On some properties of number-phase Wigner function

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Abstract

It is shown that the number-phase Wigner function defines uniquely the respective density operator. Relations between the Glauber-Sudarshan distribution $P(\alpha)$ and the number-phase Wigner function is found. This result is then generalised to the case of the Cahil-Glauber distributions $W_s(\alpha), -1 \leq s \leq 1$.

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1 Introduction

In recent works [1–3] we have developed a theory of quantum phase which is based on some enlarging of the Fock space to the Hilbert space of the square integrable functions on the circle, $L^2(S^1)$. Then the well known machinery of the Naimark projection is employed to find the respective objects in the original Fock space. Such an approach to quantum phase have been considered by many authors (for references see [1,2]). In our case we were able to define also a number-phase quasiprobability distribution which we called the number-phase Wigner function [2,3]. In short our construction can be introduced as follows: Firstly, define the self-adjoint operator $\hat{\Omega}(\phi, n)$

$$\hat{\Omega}(\phi, n) := \pi \left\{ |n\rangle \langle n| \phi \rangle \langle \phi| + |\phi\rangle \langle \phi| n\rangle \langle n| \right\} \quad (1.1)$$

where $|n\rangle$ is a normalised eigenvector of the number operator $\hat{n}$ i.e.

$$\hat{n} |n\rangle = n |n\rangle, \quad \langle n'|n\rangle = \delta_{n'n}, \quad n, n' = 0, 1, 2, \ldots \quad (1.2)$$

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and $|\phi\rangle$ stands for the phase state vector

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle$$

(1.3)

Then the number-phase Wigner function is defined as

$$g_W(\phi, n) := \frac{1}{2\pi} \text{Tr} \left\{ \hat{\rho} \hat{\Omega}(\phi, n) \right\} = \text{Re} \{ \langle \phi | \hat{\rho} | n \rangle \langle n | \phi \rangle \}$$

(1.4)

where $\hat{\rho}$ is the density operator of a given quantum system. The operator $\hat{\Omega}(\phi, n)$ given by (1.1) can serve as a number-phase Stratonovich-Weyl quantizer. Namely, for any classical number-phase function $f = f(\phi, n)$ one can assign the respective operator $\hat{f}$ according to the rule (the generalized Weyl quantization)

$$\hat{f} := \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(\phi, n) \hat{\Omega}(\phi, n) d\phi.$$  

(1.5)

Employing the well known formulae

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1, \quad \int_{-\pi}^{\pi} |\phi\rangle \langle \phi| d\phi = 1$$

(1.6)

we quickly find that if $f = f(n)$

$$\hat{f} = f(\hat{n})$$

(1.7)

and if $f = f(\phi)$ then

$$\hat{f} = \int_{-\pi}^{\pi} f(\phi) |\phi\rangle \langle \phi| d\phi.$$  

(1.8)

For any $f = f(\phi, n)$ with the corresponding operator $\hat{f}$ defined by (1.5) the expectation value $\langle \hat{f} \rangle$ in a state $\hat{\rho}$ is

$$\langle \hat{f} \rangle = \text{Tr} \left\{ \hat{\rho} \hat{f} \right\} = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(\phi, n) g_W(\phi, n) d\phi.$$  

(1.9)

Finally, the marginal distributions read

$$g(n) := \int_{-\pi}^{\pi} g_W(\phi, n) d\phi = \langle n | \hat{\rho} | n \rangle,$$  

(1.10a)

$$g(\phi) := \sum_{n=0}^{\infty} g_W(\phi, n) = \langle \phi | \hat{\rho} | \phi \rangle.$$  

(1.10b)

In particular for any function $f = f(\phi)$ Eqs (1.9) and (1.10b) give

$$\langle \hat{f} \rangle = \int_{-\pi}^{\pi} f(\phi) g(\phi) d\phi =: \langle f(\phi) \rangle$$  

(1.11)
with $\hat{f}$ given by (1.8). As it has been pointed out by many authors [2–6] the formula (1.11) gives the expectation value for the quantum phase-function $f(\phi)$ equal to the one calculated within the Pegg-Barnett approach to quantum phase [7–9]. Consequently, our choice for a number-phase Wigner function (1.4) seems to be appropriate when the Pegg-Barnett formulation is considered. A crucial question one should answer is whether the given number-phase Wigner function $V_W(\phi, n)$ defines uniquely the respective density operator $\hat{\rho}$. In Ref. [2] we have suggested that this is indeed the case (see Corrigendum in [2]) but we were not able to give any explicit formula therein. We fill this gap in Section 2 of the present paper. Another important issue which is explored in Section 3 is the relation between $V_W(\phi, n)$ and other well known in quantum optics quasiprobability distributions such as the Glauber-Sudarshan distribution $P(\alpha)$, the Husimi function $Q(\alpha)$ or the Wigner function $W(\alpha)$. We find the formula which enables one to obtain the number-phase Wigner function $V_W(\phi, n)$ from those distributions and, in general from the Cahill-Glauber function $W^{(s)}(\alpha)$, $-1 \leq s \leq 1$. We hope that the results of the present work show that the function $V_W(\phi, n)$ can be considered as some useful quasiprobability distribution depending on the photon number $n$ and on the phase $\phi$ (see Section 4 where some conclusions are given).

2 From the number-phase Wigner function $V_W(\phi, n)$ to the respective density operator $\hat{\rho}$

First, we recall some properties of the celebrated Susskind-Glogower (SG) phase operators $\hat{e}^{i\phi}$ and $\hat{e}^{-i\phi}$ [10,11,4,5,12,13]. They can be defined as

$$\hat{e}^{i\phi} = \int_{-\pi}^{\pi} e^{i\phi} \left| \phi \right\rangle \left\langle \phi \right| d\phi = \sum_{n=0}^{\infty} \left| n \right\rangle \left\langle n+1 \right|$$

and

$$\hat{e}^{-i\phi} = \int_{-\pi}^{\pi} e^{-i\phi} \left| \phi \right\rangle \left\langle \phi \right| d\phi = \sum_{n=0}^{\infty} \left| n+1 \right\rangle \left\langle n \right| = \left( \hat{e}^{i\phi} \right)^{\dagger}.$$ (2.1)

One can easily show that the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators can be expressed in terms of the SG operators as follows

$$\hat{a} = \sqrt{n+1} \hat{e}^{i\phi} = \hat{e}^{i\phi} \sqrt{n}$$

and

$$\hat{a}^\dagger = \hat{e}^{-i\phi} \sqrt{n+1} = \sqrt{n} \hat{e}^{-i\phi}.$$ (2.2)

For any operator depending on $\hat{n} f = f(\hat{n})$ one gets

$$\hat{e}^{i\phi} f(\hat{n}) = f(\hat{n}+1)\hat{e}^{i\phi}$$

$$\hat{e}^{-i\phi} f(\hat{n}+1) = f(\hat{n})\hat{e}^{-i\phi}.$$ (2.5)
(compare with \(2.3\) ad \(2.4\)). From \((2.1)\) and \((2.2)\) we quickly have
\[
\hat{e}^{\phi} |n\rangle = |n - 1\rangle, \quad \hat{e}^{-\phi} |n\rangle = |n + 1\rangle.
\]
So \(\hat{e}^{\phi}\) and \(\hat{e}^{-\phi}\) are certain lowering and raising operators, respectively. Finally, employing \((2.1), (2.2)\) and the general definition \((1.8)\) one finds the relation
\[
(\hat{e}^{\phi})^{k} (\hat{e}^{-\phi})^{m} = \begin{cases} \hat{e}^{\phi}^{k-m}, & k \geq m \\ \hat{e}^{-\phi}^{m-k}, & k < m \end{cases} \quad \int_{-\pi}^{\pi} e^{i(k-m)\phi} |\phi\rangle \langle \phi| \, d\phi = \hat{e}^{i(k-m)\phi}. \tag{2.7}
\]
Using \((2.3), (2.4), (2.5)\) and \((2.7)\), after performing straightforward calculations we get
\[
\hat{a}^{m} = \sqrt{\hat{n}+1}(\hat{n}+2)\ldots(\hat{n}+m) (\hat{e}^{\phi})^{m} \\
(\hat{a}^{\dagger})^{m} = (\hat{n}+1)(\hat{n}+2)\ldots(\hat{n}+m) \sqrt{\hat{n}+1}(\hat{n}+2)\ldots(\hat{n}+m) \tag{2.8}
\]
We express the density operator \(\hat{\rho}\) in the form
\[
\hat{\rho} = \sum_{m,k=0}^{\infty} A_{mk} \hat{a}^{m} (\hat{a}^{\dagger})^{k} \tag{2.9}
\]
i.e. as a series of operators constructed from \(\hat{a}\) and \(\hat{a}^{\dagger}\) in anti-normal ordering. Inserting \((2.8)\) into \((2.9)\) and using the last equality of \((2.7)\) one obtains
\[
\hat{\rho} = \sum_{m=0}^{\infty} \left\{ \varrho_{m}(\hat{n}) e^{im\phi} + \varrho_{m}^{*}(\hat{n}) e^{-im\phi} \right\} \tag{2.10}
\]
where \(\varrho_{m}(\hat{n})\), \(m = 0, 1, \ldots\), are some operators, depending on \(\hat{n}\). Substituting \((2.10)\) into \((1.4)\) we get the number-phase Wigner function \(\varrho_{W}(\phi, n)\) in the form
\[
\varrho_{W}(\phi, n) = \frac{1}{2\pi} \mathrm{Re} \left\{ \sum_{m=0}^{\infty} \varrho_{m}(n-m) e^{im\phi} + \sum_{m=0}^{\infty} \varrho_{m}^{*}(n) e^{-im\phi} \right\} \\
= \frac{1}{4\pi} \left\{ \sum_{m=0}^{n} \left[ (\varrho_{m}(n-m) + \varrho_{m}^{*}(n-m)) e^{im\phi} + (\varrho_{m}(n-m) + \varrho_{m}^{*}(n-m)) e^{-im\phi} \right] \right. \tag{2.11}
\]
\[
+ \sum_{m=n+1}^{\infty} \left( \varrho_{m}(n) e^{im\phi} + \varrho_{m}^{*}(n) e^{-im\phi} \right) \right\}.
\]
From (2.11) one immediately has

\[
\varrho_m(n) = 2 \int_{-\pi}^{\pi} \varrho_W(\phi, n) e^{-im\phi} d\phi, \quad m \geq n + 1, \quad (2.12a)
\]

\[
\varrho_m(n-m) + \varrho_m(n) = 2 \int_{-\pi}^{\pi} \varrho_W(\phi, n) e^{-im\phi} d\phi, \quad 1 \leq m \leq n, \quad (2.12b)
\]

\[
\varrho_0(n) + \varrho_0^*(n) = \int_{-\pi}^{\pi} \varrho_W(\phi, n) d\phi \quad (2.12c)
\]

and we quickly conclude that to have all ingredients determining \( \hat{\varrho} \) from a given \( \varrho_W(\phi, n) \)
we need \( \varrho_m(n) \) for \( 1 \leq m \leq n \) (see (2.12b)). Simple but rather tedious inductive analysis
of (2.12a) and (2.12b) leads to the result

\[
\varrho_m(n) = 2 \int_{-\pi}^{\pi} \left( \sum_{l=0}^{\left[ \frac{n}{m} \right]} (-1)^l \varrho_W(\phi, n - lm) \right) e^{-im\phi} d\phi, \quad 1 \leq m \leq n. \quad (2.13)
\]

Observe that the formula (2.13) holds true also for \( m \geq n + 1 \) as if \( m \geq n + 1 \) then Eq. (2.13) gives exactly (2.12a). Gathering, we arrive at the conclusion

\[
\hat{\varrho} = \varrho_0(\hat{n}) + (\varrho_0(\hat{n}))^\dagger + \sum_{m=1}^{\infty} \left\{ \varrho_m(\hat{n}) e^{im\phi} + e^{-im\phi} (\varrho_m(\hat{n}))^\dagger \right\} \quad (2.14)
\]

where

\[
\varrho_0(\hat{n}) + (\varrho_0(\hat{n}))^\dagger = \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \varrho_W(\phi, n) d\phi \right\} \langle n \rangle \langle n \rangle \quad (2.15a)
\]

\[
\varrho_m(\hat{n}) = \sum_{n=0}^{\infty} \left\{ 2 \int_{-\pi}^{\pi} \left( \sum_{l=0}^{\left[ \frac{n}{m} \right]} (-1)^l \varrho_W(\phi, n - lm) \right) e^{-im\phi} d\phi \right\} \langle n \rangle \langle n \rangle, \quad m \geq 1. \quad (2.15b)
\]

The formulae (2.14), (2.15a) and (2.15b) show that the density operator \( \hat{\varrho} \) is defined uniquely by the respective number-phase Wigner function \( \varrho_W(\phi, n) \). From (2.14) with (2.15a), (2.15b) and (2.6) one can derive the following useful relations

\[
\varrho_m(n) = \langle n | \varrho_m(\hat{n}) | n \rangle = \langle n | \hat{\varrho} | n + m \rangle, \quad m \geq 1,
\]

\[
\varrho_0(n) + \varrho_0^*(n) = \langle n | \varrho_0(\hat{n}) + (\varrho_0(\hat{n}))^\dagger | n \rangle = \langle n | \hat{\varrho} | n \rangle. \quad (2.16)
\]

**Example:** As a simple example consider the coherent phase states [5][14] represented by the kets

\[
| \zeta \rangle = (1 - | \zeta |^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \zeta^n | n \rangle, \quad | \zeta | < 1 \quad (2.17)
\]

which are the normalised eigenvectors of the lowering operator \( \hat{e}^{i\phi} \)

\[
\hat{e}^{i\phi} | \zeta \rangle = \zeta | \zeta \rangle, \quad \langle \zeta | \zeta \rangle = 1. \quad (2.18)
\]
The respective density operator \( \hat{\rho} \) reads

\[
\hat{\rho} = |\zeta\rangle \langle \zeta| = (1 - |\zeta|^2) \sum_{k,l=0}^{\infty} |\zeta|^{k+l} e^{i(k-l)\phi} |k\rangle \langle l| \tag{2.19}
\]

where \( \zeta = |\zeta| e^{i\varphi} \). Inserting (2.19) into (1.4) one easily gets the corresponding number-phase Wigner function

\[
\hat{\rho}_W(\phi, n) = \frac{(1 - |\zeta|^2)|\zeta|^n}{2\pi} \sum_{k=0}^{\infty} |\zeta|^k \cos [(n - k)(\phi - \varphi)]
\]

\[
= \frac{(1 - |\zeta|^2)|\zeta|^n}{2\pi} \left\{ \cos [n(\phi - \varphi)] \sum_{k=0}^{\infty} |\zeta|^k \cos [k(\phi - \varphi)] \right. \\
+ \sin [n(\phi - \varphi)] \sum_{k=0}^{\infty} |\zeta|^k \sin [k(\phi - \varphi)] \right\} 
\tag{2.20}
\]

[Compare with the number-phase Wigner function for the coherent state \(|\alpha\rangle\) (see Eq. (6.6) of [2]).] Figure 1 shows exemplary plots of the number-phase Wigner function for the coherent phase state with \( \phi = 0 \) and various values of \( n \) and \( |\zeta| \). Note that for \( n = 0 \) the \( \hat{\rho}_W(\phi, n) \) is positive for every value of \( \zeta \) (see Figure 1a).

Substituting (2.19) into (2.16) we find

\[
\hat{\rho}_m(n) = (1 - |\zeta|^2)|\zeta|^{2n+m} e^{-im\varphi}, \quad m \geq 1
\]

\[
\hat{\rho}_0(n) + \hat{\rho}_0(n)^* = (1 - |\zeta|^2)|\zeta|^{2n}.
\tag{2.21}
\]

Hence

\[
\hat{\rho}_m(\hat{n}) = \sum_{n=0}^{\infty} \hat{\rho}_m(n) |n\rangle \langle n| \\
= (1 - |\zeta|^2)|\zeta|^{m} e^{-im\varphi} \sum_{n=0}^{\infty} |\zeta|^{2n} |n\rangle \langle n|, \quad m \geq 1
\tag{2.22}
\]

\[
\hat{\rho}_0(\hat{n}) + (\hat{\rho}_0(\hat{n}))^\dagger = (1 - |\zeta|^2) \sum_{n=0}^{\infty} |\zeta|^{2n} |n\rangle \langle n|.
\]

It is an easy matter to show that inserting (2.22) into (2.14) and employing (2.6) one gets the density operator (2.19).

3 Relation between \( \hat{\rho}_W(\varphi, n) \) and other quasiprobability distributions.

We begin our considerations with searching for a relation between \( \hat{\rho}_W(\varphi, n) \) and the celebrated Glauber-Sudarshan function \( P(\alpha) \) defined by [15–18]

\[
\hat{\rho} = \int P(\alpha) |\alpha\rangle \langle \alpha| d^2\alpha, \quad \alpha \in \mathbb{C}, d^2\alpha = d\text{Re}\alpha \cdot d\text{Im}\alpha
\tag{3.1}
\]
where \(|\alpha\rangle\) stands for the normalised ket vector of the coherent state

\[
\hat{a} |\alpha\rangle = \alpha |\alpha\rangle ,
\]

\[
|\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,
\]

\[
\langle \alpha|\alpha\rangle = 1, \quad \langle \alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-\alpha^*\beta)}, \quad \alpha, \beta \in \mathbb{C}.
\]

From the last equation of (3.2) one quickly concludes that two different coherent states are non orthogonal. However, as is well known the set \(\{|\alpha\rangle\}_{\alpha \in \mathbb{C}}\) gives a resolution of the identity operator

\[
\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2\alpha = \hat{1}.
\]

Assuming \(\hat{\phi}\) in the form of (2.9) and using (3.3) and then the first formula of (3.2) one gets

\[
\hat{\phi} = \frac{1}{\pi} \int \sum_{m,k=0}^{\infty} A_{mk} \hat{a}^m |\alpha\rangle \langle \alpha| \hat{(a^\dagger)}^k d^2\alpha
\]

\[
= \int \left( \frac{1}{\pi} \sum_{m,k=0}^{\infty} A_{mk} \alpha^m (\alpha^*)^k \right) |\alpha\rangle \langle \alpha| d^2\alpha.
\]
Comparing (3.1) and (3.4) we immediately have

$$\mathcal{P}(\alpha) = \frac{1}{\pi} \sum_{m,k=0}^{\infty} A_{mk} \alpha^m (\alpha^*)^k.\quad (3.5)$$

To express \(q_w(\phi, n)\) by \(\mathcal{P}(\alpha)\) we insert (3.1) into (1.4) and we obtain

$$q_w(\phi, n) = \frac{1}{2\pi} \int \mathcal{P}(\alpha) \text{Tr} \left\{ |\alpha\rangle \langle \alpha| \hat{\Omega}(\phi, n) \right\} d^2\alpha \quad (3.6)$$

$$= \text{Re} \left\{ \int \mathcal{P}(\alpha) \langle \phi|\alpha\rangle \langle n|\phi\rangle d^2\alpha \right\}.\quad (3.7)$$

Employing (1.3) and (3.2), and performing simple calculations one gets

$$q_w(\phi, n) = \frac{1}{2\pi \sqrt{n!}} \int \mathcal{P}(\alpha)|\alpha|^{n} e^{-|\alpha|^2} \sum_{k=0}^{\infty} \frac{|\alpha|^k}{\sqrt{k!}} \cos [(n-k)(\phi - \gamma)] d^2\alpha$$

where \(\alpha = |\alpha|e^{i\gamma}, d^2\alpha = |\alpha|d|\alpha| \cdot d\gamma\).

Now we are going to study the relation between \(q_w(\phi, n)\) and the Cahill-Glauber function \(W_s(\alpha)\) defined as [19,18]

$$W_s(\alpha) = \frac{1}{\pi} \text{Tr} \left\{ \hat{Q}(s) \right\}, \quad s \in [-1, 1], \quad \alpha \in \mathbb{C}\quad (3.8)$$

where the operator \(\hat{Q}(s)\) is given by

$$\hat{Q}(s) = \frac{1}{\pi} \int e^{s\xi^* - \alpha^* \xi^s + \frac{s}{2} |\xi|^2} \hat{D}(\xi) d^2\xi, \quad \xi \in \mathbb{C}, \quad d^2\xi = d\text{Re}\xi \cdot d\text{Im}\xi\quad (3.9)$$

with \(\hat{D}(\xi)\) standing for the displacement operator

$$\hat{D}(\xi) = e^{\xi\hat{a}^\dagger - \alpha^* \hat{a}} = e^{-\frac{s}{2} |\xi|^2} e^{\xi\hat{a}^\dagger} e^{-\hat{a}^\dagger \hat{a}} = e^{\frac{s}{2} |\xi|^2} e^{-\hat{a}^\dagger \hat{a}} e^{-\hat{a}^\dagger \hat{a}}.\quad (3.10)$$

Note that for \(s = -1\), the Cahill-Glauber function \(W_{(-1)}(\alpha)\) is usually denoted by \(Q(\alpha)\) and it is well known in quantum optics as the Husimi function; for \(s = 0\), \(W^{(0)}(\alpha)\) is the famous Wigner function and for \(s = 1\) we get \(W^{(1)} = \mathcal{P}(\alpha)\) [17,21]. Given \(W_s(\alpha)\), \(-1 \leq s \leq 1\), one can find the respective density operator \(\hat{Q}\) from the following formula (see for instance [18,19])

$$\hat{Q} = \int W_s(\alpha) \hat{T}(-s)(\alpha) d^2\alpha.\quad (3.11)$$

Substituting (3.11) into (1.4) we get

$$q_w(\phi, n) = \frac{1}{2\pi} \int W_s(\alpha) \text{Tr} \left\{ \hat{T}(-s)(\alpha) \hat{\Omega}(\phi, n) \right\} d^2\alpha$$

$$= \frac{1}{4\pi} \int W_s(\alpha) \sum_{k=0}^{\infty} \left\{ \langle k|\hat{T}(-s)(\alpha)|n\rangle e^{i(n-k)\phi} + \langle n|\hat{T}(-s)(\alpha)|k\rangle e^{-i(n-k)\phi} \right\} d^2\alpha.\quad (3.12)$$
The matrix elements of $\hat{T}^{(-s)}(\alpha)$ are given by (see e.g. \cite{18,19})
\[ 
\langle j | \hat{T}^{(-s)}(\alpha) | k \rangle = \langle k | \hat{T}^{(-s)}(\alpha) | j \rangle^* = \sqrt{\frac{j!}{k!}} \left( \frac{2}{s+1} \right)^{k+j+1} \left( \frac{s-1}{s+1} \right)^j (\alpha^*)^{k-j} 
\cdot \exp \left( -\frac{2|\alpha|^2}{s+1} \right) L_j^{k-j} \left( \frac{4|\alpha|^2}{1-s^2} \right) 
\]
where $L_j^{k-j}$ stands for the associated Laguerre polynomial. Inserting (3.13) into (3.12), after some simple manipulations one arrives at the formula
\[ 
\varrho_W(\phi, n) = \frac{\sqrt{n!}(s-1)^n}{2^n(s+1)\pi} \int_0^\infty d|\alpha| \int_0^\pi d\gamma \mathcal{W}^{(s)}(|\alpha|e^{i\gamma})|\alpha'|^{-n+1} \exp \left( -\frac{2|\alpha|^2}{s+1} \right) 
\cdot \sum_{k=0}^\infty \frac{1}{\sqrt{k!}} \left( \frac{2|\alpha|}{s+1} \right)^k L_{k-n}^{k-n} \left( \frac{4|\alpha|^2}{1-s^2} \right) \cos[(k-n)(\phi-\gamma)] 
\]
where we put $\alpha = |\alpha|e^{i\gamma}$.

Although the relation (3.14) is rather involved it shows that, in principle, given the Cahill-Glauber function $\mathcal{W}^{(s)}(\alpha)$, $-1 \leq s \leq 1$, we can find the respective number-phase Wigner function $\varrho_W(\phi, n)$. Of course, since $\varrho_W(\phi, n)$ defines uniquely the density operator $\hat{\rho}$ (see section 2) and $\hat{\rho}$ defines $\mathcal{W}^{(s)}(\alpha)$ by (3.8) one concludes that given $\varrho_W(\phi, n)$ we can find the respective $\mathcal{W}^{(s)}(\alpha)$. However the relevant formula seems to be extremely involved.

4 Conclusions

We conclude that the proposed number-phase Wigner function $\varrho_W(\phi, n)$ can be used as a quasiprobability distribution in quantum optics. The natural question is what are the relations between $\varrho_W(\phi, n)$ and other number-phase Wigner functions introduced previously by several authors \cite{12,22,23}. Another interesting question is if our consideration of the number-phase Wigner function can be carried over to the case of a quantum system with finite dimensional space of states. If it can be done then the respective procedure would provide us with an approach to the Weyl-Wigner-Moyal formalism in quantum mechanics of discrete systems alternative to the one given by S. Chaturvedi et al \cite{23}. We are going to consider these problems very soon.

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