal{F} \to 0,
\]

\[
0 \to K_X \otimes \mathcal{F}^* \to \Omega^2_X \to \det \mathcal{F} \otimes \mathcal{I}_Z \to 0,
\]

where \( Z \subset X \) is a subscheme of codimension \( \geq 2 \).

If \( h^0(X, \det \mathcal{F}(-\sum_{i=1}^N X_{p_i})) > 0 \), we have

\[
K_X \sim L + \det \mathcal{F} \geq I_X^*(K_C) + \sum_{i=1}^N X_{p_i}
\]

where \( K_C + \sum_{i=1}^N X_{p_i} \) is an ample divisor on \( C \) of degree \( \geq N - 2 \).

If \( h^0(X, K_X \otimes \mathcal{F}^*(-\sum_{i=1}^N X_{p_i})) > 0 \), then we have a non-trivial map

\[
\mathcal{F} \to K_X \otimes \mathcal{O}_X(-\sum_{i=1}^N X_{p_i}).
\]
Since any torsion-free quotient of $F$ is pseudo-effective by the main result of [39], we have

$$K_X = \sum_{i=1}^{N} X_{p_i} + P$$

where $P$ is a pseudo-effective divisor.

In both cases, $(1 + \epsilon)K_V = \epsilon K_V + K_{V/X} + K_X \geq (I_X \circ g)^* H_C$ where $H_C$ is an ample divisor of degree $\geq N - 2$ on the curve $C$.

We now consider the commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{g} & X \\
\downarrow f & & \downarrow I_X \\
S & \xrightarrow{h} & C.
\end{array}$$

If the fibration $I_X \circ g : X \to C$ does not factor through $f$, a general fiber $F$ of $f$ dominates $C$. Let $g_F := (I_X \circ g)|_F : F \to C$ be the induced morphism. We then have

$$(1 + \epsilon)K_F = (1 + \epsilon)K_V|_F \geq g_F^* H_C.$$ 

It is then clear that $(1 + \epsilon)\text{vol}(F) \geq \deg H_C \geq N - 2$, which is a contradiction since $\text{vol}(F) \leq 3$ and $N \gg 0$.

Thus $I_X \circ g$ factors as $I_X \circ g : V \xrightarrow{g} X \xrightarrow{h} S \xrightarrow{f} C$. Let $Y$ be a general fiber of $I_X \circ g$ and let $Z$ be the corresponding fiber of $I_X$. Note that

$$0 \ll h^1(S, h^* \omega_X) = h^1(C, I_X^* \omega_X) + h^0(C, R^1 f^*_S h^* \omega_X).$$

Since $I_X^* \omega_X|_C$ is a nef line bundle on $C$, $h^1(C, I_X^* \omega_X) \leq 1$. Hence we have $h^0(C, R^1 f^*_S h^* \omega_X) \gg 0$ and thus

$$R^1 I_X^* \omega_X = R^1 f^*_S h^* \omega_X \oplus f^*_S R^1 h^* \omega_X$$

is non-zero. Therefore $q(Z) > 0$ and $Y$ is an irregular threefold of general type.

Since $N \gg 0$, by Theorem 4.1 either

$$H^0(V, 5K_V) \to H^0(F, 5K_F)$$

is surjective or

$$H^0(V, 5K_V) \to H^0(Y, 5K_Y)$$

is surjective. We finish the proof as before.

6.2.4. The case with $\kappa(X) = 2$. In this case, $h : X \to S$ is exactly the Iitaka fibration of $X$. Since $h^1(S, h^* \omega_X) \gg 0$, $X$ has sufficiently many non-$h$-vertical holomorphic 2 forms in the terminology of [6, Definition 1.9]. Indeed, the Hodge pairing between the conjugate of $h$-vertical holomorphic 2-forms and forms of $H^1(S, f^* \omega_X)( \subset H^1(X, K_X))$ is zero.

Note that $h : X \to S$ is a genus 1 curve fibration. There are two cases to discuss depending on whether or not the $j$-invariant of $h$ is constant.
If the $j$-invariant is constant, let $E$ be a general fiber of $h$. After a Galois base change $\hat{S} \to S$ and birational modifications, we get a trivial elliptic fibration $\hat{h} : \hat{X} \to \hat{S}$, i.e. $\hat{h}$ is birational to the projection $\hat{S} \times E \to \hat{S}$. Let $\pi : \hat{X} \to X$ be the generically finite cover, which is a Galois cover over an open dense subset of $X$ with Galois group $G$. Note that, since $\hat{S}$ is of general type, $G$ acts on $\hat{S}$ and $E$ diagonally. We may and do assume that $G$ acts on $\hat{S}$ and $E$ faithfully, hence $G \hookrightarrow \text{Aut}(E)$. Then $X$ is birationally equivalent to the diagonal quotient $(\hat{S} \times E)/G$. For simplicity, we assume that $\hat{S}$ is minimal and identify $X$ with the diagonal quotient $(\hat{S} \times E)/G$. Then $X$ is a normal threefold with quotient singularities and the quotient morphism $\pi$ is quasi-étale. It is known that the neutral component $\text{Aut}_0(E)$ of the automorphism group $\text{Aut}(E)$ of the elliptic curve $E$ is isomorphic to $E$ and $\text{Aut}(E)/\text{Aut}_0(E)$ is either $\mathbb{Z}_2$, or $\mathbb{Z}_3$, or $\mathbb{Z}_6$. Set $G_0 = G \cap \text{Aut}_0(E)$ and $\overline{G} := G/G_0$. Note that $G_0$ acts trivially on $H^0(E, \Omega_E)$ and $|\overline{G}| \leq 6$. Let $\overline{X} := (\hat{S} \times E)/G_0$ and $\overline{S} := \hat{S}/G_0$. Note that $\overline{X}$ is smooth and $\overline{S}$ is normal with quotient singularities. We have the commutative diagram:

$$
\begin{array}{cccc}
\hat{S} & \stackrel{\hat{h}}{\longrightarrow} & \hat{S} & \stackrel{\pi_0}{\longrightarrow} & S \\
\downarrow & & \downarrow & & \downarrow \\
\hat{X} & \stackrel{g}{\longrightarrow} & X & \stackrel{f}{\longrightarrow} & \overline{X} & \stackrel{\pi}{\longrightarrow} & \overline{S} & \stackrel{\overline{\pi}}{\longrightarrow} & \overline{S} \\
\end{array}
$$

Since $X$ has many non-$h$-vertical global 2-forms, $\overline{X}$ has many non-$\overline{h}$-vertical global 2-forms. Moreover, since the space of non-$\hat{h}$-vertical global 2-forms on $\hat{S} \times E$ is $h^*H^0(\hat{S}, \Omega_\hat{S}) \otimes p_E^*H^0(E, \Omega_E)$, where $p_E : \hat{S} \times E \to E$ is the second projection and $G_0$ acts trivially on $H^0(E, \Omega_E)$, we conclude that the space of non-$\overline{h}$-vertical global 2-forms on $\overline{X}$ is

$$
(h^*H^0(\hat{S}, \Omega_\hat{S}) \otimes p_E^*H^0(E, \Omega_E))^G_0 = \hat{h}^*H^0(\overline{S}, \Omega_{\overline{S}}) \otimes p_{\overline{E}}^*H^0(\overline{E}, \Omega_{\overline{E}}),
$$

where $\overline{E} = E/G_0$ is also an elliptic curve, $\overline{h} : \overline{S} \to \overline{E}$ is the natural morphism, and $\Omega_{\overline{S}} = i_*(\Omega_U)$, where $i : U \subset \overline{S}$ is the smooth locus of $\overline{S}$. In particular, we see that $q(\overline{S}) = h^0(\overline{S}, \Omega_{\overline{S}}) \gg 0$.

If the image of the Albanese morphism of $\overline{S}$ is a curve $\overline{D}$. Then $\overline{D}$ is a smooth projective curve of genus equal to $q(\overline{S}) \gg 0$. Let $D = \overline{D}/\overline{G}$.
and \( \pi_D : \overline{D} \to D \) be the quotient morphism. We have

\[
\begin{array}{c}
\xymatrix{
V \ar[d]_{\pi_V} & X \ar[dl]_{h_D} \\
S \ar[d]_{\pi_S} & \overline{D} \ar[d]_{\pi_D} & X \ar[dl]_{f_D} \\
S & \overline{D} & D
}
\end{array}
\]

where \( h_D \) and \( f_D \) are the naturally composited morphisms. By the ramification formula, we write \( K_{\overline{D}} = \pi_D^*(K_D + \sum_i (1 - \frac{1}{n_i})p_i) \), where \( p_i \) are the branched loci of \( \overline{D} \) and \( n_i \) is the ramification index over \( p_i \). Note that \( \deg_D(K_D + \sum_i (1 - \frac{1}{n_i})p_i) = \frac{1}{|G|} \left( 2g(\overline{D}) - 2 \right) \gg 0 \). Let \( \Delta := \sum_i (1 - \frac{1}{n_i})p_i \). Recall that the ramification divisor \( \mathcal{R}(h_D) := \sum_{p} (h_D^{-1}p - (h_D^*p)_{\text{red}}) \), where \( p \) goes through each point of \( D \) (see [19, Notation 2.7]). Since the horizontal morphisms in (6.2) are \( G \)-quotients, we see that \( h_D^*\Delta \leq \mathcal{R}(h_D) \). Hence \( f_D^*\Delta \leq \mathcal{R}(f_D) \). By [19, Corollary 4.5], \( K_{V/D} - f_D^*\Delta \) is pseudo-effective. Thus, \( (1 + \epsilon)K_V \geq Q f_D^*(K_D + \Delta) \). Let \( Y \) be a general fiber of \( f_D \). Since \( V \) (or \( X \)) has many non-\( f_D \)-vertical (non-\( h \)-vertical) global holomorphic 2-forms, \( Y \) is an irregular 3-fold of general type. We now apply Theorem 4.1 to conclude that either \( H^0(V, 5K_V) \to H^0(Y, 5K_Y) \) or \( H^0(V, 5K_V) \to H^0(F, 5K_F) \) is surjective.

If the Albanese image of \( \overline{S} \) is a surface but \( K_{\overline{S}} \) is not big, we have a similar picture as before. Indeed, in this case \( \kappa(\overline{S}) = 1 \) and let \( \overline{S} \to \overline{D} \) be the Iitaka fibration of \( \overline{S} \). Then \( \overline{G} \) also acts naturally on \( \overline{D} \) and \( g(\overline{D}) = q(\overline{S}) - 1 \gg 0 \). Then we prove the statement exactly similar to the last paragraph.

We then assume that the Albanese image of \( \overline{S} \) is a surface and \( K_{\overline{S}} \) is big. In this case, any smooth model of \( \overline{S} \) is also of general type. Thus, modulo birational modifications, we may assume that \( \overline{X} \) and \( \overline{S} \) are smooth and \( \overline{S} \) is a minimal surface. We write \( K_{\overline{S}} = \pi_S^*(K_S + \sum D_i (1 - \frac{1}{n_i})D_i) \), where \( D_i \) are branched divisors and \( n_i \) is the corresponding ramification index. Let \( \Delta_S := \sum D_i (1 - \frac{1}{n_i})D_i \). In particular, \( K_S + \Delta_S \) is big and nef. Moreover, \( \text{vol}(\overline{S}) = (K_S^2) \geq 2q(\overline{S}) - 4 \gg 0 \) by Debarre’s inequality and thus \( ((K_S + \Delta_S)^2) = \frac{1}{|G|} \text{vol}(\overline{S}) \gg 0 \). On the other hand, let \( C \) be an irreducible curve passing through a very general point of \( S \),

\[
((K_S + \Delta_S) \cdot C) = \frac{1}{|G|} (K_{\overline{S}} \cdot \pi_S C) \geq \frac{1}{3}.
\]
We then deduce the surjectivity of the restriction map \( H^0(V, 5K_V) \to H^0(F, 5K_F) \) by applying Corollary 4.4. It suffices to verify the assumption (1) thereof. We go back to the commutative diagram (6.1). Let \( \overline{V} \) be a resolution of \( S \times_S V \). We then have

![Diagram](attachment:image.png)

By the weak positivity of \( \overline{f} \circ \pi_S^* \otimes^N \), for any effective big \( \mathbb{Q} \)-divisor \( H \) on \( S \) and any sufficiently large and divisible integer \( N \),

\[
H^0(\overline{V}, N(K_{\overline{V}/S} + \overline{f} \circ \pi_S^* H)) \to H^0(F, NK_F)
\]

is surjective. Note that the restriction map is \( \mathcal{G} \)-equivariant and the space \( H^0(F, NK_F) \) is \( \mathcal{G} \)-invariant. Thus

\[
H^0(\overline{V}, N(K_{\overline{V}/S} + \overline{f} \circ \pi_S^* H))^{\mathcal{G}}
\]

is also surjective.

So far, we have finished the discussion when the \( j \)-invariant of \( h \) is constant.

Finally, if the \( j \)-invariant of \( h \) is non-constant, by [6, Corollary 5.4], there exists a fibration \( f_S : S \to C \) onto a smooth projective curve \( C \) and another smooth projective surface \( S' \), and an elliptic fibration \( f_{S'} : S' \to C \) such that \( S \times_C S' \) is smooth and \( X \) is birational to \( S \times_C S' \). Replacing \( S \times_C S' \) with \( X \) and let \( h' : X \to S' \) be the projection. We have the commutative diagram:

![Diagram](attachment:image.png)

By [6, (5.5)], a non-\( h \)-vertical holomorphic global 2 form on \( X \) belongs to the space

\[
h^* H^0(S, \Omega_S) \wedge h'^* H^0(S', \Omega_{S'}) \oplus h'^* H^0(S', K_{S'}). \tag{6.3}
\]
Note that $f_{S'}$ is an elliptic fibration with non-constant $j$-invariant. Thus, $H^0(S', \Omega_{S'}) = f_{S'}^*H^0(C, \Omega_C)$. Hence non-h-vertical 2-forms of $X$ can only come from the second summand of (6.3) and so $h^0(S', K_{S'}) \gg 0$.

We still denote by $Y$ a general fiber of $f_{S} \circ f : V \to C$. Note that $Y$ is an irregular 3-fold of general type, since $f_{S'}$ is an elliptic fibration.

Since $h^0(S', K_{S'}) \gg 0$, $S'$ has Kodaira dimension 1 and $f_{S'}$ is the Iitaka fibration of $S'$. In particular, there exists an ample line bundle $H$ on $C$ with $h^0(C, H) \gg 0$ such that $K_{S'} \geq Q_{f_{S'}^*} H$. Then $(1 + \epsilon)K_Y \geq Q (f_{S'} \circ h' \circ g)^* H$ and we conclude the statement of the theorem as before.

We have proved Theorem 6.1.

7. Proof of Theorem 7.2 and open questions

It is well-known, since the work of Kollár in [32], that the study of pluricanonical systems of varieties of general type with two linearly independent global top forms can be reduced to the study of pluricanonical systems of varieties of lower dimensions (see, for instance, Kollár [32, Corollary 4.8] and [13, 11]).

7.1. Varieties with global 1-forms.

The pluricanonical systems of varieties with many holomorphic 1-forms have also been studied by many authors (see, for instance, [12, 27, 8]). We are inclined to ask the following question:

**Question 7.1.** Let $X$ be an irregular variety of general type of dimension $n \geq 4$. Does $|mK_X|$ induce a birational map for each $m \geq r_{n-1}$?

When $n = 2$, the statement is due to Bombieri [3]; when $n = 3$, the affirmative answer to Question 7.1 was recently given in Chen-Chen-Chen-Jiang [8].

We have here a partial answer to this question in any dimension as follows:

**Theorem 7.2.** Let $X$ be a smooth projective variety of general type of dimension $n \geq 4$. Assume that either $q(X) > n$ or the Albanese image of $X$ is a proper subvariety of the Albanese variety. Then $|mK_X|$ induces a birational map for all $m \geq r_{n-1}$.

**Proof.** Let $a_X : X \to A_X$ be the Albanese morphism of $X$. By assumption, $a_X(X) \subset A_X$ generates $A_X$. By Ueno’s theorem (see for instance [10, Theorem 3.7]), there exists a fibration $q_B : A_X \to B$ between abelian varieties such that any smooth model of $q_B \circ a_X(X)$ is of general type. Let $X \to Z \to q_B \circ a_X(X)$ be the Stein factorization of $q_B \circ a_X$. After birational modifications, we may assume that $Z$ is a
smooth projective variety. We have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a_X} & A_X \\
\downarrow{h} & & \downarrow{q_B} \\
Z & \xrightarrow{t} & B.
\end{array}
\]

Note that \( Z \) is of maximal Albanese dimension. We denote by \( n_1 = \dim Z \). We know that \( \dim|mK_Z| \) induces a birational map of \( Z \) for each \( m \geq 3 \) (see [12, 27]). Let \( F \) be a general fiber of \( h \). Then, \( 0 \leq \dim F = n - n_1 \leq n - 1 \). Let \( a \) be the canonical stability index of \( F \). Then \( 1 \leq a \leq r_{n-n_1} \leq \nu_{n-1} \).

We first show that \( |mK_X| \) separates two general points on different fibers of \( h \) for each \( m \geq \max\{5, a\} \). Because \( |3K_Z| \) induces a birational map of \( Z \), it suffices to show that \( mK_X - 3h^*K_Z \) is effective. We write \( mK_X - 3h^*K_Z = K_X + (m - 1)K_{X/Z} + (m - 4)h^*K_Z \). Note that by Viehweg’s weak positivity, the Iitaka model of \( (m - 1)K_{X/Z} + (m - 4)h^*K_Z \) dominates \( Z \). Let \( D = (m - 1)K_{X/Z} + (m - 4)h^*K_Z \). We apply once again Viehweg’s weak positivity with the generic restriction theorem (see [38, Theorem 11.2.8]) to conclude that \( \mathcal{J}(|D|)|_F = \mathcal{J}((m - 1)K_F||) \). Thus

\[
h_*\left( \mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(|D|) \right)
\]

is a torsion-free sheaf on \( Z \) of rank equal to \( h^0(F, \mathcal{O}_F(mK_F) \otimes \mathcal{J}((m - 1)K_F||)) = p_m(F) > 0 \). By Kollár’s vanishing,

\[
H^i(Z, h_*\left( \mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(|D|) \right) \otimes t^*P) = 0
\]

for each \( i \geq 1 \) and \( P \in \text{Pic}^0(B) \). Thus,

\[
t_*h_*\left( \mathcal{O}_X(mK_X - 3h^*K_Z) \otimes \mathcal{J}(|D|) \right)
\]

is IT\(^0\) on \( B \) and it has a non-zero global section. Thus \( mK_X - 3h^*K_Z \) is effective.

We then show that \( |mK_X||F \) induces a birational map of \( F \) for \( m \geq \max\{n_1 + 2, a\} \). Since a smooth model of \( t(Z) \) is of general type, by a result of Griffiths and Harris (see, for instance, [40, Theorem 3.9]), the canonical map of a smooth model of \( t(Z) \) is generically finite. Thus the canonical map of \( Z \) is also generically finite. Let \( \phi : Z \rightarrow Z_1 \subset \mathbb{P}^N \) be the canonical map of \( Z \). Let \( z \in Z \) be a general point such that \( \phi \) induces an étale map between an open neighborhood of \( z \) with an open neighborhood of \( \phi(z) \in Z' \). Let \( H_1, \ldots, H_{n_1} \) be \( n_1 \) general hyperplane of \( \mathbb{P}^N \) through \( \phi(z) \). Then, \( (Z', \sum_{i=1}^{n_1} H_i|_{Z'}) \) is log canonical in an open neighborhood of \( \phi(z) \) and \( \phi(z) \) is a log canonical center of the pair. Let \( D_i \) be the corresponding divisor of \( K_Z \) for \( 1 \leq i \leq n_1 \). Thus \( (Z, \sum_{i=1}^{n_1} D_i) \) is log canonical in an open neighborhood of \( z \) and \( z \) is a log canonical center of this pair. Thus, by Theorem 4.3,

\[
H^0(X, mK_X) \rightarrow H^0(F, mK_F)
\]
is surjective for each \( m \geq n_1 + 2 \).

We then see that \(|mK_X|\) induces a birational map of \( X \), for each \( m \geq \max\{n + 2, r_{n-1}\} \). It suffices to see that \( r_{n-1} \geq n + 2 \). It is well known that \( r_5 \geq r_4 \geq r_3 \geq 27 \). When \( n \geq 7 \), by [20, Theorem 1.1], \( r_{n-1} \geq 2^{\frac{2n-3}{2}} > n + 2 \). \( \square \)

### 7.2. Open questions on varieties with many global \( k \)-forms.

Theorem 1.2 and Theorem 1.5 suggest that the pluricanonical system of \( n \)-folds of general type with many global two forms behave similarly to \((n - 2)\)-folds of general type. We have the following very bold conjecture:

**Conjecture 7.3.** For any \( n \geq 5 \), there exists a constant \( M(n) \) such that, for every smooth projective \( n \)-fold \( X \) of general type with \( h^{2,0}(X) \geq M(n) \), \(|mK_X|\) induces a birational map for all \( m \geq r_{n-2} \).

One can even ask a very general question about varieties with many global \( k \)-forms. Let

\[
 r^k_n := \sup \{ r_s(W) \mid W \text{ is a smooth projective } n \text{-fold of general type with } h^{k,0}(W) > 0 \}.
\]

**Question 7.4.** For any \( n \geq 4 \), does there exist a constant \( M(n) \) such that, for every smooth projective \( n \)-fold \( X \) of general type with \( h^{k,0}(X) \geq M(n) \), \(|mK_X|\) induces a birational map for each

\[
 m \geq \max\{r_{n-k}, r_{n-k+1}, \ldots, r_{n-1}\}?
\]

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