Asymptotic behavior of unstable INAR\((p)\) processes

MÁTYÁS BARCZY\(*,\diamondsuit\), MÁRTON ISPÁNY\(*\), GYULA PAP\(\#\)

* Faculty of Informatics, University of Debrecen, Pf. 12, H–4010 Debrecen, Hungary.
\# Bolyai Institute, University of Szeged, Aradi védvánk tere 1, H–6720 Szeged, Hungary.
e–mails: barczy.matyas@inf.unideb.hu (M. Barczy), ispany.marton@inf.unideb.hu (M. Ispány), papgy@math.u-szeged.hu (G. Pap).
\diamondsuit Corresponding author.

Abstract

In this paper the asymptotic behavior of an unstable integer-valued autoregressive model of order \(p\) (INAR\((p)\)) is described. Under a natural assumption it is proved that the sequence of appropriately scaled random step functions formed from an unstable INAR\((p)\) process converges weakly towards a squared Bessel process. We note that this limit behavior is quite different from that of familiar unstable autoregressive processes of order \(p\).

1 Introduction

Recently, there has been remarkable interest in integer-valued time series models and a number of results are now available in specialized monographs (e.g., MacDonald and Zucchini [42], Cameron and Trivedi [11], and Steutel and van Harn [58]) and review papers (e.g., McKenzie [46], Jung and Tremayne [33], and Weiß [60]). Reasons to introduce discrete data models come from the need to account for the discrete nature of certain data sets, often counts of events, objects or individuals. Examples of applications can be found in the analysis of time series of count data on the area of financial mathematics by analyzing stock transactions (Quoreshi [51]), insurance by modeling claim counts (Gouriéroux and Jasiak [23]), medicine by investigating disease incidence (Cardinal et al. [12]), neurobiology by change-point analysis of neuron spike train data (Bélisle et al. [4]), optimal alarm systems (Monteiro et al. [37]), psychometrics by treating longitudinal count data (Böckenholt [9, 7]), environmetrics by analyzing rainfall measurements (Thyregod et al. [59]), experimental biology (Zhou and Basawa [61]), and queueing systems (Ahn et al. [11] and Pickands III and Stine [50]).

Among the most successful integer-valued time series models proposed in the literature we mention the INteger-valued AutoRegressive model of order \(p\) (INAR\((p)\)). This model was first introduced by McKenzie [45] and Al-Osh and Alzaid [2] for the case \(p = 1\). The INAR\((1)\) model has been investigated by several authors. Franke and Seligmann [19] analyzed...
maximum likelihood estimation of parameters under Poisson innovation. Du and Li [17] and
Freeland and McCabe [21] derived the limit-distribution of the ordinary least squares estimator
of the autoregressive parameter. Silva and Oliveira [54] proposed a frequency domain based
estimator, Brännäs and Hellström [8] investigated generalized method of moment estimation,
Silva and Silva [56] considered a Yule-Walker estimator. Jung et al. [32] analyzed the finite
sample behavior of several estimators by a Monte Carlo study. Ispány et al. [27], [28] derived
asymptotic inference for nearly unstable INAR(1) models which has been refined by Drost et
al. [15] later. A Poisson limit theorem has been proved for an inhomogeneous nearly critical
INAR(1) model by Györfi et al. [24].

The more general INAR(p) processes were first introduced by Al-Osh and Alzaid [3]. In their
setup the autocorrelation structure of the process corresponds to that of an ARMA(p, p − 1)
process, see also Section 2. Another definition of an INAR(p) process was proposed indepen-
dently by Du and Li [17] and by Gauthier and Latour [22] and Latour [39], and is different
from that of Alzaid and Al-Osh [3]. In Du and Li’s setup the autocorrelation structure of an
INAR(p) process is the same as that of an AR(p) process. The setup of Du and Li [17] has been
followed by most of the authors, and our approach will also be the same, see Section 2. The
INAR(p) model has been investigated by several authors from different points of views. Drost
et al. [14] provided asymptotically efficient estimator for the parameters. Silva and Oliveira [55]
described the higher order moments and cumulants of INAR(p) processes, and Silva and Silva
[56] derived asymptotic distribution of the Yule-Walker estimator. Drost et al. [16] considered
semiparametric INAR(p) models and proposed efficient estimation for the autoregression
parameters and innovation distributions. Recently, the so called p-order Rounded INteger-valued
AutoRegressive (RINAR(p)) time series model was introduced and studied by Kachour and Yao
[35] and Kachour [34]. The broad scope of the empirical literature in which INAR models are
applied indicates its relevance. Examples of such applications include Franke and Seligmann
[19] (epileptic seizure counts), Böckenholt [7] (longitudinal count data), Thyregod et al. [59]
(rainfall measurements), Brännäs and Hellström [8] and Rudholm [53] (economics), Brännäs
and Shahiduzzaman [9] (finance), Gourieroux and Jasiak [23] (insurance), Pavlopoulos and
Karlis [48] (environmental studies) and McCabe et al. [44] (finance and mortality).

An interesting problem, which has not yet been addressed for INAR(p) models, is to in-
vestigate the asymptotic behavior of unstable INAR(p) processes, i.e., when the characteristic
polynomial has a unit root. In this paper we give a complete description of this limit behavior.
In particular, it will turn out that an INAR(p) model is unstable if and only if the sum of
its autoregression parameters equals 1, and in this case the only unit root of the character-
istic polynomial is 1 with multiplicity one. For the sake of convenience, we suppose that the
process starts from zero. Without loss of generality, we may suppose that the pth autoregres-
sion parameter is strictly positive and that the greatest common divisor of the strictly positive
autoregression parameters is 1, see Remark 2.2. Under the assumption that the second mo-
ment of the innovation distribution is finite, we prove that the sequence of appropriately scaled
random step functions formed from an unstable INAR(p) process converges weakly towards a
squared Bessel process. This limit process is a continuous branching process also known as
square-root process or Cox-Ingersoll-Ross process. We should also note that the behavior of
unstable INAR(p) models is completely different from that of familiar (real-valued) unstable
AR(p) models in at least two senses. On one hand, the characteristic polynomial of an unstable
INAR(p) model has only one unit root, namely 1, with multiplicity one, whereas for a familiar AR(p) model it may have real or complex unit roots with various different multiplicities. On the other hand, in the case of an unstable INAR(p) model there is a limit process which is a squared Bessel process, while in the case of an unstable AR(p) model in general there is no limit process, only for appropriately transformed and scaled random step functions, see Chan and Wei [13], Jeganathan [31] and van der Meer et al. [43, Theorem 3].

We remark that our result can be considered as the first step towards the comprehensive theory of nonstationary integer-valued time series and investigation of the unit root problem of econometrics in the integer-valued setup. Nonstationary time series have been playing an important role in both econometric theory and applications over the last 20 years, and a substantial literature has been developed in this field. A detailed set of references is given in Phillips and Xiao [49]. We note that Ling and Li [40], [41] considered an unstable ARMA model with GARCH errors and an unstable fractionally integrated ARMA model. Concerning relevance and practical applications of unstable INAR models we note that empirical studies show importance of these kind of models. Brännäs and Hellström [8] reported an INAR(0.98) model for the number of private schools, Rudholm [53] considered INAR(0.98) and INAR(0.99) models for the number of Swedish generic-pharmaceutical market. Hellström [25] focused on the testing of unit root in INAR(1) models and provided small sample distributions for the Dickey-Fuller test statistic under the null hypothesis of unit root in an INAR(1) model with Poisson distributed innovations. To our knowledge a unit root test for general INAR(p) models is not known, and from this point of view studying unstable INAR(p) models is an important preliminary task.

The rest of the paper is organized as follows. Section 2 provides a background description of basic theoretical results related with INAR(p) models. In Section 3 we describe the asymptotic behavior of unstable INAR(p) processes. Under the assumption that the second moment of the innovation distribution is finite, we prove that the sequence of appropriately scaled random step functions formed from an unstable INAR(p) process converges weakly towards a squared Bessel process, see Theorem 3.1. Section 4 contains a proof of our main Theorem 3.1. For the proof, we collect some properties of the first and second moments of (not necessarily unstable) INAR(p) processes, we recall a useful functional martingale limit theorem and an appropriate version of the continuous mapping theorem, see Lemma 5.1, Corollary 5.1, Theorem 5.1 and Lemma 5.2 in Appendix, respectively.

2 The INAR(p) model

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and complex numbers, respectively. For all $n \in \mathbb{N}$, let us denote by $I_n$ the $n \times n$ identity matrix. Every random variable will be defined on a fixed probability space $(\Omega, \mathcal{A}, P)$.

One way to obtain models for integer-valued data is to replace multiplication in the conventional ARMA models in such a way to ensure the integer discreteness of the process and to adopt the terms of self-decomposability and stability for integer-valued time series.
2.1 Definition. Let \((\varepsilon_k)_{k \in \mathbb{N}}\) be an independent and identically distributed (i.i.d.) sequence of non-negative integer-valued random variables, and let \(\alpha_1, \ldots, \alpha_p \in [0, 1]\). An INAR\((p)\) time series model with coefficients \(\alpha_1, \ldots, \alpha_p\) and innovations \((\varepsilon_k)_{k \in \mathbb{N}}\) is a stochastic process \((X_n)_{n \geq -p + 1}\) given by

\[
X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,1,j} + \cdots + \sum_{j=1}^{X_{k-p}} \xi_{k,p,j} + \varepsilon_k, \quad k \in \mathbb{N},
\]

where for all \(k \in \mathbb{N}\) and \(i \in \{1, \ldots, p\}\), \((\xi_{k,i,j})_{j \in \mathbb{N}}\) is a sequence of i.i.d. Bernoulli random variables with mean \(\alpha_i\) such that these sequences are mutually independent and independent of the sequence \((\varepsilon_k)_{k \in \mathbb{N}}\), and \(X_0, X_1, \ldots, X_{p-1}\) are non-negative integer-valued random variables independent of the sequences \((\xi_{k,i,j})_{j \in \mathbb{N}}, \ k \in \mathbb{N}, \ i \in \{1, \ldots, p\}\), and \((\varepsilon_k)_{k \in \mathbb{N}}\).

The INAR\((p)\) model (2.1) can be written in another way using the binomial thinning operator \(\alpha \circ \) (due to Steutel and van Harn [57]) which we recall now. Let \(X\) be a non-negative integer-valued random variable. Let \((\xi_j)_{j \in \mathbb{N}}\) be a sequence of i.i.d. Bernoulli random variables with mean \(\alpha \in [0, 1]\). We assume that the sequence \((\xi_j)_{j \in \mathbb{N}}\) is independent of \(X\). The non-negative integer-valued random variable \(\alpha \circ X\) is defined by

\[
\alpha \circ X := \begin{cases} 
\sum_{j=1}^{X} \xi_j, & \text{if } X > 0, \\
0, & \text{if } X = 0.
\end{cases}
\]

The sequence \((\xi_j)_{j \in \mathbb{N}}\) is called a counting sequence. The INAR\((p)\) model (2.1) takes the form

\[
X_k = \alpha_1 \circ X_{k-1} + \cdots + \alpha_p \circ X_{k-p} + \varepsilon_k, \quad k \in \mathbb{N},
\]

Note that the above form of the INAR\((p)\) model is quite analogous with a usual AR\((p)\) process (another slight link between them is the similarity of some conditional expectations, see (2.3)). As we noted in the introduction, this definition of the INAR\((p)\) process was proposed independently by Du and Li [17] and by Gauthier and Latour [22] and Latour [39], and is different from that of Alzaid and Al-Osh [3], which assumes that the conditional distribution of the vector \((\alpha_1 \circ X_t, \alpha_2 \circ X_t, \ldots, \alpha_p \circ X_t)\) given \(X_t = x_t\) is multinomial with parameters \((\alpha_1, \alpha_2, \ldots, \alpha_p, x_t)\) and is independent of the past history of the process. The two different formulations imply different second-order structure for the processes: under the first approach, the INAR\((p)\) has the same second-order structure as an AR\((p)\) process, whereas under the second one, it has the same one as an ARMA\((p, p-1)\) process.

An alternative representation of the INAR\((p)\) process as a p-dimensional INAR(1) process was obtained by Franke and Subba Rao [20] and see also Latour [38, formula (2.3)]. Accordingly, the INAR\((p)\) process defined in (2.1) can be written as

\[
X_k = A \circ X_{k-1} + \varepsilon_k, \quad k \in \mathbb{N},
\]
where the $p$-dimensional random vectors $X_k$, $\varepsilon_k$ and the $(p \times p)$-matrix $A$ are defined by

$$
\begin{align*}
(2.2) \quad X_k := \begin{bmatrix} X_k \\ X_{k-1} \\ X_{k-2} \\ \vdots \\ X_{k-p+2} \\ X_{k-p+1} \end{bmatrix}, \quad A := \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \varepsilon_k := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\end{align*}
$$

and for a $p$-dimensional random vector $Y = (Y_1, \ldots, Y_p)$ and for a $p \times p$ matrix $B = (b_{ij})_{i,j=1}^{p}$ with entries satisfying $0 \leq b_{ij} \leq 1$, $i, j = 1, \ldots, p$, the matricial binomial thinning operation $B \circ Y$ is defined as a $p$-dimensional random vector whose $i$-th component, $i = 1, \ldots, p$, is given by

$$
\sum_{j=1}^{p} b_{ij} \circ Y_j,
$$

where the counting sequences of all $b_{ij} \circ Y_j$, $i, j = 1, \ldots, p$, are assumed independent of each other.

In what follows for the sake of simplicity we consider a zero start INAR($p$) process, that is we suppose $X_0 = X_{-1} = \ldots = X_{-p+1} = 0$. The general case of nonzero initial values may be handled in a similar way, but we renounce to consider it. For nonzero initial values the first and second order moments of the sequence $(X_k)_{k \in \mathbb{Z}_+}$ have a more complicated form than in Lemma 5.1. Further, for proving a corresponding version of our main result (see Theorem 3.1) one needs to apply a more general version of Theorem 5.1 which is also valid for random step functions not necessarily starting from 0.

In the sequel, we always assume that $E(\varepsilon_1^2) < \infty$. Let us denote the mean and variance of $\varepsilon_1$ by $\mu_\varepsilon$ and $\sigma_\varepsilon^2$, respectively.

For all $k \in \mathbb{Z}_+$, let us denote by $\mathcal{F}_k$ the $\sigma$-algebra generated by the random variables $X_0, X_1, \ldots, X_k$. (Note that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, since $X_0 = 0$.) By (2.1),

$$
E(X_k \mid \mathcal{F}_{k-1}) = \alpha_1 X_{k-1} + \cdots + \alpha_p X_{k-p} + \mu_\varepsilon, \quad k \in \mathbb{N}.
$$

Consequently,

$$
E(X_k) = \alpha_1 E(X_{k-1}) + \cdots + \alpha_p E(X_{k-p}) + \mu_\varepsilon, \quad k \in \mathbb{N}.
$$

This can also be written in the form $E(X_k) = \mathbf{A} E(X_{k-1}) + \mu_\varepsilon \mathbf{e}_1$, $k \in \mathbb{N}$, where $\mathbf{e}_1 := [1, 0, 0, \ldots, 0, 0]^\top \in \mathbb{R}^{p \times 1}$. Consequently, we have

$$
E(X_k) = \mu_\varepsilon \sum_{j=0}^{k-1} A^j \mathbf{e}_1, \quad k \in \mathbb{N},
$$

which implies

$$
E(X_k) = E(\mathbf{e}_1^\top X_k) = \mu_\varepsilon \sum_{j=0}^{k-1} \mathbf{e}_1^\top A^j \mathbf{e}_1, \quad k \in \mathbb{N}.
$$
Hence the matrix $A$ plays a crucial role in the description of asymptotic behavior of the sequence $(X_k)_{k \geq -p+1}$. Let $\rho(A)$ denote the spectral radius of $A$, i.e., the maximum of the modulus of the eigenvalues of $A$.

In what follows we collect some known facts about the matrix $A$. First we recall the notions of irreducibility and primitivity of a matrix. A matrix $M \in \mathbb{R}^{p \times p}$ is called reducible if $p = 1$ and $M = 0$, or if $p \geq 2$ and there exist a permutation matrix $P \in \mathbb{R}^{p \times p}$ and an integer $r$ with $1 \leq r \leq p - 1$ such that

$$P^T M P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where $B \in \mathbb{R}^{r \times r}$, $D \in \mathbb{R}^{(p-r) \times (p-r)}$, $C \in \mathbb{R}^{r \times (p-r)}$, and $0 \in \mathbb{R}^{(p-r) \times r}$ is a null matrix. A matrix $M \in \mathbb{R}^{p \times p}$ is called irreducible if it is not reducible, see, e.g., Horn and Johnson [26, Definitions 6.2.21 and 6.2.22]. A matrix $M \in \mathbb{R}^{p \times p}_+$ is called primitive if it is irreducible and has only one eigenvalue of maximum modulus, see, e.g., Horn and Johnson [26, Definition 8.5.0]. By Horn and Johnson [26, Theorem 8.5.2], a matrix $M \in \mathbb{R}^{p \times p}_+$ is primitive if and only if there exists a positive integer $k$ such that all the entries of the matrix $M^k$ are positive.

Let us denote by $\varphi$ the characteristic polynomial of the matrix $A$, i.e.,

$$\varphi(\lambda) := \det(\lambda I_p - A) = \lambda^p - \alpha_1 \lambda^{p-1} - \cdots - \alpha_{p-1} \lambda - \alpha_p, \quad \lambda \in \mathbb{C}.$$

2.1 Proposition. For $\alpha_1, \ldots, \alpha_p \in [0, 1]$, $\alpha_p > 0$, let us consider the matrix $A$ defined in (2.2). Then the following assertions hold:

(i) The characteristic polynomial $\varphi$ has just one positive root, $\rho(A) > 0$, the nonnegative matrix $A$ is irreducible, $\rho(A)$ is an eigenvalue of $A$ and

$$\sum_{k=1}^{p} \alpha_k \rho(A)^{-k} = 1.$$  \hspace{1cm} (2.5)

$$\sum_{k=1}^{p} k \alpha_k \rho(A)^{-k} = \rho(A)^{-p+1} \varphi'(\rho(A)).$$  \hspace{1cm} (2.6)

Further,

$$\rho(A) \begin{cases} < 1 & \Leftrightarrow \sum_{k=1}^{p} \alpha_k < 1, \\ > 1 & \end{cases}$$  \hspace{1cm} (2.7)

(ii) If $d = 1$, then $A$ is primitive, $\rho(A)$ is an eigenvalue of $A$, the algebraic and geometric multiplicity of $\rho(A)$ equal 1 and the absolute value of the other eigenvalues of $A$ are less than $\rho(A)$. Corresponding to the eigenvalue $\rho(A)$ there exists a unique vector $u_A \in \mathbb{R}^p$ with positive coordinates such that $Au_A = \rho(A)u_A$ and the sum of the coordinates of $u_A$ is 1, namely, $u_A$ takes the form

$$u_A = \left[u_{A,1}, \ldots, u_{A,p}\right]^\top \quad \text{with} \quad u_{A,i} := \frac{\rho(A)^{-i+1}}{\sum_{k=1}^{p} \rho(A)^{-k+1}}, \quad i = 1, \ldots, p.$$
Further,\begin{equation}
  \varphi(A) \rightarrow \Pi_A := u_A v_A^T, \quad \text{as } n \to \infty,
\end{equation}
where $v_A \in \mathbb{R}^p$ is a unique vector with positive coordinates such that $A^T v_A = \varphi(A) v_A$ and $u_A^T v_A = 1$, namely $v_A$ takes the form $v_A = [v_{A,1}, \ldots, v_{A,p}]^T$ with

\begin{equation}
  v_{A,i} := \sum_{k=1}^p \frac{\varphi(A)^{-k+1}}{k \alpha_k \varphi(A)^{-k}} \sum_{\ell=1}^p \alpha_\ell \varphi(A)^{i-\ell} = \sum_{k=1}^p \frac{\varphi(A)^{-k+1}}{\varphi(A)^{-p+1} \varphi''(\varphi(A))} \sum_{\ell=1}^p \alpha_\ell \varphi(A)^{i-\ell},
\end{equation}
for $i = 1, \ldots, p$. Moreover, there exist positive numbers $c_A$ and $r_A$ with $r_A < 1$ such that for all $n \in \mathbb{N}$

\begin{equation}
  \|\varphi(A)^{-n} A^n - \Pi A\| \leq c_A r_A^n,
\end{equation}
where $\|B\|$ denotes the operator norm of a matrix $B \in \mathbb{R}^{p \times p}$ defined by $\|B\| := \sup_{\|x\|=1} \|Bx\|$.

**Proof.** (i): First we check that $\varphi$ has just one positive root, which readily yields that $\varphi(A) > 0$. The function $\lambda \mapsto 1 - \lambda^{-p} \varphi(\lambda) = \alpha_1 \lambda^{-1} + \cdots + \alpha_{p-1} \lambda^{-p+1} + \alpha_p \lambda^{-p}$ is strictly decreasing and continuous on $(0, \infty)$ with $\lim_{\lambda \downarrow 0} (1 - \lambda^{-p} \varphi(\lambda)) = \infty$ and $\lim_{\lambda \uparrow \infty} (1 - \lambda^{-p} \varphi(\lambda)) = 0$, thus it takes the value 1 at exactly one positive point, which is the only positive root of $\varphi$.

Now we turn to check that $A$ is irreducible. By Brualdi and Cvetković [10, Definition 8.1.1 and Theorem 1.2.3], a nonnegative matrix $B = (b_{i,j})_{i,j=1,\ldots,p}$ is irreducible provided that its digraph (directed graph) $D(B)$ (having $p$ vertices labeled by the numbers $1, 2, \ldots, p$ and an edge from vertex $i$ to vertex $j$ provided $b_{i,j} > 0$) is strongly connected (that is, for each pair $i$ and $j$ of distinct vertices, there is a path from $i$ to $j$ and a path from $j$ to $i$). Now $\alpha_p > 0$ implies that $D(A)$ contains a cycle $1 \to p \to (p-1) \to \cdots \to 2 \to 1$, hence $D(A)$ is strongly connected.

Using that $A$ is nonnegative and irreducible, by Horn and Johnson [26, Theorem 8.4.4], we have $\varphi(A)$ is an eigenvalue of $A$ and hence

\begin{equation}
  \varphi(A)^p - \alpha_1 \varphi(A)^{p-1} - \cdots - \alpha_{p-1} \varphi(A) - \alpha_p = 0,
\end{equation}
which yields (2.5). Since

\begin{equation}
  \varphi'(\lambda) = p\lambda^{p-1} - (p-1)\alpha_1 \lambda^{p-2} - \cdots - \alpha_{p-1}, \quad \lambda \in \mathbb{C},
\end{equation}
we have

\begin{equation}
  \varphi'(\varphi(A)) = p\varphi(A)^{-1} \sum_{k=1}^p \alpha_k \varphi(A)^{p-k} - \sum_{k=1}^{p-1} (p-k) \alpha_k \varphi(A)^{p-k-1}
  = \sum_{k=1}^p k \alpha_k \varphi(A)^{p-k-1} = \varphi(A)^{p-1} \sum_{k=1}^p k \alpha_k \varphi(A)^{-k},
\end{equation}
which yields (2.6).
Further, \( (2.5) \) yields that

\[
\begin{cases}
\varrho(A) < 1, & \text{if } \varrho(A) = 1, \\
> 1, & \text{then } 1 = \sum_{k=1}^{p} \alpha_k \varrho(A)^{-k} > \sum_{k=1}^{p} \alpha_k.
\end{cases}
\]

This readily implies \( (2.7) \).

(ii): By Brualdi and Cvetković \[10\] Definition 8.2.1 and Theorem 8.2.7, an irreducible nonnegative matrix \( B = (b_{i,j})_{i,j=1,...,p} \) is primitive provided that the index of imprimitivity of \( B \) (the greatest common divisor of the lengths of the cycles of its digraph \( D(B) \)) equals 1. Now the cycles of \( D(A) \) are \( 1 \rightarrow i \rightarrow (i - 1) \rightarrow \cdots \rightarrow 2 \rightarrow 1 \) for all \( i = 1, \ldots, p \) such that \( \alpha_i > 0 \) (not considering rotations). Since such a cycle has length \( i \), we get the index of imprimitivity of \( A \) is \( d = 1 \), which yields that \( A \) is primitive.

The other assertions of (ii) except the uniqueness of \( u_A \) and \( v_A \) follows by the Frobenius-Perron theorem, see, e.g., Horn and Johnson \[26\] Theorems 8.2.11 and 8.5.1. The uniqueness of \( u_A \) follows by Horn and Johnson \[26\] Corollary 8.2.6] using that \( \varrho(A^m) = \varrho(A)^m \) for all \( m \in \mathbb{N} \). The uniqueness of \( v_A \) can be checked as follows. Using that the irreducibility and primitivity of \( A \) yields the irreducibility and primitivity of \( A^\top \) (see, e.g., page 507 in Horn and Johnson \[26\]), by Horn and Johnson \[26\] Theorems 8.2.11, 8.5.1 and Corollary 8.2.6] we get \( \varrho(A^\top) = \varrho(A) \) is an eigenvalue of \( A^\top \), the algebraic and geometric multiplicity of \( \varrho(A) \) equal 1, corresponding to the eigenvalue \( \varrho(A) \) there exists a unique vector \( \tilde{v}_A \in \mathbb{R}^p \) with positive coordinates such that \( A^\top \tilde{v}_A = \varrho(A) \tilde{v}_A \) and the sum of the coordinates of \( \tilde{v}_A \) is 1. Further, by Horn and Johnson \[26\] page 501, Problem 1], we also have \( u_A^\top \tilde{v}_A > 0 \). Using that the geometric multiplicity of \( \varrho(A^\top) = \varrho(A) \) equals 1, we get \( v_A := \frac{1}{u_A^\top \tilde{v}_A} \tilde{v}_A \) is a unique vector with positive coordinates such that \( A^\top v_A = \varrho(A) v_A \) and \( u_A^\top v_A = 1 \).

The forms of \( u_A \) and \( v_A \) can be checked as follows. Using that they are unique it remains to verify that the imposed conditions are satisfied by the given forms. We easily have \( u_A \) has positive coordinates of which the sum is 1. Further, with the notation \( A u_A = [(A u_A)_{1, \ldots, (A u_A)_p}]^\top \), we get

\[
(A u_A)_1 = \sum_{i=1}^{p} \alpha_i u_{A,i} = \sum_{i=1}^{p} \alpha_i \varrho(A)^{-i+1} = \frac{\varrho(A)}{\sum_{k=1}^{p} \varrho(A)^{-k+1}} \sum_{i=1}^{p} \alpha_i \varrho(A)^{-i} = \frac{\varrho(A)}{\sum_{k=1}^{p} \varrho(A)^{-k+1}} \sum_{i=1}^{p} \alpha_i \varrho(A)^{-i}.
\]

where the last but one equality follows by \( (2.5) \). Similarly, for \( i = 2, \ldots, p \), we get

\[
(A u_A)_i = u_{A,i-1} = \frac{\varrho(A)^{-i+2}}{\sum_{k=1}^{p} \varrho(A)^{-k+1}} = \varrho(A) u_{A,i}.
\]

Moreover, we easily have \( v_A \) has positive coordinates and

\[
u_A^\top v_A = \frac{1}{\varrho(A)^{-p+1} \varrho'(\varrho(A))} \sum_{i=1}^{p} \varrho(A)^{-i+1} \sum_{\ell=1}^{p} \alpha_\ell \varrho(A)^{i-1-\ell} \]

\[
= \frac{1}{\varrho(A)^{-p+1} \varrho'(\varrho(A))} \sum_{i=1}^{p} \sum_{\ell=1}^{p} \alpha_\ell \varrho(A)^{-\ell} = 1,
\]
where the last equality follows by (2.3). With the notation \( A^\top v_A = [(A^\top v_A)_1, \ldots, (A^\top v_A)_p]^\top \), we get for all \( i = 1, \ldots, p-1 \),

\[
(A^\top v_A)_i = \alpha_i v_{A,i} + v_{A,i+1} = \frac{\sum_{k=1}^{p} \varrho(A)^{-k+1}}{\varrho(A)^{-p+1} \varphi' (\varrho(A))} \left( \alpha_i + \sum_{\ell=i+1}^{p} \alpha_{\ell} \varrho(A)^{-i-\ell} \right) \cdot \frac{\sum_{k=1}^{p} \varrho(A)^{-k+1}}{\varrho(A)^{-p+1} \varphi' (\varrho(A))} \sum_{\ell=i}^{p} \alpha_{\ell} \varrho(A)^{-i-\ell} \cdot \varrho(A) v_{A,i}.
\]

Finally, using that \( \sum_{k=1}^{p} \alpha_k \varrho(A)^{-k} = 1 \), we get

\[
(A^\top v_A)_p = \alpha_p v_{A,1} = \frac{\sum_{k=1}^{p} \varrho(A)^{-k+1}}{\varrho(A)^{-p+1} \varphi' (\varrho(A))} \frac{\sum_{k=1}^{p} \varrho(A)^{-k+1}}{\varrho(A)^{-p+1} \varphi' (\varrho(A))} \varrho(A) \alpha_p \varrho(A)^{-1} = \varrho(A) v_{A,p}.
\]

\[\square\]

2.1 Remark. If \( \alpha_p > 0 \), \( d = 1 \) and \( \varrho(A) = 1 \), then the unique vectors \( u_A \) and \( v_A \) defined in (ii) of Proposition 2.1 take the forms \( u_A = \frac{1}{p} 1_p \) with \( 1_p : [1, \ldots, 1]^\top \in \mathbb{R}^{p \times 1} \), and

\[
v_A = \frac{p}{\alpha_1 + 2\alpha_2 + \cdots + p\alpha_p} \begin{bmatrix} \alpha_1 + \alpha_2 + \cdots + \alpha_p \\ \alpha_2 + \cdots + \alpha_p \\ \vdots \\ \alpha_p \end{bmatrix}.
\]

\[\square\]

2.2 Definition. An INAR(\( p \)) process \((X_n)_{n\geq p+1}\) with coefficients \( \alpha_1, \ldots, \alpha_p \) is called primitive if

(i) \( \alpha_p > 0 \),

(ii) \( d = 1 \), where \( d \) is the greatest common divisor of the set \( \{ i \in \{1, \ldots, p\} : \alpha_i > 0 \} \).

2.2 Remark. If \( \alpha_p = 0 \) and there exists \( i \in \{1, \ldots, p\} \) such that \( \alpha_i > 0 \), then \((X_n)_{n\geq p+1}\) is an INAR(\( p' \)) process with coefficients \( \alpha_1, \ldots, \alpha_{p'} \) with \( \alpha_{p'} > 0 \), where \( p' = \max \{ i \in \{1, \ldots, p\} : \alpha_i > 0 \} \). If \( \alpha_p > 0 \), but \( d \geq 2 \), then the process takes the form

\[
X_k = \alpha_d \circ X_{k-d} + \cdots + \alpha_{(p/d-1)d} \circ X_{k-(p/d-1)d} + \alpha_p \circ X_{k-p} + \varepsilon_k, \quad k \in \mathbb{N},
\]

and hence the subsequences \((X_{dn-j})_{n\geq p/d+1}, j = 0, 1, \ldots, d-1\), form independent primitive INAR(\( p/d \)) processes with coefficients \( \alpha_d, \alpha_{2d}, \ldots, \alpha_p \) such that \( X_{p+d-j} = X_{p+2d-j} = \cdots = X_{-j} = 0 \). Note also that in this case not all of the coefficients \( \alpha_d, \alpha_{2d}, \ldots, \alpha_p \) are necessarily positive. Finally, we remark that an INAR(\( p \)) process \((X_n)_{n\geq p+1}\) is primitive if and only if its matrix \( A \) defined in (2.2) is primitive. Indeed, if \((X_n)_{n\geq p+1}\) is primitive, then part (ii) of Proposition 2.1 readily yields that \( A \) is primitive. Conversely (using the notations of the proof of Proposition 2.1), if \( A \) is primitive, then, by the proof of part (i) of Proposition 2.1, the
digraph \( D(\mathbf{A}) \) is strongly connected. This yields that \( \alpha_p > 0 \), since otherwise there would be no path from 1 to \( p \). Further, the primitivity of \( \mathbf{A} \) yields that the index of imprimitivity of \( \mathbf{A} \) equals 1. Using that the cycles of \( D(\mathbf{A}) \) are \( 1 \to i \to (i-1) \to \cdots \to 2 \to 1 \) for all \( i = 1, \ldots, p \) such that \( \alpha_i > 0 \) (not considering rotations) and such a cycle has length \( i \), we get \( d = 1 \).

The next proposition is about the limit behavior of \( E(X_k) \) as \( k \to \infty \). This proposition can also be considered as a motivation for the classification of INAR(\( p \)) processes, see later on.

2.2 Proposition. Let \((X_n)_{n \geq -p+1}\) be an INAR(\( p \)) process such that \( X_0 = X_{-1} = \cdots = X_{-p+1} = 0 \) and \( E(\varepsilon_1^2) < \infty \). Then the following assertions hold:

(i) If \( \varrho(\mathbf{A}) < 1 \), then

\[
\lim_{k \to \infty} E(X_k) = \frac{\mu_\varepsilon}{1 - \sum_{i=1}^p \alpha_i}.
\]

(ii) If \( \varrho(\mathbf{A}) = 1 \), then

\[
\lim_{k \to \infty} k^{-1} E(X_k) = \frac{\mu_\varepsilon}{\sum_{i=1}^p i \alpha_i} = \frac{\mu_\varepsilon}{\varphi'(1)},
\]

where \( \varphi \) is the characteristic polynomial of the matrix \( \mathbf{A} \) defined in (2.2).

(iii) If \( \varrho(\mathbf{A}) > 1 \), then

\[
\lim_{k \to \infty} \varrho(\mathbf{A})^{-kd} E(X_{kd-j}) = \frac{d \mu_\varepsilon}{(\varrho(\mathbf{A})^d - 1) \sum_{k=1}^p k \alpha_k \varrho(\mathbf{A})^{-k}} = \frac{d \mu_\varepsilon \varrho(\mathbf{A})^{p-1}}{(\varrho(\mathbf{A})^d - 1) \varphi'(\varrho(\mathbf{A}))}
\]

for all \( j = 0, 1, \ldots, d-1 \), where \( d \) is the greatest common divisor of the set \( \{i \in \{1, \ldots, p\} : \alpha_i > 0\} \).

Proof. If \( \alpha_1 = \cdots = \alpha_p = 0 \), then \( \varrho(\mathbf{A}) = 0 \) and \( X_k = \varepsilon_k, k \in \mathbb{N} \), which yields that \( \lim_{k \to \infty} E(X_k) = \mu_\varepsilon \), i.e., part (i) is satisfied in the case of \( \alpha_1 = \cdots = \alpha_p = 0 \). If not all of the coefficients \( \alpha_1, \ldots, \alpha_p \) are 0, then, by Remark 2.2, \((X_n)_{n \geq -p+1}\) is an INAR(\( p' \)) process where \( p' = \max \{i \in \{1, \ldots, p\} : \alpha_i > 0\} \). Hence in what follows we may and do suppose that the original process \((X_n)_{n \geq -p+1}\) is such that \( \alpha_p > 0 \).

First we prove the proposition in the case of \( \alpha_p > 0 \) and \( d = 1 \), i.e., in the case of \((X_n)_{n \geq -p+1}\) is primitive.

Proof of (i) in the case of \( \alpha_p > 0 \) and \( d = 1 \): In this case we verify that

\[
\lim_{k \to \infty} E(X_k) = \mu_\varepsilon e_1^\top \sum_{j=0}^\infty A^j e_1 = \mu_\varepsilon e_1^\top (I_p - A)^{-1} e_1 = \frac{\mu_\varepsilon}{1 - \alpha_1 - \cdots - \alpha_p}.
\]

By (2.3), it is enough to prove that if \( \varrho(\mathbf{A}) < 1 \), then the series \( \sum_{j=0}^\infty A^j \) is convergent and its sum is \((I_p - A)^{-1}\). By (2.2), we have

\[
\sum_{j=0}^\infty \|A^j\| \leq \sum_{j=0}^\infty \varrho(\mathbf{A})^j (\|\mathbf{A}^{-1}\| + \|\mathbf{A}\|) \leq \sum_{j=0}^\infty \varrho(\mathbf{A})^j c_{\mathbf{A}} r_{\mathbf{A}}^j + \sum_{j=0}^\infty \varrho(\mathbf{A})^j \|\mathbf{A}\| < \infty,
\]
since \( \varrho(A) < 1 \) and \( r_A < 1 \). One can give another proof for the convergence of \( \sum_{j=0}^{\infty} \| A^j \| \). Indeed, by Horn and Johnson \cite{Horn26} Corollary 5.6.14, we have \( \varrho(A) = \lim_{n \to \infty} \| A^n \|^{1/n} \) and hence comparison test yields the assertion. Finally, by Lemma 5.6.10 and Corollary 5.6.16 in Horn and Johnson \cite{Horn26}, we have \( \sum_{j=0}^{\infty} A^j = (I_p - A)^{-1} \), and hence, by Cramer’s rule,

\[
e_1^\top (I_p - A)^{-1} e_1 = \frac{1}{\det(I_p - A)} = \frac{1}{\varphi(1)} = \frac{1}{1 - \alpha_1 - \cdots - \alpha_p}.
\]

Proof of (ii) in the case of \( \alpha_p > 0 \) and \( d = 1 \): In this case we verify that

\[
\lim_{k \to \infty} k^{-1} \mathbb{E}(X_k) = \mu \varepsilon e_1^\top \Pi_A e_1 = \frac{\mu \varepsilon}{\sum_{i=1}^p i \alpha_i} = \frac{\mu \varepsilon}{\varphi'(1)}.
\]

By \eqref{2.4}, we get

\[
\mathbb{E}(X_k) = \mu \varepsilon e_1^\top \sum_{j=0}^{k-1} A^j e_1 = \mu \varepsilon e_1^\top \sum_{j=0}^{k-1} (\Pi_A + (A^j - \Pi_A)) e_1
\]

\[
= k \mu \varepsilon e_1^\top \Pi_A e_1 + \mu \varepsilon e_1^\top \sum_{j=0}^{k-1} (A^j - \Pi_A) e_1, \quad k \in \mathbb{N}.
\]

By \eqref{2.9}, we have

\[
\sum_{j=0}^{\infty} \| A^j - \Pi_A \| \leq \sum_{j=0}^{\infty} c_A r_A^j < \infty,
\]

which yields that

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} (A^j - \Pi_A) = 0,
\]

where \( 0 \) denotes the \( p \times p \) nullmatrix. This implies \( \lim_{k \to \infty} k^{-1} \mathbb{E}(X_k) = \mu \varepsilon e_1^\top \Pi_A e_1 \). By Proposition \ref{2.1} in the case of \( \alpha_p > 0 \) and \( d = 1 \) (\( \varrho(A) \) is not necessarily 1) we have

\[
e_1^\top \Pi_A e_1 = e_1^\top u_A v_A^\top e_1 = u_{A,1} v_{A,1} = \sum_{k=1}^p \alpha_k \varrho(A)^{-k} = \frac{\varrho(A)^{p-1}}{\varrho'(\varrho(A))}.
\]

By \eqref{2.7}, we have \( \alpha_1 + \cdots + \alpha_p = 1 \), and hence

\[
e_1^\top \Pi_A e_1 = \frac{1}{\varphi(1)} = \frac{1}{p - (p-1)\alpha_1 - (p-2)\alpha_2 - \cdots - 2\alpha_{p-2} - \alpha_{p-1}}
\]

\[
= \frac{1}{\sum_{i=1}^{p-1} i \alpha_i + p \left( 1 - \sum_{i=1}^{p-1} \alpha_i \right)},
\]

which yields part (ii) in the case of \( \alpha_p > 0 \) and \( d = 1 \).

Proof of (iii) in the case of \( \alpha_p > 0 \) and \( d = 1 \): In this case we verify that

\[
\lim_{k \to \infty} \varrho(A)^{-k} \mathbb{E}(X_k) = \frac{\mu \varepsilon}{\varrho(A) - 1} e_1^\top \Pi_A e_1 = \frac{\mu \varepsilon}{(\varrho(A) - 1) \sum_{k=1}^p k \alpha_k \varrho(A)^{-k}}
\]

\[
= \frac{\mu \varepsilon \varrho(A)^{p-1}}{(\varrho(A) - 1) \varphi'(\varrho(A))}.
\]
By (2.4), we get for all \( k \in \mathbb{N} \),

\[
\varrho(A)^{-k} \mathbb{E}(X_k) = \varrho(A)^{-k} \mu e_1^\top \sum_{j=0}^{k-1} A^j e_1 = \varrho(A)^{-k} \mu e_1^\top \sum_{j=0}^{k-1} (\varrho(A)^j \Pi_A + (A^j - \varrho(A)^j \Pi_A)) e_1
\]

\[
= \mu e_1^\top \sum_{j=0}^{k-1} \varrho(A)^{-k-j} \Pi_A e_1 + \mu e_1^\top \varrho(A)^{-k} \sum_{j=0}^{k-1} (A^j - \varrho(A)^j \Pi_A) e_1.
\]

Since \( \varrho(A)^{-1} < 1 \), we have

\[
\sum_{j=0}^{k-1} \varrho(A)^{-k-j} = k \left( \frac{\varrho(A)^{-1}}{\varrho(A) - 1} \right)^\ell \to \frac{1}{\varrho(A) - 1} \quad \text{as} \quad k \to \infty.
\]

Further, by (2.9), for all \( k \in \mathbb{N} \),

\[
\left\| \varrho(A)^{-k} \sum_{j=0}^{k-1} (A^j - \varrho(A)^j \Pi_A) \right\| \leq k \left( \sum_{j=0}^{k-1} \varrho(A)^{-k-j} \right) \| A^j - \varrho(A)^j \Pi_A \| \leq c_A \sum_{j=0}^{k-1} \varrho(A)^{-k-j} r_A^j.
\]

If \( \varrho(A)r_A \neq 1 \), then

\[
\left\| \varrho(A)^{-k} \sum_{j=0}^{k-1} (A^j - \varrho(A)^j \Pi_A) \right\| \leq c_A \frac{\varrho(A)^{-k} - r_A^k}{1 - \varrho(A) r_A} \to 0 \quad \text{as} \quad k \to \infty,
\]

since \( \varrho(A) > 1 \) and \( r_A < 1 \). If \( \varrho(A)r_A = 1 \), then

\[
\left\| \varrho(A)^{-k} \sum_{j=0}^{k-1} (A^j - \varrho(A)^j \Pi_A) \right\| \leq c_A \frac{k}{\varrho(A)^k} \to 0 \quad \text{as} \quad k \to \infty.
\]

Using also (2.6) and (2.10), this concludes (iii) in the case of \( \alpha_p > 0 \) and \( d = 1 \).

Now we turn to give a proof in the case of \( \alpha_p > 0 \) and \( d \geq 2 \). In this case, by Proposition 2.7, \( A \) is irreducible, \( \varrho(A) > 0 \) and, by Remark 2.2, the subsequences \( (X_{dn-j})_{n \geq -p/d+1} \) form independent primitive INAR(p/d) processes with coefficients \( \alpha_d, \alpha_{2d}, \ldots, \alpha_p \) such that \( X_{-p+d-j} = X_{-p+2d-j} = \cdots = X_{-j} = 0 \). Let us introduce the matrix

\[
\tilde{A} := \begin{bmatrix}
\alpha_d & \alpha_{2d} & \alpha_{3d} & \cdots & \alpha_{p-d} & \alpha_p \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix} \in \mathbb{R}^{(p/d) \times (p/d)},
\]

and its characteristic polynomial

\[
\tilde{\varphi}(\lambda) := \det(\lambda I_{p/d} - \tilde{A}) = \lambda^{p/d} - \alpha_d \lambda^{p/d-1} - \alpha_{2d} \lambda^{p/d-2} - \cdots - \alpha_{p-d} \lambda - \alpha_p, \quad \lambda \in \mathbb{C}.
\]

Since the greatest common divisor of the set \( \{ i \in \{1, \ldots, p/d\} : \alpha_{id} > 0 \} \) is 1, by Proposition 2.7, we have \( \tilde{A} \) is primitive. We check that \( \varrho(A)^d = \varrho(\tilde{A}) \). Since \( \varphi(\lambda) = \tilde{\varphi}(\lambda^d), \lambda \in \mathbb{C}, \)

Since
we get \(\varrho(A)^d \leq \varrho(\tilde{A})\). By Proposition 2.1, \(\varrho(\tilde{A}) > 0\) and \(\varrho(\tilde{A})\) is an eigenvalue of \(\tilde{A}\). Hence \(\varrho(\tilde{A})^{1/d}\) is an eigenvalue of \(A\), which implies that \(\varrho(A) \geq \varrho(\tilde{A})^{1/d}\) or equivalently \(\varrho(A)^d \geq \varrho(\tilde{A})\).

If \(\varrho(A) < 1\), then \(\varrho(\tilde{A}) < 1\) and using that part (i) has already been proved for primitive matrices (i.e., in the case of \(\alpha_p > 0\) and \(d = 1\)) we have for all \(j = 0, 1, \ldots, d - 1\),

\[
\lim_{n \to \infty} \frac{E(X_{n_{d-j}})}{n} = \frac{\mu_\varepsilon}{\alpha_d + 2\alpha_{d+1} + \cdots + 2\alpha_p} = \frac{d\mu_\varepsilon}{\sum_{i=1}^p i\alpha_i}.
\]

This yields that \(\lim_{n \to \infty} E(X_n)\) exists with the given limit in (i).

If \(\varrho(A) = 1\), then \(\varrho(\tilde{A}) = 1\) and using that part (ii) has already been proved for primitive matrices we have for all \(j = 0, 1, \ldots, d - 1\),

\[
\lim_{n \to \infty} \frac{E(X_{n_{d-j}})}{\varrho(A)^n} = \frac{\mu_\varepsilon}{(\varrho(A) - 1)\sum_{k=1}^{p/d} k\alpha_k\varrho(A)^{-k}} = \frac{d\mu_\varepsilon}{(\varrho(A) - 1)\sum_{\ell=1}^{p}\ell\alpha_\ell\varrho(A)^{-\ell}} = \frac{d\mu_\varepsilon \varrho(A)^{p-1}}{(\varrho(A) - 1)\varphi'(\varrho(A))},
\]

where the last equality follows by (2.6). Since

\[
\lim_{k \to \infty} \frac{E(X_{k_{d-j}})}{\varrho(A)^{kd}} = \lim_{k \to \infty} \frac{E(X_{k_{d-j}})}{\varrho(A)^{k}}, \quad j = 0, 1, \ldots, d - 1,
\]

we have (iii).

Based on the asymptotic behavior of \(E(X_k)\) as \(k \to \infty\) described in Proposition 2.2, we distinguish three cases. The case \(\varrho(A) < 1\) is called stable or asymptotically stationary, whereas the cases \(\varrho(A) = 1\) and \(\varrho(A) > 1\) are called unstable and explosive, respectively. Note also that, if \(\alpha_p > 0\), then, by (2.7) of Proposition 2.1, \(\varrho(A) < 1\), \(\varrho(A) = 1\) and \(\varrho(A) > 1\) are equivalent with \(\alpha_1 + \cdots + \alpha_p < 1\), \(\alpha_1 + \cdots + \alpha_p = 1\) and \(\alpha_1 + \cdots + \alpha_p > 1\), respectively.

### 3 Convergence of unstable INAR\((p)\) processes

A function \(f : \mathbb{R}_+ \to \mathbb{R}\) is called càdlàg if it is right continuous with left limits. Let \(D(\mathbb{R}_+, \mathbb{R})\) and \(C(\mathbb{R}_+, \mathbb{R})\) denote the space of all real-valued càdlàg and continuous functions on \(\mathbb{R}_+\), respectively. Let \(\mathcal{D}_\infty\) denote the Borel \(\sigma\)-field in \(D(\mathbb{R}_+, \mathbb{R})\) for the metric defined in (16.4) in Billingsley \[5\] (with this metric \(D(\mathbb{R}_+, \mathbb{R})\) is a complete and separable metric space). For stochastic processes \((\mathcal{Y}_t)_{t \in \mathbb{R}_+}\) and \((\mathcal{Y}_n^n)_{t \in \mathbb{R}_+}, \ n \in \mathbb{N}\), with càdlàg paths we write \(\mathcal{Y}^n \xrightarrow{\mathcal{L}} \mathcal{Y}\).
3.2 Remark. Proposition 4.1 describes the fluctuation limit behavior of nearly unstable \( \text{INAR}(1) \) processes. 

For each \( n \in \mathbb{N} \), consider the random step processes 
\[
X^n_t := n^{-1}X_{[nt]}, \quad t \in \mathbb{R}_+, \ n \in \mathbb{N},
\]
where \( [x] \) denotes the integer part of a real number \( x \in \mathbb{R} \). The positive part of \( x \in \mathbb{R} \) will be denoted by \( x^+ \).

3.1 Theorem. Let \((X_k)_{k \geq -p+1} \) be a primitive \( \text{INAR}(p) \) process with coefficients \( \alpha_1, \ldots, \alpha_p \in [0, 1] \) such that \( \alpha_1 + \cdots + \alpha_p = 1 \) (hence it is unstable). Suppose that \( X_0 = X_{-1} = \cdots = X_{-p+1} = 0 \) and \( \mathbb{E}(\varepsilon_1^2) < \infty \). Then
\[
\text{as } n \to \infty,
\]
where \((X_t)_{t \in \mathbb{R}_+} \) is the unique strong solution of the stochastic differential equation (SDE)
\[
dX_t = \frac{1}{\varphi'(1)}  \left( \mu_x \, dt + \sqrt{\sigma^2 \varphi'(1)} \, dW_t \right), \quad t \in \mathbb{R}_+,
\]
with initial value \( X_0 = 0 \), where 
\[
\varphi'(1) = \alpha_1 + 2\alpha_2 + \cdots + p\alpha_p > 0, \quad \sigma^2 := \alpha_1(1-\alpha_1) + \cdots + \alpha_p(1-\alpha_p),
\]
and \((W_t)_{t \in \mathbb{R}_+} \) is a standard Wiener process. (Here \( \varphi \) is the characteristic polynomial of the matrix \( A \) defined in (2.2).)

3.1 Remark. Note that under the conditions Theorem 3.1, if \( p \geq 2 \), then \( \sigma^2 > 0 \), and if \( p = 1 \), then \( \sigma^2 = 0 \). Indeed, if \( p \geq 2 \), then \( \alpha_p < 1 \), since otherwise \( \alpha_1 = \cdots = \alpha_{p-1} = 0 \) and hence the greatest common divisor of \( \{i \in \{1, \ldots, p\} : \alpha_i > 0\} = \{p\} \) would be \( p \), which is a contradiction. Since, by our assumption \( \alpha_p > 0 \), we get \( \sigma^2 \geq \alpha_p(1-\alpha_p) > 0 \). If \( p = 1 \), then \( \alpha_p = \alpha_1 = 1 \), and hence \( \sigma^2 = \alpha_1(1-\alpha_1) = 0 \).

Remark also that in the case of \( p = 1 \) we have \( \alpha_1 = 1 \) and hence \( X_n = \sum_{i=1}^n \varepsilon_i, \ n \in \mathbb{N}, \varphi'(1) = 1, \ \sigma^2 = 0 \) and then the limit process in Theorem 3.1 is deterministic, namely \( X_t = \mu_x t, \ t \in \mathbb{R}_+ \). To describe the asymptotic behavior of an unstable \( \text{INAR}(1) \) process one has to go one step further and one has to investigate the fluctuation limit. By Donsker’s theorem (see, e.g., Billingsley [5, Theorem 8.2]), we have
\[
\sqrt{n}(X^n_t - \mathbb{E}(X^n)) \overset{\mathcal{L}}{\to} \sigma \mathcal{W} \text{ as } n \to \infty,
\]
where \( \mathcal{W} \) is a standard Wiener process. For completeness, we remark that Ispán, Pap and Zuijlen [27 Proposition 4.1] describes the fluctuation limit behavior of nearly unstable \( \text{INAR}(1) \) processes.

3.2 Remark. The SDE (3.2) has a unique strong solution \((X^n_t)_{t \geq 0}\) for all initial values \( X^n_0 = x \in \mathbb{R} \). Indeed, since \( |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}, \ x, y \geq 0, \) the coefficient functions \( \mathbb{R} \ni x \mapsto \mu_x/\varphi'(1) \) and \( \mathbb{R} \ni x \mapsto \sqrt{\sigma^2 x^+/\varphi'(1)} \) satisfy conditions of part (ii) of Theorem 3.5 in Chapter IX in Revuz and Yor [52] or the conditions of Proposition 5.2.13 in Karatzas and Shreve [37]. Further, by the comparison theorem (see, e.g., Revuz and Yor [52, Chapter 3.7, Chapter IX]), if the initial value \( X^n_0 = x \) is nonnegative, then \( X^n_t \) is nonnegative for all \( t \in \mathbb{R}_+ \) with probability one. Hence \( X^n_+ \) may be replaced by \( X^n_t \) under the square root in (3.2). The unique strong solution of the SDE (3.2) is known as a squared Bessel process, a squared-root process or a Cox-Ingersoll-Ross (CIR) process.

\[\square\]
\textbf{3.3 Remark.} If the matrix \( A \) is not primitive but unstable, then we can suppose that \( \alpha_p > 0 \), since otherwise it is an unstable INAR(\( p' \)) process with \( p' := \max \{ i \in \{ 1, \ldots, p \} : \alpha_i > 0 \} \) (note that there exists an \( i \in \{ 1, \ldots, p \} \) such that \( \alpha_i > 0 \) because of the unstability of \( A \)). If \( \alpha_p > 0 \) and \( d \geq 2 \), then, by Remark 2.2, the subsequences \( (X_{dn-j})_{n \geq d/p+1}, j = 0, 1, \ldots, d-1 \), form independent primitive INAR(\( p/d \)) processes with coefficients \( \alpha_{d}, \alpha_{2d}, \ldots, \alpha_{p} \) such that \( X_{-p+d-j} = X_{-p+2d-j} = \cdots = X_{-j} = 0 \). Hence one can use Theorem 3.1 for these subsequences. With the notations

\[ X^m_{n,j} := \frac{1}{n} X_{d[n]_j}, \quad t \in \mathbb{R}_+, \quad n \geq \frac{p}{d} + 1, \quad j = 0, 1, \ldots, d-1, \]

by Theorem 3.1 \( X^{n,j} \xrightarrow{L} X^{(j)} \) as \( n \to \infty \), where \( (X^{(j)})_{t \in \mathbb{R}_+} \) is the unique strong solution of the SDE

\[ dX^{(j)}_t = \frac{1}{\alpha_d + 2\alpha_{2d} + \cdots + \frac{2}{d}\alpha_p} \left( \mu \xi dt + \sqrt{\sigma^2 (X^{(j)}_t)} dW^{(j)}_t \right), \quad t \in \mathbb{R}_+, \]

with initial value \( X^{(j)}_0 = 0 \) and \( (W^{(j)}_t)_{t \in \mathbb{R}_+}, j = 0, 1, \ldots, d-1 \), are independent standard Wiener processes. We note that if \( \alpha_p > 0 \) and \( d \geq 2 \), then \( X^n \) does not converge in general as \( n \to \infty \). By giving a counterexample, we show that even the 2-dimensional distributions do not converge in general. Let \( p := 4, \alpha_1 = \alpha_3 := 0, \alpha_2 = \alpha_4 := 1/2 \). Then \( d = 2 \) and using that \( X^{n,j} \xrightarrow{L} X^{(j)} \) as \( n \to \infty, j = 0, 1, \) we have

\[ (3.3) \quad [X^{n,0}_1, X^{n,0}_2] = \left[ \frac{1}{n} X_{2n}, \frac{1}{n} X_{4n} \right] \text{ converges in distribution to } [X^{(0)}_1, X^{(0)}_2] \]

as \( n \to \infty \), and

\[ (3.4) \quad [X^{n,1}_1, X^{n,1}_2] = \left[ \frac{1}{n} X_{2n-1}, \frac{1}{n} X_{4n-1} \right] \text{ converges in distribution to } [X^{(1)}_1, X^{(1)}_2] \]

as \( n \to \infty \), where \( (X^{(j)}_t)_{t \in \mathbb{R}_+} \) is the unique strong solution of the SDE

\[ dX^{(j)}_t = \frac{2}{3} \left( \mu \xi dt + \sqrt{\frac{1}{2} (X^{(j)}_t)_+} dW^{(j)}_t \right), \quad t \in \mathbb{R}_+, \]

with initial value \( X^{(j)}_0 = 0, j = 0, 1 \). However, we show that

\[ (3.5) \quad [X^n_1, X^n_2] = \left[ \frac{1}{n} X_n, \frac{1}{n} X_{2n} \right] \text{ does not converge in distribution as } n \to \infty. \]

Indeed, we have

\[ [X^{2n}_1, X^{2n}_2] = \left[ \frac{1}{2n} X_{2n}, \frac{1}{2n} X_{4n} \right] = \left[ \frac{1}{2} X^{n,0}_1, \frac{1}{2} X^{n,0}_2 \right] \]

and hence, by (3.3),

\[ [X^{2n}_1, X^{2n}_2] \text{ converges in distribution to } \left[ \frac{1}{2} X^{(0)}_1, \frac{1}{2} X^{(0)}_2 \right] \text{ as } n \to \infty. \]
Further, using that
\[ [X_{2n-1}^2, X_{2n-1}^0] = \left[ \frac{1}{2n-1} X_{2n-1}, \frac{1}{2n-1} X_{2(2n-1)} \right] = \left[ \frac{n}{2n-1} X_{1}^{n,1}, X_{1}^{2n-1,0} \right] \]
and that the subsequences \((X_{2n-1})_{n \in \mathbb{N}}\) and \((X_{2(2n-1)})_{n \in \mathbb{N}}\) are independent, by (3.3) and (3.4), we get
\[ [X_{2n-1}^2, X_{2n-1}^0] \text{ converges in distribution to } \left[ \frac{1}{2} X_{1}^{1}, X_{1}^{0} \right] \text{ as } n \to \infty. \]

Since the random variables
\[ \left[ \frac{1}{2} X_{1}^{0}, \frac{1}{2} X_{2}^{0} \right] \text{ and } \left[ \frac{1}{2} X_{1}^{1}, X_{1}^{0} \right] \]
do not have the same distributions (the coordinates of the first one are dependent, however the coordinates of the second one are independent), we get (3.5).

For proving Theorem 3.1, let us introduce the sequence
\[ (3.6) \quad M_k := X_k - E(X_k | \mathcal{F}_{k-1}) = X_k - \alpha_1 X_{k-1} - \cdots - \alpha_p X_{k-p} - \mu \varepsilon, \quad k \in \mathbb{N}, \]
of martingale differences with respect to the filtration \((\mathcal{F}_k)_{k \in \mathbb{N}}\), and the random step processes
\[ \mathcal{M}_t^n := n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} M_k, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]
First we will verify convergence
\[ (3.7) \quad \mathcal{M}^n \xrightarrow{\mathcal{L}} \mathcal{M} \quad \text{as } n \to \infty, \]
where \( (\mathcal{M}_t)_{t \in \mathbb{R}_+} \) is the unique strong solution of the SDE
\[ (3.8) \quad d\mathcal{M}_t = \sqrt{\frac{\sigma^2}{\varphi'(1)} (\mathcal{M}_t + \mu \varepsilon t)^+} d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \]
with initial value \( \mathcal{M}_0 = 0 \). The proof of (3.7) can be found in Section 4.

3.4 Remark. If \( (X_t^x)_{t \in \mathbb{R}_+} \) is a strong solution of (3.2) with initial value \( X_0^x = x \in \mathbb{R} \), then, by Itô’s formula, \( \mathcal{M}_t^x := \varphi'(1) X_t^x - \mu \varepsilon t, \quad t \in \mathbb{R}_+, \) is a strong solution of (3.8) with initial value \( \mathcal{M}_0^x = \varphi'(1)x \). On the other hand, if \( (\mathcal{M}_t^y)_{t \in \mathbb{R}_+} \) is a strong solution of (3.8) with initial value \( \mathcal{M}_0^y = y \in \mathbb{R} \), then, again by Itô’s formula,
\[ (3.9) \quad X_t^y := \frac{1}{\varphi'(1)} (\mathcal{M}_t^y + \mu \varepsilon t), \quad t \in \mathbb{R}_+, \]
is a strong solution of (3.2) with initial value \( X_0^y = \frac{1}{\varphi'(1)}y \). Hence, by Remark 3.2, the SDE (3.8) has a unique strong solution \( (\mathcal{M}_t^y)_{t \geq 0} \) for all initial values \( \mathcal{M}_0^y = y \in \mathbb{R} \). Further, if the initial value \( \mathcal{M}_0^y = y \) is nonnegative, then \( \mathcal{M}_t^y \) is nonnegative for all \( t \in \mathbb{R}_+ \) with probability one. Hence \( (\mathcal{M}_t + \mu \varepsilon t)^+ \) may be replaced by \( (\mathcal{M}_t + \mu \varepsilon t) \) under the square root in (3.8).
Moreover, from (3.6) we obtain the recursion
\[ X_k = \alpha_1 X_{k-1} + \cdots + \alpha_p X_{k-p} + M_k + \mu_\varepsilon, \quad k \in \mathbb{N}, \]
which can be written in the form \( X_k = AX_{k-1} + (M_k + \mu_\varepsilon)e_1, \quad k \in \mathbb{N} \). Consequently,
\[ X_k = \sum_{j=1}^{k} (M_j + \mu_\varepsilon)A^{k-j}e_1, \quad k \in \mathbb{N}, \]
implying
\[ (3.10) \quad X_k = e_1^T X_k = \sum_{j=1}^{k} (M_j + \mu_\varepsilon)e_1^T A^{k-j}e_1, \quad k \in \mathbb{N}. \]

In Section 4, we show that the statement (3.1) will follow from (3.7) and (3.10) using a version of the continuous mapping theorem (see Appendix).

4 Proof of Theorem 3.1

For the proof we will use Corollary 5.1, Theorem 5.1 and Lemma 5.2 which can be found in Appendix.

First we prove (3.7), i.e., \( M^n \xrightarrow{L^2} M \) as \( n \to \infty \). We will apply Theorem 5.1 for \( U = M \), \( U^n_k = n^{-1}M_k, \quad n, k \in \mathbb{N} \), and for \( (\mathcal{F}^n_k)_{k \in \mathbb{Z}^+} = (\mathcal{F}_k)_{k \in \mathbb{Z}^+}, \quad n \in \mathbb{N} \). By Remark 3.4 the SDE (3.8) has a unique strong solution for all initial values \( M_0^x = x, \quad x \in \mathbb{R} \). Now we show that conditions (i) and (ii) of Theorem 5.1 hold. We have to check that for each \( T > 0 \),
\[
\sup_{t \in [0,T]} \left| \frac{1}{n^2} \sum_{k=1}^{[nt]} E\left( M_k^2 \mid \mathcal{F}_{k-1} \right) - \frac{\sigma^2_n}{\varphi '(1)} \int_0^T (M_s^n + \mu_\varepsilon s)^+ ds \right| \xrightarrow{P} 0,
\]
\[
\frac{1}{n^2} \sum_{k=1}^{[nT]} E\left( M_k^2 \mathbb{1}_{\{|M_k| > n\theta\}} \mid \mathcal{F}_{k-1} \right) \xrightarrow{P} 0 \quad \text{for all } \theta > 0
\]
as \( n \to \infty \), where \( \xrightarrow{P} \) means convergence in probability.

By (3.6) and using also that \( \alpha_1 + \cdots + \alpha_p = 1 \), we get
\[
M_0^x + \mu_\varepsilon s = n^{-1} \sum_{k=1}^{[ns]} \left( X_k - \sum_{i=1}^{p} \alpha_i X_{k-i} - \mu_\varepsilon \right) + \mu_\varepsilon s
\]
\[
= n^{-1} \left( \sum_{k=[ns]-p+1}^{[ns]} X_k - \sum_{i=1}^{p-1} \alpha_i \sum_{k=[ns]-p+1}^{[ns]-i} X_k \right) + \frac{ns - [ns]}{n} \mu_\varepsilon
\]
\[
= \frac{1}{n} \sum_{j=1}^{p} \sum_{i=j}^{p} \alpha_i X_{[ns] - j + 1} + \frac{ns - [ns]}{n} \mu_\varepsilon.
\]
Thus \((M^n_t + \mu_x s)^+ = M^n_t + \mu_x s\), and using that
\[
\int_0^t \frac{n s - \lfloor ns \rfloor}{n} \, ds = \frac{t^2}{2} - \frac{1}{n} \left( \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor - 1} k + \left( t - \frac{\lfloor nt \rfloor}{n} \right) \lfloor nt \rfloor \right) = \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2},
\]
we get
\[
\int_0^t (M^n_t + \mu_x s)^+ \, ds = \frac{1}{n^2} \sum_{\ell=0}^{\lfloor nt \rfloor - 1} \sum_{j=1}^p \sum_{i=j}^p \alpha_i X_{\ell-j+1} + \frac{\lfloor nt \rfloor}{n^2} \sum_{j=1}^p \sum_{i=j}^p \alpha_i X_{\lfloor nt \rfloor-j+1}
\]
\[+ \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} \mu_x.
\]
Hence, using that \(\varphi'(1) = \alpha_1 + 2\alpha_2 + \cdots + p\alpha_p\), we have
\[
\int_0^t (M^n_t + \mu_x s)^+ \, ds = \frac{\varphi'(1)}{n^2} \sum_{\ell=0}^{\lfloor nt \rfloor - 1} X_{\ell} - \frac{1}{n^2} \sum_{i=1}^p \alpha_i \sum_{j=\lfloor nt \rfloor-i+1}^{\lfloor nt \rfloor-1} X_j
\]
\[+ \frac{\lfloor nt \rfloor}{n^2} \sum_{j=1}^p \sum_{i=j}^p \alpha_i X_{\lfloor nt \rfloor-j+1} + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} \mu_x.
\]
Using (5.4), we obtain
\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M^2_k \mid \mathcal{F}_{k-1}) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \left( \sum_{i=1}^p \alpha_i (1 - \alpha_i) X_{k-i} + \sigma_x^2 \right)
\]
\[= \frac{1}{n^2} \sum_{i=1}^p \left( \alpha_i (1 - \alpha_i) \sum_{j=1}^{\lfloor nt \rfloor - i + 1} X_{j-1} \right) + \frac{\lfloor nt \rfloor}{n^2} \sigma_x^2
\]
\[= \sigma_x^2 \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} - \frac{1}{n^2} \sum_{i=2}^p \left( \alpha_i (1 - \alpha_i) \sum_{j=\lfloor nt \rfloor-i+1}^{\lfloor nt \rfloor-1} X_j \right) + \frac{\lfloor nt \rfloor}{n^2} \sigma_x^2.
\]
Hence, for all \(n \in \mathbb{N}\), the randomness of the difference in (4.1) is via a linear combination of the random variables \(X_{\lfloor nt \rfloor-j}, j = 1, \ldots, p\). Then, in order to show (4.1), it suffices to prove
\[
\left(4.3\right) \quad \sup_{t \in [0,T]} \frac{1}{n^2} X_{\lfloor nt \rfloor} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.
\]
By (3.10) and (5.8),
\[
X_{\lfloor nt \rfloor} \leq \sum_{j=1}^{\lfloor nt \rfloor} |M_j + \mu_x| \cdot \|\mathbf{A}^{\lfloor nt \rfloor-j}\| \leq C_A \left( \lfloor nt \rfloor \cdot \mu_x + \sum_{j=1}^{\lfloor nt \rfloor} |M_j| \right).
\]
Consequently, in order to prove (4.3), it suffices to show
\[
\frac{1}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} |M_j| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.
\]
In fact, one can show that \( n^{-2} \sum_{j=1}^{\lfloor nT \rfloor} E(|M_j|) \to 0 \). Indeed, Corollary 5.1 yields that
\[
 n^{-2} \sum_{j=1}^{\lfloor nT \rfloor} E(|M_j|) \leq \frac{K}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} \sqrt{j} \leq \frac{K}{n^2} \lfloor nT \rfloor \sqrt{\lfloor nT \rfloor} \to 0 \quad \text{as} \quad n \to \infty,
\]
with some constant \( K \in \mathbb{R}_+ \). Thus we obtain (4.1).

To prove (4.2), consider the decomposition \( M_k = N_k + (\varepsilon_k - \mu_\varepsilon) \), where, by (5.1),
\[
 N_k := \sum_{\ell=1}^{X_{k-1}} (\xi_{k,1,\ell} - E(\xi_{k,1,\ell})) + \cdots + \sum_{\ell=1}^{X_{k-p}} (\xi_{k,p,\ell} - E(\xi_{k,p,\ell})).
\]
Clearly,
\[
 M_k^2 \leq 2 \left( N_k^2 + (\varepsilon_k - \mu_\varepsilon)^2 \right) \quad \text{and} \quad 1_{\{|N_k|>n\theta\}} \leq 1_{\{|N_k|>n\theta/2\}} + 1_{\{|\varepsilon_k - \mu_\varepsilon|>n\theta/2\}},
\]
and hence (4.2) will be proved once we show
\[
 (4.4) \quad \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} E(N_k^2 1_{\{|N_k|>n\theta\}} | F_{k-1}) \xrightarrow{P} 0 \quad \text{for all} \quad \theta > 0,
\]
\[
 (4.5) \quad \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} E((\varepsilon_k - \mu_\varepsilon)^2 | F_{k-1}) \xrightarrow{P} 0 \quad \text{for all} \quad \theta > 0,
\]
\[
 (4.6) \quad \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} E((\varepsilon_k - \mu_\varepsilon)^2 | F_{k-1}) \xrightarrow{P} 0.
\]

First we prove (4.4). Using that the random variables \( \{\xi_{k,i,j} : j \in \mathbb{N}, i \in \{1, \ldots, p\}\} \) are independent of the \( \sigma \)-algebra \( F_{k-1} \) for all \( k \in \mathbb{N} \), we get
\[
 E(N_k^2 1_{\{|N_k|>n\theta\}} | F_{k-1}) = F_k(X_{k-1}, \ldots, X_{k-p}),
\]
where \( F_k : \mathbb{Z}_+^p \to \mathbb{R} \) is given by
\[
 F_k(z_1, \ldots, z_p) := E((S_k(z_1, \ldots, z_p)^2 1_{\{|S_k(z_1, \ldots, z_p)|>n\theta\}})), \quad z_1, \ldots, z_p \in \mathbb{Z}_+,
\]
with \( S_k(z_1, \ldots, z_p) := \sum_{i=1}^{p} \sum_{\ell=1}^{z_i} (\xi_{k,i,\ell} - E(\xi_{k,i,\ell})) \). Consider the decomposition
\[
 F_k(z_1, \ldots, z_p) = A_k(z_1, \ldots, z_p) + B_k(z_1, \ldots, z_p),
\]
where
\[
 A_k(z_1, \ldots, z_p) := \sum_{i=1}^{p} \sum_{\ell=1}^{z_i} E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2 1_{\{|S_k(z_1, \ldots, z_p)|>n\theta\}}),
\]
\[
 B_k(z_1, \ldots, z_p) := \sum' E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))(\xi_{k,j,\ell'} - E(\xi_{k,j,\ell'})) 1_{\{|S_k(z_1, \ldots, z_p)|>n\theta\}}),
\]
where the sum \( \sum' \) is taken for \( i, j = 1, \ldots, p, \ell = 1, \ldots, z_i, \ell' = 1, \ldots, z_j \) with \( (i, \ell) \neq (j, \ell') \). Consider the decompositions
\[
 S_k(z_1, \ldots, z_p) = (\xi_{k,i,\ell} - E(\xi_{k,i,\ell})) + \tilde{S}_k(z_1, \ldots, z_p), \quad i = 1, \ldots, p, \ell = 1, \ldots, z_i,
\]
where
\[ \tilde{S}^i_{k,\ell}(z_1, \ldots, z_p) := \sum'' (\xi_{k,j,\ell} - E(\xi_{k,j,\ell})), \]
where the sum \( \sum'' \) is taken for \( j = 1, \ldots, p \) and \( \ell' = 1, \ldots, z_j \) with \((j, \ell') \neq (i, \ell)\).

Using that
\[ \mathbb{1}\{|S_k(z_1, \ldots, z_p)| > n\theta| \} \leq \mathbb{1}\{|\xi_{k,i,\ell} - E(\xi_{k,i,\ell})| > n\theta/2| \} + \mathbb{1}\{|\tilde{S}^i_{k,\ell}(z_1, \ldots, z_p)| > n\theta/2| \}, \]
we have
\[ A_k(z_1, \ldots, z_p) \leq A_k^{(1)}(z_1, \ldots, z_p) + A_k^{(2)}(z_1, \ldots, z_p), \]
where
\[ A_k^{(1)}(z_1, \ldots, z_p) := \sum_{i=1}^p \sum_{\ell=1}^{z_i} E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2 \mathbb{1}\{|\xi_{k,i,\ell} - E(\xi_{k,i,\ell})| > n\theta/2| \}), \]
\[ A_k^{(2)}(z_1, \ldots, z_p) := \sum_{i=1}^p \sum_{\ell=1}^{z_i} E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2 \mathbb{1}\{|\tilde{S}^i_{k,\ell}(z_1, \ldots, z_p)| > n\theta/2| \}). \]

In order to prove \((4.4)\), it is enough to show that
\[ \frac{1}{n^2} \sum_{k=1}^{[nT]} A_k^{(1)}(X_{k-1}, \ldots, X_{k-p}) \xrightarrow{P} 0, \quad \frac{1}{n^2} \sum_{k=1}^{[nT]} A_k^{(2)}(X_{k-1}, \ldots, X_{k-p}) \xrightarrow{P} 0, \]
\[ \frac{1}{n^2} \sum_{k=1}^{[nT]} B_k(X_{k-1}, \ldots, X_{k-p}) \xrightarrow{P} 0 \]
(4.7)
as \( n \to \infty \). We have
\[ A_k^{(1)}(z_1, \ldots, z_p) = \sum_{i=1}^p z_i E((\xi_{1,i,1} - E(\xi_{1,i,1}))^2 \mathbb{1}\{|\xi_{1,i,1} - E(\xi_{1,i,1})| > n\theta/2| \}), \quad k \in \mathbb{N}, \]
where
\[ E((\xi_{1,i,1} - E(\xi_{1,i,1}))^2 \mathbb{1}\{|\xi_{1,i,1} - E(\xi_{1,i,1})| > n\theta/2| \}) \to 0, \]
as \( n \to \infty \) for all \( i \in \{1, \ldots, p\} \) by the dominated convergence theorem. Thus, by Corollary 5.1 we get with some constant \( K \in \mathbb{R}_+ \),
\[ \frac{1}{n^2} \sum_{k=1}^{[nT]} E(A_k^{(1)}(X_{k-1}, \ldots, X_{k-p})) \]
\[ = \frac{1}{n^2} \sum_{k=1}^{[nT]} \sum_{i=1}^p E(X_{k-i}) E((\xi_{1,i,1} - E(\xi_{1,i,1}))^2 \mathbb{1}\{|\xi_{1,i,1} - E(\xi_{1,i,1})| > n\theta/2| \}) \]
\[ \leq \sum_{i=1}^p \left[ E((\xi_{1,i,1} - E(\xi_{1,i,1}))^2 \mathbb{1}\{|\xi_{1,i,1} - E(\xi_{1,i,1})| > n\theta/2| \}) \frac{K}{n^2} \sum_{k=i+1}^{[nT]} (k - i) \right] \]
\[ \leq K \frac{[nT][(nT) + 1]}{2n^2} \sum_{i=1}^p E((\xi_{1,i,1} - E(\xi_{1,i,1}))^2 \mathbb{1}\{|\xi_{1,i,1} - E(\xi_{1,i,1})| > n\theta/2| \}) \to 0, \]
which yields \( n^{-2} \sum_{k=1}^{nT} A_k^{(1)}(X_{k-1}, \ldots, X_{k-p}) \xrightarrow{P} 0 \).

Independence of \( \xi_{k,i,\ell} - E(\xi_{k,i,\ell}) \) and \( \tilde{S}_{k,\ell}^i(z_1, \ldots, z_p) \) implies

\[
A_k^{(2)}(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{\ell=1}^{p} E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2) P(|\tilde{S}_{k,\ell}^i(z_1, \ldots, z_p)| > n\theta/2).
\]

Here \( E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2) = \alpha_i(1 - \alpha_i), i = 1, \ldots, p, \) and, by Markov’s inequality,

\[
P(|\tilde{S}_{k,\ell}^i(z_1, \ldots, z_p)| > n\theta/2) \leq \frac{4}{n^2\theta^2} E(\tilde{S}_{k,\ell}^i(z_1, \ldots, z_p)^2)
\]

\[
= \frac{4}{n^2\theta^2} \text{Var}(\tilde{S}_{k,\ell}^i(z_1, \ldots, z_p)) = \frac{4}{n^2\theta^2} \sum_{i}^{p} \alpha_j(1 - \alpha_j) \leq \frac{4}{n^2\theta^2} \sum_{j=1}^{p} z_j \alpha_j(1 - \alpha_j).
\]

Thus we get

\[
A_k^{(2)}(z_1, \ldots, z_p) \leq \frac{4}{n^2\theta^2} \sum_{i=1}^{p} \sum_{j=1}^{p} z_i z_j \alpha_i(1 - \alpha_i) \alpha_j(1 - \alpha_j).
\]

Hence, by Cauchy-Schwarz’s inequality and Corollary 5.1, we get with some constant \( K \in \mathbb{R}_+ \),

\[
\frac{1}{n^2} \sum_{k=1}^{nT} E(A_k^{(2)}(X_{k-1}, \ldots, X_{k-p})) \leq \frac{4}{n^4\theta^2} \sum_{k=1}^{nT} \sum_{i=1}^{p} \sum_{j=1}^{p} E(X_{k-i}X_{k-j}) \alpha_i(1 - \alpha_i) \alpha_j(1 - \alpha_j)
\]

\[
\leq \frac{4K}{n^4\theta^2} \sum_{k=1}^{nT} k^2 \left( \sum_{i=1}^{p} \alpha_i(1 - \alpha_i) \right)^2 \rightarrow 0,
\]

which implies \( n^{-2} \sum_{k=1}^{nT} A_k^{(2)}(X_{k-1}, \ldots, X_{k-p}) \xrightarrow{P} 0 \).

By Cauchy-Schwarz’s inequality,

\[
|B_k(z_1, \ldots, z_p)| \leq \sqrt{B_k^{(1)}(z_1, \ldots, z_p) E(1_{\{|S_k(z_1, \ldots, z_p)| > n\theta\}})},
\]

where

\[
B_k^{(1)}(z_1, \ldots, z_p) := E\left(\left(\sum_{i,j}^\prime (\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))(\xi_{k,j,\ell'} - E(\xi_{k,j,\ell'}))^2\right)^2\right), \quad z_1, \ldots, z_p \in \mathbb{Z}_+.
\]

Using the independence of \( \xi_{k,i,\ell} - E(\xi_{k,i,\ell}) \) and \( \xi_{k,j,\ell'} - E(\xi_{k,j,\ell'}) \) for \( (i, \ell) \neq (j, \ell') \), we get

\[
B_k^{(1)}(z_1, \ldots, z_p) = \sum_{i}^\prime \alpha_i(1 - \alpha_i) \alpha_j(1 - \alpha_j)
\]

\[
= \sum_{i=1}^{p} z_i(z_i - 1)\alpha_i^2(1 - \alpha_i)^2 + \sum_{i \neq j} z_i z_j \alpha_i(1 - \alpha_i) \alpha_j(1 - \alpha_j)
\]

\[
\leq K_1(z_1 + \cdots + z_p)^2,
\]

with some constant \( K_1 \in \mathbb{R}_+ \). Further, by Markov’s inequality,

\[
E(1_{\{|S_k(z_1, \ldots, z_p)| > n\theta\}}) \leq \frac{1}{n^2\theta^2} \sum_{j=1}^{p} z_j \alpha_j(1 - \alpha_j) \leq \frac{K_2}{n^2\theta^2}(z_1 + \cdots + z_p),
\]

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with some constant \( K_2 \in \mathbb{R}_+ \). Hence
\[
|B_k(z_1, \ldots, z_p)| \leq \frac{K}{n}(z_1 + \cdots + z_p)^{3/2}, \quad z_1, \ldots, z_p \in \mathbb{Z}_+,
\]
with some constant \( K \in \mathbb{R}_+ \). Using that
\[
(z_1 + \cdots + z_p)^{3/2} \leq c_p(z_1^{3/2} + \cdots + z_p^{3/2}), \quad z_1, \ldots, z_p \in \mathbb{Z}_+,
\]
with some constant \( c_p \in \mathbb{R}_+ \), we get, in order to show (4.7), it suffices to prove
\[
n^{-3} \sum_{k=1}^{\lfloor nT \rfloor} \left( X_{k-1}^{3/2} + \cdots + X_{k-p}^{3/2} \right) \xrightarrow{P} 0.
\]
In fact, \( n^{-3} \sum_{k=1}^{\lfloor nT \rfloor} \left( E(X_{k-1}^{3/2}) + \cdots + E(X_{k-p}^{3/2}) \right) \to 0 \) since Corollary 5.1 implies \( E(X_{k-1}^{3/2}) \leq \left( E(X_1^2) \right)^{3/4} = O(\epsilon^{3/2}) \). Thus we finished the proof of (4.4).

Now we turn to prove (4.5). Using that for all \( k \in \mathbb{N} \) the random variables \( \{\xi_{k,i,j}, \varepsilon_k : j \in \mathbb{N}, i \in \{1, \ldots, p\}\} \) are independent of the \( \sigma \)-algebra \( \mathcal{F}_{k-1} \), we get \( E(N_k^2 1_{|\varepsilon_k - \mu_k| > \theta}) | \mathcal{F}_{k-1} = G_k(X_{k-1}, \ldots, X_{k-p}) \), where \( G_k : \mathbb{Z}_+^p \to \mathbb{R} \) is given by
\[
G_k(z_1, \ldots, z_p) := E(S_k(z_1, \ldots, z_p)^2 1_{|\varepsilon_k - \mu_k| > \theta}), \quad z_1, \ldots, z_p \in \mathbb{Z}_+.
\]

Using again the independence of \( \{\xi_{k,i,j}, \varepsilon_k : j \in \mathbb{N}, i \in \{1, \ldots, p\}\} \),
\[
G_k(z_1, \ldots, z_p) = P(|\varepsilon_k - \mu_k| > \theta) n \sum_{i=1}^p \sum_{\ell=1}^p E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2),
\]
where by Markov's inequality, \( P(|\varepsilon_k - \mu_k| > \theta) \leq n^{-2} t^{-2} = n^{-2} \theta^{-2} \), and \( E((\xi_{k,i,\ell} - E(\xi_{k,i,\ell}))^2) = \alpha_i(1 - \alpha_i) \). Hence, in order to show (4.5), it suffices to prove \( n^{-4} \sum_{k=1}^{\lfloor nT \rfloor} X_k \xrightarrow{P} 0 \). In fact, by Corollary 5.1 \( n^{-4} \sum_{k=1}^{\lfloor nT \rfloor} E(X_k) \to 0 \).

Now we turn to prove (4.6). By independence of \( \varepsilon_k \) and \( \mathcal{F}_{k-1}, \)
\[
\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} E((\varepsilon_k - \mu_k)^2 | \mathcal{F}_{k-1}) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} E((\varepsilon_k - \mu_k)^2) = \frac{|nT|}{n^2} \sigma_\varepsilon^2 \to 0,
\]
thus we obtain (4.6). Hence we get (4.2), and we conclude, by Theorem 5.1 convergence \( \mathcal{M}_n \xrightarrow{\mathcal{L}} \mathcal{M} \).

Now we start to prove (3.1). By (3.10), \( \mathcal{X}^n = \Psi_n(\mathcal{M}^n) \), where the mapping \( \Psi_n : \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \) is given by
\[
\Psi_n(f)(t) := \sum_{j=1}^{\lfloor nt \rfloor} \left( f \left( \frac{j}{n} \right) - f \left( \frac{j-1}{n} \right) + \frac{\mu(t)}{n} \right) e_1^T A_{\lfloor nt \rfloor - j} e_1
\]
for \( f \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}), \ t \in \mathbb{R}_+, \ n \in \mathbb{N} \). Further, \( \mathcal{X} = \Psi(\mathcal{M}) \), where, by (3.9), the mapping \( \Psi : \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \) is given by
\[
\Psi(f)(t) := \frac{1}{\varphi'(1)} (f(t) + \mu(t)), \quad f \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}), \ t \in \mathbb{R}_+.
\]

We check that the mappings \( \Psi_n, n \in \mathbb{N} \), and \( \Psi \) are measurable. Continuity of \( \Psi \) follows from the characterization of convergence in \( \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \), see, e.g., Ethier and Kurtz \[18\] Proposition...
In order to prove measurability of \( \Psi \), indeed, if \( f_n \in D(\mathbb{R}_+, \mathbb{R}) \), \( n \in \mathbb{N} \), \( f \in D(\mathbb{R}_+, \mathbb{R}) \) and the sequence \((f_n)_{n \in \mathbb{N}}\) converges in \( D(\mathbb{R}_+, \mathbb{R}) \) to \( f \), then for all \( T > 0 \) there exist continuous, increasing mappings \( \lambda_n \), \( n \in \mathbb{N} \), from \([0, \infty)\) onto \([0, \infty)\) such that

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} |\lambda_n(t) - t| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup_{t \in [0, T]} |f_n(\lambda_n(t)) - f(t)| = 0.
\]

Since for all \( t \in \mathbb{R}_+ \)

\[
|\Psi(f_n)(\lambda_n(t)) - \Psi(f)(t)| = \left| \frac{1}{\varphi'(1)}(f_n(\lambda_n(t)) + \mu_\varepsilon \lambda_n(t)) - \frac{1}{\varphi'(1)}(f(t) + \mu_\varepsilon t) \right|
\]

\[
\leq \frac{1}{\varphi'(1)}|f_n(\lambda_n(t)) - f(t)| + \frac{\mu_\varepsilon}{\varphi'(1)}|\lambda_n(t) - t|,
\]

we have for all \( T > 0 \),

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} |\Psi(f_n)(\lambda_n(t)) - \Psi(f)(t)| = 0.
\]

In order to prove measurability of \( \Psi_n \), first we localize it. For each \( N \in \mathbb{N} \), consider the stopped mapping \( \Psi^N_n : D(\mathbb{R}_+, \mathbb{R}) \to D(\mathbb{R}_+, \mathbb{R}) \) given by \( \Psi^N_n(f)(t) := \Psi_n(f(t \land N)) \) for \( f \in D(\mathbb{R}_+, \mathbb{R}) \), \( t \in \mathbb{R}_+ \), \( n, N \in \mathbb{N} \). Obviously, \( \Psi^N_n(f) \to \Psi_n(f) \) in \( D(\mathbb{R}_+, \mathbb{R}) \) as \( N \to \infty \) for all \( f \in D(\mathbb{R}_+, \mathbb{R}) \), since for all \( T > 0 \) and \( N \geq T \) we have \( \Psi^N_n(f)(t) := \Psi_n(f)(t), t \in [0, T] \), and hence \( \sup_{t \in [0, T]} |\Psi^N_n(f)(t) - \Psi_n(f)(t)| \to 0 \) as \( N \to \infty \). Consequently, it suffices to show measurability of \( \Psi^N_n \) for all \( n, N \in \mathbb{N} \). We can write \( \Psi_n = \Psi_n^{N, 2} \circ \Psi_n^{N, 1} \), where the mappings \( \Psi_n^{N, 1} : D(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}^{nN+1} \) and \( \Psi_n^{N, 2} : \mathbb{R}^{nN+1} \to D(\mathbb{R}_+, \mathbb{R}) \) are defined by

\[
\Psi_n^{N, 1}(f) := \left( f(0), f\left(\frac{1}{n}\right), f\left(\frac{2}{n}\right), \ldots, f(N) \right),
\]

\[
\Psi_n^{N, 2}(x_0, x_1, \ldots, x_{nN})(t) := \sum_{j=1}^{[n(T \land N)]} \left(x_j - x_{j-1} + \frac{\mu_\varepsilon}{n}\right) e_1^T A^{[nt]-j} e_1
\]

for \( f \in D(\mathbb{R}_+, \mathbb{R}) \), \( t \in \mathbb{R}_+ \), \( x = (x_0, x_1, \ldots, x_{nN}) \in \mathbb{R}^{nN+1} \), \( n, N \in \mathbb{N} \). Measurability of \( \Psi_n^{N, 1} \) follows from Ethier and Kurtz [18, Proposition 3.7.1]. Next we show continuity of \( \Psi_n^{N, 2} \) by checking \( \sup_{t \in [0, T]} |\Psi_n^{N, 2}(x^k)(t) - \Psi_n^{N, 2}(x)(t)| \to 0 \) as \( k \to \infty \) for all \( T > 0 \) whenever \( x^k \to x \) in \( \mathbb{R}^{nN+1} \). This convergence follows from the estimates

\[
\sup_{t \in [0, T]} |\Psi_n^{N, 2}(x^k)(t) - \Psi_n^{N, 2}(x)(t)| \leq \sum_{j=1}^{[n(T \land N)]} \left(|x_j^k - x_j| + |x_{j-1}^k - x_{j-1}|\right) e_1^T A^{[nt]-j} e_1 |
\]

since \( e_1^T A^{[nt]-j} e_1 \leq \|A^{[nt]-j}\| \leq C_A \). We obtain measurability of both \( \Psi_n^{N, 1} \) and \( \Psi_n^{N, 2} \), hence we conclude measurability of \( \Psi_n \). The aim of the following discussion is to show that there exists \( C \subset C_{\Psi_n} \), \( \Psi_n \in \mathbb{N} \) with \( C \in D_\infty \) and \( P(M \in C) = 1 \), where \( C_{\Psi_n} \) is defined in Appendix. We check that \( C := \{ f \in C(\mathbb{R}_+, \mathbb{R}) : f(0) = 0 \} \) satisfies the above mentioned conditions. First note that \( C = C(\mathbb{R}_+, \mathbb{R}) \cap \pi_0^{-1}(0) \), where \( \pi_0 : D(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R} \), \( \pi_0(f) := f(0), f \in D(\mathbb{R}_+, \mathbb{R}) \). Using that \( C(\mathbb{R}_+, \mathbb{R}) \) is a measurable subset of \( D(\mathbb{R}_+, \mathbb{R}) \) (see, e.g., Ethier and Kurtz [18, Problem 3.11.25]) and that \( \pi_0 \) is measurable (see, e.g., Ethier and Kurtz [18, Proposition 3.7.1]), we have \( C \in D_\infty \). Fix a function \( f \in C(\mathbb{R}_+, \mathbb{R}) \) and a sequence \((f_n)_{n \in \mathbb{N}}\)
in \( D(\mathbb{R}^+, \mathbb{R}) \) with \( f_n \overset{h_n}{\rightarrow} f \), where \( h_n \) is defined in Appendix. By the definition of \( \Psi \), we get \( \Psi(f) \in C(\mathbb{R}^+, \mathbb{R}) \). Further, we can write

\[
\Psi_n(f_n)(t) = \sum_{j=1}^{\lfloor nt \rfloor} \left( f_n \left( \frac{j}{n} \right) - f_n \left( \frac{j-1}{n} \right) + \frac{\mu_x}{n} \right) e_1^T \Pi_A e_1 \\
+ \sum_{j=1}^{\lfloor nt \rfloor} \left( f_n \left( \frac{j}{n} \right) - f_n \left( \frac{j-1}{n} \right) + \frac{\mu_x}{n} \right) e_1^T (A^{[nt]} - \Pi_A) e_1, \quad t \in \mathbb{R}^+.
\]

Using (2.10) and the assumption \( \varrho(A) = \alpha_1 + \cdots + \alpha_p = 1 \), we get \( e_1^T \Pi_A e_1 = \frac{1}{\varphi'(1)} \) and

\[
\sum_{j=1}^{\lfloor nt \rfloor} \left( f_n \left( \frac{j}{n} \right) - f_n \left( \frac{j-1}{n} \right) + \frac{\mu_x}{n} \right) = f_n \left( \frac{\lfloor nt \rfloor}{n} \right) - f_n(0) + \frac{\lfloor nt \rfloor}{n} \mu_x.
\]

Thus we have

\[
|\Psi_n(f_n)(t) - \Psi(f)(t)| \leq \frac{1}{\varphi'(1)} \left| f_n \left( \frac{\lfloor nt \rfloor}{n} \right) - f(t) \right| + \frac{\mu_x}{n \varphi'(1)} + \frac{|f_n(0)|}{\varphi'(1)} \\
+ \sum_{j=1}^{\lfloor nt \rfloor} \left| f_n \left( \frac{j}{n} \right) - f_n \left( \frac{j-1}{n} \right) \right| + \frac{\mu_x}{n} \| A^{[nt]} - \Pi_A \|.
\]

Here for all \( T > 0 \) and \( t \in [0, T] \),

\[
\left| f_n \left( \frac{\lfloor nt \rfloor}{n} \right) - f(t) \right| \leq \left| f_n \left( \frac{\lfloor nt \rfloor}{n} \right) - f \left( \frac{\lfloor nt \rfloor}{n} \right) \right| + \left| f \left( \frac{\lfloor nt \rfloor}{n} \right) - f(t) \right| \\
\leq \omega_T(f, n^{-1}) + \sup_{t \in [0, T]} |f_n(t) - f(t)|,
\]

where \( \omega_T(f, \cdot) \) is the modulus of continuity of \( f \) on \([0, T]\), and we have \( \omega_T(f, n^{-1}) \to 0 \) since \( f \) is continuous (see, e.g., Jacod and Shiryaev [30, Chapter VI, 1.6]). In a similar way, for all \( j = 1, \ldots, \lfloor nt \rfloor \),

\[
\left| f_n \left( \frac{j}{n} \right) - f_n \left( \frac{j-1}{n} \right) \right| \leq \omega_T(f, n^{-1}) + 2 \sup_{t \in [0, T]} |f_n(t) - f(t)|.
\]

By (2.9), since \( \varrho(A) = 1 \),

\[
\sum_{j=1}^{\lfloor nt \rfloor} \| A^{[nt]} - \Pi_A \| \leq \sum_{j=1}^{\lfloor nt \rfloor} c_A r_A^{[nt]} - j \leq \frac{c_A}{1 - r_A}.
\]

Further,

\[
|f_n(0)| \leq |f_n(0) - f(0)| + |f(0)| \leq \sup_{t \in [0, T]} |f_n(t) - f(t)| + |f(0)|.
\]

Thus we conclude \( C \subset C_{\Psi, (\Psi_n)_{n \in \mathbb{N}}}. \) Since \( \mathcal{M}_0 = 0 \) and, by the definition of a strong solution (see, e.g., Jacod and Shiryaev [30, Definition 2.24, Chapter III]), \( \mathcal{M} \) has almost sure continuous sample paths, we have \( P(\mathcal{M} \in C) = 1 \). Consequently, by Lemma 5.2 we obtain \( \mathcal{X}^n = \Psi_n(\mathcal{M}_n) \overset{L^2}{\longrightarrow} \Psi(\mathcal{M}) = \mathcal{X} \) as \( n \to \infty \). \( \square \)
5 Appendix

In the proof of Theorem 3.1 we will extensively use the following facts about the first and second order moments of the sequences \((X_k)_{k \in \mathbb{Z}_+}\) and \((M_k)_{k \in \mathbb{Z}_+}\).

5.1 Lemma. Let \((X_k)_{k \geq -p+1}\) be an INAR(p) process defined by (2.1) such that \(X_0 = X_{-1} = \cdots = X_{-p+1} = 0\) and \(\text{E}(\varepsilon_1^2) < \infty\). Then, for all \(k \in \mathbb{N}\),

\[
(5.1) \quad \text{E}(X_k) = \mu_e \sum_{\ell=0}^{k-1} e_1^\top A^\ell e_1,
\]

\[
(5.2) \quad \text{Var}(X_k) = \sigma_e^2 \sum_{\ell=0}^{k-1} (e_1^\top A^\ell e_1)^2 + \mu_e \sum_{i=1}^p \alpha_i (1 - \alpha_i) \sum_{j=0}^{k-1-i} \sum_{\ell=0}^j (e_1^\top A^{k-j-i} e_1)^2 (e_1^\top A^\ell e_1).
\]

Moreover,

\[
(5.3) \quad \text{E}(M_k \mid \mathcal{F}_{k-1}) = 0 \quad \text{for} \ k \in \mathbb{N},
\]

\[
(5.4) \quad \text{E}(M_k M_\ell \mid \mathcal{F}_{\max(k,\ell)-1}) = \begin{cases} 
\alpha_1 (1 - \alpha_1) X_{k-1} + \cdots + \alpha_p (1 - \alpha_p) X_{k-p} + \sigma_e^2 & \text{if} \ k = \ell, \\
0 & \text{if} \ k \neq \ell.
\end{cases}
\]

Further,

\[
(5.5) \quad \text{E}(M_k) = 0 \quad \text{for} \ k \in \mathbb{N},
\]

\[
(5.6) \quad \text{E}(M_k M_\ell) = \begin{cases} 
\alpha_1 (1 - \alpha_1) \text{E}(X_{k-1}) + \cdots + \alpha_p (1 - \alpha_p) \text{E}(X_{k-p}) + \sigma_e^2 & \text{if} \ k = \ell, \\
0 & \text{if} \ k \neq \ell.
\end{cases}
\]

Proof. We have already proved (5.1), see (2.4). The equality \(M_k = X_k - \text{E}(X_k \mid \mathcal{F}_{k-1})\) clearly implies (5.3) and (5.5). By (2.1) and (3.6),

\[
(5.7) \quad M_k = \sum_{j=1}^{X_{k-1}} (\xi_{k,j} - \text{E}(\xi_{k,j})) + \cdots + \sum_{j=1}^{X_{k-p}} (\xi_{k,p,j} - \text{E}(\xi_{k,p,j})) + (\varepsilon_k - \text{E}(\varepsilon_k)).
\]

For all \(k \in \mathbb{N}\), the random variables \(\{\xi_{k,j} - \text{E}(\xi_{k,i,j}) : j \in \mathbb{N}, i \in \{1, \ldots, p\}\}\) are independent of each other, independent of \(\mathcal{F}_{k-1}\), and have zero mean, thus in the case \(k = \ell\) we conclude (5.4) and hence (5.6). If \(k < \ell\), then \(\text{E}(M_k M_\ell \mid \mathcal{F}_{\ell-1}) = M_k \text{E}(M_\ell \mid \mathcal{F}_{\ell-1}) = 0\) by (5.3), and thus we obtain (5.4) and (5.6) in the case of \(k \neq \ell\).

By (3.10) and (5.1), we conclude

\[
X_k - \text{E}(X_k) = \sum_{j=1}^k M_j e_1^\top A^{k-j} e_1, \quad k \in \mathbb{N}.
\]
Now, by (5.6), (5.1),
\[
\operatorname{Var}(X_k) = \sum_{j=1}^{k} \sum_{\ell=1}^{k} E(M_j M_\ell) e_1^\top A^{k-j} e_1 \alpha e_1^\top A^{k-\ell} e_1 = \sum_{j=1}^{k} E(M_j^2)(e_1^\top A^{k-j} e_1)^2 \]
\[
= \sum_{j=1}^{k} \left( \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) E(X_{j-i}) + \sigma_x^2 \right) (e_1^\top A^{k-j} e_1)^2 \]
\[
= \sigma_x^2 \sum_{j=1}^{k} (e_1^\top A^{k-j} e_1)^2 + \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) \sum_{j=1}^{k} E(X_{j-i})(e_1^\top A^{k-j} e_1)^2, \]
and hence, using also that \( E(X_0) = E(X_{-1}) = \cdots = E(X_{-p+1}) = 0 \), we get
\[
\operatorname{Var}(X_k) = \sigma_x^2 \sum_{\ell=0}^{k-1} (e_1^\top A^\ell e_1)^2 + \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) \mu_\ell \sum_{j=1}^{k-p} \sum_{\ell=0}^{j} (e_1^\top A^\ell e_1)(e_1^\top A^{k-j-i-\ell} e_1)^2 \]
\[
= \sigma_x^2 \sum_{\ell=0}^{k-1} (e_1^\top A^\ell e_1)^2 + \mu_\ell \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) \sum_{j=0}^{k-i-1} \sum_{\ell=0}^{j} (e_1^\top A^\ell e_1)(e_1^\top A^{k-j-i-\ell} e_1)^2, \]
which yields (5.2). \( \square \)

**5.1 Corollary.** Let \((X_k)_{k \geq -p+1}\) be a primitive INAR\((p)\) process defined by (2.1) such that \( \alpha_1 + \cdots + \alpha_p = 1 \) (i.e. unstable), \( X_0 = X_{-1} = \cdots = X_{-p+1} = 0 \) and \( E(\varepsilon_i^2) < \infty \). Then
\[
E(X_k) = O(k), \quad E(X_k^2) = O(k^2), \quad E(\|M_k\|) = O(k^{1/2}). \]

**Proof.** By (5.1),
\[
E(X_k) \leq \mu_\ell \sum_{\ell=0}^{k-1} \|A^\ell\| \leq C_A \mu_\ell k, \]
where
\[
(5.8) \quad C_A := \sup_{\ell \in \mathbb{Z}_+} \|A^\ell\| < \infty. \]
Here \( C_A \) is finite since, by (2.2), \( C_A \leq c_A + \|P_A\| \). Hence we obtain \( E(X_k) = O(k) \). We remark that \( E(X_k) = O(k) \) is in fact an immediate consequence of part (ii) of Proposition 2.2.

We have, by Lyapunov’s inequality,
\[
E(\|M_k\|) \leq \sqrt{E(M_k^2)} = \left( \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) E(X_{k-i}) + \sigma_x^2 \right)^{1/2} \]
\[
\leq \left( \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) E(X_{k-i}) \right)^{1/2} + (\sigma_x^2)^{1/2}, \]
hence we obtain \( E(\|M_k\|) = O(k^{1/2}) \) from \( E(X_k) = O(k) \).

Thus we get
\[
E(X_k^2) = \operatorname{Var}(X_k) + (E(X_k))^2 = O(k^2). \]

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Indeed, by (5.2) and (5.8),
\[
\text{Var}(X_k) \leq \sigma^2 \sum_{\ell=0}^{k-1} \|A^\ell\|^2 + \mu \sum_{i=1}^{p} \alpha_i (1 - \alpha_i) \sum_{j=0}^{k-i-1} \sum_{\ell=0}^{j} \|A^\ell\| \|A^{k-j-1}\|^2
\]
\[
\leq \sigma^2 C^2 \alpha + C^3 \mu \sigma^2 k^2,
\]
where \(\sigma^2\) is defined in Theorem 3.1. Hence we obtain \(\mathbb{E}(X_k^2) = O(k^2)\). □

Next we recall a result about convergence of step processes towards a diffusion process, see Ispáný and Pap [29, Corollary 2.2]. This result is used for the proof of convergence (3.7).

5.1 Theorem. Let \(\gamma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\) be a continuous function. Assume that the SDE
\[
(5.9) \quad \text{d}U_t = \gamma(t, U_t) \text{d}W_t, \quad t \in \mathbb{R}^+,
\]
has a unique weak solution with initial value \(U_0 = u_0\) for all \(u_0 \in \mathbb{R}\), where \((W_t)_{t \in \mathbb{R}^+}\) is a standard Wiener process. Let \((U_t)_{t \in \mathbb{R}^+}\) be a solution of (5.9) with initial value \(U_0 = 0\). For each \(n \in \mathbb{N}\), let \((U^n_k)_{k \in \mathbb{N}}\) be a sequence of random variables adapted to a filtration \((\mathcal{F}^n_k)_{k \in \mathbb{Z}^+}\). Let
\[
U^n_t := \sum_{k=1}^{[nt]} U^n_k, \quad t \in \mathbb{R}^+, \quad n \in \mathbb{N}.
\]
Suppose \(\mathbb{E}((U^n_k)^2) < \infty\) and \(\mathbb{E}(U^n_k | \mathcal{F}^n_{k-1}) = 0\) for all \(n, k \in \mathbb{N}\). Suppose that for each \(T > 0\),
\[
(i) \quad \sup_{t \in [0, T]} \left| \sum_{k=1}^{[nt]} \mathbb{E}((U^n_k)^2 | \mathcal{F}^n_{k-1}) - \int_0^t \gamma(s, U^n_s)^2 \text{d}s \right| \to 0, \quad \mathbb{P}
\]
\[
(ii) \quad \sum_{k=1}^{[nT]} \mathbb{E}((U^n_k)^2 \mathbb{1}_{\{U^n_k > \theta\}} | \mathcal{F}^n_{k-1}) \to 0 \quad \text{for all } \theta > 0, \quad \mathbb{P}
\]
where \(\to \mathbb{P}\) denotes convergence in probability. Then \(U^n \rightarrow^\mathbb{L} U\) as \(n \to \infty\).

In fact, this theorem is a corollary of a more general limit theorem, see Ispáný and Pap [29, Theorem 2.1].

Now we recall a version of the continuous mapping theorem.

For a function \(f \in D(\mathbb{R}^+, \mathbb{R})\) and for a sequence \((f_n)_{n \in \mathbb{N}}\) in \(D(\mathbb{R}^+, \mathbb{R})\), we write \(f_n \rightarrow^\text{lu} f\) if \((f_n)_{n \in \mathbb{N}}\) converges to \(f\) locally uniformly, i.e., if \(\sup_{t \in [0, T]} |f_n(t) - f(t)| \to 0\) as \(n \to \infty\) for all \(T > 0\). For measurable mappings \(\Phi : D(\mathbb{R}^+, \mathbb{R}) \to D(\mathbb{R}^+, \mathbb{R})\) and \(\Phi_n : D(\mathbb{R}^+, \mathbb{R}) \to D(\mathbb{R}^+, \mathbb{R})\), \(n \in \mathbb{N}\), we will denote by \(C_{\Phi_n}(\Phi_n)_{n \in \mathbb{N}}\) the set of all functions \(f \in C(\mathbb{R}^+, \mathbb{R})\) such that \(\Phi(f) \in C(\mathbb{R}^+, \mathbb{R})\) and \(\Phi_n(f_n) \rightarrow^\text{lu} \Phi(f)\) whenever \(f_n \rightarrow^\text{lu} f\) with \(f_n \in D(\mathbb{R}^+, \mathbb{R})\), \(n \in \mathbb{N}\).

For deriving convergence (3.1) from convergence (3.7) we will need the following version of the continuous mapping theorem.
5.2 Lemma. Let \((U_t)_{t\in\mathbb{R}_+}\) and \((U^n_t)_{t\in\mathbb{R}_+}\), \(n \in \mathbb{N}\), be stochastic processes with càdlàg paths such that \(U^n \xrightarrow{L} U\) as \(n \to \infty\). Let \(\Phi : D(\mathbb{R}_+, \mathbb{R}) \to D(\mathbb{R}_+, \mathbb{R})\) and \(\Phi_n : D(\mathbb{R}_+, \mathbb{R}) \to D(\mathbb{R}_+, \mathbb{R}), \ n \in \mathbb{N}\), be measurable mappings such that there exists \(C \subset C(\Phi, (\Phi_n)_{n \in \mathbb{N}})\) with \(C \in \mathcal{D}_\infty\) and \(P(U \in C) = 1\). Then \(\Phi_n(U^n) \xrightarrow{L} \Phi(U)\) as \(n \to \infty\).

Lemma 5.2 can be considered as a consequence of Theorem 3.27 in Kallenberg [36], and we note that a proof of this lemma can also be found in Ispány and Pap [29, Lemma 3.1].

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