Abstract: Experiments with trapped atomic gases have opened novel possibilities for studying the evolution of nonequilibrium finite quantum systems, which revived the necessity of reconsidering and developing the theory of such processes. This review analyzes the basic approaches to describing the phenomena of equilibration, thermalization, and decoherence in finite quantum systems. Isolated, nonisolated, and quasi-isolated quantum systems are considered. The relations between equilibration, decoherence, and the existence of time arrow are emphasized. The possibility for the occurrence of rare events, preventing complete equilibration, are mentioned.

Equilibration and thermalization in finite quantum systems

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1. Trapped cold atoms

1.1. Introduction

In recent years, there has been a remarkable progress in experimental studies of cold trapped atoms, which has been accompanied by intensive theoretical investigations. Numerous literature can be found in the books [1–4] and review articles [5–19]. Experiments with trapped atoms possess several important features making such experiments one of the most convenient tools for studying the properties of many-body systems. First, trapped atomic systems allow for a high degree of tunability, which makes it possible to vary their parameters in a rather wide range. Second, the variation of system parameters can be realized very quickly, thus, putting a system in a nonequilibrium initial state and studying the system evolution. Third, the diluteness of the cold atomic gases and exceptionally low temperatures result in rather long timescales of dynamical effects (typically of the order of milliseconds or longer) allowing for a very precise time resolution of nonequilibrium processes. Finally, trapped atomic systems can be well isolated from environment. Therefore, it is possible to study the quantum dynamics of finite systems under different levels of isolation and to analyze the influence of the latter on the system equilibration, decoherence, and thermalization.

This review is devoted to exactly the latter problems. In the present section, the most interesting experiments and computer simulations, considering the dynamic equilibration effects of nonequilibrium trapped atoms, are surveyed. While the following sections are devoted to the general theoretical description of such processes. In order to avoid ambiguities, the review tries to use accurate mathematical formulations.

Throughout the paper, the system of units is employed, where the Planck and Boltzmann constants are set to unity ($\hbar = 1$, $k_B = 1$).

1.2. Experiments with trapped atoms

In the experiment of Greiner et al. [20] $^{87}$Rb atoms were loaded into a three-dimensional optical lattice. Varying the
ratio $U/J$ of the one-site interactions $U$ to the hopping parameter $J$, the transition between the superfluid Bose-condensed phase and the incoherent Mott insulator phase was studied. When atoms were initially in the Mott insulating phase with the ratio $U/J > 5.8z_0$, with $z_0$ being the number of nearest neighbors, and then this ratio was quickly reduced to $U/J < 5.8z_0$, corresponding to the superfluid phase, the coherence was restored during the timescale of $1/J$. In the opposite process of quench from the Bose-condensed superfluid to the Mott insulator state, the equilibration was accompanied by collapses and revivals of the coherent fraction on the shorter time scale of $1/\tilde{U}$ [21]. The equilibration, after crossing the superfluid-insulator phase transition line, evolved through the inhomogeneous state with alternating regions of incoherent Mott insulator phases and coherent superfluid phases [20].

The dynamics of pair condensate formation in a strongly interacting Fermi gas of $^6$Li, close to Feshbach resonance, was studied by Zwierlein et al. [22]. The relaxation time, after a rapid magnetic field ramp across the Feshbach resonance, was of the order of $100/\sigma_F$. Here $\sigma_F \approx \omega_0(3N)^{1/3}$ is the Fermi surface energy, $\omega_0$ is the geometric mean of the trapping frequencies, and $N$ is the total number of atoms in the trap. After a rapid ramp, molecular Bose condensate co-existed with unpaired Fermi atoms.

Trapped quasi-one-dimensional arrays of Bose gases, each containing from 40 to 250 $^8$Rb atoms, were prepared in an out-of-equilibrium state by Kinoshita et al. [23]. They observed that the system of two initially separated Bose-Einstein condensates oscillates, as in a Newton cradle, but does not noticeably equilibrate even after thousands of collisions.

Spontaneous symmetry breaking in a quenched spinor Bose-Einstein condensate was studied by Sadler et al. in a quasi-two-dimensional trap [24]. Nearly pure Bose condensate of spinor $^8$Rb was prepared in the unmagnetized $m_z = 0$ state at a high quadratic Zeeman shift. By rapidly reducing the magnitude of the applied magnetic field, the gas was quenched to low quadratic Zeeman shift, obtaining conditions that favored the ferromagnetic phase. Spontaneous symmetry breaking occurred through a heterogeneous stage, where small ferromagnetic domains of variable size and orientation were intermixed with unmagnetized regions. This could be regarded as the phase separation of magnetized and unmagnetized states in the process of nonequilibrium phase transition. Spin-vortex defects were also observed with high confidence in about one-third of all images containing significant ferromagnetism. Some images indicated as many as four vortices. All observed vortices were singly quantized, with no apparent preferred direction or circulation. The thermalization time was an order longer than the local equilibrium time.

Nonequilibrium coherence dynamics of quasi-one-dimensional Bose gas of $^8$Rb was considered by Hofferberth et al. [25]. The initial state consisted of two completely separated Bose condensates loaded into a double well, where they started interacting and developing the common coherent system. The final state was a thermalized Bose-condensed gas with a temperature slightly higher than the initial temperature of separated Bose condensates. The thermalization time, depending on the system parameters, was between 6 and 9 ms.

The spontaneous formation of vortices in the process of nonequilibrium Bose-Einstein condensation and the relation of this phenomenon to the Kibble-Zurek mechanism [26,27] was investigated by Weiler et al. [28].

In the experiment of Hung et al. [29], a $^{133}$Cs quantum Bose gas was trapped in a two-dimensional optical lattice. The lattice depth was quickly ramped to induce the superfluid to Mott-insulator phase transition. The global equilibration involved mass transport and was found to be much slower than the microscopic tunneling time scale $1/J$.

By using the high spatial resolution in-situ fluorescence imaging, Bakr et al. [30] and Sherson et al. [31] demonstrated that the superfluid to Mott insulator phase transition can be studied with single-site and single-particle resolved detection.

### 1.3. Numerical simulations with models

Numerical simulations for different models play the role of computer experiments, providing essential information on the relaxation processes in nonequilibrium finite quantum systems. In the present subsection, some of such simulations are briefly described. The detailed analysis of terminology and correct mathematical definitions will be given in the following sections. Many of simulations were done for lattice models. Such models are also of interest for cold atom physics, since periodic atomic systems are now easily created by means of optical lattices.

Probably, the first computer simulation of nonequilibrium dynamics for a finite classical system was done by Fermi et al. [32] who considered a one-dimensional chain of anharmonic oscillators and discovered that it does not equilibrate. But that was a classical system.

A quantum two-dimensional anharmonic oscillator, corresponding to the Henon-Heiles model [33], was studied by Feingold et al. [34]. They observed that the mean location and dispersion of the wave packet, first, tend to quasi-equilibrium values, but later display large fluctuations.

Jensen and Shankar [35] studied relaxation of a finite spin-half quantum chain in a magnetic field in two situations, when the model was integrable or not. They attempted to find out if there would be difference in the relaxation of the integrable and nonintegrable cases. It turned out that both integrable and nonintegrable quantum systems, with as few as seven degrees of freedom, could exhibit thermalization in finite time to the states well described by microcanonical ensemble, provided that the ini-
The quench was realized by a sudden change of the on-site interaction $U$, lower than the critical value, given by $U_c/J = 2$, that is, $u_c = 1$, the system consists of a Luttinger liquid, while for $U > U_c$, it is a charge-density-wave insulator. By a sudden change of the interaction strength it is possible to switch the initial state between metallic and insulating phases. The relaxation goes through an oscillatory stage, with the oscillations of period $2\pi/U$, and tends to a steady state on a time scale $1/J$. The reached quasistationary state is not thermal in the usual sense, since it depends on the initial conditions. The time evolutions, starting from different initial states, can be distinguished from each other, so that neither relaxation to one and the same equilibrium state nor the usual thermalization occur. The resulting quasisteady state could be described by a representative ensemble with fixed expectation values of the powers of the Hamiltonian $\langle H^n \rangle$. The choice of these additional conditions can be justified by the fact that, for a closed system, the set of all powers of the Hamiltonian constitute an infinite number of integrals of motion.

The considered one-dimensional fermion Hubbard model [48], with nearest-neighbor interactions, is integrable. The authors have also studied the effect of adding to the model next-nearest-neighbor interactions, which makes it nonintegrable. It has been found that, independently of the model integrability or criticality, the relaxation process does not change much and the system always relaxes to a nonthermal quasistationary state that can be described by a representative ensemble with additional conditions of fixed $\langle H^n \rangle$. When different initial states have the same energy and are sufficiently close to each other, the observable quantities relax to the same value, i.e., the memory of the initial state in that case is lost after the relaxation. However, the general energy distribution is defined by a representative ensemble retaining the memory of the initial states.

A chain of two-level systems interacting through the van der Waals forces, with a strength $U$, was considered by Lesanovsky et al. [49]. The transitions between the energy levels were due to the Rabi term, with a Rabi frequency $\Omega$. For weak interactions $U/\Omega \ll 1$, no equilibration was noticed. But for sufficiently strong interactions $U/\Omega > 2$, the system reached a steady state at the time of order $50/\Omega$. The resulting steady state could be well characterized by a grand canonical ensemble. Thermalization also occurs for a finite spin system with randomly distributed initial states [50].

The behavior of integrable and nonintegrable systems for one-dimensional [41–43] and two-dimensional [44] lattices of hard-core bosons, Rigol et al. [41–44] concluded that integrable systems relax to a nonstandard representative Gibbs ensemble, while the nonintegrable systems relax to a standard microcanonical ensemble. Similar behavior occurs for one-dimensional lattices of fermions [45].

Kollath et al. [46] investigated one- and two-dimensional optical lattices described by the boson Hubbard model. They studied the time evolution following a quench from the superfluid to the Mott insulator phase. The quench was realized by a sudden change of the on-site interaction $U$ under a fixed hopping parameter $J$. The superfluid-insulator line, in the case of unit filling, is characterized by the critical value $u_c$ of the ratio $u \equiv U/(Jz_0)$, where $z_0$ is the number of nearest neighbors. For one-dimensional lattices, $u_c = 1.8$ and for two-dimensional lattices, $u_c = 4.2$ (see review [18]). The quench from the superfluid to insulator phase corresponds to the change of $u$ from $u < u_c$ to the values $u > u_c$. For large values of the final interaction strength $u \gg u_c$, the system reaches a quasisteady state that is very different from the standard thermal equilibrium and retains memory of the initial state. As the final $u$ decreases, the nature of the steady state changes, and in the region where $U$ is comparable to $J$, that is, $u \sim 1/z_0$, the steady state is well approximated by a standard thermally equilibrium Gibbs ensemble. After the quench, the system exhibits a number of oscillations with a period $2\pi/U$, which relax to a quasisteady state on a time scale $1/J$. Similar results for a two-dimensional lattice of bosons were found by Natu et al. [47].

The existence of two distinct equilibration regimes seems surprising, given the nonintegrability of the boson Hubbard model for any finite values of $U$ and $J$. This tells us that the equilibration regime and the quasisteady state resulting in this process depend mainly not on the integrability of the system but rather on the system parameters. Both, integrable as well as nonintegrable quantum systems, depending on their parameters, can relax either to a standard statistical ensemble or to a nonstandard one, any of such ensembles being just particular cases of the class of representative ensembles.

The results, similar to the boson Hubbard model were found for a one-dimensional fermion Hubbard model by Mannama et al. [48]. The ground-state phase diagram of this model, at half filling, is well known. For weak on-site interactions $U$, lower than the critical value, given by $U_c/J = 2$, that is, $u_c = 1$, the system consists of a Luttinger liquid, while for $U > U_c$, it is a charge-density-wave insulator. By a sudden change of the interaction strength it is possible to switch the initial state between metallic and insulating phases. The relaxation goes through an oscillatory stage, with the oscillations of period $2\pi/U$, and tends to a steady state on a time scale $1/J$. The reached quasistationary state is not thermal in the usual sense, since it depends on the initial conditions. The time evolutions, starting from different initial states, can be distinguished from each other, so that neither relaxation to one and the same equilibrium state nor the usual thermalization occur. The resulting quasisteady state could be described by a representative ensemble with fixed expectation values of the powers of the Hamiltonian $\langle H^n \rangle$. The choice of these additional conditions can be justified by the fact that, for a closed system, the set of all powers of the Hamiltonian constitute an infinite number of integrals of motion.
Integrable as well as nonintegrable finite quantum systems, it seems, both can equilibrate. Integrability does not seem to play a crucial role on the structure of the quasistationary state. This is despite the fact that integrable and nonintegrable quantum systems display different level-spacing statistics and differently react to external perturbations [51]. Though integrable systems can equilibrate, but their main difference from nonintegrable systems can be in much longer equilibration times. This behavior is contrary to integrable classical finite systems that do not equilibrate at all. Nonintegrable classical systems can equilibrate if they are chaotic [52,53].

The influence of the vicinity of critical points on the nonequilibrium dynamics of closed quantum systems was investigated by Polkovnikov et al. [54–56] who emphasized the universality of this dynamics in gapless systems near continuous phase transitions.

2. Thermalization versus equilibration

2.1. Quantum statistical ensemble

The term statistical ensemble is constantly employed in studies of many-body systems. It is therefore important to recall its correct mathematical definition.

A quantum system is characterized by a Hilbert space \( \mathcal{F} \). The probability operator measure is given on \( \mathcal{F} \) by a statistical operator \( \hat{\rho}(t) \) parameterized by the time variable \( t \geq 0 \). Hence \( \hat{\rho}(t) \) is a positive operator normalized to one,

\[
\text{Tr} \hat{\rho}(t) = 1,
\]

where the trace is over \( \mathcal{F} \). By definition, a quantum statistical ensemble is the pair \( \{ \mathcal{F}, \hat{\rho}(t) \} \).

2.2. Algebra of local observables

On the Hilbert space \( \mathcal{F} \), one defines self-adjoint operators \( \hat{A} \) that are called the operators of local observables, provided their expectation values

\[
\langle \hat{A}(t) \rangle \equiv \text{Tr} \hat{\rho}(t) \hat{A}
\]

(1)

correspond to the measurable quantities. The set of all such operators forms the algebra of local observables \( \mathcal{A} \equiv \{ \hat{A} \} \).

2.3. System statistical state

The set of all expectation values for the operators of local observables is the statistical state

\[
\langle \mathcal{A}(t) \rangle \equiv \left\{ \langle \hat{A}(t) \rangle \right\}
\]

(2)

An important role in dynamical problems is played by the initial statistical state

\[
\mathcal{A}_0 \equiv \langle \mathcal{A}(0) \rangle = \left\{ \langle \hat{A}(0) \rangle \right\},
\]

(3)

which is the set of the expectation values for the operators of local observables,

\[
\langle \hat{A}(0) \rangle = \text{Tr} \hat{\rho}(0) \hat{A},
\]

at the initial moment of time \( t = 0 \).

2.4. Equilibration of quantum systems

In a review, devoted to equilibration and thermalization, it is crucial to follow precise definitions of these processes. In literature, one can meet different descriptions of these phenomena. Below, the definitions are given that will be used in the present review.

A statistical system, characterized by a statistical ensemble \( \{ \mathcal{F}, \hat{\rho}(t) \} \), equilibrates from an initial statistical state \( \mathcal{A}_0 \) if and only if for any \( \hat{A} \in \mathcal{A} \) there exists a limit

\[
\lim_{t \to \infty} \langle \hat{A}(t) \rangle = \text{Tr} \hat{\rho}^*(\mathcal{A}_0) \hat{A},
\]

(4)

generally, depending on the initial state \( \mathcal{A}_0 \).

If the limit (4) exists, one says that the statistical system is in an equilibrium state with a statistical ensemble \( \{ \mathcal{F}, \hat{\rho}^*(\mathcal{A}_0) \} \).

This is a general definition of equilibration as such. One also considers another form of equilibration, the equilibration on average. This will be treated in the following sections. It is worth stressing that the above definition of equilibration requires that the limits (4) would exist for all operators from the algebra of local observables. When such limits exist only for some observables, but are absent for others, it is not, strictly speaking, equilibration.

Also, when a statistical state appears to be time-independent for a finite interval of time, but not for the actual limit \( t \to \infty \), this is termed quasi-equilibration.

2.5. Thermalization of quantum systems

Thermalization is a more restrictive process, when not merely limit (4) exists, but the memory of initial conditions is, at least partially, lost.

A statistical system, characterized by a statistical ensemble \( \{ \mathcal{F}, \hat{\rho}(t) \} \), thermalizes from an initial state \( \mathcal{A}_0 \) if and only if there exists a dense set \( \mathcal{B} = \{ \mathcal{A}_0 \} \) of initial states, including the given initial state \( \mathcal{A}_0 \), such that for any \( \hat{A} \in \mathcal{A} \) the limit

\[
\lim_{t \to \infty} \langle \hat{A}(t) \rangle = \text{Tr} \hat{\rho}^* \hat{A}
\]

(5)

does not depend on \( \mathcal{A}_0 \in \mathcal{B} \).

The set \( \mathcal{B} \) is termed the attraction basin of the stationary state (5).

When the thermalization occurs from any available initial state, that is, when the attraction basin is the whole set of all admissible initial states, this is called the global thermalization.
3. Evolution of statistical states

3.1. Representative statistical ensembles

To correctly describe the system equilibration, it is necessary to be accurate in classifying the limiting stationary states. Each statistical system is characterized by a statistical ensemble. It is assumed that the statistical ensemble is defined so that to uniquely represent the considered statistical system. This implies that the definition of the ensemble must take into account all conditions and constraints that correctly represent the system. This principal point was emphasized by Gibbs [38,39], Tolman [57], and ter Haar [58]. The term *representative ensemble* was introduced by Tolman [57].

The practical realization for constructing a representative ensemble can be illustrated for an equilibrium system, following the basic Gibbs idea. Let a Hilbert space of microstates $\mathcal{F}$ be given. To define a statistical ensemble $\{\mathcal{F}, \hat{\rho}\}$, one needs to specify the operator probability measure, that is, the statistical operator $\hat{\rho}$. In constructing the latter, it is necessary to take into account those constraints that uniquely define the system. Of course, the statistical operator is to be normalized as

$$\text{Tr } \hat{\rho} = 1.$$  \hspace{1cm} (6)

An important quantity is the internal energy given by the average of the energy operator $\hat{H}$, called Hamiltonian,

$$\text{Tr } \hat{\rho} \hat{H} = E.$$  \hspace{1cm} (7)

In addition, there can be other constraints defining the expectation values of some *constraint operators* $C_i$, with $i = 1, 2, \ldots$,

$$\text{Tr } \hat{\rho} \hat{C}_i = C_i.$$  \hspace{1cm} (8)

The statistical operator of an equilibrium state is defined as the minimizer of the *information functional*

$$I[\hat{\rho}] = \text{Tr } \hat{\rho} \ln \hat{\rho} + \lambda_0 (\text{Tr } \hat{\rho} - 1) + \beta (\text{Tr } \hat{\rho} \hat{H} - E) + \beta \sum_i (\text{Tr } \hat{\rho} \hat{C}_i - C_i),$$  \hspace{1cm} (9)

in which $\lambda_0, \lambda_i$, and $\beta$ are Lagrange multipliers. This is equivalent to the maximization of the Shannon entropy under conditions (6)–(8). The result is the statistical operator

$$\hat{\rho} = \frac{1}{Z} \exp(-\beta \hat{H}),$$  \hspace{1cm} (10)

with the normalization factor being the partition function $Z \equiv \text{Tr } \exp(-\beta \hat{H})$, and with the *grand Hamiltonian*

$$H = \hat{H} + \sum_i \lambda_i \hat{C}_i.$$  \hspace{1cm} (11)

As is clear, when the sole constraint is the normalization condition (6), then the statistical operator reduces to the microcanonical form

$$\hat{\rho} = \frac{1}{Z} \left( Z = \text{Tr } 1 \right).$$

If only conditions (6) and (7) are accepted, one gets the Gibbs canonical ensemble. If one also defines the average number of atoms, one gets the Gibbs grand canonical ensemble. The general situation, with any number of additional constraints, is also termed the grand Gibbs ensemble [59,60].

The general form of the grand ensemble, with the statistical operator (10), was introduced by Gibbs [38,39]. One also calls this the generalized or conditional Gibbs ensemble [61]. But one should not forget that this ensemble was advanced by Gibbs. When deciding on a particular type of a Gibbs ensemble, one has to choose such that would be representative for the considered statistical system. General mathematical properties and applications of representative statistical ensembles have been analyzed in [13,62–64].

3.2. Nonequilibrium representative ensembles

When a quantum system equilibrates, it tends to one of the equilibrium representative statistical ensembles. A nonequilibrium state is characterized by a nonequilibrium ensemble $\{\mathcal{F}, \hat{\rho}(t)\}$. The temporal evolution of the statistical operator is governed by a unitary evolution operator $\hat{U}(t)$, such that

$$\hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^+(t).$$  \hspace{1cm} (12)

The evolution is generated by the grand Hamiltonian (11), and the evolution operator satisfies the Schrödinger equation

$$i \frac{d}{dt} \hat{U}(t) = H \hat{U}(t).$$  \hspace{1cm} (13)

This means that the grand Hamiltonian is the evolution generator.

If one assumes that the evolution is generated by the energy Hamiltonian $\hat{H}$, then the constraint operators $\hat{C}_i$ in the grand Hamiltonian (11), defining an equilibrium state, are to be the integrals of motion, commuting with $\hat{H}$. But if the evolution is generated by the grand Hamiltonian (11) itself, the constraint operators do not need to be the integrals of motion [13,62–64].

In particular, the constraint operators can fix the required initial conditions for observable quantities in the form

$$C_i = \langle \hat{C}_i \rangle = \text{Tr } \hat{\rho}(0) \hat{C}_i.$$  \hspace{1cm} (14)
These initial conditions then define the solutions to the evolution equations for the studied observables. Generally, the steady states, resulting from sudden changes of the Hamiltonian, are not thermal, in the sense that they retain information on initial conditions [65,66]. As examples of the cases, where the memory of initial conditions is retained in quasi-equilibrium states, it is possible to recall the hard-core bosons in lattices [36,37,41–44], Luttinger liquids [40], Hubbard-type models of bosons and fermions [45–48,67,68], free bosonic models [69], and one-dimensional fermionic systems with Hall states [70]. Other examples are given by the quasistationary states that are the solutions to the evolution equations describing the relaxation of average spins for different strongly nonequilibrium spin systems (nuclei, electrons, molecules, clusters), whose dynamics essentially depends on initial conditions, such as the initial spin polarization [71–80].

It is possible to conclude that closed quantum nonequilibrium systems equilibrate to quasi-equilibrium states characterized by representative statistical ensembles. In the majority of cases the resulting representative ensemble retains the memory of initial states, regardless of whether a model is integrable or not.

### 3.3. Evolution of observable quantities

What one needs and measures in experiments is the statistical state, that is, the averages of operators of observables. Therefore, equilibration has to do with the temporal behavior of such averages. An average of an operator \( \hat{A} \), because of the evolution law (12), can be written in two ways:

\[
\langle \hat{A}(t) \rangle = \text{Tr} \, \hat{\rho}(t) \hat{A} = \text{Tr} \, \hat{\rho} \hat{A}(t),
\]

in which

\[
\hat{\rho} \equiv \hat{\rho}(0), \quad \hat{A} \equiv \hat{A}(0),
\]

and the operator time evolution is given by the Heisenberg form

\[
\hat{A}(t) = \hat{U}^+(t) \hat{A}(0) \hat{U}(t).
\]

The expectation value (15) is real, as a consequence of the operator of an observable being self-adjoint.

The Heisenberg equation of motion for the operator (16) reads as

\[
i \frac{d}{dt} \hat{A}(t) = i \hat{U}^+(t) \frac{\partial \hat{A}}{\partial t} \hat{U}(t) + \left[ \hat{A}(t), H(t) \right],
\]

where

\[
H(t) \equiv \hat{U}^+(t) H(t) \hat{U}(t).
\]

The Hamiltonian, being the evolution generator, varies with time only through an explicit dependence on time, if any,

\[
i \frac{d}{dt} H(t) = i \hat{U}^+(t) \frac{\partial H}{\partial t} \hat{U}(t).
\]

If the considered operator does not explicitly depend on time, then its average evolves in time according to the law

\[
i \frac{d}{dt} \langle \hat{A}(t) \rangle = \left\langle \left[ \hat{A}(t), H(t) \right] \right\rangle = \left( \frac{\partial \hat{A}}{\partial t} = 0 \right).
\]

The evolution equation is to be complemented by an initial condition \( A(0) = \langle \hat{A}(0) \rangle \). It is the memory of such initial conditions that can be retained in the representative ensemble characterizing equilibrated finite systems.

### 3.4. Equations in matrix representation

Let \( \{ |n\rangle \} \) be an orthonormal basis in the Hilbert space of microstates \( \mathcal{F} \). Define the matrix elements for the statistical operator

\[
\rho_{mn}(t) \equiv \langle m | \hat{\rho}(t) | n \rangle
\]

and for the operator of an observable

\[
A_{mn}(t) \equiv \langle m | \hat{A}(t) | n \rangle.
\]

The related matrix elements at zero time are

\[
\rho_{mn} \equiv \langle m | \hat{\rho} | n \rangle = \rho_{mn}(0),
\]

\[
A_{mn} \equiv \langle m | \hat{A} | n \rangle = A_{mn}(0).
\]

Then the average (15) can be represented as

\[
\langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}(t) A_{mn} = \sum_{mn} \rho_{mn} A_{mn}(t).
\]

This average, in general, contains nondiagonal terms. Of course, if one chooses as the basis the set of the eigenvectors of \( \hat{A} \), then the sum would contain only the diagonal terms. However, other operators, noncommuting with \( \hat{A} \), will have nondiagonal terms. When one discusses equilibration, one should keep in mind not just one operator but the set of operators from the algebra of local observables \( \mathcal{A} \). Equilibration, by definition, presupposes that it is the whole statistical state \( \langle \hat{A} \rangle \) that equilibrates, but not merely some of its components.

### 3.5. Pure quantum states

One often considers equilibration of quantum systems prepared in a pure state described by a normalized wave function \( |\psi(0)\rangle \). With a wave function \( |\psi(t)\rangle \), the statistical operator of a pure state is

\[
\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)|.
\]

It is an idempotent operator, with the property \( \hat{\rho}^2(t) = \hat{\rho}(t) \). The wave function satisfies the Schrödinger equation

\[
i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle.
\]
Employing an expansion
\[ \psi(t) = \sum_n c_n(t)|n\rangle, \quad c_n(t) = \langle n|\psi(t) \rangle, \]  
(27)
one gets the equation for the expansion functions
\[ i \frac{d}{dt} c_n(t) = \sum_m H_{mn} c_m(t), \]  
(28)
where \( H_{mn} \equiv \langle m|H|n\rangle \), complimented by an initial condition \( c_0 = c(0) \) and the normalization condition
\[ \sum_n |c_n(t)|^2 = 1. \]
The matrix element \( H_{mn} \) can be explicitly dependent on time.

The statistical operator (25) enjoys the expansion
\[ \dot{\rho}(t) = \sum_{mn} \rho_{mn}(t)|m\rangle\langle n|, \]  
(29)
in which
\[ \rho_{mn}(t) = c_m(t)c_n^\dagger(t). \]

Average (24) takes the form
\[ \langle \dot{\hat{A}}(t) \rangle = \sum_{mn} \rho_{mn}(t)A_{nm} = \langle \psi(t)|\hat{A}|\psi(t) \rangle. \]  
(30)
The pure-state statistical operator (25) is a particular case of the general statistical operator. Because of this, it is sufficient to consider the general case.

4. Isolated quantum systems

4.1. Absence of absolute equilibrium

Let us assume that a finite quantum system can be completely isolated from its surrounding. Hence the Hamiltonian \( H \) does not explicitly depend on time. Then the evolution operator is
\[ \hat{U}(t) = \exp(-iHt) \left( \frac{\partial H}{\partial t} = 0 \right). \]  
(31)
It is admissible to take the set of the Hamiltonian eigenvectors, defined by the eigenproblem
\[ H|n\rangle = E_n|n\rangle, \]  
(32)as the basis in \( \mathcal{F} \). Then the matrix element (21) becomes
\[ \rho_{mn}(t) = \rho_{mn}\exp(-i\omega_{mn}t), \]  
(33)
with the transition frequency
\[ \omega_{mn} \equiv E_m - E_n. \]  
(34)
The operator average (24) yields
\[ \langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}A_{nm}\exp(-i\omega_{mn}t). \]  
(35)
And the equation of motion (20) reduces to
\[ i \frac{d}{dt} \langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}A_{nm}\omega_{mn}\exp(-i\omega_{mn}t). \]  
(36)
It is evident that Eq. (35) is a quasi-periodic function of time. Hence, it cannot tend to a time-independent stationary state. Any given initial state will reproduce itself after the Poincaré recurrence time [81–84]. The temporal limit (4) does not exist. A finite quantum system, completely isolated from its surrounding, does not have an absolute equilibrium in the sense of limit (4).

An exception could be the case, when the system would be initially prepared in a pure state, with a given Hamiltonian eigenvector \( |j\rangle \), when \( \rho_{mn} = \delta_{mj}\delta_{nj} \). Then the average never changes with time:
\[ \langle \hat{A}(t) \rangle = A_{jj} \quad (\rho_{mn} = \delta_{mj}\delta_{nj}). \]  
(37)
However, the dynamics of the system with this initial state is not structurally stable. If the initial condition slightly deviates from the pure state \( |j\rangle \) or there is a perturbation of the Hamiltonian, so that the initial condition gives
\[ \rho_{mn} = \delta_{mj}\delta_{nj} + \delta\rho_{mn}, \]
then the average
\[ \langle \hat{A}(t) \rangle = A_{jj} + \sum_{mn} (\delta\rho_{mn})A_{nm}\exp(-i\omega_{mn}t) \]
again becomes quasi-periodic.

4.2. Possibility of equilibrium on average

Von Neumann [85] suggested that quantum systems could exhibit a kind of ergodic behavior, similarly to that of classical systems, when the time average of the expectation value of an observable coincides with its ensemble-averaged value. For this purpose, one considers the time average
\[ \langle \hat{A}(t) \rangle \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle \hat{A}(t) \rangle dt. \]  
(38)
For an isolated system, this gives
\[ \langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}A_{nm}\Delta(\omega_{mn}). \]  
(39)
where
\[ \Delta(\omega) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \exp(-i\omega t) \, dt, \]

which gives
\[ \Delta(\omega) = \begin{cases} 1, \, \omega = 0, \\ 0, \, \omega \neq 0. \end{cases} \]

If the eigenenergies of the Hamiltonian are not degenerate, then
\[ \Delta(\omega_{mn}) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \exp(-i\omega_{mn} t) \, dt = \delta_{mn}. \quad (40) \]

On usually assumes this nondegeneracy condition, which yields
\[ \langle \hat{A}(t) \rangle = \sum_n \rho_{nn} A_{nn}. \quad (41) \]

Hamiltonians, enjoying nondegenerate spectra, are termed typical or generic.

Von Neumann assumed that, under some conditions, the right-hand side of Eq. (41) would correspond to the average over an equilibrium ensemble. In that sense, Eq. (41) would play the role of an ergodic relation, similar to that in classical systems.

It may happen, however, that, even when this ergodic-type relation is valid, but system fluctuations around the averaged value are so large that it is senseless to talk about equilibrium. In order that it would be possible to state that a quantum systems spends the most of its time close to the averaged value, having not so many and not so large deviations from it, it is necessary to analyze the strength of the system fluctuations. For an operator \( \hat{A} \), these fluctuations are quantified by the dispersion
\[ \sigma^2_{\hat{A}} \equiv \langle \hat{A}(t)^2 \rangle - \left( \langle \hat{A}(t) \rangle \right)^2, \quad (42) \]

which is also called the variance.

Considering the dispersion, one usually assumes that, in addition to the nondegeneracy condition (40), the non-resonance condition
\[ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \exp \left[ -i(\omega_{mn} + \omega_{kl}) t \right] \, dt = \]
\[ = \delta_{mn} \delta_{kl} + \delta_{ml} \delta_{nk} - \delta_{mn} \delta_{nl} \delta_{kl} \]
holds true. If so, it immediately follows that
\[ \sigma^2_{\hat{A}} = \sum_{n \neq m} |\rho_{mn} A_{nn}|^2. \quad (44) \]

One also employs the inequality
\[ \sum_n \rho^\alpha_{nn} \leq \left( \max_n \rho_{nn} \right)^{\alpha - 1}, \]

being valid for any real \( \alpha > 1 \).

Another important point is the Chebyshev inequality stating that for any random variable \( x \), with average \( \bar{x} \) and variance \( \sigma^2 \), and any given \( \epsilon > 0 \), the probability that \( x \) deviates from \( \bar{x} \) by more than \( \epsilon \) satisfies the inequality
\[ P(|x - \bar{x}| > \epsilon) < \left( \frac{\sigma}{\epsilon} \right)^2. \quad (45) \]

This inequality allows for the estimation of the operator variance (44) characterizing the deviations from the operator average [86–88].

For the operator average \( \langle \hat{A} \rangle \), assuming that the experimental resolution \( \delta_A \) of measuring the observable, associated with the operator \( \hat{A} \), is sufficiently small, and that \( \max_n \rho_{nn} \ll 1 \), Reimann [87,88] obtains
\[ P\left( \left| \langle \hat{A}(t) \rangle - \langle \hat{A}(t) \rangle \right| \geq \delta_A \right) \leq \left( \frac{\sigma_A}{\delta_A} \right)^2. \quad (46) \]

This inequality tells us that, under some conditions, the deviations of the observable quantity \( \langle \hat{A} \rangle \) from the time-averaged value \( \langle \hat{A} \rangle \) could be relatively small. Hence the system would spend its major time near the time-averaged value.

The assumption that \( \max_n \rho_{nn} \ll 1 \) presupposes a uniform distribution over many degrees of freedom, which is equivalent to ascribing to the system a high effective temperature. Such an assumption loses its validity at low effective temperatures, when it can be that \( \rho_{nn} \sim 1 \), for instance, when there is Bose-Einstein condensation.

### 4.3. Eigenstate thermalization on average

If the initial statistical operator corresponds to a fixed pure state \( |j\rangle \) that is a Hamiltonian eigenstate then, according to Eq. (37), the observable quantities do not change in time,
\[ \langle \hat{A}(t) \rangle = A_{jj} = \langle \hat{A} \rangle, \quad (47) \]

which is evident, since the Hamiltonian eigenstate is a stationary state. It is reasonable to expect that there could exist a set of states around a given eigenstate from which the system would equilibrate on average to a state close to that eigenstate.

Let us fix the eigenstate \( |j\rangle \), whose energy is \( E_j \). And let us consider an energy shell
\[ E_j \equiv \{ E_n : |E_n - E_j| < \Delta E_j \} \quad (48) \]
of energies deviating from \( E_j \) not more than by \( \Delta E_j \). The energy shell is characterized by the indicator function

\[
\xi_j(E_n) = \begin{cases} 
1, & E_n \in E_j, \\
0, & E_n \notin E_j.
\end{cases}
\]  
(49)

If we take for an initial condition an eigenvector with the energy in the energy shell \( (48) \), then we have

\[ \rho_{nn} = \sum_j \rho_{nj} \xi_j(E_n) = 1 \]  
(50)

with the normalization condition

\[ \sum_n \rho_{nn} = 1. \]

The variation of the diagonal matrix elements of an operator, in the range of the energy shell, is smaller than

\[ \Delta A_j = \max_{E_n \in B_j} A_{nn} - \min_{E_n \in B_j} A_{nn}, \]  
(51)

which is a non-negative quantity. The values \( A_{nn} \) lie in the vicinity of the average

\[ A_j = \frac{\sum_n \xi_j(E_n) A_{nn}}{\sum_n \xi_j(E_n)}. \]  
(52)

It is always possible to choose so narrow energy shell that

\[ \frac{\Delta A_j}{A_j} \ll 1. \]  
(53)

Then the elements \( A_{nn} \) are almost constant inside the chosen energy shell, so that one can invoke the theorem of average

\[ \sum_n p_{nj} \xi_j(E_n) A_{nn} \approx A_j \sum_n p_{nj} \xi_j(E_n) = A_j. \]  
(54)

Therefore, for any normalized weight \( p_{nj} \), we have

\[ \langle A(t) \rangle \approx A_j. \]  
(55)

Equality (55) implies that there exists an attraction basin in the set of initial states, from which there has an approximate equilibration on average to the value \( A_j \). Since, under condition (53), the weight \( p_{nj} \) can be taken as an arbitrary normalized distribution, it can be chosen to be a simple uniform distribution

\[ p_{nj} = \frac{1}{\sum_n \xi_j(E_n)}. \]  
(56)

This allows us to rewrite average (52) in the form

\[ A_j = \sum_n p_{nj} \xi_j(E_n) A_{nn}. \]  
(57)

which corresponds to the Gibbs microcanonical ensemble. Such an equilibration can be called the eigenstate thermalization on average. Different variants of this type of thermalization have been considered in [89–92].

It is worth emphasizing that this thermalization is approximate, it is approximate on average, initial states are assumed to be pure and sufficiently close to a fixed eigenstate, the energy shell has to be rather narrow, with the width of this shell depending on the considered observable, and also the width can depend on the statistics of the system components.

4.4. Density of quantum states

The properties of a quantum system are governed by its Hamiltonian spectrum, whose form should also be important for quantum system equilibration. Because of the existence of the correspondence principle, one often compares quantum systems with classical ones. The equilibration of the latter depends on whether the system is integrable or not. Integrable classical systems do not equilibrate. For equilibration to happen, the system has to be nonintegrable.

The notion of integrability is well defined for classical systems. If such a system, with \( n \) degrees of freedom, possesses \( n \) independent functions in involution (mutually Poisson commuting), then the system can be integrated up to quadratures [52]. This is known as Liouville theorem.

Quantum integrable systems in \( n \) dimensions are often defined analogously by requiring the existence of \( n \) mutually commuting operators. However, there is no a theorem equivalent to the classical Liouville theorem, and there is nothing resembling the reduction to quadratures. In quantum mechanics, an integrable system is understood as that for which the spectral eigenproblem can be solved exactly [93]. The class of such exactly solvable quantum problems is very limited.

Equilibration in closed classical systems is usually accompanied by the appearance of chaos [52]. The notion of quantum chaos is not well defined. One often resorts to the correspondence principle, telling that quantum chaos can exist, provided that the corresponding classical system is chaotic [94]. The latter requires that the system be nonintegrable. However, classical chaos does not necessarily imply quantum chaos [95]. Quantum chaos is assumed to be related to the properties of the energy spectra.

Gutzwiller [96] proposed that the spectra of integrable and nonintegrable quantum systems should be qualitatively different, which would result in the qualitative difference of the density of states

\[ \rho(E) = \sum_n \delta(E - E_n). \]  
(58)

It has been suggested [96–100] that, generally, the spectra of quantum systems consist of two parts, regular and irregular, the regular part being almost equidistant, while the irregular part being random. Respectively, the density of states can be represented as a sum of two terms, regular and irregular [94,96,101,102],

\[ \rho(E) = \rho_{reg}(E) + \rho_{irr}(E). \]  
(59)

The regular term, in the semiclassical approximation, reads as

\[ \rho_{reg}(E) = \frac{1}{\Gamma(d/2)} \left( \frac{m}{2\pi} \right)^{d/2} \times \]  
(60)

\[ \times \int \Theta(E - U(r))[E - U(r)]^{(d-2)/2} dr. \]
where $d$ is the space dimensionality and $U(r)$ is an external potential. The semiclassical density of states (60) is widely used for describing cold trapped atoms (see, e.g., [103]). If the external potential is a homogeneous function of degree $\alpha$, such that

$$U(\lambda r) = \lambda^\alpha U(r),$$

then the regular term is

$$\rho_{reg}(E) \equiv \frac{(2 + \alpha)d}{2\alpha} E^{(2 + \alpha)d/2\alpha - 1}.$$

In particular, for a harmonic trap, when $\alpha = 2$, one has

$$\rho_{reg}(E) \simeq E^{d-1}d \quad (\alpha = 2).$$

The irregular term of the density of states can be represented as

$$\rho_{irr}(E) \equiv \text{Im} \sum_n A_n \exp \left\{ i \left( S_n - \frac{\pi}{2} \beta_n \right) \right\}, \quad (61)$$

where the sum is over semiclassical trajectories, $S_n$ is an action integral, and $\beta_n$ is the Maslov index [94,96,104]. The amplitude $A_n$ and the action integral $S_n$ are continuous functions of the energy levels. The irregular term of the density of states is a fastly oscillating function of energy.

4.5. Statistics of level spacing

The most evident difference between integrable and nonintegrable quantum systems is seen in the statistics of their level spacing. Integrable quantum systems is characterized by level repulsion.

Thus, integrable and nonintegrable quantum systems display essentially different statistics of their level spacing. Therefore equilibration can proceed differently for these systems. However, as is discussed in the previous subsections, integrable quantum systems can exhibit equilibration on average to a representative ensemble, though, may be, slower than nonintegrable systems. This is contrary to classical integrable systems that do not equilibrate at all.

5. Characteristic time scales

5.1. Interaction time

The process of equilibration of quantum systems from a strongly nonequilibrium state goes through several stages characterized by the corresponding time scales [7,62,82,107].

The shortest characteristic time is the time, during which two particles interact with each other. For atomic gases, this time is of order $t_{int} \sim a_s/v$, with $a_s$ being a scattering length and $v \sim 1/m a_s$, atomic velocity, where $m$ is mass. Thence, the interaction time reads as

$$t_{int} = m a_s^2. \quad (66)$$

For Hubbard-type models with the on-site interaction $U$, the interaction time is

$$t_{int} = \frac{1}{U}. \quad (67)$$

Generally, the interaction time is inverse to the typical strength of the strongest particle interactions. This time defines the first stage of relaxation

$$0 < t < t_{int} \quad \text{(interaction stage)}, \quad (68)$$

when particles move being yet not correlated with each other.

5.2. Local-equilibrium time

After the interaction time, particles start developing mutual correlations that are growing till the local-equilibrium time that, for atomic gases, is of the order $t_{loc} \sim \lambda/v$, where $\lambda \sim 1/\rho a_s^2$ is the mean free path and $\rho$ is an average particle density. Then the local-equilibrium time is

$$t_{loc} = \frac{m}{\rho a_s}. \quad (69)$$

The interval of time between the interaction time and local-equilibrium time is the kinetic stage:

$$t_{int} < t < t_{loc} \quad \text{(kinetic stage)}, \quad (70)$$

when the motion can be described by kinetic equations.
5.3. Heterophase time

After the local-equilibrium time $t_{loc}$, the evolution enters hydrodynamic stage that can be subdivided into substages, when the system dynamics crosses a phase transition line. If the system starts from one phase, but is quenched into the conditions supporting another phase, then, after $t_{loc}$, there appear the nuclei of the new phase, which are also called clusters, or droplets, of this new phase, or mesoscopic topological defects. These are randomly intermixed in space with the regions of the former phase. Frenkel [108] called such systems heterophase. The nuclei of the competing phase can be of various shapes. In particular, they can be in the form of vortices [26,27,56]. Such a heterophase mixture exists during the heterophase stage

$$ t_{loc} < t < t_{het} \quad \text{(heterophase stage)}, \quad (71) $$

till the heterophase time $t_{het}$, when the system becomes uniform, with the more stable phase filling the system volume. Equilibration passing through the heterophase stage has been observed, e.g., in experiments [20,22,24]. The heterophase stage can be of different duration, with $t_{het}$ being defined by the system properties [56,62]. In some cases, the heterophase time can be so long that the system can be treated as quasi-equilibrium during the whole stage (71), which happens for many condensed-mater systems [62,107,109–118].

5.4. Relaxation time

The second substage of the hydrodynamic stage is that during which the system relaxes to an equilibrium or quasi-equilibrium state. This is the relaxation stage that lasts in the interval of time

$$ t_{het} < t < t_{rel} \quad \text{(relaxation stage)}, \quad (72) $$

if there exists the heterophase stage (71). If the latter does not exist, then the relaxation stage starts from the local-equilibrium time (69). The heterophase stage may be absent, when the initial nonequilibrium conditions correspond to the same phase to which the system is assumed to relax, so that no transition line is crossed [119].

In numerical simulations for the Hubbard model, corresponding to atoms in optical lattices, the relaxation time was found to be

$$ t_{rel} = \frac{1}{J}, \quad (73) $$

for both bosons [46,47] as well as fermions [48]. Introducing [18] the effective mass

$$ m^* \equiv \frac{1}{2Ja^2}, \quad (74) $$

where $a$ is the distance between the nearest-neighbor sites, transforms the relaxation time (73) to the form

$$ t_{rel} = 2m^* a^2. \quad (75) $$

The effective mass, of course, can be very different from the atomic mass $m$.

For quasi-one-dimensional Bose gases with local interactions, the relaxation rates were calculated by Mazets and Schmiedmayer [120]. The rate of populating the radially excited modes by pairwise atomic collisions, at $T < \omega_\perp$ was found to be

$$ \Gamma_2 = \alpha \omega_\perp \exp \left( -\frac{2\omega_\perp}{T} \right), \quad (76) $$

with the parameter

$$ \alpha \equiv \frac{Na^2}{2l_\perp^2}. \quad (77) $$

Here $N$ is the number of atoms in the trap, $l_\perp \equiv 1/\sqrt{m\omega_\perp}$ and $l_\parallel \equiv 1/\sqrt{m\omega_\parallel}$ are the effective trap lengths in the longitudinal and transverse directions, while $\omega_\parallel$ and $\omega_\perp$ are the related trap frequencies. The relaxation rate, due to three-body collisions [120], is

$$ \Gamma_3 = 2\alpha^2 \omega_\perp. \quad (78) $$

The relaxation time is defined by the minimal inverse rate:

$$ t_{rel} = \min \left\{ \frac{1}{\Gamma_2}, \frac{1}{\Gamma_3} \right\}, \quad (79) $$

that is, by the maximal relaxation rate. Atomic correlations, described by the two-body and three-body correlation functions $g_n(0)$ at zero distance, can diminish [120] the relaxation rates to

$$ \tilde{\Gamma}_n = \Gamma_n g_n(0) \quad (n = 2,3), \quad (80) $$

resulting in the increase of the relaxation time.

After the relaxation time, the system enters a quasi-equilibrium regime, lasting till the recurrence time.

5.5. Recurrence time

As is seen from Eq. (35), the operator average is quasi-periodic, hence it always returns arbitrarily close to any given initial value after the recurrence time that can be estimated as

$$ t_{rec} \approx \frac{2\pi}{\Delta E}, \quad (81) $$

where $\Delta E$ is the mean spacing between energy eigenvalues near $E$. The mean spacing is of the order $\Delta E \sim E/\delta E$, with $\delta E$ being the number of states in the interval $\Delta E$. The entropy, as is known, is related to $\delta E$ as $S \sim \ln \delta E$, hence
If \( \delta E \sim \exp(S) \). For systems with \( N \gg 1 \), one has \( S \sim N \) and \( \delta E \sim \epsilon N \), with \( \epsilon \) being energy per atom. Therefore the recurrence time can be represented by the expression

\[
t_{\text{rec}} = \frac{2\pi}{\epsilon N} \exp(N).
\]  
(82)

If the reduced energy \( \epsilon \) defines the interaction time \( t_{\text{int}} \sim 1/\epsilon \), then

\[
t_{\text{rec}} = \frac{2\pi}{N} \exp(N) t_{\text{int}}.
\]  
(83)

The recurrence time, for large \( N \), can be extremely long. The quasistationary stage is associated with the time interval

\[
t_{\text{rel}} < t < t_{\text{rec}} \quad (\text{quasistationary stage}).
\]  
(84)

Generally, the typical relations between the characteristic times satisfy the inequalities

\[
t_{\text{int}} < t_{\text{loc}} < t_{\text{het}} < t_{\text{rel}} < t_{\text{rec}}.
\]  
(85)

Numerical values of these time scales, for condensed-matter systems and atomic gases, are given in [7,62].

### 6. Nonisolated quantum systems

#### 6.1. Quantum system in a bath

If a finite quantum system is not isolated, one needs to treat the coupled pair of the system and its surrounding, called bath. The composite system is characterized by the space of microstates

\[
\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B,
\]  
(86)

being the tensor product of the Hilbert space \( \mathcal{H}_A \), related to the studied system, and of the Hilbert space \( \mathcal{H}_B \), related to the bath. The dimension of the composite space is

\[
\dim \mathcal{H}_{AB} = d_A d_B,
\]  
(87)

where the factor space dimensions are

\[
d_A \equiv \dim \mathcal{H}_A, \quad d_B \equiv \dim \mathcal{H}_B.
\]  
(88)

The total Hamiltonian is the sum

\[
H_{AB} = H_A \otimes 1_B + 1_A \otimes H_B + H_{\text{int}},
\]  
(89)

where \( 1_A \) and \( 1_B \) are the unity operators defined on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. To simplify the notation, one usually omits the unity operators, writing

\[
H_{AB} = H_A + H_B + H_{\text{int}},
\]  
(90)

which implies Eq. (89).

The statistical operator of the composite system is denoted by \( \hat{\rho}_{AB}(t) \), which can be pure or not. One introduces partial statistical operators

\[
\hat{\rho}_A(t) \equiv \text{Tr}_B \hat{\rho}_{AB}(t), \quad \hat{\rho}_B(t) \equiv \text{Tr}_A \hat{\rho}_{AB}(t),
\]  
(91)

in which \( \text{Tr}_A \) means the trace over the space \( \mathcal{H}_A \) and \( \text{Tr}_B \), the trace over \( \mathcal{H}_B \).

Observable quantities are related to the system defined on \( \mathcal{H}_A \). The expectation value of an operator \( \hat{A} \) of a local observable, acting on \( \mathcal{H}_A \), is

\[
\langle \hat{A}(t) \rangle \equiv \text{Tr}_A \hat{\rho}_{AB}(t) \hat{A} = \text{Tr}_A \hat{\rho}_A(t) \hat{A},
\]  
(92)

where \( \text{Tr}_{AB} \) is the trace over the space \( \mathcal{H}_{AB} \).

#### 6.2. Distance between statistical operators

If one is interested in the equilibration on average, one needs to consider the time-averaged statistical operator of the composite object,

\[
\overline{\rho}_{AB} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \hat{\rho}_{AB}(t) \, dt,
\]  
(93)

and the time-averaged partial statistical operators

\[
\overline{\rho}_A \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \hat{\rho}_A(t) \, dt, \quad \overline{\rho}_B \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \hat{\rho}_B(t) \, dt.
\]  
(94)

The difference between two statistical states, described by the operators \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \), defined on the same Hilbert space \( \mathcal{H} \), can be characterized by the Hilbert-Schmidt trace distance

\[
\text{dist}[\hat{\rho}_1, \hat{\rho}_2] \equiv \text{Tr}_H \sqrt{ (\hat{\rho}_1 - \hat{\rho}_2)^2 }.
\]  
(95)

Respectively, the time-averaged distance is

\[
\overline{\text{dist}}[\hat{\rho}_1, \hat{\rho}_2] \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \text{dist}[\hat{\rho}_1(t), \hat{\rho}_2(t)] \, dt.
\]  
(96)

It has been shown [121–123] that

\[
\overline{\text{dist}}[\hat{\rho}_A(\tau), \overline{\rho}_A] \leq \sqrt{d_A \text{Tr}_H \overline{\rho}_B^2}.
\]  
(97)

In the proof, one uses the Jensen inequality, telling that for a concave function, for which \( d^2 f(x)/dx^2 \leq 0 \), one has \( f(x) \leq f(\overline{x}) \), and the inequality

\[
\text{Tr}_A \sqrt{ (\hat{\rho}_1 - \hat{\rho}_2)^2 } \leq \sqrt{d_A \text{Tr}_H (\hat{\rho}_1 - \hat{\rho}_2)^2 }.
\]  
(98)

If the bath is large, with a great number of uniformly distributed states, such that

\[
\text{Tr} \overline{\rho}_B^2 \ll \frac{1}{d_A},
\]  
(99)

then the system equilibrates on average in the sense of the small trace distance (97).
6.3. Decoherence induced by environment

For the problem of equilibration, a large bath, representing an environment in thermodynamic limit, is of special interest. The standard definition of thermodynamic limit for a system of \( N \) particles in a volume \( V \) reads as

\[
N \to \infty, \quad V \to \infty, \quad \frac{N}{V} \to \text{const}.
\]

This definition, however, is not applicable for nonuniform quantum systems confined in an external potential, such as trapped atoms, since the system volume \( V \) is not well defined in this case. A general definition of thermodynamic limit, which is valid for arbitrary systems, including confined ones, is given as follows [18,103]. For any extensive observable quantity \( A_N \equiv \langle \hat{A} \rangle \), corresponding to a system of \( N \) particles, it should be:

\[
N \to \infty, \quad A_N \to \infty, \quad \frac{A_N}{N} \to \text{const}.
\]

As is obvious, for a uniform system, where the volume \( V \) is well defined, the latter definition reduces to the standard one.

But we are interested in thermodynamic limit for a bath of volume \( V_B \) and described by an effective number of degrees of freedom \( N_B \), while the considered quantum system is kept finite, with a finite number of atoms \( N \). Then thermodynamic limit for the bath can be defined [124] in the standard way

\[
N_B \to \infty, \quad V_B \to \infty, \quad \frac{N_B}{V_B} \to \text{const}.
\]

The role of the bath can be played by other parts of the same physical system, where a finite part of it is fixed. In that case, thermodynamic limit is applied to the whole system [125].

The effect of decoherence assumes that at long times the average (92) yields

\[
\lim_{t \to \infty} \lim_{N_B \to \infty} \langle \hat{A}(t) \rangle = \sum_n \rho^\ast_{nn} A_{nn},
\]

where the bath thermodynamic limit (99) is implied. The thermodynamic and time limits do not commute with each other. Decoherence, as is defined in Eq. (100), also means equilibration, according to definition (4).

Decoherence, as a general phenomenon, is supposed to hold for any operator \( \hat{A} \) from the algebra of local observables. Condition (100) may be trivially valid for some operators that commute with the system Hamiltonian \( H_A \), which has nothing to do with decoherence.

The reduction of an operator average to the diagonal sum (100) is called decoherence or dephasing [126,127] because of the evident reason. In quantum mechanics, the nondiagonal terms characterize interference effects that are typical of coherent processes. In radiating systems, such nondiagonal terms are responsible for the appearance of coherent radiation [71–80]. Hence, the disappearance of the coherent terms is nothing but decoherence.

The properties of the environment are often characterized by random distributions. For examples, expanding the wave function of the composite system (system plus bath) over a basis, with the real and imaginary parts of the expansion coefficients being independent Gaussian random variables with zero mean, one comes to the reduced statistical operator for the system, having canonical form [128,129].

Assuming a random distribution of the wave-function expansion coefficients is equivalent to assuming infinite number of degrees of freedom of environment. It is therefore possible, to impose the condition of randomness on the wave function of the considered system, without explicitly invoking a bath [130–132].

Similarly, equilibration and decoherence occur, if one assumes that the system correlation functions exponentially decay [133,134]. This assumption is analogous to the random-phase averaging and can be interpreted as a random influence of environment.

7. Quasi-isolated quantum systems

7.1. Absence of complete isolation

The assumption of the possibility of an absolute system isolation is, evidently, an idealization of the real situation. In reality, no one system can be completely isolated. There always exists some random influence of environment, though may be weak. Moreover, the notion of absolute isolation is self-contradictory, since in order to state that the given system has been isolated during a period of time, it is necessary to accomplish a series of measurements with this system for proving that the system has really been isolated during that time span [60,135,136]. But each measurement, even if it is a nondemolition measurement [137], disturbs the system, making it not isolated. Measuring devices play the role of environment acting on the system. And, vice versa, the influence of environment is analogous to the action of measuring procedures.

In computer modeling, the role of external noise is played by numerical errors accompanying calculations. With time, there is the accumulation of errors, analogous to the increasing influence on the system of random environment. Thus, a complete isolation is merely a theoretical abstraction not existing in nature. There can exist only quasi-isolated systems.

7.2. Properties of quasi-isolated systems

To formulate rigorously the notion of quasi-isolated systems, let us proceed to mathematical definitions. Let \( H_A \)
be a system Hamiltonian defined on the Hilbert space $\mathcal{H}_A$ and let $H_B$, defined on $\mathcal{H}_B$, be the Hamiltonian of surrounding. The total space of microstates for the composite object “system plus environment” is the tensor product
\[
\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B.
\]
(101)

The Hamiltonian of the composite object is
\[
H_{AB} = H_A + H_B + H_{int},
\]
(102)
where $H_{int}$ characterizes interactions between the system and environment and where the unity operators, for the brevity of notation, are omitted.

It is reasonable to call a system quasi-isolated, when the matrix representation for the observables of the composite object “system plus environment” is not changed under the influence of the environment, so that this matrix representation coincides with the matrix representation of these operators in the basis of the eigenvectors of the total Hamiltonian (102). This means that $H_A$ is to be the integral of motion with respect to $H_{AB}$,
\[
[H_A, H_{AB}] = 0.
\]
(103)

Condition (103) is the mathematical definition of a quasi-isolated system. The latter also implies the commutation relation
\[
[H_A, H_{int}] = 0.
\]
(104)

The property of quasi-isolatedness defines nondestructive, or nondestroying, action of environment on the system, similarly to the minimally disturbing action of measurements [60,135,136]. This property should not be confused with the definition of nondemolition measurements for a given observable [137], where the commutation of the operator of the measured quantity with the total Hamiltonian is required: $[\hat{A}, H_{AB}] = 0$.

When the system Hamiltonian satisfies the eigenproblem
\[
H_A|n\rangle = E_n|n\rangle,
\]
(105)
and the system is quasi-isolated, then the eigenproblem for $H_{AB}$ has the form
\[
H_{AB}|nk\rangle = (E_n + \varepsilon_{nk})|nk\rangle,
\]
(106)
where the vector $|nk\rangle \equiv |n\rangle \otimes |k\rangle$ pertains to the space $\mathcal{H}_{AB}$.

For the statistical ensemble $\{\mathcal{H}_{AB}, \hat{\rho}_{AB}(t)\}$, the evolution of the statistical operator is given by the law
\[
\dot{\hat{\rho}}_{AB}(t) = \hat{U}_{AB}(t)\hat{\rho}_{AB}(0)\hat{U}_{AB}^+(t).
\]
(107)

If $H_{AB}$ does not explicitly depend on time, then
\[
\hat{U}_{AB}(t) = \exp(-iH_{AB}t).
\]
(108)

As an initial condition for the evolution of the statistical operator, we accept
\[
\hat{\rho}_{AB}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_B(0),
\]
(109)
which means that the interaction between the considered system and the bath is switched on at $t=0$. This initial condition is not principal, but it is assumed here merely for simplicity.

### 7.3. Equilibration of quasi-isolated systems

We are interested in the temporal behavior of operators of local observables, associated with the system. Say, we consider the average of an operator $\hat{A}$ defined on $\mathcal{H}_A$. For this average, we have the expression
\[
\langle \hat{A}(t) \rangle \equiv \text{Tr}_{AB}\hat{\rho}_{AB}(t)\hat{A} = \text{Tr}_{A}\hat{\rho}_A(t)\hat{A},
\]
(110)
in which
\[
\hat{\rho}_A(t) = \text{Tr}_{B}\hat{\rho}_{AB}(t).
\]
(111)

In the matrix representation, this yields
\[
\langle \hat{A}(t) \rangle = \sum_{mn} A_{nm} \langle m|\hat{\rho}_A(t)|n\rangle = \sum_k \langle nk|\hat{\rho}_{AB}(t)|nk\rangle.
\]
(112)

where
\[
\rho_{mn}^A(t) = \langle m|\hat{\rho}_A(t)|n\rangle = \sum_k \langle nk|\hat{\rho}_{AB}(t)|nk\rangle.
\]

In view of the evolution operator (108), Eq. (113) can be written as
\[
\rho_{mn}^A(t) = \sum_k \rho_{mnk}^AB \exp\left\{-i(\omega_{mn} + \varepsilon_{mn})t\right\},
\]
(114)
with the notation
\[
\rho_{mnk}^AB \equiv \langle nk|\hat{\rho}_{AB}(0)|nk\rangle
\]
(115)
and
\[
\omega_{mn} \equiv E_m - E_n, \quad \varepsilon_{mn} \equiv \varepsilon_{mk} - \varepsilon_{nk}.
\]
(116)

Thus for the average (110), we get
\[
\langle \hat{A}(t) \rangle = \sum_{mn} \sum_k \rho_{mnk}^AB A_{nm} \exp\left\{-i(\omega_{mn} + \varepsilon_{mn})t\right\}.
\]
(117)

Taking into account the initial condition (109) yields
\[
\rho_{mnk}^AB = \langle m|\hat{\rho}_A(0)|n\rangle \langle k|\hat{\rho}_B(0)|k\rangle.
\]
(118)

The latter, employing the notation
\[
\rho_{mn} \equiv \langle m|\hat{\rho}_A(0)|n\rangle, \quad p_k \equiv \langle k|\hat{\rho}_B(0)|k\rangle,
\]
(119)
reduces to
\[
\rho_{mnk}^AB = \rho_{mn} p_k.
\]
(120)

The forms $\rho_{mn}$ and $p_k$ satisfy the normalization conditions
\[
\sum_n \rho_{nn} = 1, \quad \sum_k p_k = 1.
\]
(121)
Introducing the decoherence factor

\[ D_{mn}(t) \equiv \sum_k p_k \exp(-i\varepsilon_{mnk}t), \quad (122) \]

we come to the expression

\[ \langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}(t)A_{nm}D_{mn}(t), \quad (123) \]

in which

\[ \rho_{mn}(t) \equiv \rho_{mn} \exp(-i\omega_{mn}t). \quad (124) \]

Separating in summation (123) the diagonal and non-diagonal terms gives the average

\[ \langle \hat{A}(t) \rangle = \sum_n \rho_{nn}A_{nn} + \sum_{m \neq n} \rho_{mn}(t)A_{nm}D_{mn}(t), \quad (125) \]

where we take into account that, according to definition (116),

\[ \varepsilon_{nnk} = 0. \quad (126) \]

Introducing the density of states

\[ p_{mn}(\varepsilon) \equiv \sum_k p_k \delta(\varepsilon - \varepsilon_{mnk}), \quad (127) \]

satisfying the normalization condition

\[ \int_{-\infty}^{\infty} p_{mn}(\varepsilon)\,d\varepsilon = 1, \quad (128) \]

transforms the decoherence factor (122) to

\[ D_{mn}(t) = \int_{-\infty}^{\infty} p_{mn}(\varepsilon) \exp(-i\varepsilon t)\,d\varepsilon. \quad (129) \]

For the diagonal elements, we have

\[ p_{nn}(\varepsilon) = \sum_k p_k \delta(\varepsilon), \quad D_{nn}(t) = 1. \]

Therefore the decoherence factor is nontrivial only for the non-diagonal elements of Eq. (129).

Assume that the surrounding environment is large, such that the spectrum of \( H_{AB} \), with respect to the index \( k \), can be treated as continuous and the summation over \( k \) can be replaced by integration. Then the density of states (127) may be considered as a smooth function of \( \varepsilon \). In that case, we can invoke the Riemann-Lebesgue lemma \[138\] telling that, if \( p_{mn}(\varepsilon) \) is a measurable \( L^1 \)-integrable function, then the time limit of Eq. (129), for \( m \neq n \), is zero:

\[ \lim_{t \to \infty} D_{mn}(t) = 0 \quad (m \neq n). \quad (130) \]

Hence for average (125), we obtain

\[ \lim_{t \to \infty} \langle \hat{A}(t) \rangle = \sum_n \rho_{nn}A_{nn}. \quad (131) \]

Therefore, a quasi-isolated system in a large environment equilibrates. This has been proven here for an arbitrary quasi-isolated system, which makes the given proof principally different from the considerations, where the property of quasi-isolatedness was not used, but instead, concrete systems were studied. For instance, equilibration for quadratic Hamiltonians was demonstrated \[124,134\]. Also, we have considered equilibration in the strict sense of definition (4), but not equilibration on average \[86–92\], as described in Sec. 4.2 and Sec. 4.3.

7.4. Examples of decoherence factors

To illustrate explicitly admissible forms of the decoherence factor (129), let us consider several concrete cases.

When the density of states enjoys the Gaussian form

\[ p_{mn}(\varepsilon) = \frac{1}{\sqrt{2\pi\gamma_{mn}}} \exp\left(-\frac{\varepsilon^2}{2\gamma_{mn}^2}\right), \quad (132) \]

then the decoherence factor is also Gaussian,

\[ D_{mn}(t) = \exp\left(-\frac{\varepsilon_{mn}^2}{2}\right). \quad (133) \]

For a uniform density of states

\[ p_{mn}(\varepsilon) = \frac{1}{2\gamma_{mn}} \Theta(\gamma_{mn} - \varepsilon) \Theta(\gamma_{mn} + \varepsilon), \quad (134) \]

where \( \Theta(\varepsilon) \) is a unit-step function, the decoherence factor is

\[ D_{mn}(t) = \frac{\sin(\gamma_{mn}t)}{\gamma_{mn}t}. \quad (135) \]

Assuming the Poisson distribution

\[ p_{mn}(\varepsilon) = \frac{1}{2\gamma_{mn}} \exp\left(-\frac{|\varepsilon|}{\gamma_{mn}}\right), \quad (136) \]

yields the Lorentz-type decoherence factor

\[ D_{mn}(t) = \frac{1}{1 + (\gamma_{mn}t)^2}. \quad (137) \]

Conversely, for the Lorentz density of states

\[ p_{mn}(\varepsilon) = \frac{\gamma_{mn}}{\pi(\varepsilon^2 + \gamma_{mn}^2)}, \quad (138) \]

the decoherence factor is exponential,

\[ D_{mn}(t) = \exp(-\gamma_{mn}t). \quad (139) \]
In the above examples, the density of states is represented by monotonic nonincreasing functions of $\varepsilon$. For nonmonotonic functions, the situation is similar, just formulas can become a bit more cumbersome. For instance, if we take the density of states in the Wigner form

$$p_{mn}(\varepsilon) = \frac{\pi|\varepsilon|}{4\gamma_{mn}} \exp\left(-\frac{\pi\varepsilon^2}{4\gamma_{mn}}\right),$$  \hspace{1cm} (140)

then the decoherence factor reads as

$$D_{mn}(t) = 2\int_0^\infty x \exp\left(-x^2\right) \cos\left(2\gamma_{mn} t \sqrt{\pi} x\right) dx.$$  \hspace{1cm} (141)

The integral

$$2\int_0^\infty x \exp\left(-x^2\right) \cos(qx) dx = 1 - qF\left(\frac{q}{2}\right)$$

is expressed through the Dawson function

$$F(z) = \exp\left(-z^2\right) \int_0^z \exp\left(x^2\right) dx.$$\hspace{1cm} (142)

At large variable, the Dawson function behaves as

$$F(z) \simeq \frac{1}{2z} + \frac{1}{4z^2} \hspace{1cm} (z \to \infty).$$

Consequently, the decoherence factor decays as

$$D_{mn}(t) \simeq -\frac{\pi}{2\gamma_{mn}^2 t^2} \hspace{1cm} (t \to \infty).$$  \hspace{1cm} (143)

It is seen from these examples that the typical decoherence time is

$$t_{\text{dec}} = \frac{1}{\gamma},$$  \hspace{1cm} (144)

where $\gamma$ is an average $\gamma_{mn}$.

7.5. Occurrence of rare events

The property of asymptotic decoherence (130) and, consequently, of the equilibration limit (131), are based on the use of the Riemann-Lebesgue lemma assuming that the density of states $p_{mn}(\varepsilon)$ is a measurable $L^1$ integrable function. By its definition, the density of states is a non-negative function that is normalized as in Eq. (128). Hence, it is always $L^1$ integrable. But it may happen that it can contain nonmeasurable parts. Then the properties of decoherence and equilibration are not guaranteed.

For instance, the density of states can be formed of two parts,

$$p_{mn}(\varepsilon) = \lambda_1 p_{mn}^{\text{rel}}(\varepsilon) + \lambda_2 p_{mn}^{\text{osc}}(\varepsilon),$$  \hspace{1cm} (145)

in which the first term is a measurable function leading to a relaxation process accompanied by decoherence and equilibration, while the second term is a nonmeasurable function resulting in decoherence oscillations, and where

$$\lambda_1 + \lambda_2 = 1 \hspace{1cm} (\lambda_1 > 0, \lambda_2 > 0).$$  \hspace{1cm} (146)

Thence the decoherence factor is also a sum

$$D_{mn}(t) = \lambda_1 D_{mn}^{\text{rel}}(t) + \lambda_2 D_{mn}^{\text{osc}}(t)$$  \hspace{1cm} (147)

of the related terms, the first of which satisfies property (130).

Let the second term of the density of states have the form

$$p_{mn}^{\text{osc}}(\varepsilon) = \frac{1}{2} \sum_j \left[ \delta(\varepsilon - \alpha_j) + \delta(\varepsilon + \alpha_j) \right],$$  \hspace{1cm} (148)

where $\alpha_j$ are real quantities. Then the oscillating part of the decoherence factor is

$$D_{mn}^{\text{osc}}(t) = \sum_j \cos(\alpha_j t).$$  \hspace{1cm} (149)

Therefore the level of decoherence will oscillate, being a quasi-periodic function characterized by the periods

$$t_j = \frac{2\pi}{\alpha_j}.$$  \hspace{1cm} (150)

Respectively, the operator average (123) does not decohere and the system does not equilibrate. However it may look as equilibrium on average, keeping in mind temporal averaging.

If the parameters $\alpha_j$ are small, nonequilibrium oscillations will be rare. But if they are strong, they cannot be neglected, since their presence can essentially influence the system properties. Such rare, but strong, events can support the existence of heterophase fluctuations [62,107,109–118].

8. Equilibration under nondestructive measurements

8.1. Definition of nondestructive measurements

As has been mentioned above, measurements accomplished with the considered system act similarly to the action of environment. The main difference is that the measurement procedures, generally, depend on time, which makes the consideration more complicated.
The system be characterized by a Hamiltonian $H_A$, defined on a Hilbert space $\mathcal{H}_A$, and a measuring device be described by a Hamiltonian $H_D$, defined on a Hilbert space $\mathcal{H}_D$. These Hamiltonians $H_A$ and $H_D$ are assumed to be time independent.

The total space of microstates is

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_D.$$  \hspace{1cm} (151)

The total Hamiltonian can be written as

$$H = H_A + H_M,$$  \hspace{1cm} (152)

where the measurement Hamiltonian

$$H_M = H_D + H_{int}$$  \hspace{1cm} (153)

is the sum of the Hamiltonian $H_D$ of the measuring device and of $H_{int}$ characterizing the interactions between the studied system and the measuring device. The system Hamiltonian $H_A$ and the device Hamiltonian $H_D$ do not explicitly depend on time, but the interaction Hamiltonian $H_{int}$ is a function of time. Consequently, the measurement Hamiltonian is an explicit function of time, $H_M = H_M(t)$.

As usual, we omit in the formulas the unity operators, keeping in mind that the rigorous expression for Hamiltonian (152) would be

$$H = H_A \otimes \hat{1}_D + \hat{1}_A \otimes H_D + H_{int}.$$  \hspace{1cm} (154)

We consider measurements that minimally disturb the system [60,135,136], because of which they are named nondestructive. The observed system has to be quasi-isolated from the measuring device. A measurement is called nondestructive, if the system Hamiltonian, as well as the device Hamiltonian do not disturb the properties of each other, being the integrals of motion:

$$[H_A, H] = 0, \quad [H_D, H] = 0.$$  \hspace{1cm} (155)

It follows from Eq. (155) that

$$[H_A, H_M] = 0, \quad [H_D, H_M] = 0.$$  \hspace{1cm} (156)

If the eigenproblem for $H_A$ has the form

$$H_A |n\rangle = E_n |n\rangle,$$  \hspace{1cm} (157)

then the measurement Hamiltonian satisfies the eigenproblem

$$H_M(t)|nk\rangle = \varepsilon_{nk}(t)|nk\rangle,$$  \hspace{1cm} (158)

where $|nk\rangle \equiv |n\rangle \otimes |k\rangle$, with $\{|n\rangle\}$ being a basis in $\mathcal{H}_A$ and $\{|k\rangle\}$, a time-independent basis in $\mathcal{H}_D$.

The statistical operator for the composite system is

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^+(t),$$  \hspace{1cm} (159)

in which the initial condition

$$\hat{\rho}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_D(0)$$  \hspace{1cm} (160)

is assumed. This means that measurements start at the moment of time $t = +0$.

In view of eigenproblem (158), it follows that the measurement Hamiltonian satisfies the Lyappo-Danilevsky condition

$$\left[ H_M(t), \int_{0}^{t} H_M(t') dt' \right] = 0.$$  \hspace{1cm} (161)

Since the system Hamiltonian $H_A$ does not explicitly depend on time, the previous property (161) gives

$$\left[ H(t), \int_{0}^{t} H(t') dt' \right] = 0.$$  \hspace{1cm} (162)

Therefore the evolution operator has the form

$$\hat{U}(t) = \exp \left\{ -i \int_{0}^{t} H(t') dt' \right\}.$$  \hspace{1cm} (163)

With the total Hamiltonian (152), this becomes

$$\hat{U}(t) = \exp \left\{ -i H_A t - i \int_{0}^{t} H_M(t') dt' \right\}.$$  \hspace{1cm} (164)

In view of eigenproblems (157) and (158), we have

$$H|nk\rangle = (E_n + \varepsilon_{nk}(t))|nk\rangle.$$  \hspace{1cm} (165)

Therefore the action of the evolution operator on the vector $|nk\rangle$ yields

$$\hat{U}(t)|nk\rangle = \exp \left\{ -iE_n t - i \int_{0}^{t} \varepsilon_{nk}(t') dt' \right\} |nk\rangle.$$  \hspace{1cm} (166)

### 8.2. Temporal dependence of observables

Observable quantities are represented by operator averages. Let us consider the evolution of an observable quantity represented by an operator $\hat{A}$ acting on $\mathcal{H}_A$. This observable is given by the average

$$\langle \hat{A}(t) \rangle \equiv \text{Tr}_N \hat{\rho}(t)\hat{A} = \sum_{mnk} \rho_{mnk}(t) A_{nm},$$  \hspace{1cm} (167)

in which

$$\rho_{mnk}(t) \equiv \langle mk| \hat{\rho}(t) |nk\rangle.$$  \hspace{1cm} (168)
Under the evolution operator (164), we have
\[ \rho_{mnk}(t) = \rho_{mnk}(0) \exp \left\{ -i \omega_{mn} t - i \int_0^t \varepsilon_{mnk}(t') dt' \right\} , \]
with
\[ \rho_{mnk}(0) \equiv \langle mk | \hat{\rho}(0) | nk \rangle , \]
and where the notation is used for the energy differences
\[ \varepsilon_{mnk}(t) \equiv \varepsilon_{mk}(t) - \varepsilon_{nk}(t) \quad \omega_{mn} \equiv E_m - E_n . \]

Notice that
\[ \varepsilon_{nnk}(t) = 0 . \]

Because of the initial condition (160), Eq. (170) reduces to
\[ \rho_{mnk}(0) = \rho_{mn}(0)p_k , \]
where
\[ \rho_{mn}(0) \equiv \langle m | \hat{\rho}(0) | n \rangle , \quad p_k \equiv \langle k | \hat{\rho}(0) | k \rangle . \]

The normalization conditions
\[ \sum_n \rho_{nn}(0) = 1 , \quad \sum_k p_k = 1 \]
are valid. In that way, Eq. (169) becomes
\[ \rho_{mnk}(t) = \rho_{mn}(0)t \rho_{mn}(0) \exp \left\{ -i \int_0^t \varepsilon_{mnk}(t') dt' \right\} , \]
with
\[ \rho_{mn}(t) = \rho_{mn}(0) \exp(-i \omega_{mn} t) . \]

Introducing the decoherence factor
\[ D_{mn}(t) \equiv \sum_k p_k \exp \left\{ -i \int_0^t \varepsilon_{mnk}(t') dt' \right\} , \]
allows us to represent average (167) in the form
\[ \langle \hat{A}(t) \rangle = \sum_{mn} \rho_{mn}(t) A_{nm} D_{mn}(t) . \]

Taking into account properties (171) and (172), we can rewrite the above average as
\[ \langle \hat{A}(t) \rangle = \sum_n \rho_{nn} A_{nn} + \sum_{m \neq n} \rho_{mn}(t) A_{nm} D_{mn}(t) , \]
with \( \rho_{nn} \equiv \rho_{nn}(0) \). This expression is similar to the average for a quasi-isolated system, but with a different decoherence factor.

8.3. Specification of measurement procedure

To better understand the structure of the decoherence factor (177), we need to specify the action of the measurement Hamiltonian. Let this Hamiltonian, describing \( M \) measurements, have the form
\[ H_M(t) = \sum_{j=1}^M \hat{X}_j f_j(t) , \]
where \( f_j(t) \) is a real function and \( \hat{X}_j \) is a self-adjoint operator, not explicitly depending on time. This Hamiltonian satisfies the Lyappo-Danilevsky condition (161). According to Eq. (158), the operator \( H_M \) and, hence, \( \hat{X}_j \), possess the eigenvectors \( | nk \rangle \). That is, we can write
\[ \hat{X}_j | nk \rangle = x_{jnk} | nk \rangle , \]
where \( x_{jnk} \) is a real number, characterizing the measurement impact. As an explicit illustration, we can define \( \hat{X}_j \) as a spectral resolution over projector operators:
\[ \hat{X}_j = \sum_{nk} x_{jnk} \hat{P}_{nk} \quad (\hat{P}_{nk} \equiv | nk \rangle \langle nk |) . \]

The eigenvalues of \( H_M \) acquire the form
\[ \varepsilon_{nk}(t) = \sum_{j=1}^M x_{jnk} f_j(t) . \]

Respectively, for the transition frequency, defined in Eq. (171), we get
\[ \varepsilon_{mnk}(t) = \sum_{j=1}^M \Delta_{jmnk} f_j(t) , \]
where
\[ \Delta_{jmnk} \equiv x_{jnk} - x_{jnk} . \]

Thence the decoherence factor (177) becomes
\[ D_{mn}(t) = \sum_k p_k \exp \left\{ -i \sum_{j=1}^M \Delta_{jmnk} \varphi_j(t) \right\} , \]
with the phase function
\[ \varphi_j(t) \equiv \int_0^t f_j(t') dt' . \]

Being interested in the long-time behavior of the decoherence factor, let us assume that the phase function, after the last measurement, occurring at the time \( t_M \), does not depend on the measurement index \( j \),
\[ \varphi_j(t) = \varphi(t) \quad (t > t_M) . \]
Then, after \( t > t_M \), the decoherence factor takes the form
\[
D_{mn}(t) = \sum_k p_k \exp \left\{ - i \Delta_{mn} \varphi(t) \right\}, \tag{188}
\]
in which the definition of the mean impact
\[
\Delta_{mn} = \frac{1}{M} \sum_{j=1}^{M} \Delta_{jn} \tag{189}
\]
is used. Finally, introducing the distribution of the measurement impacts
\[
p_{mn}(x) \equiv \sum_k p_k \delta \left( x - \Delta_{mn} \right), \tag{190}
\]
we come to the decoherence factor
\[
D_{mn}(t) = \int p_{mn}(x) \exp \left\{ - ix \varphi(t) \right\} dx. \tag{191}
\]

The measuring device is a macroscopic object, because of which the spectrum of the measurement Hamiltonian, with respect to the index \( k \) should be treated as continuous. This means that the notation \( \Sigma_k \) has to be understood as integration over \( k \). Consequently, the impact distribution (190) can be considered as a measurable function. By its definition, it is also \( L^1 \) integrable. Therefore, by the Riemann-Lebesgue lemma,
\[
D_{mn}(t) \to 0 \quad (M \varphi(t) \to \infty). \tag{192}
\]
That is, nondestructive measurements lead, at large times, to the system decoherence, provided that either the number of measurements \( M \) or the phase function \( \varphi(t) \) increases with time, so that their product also tends to infinity.

### 8.4. Decoherence under nondestructive measurements

To obtain an explicit form of the decoherence factor, we have to specify the impact distribution (190). Taking for the latter a Gaussian distribution
\[
p_{mn}(x) = \frac{1}{\sqrt{2\pi} \sigma_{mn}} \exp \left\{ - \frac{x^2}{2\sigma_{mn}^2} \right\}, \tag{193}
\]
yields the decoherence factor
\[
D_{mn}(t) = \exp \left\{ - \frac{\sigma_{mn}^2}{2} M^2 \varphi^2(t) \right\}. \tag{194}
\]

In the case of the Lorentz distribution
\[
p_{mn}(x) = \frac{\sigma_{mn}}{\pi (x^2 + \sigma_{mn}^2)}, \tag{195}
\]
we have the decoherence factor
\[
D_{mn}(t) = \exp \left\{ - \frac{\sigma_{mn}}{2} M \varphi(t) \right\}. \tag{196}
\]

Let us consider two opposite types of measurements, instantaneous and continuous. For instantaneous measurement, we have
\[
f_j(t) = \delta(t - t_j), \quad \varphi_j(t) = \Theta(t - t_j). \tag{197}
\]
Hence,
\[
\varphi(t) = \varphi_j(t) \bigg|_{t > t_M} = 1. \tag{198}
\]
And continuous measurement implies
\[
f_j(t) = 1, \quad \varphi_j(t) = t. \tag{199}
\]
In any case, decoherence happens when the number of measurements \( M \) increases [60,135,136].

If measurements are accomplished during all period of observation, the number of measurements is proportional to the time elapsed. This can be formalized by the relation
\[
M = \frac{t}{\Delta t}, \quad \Delta t \equiv \frac{1}{M} \sum_{j=1}^{M} (t_{j+1} - t_j). \tag{200}
\]

To summarize the results, we omit, for brevity, the indices \( m \) and \( n \), marking instead the form of the impact distribution \( p(x) \), Gaussian (193) or Lorentzian (195), and the type of the measurement, instantaneous or continuous. We assume relation (200) and define the decoherence time
\[
t_{dec} \equiv \frac{\sqrt{\Delta t}}{\sigma}. \tag{201}
\]

Then we obtain the decoherence factors, under instantaneous measurements, for the Gaussian distribution,
\[
D_{\text{inst}}^G(t) = \exp \left\{ - \frac{1}{2} \left( \frac{t}{t_{dec}} \right)^2 \right\}, \tag{202}
\]
and for the Lorentzian distribution,
\[
D_{\text{inst}}^L(t) = \exp \left\{ - \frac{t}{t_{dec}} \right\}. \tag{203}
\]

While in the case of continuous measurements, we find for the Gaussian distribution,
\[
D_{\text{cont}}^G(t) = \exp \left\{ - \frac{1}{2} \left( \frac{t}{t_{dec}} \right)^4 \right\}, \tag{204}
\]
and for the Lorentzian distribution,
\[
D_{\text{cont}}^L(t) = \exp \left\{ - \frac{t}{t_{dec}}^2 \right\}. \tag{205}
\]

Continuous measurements result in much faster decoherence than instantaneous ones.

When decoherence leads to the disappearance of the second term in the operator average (179), this also implies equilibration. Therefore, equilibration of a finite quantum system can be achieved by the measurement procedure. As has been emphasized above, the notion of complete isolation is self-contradictory, since to state that the system is isolated requires accomplishing measurements confirming the fact of isolation [60,135,136]. But measurements, even the nondestructive ones, produce decoherence in the observable quantities. In the long run, decoherence, induced by measurements, results in the system equilibration.
8.5. Existence of time arrow

The real time in the world, as is known, flows in one direction, from the past to future. The real time arrow cannot be reversed. But the microscopic equations of motion are time-symmetric, allowing one to reverse the direction of time. The problem why this time symmetry becomes broken, when passing from the microscopic equations to the observable quantities, is the long-standing puzzle attracting the attention of many. It is not our aim to give here a discussion of various approaches to this problem. But we want to stress that this problem finds a simple resolution in the frame of the equilibration theory of quasi-isolated systems.

The explanation for the existence of the unidirected time arrow lies in the fact that completely isolated systems do not exist in nature [139–141]. In the best case, they can be quasi-isolated, interacting with environment that, even if it does not destroy and does not essentially disturb the system, anyway, influences the temporal behavior of observable quantities. Even if the influence of environment is so weak that it can be neglected, the considered system is subject to measurement procedures. An absolutely isolated system is an abstraction that cannot be observed. As soon as one passes from time-reversible microscopic equations of motion to the consideration of observables, on has to take into account that the studied system can only be quasi-isolated. And the statement that the system is quasi-isolated necessarily presupposes that this fact is to be confirmed by measurements. In any case, whether there exists environment acting on a quasi-isolated system or the latter is subject to at least nondestructive measurements, there appears decoherence and equilibration. Each of the latter phenomena influences the time dependence of observable quantities, making their evolution irreversible.

The existence for any quasi-isolated system of attenuation, caused either by environment or by measurements, or by both, leads to equilibration in the strict sense of limit (4), but not merely in the sense of time averaged quantities. Even infinitesimally small such an attenuation principally changes the situation, resulting in the existence of time arrow.

In order to emphasize the importance of even an infinitesimally weak nonisolatedness, let us consider two limits: One limit is when the measurement dispersion \( \sigma \) tends to zero, which corresponds to neglecting the influence of measurements or environment. In the case of the environment influence, the role of \( \sigma \) is played by the environment attenuation rate \( \gamma \). Another limit is when time tends to infinity. For the decoherence factor, in the first case, we have

\[
\lim_{\sigma \to 0} D(t) = 1 .
\] (206)

While in the second case,

\[
\lim_{t \to \infty} D(t) = 0 .
\] (207)

These properties lead to the noncommutativity of the limits, as far as

\[
\lim_{\sigma \to 0} \lim_{t \to \infty} D(t) = 0 ,
\] (208)

while reversing the limits, we get

\[
\lim_{t \to \infty} \lim_{\sigma \to 0} D(t) = 1 .
\] (209)

This noncommutativity of the limits for the decoherence factor results in the noncommutativity of the limits for the observable quantities:

\[
\lim_{\sigma \to 0} \lim_{t \to \infty} \langle \hat{A}(t) \rangle \neq \lim_{t \to \infty} \lim_{\sigma \to 0} \langle \hat{A}(t) \rangle .
\] (210)

The noncommutativity of these limits is a pivotal property of quasi-isolated systems, exhibiting the existence of time arrow [139–141].

We may also notice that, if we would like to formally consider negative time, then the exponentials in the decoherence factors (139) and (203) should be treated as depending on \( |t| \). Therefore, in all cases the average \( \langle \hat{A}(t) \rangle \) equals \( \langle \hat{A}(-t) \rangle \). That is, formally the observable quantities are symmetric with respect to time inversion, in agreement with microscopic equations of motion. However, any nonequilibrium statistical state of a quasi-isolated system always tends with time to an equilibrium or, at least, to a quasi-equilibrium state. This tendency explains the existence of time arrow.

9. Summary

Finite quantum systems, because of their important and widespread technological applications, have become the topic of intense investigations, both experimental and theoretical. Nonequilibrium properties of such systems are of special interest, being less understood than equilibrium ones. The principal question is whether finite quantum systems equilibrate and if so, when and how?

Cold trapped atoms provide a very convenient laboratory for the study of nonequilibrium phenomena. The most interesting experiments with nonequilibrium trapped atoms are reviewed in the present survey, and the related numerical simulations are discussed.

Finite quantum systems, strictly speaking, cannot be absolutely equilibrium, as far as their statistical states are quasi-periodic. But, the studies show that isolated integrable as well as nonintegrable quantum systems can equilibrate on average to a quasi-equilibrium state with a rather long lifetime, since the recurrence time is usually very long. Equilibration to such quasi-stationary states can be faster or longer depending on the system integrability. The resulting quasi-stationary states are, generally, described by representative Gibbs ensembles. In particular cases, these can be microcanonical ensembles.
Nonisolated quantum systems, interacting with a macroscopic bath, equilibrate in the strict sense. Equilibration is accompanied by decoherence. The time dependence of the decoherence factor is defined by the density of states that, in turn, depends on the system parameters. The relaxation can be exponential or Gaussian [142].

Actually, completely isolated finite systems do not exist, but there are only quasi-isolated systems. Any finite system is always influenced by its environment, maybe weakly, but, anyway, noticeably. Moreover, the concept of isolated systems is self-contradictory, since in order to state that a system is isolated during a period of time, it is necessary to prove this by a series of measurements, whose influence makes the system not isolated. Such quasi-isolated systems can equilibrate in the strict sense. In some cases, there can exist rare events making the resulting state quasi-equilibrium. The latter is equilibrium on average.

In any case, quasi-isolated systems do relax to an either equilibrium or a quasi-equilibrium state. This irreversible behavior explains the existence of time arrow.

In this review, the equilibration processes have been described, starting from a strongly nonequilibrium state and tending to a steady state. Another problem of great interest is the opposite process of the development of nonequilibrium states from the given equilibrium one. For example, acting on trapped Bose atoms by external alternating fields [18,143,144], it is possible to generate dynamic transitions from the ground-state Bose-Einstein condensate to a vortex superfluid and then to a turbulent superfluid [145–147]. Investigating such transformations from a given equilibrium state to a highly nonequilibrium one would allow for the better understanding of the development and properties of nonequilibrium states in finite quantum systems.

Nonequilibrium effects in finite quantum systems can find numerous applications, ranging from various electronic devices to quantum computers. As has been noticed long time ago by Schrödinger [148] and Bohr [149], even the functioning of humans may require, for their correct description, the necessity of employing the theory of finite quantum systems. For instance, this could be necessary for understanding the functioning of the brain. In turn, understanding the brain activity would allow for the creation of thinking quantum systems [150].

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References

[1] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Clarendon, Oxford, 2003).
[2] E.H. Lieb, R. Seiringer, J.P. Solovej, and J. Yngvason, The Mathematics of the Bose Gas and Its Condensation (Birkhauser, Basel, 2005).
[3] V. Letokhov, Laser Control of Atoms and Molecules (Oxford University, New York, 2007).
[4] C.J. Pethik and H. Smith, Bose-Einstein Condensation in Dilute Gases (Cambridge University, Cambridge, 2008).
[5] Ph.W. Courteille, V.S. Bagnato, and V.I. Yukalov, Laser Phys. 11, 659 (2001).
[6] J.O. Andersen, Rev. Mod. Phys. 76, 599 (2004).
[7] V.I. Yukalov, Laser Phys. Lett. 1, 435 (2004).
[8] K. Bongs and K. Sengstock, Rep. Prog. Phys. 67, 907 (2004).
[9] V.I. Yukalov and M.D. Girardeau, Laser Phys. Lett. 2, 375 (2005).
[10] A. Posazhennikova, Rev. Mod. Phys. 78, 1111 (2006).
[11] V.I. Yukalov, Laser Phys. Lett. 4, 632 (2007).
[12] N.P. Proukakis and B. Jackson, J. Phys. B 41, 203002 (2008).
[13] V.I. Yukalov, Ann. Phys. (N.Y.) 323, 461 (2008).
[14] V.A. Yurovsky, M. Olshanii, and D.S. Weiss, Adv. At. Mol. Opt. Phys. 55, 61 (2008).
[15] W. Ketterle, and M.W. Zwierlein, Riv. Nuovo Cimento 31, 247 (2008).
[16] C. Moseley, O. Fialko, and K. Ziegler, Ann. Phys. (Berlin) 17, 561 (2008).
[17] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).
[18] V.I. Yukalov, Laser Phys. 19, 1 (2009).
[19] A.L. Fetter, Rev. Mod. Phys. 81, 647 (2009).
[20] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch, and I. Bloch, Nature 415, 39 (2002).
[21] M. Greiner, O. Mandel, T.W. Hänsch, and I. Bloch, Nature 419, 51 (2002).
[22] M.W. Zwierlein, C.H. Schunck, C.A. Stan, S.M.F. Rupach, and W. Ketterle, Phys. Rev. Lett. 94, 180401 (2005).
[23] T. Kinoshita, T. Wenger, and D.S. Weiss, Nature 440, 900 (2006).
[24] L.E. Sadler, J.M. Highie, S.R. Leslie, M. Vengalattore, and D.M. Stapper-Kurn, Nature 443, 312 (2006).
[25] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature 440, 324 (2007).
[26] T.W. Kibble, J. Phys. A 9, 1387 (1976).
[27] W.H. Zurek, Nature 317, 505 (1985).
[28] C.N. Weller, T.W. Neely, D.R. Scherer, A.S. Bradley, M.J. Davis, and B.P. Anderson, Nature 455, 948 (2008).
[29] C.-L. Hung, X.B. Zhang, N. Gemelke, and C. Chin, Phys. Rev. Lett. 104, 160403 (2010).
[30] W.S. Bakr, A. Peng, M.E. Tai, R. Ma, and J. Simon, arXiv:1006.0754 (2010).
[31] J.F. Sherson, C. Weitenberg, M. Endres, M. Cheneau, I. Bloch, and S. Kuhr, Nature 467, 68 (2010).
[32] E. Fermi, J. Pasta, and S. Ulam, Los Alamos Report LA-1940 (1955).
[33] M. Henon and C. Heiles, Astron. J. 69, 73 (1964).
[34] M. Feingold, N. Moiseyev, and A. Peres, Phys. Rev. A 30, 509 (1984).
[35] R.V. Jensen and R. Shankar, Phys. Rev. Lett. 54, 1879 (1985).
[36] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. 98, 050405 (2007).
[37] A. Cassidy, C.W. Clark, and M. Rigol, arXiv:1008.4794 (2010).
[124] T. Barthel and U. Schollw"ock, Phys. Rev. Lett. 100, 100601 (2008).
[125] M. Cramer, C.M. Dawson, J. Eisert, and T.J. Osborne, Phys. Rev. Lett. 100, 030602 (2008).
[126] W.H. Zurek, Rev. Mod. Phys. 75, 715 (2003).
[127] M. Schlosshauer, Rev. Mod. Phys. 76, 1267 (2004).
[128] E. Schr"odinger, Statistical Thermodynamics (Cambridge University, Cambridge, 1952).
[129] S. Goldstein, J.L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. 96, 050403 (2006).
[130] P. Reimann, Phys. Rev. Lett. 99, 160404 (2007).
[131] P. Reimann, J. Stat. Phys. 132, 921 (2008).
[132] B.V. Fine and F. Hantschel, arXiv:1010.4673 (2010).
[133] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).
[134] M. Cramer and J. Eisert, New J. Phys. 12, 055020 (2010).
[135] V.I. Yukalov, Moscow Univ. Phys. Bull. 25, 49 (1970).
[136] V.I. Yukalov, Moscow Univ. Phys. Bull. 26, 22 (1971).
[137] V.B. Braginsky and F.Ya. Khalili, Rev. Mod. Phys. 68, 1 (1996).
[138] S. Bochner and K. Chandrasekharan, Fourier Transforms (Princeton University, Princeton, 1949).
[139] V.I. Yukalov, Physica A 234, 725 (1997).
[140] V.I. Yukalov, Phys. Rev. E 65, 056118 (2002).
[141] V.I. Yukalov, Phys. Lett. A 308, 313 (2003).
[142] S. Genway, A.F. Ho, and D.K. Lee, Phys. Rev. Lett. 105, 260402 (2010).
[143] V.I. Yukalov, E.P. Yukalova, and V.S. Bagnato, Phys. Rev. A 56, 4845 (1997).
[144] V.I. Yukalov and V.S. Bagnato, Laser Phys. Lett. 6, 399 (2009).
[145] E.A. Henn, J.A. Seman, G. Roati, K.M. Magalhães, and V.S. Bagnato, Phys. Rev. Lett. 103, 045301 (2009).
[146] V.I. Yukalov, Laser Phys. Lett. 7, 467 (2010).
[147] J.A. Seman, E.A. Henn, R.F. Shiozaki, G. Roati, F.J. Poveda-Cuevas, K.M. Magalhães, V.I. Yukalov, M. Tsubota, M. Kobayashi, K. Kasamatsu, and V.S. Bagnato, arXiv:1007.4953 (2010).
[148] E. Schr"odinger, What is Life (Cambridge University, Cambridge, 1945).
[149] N. Bohr, La Physique Atomique et la Connaissance Humaine (Gontier, Genève, 1961).
[150] V.I. Yukalov and D. Sornette, Laser Phys. Lett. 6, 833 (2009).

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