Quaternion-Based Conversion Formulas for Kinematic Attitude of Floating Offshore Wind Turbines (FOWT)

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Abstract. A floater allowing large-angle motion supporting a large payload (wind turbine and nacelle) with large aerodynamic loads high above the water surface is a great challenge because of the raised center of gravity and large overturning moment. In this paper, the conversion formulas between Euler angles and quaternions were derived, the research offered an efficient methodology without singularity to compute large-angle rigid body rotations of a FOWT, which laid the foundation for quaternion-based attitude kinematic model introduced to describe the dynamic response of the FOWT system and further solution.

1. Introduction

As a kind of clean and renewable energy, offshore wind energy has attracted more and more attention all over the world. Among them, large power wind turbines and floating platforms are the two major technological trends in developing offshore wind energy industry [1]. Be relative to the grounding-type wind turbines, the whole floating structure is upset by ocean waves. Therefore, rotor surfaces of revolution of the wind turbines always vibrate [2], but there are extremely few examples which really confirmed those influence on oscillation of the floating structure by the rotation of the wind turbine.

The compliant floating wind turbine system can be considered as a multi-body system including platform, mooring lines, tower, rotor and nacelle, which are mechanically connected by the yaw bearing, hub, etc. [3]. The convention analytical method to simulate the dynamics motions of such a system would be the Newton–Euler (NE) equations or Euler–Lagrange (EL) equations [4]. The traditional NE equation is used for the dynamic analysis of general parallel manipulators and the Stewart-Gough platform [5], which inevitably leads to a large number of equations and less computational efficiency. The EL equation eliminates all the unwanted reaction forces at the outset, and it is usually more efficient [6]. However, because of the constraints imposed by the closed loop kinematic chains of a parallel manipulator, deriving explicit equations of motion in terms of a set of independent generalized coordinates is a prohibitive task [7]. Kane’s method combines the advantages of both the NE and EL methods. As the well-recognized wind turbine dynamics analysis software, the NREL FAST aero-elastic simulator [8] uses Kane’s method to derive the EOMs for the floating wind turbine system with rotations of platform less than20° [9]. A floating wind turbine system is typically subject to wind forcing, wave forcing and mooring forcing, each of which is coupled with the dynamic response of the structure. In the case of large-amplitude motion, this coupling is highly nonlinear and should not be ignored. Reasonable quantification of gyroscopic effects requires the establishment of the EOMs applicable to the large-amplitude rotation. Generally, the visual description of the FOWT system according to the Euler motion moment of attitude is the most easy to be understand [10].
However, the application of Kane method based on Euler angle description is usually limited by singularity, or “gimbal lock”. Any set of Euler angles where the second rotation makes the first and third rotational axes align causes a singularity [11].

Quaternion expressions are linear differential equations, which are simpler and higher calculation speed. Quaternion rotation will not lead to the so-called gimbal lock phenomenon. Since 1960s, with the rapid development of space technology and industrial robot technology, it gradually began to be widely used to describe the attitude motion of a rigid body and multi-rigid body system, especially the robot spacecraft attitude maneuver, etc. [12,13]. Since the attitude of the system is unique, the two expression models can be converted to each other. In this paper, we will focus on the transformation of Quaternion and Euler angles.

2. The Concept of Euler Angles and Quaternion

2.1. Euler Angles

The basic idea stated by Euler is that the relative orientation of two coordinate systems may be specified by a set of three independent angles. The proof presented by Euler, although directed toward showing that a particular set of three angles was adequate, parallels the following argument phrased in modern terms. Rotations about the coordinate axes are easy to define and work with. Rotation about the x-axis by angle $\varphi$ is

$$R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{bmatrix}$$

(1)

Rotation about the y-axis by angle $\theta$ is

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

(2)

Rotation about the z-axis by angle $\psi$ is

$$R_z(\psi) = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

2.2. Quaternion

A unit quaternion can be described as:

$$q = (q_0, q_1, q_2, q_3)^T = [q_w, q_x, q_y, q_z]^T$$

(4)

$$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1$$

(5)

We can associate a quaternion with a rotation around an axis by the following expression

$$q_0 = q_w = \cos\left(\frac{\alpha}{2}\right)$$

(6)
Where $\alpha$ is a simple rotation angle (the value in radians of the angle of rotation) and $\cos (\beta x)$, $\cos (\beta y)$ and $\cos (\beta z)$ are the "direction cosines" locating the axis of rotation (Euler's Theorem).

### 3. Euler Angles to Quaternion Conversion

#### 3.1. 3-1-3 Euler Angle Sequence

Characteristics: the solution of the equation is simple and it can describe the gyroscopic effect of blade more vividly, but the rigid body with the 3-1-3 Euler angle sequence to describe its rotation attitude will appear singularity when the tower is vertical (0 degree elevation angle)

\[
q = \cos \frac{\psi}{2} + \sin \frac{\psi}{2} k \Rightarrow q_0 = \cos \frac{\psi}{2}, q_1 = q_2 = 0, q_3 = \sin \frac{\psi}{2}
\]

\[
q' = \cos \frac{\phi}{2} + \sin \frac{\phi}{2} i \Rightarrow q_0' = \cos \frac{\phi}{2}, q_1' = q_2' = 0, q_3' = \sin \frac{\phi}{2}
\]

\[
q'' = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} j \Rightarrow q_0'' = \cos \frac{\theta}{2}, q_1'' = q_2'' = 0, q_3'' = \sin \frac{\theta}{2}
\]

\[
q^w = \begin{bmatrix} q_0^w \\ q_1^w \\ q_2^w \\ q_3^w \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} & 0 & 0 & -\sin \frac{\theta}{2} \\ 0 & \cos \frac{\phi}{2} & \sin \frac{\phi}{2} & 0 \\ 0 & -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \\ \sin \frac{\phi}{2} & 0 & 0 & \cos \frac{\phi}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\psi}{2} & -\sin \frac{\psi}{2} & 0 & 0 \\ 0 & \cos \frac{\psi}{2} & \sin \frac{\psi}{2} & 0 \\ 0 & -\sin \frac{\psi}{2} & \cos \frac{\psi}{2} & 0 \\ \sin \frac{\psi}{2} & 0 & 0 & \cos \frac{\psi}{2} \end{bmatrix}
\]

\[
\begin{bmatrix} 
\cos \frac{\theta}{2} \cos \frac{\phi}{2} \cos \frac{\psi}{2} - \sin \frac{\theta}{2} \cos \frac{\phi}{2} \sin \frac{\psi}{2} \\
\cos \frac{\theta}{2} \sin \frac{\phi}{2} \cos \frac{\psi}{2} + \sin \frac{\theta}{2} \sin \frac{\phi}{2} \sin \frac{\psi}{2} \\
-\sin \frac{\theta}{2} \sin \frac{\phi}{2} \cos \frac{\psi}{2} + \cos \frac{\theta}{2} \sin \frac{\phi}{2} \sin \frac{\psi}{2} \\
\sin \frac{\theta}{2} \cos \frac{\phi}{2} \cos \frac{\psi}{2} + \sin \frac{\theta}{2} \sin \frac{\phi}{2} \sin \frac{\psi}{2}
\end{bmatrix}
\]

#### 3.2. 1-2-3 Euler Angle Sequence

Advantages: the rigid body with the 1-2-3 Euler angle sequence to describe its rotation attitude is more consistent with the actual motion characteristics of FOWR pitch-roll-yaw.
\[ q = \cos \frac{\varphi}{2} + \sin \frac{\varphi}{2} i \Rightarrow q_0 = \cos \frac{\varphi}{2}, q_2 = q_3 = 0, q_1 = \sin \frac{\varphi}{2} \]

\[ q' = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} j \Rightarrow q'_0 = \cos \frac{\theta}{2}, q'_1 = 0, q'_2 = \sin \frac{\theta}{2} \] (12)

\[ q'' = \cos \frac{\psi}{2} + \sin \frac{\psi}{2} k \Rightarrow q''_0 = \cos \frac{\psi}{2}, q''_1 = q''_2 = 0, q''_3 = \sin \frac{\psi}{2} \]

\[
q'' = \begin{pmatrix}
q''_0 \\
q''_1 \\
q''_2 \\
q''_3
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{\psi}{2} & 0 & 0 & -\sin \frac{\psi}{2} \\
0 & \cos \frac{\psi}{2} & \sin \frac{\psi}{2} & 0 \\
0 & -\sin \frac{\psi}{2} & \cos \frac{\psi}{2} & 0 \\
\sin \frac{\psi}{2} & 0 & 0 & \cos \frac{\psi}{2}
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\theta}{2} & 0 & -\sin \frac{\theta}{2} & 0 \\
0 & \cos \frac{\theta}{2} & 0 & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & 0 & \cos \frac{\theta}{2} & 0 \\
0 & \sin \frac{\theta}{2} & 0 & \cos \frac{\theta}{2}
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\varphi}{2} \\
\sin \frac{\varphi}{2} \\
0 \\
0
\end{pmatrix}
\]

(13)

In summary, the 3-1-3 sequence and the 1-2-3 sequence can be written by Quaternion as follows:

\[ q_0 = q_0(\varphi, \psi, \theta) \]
\[ q_1 = q_1(\varphi, \psi, \theta) \]
\[ q_2 = q_2(\varphi, \psi, \theta) \]
\[ q_3 = q_3(\varphi, \psi, \theta) \]

(14)

\[ q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \]

The initial conditions of motion represented with Euler angles can be transformed into the initial conditions of motion expressed with Quaternion using the upper form.

4. Quaternion to Euler Angles Conversion

4.1. 3-1-3 Euler Angle Sequence

The attitude matrix of 3-1-3 Euler angle sequence is as follows:

\[
R_z(\psi)R_y(\varphi)R_z(\theta) =
\begin{pmatrix}
\cos \psi \cos \theta - \sin \psi \sin \theta \cos \varphi & \cos \psi \sin \theta + \sin \psi \cos \varphi \cos \theta & \sin \psi \sin \varphi \\
-\sin \psi \cos \theta - \cos \psi \cos \varphi \sin \theta & -\sin \psi \sin \theta + \cos \psi \cos \varphi \cos \theta & \cos \psi \sin \varphi \\
\sin \varphi \sin \theta & -\sin \varphi \cos \theta & \cos \varphi
\end{pmatrix}
\]

(15)

The attitude matrix of Quaternion is as follows:
\[
A = \begin{bmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) & 2(q_1q_3 + q_0q_2) \\
2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_1 - q_0q_3) \\
2(q_1q_3 - q_0q_2) & 2(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{bmatrix}
\]

(16)

According to the corresponding relation of attitude matrix,

\[
\cos \varphi = A(3,3) = q_0^2 - q_1^2 - q_2^2 + q_3^2, \quad \varphi = \arccos(q_0^2 - q_1^2 - q_2^2 + q_3^2)
\]

(17)

\[
\frac{A(3,1)}{A(3,2)} = \tan \theta = \frac{2(-q_2q_3 + q_0q_1)}{2(q_2q_3 + q_0q_1)}, \quad \theta = \arctan\left(\frac{-q_2q_3 + q_0q_1}{q_2q_3 + q_0q_1}\right)
\]

(18)

\[
\frac{A(1,3)}{A(2,3)} = \tan \psi = \frac{2(q_1q_3 - q_0q_2)}{2(q_2q_1 + q_0q_3)}, \quad \psi = \arctan\left(\frac{q_1q_3 - q_0q_2}{q_2q_1 + q_0q_3}\right)
\]

(19)

4.2. 1-2-3 Euler Angle Sequence

The attitude matrix of 1-2-3 Euler angle sequence is as follows:

\[
R_z(\psi)R_y(\theta)R_x(\varphi) = \begin{bmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \varphi \sin \theta \cos \psi - \cos \varphi \sin \psi & \sin \varphi \sin \theta \cos \psi + \cos \varphi \cos \psi & \sin \varphi \cos \theta \\
\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi & \cos \varphi \sin \theta \cos \psi - \sin \varphi \cos \psi & \cos \varphi \cos \theta
\end{bmatrix}
\]

(20)

According to the corresponding relation of attitude matrix,

\[
-\sin \theta = A(1,3) = 2(q_1q_3 - q_0q_2), \quad \theta = \arcsin 2(-q_1q_3 + q_0q_2)
\]

(21)

\[
\frac{A(3,3)}{A(2,3)} = \tan \phi = \frac{2(q_2q_3 + q_0q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}, \quad \phi = \arctan\left(\frac{2(q_2q_3 + q_0q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}\right)
\]

(22)

\[
\frac{A(1,2)}{A(1,1)} = \tan \psi = \frac{2(q_1q_3 + q_0q_2)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}, \quad \psi = \arctan\left(\frac{2(q_1q_3 + q_0q_2)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}\right)
\]

(23)

In summary, the 3-1-3 sequence and the the 1-2-3 sequence expressed by the Quaternion can be written as follows Euler angles

\[
\phi = \phi(q_0, q_1, q_2, q_3)
\]

\[
\psi = \psi(q_0, q_1, q_2, q_3)
\]

\[
\theta = \theta(q_0, q_1, q_2, q_3)
\]

(24)

The Euler angle in the equation of motion can be eliminated by using the upper formula, and the Euler angle can be converted to the Euler angle according to the equation of Quaternion.

5. Conclusion

Euler angle and Quaternion are the two most commonly used methods of attitude representation in engineering. Euler angle representation has the advantages of simplicity and obvious geometric meaning, but the attitude description method using Euler angle has the singular problem, and it needs many trigonometric operations, while the Quaternion representation method can avoid these problems, this paper used Quaternion to describe the FOWT motion and dynamics equation of the attitude, but there is a transformation between quaternions and Euler angles. Using the numerical calculus and derivation, this paper established the transformation relation between quaternions and Euler angles for 1-2-3 sequences and 3-1-3 sequences, it laid a foundation for further solving.
6. Acknowledgement
Supported by “the National Natural Science Foundation of China (Grant No. 51409040), “the Fundamental Research Funds for the Central Universities (Grant No. DUT17JC13), “the Innovation Fund for Young scholars of State Key Laboratory of Coastal and Offshore Engineering (Grant No. LY1701)”

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