A Characterization Theorem for a Modal Description Logic

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Abstract

Modal description logics feature modalities that capture dependence of knowledge on parameters such as time, place, or the information state of agents. E.g., the logic \( S5_{ALC} \) combines the standard description logic \( ALC \) with an \( S5 \)-modality that can be understood as an epistemic operator or as representing (undirected) change. This logic embeds into a corresponding modal first-order logic \( S5_{FOL} \). We prove a modal characterization theorem for this embedding, in analogy to results by van Benthem and Rosen relating \( ALC \) to standard first-order logic. We show that \( S5_{ALC} \) with only local roles is, both over finite and over unrestricted models, precisely the bisimulation invariant fragment of \( S5_{FOL} \), thus giving an exact description of the expressive power of \( S5_{ALC} \) with only local roles.

1 Introduction

Modal description logics extend the static knowledge model of standard description logics by adding modalities capturing, e.g., the temporal evolution of the state of the world or the dependence of knowledge on the information available to individual agents. Their semantics is typically two-dimensional [Gabbay et al., 2003], i.e., it is defined over interpretations involving two sets of individuals and worlds, respectively, and concepts are interpreted as subsets of the Cartesian product of these two sets. For instance, temporal description logics (surveyed, e.g., by Lutz et al. [2008]) have a frame structure on the set of worlds, in the same way as in the semantics of standard temporal logics such as CTL; they support statements such as ‘every person that is currently a child will eventually become an adult in the future’.

A simpler variant of the same idea is to give up directedness of temporal evolution and instead introduce a modality that reads ‘at some other point in time’, so that, continuing the previous example, one can express only that that every person that is currently a child is an adult at some other time. This coarser granularity buys a simplification of the semantics in which the set of worlds is just a set (equivalently, a frame whose transition relation is an equivalence), i.e. a model of the modal logic \( S5 \). Modal description logics with an \( S5 \)-modality have been used prominently as description logics of change, and are able to encode a restricted form of temporal entity-relationship models if the description logic is strong enough (specifically, contains \( ALCQI \)) [Artale et al., 2007].

One of the simplest description logics of change in this sense is \( S5_{ALC} \), i.e. the extension of the standard description logic \( ALC \) [Baader et al., 2003] with an \( S5 \)-change modality. In fact, there are many other readings for the \( S5 \)-modality. In particular, \( S5 \)-modalities standardly feature in epistemic logics, and indeed \( S5_{ALC} \) was originally introduced as an epistemic description logic [Wolter and Zakharyaschev, 1999b]. As a variant of this view, \( S5_{ALC} \) and its \( EL \) fragment have been considered as a corner case of probabilistic description logics for subjective uncertainty, with probabilities mentioned in concepts restricted to 0 or 1 [Gutiérrez-Basulto et al., 2017]. In the current work, we focus on \( S5_{ALC} \) as one of the most basic modal description logics, and use it as a starting point for the correspondence theory of modal description logics.

Specifically, \( S5_{ALC} \) embeds as a fragment into the modal first-order logic \( S5_{FOL} \), which extends standard first-order logic with an \( S5 \)-modality and lives over the same type of semantic structures as \( S5_{ALC} \). This situation is analogous to the one with \( ALC \) itself, which embeds as a fragment into ordinary first-order logic (FOL). For \( ALC \), it is straightforward to check that its concepts are bisimulation invariant, i.e., bisimilar individuals satisfy the same \( ALC \)-concepts. This constitutes in effect an upper bound on the expressivity of \( ALC \): any property that fails to be bisimulation invariant (such as ‘individual \( x \) is related to itself under role \( r \)’) is not expressible in \( ALC \). Remarkably, it can be shown that this is also a lower bound: every bisimulation invariant first-order property can be expressed in \( ALC \), a fact first proved by van Benthem [1976] and later shown to hold true also over finite structures by Rosen [1997]. In other words, \( ALC \) is precisely the bisimulation invariant fragment of FOL: we refer to theorems of this type as modal characterization theorems. In this terminology, the object of this paper is to establish a modal characterization theorem for \( S5_{ALC} \), the fragment of \( S5_{ALC} \) determined by admitting only local (i.e. non-modalized) roles: We show that both over unrestricted and over finite interpretations, \( S5_{ALC} \) is precisely the bisimulation invariant fragment of \( S5_{FOL} \), where both bisimulation
We recall the syntax of modalized $\mu$-calculus, where as usual $\top$ and $\bot$ are interpreted by relations that move only in one dimension of the world. In fact, the main challenge in available van Benthem / Rosen characterization theorems for unrestricted frames in the modal dimension, i.e., in the interpretation language (Corollary 4.13 below).

**2 $\mu$5-modalized ALC and FOL**

We recall the syntax of modalized $\mu$-calculus as introduced by Wolter and Zakharyaschev [1999b], restricting to a single modality: Concepts $C, D$ of $\mu$ALC (5-modalized ALC with only local roles) are given by the grammar

$$C, D ::= A | \lnot C | C \land D | \exists r C | \Box C$$

where as usual $A$ ranges over a set $N_\text{C}$ of (atomic) concept names and $r$ over a set $N_\text{R}$ of role names. The remaining

Boolean connectives $\lor, \land, \perp$, as well as universal restrictions $\forall r, C$, are encoded as usual. The rank of an $\mu$ALC-concept $C$ is the maximal nesting depth of modalities $\Box$ and existential restrictions $\exists r$ in $C$ (e.g. $\exists r. \Box A$ has rank 2).

An $\mu$5-interpretation

$$\mathcal{I} = (W^5, \Delta^5, ((-)^{\Box}, w)_{w \in W^5})$$

consists of nonempty sets $W^5$, $\Delta^5$ of of worlds and individuals, respectively, and for each world $w \in W^5$ a standard ALC interpretation $(-)^{\Box}, w$ over $\Delta^5$, i.e. for each concept name $A \in N_\text{C}$ a subset $A^{\Box}, w \subseteq \Delta^5$, and for each role name $r \in N_\text{R}$ a binary relation $r^{\Box}, w \subseteq \Delta^5 \times \Delta^5$. We refer to $\Delta^5$ as the domain of $\mathcal{I}$. The interpretation $C^{\Box}, w \subseteq \Delta^5$ of a composite concept $C$ at a world $w$ is then defined recursively by the usual clauses for the ALC constructs $((\neg C)^{\Box}, w = \Delta^5 \setminus C^{\Box}, w)$, $(C \land D)^{\Box}, w = C^{\Box}, w \cap D^{\Box}, w$, $(\exists r. C)^{\Box}, w = \{d \in \Delta^5 \mid \exists e \in C^{\Box}, w . (d, e) \in r^{\Box}, w\}$, and by

$$(\Box C)^{\Box}, w = \{d \in \Delta^5 \mid \forall v \in W^5 . d \notin C^{\Box}, v\}$$

In words, $\Box C$ denotes the set of individuals that belong to $C$ in all worlds. As usual, we write $\Diamond$ for the dual of $\Box$, i.e. $\Diamond C$ abbreviates $\neg \Box \neg C$ and denotes the set of all individuals that belong to $C$ in some world. We write $I, w, d \models C$ if $d \in C^{\Box}, w$.

$\mu$5-interpretations are two-dimensional in the sense that concepts are effectively interpreted as subsets of Cartesian products $W^5 \times \Delta^5$, and the modalities $\Box$ and $\exists r$ are interpreted by relations that move only in one dimension of the product: $\Box$ moves only in the world dimension and keeps the individual fixed, and vice versa for $\exists r$. Thus, $\mu$ALC and $\mu$ACC (Remark 2.2) are examples of many-dimensional modal logics [Marx and Venema, 1996; Gabbay et al., 2003].

As indicated in the introduction, there are various readings that can be attached to the modality $\Box$. E.g. if we see $\Box$ as a change modality [Artale et al., 2007], and, for variety, consider spatial rather than temporal change, then the concept

$\exists C. \text{MarriedTo} \neg C$ (where $C = \exists r. \text{WantTo} \Diamond$) describes persons married to fugitives from the law, i.e. to persons that are wanted by the police in some place but not here. As an example where we read $\Box$ as an epistemic modality ‘I know that’ [Wolter and Zakharyaschev, 1999b; Gabbay et al., 2003], the concept

$\Box \exists C. \Diamond \text{Has} \lor \Diamond \text{Concealed} \lor \Diamond \text{Loaded}$

applies to people who I know are armed with a concealed gun that as far as I know might be loaded. A more expressive modal language is $\mu$5-modalized first-order logic with constant domains, which we briefly refer to as $\mu$5FOL. Formulas $\phi, \psi$ of $\mu$5FOL are given by the grammar

$$\phi, \psi ::= R(x_1, \ldots, x_n) \mid x = y \mid \neg \phi \mid \phi \land \psi \mid \exists x. \phi \mid \Box \phi$$

where $x, y$ and the $x_i$ are variables from a fixed countably infinite reservoir and $R$ is an $n$-ary predicate from an underlying language of predicate symbols with given arities. The
quantifier $\exists x$ binds the variable $x$, and we have the usual notions of free and bound variables in formulas. The rank of a formula $\phi$ is the maximal nesting depth of modalities $\Box$ and quantifiers $\exists x$ in $\phi$; e.g. $\exists x. \Box(\exists y. r(x,y))$ has rank 3. This is exactly the $S_5$ modal first-order logic called QML by Sturm and Wolter [2001]. From now on we fix the language to be the correspondence language of $S_{5\text{ACC}}$, which has a unary predicate symbol $A$ for each concept name $A$ and a binary predicate symbol $r$ for each role name $r$. The semantics of $S_{5\text{FOL}}$ is then defined over $S_5$-interpretations, like $S_{5\text{ACC}}$. It is given in terms of a satisfaction relation $|=\models$ that relates an interpretation $I$, a world $w \in W^I$, and a valuation $\eta$ assigning a value $\eta(x) \in \Delta^I$ to every variable $x$ on the one hand to a formula $\phi$ on the other hand. The relation $|=\models$ is defined by the expected clauses for Boolean connectives, and $I, w, \eta \models R(x_1, \ldots, x_n) \iff (\eta(x_1), \ldots, \eta(x_n)) \in R^{I,w}$ $I, w, \eta \models x = y \iff \eta(x) = \eta(y)$ $I, w, \eta \models \exists x. \phi \iff I, w, [x \mapsto d] \models \phi$ for some $d \in \Delta^I$ $I, w, \eta \models \Box \phi \iff I, v, \eta \models \phi$ for all $v \in W^I$ (where $[x \mapsto d]$ denotes the valuation that maps $x$ to $d$ and otherwise behaves like $\eta$). That is, the semantics of the first-order constructs is as usual, and that of $\Box$ is as in $S_{5\text{ACC}}$. We often write valuations as vectors $d = (d_1, \ldots, d_n) \in (\Delta^I)^n$, which list the values assigned to variables $x_1, \ldots, x_n$ if the free variables of $\phi$ are contained in $\{x_1, \ldots, x_n\}$.

To formalize the obvious fact that $S_{5\text{ACC}}$ is a fragment of $S_{5\text{FOL}}$, we extend the usual standard translation to $S_{5\text{ACC}}$: a translation $ST_x$ that maps $S_{5\text{ACC}}$-concepts $C$ to $S_{5\text{FOL}}$-formulas $ST_x(C)$ with a single free variable $x$ is given by

$$ST_x(A) = A(x)$$
$$ST_x(\exists r. C) = \exists y. (r(x,y) \cap ST_y(C)) \quad (y \text{ fresh})$$

and commutation with all other constructs. Then $ST_x$ preserves the semantics, i.e.

**Lemma 2.1.** For every $S_{5\text{ACC}}$-concept $C$, interpretation $I$, $w \in W^I$, and $d \in \Delta^I$, we have

$I, w, d \models C \iff I, w, d \models ST_x(C)$.

**Remark 2.2.** Modalized $\lambda$C is extended with modalized roles [Wolter and Zakharyaschev, 1999], i.e. roles of the form $\Box r$ or $\Diamond r$, interpreted as

$$(\Box r)^{I,w} = \{(d, e) \mid \forall w \in W^I, (d, e) \in r^{I,w}\}$$

$$(\Diamond r)^{I,w} = \{(d, e) \mid \exists w \in W^I, (d, e) \in r^{I,w}\}.$$  

The $S_5$-modalized description logic in this extended sense has been termed $S_{5\text{ACC}}$ by Gabbay et al. [2003]; so in our notation $S_{5\text{loc}}$ is the fragment of $S_{5\text{ACC}}$ without modalized roles. Since modalized roles $\Box r$ or $\Diamond r$ have an interpretation that is independent of the world while that of basic roles $r$ varies between worlds, the latter are called local roles, explaining the slightly verbose terminology used above. We will see that $S_{5\text{ACC}}$ fails to be bisimulation invariant, and is therefore strictly more expressive than $S_{5\text{ACC}}$.

### 3 Bisimulation and Invariance

We proceed to introduce the relevant notion of bisimulation for $S_5$-interpretations. This is just the usual notion of bisimilarity, specialized to the two-dimensional shape of $S_5$-interpretations and the $S_5$ structure of the world dimension; explicitly:

**Definition 3.1** (Bisimulation). A bisimulation between interpretations $I, J$ is a relation

$$R \subseteq (W^I \times \Delta^I) \times (W^J \times \Delta^J)$$

such that whenever $(w, d) R (v, e)$, then

1. $d \in A^I, w$ iff $e \in A^J, v$ for all $A \in NC$;
2. for every $w' \in W^I$ there is $v' \in W^J$ such that $(w', d) R (v', e)$;
3. Same with the roles of $I$ and $J$ interchanged.
4. for every $(d, d') \in r^{I,w} (r \in N_R)$ there is $e'$ such that $(e, e') \in r^{J,v}$ and $(w, d') R (v, e')$
5. Same with the roles of $I$ and $J$ interchanged.

We say that $I, w, d$ and $J, v, e$ are bisimilar, and write

$I, w, d \approx J, v, e$

if there exists a bisimulation $R$ such that $(w, d) R (v, e)$.

We record explicitly that $S_{5\text{loc}}$ is bisimulation invariant, a fact that is immediate from bisimulation invariance of basic multi-modal logic (over all interpretations, including $S_5$-interpretations). As a general manner of speaking, whenever $P$ is any property that applies to triples $I, w, d$ consisting of an $S_5$-interpretation $I$, $w \in W^I$, and $d \in \Delta^I$ (e.g. $P$ could be an $S_{5\text{ACC}}$-concept or an $S_{5\text{FOL}}$-formula with one free variable), then we say that $P$ is bisimulation invariant, or just $\approx$-invariant, if whenever $I, w, d \approx J, v, e$ then $I, w, d$ has property $P$ iff $J, v, e$ has property $P$. We will extend this terminology without further comment to other notions of equivalence that we introduce later, such as bisimilarity up to finite depth and Ehrenfeucht-Fraïssé equivalence. Moreover, we will consider restrictions of these notions to finite $S_5$-interpretations; e.g. bisimulation-invariance over finite $S_5$-interpretations of a property $P$ is defined like bisimulation-invariance of $P$ above but with $I$ and $J$ assumed to be finite. In these terms, we have

**Lemma 3.2** (Bisimulation invariance). Every $S_{5\text{ACC}}$-concept is $\approx$-invariant.

**Example 3.3.** As indicated in the introduction, bisimulation invariance is an upper bound on the expressivity of $S_{5\text{ACC}}$. As an extremely simple example, the formula $r(x, y)$ of $S_{5\text{FOL}}$ fails to be $\approx$-invariant and is therefore, by Lemma 3.2, not equivalent to (the standard translation of) any $S_{5\text{loc}}$-concept. Bisimulation invariance also separates $S_{5\text{loc}}$ from $S_{5\text{ACC}}$ (Remark 2.2): the $S_{5\text{ACC}}$-concept $\exists \Diamond r$. A fails to be $\approx$-invariant and is therefore not expressible in $S_{5\text{loc}}$.

**Bisimulation games** As usual, bisimilarity can equivalently be captured in terms of games. Explicitly:
**Definition 3.4 (Bisimulation game).** Let $I, J$ be $S5$-interpretations, and let $(w_0, d_0) \in W \times \Delta I$, $(v_0, e_0) \in W J \times \Delta J$. The bisimulation game for $I, w_0, d_0$ and $J, v_0, e_0$ is played by players $S$ (Spoiler) and $D$ (Duplicator), where $D$ means to establish bisimilarity and $S$ aims to disprove it. A configuration of the game is a quadruple $((w, d), (v, e)) \in (W I \times \Delta I) \times (W J \times \Delta J)$, with $((w_0, d_0), (v_0, e_0))$ being the initial configuration. A round consists of one move by $S$ and a subsequent move by $D$, with the following alternatives in the current configuration $((w, d), (v, e))$:

1. $S$ may pick a world $w' \in W I$, and then $D$ needs to pick a world $v' \in W J$; the new configuration then is $((w', d), (v, e))$.
2. Same with the roles of $I$ and $J$ interchanged.
3. $S$ may pick a role $r \in \mathbb{N}_R$ and an individual $d' \in \Delta I$ such that $(d, d') \in r^I_{w,d}$. Then $D$ needs to pick an individual $d' \in \Delta J$ such that $e(e', d') \in r^J_{v,e}$; the new configuration reached is $((w, d'), (v, e'))$.
4. Same with the roles of $I$ and $J$ interchanged.

We will call the first two kinds of moves $W$-moves and the other two kinds $\Delta$-moves. If one of the players cannot move, then the other one wins. A configuration $((w, d), (v, e))$ is winning for $S$ if $d \in A^I_{w,d}$ and $e \notin A^J_{v,e}$ for some concept name $A \in \mathcal{N}_C$, or vice versa; and $S$ wins if a winning configuration for $S$ is reached. Infinite plays that do no visit a winning configuration for $S$ are won by $D$.

The following is then standard:

**Lemma 3.5.** We have $I, w, d \approx_a J, v, e$ if and only if $D$ wins the bisimulation game for $I, w, d$ and $J, v, e$.

The bisimulation game can be restricted to a finite number of rounds, capturing bisimilarity up to finite depth:

**Definition 3.6 (Finite-depth bisimulation).** The $n$-round bisimulation game for $n \geq 0$ is played in the same way as the bisimulation game but only for at most $n$ rounds. The winning conditions are the same as in the bisimulation game except that $D$ now wins if no winning configuration for $S$ has been reached after $n$ rounds. We say that $I, w, d$ and $J, v, e$ are depth-$n$ bisimilar and write $I, w, d \approx_n J, v, e$

if $D$ wins the $n$-round bisimulation game for $I, w, d$ and $J, v, e$.

Again, the following is then standard:

**Lemma 3.7 (Invariance under finite-depth bisimulation).** Every $S5_{\mathcal{L}C}$-concept $C$ of rank at most $n$ is $\approx_n$-invariant.

For technical purposes, we shall need a normalization of the bisimulation game based on the observation that due to the $S5$ structure of the worlds, $S$ can never gain an advantage from playing more than one consecutive $W$-move. Formally:

**Definition 3.8 (Alternating bisimulation game).** The alternating bisimulation game is played like the bisimulation game but with a restriction on the sequence of moves: Each round in the alternating bisimulation game consists of two phases,

1. $S$ may decide to make a $W$-move, and in this case $D$ also makes a $W$-move, according to Item 1 or Item 2 of Definition 3.4, and then

2. $S$ and $D$ each play exactly one $\Delta$-move according to Item 3 or Item 4 of Definition 3.4

(where $D$ needs to avoid winning configurations for $S$ at all times). The alternating bisimulation game also comes in two variants, the unbounded and the $n$-round game, with the proviso that at the end of an $n$-round game, there may be one extra phase of type 1 above. We write

$I, w, d \approx_n a J, v, e$ and $I, w, d \approx^a_n J, v, e$

if $D$ has a winning strategy in the alternating and in the $n$-round alternating bisimulation game for $I, w, d$ and $J, v, e$, respectively.

The unrestricted game is equivalent to the alternating game in the following sense:

**Lemma 3.9.** For interpretations $I, J$ and $(w, d) \in W I \times \Delta I$, $(v, e) \in W J \times \Delta J$:

1. If $I, w, d \approx_{n+1} J, v, e$, then $I, w, d \approx_{n} J, v, e$.
2. If $I, w, d \approx_{n} J, v, e$, then $I, w, d \approx_{n} J, v, e$.
3. $I, w, d \approx J, v, e$ if and only if $I, w, d \approx_{n} J, v, e$.

4. **The Modal Characterization Theorem.**

We proceed to state our main result and sketch its proof: $S5_{\mathcal{L}C}$ is the bisimulation-invariant fragment of $S5_{\mathcal{F}O \mathcal{L}}$, both over finite and over unrestricted $S5$-interpretations. Formally,

**Theorem 4.1 (Modal characterization).** Let $\phi = \phi(x)$ be an $S5_{\mathcal{F}O \mathcal{L}}$-formula with one free variable $x$. If $\phi$ is $\approx_n$-invariant (over finite $S5$-interpretations), then there exists an $S5_{\mathcal{L}C}$-concept $C$ such that $\phi$ is logically equivalent to $\tau(C)$ (over finite $S5$-interpretations). Moreover, the rank of $C$ is exponentially bounded in the rank of $\phi$.

While modal characterization theorems over unrestricted structures can often be proved using model-theoretic tools such as compactness [van Benthem, 1976], proofs that apply also to finite structures typically need to work with some form of locality [Otto, 2004]. In the basic, one-dimensional case, this is Gaifman locality [Gaifman, 1982], which is based on the notion of Gaifman distance in a first-order model. The Gaifman graph of the model connects two of its points if they occur together in some tuple that is in the interpretation of some relation in the model, and the Gaifman distance is then just the graph distance in the Gaifman graph. We adapt these notions for our purposes as follows:

**Definition 4.2.** The Gaifman graph of an $S5$-interpretation $I$ is the undirected graph with vertex set $\Delta I$ that has an edge between $d$ and $e$ if $d \neq e$ and either $(d, e) \in r^I_{w,d}$ or $(e, d) \in r^I_{w,d}$ for some role name $r$ and some $w \in W I$. The Gaifman distance $D : \Delta I \times \Delta I \rightarrow \mathbb{N} \cup \{\infty\}$ is just graph distance (length of the shortest connecting path) in the Gaifman graph, and for any tuple $d = (d_1, \ldots, d_k) \in (\Delta I)^k$, the neighbourhood $U^\ell(d)$ of $d$ with radius $\ell$ is given by

$$U^\ell(d) = \{e \in \Delta I : \min_{i=1}^k D(d_i, e) \leq \ell\}.$$
Remark 4.3. It may be slightly surprising that Gaifman graphs for \( S^5 \)-interpretations live only in the individual dimension, so that implicit steps between worlds are effectively discounted (a point where the \( S^5 \) structure on worlds becomes important). The technical reason for this is that it does not seem easily possible to include the worlds in the Gaifman graph without creating unduly short paths. The fact that world steps count 0 in the Gaifman distance creates a certain amount of tension with the fact that bisimulation games do feature explicit \( W \)-moves (Definition 3.4). Our alternating bisimulation games (Definition 3.3) serve mainly to address this point.

Definition 4.4 (Locality). The restriction \( \mathcal{I}|_U \) of an \( S^5 \)-interpretation \( \mathcal{I} \) to a subset \( U \subseteq \Delta^2 \) is given by \( W^2|_U = W^2, \Delta^2|_U = U, A^2|_U.w = A^2w \cap U \) for \( A \in \mathcal{N}_c \), and \( r^2|_U.w = r^2w \cap (U \times U) \) for \( r \in \mathcal{N}_o \). An \( S^5_{\text{FOL}} \)-formula \( \phi \) with \( k \) free variables is \( \ell \)-local for \( \ell \geq 0 \) if for every \( S^5 \)-interpretation \( \mathcal{I} \), \( w \in W^2 \), and \( \bar{d} \in (\Delta^2)^k \),

\[
\mathcal{I}, w, \bar{d} \models \phi \quad \text{iff} \quad \mathcal{I}|_{U^{\ell}(\bar{d})}, w, \bar{d} \models \phi.
\]

In these terms, we organize the proof of our main result as follows, following a generic strategy proposed by Otto [2004].

Proof of Theorem 4.4 (Sketch). For \( \phi \approx \)-invariant of rank \( n \), we prove the following steps in order:

- \( \phi \) is \( \ell \)-local, where \( \ell = 3^n \) (Lemma 4.7).
- \( \phi \) is \( 2\ell+1 \)-invariant (Lemma 4.9).
- \( \phi \) is equivalent to a concept of rank \( 2\ell+1 \) (Lemma 4.12).

(The locality bound is slightly generous, for simplicity.)

Standard FOL comes with its own notion of invariance, with respect to Ehrenfeucht-Fraïssé equivalence [Libkin, 2004]. This notion has been extended to \( S^5_{\text{FOL}} \) by complementing it with bisimilarity in the world dimension [van Benthem, 2001; Sturm and Wolter, 2001]. Here, we introduce a bounded version of this equivalence, which we phrase in game-theoretic terms; this will be instrumental in the proof of locality:

Definition 4.5 (Bounded Ehrenfeucht-Fraïssé game for \( S^5_{\text{FOL}} \)). Let \( \mathcal{I}, \mathcal{J} \) be \( S^5 \)-interpretations, let \( (w_0, d_0) \in W^2 \times \Delta^2 \), \( (v_0, e_0) \in W^2 \times \Delta^2 \), and let \( n \geq 0 \). The \( n \)-round Ehrenfeucht-Fraïssé game for \( \mathcal{I}, \mathcal{J}, v_0, e_0 \) is played by players \( S \) and \( D \). The configurations are quadruples \( ((w, \bar{d}), (v, \bar{e})) \), where \( w \in W^2, v \in W^2 \), \( \bar{d} \) and \( \bar{e} \) are finite sequences over \( \Delta^2 \) and \( \Delta^2 \), respectively. The initial configuration is \( ((w_0, d_0), (v_0, e_0)) \). The possible moves from configuration \( ((w, \bar{d}), (v, \bar{e})) \) are:

1. \( S \) may pick a world \( w' \in W^2 \) and \( D \) then needs to pick a world \( v' \in W^2 \); the new configuration is \( ((w', \bar{d}), (v', \bar{e})) \).
2. Same with the roles of \( \mathcal{I} \) and \( \mathcal{J} \) interchanged.
3. \( S \) may pick some \( d \in \Delta^2 \) and \( D \) then needs to pick \( e \in \Delta^2 \). The new configuration is \( ((w, \bar{d}d), (v, \bar{e}e)) \).
4. Same with the roles of \( \mathcal{I} \) and \( \mathcal{J} \) interchanged.

The winning conditions are as in the \( n \)-round bisimulation game, except that a configuration is now winning for \( S \) if it fails to be a partial isomorphism. Here, \( ((w_0, d_0, \ldots, d_k), (v_0, e_0, \ldots, e_k)) \) is a partial isomorphism if

- for all \( 0 \leq i, j \leq k \), \( d_i = d_j \Leftrightarrow e_i = e_j \); and
- for all \( 0 \leq i_1, \ldots, i_m \leq k \) and \( m \)-ary relation symbols \( R, (d_{i_1}, \ldots, d_{i_m}) \in R^{\Delta^2, w} \Leftrightarrow (e_{i_1}, \ldots, e_{i_m}) \in R^{\Delta^2, v} \).

We say that \( \mathcal{I}, w_0, d_0 \) and \( \mathcal{J}, v_0, e_0 \) are \( S^5 \)-Ehrenfeucht-Fraïssé equivalent up to \( \ell \)-depth, and write

\[
\mathcal{I}, w_0, d_0 \cong_{\ell} \mathcal{J}, v_0, e_0,
\]

if \( D \) has a winning strategy in this game.

As announced, \( S^5_{\text{FOL}} \) is invariant under \( S^5 \)-Ehrenfeucht-Fraïssé equivalence. For the unbounded variant, this has been shown in earlier work [Sturm and Wolter, 2001]; for our bounded variant, invariance takes the following shape:

Lemma 4.6 (Bounded \( S^5 \)-Ehrenfeucht-Fraïssé invariance). Every \( S^5_{\text{FOL}} \)-formula of rank at most \( n \) with one free variable is \( \cong_{\ell}\)-invariant.

We use this to prove locality:

Lemma 4.7. Let \( \phi \) be a \( \cong_{\ell}\)-invariant \( S^5_{\text{FOL}} \)-formula of rank \( n \). Then \( \phi \) is \( \ell \)-local for \( \ell = 3^n \).

Proof (sketch). Let \( \mathcal{I} \) be an \( S^5 \)-interpretation and \( (w_0, d_0) \in W^2 \times \Delta^2 \). Put \( \mathcal{J} = \mathcal{I}|_{U^{\ell}(d_0)} \); we need to show that \( \mathcal{I}, w_0, d_0 \models \phi \Leftrightarrow \mathcal{J}, w_0, d_0 \models \phi \). By \( \cong_{\ell}\)-invariance, we can disjointly extend the domains of \( \mathcal{I} \) and \( \mathcal{J} \) without affecting satisfaction of \( \phi \). We thus extend both \( \mathcal{I} \) and \( \mathcal{J} \) with \( n \) copies of both \( \mathcal{I} \) and \( \mathcal{J} \) each, obtaining \( \mathcal{I}' \) and \( \mathcal{J}' \), respectively.

By Lemma 4.6 it suffices to show that \( \mathcal{I}', w_0, d_0 \cong_{\ell} \mathcal{J}', w_0, d_0 \). The winning strategy for \( D \) is to maintain the following invariant, where we put \( \ell_i = 3^{n-i} \) for \( 0 \leq i \leq n \):

If \( ((w, \bar{d}), (v, \bar{e})) \) is the current configuration, with \( \bar{d} = (d_0, \ldots, d_i) \) and \( \bar{e} = (e_0, \ldots, e_i) \), then \( w = v \) and there is an isomorphism between \( \mathcal{I}|_{U^{\ell_i}(\bar{d})} \) and \( \mathcal{J}'|_{U^{\ell_i}(\bar{e})} \) mapping each \( d_j \) to \( e_j \).

D maintains the invariant as follows: Whenever \( S \) picks a new world in either interpretation, \( D \) can just pick the same world in the other interpretation, as \( \mathcal{I}' \) and \( \mathcal{J}' \) have the same set of worlds. Whenever \( S \) picks a new individual \( d \) in \( U^{2\ell_i+1}(\bar{d}) \) or \( U^{2\ell_i+1}(\bar{e}) \) (where \( \bar{d} = (d_0, \ldots, d_i) \) and \( \bar{e} = (e_0, \ldots, e_i) \), then \( d \) is in the domain or range of the isomorphism in the invariant, and \( D \) picks his response according to the isomorphism. Otherwise, \( D \) picks a ‘fresh’ copy of the appropriate type (\( \mathcal{I} \) or \( \mathcal{J} \), depending on where \( d \) lies) in the other interpretation and responds with \( d \) in that copy.

Having proved locality of \( \cong_{\ell}\)-invariant formulas, we next establish invariance even under finite-depth bisimilarity. To this end, we need tree unravellings of \( S^5 \)-interpretations:

Definition 4.8 (Tree unravelling). Let \( \mathcal{I} \) be an interpretation and \( d_0 \in \Delta^2 \). The tree unravelling \( \mathcal{I}_0 \) of \( \mathcal{I} \) is the interpretation with set \( W^{2\Delta^2} = W^2 \) of worlds; with domain \( \Delta^2 \) consisting of all paths of the form \( (d_0, d_1, \ldots) \) such that for
each \( i \in \{0, \ldots, k-1\} \), \((d_i, d_{i+1}) \in \pi^{T,w} \) for some role name \( r \) and some world \( w \); and with the following interpretations of concept and role names:

\[
A^{T_d,w}_0 = \{ \bar{d} \in \Delta^{T_d}_0 \mid \pi(\bar{d}) \in A^{T,w}_0 \}
\]

\[
r^{T_d,w}_0 = \{ (\bar{d}, d) \mid \bar{d} \in \Delta^{T_d}_0, (\pi(\bar{d}), d) \in \pi^{T,w} \}
\]

where \( \pi : (d_0, \ldots, d_k) \mapsto d_k \) is projection to the last entry.

It is then easy to show that \( I, w, d \approx I_d, w, d \). In fact, a bisimulation is given by the function \( \pi \) (identity on the set of worlds). Also, \( I_d, w, d \approx I_d^{alt}(w, d), w, d \).

**Lemma 4.9.** Let \( \phi = \phi(x) \) be \( \approx \)-invariant and \( \epsilon \)-local. Then \( \phi \approx_{2\epsilon+1} \)-invariant.

**Proof (sketch).** Let \( I, w, d \approx 2\epsilon+1 \) \( \mathcal{J}, v, e \) and \( I, w, d \models \phi \). We need to show that \( \mathcal{J}, v, e \models \phi \). By Lemma 3.9, \( I, w, d \approx_{2\epsilon} \mathcal{J}, v, e \). By \( \approx \)-invariance of \( \phi \), we may pass from \( I \) and \( \mathcal{J} \) to their unravellings, and by \( \epsilon \)-locality of \( \phi \), we may then restrict those to the radius \( \epsilon \) neighbourhoods of \( d \) and \( e \), respectively. The resulting interpretations \( I_d^{alt}(w,d) \) and \( J^{alt}_e(v,e) \) are then trees of height at most \( \epsilon \) in the individual dimension.

Now \( J^{alt}_e(v,e) \), \( w, d \approx_{alt} J^{alt}_{e(v,e)}, v, e \), i.e. \( D \) wins the alternating \( \epsilon \)-round bisimulation game. Due to the tree structure on the domains, \( D \)'s winning strategy is also winning for the unbounded alternating bisimulation game, as eventually a leaf node will be reached and \( S \) will not have a legal move in the second phase of a round. So, using Lemma 3.9 again, \( D \) wins the unbounded ordinary bisimulation game, and therefore \( \mathcal{J}, v, e \models \phi \) by \( \approx \)-invariance of \( \phi \). \( \square \)

**Remark 4.10.** In the case of finite interpretations (the ‘Rosen’ part of the characterization theorem), there is a caveat: the tree unravelling of a finite interpretation is not finite in general, so we cannot use \( \approx \)-invariance over finite interpretations to pass from interpretations to their unravellings. To remedy this, we work with partial unravellings up to level \( \epsilon \) instead. Such a partial unravelling is constructed by restricting the tree unravelling \( I_d \) to the root \( \epsilon + 1 \) neighbourhood of \( d_0 \) and then identifying each leaf node \( \bar{d} \) with the corresponding element \( \pi(\bar{d}) \) in a fresh disjoint copy of \( I \). The resulting interpretation is clearly finite if \( I \) is finite, and readily shown to be bisimilar to \( I \). Also, the radius \( \epsilon \) neighbourhood of \( d_0 \) in the partial unravelling is a tree.

Finally, we construct an equivalent \( S5_{ALC} \)-concept for a given formula that is invariant under finite-depth bisimulation. We will make use of normal forms, as introduced by Fine [1975].

Since the formula \( \phi \) is fixed, we can assume w.l.o.g. that \( N_c \) and \( N_r \) are finite sets \( N_c = \{ A_1, \ldots, A_s \} \) and \( N_r = \{ r_1, \ldots, r_t \} \).

**Definition 4.11.** The sets \( nf_k \) and \( at_k \) of normal forms and atoms of rank \( k \geq 0 \), respectively, are defined by induction:

\( at_k = \{ A_1, \ldots, A_s \} \cup \{ 3r_i.C \mid 1 \leq i \leq t, C \in nf_{k-1} \} \cup \{ \langle C \mid C \in nf_{k-1} \} \)

and \( nf_k \) is the set of finite conjunctions of the form \( \bigwedge_{B \in at_k} \bar{B} \in B \) (according to some fixed total ordering on \( at_k \) where each \( \varepsilon_B \) is either nothing or negation. Moreover, \( nf_{-1} = \emptyset \) for convenience.

These normal forms have the following properties:

- For any \( I, w, d \), there is exactly one normal form \( C^{i}_{I,w,d} \) of rank \( k \) such that \( I, w, d \models C^{i}_{I,w,d} \).
- We have \( I, w, d \models C^{i}_{I,w,d} \iff C^{i}_{I,w,d} = C^{i}_{J,v,e} \).

**Lemma 4.12.** Every \( \approx \)-invariant \( S5_{FOL} \)-formula \( \phi \) with one free variable \( x \) can be expressed as an \( S5_{ALC} \)-concept of rank \( k \), namely

\( \phi \equiv ST_x(\bigvee_{I,w,d} C^{i}_{I,w,d}) \).

**Proof.** First, note that the above disjunction is finite, even though there may be infinitely many interpretations satisfying \( \phi \). We denote the arising \( S5_{ALC} \)-concept by \( C^{i} \).

For the implication from \( \phi \) to \( ST_x(C) \), just note that if \( I, w, d \models \phi \), then \( C^{i}_{I,w,d} \) is one of the disjuncts in \( C \).

For the reverse implication, let \( I, w, d \models C \) and let \( C^{i}_{J,v,e} \) be a disjunct in \( C \) such that \( I, w, d \models C^{i}_{J,v,e} \). By the above properties of normal forms, it follows that \( C^{i}_{I,w,d} = C^{i}_{J,v,e} \) and therefore \( I, w, d \models C^{i} \). By definition, \( J, v, e \models \phi \), so \( I, w, d \models \phi \) by \( \approx \)-invariance of \( \phi \), as desired.

This completes the proof of Theorem 4.1 as outlined above.

**Characterization within two-sorted FOL**

The natural first-order correspondence language for \( S5_{FOL} \) [Sturm and Wolter, 2001] is a two-sorted language with sorts \( \text{domain} \) and \( \text{world} \); for every \( n \)-ary predicate \( R \) in the \( S5_{FOL} \) language, the two-sorted language has an \( n + 1 \)-ary predicate \( R \) with \( n \) arguments of sort \( \text{domain} \) and one additional argument of sort \( \text{world} \). This language \( SL \) is interpreted in the standard way over two-sorted first-order structures; for the two-sorted language induced by the correspondence language of \( S5_{ALC} \), these are just \( S5 \)-interpretations. One has a translation \((-)^{v}_{v} \) of \( S5_{FOL} \) into the two-sorted first-order language, given by \( R(x_1, \ldots, x_n)^{v}_{v} = R(x_1, \ldots, x_n, v) \) and \( (\langle \phi \rangle)^{v}_{v} = \forall v. (\phi)^{v}_{v} \), and commutation with all other constructs, where \( v \) is a variable of sort \( \text{world} \). Sturm and Wolter [2001] show that \( S5_{FOL} \) is, over unrestricted \( S5 \)-interpretations, precisely the fragment of \( SL \) that is determined by invariance under potential \( S5 \)-isomorphisms, i.e. unbounded \( S5 \)-Ehrenfeucht-Fraïssé equivalence, defined as above but without a bound on the number of rounds. Since every potential \( S5 \)-isomorphism is a bisimulation, we can combine this result with Theorem 4.1 to obtain that \( S5_{ALC} \) is the bisimulation invariant fragment of \( SL \):

**Corollary 4.13** (Modal characterization within \( SL \)). Let \( \phi = \phi(x, v) \) be a \( \approx \)-invariant formula with one variable \( x \) of sort \( \text{domain} \) and one free variable \( v \) of sort \( \text{world} \), in the two-sorted first-order language \( SL \). Then there exists an \( S5_{ALC} \)-concept \( C \) such that \( \phi \) is logically equivalent to \( (ST_x(C))^{v}_{v} \).

(Unlike for Theorem 4.1 there is as yet no version of Corollary 4.13 for finite \( S5 \)-interpretations, as the characterization of \( S5_{FOL} \) within \( SL \) is known only for the unrestricted case.)
5 Conclusions

We have proved a modal characterization theorem for the modal description logic $S5_{\text{loc}}^{\text{ACC}}$, i.e. $S5_{\text{loc}}^{\text{ACC}}$ [Gabbay et al., 2003] with only local roles. Specifically, we have shown that $S5_{\text{loc}}^{\text{ACC}}$, one of the modal description logics originally introduced by Wolter and Zakharyaschev [1999b], is, both over finite and over unrestricted models, the bisimulation-invariant fragment of $S5$-modal first-order logic. By a result of Sturm and Wolter [2001], it follows moreover that $S5_{\text{loc}}^{\text{ACC}}$ is, over unrestricted models, the bisimulation-invariant fragment of two-sorted FOL with explicit worlds. To our knowledge, these are the first modal characterization theorems in modal description logic.

It remains a topic of interest to obtain similar characterization theorems for other modal description logics or many-dimensional modal logics. Notably, this concerns logics whose modal dimension differs from the comparatively simple structure of $S5$, e.g. $K_{\text{ACC}}$. Also, one may investigate the possibility of a modal characterization of full $S5_{\text{ACC}}$, then of course with respect to a different notion of equivalence.

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A Details and Proofs

Details for Example 5.3

We show that the $S_5^{ACI}$-concept $\exists r. A$ fails to be invariant under bisimulation. Define an $S_5$-interpretation $I$ by taking $\Delta^I = \{a, b\}$, $W^I = \{v_1, v_2\}$, $r^I,v_1 = \{a, b\}$, $r^I,v_1 = \emptyset$, $A^I,v_1 = \{b\}$, and $A^I,v_2 = \emptyset$. Moreover, define an $S_5$-interpretation $J$ by $\Delta^J = \{a, b\}$, $W^J = \{w_1, w_2, w_3\}$, $r^J,v_1 = r^J,w_3 = \emptyset$, $A^J,v_3 = \{b\}$, and $A^J,v_3 = A^J,w_2 = \emptyset$. Then we have $(v_1, a) \approx (w_1, a)$, as

$$R = \{ ((v_1, a), (w_1, a)), ((v_2, a), (w_2, a)), ((v_1, b), (w_3, b)), ((v_2, b), (w_1, b)), ((v_2, b), (w_2, b)) \}$$

is a bisimulation. But $a \in (\exists r. A)^I,v_1$ while $a \notin (\exists r. A)^J,w_1$.

Proof of Lemma 3.9

For item 1 and the ‘only if’ direction of item 3, we note that in the alternating game only the options for $S$ are restricted when compared to the ordinary game, in the sense that he is forced to make $\Delta$-moves at certain times. Also, the total number of pairs of moves in the $n$-round alternating game is at most $2n + 1$. Therefore $D$ can use his winning strategy for the $\approx_{2n+1}$ game to win the $\approx_m^n$ game.

For item 2 and the ‘if’ part of item 3, if $D$ has a winning strategy in the alternating game, then for every winning configuration $((w, d), (v, e))$ that can occur before the first phase of a round there must exist functions $f : W^I \to W^J$ and $g : W^J \to W^I$ such that for any $W$-move to some $w' \in W^I$ the answer by $D$ (according to the strategy) is $f(w')$ and for any $W'$-move to some $v' \in W^J$ the answer by $D$ is $g(v')$. Now, as long as $S$ keeps making $W$-moves, $D$ can just reply according to the functions $f$ and $g$. If $S$ does so indefinitely (in the unbounded game), then $D$ wins. Otherwise, eventually either $S$ makes a $\Delta$-move or the game ends. In the latter case, $D$ wins immediately. In the former case, there are two subcases:

- There were no $W$-moves played. Then $D$ is in the same situation that arises in the alternating game when $S$ decides against moving in the first phase of a round. $D$ plays his winning reply for that situation.
- There was at least one pair of $W$-moves, w.l.o.g. the last one was $S$ picking $w' \in W^I$ and $D$ replying with $f(w')$. This is the same configuration that arises in the alternating game when $S$ plays $w'$ during the first phase, by definition of $f$. So $D$ has a winning reply for $S$’s $\Delta$-move.

For the finite case it should be noted that every configuration that can be reached in $n$ rounds of the ordinary game according to this strategy can also be reached in $n$ rounds of the alternating game (while following the winning strategy).

Proof of Lemma 4.6

The proof will proceed by induction on the structure of $\phi$. However, we first need to generalize some notions for the purpose of the proof:
First, we generalize the Ehrenfeucht-Fra"issé game to allow for more possible starting configurations:

**Definition A.1.** Let $\mathcal{I}, \mathcal{J}$ be $S_5$-interpretations, let $w \in W^\mathcal{I}$, $v \in W^\mathcal{J}$, let $d$ and $\bar{e}$ be finite sequences of equal length over $\Delta^\mathcal{I}$ and $\Delta^\mathcal{J}$, respectively, and let $n \geq 0$. The $n$-round Ehrenfeucht-Fra"issé game for $\mathcal{I}, w, d$ and $\mathcal{J}, v, \bar{e}$ is played with the same rules as in Definition 4.5, but the starting configuration is now $((w, d), (v, \bar{e}))$. We also write $\mathcal{I}, w, d \equiv_n \mathcal{J}, v, \bar{e}$ when $D$ has a winning strategy for this game.

We can now also generalize the notion of $\equiv_n$-invariance to formulas that may have more than one free variable:

**Definition A.2.** Let $\phi$ be an $S_5_{\text{FOL}}$-formula with free variables contained in $\{x_1, \ldots, x_k\}$. $\phi$ is $\equiv_n$-invariant, if for all $S_5$-interpretations $\mathcal{I}, \mathcal{J}$ and all $w \in W^\mathcal{I}$, $v \in W^\mathcal{J}$, $d \in (\Delta^\mathcal{I})^k$ and $\bar{e} \in (\Delta^\mathcal{J})^k$ such that $\mathcal{I}, w, d \equiv_n \mathcal{J}, v, \bar{e}$,

$$\mathcal{I}, w, d \models \phi \iff \mathcal{J}, v, \bar{e} \models \phi.$$  

We are now set to prove the following more general version of Lemma 4.6. We will denote the rank of a formula $\phi$ by $\text{rk}(\phi)$.

**Lemma A.3.** Every $S_5_{\text{FOL}}$-formula $\phi$ of rank at most $n$ with free variables contained in $\{x_1, \ldots, x_k\}$ is $\equiv_n$-invariant.

**Proof.** The proof will be by induction on the structure of $\phi$.

- For the base cases where $\phi$ is of the form $y_1 = y_2$ or $R(y_1, \ldots, y_m)$ the $\equiv_n$-invariance follows from the fact that the starting configuration is a partial isomorphism.

- The Boolean cases ($\phi = \psi \lor \chi$ or $\phi = \neg \psi$) are straightforward.

- Suppose $\phi = \square \psi$. Since $\text{rk}(\phi) \leq n$, we get that $\text{rk}(\psi) \leq n - 1$ and therefore $\psi$ is $\equiv_{n-1}$-invariant by the induction hypothesis. Let $\mathcal{I}, w, d \equiv_n \mathcal{J}, v, \bar{e}$ and $\mathcal{I}, w, d \models \square \psi$. We then need to show that $\mathcal{J}, v, \bar{e} \models \psi$, so let $v' \in W^\mathcal{J}$ and we need to show that $\mathcal{J}, v', \bar{e} \models \psi$. By assumption, $D$ has a winning response if $S$ plays the $W$-move $v'$, let this response be $w'$. Then $\mathcal{J}, v', d \equiv_{n-1} \mathcal{J}, v', \bar{e}$, because $D$ has a winning strategy for the remaining $n - 1$ rounds. Since $\mathcal{I}, w, d \models \square \psi$, we get $\mathcal{I}, w', d \models \psi$, and $\equiv_{n-1}$-invariance of $\psi$ then yields $\mathcal{J}, v', \bar{e} \models \psi$, as desired.

- Suppose $\phi = \exists k_1+1. \psi$ (w.l.o.g. we can substitute the variable bound by the quantifier). Since $\text{rk}(\phi) \leq n$, we get that $\text{rk}(\psi) \leq n - 1$ and therefore $\psi$ is $\equiv_{n-1}$-invariant by the induction hypothesis. Let $\mathcal{I}, w, d \equiv_n \mathcal{J}, v, \bar{e}$ and $\mathcal{I}, w, d \models \exists k_1+1. \psi$. We then need to show that $\mathcal{J}, v, \bar{e} \models \exists x_{k+1}. \psi$. Let $\mathcal{I}, w, d \equiv \exists x_{k+1}. \psi$, so by definition there must exist some $d' \in \Delta^\mathcal{I}$ such that $\mathcal{I}, w, d' \models \psi$. By assumption, $D$ has a winning response if $S$ plays the $\Delta$-move $d'$, let this response be $e$. Then $\mathcal{I}, w, d' \equiv_{n-1} \mathcal{J}, v, \bar{e}$, because $D$ has a winning strategy for the remaining $n - 1$ rounds. Because of this, it follows that $\mathcal{J}, v, \bar{e} \models \psi$ and thus also $\mathcal{J}, v, \bar{e} \models \phi$, as desired.

**Proof Details for Lemma A.7**

We first note that $\mathcal{I}, w, d \models I, w, d$ and $\mathcal{J}, v, e \models \mathcal{J}', v, e$, where in both cases the bisimulation is given by the embedding into the disjoint union (and identity on the set of worlds). So at the end of the proof we can combine $\approx$-invariance of $\phi$ with $\mathcal{I}', w, d \equiv_n \mathcal{J}', v, e$ to prove:

$$\mathcal{I}, w, d \models \phi \iff \mathcal{J}', v, e \models \phi \iff \mathcal{J}, v, e \models \phi$$

Now we recall the invariant that $D$ needs to maintain:

If $((w, d), (v, \bar{e}))$ is the current configuration, and $d = (d_0, \ldots, d_i), e = (e_0, \ldots, e_i)$, then $w = v$ and there is an isomorphism between $\mathcal{I}'|_{U^i(d)}$ and $\mathcal{J}'|_{U^i(e)}$ mapping each $d_j$ to $e_j$.

First, the invariant clearly holds at the beginning of the game, as the starting configuration is $((w_0, d_0), (v_0, d_0))$, and since $\ell_0 = 1$, both interpretations from the invariant are isomorphic to $\mathcal{J}$ and that isomorphism maps $d_0$ to itself.

Second, whenever the invariant holds after at most $n$ rounds, the current configuration is a partial isomorphism as defined in Definition 4.5 i.e. actually ensures that $D$ wins. Using the names from the invariant, this follows directly from the fact that the isomorphism maps every $d_j$ to the corresponding $e_j$, where for the second item in the definition of partial isomorphism we note that $w = v$.

Finally, we show that the invariant is actually invariant with respect to the strategy described in the proof sketch. For the case of a $W$-move, this is clear. In the following, we treat the case of a $\Delta$-move, with notation as in the invariant. There are two cases:

First, suppose that $S$ picks $d \in U^{2\ell+1}(d)$ and $e$ is $D$'s response according to the isomorphism. Note that, using the triangle inequality for the Gaifman distance, $U^{\ell+1}(d) \subseteq U^{\ell}(d)$ (since $2\ell+1 + \ell+1 = 3\ell+1 = 3^3 - 1 = 3^n - 1 = \ell_4$) and thus also $U^{\ell+1}(e) \subseteq U^{\ell}(e)$ by isomorphism. This implies that the domain $U^{\ell+1}(dd)$ and range $U^{\ell+1}(ee)$ of the putative new isomorphism are contained in those of the old one. Therefore, the new isomorphism can be taken to be the restriction of the old isomorphism to the new domain and range. The same argument works if $S$ picks some $e \in U^{2\ell+1}(e)$ instead.

Otherwise, $S$ picks a $d$ such that $D(d_j, d) > 2\ell+1$ for all $0 \leq j \leq i$. Then, $U^{\ell+1}(d) \cap U^{\ell+1}(d) = \emptyset$, again by the triangle inequality. Now $D$ picks $e$ in $\mathcal{J}'$ from a fresh copy (which means that it contains none of the $e_j$ ($0 \leq j \leq i$) of the same type ($\mathcal{I}$ or $\mathcal{J}$) that $d$ lies in. Such a copy always exists, because $\mathcal{J}'$ contains $n$ copies of both types and in each of the $n$ rounds at most one of them is visited. Now we obtain two isomorphisms of $S_5$-interpretations. The radius-$\ell+1$-neighbourhoods of $d$ and $e$ are isomorphic by restriction of the old isomorphism, as in the first case. The radius-$\ell+1$-neighbourhoods of $d$ and $e$ are isomorphic because $d$ and $e$ are the same element in isomorphic copies of the same type ($\mathcal{I}$ or $\mathcal{J}$). Now, since the domains and ranges of the two isomorphisms are disjoint, we can combine them into a new isomorphism which satisfies the constraints from the invariant. Again, the same argument applies for the case where $S$ picks an element $e$ in $\mathcal{J}'$ instead.
Proof Details for Lemma 4.9

We first show the following lemma:

**Lemma A.4.** Let \( I \) be an S5-interpretation, \( (w, d) \in W^Z \times \Delta^Z \). Then \( I, w, d \approx_{\it{alt}} I|_{U^i(d)}, w, d \).

**Proof.** The winning strategy for \( D \) is to copy every move by \( S \). Clearly, no winning configuration for \( S \) can be reached in this way, so we just need to show that this is a valid strategy, i.e. that copying \( S \)’s move is always a legal move. But this easily follows from the fact that after \( k \) rounds of the game, if the current configuration is \(((w', d'), (w', d'))\) then \( D(d, d') \leq k \). Note that any \( W \)-moves \( S \) elects to play (including the one after the \( \ell \)-th round) do not affect the Gaifman distance.

Now let \( I, w, d \) and \( J, v, e \) be as in Lemma 4.9 so \( I, w, d \approx_{\it{alt}} J, v, e \). Because every \( \approx_{\it{alt}} \)-interpretation is bisimilar to its tree unravelling, and bisimilarity implies alternating bisimilarity up to depth \( \ell \) by Lemma 3.9, \( I, w, d \approx_{\it{alt}} J|_{U^i(d)}, w, d \) and \( J, v, e \approx_{\it{alt}} J|_{U^i(e)}, v, e \). By transitivity of \( \approx_{\it{alt}} \), \( I, w, d \approx_{\it{alt}} J|_{U^i(e)}, v, e \).

Using the above lemma and again transitivity of \( \approx_{\it{alt}} \), we obtain \( I|_{U^i(d)}, w, d \approx_{\it{alt}} J|_{U^i(e)}, v, e \).

Now we show that the winning strategy for \( D \) in the \( \approx_{\it{alt}} \) game between \( I|_{U^i(d)}, w, d \approx_{\it{alt}} J|_{U^i(e)}, v, e \) and \( J|_{U^i(e)}, v, e \) is also winning for the \( \approx_{\it{alt}} \) game. The tree structure of the interpretations guarantees that, regardless of strategy, after round \( k \), if the current state is \(((w', d'), (v', e'))\), then \( d' \) and \( e' \) are at distance \( k \) from the root.

So, if \( D \) follows his winning strategy, either \( S \) loses within \( \ell \) rounds or at least \( \ell \) rounds are played. Going into round \( \ell + 1 \), there are two cases for phase 1 of this round:

1. \( S \) chooses to make a \( W \)-move. In this case, there exists a winning reply for \( D \), remembering that the \( \approx_{\it{alt}} \) game allows for a last pair of \( W \)-moves to be played after round \( \ell \), so this case is covered by the existing strategy.
2. \( S \) does not choose to make a \( W \)-move. In this case, we go straight to phase 2.

Now, \( S \) is forced to make a \( \Delta \)-move, but both individuals in the current configuration are at distance \( \ell \) from their respective roots \( d \) and \( e \), so they do not have any \( r \)-successors for any role \( r \). Therefore \( S \) cannot make a legal move, and \( D \) wins the game.

By item 3 of Lemma 3.9, \( T_d|_{U^i(d)}, w, d \approx J|_{U^i(e)}, v, e \), so to finish the proof, we combine the \( \approx \)-invariance and \( \ell \)-locality of \( \phi \) as follows:

\[ I, w, d \models \phi \iff T_d, w, d \models \phi \iff T_d|_{U^i(d)}, w, d \models \phi \iff J|_{U^i(e)}, v, e \models \phi \iff J, v, e \models \phi \]

Details for Remark 4.10

Let \( I \) be an S5-interpretation and \((w_0, d_0) \in W^Z \times \Delta^Z \). Let \( J, w_0, d_0 \) be the partial unravelling of \( I \) up to level \( \ell \). Then we can define a map \( \rho : \Delta^Z \to \Delta^Z \) as follows: every element from a copy of \( I \) is mapped to itself and every path \( d \) from the tree unravelling is mapped to its last element \( \pi(d) \) (note that this is well-defined, because any elements that were identified to form the partial unravelling have the same image under this map).

A bisimulation is then given by \( \rho \) in the individual dimension and identity in the world dimension, i.e.

\[(w, d)R(v, e) \Leftrightarrow v = w \text{ and } \rho(e) = d\]

Proofs of the Properties of Normal Forms

We prove the following two properties of normal forms:

- For any \( I, w, d \), there is exactly one normal form \( C_k^I \) of rank \( k \) such that \( I, w, d \models C_k^I \).
- We have \( I, w, d \models C_k^I \) iff \( C_k^I,w,d \models C_k^I,v,e \).

In what follows, we will sometimes refer to the \( \varepsilon_B \) from Definition 4.11 as signs where ‘nothing’ is the positive sign and negation the negative sign.

For the first property: For every \( B \in \Delta_k \), either \( I, w, d \models B \) or \( I, w, d \models \neg B \), and we put \( \varepsilon_B \) to be nothing in the first case and negation in the second. Together, this gives a normal form \( C_k^I,w,d \). For uniqueness, we note that if we defined any of the \( \varepsilon_B \) differently, \( I, w, d \) would fail to satisfy the resulting normal form.

For the second property: For the ‘only if’ direction, we note that if \( I, w, d \models C_k^I \), \( J, v, e \models C_k^I \) by bisimilarity (and the fact that \( C_k^I \) is of rank \( k \)), but then \( C_k^I,w,d = C_k^I,v,e \) by uniqueness of normal forms (the first property).

For the ‘if’ direction, we proceed by induction on \( k \). So suppose \( C_k^I,w,d = C_k^I,v,e =: C \). First we ensure that the configuration \((w, d, (v, e))\) is not winning for \( S \) for every atomic concept \( A \), the sign of \( A \) in \( C \) determines for both sides whether they satisfy \( A \) or not, so \( I, w, d \models C \), \( J, v, e \models C \) satisfy the same atomic concepts. If \( k = 0 \), we are done because the game ends immediately. So suppose \( k > 0 \), and we now need to give a winning response for \( D \) for all possible moves by \( S \):

1. Suppose \( S \) picks some \( w' \in W^Z \). Then \( I, w, d \models \Diamond C_k^{I-1,w,d} =: B \), so the sign \( \varepsilon_B \) in \( C \) is positive and so also \( J, v, e \models \Diamond C_k^{I-1,w,d} \) By definition, there must exist some \( w' \in W^Z \) such that \( J, w', e \models C_k^{I-1,w,d} \). By uniqueness of normal forms, \( C_k^{I-1,w,d} = C_k^{I-1,v,e} \), and by the induction hypothesis, \( I, w', d \models C_k^{I-1} \). This means that \( D \) has a winning strategy for the remaining \( k - 1 \) rounds, and thus wins the bisimulation game.

2. The case where \( S \) picks some \( w' \in W^Z \) is analogous.

3. Suppose \( S \) picks the role \( r \in \Delta_R \) and \( d' \in \Delta^Z \) such that \( (d', d') \in \Delta^Z \). Then \( I, w, d \models \exists r.C_k^{I-1,w,d} =: B \), so the sign \( \varepsilon_B \) in \( C \) is positive and so also \( J, v, e \models \exists r.C_k^{I-1,w,d} \).

By definition, there must exist some \( r' \in \Delta^Z \) such that \( J, v, e' \models C_k^{I-1,w,d} \). By uniqueness of normal forms, \( C_k^{I-1,w,d} = C_k^{I-1,v,e'} \), and by the induction hypothesis, \( I, w', d' \models C_k^{I-1} \). This means that \( D \) has a winning strategy for the remaining \( k - 1 \) rounds, and thus wins the bisimulation game.

4. The case where \( S \) picks a role \( r \in \Delta_R \) and \( e' \in \Delta^Z \) such that \((e, e') \in rJ^\uparrow \) is analogous.