Graph and Union of Graphs Compositions

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Abstract

The graph compositions’ notion was introduced by A. Knopfmacher and M. E. Mays [1]. In this note we add to these a new construction of tree-like graphs where nodes are graphs themselves. The first examples of these tree-like compositions, a corresponding theorem and resulting conclusions are provided.

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1 Introduction

Graphs considered here are finite, undirected and labeled graphs, with no loops or multiple edges. The edge between \(v_1, v_2\) is \((v_1, v_2)\). The set of vertices of graph \(G\) is denoted by \(V(G)\) and the set of edges - by \(E(G)\).

Let \(G\) be a graph and \(S \subseteq G\) be a subgraph of \(G\). We say that \(S\) is a maximal subgraph of \(G\) iff \(E(S) = \{(x, y) \in E(G) : x, y \in V(S)\}\). Once we have this notion we can introduce definition of the composition (equivalent to that from [1]):

**Definition 1** The composition of graph \(G\) is a partition \(\{V_1, V_2, ..., V_n\}\) of the set \(V(G)\), such that each maximal subgraph of \(G\) induced by \(V_i\) is connected for \(1 \leq i \leq n\).

Let \(C(G)\) denotes the number of compositions of the graph \(G\). In particularly simple cases, the number of composition may be counted immediately. For example, let \(E_0\) be a graph with empty set of edges. Then \(C(E_0) = 1\).

For cycle graph \(C_4\) with 4 vertices and 4 edges (tetragon) \(C(C_4) = 12\).

Next - the graphs such as point, interval and triangle, are special cases of complete graphs \(K_n\) with \(n\) vertices \((n = 1, 2, 3)\). Since \((x, y) \in E(K_n)\) for any two vertices \(x, y \in V(K_n)\), then the number of compositions of \(K_n\) is equal to the number of partitions of a set with \(n\) elements, i.e to the \(n\)-th Bell number \(B(n)\) ([1] theorem 2):

\[
C(K_n) = B(n) \quad (1)
\]
2 Compositions of the union of graphs

Let us consider at first the union $G_1 \cup G_2$ of two graphs which are disconnected or are connected by one common vertex. Since each composition of graph $G_1 \cup G_2$ is in one to one correspondence with a pair of compositions of $G_1$ and $G_2$, we have ([1] theorem 3):

**Theorem 1** Let $G_1, G_2$ be graphs with no common edges and at most one common vertex. Then

$$C(G_1 \cup G_2) = C(G_1) \cup C(G_2)$$

(2)

The following formula was independently proved in [1] (theorem 4), but it immediately follows from above theorem and the equation $C((x_1, x_2)) = 2$.

**Conclusion 1** Let $G_1, G_2$ be a disconnected graphs and $x_1 \in V(G_1), x_2 \in V(G_2)$. Then

$$C(G_1 \cup (x_1, x_2) \cup G_2) = 2 \cdot C(G_1) \cdot C(G_2)$$

(3)

Consider now another example of applying the above. In [1] was shown, that for tree $T_n$ with $n$ edges ($n - 1$ vertices), the number of compositions is $2^n$. As any tree may be constructed inductively from one edge by adding succeeding edges, the proof of formula is by induction. By analogous construction we can build a tree $T_n(G)$ of $n$ copies of any graph $G$, where in succeeding steps we connect copy of $G$ with constructed tree by common vertex. Then $C(T_n(G)) = (C(G))^n$.

For $A \subseteq V(G)$ let $C^+(G, A)$ denotes the number of compositions of a graph $G$ such that set $A$ is in one part of composition while $C^-(G, A)$ denotes the number of compositions of the graph $G$ such that each element of a set $A$ is in another part of a composition. Then we state.

**Lemma 1** Let $G, H$ be a graphs, $V(G) \cap V(G) = \{x, y\}$ and $(x, y) \in E(G) \cap E(H)$. Then

$$C(G \cup H) = C^+(G, \{x, y\}) \cdot C^+(H, \{x, y\}) + C^-(G, \{x, y\}) \cdot C^-(H, \{x, y\})$$

PROOF. In order to prove this equation, it is sufficient to observe that

$$C^+(G \cup H, \{x, y\}) = C^+(G, \{x, y\}) \cdot C^+(H, \{x, y\}) \quad \text{and}$$

$$C^-(G \cup H, \{x, y\}) = C^-(G, \{x, y\}) \cdot C^-(H, \{x, y\}).$$

Hence the above equation follows by

$$C(G \cup H) = C^+(G \cup H, \{x, y\}) + C^-(G \cup H, \{x, y\}). \quad \square$$

**Definition 2** Let $G$ be a graph and $k = (x, y) \in E(G)$. Then $G/k$ denotes graph obtained from $G$ by removing $k$ from edges, identifying vertices $x, y$ and identifying edges $(x, z)$ with $(y, z)$ for vertices $z \in V(G)$ such that $(x, z), (y, z) \in E(G)$.

With this notion applied all together with Lemma 1. we arrive at the following theorem.

**Theorem 2** Let $G, H$ be graphs with exactly two common vertices $x, y$ and one common edge $(x, y)$. Then

$$C(G \cup H) = C(G) \cdot C(H) + 2 \cdot C(G/k) \cdot C(H/k) - C(G) \cdot C(H/k) - C(G/k) \cdot C(H)$$

(4)
Proof. The thesis follows from equations for $G$:
\[
C(G) = C^+(G, \{x, y\}) + C^-(G, \{x, y\}).
\]
and analogical equations for $H$.

Theorems above make it possible (in some cases) to count the number of compositions of graphs constructed inductively just by attaching succeeding graphs with common vertex or common edge - similarly like in the construction of trees.

3 Trees of graphs

In this section we use theorems the preceding section in some special cases, when $n$ copies of graph $G$ are connected into a tree-like structure i.e. nodes of a tree are graphs themselves.

Definition 3 The V-tree (respectively E-tree) $T$ of graphs $G_1, G_2, ..., G_n$ is a graph $T = T_n$ constructed as following:

1. $T_1 = G_1$.
2. If V-tree (E-tree) $T_k$ is defined for some $k < n$, then $T_{k+1}$ is obtained from graphs $T_k$ and $G_{k+1}$ by identifying some vertex (edge) in $T_k$ and $G_{k+1}$.

Immediately from equation 2 the simple conclusion follows.

Conclusion 2 If $T$ is a V-tree of graphs $G_1, G_2, ..., G_n$, then

\[
C(T) = \prod_{i=1}^{n}(C(G_i))
\]  

(5)

E-trees are more complicated, nevertheless in this case we can obtain an interesting insight too into the matters via the Theorem 3. and resulting consequences.

Theorem 3 (technical) Let $G$ be a graph and $T$ be an E-tree of $n$ copies of graph $G$. Let $T_1, T_2, ..., T_n$ be the sequence of E-trees used to construction $T$ and $k_1, k_2, ..., k_{n-1}$ be a sequence of edges in $T$, such that $T_{r+1}$ is union of graphs $T_r$ and copy of $G$ connected by edge $k_r$ for $1 \leq r < n$. Then

\[
C(T_{r+1}) = (C(G) - C(G/k_r) + (2 \cdot C(G/k_r) - C(G)) \cdot C(T_r/k_r)
\]

\[
C(T_{r+1}/k_{r+1}) = \begin{cases} 
2 \cdot C(G/k_{r+1}/k_r) - C(G/k_{r+1}) \cdot C(T_r/k_r) \\
+(C(G/k_{r+1}) - C(G/k_{r+1}/k_r)) \cdot C(T_r) \\ C(G/k_r) \cdot C(T_r/k_r) 
\end{cases} \quad k_{r+1} \neq k_r
\]

\[
C(T_{r+1}/k_{r+1}) = \begin{cases} 
2 \cdot C(G/k_{r+1}/k_r) - C(G/k_{r+1}) \cdot C(T_r/k_r) \\
+(C(G/k_{r+1}) - C(G/k_{r+1}/k_r)) \cdot C(T_r) \\ C(G/k_r) \cdot C(T_r/k_r) 
\end{cases} \quad k_{r+1} = k_r
\]

for $1 \leq r < n$.

Proof. The thesis follows by 4. □

The above equation may be simplified for "regular" trees.
Conclusion 3 Let $K_n$ be the complete graph on $n > 2$ vertices and let be given a sequence $T_1, T_2, \ldots, T_m$ of $E$-trees used to construction of $E$-tree $T = T_m$ of $m$ copies of $K_n$, such that any three different copies of $K_n$ have no common edge in $T$. Then for $r < m$ holds

$$C(T_{r+1}) = (B_n - 2 \cdot B_{n-1} + 2 \cdot B_{n-2}) \cdot C(T_r) + ((B_{n-1})^2 - B_n \cdot B_{n-2}) \cdot C(T_{r-1})$$

(6)

Proof. Let $k_1, k_2, \ldots, k_m - 1$ be the sequence of common edges in construction of sequence $T_1, T_2, \ldots, T_m$. At first observe that for any edge $k \in E(K_n), K_n/k$ is complete graph on $n - 1$ vertices. Therefore by theorem 3 and equation 1

$$C(T_{r+1}) = (B_n - B_{n-1}) \cdot C(T_r) + (2B_{n-1} - B_n)C(T_r/k_r),$$

$$C(T_{r+1}/k_{r+1}) = (2B_{n-2} - B_{n-1})C(T_r/k_r) + B_{n-2})C(T_r).$$

Moreover $T_r/k_r = T_{r-1} \cup K_n/k_r = T_{r-1} \cup K_{n-1}$ and graphs $T_{r-1}, K_{n-1}$ have common exactly one edge $k_{r-1}$ for $r > 1$. From first equation we can compute $C(T(r - 1) \cup K_{n-1})$ and use it in second equation to obtain the thesis. □

Similarly, although more easily, the following conclusion becomes apparent.

Conclusion 4 Let $K_n$ be the complete graph on $n > 2$ vertices and let be given a sequence $T_1, T_2, \ldots, T_m$ of $E$-trees used to construction $E$-tree $T$ of $m$ copies of $K_n$ with property: there is an edge $k \in E(T)$ such that every two different copies of $K_n$ have common edge $k$. Then for $r < m$ holds

$$C(T_{r+1}) = (B_n - B_{n-1}) \cdot C(T_r) + (2 \cdot B_{n-1} - B_n) \cdot (B_{n-1})^r$$

(7)

The cycle graph $C_n$ with $n$ vertices and $n$ edges, with vertex $i$ connected to vertices $i \pm 1 \pmod n$ has similar properties to those of the complete graph $K_n$ has. Namely, for any edge $k$, the graph $C_n/k = C_{n-1}$. The "only" difference in between $C_n$ and $K_n$ shows up in well known expressions for number of objects $C(C_n) = 2^n - n$ (see [1], theorem 9), $C(K_n) = B_n$. Therefore one may exchange $B_n$ with $2^n - n$ in equations 6 and 7 in order to obtain formulas for trees of cycle graphs.

Conclusion 5 Let $C_n$ be the cycle graph on $n > 2$ vertices and let $T_1, T_2, \ldots, T_m$ be a sequence of $E$-trees used to construction of $E$-tree $T = T_m$ of $m$ copies of $C_n$, such that any three different copies of $C_n$ have no common edge in $T$. Then for $r < m$ holds

$$C(T_{r+1}) = (2^{n-1} - n + 2) \cdot C(T_r) + ((n - 4) \cdot 2^{n-2} + 1) \cdot C(T_{r-1})$$

(8)

For the ladder $L_n$, which is a case of $E$-tree (more precisely - the chain) of $n - 1$ copies of $C_4$ we obtain $C(L_{n+1}) = 6 \cdot C(L_n)$ (like in [1], theorem 9).

For the broken wheel $W^*_{n+1}$, which is a case of $E$-tree (the chain like in the ladder) of $n - 2$ copies of $C_3$ we obtain $C(W^*_{n+1} = 3 \cdot W^*_n - W^*_{n-1}$ (compare with [2], proposition 1.2).

Conclusion 6 Let $K_n$ be the complete graph on $n > 2$ vertices and let be given a sequence $T_1, T_2, \ldots, T_m$ of $E$-trees used to construction $E$-tree $T$ of $m$ copies of $K_n$ with property: there is an edge $k \in E(T)$ such that every two different copies of $K_n$ have common edge $k$. Then for $r < m$ holds

$$C(T_{r+1}) = (2^{n-1} - 1) \cdot C(T_r) - (n - 1) \cdot (2^{n-1} - n + 1)^r$$

(9)
4 Recapitulation

The main idea of this note i.e. the tree of connected graphs concept allows one to construct a quite a big class of examples of graphs, for which the number of compositions may be computed in the way presented above. Of course "plenty" of graphs are beyond the reach of the method.

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http://ii.uwb.edu.pl/akk/index.html - are highly appreciated.

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