Position–momentum correlations in matter waves double-slit experiment

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Received 18 August 2014, revised 16 November 2014
Accepted for publication 19 January 2015
Published 19 February 2015

Abstract
We present a treatment of the double-slit interference of matter-waves represented by Gaussian wavepackets. The interference pattern is modelled with Green’s function propagator which emphasizes the coordinate correlations and phases. We explore the connection between phases and position–momentum correlations in the intensity, visibility and predictability of the wavepacket interference. This formulation will indicate some aspects that can be useful for theoretical and experimental treatment of particle, atom or molecule interferometry and can be discussed in introductory quantum mechanics courses.

Keywords: matter waves, double-slit experiment, position–momentum correlations

1. Introduction
The double-slit experiment illustrates the essential mystery of quantum mechanics [1]. Under different circumstances, the same physical system can exhibit either a particle-like or a wave-like behaviour, otherwise known as wave–particle duality [2]. Experiments with matter waves using double-slit were performed by Möllenstedt and Jössen for electrons [3], by Zeilinger et al for neutrons [4], by Carnal and Mlynek for atoms [5], and using diffraction gratings by Schrödingerkopf and Toennies for small molecules [6] and by Zeilinger et al for macromolecules [7]. In addition, electron double-slit diffraction has been recently experimentally observed in [8]. Position–momentum correlations have been studied and interpreted in some textbooks, while the most common example is the simple Gaussian or minimum-uncertainty wavepacket solution for the Schrödinger equation for a free particle. Such wavepacket presents no position–momentum correlations at \( t = 0 \), which appear only for latter times [9, 10]. The mechanisms of how the phases of the wave function influences the existence of position–
momentum correlations is also explained in [9]. Posteriorly, it has been shown that squeezed states or linear combination of Gaussian states can exhibit initial correlations, i.e., correlations that do not depend on the time evolution [11–14].

The qualitative changes in the interference pattern as a function of the increasing in the position–momentum correlations was studied in [15]. In addition, other studies have shown that the Gouy phase of matter waves is directly related to the position–momentum correlations, as studied for the first time in [16, 17]. The Gouy phase of matter waves was experimentally observed in different systems, such as Bose–Einstein condensates [18], electron vortex beams [19], and astigmatic electron matter waves using in-line holography [20]. More recently, it was observed that the position–momentum correlations can provide further insight into the formation of above-threshold ionization spectra in the electron–ion scattering in strong laser fields [21].

In this work, we use the previously developed ideas on position–momentum correlations to analyse the Gaussian features of the wavepacket and the interference pattern, as well as the wave-like and particle-like behaviours, in double-slit experiment with matter waves. Before reaching the double-slit setup, the particle is represented by a simple Gaussian wavepacket and, after the double-slit apparatus, the particle is represented by a linear combination of two identical Gaussian wavepackets coming from the two slits. After the double-slit, the position and momentum of the particle will be correlated even if the time evolution from the source to the double-slit is zero. The correlations will be changed by the evolution, enabling us to extract some information about the interference pattern.

The way the phases of the wavefunction evolve in the double-slit experiment is not treated in introductory textbooks of quantum mechanics, although it is known that it is the presence of the phases that govern the existence and quality of interference process. In this paper we use the connection, previously pointed out by Bohm, between the phases and the position–momentum correlations showing that there are not position–momentum correlations for a real wavefunction [9]. Therefore, these correlations carry information about the phases and they can help us to better understand the role of the phases in the construction of the interference pattern. We calculate the position–momentum correlations in the screen taking into account the superposition principle, which is taught in introductory quantum mechanics to obtain the total wavefunction. The position–momentum correlations obtained here give us information about the phase difference between the two wavefunctions at a time and thus give us information about the coherence, which is a basic knowledge to understand the existence of interference [22]. Another point that we consider important to discuss in introductory quantum mechanics courses is what happens with the particle in the double-slit experiment. Here we characterize the ‘particle’ behaviour and the ‘wave’ behaviour in terms of the predictability and visibility, respectively, as well as we explore these quantities in terms of the position–momentum correlations.

In section 2 we present the model for the double-slit experiment considering that the matter wave propagates during the time $t$ from the source to the double-slit and during the time $\tau$ from the double-slit to the screen. Furthermore, we calculate the wave functions for the passage through each slit using the Green’s function for the free particle. In section 3, we calculate the position–momentum correlations and the generalized Robertson–Schrödinger uncertainty relation for the state that is a linear combination of the states which passed through each slit. In section 4, we calculate the intensity, visibility and predictability to analyse the interference pattern in terms of the knowledge of the position–momentum correlations.
2. Double-slit experiment

In this section we return to the double-slit experiment and analyse the effect of the position–momentum correlations in the interference pattern. We consider that a coherent Gaussian wavepacket of initial width $\sigma_0$ propagates during a time $t$ before arriving at a double-slit that divides it into two Gaussian wavepackets. After the double-slit, the two wavepackets propagate during a time $\tau$ until they reach the detection screen, where they are recombined and the interference pattern is observed as a function of the transverse coordinate $x$. As we will see, the number of interference fringes and its quality are dramatically influenced by the propagation times $t$ and $\tau$. In particular, there is a value of time $t_{\text{max}}(\tau)$ for which the number of fringes tends to be minimum. Such a value corresponds to a maximum separation of the wavepackets on the screen and it is associated with one maximum of the position–momentum correlations. On the other hand, if the source of particles is positioned in such a way that, before arriving the screen, the particles travel during a time interval which is not close to $t_{\text{max}}(\tau)$, the number of interference fringes and its quality are increased significantly. We expect that the change in the fringe visibility, as a function of the source position, also occurs in optics. In addition, we also expect that the increase in the transversal coherence length with increase in the distance between the source and the double-slit, always observed in the previous double-slit experiments with light wave, occurs due to the fact that the source was always positioned before the point of minimum transversal coherence length for these experiments, presenting an increase when the source was displaced to the left of this position.

The wavefunction at the time when the wave passes through the slit 1(+) or the slit 2(−) is given by [15]

$$\psi_{1,2}(x, t, \tau) = \int_{-\infty}^{+\infty} dx_j \int_{-\infty}^{+\infty} dx_i G_2(x, t + \tau; x_j, t) \times F(x_j \pm d/2) G_1(x_j; t, x_i) \psi_0(x_i),$$

where

$$G_1(x_j; t, x_i) = \sqrt{\frac{m}{2\pi i\hbar}} \exp\left[\frac{im(x_j - x_i)^2}{2\hbar}\right],$$

$$G_2(x, t + \tau; x_j, t) = \sqrt{\frac{m}{2\pi i\hbar\tau}} \exp\left[\frac{im(x - x_j)^2}{2\hbar\tau}\right],$$

$$F(x_j \pm d/2) = \frac{1}{\sqrt{\beta \sqrt{\pi}}} \exp\left[-\frac{(x_j \pm d/2)^2}{2\beta^2}\right],$$

and

$$\psi_0(x_i) = \sqrt{\frac{1}{\sqrt{\sigma_0 \sqrt{\pi}}} \exp\left(-\frac{x_i^2}{2\sigma_0^2}\right)}.$$
slit (double-slit) to the double-slit (screen). We decided to use a Gaussian transmission function instead of a top-hat transmission function because a Gaussian transmission function represents a good approximation to the experimental reality and also because the Gaussian transmission function is mathematically simpler to treat than a top-hat transmission function, enabling us to obtain analytic expressions for the intensity, visibility, predictability and specially for the position–momentum correlations. The same treatment using the Gaussian transmission filters to describe the double-slit experiment was given in [23, 24]. As a consequence of the free propagation, which decouples the $x$, $y$ and $z$ dimensions for a given longitudinal location, we can write $z = vt$. This model is presented in figure 1 together with a qualitative illustration of the interference pattern for three different values of time $t$, maintaining $\tau$ constant.

After some algebraic manipulations, we obtain the following result for the wave that passed though the slit 1

$$
\psi_1(x, t, \tau) = \frac{1}{\sqrt{B J/R}} \exp \left[ -\frac{(x + D/2)^2}{2B^2} \right] 
\times \exp \left( \frac{i m x^2}{2\hbar R} + i\Delta x + i\theta + i\mu \right),
$$

where

$$
B^2(t, \tau) = \left( \frac{1}{\beta^2} + \frac{1}{\beta'}^2 \right)^2 + \frac{m^2}{\hbar^2} \left( \frac{1}{\tau^2} + \frac{1}{\tau'}^2 \right)^2
\times \left( \frac{m}{\hbar\tau} \left( \frac{1}{\beta^2} + \frac{1}{\beta'}^2 \right) \right).
$$
\[ R(t, \tau) = \frac{\left( \frac{1}{\beta^2} + \frac{1}{b^2} \right)^2 + \frac{m^2}{\beta^2} \left( \frac{1}{\tau} + \frac{1}{\tau + \gamma} \right)^2}{\left( \frac{1}{\beta^2} + \frac{1}{b^2} \right)^2 + \frac{\tau}{\frac{m^2}{\beta^2} \left( \frac{1}{\tau} + \frac{1}{\tau + \gamma} \right)^2}}. \]  

(8)

\[ \Delta(t, \tau) = \frac{\tau \sigma_0^2 \xi}{2\tau_0 \beta^2 B^2}. \]  

(9)

\[ D(t, \tau) = \frac{(1 + \frac{\tau}{\tau_0})}{(1 + \frac{\tau}{\tau_0})} d, \]  

(10)

\[ \theta(t, \tau) = \frac{md^2 \left( \frac{1}{\tau} + \frac{1}{\tau + \gamma} \right)}{8\hbar\beta^4 \left( \frac{1}{\beta^2} + \frac{1}{b^2} \right)^2 + \frac{m^2}{\beta^2} \left( \frac{1}{\tau} + \frac{1}{\tau + \gamma} \right)^2}. \]  

(11)

\[ \mu(t, \tau) = -\frac{1}{2} \arctan \left[ \frac{\left( \frac{1}{\tau_0} + \frac{1}{\tau + \gamma} \right)}{1 - \frac{1}{m} \left( \frac{1}{\tau_0} \right) \left( \frac{1}{\tau + \gamma} \right) \left( \frac{1}{\beta^2} + \frac{1}{b^2} \right)} \right]. \]  

(12)

\[ b^2(t) = \sigma_0^2 \left[ 1 + \left( \frac{t}{\tau_0} \right)^2 \right]. \]  

(13)

and

\[ r(t) = t \left[ 1 + \left( \frac{\tau_0}{t} \right)^2 \right]. \]  

(14)

In order to obtain the expressions for the wave passing through slit 2, we just have to substitute the parameter \( d \) by \(-d\) in the expressions corresponding to the wave passing through the first slit. Here, the parameter \( B(t, \tau) \) is the beam width for the propagation through one slit, \( R(t, \tau) \) is the radius of curvature of the wavefronts for the propagation through one slit, \( b(t) \) is the beam width for the free propagation and \( r(t) \) is the radius of curvature of the wavefronts for the free propagation. \( D(t, \tau) \) is the separation between the wavepackets produced in the double-slit. \( \Delta(t, \tau) \) is a phase which varies linearly with the transverse coordinate. \( \theta(t, \tau) \) and \( \mu(t, \tau) \) are the time dependent phases and they are relevant only if the slits have different widths. \( \mu(t, \tau) \) is the Gouy phase for the propagation through one slit. The knowledge of how this phase depends on time, and particularly on the slit width, can provide us with some understanding in the new design of the double-slit experiment with matter waves. \( \tau_0 = \frac{m\sigma_0^2}{\hbar} \) is one intrinsic time scale which essentially corresponds to the time during which a distance of the order of the wavepacket extension is traversed with a speed corresponding to the dispersion in velocity. It is viewed as a characteristic time for the ‘aging’ of the initial state [15] since it is a time from which the evolved state acquires properties completely different from the initial state.
3. Phase of the wavefunction and position–momentum correlations

In this section we calculate the position–momentum correlations $\sigma_{xp}$ at the screen and study how they behave as a function of the propagation times $t$ and $\tau$. We find out that the correlations present a point of maximum for the propagation time from the source to the double-slit $t$, whose value depends on $\tau$, the propagation time from the double-slit to the screen. This point of maximum express one instability of the phases of the wave function, which we can associate with incoherence and lack of interference. In addition, we find that the higher the correlations are, the smaller the region of overlap between the packets sent from each slit will be, i.e., the maximum of the correlations is associated with a maximum separation between the two wavepackets when they arrive at the screen. Here we show that only the phases that are dependent of the transverse position, which are the only phases that influence in the interference pattern when the slit apertures are equal, contribute to the correlations and this relation is important when discussing quantum mechanics in introductory courses.

In the quantum mechanical description of the double-slit experiment, the total wavefunction at the screen is given by the linear superposition, which enable us to obtain the following normalized wavefunction at the screen

$$\psi(x, t, \tau) = \frac{\psi_1(x, t, \tau) + \psi_2(x, t, \tau)}{\sqrt{2 + 2 \exp\left[-\left(\frac{D}{2B}\right)^2 - (\Delta B)^2\right]}}.$$ (15)

The state (15) is a superposition of two Gaussians and therefore presents position–momentum correlations even when $t = 0$ [11, 12]. For this state we calculate the correlations and obtain

$$\sigma_{xp}(t, \tau) = \frac{1}{2} \left\{ \hat{x} \hat{\rho} + \hat{\rho} \hat{x} \right\} - \left\{ \hat{x} \right\} \left\{ \hat{\rho} \right\}$$

$$= \frac{mB^2}{2R} + \frac{(mD^2/R)}{4 + 4 \exp\left[-\left(\frac{D}{2B}\right)^2 - (\Delta B)^2\right] - \frac{\hbar \Delta D}{2} \exp\left[-\left(\frac{D}{2B}\right)^2 + (\Delta B)^2\right]}.$$ (16)

We observe that the position–momentum correlations are not dependent on the terms $\theta$ and $\mu$ and its existence is exclusively due to the phase dependent on the transverse position $x$. As associated for the first time by Bohm [9], the four terms appearing in the expression for the correlations can be understood as the product of one ‘momentum’ by one ‘position’ for each time $t$ and $\tau$. For example, the first term is the product of the momentum $(mB/R)$ by the position $B$. The second term is the product of the momentum $(mD/R)$ by the position $D$. The third term is the product of the momentum $(\hbar \Delta)$ by the position $D$ and the fourth term is the product of the momentum $(m\Delta^2 B^3/R)$ by the position $B$. This connection allows us to understand that the higher the ‘position’ $B$ or $D$ is, the higher the associated ‘momentum’ and the contribution to the position–momentum correlations will be. Therefore, this appears to be a very simple way to characterize the particle when it arrives at the screen, allowing us to take a lot of information about its behaviour, an additional aspect which also can be discussed in introductory textbooks of quantum mechanics as made by Bohm for free propagation [9].
In the following, we plot the curves for the position–momentum correlations as a function of the times $t$ and $\tau$ for neutrons. The reason to consider neutrons relies in their experimental reality, which is most close to our model for interference with completely coherent matter waves, although we still have loss of coherence as discussed in [24]. We adopt the following parameters: mass $m = 1.67 \times 10^{-27}$ kg, initial width of the packet $\sigma_0 = 7.8 \mu$m (which corresponds to the effective width of $2\sqrt{2}\sigma_0 \approx 22 \mu$m), slit width $\beta = 7.8 \mu$m, separation between the slits $d = 125 \mu$m and de Broglie wavelength $\lambda = 2$ nm. These same parameters were used previously in double-slit experiments with neutrons by Zeilinger et al [4]. In figure 2(a), we show the correlations as a function of $t/\tau_0$ for $\tau = 18\tau_0$, where we observe the existence of a point of maximum. In figure 2(b), we show the absolute value of each term from equation (16) as a function of $t/\tau_0$ for $\tau = 18\tau_0$, where we see that the larger contribution for the position–momentum correlations comes from the second term, which is directly dependent on the separation $D(t, \tau)$ between the wavepackets at the screen. Therefore, a higher separation between the wavepackets at the screen implies higher position–momentum correlations, i.e., the maximum of the correlations is associated with a small region of the overlap between the two packets.

In figure 3, we show the position–momentum correlations as a function of $t/\tau_0$ and $\tau/\tau_0$. We observe that the region around the point of maximum, or region of phase instability, tends to stay narrower when the propagation time from the double-slit to the screen $\tau$ increases. We also observe that the point of maximum is displaced from the left when $\tau$ increases. In the next section we will show a table in which we clearly see the dependence of the time for the maximum of the correlations $t_{\text{max}}$ with the value of $\tau$, i.e., $t_{\text{max}} = t_{\text{max}}(\tau)$. Therefore, the dynamics after the double-slit also influences the interference pattern and should be taken into account in the analysis of double-slit experiments. Taking into account only the dynamics before the double-slit is not sufficient to obtain all the information about the interference pattern on the screen.

![Figure 2](image_url)
4. Schrödinger uncertainty relation

It is known that the uncorrelated free particle Gaussian wavepackets are states of minimum uncertainty both in position and in momentum. For this case, the position–momentum correlations appear only with the time evolution and they are followed by a spreading of the associated position distribution, while the momentum uncertainty is maintained constant for all times. For the most general Gaussian wavepacket, in which the initial position–momentum correlations are present, the uncertainty in position is minimum at $t = 0$. However, this is not true for the uncertainty in momentum [12]. Therefore, the position–momentum correlations indicate that the uncertainty in one or in both the quadratures is not a minimum. For the problem treated here, we have a superposition of two Gaussian wavepackets at the screen, for which the position–momentum correlations are present indicating that the uncertainty in both the quadratures is not minimum. In order to study the behaviour of the correlations together with the behaviour of the uncertainties in position and in momentum, we now calculate the determinant of the covariance matrix defined by

$$M_C = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp}^2 \end{bmatrix},$$

where $\sigma_{xx}^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$, $\sigma_{pp}^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$ are the squared variances in position and momentum, respectively, and $\sigma_{xp}$ are the position–momentum correlations. The expression for $\sigma_{xp}$ was obtained previously in equation (16) and for the other quantities we obtain the following results

$$\sigma_{xx}^2(t, \tau) = \frac{B^2}{2} + \frac{D^2 - 4\Delta^2 B^4 \exp\left[-\left(\frac{D}{4\Delta^2 B^2}\right)^4 - \left(\frac{\Delta}{2B}\right)^4 - \left(\Delta B\right)^2\right]}{4 + 4 \exp\left[-\left(\frac{D}{4\Delta^2 B^2}\right)^4 - \left(\Delta B\right)^2\right]},$$

(17)

Figure 3. Position–momentum correlation as a function of $t/\tau_0$ and $\tau/\tau_0$. The maximum is displaced to the left and the region around it tends to stay narrower as $\tau$ increases.
The determinant of the covariance matrix is the generalized Robertson–Schrödinger uncertainty relation and it is given by

\[
\sigma_{xx}^2 - \sigma_{pp}^2 = -\frac{\hbar^2}{D C^2}.
\]  

In figure 4(a) we show the curves of the uncertainties \(\sigma_{xx}, \sigma_{pp}\) and the correlations \(\sigma_{xp}\) normalized to the same scale as a function of \(t/t_0\) for \(\tau = 18t_0\) and in figure 4(b) we show the determinant \(D_C/\hbar^2\) (solid line) as a function of \(t/t_0\) for \(\tau = 18t_0\), where we compare it with the value 1/4 (dashed line). The determinant is practically constant at the extremes but different from the value \(\hbar^2/4\) and varies rapidly in the region where the position–momentum correlations have a maximum.

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extreme $t \approx 4t_0$ corresponds to the point for which the correlations have the same value when $t = 0.50t_0$, i.e., $\sigma_{xp}(t = 0.50t_0) \approx \sigma_{xp}(t = 4t_0) \approx 82\hbar$. Finally, the determinant varies slowly in the regions where the correlations tend to be minima, more specifically the regions $0 < t < 0.50t_0$ and $t > 4t_0$. At the interval $0 < t < 0.50t_0$, the uncertainty in position and in momentum increases, in practical terms, by the same rate and at the interval $t > 4t_0$ the uncertainty in position decreases more slowly than the uncertainty in momentum. The determinant tends to be a constant value in both intervals, but at the first interval, $0 < t < 0.50t_0$, the curve of correlations has a concavity upwards in which the value of the determinant tends to the minimum value $D_C \approx 16\hbar^2$. At the second interval, $t > 4t_0$, the curve of correlations has a concavity turned down (tending to a constant function for $t \gg t_{\text{max}}$) in which the determinant tends to the maximum value $D_C \approx 33\hbar^2$. Then, we observe that $D_C > \hbar^2/4$ for all time. This characterizes the non-Gaussianity of the state (15), since for Gaussian states, initially correlated or not, the generalized Robertson–Schrödinger uncertainty relation is constant and equal to $\hbar^2/4$ for all times. Therefore, for states obtained from the superposition of two Gaussian states, as the case treated here, the determinant of the covariance matrix is larger than $\hbar^2/4$ for all times and it is constant only for values of time outside the region around which the correlations have a point of maximum, showing that the Gaussian features are strictly changed by the evolution of the position–momentum correlations. Thus, if we construct one state that has correlations with a point of minimum, for which the determinant can tend to the value $\hbar^2/4$ at the screen, the number of interference fringes and its visibility can be increased significantly. It is possible to do this by considering one double-slit experiment in which the initial state is the correlated Gaussian state or by putting an atomic convergent lens next to the double-slit as similarly has been proposed for light waves [25].

In table 1 we show some values of time $t_{\text{max}}$ that we calculate numerically, for which the correlations $\sigma_{xp}$, the uncertainty in position $\sigma_{xx}$ and the uncertainty in momentum $\sigma_{pp}$ are maxima and the point of inflexion of the correlations as a function of time $\tau$. We observe that when $\tau$ increases, the time $t_{\text{max}}$ of the correlations is dislocated to the left and that this time is always localized between the times for which the uncertainties in position and in momentum are maxima. We also observe that the times of maxima tend to coincide for $\tau > 1000t_0$ and that the time of maximum for $\sigma_{pp}$ is independent of $\tau$ as a consequence of the free propagation from the double-slit to the screen.

5. Intensity, visibility and predictability

In this section we calculate the relative intensity, visibility and predictability to analyse the interference pattern, the wave-like and particle-like behaviour from knowledge of the position–momentum correlations. Such analysis is important because it allows us to choose the set of parameters that provides the better interference pattern in the double-slit experiment. Knowledge of the correlations tells us if the particle sent by the source will behave more as wave-like or particle-like on the screen. In other words, if the particle is sent by one position for which the time of flight until the double-slit is within the interval around the maximum of the correlations, it will behave mostly as a particle for most values of $x$, excluding only the values near $x = 0$. The developed theoretical model shows the intensity function in closed form, thus directly meets the analytical behaviour. This behaviour helps students, in a didactic way, to notice the functional aspects of intensity and associate them with combinations of elementary mathematical functions. This analytical aspect is of fundamental importance to
compute the visibility and predictability, and the closed form of the intensity greatly simplifies the interpretation of results.

The intensity on the screen, defined as $I(x, t, \tau) = |\psi(x, t, \tau)|^2$, is given by

$$I(x, t, \tau) = F(x, t, \tau) + F(x, t, \tau) \frac{\cos(2\Delta x)}{\cosh \left(\frac{Dx}{B^2}\right)},$$

(20)

where

$$F(x, t, \tau) = I_0 \exp \left[-\frac{x^2 + \left(\frac{\tau}{\sqrt{2}}\right)^2}{B^2}\right] \cosh \left(\frac{Dx}{B^2}\right).$$

(21)

The first term in equation (20) is the single slits envelope and the second term is the interference.

To characterize the ‘particle’ behaviour and the ‘wave’ behaviour for each position in the screen, Greenberger and Yasin define the quantities predictability and visibility, respectively, as a function of a parameter ($\beta$) which varies from full particle to full wave knowledge preserving the general case in which one can have considerable knowledge of both, i.e., $P(\beta)^2 + V(\beta)^2 = 1$, which is the mathematical expression for the Bohr’s complementarity principle for pure quantum mechanical states [26]. In the double-slit experiment for which the slits are treated as Gaussian transmission functions and for pure quantum mechanical states, Bramon et al [23] obtained for the predictability and visibility, respectively, the following results

$$P(x) = \left|\frac{\psi_1^2 - \psi_2^2}{\psi_1^2 + \psi_2^2}\right| = \left|\tanh \left(\frac{Dx}{B^2}\right)\right|,$$

(22)

and

$$V(x) = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{1}{\cosh \left(\frac{Dx}{B^2}\right)}.$$

(23)

where $I_{\text{max}}$ is the intensity for $\cos(2\Delta x) = 1$ and $I_{\text{min}}$ is the intensity for $\cos(2\Delta x) = -1$.

The Bohr’s complementarity principle established, by the relation of Greenberger and Yasin for pure quantum mechanical states, that $P(x)^2 + V(x)^2 = 1$ is satisfied for all values

### Table 1. Times of maxima $t_{\text{max}}$ and inflexion $t_{\text{inf}}$ as a function of $\tau_0$.

| $\tau$ | $t_{\text{max}}$ of $\sigma_{\text{sp}}$ | $t_{\text{max}}$ of $\sigma_{\text{ss}}$ | $t_{\text{max}}$ of $\sigma_{\text{pp}}$ | $t_{\text{inf}}$ of $\sigma_{\text{sp}}$ |
|---|---|---|---|---|
| 2 | 1.568 109 061 | 1.984 545 314 | 1.392 356 020 | 0.472 034 9103 |
| 8 | 1.450 312 552 | 1.525 841 616 | 1.392 356 020 | 0.499 024 0822 |
| 18 | 1.419 651 602 | 1.450 522 331 | 1.392 356 020 | 0.504 918 7153 |
| 50 | 1.402 487 095 | 1.413 088 513 | 1.392 356 020 | 0.508 073 7518 |
| 100 | 1.397 465 783 | 1.402 693 625 | 1.392 356 020 | 0.508 978 9150 |
| 1000 | 1.392 871 030 | 1.393 387 225 | 1.392 356 020 | 0.509 800 4574 |
of $x$ [26]. As we can see, the densities of probabilities for each slit (each path) $|\psi_1(x, t, \tau)|^2$ and $|\psi_2(x, t, \tau)|^2$ depend on the position $x$ which makes the path predictability be $x$ dependent and consequently the fringe visibility which is necessary to satisfy the Greenberger and Yasin’s relation.

The visibility and predictability depend on the ratio $D/B^2$, showing the influence of the parameter $D$ (the separation between the wavepackets at the time), equivalently to the position–momentum correlations, on the interference pattern. Therefore, for higher values of $D$ and smaller values of $B$, the particle-like behaviour will be dominant and less visible will be the interference fringes for the values of $x$ far away from from $x = 0$. As we will see, there is a value of time $t$, within the interval of maximum correlations, for which the visibility is minimum and the predictability is maximum. Previously, the effective number of fringes for light waves in the double-slit experiment was characterized in [23] for a given distance (or time) of propagation from the double-slit to the screen while neglecting the propagation from the source to the double-slit. According to [23], the number of fringes was estimated by a new index defined by $\nu = 0.264/R$. For the problem treated here, we have $R = D/2\Delta B^2$, indicating that the higher the value of $D$ is, the smaller the number of fringes is.

In figure 5(a), we show half of the symmetrical plot for the relative intensity (black line) and predictability (red line) as functions of $x$ for three different values of $t$, one of them being the time for which the correlations have a maximum, with $\tau$ fixed to $\tau = 18\tau_0$, where $\tau_0 = m\alpha^2/\hbar$. The corresponding values of $t$ are, respectively, $t = 0.2\tau_0$ (solid line), $t_{\text{max}} \approx 1.42\tau_0$ (dotted line) and $t = 18\tau_0$ (dashed line). We observe that for $t_{\text{max}} \approx 1.42\tau_0$, the number of interference fringes is minimum and the visibility extends over a small range of the $x$ axis behind the double-slit. In addition, the predictability dominates, extending over a wide range of the $x$ axis. For $t = 0.2\tau_0$ or $t = 18\tau_0$ we have a large number of fringes and the visibility extends over a larger range of the $x$ axis behind the double-slit comparatively with the results for $t_{\text{max}} \approx 1.42\tau_0$. The predictability dominates only in a range outside that.

Figure 5. (a) Relative intensity (black line) and (b) visibility (blue line) and predictability (red line) as a function of $x$ for three different values of $t$ and $\tau = 18\tau_0$. The corresponding values of $t$ are, respectively, $t = 0.2\tau_0$ (solid line), $t_{\text{max}} \approx 1.42\tau_0$ (dotted line) and $t = 18\tau_0$ (dashed line). For these values of time, the time for which the correlations have a maximum $t_{\text{max}} \approx 1.42\tau_0$ presents the least number of fringes and visibility. Moving the source of particles to the left or to the right from the region around the maximum of the correlations, the number of fringes and visibility increases.
immediately behind the double-slit. This shows that a displacement of the source either to the
left or to the right, so that the particles
flights a different time from the times around which the
correlations have a maximum
tmax, most specifically the times in the interval
ττ << t0.50 4,
the number of fringes increases and the interference pattern has a better
quality. We have to focus on the region for which the correlations have a maximum and not
specifically at the time of maximum. This is because, although
τmax really appears as the time
for which the number of fringes is a minimum, the visibility has a minimum in the region of
maximum correlations. However, it does not coincide with τmax being displaced a little from
this point to the right, as we can see in
figure 6. In fact, for τ= t0.2 0
and τ= t18 0,
the position–momentum correlations assume values close to each other and the number of fringes
is nearly the same. However, the visibility is larger for τ= t0.2 0
in comparison with τ= t18 0. This shows that the wave-like behaviour
becomes more evident, comparatively, when the particle is released from a position such that
the
flight time until the double-slit is most distant from the time for which the correlations
have a maximum. On the other hand, we can say that our ignorance about which slit the

Figure 6. Visibility (blue line) and predictability (red line) as a function of τ/τ0 for three
different values of x. The corresponding values of x are x = 0.01 mm (dotted line),
x = 0.05 mm (solid line) and x = 0.1 mm (dashed line). We present figures for
τ= 10τ0, τ= 30τ0 and τ= 60τ0. The values of V and P for each value of x are
strongly influenced by the values of τ and t. For example, there is a value of time t for
which the visibility is minimum and the predictability is maximum and for τ > 60τ0,
the values of V are higher than the values of P.

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particle passed, when it is launched from the position \( \tau = z_0 v_t \) (8), is smaller when the screen is positioned at \( \tau = z_1 v_t \) (10) than when it is positioned at \( \tau = z_2 v_t \) (30). Again, we see the influence of the times \( t \) and \( \tau \) over the quantities \( t(\tau) \) and \( \tau(\tau) \), although the result \( P^2 + V^2 = 1 \) is maintained for all \( x \) values independent of the time.

The results above were obtained for neutrons treated as wavepackets of initial transverse width \( \sigma = 7.8 \text{ \mu m} \). For this parameter, the time scale is given by \( \tau_0 = m\sigma^2/\hbar = 1.02 \text{ ms} \). We can note a good quality in the interference pattern for \( \tau = t_{18.02} \) and \( \tau = t_{18.02} \), whose velocity around \( v = 200 \text{ m s}^{-1} \), corresponds to distances \( z_t = 3.6 \text{ m} \) and \( z_t = 3.6 \text{ m} \). These parameters were used by Zeilinger et al and they correspond to distances within experimental viability [4]. Now, if we take, for instance, the mass of the order of \( m = 1.2 \times 10^{-24} \text{ kg} \), which is next to the mass of the fullerene molecules, and build a package of the same width of the neutrons, we will have \( \tau_0 = 0.73 \text{ s} \). In this case, \( t = 18\tau_0 = 13.14 \text{ s} \) and \( \tau = 18\tau_0 = 13.14 \text{ s} \). Considering one velocity of \( 200 \text{ m s}^{-1} \), we will have \( z_t = 2.63 \times 10^3 \text{ m} \) and \( z_t = 2.63 \times 10^3 \text{ m} \), which are distances outside the experimental reality. Therefore, by analysing the behaviour of the correlations, we can also capture information about how hard it is to observe interference with macroscopic objects. We have to remember that the model described here can be applied to treat macromolecules, as fullerenes, only if the effects of loss of coherence are negligible, which is an idealized situation. In a realistic case, to describe fullerenes or other type of macromolecules, we have to adapt the model in such a way to include the effects of loss of coherence.

6. Conclusions

In this work, we studied the double-slit experiment as an attempt to find parameters that produce the maximum number of interference fringes and with the highest possible quality on the screen. Our results show that we can take information about the interference pattern by...
looking at the behaviour of the position–momentum correlations that are installed with the quantum dynamics. We observe that both the dynamics before and after the double-slit are important for the existence and quality of the interference fringes on the screen. In particular, we observe that there is a value of the propagation time from the source to the double-slit for which the correlations have a point of maximum, so that particles released by a source at the region around this point produce interference fringes on the screen with the worst quality. The wave-like and particle-like behaviours expressed by the complementary relation of Greenberger and Yasin $P^2 + q^2 = 1$ is also strongly influenced at each point $x$ by the times $t$ and $\tau$, i.e., depending where the particle came from and where the screen was positioned, it will behave mostly as a wave or mostly as a particle at the screen. Satisfying the requisites of completely coherent matter waves, knowledge of the point of maximum of the position–momentum correlations can also help us to choose the best parameters which allow us to observe interference effects with macromolecules, such as fullerenes. From the determinant of the covariance matrix it is possible to observe how the Gaussian properties of the state produced on the screen by the superposition of two Gaussian states are changed when the uncertainties in position and in momentum and the position–momentum correlations vary with the times $t$ and $\tau$.

Acknowledgments

We would like to thank Professor E C Girão for carefully reading the manuscript. IGdP and LAC acknowledge useful discussions with M C Nemes. JSMN thanks the CAPES for financial support under grant number 210010114016P3. IGdP is grateful for support from the program PROPESQ (UFPI/PI) under grant number PROPESQ 23111.011083/2012-27.

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