THE MECHANICS OF TURBULENCE FORMATION VIA AN INTERACTION POTENTIAL: A SIMPLE TWO-PARTICLE MODEL

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ABSTRACT. In a recent work, we proposed a hypothesis that the turbulence in fluids could be produced by fluid particles interacting via a potential – for example, the interatomic potential at short ranges, and the electrostatic potential at long ranges. Here, we examine the basic mechanics of turbulence formation by studying a simple model, which consists of only two particles interacting via a potential. We start with the equations of motion for this pair of particles, then consider the corresponding Liouville equation for these dynamics, and subsequently derive the velocity moment transport equations. We close these equations using the hydrostatic and thermostatic approximations, and arrive at the evolution equation for the velocity variable alone. We then further decouple the dynamics of a generic inertial range Fourier coefficient of the velocity from all other Fourier coefficients, except those of the strong large scale flow. The resulting simplified dynamics possess what could be the fundamental mechanism of turbulence formation – the cubic oscillator with a linear term, forced by the strong large scale flow. We find that the dependence of certain solutions of this oscillator on their Fourier wavenumber is consistent with Kolmogorov’s ubiquitous “five-thirds” scaling law.

1. Introduction

In our recent work [1], we proposed a hypothesis that the turbulence in fluids could be produced by their particles interacting via a potential, which decays inversely proportionally to the distance between the particles. We speculated that, at short ranges, it could be the interatomic potential (such as the Thomas–Fermi [8, 19], or the Ziegler–Biersack–Littmark [20] potentials), while at long ranges it could be the electrostatic potential.

The reason why we proposed such an unusual hypothesis is the following. In our work [1], we investigated the full Liouville equation for \( N \) particles interacting via a generic potential, without reducing it to a single-particle equation as typically done in the conventional kinetic theory. We found that, due to the presence of the potential, a strong large scale flow creates the forcing in the 3-dimensional bundles of the full 3\( N \)-dimensional coordinate space, with each bundle belonging to a pair of particles. These bundles are destroyed in the course of the standard Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) formalism [3, 5, 11], so that no such forcing manifests in the Boltzmann equation [4, 7], and, subsequently, in the Euler [2] and Navier–Stokes [9] equations.

However, we noted that the direct observations and measurements of a turbulent fluid can register some bulk properties of the induced flow in these particle-pair bundles, such as the power scaling of the Fourier coefficients of the kinetic energy of the flow. In our

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work [1], we made crude estimates of the power scaling of the kinetic energy in the inertial range, induced by a strong large scale flow via the interaction potential. These estimates suggested that the kinetic energy of such flow should scale, in the Fourier space, as $|k|^{-3}$ at large scales (if the forcing was driven by the long-range electrostatic potential), and as $|k|^{-5/3}$ at small scales, where $k$ is the three-dimensional Fourier wavevector. Remarkably, we found our crude estimates to be consistent with direct measurements and observations [6, 15].

Our results [1] further motivated us to look for a more detailed explanation of how the turbulence could be induced by a strong large scale flow via the interaction potential. In the current work, we study our hypothesis in more detail; to simplify the setting, we consider only a single pair of particles which interact via the potential. The paper is organized as follows. In Section 2, we start with the equations of motion of the two particles, and change the variables to those which correspond to the motion of the center of mass of the pair of particles, and the differences between the two particles. The dynamics for these two variables are completely decoupled, which allows us to formulate the Liouville equation for the difference variables alone, and subsequently derive the corresponding velocity moment equations (just as done in the conventional kinetic theory for a single particle). We then combine the equations for the mass, momentum and energy transport to obtain a single equation for the velocity variable.

However, the equation for the velocity involves other variables, such as the density, temperature, shear stress and heat flux. In order to exclude these variables and obtain the closed equation for the velocity alone, in Section 3 we use the hydrostatic balance approximation for the transport of the momentum, and the thermostatic balance approximation for the transport of the energy. The hydrostatic balance approximation assumes that, for a nonstationary solution, the potential forcing term for the momentum transport remains balanced by the pressure gradient (the so-called hydrostatic equation). The thermostatic balance approximation assumes that, for a nonstationary solution, the emerging advective heat flux is automatically balanced by the conductive heat flux, so that the net heat flux of the system remains preserved. These two closures, along with some minor simplifications, result in a fully closed turbulent velocity equation.

To examine the scaling of the velocity in the Fourier space, in Section 4 we express the turbulent velocity equation in terms of the Fourier coefficients. This results in the infinite-dimensional system of nonlinear ordinary differential equations of the second order in time. We simplify this system by removing the majority of the coupling terms, and retaining only the coupling to the strong large scale flow, which is presumed to be the triggering mechanism for the turbulence formation. The latter simplification exposes what could be the underlying mechanism of turbulence formation – the cubic oscillator with a linear term, coupled to the large scale velocity forcing via the interaction potential.

In Section 5, we examine the behavior of the discovered oscillator. Assuming that the strong large scale flow is a jet, we identify the following regimes of its behavior:

a) The linear oscillator regime, where the Fourier wavevector is parallel to the strong large scale flow;

b) The cubic oscillator regime, where the Fourier wavevector is nearly orthogonal to the strong large scale flow.
In the latter case, we find that the Fourier coefficients of the velocity develop periodic motions with large amplitude. This amplitude scales as the inverse two-thirds of the length of the Fourier wavevector.

In Section 6, we show that the corresponding time-windowed averages of the kinetic energy scale as the inverse five-thirds of the length of the Fourier wavevector. This coincides with the famous Kolmogorov scaling [12–14, 16, 17]. The results of this work are summarized in Section 7.

2. The transport equation for the turbulent velocity

2.1. The equations of motion. We start with the dynamical system consisting of two identical particles, with coordinates $x_1$ and $x_2$, and velocities $v_1$ and $v_2$, respectively. These particles interact with each other via a potential $\phi(r)$. Optionally, the particles may also be forced by an external acceleration $g$ (e.g. the Earth gravity). The corresponding equations of motion for the coordinates and velocities of these particles are given via

\begin{align}
\frac{dx_1}{dt} &= v_1, & \frac{dv_1}{dt} &= -\frac{\partial}{\partial x_1} \phi(||x_2-x_1||) + g, \\
\frac{dx_2}{dt} &= v_2, & \frac{dv_2}{dt} &= -\frac{\partial}{\partial x_2} \phi(||x_2-x_1||) + g.
\end{align}

Above, observe that the forcing terms in both equations depend only on the differences in coordinates of the two particles. Thus, it is convenient to change the variables of the dynamics so that the center of mass of the system is one of the variables, and the difference in coordinates is the other one. First, we observe that

\begin{equation}
\frac{\partial}{\partial x_1} \phi(||x_2-x_1||) = -\frac{\partial}{\partial x_2} \phi(||x_2-x_1||),
\end{equation}

which allows us to write

\begin{align}
\frac{d}{dt} \left( \frac{x_1 + x_2}{2} \right) &= \frac{v_1 + v_2}{2}, & \frac{d}{dt} \left( \frac{v_1 + v_2}{2} \right) &= g, \\
\frac{d}{dt} (x_2 - x_1) &= v_2 - v_1, & \frac{d}{dt} (v_2 - v_1) &= -2 \frac{\partial}{\partial x_2} \phi(||x_2-x_1||).
\end{align}

Next, we note that the motion of the center of mass of the system is affected solely by the external acceleration. At the same time, the dynamics for the differences of coordinates and velocities are fully decoupled from the dynamics of the center of mass of the system, and are affected solely by the interaction potential. Thus, denoting $x = x_2 - x_1$ and $v = v_2 - v_1$, we can consider the dynamics for $x$ and $v$ separately from the dynamics of the center of mass:

\begin{align}
\frac{dx}{dt} &= v, & \frac{dv}{dt} &= -2 \frac{\partial}{\partial x} \phi(||x||).
\end{align}
2.2. The Liouville equation and the moment equations. Below we use the usual divergence notation for the Jacobian trace of a vector field whenever stylistically appropriate:

\[ \text{div} a \overset{\text{def}}{=} \frac{\partial}{\partial x} \cdot a = \frac{\partial}{\partial x} a^T. \]

We will also use the following notation for the Jacobian itself:

\[ \frac{\partial a}{\partial x} \overset{\text{def}}{=} \left( \frac{\partial}{\partial x} \otimes a \right)^T, \]

that is, in contrast to the divergence, the Jacobian is the “outer product”, with \( a \) being the column vector, and the differentiation operator being the “row vector”. These operations are easily extended onto matrices – for example, the divergence of a matrix is the contraction of the differentiation operator along each column of the matrix.

Let \( f(t,x,v) \) describe the distribution density of the differences of the coordinates and velocities, independently of the distribution of the coordinate and velocity of the center of mass. The corresponding Liouville equation for this probability density is given, via the vector field of (2.4), as

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} = 2 \frac{\partial \phi}{\partial x} \cdot \frac{\partial f}{\partial v}. \]

Observe that (2.7) is, effectively, the single-particle transport equation with a forcing term, even though \( x \) and \( v \) are not the actual coordinate and velocity, but rather the difference between the coordinates and velocities of the two particles. As usual, we denote the velocity moments via the density \( \rho \), average velocity \( u \), and energy \( E \),

\[ \rho = \int_{\mathbb{R}^3} f \, dv, \quad \rho u = \int_{\mathbb{R}^3} vf \, dv, \quad \rho E = \int_{\mathbb{R}^3} v^2 f \, dv, \]

where \( v^2 = vv^T \) is the outer product of \( v \) with itself. Then, for the moments of (2.7), we integrate by parts the potential forcing terms with \( v \)-derivatives, and obtain

\[\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0, \quad \frac{\partial (\rho u)}{\partial t} + \text{div}(\rho E) = -2\rho \frac{\partial \phi}{\partial x}, \]

\[\frac{\partial (\rho E)}{\partial t} + \frac{\partial}{\partial x} \cdot \int_{\mathbb{R}^3} v^3 f \, dv = -2\rho \left( u^T \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} u^T \right). \]

where the symmetric 3-rank tensor \( v^3 \) is the outer product of \( v \) with itself, computed twice.

2.3. The equation for the turbulent velocity. According to our hypothesis [1], the turbulent velocity motions are induced by a strong large scale flow when the latter communicates its presence onto the inertial scales via the interaction potential \( \phi \). Thus, it is convenient to study this process in the form of an equation which governs the time evolution of the velocity, and, at the same time, includes the terms which couple the velocity itself and the interaction potential. The moment equations (2.9) do not, however, satisfy this requirement – indeed, observe that the potential term in the momentum
equation (2.9a) is not coupled to the velocity \( u \); instead, such a coupling is present in the energy transport equation (2.9b).

In order to combine the time-derivative of the velocity and its coupling to the potential into a single equation, we apply the following series of manipulations:

1. Differentiate the momentum equation (2.9a) in \( t \),

\[
\frac{\partial^2 (\rho u)}{\partial t^2} + \frac{\partial}{\partial x} \cdot \left( \frac{\partial (\rho E)}{\partial t} \right) = 2 \text{div}(\rho u) \frac{\partial \phi}{\partial x},
\]

where in the right-hand side we used the mass equation (2.9a) to replace the time-derivative of \( \rho \) with the divergence of the momentum.

2. Differentiate the energy equation (2.9b) in \( x \):

\[
\frac{\partial}{\partial x} \cdot \left( \frac{\partial (\rho E)}{\partial t} \right) + \frac{\partial^2}{\partial x^2} : \int_{\mathbb{R}^3} v^3 f \, dv = -2 \frac{\partial}{\partial x} \cdot \left[ \rho \left( \frac{u \partial \phi}{\partial x}^T + \frac{\partial \phi}{\partial x} u^T \right) \right],
\]

where "::" denotes the "Frobenius product" of the double-differentiation operator with the two indices of a symmetric 3-rank tensor.

3. Combine the resulting equations using their common mixed derivative of the energy:

\[
\frac{\partial^2 (\rho u)}{\partial t^2} - \frac{\partial^2}{\partial x^2} : \int_{\mathbb{R}^3} v^3 f \, dv = 2 \frac{\partial}{\partial x} \cdot \left[ \rho \left( \frac{u \partial \phi}{\partial x}^T + \frac{\partial \phi}{\partial x} u^T \right) \right] + 2 \text{div}(\rho u) \frac{\partial \phi}{\partial x}.
\]

4. Expand the second time-derivative of the momentum as

\[
\frac{\partial^2 (\rho u)}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial \rho}{\partial t} \frac{\partial u}{\partial t} + \frac{\partial^2 \rho}{\partial t^2} u,
\]

and express the time derivatives of the density and velocity via the transport equations for the mass and momentum (2.9a):

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \cdot \left( \frac{\partial (\rho u)}{\partial t} \right) = \frac{\partial^2}{\partial x^2} : (\rho E) + 2 \frac{\partial}{\partial x} \cdot \left( \frac{\partial \rho}{\partial x} \frac{\partial \phi}{\partial x} \right),
\]

\[
\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial u}{\partial t} = \frac{1}{\rho} \left( \frac{\partial (\rho u)}{\partial t} - \frac{\partial \rho}{\partial t} u \right) = \frac{1}{\rho} \left( \text{div}(\rho E) + 2 \rho \frac{\partial \phi}{\partial x} - \text{div}(\rho u) u \right).
\]

The substitution of the above expressions into (2.12) leads to

\[
\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2} : \int_{\mathbb{R}^3} v^3 f \, dv + 2 \frac{\rho}{\partial t} \text{div}(\rho u) \left( \text{div}(\rho E) + 2 \rho \frac{\partial \phi}{\partial x} - \text{div}(\rho u) u \right) +
\]

\[
\left( \frac{\partial^2}{\partial x^2} : (\rho E) + 2 \frac{\partial}{\partial x} \cdot \left( \frac{\partial \rho}{\partial x} \frac{\partial \phi}{\partial x} \right) \right) u = 2 \frac{\partial}{\partial x} \cdot \left[ \rho \left( \frac{u \partial \phi}{\partial x}^T + \frac{\partial \phi}{\partial x} u^T \right) \right] + 2 \text{div}(\rho u) \frac{\partial \phi}{\partial x}.
\]

The transport equation above has the desired form – the only time derivative it contains is the one for the velocity (which happens to be of the second order), and it also has the forcing terms which couple the velocity to the potential. Hence, we refer to the equation (2.15) as the turbulent velocity equation.
3. The closure for the turbulent velocity equation

Observe that the turbulent velocity equation (2.15) is not closed with respect to its prognostic variable $u$, as it is also coupled to the density $\rho$ and other velocity moments. In what follows, we introduce suitable approximations which allow us to discard the unwanted couplings. We note, however, that the approximations to be introduced cannot be differentiated in time – while the involved terms themselves are presumed to maintain certain approximate relations over time, no such assumptions can be made, generally, about their instantaneous rates of change.

Additionally, given the fact that the motion of particles in a fluid is chaotic, it is clear that the differences between the coordinates and velocities of a pair of particles behave statistically similarly to the coordinate and velocity of a single particle. Thus, below we assume that the physical and statistical properties of the $v$-moments of $f$ are largely same as those of the corresponding velocity moments of a single particle.

Recall that the energy and the cubic moment can be expressed, respectively, via

\begin{align}
E &= u^2 + T, \\
\int_{\mathbb{R}^3} v^3 f dv &= \rho (u^3 + u \otimes T + (u \otimes T)^T + (u \otimes T)^{TT} + Q),
\end{align}

where $T$ and $Q$ are the kinetic temperature matrix and the skewness tensor, respectively:

\begin{align}
\rho T &= \int_{\mathbb{R}^3} (v - u)^2 f dv, \\
\rho Q &= \int_{\mathbb{R}^3} (v - u)^3 f dv.
\end{align}

Above, the subscripts “$T$” and “$TT$” denote the two possible permutations of indices of a 3-rank tensor (just as for a matrix, where the transposition is a single permutation of its two indices). The quantity $\rho T$ is known as the pressure tensor. Additionally, we separate $T$ into the sum of its own trace, and the remainder of the matrix $S$:

\begin{align}
T &= \frac{1}{3} \text{tr}(T) I + S = \theta I + S, \\
\theta &= \frac{1}{3} \text{tr}(T).
\end{align}

Above, $\theta$ is the kinetic temperature, and $S$ is the shear stress, whose trace is zero, by construction. The product $\rho \theta$ is known as the pressure.

In a typical fluid dynamics scenario, the standard closure of the moment equations is either the Euler closure ($S$ and $Q$ set to zero), or the Navier–Stokes closure ($S$ and $Q$ are expressed via the Newton law of viscosity, and the Fourier law of heat conduction, respectively). However, these laws are the consequences of the presence of the collision integral in the Boltzmann equation (for details, see, e.g. Grad [10]). This collision integral is time-irreversible, and, in effect, describes the irreversible dissipation of the kinetic properties of the flow into its heat properties (particularly, the Newton law of viscosity describes the momentum dissipation, while the Fourier law of heat conduction describes the energy dissipation). The current setting is, however, very simplified, and the irreversible collision integral is not present. In what follows, we impose a different closure on (2.15), which forces our model to behave in a realistic fashion despite its simplicity.

3.1. The closure for the shear stress. In practical scenarios, the shear stress is often known to be much smaller than the square of the velocity, that is, $\|S\| \ll \|u^2\|$. However, our simple model lacks the physical mechanism to dissipate the shear stress (such as the irreversible collision integral in the Boltzmann equation for a single particle).
Therefore, in order to satisfy this requirement, we set $S$ to zero akin to what is done in the conventional Euler equations [2, 9]. This closure artificially forces the model to behave realistically, in spite of the absence of a physical reason for the shear stress to be small. Discarding $S$ from the expressions for the energy and the cubic moment leads to

\[ E = u^2 + \theta I, \quad \int_{\mathbb{R}^3} v^3 f \, dv = \rho (u^3 + \theta (u \otimes I + (u \otimes I)^T + (u \otimes I)^{TT}) + Q). \]

3.2. The hydrostatic balance approximation. The second approximation we introduce is known as the “hydrostatic balance” – namely, we assume that, in the momentum equation (2.9a), the gradient of the pressure $\rho \theta$ is balanced by the potential forcing in the momentum equation:

\[ \frac{\partial (\rho \theta)}{\partial x} = -2\rho \frac{\partial \phi}{\partial x}. \]

Observe that this relation is exact in the momentum equation (2.9a) in the case of the steady state with $u = 0$, and all we do is extend it onto an unsteady flow near this steady state. This approximation is known to hold very well in practice for typical subsonic flows (many operational atmospheric models are, in fact, hydrostatic). Together with the approximation of the vanishing shear stress, the hydrostatic balance approximation leads to

\[ \text{div}(\rho E) + 2\rho \frac{\partial \phi}{\partial x} = \text{div}(\rho u^2) + \frac{\partial (\rho \theta)}{\partial x} + 2\rho \frac{\partial \phi}{\partial x} = \text{div}(\rho u^2), \]

and, therefore, the turbulent velocity equation (2.15) becomes

\[ \rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2} \left( \int_{\mathbb{R}^3} v^3 f \, dv + \rho \frac{\partial u}{\partial x} + \left( \frac{\partial^2}{\partial x^2} : (\rho u^2) \right) u \right) = 2 \frac{\partial}{\partial x} \cdot \left[ \rho \left( \frac{\partial \phi^T}{\partial x} + \frac{\partial \phi}{\partial x} u^T \right) \right] + \text{div}(\rho u) \frac{\partial \phi}{\partial x}. \]

3.3. The thermostatic balance approximation. In order to close the system with respect to the skewness tensor, we will use a similar principle as the hydrostatic balance above, except that it will be applied to the net heat flux of the system.

The conductive heat flux $q$ is defined via the following velocity-centered cubic moment:

\[ \rho q = \frac{1}{2} \int_{\mathbb{R}^3} \|v - u\|^2 (v - u) f \, dv. \]

Observe that $2q$ is the contracted, along any pair of indices, skewness tensor $Q$:

\[ 2(q)_i = \sum_{j=1}^{3} (Q)_{iji}. \]

It is easy to see that the full skewness tensor $Q$ can be written in the form

\[ Q = \frac{2}{5} (q \otimes I + (q \otimes I)^T + (q \otimes I)^{TT}) + R, \]
where \( R \) is a fully symmetric 3-rank tensor whose contractions along any pair of indices are zero. The skewness deviator \( R \) is related to \( Q \) in the same way as the shear stress \( S \) is related to the kinetic temperature matrix \( T \) – namely, \( R \) quantifies the deviation of \( Q \) from its own trace along any pair of its indices. In the current work, we are going to make the same assumption for \( R \) as we made above for the shear stress \( S \), that is, \( \| R \| \ll \| u^3 \| \), and thus it can be discarded from the cubic moment (3.4):

\[
\int_{\mathbb{R}^3} v^3 f \, dv = \rho \left\{ u^3 + \left( \theta u + \frac{2}{5}q \right) \otimes I + \left[ \left( \theta u + \frac{2}{5}q \right) \otimes I \right]^T + \left[ \left( \theta u + \frac{2}{5}q \right) \otimes I \right]^{TT} \right\}.
\]

The same assumption of negligible \( R \) is used in Grad’s 13-moment system [10], as well as its regularization [18].

Next, we are going to introduce the appropriate closure with respect to the heat flux. Recall that, in practical real-world settings, the moment transport equations for the probability density of a single particle can be endowed with external heat fluxes. In particular, this can be seen in the atmospheric models, which exchange the heat energy with the surface of the Earth, absorb the energy arriving from the Sun, and also drain the heat energy into the outer space in the form of the electromagnetic emission. Thus, the dynamics of the atmosphere are closed with respect to the mass transport (as the mass is not exchanged with the outside medium), largely closed with respect to the momentum transport (as the momentum is not exchanged with the outside medium, except for the fictitious Coriolis force due to Earth rotation), but completely open with respect to the heat exchange. Thus, the balance of the heat fluxes is the important benchmark of any realistic model of the atmosphere.

Here, we are also going to assume that our system is thermally open, and thus must balance the heat fluxes. For simplicity, we set the net external heat flux in our system to zero. Now, assume that, under the above condition, together with the hydrostatic balance approximation (3.5), the system of the moment equations (2.9) has a steady solution with \( u = 0 \) (that is, the gas is at rest). If this is possible, it implies that \( q = 0 \), otherwise there would be a nonzero forcing term in the energy transport equation.

Next, we introduce a nonzero velocity \( u \) into the system. This velocity creates the corresponding advective heat flux \( q_a \), given via

\[
q_a = c_p \theta u = \frac{5}{2} \theta u,
\]

where \( c_p = 5/2 \) is the heat capacity at constant pressure. Observe that \( c_p \) depends only on the number of degrees of freedom which can store the kinetic energy, and since, in our variables, the pair of particles has three degrees of freedom, its \( c_p \) is the same as that of the usual ideal gas. The advective heat flux \( q_a \) is the amount of the heat energy mechanically transported by the flow with the velocity \( u \) and temperature \( \theta \). In the current work, we assume that the heat flux remains balanced, which means that the system responds to the change in velocity via the “automatic” creation of the conductive heat flux \( q \). The latter balances \( q_a \) so that the net heat flux in the system remains preserved:

\[
q = -q_a = -\frac{5}{2} \theta u.
\]
Thermodynamically, it means that, even though the advective heat flux is expected to carry the heat energy by transporting the mean temperature along the mean flow, the distribution \( f \) “automatically” adjusts its skewness in the opposite direction, so that the net heat flux of the system remains preserved. We call the relation in (3.13) the thermostatic balance approximation. Upon the substitution of (3.13) into (3.11), the terms with \( \theta u \) and \( q \) cancel out, and the expression for the cubic moment simplifies to

\[
\int v^3 f \, dv = \rho u^3.
\]

Further observing that

\[
\left( \frac{\partial^2}{\partial x^2} : (\rho u^2) \right) u + 2 \text{div}(\rho u) \frac{\partial u}{\partial x} - \frac{\partial^2}{\partial x^2} : (\rho u^2) = -2\rho \left( \frac{\partial u}{\partial x} \right)^2 u - \rho \frac{\partial^2 u}{\partial x^2} : u^2,
\]

we arrive at the following form of the turbulent velocity equation (2.15):

\[
\rho \left[ \frac{\partial^2 u}{\partial t^2} - 2 \left( \frac{\partial u}{\partial x} \right)^2 u - \frac{\partial^2 u}{\partial x^2} : u^2 \right] = 2 \frac{\partial}{\partial x} \left[ \rho \left( \frac{\partial \phi^T}{\partial x} + \frac{\partial \phi}{\partial x} u^T \right) \right] + 2 \text{div}(\rho u) \frac{\partial \phi}{\partial x}.
\]

### 3.4. The closed form of the turbulent velocity equation.

At this point, what remains to be done is to exclude the density \( \rho \) from the turbulent velocity equation. Note that the left-hand side is now a multiple of \( \rho \). In the right-hand side, we express

\[
\frac{\partial}{\partial x} \left[ \rho \left( \frac{\partial \phi^T}{\partial x} + \frac{\partial \phi}{\partial x} u^T \right) \right] + \text{div}(\rho u) \frac{\partial \phi}{\partial x} = \rho \left( \Delta \phi \right) I + \rho \left( 2 \text{div} u \right) I + \rho \left( 2 \frac{\partial \phi}{\partial x} I + \frac{\partial \phi^T}{\partial x} \right) \frac{\partial \phi}{\partial x},
\]

where \( \Delta \) is the Laplace operator (that is, the trace of the second derivative). Next, we assume that, for the purposes of approximating the potential forcing term, the derivative of the density can be expressed by rearranging the hydrostatic balance relation (3.5):

\[
\frac{\partial \rho}{\partial x} = -\frac{\rho}{\theta} \frac{\partial}{\partial x} (\theta + 2\phi).
\]

Upon the substitution of (3.18) into the potential forcing term, all terms of the turbulent velocity equation become multiples of \( \rho \), which we subsequently cancel out:

\[
\frac{\partial^2 u}{\partial t^2} - 2 \left( \frac{\partial u}{\partial x} \right)^2 u - \frac{\partial^2 u}{\partial x^2} : u^2 = 2 \left[ \left( \Delta \phi \right) I + \frac{\partial^2 \phi}{\partial x^2} \right] u + \left( 2 \mathcal{L} \cdot u + \mathcal{L} u \right) \frac{\partial \phi}{\partial x},
\]

\[
\mathcal{L} \cdot = \frac{\partial}{\partial x} \cdot - \frac{1}{\theta} \frac{\partial (\theta + 2\phi)}{\partial x} \cdot.
\]

In (3.19b), the first term differentiates the argument, while the second term multiplies it.

For the final simplification, observe that, for a flow at normal conditions, the kinetic temperature \( \theta \) has the form \( \theta = \theta_0 + \theta' \), where the base value \( \theta_0 \) is a large constant (usually amounting to molecular velocities in excess of 500 m/s), while the fluctuating part \( \theta'(t, x) \) and its derivatives are of the same order as the rest of the terms in (3.19a). Additionally, in the inertial range, where turbulence is typically registered, the effect
of the interaction potential $\phi$ is weak in comparison with $\theta_0$. Observe, however, that, while the above conditions are observed in nature, our simplified model here does not necessarily possess relevant physical effects to fulfill them. Therefore, here we treat the kinetic temperature in the same fashion as the shear stress is treated in the conventional Euler equations – namely, we simply postulate that $\theta$ has the form above, despite the possible lack of physical effects in our model which cause this form to manifest itself.

Under such conditions, it is clear that the contribution of the second term in the expression (3.19b) for $L\mathbf{u}$ is much smaller than the contribution of the rest of the terms in the right-hand side of (3.19a). Therefore, the former can be discarded, which leads to the turbulent velocity equation in the following closed form:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - 2 \left( \frac{\partial \mathbf{u}}{\partial x} \right)^2 \mathbf{u} - \frac{\partial^2 \mathbf{u}}{\partial x^2} : \mathbf{u}^2 = 2 \left[ \left( (\Delta \phi) \mathbf{I} + \frac{\partial^2 \phi}{\partial x^2} \right) \mathbf{u} + \left( 2(\text{div}\mathbf{u}) \mathbf{I} + \frac{\partial \mathbf{u}}{\partial x} \right) \frac{\partial \phi}{\partial x} \right].$$

Observe that, in the present form, the closed turbulent velocity equation (3.20) is a nonlinear PDE, which was derived purely from the “physical principles”, without taking into account any of its mathematical properties. As a result, it may have solutions which blow up in finite time, or otherwise behave irregularly. However, for the purpose of this work, we will presume that finite, physically meaningful solutions of (3.20) also exist, at least on the time scale of interest.

4. Simplified dynamics for the Fourier coefficients of turbulent velocity

Here, for simplicity, we assume that the turbulent velocity equation is solved in a periodic cube of size $L$, with the corresponding volume $V = L^3$. The standard Fourier expansion of $\mathbf{u}(t, x)$ in such a cube is given via

$$\mathbf{u}(t, x) = \sum_{k \in \mathbb{Z}^3} \hat{\mathbf{u}}(t, k) e^{2\pi i k \cdot x / L}, \quad \hat{\mathbf{u}}(t, k) = \frac{1}{V} \int_V \mathbf{u}(t, x) e^{-2\pi i k \cdot x / L} \, dx.$$

In what follows, for convenience, we replace $k \to 2\pi k / L$, so that the components of the vector $k$ are no longer integers, and its units are $[L]^{-1}$. Also, we will assume that, on the scales of interest, the potential $\phi(r) \sim 1/r$, and its Fourier coefficient has the form

$$\hat{\phi}(k) = \frac{\phi_0}{\|Lk\|^2},$$

where $\phi_0$ is a real constant. These are the same assumptions as in our previous work; in particular, the Thomas–Fermi [8, 19] and Ziegler–Biersack–Littmark [20] interatomic potentials, as well as the electrostatic potential, have the Fourier coefficients of the form (4.2) in the range of interest (for details, see Abramov [1]). We place the factor $L^2$ in the denominator of (4.2) to ensure that the constant $\phi_0$ has the kinetic energy units.

In terms of the velocity Fourier coefficients $\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, k)$, the closed turbulent velocity equation (3.20) is given via

$$\frac{d^2 \hat{\mathbf{u}}}{dt^2} + 2(\hat{\mathbf{u}} k^T) * (\hat{\mathbf{u}} k^T) * \hat{\mathbf{u}} + (\hat{\mathbf{u}} \otimes k^2) * (\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) =$$

$$= -2 \left[ \left( \|k^2 \mathbf{I} + k^2 \right) \hat{\phi} * \hat{\mathbf{u}} + \left( 2(k \cdot \hat{\mathbf{u}}) \mathbf{I} + \hat{\mathbf{u}} k^T \right) * (\hat{\phi} k) \right],$$
where "\(*\)" denotes a convolution,
\[(4.4) \quad (\hat{a} \ast \hat{b})(k) = \sum_m \hat{a}(k - m)\hat{b}(m),\]
and "\(\otimes\)" is the combination of the convolution with an outer product. In what follows, we make the same assumptions as in our previous work [1], namely, that the solution of (3.20) is initially placed near the state \(u = 0\), perhaps, with small random fluctuations. At the initial time, a strong jet \(u_0\) is instantaneously created at the large scales – that is, it is confined to a small subset of large scale wavenumbers, \(|k| \sim L^{-1}\). By “strength”, here we refer to the average energy of \(u_0\), which we assume to be much greater than the energy \(\phi_0\) of the interaction potential (4.2):
\[(4.5) \quad \frac{1}{V} \int_V ||u_0||^2 dx = \sum_k ||\hat{u}_0(k)||^2 \gg |\phi_0|.

Above, the equality follows from Plancherel’s theorem. This condition ensures that \(u_0\) effectively obeys the moment equations (2.9) without the potential forcing, and thus behaves in the same way as a typical large scale flow in a fluid dynamics setting.

The wavenumber of interest \(k\) corresponds to small scales, \(|k| \gg L^{-1}\), where the velocity dynamics evolve rapidly in time. We assume that, on the characteristic time scale of \(k\), the Fourier coefficients \(\hat{u}_0\) of the strong large scale flow can be regarded as constants. Our hypothesis of turbulence formation [1] is that the turbulent motions in the inertial range are induced by the coupling of the velocity Fourier coefficients to the strong large scale flow. In order to study this hypothesis, here we further simplify (4.3) by discarding the coupling of the given Fourier coefficient \(\hat{u}(k)\) to all other Fourier coefficients, except those of the strong large scale flow \(\hat{u}_0\). In the nonlinear terms with convolutions, we also retain the terms with the self-coupling, and the coupling to the corresponding complex-conjugate Fourier coefficient \(\hat{u}^*(k) = \hat{u}(-k)\).

For such a simplification, the terms with convolutions in (4.3) become
\[(4.6a) \quad (\hat{u}_k^T) \ast (\hat{u}_k^T) \ast \hat{u} \approx -(k \cdot \hat{u})^2 \hat{u}^* - \sum_m (m \cdot \hat{u}(k))(m \cdot \hat{u}_0^*(m))\hat{u}_0(m) + \sum_m (k \cdot \hat{u}_0^*(m))(\hat{u}_0(m)\hat{u}^T(k) - \hat{u}(k)\hat{u}_0^T(m))m,\]
\[(4.6b) \quad (\hat{u} \otimes k^2) \ast (\hat{u} \otimes \hat{u}) \approx 2|k \cdot \hat{u}|^2 \hat{u} + (k \cdot \hat{u})^2 \hat{u}^* + \sum_m |k \cdot \hat{u}_0(m)|^2 \hat{u}(k) + 2\sum_m (m \cdot \hat{u}(k))(m \cdot \hat{u}_0^*(m))\hat{u}_0(m).\]

For the sum of the two terms, we subsequently have
\[(4.7) \quad 2(\hat{u}_k^T) \ast (\hat{u}_k^T) \ast \hat{u} + (\hat{u} \otimes k^2) \ast (\hat{u} \otimes \hat{u}) \approx 2|k \cdot \hat{u}|^2 \hat{u} - (k \cdot \hat{u})^2 \hat{u}^* + \sum_m |k \cdot \hat{u}_0(m)|^2 \hat{u}(k) + 2\sum_m (k \cdot \hat{u}_0^*(m))(\hat{u}_0(m)\hat{u}^T(k) - \hat{u}(k)\hat{u}_0^T(m))m.\]

Clearly, the second sum in the right-hand side above is much smaller than the first sum, because the latter is of the order \(|k|^2\), while the former is of the order \(|k|\|m\|. Thus,
we further drop the second sum from the approximation. Finally, we note that

\[ \sum_m |k \cdot \hat{u}_0(m)|^2 = k^2 : \sum_m \hat{u}_0(m) \hat{u}_0^T(-m) = k^2 : \frac{1}{V} \int_V u_0^2 \, dx, \]

where \( V \) is the volume of the physical domain, and the last identity results from Plancherel’s theorem. Denoting the bulk energy matrix of the large scale flow via \( \mathbf{E}_0 \),

\[ \mathbf{E}_0 = \frac{1}{V} \int_V u_0^2 \, dx, \]

we have

\[ 2(\hat{u}k^T) \ast (\hat{u}k^T) \ast \hat{u} + (\hat{u} \otimes k^2) \ast (\hat{u} \otimes \hat{u}) \approx (k^2 : \mathbf{E}_0 + 2|k \cdot \hat{u}|^2) \hat{u} - (k \cdot \hat{u})^2 \hat{u}^*. \]

The two terms with the potential forcing in the right-hand side of (4.3) become

\[ (2(k \cdot \hat{u})I + \hat{u}k^T) \ast (\hat{\phi}k) = \sum_m \hat{\phi}(k - m)(2(m \cdot \hat{u}(m))I + \hat{u}(m)m^T)(k - m) = O(\phi_0L^{-3}\|k\|^{-1}\|u_0\|), \]

\[ ((\|k\|^2I + k^2)\hat{\phi}) \ast \hat{u} = \sum_m \hat{\phi}(k - m)((\|k - m\|^2I + (k - m)^2)\hat{u}(m) = \]

\[ = (\|k\|^2I + k^2)(\hat{\phi} \ast \hat{u}_0) + O(\phi_0L^{-3}\|k\|^{-1}\|u_0\|), \]

where we observe that \( \|m\| \sim L^{-1} \), and take into account the form of \( \hat{\phi} \) in (4.2). Thus, retaining only the dominant term in the right-hand side of (4.3), we arrive at

\[ \frac{d^2 \hat{u}}{dt^2} + (k^2 : \mathbf{E}_0 + 2|k \cdot \hat{u}|^2) \hat{u} - (k \cdot \hat{u})^2 \hat{u}^* = -2(\|k\|^2I + k^2)(\hat{\phi} \ast \hat{u}_0). \]

5. Behavior of Turbulent Velocity in the Simplified Model

Recall that, under our prior assumptions [1], at the time when the strong large scale flow \( u_0 \) is introduced into the system, the initial state \( \hat{u} \) of (4.12) and its time derivative are zero. Additionally, in this work we assume that \( u_0 \) has the form of a jet – that is, a one-dimensional flow in a given direction. If we let \( n_0 \) denote the unit vector in the direction of this jet, then \( u_0, \hat{u}_0, \) and the related quantities in the Fourier space become

\[ u_0 = u_0n_0, \quad \hat{u}_0 = \hat{u}_0n_0, \quad \hat{\phi} \ast \hat{u}_0 = (\hat{\phi} \ast \hat{u}_0)n_0, \quad \mathbf{E}_0 = \mathbf{E}_0n_0^2. \]

As a result, the forcing in the right-hand side of (4.12) acquires the form

\[ -2(\|k\|^2I + k^2)(\hat{\phi} \ast \hat{u}_0) = -2\|k\|^2\hat{\phi} \ast \hat{u}_0(I + n^2)n_0, \]

where \( n = k/\|k\| \) is the unit vector in the direction of \( k \). It is, therefore, clear that, with zero initial condition, the solution \( \hat{u} \) of (4.12) remains collinear to this one-dimensional forcing, and, together with its dot-product with \( k \), can be written in the form

\[ \hat{u} = \frac{1}{2}\hat{u}(I + n^2)n_0, \quad k \cdot \hat{u} = \hat{u}(k \cdot n_0), \]
where the coefficient $1/2$ in front of $\dot{u}$ is introduced for convenience. Substituting the above expressions into (4.12), we arrive at the equation for the scalar variable $\dot{u}$,

$$
\frac{d^2 \dot{u}}{dt^2} + (k \cdot n_0)^2 (E_0 + |\dot{u}|^2) \dot{u} = -4\|k\|^2 \hat{\phi} \ast \dot{u}_0,
$$

where we note that $\ddot{u}^2 \dot{u} = |\dot{u}|^2 \dot{u}$. Next, observe that the coefficient in front of $\dot{u}$ in the second term of the left-hand side is purely real and nonnegative, and thus, in the complex plane, $\dot{u}$ remains collinear to $-\hat{\phi} \ast \dot{u}_0$ at all times. The latter means that we only need to evaluate the modulus $|\dot{u}|$ of $\dot{u}$. Denoting $u = |\dot{u}|$, we obtain the following differential equation for the nonnegative scalar variable $u(t)$:

$$
\frac{d^2 u}{dt^2} + (k \cdot n_0)^2 (E_0 + u^2) u = 4\|k\|^2 |\hat{\phi} \ast \dot{u}_0|.
$$

The choice of the orientation of $k$ relative to $n_0$ results in different types of behavior of the solution of (5.5). An obvious special case of such an orientation occurs when $k \cdot n_0 = 0$, leading to the constant acceleration dynamics

$$
\frac{d^2 u}{dt^2} = 4\|k\|^2 |\hat{\phi} \ast \dot{u}_0|, \quad u(t) = 2\|k\|^2 |\hat{\phi} \ast \dot{u}_0| t^2.
$$

The direction of $\dot{u}(t, k)$ in this case is strictly parallel to the strong large scale jet $u_0$.

For $k \cdot n_0 \neq 0$, the equation (5.5) becomes the cubic oscillator with a linear term, whose solution varies between zero and some maximal state. The equilibrium state $u_{eq}$ of such an oscillator is given via the depressed cubic equation

$$
u_{eq}^3 + E_0 u_{eq} = \frac{4|\hat{\phi} \ast \dot{u}_0|}{(n \cdot n_0)^2}.
$$

It is easy to see that this equation has a unique real positive root. Indeed, denoting

$$p = \frac{1}{3}E_0, \quad q = \frac{2|\hat{\phi} \ast \dot{u}_0|}{(n \cdot n_0)^2},
$$

we find, via Cardano’s formula,

$$u_{eq} = \left(q + \sqrt{q^2 + p^3}\right)^{1/3} + \left(q - \sqrt{q^2 + p^3}\right)^{1/3}.
$$

The balance of terms above in (5.9) depends on how $p^3$ is related to $q^2$. Observe that

$$|\hat{\phi} \ast \dot{u}_0| \sim |\hat{\phi}| \frac{E_0^{1/2}}{\|k\|^2},
$$

and thus

$$\frac{q^2}{p^3} = \frac{108|\hat{\phi} \ast \dot{u}_0|^2}{(n \cdot n_0)^4 E_0^3} \sim \left(\frac{|\phi_0|}{(Lk \cdot n_0)^2 E_0}\right)^2 = \left(\frac{|\phi_0|}{(Lk^2 : E_0)}\right)^2.
$$

Therefore, choosing the orientation of $k$ relative to $n_0$ affects the balance of terms in (5.9), which further results in a different behavior of the solution of (5.5). Here we examine two limiting cases of such a behavior – the linear and cubic oscillators.
5.1. **Linear oscillator**. Choosing $k$ to be parallel to $u_0$ yields

\[ \frac{q^2}{p^3} \sim \left( \frac{|\phi_0|}{\|Lk\|^2 \|\bar{E}_0\|} \right)^2 \ll 1 \]

due to (4.5) and the fact that $L\|k\| \gg 1$. This leads to the linear oscillator regime, with

\[ u_{eq} \approx (q + p^{3/2})^{1/3} + (q - p^{3/2})^{1/3} \approx p^{1/2} \left[ \frac{1}{3} \left( \frac{q}{p^{3/2}} + 1 \right) + \frac{1}{3} \left( \frac{q}{p^{3/2}} - 1 \right) \right] = \frac{2q}{3p}. \]

The corresponding scaling estimate of $\hat{u}_{eq}$ in this case is given via

\[ \|\hat{u}_{eq}\| \approx \frac{4|\hat{\phi} \star \hat{u}_0|}{\|\bar{E}_0\|} \sim \frac{|\phi_0|}{\|Lk\|^2 \|\bar{E}_0\|^{1/2}}. \]

If the cubic term in (5.5) is discarded, the solution to the resulting linear oscillator equation is given via

\[ u(t) = \frac{4|\hat{\phi} \star \hat{u}_0|}{\|\bar{E}_0\|} \left[ 1 - \cos \left( \|k\| \|\bar{E}_0\|^{1/2} t \right) \right]. \]

The direction of $\hat{u}(t, k)$ in this case is also parallel to the direction of $u_0$, and coincides with the direction of $k$. The period of oscillations $T$ in the linear regime is given via

\[ T = \frac{2\pi}{\|k\| \|\bar{E}_0\|^{1/2}}. \]

Observe, however, that the magnitude of oscillations in the linear regime decays with increasing $\|u_0\|$, which suggests that it is likely difficult to practically detect the motions in such a regime.

5.2. **Cubic oscillator**. Here, we choose $k$ to be *almost* orthogonal to $u_0$, with the condition

\[ 0 < (Lk)^2 : \bar{E}_0 \ll |\phi_0|, \quad \text{or} \quad \frac{q^2}{p^3} \gg 1. \]

This allows the linear term in (5.5) to be discarded, leading to the regime of a purely cubic oscillator, with

\[ u_{eq} = q^{1/3} \left( 1 + \sqrt{1 + p^3/q^2} \right)^{1/3} + q^{1/3} \left( 1 - \sqrt{1 + p^3/q^2} \right)^{1/3} \approx (2q)^{1/3}. \]

The corresponding scaling estimate for $\hat{u}_{eq}$ is given via

\[ \|\hat{u}_{eq}\| \approx \frac{1}{2} \left( \frac{4|\hat{\phi} \star \hat{u}_0|}{(n \cdot n_0)^2} \right)^{1/3} \sim \frac{1}{\|Lk\|^{2/3}} \left( \frac{|\phi_0| \|\bar{E}_0\|^{3/2}}{(n^2 : \bar{E}_0)} \right)^{1/3}. \]

Observe that, in the cubic regime, the magnitude of $\|\hat{u}_{eq}\|$ can be quite large – for a given potential $\phi$, the right-hand side is multiplied by $\|\bar{E}_0\|^{1/2}$ (which is large), and divided
by \((n^2 : \mathbf{E}_0)^{1/3}\) (which is small, due to (5.17)). In particular, from (5.17), we can estimate the magnitude of \(\hat{u}_{eq}\) below via

\[(5.20)\]
\[\|\hat{u}_{eq}\| \gg \frac{\|E_0\|^{1/2}}{Lk^{2/3}},\]

that is, we can expect the Fourier coefficients \(\hat{u}\) of (4.12), whose wavevector \(k\) is almost orthogonal to the large scale flow \(u_0\), but \(\hat{u}\) itself is nearly aligned with \(u_0\) to scale, at the very least, as the magnitude of the large scale flow, scaled by the negative two-thirds power of the nondimensionalized wavenumber.

Let us also compute the estimate for the period of oscillations of \(u\) in the regime of the cubic oscillator. To accomplish this, we center the cubic oscillator at zero,

\[(5.21)\]
\[\frac{d^2 u}{dt^2} + (k \cdot n_0)^2 u^3 = 0,\]

and instead assume that the initial condition is offset from zero by \(u_{eq}\), while its starting time derivative remains zero. This, of course, is not an invariant transformation (unlike the one for the linear oscillator), because, even though the oscillations around zero are symmetric, the oscillations in (5.5), which occur between zero and some maximal state, are asymmetric due to nonlinearity. Nonetheless, we disregard this observation to obtain a crude estimate of the period.

The equation above can be integrated at least once, and the period of oscillations can be computed from there. Multiplying both sides by \(2\dot{u}\) and integrating, we arrive at

\[(5.22)\]
\[\frac{du^2}{dt} = -\frac{(k \cdot n_0)^2 du^4}{2}, \quad \dot{u}^2 = \frac{(k \cdot n_0)^2}{2} (u_{eq}^4 - u^4),\]

where the constant of integration is chosen so as to ensure that \(\dot{u}\) at the initial time is zero. Next, we introduce the change of variables

\[(5.23a)\]
\[u = u_{eq} \cos z, \quad \dot{u} = -u_{eq} \sin z \cdot \dot{z},\]

\[(5.23b)\]
\[u_{eq}^4 - u^4 = (u_{eq}^2 - u^2)(u_{eq}^2 + u^2) = u_{eq}^4 \sin^2 z (1 + \cos^2 z).\]

Upon substitution, the sines cancel out on both sides, and we obtain

\[(5.24)\]
\[\dot{z} = \pm |k \cdot n_0| u_{eq} \sqrt{\frac{1 + \cos^2 z}{2}}.\]

Clearly, \(z\) increases from zero to \(\pi/2\) during one-quarter of the period, which allows us to separate the variables, integrate, and arrive at

\[(5.25)\]
\[T = \frac{4K(1/2)}{|k \cdot n_0| u_{eq}}, \quad K(1/2) = \int_0^{\pi/2} \frac{dz}{\sqrt{1 - \sin^2 z/2}},\]

where \(K\) denotes the complete elliptic integral of the first kind. The corresponding scaling of \(T\) with \(\|k\|\) can be estimated as

\[(5.26)\]
\[T \sim \left(\frac{L^2}{|\phi_0||k||(n^2 : \mathbf{E}_0)^{1/2}}\right)^{1/3}.\]
Similarly to the linear regime, for a fixed \( n \), here the period of oscillations decreases with increasing \( \| k \| \) and \( \| u_0 \| \). However, the rate of increase is slowed down by the relatively low powers of \( \| k \| \) and \( \| u_0 \| \). Thus, while the oscillations keep becoming faster, they do so at a much decreased rate.

6. The Scaling of the Windowed Time Averages of the Kinetic Energy

So far, we have estimated the scaling of the equilibrium states \((5.14)\) and \((5.19)\) of the decoupled and simplified turbulent velocity equation \((4.12)\). These different states correspond to different combinations of orientations of the wavevector \( k \), and the solution \( \hat{u}_k \), relative to the direction of the strong large scale flow \( u_0 \):

a) The scaling estimate \((5.14)\) corresponds to both \( k \) and \( \hat{u}_k \) aligned with \( u_0 \);

b) The scaling estimate \((5.19)\) corresponds to \( k \) nearly orthogonal to \( u_0 \), and \( \hat{u}_k \) nearly aligned with \( u_0 \). This combination has the strongest magnitude of \( \hat{u}_k \), estimated via \((5.20)\).

However, recall that, in the experiments, the measured quantity is the windowed time average of the kinetic energy of the flow instead [6, 15]. Here, we are going to relate the latter to the former.

To estimate the scaling of a windowed time average of the kinetic energy, we refer to the momentum equation \((2.9a)\), where we additionally use the hydrostatic balance approximation \((3.5)\). The momentum equation thus becomes

\[
\frac{\partial (\rho u)}{\partial t} = -\text{div} (\rho u^2).
\]

Let the time averaging window be denoted via \( \tau \). Integrating the above equation between \( t \) and \( t + \tau \), and dividing by \( \tau \), we have

\[
\frac{\rho(t + \tau)u(t + \tau) - \rho(t)u(t)}{\tau} = -\text{div} \frac{1}{\tau} \int_t^{t+\tau} \rho(s)u^2(s) \, ds.
\]

In the right-hand side, we use the mean value theorem on \( \rho \):

\[
\frac{\rho(t + \tau)u(t + \tau) - \rho(t)u(t)}{\tau} = -\text{div} \left( \rho(t_1) \langle u^2 \rangle \right), \quad \langle u^2 \rangle = \frac{1}{\tau} \int_t^{t+\tau} u^2(s) \, ds,
\]

where \( t \leq t_1 \leq t + \tau \). Next, in the same way as in Section 3, via the hydrostatic balance relations in \((3.5), (3.18)\) and \((3.19b)\), we approximate

\[
\frac{\rho(t + \tau)u(t + \tau) - \rho(t)u(t)}{\tau} \approx -\rho(t_1) \mathcal{L} \cdot \langle u^2 \rangle.
\]

Here, we are going to presume that the density does not change much over time, that is, \( \rho(t) \approx \rho(t_1) \approx \rho(t + \tau) \). Note that this assumption does not amount to incompressibility and cannot be differentiated in time – particularly, it is natural for \( \rho \) to rapidly oscillate in time with a large time derivative, yet not drift too far away from its base value over time (e.g. acoustic waves). If we cancel out the density on both sides above, we arrive at

\[
\frac{u(t + \tau) - u(t)}{\tau} \approx -\mathcal{L} \cdot \langle u^2 \rangle.
\]
For the same assumptions for \( L \) as above in Section 3, with a large base value of the kinetic temperature, we can neglect the second term in the expression (3.19b) for \( L \):

\[
\frac{u(t + \tau) - u(t)}{\tau} \approx - \text{div} \langle u^2 \rangle, \quad \text{or} \quad i \frac{\tau}{\tau} (\hat{u}_k(t + \tau) - \hat{u}_k(t)) \approx k \cdot \langle u^2 \rangle_k.
\]

If \( \tau \sim T \) (where \( T \) is the period of the oscillations for the given scale), then we can expect that, on average, the registered magnitude of the right-hand side will be

\[
\| \langle u^2 \rangle_k \| \sim \frac{\| \hat{u}_{eq} \|}{\tau \| k \|}.
\]

The kinetic energy scalings for the linear (5.14) and cubic (5.19) regimes are, respectively,

\[
\| \langle u^2 \rangle_k \|_{\text{lin}} \sim \| k \|^{-3}, \quad \| \langle u^2 \rangle_k \|_{\text{cub}} \sim \| k \|^{-5/3}.
\]

Remarkably, the kinetic energy scaling for the estimate with the strongest magnitude (5.19) matches Kolmogorov’s “five-thirds” law [12–14, 16, 17].

7. Summary

In the current work, we study the dynamics of two particles, which interact via a potential. Upon formulating the equations of motion for this pair of particles, we decouple the dynamics by considering separately the motion of the center of mass of the system, and the difference of the coordinates of the particles. We formulate the Liouville equation for the coordinate and velocity difference variables, and then derive the velocity moment transport equations in the same manner as in the conventional fluid mechanics. We close these moment equations using the hydrostatic and thermostatic approximations, and arrive at the closed equation for the velocity variable. In the course of the derivation, we use the following assumptions and approximations:

i) Small shear stress and skewness deviator in the transport of the momentum and the energy, in comparison to the corresponding power of the velocity;

ii) The hydrostatic balance approximation for the transport of the momentum;

iii) The thermostatic balance approximation (the balance of the advective and conductive heat fluxes) for the transport of the energy;

iv) The assumption that the base kinetic temperature of the system is much greater than the energy of the interaction potential.

Next, we focus on the corresponding system of equations for the Fourier coefficients of the velocity. In order to study what we believe is the underlying mechanism of the turbulence formation, we discard the coupling of a given Fourier coefficient of the velocity in the inertial range to all other Fourier coefficients, with the exception of the coupling to the strong large scale flow. On the characteristic time scale of the inertial range, the strong large scale flow is treated as a constant in time. As a result, we obtain a simplified equation for the given Fourier coefficient of the velocity, which is easy to examine.

The simplified dynamics reveal what could be the fundamental mechanism of turbulence formation. It is comprised of the cubic oscillator with a linear term, forced by the strong large scale flow via the interaction potential. When this strong large scale flow is a jet, we identify two distinct regimes of behavior of this oscillator (linear and
cubic), whose manifestation depends on the orientation of the wavevector of the given Fourier coefficient relative to the direction of the large scale forcing jet. We find that the cubic regime corresponds to the largest amplitude of oscillations, and the scaling of the time-windowed averages of its kinetic energy matches Kolmogorov’s “five-thirds” law.

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