Conformal geometrodynamics: True degrees of freedom in a truly canonical structure

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The standard geometrodynamics is transformed into a theory of conformal geometrodynamics by extending the ADM phase space for canonical general relativity to that consisting of York’s mean exterior curvature time, conformal three-metric and their momenta. Accordingly, an additional constraint is introduced, called the conformal constraint. In terms of the new canonical variables, a diffeomorphism constraint is derived from the original momentum constraint. The Hamiltonian constraint then takes a new form. It turns out to be the sum of an expression that previously appeared in the literature and extra terms quadratic in the conformal constraint. The complete set of the conformal, diffeomorphism and Hamiltonian constraints are shown to be of first class through the explicit construction of their Poisson brackets. The extended algebra of constraints has as subalgebras the Dirac algebra for the deformations and Lie algebra for the conformorphism transformations of the spatial hypersurface. This is followed by a discussion of potential implications of the presented theory on the Dirac constraint quantization of general relativity. An argument is made to support the use of the York time in formulating the unitary functional evolution of quantum gravity. Finally, the prospect of future work is briefly outlined.

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I. INTRODUCTION

Finding the correct theory to unify general relativity (GR) and quantum mechanics is amongst the most important outstanding issues in fundamental physics. For almost a century countless ingenious ideas have been put forward to this end. The string theory exemplifies those based on a drastic modification of matter and spacetime structure. There have also been significant recent developments, notably the decoherent histories theories, that advocate the revision of the quantization procedure itself. A less radical strategy falls into the broad category of canonical quantum GR. This strategy often requires extensive classical analysis in terms of, e.g. sophisticated canonical transformations of the gravitational variables, so as to reach an appropriate point of departure for quantization. To gain advantage, the original phase space for canonical GR may undergo symplectic reduction or extension by exploiting symmetry principles. With a minimum of alternation to established physical principles, it offers a natural approach to quantum gravity and has a long history of investigations.

By leaving aside the conceptual issue of time and technical difficulties in deparametrizing GR, important progress of canonical quantum GR has been made on the functional-analytic approach to the gravitational constraints over the last decade or so. This is made possible by enlarging the phase space of GR to allow for a spin gauge symmetry that is amiable to the background independent “loop quantization” technique. The development is also based on the recognition of various technical issues in implementing the Dirac constraint quantization. These include the need to regularize the Wheeler-DeWitt equation, as it involves second order functional derivatives that may give rise to divergence. Despite the above encouraging progress, however, the loop quantum gravity programme has yet a number of major challenges to overcome. Amongst them are the well known Barbero-Immirzi ambiguity in choosing the canonical variables and the persistent lack of preferred time variable to generate unitary evolution of quantum gravity. While the former difficulty has recently been related to a conformal freedom in the Ashtekar type gauge treatment of the ADM geometrodynamical variables, the latter has long been suspected to reflect the need to quantize the true gravitational degrees of freedom.

In classical GR, a promising candidate to carry the true dynamics of the gravitational field has already been identified to be the conformal three-geometry by York over three decades ago [1]. Since then, considerable efforts have been made to construct a canonical approach to GR in which the scaling part of kinematics naturally generates the functional time evolution of the conformal part. Indeed, one of the most compelling physical reasons for such a construction is the anticipation for the transverse-traceless momentum of the conformal three-geometry to give rise to a spin-2 graviton description on quantization. Early works in this enterprise involve normalizing the conformal metric

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to have unit scale factor locally. The imposition of this condition inevitably introduces a nontrivial projection factor into the Poisson bracket (PB) relations of the traceless metric momenta \( \Omega^a_b \). This leads tantalizingly to an “almost Hamiltonian” formulation of GR, which is unfortunately not readily amicable to standard canonical quantization.

Recently a coordinate independent normalization scheme has been considered \( \mathbb{A} \) in which the conformal metric on a slice with constant mean curvature (CMC) has been rescaled in the fashion of Yamabe, so that the scale curvature \( \pm 1 \) or 0. It has been successfully demonstrated that at least in the case of hyperbolic compact three-spaces with \( R = -1 \), a full classical Hamiltonian reduction of GR can be accomplished. Formally, the reduced Hamiltonian density is simply the scale factor. However it is implicitly defined to satisfy a nonlinear elliptic equation and therefore is hard to be turned into a quantum operator. Besides, it is not clear whether general covariance could be retained if GR is to be quantized in a formalism relying on a privileged foliation.

In this paper we present how a genuine canonical evolution of conformal three-geometry for arbitrary spacetime foliations can be achieved by explicit construction of a new form of Hamiltonian for GR. By judicious use of Dirac’s theory of first class constraints, we show how various concerns over previous approaches described above are transcended. Although the main results of this paper lie in the classical domain, it provides the necessary canonical theory of first class constraints, we show how various concerns over previous approaches described above are transcended. Although the main results of this paper lie in the classical domain, it provides the necessary canonical theory of first class constraints, we show how various concerns over previous approaches described above are transcended.

II. CONVENTIONAL GEOMETRODYNAMICS

To establish convention, we start by recapitulating the standard ADM paradigm of canonical gravity \( \mathbb{B} \) with metric signature \((-+,++,+)\) and compact spatial sectors. The three-metric, its inverse, scale factor (unit density function) and associated Levi-Civita connection are respectively denoted by \( g_{ab}, g^{ab}, \mu := \sqrt{\det g_{ab}} \) and \( \nabla \), for \( a, b, \cdots = 1, 2, 3 \).

The extrinsic curvature is given by

\[
K_{ab} = \frac{1}{2N} (\dot{g}_{ab} - \nabla_a N_b - \nabla_b N_a)
\]

where \( N \) is the lapse function and \( N^a \) the shift vector. In terms of \( \Omega \) and the mean curvature \( K := g^{ab} K_{ab} \), the canonical momentum of \( g_{ab} \) is given by

\[
p^{ab} = \mu (K_{ab} - g^{ab} K).
\]

Note that, although for convenience, our sign convention for the exterior curvature differs from that of \( \mathbb{B} \), the expression of \( p^{ab} \) in terms of the metric and its derivatives is the same. We denote by \( G \) the ADM phase space for canonical GR with coordinates \((g_{ab}, p^{ab})\). These variables will also be referred to as \( G \)-variables. The canonical form of Einstein’s equation is generated by the well established ADM action given by

\[
S_G = \int (p \cdot \dot{g} - H_G) \, dt
\]

where \( p \cdot \dot{g} := \int p^{ab} \dot{g}_{ab} \, d^3x \) with the over dot denoting a \( t \)-derivative, and

\[
H_G := \int (N \mathcal{H} + N^a \mathcal{H}_a) \, d^3x.
\]

This “Hamiltonian” consists of the momentum constraint:

\[
\mathcal{H}_a = -2 \nabla_b p^b_a
\]

and Hamiltonian constraint:

\[
\mathcal{H} = \mu^{-1} g_{abcd} p^{ab} p^{cd} - \mu R
\]

were \( R \) is the (Ricci) scalar curvature of \( g_{ab} \) and

\[
g_{abcd} := \frac{1}{2} (g_{ac} g_{bd} + g_{ad} g_{bc} - g_{ab} g_{cd}).
\]

The Poisson bracket for any functionals \( A \) and \( B \) with respect to the \( G \)-variables \((g_{ab}, p^{ab})\) is defined by

\[
\{A, B\}_G := \int \left[ \frac{\delta A}{\delta g_{ab}(x)} \frac{\delta B}{\delta p^{ab}(x)} - \frac{\delta B}{\delta g_{ab}(x)} \frac{\delta A}{\delta p^{ab}(x)} \right] \, d^3x.
\]
The canonical PB relations follows as:

\[ \{g_{ab}(x), p_{cd}(x')\}_\Gamma = \delta_{ab} \delta(x, x') \tag{9} \]

\[ \{g_{ab}(x), g_{cd}(x')\}_\Gamma = \{p^{ab}(x), p^{cd}(x')\}_\Gamma = 0. \tag{10} \]

The PBs of the momentum and Hamiltonian constraints satisfy the well known Dirac algebra [5, 6]:

\[ \{H_a(x), H_b(x')\}_G = H_b(x) \delta_a(x, x') - (ax \leftrightarrow bx') \tag{11} \]

\[ \{H(x), H(x')\}_G = (g^{ab}H_a)(x) \delta_b(x, x') - (x \leftrightarrow x') \tag{12} \]

\[ \{H_a(x), H(x')\}_G = H(x) \delta_a(x, x'). \tag{13} \]

The structure of this algebra guarantees that the evolution of the intrinsic and extrinsic three-geometry using the canonical equations of motion

\[ \dot{g}_{ab} = \{g_{ab}, H_G\}_G \tag{14} \]

\[ \dot{p}^{ab} = \{p^{ab}, H_G\}_G \tag{15} \]

subject to the constraint equations

\[ H_a = 0 \tag{16} \]

\[ H = 0 \tag{17} \]

is independent of how the spatial hypersurface is deformed and coordinatized from the initial to final configurations for arbitrary \(N(x^a, t)\) and \(N^a(x^a, t)\) \[7\].

### III. EXTENDED PHASE SPACE OF GR WITH CONFORMAL SYMMETRY

We shall now reformulate the canonical dynamics of GR to explicitly depend on the underlying conformal spatial metric. This is motivated by previous works in the conformal treatment of GR (see e.g. [1, 2, 4, 8, 9, 10]) as well as recent efforts in addressing the unitary evolution and the conformal approach to quantum gravity [11]. The present approach features a number of new perspectives and generalities: (i) it is free from the constant mean curvature (CMC) condition, (ii) no normalization condition on the conformal metric will be imposed, and (iii) instead, Dirac’s theory of first class constraints will be employed to explore and exploit the conformal symmetry of an extended phase space for canonical GR.

Consider the decomposition of the metric momentum \(p^{ab}\) into its trace-free part and trace part according to

\[ p^{ab} = p_t^{ab} + \frac{1}{3} pg^{ab} \tag{18} \]

where \(p := g_{ab}p_t^{ab} = -2\mu K\) such that \(g_{ab}p_t^{ab} = 0\). Conformal transformations may be applied to the two irreducible pieces of \(p_t^{ab}\) in [13]. The metric \(g_{ab}\) and traceless part of the metric momentum \(p_t^{ab}\) are then conformally related to a symmetric tensor \(\gamma_{ab}\) and symmetric tensor density \(\pi^{ab}\) respectively as follows:

\[ g_{ab} = \phi^4 \gamma_{ab} \tag{19} \]

\[ p_t^{ab} = \phi^{-4} \pi^{ab} \tag{20} \]

using a positive conformal factor \(\phi\). The trace-free condition of \(p_t^{ab}\) implies a similar condition of \(\pi^{ab}\) with respect to \(\gamma_{ab}\). However, rather than restricting the independent components of \(\pi^{ab}\) from the outset, we choose to implement the trace-free condition of \(\pi^{ab}\) “weakly” a la Dirac [12] by introducing a new constraint

\[ C := \gamma_{ab} \pi^{ab} \tag{21} \]

which will only be required to vanish alongside the Hamiltonian and momentum constraints. The advantage of this approach will become evident below. The tensor \(\gamma_{ab}\) will play the role of the conformal metric, whereas the original \(g_{ab}\) will stay as the physical metric. Introducing the conformal scale factor

\[ \mu := \sqrt{\det \gamma_{ab}} \tag{22} \]
we see from (19) that \( \phi \) may be expressed as a function of \( \gamma_{ab} \) and \( \mu \) given by

\[
\phi = \phi(\gamma_{ab}, \mu) = \left( \frac{\mu}{\mu_\gamma} \right)^{1/6}.
\tag{23}
\]

Using (24) we can rewrite (18) as

\[
p^{ab} = \phi^{-4} \pi^{ab} - \frac{1}{2} \phi^2 \mu_\gamma \gamma^{ab} \tau
\tag{24}
\]

where \( \tau := \frac{4}{3} K \) which will be designated as the “York time”. Note that all components of the symmetric tensors \( \gamma_{ab} \) and \( \pi^{ab} \) are treated as independent.

By virtue of the expression (24) we may calculate the time derivative terms in the ADM action (3), resulting in

\[
p^{ab} g_{ab} = -\tau \dot{\mu} + \pi^{ab} \gamma_{ab} + 4(\ln \phi)' \mathcal{C}.
\tag{25}
\]

This way the pairs \( (\gamma_{ab}, \pi^{ab}) \) and \( (\tau, \mu) \) emerge as new canonical variables in place of \( (g_{ab}, \pi^{ab}) \) so long as \( \mathcal{C} \) can be consistently treated as a constraint. To see that this is indeed the case, we further show that the momentum and Hamiltonian constraints now take the following forms:

\[
\mathcal{H}_a = \tau_a \mu - 2 \nabla_{\gamma b} \pi^b_a + 4(\ln \phi)'_a \mathcal{C}
\tag{26}
\]

\[
\mathcal{H} = -\frac{3}{8} \tau^2 \mu + \frac{1}{\mu} \pi_{ab} \pi^{ab} - \mu R + \frac{\tau}{2} \mathcal{C} - \frac{1}{2 \mu} \mathcal{C}^2.
\tag{27}
\]

Here both \( \mathcal{C} \) and \( \phi \) in turn depend on the new variables as specified in (21) and (24), respectively. The appearance of \( \mathcal{C} \) as subexpressions of \( \mathcal{H}_a \) and \( \mathcal{H} \) above is a necessary consequence the “weak” treatment of this constraint. The indices of \( \pi^{ab} \) and \( \pi_{ab} \) are raised or lowered by the conformal metric \( \gamma_{ab} \) and its inverse \( \gamma^{ab} \) respectively. The scalar curvature \( R \) of \( g_{ab} \) in (27) may be expressed in terms of \( \mu \) and \( \gamma_{ab} \) as

\[
R = \phi^{-4} R_\gamma - 8 \phi^{-5} \Delta \phi
\tag{28}
\]

using the Levi-Civita connection \( \nabla_\gamma \), scalar curvature \( R_\gamma \) and Laplacian \( \Delta \gamma := \gamma_{ab} \nabla_\gamma a \nabla_\gamma b \), associated with the conformal metric \( \gamma_{ab} \).

Based on the above expressions, we can construct an action integral for GR equivalent to (3) as follows:

\[
S_\Gamma = \int (\mu \cdot \dot{\tau} + \tau \cdot \dot{\gamma} - H_\Gamma) \, dt
\tag{29}
\]

where (26) has been used with a total time derivative dropped. The Hamiltonian above is given by

\[
H_\Gamma := \int (N \mathcal{H} + N^a \mathcal{H}_a + \Phi \mathcal{C}) \, d^3 x
\tag{30}
\]

with an additional term \( \Phi \mathcal{C} \) where \( \Phi = \Phi(x, t) \) is a new Lagrange multiplier used to effect weakly vanishing of \( \mathcal{C} \). The presence of this additional constraint offsets the redundancy of \( \gamma_{ab} \) and \( \pi^{ab} \) due to the fact that they are unique up to a local rescaling. We denote by \( \Gamma \) the extended phase space, having the new “conformo-geometrodynamical variables” \( (\gamma_{ab}, \pi^{ab}; \tau, \mu) \) as coordinates. They will also be referred to as the \( \Gamma \)-variables.

IV. STRUCTURE OF THE NEW CONSTRAINTS AND THE NEW HAMILTONIAN

To determine the main mathematical structure and physical interpretation of the enlarged set of constraints including \( \mathcal{C} \), it is necessary to compute the PBs of various quantities involved in the analysis with respect to the \( \Gamma \)-variables \( (\gamma_{ab}, \pi^{ab}; \tau, \mu) \) given by

\[
\{A, B\}_\Gamma := \int \left[ \frac{\delta A}{\delta \gamma_{ab}(x)} \frac{\delta B}{\delta \pi^{ab}(x)} - \frac{\delta B}{\delta \gamma_{ab}(x)} \frac{\delta A}{\delta \pi^{ab}(x)} + \frac{\delta A}{\delta \tau(x)} \frac{\delta B}{\delta \mu(x)} - \frac{\delta B}{\delta \tau(x)} \frac{\delta A}{\delta \mu(x)} \right] \, d^3 x.
\tag{31}
\]

The corresponding canonical PB relations follow as:

\[
\{\gamma_{ab}(x), \pi^{cd}(x')\}_\Gamma = \delta^c_b \delta^d_a \delta(x, x')
\tag{32}
\]

\[
\{\gamma_{ab}(x), \mu(x')\}_\Gamma = \delta(x, x')
\tag{33}
\]

\[
\{\gamma_{ab}(x), \gamma_{cd}(x')\}_\Gamma = \{\pi^{ab}(x), \pi^{cd}(x')\}_\Gamma = 0
\tag{34}
\]

\[
\{\tau(x), \tau(x')\}_\Gamma = \{\mu^{ab}(x), \mu^{cd}(x')\}_\Gamma = 0
\tag{35}
\]
For brevity we shall abbreviate \( \{ \mathcal{A}, \mathcal{B} \}_T \) by \( \{ \mathcal{A}, \mathcal{B} \} \) in the following discussion. Using \( \text{(19)} \), \( \text{(24)} \) and the variational identity
\[
\delta \phi = \frac{1}{6} \phi \delta \mu - \frac{1}{12} \phi \gamma^{ab} \delta \gamma_{ab} \tag{36}
\]
we see the following PB relations
\[
\{g_{ab}(x), p^{cd}(x')\} = \delta_{ab}^{cd} \delta(x, x') \tag{37}
\]
\[
\{g_{ab}(x), \gamma_{cd}(x')\} = \langle p_{ab}(x), p^{cd}(x') \rangle = 0 \tag{38}
\]
inherited from the PB relations \( \text{(19)} \) and \( \text{(24)} \) for the original geometrodynamical variables. An important consequence of this is that the Dirac algebra in \( \text{(11)} - \text{(13)} \) is “strongly” preserved by regarding all quantities involved as functions of the \( \Gamma \)-variables rather than the \( G \)-variables. For instance, the factor \( g^{ab} \) on the RHS of \( \text{(12)} \) is given by
\[
g^{ab} = \phi^{-4} \gamma^{ab} = \left( \frac{\mu_2}{\mu} \right)^{2/3} \gamma^{ab} \tag{39}
\]
as per \( \text{(19)} \) and \( \text{(24)} \). A similar statement can equally be made for the PBs of any functionals depending on \( (\gamma_{ab}, \pi^{ab}; \tau, \mu) \) through \( (g_{ab}, p^{ab}) \). Straight from the definition of \( C \) we see that
\[
\{C(x), C(x')\} = 0 \tag{40}
\]
\[
\{\gamma_{ab}(x), C(x')\} = \gamma_{ab}(x) \delta(x, x') \tag{41}
\]
\[
\{\pi^{ab}(x), C(x')\} = -\pi^{ab}(x) \delta(x, x') \tag{42}
\]
which reveal \( C \) as the canonical generator of the conformal transformations. Accordingly, \( C \) will be referred to as the “conformal constraint”. This interpretation is consistent with \( \tau \) and \( \mu \) being conformally invariant, i.e.
\[
\{\tau(x), C(x')\} = 0, \quad \{\mu(x), C(x')\} = 0. \tag{43}
\]
By virtue of \( \text{(11)}, \text{(12)}, \text{(13)} \) and
\[
\{\phi(x), C(x')\} = -\frac{1}{4} \phi(x) \delta(x, x') \tag{44}
\]
derived from \( \text{(36)} \) we can reaffirm that the physical metric \( g_{ab} \) and momentum \( p^{ab} \) are themselves conformally invariant, i.e.
\[
\{C(x), g_{ab}(x')\} = 0, \quad \{C(x), p^{ab}(x')\} = 0. \tag{45}
\]
Eq. \( \text{(44)} \) implies that the quantity \( Q \) defined by \( \phi^4 = e^{-Q} \) is canonically conjugate to \( C \) since
\[
\{-4 \ln(\phi(x)), C(x')\} = \delta(x, x'). \tag{46}
\]
It is useful to observe that
\[
\{\phi(x), \phi(x')\} = 0 \tag{47}
\]
\[
\{\phi(x), g_{ab}(x')\} = 0 \tag{48}
\]
\[
\{\phi(x), p^{ab}(x')\} = 0 \tag{49}
\]
by using the identities
\[
\{\mu(x), p^{ab}(x')\} = \frac{1}{2} (\phi^2 \mu_2 \gamma^{ab}(x)) \delta(x, x') \tag{50}
\]
\[
\{\mu_2(x), p^{ab}(x')\} = \frac{1}{2} (\phi^{-4} \mu_2 \gamma^{ab}(x)) \delta(x, x'). \tag{51}
\]
Hence, as well as \( C \), the PBs of \( \phi \) with any functionals depending on and \( (\gamma_{ab}, \pi^{ab}; \tau, \mu) \) through \( (g_{ab}, p^{ab}) \) vanish strongly. From this, a host of mathematical niceties can be derived. Of particular interest in the present analysis are the vanishing PBs for the Hamiltonian and momentum constraints with \( C \) and \( \phi \):
\[
\{\mathcal{H}(x), C(x')\} = \{\mathcal{H}_\phi(x), C(x')\} = 0 \tag{52}
\]
\[
\{\mathcal{H}(x), \phi(x')\} = \{\mathcal{H}_\phi(x), \phi(x')\} = 0. \tag{53}
\]
Consequently, the expression $4(\ln \phi)_a C$ as the last term of $\mathcal{H}_a$ in (20) has zero PBs with the Hamiltonian and momentum constraints. Therefore, if we introduce the “diffeomorphism constraint”:

$$C_a := \mathcal{H}_a - 4(\ln \phi)_a C = \tau_{a\beta} \mu - 2 \nabla_{\gamma} \pi^a_{\alpha}$$

then it follows from the Dirac algebra, (11), (12), (52) and (53) that the complete set of independent constraints $\{\mathcal{C}, C_a, \mathcal{H}\}$ is first class satisfying the following algebra:

$$\{\mathcal{C}(x), \mathcal{C}(x')\} = 0$$

$$\{C_a(x), C_b(x')\} = C_b(x) \delta_{a}(x, x') - (ax \leftrightarrow bx')$$

$$\{C_a(x), \mathcal{C}(x')\} = \mathcal{C}(x) \delta_{a}(x, x')$$

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = \phi^{-4} \gamma^{ab}(C_a + 4(\ln \phi)_a C)(x) \delta_{b}(x, x') - (x \leftrightarrow x')$$

$$\{\mathcal{C}(x), \mathcal{H}(x')\} = 0$$

$$\{C_a(x), \mathcal{H}(x')\} = \mathcal{H}(x) \delta_{a}(x, x')$$

$$\{C_a(x), C_b(x')\} = C_b(x) \delta_{a}(x, x')$$

where $\phi$ depends on $\gamma_{ab}$ and $\mu$ according to $\phi = (\mu/\mu_0)^{1/6}$ as specified in (26).

It is clear that $\mathcal{C}$ and $C_a$ are the canonical generators of the conformal and diffeomorphism transformations, with the corresponding groups denoted by $P$ and $D$ respectively. Furthermore, the subset consisting of (34), (35) and (37) is a manifestation of the Lie algebra of the conformorphism group $C$ as the semi-direct product of $P$ and $D$. Such a group structure goes back to the original study of the initial value problem in GR.

In terms of the constraints $\mathcal{C}$, $C_a$ and $\mathcal{H}$, we are finally led to the canonical action and its Hamiltonian of the form

$$S = \int (\mu \cdot \dot{\tau} + \pi \cdot \dot{\gamma} - H) \, dt$$

and

$$H := \int (N\mathcal{H} + N^a C_a + \Phi \mathcal{C}) \, d^3 x$$

which is equivalent to (30) with a slightly redefined $\Phi$ due to (34).

This Hamiltonian generates the following canonical equations of motion

$$\dot{\tau} = \{\tau, H\}$$

$$\dot{\mu} = \{\mu, H\}$$

$$\dot{\gamma}_{ab} = \{\gamma_{ab}, H\}$$

$$\dot{\pi}^{ab} = \{\pi^{ab}, H\}$$

subject to the conformal, diffeomorphism and Hamiltonian constraint equations:

$$\mathcal{C} = 0$$

$$C_a = 0$$

$$\mathcal{H} = 0$$

respectively. The dynamical consistency of these equations is guaranteed by the first class nature of the constraints $\mathcal{C}$, $C_a$ and $\mathcal{H}$.

V. HAMILTONIAN REDUCTION TO THE ADM FORMALISM OF GR

We shall demonstrate that the new system of equations (34) - (66) and (11) - (17) does indeed imply the original ADM system of equations for GR, i.e. (14), (15), (16) and (17). Equations (19) and (34) augmented by (23) enable the mapping of solutions in the $\Gamma$-variables to equivalent solutions in the $G$-variables.

To show that the ADM canonical equations of motion are satisfied by the recovered metric and its momentum we evaluate the time derivative of $g_{ab}$ on the constraint surface where $C = 0$ as follows:

$$g_{ab}(x) = \int \left[ \delta g_{ab}(x) \frac{\delta H}{\delta \gamma^{cd}(x')} + \delta g_{ab}(x) \frac{\delta H}{\delta \pi^{cd}(x')} \delta \gamma^{cd}(x') \delta \pi^{cd}(x') \right] \, d^3 x'$$

$$= \int \left[ \frac{\delta g_{ab}(x)}{\delta \gamma^{cd}(x')} \frac{\delta H}{\delta \gamma^{cd}(x')} + \frac{\delta g_{ab}(x)}{\delta \pi^{cd}(x')} \frac{\delta H}{\delta \pi^{cd}(x')} \right] \, d^3 x'$$

$$= \int \left[ \frac{\delta g_{ab}(x)}{\delta \gamma^{cd}(x')} \frac{\delta H}{\delta \gamma^{cd}(x')} + \frac{\delta g_{ab}(x)}{\delta \pi^{cd}(x')} \frac{\delta H}{\delta \pi^{cd}(x')} \right] \, d^3 x'$$
where the new canonical equations of motion have been used.

Note that on the constraint surface of \( C \) we have the following relation

\[
\frac{\delta H}{\delta \pi^{cd}(x')} = \int \left[ \frac{\delta g_{ef}(x'')}{\delta \pi^{cd}(x')} \frac{\delta H_G}{\delta g_{ef}(x'')} + \frac{\delta p_{ef}(x'')}{\delta \pi^{cd}(x')} \frac{\delta H_G}{\delta p_{ef}(x'')} \right] d^3 x''
\]

(72)
as well as similar relations obtained by replacing \( \pi^{cd}(x') \) above with \( \gamma_{cd}(x') \), \( \tau(x') \) and \( \mu(x') \) respectively. Substituting these relations into (71) we see that

\[
\dot{g}_{ab}(x) = \int \left\{ \left[ g_{ab}(x), g_{ef}(x') \right] \frac{\delta H_G}{\delta g_{ef}(x')} + \left[ g_{ab}(x), p_{ef}(x') \right] \frac{\delta H_G}{\delta p_{ef}(x')} \right\} d^3 x'
\]

\[
= \frac{\delta H_G}{\delta g_{ab}(x)} = \{ g_{ab}(x), H_G \}_G
\]

(73)

where the PB relations and have been used. The above result is identical to (14).

Following arguments similar to those leading from (70) to (73) we can show that

\[
\dot{p}^{ab}(x) = -\frac{\delta H_G}{\delta g_{ab}(x)} = \{ p^{ab}(x), H_G \}_G
\]

(74)

which is identical to (15).

By taking into account (74) we see immediately that if the new constraint equations (67)–(69) are satisfied, the ADM constraint equations (16) and (17) are satisfied as well. We have therefore recovered the ADM description of GR as expected.

**VI. DISCUSSION ON QUANTIZATION ISSUES AND OUTLOOK**

Having reformulated the conventional geometrodynamics to conformal geometrodynamics, we shall be interested in whether the new formalism of canonical GR bears any implication on the quantum gravity programme. Would this approach to the York time for arbitrary spacetime foliations shed new light on the problem of time \( \tau \) and hence on the issue of unitary evolution in quantum gravity? Let us briefly address these issues using somewhat heuristic arguments with a view to more complete analysis.

We will for a moment work in the \( (\gamma_{ab}, \tau) \) representation and aim to treat \( \tau \) as the functional time. Let \( M \) be the space of Riemannian metrics with \( \gamma_{ab} \) as elements. Due to the presence of the conformorphism symmetry, the effective configuration space under consideration is the quotient \( M/P/D \), i.e. the (quantum) conformal superspace \( S \). Consider the state functional \( \Psi = \Psi[\gamma_{ab}, \tau] \) and adopt the formal functional integral

\[
\langle \Psi_1, \Psi_2 \rangle := \int_{M/P/D} \Psi_1^* \Psi_2 \delta \gamma
\]

(75)
as the inner product of two state functionals \( \Psi_1 \) and \( \Psi_2 \). In proceeding with the Dirac constraint quantization scheme, we will omit the “hat” over operators converted from classical variables for brevity. Specifically \( \mu = -i \frac{\delta}{\delta \tau} \) and \( \pi^{ab} = -i \frac{\delta}{\delta \gamma_{ab}} \) should be understood. With this in mind, the constraints and are formally turned into the following quantum constraint equations:

\[
C \Psi = 0 \quad (76)
\]

\[
C_a \Psi = 0 \quad (77)
\]

\[
\mathcal{H} \Psi = 0 \quad (78)
\]

respectively. Concerning factor ordering, we shall pay some attention to the following choice:

\[
C = S \left( \gamma_{ab} \pi^{ab} \right) \quad (79)
\]

\[
C_a = \tau, a \mu + S \left( -2 \nabla_{\gamma_{b}} \pi^{b}_{a} \right) \quad (80)
\]

\[
\mathcal{H} = -\frac{3}{8} \tau^2 \mu + S \left( \frac{1}{\mu} \pi_{ab} \pi^{ab} - \mu R + \frac{\tau}{2} C - \frac{C^2}{2 \mu} \right) \quad (81)
\]

where \( S \) denotes symmetrization with respect to \( \gamma_{ab} \) and \( \pi^{ab} \). Thus the term \( S(\pi_{ab} \pi^{ab}) \) may naturally be chosen to be the Laplacian of the conformal superspace. Note that the leading terms in and have \( \mu \) consistently standing
to the right. The usefulness of this is as follows: Take a “physical” state functional $\Psi$ satisfying (76)–(78). Owing to the described factor ordering, the Hermiticity (with respect to the inner product (75)) of the second term on the RHS of (80) alone is sufficient for one to conclude that

$$
\langle \Psi, i \frac{\delta \Psi}{\delta \tau(x)} \rangle = \langle i \frac{\delta \Psi}{\delta \tau(x)}, \Psi \rangle
$$

so long as it is evaluated at a space point $(x^a)$ with nonzero gradient of $\tau(x)$, i.e.

$$\tau_a(x) \neq 0.$$  

Take now a strictly non-CMC foliation of spacetime so that nowhere $\tau_a(x)$ vanishes. Using (82) and the relation $\Psi = \int \tau(x) \frac{\delta \Psi}{\delta \tau(x)} d^3x$, a straightforward formal calculation shows that $\langle \Psi, \Psi \rangle' = 0$ thereby implying the conservation of the “total probability”. For any canonical transformation of the form $(\gamma_{ab}, \pi^{ab}; \tau, \mu) \rightarrow (X^A, P_A; \tau, \mu)$, the above discussion can in principle be repeated in the $(X^A, \tau)$ representation.

It would be desirable to fully justify (82) without evoking the caveat (83). This will form a subject for further investigation. Nevertheless, our present discussion strongly suggests that an implementation of the Dirac quantization of GR using the described conformal formalism may lead to the unitary description of quantum gravity. A particular strategy along this line currently pursued by the author is to assimilate the connection approach into the conformal framework, with an aim to construct a theory of “conformal loop quantum gravity”. The progress of this work will be reported elsewhere [15].

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