BERGMAN TYPE METRICS IN TOWER OF COVERINGS

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The aim of this note is to prove a version of a statement attributed to Kazhdan by Yau [Y]. For a survey of known results and historical remarks, see the recent article by Ohsawa [O, Section 5] as well as articles by Kazhdan ([Ka1], [Ka2]) and McMullen [M, Appendix]. As a corollary, we obtain a more transparent form of the uniformization theorem in [T1].

1. Preliminaries

1.1. Throughout the note, $X \subset \mathbb{P}^\ell$ denotes a nonsingular $n$-dimensional connected projective variety with large (see [Kol]) and residually finite fundamental group, and without general elliptic curvilinear sections. Let $U$ denote its universal covering.

We assume the canonical bundle $\mathcal{K}_X$ on $X$ is ample. Set $\mathcal{K}^\ast_X := \mathcal{K}_X^{\otimes s}$. We consider a tower of Galois coverings with each $Gal(X_i/X)$ a finite group:

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow U,$$

$\bigcap_i Gal(U/X) = \{1\}$ (0 $\leq i < \infty$).

It is known that for all sufficiently large $s$, the bundle $\mathcal{K}^\ast_X$ and its pullbacks on each $X_i$, denoted by $\mathcal{K}_{X_i}^\ast$, are very ample (see [Kol, 16.5]). In the sequel, we fix the smallest integer $m$ with this property and assume $X \subset \mathbb{P}^\ell$ is given by $\mathcal{K}_X^m$. The pullback of $\mathcal{K}_X^s$ on $U$ will be denoted by $\mathcal{K}_U^s$.

In [T2, (1.1)], we have defined a Kahler metric (called original metric below) on $U$ depending on the embedding of $X \subset \mathbb{P}^\ell$. This metric defines the volume form $dv_U = \rho d\mu$, where $d\mu$ is the Euclidean volume form. The metric and the volume form are invariant under the action of $\pi_1(X)$ so we get the volume form $dv_U$ on $X_i$.

The bundle $\mathcal{K}_U^m$ is equipped with the Hermitian metric $h_{\mathcal{K}_U^m} := h_{\mathcal{K}_U}^m$, where $h_{\mathcal{K}_U}$ is the standard Hermitian metric on $\mathcal{K}_U$ [Kol, 5.12, 5.13, 7.1]. Similarly, each bundle $\mathcal{K}_{X_i}^m$ is equipped with the Hermitian metric $h_{\mathcal{K}_{X_i}^m} := h_{\mathcal{K}_{X_i}}^m$.

The Calabi diastasis was introduced in [C] (see also [U, Appendix]). For properties of infinite-dimensional projective spaces, see [C, Chap.4] and [Kob, Sect. 7].

1.2. Definition. We will define the $q$-Bergman metric on $U$, where $q \geq m$ is an integer. Let $H$ be the Hilbert space of weight $q$ square-integrable with respect to $dv_U$ holomorphic differential forms $\omega$ on $U$, where by square-integrable we mean

$$\int_U h_{\mathcal{K}_U^m}(\omega, \omega)\rho^{-q}dv_U < \infty.$$
We assume $H \neq 0$. This Hilbert space has a reproducing kernel as, e.g., in [FK, pp. 8-10 or pp. 187-189]. Let $\mathbf{P}(H^*)$ denote the corresponding projective space with its Fubini-Study metric. If the natural evaluation map

$$e_U : U \rightarrow \mathbf{P}(H^*)$$

is an embedding then the metric on $U$, induced from $\mathbf{P}(H^*)$, is called the $q$-Bergman metric.

Similarly, one defines the Euclidean space $V_i$, $X_i \hookrightarrow \mathbf{P}(V_i^*)$, and the $q$-Bergman metric on $X_i$. We consider every $V_i$ with the normalized inner product in order to get the inclusion of Euclidean spaces $V_i \hookrightarrow V_{i+1}$ induced by pullbacks of forms.

2. Proposition and Corollary

**Proposition.** With the above notation and assumptions, we assume, in addition, that $U$ has the $q$-Bergman metric for an integer $q \geq m$. Then the $q$-Bergman metric on $U$ equals the limit of pullbacks of the $q$-Bergman metrics from $X_i$’s.

**Proof.** (2.1) Let $b_{U,q}$ denote the $q$-Bergman metric on $U$. Set \( \tilde{b}_{U,q} := \limsup b_i \), where $b_i$ is the pullback on $U$ of the $q$-Bergman metric on $X_i$.

By an argument similar to [Ka2, Lemma], now we will establish the inequality $\tilde{b}_{U,q} \leq b_{U,q}$ (in [Ka2], $U$ is a bounded symmetric domain and $q = 1$).

For $r > 0$, we set $U_r := \{ x \in U | |P(x)| \leq r \}$, where $P(x)$ is the function on $U$ generated by the diastasis of the original Kähler metric on $U$; $U_r$ is a compact subset in $U$ [T2, p. 2, (1.2)]. Let $b(r)$ denote the $q$-Bergman metric on $U_r$.

It is well known that $b_{U,q} = \lim_{r \to \infty} b(r)$. Given $U_r$, the restriction on $U_r$ of the projection $U \to X_i$ is one-to-one for $i \gg 0$ because $\pi_1(X)$ is residually finite. Further, $b(r) > b_i|U_r$ for all $i > i(r)$. This establishes the inequality $\tilde{b}_{U,q} \leq b_{U,q}$.

We have $\tilde{b}_{U,q} = \lim b_i$. In the sequel, by a local embedding we mean an embedding of a neighborhood. The above argument suggests the following proof.

(2.2) Let $V$ be the completion of the Euclidean space $E := \cup V_i$. We will show that $U$, with the metric $b_{U,q}$, can be naturally isometrically embedded in the infinite-dimensional projective space $\mathbf{P}(V^*)$ with its Fubini-Study metric.

First, let $p(w) \in U$ be a point, and $\Omega = f(dw_1 \wedge \cdots \wedge dw_n)^q \in H$ in local coordinates $w = (w_1, \ldots, w_n)$ around $p(w)$. Recall that the $b_{U,q}$-isometric embedding $U \hookrightarrow \mathbf{P}(H^*)$ was given by the $q$-Bergman kernel section $B := B_{U,K^q}$ (see [T1]; we do not assume $K^q_U$ is a trivial bundle). The map $p(w) \mapsto \langle f, B_w \rangle$ defines the local $b_{U,q}$-isometric embedding $U \hookrightarrow \mathbf{P}(H^*)$ (see [Kob], [FK, pp. 5-13 or p. 188]) hence the global $b_{U,q}$-isometric embedding (see [C, Theorems 10, 11]).

(2.3) Now, let $\Omega = f(dw_1 \wedge \cdots \wedge dw_n)^q$ be a form on $U$ that is the pullback of a form from $V_i$. As above, one can define locally the map $p(w) \mapsto \langle f, B_w \rangle$ because $B = \lim_{r \to \infty} B(r)$ [A, Part I, Sect. 9]. The above map yields an analytic, local $b_i$-isometric embedding $U \hookrightarrow \mathbf{P}(V_i^*)$. Hence we get an analytic, local $b_{U,q}$-isometric embedding $U \hookrightarrow \mathbf{P}(V^*)$ (see [FK, Proposition I.1.6 on p. 13]).

As in (2.2), by Calabi [C, Theorems 10, 11], the local $b_{U,q}$-isometric embedding yields the global one.

(2.4) Clearly, $U$ with the metric $b_{U,q}$ is $\mathbf{P}(H^*)$-resolvable at a point $p$ (1-resolvable of rank $N = \infty$ in Calabi’s terminology [C, p. 19, Definition]); the image of $U$ does not lie in a proper subspace of $\mathbf{P}(H^*)$. 2
Similarly, $U$ with the metric $\tilde{b}_{U,q}$ is $P(V^*)$-resolvable at $p$. By the construction of the embedding $U \hookrightarrow P(V^*)$, we get the embedding $P(V^*) \hookrightarrow P(H^*)$. It follows the latter embedding is surjective and $P(V^*) = P(H^*)$.

This concludes the proof of the proposition.

**Corollary.** With the notation and assumptions of (1.1), $U$ is a bounded domain in $\mathbb{C}^n$ provided $U$ has the $q$-Bergman metric for a sufficiently large integer $q$.

**Proof.** With all the assumptions of the corollary, in the uniformization theorem [T1, Theorem] we can replace “very large” by a weaker assumption “large”. Indeed, we now get the embedding corresponding to the metric $\tilde{b}_{U,q}$, as in [T1, Sect. 2.2]:

$$U \hookrightarrow P(V^*).$$

The above embedding is required for the proof of the uniformization theorem (see [T1, Sect. 3]); recall that the assumption “very large” (or “$U$ has the $q$-Bergman metric”) does not follow from other assumptions [T1, Remark 4.3].

**Remark 1.** Let $\dim X = 1$ and $K_X$ is very ample. In the above proposition, one can take $m = 1$, an arbitrary integer $q \geq m$, and the Poincaré metric on the disk in place of the original metric.

Other $\pi_1$-invariant volume forms on $U$ can be used in place of $dv_U$.

**Remark 2.** We conjecture that, in the above corollary, one can replace “$U$ has the $q$-Bergman metric” by “the fundamental group $\pi_1(X)$ is nonamenable”.

Assume the fundamental group is nonamenable. Perhaps, one has to employ the Bergman type reproducing kernel of a space of harmonic forms.

**References**

[A] N. Aronszajn, Theory of Reproducing Kernels, Trans. Amer. Math. Soc. 68 (1950), 337–404.
[C] E. Calabi, Isometric imbedding of complex manifolds, Ann. of Math. 58 (1953), 1–23.
[FK] J. Faraut, S. Kaneyuki, A. Korányi, Q.-k. Lu, G. Roos, Analysis and geometry on complex homogeneous domains, Birkhäuser, Boston, 2000.
[Ka1] D. A. Kazhdan, On arithmetic varieties. in Lie Groups and Their Representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, 1975, pp. 151–217.
[Ka2] ______, email to Curtis T. McMullen.
[Kob] S. Kobayashi, Geometry of bounded domains, Trans. Amer. Math. Soc. 92 (1959), 267–290.
[Kol] J. Kollár, Shafarevich maps and automorphic forms, Princeton Univ. Press, Princeton, 1995.
[M] C.T. McMullen, Entropy on Riemann surfaces and the Jacobians of finite covers, Comment. Math. Helv. (to appear).
[O] T. Ohsawa, Review and Questions on the Bergman Kernel in Complex Geometry, Preprint. (2010).
[T1] R. Treger, Uniformization, arXiv:math.AG/1001.1951v4.
[T2] ______, Remark on a conjecture of Shafarevich, arXiv:math.AG/1008.1745v2.
[U] M. Umehara, Kaehler Submanifolds of Complex Space Forms 10 (1987), 203-214.
[Y] S.-T. Yau, Nonlinear Analysis in Geometry, Eiseignement Mathématique 33 (1987), 108-159.

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