A Weak-Coupling Treatment of Nonperturbative QCD Dynamics to Heavy Hadrons

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Abstract

Based on the recently developed light-front similarity renormalization group approach and the light-front heavy quark effective theory, we derive analytically from QCD a heavy quark light-front Hamiltonian which contains explicitly a confinement interaction at long distances and a Coulomb-type interaction at short distances. With this light-front QCD Hamiltonian, we further demonstrate that the nonperturbative QCD dynamics of the strongly interacting heavy hadron bound states can be treated as a weak-coupling problem.

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I. INTRODUCTION

There are two fundamental features in nonperturbative QCD, the quark confinement and the spontaneous breaking of chiral symmetry. These two features are the basis for solving hadronic bound states from QCD but none of them has been completely understood. Recently, Wilson et al. proposed a new approach to determine hadronic bound states from nonperturbative QCD on the light-front with a weak-coupling treatment (WCT) [1]. The key to eliminating necessarily nonperturbative effects is to construct an effective QCD Hamiltonian in which quarks and gluons have nonzero constituent masses rather than the zero masses of the current picture. The use of constituent masses cuts off the growth of the running coupling constant and makes it conceivable that the running coupling never leaves the perturbative domain. The WCT approach potentially reconciles the simplicity of the constituent quark model with the complexities of QCD. The penalty for achieving this weak-coupling picture is the necessity of formulating the problem in light-front coordinates and of dealing with the complexities of renormalization.

Succinctly, this new approach of achieving a QCD description of hadronic bound states can be summarized as follows [1]: Using a new renormalization scheme, called similarity renormalization group (SRG) scheme [2], one can obtain an effective QCD Hamiltonian $H_{\lambda}$ which is a series of expansion in terms of the QCD coupling constant, where $\lambda$ is a low energy scale ($\simeq 1$ GeV). Then one may solve from $H_{\lambda}$ the strongly interacting bound states as a weak-coupling problem. The WCT scheme contains the following steps: (i) Compute explicitly from SRG the $H_{\lambda}$ up to the second order and denote it by $H_{\lambda,0}$ as a nonperturbative part of $H_{\lambda}$. The remaining higher order contributions in $H_{\lambda}$ are considered as a perturbative part $H_{\lambda,I}$. (ii) Introduce a constituent picture which allows one to start the hadronic bound states with the valence constituent Fock space. The constituent quarks and gluons have masses of a few hundreds MeV, and these masses are functions of the scale $\lambda$ that must vanish when the effective theory goes back to the high energy region. (iii) Solve hadronic bound states with $H_{\lambda,0}$ nonperturbatively in the constituent picture and determine the scale dependence of the constituent masses and the coupling constant. The coupling constant $g$ now becomes an effective one, denoted by $g_{\lambda}$. If we could show that with a suitable choice of $\lambda$ at the hadronic mass scale, the effective coupling constant $g_{\lambda}$ can be arbitrarily small, then WCT could be applied to $H_{\lambda}$ such that the corrections from $H_{\lambda,I}$ can be truly computed perturbatively. If everything listed above works well, we may arrive at a weak-coupling QCD theory of the strong interaction for hadronic bound states.

With the idea of SRG and the concept of coupling coherence [3], Perry has shown that upon a calculation to the second order, there exists a logarithmic confining potential in the resulting light-front QCD effective Hamiltonian [4]. This is a crucial finding to light-front nonperturbative QCD. However, the general strategy of solving hadrons through the WCT scheme has not been examined. In this paper, I will use SRG to analytically derive from the light-front heavy quark effective theory (HQET) [5-6] a heavy quark QCD Hamiltonian which is responsible to heavy hadron bound states. The resulting Hamiltonian explicitly contains a confining interaction between heavy quark and antiquark at long distance plus a Coulomb-type interaction at short distance. Based on this effective QCD Hamiltonian, I then study the strongly interacting heavy hadronic bound states, from which I can qualitatively provide a realization of WCT to nonperturbative QCD on the light-front.
The reason of choosing heavy hadron systems as a starting example in this investigation is that for light quark systems, both quark confinement and spontaneously chiral symmetry breaking play an essential role to the quark dynamics in hadrons. In other words, to provide a good QCD description for light quark systems, it is necessary to understand the underlying mechanism for quark confinement as well as for chiral symmetry breaking. This will certainly complicate the study of nonperturbative QCD. However, for heavy quark systems, chiral symmetry is explicitly broken so that confinement is the sole nontrivial feature influencing heavy quark dynamics. One may argue that the mass scales for heavy and light quark systems are different. The heavy quark energy cannot run down to the usual hadronic scale of light quark systems due to heavy quark mass. Meanwhile, confining interactions must be energy scale dependent. Apparently, confinement for heavy quark systems could be very different from light quark systems. However, despite heavy and light quarks, confinement arises only from low energy gluon interactions. In other words, the confinement mechanism must be the same for both heavy and light quark systems. We thus choose heavy hadron systems without any loss of generality.

In order to avoid the possible confusion arisen from different mass scales and to correctly extract confining interactions in heavy quark dynamics, it is convenient to work with heavy quark effective theory (HQET). HQET is a theory of QCD in $1/m_Q$ expansion \[^{[10]}\], where $m_Q$ is the heavy quark mass. In HQET, the nonperturbative dynamics is determined through the interacting gluons and heavy quarks by exchanging a small residual momentum of heavy quarks, which is of order $\Lambda_{QCD}$. As a result, within HQET we can indeed explore the nonperturbative QCD dynamics for heavy quark systems in the same scale as that for light quark systems. Meanwhile, the extension of the study to light quark systems is straightforward, although undoubtedly the corresponding result must be very complicated due to the spin dependence of the nonperturbative interacting Hamiltonian. The spin independent interactions on the light-front are essentially related to the chiral symmetry breaking. These spin dependent interactions in HQET are suppressed in the leading order approximation because they are $1/m_Q$ corrections and can be treated perturbatively with respect to heavy hadron states. This is why for heavy quark systems the chiral symmetry breaking can be treated separately from the confinement.

In fact, the model-based theoretical investigations on heavy quarkonia lasted for one and half decades is recently replacing by first-principles exploration on QCD. The lattice QCD simulation may give an acceptable description for heavy quarkonium spectroscopy with manageable control over all the systematic errors \[^{[7]}\]. The development of nonrelativistic QCD provides a general factorization formula to quarkonium annihilation and production processes so that a rigorous QCD analysis may become possible \[^{[8]}\]. Meanwhile, in the past five years considerable progress has been made for heavy hadrons containing one heavy quark, due mainly to the discovery of the heavy quark symmetry (HQS) \[^{[9]}\] and the development of HQET \[^{[10]}\] from QCD. The HQS and HQET have in certain contents put the description of heavy hadron physics on a QCD-related and model-independent basis. Moreover, HQET has also been extended to describe heavy quarkonia \[^{[11]}\]. Yet, a truly first-principles QCD understanding of heavy hadrons is still lacking because so far none is able to give a direct computation of heavy hadron bound states from QCD. On the other hand, in the last decade, the investigations of the light-front field theory on nonperturbative bound state problems have made some progress. Starting with heavy hadrons may provide a possible explicit
solution of hadronic bound states in light-front QCD. Hence, in this paper we shall mainly concentrate on heavy quark systems, especially for heavy quarkonia, but the generality of nonperturbative QCD dynamics obtained in this formalism will also be discussed.

The paper is organized as follows. In Section 2, the general procedure of constructing a renormalized effective Hamiltonian $H_\lambda$ in the light-front SRG is briefly reviewed and the possible existence of light-front confining interactions in this formulation is discussed. In Section 3, by applying the general procedure to the light-front HQET of QCD, a light-front heavy quark confining Hamiltonian is analytically derived. A light-front picture of quark confinement is illustrated. In Section 4, the light-front heavy hadronic bound states are explored within the WCT scheme. As an example, the light-front heavy quarkonium bound state equation is solved by the use of a Gaussian-type wavefunction ansatz, from which the WCT to nonperturbative QCD is explicitly explored. Finally, the physical implications in the realization of WCT are discussed in Section 5.

II. EFFECTIVE QCD HAMILTONIAN IN THE SRG SCHEME

A. Light-front SRG Scheme

We begin with the general formulation of the similarity renormalization group approach to construct a low energy QCD Hamiltonian which was first proposed by Glazek and Wilson \[2\]. The basic idea of the SRG approach is to develop a sequence of infinitesimal unitary transformations $S_\lambda$ that transform an initial bare Hamiltonian $H_B$ to an effective Hamiltonian $H_\lambda$ in a band-diagonal form relative to an arbitrarily chosen energy scale $\lambda$:

$$H_\lambda = S_\lambda H_B S_\lambda^\dagger. \quad (2.1)$$

Here the band-diagonal form means that the matrix elements of $H_\lambda$ involving energy jumps much larger than $\lambda$ will all be zero, while matrix elements involving smaller jumps or two nearby energies remain in $H_\lambda$. The similarity transformation should satisfy the condition that for $\lambda \to \infty$, $H_\lambda \to H_B$ and $S_\lambda \to 1$.

In this paper, we shall follow the formulation of SRG developed on the light-front \[1\]. The effective Hamiltonian we seek is $H_\lambda$ with $\lambda$ being of order a hadronic mass ($\sim 1$ GeV). We begin with a given bare Hamiltonian which can be written by $H_B = H_0 + H_B^I$, where $H_0$ is a bare free Hamiltonian and $E_i$ is its eigenvalue. Consider an infinitesimal transformation, then Eq.(2.1) is reduced to

$$\frac{dH_\lambda}{d\lambda} = [H_\lambda, T_\lambda] \quad (2.2)$$

which is subject to the boundary condition $\lim_{\lambda \to \infty} H_\lambda = H_B$, where $T_\lambda$ is a generator of the similarity transformation.

To force the Hamiltonian $H_\lambda$ becoming a band-diagonal form in energy space, we need to specify the action of $T_\lambda$. This can be done by introducing the scale $\lambda$ with $x_{\lambda ij} = \frac{E_i - E_j}{E_i + E_j + \lambda}$ into a smearing function $f_{\lambda ij} = f(x_{\lambda ij})$ such that

$$f(x) = \begin{cases} 1 & x \leq 1/3 \\ \text{smoothly from 1 to 0} & 1/3 < x < 2/3 \\ 0 & x \geq 2/3 \end{cases} \quad (2.3)$$
and reexpressing Eq.(2.2) by
\[
\frac{dH_{\lambda ij}}{d\lambda} = f_{\lambda ij}[H_{I\lambda}, T_\lambda]_{ij} + \frac{d}{d\lambda}(\ln f_{\lambda ij})H_{\lambda ij},
\]
\[
T_{\lambda ij} = \frac{1}{E_j - E_i}\left\{ (1 - f_{\lambda ij})[H_{I\lambda}, T_\lambda]_{ij} - \frac{d}{d\lambda}(\ln f_{\lambda ij})H_{\lambda ij} \right\}. \tag{2.4}
\]

Here we have written \( H_\lambda = H_0 + H_{I\lambda} \) because \( H_0 \) is invariant under transformations. And we have also used the notation \( A_{ij} = \langle i|A|j \rangle \), where \( |i\rangle \) and \( |j\rangle \) are eigenstates of \( H_0 \). Since \( f(x) \) vanishes when \( x \geq 2/3 \), one can see that \( H_{\lambda ij} \) does indeed vanish in the far off-diagonal region. It also can be seen that \( T_{\lambda ij} \) is zero in the near-diagonal region. The solutions for \( H_{I\lambda} \) and \( T_\lambda \) are
\[
H_{I\lambda} = H_{I\lambda}^B + [H_{I\lambda}, T_\lambda]_R, \quad T_\lambda = H_{I\lambda}^T + [H_{I\lambda}, T_\lambda]_T, \tag{2.5}
\]
where \( H_{I\lambda}^B = f_{\lambda ij}H_{ij}^B, \quad H_{I\lambda}^T = -\frac{1}{E_j - E_i}\left( \frac{d}{d\lambda}f_{\lambda ij} \right)H_{ij}^T, \) and
\[
X_{\lambda ij}^R = -f_{\lambda ij}\int_{\lambda}^{\infty} d\lambda X_{\lambda ij}, \quad X_{\lambda ij}^T = -\frac{1}{E_j - E_i}\left( \frac{d}{d\lambda}f_{\lambda ij} \right)\int_{\lambda}^{\infty} d\lambda X_{\lambda ij} + \frac{1}{E_j - E_i}(1 - f_{\lambda ij})X_{\lambda ij}. \tag{2.7}
\]

From eq.(2.3), one obtains an iterated solution for \( H_\lambda \),
\[
H_\lambda = \left( H_0 + H_{I\lambda}^B \right) + \left( [H_{I\lambda}^B, H_{I\lambda}^T]_R \right)
+ \left( [H_{I\lambda}^B, H_{I\lambda}^T]_R, H_{I\lambda}^T \right)_R
+ \left( [H_{I\lambda}^B, H_{I\lambda}^T]_R, [H_{I\lambda}^B, H_{I\lambda}^T]_T \right)_R
+ \cdots
= H_\lambda^{(0)} + H_\lambda^{(2)} + H_\lambda^{(3)} + \cdots. \tag{2.8}
\]

Thus, through SRG, we eliminate the interactions between the states well-separated in energy and generate the effective Hamiltonian of eq.(2.8). The expansion of eq.(2.8) in terms of the interaction coupling constant brings in order by order the full theory corrections to the band diagonal low energy Hamiltonian.

Practically, we do not really compute \( H_\lambda \) to all the orders in the expansion of eq.(2.8). In the WCT scheme, we consider the first two terms in (2.8) as the nonperturbative part of \( H_\lambda \) in the determination of hadronic bound states. Thus, in the fist step of WCT, we shall only compute \( H_\lambda \) up to the second order in coupling constant. We hope that the remaining part can be handled perturbatively, as we have mentioned in the introduction and will be discussed more later. Up to the second order, eq.(2.8) can be further expressed explicitly by
\[
H_{\lambda ij} = f_{\lambda ij}\left\{ H_{ij} + \sum_k H_{ik}^B H_{kj}^B \left[ \frac{g_{\lambda ij k}}{\Delta E_{ik}} + \frac{g_{\lambda ij k}}{\Delta E_{jk}} \right] + \cdots \right\}. \tag{2.9}
\]

The front factor in the above equation indicates that \( H_\lambda \) only describes long distance interactions (with respect to the scale \( \lambda \)) which is responsible to hadronic bound states. The function \( g_{\lambda ij} \) is given by
\[ g_{\lambda ij k} = \int_{\lambda}^{\infty} d\lambda' f_{\lambda' ik} \frac{d}{d\lambda'} f_{\lambda' jk}. \] (2.10)

More explicitly, the bare Hamiltonian \( H_B \) input in the above formulation can be obtained from the canonical Lagrangian with a high energy cutoff that removes the usual UV divergences. For light-front QCD dynamics, the bare Hamiltonian in our consideration is the canonical light-front QCD Hamiltonian that can be either obtained from the canonical procedure in the light-front gauge \([12,13]\) or generated from the light-front power counting rules \([1]\). Instead of the cutoff on the field operators which is introduced in ref. \([1]\), we shall use in this paper a vertex cutoff to every vertex in the bare Hamiltonian:

\[ \theta(\Lambda^2 / P^+ - |p^-_i - p^-_j|), \] (2.11)

where \( p^-_i \) and \( p^-_j \) are the initial and final state light-front energies respectively between the vertex, \( \Lambda \) is the UV cutoff parameter, and \( P^+ \) the total light-front longitudinal momentum of the system we are interested in. The theta function is defined as: \( \theta(x) = 1, 1/2 \) and 0 for \( x > 0, = 0 \) and \( < 0 \), respectively. Eq.\((2.11)\) is also called the local cutoff in light-front perturbative QCD \([17]\). All the \( \Lambda \)-dependences in the final bare Hamiltonian are removed by the counterterms. The use of eq.\((2.11)\) largely simplifies the analysis on the original cutoff scheme in ref. \([1]\).

Meanwhile, for a practical SRG calculation, we also have to give an explicit form of the smearing function \( f_{\lambda ij} \). One of the simplest smearing functions that satisfies the requirements of SRG is a theta-function:

\[ f_{\lambda ij} = \theta(\frac{1}{2} - x_{\lambda ij}). \] (2.12)

On the light-front, it is convenient to redefine \( x_{\lambda ij} = \frac{|p^-_i - p^-_j|}{P^-_i + P^-_j + \lambda^2 / P^+} \) on the light-front. Then we can further replace the above smearing function by the following form:

\[ f_{\lambda ij} = \theta(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|), \] (2.13)

where \( \Delta P^-_{ij} = P^-_i - P^-_j \) is the light-front free energy difference between the initial and final states of the physical processes. The light-front free energies of the initial and final states are defined as sums over the light-front free energies of the constituents in the states.

Note that the choice of a specific smearing function corresponds to picking up a particular renormalization scheme to specify the renormalization scale in SRG. This is similar to the specification of the renormalization scale in usual RG by picking up a particular regularization scheme. As we will see later, physics must be independent of the choice of the smearing function. With the smearing function given by eq.\((2.13)\), eq.\((2.9)\) can be simply reduced to

\[ H_{\lambda ij} = \theta(\lambda^2 / P^+ - |\Delta P^-_{ij}|) \left\{ H_{ij}^B + \sum_k H_{ik}^B H_{kj}^B \left[ \theta(|\Delta P^-_{ik}| - \lambda^2 / P^+) \theta(|\Delta P^-_{ik}| - |\Delta P^-_{jk}|) \frac{\Delta P^-_{ik}}{\Delta P^-_{jk}} + \theta(|\Delta P^-_{jk}| - \lambda^2 / P^+) \theta(|\Delta P^-_{jk}| - |\Delta P^-_{ik}|) \frac{\Delta P^-_{jk}}{\Delta P^-_{ik}} \right] + \cdots \right\}. \] (2.14)
B. Application to light-front QCD

Next, we shall calculate explicitly the low energy effective light-front QCD Hamiltonian in the similarity renormalization group scheme. Light-front QCD is the theory of QCD defined on the light-front with the light-front gauge. The canonical theory of the light-front QCD was formulated in 70’s [13,14]. In early 80’s, Lepage and Brodsky developed the Hamiltonian formulation of light-front QCD and explored extensively the asymptotic behavior of hadrons and the perturbative expansion of various exclusive processes [17]. Here we shall use the two-component Hamiltonian formulation of the light-front QCD [13] that has been used in illustrating the ideas of the WCT scheme to nonperturbative QCD in the previous publication [1].

The light-front coordinates are defined as follows: \( x^\pm \equiv x^0 \pm x^3, x^i = x^i (i = 1, 2) \), with \( x^- \) and \( x_\perp \) become naturally the longitudinal and transverse coordinates. The inner product of any two four-vectors is given by

\[
\langle a, b \rangle = \frac{1}{2} (a^+ b^- + a^- b^+) - a_\perp \cdot b_\perp,
\]

and the time and space derivatives (\( \partial^\mu = \partial / \partial x^\mu \)) and the 4-dimensional volume element are written as

\[
\partial^+ = 2 \partial / \partial x^-, \quad \partial^- = 2 \partial / \partial x^+, \quad \partial^i = - \partial / \partial x^i, \quad \text{and} \quad d^4x = \frac{1}{2} dx^+ dx^- d^2x_\perp.
\]

In the above coordinate system, the fermion field can be separated by

\[\psi = \psi_+ + \psi_- \]

with \( \psi_+ = \frac{1}{2} \gamma^0 \gamma^\pm \psi \) and \( \psi_+ \) and \( \psi_- \) are the so-called dynamical and constraint components of the quark field on the light-front, respectively. Due to the gauge symmetry, only two components of the 4-vector gauge field \( A^\mu_a \) are dynamical field variables. With the choice of the light-front gauge \( A^+_a = 0 \), we can explicitly solve the unphysical components of the quark and gauge fields \( (A^-_a, \psi_-) \) from the constraint equations which are obtained from QCD Lagrangian \( \mathcal{L}_{QCD} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \bar{\psi}(i \partial + m_q) \psi \):

\[
\psi_- = \left( \frac{1}{i \partial^+} \right) (i \alpha_\perp \cdot D_\perp + \beta m_q) \psi_+, \quad \alpha_\perp = \gamma^0 \gamma_\perp, \quad \beta = \gamma^0 
\]

\[
A^-_a = 2 \left[ \left( \frac{1}{\partial^+} \right) (\partial^+ A^+_a) + \left( \frac{1}{\partial^+} \right)^2 (f^{abc} A^i_b \partial^+ A^i_c + 2 \psi_+ T_a \psi_+) \right], \quad (2.16)
\]

where \( T_a \) is the generator of SU(3) color group: \( [T_a, T_b] = if^{abc} T_c \). Then the QCD Lagrangian on the light front can be expressed in terms of only the physical field variables \( (A^+_a, \psi_+) \):

\[
\mathcal{L} = \frac{1}{2} (\partial^+ A^+_a \partial^- A^+_a) + i \psi_+^\dagger \partial^- \psi_+ - \mathcal{H},
\]

where \( \mathcal{H} \) is the light-front QCD Hamiltonian density [13]:

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \quad (2.18)
\]

and

\[
\mathcal{H}_0 = \frac{1}{2} (\partial^+ A^+_a) (\partial^+ A^+_a) + \xi \frac{- \partial^2 + m^2}{i \partial^+} \xi, \quad (2.19)
\]

\[
\mathcal{H}_I = \mathcal{H}_{qqq} + \mathcal{H}_{ggg} + \mathcal{H}_{qqgg} + \mathcal{H}_{qggg} + \mathcal{H}_{gggg}, \quad (2.20)
\]

with
\[ \mathcal{H}_{qgg} = g \xi^\dagger \left\{ -2 \left( \frac{1}{\partial^+} \right) (\partial \cdot A_\perp) + \tilde{\sigma} \cdot A_\perp \left( \frac{1}{\partial^+} \right) (\tilde{\sigma} \cdot \partial_\perp + m) \right. \]
\[ \left. + \left( \frac{1}{\partial^+} \right) (\tilde{\sigma} \cdot \partial_\perp - m) \tilde{\sigma} \cdot A_\perp \right\} \xi, \]  
(2.21)

\[ \mathcal{H}_{ggg} = gf^{abc} \left\{ \partial^i A^j_a A^k_c + (\partial^i A^k_c) \left( \frac{1}{\partial^+} \right) (A^j_a \partial^+ A^k_c) \right\}, \]  
(2.22)

\[ \mathcal{H}_{qggg} = g^2 \left\{ \xi^\dagger \tilde{\sigma} \cdot A_\perp \left( \frac{1}{i\partial^+} \right) \tilde{\sigma} \cdot A_\perp \xi \right. \]
\[ \left. + 2 \left( \frac{1}{\partial^+} \right) (f^{abc} A^i_a \partial^+ A^i_c) \left( \frac{1}{\partial^+} \right) (\xi^\dagger T^a \xi) \right\}, \]  
(2.23)

\[ \mathcal{H}_{qqgg} = 2g^2 \left\{ \left( \frac{1}{\partial^+} \right) (\xi^\dagger T^a \xi) \left( \frac{1}{\partial^+} \right) (\xi^\dagger T^a \xi) \right\}, \]  
(2.24)

\[ \mathcal{H}_{gggg} = \frac{g^2}{4} f^{abc} f^{ade} \left\{ A^i_b A^j_c A^k_d A^l_e \right. \]
\[ \left. + 2 \left( \frac{1}{\partial^+} \right) (A^i_b \partial^+ A^i_c) \left( \frac{1}{\partial^+} \right) (A^j_d \partial^+ A^j_e) \right\}. \]  
(2.25)

Here, we denoted \( A_\perp = T^a A^a_\perp, A_\perp^a = (A^1_\perp, A^2_\perp) \). The notation \( \tilde{\sigma} \) is defined by: \( \tilde{\sigma}^1 = \sigma^2, \tilde{\sigma}^2 = -\sigma^1 \) (the \( 2 \times 2 \) Pauli matrices), and \( \xi \) is the two-component form of the light-front quark field:

\[ \psi_+ = \Lambda^+ \psi = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \quad \psi_- = \Lambda^- \psi = \begin{bmatrix} 0 \\ \left( \frac{1}{i\partial^+} \right) [\tilde{\sigma}^i (i\partial^i + gA^i) + im] \xi \end{bmatrix}, \]  
(2.26)

which comes from the use of the light-front \( \gamma \)-representation:

\[ \gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \]

\[ \gamma^1 = \begin{bmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{bmatrix}, \quad \gamma^2 = \begin{bmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{bmatrix}. \]  
(2.27)

Thus the bare light-front QCD Hamiltonian is given by

\[ H^B = \int_c dx^+ d^2 x_\perp (\mathcal{H}_0 + \mathcal{H}_I) + \text{counterterms}, \]  
(2.28)

where \( f_c \) means that the local cutoff, eq.(2.11), has been imposed. Also, counterterms are added to eq.(2.28) in order to remove all the \( \Lambda \)-dependence. Now, we can see that the canonical light-front QCD Hamiltonian is expressed purely in terms of the transverse gauge field components \( A_\perp \) and the physical two-component quark fields \( \xi \).

Upon the calculation to the second order in the coupling constant, the effective Hamiltonian [2.14] in \( g \bar{q} q \) sector (with the initial and final states, \(|i\rangle = \bar{b}^i(p_1, \lambda_1) d^i(p_2, \lambda_2)|0\rangle \) and \(|j\rangle = \bar{b}^i(p_3, \lambda_3) d^i(p_4, \lambda_4)|0\rangle \), respectively, where \( p_i \) and \( \lambda_i \) denote the respective momentum and helicity of a quark on the light-front) becomes

\[ H_{\lambda_{ij}} = \theta(\lambda^2 / P^+ - |\Delta P_{ij}^-|) \left\{ \langle j \rangle : H^B : |i\rangle - \frac{g^2}{2\pi^2} \lambda^2 C_f \frac{1}{P^+} \ln \epsilon + \text{mass counterterms} \right\} \]
All light-front energies in eq.(2.30) are on mass-shell: $p_g$ small coupling constant Hamiltonian. The basic idea to realize a weak-coupling treatment of QC D for hadrons is to consider the leading and the next-to-leading terms, i.e., eq.(2.29), as a starting effective Hamiltonian is the exact QCD Hamiltonian at the energy scale $\lambda$ (2.29).

However, Perry has recently found that upon to the second order calculation of the low energy Hamiltonian, a logarithmic confining potential has already occurred [4].

In eq.(2.29), $P^+ = p_1^+ + p_2^+ = p_2^+ + p_4^+$, $\Delta P_{ij} = p_1^- + p_2^- - p_3^- - p_4^-$, $: H^B :$ represents a normal ordering, where the instantaneous interaction contribution to the quark self-energy has been included in the self-energy calculation which is given by the mass counterterms and the logarithmic divergence, the color factor $C_f = (T^a T^a) = (N^2 - 1)/2N$, $N = 3$ the total numbers of colors, and $\epsilon$ is an infrared longitudinal momentum cutoff. Since $\ln \epsilon$ is an infrared divergence, it cannot be removed by mass counterterms. In gauge symmetry, this divergence must be canceled in the physical sector (and this is true as we will see later). The last term in (2.29) corresponds to one transverse gluon exchange contribution in the second order of the low energy Hamiltonian expansion (2.14). The momentum $q$ is carried by the exchange gluon: $q^+ = p_1^+ - p_3^+ = p_4^+ - p_2^+$, $q_\perp = p_1\perp - p_3\perp = p_4\perp - p_2\perp$, $\chi_{\lambda_i}$ denotes a helicity eigenstate. The factor $F_{rij}$ arises from the similarity renormalization transformation:

$$F_{rij} = \left\{ \theta(\lambda^2/P^+ - |p_1^- - p_3^- - q^-|)\theta(\lambda^2/P^+ - |p_4^- - p_2^- - q^-|) \times \frac{\theta(|p_1^- - p_3^- - q^-| - |p_4^- - p_2^- - q^-|)}{p_1^- - p_3^- - q^-} \right.$$

$$+ \frac{\theta(|p_4^- - p_2^- - q^-| - |p_1^- - p_3^- - q^-|)}{p_4^- - p_2^- - q^-} \left. \right\}.$$  

2.30

All light-front energies in eq.(2.30) are on mass-shell: $p_i^- = \frac{p_{1\perp}^2 + m_i^2}{p_i^+}$ and $q^- = \frac{q_i^2}{q^+}$.

If we continue to evaluate all the terms in the expansion of eq.(2.14), the resulting Hamiltonian is the exact QCD Hamiltonian at the energy scale $\lambda$. In practice, we only consider the leading and the next-to-leading terms, i.e., eq.(2.29), as a starting effective Hamiltonian. The basic idea to realize a weak-coupling treatment of QCD for hadrons is whether we can solve hadron states from this effective Hamiltonian (2.29) with an arbitrary small coupling constant $g$ such that the higher order corrections in (2.14) can be handled perturbatively. From the success of the constituent quark model, we understand that a necessity for such a realization might depend on the existence of a confining interaction in (2.29).

C. Quark confining interaction on the light-front

Naively, we know that any weak-coupling Hamiltonian derived from QCD will have only Coulomb-like interactions, and confinement can only be exhibited in a strong-coupling theory. However, Perry has recently found that upon to the second order calculation of the low energy Hamiltonian, a logarithmic confining potential has already occurred [4].
Explicitly, the two-body quark-antiquark interaction from the first term of eq.(2.29) is the instantaneous gluon exchange interaction $H_{qqq}$ which has the form in the momentum space:

$$\frac{-1}{(q^+)^2}. \quad (2.31)$$

The confinement must be associated with the interaction where $q^+ \to 0$. This is because only these particles with zero longitudinal momentum can occupy the light-front vacuum state $|1\rangle$. Thus, we need to analyze the feature of the effective Hamiltonian when $q^+ \to 0$. For $q^+ \to 0$, the dominant second order contribution from one transverse gluon exchange interaction in eq.(2.29) is given by

$$\left[ \frac{1}{(q^+)^2} \frac{q_\perp^2}{q^+} + O(\frac{1}{q^+}) \right] \theta(q^- - \lambda^2/P^+). \quad (2.32)$$

In the usual perturbative calculation, no such a $\theta$-function (coming from the smearing function) is attached in the above equation. Thus these two dominant contributions from the instantaneous and one-gluon exchange interactions are exactly cancelled when $q^+ \to 0$. Only Coulomb-type interaction (the terms $\sim O(\frac{1}{q^+})$) remains. However, in the light-front similarity renormalization group scheme, the corresponding one-gluon exchange contribution of eq.(2.32) only contains these gluons with energy being greater than the energy cutoff $\lambda^2/P^+$. As a result, the instantaneous gluon exchange term $1/(q^+)^2$ remains uncancelled if the gluon energy, $q_\perp^2/q^+$, is less than $\lambda^2/P^+$. The remaining uncancelled instantaneous interaction contains an infrared divergence and a finite part contribution (for a detailed derivation, see the next section). The divergence part is cancelled precisely for physical states by the same divergence in the quark self-energy correction [see (2.29)]. The remaining finite part corresponds to a logarithmic confining potential:

$$b_\lambda \ln(P^+ |x^-|) + c_\lambda \ln(\lambda^2 |x_\perp|^2).$$

The above result is indeed first obtained by Perry with the use of the concept of coupling coherence and a slightly different renormalization scheme [4] (also see a oversimple derivation given by Wilson [18]). Here the derivation is purely based on the light-front similarity renormalization scheme.

One may argue that the existence of such a confining potential in $H_\lambda$ up to the second order may only be an artificial effect designed in the renormalization scheme we used here. If we included the interaction with the exchange gluon energy below the cutoff, then the instantaneous interaction would be completely cancelled, and no such confining potential should exist, as expected in the usual perturbation computation. Wilson has pointed out that the set up of the new renormalization scheme is motivated by the idea that the gluon mass must be nonzero in the low energy domain (a constituent picture). The massive gluon can be originated from the nonlinear interactions in non-abelian gauge theory. Therefore, the non-cancellation of the instantaneous interaction in the low energy domain is indeed a consequence of the existence of the constituent massive gluons due to the non-abelian gauge interactions. It is independent of any particular renormalization scheme. The use of the low energy cutoff $\lambda$ (namely, taking the smearing function as a theta function) just gives us a simple realization of this confining picture that the exchange energy of massive dynamical gluons cannot run down to the zero value in nonperturbative QCD.
Yet, although there exists a confining interaction in $H_\lambda$ even up to the second order, this effective QCD Hamiltonian is already very complicated, due to the spin dependent part in the interactions. As we know the spin dependent interactions on the light-front are essentially related to the chiral symmetry breaking. In order to examine the above picture of confining mechanism on the light-front and to develop explicitly a weak-coupling treatment approach of nonperturbative QCD to hadronic bound states, in the next section we shall utilize the above formulation to heavy quark systems. We find that the low-energy QCD Hamiltonian $H_\lambda$ for heavy quarks can be largely simplified and an analytic form consisting of the confining and Coulomb potentials emerges.

III. HEAVY QUARK CONFINING HAMILTONIAN

In the past few years, QCD has been made a numerous progresses in understanding the heavy hadron structure, due mainly to the discovery of heavy quark symmetry by Isgur and Wise [9], and the development of HQET by Georgi et al. [10]. However, to completely understand the nonperturbative QCD dynamics of heavy hadrons, one should solve heavy hadron bound states directly from QCD, which is still an open question. Next we will use SRG to the light-front HQET to derive a heavy quark confining Hamiltonian, from which we may solve from QCD the heavy hadron bound states directly.

A. Light-front HQET

The light-front heavy quark effective Lagrangian derived from QCD lagrangian $\mathcal{L} = \bar{Q}(i\not{D} - m_Q)Q$ as a $1/m_Q$ expansion is given in refs. [5,6]:

$$\mathcal{L} = \frac{2}{v^+} Q_{v+}^\dagger (iv \cdot D) Q_{v+} - \sum_{n=1}^{\infty} \left( \frac{1}{m_Q v^+} \right)^n Q_{v+}^\dagger \left\{ (i\vec{\alpha} \cdot \vec{D}) (-iD^+)^{n-1} (i\vec{\alpha} \cdot \vec{D}) \right\} Q_{v+},$$

(3.1)

where $Q_{v+}$ is the light-front dynamical component of the heavy quark field after the phase redefinition: $Q(x) = e^{-im_Qv^+x} (Q_{v+}(x) + Q_{v-}(x))$, $v^\mu$ the four velocity of the heavy hadrons, $P^\mu = M_H v^\mu$ with $v^2 = 1$ and $M_H$ the heavy hadron mass, $\vec{\alpha} \cdot \vec{D} \equiv \alpha_\perp \cdot D_\perp - \frac{1}{v^+} (\alpha_\perp \cdot v_\perp + \beta) D^+ + D^\mu$, and $D^\mu$ the usual covariant derivative. The corresponding light-front heavy quark bare Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{v^+} Q_{v+}^\dagger (v^- \partial^+ - 2v_\perp \cdot \partial_\perp) Q_{v+} - \frac{g}{v^+} Q_{v+}^\dagger (v \cdot A) Q_{v+}$$

$$+ \sum_{n=1}^{\infty} \left( \frac{1}{m_Q v^+} \right)^n Q_{v+}^\dagger \left\{ (i\vec{\alpha} \cdot \vec{D}) (-iD^+)^{n-1} (i\vec{\alpha} \cdot \vec{D}) \right\} Q_{v+}.$$

(3.2)

The heavy antiquark Hamiltonian has the same form except for replacing $v$ by $-v$.

In the large $m_Q$ limit, only the leading (spin and mass independent) Hamiltonian is remained. The $1/m_Q^n$ terms ($n \geq 1$) in (3.2) can be regarded as perturbative corrections to the leading order operators and states. To determine confining interactions in heavy quark systems, the leading heavy quark Hamiltonian plays an essential role. With the light-front gauge $A^+ = 0$, the leading-order bare QCD Hamiltonian density is
\[ H_{ld} = \frac{1}{iv^+} Q^I_{v^+} (v^- \partial^+ - 2v_\perp \cdot \partial_\perp) Q_{v^+} - \frac{2g}{v^+} Q^I_{v^+} \left\{ v^+ \left[ \left( \frac{1}{\partial^+} \right) \partial_\perp \cdot A_\perp \right] - v_\perp \cdot A_\perp \right\} Q_{v^+} + 2g^2 \left( \frac{1}{\partial^+} \right) (Q_{v^+}^T A^T Q_{v^+}) \left( \frac{1}{\partial^+} \right) (\psi^I_{v^+} A^T \psi_{v^+}), \quad (3.3) \]

where \( \psi_+ \) is either the heavy antiquark field or the light-front quark field operator in the present consideration. Note that besides the leading term in eq.(2.12), the above bare Hamiltonian has also already included the relevant terms from the gauge field part, \(-\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu})\), of the QCD Lagrangian. These terms come from the elimination of the unphysical gauge degrees of freedom, the longitudinal component \( A_\perp \) \[13\]. Eq.(3.3) has obviously the spin and flavour heavy quark symmetry, or simply the heavy quark symmetry.

The above leading Hamiltonian (or Lagrangian) is the basis of the QCD-based description for heavy hadrons containing a single heavy quark, such as \( B \) and \( D \) mesons. As recently pointed out by Mannel et al. \[19,1,11\] the purely heavy quark leading Lagrangian may be not appropriate to describe heavy quarkonia. This is because the anomalous dimension of QCD radiative correction to \( Q\overline{Q} \) currents contains an infrared singularity in the limit of two heavy constituents having equal velocity. Such an infrared singularity is a long distance effect and should be absorbed into quarkonium states. To solve this problem, they show that one should incorporate the effective Hamiltonian with the first order kinetic energy term into the leading Hamiltonian \[11\]. The light-front kinetic energy can be obtained from eq.(3.2),

\[ H_{\text{kin}} = -\frac{1}{m_Q v^+} Q^I_{v^+} \left\{ \partial_\perp^2 - \frac{2v_\perp \cdot \partial_\perp}{v^+} \partial^+ + \frac{v^-}{v^+} \partial^+ v^+ \right\} Q_{v^+}. \quad (3.4) \]

Indeed, as we will see in Section IV.C, for heavy quarkonia the heavy quark carries a relative momentum that is proportional to heavy quark mass. Then the first order kinetic energy term is of the same order as the leading Hamiltonian. Other terms in the \( 1/m_Q \) are really suppressed by the power of \( 1/m_Q \). This is why one have to incorporate the leading Hamiltonian with the first order kinetic energy term. As a consequence, in the heavy mass limit, quarkonia have spin symmetry but no flavour symmetry.

**B. Confining Hamiltonian for heavy quarkonia**

Within light-front HQET, we now follow the procedure described in the previous section to find an effective QCD Hamiltonian for \( Q\overline{Q} \) systems. The bare Hamiltonian for \( Q\overline{Q} \) systems contains (3.3) and (3.4) for both heavy quark and antiquark plus the full QCD Hamiltonian for gluons and light quarks \[13\]. Since the kinetic energy (3.4) is of order \( \Lambda_{QCD}/m_Q \), which is at most of the same order of the Coulomb interaction, we can treat the kinetic energy in the same way as the instantaneous \( Q\overline{Q} \) interaction [the last term in eq.(3.3)]. Thus, the free Hamiltonian \( H_0 \) used in SRG is given only by the first term in eq.(3.3) plus the free gluon Hamiltonian.

Keeping the above consideration in mind, it is easy to compute the effective Hamiltonian eq.(2.14) for \( Q\overline{Q} \) systems. Following the WCT ideas, we shall calculate \( H_\lambda \) for \( Q\overline{Q} \) systems up to the second order in the initial and final states defined by \( |i\rangle = b^\dagger_{v}(k_1, \lambda_1) d^\dagger_{v}(k_2, \lambda_2) |0\rangle \) and \( |j\rangle = b^\dagger_{v}(k_3, \lambda_3) d^\dagger_{v}(k_4, \lambda_4) |0\rangle \), respectively, where \( k_i \) is the residual momentum of heavy quarks, \( p_i^\mu = m_Q v^\mu + k_i^\mu \), and \( \lambda_i \) its helicity. The operator \( b^\dagger_{v}(k_i, \lambda_i) \) \( [d^\dagger_{v}(k_i, \lambda_i)] \) creates a heavy quark [antiquark] with velocity \( v \),
\{b_{\nu}(k, \lambda), \ b_{\nu}^+(k', \lambda') = \{d_{-\nu}(k, \lambda), \ d_{-\nu}^+(k', \lambda') \} = 2(2\pi)^3\delta_{\nu\nu'}\delta_{\lambda\lambda'}\delta^3(\vec{k} - \vec{k}'), \quad (3.5)

and \(\delta^3(\vec{k} - \vec{k}') \equiv \delta(k^+ - k'^+)\delta^2(k_\perp - k'_\perp)\). The result is

\[H_{\lambda\nu ij} = H_{Q\bar{Q}\text{freeij}} + V_{Q\bar{Q}ij}, \quad (3.6)\]

where

\[
H_{Q\bar{Q}\text{freeij}} = [2(2\pi)^3]^2\delta^3(\vec{k}_1 - \vec{k}_3)\delta^3(\vec{k}_2 - \vec{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
\times \left\{ \frac{\Lambda}{m_Q} \left[ 2\kappa_1^2 + \Lambda^2(2y^2 - 2y + 1) \right] - \Lambda^2 - \frac{g^2}{4\pi^2}C_f\frac{\Lambda^2}{K^+}\ln\epsilon \right\}, \quad (3.7)\]

\[
V_{Q\bar{Q}ij}(y - y', \kappa_\perp - \kappa'_\perp) = 2(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
\times \frac{-4g^2(T^a)(T^a)}{(K^+)^2} \left\{ \frac{1}{(y - y')^2} \left( 1 - \theta(A(y - y', \kappa_\perp - \kappa'_\perp, \Lambda) - \lambda^2) \right) \right\} \frac{\Lambda^2}{(\kappa_\perp - \kappa'_\perp)^2 + (y - y')^2\Lambda^2} \theta(A(y - y', \kappa_\perp - \kappa'_\perp, \Lambda) - \lambda^2). \quad (3.8)\]

Here we have introduced the longitudinal residual momentum fractions and the relative transverse residual momenta,

\[
y = k_1^+ / K^+, \quad \kappa_\perp = k_\perp - yK_\perp, \quad (3.9)\]

\[
y' = k_1'^+ / K^+, \quad \kappa'_\perp = k_\perp - y'K_\perp, \quad (3.9)\]

where \(K^\mu\) is defined as the residual center mass momentum of the heavy quarkonia: \(K^\mu = \Lambda v^\mu\), and \(\Lambda = M_H - m_Q - m_{\bar{Q}}\) is a residual heavy hadron mass. It follows that \(K^+ = k_1^+ + k_2^+ = k_3^+ + k_4^+, \ K_\perp = k_{1\perp} + k_{2\perp} = k_{3\perp} + k_{4\perp}\). Since \(0 \leq p_1^+ = m_Qv^+ + k_1^+ \leq M_Hv^+\), in the heavy quark mass limit, we have \(M_H \to 2m_Q\) so that \(-m_Qv^+ \leq k_1^+, \ k_3^+ \leq m_Qv^+.\) Hence, the range of \(y\) and \(y'\) are given by \(-\infty < y, y' < \infty\). We have also defined in eq.\( (3.8)\)

\[
A(y - y', \kappa_\perp - \kappa'_\perp, \Lambda) = \frac{(\kappa_\perp - \kappa'_\perp)^2}{|y - y'|} + |y - y'|\Lambda^2. \quad (3.10)\]

Eq.\( (3.9)\) is the nonperturbative part of the effective Hamiltonian for heavy quarkonia in the WCT scheme, in which we have already let UV cutoff parameter \(\Lambda \to \infty\) and the associated divergence has been put in the mass correction. The kinetic energy \(\frac{1}{m_Q}\) is now included in the above effective Hamiltonian [the \(1/m_Q\) term in eq.\( (3.7)\)]. Note that there is an infrared divergent term in eq.\( (3.7)\) which comes from the quark self-energy correction in SRG,

\[
2\Sigma = -\frac{g^2}{2\pi^2}\lambda^2C_f\ln\epsilon + 2\delta m_Q^2, \quad (3.11)\]

where \(\epsilon\) is an infrared cutoff of the momentum fraction \(q^+ / K^+\), and \(q^+\) the longitudinal momentum carried by gluon in the quark self-energy loop. The usual mass correction \(\delta m_Q^2 = \frac{g^2}{4\pi^2}C_f\Lambda^2\ln\frac{\Lambda^2}{\kappa^2}\), has been renormalized away in eq.\( (3.7)\), where the color factor \(C_f = (T^aT^a) = (N^2 - 1) / 2N\), \(N = 3\) the total numbers of colors. In the WCT scheme, by removing away this
mass correction, we should assign the corresponding constituent quark mass in $H_{\lambda 0}$ being $\lambda$-dependent. But, the heavy quark mass is larger than the low energy scale. Its dependence on $\lambda$ should be very weak and could be neglected. Meanwhile, the $Q\bar{Q}$ interaction (3.8) contains two contributions: the instantaneous interaction plus the second order contribution in eq. (2.14) [i.e. the terms proportional to the theta function in eq. (3.8)]. We shall show next that the above $V_{Q\bar{Q}}$ is indeed a combination of a confining interaction plus a Coulomb-type interaction.

C. A simple picture of quark confinement on the light-front

In the conventional picture, QCD has a complex vacuum that contains infinite quark pairs and gluons necessary for confinement and chiral symmetry breaking. On the light-front, the longitudinal momentum of physical particles is always positive, $p^+ = p^0 + p^3 \geq 0$. Consequently, only those constituents with zero longitudinal momentum (called zero modes [20]) can occupy the light-front vacuum. The zero modes carry an extremely high light-front energy which has been integrated out in SRG. As a result, some equivalent effective interactions are generated in $H_{\lambda}$ so that the light-front QCD vacuum becomes trivial. The nature of nontrivial QCD vacuum structure, the confinement as well as the chiral symmetry breaking, is then made manifestly in $H_{\lambda}$ in terms of these new effective interactions. We will see that $H_{\lambda 0}$ explicitly contains a confining interaction at long distances. The interactions associated with the chiral symmetry breaking may be manifested in the fourth order computation of $H_{\lambda}$ for light quark systems [18], but these interactions are not important in the study of heavy hadrons here.

The confining interaction can be easily obtained by applying the Fourier transformation to the first term in (3.8). It is convenient to perform the calculation in the frame $K_\perp = 0$, in which

$$
\int \frac{dq^+ dq_\perp}{(2\pi)^2} e^{i(q^+ x^+ + q_\perp x_\perp)} \left\{ - \frac{4g^2\Lambda^2}{K^2} \frac{1}{(y - y')^2} \theta(\lambda^2 - A(y - y', \kappa_\perp - \kappa'_{\perp}, \Lambda)) \right\}
= - \frac{g^2\Lambda^2}{2\pi^2} \int_0^{1/\Lambda} dq^+ e^{iq^+ x^+ - q^+_m} \frac{2J_1(|x_\perp|q_{\perp m})}{|x_\perp|q_{\perp m}}
$$

(3.12)

where $q^+ = k^+_1 - k^+_3 = K^+(y - y')$, $q_\perp = k_{\perp 1} - k_{\perp 3} = \kappa_{\perp} - \kappa'_{\perp}$ for $K_\perp = 0$, $q_{\perp m} = \sqrt{\frac{\lambda^2}{K^2} q^+ - \frac{\lambda^2}{K^2} q^+}$, and $J_1(x)$ is a Bessel function. An analytic solution to the integral (3.12) may be difficult to carry out. However, the nature of confining interactions is a large distance QCD behavior. We may consider the integral for large $x^-$ and $x_\perp$. In this case, if $q^+ x^-$ and/or $|x_\perp|q_{\perp m}$ are large, the integration vanishes, yet $J_1(x) = \frac{x}{2} + \frac{x^3}{16} + \cdots$ for small $x$. The dominant contribution of the integral (3.12) for large $x^-$ and $x_\perp$ comes from the small $q^+$ such that $q^+ x^-$ and/or $|x_\perp|q_{\perp m}$ must remain small, which leads to $e^{iq^+ x^+ - 2J_1(|x_\perp|q_{\perp m})} \approx 1$. This corresponds to $q^+ < \frac{1}{x^-}$ and/or $q^+ < \frac{K^+}{|x_\perp|\Lambda}$. If $q^+ < \frac{1}{x^-} < \frac{K^+}{|x_\perp|\Lambda}$, eq. (3.12) is reduced to

$$
- \frac{g^2\Lambda^2}{2\pi^2} \int_0^{1/\Lambda} dq^+ \frac{1}{q^+} \left( \frac{\lambda^2}{K^2} q^+ - \frac{\lambda^2}{K^2} q^+ \right) = \frac{g^2\lambda^2}{2\pi^2 K^+} \left( \ln(K^+ |x^-|) + \ln \epsilon \right).
$$

(3.13)
where a term $\sim \frac{1}{x}$ is neglected since $x^-$ is large, and $\epsilon$ is an infrared cutoff of the momentum fraction $q^+/K^+$. It is the same as in eq. (3.11) so that the above infrared logarithmic divergence ($\sim \ln \epsilon$) exactly cancels the divergence in eq. (3.7) for color single states. What remains is a logarithmic confining interaction (except for a color factor):

$$V_{\text{conf.}}(x^-, x_{\perp}) \sim \frac{g_0^2 \lambda^2}{2 \pi^2 K^+} \ln(K^+ |x^-|).$$  \hspace{1cm} (3.14)

Similarly, when $q^+ < \frac{K^+}{|x_{\perp}|^2 x^2} < \frac{1}{x}$, we have

$$- \frac{g_0^2}{2 \pi^2} \int_0^{\frac{K^+}{|x_{\perp}|^2 x^2}} dq^+ \frac{1}{q^+} \left( \frac{\lambda^2}{K^+} q^+ - \frac{\lambda^2}{K^+} q^+ \right) = \frac{g_0^2 \lambda^2}{2 \pi^2 K^+} \left( \ln(\lambda^2 |x_{\perp}|^2) + \ln \epsilon \right), \hspace{1cm} (3.15)$$

where the term $\sim \frac{1}{x}$ has also been ignored because of the large $x_{\perp}^2$. Again, the infrared divergence ($\sim \ln \epsilon$) is cancelled in $H_0$ for physical states, and we obtain the following confining interaction:

$$V_{\text{conf.}}(x^-, x_{\perp}) \sim \frac{g_0^2 \lambda^2}{2 \pi^2 K^+} \ln(\lambda^2 |x_{\perp}|^2).$$  \hspace{1cm} (3.16)

Hence, the effective Hamiltonian $H_0$ exhibits a logarithmic confining interaction between a heavy quark and a heavy antiquark in all the directions of $x^-$ and $x_{\perp}$ space.

The Coulomb interaction corresponds to the second term in (3.8), its Fourier transformation (except for the color factor) is

$$\frac{\lambda^2}{(\kappa_{\perp} - \kappa_{\perp}')^2 + (y - y')^2 \lambda^2} \sim \frac{1}{4\pi} \int dx^- dx_{\perp} e^{i(x^- q^+ + q_{\perp} x_{\perp})} \left( \frac{\lambda}{K^+} \right) \frac{1}{r_t},$$  \hspace{1cm} (3.17)

where $r_t \equiv \sqrt{x_{\perp}^2 + (\frac{\lambda}{K^+})^2 (x^-)^2}$ which is defined as a “radial” variable in the light-front space $\Pi$. Eq. (3.14) shows that the Coulomb interaction on the light-front has the form

$$V_{\text{Coul.}}(x^-, x_{\perp}) \sim \frac{g_0^2 \lambda}{4\pi K^+ r_t}. \hspace{1cm} (3.18)$$

Thus, we have explicitly shown that $H_0$ contains a Coulomb interaction at short distances and a confining interaction at long distances.

Moreover, a clear light-front picture of quark confinement emerges here. To be specific, we define quark confinement as follows: i) There is a confining interaction between quarks such that quarks cannot be well-separated; ii) No color non-singlet bound states exist in nature, only color singlet states with finite masses can be produced and observed; and iii) The conclusions of i–ii) are only true for QCD but not for QED.

We have shown explicitly the existence of a confining interaction in $H_0$. One can also easily see from $H_0$ the non-existence of color non-singlet bound states. This is essentially related to the infrared divergences in $H_0$. From eqs. (3.13) and (3.15), we find that the uncancelled instantaneous interaction contains a logarithmic infrared divergence. Except for the color factor, this infrared divergence has the same form as the divergence in eq. (3.7). Thus, we immediately obtain the following conclusions.
(a). For a single (constituent) quark state, the interaction part of $H_{\lambda}$ does not contribute to its energy. The remaining infrared divergence from quark self-energy correction implies that the dynamical quark mass for a single quark state is infinite (infrared divergent) and cannot be renormalized away in the spirit of gauge invariance. Equivalently speaking, single quark states carry an infinitely large mass and therefore they cannot be produced.

(b). For color non-singlet composite states, the color factor $(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}$ in the $Q\bar{Q}$ interaction is different from the color factor $C_f = \text{Tr}(T^a T^a)$. Therefore, the infrared divergence in the self-energy correction also cannot be cancelled by the corresponding divergence from the uncancelled instantaneous interaction. As a result, color non-singlet composite states are infinitely heavy that they cannot be produced as well.

(c). For color singlet $QQ$ states, the color factor $(T^a)(T^a) \rightarrow C_f$. Thus, the infrared divergences are completely cancelled and the resulting effective Hamiltonian is finite. In other words, only color singlet composite are physically observable.

Finally, we argue that the above mechanism of quark confinement is indeed only true for QCD. As we have seen the light-front confinement interaction is just an effect of the non-cancellation between instantaneous interaction and one transverse gluon interaction generated in SRG. Such a non-cancellation arises in SRG because we introduce the energy scale $\lambda$ through the smearing function. Introducing the energy scale $\lambda$ in SRG forces the transverse gluon energy involved in the $Q\bar{Q}$ effective interaction never be less than a certain value (the energy scale $\lambda$). This implies that the gluon may become massive at the hadronic mass scale. Of course, such a gluon mass must be a dynamical mass generated from the highly nonlinear gluon interactions. In other words, the above confining picture is indeed a dynamical consequence of non-Abelian gauge theory. This confinement mechanism is not valid in QED. In QED, since photon mass is always zero, the photon energy covers the entire range from zero to infinity. Thus, in QED, we can always choose the energy scale $\lambda$ being zero. With $\lambda = 0$, the infrared divergences do not occur in the electron self-energy correction. As a result, the renormalized single electron mass is finite, in contrast to the divergent mass of single quark states. For the same reason, with $\lambda = 0$, the instantaneous interaction in the effective QED Hamiltonian is also exactly cancelled by the same interaction from one transverse photon exchange so that only one photon exchange Coulomb interaction remains. Thus, applying SRG to QED and let $\lambda = 0$ in the end of procedure, we obtain a conventional QED Hamiltonian which only contains the Coulomb interaction.

**D. Generality of the confining Hamiltonian in SRG**

As we have pointed out in the previous section, the existence of quark confining interaction in this formalism of nonperturbative QCD should be independent of the choice of a particular renormalization scale. In this subsection, we shall examine if such a light-front confining picture could be an artificial effect designed in our SRG renormalization scheme, especially if it is obtained from our specific choice of the smearing function (2.13). To answer this question, we shall rederive $H_{\lambda}$ without specifying the detailed form of the smearing function $f_{\lambda ij}$.

For heavy quarkonia in the HQET, $\Delta K_{ik}^- = \Delta K_{jk}^- = \frac{1}{K^+ (y - y')^2} ((\kappa_1 - \kappa_1')^2 + (y - y')^2 \Lambda^2)$. This immediately leads to $f_{\lambda ik} = f_{\lambda jk}$ because $x_{\lambda ik} = x_{\lambda jk} = \frac{|\Delta K_{ik}^-|}{K_i + K_j - \Delta K_{ik}^- + \lambda^2/K^+}$. Thus
eq. (2.10) is reduced to
\[ g_{\lambda ijk} = 1 - f_{\lambda ik}^2 = 1 - f_{\lambda jk}^2. \tag{3.19} \]
Following the same calculation of Sec.III.B, we can find that
\[ H_{\lambda iji} = f_{\lambda ij} \left( H_{\bar{Q}Q_{\text{freeij}}} + V_{Q\bar{Q}_{ij}} \right), \tag{3.20} \]
where
\[
H_{Q\bar{Q}_{\text{freeij}}} = \left[ 2(2\pi)^3 \right]^2 \delta^3(\vec{k}_1 - \vec{k}_3) \delta^3(\vec{k}_2 - \vec{k}_4) \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
\times \left\{ \frac{\Lambda}{m_Q} \left[ 2\kappa_{\perp}^2 + \Lambda^2 (2y^2 - 2y + 1) \right] - \frac{\Lambda^2}{4\pi^2} C_f \frac{\lambda^2}{K} f_s \ln \epsilon \right\}, \tag{3.21} \\
V_{Q\bar{Q}_{ij}}(y - y', \kappa_{\perp} - \kappa'_{\perp}) = 2(2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
\times -\frac{4g^2(T^a)(T^b)}{(K^+)^2} \left\{ \frac{1}{(y - y')^2} f_{\lambda ik}^2 + \frac{\Lambda^2}{(\kappa_{\perp} - \kappa'_{\perp})^2 + (y - y')^2 \Lambda^2} (1 - f_{\lambda ik}^2) \right\}. \tag{3.22}
\]
In Eq. (3.21), \( f_s \) depends on the detailed form of \( f_{\lambda ik} \).
If we took the smearing function \( f_{\lambda ik} \) by the theta function of Eq. (2.13), then Eq. (3.22) will immediately be reduced to Eq. (3.3). Now we shall consider the general case. Note that \( K_i^- + K_j^- \ll \lambda^2/K^+ \) since \( \Lambda \) is of the same order \( \Lambda_{QCD} \simeq 0.1 \sim 0.4 \) GeV (for heavy quarkonia, \( \Lambda \) is indeed about 0.05 \( \sim 0.20 \) GeV) but \( \lambda \) is of the order 1 GeV. Meanwhile, for confining interaction, we shall only concentrate in the region \( q^+ = (y - y')K^+ \rightarrow 0 \). Thus,
\[
x_{\lambda ik} = \frac{|\Delta K^+_{ik}|}{\lambda^2/K^+ - \Delta K^-_{ik}} \simeq \frac{(\kappa_{\perp} - \kappa'_{\perp})^2}{(\kappa_{\perp} - \kappa'_{\perp})^2 + (y - y')^2 \lambda^2} = \frac{q_{\perp}^2}{q_{\perp}^2 + \lambda^2 q^+ / K^+}, \tag{3.23}
\]
where \( q_{\perp} \) and \( q^+ \) are the transverse and longitudinal momentum carried by the exchanged gluons (see after Eq. (3.12)). According to the properties of \( f_{\lambda ij} \),
\[
\begin{align*}
q_{\perp}^2 &\leq \lambda^2 q^+ / 2K^+ \rightarrow x_{\lambda ik} \leq \frac{1}{3} \rightarrow f_{\lambda ik} = 1 \\
\lambda^2 q^+ / 2K^+ < q_{\perp}^2 < 2\lambda^2 q^+ / K^+ \rightarrow \frac{1}{3} < x_{\lambda ik} < \frac{2}{3} \rightarrow f_{\lambda ik} \text{ smoothly goes from 1 to 0 .} \\
q_{\perp}^2 &\geq 2\lambda^2 q^+ / K^+ \rightarrow x_{\lambda ik} \geq \frac{2}{3} \rightarrow f_{\lambda ik} = 0
\end{align*}
\]
(3.24)
Then the dominant contribution of the Fourier transformation of \( V_{Q\bar{Q}_{ij}} \) for \( q^+ \rightarrow 0 \) is given by
\[
\int \frac{dq^+ d^2q_{\perp}}{(2\pi)^2} e^{i(q^+ x - q_{\perp} \cdot x_{\perp})} \left\{ -\frac{4g^2}{K^+ + q_{\perp}^2} \frac{1}{(y - y')^2} f_{\lambda ik}^2 \right\} \\
= -\frac{g^2}{2\pi^2} \int dq^+ e^{i q^+ x} - \frac{q_{\perp}^2}{q^+} \frac{2J_1(|x_{\perp}|q_{\perp})}{|x_{\perp}|q_{\perp}} + \ldots, \tag{3.25}
\]
where \( q_{\perp}^2 = \lambda^2 q^+ / 2K^+ \), the dots denote the contribution from the region of \( \frac{1}{3} < x_{\lambda ik} < \frac{2}{3} \). Introducing a smooth function in this region is motivated for an easy control of possible
E. Extension to heavy-light quark systems

The heavy-light quark system (heavy hadrons containing one heavy quark) is one of the most interesting topics in the current study of heavy hadron physics. We now apply SRG to such systems. Indeed, Perry and Wilson have used different ways to show the existence of such a confining interaction\cite{4,18}. The crucial point is that the above confining picture is indeed model independent since the parameter \( \lambda \) is a renormalization scale. Under the consideration of SRG (see Section V), physical observables are independent of \( \lambda \). In other words, the confining interactions obtained here are derived from QCD without introducing new free parameters in SRG scheme.

The heavy-light quark system (heavy hadrons containing one heavy quark) is one of the most interesting topics in the current study of heavy hadron physics. We now apply SRG to such systems. Following the general procedure, it is easy to find the nonperturbative part of the effective Hamiltonian for heavy-light quark systems,

\[
H_{\lambda ij} = \theta\left( \frac{\lambda^2}{K^+} - |\Delta K_{ij}| \right) \left\{ H_{Qij}^{\text{free}} + V_{Qij} \right\},
\]

where

\[
H_{Qij}^{\text{free}} = \left[ 2(2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2) \delta^3(p_1 - p_2) \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \right. \\
\times \left\{ (y - 1)\vec{\Lambda}^2 + \frac{\kappa_1^2 + m_2^2}{y} - \frac{g^2}{2\pi^2} C_f \frac{\lambda^2}{K^+} \ln \epsilon \right\},
\]

\[
V_{Qij}(y - y', \kappa_\perp - \kappa'_\perp) = 2(2\pi)^3 \delta^3(\vec{k}_1 + \vec{p}_1 - \vec{k}_2 - \vec{p}_2) \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \frac{-2g^2(T_a)(T_a)}{(K^+)^2} \\
\times \left\{ \frac{2}{(y - y')^2} - \left[ \frac{(\kappa_\perp - \kappa'_\perp)^2}{(y - y')^2} - \frac{\kappa_\perp^2 - \kappa'_\perp \cdot \kappa'_\perp}{y(y - y')} - \frac{\kappa_\perp \cdot \kappa'_\perp - (\kappa'_\perp)^2}{y'(y - y')} \right] \\
\times \left[ \frac{\theta(B - \lambda^2)\theta(B - A)}{(\kappa_\perp - \kappa'_\perp)^2 - (y - y')(\frac{\kappa_\perp^2}{y} - (\kappa'_\perp)^2)} + \frac{\theta(A - \lambda^2)\theta(A - B)}{(\kappa_\perp - \kappa'_\perp)^2 + (y - y')^2} \right] \right\},
\]

sighnatures in the further numerical computations\cite{1}. However, we found that the use of (2.13) for \( f_{ik} \) does not cause any new singularity for heavy quarkonium systems. It is more interesting to see that the dominant contribution of \( V_Q\) (the first term in eq. (3.23)) has the same form of eq. (3.12) and results in the same confining interactions of eqs. (3.14) and (3.16) (except for a factor \( 1/\sqrt{2} = f_s \)) that are obtained by using eq. (2.13). This shows that the existence of confining interactions in SRG is a general feature in our formulation. The choice of eq. (2.13) is just a simple way to introduce a convenient low-energy renormalization scale to nonperturbative QCD that makes the confining interaction explicit for heavy quarkonium systems. Indeed, Perry and Wilson have used different ways to show the existence of such a confining interaction\cite{4,18}. The crucial point is that the above confining picture is indeed model independent since the parameter \( \lambda \) is a renormalization scale. Under the consideration of SRG (see Section V), physical observables are independent of \( \lambda \). In other words, the confining interactions obtained here are derived from QCD without introducing new free parameters in SRG scheme.
with $B \equiv \frac{(\kappa - \kappa')^2}{y - y'} - \frac{\kappa^2}{y} + \frac{(\kappa')^2}{y'}$ and the function $A$ has the same form as in quarkonium case. Here we have also introduced $y = p_1^+/K^+$, $\kappa_\perp = p_{1\perp} - yK_\perp$, but the range of $y$ is now given by $0 < y = m^+ \lambda^+ \eta < \infty$.

The heavy-light quark effective Hamiltonian is $m_Q$-independent. This is because in heavy-light quark systems the heavy quark kinetic energy can be treated as a perturbative correction to $H_{\lambda 0}$. Obviously the above $H_{\lambda 0}$ has the heavy quark spin and flavour symmetry. Compared to the $V_{Q\bar{Q}}, V_{Q\bar{q}}$ interactions are much more complicated. But it is not difficult to check that the above $V_{Q\bar{q}}$ contains a confining interaction when $q^+$ is small. The confining mechanism is the same for $Q\bar{Q}$ and $Q\bar{q}$ systems, as well as for $q\bar{q}$ systems.

In conclusion, we have obtained in this section the nonperturbative part of a confining QCD Hamiltonian for heavy-heavy and heavy-light quark systems in the WCT scheme. We have also shown that the confining mechanism is independent of a specific choice of the smearing functions. It is indeed also independent of the HQET, namely the same confining picture occurs when we include all the $1/m_Q$ corrections (see the discussion in the previous section). We are now ready to study hadron states on the light-front and to explore how the WCT scheme works in the present formulation.

**IV. THE WCT TO HEAVY HADRON BOUND STATES**

As we mentioned in the Introduction, the ideas of WCT to nonperturbative QCD is to begin with the effective QCD Hamiltonian $H_\lambda = H_{\lambda 0} + H_M$. Then using the constituent picture to solve nonperturbatively the hadronic bound state equations governed by $H_{\lambda 0}$ and to determine the running coupling constant $g_\lambda$. If one could properly choose the nonperturbative $H_{\lambda 0}$ such that $g_\lambda$ is arbitrarily small, then the corrections from $H_M$ could be computed perturbatively, and we would say that a WCT to nonperturbative QCD is realized. In this section, we shall study such a WCT to heavy hadron bound states.

**A. General structure of light-front bound state equations**

In general, a hadronic bound state on the light-front can be expanded in the Fock space composed of states with definite number of particles [17,23]. Formally, it can be expressed as follows

$$|\Psi(P^+, P_\perp, \lambda_s)\rangle = \sum_{n,\lambda_i} \int \left( \prod_i [d^3 \bar{p}_i] \right) 2(2\pi)^3 \delta^3(\bar{P} - \sum_i \bar{p}_i) |n, \bar{p}, \lambda_i\rangle \Phi_n(x_i, \kappa_\perp i, \lambda_i), \quad (4.1)$$

where $P^+, P_\perp$ are its total longitudinal and transverse momenta respectively and $\lambda_s$ its total helicity, $|n, \bar{p}, \lambda_i\rangle$ is a Fock state consisting of $n$ constituents, each of which carries momentum $\bar{p}_i$ and helicity $\lambda_i (\sum_i \lambda_i = \lambda_s)$; $\Phi(x_i, \kappa_\perp i, \lambda_i)$ the corresponding amplitude which only depends on the helicities $\lambda_i$, the longitudinal momentum fractions $x_i$, and the relative transverse momenta $\kappa_\perp i$: $x_i = \frac{\bar{p}_i^\perp}{P^+}$, $\kappa_{\perp i} = p_{i\perp} - x_i P_\perp$.

The eigenstate equation that the wave functions obey on the light-front is given by

$$H_{\text{LF}}|P^+, P_\perp, \lambda_s\rangle = \frac{P^2 + M^2}{P^+}|P^+, P_\perp, \lambda_s\rangle, \quad (4.2)$$
where $H_{LF} = P^-$ the light-front Hamiltonian. Explicitly, for a meson wave function, the corresponding light-front bound state equation is:

$$
\left( M^2 - M_0^2 \right) \left[ \begin{array}{c} \Phi_{q\bar{q}} \\ \Phi_{q\bar{g}} \end{array} \right] = \left[ \begin{array}{c} \langle q\bar{q}|H_{int}|q\bar{q} \rangle \\ \langle q\bar{g}|H_{int}|q\bar{q} \rangle \\ \vdots \\ \langle q\bar{g}|H_{int}|q\bar{g} \rangle \\ \vdots \\ \langle q\bar{g}|H_{int}|q\bar{g} \rangle \end{array} \right] \left[ \begin{array}{c} \Phi_{q\bar{q}} \\ \Phi_{q\bar{g}} \end{array} \right],
$$

(4.3)

where $M_0^2 = \sum_i \frac{\kappa_i^2 + m_i^2}{\xi_i}$ the so-called invariant mass, $H_{int}$ the interaction part of $H_{LF}$.

Obviously, solving eq. (4.3) from QCD with the entire Fock space is impossible. A basic motivation of introducing the WCT scheme is to simplify the complexities in solving the above equation. In the present framework, $H_{LF} = H_\lambda$, where $H_\lambda$ has already decoupled from high energy states. Furthermore, the reseparation $H_\lambda = H_{\lambda 0} + H_M$ is another crucial step in WCT, where only $H_{\lambda 0}$ is assumed to have the nonperturbative contribution to bound states through eq. (4.3), and $H_M$ is supposed to be a perturbative term which should not be considered when we try to solve eq. (4.3) nonperturbatively.

The next important step in the WCT scheme is the use of a constituent picture. The success of the constituent quark model suggests that we may only consider the valence quark Fock space in determining the hadronic bound states from $H_{\lambda 0}$. In this picture, quarks and gluons must have constituent masses. This constituent picture can naturally be realized on the light-front [1]. However, an essential difference from the phenomenological constituent quark model description is that the constituent masses introduced here are $\lambda$ dependent. The scale dependence of constituent masses (as well as the effective coupling constant) is determined by solving the bound states equation and fitting the physical quantities with experimental data. But for heavy quark mess, this $\lambda$-dependence can be ignored. Once the constituent picture is introduced, we can truncate the general expression of the light-front bound states to only including the valence quark Fock space. The higher Fock space contributions can be recovered as a perturbative correction through $H_M$. Thus, eq. (4.1) for heavy quarkonia can be approximately written as:

$$
|\Psi(K^+, K_\perp, \lambda_s)\rangle = \sum_{\lambda_1 \lambda_2} \int [d^3 \bar{k}_1][d^3 \bar{k}_2] 2(2\pi)^3 \delta^3(\bar{K} - \bar{k}_1 - \bar{k}_2) \times \phi_{Q\bar{Q}}(y, \kappa_\perp) b_1^\dagger(k_1, \lambda_1) d_{-\lambda_2}^\dagger(k_2, \lambda_2) |0\rangle,
$$

(4.4)

where the wavefunction $\phi_{Q\bar{Q}}(y, \kappa_\perp)$ may be mass dependent due to the kinetic energy in $H_{\lambda 0}$ [see (3.7)] but it is spin independent in heavy mass limit. Also note that the heavy quarkonium states in heavy mass limit are labelled by the residual center mass momentum $K^\mu$. We may normalize eq. (4.4) as follows:

$$
\langle \Psi(K'^+, K'_\perp, \lambda'_s)|\Psi(K^+, K_\perp, \lambda_s)\rangle = 2(2\pi)^3 K^+ \delta^3(\bar{K} - \bar{K}') \delta_{\lambda'_s\lambda_s},
$$

(4.5)

which leads to

$$
\int \frac{dyd^2\kappa_\perp}{2(2\pi)^3} |\phi_{Q\bar{Q}}(y, \kappa_\perp)|^2 = 1.
$$

(4.6)

With the above analysis on the quarkonium states, it is easy to derive the corresponding bound state equation. Let $H_{LF} = H_{\lambda 0}$ of eq. (3.6), eq. (4.3) is reduced to

20
\[
\left\{ 2\Lambda^2 - \frac{\Lambda^2}{m_Q} \left[ 2\kappa_+^2 + \Lambda^2 (2y^2 - 2y + 1) \right] \right\} \phi_{Q\bar{Q}}(y, k_\perp) = \left( - \frac{g_3^2}{2\pi^2} \lambda^2 C_f \ln \epsilon \right) \phi_{Q\bar{Q}}(y, k_\perp)
- 4g_3^2 (T^a)(T^a) \int \frac{dy'd^2\kappa'_+}{2(2\pi)^3} \left\{ \frac{1}{(y - y')^2} \theta(\lambda^2 - A) \right. \\
+ \frac{\Lambda^2}{(\kappa_- - \kappa'_-)^2 + (y - y')^2} \theta(\Lambda^2 - \lambda^2) \left. \right\} \phi_{Q\bar{Q}}(y', k'_\perp). \tag{4.7}
\]

This is the light-front bound state equation for heavy quarkonia in the WCT scheme.

For the heavy mesons containing one heavy quark, similar consideration leads to

\[
\left\{ 2\Lambda^2 - \Lambda \left[ y\Lambda - \frac{\kappa_+^2 + m_q^2(\lambda)}{y\Lambda} \right] \right\} \Phi_{Q\bar{F}}(y, k_\perp, \lambda_1, \lambda_2)
= \left( - \frac{g_3^2}{2\pi^2} \lambda^2 C_f \ln \epsilon \right) \Phi_{Q\bar{F}}(y, k_\perp, \lambda_1, \lambda_2)
+ (K^+)^2 \int \frac{dy'd^2\kappa'_+}{2(2\pi)^3} V_{Q\bar{F}}(y - y', \kappa_- - \kappa'_-) \Phi_{Q\bar{F}}(y', k'_\perp, \lambda_1, \lambda_2), \tag{4.8}
\]

where \( V_{Q\bar{F}} \) is given by eq.\{3,28\}. Note that the light antiquark here is a brown muck, a current light antiquark surround by infinite gluons and \( q\bar{q} \) pairs that results in a constituent quark mass \( m_q \) which is a function of \( \lambda \).

**B. A general analysis of light-front wavefunctions**

A numerical computation to the bound state equations, eqs.(4.7) and (4.8), is actually not too difficult. However, to have a deeper insight about the internal structure of light-front bound states and to determine the scale dependence of the effective coupling constant in \( H_\lambda \), it is better to have an analytic analysis. For this propose, we would like to present a general analysis of light-front hadronic wavefunctions and then use variational approach to solve the bound state equations.

The heavy hadronic wavefunctions in the heavy mass limit are rather simple. First of all, the heavy quark kinematics have already added some constraints on the general form of the light-front wavefunction \( \phi(x, \kappa_\perp) \). When we introduce the residual longitudinal momentum fraction \( y \) for heavy quarks, the longitudinal momentum fraction dependence in \( \phi \) is quite different for the heavy-heavy, heavy-light and light-light mesons.

For the light-light mesons, such as pions, rhos, kaons etc., the wavefunction \( \phi_{q\bar{q}}(x, \kappa_\perp) \) must vanish at the endpoint \( x = 0 \) or 1. This can be seen from the kinetic energy term in eq.(4.3), where \( M_0^2 = \frac{\kappa_+^2 + m_q^2}{x} - \frac{\kappa_+^2 + m_q^2}{1-x} \) for the valence Fock space. To ensure that the bound state equation is well defined in the entire range of momentum space, \( |\phi_{q\bar{q}}(x, \kappa_\perp)|^2 \) must fall down to zero in the longitudinal direction not slower than \( 1/x \) and \( 1/(1-x) \) when \( x \to 0 \) and 1, respectively. In other words, at least \( \phi_{q\bar{q}}(x, \kappa_\perp) \sim \sqrt{x(1-x)} \). For heavy-light quark mesons, namely the \( B \) and \( D \) mesons, the wavefunction \( \phi_{Q\bar{F}}(y, \kappa_\perp) \) is required to vanish at \( y = 0 \), where \( y \) is the residual longitudinal momentum fraction carried by the light quark. This is because the kinetic energy in eq.(4.8) only contains a singularity at \( y = 0 \). On the other hand, since \( 0 \leq y \leq \infty \), \( \phi_{Q\bar{F}}(y, \kappa_\perp) \) should also vanish when \( y \to \infty \). Hence, a possible
simple solution is \( \phi_{Q\bar{Q}}(y, \kappa_\perp) \sim \sqrt{ye^{-\alpha y}} \) or \( \sqrt{ye^{-\alpha y^2}} \). Other form to suppress the singularity at \( y = 0 \), like \( e^{-\alpha/y} \) instead of \( \sqrt{y} \), is also possible. For heavy quarkonia, \(-\infty < y < \infty\), the normalization forces \( \phi_{Q\bar{Q}}(y, \kappa_\perp) \) to vanish as \( y \to \pm \infty \). Thus, a simple solution may be \( \phi_{Q\bar{Q}}(y, \kappa_\perp) \sim e^{-\alpha y^2} \).

On the other hand, the transverse momentum dependence in these light-front wavefunctions should be more or less similar. They all vanish at \( \kappa_\perp \to \pm \infty \). A simple form of the \( \kappa_\perp \) dependence for these wavefunctions is a Gaussian function: \( e^{-\kappa_\perp^2/2\omega^2} \).

The above analysis of light-front wavefunctions is only based on the kinetic energy properties of the constituents. Currently, many investigations on the hadronic structures use phenomenological light-front wavefunctions. One of such wavefunctions that has been widely used in the study of heavy hadron structure is the BSW wavefunction [24],

\[
\phi_{BSW}(x, \kappa_\perp) = \mathcal{N}\sqrt{x(1-x)} \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right) \exp\left[-\frac{M_H^2}{2\omega^2}(x-x_0)^2\right], \tag{4.9}
\]

where \( \mathcal{N} \) is a normalization constant, \( \omega \) a parameter of order \( \Lambda_{QCD} \), \( x_0 = (\frac{1}{2} - \frac{m_1^2-m_2^2}{2M_H^2}) \), and \( M_H, m_1, \) and \( m_2 \) are the hadron, quark, and antiquark masses respectively. In the heavy mass limit, the BSW wavefunction can be produced from our analysis based on the light-front bound state equations.

Explicitly, for heavy-light quark systems, such as the \( B \) and \( D \) mesons, one can easily find that in the heavy mass limit, \( m_1 = m_Q \sim M_H, m_q << m_Q \) so that \( x_0 = 0 \). Meanwhile, we also have \( M_Hx = \bar{\Lambda}y \). Furthermore, the factor \( \sqrt{x(1-x)} \) can be rewritten by \( \sqrt{y} \) in according to the corresponding bound state equation discussed above. Thus, the BSW wavefunction is reduced to

\[
\phi_{Q\bar{Q}}(y, \kappa_\perp) = \mathcal{N}\sqrt{\bar{\Lambda}y} \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right) \exp\left(-\frac{\bar{\Lambda}^2 y^2}{2\omega^2}\right). \tag{4.10}
\]

This agrees with our qualitative analysis given above. Indeed, using such a wavefunction we have already computed the universal Isgur-Wise function in \( B \to D, D^* \) decays [3]: \( \xi(v \cdot v') = \frac{1}{v' v} \), and from which we obtained the slope of \( \xi(v \cdot v') \) at the zero-recoil point, \( \rho^2 = -\xi'(1) = 1 \), in excellent agreement with the recent CLCO result [25] of \( \rho^2 = 1.01 \pm 0.15 \pm 0.09 \).

For heavy quarkonia, such as the \( \bar{b}b \) and \( c\bar{c} \) states, \( m_1 = m_2 = m_Q \) which leads to \( x_0 = 1/2 \) in eq.(4.9). Also note that \( M_H(x-1/2) = \bar{\Lambda}y \), and the factor \( \sqrt{x(1-x)} \) must be totally dropped as we have discussed form the quarkonium bound state equation. Then the BSW wavefunction for quarkonia is reduced to

\[
\phi_{Q\bar{Q}}(y, \kappa_\perp) = \mathcal{N}\exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right) \exp\left(-\frac{\bar{\Lambda}^2 y^2}{2\omega^2}\right), \tag{4.11}
\]

which is the exact form as we expected from the qualitative analysis. Here we have not taken the limit of \( m_Q \to \infty \) for heavy quarkonia. Thus a possible \( m_Q \) dependence in wavefunction may be hidden in the parameter \( \omega \).

In the following, we shall use the above Gaussian-type wavefunction ansatz to solve the light-front quarkonium bound state equation, and from which to determine qualitatively the nonperturbative scaling dynamics and to examine the ideas of WCT to heavy hadron bound states.
C. WCT of nonperturbative QCD description to heavy quarkonia

We take the normalized wavefunction ansatz of (4.11),
\[ \psi_{Q\bar{Q}}(y, \kappa_{\perp}) = 4\sqrt{\Lambda} \left( \frac{\pi}{\omega_\lambda^2} \right)^{3/4} \exp \left( -\frac{\kappa_{\perp}^2}{2\omega_\lambda^2} \right) \exp \left( -\frac{\Lambda^2}{2\omega_\lambda^2} y^2 \right), \]  
(4.12)
as a quarkonium trial wavefunction, where \( \omega_\lambda \) means that the wavefunction is also scale dependent. Substituting the above wavefunction into the quarkonium bound state equation (4.7), we have
\[ 2\Lambda^2 = \mathcal{E}_{kin} - \frac{g_\lambda^2}{2\pi^2} \lambda^2 C_f \ln \epsilon + \mathcal{E}_{nonc} + \mathcal{E}_{Coul}, \]  
(4.13)
where \( \mathcal{E}_{kin} = \frac{\Lambda}{m_Q} (3\omega_\lambda^2 + \Lambda^2) \) represents the kinetic energy, \( \mathcal{E}_{nonc} \) is the contribution of the noncancellation of the instantaneous interaction,
\[ \mathcal{E}_{nonc} = \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ \gamma + \ln \frac{\lambda^2 \epsilon}{4\omega_\lambda^2} + E_1(\varpi^2) + \frac{\sqrt{\pi}}{\varpi} \text{Erf}(\varpi) \right\}, \]  
(4.14)
and \( \mathcal{E}_{Coul} \) from the Coulomb interaction,
\[ \mathcal{E}_{Coul} = -\frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ \frac{\sqrt{\pi}}{\varpi} \left[ 1 - \text{Erf}(\varpi) \right] + \frac{1}{\varpi^2} \left[ 1 - e^{-\varpi^2} \right] \right\}, \]  
(4.15)
where \( \gamma = 0.57721566... \) the Euler constant, \( \epsilon \) the small longitudinal momentum cutoff, the dimensionless \( \varpi \) is defined by \( \varpi = \frac{\lambda^2}{2\omega_\lambda^2} \), and \( E_1 \) and Erf are the exponential integral function and the error function, respectively.

We may rewrite the term \( \ln \frac{\lambda^2 \epsilon}{4\omega_\lambda^2} \) in \( \mathcal{E}_{nonc} \) by \( \ln \frac{\lambda^2 \epsilon}{4\omega_\lambda^2} = \ln \epsilon + \ln \varpi^2 + \ln \frac{\Lambda^2}{\lambda^2} \). It shows that \( \mathcal{E}_{nonc} \) contains a logarithmic divergence \( \ln \epsilon \) which exactly cancels the same divergence from the self-energy correction, as expected, and the term \( \ln \varpi^2 \) is the logarithmic confining energy.

After the cancellation of the infrared \( \ln \epsilon \) divergences in eq.(4.13), the binding energy for heavy quarkonia is given by the kinetic energy plus the interaction energy:
\[ 2\Lambda^2 = \mathcal{E}_{kin} + \mathcal{E}_{conf} + \mathcal{E}_{Coul} = \frac{\Lambda}{m_Q} \left\{ 3\omega_\lambda^2 + \Lambda^2 \right\} + \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ F(\varpi) + \ln \frac{\Lambda^2}{\lambda^2} \right\}, \]  
(4.16)
where
\[ F(\varpi) = \gamma + \ln \varpi^2 + E_1(\varpi^2) - \frac{\sqrt{\pi}}{\varpi} \left[ 1 - 2\text{Erf}(\varpi) \right] - \frac{1}{\varpi^2} \left[ 1 - e^{-\varpi^2} \right]. \]  
(4.17)
In Fig.1, we plot the confining energy, the Coulomb energy and the totally interaction energy as functions of \( \varpi \) which is proportional to the radial variable in the light-front space,
\[ \varpi \sim \frac{1}{\omega_\lambda} \sim r_l. \]  
(4.18)
Fig. 1 shows that the total interaction energy is a typical combination of the Coulomb interaction at short distance and a confining interaction at long distance that has been widely used in previous phenomenological description of hadronic states, but it is now analytically derived from QCD without introducing any free parameter except for the QCD running coupling constant. Furthermore, eq. (4.16) also indicates that without considering the kinetic energy, we cannot find a stable quarkonium bound state. The kinetic energy balances the interaction energy and ensures the existence of a stable solution for (4.16). Therefore, the first order kinetic energy in HQET is an important nonperturbative effect in binding two heavy quarks, as noticed first by Mannel et al. [11].

If we know the experimental value of the quarkonium binding energy \( \Lambda \), minimizing eq. (4.16) can completely determine the parameter \( \omega_{\lambda} \) and the coupling constant \( g_{\lambda} \). The precise value of quarkonium binding energy that can be compared with the data in Particle Data Group [26] must include the spin-splitting energy (\( 1/m_Q \) corrections) which we will present in the forthcoming paper [27]. Here, to justify whether a WCT of nonperturbative QCD can become possible in the present formulation, we will give a schematic calculation. It is known that \( \Lambda \) is of the same order \( \Lambda_{QCD} \) which is about 100–400 MeV. We choose \( \lambda \) as a typical hadronic mass, \( \lambda = 1 \) GeV. The charmed and bottom quark masses used here are \( m_c = 1.4 \) GeV and \( m_b = 4.8 \) GeV. From the particle data [20], the lowest charmonium states \( M(\eta_c(1S)) = 2.98 \) GeV and \( M(J/\Psi(1S)) = 3.10 \) GeV, the bottomonium state \( M(\Upsilon(1S)) = 9.46 \) GeV. Hence the binding energies of quarkonia \( \Lambda < 400 \) MeV in the above choice of heavy quark mass. To solve (4.16) we shall take several values of \( \Lambda \) within the above range. (In principle, it is not necessary to must use a value of \( \Lambda \) below 400 MeV.) The results are listed in Tables I and II for charmonium and bottomonium, respectively, where \( \omega_{\lambda 0} \) denotes the minimum point of the binding energy \( \Lambda \) (4.16).

We see from the Tables I–II that the coupling constant \( \alpha_{\lambda} = g_{\lambda}^2/4\pi \) is very small. For instance, with \( \Lambda = 200 \) MeV, we obtain

\[
\alpha_{\lambda} = \begin{cases} 
0.02665 & \text{charmonium,} \\
0.06795 & \text{bottomonium,}
\end{cases}
\]

which is much smaller than that extrapolated from the running coupling constant in the naive perturbative QCD calculation. The parameter \( \omega_{\lambda 0} \) is the mean value of the (transverse) momentum square of heavy quarks inside quarkonia:

\[
\langle k_{\perp}^2 \rangle = \omega_{\lambda}^2.
\]

For charmonium, we can see that the resulting \( \omega_{\lambda} \) are typical values of \( \Lambda_{QCD} \sim \Lambda \). The kinetic energy is about a half of the interaction energy. For bottomonium, we find that the binding energy \( \Lambda \) cannot be too large. In fact, when \( \Lambda \) is over about 260 MeV, eq. (4.16) has no solution. Meanwhile, compared to charmonium, the effective coupling constant is relatively large (in contrast to the perturbative running coupling constant which is smaller with increasing \( m_Q \) if it is taken as the energy scale). Also the values of \( \omega_{\lambda} \) in bottomonium wavefunctions are larger than that in charmonium. The difference between charmonium and bottomonium in the above nonperturbative calculation can be understood as follows. As we know, in the nonrelativistic quark model, the quark momentum in quarkonia is proportional to the quark mass, \( \omega_{\lambda}^2 \sim m_Q \) [21]. Our relativistic QCD bound state solution exhibits such a property. This also indicates the first order kinetic energy has of the same order the
leading Hamiltonian while all other terms are suppressed by the power of $1/m$ [see eq.(3.2)]. Therefore, the kinetic energy must be treated nonperturbatively when we apply HQET to heavy quarkonia. As a result, the bottomonium kinetic energy ($\sim \omega^2$) becomes large as well. To have a nonperturbative balance between the kinetic energy and the interaction energy in the bound states, the coupling constant in bottomonium must be larger than that in charmonium. All these properties now have been manifested in the solution of eq.(4.16). A more precise determination of $\alpha_\lambda$ (i.e., $g_\lambda$) requires an accurate computation of the low-lying quarkonium spectroscopy with the $1/m_Q$ corrections included [27]. Nevertheless, it has been shown that the effective coupling constant in $H_\lambda$ is very small at the hadronic mass scale.

In order to see how this weak coupling constant varies with the scale $\lambda$, we take $\Lambda = 200$ MeV and vary the value of $\lambda$ around 1 GeV. The result is listed in Table III. We find that the coupling constant is decreased very faster with increasing $\lambda$. In other words, with a suitable choice of the hadronic mass scale $\lambda$ in SRG, we can make the effective coupling constant $\alpha_\lambda$ in $H_\lambda$ arbitrarily small, and therefore the WCT of nonperturbative QCD can be achieved in terms of $H_\lambda$ such that the corrections from $H_{\lambda I}$ can be truly computed perturbatively.

V. DISCUSSION AND SUMMARY

Thus far, the main ideas of the WCT to nonperturbative QCD proposed in the recent publication [1] have, at least qualitatively, been achieved for heavy quarkonia when $\lambda$ is around hadronic mass scale ($\sim 1$ GeV). To have a deep understanding of how WCT works to nonperturbative QCD, we shall study the $\lambda$-dependence of $\alpha_\lambda$ and discuss the physical implication of the WCT in this last section.

A. Running effective coupling constant $\alpha_\lambda$ in SRG

From Tables I to III, we find that values of the dimensionless parameter $\varpi = \frac{\lambda^2}{2\Lambda^2 \omega_{\lambda_0}}$ are greater than 2.5. When $x > 2.5$, the exponential integral function and the error function are simply reduced to $E_1(x) = 0$ and $\text{Erf}(x) = 1$. Then, eq.(4.1) is reduced to

$$2\Lambda^2 = \frac{3\Lambda}{m_Q} \omega_\lambda^2 - \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ \frac{4\Lambda^2 \omega_\lambda^2}{\lambda^4} - \sqrt{\pi} \frac{2\Lambda \omega_\lambda}{\lambda^2} - \left( \gamma + \ln \frac{\lambda^2}{4} - \ln \omega_\lambda^2 \right) \right\} + \frac{\Lambda^3}{m_Q} ,$$

with an error less than $10^{-5}$. Minimizing $\Lambda$ with respect to $\omega_\lambda$, we obtain

$$\left\{ \frac{3\Lambda}{m_Q} - \frac{g_\lambda^2}{2\pi^2} C_f \frac{4\Lambda^2}{\lambda^2} \right\} \omega_\lambda^2 = \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ 1 - \sqrt{\pi} \frac{\Lambda \omega_\lambda}{\lambda^2} \right\} .$$

Therefore, eq.(5.1) becomes

$$2\Lambda^2 = \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ 1 + \gamma + \ln \frac{\lambda^2}{4} - \ln \omega_{\lambda_0}^2 + \sqrt{\pi} \frac{\Lambda \omega_{\lambda_0}}{\lambda^2} \right\} + \frac{\Lambda^3}{m_Q} ,$$

where $\omega_{\lambda_0}$ is a solution of (5.2). Since $\Lambda$ is the binding energy of quarkonia, it should be $\lambda$-independent. Thus, eqs.(5.2) and (5.3) determine the $\lambda$-dependence of $g_\lambda$. The result is
\[\alpha_\lambda = \frac{g_\lambda^2}{4\pi} = \frac{\pi}{C_f} \left( \frac{\Lambda^2}{\lambda^2} \right) \frac{1}{a + b \ln \frac{\lambda^2}{\Lambda^2}},\]
\sim \frac{\Lambda^2}{\lambda^2} \text{ (for a relatively large } \lambda \gg \bar{\Lambda}) \tag{5.4}\]

where the coefficients \(a\) and \(b\) can be obtained by numerically solving eqs. (5.2) and (5.3). The coefficient \(b\) is almost a constant (with a weak dependence on \(m_Q\) but independence on \(\bar{\Lambda}\) and \(\lambda\)), while \(a\) depends on both \(\bar{\Lambda}\) and \(m_Q\), and also slightly on \(\lambda\). For \(\lambda \geq 0.6\) GeV, the \(\lambda\)-dependence in the parameter \(a\) is negligible. In Fig. 2, we plot the \(\lambda\)-dependence of the effective coupling constant \(\alpha_\lambda\) for charmonium. The dots are the numerical solutions of (4.16) and the solid line is given by the analytical form (5.4) with \(b = 1.15\), and \(a = -0.25\) for \(\bar{\Lambda} = 0.2\) GeV and \(a = 1.1\) for \(\bar{\Lambda} = 0.4\) GeV. We can see that (5.4) is a very good analytical solution of the eqs. (5.2) and (5.3) [or of the minimizing eq. (4.16)].

To understand the running behavior of \(\alpha_\lambda\), we calculate its SRG \(\beta\)-function. Denote the binding energy
\[
\bar{\Lambda} = \bar{\Lambda}(g_\lambda, \omega_\lambda, \lambda). \tag{5.5}
\]
The invariance of the binding energy \(\bar{\Lambda}\) under similarity renormalization group transformation means that \(\bar{\Lambda}\) determined from \(H_{\lambda 0}\) and \(H'_{\lambda' 0}\) must be the same for \(\lambda \neq \lambda'\). Let \(\lambda' = \lambda + \delta\lambda\), we obtain the corresponding similarity renormalization group equation
\[
(\lambda \frac{\partial}{\partial \lambda} + \beta \frac{\partial}{\partial g_\lambda} + \gamma_\omega \frac{\partial}{\partial \omega_\lambda}) \bar{\Lambda}(g_\lambda, \omega_\lambda, \lambda) = 0, \tag{5.6}
\]
where the quantity \(\beta\) is the similarity renormalization group \(\beta\) function which is defined by
\[
\beta(g_\lambda) = \lambda \frac{dg_\lambda}{d\lambda} \bigg|_{\lambda = \lambda(g_\lambda)}, \tag{5.7}
\]
and \(\gamma_\omega\) is an anomalous dimension that describes the running properties of the bound state wavefunction. The \(\beta\) function can be determined by solving the SRG equation (5.7) or directly computed from eq. (5.4),
\[
\beta = -g_\lambda \left( 1 + \frac{2b}{a + 2b \ln \frac{\lambda^2}{\bar{\Lambda}^2}} \right) \bigg|_{\lambda = \lambda(g_\lambda)} \approx -g_\lambda \text{ (for a relatively large } \lambda >> \bar{\Lambda}). \tag{5.8}
\]
As we will see later, this \(\beta\)-function also indicates the existence of confining interaction in low energy region.

**B. Why a WCT in low energy QCD region**

As we have pointed out in the last section we can always make the effective coupling constant \(\alpha_\lambda\) small with a suitable choice of the low energy scale \(\lambda\) such that a WCT to nonperturbative QCD can be realized on the light-front. The above dependence of \(\alpha_\lambda\) with
\(\lambda\) shows that the effective coupling constant \(\alpha_\lambda\) becomes smaller with larger \(\lambda\). Thus, to have a WCT to nonperturbative QCD, \(\lambda\) cannot be too small. In practice, \(\lambda\) is of the order 1 GeV. Now, the question is what is the physical picture behind this dependence of \(\alpha_\lambda\) with \(\lambda\).

As we discussed in the previous sections, the effective low energy Hamiltonian \(H_{\lambda 0}\) is derived by integrating out the light-front high energy modes above \(\lambda\) through the SRG. The existence of an explicit quark confining interaction in \(H_{\lambda 0}\) (like to extract a mean-field in strongly correlated systems) is crucial to have a WCT in low-energy QCD region. Formally, as we have seen that the confining interaction in \(H_{\lambda 0}\) comes from the noncancellation of instantaneous gluon exchange with energy below the scale \(\lambda\). With the larger \(\lambda\), the more the instantaneous interaction contributes to \(H_{\lambda 0}\) so that the confining interaction becomes stronger. A physical interpretation of the low energy scale \(\lambda\), given by Wilson [18], is that \(\lambda\) corresponds to a up-bounded value of dynamical gluon mass in low-energy region. Introducing \(\lambda\) in SRG forces the gluon exchange energy in the \(q\bar{q}\) effective interaction of \(H_{\lambda 0}\) to be never less than \(\lambda\). This is equivalent to say that dynamical gluons in low energy region have a mass of order \(\lambda\) such that the energy of gluons propagating in \(q\bar{q}\) channel cannot be smaller than this dynamical mass. A large \(\lambda\) implies a large dynamical gluon mass which, on one hand, results in an explicit strong quark confining interaction in \(H_{\lambda 0}\), and on the other hand, makes the residual quark-gluon interactions weak. The effective coupling constant \(\alpha_\lambda\) characterizes these residual interactions (\(H_{\lambda f}\) is expanded in terms of \(\alpha_\lambda\)). While, the smaller \(\lambda\) implies to take the smaller dynamical gluon mass in low energy region. Then the explicit confining interactions in \(H_{\lambda 0}\) is weakened and the residual interactions become stronger (i.e., some of the confining interaction is hidden in the residual interactions). The limit \(\lambda \to 0\) implies to keep massless gluons in the low energy region. Then, our effective QCD theory turns back to the canonical form where the confining interaction is completely hidden in the beautiful but complicated quark-gluon interactions and the coupling constant becomes very strong in low energy region, as we expected from the canonical theory. This is a physical interpretation of the above dependence of \(\alpha_\lambda\) with \(\lambda\). It also provides a physical picture why a WCT can be applied to low energy QCD region for a finite \(\lambda\) in our description.

To further examine the above understanding, we study the parameter \(\omega_{\lambda 0}\) (the mean value of the transverse momentum quarks carried inside the hadrons) as a function of \(\lambda\). The result is plotted in Fig. 3. From Fig. 3, we see that with increasing \(\lambda\), \(\omega_{\lambda 0}\) is decreased. Correspondingly, the distance between two quarks inside the quarkonia, \(r_l \sim \frac{1}{\omega_{\lambda 0}}\), is increased. This ensures that confining interaction becomes stronger with larger \(\lambda\) (as also shown in Table IV). Besides, Fig.2 also shows that \(\lambda\) determined from the bound state equation can never go to zero. It implies that the dynamical gluons in low energy region are most likely massive around a few hundreds MeV.

It may also be worth noting that the low energy scale \(\lambda\) introduced here is totally different from the high energy scale \(Q^2\) used in perturbative QCD. The use of the scale \(Q^2\) in perturbative QCD is to separate high and low energy region such that one should in principle integrate out the physics below the energy scale \(Q^2\) [practically, instead of doing this (since none is able to do it), one usually uses factorization theorem to experimentally fit low energy physics] and then perturbatively compute the high energy physics above \(Q^2\) from QCD. Here, the procedure seems to be an inverse processes of the perturbative QCD treatment. We introduce the low-energy scale \(\lambda\) and integrate out the high energy...
dynamics above $\lambda$. The left is just the low energy QCD theory which describes the low energy physics. Normally, by integrating out high energy modes, the resulting low energy QCD is not necessary to lead to the explicit quark confining interaction we obtained in $H_{\lambda_0}$. Therefore the theory is still nonperturbative. It is only on the light-front that the high energy modes contain long distance interactions which come from small longitudinal momentum of quarks and gluons [see the light-front dispersion relation $k^- = \frac{1}{k^+}(k^2 + m^2)$, where a large light-front energy $k^-$ can be obtained from a small $k^+$]. Thus, integrating out light-front high energy modes results in an explicit quark confining interactions in $H_{\lambda_0}$. Hence, the light-front formulation of the effective theory derived from SRG may be another crucial point for making the WCT to nonperturbative QCD to be possible.

As we know in the canonical QCD theory, the confining interaction should become more important if the scale $Q^2$ would be smaller. Correspondingly the running coupling constant $\alpha_s(Q^2)$ becomes larger. Compare this fact with the above analysis, there may be an inverse correspondence between the effective coupling constant $\alpha_\lambda$ in $H_{\lambda}$ and the running coupling constant $\bar{\alpha}(Q^2)$ in the full QCD theory:

$$\alpha_\lambda \sim \frac{1}{\bar{\alpha}(Q^2)}, \quad \text{with} \quad \lambda^2 \sim \frac{1}{Q^2}. \quad (5.9)$$

In other words, the weak-coupling treatment of the confining Hamiltonian $H_{\lambda}$ may correspond to an inverse expansion of the original strong-coupling in the full QCD theory. SRG just provides an implicative realization for such an expansion.

More specifically, the running coupling constant in full QCD theory is given by

$$t = \int \frac{d g'}{\beta(g')}, \quad (5.10)$$

where $t = \frac{1}{2} \ln \frac{Q^2}{\mu^2}$, and $Q^2$ is a space-like momentum (the same as $\lambda^2$). Since the similarity renormalization group $\beta$ function of eq. (5.8) is determined in the physical sector of low energy QCD dynamics, the low energy $\beta$ function of the running coupling constant $\bar{\alpha}(Q^2)$ in the full theory should behave qualitatively the same. Then we may take the $\beta(g)$ function in the above equation the same form as eq. (5.8) for $Q^2 << \Lambda_{QCD}^2$ (in low energy region). This leads to

$$\bar{\alpha}(Q^2) = \alpha_s(\mu^2) \frac{\mu^2}{Q^2} \equiv c_0 \frac{\Lambda_{QCD}^2}{Q^2}, \quad Q^2 << \Lambda_{QCD}^2, \quad (5.11)$$

where $c_0 = \alpha_s(\mu^2) \mu^2 / \Lambda_{QCD}^2$. This is consistent with eq. (5.4) via eq. (5.9). To give a qualitative determination of the coefficient $c_0$ in (5.11), we consider $\lambda = 0.75 \sim 1.5$ GeV and $\Lambda = 0.2$ GeV, then $\lambda^2 = (14 \sim 56) \Lambda^2 >> \bar{\lambda}^2$. From (5.4), we have:

$$\alpha_\lambda = (1.0 \sim 1.5) \frac{\bar{\lambda}^2}{\lambda^2}. \quad (5.12)$$

The corresponding $Q^2 \sim \frac{1}{\lambda^2} << \bar{\lambda}^2 \sim \Lambda_{QCD}^2$. From (5.11), it follows that for $Q^2 << \Lambda_{QCD}^2$ the running coupling constant in the full QCD theory may be approximately given by

$$\bar{\alpha}(Q^2) = (1.0 \sim 1.5) \frac{\Lambda_{QCD}^2}{Q^2}. \quad (5.13)$$
This is just a qualitative estimation of the running coupling constant in the full QCD theory in low energy region that is obtained by the use of the same β-function determined in SRG from our effective low energy QCD description.

Up to date, no one precisely knows how the full QCD coupling constant $\alpha_s$ varies in low energy region. However, it is interesting to see that the running coupling constant given by eq.(5.11) for small $Q^2$ is indeed the basic assumption of the Richardson $Q\bar{Q}$ potential [29]:

$$V(Q^2) = -C_f \frac{\alpha(Q^2)}{Q^2}, \quad (5.14)$$

where

$$\alpha(Q^2) = \frac{12\pi}{(33 - 2N_f) \ln(1 + Q^2/\Lambda_{QCD}^2)}. \quad (5.15)$$

The Richardson $Q\bar{Q}$ potential is proposed to exhibit the asymptotic freedom of QCD in short distance and a confining potential in large distance limit. From eq.(5.14), we see that for small $Q^2$ ($Q^2 \ll \Lambda_{QCD}^2$),

$$\alpha(Q^2) \sim \frac{12\pi}{33 - 2N_f} \frac{\Lambda_{QCD}^2}{Q^2}, \quad (5.16)$$

Comparing with eqs.(5.11) and (5.16), we have from the Richardson $Q\bar{Q}$ potential(with $N_f = 3$ [29])

$$c_0 = \frac{12\pi}{33 - 2N_f} = 1.4. \quad (5.17)$$

This result agrees very well with eq.(5.13). Hence, the dependence of $\alpha_\lambda$ with $\lambda$ determined in our description is essentially a consequence of exhibiting an explicit quark confining interaction in our low energy QCD Hamiltonian. The possible inverse relation between $\alpha_\lambda$ with $\alpha_s(Q^2)$ discussed above can be considered as another reason why the present effective theory to low energy of QCD can be treated as a weak-coupling problem.

Finally, we should also note that there is an very interesting question remained to be answered: where is the cross-point (or a cross region) between low energy and high energy domains that let $\alpha_\lambda$ and $\alpha_s$ match at this point (or region). At this point, we have no answer for it but this is a very important open question for a complete understanding of QCD dynamics [1].

C. A possible criterion for the $\lambda$ scale-fixing

Because of the dependence of $\alpha_\lambda$ with $\lambda$ (5.4), the realization of WCT to nonperturbative QCD in our description depends on a suitable choice of $\lambda$, as we have mentioned. One may further ask what is a suitable choice of $\lambda$ so that we can always treat low energy QCD as a weak-coupling problem, and whether there is any ambiguous in the choice of $\lambda$.

Table IV tells us that the confining interaction plays a more important role than the Coulomb interaction in the determination of the quarkonium bound states on the light-front.
This result is different from the usual understanding in the nonrelativistic phenomenological description that the dominant contribution in heavy quarkonium spectroscopy is the Coulomb interaction. This discrepancy can be understood as follows. The currently relativistic light-front description for heavy quark systems mostly uses the heavy quark masses of $m_c = 1.3 \sim 1.4$ GeV and $m_b = 4.7 \sim 4.8$ GeV or less. In Particle Data Group [20], $m_b = 1.0 \sim 1.6$ GeV and $m_c = 4.1 \sim 4.5$ GeV. Thus, the heavy quarkonium binding energies, $\Lambda = M_H - 2m_Q$, might be positive [the lowest charmonium ground state $M(\eta_c(1S)) = 2.98$ GeV, and the bottomonium $M(\Upsilon(1S)) = 9.46$ GeV]. Therefore, the Coulomb energy becomes not important for a relativistic description in quarkonia. The dominant contribution for binding quarkonium states must come from the nonperturbative balance between the kinetic energy and the confining energy. While, in the nonrelativistic phenomenological description, one used larger quark masses, $m_c > 1.8$ GeV and $m_b > 5.1$ GeV [28], such that the binding energy is forced to be negative. As a result, the Coulomb interactions must be dominant in this picture. Of course, on the light-front, the structure of the bound state equation is different from the nonrelativistic Schrödinger equation. There is no direct comparison. A real solution to the above discrepancy can be obtained after including the spin-splitting interactions ($1/m_Q$-corrections).

Nevertheless, from the above analysis, we have seen that the quark confining energy is a dominant contribution to the binding energy of bound states. A large binding energy $\Lambda$ requires a large contribution from the confining interaction and therefore a large value of $\lambda$ is needed. A small $\Lambda$ requires a relatively small effect from the confining interaction so that a small $\lambda$ can be chosen. In all the cases, the effective coupling constant $\alpha_\lambda$ is kept at a small value. Thus, by choosing $\lambda$ to fix the rate $\Lambda/\lambda$ at a certain value, the resulting $\alpha_\lambda$ can remain to be small. The WCT to low energy QCD dynamics is then always applicable. This is a criterion for fixing $\lambda$ in our description, similar to the scale-fixing procedure of Brodsky-Lepage-Mackenzie in solving scale ambiguity for the usual perturbation theory [30]. Of course, in principle one can choose an arbitrary value of $\lambda$, physics should not be changed since the dependence of $\alpha_\lambda$ with $\lambda$ is determined by the SRG invariance. The above criterion for the scale-fixing is useful for the realization of a WCT to perturbative QCD in our description.

In practice, we also find that there is no much freedom in the choice of $\lambda$ value. The scale dependence of wavefunctions also provides a restriction on the range of $\lambda$. For quarkonia, $\omega_\lambda$ is decreased with increasing $\lambda$. However, $\omega_\lambda$ is proportional to the mean value of the (transverse) momentum square of quarks inside quarkonia, which characterizes the size of hadrons. Therefore $\lambda$ should not be too large for the best description of bound states.

On the other hand, as we argued that gluons are massive at the hadronic scale. Hence, the lowest bound value of $\lambda$ can be considered as the constituent gluon mass. As an example, one may take $\lambda$ to be a constituent gluon mass (about a half of the glueball masses, such as the recent possible evidences of $f_0(1500)$ and $\xi(2230)$ [22]),

$$\lambda : (0.75 \sim 1.5) \text{ GeV.} \quad (5.18)$$

Then the effective coupling constant (with $\Lambda = 200$ MeV) is

$$\alpha_\lambda = 0.06 \sim 0.01 \quad (5.19)$$

which is very small. Only a finite $\lambda$ at the order of hadronic mass scale can effectively
turn the nonperturbative contribution in the higher order processes into the long distance two-body confining interactions through SRG. For the range of eq. (5.18), we further have,

\[ \omega_{\lambda_0} = 0.24 \sim 0.2 \text{ GeV}, \]

which leads to

\[ \langle r \rangle \sim \frac{1}{\omega_{\lambda_0}} = 0.8 \sim 1.0 \text{ fm}. \]

This gives a reasonable quarkonium size. Thus, we have provided a qualitative criterion and a quantitative analysis on the choice of \( \lambda \) such that a true WCT to nonperturbative QCD can always be realized by \( H_\lambda \).

D. Summary

In summary, to examine the weak-coupling treatment of nonperturbative QCD recently proposed in Ref. [1], we have studied explicitly the heavy quark bound state problem, based on the light-front heavy quark effective theory of QCD [5,6]. Firstly, we have used the similarity renormalization group approach [1,2] to derive the effective confining Hamiltonian in the low energy scale for heavy quarks in heavy mass limit. To make the similarity renormalization approach practically manageable, we have introduced a local cutoff scheme (2.11) to the bare QCD (and the effective heavy quark) Hamiltonian, which simplifies the cutoff scheme in [1]. Meanwhile we have also introduced a simple smearing function \( f_{\lambda ij} \) (2.13) to the similarity renormalization group approach which further simplifies the original formulation of [1]. The resulting low-energy effective QCD Hamiltonian of heavy quark interactions exhibits the coexistence of a confining interaction and a Coulomb interaction on the light-front without introducing any additional free parameter except for the effective QCD coupling constant.

The weak-coupling treatment can be realized for nonperturbative QCD because the light-front similarity renormalization group approach with a finite \( \lambda \) extracts an explicit quark confining interaction from the higher order nontrivial quark-gluon interactions into \( H_{\lambda_0} \) so that the residual quark-gluon interactions become weak even in low energy region. The weak-coupling treatment of nonperturbative QCD is manageable for \( \lambda \) being around the hadronic mass scale. The similarity renormalization group invariance can remove the \( \lambda \)-dependence in all the physical observables obtained from the effective \( H_\lambda \). The well-defined bound state description in QED is a special case (\( \lambda \to 0 \)) of the SRG approach. The whole idea of the WCT to nonperturbative QCD is originally motivated from the bound state description of QED [1]. Now, a consistent connection between QCD and QED and their differences in determining bound states is explicitly examined on the light-front.

The applications of the present theory to heavy quarkonium spectroscopy and various heavy quarkonium annihilation and production processes can be simply achieved by numerically solving the bound state equations (4.7), and by further including the \( 1/m_Q \) corrections (which naturally leads to the spin splitting interactions). The extension of the computations to heavy-light quark systems is straightforward. The extension of the present work to light-light hadrons requires the understanding of chiral symmetry breaking in QCD which is
a new challenge to nonperturbative QCD on the light-front. Nevertheless, the present work has provided a preliminary realization to the weak-coupling treatment of nonperturbative QCD proposed recently by Wilson et al. [1]. The new research along this direction is in progress.

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FIGURES

FIG. 1. A plot of the confining energy, the Coulomb energy and the total interaction energy as functions of the dimensionless variable \( \varpi \) but \( \varpi \) is proportional to \( \sim r_l \) via \( \omega_\lambda \). The energies are scaled by the factor \( \frac{g_3^2 \lambda^2}{2\pi^2} C_f \).

FIG. 2. The \( \lambda \)-dependence of the effective coupling constant \( \alpha_\lambda \). The solid line is given by the analytical result (5.4), and the dots are obtained by numerically minimizing the quarkonium binding energy (4.16). Here \( \lambda \) is given in units of GeV.

FIG. 3. The \( \lambda \)-dependence of the wavefunction parameter \( \omega_{\lambda 0} \) which is the solution of minimizing the quarkonium binding energy (4.16), where \( \lambda \) is given in units of GeV.
Table I. Solution for charmonium ground state with $m_c = 1.4$ GeV

| $\bar{\Lambda}$ (GeV) | $\alpha_\lambda$ | $\omega_{\lambda 0}$ (GeV) | $\mathcal{E}_{\text{kin}}$(GeV$^2$) + $\mathcal{E}_{\text{pot}}$(GeV$^2$) = 2$\bar{\Lambda}^2$(GeV$^2$) |
|------------------------|------------------|---------------------------|----------------------------------|
| 0.2                    | 0.02665          | 0.222                     | 0.026836                         |
|                        |                  |                           | 0.053173                         |
| 0.3                    | 0.06480          | 0.275                     | 0.067902                         |
|                        |                  |                           | 0.112018                         |
| 0.4                    | 0.11831          | 0.314                     | 0.130225                         |
|                        |                  |                           | 0.189781                         |

Table II. Solution for bottomonium ground state with $m_b = 4.8$ GeV

| $\bar{\Lambda}$ (GeV) | $\alpha_\lambda$ | $\omega_{\lambda 0}$ (GeV) | $\mathcal{E}_{\text{kin}}$(GeV$^2$) + $\mathcal{E}_{\text{pot}}$(GeV$^2$) = 2$\bar{\Lambda}^2$(GeV$^2$) |
|------------------------|------------------|---------------------------|----------------------------------|
| 0.15                   | 0.029965         | 0.492                     | 0.023397                         |
|                        |                  |                           | 0.021602                         |
| 0.20                   | 0.06795          | 0.623                     | 0.050183                         |
|                        |                  |                           | 0.029816                         |
| 0.25                   | 0.1385           | 0.779                     | 0.098074                         |
|                        |                  |                           | 0.026930                         |

Table III. Some numerical solution on the $\lambda$-dependence of the weak coupling constant $\alpha_\lambda$.

| $\lambda$ (GeV) | charmonium | bottomonium |
|-----------------|------------|-------------|
|                 | $\alpha_\lambda$ | $\omega_{\lambda 0}$ (GeV) | $\alpha_\lambda$ | $\omega_{\lambda 0}$ (GeV) |
| 0.75            | 0.05960    | 0.241       | 0.06795          | 0.623 |
| 1.0             | 0.02665    | 0.222       | 0.01607          | 0.478 |
| 1.5             | 0.00912    | 0.199       | 0.00695          | 0.427 |
| 2.0             | 0.00441    | 0.185       | 0.00695          | 0.427 |

Table IV. The $\lambda$-dependence of various interactions to the binding energy

| $\lambda$ (GeV) | $\mathcal{E}_{\text{kin}}$ (GeV$^2$) | $\mathcal{E}_{\text{conf}}$ (GeV$^2$) | $\mathcal{E}_{\text{Col}}$ (GeV$^2$) |
|-----------------|-----------------------------------|-----------------------------------|-----------------------------------|
| 0.5             | 0.04                              | 0.049                             | -0.009                            |
| 0.75            | 0.031                             | 0.050                             | -0.001                            |
| 1.0             | 0.027                             | 0.053                             | -0.001                            |
| 1.2             | 0.025                             | 0.055                             | 0.000                             |
| 1.4             | 0.023                             | 0.057                             | 0.000                             |
| 1.8             | 0.021                             | 0.059                             | 0.000                             |
| 2.0             | 0.02                               | 0.06                              | 0.000                             |
| 3.0             | 0.018                             | 0.062                             | 0.000                             |
Fig. 2

\[ \Lambda = 0.4 \text{ (GeV)} \]

\[ \bar{\Lambda} = 0.2 \text{ (GeV)} \]
Fig. 3
Confining Energy

Coulomb Energy

Total Potential Energy

\[ \omega \sim r \]

Fig. 1