HAMILTONIAN TYPE OPERATORS IN REPRESENTATIONS OF THE QUANTUM ALGEBRA $U_q(su_{1,1})$

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Abstract

We study some classes of symmetric operators for the discrete series representations of the quantum algebra $U_q(su_{1,1})$, which may serve as Hamiltonians of various physical systems. The problem of diagonalization of these operators (eigenfunctions, spectra, overlap coefficients, etc.) is solved by expressing their overlap coefficients in terms of the known families of $q$-orthogonal polynomials. We consider both bounded and unbounded operators. In the latter case they are not selfadjoint and have deficiency indices $(1,1)$, which means that they may have infinitely many selfadjoint extensions. We find possible sets of point spectrum (which depend on the representation space under consideration) for one of such symmetric operators by using the orthogonality relations for $q$-Laguerre polynomials. In another case we are led to new orthogonality relations for $3\phi_1$-hypergeometric polynomials. Many new realizations for the discrete series representations are constructed, which follow from the diagonalization of the operators considered. In particular, a new system of orthogonal functions on a discrete set is shown to emerge.

I. INTRODUCTION

The wealthy theory of representations of the Lie group $SU(1,1) \simeq SL(2,\mathbb{R})$ and its Lie algebra (see, for example, Refs. 1, 2, Chapter 7, and 3) has been extensively employed in various branches of physics and mathematics. Representations of the Lie algebra $su(1,1)$ have been particularly useful in studying the isotropic harmonic oscillator, non-relativistic Coulomb problem, relativistic Schrödinger equation, Dirac equation with the Coulomb interaction, and so on. The Hamiltonian $H$ in the interacting boson model is represented as a linear combination of the operators, corresponding to generating elements $J^{cl\pm}, J^{cl0}$ of the Lie algebra $su(1,1)$. For this reason, the diagonalization of representation operators, corresponding to such linear combinations, is an important problem.

Diagonalization of representation operators is also important for solving various other problems. In particular, matrices of a transition between bases, diagonalizing two representation operators, describe (on the representation level) automorphisms of the group $SU(1,1)$, if those two operators correspond to one-parameter subgroups (see Ref. 4 and references therein). The most recent papers on the diagonalization of representation operators for $SU(1,1)$ and their use for elucidating properties of special functions are Refs. 5–10.

After the appearance of quantum groups and quantum algebras$^{11–13}$, most problems of the representation theory for Lie groups and Lie algebras were transferred to the
representation theory of quantized groups and algebras. This development is also very important from the point of view of possible applications both in mathematics and in physics. In particular, the diagonalization of representation operators for simplest quantum groups and algebras (especially, such as $U_q(su_2)$ and $U_q(su_{1,1})$) is of great significance. These results are essential for deeper understanding the theory of special functions and orthogonal polynomials.

It was shown in Ref. 14 that for irreducible representations of $U_q(su_{1,1})$ a closure of a representation operator may be not a selfadjoint operator (whereas its classical counterpart is selfadjoint). In particular, the operator $A = q^{J_0/4}(J_++J_-)q^{-J_0/4} + b q^{J_0}$ in the positive discrete series representation of $U_q(su_{1,1})$ was diagonalized for $|b| < 2/(q^{-1/2} - q^{1/2})$. Later on, it turned out that this operator is of the great importance for studying harmonic analysis on the quantum group $SU_q(1,1)$ and for elucidating properties of $q$-orthogonal polynomials. Namely, convolution identities for Al-Salam–Chihara polynomials (which are very closely related to the $q$-Meixner–Pollaczek polynomials) and for Askey–Wilson polynomials were derived from the quantum algebra $U_q(su_{1,1})$ in Ref. 6. The linearization coefficients for a two-parameter family of Askey–Wilson polynomials were obtained. In Ref. 7, the Poisson kernel for Al-Salam–Chihara polynomials is derived. By using the operator $A$, a bilinear generating function for Askey–Wilson polynomials is obtained. Diagonalization of representation operators for $U_q(su_{1,1})$ is utilized to interpret Askey–Wilson polynomials and $W_7$ series as matrix elements of representation operators in mixed bases. Spectral analysis of operators of irreducible representations of $U_q(su_{1,1})$ is employed in Ref. 16 to develop harmonic analysis on $SU_q(1,1)$ (see also Refs. 17 and 18).

Diagonalization of representation operators for the quantum algebras $U_q(su_2)$ and $U_q(su_{1,1})$ finds wide applications in physics. For example, some models in quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians, which are representation operators for $U_q(su_2)$ or $U_q(su_{1,1})$ (see, for example, Ref. 19 and references therein).

A great interest to spectral analysis of the operators for the positive discrete series of $U_q(su_{1,1})$ appears in the analysis on noncommutative (quantum) spaces. For example, the Laplace operator and the squared radius (together with the third operator, which serves as the operator $J_0$) generate the algebra $U_q(su_{1,1})$ acting on the space of polynomials on the $n$-dimensional Manin space or on the quantum complex vector space (see Refs. 20–22). These operators realize irreducible representations of the algebra $U_q(su_{1,1})$, which belong to the positive discrete series and form a $q$-analogue of the oscillator representations of the Lie algebra $su_{1,1}$. To construct Hamiltonians of physical systems, existing in the Manin space or in the quantum complex vector space (for example, Hamiltonians for harmonic oscillators in these spaces), one thus needs to deal with operators of the positive discrete series representations of $U_q(su_{1,1})$. Consequently, the diagonalization of these Hamiltonians reduces to the problem of diagonalization of representation operators for the quantum algebra $U_q(su_{1,1})$.

In the present paper we study the diagonalization (eigenfunctions, spectra, transition coefficients, etc.) of some classes of operators for the discrete series representations of the quantum algebra $U_q(su_{1,1})$, related to $q$-orthogonal polynomials. We restrict
ourselves by the discrete series representations of $U_q(su_{1,1})$, because just these representation operators are often used as Hamiltonians of physical models and these representations are related to the $q$-oscillator algebra. The present paper is a continuation of our study of representation operators, undertaken in Ref. 23 (note that this paper has an essential overlap with Refs. 6, 7 and 15).

The paper is organized as follows. In section 2 we assemble some definitions and formulas on discrete series representations of the algebras $su_{1,1}$ and $U_q(su_{1,1})$, which are necessary in the sequel. In section 3 we diagonalize the representation operator $I_1$ with a bounded continuous spectrum, that is a $q$-extension of the $su_{1,1}$-operator $J^{cl}_0 - J^{cl}_1$. Then we briefly describe eigenfunctions and spectra of a more general class of representation operators $I^{(\phi)}_1$. In section 4 we consider isometry between the standard Hilbert spaces $\mathcal{H}_I$, on which the discrete series representations are realized, and the Hilbert space, on which the operator $I_1$ is a multiplication operator. The latter Hilbert space is the space of square integrable functions on the spectrum of the operator $I_1$. We realize the representation $T^{+}_{1}$ on this Hilbert space. In section 5 we study representation operator with bounded discrete spectrum. We find this spectrum explicitly. This diagonalization is also related to a new realization of the representations of $U_q(su_{1,1})$, which is considered in section 6. In section 7 we consider the unbounded operator $I_3$. This operator is symmetric, but not selfadjoint. Its deficiency indices are $(1, 1)$ and this means that $I_3$ has infinitely many selfadjoint extensions. We find possible sets of point spectrum for this symmetric operator by using the orthogonality relations for $q$-Laguerre polynomials. These sets depend on the space, on which the corresponding representation of the algebra $U_q(su_{1,1})$ is realized. In section 8 we consider the diagonalization of a one-parameter family of unbounded symmetric representation operators. These operators also have deficiency indices $(1, 1)$ and we find their eigenfunctions. In order to find selfadjoint extensions of the operators $I_3$ and $I_4$, it is necessary to know extremal orthogonality measures for the orthogonal polynomials, related to these operators. To the best of our knowledge, these measures are not explicitly known. Calculation of their orthogonality relations is a very complicated problem, which goes beyond the scope of the present paper.

Note that our results are well suited for studying properties of $q$-orthogonal polynomials by different algebraic methods (including the methods of Refs. 6–8 and 15). This topic will be discussed separately.

Throughout the sequel we always assume that $q$ is a fixed positive number such that $q < 1$. We extensively use the theory of $q$-special functions and notations of the standard $q$-analysis (see, for example, Refs. 24 and 25). In particular, we assume that

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}, \quad (1.1)$$

where $a$ can be a number or an operator.

II. DISCRETE SERIES REPRESENTATIONS OF $su_{1,1}$ AND $U_q(su_{1,1})$

A. The classical case

The classical Lie algebra $su_{1,1}$ is generated by the elements $J^{cl}_0, J^{cl}_1, J^{cl}_2$, satisfying
One defines a scalar product on \( A \) denoted by \( \langle \cdot, \cdot \rangle \). In terms of the raising and lowering operators \( J^\pm = J^1 \pm i J^2 \) these commutation relations can be written as
\[
[J^0, J^\pm] = \pm J^\pm,
\]
\[
[J^\pm, J^\mp] = 2 J^0.
\] (2.1)

The discrete series representations \( T_i^+ \) of \( su_{1,1} \) with lowest weights are given by a positive number \( l \) and they are realized on the spaces \( L_i \) of polynomials in \( x \). The basis of \( L_i \) consists of the monomials
\[
f^l_n(x) = c^l_n x^n, \quad n = 0, 1, 2, 3, \ldots
\] (2.2)

where \( c^l_n = \{(2l)_n/n!\}^{1/2} \). Assuming that this basis consists of orthonormal elements, one defines a scalar product on \( L_i \). The closure of \( L_i \) leads to a Hilbert space, on which the representation \( T_i^+ \) acts. The operators \( A \), acting on spaces of functions in \( x \), will be denoted by \( A(x) \).

One can consider an explicit realization of the representation operators \( J_i^\pm \), \( i = 0, 1, 2 \), in terms of the first-order differential operators:
\[
J^0(x) = x \frac{d}{dx} + l,
\]
\[
J^1(x) = \frac{1}{2} (1 + x^2) \frac{d}{dx} + lx,
\]
\[
J^2(x) = \frac{1}{2} (1 - x^2) \frac{d}{dx} - ilx. \] (2.3)

Then the action of the generators \( J^0, J^\pm \) in the standard, or canonical, basis (2.2), consisting of the eigenfunctions of the operator \( J^0 \), is given by
\[
J^0(x) f^l_n(x) = (l + n) f^l_n(x),
\]
\[
J^\pm(x) f^l_n(x) = \sqrt{(2l + n)(n + 1)} f^l_{n+1}(x),
\]
\[
J^\mp(x) f^l_n(x) = \sqrt{(2l + n - 1)n} f^l_{n-1}(x),
\]
In some cases it is of interest (see, for example, Ref. 4) to consider a basis, associated with the eigenfunctions of the selfadjoint operator \( J^0(x) - J^1(x) \), namely,
\[
[J^0(x) - J^1(x)] \eta^l_\lambda(x) = \lambda \eta^l_\lambda(x), \] (2.4)
\[
\eta^l_\lambda(x) = (1 - x)^{-2l} \exp \left( \frac{2\lambda x}{x - 1} \right). \] (2.5)

Here is an outline of a proof of (2.5); as we shall see in the sequel, one can treat the quantum case quite similarly. Look for the eigenfunctions \( \eta^l_\lambda(x) \) in the form of an expansion
\[
\eta^l_\lambda(x) = \sum_{n=0}^{\infty} a^l_n(\lambda) f^l_n(x) = \sum_{n=0}^{\infty} b^l_n(\lambda) x^n,
\] (2.6)
where \( b^l_n(\lambda) := a^l_n(\lambda) c^l_n \). Then
\[
[J^0(x) - J^1(x)] \eta^l_\lambda(x) = \sum_{n=0}^{\infty} a^l_n(\lambda) [J^0(x) - J^1(x)] f^l_n(x)
\]
\[ \sum_{n=0}^{\infty} a_n^l(\lambda) \left\{ (n+l)f_n^l(x) - \frac{1}{2}\sqrt{(2l+n)(n+1)} f_{n+1}^l(x) - \frac{1}{2}\sqrt{(2l+n-1)n} f_{n-1}^l(x) \right\} . \]  
\( (2.7) \)

This means that (2.4) leads to a three-term recurrence relation
\[ 2(n+l-\lambda)b_n^l(\lambda) = (n+1)b_{n+1}^l(\lambda) + (2l+n-1)b_{n-1}^l(\lambda) \]  
\( (2.8) \)

for the coefficients \( b_n^l(\lambda) \). Since (2.8) represents the recurrence relation for the Laguerre polynomials \( L_n^{(2l-1)}(2\lambda) \), one obtains that
\[ \eta_l^\alpha(x) = \sum_{n=0}^{\infty} b_n^l(\lambda) x^n = \sum_{n=0}^{\infty} L_n^{(2l-1)}(2\lambda) x^n. \]  
\( (2.9) \)

Formula (2.5) now follows immediately from (2.9) upon using the generating function
\[ \sum_{n=0}^{\infty} L_{(\alpha)}^n(x) t^n = (1-t)^{-\alpha-1} \exp \left( \frac{xt}{t-1} \right) \]  
\( (2.10) \)

for the Laguerre polynomials
\[ L_n^{(2l-1)}(x) := \frac{(2l)_n}{n!} F_1(-n, 2l; x) = \frac{(2l)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{(2l)_k k!}. \]  
\( (2.11) \)

Observe that since the Laguerre polynomials (2.11) satisfy the orthogonality relation
\[ \int_0^{\infty} e^{-x} x^{2l-1} L_m^{(2l-1)}(x) L_n^{(2l-1)}(x) dx = \frac{(n+2l-1)!}{n!} \delta_{mn}, \]  
\( (2.12) \)

the spectrum of the operator \( J_0^{cl}(x) - J_1^{cl}(x) \) is simple and it covers the interval \([0, \infty)\).

**B. The quantum case**

The quantum algebra \( U_q(\text{su}_{1,1}) \) is defined as the associative algebra, generated by the elements \( J_+, J_- \), and \( J_0 \), which satisfy the commutation relations
\[ [J_0, J_\pm] = \pm J_\pm, \]  
\[ [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}} \equiv [2J_0]_q, \]  
\( (2.13) \)

and the conjugation relations
\[ J_0^* = J_0, \quad J_+^* = J_. \]  
\( (2.14) \)

(Observe that here we have replaced \( J_- \) by \( -J_- \) in the usual definition of the algebra \( U_q(\text{sl}_2) \).) For convenience, in what follows we denote the algebra \( U_q(\text{su}_{1,1}) \) by \( \text{su}_q(1,1) \).

The Casimir element of the algebra \( \text{su}_q(1,1) \) is given by the formula
\[ C_q := [J_0 - 1/2]_q^2 - J_+ J_- = [J_0 + 1/2]_q^2 - J_- J_+. \]
We are interested in the discrete series representations of $\mathfrak{su}_q(1,1)$ with lowest weights. These irreducible representations will be denoted by $T^+_l$, where $l$ is a lowest weight, which can take any positive number (see, for example, Ref. 26). These representations are obtained by deforming the corresponding representations of the Lie algebra $\mathfrak{su}_{1,1}$.

As in the classical case, the representation $T^+_l$ can be realized on the space $\mathcal{L}_l$ of all polynomials in $x$. We choose a basis for this space, consisting of the monomials

$$f^l_n \equiv f^l_n(x; q) := c^l_n(q) x^n, \quad n = 0, 1, 2, \ldots,$$

where

$$c^l_0(q) = 1, \quad c^l_n(q) = \prod_{k=1}^{n} \frac{(2l + k - 1)^{1/2}}{[k]_{q}^{1/2}} = q^{(1-2l)n/4} \frac{(q^{2l}; q)_{n}^{1/2}}{(q; q)_{n}^{1/2}}, \quad n = 1, 2, 3, \ldots,$$

and $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$. The representation $T^+_l$ is then realized by the operators

$$J_0(x) = x \frac{d}{dx} + l, \quad J_{\pm}(x) = x^{\pm 1}[J_0(x) \pm l]_q.$$

As a result of this realization, we have

$$J_+(x) f^l_n(x; q) = \sqrt{[2l + n]_q[n + 1]_q} f^l_{n+1}(x; q),$$

$$J_-(x) f^l_n(x; q) = \sqrt{[2l + n - 1]_q[n]_q} f^l_{n-1}(x; q),$$

$$J_0(x) f^l_n(x; q) = (l + n) f^l_n(x; q).$$

Obviously, these operators satisfy the commutation relations (2.13). The basis functions $f^l_n(x; q)$ are eigenfunctions of the operators $J_0(x)$ and $C_q(x)$: $C_q(x) f^l_n(x; q) = [l - 1/2]_q^2 f^l_n(x; q)$. We know that the discrete series representations $T^+_l$ can be realized on a Hilbert space, on which the conjugation relations (2.14) are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f^l_n(x; q), \ n = 0, 1, 2, \ldots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle \cdot, \cdot \rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain the Hilbert space, which will be denoted by $\mathcal{H}_l$. The Hilbert space $\mathcal{H}_l$ consists of functions (series)

$$f(x) = \sum_{n=0}^{\infty} b_n f^l_n(x; q) = \sum_{n=0}^{\infty} b_n c^l_n(q) x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = b_n c^l_n(q)$. Since $\langle f^l_m, f^l_n \rangle = \delta_{mn}$ by definition, for $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{a}_n x^n$ we have $\langle f(x), \tilde{f}(x) \rangle = \sum_{n=0}^{\infty} a_n \tilde{a}_n / |c^l_n(q)|^2$, that is, the Hilbert space $\mathcal{H}_l$ consists of analytical functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$, such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n / c^l_n(q)|^2 < \infty.$$
It is directly checked that for an arbitrary function \( f(x) \in \mathcal{H}_t \) we have
\[
q^{\frac{d}{dx}} f(x) = f(q^c x).
\]
Therefore, taking into account formulas (2.17), we conclude that
\[
q^{J_0(x)/2} f(x) = q^{\frac{1}{2}(x^2 + 1)} f(x) = q^{1/2} f(q^{1/2} x),
\] \( (2.19) \)
\[
J_+(x) f(x) = \frac{x}{q^{1/2} - q^{-1/2}} \left[ q^f(q^{1/2} x) - q^{-f}(q^{-1/2} x) \right],
\] \( (2.20) \)
\[
J_-(x) f(x) = \frac{1}{(q^{1/2} - q^{-1/2})x} \left[ f(q^{1/2} x) - f(q^{-1/2} x) \right].
\] \( (2.21) \)

These relations will be used in the sequel for finding eigenfunctions of representation operators.

We shall study spectra, eigenfunctions and overlap functions for operators of the representations \( T_i^+ \), which correspond to elements of the quantum algebra \( \text{su}_q(1,1) \) of the form
\[
A := a q^{J_0} \alpha J_+ + \alpha^* J_- + b q^{J_0}, \quad a, b \in \mathbb{R}, \quad \alpha \in \mathbb{C}, \quad |\alpha| = 1, \quad (2.22)
\]
where \( f \) is some function. These operators are representable in the basis (2.15) by a Jacobi matrix. This circumstance allows one to apply the theory of \( q \)-orthogonal polynomials for diagonalizing these operators.

A study of operators of the type (2.22) in representations of the quantum algebra \( U_q(\text{su}_2) \) was started by T. Koornwinder \( ^{27} \) (see also Refs. 28 and 29), who applied them to investigation of the Askey–Wilson polynomials.

For the appropriate choice of the function \( f \), the operators (2.22) are symmetric, so one can close them and thus obtain closed operators. If \( A \) is bounded (for some particular choice of the constants \( a, b \) and \( \alpha \)), then it is a selfadjoint operator. If \( A \) is not bounded, then its closure may give symmetric operator, which is not selfadjoint (see Ref. 14, note that its classical counterpart is a selfadjoint operator). Deficiency indices of such closed operator in this case are \((1,1)\) and it has infinitely many self-adjoint extensions. Then the corresponding overlap coefficients (which are orthogonal polynomials) may have many orthogonality relations. (We deal with operators of this type in sections 7 and 8.) Only orthogonality relations, corresponding to extremal orthogonality measures, lead to selfadjoint extensions of the operator \( A \).

III. REPRESENTATION OPERATORS WITH BOUNDED CONTINUOUS SPECTRA

In this section we are interested in the operator
\[
I_1 := \frac{a}{2} q^{J_0} - b - q^{J_0/4} J_1 q^{J_0/4} = \frac{a}{2} q^{J_0} - b - \frac{1}{2} \left[ q^{1/4} J_+ + q^{-1/4} J_- \right] q^{J_0/2}
\] \( (3.1) \)
of the discrete series representation \( T_i^+ \), where
\[
a = (q^{1/4} + q^{-1/4}) b, \quad b = (q^{1/2} - q^{-1/2})^{-1}.
\]
The representation operator \( (q^{1/4}J_+ + q^{-1/4}J_-)q^{l_0/2} \) is bounded (see [14]). Since \( J_0 \) has the eigenvalues \( m = l, l + 1, l + 2, \ldots \), the operator \( q^{l_0} \) is also bounded (recall that \( 0 < q < 1 \)). Thus, the operator \( I_1 \) is bounded. It is easy to check that \( I_1 \) is a selfadjoint operator since
\[
I_1 f^I_k = \beta_k(q) f^I_k - \alpha_k(q) f^I_{k+1} - \alpha_{k-1}(q) f^I_{k-1},
\]
where
\[
\alpha_k(q) := \frac{1}{2} \left( q^{k+k+1/2} [2l+k]_q [k+1]_q \right)^{1/2}, \quad \beta_k(q) = \frac{(q^{1/4} + q^{-1/4}) q^{l+k} - 2}{2(q^{1/2} - q^{-1/2})}.
\]
The constants \( a \) and \( b \) in (3.1) are chosen in such a way that in the limit as \( q \to 1 \) the operator \( I_1 \) reduces to the \( su_{1,1} \)-operator \( J^0_{11} - J^1_{11} \) (see formula (2.4)).

Eigenfunctions of the operator \( I_1(x) \),
\[
I_1(x) \xi^I_\lambda(x; q) = \lambda(q) \xi^I_\lambda(x; q),
\]
and its spectrum can be found exactly in the same way as in the case of the classical operator \( J^0_{11}(x) - J^1_{11}(x) \) (observe that these functions do not belong to the Hilbert space \( \mathcal{H}_t \), if a spectrum of the operator \( I_1(x) \) is continuous). Namely, one can expand these functions into Taylor series in \( x \),
\[
\xi^I_\lambda(x; q) = \sum_{n=0}^\infty a^I_n(q) f^I_n(x; q) = \sum_{n=0}^\infty b^I_n(q) x^n, \quad b^I_n(q) := c^I_n(q) a^I_n(q),
\]
with coefficients of the expansion \( b^I_n(q) \), and show that the \( b^I_n(q) \) are expressed in terms of the continuous \( q \)-Laguerre polynomials. Then the orthogonality relation for these polynomials determines a spectrum of the operator \( I_1(x) \).

However, we illustrate in this section a more ”direct” way of deriving eigenfunctions of the operator \( I_1(x) \) by using the explicit realization (2.17) for the generators \( J_0(x) \) and \( J_{\pm}(x) \) (see Refs. 23, 28 and 29). Indeed, from formulas (2.19)–(2.21) it follows that
\[
I_1(x) f(x) = \frac{b}{2x} q^{(2l-1)/4} [(1 - q^{(1-2l)/4} x)^2 f(x) - (1 - q^{l/2} x)(1 - q^{(l+1)/2} x) f(qx)]
\]
for an arbitrary function \( f(x) \). It is thus natural to look for the eigenfunctions \( \xi^I_\lambda(x; q) \) of the operator \( I_1(x) \) in the form
\[
\xi^I_\lambda(x; q) = (\alpha x; q)_\infty (\beta x; q)_\infty (\gamma x; q)_\infty (\delta x; q)_\infty,
\]
where \( (a; q)_\infty = \prod_{r=0}^\infty (1 - aq^r) \). Since \( (a; q)_\infty = (1 - a)(aq; q)_\infty \), we have
\[
\xi^I_\lambda(qx; q) = \frac{(1 - \gamma x)(1 - \delta x)}{(1 - \alpha x)(1 - \beta x)} \xi^I_\lambda(x; q).
\]
Substituting (3.4) and (3.5) into (3.3), one gets
\[
I_1(x) \xi^I_\lambda(x; q) = \frac{b}{2x} q^{(2l-1)/4} \left\{ (1 - q^{(1-2l)/4} x)^2
\right\}
\]

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and the corresponding eigenvalues are

\[-(1 - q^{l/2} x) (1 - q^{(l+1)/2} x) \left(\frac{1 - \gamma x (1 - \delta x)}{1 - \alpha x (1 - \beta x)}\right) \xi_l^1(x; q). \quad (3.6)\]

This equation can be written as

\[I_1(x) \xi_l^1(x; q) = \frac{b}{2} q^{(2l-1)/4} \frac{Ax^3 + Bx^2 + Cx + D}{(1 - \alpha x)(1 - \beta x)} \xi_l^1(x; q), \quad (3.7)\]

where the constant coefficients \(A, B, C,\) and \(D\) are equal to

\[A = q^{1/2}(q^{-l}\alpha\beta - q^l\gamma\delta),\]

\[B = q^{1/2}[q^l(\gamma + \delta) - q^{-l}(\alpha + \beta)] + (1 + q^{1/2})q^{l/2}\gamma\delta - 2q^{(1-2l)/4}\alpha\beta,\]

\[C = \alpha\beta - \gamma\delta + 2q^{(1-2l)/4}(\alpha + \beta) - (1 + q^{1/2})q^{l/2}(\gamma + \delta) - q^{l/2}(q^l - q^{-l}),\]

\[D = \gamma + \delta - \alpha - \beta + (1 + q^{1/2})q^{l/2} - 2q^{(1-2l)/4}.\]

It is clear from (3.7) that the \(\xi_l^1(x; q)\) is an eigenfunction of the operator \(I_1(x)\) if the factor in front of \(\xi_l^1(x; q)\) on the right-hand side of (3.7) does not depend on \(x\). It is the case if

\[A = 0, \quad B = \alpha\beta D, \quad C = -(\alpha + \beta)D. \quad (3.8)\]

Then eigenvalues of the operator \(I_1\) on the right-hand side of (3.7) will be equal to \(\lambda = q^{(2l-1)/4}D/2(q^{l/2} - q^{-l/2})\). Requirements (3.8) are equivalent to the following three relations between the parameters \(\alpha, \beta, \gamma, \delta:\)

\[\alpha\beta = q^{2l}\gamma\delta, \quad (q^{1/2} - \alpha\beta)(\alpha + \beta) = (q^{l+1/2} - \alpha\beta)(\gamma + \delta) - (1 + q^{1/2})q^{-l/2}(q^l - q^{-l})\alpha\beta,\]

\[(q^l - q^{-l})(q^{l/2} - q^{-l}\alpha\beta) = [\alpha + \beta - (1 + q^{1/2})q^{l/2}](\gamma + \delta - \alpha - \beta).\]

From these relations it follows that

\[\alpha = q^{l/2}, \quad \beta = q^{(l+1)/2}, \quad \gamma = q^{(1-2l)/4} e^{i\theta}, \quad \delta = q^{(1-2l)/4} e^{-i\theta},\]

where \(\theta\) is an arbitrary angle. Consequently, the eigenfunctions of the operator \(I_1(x)\) are equal to

\[\xi_l^\lambda(x; q) = \frac{(q^{l/2}x; q)_\infty (q^{l+1/2}x; q)_\infty}{(q^{1-2l}/4 e^{i\theta} x; q)_\infty (q^{1-2l}/4 e^{-i\theta} x; q)_\infty} = \frac{(q^{l/2}x; q^{l/2})_\infty}{(q^{1-2l}/4 e^{i\theta} x; q)_\infty (q^{1-2l}/4 e^{-i\theta} x; q)_\infty}, \quad (3.9)\]

and the corresponding eigenvalues are

\[\lambda(q) = \frac{1 - \nu}{q^{-1/2} - q^{1/2}},\]

where \(\nu = \cos \theta\) is a real parameter, see below.

The eigenfunctions (3.9) are in fact the generating functions for the continuous \(q\)-Laguerre polynomials

\[P_n^{(\alpha)}(y|q) = \frac{(q^{(2\alpha+3)/4} e^{-i\theta}; q)_n}{(q; q)_n} q^{(2\alpha+1)n/4} e^{-i\theta}\]
Laguerre polynomials $P_n$ in Ref. 31). The spectral measure of this operator; the spectrum of $I$ with respect to some measure $\mu$ is the basis elements $f_n(x)$ in the basis

$$
(1.2.14) \text{in Ref. 24. In order to make this evident, one needs to represent (3.9) in the form}
$$

$$
\xi^l_\lambda(x; q) = \frac{(q^{2l-1/2}ax; q)_\infty(q^{2l}ax; q)_\infty}{(q^{1-1/4}e^{i\theta}ax; q)_\infty(q^{1-1/4}e^{-i\theta}ax; q)_\infty},
$$

(3.10)

where $a = q^{(1-3l)/2}$. Consequently, due to formula (3.19.11) in Ref. 30, the desired connection is

$$
\xi^l_\lambda(x; q) = \sum_{n=0}^\infty q^{n(1-3l)/2} P_n^{(2l-1)}(\cos \theta | q) x^n = \sum_{n=0}^\infty q^{n(1-3l)/2} \frac{c_l}{c_n} P_n^{(2l-1)}(\cos \theta | q) f_n^l(x),
$$

where $\cos \theta = 1 - (q^{-1/2} - q^{1/2})\lambda$.

Thus, we have proved that the eigenfunctions $\xi^l_\lambda(x; q)$ are connected with the basis elements $f_n^l(x; q)$ by the formula

$$
\xi^l_\lambda(x; q) = \sum_{n=0}^\infty p_n(\lambda) f_n^l(x; q),
$$

(3.11)

where the overlap coefficients $p_n(\lambda)$ are given by

$$
p_n(\lambda) = \frac{q^{(1/4-l)n}(q; q)_n^{1/2}}{(q^{2l}; q)_n^{1/2}} P_n^{(2l-1)}(\nu | q) = \frac{q^{(1/4-l)n}(q; q)_n^{1/2}}{(q^{2l}; q)_n^{1/2}} P_n^{(2l-1)}(1-(q^{-1/2} - q^{1/2})\lambda | q)
$$

(3.12)

and, for convenience, $\lambda(q)$ is denoted by $\lambda$.

To find the spectrum of $I_1(x)$ (that is, a range of the parameter $\nu$), we take into account the following. The selfadjoint operator $I_1(x)$ is represented by a Jacobi matrix in the basis $f_n^l(x; q)$, $n = 0, 1, 2, \cdots$. As is evident from (3.11), the eigenfunctions $\xi^l_\lambda(x; q)$ are expanded in the basis elements $f_n^l(x; q)$ with the coefficients (3.12). According to the results of Chapter VII in Ref. 31, these polynomials $p_n(\lambda)$ are orthogonal with respect to some measure $d\mu(\lambda)$ (this measure is unique, up to a multiplicative constant, since the operator $I_1(x)$ is bounded). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator $I_1(x)$ and $d\mu(\lambda)$ determines the spectral measure of this operator; the spectrum of $I_1(x)$ is simple (see Chapter VII in Ref. 31).

We thus remind the reader that the orthogonality relation for the continuous $q$-Laguerre polynomials $P_n^{(2l-1)}(y|q)$ has the form

$$
\frac{1}{2\pi} \int_{-1}^1 P_m^{(2l-1)}(y|q) P_n^{(2l-1)}(y|q) \frac{w(y)dy}{\sqrt{1-y^2}} = \frac{(q^{2l}; q)_n q^{(2l-1)/2}n}{(q; q)_\infty(q^{1/2}; q)_\infty(q^{1/2}; q)_n} \delta_{mn},
$$

where

$$
w(y) = \left| \frac{(e^{i\theta}; q^{1/2})_\infty(-e^{i\theta}; q^{1/2})_\infty}{(q^{1-1/4}e^{i\theta}; q^{1/2})_\infty} \right|^2, \quad y = \cos \theta
$$
proved the following theorem. Theorem 1. The selfadjoint operator $I_1(x)$ has the continuous and simple spectrum, which covers the finite interval $[0, 2q^{1/2}/(1-q)]$. The eigenfunctions $\xi_l^\alpha(x; q)$ are explicitly given by (3.9) and they are related to the basis (2.15) by formula (3.11).

As we remarked at the beginning of this section, the operator $I_1(x)$ represents a $q$-extension of the $su_{1,1}$-operator $J_0^\alpha(x) - J_1^1(x)$. In the limit as $q \to 1$ the finite interval $[0, 2q^{1/2}/(1-q)]$ of the eigenvalues of the operator $I_1(x)$ extends to the infinite interval $[0, \infty)$. So if one puts $\nu = q^\mu$, then

$$\lim_{q \to 1} \lambda(q^\mu; q) = \mu.$$ (3.15)

Besides, it is known that the continuous $q$-Laguerre polynomials $P_n^{(\alpha)}(y|q)$ have the following limit property (see [30], formula (5.19.1))

$$\lim_{q \to 1} P_n^{(\alpha)}(q^\nu|q) = I_n^{(\alpha)}(2\lambda).$$ (3.16)

Thus, the coefficients of the series expansion (3.11) in $x$ of the eigenfunctions $\xi_l^\alpha(x; q)$, $\nu = q^\mu$, coincide with the coefficients of the corresponding expansion of the $su_{1,1}$-eigenfunctions $\eta_{\mu}^l(x)$ (see (2.9)) in the limit as $q \to 1$.

There exists another, more complicated, family of selfadjoint operators, closely related to $I_1$. They are defined as

$$I_1^{(\varphi)} := \frac{a}{2} q^{j_0} - b - q^{j_0}/4 \left[ \cos \varphi J_1 - \sin \varphi J_2 \right] q^{-j_0/4} \quad \frac{a}{2} q^{j_0} - b - \frac{1}{2} \left[ q^{1/4} e^{i\varphi} J_+ + q^{-1/4} e^{-i\varphi} J_- \right] q^{-j_0/2},$$ (3.17)

where $0 \leq \varphi < 2\pi$, $J_{\pm} = J_1 \pm iJ_2$ and $a$, $b$ are such as in (3.1). These operators are bounded and selfadjoint. Repeating the same reasoning, as for the operator $I_1$, we arrive at the following theorem.

Theorem 2. The eigenfunctions of the operator $I_1^{(\varphi)}(x)$ are

$$\xi_l^\alpha(e^{i\varphi}x; q) = \frac{(q^{1/2} e^{i\varphi}x; q)\infty (q^{(l+1)/2} e^{i\varphi}x; q)\infty}{(q^{(1-2l)/4} e^{i(\theta+\varphi)}x; q)\infty (q^{(1-2l)/4} e^{-i(\theta-\varphi)}x; q)\infty}, \quad \nu = \cos \theta,$$
where $\lambda = (1 - \nu)/(q^{-1/2} - q^{1/2})$. Its spectrum is simple and covers the interval $[0, 2q^{1/2}/(1 - q)]$, and the corresponding eigenvalues $\lambda(\nu; q)$ are the same as for the operator $I_1(x)$.

The operators $I_1^{(x)}$ are $q$-extensions of $\text{su}_{1,1}$-family of operators $J_0^* - \cos \varphi J_1^* + \sin \varphi J_2^*$.

IV. REALIZATION OF REPRESENTATIONS $T_i^+$, RELATED TO THE OPERATOR $I_1$

Let $\mathcal{L}^2_{0,a} := \mathcal{L}^2([0, a], \tilde{w}(\lambda)d\lambda)$ be the Hilbert space of functions $F(\lambda)$ on the interval $[0, a]$, $a = 2q^{1/2}/(1 - q)$, with the scalar product
\[
\langle F_1, F_2 \rangle_a = \int_0^a F_1(\lambda)\overline{F_2(\lambda)}\tilde{w}(\lambda)d\lambda,
\]
where $\tilde{w}(\lambda)$ is determined by (3.14). Let us construct a realization of the representation $T_i^+$ on the space $\mathcal{L}^2_{0,a}$. Our reasoning in this section is close to the Favard theorem (see, for example, Ref. 24).

Due to Theorem 1.6 of Chapter VII in Ref. 31, the space $\mathcal{D}$ of all polynomials in $\lambda$ is everywhere dense in the Hilbert space $\mathcal{L}^2_{0,a}$. Due to formula (3.13), the polynomials (3.12) form an orthonormal basis of $\mathcal{L}^2_{0,a}$.

Let $\mathcal{H}_t$ be the Hilbert space from section 2 and $f(x) = \sum_{n=0}^{\infty} a_n f_n^q(x)$ be an expansion of $f \in \mathcal{H}_t$ with respect to the orthonormal basis (2.15). With every $f \in \mathcal{H}_t$ we associate a function $F(\lambda)$ on the spectrum of the operator $I_1$, such that
\[
F(\lambda) = \langle f(x), \xi_n^q(x; q) \rangle \equiv \sum_{n=0}^{\infty} a_n \frac{q^{(1/4-n)}(q^2;q)_n^{1/2}}{(q^2;q)_n^{1/2}} P_n^{(2l-1)}(1 - (q^{-1/2} - q^{1/2})\lambda|q) (4.2)
\]
(we have taken into account the expansion, given by formula (3.11)). Note that the function $\xi_n^q(x; q)$ (considered as a function of $x$) does not belong to $\mathcal{H}_t$. This means that we consider $\langle f(x), \xi_n^q(x; q) \rangle$ as a formal expression, analogous to an integral transform with a kernel, which does not belong to the corresponding Hilbert space. By expanding $f(x)$ and $\xi_n^q(x; q)$ with respect to the basis $f_n^q(x)$ (see (3.11)), we formally obtain the right-hand side of (4.2). Clearly, the sum in (4.2) has a strict sense at least for functions $f(x)$ with a finite number of nonzero coefficients $a_n$. For $f(x) \in \mathcal{H}_t$ this sum converges in the topology of the Hilbert space $\mathcal{L}^2_{0,a}$.

**Proposition 1.** The mapping $\Phi : f(x) \rightarrow F(\lambda)$, given by formula (4.2), establishes an invertible isometry between the Hilbert spaces $\mathcal{H}_t$ and $\mathcal{L}^2_{0,a}$.

**Proof.** If $f(x) = \sum_n a_n f_n^q(x) \in \mathcal{H}_t$, then due to the orthogonality relation (3.13) for the continuous $q$-Laguerre polynomials from (4.2), we have $\langle F, F \rangle_a = \sum_n |a_n|^2 = \langle f, f \rangle$. Invertibility of the mapping $\Phi$ is evident. Proposition is proved.

It is easy to see that the isometry $\Phi$ maps basis elements $f_n^q$ of $\mathcal{H}_t$ to the basis elements $p_n(\lambda)$ of $\mathcal{L}^2_{0,a}$ from (3.12), respectively.

The results of the previous section allow us to realize the discrete series representation $T_i^+$ of $\text{su}_q(1,1)$ on the Hilbert space $\mathcal{L}^2_{0,a}$. Taking into account the self-adjointness
of the operator $I_1(x)$ and the fact that $I_1(x)\xi^l_\lambda(x;q) = \lambda \xi^l_\lambda(x;q)$, for functions (4.2) we formally obtain

$$I_1 F(\lambda) = \langle I_1(x)f(x), \xi^l_\lambda(x;q) \rangle = \langle f(x), I_1(x)\xi^l_\lambda(x;q) \rangle = \lambda F(\lambda).$$ (4.3)

For this reason, we define an action of $I_1$ on the Hilbert space $L^2_{0,a}$ by the formula $I_1 f(\lambda) = \lambda f(\lambda)$. Therefore, for the basis elements (3.12) we get

$$I_1 p_k(\lambda) = \lambda p_k(\lambda).$$ (4.4)

Let us show that the operator $I_1$ acts on the basis $p_k(\lambda)$, $k = 0, 1, 2, \cdots$, by the formula

$$I_1 p_k(\lambda) = \frac{q^{l+k}(q^{-1/4} + q^{-1/4}) - 2}{2(q^{1/2} - q^{-1/2})} p_k(\lambda) - a_k p_{k+1}(\lambda) - a_{k-1} p_{k-1}(\lambda),$$ (4.5)

where

$$a_k = \frac{1}{2} (q^{k+l+1/2} [k+1]_q [k+2]_q)^{1/2},$$

that is, $I_1$ acts upon the basis functions $p_k(\lambda)$ of the space $L^2_{0,a}$ by the same formulas as $I_1(x)$ acts upon the basis elements $f^l_k$ of the space $\mathcal{H}_t$. In order to prove formula (4.5), we replace (according to (4.4)) the left-hand side by $\lambda p_k(\lambda)$ and substitute the expression (3.12) for $p_{k-1}(\lambda), p_k(\lambda),$ and $p_{k+1}(\lambda)$. After simple transformations we obtain from (4.5) the recurrence relation (3.19.3) of Ref. 30 for the continuous $q$-Laguerre polynomials. This proves the formula (4.5).

As in the case of formula (4.3), for the action of the operator $q^{-j_0}$ we have

$$q^{-j_0} p_k(\lambda) = \langle q^{-j_0} f^l_k(x; q), \xi^l_\lambda(x;q) \rangle = q^{-l-k} \langle f^l_k(x; q), \xi^l_\lambda(x; q) \rangle = q^{-l-k} p_k(\lambda),$$

where $p_k(\lambda)$ are the basis elements (3.12), that is,

$$q^{-j_0} p_k(\lambda) = q^{-l-k} p_k(\lambda).$$

This means that $q^{-j_0}$ acts on the basis $p_k(\lambda)$, $k = 0, 1, 2, \cdots$, by the same formula as it acts on the basis (2.15) of the space $\mathcal{H}_t$.

It is easy to see that the operators $I_1$ and $q^{-j_0}$ determine uniquely all other operators of the representation $T^+_t$. Thus, we have obtained a realization of $T^+_t$ on the Hilbert space $L^2_{0,a}$.

Now we show that the operator $q^{-j_0}$ acts on $L^2_{0,a}$ by the formula

$$q^{-j_0} F(\lambda) = q^{-l} \left( 1 - \frac{1}{4\hat{w}(\lambda; q^{2l})} D_q \hat{w}(\lambda; q^{2l+1}) D_q \right) F(\lambda),$$ (4.6)

where $\hat{w}(\lambda; q^{2l}) := \hat{w}(\lambda)$ ($\hat{w} (\lambda)$ is given by (3.14)) and $D_q f(\lambda) := \frac{f(\lambda) - f(q\lambda)}{\lambda - q\lambda}$. In order to prove formula (4.6), we take into account the $q$-difference equation (3.19.5) from Ref. 30 for the continuous $q$-Laguerre polynomials $P_n(y) := P_n^{(2l-1)}(y|q)$. From this $q$-difference equation it follows that

$$q^{-n} \hat{w}(y, q^{2l}) P_n(y) = \hat{w}(y, q^{2l}) P_n(y) - \frac{1}{4} D_q \hat{w}(y, q^{2l+1}) D_q P_n(y).$$
Since \( q^{-l_0} p_k(\lambda) = q^{-l-k} p_k(\lambda) \), formula (4.6) does hold for the basis elements \( p_k(\lambda) \) and consequently for all \( F \in \mathcal{L}^2_{0,a} \).

**Theorem 3.** Let \( \mathcal{L}^2_{0,a} \) be the Hilbert space, introduced above. Then the representation \( T^+_l \) can be determined on it. The formulas (4.4) and (4.6) give the action of the operators \( I_1 \) and \( q^{-l_0} \) on \( \mathcal{L}^2_{0,a} \). The operators \( J_{\pm} \) and \( J_0 \) act on the basis \( p_n(\lambda) \), \( n = 0, 1, 2, \ldots \), of this space as \( J_0 p_n(\lambda) = (l + n) p_n(\lambda) \) and

\[
J_+ p_n(\lambda) = \sqrt{[2l + n]q [n + 1]q} p_{n+1}(\lambda), \quad J_- p_n(\lambda) = \sqrt{[2l + n - 1]q [n]q} p_{n-1}(\lambda).
\]

The results of this section allow to prove the following assertion.

**Proposition 2.** Let \( p_n(\lambda) \) be the polynomials (3.12). Then in the space \( \mathcal{H}_l \) we have

\[
p_n(I_1(x)) f_0^l = f_n^l.
\]

**Proof.** The isometry \( \Phi : \mathcal{H}_l \rightarrow \mathcal{L}^2_{0,a} \) maps \( f_0^l \equiv 1 \) to \( p_0(\lambda) \equiv 1 \). By formula (4.4) we have \( I_1^l p_0 \equiv I_1^l 1 = \lambda^k \). Therefore, \( p_n(I_1^l) p_0 = p_n(\lambda) \). Applying the mapping \( \Phi^{-1} \) to this identity, one obtains the desired relation (4.7). Proposition is proved.

**V. REPRESENTATION OPERATORS WITH BOUNDED DISCRETE SPECTRA**

In this section we consider the operators

\[
I_2^{(\psi)} = q^{3l_0/4} (e^{i\psi} J_+ + e^{-i\psi} J_-) q^{3l_0/4} - ([J_0 - l]q q^{l/2} + [J_0 + l]q q^{-l/2}) q^{3l_0/2}
\]

\[
= (q^{3/4} e^{i\psi} J_+ + q^{-3/4} e^{-i\psi} J_- - [J_0 - l]q q^{l/2} - [J_0 + l]q q^{-l/2}) q^{3l_0/2},
\]

where \( 0 < \psi \leq 2\pi \) (note that these operators depend on the index \( l \) of the representation \( T^+_l \)). They act on the basis elements (2.15) by the formula

\[
I_2^{(\psi)} f_k^l(x; q) = e^{i\psi} q^{3(l+k)/2+3/4} \sqrt{[k+1]q [2l+k]q} f_{k+1}^l(x; q)
\]

\[
+ e^{-i\psi} q^{3(l+k)/2-3/4} \sqrt{[k]q [2l+k-1]q} f_{k-1}^l(x; q)
\]

\[
- q^{3(l+k)/2} ([k]q q^{(l-1)/2} + [2l+k]q q^{-(l-1)/2}) f_k^l(x; q).
\]

(5.1)

By using this action it is easy to check that the \( I_2^{(\psi)} \) are bounded selfadjoint operators for any value of \( \psi \in (0, 2\pi] \). For a fixed value of \( \psi \) we look for eigenfunctions of the operator \( I_2^{(\psi)} \),

\[
I_2^{(\psi)} \chi^l_{\lambda}(x; q) = \lambda \chi^l_{\lambda}(x; q),
\]

in the form

\[
\chi^l_{\lambda}(x; q) := \sum_{k=0}^{\infty} P_k(\lambda) f_k^l(x; q).
\]

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As in section 2A, the equation

\[ I_2^{(\psi)} l(x; q) = \sum_{k=0}^{\infty} P_k(\lambda) I_2^{(\psi)} f_k(x; q) = \lambda \sum_{k=0}^{\infty} P_k(\lambda) f'_k(x; q) \]

leads to the following recurrence relation for the polynomials \( P_k(\lambda) \), which after simple transformations can be written as

\[
-e^{-i\psi} q^{k+l} [(1 - q^{k+1})(1 - q^{2l+k})]^{1/2} P_{k+1}(\lambda) - e^{i\psi} q^{k+l-1} [(1 - q^k)(1 - q^{2l+k-1})]^{1/2} P_{k-1}(\lambda) \\
+ (q^k - q^{2k+2l} + q^{2l+k-1} - q^{2k+2l-1}) P_k(\lambda) = (1 - q^{-1}) \lambda P_k(\lambda).
\]

(5.2)

Upon making the substitution

\[ P_k(\lambda) = e^{ik\psi} \left( \frac{(q^2; q)_k}{(q; q)_k} \right)^{1/2} q^{-lk} P'_k(\lambda) \]

in this recurrence relation, one derives the equation

\[
-q^k (1 - q^{2l+k}) P'_{k+1}(\lambda) - q^{k+l-1} (1 - q^k) P'_{k-1}(\lambda) \\
+ (q^k - q^{2k+2l} + q^{2l+k-1} - q^{2k+2l-1}) P'_k(\lambda) = (1 - q^{-1}) \lambda P'_k(\lambda).
\]

This is the recurrence relation for the little \( q \)-Laguerre (Wall) polynomials

\[ p_k(q^y; q^{2l-1}|q) = 2\phi_1(q^{-k}, 0; q^{2l}; q, q^{y+1}) \]

\[ = (q^{1-2l-k}; q)_k 2\phi_0(q^{-k}, q^{-y}; -; q; q^{y+2l+1}) \]

with \( q^y = (1 - q^{-1})\lambda \). Thus, we have

\[ P'_k(\lambda) = p_k(q^y; q^{2l-1}|q), \quad q^y = (1 - q^{-1})\lambda, \]

and, consequently,

\[ P_k(\lambda) = e^{ik\psi} \left( \frac{(q^2; q)_k}{(q; q)_k} \right)^{1/2} q^{-lk} p_k(q^y; q^{2l-1}|q). \]

(5.3)

This means that eigenfunctions of the operator \( I_2^{(\psi)} \) are of the form

\[ \chi_\lambda^l(x; q) = c_l \sum_{k=0}^{\infty} q^{(1/4-3l)k} e^{ik\psi} \left( \frac{q^2; q)_k}{(q; q)_k} \right) p_k(q^y; q^{2l-1}|q) x^k, \quad q^y = (1 - q^{-1})\lambda. \]

To sum up the right-hand side of this relation, one needs to know a generating function

\[ F(x; t; a|q) := \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(q; q)_n} p_n(x; a|q) t^n \]

(5.4)

for the little \( q \)-Laguerre polynomials

\[ p_n(x; a|q) := 2\phi_1(q^{-n}, 0; aq; q; qx) = \frac{1}{(a^{-1}q^{-n}; q)_n} 2\phi_0(q^{-n}, x^{-1}; -; q; x/a). \]

(5.5)
To evaluate (5.4), we start with the second expression in (5.5) in terms of the basic hypergeometric series \( \phi_0 \). Substituting it into (5.4) and using the relation

\[
\frac{(q^{-n}; q)_k}{(q; q)_k} = (-1)^k q^{-kn+1/2} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.
\]

one obtains that

\[
F(x; t; a|q) := \sum_{n=0}^{\infty} (-aq^n) q^{n(n-1)/2} x^n \sum_{k=0}^{n} \frac{(x^{-1}; q)_k}{(q; q)_k(q; q)_{n-k}} (q^{-n}x/a)^k.
\]

Interchanging the order of summations in (5.6) leads to the desired expression

\[
F(x; t; a|q) := E_q(-aqt) \phi_0(x^{-1}, 0; -; q; xt),
\]

where \( E_q(z) = (-z; q)_\infty \) is the q-exponential function of Jackson.

Similarly, if one substitutes into (5.4) the explicit form of the little q-Laguerre polynomials in terms of \( \phi_1 \) from (5.5), this yields an expression

\[
F(x; t; a|q) := \frac{E_q(-aqt)}{E_q(-t)} \phi_1(0, 0; q/t; q; qx).
\]

Using the explicit form of the generating function (5.7) for the little q-Laguerre polynomials, we arrive at

\[
\chi^l_\lambda(x; q) = (-e^{i\psi} q^{(2l-1)/4} x; q)_{\infty} \phi_0(q^{-y}, 0; -; q; e^{i\psi} q^{-y-(6l-1)/4} x),
\]

where, as before, \( q^y = (1-q^{-1})\lambda \). Another expression for \( \chi^l_\lambda(x; q) \) can be written by using formula (5.8).

Due to the orthogonality relation

\[
(q^{2l}; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2lk}}{(q; q)_k} p_m(q^{k}; q^{2l-1}|q) p_n(q^{k}; q^{2l-1}|q) = \frac{q^{2ln}(q; q)_n}{(q^{2l}; q)_n} \delta_{mn}
\]

for little q-Laguerre polynomials (see formula (3.20.2) in Ref. 30), spectrum of the operator \( I_2^{(\psi)} \) coincides with the set of points \( q^n/(1-q^{-1}), n = 0, 1, 2, \cdots \). This means that the eigenfunctions

\[
\chi^l_{\lambda_n}(x; q) \equiv \Xi^l_n(x), \quad n = 0, 1, 2, \cdots, \quad \lambda_n = \frac{q^n}{1-q^{-1}},
\]

constitute a basis of the representation space. We thus proved the following theorem.

**Theorem 4.** The operator \( I_2^{(\psi)} \) has a simple discrete spectrum, which consists of the points \( q^n/(1-q^{-1}), n = 0, 1, 2, \cdots \). The corresponding eigenfunctions \( \Xi^l_n(x) \) constitute an orthogonal basis in the space \( \mathcal{H}_l \).

The basis (5.11) is orthogonal, but not orthonormal. The functions

\[
\hat{\Xi}^l_n(x) = c_n \Xi^l_n(x), \quad c_n = q^{ln} \left( \frac{(q^n; q)_\infty}{(q; q)_n} \right)^{1/2}, \quad n = 0, 1, 2, \cdots,
\]

constitute an orthonormal basis in the space \( \mathcal{H}_l \).
form an orthonormal basis of $\mathcal{H}_l$. This follows from the fact that a matrix $(a^l_{mn})$ with the entries $a^l_{mn} = e_n P_m(\lambda_n)$, which connects the bases $\{J^l_m\}$ and $\{\hat{\Xi}^l_n(x)\}$, is unitary (due to the orthogonality relation for the little $q$-Laguerre polynomials).

The fact that the matrix $(a^l_{mn})$ is unitary and real means that

$$\sum_{n=0}^{\infty} a^l_{mn} a^l_{m'n} = \delta_{mm'}, \quad \sum_{m=0}^{\infty} a^l_{mn} a^l_{mn'} = \delta_{nn'}.$$  

So, if one takes into account the explicit expression for $a^l_{mn}$, then the first relation is actually the orthogonality relation for the little $q$-Laguerre polynomials. The second relation corresponds to the orthogonality relation for the polynomials

$$\hat{p}_n(q^{-m}; q^{2l-1}|q) := (q^{-2l-m-1}; q)_{m} 2\phi_0(q^{-m}, q^{-n}; - : q, q^{-2l+1}q^n). \quad (5.13)$$

This relation has the form

$$\sum_{m=0}^{\infty} q^{-lm} \frac{(q^2; q)_m}{(q; q)_m} \hat{p}_n(q^{-m}; q^{2l-1}|q) \hat{p}_{n'}(q^{-m}, q^{2l-1}|q) = q^{-2ln} \frac{(q; q)_n}{(q^{2l}; q)_n} \delta_{nn'}, \quad (5.14)$$

Comparing (5.13) with the polynomials (3.15.1) in Ref. 30, we see that these polynomials are multiple to the Al-Salam–Carlitz polynomials $V^{(2l-1)}_n(q^{-m}; q)$ and (5.14) is the orthogonality relation for them (compare with the relation (3.25.5) in Ref. 30). This means that the Al-Salam–Carlitz polynomials $V^{(2l-1)}_n(q^{-m}; q)$ are in fact dual with respect to the little $q$-Laguerre polynomials. Thus, we see that our study of the operator $I^0_2$ with the aid of the little $q$-Laguerre polynomials led us to the orthogonality relation for the Al-Salam–Carlitz polynomials $V^{(2l-1)}_n(q^{-m}; q)$.

VI. A REALIZATION OF $T^+_l$, RELATED TO THE OPERATOR $I^0_2$  

We know that the operator $I^0_2$ acts upon the basis functions $\hat{\Xi}^l_n(x)$ as

$$I^0_2 \hat{\Xi}^l_n(x) = \frac{q^n}{1 - q^{-1}} \hat{\Xi}^l_n(x), \quad n = 0, 1, 2 \ldots. \quad (6.1)$$

We can also find how the operator $q^{-J_0}$ acts upon the basis (5.12). From $q$-difference equation (3.20.5) in Ref. 30 for the little $q$-Laguerre polynomials, one readily derives the following difference relation for the polynomials $p_k(q^y) \equiv p_k(q^y; q^{2l-1}|q)$:

$$q^{-k-l} p_k(q^y) = -q^{-y-1} p_k(q^{y+1}) + q^{-l}(1 - q^{-y}) p_k(q^{y-1}) + q^{-y}(q^{-1} + q^{-l}) p_k(q^y).$$

As in the case of the representations of the algebra $\mathfrak{su}(2)$ (see Ref. 29), we derive from it the action formula

$$q^{-J_0} \Xi^l_n(x) = -q^{-n-1} \Xi^l_{n+1}(x) + q^{-l}(1 - q^{-n}) \Xi^l_{n-1}(x) + q^{-n}(q^{-1} + q^{-l}) \Xi^l_n(x).$$

In the orthonormal basis (5.12) this formula takes the form

$$q^{-J_0} \hat{\Xi}^l_n(x) = q^{-n} \left[ (q^{-1} + q^{-l}) \hat{\Xi}^l_n(x) - q^{-1}(1 - q^{n+1})^{1/2} \hat{\Xi}^l_{n+1}(x) - (1 - q^n)^{1/2} \hat{\Xi}^l_{n-1}(x) \right]. \quad (6.2)$$
The operators (6.1) and (6.2) completely determine the action of other operators of the representation $T^+_l$ on the basis (5.12). However, the corresponding formulas are not simple and we do not present them here.

Now we introduce the Hilbert space $l^2_0$, which consists of infinite sequences $a = \{a_k|k \in \mathbb{Z}_+\}$ (note that $\mathbb{Z}_+ = \{0,1,2,\cdots\}$), such that $\sum_{k=0}^{\infty} q^{2k}|a_k|^2/(q;q)_k < \infty$. The scalar product in this Hilbert space is naturally defined as

$$\langle a, a' \rangle_0 = (q^2;q)_\infty \sum_{k=0}^{\infty} \frac{q^{2k}}{(q;q)_k}a_k\overline{a_k}.$$  

Then by (5.10) the sequences of values of the polynomials from (5.3),

$$P_n(\lambda) = \left(\frac{(q^2;q)_n}{(q;q)_n}\right)^{1/2} q^{-ln}p_n((1-q^{-1})\lambda; q^{2l-1}|q),$$

on the set $\{\lambda_k = \frac{q^k}{1-q^{-1}}|k \in \mathbb{Z}_+\}$ form an orthonormal basis of $l^2_0$. We denote these sequences by $\{P_n(\lambda_k) | k \in \mathbb{Z}_+\}$.

Let $\mathcal{H}_l$ be the Hilbert space from section 2 and $f(x) = \sum_{n=0}^{\infty} a_n f_n(x)$ be an expansion of $f \in \mathcal{H}_l$ with respect to the orthonormal basis (2.15). As in section 4, with every function $f \in \mathcal{H}_l$ we associate the sequence $\{F(\lambda_k) | k \in \mathbb{Z}_+\}$, $\lambda_k = \frac{q^k}{1-q^{-1}}$, such that

$$F(\lambda_k) = \langle f(x), \psi_{\lambda_k}(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}_l$. This defines a linear mapping $\Phi : f(x) \rightarrow \{F(\lambda_k) | k \in \mathbb{Z}_+\}$ from $\mathcal{H}_l$ to the Hilbert space $l^2_0$. The following proposition is proved in the same manner as Proposition 1.

**Proposition 3.** The mapping $\Phi : f(x) \rightarrow \{F(\lambda) | k \in \mathbb{Z}_+\}$ establishes an invertible isometry between the Hilbert spaces $\mathcal{H}_l$ and $l^2_0$.

It is easy to see that the isometry $\Phi$ maps basis elements $f_n^l$ of the space $\mathcal{H}_l$ to the basis elements $\{P_n(\lambda_k) | k \in \mathbb{Z}_+\}$ of the space $l^2_0$.

As in section 4, we obtain a realization of the representation $T^+_l$ on the space $l^2_0$, such that

$$I_2^{(v)} \{F(\lambda_k) | k \in \mathbb{Z}_+\} = \{\lambda_k F(\lambda_k) | k \in \mathbb{Z}_+\}, \quad \lambda_k = \frac{q^k}{1-q^{-1}}.$$

In particular, $I_2^{(0)} \{P_n(\lambda_k) | k \in \mathbb{Z}_+\} = \{\lambda_k P_n(\lambda_k) | k \in \mathbb{Z}_+\}$. The operator $I_2^{(0)}$ acts on $k$-th coordinate of the sequence $\{P_n(\lambda_k) | k \in \mathbb{Z}_+\}$ as $I_2^{(0)}P_n(\lambda_k) = \lambda_k P_n(\lambda_k)$. Taking into account formula (5.2), we deduce that

$$I_2^{(0)} \{P_n(\lambda_k) | k \in \mathbb{Z}_+\} = q^{3(n+1)/2}q^{-3/4}\left([n+1]q[2l+n]_q\right)^{1/2}\{P_{n+1}(\lambda_k) | k \in \mathbb{Z}_+\}$$

$$+q^{3(n+1)/2}q^{-3/4}\left([n]q^2[2l+n-1]_q\right)^{1/2}\{P_{n-1}(\lambda_k) | k \in \mathbb{Z}_+\}$$

$$-q^{3(n+1)/2}\left([n]q^{-l/q}\right)^{1/2}\{P_n(\lambda_k) | k \in \mathbb{Z}_+\}. \quad (6.4)$$
that is, \( I_2^{(0)} \) acts upon the basis elements \( \{ P_n(\lambda) \mid k \in \mathbb{Z}_+ \} \) by the same formulas as upon the basis functions \( f_n^l(x) \) of the space \( H_l \). We also have
\[
q^{-J_0} \{ P_n(\lambda_k) \mid k \in \mathbb{Z}_+ \} = q^{-l-n} \{ P_n(\lambda_k) \mid k \in \mathbb{Z}_+ \}.
\] (6.5)

The operators (6.4) and (6.5) determine uniquely all other operators of the representation \( T_i^+ \) on \( l_0^q \). In particular, we have
\[
J_+ \{ P_n(\lambda_k) \mid k \in \mathbb{Z}_+ \} = \sqrt{[2l + n]_q} \{ P_{n+l}(\lambda_k) \mid k \in \mathbb{Z}_+ \},
\]
\[
J_- \{ P_n(\lambda_k) \mid k \in \mathbb{Z}_+ \} = \sqrt{[2l + n - 1]_q} \{ P_{n-1}(\lambda_k) \mid k \in \mathbb{Z}_+ \}.
\]

The results of this section allow to prove (exactly in the same way as Proposition 2) the following assertion.

**Proposition 4.** Let \( P_n(\lambda) \) be the polynomials determined above. Then in the Hilbert space \( H_l \) we have
\[
P_n(I_2^{(\psi)}(x))f_0^l = f_n^l.
\] (6.6)

**VII. UNBOUNDED REPRESENTATION OPERATORS**

In this section we deal with the operator
\[
I_3 := -q^{-3J_0/4}(J_+ + J_-)q^{-3J_0/4} + \frac{1 + q}{2(1 - q)} q^{-2J_0} - \frac{q^l + q^{l-1}}{2(1 - q)} q^{-J_0}.
\] (7.1)

Since the operators \( q^{-2J_0} \) and \( q^{-J_0} \) are unbounded, the \( I_3 \) is also an unbounded operator. We close the operator \( I_3 \) in the space \( H_l \) and assume in what follows that \( I_3 \) is a closed operator. It is easy to check that \( I_3 \) is a symmetric operator. We shall see below that \( I_3 \) is not a selfadjoint operator and has deficiency indices \((1, 1)\), that is, it has infinite number of selfadjoint extensions.

In the limit as \( q \to 1 \) the operator \( I_3 \) reduces to the \( su_{1,1} \)-operator \( J_0^{cl} - J_1^{cl} + 1/2 \). In other words, (7.1) represents another \( q \)-extension of essentially the same classical operator \( J_0^{cl} - J_1^{cl} \) (see section 3).

To find eigenfunctions of the operator \( I_3 \),
\[
I_3(x) \zeta^l_\lambda(x; q) = \lambda \zeta^l_\lambda(x; q),
\] (7.2)
we evaluate first, by using (7.1) and (2.18), that
\[
I_3(x) f_n^l(x; q) = \frac{q^{-2(n+l)}}{2(1 - q)} \{ \beta_n f_n^l(x; q) - q^{-1/2} \alpha_n f_{n+1}^l(x; q) - q^{3/2} \alpha_{n-1} f_{n-1}^l(x; q) \}.
\] (7.3)

where \( \alpha_n = \sqrt{(1 - q^{n+1})(1 - q^{2l+n})} \) and \( \beta_n = 1 - q^{n+1} + q(1 - q^{2l+n-1}) \). Substituting the expansion
\[
\zeta^l_\lambda(x; q) := \sum_{n=0}^{\infty} P_n(\lambda) f_n^l(x; q) = \sum_{n=0}^{\infty} b_n^l x^n, \quad b_n^l := P_n(\lambda) e_n^l(q),
\] (7.4)
into (7.2) and taking into account (7.3), one obtains the three-term recurrence relation

\[ q^{-1/2} \sqrt{(1 - q^{n+1})(1 - q^{2l+n})} P_{n+1}^l(\lambda) + q^{3/2} \sqrt{(1 - q^n)(1 - q^{2l+n-1})} P_{n-1}^l(\lambda) = \left[ 1 - q^{n+1} + q(1 - q^{2l+n-1})(1 + q) - 2(1 - q) q^{2(l+n)} \lambda(q) \right] P_n^l(\lambda) \] (7.5)

for the coefficients \( P_n^l(\lambda) \). Now multiplying both sides of (7.5) by \( c_n^l(q) \) and using the relation \( \sqrt{1 - q^{2l+1}} c_n(q) = q^{(2l-1)/4} \sqrt{1 - q^{n+1}} c_{n+1}(q) \), one finally arrives at the following recurrence relation for the coefficients \( b_n^l \):

\[ q^{(2l-3)/4} (1 - q^{n+1}) b_{n+1}^l + q^{(7-2l)/4} (1 - q^{2l+n-1}) b_{n-1}^l = \left[ 1 - q^{n+1} + q (1 - q^{2l+n-1}) - 2(1 - q) \lambda(q) q^{2(l+n)} \right] b_n^l. \] (7.6)

Up to the multiplicative factor \( q^{3-2l)n/4} \), this is the recurrence relation for \( q \)-Laguerre polynomials \( L_n^{(\alpha)}(y; q) \), \( \alpha = 2l - 1 \), defined as

\[ L_n^{(\alpha)}(y; q) := \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \phi_1(q^{-n}; q^{\alpha+1}; q; -q^{n+\alpha+1} y) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k y^k}{(q^{\alpha+1}; q)_k (q; q)_k} q^{k[n+\alpha+(k+1)/2]} \cdot \]

Consequently, we have

\[ b_n^l = q^{(3-2l)n/4} L_n^{(2l-1)}(\nu; q), \quad \lambda(q) = \lambda(\nu; q) := \frac{\nu}{2(1 - q)}, \] (7.7)

where \( \nu \) is a real constant and the polynomials \( P_n(\lambda) \) in (7.4) are expressed in terms of the \( q \)-Laguerre polynomials as

\[ P_n(\lambda) = q^{n/2} \frac{(q; q)_n^{1/2}}{(q^{2l}; q)_n^{1/2}} L_n^{(2l-1)}(2(1 - q) \lambda; q). \] (7.8)

This means that the eigenfunctions \( \zeta^l_\lambda(x; q) \) of the operator \( I_3(x) \) have the form

\[ \zeta^l_\lambda(x; q) = \sum_{n=0}^{\infty} b_n^l x^n = \sum_{n=0}^{\infty} L_n^{(2l-1)}(\nu; q) \left( q^{(3-2l)/4} x \right)^n. \] (7.9)

With the aid of the generating function

\[ \sum_{n=0}^{\infty} L_n^{\alpha}(x; q) t^n = \frac{1}{(t; q)_\infty} \phi_1(-x; 0; q; q^{\alpha+1} t) \] (7.10)

for the \( q \)-Laguerre polynomials (see formula (3.21.12) in [30]), one can write these eigenfunctions as

\[ \zeta^l_\lambda(x; q) = \frac{1}{(q^{(3-2l)/4}; q)_\infty} \phi_1(-\nu; 0; q; q^{3(1-2l)/4} x), \] (7.11)
where $\phi_1$ is the basic hypergeometric function.

Thus, the eigenfunctions $\zeta_k^j(x; q)$ can be expanded in the basis (2.15) as

$$\zeta_k^j(x; q) = \sum_{n=0}^{\infty} P_n(\lambda) f_n^j(x; q),$$

where the polynomials $P_n(\lambda)$ are given by the formula (7.8). As we can see from section 3.21 in Ref. 30, the polynomials $P_n(\lambda)$ have many different orthogonality relations. Taking into account the results of Chapter VII in Ref. 31, we conclude from this fact that the closed symmetric operator $I_3(x)$ is not selfadjoint and has deficiency indices $(1, 1)$. It has infinitely many selfadjoint extensions. In order to find these extensions, it is necessary to know extremal orthogonality measures for the $q$-Laguerre polynomials. To the best of our knowledge, they are not known (see, for example, Refs. 32 and 33).

The $q$-Laguerre polynomials have the discrete orthogonality relation

$$b_c \sum_{k=-\infty}^{\infty} \frac{q^{2lk}}{(-cq^k; q)_\infty} L_m^{(2l-1)}(cq^k; q) L_n^{(2l-1)}(cq^k; q) = \frac{(q^{2l}; q)_n}{(q; q)_n q^n} \delta_{mn}, \quad (7.12)$$

where $c$ is some positive number and

$$b_c = c^{2l} q^{l(l-1)} (q; q)_{2l-1}$$

(see formula (3.21.3) in Ref. 30). Note that to each positive number $c$ there corresponds an orthogonality relation.

We fix this positive number $c$ and introduce the Hilbert space $l_2^c$, which consists of infinite sequences $a = \{a_k| k \in \mathbb{Z}\}$, such that $\sum_{k=-\infty}^{\infty} q^{-2k}\cdot |a_k|^2/(cq^k; q)_\infty < \infty$. The scalar product $\langle \cdot, \cdot \rangle_c$ in $l_2^c$ is given by

$$\langle a, a' \rangle_c = b_c \sum_{k=-\infty}^{\infty} \frac{q^{2lk}}{(-cq^k; q)_\infty} a_k \overline{a'_k}.$$

Then by (7.8) and (7.12) the sequences, consisting of values of the polynomials $P_n(\lambda)$, $n = 0, 1, 2, \cdots$, on the set $\{\lambda_k = cq^k/(1 - q) | k \in \mathbb{Z}\}$, form an orthogonal system of elements in $l_2^c$. We denote these sequences by $\{P_n(\lambda_k) | k \in \mathbb{Z}\}$. However, as we know from the results of Ref. 34, the polynomials $\{P_n(\lambda_k) | k \in \mathbb{Z}\}$, $n = 0, 1, 2, \cdots$, do not constitute a basis in the space $l_2^c$. We denote by $H_c$ the closed subspace of $l_2^c$, spanned by the polynomials $\{P_n(\lambda_k) | k \in \mathbb{Z}\}$, $n = 0, 1, 2, \cdots$.

Let $H_l$ be the Hilbert space from section 2 and $f(x) = \sum_{n=0}^{\infty} a_n f_n^l(x; q)$ be an expansion of $f \in H_l$ with respect to the orthonormal basis (2.15). With every function $f \in H_l$ we associate the sequence $\{F(\lambda_k) | k \in \mathbb{Z}\}$, $\lambda_k = cq^k/(1 - q)$, such that

$$F(\lambda_k) = \langle f(x), \zeta_k^l(x; q) \rangle = \sum_{n=0}^{\infty} a_n P_n(\lambda_k), \quad (7.13)$$

This yields the linear mapping $\Phi : f(x) \to \{F(\lambda_k) | k \in \mathbb{Z}\}$ from $H_l$ to $H_c$. The following proposition is proved in the same way as Proposition 1.
The representation $T$ on the space $H$ is defined on $H$ as the multiplication operator, $\hat{L}$ Laguerre polynomials one obtains that $I_q$ isometry from $H$ constitute the point spectrum of $I_q$.

However, the self-adjointness of the former operator does not mean that the latter $3$-Laguerre polynomials that the set of points $(k)$, that is, by the formula (7.3). To the element $q$ there corresponds the operator $q^{jo}$ on the space $H_c$. The operators $I_3^{(c)}$ and $q^{jo}$ determine uniquely all other operators of the representation $T^+_I$ on the space $H_c$. Thus, we have proved the following theorem.

**Theorem 5.** Let $H_c$ be the Hilbert space, defined above. Then the representation $T^+_I$ is defined on $H_c$. In particular, the operators $q^{jo}$ and $I_\pm$ act on the basis $\{P_n(\lambda_k) \mid k \in \mathbb{Z}\}$ of this space as $q^{jo}\{P_n(\lambda_k) \mid k \in \mathbb{Z}\} = q^{jo+n}\{P_n(\lambda_k) \mid k \in \mathbb{Z}\}$ and

\[
J_+\{P_n(\lambda_k) \mid k \in \mathbb{Z}_+\} = \sqrt{[2l+n]_q [n+1]_q}\{P_{n+1}(\lambda_k) \mid k \in \mathbb{Z}_+\},
\]

\[
J_-\{P_n(\lambda_k) \mid k \in \mathbb{Z}_+\} = \sqrt{[2l+n-1]_q [n]_q}\{P_{n-1}(\lambda_k) \mid k \in \mathbb{Z}_+\}.
\]

The operator of multiplication by independent variable on the space $l_2^{(c)}$ is selfadjoint (see section 48, Chapter 4, in Ref. 35) and its restriction to $H_c$ gives the operator $I_3^{(c)}$. However, the self-adjointness of the former operator does not mean that the latter operator $I_3^{(c)}$ is selfadjoint. However, we know from the orthogonality relation for the $q$-Laguerre polynomials that the set of points $cq^k/2(1-q)$, $k = 0, \pm 1, \pm 2, \cdots$, constitute the point spectrum of $I_3^{(c)}$.

The functions

\[\zeta_{\lambda_k}(x; q) \equiv \Omega_k^{(c)}(x), \quad \lambda_k = cq^k/2(1-q), \quad k = 0, \pm 1, \pm 2, \cdots,\]
defined by (7.11), constitute a basis of the Hilbert space \( \mathcal{H}_t \). This basis is orthogonal, but not orthonormal. The functions
\[
\hat{\Omega}^{l,c}_k(x) = d_k \Omega^{l,c}_k(x), \quad k = 0, \pm 1, \pm 2, \cdots, \tag{7.15}
\]
with \( d_k = b^{1/2} q^{ik}/(-cq^k; q)_{\infty}^{1/2} \) form an orthonormal basis in \( \mathcal{H}_t \). This can be shown in the same way as in the case of the basis (5.12).

We know that the operator \( I_3(x) \) acts upon the basis functions \( \hat{\Omega}^{l,c}_n(x) \) as
\[
I_3(x) \hat{\Omega}^{l,c}_n(x) = \frac{cq^n}{2(1 - q)} \hat{\Omega}^{l,c}_n(x). \tag{7.16}
\]
We can find how the operator \( q^{J_0} \) acts upon this basis. From \( q \)-difference equation (3.21.6) in Ref. 30 for the \( q \)-Laguerre polynomials one finds the following difference equation for the polynomials \( p_n(y) \equiv L^{2l-1}_n(y; q) \):
\[
q^n y p_n(y) = (1 + y)p_n(qy) - (q^{-2l+1} + 1)p_n(y) + q^{-2l+1} p_n(q^{-1} y).
\]
As in the case of the representations of the algebra \( su_q(2) \) (see Ref. 29), this leads to the action formula
\[
q^{J_0} \Omega^{l,c}_k(x) = q^l(c^{-1} q^{-k} + 1) \Omega^{l,c}_{k+1}(x) + c^{-1} q^{1-l-k} \Omega^{l,c}_{k-1}(x) - c^{-1} q^{-l-k}(1 + q^{-2l+1}) \Omega^{l,c}_k(x).
\]
In the orthonormal basis \( \{ \hat{\Omega}^{l,c}_n(x) \} \) this formula takes the form
\[
q^{J_0} \hat{\Omega}^{l,c}_n(x) = \frac{1}{cq^k} \left[ (1 + cq^k)^{1/2} \hat{\Omega}^{l,c}_{k+1}(x) + q(1 + cq^{k-1})^{1/2} \hat{\Omega}^{l,c}_{k-1}(x) - (q^l + q^{1-l}) \hat{\Omega}^{l,c}_k(x) \right]. \tag{7.17}
\]
The formulas (7.16) and (7.17) completely determine other operators of the representations \( T^+_i \) in the basis \( \{ \hat{\Omega}^{l,c}_n(x) \} \).

There exists another, more complicated, family of symmetric operators, closely related to \( I_3 \). They are of the form
\[
I^{(\psi)}_3 := -q^{-3J_0/4}(e^{i\psi} J_+ + e^{-i\psi} J_-)q^{-3J_0/4} + \frac{1 + q}{2(1 - q)} q^{-2J_0} - \frac{q^l + q^{1-l}}{2(1 - q)} q^{-J_0}
\]
and can be diagonalized in exactly the same way as above.

At the end of this section we mention that the results obtained above lead to a new system of orthogonal functions on a discrete set. They can be obtained from the entries of the matrix \( (a_{kn}) \), which connects the orthonormal bases \( f^l_n(x), n = 0, 1, 2, \cdots, \) and \( \hat{\Omega}^{l,c}_k(x), k = 0, \pm 1, \pm 2, \cdots \). These entries are of the form
\[
a_{kn} = d_k q^{n/2} (q; q)_{\infty}^{1/2} (q^2; q)_{\infty}^{1/2} L^{2l-1}_n(cq^k; q),
\]
where \( d_k \) are defined in (7.15). Orthonormality of the both bases means that
\[
\sum_{k=-\infty}^{\infty} a_{kn} a_{kn'} = \delta_{nn'}, \quad \sum_{n=0}^{\infty} a_{kn} a_{k'n} = \delta_{kk'}.
\]
The first relation in fact coincides with the orthogonality relation (7.12) for the $q$-Laguerre polynomials. The second relation corresponds to the orthogonality relations for the set of functions

$$F_k(q^{-n}; c, 2l - 1|q) := (q; q)_n \phi_1(q^{-n}, -cq^k; 0; q, q^{n+2l}), \quad k = 0, \pm 1, \pm 2, \ldots. \quad (7.18)$$

This orthogonality relation is of the form

$$\sum_{n=0}^{\infty} \frac{q^n(q; q)_n}{(q^{2l}; q)_n} F_k(q^{-n}; c, 2l - 1|q) F_{k'}(q^{-n}; c, 2l - 1|q) = \frac{(-cq^k; q)_{\infty}}{q^{2lk} b_c} \delta_{kk'}. \quad (7.19)$$

Thus, in the study of the operator $I_3$ with the aid of the $q$-Laguerre polynomials a new system of functions, orthogonal on a discrete set, emerges.

VIII. ANOTHER EXAMPLE OF THE UNBOUNDED CASE

In this section we wish to find eigenfunctions of the symmetric operator

$$I_4^{(\psi)} = q^{-J_0/4}(e^{i\psi} J_+ + e^{-i\psi} J_-) q^{-J_0/4} = (e^{i\psi} q^{-1/4} J_+ + e^{-i\psi} q^{1/4} J_-) q^{-J_0/2}. \quad (19)$$

This operator acts upon the basis elements $f^l_k$ as

$$I_4^{(\psi)} f^l_k = q^{-l-n-1/2} \left( e^{i\psi} \sqrt{(1-q^{k+1})(1-q^{2l+k})} f^n_{l+1} + q e^{-i\psi} \sqrt{(1-q^k)(1-q^{2l+k-1})} f^n_{l-1} \right). \quad (20)$$

Since the coefficients here tend to $\infty$ when $n \to \infty$, the operators $I_4^{(\psi)}$ are unbounded. Let us show that the closure of $I_4^{(\psi)}$ is not a selfadjoint operator and has selfadjoint extensions (actually, there are many such extensions). Changing the basis $\{f^l_k\}$ by the basis $\{\tilde{f}^l_k\}$, where $\tilde{f}^l_k = e^{ik\psi} f^l_k$, we obtain the matrix form of the operator $I_4^{(\psi)}$ in the new basis:

$$I_4^{(\psi)} \tilde{f}^l_k = a_k \tilde{f}^l_{k+1} + a_{k-1} \tilde{f}^l_{k-1}, \quad a_k = q^{-l-k-1/2} \sqrt{(1-q^{k+1})(1-q^{2l+k})}. \quad (21)$$

According to Theorem 1.5 in Chapter VII of Ref. 31, the closed operator $I_4^{(\psi)}$ is not selfadjoint and has deficiency indices $(1,1)$ if $a_{k-1}a_{k+1} \leq a^2_k$ and $\sum_{k=0}^{\infty} a^{-1}_k < \infty$. Let us show that these conditions are fulfilled for the operator $I_4^{(\psi)}$.

Since $q + q^{-1} \geq 2$ for positive values of $q$ (and $q + q^{-1} = 2$ only if $q = 1$), then

$$(1-q^{k+1})(1-q^{-k-1}) \leq (1-q^k)^2. \quad (22)$$

This leads to $a_{k-1}a_{k+1} \leq a^2_k$. Since $a_k/a_{k+1} \to q < 1$ when $k \to \infty$ and $0 < q < 1$, then $\sum_{k=0}^{\infty} a^{-1}_k < \infty$. Thus, $I_4^{(\psi)}$ is not selfadjoint operator and has deficiency indices $(1,1)$.

For eigenfunctions $\eta_\lambda(x; q)$ of the operator $I_4^{(\psi)}(x)$ we have

$$I_4^{(\psi)}(x) \eta_\lambda(x; q) = \lambda \eta_\lambda(x; q)$$

and these functions can be represented as

$$\eta_\lambda(x; q) = \sum_{k=0}^{\infty} P_k(\lambda) f^l_k(x; q). \quad (8.1)$$
Then for polynomials $P_k(\lambda)$ we obtain the recurrence relation, which is equivalent to the following one:

$$
e^{-i\psi} \sqrt{(1 - q^{k+1})(1 - q^{2l+k})} P_{k+1}(\lambda) + q e^{i\psi} \sqrt{(1 - q^k)(1 - q^{2l+k-1})} P_{k-1}(\lambda) = q^{l+k-1/2}(q^{-1/2} - q^{1/2}) \lambda P_k(\lambda).$$  \hspace{1cm} (8.2)

Upon making the substitution

$$P_k(\lambda) = e^{ik\psi} \frac{(q; q)_k^{1/2}}{(q^{2l}; q)_k^{1/2}} P'_k(\lambda),$$  \hspace{1cm} (8.3)

one obtains

$$(1 - q^{k+1}) P'_{k+1}(\lambda) + q(1 - q^{2l+k-1}) P'_{k-1}(\lambda) = q^{k} d \lambda P'_k(\lambda),$$  \hspace{1cm} (8.4)

where $d = (1 - q) q^{l-1}$.

Now we substitute the expression (8.3) for $P_k(\lambda)$ and the expression (2.15) for $f'_k(x; q)$ into (8.1). After simple transformation this yields

$$\eta'_k(x; q) = \sum_{k=0}^{\infty} P'_k(\lambda)(q^{(1-2l)/4} e^{i\psi} x)^k.$$  \hspace{1cm} (8.5)

Multiply both sides of (8.4) by $y^{k+1}, y := q^{(1-2l)/4} e^{i\psi} x$, and sum up over $k$:

$$\eta'_k(x; q) - \eta'_k(qx; q) + y^2 q \eta'_k(x; q) - y^2 q^{2l+1} \eta'_k(qx; q) = y d \lambda \eta'_k(qx; q).$$

This gives

$$\eta'_k(x; q) = \frac{1 + y y^2 q^{2l+1}}{1 + q y^2} \eta'_k(qx; q).$$  \hspace{1cm} (8.6)

Then we set $d \lambda = -2 q^{l-1/2} \cosh \theta$ and iterate (8.6):

$$\eta'_k(x; q) = \frac{(y q^{l+1/2} e^\theta; q)_n (y q^{l+1/2} e^{-\theta}; q)_n}{(iy q^{1/2}; q)_n (-iy q^{1/2}; q)_n} \eta'_k(q^n x; q).$$

Passing to the limit $n \to \infty$ and taking into account that $q^n \to 0$ when $n \to \infty$, we arrive at the following explicit expression for $\eta'_k(x; q)$:

$$\eta'_k(x; q) = \frac{(y q^{l+1/2} e^\theta; q)_\infty (y q^{l+1/2} e^{-\theta}; q)_\infty}{(iy q^{1/2}; q)_\infty (-iy q^{1/2}; q)_\infty} \frac{(e^{i\psi} q^{2l+3/4} x e^\theta; q)_\infty (e^{i\psi} q^{2l+3/4} x e^{-\theta}; q)_\infty}{(ie^{i\psi} q^{(3-2l)/4} x; q)_\infty (-ie^{i\psi} q^{(3-2l)/8} x; q)_\infty}.$$  \hspace{1cm}

Employing here the $q$-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} t^n = \frac{(at; q)_\infty}{(t; q)_\infty}, \quad |t| < 1, \ |q| < 1,$$

and comparing the obtained expression with formula (8.5), we deduce that

$$P'_k(\lambda) = (-i)^k q^{k/2} \sum_{n=0}^{k} \frac{(-1)^n (-i q^l e^\theta; q)_n (iq^l e^{-\theta}; q)_{k-n}}{(q; q)_n (q; q)_{k-n}}.$$
It is a polynomial in \( \cosh \theta = -\frac{1}{2} dq^{-l+1/2} \lambda \). This expression can be represented in terms of a basic hypergeometric functions. To this end we take into account the relation

\[
(a; q)_{k-n} = \frac{(a; q)_k q^{n(n+1)/2}}{(-aq^k)^n (q^{1-k} a^{-1}; q)_n}.
\]

As a result, we obtain

\[
P'_k(\lambda) = c' \sum_{n=0}^{k} \frac{(-iq^l e^{\theta}; q)_n (q^{-k}; q)_n}{(q; q)_n (-iq^{-k+l+1} e^{\theta}; q)_n} (iq^{-l+1} e^{\theta})^n
\]

where \( c' = (-iq^{1/2})^k (iq^l e^{-\theta}; q)_k/(q; q)_k \). Applying here the relation (III.8) and then (I.8) from Ref. 24 we obtain the expression

\[
P'_k(\lambda) = (\lambda - i q^k - q^{-k+l+3/2}) (q^2; q)_k \frac{(q^2; q)_k}{(q; q)_k} 3 \phi_1(q^{-k}, -iq^l e^{\theta}, -iq^l e^{-\theta}; q^2 q; q, -q^k) \tag{8.7}
\]

for \( P'_k(\lambda) \), which explicitly exhibits a polynomial dependence on \( \cosh \theta \).

These polynomials in \( \lambda \) have many orthogonality relations, since a closure of the operator \( I_4^{(\psi)} \) is not selfadjoint. It is difficult problem to find orthogonality relations explicitly. For this reason, we cannot construct selfadjoint extensions of the operator \( I_4^{(\psi)} \) and their spectra. Note that the polynomials (8.7) are very similar to (but not coinciding with) the polynomials (5.17) in Ref. 36.

As in the previous case, we can realize the representation \( T_l^+ \) on the linear space of all polynomials \( p(\lambda) \) in \( \lambda \), such that

\[
I^{(\psi)}_4 p(\lambda) = \lambda p(\lambda).
\]

The polynomials \( P'_k(\lambda), k = 0, 1, 2, \cdots \) (and also the polynomials \( P_k(\lambda), k = 0, 1, 2, \cdots \)), form a basis in this space. It follows from (8.2) and (8.3) that

\[
I^{(\psi)}_4 P_k(\lambda) = q^{-l-n-1/2} \left( e^{i\psi} \sqrt{(1-q^k+1)(1-q^{2l+k})} P_{k+1}(\lambda) + e^{-i\psi} \sqrt{(1-q^k)(1-q^{2l+k-1})} P_{k-1}(\lambda) \right),
\]

that is, \( I^{(\psi)}_4 \) acts on the basis \( P_k(\lambda), k = 0, 1, 2, \cdots \), by the same formula as it acts on the basis (2.15).

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