On nonlinear evolution and supraluminal communication between finite quantum systems

M. Ferrero  
Dpto. Física, Universidad de Oviedo, Spain  
maferrero@uniovi.es

D. Salgado & J.L. Sánchez-Gómez  
Dpto. Física Teórica, Universidad Autónoma de Madrid, Spain  
david.salgado@uam.es & jl.sanchezgomez@uam.es

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We revise the 'no-signaling' condition for the supraluminal communication between two spatially separated finite quantum systems of arbitrary dimensions, thus generalizing a similar preceding approach for two-qubits: non-linear evolution does not necessarily imply the possibility of supraluminal communication between any sort of finite quantum systems.

1 Introduction

Though up to now there is no experimental indication why quantum evolution may be nonlinear, it has been traditionally considered both as a possible way out to the measurement problem or as matter of theoretical considerations to be contrasted with high-finesse experiments. One of the most remarkable consequences of these considerations was the possibility, under the nonlinearity assumption, of supraluminal communication between two spatially separated parties. This soon led some authors to conclude that any nonlinear quantum evolution would necessarily entail the possibility of such a communication and even to consider the relativistic postulate of 'no-faster-than-light' phenomena as the theoretical basis for the quantum evolution to be linear. Recently we have proven that this implication is not strict, i.e. that there exist possible nonlinear quantum evolutions not implying this fatal supraluminal communication.

Here we extend our previous result to finite quantum systems of arbitrary dimensions. We formulate the 'no-signaling' condition for these systems and show a full-fledged infinity of examples fulfilling this condition. Everything is
expressed in Bloch space language\cite{8, 9}, i.e. the states of quantum systems are expressed as

$$\rho(t) = \frac{1}{N} (I_N + r(t) \cdot \sigma) \tag{1}$$

and orthogonal projectors as

$$P = P_0 I_N + P \cdot \sigma \tag{2}$$

where \(r(t)\) is a time-dependent so-called Bloch vector belonging to a particular convex subset of \(\mathbb{R}^{N^2 - 1}\), \(\sigma \equiv (\sigma_1, \ldots, \sigma_{N^2 - 1})\) are the traceless orthogonal generators of \(SU(N)\) and \((P_0, P) \equiv (P_0, P_1, \ldots, P_{N^2 - 1})\) are real numbers subjected to certain restrictions (cf. \cite{8} for the details).

2 The 'no-signaling' condition

As remarked in \cite{7}, the impossibility of communication through the projection postulate, i.e. at a speed faster than that of light, is obtained only after imposing that the probability distribution of any observable of one subsystem only depends on its own reduced state. The mathematical translation of this criterion is straightforward provided one is familiar with the preceding language. Let us consider a two-partite system of subsystems 1 and 2, which have dimensions \(N_1\) and \(N_2\), respectively. Their common density matrix, using a tensor product basis, will be given by

$$\rho_{12} = \frac{1}{N_1 N_2} \left( I_{N_1 N_2} + r^{(1)} \cdot \sigma \otimes I_{N_2} + I_{N_1} \otimes r^{(2)} \cdot \lambda + \sum_{ij} r^{(1)(2)}_{ij} \sigma_i \otimes \lambda_j \right) \tag{3}$$

and an orthogonal projector for each of them by

$$P^{(1)} = P_0^{(1)} I_{N_1} + P^{(1)} \cdot \sigma \quad P^{(2)} = P_0^{(2)} I_{N_2} + P^{(2)} \cdot \lambda \tag{4}$$

respectively, where \(\sigma (\lambda)\) stands for the traceless orthogonal generators of \(SU(N_1)\) (\(SU(N_2)\)) and \(P^{(1)} (P^{(2)})\) is a \((N_1^2 - 1)\)-\((N_2^2 - 1)\)-dimensional vector restricted to some given subset\(^1\).

Suppose now that an orthogonal projector \((u_0, u)\) is measured upon subsystem 2. Then \(N_2\) possible outcomes \((u_0^{(k)}, u^{(k)})\) will result with probabilites \(p_k = u_0^{(k)} + u^{(k)} \cdot r^{(2)}\) given by the trace rule. Also, the projection postulate allows us to conclude that after such a measurement, the reduced density operator for its partner, subsystem 1 will be given by

$$\rho^{(1)}_k (0) = \frac{1}{N_1} \left( I_{N_1} + r^{(1);k} \cdot \sigma \right) \tag{5}$$

\(^1\)Namely, \(P_0 = P_0^2 + P \cdot P\) and \(2P_0 P_n + z_{ijn} P_i P_j = P_n\), where \(z_{ijk} \equiv g_{ijk} + if_{ijk}\), the latter denoting the completely symmetric and antisymmetric tensors of the Lie algebra \(su(N_j)\), respectively.
where $\mathbf{r}^{(1;k)}$ is an $(N_1^2 - 1)$-dimensional vector $(k = 1, \ldots, N_2$ possible outcomes) dependent on the joint state $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \mathbf{r}^{(12)}_{ij}$ and on the measured observable $(u_0, \mathbf{u})$:

$$
\mathbf{r}^{(1;k)}_j = \frac{u_0^{(k)} r^{(1)}_j + \sum_{n=1}^{N_2^2-1} r^{(12)}_{jn} u_n^{(k)}}{u_0^{(k)} + \mathbf{u}^{(k)} \cdot \mathbf{r}^{(2)}} \equiv r^{(1;k)}_j(0) \quad (6)
$$

In these conditions, the probability distribution $P$ of an arbitrary orthogonal projector $(\mathbf{v}_0, \mathbf{v})$ with $p = 1, \ldots, N_1$ possible outcomes $(\mathbf{v}_0^{(p)}, \mathbf{v}^{(p)})$ at time $t$ of subsystem 1 will be given by

$$
P^{(1)}(t; \mathbf{v}^{(p)}) = \sum_{k=1}^{N_2} (u_0^{(k)} + \mathbf{r}^{(2)} \cdot \mathbf{u}^{(k)})(\mathbf{v}_0^{(p)} + \mathbf{v}^{(p)} \cdot \mathbf{r}^{(1)}(t; \mathbf{r}^{(1;k)}(0)) \quad (7)
$$

where $\mathbf{r}^{(1)}(t; \mathbf{r}^{(1;k)}(0))$ denotes the Bloch vector of subsystem 1 at time $t$ with initial condition $\mathbf{r}^{(1;k)}(0)$.

The 'no-signaling' condition can then be easily formulated. The independence with respect to other partners’ reduced state and their mutual correlations will be expressed as

$$
\frac{\partial P^{(1)}(t; \mathbf{v}^{(p)})}{\partial r^{(2)}_k} = 0 \quad (8)
$$

$$
\frac{\partial P^{(1)}(t; \mathbf{v}^{(p)})}{\partial r^{(12)}_{ij}} = 0 \quad (9)
$$

Finally, the independence with respect to observables to be measured in spatially separated subsystems will be expressed as

$$
\frac{\partial P^{(1)}(t; \mathbf{v}^{(p)})}{\partial u^{(k)}_{\mu}} = 0 \quad \mu = 0, 1, \ldots, N_1^2 - 1 \quad (10)
$$

These three conditions are the mathematical translation of the previously formulated 'no-signaling' condition. The reader may check for himself that, as expected, the usual linear quantum evolution fulfills each of them (see also below).

### 3 Consequences

One of the main consequences of eqs. (8), (9) and (10) arises after noticing that they must be valid for any particular value of the parameters involved, which implies $\mathbf{r}^{(i)}(t; \mathbf{r}_k) = A^{(i)}(t) \mathbf{r}_k$, where $A^{(i)}(t)$ is a time-dependent matrix. In
other words, the reduced dynamics in absence of interactions (spatial separation) must be linear. Note that this does not exhaust the possibility of having nonlinear joint evolution. Indeed reduced linearity in absence of interactions entails neither joint linearity nor even reduced unitarity. Expressing this in Bloch vector language, if \( (r^{(1)}(t), r^{(2)}(t), r^{(12)}_{ij}(t)) \) denotes the Bloch vector of a two-partite system and if \( H = H_0 I_{N_1 N_2} + \mathbf{H} \cdot \sigma_{12} \) \( (\mathbf{H} = (H^{(1)}, H^{(2)}, H^{(12)})) \) and \( \sigma_{12} = (\sigma \otimes I_{N_2}, I_{N_1} \otimes \lambda, \sigma \otimes \lambda) \) denotes its joint Hamiltonian, then any evolution given by

\[
\begin{align*}
    r^{(1)}(t) &= F_1(t; H, r(0)) \\
    r^{(2)}(t) &= F_2(t; H, r(0)) \\
    r^{(12)}(t) &= F_{12}(t; H, r(0))
\end{align*}
\]

such that in absence of interactions \( (H^{(12)} = 0) \) satisfies

\[
\begin{align*}
    r^{(1)}(t) &= M^{(1)}(t; H^{(1)}) r^{(1)}(0) \\
    r^{(2)}(t) &= M^{(2)}(t; H^{(2)}) r^{(2)}(0)
\end{align*}
\]

where \( M^{(k)}(t; H^{(k)}) \) denotes a time-dependent matrix depending only on the Hamiltonian of the \( k \)th subsystem, is free of supraluminal communication.

It should be clear that this nonlinearity only affects the evolution and never the static structure of the theory, i.e. the principle of superposition of quantum states at a given instant of time is still valid, only the evolution of these states is affected.

Alternatively, one can express these nonlinearities through the evolution equations:
\[ \frac{dr_{ij}^{(2)}}{dt} = \left( \sum_{m,n=1}^{N_2-1} f_{imn}^{(2)} H_m^{(2)} r_{jn}^{(2)} + \sum_{j,m,n=1}^{N_2-1} f_{ijm}^{(2)} H_j^{(12)} r_{nm}^{(12)} \zeta_{ij,jn}^{(12)} (r^{(1)}, r^{(2)}, r^{(12)}) \right) \] (17)

\[ \frac{dr_{pq}^{(12)}}{dt} = 2 \left( \sum_{i,j=1}^{N_2-1} f_{jip}^{(1)} H_j^{(1)} r_{iq}^{(1)} + \sum_{i,j=1}^{N_2-1} f_{jip}^{(2)} H_j^{(12)} r_{iq}^{(12)} \right) + \] 

\[ + \sum_{i,j=1}^{N_2-1} \sum_{m,n=1}^{N_2-1} \text{Im} \left[ H_{im}^{(12)} r_{jn}^{(12)} \zeta_{pq,im}^{(12)} (r^{(1)}, r^{(2)}, r^{(12)}) \right] + \] 

\[ + \sum_{i,j=1}^{N_2-1} f_{jip}^{(1)} H_j^{(12)} r_{iq}^{(1)} \zeta_{pq,ij}^{(12)} (r^{(1)}, r^{(2)}, r^{(12)}) + \sum_{i,j=1}^{N_2-1} f_{jip}^{(2)} H_j^{(12)} r_{iq}^{(12)} \zeta_{pq,ij}^{(12)} (r^{(1)}, r^{(2)}, r^{(12)}) \right) \] (18)

where the functions \( \xi \) are completely arbitrary. Notice that in absence of interactions \( (H^{(12)} = 0) \), one recovers the usual well-known quantum evolution.

### 4 Conclusions

The main two conclusions to be drawn are that (i) nonlinear evolution does not necessarily imply the possibility of supraluminal communication between two arbitrary finite quantum systems, and (ii) non linear terms, in order to fulfill the no-signaling condition, must be necessarily associated to interactions.

This reopens a door, originally suggested by Wigner, to explore possible solutions to the measurement problem without contradicting other well contrasted theories.

A third generalization of this approach can be undertaken by focusing on non-projective measurements, but on generalized measurements, i.e. on POVM’s [10].

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