The \( q \)-golden ratio, Catalan numbers, and an identity of Sauermann–Wigderson

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Abstract

In this note, we present some basic properties of \( q \)-Fibonacci numbers and their relationship to the \( q \)-golden ratio and Catalan numbers. We then use this relationship to give a short proof of a combinatorial identity.

1 \( q \)-Fibonacci numbers

Definition 1. The \( q \)-Fibonacci numbers are defined recursively by

\[
F_0(q) = F_1(q) = 1, \quad F_n(q) = F_{n-1}(q) + qF_{n-2}(q).
\]

Note that the original Fibonacci numbers are retrieved by setting \( q = 1 \).

These \( q \)-Fibonaccis have a nice non-recursive form.

Proposition 2.

\[
F_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k.
\]

Proof. We carry out a quick proof by induction. Clearly this holds for \( F_0(q) = F_1(q) = 1 \). Assuming the statement for \( n-1 \) and \( n-2 \), we have

\[
F_n(q) = F_{n-1}(q) + qF_{n-2}(q) = \sum_k \left( \binom{n-1-k}{k} + \binom{n-2-(k-1)}{k-1} \right) q^k
\]

\[
= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k
\]

as desired. \( \square \)

From this, by plugging in \( q = 1 \), we immediately have an identity relating the standard Fibonacci numbers to binomial coefficients:

\[
F_n = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}.
\]
The $q$-golden ratio

It is well known that the ratio of successive Fibonacci numbers approaches the golden ratio $\varphi = \frac{1 + \sqrt{5}}{2}$. We prove the following $q$-analog:

**Theorem 3.** For $n \geq 0$, we have as a power series around $q = 0$

$$\frac{F_{n+1}(q)}{F_n(q)} = \frac{1 + \sqrt{1 + 4q}}{2} + O(q^{n+1}).$$

In particular, as $n \to \infty$, the ratio approaches $\varphi(q) = \frac{1 + \sqrt{1 + 4q}}{2}$, which itself equals the golden ratio $\varphi$ when $q = 1$.

**Proof.** Before we begin, note that the theorem is equivalent (by taking reciprocals) to

$$\frac{F_n(q)}{F_{n+1}(q)} = -\frac{1 - \sqrt{1 + 4q}}{2q} + O(q^n).$$

The theorem itself will be a straightforward proof by induction on $n$. The base case of $n = 0$ is immediate. We will now show that if the statement is true for $n - 1$, it is also true for $n$.

By the defining recurrence for the $q$-Fibonacci, we have

$$\frac{F_{n+1}(q)}{F_n(q)} = \frac{F_n(q) + qF_{n-1}(q)}{F_n(q)} = 1 + q\frac{F_{n-1}(q)}{F_n(q)}.$$

By the inductive hypothesis and the equivalent reciprocal formulation, this is equal to

$$1 + q\left(-\frac{1 - \sqrt{1 + 4q}}{2q} + O(q^n)\right) = \frac{1 + \sqrt{1 + 4q}}{2} + O(q^{n+1}),$$

as desired. $\square$

The formula appearing in this theorem requires no guessing to discover. Ignoring the error term, the recurrence for the $q$-Fibonacci tells us that the $q$-golden ratio $\varphi(q)$ should satisfy $\varphi(q) = 1 + \frac{2q}{\varphi(q)}$ or $\varphi(q)^2 - \varphi(q) - q = 0$, the $q$-analog of the standard quadratic satisfied by $\varphi$. By the quadratic formula and appropriate choice of signs, we immediately get $\varphi(q) = \frac{1 + \sqrt{1 + 4q}}{2}$. This all parallels how one might discover and prove that $F_{n+1}/F_n \to \varphi$ for the standard Fibonacci numbers.

Catalan numbers and the Sauermann–Wigderson identity

In the proof, we took the reciprocal

$$\frac{F_n(q)}{F_{n+1}(q)} = -\frac{1 - \sqrt{1 + 4q}}{2q} + O(q^n).$$
Ignoring the error term, the right hand side is, upon replacing \( q \) by \(-q\), the generating function for the Catalan numbers.

We therefore have the following:

**Corollary 4.** The coefficients on \(1, q, q^2, \ldots, q^n\) of \(F_n(q)/F_{n+1}(q)\) are, up to sign, the Catalan numbers: \(1, -1, 2, -5, 14, \ldots, (-1)^nC_n\).

As a further corollary, we get a stronger version of an identity needed by Sauermann–Wigderson in [1]. (This strengthening was noted and proved differently in [2].)

**Corollary 5.** Let \(m \leq n\). Then

\[
\sum_{(m_1, \ldots, m_t)} (-1)^t \binom{n - m_1}{m_1 - 1} \binom{n - m_2}{m_2} \cdots \binom{n - m_t}{m_t} = (-1)^m C_{m-1},
\]

where the sum is over all tuples (of varying length) of positive integers summing to \(m\).

**Proof.** Note that

\[
\sum_k \binom{n - k}{k - 1} q^k = \sum_k \binom{n - 1 - (k - 1)}{k - 1} q^k = qF_{n-1}(q).
\]

The other binomial coefficients appearing in the original identity are of the form \(\binom{n - k}{k}\), which are directly the coefficients in \(F_n(q) - 1\). (The requirement of positive \(k\) means we need to subtract off the constant term.)

The desired quantity, therefore, is the \(q^m\) coefficient in

\[
-qF_{n-1}(q) \left(1 - (F_n(q) - 1) + (F_n(q) - 1)^2 - \ldots\right) = -q \frac{F_{n-1}(q)}{F_n(q)}.
\]

Because of the \(-q\) out front, we are seeking the negative of the \(q^{m-1}\) coefficient in \(\frac{F_{n-1}(q)}{F_n(q)}\). By the \(q\)-golden ratio Catalan corollary, our final result is

\[-(-1)^{m-1} C_{m-1} = (-1)^m C_{m-1},\]

as desired.

\[\Box\]

**References**

[1] L. Sauermann and Y. Wigderson, *Polynomials that vanish to high order on most of the hypercube*. arXiv:2010.00077 [math.CO] (2020).

[2] D. Zeilberger, *Two Quick Proofs of a Catalan Lemma Needed by Lisa Sauermann and Yuval Wigderson*, Personal Journal of Shalosh B. Ekhad and Doron Zeilberger (2020).