Bounding the edge cover of a hypergraph

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Abstract

Let $H = (V, E)$ be a hypergraph. Let $C \subseteq E$, then $C$ is an edge cover, or a set cover, if $\cup_{e \in C} \{v | v \in e\} = V$. A subset of vertices $X$ is independent in $H$, if no two vertices in $X$ are in any edge. Let $c(H)$ and $\alpha(H)$ denote the cardinalities of a smallest edge cover and largest independent set in $H$, respectively. We show that $c(H) \leq \hat{m}(H)c(H)$, where $\hat{m}(H)$ is a parameter called the mighty degeneracy of $H$. Furthermore, we show that the inequality is tight and demonstrate the applications in domination theory.

1 Introduction

We assume the reader is familiar with standard graph theory [5], hypergraph theory [1], [3], domination theory [11], and algorithm analysis [6]. Throughout this paper we denote by $H = (V, E)$ a hypergraph on vertex set $V$ and the edge set $E$. So any $e \in E$ is a subset of $V$. We do not allow multiple edges in our definition of a hypergraph, unless explicitly stated. Every hypergraph can be represented by its incidence bipartite graph $B$ whose vertex set is $V \cup E$. If $x \in V$ and $e \in E$, then $xe$ is an edge in $B$, provide that $x \in e$. Let $C \subseteq E$, then $C$ is an edge cover, or a set cover, if $\cup_{e \in C} \{v | v \in e\} = V$. A subset of vertices $X$ is independent in $H$, if no two vertices in $X$ are in any edge. Let $c(H)$ and $\alpha(H)$ denote the cardinalities of a largest independent set and a smallest edge cover in $H$, respectively. It is known that computing $\alpha(H)$ and $c(H)$ are NP hard problems [10]. Clearly, $c(H) \geq \alpha(H)$. Furthermore, it is known that $c(H)$ can not bounded above by a function of $\alpha(H)$, only. However, an important result in this area is known. Specifically, it is a consequence of the result in [9] that

$$c(H) = \alpha(H)^{O(2^v)}$$

where $v$ denotes the vc dimension of $H$ [16]. Design of approximation algorithms for the edge cover problem has been an active and ongoing research in computer science. A greedy algorithm [7], [13] is known to approximate $c(H)$ within $O(\log(n))$ from its optimal value. Moreover, there are examples of hypergraphs that show the worst case approximation scenario of $O(\log(n))$ can not be improved [4].

The main result of this paper is to show that

$$c(H) \leq \hat{m}(H)\alpha(H)$$

where the multiplicative factor $\hat{m}(H)$ is a parameter called the mighty degeneracy of $H$ which we introduce here. Recall that a set $S \subseteq V$ is a transversal set (hitting set) in the hypergraph $H = (V, E)$, if every $e \in E$ has a vertex in $S$. A set $M \subseteq E$ is a matching in $H$, if every two edges in $M$ are disjoint. Let $\tau(H)$ and $\rho(H)$ denote the sizes of a smallest transversal and a largest matching in $H$, respectively, and note that $\tau(H) \geq \rho(H)$.

A direct consequence of (2) is that

$$\tau(H) \leq \hat{m}(H^{d})\rho(H)$$

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where \( \hat{m}(H^d) \) is the mighty degeneracy of the dual hypergraph of \( H \), defined as \( H^d = \{ E, V \} \).

This paper is organized as follows. In Section Two we introduce some terms and concepts and set up our notations. Particularly, we introduce the strong degeneracy of a hypergraph, denoted by \( \hat{s}(H) \), which is an upper bound on \( \hat{m}(H) \). In Section Three we derive (2) which is the main result, and also present a linear time algorithm for computing \( \hat{s}(H) \). Section Four contains the applications to domination theory of graphs. Specifically, we show \( \hat{s}(H) = 1 \) (and hence \( \hat{m}(H) = 1 \)), when the underlying graph \( G \) is a tree and \( H \) is the so called closed or open neighborhood hypergraph of \( G \). Consequently, we provide new proofs (and algorithms) for two classical results in domination theory [14], [15], by showing that in any tree the size of a smallest dominating (total domination) set equals to the size of a largest 2-packing (open 2-packing). The results in Section Four are conveniently derived utilizing concept of strong degeneracy, instead of mighty degeneracy, however generally speaking, the former can be much larger than the latter. In Section Five we give examples of hypergraphs with bounded mighty degeneracy, whose strong degeneracy is a linear function of number of vertices. Section Six contains our suggestions for future research.

2 Preliminaries

Let \( H = (V, E) \), let \( S \subseteq V \) and \( e \in E \), then \( e \cap S \) is the trace of \( e \) on \( S \). The restriction of \( H \) to \( S \), denoted by \( H[S] \), is the hypergraph on vertex set \( S \) whose edges are the set of all distinct traces of edges in \( E \) on \( S \). \( H[S] \) is also referred to as the induced subhypergraph of \( H \) on \( S \). In general, a hypergraph \( H \) is a subhypergraph of \( H \), if it can be obtained by removing some vertices and some edges from \( H \). \( S \) is shattered in \( H \), if any \( X \subseteq S \) is a trace. Thus if \( S \) is shattered, then it has \( 2^{|S|} \) traces. The Vapnik–Chervonenkis (VC) dimension of a hypergraph \( H \), denoted by \( vc(H) \), is the cardinality of the largest subset of \( V \) which is shattered in \( H \). Let \( H = (V, E) \) and let \( x \in V \). The degree of \( x \) denoted by \( d_H(x) \) is the number of edges that contain \( x \). The strong degree of \( x \) in \( H \), denoted by \( s_H(x) \), is the number of distinct maximal edges that contain \( x \). An edge is maximal, if it is not properly contained in another edge.) Let \( \delta(H) \) and \( s(H) \) denote the smallest degree and smallest strong degree, respectively, of any vertex in \( H \). The degeneracy and strong degeneracy of \( H \), denoted by \( \hat{\delta}(H) \) and \( \hat{s}(H) \), respectively, are the largest minimum degree and largest minimum strong degree of any induced subhypergraph of \( H \). Let \( R \subseteq S \). A strong subset of \( V \) in \( H \) is a non empty subset of \( V \) which is obtained by removing all vertices in \( R \) from \( H \), as well as all vertices in the edges that have nonempty intersection with \( R \) and all vertices in such edges.

The mighty degeneracy of \( H \), denoted by \( \hat{m}(H) \), is the largest minimum strong degree of any strong subhypergraph of \( H \). Clearly, for any \( x \in V \) one has \( s_H(x) \leq d_H(x) \) and consequently

\[
\hat{m}(H) \leq \hat{s}(H) \leq \hat{\delta}(H) 
\]  

3 Our Greedy Algorithms

Our next result is the main result of this paper.

**Theorem 3.1.** Let \( H = (V, E) \) be a hypergraph, then there is an edge cover \( C \), and an independent set \( X \) in \( H \) so that

\[
|C| \leq \hat{m}(H)|X| 
\]  

Consequently

\[
|C| \leq \hat{s}(H)|X| 
\]  

\[1\] When a vertex set is removed from \( H \), the edges of \( H \) will also be updated accordingly.
Moreover, $X$ and $C$ can be constructed in $O(|V| + \sum_{e \in E} |e|)$ time.

**Proof.** Consider the following algorithm. Initially, set $i \leftarrow 1$, $I \leftarrow H, W \leftarrow V$ and $K \leftarrow E$. While there are vertices in $W$ repeat the following steps: Remove the vertex of minimum strong degree, denoted by $x$, from $W$, remove the set of all distinct maximal edges containing $x$, from $K$, then, remove and the set of all vertices contained in these edges from $W$ and finally set $i \leftarrow i + 1$.

Clearly, the algorithm terminates. Now let $t$ be the number of iterations of the algorithm and at any iteration $i = 1, 2, \ldots, t$, let $I_i$ denote the constructed hypergraph, (which is strongly induced), and let $W_i$ (which is a strong subset) and $K_i$ denote, respectively, the vertices and edges of this hypergraph. Let $X = \{x_1, x_2, \ldots, x_t\}$ be the set of all vertices removed from $H$ when the algorithm terminates. Clearly, $X$ is an independent set in $H$. We denote by $K_{x_i}$ the set of all distinct maximal edges containing the vertex $x_i$ in the hypergraph $I_i$ at iteration of $i$ of the algorithm and note that $|K_{x_i}| \leq \hat{m}(H)$, since $x_i$ is the vertex of minimum strong degree in $I_i$. Consequently,

$$\sum_{i=1}^{t} |K_{x_i}| \leq \hat{m}(H) \times t = \hat{m}(H) \times |X| \quad (7)$$

Now for $i = 1, 2, \ldots, t$, let $C_{x_i}$ be the set of all edges in $H$ obtained by extending each edge of $K_{x_i}$ in $I_i$ to an edge in $H$ and let $C = \cup_{i=1}^{t} C_{x_i}$. Clearly, $C$ is an edge cover and furthermore $|C| = \lvert \cup_{i=1}^{t} F_{x_i} \rvert$, and therefore the first claim follows from (7). To verify the second inequality note that $\hat{m}(H) \leq \hat{s}(H)$. We omit the details of claims regarding time complexity that involves representing $H$ as a bipartite graph. \qed

To use Theorem 3.1 we really need to know $\hat{m}(H)$. Alternatively, we can use $\hat{s}(H)$ which is an upper bound for $\hat{m}(H)$. At this time, we still do not know how to efficiency compute $\hat{m}(H)$. We finish this section by presenting a simple greedy algorithm for computing $\hat{s}(H)$ which is similar to the known algorithm for computing degeneracy of $H$, or $\hat{d}(H)$. The properties of the output of algorithm will be used to prove our results in the next section.

**Theorem 3.2.** Let $H = (V, E)$ be a hypergraph on $n$ vertices, then $\hat{s}(H)$ can be computed in $O(|V| + \sum_{e \in E} |e|)$ time.

**Proof.** Consider the following algorithm. For $i = 1, 2, \ldots, n$, select a vertex $x_i$ of of minimum strong degree $s_i = s(H_i)$ in the induced subhypergraph $H_i = H[V_i]$ whose vertex set is $V_i = V - \{x_1, x_2, \ldots, x_{i-1}\}$ and whose edge set is denoted by $E_i$. Let $s = \max\{s_i, i = 1, 2, \ldots, n\}$. We claim that $\hat{s}(H) = s$. Note that $\hat{s}(H) \geq s$. We will show that $\hat{s}(H) \leq s$. Now let $I = (W, F)$ be an induced subhypergraph of $H$ whose minimum strong degree equals $\hat{s}(H)$ and let $j, 1 \leq j \leq n$, be the smallest integer so that $x_j \in W$. Then $s_{f}(x_j) \leq s_j = s(H_j) \leq s$, since $W \subseteq E_j$ and consequently the claim is proved. To verify the claim for time complexity, one needs to represent $H$ as a bipartite graph $H$ as the input of algorithm. The details are omitted. \qed

### 4 Applications in domination theory

For a graph $G$ on vertex set $V$ and $x \in V$ let $N(x)$ denote the open neighborhood of $x$, that is the set of all vertices adjacent to $x$, not including $x$. The closed neighborhood of $x$ is $N[x] = N(x) \cup \{x\}$. The closed (open) neighborhood hypergraph of an $n$ vertex graph $G$ is a hypergraph on the same vertices as $G$ whose edges are all $n$ closed (open) neighborhoods of $G$. A subset of vertices $S$ in $G$ is a dominating set \cite{11}, if for every vertex $x$ in $G$, $N[x] \cap S \neq \emptyset$. $S$ is a total or open domination set if, $N(x) \cap S \neq \emptyset$. $S$ is a 2-packing (packing), if for any distinct pair $x, y \in S$, $N[x]$ and $N[y]$ do not intersect. $S$ is an open 2-packing(packing), if for any distinct
pair \( x, y \in S \), \( N(x) \) and \( N(y) \) do not intersect. Let \( \gamma(G), \gamma^0(G), \alpha_2(G) \) and \( \alpha_0^2(G) \) denote the sizes of a smallest dominating, a smallest open domination, a largest packing and a largest open packing, respectively, in \( G \). Computing \( \gamma(G), \gamma^0(G), \alpha_2(G) \) and \( \alpha_0^2(G) \) are known to be NP-hard. \( \gamma(G) \) can be approximated within a factor of \( O(\log(n)) \) times form its optimal solution in \( O(n + m) \) time, where \( n \) and \( m \) are the number of vertices and edges of \( G \). The approximation algorithm is arising from the approximation algorithm for the set cover problem\[13\] \[7\]. It is known that one can not improve the approximation factor of \( O(\log(n)) \) asymptotically.

Let \( G \) be a graph on vertex set \( V \). The closed neighborhood hypergraph, of \( G \) is a hypergraph on vertex set \( V \) and edge set \( \{N[x], x \in V\} \). The open neighborhood hypergraph of graph \( G \) is a hypergraph on the vertex set \( V \) and the edge set \( \{N(x), x \in V\} \). The following summarizes basic properties of neighborhood hypergraphs as they relate to our work.

**Observation 4.1.** Let \( H \) the closed neighborhood hypergraph of a graph \( G \) with the vertex set \( V \).

(i) Let \( S \subseteq V \), then \( S \) is a dominating set in \( G \) if and only if \( S \) is an edge cover in \( H \).

(ii) Let \( S \subseteq V \), then \( S \) is a packing in \( G \) if and only if \( S \) is an independent set in \( H \).

(iii) Let \( x \in V \), then \( s_H(x) \leq \deg(x) + 1 \), where \( \deg(x) \) is degree of \( x \) in \( G \). Consequently, \( \hat{s}(H) \leq \Delta(G) + 1 \), where \( \Delta(G) \) is the maximum degree of \( G \).

(iv) If \( G \) is a tree and \( x \in V \) is a leaf, then \( s_H(x) = 1 \).

**Remark 4.1.** Observation 4.1 is valid if \( H \) is the open neighborhood hypergraph of \( G \), with the exception that in item (iii), one has \( s_H(x) \leq \deg(x) \) and consequently \( \hat{s}(H) \leq \Delta(G) \).

By the above observation, if we apply the greedy algorithm in Theorem 3.1 to the neighborhood hypergraph of a graph \( G \), we obtain a dominating (total domination) set \( C \) and a packing (open packing) \( X \) so that \(|C| \leq \hat{s}(H)|X|\). To determine how small is \( C \), we need to estimate \( \hat{s}(H) \), for the hypergraph \( H \). As stated above, we only know \( \hat{s}(H) \leq \Delta(G) + 1 \), where \( \Delta(G) \) is the maximum degree of \( G \). For trees one can get a significantly better result.

Let \( T \) be a tree and let \( T_1 \) be a tree which is obtained after removing all leaves of \( T \). Then each leaf in \( T_1 \) is a support vertex in \( T \) (attached to a leaf) and is called a canonical support vertex in \( T \).

Next we derive two classical results in domination theory that were proved first proved in \[14\] and \[15\], respectively.

**Theorem 4.1.** Let \( T \) be a tree on the vertex set \( V \) whose closed and open neighborhood hypergraphs are \( H \) and \( H^o \), respectively. Then, the following hold.

(i) \( \hat{s}(H) = \hat{m}(H) = 1 \) and consequently \( \gamma(T) = \alpha_2(T) \).

(ii) \( \hat{s}(H^o) = \hat{m}(H^o) = 1 \) and consequently \( \gamma^0(T) = \alpha_0^2(T) \).

Moreover, the domination and packing sets can be obtained in \( O(V) \) time

**Proof.** We first verify that at each iteration of the greedy algorithm in Theorem 3.2 a vertex of strong degree one is detected. This shows \( \hat{s}(H) = 1 \). We then apply the greedy algorithm in Theorem 3.1 to obtain the equality of packing and domination numbers.

To prove the first the claim, note that algorithm in Theorem 3.2 can break the ties arbitrary. So assume that the algorithm selects the leaves in \( T \) which as stated in 4.1 have strong degree one in \( H \). Now Consider the execution of algorithm on Tree \( T_1 \) which is obtained after removing all leaves of \( T \). If \( T_1 \) is empty we are done, since all vertices have already had degree one. So assume \( T_1 \) is not empty.
Claim. Let $x$ be a leaf in $T_1$, then $s_I(x) = 1$, where $I$ is the closed induced neighborhood hypergraph which is obtained after removal of all leaves of $T$.

Proof of claim. Since $x$ is leaf in $T_1$, there is exactly one vertex $z$ adjacent to $x$ in $T_1$. Now let $Y \subseteq V$ be the set of leaves of $T$ adjacent to $x$ (in $T$) and $N_I[y]$ denote the closed neighborhood of $y \in Y$ in $I$ after removal of $y$. Then, we have $N_I[y] = x$. Additionally, note that $N_I[x] = \{x, z\} \subseteq N_I[z]$, since $x \in N_I[z]$, and consequently $s_I(x) = 1$.

Coming back to the proof, now let algorithm select leaves of $T_1$, then, delete all these leaves and continue the process with the tree obtained after removal of these leaves. This proves $\hat{s}(H) = 1$, consequently $\hat{m}(H) = 1$. Now run the algorithm in Theorem 3.1 on $T$ to prove $\gamma(T) = \alpha_2(T)$.

Proof of second the claim is similar to the first and is omitted. The claim on the time complexity follows from running times stated in Theorems 3.1, 3.2. □

5 The gap between $\hat{m}(H)$ and $\hat{s}(H)$

In the proof of Theorem 4.1, we were able to effectively use $\hat{s}(H)$ instead of $\hat{m}(H)$. However, in general this may not be possible since $\hat{s}(H)$ can be much larger than $\hat{m}(H)$ as demonstrated in the following.

Theorem 5.1. For any integer $n \geq 3$ there is an $n$ vertex hypergraph such that $\hat{m}(H) = 2$ and $\hat{s}(H) = n - 2$.

Proof. Let $G$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ composed of a clique on vertex set $\{v_2, v_3, \ldots, v_n\}$ so that vertex $v_1$ (which is not in the clique) is adjacent to vertex $v_2$ (which is in the clique). Now define a hypergraph $H = (V, E)$ with $E = N[v_1] \cup_{i=2}^{n} N(v_i)$. Note that

$$s_H(v_1) = 2$$

since $N[v_1]$ and $N(v_2)$ are maximal edges of $H$ containing $v_1$. It is also easy to verify that

$$s_H(v_2) = n - 1 \quad \text{and} \quad s_H(v_i) = n - 2 \quad \text{for} \quad i = 3, 4, \ldots, n$$

Next note that the only strong subset of $V$ in $H$ is $V$ itself and thus equations 8 and 9 imply $\hat{m}(H) = s_H(v_1) = 2$ as claimed for mighty degeneracy.

Now consider the induced hypergraph $I$ on vertex set $W = V - \{v_1\}$, whose edges are obtained by removing $v_1$ from those edges of $H$ that contains $v_1$ (these edges are $N[v_1]$ and $N(v_2)$). One can verify that

$$s_I(v_i) = n - 2 \quad \text{for} \quad i = 2, 3, \ldots, n$$

which implies $\hat{s}(H) = n - 2$ as claimed. □

6 Future Work

This paper contains our preliminary results and we suggest several directions for future research. It is not known to us yet, if $\hat{m}(H)$ can be computed in polynomial time or not. We suspect that a variation of the algorithm in Theorem 3.1 can actually compute $\hat{m}(H)$, but have not been able to prove it.

The connections between the $vc(H)$ and $\hat{m}(H)$ ($\hat{s}(H)$) needs to be explored further. Is it true that one can always be bounded by a function of the other?
The most recent results for approximation of $\gamma(G)$ (domination number of a graph $G$) in sparse graphs require solving the linear programming relaxations (fractional versions) of the problem and then rounding the solutions [2],[8]. For a recent survey see [12]. We suspect that proper modification of our method in Section Four would give similar results without the need to actually solve the linear programming problems.

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