On the solution of the coupled steady-state dual-porosity-Navier-Stokes fluid flow model with the Beavers-Joseph-Saffman interface condition

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Abstract

In this work, we propose a new analysis strategy to establish an a priori estimate of the weak solutions to the coupled steady-state dual-porosity-Navier-Stokes fluid flow model with the Beavers-Joseph-Saffman interface condition. The most advantage of our proposed method is that the a priori estimate and the existence result are independent of small data and the large viscosity restriction. Therefore the global uniqueness of the weak solution is naturally obtained.

Keywords: weak solution, dual-porosity-Navier-Stokes, Beavers-Joseph-Saffman interface condition, a priori estimate, existence, global uniqueness

1. Introduction

Coupled free flow and porous medium flow systems play an important role in many practical engineering fields, e.g., the flood simulation of arid areas in geological science [20], filtration treatment in industrial production [28, 37], petroleum exploitation in mining, and blood penetration between vessels and organs in life science [19]. Specifically, the systems are usually described by Navier-Stokes equations (or Stokes equations) coupled with Darcy’s equation, and there are amounts of achievements such as [12, 13, 10, 16, 26, 36, 48]. However, the standard Darcy’s equation describes fluids flowing through only a single porosity medium, which is not accurate to deal with the complicated multiple porous media similar to naturally fractured reservoir. Actually, the naturally fractured reservoir is comprised of low permeable rock matrix blocks surrounded by an irregular network of natural microfractures, and further they have different fluid storage and conductivity properties [43, 47]. In 2016, Hou et al. [31] proposed and numerically solved a coupled dual-porosity-Stokes fluid flow model with four multi-physics interface conditions. The authors used the dual-porosity equations over Darcy’s region to describe fluid flowing through the multiple porous medium. Recently, several related research on the above model can be found in the literatures [2, 3, 24, 29, 44]. In particular, Gao and Li [24] proposed a decoupled stabilized finite element method to solve the coupled dual-porosity-Navier-Stokes fluid flow model in the numerical field.

The steady-state dual-porosity-Navier-Stokes fluid flow model has distinct features and difficulties in mathematical analysis. Many numerical methods have been studied for the well-known stationary or time-dependent Navier-Stokes/Darcy model with Beavers-Joseph or Beaver-Joseph-Saffman interface condition, including coupled finite element methods [6, 11, 21, 22, 49], discontinuous Galerkin methods [17, 26, 27], domain decomposition methods [12, 20, 21, 30, 38], and decoupled methods based on two-grid finite element [10, 23, 48, 49]. In spite of the above great contributions to numerical simulation, the
existence of a weak solution to the coupled dual-porosity-Navier-Stokes fluid flow model with Beavers-Joseph-Saffman interface condition for general data keeps unresolved. In many literatures [6, 10, 17, 20, 21, 26, 27, 30, 49], a priori estimates and existence of a weak solution need suitable small data and/or large viscosity restrictions, and therefore only local uniqueness can be established when the data satisfy additional restrictions. In [26], the authors pointed out that the difficulty for a priori estimates and existence with general data is stemmed from the transmission interface condition, which does not completely compensate the nonlinear convection term from the Navier-Stokes equations in the energy balance.

Therefore in this paper, stemming from resolving steady-state Navier-Stokes equations with mixed boundary conditions in [32], we shall establish a new a priori estimate of the weak solutions by coupling the model problem with an auxiliary problem in order to completely compensate the nonlinear convection term from the Navier-Stokes equations. In addition, we shall also prove existence of a weak solution without small data or large viscosity restriction. As a result, the global uniqueness of the weak solution is naturally obtained.

The rest of this paper is organized as follows. In Section 2, we specify the steady-state dual-porosity-Navier-Stokes fluid flow model with Beavers-Joseph-Saffman interface condition and provide its Galerkin variational formulation. In Section 3, we establish a new a priori estimate of the weak solutions by coupling the model problem with an auxiliary problem, which is designed subject to the model problem. Finally, in Section 4, we prove existence of the weak solution without small data and/or large viscosity restriction by the Galerkin method and Brouwer’s fixed-point theorem, and global uniqueness of all variables by the inf-sup condition and Babuška–Brezzi’s theory.

2. Model specification

2.1. Setting of the problem

In this section, we consider the steady-state dual-porosity-Navier-Stokes fluid flow model in a bounded open polygonal domain \( \Omega \subset \mathbb{R}^N (N = 2, 3) \) with four physically valid interface conditions.

The domain \( \Omega \) consists of a fluid flow region \( \Omega_s \) and a dual-porous medium region \( \Omega_d \), with interface \( \Gamma = \overline{\Omega_s} \cap \overline{\Omega_d} \) (see Figure 1). Both \( \Omega_s \) and \( \Omega_d \) are open, regular, simply connected, and bounded with Lipschitz continuous boundaries \( \Gamma_i = \partial \Omega_i \setminus \Gamma \) (i = s, d), respectively. Here, \( \Omega_s \cap \Omega_d = \emptyset, \overline{\Omega_s} \cup \overline{\Omega_d} = \Omega \).

The unit normal vector of the interface \( \Gamma \) pointing from \( \Omega_s \) to \( \Omega_d \) (from \( \Omega_d \) to \( \Omega_s \)) is denoted by \( n \) (resp. \( n_d \)), and the corresponding unit tangential vectors are denoted by \( \tau, \tau_i, i = 1, \cdots, N - 1 \).

In two-dimensional case, if we write \( n_s = (n_{1s}, n_{2s}) \in \mathbb{R}^2 \), then \( \tau_s = n_{s}^\top = (-n_{2s}, n_{1s}) \).

In \( \Omega_s \), the fluid flow is governed by the Navier-Stokes equation [11, 12, 17, 20, 21, 26, 27]:

\[
\begin{align*}
(u_s \cdot \nabla) u_s - \nabla \cdot T(u_s, p_s) &= f_s \quad \text{in } \Omega_s, \\
\nabla \cdot u_s &= 0 \quad \text{in } \Omega_s, \\
\nabla \cdot u_s &= 0 \quad \text{on } \Gamma_s,
\end{align*}
\]

where \( T(u_s, p_s) = -p_s I + 2\nu D(u_s) \) is the stress tensor, \( \nu \) is the kinetic viscosity, \( D(u_s) = \frac{1}{2} [\nabla u_s + (\nabla u_s)^\top] \) is the velocity deformation tensor, \( u_s(x) \) denotes the fluid velocity in \( \Omega_s \), \( p_s(x) \) denotes the kinematic pressure in \( \Omega_s \), and \( f_s(x) \) denotes a general body force term that includes gravitational acceleration.

As usual, we write formally:

\[
(v \cdot \nabla) w = \sum_{i=1}^{N} v_i \frac{\partial w}{\partial x_i}, \quad \nabla \cdot v = \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i},
\]
The filtration of an incompressible fluid through porous media is often described using Darcy’s law. So in dual-porous medium domain $\Omega_d$, the flow is governed by a traditional dual-porosity model, which is composed of matrix and microfracture equations as follows:

\[
\begin{align*}
-\nabla \cdot \left( k_m \mu \nabla \phi_m \right) + \frac{\sigma k_m}{\mu} (\phi_m - \phi_f) &= 0 \quad \text{in } \Omega_d, \\
-\nabla \cdot \left( \frac{k_f}{\mu} \nabla \phi_f \right) + \frac{\sigma k_m}{\mu} (\phi_f - \phi_m) &= f_d \quad \text{in } \Omega_d, \\
\phi_m &= 0 \quad \text{on } \Gamma_d, \\
\phi_f &= 0 \quad \text{on } \Gamma_d,
\end{align*}
\]

(2.2)

where $\phi_m(x)$, $\phi_f(x)$ denote the matrix and microfracture flow pressure, $\mu$ is the dynamic viscosity, $k_m$, $k_f$ are the intrinsic permeability of the matrix and microfracture regions, $\sigma$ is the shape factor characterizing the morphology and dimension of the microfractures, $f_d$ is the sink/source term for the microfractures, and the term $\frac{\sigma k_m}{\mu} (\phi_m - \phi_f)$ describes the mass exchange between matrix and microfractures.

Based on the fundamental properties of the dual-porosity fluid flow model and traditional Stokes–Darcy flow model, Hou et al. [31] introduced four physically valid interface conditions as follows to couple appropriately the dual-porosity-Stokes model, which are also adapted to our problem.

- No-exchange condition between the matrix and the conduits/macrofractures:

\[
- \frac{k_m}{\mu} \nabla \phi_m \cdot n_d = 0 \quad \text{on } \Gamma.
\]

(2.3)

- Mass conservation:

\[
\mathbf{u}_s \cdot n_s = \frac{k_f}{\mu} \nabla \phi_f \cdot n_d \quad \text{on } \Gamma.
\]

(2.4)
• Balance of normal forces:
\[
- \mathbf{n}_s \cdot (\mathbf{T}(u_s, p_s)\mathbf{n}_s) = \frac{\phi_f}{\rho} \quad \text{on } \Gamma.
\]  
(2.5)

• The Beavers-Joseph-Saffman interface condition [7, 33, 35, 41]: for \( i = 1, \cdots, N - 1 \),
\[
- \mathbf{r}_i \cdot (\mathbf{T}(u_s, p_s)\mathbf{n}_s) = \frac{\alpha \nu \sqrt{N}}{\sqrt{\text{trace}(\Pi)}} u_s \cdot \mathbf{r}_i \quad \text{on } \Gamma.
\]  
(2.6)

In (2.3)–(2.6), \( \alpha \) is the Beavers-Joseph constant depending on the properties of the dual-porous medium, \( \Pi \) represents the intrinsic permeability that satisfies the relation \( \Pi = k_f\mathbb{I} \), \( \mathbb{I} \) is the \( N \times N \) identity matrix, and \( \rho \) is the fluid density.

2.2. Galerkin variational formulation

Throughout this paper we use the following standard function spaces. For a Lipschitz domain \( \mathcal{D} \subset \mathbb{R}^N \), \( N \geq 1 \), we denote by \( W^{k,p}(\mathcal{D}) \) the Sobolev space with indexes \( k \geq 0 \), \( 1 \leq p \leq \infty \) of real-valued functions defined on \( \mathcal{D} \), endowed with the seminorm \( |\cdot|_{W^{k,p}(\mathcal{D})} \) denoted by \( |\cdot|_{k,p,\mathcal{D}} \) and norm \( \|\cdot\|_{W^{k,p}(\mathcal{D})} \) denoted by \( \|\cdot\|_{k,p,\mathcal{D}} \) [1]. When \( p = 2 \), \( W^{s,2}(\mathcal{D}) \) is denoted as \( H^s(\mathcal{D}) \) and the corresponding seminorm and norm are written as \( |\cdot|_{k,s} \) and \( \|\cdot\|_{k,s,\mathcal{D}} \), respectively. In addition, with \( |\mathcal{D}| \) we denote the \( N \)-dimensional Hausdorff measure of \( \mathcal{D} \).

To perform the variational formulation, we define some necessary Hilbert spaces given by

\[
X_s := \{ v \in H^1(\Omega_s) : v = 0 \text{ on } \Gamma_s \},
\]
\[
X_d := \{ \psi \in H^1(\Omega_d) : \psi = 0 \text{ on } \Gamma_d \},
\]
\[
Q_s := L^2(\Omega_s).
\]

We also need the trace space \( H^{1/2,0}_{00}(\Gamma) := X_s|_\Gamma \) (resp. \( H^{1/2}_{00}(\Gamma) := X_d|_\Gamma \)), which is a nonclosed subspace of \( H^{1/2}(\Gamma) \) (resp. \( H^{1/2}(\Gamma)_s \)) and has a continuous zero extension to \( H^{1/2}(\partial \Omega_s) \) (resp. \( H^{1/2}(\partial \Omega_d) \)) [14, 15, 20]. For the trace space \( H^{1/2}_{00}(\Gamma) \) and its dual space \( (H^{1/2}_{00}(\Gamma))^\prime \), we have the following continuous imbedding result [15]:
\[
H^{1/2}_{00}(\Gamma) \subset \subset H^{1/2}(\Gamma) \subset \subset L^2(\Gamma) \subset \subset H^{-1/2}(\Gamma) \subset \subset (H^{1/2}_{00}(\Gamma))^\prime.
\]
(2.7)

One can see more details in [15, 34, 45] and the references therein. For any bounded domain \( \mathcal{D} \subset \mathbb{R}^N \), \( (\cdot, \cdot)_\mathcal{D} \) denotes the \( L^2 \) inner product on \( \mathcal{D} \), and \( (\cdot, \cdot)_{\partial \mathcal{D}} \) denotes the \( L^2 \) inner product (or duality pairing) on the boundary \( \partial \mathcal{D} \). We also consider the following product Hilbert space \( \mathbf{Y} := X_s \times X_d \times X_d \) with norm
\[
\| \mathbf{w} \|_{\mathbf{Y}} = \left( \| w_s \|_{1,\Omega_s}^2 + \| \psi_m \|_{1,\Omega_d}^2 + \| \psi_f \|_{1,\Omega_d}^2 \right)^{1/2}, \quad \forall \mathbf{w} = (w_s, \psi_m, \psi_f) \in \mathbf{Y}.
\]

In addition, based on the following formula:
\[
((u_\cdot \nabla)v,w)_\mathcal{D} = (u_\cdot n,v \cdot w)_{\partial \mathcal{D}} - ((u_\cdot \nabla)w,v)_{\mathcal{D}} - ((\nabla \cdot u)v,w)_{\mathcal{D}}, \quad \forall u,v,w \in H^1(\mathcal{D}),
\]
we introduce the trilinear form \( b(\cdot; \cdot; \cdot) \) given by \( \forall u_s,v_s,w_s \in X_s \),
\begin{equation}
\begin{aligned}
b(u_s; v_s, w_s) &= ((u_s \cdot \nabla) v_s, w_s)_{\Omega_s} + \frac{1}{2} ((\nabla \cdot u_s) v_s, w_s)_{\Omega_s} \\
&= \frac{1}{2} (u_s \cdot n_s, v_s \cdot w_s)_{\Gamma} + \frac{1}{2} ((u_s \cdot \nabla) v_s, w_s)_{\Omega_s} - \frac{1}{2} ((u_s \cdot \nabla) w_s, v_s)_{\Omega_s}.
\end{aligned}
\end{equation}

Hence, the Galerkin variational formulation of the coupled problem (2.1)–(2.6) is proposed that: to find \((u, p_s) \in Y \times Q_s\) such that

\begin{equation}
a(u, v) + d(v, p_s) - d(u, q_s) + b(u; u_s, v_s) = \rho(f_s, v_s)_{\Omega_s} + (f_d, \psi_f)_{\Omega_d}, \quad \forall (v, q_s) \in Y \times Q_s,
\end{equation}

where \(u = (u_s, \phi_m, \phi_f), v = (v_s, \psi_m, \psi_f)\), the bilinear forms \(a(\cdot, \cdot)\) and \(d(\cdot, \cdot)\) are defined as

\begin{align*}
a_s(u, v) &= 2 \rho \nu (\mathbb{D}(u_s), \mathbb{D}(v_s))_{\Omega_s}, \\
a_d(u, v) &= \frac{k_m}{\mu} (\nabla \phi_m, \nabla \psi_m)_{\Omega_d} + \frac{k_f}{\mu} (\nabla \phi_f, \nabla \psi_f)_{\Omega_d} + \frac{\sigma k_m}{\mu} (\phi_m - \phi_f, \psi_m)_{\Omega_d} + \frac{\sigma k_m}{\mu} (\phi_f - \phi_m, \psi_f)_{\Omega_d}, \\
a_{\Gamma}(u, v) &= \langle \phi_f, v_s \cdot n_s \rangle_{\Gamma} - \langle \psi_f, u_s \cdot n_s \rangle_{\Gamma} + \sum_{i=1}^{N-1} \frac{\alpha \rho \nu}{\sqrt{k_f}} \langle u_s \cdot \tau_i, v_s \cdot \tau_i \rangle_{\Gamma}, \\
a(u, v) &= a_s(u, v) + a_d(u, v) + a_{\Gamma}(u, v), \\
d(v, q_s) &= -\rho(\nabla \cdot v_s, q_s)_{\Omega_s}.
\end{align*}

**Remark 2.1** We note that the term \(\frac{1}{2} ((\nabla \cdot u_s) v_s, w_s)_{\Omega_s}\) vanishes in (2.8) if \(\nabla \cdot u_s = 0\).

**Remark 2.2** For the sake of clarity, in our analysis we shall adopt homogenous boundary conditions. In addition, for the general Dirichlet boundary conditions:

\[ u|_{\Gamma_s} = u_{\text{dir}}, \quad \phi_m|_{\Gamma_d} = \phi_{m, \text{dir}}, \quad \phi_f|_{\Gamma_d} = \phi_{f, \text{dir}}, \]

the standard homogenization technique and the lifting operators employed in [20] can be used to obtain an equivalent system with the homogeneous Dirichlet boundary conditions.

Thanks to [17, 26], we have the following important result:

**Lemma 2.1** Assume that \(f_s \in L^2(\Omega_s)\) and \(f_d \in L^2(\Omega_d)\). Then if \((u_s, \phi_f, \phi_m, p_s) \in X_s \times X_d \times X_d \times Q_s\) satisfies (2.1)–(2.6), it is also a solution to problem (2.9). Conversely, any solution of problem (2.9) satisfies (2.1)–(2.6).

### 3. An a priori estimate

In this section, we shall propose an a priori estimate for possible solutions of (2.9).

#### 3.1. Some technique inequalities

Firstly, throughout this paper we use \(C\) to denote a generic positive constant independent of discretization parameters, which may take different values in different occasions. Then, based on general Sobolev inequalities, the trace theorem and the Sobolev embedding theorem [1], we have that for any bounded open set \(D \subset \mathbb{R}^N\) with Lipschitz continuous boundary \(\partial D\) and for all \(v \in H^1(D)\),
\[\|v\|_{0,q,D} \leq C\|v\|_{1,D}, \quad 1 \leq q \leq 6, \quad (3.1)\]
\[\|v\|_{0,0,D} \leq C\|v\|_{0,0,D}^{1/2}v_{1,1,D}^{1/2} \leq C\|v\|_{1,D}, \quad (3.2)\]
\[\|v\|_{0,4,0,D} \leq C\|v\|_{1,D}, \quad (3.3)\]

which can be also found in [17, 26]. We also need Poincaré inequality and Korn’s inequality [15] that for all \(v_s \in X_s\) and for all \(\psi \in X_d,\)

\[\|v_s\|_{0,\Omega_s} \leq C|v_s|_{1,\Omega_s}, \quad (3.4)\]
\[\|\psi\|_{0,\Omega_d} \leq C|\psi|_{1,\Omega_d}, \quad (3.5)\]
\[|v_s|_{1,\Omega_s} \leq C\|\mathcal{D}(v_s)\|_{0,\Omega_s}. \quad (3.6)\]

Moreover, for the trilinear form \(b(\cdot,\cdot,\cdot),\) the following lemma holds with the help of [4, 8], (3.1), (3.4) and (3.6).

**Lemma 3.1** For any functions \(u_s, v_s, w_s \in X_s,\) we have

\[\|(u_s \cdot \nabla)v_s, w_s\|_{\Omega_s} \leq C\|u_s\|_{0,\Omega_s}^{1/2}\|\mathcal{D}(u_s)\|_{0,\Omega_s}^{1/2}\|\mathcal{D}(v_s)\|_{0,\Omega_s}\|\mathcal{D}(w_s)\|_{0,\Omega_s}, \quad (3.7)\]
\[\|\nabla \cdot u_s)v_s, w_s\|_{\Omega_s} \leq C\|v_s\|_{0,\Omega_s}^{1/2}\|\mathcal{D}(v_s)\|_{0,\Omega_s}^{1/2}\|\mathcal{D}(u_s)\|_{0,\Omega_s}\|\mathcal{D}(w_s)\|_{0,\Omega_s}. \quad (3.8)\]

**Proof.** First, let us recall the standard interpolation inequality [4, 8]: for any bounded open set \(\mathcal{D} \subset \mathbb{R}^N\) with Lipschitz continuous boundary \(\partial \mathcal{D}, 1 \leq p < q < \infty\) and \(\theta = N(1/p - 1/q) \leq 1,\)

\[\|v\|_{0,q,\mathcal{D}} \leq C\|v\|_{0,p,\mathcal{D}}^{1-\theta}\|v\|_{1,p,\mathcal{D}}^\theta, \quad \forall v \in W^{1,p}(\mathcal{D}). \quad (3.9)\]

In (3.9), we set \(\mathcal{D} = \Omega_s,\) and

- if \(N = 2, p = 2, q = 4,\) we then have
  \[\|v_s\|_{0,4,\Omega_s} \leq C\|v_s\|_{0,\Omega_s}^{1/2}\|v_s\|_{1,\Omega_s}^{1/2}, \quad \forall v_s \in X_s; \quad (3.10)\]

- if \(N = 3, p = 2, q = 3,\) we then have
  \[\|v_s\|_{0,3,\Omega_s} \leq C\|v_s\|_{0,\Omega_s}^{1/2}\|v_s\|_{1,\Omega_s}^{1/2}, \quad \forall v_s \in X_s. \quad (3.11)\]

Hence, the results (3.7) and (3.8) then follow from Hölder inequality; (3.1), (3.4), (3.6), (3.10) and (3.11) when \(N = 2\) or 3.

\[\square\]

3.2. The equivalent problem

Let us introduce the following divergence-free space:

\[V_s := \{v_s \in X_s : \nabla \cdot v_s = 0\},\]

and we consider the new product Hilbert space:

\[W := V_s \times X_d \times X_d,\]

6
which equipped with the same norm as $\mathbf{Y}$.

Thanks to [26], it provides a lower constant bound $\beta_0 > 0$ depending only on $\Omega$ to guarantee the following inf-sup condition:

$$\inf_{q_s \in Q_s} \sup_{\mathbf{v} = (\mathbf{v}_s, \mathbf{\psi}_m, \mathbf{\psi}_f) \in \mathbf{Y}} \frac{d(\mathbf{v}_s, q_s)}{\|q_s\|_{0, \Omega_s} \|\mathbf{v}\|_{\mathbf{Y}}} \geq \beta_0. \quad (3.12)$$

Hence, by (3.12) and the same argument in [25], the Galerkin variational formulation (2.9) is equivalent to the following problem: to find $\mathbf{u} \in \mathbf{W}$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}_s; \mathbf{u}_s, \mathbf{v}_s) = \rho(\mathbf{f}_s, \mathbf{v}_s)_{\Omega_s} + (f_d, \mathbf{\psi}_f)_{\Omega_d}, \quad \forall \mathbf{v} \in \mathbf{W}. \quad (3.13)$$

### 3.3. Discretization and an auxiliary problem

Let $\mathcal{R}_{s,h}$ be a quasi-uniform regular triangulation of the domain $\Omega_s$, $i = s, d$, respectively. If assuming that the two meshes $\mathcal{R}_{s,h}$ and $\mathcal{R}_{d,h}$ coincide along the interface $\Gamma$, then we can define $\mathcal{R}_h := \mathcal{R}_{s,h} \cup \Gamma \cup \mathcal{R}_{d,h}$, which is also a quasi-uniform regular triangulation of $\Omega$. The diameter of element $K \in \mathcal{R}_h$ is denoted as $h_k$, and we set the mesh parameter $h = \max_{K \in \mathcal{R}_h} h_k$.

Then we denote by $\mathbf{X}_h \subset H^1_0(\Omega)$ a finite element space defined on $\Omega$. The discrete space $\mathbf{X}_h$ can be naturally restricted to $\Omega_s$, so we set $\mathbf{X}_{s,h} = \mathbf{X}_h|_{\Omega_s} \subset \mathbf{X}_s$. Following the same technique, we establish other finite element spaces $Q_{s,h} \subset Q_s$ and $X_{d,h} \subset X_d$. We assume that $(\mathbf{X}_{s,h}, Q_{s,h})$ is a stable finite element space pair. Further we define the following vector-valued Hilbert space on $\Omega_d$:

$$\mathbf{X}_d := \{ \mathbf{v} \in H^1(\Omega_d) : \mathbf{v} = 0 \text{ on } \Gamma_d \}.$$  

In addition, we need a finite element subspace $\mathbf{V}_h \subset \mathbf{X}_h$ defined on $\Omega$ given by

$$\mathbf{V}_h := \{ v_s \in \mathbf{X}_h : d(v_s, q_{s,h}) = 0, \forall q_{s,h} \in Q_{s,h} \}.$$  

Similarly, we have

$$\mathbf{V}_{s,h} = \mathbf{V}_h|_{\Omega_s} \subset \mathbf{X}_s, \quad \mathbf{X}_{d,h} = \mathbf{V}_h|_{\Omega_d} \subset \mathbf{X}_d,$$

where $\mathbf{V}_{s,h}$ is a weakly divergence-free finite element space on $\Omega_s$, and any function $\mathbf{v}_{d,h} \in \mathbf{X}_{d,h}$ has an implicit restriction that $\int_{\Gamma} \mathbf{v}_{d,h} \cdot \mathbf{n}_d ds = 0$ because of the continuity of $\mathbf{V}_h$ across the interface $\Gamma$.

According to [26], the difficulty of obtaining an a priori estimate of the Navier-Stokes/Darcy fluid flow model comes from the energy unbalance due to the nonlinear convection term ($\mathbf{u}_s \cdot \nabla \mathbf{u}_s$) in (2.1). It motivates us to construct an auxiliary discrete Galerkin variational problem defined on $\Omega_d$ with compatible boundary conditions, so that the auxiliary problem can compensate the nonlinear convection term in the energy balance of the Navier-Stokes equations.

To fix ideas, we first define a lifting operator $\mathcal{L} : H^{1/2}_{00}(\Gamma) \rightarrow \mathbf{X}_d$ as follows: for any $\mathbf{\eta} \in H^{1/2}_{00}(\Gamma)$ with $\int_{\Gamma} \mathbf{\eta} ds = 0$ such that

$$\mathcal{L} \mathbf{\eta} \in \mathbf{X}_d, \quad \langle \mathcal{L} \mathbf{\eta} \rangle_{\Gamma} = \mathbf{\eta}, \quad \nabla \cdot (\mathcal{L} \mathbf{\eta})|_{\Omega_d} = 0.$$

Then, we introduce the Scott-Zhang interpolator $\Pi_{s,h} : \mathbf{X}_s \rightarrow \mathbf{X}_{s,h}$ satisfying the following properties [42]:

$$\|\mathbf{v}_s - \Pi_{s,h} \mathbf{v}_s\|_{0, \Omega_s} \leq C h \|\mathbf{D}(\mathbf{v}_s)\|_{0, \Omega_s}, \quad \forall \mathbf{v}_s \in \mathbf{X}_s, \quad (3.14)$$

$$\|\Pi_{s,h} \mathbf{v}_s\|_{1, \Omega_s} \leq C \|\mathbf{D}(\mathbf{v}_s)\|_{0, \Omega_s}, \quad \forall \mathbf{v}_s \in \mathbf{X}_s. \quad (3.15)$$
Now, with the mesh parameter $h$, the interpolator $\Pi_{s,h}$ and any given $u_s \in V_s$, we consider the following auxiliary discrete Galerkin variational problem: to find $u_{d,h} \in X_{d,h}$ with $u_{d,h}|_{\Gamma} = (\Pi_{s,h} u_s)|_{\Gamma}$ such that for all $v_{d,h} \in X_{d,h}$,

$$2\kappa \left( \mathbb{D}(u_{d,h}), \mathbb{D}(v_{d,h}) \right)_{\Omega_d} + \left( (u_{d,h}^0 \cdot \nabla) u_{d,h}, v_{d,h} \right)_{\Omega_d} - \kappa \left( \frac{\partial u_{d,h}}{\partial n_d}, v_{d,h} \right)_{\Gamma} = 0,$$

(3.16)

where $u_{d,h}^0 = \mathcal{L}(u_s|_{\Gamma}) \in X_d$, and $\kappa > 0$ is a certain positive constant specified later. Furthermore, the discrete problem (3.16) is uniquely solvable in $X_{d,h}$ from Lemma 3.1 in [32].

**Remark 3.1** The discrete problem (3.16) is the conforming Galerkin approximation of the following convection-diffusion equation defined on $\Omega_d$: for any given $u_s \in V_s$, to find $u_d \in X_d$ with $u_d|_{\Gamma} = u_s|_{\Gamma}$ such that

$$
\begin{cases}
-2\kappa \nabla \cdot \mathbb{D}(u_d) + (u_d^0 \cdot \nabla) u_d = 0 & \text{in } \Omega_d, \\
u_d = 0 & \text{on } \Gamma_d.
\end{cases}
$$

(3.17)

Clearly, for any constant $\kappa > 0$ and $u_s \in V_s$, the solution of (3.17) is well-posed [40].

### 3.4. An a priori estimate of weak solutions

We realize that (3.13) and (3.16) can form a larger coupled system because of the convection term $u_d^0$ and the constraint $u_{d,h}|_{\Gamma} = (\Pi_{s,h} u_s)|_{\Gamma}$ for the unknown variable $u_{d,h}$ over the interface $\Gamma$, and we also stress that the new coupled system has no energy exchange via the interface $\Gamma$. It implies that (3.16) is subjected to (3.13) but any possible solution of (3.13) has nothing to do with the auxiliary problem (3.16).

**Theorem 3.2** Assume that the data in the auxiliary discrete problem (3.16) satisfy $h$ small enough and $0 < \kappa \leq C\rho\mu h$. If problem (3.13) exists a possible solution $(u_s, \phi_m, \phi_f) \in W$, we then have the following a priori estimate that

$$\rho \nabla \mathbb{D}(u_s)\|_{0,\Omega_d}^2 + \frac{2\sigma k_m}{\mu} \|\nabla \phi_m\|_{0,\Omega_d}^2 + \frac{\sigma k_f}{\mu} \|\nabla \phi_f\|_{0,\Omega_d}^2 \leq C^2,$$

(3.18)

where

$$C^2 = \rho \mu^{-1} \|f_s\|_{0,\Omega_d}^2 + \mu \sigma^{-1} k_f^{-1} \|f_d\|_{0,\Omega_d}^2.$$

Here we stress that there are no assumptions for the data and physical parameters of the model problem.

**Proof.** We denote by $\underline{u} = (u_s, \phi_m, \phi_f) \in W$ a possible solution of problem (3.13), and denote by $u_{d,h} \in X_{d,h}$ the solution of problem (3.16). Then, we assume that there is a positive finite constant $M_s < \infty$ such that $\|\mathbb{D}(u_s)\|_{0,\Omega_d} \leq M_s$. Taking $v = \underline{u}$ in (3.13), and noting that the terms $a_{\Gamma}(\underline{u}, \underline{u})$ and $\frac{\sigma k_m}{\mu} (\phi_m - \phi_f, \phi_m)_{\Omega_d} + \frac{\sigma k_f}{\mu} (\phi_f - \phi_m, \phi_f)_{\Omega_d}$ are non-negative, we have

$$
2\rho \nabla \mathbb{D}(u_s)\|_{0,\Omega_d}^2 + \frac{\sigma k_m}{\mu} \|\nabla \phi_m\|_{0,\Omega_d}^2 + \frac{\sigma k_f}{\mu} \|\nabla \phi_f\|_{0,\Omega_d}^2 + \frac{1}{2} \left( u_s \cdot n_s, |u_s|^2 \right)_{\Gamma} \leq \rho (f_s, u_s)_{\Omega_d} + (f_d, \phi_f)_{\Omega_d}.
$$

(3.19)

In addition, we note that $\nabla \cdot u_d^0 = 0$ in $\Omega_d$, and the $u_{d,h}|_{\Gamma} = u_s|_{\Gamma}$ by the definition of $\mathcal{L}$. Hence, the identity (2.8) can also be applied to the second term in the left hand of (3.16), that is for all $v_{d,h} \in X_{d,h}$,
\[ ((u_d^0 \cdot \nabla) u_{d,h}, v_{d,h})_{\Omega_d} = (u_d^0 \cdot \nabla) u_{d,h}, v_{d,h})_{\Omega_d} + \frac{1}{2} (u_d^0 \cdot \nabla) u_{d,h}, v_{d,h})_{\Omega_d} \]
\[ = -\frac{1}{2} \left\langle u_s \cdot n_s, [\Pi_{s,h} u_s]^2 \right\rangle + \frac{1}{2} ((u_d^0 \cdot \nabla) u_{d,h}, v_{d,h})_{\Omega_d} - \frac{1}{2} ((u_d^0 \cdot \nabla) v_{d,h}, u_{d,h})_{\Omega_d}. \]  
(3.20)

So, taking \( v_{d,h} = u_{d,h} \) in (3.16) and using (3.20), we obtain
\[ 2\kappa||\mathbb{D}(u_{d,h})||_{0,\Omega_d}^2 - \frac{1}{2} \left\langle u_s \cdot n_s, [\Pi_{s,h} u_s]^2 \right\rangle = \kappa \left\langle \frac{\partial u_{d,h}}{\partial n_{d,h}}, \Pi_{s,h} u_s \right\rangle \]  
(3.21)

It follows from (3.2), (3.3), (3.4), (3.6), (3.14), (3.15), Hölder inequality and the triangle inequality that
\[ \frac{1}{2} \left\langle u_s \cdot n_s, |u_s|^2 \right\rangle - \frac{1}{2} \left\langle u_s \cdot n_s, [\Pi_{s,h} u_s]^2 \right\rangle \leq \frac{1}{2} ||u_s||_{0,\Omega}^2 ||u_s + \Pi_{s,h} u_s||_{0,\Gamma} ||u_s - \Pi_{s,h} u_s||_{0,\Gamma} \]
\[ \leq C h^{1/2} M \|\mathbb{D}(u_s)\|_{0,\Omega}. \]  
(3.22)

As follows, we now define the dual norms of \( X_s \) and \( X_d \), which are denoted as \( ||\cdot||_{-1,\Omega_s} \) and \( ||\cdot||_{-1,\Omega_d} \), respectively.
\[ ||f_s||_{-1,\Omega_s} := \inf_{v_s \in X_s} \left( \frac{\langle f_s, v_s \rangle_{\Omega_s}}{\|\mathbb{D}(v_s)\|_{0,\Omega_s}} \right), \quad ||f_d||_{-1,\Omega_d} := \inf_{\psi \in X_d} \left( \frac{\langle f_d, \psi \rangle_{\Omega_d}}{\|\nabla \psi\|_{0,\Omega_d}} \right). \]

Hence, by using Hölder inequality and Young’s inequality, we have
\[ \rho(f_s, u_s)_{\Omega_s} + (f_d, \phi_d)_{\Omega_d} \leq \frac{\rho \nu}{2} ||\mathbb{D}(u_s)||_{0,\Omega_s}^2 + \frac{\sigma K_f}{2 \mu} ||\nabla \phi_d||_{0,\Omega_d}^2 + \frac{\rho}{2 \nu} ||f_s||_{-1,\Omega_s}^2 + \frac{\mu}{2 \sigma K_f} ||f_d||_{-1,\Omega_d}^2. \]  
(3.23)

In addition, based on the imbedding result (2.7), the standard trace theorem [1], Hölder inequality, Korn’s inequality, Young’s inequality, (3.15) and the following inverse inequality [39]: for any polynomial \( v \) on \( K \),
\[ ||(\nabla v)n||_{0,e} \leq C h^{-1/2} ||v||_{1,K}, \quad \forall e \subset \partial K, \forall K \in \mathcal{R}_h, \]
we have
\[ \kappa \left\langle \frac{\partial u_{d,h}}{\partial n_{d,h}}, \Pi_{s,h} u_s \right\rangle \leq \kappa \left\langle (\nabla u_{d,h})n_d, (H^{1/2}_0(\Gamma)) \right\rangle \left\langle \Pi_{s,h} u_s, H^{1/2}_0(\Gamma) \right\rangle \]
\[ \leq C \kappa \left\langle (\nabla u_{d,h})n_d, \Pi_{s,h} u_s \right\rangle \leq C \kappa \left( \sum_{e \in \Gamma} \left\langle (\nabla u_{d,h})n, |e| \right\rangle \right)^{1/2} ||\mathbb{D}(u_s)||_{0,\Omega_s} \]
\[ \leq C h^{-1/2} ||\mathbb{D}(u_{d,h})||_{0,\Omega_d} ||\mathbb{D}(u_s)||_{0,\Omega_s} \leq C h^{1/2} M ||\mathbb{D}(u_{d,h})||_{0,\Omega_d} + \frac{\rho \nu}{2} ||\mathbb{D}(u_s)||_{0,\Omega_s}. \]  
(3.24)

Finally, with the assumptions that \( h \) is small enough such that \( Ch^{1/2} M < \rho \nu / 2 \), and \( \kappa \) satisfies \( 0 < \kappa \leq C \rho h \), gathering (3.19), (3.21), (3.22), (3.23) and (3.24) yields (3.18), where
\[ C^2 = \rho \nu^{-1} ||f_s||_{-1,\Omega_s}^2 + \mu \sigma^{-1} K_f^{-1} ||f_d||_{-1,\Omega_d}^2. \]  
\[ \square \]
4. Existence and global uniqueness of the solution

In this section, we shall use the technique of the Galerkin method to verify that problem (3.13) has at least one weak solution, and then we can prove the global uniqueness of the weak solution due to the a priori estimate (3.18) obtained in Section 3.

4.1. The solvability of the conforming Galerkin approximation problem

We denote by $W_h$ the product finite element space $V_{s,h} \times X_{d,h} \times X_{d,h}$. Then, we consider the following problem (3.13): to find $\mathbf{u}_h = (u_{s,h}, \phi_{m,h}, \phi_{f,h}) \in W_h$ such that $\forall \mathbf{v}_h = (v_{s,h}, \psi_{m,h}, \psi_{f,h}) \in W_h$,

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(u_{s,h}, u_{s,h}, v_{s,h}) = \rho(f_s, v_{s,h})\Omega_s + (f_d, \psi_{f,h})\Omega_d.$$  (4.1)

To show the solvability of (4.1), stemming from Theorem 3.2, we shall consider the following constructed coupled discrete system: to find $(\mathbf{u}_{h}, \mathbf{u}_{d,h}) \in W_h \times X_{d,h}$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(u_{s,h}, u_{s,h}, v_{s,h}) = \rho(f_s, v_{s,h})\Omega_s + (f_d, \psi_{f,h})\Omega_d \quad \forall \mathbf{v}_h \in W_h,$$  (4.2)

$$2\xi \left( \mathbf{D}(u_{d,h}), \nabla \mathbf{v}_{d,h} \right)_{\Omega_d} = 0 \quad \forall \mathbf{v}_{d,h} \in X_{d,h},$$  (4.3)

$$u_{d,h}|_{\Gamma} = u_{s,h}|_{\Gamma},$$  (4.4)

where $\xi > 0$ is a positive constant specified later, $\mathbf{D} := \mathcal{L}(u_{s,h}|_{\Gamma})$, and $\nabla \cdot \mathbf{D} = 0$ in $\Omega_d$ by the definition of the lifting operator $\mathcal{L}$. Furthermore, it follows the head statements of Section 3.4 that if $(\mathbf{u}_h, \mathbf{u}_{d,h}) \in W_h \times X_{d,h}$ is a solution of problem (4.2)–(4.4), then $\mathbf{u}_h \in W_h$ will solve (4.1).

Lemma 4.1 If problem (4.1) exists a possible solution $\mathbf{u}_h = (u_{s,h}, \phi_{m,h}, \phi_{f,h}) \in W_h$, we have the following a priori estimate that

$$\rho \nu \|\mathbf{D}(u_{s,h})\|_{\Omega_s}^2 + \frac{2\sigma_k m}{\mu} \|\nabla \phi_{m,h}\|_{\Omega_s}^2 + \frac{\sigma_k f}{\mu} \|\nabla \phi_{f,h}\|_{\Omega_d}^2 \leq C^2,$$  (4.5)

where $C^2$ is defined in Theorem 3.2.

PROOF. The proof is quite close to the proof of Theorem 3.2. To avoid repeating, we just present the differences. Since $\mathbf{D} = \mathcal{L}(u_{s,h}|_{\Gamma})$, we have

$$\left( (\mathbf{D} \cdot \nabla) u_{d,h}, u_{d,h} \right)_{\Omega_d} = -\frac{1}{2} \left( u_{s,h} \cdot n_s, |u_{s,h}|^2 \right)_{\Gamma},$$

and thus the term (3.22) vanishes here. As a result, some subtle differences occur to the following estimate:

$$\xi \left( \left\langle \frac{\partial u_{d,h}}{\partial n_d}, u_{s,h} \right\rangle \right)_{\Gamma} \leq \frac{C^2}{2\rho \nu h} \|\mathbf{D}(u_{d,h})\|_{\Omega_d}^2 + \rho \nu \|\mathbf{D}(u_{s,h})\|_{\Omega_s}^2.$$

Thus, for any given mesh parameter $h$, the result (4.5) follows the assumption that $0 < \xi \leq C \rho \nu h$. □

Now we start to verify the solvability of problem (4.2)–(4.4). For any $\tilde{\mathbf{v}}_h \in V_h$, following the similar technique proposed in [32], we denote $\mathbf{v}_h^d = \tilde{\mathbf{v}}_h|_{\Omega_d} \in V_{d,h}$, $\mathbf{v}_h^d = \tilde{\mathbf{v}}_h|_{\Omega_d} \in X_{d,h}$, and

$$\hat{\mathbf{v}}_h := \begin{cases} \mathbf{v}_h^s & \text{in } \Omega_s, \\ \mathbf{v}_h^d & \text{in } \Omega_d. \end{cases}$$
Furthermore, the Sobolev imbedding implies that the above convergence results are strong in $L^q$.

There exists a $3$-tuple function $(u, f, v) \in \mathbb{V}$ such that $\forall \, h \in \mathbb{Z}_h$ such that

$$
(F_h \mathbf{u}_h, \mathbf{w}_h)_{\mathbb{Z}_h} := 2\mu \left( \mathbf{D}(u_h), \mathbf{D}(w_h) \right)_{\Omega} + \frac{k_m}{\mu} \left( \nabla \psi_{m,h}, \nabla v_{m,h} \right)_{\Omega} + \frac{k_f}{\mu} \left( \nabla f_{s,h}, \nabla \varphi_{f,h} \right)_{\Omega}
$$

+ $\sigma k_m \mu \left( \psi_{m,h} - \psi_{f,h}, \varphi_{m,h} \right)_{\Omega} + \sigma k_m \mu \left( \psi_{f,h} - \psi_{m,h}, \varphi_{f,h} \right)_{\Omega} + 2\xi \left( \mathbf{D}(u_h), \mathbf{D}(w_h) \right)_{\Omega}

+ \left( \mathbf{v}_h - \nabla \mathbf{v}_h \right)_{\Omega} + \frac{1}{2} \left( \left( \nabla \cdot \mathbf{v}_h \right) \mathbf{v}_h, \mathbf{w}_h \right)_{\Omega} - \rho(f_s, w_h)_{\Omega} - (f_d, \varphi_{f,h})_{\Omega}.

(4.6)

Clearly, $F_h$ define a mapping from $\mathbb{Z}_h$ into itself, and a zero of $F_h$ is a solution of the coupled system (4.2)–(4.4). Further we introduce the Brouwer’s fixed-point theorem:

**Lemma 4.2** [18] Let $U$ be a nonempty, convex, and compact subset of a normed vector space and let $F$ be a continuous mapping from $U$ into $U$. Then $F$ has at least one fixed point.

Based on Lemma 4.1, we define $U_h$ a subset of $\mathbb{Z}_h$ as:

$$
U_h := \left\{ \mathbf{u}_h \in \mathbb{Z}_h : \rho \left\| \mathbf{D}(u_h) \right\|^2_{\Omega,\Omega} + \frac{2\sigma k_m}{\mu} \left\| \nabla \psi_{m,h} \right\|^2_{\Omega,\Omega} + \frac{\sigma k_f}{\mu} \left\| \nabla f_{s,h} \right\|^2_{\Omega,\Omega} \leq C \right\},
$$

where $C^2$ is defined in Theorem 3.2. Then, taking $\mathbf{w}_h = \mathbf{u}_h \in U_h$ in (4.6), and following the steps of proving Theorem 3.2 and Lemma 4.1, we obtain that

$$
(F_h \mathbf{u}_h, \mathbf{u}_h)_{\mathbb{Z}_h} \geq 0, \quad \forall \, \mathbf{u}_h \in U_h.
$$

Hence, it follows Lemma 4.2 that there is at least one zero of $F_h$ in the ball $U_h$ centered at the origin.

Gathering all the above results, we conclude the following theorem:

**Theorem 4.3** For any given mesh parameter $h > 0$, the conforming Galerkin approximation problem (4.1) has at least one solution $(u_{s,h}, \phi_{m,h}, \phi_{f,h}) \in \mathbb{V}_{s,h} \times X_{d,h} \times X_{d,h}$ with the following estimate:

$$
\rho \left\| \mathbf{D}(u_{s,h}) \right\|^2_{\Omega,\Omega} + \frac{2\sigma k_m}{\mu} \left\| \nabla \phi_{m,h} \right\|^2_{\Omega,\Omega} + \frac{\sigma k_f}{\mu} \left\| \nabla \phi_{f,h} \right\|^2_{\Omega,\Omega} \leq C^2,
$$

(4.7)

where $C^2$ is defined in Theorem 3.2.

4.2. Existence and global uniqueness

It stems from Theorem 4.3 and the conforming property $\mathbb{V}_{s,h} \times X_{d,h} \times X_{d,h} \subset \mathbb{V}_s \times X_d \times X_d$ that there exists a $3$-tuple function $(u_s, \phi_m, \phi_f)$ in the Hilbert space $\mathbb{W} = \mathbb{V}_s \times X_d \times X_d$, and a uniformly bounded subsequence $\{(u_{s,h}, \phi_{m,h}, \phi_{f,h})\}_{h>0}$ such that

$$
\lim_{h \to 0} (u_{s,h}, \phi_{m,h}, \phi_{f,h}) = (u_s, \phi_m, \phi_f) \quad \text{weakly in} \, \mathbb{W}.
$$

(4.8)

Furthermore, the Sobolev imbedding implies that the above convergence results are strong in $L^q(\Omega)$ for any $1 \leq q < 6$ whenever $N = 2$ or $3$. In particular, by extracting another subsequence, still denoted by $h$, we obtain

$$
\lim_{h \to 0} (u_{s,h}, \phi_{m,h}, \phi_{f,h}) = (u_s, \phi_m, \phi_f) \quad \text{strongly in} \, L^2(\Omega_s) \times L^2(\Omega_d) \times L^2(\Omega_d).
$$

(4.9)
Theorem 4.4 If the data satisfies that
\[ N \left( \rho^{-1} \nu^{-2} \| f_s \|_{-1, \Omega_s} + \rho^{-3/2} \nu^{-3/2} \mu^{1/2} \sigma^{-1/2} \kappa_f^{1/2} \| f_d \|_{-1, \Omega_d} \right) < 1, \]  
the problem (3.13) then admits a unique solution \((u_s, \phi_m, \phi_f)\) in \(V_s \times X_d \times X_d\) such that
\[ \rho \nu \| D(u_s) \|^2_{0, \Omega_s} + \frac{2 \sigma k_m}{\mu} \| \nabla \phi_m \|^2_{0, \Omega_d} + \frac{\sigma k_f}{\mu} \| \nabla \phi_f \|^2_{0, \Omega_d} \leq C^2, \]
where \(C^2\) is defined in Theorem 3.2.

PROOF. For \((u_s, \phi_m, \phi_f)\) denoted as \(u\) and the subsequences \(\{(u_{s,h}, \phi_{m,h}, \phi_{f,h})\}_{h>0}\) denoted as \(\{u_h\}_{h>0}\) defined in (4.8) and (4.9), we can easily obtain
\[ \lim_{h \to 0} \{a_s(u_h, v) + a_d(u_h, v)\} = a_s(u, v) + a_d(u, v), \quad \forall \ v = (v_s, \psi_m, \psi_f) \in W, \]  
(4.11)
because of (4.8). Then, for the trace bilinear term \(a_T(u_h, v)\), it follows from Hölder inequality, (3.2), and (3.4)–(3.6) that
\[ a_T(u_h, v) = \langle \phi_f, \psi_s \cdot n_s \rangle_{\Gamma} - \langle \psi_f, (u_{s,h} - u_s) \cdot n_s \rangle_{\Gamma} + \sum_{i=1}^{N-1} \alpha \rho \nu \frac{1}{k_f^{1/2}} \langle (u_{s,h} - u_s) \cdot \tau_i, v_s \cdot \tau_i \rangle_{\Gamma} \]
\[ + \langle \phi_f, v_s \cdot n_s \rangle_{\Gamma} - \langle \psi_f, u_s \cdot n_s \rangle_{\Gamma} + \sum_{i=1}^{N-1} \alpha \rho \nu \frac{1}{k_f^{1/2}} \langle u_s \cdot \tau_i, v_s \cdot \tau_i \rangle_{\Gamma} \]
\[ \leq C \| \phi_{f,h} - \phi_f \|_{0, \Omega_d}^{1/2} \| \nabla \phi_{f,h} - \phi_f \|_{0, \Omega_d}^{1/2} \| D(v_s) \|_{0, \Omega_s} + C \| u_{s,h} - u_s \|_{0, \Omega_s} \| D(u_{s,h} - u_s) \|_{0, \Omega_s} \| \nabla \phi_f \|_{0, \Omega_d} \]
\[ + C \| u_{s,h} - u_s \|_{0, \Omega_s} \| D(u_{s,h} - u_s) \|_{0, \Omega_s} \| D(v_s) \|_{0, \Omega_s} + a_T(u, v). \]

Since the uniform boundedness of \((u_{s,h}, \phi_{m,h}, \phi_{f,h})\) shown in (4.7) and the strong \(L^2\)-convergence result (4.9), we derive that
\[ \lim_{h \to 0} a_T(u_h, v) = a_T(u, v), \quad \forall \ v \in W. \]  
(4.12)

In addition, for the limit of the trilinear form \(b(u_{s,h}, u_{s,h}, v_s)\), gathering (2.8) and Lemma 3.1 yields
\[ \left| b(u_{s,h}, u_{s,h}, v_s) - b(u_s, u_s, v_s) \right| \]
\[ \leq \left| \left( \left( u_{s,h} - u_s \right) \cdot \nabla \right) u_{s,h}, v_s \right|_{\Omega_s} + \frac{1}{2} \left| \left( \nabla \cdot u_{s,h}, \left( u_{s,h} - u_s \right) \cdot v_s \right)_{\Omega_s} \right| \]
\[ + \left| \left( u_s \cdot \nabla \right) \left( u_{s,h} - u_s \right), v_s \right|_{\Omega_s} + \frac{1}{2} \left| \left( \nabla \cdot \left( u_{s,h} - u_s \right), u_s \cdot v_s \right)_{\Omega_s} \right| \]
\[ \leq C \left\| u_{s,h} - u_s \right\|_{0, \Omega_s}^{1/2} \| D(u_{s,h} - u_s) \|_{0, \Omega_s}^{1/2} \| D(u_{s,h}) \|_{0, \Omega_s} \| D(v_s) \|_{0, \Omega_s} \]
\[ \left. \frac{I_h}{I_h} \right] + \left| \left( u_s \cdot \nabla \right) \left( u_{s,h} - u_s \right), v_s \right|_{\Omega_s} + \frac{1}{2} \left| \left( \nabla \cdot \left( u_{s,h} - u_s \right), u_s \cdot v_s \right)_{\Omega_s} \right| \].

We can easily obtain that \(\lim_{h \to 0} I_h = 0\) by (4.7) and (4.9), and \(\lim_{h \to 0} I_h = 0\) by (4.8). Therefore the following limit holds:
\[
\lim_{h \to 0} b(u_{s,h}; u_{s,h}, v) = b(u_s; u_s, v), \quad \forall v \in V_s. \tag{4.13}
\]

It follows from (4.11)–(4.13) that

\[
a(u, v) + b(u_s; u_s, v) = \rho(f_s, v)_{\Omega_s} + (f_d, \psi_f)_{\Omega_d}, \quad \forall v \in W,
\]

which implies that \( u = (u_s, \phi_m, \phi_f) \in W \) is a solution of (3.13).

Finally, we assume that there are two solutions \( u^1, u^2 \in W \) to (3.13). Then, there differences \( e_s = u^1_s - u^2_s, e_m = \phi^1_m - \phi^2_m \) and \( e_f = \phi^1_f - \phi^2_f \) satisfy that \( \forall (v_s, \psi_m, \psi_f) \in V_s \times X_d \times X_d, \)

\[
2\rho \nu (\mathcal{D}(e_s), \mathcal{D}(v_s))_{\Omega_s} + \frac{k_m}{\mu} (\nabla e_m, \nabla \psi_m)_{\Omega_d} + \frac{k_f}{\mu} (\nabla e_f, \nabla \psi_f)_{\Omega_d} + (\langle (e_s \cdot \nabla) u^1_s, v_s \rangle_{\Omega_s} + ((u^2_s \cdot \nabla)e_s, v_s)_{\Omega_s} \\
+ \frac{\sigma k_m}{\mu} (e_m - e_f, \psi_m)_{\Omega_d} + \frac{\sigma k_m}{\mu} (e_f - e_m, \psi_f)_{\Omega_d} + \langle e_f, v_s \cdot n_s \rangle_{\Gamma} - \langle e_f, \psi_f, e_s \cdot n_s \rangle_{\Gamma} + \sum_{i=1}^{N-1} \langle e_s \cdot \tau_i, v_s \cdot \tau_i \rangle_{\Gamma} = 0.
\]

Taking \( v_s = e_s, \psi_m = e_m \) and \( \psi_f = e_f \) in (4.14) and using another version of (3.7), which is

\[
\left| (v_s \cdot \nabla) w_s, z_s \right|_{\Omega_s} \leq \mathcal{N} \left| \mathcal{D}(v_s) \right|_{0,\Omega_s} \left| \mathcal{D}(w_s) \right|_{0,\Omega_s} \left| \mathcal{D}(z_s) \right|_{0,\Omega_s}, \quad \forall v_s, w_s, z_s \in X_s,
\]

we obtain

\[
[2\rho \nu - \mathcal{N} (\left| \mathcal{D}(u^1_s) \right|_{0,\Omega_s} + \left| \mathcal{D}(u^2_s) \right|_{0,\Omega_s})] \left| \mathcal{D}(e_s) \right|_{0,\Omega_s}^2 + \frac{k_m}{\mu} \left| \nabla e_m \right|_{0,\Omega_d}^2 + \frac{k_f}{\mu} \left| \nabla e_f \right|_{0,\Omega_d}^2 + \frac{\sigma k_m}{\mu} \left| e_m - e_f \right|_{\Omega_d}^2 \leq 0.
\]

Hence, if we assume (4.10) holds, then (4.16) shows the unique solution to (3.13) based on a priori estimate (3.18).

□

In order to prove the existence and uniqueness of the solution \((u, p_s) \in \mathcal{Y} \times Q_s\) to the model problem (2.9), we shall use the inf-sup condition (3.12) and the Babuška–Brezzi’s theory \([5, 9, 25, 46]\).

**Theorem 4.5** Under the assumption (4.10) of Theorem 4.4, the model problem (2.9) admits a unique solution \((u, p_s) \in \mathcal{Y} \times Q_s\) such that

\[
\left| \rho \nu \left| \mathcal{D}(u_s) \right|_{0,\Omega_s}^2 + \frac{\sigma k_m}{\mu} \left| \nabla \phi_m \right|_{0,\Omega_d}^2 + \frac{\sigma k_f}{\mu} \left| \nabla \phi_f \right|_{0,\Omega_d}^2 \right| \leq \mathcal{C}^2,
\]

\[
\left| p_s \right|_{0,\Omega_s} \leq \beta_0^{-1} \left( 2\rho^{1/2} \nu^{1/2} \mathcal{C} + \rho^{-1} \nu^{-1} \mathcal{N} \mathcal{C}^2 + \rho \left| f_s \right|_{-1,\Omega_s} + \left| f_d \right|_{-1,\Omega_d} \right),
\]

where \(\mathcal{C}^2\) is defined in Theorem 3.2.

**Proof.** For the solution \( u = (u_s, \phi_m, \phi_f) \in W \) to (3.13), the following mapping:

\[
\mathbf{v} = (v_s, \psi_m, \psi_f) \in \mathcal{Y} \mapsto a(u, v) + b(u_s; u_s, v) - \rho(f_s, v)_{\Omega_s} - (f_d, \psi_f)_{\Omega_d}
\]

defines an element \( L(\mathbf{v}) \) of the dual space \( \mathcal{Y}' \), and furthermore, \( L \) vanishes on \( W \). As a result, the inf-sup condition (3.12) implies that there exists exactly one \( p_s \in Q_s \) such that

\[
L(\mathbf{v}) = d(v_s, p_s), \quad \forall \mathbf{v} = (v_s, \psi_m, \psi_f) \in \mathcal{Y}. \tag{4.19}
\]
Therefore the fact \( \mathbf{u}_s \in X_s \) and (4.19) show that the model problem (2.9) admits a unique solution \((\mathbf{u}_s, p_s) \in Y \times Q_s\). Finally, the result (4.18) is a straightforward application of the inf-sup condition (3.12) with the help of (4.15) and (4.17).

\[\square\]

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