Transverse Momentum Dependent Light-Cone Wave Function of 
$B$-Meson and Relation to the Momentum Integrated One

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Abstract

A direct generalization of the transverse momentum integrated (TMI) light-cone wave function to define a transverse momentum dependent (TMD) light-cone wave function will cause light-cone singularities and they spoil TMD factorization. We motivate a definition in which the light-cone singularities are regularized with non-light like Wilson lines. The defined TMD light-cone wave function has some interesting relations to the corresponding TMI one. When the transverse momentum is very large, the TMD light-cone wave function is determined perturbatively in term of the TMI one. In the impact $b$-space with a small $b$, the TMD light-cone wave function can be factorized in terms of the TMI one. In this letter we study these relations. By-products of our study are the renormalization evolution of the TMI light-cone wave function and the Collins-Soper equation of the TMD light-cone wave function, the later will be useful for resumming Sudakov logarithms.

Exclusive $B$-decays play an important role for testing the standard model and seeking for new physics. Experimentally they are studied intensively. Theoretically, there are two approaches of QCD factorization for studying these decays. One is based on the collinear factorization\cite{1}, in which the transverse momenta of partons in a $B$-meson are integrated out and their effect at leading twist is neglected. The collinear factorization has been proposed for other exclusive processes for long time\cite{2}. Another one is based on the transverse momentum dependent (TMD) factorization\cite{3}, where one takes the transverse momenta of partons into account at leading twist by meaning of TMD light-cone wave function. The advantage of the TMD factorization is that it may eliminate end-point singularities in collinear factorization\cite{4} and some higher-twist effects are included. The knowledge of the TMD light-cone wave function will provide a 3-dimensional picture of a $B$-meson bound state. However, it is not clear how to define the TMD light-cone wave function in a consistent way to perform a TMD factorization because of light-cone singularities\cite{5}.

In the collinear factorization the light-cone wave function for a $B$-meson moving in the $z$-direction with the four velocity $v$ is defined as\cite{6}:

$$
\Phi_+(k^+, \mu) = \int \frac{dz^-}{2\pi} e^{ik^+z^-} \langle 0|\bar{q}(z^- n)L_n^\dagger(\infty, z^- n)\gamma^+\gamma_0 L_n(\infty, 0) h(0)|\bar{B}(v)\rangle,
$$

where we used the light-cone coordinate system, in which a vector $a^\mu$ is expressed as $a^\mu = (a^+, a^-, \vec{a}_\perp) = ((a^0 + a^3)/\sqrt{2}, (a^0 - a^3)/\sqrt{2}, a^1, a^2)$ and $a^2_\perp = (a^1)^2 + (a^2)^2$. The field $h(x)$ is the field for $b$-quark in the heavy quark effective theory (HQET) and $q(x)$ is the Dirac field for a light quark in QCD. The gauge link is $L_n$ defined with the light-cone vector $n^\mu = (0, 1, 0, 0)$ as:

$$
L_n(\infty, z) = P \exp \left( -ig_s \int_0^\infty d\lambda n \cdot G(\lambda n + z) \right).
$$
In the definition the transverse momentum of $\vec{q}$ is integrated, resulting in that the field $\vec{q}$ and $h$ are only separated in the light-cone direction. We will call $\Phi_+(k^+, \mu)$ as transverse momentum integrated(TMI) light-cone wave function. A direct generalization of Eq.(1) by undoing the integration to define the TMD light-cone wave function causes serious problems because it has light-cone singularities, if quarks emit gluons carrying momenta which are vanishing small in the $+\text{-direction}$ but large in other directions. Similar problems also appear in defining TMD parton distributions with gauge links slightly off the light-cone and study inclusive processes can be done without light-cone singularities\cite{7, 8, 9}. In this letter we propose to define the TMD light-cone wave function with gauge links slightly off the light-cone and study its relation to the TMI light-cone wave function.

We introduce a vector $u^\mu = (u^+, u^-, 0, 0)$ and define the TMD light-cone wave function in the limit $u^+ << u^-:$

$$\phi_+(k^+, k_\perp, \zeta, \mu) = \int \frac{dz^- d^2 k \perp e^{ik^+z^- - i\vec{k} \cdot \vec{z}}}{2\pi^2} \langle 0 | \bar{q}(z)L_u(\infty, z)\gamma^+ \gamma_5 L_u(\infty, 0) h(0) | \bar{B}(v) \rangle | z^+ = 0,$$  \hspace{1cm} (3)

the gauge link $L_u$ is defined by replacing the vector $n$ with $u$ in $L_n$. This definition has not the mentioned light-cone singularity, but it has an extra dependence on the momentum $k^+$ through the variable $c^2 = 4(u \cdot k)^2/u^2$. This extra dependence is useful. The evolution in $\zeta$ is controlled by the Collins-Soper equation\cite{7} which leads to the so-called CSS resummation formalism\cite{8}, and it will be derived here. The evolution with the renormalization scale $\mu$ is simple:

$$\frac{\partial \phi_+(k^+, k_\perp, \zeta, \mu)}{\partial \mu} = (\gamma_q + \gamma_Q) \phi_+(k^+, k_\perp, \zeta, \mu),$$  \hspace{1cm} (4)

where $\gamma_q$ and $\gamma_Q$ is the anomalous dimension of the light quark field $q$ and the heavy quark field $h$ in the axial gauge $u \cdot G = 0$, respectively.

If one integrates $k_\perp$ in $\phi_+(k^+, k_\perp, \zeta, \mu)$, the TMD light-cone wave function will in general not reduce to the TMI one. The reason is that the integration over $k_\perp$ in Eq.(1) has ultraviolet divergences, a renormalization is needed. Hence the integration over the transverse momentum in Eq.(1) is done in $d - 2$ dimension, if one uses $d$-dimensional regularization, and then a UV subtraction is performed. In contrast, the transverse momentum $k_\perp$ in $\phi_+(k^+, k_\perp, \zeta, \mu)$ is in the physical space with $d = 4$, ultraviolet divergences will be generated if one integrates over $k_\perp$ and they are not subtracted in Eq.(3) because it is a distribution of $k_\perp$. Therefore, one can not simply relate $\phi_+(k^+, k_\perp, \zeta, \mu)$ by integrating $k_\perp$ to the TMI light-cone wave function in Eq.(1). However, the TMD light-cone wave function has some interesting relations to the TMI one. If the transverse momentum $k_\perp$ carried by the parton $\vec{q}$ is much larger than the soft scale $\Lambda_{QCD}$, the $b$-quark as a parton will also carry large transverse momentum because the momentum conservation. This can only happen if hard gluons are exchanged between the two partons and the exchange can be studied with perturbative QCD. Without the exchange of hard gluons, one expects that the partons will carry $k_\perp$ with a typical value at order of $\Lambda_{QCD}$. In the case with large $k_\perp$ the TMD light-cone wave function is determined in term of the TMI one as:

$$\phi_+(k^+, k_\perp, \zeta, \mu) = \int_0^\infty dq^+ C_\perp(k^+, q^+, k_\perp, \zeta) \Phi_+(q^+, \mu) + O((k_\perp^2)^{-2}),$$  \hspace{1cm} (5)

where the function $C_\perp$ can be determined by perturbative QCD. By power counting\cite{11} $C_\perp$ is proportional to $(k_\perp^2)^{-1}$. When we consider the Fourier-transformed TMD light-cone wave function
into the impact $b$-space:

$$\phi_+(k^+, b, \zeta, \mu) = \int d^2k_\perp e^{i\vec{k}_\perp \cdot \vec{b}} \phi_+(k^+, k_\perp, \zeta, \mu),$$

(6)

$\phi_+(k^+, b, \zeta, \mu)$ can be related to the TMI one for small $b$ as:

$$\phi_+(k^+, b, \zeta, \mu) = \int_0^\infty dq^+ C_B(k^+, q^+, b, \zeta, \mu) \Phi_+(q^+, \mu) + O(b),$$

(7)

where $C_B$ can also be calculated with perturbative QCD, i.e., it does not contain any soft divergence. Hence the relation represents a factorization. The leading order result is $C_B^{(0)} = \delta(k^+ - q^+)$. A similar factorization between TMD- and usual parton distribution was proven in [7]. We will determine the relation in Eq.(5) and Eq.(7) up to order of $\alpha_s$ and show that the factorization in Eq.(7) holds at one-loop level. In determination of these relations we also derive the $\mu$-evolution equation of the TMI light-cone wave function and the Collins-Soper equation of the TMD one.

The relation in Eq.(5) is useful for constructing models of the TMD light-cone wave function. The importance of the relation in the $b$-space in Eq.(7) and the Collins-Soper equation is the following: When the TMD factorization is formulated in the $b$-space, the relation allows to use the TMI light-cone wave function, while the Collins-Soper equation is used to resum large logarithms. Hence it is possible to have relations between two factorization approaches under certain conditions. It should be noted that other definitions of a TMD light-cone wave function are possible. A different definition is given in [10] with a complicated structure of gauge links. With this definition the relations in Eq.(5) and Eq.(7) can also be studied.

The functions $C_\perp$ and $C_B$ are free from any soft divergence, i.e., infrared- and collinear singularities, we can use a partonic state instead of a B-meson state to determine them. We use the partonic state $|b(m_b, \bar{v})\rangle$, the momenta are given as $k_\mu^u = (k_\perp^+, k_\perp^-, \vec{k}_\perp)$ and $k_\nu^b = (k_b^+, k_b^-, -\vec{k}_q^\perp)$. These partons are on-shell, i.e., $k_\perp^2 = m_q^2$ and $v \cdot k_b = 0$ in HQET. The variable $k^+$ of the wave functions is from 0 to $\infty$. Actually, from the momentum conservation, it is from 0 to $P^+ = m_b + k_b^+ + k_\perp^+$. Under the limit $m_b \to \infty$ we have $P^+ \to \infty$. If we set $P^+$ to be $\infty$ at the beginning, it will results in an ill-defined distribution like $1/((k^+ - k_\perp^+)_+)$ for $k^+$ going to $\infty$. Therefore we will take a finite $P^+$ in the calculation and take the limit $P^+ \to \infty$ in the final result. We illustrate this in detail for $\Phi_+$.

![Figure 1: Diagrams of one-loop contributions. Thick lines stand for $b$-quark, double lines represent gauge links.](image-url)
regularization with $d = 4 - \varepsilon$ for U.V. divergence and give gluons a small mass $\lambda$ for infrared divergences. There are light-cone singularities in diagrams where the gluon is attached to a gauge link. They are cancelled between different diagrams. To show this, we take Fig.1c. as an example. The contribution from Fig.1c after the integration of the gluon momentum is:

$$\Phi_+(k^+, \mu)|_{1c} = -\frac{2\alpha_s}{3\pi^2} \bar{v}(k_q) \gamma^+ \gamma_5 u(k_b) \theta(k^+ - k_q^+) (4\pi\mu^2)^{\varepsilon/2} \Gamma(\varepsilon/2) \frac{(\lambda^2 + \zeta_0^2(1-x)^2)^{-\varepsilon/2}}{k_q^+ - k^+},$$

(8)

where $x = k_q^+ / k^+$ and $\zeta_0^2 = 4(v^{-k_q})^2$. This contribution has the light-cone singularity at $k^+ = k_q^+$. The contribution from Fig.2c reads:

$$\Phi_+(k^+, \mu)|_{2c} = -\delta(k^+ - k_q^+) \bar{v}(k_q) \gamma^+ \gamma_5 u(k_b) \frac{2\alpha_s}{3\pi} (4\pi\mu^2)^{\varepsilon/2} \Gamma(\varepsilon/2) \int_0^\infty dp^+ \frac{(\lambda^2 + \zeta_0^2(p^+/k_q^+)^2)^{-\varepsilon/2}}{p^+}.$$  

(9)

The integral is divergent for $p^+ \to \infty$ and $p^+ \to 0$. The divergence at $p^+ = 0$ is the light-cone singularity and will be cancelled with that from Fig.1c. If we set $P^+ \to \infty$ at the beginning, the sum of these two contributions integrated with a test function $f(k^+)$ is proportional to

$$\int_{k_q^+}^\infty dk^+ f(k^+) - f(k_q^+) \ln \frac{(\mu k_q^+)^2}{\zeta_0^2(k^+ - k_q^+)^2}.\,$$

(10)

This integral is divergent, because the divergence at $p^+ \to \infty$ in Eq.(9) is still there. The divergence has a geometrical reason[12]. In HQET the $b$-quark field $h$ in Eq.(1) can be represented as a gauge link along the direction $v$ and forms a Wilson line combining the gauge link $L_n$. This Wilson line has a cusp singularity at the origin where the two gauge links join[14]. This divergence needs to be renormalized. With a finite $P^+$ the sum becomes proportional to the expression instead of the integral in Eq.(10):

$$\int_{k_q^+}^{P^+} dk^+ f(k^+) - f(k_q^+) \ln \frac{(\mu k_q^+)^2}{\zeta_0^2(k^+ - k_q^+)^2} = \frac{f(k_q^+)}{4} \left[ \ln^2 \frac{(\mu k_q^+)^2}{\zeta_0^2(P^+ - k_q^+)^2} + \xi(2) \right].$$

(11)

One can easily check that the expression is finite under the limit $P^+ \to \infty$. Evaluating all diagrams we have the result at one loop with $x_P = P^+ / k_q^+$:

$$\Phi_+(k^+, \mu) = \bar{v}(k_q) \gamma^+ \gamma_5 u(k_b) \delta(k^+ - k_q^+) + \Phi_+(k^+, \mu)|_{1a}$$

$$+ \frac{2\alpha_s}{3\pi} \bar{v}(k_q) \gamma^+ \gamma_5 u(k_b) \{ \delta(k^+ - k_q^+) \left[ \frac{1}{4} \ln \frac{\mu^2}{m_q^2} - \ln \frac{\lambda^2}{m_q^2} - 1 - \frac{1}{4} \ln^2 \frac{\mu^2}{\zeta_0^2(x_P - 1)^2} - \frac{\pi^2}{24} \right] \right. + \left. \frac{k^+ \theta(k_q^+)}{k_q^+ (k^+ - k_q^+)} \ln \frac{\mu^2}{m_q^2(1-x)^2} \} + \theta(k^+ - k_q^+) \ln \frac{\zeta_0^2(x_P - 1)^2}{\mu^2} \right\},$$

(12)

the contribution from Fig.1a is U.V. finite and is not needed for deriving our final results, as explained later. The above expression is a distribution for $0 < k^+ < P^+$. From the above result one derives the renormalization evolution under the limit $P^+ \to \infty$:

$$\mu \frac{\partial}{\partial \mu} \phi_+(k^+, \mu) = \int_0^\infty dq^+ \gamma_+(k^+, q^+, \mu) \phi_+(q^+, \mu);$$

$$\gamma_+(k^+, q^+, \mu) = \frac{4\alpha_s}{3\pi} \left\{ \left( \frac{5}{4} - 2\ln \frac{\mu}{2v-k^+} \right) \delta(k^+ - q^+) + \frac{k^+ \theta(q^+ - k^+)}{q^+ (q^+ - k^+)} \right\},$$

(13)
In deriving this one should be careful with the plus description acting on different distribution variables. The plus prescription above is for $q^+$. This result is in agreement with that in \[12\] by noting the fact that the wave function defined in \[12\] is with the decay constant in HQET. From our explicit result it is observed that the wave function can not be normalized under the limit $P^+ \to \infty$ as first observed in \[6\] and in other studies \[13\].

To determine the mentioned relations we need to calculate the wave function $\phi_+$ at one-loop order. All diagrams in Fig.1 and Fig.2 give contributions. The calculation is straightforward in the momentum space. The light-cone singularity is regularized by a small but finite $u^+$. The detailed calculation and result will be presented elsewhere and we will only give final results here. To determine the function $C_{\perp}$ we only need to calculate Fig.1b, Fig.1c and Fig.1d. The contribution from Fig.1a is proportional to $1/(k_\perp^2)^2$ for large $k_\perp$ and will not contribute to $C_{\perp}$. The function is determined by taking the limit of large $k_\perp$ and then $P^+ \to \infty$. We obtain:

$$C_{\perp}(k^+, q^+, k_\perp, \zeta) = \frac{2\alpha_s}{3\pi} \frac{1}{k_\perp^2} \left[ \left( \frac{k^+ \theta(q^+ - k^+)}{q^+(q^+ - k^+)} + \frac{\theta(k^+ - q^+)}{k^+ - q^+} \right) \right] + \delta(k^+ - q^+)(\ln \frac{\zeta^2}{k_\perp^2} - 1) \right] \ . \ (14)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{The self-energy corrections.}
\end{figure}

To determine the function $C_B$ we do not need to calculate the contribution from Fig.1a, again. The reason is that the contribution $\phi_+(k^+, k_\perp, \zeta, \mu)_{1a}$ is U.V. finite when integrated over $k_\perp$. That means that $\phi_+(k^+, b=0, \zeta, \mu)_{1a} = \phi_+(k^+, b=0, \zeta, \mu)_{1a} + \mathcal{O}(b)$ for $b \to 0$. For small $b$ we have:

$$\phi_+(k^+, b, \zeta, \mu) = \bar{v}(k_q)\gamma^+ \gamma_5 u(k_b) \left\{ \delta(k^+ - k_q^+) + \frac{2\alpha_s}{3\pi} \left[ \delta(k^+ - k_q^+)(-\frac{1}{4} \ln^2 \zeta^2 b^2 - \frac{1}{2} \ln \mu^2 b^2 \ln \frac{\zeta^2}{\zeta^2 v^2} \right. \right. \right.$$

$$-\frac{1}{4} \ln^2 \zeta^2 b^2(x_P - 1)^2 + 2 \ln \mu^2 b^2 + \frac{1}{2} \ln \frac{\zeta^2}{\mu^2} - \ln \frac{\chi^2}{m_q^2} + \frac{1}{4} \ln \frac{\mu^2}{m_q^2} - 2 - \frac{\pi^2}{3} \right)$$

$$+ \left. \left. \left. \left. \frac{\theta(k^+ - k_q^+)}{k_q^+ - k^+} \ln(b^2 \zeta_v^2(1 - x)^2) - \ln(b^2 m_q^2(1 - x)^2) \frac{k^+ \theta(k_q^+ - k^+)}{k_q^+(k_q^+ - k^+)} \right) \right] \right\}$$

$$+ \delta(k^+ - q^+)(\ln \frac{\zeta^2}{k_\perp^2} - 1) \right] \ . \ (15)$$
with $\bar{b}^2 = b^2e^{2\gamma}/4$. This result is a distribution for $0 < k^+ < P^+$. From the above result one can derive the Collins-Soper equation:

$$\zeta \frac{\partial}{\partial \zeta} \phi_+(k^+, b, \zeta, \mu) = \left[ -\frac{4\alpha_s}{3\pi} \ln \frac{\zeta^2b^2e^{2\gamma}-1}{4} - \frac{2\alpha_s}{3\pi} \ln \frac{\mu^2e}{\zeta^2} \right] \phi_+(k^+, b, \zeta, \mu). \quad (16)$$

The first factor is the famous factor $K + G^{[7,8]}$, the last factor comes because we used HQET for the heavy quark.

Comparing the result in Eq.(12) and Eq.(15) and noting the fact that $\phi_+(k^+, b = 0, \zeta, \mu)_{1a}$ is just $\Phi_+(k^+, \mu)_{1a}$, we can derive the function $C_B$ under the limit $P^+ \to \infty$:

$$C_B(k^+, q^+, b, \zeta) = \delta(k^+ - q^+) + \frac{2\alpha_s}{3\pi} \left\{ \ln(\mu^2 \bar{b}^2) \left[ -\frac{k^+\theta(q^+ - k^+)}{q^+(q^+ - k^+)} + \frac{\theta(k^+ - q^+)}{q^+ - k^+} \right] + \delta(k^+ - q^+) \left[ \frac{1}{4} \ln^2 \frac{\mu^2}{\zeta^2} - \frac{1}{2} \ln^2(\zeta^2 \bar{b}^2) + \ln(\mu^2 \bar{b}^2) + \frac{1}{2} \ln \frac{\zeta^2}{\mu^2} - 1 - \frac{7\pi^2}{24} \right] \right\}, \quad (17)$$

which does not contain any soft divergence and does not depend on $v$. In the above the plus prescription is for $q^+$.

To summarize: We proposed the definition in Eq.(3) for the TMD light-cone wave function of a $B$-meson. The definition does not contain the light-cone singularity and can be used for performing TMD factorization in a consistent way. Two relations between TMD- and TMI light-cone wave function are found. One is that the TMD light-cone wave function with large $k_\perp$ is determined by the TMI one. This relation will be important for constructing models of the TMD light-cone wave function. Another one is the factorization relation between the TMI light-cone wave function and TMD one in the impact $b$ space with the small $b$. In studying these relations we also obtained the renormalization evolution of the TMI light-cone wave function and the Collins-Soper equation of the TMD one. The equation and the relation in the impact space are important. When TMD factorization is formulated in the impact space, the relation allows us to use the TMI light-cone wave function and the equation allows us to resum large logarithms. These issues will be discussed in another publication in the near future.

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