Extreme Points and Factorizability for New Classes of Unital Quantum Channels

Uffe Haagerup, Magdalena Musat and Mary Beth Ruskai

Abstract. We introduce and study two new classes of unital quantum channels. The first class describes a 2-parameter family of channels given by completely positive (CP) maps $M_3(C) \mapsto M_3(C)$ which are both unital and trace-preserving. Almost every member of this family is factorizable and extreme in the set of CP maps which are both unital and trace-preserving, but is not extreme in either the set of unital CP maps or the set of trace-preserving CP maps. We also study a large class of maps which generalize the Werner-Holevo channel for $d=3$ in the sense that they are defined in terms of partial isometries of rank $d-1$. Moreover, we extend this to maps whose Kraus operators have the form $t |e_j\rangle \langle e_j| \oplus V$ with $V \in M_{d-1}(C)$ unitary and $t \in (-1, 1)$. We show that almost every map in this class is extreme in both the set of unital CP maps and the set of trace-preserving CP maps. We analyze in detail a particularly interesting family which is extreme unless $t = \frac{-1}{d-1}$. For $d=3$, this includes a pair of channels which have a dual factorization in the sense that they can be obtained by taking the partial trace over different subspaces after using the same unitary conjugation in $M_3(C) \otimes M_3(C)$.

Contents

1. Introduction 3456
2. High Rank Extreme Points of Unital Quantum Channels 3459
   2.1. Background 3459
   2.2. A Factorizable Family of Extreme UCPT Maps 3459
   2.3. Remarks 3461
     2.3.1. Non-extreme Cases 3461
     2.3.2. Entanglement of Formation 3462

Uffe Haagerup initiated and contributed significantly to this work until his untimely death on July 5, 2015. He was Professor of Mathematics at the University of Copenhagen and the University of Southern Denmark.
1. Introduction

It is by now well-established that completely positive, trace-preserving (CPT) maps on matrix algebras play an important role in quantum information theory because they describe the effect of noise on a quantum system. The set of CPT maps $M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ is convex, as is its dual, the set of unital completely positive (UCP) maps, and their intersection, for which we use the acronym UCPT maps.

By the Choi-Jamiolkowski [4,11] isomorphism the set of UCPT maps from $M_d(\mathbb{C})$ to $M_d(\mathbb{C})$ is in one-to-one correspondence with the set of bipartite states on $\mathbb{C}_d \otimes \mathbb{C}_d$ for which both quantum marginals (also called reduced density matrices) are given by the maximally mixed state, $\frac{1}{d^2}I_d$. From an operator algebra perspective, every density matrix, i.e., every positive operator $\rho \in M_d(\mathbb{C})$ generates a state $\phi$ on the operator algebra $M_d(\mathbb{C})$ given...
by \(\phi(A) = \text{Tr } A \rho\). The maximally mixed state \(\frac{1}{d} I_d\) corresponds to the “tracial state” \(\tau(A) = \frac{1}{d} \text{Tr } A\) (where \(\tau\) denotes the trace normalized so that \(\tau(I_d) = 1\)).

For qubits, i.e., two-dimensional systems, Kümmerrer [15] showed that the extreme points of the set of UCPT maps on \(M_2(\mathbb{C})\) are precisely those that correspond to unitary conjugations. For \(d \geq 3\), the convex structure of the UCPT maps is much more complex. However, relatively little was known about the extreme points other than unitary conjugations.

Despite some evidence [17] that there are extreme points of the UCPT maps which are not extreme for either the CPT or UCP maps, no explicit examples were given in the literature until Ohno [20] presented two examples which he attributed to Arveson. We present a new family of UCPT maps on \(\mathcal{M}\) for \(d \geq 3\), and show that they are extreme in both the set of CPT and UCP maps. Moreover, we show that almost all such maps are extreme points for both the CPT and UCP maps.

We show that all maps in this family are factorizable [6, 7]. When mapped to bipartite states on \(\mathbb{C}^3 \otimes \mathbb{C}^3\) via the Choi matrix, both quantum marginals are \(\frac{1}{3} I_3\) for which the von Neumann entropy is \(\log 3\). We give upper bounds on the entanglement of formation (EoF) demonstrating that, although these states are entangled, they are not maximally entangled states on \(\mathbb{C}^3 \otimes \mathbb{C}^3\).

In a complementary direction we study a class of extreme points of the set of UCPT maps for \(d \geq 3\) whose Choi–Jamiolkowski matrix has rank \(d\) (sometimes known as Choi rank). These maps are not given by unitary conjugations and are, in general, not factorizable. Following a suggestion in [21, Section 2], we consider partial isometries on \(M_d(\mathbb{C})\) constructed from unitary matrices in \(M_{d-1}(\mathbb{C})\) and extend this construction in a natural way to a much larger class of maps whose Kraus operators have the form \(t |e_j\rangle \langle e_j| \otimes V\) with \(V \in M_{d-1}(\mathbb{C})\) unitary and \(t \in (-1,1)\).

We show that almost all such maps are extreme points for both the UCP and CPT maps. We analyze in detail maps constructed from the unitary operator \(2 |1_{d-1}\rangle \langle 1_{d-1}| - I_{d-1}\), where \(|1_d\rangle\) is the vector whose elements are all \(d^{-1/2}\), and show that they are extreme in both the set of CPT and UCP maps unless \(t = \frac{1}{d-1}\). When \(d\) is odd, we construct a quite different family from rank two permutations in \(M_{d-1}(\mathbb{C})\); these maps are extreme for all \(t \in (-1,1)\) when \(d > 3\).

One of several equivalent ways (described in [14, Appendix A]) of formulating the Stinespring representation for a CPT map \(\Phi : M_{d_A}(\mathbb{C}) \mapsto M_{d_B}(\mathbb{C})\) on matrix algebras uses an auxiliary space \(\mathcal{H}_E = \mathbb{C}_{d_E}\) called the environment and a unitary matrix \(U\) in \(M_{d_B}(\mathbb{C}) \otimes M_{d_E}(\mathbb{C})\) such that

\[
\Phi(\rho) = (\mathcal{I} \otimes \text{Tr}_E) U^* (\rho \otimes |\phi_E\rangle \langle \phi_E|) U.
\]

This is a natural model for noise when \(d_A = d_B\), the system is initially in a pure product state \(|\psi_A\rangle \otimes |\phi_E\rangle\) on \(\mathbb{C}_{d_A} \otimes \mathbb{C}_{d_E}\), \(U(t)\) describes the time evolution of the interacting system and environment, and \(U = U(t_f)\) corresponds to some later time \(t_f\). The question of factorizability of a UCPT map (over a matrix algebra) roughly asks if the ancilla state \(|\phi_E\rangle \langle \phi_E|\) can be replaced by the maximally mixed state \(\frac{1}{d_E} I_{d_E}\). When \(d_A = d_E\), this is a natural model for
noise when the system is initially decoupled from the environment which is in a maximally mixed or “thermal” state. See, e.g., [10,18,22].

The concept of a factorizable map was introduced by Anantharaman-Delaroche in [1] in a more general mathematical setting. It was further studied extensively in the context of unital quantum channels on $M_d(C)$ in [6,7]. Following [6], we say that a UCPT map $\Phi$ on $M_d(C)$ has an exact factorization through $M_d(C) \otimes \mathcal{N}$, where $\mathcal{N}$ is a von Neumann algebra with a normalized faithful trace $\tau$ (so that $\tau(I_N) = 1$), if there is a unitary $U \in M_d(C) \otimes \mathcal{N}$ such that for all $\rho \in M_d(C)$,

$$\Phi(\rho) = (I \otimes \tau)U^*(\rho \otimes I_N)U.$$  \hspace{1cm} (2)

When $\mathcal{N} = M_\nu(C)$ is a matrix algebra, this is equivalent to $\Phi(\rho) = (I \otimes \text{Tr})U^*(\rho \otimes 1_\nu I_\nu)U$. In [19] it was shown that for every $d \geq 11$, there are UCPT maps with an exact factorization through $M_d(C) \otimes \mathcal{N}$, where $\mathcal{N}$ is a von Neumann algebra of type II$_1$ which cannot be replaced by any finite dimensional von Neumann algebra. In view of [6, Theorem 3.7] and the announcement in [12] that the Connes Embedding Problem has a negative answer, there are factorizable maps for some (presumably very large) $d$ which cannot even be approximated by maps with exact factorizations through matrix algebras.

Most of the maps considered in Sect. 3 are extreme in the set of UCP or CPT maps and, hence, not factorizable. However, our work led us to a pair of channels, which have exact factorizations through $M_3(C) \otimes M_3(C)$ that are dual in the sense that they can be obtained from the same unitary operator by exchanging the roles of the system and auxiliary spaces.

This paper is organized as follows. In Sect. 2, we describe and study a family of UCPT maps which are not extreme in either the set of UCP maps or the set of CPT maps, but are extreme in the set of UCPT maps. In Sect. 3.1 we introduce UCPT maps $\Phi(\rho) = \frac{1}{d-1+t^*} \sum_{m=1}^d A_m^* \rho A_m$ with $A_m = t |e_j\rangle\langle e_j| \oplus V_m$, with $V_m \in M_{d-1}(C)$ unitary, and $t \in (-1,1)$. In Sect. 3.2 we prove several theorems which imply that, in general, maps of this form are extreme in both the set of UCP and the set of CPT maps using the equivalent condition from [4, Theorem 5] of linear independence of the sets $\{A_m^* A_n\}$ and $\{A_m A_n^*\}$, respectively. In Sect. 3.3 we consider explicit examples and subclasses of maps of this type. Section 4 is devoted to the special case in which $V_m = 2|1_{d-1}\rangle\langle 1_{d-1}| - 1_{d-1}$. Sections 4.2–4.4 present the details of our analysis for this case when $d > 3$ and $t \neq \frac{1}{d-1}$; Sect. 4.5 deals with the case $d = 3$; and Sect. 4.6 deals with $t = \frac{1}{d-1}$ when $d > 3$.

Appendix A.1 discusses exact factorizations; Appendix A.2 considers dual pairs of channels $\Phi, \Psi$ associated with a unitary $U \in M_p(C) \otimes M_q(C)$; Appendix A.3 extends a result in [6] to show that if a channel has Choi rank $\leq 4$, then both $\Phi \circ \Phi^*$ and $\Phi^* \circ \Phi$ are factorizable. Finally, Appendix B presents an example to show that a set $\{A_m^* A_n\}$ can be linearly independent when $\{A_m A_n^*\}$ is linearly dependent.
2. High Rank Extreme Points of Unital Quantum Channels

2.1. Background

It is a fundamental result of Choi [4] that when a CP map $\Phi : M_{d_1}(C) \mapsto M_{d_2}(C)$ is written in the form $\Phi(\rho) = \sum_k A_k^\dagger \rho A_k$, then it is extreme in the convex set of CP maps for which $\Phi(1_{d_1}) = B$ for some fixed $B \in M_{d_2}$ if the operators $\{A_k\}$ satisfy $\sum_k A_k^\dagger A_k = B$ and can be chosen so that the set $\{A_m^* A_n\}$ is linearly independent. Thus, a UCP map is extreme if and only if $\{A_m^* A_n\}$ is linearly independent, and a CPT map is extreme if and only if $\{A_n A_m^*\}$ is linearly independent. This implies that a UCPT map whose Choi-rank is $> d_1$ can not be an extreme point of the UCP maps and one whose Choi-rank is $> d_2$ can not be an extreme point of the CPT maps. Thus a UCPT map $\Phi : M_d(C) \mapsto M_d(C)$ with Choi rank $> d$ can not be an extreme point of either the UCP or CPT maps.

Nevertheless this does not preclude the possibility that a UCPT map with Choi-rank greater than $d$ can be extreme in the set of UCPT. We present a new family of such maps for $d = 3$ with Choi-rank equal to 4, which are parameterized by $\alpha, \beta \in C$ with $|\alpha|^2 + |\beta|^2 = 1$. When $|\alpha|^2 \neq 0, 1$, these maps are extreme in the set of UCPT maps. Moreover, all of the maps in this family have exact factorizations through $M_3(C) \otimes M_2(C)$.

Although earlier work [17] suggested the existence of such maps, the only explicit examples in the literature are due to Arveson, as presented by Ohno [20] for $d = 3$ and $d = 4$. For comparison with our results, we note that in the $d = 3$ case the Choi–Kraus operators for the Arveson–Ohno map are

$$A_1 = |e_1\rangle\langle e_1| \quad \quad A_2 = |e_1\rangle\langle e_2| + \sqrt{2}|e_2\rangle\langle e_3| \quad \quad A_3 = \sqrt{2}|e_2\rangle\langle e_1| + \sqrt{3}|e_3\rangle\langle e_2| \quad \quad A_4 = |e_3\rangle\langle e_1| + \sqrt{2}|e_1\rangle\langle e_3|$$

In Sect. 2.3 we show that the map $\Phi(\rho) = \sum_{k=1}^4 A_k^\dagger \rho A_k$ is not factorizable, i.e., it does not have an exact factorization through $M_3(C) \otimes N$ for any von Neumann algebra $N$. However, it follows from Proposition A.3, which is a straightforward generalization of [6, Remark 5.6], that the maps $\Phi \circ \Phi^*$ and $\Phi^* \circ \Phi$ have exact factorizations through $M_3(C) \otimes M_4(C)$.

2.2. A Factorizable Family of Extreme UCPT Maps

We present a new family of UCPT maps $M_3(C) \mapsto M_3(C)$ which have Choi-rank equal to 4 so that they can not be extreme in either the set of UCP or CPT maps. However, these maps are extreme in the set of UCPT maps. All maps in this family, including those which are not extreme, have exact factorizations through $M_3(C) \otimes M_2(C)$.

**Theorem 2.1.** Let $\alpha, \beta \in C$ with $|\alpha|^2 + |\beta|^2 = 1$ and let

$$A_1 = \alpha|e_1\rangle\langle e_1| + |e_2\rangle\langle e_3| \quad \quad A_2 = \beta|e_1\rangle\langle e_3| + |e_3\rangle\langle e_2| \quad \quad A_3 = -|e_1\rangle\langle e_2| - \overline{\beta}|e_3\rangle\langle e_1| \quad \quad A_4 = |e_2\rangle\langle e_1| + \overline{\alpha}|e_3\rangle\langle e_3|. \quad (4)$$
Then \( \Phi_{\alpha,\beta}(\rho) = \frac{1}{2} \sum_{k=1}^{4} A_k^* \rho A_k \) is an extreme point in the set of UCPT maps if \( |\alpha|^2 \neq 0, \frac{1}{2}, 1 \). Moreover, \( \Phi_{\alpha,\beta} \) has an exact factorization through \( M_3(\mathbb{C}) \otimes M_2(\mathbb{C}) \) for all \( \alpha, \beta \).

**Proof.** To show that \( \Phi_{\alpha,\beta} \) has an exact factorization through \( M_3(\mathbb{C}) \otimes M_2(\mathbb{C}) \) let

\[
U = \sum_{j,k \in \{1,2\}} A_{2(j-1)+k} \otimes |e_j \rangle \langle e_k| = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.
\]

Then \( U \in M_3(\mathbb{C}) \otimes M_2(\mathbb{C}) \simeq M_6(\mathbb{C}) \) is unitary and

\[
\Phi_{\alpha,\beta}(\rho) = \frac{1}{2} \sum_{k=1}^{4} A_k^* \rho A_k = (I_3 \otimes \text{Tr})(U^* (\rho \otimes \frac{1}{2} I_2) U).
\]

To prove the rest of this theorem, recall that Landau and Streater [16] showed that a necessary and sufficient conditions for \( \Phi_{\alpha,\beta} \) to be an extreme UCPT map is that the set \( \{ B_{jk} = A_j^* A_k + A_k^* A_j^* \} \) is linearly independent in \( M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \). First, observe that the diagonal of \( B_{jk} \) is zero if \( j \neq k \), and that each \( B_{jj} \) is diagonal with diagonal elements corresponding to the rows of the matrix

\[
\begin{pmatrix}
|\alpha|^2 & 0 & 1 & |\alpha|^2 & 1 & 0 \\
0 & 1 & |\beta|^2 & |\beta|^2 & 0 & 1 \\
|\beta|^2 & 1 & 0 & 1 & 0 & |\beta|^2 \\
1 & 0 & |\alpha|^2 & 0 & 1 & |\alpha|^2
\end{pmatrix}.
\]

After deleting the third and fourth columns one obtains a matrix whose determinant is \(-|\alpha|^2|\beta|^2\). This implies that if \( |\alpha| \neq 0,1 \) the rows are linearly independent and, hence, the set \( \{ B_{jj} \}_{j=1}^{4} \) is linearly independent.

Next observe that

\[
\begin{align*}
B_{12} &= A_1^* A_2 + A_2 A_1^* = \overline{\alpha}\beta |e_1 \rangle \langle e_2| + \beta 0_3 + \alpha 0_3 \oplus |e_1 \rangle \langle e_2| \\
B_{14} &= A_1^* A_4 + A_4 A_1^* = |e_1 \rangle \langle e_3| \oplus 0_3 + \alpha 0_3 \oplus |e_1 \rangle \langle e_2| + \alpha 0_3 \oplus |e_2 \rangle \langle e_3| \\
B_{34} &= A_3^* A_4 + A_4 A_3^* = -\overline{\alpha}\beta |e_1 \rangle \langle e_3| \oplus 0_3 - \overline{\beta} 0_3 \oplus |e_2 \rangle \langle e_3|
\end{align*}
\]

which are clearly linearly independent if and only if

\[
\det \begin{pmatrix}
\overline{\alpha} \beta & \beta & 0 \\
1 & \alpha & 0 \\
\overline{\beta} & 0 & \beta
\end{pmatrix} = |\beta|^2(2|\alpha|^2 - 1) \neq 0
\]

Since \( B_{jk} = B_{kj}^* \), the matrices \( \{ B_{21}, B_{14}, B_{43} \} \) are also linearly independent under the same conditions. Next, we similarly treat

\[
\begin{align*}
B_{24} &= A_2^* A_4 + A_4 A_2^* = \overline{\alpha} |e_2 \rangle \langle e_3| \oplus 0_3 + \overline{\beta} 0_3 \oplus |e_3 \rangle \langle e_1| \\
B_{32} &= A_3^* A_2 + A_2 A_3^* = -\beta |e_1 \rangle \langle e_2| \oplus 0_3 + \beta |e_2 \rangle \langle e_3| \oplus 0_3 - 0_3 \oplus |e_3 \rangle \langle e_1| \\
B_{13} &= A_1^* A_3 + A_3 A_1^* = -\overline{\alpha} |e_1 \rangle \langle e_2| \oplus 0_3 - \overline{\beta} 0_3 \oplus |e_3 \rangle \langle e_1|
\end{align*}
\]
which are linearly independent if and only if
\[
\det \begin{pmatrix} 0 & \alpha & \alpha \beta \\ \beta & \beta & 1 \\ \alpha & 0 & \alpha \beta \end{pmatrix} = \alpha^2 (1 - 2|\beta|^2) \neq 0.
\]

After again observing that the adjoints are linearly independent under the same conditions, we can conclude that if $|\alpha|^2 \neq 0, 1, \frac{1}{2}$ then each of the four sets
\[
\{B_{12}, B_{41}, B_{34}\}, \quad \{B_{21}, B_{14}, B_{43}\}, \quad \{B_{24}, B_{32}, B_{13}\}, \quad \{B_{42}, B_{23}, B_{31}\}
\]
consists of linearly independent matrices in $M_6(C)$. Moreover, the only common point in the spans of each of these four sets is the zero matrix $0_6$. Therefore $\{A_j^* A_k \oplus A_k A_j^*\}_{j \neq k}$ is linearly independent when $|\alpha|^2 \neq 0, \frac{1}{2}, 1$. Combining this with our observations above for $B_{kk}$, implies that $\{B_{jk} : j, k \in \{1, 2, 3, 4\}\}$ is linearly independent.

Because the maps $\Phi_{\alpha,\beta}$ are parameterized by a unit vector in $C_2$, it is tempting to associate each channel with a qubit state. However, the vectors $|v\rangle$ and $e^{i\theta} |v\rangle$ represent the same physical state. But the channels associated with, e.g., $\alpha = 1$ and $\alpha = -1$ are not the same.

2.3. Remarks

2.3.1. Non-extreme Cases. When $|\alpha| = 1$ or $|\beta| = 1$, the map $\Phi_{\alpha,\beta}$ can be written as a convex combination of unitaries, e.g., when $\alpha = 1$, $\Phi_{1,0} = \frac{1}{4} \sum_{k=1}^{4} U_k^* \rho U_k$ where
\[
U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

However, when $\alpha = \beta = \frac{1}{\sqrt{2}}$, the map $\Phi_{\alpha,\beta}$ is not a convex combination of unitaries. Let $|1_4\rangle$ be the vector with all elements $\frac{1}{2}$. Then $W = 2 |1_4\rangle \langle 1_4| - 1_4$ is unitary. Now let $X_j = \frac{1}{\sqrt{2}} \sum_{k=1}^{4} w_{jk} A_k$, then
\[
X_1 = \frac{1}{4} \begin{pmatrix} -1 - \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ -1 & \sqrt{2} & 1 \end{pmatrix}, \quad X_2 = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ -1 & -\sqrt{2} & 1 \end{pmatrix},
\]
\[
X_3 = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \end{pmatrix}, \quad X_4 = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ -1 & -\sqrt{2} & 1 \end{pmatrix}.
\]

Since $W$ is unitary, $\sum_j X_j^* \rho X_j = \frac{1}{2} \sum_k A_k^* \rho A_k = \Phi(\rho)$. Although $\tilde{X}_2 = 2 X_2$ is unitary, this is not true for $j = 1, 3, 4$. In fact, both $\{X_j^* X_k\}_{j,k=1,3,4}$ and $\{X_j X_k^*\}_{j,k=1,3,4}$ are linearly independent sets so that $\Psi(\rho) = \frac{1}{4} \sum_{k=1,3,4} X_j^* \rho X_j$ is an extreme point of both the UCP and the CPT maps. Thus, $\Phi(\rho) = \frac{1}{4} \tilde{X}_2^* (\rho) \tilde{X}_2 + \frac{3}{4} \Psi(\rho)$ is a convex combination of a unitary conjugation and a
UCPT map with Choi rank 3. (It may be worth noting that $\Psi$ is an extreme point of both the UCP and CPT maps which has Choi-rank 3, but is not of the form considered in Sect. 3.)

The argument above easily extends to $\alpha = \pm \beta$ with $2X_3$ unitary. It seems reasonable to conjecture that, whenever $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$, the channel $\Phi_{\alpha,\beta}$ is a convex combination of a unitary conjugation and a map with Choi-rank 3 which is extreme in both the UCP and CPT maps. However, one would need a different unitary transformation relating $\{A_k\}$ to $\{X_k\}$.

2.3.2. Entanglement of Formation. Recall that the Choi matrix associates any CP map with a bipartite state $\rho$ on $C_d \otimes C_d$ and those associated with UCPT maps have quantum marginals given by the maximally mixed state $\frac{1}{d}I_d$. Since the UCP maps considered here are not associated with pure maximally entangled states, any measure of entanglement based on entropy will be less than the maximum value of $\log d$. A natural one to use in this situation is the entanglement of formation [3], defined as

$$\text{EoF}(\rho) = \inf \left\{ \sum_k p_k E(\psi_k) : \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \right\}$$

where $E(\psi_k) = S(\text{Tr}_B |\psi_k\rangle \langle \psi_k|)$ and $S(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy. An upper bound on the EoF (which is probably optimal) for the Choi matrix of the UCPT maps is readily calculated for the examples given here since the Kraus operators, which correspond to eigenvectors of the Choi matrix, are all associated with pure bipartite states.

For our family of channels, the Choi matrix is

$$\rho = \frac{1 + |\alpha|^2}{6} |\psi_1\rangle \langle \psi_1| + \frac{1 + |\beta|^2}{6} |\psi_2\rangle \langle \psi_2| + \frac{1 + |\beta|^2}{6} |\psi_3\rangle \langle \psi_3| + \frac{1 + |\alpha|^2}{6} |\psi_4\rangle \langle \psi_4|,$$

where

$$\psi_1 = \frac{1}{\sqrt{1 + |\alpha|^2}} (\alpha |e_1 \otimes e_1\rangle + |e_2 \otimes e_3\rangle), \quad \psi_2 = \frac{1}{\sqrt{1 + |\beta|^2}} (\beta |e_1 \otimes e_3\rangle + |e_3 \otimes e_2\rangle)$$

$$\psi_3 = \frac{1}{\sqrt{1 + |\beta|^2}} (|e_1 \otimes e_2\rangle + \overline{\beta} |e_3 \otimes e_1\rangle), \quad \psi_4 = \frac{1}{\sqrt{1 + |\alpha|^2}} (|e_2 \otimes e_1\rangle + \overline{\alpha} |e_3 \otimes e_3\rangle).$$

This implies that

$$\text{EoF}(\rho_{AB}) \leq \left( \frac{1 + |\alpha|^2}{3} h \right) \left( \frac{1}{1 + |\alpha|^2} \right) + \frac{1 + |\beta|^2}{3} h \left( \frac{1}{1 + |\beta|^2} \right)$$

where $h(x) = -x \log x - (1 - x) \log(1 - x)$ is the binary entropy. For $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$ the expression above takes its largest value of $h(\frac{1}{3}) = 0.918296 < 1 = \log 2$, and for $|\alpha|^2 = 0.1$ its smallest value of $\frac{2}{3} h(\frac{1}{3}) = \frac{2}{3} \approx 0.66667$. This is considerably less than the EoF of a maximally entangled state which is $\log 3 \approx 1.58496$. Thus, we have a large family of bipartite states on $C_3 \otimes C_3$ which are extreme in the convex set of states whose quantum marginals are $\frac{1}{3}I_3$, but whose entanglement as measured by the EoF can be less than half that of a maximally entangled state.
For comparison, we can bound the EoF for the Arveson–Ohno example (3) by similarly observing that its Choi matrix is
\[
\rho = \frac{1}{12} e_1 \otimes e_1 |\psi_2\rangle + \frac{1}{2} |\psi_2\rangle \langle \psi_2 | + \frac{5}{12} |\psi_3\rangle \langle \psi_3 | + \frac{1}{4} |\psi_4\rangle \langle \psi_4 |,
\]
with \( \psi_2 = \frac{1}{\sqrt{3}} \left( |e_1 \otimes e_2 \rangle + \sqrt{2} |e_2 \otimes e_3 \rangle \right) \), \( \psi_3 = \frac{1}{\sqrt{3}} \left( \sqrt{\frac{2}{3}} |e_1 \otimes e_2 \rangle + \sqrt{\frac{3}{3}} |e_3 \otimes e_2 \rangle \right) \), and \( \psi_4 = \frac{1}{\sqrt{3}} \left( |e_3 \otimes e_1 \rangle + \sqrt{2} |e_1 \otimes e_3 \rangle \right) \). This implies that
\[
\text{EoF}(\rho_{AB}) \leq \frac{1}{4} h \left( \frac{1}{3} \right) + \frac{5}{12} h \left( \frac{2}{3} \right) + \frac{1}{4} h \left( \frac{1}{3} \right) \approx 0.8637.
\]

**2.3.3. The Arveson–Ohno Channel is Not Factorizable.**

**Proposition 2.2.** The UCPT channel \( \Phi(\rho) = \sum_{k=1}^{4} A_k^* \rho A_k \) with \( A_k \) given by (3) is not factorizable.

**Proof.** First observe that the matrices \( \{A_1, A_2, A_3, A_4\} \) are linearly independent, so that we can use Proposition A.1 in Appendix A.1. Let \( \mathcal{N} \) be a von Neumann algebra with normal faithful trace satisfying \( \tau(I_{\mathcal{N}}) = 1 \). We assume that we can find a set of matrices \( \{Y_1, Y_2, Y_3, Y_4\} \) in \( \mathcal{N} \) which satisfy \( \tau(Y_{j^*} Y_k) = \delta_{jk} \), and find a contradiction. A straightforward calculation shows that \( \langle e_3, A_j^* A_k e_2 \rangle = 0 \) unless \( j = 4, k = 2 \) which implies \( 0 = \frac{1}{\sqrt{3}} Y_4^* Y_2 \). Multiplying this by \( Y_4 \) on the left and \( Y_2^* \) on the right implies \( Y_4 Y_2^* Y_4 Y_2 = 0 \).

Next, observe that \( \langle e_m, A_j^* A_k e_m \rangle = 0 \) if \( j \neq k \), and
\[
\langle e_2, A_j^* A_j e_2 \rangle = \begin{cases} \frac{1}{2} & j = 2 \\ \frac{1}{2} & j = 3 \\ 0 & j = 1, 4 \end{cases}
\]
\[
\langle e_3, A_j^* A_k e_3 \rangle = \begin{cases} \frac{3}{4} & j = 3 \\ \frac{1}{4} & j = 4 \\ 0 & j = 1, 2 \end{cases}
\]
from which it follows that
\[
\frac{1}{2} Y_2 Y_2^* + \frac{1}{2} Y_3 Y_3^* = I_{\mathcal{N}} \quad \frac{3}{4} Y_3 Y_3^* + \frac{1}{4} Y_4 Y_4^* = I_{\mathcal{N}}.
\]
Combining these to eliminate \( Y_3 Y_3^* \) gives \( 3 Y_2 Y_2^* - Y_4 Y_4^* = 2 I_{\mathcal{N}} \). Then multiplying by \( Y_4 Y_4^* \) and using \( Y_2 Y_2^* Y_2 Y_2^* = 0 \) implies \( Y_4 Y_4^* = - (Y_4 Y_4^*)^2 \) which implies \( Y_4 = 0 \). Thus, we have shown \( Y_2 Y_2^* = \frac{2}{3} I_{\mathcal{N}} \).

Finally, we observe that \( \langle e_3, A_j^* A_k e_3 \rangle = 0 \) if \( j \neq k \) and \( \langle e_3, A_j^* A_j e_3 \rangle = \begin{cases} \frac{1}{2} & j = 2, 4 \\ 0 & j = 1, 3 \end{cases} \) which implies \( \frac{1}{2} Y_2 Y_2^* + \frac{1}{2} Y_4 Y_4^* = I_{\mathcal{N}} \). However, since \( Y_4 = 0 \), this implies \( Y_2 Y_2^* = 2 I_{\mathcal{N}} \) which is not consistent with \( Y_2 Y_2^* = \frac{2}{3} I_{\mathcal{N}} \). (Since, e.g., it would give \( \tau(Y_2 Y_2^*) \neq \tau(Y_2 Y_2^*) \).)

**\( \square \)**

**3. Extreme Points from Partial Isometries**

**3.1. Definitions and Basic Properties**

It is well-known that the extreme points of the convex set of UCPT maps include conjugation with a single unitary, i.e., \( \rho \mapsto U^* \rho U \), and that these are extreme in both the set of UCP maps and the set of CPT maps. It is also known that there are maps which are not unitary conjugation but, nonetheless,
contrary, it is factorizable. When $U$ linear independence of the sets operator $S$ of a partial isometry of rank studied. These channels have a generalization of this, was proposed in [21, Section 4.2], but not widely extreme in both the set of UCP and the set of CPT maps. The simplest example is the Werner-Holevo channel [24] for $d = 3$ and its symmetric counterpart.

One interesting class of quantum channels, which can be regarded as a generalization of this, was proposed in [21, Section 4.2], but not widely studied. These channels have $d$ Kraus operators, each of which is a multiple of a partial isometry of rank $d - 1$. To define them, we will use the cyclic shift operator $S = \sum_{k} |e_k\rangle\langle e_{k+1}|$. Let $\{V_1, V_2, \ldots V_d\}$ be a set of unitary matrices in $M_{d-1}(C)$, and let

$$A_m = S^{-m+1} \begin{pmatrix} 0 & 0 \\ 0 & V_m \end{pmatrix} S^{m-1}, \quad m = 1, 2, \ldots d. \quad (10)$$

Then it is easy to check that $A_m^* A_m = A_m A_m^* = (I_d - |e_m\rangle\langle e_m|)$, so that $\sum_m A_m^* A_m = \sum_m A_m A_m^* = (d - 1)I_d$ and the map $\Phi(\rho) = \frac{1}{d-1} \sum_m A_m^* \rho A_m$ is both CPT and UCP. One can, more generally, consider operators of the form

$$A_m = S^{-m+1} \begin{pmatrix} t & 0 \\ 0 & V_m \end{pmatrix} S^{m-1}, \quad m = 1, 2, \ldots d \quad (11)$$

with $t \in [-1, 1]$. Since $A_m^* A_m = A_m A_m^* = I_d - (1 - t^2)|e_m\rangle\langle e_m|$, $\sum_m A_m^* A_m = \sum_m A_m A_m^* = (d - 1 + t^2)I_d$ which implies that the map $\Phi$ given by

$$\Phi(\rho) = \frac{1}{d-1+t^2} \sum_{m=1}^{d} A_m^* \rho A_m \quad (12)$$

is both UCP and CPT. Note that the Kraus operators for the channel are $\tilde{A}_m \equiv \frac{1}{\sqrt{d-1+t^2}} A_m$ which satisfy $\sum_m \tilde{A}_m^* \tilde{A}_m = \sum_m \tilde{A}_m \tilde{A}_m^* = I_d$.

**Remark 3.1.** When $t = \pm 1$, the matrices $A_m$ in (11) are unitary and the channel (12) can not be extreme in either the UCP or CPT maps. On the contrary, it is factorizable. When $U = \oplus_{m=1}^{d} A_m = \sum_{m=1}^{d} A_m \otimes |e_m\rangle\langle e_m|$ is a block diagonal $d^2 \times d^2$ unitary matrix,

$$\Phi(\rho) = \sum_{m} A_m^* \rho A_m = (I_d \otimes \text{Tr}) \left( U^* \left( \rho \otimes \frac{1}{d} I \right) U \right). \quad (13)$$

We now focus on the case $t \in (-1, 1)$, for which we are interested in the linear independence of the sets $\{A_m^* A_n\}$ and $\{A_m A_n^*\}$, as these are precisely the conditions for a map $\Phi$ given by (12) to be an extreme point of the set of UCP and CPT maps respectively.

**Proposition 3.2.** Let $t \in (-1, 1)$ and for $m = 1, 2, \ldots d$ let $\{A_m\}$ be as in (9) with $\{V_m\}$ arbitrary unitary matrices in $M_{d-1}(C)$. Then each $A_m$ is a normal matrix in $M_d(C)$, and $\{A_m^* A_m\}_{m=1}^{d}$ is a linearly independent set of diagonal matrices. Moreover it is a basis for the set of diagonal matrices in $M_d(C)$. 


Proof. First, observe that
\[ A_m^* A_m = A_m A_m^* = S^{-m+1} \begin{pmatrix} t^2 & 0 \\ 0 & I_{d-1} \end{pmatrix} S^{m-1} = I_d - (1 - t^2) |e_m\rangle\langle e_m|, \]
which implies that \( \sum_{m=1}^d A_m^* A_m = (d - 1 + t^2)I_d \). Thus, for the purpose of determining linear independence, one can replace \( A_m^* A_m - I_d = -(1 - t^2) |e_m\rangle\langle e_m| \). This immediately implies the desired linear independence. \( \square \)

Remark 3.3. It follows immediately from the proof of Proposition 3.2 that when \( t \in (-1, 1) \)
\[ \text{span}\{A_m A_m^* \}_{m=1}^d = \text{span}\{A_m^* A_m \}_{m=1}^d = \text{span}\{|e_m\rangle\langle e_m| \}_{m=1}^d. \tag{14} \]
Therefore, for the purpose of determining the linear independence of the set \( \{A_m^* A_n\} \), one can make arbitrary changes to the diagonal elements of \( A_m^* A_n \) and it suffices to show that the set \( \{A_m^* A_n\}_{m\neq n} \) is linearly independent.

We study here only maps generated by unitaries in \( M_{d-1}(\mathbb{C}) \). If one allows more general partial isometries, many more examples of maps which are extreme in both the set of UCP and CPT maps can be constructed. For \( d = 6 \), one could construct three Kraus operators
\[ A_1 = \frac{1}{2} \begin{pmatrix} X & Y & 0_2 \\ Y & X & 0_2 \\ 0_2 & 0_2 & 0_2 \end{pmatrix}, \quad A_2 = \frac{1}{2} \begin{pmatrix} X & 0_2 & Y \\ 0_2 & 0_2 & 0_2 \\ Y & 0_2 & X \end{pmatrix}, \quad A_3 = \frac{1}{2} \begin{pmatrix} 0_2 & 0_2 & 0_2 \\ 0_2 & X & Y \\ 0_2 & Y & X \end{pmatrix} \]
based on the unitary \( \frac{1}{2} \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \) with \( X = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \), \( Y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and related by \( A_m = S^{-2} A_{m-1} S^2 = S^{-2(m-1)} A_1 S^{2(m-1)}. \)

We note that Ohno [20] considered the channel with
\[ A_1 = \sqrt{\frac{d-2}{d-1}} (I_d - |e_1\rangle\langle e_1|), \quad A_k = \frac{1}{\sqrt{d-1}} (|e_1\rangle\langle e_j| + |e_j\rangle\langle e_1|), \quad k = 2, \ldots, d, \]
which also has Choi rank \( d \) and is constructed from partial isometries, albeit with lower rank than \( d - 1 \).

3.2. Extreme Points are Generic

3.2.1. Overview. In this section we show that channels generated using (12) and (11) are usually extreme in both the UCP and CPT maps in two ways. First, we consider the \( V_m \) fixed and show that, in general, for all but a finite number of \( t \in (-1, 1) \) the corresponding maps are extreme. Then, we fix \( d \) and \( t \) and show that almost all choices of \( \{V_m\} \) generate maps which are extreme.

The latter implies that, roughly speaking, if we can find one choice of \( V_k \) in (11) for which the channel given by (12) is extreme in the UCP or CPT maps, then almost every choice of \( V_k \) also generates a channel which is extreme. However (as discussed after Theorem 3.8) if the example has all \( V_j \neq V_k \), this does not preclude the possibility that no channels with, e.g., \( V_1 = V_2 \) are extreme. But an example with \( V_1 = V_2 = \cdots = V_d \) which is extreme implies that almost every channel with unconstrained \( V_j \) is also extreme. In Sect. 3.3.1
we present a simple example of this type which is extreme unless $t = \frac{-1}{d-1}$ and for which we also have $V_j = V_j^*$. In Sect. 3.3.2 we provide additional evidence for generic extremality even when $t = \frac{-1}{d-1}$.

**Remark 3.4.** Before presenting these results precisely, we make some observations about maps on $M_d(C)$ that will be used in both settings.

(A) A map $\Phi$ is extreme in the set of UCP maps if and only if its adjoint $\Phi^*$ is extreme in the set of CPT maps. Thus, a UCPT map with $\Phi = \Phi^*$ is extreme in the UCP maps if and only if it is extreme in the CPT maps.

(B) It follows from [4, Theorem 5] that a map $\Phi(\rho) = \sum_k A_k^* \rho A_k$ with $\sum_k A_k^* A_k = I_d$ is extreme in the set of UCP maps if and only if the set of matrices $\{A^*_n A_n\}$ is linearly independent.

(C) It then follows from [6, Corollary 2.3] that if a UCPT map is extreme in the set of matrices $\{A^*_m A_n\}$, then $\{A^*_m A_n\}^*$ is non-singular. Hence, the set of matrices $\{A^*_m A_n\}$ is linearly independent if and only if its Gram matrix $G = \text{Tr} A_j^* A_k (A^*_m A_n)^*$ is non-singular.

In view of (B) above, we present our results in terms of the linear independence of the sets $\{A^*_m A_n\}$ and $\{A_m A^*_n\}$. The implications for extremality of UCP and CPT maps is straightforward.

When $V_m \neq V_m^*$ it is possible that $\{A^*_m A_n\}$ is a linearly independent set, but $\{A_m A^*_n\}$ is linearly dependent. Therefore, a UCPT map $\Phi$ can be extreme in the set of CPT maps, but not in the set of UCP maps (or vice versa). An explicit example for $d = 4$ is given in Appendix B. However, Theorem 3.5 below implies that, unless one of these sets is linearly dependent for all $t \in \mathbb{R}$, this can happen for, at most, a finite number of values of $t$.

**3.2.2. Generic in $t$ for $V_k$ Fixed.**

**Theorem 3.5.** Let $V_1, V_2, \ldots, V_d$ be a fixed set of unitary matrices in $M_d(C)$ and define $A_m$ as in (11) with $t \in (-1, 1)$. Then either $\{A^*_m A_n\}$ is a linearly dependent set for all $t \in \mathbb{R}$ or it is a linearly independent set except for a finite number of values of $t \in (-1, 1)$. The same holds for $\{A_m A^*_n\}$.

**Proof.** We present two arguments.

(i) With $A_m$ generated from $V_m$ as in (11), let $G(t)$ denote the Gram matrix in part (D) of Remark 3.4. Then det $G(t)$ is a polynomial in $t$ and $\{A^*_m A_n\}$ is a linearly independent set if and only if $t$ is not a root of det $G(t) = 0$. The maximum degree is determined by the diagonal which has $d$ elements $g_{kk,kk}$ of order $t^4$ and $d^2 - d$ elements $g_{jk,jk}$ ($j \neq k$) of order $t^2$. Thus det $G(t)$ is a polynomial of degree at most $4d + 2(d^2 - d) = 2d(d + 1)$.

(ii) Every $d \times d$ matrix $B$ with elements $b_{jk}$ can be associated with a $1 \times d^2$ vector whose elements are $v_{d(j-1) + k} = b_{jk}$. Let $F(t)$ be the $d^2 \times d^2$ matrix whose rows are given by the vectors associated with the matrices $A^*_m A_n$.
in this way. Then \( \{A^*_m A_n \} \) is a linearly independent set if and only if \( \det F(t) \neq 0 \). Now \( A^*_m A_n \) has exactly one element \( t^2 \) (and all others 0, 1). When \( m \neq n \) the elements of \( A^*_m A_n \) are of order 0 or 1 in \( t \). Thus, \( \det F(t) \) is a polynomial of degree \( \leq 2d + d^2 - d = d(d + 1) \). Since \( t = \pm 1 \) are always roots, the maximum number of distinct roots in \((-1, 1)\) is 
\[
d^2 + d - 2 = (d + 2)(d - 1).
\]

Although the polynomials \( \det G(t) \) and \( \det F(t) \) are not identical, they must have the same roots. However, these roots need not have the same degeneracy. Numerical work suggests that the number of distinct roots which lie in \((-1, 1)\) is often much less than \((d + 2)(d - 1)\) because some roots are degenerate, imaginary, or lie outside \((-1, 1)\). For the example in Sect. 3.3.1, there is a single highly degenerate root at \( t = \frac{1}{d-1} \in (-1, 1) \) for each integer \( d \geq 3 \). For the example in Sect. 3.3.3, there are no roots in \((-1, 1)\) when \( d > 3 \).

The simplest example of a situation in which \( \det G(t) = \det F(t) = 0 \ \forall \ t \) is when all \( V_m \) in (11) are diagonal unitaries. At the end of Sect. 3.3.3, we show that when \( d = 2\nu \) is even and all \( V_m \) are given by (20), the \( A_m \) generated from (11) have \( \det G(t) = \det F(t) = 0 \ \forall \ t \). Other cases are discussed in Sect. 3.3.4.

### 3.2.3. Algebraic Geometry Preliminaries.
We now present some results from algebraic geometry, which are needed for the rest of this section. All algebraic varieties and manifolds in what follows will be assumed to be real. The terminology *algebraic manifold* is used for an algebraic variety which is also a smooth manifold, i.e., without singularities.

A measure \( \mu \) on an algebraic manifold \( M \subset \mathbb{R}_m \) of dimension \( d \leq m \) is said to be *locally equivalent* to the \( d \)-dimensional Lebesgue measure if for each open subset \( V \) of \( M \) which is diffeomorphic to an open ball \( B \) in \( \mathbb{R}_d \), the restriction of \( \mu \) to \( V \) is equivalent to the restriction of Lebesgue measure to \( B \), i.e., they have the same null sets.

**Lemma 3.6.** Let \( M \subset \mathbb{R}_m \) be an algebraic manifold of dimension \( d \), and let \( N \) be an algebraic sub-variety of \( M \) of dimension at most \( d - 1 \). Let \( \mu \) be a measure on \( M \) which is locally equivalent to the Lebesgue measure. Then \( \mu(N) = 0 \).

**Proof.** The set \( N_1 \) of singular points of \( N \) is an algebraic variety of dimension at most \( d - 2 \) (if not empty), and it is a Zariski closed subset of \( M \). For \( x \in N \setminus N_1 \), choose an open neighborhood \( V_x \subseteq M \) of \( x \) diffeomorphic to an open ball in \( \mathbb{R}_d \), and such that \( V_x \cap N \) is diffeomorphic to a smooth submanifold in \( V_x \) of dimension \( d' = \dim(N) < d \). Since the \( d \)-dimensional Lebesgue measure of any smooth submanifold in \( \mathbb{R}_d \) of dimension less than \( d \) is zero, and since the restriction of \( \mu \) to \( V_x \) is equivalent to the \( d \)-dimensional Lebesgue measure, we deduce that \( \mu(V_x \cap N) = 0 \). Since \( N \setminus N_1 \subseteq \bigcup_{j=1}^{\infty} (V_{x_j} \cap N) \) for some countable set \( \{x_j\} \) of points in \( N \setminus N_1 \) (because \( N \setminus N_1 \) is a Lindelöf space, being second countable), we conclude that \( \mu(N \setminus N_1) = 0 \).

Let \( N_2 \) be the set of singular points of \( N_1 \), and define, successively, \( N_{j+1} \) to be the set of singular points of \( N_j \). Then \( N_k \) is empty for some \( k \leq d + 1 \). Repeating the argument above, with \( N_{j+1} \subset N_j \subset M \), \( j = 1, 2, \ldots, k-1 \), in the
place of $N_1 \subset N \subset M$, we see that $\mu(N) = \mu(N_1) = \mu(N_2) = \cdots = \mu(N_k) = 0$, as desired. \hfill \Box

Let $\mathcal{U}(d)^m$ denote the set of $m$-tuples $(U_1, U_2, \ldots U_m)$ of unitary matrices $U_j \in M_d(\mathbb{C})$.

**Lemma 3.7.** Let $d, k, m \geq 1$ be integers, and let $P : \mathcal{U}(d)^m \to \mathbb{C}$ be a function that arises from evaluating a polynomial in the $2md^2$ real variables given by the real and imaginary parts of the entries of the elements in $\mathcal{U}(d)^m$. If $P$ is not identically zero, then

$$Z := \{(U_1, U_2, \ldots, U_m) \in \mathcal{U}(d)^m : P(U_1, U_2, \ldots, U_m) = 0\}$$

is a null set with respect to the normalized Haar measure on $\mathcal{U}(d)^m$.

**Proof.** Let $(U_1, U_2, \ldots, U_m)$ be an $m$-tuple of matrices with each $U_j \in M_d(\mathbb{C})$. Consider the natural embedding of this $m$-tuple into $\mathbb{R}^{2md^2}$, obtained by taking the real and the imaginary part of each entry of each $U_j \in M_d(\mathbb{C})$. Such an $m$-tuple belongs to $\mathcal{U}(d)^m$ if it is in the zero set of finitely many polynomials in these $2md^2$ real variables. Hence $\mathcal{U}(d)^m$ is a (real) algebraic variety. Moreover, it is an algebraic manifold (i.e., it has no singularities), because it is homogeneous, being a group. Thus, if it had one singularity, then all points would be singularities by homogeneity, which is impossible. Moreover, $\mathcal{U}(d)^m$ is connected in the usual Euclidean topology, and hence also in the Zariski topology. This implies that $\mathcal{U}(d)^m$ is an irreducible algebraic manifold.

It is a standard fact from algebraic geometry that $Z$ is an algebraic subvariety of $\mathcal{U}(d)^m$ of dimension strictly less than the dimension of $\mathcal{U}(d)^m$. It therefore follows from Lemma 3.1 that the Haar measure of $Z$ is zero. \hfill \Box

### 3.2.4. Almost all Channels are Extreme.

We are now ready to prove the key result which implies that almost all channels generated using (12) and (11) are extreme. Although both (a) and (c) are special cases of (b) in the theorem below, we consider them separately for ease of exposition.

**Theorem 3.8.** Let $d \geq 3$ be an integer and fix $t \in (-1, 1)$. Suppose that there is a unitary matrix $W$ in $\mathcal{U}(d-1)$ such that when all $V_m = W$ in (11), then the resulting set $\{A_m^* A_n\}_{n,m=1}^{d}$ is linearly independent. Then we can conclude the following:

(a) The set of $d$-tuples $(V_1, V_2, \ldots, V_d)$ in $\mathcal{U}(d-1)^d$ for which the associated matrices $A_1, A_2, \ldots, A_d \in M_d(\mathbb{C})$ defined in (10) satisfy that $\{A_m^* A_n\}_{n,m=1}^{d}$ is linearly independent is a co-null set in $\mathcal{U}(d-1)^d$ with respect to the Haar measure on $\mathcal{U}(d-1)^d$.

(b) Fix a partition $\{J_1, J_2, \ldots, J_\kappa\}$ of $\{1, 2, \ldots, d\}$. For each set of unitaries $W_1, W_2, \ldots, W_\kappa$ in $\mathcal{U}(d-1)$, let $(V_1, V_2, \ldots, V_d) \in \mathcal{U}(d-1)^d$ be given by $V_m = W_j$ when $m \in J_j$, and let $A_1, A_2, \ldots, A_d$ be the matrices in $M_d(\mathbb{C})$ associated to $V_1, V_2, \ldots, V_d$. The set of $\kappa$-tuples $(W_1, W_2, \ldots, W_\kappa)$ in $\mathcal{U}(d-1)^\kappa$ for which $\{A_m^* A_n\}_{n,m=1}^{d}$ is linearly independent is a co-null set in $\mathcal{U}(d-1)^\kappa$ with respect to the Haar measure on $\mathcal{U}(d-1)^\kappa$. 


(c) For each unitary \( V \in \mathcal{U}(d-1) \), let \( A_1, A_2, \ldots, A_d \in M_d(\mathbb{C}) \) be as defined in (11) with all \( V_m = V \). The set of \( V \in \mathcal{U}(d-1) \) for which the set \( \{A_mA_n^*\}_{n,m=1}^d \) is linearly independent is a co-null set in \( \mathcal{U}(d-1) \) with respect to the Haar measure on \( \mathcal{U}(d-1) \).

If, instead, \( \{A_mA_n^*\}_{n,m=1}^d \) is linearly independent, the conclusions above hold for \( \{A_mA_n^*\}_{n,m=1}^d \).

**Proof.** (a) Let \( G \) be the Gram matrix defined in part (D) of Remark 3.4. The function \( P: \mathcal{U}(d-1)^d \to \mathbb{C} \) given by \( P(V_1, V_2, \ldots, V_d) = \det G \) is a polynomial in the real and imaginary entries of the \( V_m \), as in Lemma 3.7. A \( d \)-tuple \( V_1, V_2, \ldots, V_d \) belongs to the null-set \( Z \) if and only if the set \( \{A_m^*, A_n\} \) is not linearly independent. By assumption, there exists a unitary \( W \) such that \( P(W, W, \ldots, W) \neq 0 \). Lemma 3.7 therefore implies that \( Z \) is a null-set with respect to the Haar measure.

(c) Let \( G \) again be the Gram matrix defined in part (D) of Remark 3.4. Then the function \( P: \mathcal{U}(d-1) \to \mathbb{C} \) given by \( P(V) = \det G \) is a polynomial in the real and imaginary entries of \( V \), as in Lemma 3.7. By assumption, there exists a unitary \( W \) such that \( P(W) \neq 0 \). Lemma 3.7 therefore implies that \( Z \) is a null-set with respect to the Haar measure.

(b) Although (a) and (c) are special cases of (b), we proved them first to avoid cumbersome notation. The proof of (b) is similar and further details are omitted. \( \square \)

It is worth noting that, in Theorem 3.8 above, (a) does not imply (c) because the set of \( d \)-tuples of the form \( (V, V, \ldots V) \in \mathcal{U}(d-1)^d \) is a null set with respect to the Haar measure on \( \mathcal{U}(d-1)^k \). For the same reason, (a) does not imply (b).

For the next result observe that when \(|x\rangle\) is a vector on the unit sphere in \( \mathbb{C}_d \), then the matrix \( 2|x\rangle\langle x| - I_d \) is unitary and self-adjoint.

**Theorem 3.9.** Let \( d \geq 3 \) be an integer and fix \( t \in (-1, 1) \). Suppose that there is a vector \(|w\rangle\) on the unit sphere in \( \mathbb{C}_{d-1} \) such that when all \( V_m = 2|w\rangle\langle w| - I_{d-1} \) in (11), then the resulting set \( \{A_mA_n\}_{n,m=1}^d \) is linearly independent. Then the following hold:

(a) For almost all \(|x\rangle\) on the unit sphere in \( \mathbb{C}_{d-1} \), when \( V_m = 2|x\rangle\langle x| - I_{d-1} \) \( \forall m \) in (11) the resulting set \( \{A_mA_n\}_{n,m=1}^d \) is linearly independent.

(b) The conclusions (a)–(c) of Theorem 3.8 hold for both \( \{A_m^*, A_n\} \) and \( \{A_mA_n^*\} \).

**Proof.** View the unit sphere \( S_{2d-1} \) of \( \mathbb{C}_d \) as a real submanifold of \( \mathbb{R}_{2d} \). As \( S_{2d-1} \) is the zero-set of a polynomial (in \( 2d \) real variables), it is a real algebraic variety. The sphere is also without singularities, being homogeneous, so it is a real algebraic manifold. Hence we can apply Lemma 3.6.

For each fixed \(|x\rangle \in S_{2d-1} \), set \( V_m = 2|x\rangle\langle x| - I_{d-1} \) and let \( A_m \in M_d(\mathbb{C}) \) be given as in (10), for \( m = 1, 2, \ldots, d \). The \((j,k)\)th entry of \( V_m \) is \( 2x_jx_k - \delta_{jk} \delta_{jm} \), which is a polynomial of degree two in the real variables \( \text{Re}(x_j), \text{Im}(x_j), \text{Re}(x_k), \text{Im}(x_k) \). Hence each entry of \( A_m \) is a polynomial (of degree two) with respect to these variables.
Consider the polynomial $Q(x) = \det(G)$, for $|x\rangle \in S_{2d-1}$, where $G$ is the Gram matrix defined in part (C) of Remark 3.4. Then, as in the proof of Theorem 3.8, $\{A_m^* A_n\}$ is linearly independent if and only if $Q(x) \neq 0$. By assumption, there exists $|w\rangle \in S_{2d-1}$ with the corresponding $\{A_m^* A_n\}$ linearly independent which implies that $Q(w)$ is not identically zero. Hence, by Lemma 3.6, the zero-set of $Q$ is a Lebesgue null-set. (By the comments above Lemma 3.6, the standard Lebesgue measure on the sphere $S_{2d-1}$ is locally equivalent to the Lebesgue measure on $\mathbb{R}^{2d-1}$.) This proves part (a) of the theorem. Part (b) is an immediate consequence of Theorem 3.8 and the fact that $A_m = A_m^*$.

Remark 3.10. We could also consider $V_m = 2|x_m\rangle\langle x_m| - I_{d-1}$ in (11) using different vectors $|x_m\rangle$ on the unit sphere in $\mathbb{C}^{d-1}$ and prove results analogous to (a) and (b) in Theorem 3.8.

Let $d \geq 3$, fix $t \in (-1,1)$, and assume that we can find a unit vector $|x\rangle \in \mathbb{C}^{d-1}$ such that the map $\Phi$ given by (12) with all $V_m = 2|x\rangle\langle x| - I_{d-1}$ in (11) is extreme in either the set of UCP or CPT maps. Since $V_m$ is self-adjoint, it then follows from part (B) of Remark 3.4 and the theorems above that almost every choice of $(V_1, V_2, \ldots V_d)$ in (11) generates a UCPT map $\Phi$ that is extreme in both the set of UCP maps and the set of CPT maps, for all the scenarios described in Theorem 3.8.

Thus, we are motivated to find unit vectors $|x\rangle$ which generate extreme UCP (or CPT) maps in this way. This is done in the following sections.

- In Sect. 3.3.1 we introduce a vector $|w\rangle$ for which the hypothesis in Theorem 3.9 holds for all $d \geq 3$ if $t \neq \frac{-1}{d-1}$.
- In Sect. 3.3.2 we describe evidence that the hypothesis in Theorem 3.9 holds for all $t \in (-1,1)$ when $d = 3, 4, 5, 6, 7$, and conjecture that it holds for all $d \geq 3$.
- In Sect. 3.3.3 we introduce a different type of unitary $W = W^* \in U(d-1)$ for which the hypothesis in Theorem 3.8 holds for all $t \in (-1,1)$ when $d \geq 5$ is odd.

Although Theorem 3.8 only requires $W$ to be unitary, in all of our examples $W = W^*$. Similarly, in all of our examples which satisfy the hypothesis of Theorem 3.9, the unit vector $|x\rangle \in \mathbb{C}^{d-1}$ is in $\mathbb{R}^{d-1}$.

If, however, $W$ in Theorem 3.8 is unitary, but not self-adjoint, then our conclusions are more restricted. Let $d \geq 3$ and fix $t \in (-1,1)$.

(a) If (11) and (12) generate a map $\Phi$ which is extreme in the set of CPT maps, then almost every corresponding choice of $(V_1, V_2, \ldots V_d)$ in (11) generates a map $\Phi$ that is extreme in the set of CPT maps, for all the scenarios described in Theorem 3.8.

(b) If (11) and (12) generate a map $\Phi$ which is extreme in the set of UCP maps, then almost every corresponding choice of $(V_1, V_2, \ldots V_d)$ in (11) generates a map $\Phi$ that is extreme in the set of UCP maps, for all the scenarios described in Theorem 3.8.
3.3. Critical Examples

3.3.1. Key Example. Recall that $|\mathbf{1}_d\rangle$ denotes the vector whose elements are all $d^{-1/2}$ and define $W_d \in M_d(\mathbb{C})$ as

$$W_d = 2|\mathbf{1}_d\rangle\langle \mathbf{1}_d| - \mathbf{I}_d. \quad (15)$$

It is easy to verify directly that $W_d$ is unitary and self-adjoint; in fact, its eigenvalues are $-1$ with multiplicity $d-1$, and $+1$ (non-degenerate).

For our first, and most important, example we choose $V_m$ in (11) to be $W_{d-1}$ so that $A_1 = t|e_1\rangle\langle e_1| \oplus W_{d-1}$ and $A_m = S^{-1}A_{m-1}S = S^{-m+1}A_1S^{m-1}$ for $m = 2, 3, \ldots, d$. Then $A_m$ can be written in block form as

$$A_m = S^{-m+1} \begin{pmatrix} t & 0 \\ 0 & W_{d-1} \end{pmatrix} S^{m-1}, \quad m = 1, 2, \ldots, d. \quad (16)$$

For this example we will prove the following

**Theorem 3.11.** For $d \geq 3$ and $t \in (-1, 1)$, let $A_m$ be the matrices defined in (16). Then $A_m^\ast = A_m$ and when $t \neq -1/d - 1$,

(a) the set of matrices $\{A_mA_n\}_{m,n=1}^d$ is linearly independent,

(b) the map $\Phi(\rho) = \frac{1}{d-1+t^2} \sum_{m=1}^d A_m \rho A_m$ is an extreme point of both the CPT and UCP maps, and

(c) $\Phi$ is not factorizable.

Since $A_m = A_m^\ast$, part (B) of Remark 3.4 implies that part (b) follows immediately from part (a). Part (c) then follows from (b) because, as stated in part (C) of Remark 3.4, an extreme point of the UCP maps is never factorizable.

The proof of (a) is postponed to Sect. 4, in part because the argument is fairly long, but also because in the cases $d = 3$ and $t = -1/d - 1$ we prove some related results of independent interest. The proof of (a) is given in Sects. 4.1–4.4 for $d \geq 4$ and in Sect. 4.5 for $d = 3$. For $d = 3$, we also prove that the channels for $t = 1$ and $t = -1/d - 1 = -\frac{1}{2}$ have exact factorizations which are dual in the sense that they can be obtained from the same unitary operator in $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ by switching the roles of the two algebras. In Sect. 4.5, we consider the special case $t = -1/d - 1$ and show that the anti-symmetric and symmetric sets $\{A_mA_n - A_nA_m\}_{m<n}$ and $\{A_mA_n + A_nA_m\}_{m<n}$ are each separately linearly dependent.

**Remark 3.12.** We also note that our proof (presented in Sect. 4) implies the following

(a) For $d \geq 3$ and $t = 1$, the sets $\{A_m^2\}_{m=1}^d$ and $\{A_mA_n - A_nA_m\}_{m<n}$ are each separately linearly dependent.

(b) For $d \geq 3$ and $t = -1$, the set $\{A_m^2\}_{m=1}^d$ is linearly dependent, but the set $\{A_mA_n\}_{m \neq n}$ is linearly independent.
3.3.2. Rank One Projections. The unitary $W_{d-1} = 2|\mathbb{1}_{d-1}\rangle\langle \mathbb{1}_{d-1}| - I_{d-1}$ used in the previous section is invariant under permutations, i.e., $P^{-1}W_{d-1}P = W_{d-1}$ for all permutations $P$. This symmetry allows us to present a full analysis of the linear independence of $\{A_mA_n\}$ in Sect. 4. However, this same symmetry gives rise to a highly degenerate linear dependency when $t = \frac{-1}{d-1}$.

If $|\mathbb{1}_{d-1}\rangle$ is replaced by any unit vector $|x\rangle \in \mathbb{C}_{d-1}$, then $V = 2|x\rangle\langle x| - I_{d-1}$ is also unitary. One would expect that most unit vectors $|x\rangle$ give a unitary $V$ which, when used in (11), generates a set of matrices for which $\{A_mA_n\}$ is linearly independent when $t = \frac{-1}{d-1}$.

**Conjecture 3.13.** For all integers $d \geq 3$, one can find a vector $|x\rangle$ on the unit sphere in $\mathbb{R}_{d-1}$ such that with $V_m = 2|x\rangle\langle x| - I_{d-1}$ $\forall m$ in (11), the set of matrices $\{A_mA_n\}$ is linearly independent when $t = \frac{-1}{d-1}$.

Exact numerical results obtained using Maple for $d = 3, 4, 5, 6, 7$ not only support this conjecture, but suggest that it is easy to find such $|x\rangle$. Indeed, any vector for which $x_j \neq 0 \forall j$ and some $|x_j| \neq (d - 1)^{-1/2}$ seems to satisfy this conjecture. Moreover, when $d$ was small enough to find all roots of $\det G(t) = 0$, there was high degeneracy at $t = \pm 1$, and many complex roots. In view of Theorem 3.5, it hardly seems plausible that every $|x\rangle \in \mathbb{C}_d$ generates a Gram matrix from $\{A_mA_n\}$ which has a root at $t = \frac{-1}{d-1}$.

3.3.3. $d = 2\nu + 1 > 3$ Odd. In this section we consider a completely different example which is not associated with a rank one projection. Instead, it is based on a permutation matrix with $\nu = \frac{1}{2}(d - 1)$ swaps, which can be viewed as a projection of rank $\nu = \frac{1}{2}(d - 1)$.

All addition of indices in what follows will be mod $d$. For $m = 1, 2, \ldots d$, define

$$V_m = \sum_{k=1}^{\nu} (|e_{m+k}\rangle\langle e_{m-k}| + |e_{m-k}\rangle\langle e_{m+k}|) = \left(\sum_{j=1}^{d} |e_j\rangle\langle e_{2m-j}|-|e_m\rangle\langle e_m|\right). \quad (17)$$

Then $V_m = V_m^*$ and its restriction to $\mathbb{C}_d\setminus\text{span}\{|e_m\rangle\} \simeq \mathbb{C}_{d-1}$ is a unitary whose effect on a vector $|w\rangle$ is to swap its elements in $\nu$ pairs $w_{j+\nu} \leftrightarrow w_{j-\nu}$, and $A_m = V_m \oplus t|e_m\rangle\langle e_m|$ has the form (11).

**Theorem 3.14.** When $A_m = V_m \oplus t|e_m\rangle\langle e_m|$, then $A_m = A_m^*$, the set $\{A_m^* A_n\}_{m,n=1}^d$ is linearly independent and the channel $\Phi(\rho) = \sum_{k=1}^{d} A_k \rho A_k^*$ is an extreme point of both the set of CPT maps and the set of UCP maps

(a) for all $t \in (-1, 1)$ when $d = 2\nu + 1 > 3$, and
(b) for all $t \neq \frac{-1}{2} \in (-1, 1)$ when $d = 3$.

**Proof.** It follows from Proposition 3.2 and the fact that $A_m = A_m^*$ that it suffices to show that the set $\{A_m A_n\}_{m \neq n}$ is linearly independent. Observe that
\[
(V_m + |e_m\rangle\langle e_m|)(V_n + |e_n\rangle\langle e_n|) = \sum_{j=1}^{d} \sum_{k=1}^{d} |e_j\rangle\langle e_{2m-j}|e_k\rangle\langle e_{2n-k}|
\]

\[
= \sum_{j=1}^{d} |e_j\rangle\langle e_{2(n-m)+j}| = S^{2(n-m)}
\]

(18)

where \( S = \sum_k |e_k\rangle\langle e_{k+1}| \) is the cyclic shift. It then follows that

\[
A_mA_n = S^{2(n-m)} - (1 - t)(|e_{2m-n}\rangle\langle e_n| + |e_m\rangle\langle e_{2n-m}|) + \delta_{mn}(1 - t)^2|e_m\rangle\langle e_m|
\]

or, equivalently, with \( \ell = n - m \)

\[
A_mA_{m+\ell} = S^{2\ell} - (1 - t)(|e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|) + \delta_{\ell0}(1 - t)^2|e_m\rangle\langle e_m|.
\]

(19)

This implies that \( \{A_mA_n\}_{m \neq n} \) can be decomposed into \( d - 1 \) disjoint sets of the form \( \mathcal{C}_\ell = \{A_mA_{m+\ell}\}_{m=1}^{d} \), with \( \ell = 1, \ldots, (d - 1) \), i.e., \( \text{span}\{\mathcal{C}_\ell\} \cap \text{span}\{\mathcal{C}_\ell\} = \{0\} \) whenever \( k \neq \ell \). Thus it suffices to show that each of the sets \( \mathcal{C}_\ell \) contains \( d \) linearly independent matrices.

For \( \ell \neq 0 \), the matrices in \( \mathcal{C}_\ell \) are nonzero only where \( S^{2\ell} \) is nonzero and

\[
\sum_{m=1}^{d}A_mA_{m+\ell} = [d - 2(1 - t)]S^{2\ell} \text{ (which is nonzero unless } d = 3, t = -\frac{1}{2})\]

Thus, it suffices to show that for each fixed \( \ell \in \{1, 2, \ldots, d - 1\} \), the matrices

\[
F_{m\ell} \equiv \frac{1}{1 - t}(S^{2\ell} - A_mA_{m+\ell}) = |e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|
\]

\( m = 1, 2, \ldots, d \)

are linearly independent. Now, after a shift which does not affect linear independence, we can associate each \( F_{m\ell} \) with the vector \( |v_{m\ell}\rangle = |e_m\rangle + |e_{m-\ell}\rangle \) in \( \mathbb{C}_d \) where \( m = 1, 2, \ldots, d \) and \( \ell \in \{1, 2, \ldots, d - 1\} \) is fixed.

One way to show linear independence is to consider the matrix \( M_\ell \) in which the vectors \( |v_{m\ell}\rangle \) are the rows. Then \( M_\ell = I + S^{-\ell} \) is a circulant matrix whose eigenvalues are well-known to be \( \lambda_k = 1 + e^{-2\pi ik\ell/d} \neq 0 \) [9, Section 2.5, Problem 21]. For odd, no \( d \)-th root of unity can be \( -1 \) which implies \( \lambda_k \neq 0 \) and \( M_\ell \) is non-singular. Therefore, the set of matrices \( \{F_{m\ell}\}_{m=1}^{d} \) or, equivalently, \( \{A_mA_{m+\ell}\}_{m=1}^{d} \) is linear independent for each fixed \( \ell \).

Alternatively, observe that since \( |e_j\rangle = |e_{j-d\ell}\rangle \), one can write

\[
2|e_j\rangle = |e_j\rangle + |e_{j-d\ell}\rangle = \sum_{k=1}^{d}(-1)^{k+1}|e_{j-(k-1)\ell}\rangle + |e_{j-k\ell}\rangle
\]

\[
= \sum_{k=1}^{d}(-1)^{k+1}|v_{(j-(k-1)\ell)}\rangle, \ell).
\]

This implies that, for each fixed \( \ell \), every vector in the orthonormal basis \( \{|e_j\rangle\}_{m=1}^{d} \) is in the span of \( \{|v_{m\ell}\rangle\}_{m=1}^{d} \) which implies that the vectors \( \{v_{m\ell}\}_{m=1}^{d} \) form a basis for \( \mathbb{C}_d \) and are, hence, linearly independent.

To see why this example is the opposite of a rank one projection, recall that \( E = E^* = E^2 \in \mathbb{C}_d \) is a projection of rank \( r \) if and only if \( I_d - E \) is a projection of rank \( d - r \). Whenever \( E \) is a projection, \( V = 2E - I \) is unitary and
the replacement $E \mapsto (I - E)$ takes $V \mapsto -V$. Thus any unitary matrix that can be constructed with a projection of rank $> \lfloor d/2 \rfloor$ can also be constructed with a projection of rank $\leq \lfloor d/2 \rfloor$.

When $V_m$ is given by (17), there is a projection $E_m = \sum_{k=1}^\nu |x_{mk}\rangle \langle x_{mk}|$ of rank $\nu = \frac{1}{2}(d - 1)$ formed from the vectors $|x_{mk}\rangle = \frac{1}{\sqrt{2}}(|e_{m+k}\rangle + |e_{m-k}\rangle)$ so that $V_m = 2E_m - I_{d-1}$. Thus $V_m$ comes from a projection with the maximal rank on $C_{d-1}$.

When $d = 2\nu$ is even, $d - 1$ is odd and it is not possible to construct a unitary consisting only of $\nu$ swaps. If one uses (17), the term with $k = \nu$ becomes $|m + \nu\rangle\langle m - \nu| = |m + \nu\rangle\langle m + \nu|$ (mod $d = 2\nu$). When $d$ is odd, $V_m$ is essentially a projection $W \in C_{d-1}$ with 1’s on the skew diagonal and 0 elsewhere. Extending this to $d = 2\nu$, i.e., generating $A_m$ in (11) from a projection $W \in C_{d-1}$ with 1’s on the skew diagonal, gives

$$V_m = \sum_{k=1}^{\nu-1} (|e_{m+k}\rangle \langle e_{m-k}|e_{m+k}\rangle \langle e_{m-k}|) + |m + \nu\rangle \langle m + \nu|,$$

(20)

However, when $A_m = t|e_m\rangle \langle e_m| \oplus V_m,$

$$S^{-\nu}A_mS^{\nu} - A_m = (1 - t)(|e_m\rangle \langle e_m| - |e_{\nu}\rangle \langle e_{\nu}|)$$

which implies that $A_mA_{m+\nu}$ is diagonal so that $A_mA_{m+\nu} \in \text{span}\{A_n^{2}\}$. There does not seem to be a natural generalization of (17) to $d = 2\nu$ which yields a linearly independent set of $\{A_mA_n\}$.

3.3.4. Band Width. We have focused on finding examples of matrices $V \in C_{d-1}$ that generate linearly independent sets of $\{A_m^*A_n\}$ via (11) because these imply that almost all choices of $V_k$ also generate linearly independent sets as described in Theorem 3.8. Nevertheless, a null set with respect to Haar measure does allow for infinitely many choices of $V_k$ that generate linearly dependent sets. We now consider what those sets might look like.

First, if all of the $V_m$ in (11) are diagonal, then $A_m^*A_n$ is also diagonal. It then follows immediately from Theorem 3.2 and Remark 3.3 that $\{A_m^*A_n\}_{m=1}^d$ are linearly dependent. In fact, if $A_m^*A_n$ is diagonal for even one pair of $m \neq n$, the set of $\{A_mA_n\}$ is linearly dependent. One can have such pairs even when none of the $V_m$ are diagonal. This is the case for the $V_m$ in (20), which generate $A_m$ for which $A_m^*A_{m+\nu}$ is diagonal when $d = 2\nu$ is even.

We can extend the diagonal examples by introducing the notion of band width. We first observe that when $S$ is the cyclic shift defined above (10), every matrix $B \in M_d(C)$ can be written uniquely as

$$B = \sum_{k=-\xi_-}^{\xi_+} D_k S^k$$

(21)

where $D_k$ is diagonal, $\xi_- = \lfloor (d - 1)/2 \rfloor$ and $\xi_+ = \lceil (d - 1)/2 \rceil$.

Definition 3.15. Let $d \geq 3$ be an integer. For each matrix $B$ in $M_d(C)$ with elements $b_{jk}$ define its band width to be $\beta(B) = \max\{|j - k| : b_{jk} \neq 0\}$ (with
0 the band width of the zero matrix). Define its cyclic band width \( \mu \) as the smallest positive integer such that \( B = \sum_{k=-\mu}^{\mu} D_k S^k \) with each \( D_k \) diagonal.

The following properties are straightforward to verify for all \( A, B \in M_d(\mathbb{C}) \).

(a) \( \mu(B) \leq \beta(B) \) and \( \mu(B) \leq \lfloor (d - 1)/2 \rfloor \).

(b) A matrix \( D \) is diagonal if and only if \( \mu(D) = 0 \) if and only if \( \beta(D) = 0 \).

(c) \( \beta(A^*) = \beta(A) \); \( \mu(A^*) = \mu(A) \) and \( \mu(S^* A S) = \mu(A) \);

(d) \( \beta(AB) \leq \beta(A) + \beta(B) \), and \( \mu(AB) \leq \mu(A) + \mu(B) \).

We also note that one can find \( B \) with \( \beta(B) = d - 1 \). Moreover, \( \beta(B) \) is not invariant under cyclic permutations. Indeed, \( S^* |e_{d-1}\rangle \langle e_d| S = |e_d\rangle \langle e_1| \) so that a single cyclic shift can map the matrix \( B = |e_{d-1}\rangle \langle e_d| \) with \( \beta(B) = 1 \) to one with \( \beta(S^* B S) = d - 1 \), which is the maximal value of \( \beta \).

**Proposition 3.16.** Let \( d \geq 3 \) be an integer and let \( V_1, \ldots, V_d \) be unitaries in \( M_{d-1}(\mathbb{C}) \) with band width \( \beta(V_m) < (d - 1)/4 \), for each \( m \). Then, for the associated matrices \( A_1, \ldots, A_d \) in \( M_d(\mathbb{C}) \), defined in (10), it follows that the set \( \{A^*_m A_n\} \) is not linearly independent for any value of \( t \in [-1, 1] \).

**Proof.** Let \( B_m = t |e_1\rangle \langle e_1| \oplus V_m = \begin{pmatrix} t & 0 \\ 0 & V_m \end{pmatrix} \), \( m = 1, 2, \ldots, d \). Then \( \beta(B_m) = \beta(V_m) < (d - 1)/4 \) and \( A_m = S^{-m+1} B_m S^{m-1}, \) which implies \( \mu(A_m) = \mu(B_m) < (d - 1)/4, \) for all \( m \), so that \( \mu(A^*_m A_n) < (d - 1)/2, \) for all \( m, n \).

Since the set of matrices \( A \) in \( M_d(\mathbb{C}) \) satisfying \( \mu(A) < (d - 1)/2 \) is a proper linear subspace of \( M_d(\mathbb{C}) \), we conclude that \( \{A^*_m A_n\} \) cannot be a basis for \( M_d(\mathbb{C}) \), and hence not linearly independent. \( \square \)

Note, however, that \( \mu(V_m) > (d - 1)/4 \) does not necessarily imply that \( \{A^*_m A_n\} \) generated as in (11) are linearly independent. When \( d = 2 \nu \geq 8 \), the \( V_m \) in (20) have the maximal cyclic band width of \( \nu = \lceil \frac{1}{2}(d - 1) \rceil = \frac{1}{2} d \), but nevertheless generate sets of \( \{A^*_m A_n\} \) which are linearly dependent for all \( t \in \mathbb{R} \).

4. Analysis of Key Example

4.1. Overview

We begin with an outline of the steps in the rather long analysis of the key example in Sect. 3.3.1. In the next section we describe \( A_m A_n \) explicitly and find some properties which allow us to reduce the linear independence of \( \{A_m A_n\}_{m \neq n} \) to that of \( \{X_{mn}\}_{m \neq n} \) where the matrices \( X_{mn} \) each have only two nonzero rows and columns.

In Sect. 4.3 we observe that it suffices to consider the linear independence of the sets \( \{X_{mn}^+\}_{m < n} \) and \( \{X_{mn}^-\}_{m < n} \) where \( X_{mn}^\pm = X_{mn} \pm X_{mn}^* \). We then observe that we can remove a common factor so that the nonzero elements \( x_{jk} \) of \( X_{mn}^\pm \) are \( \pm 1 \) with the exception of \( x_{mn} \) and \( x_{nm} \). The linear independence of the sets \( \{X_{mn}^\pm\}_{m < n} \) depends only on the elements above the diagonal. We
construct a larger matrix $\Omega^\pm$ whose rows are these elements $x_{jk}$ of $X_{mn}$ arranged in lexicographic order. Remarkably, the diagonal of $\Omega^\pm$ is a multiple of the identity, whose value depends on $t$. This allows us to reduce the linear independence of $\{X_{mn}\}_{m<n}$ to an eigenvalue problem for $\Omega^\pm$.

The eigenspaces of $\Omega^\pm$ are associated with representations of the symmetric group, which allows us to find all of the eigenvalues and eigenspaces explicitly. This is done in Sect. 4.6 for $d \geq 4$. The results imply linear independence of $\{A_mA_n\}_{m \neq n}$ when $t \neq \frac{-1}{d-1}$.

The reduction process used in Sect. 4.2 is not valid when $d = 3$ or when $t = \frac{-1}{d-1}$. Therefore the case $d = 3$ is analyzed separately in Sect. 4.5 in which we also present a pair of novel factorizability results relating the channels with $t = 1$ and $t = \frac{-1}{2}$. The linear dependence of $\{A_mA_n\}_{m \neq n}$ when $d > 3$ and $t = \frac{-1}{d-1}$ is analyzed in Sect. 4.6.

4.2. Structure of $A_mA_n$

We begin our analysis of our key example in Sect. 3.3.1 by observing that the entries of the matrix $A_1 = t|e_1\rangle\langle e_1| \oplus W_{d-1} = t|e_1\rangle\langle e_1| \oplus (2|1_{d-1}\rangle\langle 1_{d-1}| - I_{d-1})$ are

$$a_{jk} = \begin{cases} t & j = k = 1 \\ v_{jk} = \frac{2}{d-1} & j \neq k \in \{2,3\ldots d\} \\ 0 & j = 1 \text{ or } k = 1, j \neq k \\ v_{jj} = \frac{3-d}{d-1} & j = k \neq 1 \end{cases}$$

(22)

Since $A_m = A_m^*$, and $A_mA_n^* = A_m^*A_n = A_mA_n$ a straightforward calculation gives

$$(d-1)^2\langle e_j, A_mA_ne_k \rangle = \begin{cases} -t(d-1)(d-3) & j = k = m, j = k = n \\ 2t(d-1) & j = m, k \neq m, n \text{ or } k = n, j \neq m, n \\ 2(d-3) & j = n, k \neq m, n \text{ or } k = m, j \neq m, n \\ 0 & j = m, k = n \\ 4(d-2) & j = n, k = m \\ (d-3)(d+1) & j = k \neq m, n \\ -4 & j \neq k, j, k \neq m, n \end{cases}$$

(23)

so that, e.g.,

$$A_1A_d = \frac{1}{(d-1)^2} \begin{pmatrix} \tau & b & \ldots & b & \ldots & b & 0 \\ a & & & & & & b \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & b \\ a & \tilde{V}_{d-2} & b & & & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & b \\ a & b & \ldots & b & & & \\ a & b & \ldots & a & \ldots & \tau \end{pmatrix}$$

U. Haagerup et al. Ann. Henri Poincaré
where \( a = 2(d - 3) \), \( b = 2t(d - 1) \), \( \tau = -t(d - 1)(d - 3) \), \( u = 4(d - 2) \) and \( \tilde{V}_{d-2} \) is the matrix in \( M_{d-2}(C) \) obtained by removing the last row and column from \( V \).

By Remark 3.3 we do not need to consider the diagonal elements of \( A_m A_n \); therefore, it suffices to determine whether or not there exists a matrix \( C \) with elements \( c_{jk} \) such that \( \langle e_j, \sum_{m \neq n} c_{mn} A_m A_n e_k \rangle = 0 \) when \( j \neq k \). It follows immediately from (23) that this holds if and only if

\[
0 = 4(d - 2)c_{jk} + a \sum_{m \neq j, k} (c_{jm} + c_{mk}) + b \sum_{m \neq j, k} (c_{mj} + c_{km}) - 4 \sum_{m, n \neq j, k} c_{mn}.
\]

(24)

We can simplify the conditions on \( c_{jk} \) by first observing that it also follows immediately from (23) that

\[
(d - 1)^2 \sum_{m \neq n} \langle e_j, A_m A_n e_k \rangle = 4(d - 2)[1 + t(d - 1)] \quad \forall j \neq k,
\]

and

\[
(d - 1)^2 \sum_{m \neq n} \langle e_k, A_m A_n e_k \rangle = (d - 1)(d - 3)[(d - 2)(d + 1) - 2t(d - 1)] \quad \forall k
\]

so that

\[
(d - 1)^2 \sum_{m=1}^{d} \sum_{n=1}^{d} A_m A_n = p_d(t) |1_d\rangle\langle 1_d| + [q_d(t) - d \cdot p_d(t)] I_d
\]

(25)

where

\[
p_d(t) = 4d(d - 2)[1 + t(d - 1)]
\]

(26a)

and

\[
q_d(t) = (d - 1)(d - 3)[(d - 2)(d + 1) - 2t(d - 1)].
\]

(26b)

In the special case, \( t = \frac{-1}{d-1} \),

\[
\sum_{m \neq n} A_m A_n = \begin{cases} 0 & d = 3 \\ q_d \left( \frac{-1}{d-1} \right) I_d & d \geq 4 \end{cases}
\]

(27)

with \( q_d \left( \frac{-1}{d-1} \right) = d(d - 1)^2(d - 2) \). For \( d = 3 \), this immediately implies that \( \{A_m A_n\}_{m \neq n} \) is linearly dependent when \( t = \frac{-1}{d-1} \).

If \( t \neq \frac{-1}{d-1} \), then after using the freedom from Remark 3.2 to adjust the diagonal elements arbitrarily, we can proceed as if \( \sum_{mn} A_m A_n \) is a multiple of \( |1_d\rangle\langle 1_d| \). Then we can remove \( \tilde{V}_{d-2}^2 \) by replacing \( A_m A_n \) by \( (d - 1)^2 A_m A_n + 4d|1_d\rangle\langle 1_d| \). Thus we can conclude that

**Proposition 4.1.** For \( d \geq 4 \) with \( t \in (-1, 1) \) and \( t \neq \frac{-1}{d-1} \), the set \( \{A_m A_n\}_{m,n=1}^d \) is linearly independent if and only if \( \{X_{mn}\}_{m \neq n} \) is linearly independent where

\[
X_{mn} = (d - 1)^2 A_m A_n + 4d|1_d\rangle\langle 1_d| - D_{mn}, \quad m, n = 1, 2, \ldots d
\]

(28)
and $D_{mn} = D_{nm}$ is a diagonal matrix chosen so that the diagonal of $X_{mn}$ is identically zero. In particular, one can find a constant $\lambda$ such that

$$D_{mn} = \lambda I_d + (\tau - \lambda)(|e_m\rangle\langle e_m| + |e_n\rangle\langle e_n|).$$  \hspace{1cm} (29)$$

In Sect. 4.6, we consider the case $t = \frac{\tau}{d^2} - 1$ in detail for $d \geq 4$. Although we can not write $|1_d\rangle\langle 1_d|$ as a linear combination of $A_m A_n$ in that case, we can still use $X_{mn}$ to reach some conclusions about the linear independence of $\{A_m A_n\}_{m \neq n}$. Moreover, we give a simple proof of the linear dependence conditions in this case which does not use Proposition 4.1. The case $d = 3$ must be treated separately, which is done in Sect. 4.5.

The matrix $X_{mn}$ obtained in (28) has elements

$$\langle e_j, X_{mn} e_k \rangle = \begin{cases} \hat{a} \equiv 2(d - 1) & j = n, k \neq m, n \text{ or } k = m, j \neq m, n \\ \hat{b} \equiv 4 + 2t(d - 1) & j = m, k \neq n, j \neq m, n \\ \hat{u} \equiv 4(d - 1) & j = n, k = m \\ 4 & j = m, k = n \\ 0 & \text{otherwise} \end{cases}$$

or, equivalently,

$$X_{mn} = \hat{u}|e_n\rangle\langle e_m| + 4|e_m\rangle\langle e_n| + \hat{a} \sum_{j \neq m, n} (|e_j\rangle\langle e_m| + |e_n\rangle\langle e_j|) + \hat{b} \sum_{j \neq m, n} (|e_m\rangle\langle e_j| + |e_j\rangle\langle e_n|)$$  \hspace{1cm} (30)$$

where $\hat{a} = 2(d - 1), \hat{u} = 4(d - 1), \hat{b} = 2t(d - 1) + 4$. Thus we can write, showing only rows and columns with nonzero elements,

$$X_{mn} = \begin{pmatrix} \hat{a} & \hat{b} & \cdots & \hat{a} & \hat{b} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{a} & \hat{b} & \cdots & \hat{a} & \hat{b} \\ \hat{b} & 0 & \hat{b} & \cdots & \hat{b} \\ \hat{a} & 0 & \hat{a} & \hat{b} & \cdots \\ \vdots & \vdots & \vdots & \hat{b} & \cdots \\ \hat{a} & \hat{b} & \cdots & \hat{a} & \hat{b} \\ \vdots & \vdots & \vdots & \hat{b} & \cdots \\ \hat{b} & \cdots & \hat{a} & \hat{b} & \cdots \\ \vdots & \vdots & \vdots & \hat{b} & \cdots \\ \hat{a} & \hat{b} & \cdots & \hat{a} & \hat{b} \end{pmatrix}.$$  

It follows from (30) that the set $\{X_{mn}\}_{m \neq n}$ is linearly dependent if and only if there is a matrix $C$ with elements $c_{mn}$ such that

$$0 = \sum_{mn} c_{mn} \langle e_j, X_{mn} e_k \rangle \quad \forall j, k.$$  \hspace{1cm} (31)$$
Remark 4.2. The set of matrices satisfying (31) or, equivalently, (24) is a subspace \( \mathcal{N} \) of \( M_d(\mathbb{C}) \) which is invariant under permutations and under the adjoint map, i.e., \( C \in \mathcal{N} \) implies \( C^* \in \mathcal{N} \) and \( P^*CP \in \mathcal{N} \) for all permutation matrices \( P \).

We chose \( \mathcal{N} \) to denote this subspace because it is the null space of the \( d^2 \times d^2 \) matrix whose rows are the elements of \( X_{mn} \). However, this formulation does not give much insight. Instead we will exploit invariance under the adjoint map to reformulate the linear dependence as pair of eigenvalue problems. Moreover, the invariance under permutations will allow us to identify the eigenspaces with irreducible representations of the symmetric group \( S_d \).

### 4.3. Reformulation as an Eigenvalue Problem

We can replace the pair \( X_{mn}, X_{nm} \) with \( m < n \) by the pair of matrices

\[
X_{mn}^\pm \equiv X_{mn} \pm X_{nm}^* = X_{mn} \pm X_{nm}
\]

so that \( (X_{mn}^+)^* = X_{mn} - X_{nm} \) is symmetric and \( (X_{mn}^-)^* = -X_{mn} \) skew symmetric. Since \( \text{Tr} \ X_{mn}^+ X_{jk}^- = \text{Tr} (X_{mn}^+ X_{jk}^-)^T = -\text{Tr} X_{mn}^+ X_{jk}^- \) for all \( j < k \), and \( m < n \), the sets \( \{X_{mn}^+\}_{m<n} \) and \( \{X_{mn}^-\}_{m<n} \) are orthogonal. Thus the set \( \{X_{mn}\}_{m\neq n} \) is linearly independent if and only if both of the sets \( \{X_{mn}^+\}_{m<n} \) and \( \{X_{mn}^-\}_{m<n} \) are linearly independent.

For \( d \geq 4 \), we observe that when \( t = \frac{d-3}{d-1} \), \( \hat{a} = \hat{b} \) and

\[
X_{mn}^- = -(\hat{a} - 4)(|e_m\rangle\langle e_n| - |e_n\rangle\langle e_m|) = 4(2 - d)(|e_m\rangle\langle e_n| - |e_n\rangle\langle e_m|)
\]

from which the linear independence of \( \{X_{mn}^-\} \) follows immediately. For all other choices of \( t \), \( \hat{a} - \hat{b} \neq 0 \) and \( \hat{a} + \hat{b} \neq 0 \) for any choice of \( t \). Thus, in what follows, we can assume that \( \hat{a} \pm \hat{b} \neq 0 \), and modify the definition of \( X_{mn}^\pm \) above by removing the common nonzero factor of \( \hat{a} \pm \hat{b} \) and defining (with \( t \neq \frac{d-3}{d-1} \) for \( X_{mn}^- \))

\[
X_{mn}^\pm \equiv \frac{1}{\hat{b} \pm \hat{a}} (X_{mn} \pm X_{nm}^*) = \frac{1}{\hat{b} \pm \hat{a}} (X_{mn} \pm X_{nm})
\]

\[
= w_d^\pm(|e_m\rangle\langle e_n| \pm |e_n\rangle\langle e_m|) + \sum_{j \neq m,n} (|e_j\rangle\langle e_m| + |e_n\rangle\langle e_j|)
\]

\[
\pm (|e_m\rangle\langle e_k| \pm |e_k\rangle\langle e_m|)
\]

(32)

where we used (30) and

\[
w_d^+ (t) = \frac{\hat{a} + 4}{\hat{a} + \hat{b}} = \frac{2d}{d + t(d-1)} \in (1, d) - 1 < t < 1
\]

\[
w_d^- (t) = \frac{\hat{a} - 4}{\hat{a} - \hat{b}} = \frac{2(d - 2)}{(d - 3) - t(d-1)} \in \left\{ \begin{array}{ll} (1, \infty) & -1 < t < \frac{d-1}{d-3} \\ (-\infty, 2 - d) & \frac{d-3}{d-1} < t < 1 \end{array} \right.
\]

Note that \( w_d^\pm \) depends on \( t \) and when this is important we write \( w_d^\pm (t) \); otherwise we suppress this dependence to avoid cumbersome notation. In particular

\[
w_d^+ (-1) = d \quad w_d^+ (\frac{-1}{d-1}) = 2 \quad w_d^+ (0) = \frac{2d}{d+1} \quad w_d^+ (\frac{-3}{d-1}) = \frac{d}{d-1} \quad w_d^+ (1) = 1
\]
Although the replacement (28) does not preserve linear independence when $t = \frac{-1}{d-1}$, we allow $w^\pm_d = 2$ in order to obtain a complete set of eigenvectors for the eigenvalue problem for $\Omega^\pm_d(0)$ described below.

It will be useful to consider $X^\pm_{mn}$ as a function of $x$, where $x$ replaces $w^\pm_d(t)$, and allow $x \in \mathbb{R}$ even when there is no $t \in [-1,1]$ for which $x = w^\pm_d(t)$. Thus,

$$X^\pm_{mn}(x) = x (|e_m\rangle\langle e_n| \pm |e_n\rangle\langle e_m|) + \sum_{j \neq m,n} (|e_m\rangle\langle e_j| \pm |e_j\rangle\langle e_m| + |e_n\rangle\langle e_j| \pm |e_j\rangle\langle e_n|)$$

or, equivalently, again showing only the nonzero rows and columns.

$$X^\pm_{nm}(x) = \begin{pmatrix}
\pm1 & 1 \\
\vdots & \vdots \\
\pm1 & 1 \\
1 & \ldots & 1 & 0 & 1 & x & 1 & \ldots \\
\pm1 & 1 \\
\vdots & \vdots \\
\pm1 & 1 \\
\pm1 & \ldots & \pm1 & \pm x & \pm 1 & 0 & \pm 1 & \ldots \\
\pm1 & 1 \\
\vdots & \vdots 
\end{pmatrix}$$

We now define a $\frac{1}{2}d(d-1) \times \frac{1}{2}d(d-1)$ matrix $\Omega^\pm_d(x)$ whose rows are given by the elements of $X^\pm_{mn}(x)$ above the diagonal, arranged in lexicographic order so that its elements are as follows with $m < n, j < k$

$$\omega_{mn,jk} = (x \mp 2)\delta_{jm}\delta_{kn} + \delta_{jm} + \delta_{mk} \pm \delta_{jn} \pm \delta_{nk}$$

$$= \begin{cases} 
\pm 1 & j = m, k = n \\
+1 & j = m, k \neq n, n, j \neq m, n \\
-1 & j = n, k \neq m, n, k = m, j \neq m, n \\
0 & \text{otherwise}
\end{cases}$$

(34)

Since $\Omega^\pm_d(x) = \Omega^\pm_d(0) + xI_d$ we can conclude that

**Theorem 4.3.** The elements of each of the sets $\{X^\pm_{mn}(w^\pm_d)\}_{m<n}$ are linearly independent if and only if $-w^\pm_d$ is not an eigenvalue of $\Omega^\pm_d(0)$.

We will show that

**Proposition 4.4.** The eigenvalues of $\Omega^\pm_d(0)$ are

- $d - 2$ with multiplicity $d - 1$, and
- $-2$ with multiplicity $\frac{1}{2}(d - 2)(d - 1)$. 
Proposition 4.5. The eigenvalues of $\Omega^\pm_d(0)$ are

- $2(d - 2)$ non-degenerate,
- $d - 4$ with multiplicity $d - 1$, and
- $-2$ with multiplicity $(\frac{d(d - 1)}{2}) - 1 = \frac{1}{2}d(d - 3)$.

These results were conjectured after using Mathematica for $d = 3, 4, 5, 6$. We prove them by giving, for each eigenvalue, a linearly independent set of eigenvectors that is at least as large as the claimed multiplicity. Since the total number of eigenvectors of $\Omega^\pm_d$ cannot be larger than its dimension, which is $\frac{1}{2}d(d - 1)$, both lists of eigenvalues above are exhaustive.

4.4. Description of the Eigenspaces for $d \geq 4$

Since each eigenspace of $\Omega^\pm_d(0)$ is the null space of $\Omega^\pm_d(x)$ for the eigenvalue $-x$, we will describe them as these null spaces. Although each eigenvector is an element of $C_d(d - 1)/2$, we write them as the associated $d \times d$ symmetric or skew symmetric matrices (both with zero diagonal), in the case of $\Omega^+_d$ and $\Omega^-_d$ respectively. This facilitates associating each eigenspace with an irreducible representation of $S_d$. It also allows us to translate linear dependence relations for elements of the sets $\{X^\pm_{mn}\}_{m<n}$ to relations for elements of $\{A_m \mp A_n\}_{m<n}$, which we will write using commutators and anti-commutators defined, respectively, as

$$[A, B] = AB - BA \quad \{A, B\} = AB + AB.$$  \hspace{1cm} (35)

For each $x$ for which $\Omega^\pm_d(x)$ is singular, we will exhibit enough linearly independent eigenvectors of the null space to demonstrate that the multiplicity of the eigenvalue $-x$ is at least as large as claimed above. Although Proposition 4.1 also does not hold when $t = \frac{1}{d - 1}$ (which corresponds to $x = 2$) the associated eigenspaces are still needed to complete the proofs of Propositions 4.4 and 4.5. Moreover, as explained in Sect. 4.6, the eigenspaces we find can still be associated with linear dependence of certain subsets of $\{A_mA_n\}$.

Because Proposition 4.1 does not hold for $d = 3$, that case is analyzed separately in Sect. 4.5 (where it is also observed that the results below do hold.) Therefore, in what follows we assume that $d \geq 4$.

Skew symmetric, $x = 2 - d = w_d^{-1}(1)$ The null space is spanned by skew-symmetric matrices $C_k$, $k = 1, 2, \ldots d$ with

$$C_k = \sum_{j \neq k} (\langle e_k | e_j \rangle - \langle e_j | e_k \rangle).$$  \hspace{1cm} (36)

One readily verifies that $\sum_{k=2}^{d} C_k = -C_1$ and that $C_k$ with $k = 2, 3, \ldots d$ give a basis of $d - 1$ linearly independent vectors associated with the $d - 1$ Young tableaux $\begin{array}{cccc} 1 \\ \hline k \end{array}$. This corresponds to the linear dependence

$$\sum_{m \neq n} [A_mA_n] = 0 \quad \text{for each } n = 1, 2 \ldots d$$  \hspace{1cm} (37)
when \( t = 1 \), and proves part (a) of Remark 3.12.

**Skew symmetric,** \( x = 2 = w_d^-(\frac{-1}{d-1}) \) Although Proposition 4.1 does not hold in this case, \( X^d_{mn} \) is a multiple of \([A_m, A_n]\) so that the null space of \( \Omega_d^-(2) \) describes the linear dependence of the matrices \( \{A_m A_n - A_n A_m\}_{m<n} \). In this case, the null space is spanned by \( \frac{1}{2}(d-1)(d-2) \) skew symmetric \( d \times d \) matrices \( C_{jk} \) with \( 1 < j < k \leq d \) given by

\[
C_{jk} = |e_1\rangle\langle e_j| - |e_j\rangle\langle e_1| - |e_1\rangle\langle e_k| + |e_k\rangle\langle e_1| + |e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|.
\]

Since the term \( |e_j\rangle\langle e_k| \) occurs in one and only one \( C_{jk} \) with \( 1 < j < k \leq d \) these \( \frac{1}{2}(d-1)(d-2) \) matrices are clearly linearly independent. In fact, they are associated with Young tableaux of the form

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots & d \\
j & & & & & k
\end{array}
\]

this generates the linear dependence relations

\[
[A_j, A_k] + [A_k, A_m] + [A_m, A_j] = 0
\]

with \( j, k, \ell \) distinct.

**Symmetric,** \( x = 2(2 - d) \) The null space of \( \Omega_d^+(2(2 - d)) \) is readily seen to be \( \{|1_d\rangle\langle 1_d|\} \) (or \( d|1_d\rangle\langle 1_d| - I_d \) if one wants to keep the diagonal zero) which corresponds to the trivial representation of \( S_d \) with Young tableaux

\[
\begin{array}{cccc}
1 & 2 & \cdots & d \end{array}
\]

This gives \( \sum_{m\neq n} X_{mn}^+ = 0 \) when \( x = 2(2 - d) \). Although \( w_d^+(t) \neq 2(2 - d) \) when \( t \in [-1, 1] \), this reflects the fact that \( \sum_{m\neq n} A_m A_n \) has the form (25).

**Symmetric,** \( x = 4 - d \) \( w_d^+(t) = 4 - d \) in the domain \( [-1, 1] \) under consideration only in the case \( d = 3 \), which is studied in Sect. 4.5. For \( d = 4 \), \( w_d^+(t) \neq 0 \) for any \( t \in \mathbb{R} \) and for \( d \geq 5 \), \( w_d^+(t) = 4 - d \) only for \( t \in [-4, -1] \).

The null space of \( \Omega_d^+(4 - d) \) is spanned by the symmetric matrices

\[
C_{k\ell} = \sum_{j \neq k, \ell} (|e_k\rangle\langle e_j| + |e_j\rangle\langle e_k| - |e_\ell\rangle\langle e_j| - |e_j\rangle\langle e_\ell|) \quad k \neq \ell.
\]

The matrices \( C_{1k} \) with \( k = 2, 3, \ldots, d \) are readily verified to be linearly independent, giving a basis of \( d - 1 \) matrices corresponding to the \( d - 1 \) Young tableaux

\[
\begin{array}{cccc}
1 & 2 & \cdots & k \\
1 & 2 & \cdots & d \\
k & & &
\end{array}
\]

For \( d \geq 4 \), this null space corresponds to linear dependence relations

\[
\sum_m (X_{jm}^+ - X_{jm}^+) = 0
\]

for each fixed choice of \( j < k \). Then (28) implies

\[
\sum_{m \neq j, k} (\{A_j, A_m\} - \{A_k, A_m\}) = \tilde{D}_{jk}
\]

where \( \langle e_n, \tilde{D}_{jk} e_n \rangle = 0 \) if \( n \neq j, k \).
The null space of $\Omega_d^+(2)$ is spanned by symmetric matrices of the form
\[ C_{jk,mn}^+ = B_{jk,mn} + B_{jk,mn}^*, \]
where $j, k, m, n$ are distinct in \{1, 2, ..., $d$\}, and
\[ B_{jk,mn} \equiv |e_m\rangle\langle e_j| - |e_m\rangle\langle e_k| - |e_n\rangle\langle e_j| + |e_n\rangle\langle e_k| \tag{42} \]
and can be identified with the irreducible representation of $S_d$ described by the Young diagram
\[ \begin{array}{cccc} 1 & 2 & 3 & \ldots \\ n & k & \end{array} \]
which can readily be shown (using standard hook length arguments, e.g., [23, Section 2.8]) to have dimension $\frac{1}{2}d(d-3)$. We can verify this independently by observing that the matrices
\[ \{B_{2k,1n} : 3 \leq n < k \leq d\} \cup \{B_{2k,13} : k = 4, 5, \ldots d\} \tag{43} \]
are linearly independent. There are $\sum_{k=4}^{d}(k-3) = \frac{1}{2}(d-3)(d-2)$ matrices in the first group and $d-3$ in the second for a total of $\frac{1}{2}d(d-3)$ matrices. (Or one can observe that each $k$ is associated with a total of $k-2$ matrices and $\sum_{k=4}^{d}(k-2) = \sum_{j=2}^{d-2} j = \frac{1}{2}(d-2)(d-1)-1$.) The matrices in (43) do not belong to the null space; however, the set of adjoints of the matrices in (43) also form a linearly independent set whose elements are linearly independent of those in (43). Hence the corresponding sets $\{C^+_{jk,mn} \equiv B_{jk,mn} \pm B_{jk,mn}^*\}$ are each linearly independent. Thus we have found that the set
\[ \{C^+_{2k,1n} : 3 \leq n < k \leq d\} \cup \{C^+_{3k,12} : k = 4, 5, \ldots d\} \]
gives $\frac{1}{2}d(d-3)$ linearly independent elements of the null space of $\Omega_d^+(2)$.

Note that this basis is formed from the Young tableau
\[ \begin{array}{cccc} 1 & 2 & \ldots & \\ n & k \end{array} \]
and
\[ \begin{array}{cccc} 1 & 3 & \ldots & \\ 2 & k \end{array} \]
in standard form. This gives linear dependence relations of the form
\[ X^+_{jk} - X^+_{k,m} + X^+_{m,n} - X^+_{n,j} = 0 \tag{44} \]
which hold for any choice of $j, k, m, n$ distinct.

### 4.5. Analysis for $d = 3$

When $d = 3$, the results of the previous section hold formally, however, the arguments given there are not valid because they depend on Proposition 4.1 which does not hold for $d = 3$. Moreover, in addition to showing that a channel with $t \in (-1, 1)$ is extreme unless $t = -\frac{1}{d-1} = -\frac{1}{2}$, we show that when $t = -\frac{1}{2}$ it is factorizable. Furthermore, the channel with $t = 1$ has an exact factorization through $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ which uses a different unitary $U$ than in (13), and which is dual to the factorization for $t = -\frac{1}{2}$ in the sense that these channels can be obtained using the same unitary conjugation after exchanging the roles of the subalgebras.
When $d = 3$, instead of (28), we simply define $X_{mn}^{\pm} = \pm \frac{1}{7}(A_m A_n \pm A_n A_m)$, in which case $\pm \frac{1}{7}$ plays the role of $w_3^{\pm}(t)$. Then $\Omega_{3}^{\pm}(x) = \begin{pmatrix} x & 1 & \pm 1 \\ 1 & x & 1 \\ \pm 1 & 1 & x \end{pmatrix}$ so that $\det \Omega_{3}^{\pm}(x) = x^3 - 3x \pm 2 = (x \pm 2)(x \mp 1)^2$, which is consistent with Propositions 4.4 and 4.5 above. In particular,

- For $t = -\frac{1}{2}$, $x = -2$, the matrix $\Omega_{3}^+(−2)$ has a one-dimensional null space.
- For $t = -\frac{1}{2}$, $x = +2$, the matrix $\Omega_{3}^+(+2)$ has a one-dimensional null space.
- For $t = 1$, $x = 1 = 4 - d$, the matrix $\Omega_{3}^+(+1)$ has a two-dimensional null space.
- For $t = 1$, $x = -1 = 2 - d$, the matrix $\Omega_{3}^−(−1)$ has two-dimensional null space.

We first consider the case $t = \frac{-1}{d-1} = -\frac{1}{2}$, for which the situation for $d = 3$ differs slightly from that for $d \geq 4$.

**Proposition 4.6.** For $d = 3$ and $t = \frac{-1}{d-1} = -\frac{1}{2}$, the set $\{A_{m}^{2}\}_{m=1}^{d}$ is linearly independent, but

(a) $\sum_{m \neq n} A_{m} A_{n} = \sum_{m<n} \{A_{m}, A_{n}\} = 0$, which implies that $\{A_{m} A_{n} + A_{n} A_{m}\}_{m<n}$ is linearly dependent,

(b) $[A_1, A_2] + [A_2, A_3] + [A_3, A_1] = 0$, which implies that $\{A_{m} A_{n} - A_{n} A_{m}\}_{m<n}$ is linearly dependent, and

(c) $A_1 A_2 + A_2 A_3 + A_3 A_1 = A_1 A_3 + A_3 A_2 + A_2 A_1 = 0$.

Moreover, the map $\Phi(\rho) = \frac{4}{9} \sum_{m=1}^{3} A_{m}^{*} \rho A_{m}$ has an exact factorization through $M_{3}(C) \otimes M_{3}(C)$.

**Proof.** Part (a) follows immediately from (27). One can also observe that the null space of $\Omega_{3}^+(−2)$ is spanned by $|e_1\rangle\langle e_2| - |e_1\rangle\langle e_3| + |e_2\rangle\langle e_3|$, which implies (a). To prove (b) observe that the null space of $\Omega_{3}^+(+2)$ is spanned by $|e_1\rangle\langle e_2| - |e_2\rangle\langle e_3| + |e_3\rangle\langle e_1|$. Part (c) follows immediately from (a) and (b). The final assertion follows immediately from Theorem 4.8 below. \(\square\)

To distinguish the cases $t = -\frac{1}{2}$ and $t = 1$, we now use $B_k$ to denote the matrices associated with $t = 1$. The matrices $B_k$ are unitary which implies that $\{B_{m}^{2}\}_{m=1}^{3}$ are linearly dependent. However, as noted above, $\Omega_{3}^{\pm}(x)$ has a pair of two dimensional null spaces when $t = 1$, which leads to additional linear dependence relations. Thus, we find

**Proposition 4.7.** For $d = 3$ and $t = 1$, and $j, k, \ell$ distinct

(a) $\{B_j, B_k\} - \{B_j B_\ell\} = 0$, which implies that $\{B_n B_n + B_n B_n\}_{m<n}$ is linearly dependent,

(b) $[B_j, B_k] + [B_j, B_\ell] = 0$, which implies that $\{B_m B_n - B_n B_m\}_{m<n}$ is linearly dependent, and

(c) $B_1 B_2 = B_2 B_3 = B_3 B_1 = Q$ and $B_2 B_1 = B_3 B_2 = B_1 B_3 = Q^T$ with

$Q = \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. 


Moreover, the channel \( \Psi(\rho) = \frac{1}{3} \sum_{j=1}^{3} B_j \rho B_j \) has an exact factorization through \( M_3(\mathbb{C}) \otimes M_3(\mathbb{C}) \) which uses a different unitary \( U \) than in Remark 3.1.

**Proof.** To show (a), it suffices to observe that the null space of \( \Omega_3^+ (1) \) is spanned by the pair of vectors \(|e_1\rangle\langle e_2| - |e_1\rangle\langle e_3|\) and \(|e_1\rangle\langle e_2| - |e_2\rangle\langle e_3|\). To show (b), it suffices to observe that the null space of \( \Omega_3^- (-1) \) is spanned by the pair of vectors \(|e_1\rangle\langle e_2| + |e_1\rangle\langle e_3|\) and \(|e_1\rangle\langle e_3| + |e_2\rangle\langle e_3|\). The equivalences in part (c) follow immediately from parts (a) and (b), and simple computation or (23) gives \( Q \). (Note that (a) and (b) are exactly what one would get by formally using (37) and (41) when \( d = 3 \).) The final assertion follows immediately from the next result. \( \square \)

**Theorem 4.8.** The channels \( \Phi \) and \( \Psi \), defined in part (d) of Propositions 4.6 and 4.7 respectively, have exact factorizations through \( M_3(\mathbb{C}) \otimes M_3(\mathbb{C}) \) which are dual in the sense that there is a unitary matrix \( U \in M_3(\mathbb{C}) \otimes M_3(\mathbb{C}) \) such that

\[
\Phi(\rho) = \frac{4}{9} \sum_{k=1}^{3} A_k^\ast \rho A_k = (I_3 \otimes \text{Tr})(U^\ast (\rho \otimes \frac{1}{3} I_3) U), \quad \text{and} \quad (45)
\]

\[
\Psi(\rho) = \frac{1}{3} \sum_{k=1}^{3} B_k^\ast \rho B_k = (\text{Tr} \otimes I_3)(U^\ast (\rho \otimes \frac{1}{3} I_3) U). \quad (46)
\]

**Proof.** We first define

\[
U = \frac{2}{3} \begin{pmatrix} A_1 & A_3 & A_2 \\ A_3 & A_2 & A_1 \\ A_2 & A_1 & A_3 \end{pmatrix} = \frac{2}{3} \sum_{k=1}^{3} A_k \otimes B_k \quad (47)
\]

\[
W = \frac{1}{3} \begin{pmatrix} -B_1 & 2B_3 & 2B_2 \\ 2B_3 & -B_2 & 2B_1 \\ 2B_2 & 2B_1 & -B_3 \end{pmatrix} = \frac{2}{3} \sum_{k=1}^{3} B_k \otimes A_k. \quad (48)
\]

It follows immediately from part (c) of Propositions 4.6 and 4.7 respectively, that both \( U \) and \( W \) are unitary, and that (45) and (46) hold. \( \square \)

We note that since \( U \) and \( W \) are identical except for the exchange \( A_k \leftrightarrow B_k \) we also have

\[
\Phi(\rho) = (\text{Tr} \otimes I_3)(W^\ast (\rho \otimes \frac{1}{3} I_3) W) \quad \text{and} \quad \Psi(\rho) = (I_3 \otimes \text{Tr})(W^\ast (\rho \otimes \frac{1}{3} I_3) W).
\]

The interplay between the channels for \( t = -\frac{1}{2} \) and \( t = 1 \) is interesting. They are dual in the sense that they can be obtained by simply switching the subspace over which one takes the partial trace of \( U^\ast (\rho \otimes \frac{1}{3} I) U \) with the same unitary for both channels.

4.6. The Special Case \( t = \frac{-1}{d-1} \) with \( d \geq 4 \)

4.6.1. Linear Dependence Relations. In this case, \( p_d(t) = 0 \) so that (25) does not imply that \( |1_d\rangle\langle 1_d| \) is a linear combination of elements of \( \{A_m A_n\} \). Therefore, it is not clear that we can draw any conclusions about the linear independence of the set \( \{A_m \pm A_n\}_{m<n} \) from that of \( \{X^\pm_{mn}\}_{m<n} \). However, the
resulting linear dependence relations for \( \{A_m \pm A_n\}_{m<n} \) can also be proved directly, as observed after the following

**Proposition 4.9.** For \( d \geq 4 \) and \( t = \frac{-1}{d-1} \), the set \( \{A_m^2\}_{m=1}^d \) is linearly independent, but \( \sum_{m \neq n} A_mA_n = 4I_d = \frac{d}{2} \sum_{m=1}^d A_m^2 \), which implies that \( \{A_mA_n\}_{m,n=1}^d \) is linearly dependent. Moreover, each of the sets \( \{A_mA_n + A_mA_n\}_{m<n} \) and \( \{A_mA_n - A_mA_n\}_{m<n} \) is linearly dependent, and the following relations hold for any choice of \( j, k, m, n \) distinct

\[
0 = [A_j, A_k] + [A_k, A_m] + [A_m, A_j] \quad (49a)
0 = [A_j, A_k] + [A_k, A_m] + [A_m, A_n] + [A_n, A_j] \quad (49b)
0 = \{A_j, A_k\} - \{A_k, A_m\} + \{A_m, A_n\} - \{A_n, A_j\} \quad (49c)
0 = (A_j - A_m)(A_k - A_m).
\]

**Proof.** It is straightforward to verify (49d) directly. For example, observe that for \( d = 5 \), the matrices \( A_1 - A_2 \) and \( A_5 - A_4 \) are, respectively,

\[
\frac{1}{4} \begin{pmatrix}
  d-4 & 0 & -2 & -2 & -2 \\
  0 & d-4 & 2 & 2 & 2 \\
 -2 & 2 & 0 & 0 & 0 \\
 -2 & 2 & 0 & 0 & 0 \\
 -2 & 2 & 0 & 0 & 0
\end{pmatrix}
\text{ and }
\frac{1}{4} \begin{pmatrix}
  0 & 0 & 0 & 2 & -2 \\
  0 & 0 & 0 & 2 & -2 \\
 2 & 2 & 0 & 0 & 0 \\
-2 & -2 & 0 & 0 & 0 \\
-2 & -2 & 0 & 0 & 0
\end{pmatrix}.
\]

Then \( (A_j - A_m)(A_k - A_m) \pm (A_k - A_m)(A_j - A_m) = 0 \), gives (49c) and (49b), respectively. Moreover, (49a) can be obtained from linear combinations of permutations of (49b). For example, let \( X_{jk,mn} \) denote the expression on the right in (49b). Then \( X_{1234} + X_{1324} + X_{1243} \) yields (49a) with \( j, k, m \simeq 1, 2, 4 \).

Although the argument above does not use the results from Sect. 4.4, it is worth describing the connection. In the skew symmetric case, (28) implies that \([A_m, A_n] = \frac{b-\hat{a}}{(d-1)!} X_{mn}^-\) so that (39) implies (49a). Then, by using (49a) with \( j, k, m \) and with \( j, m, n \) together with the fact that \([A_j, A_m] = -[A_m, A_j]\), one can prove (49b).

The symmetric case requires slightly more work. Because the signs in (44) alternate, it follows immediately from (28) that the terms with \(|1_d\rangle\langle1_d|\) cancel so that

\[
\{A_j, A_k\} - \{A_k, A_m\} + \{A_m, A_n\} - \{A_n, A_j\} = D_{jk,mn}
\]

where \( D_{jk,mn} \) is a diagonal matrix. Moreover, it follows from (29) that \( D_{jk,mn} \) is identically zero. Combining this with (49b) implies \((A_j - A_k)(A_m - A_n) = D_{jk,mn} = 0\). (Instead of using (29), one could use (49d) to conclude that \( D_{jk,mn} = 0\).)

As described after (38) and (43) respectively, there are \( \frac{1}{2}(d-1)(d-2) \) independent constraints of the form (49a) or (49b) and \( \frac{1}{2}d(d-3) \) independent constraints of the form (49c). In addition, (27) implies that \( \sum_{m \neq n} A_mA_n \) is a multiple of \( \sum_{j=1}^d A_j^2 \). Thus, the subspace of \( M_d(\mathbb{C}) \) spanned by \( \{A_mA_n\}_{m,n=1}^d \) has dimension \( 3d - 2 \).
4.6.2. Factorizability When $d = 4$. The very high level of linear dependence when $t = \frac{-1}{d-1}$ makes it natural to conjecture that the associated channels are factorizable, as is the case when $d = 3$. Therefore, we considered this question when $d = 4$. Although we did not fully resolve it, we have some partial results. Moreover, we can show that a necessary condition is that $Y_j^* Y_j$ and $Y_j^* Y^*_j$ in Proposition (A.1) are both independent of $j$. This suggests that if the channel is factorizable, then $Y_j = U_j$ is unitary. In this section, we give the conditions on $U_j$ and discuss their implications.

Remark 4.10. For $d = 4$, $t = -\frac{1}{3}$ and $A_k$ given by (16), let $\Phi$ be the channel in part (b) of Theorem 3.11. Then $\Phi$ has an exact factorization through $M_d(\mathbb{C}) \otimes \mathcal{N}$ with $U = \sum_{k=1}^4 A_k \otimes U_k$ with $U_k \in \mathcal{N}$ unitary and $\tau(U_j U_k^*) = \delta_{jk}$ if and only if the following conditions hold:

$$Q^+_{jk} = Q^+_{mn} \quad \text{and} \quad R^+_{jk} = R^+_{mn} \quad j, k, m, n \text{ distinct}$$

$$Q^+_{jk} + Q^+_{km} + Q^+_{mj} = 0 \quad \text{and} \quad R^+_{jk} + R^+_{km} + R^+_{mj} = 0 \quad j, k, m \text{ distinct}$$

$$Q^+_{jk} + Q^+_{km} + Q^+_{mj} = 0 \quad \text{and} \quad R^+_{jk} + R^+_{km} + R^+_{mj} = 0 \quad j, k, m \text{ distinct}$$

with $Q^\pm_{jk} \equiv U_j U_k^* \pm U_k U_j^*$ and $R^\pm_{jk} \equiv U_j^* U_k \pm U_k^* U_j$.

Before giving the proof, we observe that (51) and (52) are equivalent to the asymmetric pair of conditions

$$U_j U_k^* + U_k U_j^* + U_m U_j^* = 0 \quad \text{and} \quad U_j^* U_k + U_k^* U_m + U_m^* U_j = 0. \quad (53)$$

Adding (51) and (52) gives (53). Conversely, by combining (53) with its adjoints, we can recover (51) and (52).

Proof. We begin without the assumption that $Y_j$ is unitary and (with a slight abuse of notation) let $Q^\pm_{jk} \equiv Y_j Y_k^* \pm Y_k Y_j^*$. When $s \neq t$, the conditions in (65) for $Y_j Y_k^*$ are formally the same as those on $c_{jk}$ in (24). Thus, a necessary condition for factorizability is

$$4Y_k Y_j^* = 2Q^+_{mn} + \sum_{\ell \neq j, k} Q^-_{\ell k} - Q^-_{j \ell} \quad j, k, m, n \text{ distinct}. \quad (54)$$

Then $Q^-_{jk} = -Q^-_{\ell j}$ implies that $4(Y_k Y_j^* + Y_j Y_k^*) = 4Q^+_{jk} = 4Q^+_{mn}$ as in (50). Next, observe that $Y_j Y_k^* - Y_k Y_j^*$ gives $T_{jk} \equiv 2Q^-_{jk} + Q^-_{km} + Q^-_{mj} - Q^-_{jn} - Q^-_{nk} = 0$. Then taking $T_{jk} + T_{mn}$ (after $j \mapsto m$, $k \mapsto n$, $m \mapsto k$, $n \mapsto j$) we obtain

$$Q^-_{jk} + Q^-_{km} + Q^-_{mn} + Q^-_{nj} = 0 \quad j, k, m, n \text{ distinct} \quad (55)$$

which has the same form as (49b). Then, just as (49b) implies (49a), (55) formally implies (52).

When $s = t$, the conditions in (65) become (for each fixed $j = 1, 2, 3, 4$)
\[ 28I_N = Y_j Y_j^* + 9 \sum_{k \neq j} \left( Y_k Y_k^* + Q_{jk}^+ + 5Q_{mn}^+ \right) \]  
\[ = Y_j Y_j^* + 9 \sum_{k \neq j} \left( Y_k Y_k^* + 6Q_{jk}^+ \right) \]  
(56)  

Taking the difference between (56) with \( j = 1 \) and with \( j = 2 \) implies  
\[ 0 = 8(Y_2 Y_2^* - Y_1 Y_1^*) + 6(Q_{13}^+ + Q_{14}^+ - Q_{23}^+ - Q_{24}^+) = 8(Y_2 Y_2^* - Y_1 Y_1^*) \]  
(57)  

Repeating for other pairs of \( j \neq k \), we conclude that \( Y_j Y_j^* \) is independent of \( j \). Thus, using a variant of the polar decomposition theorem, we write \( Y_j = M U_j \) for some positive definite operator \( M \) and \( U_j \) unitary. Then \( \tau(Y_j Y_j^*) = \tau(M^2 U_j U_j^*) = \delta_{jk} \), implies \( \tau(M^2) = \tau(I_N) \). Moreover, since \( Q_{jk}^+ = MU_j U_k^* M \), one can multiply by \( M^{-1} \) to conclude that the homogenous equations (50) to (52) hold for the \( U_j \) in \( Y_j = MU_j \) even if \( Y_j \) is not unitary. Moreover, they hold for \( U_j U_k^* \pm U_k U_j^* \) if and only if they hold for \( Y_j Y_j^* \pm Y_k Y_k^* \), justifying our ambiguous use of \( Q_{jk}^+ \).  

Applying the argument above to \( Y_j Y_j^* \) implies \( Y_j Y_j^* \) is independent of \( j \) so that \( U_j^* M^2 U_j = U_k^* M^2 U_k \). This implies that \( U_j U_k^* \) commutes with \( M^2 \) for all pairs \( j \neq k \) so that  
\[ 28I_N = 28 U_j^* M^2 U_j + 6 \sum_{k \neq j} (U_j^* M^2 U_k + U_k^* M^2 U_j) \]  
(58)  
\[ = 28 M^2 + 6M^2 \sum_{k \neq j} (U_k U_j^* + U_j U_k^*) \]  
\[ j = 1, 2, 3, 4. \]  
(59)  

When \( M^2 = I_N \), (56) implies \( \sum_{k \neq j} Q_{jk}^+ = 0 \) and (58) implies \( \sum_{k \neq j} R_{jk}^+ = 0 \) for each fixed \( j = 1, 2, 3, 4 \). Combining this with (50) gives (51). When \( M^2 \neq I_N \), we obtain (59). \( \square \)  

Even though it seems unlikely, we can not exclude the possibility that \( \Phi \) is factorizable when \( M^2 \neq I_N \). Nevertheless, we can state necessary and sufficient conditions for factorizability in terms of unitary operators.  

Remark 4.11. The channel \( \Phi \) in Remark 4.10 has an exact factorization through \( M_d(C) \otimes \mathcal{N} \) with \( U = \sum_{k=1}^4 A_k \otimes Y_k \) if and only if there are unitary operators \( U_k \in \mathcal{N} \) such that the following conditions hold:  
(a) \( I_N + \frac{3}{14} \sum_{k \neq j} Q_{jk}^+ \) is positive definite, \( \tau \left( (I_N + \frac{3}{14} \sum_{k \neq j} Q_{jk}^+)^{-1} U_m U_n^* \right) = \delta_{mn} \), and  
(b) both (50) and (52) hold.  

When these conditions hold, one can choose \( M^2 = (I_N + \frac{3}{14} \sum_{k \neq j} Q_{jk}^+)^{-1} \) and \( Y_n = MU_n \). Moreover, (50) to (52) hold if and only if they also hold when \( U_j \) is replaced by \( Y_j \) in \( R_{jk}^+ \).  

Proof. Both (56) and (59) are equivalent to \( M^{-2} = I_N + \frac{3}{14} \sum_{k \neq j} (U_k U_j^* + U_j U_k^*) \). Then \( \tau(Y_m Y_n^*) = \tau(M^2 U_m U_n^*) = \delta_{mn} \) gives (a). Although (51) does
not hold, (50) implies that \( \sum_{k \neq j} Q_{jk}^+ \) is independent of \( j \) so that the definition of \( M^2 \) is also.

However, relating conditions on \( Y_j^* Y_k \) to those on \( U_j^* U_k \), requires more work. Using the conditions for \( s \neq t \) on \( Y_j^* Y_k \), we can conclude that \( Y_j^* Y_k + Y_k^* Y_j = Y_m^* Y_n + Y_n^* Y_m \) when \( j, k, m, n \) are distinct, which is equivalent to

\[
U_j^* M^2 U_k + U_k^* M^2 U_j = U_m^* M^2 U_n + U_n^* M^2 U_m.
\]

(60)

Now observe that, since \( M^2 \) commutes with \( U_k U_\ell^* \),

\[
U_j^* M^2 U_k = U_j^* M^2 U_k U_\ell^* U_\ell = U_j^* U_k U_\ell^* M^2 U_\ell.
\]

(61)

Applying (61) to all terms in (60) and then multiplying on the right by \( U_\ell^* M^{-2} U_\ell^* \) gives \( U_j^* U_k + U_k^* U_j = U_m^* U_n + U_n^* U_m \). One can similarly show that (52) holds for \( R_{jk}^- \).

We have found \( U_j \in M_4(\mathbb{C}) \) which satisfy the conditions for \( Q_{jk}^+ \). They have the form \( U_j = U_j^* = 2E_j - I_4 \) with each \( E_j \) a rank two projection from one of the 4 mutually unbiased bases [15] for \( C_4 \) (excluding the standard basis). However, they do not satisfy (52). It is plausible that one could use the fact that (52) is associated with an irreducible representation of the symmetric group, as in (39), to find \( U_j \) which satisfy (52). If so, the question of factorizability depends on whether or not the conditions for \( Q_{jk}^+ \) and \( Q_{jk}^- \) are compatible. One can use the asymmetric condition (53) to test this without finding \( Q_{jk}^+ \) or \( Q_{jk}^- \).

**Remark 4.12.** The conditions in Remark 4.10 can not hold for \( U_k \in M_\nu(\mathbb{C}) \) unless \( \nu = 3n \) is a multiple of 3. This implies that the channel \( \Phi \) in Remark 4.10 can not have an exact factorization through \( M_4(\mathbb{C}) \otimes M_\nu(\mathbb{C}) \) when \( \nu \neq 3n \).

**Proof.** For \( j, k, m = 1, 2, 3 \) multiplying the first equation in (53) by \( U_2 \) on the right and the second by \( U_3 \) on the left gives

\[
U_1 + U_2 U_3^* U_2 + U_3 U_1^* U_2 = 0 \quad \text{and} \quad U_3 U_1^* U_2 + U_3 U_2^* U_3 + U_1 = 0
\]

(62)

which implies \( U_2 U_3^* U_2 = U_3 U_2^* U_3 \) or, equivalently \( (U_3^* U_2)^2 = U_2^* U_3 \). Multiplying this by \( U_3^* U_2 \) gives \( (U_3^* U_2)^3 = I_\nu \). Thus, we can conclude

\[
\text{Tr} U_3^* U_2 = 0 \quad \text{Tr} (U_3^* U_2)^2 = \text{Tr} U_2^* U_3 = 0 \quad \text{Tr} (U_3^* U_2)^3 = \text{Tr} I_\nu = \nu
\]

(63)

Let \( \lambda_k \) be the eigenvalues of \( U_3^* U_2 \) which is unitary. Then we have

\[
|\lambda_k| = 1 \quad \forall \ k \quad \sum_{k=1}^{\nu} \lambda_k = 0 \quad \sum_{k=1}^{\nu} \lambda_k^2 = 0 \quad \sum_{k=1}^{\nu} \lambda_k^3 = \nu.
\]

(64)

The last equation implies \( \lambda_k^3 = 1 \) \( \forall \ k \). Thus, the only possible choices for \( \lambda_k \) are \( 1, e^{\pm 2\pi i/3} \). Then \( \sum_k \lambda_k = \sum_k \lambda_k^2 = 0 \) holds if and only if the eigenvalues of \( U_3^* U_2 \) are \( 1, e^{2\pi i/3}, e^{-2\pi i/3} \), each with the same multiplicity. Thus, \( \nu \) must be a multiple of 3. \( \square \)
This does not resolve the question of factorizability in the case $d = 4, t = -\frac{1}{3}$, although it seems unlikely. We have not excluded the possibility of satisfying (50) to (52) with unitary matrices $U_k \in M_{3n}(\mathbb{C})$. Moreover, we have not excluded the possibility of satisfying the conditions in Remark 4.11 when $M^2 \neq I_N$. Nor have we excluded the possibility that $\Phi$ does have an exact factorization through $M_d(\mathbb{C}) \otimes \mathcal{N}$ with $\mathcal{N}$ not a matrix algebra, as in [19].

Remark 4.13. When $d = 4$, the channel $\Phi$ is factorizable at most when $t = \pm 1$ or $t = -\frac{1}{3}$. Nevertheless, the channel $\Phi \circ \Phi$ is factorizable for all $t \in [-1, 1]$.

Since $\Phi = \Phi^*$, this follows immediately from [6, Remark 5.6] as observed in Appendix A.3.

Acknowledgements

It is a pleasure to thank Matthias Christandl, Mikael Rørdam and Anders Thorup for helpful discussions, and the graduate student Jon Lindegaard Holmberg for skillfully performing insightful numerical work. The second named author was supported by a grant from The Independent Research Fund Denmark (FNU).

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Factorizability

A.1. Conditions for Factorizability

Let $\Phi: M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a UCPT map and $\mathcal{N}$ a von Neumann algebra with a faithful trace $\tau$ normalized so that $\tau(I_{\mathcal{N}}) = 1$. Following [6], we say (as noted in the introduction) that $\Phi$ has an exact factorization through $M_d(\mathbb{C}) \otimes \mathcal{N}$ if there is a unitary $U \in M_d(\mathbb{C}) \otimes \mathcal{N}$ such that for all $\rho \in M_d(\mathbb{C})$ (2) holds, i.e., $\Phi(\rho) = (I \otimes \tau)U^*(\rho \otimes I_{\mathcal{N}})U$. Furthermore, $\Phi$ is called factorizable if it has an exact factorization through $M_d(\mathbb{C}) \otimes \mathcal{N}$ for some $(\mathcal{N}, \tau)$.

When $\mathcal{N} = M_d(\mathbb{C})$ is a matrix algebra, (2) becomes $\Phi(\rho) = (I \otimes \text{Tr})U^*(\rho \otimes \frac{1}{d}I_d)U$. Following standard practice used elsewhere in this paper, when $\mathcal{N} = M_d(\mathbb{C})$, we denote the identity by $I_d$ rather than the awkward $I_{M_d(\mathbb{C})}$. We also follow the standard convention for type I algebras that $\text{Tr}$ is normalized so that $\text{Tr}|v\rangle\langle v| = 1$ when $v$ with $\|v\|^2 = \langle v, v \rangle = 1$.

The following useful result follows from the factorizability criteria in [5, Theorem 2.2].

Proposition A.1. Necessary and sufficient conditions for a UCPT map $\Phi$ on $M_d(\mathbb{C})$ of the form $\Phi(\rho) = \sum_{k=1}^{n} A_k^*\rho A_k$, with $\{A_1, A_2, \ldots, A_k\}$ linearly independent to have an exact factorization through $M_d(\mathbb{C}) \otimes \mathcal{N}$ are that there
are $Y_1, Y_2, \ldots, Y_κ \in N$ such that $τ(Y_jY_k^*) = δ_{jk}$ and (with $\{|e_s\}_s$ the standard basis for $C_d$) the following conditions hold for all $1 ≤ s, t ≤ d$:

$$\sum_{j,k} \langle e_s, A_j^*A_k^*e_t \rangle Y_j^*Y_k^* = δ_{st} I_N, \quad \sum_{j,k} \langle e_s, A_j^*A_k^*e_t \rangle Y_j^*Y_k = δ_{st} I_N.$$  \hfill (65)

**Proof.** By [5, Theorem 2.2], if $Φ$ has an exact factorization through $M_d(C)⊗N$, then one can find $Y_1, Y_2, \ldots, Y_n \in N$ such that $U := \sum_{κ=1}^n A_κ ⊗ Y_κ \in M_d(C)⊗N$ is unitary. Hence

$$UU^* = I_d ⊗ I_N = \sum_{j,k} A_j^*A_k ⊗ Y_j^*Y_k^*, \quad U^*U = I_d ⊗ I_N = \sum_{j,k} A_j^*A_k ⊗ Y_j^*Y_k.$$  \hfill (66)

Now observe that for any $A ∈ M_d(C)$ and $Y ∈ N$ one has

$$A ⊗ Y = \sum_{s,t} \langle e_s, Aε_t \rangle |e_s⟩⟨ε_t| ⊗ Y = \sum_{s,t} |e_s⟩⟨ε_t| ⊗ ⟨e_s, Aε_t⟩ Y.$$  \hfill (67)

Applying this to both sides of the equations in (66) and using the fact that $\{|e_s⟩⟨ε_t|\}_{s,t}$ is a basis for $M_d(C)$, gives the equations in (65). \hfill \Box

### A.2. Pairs of Factorizable Maps

**Definition A.2.** Let $d = p \cdot q$ be a product of primes and $U ∈ M_d(C) ≃ M_p(C) ⊗ M_q(C)$ unitary. Then $Φ(ρ) = (I ⊗ Tr)U^*(ρ ⊗ \frac{1}{q} I_q)U$ and $Ψ(γ) = (Tr ⊗ I)U^*(\frac{1}{p} I_p ⊗ γ)U$ are UCPT maps on $M_p(C)$ and $M_q(C)$, respectively.

The channels $Φ, Ψ$ are said to be dual channels associated with the unitary $U$.

**Theorem 4.8** essentially says that the channels $Φ, Ψ$ in Sect. 4.5 are dual channels associated with the unitary $U$ given by (47) and $p = q = 3$.

Let $\{U_j\}$ be a set of $q$ unitary matrices in $M_p(C)$ and $X = \bigoplus_{j=1}^q U_j = \sum_{j=1}^q U_j ⊗ |e_k⟩⟨ε_k|.$

Then $X ∈ M_p(C) ⊗ M_q(C)$ is unitary and, as essentially observed in [6, Proposition 2.8], the associated pair of dual channels are $Φ(ρ) = \frac{1}{q} \sum_{j=1}^q U_j^*ρU_j$ and $Ψ(γ) = C ⊗ γ$, where $⊗$ denotes the Schur or Hadamard product and $C ∈ M_q(C)$ is the matrix with elements $c_{jk} = \frac{1}{p} TrU_j^*U_k$. When the $U_j$ are also orthogonal so that $TrU_j^*U_k = p δ_{jk}$, then $Ψ(γ) = I_q ⊗ γ = \sum_{j=1}^q γ_{jj}|ej⟩⟨e_j|.$ is diagonal.

Whenever a UCPT map $Φ$ has an exact factorization through $M_d(C) ⊗ M_p(C)$, there is a unitary $U ∈ M_d(C) ⊗ M_p(C)$ for which $Φ(ρ) = (I ⊗ Tr)U^*(ρ ⊗ \frac{1}{q} I_q)U$ so that there is another UCPT map $Ψ$ such that $Φ, Ψ$ are the dual pair associated with that unitary. However, a UCPT map can have an exact factorization in more than one way. An example is given by the channel $Ψ$ in Sect. 4.5 which corresponds to $t = 1$ in (16). This channel has an exact factorization through $M_3(C) ⊗ M_3(C)$ with the unitary $W$ given by (48) and another with $X = \sum_{k=1}^3 B_j ⊗ |e_j⟩⟨e_j|$, as in Remark 3.1. The dual pair associated with $W$ is $Ψ, Φ$ as in Section 4.5; the dual pair associated with $X$ is $Ψ, Y$ where $Y(γ) = I_3 ⊗ γ$ as above.
One can extend this to situations in which $d$ is a product of more than two primes. However, each way of writing $d$ as a product of two integers will give a different pair of dual channels. Finally, we note that when $p = q$, it is possible to have a self-dual channel. For example, when $U = \frac{1}{\sqrt{2}} (\sigma_x \otimes \sigma_z + i \sigma_z \otimes \sigma_x)$, one finds $\Phi(\rho) = \Psi(\rho) = \frac{1}{2} (\sigma_x \rho \sigma_x + \sigma_z \rho \sigma_z)$, with $\sigma_{x,y,z}$ denoting the usual Pauli matrices.

It is worth noting that $U = \sum_k A_k \otimes B_k$ with $A_k \in M_p(C)$ and $B_k \in M_q(C)$ does not imply that $\Phi(\rho) = \sum_k A_k^* \rho A_k$ and $\Psi(\gamma) = \sum_k B_k^* \gamma B_k$. This only holds if $\text{Tr} B_j^* B_k = q \delta_{jk}$ in the first case and $\text{Tr} A_k^* A_k = p \delta_{jk}$ in the latter. Thus, for example, when $U$ is given by (5), $\Psi$ in the dual pair $\Phi_{\alpha,\beta}, \Psi_{\alpha,\beta}$ is not given by $\Psi(\gamma) = \frac{1}{2} \sum_{j,k=1}^2 |e_j \rangle \langle e_k| e_j \langle e_k| = \frac{1}{2} (\text{Tr} \gamma) I_2$, but by

$$\Psi_{\alpha,\beta} = \frac{|\alpha|^2 + 1}{3} (|e_1 \rangle \langle e_1| |\gamma| e_1 \rangle \langle e_1| + |e_2 \rangle \langle e_2| |\gamma| e_2 \rangle \langle e_2|) + \frac{|\beta|^2 + 1}{3} (|e_1 \rangle \langle e_2| |\gamma| e_2 \rangle \langle e_1| + |e_2 \rangle \langle e_1| |\gamma| e_1 \rangle \langle e_2|) = \frac{1}{3} (\text{Tr} \gamma) I_2 + \frac{1}{3} \begin{pmatrix} a^2 \gamma_{11} + b^2 \gamma_{22} & 0 \\ 0 & b^2 \gamma_{11} + a^2 \gamma_{22} \end{pmatrix}.$$

It should be emphasized that this duality is not equivalent to the notion of a pair of “complementary channels” used in the quantum information literature [5, 8, 14], which is defined in terms of the Stinespring representation, and goes back to Arveson [2, Section 1.3] who used the term “lifting”. In that case, the auxiliary space is interpreted as the environment and the complementary channel maps the input state $\rho$ to a state for the environment. For a channel $\Phi : C_{d_A} \mapsto C_{d_B}$ of the form $\Phi(\rho_A) = \sum_{k=1}^{d_E} A_k^* \rho A_k$, one can regard the Stinespring representation as mapping $\rho_A \mapsto \rho_{BE} = \sum_{j,k} A_j^* \rho_A A_k \otimes |e_j \rangle \langle e_k|$ with $|e_j \rangle$ the standard basis for $C_{d_E}$. Then $\Phi(\rho_A) = \text{Tr}_E \rho_{BE}$ and the complementary channel $\Phi^C : C_{d_A} \mapsto C_{d_E}$ is

$$\Phi^C(\rho_A) = \text{Tr}_B \rho_{BE} = \sum_{jk} \text{Tr} (A_j^* \rho_A A_k) |e_j \rangle \langle e_k|.$$

Even when $d_A = d_B$ these concepts are quite different. Complementary channels are defined with the implicit assumption of a pure ancilla rather than a maximally mixed ancilla. In the notion of dual pairs introduced above, the roles of the input and environment are interchanged in terms of both the subspace over which the trace is taken and the space in which the (maximally mixed) ancilla resides.

### A.3. Factorizability of $\Phi \circ \Phi^*$ with Choi Rank $\leq 4$

When a UCPT map $\Phi$ is factorizable, the adjoint $\Phi^*$ is also factorizable. Moreover, the maps $\Phi \circ \Phi$, $\Phi^* \circ \Phi$, and $\Phi \circ \Phi^*$ are also factorizable. However, there are some special circumstances in which $\Phi$ is not factorizable, but $\Phi^* \circ \Phi$ and $\Phi \circ \Phi^*$ are factorizable. This includes the Arveson–Ohno channel (3) and the channels in Sect. 3.3.1 for $d = 4$ and all $t \in (-1, 1)$.

The next result is a straightforward generalization of [6, Remark 5.6]. It follows from [6, Lemma 5.5] that the condition Choi rank $\leq 4$ is critical.
Proposition A.3. Let $\Phi : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ be a UCPT map with Choi rank $\leq 4$. Then the maps $\Phi \circ \Phi^*$ and $\Phi^* \circ \Phi$ each have exact factorizations through $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$. 

Proof. Let $\{A_k \in M_d(\mathbb{C}) : k = 1, 2, 3, 4\}$ satisfy $\sum_{k=1}^{4} A_k^* A_k = \sum_{k=1}^{4} A_k A_k^* = I_d$ and define the UCPT map $\Phi(\rho) = \sum_{k=1}^{4} A_k^* \rho A_k$. As observed in [6], one can always choose some $A_k = 0$ so that the result follows if there is a unitary map $U \in M_{4d}(\mathbb{C})$ such that

$$ (I \otimes \text{Tr}) U^* (\rho \otimes \frac{1}{4} I_4) U = (\Phi \circ \Phi^*)(\rho) = \sum_{j=1}^{4} \sum_{k=1}^{4} A_j^* A_k \rho A_k^* A_j. \quad (67) $$

Following the strategy in [6, Remark 5.6], define $U = \sum_{j,k=1}^{4} A_j^* A_k \otimes (2|e_j\rangle \langle e_k| - \delta_{jk} I_4)$. Then, by repeatedly using $\sum_{k=1}^{4} A_k^* A_k = \sum_{k=1}^{4} A_k A_k^* = I_d$ one finds

$$ UU^* = \sum_{j,k=1}^{4} \sum_{m,n=1}^{4} A_j^* A_k A_n^* A_m \otimes (2|e_j\rangle \langle e_k| - \delta_{jk} I_4)(2|e_n\rangle \langle e_m| - \delta_{mn} I_4) $$

$$ = 4 \sum_{jkm} A_j^* A_k A_n^* A_m \otimes |e_j\rangle \langle e_m| - 2 \sum_{j,m,n} A_j^* A_j A_n^* A_m \otimes |e_k\rangle \langle e_m| $$

$$ - 2 \sum_{jkm} A_j^* A_k A_n^* A_m \otimes |e_j\rangle \langle e_k| + \sum_{k,n} A_k^* A_k A_n^* A_n \otimes I_4 $$

$$ = 4 \sum_{jm} A_j^* A_m \otimes |e_j\rangle \langle e_m| - 2 \sum_{m,n} A_n^* A_m \otimes |e_n\rangle \langle e_m| $$

$$ - 2 \sum_{jk} A_j^* A_k \otimes |e_j\rangle \langle e_k| + I_d \otimes I_4 $$

$$ = I_d \otimes I_4 $$

so that $U$ is unitary. Similarly, one finds (using $\text{Tr} |e_j\rangle \langle e_k| = \delta_{jk}$ and $\text{Tr} I_4 = 4$)

$$ (I \otimes \text{Tr}) U^* (\rho \otimes \frac{1}{4} I_4) U = \frac{1}{4} \sum_{jkmn} A_j^* A_k \rho A_n^* A_m (4\delta_{jm}\delta_{kn} - 2\delta_{jk}\delta_{mn} - 2\delta_{j}\delta_{mn} + 4\delta_{jk}\delta_{mn}) $$

$$ = \sum_{jk} A_j^* A_k \rho A_k^* A_j = (\Phi \circ \Phi^*)(\rho) $$

\[\square\]

B. Linear Dependence of $\{A_m^* A_n\}$ versus $\{A_m A_n^*\}$

We give an explicit example to show that $\{A_m^* A_n\}$ can be linearly independent, but $\{A_m A_n^*\}$ linearly dependent. Let $d = 4$ and $W = \frac{1}{27} \begin{pmatrix} 8 & -11 & 16 \\ -19 & -8 & 4 \\ -4 & 16 & 13 \end{pmatrix}$.

When $A_m$ is constructed as in (11) with all $V_m = W$, we found that
\[ \{ A^*_m A_n \} \text{ is linearly independent unless } t = \pm 1, t = \frac{-13}{3}, t = \frac{-59}{84}, t = \frac{19}{21}, \text{ or } t = \frac{107}{21}. \]

\[ \{ A_m A^*_n \} \text{ is linearly independent unless } t = \pm 1, t = \frac{-59}{84}, t = \frac{-1}{7}, t = \frac{19}{21}, \text{ or } t = \frac{107}{21}. \]

Thus, when \( t = \frac{-1}{7} \), \( \{ A^*_m A^*_n \} \) is linearly independent but \( \{ A_m A^*_n \} \) is linearly dependent. When \( t = \frac{-13}{3} \), \( \{ A_m A^*_n \} \) is linearly independent but \( \{ A^*_m A_n \} \) is linearly dependent. At \( t = \pm 1, t = \frac{-59}{84}, t = \frac{19}{21}, t = \frac{107}{21} \) both sets are linearly dependent. For all other values of \( t \in \mathbb{R} \) both sets are linearly independent.

This result might seem counter-intuitive because the cyclicity of the trace implies

\[ \text{Tr} \ A^*_j A_k (A^*_m A_n)^* = \text{Tr} \ A_m A^*_j A_k A^*_n = \text{Tr} \ A_m A^*_j (A_n A^*_k)^* \]

so that the Gram matrices for the two sets have the same elements, albeit arranged differently. Let \( G \) and \( H \) denote these Gram matrices and consider the elements \( g_{11, kk} = \text{Tr} \ A^*_1 A_1 (A^*_k A_k)^* = \text{Tr} \ A_k A^*_1 (A_k A^*_1)^* = h_{k1, k1} \). Since \( g_{11, kk} \) all lie in the first row of \( G \), \( \det G \) will not contain any terms with \( g_{11, kk} \cdot g_{11, jj} \) when \( j \neq k \). However, since \( h_{k1, k1} \) lies on the diagonal of \( H \), \( \det H \) will contain a term which includes \( \prod_{k=2}^{d} h_{k1, k1} = \prod_{k=2}^{d} g_{11, kk} \).

We conjecture that if \( \{ A_m A^*_n \} \) is linearly dependent for all \( t \), then \( \{ A_m A^*_n \} \) should also be linearly dependent for all \( t \).

References

[1] Anantharaman-Delaroche, C.: On ergodic theorems for free group actions on noncommutative spaces. Probab. Theory Related Fields 135, 520–546 (2006)
[2] Arveson, W.: Subalgebras of \( C^* \)-algebras. Acta Mathematica 123, 141–224 (1969)
[3] Bennett, C.H., DiVincenzo, D.P., Smolin, J., Wootters, W.K.: Mixed state entanglement and quantum error correction. Phys. Rev. A 54, 3824 (1996)
[4] Choi, M.-D.: Completely positive linear maps on complex matrices. Linear Algebra Appl. 10, 285–290 (1975)
[5] Devetak, I., Shor, P.: The capacity of a quantum channel for simultaneous transmission of classical and quantum information. Commun. Math. Phys. 256, 287–303 (2005)
[6] Haagerup, U., Musat, M.: Factorization and dilation problems for completely positive maps on von Neumann algebras. Commun. Math. Phys. 303, 555–594 (2011)
[7] Haagerup, U., Musat, M.: An asymptotic property of factorizable completely positive maps and the Connes embedding problem. Commun. Math. Phys. 338, 721–752 (2015)
[8] Holevo, A.S.: On complementary channels and the additivity problem. Prob. Theory Appl. 51, 133–143 (2006)
[9] Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge Press, Cambridge (1991)
[10] Horodecki, M., Horodecki, P., Oppenheim, J.: Reversible transformations from pure to mixed states and the unique measure of information. Phys. Rev. A 67 (2003)

[11] Jamiołkowski, A.: Linear transformations which preserve trace and positive semi-definiteness of operators. Rep. Math. Phys. 3, 275–278 (1972)

[12] Ji, Z., Natarajan, A., Vidick, T., Wright, J., Yuen, H.: “MIP*=RE”. arXiv:2001.04383v2

[13] Kantor, W.K.: MUBs inequivalence and affine planes. J. Math. Phys. 53 (2012)

[14] King, C., Matsumoto, K., Nathanson, M., Ruskai, M.B.: Properties of conjugate channels with applications to additivity and multiplicity. Markov Process. Relat. Fields 13, 391–423 (2007)

[15] Kümmeler, B.: Markov dilations on the 2 × 2 matrices. In: Araki, H., Moore, C.C., Stratila, S., Voiculescu, D. (eds) Operator Algebras and their Connections with Topology and Ergodic Theory (Proceedings, 1983). Lecture Notes in Math., vol. 1132, pp. 312–323. Springer, Berlin (1985)

[16] Landau, L.J., Streater, R.F.: On Birkhoff’s theorem for doubly stochastic completely positive maps of matrix algebras. Linear Algebra Appl. 193, 107–127 (1993)

[17] Mendl, C.B., Wolf, M.M.: Unital quantum channels–convex structure and revival of Birkhoff’s theorem. Commun. Math. Phys. 289, 1057–1096 (2009)

[18] Musz, M., Kus, M., Zyczkowski, K.: Unitary quantum gates, perfect entanglers, and unistochastic maps. Phys. Rev. A 87 (2013)

[19] Musat, M., Rørdam, M.: Non-closure of quantum correlation matrices and factorizable channels that require infinite dimensional ancilla. Commun. Math. Phys. 375, 1761–1176 (2020)

[20] Ohno, H.: Maximal rank of extremal marginal tracial states. J. Math. Phys. 51 (2010)

[21] Ruskai, M.B.: Some Open Problems in Quantum Information Theory. arXiv:0708.1902

[22] Scharlau, J., Müller, M.P.: Quantum Horn lemma, finite heat baths, and the third law of thermodynamics. Quantum 2, 54 (2018)

[23] Sternberg, S.: Group Theory and Physics. Cambridge Press, Cambridge (1994)

[24] Werner, R.F., Holevo, A.S.: Counterexample to an additivity conjecture for output purity of quantum channels. J. Math. Phys. 43, 4353 (2002)
Mary Beth Ruskai  
Department of Mathematics  
University of Vermont  
Burlington VT 05405  
USA  
e-mail: mbruskai@gmail.com

Communicated by Alain Joye.  
Received: July 6, 2020.  
Accepted: May 25, 2021.