The Paradox of the Scale-Free Disks

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21 March 2022

ABSTRACT
Scale-free disks have no preferred length or time scale. The question has been raised whether such disks have a continuum of unstable linear modes or perhaps no unstable modes at all. We resolve this paradox by analysing the particular case of a gaseous, isentropic disk with a completely flat rotation curve (the Mestel disk) exactly. The heart of the matter is this: what are the correct boundary conditions to impose at the origin or central cusp? We argue that the linear stability problem is ill-posed. From any finite radius, waves reach the origin after finite time but with logarithmically divergent phase. Instabilities exist, but their pattern speeds depend upon an undetermined phase with which waves are reflected from the origin. For any definite choice of this phase, there is an infinite but discrete set of growing modes. Similar ambiguities may afflict general disk models with power-law central cusps. The ratio of growth rate to pattern speed, however, is independent of the central phase. This ratio is derived in closed form for non self-gravitating normal modes. The ratio for self-gravitating normal modes is found numerically by solving recurrence relations in Mellin-transform space.

Key words: instabilities – hydrodynamics – galaxies: kinematics and dynamics – galaxies: spiral

1 INTRODUCTION
A scale-free disk, whether gaseous or stellar, has no characteristic frequency. It is therefore puzzling how such a disk can distinguish a characteristic frequency for its marginally stable modes. This paradox was first pointed out by Zang (1976) in a famous Massachusetts Institute of Technology Ph. D. thesis (supervised by Alar Toomre), which explored the stability of the scale-free stellar dynamical disks with completely flat rotation curves. Zang showed that if a scale-free disk admits one unstable mode, then a two-dimensional continuum can be constructed by self-similar scaling. Thinking this improbable, Zang argued that it is more likely that scale-free disks admit no normal modes whatsoever. If true, this is astonishing, as it must hold good even for the completely cold disk. Subsequently, the problem was re-visited by Lynden-Bell & Lemos (1993), who took the opposite view and argued that all the modes become unstable together. The possibility that an entire continuum of modes, each with a different frequency, might all become simultaneously unstable had earlier been envisaged by Birkhoff in his classic 1960 textbook on Hydrodynamics. If true, this also is astonishing, as the disks would become unstable at all scales to normal modes of all frequencies.

This paper resolves the paradox by showing how to construct an infinite but discrete set of normal modes. Section 2 introduces the disks under scrutiny and sets up the mode equations for the perturbed quantities. Section 3 examines the easier but still ambiguous problem of modes without self-gravity, whereas Section 4 deals with the fully self-gravitating case.

2 EQUILIBRIA AND PERTURBATIONS
2.1 Equilibria
Our disks are scale-free in every respect. For definiteness, we suppose that they are razor-thin and fully self-gravitating in equilibrium. In fact, these properties are inessential as long as there is no preferred scale. Wherever possible, we use the same notation as in the recent stability analysis of the stellar dynamical power-law disks carried out by Evans & Read (1998a,b). Some basic formulae are repeated here for convenience and completeness.

In equilibrium, the surface density and self-consistent potential are
\begin{align}
\Sigma_{eq}(R) &= \Sigma_0 \left( \frac{R}{R_0} \right)^{-\beta-1}, \\
\psi_{eq}(R) &= -\frac{v_0^2}{\beta} \left( \frac{R}{R_0} \right)^{-\beta},
\end{align}

in which $v_0$ is a constant reference velocity:
\[ v_\beta^2 = 2\pi G\Sigma_0 R_0 \frac{\Gamma[\frac{1}{2}(1 - \beta)] \Gamma[\frac{1}{2}(2 + \beta)]}{\Gamma[\frac{1}{2}(1 + \beta)] \Gamma[\frac{1}{2}(2 - \beta)]}. \]  

Here, \( \psi \) is a positive quantity; the gravitational acceleration is \( \nabla \psi \). As \( \beta \to 0 \), \( v_\beta^2 \to 2\pi G\Sigma_0 R_0 \equiv v_0^2 \), and the potential becomes that of the Mestel disk \( \text{Mestel 1963} \). \[ \psi_{eq}(R) = -v_0^2 \ln \left( \frac{R}{R_0} \right). \]  

The reference radius \( R_0 \) and surface density \( \Sigma_0 \) appear for the sake of dimensional consistency but do not introduce preferred scales, since the replacements \( R_0 \to s R_0 \), \( \Sigma_0 \to s^{-\beta-1} \Sigma_0 \), \( v_\beta^2 \to s^{-\beta} v_\beta^2 \), \( v_R \to 0 \) \( \Delta \psi \), \( \psi_{eq} \), \( \psi_{circ} \), \( \psi_{circ\theta} \), and \( \psi_{circ\varphi} \), preserve the form of equations \( \text{[1]} \) - \( \text{[3]} \) [except in the case \( \beta = 0 \), where the potential \( \text{[3]} \) gains a physically unimportant additive constant]. \[ \psi_{eq}(R) \to -v_0^2 \ln \left( \frac{R}{R_0} \right). \]  

The disks studied here are gaseous and isentropic, unlike their stellar dynamical counterparts explored by Evans \& Read (1998a,b). The axisymmetric and the neutral modes of the gaseous power-law disks have already been the subject of recent attention (e.g., Schmitz \& Ebert 1987; Lemos, Kalnajs \& Lynden-Bell 1991; Syer \& Tremaine 1996). The square of the sound speed varies with radius as \[ c^2 = \frac{v_\beta^2}{\beta + 1} \left( \frac{R}{R_0} \right)^{-\beta}. \]  

For convenience, let us introduce a dimensionless “temperature” \[ \Theta \equiv \frac{c^2}{|v_{\psi_{eq}}/d \ln R|}. \]  

Radial hydrostatic equilibrium requires the square of the circular velocity to be \[ v_{circ}^2 = (1 - \Theta) v_\beta^2 \left( \frac{R}{R_0} \right)^{-\beta}. \]  

The Mach number is defined as: \[ M \equiv \frac{v_{circ}}{c} = \sqrt{1 - \Theta / \Theta}. \]  

The angular velocity and epicyclic frequency, \[ \Omega \equiv \frac{v_{circ}}{R}, \quad \kappa \equiv \left[ \frac{1}{R} \frac{d}{dR} (R v_{circ})^2 \right]^{1/2}, \]  

both scale as \( R^{-\beta/2 - 1} \), and their ratio is \[ \frac{\kappa}{\Omega} = \sqrt{2 - \beta}. \]  

We assume \( -1 \leq \beta < 1 \) so that \( \kappa \) is always real. Toomre’s condition \( Q \geq 1 \) for local axisymmetric stability (Toomre 1964) relate to \( \Theta \) by \[ Q \equiv \frac{c_0}{\pi G \Sigma} = 2 \left( \frac{v_\beta^2}{2\pi G \Sigma_0 R_0} \right) \sqrt{(2 - \beta) \Theta (1 - \Theta)}. \]  

In terms of the Mach number, this becomes: \[ Q = 2\sqrt{2 - \beta} \left( \frac{v_\beta^2}{2\pi G \Sigma_0 R_0} \right) \frac{M}{1 + M^2}. \]  

The factor in parentheses, which can be read off from equation \( \text{[4]} \), reduces to unity in the Mestel disk (\( \beta = 0 \)).

### 2.2 Perturbations

We consider time-dependent first-order Eulerian perturbations of the equilibria described above and adopt cylindrical polar coordinates \( (R, \theta, z) \). Thus, \[ \Sigma \to \Sigma_{eq}(R) + \delta \Sigma(R, \theta, t), \quad \psi \to \psi_{eq}(R) + \delta \psi(R, \theta, z, t), \quad v_R \to 0 + \delta v_R(R, \theta, t), \quad v_\theta \to v_{circ} + \delta v_\theta(R, \theta, t). \]  

Strictly speaking, \( \psi_{eq} \) depends upon \( z \) as well as \( R \), but we have no occasion to evaluate it outside the plane \( z = 0 \). As the equilibrium is independent of angle \( \theta \) and time \( t \), the stability analysis is simplified by assuming that the perturbed quantities have the \( (\theta, t) \) dependence characteristic of normal modes \[ \delta \propto e^{i \theta - \omega t}. \]  

Henceforth, we take this factor as read and often write, for example, \( \delta \Sigma(R) \) rather than \( \delta \Sigma(R, \theta, t) \).

The linearized dynamical equations are [cf. Binney \& Tremaine, 1987, chap. 5] \[ -i \omega \delta v_R - 2\Omega \delta v_\theta = \frac{\partial}{\partial R} \left( \delta \psi - \frac{c_0^2 \delta \Sigma}{\Sigma} \right), \]
\[ -i \omega \delta v_\theta + \frac{\kappa^2}{2 H} \delta v_R = \frac{im}{R} \left( \delta \psi - \frac{c_0^2 \delta \Sigma}{\Sigma} \right), \]
\[ -i \omega \delta \Sigma - \beta \delta v_R \frac{\partial}{\partial R} = \frac{\partial}{\partial R} (\delta v_R - \frac{im}{R} \delta v_\theta), \]  

together with Poisson’s equation \[ \nabla^2 \delta \psi = 4\pi G \delta \Sigma \delta(z). \]  

We have used the abbreviation \( \tilde{\omega} \equiv \omega - m \Omega \) for the Doppler-shifted frequency in the local rest frame of the gas. Taking the curl of the first two of equations \( \text{[14]} \) and using the third to eliminate the derivative of \( \delta v_R \) produces \[ i \omega \left[ \frac{1}{R} \frac{\partial}{\partial R} (R \delta v_\theta) - \frac{im}{R} \delta v_R - \frac{\kappa^2}{2 H} \frac{\partial}{\partial R} \left( \frac{\Sigma^2}{\Omega^2} \right) \right] = \delta \Sigma \frac{d}{dR} \left( \frac{\Sigma^2}{\Omega^2} \right). \]  

The quantity \[ \zeta_{eq} \equiv \frac{\kappa^2}{2 H \Sigma} \]  

is the equilibrium distribution of potential vorticity (or the curl of the velocity divided by surface density). Equation \( \text{[17]} \) states that fluid elements conserve their potential vorticity in a Lagrangian sense. This is easy to understand physically, as the fluid elements must preserve both their mass and their circulation (Kelvin’s theorem).

We expect a continuum of modes involving a net change in potential vorticity. Such modes will be discontinuous in \( \delta v_R \) at the corotation radius where \( \tilde{\omega} = 0 \). This can be anywhere in the disk. The eigenfrequencies \( \{ \omega \} \) are continuously distributed. We are not interested in these modes because they can never be unstable (\( \omega \) must be real). If any such mode were to grow or decay, the net potential vorticity of the...
disk would have to change, which is not possible for our inviscid equations. The issue of interest to us is whether there exist growing, non-vortical modes; that is, disturbances with the \((\theta, t)\) dependence \((13)\) and vanishing total potential vorticity. The equations governing the non-vortical modes simplify in the Mestel disk because \(\zeta_c \propto R^{3/2}\), which is constant when \(\beta = 0\) so that equation \((17)\) reduces to
\[
\frac{1}{R} \frac{\partial}{\partial R} \left( R \delta v_R \right) - \frac{im}{R} \delta v_R - \frac{\kappa^2}{2\Omega} \frac{\delta \Sigma}{\Sigma} = 0. \tag{19}
\]
Henceforth, we restrict our attention to the Mestel disk.

3 MODES WITHOUT SELF-GRAVITY

In this section, on top of the restriction to \(\beta = 0\), we neglect the perturbation in the gravitational potential: that is, we force \(\delta \psi / \rho \rightarrow 0\). This is called the “Cowling approximation” in the context of stellar oscillations, where it is justified by the central concentration of the star’s mass or by the short wavelength of the modes of interest. Similar justifications could be offered for the disk modes in the limit \(\beta = 1\) or for general \(\beta\) in the limit of large \(M\). But we invoke neither limit, because we make no pretense of quantitative accuracy until Section 4. The true justification for neglecting the gravitational perturbation is that we thereby obtain equations that can be solved exactly. This simplified problem retains the elements that make the stability problem ambiguous in the self-gravitating case.

3.1 A Second-Order Differential Equation

With \(\delta \psi / \rho\) neglected, the second of equations \((14)\) and equation \((19)\) can be solved for \(\delta v_R\) and \(\delta \Sigma\) in terms of the variable
\[
y = R \delta v \tag{20}
\]
and its radial derivative:
\[
\delta v_R = - \frac{i}{D} \left( m c^2 \frac{\partial y}{\partial R} - \omega \delta v_{\text{circ}} y \right),
\]
\[
\frac{\delta \Sigma}{\Sigma} = \frac{1}{D} \left( \delta v_{\text{circ}} \frac{\partial y}{\partial R} + m \omega \delta v \right), \tag{21}
\]
where the denominator
\[
D \equiv v_{\text{circ}}^2 + m^2 c^2 \tag{22}
\]
is constant. Eliminating \(\delta v_R\) and \(\delta \Sigma\) from the third of equations \((14)\) produces a second-order differential equation for \(y\), which we write in the equivalent forms
\[
\frac{\partial^2 y}{\partial R^2} + \left( \frac{\omega^2 - \kappa^2}{c^2} - \frac{m^2}{R^2} \right) y = 0,
\]
\[
\frac{\partial^2 y}{\partial R^2} + \left( \frac{(\omega - m \Omega)^2 - 2\kappa^2}{c^2} - \frac{m^2}{R^2} \right) y = 0. \tag{23}
\]
Equation \((23)\) is non-singular. In the general \(\beta \neq 0\) case, second-order equations can also be obtained for the various fluid variables, but at least one coefficient will be singular at corotation \((\omega \rightarrow 0)\) because of the non-zero potential vorticity gradient. In all other respects, equation \((23)\) is typical of the general case. The wave is evanescent between the Lindblad resonances, \(\omega = \pm \kappa\) or \(\omega = m \Omega \pm \kappa\). In fact, the WKBJ radial wavenumber becomes real somewhat beyond the Lindblad resonances (further from corotation) because of the \(\omega^2 / c^2\) term. If we had allowed for the self-gravity of the mode, propagation would be possible between the Lindblad resonances but not in the immediate vicinity of corotation unless \(Q < 1\) (see, for example, Binney & Tremaine 1987 or Shin 1992).

Equation \((23)\) can be reduced to a confluent hypergeometric function. Let us make the following transformations of the independent and dependent variables:
\[
z \equiv \frac{2i \omega}{c} R, \quad y \equiv R^\pm \text{e}^{i \omega R / c} w(z) \propto z^{i \mu} \text{e}^{-z^2 / 2} w(z), \tag{24}
\]
where \(\mu\) is the following dimensionless function of \(m\) and the Mach number \(\delta\):
\[
\mu \equiv \frac{1}{2} \left[ 4(m^2 - 2) \mathcal{M}^2 - 4 m^2 - 1 \right]^{1/2}. \tag{25}
\]
There should be no confusion between the dimensionless variable \(z\) and the original cylindrical polar coordinate of the same name. Equation \((23)\) becomes
\[
z \frac{d^2 w}{dz^2} + \left( 1 + 2i \mu - z \right) \frac{dw}{dz} - \left( i \mu + im \mathcal{M} + \frac{1}{2} \right) w = 0, \tag{26}
\]
which is Kummer’s equation (Abramowitz & Stegun 1970). The solution regular at \(z = 0\) is Kummer’s function \(M(\frac{1}{2}, \frac{1}{2} + i \mu + im \mathcal{M}, 1 + 2i \mu, z)\). In fact, if one uses \(-\mu\) instead of \(\mu\) in the transformation \((24)\) of the dependent variable, one obtains an equation identical to \((23)\) except in the sign of \(\mu\), whose regular solution is \(M(\frac{1}{2}, \frac{1}{2} - i \mu + im \mathcal{M}, 1 + 2i \mu, z)\). Hence, the two independent solutions of equation \((23)\) are
\[
y_\pm(z) \equiv z^{i \mu} \text{e}^{-z^2 / 2} M \left( \frac{1}{2} \pm i \mu + im \mathcal{M}, 1 \pm 2i \mu, z \right). \tag{27}
\]
Let us note that Drury (1980) already derived a differential equation equivalent to \((23)\) and used it to find the reflection and transmission coefficients for the Mestel disk. He did not, however, construct the normal modes, which we now proceed to do.

3.2 Boundary Conditions

The desired eigenfunction is a linear combination of the independent solutions \((27)\) that satisfies appropriate boundary conditions at large and small \(R\). The boundary condition at large radius is straightforward. We insist on purely outgoing disturbances, otherwise it is all too easy to manufacture counterfeit instabilities whose exponential “growth” is due solely to increasingly powerful transmissions from sources at \(R \rightarrow \infty\). One can see directly from the second form of equation \((23)\) that the radial wavenumber \(k_R \rightarrow \pm \omega / c\) as \(R \rightarrow \infty\) for any value of \(m\) provided only that \(\text{Real}(\omega) \neq 0\). The outgoing wave is proportional to
\[
e^{i \omega R / c} \propto e^{-z^2 / 2},
\]
and the signs of the exponents are reversed for the ingoing wave. Using standard results (Abramowitz & Stegun 1970), one sees that the asymptotic behavior of \(y_\pm(z)\) as \(|z| \rightarrow \infty\) in the cut plane \(-\frac{\pi}{2} < \text{arg}(z) < \frac{\pi}{2}\) is
\[
y_\pm(z) \sim \Gamma(1 \pm 2i \mu) \left[ \frac{e^{i (\frac{1}{2} + im \mathcal{M}) z} - e^{-z^2 / 2}}{\Gamma(\frac{1}{2} \pm i \mu - im \mathcal{M})} \right]. \tag{28}
\]
we have to cancel the term in $e^{+z/2}$, so the desired linear combination is
\[(28)\]
\[y(z) = \frac{\Gamma(1 - 2i\mu)y_{\text{in}}(z)}{\Gamma(\frac{1}{2} - i\mu + \frac{m\Omega}{m})} - \frac{\Gamma(1 + 2i\mu)y_{\text{out}}(z)}{\Gamma(\frac{1}{2} + i\mu + \frac{m\Omega}{m})},\]
or any fixed multiple thereof.

The boundary condition at the origin requires more thought. First, it is important to realize that an ingoing wave reaches the origin – where there is a density cusp – in finite time. Propagation to the very centre requires $m > m/\sqrt{m^2 - 2}$, or equivalently $0 \leq \Theta < (m^2 - 2)/(2m^2 + 2)$. This is never possible for $m = 1$ disturbances when self-gravity is neglected, but can happen in sufficiently cold disks for all other azimuthal wavenumbers. The WKBJ dispersion relation implied by equation $\text{(23)}$ is
\[(30)\]
k_{R}^{2} = \frac{(\omega - m\Omega)^{2} - 2\Omega^{2}}{c^{2}} = \frac{m^{2} \omega^{2}}{R^{2}}.

At sufficiently small radius, where $\Omega(R) \gg |\omega|$, this reduces to
\[(31)\]
k_{R}^{2} \approx \frac{m^{2} - 2}{R^{2}} - \frac{2mM}{Rc} \omega + O(\omega^{2}).

The numerator of the first term on the right is almost $4\mu^{2}$ [cf. eq. $\text{(23)}$]. A comparison with the exact solutions $\text{(24)}$ shows indeed that it should be $4\mu^{2}$. Asymptotically, therefore, the group velocity is
\[(32)\]
V_{g} \equiv \frac{\partial \omega}{\partial k_{R}} \approx \pm \frac{2\mu}{m \Omega} c,

the upper signs applies if $k_{R} > 0$, and we have assumed $m \geq 2$. Thus, the propagation speed is of order the sound speed, and an ingoing wave can reach the centre in finite time.

Allowance must therefore be made for a reflected wave. By analogy with the situation at large $R$, we require that the origin must neither absorb nor emit wave energy. We proceed to implement this principle.

At sufficiently small $R$, the Kummer functions $M(\frac{1}{2} \pm i\mu + \frac{m\Omega}{m}, 1 \pm 2i\mu, z) \rightarrow 1$, and we can read off the ingoing and outgoing parts of the general solution $\text{(24)}$ that is compatible with the large-$R$ boundary condition:
\[(33)\]
y_{\text{in}} \sim - \frac{\Gamma(1 + 2i\mu)}{\Gamma(\frac{1}{2} + i\mu + \frac{m\Omega}{m})} \left( \frac{2i\omega}{c} R \right)^{-i\mu + \frac{1}{2}},
y_{\text{out}} \sim \frac{\Gamma(1 - 2i\mu)}{\Gamma(\frac{1}{2} - i\mu + \frac{m\Omega}{m})} \left( - \frac{2i\omega}{c} R \right)^{i\mu + \frac{1}{2}}.

(In this paper, the symbol “$\sim$” means “asymptotically approaches”. ) Let us define the flux as the total cross section of a cylinder centred on the rotation axis. Then, the fluxes carried by $y_{\text{in}}$ and $y_{\text{out}}$ are
\[(34)\]
F_{\text{in}} = C_{m,\omega}(R)|y_{\text{in}}|^{2} \quad \text{and} \quad F_{\text{out}} = -C_{m,\omega}(R)|y_{\text{out}}|^{2},
\[C_{m,\omega}(R) \equiv \frac{\mu \Sigma(R) \text{Real}(\omega)}{M^{2} + m^{2}},\]
in the limit $\Omega \gg |\omega|$. The corresponding angular momentum fluxes can be obtained by dividing by the pattern speed $\Omega_{p} = \text{Real}(\omega)/m$. The coefficient $C_{m,\omega}(R)$ can be derived by adapting results from Appendix A of Narayan, Goldreich \\
\[& \text{& Goodman (1987), but for our purposes, all that matters is that the ratio of these fluxes is}
\]
\[
F_{\text{in}} \quad \text{and} \quad F_{\text{out}} \quad \text{is}
\]
\[
\frac{|y_{\text{in}}|^{2}}{|y_{\text{out}}|^{2}} \rightarrow 1 \quad \text{as} \quad R \rightarrow 0.
\]

\textbf{3.3 Dispersion Relation and Growth Rates}

The ratio of the moduli of ingoing and outgoing parts of the eigenfunction $\text{(23)}$ is
\[(35)\]
\[
\frac{|y_{\text{out}}|}{|y_{\text{in}}|} = \exp\left[\mu\pi - 2\mu \arg(\omega)\right] \left( \frac{\Gamma\left(\frac{1}{2} + im\Omega + i\mu\right)}{\Gamma\left(\frac{1}{2} + im\Omega - i\mu\right)} \right).
\]

Therefore, the inner boundary condition $\text{(33)}$ requires
\[
\arg(\omega) \equiv \tan^{-1}\left( \frac{s}{m\Omega_{p}} \right) = \frac{\pi}{2} - \frac{1}{4\mu} \ln \left( \frac{\cosh \pi(m\Omega_{p} - \mu)}{\cosh \pi(m\Omega_{p} + \mu)} \right).
\]

Here, the complex eigenfrequency $\omega$ has been written as $m\Omega_{p} + is$, where $s$ is the growth rate and $\Omega_{p}$ is the pattern speed of the mode. We have used some identities $\text{(Abramowitz & Stegun 1970)}$ to reduce moduli of complex gamma functions to elementary functions in the dispersion relation $\text{(23)}$. Fig. 1 shows the variation of the ratio of growth rate to pattern speed with the hotness of the disks for various azimuthal wavenumbers. For any temperature or Mach number, only this ratio is fixed by the dispersion relation. The actual values of $s$ or $\Omega_{p}$ are indeterminate. This is because there is insufficient physical information to deduce the phase shift of waves reflected from the origin.
It may help to give an example of the sort of inner boundary condition that would fix the required phase and thereby yield a definite discrete spectrum of modes. Suppose the origin is enclosed by a rigid wall of small radius $R_0 \ll \omega_{\text{circ}}/\omega$. Then since no gas penetrates the wall from outside, $\delta u = 0$ at $R = R_0$. Using eq. (21) to write $\delta u$ in terms of $y = y_{\text{in}} + y_{\text{out}}$, and using eq. (33) to estimate $\partial y / \partial R$, one finds the boundary condition

$$\frac{y_{\text{out}}(R_0)}{y_{\text{in}}(R_0)} = \frac{\frac{1}{2} + mM - i\mu}{\frac{1}{2} + mM + i\mu} \equiv e^{i\phi_0},$$

(38)

to leading order in $R_0 \ll \omega_{\text{circ}}/\omega$. Notice that the intermediate expression above indeed has unit modulus, so that $\phi_0$ is real: this is a consequence of the fact that a rigid wall absorbs no mechanical energy. On the other hand, the asymptotic expressions (33) yield

$$\arg \left( \frac{y_{\text{out}}}{y_{\text{in}}} \right) = F(m,M) + 2\mu \ln \frac{2\omega R}{c},$$

(39)

where $F$ is independent of $R$ and $\omega$. Comparing these two equations, we find that

$$\ln |\omega| = \phi_0 + 2\pi n - F(m,M) - \ln R_0,$$

(40)

where $n$ can be any integer. Thus the eigenfrequencies form a discrete geometric progression: $\omega_{n+1} \sim e^{\pi / \omega_{\text{circ}}}$. The actual frequencies depend upon the choice of boundary radius $R_0$, although the ratio of their real to imaginary parts does not. For a different kind of physical boundary condition (perhaps a free inner edge, for example), one would have a different $\phi_0$ and hence a different correspondence between $R_0$ and the eigenfrequencies. However, $\arg(\omega_n)$ is always given by eq. (37), provided $\omega_n \ll \omega_{\text{circ}}/R_0$.

4 MODES WITH SELF-GRAVITY

Let us now return to the fully self-gravitating case. In Section 4.1, we derive a recurrence relation between the Mellin transforms of the perturbed fluid quantities. This relates Mellin transforms with arguments with the same real part but differing by integers in their imaginary part. The boundary conditions for this recurrence relation are discussed with some considerable care in Section 4.2 prior to the numerical construction in Section 4.3.

4.1 The Recursion Relation

As the unperturbed physical quantities vary like powers of the cylindrical radius, the Mellin transform (e.g., Carrier, Krook & Pearson 1966) is a natural tool for simplifying the linear stability equations (14). First, we collect the real-space variables into a dimensionless four-component vector

$$(U_1, U_2, U_3, U_4)^T \equiv \left( \frac{\delta \Sigma}{\Sigma} \frac{\partial u}{c} \frac{\partial v}{c} \frac{\partial R}{c} \right)^T \equiv U(R),$$

(41)

with the corresponding Mellin transforms $u(\alpha) \equiv (u_1, u_2, u_3, u_4)^T$ defined by

$$u(\alpha) \equiv \int_0^\infty dR R^{-1/2-i\alpha} U(R).$$

(42)

Both sides have the implicit ($\theta, t$) dependence $\exp(i m \theta - i \omega t)$ of course. This is not quite the usual definition of a Mellin transform (Carrier, Krook & Pearson 1966), but the factor of $R^{-1/2}$ in the integrand makes for a real value of $\alpha_0$ as defined by eq. (16) below.

From the second and third of eqs. (16), eq. (17), and from eq. (21), we have the following system:

$$\frac{\omega}{ic} u_1 + i = -i[Mu_2(\alpha) + u_2(\alpha)],$$

$$\frac{\omega}{ic} u_2 + i = -i[Mu_2(\alpha) + u_1(\alpha) - u_3(\alpha)]$$

$$\frac{\omega}{ic} u_3 = (M^2 + 1)K(\alpha, m)u_1(\alpha),$$

$$\frac{\omega}{ic} u_4 + i = (i + \frac{1}{2})u_2(\alpha) - Mu_1(\alpha).$$

(43)

Here, $K(\alpha, m)$ is the famous Kalnajs function (Snow 1952; Kalnajs 1971; Binney & Tremaine 1987),

$$K(\alpha, m) = \frac{1}{2} \frac{\Gamma \left[ \frac{1}{2} \left( \frac{1}{2} + m + i\alpha \right) \right]}{\Gamma \left[ \frac{1}{2} \left( \frac{1}{2} + m - i\alpha \right) \right]} \Gamma \left[ \frac{1}{2} \left( \frac{1}{2} + m + i\alpha \right) \right] \Gamma \left[ \frac{1}{2} \left( \frac{1}{2} + m - i\alpha \right) \right],$$

(44)

which relates the Mellin transforms of the density and the potential perturbations. Note the distinction between $v_0^2 \equiv 2\pi\Sigma_0 R_0$ and $v_{\text{circ}}^2 = v_0^2 M^2/(M^2 + 1)$. Only the first two of eqs. (13) couple different values of the Mellin variable $\alpha$. So, after elimination of $u_1$ and $u_4$ using the last two equations, one has a system of the form

$$\frac{\omega}{ic} \hat{u}(\alpha + i) = M(\alpha) \cdot \hat{u}(\alpha),$$

(45)

where $\hat{u} = (u_1, u_2)^T$ and $M(\alpha)$ is the $2 \times 2$ matrix given by:

$$M = \begin{pmatrix} \frac{M}{m} (\frac{i}{2} + \alpha - im^2) & \frac{\alpha^2 + m^2 + 1/4}{im} \\ \frac{m^2 + M^2 - m^2(M^2 + 1)K}{im} & \frac{M}{m^2} (\frac{i}{2} + \alpha - im^2) \end{pmatrix}$$

It is helpful to view eq. (43) as a recursion relation relating Mellin transforms with arguments with the same real part, but differing by imaginary increments. We shall refer to such an arrangement as a “ladder”.

For neutral modes, $\omega \to 0$, there is no coupling between different values of $\alpha$ in eqs. (14). Equivalently, there is only one rung of the ladder. As realised by a number of investigators (e.g., Zang 1976; Lemos & Lynden-Bell 1993; Syer & Tremaine 1996; Evans & Read 1998b), exact equiangular neutral modes are possible. In fact, a solution exists in the form of an exact logarithmic spiral provided $\det M = 0$, or

$$[1 - (M^2 + 1)K(\alpha, m)] \frac{\alpha^2 + m^2 + 1/4}{M^2(m^2 - 2)} = 0.$$

(46)

The roots of this equation are real. Let us call them $\alpha_0$. If we artificially set $K \to 0$ thereby turning off the self-gravity, $\alpha_0$ reduces to the quantity $\mu$ defined in eq. (25).

4.2 Boundary Conditions

Let us now consider two complementary, semi-infinite ladders differing in the sign of the real part of the Mellin argument: $\alpha = +\alpha_0 + ni$ and $\alpha = -\alpha_0 + ni$, where $n$ is a nonnegative integer that can become arbitrarily large. What are
the boundary conditions for the recurrence relation? This is a delicate matter and we will consider the bottom and top rungs of the ladders in turn. In physical terms, these correspond to small-$R$ and large-$R$ respectively.

4.2.1 The Top Rung

At large radii, the WKBJ dispersion relation
\[ \left( \omega - \frac{m\omega_{\text{circ}}}{R} \right)^2 = \frac{2\kappa^2}{R^2} \left( \frac{\omega}{c} \right)^2 k^2, \]
(47)
indicates that self-gravity becomes negligible and the radial wave number becomes constant, viz.,
\[ k \sim \frac{\omega}{c} + \left( \frac{M^2 + 1}{2} - m\mathcal{M} \right) \frac{O(R^{-2})}{R} \text{ if } R \gg \frac{\omega_{\text{circ}}}{|\omega|}. \]
(48)
The sign of $k$ has been chosen appropriately for the outgoing wave, and the subdominant $O(R^{-1})$ terms have been explicitly retained as they contribute logarithmically to the phase of the eigenfunction. It follows from equations (20) - (23) that
\[ U(R) \sim \exp \left( \frac{i\omega R}{c} \right) R^{i(\mathcal{M}^2 + 1)/2 - m\mathcal{M}} U_{\infty} \]
(49)
as $R \to \infty$, where $U_{\infty}$ is another constant vector. This can be compared with the eigenfunction in the non self-gravitating case [defined by eqs (29) - (31)], where the exponent is only $-im\mathcal{M}$. The term $(\mathcal{M}^2 + 1)/2$ in eq. (49) stems ultimately from the self-gravity in the WKBJ dispersion relation. At the $n$th rung of the ladders, the Mellin variable has had its imaginary part incremented $n$ times and has become
\[ \alpha_n = \pm \alpha_0 + in \]
(50)
As $n \to \infty$, the Mellin transform is dominated by the large-$R$ behavior above, so that
\[ \hat{u}(\alpha_n) \sim \hat{u}_{\infty}(s_n) \left( \frac{ic}{\omega} \right)^{s_n}, \]
(51)
\[ s_n \equiv n + \frac{1}{2} - i \left( \pm \alpha_0 + m\mathcal{M} - \frac{\mathcal{M}^2 + 1}{2} \right) \]
The direction of the constant vector $\hat{u}_{\infty}$ is determined by the relations (20) - (22). The second component of $\hat{u}_{\infty}$ is $O(n^{-1})$ smaller than the first component because $\delta\nu_0 = y/R \sim R^{-1}(\delta\Sigma/\Sigma)$ asymptotically. As the overall normalization of the eigenfunction is arbitrary, the simplest approximation is
\[ \hat{u}_{\infty} = (1, 0)^T. \]
(52)

In fact, though the WKBJ analysis gives good physical insight into what is happening at the top rung of the ladders, the boundary conditions (24) - (25) are insufficiently accurate for numerical construction of the modes. This compels us to develop an asymptotic analysis to find the higher order corrections. The following paragraphs accomplish this task. They are primarily matters of detail rather than principle. Readers interested mainly in the latter can skip to the next sub-section without losing the thread of our argument.

Rather than deal with two coupled first-order difference equations, it proves convenient to recast (43) as a single second-order difference equation for which standard asymptotic techniques are readily available in the literature (e.g., Bender & Orszag 1978, chap. 5). Let us write the two-component vector $\hat{u}(\alpha_n)$ as
\[ \hat{u}(\alpha_n) = \left[ \frac{ic}{\omega} \right]^n (x_n, y_n). \]
(53)
Substitution into eq. (42) yields a coupled pair of first-order difference relations relating $(x_{n+1}, y_{n+1})$ to $(x_n, y_n)$. By eliminating the $y$s, one has a second-order difference equation:
\[ x_{n+1} = \left( A_n + \frac{B_nD_{n-1}}{B_{n-1}} \right) x_n + \frac{B_n\Delta_{n-1}}{B_{n-1}} x_{n-1} = 0. \]
(54)
with
\[ A_n = \frac{\mathcal{M}}{m} \left( \frac{\alpha_0 - \alpha_n}{\delta^2} \right) \sim O(n) \]
\[ B_n = \frac{\alpha_n^2 + m^2 + \Omega^2}{\Omega^2} \sim O(n^2) \]
\[ C_n = \frac{m^2 + \mathcal{M}^2 - m^2(\mathcal{M}^2 + 1)K(\alpha_n, \mathcal{M})}{\delta^2} \sim O(1) \]
\[ D_n = \frac{\frac{\Omega^2}{m} \left( \frac{\alpha_0 - \alpha_n}{\delta^2} \right)}{\delta^2} \sim O(n) \]
(55)
This notation is used because the matrix $M_n = M(\alpha_n)$ which takes us from the $nt$th to $(n+1)$th rung of the ladder is just
\[ M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}. \]
(56)
It has determinant
\[ \Delta_n = A_nD_n - B_nC_n \]
(57)
For future use, let us also note that the second component $y_{n+1}$ can be derived via
\[ y_{n+1} = \left( D_n x_{n+1} - \Delta_n x_n \right)/B_n \]
(58)
The coefficients of $x_{n+1}, x_n$ and $x_{n-1}$ in (54) are $O(1), O(n)$ and $O(n^2)$ respectively. Therefore, this suggests the ansatz (c.f. Bender & Orszag 1978)
\[ \frac{x_{n+1}}{x_n} = P_0 \alpha_n + P_1 \alpha_n^2 + P_2 \alpha_n^3 + \ldots \]
(59)
where the $\{P_k\}$ are independent of $n$. By substituting this ansatz into the difference equation (54) and matching the orders of the asymptotic expansion, it is possible to solve recursively for the $\{P_k\}$. A second-order linear recursion relation should have two independent solutions, hence there are two roots for the leading coefficient: $P_1 = \mp i$. Only the upper sign is consistent with the required behaviour (54), and the subsequent terms are then determined. From the first two terms, we have
\[ \frac{x_{n+1}}{x_n} = n + \frac{1}{2} - i \left( \pm \alpha_0 + m\mathcal{M} - \frac{\mathcal{M}^2 + 1}{2} \right) + O \left( \frac{1}{n} \right) \]
\[ \frac{y_{n+1}}{x_{n+1}} = -\frac{\mathcal{M} + im}{n} + O \left( \frac{1}{n^2} \right), \]
(60)
where the second line has been derived with the aid of (58). This suggests that as $n \to \infty$, the solution on the $nt$th rung approaches
\[ \hat{u}(\alpha_n) \sim \Gamma(s_n) \left( \frac{ic}{\omega} \right)^{s_n} (1, -\mathcal{M} + im/n)^T. \]
(61)
This is recognised as a more accurate version of eqs (51) - (52).

From a computational point of view, it is best to carry out the calculation (eased by computer algebra) to still higher order. We find that:
\[ x_n \sim \Gamma(-i\alpha_n)(-i\alpha_n)^\nu \left[ 1 + \frac{i f_1}{\alpha_n} + O\left(\frac{1}{\alpha_n^2}\right) \right] \]  
with 
\[ \nu = \frac{1}{2} - i \left( m\mathcal{M} - \mathcal{M}^2 + \frac{1}{2} \right) \]
\[ f_1 = \frac{1}{2} \left[ (2 - m^2)\mathcal{M}^2 - \frac{3}{4} + m^2 + 2i(m\mathcal{M} - \mathcal{M}^2) \right. \]
\[ \left. + 2\nu(\nu + im\mathcal{M}) \right] \]

The second component \( y_n \) can be found, for example, by use of eq. (58). These are the boundary conditions that are employed in our numerical implementation in Section 4.3 below.

### 4.2.2 The Bottom Rung

Now, let us see how the bottom rungs of these ladders \( \alpha = \pm \alpha_0 \) correspond to the ingoing and outgoing waves far inside the corotation radius. The Mellin transforms have poles at \( \alpha = \pm \alpha_0 \) because there is an infinite range of \( \text{Im} \, R \) over which the eigenfunction is asymptotically equal to a sum of two power laws. In fact, this representation is exact for a neutral mode \( (\omega = 0) \), where the Mellin transforms vanish on the higher rungs of the ladder. In the general case \( \omega \neq 0 \), one expects that the growing modes are superpositions of the neutral modes at least for \( R \ll v_{\text{circ}}/|\omega| \).

In other words,
\[ U(R) \sim U_+ R^{-1/2+i\alpha_0} + U_- R^{-1/2-i\alpha_0}, \]
where the coefficients \( U_\pm \) are constant vectors. Hence, if the range of integration in (64) is divided into two intervals \( 0 < R \leq R_0 \) and \( R_0 \leq R < \infty \), with \( R_0 \ll v_{\text{circ}}/|\omega| \), then the contribution from the first interval is approximately
\[ U_+ R^{+i\alpha_0-i\alpha} \]
\[ + U_- R^{-i\alpha_0+i\alpha}, \]
provided \( \text{Im}(\alpha) > 0 \). Guided by this, we expect that \( \hat{u}(\alpha) \) is composed of both a regular part and a singular part with simple poles:
\[ \hat{u}(\alpha) \rightarrow \hat{u}_{\text{reg}}(\pm \alpha_0) + \frac{\hat{u}_\pm}{\alpha \mp \alpha_0} \text{ as } \alpha \rightarrow \pm \alpha_0 + i0^+. \]

Here, \( \hat{u}_\pm \) are two-component constant vectors linearly related to \( U_\pm \), while \( \hat{u}_{\text{reg}}(\alpha) \) is a vectorial function that is regular at the poles. Furthermore, it must be true that
\[ M(\alpha_0)\hat{u}_+ = 0 = M(-\alpha_0)\hat{u}_-, \]
so that \( \hat{u}_{\pm}(\pm \alpha_0 + i) \) is finite:
\[ -i \frac{\omega}{c} \hat{u}_{\pm}(\pm \alpha_0 + i) = \lim_{\epsilon \to 0^+} M(\pm \alpha_0 + i\epsilon) \cdot \hat{u}_{\pm}(\pm \alpha_0 + i\epsilon) \]
\[ = \frac{dM}{d\alpha} \bigg|_{\pm \alpha_0} \cdot \hat{u}_{\pm} + M(\pm \alpha_0) \cdot \hat{u}_{\text{reg}}(\pm \alpha_0). \]

Owing to the conditions (65), the directions of the vectors \( \hat{u}_\pm \) are fixed. Their complex normalization factors are related by the requirement that the ingoing and outgoing parts of the asymptotic form (63) have equal moduli. Therefore, \( |\hat{u}_+| = |\hat{u}_-| \).

In other words, the residues of the poles at \( \alpha = \pm \alpha_0 \) are related by the boundary conditions at small \( R \). Since the ingoing and outgoing wave must carry equal energies, the moduli of these two residues must be equal. The relative phase of the two residues is arbitrary in the absence of a condition on the phases of the ingoing and outgoing waves.

Thus the large-\( R \) boundary condition gives us two complex constraints, or four real ones, at large \( n \) (one complex constraint for each of the two ladders). The small-\( R \) boundary condition gives a single real constraint. Our degrees of freedom are the complex residues of the poles at the bottom rung of each ladder (two complex or four real parameters), plus the complex parameter \( \omega \). Hence we have five real constraints and six real free parameters. As in the non self-gravitating case, the problem is indeterminate unless we impose an additional constraint. This could be the relative phase of the residues at \( \pm \alpha_0 \), or else the real part of \( \omega \).

At the risk of repetition, we now show that just as in the non self-gravitating case of Section 3, so here also the ratio of moduli of the ingoing and outgoing waves is determined by \( \arg(\omega) \), while the relative phase is determined by \( |\omega| \). On the one hand, we have seen that the complex amplitudes of the outgoing and ingoing waves at small \( R \) are proportional to \( \hat{u}_+ \) and \( \hat{u}_- \). On the other hand, the recursion relations (64)-(65) for the quantities \( \{x_n, y_n\} \) defined by eq. (53) are independent of \( \omega \). Therefore the ratio of ingoing to outgoing complex amplitudes is
\[ \frac{\hat{u}_+}{\hat{u}_-} \propto \frac{\hat{u}_+(\alpha_0 + im)}{\hat{u}_-(\alpha_0 - im)} \propto e^{2i\alpha_0} = \exp[-2\alpha_0 \arg(\omega)] \cdot \exp[2i\alpha_0 \ln |\omega|], \]
where the unwritten constants of proportionality are independent of \( \omega \) and \( n \) as \( n \to \infty \) [cf. eq. (40)]. Therefore all boundary conditions that neither absorb nor emit energy produce the same ratio of real to imaginary parts for all eigenfrequencies, but inasmuch as they fix different phase shifts at the inner boundary, they produce different discrete values for the moduli of the eigenfrequencies. Also, successive eigenfrequencies are always separated by \( \pi/\alpha_0 \) in the natural logarithm.

### 4.3 Numerical Strategy and Results

To construct the normal modes, we must adjust the complex parameter \( \omega \) so that all the boundary conditions are satisfied. For a large value of \( n \), the starting value of \( \hat{u}(\alpha_0) \) is fixed on the two ladders using eq. (62). The recurrence relation (63) is successively applied to reduce the imaginary part of the Mellin transform variable to unity. This calls for the evaluation of the Kalnajs function (44) for complex values of \( \alpha_0 \) with large imaginary parts. The best way to proceed is to use Stirling’s formula to derive the asymptotic result
\[ K(\alpha_0, m) \sim \pm \frac{1}{\alpha_0} \left[ 1 + \frac{1 - 4m^2}{8\alpha_0^2} \right] \]
and then use the recurrence relation
\[ K(\alpha_{n-2}, m) = \frac{m^2 + (\alpha_{n-2} + \frac{4}{m})^2}{m^2 + (\alpha_{n-2} + \frac{4}{m})^2} K(\alpha_n, m) \]
to work down the ladders. The final step to the bottom rung warrants further discussion. From eq. (53), the quan-
Now, growing modes are possible provided the disk is cold pattern speed for self-gravitating modes in the Mestel disk.

Of the work in Section 3.

Here, the exact answer is already known by virtue of the Mach number $M$ being a function of the Mach number $M$.

Terms (which lie in the null eigenspace) and the non-singular terms (which lie in its complement). Let $n$ be the null eigenvector so that $M \cdot n = 0$ and let $p$ be a projection operator that annihilates the non-null eigenvector. Then the residues we seek on each ladder are

$$\hat{u}_\pm = \frac{p \cdot \hat{u}(\pm \alpha_0 + i)}{p^2 \cdot M(\pm \alpha_0) \cdot n}$$

To satisfy the inner boundary condition, these residues must be equal in modulus. The entire numerical algorithm can be tested in the non self-gravitating instance by the obvious stratagem of setting the Kalnajs gravity factor $K(\alpha, m)$ to vanish. Here, the exact answer is already known by virtue of the work in Section 3.

Fig. 2 shows a graph of the ratio of growth rate to pattern speed for self-gravitating modes in the Mestel disk. Now, growing modes are possible provided the disk is cold enough. The condition for instability can be worked out exactly as

$$\mathcal{M}^2 \geq \frac{[m^2 + \frac{i}{2}][1 - K(0, m)]}{m^2 - 2 + (m^2 + \frac{1}{2})K(0, m)}$$

where $K(0, m)$ is readily deduced from eq. (44) as

$$K(0, m) = \frac{1}{2} \frac{1^2 - \frac{1}{2} (\frac{1}{2} + m)}{1^2 - \frac{1}{2} (\frac{1}{2} + m)}$$

So, for example, $m = 1$ growing modes are possible only if $\mathcal{M} \geq 0.869$, and $m = 2$ modes only if $\mathcal{M} \geq 0.733$. At a fixed temperature or Mach number, the ratio of growth rate to pattern speed is fixed. But, just as in the non self-gravitating case of Section 3.3, the magnitudes of these quantities are not fixed as a consequence of the indeterminate nature of the phase shift at the origin.

5 CONCLUSIONS

This paper has resolved the paradox of the scale-free disks. By itself, the linear stability problem is ill-posed. Normal modes do exist, but the moduli of the eigenfrequencies depend on the undetermined phase shift with which waves are reflected from the centre. For any definite choice of this phase, there can be a discrete set of growing modes for some disks. On the other hand, the argument of the eigenfrequencies—or equivalently the ratio of growth rate to pattern speed—is independent of the phase shift. This paper has explicitly calculated such ratios with and without self-gravity.

The paradox of the scale-free disks is of interest for two reasons. First, as Birkhoff (1960) has indicated, paradoxes are of inherent pedagogical value. It sharpens the insight to track down the flaw in a plausible physical argument that nonetheless leads to an apparent inconsistency. The paradox of the scale-free disks is a paradox of oversimplification. In theoretical work, it is customary to develop as simple a model as possible to describe objects or phenomena. The power-law disks, in which all physical quantities scale like powers, are attractive candidates for representing both galactic and accretion disks. But, they are not sufficiently rich to act as reasonable models for stability problems unless extra ingredients such as reflecting boundaries or cut-outs are added. This is the procedure that was followed in Zang (1976) and Evans & Read (1998a,b) to yield normal modes. Second, the paradox draws attention to the importance of inner boundary conditions in all models with central cusps. The stability properties are crucially affected by the manner in which impinging waves are reflected off the cusp. Therefore, the correct boundary condition at the centre—whether applied directly in a linear stability analysis or indirectly in a computer simulation—needs careful thought. Thinking physically, central cusps are usually a consequence of black holes. Probably the most astronomically relevant boundary condition is to allow the wave to reflect off a central black hole.

ACKNOWLEDGMENTS

We thank the Isaac Newton Institute for hospitality during its program on the Dynamics of Astrophysical Disks, when this work began. JG is supported by NASA Astrophysical Theory Grant NAG5-2796, and NWE by the Royal Society.

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