Geometric Interpretation of the Quantum Master Equation in the BRST–anti-BRST Formalism

Marc Henneaux
Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

(*)Maître de Recherches au Fonds National de la Recherche Scientifique. Also at Centro de Estudios Científicos de Santiago, Chile.
Abstract

The geometric interpretation of the antibracket formalism given by Witten is extended to cover the anti-BRST symmetry. This enables one to formulate the quantum master equation for the BRST–anti-BRST formalism in terms of integration theory over a supermanifold. A proof of the equivalence of the standard antibracket formalism with the antibracket formalism for the BRST–anti-BRST symmetry is also given.
1 Introduction

The most powerful method for covariantly quantizing gauge theories is based on the antibracket-antifield formalism \[1, 2, 3\]. In that formalism, one associates with each field $\Phi^A$ one antifield $\Phi^*_A$ of opposite Grassmann parity,

$$\epsilon(\Phi^*_A) = \epsilon_A + 1, \quad \epsilon_A = \epsilon(\Phi^A).$$

One then solves the quantum master equation for the “quantum action” $W(\Phi^A, \Phi^*_A)$,

$$i\Delta W - \frac{1}{2}(W, W) = 0 \iff \Delta(exp iW) = 0,$$

where we have set $\hbar = 1$. In (2), the operator $\Delta$ is defined by

$$\Delta = -(-)^{\epsilon_A} \frac{\delta^R}{\delta \Phi^A} \frac{\delta^R}{\delta \Phi^*_A}$$

and is nilpotent,

$$\Delta^2 = 0,$$

while the antibracket is given by

$$(F, G) = \frac{\delta^R F \delta^L G}{\delta \Phi^A} \frac{\delta^R F \delta^L G}{\delta \Phi^*_A}.$$

The generating functional of the Green functions is equal to

$$Z(j) = \int [D\Phi] exp\{i[W_\psi + j_A \Phi^A]\}$$

with

$$W_\psi = W(\Phi, \Phi^* = \frac{\delta \psi}{\delta \Phi}).$$

The vacuum functional $Z(0)$ does not depend on the choice of $\psi$ because of the master equation.

As shown by Witten in \[6\], the antibracket formalism admits a geometric interpretation if one identifies the antifields with the vectors $\delta/\delta \Phi^A$ tangent

\footnote{For recent reviews developing the approach followed in this paper, see \[4, 5\].}
to the coordinate lines in field space. The function \( W(\Phi, \Phi^*) \) can be seen as a multivector,

\[
W(\Phi, \Phi^*) = W_0(\Phi) + W^A(\Phi)\Phi^*_A + \frac{1}{2} W^{AB}(\Phi)\Phi^*_A\Phi^*_B + \cdots
\]  

(8)

where \( W_0(\Phi) \) is a scalar, \( W^A(\Phi)\Phi^*_A \) a 1-vector, \( \frac{1}{2} W^{AB}(\Phi)\Phi^*_A\Phi^*_B \) a bivector...
The operator \( \Delta \) becomes the divergence of p-vectors, so that (2) expresses that \( exp iW \) is divergenceless. One can then interpret (6) in terms of integration theory on a supermanifold and the \( \psi \)-independence of \( Z(0) \) is just Stokes theorem in field space (see below),

\[
\int \Delta F [D\Phi] = 0
\]  

(9)

(assuming a fast decrease of \( F \) at infinity in field space).

Recently, Batalin, Lavrov and Tyutin have extended the antibracket-antifield formalism in order to include the anti-BRST symmetry \([\mathbb{1} \ \mathbb{8} \ \mathbb{1} \ \mathbb{1}]\). To that end, they associate with each field not just one antifield, but rather three antifields \( \Phi^*_A(a = 1, 2) \) and \( \overline{\Phi}_A \) of parity

\[
\epsilon(\Phi^*_a) = \epsilon_A + 1, \ \epsilon(\overline{\Phi}_A) = \epsilon_A.
\]  

(10)

They replace the quantum master equation (2) by

\[
\frac{1}{2}(W,W)^a + V^a = i \Delta^a W \ (a = 1, 2)
\]  

(11)

with

\[
(F,G)^a = \frac{\delta^R F}{\delta \Phi^*_A} \frac{\delta^L G}{\delta \Phi_A} - \frac{\delta^R F}{\delta \Phi^*_A} \frac{\delta^L G}{\delta \Phi_A}.
\]  

(12)

\[
V^a = \epsilon^{ab} \Phi^*_{A_B} \frac{\delta^L}{\delta \Phi^*_A}, \ \epsilon^{ab} = -\epsilon^{ba}, \ \epsilon^{12} = 1,
\]  

(13)

\[
\Delta^a = -(-)^{\epsilon_A} \frac{\delta^R}{\delta \Phi^*_A} \frac{\delta^R}{\delta \Phi^*_A}.
\]  

(14)

In this extended formalism, the path integral having both BRST-invariance and anti-BRST invariance is given by \([\mathbb{1} \ \mathbb{8} \ \mathbb{1} \ \mathbb{1}]\)

\[
Z(0) = \int [D\Phi][Uexp iW]_{\Phi^*_a=0,\overline{\Phi}_A=0}
\]  

(15)
where the operator $\hat{U}$ is equal to

$$
\hat{U} = \exp \left[ \frac{\delta R_F}{\delta \Phi_A} \frac{\delta L}{\delta \Phi_A} + \frac{i}{2} \epsilon_{ab} \frac{\delta L}{\delta \Phi_A} \frac{(\delta R)^2 F}{\delta \Phi_A} \frac{\delta L}{\delta \Phi_B} \right]
$$

(16)

and involves an arbitrary bosonic gauge fixing fermion $F(\Phi)$.

The purpose of this letter is to show that in spite of the increase in the number of antifields, the formalism can still be given Witten’s geometric interpretation in terms of multivectors on the supermanifold of the fields. The crucial new feature is that the multivectors are now described in an overcomplete basis. A byproduct of our analysis is a direct proof of the equivalence of the extended BRST formalism based on (11) with the original antibracket formalism based on the single quantum master equation (2).

2 Overcomplete Sets

On the supermanifold $M$ of the fields $\Phi^A$, let us introduce the redundant basis of vectors

$$
\Phi^*_{A1} = \frac{\delta}{\delta \Phi_A}, \quad \Phi^*_{A2} = \frac{\delta}{\delta \Phi_A},
$$

(17)

that is, let us duplicate each basis vector $\delta/\delta \Phi_A$. The idea of systematically duplicating ghost-like variables in the construction of the anti-BRST symmetry has been introduced previously in [11]. Let us also introduce a vector bidegree that distinguishes between $\Phi^*_{A1}$ and $\Phi^*_{A2}$ by setting

$$
\text{vect}(\Phi^A) = (0, 0); \quad \text{vect}(\Phi^*_{A1}) = (1, 0); \quad \text{vect}(\Phi^*_{A2}) = (0, 1).
$$

(18)

With the identification (17), every polynomial in $\Phi^A$, $\Phi^*_{A1}$ and $\Phi^*_{A2}$ becomes a multivector on $M$. Conversely, every multivector on $M$ can be represented by at least one polynomial in $\Phi^A$, $\Phi^*_{A1}$ and $\Phi^*_{A2}$. However, that polynomial is not unique since both $\Phi^*_{A1}$ and $\Phi^*_{A2}$ are identified with the same vector $\delta/\delta \Phi_A$. Hence, a multivector has more than one expansion in $\Phi^A$, $\Phi^*_{Aa}$. In order to identify the algebra of multivectors with the algebra of polynomials in $\Phi^A$ and $\Phi^*_{Aa}$, it is necessary to set $\Phi^*_{A1} - \Phi^*_{A2} = 0$ in that latter algebra.

This task can be achieved by hand, but a more elegant procedure, used repeatedly in the BRST context [11, 12], is to introduce a nilpotent operator that does the job. The searched-for operator, denoted by $V$, is defined as
follows. First, one introduces extra variables $\Phi_A$ of degree $(1,1)$. Second, one sets

$$V \Phi_A = \Phi_A^* - \Phi_{A1}^*, \quad VJ \Phi_{Aa} = 0, \quad VJ \Phi^A = 0,$$  \hspace{1cm} (19)

i.e.,

$$V = V^1 + V^2,$$  \hspace{1cm} (20)

with $V^a$ given by (13), and one extends $V$ as a derivation. It is clear that $V$ is nilpotent,

$$V^2 = 0,$$  \hspace{1cm} (21)

and that the cohomology of $V$ in the algebra of polynomials in $\Phi_{Aa}$ and $\Phi_A$ with coefficients that are functions of $\Phi^A$, is just the algebra of multivectors on $M$. Indeed, $\Phi_A$ does not contribute to the cohomology because it is not closed ($V \Phi_A \neq 0$), while $\Phi_A^* - \Phi_{A1}^*$ is killed in cohomology because it is exact. Hence, only $\Phi^A$ and $\Phi_{A1}^* + \Phi_{A2}^*$ (say) remain in the cohomology of $V$. One says that the differential complex of polynomials in $\Phi_{Aa}$, $\Phi_A$ with coefficients that are functions on $M$, equipped with the differential $V$, is a resolution of the algebra of multivectors on $M$,

$$H^*(V) \simeq \{\text{algebra of multivectors on } M\}.$$  \hspace{1cm} (22)

In that picture, the extra antifields $\Phi_A$ of degree $(1,1)$ just appear as the variables that implement the identification of $\Phi_A^* - \Phi_{A1}^*$ with zero through the cohomology of $V$.

It is convenient to redefine the variables as

$$u_A^* = \frac{\Phi_{A2}^* + \Phi_{A1}^*}{2}, \quad v_A^* = \frac{\Phi_{A2}^* J - \Phi_{A1}^*}{2}$$  \hspace{1cm} (23)

and to introduce a new degree such that $\Phi^A$ and $u_A^*$ are in degree zero, $v_A^*$ in degree one, and $\Phi_A$ in degree two. In terms of that degree, the cohomology of $V$ lies in degree zero and Equ. (22) becomes

$$H^0(V) = C^\infty(M) \otimes C(u_A^*), \quad H^k(V) = 0 \ (k \neq 0),$$  \hspace{1cm} (24)

since the algebra of multivectors on $M$ is isomorphic with the algebra $C^\infty(M) \otimes C(u_A^*)$. 

5
3 Divergence of Multivectors in the Over-complete Representation

We have seen that for multivectors, one can introduce the divergence operator given by

$$\Delta = -(-)^{e_A} \frac{\delta^2}{\delta \Phi^A \delta (\delta/\delta \Phi^A)}.$$

Can one extend $\Delta$ to the algebra of polynomials in $\Phi^*_{Aa}$ and $\Phi_A$ with coefficients that are functions on $M$ in such a way that its cohomology is unchanged? The answer is affirmative, as it follows again from techniques familiar from BRST theory [12]. The required extension, which we denote by $\overline{\Delta}$ and still call the divergence operator, is given by

$$\overline{\Delta} = \overline{\Delta}^1 + \overline{\Delta}^2 \quad (25)$$

with

$$\overline{\Delta}^1 = \Delta^1 + i V^1, \quad \overline{\Delta}^2 = \Delta^2 + i V^2. \quad (26)$$

That this is the correct $\overline{\Delta}$ can be seen as follows\footnote{The inclusion of the factor $i$ in (26) is purely conventional and can be absorbed in a redefinition of the variables. Our conventions follow those of [7, 8].}

1. It is easily verified that $\overline{\Delta}$ is nilpotent,

$$\overline{\Delta}^2 = 0. \quad (27)$$

2. In terms of the degree introduced in Equ.(24), $\overline{\Delta}$ splits as

$$\overline{\Delta} = \hat{\Delta} + i V \quad (28)$$

where $V$ is in degree $-1$ and $\hat{\Delta} = \Delta^1 + \Delta^2$ is in degree 0. By standard spectral sequence arguments, the cohomology of $\overline{\Delta}$ is given by the cohomology of the operator induced by $\overline{\Delta}$ in the cohomology $H^*(V)$ of $V$, that is, it is the cohomology of the divergence operator in the algebra of multivectors on $M$, as required.
The two equations in (11) imply
\[ \Delta e^{iW} = 0 \] (29)

Thus, the quantum master equation of the extended BRST-anti-BRST formalism expresses that the multivector \( e^{iW} \) is divergence-free, just as in the ordinary case considered in [6].

4 Integration Theory

The geometric point of view enables one to get a better understanding of the formalism. In particular, it sheds a new light on why the path integral (15) does not depend on the choice of the gauge fixing function and coincides with the path integral \( Z(0) \) (Equ.(6)) of the non-extended formalism.

The natural definition of the integral of a multivector \( A \) over \( M \),
\[ A = A_0 + A_1 + A_2 + \ldots \] (30)
(where \( A_0 \) is the scalar part of \( A \), \( A_1 \) its 1-vector part . . . ) consists in setting
\[ \int A \, [D\Phi] \equiv \int A_0 \, [D\Phi], \] (31)

where \( \int A_0 \, [D\Phi] \) is the standard integral of functions \( \Phi \). Thus, the integral of a \( p \)-vector over \( M \) is zero if \( p \neq 0 \). One can rewrite (31) as
\[ \int A \, [D\Phi] \equiv \int A(\Phi, \Phi^* = 0, \Phi = 0). \] (32)

A crucial property of the integral (31) is given by Stokes theorem,
\[ \int \text{divergence}(B) \, [D\Phi] = 0, \] (33)

where \( B \) is an arbitrary multivector decreasing fast enough at infinity in field space. Equ. (33) plays a fundamental role in quantum field theory.

\[ \text{The single equation (29) implies in turn the two equations in (11) if one requires } W \text{ to be of appropriate ghost bidegree, see [6] for more information. This point will not be needed in the sequel.} \]

\[ \text{We assume that a measure (superdensity of weight one) has been given and is equal to 1 in the given } \Phi^A \text{-coordinate system on } M. \]
and contains the Schwinger-Dyson equations (see e.g. [3]). In terms of the redundant description of multivectors, (33) becomes

$$\int \overline{\Delta} B [D\Phi] = 0.$$  \hspace{1cm} (34)

Because of Stokes theorem, the integral is defined in the cohomology algebra $H^*(\overline{\Delta})$ of the divergence operator $\overline{\Delta}$.

The simple definition (31) is too restrictive and must be generalized for gauge theories. Indeed, one finds in that case that the integral of $exp i W$ is ill-defined: the integral over the gauge modes and the ghosts yields 0 $\delta(0)$. The way to make sense out of the integral of $exp i W$ is to regularize it by means of a total divergence. More precisely, one replaces $exp i W$ by

$$\hat{U}_K exp i W,$$  \hspace{1cm} (35)

$$exp i W \rightarrow \hat{U}_K exp i W,$$  \hspace{1cm} (36)

where $\hat{U}_K$ is the operator

$$\hat{U}_K = exp [\hat{K}, \overline{\Delta}].$$  \hspace{1cm} (37)

One has

$$exp [\hat{K}, \overline{\Delta}] = 1 + [\hat{J}, \overline{\Delta}]$$  \hspace{1cm} (38)

from which it follows

$$\hat{U}_K exp i W = exp i W + \overline{\Delta} (\hat{J} exp i W)$$  \hspace{1cm} (39)

for some operator $\hat{J}$ since $\overline{\Delta}$ is nilpotent and $exp i W$ is $\overline{\Delta}$-closed.

Because the operator $\hat{U}_K$ differs from the identity by a $\overline{\Delta}$-exact operator, one defines more generally the integral of a $\overline{\Delta}$-closed multivector $A$ by

$$\int [D\Phi] A \equiv \int [D\Phi] (\hat{U}_K A)_0$$  \hspace{1cm} (40)

where the operator $\hat{K}$ determines the gauge fixing procedure and is chosen in such a way that the right-hand side of (40) is well-defined\footnote{If the right-hand side of (31) is already well-defined, one can take $\hat{K} = 0$. For $\int [D\Phi] exp i W$ in a gauge theory, one must take $\hat{K}$ different from zero.}. This definition of the integral possesses the following good properties:
1. It reduces to the usual definition (31) when this latter is applicable.

2. Because of Stokes theorem, (40) is formally independent on the choice of \( \hat{K} \) ("Fradkin-Vilkovisky theorem"). Indeed, one has

\[
\hat{U}_{K'} A = \hat{U}_{K} (1 + [\hat{J}_{K,K'}, \hat{\Delta}]) A = \hat{U}_{K} A + \hat{\Delta} C
\]

for some \( C \), since \( \Delta A = 0 \).

3. The integral is actually a function of the cohomological classes of \( \hat{\Delta} \), since \( \hat{U}_K \) commutes with \( \hat{\Delta} \).

For these reasons, the definition (40) is quite reasonable.

Now, the integral (15) considered in [7, 8] is precisely of the form (40). Indeed, one can write (16) as

\[
\exp [\hat{K}, \hat{\Delta}]
\]

with

\[
i\dot{\hat{K}} = \frac{\delta F}{\delta \Phi^B} \frac{\delta}{\delta \Phi^*_{B1}} - \frac{\delta F}{\delta \Phi^B} \frac{\delta}{\delta \Phi^*_{B2}}.
\]

The path integral (15) is thus quite natural from the point of view of the integration theory on \( M \). It fits within the above construction and can be viewed as a regularization of the naive definition (31) by means of the addition of a particular \( \hat{\Delta} \)-exact term.

The choice of \( \hat{K} \) in (43) yields a path integral that is both BRST and anti-BRST invariant [7, 8]. However, one may take a gauge fixing \( \hat{K} \) that has a different structure without changing the integral. For instance, one may take a \( \hat{K} \) that provides a direct link with the original antifield formalism. This is done through

\[
\hat{K} = -\psi(\Phi^A)
\]

where \( \psi \) is the operator of multiplication by \( \psi \). One has

\[
[\hat{K}, \hat{\Delta}] = \frac{\delta \psi}{\delta \Phi^A} \frac{\delta}{\delta u^*_A}
\]

and thus

\[
\hat{U}_{K} A = A(\Phi, u^* + \frac{\delta \psi}{\delta \Phi}, v^*, \Phi).
\]
This yields the path integral (6) if one identifies $u^*_A$ with $\Phi^*_A$. Indeed, the function $W(\Phi, u^* = \Phi^*, v^* = 0, \overline{\Phi} = 0)$ is a solution of the quantum master equation (2). This is a direct consequence of (11). Furthermore, $W(\Phi, u^* = \Phi^*, v^* = 0, \overline{\Phi} = 0)$ can easily be seen to be a (non minimal) proper solution of (2); the ghost spectrum of [7, 8] merely corresponds to a duplication of the gauge symmetries [13]. Hence, there is complete equivalence with the standard formalism, because the path integral does not depend on the choice of the non minimal sector. For another equivalence proof, see [14].

5 Conclusion

We have shown in this letter that the antifield formalism for the combined BRST-anti-BRST symmetry developed in [7, 8, 9, 10] can be given a geometric interpretation in terms of multivectors on the supermanifold of the fields. This extends the work of [6] to the anti-BRST context. The crucial point was to introduce a redundant description of the multivectors and to introduce further variables ($\overline{\Phi}^A$) that kill the redundancy in cohomology. We have also given a direct geometrical proof of equivalence with the standard antifield formalism in terms of integration theory.

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