I. INTRODUCTION

In recent years there has been growing interest in the observational and theoretical consequences of time variations in the values of the traditional constants of Nature, notably of the fine structure constant, \( \alpha \), [1–10], the electron-proton mass ratio, \( \mu = m_e/m_p \), [11–13], and Newton’s ‘constant’ of gravitation, \( G \), [14]. In all cases the experimental evidence that can be brought to bear on the problem is a combination of local (laboratory, terrestrial, and solar system) and global (astronomical and cosmological) observations [15–19]. The first theoretical challenge is to develop self-consistent extensions of general relativity which incorporate varying ‘constants’ rigorously by including the gravitational effects of the variations and ensuring that energy and momentum are totally conserved by the variations that replace the former constants. This is achieved by regarding the ‘constant’ as a scalar field with particular couplings and self-interaction. In the case of varying \( G \), the Brans-Dicke theory [20, 21] provides the paradigm for a scalar-tensor theory of this type. Recently, the same philosophy has been applied to produce simple extensions of general relativity which self-consistently describe the spacetime variation of \( \alpha \), [22, 23], and \( \mu \), [24]. It is also possible to extend these studies to include simultaneous variations of several constants, and to include the weak coupling by a generalisation of the Weinberg-Salam theory to include spacetime varying coupling ‘constants’ [25, 26]. In these theories the analysis of the behaviour of their solutions is simplified because we know that the allowed variations in constants like \( \alpha \) are constrained already to be ‘small’ and will not have any significant effect on the expansion dynamics of the universe in recent times. The Brans-Dicke theory is different. Small variations in \( G \) will always have direct consequences for the expansion dynamics of the universe. Typically, a power-law time variation of \( G \propto t^{-n} \) creates a variation of the expansion scale factor that goes as \( a(t) \propto t^{(2-n)/3} \) in a dust-dominated Friedmann universe [27].

These theories are confronted with a variety of laboratory, geochemical, and astronomical observations. In the case of variations of \( \alpha \) we have laboratory constraints on atomic lines, indirect bounds from the Oklo natural reactor operation 1.8 billion years ago, radioactive decay products in meteoritic data back to 4.6 billion years ago, and quasar spectra out to redshifts \( z \lesssim O(4) \), as the prime sources of observational evidence against which to test theories which permit spacetime variations [15–19]. Generally, the data from all these diverse physical scales are lumped together and used to test time variations of \( \alpha \). Thus laboratory or solar system evidence is compared directly with quasar data and used to constrain the allowed cosmological variations of \( \alpha \). Similar tactics are used to constrain the allowed variations of other constants, like \( G \) or \( \mu \).

This simultaneous use of terrestrial and astronomical bounds on constants assumes implicitly the unproven requirement that any variation of a constant on cosmological scales is ‘seen’ locally inside virialised structures like galaxies or solar systems, and has a measurable effect in laboratory experiments on Earth. It is not obvious a priori that this need be the case: we would not expect to test the expansion of the universe by measuring the expansion of the Earth. The central question that this paper addresses is the extent to which global variations of ‘constants’ on cosmological scales that take part in the Hubble expansion of the universe are seen locally on the surface of gravitationally-bound structures, like planets, or inside bound systems of stars like galaxies [28]. Only if we can show that cosmological variations have calculable local effects will it be legitimate to use laboratory and solar system observations to constrain
theories of varying constants in the way that is habitually done, without proof, in the literature. So far, detailed analyses of spatial variations of constants have been made only for small variations, where the isotropy of the microwave background places very strong limits on spatial variations because of the associated Sachs-Wolfe effects created by the gravitational potential perturbations that accompany spatial fluctuations in ‘constants’ via their associated scalar fields because of the coupling of the latter to matter [29, 30].

We are concerned with the dynamics of spacetime scalar fields that are weakly coupled to gravity and matter. We will not consider theories where there are two or more scalar fields interacting amongst themselves, although the method we use here could also be easily extended to that scenario. Theories which introduce varying constants self-consistently into Einstein’s conception of a gravitation theory do so by associating the ‘constant’ C to be varied with a new scalar field, so C → C(φ). The variations of this scalar field gravitate and contribute to the curvature of spacetime like any other form of mass-energy. They must also conserve energy and momentum and so their forms are constrained by a covariant conservation equation for the scalar field. Typically, this results in a wave equation of the form

\[ \Box \phi = \lambda f(\phi)L(\rho, p), \]  

where \( \varphi \) is a scalar field associated with the variation of some ‘constant’ C via a relation \( C = f(\varphi), \lambda \) is a dimensionless measure of the strength of the space-time variation of C, \( f(\varphi) \) is a function determined by the definition of \( \varphi \), and \( L(\rho, p) \) is some linear combination of the density, \( \rho \), and pressure, \( p \), of the matter that is coupled to the field \( \varphi \) and \( f(\varphi) \approx 1 \) for small variations in \( \varphi \) and C. This form includes all the standard theories for varying constants, like \( G, \alpha, \) and \( \mu \), of refs. [20, 22–24].

We shall refer to our scalar field as the ‘dilaton’, denote it by \( \phi \), and analyse the form of the standard equation (1) a little further by assuming that \( \phi(\vec{x}, t) \) satisfies a conservation equation that can be decomposed into the form

\[ \Box \phi = B(\phi)\kappa T - V(\phi) \]  

where \( T \) is the trace of the energy momentum tensor, \( T = T^\mu_\mu \) (with the contribution from any cosmological constant neglected). We absorb any dilaton-to-cosmological constant coupling into the definition of \( V(\phi) \). The dilaton to matter coupling \( B(\phi) \) and the self-interaction potential, \( V(\phi) \), are arbitrary functions of \( \phi \) and \( \kappa = 8\pi G \) and \( c = \hbar = 1 \). This covers a wide range of theories which describe the spacetime variation of ‘constants’ of Nature, and includes Einstein-frame Brans-Dicke (BD) and all other, single field, scalar-tensor theories of gravity. In cosmologies that are composed of dust, cosmological constant and radiation it will also contain the Bekenstein-Sandvik-Barrow-Magueijo (BSBM) of varying \( \alpha \), [23], and other (single dilaton) theories which describe the variation of standard model couplings, [19]. Note that one could, for example, generalise the form of this conservation equation, (2), whilst maintaining relativistic invariance, by adding a coupling to \( \sqrt{T^{\alpha\beta}T_{\alpha\beta}} \) or we could also break local Lorentz invariance by adding extra couplings to the pressure, \( P \), defined w.r.t. to some preferred coordinate system. In this paper we will mostly be considering spacetimes in which the pressure of the matter vanishes, \( P = 0 \), and so all of the potential extra couplings mentioned above will reduce to a form that is included in the conservation equation (2) that we assume.

We will assume that the background cosmology is isotropic and homogeneous and work with a Friedmann-Robertson-Walker (FRW) background metric:

\[ ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 \{d\theta^2 + \sin^2 \theta d\phi^2\} \right), \]  

where \( k \) is the spatial curvature parameter. In the background universe the dilaton field is also assumed to be homogeneous (so \( \phi = \phi_c(t) \)) and therefore satisfies the ordinary differential equation (ODE):

\[ \frac{1}{a^2} \left( a^2 \dot{\phi}_c \right) = B(\phi)\kappa (\epsilon_c - 3P_c) - V(\phi) (\phi_c). \]  

The dilaton conservation equation also reduces to a similar ODE if we are only interested in its time-independent mode \( \phi(r) \) in a static spherically symmetric background, like the Schwarzschild metric. In the actual universe, however, spacetimes that look static in some locality must still match smoothly on to the cosmological background spacetime on large scales. Hence, if we are to model the evolution of the dilaton field accurately in some inhomogeneous region, embedded in a homogeneous background, we must demand that at large distances \( \phi \rightarrow \phi_c \) in some appropriate way. Even if spherical symmetry is assumed, the conservation equation is generically a non-linear, second-order, partial differential equation (PDE) and its solution is far from straightforward to find, either exactly or approximately. Even numerical models are technically difficult to set-up, see ref. [34], and only allow us to consider one particular choice of \( B(\phi) \), \( V(\phi) \), and the spacetime background at a time. However, as we have mentioned above, if we are to bring all of our experimental evidence to bear on these models, we need to know how to interpret local observations in the light
of our cosmological ones. Perhaps the most important piece of knowledge we would like to have is the correlation between the local and global (or cosmological) time variation of the dilaton. In particular, we would like to find the criteria under which it is true that

\[ \dot{\phi}(x, t) \approx \dot{\phi}_c(t), \]  

(5)

that is, when does the local time-variation of \( \phi \) track the cosmological one?

In this paper we will try to answer this question by applying asymptotic methods commonly used in fluid dynamics in order to construct asymptotic approximations to the behaviour of \( \phi \) close to a spherical static mass, that match to the cosmological solution, \( \phi_c \), at large distances. From this analysis, we can derive a sufficient condition for eq. (5) to hold. We limit ourselves in this paper to considering spacetime backgrounds that are a spherical symmetric. In a subsequent work we will generalise our results to deal with more general non-spherical backgrounds, [49]. Throughout our analysis we will refer to the spherical, static mass as a ‘star’, however it could be taken to be a planet (e.g. the Earth), a black-hole, or even a galaxy or cluster of galaxies. We are mostly interested in the (realistic) case where the surface of our ‘star’ lies far-outside its own Schwarzschild-radius. By applying our results to the black-hole case, however, we will comment on the problem of ‘gravitational memory’: that is, is the cosmological background value of \( \phi \) on the horizon at the time when a black hole forms, frozen-in, or ‘rememred’, when the black-hole forms, or does it continue to track the background cosmological evolution?

This paper is organised as follows: in section II we review some of the previous studies into the problem of local vs. global dilaton evolution and note where our work extends and improves these studies. In section III we will introduce the method of matched asymptotic expansions which we will use to carry out our study, and provide some simple examples of its application. In section IV we define our geometrical set-up of a star in a background cosmology, and detail our particular choices of possible spacetime backgrounds. Then, in section V, we construct overlapping asymptotic expansions to study the constraints required if the local and global evolution is to match together. In section VI we derive the conditions that must be satisfied for our method to be valid. Taking these into account, we interpret and generalise our results in section VII. In this section we shall also make a conjecture about a general condition that is sufficient for eq. (5) to hold, which will apply to more general spacetime backgrounds that those explicitly considered in this paper. Finally, in section VIII we use our sufficient condition to show that we do expect eq. (5) to hold here on Earth.

II. PAST WORK

Several authors have, in the past, claimed to have shown that condition (5) holds close to the surface at, \( r = R_s \), of a spherically-symmetric, massive body embedded, in some particular way, into an expanding universe. The results derived are usually only valid for a particular choice of \( B(\phi) \) and \( V(\phi) \) and under some, usually restrictive, assumptions about the background distribution of matter. In this section we will review the pioneering analyses carried out by Wetterich in [31] and Jacobson in [35] and describe how our study will go further.

A. Review of Wetterich’s analysis

In ref. [31] Wetterich claimed that condition (5) would hold for any potential, \( V^{(w)}(\phi) \), with the properties \( V_\phi(\phi_c) \gg B_\phi(\phi_c) \kappa \epsilon_c \). The demonstration was confined to a given background and particular potential \( V^{(w)}(\phi) \propto \exp(-\lambda \phi) \). It was argued that similar behaviour should be found whenever the potential dominates the cosmic evolution of the dilaton field i.e. \( V_\phi(\phi_c) \gg B_\phi(\phi_c) \kappa \epsilon_c \). In what follows we will show that condition (5) also holds in many situations where the potential does not dominate the cosmic evolution of the dilaton field. It is our belief, however, that the reasoning given in ref. [31] is incomplete and does not show that condition (5) holds even under the restrictive conditions specified there. We shall briefly outline the arguments given in ref, [31] below, and show where we believe they fail.

Wetterich considered a universe filled with pressureless dust and a cosmological constant. Under these assumptions we have \( T = \epsilon_{dust} \), the dust density, and:

\[ \square \phi = B_\phi(\phi) \kappa \epsilon_{dust} - V^{(w)}_\phi(\phi) , \]

where \( \epsilon \) is the local matter density and it was assumed that the potential is

\[ V^{(w)}(\phi) = \omega e^{-\phi} , \]
The function $B,\phi(\phi)$ represents the coupling of the dilaton field to the dust. Experimental bounds on the largest allowed violations of the Weak Equivalence Principle (WEP) that will be created by a dilaton field that couples to the electromagnetic energy of matter suggest that $|B,\phi(\phi)| < 10^{-4}$. Wetterich considered a particularly simple example of a spherically symmetric massive body superimposed onto the cosmological background; table I shows the local energy density budget in this model. For $r \gg r_c$, $\phi$ takes it cosmological value $\phi_c(t)$, which satisfies:

$$\ddot{\phi}_c + 3H\dot{\phi}_c = \omega e^{-\phi} B,\phi(\phi_c) \kappa e_c.$$

Wetterich only considered the case where $\omega e^{-\phi} \gg B,\phi(\phi_c) \kappa e_c$. Assuming that the scale factor $a \propto t^n$, we find:

$$\phi_c(t) = \phi_0 + 2 \ln(t/t_0),$$

where $t = t_0$ today, and $\phi(t_0) = \phi_0 = -\ln(6n + 2)/\omega t_0 \approx 140 + \ln\left(\omega/M^2_{pl}\right)$ is the present value of the scalar field. For $r \approx R_s$, dilaton field in ref. [31] is written as:

$$\phi = \phi_c(t) + \phi_t(r,t) + \phi_e(r,t),$$

where this defines $\phi_e$ and $\phi_t(r,t)$ is the ‘local’, quasi-static field configuration satisfying:

$$\nabla^2 \phi_t - \mu^2 \phi_t = -B,\phi(\phi_e) \kappa e_E \theta(R_s - r),$$

with $\nabla^2$ the 3-D Laplacian and $\theta(R_s - r)$ the Heaviside function; $\mu^2 = V^{(w)}(\phi_e)$ and $\mu^2 R_s^2 \ll 1$. Near $r = R_s$ it was found that $\phi_t/\phi_c \sim B,\phi(\phi_c) 2M/R_s \ll 1$. If $\phi_e(r,t)/\phi_c(t) \ll 1$ for $r \sim R_s$ then the dilaton field will satisfy condition (5). If $\Box \phi_e/\Box \phi_c \ll 1$ locally then, as stated in ref. [31] we will have the required result. Wetterich argues that this is indeed the case in his paper. However, this does not appear to be the case. Assuming $2m/R_s \ll 1$, from $r = R_s$ spacetime is approximately Minkowski and so:

$$\Box \phi_e = \ddot{\phi}_e - \nabla^2 \phi_e \approx -\mu^2 \phi_e - \ddot{\phi}_e + 3H \dot{\phi}_e - B,\phi(\phi_e) \kappa e_c + B,\phi(\phi_e) \kappa e_E \theta(R_s - r) (\dot{\phi}_t + \phi_e),$$

$$\Box \phi_c = \ddot{\phi}_c - 3H \dot{\phi}_c - V^{(w)}(\phi_e).$$

Now, $\ddot{\phi}_c \ll \ddot{\phi}_e$ and $\mu^2 R_s^2 \ll 1$ and so, near the surface of our body, the $\mu^2 \phi_e$ term represents a negligible correction to the $\phi_e$ dynamics. Therefore, if $\Box \phi_e \gg \Box \phi_c$ we must have:

$$\left(3H \dot{\phi}_e - B,\phi(\phi_e) \kappa e_c + B,\phi(\phi_e) \kappa e_E (\dot{\phi}_t + \phi_e)\right) \ll \ddot{\phi}_e.$$

However, under Wetterich’s assumptions that $B,\phi(\phi_e) \sim O(B,\phi(\phi_c))$ and $\kappa e \sim 10^{30} e_c$, as is the case for the density of the Earth, we find:

$$\frac{B,\phi(\phi_e) \kappa e E \phi_t}{\phi_e} \approx 10^{22} \left(\frac{B,\phi(\phi_e)}{10^{-4}}\right)^2 \frac{2m}{R_s} \approx 10^{13} \left(\frac{B,\phi(\phi_e)}{10^{-4}}\right)^2 \gg 1,$$

where we have taken $\frac{2m}{R_s} \sim 10^{-9}$ in the final deduction. Hence, we have shown that in general the condition $\Box \phi_e/\Box \phi_c \ll 1$ does not hold; indeed $\Box \phi_e \gg \Box \phi_c$. This result is opposite to the one found in ref. [31]. We conclude then that Wetterich’s analysis does not prove that condition (5) holds. If we approximate our local solution by $\phi = \phi_t + \phi_e$ then we have seen that correction terms, $\phi_e$, to this solution vary on scales much smaller than $1/\phi_c$. As a result we cannot conclude that $\phi \approx \phi_c$. In fact, at the epoch $t = t_0$, the asymptotic expansion of the local solution should be correctly written as:

$$\phi \sim \phi_0 + \phi_t(r,t_0),$$

We shall see later by more detailed methods that, even though the above analysis fails to show it, that we should expect condition (5) to hold for this set-up.

**TABLE I: Density distribution in Wetterich’s model**

| Region | Range   | $\epsilon_{\text{dust}}$ | Description     |
|--------|---------|--------------------------|-----------------|
| a      | $r < R_s$ | $\epsilon = \epsilon_E$ | local planet    |
| b      | $R_s < r < r_c$ | $\epsilon = 0$ | intermediate space |
| c      | $r > r_c$    | $\epsilon = \epsilon_e(t)$ | Hubble flow      |
In ref. [35], the problem of gravitational memory [32] is considered in the context of Brans-Dicke (varying-\(G\)) theory. If Newton’s constant, \(G\), can and does vary over time and space then one must ask which value of \(G\) is appropriate for use on the horizon of a black hole after it forms. One motivation was to discover if the black hole possesses a type of gravitational memory, freezing-in the value of \(G\) that existed cosmologically at the moment when it formed in the early universe, or whether the value of Newton’s ‘constant’ on the horizon changes over time so as to track its changing cosmological value in the background universe. This has been investigated by several different methods and found to be a small effect [33] but Jacobson’s argument was that if the cosmological variation in \(G\), and the related Brans-Dicke field, \(\phi \propto G^{-1}\), is slow over time-scales of the order of the intrinsic length scale of the black-hole (\(~2GM\)) then, at each epoch \(t = t_0\), one can expand the cosmological value of \(\phi\) as a Taylor series in \((T = t - t_0)\) and, to a good approximation drop all \(O(T^2)\) and higher-order terms, so

\[
\phi_c \approx \phi_1 + \dot{\phi}_c(t_0)T.
\]

Jacobson noted that \(\phi_1\) is a complementary solution to the Brans-Dicke field conservation equation in an (empty) Schwarzschild background:

\[
\Box_s \phi_1 = 0,
\]

where \(\Box_s\) is the d’Alembertian operator for the Schwarzschild metric. Therefore \(\phi_1\) can be added to any known Schwarzschild-background solution of Brans-Dicke field equations to gain a new solution. Jacobson took \(T\) to be the ‘curvature’ (or Schwarzschild) time so that the metric is given by:

\[
ds^2 = \left(1 - \frac{2m}{R}\right)dt^2 - \left(1 - \frac{2m}{R}\right)^{-1} dr^2 - R^2\{d\theta^2 + \sin^2 \theta d\phi^2\}.
\]

This time coordinate diverges as we move towards the horizon, hence \(\phi_1(T)\) also diverges in this limit. The unique static solution of the dilaton conservation equation, which vanishes as \(r \to \infty\), is given by:

\[
\phi_2 = \ln \left(1 - \frac{2m}{r}\right)
\]

Whilst both \(\phi_1\) and \(\phi_2\) diverge on the horizon, Jacobson found that there is a unique linear combination of them that is well-defined as \(r \to 2m\):

\[
\phi_{jac} = \phi_1(t) + 2m \dot{\phi}_c(t_0) \phi_2(r) = \dot{\phi}_c(v - r - 2m \ln(r/2m))
\]

(8)

where \(v = t + r + 2m \ln(r/2m - 1)\) is the advanced time coordinate. By construction, it can then be shown that there exists a unique solution for \(\phi\) in the Schwarzschild background that is non-singular on the horizon and has \(\phi_{jac}(r = \infty, t) \approx \phi_c(t)\) for the particular case where \(\phi_c(t) \propto t \propto G^{-1}\). If the approximation used is valid, then this suggests that the value of \(G\) on the horizon lags slightly behind the cosmological value, but that over time-scales much larger than \(m\) there is no gravitational memory when \(G \propto t^{-1}\) falls in this extreme fashion.

Jacobson’s result assumes that the cosmological region can be taken to be infinitely far away, so that the entire spacetime is Schwarzschild; in reality the cosmological matter will become gravitational dominant over the black-hole at some finite-distance. Another issue with Jacobson’s method is that, since \(T\) diverges on the horizon, the expansion of \(\phi_c\) as a Taylor series in small \(T\) will not be valid near the horizon. In realistic models the space surrounding the black hole will also not be totally empty, indeed there will be an accretion disk surrounding the black-hole. The time-scale for matter to fall into the black-hole is order \(\sqrt{r^3/2m}\) and this is relatively short compared with the time-scale over which cosmological expansion occurs. Accretion of matter into the black-hole, therefore, might well result in a significant difference in the time-variation of \(G\) close to the horizon or cosmologically. In this paper we will consider a more realistic embedding of a Schwarzschild mass in an expanding universe; and build our asymptotic expansions in such a way that they are well-defined on the horizon. We will conclude that Jacobson’s result [35] (see also refs. [33, 34] for similar conclusions arrived at by different methods) does indeed give the correction behaviour of \(\phi\) whenever the energy-density in the region surrounding the black-hole is low enough and for more general time-variations of \(G(t)\) in the background universe.

### III. MATCHED ASYMPTOTIC EXPANSIONS

The method of matched asymptotic expansions that we will use in this paper was first developed to solve systems of PDEs that involve multiple length scales. It is often used in the field of fluid dynamics to study systems where there is
thin boundary layer, inside which the length scale of variation is much smaller than outside it. It also has applications
in the study of slender bodies, with widths much smaller than their lengths. Other problems with multiple length
scales include the interaction of greatly separated particles and the evaluation of the influence of a slowly changing
background field on the dynamics of a small body. The books by Cole, [36], and Hinch, [37], provide a more detailed
treatment of this subject. In this section we will briefly introduce the method and give some simple examples of its
applications in order to fix ideas.

A. Asymptotic expansions

Critical to this method is the requirement that the approximations we will work with are asymptotic expansions
rather than convergent (Taylor-like) power-series. In general, an asymptotic expansion will not be convergent. Thus,
we define asymptotic approximations and expansions as follows:

**Definition III.1.** $\sum_{n=0}^{M} f_n(\delta)$ is an asymptotic approximation to $f(\delta)$ as $\delta \to 0$, if for each $M \leq N$ the remainder
term is much smaller than the last included term:

$$\frac{f(\delta) - \sum_{n=0}^{M} f_n(\delta)}{f_M(\delta)} \to 0 \text{ as } \delta \to 0.$$ 

One then writes:

$$f(\delta) \sim \sum_{n=0}^{N} f_n(\delta) \text{ as } \delta \to 0.$$ 

**Definition III.2.** If definition III.1 holds, in principle, for all $N$, i.e. we can take $N = \infty$, then the we deem the
approximation to be an asymptotic expansion of $f(\delta)$:

$$f(\delta) \sim \sum_{n=0}^{\infty} f_n(\delta) \text{ as } \delta \to 0.$$  \hfill (9)

In many cases $f_n(\delta) \propto (\delta^p)^n$ for some constant $p$ and we will have an asymptotic power series. It is also quite
common to find $f_i(\delta) \propto \delta^i \ln(\delta)$, for some $i$ and $j$. The sum in eq. (9) is a formal one, and it is not required that it
does converge. The property required by definition III.1 is, however, more useful that convergence in many cases; it
ensures that one only needs the first few terms of the expansion to create a good numerical approximation to $f(\delta)$.

B. Singular problems and ones with multiple scales

Consider a differential operator $L_x(\delta)$ which defines some function $f(x,\delta)$ by $L_x(\delta)f(x,\delta) = 0$. We can attempt to
solve this equation by making an asymptotic expansion of $f(x,\delta)$ and solving the resultant equation order-by-order
in $\delta$:

$$f(x,\delta) \sim \sum_{n=0}^{\infty} f_n(x)\gamma_n(\delta) \text{ as } \delta \to 0,$$

with $x$ fixed. A ‘singular’ problem is one where the above asymptotic expansion is not uniformly valid, i.e. it breaks
down for certain ranges of $x$, typically $x = O(\delta)$ or $x = O(1/\delta)$. Singular behaviour such as this can be divided
into two distinct classes. In the first, the highest derivative in $L_x$ is multiplied by some power of $\delta$ and so can be
ignored everywhere apart from in those regions where the variation in $f(x,\delta)$ is fast enough to ensure that the highest
derivative make a significant contribution. In the second class, the problem has more than one intrinsic length (or
time) scale, one much larger than the other. The application of this method to physical problems generally tends to
fall into this latter class. In both cases one proceeds by constructing two (or more) asymptotic approximations to the
solutions which are valid for different ranges of $x$, e.g. for $x \sim O(1)$ and $x/\delta = \xi \sim O(1)$, with

$$f(x,\delta) \sim \sum_{n=0}^{\infty} f_n(x)\delta_n \text{ as } \delta \to 0, \text{ } x \text{ fixed}, \hfill (10)$$

$$f(x,\delta) \sim \sum_{n=0}^{\infty} g_n(\xi)\delta_n \text{ as } \delta \to 0, \text{ } \xi = x/\delta \text{ fixed}, \hfill (11)$$

and solving $L_x(\delta)f(x,\delta)$ order by order in $\delta$ for both expansions w.r.t. to some boundary conditions. We will call
expansion (10) the outer solution, and (11) the inner solution. The inner expansion is not uniformly valid in the
region $\xi = O(1/\delta)$, as the outer one is not valid where $x = O(\delta)$. Because of these restrictions on the size of $x$, we
will only be able to apply a subset of the boundary conditions to each expansion; in general we will, therefore, be left
with unknown coefficients in our asymptotic approximations. This ambiguity can be lifted by matching the inner and
outer solutions in an intermediate region where they are both valid.
C. Matching

We can match the inner and outer solutions together if there exists some intermediate range of $x$ where both (11) and (10) are valid asymptotic approximations to $f(x, \delta)$. By the uniqueness properties of asymptotic expansions, see [37] for a proof, if they are both valid in some region then the two expansions must be equal. We take $x$ to scale as some intermediate function, $\eta(\delta)$, with magnitude between 1 and $\delta$, e.g. $\eta(\delta) = \delta^\alpha$ where $0 < \alpha < 1$, and define a new $X$ coordinate appropriate for the intermediate region by

$$X = x/\eta(\delta) = \left(\frac{\delta}{\eta(\delta)}\right) \xi.$$ 

We then write both the inner and outer approximations in terms of $X$ and, keeping $X$ fixed, make an asymptotic expansion of each of them in the limit $\delta \to 0$; this is called the intermediate limit. We must be careful to neglect any terms in the approximations, that we find in this way, which would be smaller than the first excluded term; equality is then demanded between the remaining terms:

$$\sum_{n=0}^{Q} f_n(x(X, \eta)) \delta_n \sim \sum_{n=0}^{S} h_n^{\text{out}}(X) \gamma_n(\delta) \quad \text{as } \delta \to 0; \quad X \text{ fixed,}$$

$$\sum_{n=0}^{P} g_n(x(X, \eta)) \delta_n \sim \sum_{n=0}^{S} h_n^{\text{in}}(X) \gamma_n(\delta) \quad \text{as } \delta \to 0; \quad X \text{ fixed,}$$

$$h_n^{\text{out}}(X) = h_n^{\text{in}}(X) \quad \text{for } 0 \leq n \leq S. \quad (12)$$

Usually, the precise form we choose for $\eta(\delta)$ is unimportant and the matching can be done regardless; there are some cases, however, where we must be more careful (see Hinch [37] for more).

D. Simple Examples

1. Example 1: Simple matching

We will start with a boundary layer example where the exact solution is known. Consider $y(x, \delta)$ which satisfies:

$$2\delta y'' + (1 + 2\delta) y' + y = 0,$$

in $0 \leq x \leq 1$, with boundary conditions $y = 0$ at $x = 0$ and $y = e^{-1}$ at $x = 1$ and where $' = d/dx$. The exact solution is:

$$y(x, \delta) = e^{-1} \left( \frac{e^{-x} - e^{-x/2\delta}}{e^{-1} - e^{-1/2\delta}} \right). \quad (13)$$

If we apply the method of matched asymptotic expansions to this, we would find that the outer approximation is given (to all orders) by:

$$y(x, \delta) \sim e^{-x} \neq 0 \text{ fixed.}$$

We have been able to apply the boundary condition at $x = 1$; the $x = 0$ boundary condition cannot be reached given our assumptions about the size of $x$. For the inner approximation we keep $\xi = x/\delta$ fixed. In terms of $\xi$ our differential equation reads:

$$2y,\xi + (1 + 2\delta)y,\xi + \delta y = 0.$$ 

The inner solution is found to be:

$$y(x, \delta) \sim a_0 \left( 1 - e^{-\xi/2} \right) + \delta \left( a_1 \left( 1 - e^{-\xi/2} \right) - a_0 \xi \right) + \delta^2 \left( a_2 \left( 1 - e^{-\xi/2} \right) - a_1 \xi + \frac{1}{2} a_0 \xi^2 \right) + \mathcal{O} \left( \delta^3 \right),$$

with $\xi = x/\delta$ held fixed. Again we can only apply one boundary condition, this time the one at $x = 0$. The boundary condition at $x = 1$ is beyond our reach in this approximation. In this problem the choice of the intermediate scale turns out to be unimportant; we take $\eta = \delta^{1/2}$ and define $X = \delta^{-1/2} x = \delta^{1/2} \xi$. The intermediate limit of the inner approximation is:

inner approx. $\sim a_0 - \delta^{1/2} a_0 X + \delta \left( a_1 + \frac{1}{2} a_0 X^2 \right) + \mathcal{O} \left( \delta^{3/2} \right).$
The first excluded term is order $\delta^{3/2}$; we have neglected all terms of, or below, this order including the exponentially small $e^{-X/2\delta^{3/2}}$ terms. Expanding the outer approximation we have:

$$\text{outer approx. } \sim 1 - \frac{1}{2}X + \frac{1}{4}X^2 + \mathcal{O}\left(\delta^{3/2}\right).$$

By our matching criteria, eq. 12, we must have:

$$a_0 = 1,$$

$$a_1 = 0.$$ 

If we were to perform this process to all orders we would find that all $a_i = 0$ for $i \geq 1$. The fully specified inner approximation is therefore

$$y(x, \delta) \sim -e^{-\xi/2} + \sum_{n=0}^{\infty} \delta^n (-1)^n \xi^n.$$ 

This is precisely what we would have found by performing a Taylor series expansion of the exact solution, (13), in the inner limit and then dropped all exponentially small terms i.e. $e^{-1/8}$. By performing the matching we have been able to lift the ambiguity in the coefficients, $a_i$, and fully specify the inner approximation.

2. Example 2: Scalar field in an expanding universe

Consider a spherically-symmetric, scalar field, $\phi(r, t)$, like the dilaton, in a flat FRW cosmology with metric

$$ds^2 = dt^2 - a^2(t) \left(dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right).$$ 

To make contact with our problem we will assume that the only source term in the $\phi$ evolution equation is homogeneous and proportional to the background dust density, so

$$\Box \phi = \frac{\left(a^3(t)\phi\right)'}{a^3(t)} - \frac{(r\phi)''}{a^2(t)r} = \frac{\lambda H_0^2}{a^3(t)}$$

where $' = \partial/\partial t$ and $' = \partial/\partial r$ and with $H_0$ the Hubble parameter at some arbitrary time $t = t_0$ and $\lambda$ is a constant parameter of the theory. We simplify the problem further by considering only the matter era, where $a(t) \propto t^{2/3}$. As boundary conditions we take $\phi \rightarrow \phi_0(t)$ as $r \rightarrow \infty$ and $\phi'/a(t) |_{r=r_0/a(t)} = \mu 2m/r_0^2 = \text{const}$; $\mu$ is a dimensionless

FIG. 1: The leading order, inner and outer approximations to the solution of problem given in section III D 1, with $\delta = 0.00001$. The exact solution is not visible on this plot since it lies underneath one or other of the approximations everywhere.
constant, and \( m \) is a length scale such that \( \delta = 2mH_0 \ll 1 \). We assume that \( r_0/a(t_0) \sim \mathcal{O}(2m) \). We could visualise this problem arising from the presence of a massive body in the region \( a(t)r < r_0 \) that creates a spatial gradient in \( \phi \). Near \( a(t)r = r_0 \) we have \( \phi' \gg \phi \). We take our problem to be similar to one we will consider later and determine the relation between \( \phi_0 \) and \( \phi(r_0/a(t), t) \).

The small parameter in this problem is \( \delta = 2mH_0 \). We work at the epoch when \( t = t_0 \); for simplicity we take \( a(t_0) = 1 \). In the inner approximation we define \( 2m\xi = a(t)r \), and \( \xi_0 := r_0/2m \sim \mathcal{O}(1) \). We assume that the inner solution is quasi-static, so it depends on time only through the slowly increasing cosmological \( t \), rather than \( (t - t_0)/2m \). We define a dimensionless time coordinate by \( \tau = H_0t \), \( \tau_0 = H_0t_0 = 2/3 \). In this interior region \( \phi = \phi_I(\xi, \tau; \delta) \) which satisfies

\[
\frac{1}{\xi} \partial^2_{\xi} (\xi \phi_I) = \delta^2 \left[ \frac{1}{\tau^2} \partial_{\tau} \left( \tau^2 \partial_{\tau} \phi_I \right) - \frac{4\lambda}{9\tau^2} + \frac{2}{3\tau^2} \xi \partial_{\xi} \phi_I + \frac{4}{9\tau^2} (\xi \partial_{\xi})^2 \phi_I + \frac{4}{3\tau^2} \xi \partial_{\xi} \partial_{\tau} \phi_I \right].
\]

Solving this to order \( \mathcal{O}(\delta^2) \) we find:

\[
\phi_I(\tau, \rho) \sim \phi_0^\prime(\tau) - \frac{\mu}{\xi} + \delta^2 \left( \phi_0^\prime(\tau) + \frac{\mu}{9\tau^2} \left( \xi + \frac{\xi_0^2}{\xi} \right) + \frac{g(\tau)}{4\xi^3} \right) + \mathcal{O}(\delta^3)
\]

where \( \phi_0^\prime(\tau) \) and \( \phi_0^\prime(\tau) \) are ‘constants’ of integration, to be determined via the matching procedure. \( g(\tau) = \frac{\partial}{\partial \tau} \left( \tau^2 \partial_{\tau} \phi_0^\prime \right) - \frac{4\lambda}{9\tau^2} \). In the exterior we define \( \rho = H_0r \) to be our dimensionless radial coordinate, and \( \phi = \phi_E \). Our boundary conditions determine the leading-order exterior term to be \( \phi_0(\tau) = \phi_0(t) \) with some abuse of notation. The sub-leading order terms, \( \delta^i \phi_E^{(i)} \) are then given by \( \Box \phi_E^{(i)} = 0 \). So, to order \( \delta \),

\[
\phi_E(\tau, \rho) \sim \phi_0(\tau) + \delta \int_{\infty}^{\infty} \frac{d\gamma T_\gamma(\alpha)}{\rho} + \mathcal{O}(\delta^2)
\]

where

\[
\begin{align*}
X_\gamma(\rho) &= A(\beta) \cos(\beta \rho) + B(\beta) \sin(\beta \rho) \quad \text{where} \quad \gamma = -\beta^2 < 0, \\
X_\gamma(\rho) &= A(0) \quad \text{where} \quad \gamma = 0, \\
X_\gamma(\rho) &= C(\alpha)e^{-\alpha \rho} \quad \text{where} \quad \gamma = \alpha^2 > 0,
\end{align*}
\]

\[
\tau^2 T_\gamma(\tau),_{\tau\tau} + 2\tau T_\gamma(\tau),_{\tau} = \gamma \tau^{2/3} T_\gamma(\tau),
\]

and \( A(\beta), B(\beta), C(\alpha) \) are all to be determined by the matching procedure. As in the previous example, the precise position of the intermediate region is not important. We choose an intermediate coordinate \( \xi = \delta^{-1/2} \rho = \delta^{1/2} \xi \) and take the intermediate limit of both the interior and exterior approximations. By equating our two approximations in the intermediate region we find \( B(\beta) = C(\beta) = 0 \) from the \( \mathcal{O}(\delta) \) terms and

\[
\begin{align*}
&\mathcal{O}(1): \quad \phi_0^\prime(\tau) = \phi_0(\tau) \rightarrow g(\tau) = 0, \\
&\mathcal{O}(\delta^{1/2}): \quad \int_{0}^{\infty} d\beta A(\beta) T_{-\beta^2}(\tau) = \frac{\mu}{\tau^{2/3}}, \\
&\mathcal{O}(\delta^{3/2}): \quad \int_{0}^{\infty} d\beta \beta^2 A(\beta) T_{-\beta^2}(\tau) = \frac{2\mu}{9\tau^{4/3}}.
\end{align*}
\]

By differentiating twice and applying (14) we can check that conditions (17) and (18) can be simultaneous satisfied for some choice of \( A(\beta) \). The \( T_\gamma(\tau) \) can be made orthonormal w.r.t. to some inner product and so eq. (17) can, in principle, be inverted to find \( A(\beta) \). We omit this step, however, since we are mostly concerned with the effect of the exterior on the behaviour of \( \phi \) in the interior rather than vice versa. By finding the interior solution to \( \mathcal{O}(\delta^4) \) and the exterior to \( \mathcal{O}(\delta^2) \) we can show that \( \phi_0^\prime(\tau) = 0 \); we have now fully specified the interior solution to \( \mathcal{O}(\delta^2) \):

\[
\phi_I(t, R = a(t)r) \sim \phi_0(t) - \frac{2m\mu}{R} + \frac{4m\mu}{9R^2} \left( R + \frac{r_0^2}{R} \right) + \mathcal{O}(\delta^4)
\]

We have that:

\[
\frac{\dot{\phi}_I|_{R=r_0}}{\phi_0} = 1 \approx \frac{16m\mu r_0}{9R^3}.
\]
and $\dot{\phi}_0 \sim 4\lambda/9t$ for large $t$. Therefore we have shown that for the time variation of $\phi$ at $ar = r_0$ to track the cosmological time variation we need

$$\lambda \gg \frac{2mr_0}{2} = O(\delta^2)$$

It is clear that whatever the value of $\lambda$, the rate of time variation in $\phi$ will tend to homogeneity as $t \rightarrow \infty$. This example is a greatly simplified version of the problem we will consider in the rest of this paper.

E. Application to General Relativity

The use of matched asymptotic expansions in general relativity was pioneered by Burke and Thorne [38], Burke [39], and D’Eath [40, 41] in the 1970s. These authors used them to the study how the laws of motion of a test body were affected by the background spacetime. We shall now outline how matched asymptotic expansions are applied in general relativity.

We assume that the universe

$$C = (\mathcal{M}, g_{ab}, T^{ab}, \phi),$$

with $T_{ab}$ is the energy-momentum tensor and $\phi$ the dilaton, can be viewed as a background cosmology,

$$C_0 = (\mathcal{M}_0, g_{ab}^0, T_{0}^{ab}, \phi_0),$$

onto which some localised, interior configuration has been superimposed in a non-linear fashion. In what follows, for simplicity, we will require the interior configuration to be static in some coordinate system. We demand that the ‘size’ of the interior region be given by a single parameter $m$; and that as $m \rightarrow 0$, with $\{x^\mu\}$ fixed, the interior region disappears and $C \rightarrow C_0$. We shall formalise this statement later.

1. Length scales

The length scale of the interior region, as defined by its curvature invariants is denoted by $L_I(m)$, with $L_I(m) \rightarrow 0$ as $m \rightarrow 0$. The length scale of the background (exterior) region is written $L_E$. For the asymptotic expansion method to be viable we need $\delta = L_I/L_E \ll 1$. The effect of the interior on the background configuration can then be treated as a linear perturbation to $C_0$, with $\delta$ playing the role of a small parameter. We can similarly treat the effect of the background on the interior as a linear perturbation.

2. Five-dimensional manifold

For each $m$, in some interval $[0, m_{\text{max}})$, we write the global configuration as $C_m = (\mathcal{M}_m, g_{ab}(m), T^{ab}(m), \phi(m))$. Following Geroch [42] and D’Eath [40], we consider a five-dimensional manifold with boundary, $\mathcal{N}$, that is built up from spacetimes $(\mathcal{M}_m, g_{ab}(m))$ for $m \in [0, m_{\text{max}})$; $g_{ab}(0) = g_{ab}^{(0)}$. As D’Eath noted we should properly exclude from $\mathcal{M}_0$ a smooth time-like world line, $l_0$, that corresponds to the ‘position’ of the interior region; this is illustrated in figure 2. We require that the contravariant metrics $g^{ab}(m)$ define a smooth tensor field on $\mathcal{N}$; in addition, we require that the dilaton, $\phi(m)$, defines a smooth scalar field on $\mathcal{N}$. These conditions are required by our assumption that the interior region has only a small perturbing effect over length scales $\gg L_I(m)$.

In an open subset on $\mathcal{N}$, we choose a coordinate chart $(t, r, \theta, \phi, m)$ such that the aforementioned world line $l_0$ is given by $(r = 0, m = 0)$. In the limit $(tL_E^{-1}, rL_E^{-1}, \theta, \phi, m) \rightarrow \text{constants}$ and $m \rightarrow 0$ we can give $g_{ab}(m)$, $T^{ab}(m)$ and $\phi(m)$ as asymptotic expansions about $g_{ab}^{(0)}$, $T_{(0)}^{ab}$ and $\phi_0$, respectively. This is the exterior approximation, and is appropriate for considering the perturbing effect that the interior has on the background universe.

For each epoch $t = t_0$, we shall also define an interior approximation. For this, we take the asymptotic expansions of $g_{ab}(m)$, $T^{ab}(m)$ and $\phi(m)$ in the limit $((t - t_0)L_I^{-1}(m), rL_I^{-1}(m), \theta, \phi) \rightarrow \text{constants}$, and $m \rightarrow 0$. This approximation is appropriate for considering the perturbations produced in the interior region by the background cosmology. This is the problem that we are most interested in. We shall assume that we know the leading-order configuration in this limit and in what follows we take it to be a Schwarzschild metric with a quasi-static dilaton field.
FIG. 2: The 5-D manifold $\mathcal{N}$ is built up from spacetimes $(\mathcal{M}_m, g_{ab}(m))$ for $m \in [0, m_{\text{max}})$.

3. Intermediate region and matching

We match the interior and exterior approximations in some intermediate region, where $(t - t_0)L_{\text{int}}^{-1}, r L_{\text{int}}^{-1}) \to \text{const}$ as $m \to 0$, and where $L_{\text{int}} = L E^{\delta^{1-\alpha}} = L_1 \delta^{-\alpha}$, with $0 < \alpha < 1$. In many cases the precise value of $\alpha$ is not important (see section III C). Following D’Eath, [40], we assume that all the functions in our approximations are sufficiently well-behaved in this region so as to admit a power-series expansion in the radial coordinate; given this we shall not need to examine this region explicitly. A typical term in the interior expansion will, in this matched region, look like $\delta^i \xi^j f_{ij}(t_0; L_{\text{int}}^{-1}(t - t_0))$ where $\xi = r L_{\text{int}}^{-1}$ and, $i$ and $j$ are rational numbers; $i \geq 0$. In the exterior region, a typical term will look like $\delta^k \rho^l g_{kl}(t_0; L_{\text{int}}^{-1}(t - t_0))$, again with $k$ and $l$ rational and $k \geq 0$ and $\rho = L E^{1} r$. Matching will require that $g_{(k+1)l} = f_{ij}$. When we consider the Tolman-Bondi class of background metrics we will modify this procedure slightly. For these metrics there are two choices for radial coordinate:

- the physical radial coordinate $R$ defined by surfaces of $(t, R) = \text{const}$ with surface area $4\pi R^2$,
- the radial coordinate which is constant on the world-lines of dust particles, $r$. There is an ambiguity in this definition of $r$ in that the defining property is also satisfied by any arbitrary function $h(r)$ of $r$. We lift this ambiguity by demanding that $R = r$ at $t = t_0$. At later times the Einstein equations give us $R = R(r, t)$.

The functions in our interior expansion will generally be quasi-static when written as functions of $R$ and $r$. We will choose to expand the interior and exterior approximations as power series in the physical radius, $R$, when taking the intermediate limit.

IV. GEOMETRICAL SET-UP

A. General Picture

We shall assume that the dilaton field is only weakly coupled to gravity, and so its energy density has a negligible effect on the background spacetime geometry. If we ignore this back-reaction then we can, rather than solving the full set of Einstein-matter-dilaton equations, simply consider the dilaton field’s evolution on a fixed background with given matter density. We will take the background to be some known exact solution to Einstein’s equations with matter possessing the following properties:

- The metric is approximately Schwarzschild, with mass $m$, inside some closed region of spacetime outside a surface at $r = R_s$. The metric for $r < R_s$ is left unspecified.
- Asymptotically, the metric must approach FRW and the whole spacetime should tend to pure FRW in the limit $m \to 0$. 

As the local inhomogeneous energy density of asymptotically FRW spacetimes tends to zero, the spacetime metric exterior to \( r = R_s \) must tend to a Schwarzschild metric with mass \( m \).

This list of requirements is far from exhaustive, and they shall be re-expressed in more rigorous fashion later. In the local, approximately Schwarzschild, region the intrinsic length scale, \( L_I \), of a sphere centred on the Schwarzschild mass with surface area \( 4\pi R_s^2 \), is given by considering the curvature invariant:

\[
L_I = \left( \frac{1}{12} R_{abcd} R^{abcd} \right)^{-1/4} = \frac{R_s^{3/2}}{(2m)^{1/2}}.
\]  

(19)

In the asymptotically FRW region, the intrinsic length scale is proportional to the inverse root of the local energy density: \( 1/\sqrt{\kappa\epsilon + \Lambda} \). We shall assume that, in line with current observations, that the FRW region is approximately flat, and so we set our exterior length scale appropriate for the epoch at \( t = t_0 \) is the inverse Hubble parameter at that time:

\[
L_E = 1/H_0.
\]

Our small parameter is then defined to be

\[
\delta = L_I / L_E.
\]

Formally we require that, as \( \delta \to 0 \), our choice of spacetime background should be FRW at zeroth-order in the exterior limit and Schwarzschild to lowest order in the interior limit. In section VI we will say more about what is required of the background for this matching procedure to be valid. In this paper we shall, in addition to the criteria given above, restrict ourselves to the subcase where the spacetime is spherically symmetric. All spherically-symmetric solutions to Einstein’s equations with matter, where the matter is pressureless dust and a cosmological constant and the flow lines of the matter particles are geodesic, fall into the Tolman-Bondi class of metrics (for a review of these and other inhomogeneous spherically symmetric metrics see ref [43]). This class of possible Tolman-Bondi spacetimes is parametrised by two arbitrary functions of one spatial variable, \( r \). We will only look at the flat, Tolman-Bondi models with non-simultaneous initial singularities (which where rediscovered by Gautreau in [43, 44]), and the non-flat Tolman-Bondi models with simultaneous initial singularities. These two subcases are fully specified by prescribing only a single function: the matter density on some initial space-like hypersurface. Another metric that has often been used to study the effect of the universe’s expansion on solar system dynamics is the McVittie metric [43, 45]. In the exterior limit the McVittie metric asymptotes to a dust + \( \Lambda \) FRW cosmology. For radii \( r \) where \( r \geq 2m, r \ll H_0^{-1} \), the McVittie metric looks like Schwarzschild spacetime; the horizon in the McVittie metric possesses a curvature singularity however, and it cannot therefore be used to model a black hole in an expanding spacetime.
B. Case I: The McVittie background

The earliest studies of the gravitational field produced by a spherically-symmetric, massive body in an expanding universe were based upon McVittie’s solution to Einstein’s equations [43, 45]. McVittie’s solution is a superposition of the Schwarzschild metric and a FRW background. In isotropic coordinates the metric, is

\[ ds^2 = \left[ 1 - \frac{\mu(t,r)}{1 + \mu(t,r)} \right]^2 dt^2 - \left[ 1 + \frac{\mu(t,r)}{1 + \mu(t,r)} \right]^4 \frac{dr^2}{(1 + \frac{1}{4} kr^2)^2} - r^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right], \]  

(20)

where

\[ \mu(t,r) = \frac{m}{2\alpha(t)} (1 + \frac{1}{4} kr^2)^{1/2}, \]  

(21)

and \( m \) and \( k \) are arbitrary constants. In the limit \( \alpha = 1 \), \( m \) is the Schwarzschild mass. \( k \) is the curvature of the surfaces \((t,r) = \text{const}\) in the \( m = 0 \) (i.e. FRW) limit. As a model of a black hole in an expanding universe it has the distinct disadvantage that the ‘horizon’ is a naked curvature singularity. This defect aside, the McVittie metric is believed to be a good approximation to the exterior metric of a massive spherical body with a physical radius much larger than its Schwarzschild radius.

In the flat \((k = 0)\) case, the local energy density depends only on time and is the same as the FRW energy density to which the solution matches smoothly as \( r \to \infty \). The pressure, however, is not the same as the FRW pressure. Apart from the vacuum-energy dominated case (i.e. where \( c = -P \)), it is not possible to prescribe a dust, \( P = 0 \), or any other, non-vacuum, barotropic, \( P = P(\rho) \), equation of state to hold apart from in the asymptotic FRW limit.

1. Exterior Metric

We shall work in the \((r,t)\) coordinates introduced above. In the exterior \( dr \sim H_0^{-1} \sim dt \). We define dimensionless exterior coordinates \( \tau = H_0 t \) and \( \rho = H_0 \). Then, with \( \delta = L_I/L_E = H_0 Rs^{3/2}/(2m)^{1/2} \ll 1 \) we have:

\[ ds^2_{\text{ext}} \sim H_0^{-2} \left\{ g_{ab}^{(0)} + \delta g_{ab}^{(1)} + O\left(\delta^2\right) \right\} dx^a dx^b \]

(22)

\[ = H_0^{-2} \left[ \frac{1}{1 - \delta \mu(\tau,\rho)} \right]^2 dt^2 - \left[ \frac{1}{1 + \delta \mu(\tau,\rho)} \right]^4 \left[ d\rho^2 + \rho^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] \right], \]

where \( \mu(\tau,\rho) = \frac{1}{4} (2m/Rs)^{3/2} (\rho a(\tau))^{-1} \left[ 1 + \frac{1}{4} \Omega_k \rho^2 \right] \), \( a(\tau) := a(t) \) and \( \Omega_k = k H_0^2 \).

2. Interior Metric

In the interior it is most convenient to work with

\[ R = [1 + \mu] \left( 1 + \frac{1}{4} kr^2 \right)^{-1/2} a(t) r \]

as the radial coordinate. If \( k = 0 \), then the surface \((t,R) = \text{const}\) has area \( 4\pi R^2 \). When \( t = t_0 \) the interior geometry will be approximately Schwarzschild over scales where \( dR \sim O(R_s) \) and \( d(t-t_0) \sim O(L_I) \). We define new dimensionless coordinates by

\[ T = (t-t_0)/L_I \quad \text{and} \quad \xi = R/R_s. \]

In these coordinates the interior metric is:

\[ ds^2_{\text{int}} \sim R_0^2 \left\{ g_{ab}^{(0)}(\xi) + \delta g_{ab}^{(1)}(\delta T, \xi) + O\left(\delta^2\right) \right\} dx^a dx^b \]

(23)

\[ = R_0^2 \left\{ A(\xi) \left( \frac{R_s}{2m} \right) dT^2 - \left( 1 - \frac{\delta^2 \Omega_k(\delta T) 2m R_s X^2(\xi)}{4} \right)^{-1} A(\xi)^{-1} \left[ \psi^2 + \xi^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] \right] \right\}, \]
where \( A(\xi) = 1 - 2m/(R_s \xi) \),

\[
\psi = d\xi - \delta h(\delta T)\xi \left( \frac{R_s}{2m} \right)^{1/2} A(\xi)^{1/2} dT,
\]

and \( h(\delta T) = \tilde{a}'(\tau = \delta T)/\tilde{a}(\tau) \), \( \Omega_k(\delta T) = \Omega_k^0 \tilde{a}^{-2}(\tau) \). We have defined \( X(\xi) = \frac{A}{2} (1 + A(\xi)^{1/2})^2 \). Note that the interior metric functions \( j^0_{ab}, j^1_{ab} \) etc. can, at each order, be written in quasi-static form, as functions only of \( \xi \) and \( \delta T \).

3. Energy-Momentum Tensor

The dilaton field theories we are considering couple to the matter sector through the trace of the energy momentum tensor, \( \kappa T^\mu_\mu = \kappa(\epsilon - 3P) \). In the exterior expansion:

\[
H_0^{-2} \kappa T^\mu_\mu = \varepsilon_0(\tau) + \delta\varepsilon_1(\tau, \rho) + \mathcal{O}(\delta^2) \tag{24}
\]

\[
= 12h^2(\tau) + 6h'(\tau) \left( \frac{1 + \delta\mu(\tau, \rho)}{1 - \delta\mu(\tau, \rho)} \right) + 3\Omega_k(\tau) \left( \frac{2 - \delta\mu(\tau, \rho)}{1 - \delta\mu(\tau, \rho)} \right) (1 + \delta\mu(\tau, \rho))^{-5} - 4\Omega_k^0
\]

In the interior expansion we have:

\[
R_s^2 \kappa T^\mu_\mu = \delta^2\varepsilon_0(\delta T, \xi) \tag{25}
\]

\[
= 12\delta^2 \left( \frac{2m}{R_s} \right) h^2(\delta T) + 6\delta^2 \left( \frac{2m}{R_s} \right) h'(\delta T)A(\xi)^{-1/2}
\]

\[
+ 6\delta^2 \left( \frac{2m}{R_s} \right) \Omega_k(\delta T) \left( 3 + A^{-1/2}(\xi) \right) \left( \frac{1 + A^{1/2}(\xi)}{2} \right)^5 - \delta^2 A \left( \frac{2m}{R_s} \right) \Omega_k^0
\]

When rewritten in quasi-static form, the only term in the interior expansion of \( T \) is at \( \mathcal{O}(\delta^2) \).

C. Case II: The flat Gautreau-Tolman-Bondi background

Gautreau considered the metric outside a massive body embedded in a universe containing inhomogeneous, spherically symmetric dust and cosmological constant, \( \Lambda \). Unlike the McVittie solution, the dust is in this case pressureless everywhere. The Gautreau solution is the flat limit of the Tolman-Bondi model written in curvature coordinates. It is also the \( \kappa P = -\Lambda \) limit of the Leibovitz [46], solutions. Although not given here, we expect a treatment of dilaton evolution in asymptotically FRW, locally Schwarzschild, Leibovitz backgrounds to proceed along similar lines to this case. In co-moving coordinates, which are the most appropriate for considering the exterior expansion, the metric can be written as

\[
ds^2 = dt^2 - R^2_{,r}(t, r)dr^2 - R^2(t, r)\{d\theta^2 + \sin^2 \theta d\phi^2\}, \tag{26}
\]

where

\[
R^2_{,r} = \frac{2m + 2Z(r)}{R} + \frac{1}{3}\Lambda R^2. \tag{27}
\]

The energy density of the dust is given by:

\[
\kappa \epsilon = 2Z,/(R^2 R_{,r}). \tag{28}
\]

We use the remaining freedom we have in the definition of \( R \) to prescribe that at some epoch of interest, \( t = t_0 \), \( r \) is the physical radial coordinate, so \( R(t_0, r) = r \) and the surface \( (t = t_0, r = \text{const}) \) has area \( 4\pi r^2 \). The surface of our spherical massive body is at \( R = R_s \). The quantity \( m + Z(r) \) is then defined to be the active gravitational mass contained inside the shell \( (t = t_0, r = \text{const}) \) and \( m \) is the active gravitational mass of the central massive body at \( t = t_0 \). In general, the mass of the central object will change over time due to accretion of external material. We
require that as \( t \to \infty \) the central mass remains strictly positive. The dust density must also be everywhere positive and tend to homogeneity for large \( r \). Therefore we have the conditions:

\[
\begin{align*}
Z(R_s) &= 0 \\
Z_r &\geq 0 \\
\lim_{r \to \infty} Z(r) &= \frac{1}{2} \Omega_{\text{dust}}^0 H_0^2 r^3 \\
\lim_{r \to -\infty} Z(r) &> -m
\end{align*}
\]

The last requirement is only strictly necessary if we wish the interior to have the required properties for \( t \gg t_0 \). All we actually require for this analysis to hold is that the interior remains Schwarzschild to leading order over the time-scales appropriate to our intermediate matching region. Under our assumptions, equation (27) has exact solution:

\[
\begin{align*}
R(t, r) &= \left( \frac{6(m + Z)}{\Lambda} \sinh^2 \left( \frac{\sqrt{3} \Lambda}{2} (t - t_1(r)) \right) \right)^{1/3} \\
t_1(r) &= t_0 - \frac{2}{\sqrt{3} \Lambda} \sinh^{-1} \left( \sqrt{\frac{r^3 \Lambda}{6(m + Z)}} \right).
\end{align*}
\]

1. Exterior Metric

As in the McVittie case, we define dimensionless coordinates \( \rho = H_0 r \) and \( \tau = H_0 t \). With respect to these coordinates we can then asymptotically expand \( Z(r) \), order by order, in the small parameter \( \delta \). First, we write:

\[ H_0 Z(r) \sim \frac{1}{2} \Omega_{\text{dust}}^0 r^3 + \delta^p z_1(\rho) + o(\delta^p). \]

Next, we require \( 2 \delta^p z_1(\rho)/\Omega_{\text{dust}}^0 r^3 \ll 1 \) for \( \rho \sim \mathcal{O}(1) \) so that this is valid asymptotic expansion; this ensures \( p > 0 \). The unperturbed spacetime is then FRW. Within the framework of a given model it will be possible to find the value of \( p \). Given the expansion of \( Z \) we can then expand \( R \) order-by-order using equation (33). Putting this expansion back into the metric, eq. (26), we will have the expanded exterior metric in the form:

\[
\begin{align*}
\text{d}s^2_{\text{ext}} &\sim H_0^{-2} \left( g^{(0)}_{ab}(\tau, \rho) + \delta^p g^{(1)}_{ab}(\tau, \rho) + o(\delta^p) \right) \text{d}x^a \text{d}x^b,
\end{align*}
\]

where \( g^{(0)}_{ab} \) is the FRW metric.

2. Interior Metric

We define \( T = (t - t_0)/L_1 \), and \( \xi = R/R_s \) to be our coordinates in the interior. We express \( Z(r) = Z(\xi, T) \) and expand out in power of the small parameter \( \delta \):

\[ Z(r)/m \sim \delta^q \mu_1(\chi) + o(\delta^q). \]

Where, from eq. (33), \( \chi = (\xi^{3/2} - 3T/2)^{2/3}; \ R_s \chi = r + \mathcal{O}(\delta^q) \). Putting this expansion back into the metric, eq. (26) gives the first two terms in the asymptotic expansion of the interior metric:

\[
\begin{align*}
\text{d}s^2_{\text{int}} &\sim R_s^2 \left( j^{(0)}_{ab}(\xi, \chi) + \delta^q j^{(1)}_{ab}(\xi, \chi) + o(\delta^q) \right) \text{d}x^a \text{d}x^b,
\end{align*}
\]

where \( j^{(0)}_{ab} \) is the Schwarzschild metric with mass \( m/R_s \) in Painlevé-Gullstrand coordinates (with a re-scaling of the time coordinate):

\[ j^{(0)}_{ab} \text{d}x^a \text{d}x^b = \frac{R_s}{2m} \text{d}T^2 - \left( \text{d}\xi - \xi^{-1/2} \text{d}T \right)^2 - \xi^2 \{ \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \}. \]
and \( j_{ab}^{(1)} \) is given by:

\[
j_{ab}^{(1)} \, dx^a \, dx^b = \frac{\mu_1(\chi)}{\xi^{1/2}} \, d\xi \, dT - \frac{\mu_1(\chi)}{\xi} \, dT^2.
\]

Unlike in the McVittie solution, the next-to-leading order term in the interior metric here is not quasi-static, however by defining \( \dot{T} = \sqrt{R_s/2mT} \) to be the time coordinate, the metric can be seen to be quasi-static in a different sense. Instead of only depending on \( T \) through \( \delta T \), it depends on \( \dot{T} \) only through the combination \( 2m\dot{T}/R_s \), and for most bodies, with the exception of black holes and neutron stars, \( 2m/R_s \ll 1 \). For eq. (35) to be a valid asymptotic expansion we require \( \delta T \mu_1(\chi) \ll 1 \) for \( \chi \sim \mathcal{O}(1) \).

### 3. Energy-Momentum Tensor

The Gautreau model is a subcase of the Tolman-Bondi dust models and so the pressure vanishes, \( P = 0 \), and \( T_{\mu}^{\mu} = \epsilon \). In the interior it is given by:

\[
R_s^2 \kappa T_{\mu}^{\mu} = \delta^q \varepsilon^1(\xi, \chi) + O(\delta^q)
\]

\[
= \left( \frac{2m\delta^q}{R_s} \right) \frac{\mu_1(\chi)}{\xi^{3/2} \chi^{1/2}}
\]

An expression for the exterior expansion of \( T_{\mu}^{\mu} \) is not given here, but it can be found, with reference to the exact solutions for \( R(r, t) \), in the same way as the exterior expansion for the metric.

### D. Case III: Tolman-Bondi models with simultaneous big bang

The final class of specific model that we will consider in this paper is, like the Gautreau solution, also a subcase of the Tolman-Bondi models [43]. This class is distinct from the Gautreau models in that we demand the big-bang to be simultaneous for all observers, and we do not require the hyper-surfaces of constant \( t \) to be flat. In the limit of spatial homogeneity these models can therefore reproduce all dust-plus-\( \Lambda \) FRW models. In the Gautreau model \( R_t > 0 \) everywhere. This implies that the world lines of the matter particles will stream out of the surface \( R = R_s \); swept along by the cosmic expansion. In contrast, in this model we can, and will, require \( R_t < 0 \) near \( R = R_s \). As a result, dust particles will fall onto our massive body; this seems a more physical reasonable scenario than that of the Gautreau case. In co-moving coordinates the metric can be written as

\[
ds^2 = dt^2 - \frac{R_s^2(t, r)}{1 - k(r)} \, dr^2 - R^2(t, r) \{ d\theta^2 + \sin^2 \theta d\phi^2 \},
\]

where

\[
R_s^2 = -k(r) + \frac{2m + 2Z(r)}{R} + \frac{1}{3} \Lambda R^2.
\]

The matter content of these models is pressureless dust with a cosmological constant. The dust energy density is, as in the Gautreau models, given by equation (27). As in the previous example we use the freedom we have in the definition of \( R \) to prescribe that at some epoch of interest, \( t = t_0 \), \( R \) is the physical radial coordinate i.e. \( R(t_0, r) = r \) and the surface \( (t, R) = \text{const} \) always has area \( 4\pi R^2 \); this requirement, combined with the simultaneity of the initial big-bang curvature singularity, determines the form of \( k(r) \). The surface of our massive spherical body is at \( R = R_s \). As before, \( m + Z(r) \) is the active gravitational mass interior to the surface \( (t = t_0, r) = \text{const} \) and \( m \) is the active gravitational mass of the central body at \( t = t_0 \). The mass of the central object will grow over time as a result of accretion. We also require the dust density to be everywhere positive and tend to spatial homogeneity for large \( r \), hence we need

\[
Z(R_s) = 0,
\]

\[
Z_r \geq 0,
\]

\[
\lim_{r \to \infty} Z(r) = \frac{1}{2} \Omega_{\text{dust}}^{(0)} \dot{H}_s^2 r^3.
\]

We want the zeroth-order, exterior limit to be a FRW spacetime with curvature parameter \( k \); this requires \( \lim_{r \to \infty} k(r) = kr^2 \). We must require \( k(r) > 0 \) in the interior region; however we do not require \( k(r) > 0 \) everywhere. The exact solution for \( R(r, t) \) was found by Barrow and Stein-Schabes and is given in ref. [47].
1. Exterior Metric

As in the previous examples, we define dimensionless coordinates \( \rho = H_0 r \) and \( \tau = H_0 t \), and expand \( Z(r) \) order by order in the small parameter \( \delta \). We write:

\[
H_0 Z(r) \sim \frac{1}{2} \Omega_{dust}^{(0)} \rho^3 + \delta^p z_1(\rho) + o(\delta^p).
\]

We require \( 2\delta^p z_1(\rho)/\Omega_{dust}^{(0)} \rho^3 \ll 1 \) for \( \rho \sim \mathcal{O}(1) \) to ensure that this is valid asymptotic expansion; i.e. \( p > 0 \). The unperturbed spacetime is then FRW. The value of \( p \) is model dependent. With \( Z(r) \) specified, we can use the exact solutions of Barrow and Stein-Schabes, \([47, 48]\), for \( R(t, r) \), and the requirement that \( R(r, t_0) = r \) to find the expansion of \( k(r) \) and from there the expansion of the metric; schematically we have:

\[
k(r) \sim \Omega_k^{(0)} \rho^2 + \delta^p E_1(\rho) + o(\delta^p),
\]

\[
ds^2_{\text{ext}} \sim H_0^{-2} \left( g_{ab}^{(0)}(\tau, \rho) + \delta^p g_{ab}^{(1)}(\tau, \rho) + o(\delta^p) \right) dx^a dx^b,
\]

where \( g_{ab}^{(0)} \) is the FRW metric.

2. Interior Metric

We take \( T = (t - t_0)/L_1 \), and \( \xi = R/R_s \) to be our coordinates in the interior, and:

\[
Z(r)/m \sim \delta^9 \mu_1(\eta) + o(\delta^9),
\]

where \( \eta = (\xi^{3/2} + 3T/2)^{2/3}; R_s \eta = r + \mathcal{O}(\delta^9, \delta^{2/3}) \). From the exact solutions we find:

\[
k(\eta) = \delta^{2/3} k_0 (1 + \delta^9 \mu_1(\eta) + o(\delta^9)) + \mathcal{O}(\delta^{5/3}),
\]

where

\[
k_0(\delta T) = \frac{2m}{R_s} \left( \frac{\pi}{H_0 t_0 + \delta T} \right)^{2/3}.
\]

However, by a redefinition of the \( T \) coordinate we can transform away this \( \mathcal{O}(\delta^{2/3}) \) term. We take \( T \rightarrow T^* \) where:

\[
\sqrt{1 - \delta^{2/3}k_0}T^* = T + \int_{\xi}^{\xi'} \sqrt{\frac{2m}{R_s \xi^2}} \left( 1 - \frac{1 - (\delta^{2/3} \xi)^2}{1 - \frac{2m}{R_s \xi^2}} \right) d\xi'.
\]

To leading order, we find \( T \sim T^* \). The interior expansion of the metric is written:

\[
ds^2_{\text{int}} \sim R_s^2 \left( j_{ab}^{(0)}(\xi) + \delta^9 j_{ab}^{(1)}(\xi, \chi) + o(\delta^9) \right) dx^a dx^b + o(\delta^9),
\]

where \( j_{ab}^{(0)} \) and \( j_{ab}^{(1)} \) are given by:

\[
j_{ab}^{(0)} dx^a dx^b = \frac{R_s}{2m} dT^* - \left( d\xi + \xi^{1/2} dT^* \right)^2 - \xi^2 \{ d\theta^2 + \sin^2 \theta d\phi^2 \},
\]

\[
j_{ab}^{(1)} dx^a dx^b = -\frac{\mu_1(\chi)}{\xi^{1/2}} d\xi dT - \frac{\mu_1(\chi)}{\xi} dT^2.
\]

As with the Gautreau case, \( j_{ab}^{(0)} \) is the Schwarzschild metric in Painlevé-Gullstrand coordinates. The discussion about the quasi-static nature of the Gautreau \( j_{ab}^{(1)} \) term, given at the end of section IV C 2, applies equally well here. For the approximation above to be a valid asymptotic expansion we require \( \delta^9 \mu_1(\chi) \ll 1 \) for \( \chi \sim \mathcal{O}(1) \).
3. Energy - Momentum Tensor

The energy density of the dust locally is given by:

\[ R_s^2 \kappa T^\mu_\mu = \delta \epsilon_1^\prime (\xi, \chi) + o (\delta^\theta) \]
\[ = \left( \frac{2m \delta^\theta}{R_s} \right) \mu_1 (\chi) \chi \frac{\delta^\theta}{\sqrt{\chi}} + o (\delta^\theta). \]

We will not give the exterior expansion of \( T^\mu_\mu \) here explicitly, although it can in principle be found with reference to the exact solutions for \( R(r, t) \) if required.

E. Boundary Conditions

As the physical radius tends to infinity, \( R \rightarrow \infty \), we demand that the dilaton tends to its homogeneous cosmological value: \( \phi(R, t) \rightarrow \phi_c(t) \). We can apply this boundary condition in the exterior region but not in the interior. To solve the interior dilaton field equations we need to specify the dilaton-flux passing out from the surface of our ‘star’ at \( R = R_s \). At leading order we parametrise this by:

\[ -R_s^2 \left( 1 - \frac{2m}{R_s} \right) \partial_R \tilde{\phi}_0|_{R=R_s} = 2mF(\tilde{\phi}_0) = \int_0^{R_s} dR' R'^2 B_{,\phi} (\phi_0 (R')) \kappa \epsilon (R'), \]

where \( \tilde{\phi}_0 = \phi_0 (R = R_s) \). The function \( F(\phi) \) can be found by solving the dilaton field equations to leading order in the \( R < R_s \) region. If the interior region is a black-hole (\( R_s = 2m \)) then we must have \( F(\phi) = 0 \); otherwise we expect \( F(\phi) \sim B_{,\phi}(\phi) \). Without considering the sub-leading order dilaton evolution inside our ‘star’, i.e. at \( R < R_s \), we cannot rigorously specify any boundary conditions beyond leading order. Despite this, we can guess at a general boundary condition by perturbing eq. (45):

\[ -R_s^2 \left( 1 - \frac{2m}{R_s} \right) \partial_R \hat{\delta}(\phi)|_{\xi=R_s} = -\hat{\delta} (\sqrt{-g} g^{RR}) \partial_R \phi_0|_{R=R_s} + 2 \hat{\delta}(M) F(\tilde{\phi}_0) + 2m F_{,\phi}(\tilde{\phi}_0) \hat{\delta}(\tilde{\phi}_0) + \text{smaller terms}, \]

where \( \hat{\delta}(X) \) is the first sub-leading order term in the interior expansion of \( X \); \( M \) is the total mass contained inside \( \xi < R_s \) and is found by requiring the conservation of energy; at \( t = t_0 \) we have \( M = m \). Only \( \hat{\delta}(\tilde{\phi}_0) \) remains unknown, however we shall assume it to be the same order as \( \hat{\delta}(\phi) \) and see that this unknown term is usually suppressed by a factor of \( 2m/R_s \) relative to the other terms in eq. (46).

V. APPLICATIONS

A. Zeroth-order solutions

In accord with our prescription, all of the models that we have considered share the property that, to lowest order in \( \delta \), the interior is pure Schwarzschild, and the exterior is pure FRW.

1. Exterior

In the exterior, the dilaton field takes its cosmological value: \( \phi = \phi(t)_c \) to zeroth order and so satisfies the homogeneous conservation equation:

\[ \partial_t^2 \phi(t)_c + 3H \partial_t \phi(t)_c = B_{,\phi} (\phi_c) \kappa \epsilon_{\text{dust}} (t) - V_{,\phi} (\phi_c), \]

The effect of the interior region on the exterior should, even for finite \( \delta \), become increasingly smaller as \( r \rightarrow \infty \). As a result there will be no sub-leading order, \( r \)-independent, terms in the exterior expansion of \( \phi \). Equivalently, the homogeneous mode of \( \phi \) in the exterior will be given by the cosmological term, \( \phi_c \), alone to all orders
2. Interior

In the interior the dilaton field satisfies the wave equation on a Schwarzschild background at zeroth order in $\delta$; we take $\phi$ to be quasi-static to leading order. By applying the zeroth-order boundary condition, eq. (45), we find:

$$\phi_0 = \phi_c (\delta T) - F (\bar{\phi}_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right),$$

where $\phi_c (\delta T)$ is a constant of integration to be determined by the matching procedure.

3. Matching

As specified above, we should rewrite the exterior and interior expansion of $\phi$ in some intermediate region, with length scale, $L_I < L_{\text{int}} < L_E$. We define

$$L_{\text{int}} T = (t - t_0).$$

The only homogeneous mode in the exterior expansion was the cosmological term, $\phi_c (t) = \phi_{\text{hom}} (L_{\text{int}} T / L_E)$. This term will therefore appear at leading order in the intermediate expansion. Any other homogeneous terms that result from taking the intermediate limit of the exterior expansion must be sub-leading order. Therefore, we conclude that $\phi_{\text{hom}} = \phi_c (t)$ is the only leading order, homogeneous term in the intermediate expansion of $\phi$. In addition, all sub-leading-order homogeneous terms must also depend on time only $t$, and so will be quasi-static ($L_{\text{int}} T / L_E$ dependent) in the intermediate regions.

When the intermediate limit of the interior approximation is taken, it is clear that homogeneous terms in the interior will map to homogeneous terms in the intermediate zone. The matching criteria therefore implies that all homogeneous terms in the interior must be quasi-static, i.e. only depending on time through $\delta T$, and that at leading order:

$$\phi_c (\delta T) = \phi_c (L_{\text{int}} T / L_E) = \phi_{\text{hom}} (L_{\text{int}} T / L_E) = \phi_c (t)$$

The interior solution therefore reads:

$$\phi_0 = \phi_c (t) - F (\bar{\phi}_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right),$$

and $\bar{\phi}_0 = \phi_c (t) - F (\bar{\phi}_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right)$. We see directly the effect of the cosmological evolution of $\phi$ on the local region.

B. Case I: The McVittie Background

When the matched asymptotic expansion method is applied to the McVittie background, the analysis goes through in much the same way as it did in example 2 above. In the interior we find:

$$\phi_I = \phi_I^{(0)} + \delta^2 \frac{2m}{R_s} \phi_I^{(1)} + \mathcal{O} (\delta^4),$$

where $\phi_I^{(1)}$ satisfies:

$$\frac{1}{\xi^2} \partial_\xi \left( \xi^2 A(\xi) \partial_\xi \phi_I^{(1)} \right) = \frac{\phi_E'' (\delta T)}{A(\xi)} + h (\delta T) \left( \frac{3}{A^{1/2}(\xi)} + \frac{1}{2} \frac{2m}{R_s R_A^{3/2}(\xi)} \right) \phi_E' (\delta T) - \frac{2m}{R_s A^{1/2}(\xi) \xi} F (\bar{\phi}_0) + \frac{2m}{R_s A^{1/2}(\xi) \xi} \partial_\xi \left( \frac{\Omega_k (\delta T) X^2}{A(\xi)} \right) - \frac{h^2 \xi^2}{A(\xi)} F (\bar{\phi}_0)$$

$$- B_{,\phi} \left( \phi_I^{(0)} \right) \left[ 12 h^2 + 6 h' A(\xi)^{1/2} + 3 \Omega_k (\delta T) \left( \frac{1 + 3 A^{1/2}}{A^{1/2}} \right) \left( \frac{1 + A^{1/2}}{2} \right)^5 - 12 \Omega_A \right] + R_s^2 V_{,\phi} \left( \phi_I^{(0)} \right).$$
This can be integrated once the functions $B_\phi$ and $V_\phi$ are specified. If we also have
\[
\begin{align*}
\frac{B_{\phi\phi}(\phi_E)F(\bar{\phi}_0)^2}{B_\phi(\phi_E)} & \ll 1, \\
\frac{V_{\phi\phi}(\phi_E)F(\bar{\phi}_0)^2}{V_\phi(\phi_E)} & \ll 1,
\end{align*}
\]}
then we can solve for $\phi^{(1)}_I$ as an asymptotic series in $2m/R_s$ (provided this quantity is small). These relations will almost always hold provided that $2m/R_s \xi \ll 1$ and the cosmological value of $\phi$ does not lie near the minimum of $V$ or $B$. To $O(2m/R_s)$ we find:
\[
\begin{align*}
\phi^{(1)}_I & \sim \frac{2m\xi}{R_s} \left( \frac{1}{2} \left( \phi^{(0)}_E + \frac{h\phi^{(0)}_E}{h'} - h ' F(\bar{\phi}_0) \right) + \frac{1}{2} \Omega_k(\delta T)F(\bar{\phi}_0) - h^2F(\bar{\phi}_0) \\
& \quad \quad \quad + \frac{3}{2} B_\phi(\phi_E) \left( h'(\delta T) + \frac{1}{4} \Omega_k(\delta T) \right) - \frac{1}{2} F(\bar{\phi}_0) B_{\phi\phi}(\phi_E) R_s^2 \kappa_{dust}^{(0)} + F(\bar{\phi}_0) R_s^2 V_{\phi\phi}(\phi_E) \right) \\
& \quad \quad \quad + \frac{2m}{R_s} \left( C(\delta T) + D(\delta T) \right) + O \left( \frac{2m}{R_s} \right),
\end{align*}
\]
where $C(\delta T)$ is determined by the boundary conditions on $R = R_s$ and $D(\delta T)$ comes from the matching conditions.

The matching procedure would set $D(\delta T) = 0$ if we were to continue our asymptotic expansions to an order smaller than we do here. We do not expect $C(\delta T)$ to be any larger than the other terms in this expression when $\xi = 1$. The exterior expansion, to $O(\delta)$, along with the matching conditions that determine the otherwise unknown functions in it, is given in Appendix A. In this appendix, we show that these matching conditions are self-consistent, and hence that, to the order considered, the matching procedure works correctly. As with the simple example we considered in section III D 2, we see that when the matching conditions are applied, the background sets the form of the homogeneous terms in the interior solution, and the interior solution gives us the behaviour of the inhomogeneous terms in the exterior expansion. This is to be expected, since in these models the interior region is the sole source of the inhomogeneities in both the spacetime and the dilaton field. This picture would be more complicated if we were to consider more than one interior region, as in the case of a nested series of shells of matter of differing densities.

Our original concern was the behaviour of the time derivative of $\phi$ in the interior and we have shown that in the McVittie background:
\[
\frac{\partial_I \phi_I}{\partial_I \phi_c} \sim 1 - F_\phi(\bar{\phi}_0) \ln(1 - 2m/R) + O \left( \delta^2 \left( \frac{2m}{R} \right)^2 \right).
\]
Since, in most cases of interest, the dilaton to matter coupling is weak and we are far from the Schwarzschild radius of our ‘star’, both $F(\bar{\phi}_0)$ and $2m/R$ are $\ll 1$. Hence: $\partial_I \phi_I / \partial_I \phi_c \approx 1$, and the local time evolution of the dilaton tracks the cosmological one. The strength of this result arises partly from our restrictive choice of background metric. If the background is taken to be Tolman-Bondi rather than McVittie, we can find quite different behaviour.

### C. Case II: The Gautreau-Tolman-Bondi Background

We assume that we have specified some $Z(r; \delta)$ that has outer and inner approximations as given in sections IV C 1 and IV C 2 respectively. But we must be careful to ensure that the form of $Z(r; \delta)$ is such that these two approximations can be matched in some intermediate region. We can now proceed to solve the dilaton evolution equation in the inner and outer limits and then apply our matching procedure. We have already seen that at zeroth order the matching conditions imply that $\phi$ is approximated by
\[
\begin{align*}
\text{inner approx} & \sim \phi^{(0)}_I(\xi; \delta T) + O(\delta^0) = \phi_c(\tau = \tau_0 + \delta T) - F(\bar{\phi}_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right) + O(\delta^0), \\
\text{outer approx} & \sim \phi_c(\tau) + O(\delta^0).
\end{align*}
\]
We now consider the next-to-leading order terms with a view to determining how time variation of $\phi$ in the interior is related to its cosmological rate of change.

In the interior the local energy density and metric are, to $O(\delta^0)$, depend on $T$ only through $\chi$ and are written as functions of $\xi$ and $\chi$. The calculation is easiest if we move from $(T, \xi)$ coordinates to $(T, \chi)$ ones and write:
\[
\phi \sim \phi^{(0)}_I(\xi; \delta T) + \delta^0 \phi^{(1)}_I(T, \chi; \delta T) + o(\delta^0),
\]
where $\phi_I^1$ satisfies (for $q < 2$):

$$\frac{2m}{R_s} \left( \xi^{3/2} \phi_{1,TT}^{(1)} - \frac{3}{2} \phi_{1,T}^{(1)} \right) + \frac{1}{\chi^{1/2}} \left( \frac{\xi^{5/2}}{\chi^{1/2}} \phi_{1,\chi}^{(1)} \right)_{,\chi} \equiv - \frac{2m}{R_s} B(\phi) \left( \phi_0 \right) \frac{\mu_1(\chi)}{\chi^{1/2}}$$

$$+ \left( \frac{2m}{R_s} \right)^2 F(\phi_0) \left( \frac{\mu_1(\chi)}{\xi^{1/2}} \left( \frac{1}{1 - \frac{2m}{R_s}} \right) - \frac{2\mu_1(\chi)}{\xi^{5/2}} \left( \frac{1}{1 - \frac{2m}{R_s}} \right)^2 \right).$$

(51)

(52)

There is also a single term that appears at order $\delta$ which we have omitted from the above expression; we shall deal with this later. In general, the above equation is difficult to solve exactly, however, in most of the cases of interest the surface of our ‘star’ will be far outside its Schwarzschild radius and so $2m/R_s \ll 1$. We may then solve eq. (51) as an asymptotic series in $2m/R_s$, valid where $\xi \gg \frac{2m}{R_s}$. Given the nature of our problem we are only interested in the leading term:

$$\phi_I^{(1)} \sim - \frac{2m}{R_s} B(\phi) \left( \int d\chi \frac{\mu_1(\chi')}{\xi(\chi',T)} \frac{\mu_1(\chi)}{\xi} + \frac{A(T)}{\xi} + C(T) \right) + \mathcal{O} \left( \left( \frac{2m}{R_s} \right)^2 \right).$$

Here, $A(T)$ and $B(T)$ are ‘constants’ of integration; $A(T)$ should be determined by a boundary condition on the surface at $R = R_s$ i.e. at $\xi = 1$, and $B(T)$ will come out of the matching. We argued in section VA that the matching requires that all homogeneous terms in the interior expansion be quasi-static, thus $B(T) = \bar{B}(\delta T)$. Without knowing more about the region $\xi < 1$, we do not have a boundary condition capable of determining $A(T)$. If, however, we assume that the prescription given by eq. (46) is at least approximately correct then we can proceed. We assume that the next-to-leading order perturbation of $\phi_0$ occurs at the same order in $\delta$ and $2m/R_s$ as the next-to-leading order term in $\phi$. Given these assumptions we prescribe our full boundary condition at $R = R_s$ to be given by eq. (45) but with $\phi_0 \to \phi_0 + \mathcal{O}(\delta^2 2m/R_s)$ and $m \to m + Z|_R=R_s$. The resultant boundary condition on $\phi_I^{(1)}$ is then remarkably simple:

$$\partial_\xi \phi_I^{(1)}|_{\xi=1} = - \frac{2m}{R_s} F(\phi_0) \mu_1(\chi(\xi = 1, T)).$$

With this choice of boundary condition,

$$A(T) = \left( 1 - F(\phi_0)/B(\phi) \right) \mu_1(\chi = 1, T)$$

and the interior solution is:

$$\phi_I^{(1)} \sim \frac{2m}{R_s} B(\phi) \left( \int d\chi \frac{\mu_1(\chi')}{\xi(\chi',T)} \frac{\mu_1(\chi)}{\xi} + \left( \frac{1 - F(\phi_0)}{B(\phi)} \right) \frac{\mu_1(\chi(\xi = 1, T))}{\xi} + \bar{B}(\delta T) \right) + \mathcal{O} \left( \left( \frac{2m}{R_s} \right)^2 \right).$$

The unknown quasi-static term, $\bar{B}(\delta T)$, will not affect the leading-order time-variation of $\phi_I^{(1)}$ and so will not have any bearing on our result. We could, in principle, solve for $\phi$ in the outer approximation, however, since we are interested only in dynamics of the dilaton field in the interior, but we will not do so.

Recall that we dropped an order-$\delta$ term in writing down eq. (51); this perturbation decouples from the one at order $\delta^0$, and is quasi-static. Writing

$$\phi_I \sim \phi_I^{(0)}(\delta T) + \delta^0 \phi_I^{(1)} + \delta^0 \phi_I^{(0)}(\xi; \delta T)$$

we find:

$$\frac{1}{\xi^2} \left( \xi(\xi - 2m/R_s) \phi_{I,\xi}^{(0)} \right)_{,\xi} = \frac{2m}{R_s} \left( \sqrt{\frac{R_s}{2m}} \ln \left| \frac{R_s - \xi}{R_s - \frac{2m}{R_s}} \right| \phi_{I,\xi}^{(0)}(\delta T) - \left( \frac{2m}{R_s} \right)^{3/2} l(\xi) F(\phi_0) \phi_{0,\xi}(\delta T) \right).$$

Solving this we find:

$$\phi_I^{(0)} = 2 \left( \frac{2m}{R_s} \right)^{3/2} \left( \sqrt{\frac{R_s - \xi}{2m}} \ln \left| \frac{R_s - \xi}{R_s - \frac{2m}{R_s}} \right| \phi_{I,\xi}^{(0)}(\delta T) - \left( \frac{2m}{R_s} \right)^{3/2} l(\xi) F(\phi_0) \phi_{0,\xi}(\delta T) \right),$$

$$- \left( \frac{2m}{R_s} \right)^{3/2} A \ln \left( 1 - \frac{2m}{R_s} \right).$$

(53)
where \( A \) is a constant of integration and
\[
l(\xi) = -\int_{\xi}^{e} d\xi \xi^{-2} \left( 1 - \frac{2m}{R_s \xi} \right)^{-1} \left( \frac{R_s \xi}{2m} \right)^{3/2} \ln \left( 1 - \frac{2m}{R_s \xi} \right) - 2 \left( \frac{R_s \xi}{2m} \right)^{1/2} - 2 \ln \left| 1 + \sqrt{1 - \frac{2m}{R_s \xi}} + \sqrt{\frac{2m}{R_s \xi}} \right|.
\]

The term proportional to \( l(\xi) \) dies off as \( \xi^{1/2} \), and is suppressed by a factor of \( 2m/R_s \xi \) relative to the first term in eq. (53). Both these terms only depend on time through \( \delta T \) and are thus deemed quasi-static. These terms will not, therefore, change the leading-order behaviour of the time derivative of \( \phi \) in the interior.

Although we are concerned mostly with objects which are much larger than their Schwarzschild radii it would be nice to be able to address the problem of black-hole gravitational memory via this method. For the zeroth-order approximation to \( \phi \) to be well-defined on the horizon we need \( F(\phi_0) = 0 \) (this is just a statement of the ‘no-hair’ theorem for Schwarzschild black holes). We can remove any divergence in \( \phi^{(1)}_t \) near the horizon by an appropriate choice of the constant of integration, \( A \). We take \( A = \phi^{(1)}(\delta T) \). This is analogous to what was done by Jacobson in [35]. We can now absorb the first and last terms on the RHS of eq. 53 into the definition of \( \phi_t(\delta T) \) by a definition of the time coordinate:
\[
\phi_t(\delta T) \to \phi_t(\delta T) \quad \text{where} \quad \bar{T} = T + 2 \left( \frac{2m}{R_s} \right)^{3/2} \left( \frac{R_s \xi}{2m} - \ln \left| 1 + \sqrt{\frac{2m}{R_s \xi}} \right| \right).
\]

The zeroth-order matching is not affected since \( \delta(\bar{T} - T) \) will remain sub-leading order in any intermediate region. We note that \( L_I \bar{T} = v - R - 2m \ln(R/2m) \) where \( v = t_s + R + 2m \ln(R/2m - 1) \) and \( t_s \) is the usual Schwarzschild time coordinate. Thus the leading-order homogeneous term in \( \phi_I \) has the same behaviour as that predicted by Jacobson (see ref. [35] and section II B above).

By expanding \( \phi_e \) w.r.t. the co-moving time, \( L_I T \), rather than the Schwarzschild time, \( t_s \), as Jacobson did, we avoid one of the problems encountered in his analysis: that the time coordinate use to make the expansion diverges on the horizon.

At lower orders we can always ensure, by choice of the constants of integration, that \( \lim_{R \to 0} (1 - 2m/R) \partial R \phi \big|_{T} = 0 \). This boundary condition ensures that the inner approximation does not diverge as \( R \to 2m \), and remains valid on the horizon. This justifies the statement that, if \( p > 1 \), Jacobson’s prediction is itself a valid asymptotic approximation to \( \phi \) on the horizon. The condition \( p > 1 \) is equivalent to: \( \epsilon_{local}/\epsilon_c \ll 1/(2mH_0)(\gg 1) \), with \( \epsilon_i \) being the average value of the energy density, outside the black-hole, over length scales \( \mathcal{O}(2m) \) from the horizon.

We are now in a position to study the conditions under which the local time variation of \( \phi \) tracks its cosmological value. In the interior we have:
\[
\phi_I(\tau) \approx \delta \phi_e(\delta T) \left( 1 - \frac{F(\phi_0)}{\phi_0} \ln \left( 1 - \frac{2m}{R_s \xi} \right) \right) + \delta^9 B_{\phi_e(\phi_e)} \frac{2m}{R_s} \int_{\xi}^{e} d\chi \left[ \mu_1(\chi')_{\chi}(\chi', T) + \left( 1 - \frac{F(\phi_0)}{B(\phi_e)} \right) \frac{\mu_1(\chi = 1, T)}{\xi^{3/2}} \right] + ...
\]

The excluded terms are then certainly smaller than the second term, however, since they might be larger than the first term, both numerically and in the limit \( \delta \to 0 \), the above expression is not a formal asymptotic approximation. When the first term in eq. (54) dominates, condition (5) holds, and the local time evolution of the scalar field is, to leading order, the same as the cosmological one. Assuming our background choice is suitable for the application of this method, condition (5) fails to hold if, and only if, the second term in this expression dominates over the first.

D. Case III: Tolman-Bondi models with simultaneous big bang

In the interior, the leading-order corrections to the metric occur at either order \( \delta^9 \) or \( \delta^{2/3} \). The order \( \delta^{2/3} \) correction comes from the leading, quasi-static, and \( \xi \)-independent term in the expansion of \( k(r) \); we have seen that this can be transformed away by a redefinition of the local time coordinate, \( T \to T^* \), as done in section IV D 2.

In the intermediate region \( T \sim \sqrt{1 - \delta^{2/3}k_0 T^*} \), the zeroth-order matching goes through in the same way as it did the two previous cases. The zeroth-order interior solution is
\[
\phi_0 = \phi_e \left( \delta \sqrt{1 - \delta^{2/3}k_0 T^*} \right) - F(\phi_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right).
\]
The analysis of the $\mathcal{O}(\delta^0)$ terms then follows through in much the same way as for the Gautreau case, but with
\[ \chi = \left(\xi^{3/2} - 3T/2\right)^{2/3} \rightarrow \eta = \left(\xi^{3/2} + 3T/2\right)^{2/3}. \]

One finds that the $\mathcal{O}(\delta^0)$ correction to the interior solution is given by
\[ \phi_I^{(1)} \sim -\frac{2m}{R_s} B_{\phi}(\phi_c) \left( \int^\eta \frac{d\eta'}{\xi(\eta',T)} - \frac{\mu_1(\eta)}{\xi} + \left( 1 - \frac{F(\phi_0)}{B(\phi_c)} \frac{\mu_1(\eta(\xi = 1, T))}{\xi} + \tilde{B}(\delta T) \right) + \mathcal{O}\left( \frac{2m}{R_s} \right)^2. \]

There is also a quasi-static order $\delta$ correction to $\phi_I$, just as for the Gautreau case. We find
\[ \phi_I^{(\delta)} = -\sqrt{1 - \delta^2/k_0} \omega \left( \frac{2m}{R_s} \right)^{3/2} \left( \sqrt{\frac{R_s}{2m}} + \frac{1}{2} \ln \left[ \frac{\sqrt{\xi} - 1}{\sqrt{\xi} + 1} \right] \right) R_{\phi}(\delta T) + \left( \frac{2m}{R_s} \right)^{3/2} A \ln \left( 1 - \frac{2m}{R_s} \right), \]

with $l(\xi)$ as given in section V C. As above, we shall choose the constant of integration, $A$, so as to make $\phi_I^{(\delta)}$ well-defined as $R_s \xi \rightarrow 2m$ whenever $F(\phi_0) = 0$; we can then absorb this correction into the definition of $\phi_c$:
\[ \phi_c \left( \sqrt{1 - \delta^2/k_0} \omega \left( \frac{2m}{R_s} \right)^{3/2} \right) \rightarrow \phi_c \left( \sqrt{1 - \delta^2/k_0} \omega \left( \frac{2m}{R_s} \right)^{3/2} \right). \]

The transform $T^* \rightarrow \tilde{T}$ is the same as the one for $T \rightarrow \tilde{T}$ given in section V C: $L_1\tilde{T} = v - R - 2m\ln(R/2m)$. As in previous case, we can now analyse the conditions under which the local time variation of $\phi$ tracks its cosmological value:
\[ \phi_{I,\tilde{T}} \approx \sqrt{1 - \delta^2/k_0} \omega \left( \frac{2m}{R_s} \right)^{3/2} \left( 1 - F(\phi_0) \ln \left( 1 - \frac{2m}{R_s} \right) \right) - \delta^0 B_{\phi}(\phi_c) \left( \frac{2m}{R_s} \right)^{3/2} \left( \int^\eta \frac{d\eta'}{\xi^{3/2}(\eta',T)} + \left( 1 - \frac{F(\phi_0)}{B(\phi_c)} \frac{\mu_1(\eta(\xi = 1, T))}{\xi^{3/2}} \right) \right) + \ldots \]

This not a formal asymptotic approximation, but the excluded terms are certainly smaller than at least one of the two terms in the above expansion. The effect of the transform $T \rightarrow T^*$ is to induce a slight, sub-leading order, lag in the time evolution of the scalar field, that increases as $\xi$ decreases. As in the Gautreau case above, condition (5) will fail to hold if, and only if, the second term in this expansion dominates. We will interpret this condition, and the similar one which arose from the analysis of the Gautreau case, in terms of what it requires for the local energy-density in section VII below.

**VI. CONDITIONS FOR THE ASYMPOTIC EXPANSIONS**

By considering the growing modes in the interior expansion for $\phi$, and the singular modes in the exterior expansion, we can say something about the position of the matching region, and in some cases show that a matching region could not exist.

**A. Case I: McVittie background**

At $\mathcal{O}(\delta^2)$, the interior expansion of $\phi$ in the McVittie metric has a mode that grows like $\xi$. This asymptotic expansion will certainly cease to be valid if, when rewritten in some intermediate scaling region, where $\xi \propto \delta^{-\alpha}$ $\xi$ say with $0 < \alpha < 1$, terms appear that scale as inverse powers of $\delta$. In such a region $\delta^{2}\xi \sim \delta^{2-\alpha}\xi$, and so such terms will always be suppressed by at least a single power of $\delta$. The zeroth-order term, $\phi_I^{(0)}$, in the interior will itself have ceased to be a valid asymptotic approximation to $\phi$ when the $\delta^2\xi$ terms dominate over the 1/$\xi$ term. This will occur only when $\xi \propto \delta^{-1}$, i.e. in the exterior region. At order $\delta^3$, the fastest growing mode will go like $\xi^3$, and this will also only conflict with the first two terms in the expansion when $\xi \propto \delta^{-1}$. We conclude that eq. (50) is a valid asymptotic approximation to $\phi$ everywhere outside the exterior region. Similarly we note that the most singular term
in the first two terms of the exterior expansion, as given in Appendix A, goes like \( \delta / \rho \). At \( \mathcal{O}(\delta^2) \) we would find terms that behave as \( \delta^2 / \rho^2 \). The exterior expansion will therefore only cease to be valid when \( \rho \sim \delta \), i.e. in the interior region. Therefore, in the McVittie case, the position of the intermediate region is not important; we can choose any intermediate scaling and the matching procedure will be valid. As a final check on our method, we have explicitly performed the matching for the McVittie background in Appendix A, and shown it to be self-consistent.

B. Cases II and III: Tolman-Bondi models

We assume that as \( \mu_1(\chi) \sim \chi^n \) as \( \chi \to \infty \) for some \( n > 0 \). At order \( \delta^p \), the growing mode in the interior approximation will grow like \( \delta^p \chi^n / \xi \) in the Gautreau case, or as \( \delta^p \eta^n / \xi \) for the simultaneous big-bang models. Assuming that in some intermediate region \( \eta, \chi, \xi \sim \delta^{-\alpha} \) with \( 0 < \alpha < 1 \) we can see that this growing mode will dominate over the zeroth order \( 1 / \xi \) when \( \alpha = q/n \), and the asymptotic approximate will fail altogether if \( n > 1 \) and \( \alpha > q/(n-1) \). We did not explicitly find the exterior expansion of \( \phi \), since we were only really concerned with its behaviour in the interior, however the first non-homogeneous mode should behave as \( \delta^p z_1(\rho) / \rho \) if \( p \leq 1 \) or \( \delta / \rho \) if \( p > 1 \). If \( z_1(\rho) \sim \rho^m \) as \( \rho \to 0 \), then the exterior expansion will break down if

\[
\begin{align*}
  m &< 1 - p \quad \alpha \leq 1 - \frac{p}{1 - m}, \\
  m &\geq 1 - p \quad \alpha = 0.
\end{align*}
\]

In addition to this consideration, we note that the transformation \( T \to T^* \) used in case III will not be well-defined if \( \xi \propto \delta^{-\alpha} \) where \( \alpha \geq 2/3 \). Just by considering the behaviour of the next-to-leading order terms we can say that if both the interior and exterior zeroth-order approximations are to be simultaneously valid in some intermediate scaling region we need, for the Gautreau case:

\[
\max \left( 0, 1 - \frac{p}{1 - m} \right) < \alpha < \frac{q}{n},
\]

and for the simultaneous-big-bang case:

\[
\max \left( 0, 1 - \frac{p}{1 - m} \right) < \alpha < \min \left( \frac{q}{n}, 2/3 \right).
\]

If such an intermediate region does indeed exist then the zeroth-order matching performed in section VA will be valid. The general form of the interior approximation to \( \mathcal{O}(\delta^p) \) will then be correct; the only unknown function in this term is \( B(\delta T) \). If the matching works to order \( \delta^p \), as well as zeroth-order, then we have argued that \( B(\delta T) \) will be quasi-static. If the matching procedure does not work to this order then its quasi-static character may be lost. However, we would not expect it to vary in time any faster than the other \( \mathcal{O}(\delta^p) \) terms. So long as we can match the zeroth-order approximations in some region, we can find the circumstances under which condition (5) holds by comparing the sizes of the two terms in (54) and (55), for the Gautreau and simultaneous big bang cases respectively.

VII. INTERPRETATION AND GENERALISATION

We assume that \( 2mF(\bar{\phi}_0)/R_s \ll 1 \) in the Tolman-Bondi cases II and III considered above and that the matched asymptotic expansion method is valid. Condition (5) will then only fail to hold if the \( \mathcal{O}(\delta^p) \) term in expressions (54) and (55) is larger that the order \( \delta \phi^0_e \) term. For condition (5) to hold we require:

\[
\frac{B_{\phi_e}(\bar{\phi}_0)(\int r^3 \, dv)(2m)^{3/2}}{R(r,t)} \frac{R^2}{(\bar{\phi}_0)^2} \frac{\delta(\mu_1, \phi_e)}{\mu_1} \left( \frac{(2m)^{3/2}}{R} \right)^{\delta \phi_e(t)} \left( \frac{x^{1/2}}{Rite=R_s} \right) \ll 1,
\]

where \( R_s \chi, R_s \eta \sim r \) and \( R_s \xi = R_s \). For any given model this can be evaluated. As the expression is currently written its physical meaning is somewhat obscured. Firstly, the lower limit on the integral is not specified; we merely specified that the ‘constant’ of integration that arises from this lower limit should be quasi-static. We should therefore take the lower limit to be some value of \( r \) such that the contribution to the integral from that lower limit vanishes. If \( p > 1 \), the above criterion (57) will certainly hold. Therefore, in looking for where (5) fails to hold, we are concerned only with the cases where \( p \leq 1 \). In these situations, and indeed also if \( 1 < \delta < 2 \), the local
matter density close to $R = R_s$ will be much larger than the cosmological dust density. We note that in the interior the energy density, $\epsilon(r, t)$ and also energy overdensity, $\Delta \epsilon$ (as $p < 2$), is given by

$$\kappa(\epsilon, t) \sim \kappa(\Delta \epsilon) = \kappa(\epsilon - \epsilon_c) \sim 2m / R^{3/2} \frac{\mu_1(r)}{R^{3/2} \xi^{1/2}}.$$ 

We also note that $R_t = R_s = \pm (2m/R)^{1/2}$ and $R_r = R^{1/2} / \xi^{1/2}$. Therefore, in the interior region, we can substitute $\pm (2m/R)^{1/2} \mu_1(r)$ for $\Delta(R, \kappa \epsilon) := \kappa(R_t \epsilon - H R \epsilon_c)$. Doing this we find that our criterion, eq. (57), becomes:

$$\left| B_{B, \phi}(\phi_e(t)) \int \int \frac{d^3 r}{R r} \frac{\Delta(\epsilon'(r', t))}{\phi_c(t)} \right| \ll 1.$$ 

The two expressions, (57) and (58), are equivalent whenever the interior approximation holds. These should be checked in the context of a given model. We can further simplify expression (58) by noting that, to zeroth order in $2m/R_s$, we should expect $F(\phi_0) \sim B_{B, \phi}(\phi_e)$, and so the second term in the denominator will be a factor of $2m/\xi s$ smaller than the first and so can safely be neglected.

Expression (58) is especially useful since it is written in terms of the matter overdensity, $\kappa \Delta \epsilon$. Using this expression we will now conjecture a more general criterion, that will coincide with (57) and (58) whenever the matching procedure would itself fail.

In the Tolman-Bondi models the radial velocity of the dust particles, as measured by an observer at fixed $R$, is given by $\gamma(R)$. Assuming that the second term in denominator of (57) and (58) is negligible compared to the first, we can rewrite our criterion for condition (5) in the more concise and revealing form by replacing $R_t$ with the dust particle velocity, $v$:

$$\int \int \frac{d^3 r}{R r} \Delta(R, \kappa \epsilon'(r', t)) \ll 1.$$ 

(59)

will hold provided $\kappa \Delta \epsilon, \sim o(1/R^2)$ as $R \to \infty$: this is certainly true whenever the extra mass contained in the over-dense interior region is finite. The integrals in (57) and (58) are specified by the condition that any constant term that remains as $R \to \infty$ should be neglected. So long as $\kappa \mu_1, \chi \sim n < 5/2$, these integrals will be equivalent to the expression given in eq. (59), at least in the interior region. The analysis of section 4B suggests that if $n \geq 5/2$ and $q \leq 1$, the size of any potential intermediate matching region will be strongly bounded; thus, for most cases where we expect the matching procedure to work, $n < 5/2$.

In the Tolman-Bondi models we will now conjecture a more general criterion, that will coincide with (57) and (58) whenever the matching procedure works, but which we believe should also apply to many cases where the matching would not have worked. Eq. (58) only features terms that are well defined for all Tolman-Bondi models, i.e. $R_r, \xi, t$ and $\kappa \Delta \epsilon$. The condition

$$\frac{2}{1} B_{B, \phi}(\phi_e) \int_{\gamma(R)} dl \left| \frac{\Delta(\epsilon \kappa \epsilon)}{\phi_e(t)} \right| \ll 1.$$ 

(60)

where $dl = d^3 r / R r$ and the path of integration, $\gamma(R)$, runs from $R_{0} \to \infty$, and $\Delta(\epsilon \kappa \epsilon) = v \kappa \epsilon - H R \epsilon_c$. Since we expect that $\phi$ will only feel the effects of events that happened in its causal past, we should take $\gamma(R)$ to run from $R$ to spatial infinity along a past, radially directed light-ray. For the Tolman-Bondi models we have considered, the LHS of (60) will coincide (at least in magnitude) with the LHSs of (57) and (58).

Eq. (60) has the form that we might expect for condition (5) to hold inside a local, spherically-symmetric, inhomogeneous region produced by an embedded Schwarzschild mass in an asymptotically FRW universe: the LHS of eq. (60) vanishes as we approach the cosmological region and the local time variation of the dilaton field is driven by the time variation of the energy density along its past light-cone.

We shall therefore conjecture that eq. (60) is a sufficient condition for eq. (5) to hold: applicable even to spherically-symmetric (or approximately spherically-symmetric), dust plus cosmological constant, backgrounds in which the matching procedure would itself fail.

If the cosmic evolution of the dilaton field is dominated by its coupling to matter, so $|B_{B, \phi}(\phi_e)| \gg |V_{\phi}(\phi_e)|$, then $\phi_e(t) \propto (B_{B, \phi}(\phi_e) \kappa \epsilon_c(t)/H(t))$ and condition (5) therefore holds, at the epoch $t = t_0$, whenever

$$\int_{\gamma(R)} dl H(t_0) \left| \frac{\Delta(\epsilon \kappa \epsilon)}{\epsilon_c(t_0)} \right| \ll 1.$$ 

If, alternatively, the cosmic evolution is potential-driven, so $|V_{\phi}(\phi_e)| \gg |B_{B, \phi}(\phi_e) \kappa \epsilon_c|$, then the LHS of the above expression will be suppressed by an additional factor of $|B_{B, \phi}(\phi_e) \kappa \epsilon_c|/V_{\phi}(\phi_e) \ll 1$. 

(61)
In section II A we reviewed Wetterich’s conclusion of [31] that condition (5) holds true when the cosmic evolution of the dilaton is potential dominated. Whilst we have identified gaps in the analysis performed in that paper, we have been able to establish a similar result. For a given evolution of the background matter density, we have seen that condition (5) is more likely to hold (or will hold more strongly) when \( |B_{\phi}(\phi_c)\epsilon_c/V_{\phi}(\phi_c)| \ll 1 \). The domination by the potential term in the cosmic evolution of the dilaton has a homogenising effect on the time variation of \( \phi \). In the final section we shall apply eq. (61) to some physically relevant scenarios and show that, on the basis of the above analysis, we should expect \( \dot{\phi}(x,t) \approx \dot{\phi}_c(t) \) in the solar system.

VIII. OBSERVATIONS IN OUR SOLAR SYSTEM

We would like to apply out results to solve our original problem: whether or not condition (5) holds when the local region is the Earth or the solar system. We will consider a star (and associated planetary system) inside a galaxy that is itself embedded in a large galactic cluster. The cluster is assumed to have virialised and be of size \( R_{\text{clust}} \). Close to the edge of the cluster we allow for some dust to be unvirialised and still undergoing collapse. Since we have only performed our calculations for spherically-symmetric backgrounds, we should, strictly speaking, also require spherical symmetry about our star; for a realistic model this seems contrived. In an upcoming paper, [49], we shall consider more fully the effect that deviations from pure spherical symmetry have on our results; for now we assume that these effects are small, and we can relax the spherical symmetry requirement somewhat without invalidating our analysis. We consider the different contributions to the LHS of condition (61) in this astronomical set-up:

\[
I := \int_{(R)} dl H(t_0) \frac{|\Delta(\nu_c)|}{\epsilon_c(t_0)} = I_{\text{clust}} + I_{\text{gal}} + I_{\text{star}}. \tag{62}
\]

For illustration we evaluate \( I \) for Brans-Dicke theory. For other theories, \( I \) will still take very similar values. We assume that the density of non-virialised matter, just outside the virialised cluster, is no greater than the average density of the cluster and if we move away from the edge of the virialised region, the over-density drops off quickly, i.e. as \( R^{-s}, s > 1/2 \). In this case, the Tolman-Bondi result, eq. (55), applies and the magnitude of \( I \) is bounded by:

\[
I_{\text{clust}} \lesssim t_0^{-1}(s-1/2)^{-1} \sqrt{2M_{\text{clust}}R_{\text{clust}} \epsilon_{\text{clust}}/\epsilon_c} \approx 1.5 \times 10^{-3} \Omega^{-1}_{m}(1 + z_{\text{vir}})^3 \ll 1,
\]

where we have used \( 3GM/R_{\text{clust}} = v_{\text{clust}}^2 \). \( R_{\text{clust}} \) is the scale of the cluster post-virialisation, and \( v_{\text{clust}} \) is the average velocity of the virialised dust particles in it; \( t_0 = 13.7 \text{Gyr} \) is the age of the universe. In the final approximation we used representative values \( R_{\text{clust}} = 100 \text{Mpc} \) and \( v_{\text{clust}} = 200 \text{km s}^{-1} \) appropriate for a cluster like Coma. Taking a cosmological density parameter equal to \( \Omega_m = 0.27 \), in accordance with WMAP, and \( h = 0.71 \), we expect that for a typical cluster which virialised at a redshift \( z_{\text{vir}} \ll 1 \), we would have \( I_{\text{clust}} \approx 5.7 \times 10^{-3} \). The term in \( \theta \) is unity when \( s = 2 \), i.e. \( 2GM/r \rightarrow \text{const} \); such a matter distribution is characteristic of dark matter halos. Different choices of \( s > 1/2 \) can be seen to only change this estimate by an \( O(1) \) factor. As \( s \rightarrow 1/2 \) the matched asymptotic expansion method, and hence this particular evaluation, breaks down. We believe that the generalised formula for \( I \) will, however, still give accurate results. We have assumed that the baryon-to-dark matter ratio inside the cluster is the same as in the universe on average. If this is not the case, and the dilaton couples with vastly different strengths to baryons and dark matter, then \( I_{\text{clust}} \) could be made as large as \( 3 \times 10^{-2} \), for these values of \( R_{\text{clust}} \) and \( v_{\text{clust}} \); this scenario would require the cluster to be comprised completely of baryons, and the dilaton to have zero coupling to dark matter.

We now consider the galactic contribution. This will come about as a result of the galaxy slowly accreting matter from the intergalactic medium (IGM). We shall assume that the properties of IGM are the same as the average properties of the cluster. In this case the density of the IGM will be approximately constant and as such eq. (55) will not be strictly applicable. However, if we assume that our conjecture of the previous section holds then we can evaluate \( I_{\text{gal}} \). The IGM has an average particle velocity \( v_{\text{IGM}} \); as a result of this, the gravitational influence of our Galaxy upon it will only be felt significantly out to a radius \( R_G^{(\text{gal})} = 2GM_{\text{gal}}/v_{\text{IGM}}^2 \). We therefore take \( R = R_G^{(\text{gal})} \),
where \( r \sim \epsilon \) where we have taken \( \epsilon = m/200 \text{km s}^{-1} \) rather than \( \epsilon = \infty \).

This calculation proceeds in the same way as the one for the back reaction, is large then the scope for violations of condition (5) increases; if the coupling to gravity of the dilaton cluster will tend to be the dominant contribution to the LHS of eq. (6.1). The estimate of \( \epsilon_{\text{clust}} \) given above should be viewed as an upper bound on its value; even still we have seen that is small compared with 1, and hence that we must hold in the solar system in general, and on Earth in particular. We have seen that if the cosmic evolution of \( \phi \) is potential dominated then this only serves to strengthen this result. The only caveat is that since we have ignored the back reaction of \( \phi \) on the background cosmological geometry we are limited to \( |B_{\phi}| \ll 1 \). If \( B_{\phi} \), and hence the back reaction, is large then the scope for violations of condition (5) increases; if the coupling to gravity of the dilaton is large, then dilaton field tends towards homogeneity. Contrary to what has been claimed before in the literature, virialisation does not stabilise the value of the dilaton, and protect it from any cosmological variation.

We can also conclude, from the above analysis, that, even if condition (5) is violated during the collapse of an overdense region of matter, once the region stops collapsing as a result of its virialisation, the time evolution of the dilaton field tends towards homogeneity. Contrary to what has been claimed before in the literature, virialisation does not stabilise the value of the dilaton, and protect it from any cosmological variation.

In addition whilst we did not explicitly calculate the interior solution for \( 2m/R_s = O(1) \), the evolution eq. (51) is still valid in this case. We should therefore expect the magnitude of the rate of time variation in \( \phi^{(1)} \) to be similar to that was found for the \( 2m/R_s \ll 1 \) case. We therefore conclude that over time-scales that are large compared to \( 2Gm \), condition (5) will hold whenever condition (61) does, and there will be no significant gravitational memory of the value of \( G \) from the epoch when black holes or gravitationally bound structures first formed. This result agrees with the prediction made by Jacobson, [35], the inhomogeneous models of [33], and the numerical calculations of Harada et. al., [34], who also studied the Tolman-Bondi background. Indeed we have seen that whenever the, not very restrictive, condition, \( \epsilon_l/\epsilon_c \ll 1/(2mH_0)(\gg 1) \), holds then eq. (8), gives a true asymptotic approximation to the behaviour of the dilaton close to the horizon.

\[ I_{\text{gal}} \approx -2t_0^{-1} \sqrt{2GM_{\text{gal}}R_G^{(\text{gal})}/\epsilon_c} \]
\[ = -2.4 \times 10^{-15} \Omega_m^{-1} M_{\text{gal}}/M_\odot (1 + z_{\text{vir}})^3 (\text{km s}^{-1}) \]
\[ \approx -4.4 \times 10^{-6}(1 + z_{\text{vir}})^3 \ll 1, \]

where we have taken \( \epsilon_{\text{IGM}} \approx \epsilon_{\text{clust}} \) and \( v_{\text{IGM}} \approx v_{\text{clust}} \). In the last line we have taken \( M_{\text{gal}} = 10^{12} M_\odot \) and \( v_{\text{clust}} = 200 \text{km s}^{-1} \) as above.

Finally we consider the contribution that results from our star accreting matter from the interstellar medium (ISM). This calculation proceeds in the same way as the one for \( I_{\text{gal}} \).

\[ I_{\text{star}} \approx -2t_0^{-1} \sqrt{2Gm/v_{\text{ISM}}} \epsilon_c \]
\[ = -1.2 \times 10^{-13} \Omega_m^{-1} m/M_\odot (v_{\text{ISM}}/h)h^{-2}n \]
\[ \approx -1.8 \times 10^{-14} n \ll 1, \]

where \( r_G = 2Gm/v_{\text{ISM}}^2 \), and \( \epsilon_{\text{ISM}} = n \) protons cm\(^{-3} \) where \( n \approx 1 - 10^4 \). We have taken \( v_{\text{ISM}} = 5 \text{km s}^{-1} \) and \( m = M_\odot \) to give the final approximation. It is clear that, in general, \( I_{\text{star}} \ll I_{\text{gal}} \ll I_{\text{clust}} \). The infall of dust into cluster will tend to be the dominant contribution to the LHS of eq. (61). The estimate of \( I_{\text{clust}} \) given above should be viewed as an upper bound on its value; even still we have seen that is small compared with 1, and hence that we should expect condition (5) to hold near our star; with deviations from that behaviour bounded by the 0.6% level (if the dilaton couples only to baryonic matter this could rise to as high as 3%). Assuming that the conditions in our solar system are not too different from those considered above, we conclude that irrespective of the value of the dilaton-to-matter coupling, and what dominates the cosmic dilaton evolution, that

\[ \dot{x}(x, t) \approx \dot{x}_c(t) \]

will hold in the solar system in general, and on Earth in particular. We have seen that if the cosmic evolution of \( \phi \) is potential dominated then this only serves to strengthen this result. The only caveat is that since we have ignored the back reaction of \( \phi \) on the background cosmological geometry we are limited to \( |B_{\phi}| \ll 1 \). If \( B_{\phi} \), and hence the back reaction, is large then the scope for violations of condition (5) increases; if the coupling to gravity of the dilaton is large, then dilaton field may itself undergo gravitational collapse.

IX. SUMMARY

In this paper we have considered the extent to which a cosmological time variation in a scalar field (dilaton) would be detectable on the surface of gravitational-bound systems that are otherwise disconnected from changes that occur over cosmological scales. This problem is of particular relevance when the dilaton defines the local value of one of the traditional ‘constants’ of Nature, and when the system is the Earth or our solar system. Several scalar-tensor theories have already been developed which self-consistently describe the variations of traditional ‘constants’ of Nature, like
\( \alpha \) and \( G \). By matching rigorously constructed asymptotic expansions of the associated scalar field found in different limits, close to the Earth and far away from it at cosmological scales, we have been able to derive approximate, analytical expressions for the scalar field near the surface of such bound systems. This result was found under two major assumptions: the physically realistic condition that the scalar field should be weakly coupled to matter and gravity (in effect the variations of ‘constants’ on large scales occur more slowly than the universe is expanding) and thus have a negligible back-reaction on the cosmological background, and the less realistic one of spherical symmetry. We do not expect the relaxation of the spherical symmetry condition to greatly alter the qualitative nature of our conclusions, and we shall present a rigorous treatment of the problem when no symmetry is present in a subsequent work.

Finally we have extracted from our analysis a sufficient condition for the local time-variation of a scalar field, or varying physical ‘constant’, to track the cosmological one, and we have proposed a generalisation of this condition that is applicable to scenarios more general than those explicitly considered here. By evaluating the condition for an astronomical scenario similar to the one appropriate for our solar system, we have concluded that almost irrespective of the form of the dilaton-to-matter, and the form of the dilaton self-interaction, its time variation in the solar system will track the cosmological one. We have therefore provided a general proof of what was previously merely assumed: that terrestrial and solar system based observations can legitimately, be used to constrain the cosmological time variation of supposed ‘constants’ of Nature.

Acknowledgements DS is supported by a PPARC studentship. We would like to thank P.D. D’Eath and T. Clifton for helpful discussions.

**APPENDIX A: EXPLICIT MATCHING FOR THE MCVITTIE BACKGROUND**

1. Exterior Solution

In the exterior region \( \phi_E = \phi_E^{(0)} + \delta \phi_E^{(1)} + O(\delta^2) \) where:

\[
\frac{1}{a^3} \partial_\tau \left( a^3 \partial_\tau \phi_E \right) = \left( 1 + \frac{1}{4} \Omega_k r^2 \right)^3 \frac{1}{a^2 \rho^2} \partial_\rho \left( \frac{\rho^2}{1 + \frac{1}{4} \Omega_k \rho^2} \partial_\rho \phi_E^{(1)} \right)
\]

\[
= \left( \frac{2m}{R_s} \right)^{3/2} \left( 1 + \frac{1}{4} \Omega_k r^2 \right)^{1/2} \left( \phi_E^{(0)}'' + h_\phi \phi_E^{(0)}' + 3B_{\phi}(\phi_E) \left( h' + \frac{11}{4} \Omega_k \right) \right)
\]

\[+ B_{\phi}(\phi_E)^2 \kappa \epsilon_{dust} \phi_E^{(1)} - V_{\phi}(\phi_E) \phi_E^{(1)} \, \right) \, \phi_E^{(1)}.
\]

Despite the complexity of this formula we can solve it admits separable solutions in \( \tau \) and \( \rho \):

\[
\phi_E^{(1)} = \left( \frac{2m}{R_s} \right)^{3/2} \left( 1 + \frac{1}{4} \Omega_k r^2 \right)^{1/2} \phi_\tau(\tau) \phi_\rho(\rho).
\]

where \( \phi_\tau(\tau) \) satisfies the ODE:

\[
\frac{1}{a^3} \partial_\tau \left( a^3 \partial_\tau \phi_\tau \right) = \frac{3 \Omega_k(\tau)}{4} \phi_\tau - B_{\phi}(\phi_E^{(0)}) \phi_\tau H_0^{-2} \kappa \epsilon_{dust} \phi_\tau + H_0^{-2} V_{\phi}(\phi_E^{(0)}) \phi_\tau
\]

\[= - \frac{1}{a} \left( \phi_\tau^{(0)}'' + h_\phi \phi_\tau^{(0)}' + 3B_{\phi}(\phi_E) \left( h' + \frac{11}{4} \Omega_k \right) \right). \]

When \( k > 0, \phi = \sum_{\gamma = -\infty}^{\infty} \sqrt{\Omega_k} c_\gamma \cdot T^+_\gamma(\tau) X^+_\gamma(\rho), c_\gamma \in \mathbb{C}^2 \):

\[
X_\gamma = \frac{e^{i \gamma \alpha(\rho)}}{\sin \alpha},
\]

\[
\alpha(\rho) = \sin^{-1} \left( \frac{\sqrt{\Omega_k \rho}}{1 + \frac{1}{4} \Omega_k \rho^2} \right),
\]

and \( T^+_\gamma = (T^+_\gamma, T^+_\gamma, T^+_\gamma) \), where \( T^+_\gamma \) and \( T^+_\gamma \) are linearly independent solutions of the following ODE:

\[
\frac{1}{a^3} \partial_\tau \left( a^3 \partial_\tau T^+_\gamma(\tau) \right) + (\gamma^2 - 1) \Omega_k(\tau) T^+_\gamma(\tau) = B_{\phi}(\phi_E^{(0)}) H_0^{-2} \kappa \epsilon_{dust} T^+_\gamma(\tau) - H_0^{-2} V_{\phi}(\phi_E^{(0)}) T^+_\gamma(\tau).
\]
If $k = 0$, then $Φ = \int_{-∞}^{∞} dγ \, c(γ) \cdot T_γ^0(τ)X_γ^0(ρ)$, $c(γ) \in C^2$, and $X_γ^0(ρ) = e^{iγ/γρ}$. $T_γ^0(τ) = (T_γ^0,1, T_γ^0,2)$ satisfies:

$$1/a^3 \partial_τ(\bar{a}^3 \partial_τ T_γ^0,i(τ)) + \gamma^2 T_γ^0,i(τ) = B,ϕφ(ϕ_E^0)H_0^{-2}κ_{dust}^0 T_γ^0,i(τ) - H_0^{-2}V,ϕφ(ϕ_E^0)T_γ^0,i(τ),$$

where $i = 1, 2$. Finally if $k < 0$ then $Φ = \int_{-∞}^{∞} dγ \, \sqrt{-Ω_k^0}c(γ) \cdot T_γ^0(τ)X_γ^0(ρ)$, $c(γ) \in C^2$:

$$X_γ^0(ρ) = \frac{e^{iγ/γρ}}{\sinh α},$$

$$α = \sinh^{-1}\left(\frac{\sqrt{-Ω_k^0}/γρ}{1 + \frac{1}{4}Ω_k^0γ^2}\right),$$

and $T_γ^0(τ) = (T_γ^{-1}, T_γ^{-2})^T$:

$$1/a^3 \partial_τ(\bar{a}^3 \partial_τ T_γ^{-i}(τ)) - (γ^2 + 1)Ω_k(τ)T_γ^{-i}(τ) = B,ϕφ(ϕ_E^0)H_0^{-2}κ_{dust}^0 T_γ^{-i}(τ) - H_0^{-2}V,ϕφ(ϕ_E^0)T_γ^{-i}(τ),$$

where $i = 1, 2$. In all cases we shall fix our definition of the $T_γ$ by the normalisation: $T_γ(τ_0) = 1$.

2. Matching Conditions

By making our interior and exterior solutions for $φ$ we see that: $c_γ = c_{−γ}$ for $k > 0$, and $c(γ) = \bar{c}(−γ)$; $\bar{ε}$ is the complex conjugate of $ε$. Defining $A_n(τ) = \sum_γ γ^n c_γ \cdot T_γ^+ \text{ or } A_n(τ) = \int dγ \, γ^n c(γ) \cdot T_0^−/γ$ for $k > 0$ and $k ≤ 0$ respectively we see that:

$$A_0(τ) = \frac{F(\bar{φ}_0)}{a} - Υ(τ) \quad \text{(A2)}$$

$$A_1(τ) = 0 ⇔ c(γ) = \bar{c}(−γ) \quad \text{(A3)}$$

$$|Ω_k^0| A_2(τ) = \left(h' + 2h^2)\bar{a}F(\bar{φ}_0) + \left(B,ϕφ(ϕ_E^0)\bar{κ}_{dust}^0 - V,ϕφ\right)\bar{a}F(\bar{φ}_0) + Ω_k(τ)\bar{a}F(\bar{φ}_0) \quad \text{(A4)}$$

$$−\bar{a} \left(ϕ_E^0,0'' + hϕ_E^0,0' + 3B,ϕ (h' + \frac{11}{2}Ω_k(τ))\right) - \frac{1}{4}Ω_k^0Υ.$$

In principle we can invert the equation for $A_0(τ)$ to find the $c(γ)$.

Given the definition of the $A_n$, the expressions for $A_0$ and $A_2$ must satisfy a consistency relation (if there were not to satisfy this, the matching procedure would be invalid). We have checked that this relation does indeed hold here. The relation is:

$$|Ω_k(τ)| A_2 - Ω_k(τ)A_0 = -\frac{1}{a^3} \partial_τ(\bar{a}^3 \partial_τ A_0) + \left(B,ϕφ(ϕ_E^0)\bar{κ}_{dust}^0 - V,ϕφ\right)A_0.$$

We can see, explicitly, that the matched asymptotic expansion method works for the McVittie background.

[1] J.K. Webb et al, Phys. Rev. Lett. 82, 884 (1999); M. T. Murphy et al, Mon. Not. Roy. Astron. Soc. 327, 1208 (2001); J.K. Webb et al, Phys. Rev. Lett. 87, 091301 (2001); M.T. Murphy, J.K. Webb and V.V. Flambaum, Mon. Not R. astron. Soc. 345, 609 (2003).
[2] H. Chand et al., Astron. Astrophys. 417, 853 (2004); R. Srianand et al., Phys. Rev. Lett. 92, 121302 (2004).
[3] J. Bahcall, C.L. Steinhardt, and D. Schlegel, Astrophys. J. 600, 520 (2004).
[4] R. Quast, D. Reimers and S.A Levshakov, A. & A. 386, 796 (2002).
[5] S.A. Levshakov, et al., A. & A. 434, 827 (2005).
[6] S.A. Levshakov, et al., astro-ph/0511765.
[7] G. Rocha, R. Trotta, C.J.A.P. Martins, A. Melchiorri, P.P. Avelino, R. Bean, and P.T.P. Viana, Mon. Not. astron. Soc. 352, 20 (2004).
[8] J. Darling, Phys. Rev. Lett. 91, 011301 (2003).
[9] J. Darling, Astrophys. J. 612, 58 (2004).
[10] M.J. Drinkwater, J.K. Webb, J.D. Barrow and V.V. Flambaum, Mon. Not. R. Astron. Soc. 295, 457 (1998).
[11] W. Ubachs and E. Reinhold, Phys. Rev. Lett. 92, 101302 (2004).
[12] R. Petitjean et al, Comptes Rendus Acad. Sci. (Paris) 5, 411 (2004).
[13] P. Tzanavaris, J.K. Webb, M.T. Murphy, V.V. Flambaum, and S.J. Curran, Phys.Rev.Lett. 95, 041301 (2005).
[14] B. Bertotti, L. Iess and P. Tortora, Nature 425, 374 (2003).
[15] J.P. Uzan, Rev. Mod. Phys. J.P. Uzan, Rev. Mod. Phys. 75, 403 (2003): J.-P. Uzan, astro-ph/0409424.
[16] K.A. Olive and Y-Z. Qian, Physics Today, pp. 40-5 (Oct. 2004).
[17] J.D. Barrow, *The Constants of Nature: from alpha to omega*, (Vintage, London, 2002).
[18] J.D. Barrow, Phil. Trans. Roy. Soc. 363, 2139 (2005).
[19] J.D. Barrow, D. Kimberly and J. Magueijo, Class. Quantum Grav. 21, 4289 (2004).
[20] J.D. Barrow, Mon. Not. R. astr. Soc., 282, 1397 (1996).
[21] J.D. Barrow, J. Magueijo, Phys. Rev. Lett. 88, 031302 (2002); J.D. Barrow, H. B. Sandvik and J. Magueijo, Phys. Rev. D 65, 063504 (2002); J.D. Barrow, H. B. Sandvik and J. Magueijo, Phys. Rev. D 65, 123501 (2002); J.D. Barrow, J. Magueijo and H. B. Sandvik, Phys. Rev. D 66, 043515 (2002); J. Magueijo, J. D. Barrow and H. B. Sandvik, Phys. Lett. B 541, 201 (2002); H. Sandvik, J.D. Barrow and J. Magueijo, Phys. Lett. B 549, 284 (2002).
[22] J.D. Barrow and J. Magueijo, Phys. Rev. D 72, 043521 (2005).
[23] J.D. Barrow and J. Magueijo, Phys. Lett. B 584, 8 (2004).
[24] J.D. Barrow, Mon. Not. R. astr. Soc., 282, 1397 (1996).
[25] J.D. Barrow, D. Kimberly and J. Magueijo, Class. Quantum Grav. 21, 4289 (2004).
[26] J.D. Barrow, Phys. Rev. D 71, 063525 (2005).
[27] J.D. Barrow, Mon. Not. Roy. Astron. Soc. 349, 281 (2004); J.D. Barrow and D. Mota, Phys. Lett. B 581, 141 (2004); T. Clifton, D. Mota and J.D. Barrow, Mon. Not. R. Astron. Soc. 358, 601 (2005).
[28] J.D. Barrow and C. O’Toole, Mon. Not. Roy. Astron. Soc. 322, 585 (2001).
[29] J.D. Barrow, Phys. Rev. D 71, 083520 (2005).
[30] C. Wetterich, JCAP 10, 002 (2003).
[31] J.D. Barrow, Phys. Rev. D 46, R3227 (1992); J.D. Barrow, Gen. Rel. Gravitation 26, 1 (1992); J.D. Barrow and B.J. Carr, Phys. Rev. D 54, 3920 (1996).
[32] H. Saida and J. Soda, Class. Quantum Gravity 17, 4967 (2000); J.D. Barrow and N. Sakai, Class. Quantum Gravity 18, 4717 (2001).
[33] T. Harada, C. Goymer and B. J. Carr, Phys. Rev. D. 66, 104023 (2002).
[34] T. Jacobson, Phys. Rev. Lett. 83, 2699 (1999).
[35] J. D. Cole, *Perturbation methods in applied mathematics*, (Blaisdell, Waltham, Mass., 1968).
[36] E. J. Hinch, *Perturbation methods*, (Cambridge UP, Cambridge, 1991).
[37] W. L. Burke and K. Thorne, in *Relativity*, edited by M. Carmeli, S. Fickler, and L. Witten (Plenum Press, New York and London, 1970), pp. 209-228.
[38] W. L. Burke, J. Math. Phys. 12, 401 (1971).
[39] P. D. D’Eath, Phys. Rev. D. 11, 1387 (1975).
[40] P. D. D’Eath, Phys. Rev. D. 12, 2183 (1975).
[41] R. Geroch, Commun. Math. Phys. 13, 180 (1969).
[42] A. Krasiński, *Inhomogeneous Cosmological Models*, (Cambridge UP, Cambridge, 1996).
[43] R. Gautreau, Phys. Rev. D. 29, 198 (1984).
[44] G.C. McVittie, Mon. Not. Roy. Astron. Soc. 93, 325 (1933); G.C. McVittie, *General Relativity and Cosmology* (Chapman and Hall, London, 1965).
[45] C. Leibovitz, Phys. Rev. D 4, 2949 (1971).
[46] J. D. Barrow and J. Stein-Schabes, Phys. Lett. 103A, 315 (1984).
[47] U. Debnath, S. Chakraborty, and J.D. Barrow, Gen. Rel. Gravitation 36, 231 (2004).
[48] D. J. Shaw and J. D. Barrow, gr-qc/0601056.