From Strings to Membranes

String theory replaces point particles with strings, one-dimensional objects whose tension $T_F \equiv 1/2\pi l_s^2$ defines a dimensionful parameter $l_s$, known as the string length (which in conventional models is expected to be of order the Planck length, $l_s \sim 10^{-32}$ cm). Besides moving as a whole, a closed string can oscillate in different ways, and upon quantization these internal modes give rise to a perturbative spectrum consisting of an infinite tower of states with masses $m = 2\sqrt{n}/l_s$, $n = 0, 1, \ldots$ For the specific string theory known as Type IIA, the massless states at the bottom of the tower correspond to fluctuations of a metric field $g_{\mu\nu}$, a rank-2 antisymmetric tensor gauge field $B_{\mu\nu}$, a scalar field $\phi$ (the dilaton) whose vacuum expectation value determines the string coupling constant $g_s = \exp \phi$, a vector field $C_\mu$, a rank-3 antisymmetric tensor $C_{\mu\nu\lambda}$, and the corresponding superpartners. At low energies ($E \ll l_s^{-1}$) these are the only relevant modes; the effective field-theoretic description that they provide is known as ten-dimensional Type IIA supergravity, with Newton’s constant $G_N \sim g_s^2 l_s^8$.

The non-perturbative spectrum of string theory includes objects known as D-branes [1]. A $D_p$-brane is a solitonic object extended along $p$ spatial dimensions, whose tension (mass per unit $p$-volume) is inversely proportional to $g_s$. Its excitations are described by...
open strings whose endpoints are constrained to lie on the brane; quantization of these strings gives rise to another infinite tower of states, at the bottom of which there are massless states, including a vector gauge field. The effective low-energy description is consequently in terms of a \((p+1)\)-dimensional supersymmetric gauge theory.

Type IIA string theory contains D-branes with even \(p\): a D0-brane (a particle), a D2-brane (a membrane), and so on, which are respectively charged under the gauge fields \(C_\mu, C_{\mu\nu\lambda}\), etc. The general principle at work here is that, just like a point particle (e.g., the electron) couples to an ordinary vector gauge field \((A_\mu)\), an object extended in \(p\) spatial dimensions naturally couples to a rank-\((p+1)\) antisymmetric tensor gauge field. In particular, the string itself is charged under \(B_{\mu\nu}\).

Consider a D0-brane. It has a mass \(m = 1/g_s l_s\) (known exactly because of supersymmetry), and so is very heavy for \(g_s \ll 1\), as expected for a solitonic object. On the other hand, for \(g_s \gg 1\) the D0 mass is the smallest energy scale in the theory. The dynamics of D-particles are such that \(n\) D0-branes can form a bound state, with mass \(m_n = n/g_s l_s\). This evenly-spaced tower of states gives rise to a continuum as \(g_s \to \infty\), a phenomenon which is reminiscent of Kaluza-Klein compactification. Indeed, if we consider an \(eleven\)-dimensional theory in which the \(x^{10}\) direction is a circle of radius \(R_{10}\), we know that the corresponding momentum must be quantized, \(p_{10} = n/R_{10}\). A massless eleven-dimensional field can thus be expanded in a Fourier series,

\[
\phi(x^\mu, x^{10}) = \sum_n e^{inx^{10}/R_{10}} \phi_n(x^\mu),
\]

giving rise to an infinite tower of ten-dimensional fields \(\phi_n\), with masses \(m_n^2 \equiv p_\mu p^{\mu} = (n/R_{10})^2\). This would precisely match the D0-brane bound state spectrum if it turned out to be the case that

\[
R_{10} = g_s l_s.
\]

The above agreement is, in fact, more than a coincidence. Ten-dimensional Type IIA supergravity, which as mentioned before is the low-energy approximation to Type IIA string theory, has been known for many years to be directly related to supergravity in \(eleven\) dimensions, with the additional dimension a circle of radius \(R_{10}\). More precisely, Type IIA supergravity can be obtained by restricting the fields of eleven-dimensional supergravity (a metric \(g_{MN}\), a rank-3 gauge field \(A_{MNP}\), and a gravitino \(\Psi^M_\alpha\)) to be constant along \(x^{10}\), i.e., truncating their Kaluza-Klein expansions down to the \(p_{10} = 0\) modes. When the circle is small, these modes have masses much lower than the rest, and so the truncation in question (known as dimensional reduction) is physically justified. It is then natural to wonder whether this correspondence at the level of Type IIA supergravity could extend somehow to the regime where \(R_{10}\) is not small, where it would necessarily have to involve the \(p_{10} = n/R_{10} \neq 0\) modes.

The answer to this question lies in the precise form of the mapping between the various supergravity fields. In particular, the \(\mu\)-10 component of the eleven-dimensional metric, which from the ten-dimensional perspective is the gauge field that couples to the Kaluza-Klein charge \(n = p_{10} R_{10}\), corresponds in Type IIA language to the gauge field \(C_\mu\), which as we saw before, couples to D0-brane charge. So the D-particle bound states present in Type IIA string theory have precisely the right properties to match the full Kaluza-Klein tower of eleven dimensional supergravity. Moreover, the 10-10 component
FIGURE 1. Summary of the connections between the ten- and eleven-dimensional theories discussed in the main text. The most important point is that the strong-coupling limit of ten-dimensional Type IIA string theory is in fact a theory in eleven dimensions.

of the metric, which controls the size of the eleventh dimension and is a scalar from the ten-dimensional perspective, translates into the dilaton field \( \phi \), which determines the string coupling constant. The precise relation is in fact (2). Finally, the Type IIA field \( B_{\mu \nu} \), which couples to the fundamental string, descends from the eleven-dimensional rank-3 tensor gauge field \( A_{\mu \nu [10]} \), which would naturally couple to a membrane.

The conclusion then is that Type IIA string theory is secretly eleven-dimensional, and its fundamental degree of freedom, the string, is in fact a membrane (the ‘M2-brane’) wrapped around the hidden dimension \([2]\). The well-known connection at the level of supergravity extends to the full string theory, which is understood then to be a special (small \( R_{10} \)) limit of an eleven-dimensional theory. This larger theory has been provisionally baptized M-theory (with ‘mystery’ one of the intended meanings); from the preceding discussion we know that eleven-dimensional supergravity gives its effective low-energy description. The situation is summarized in Fig. 1. M-theory can be shown to englobe not only Type IIA but also all of the other known string theories, which are thus understood to be part of a single unified framework.

In the effort to understand this mysterious theory, the obvious first thing to try is to quantize the M2-brane. As the membrane moves about in eleven-dimensional spacetime, it sweeps out a three-dimensional ‘worldvolume,’ which can be described through an embedding function \( X^M(\tau, \sigma, \rho) \). Unfortunately, the natural (bosonic part of the) action for the M2-brane, its tension times the volume it sweeps out,

\[
S_{M2} = -T_{M2} \int d\tau d\sigma d\rho \sqrt{-\det g_{MN} \partial_\alpha X^M \partial_\beta X^N},
\]

is a complicated non-linear constrained system which has proven extremely difficult to quantize. Essentially all of the progress that has been made is based on a discretized version of \( S_{M2} \) that employs \( N \times N \) matrices (the continuous membrane being approached in the \( N \to \infty \) limit) \([3]\). Quantization of this model was found to yield a continuous spectrum \([4]\). This can be understood at an intuitive level by noting that the action \( (3) \) assigns to the membrane an energy proportional to its area. As a consequence, the membrane can develop arbitrarily long spikes of infinitesimal area, at zero energy cost. It is the existence of these ‘flat directions’ in the membrane potential (together with the supersymmetry-induced cancellation of zero-point energies) that gives rise to a contin-
uous spectrum. This result is in sharp contrast with the discrete spectrum of the string, and was initially a source of disappointment.

Years later, and following a quite independent line of development, the discretized membrane model of [3] resurfaced (under the name Matrix theory) as a proposal for a non-perturbative definition of M-theory, restricted to the specific kinematic setup known as the infinite momentum frame [5]. In this context, the continuous spectrum of the model, previously believed to be a flaw, was recognized as a virtue: it is a sign that the membrane yields a second-quantized description, with a spectrum that includes multiple-particle states. An \( n \)-particle state is obtained by deforming the membrane into \( n \) blobs connected by infinitesimally thin tubes, which carry no energy. In this way, a single membrane leads to configurations which are indistinguishable from multiple-membrane states.

Despite the success of the Matrix proposal, the search is on for new ideas which could lead to a less kinematically-restrictive (and hopefully more manageable) formulation of M-theory. In particular, the desire to obtain a covariant definition of M-theory naturally fuels the ongoing attempts to quantize the membrane covariantly (see [6] for some interesting recent developments). Given the complexity of this task, an alternative strategy would be to look for an interesting limit in which \( S_M^2 \) simplifies. We will describe such a limit in the next section.

**OM/WM2 THEORY**

Consider a D4-brane (or a stack of them) in Type IIA string theory at low energies, \( E \ll l_s^{-1} \). Whereas ordinarily this system would have an effective description in terms of a standard (supersymmetric) five-dimensional gauge theory, it was discovered in [7] that, in the presence of a constant background \( B_{01} \) field (and with appropriately adjusted values of this \( B \)-field, the metric, and \( g_s \)), the description is in terms of a five-dimensional Noncommutative Open String (NCOS) theory (which displays noncommutativity between space and time). What is most remarkable about this is that, despite the fact that we are considering a low energy regime, by fine-tuning the relevant parameters we manage to retain not just the massless modes but the whole infinite tower of open string excitations. On the other hand, the low-energy limit does remove from the spectrum the usual closed strings, and in particular the graviton. The result was therefore initially believed to be a non-gravitational purely open string theory.

Subsequent work showed that the story is more complicated than that. If \( x^1 \) (the direction of the ‘electric’ \( B \)-field) is not the real line, but a circle of radius \( R \), then closed strings are in fact present in the spectrum of the theory, but only if they wind around the circle in the positive direction (i.e., if they have strictly positive ‘winding number,’ \( w > 0 \)) [8]. Notwithstanding the fact that they coexist with relativistic open strings, these wound closed strings obey a non-relativistic dispersion relation, \( p_0 \propto p^2 /2wR + \text{oscillators} \). For finite \( R \) these closed strings are able to leave the D4-brane(s) and move about freely in ten-dimensional spacetime, which allows them to be studied even in the absence of the brane(s). The conclusion is that five-dimensional NCOS is actually part of a larger, ten-dimensional string theory, which in addition to D4-branes
extending along direction 1 contains other objects, including (wound) closed strings and Dp-branes oriented in various ways \([9, 10]\). Gravity also turns out to be present, but in a more rudimentary form: it is Newtonian when the theory is formulated on a flat background \([10, 11]\), and ‘asymptotically Newtonian’ in a more general background \([11, 12]\). The theory in question is thus a drastically simplified version of Type IIA string theory; it is known as Type IIA Wound (WIIA) or Non-relativistic string theory.

Given the connection between Type IIA string theory and M-theory, it is natural to inquire about the eleven-dimensional origin of the above setup. A D4-brane, whose excitations are described by open strings, turns out to have as its M-theoretic counterpart a fivebrane with one direction wrapped around the \(x^{10}\) circle, an object whose excitations are described by open M2-branes ending on it. In addition, we know that the \(B_{01}\) field included in the NCOS setup descends from \(A_{01[10]}\) in eleven dimensions. Assembling these facts together, five-dimensional NCOS theory is understood to be a special limit of a six-dimensional theory, which is expected to admit a description in terms of open M2-branes terminating on the fivebrane(s), and to possess a generalized form of noncommutativity. This M-theoretic structure, known as Open Membrane (OM) theory \([13]\), plays a role analogous to that of M theory itself: it underlies and unifies all of the noncommutative theories which originate from string theory, be they of the open brane \([7, 13, 14]\) or of the purely field-theoretic \([15]\) type.

The embedding of NCOS into Wound string theory implies of course an analogous embedding for the OM case. Indeed, Wound IIA string theory can be lifted to eleven dimensions to obtain what is known as Wrapped \([9]\) or Galilean \([10]\) M2-brane (WM2) theory, an M-theoretic constrict which contains OM theory as a special class of states \([9]\). To be precise, OM theory corresponds to those states of WM2 theory that involve M5-branes extended along the ‘longitudinal’ directions 1-10 (the directions singled out by the background \(A\) field). WM2 theory contains in addition (partially or fully) transverse M5-branes, closed M2-branes, and Newtonian gravity \([9, 10, 11]\), and includes all Wound string and Wrapped brane theories \([9, 10]\) (and consequently all noncommutative open brane theories) as special limits. It is clearly desirable to increase our knowledge about this rich theoretical structure, which constitutes a simplified model of M theory. The question for us then becomes, what happens to \(S_{M2}\) in the OM/WM2 theory limit?

**OM/WM2 ACTION**

The bosonic part of the action for an M2-brane in a background \(A_{012}\) field is

\[
S_{M2} = -T_{M2} \int d^3 \sigma \left[ \sqrt{-\det g_{MN} \partial_\alpha X^M \partial_\beta X^N - A_{012} \varepsilon^{\alpha\beta\gamma} \partial_\alpha X^0 \partial_\beta X^1 \partial_\gamma X^2} \right],
\]

where \(T_M = 1/(2\pi)^2 l_P^3\) (with \(l_P\) the eleven-dimensional Planck length) is the membrane tension, the worldvolume coordinates \(\sigma^\alpha \equiv (\tau, \sigma, \rho)\), \(\varepsilon^{012} = +1\), and the spacetime indices \(M, N = 0, \ldots, 10\). Notice that, for ease of notation, we have made a slight change of conventions, relabeling as \(\sigma^2\) the coordinate which in the previous sections was denoted \(x^{10}\). In the following, directions 1 and 2 will both be assumed to be circles, with respective radii \(R_1\) and \(R_2\).
In the OM/WM2 limit (and after partial gauge-fixing), the action (4) can be shown to reduce to \[ S_W = -T_W \int d^3 \sigma \left[ -\frac{1}{2} \dot{Y}^2 + \frac{1}{2} (X'^2 \dot{Y}^2 - 2X' \cdot \dot{X} \cdot \dot{Y} + \dot{X}^2 Y'^2) + l_a (\dot{X}^a - \epsilon^{abc} \dot{X}^b c) + \lambda \epsilon^{\alpha \beta \gamma} \partial_\alpha X^0 \partial_\beta X^1 \partial_\gamma X^2 \right] , \] (5)

where \( T_W \equiv 1/(2\pi)^2 L_p^3 \) is the effective membrane tension, \( X^a (a = 0, 1, 2) \) and \( Y^i (i = 3, \ldots, 10) \) stand respectively for the longitudinal and transverse coordinates, \( l_a \) are Lagrange multipliers enforcing the gauge conditions, and dots, primes and hats denote \( \tau \)-, \( \sigma \)- and \( \rho \)-derivatives, respectively. This action must be supplemented with the constraint that the corresponding energy-momentum tensor vanish, \( T_{\alpha \beta} = 0 \). (See [16] for further details.)

The interesting question now is whether it is any easier to quantize the WM2 membrane action (5) than the original action (4). The answer turns out to be yes and no [16]. The first thing to note is that there are in fact several distinct cases to consider, depending on whether the membrane is closed or open, and if open, whether it ends on (and thus describes the dynamics of) a longitudinal, partially transverse, or fully transverse M5-brane (e.g., fivebranes extending along directions 012345, 013456, and 034567, respectively). For the closed membrane, and for the open membrane associated with a partially transverse\(^2\) fivebrane, the boundary conditions allow one to completely fix the gauge via the ‘static gauge’ choice

\[ X^0 = c \tau, \quad X^1 = w_1 R_1 \sigma, \quad X^2 = w_2 R_2 \rho, \] (6)

with \( c \) an arbitrary constant and \( w_1, w_2 \in \mathbb{Z} \), thereby reducing (5) to the free-field action

\[ S^{(s)}_W = -T_W \int d^3 x \left[ \frac{1}{2} \partial_a Y \cdot \partial^a Y + \lambda \right] , \] (7)

which describes a non-relativistic membrane. The resulting energy spectrum is [16]

\[ p_0 = \lambda \frac{w R_1 R_2}{L_p^3} + \frac{L_p^3 p_\perp^2}{w R_1 R_2} + \frac{\mathcal{N}}{w R_1 R_2} , \] (8)

where \( w \equiv w_1 w_2 > 0 \) is the membrane wrapping number, and we have defined a number operator

\[ \mathcal{N} \equiv \sum_\vec{n} \sqrt{(n_1 w_1 R_1)^2 + (n_2 w_2 R_2)^2} a_\vec{n}^\dagger a_\vec{n} . \] (9)

We thus learn that, in contrast with the standard membrane, the spectrum of the closed WM2 membrane (and that of the open membrane ending on a partially transverse M5-brane) is discrete. This is of course due to the non-relativistic character of \( S^{(s)}_W \) as is

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\(^2\) Notice this corrects an erroneous statement in [16].
evident in (7), the membrane potential does not have flat directions. As a check on the result (8)-(9), one can verify that, under reduction to ten dimensions, the expected NCOS/WIIA spectra [7, 9, 10] are correctly reproduced. (In addition, one obtains an interesting prediction for the ‘longitudinal NS5-brane’ of WIIA theory.)

The remaining two cases involve an open membrane that ends on either a longitudinal or a fully transverse M5-brane. These cases are particularly interesting: the first because it is precisely the OM theory setup, the second because the fivebrane in question is tensionless [16], and would therefore be expected to play an important role in the dynamics of the theory. Unfortunately, for these cases the boundary conditions are incompatible with the choice of static gauge, and so the system remains complicated. One can actually show that the potential following from (5) has flat directions; just like (4), it assigns zero energy to arbitrarily long but infinitesimally thin spikes [16]. Our expectation is then that the spectrum of excitations is continuous (and in particular includes multi-particle states).

**CONCLUSIONS**

We have reviewed the story of the passage from strings in ten dimensions to membranes in eleven dimensions [2]. The punchline of this story is the existence of a mysterious eleven-dimensional structure known as M-theory, which underlies and unifies all of the known string theories. We have then argued that OM/WM2 theory [13, 9, 10] is an interesting simplified version of M-theory, and consequently a good setting to try to improve our understanding of the basic M-theoretic degrees of freedom.

Our main result in this direction is the derivation of an explicit membrane action, Eq. (5), for OM/WM2 theory [16]. After gauge-fixing, this action was seen to yield discrete excitation spectra for the closed membrane and for the ‘partially transverse’ fivebrane (in the approach we adopt, the latter is described through open membranes ending on it). Upon their reduction to ten dimensions and their reinterpretation in the language of the corresponding (NCOS/WIIA) string theory [7, 9, 10], these spectra correctly reproduce known results (and yield an interesting prediction for the ‘longitudinal NS5-brane’ spectrum). For the ‘longitudinal’ and ‘fully transverse’ fivebranes, on the other hand, progress is hampered by the more complicated form of the membrane boundary conditions. These two cases are particularly interesting—the former because it is precisely the OM-theory setup [13], and is consequently directly related to various noncommutative theories; the latter because it involves a fivebrane which is known to be tensionless [16], and is therefore expected to play an important role in the dynamics of WM2 theory. The membrane potential for these cases can be seen to possess flat directions, suggesting continuous excitation spectra. These intriguing systems clearly deserve further study.
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