GPDs with \( \zeta \neq 0 \)

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We revisit the light-cone wave function representation for generalized parton distributions with \( \zeta \neq 0 \). After translating the \( t \)-slope into a \( \Delta \zeta \)-slope, the two-dimensional Fourier transform of GPDs is interpreted as the transition matrix element as a function of the separation between the active quark and the center of momentum of the spectators. In the limit \( x \to \zeta \) it is discussed how this information can be used to learn about the dependence of the mean separation between the active quark and the spectators on the momentum fraction carried by the active quark.

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I. INTRODUCTION

Hard exclusive processes, such as Deeply Virtual Compton Scattering (DVCS), \( \gamma^* p \to \gamma p \), where \( \gamma^* \) is a virtual photon with virtuality \( q^2 = -Q^2 < 0 \), have emerged as a novel probe for hadron structure. In the Bjorken limit, and for a momentum transfer \( t \) to the proton that is much less than \( Q^2 \), the DVCS amplitude factorizes into a convolution of the Compton amplitude off a quark, constituting the hard part, and a quark correlation function, constituting the soft part \( \ref{DVCS} \). The latter is parameterized by Generalized Parton Distributions GPDs, through their dependence on the Bjorken variable \( \zeta = \frac{Q^2}{2 p \cdot q} \), and the momentum fraction \( x \) of the active quark before being struck by the virtual photon. A physical interpretation for GPDs is most easily available in the light-cone framework \( \ref{LC} \) where, for \( x > \zeta \) they represent the probability amplitude that the proton remains intact after a quark carrying momentum fraction \( x \) absorbs a longitudinal momentum \( -\zeta \) (in units of the initial proton momentum) and an invariant momentum transfer \( t \). As in Deep-Inelastic Scattering (DIS), the variable \( Q^2 \) has the interpretation of the spatial resolution. However, since GPDs provide information about the distribution of partons in impact parameter space \( \ref{impact} \), the \( Q^2 \) dependence in DVCS not only provides the scale dependence, but also the ‘pixel-size’ for the spatial images obtained from Fourier transforming GPDs. Unfortunately, a probabilistic interpretation for the Fourier transforms of GPDs is restricted to \( \zeta = 0 \) \( \ref{prob} \). Since the probabilistic interpretation facilitates the development of intuitive models for GPDs, most phenomenological models for GPDs are more reliable for \( \zeta = 0 \), and utilizing these models for \( \zeta \neq 0 \) gives rise to uncertainties that are difficult to quantify. This is very unfortunate since DVCS typically provides constraints only for GPDs with \( \zeta \neq 0 \). In particular, the imaginary part of the DVCS amplitude is only sensitive to \( x = \zeta \)

\[
\Im \left\{ T^{DVCS} \right\} \propto GPD(x = \zeta, t, Q^2),
\]

(1)

The real part appears in a convolution integral

\[
\Re \left\{ T^{DVCS} \right\} \propto \int dx GPD(x, \zeta, t, Q^2) \frac{1}{x \pm \zeta},
\]

(2)

where the factor \( \frac{1}{x \pm \zeta} \) emphasizes the regions \( x \approx \pm \zeta \). In either case, understanding the vicinity of \( x \approx \pm \zeta \) appears to be crucial for understanding the DVCS amplitude. The goal of this note is to develop some intuition about GPDs in this important regime. More specifically, we will consider the \( t \)-slopes of GPDs for \( x \approx \zeta \) and what can be learned from them.

II. LIGHT-CONE WAVE FUNCTION REPRESENTATION FOR GPDS

Although the primary focus of this work is \( x \approx \zeta \), we consider first \( x > \zeta \), where GPDs are diagonal in Fock space. The regime \( x = \zeta \) is then approached through a limiting procedure. For \( x > \zeta \) simple overlap representations for GPDs in terms of light-cone wave functions exist that resemble overlap integrals for form factors in non-relativistic systems \( \ref{overlap} \).

\[
GPD(x, \zeta, t) = \sum_{n, \lambda_i} (1 - \zeta)^{1 - \frac{2}{D}} \int \prod_{i=1}^n \frac{dx_i dk_{\perp i}}{16\pi^3} 16\pi^3 \delta \left( 1 - \sum_{j=1}^n x_j \right) \delta \left( \sum_{j=1}^n k_{\perp j} \right) \delta(x - x_1) \psi^{*}(n)(x_i, k_{\perp i}, \lambda_i) \psi(n)(x_i, k_{\perp i}, \lambda_i),
\]

(3)

(4)
where

$$GPD(x, \zeta, t) = \frac{\sqrt{1-\zeta}}{1-\frac{x}{2}} H(x, \zeta, t) - \frac{\zeta^2}{4\left(1-\frac{x}{2}\right)\sqrt{1-\zeta}} E(x, \zeta, t)$$  \hspace{1cm} (5)$$

for \(s' = s\), and

$$GPD(x, \zeta, t) = \frac{1}{\sqrt{1-\zeta}} \frac{\Delta^1 - i \Delta^2}{2M} E(x, \zeta, t)$$  \hspace{1cm} (6)$$

for \(s' = \uparrow\) and \(s = \downarrow\), and \(\Delta\) is the transverse momentum transfer. The arguments of the final state wave function are \(x'_1 = \pi \frac{1-\zeta}{1-\xi}\) and \(k'_{1,1} = k_{1,1} - \frac{x_1-x_2}{1-\zeta} \Delta\) for the active quark, and \(x'_1 = \pi \frac{1-\zeta}{1-\xi}\) and \(k'_{1,1} = k_{1,1} + \frac{x_1-x_2}{1-\zeta} \Delta\) for the spectators.

In order to elucidate the essential steps, we study first the simple case of a two-particle system (e.g. quark plus diquark), where we consider light-cone wave functions as a function of the distance between the active quark and the spectator

$$\tilde{\psi}^s(x, r_\perp) = \int \frac{d^2k_\perp}{2\pi} \psi^s(x, k_\perp) e^{i k_\perp \cdot r_\perp}.$$  \hspace{1cm} (7)$$

Inserting the position space wave function \((7)\) diagonalizes the transverse part of the overlap integral in Eq. \((4)\), yielding

$$\int d^2k_\perp \psi^s(x', k_\perp')^* \psi^s(x, k_\perp) = \int d^2r_\perp \psi^s(x', r_\perp')^* \tilde{\psi}^s(x, r_\perp) e^{-i \frac{\pi}{\zeta} \frac{\Delta}{\sqrt{2}r_\perp}}\Delta.$$  \hspace{1cm} (8)$$

For \(\zeta \to 0\) one recovers the known result \([3]\) that GPDs are Fourier transforms of the distribution of partons in impact parameter space, where the impact parameter \(b_\perp = (1-x)r_\perp = r_{1,1} - R_\perp\) is the separation of the active quark from the center of momentum \(R_\perp \equiv x r_{1,1} + (1-x) r_{2,2}\).

For the general case, we also switch to transverse position

$$\psi^s_n(x_i, k_{1,1}, \lambda_i) = \int \prod_{i=1}^n d^2r_{1,1} e^{-i k_{1,1} \cdot r_{1,1}} \tilde{\psi}^s_n(x_i, r_{1,1}, \lambda_i).$$  \hspace{1cm} (9)$$

Since we are dealing with plane wave states, one needs to be careful with the normalization of these states and a more careful treatment should involve working with wave packets. Here we will skip these tedious steps that have been studied carefully in Refs. \([3, 6]\) and immediately insert \((9)\) into \((4)\), yielding

$$GPD(x, \zeta, t) = \sum_{n, \lambda} (1-\zeta)^{1-\frac{1}{2}} \int \prod_{i=1}^n d^2r_{1,1} e^{i k_{1,1} \cdot r_{1,1}} \psi^s_n(x'_i, r_{1,1}, \lambda_i)^* \tilde{\psi}^s_n(x_i, r_{1,1}, \lambda_i) e^{-i \frac{\pi}{\zeta} \frac{\Delta}{\sqrt{2}r_\perp}}(r_{1,1} - R_{1,1}) \Delta.$$  \hspace{1cm} (10)$$

where \(R_\perp = \sum_i x_i r_{1,1}\) is the transverse center of momentum of all partons in the initial state.

Since the transverse center of momentum changes in the process \([6]\), it is useful to replace it by the separation between the active quark and the center of momentum of the spectators \(R_{1,1}\), using

$$r_\perp \equiv r_{1,1} - R_{1,1} = \frac{1}{1-x}(r_{1,1} - R_\perp)$$  \hspace{1cm} (11)$$

and one finds

$$GPD(x, \zeta, t) = \sum_{n, \lambda} (1-\zeta)^{1-\frac{1}{2}} \int \prod_{i=1}^n d^2r_{1,1} e^{i k_{1,1} \cdot r_{1,1}} \psi^s_n(x'_i, r_{1,1}, \lambda_i)^* \tilde{\psi}^s_n(x_i, r_{1,1}, \lambda_i) e^{-i \frac{\pi}{\zeta} \frac{\Delta}{\sqrt{2}r_\perp}}(r_{1,1} - R_{1,1}) \Delta.$$  \hspace{1cm} (12)$$

While GPDs for \(x > \zeta > 0\) are still diagonal in the absolute transverse positions of all partons, they appear off-diagonal when positions are measured relative to the \(\perp\) center of momentum \([6]\). However, as the momentum carried by the active quark changes between initial and final state, so does the location of the transverse center of momentum \([6]\). Therefore, even though the (absolute) \(\perp\) positions of the active quark/spectators remain unchanged, their separation from the \(\perp\) center of momentum changes since the latter does. For the physical interpretation of GPDs in the case of \(\zeta \neq 0\), working with relative \(\perp\) position coordinates (i.e. relative to each other) rather than impact parameter (measured relative to the \(\perp\) center of momentum may thus be preferable. Indeed, the discussion above illustrates that, for nonzero \(\zeta\), the Fourier transform of GPDs w.r.t. the transverse momentum transfer \(\Delta_\perp\) yields information about the transition matrix element between the initial and final state, when the \(\perp\) distance between the active quark and the center of momentum of the spectators is \(r_\perp\). More precisely, \(\frac{\pi}{\zeta} \frac{\Delta}{\sqrt{2}r_\perp}\) is Fourier conjugate to \(\Delta_\perp\), and for \(x = \zeta\), the variable conjugate to the \(\Delta_\perp\) is just \(r_\perp\).
III. GPDS FOR $x \to \zeta$

When $x = \zeta$, the coefficient multiplying $r_{\perp} \cdot \Delta_{\perp}$ in the exponent in Eq. (12) becomes equal to one, i.e. in that limit the Fourier transform of GPDs w.r.t. $\Delta_{\perp}$ yields the dependence of the overlap matrix element on the separation $r_{\perp}$ between the active quark and the center of momentum $R_{\perp}$ of the spectators. While for $\zeta = 0$ it is the separation from the center of momentum of the whole hadron that sets the scale, it is the separation from then center of momentum of the spectators that matters for $x = \zeta$. In order to utilise the above observations in the interpretation of GPDs, we note that \[ \frac{1}{1 - \zeta} \]

Therefore, if the $t$-dependence of GPDs is parameterized in the form

$$GPD \propto e^{B t}$$

one finds for the $\Delta_{\perp}^2$-dependence

$$GPD \propto e^{-B_{\perp} \Delta_{\perp}^2}$$

with $B_{\perp} = \frac{1}{1 - \zeta} B$. Thus, even if the $\Delta_{\perp}^2$-slope (described by $B_{\perp}$) remains finite as $\zeta \to 1$, the $t$-slope (described by $B$) goes to zero. This purely kinematical effect arises from the relation between $t$ and $\Delta_{\perp}^2$ \[ (13) \] with $\zeta = 1 - \frac{p_{\perp}^2}{p^2}$ fixed.

Since $\Delta_{\perp}$ is the momentum space variable conjugate to $r_{\perp} = r_{\perp 1} - R_{\perp}$ (for $x = \zeta$), it is thus important to translate the $t$-dependence of GPDs first into a $\Delta_{\perp}^2$-dependence before attempting to interpret the data.

What should one thus expect for the $\zeta$-dependence of GPDs at $x = \zeta$? The relevant GPDs are proportional to the overlap between on initial state where the active quark carries momentum fraction $\zeta$ and a final state where the active quark carries almost no momentum. Intuitively one would expect that the average separation between active quark and the spectators increases as the momentum fraction of the active quark decreases, i.e. in this case the final state wave function should be smaller than the initial state wave function.

In general, the overlap integral describing the GPDs \[ (14) \] depends not only on the distribution of the active quark but also on that of the spectators. However, it appears reasonable to assume that the spectator wave function (for a given position of the spectator center of momentum) does not depend very strongly on the position of the active quark when the active quark is far away from the spectators. In the following we will thus make the simplifying assumption that the overlap integral for the spectators (at fixed $x$ and $\zeta$) does not depend on the separation of the active quark from the spectators. This does not mean that the spectators wave function is point-like!

In order to qualitatively understand how the above overlap integrals depend on $\zeta$ (for , we rescale all momentum fractions in units of the final state momentum, i.e. the initial state hadron carries momentum $1/(1 - \zeta)$ and the final state hadron carries momentum 1. As the active quark carries momentum fraction 0, nothing in the final state depends on $\zeta$ and hence the $\zeta$-dependence arises from the change in the initial state wave function, and the resulting change in the overlap integrals. This observation suggests the following interpretation for the $\zeta$ dependence of the $\Delta_{\perp}^2$-slope of GPDs. For instance, if the $\Delta_{\perp}^2$-slope decreases with increasing $\zeta$, that would be an indication that the mean separation between the active quark and the spectators decreases with the momentum fraction carried by the active quark.

If one neglects the $\Delta_{\perp}^2$-dependence of the overlap integral for the spectators, one can use this reasoning to extract the ‘size’ (mean separation of the active quarks from the spectators) as a function of the momentum fraction carried by the active quark. For example, when the initial and final state wave function are proportional to $e^{-\frac{r_{\perp}^2}{\Xi^2}}$ and $e^{-\frac{r_{\perp}^2}{\Xi^2}}$ then the effective radius appearing in the product is the harmonic mean of the rms radii of the individual wave functions squared $\frac{1}{1 - \zeta} = \frac{1}{2} \left( \frac{1}{\Xi_1^2} + \frac{1}{\Xi_2^2} \right)$.

IV. SUMMARY

For $\zeta \neq 0$, the two dimensional Fourier transform of GPDs is more easily interpretable if one introduces the separation $r_{\perp}$ between the active quark and the center of momentum of the spectators, as this variable is the same in the initial and final state of the hadronic matrix element defining the GPDs. The $r_{\perp}$ dependence of the matrix element is obtained by Fourier transforming GPDs with a factor $e^{-\frac{r_{\perp}^2}{\Xi^2} r_{\perp}}$, i.e. for $x = \zeta$ the variable $r$ is Fourier conjugate to $\Delta_{\perp}$.
The mean $r^2$, and hence the $\Delta^2 - \perp$-slope of GPDs should be a typical hadronic scale. Therefore the $t$-slope, which is related to the $\Delta^2 - \perp$-slope by a kinematic factor of $1 - \zeta$, should go to zero as $\zeta \to 1$, even if the wave function does not become point-like. The $t$-slope divided by $1 - \zeta$ can be used to study how the mean separation of the active quark from the center of momentum of the spectators varies with $\zeta$. Intuitively, one would expect this ‘size’ to decrease with $\zeta$. Application of the above procedure to deeply-virtual meson production indeed yields a size that decreases with increasing $\zeta$ \[11\]. DVCS data for the $t$-slope \[12\] also shows a decrease with increasing $\zeta$.

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