Generalized fusion potentials

Ofer Aharony

School of Physics and Astronomy
Beverly and Raymond Sackler
Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv, Tel-Aviv, 69978, Israel

ABSTRACT

Recently, DiFrancesco and Zuber have characterized the RCFTs which have a description in terms of a fusion potential in one variable, and proposed a generalized potential to describe other theories. In this note we give a simple criterion to determine when such a generalized description is possible. We also determine which RCFTs can be described by a fusion potential in more than one variable, finding that in fact all RCFTs can be described in such a way, as conjectured by Gepner.

† Work supported in part by the US-Israel Binational Science Foundation and the Israel Academy of Sciences.
1. Introduction

Rational conformal field theories (RCFT’s) are characterized by the fusion rules of their (finite number of) fields:

\[ \phi_i \phi_j = \sum_k N_{ij}^k \phi_k \]  

(1.1)

where the indices \( i, j, k \) run over the \( N \) primary chiral fields in the operator algebra of the RCFT, and the \( N_{ij}^k \) are non-negative integers. This algebra is commutative and associative, giving symmetry and crossing constraints on the structure constants \( N_{ij}^k \). As shown by E. Verlinde \[^{[3]}\], the fusion rules can be expressed in terms of the unitary modular matrix \( S \) of the RCFT:

\[ N_{ij}^k = \sum_l S_{il} S_{jl} S_{kl}^{*} \]  

(1.2)

where \( \phi_1 \) is the identity operator.

A trivial representation of this algebra via \( N \times N \) matrices can be obtained by representing the field \( \phi_i \) as the matrix

\[ (\phi_i)_{jk} = N_{ij}^k. \]  

(1.3)

These matrices form a representation of the fusion algebra as can be checked using the associativity conditions. It follows from (1.2) that the matrix \( S \) diagonalizes all the matrices \( \phi_i \), and their eigenvalues are of the form:

\[ \lambda_i^{(l)} = \frac{S_{il}}{S_{1l}} \]  

(1.4)

where \( \lambda_i^{(l)} \) designates the \( l \)th eigenvalue of \( \phi_i \).
It was shown by Gepner\textsuperscript{[1]} that any such algebra can be represented as a ring of polynomials in variables $x_1, \ldots, x_n$ modulo some ideal, such that the multiplication of fields becomes simple polynomial multiplication modulo several polynomials. Moreover, Gepner conjectured\textsuperscript{[1]} that this ideal of constraints can always be derived from a potential, and has explicitly shown that this is true for the fusion rules of $SU(N)_k$ (this has since been shown for other Kac-Moody theories as well\textsuperscript{[5],[4]}). Such a representation is useful because it has a simple geometrical interpretation\textsuperscript{[1]} as well as a physical interpretation (such as in Landau-Ginzburg models). The demands we have from such a representation is that the trivial polynomials $x_i$ will represent actual primary fields of the theory (for $i = 1, \ldots, n$), that the representatives of different fields will be linearly independent, and of course that the correct fusion rules will be recovered from the ring of polynomials modulo the derivatives of this potential. To recover the fusion rules of the RCFT from such a representation we need to know the way all primary fields are represented as polynomials in addition to knowing the potential.

Di Francesco and Zuber\textsuperscript{[2]} have examined which theories have a representation in terms of a fusion potential in one variable. They found a simple criterion to determine whether this is possible: it is possible whenever the field we choose to be represented by $x$ has no degenerate eigenvalues (otherwise there is no polynomial representation at all, let alone a potential representation). They have also proposed a generalized representation in which one does not demand that all fields are linearly independent over the real numbers $\mathbb{R}$, but only that they are linearly independent over the rational numbers $\mathbb{Q}$. This is enough to enable recovering the fields’ fusion rings from the polynomial ring. In ref. [2] several examples of such representations were given. In section 2 of this note we will give a simple criterion determining whether such a representation is possible or not (and in fact providing a simple way to construct all such representations whenever they exist).

In section 3, we examine fusion potentials using polynomials in more than one variable. We find that the generalization of the results of ref. [2] to this case is quite straightforward, and that whenever we have fields $x_1, \ldots, x_n$ such that no
two eigenvectors are degenerate for all of them, we can represent the theory by
a potential in these variables. This will prove Gepner’s conjecture, since in any
RCFT there are no eigenvectors which are degenerate for all fields. Unfortunately,
we will find that there exist many different potential descriptions for a single RCFT,
meaning that the potential of a theory is probably not the best way to characterize
it.

2. Generalized one-variable fusion potentials

As mentioned above, all the matrices representing the fields in their matrix
representation have the same eigenvectors, which we shall denote below by $v_j$ (for
$j = 1, ..., N$). We can look at these vectors as combinations of fields, in which case
they are given by

$$v_j = \sum_i S_{ij}^* \phi_i$$  \hspace{1cm} (2.1)

or equivalently

$$\phi_i = \sum_j S_{ij} v_j.$$  \hspace{1cm} (2.2)

By definition the eigenvectors satisfy

$$\phi_i v_j = \lambda_i^{(j)} v_j,$$  \hspace{1cm} (2.3)

where $\lambda_i^{(j)}$ are given by (1.4), and from these equations one can easily check that
fusion of two eigenvectors gives

$$v_i v_j = \delta_{ij} v_j \frac{\lambda_i^{(j)}}{S_{ij}},$$  \hspace{1cm} (2.4)

or

$$v_i v_j = \delta_{ij} v_j \frac{1}{S_{ij}}.$$  \hspace{1cm} (2.5)

In this section we will be interested in representations of the fusion algebra
in terms of a fusion potential in one variable. It is obvious that in this case
we get just one polynomial constraint on our variable \( x \) \( \left( \frac{\partial V(x)}{\partial x} = 0 \right) \) and that any such constraint can be derived from a potential (by integration). From the matrix representation described above, it is clear that the constraint of minimal degree satisfied by a field \( x \) is exactly the minimal polynomial of the matrix that represents it. In [2] it was shown that when this polynomial is of degree \( N \) (so that all eigenvalues of \( x \) are different) we can represent the fusion algebra as the algebra of polynomials in \( x \) modulo the constraint given by this minimal polynomial (which can obviously be derived from a potential). If the eigenvalues of \( x \) are denoted by \( \mu_j \) the representation of \( \phi_i \) is given simply by the polynomial of degree \( N - 1 \) that transforms \( \mu_j \) to \( \lambda_i^{(j)} \) for \( j = 1, \ldots, N \). Obviously, if we do not demand that \( x \) be a field from the theory, we can always build such a representation in terms of any \( x \) which has no degenerate eigenvalues, but usually we want \( x \) to be one of our fields and in [2] it was shown that this is only possible when \( x \) has no degenerate eigenvalues. Theories which have no field without degenerate eigenvalues (such as the unitary minimal models excluding the Ising model, or general \((p,q)\) minimal models with \( p,q > 3 \)) cannot be, therefore, represented by a fusion potential in one variable, in the usual way.

In the next section we will show that such theories can always be represented by a potential in more than one variable. For now let us stay with the one variable case and, following [2], try to relax some of our requirements from the representation to enable such a description after all. Let us look at a certain field \( x \) from which we wish to generate our polynomial algebra. If \( x \) has \( m \) different eigenvalues, it’s minimal polynomial is of degree \( m \). Hence, we can only have \( m \) different polynomials which are not linearly dependent over \( \mathbb{R} \) (for instance, we can take them to be \( 1, x, \ldots, x^{m-1} \)). However, to reconstruct the fusion algebra from the potential, all we need is that the polynomials representing the different fields be linearly independent over \( \mathbb{Q} \), since we know that the coefficients \( N_{ik}^{j,k} \) are all integers. In [2], Di Francesco and Zuber have analyzed some examples for which such a generalized representation is possible and some for which it is not. We shall now analyze the general case and construct all possible representations of this form for
a given theory.

Suppose we wish to know if we can find such a representation for a given field $x$ with eigenvalues $\lambda_i$, (for $i = 1, ..., m$) such that the minimal polynomial of $x$ (which obviously still must be the constraint we impose on the polynomial algebra) is given by

$$V'(x) = \prod_{i=1}^{m} (x - \lambda_i). \tag{2.6}$$

We are looking for polynomial representations of all other fields $\phi_i$ which will satisfy the fusion algebra (modulo $V'(x)$) and which will be linearly independent over $\mathbb{Q}$. Since there is a non-singular linear transformation between the fields and the eigenvectors (given by (2.1),(2.2)), this is equivalent to finding a polynomial representation for the eigenvectors $v_j$, which should satisfy the same requirements.

Suppose that $v_i$ is an eigenvector with an eigenvalue $\lambda_k$ for $x$: in this case one of the equations which should be satisfied modulo $V'(x)$ is $x v_i = \lambda_k v_i$ or $(x - \lambda_k) v_i = 0$. From the form of $V'(x)$ (2.6) it is clear that the only possible form for the polynomial representing $v_i$ (to be denoted by $V_i$) is

$$V_i(x) = \alpha_i \prod_{j \neq k \atop \lambda_k - \lambda_j} \frac{x - \lambda_j}{\lambda_k - \lambda_j} \tag{2.7}$$

where the product goes over all different eigenvalues of $x$, and $\alpha_i$ is a normalization constant (chosen so that $V_i(\lambda_k) = \alpha_i$). Now let us use equation (2.5) for $i = j$. Since the equation should be satisfied exactly at the point $x = \lambda_k$ where $V'(x)$ vanishes, we find that

$$\alpha_i^2 = \frac{1}{S_{1i}} \alpha_i. \tag{2.8}$$

Thus, for all $i$, either $\alpha_i = 0$ or $\alpha_i = \frac{1}{S_{1i}}$. For all such choices equation (2.5) is trivially satisfied if $v_i$ and $v_j$ correspond to different eigenvalues of $x$. However, if it is to be satisfied for two eigenvectors $v_i$ and $v_j$ corresponding to the same eigenvalue it is necessary that either $\alpha_i$ or $\alpha_j$ be zero. This proves that at most
one of the $\alpha_i$ corresponding to some eigenvalue $\lambda_k$ of $x$ can be different than zero. In fact, exactly one must be different from zero. This can be seen from the equation

$$1 = \sum_j S_{1j} V_j(x)$$

which is equation (2.2) taken for $i = 1$, and which must be satisfied exactly for $x = \lambda_k$. We have thus found which polynomial representations in terms of a field $x$ we can build in a certain theory (without yet demanding linear independence of the polynomials representing the fields). For every eigenvalue $\lambda_k$ of $x$ we should choose one eigenvector which has this eigenvalue, set it’s polynomial to be (2.7) with $\alpha_i = \frac{1}{S_{ii}}$, and set the polynomials of all other eigenvectors with the same eigenvalue to zero. From the polynomials of the eigenvectors we get via equation (2.2) the polynomials representing the fields. The number of different possible representations utilizing a certain field $x$ is therefore the product of the multiplicities of it’s eigenvalues, and we have given a prescription for the construction of all such representations.

Next we should check when this construction gives a faithful polynomial representation, with fields which are linearly independent over $\mathbb{Q}$. We, therefore, check if there exists a rational linear combination of the polynomials which is zero. Since the transformation from the fields to the eigenvectors is nonsingular, a linear combination of fields is equivalent to a linear combination of eigenvectors, and from our construction it is clear that such a combination is zero if and only if it contains only eigenvectors which we have set to zero (there are $N - m$ such eigenvectors). The question, therefore, is whether in the vector space generated by the eigenvectors which we have set to zero there is a vector with rational coefficients (in terms of the fields) or not. This can easily be checked for a given theory, as demonstrated below for some examples, but we have not been able to obtain a simpler criterion to determine when this is possible without constructing all eigenvectors.

Let us now analyze two simple examples which were also analyzed in [2] from this point of view. Let us start with the $D_{2\nu+2}$ which has been shown in [2] to
have such a representation whenever $2\nu + 1$ is not a square of an integer. From the considerations above this follows straightforwardly. It turns out that the field $x = \phi_1$ has exactly one degenerate eigenvalue, $\lambda = -1$, which has two corresponding eigenvectors given (in the basis $\phi_0, \phi_1, ..., \phi_{\nu-1}, \phi_{\nu}^+, \phi_{\nu}^-$ and in a convenient normalization) by

\[
v_1 = (1, -1, 1, ..., (-1)^{\nu-1}, \frac{1}{2}(1 + \sqrt{2\nu + 1}), \frac{1}{2}(1 - \sqrt{2\nu + 1}))
\]
\[
v_2 = (1, -1, 1, ..., (-1)^{\nu-1}, \frac{1}{2}(1 - \sqrt{2\nu + 1}), \frac{1}{2}(1 + \sqrt{2\nu + 1}))
\] (2.10)

A potential representation of the algebra in terms of $x$ is therefore obtained by setting one of these eigenvectors to zero (and no other eigenvector), and this vector is a rational combination of the fields if and only if $2\nu + 1$ is the square of an integer, as was derived in [2]. For other $\nu$ we obtain in this way exactly the two possible representations given in [2].

Our second example will be the $(4,5)$ minimal model (the tricritical Ising model) for which several such representations were obtained in [2]. The eigenvectors for this case are (written in the basis $\phi_{(1,1)}, \phi_{(1,2)}, \phi_{(1,3)}, \phi_{(1,4)}, \phi_{(2,1)}, \phi_{(2,2)}$ for the fields) :

\[
v_1 = (1, \mu_1, \mu_1, 1, \sqrt{2}, \sqrt{2}\mu_1)
\]
\[
v_2 = (1, \mu_1, \mu_1, 1, -\sqrt{2}, -\sqrt{2}\mu_1)
\]
\[
v_3 = (1, -\mu_1, \mu_1, -1, 0, 0)
\]
\[
v_4 = (1, \mu_2, \mu_2, 1, \sqrt{2}, \sqrt{2}\mu_2)
\]
\[
v_5 = (1, \mu_2, \mu_2, 1, -\sqrt{2}, -\sqrt{2}\mu_2)
\]
\[
v_6 = (1, -\mu_2, \mu_2, -1, 0, 0)
\] (2.11)

where $\mu_{1,2}$ are the two roots of the equation $x^2 - x - 1 = 0$, and the eigenvectors
of the various fields are given (in the above order for the eigenvectors) by:

\[
\begin{align*}
\lambda_{(1,1)}^i &= (1, 1, 1, 1, 1) \\
\lambda_{(1,2)}^i &= (\mu_1, -\mu_1, \mu_2, -\mu_2) \\
\lambda_{(1,3)}^i &= (\mu_1, \mu_1, \mu_2, \mu_2) \\
\lambda_{(1,4)}^i &= (1, 1, 1, 1, -1) \\
\lambda_{(2,1)}^i &= (\sqrt{2}, 0, \sqrt{2}, 0) \\
\lambda_{(2,2)}^i &= (\sqrt{2}, -\mu_1, \sqrt{2}, -\mu_2) \\
\end{align*}
\]  

(2.12)

We can see that any field (except \(\phi_{(1,1)}\)) can be chosen as \(x\). This follows from the observation that the only combinations of eigenvectors which have rational coefficients in terms of the fields include \(v_3\) and \(v_6\) or \(v_1, v_2, v_4\) and \(v_5\). For example, if we wish \(x\) to be \(\phi_{(1,1)}\) we have 4 possibilities, since either \(v_1\) or \(v_2\) and either \(v_4\) or \(v_5\) must be set to zero, giving exactly the 4 representations given in [2] for this case.

For \(x = \phi_{(2,1)}\) we have 8 different representations which are also all faithful. For \(x = \phi_{(2,2)}\) we must set to zero \(v_3\) or \(v_6\), and we obtain a representation with a potential of the highest possible degree. The potential of the lowest possible degree is obtained if we wish to take \(x = \phi_{(1,4)}\). In this case only one of \(v_1, v_2, v_4, v_5\) is different from zero, and only one of \(v_3, v_6\), giving altogether 8 possible representations. For example, if we choose \(v_1\) and \(v_6\) to be non-zero we obtain the representation:

\[
\begin{align*}
\phi_{(1,1)} &= 1 \\
\phi_{(1,2)} &= \frac{1}{2}x + \frac{\sqrt{5}}{2} \\
\phi_{(1,3)} &= \frac{\sqrt{5}}{2}x + \frac{1}{2} \\
\phi_{(1,4)} &= x \\
\phi_{(2,1)} &= \frac{1}{\sqrt{2}}(x + 1) \\
\phi_{(2,2)} &= \frac{\mu_1}{\sqrt{2}}(x + 1)
\end{align*}
\]  

(2.13)

which satisfies the algebra when taken modulo the constraint \(V'(x) = x^2 - 1 = 0\).
3. Multi-variable fusion potentials

As Gepner has shown in ref. [1], any RCFT can be represented as a ring of polynomials modulo some ideal of polynomials. This ideal is exactly the ideal of polynomials which vanish at all of the points \((\lambda_1^{(i)}, \lambda_2^{(i)}, ..., \lambda_n^{(i)})\) where we have chosen the polynomials to be polynomials in variables \(x_1, ..., x_n\) corresponding to the fields \(\phi_1, ..., \phi_n\), and where \(i\) goes over the \(N\) eigenvectors of the theory. We wish to find a potential \(V(x_1, ..., x_n)\) whose derivatives will generate this ideal, meaning that any function vanishing at all of the above points can be written as a (polynomial) linear combination of the derivatives of the potential \(V\). Since all the fusion rules are exactly satisfied at the above points (as seen from the matrix representation of the algebra), they will all be generated by the potential and vice versa. Obviously this is only possible when all of the above points are different (since if all of the fields \(\phi_i\) have degenerate eigenvalues for some pair of eigenvectors, there is no way to represent a field which is not degenerate for these eigenvectors as a polynomial in them). This is a necessary condition, and we will show that it is also sufficient for the existence of a polynomial representation.

We will start by analyzing the simple case of a theory represented by polynomials in two fields, which we shall denote by \(x\) and \(y\), and where \(x\) has no degenerate eigenvalues. Of course in this case there is a representation of the theory in terms of polynomials in \(x\) alone\(^2\), but we will later be able to generalize to other cases. Let us denote the aforementioned points by \((\lambda_i, \mu_i)\) : we will look for a potential of the form \(V(x, y) = P(x) + yQ(x)\) which gives constraints of the form

\[
\begin{align*}
Q(x) &= 0 \\
P'(x) + yQ'(x) &= 0
\end{align*}
\]  

(3.1)

which must be satisfied only at the above points. This can be trivially solved by choosing \(Q(x)\) to be a polynomial vanishing only at the points \(\lambda_i\), namely \(Q(x) = \prod_i (x - \lambda_i)\), and by choosing \(P(x)\) to satisfy the \(N\) equations

\[
\begin{align*}
P'(&\lambda_i) + \mu_i Q'(\lambda_i) = 0
\end{align*}
\]  

(3.2)
for $i = 1, \ldots, N$. For example, we could choose $P(x)$ to be the polynomial of the lowest degree satisfying these constraints, which is

$$P(x) = \sum_i (-\mu_i) \int_{x_0}^x dx' \prod_{j \neq i} (x' - \lambda_j)$$

(3.3)

for any $x_0$. For this choice of the potential, it is obvious that the points in which it’s derivatives vanish are exactly the desired points, and we will show explicitly that the algebra satisfied by the matrices $x, y$ can indeed be represented as the algebra of polynomials modulo the derivatives of this potential.

In general, the polynomials in this algebra are linear combinations of the polynomials $x^n y^m$ for all $n, m$, and only $N$ of these polynomials (viewed as matrix polynomials) are linearly independent. Since we chose $x$ not to have degenerate eigenvalues these can be chosen to be the polynomials $1, x, x^2, \ldots, x^{N-1}$. We need to show that all other polynomials can be represented as linear combinations of these (modulo the derivatives of the potential) and that they give rise to the correct algebra. For polynomials of $x$ alone this is obvious since $Q(x)$ is the characteristic polynomial of the matrix $x$ and all higher powers of $x$ can be expressed as combinations of the above basis modulo $Q(x)$ alone. Let us show that $y$ can also be expressed as a combination of polynomials from the above basis. It is enough to show that it can be expressed as a general polynomial in $x$ modulo the derivatives (3.1):

$$y = h(x) \pmod{Q(x), P'(x) + yQ'(x)}$$

(3.4)

or equivalently that there exist polynomials $f(x, y), g(x, y), h(x)$ such that

$$y = h(x) + f(x, y)Q(x) + g(x, y)(P'(x) + yQ'(x)).$$

(3.5)

But, since we chose $Q(x)$ to have no degenerate zeroes, $Q(x)$ and $Q'(x)$ have no
common divisor, so that there exists a solution \( p(x), q(x) \) to the equation

\[
p(x)Q'(x) + q(x)Q(x) = 1 \tag{3.6}
\]

and thus we can take \( f(x, y) = yq(x), g(x, y) = p(x) \) and \( h(x) = -g(x)P'(x) \) to be the solution to (3.5). We have thus proven that all polynomials in \( x \) and \( y \) can be expressed (modulo the derivatives of the potential) as linear combinations of the basis elements \( 1, x, ..., x^{N-1} \). Since the equations are satisfied by the matrices corresponding to \( x \) and \( y \), and since the basis elements have the correct fusion rules, this representation is indeed a good representation of the desired algebra.

Now, let us continue to the more interesting case when neither \( x \) nor \( y \) have degenerate eigenvalues. To handle this case we notice that a linear change of variables from \((x, y)\) to new variables of the form \((\tilde{x} = ax + by, \tilde{y} = cx + dy)\), which is non-singular (namely, \(ad - bc\) is non-zero), does not change the set of points where the derivatives of \( V \) (as expressed in terms of the new variables) vanish. Explicitly, if \( V(x, y) \) satisfies \( \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0 \) only at the points \((\lambda_i, \mu_i)\), then \( V(\tilde{x}, \tilde{y}) \) will satisfy \( \frac{\partial V}{\partial \tilde{x}} = \frac{\partial V}{\partial \tilde{y}} = 0 \) only at the points \((a\lambda_i + b\mu_i, c\lambda_i + d\mu_i)\). All we need to do, therefore, is to find \(a, b, c, d\) that satisfy \(ad - bc \neq 0\) and such that all points \(a\lambda_i + b\mu_i\) are different. Then we can transform to the variables \(\tilde{x} = ax + by, \tilde{y} = cx + dy\) and build the potential in exactly the same way as above. The proof that the representation is faithful works in exactly the same way as above (working in the transformed basis \( 1, ax + by, (ax + by)^2, ..., (ax + by)^{N-1} \) instead of the one above) and can also serve to obtain explicit fusion rules from the potential in any desired basis (although since we have used the fusion rules to build the potential in the first place, this does not give us additional information).

It should be mentioned, that although the eigenvalues \( \lambda_i \) and \( \mu_i \) are in general non-rational, the potential we have obtained will have rational coefficients (obviously we can then write it with integer coefficients as well) as long as the parameters \(a, b, c, d\) are rational. Before the transformation to the new variables, it is trivial that \( Q(x) \) described above has integer coefficients, since it is the characteristic polynomial of the integer valued matrix representing \( x \). The coefficients of
$P'(x)$ described above can also be written in terms of traces and determinants of matrices of the form $x^ny^m$ which are of course also integers. Thus, $P(x)$ will also have rational coefficients. It is then obvious that if $a, b, c, d$ are all rational, this property remains true after the transformation to $(\tilde{x}, \tilde{y})$ as well.

As an example, let us analyze the $(4, 5)$ minimal model (the tri-critical Ising model) which was also analyzed as an example in the previous section, where the eigenvalues of it’s fields were given. From the eigenvalues we can easily see which pair of fields can generate a two-variable fusion potential representation. For example we can choose $x$ to be $\phi_{(1,2)}$ and $y$ to be $\phi_{(2,1)}$. The simplest choice for the change of variables is $a = b = d = 1, c = 0$ so that $\tilde{x} = x + y, \tilde{y} = y$, and in this case the polynomials we get from the above procedure are

$$Q(\tilde{x}) = (\tilde{x}^4 - 2\tilde{x}^3 - 5\tilde{x}^2 + 6\tilde{x} - 1)(\tilde{x}^2 + \tilde{x} - 1) \quad (3.7)$$

and

$$P'(\tilde{x}) = -4(\tilde{x}^2 + \tilde{x} - 1)(2\tilde{x}^2 - 2\tilde{x} - 1) \quad (3.8)$$

so that the potential turns out to be

$$V(x, y) = -\frac{8}{5}(x + y)^5 + \frac{20}{3}(x + y)^3 - 2(x + y)^2 - 4(x + y) +$$

$$y((x + y)^4 - 2(x + y)^3 - 5(x + y)^2 + 6(x + y) - 1) \cdot ((x + y)^2 + (x + y) - 1) \quad (3.9)$$

with constraints (which are linear combinations of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$) of the form

$$((x + y)^4 - 2(x + y)^3 - 5(x + y)^2 + 6(x + y) - 1)((x + y)^2 + (x + y) - 1) = 0 \quad (3.10)$$

and

$$-4((x + y)^2 + (x + y) - 1)(2(x + y)^2 - 2(x + y) - 1) +$$

$$y(4(x + y)^3 - 6(x + y)^2 - 10(x + y) + 6)((x + y)^2 + (x + y) - 1) +$$

$$y((x + y)^4 - 2(x + y)^3 - 5(x + y)^2 + 6(x + y) - 1)(2(x + y) + 1) = 0 \quad (3.11)$$

By the procedure analyzed above, we can represent all fields $x^n y^m$ as linear combi-
nations of 1, \((x+y), (x+y)^2, \ldots, (x+y)^5\) modulo the above constraints. To work in another basis (in this case a comfortable basis is 1, \(x, x^2, x^3, y, xy\)) we have to obtain the transformation between the two bases which enables us to write any field in our preferred basis. In this way we can systematically obtain the constraints associated with the fusion rules in their more recognizable form (recall \(x = \phi_{(1,2)}, y = \phi_{(2,1)}\))

\[
\begin{align*}
x^4 &= 3x^2 - 1 \\
x^2y &= xy + y \\
y^2 &= x^3 - 2x + 1
\end{align*}
\]

where all equalities are satisfied modulo the above constraints. We should note that the constraints (3.12) cannot be derived directly from a potential.

For all minimal models, polynomials in two variables (namely, \(\phi_{(1,2)}\) and \(\phi_{(2,1)}\)) are sufficient to obtain a potential description of the fusion rules of the theory, but in other models this is not necessarily the case. The construction described above can quite easily be generalized to the case of more than two variables: let us denote the fields we want to generate the polynomial algebra as \(x^{(i)}\) for \(i = 1, \ldots, n\) and their eigenvalues by \(\lambda_{j}^{(i)}\) for \(j = 1, \ldots, N\). Again we may assume that \(x^{(1)}\) has no degenerate eigenvalues, otherwise we can make a linear transformation to new variables in which that will be the case. For this case a suitable potential is:

\[
V(x^{(i)}) = P(x^{(1)}) + x^{(2)} \prod_{i=1}^{N} (x^{(1)} - \lambda_{i}^{(1)}) + \\
\sum_{l=3}^{n} \sum_{i=1}^{N} \frac{1}{2} (x^{(l)} - \lambda_{i}^{(l)})^2 \prod_{j \neq i} (x^{(1)} - \lambda_{j}^{(1)})
\]

where \(P(x^{(1)})\) is chosen so that the polynomial \(\frac{\partial V}{\partial x^{(1)}}\), which is linear in \(x^{(2)}\), will vanish at \(x^{(2)} = \lambda_{i}^{(2)}\) and \(x^{(j)} = \lambda_{i}^{(j)}\) for all other variables. This potential works because the derivative with respect to \(x^{(2)}\) vanishes only when \(x^{(1)}\) is at one of it’s eigenvalues, then the derivative with respect to \(x^{(l)}\) (for \(l = 3, \ldots, n\)) vanishes only when \(x^{(l)}\) is at it’s corresponding eigenvalue, and the derivative with respect
to $x^{(1)}$ forces that to be the case for $x^{(2)}$ as well. The proofs that this potential has rational coefficients, and that it gives a faithful representation of the fusion algebra, work for this case in the same way as in the two variable case. Since for $n$ large enough such a representation is available for any fusion algebra, this proves Gepner’s conjecture$^{[1]}$.

4. Summary and conclusions

In this paper we analyzed various forms of potential representations of fusion algebras. We started by analyzing the generalized one-variable representation suggested in ref. [2], giving a simple criterion to determine (given the fusion rules) when such a representation is possible. We have also given a simple way to construct all such representations (there is always a finite number of them). Representations of this sort, however, do not appear to have a simple ”physical” meaning, since in physical theories we usually allow fields to be multiplied by any real number and not just by rational numbers.

We then went on to analyze usual fusion representations in more than one variable, showing that any theory can be represented in such a way. In fact, we have shown that whenever there is any polynomial representation for an algebra where the variable $x_i$ represent certain fields $\phi_i$ (which is always available given enough fields as shown in ref. [1]) we can find a potential representation in the same variables. Unfortunately, the representation of this sort is far from unique. Even for a given choice of fields as generators of the algebra there is an infinite number of representations of this sort, corresponding for example to different solutions of equation (3.2). The geometrical meaning of this kind of representations was analyzed in [1], where the algebra was interpreted as the algebra of modular transformations of the hyper-surface $V = 0$. Their physical meaning can perhaps be derived from Landau-Ginzburg models. For this we would like to interpret them as being perturbed from some conformal point, with the fields themselves (which have to be given externally in the simple potential representation) given as the
derivatives of the potential with respect to the various available perturbations. It is not yet clear to us when such an interpretation is possible, and perhaps it can serve to limit the number of possible potential representations.

I would like to thank Prof. S. Yankielowicz for suggesting to me the subject of this work and for discussions on this subject.

REFERENCES

1. D. Gepner, *Comm. Math. Phys.* **141** 381-411 (1991)

2. P. Di Francesco and J.-B. Zuber, ”Fusion potentials I”, SPhT 92/138, [hep-th/9211138](http://arxiv.org/abs/hep-th/9211138)

3. E. Verlinde, *Nucl. Phys.* **B300** [FS22] 360-376 (1988)

4. D. Gepner and A. Schwimmer, *Nucl. Phys.* **B380** 147-167 (1992)

5. M. Bourdeau, E.J. Mlawer, H. Riggs, H.J. Schnitzer, *Mod. Phys. Lett.* **A7** 689-700 (1992)