FITTING PHASE-TYPE FRAILTY MODELS

JORGE YSLAS

Abstract. Frailty models are survival analysis models which account for heterogeneity and random effects in the data. In these models, the random effect (the frailty) is assumed to have a multiplicative effect on the hazard. In this paper, we present frailty models using phase-type distributions as the frailties. We explore the properties of the proposed frailty models and derive expectation-maximization algorithms for maximum-likelihood estimation. The algorithms' performance is illustrated in several numerical examples of practical significance.

1. Introduction

A phase-type (PH) distribution is defined as the distribution of the time until absorption of an otherwise transient time-homogeneous pure-jump Markov process. These distributions have a long history going back to Erlang [16], Jensen [18], and more recently Neuts [21]. Examples of PH distributions include several classical distributions such as the exponential, Erlang, Coxian, hyperexponential, and any finite mixture of them. PH distributions have been employed in various contexts since they often provide exact, or even explicit, solutions to important problems in complex stochastic models. This is the case, for example, in fields as diverse as biology, operational research, queuing theory, renewal theory, and risk theory, cf. e.g., [10, 14, 17]. Furthermore, the class of PH distributions is known to be dense in the set of distributions with support on the positive real numbers in the sense of weak convergence (see [11, Section 3.2.1]). This means that any distribution with support on the positive half-line can be approximated arbitrarily well by a PH distribution (of sufficiently large dimension). Moreover, statistical inference for PH distributions is a topic well developed in the literature. For instance, maximum-likelihood estimation was proposed in [8] using an expectation-maximization (EM) algorithm, which was subsequently extended in [22] for the case of censored observations.

A core concept in survival analysis is that of the hazard function. This function specifies the instantaneous risk of the event of interest for an individual, given that the individual has not experienced the event previously. Models based on the hazard function are fundamental tools in survival analysis, being Cox’s proportional hazards model ([15]) one of the most influential models. This model assumes that the ratio of the hazards between any two individuals is constant over time. Implicitly, this means that if the model is perfectly specified so that all possible relevant covariates are accounted for, then all individuals in a group with the same covariates have the same risk of the event of interest. However, in practice, it is impossible to include all relevant risk factors. This unaccounted part is usually known as the unobserved heterogeneity. The frailty model addresses this problem of unobserved heterogeneity in events. The first account of univariate frailty models can be traced back to the work in [9], although the term frailty itself was introduced in [24]. This model assumes that an individual’s hazard function depends on an unobservable, time-independent random variable known as the frailty. A popular choice for the frailty is the Gamma distribution. This choice is mainly due to its mathematical tractability and computational
feasibility, although there is no biological reason that makes the Gamma distribution more appealing than other distributions.

Regarding the use of PH distributions in survival analysis, the first exploration can be found in [1], where the author showed the flexibility of the hazard function for PH distributions. In [20, 23] phase-type distributions with Coxian structure were considered as survival regression models. More recently in [12], the authors proposed a generalization of the proportional hazards model based on inhomogeneous phase-type distributions (see [3]) and showed how maximum-likelihood estimation can be performed via an EM algorithm.

In this paper, we propose the use of PH distributions as frailties in the frailty model. We show that the resulting model has similar properties to the Gamma frailty model, in addition to closed-form expressions for its different functionals, making this model very appealing from a mathematical and computational point of view. Moreover, due to the denseness of PH distributions, any other frailty model can be approximated by this model. For application purposes, estimation of these models is essential. Thus, we proceed to develop an EM algorithm for maximum-likelihood estimation of this model, including the case of right-censoring data and covariates effect. Furthermore, we laid out the specific changes to extend the model to the shared frailty and correlated frailty frameworks.

The rest of the paper is organized as follows. In Section 2, we present an overview of the class of PH distributions and some important properties for our purposes. In Section 3, we review the frailty model, show how PH distributions can be employed as frailties, and derive some relevant properties. In Section 4, we derive an EM algorithm for maximum-likelihood estimation of the proposed model. In Section 5, we present several numerical illustrations. In Section 6, we derive extensions to the shared and correlated frailty cases. Finally, in Section 7, we provide a summary of our findings.

2. Phase-Type Distributions

Let \((J_t)_{t \geq 0}\) denote a Markov jump process on a state-space \(\{1, \ldots, p, p+1\}\), where states 1, \ldots, \(p\) are transient and state \(p+1\) is absorbing. In this way, \((J_t)_{t \geq 0}\) has an intensity matrix of the form

\[ \Lambda = \begin{pmatrix} T & t \\ 0 & 0 \end{pmatrix}, \]

where \(T\) is a \(p \times p\) sub-intensity matrix and \(t\) is a \(p\)-dimensional column vector. Since the rows of \(\Lambda\) sum to zero, we have that \(t = -T e\), where \(e\) is the \(p\)-dimensional column vector of ones. Assume that the process starts in state \(k\) with probability \(\pi_k\), that is \(\pi_k = \mathbb{P}(J_0 = k), k = 1, \ldots, p\), and let \(\pi = (\pi_1, \ldots, \pi_p)\) be the vector of initial probabilities. Here, we assume that \(\mathbb{P}(J_0 = p+1) = 0\). Then, we say that the time until absorption

\[ Z = \inf\{t \geq 0 \mid J_t = p + 1\} \]

has a phase-type distribution with representation \((\pi, T)\) and we write \(Z \sim \text{PH}(\pi, T)\). The density \(f_Z\) and survival function \(S_Z\) of \(Z \sim \text{PH}(\pi, T)\) are given by

\[ f_Z(z) = \pi \exp(Tz)t, \quad z > 0, \]
\[ S_Z(z) = \pi \exp(Tz)e, \quad z > 0, \]

where the exponential of a matrix \(A\) is defined by

\[ \exp(A) = \sum_{i=1}^{\infty} \frac{A^i}{i!}. \]

Not only are there closed-form expressions for the distribution and density functions, but this also applies to the Laplace transform and moments. More specifically, the moments are given by

\[ \mathbb{E}(Z^n) = n! \pi (-T)^{-n} e, \quad n \in \mathbb{N}, \]
and the Laplace transform is given by
\[
L_Z(u) = \mathbb{E}(e^{-uZ}) = \pi(uI - T)^{-1}t = \pi(u(-T)^{-1} + I)^{-1}e,
\]
which is well-defined for \( u \) larger than the largest real eigenvalue of \( T \), in particular for \( u \geq 0 \). For a comprehensive reading on phase-type distributions we refer to [11].

3. Phase-type frailty model

Recall that for a continuous random variable \( Y \), the hazard function \( \mu_Y \) is given by
\[
\mu_Y(t) = \frac{f_Y(t)}{S_Y(t)}.
\]
A related function is the cumulative hazard function \( M_Y \), which is given by
\[
M_Y(t) = \int_0^t \mu_Y(s)ds = -\log(S_Y(t)).
\]
In the frailty model one assumes that the hazard function of an individual is proportional to an unobservable random variable \( Z \). More specifically, let \( Y \) be a random variable with conditional hazard given \( Z \) of the form
\[
\mu(t; Z) = Z\mu(t).
\]
(3.1)
Here, \( \mu \) is known as the baseline hazard function and the random variable \( Z \) as the frailty. Note that the conditional survival function of \( Y \mid Z = z \) is given by
\[
S_Y(y \mid z) = \exp\left(-z \int_0^y \mu(t)dt\right) = \exp(-zM(y)).
\]
Thus, the survival function of \( Y \) is given by
\[
S_Y(y) = \int_0^\infty S_Y(y \mid z)dF_Z(z) = \int_0^\infty \exp(-zM(y))dF_Z(z) = L_Z(M(y)),
\]
where \( L_Z \) is the Laplace transform of \( Z \). Furthermore, the model can incorporate predictor variables \( X = (X_1, \ldots, X_d) \) via
\[
\mu(t; Z, X) = Z\mu(t)\exp(X\beta),
\]
where \( \beta \) is a \( d \)-dimensional column vector. Consequently, a frailty model is a generalization of the well-known proportional hazards model. For the sake of simplicity, we will restrict our treatment to the model (3.1) sometimes to focus on the main ideas of the frailty model. Next, we present two classic examples of frailties.

Example 3.1 (Gamma frailty). Assume that \( Z \sim \text{Gamma}(\alpha, \beta) \). Then, the Laplace transform of \( Z \) is given by
\[
L_Z(u) = (1 + u/\beta)^{-\alpha}.
\]
Thus, the survival function of \( Y \) is given by
\[
S_Y(y) = (1 + M(y)/\beta)^{-\alpha}.
\]
To avoid identifiability issues, one typically assumes that \( \mathbb{E}(Z) = 1 \). In the Gamma frailty case, this can be done by taking \( \alpha = \beta = 1/\sigma^2 \), \( \sigma > 0 \), so that \( \mathbb{E}(Z) = 1 \) and \( \text{var}(Z) = \sigma^2 \).

Example 3.2 (Positive stable frailty). Consider positive stable frailty \( Z \) with corresponding Laplace transform \( L_Z(u) = e^{-u^\alpha} \), where \( \alpha \in (0, 1] \). This yields
\[
S_Y(y) = \exp(-M(y)^\alpha).
\]
We refer to [25] for a comprehensive account on frailty models in survival analysis.

3.1. **Phase-type frailty.** We now introduce our novel model by considering phase-type distributions as frailties. More specifically, let $Z \sim \text{PH}(\mathbf{\pi}, \mathbf{T})$ with Laplace transform (2.1). Then, the survival function of $Y$ is given by

$$S_Y(y) = \pi(M(y)\mathbf{I} - \mathbf{T})^{-1}t,$$

with corresponding density function

$$f_Y(y) = \mu(y)\pi(M(y)\mathbf{I} - \mathbf{T})^{-2}t.$$

In particular, this implies the hazard function is given by

$$\mu_Y(y) = \mu(y)\pi(M(y)\mathbf{I} - \mathbf{T})^{-2}t / \pi(M(y)\mathbf{I} - \mathbf{T})^{-1}t.$$

**Remark 3.1.** Note that the Gamma frailty model with integer shape values, that is Gamma($k, \beta$) with $k \in \mathbb{N}$, is a particular case. Furthermore, we know that for non-negative random variables, weak convergence implies convergence of the Laplace transform. Then, the denseness of the PH class of distributions implies weak convergence of the set of PH frailty models to any other frailty model, which means that any frailty model can be approximated arbitrarily well by a PH frailty model.

**Remark 3.2.** It is well-known that PH distributions have identifiability issues. Namely, different dimension and parameter configurations may lead to very similar or exactly the same density shapes. Thus, while one can assume without loss of generality that the mean of the PH distribution is one (the class of PH distributions is closed under scalar constant multiplication), the identifiability issues will be preserved. Hence, we do not make any assumptions on the mean of the PH frailty here.

We now show a series of properties of this model that resemble those of the Gamma frailty model, for instance, those found in [25, Section 3.3]:

i) It is a standard result that the expectation of $Z \mid Y > y$ is given by

$$E(Z \mid Y > y) = -\frac{L_Z'(M(y))}{L_Z(M(y))}.$$

For $Z \sim \text{PH}(\mathbf{\pi}, \mathbf{T})$, the expression above takes the form

$$E(Z \mid Y > y) = \frac{\pi(M(y)\mathbf{I} - \mathbf{T})^{-2}t}{\pi(M(y)\mathbf{I} - \mathbf{T})^{-1}t}.$$

Note that this is a decreasing function that reaches its maximum $E(Z)$ at $y = 0$. Then, it is immediate that $\mu_Y(y) \leq \mu(y)E(Z)$.

ii) It turns out that, given that an individual has survive a time $t > 0$, the remaining survival time is also PH frailty distributed. Indeed, for $t > 0$, we have that

$$P(Y > y + t \mid Y > t)$$

$$= \frac{P(Y > y + t)}{P(Y > t)}$$

$$= \frac{\pi(M(y + t)\mathbf{I} - \mathbf{T})^{-1}t}{\pi(M(t)\mathbf{I} - \mathbf{T})^{-1}t}$$

$$= \frac{\pi(M(t)\mathbf{I} - \mathbf{T})^{-1}(M(t)\mathbf{I} - \mathbf{T})(-\mathbf{T})(M(y + t)\mathbf{I} - \mathbf{T})^{-1}e}{\pi(M(t)\mathbf{I} - \mathbf{T})^{-1}t}.$$
\[
= \frac{\pi(M(t)I - T)^{-1}(-T)}{\pi(M(t)I - T)^{-1}t} ((M(y + t) - M(t))I - (T - M(t)I)^{-1}(- (T - M(t)I)) e, \]
which corresponds to a PH frailty model with vector of initial probabilities
\[
\tilde{\pi} = \frac{\pi(M(t)I - T)^{-1}(-T)}{\pi(M(t)I - T)^{-1}t},
\]
sub-intensity matrix
\[
\tilde{T} = T - M(t)I,
\]
and hazard rate \( \tilde{\mu}(y) = \mu(y + t) \).

iii) Regarding the tail behavior of the PH frailty model, this resembles the one of a Gamma frailty with integer shape parameter. More specifically, it is to see that
\[
L_Z(u) \sim C u^{-k}, \quad u \to \infty,
\]
where \( k \) is the size of the Jordan block of the largest real eigenvalue of \( T \).

4. Parameter estimation

The EM algorithm is an iterative method for maximum-likelihood estimation. This algorithm is particularly suitable for situations best described as incomplete-data problems. It solves indirectly the problem of maximizing the incomplete-data likelihood by alternating between an expectation (E) step, consisting of computing the conditional expectation of the complete log-likelihood given the observed data, and a maximization (M) step, which requires the maximization of the the expected log-likelihood found on the E-step. Given that the PH component is not observed for a replication of the PH frailty model, we are clearly in an incomplete-data set-up and the EM algorithm shall be employed. For such a purpose, we assume that \( M(\cdot; \alpha) \) is a parametric function depending on the vector \( \alpha \).

In many applications, a large proportion of the data is either not entirely observed or censored. Here, we consider only the case of right-censoring since it is the most common scenario in the context of its applications in survival analysis. However, the cases of left-censoring and interval-censoring can be treated by similar means. Let \( Y^* \) be a survival time with density (3.2), and let \( C \) be a censoring time. Thus, an observation would consist of \((Y, \Delta)\), where \( Y = \min(Y^*, C) \) and \( \Delta = 1(Y^* \leq C) \). Consider \((y_1, \delta_1), \ldots, (y_N, \delta_N)\) an iid sample from this model, which is also denoted by \((y, \delta)\). In this case, our complete data consist of observations \((Y, \Delta)\) and the phase-type frailty \( Z \sim \text{PH}(\pi, T) \). Hence, the complete likelihood \( L_c \) is given by
\[
L_c(\alpha, \pi, T; (y, \delta)) = \prod_{n=1}^{N} (z_n \mu(y_n; \alpha))^{\delta_n} \exp(-z_n M(y_n; \alpha)) f_Z(z_n; \pi, T). \]
Consequently, the log-likelihood (disregarding the terms which do not depend on any parameters) is given by
\[
l_c(\alpha, \pi, T; (y, \delta)) = \sum_{n=1}^{N} \delta_n \log(\mu(y_n; \alpha)) - z_n M(y_n; \alpha) + \log(f_Z(z_n; \pi, T)). \]

**E-step**

We first require computing the conditional expectation of the log-likelihood given the observed data. In particular, we need to compute \( \mathbb{E}(Z \mid (y, \delta)) \) and \( \mathbb{E}(\log(f_Z(Z; \pi, T)) \mid (y, \delta)) \). For that purpose, we consider first one (generic) data point \((M = 1)\) and let \((y, \delta) = (y_1, \delta_1)\). Then, for the non-censored case \((\delta = 1)\), we have that
\[
\mathbb{E}(Z \mid (y, 1)) = \mathbb{E}(Z \mid Y = y)
\]
\[
\int_0^\infty zf_{Z|Y}(z|y)dz
\]

\[
= \int_0^\infty \frac{f_{Y|Z}(y|z)f_{Z}(z)}{f_{Y}(y)}dz
\]

\[
= \int_0^\infty z\mu(y)\exp(-zM(y))f_{Z}(z)dz
\]

\[
= \frac{\mu(y)}{f_{Y}(y)} \int_0^\infty z^2\exp(-zM(y))f_{Z}(z)dz
\]

\[
= \frac{\mu(y)}{f_{Y}(y)} \mathcal{L}''_{Z}(M(y))
\]

\[
= \frac{\mu(y)}{f_{Y}(y)} 2\pi(M(y)I - T)^{-3}t
\]

\[
= \frac{2\pi(M(y)I - T)^{-3}t}{\pi(M(y)I - T)^{-3}t}.
\]

Now, for the right-censoring case (\(\delta = 0\)) we have that \(E(Z \mid (y, 0)) = E(Z \mid Y > y)\) is given by (3.3).

Regarding the logarithmic term, we have that

\[
E(\log(f_{Z}(Z; \pi, T)) \mid Y = y) = \int \log(f_{Z}(z; \pi, T))f_{Z|Y}(z|y)dz
\]

\[
= \int \log(f_{Z}(z; \pi, T))\frac{f_{Y|Z}(y|z)f_{Z}(z)}{f_{Y}(y)}dz,
\]

and

\[
E(\log(f_{Z}(Z; \pi, T)) \mid Y > y) = \int \log(f_{Z}(z; \pi, T))\frac{\exp(-zM(y))f_{Z}(z)}{S_{Y}(y)}dz.
\]

In general, closed-form solutions for these integrals are not available. However, numerical approximations can be employed for the subsequent maximization needed in the M-step.

Finally, for \(N > 1\) we sum over \((y_n, \delta_n), n = 1, \ldots, N\), in the formulas above.

**M-step**

Having found the required expectations, the next step is to maximize the conditional expected log-likelihood with respect to the parameters \(\alpha\) and \((\pi, T)\), which will be done separately.

In full generality, for the parameter \(\alpha\) we write

\[
\hat{\alpha} = \arg\max_{\alpha} \sum_{n=1}^{N} \delta_n \log(\mu(y_n; \alpha)) - E(Z \mid (y_n, \delta_n))M(y_n; \alpha),
\]

which can be evaluated numerically.

Regarding the parameters \((\pi, T)\) of the phase-type frailty, we will employ the EM algorithm in [8] as follows. Recall that such an algorithm can be employed to fit a PH distribution to a given theoretical distribution. More specifically, let us denote by \(f_{(\pi, T)}\) the density of the phase-type and \(h\) the density of the given theoretical distribution. By fitting \(f_{(\pi, T)}\) to \(h\), one minimizes the Kullback-Leibler information, or equivalently, maximizes \(\int \log(f_{(\pi, T)}(w))h(w)dw\) with respect to \((\pi, T)\). Now note that maximizing

\[
E(\log(f_{Z}(Z; \pi, T)) \mid (y, \delta)) = \sum_{n : \delta_n = 1}^{N} \int \log(f_{Z}(z; \pi, T))f_{Z|Y}(z|y_n)dz
\]
with respect to \((\pi, T)\) is equivalent to maximizing
\[
\frac{1}{N} \mathbb{E} \left( \log \left( f_Z(Z; \pi, T) \right) \mid (y, \delta) \right)
\]
\[
= \int \log \left( f_Z(z; \pi, T) \right) \left( \frac{1}{N} \sum_{n=1}^{N} \delta_n f_{YZ}(z \mid y_n) + (1 - \delta_n) \frac{\exp(-zM(y_n))f_Z(z)}{S_Y(y_n)} \right) dz .
\]

Moreover,
\[
\frac{1}{N} \sum_{n=1}^{N} \delta_n f_{ZY}(z \mid y_n) + (1 - \delta_n) \frac{\exp(-zM(y_n))f_Z(z)}{S_Y(y_n)}
\]

is a density function. Thus, this part of the M-step reduces to fitting a PH distribution to the density (4.1). In general, the EM algorithm does not necessarily require finding estimators that maximize the complete likelihood to guarantee that the incomplete likelihood increases. It suffices to find parameters such that the complete likelihood increases. Hence, we can apply as many iterations as necessary of the EM algorithm in [8].

Remark 4.1. When considering predictor variables \(X\), we have that the conditional density and survival function of \(Y \mid Z = z\) are given by
\[
S_Y(y \mid z; X) = \exp(-z \exp(X\beta)M(y; \alpha))
\]
\[
f_{Y \mid Z}(y \mid z; X) = z \exp(X\beta)\mu(y; \alpha) \exp(-z \exp(X\beta)M(y; \alpha)) .
\]

Then, the complete log-likelihood is given by
\[
l_c(\alpha, \pi, T; y, X) = \sum_{n=1}^{N} \delta_n (X\beta + \log(\mu(y_n; \alpha))) - z_n \exp(X\beta)M(y_n; \alpha) + \log(f_Z(z_n; \pi, T)) .
\]

In this scenario, the only change in the E-step is that the conditional expectations should be modified accordingly to include the factor \(\exp(X\beta)\) in the formulas. Regarding the M-step, the maximization of \(\alpha\) and \(\beta\) should be done conjunctly. More specifically, we need to compute
\[
(\hat{\alpha}, \hat{\beta}) = \arg\max_{(\alpha, \beta)} \sum_{n=1}^{N} \delta_n (X\beta + \log(\mu(y_n; \alpha))) - \mathbb{E}(Z \mid (y_n, \delta_n)) \exp(X\beta)M(y_n; \alpha) .
\]

5. Examples

5.1. Lognormal frailty. One downside of using the lognormal distribution as frailty is that there is no closed-form solution for the Laplace transform, and consequently, neither for the distribution and density functions of \(Y\). Hence, numerical approximations of this Laplace transform must be employed (see, e.g., [7] for an approach in terms of the Lambert W function). In the following we illustrate the practical feasibility of the PH frailty model by fitting it to a simulated dataset from a lognormal frailty.

Consider the Gompertz baseline hazard \(\mu(y) = be^{cy}\) with \(b = 0.01\) and \(c = 1\), and lognormal frailty with density
\[
f_Z(z) = \frac{1}{z\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log(z) - \mu)^2}{2\sigma^2}\right) , \quad z > 0 ,
\]
and parameters $\mu = -0.35$ and $\sigma = 0.8$. Assume a simple two-group setup of equal size, 500 each. For the categorical covariate, we take $\beta = 0.5$. Then, we fit a PH frailty model of dimension 2 with Gompertz baseline hazard to the simulated data. The estimated parameters are

$$\hat{\pi} = (1, 0),$$
$$\hat{T} = \begin{pmatrix} -2.7076 & 2.7076 \\ 0.2168 & -1.5515 \end{pmatrix},$$
$$\hat{b} = 0.0065 \quad \hat{c} = 1.0377 \quad \hat{\beta} = 0.5245.$$

The quality of the fit is supported by Figure 5.1, which shows that the algorithm recovers the data structure for both groups.

**Figure 5.1.** QQ-plot of simulated sample from the lognormal frailty versus fitted PH frailty. $X = 0$ (left) and $X = 1$ (right).

Alternatively, given a theoretical lognormal (or any other) frailty model, we can approximate such a model by simply fitting a PH distribution to the theoretical given distribution. We refer to [8] for details and illustrations, which include the lognormal distribution.

5.2. **Loss insurance data.** The aim of this example is to show that the PH frailty model leads to parsimonious distributions that can be employed in other contexts different to survival analysis. Consider $\mu(y) = \theta y^{\theta-1}$, so that

$$S_Y(y) = \pi(y^\theta I - T)^{-1}t = \pi(y^\theta(-T)^{-1} + I)^{-1}e.$$  

We call this the *matrix-Pareto type III* distribution due to its similarity with the conventional Pareto type III distribution, and also to distinguish it from the matrix-Pareto models introduced in [3] and [5]. For this case, (3.4) yields

$$S_Y(y) \sim C y^{-k\theta}, \quad y \to \infty,$$

meaning that the distribution is regularly varying with index $k\theta$. As noted earlier in [5] for the matrix-Pareto type I and II models, a primary difference among the matrix-Pareto models is how the tail behavior is specified. For the matrix-Pareto type I in [3], the tail behavior is determined by the largest real eigenvalue of $T$. In contrast, for matrix-Pareto type II in [5] the tail behavior
is specified by a scalar (shape) parameter. Finally, for the matrix-Pareto type III introduced here, the tail behavior depends on a scalar parameter and the size of the Jordan block of the largest eigenvalue of $T$. Another key difference arises when working with covariate information. In the matrix-Pareto type III, the covariates can be incorporated to act multiplicative in scale. In contrast, the matrix-Pareto type I distribution can include covariates that act multiplicative in shape (tail) via the survival regression model in [12].

We now illustrate this model’s use in a real-life data set, namely the loss insurance claim data available in the R-package copula. The data consist of 1500 insurance claims from a real-life insurance company, where each data point is conformed of an indemnity payment (loss) and an allocated loss adjustment expense (ALAE). For this analysis, we consider only the loss component (scaled by a factor of $10^{-4}$), for which 34 observations are right-censored. Then, we fit a matrix-Pareto type III to the resulting sample. To reduce the number of parameters, we consider a Coxian structure of dimension 4 in the PH frailty, obtaining in this way the following (rounded) estimated parameters

$$\hat{\pi} = (1,0,0,0),$$
$$\hat{T} = \begin{pmatrix}
-19.6942 & 16.2647 & 0 & 0 \\
0 & -2.1502 & 0.7218 & 0 \\
0 & 0 & -0.5011 & 0.5010 \\
0 & 0 & 0 & -0.5011
\end{pmatrix},$$
$$\hat{\theta} = 1.3705.$$

Figure 5.2 shows that the cumulative hazard of the matrix-Pareto type III model is close to the non-parametric Nelson-Aalen estimator, indicating that the fitted distribution provides an adequate model for the sample. We would also like to mention that an analysis of the same data set employing the matrix-Pareto type I distribution can be found in [13], and an analysis with the matrix-Pareto type II was done in [4].

![Figure 5.2. Cumulative hazard function of the fitted matrix-Pareto type III distribution versus the non-parametric Nelson-Aalen estimator of the sample.](image)
6. Extensions

6.1. Shared PH frailty. As a first extension, we show how PH distributions can be employed in the shared frailty model. In the shared frailty model, one assumes that a group of individuals is conditional independent given the frailty. In this way, the conditional joint survival function of \( Y \mid Z = z, Y = (Y_1, \ldots, Y_N) \), is given by

\[
S_{Y \mid Z}(y_1, \ldots, y_N \mid z) = \prod_{j=1}^{N} \exp(-z M(y_j)) = \exp \left( -z \sum_{j=1}^{N} M(y_j) \right),
\]

with corresponding conditional joint density

\[
f_{Y \mid Z}(y \mid z) = \prod_{j=1}^{N} z \mu(y_j) \exp(-z M(y_j)).
\]

Then, the joint survival function of \( Y \) is given by

\[
S_Y(y) = \mathcal{L}_Z \left( \sum_{j=1}^{N} M(y_j) \right).
\]

In the PH particular case, we have that

\[
S_Y(y) = \pi \left( \sum_{j=1}^{N} M(y_j) I - T \right)^{-1} t,
\]

and

\[
f_Y(y) = N! \left( \prod_{j=1}^{N} \mu(y_j) \right) \pi \left( \sum_{j=1}^{N} M(y_j) I - T \right)^{-1-N} t.
\]

6.1.1. Estimation. We now show how the estimation of the shared PH frailty model can be performed via an EM algorithm. We consider a sample with \( N \) clusters of size \( N_i, i = 1, \ldots, N \). Then, the complete log-likelihood is given by

\[
l_c(\alpha, \pi, T; (\tilde{y}, \tilde{\delta})) = \sum_{i=1}^{N} \sum_{j=1}^{N_i} \delta_{i,j} \log(\mu(y_{i,j}; \alpha)) - z_i M(y_{i,j}; \alpha) + \log(f_Z(z_i; \pi, T)).
\]

where \((\tilde{y}, \tilde{\delta})\) denotes the whole dataset.

The main change with respect to the algorithm in Section 3 is in the E-step. We consider one (generic) group of size \( N \). First, let us illustrate the calculations by considering only non-censored observations. In such a case, we have that

\[
\mathbb{E}(Z \mid Y = y) = \int_{0}^{\infty} z f_{Z \mid Y}(z \mid y) dz.
\]

\[
= \int_{0}^{\infty} z \frac{f_{Y \mid Z}(y \mid z) f_Z(z)}{f_Y(y)} dz.
\]

\[
= \frac{(N + 1)! \prod_{j=1}^{N} \mu(y_j))^{\delta_j}}{f_Y(y)} \pi \left( \sum_{j=1}^{N} M(y_j) I - T \right)^{-2-N} t.
\]
\[
(N + 1)\pi \left( \sum_{j=1}^{N} M(y_j)I - T \right)^{-2-N} t \over \pi \left( \sum_{j=1}^{N} M(y_j)I - T \right)^{-1-N} t .
\]

For the general case, where censored observations are present, the likelihood of one observation \((y, \delta)\) is given by
\[
\int_0^\infty \prod_{j=1}^{N} (z\mu(y_j))^{\delta_j} \exp (-z M(y_j)) f_Z(z)dz = \left( \prod_{j=1}^{N} (\mu(y_j))^{\delta_j} \right) \int_0^\infty z^\kappa \exp (-z \sum_{j=1}^{N} M(y_j)) f_Z(z)dz
= \kappa! \left( \prod_{j=1}^{N} (\mu(y_j))^{\delta_j} \right) \pi \left( \sum_{j=1}^{N} M(y_j)I - T \right)^{-1-N} t ,
\]

where \(\kappa\) is the number of non-censored observations. This yields,
\[
\mathbb{E}(Z \mid (y, \delta)) = \frac{(\kappa + 1)\pi \left( \sum_{i=1}^{N} M(y_i)I - T \right)^{-2-\kappa} t}{\pi \left( \sum_{k=1}^{N} M(y_k)I - T \right)^{-1-\kappa} t} .
\]

This last expression can then be employed in the M-step to find the estimator for \(\alpha\). Regarding the logarithmic term, the computation of the conditional expectation is similar, which can then be maximized in the same way as in the univariate PH frailty to obtain estimators for \((\pi, T)\).

6.2. Correlated PH frailty. Next, we show how correlated frailty can be performed using multivariate PH distributions. For illustration purposes, we consider the bivariate class of PH distributions for which the joint density is given by
(6.1)
\[
f_Z(z_1, z_2) = \eta e^{T_{11}z_1}T_{12}e^{T_{22}z_2}(-T_{22})e ,
\]

where \(T_{11}\) and \(T_{22}\) are sub-intensity matrices of dimension \(p_1\) and \(p_2\), respectively, and \(T_{11}e + T_{12}e = 0\). From (6.1) it is easy to see that
\[
\mathbb{E}(e^{-u_1Y_1-u_2Y_2}) = \eta (u_1I - T_{11})^{-1}T_{12}(u_2I - T_{22})^{-1}(-T_{22})e .
\]

This class of bivariate distributions is dense on the set of distributions on \(\mathbb{R}_+^2\) and allows for estimation via an EM algorithm (see [2]). Moreover, this class can be extended to larger dimensions with a similar construction principle (see [3]). However, to focus on the main ideas of the model, we consider only the bivariate case.

The key idea of the correlated frailty model is that the frailties of individuals in a group are correlated but not necessarily shared. Considering a bivariate correlated frailty model. Under this model, the conditional joint survival function is given by
\[
S(y_1, y_2 \mid Z_1, Z_2) = S_1(y_1 \mid Z_1)S_2(y_2 \mid Z_2) = \exp(-Z_1M_1(y_1))\exp(-Z_2M_2(y_2)) ,
\]

where \(Z_1\) and \(Z_2\) are the two correlated frailties and \(M_1\) and \(M_2\) denote the baseline cumulative hazards. Here, the distribution of the random vector \((Z_1, Z_2)\) determines the association structure of the events in the model.

We now take \(Z = (Z_1, Z_2)\) with joint density given by (6.1). In this way,
\[
S(y_1, y_2) = \int \int \exp(-z_1M_1(y_1))\exp(-z_2M_2(y_2))f_Z(z_1, z_2)dz_1dz_2
= \eta (M_1(y_1)I - T_{11})^{-1}T_{12}(M_2(y_2)I - T_{22})^{-1}(-T_{22})e .
\]

From this last expression, it follows that
\[
f(y_1, y_2) = \mu_1(y_1)\mu_2(y_2)\eta (M_1(y_1)I - T_{11})^{-2}T_{12}(M_2(y_2)I - T_{22})^{-2}(-T_{22})e .
\]
6.2.1. Estimation. Estimation of this model can also be performed via an EM algorithm. For the sake of simplicity, we assume that all observations are non-censored. Thus, for a sample of size $N$, we have that the complete log-likelihood is given by

$$l_c(\alpha, \eta, \pi, T; y)$$

$$= \sum_{n=1}^{N} \log(\mu_1(y_1, n; \alpha_1)) + \log(\mu_2(y_2, n; \alpha_2)) - z_{1,n}M_1(y_1, n; \alpha_1) - z_{2,n}M_2(y_2, n; \alpha_2)$$

$$+ \log(f_Z(z_{1,n}, z_{2,n}; \eta, \pi, T)),$$

where $\alpha = (\alpha_1, \alpha_2)$. First, we compute the conditional expectations given the observed data by noting that

$$\mathbb{E}(Z_1 \mid Y = y)$$

$$= \int_0^\infty \int_0^\infty z_1 \frac{f_{Y_1,Z}(y_1, y_2 \mid z_1, z_2)}{f_Y(y_1, y_2)} f_Z(z_1, z_2) dz_1 dz_2$$

$$= \frac{\mu_1(y_1)\mu_2(y_2)}{f_Y(y_1, y_2)} \int_0^\infty \int_0^\infty z_1^2 z_2 \exp(-z_1M_1(y_1)) \exp(-z_2M_2(y_2)) f_Z(z_1, z_2) dz_1 dz_2$$

$$= \frac{\mu_1(y_1)\mu_2(y_2)}{f_Y(y_1, y_2)} 2\eta(M_1(y_1)I - T_{11})^{-3}T_{12}(M_2(y_2)I - T_{22})^{-2}(-T_{22})e$$

$$= \frac{2\eta(M_1(y_1)I - T_{11})^{-3}T_{12}(M_2(y_2)I - T_{22})^{-2}(-T_{22})e}{\eta(M_1(y_1)I - T_{11})^{-2}T_{12}(M_2(y_2)I - T_{22})^{-2}(-T_{22})e}.$$

Similarly,

$$\mathbb{E}(Z_2 \mid Y = y) = \frac{2\eta(M_1(y_1)I - T_{11})^{-2}T_{12}(M_2(y_2)I - T_{22})^{-3}(-T_{22})e}{\eta(M_1(y_1)I - T_{11})^{-2}T_{12}(M_2(y_2)I - T_{22})^{-2}(-T_{22})e}.$$  

For the logarithmic term, we have that

$$\mathbb{E}(\log(f_{Z_1, Z_2; \eta, \pi, T}) \mid Y = y) = \int_0^\infty \int_0^\infty \log(f_Z(z_{1}, z_{2}; \eta, \pi, T)) f_Z(y_1, y_2) dz_1 dz_2.$$

Once computed the conditional expectations, the estimation of $\alpha$ can be performed numerically as in the conventional PH frailty model. For parameters of the bivariate PH component, the maximization reduces to fitting a bivariate PH distribution to an appropriate joint density. In [6] it is shown how the algorithm in [2] can be modified for such a purpose. We omit further details.

7. Conclusion

We proposed a novel frailty model where phase-type distributions are employed as frailties. We showed that this mode has similar properties to the Gamma frailty model with closed-form expressions for different functionals. We derived an EM algorithm for maximum-likelihood estimation, which was then employed in some numerical illustrations. We showed that the PH frailty could lead to distributions that can be used in other contexts different from survival analysis. We explored some extensions of the model, namely the shared frailty and correlated frailty model. We derived several properties of these extended models and indicated the necessary changes to perform estimation via EM algorithms. A line of work for future research is on generalizing further the correlated PH frailty model by employing the class of multivariate phase-type distributions introduced in [19].

References

[1] O. O. Aalen. Phase type distributions in survival analysis. Scandinavian Journal of Statistics, pages 447–463, 1995.
[2] L. Åhlström, M. Olsson, and O. Nerman. A parametric estimation procedure for relapse time distributions. Lifetime Data Analysis, 5(2):113–132, 1999.
[3] H. Albrecher and M. Bladt. Inhomogeneous phase-type distributions and heavy tails. *Journal of Applied Probability*, 56(4):1044–1064, 2019.

[4] H. Albrecher, M. Bladt, and M. Bladt. Multivariate matrix Mittag–Leffler distributions. *Annals of the Institute of Statistical Mathematics*, pages 1–26, 2020.

[5] H. Albrecher, M. Bladt, M. Bladt, and J. Yslas. Continuous scaled phase-type distributions. *arXiv preprint arXiv:2103.02457*, 2021.

[6] H. Albrecher, M. Bladt, and J. Yslas. Fitting inhomogeneous phase-type distributions to data: The univariate and the multivariate case. *Scandinavian Journal of Statistics*, pages 1–34, 2020.

[7] S. Asmussen, J. L. Jensen, and L. Rojas-Nandayapa. On the Laplace transform of the lognormal distribution. *Methodology and Computing in Applied Probability*, 18(2):441–458, 2016.

[8] S. Asmussen, O. Nerman, and M. Olsson. Fitting phase-type distributions via the EM algorithm. *Scandinavian Journal of Statistics*, 23(4):419–441, 1996.

[9] R. E. Beard. Note on some mathematical mortality models. In *Ciba Foundation Symposium-The Lifespan of Animals (Colloquia on Ageing)*, volume 5, pages 302–311. Wiley Online Library, 1959.

[10] M. Bladt. A review on phase-type distributions and their use in risk theory. *ASTIN Bulletin: The Journal of the IAA*, 35(1):145–161, 2005.

[11] M. Bladt and B. F. Nielsen. *Matrix-Exponential Distributions in Applied Probability*. Springer, 2017.

[12] M. Bladt and J. Yslas. Inhomogeneous Markov survival regression models. *arXiv preprint arXiv:2011.03219*, 2020.

[13] M. Bladt and J. Yslas. matrixdist: An R package for inhomogeneous phase-type distributions. *arXiv preprint arXiv:2101.07987*, 2021.

[14] P. Buchholz, J. Kriege, and I. Felko. *Input Modeling with Phase-Type Distributions and Markov Models: Theory and Applications*. Springer, 2014.

[15] D. R. Cox. Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)*, 34(2):187–202, 1972.

[16] A. K. Erlang. Sandsynlighedsregning og telefonsamtaler. *Nyt tidsskrift for Matematik*, 20:33–39, 1909.

[17] A. Hobolth, A. Siri-Jegousse, and M. Bladt. Phase-type distributions in population genetics. *Theoretical Population Biology*, 127:16–32, 2019.

[18] A. Jensen. *A Distribution Model, Applicable to Economics*. Munksgaard, 1954.

[19] V. G. Kulkarni. A new class of multivariate phase type distributions. *Operations Research*, 37(1):151–158, 1989.

[20] C. A. McGrory, A. N. Pettitt, and M. J. Faddy. A fully Bayesian approach to inference for Coxian phase-type distributions with covariate dependent mean. *Computational Statistics & Data Analysis*, 53(12):4311–4321, 2009.

[21] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Courier Corporation, 1994.

[22] M. Olsson. Estimation of phase-type distributions from censored data. *Scandinavian Journal of Statistics*, 23(4):443–460, 1996.

[23] X. Tang, Z. Luo, and J. C. Gardiner. Modeling hospital length of stay by Coxian phase-type regression with heterogeneity. *Statistics in Medicine*, 31(14):1502–1516, 2012.

[24] J. W. Vaupel, K. G. Manton, and E. Stallard. The impact of heterogeneity in individual frailty on the dynamics of mortality. *Demography*, 16(3):439–454, 1979.

[25] A. Wienke. *Frailty Models in Survival Analysis*. CRC press, 2010.