h–analogue of Newton’s binomial formula

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Abstract

In this letter, the h–analogue of Newton’s binomial formula is obtained in the h–deformed quantum plane which does not have any q–analogue. For h = 0, this is just the usual one as it should be. Furthermore, the binomial coefficients reduce to \( \frac{n!}{(n-k)!} \) for h = 1. Some properties of the h–binomial coefficients are also given. Finally, I hope that such results will contribute to an introduction of the h–analogue of the well–known functions, h–special functions and h–deformed analysis.

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The study of $q$–analysis appeared in the literature very long time ago [1]. In particular, a $q$–analogue of the Newton’s formula, well–known functions like $q$–exponential, $q$–logarithm, · · · etc, and the special functions arena’s [1, 5, 6] have been introduced and studied intensively.

Such $q$–analogue of these was obtained by taking $q$–commuting variables $x, y$ satisfying the relation $xy = qyx$, i.e. $(x, y)$ belongs to the Manin plane.

In this letter, I will take another direction by introducing the analogue of Newton’s formula in the $h$–deformed quantum plane [8, 7] (i.e. $h$–Newton binomial formula). As far as I know, such a $h$–analogue does not exist in the litterature till now and the result will permit in the future the introduction of the $h$–analogue of well–known functions, $h$–special functions and $h$–deformed analysis.

Newton’s binomial formula is defined as follows:

$$(x + y)^n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) y^k x^{n-k} \quad (1)$$

where $\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$ and it is understood here that the coordinate variables $x$ and $y$ commute, i.e. $xy = yx$.

A $q$–analogue of (1) for the $q$–commuting coordinates $x$ and $y$ satisfying $xy = qyx$ was first stated by Rothe, although its special cases were known to L. Euler, see [3], found again by Schützenberger [2] long time ago and has been rediscovered many times subsequently [4].

A $q$–analogue of (1) becomes:

$$(x + y)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q y^k x^{n-k} \quad (2)$$

where the $q$–binomial coefficient is given by:

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

with

$$(a; q)_k = (1 - a)(1 - qa) \cdots (1 - q^{k-1}a), \quad a \in \mathbb{C}, k \in \mathbb{N}$$
Now consider Manin’s $q$–plane $x'y' = qyx'$. By the following linear transformation (see [8] and references therein):

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix} 1 & \frac{k}{q-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix} x \\
y
\end{pmatrix}
$$

Manin’s $q$–plane changes to $xy - qyx = hy^2$ which for $q = 1$ gives the $h$–deformed plane:

$$
xy = yx + hy^2 \quad (3)
$$

Even though the linear transformation is singular for $q = 1$, the resulting quantum plane is well-defined.

**Proposition 1:**
Let $x$ and $y$ be coordinate variables satisfying (3), then the following identities are true:

$$
\begin{align*}
x^ky &= \sum_{r=0}^{k} \frac{k!}{(k-r)!} h^r y^{r+1} x^{k-r} \\
x^k y &= y^k x + khy^{k+1}
\end{align*}
\quad (4)
$$

These identities are easily proved by successive use of (3).

**Proposition 2:** ( $h$–binomial formula )
Let $x$ and $y$ be coordinate variables satisfying (3), then we have:

$$
(x + y)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_h y^k x^{n-k} \quad (5)
$$

where $\left[ \begin{array}{c} n \\ k \end{array} \right]_h$ are the $h$–binomial coefficients given as follows:

$$
\left[ \begin{array}{c} n \\ k \end{array} \right]_h = \left( \begin{array}{c} n \\ k \end{array} \right) h^k (h^{-1})_k. \quad (6)
$$

with $\left[ \begin{array}{c} n \\ 0 \end{array} \right]_h = 1$ and $(a)_k = \Gamma(a + k)/\Gamma(a)$ is the shifted factorial.
Proof:
We will prove this proposition by recurrence. Indeed for \( n = 1, 2 \), it is verified.
Suppose now that the formula is true for \( n - 1 \), which means:

\[
(x + y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k},
\]

with \( \binom{n-1}{0} h = 1 \).
To show this for \( n \), let first consider the following expansion:

\[
(x + y)^n = \sum_{k=0}^{n} C_{n,k} y^k x^{n-k}
\]

where \( C_{n,k} \) are coefficients depending on \( h \).
Then, we have:

\[
(x + y)^n = (x + y)(x + y)^{n-1} = (x + y)\sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k}
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} x y^k x^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k}.
\]

Using the result of the first proposition, we obtain:

\[
(x + y)^n = \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k} - (1 + (k-1)h)y^k x^{n-k}.
\]

which yields respectively:

\[
C_{n,0} = \binom{n-1}{0} h = 1,
\]
\[
C_{n,k} = \binom{n-1}{k} h + (1 + (k-1)h) \binom{n-1}{k-1} h = \binom{n}{k} h,
\]
\[
C_{n,n} = \binom{n-1}{n-1} h (1 + (n-1)h) = \binom{n}{n} h.
\]
This completes the Proof.

Moreover, the $h$–binomial coefficients obey to the following properties:

$$
\left[ \begin{array}{c} n \\ k \end{array} \right]_h + (1 + (k - 1)h) \left[ \begin{array}{c} n \\ k - 1 \end{array} \right]_h = \left[ \begin{array}{c} n + 1 \\ k \end{array} \right]_h.
$$

(7)

and

$$
\left[ \begin{array}{c} n + 1 \\ k + 1 \end{array} \right]_h = \frac{n + 1}{k + 1}(1 + kh) \left[ \begin{array}{c} n \\ k \end{array} \right]_h.
$$

(8)

In fact, these properties follow from the well–known relations of the classical binomial coefficients:

$$
\left( \begin{array}{c} n + 1 \\ k \end{array} \right) = \left( \begin{array}{c} n \\ k \end{array} \right) + \left( \begin{array}{c} n \\ k - 1 \end{array} \right)
$$

and

$$
\left( \begin{array}{c} n + 1 \\ k \end{array} \right) = \frac{n + 1}{k} \left( \begin{array}{c} n \\ k - 1 \end{array} \right)
$$

upon using $(a)_k = (a + k - 1)(a)_{k-1}$, which means that (7) and (8) are just a consequence of the known properties of the classical coefficients and the shifted factorial.

Now, we make the following remarks. First, for $h = 0$ the Newton’s binomial formula is just the usual one for commuting variables $xy = yx$ as it should be.

Second, for $h = 1$ the $h = 1$–binomial coefficients are:

$$
\left[ \begin{array}{c} n \\ k \end{array} \right]_{h=1} = \frac{n!}{(n - k)!}
$$

(9)

and therefore the $h = 1$–analogue Newton’s binomial formula becomes:

$$
(x + y)^n = \sum_{k=0}^{n} \frac{n!}{(n - k)!} y^k x^{n-k}
$$

(10)

provided that $xy = yx + y^2$.

To conclude, we see that the $h$–analogue of Newton’s formula in the $h$–deformed plane has no $q$–analogue. It seems from the structures of the $h$–binomial coefficients that the $h$–deformed plane is somewhat "more classical" than the $q$–deformed plane.
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