Construction of hyperbolic Riemann surfaces with large systoles

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Abstract

Let $S$ be a compact hyperbolic Riemann surface of genus $g \geq 2$. We call a systole a shortest simple closed geodesic in $S$ and denote by $\text{sys}(S)$ its length. Let $\text{msys}(g)$ be the maximal value that $\text{sys}(\cdot)$ can attain among the Riemann surfaces of genus $g$. We call a (globally) maximal surface $S_{\text{max}}$ a hyperbolic Riemann surface of genus $g$ whose systole has length $\text{msys}(g)$. Using cutting and pasting techniques we construct hyperbolic Riemann surfaces with large systoles from maximal surfaces. This enables us to prove:

$$\text{msys}(k(g-1)+1) > \text{msys}(g) \quad \text{for} \quad k \in \mathbb{N}\{0,1\}.$$ 

As a consequence we greatly enlarge the number of genera $g$ for which the bound $\text{msys}(g) \gtrsim \frac{4}{3}\log(g)$ is valid. Furthermore, we also prove that:

If $\text{msys}(g_2) \geq \text{msys}(g_1)$, then $\text{msys}(g_1 + g_2 - 1) > \min\left(\frac{\text{msys}(g_2)}{2}, \text{msys}(g_1)\right)$

and

$$\text{msys}(g + 1) > \frac{\text{msys}(g)}{2}.$$ 

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1 Introduction

A systole of a hyperbolic Riemann surface $S$ is a shortest simple closed geodesic. We denote by $\text{sys}(S)$ its length. Let $\text{msys}(g)$ be the value

$$\text{msys}(g) = \sup\{\text{sys}(S) \mid S \text{ Riemann surface of genus } g \geq 2\}.$$ 

Due to Mumford’s generalization of Mahler’s compactness theorem in [Mu], this supremum is a maximum. The exact value of $\text{msys}(g)$ is only known for $g = 2$. We also have the following estimates from [BS]:

There exists a universal, but unknown, constant $C > 0$ such that for all genera $g \geq 2$

$$C \cdot \log(g) \leq \text{msys}(g) \leq 2\log(4g - 2). \quad (1)$$ 

Here the upper bound follows from a simple area argument (see [BS]). The first account of a lower bound is due to Buser in [Bu1]. Here an infinite sequence of surfaces with lower bound
of order $\sqrt{\log(g)}$ is constructed. It was then shown in [BS] that there is a infinite sequence of genera $(g_k)_k$ with

$$\text{msys}(g_k) \geq \frac{4}{3} \log(g_k) - c_0$$

where $c_0$ is constant. More examples of families of hyperbolic Riemann surfaces satisfying the above inequality can be found in [KSV1] and [KSV2].

The study of surfaces whose systole length is a global or local maximum in the moduli space $\mathcal{M}_g$ of compact hyperbolic Riemann surfaces of genus $g \geq 2$ was initiated by Schmutz (see [Sc1], [Sc2] and [Sc3]). Here he also provides a number of interesting properties of these surfaces. The characterization of maximal surfaces was continued in [Ba], [Ak], [Ge] and [Pa1]. Here it was shown that

- A (locally) maximal surface of genus $g$ has at least $6g - 5$ systoles [Sc2], [Ba].
- There is only a finite number of maximal surfaces of genus $g$ [Sc2], [Ba].
- The systole function $\text{sys}(\cdot)$ is a topological Morse function on the moduli space [Ak].
- All systoles of maximal surfaces are non-separating (see [Pa1], Claim on p. 336).

An open question is, whether $\text{msys}(\cdot)$ is a monotonously increasing function with respect to the genus. Though we can not prove or disprove this result, we can at least show the following intersystolic inequalities:

**Theorem 1.1.** Let $\text{msys}(g)$ be the maximal value that $\text{sys}(\cdot)$ can attain among the hyperbolic Riemann surfaces of genus $g \geq 2$.

1. $\text{msys}(k(g - 1) + 1) > \text{msys}(g)$ for $k \in \mathbb{N}\setminus\{0, 1\}$.
2. $\text{msys}(g + 1) > \frac{\text{msys}(g)}{2}$.
3. If $\text{msys}(g_2) \geq \text{msys}(g_1)$, then $\text{msys}(g_1 + g_2 - 1) > \min\{\frac{\text{msys}(g_2)}{2}, \text{msys}(g_1)\}$.

This result is obtained by cutting and pasting maximal surfaces to construct hyperbolic Riemann surfaces with large systoles. As a result we obtain from **Theorem 1.1**-1: If $S$ is a surface of genus $g$, such that $\text{sys}(S) \geq \frac{4}{3} \log(g) - c_0$, then

$$\text{msys}(l) \geq \frac{4}{3} \log(l) - c_1$$

for all $l = k \cdot (g - 1) + 1, k \ll g$.

Furthermore, by construction, we obtain a continuous parameter family $(S_t)_{t \in (-\frac{1}{2}, \frac{1}{2})^k}$ of hyperbolic Riemann surfaces of genus $k(g - 1) + 1$, such that

$$\text{sys}(S_t) = \text{msys}(g).$$

This shows that though the systoles of these surfaces are large, these surfaces can not be maximal, as this would be a contradiction to the finiteness of the number of these surfaces.

In the following table Tab.1 we give a summary of hyperbolic Riemann surfaces of genus $g \leq 25$ with maximal known systoles. Most of these are constructed from the examples presented in [Ca], [KSV1], [Sc2] and [Sc3] using **Theorem 1.1**.
Table 1: hyperbolic Riemann surfaces with a large systole of genus $g \leq 25$.

It follows furthermore from the known examples in genus 2 and 3 and Theorem 1.1-1:

**Corollary 1.2.** Let $\text{msys}(g)$ be the maximal value that $\text{sys}(-)$ can attain among the hyperbolic Riemann surfaces of genus $g \geq 2$. Then:

$$3.06 \simeq \text{msys}(2) < \text{msys}(g) \text{ for all } g \text{ and } 3.98 \leq \text{msys}(3) < \text{msys}(g) \text{ if } g \text{ odd.}$$

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2 Construction of hyperbolic Riemann surfaces with large systoles from maximal surfaces

By abuse of notation we will denote the length of a geodesic arc by the same letter as the arc itself in the following section.

Let \( \eta \) be a simple closed geodesic on \( S \). Let \( \omega_\eta \) be the supremum of all \( w \), such that the geodesic arcs of length \( w \) emanating perpendicularly from \( \eta \) are pairwise disjoint. We define a collar \( C_w(\eta) \) around \( \eta \) of width \( w < \omega_\eta \) or cylinder by

\[
C_w(\eta) = \{ p \in M \mid \text{dist}(p, \eta) < w \}.
\]

We call a collar \( C_w(\eta) \) of width \( w = \omega_\eta \) the maximal collar of \( \eta \).

Let \( \alpha \) be a systole of a maximal hyperbolic Riemann surface \( S_{\text{max}} \) of genus \( g \). We first show:

**Lemma 2.1.** Let \( \alpha \) be a systole of a hyperbolic Riemann surface \( S \) of genus \( g \geq 2 \). Then the maximal collar \( C_{\omega_\alpha}(\alpha) \) of \( \alpha \) has width

\[
\omega_\alpha > \frac{\alpha}{4}.
\]

**Proof.** The closure \( \overline{C_{\omega_\alpha}(\alpha)} \) of the maximal collar of \( \alpha \) self-intersects in a point \( p \). There exist two geodesic arcs \( \delta' \) and \( \delta'' \) of length \( \omega_\alpha \) emanating from \( \alpha \) and perpendicular to \( \alpha \) having the endpoint \( p \) in common. These two arcs form a smooth geodesic arc \( \delta \). The endpoints of \( \delta \) on \( \alpha \) divide \( \alpha \) into two parts. We denote these two arcs by \( \alpha' \) and \( \alpha'' \). Let without loss of generality \( \alpha' \) be the shorter arc of these two. We have that

\[
\alpha' \leq \frac{\alpha}{2}.
\]

Let \( \beta \) be the simple closed geodesic in the free homotopy class of \( \alpha' \cdot \delta \). We have that

\[
\beta < \alpha' + \delta \leq \frac{\alpha}{2} + 2\omega_\alpha.
\]

Now if \( \omega_\alpha \leq \frac{\alpha}{4} \) then it follows from this inequality that \( \beta < \alpha \). A contradiction to the minimality of \( \alpha \).

Hence each systole \( \alpha \) in a hyperbolic Riemann surface \( S \) has a collar \( C_{\frac{\alpha}{4}}(\alpha) \) of width \( \frac{\alpha}{4} \), which is embedded in \( S \). From this fact we also obtain an upper bound for the length of a systole via an area argument. However, this estimate is not better than the one given in the introduction (see inequality \( 
\)

**Proof of Theorem 1.1**

1. \( \text{msys}(k(g-1)+1) > \text{msys}(g) \).

Let \( S_{\text{max}} \) be a globally maximal hyperbolic Riemann surface of genus \( g \geq 2 \). To prove the first part of the theorem we construct a new surface \( S' \) of genus \( k(g-1)+1 \) from \( S_{\text{max}} \) such that

\[
\text{msys}(g) = \text{sys}(S_{\text{max}}) = \text{sys}(S').
\]

To this end we cut open \( S_{\text{max}} \) along a systole \( \alpha \) and call the surface obtained this way \( S^c \).

All systoles of maximal surfaces are non-separating (see \[Pa1\], **Claim** on p. 336). Hence the
signature of \( S^c \) is \((g - 1, 2)\). Take \( k \) copies \((S^c_i)_{i=1,\ldots,k}\) of the surface \( S^c \) and let \( \alpha^1_i \) and \( \alpha^2_i \) be the boundary geodesics of \( S^c_i \). We identify the boundaries of the different \((S^c_i)_{i=1,\ldots,k}\) in the following way

\[
\alpha^1_1 \sim \alpha^2_2 \quad \text{and} \quad \alpha^i_{i+1} \sim \alpha^i_i \quad \text{for} \quad i = 1, \ldots, k - 1
\]

(2) choosing the \( k \) twist parameters

\[(t_j)_{j=1,\ldots,k}, \quad t_j \in (-\frac{1}{2}, \frac{1}{2})\]

(see \([Bu2]\), p. 27-30)

freely to obtain a closed surface.

We denote the surface of genus \( k(g - 1) + 1 \) obtained according to this pasting scheme as

\[S' = S^c_1 + S^c_2 + \ldots + S^c_k \mod 2\]

(see Fig. 1)

We denote by \( \alpha_i \) the image of \( \alpha^1_i \subset S_i \) in \( S' \). Due to our construction each \( \alpha_i \) has an embedded collar \( C_i \) of width \( \frac{\alpha}{4} \).

\[\text{Figure 1: Four copies } (S^c_i)_{i=1,\ldots,4} \text{ of } S^c \text{ with identified boundaries.}\]

We now show that any simple closed geodesic \( \eta \) in \( S' \) has length bigger than or equal to \( \alpha \). Therefore we distinguish two cases: either \( \eta \) intersects at least one of the \((\alpha_i)_{i=1,\ldots,k}\) transversally or not.

Consider the first case. If \( \eta \) intersects one of the \((\alpha_i)_{i=1,\ldots,k}\), say \( \alpha_j \), then, due to the geometry of hyperbolic cylinders, \( \eta \) traverses \( C_j \). Now, to be a closed curve \( \eta \) has to traverse at least two times the same cylinder \( C_j \) or has to traverse \( C_j \) and a different cylinder \( C_l \). Therefore its length is bigger than \( 2 \cdot 2\omega_\alpha = 2 \cdot \frac{\alpha}{2} = \alpha \). In this case we have that

\[\eta \geq \alpha\]

which implies that \( \text{sys}(S') = \alpha \).

In the second case, a simple closed geodesic that intersects none of the \((\alpha_i)_{i=1,\ldots,k}\) transversally
is either contained in the interior of one of the \((S^i)_{i=1,...,k}\) or is one of the \((\alpha_i)_{i=1,...,k}\). In any case we have that
\[
\eta \geq \alpha
\]
thus \(\text{sys}(S') = \alpha\).

Hence in any case we have that \(\text{msys}(g) = \text{sys}(S_{\text{max}}) = \alpha = \text{sys}(S')\).

Setting the twist parameters \(t = (t_1,..,t_k)\) arbitrarily, we see that \(S'\) belongs to a continuous family \((S_t)_{t \in (-\frac{1}{2},\frac{1}{2})^k}\) of hyperbolic Riemann surfaces of genus \(k(g-1) + 1\), such that
\[
\text{sys}(S_t) = \text{msys}(g).
\]

This shows that, though the systoles of the surfaces in this family are large, these can not be maximal, as this would be a contradiction to the finiteness of the number of these surfaces. Hence \(\text{msys}(k(g-1) + 1) > \text{sys}(S_t) = \text{msys}(g)\).

2. \(\text{msys}(g + 1) > \frac{\text{msys}(g)}{2}\).

We call a surface of signature \((1,2)\) a \(F\)-piece. Let \(b_1\) and \(b_2\) be the boundary geodesics of a \(F\)-piece. Let \(F_{\text{max}}\) be a surface whose interior systole has maximal length among all \(F\)-pieces with boundaries \(b_1\) and \(b_2\) of equal length
\[
b_1 = b_2 = b.
\]

Denote by \(s = \text{sys}(F_{\text{max}})\) the length of the interior systole of \(F_{\text{max}}\). It was shown in [Sc2] that

\[
2 \cosh\left(\frac{s}{2}\right)^3 - 3 \cosh\left(\frac{s}{2}\right)^2 - (\cosh\left(\frac{b}{2}\right) + 1) \cosh\left(\frac{s}{2}\right) - \cosh\left(\frac{b}{2}\right) = 0. \tag{3}
\]

This implies that \(\text{sys}(F_{\text{max}}) = s > \frac{b}{2}\).

To prove the second part of the theorem we construct a new surface \(S''\) of genus \(g + 1\) from a globally maximal surface \(S_{\text{max}}\) of genus \(g \geq 2\) and \(F_{\text{max}}\) such that
\[
\text{sys}(S'') \geq \frac{\text{sys}(S_{\text{max}})}{2}.
\]

To construct \(S''\), we take the surface \(S^c\) of signature \((g-1,2)\) obtained by cutting open \(S_{\text{max}}\) along a systole \(\alpha\) and paste the surface \(F'_{\text{max}}\) with boundary geodesics of length \(\alpha\) along the two boundary geodesics of \(S^c\).

We denote by \(\alpha_1\) and \(\alpha_2\) the image of the two boundary geodesics of the embedded \(F\)-piece in \(S''\), which is isometric to \(F'_{\text{max}}\). Due to our construction each \(\alpha_i\) has an embedded half-collar \(H_i \subset S^c\) of width \(\frac{\alpha}{4}\).

We now show that any simple closed geodesic \(\eta\) in \(S''\) has length bigger than \(\frac{\alpha}{2}\).

Therefore we distinguish two cases: either \(\eta\) intersects either \(\alpha_1\) or \(\alpha_2\) transversally or not. Consider the first case. If \(\eta\) intersects one of the \((\alpha_i)_{i=1,2}\), say \(\alpha_1\), transversally, then, due to the geometry of the half-collar, \(\eta\) traverses \(H_1\). Furthermore \(\eta\) has to traverse at least two times the same half-collar \(H_1\) or has to traverse \(H_1\) and \(H_2\) to be a closed curve. In this case its length is bigger than or equal to \(2 \cdot \omega_\alpha = 2 \cdot \frac{\alpha}{4} = \frac{\alpha}{2}\). Hence in this case we obtain
\[
\eta \geq \frac{\alpha}{2}
\]
and therefore $\text{sys}(S'') \geq \frac{\alpha}{2}$. (In fact this inequality is strict, because the part of $\eta$ in the F-piece must have strictly positive length.)

We now consider the second case. Any simple closed geodesic that intersects neither $\alpha_1$ nor $\alpha_2$ transversally is either contained in the interior of $S^c$, the interior of $F_{\text{max}}'$ or is one of the $(\alpha_i)_{i=1,2}$. Here we obtain the lower bound for $\eta$ from the lower bound on $\text{sys}(F_{\text{max}}')$. It follows from Equation (3) that

$$\eta > \frac{\alpha}{2}$$

thus $\text{sys}(S') \geq \frac{\alpha}{2}$.

In any case we obtain that $\text{msys}(g+1) > \text{sys}(S'') \geq \frac{\text{sys}(S_{\text{max}})}{2} = \frac{\text{msys}(g)}{2}$.

As in the previous part, the inequality in Theorem 1.1.2 is strict due to the fact that the construction does not depend on the twist parameters.

3. If $\text{msys}(g_2) \geq \text{msys}(g_1)$, then $\text{msys}(g_1 + g_2 - 1) > \min\{\frac{\text{msys}(g_2)}{2}, \text{msys}(g_1)\}$.

To prove the final statement we take two maximal surfaces $S_{\text{max}}^1$ and $S_{\text{max}}^2$ of genus $g_1$ and $g_2$, respectively. For $i \in \{1,2\}$ we cut $S_{\text{max}}^i$ open along a systole $\alpha_i$ and call the surface obtained in this way $S^c_i$. As $\text{msys}(g_2) \geq \text{msys}(g_1)$ we have that $\alpha_2 \geq \alpha_1$.

Now, we can not directly paste these surfaces together, as the boundary length is different. However, by [Pa2], Theorem 1.1., we can construct a comparison surface $S^{p1}$ for $S^{c1}$ of signature $(g_1 - 1,2)$, such that

- All interior geodesics of $S^{p1}$ are longer than $\alpha_1$.
- The boundary geodesics $\gamma_1$ and $\gamma_2$ of $S^{p1}$ have length $\alpha_2$.

We identify the open boundaries of $S^{p1}$ and $S^{c2}$ to obtain the surface $S^{12}$ of genus $g_1 + g_2 - 1$. Furthermore the two boundary geodesics $\alpha_1^2$ and $\alpha_2^2$ of $S^{c2}$ have both an embedded half-collar of width $\frac{\alpha}{2}$ in $S^{c2}$. Due to the properties of $S^{12}$ we can apply similar arguments as in the previous case of the surface $S''$ to show that

$$\text{msys}(g_1 + g_2 - 1) > \text{sys}(S^{12}) \geq \min\{\frac{\alpha_2^2}{2}, \alpha_1\} = \min\{\frac{\text{msys}(g_2)}{2}, \text{msys}(g_1)\}.$$  

This concludes the proof of Theorem 1.1. □

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