Action of the restricted Weyl group on the $L$-invariant vectors of a representation

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This note constitutes a brief survey of our recent work on the problem of determining, for a given real Lie group $G$, the set of representations $V$ in which the longest element $w_0$ of the restricted Weyl group $W$ acts nontrivially on the subspace $V^L$ of $V$ formed by vectors that are invariant by $L$, the centralizer of a maximal split torus of $G$.

1 Introduction and motivation

1.1 Basic notations and statement of problem

Let $G$ be a semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^\mathbb{C}$ the complexification of $\mathfrak{g}$. We start by establishing the notations for some well-known objects related to $\mathfrak{g}$.

- We choose in $\mathfrak{g}$ a Cartan subspace $\mathfrak{a}$ (an abelian subalgebra of $\mathfrak{g}$ whose elements are diagonalizable over $\mathbb{R}$ and which is maximal for these properties).
- We choose in $\mathfrak{g}^\mathbb{C}$ a Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ (an abelian subalgebra of $\mathfrak{g}^\mathbb{C}$ whose elements are diagonalizable and which is maximal for these properties) that contains $\mathfrak{a}$.
- We denote $L := Z_G(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $G$, $\mathfrak{l}$ its Lie algebra.
- We choose on $\mathfrak{h}(\mathbb{R})$ a lexicographical ordering that “puts $\mathfrak{a}$ first”, i.e. such that every vector whose orthogonal projection onto $\mathfrak{a}$ is positive is itself positive. We call $\Delta^+$ the set of roots in $\Delta$ that are positive with respect to this ordering, and we let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots in $\Delta^+$. Let $\varpi_1, \ldots, \varpi_r$ be the corresponding fundamental weights.
• We introduce the dominant Weyl chamber

\[ \mathfrak{h}^+ := \{ X \in \mathfrak{h} \mid \forall i = 1, \ldots, r, \quad \alpha_i(X) \geq 0 \}, \]

and the dominant restricted Weyl chamber

\[ a^+ := \mathfrak{h}^+ \cap a. \]

• We introduce the restricted Weyl group \( W := N_G(a)/Z_G(a) \) of \( G \). Then \( a^+ \) is a fundamental domain for the action of \( W \) on \( a \). We define the longest element of the restricted Weyl group as the unique element \( w_0 \in W \) such that \( w_0(a^+) = -a^+ \).

Our goal is to study the action of \( W \), and more specifically of \( w_0 \), on various representations \( V \) of \( G \). Note however that this action is ill-defined: indeed if we want to see the abstract element \( w_0 \in W = N_G(a)/Z_G(a) \) as the projection of some concrete map \( \tilde{w}_0 \in N_G(a) \in G \), then \( \tilde{w}_0 \) is defined only up to multiplication by an element of \( Z_G(a) = L \), whose action on \( V \) can of course be nontrivial.

This naturally suggests the idea of restricting to \( L \)-invariant vectors. Given a representation \( V \) of \( g \), we denote

\[ V^L := \{ v \in V \mid \forall l \in L, \quad l \cdot v = v \} \]

the \( L \)-invariant subspace of \( V \): then \( W \), and in particular \( w_0 \), has a well-defined action on \( V^L \).

Our goal is to solve the following problem:

**Problem 1.1.** Given a semisimple real Lie group \( G \), characterize the representations \( V \) of \( G \) for which the action of \( w_0 \) on \( V^L \) is nontrivial.

In this note, we shall present our recent work on this problem.

1.2 Background and motivation

This problem arose from the author’s work in geometry. The interest of this particular algebraic property is that it furnishes a sufficient, and presumably necessary, condition for another, geometric property of \( V \). Namely, the author obtained the following result:

**Theorem 1.2.** \([\text{Smi21a}]\) Let \( G \) be a semisimple real Lie group, \( V \) a representation of \( G \). Suppose that the action of \( w_0 \) on \( V^L \) is nontrivial. Then there exists, in the affine group \( G \ltimes V \), a subgroup \( \Gamma \) whose linear part is Zariski-dense in \( G \), which is free of rank at least 2, and acts properly discontinuously on the affine space corresponding to \( V \).

He, and other people, also proved the converse statement in some special cases:

**Theorem 1.3.** The converse holds, for irreducible \( V \):

- \([\text{Smi20}]\) if \( G \) is split, but not of type \( A_n \) (\( n \geq 2 \)), \( D_{2n+1} \) or \( E_6 \);
• [Smi20] if $G$ is split, has one of these types, and $V$ satisfies a very restrictive additional assumption (see [Smi20] for the precise statement);

• [AMS11] if $G = \text{SO}(p,q)$ for arbitrary $p$ and $q$, and $V = \mathbb{R}^{p+q}$ is the standard representation.

Moreover, it seems plausible that, by combining the approaches of [Smi20] and [AMS11], we might prove the converse in all generality. This geometric property is related to the so-called Auslander conjecture [Aus64], which is an important conjecture that has stood for more than fifty years and generated an enormous amount of work: see e.g. [Mil77, Mar83, FG83, AMS02, DGK16] and many many others. For the statement of the conjecture as well as a more comprehensive survey of past work on it, we refer to [Abe01].

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2 Basic properties

2.1 Reduction from groups to algebras

First of all, note that, without loss of generality, we may restrict our attention to irreducible representations: indeed, plainly, the condition $w_0|_{V^L} \neq \text{Id}$ holds for a direct sum $V = V_1 \oplus \cdots \oplus V_k$ if and only if it holds for at least one of the direct summands $V_i$.

Let us start by recalling the classification of the irreducible representations of $G, \mathfrak{g}$ and $\mathfrak{g}^\mathbb{C}$. We introduce the notation $P$ (resp. $Q$) for the weight lattice (resp. root lattice), i.e. the abelian subgroup of $\mathfrak{h}(\mathbb{R})$ (or, technically, its dual) generated by $\varpi_1, \ldots, \varpi_r$ (resp. by $\Delta$).

Theorem 2.1 (see e.g. [Kna96, Theorem 5.5] or [Hal15, Theorems 9.4 and 9.5]). To every irreducible representation of $\mathfrak{g}^\mathbb{C}$ we may associate, in a bijective way, a vector $\lambda \in P \cap \mathfrak{h}^+$ called its highest weight.

Theorem 2.2 (see e.g. [Kna96, Proposition 7.15]). The restriction map $\rho \mapsto \rho|_\mathfrak{g}$ induces a bijection between irreducible representations of $\mathfrak{g}^\mathbb{C}$ and those of $\mathfrak{g}$.

Theorem 2.3. Every irreducible representation of $G$ yields, by derivation, an irreducible [Kna96, Proposition 7.15] representation of $\mathfrak{g}$. The converse is true if $G = \tilde{G}$ is simply-connected. For arbitrary $G$, the representation $\rho_\lambda(\mathfrak{g})$ lifts to $G$ if and only if $\lambda$ lies in some lattice $\Lambda_G$ satisfying $Q \subset \Lambda_G \subset P$.

For every dominant weight $\lambda \in P \cap \mathfrak{h}^+$, we denote by $V_\lambda$ the representation of $\mathfrak{g}^\mathbb{C}$, of $\mathfrak{g}$ or (if it exists) of $G$ with highest weight $\lambda$.

To reformulate Problem 1.1 in terms of algebras, it remains to note two things. First, we note that the action of $w_0$ on $V^l$ depends only on $\mathfrak{g}$, not on $G$: indeed $G$ is in general
the quotient of the (unique) simply-connected group $\tilde{G}$ with algebra $\mathfrak{g}$ by some subgroup of its center; but the center of $\tilde{G}$ is in particular contained in the centralizer $L$ of $\mathfrak{a}$, so is irrelevant when acting on $V^L$. Second, it is easy to see that the space $V^L$ always coincides with the space

$$V^I := \{ v \in V \mid \forall X \in I, \ X \cdot v = 0 \}$$

of $I$-invariant vectors of $V$. So Problem 1.1 is in fact equivalent to the following:

**Problem 2.4.** Given a semisimple real Lie algebra $\mathfrak{g}$, characterize the set of weights $\lambda \in P \cap \mathfrak{h}^+$ for which the action of $w_0$ on the $I$-invariant subspace $V^I_\lambda$ of the representation $V_\lambda$ of $\mathfrak{g}$ with highest weight $\lambda$ is nontrivial.

### 2.2 Additivity properties

A first step towards the solution of Problem 2.4 is given by the following characterization:

**Theorem 2.5.**

(i) The set of weights $\lambda \in P \cap \mathfrak{h}^+$ such that $V^I_\lambda \neq 0$ is contained in $Q \cap \mathfrak{h}^+$.

(ii) The set of weights $\lambda$ such that $V^I_\lambda \neq 0$ is in fact a submonoid of the additive monoid $Q \cap \mathfrak{h}^+$, i.e.

$$\forall \lambda, \mu \in Q \cap \mathfrak{h}^+, \begin{cases} V^I_\lambda \neq 0 \\ V^I_\mu \neq 0 \end{cases} \implies V^I_{\lambda+\mu} \neq 0.$$

(iii) In this monoid, the subset of weights $\lambda$ such that $w_0|_{V^I_\lambda} \neq \pm \text{Id}$ is an ideal, i.e.

$$\forall \lambda, \mu \in Q \cap \mathfrak{h}^+, \begin{cases} w_0|_{V^I_\lambda} \neq \pm \text{Id} \\ V^I_\mu \neq 0 \end{cases} \implies w_0|_{V^I_{\lambda+\mu}} \neq \pm \text{Id}.$$

**Proof.** Point (i) is straightforward: indeed, since $\mathfrak{h}$ is an abelian algebra containing $\mathfrak{a}$, we have $\mathfrak{h} \subset I$, hence $V^I_\lambda$ is contained in $V^0_\lambda$, which is just the zero-weight space of $V_\lambda$. By well-known theory (see e.g. [Hall15], Theorem 10.1), the latter is nontrivial, or in other terms 0 is a weight of $V_\lambda$, if and only if $\lambda$ lies in the root lattice $Q$.

Points (ii) and (iii) rely on the following classical theorem.

Let $G$ be a simply-connected complex Lie group with Lie algebra $\mathfrak{g}^\mathbb{C}$ and $N$ a maximal unipotent subgroup of $G$. Define $\mathbb{C}[G/N]$ the space of regular (i.e. polynomial) functions on $G/N$. Pointwise multiplication of functions is $G$-equivariant and makes $\mathbb{C}[G/N]$ into a $\mathbb{C}$-algebra without zero divisors (because $G/N$ is irreducible as an algebraic variety).

**Theorem 2.6** (see e.g. [PV94], (3.20)-(3.21)). Each finite-dimensional representation of $G$ (or equivalently of its Lie algebra $\mathfrak{g}^\mathbb{C}$) occurs exactly once as a direct summand of the representation $\mathbb{C}[G/N]$. The $\mathbb{C}$-algebra $\mathbb{C}[G/N]$ is graded by the highest weight $\lambda$, in the sense that the product of a vector in $V^I_\lambda$ by a vector in $V^I_\mu$ lies in $V^I_{\lambda+\mu}$ (where $V^I_\lambda$ stands here for the subrepresentation of $\mathbb{C}[G/N]$ with highest weight $\lambda$).
For given \( \lambda \) and \( \mu \), we call \textit{Cartan product} the induced bilinear map \( \odot : V_\lambda \times V_\mu \to V_{\lambda+\mu} \). Given \( u \in V_\lambda \) and \( v \in V_\mu \), this defines \( u \odot v \in V_{\lambda+\mu} \) as the projection of \( u \odot v \in V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \ldots \). Since \( \mathbb{C}[G/N] \) has no zero divisor, \( u \odot v \neq 0 \) whenever \( u \neq 0 \) and \( v \neq 0 \).

We may now finish the proof:

\textit{Proof of Theorem 2.5 continued.}

(ii) Let \( \lambda \) and \( \mu \) be two weights with this property. Choose any two nonzero vectors \( u_1 \) and \( u_2 \) in \( V_{\lambda_1} \) and \( V_{\lambda_2} \) respectively. Then the vector \( u_1 \odot u_2 \) is in \( V_{\lambda_1+\lambda_2} \), is invariant by \( I \), and is still nonzero.

(iii) Let \( \lambda \) be such that \( w_0|_{V_\lambda} \neq \pm \Id \), and \( \mu \) be such that \( V_\mu \neq 0 \). We can then choose nonzero vectors \( u_+ \) and \( u_- \) in \( V_\mu \) such that \( w_0 \cdot u_+ = u_+ \) and \( w_0 \cdot u_- = -u_- \), and a nonzero vector \( v \in V_\mu \) such that \( w_0 \cdot v = \pm v \) for some sign (indeed since \( w_0^2 = \Id \) in the Weyl group and the action of the Weyl group on \( V^I \) is well-defined, \( w_0|_{V^I} \) is a linear involution). Then \( u_+ \odot v \) and \( u_- \odot v \) are nonzero elements of \( V_{\lambda+\mu} \) on which \( w_0 \) acts by opposite signs. \( \Box \)

\subsection{2.3 Reduction from semisimple to simple algebras}

Theorem 2.5 suggests the decomposition of Problem 2.4 into two subproblems, as follows:

\textbf{Problem 2.7.} Given a semisimple Lie algebra \( \mathfrak{g} \) and a weight \( \lambda \in P \cap \mathfrak{h}^+ \), give a simple necessary and sufficient condition for having \( V_\lambda \neq 0 \).

\textbf{Problem 2.8.} Given a semisimple Lie algebra \( \mathfrak{g} \) and a weight \( \lambda \in P \cap \mathfrak{h}^+ \), assuming that \( V_\lambda \neq 0 \), give a simple necessary and sufficient condition for having \( w_0|_{V_\lambda} = \Id \).

Let us now reduce these two problems to the case where \( \mathfrak{g} \) is simple. This can be done by using the following theorem, whose proof is straightforward:

\textbf{Theorem 2.9.} Let \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \) is a semisimple Lie algebra. Let \( \lambda = \lambda_1 + \cdots + \lambda_k \) be a weight of \( \mathfrak{g} \), with every component \( \lambda_i \) lying in the Cartan subalgebra of \( \mathfrak{g}_i \). Then we have, with the obvious notations:

\[
\begin{align*}
& I = I_1 \oplus \cdots \oplus I_k; \\
& V_\lambda = V_{1,\lambda_1} \otimes \cdots \otimes V_{k,\lambda_k}; \\
& W = W_1 \times \cdots \times W_k; \\
& w_0 = w_{0,1} \cdots w_{0,k}; \\
& w_0|_{V_\lambda} = w_{0,1}|_{V_{1,\lambda_1}} \otimes \cdots \otimes w_{0,k}|_{V_{k,\lambda_k}}.
\end{align*}
\]

So Problem 2.7 completely reduces to the simple case, as \( V_\lambda \) is nontrivial if and only if each of its factors \( V_{1,\lambda_1} \) is nontrivial. For Problem 2.8 things are slightly more subtle: if the tensor product of \( k \) linear maps is the identity map, this does not imply that all factors are identity maps - only that all factors are scalars, with the coefficients having product 1. Since \( w_0^2 = \Id \) in the Weyl group, these coefficients can only be \( \pm 1 \). So Problem 2.8 reduces to the following, slightly more general problem in the simple case:
Problem 2.10. Given a simple Lie algebra \( g \) and a weight \( \lambda \in P \cap h^+ \), assuming that \( V_\lambda \neq 0 \), give a simple necessary and sufficient condition for having \( w_0|_{V_\lambda} = \pm \text{Id} \), as well as a criterion to determine the actual sign.

3 Known results

3.1 Case where \( g \) is split

We start by focusing on the particular case where \( g \) is split. In this case, we have \( a = h \) hence \( l = h = a \) (by maximality of \( h \)), so that \( V_\lambda^l \) is simply the zero-weight space of \( V_\lambda \), that we shall denote by \( V_\lambda^0 \). As for the restricted Weyl group, it coincides in this case with the ordinary Weyl group. So we may actually forget that \( g \) is a real Lie algebra, and simply work over \( \mathbb{C} \).

Problem 2.7 then becomes trivial: in fact, the containment given in Theorem 2.5 (i) now becomes an equality, so the condition is just that \( \lambda \in Q \).

As for Problem 2.10, it has been solved by the author together with Le Floch; this was the work presented at the conference talk from which this proceedings paper is derived. We proved the following result:

Theorem 3.1. [LFS18] Let \( g \) be a simple Lie algebra, \( \lambda \in Q \cap h^+ \) a weight of \( g \) that lies in the root lattice.

- Suppose that \( \lambda \) is of the form \( \lambda = kp_i \varpi_i \), where \( \varpi_i \) is one of the fundamental weights of \( \lambda \), \( p_i \) is the smallest positive integer such that \( p_i \varpi_i \in Q \), and \( k \) does not exceed some constant \( m_i \in \{0, 1, 2, +\infty\} \) depending on \( g \) and \( \varpi_i \). Then \( w_0|_{V_\lambda^0} = \sigma_i^k \text{Id} \), where \( \sigma_i \) is some sign depending on \( g \) and \( \varpi_i \). The specific values of the constants \( p_i, m_i \) and \( \sigma_i \) are all listed in Table 1 of [LFS18].

- If \( \lambda \) does not have this form, then \( w_0|_{V_\lambda^0} \neq \pm \text{Id} \).

3.2 Case where \( g \) is arbitrary

In the general case, Problem 2.7 has just recently been solved by the author [Smi21b]. The answer is as follows:

Theorem 3.2. [Smi21b] Let \( g \) be a real simple Lie algebra. Then the set \( X \) of dominant weights \( \lambda \in P \cap h^+ \) such that \( V_\lambda \neq 0 \) has of one of the following forms:

\[
X = Q \cap C \quad \text{or} \quad X = \Lambda \cap C \quad \text{or} \quad X = (Q \cap h^+ \setminus C) \cup (\Lambda \cap C),
\]

where \( C \) is a closed polyhedral convex cone (i.e. a set determined by a finite number of inequalities of the form \( \phi_i(\lambda) \geq 0 \) where each \( \phi_i \) is a linear form) contained in the dominant Weyl chamber \( h^+ \), and \( \Lambda \) (when applicable) is a sublattice of \( Q \) of index 2.

We refer to [Smi21b] for an exhaustive table listing the sets \( C \) and \( \Lambda \) for all the real simple Lie algebras. Here we will just give a brief overview:
• The sublattice $\Lambda$ intervenes only when $g$ is isomorphic to some $\mathfrak{so}(p,q)$ with $p+q$ odd. In all other cases, we simply have $X = Q \cap \mathcal{C}$.

• For split groups, quasi-split groups, all non-compact exceptional groups and some of the classical groups, $\mathcal{C}$ is actually the whole dominant Weyl chamber.

• However in the remaining cases, $\mathcal{C}$ does not always have nonempty interior. In fact, often $\mathcal{C}$ is the intersection of the dominant Weyl chamber with a vector subspace of $\mathfrak{h}$. When $g$ is compact (and only then), we have $\mathcal{C} = \{0\}$.

For Problem 2.8 we only have experimental results so far. Explicit computation of $w_0|_{V_\lambda}$ for all sufficiently low-dimensional representations $V$ of all sufficiently low-rank simple Lie algebras, combined with Theorem 2.5.(iii), suggests the following generalization of Theorem 3.1:

**Conjecture 3.3.** Let $g$ be any simple real Lie algebra of rank $r$, and $\lambda \in Q \cap h^+$ a dominant weight of $g$. Let us decompose $\lambda$ into its coordinates on the basis formed by the fundamental weights: $\lambda = \sum_{i=1}^r c_i \varpi_i$. Suppose that $V_{\lambda}^1 \neq 0$: then $w_0|_{V_{\lambda}^1} = \pm \text{Id}$ can happen only when at most $K$ of the coordinates $c_i$ can be nonzero, where $K = 3$.

(Compare this with the split case, where this statement holds for $K = 1$.) Moreover, experimental results allow us to conjecture an explicit description of the set of weights satisfying this property, for almost all simple Lie groups $g$. Here are a few examples.

**Conjecture 3.4.** Let $g$ and $\lambda = \sum_{i=1}^r c_i \varpi_i$ be as previously, with the fundamental weights $\varpi_1, \ldots, \varpi_r$ labelled in the Bourbaki ordering [Bou68]. Then:

(i) If $g = \mathfrak{so}(2,q)$ with $q = 7$ or $q \geq 9$, we have:
   - [Smi21b] tells us that $V_{\lambda}^1 \neq 0$ if and only if $\lambda \in Q$ and $c_i = 0$ for all $i > 4$;
   - assuming this is the case, $w_0|_{V_{\lambda}^1} = \pm \text{Id}$ if and only if $\lambda = x\varpi_1 + y\varpi_4$ with $i \in \{1, 2, 3\}$ and $x, y$ arbitrary nonnegative integers.

(ii) If $g = EIV$ (the real form of $E_6$ with maximal compact subalgebra $F_4$, also known as $E_6^{\text{iv}}$), we have:
   - [Smi21b] tells us that $V_{\lambda}^1 \neq 0$ always holds for $\lambda \in Q$;
   - assuming this is the case, $w_0|_{V_{\lambda}^1} = \pm \text{Id}$ if and only if $\lambda = x\varpi_1 + y\varpi_2$ with $i \in \{1, 3, 5, 6\}$ and $x, y$ arbitrary nonnegative integers.

(iii) If $g$ is the Lie algebra variously called $\mathfrak{sp}(12,4)$ (by some authors, such as Bourbaki [Bou68]) or $\mathfrak{sp}(6,2)$ (by some authors, such as Knapp [Kna96]), we have:
   - [Smi21b] tells us that $V_{\lambda}^1 \neq 0$ always holds for $\lambda \in Q$;
   - assuming this is the case, $w_0|_{V_{\lambda}^1} = \pm \text{Id}$ if and only if $\lambda = x\varpi_1 + y\varpi_8$ with $1 \leq i \leq 7$ and $x, y$ some nonnegative integers, with the additional condition $x \leq 2$ if $i = 3, 4$ or 5.
However, there are a few pairs \((\mathfrak{g}, V)\) where the dimension of \(V\) is so large that brute-force computations become intractable, but where analogous representations in smaller rank are not sufficiently well-behaved to establish a general pattern. For instance, take \(\mathfrak{g} = \mathfrak{sp}(9,3)\) (or \(\mathfrak{sp}(18,6)\) with the other convention) and \(\lambda = 4\sigma_{11}\). We know, from [Smi21b], that we then have \(V^1_\lambda \neq 0\); but we do not currently know (and cannot easily guess) whether \(w_0 V^1_\lambda\) is scalar or not.

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