Chiral Current induced by Torsional Weyl Anomaly in Dirac and Weyl Semimetals

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Abstract

Torsion can be realized as dislocation in the crystal lattice of material. It is particularly interesting if the material has fermions in the spectrum, such as graphene, topological insulators, Dirac and Weyl semimetals, as its transport properties can be affected by the torsion. In this letter, we find that, due to Weyl anomaly, “torsion” in Dirac and Weyl semimetals can induce novel chiral currents, either near a boundary or in a “conformally flat space”. We briefly discuss how to measure this interesting effect in experiment. It is remarkable that these experiments can help to clarify the theoretical controversy of whether an imaginary Pontryagin density could appear in the Weyl anomaly.

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1 Introduction

Although it was originally discovered in particle physics, due to its universal nature, quantum anomaly has non-trivial implications to a large number of physical phenomena ranging over vastly different scales. For example, the chiral anomaly of nonabelian gauge theory imposes nontrivial constraints on the fundamental interaction of chiral fermions in the standard model [1]. The chiral anomaly also affects the transport dynamics of systems with chiral fermions [2–12]. This kind of anomalous transport occurs only in a material system since nonvanishing chemical potentials are required. As anomaly itself is intrinsic to the quantum vacuum, it is an interesting question to ask if anomaly induced transport can occur independent of the chemical potentials.

The answer is positive. Recently a new kind of induced transport was predicted for boundary vacuum system as a result of the Weyl anomaly. It was found that in the quantum field theory of Dirac fermions coupled to external electromagnetic (EM) field

$$S = \int_M \sqrt{-g}(\bar{\psi}i\gamma^\mu \nabla_\mu \psi + \bar{\psi}\gamma^\mu A_\mu \psi),$$

Weyl anomaly can give rises to an induced magnetization current in the vicinity of the boundary of the vacuum system [13,14]

$$\langle J^\mu \rangle = \frac{-2\beta F^{\mu\nu} n_\nu}{x} + \cdots, \ x \sim 0$$
if an electric magnetic field is turned on. Here $\beta$ is the beta function, $x$ is the proper distance to the boundary, $n_\mu$ is the inner normal vector and ... denote higher order terms in $O(x)$. Note that the universal results \[2\] works for general quantum field theory and not just conformal field theory. It was also found that in a conformally flat spacetime $ds^2 = e^{2\sigma} \eta_{\mu\nu} dx^\mu dx^\nu$ without boundaries, the anomalous current is given by \[15\], \[16\]

$$\langle J^\mu \rangle = -2\beta F^{\mu\nu} \partial_\nu \sigma + O(\sigma^2),$$

(3)

to the leading order of small $\sigma$. Generalization of the result \[2\] to higher dimensions and the result \[3\] for arbitrary finite $\sigma$ can be found in \[17\], \[18\] and \[19\] respectively. In addition, there is also a novel effect of induced spin transport \[20\].

Central to these results is the fact that the anomalous currents \[2\], \[3\] emerge as direct response of the Weyl anomaly to the introduced scaling symmetry breaking EM background. Motivated by this observation, it is then natural to expect that similar induced phenomena may occur if the fermions are allowed to couple to other external backgrounds. One particular interesting case to consider is the coupling of fermions to the torsion of a curved spacetime. In this paper, we show that Weyl anomaly can give rises to both vector current and chiral current if a background torsion field is turned on. According to the current observational limit, torsion in our 4-dimensional spacetime is tiny and the possibility of observing these currents is slim. However, very interestingly, torsion can also appear in materials as a consequence of lattice dislocations \[21\], \[22\]. As such it is particularly interesting to study torsional defects in systems such as graphene, topological insulators, Dirac and Weyl semimetal since there are fermions in their spectrum. Recently it has been found that torsion can produce the so-called ‘Torsional Chiral Magnetic Effect’ \[23\]–\[25\] and several other interesting phenomena \[26\], \[27\]. In this paper, we first derive the Weyl anomaly for Dirac fermions coupled to torsion (section 2). Using this result, we derive the Weyl anomaly induced chiral currents in section 3. The discussion is extended to Weyl fermions in section 4. We propose that measurements of the induced currents in Weyl semimetal could help to resolve the theoretical controverse of whether the Pontryagin density appears in the Weyl anomaly.

2 Torsion and Weyl Anomaly

Spacetime is equipped with a metric which fixes the causal structure and its metric relations, and a connection which allows the parallel transport of tensors on the manifold. In Einstein general relativity (GR), the metricity condition and a symmetric connection are adopted so that connection is not independent but given by the metric. In general, departure from GR is characterized \[28\] by the non-metricity tensor $Q_{\mu\nu\rho} := \nabla_\mu g_{\nu\rho}$ and the torsion tensor

$$T^\rho_{\mu\nu} := \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu},$$

(4)
defined as the antisymmetric part of the connection. For simplicity, we will focus in this paper the particular interesting generalization of GR called the Einstein-Cartan theory where $Q = 0$. 

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and the connection is independently characterized by the torsion tensor. With the connection, minimal coupling of Dirac fermions to gravity can be constructed. For the Einstein-Cartan theory, the gravitational coupling can be simplified and shown to take the form (see [21] for a review on this),

$$S = \int_M \sqrt{-g} \bar{\psi} i \gamma^\mu (\nabla_\mu - ig_1 V_\mu - ig_2 \gamma_5 S_\mu) \psi,$$

(5)

where $\nabla_\mu$ is the covariant derivative defined with the Levi-Civita metric connection. Here the components of torsion $V_\mu := T^\rho_\rho_\mu, S_\mu := \epsilon^{\rho\nu\sigma\tau} T_{\nu\sigma\tau}$,

(6)

behave effectively as vectors and axial vectors. It is $g_1 = 0$ (resp. $g_2 = 0$) and there is no vector (resp. axial vector) coupling in 4-dimensions (resp. 3-dimensions). We use the mostly negative convention for the signature of the metric and the torsion will be taken as a background. From now on, we will absorb the coupling constants into the definition of $V_\mu$ and $S_\mu$.

The action (5) is classically Weyl invariant under the local scaling transformation: $\psi \rightarrow e^{-3\sigma/2} \psi, g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}, V_\mu \rightarrow V_\mu$ and $S_\mu \rightarrow S_\mu$. However, quantum mechanically there is an anomaly. For a manifold with a boundary, boundary conditions should be imposed on half of the spinor components. It can be shown that [29] Hermicity of the Dirac operator selects out of the general chiral bag boundary conditions the following specific ones:

$$(1 \pm i\gamma^n \gamma_5) \psi|_{\partial M} = 0$$

(7)

where $n$ denote the normal direction. The one-loop Weyl anomaly can be obtained by applying the heat kernel expansion [29]. Let us focus on 4-dimensions, the Weyl anomaly reads

$$A = \frac{1}{24\pi^2} \int_M \sqrt{-g} [F_{\mu\nu} F^{\mu\nu} + H_{\mu\nu} H^{\mu\nu}] + \frac{1}{12\pi^2} \int_{\partial M} \sqrt{-h} [B_1 - B_2 - \frac{1}{5} \bar{k}_{\mu\nu} S^\mu S^\nu],$$

(8)

where $F = dV, H = dS, h_{\mu\nu}$ is the induced metric on the boundary $\partial M, k_{\mu\nu} = h_{\mu\nu} h_{\rho\sigma} \nabla_{\rho} n_{\sigma}$ is the extrinsic curvature, $\bar{k}_{\mu\nu}$ and $k$ denote the traceless part and the trace of extrinsic curvatures respectively, $B_1 = \frac{2}{3} k (h^{\mu\nu} + n^{\mu} n^{\nu}) S_\mu S_\nu + S_n \nabla_{\mu} S^\nu + 2 S_\mu h^{\mu\nu} \nabla_n S^\nu, B_2 = \frac{4}{3} \bar{k} S_\mu S^\nu + n^{\mu} S^\nu \nabla_\mu S_\nu$. Here we choose the normal vector so that $n^\mu = -n_\mu = (0, -1, 0, 0)$ in a flat half space. The Weyl anomaly [8] is Weyl invariant and satisfies the Wess-Zumino consistency condition [30]. Note that the bulk contribution to the torsional Weyl anomaly is discussed in [32-34]. To the best of our knowledge, the boundary contribution to the torsional Weyl anomaly (8) is new.

We are interested in the expectation value of the chiral current and vector current in the theory. In 4-dimensions, the renormalized vacuum expectation value of the chiral current derived by the variation of effective action with respect to the background ‘axial vector’

$$J^\mu_S = \langle \bar{\psi} \gamma^\mu \gamma_5 \psi \rangle = \frac{1}{\sqrt{-g}} \frac{\delta I_{\text{eff}}}{\delta S_\mu},$$

(9)

In the following, we show that the knowledge of the Weyl anomaly [8] allows one immediately to determine (9) in closed analytic form.
3 Chiral Current

3.1 Boundary Theory

Let us first study the chiral current in 4-dimensional spacetime with a boundary, say, at \( x = 0 \) of the coordinate system. We follow the methods of [13,31], where we have studied the expectation value of current and stress tensor in boundary quantum field theories [35]. Since the mass dimension of chiral current is 3, it takes the asymptotic form [36]

\[
J_\mu^S = \frac{J_\mu^0}{x^3} + \frac{J_\mu^1}{x^2} + \frac{J_\mu^2}{x} + O(\ln x) \tag{10}
\]

near the boundary. Here \( x \) is the proper distance from the boundary, \( J_\mu^n \) have mass dimension \( n \) and depend on only the background geometry and the background torsion. \( J_\mu^0 \) can be solved by imposing the conservation law \( \nabla_\mu J_\mu^S = O(1) \) [13], where \( O(1) \) denotes the finite part of the chiral anomaly which is irrelevant to the divergent part of renormalized current. We obtain

\[
J_\mu^0 = 0, \quad J_\mu^1 = \lambda h^{\mu\nu} S_\nu, \tag{11}
\]

where \( \lambda \) is some constant. The key point in the above derivations is that the leading term of chiral current cannot be proportional to the normal vector \( n^\mu \), otherwise it cannot satisfy the conservation law \( \nabla_\mu (n^\mu / x^3) \sim 1/x^4 \neq O(1) \). We note that unlike the case of gauge field [13], the transformation \( \delta S_\mu = \partial_\mu \alpha \) changes the torsion and is not required to be a symmetry of the theory, therefore a non-vanishing \( J_\mu^1 \) term as in (11) is allowed.

Now since Weyl anomaly is related to the Logarithmic UV divergent term of effective action, one can follow the same analysis performed in [13,31] and obtain the following “integrability” relation [37]

\[
(\delta A)_{\partial M} = \left( \int_M \sqrt{-g} J_\mu^S \delta S_\mu \right)_{\log \epsilon} \tag{12}
\]

between the renormalized chiral current and the boundary part of the variation of the Weyl anomaly. Here a regulator \( x \geq \epsilon \) to the boundary has been introduced for the integral on RHS of (12). For our purpose, we turn only on the variation of the ‘axial vector’. Using (12), one can derive chiral current near the boundary from the boundary terms of variations of the Weyl anomaly. To proceed, let us employ the Gauss normal coordinates to express the metric \( ds^2 = -dx^2 + (h_{ij} - 2xk_{ij}) + \cdots)dy^i dy^j \) and expand \( S_\mu(x) = S_\mu^0 + xS_\mu^1 + O(x^2) \), where \( x \in [0, +\infty) \) and \( S_i \) give the \( i^{th} \) derivatives of \( S \) at \( x = 0 \). Substituting (8),(10), (11) into (12), after some calculations we obtain the chiral current near the boundary:

\[
J_a^S = \frac{S_a^0}{6\pi^2 x^2} + \frac{2k^a_{b,c} S_b^0 + k S_a^0}{10\pi^2 x} + O(\ln x),
\]

\[
J_n^S = \left( D_a S^a \right)_0 + O(\ln x), \quad x \sim 0, \tag{13}
\]
Figure 1: Chiral current $\vec{J}_S \sim \vec{b}/x^2$ induced by Screw dislocation, where $\vec{b}$ is the Burgers vector.

where $n$ and $a$ denote respectively the normal and tangential directions, $D_a$ is the covariant derivative on the boundary, and we denote for any function $F(x)$ that $F_0 = F(x = 0)$. We remark that as in the discussion [13,31], the total chiral current is finite since there are boundary contributions to the chiral current which cancel the divergence from the bulk contribution (13).

The result (13) applies not only to conformal field theory (CFT) but also the general quantum field theory (QFT) since the Weyl anomaly is well-defined for general quantum field theories [38,39]. Finally we remark that (13) can be verified by the Green’s function method [40,41]. In a flat half space with $k_{ab} = 0$, the correction of Green function due to torsion is given by

$$G_c = -\int_M \sqrt{|g|} G_0 \gamma^\mu \gamma_5 S_\mu G_0 + O(S^2)$$ (14)

where $G_0$ is Green’s function without torsion. From (14), we can obtain the chiral current by

$$J^c_\mu = -i \lim_{x' \to x} \text{Tr}_{\text{reg}} \left[ \gamma^\mu \gamma_5 G_c(x, x') \right],$$ (15)

which agrees with (13). Here Tr$_{\text{reg}}$ means we have subtracted the reference current without boundaries.

Let us briefly comment on how the chiral current (13) may be measured in Dirac Semimetals. As shown in Fig. 1, we perform the Screw dislocation of lattices so that the red parallelogram does not close, and the missing part is defined by the blue Burgers vector. The density of the Burgers vector $\vec{b}$ behaves as the axial vector in Dirac and Weyl Semimetals [21]. From (13), we draw conclusions that the Screw dislocation induces an anomalous chiral current near the boundary in Dirac and Weyl Semimetals

$$\vec{J}_S \sim \frac{\vec{b}}{x^2}, \ x \sim a$$ (16)
where $a$ denotes the lattice length. Note that, to obey the bag boundary condition (7), we should place an insulator on the boundary of the materials so that no current can flow out of the boundary $x = 0$.

Finally we make a remark for 3-dimensions. Following the same analysis as for the derivation of (10), (11), the renormalized expectation value of the vector current $J^\mu_V = \langle \bar{\psi} \gamma^\mu \psi \rangle$ takes the form

$$J^\mu_V = F^{\mu \nu} (\alpha + \beta \ln x),$$

(17)

where $\alpha, \beta$ are constant parameters which are sensitive to the boundary condition. We note that in dimensions $d < 4$, the current is not related to the Weyl anomaly. Hence the parameters $\alpha, \beta$ are not determined just by the central charges, but by further specific details of theory.

3.2 Conformally Flat Spacetime

There are also novel chiral current in 4-dimensional conformally flat spacetime without boundaries. To demonstrate this, let us start by deriving the anomalous transformation rule for the chiral current. Consider the theory (5) with metric and chiral vector field given by $(g_{\mu \nu}, S_\mu)$. Due to the anomaly, the renormalized effective action $I_{\text{eff}}$ is not invariant under the Weyl transformation. Generally, we have [42]

$$\frac{\delta}{\delta \sigma} I_{\text{eff}}(e^{-2\sigma} g_{\mu \nu}) = A(e^{-2\sigma} g_{\mu \nu})$$

(18)

for arbitrary finite $\sigma(x)$. This can be integrated to give the effective action [30, 43, 44]. Using the fact that the anomaly [8] is Weyl invariant, we obtain immediately the transformation rule for the effective action:

$$I_{\text{eff}}(e^{-2\sigma} g_{\mu \nu}) = I_{\text{eff}}(g_{\mu \nu}) + \frac{1}{24\pi^2} \int_M \sqrt{-g} H_{\mu \nu \sigma} H^{\mu \nu \sigma},$$

(19)

plus a boundary term $\frac{1}{12\pi^2} \int_{\partial M} \sqrt{-k} \left[ \frac{1}{5} k_{\mu \nu} S^\mu S^\nu + B_1 - B_2 \right] \sigma$, which we drop in spacetime without boundaries. One can check that the dilaton effective action satisfies Wess-Zumino consistency $[\delta_{\sigma_1}, \delta_{\sigma_2}] I_{\text{eff}} = 0$. Using (19), we obtain finally the transformation rule for the chiral current (9) under Weyl transformation $g_{\mu \nu} \rightarrow g'_{\mu \nu} = e^{-2\sigma} g_{\mu \nu}, S_\mu \rightarrow S'_\mu = S_\mu$

$$J^\mu_S = \frac{1}{6\pi^2} \nabla_\nu (H^{\nu \mu \sigma}),$$

(20)

plus a trivial term $e^{-4\sigma} J^\mu_S$. Here $J^\mu_S$ (resp. $J^\mu_S'$) denotes the vev of the chiral current of the theory [5] in the background spacetime $g_{\mu \nu}$ (resp. $g'_{\mu \nu}$). Taking $g'_{\mu \nu}$ to be the flat spacetime metric and assuming that the chiral current vanishes in some region of the flat spacetime, we finally obtain (20) as the chiral current in conformally flat spacetime

$$ds^2 = e^{2\sigma} \eta_{\mu \nu} dx^\mu dx^\nu.$$  

(21)
Note that the conformal factor $\sigma$ in (20) is arbitrary and needs not to be small. Therefore we can use (20) to calculate the current in general conformally flat spacetimes such as Anti-de-Sitter space, de-Sitter space and Robertson-Walker universe. For Robertson-Walker universe $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$, we have at time $t = t_*$

$$J^\mu_s = \frac{1}{6\pi^2} H^0 \mu H$$

(22)

where $H = \dot{a}/a$ is the Hubble parameter. For simplicity we have chosen $a(t_*) = 1$. In materials, curvature and torsion can be mimicked by disclinations and dislocations, respectively. Thus, one may measure the effect (20) in Dirac semimetals with suitable disclinations and dislocations.

4 Weyl Fermions

So far we focus on Dirac fermions. The discussions can be generalized to Weyl fermions straightforwardly. The real part of Weyl anomaly for Weyl fermions is half of that of Dirac fermions (8). As a result, the anomalous chiral current is also half of that Dirac fermions (13), (20). The imaginary part of Weyl anomaly is parity odd and it is controversial whether such term exists [45, 46]. This imaginary part implies that the theory is non-unitary or there is absorption and dissipation in materials. For simplicity, let us take the vector parts of torsion $V_\mu$ as an example. The discussion for axial vector parts of torsion $S_\mu$ is similar. The Weyl anomaly of Weyl fermions related to $V_\mu$ is

$$A = \frac{1}{48\pi^2} \int_M \sqrt{-g} \left[ F_{\mu\nu} F^{\mu\nu} + i \frac{3}{2} \eta \ast F_{\mu\nu} \ast F^{\mu\nu} \right]$$

(23)

where $\ast F_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, $\eta = 0$ or 1 denote the controversy. Following the above approach, we derive the currents

$$J^\mu_V = \frac{n_\nu F^{\nu\mu} + i \frac{3}{2} \eta n_\nu \ast F^{\nu\mu}}{12\pi^2 x} + O(\ln x),$$

(24)

near a boundary and

$$J^\mu_V = e^{-4\sigma} J^\mu_{V'} + \frac{1}{12\pi^2} \nabla_\nu \left( F^{\mu\nu} \sigma + i \frac{3}{2} \eta \ast F^{\nu\mu} \right),$$

(25)

in a conformally flat space without boundaries. By applying Green’s function, we can derive the current by

$$J^\mu_V = i \lim_{x' \to x} \text{Tr}_{\text{reg}} \left[ G_0 \gamma^\nu V_\nu G_0 \gamma^\mu \frac{1 + \gamma_5}{2} \right],$$

(26)

which agrees with results without imaginary part $\eta = 0$. The vector $V_\mu$ can be realized by either an electromagnetic field or suitable dislocations in Weyl semimetals. It is interesting to measure the predicted current (24), (25) in Weyl semimetals, which can help to clarify the theoretical controversy that if an imaginary Pontryagin density could appear in the Weyl anomaly [45, 46].
5 Conclusions

In this paper, we have shown that, due to Weyl anomaly, torsion can lead to novel currents and chiral currents for Dirac and Weyl fermions. We propose to measure these interesting effects in Dirac and Weyl semimetals with suitable dislocations. These experiments can help to clarify the theoretical controversy that if an imaginary Pontryagin density could appear in the Weyl anomaly.

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We have find it conveniently to consider the Weyl transformation $g_{\mu\nu} \rightarrow e^{-2\sigma} g_{\mu\nu}$ here.