Absorption of angular momentum by black holes and D-branes

Samir D. Mathur

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

We consider the absorption of higher angular momentum modes of scalars into black holes, at low energies, and ask if the resulting cross sections are reproduced by a D-brane model. To get the correct dependence on the volume of the compactified dimensions, we must let the absorbing element in the brane model have a tension that is the geometric mean of the tensions of the D-string and an effective stringlike tension obtained from the D-5-brane; this choice is also motivated by T-duality. In a dual model we note that the correct dependence on the volume of the compact dimensions and the coupling arise if the absorbing string is allowed to split into many strings in the process of absorbing a higher angular momentum wave. We obtain the required energy dependence of the cross section by carrying out the integrals resulting from partitioning the energy of the incoming quantum into vibrations of the string.

April, 1997

1 E-mail: me@ctpdown.mit.edu
1. Introduction.

With the development of string theory and the ideas of duality, there has been considerable progress in our understanding of black holes. Following suggestions of Susskind, the number of string theory states at weak coupling have been found to agree with the number of states expected from the Bekenstein entropy of the hole that would form at strong coupling [1][2][3][4]. Further, it was found that the rates of absorption and emission of minimal scalars computed at weak coupling matched the Hawking radiation rates expected from the black hole at strong coupling [5][6]. This calculation has been extended to emission of charged quanta [7], to higher orders in the energy of the incident quantum under certain conditions [8], and to nonminimal ‘fixed’ scalars [9].

To make contact with the black hole information paradox, we need to understand the absorption of quanta that are small in size compared to the horizon, so that they can fall into the horizon through a reasonably localised direction. This implies that we understand the absorption of higher angular momentum modes, since a wavepacket that is to be localised in the angular directions must be composed of several components of angular momentum $l$. It is also important to understand the absorption at wavelengths small compared to the size of the horizon, since that too is required to well localise the infalling quantum.

In this paper we discuss the absorption of low energy higher $l$ modes for minimal scalars, by the classical black hole and by the D-brane model of the hole. For the case $l = 0$ it had turned out to be adequate to use a model where a D-string absorbed an incoming scalar by converting its energy into vibration modes on the D-string [5]. An equivalent result was obtained in the S-dual model where the absorbing element was an elementary string, and the absorption amplitude was computed using standard perturbative string theory [8].

But there are difficulties with naively extending these models to the absorption of quanta with higher $l$. Let us consider the model of the 4+1 dimensional black hole introduced in [9]. The spacetime has total dimension 10, of which 5 space dimensions are compactified on a 5-torus $T^5 = T^4 \times S^1$. The $S^1$ is the direction in which the absorbing string is wound, while the 5-branes in the model wrap all over the $T^5$. Let the volume of the $T^4$ be $V_4$, and the length of the $S^1$ be $L$.

The incident quantum is expected to convert its energy into modes that travel in the direction of the circle $S^1$. As the angular momentum $l$ of the incident quantum is
increased, we expect that more and more such (fermionic) modes will be created. But if these modes are vibrations of the D-string, then the absorption calculation will be confined to the vicinity of this string, and will not be sensitive to the volume $V_4$ that is available transverse to the D-string. Thus the $V_4$ dependence of the cross section will not change with $l$. But the classical cross section does depend on $V_4$; this dependence is $\sim V_4^{-1-l}$.

If we use a dual model where the absorbing element is an elementary string, and consider the absorption process as a fundamental string interaction diagram, then we see that the incoming massless quantum can create no more than 4 new fermions in the final state, if we use the three point tree vertex. This is because the world sheet conformal theory is a free theory, and the massless scalar has at most two fermionic oscillator to contribute to each of the left and right sides. But we need a number of fermions that increases without bound with increasing $l$. If we allow loops, then we get additional powers of $g^4$ in the cross section for every extra loop, while the classical cross section is seen to increase by powers of $g^2$ as $l$ increases by one unit.

In this paper we do the following:

(a) We observe that in the D-brane model we get the correct dependence of the absorption cross section on $V_4$ if we let the absorbing element be a long string with a tension that is the geometric mean of the tension of the D-string and the effective tension obtained for vibrations of the 5-D-brane which travels in the long direction $S^1$. We note that such a choice is also T-duality symmetric.

We also note that in the dual elementary string model, the correct dependence on $V_4$ and $g$ is obtained if we allow the initial string to split into $l+1$ strings when absorbing a quantum of angular momentum $l$. The details of the amplitude calculation are however not very clear in such a model.

(b) The energy of the incoming quantum is expected to be shared between a pair of bosonic quanta and $2l$ fermionic quanta, travelling in the direction of the circle $S^1$. We carry out the integrals over momenta, and obtain the energy dependence that is required by the classical cross section.

Recently the absorption of higher angular modes has been considered for 3-branes \cite{10} and for the 4+1 dimensional black hole with three charges through an effective conformal theory \cite{11}. There now exist a large number of results pertaining to the black hole - D-brane comparison. The behavior of the D-brane as a black body was discussed in \cite{12}, where it was shown that emissions of quanta are proportional to the classically expected
emissions. The issue of higher orders in coupling was discussed in [13]. Comparisons of brane and classical absorption were discussed in [14] [15].

The plan of this paper is as follows. In section 2 we discuss the classical cross section. In section 3 we discuss the issue of $V_4$ dependence in the D-brane model. Section 4 discusses dependence on other parameters of the model. In section 5 we discuss a possible description in the dual model where we use the elementary string Polyakov amplitudes. In section 6 we discuss the energy dependence of the amplitudes. Section 7 is a discussion.

2. The classical absorption cross section.

The metric of the 5-dimensional hole is

$$
\text{ds}_5^2 = -f^{-2/3} h dt^2 + f^{1/3} \left( h^{-1} dr^2 + r^2 d\Omega_3^2 \right),
$$

(2.1)

where

$$
h(r) = \left( 1 - \frac{r_0^2}{r^2} \right), \quad f(r) = \left( 1 + \frac{r_p^2}{r^2} \right) \left( 1 + \frac{r_1^2}{r^2} \right) \left( 1 + \frac{r_5^2}{r^2} \right).
$$

(2.2)

Let

$$
r_p^2 = r_0^2 \sinh^2 \sigma_p
$$

(2.3)

We will be in the region of parameter space where $r_p \sim r_0 \ll r_1, r_5$. Thus only the momentum-antimomentum excitations on the string are excited; the excitations of strings-antistrings and 5-branes-anti 5-branes is suppressed.

We will consider the absorption of a graviton that is a scalar from the 5-dimensional point of view. This is a minimally coupled scalar, and satisfies the free wave equation on the 5-dimensional black hole metric. The absorption probability for a spherical wave of angular momentum $l$ was computed in [16] [11]. In the limit where $r_1, r_5 \gg r_0, r_p$ this probability is

$$
a^l = \frac{\omega^3}{4\pi} \frac{A_H}{[l!(l+1)!]^2} \left( \omega r_0 \right)^{2l} \left| \frac{\Gamma \left( \frac{l+2}{2} - i \frac{\omega}{4\pi T_L} \right) \Gamma \left( \frac{l+2}{2} - i \frac{\omega}{4\pi T_R} \right)}{\Gamma \left( 1 - i \frac{\omega}{2\pi T_H} \right)} \right|^2
$$

(2.4)

where

$$
A_H = 2\pi^2 r_1 r_5 r_p \cosh \sigma_p
$$

(2.5)

is the area of the horizon. The temperature of the black hole is

$$
T_H = \frac{r_0}{2\pi r_1 r_5 \cosh \sigma_p}
$$

(2.6)
The left and right temperatures are

\[ T_L = \frac{r_0 e^{\sigma p}}{2\pi r_1 r_5}, \quad T_R = \frac{r_0 e^{-\sigma p}}{2\pi r_1 r_5} \] (2.7)

The absorption cross section for angular momentum \( l \) is (see Appendix C for details)

\[ \sigma^l = (l + 1)^2 \frac{4\pi}{\omega^3} a_l \] (2.8)

We have, for \( l \) odd

\[ \sigma^l = (l + 1)^2 \frac{\pi^3 (r_1 r_5)^{2l+2} \omega^{2l-1}}{2^{2l} [l!(l+1)!]^2} \]
\[ \frac{[\omega^2 + (2\pi T_L)^2 2^2] [\omega^2 + (2\pi T_L)^2 3^2] \ldots [\omega^2 + (2\pi T_L)^2 l^2]}{[\omega^2 + (2\pi T_R)^2 2^2] [\omega^2 + (2\pi T_R)^2 3^2] \ldots [\omega^2 + (2\pi T_R)^2 l^2]} \] (2.9)
\[ \frac{e^{\omega/T_H} - 1}{(e^{\pi T_R} + 1)(e^{\pi T_L} + 1)} \]

For \( l \) even

\[ \sigma^l = (l + 1)^2 \frac{\pi^3 (r_1 r_5)^{2l+2} \omega^{2l+1}}{2^{2l} [l!(l+1)!]^2} \]
\[ \frac{[\omega^2 + (2\pi T_L)^2 2^2] [\omega^2 + (2\pi T_L)^2 4^2] \ldots [\omega^2 + (2\pi T_L)^2 l^2]}{[\omega^2 + (2\pi T_R)^2 2^2] [\omega^2 + (2\pi T_R)^2 4^2] \ldots [\omega^2 + (2\pi T_R)^2 l^2]} \] (2.10)
\[ \frac{e^{\omega/T_H} - 1}{(e^{\pi T_R} - 1)(e^{\pi T_L} - 1)} \]

As a check we note that for \( l = 0, \omega \rightarrow 0 \), we obtain \( \sigma = A_H \) in accordance with the universal form of the low energy cross section for minimal scalars [17].

3. \( V_4 \) dependence.

3.1. The behavior of the classical cross section.

In terms of microscopic variables, we have for the D-brane model [18]:

\[ r_1 = \left[ \frac{g m_1}{(2\pi)^2 V_4} \right]^{1/2} L^{(S)} \] (3.1)

\[ r_5 = \left[ \frac{g m_5}{(2\pi)^2} \right]^{1/2} L^{(S)} \] (3.2)
Here $n_1$, $n_5$ are the numbers of D-strings and D-5-branes respectively. $V_4$ is the volume of the 4-torus transverse to the direction in which the D-string is wound. $L^{(S)}$ is the string length, defined so that under T-duality, a circle of circumference $AL^{(S)}$ goes to a circle of circumference $A^{-1}L^{(S)}$. $g = e^\phi$ is the elementary string coupling. The tension of the elementary string is $T^{(S)} = 2\pi L^{(S)}^{-2}$. The tension of the D-string is

$$T^{(D)} = g^{-1}T^{(S)} = 2\pi L^{(S)}^{-2} g^{-1}$$

(3.3)

The tension of the 5-D-brane is

$$T_5^{(D)} = g^{-1}T^{(S)} L^{(S)}^{-4} = 2\pi L^{(S)}^{-6} g^{-1}$$

(3.4)

The cross section (2.9), (2.10) is seen to depend on $V_4$ as

$$\sigma^l \sim V_4^{-l(l+1)}$$

(3.5)

(The product $r_1 r_5$ depends on $V_4$ as $\sim V_4^{-1/2}$.)

Suppose we have a bound state of 5-D-branes and D-strings. In the effective string model of [19] the effect of the 5-D-branes can be taken into account through a fractionation of the tension of the D-string:

$$T_{eff} = T^{(D)} n_5^{-1}$$

(3.6)

This model was motivated by performing a duality on the case studied in [20]. In the latter case it was shown by using S-duality that the momentum modes that travel on a D-string can, under some conditions, be ‘fractional’ (i.e. go in units of $2\pi/(n_1 L)$ rather than units of $2\pi/L$), though the total momentum on the collection of strings must be still quantised in integer units (i.e. must be an integer multiple of $2\pi/L$). A sequence of dualities can map the D-string to a 5-D-brane, and the momentum mode to a D-string bound to the 5-D-brane. Then it is a plausible conjecture that at least for some dynamical purposes the D-strings bound to D-5-branes should be ‘fractional’, with tension $T^{(D)}/n_5$.

Now suppose we compute the absorption of a minimally coupled scalar by the D-string. If the only effect of the 5-D-branes comes through (3.6), then we see that the physics of oscillations on the D-string is not sensitive to the volume $V_4$. In the absorption calculation, the only way that $V_4$ will enter will be through the normalisation factor for the incoming scalar, which propagates in the full 10-dimensional spacetime and has in its
normalisation a factor $V_4^{-1/2}$. This would yield $V_4^{-1}$ in the cross section, which is indeed appropriate to the $l = 0$ cross section:

$$\sigma^0 = A_H = 4G(5)S_{Bek} = 8\pi\sqrt{n_1n_5n_p}G(5) \sim V_4^{-1} \quad (3.7)$$

where we have noted that

$$G(5) = G(10)V_4^{-1}L^{-1} \sim V_4^{-1} \quad (3.8)$$

But for the case of $l > 0$ we would continue to find the $V_4^{-1}$ dependence, since the quanta created on the D-string would not see the size of the transverse $T^4$. This is in contradiction with (3.5).

3.2. Duality considerations, and the ‘mean string’.

Let us take one 5-D-brane bound to one D-string. Consider a situation where the length $L$ of the circle where the D-string is wrapped is very long compared to the sides of the compact torus perpendicular to the D-string, which we take to be of order $V_4^{-1/4}$ each. The 5-brane is wrapped on $T^4 \times S^1$, so it also sees the length $L$. Further, in this case we can consider excitations of wavelength

$$V_4^{-1/4} << \lambda \sim L \quad (3.9)$$

There are two kinds of such excitations that we can naively see in this system. If we oscillate the D-string, we get vibrations on a string with tension:

$$T_1 = T^{(D)} \quad (3.10)$$

If we oscillate the 5-D-brane, then we expect that this will behave as a string with tension

$$T_2 = V_4T_5^{(D)} = T^{(D)}\frac{V_4}{L(S)^4} \quad (3.11)$$

If we perform four T-dualities, in the four directions of the torus $T^4$, then the D-string will become a 5-D-brane, and the 5-D-brane will become a D-string. The new coupling will be

$$g' = g\left(\frac{V_4}{L(S)^4}\right)^{-1} \quad (3.12)$$

and the new volume of $T^4$ will be

$$V_4' = \left(\frac{V_4}{L(S)^4}\right)^{-1}L(S)^4 \quad (3.13)$$
The 5-D-brane becomes a D-string, which has the tension
\[ T^{(D)'} = T'_1 = T^{(S)}(g')^{-1} = T^{(D)} \left( \frac{V_4}{L^{(S)}} \right) \] (3.14)
which agrees with (3.11) as it should. The original D-string becomes a 5-D-brane with tension
\[ T^{(D)'}_5 = T^{(S)}(g')^{-1}L^{(S)}L_4 = T^{(D)} \left[ \frac{V_4}{L^{(S)}} \right] L^{(S)}^{-4} \] (3.15)
so the effective tension for the long wavelength modes considered here is
\[ T'_2 = T^{(D)'}_5 V_4 = T^{(D)} \] (3.16)
Thus the two tensions \( T_1, T_2 \) get interchanged under T-duality. From the point of view of the noncompact spacetime, both tensions \( T_1, T_2 \) that appear are on a symmetrical footing. Thus it is not natural to choose either as the effective tension for the vibrations that are excited on the system. We take instead the geometric mean of the two tensions
\[ T_m = \sqrt{T_1 T_2} = T^{(D)} \left( \frac{V_4}{L^{(S)}} \right)^{1/2} \] (3.17)
We will have \( 4n_1n_5 \) bosonic degrees of freedom on this string, together with their \( 4n_1n_5 \) fermionic superpartners. Let us call the string with this tension the ‘mean’ string to differentiate it from the string with tension given by (3.6), which is usually termed the ‘effective string’. We can still have the case that the \( \sim n_1n_5 \) degrees of freedom in certain domain of parameters give 4 bosonic and 4 fermionic degrees of freedom on a circle of length \( n_1n_5L \), just as was the case for the effective string model. But note that the tension \( T_m \) does not give in any simple way the mass of either the D-strings or the 5-D-branes in the system. It is an effective parameter for the excitations of the D-string - 5-D-brane bound state.

3.3. Disc diagram calculations.

When the incoming scalar is absorbed in the brane model, we expect that there is one bosonic excitation created on each of the left and right sides; these bosons carry the spins of the scalar. There are also \( l \) fermions on each side, for absorption of angular momentum \( l \). (Some details of the group theoretic structure of partial waves is given in Appendix B. The above kinds of excitations were also involved in the effective conformal theory description used in [11].)
Let us see what a calculation using open string disc diagrams would look like, if we wish to obtain the $V_4$ dependence required by the classical absorption cross section. In the absorption of a $l = 0$ minimal scalar by a D-string we had one power of $g_{\text{closed}}$ from the scalar at the center of the disc, two powers of $g_{\text{open}}$ from the two open strings created on the D-string, and two negative powers of $g_{\text{open}}$ from the disc amplitude itself. Thus we were left only with $g_{\text{closed}}$ in the amplitude. Since $g_{\text{open}}^2 \sim g_{\text{closed}}$, there was however no real significance to separating the powers to $g_{\text{open}}$ and $g_{\text{closed}}$ in this way.

But if we are computing the absorption of $l > 0$ partial waves, then there are $2l + 2$ open strings on the disc boundary, besides the closed string at the center. We obtain the required $V_4$ dependence by using instead of $g_{\text{open}}$ the coupling

$$g_{\text{open}}^m = g_{\text{open}}\left[\frac{V_4}{L(S)^4}\right]^{-1/4}$$

(3.18)

while leaving $g_{\text{closed}}$ the same as before. This choice corresponds to the string tension (3.17) in the same way that we have the usual correspondence $T^{(D)} \sim g_{\text{open}}^{-2}$.

Equivalently, we can still use $g_{\text{open}}$ as the coupling but alter the normalisation factors for the vibrations that are created. Naively we have two kinds of vibrations: one where we produce open strings attached to the D-string (1-1 strings) and one where we have open strings attached to the 5-brane (5-5 strings). The latter will be taken to have momentum only along the long direction $S^1$. But these 5-5 strings will still have a normalisation factor $\sim V_4^{-1/2}$, which the 1-1 strings did not have. Since these two kinds of open strings are interchanged under T-duality, we would not know a priori which to use. (Note that 1-5 strings go to themselves under the T-duality considered here.)

Again following the path of taking geometric means, we take an effective transverse volume of the interaction region equal to $\left[\frac{V_4}{L(S)^4}\right]^{1/2} L(S)^4$. This gives normalisation factors for each open string equal to

$$\mathcal{N} \sim \left[\frac{V_4}{L(S)^4}\right]^{-1/4} L(S)^{-2}$$

(3.19)

Again note that the $l = 0$ case is not altered, since the change in the normalisation of the two open strings is compensated by the change in the volume of the interaction region, which also appears in the amplitude.
3.4. Obtaining the $V_4$ dependence.

In any of the above ways of taking into account the effective tension for the vibrations, we get the desired $V_4$ dependence. The power of $g_{\text{open}}$ is $2l + 2 - 2 = 2l$ in the amplitude, $4l$ in the cross section, so that we get an additional factor $V_4^{-l}$ in the cross section apart from the $V_4^{-1}$ that arises from the normalisation of the incoming scalar.

Equivalently, by using the proposed change of the volume of the interaction region, we get a factor $\left[ V_4^{-1/4} \right]^{2l+2} V_4^{1/2}$ in the amplitude, where we have used the fact that there are $2l+2$ open strings and that there is one factor of the volume of the interaction region. This gives $V_4^{-l}$ in the cross section, again apart from the $V_4^{-1}$ that arises from the normalisation of the incoming scalar.

Thus in each case we get the desired $V_4$ dependence.

4. $g$, $n_1$, $n_5$, $L$ dependence.

In the classical cross section, we note that with regard to $g$, $L$ dependence,

$$r_1 \sim \left[ G^{(5)}LT^{(S)} g^{-1} \right]^{1/2} \sim g^{1/2} \quad (4.1)$$

$$r_5 \sim \left[ G^{(5)}V_4LT^{(S)} g^{-1} \right]^{1/2} \sim g^{1/2} \quad (4.2)$$

$$r_1 r_5 \sim g \quad (4.3)$$

Further with regard to the dependence on the number of 1-branes and 5-branes,

$$(r_1 r_5)^2 \sim n_1 n_5, \quad \sigma^l \sim (n_1 n_5)^{l+1} \quad (4.4)$$

Thus

$$\sigma^l \sim g^{2l+2} (n_1 n_5)^{l+1} \quad (4.5)$$

All these dependences are seen to result if we assume that we have $\sim n_1 n_5$ degrees of freedom on a very long string. The local nature of the interaction says that the cross section is not sensitive to the length $L$ of the string, apart from the $l$ independent factors that were found in [6] in the case for $l = 0$. A disc diagram with $2l+2$ 1-5 open strings gives a cross section that goes like $(n_1 n_5)^{l+1}$ from the sum over flavours of the open strings, after we note that 1-5 and 5-1 open strings alternate around the disc boundary, with flavours agreeing at the junctions where two open strings meet. Further the disc amplitude goes like $g_{\text{open}}^{2l+2} g_{\text{closed}} g_{\text{open}}^{-2}$, which gives $\sim g_{\text{closed}}^{2l+2}$ in the cross section. (These dependences on $g$ and $n_1 n_5$ were noted in [11].)

Thus note that here we seem to need that the different strands of the string interact locally with each other, and the essential physics is not contained in just the vibrations of one long effective string.
5. The dual model.

In [6] we had seen that the leading term in the absorption of minimal scalars could be reproduced from a calculation where the absorption of the incident scalar by the string present in the black hole model was viewed as a three point vertex of ordinary string theory. For convenience we use the S-dual model to the brane model used in the preceding sections, though the same method could be applied to either model with suitable changes of string tensions and couplings. We wish to see if some fundamental string interaction vertex reproduces the $V_4$ and $g$ dependences required by the classical cross section.

5.1. $V_4$, $g$ dependence.

We consider the S-dual model where the black hole is composed of solitonic 5-branes, elementary strings, and momentum along the elementary strings. In this case with regard to $V_4$ and $g$ dependence

\begin{align}
  r_1 &\sim [G^{(5)} LT^{(S)}]^{1/2} \sim V_4^{-1/2} g^{1/2} \\
  r_5 &\sim [G^{(5)} V_4 LT^{(S)}]^{1/2} \sim 1 \\
  r_1 r_5 &\sim V_4^{-1/2} g \\
  \sigma_l &\sim (r_1 r_5)^{2l+2} \sim V_4^{-1-l} g^{2l+2}
\end{align}

The dependence in (5.4) is the same as that in the D-brane model.

Let us postulate that when the incoming scalar is absorbed then the initial string bound to the 5-brane splits into a total of $l + 1$ strings, all bound to the 5-brane. (Thus for the case $l = 0$ we have just one string in the final state, as was the case in [6].) The total number of strings involved in the interaction is $l + 3$, because the initial state had a masslesss scalar and the initial string bound to the 5-brane. Each string has a normalisation factor $V_4^{-1/2}$, and the amplitude also has a factor $V_4$ from the volume of the interaction region. The cross section contains then the square of the resulting $V_4$ dependence:

\begin{align}
  \sigma_l &\sim [V_4^{-(l+3)/2} V_4]^{2} \sim V_4^{-l-1}
\end{align}

which agrees with (5.4).

Note that we have assumed here that the volume $V_4$ is small, and the wavelength of the incoming scalar is large, so that there is no energy to excite momentum modes of the strings in the directions of the torus $T^4$. In the opposite limit, where such momentum
modes are in fact continuous, we would have a sum \( \sum_n \sim V_4 \int d^4k \) for each string in the final state, and we would obtain that \( \sigma^l \sim V_4^{-1} \) for all \( l \), which is not in agreement with (5.4).

Now note that the amplitude depends on \( g \) as \( g^{l+1} \), for a tree vertex involving \( l + 3 \) closed strings, which gives in the cross section

\[
\sigma^l \sim g^{2l+2}
\]

which also agrees with (5.4). Thus we get the powers of both \( V_4 \) and \( g \) to agree at the same time, which a priori need not have been the case.

5.2. Spin dependence.

Let us see how the strings on the 5-brane world volume carry the angular momentum of the 4+1 dimensional transverse space. The rotation group of the transverse space is \( SO(4) = SU(2) \times SU(2) \). The string is confined to the 5-brane, and its low energy bosonic excitations are thus vibrations in the compact directions, labelled by an index \( i = 6, 7, 8, 9 \) for the 4 directions in the 5-brane transverse to the string. If we quantise the string by an NSR prescription, we would take fermions \( \psi^i, i = 6, 7, 8, 9 \), and it is not immediately clear where the angular behavior in the directions \( X^1, X^2, X^3, X^4 \) would come from.

But we can rigorously prove that the ground states of a string bound to a 5-brane can carry spin for the directions \( X^1, X^2, X^3, X^4 \). By a sequence of S dualities and T dualities in the compact directions, we can map the D 5-brane bound to a D-string to a D-string carrying say a left moving momentum mode. The 5-brane has been transformed to the D-string, and the D-string has been transformed to the momentum mode. But the momentum mode can be one of 8 bosonic and 8 fermionic states, which were described in [20]. Out of the bosonic modes, 4 are in directions perpendicular to the compact directions, so we see that there should be a bosonic vector state of the transverse \( SO(4) \) among the ground states of the D-string bound to the D-5-brane. Similarly we find that the fermions are spinors of one of the two \( SU(2) \) components of this transverse \( SO(4) \).

An analysis of the spin properties of the string bound to the 5-brane can be found in [21]. (Since the string [21] was quantised while ignoring the fact that it was not a critical string, one may argue that this is not a rigorously correct derivation of the degrees of freedom. But the argument of duality in the above paragraph shows rigorously that the obtained spin properties of the ground state are correct.) There is indeed a bosonic vector
state, and fermionic states that are spinors, for the transverse $SO(4)$. Thus the ground states can be written as

$$ (|\alpha, k_L > \oplus |a, k_L >) \otimes (|\dot{\beta}, k_R > \oplus |b, k_R >) $$ \hspace{1cm} (5.7)

Here $|\alpha, k_L >$ are the two left moving bosonic ground states, while $|a, k_L >$ are the two left moving fermionic ground states. Overall we get 16 ground states, of which 8 are bosonic and 8 are fermionic, just as expected from the above argument through duality.

The vibrations of the string are the following. There are bosonic modes $X^i, i = 6, 7, 8, 9$ that can travel left or right on the string. There are fermionic modes $\lambda^\alpha_a$ which travel left on the string and $\lambda^{\dot{\beta}}_b$ that travel right on the string. Note that the ground states of the string on the left side, say, carry either the index $a$ or the index $\alpha$, while the travelling fermionic modes carry two indices $\alpha, a$.

To get the spin required of the final state, we postulate that each time a new closed string is produced by splitting, one left and one right moving fermion wave is produced on the initial string. The new closed string is taken to be in its ground state, with polarisations given by $(a, \dot{a})$ so that there is no spin of the transverse $SO(4)$ carried by this string. One possible form of the interaction, for the case $l = 1$, is

$$ p^i \gamma^i_{\alpha \dot{\beta}} \lambda^\alpha_a \lambda^{\dot{\beta}}_b \lambda'_a \lambda'_b $$ \hspace{1cm} (5.8)

where the $\lambda'$ refer to the polarisations of the new closed string in its ground state, and the $\lambda$ are the fermionic waves that are created on the (long) initial string during the process of absorption of the scalar. (The spatial momentum $p^i$ of the incident scalar has components only in the transverse directions, since we are considering neutral scalars.)

The details of such interactions are, however, not clear. In particular normalisation factors suggest that the new strings that are produced will have small winding number, also such strings will prefer to be in their ground states because they cannot support very low energy excitations. Summing over winding numbers may give rise to additional logarithms, not present in the classical cross section (2.9), (2.10).
6. The $\omega$ dependence.

6.1. Sources of $\omega$ dependence.

We find $\omega$ dependence of the absorption cross section from the following sources:

(a) As explained in [9], the absorption cross section is not given by $\Gamma(\omega)$, the absorption when unit flux is incident, but by

$$\sigma(\omega) = \Gamma(\omega) - \Gamma(-\omega)$$ (6.1)

The reason is that while the system can absorb from the incident flux, it can also radiate at the same time, and the absorption cross section only measures the net amount of absorption. Thus we will apply (6.1) to find $\sigma(\omega)$ after computing $\Gamma(\omega)$; the steps below pertain to the calculation of the latter quantity.

(b) The amplitude contains a factor of the energy $|\omega_1|$ of each boson that is produced, and also the normalisation factor for the boson which is $\omega_1^{-1/2}$. So we get a factor $\omega_1^{1/2}$ in the amplitude for each boson, and thus a factor $|\omega_1|$ in the cross section. There is one left moving and one right moving boson.

(c) The incoming scalar contributes a normalisation factor $\omega^{-1/2}$ in the amplitude, which gives $\omega^{-1}$ in the cross section.

(d) The excitations on the string have a left temperature $T_L = \beta_L^{-1}$ and a right temperature $T_R = \beta_R^{-1}$. The incoming quantum interacts with the string through a vertex that involves one boson and $l$ fermions on each side, when the angular momentum absorbed is $l$. These bosons and fermions can either be added to the initial state of the string or can be absorbed from the initial state. The analysis of weight factors for these two cases was carried out for bosonic excitations in [9]. We repeat such an analysis for our case here.

Consider either the left or the right set of variables, and let the inverse temperature be denoted by $\beta$. The distribution function for bosons is $\rho_B = (e^{\beta \omega} - 1)^{-1}$ and for fermions is $\rho_F = (e^{\beta \omega} + 1)^{-1}$. If the boson appears in the final state then $\omega > 0$ and the weightage factor is $1 + \rho_B(\omega) = -\rho_B(-\omega)$. If the boson was absorbed from the initial state then $\omega < 0$ and the weightage factor is $\rho_B(-\omega)$. Note that the weight factor from part (b) above is always positive; as a consequence the two cases of the boson being in the final and in the initial state can be combined to have an integral

$$- \int_{-\infty}^{\infty} d\omega \omega \rho_B(-\omega)$$ (6.2)
Similarly, a fermion in the final state has $\omega > 0$ and a weight $1 + \rho_F(\omega) = \rho_F(-\omega)$. A fermion in the initial state has $\omega < 0$ and a weight $\rho_F(-\omega)$. The two cases can thus be combined to an integral

$$\int_{-\infty}^{\infty} d\omega \rho_F(-\omega)$$

(6.3)

(e) We have on each of the left and right sides the energy conservation delta function

$$\delta\left(\frac{\omega}{2} - \sum_{i=1}^{l+1} \omega_i\right)$$

(6.4)

where $\omega_1$ is the energy of the boson and $\omega_i, i = 2 \ldots l + 1$ are the energies of the fermions. Note that we are considering the absorption of neutral quanta, so half the energy $\omega$ goes to left movers and half to right movers.

(f) We assume that the absorption vertex has a factor $\omega/2$ for each pair of fermions (one left and one right) that are involved in the interaction. The factor $1/2$ is added for convenience; since we are not computing the actual numerical amplitude here it is of no real significance. But the factor $\omega$ can be seen in the disc amplitude. In a Green-Schwarz formalism, the fermions either have the form $\sim S$, or the form $\sim (\partial X)S$. The term $\partial X$ is contracted with the factor $e^{ikX}$ in the incident scalar vertex, and gives a factor $|k| = \omega$. The absorption of angular momentum $l$ needs $l$ fermion vertices of each type, giving $(\omega/2)^l$ in the amplitude, and thus $(\omega/2)^{2l}$ in the cross section.

(g) When we integrate over the energies of the $l$ fermions on each of the left and right sides, we overcount possibilities because the fermions are actually indistinguishable. Thus we must correct by a factor $[l!]^{-2}$. In more detail, we note that if we consider the fermions travelling on a D-string bound to a 5-D-brane, then these fermions carry spin indices which may distinguish them. We can choose coordinates and consider the absorption of a suitable partial wave such that all the fermions have the same spin in the transverse $SU(2) \times SU(2)$. But the left fermions also carry an index $a = 1, 2$ for the spin within the directions of the 5-D-brane. (The right fermions similarly carry an index $\dot{a} = 1, 2$.) Thus for the left fermions we have to symmetrise the fermions with $a = 1$ among themselves and the fermions with $a = 2$ among themselves. Thus we get the sum

$$\sum_{j=0}^{l} \frac{1}{j!} \frac{1}{(l-j)!} = \frac{1}{l!} \sum_{j=0}^{l} l C_j = \frac{2^l}{l!}$$

(6.5)

so that we still get the desired factorial, with the extra freedom of the index $a$ giving the factor $2^l$. 

14
6.2. Example: \( l = 1 \)

We have one boson and one fermion on each of the left and right sides. Following steps (b)-(g) above we obtain

\[
\omega \left[ \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \delta(\omega/2 - \omega_1 - \omega_2) \omega_1 (-\rho_B^{\beta_L}(-\omega_1)) \rho_F^{\beta_L}(-\omega_2) \right] \\
\left[ \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \delta(\omega/2 - \omega_1 - \omega_2) \omega_1 (-\rho_B^{\beta_R}(-\omega_1)) \rho_F^{\beta_R}(-\omega_2) \right] \tag{6.6}
\]

where we have noted the temperature dependence of the distribution functions.

But

\[
\int_{-\infty}^{\infty} d\omega_1 \omega_1 (-\rho_B^{\beta_L}(-\omega_1)) \rho_F^{\beta_L}(\omega/2 - \omega_1) = \int_{-\infty}^{\infty} d\omega_1 \omega_1 \frac{1}{1 - e^{-\beta_L \omega_1}} \frac{1}{1 + e^{-\beta_L (\omega/2 - \omega_1)}} \\
= J_{BF}(\beta_L, \omega/2) = \frac{(\omega^2 + (2\pi T_L)^2)}{2!2^2(1 + e^{-\frac{\omega}{2T_L}})} 
\tag{6.7}
\]

where \( J_{BF} \) is defined in Appendix A.

Using (6.1) we get the contribution

\[
\frac{e^{\omega/T_H} - 1}{(e^{\frac{\omega}{2T_R}} + 1)(e^{\frac{\omega}{2T_L}} + 1)} \cdot \frac{\omega}{2^!2^1} \left( \omega^2 + (2\pi T_L)^2 \right) \left( \omega^2 + (2\pi T_R)^2 \right) \tag{6.8}
\]

6.3. Example: \( l = 2 \)

We have one boson and two fermions on each of the left and right sides. Following steps (b)-(g) above we obtain

\[
\omega^3 \left[ \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \delta(\omega/2 - \omega_1 - \omega_2 - \omega_3) \omega_1 (-\rho_B^{\beta_L}(-\omega_1)) \rho_F^{\beta_L}(-\omega_2) \rho_F^{\beta_L}(-\omega_3) \right] \\
\left[ \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \delta(\omega/2 - \omega_1 - \omega_2 - \omega_3) \omega_1 (-\rho_B^{\beta_R}(-\omega_1)) \rho_F^{\beta_R}(-\omega_2) \rho_F^{\beta_R}(-\omega_3) \right] \tag{6.9}
\]

But

\[
\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \omega_1 (-\rho_B^{\beta_L}(-\omega_1)) \rho_F^{\beta_L}(\omega_2) \rho_F^{\beta_L}(\omega/2 - \omega_1 - \omega_2) \\
= \int_{-\infty}^{\infty} d\omega_1 \omega_1 \frac{1}{1 - e^{-\beta_L \omega_1}} \frac{1}{1 + e^{-\beta_L \omega_2}} \frac{1}{1 + e^{-\beta_L (\omega/2 - \omega_1 - \omega_2)}} \\
= J_{BF^2}(\beta_L, \omega/2) = \frac{\omega \left( \omega^2 + (2\pi T_L)^22^2 \right)}{3!2^3(1 - e^{-\frac{\omega}{2T_L}})} 
\tag{6.10}
\]
Using (6.1) we thus obtain the contribution
\[
\frac{e^{\omega/T_H} - 1}{(e^{\pi T_L} - 1)(e^{\pi T_R} - 1)} \frac{\omega^5}{4(3!)^2(2!)^2} \frac{1}{2^4} (\omega^2 + (2\pi T_L)^2)(\omega^2 + (2\pi T_R)^2)
\] (6.11)

6.4. General form of \(\omega\) dependence.

For \(l\) odd we obtain
\[
\frac{1}{4} \frac{2^{2l}}{e^{\pi T_R} + 1} \frac{1}{e^{\pi T_L} + 1} \frac{e^{\omega/T_H} - 1}{(l!)^2((l + 1)!)^2} \frac{\omega^{2l-1}}{2^l} [\omega^2 + (2\pi T_L)^2]^2 \ldots [\omega^2 + (2\pi T_L)^2]^2]
\] (6.12)

For \(l\) even we obtain
\[
\frac{1}{4} \frac{2^{2l}}{e^{\pi T_R} - 1} \frac{1}{e^{\pi T_L} - 1} \frac{e^{\omega/T_H} - 1}{(l!)^2((l + 1)!)^2} \frac{\omega^{2l+1}}{2^l} [\omega^2 + (2\pi T_R)^2]^2 \ldots [\omega^2 + (2\pi T_R)^2]^2]
\] (6.13)

We see that these dependences on \(\omega\) agree with the dependences required by the classical cross section (2.9), (2.10).

7. Discussion.

We have seen that to have the classical absorption agree with the brane models, we need to obtain a dependence on \(V_4\) (the volume of the compact torus perpendicular to the string) in the brane model. While this may show up in different ways in different treatments of the string dynamics, it is possible that these differences are due to the different coupling regimes appropriate to these calculations, and not to an error in either description. Recently it has been shown that there is ‘stringy dynamics’ in all higher branes, in some domain of parameters [22].

The issue of \(V_4\) dependence may appear in other calculations, for example that for the fixed scalars in [3]. In this calculation it was assumed that \(r_1 = r_5\), which is equivalent to choosing a particular value for \(V_4\). It would be interesting to see the details of the agreement when \(r_1 \neq r_5\).
Regardless of the details of the absorption, we note that the desired \( \omega \) dependence arises from a partitioning of the energy of the incoming scalar into the energy of a certain number of momentum modes, with this number being determined by the angular momentum of the partial wave that is absorbed. The calculation also provides naturally the factorials present in the relative cross sections for different \( l \) (though since we have not computed the actual disc amplitudes themselves, we cannot know that there will be no other factorials from that source.) The argument using an ‘effective conformal theory’ carried out in \([11]\) yielded an \( \omega \) dependence that was the desired one, but there was no known way to normalise the amplitude. The energy dependence calculation of the effective conformal theory calculation is plausibly equivalent to summing over ways of sharing energy between \( l + 1 \) quanta of the left and right sides when the quanta are at temperatures \( T_L, T_R \) respectively, since one has to evaluate correlators of free fields at the appropriate temperatures.

In \([16]\) it was noted that when we are working in a domain \( r_5 >> r_1, r_p, r_0 \) then the absorption of the \( l \)th partial wave becomes significant when \( \omega r_5 = l + 1 \). In particular the \( l = 1 \) partial wave is significantly absorbed starting at the energy where the first massive mode of the effective string can be created. Creation of a new string state would bring in the required factor of \( V_4 \) as we have seen. It would be interesting to connect the present calculations to this domain of parameters where \( r_1 \) is also small, and so winding modes and momentum modes play a more symmetric role.

It may be thought that the agreement of cross sections for D-branes at low energy with the black hole cross sections implies that for low energy quanta we understand the mechanism by which the Hawking paradox is to be resolved in string theory. This is not the case. A black hole geometry takes an incident low energy quantum and, in the Schwarzschild coordinate system, converts it to a high frequency mode close to the horizon. This high frequency mode is then eaten by the hole. Starting with a quantum of even lower energy just means that we have to follow the mode closer to the horizon before we see it attaining a short wavelength. (Of course most of the low energy quanta escape falling into the hole, but the cross section we compute relates to those that are in fact swallowed by the hole in the above fashion.)

Equivalently, if we study the deflection of a particle trajectory by the gravitational field of the hole, and consider the deflection to be expanded in powers of the mass of the hole, then we would see a divergence of the series when the impact parameter approaches the value where the particle will be swallowed by the hole. When we study the D-brane
calculation we need to understand whether or not such a divergence occurs, when the number of branes and the coupling are increased to a value large enough to give a classical sized black hole. In the classical calculation taking low energy while keeping other parameters fixed simply pushes the growth of the perturbation series towards higher terms in the series; thus there may be a similar phenomenon for the D-brane calculation as well.

If string theory is to resolve the Hawking paradox, then we either need to see that the effective size of the solitonic bound state is comparable to the horizon size, so that there is really no black hole, or we need to see that loops of virtual quanta in a theory with strings and higher dimensional branes are quite different from loops in particle theory, and give nonlocal effects that take information from near the singularity and send it out with the Hawking radiation. The latter is equivalent to finding a length scale in string theory that is not plank length but is a length that grows with the number of branes involved.

Thus it appears that the agreements found between cross sections of branes and for black holes are to some extent both mysterious and interesting, and provide a strong suggestion that the black hole paradox may actually be resolved in string theory. One needs to better understand the bound states of many branes at strong coupling; for the non-BPS interactions that are involved the values of moduli also affect the nature of energy levels and interaction properties [23][16].

**Acknowledgements**

I would like to thank S.P. deAlwis, K. Johnson, I. Klebanov and G. Lifschytz for discussions. I am especially grateful to S.R. Das for discussions and a critical reading of the manuscript. This work is partially supported by cooperative agreement number DE-FC02-94ER40818.
Appendix A. The basic integrals.

Let us define the following two basic integrals

\[ I^n_{BF} \equiv \int_{-\infty}^{\infty} x^n dx \frac{x}{(e^x - 1)(e^{-x} + A)} \]  
(A.1)

\[ I^m_{FF} \equiv \int_{-\infty}^{\infty} x^n dx \frac{x}{(e^x + 1)(e^{-x} + A)} \]  
(A.2)

(The subscripts \( B \) and \( F \) stand for bose and fermi type distributions respectively.)

These integrals can be calculated in the following way. We take a contour in the complex \( x \) plane, running along the real \( x \)-axis, and backwards along the line \( x + 2\pi i \).

Consider the integral

\[ \hat{I}^n_{BF} \equiv \int_C x^n dx \frac{(x - 2\pi i)}{(e^x - 1)(e^{-x} + A)} \]  
(A.3)

The segments at \( \Re x = \pm\infty \) do not contribute. So we have

\[ \hat{I}^n_{BF} = \int_{-\infty}^{\infty} \frac{x^n(x - 2\pi i)dx}{(e^x - 1)(e^{-x} + A)} \]  
(A.4)

where now \( x \) runs along the real line. There is one pole in the contour, at \( x = i\pi - \log A \).

Then we find

\[ (n + 1)I^m_{BF} + \sum_{m=0}^{n-2} nC_nI^{m+1}_{BF}(2\pi i)^n x^n m^{-1} = \frac{(\pi^2 + (\log A)^2)(i\pi - \log A)^{n-1}}{A + 1} \]  
(A.5)

It will turn out that we will only need \( I^m_{BF} \) for \( n \) odd. From (A.3) relation we find

\[ I^1_{BF} = \frac{(\pi^2 + (\log A)^2)}{2(A + 1)} \]  
(A.6)

\[ I^3_{BF} = \frac{(\pi^2 + (\log A)^2)^2}{4(A + 1)} \]  
(A.7)

\[ I^5_{BF} = \frac{(\pi^2 + (\log A)^2)(3\pi^2 + (\log A)^2)}{6(A + 1)} \]  
(A.8)

More generally, for \( n \) odd

\[ I^n_{BF} = \frac{(i\pi)^{n-1}(\pi^2 + (\log A)^2)[(\log A)^{n-1} + \frac{(n + 3)(n - 2)(\log A)^{n-3}}{6}]}{(n + 1)(A + 1)} \]  
(A.9)

\[ + \frac{(n - 2)(n - 4)(7n^2 + 28n + 45)}{360}(\log A)^{n-5} + \ldots \]
Similarly, we define

\[ I_{FF}^n \equiv \int_C \frac{x^{n+1}dx}{(e^x + 1)(e^{-x} + A)} = \int_{-\infty}^{\infty} \frac{[x^{n+1} - (x + 2\pi i)^{n+1}]dx}{(e^x + 1)(e^{-x} + A)} \]  

(A.10)

where the contour \( C \) is the same as described above, and in the last integral \( x \) runs over the real line. There are two poles, at \( x = i\pi \) and at \( x = i\pi - \log A \). We get the relation

\[ \sum_{m=0}^{n} n^{m+1} C_m I_{FF}^m (2\pi i)^{n-m} = -\frac{1}{A-1}[(i\pi - \log A)^{n+1} - (i\pi)^{n+1}] \]

(A.11)

We will need \( I_{FF}^n \) only for even \( n \). We find

\[ I_{FF}^0 = \frac{\log A}{A-1} \]  

(A.12)

\[ I_{FF}^2 = \frac{\log A(\pi^2 + (\log A)^2)}{3(A-1)} \]  

(A.13)

\[ I_{FF}^4 = \frac{\log A(\pi^2 + (\log A)^2)(\frac{7}{3}\pi^2 + (\log A)^2)}{5(A-1)} \]  

(A.14)

More generally, for \( n \) even

\[ I_{FF}^n = \frac{(i\pi)^n(\pi^2 + (\log A)^2)}{(n+1)(A-1)}[(\log A)^n + \frac{n(n+1)}{6}(\log A)^{n-2}} \]

\[ + (n+1)n(n-1)(n-2)\frac{7}{360}(\log A)^{n-4} + \ldots] \]

(A.15)

We now define the integrals \( J_{BF^n}(\beta, \omega) \). The arguments of \( J_{BF^n} \) will not be explicitly written below, for convenience. (We also write \( J_{BF} \equiv J_{BF^1} \).)

The integral \( J_{BF} \) arises if we wish to partition energy \( \omega \) between one boson and one fermion. The integral is

\[ J_{BF} = -\int_{-\infty}^{\infty} d\omega_1 \omega_1 \rho_B(-\omega_1) \rho_F(-\omega_2) \delta(\omega_1 + \omega_2 - \omega) = -\int_{-\infty}^{\infty} \frac{d\omega_1 \omega_1}{(e^{-\beta \omega_1} - 1)(e^{-\beta(\omega - \omega_1)} + 1)} \]  

(A.16)

Define \( x = -\beta \omega_1 \). Then we see that

\[ J_{BF} = \frac{1}{\beta^2 e^{-\beta \omega}} \int_{-\infty}^{\infty} \frac{xdx}{(e^x - 1)(e^{-x} + A)} \]  

(A.17)

where

\[ A = e^{\beta \omega} \]  

(A.18)
But from (A.6)
\[
J_{BF} = \frac{1}{\beta^2 e^{-\beta \omega}} I_{BF} = \frac{(\omega^2 + \frac{\pi^2}{\beta^2})}{2!(1 + e^{-\beta \omega})}
\] (A.19)

Similarly, suppose we wish to partition the energy \(\omega\) between one boson and two fermions. Then the integral we obtain is
\[
J_{BF}^2 = \int_{-\infty}^{\infty} \frac{\omega_1 d\omega_1 d\omega_2}{(1 - e^{-\beta \omega})(1 + e^{-\beta \omega})(1 + e^{-\beta (\omega - \omega_1 - \omega_2)})}
\] (A.20)

First we compute
\[
\int_{-\infty}^{\infty} \frac{\omega_1 d\omega_1}{(1 - e^{-\beta \omega})(1 + e^{-\beta (\omega - \omega_1)})} = \frac{1}{\beta^2} I_{BF}^1(e^{\beta \omega}) = \frac{1}{\beta^2} \frac{(\pi^2 + (\omega' \beta)^2)}{2(1 + e^{-\beta \omega'})}
\] (A.21)

Then we find
\[
J_{BF}^2 = \frac{1}{2\beta^3} \left[ \pi^2 I_{BF}^0(e^{\beta \omega}) + \beta^2 I_{BF}^2(e^{\beta \omega}) \right] = \frac{\omega}{6(e^{\beta \omega} - 1)} \left( \frac{4\pi^2}{\beta^2} + \omega^2 \right)
\] (A.22)

For the energy to be shared between one boson and 3 fermions, we get
\[
J_{BF}^3 = \frac{1}{6} \left[ \frac{\pi^2}{\beta^2} I_{BF}^1(e^{\beta \omega}) + \frac{\pi^2}{\beta^4} I_{BF}^3(e^{\beta \omega}) \right] = \frac{(\pi^2 + \omega^2)(9\pi^2 + \omega^2)}{4!(e^{\beta \omega} + 1)}
\] (A.23)

Similarly we find
\[
J_{BF}^4 = \frac{\omega}{5!(e^{\beta \omega} - 1)} (4\pi^2 + \omega^2)(16\pi^2 + \omega^2)
\] (A.24)

\[
J_{BF}^5 = \frac{1}{6!(e^{\beta \omega} + 1)} \left( \frac{\pi^2}{\beta^2} + \omega^2 \right)(9\pi^2 + \omega^2)(25\pi^2 + \omega^2)
\] (A.25)

\[
J_{BF}^6 = \frac{\omega}{7!(e^{\beta \omega} - 1)} (4\pi^2 + \omega^2)(16\pi^2 + \omega^2)(36\pi^2 + \omega^2)
\] (A.26)

While we have not solved for the general term \(J_{BF}^n\) these cases provide a pattern that we will assume holds for all \(n\). In particular we note the appearance of the factor \(1/(n + 1)!\) in the integrals.
Appendix B. Angular variables.

We have the transverse rotation group $SO(4) = SU(2) \times SU(2)$. Let the generators of $SO(4)$ be $M_{ij}, i, j = 1 \ldots 4$. Then the generators for the two $SU(2)$ components are described through

\[
J_1 = \frac{1}{2}(M_{12} + M_{34})
\]
\[
J_2 = \frac{1}{2}(M_{13} + M_{42})
\]
\[
J_3 = \frac{1}{2}(M_{32} + M_{41})
\]

and

\[
K_1 = \frac{1}{2}(M_{12} - M_{34})
\]
\[
K_2 = \frac{1}{2}(M_{13} - M_{42})
\]
\[
K_3 = \frac{1}{2}(M_{32} - M_{41})
\]

We have

\[
[J_a, J_b] = \epsilon_{abc}J_c, \quad [K_a, K_b] = \epsilon_{abc}K_c, \quad [J_a, K_b] = 0
\]

The spinor of $SO(4)$ is constructed as $(1/2, 0) \oplus (0, 1/2)$. Letting the spinor components in each $SU(2)$ be represented by $\{+, -\}$, we have the four components $\{++, +-, -+, --\}$. The rotation generators are $M_{ij} = \frac{1}{4} [\Gamma_i, \Gamma_j]$. This gives

\[
\frac{1}{2}(M_{12} \pm M_{34}) = \frac{i}{4}(\sigma^3 \otimes 1 \pm 1 \otimes \sigma^3)
\]
\[
\frac{1}{2}(M_{13} \pm M_{42}) = -\frac{i}{4}(\sigma^2 \otimes \sigma^1 \pm \sigma^1 \otimes \sigma^2)
\]
\[
\frac{1}{2}(M_{32} \pm M_{41}) = -\frac{i}{4}(\sigma^1 \otimes \sigma^1 \pm \sigma^2 \otimes \sigma^2)
\]

The vector of $SO(4)$ decomposes as $(1/2, 1/2)$ under the two $SU(2)$ components. Let us write the following combinations of the four components of the vector

\[
X^1 \pm iX^2 = X^\pm_A
\]
\[
X^3 \pm iX^4 = X^\pm_B
\]

Then we have the identifications

\[
X^+_A = (++)(+-)
\]
\[
X^-_A = (-+)(-+)
\]
\[
X^+_B = (++)(-+)
\]
\[
X^-_B = (-+)(+-)
\]
If we tensor together two vectors we have \((1/2, 1/2) \otimes (1/2, 1/2) = (0, 1) \oplus (1, 1) \oplus (1, 0)\). The component \((1, 1)\) we identify with the tensors \(C_{ij}X^iX^j\) with \(C_{ij}\) symmetric and traceless. Thus this \((1, 1)\) component is all that we expect to pick up in the absorption of higher partial waves of scalars.

More generally, we identify the component \((l/2, l/2)\) with the partial wave components for angular momentum \(l\). The former description is seen to have \((l+1)^2\) components. To see that the latter also has the same number of components, note that the number of symmetric tensors that can be made from \(l\) copies of the vector \(X^i\) is \((l+3)C_3 = (l+3)(l+2)(l+1)/6\). We must remove all those components that can arise when any two of the \(X^i\) components are traced over, and this gives \((l+1)C_3 = (l+1)(l)(l-1)/6\) components. The difference is \((l+1)^2\), as required.

Thus in absorbing the \(l\)th partial wave of the scalar we will create spins on the brane system that total to a value \(l/2\) in each of the \(SU(2)\) components.

**Appendix C. Relating plane waves to partial wave components.**

We wish to find the partial wave components in a plane wave. In 3 space dimensions, we have the decomposition

\[
e^{i\omega z} = e^{i\omega r \cos \theta} \to \sum_{l \geq 0} \frac{e^{-i\omega r}}{(-i\omega r)}(-1)^l[\pi(2l + 1)]^{1/2}Y_{l,0}(\cos \theta)
\]

where \(Y_{l,0}\) is a function of the angular coordinates only and is normalised to satisfy \(\int |Y_{l,0}|^2d\Omega = 1\). Thus if the absorption probability of the \(l\)th partial wave is \(a_l\), then the cross section for this partial wave is

\[
\sigma^l = \frac{\pi}{\omega^2}(2l+1)a_l
\]

Note in particular that the conversion factor from \(a_l\) to \(\sigma^l\) does have a factor \(2l + 1\) even though only one azimuthal component is seen in \((C.1)\). One way to understand the factor \(2l + 1\) is to let the incident wave be averaged over all directions, at which stage we still expect the same cross section since we have just done an averaging. But now we can regard the incoming quanta as decomposed into partial waves, and study the radial motion of the wave as a problem in \(1 + 1\) dimensions. There are \(2l + 1\) ‘species’ of radially incoming quanta, and so the cross section has a factor \(2l + 1\).
In 4 space dimensions we have the decomposition

\[ e^{i\omega X^4} = e^{i\omega r \cos \theta} \rightarrow \sum_{l \geq 0} \frac{e^{-i\omega r}}{(\omega r)^{3/2}} e^{i3\pi/4}(-1)^l \sqrt{4\pi}(l + 1) Z_{l,0}(\cos \theta) \]  

(C.3)

where

\[ Z_{l,0}(\cos \theta) = \frac{1}{\sqrt{2\pi^2}} U_l(\cos \theta) = \frac{1}{\sqrt{2\pi^2}} \frac{\sin[(l + 1)\theta]}{\sin \theta} \]  

(C.4)

where the \( U_l \) are Chebyshev polynomials. The \( Z_{l,0} \) are normalised according to

\[ \int |Z_{l,0}|^2 d\Omega = \int_0^\pi |Z_{l,0}|^2 4\pi \sin^2 \theta d\theta = 1 \]  

(C.5)

The cross section for absorption of the \( l \)th partial wave is given in terms of its absorption probability \( a_l \) through

\[ \sigma^l = \frac{4\pi}{\omega^3}(l + 1)^2 a_l \]  

(C.6)

Similar to the case of 3 space dimensions, the number \((l + 1)^2\) is the number of azimuthal components for angular momentum \( l \), as was noted above.
References

[1] L. Susskind, [hep-th/9309145]; J. Russo and L. Susskind, Nucl. Phys. B437 (1995) 611.
[2] A. Sen, Nucl. Phys. B440 (1995) 421 and Mod. Phys. Lett. A10 (1995) 2081.
[3] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99, [hep-th/9601029].
[4] C.G. Callan and J.M. Maldacena, Nucl. Phys. B472 (1996) 591, [hep-th/9602043].
[5] S.R. Das and S.D. Mathur, Nucl. Phys. B478 (1996) 561, [hep-th/9606185].
[6] S.R. Das and S.D. Mathur, Nucl. Phys. B482 (1996) 153, hepth 9607149.
[7] S.S. Gubser and I.R. Klebanov, Nucl. Phys. B482 (1996) 173, [hep-th/9608108].
[8] J.M. Maldacena and A. Strominger, Rutgers preprint RU-96-78, [hep-th/9609026].
[9] C.G. Callan, Jr., S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, [hep-th/9610172].
[10] S.S. Gubser, I. Klebanov and A.A. Tseytlin, hepth 9703040.
[11] J. Maldacena and A. Strominger, hepth 9702015.
[12] A. Dhar, G. Mandal and S.R. Wadia, hepth 9605234.
[13] S.R. Das, hepth 9703146.
[14] F. Dowker, D. Kastor and J. Traschen, hepth 9702109.
[15] S.P. de Alwis and K. Sato, hepth 9611189.
[16] I. Klebanov and S.D. Mathur, hepth 9701187.
[17] S. Das, G. Gibbons and S.D. Mathur, [hep-th/9609052].
[18] G. Horowitz, J. Maldacena and A. Strominger, Phys. Lett B383 (1996) 151, [hep-th/9603109].
[19] J.M. Maldacena and L. Susskind, Stanford preprint SU-ITP-96-12, [hep-th/9604042].
[20] S.R. Das and S.D. Mathur, Phys. Lett. B375 (1996) 103, [hep-th/9601152].
[21] R. Dijkgraaf, E. Verlinde and H. Verlinde, hepth 9603126.
[22] U. Lindstrom and R von Unge, hepth 9704051.
[23] S.D. Mathur, hepth 9609053.