The Renormalization Group According to Balaban
I. Small Fields

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Abstract

This is an expository account of Balaban’s approach to the renormalization group. The method is illustrated with a treatment of the ultraviolet problem for the scalar \( \phi^4 \) model on a toroidal lattice in dimension \( d = 3 \). This yields another proof of the stability bound. In this first paper we analyze the small field contribution to the partition function.

1 Introduction

1.1 overview

Balaban has developed a very powerful renormalization group method for analyzing lattice quantum field theories. The characteristic feature is that after each renormalization group transformation a split is introduced into regions where the fields are large and regions where the fields are small. Then one sums over all possible splittings to cover the entire function space. In small field regions one can perform a detailed analysis of the effective actions, for example carrying out perturbative renormalization procedures. The large field region is treated crudely but makes a small contribution due to the fundamental stability of the interaction.

Using this approach Balaban has been able to treat some important problems in quantum field theory. These include an analysis of the ultraviolet problem for scalar QED in \( d = 3 \), \([1]-[4]\), the ultraviolet problem for Yang Mills in \( d = 3, 4 \), \([6]-[16]\), and the infrared problem for an \( N \)-component scalar field in \( d \geq 3 \) with a potential which has a deep minimum on the surface of a sphere, \([17]-[23]\) (known as the "N-vector model" or "linear \( \sigma \)-model").

These are all very difficult problems and for each problem the analysis extends over many papers. As a result others have been slow to adopt the approach (exceptions are \([38], [39], [24], [26], [27], [34]\)). But the basic strategy is fairly straightforward and it seems worthwhile to expose it in a simpler model. That is the purpose of this paper.

The model we choose is the scalar \( \phi^4 \) model in dimension \( d = 3 \) and in a finite volume. This is a special case of the scalar QED model treated by Balaban, \([1]-[5]\). However we also incorporate many of the improvements which can be found in \([17]-[23]\). The treatment is also more efficient, avoiding the analysis of large orders of perturbation theory. Indeed we avoid perturbation theory entirely and use a dynamical systems approach to renormalization.

The analysis stretches over three papers. In this first paper we study a modified model in which the fields are all small (i.e bounded). The second paper develops an expansion in small and large field
regions in which the results of the first paper are the leading terms. The third paper establishes the convergence of the expansion. The treatment is mostly self-contained, however we sometimes refer to the original papers for technical results.

1.2 the model

The basic torus is $T_M = (\mathbb{R}/L^M\mathbb{Z})^3$ where $L$ is a fixed large positive number and $M \geq 0$ is a fixed nonnegative integer. It has volume $\text{Vol}(T_M) = L^{3M}$. In this torus we consider lattices with spacing $L^{-N}$ defined by

$$T_M^{-N} = (L^{-N}\mathbb{Z}/L^M\mathbb{Z})^3$$

also with volume $\text{Vol}(T_M^{-N}) = \text{Vol}(T_M) = L^{3M}$. If $N < N'$ then $T_M^{-N} \subset T_M^{-N'} \subset T_M$.

Let $\phi : T_M^{-N} \rightarrow \mathbb{R}$ be a scalar field on the lattice. The lattice version of the $\phi^4$ model is defined by the density

$$\rho_N^M(\phi) = \exp \left(-\frac{1}{2} < \phi, (-\Delta + \bar{\mu})\phi > + V_N^M(\phi) \right) = \exp \left(-\frac{1}{2}\| \partial\phi \| ^2 + \bar{\mu}\| \phi \|^2 + V_N^M(\phi) \right)$$

Here the inner product is defined with a weighted sum written as an integral:

$$< u, v > = \int u(x)v(x)dx \equiv L^{-3N}\sum_{x \in T_M^{-N}} u(x)v(x)$$

If $\{ e_\mu \} = \{ e_1, e_2, e_3 \}$ are oriented unit basis vectors the derivative in the direction $e_\mu$ is

$$(\partial_\mu \phi)(x) = (\phi(x + L^{-N}e_\mu) - \phi(x))/L^{-N}$$

and the Laplacian is $\Delta = -\partial^*\partial$. The parameter $\bar{\mu} > 0$ is a fixed mass-squared. The potential has the form

$$V_N^M(\phi) = \varepsilon^N\text{Vol}(T_M) + \frac{1}{2}\mu^N \int \phi^2(x)dx + \frac{1}{4}\lambda \int \phi^4(x)dx$$

Here $\lambda > 0$ is a fixed coupling constant. The parameters $\varepsilon^N, \mu^N$ are energy and mass-squared counterterms which we allow to depend on the lattice spacing $L^{-N}$. This is renormalization. The coupling constant $\lambda$ requires no renormalization in this model.

We use our renormalization group method to study the partition function

$$Z_{M,N} = \int \rho_N^M(\phi) \, d\phi^{M,N} \quad \quad \quad \quad \quad \quad \quad \quad \quad \rho^{M,N} = \prod_{x \in T_M^{-N}} \rho(x)$$

Actually it is convenient to study the relative partition function $Z_{M,N}/Z_{M,N}(0)$ where $Z_{M,N}(0)$ is the free field partition with $T^{-N} = 0$.

A result of the analysis is the following stability bound, whose proof comes in the final paper.

**Theorem 1.** For any $\lambda, \bar{\mu} > 0$ there is a choice of renormalization counterterms $E_N, \mu^N$ and a constant $c$ such that such that

$$\exp \left(-c\text{Vol}(T_M) \right) \leq \frac{Z_{M,N}}{Z_{M,N}(0)} \leq \exp \left(c\text{Vol}(T_M) \right)$$

for all $M, N$.

This result is not new. The upper bound was first obtained by Glimm and Jaffe [36]. Their results were extended by Feldman and Osterwalder [35], who established various infinite volume limits for weak coupling. We have already mentioned the analysis of Balaban [11, 5] which includes this result. An alternative renormalization group treatment for this problem was given by Brydges, Dimock, and Hurd [31] where further references can be found.

2
1.3 the scaled model

Before proceeding we scale the problem up to a unit lattice with large volume. This changes our ultraviolet problem to an infrared problem and puts us in the natural home for the renormalization group. The new lattice is $T_{M+N}^3$ with volume $L^{3(M+N)}$ and unit lattice spacing. For fields $\Phi : T_{M+N}^3 \to \mathbb{R}$ we define

$$\rho_0^N(\Phi) = \rho^N(\Phi_{L^{-N}})$$

where $\Phi_{L^{-N}} : T_M^3 \to \mathbb{R}$ is defined by

$$\Phi_{L^{-N}}(x) = L^{N/2}\Phi(L^N x)$$

Making the change of variables $\phi = \Phi_{L^{-N}}$ in the partition function we have

$$Z_{M,N} = \int \rho_0^N(\Phi) d\Phi^{M,N} \quad d\Phi^{M,N} = \prod_{x \in T_{M+N}} d\Phi(x)$$

There should actually be a factor $(L^{N/2})^{\dim_{T_{M+N}}}$ here, but since it makes no contribution to the relative partition function we have dropped it.

This scaling preserves the Laplacian term and we have

$$\rho_0^N(\Phi) = \exp \left( -\frac{1}{2} \langle \Phi, (-\Delta + \bar{\mu}_0^N)\Phi \rangle - V_0^N(\Phi) \right)$$

where the inner product is now on the unit lattice and

$$V_0^N(\Phi) = \varepsilon_0^N \dim_{T_{M+N}} + \frac{1}{2} \mu_0^N \sum_x \Phi^2(x) + \frac{1}{4} \lambda_0^N \sum_x \Phi^4(x)$$

The fixed coupling constants have scaled to

$$\bar{\mu}_0^N = L^{-2N}\bar{\mu} \quad \lambda_0^N = L^{-N}\lambda$$

and the counterterms have scaled to

$$\varepsilon_0^N = L^{-3N}\varepsilon \quad \mu_0^N = L^{-2N}\mu$$

The subscripts "zero" indicate that we are at the starting point of our renormalization group iteration. In the following we generally omit the superscript $N$. Thus $\rho_0^N, V_0^N$ are denoted $\rho_0, V_0$, and $\lambda_0^N, \bar{\mu}_0^N, \mu_0^N, \varepsilon_0^N$ are denoted $\lambda_0, \bar{\mu}_0, \mu_0, \varepsilon_0$.

2 the RG transformation

2.1 block averaging

The renormalization group (RG) is a series of transformations which average out the short distance features of the model, leaving only the the long distance properties in which we are interested (now that we have scaled the model).

First we define averaging operators. On the lattice $L^{-k}\mathbb{Z}^3$, or any associated toroidal lattice, the averaging operator $Q$ takes functions $f$ on $L^{-k}\mathbb{Z}^3$ to functions $Qf$ on $L^{-k+1}\mathbb{Z}^3$ by

$$(Qf)(y) = L^{-3} \sum_{x \in B(y)} f(x)$$
Here $B(y)$ is cubes of $L^3$ sites ($L$ on a side) in $L^{-k}\mathbb{Z}^3$ centered on $y \in L^{-k+1}\mathbb{Z}^3$. It can be written

$$B(y) = \{ x \in L^{-k}\mathbb{Z}^3 : |x - y| \leq L^{-k+1}/2 \}$$ (16)

The distance is $|x - y| = \sup_\mu |x_\mu - y_\mu|$ and we assume $L$ is odd. The transpose operator $Q^T$ with respect to the inner product \[\text{[3]}\] takes functions on $L^{-k+1}\mathbb{Z}^3$ to functions on $L^{-k}\mathbb{Z}^3$. It is computed to be

$$(Q^T f)(x) = f(y) \quad \text{if} \quad x \in B(y)$$ (17)

Then $QQ^T = I$ while $Q^T Q$ is a projection operator onto the range of $Q^T$ which is functions constant on the cubes.

Now starting with the density $\rho_0$ on functions $\Phi_0 : T^0_{M+N} \to \mathbb{R}$ we define a transformed density $\tilde{\rho}_1$ on functions $\Phi_1 : T^1_{M+N} \to \mathbb{R}$ by

$$\tilde{\rho}_1(\Phi_1) = N_{aL,T^1_{M+N}}^{-1} \int \exp\left( - \frac{a}{2L^2} \| \Phi_1 - Q\Phi_0 \|^2 \right) \rho_0(\Phi_0) \, d\Phi_0$$

$$= N_{aL,T^1_{M+N}}^{-1} \int \exp\left( - \frac{a}{2L} \| \Phi_1 - Q\Phi_0 \|^2 \right) \rho_0(\Phi_0) \, d\Phi_0$$ (18)

Here in the first expression norms are taken with the natural metric for the lattice so $\| \Phi_1 \|^2 = L^3 \sum_{x \in T^0_{M+N}} |\Phi_1(x)|^2$. In the second expression we use an unweighted sum $|\Phi_1|^2 = \sum_{x \in T^0_{M+N}} |\Phi_1(x)|^2$.

The positive constant $a$ is arbitrary and the normalization constant is \[\text{[3]}\]

$$N_{aL,T^1_{M+N}} = \int \exp\left( - \frac{1}{2} aL |\Phi_1|^2 \right) d\Phi_1 = (\frac{2\pi}{aL})^{\frac{|T^1_{M+N}|}{2}}$$ (19)

the constant is chosen so that

$$\int \tilde{\rho}_1(\Phi_1) \, d\Phi_1 = \int \rho_0(\Phi_0) \, d\Phi_0$$ (20)

Now one scales back to the unit lattice. A function $\Phi_1 : T^0_{M+N-1} \to \mathbb{R}$ scales up to $\Phi_{1,L} : T^1_{M+N} \to \mathbb{R}$ defined by

$$\Phi_{1,L}(x) = L^{-1/2} \Phi_1(x/L)$$ (21)

We define

$$\rho_1(\Phi_1) = \tilde{\rho}_1(\Phi_{1,L}) L^{-1/2} N_{a,T^1_{M+N-1}}$$

This preserves the integral

$$\int \rho_1(\Phi_1) d\Phi_1 = \int \rho_0(\Phi_0) \, d\Phi_0$$ (23)

We compute, taking account that $\tilde{N}_{aL,T^1_{M+N}}^{-1} L^{-|T^1_{M+N}|/2} = N_{a,T^1_{M+N}}^{-1}$,

$$\rho_1(\Phi_1) = N_{a,T^1_{M+N}}^{-1} \int \exp\left( - \frac{a}{2L^2} \| \Phi_{1,L} - Q\Phi_L \|^2 \right) \rho_0(\Phi_L) \, d\Phi_L$$

$$= N_{a,T^1_{M+N}}^{-1} \int \exp\left( - \frac{a}{2L^2} \| \Phi_{1,L} - Q\Phi_L \|^2 \right) \rho_0(\Phi_L) \, d\Phi_L$$ (24)

In general if $\Omega$ is a set and $\Phi : \Omega \to \mathbb{R}$ we define

$$N_{a,\Omega} = \int \exp\left( - \frac{1}{2} a |\Phi|^2 \right) d\Phi = (\frac{2\pi}{a})^{\frac{|\Omega|}{2}}$$
In the second step we have made the change of variables by \( \Phi_0 = \phi_L \) where \( \phi : \mathbb{T}_{M+N-1}^1 \rightarrow \mathbb{R} \). Then

\[
d\Phi_0 = L^{-|\mathbb{T}_{M+N-1}^1|/2}d\phi \equiv d\phi^{(1)}
\]  

(25)

In the last step we use that \( Q \) is scale invariant: \( Q\phi_L = (Q\phi)_L \).

We repeat this step a number of times. After \( k \) steps we will have a density \( \rho_k(\Phi_k) \) defined on functions \( \Phi_k : \mathbb{T}_{M+N-k}^0 \rightarrow \mathbb{R} \). The next step is to define a density on functions \( \Phi_{k+1} : \mathbb{T}_{M+N-k-1}^0 \rightarrow \mathbb{R} \) by

\[
\tilde{\rho}_{k+1}(\Phi_{k+1}) = N_{aL}^{-1} \mathbb{T}_{M+N-k}^0 \int \exp \left( -\frac{a}{2L^2} \| \Phi_{k+1} - Q\Phi_k \|^2 \right) \rho_k(\Phi_k) \ d\Phi_k 
\]

\[

(26)
\]

Then one scales back to the unit lattice. If \( \Phi_{k+1} : \mathbb{T}_{M+N-k}^0 \rightarrow \mathbb{R} \) then \( \Phi_{k+1,L} : \mathbb{T}_{M+N-k}^1 \rightarrow \mathbb{R} \) and we define

\[
\rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_{k+1}(\Phi_{k+1,L}) L^{-|\mathbb{T}_{M+N-k}^1|/2}
\]

(27)

Then we still have the normalization

\[
\int \rho_{k+1}(\Phi_{k+1}) d\Phi_{k+1} = \int \rho_k(\Phi_k) d\Phi_k = \int \rho_0(\Phi_0) d\Phi_0
\]

(28)

The various averaging operators can be composed into a single averaging operation over large cubes. Let \( Q_k = Q^k \) be an averaging operator over cubes \( B_k(y) \) with \( L^3 \) sites \( (L^k \) on a side). The operator \( Q_k \) maps functions on \( \mathbb{T}_{M+N-k}^{-k} \) to functions on \( \mathbb{T}_{M+N-k}^0 \) and is given by

\[
(Q_k f)(y) = L^{-3k} \sum_{x \in B_k(y)} f(x) = \int_{|x-y| < 1/2} f(x) dx
\]

(29)

**Lemma 2.** \( \rho_k(\Phi_k) \) can be written

\[
\rho_k(\Phi_k) = N_{a_k}^{-1} \mathbb{T}_{M+N-k}^0 \int \exp \left( -\frac{a_k}{2} \| \Phi_k - Q_k \phi \|^2 \right) \rho_0(\Phi_{L^k}) \ d^{(k)} \phi
\]

(30)

where

\[
\phi : \mathbb{T}_{M+N-k}^{-k} \rightarrow \mathbb{R} \quad \text{and} \quad d\phi^{(k)} = L^{-k|\mathbb{T}_{M+N-k}^{-k}|/2}d\phi
\]

(31)

and

\[
a_k = a \frac{1 - L^{-2}}{1 - L^{-2k}}
\]

(32)

**Remark.** The following proof is maybe not the shortest, but it develops some machinery we want to use later.

**Proof.** The proof is by induction. Assuming it is true for \( k \) we compute

\[
\tilde{\rho}_{k+1}(\Phi_{k+1}) = \text{const} \int \exp \left( -\frac{1}{2L^2} \| \Phi_{k+1} - Q\Phi_k \|^2 - \frac{a_k}{2} \| \Phi_k - Q_k \phi \|^2 \right) \rho_0(\Phi_{L^k}) \ d^{(k)} \phi \ d\Phi_k
\]

(33)

The expression inside the exponential has a minimum in \( \Phi_k \) when

\[
\left( a_k + \frac{a}{L^2} Q^T Q \right) \Phi_k = a_k Q_k \phi + \frac{a}{L^2} Q^T \Phi_{k+1}
\]

(34)
This has the solution $\Phi_k = \Psi_k$ where

$$
\Psi_k(\Phi_{k+1}, \phi) = a_k^{-1} \left( I - \frac{aL^2}{ak + aL^2} Q^T Q \right) \left( a_k Q_k \phi + \frac{aL^2}{L^2} Q^T \Phi_k \right)
$$

$$
= Q_k \phi - \frac{aL^2}{ak + aL^2} Q^T Q_{k+1} \phi + \frac{aL^2}{ak + aL^2} Q^T \Phi_{k+1}
$$

(35)

We compute using $QQ^T = 1$

$$
Q \Psi_k = Q_{k+1} \phi - \frac{aL^2}{ak + aL^2} Q_{k+1} \phi + \frac{aL^2}{ak + aL^2} \Phi_{k+1}
$$

$$
= \frac{a_k}{ak + aL^2} Q_{k+1} \phi + \frac{aL^2}{ak + aL^2} \Phi_{k+1}
$$

(36)

Thus

$$
\Phi_{k+1} - Q \Psi_k = \frac{a_k}{ak + aL^2} (\Phi_{k+1} - Q_{k+1} \phi)
$$

(37)

and so we have

$$
\frac{a}{2L^2} ||\Phi_{k+1} - Q \Psi_k||^2 = \frac{a}{2L^2} \left( \frac{a_k}{ak + aL^2} \right)^2 ||\Phi_{k+1} - Q_{k+1} \phi||^2
$$

$$
= \frac{a_{k+1}}{2L^2} \frac{a_k}{ak + aL^2} ||\Phi_{k+1} - Q_{k+1} \phi||^2
$$

(38)

Here we use the identity

$$
a_{k+1} = \frac{a_k a}{ak + aL^2}
$$

(39)

On the other hand from (35) and $||Q^T \phi||^2 = ||\phi||^2$

$$
\frac{1}{2} a_k ||\Psi_k - Q_k \phi||^2 = \frac{1}{2} a_k \left( \frac{aL^2}{ak + aL^2} \right)^2 ||\Phi_{k+1} - Q_{k+1} \phi||^2
$$

$$
= \frac{a_{k+1}}{2L^2} \frac{a_k}{ak + aL^2} ||\Phi_{k+1} - Q_{k+1} \phi||^2
$$

(40)

Combining (38) and (40) gives the value at the minimum as

$$
\frac{a}{2L^2} ||\Phi_{k+1} - Q \Psi_k||^2 + \frac{1}{2} a_k ||\Psi_k - Q_k \phi||^2 = \frac{a_{k+1}}{2L^2} ||\Phi_{k+1} - Q_{k+1} \phi||^2
$$

(41)

Now in the integral (33) expand around the minimizer. We write $\Phi_k = \Psi_k + Z$ and integrate over $Z$. The term with no $Z$’s is (41). The linear terms in $Z$ vanish and the terms quadratic in $Z$ when integrated over $Z$ yield a constant. Thus we have

$$
\tilde{\rho}_{k+1}(\Phi_{k+1}) = \text{const} \int \exp \left( -\frac{a_{k+1}}{2L^2} ||\Phi_{k+1} - Q_{k+1} \phi||^2 \right) \rho_0(\phi_L) \ d(k) \phi
$$

(42)

Replacing $\Phi_{k+1}$ by $\Phi_{k+1, L}$ now with $\Phi_{k+1} : \mathbb{T}_{M+N-k-1}^0 \to \mathbb{R}$, and replacing $\phi$ by $\phi_L$ with now $\phi : \mathbb{T}_{M+N-k-1}^{k-1} \to \mathbb{R}$ we find

$$
\rho_{k+1}(\Phi_{k+1}) = \text{const} \int \exp \left( -\frac{a_{k+1}}{2} ||\Phi_{k+1} - Q_{k+1} \phi||^2 \right) \rho_0(\phi_{L+1}) \ d(k+1) \phi
$$

(43)

But the constant must be $N_{a_{k+1}, \mathbb{T}_{M+N-k-1}^{k-1}}^{-1}$ in order to preserve the identity (28). This completes the proof.
2.2 free flow

Now suppose we only keep the quadratic part of $\rho_0$ so that

$$\rho_0(\Phi_0) = \exp\left( -\frac{1}{2} < \Phi_0, (-\Delta + \bar{\mu}_0)\Phi_0 > \right)$$

Inserting this in (30) yields

$$\rho_k(\Phi_k) = N_{a_k, T_{M+N-k}}^{-1} \int \exp(-S_k(\Phi_k, \phi)) \ d^{(k)}\phi$$

where

$$S_k(\Phi_k, \phi) = \frac{a_k}{2}||\Phi_k - Q_k\phi||^2 + \frac{1}{2} < \phi, (-\Delta + \bar{\mu}_k)\phi >$$

with

$$\bar{\mu}_k = L^{2k}\bar{\mu}_0 = L^{-2(N-k)}\bar{\mu}$$

To compute this we look for the minimizer of $S_k(\Phi_k, \phi)$ in $\phi$. The minimizer satisfies the equation

$$(-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)\phi = a_k Q_k^T \Phi_k$$

The solution involves the inverse

$$G_k = (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)^{-1}$$

and has the form $\phi = \phi_k(\Phi_k)$ defined by

$$\phi_k(\Phi_k) = a_k G_k Q_k^T \Phi_k$$

We shift the integration in (45) so it is centered on the minimum. We take $\phi = \phi_k + Z$ where $Z : \mathbb{T}_{M+N-k} \rightarrow \mathbb{R}$ is the new integration variable. The cross terms vanish and so

$$S_k(\Phi_k, \phi_k + Z) = S_k(\Phi_k, \phi_k) + \frac{1}{2} < Z, (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)Z >$$

Then (45) becomes

$$\rho_k(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k))$$

where

$$Z_k = N_{a_k, T_{M+N-k}}^{-1} \int \exp\left( -\frac{1}{2} < Z, (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)Z > \right) \ d^{(k)}Z$$

Here is another representation of $S_k(\Phi_k, \phi_k)$. With $\phi_k = a_k G_k Q_k^T \Phi_k$ we have

$$S_k(\Phi_k, \phi_k) = \frac{a_k}{2}||\Phi_k||^2 - a_k < \phi_k, Q_k^T \Phi_k > + \frac{1}{2} < \phi_k, (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k)\phi_k >$$

$$= \frac{a_k}{2}||\Phi_k||^2 - a_k^2 < \Phi_k, Q_k G_k Q_k^T \Phi_k > + \frac{a_k^2}{2} < G_k Q_k^T \Phi_k, (-\Delta + \bar{\mu}_k + Q_k^T Q_k)G_k Q_k^T \Phi_k >$$

$$= \frac{1}{2} < \Phi_k, \Delta_k \Phi_k >$$

where $\Delta_k = a_k - a_k^2 Q_k G_k Q_k^T$. 

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2.3 single step free flow

Now we rederive the identity \( \rho_k(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k)) \) for the free case by an inductive procedure. We assume that the identity holds for \( \rho_k \) and show it holds for \( \rho_{k+1} \). This approach will generate some useful identities and provide guidance for the treatment of the general case with the potential added.

Starting with the identity for \( k \) we have

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = \frac{1}{aL^2} \int_{n-1}^{n} Z_k \exp \left( -\frac{a}{2L^2} \| \Phi_{k+1} - Q \Phi_k \|^2 - S_k(\Phi_k, \phi_k) \right) d\Phi_k
\]

\[
= \frac{1}{aL^2} \int_{n-1}^{n} Z_k \exp \left( -J(\Phi_{k+1}, \Phi_k, \phi_k) \right) d\Phi_k
\]

where

\[
J(\Phi_{k+1}, \Phi_k, \phi) = \frac{a}{2L^2} \| \Phi_{k+1} - Q \Phi_k \|^2 + \frac{1}{2} a_k \| \Phi_k - Q \phi_k \|^2 + \frac{1}{2} \| \phi \|^2 + \frac{1}{2} \mu_k \| \phi \|^2
\]

Here \( \Phi_{k+1}, \Phi_k, \phi \) are fields on \( \mathbb{T}_{M+N-k}^{1}, \mathbb{T}_{M+N-k}^{0}, \mathbb{T}_{M+N-k}^{-k} \) respectively.

To compute the integral we want to minimize \( J(\Phi_{k+1}, \Phi_k, \phi_k) \) in \( \Phi_k \). But \( J(\Phi_{k+1}, \Phi_k, \phi_k) \) is the minimum value of \( J(\Phi_{k+1}, \Phi_k, \phi) \) in \( \phi \). This suggests we study the minimizer of \( J(\Phi_{k+1}, \Phi_k, \phi) \) in \( \Phi_k, \phi \) simultaneously.

For the next lemma we need the operator

\[
G_{k+1}^0 = \left( -\Delta + \frac{1}{2L^2} \Phi_{k+1} + \frac{1}{2} \| \phi \|^2 \right)^{-1}
\]

defined on functions on \( \mathbb{T}_{M+N-k}^{-k} \). This scales to \( G_{k+1} \) as we will see.

Lemma 3.

1. Given \( \Phi_{k+1} \) the unique minimum of \( J(\Phi_{k+1}, \Phi_k, \phi) \) comes at \( (\Phi_k, \phi) = (\Psi_k, \phi_k^0) \) where

\[
\phi_{k+1}^0 = \phi_{k+1}(\Phi_{k+1}) = L^{-2}a_k+1 C_k^{0} Q_k^{T} \Phi_k+1
\]

and where

\[
\Psi_k = \Psi_k(\Phi_{k+1}, \phi_{k+1}) = Q_k \phi_{k+1} - \frac{aL^{-2}}{a_k+1} Q_k^{T} Q_k+1 \phi_{k+1} + \frac{aL^{-2}}{a_k+1} Q_k^{T} \phi_k+1
\]

2. The minimizer in \( \phi \) can also be written \( \phi_k(\Psi_k) \) so we have the identity between functions of \( \Phi_{k+1} \)

\[
\phi_{k+1}^0 = \phi_k(\Psi_k)
\]

3. The value of \( J(\Phi_{k+1}, \Phi_k, \phi) \) at the minimizers is

\[
S_{k+1}^0(\Phi_{k+1}, \phi_{k+1}) = \frac{1}{2} a_k L^{-2} \| \Phi_{k+1} - Q_k \phi_{k+1} \|^2 + \frac{1}{2} \| \phi \|^2 + \frac{1}{2} \mu_k \| \phi \|^2
\]

Proof. The variational equations for the minimizer in \( \Phi_k, \phi \) are

\[
\left( a_k + \frac{a}{L^2} Q_k^{T} Q \right) \Phi_k = a_k Q_k \phi + \frac{a}{L^2} Q_k^{T} \Phi_k+1
\]

\[
(-\Delta + \frac{1}{2L^2} \Phi_k + \frac{1}{2} \| \phi \|^2) \phi_k = a_k Q_k \Phi_k+1
\]

\[
\inf_{y \in B} \left( \inf_{x \in A} f(x, y) \right) = \inf_{x \in A} \left( \inf_{y \in B} f(x, y) \right)
\]

If the minimizers are unique then they must come at the same point.
both of which we have seen before. The first is solved by \( \Phi_k = \Psi_k = \Psi_k(\Phi_{k+1}, \phi) \). Substituting this into the second and solving for \( \phi \) we find that the minimizer comes at \( \phi = a_k G_k Q_k^T \Psi_k \equiv \phi_k(\Psi_k) \).

We further analyze the second equation at \( \Phi_k = \Psi_k \). Using (59) the right side is evaluated as

\[
a_k Q_k^T \Psi_k = a_k Q_k^T Q_k \phi - a_{k+1} L^{-2} Q_{k+1}^T Q_{k+1} \phi + a_{k+1} L^{-2} Q_{k+1}^T \Phi_{k+1}
\]

(63)

The second equation then becomes

\[
(-\Delta + \mu_k + a_{k+1} L^{-2} Q_{k+1}^T Q_{k+1}) \phi = a_{k+1} L^{-2} Q_{k+1}^T \Phi_{k+1}
\]

(64)

which has the solution \( \phi = \phi_0^{k+1} = L^{-2} a_{k+1} G_k Q_k^T \Phi_{k+1} \). This establishes (58), (59), and (60).

The value at the minimum is

\[
J(\Phi_{k+1}, \Psi_k, \phi_0^{k+1}) = \frac{a}{2L^2} ||\Phi_{k+1} - Q_k \Psi_k||^2 + \frac{1}{2} a_k ||\Psi_k - Q_k \phi_k||^2 + \frac{1}{2} \mu_k ||\phi_k||^2 + \frac{1}{2} \overline{\mu_k} ||\phi_k||^2
\]

= \frac{a_{k+1}}{2L^2} ||\Phi_{k+1} - Q_{k+1} \phi_0^{k+1}||^2 + \frac{1}{2} ||\partial \phi_0^{k+1}||^2 + \frac{1}{2} \mu_k ||\phi_0^{k+1}||^2
\]

(65)

Here we used (11). This completes the proof.

**Lemma 4.** Let \( \rho_k(\Phi_k) = Z_k \exp(-S_k(\Phi_k, \phi_k)) \). Then \( \hat{\rho}_{k+1} \) is given by

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = Z_k N^{-1} L_{\pi_k}^{M-N-k} \exp \left( \frac{a}{2L^2} ||Q_k Z_k||^2 + S_k(\Phi_k) \right) \exp \left( -S_k(N^{-1} L_{\pi_k}^{M-N-k} (\det C_k)^{1/2} \exp \left( -S_k(\Phi_{k+1}, \phi_0^{k+1}) \right) \right)
\]

where

\[
C_k = \left( \Delta_k + \frac{a}{L^2} Q_k^T Q_k \right)^{-1}
\]

(66)

**Proof.** We calculate \( \hat{\rho}_{k+1} \) given in (55) by expanding around the minimum is \( \Phi_k \). We we write \( \Phi_k = \Psi_k + Z \) and integrate over \( Z : T_{M+N-k} \rangle \rightarrow \mathbb{R} \) instead of \( \Phi_k \). We have from (60)

\[
\phi_k(\Psi_k + Z) = \phi_0^{k+1} + Z
\]

(68)

where \( Z : T_{M+N-k} \rangle \rightarrow \mathbb{R} \) is defined by

\[
Z_k = \phi_k(Z) = a_k G_k Q_k^T Z
\]

(69)

Thus we are expanding \( J(\Phi_{k+1}, \Phi_k, \phi_k) \) around the minimum in the last two variables. We claim that

\[
J(\Phi_{k+1}, \Psi_k + Z, \phi_0^{k+1} + Z_k) = S_k(\Phi_{k+1}, \phi_0^{k+1}) + \frac{a}{2L^2} ||Q_k Z||^2 + S_k(Z, Z_k)
\]

\[
= S_k(\Phi_{k+1}, \phi_0^{k+1}) + \frac{1}{2} \left< Z, \left( \Delta_k + \frac{a}{L^2} Q_k^T Q_k \right) Z \right>
\]

(70)

Indeed we have already seen that \( S_k(\Phi_{k+1}, \phi_0^{k+1}) \) is the minimum value. The linear terms in \( Z, Z_k \) must vanish. The quadratic terms in \( Z, Z_k \) are as indicated. The second form follows from (54).

Inserting this last expression into (55) yields

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = N^{-1} L_{\pi_k}^{M-N-k} Z_k \exp(-S_k(\Phi_{k+1}, \phi_0^{k+1})) \int \exp \left( -\frac{1}{2} \left< Z, \left( \Delta_k + \frac{a}{L^2} Q_k^T Q_k \right) Z \right> \right) dZ
\]

(71)

We evaluate the last integral as \((2\pi)^{1/2} (\det C_k)^{1/2} \) which gives the result.
Lemma 5. (scaling) With $\hat{\rho}_{k+1}$ given by (66), the scaled density $\rho_{k+1}$ as defined by (27) is for $\Phi_{k+1} : \mathbb{T}_{M+N-k-1}^m \to \mathbb{R}$

$$\rho_{k+1}(\Phi_{k+1}) = Z_{k+1} \exp \left( - S_{k+1}(\Phi_{k+1}, \phi_{k+1}) \right)$$

(72)

Furthermore

$$\phi^0_{k+1}(\Phi_{k+1}, L) = [\phi_{k+1}(\Phi_{k+1})]_L$$

and

$$Z_{k+1} = Z_k N^{-1}_{a^{\Phi_{k+1}}_{\mathbb{T}_{M+N}}} (2\pi)^{\frac{m}{2}} \sqrt{\det C_k}$$

(73)

(74)

Proof. We scale by $f_L(x) = L^{-1/2} f(x/L)$. The averaging operator $Q$ is scale invariant and $\bar{\mu}_k = L^{-2} \bar{\mu}_{k+1}$ so we compute

$$\left( - \Delta + \bar{\mu}_k + a_{k+1} L^{-2} Q_{k+1}^T Q_k \right) f_L = L^{-2} \left[ ( - \Delta + \tilde{\mu}_{k+1} + a_{k+1} Q_{k+1}^T Q_k ) f \right]_L$$

(75)

It follows that the inverses satisfy $G_{k+1}^0 f_L = L^2 [G_{k+1} f]_L$ and so

$$\phi^0_{k+1}(\Phi_{k+1}, L) = L^{-2} a_{k+1} G_{k+1}^0 Q_{k+1}^T \Phi_{k+1} = [a_{k+1} G_{k+1} Q_{k+1}^T \Phi_{k+1}]_L = [\phi_{k+1}(\Phi_{k+1})]_L$$

(76)

Now in $\rho_{k+1}(\Phi_{k+1})$ we have

$$S_{k+1}^0(\Phi_{k+1}, L, \phi_{k+1}, L) = \frac{1}{2} a_{k+1} \parallel \Phi_{k+1} - Q_{k+1}^T \Phi_{k+1} \parallel^2 + \frac{1}{2} \parallel \partial \phi_{k+1} \parallel^2 + \frac{1}{2} \bar{\mu}_k \parallel \phi_{k+1} \parallel^2$$

$$= \frac{1}{2} a_{k+1} \parallel \Phi_{k+1} - Q_{k+1}^T \Phi_{k+1} \parallel^2 + \frac{1}{2} \parallel \partial \phi_{k+1} \parallel^2 + \frac{1}{2} \bar{\mu}_{k+1} \parallel \phi_{k+1} \parallel^2$$

(77)

Thus

$$\rho_{k+1}(\Phi_{k+1}) = Z_k N^{-1}_{a^{\Phi_{k+1}}_{\mathbb{T}_{M+N}}} (2\pi)^{\frac{m}{2}} \sqrt{\det C_k} \exp \left( - S_{k+1}(\Phi_{k+1}, \phi_{k+1}) \right)$$

(78)

and the constant is identified as $Z_{k+1}$.

2.4 random walk expansion

We develop a random walk expansion for the the Green’s function $G_k = ( - \Delta + \bar{\mu}_k + a_k Q_k^T Q_k )^{-1}$ on $\mathbb{T}_{M+N-k}^m$. This will give us estimates on $G_k$ and also provide the basis of localized approximations to $G_k$.

The random walk expansion is based on localized inverses which we now define. Let $M = L^m$ for some positive integer $m$, and let $\Box_z$ be a large cube in $\mathbb{T}_{M+N-k}^m$ of linear size $M$ centered on points $z \in \mathbb{T}_{M+N-k}^m$. (Warning: $M$ is not the same as the volume parameter $M$.) The centers are a distance $M$ apart so the $\Box_z$ partition the lattice. Also let $\tilde{\Box}_z$ be the union of all $M$-cubes touching $\Box_z$. The $\tilde{\Box}_z$ are overlapping cubes in $\mathbb{T}_{M+N-k}^m$ of linear size $3M$ still centered on points $z \in \mathbb{T}_{M+N-k}^m$. For $\Box = \Box_z$ let $\Delta_{\Box}$ be the Laplacian on $\Box$ with Neumann boundary conditions. This means that in $< \phi, -\Delta_{\Box} \phi > = \sum_x \int |\partial_x \phi(x)|^2 dx$ only terms with both $x, x + L^{-k} e_\mu \in \Box$ contribute. Now restrict the operator $-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k$ to $\Box$ with the Neumann conditions and take the inverse defining

$$G_k(\Box) = [ - \Delta + \bar{\mu}_k + a_k Q_k^T Q_k ]_{\Box}^{-1}$$

(79)

We give some estimates on these operators. Here and throughout the paper we employ the convention that $C$ stands for a constant depending on $L$, but no other parameters. It may change from line to line. Also $O(1)$ stands for a constant independent of all parameters.
Let $\Delta_y = B_k(y)$ be unit cubes in $\mathbb{T}_{M+N-k}$ centered on points $y \in \mathbb{T}_{M+N-k}$. These partition the lattice and any large cube $\Box$ or $\tilde{\Box}$. Let $y, y' \in \Box$ and let $x \in \Delta_y \subset \Box$ and supp$f \subset \Delta_{y'} \subset \tilde{\Box}$. Then for some constants $C$ and $\gamma_0 = O(L^{-2})$

$$\|(G_k(\tilde{\Box}))f(x)\| \leq Ce^{-\gamma_0 d(y,y')}\|f\|_{\infty}$$

$$\|\partial G_k(\tilde{\Box})f(x)\| \leq Ce^{-\gamma_0 d(y,y')}\|f\|_{\infty}$$

(80)

We also estimate the Holder derivative of $\partial G_k(\tilde{\Box})$ of order $1/2 < \alpha < 1$. For $x, x' \in \Delta(y)$ and supp$f \subset \Delta(y')$

$$\|(\delta_{\alpha}\partial G_k(\tilde{\Box}))f(x, x')\| \leq Ce^{-\gamma_0 d(y,y')}\|f\|_{\infty}$$

(81)

Here $\delta_{\alpha}$ is defined for $d(x, x') \leq 1$ by

$$(\delta_{\alpha}f)(x, x') = \frac{f(x) - f(x')}{d(x, x')^\alpha}$$

(82)

and $C$ does depend on $\alpha$ in (81). We give proofs of these estimates in Appendix D. References are [4], [7], [19].

A random walk or path is a sequence of points in the lattice $\mathbb{T}_{M+N-k}$ with spacing $M = L^m$ written

$$\omega = (\omega_0, \omega_1, \ldots, \omega_n)$$

(83)

such that $\omega_j, \omega_{j+1}$ are neighbors in the sense that for each component $|\omega_{j, \mu} - \omega_{j+1, \mu}| \leq M$. Thus $\omega_j$ has $3^d = 9$ neighbors counting itself. The number of steps in the walk is $|\omega| = n$.

**Lemma 6.** The Green’s function $G_k$ has a random walk expansion of the form

$$G_k = \sum_{\omega} G_{k, \omega}$$

(84)

where the sum is over all paths $\omega$. If $M$ is sufficiently large the series for $G_k, \partial G_k, \delta_{\alpha}\partial G_k$ all converge and give for $x, x' \in \Delta_y$ and supp$f \subset \Delta_{y'}$

$$\|(G_k f)(x)\| \leq Ce^{-\frac{1}{2} \gamma_0 d(y,y')}\|f\|_{\infty}$$

$$\|\partial G_k f(x)\| \leq Ce^{-\frac{1}{2} \gamma_0 d(y,y')}\|f\|_{\infty}$$

$$\|(\delta_{\alpha}\partial G_k f)(x, x')\| \leq Ce^{-\frac{1}{2} \gamma_0 d(y,y')}\|f\|_{\infty}$$

(85)

**Proof.** We construct a partition of unity. Let $h \in C^\infty_0((-2/3, 2/3))$ satisfy $h \geq 0$ and $h = 1$ on $(-1/3, 1/3)$ and

$$1 = \sum_{n \in \mathbb{Z}} h^2(x - n)$$

(86)

Then for $z \in \mathbb{T}_{N+M-k}$ define $h_z$ on $\mathbb{T}_{N+M-k}$ by

$$h_z(x) = \prod_{\mu=1}^d h \left( \frac{x_\mu - z_\mu}{M} \right)$$

(87)

Then $h_z$ has support in $\tilde{\Box}_z$, and in fact in the smaller set $\{x : |x_\mu - z_\mu| \leq \frac{1}{3}M\}$. Hence $h_z h_{z'} = 0$ unless $z, z'$ are neighbors. We have

$$1 = \sum_z h_z^2(x)$$

(88)
Furthermore
\[ |\partial h| \leq \mathcal{O}(1)M^{-1} \quad |\partial^2 h| \leq \mathcal{O}(1)M^{-2} \quad (89) \]

Define a parametrix \( G^*_k \) by
\[ G^*_k = \sum_z h_z G_k(\bar{\Delta}_z)h_z \quad (90) \]

Then we have
\[ (-\Delta + \bar{\mu}_k + a_k Q^T_k Q_k)G^*_k = I - \sum_z K_z G(\bar{\Delta}_z)h_z \equiv I - K \quad (91) \]
where
\[ K_z = -[(\Delta + a_k Q^T_k Q_k), h_z] \quad (92) \]

The solution is now
\[ G_k = G^*_k(I - K)^{-1} = G^*_k \sum_{n=0}^{\infty} K^n \quad (93) \]
provided the series converges. This can also be written as the random walk expansion
\[ G_k = \sum_{n=0}^{\infty} \sum_{\omega/\omega_1,\ldots,\omega_n} (h_{\omega_0} G_k(\bar{\Delta}_{\omega_0})h_{\omega_0}) \left( K_{\omega_1} G_k(\bar{\Delta}_{\omega_1})h_{\omega_1} \right) \cdots \left( K_{\omega_n} G_k(\bar{\Delta}_{\omega_n})h_{\omega_n} \right) \equiv \sum_{\omega} G_{k,\omega} \quad (94) \]

Now we claim that for \( x \in \Delta_y \subset \bar{\Delta} \)
\[ |(K_z f)(x)| \leq \mathcal{O}(1)M^{-1} \left( \|1_{\Delta_y} f\|_{\infty} + \|1_{\Delta_y} \partial f\|_{\infty} \right) \quad (95) \]

Indeed the term \([-\Delta, h_z]\) is local and involves derivatives of \( h \) so we get the factor \( M^{-1} \) from \( 89 \). The term \([Q^T_k Q_k, h_z]\) is also local and also can be expressed in term of derivatives of \( h_z \) since it can be written
\[ \left( [Q^T_k Q_k, h_z] f \right)(x) = \int_{x' \in \Delta_y} (h_z(x') - h_z(x)) f(x') dx' \quad (96) \]

Combining the bound on \( K_z \) with the basic bound \( 80 \) on \( G_k(\bar{\Delta}) \) yields for \( x \in \Delta_y \subset \bar{\Delta} \) and \( \text{supp} f \subset \Delta_{y'} \subset \bar{\Delta} \):
\[ |(K_z G_k(\bar{\Delta}_z)f)(x)| \leq CM^{-1} \left( \|1_{\Delta_z} G_k(\bar{\Delta}_z)f\|_{\infty} + \|1_{\Delta_z} \partial G_k(\bar{\Delta}_z)f\|_{\infty} \right) \]
\[ \leq CM^{-1} \exp(-\gamma_0 d(y, y')) \|f\|_{\infty} \quad (97) \]

(Only interior derivatives of \( G_k(\bar{\Delta}_z) \) appear since \( \text{supp} h_z \) is well-inside \( \bar{\Delta} \).) We use this bound repeatedly on \( G_{k,\omega} \) with \( |\omega| = n \). We have for \( x \in \Delta_y \) and \( \text{supp} f \subset \Delta_{y'} \) with \( y_0 = y, y_{n+1} = y' \)
\[ |(G_{k,\omega} f)(x)| = \left| \left( (h_{\omega_0} G_k(\bar{\Delta}_{\omega_0})h_{\omega_0}) \left( K_{\omega_1} G_k(\bar{\Delta}_{\omega_1})h_{\omega_1} \right) \cdots \left( K_{\omega_n} G_k(\bar{\Delta}_{\omega_n})h_{\omega_n} \right) f \right)(x) \right| \]
\[ \leq \sum_{y_1,\ldots,y_n} \left| \left( (h_{\omega_0} G_k(\bar{\Delta}_{\omega_0})h_{\omega_0}) \left( K_{\omega_1} G_k(\bar{\Delta}_{\omega_1})h_{\omega_1} \right) \cdots \left( K_{\omega_n} G_k(\bar{\Delta}_{\omega_n})h_{\omega_n} \right) f \right)(x) \right| \quad (98) \]
\[ \leq C(CM)^{-n} \sum_{y_1,\ldots,y_n} \prod_{j=0}^{n} e^{-\gamma_0 d(y_j, y_{j+1})} \|f\|_{\infty} \]
\[ \leq C(CM)^{-n} e^{-\frac{1}{2} \gamma_0 d(y, y')} \|f\|_{\infty} \]

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For convergence of the random walk expansion we have for $M$ sufficiently large
\[
|G_k f(x)| \leq \sum_{\omega} |G_{k,\omega} f(x)|
\leq \sum_{n=0}^{\infty} \sum_{|\omega|=n} C \left( CM^{-1} \right)^n e^{-\frac{1}{2} \gamma_0 d(y,y')} \|f\|_\infty
\leq \sum_{n=0}^{\infty} C \left( CM^{-1} \right)^n (3d)^n e^{-\frac{1}{2} \gamma_0 d(y,y')} \|f\|_\infty
\leq C e^{-\frac{1}{2} \gamma_0 d(y,y')} \|f\|_\infty
\]
(99)
This establishes the bound on $G_k$ and the bounds on $\partial G_k$ and $\delta_\alpha \partial G_k$ are similar. This completes the proof.

The bounds of the lemma yield (more elementary) global estimates:

**Corollary 7.** For any $f : T_{M+k} \rightarrow \mathbb{R}$
\[
|G_k f|, |\partial G_k f|, |\delta_\alpha \partial G_k f| \leq C \|f\|_\infty
\]
(100)

**Proof.** For $x \in \Delta_y$
\[
|(G_k f)(x)| \leq \sum_{y'} |(G_{k,1\Delta_y} f)(x)| \leq C \sum_{y'} e^{-\frac{1}{2} \gamma_0 d(y,y')} \|f\|_\infty \leq C \|f\|_\infty
\]
(101)
The others are similar.

### 2.5 decoupling

We also need a version of $G_k$ in which the communication between sites is systematically weakened.

For each $M$-cube $\Box$ introduce a variable $s_\Box$ with $0 \leq s_\Box \leq 1$. Then define for $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$
\[
s_\omega = \prod_{\Box \subset X_\omega} s_\Box \quad X_\omega = \bigcup_{j=1}^{n} \Box_{\omega_j}
\]
(102)
Note that $\Box_{\omega_0}$ is omitted from $X_\omega$. Hence if $\omega$ is only a single point $\omega_0$ (i.e. $|\omega| = 0$) then $X_\omega$ is empty and in this case we set $s_\omega = 1$.

Now we define for $s = \{s_\Box\}$
\[
G_k(s) = \sum_{\omega} s_\omega G_{k,\omega}
\]
(103)
Then we have
\[
G_k(1) = \sum_{\omega} G_{k,\omega} = G_k
\]
\[
G_k(0) = \sum_{\omega : |\omega| = 1} G_{k,\omega} = G_k^*
\]
(104)
Thus $G_k(s)$ interpolates between an operator for which all sites are coupled and an operator for which keeps thing localized in each cube $\Box$.

Note that $G_k(s)$ can be defined and bounded for $s_\Box$ complex and in a much large domain. We can take for example $|s_\Box| \leq M^{1/2}$. Then in (99) instead of $(CM^{-1})^n$ we have $(C|s_\Box| M^{-1})^n \leq (CM^{-1/2})^n$. The random walk expansion still converges if $M$ is sufficiently large. The bounds (85) and (100) still hold for $G_k(s)$ with $|s_\Box| \leq M^{1/2}$. 

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3 Localized functionals of the field

3.1 overview

Our main goal is to follow the flow of the renormalization group with the potential included. The detailed analysis is in the next section. Here we do some preliminary work.

After $k$ steps with certain small field assumptions we will find that the density can be written in the following form:

$$
\rho_k(\Phi_k) = \text{const exp} \left( -S_k(\Phi_k, \phi_k) + \varepsilon_k \text{Vol}(T_{M+N-k}) - \frac{1}{2} \mu_k \int \phi_k^2 - \frac{1}{4} \lambda_k \int \phi_k^4 + E_k(\phi_k) \right)
$$

(105)

Here $\phi_k = a_k G_k Q_k^T \Phi_k$ as before and $S_k(\Phi_k, \phi_k)$ is the free action as defined in (46). In the next terms we track the growth of energy density $\varepsilon_k$ and the mass-squared $\mu_k$. We could do this as well with the coupling constant, but for this model it is sufficient to just let it scale and define

$$
\lambda_k = L^k \lambda_0 = L^{-(N-k)} \lambda
$$

(106)

We will only be interested in $N, k, \lambda$ such that $\lambda_k$ is small.

The term $E_k$ is real-valued and contains all non-leading and non-local corrections to the simple local form we have isolated. However it has some local structure which we now explain. Again consider cubes $\square$ with side of length $M = L^m$ centered on points of the lattice $T_{M+N-k}$ which partition the lattice $T_{M+N-k}$. An $M$-polymer $X$ is a connected unions of such cubes. Here connected means that for any two cubes $\square, \square'$ in $X$ there is a sequence $\square = \square_0, \square_1, \square_2, \ldots, \square_m = \square'$ such that $\square_j \subset X$ and $\square_j$ and $\square_{j+1}$ have a $d - 1 = 2$ dimensional face in common.

Now we define

$$
\mathcal{D}_k = \text{ all } M\text{-polymers } X \text{ in } T_{M+N-k}^{-k}
$$

(107)

We will assume the local structure

$$
E_k(\phi_k) = \sum_{X \in \mathcal{D}_k} E_k(X, \phi_k)
$$

(108)

where $E_k(X, \phi_k)$ only depend on the restriction of $\phi_k$ to $X$.

3.2 small fields

Our small field assumption is chosen so that when it is violated either the term exp $(-S_k(\Phi_k, \phi_k))$ or the term exp $(-\frac{1}{4} \lambda_k \int \phi_k^4)$ in the density is tiny. This is arranged as follows. Let

$$
p_k = p(\lambda_k) = (- \log \lambda_k)^p = ((N - k) \log L - \log \lambda)^p
$$

(109)

for some positive integer $p$. We assume always $\lambda_k < 1$ so the quantity we are exponentiating is positive.

**Definition 1.** $S_k$ is all functions $\Phi_k : T_{M+N-k}^{-k} \rightarrow \mathbb{R}$ such that with $\phi_k = a_k G_k Q_k^T \Phi_k$ on $T_{M+N-k}^{-k}$

$$
|\Phi_k - Q_k \phi_k| \leq p_k
$$

$$
|\partial \phi_k| \leq p_k
$$

$$
|\phi_k| \leq \lambda_k^{-1/4} p_k
$$

(110)

Since $\lambda_k$ is assumed small, $p_k$ is large, and these ”small field” conditions actually allow rather large fields. If one of these conditions fails then we gain a tiny factor $O(e^{-p_k})$.

In fact $E_k(X, \phi_k)$ will be the restriction of more general complex-valued functions $E_k(X, \phi)$ defined for complex fields $\phi : T_{M+N-k}^{-k} \rightarrow \mathbb{C}$. We want to choose weaker restrictions on $\phi$ so that if $\phi = \phi_k \in S_k$ then the new conditions are satisfied. We would also like bounds on $\phi, \partial \phi, \delta_x \partial \phi$ to be all about the same size, however it is convenient to allow a little deviation. This motivates the following definition:
Definition 2. Let $\epsilon$ be a fixed small positive number. $R_k$ is all functions $\phi : \mathbb{T}^{k}_{N+M-k} \to \mathbb{C}$ such that:

\[
|\phi| < \lambda_k^{-1/4-3\epsilon} \\
|\partial \phi| < \lambda_k^{-1/4-2\epsilon} \\
|\delta_\alpha \partial \phi| < \lambda_k^{-1/4-\epsilon}
\] (111)

This does the job for we have:

Lemma 8. Let $\Phi_k \in S_k$. Then

1. $|\Phi_k| \leq 2p_k \lambda_k^{-1/4}$ and $|\partial_\mu \Phi_k| \leq 3p_k$.
2. For $\lambda_k$ sufficiently small $\phi_k = a_k G_k Q_k^T \Phi_k \in R_k$.

Proof. For the first point we have

\[
|\Phi_k| \leq |\Phi_k - Q_k \phi_k| + |Q_k \phi_k| \leq p_k + p_k \lambda_k^{-1/4} \leq 2p_k \lambda_k^{-1/4}
\] (112)

and also

\[
|\partial_\mu \Phi_k(x)| = |(\Phi_k)(x + e_\mu) - (\Phi_k)(x)| \\
\leq |Q_k \phi_k(x + e_\mu) - Q_k \phi_k(x)| + 2p_k \\
\leq \|\partial \phi_k\|_\infty + 2p_k \leq 3p_k
\] (113)

For the second point if $\lambda_k$ is small we have have $p_k \leq \lambda_k^{-\epsilon}$ since

\[
p_k = (-\log \lambda_k)^p \leq p! \left(\frac{2}{\epsilon}\right)^p e^{2(-\log \lambda_k)} = p! \left(\frac{2}{\epsilon}\right)^p \lambda_k^{-\epsilon/2} \leq \lambda_k^{-\epsilon}
\] (114)

The bounds on $\phi_k, \partial \phi_k$ follow directly. Furthermore by (100) and $\|Q_k^T \Phi_k\|_\infty \leq \|\Phi_k\|_\infty$ and $p_k \leq O(1)\lambda_k^{-\epsilon/2}$

\[
|\delta_\alpha \partial \phi_k| = |a_k \delta_\alpha \partial G_k Q_k^T \Phi_k| \leq C\|\Phi_k\|_\infty \leq C p_k \lambda_k^{-1/4} \leq \lambda_k^{-1/4-\epsilon}
\] (115)

This completes the proof.

3.3 norms

The functions $E(X, \phi)$ form a complex vector space. We add a few more conditions and define a subspace:

Definition 3. $K_k = \text{all } E : D_k \times R_k \to \mathbb{C}$ such that

(a.) $E(X, \phi)$ only depends on $\phi$ in $X \in D_k$
(b.) $E(X, \phi)$ is analytic and bounded in $\phi \in R_k$.
(c.) $E(X, \phi)$ is even in $\phi$
(d.) $E(X, \phi)$ is invariant under lattice symmetries (translations, rotations by $\pi/2$, reflections).

(116)
In fact we are mainly interested in the real subspace
\[ \text{Re}(\mathcal{K}_k) = \{ E \in \mathcal{K}_k : \overline{E(\phi)} = E(\overline{\phi}) \} \]  
(117)
Elements \( E(X, \phi) \) of this space are real for real fields \( \phi \).

We introduce a norm on these spaces. For the \( \phi \) dependence we define for each \( X \in \mathcal{D}_k \):
\[ \| E(X) \|_k = \sup_{\phi \in \mathcal{R}_k} \| E(X, \phi) \| \]  
(118)
We also need to describe how \( E(X) \) decays in \( X \). For any \( X \in \mathcal{D}_k \) define \( d_M(X) \) by:
\[ M d_M(X) = \text{the length of the shortest tree in } X \text{ joining the } M\text{-cubes in } X. \]  
(119)
Here the tree is in the continuum torus \( \mathbb{T}_{M+N-k} = (\mathbb{R}/(\mathbb{L}^{M+N-k}\mathbb{Z}))^3 \). We expect \( E(X) \) to decay exponentially in \( d_M(X) \) so we define our norm by
\[ \| E \|_{k, \kappa} = \sup_X \| E(X) \|_k e^{\kappa d_M(X)} \]  
(120)
The norm depends on a parameter \( \kappa > 0 \). With any of these norms the space \( \mathcal{K}_k \) is complete and hence a complex Banach space. The space \( \text{Re}(\mathcal{K}_k) \) is a real Banach space.

We elaborate a bit on the these definitions. First define
\[ |X|_M = \text{Vol}(X)/M^3 = \text{number of } M\text{-cubes in } X. \]  
(121)
Then we have the inequalities \( \sum_{X \supset \Box} e^{-\kappa_0 d_M(X)} \leq K_0 \) \( \sum_{X \supset \Box} |E(X, \phi)| \leq \sum_{X \supset \Box} \| E(X) \|_k \leq \| E \|_{k, \kappa} \sum_{X \supset \Box} e^{-\kappa d_M(X)} \leq K_0 \| E \|_{k, \kappa} \) \( \sum_{X \supset \Box} \| E(X) \|_k \leq \| E \|_{k, \kappa} \sum_{X \supset \Box} e^{-\kappa d_M(X)} \leq K_0 \| E \|_{k, \kappa} \)  
(124)

3.4 scaling and reblocking
We want to know how these localized functionals scale. First some definitions.

An \( LM\)-polymer \( Y \) in \( \mathbb{T}_{M+N-k} \) is a connected union of \( LM = L^{m+1} \) cubes centered on the points of \( \mathbb{T}_{M+N-1}^{m+1} \). The set of all \( LM \) polymers is denoted \( \mathcal{D}_k^{0} \). For such \( Y \) we have that \( L^{-1}Y \) is an \( M\)-polymer in \( \mathbb{T}_{M+N-k-1}^{k-1} \) so \( L^{-1}\mathcal{D}_k^{0} = \mathcal{D}_k^{k-1} \). We let \( \mathcal{K}_k^{0} \) be the space of all \( F : \mathcal{D}_k^{0} \times \mathcal{R}_k \to \mathbb{C} \) satisfying conditions like \( \text{[116]} \).

Now for \( F \in \mathcal{K}_k^{0} \) define the scaled down functional \( F_{L^{-1}} \in \mathcal{K}_k^{k+1} \) by
\[ F_{L^{-1}}(X, \phi) = F(LX, \phi L) \]  
(125)
This is well-defined since \( X \in \mathcal{D}_k^{k+1} \) implies \( LX \in \mathcal{D}_k^{0} \) and \( \phi \in \mathcal{R}_{k+1} \) implies \( \phi L \in \mathcal{R}_k \) as the following lemma shows.
Lemma 9. (scaling)

1. If \( \phi \in \mathcal{R}_{k+1} \) then \( \phi_L \in L^{-3/4-3\epsilon} \mathcal{R}_k \)

2. \( \|F_{L^{-1}}(X)\|_{k+1} \leq \|F(LX)\|_k \)

Proof. The first item follows from \( \lambda_{k+1} = L \lambda_k \) and

\[
|\phi_L(x)| = L^{-1/2} |\phi(x/L)| \leq L^{-1/2} \lambda_k^{-1/4-3\epsilon} = [L^{-3/4-3\epsilon}] \lambda_k^{-1/4-3\epsilon}
\]

\[
|\partial \phi_L(x)| = L^{-3/2} |\partial \phi(x/L)| \leq L^{-3/2} \lambda_k^{-1/4-2\epsilon} = [L^{-7/4-2\epsilon}] \lambda_k^{-1/4-2\epsilon}
\]

\[
|(\delta_a \partial \phi)_{L}(x)| = L^{-3/2-\alpha} |(\delta_a \partial \phi(x/L)| \leq L^{-3/2-\alpha} \lambda_k^{-1/4-\epsilon} = [L^{-7/4-\alpha-\epsilon}] \lambda_k^{-1/4-\epsilon}
\]

The second is immediate. This completes the proof.

To prepare for scaling we need a reblocking operation. If \( X \in D_k \) let \( \bar{X} \in D^0_{k+1} \) be the union of all \( LM \) cubes intersecting \( X \). Given \( E \in \mathcal{K}_k \) we define functionals \( BE \in \mathcal{K}^0_{k+1} \) by

\[
(BE)(Y, \phi) = \sum_{X \in D_k : X = Y} E(X, \phi)
\]

Then we have

\[
\sum_{X \in D_k} E(X, \phi) = \sum_{Y \in D^0_{k+1}} (BE)(Y, \phi)
\]

Lemma 10. (reblocking) For \( \kappa' = L(\kappa - \kappa_0 - 1) \)

\[
\|BE\|_{k, \kappa'} \leq 9K_0 L^3 \|E\|_{k, \kappa}
\]

where the norm on the left is defined with \( d_{LM} \).

Proof. If \( \bar{X} = Y \) then a minimal tree on the \( M \) blocks in \( X \) is also a tree on the \( LM \) blocks in \( X \) and so \( Md_M(X) \geq LMd_{LM}(Y) \) or \( d_M(X) \geq Ld_M(Y) \). Therefore

\[
\|BE(Y)\|_k \leq \sum_{X = Y} \|E(X)\|_k \leq \|E\|_{k, \kappa} \sum_{X = Y} e^{-\kappa \alpha d_M(X)}
\]

\[
\leq \|E\|_{k, \kappa} e^{-(\kappa - \kappa_0) Ld_M(Y)} \sum_{X = Y} e^{-\kappa \alpha d_M(X)}
\]

(130)

If \( \bar{X} = Y \) there must be an \( M \)-cube \( \square \) so \( \square \subset X \subset Y \). Using this and (123) yields

\[
\sum_{X : X = Y} e^{-\kappa \alpha d_M(X)} \leq \sum_{\square \subset Y} \sum_{X > \square} e^{-\kappa \alpha d_M(X)} \leq |Y|_M \sum_{X > \square} e^{-\kappa \alpha d_M(X)} \leq K_0 L^3 |Y|_{LM}
\]

(131)

But by (122)

\[
|Y|_{LM} \leq 9(1 + d_{LM}(Y)) \leq 9e^{d_{LM}(Y)}
\]

so we have

\[
\|BE(Y)\|_k \leq 9K_0 L^3 \|E\|_{k, \kappa} e^{-\kappa(\kappa - \kappa_0 - 1) d_M(X)} \leq 9K_0 L^3 \|E\|_{k, \kappa} e^{-\kappa' d_{LM}(Y)}
\]

(132)

This gives the result.

Remark. If we combine them we have a map \( E \to (BE)_{L^{-1}} \) from \( \mathcal{K}_k \) to \( \mathcal{K}_{k+1} \). Since \( d_{LM}(LX) = d_M(X) \) we have:

\[
\|(BE)_{L^{-1}}\|_{k+1, \kappa'} = \sup_{X \in D_{k+1}} \|((BE)_{L^{-1}}(X))\|_{k+1} e^{\kappa' d_M(X)} \leq \sup_{X \in D_{k+1}} \|(BE)(LX)\|_k e^{\kappa' d_{LM}(LX)}
\]

\[
= \sup_{Y \in D^0_{k+1}} \|(BE)(Y)\|_k e^{\kappa' d_{LM}(Y)} = \|BE\|_{k, \kappa'} \leq 9K_0 L^3 \|E\|_{k, \kappa}
\]

(133)

Assuming \( \kappa \leq \kappa' \) (a condition that \( \kappa \) be large) we have

\[
\|(BE)_{L^{-1}}\|_{k+1, \kappa} \leq 9K_0 L^3 \|E\|_{k, \kappa}
\]

(134)
3.5 normalization

The previous estimate has a growth factor \( O(1)L^3 \). We can cancel some or all of this if we remove relevant terms from \( E \). Such a functional will be called normalized. We give a definition appropriate for our model. The following treatment roughly follows [32].

The functional \( E \in \mathcal{K}_k \) is said to be normalized if

\[
\begin{align*}
E(X, 0) &= 0 \\
E_2(X, 0; 1, 1) &= 0 \\
E_2(X, 0; 1, x_\mu) &= 0
\end{align*}
\]

(135)

Here the derivatives can be evaluated by

\[
E_n(X, \phi; f_1, \ldots, f_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} E_n(X, \phi + t_1 f_1 + \cdots + t_n f_n)|_{t_i=0}
\]

(136)

Note that all odd derivatives at zero vanish. This is due to our assumption that \( E(X, \phi) \) is even in \( \phi \).

It is convenient to make a distinction between small polymers \( X \) which have \( d_M(X) < L \) and large polymers which have \( d_M(X) \geq L \). (A similar distinction was first exploited in [28]). We generally only require normalization for small polymers. The set of all small polymers is denoted \( S \) and the large polymers are denoted \( \bar{S} \).

Now define

\[
\mathcal{K}^\text{norm}_k = \{ E \in \mathcal{K}_k : E(X, \phi) \text{ is normalized for small } X \}
\]

(137)

This is a closed subspace of \( \mathcal{K}_k \). We also need the real closed subspace \( \text{Re}(\mathcal{K}^\text{norm}_k) = \mathcal{K}^\text{norm}_k \cap \text{Re}(\mathcal{K}_k) \).

Then we have the following improvement of (134).

**Lemma 11.** Let \( E \in \mathcal{K}^\text{norm}_k \). Then for \( L \) sufficiently large and \( \lambda_k \) sufficiently small (depending on \( L, M \))

\[
\| (BE)_{L^{-1}} \|_{k+1, \kappa} \leq O(1)L^{-\epsilon} \| E \|_{k, \kappa}
\]

(138)

**Proof.** Let \( 1_S, 1_{\bar{S}} \) be the characteristic functions of small polymers and large polymers. We write \( (BE)_{L^{-1}} = (B1_{\bar{S}} E)_{L^{-1}} + (B1_S E)_{L^{-1}} \).

For the large set term we follow the proof of the previous lemma. In (131) we can arrange to have \( \kappa_0 + 1 \) instead of \( \kappa_0 \) (with a change in \( \kappa' \)). Then since \( d_M(X) \geq L \)

\[
\sum_{X \in \bar{S} : X = Y} e^{-(\kappa_0 + 1)d_M(X)} \leq e^{-L} \sum_{X = Y} e^{-\kappa_0 d_M(X)} \leq e^{-L} K_0 L^3 |Y|_{LM}
\]

(139)

Since \( e^{-L} K_0 L^3 \leq L^{-1} \) for \( L \) sufficiently large, we get instead of (134)

\[
\| (B1_{\bar{S}} E)_{L^{-1}} \|_{k+1, \kappa} \leq L^{-1} \| E \|_{k, \kappa}
\]

(140)

Now consider the contribution of small polymers which is

\[
(B1_S E)_{L^{-1}}(Z, \phi) = \sum_{X \in S : X = LZ} E(X, \phi_L)
\]

(141)

We will show that for \( X \in S \) and \( \phi \in \mathcal{R}_{k+1} \) we have

\[
\| E(X, \phi_L) \| \leq O(1)L^{-3-\epsilon} \| E(X) \|_k
\]

(142)

\( ^31 \) means the function \( x \to 1 \) and \( x_\mu \) means the projection \( x \to x_\mu \).
Then
\[ \|(B_1S_0)_{L^{-1}}(Z)\|_{k+1} \leq O(1)L^{-3-\epsilon} \sum_{X=LZ} \|E(X)\|_k \]  
\tag{143}

This is the same as \([130]\), except that \(Y = LZ\) and there is the extra factor \(L^{-3-\epsilon}\). Following the argument \([130] - [132]\) the factor \(L^3\) there is canceled. Using also \(d_{LM}(LZ) = d_M(Z)\) and \(\kappa < \kappa'\) we have that \((143)\) is bounded by \(O(1)L^{-\epsilon}\|E(X)\|_{k,\kappa}e^{-\kappa M(Z)}\), and therefore instead of \((133)\)
\[ \|(B_1S_0)_{L^{-1}}\|_{k+1,\kappa} \leq O(1)L^{-\epsilon}\|E\|_{k,\kappa} \]  
\tag{144}

To establish \((142)\) we make a Taylor expansion of \(E(X, \phi_L)\) in the field. At \(\phi_L = 0\) we get zero by the normalization condition. Also odd derivatives vanish since the functional is even in \(\phi_L\). Therefore
\[ E(X, \phi_L) = \frac{1}{2}E_2(X, 0; \phi_L, \phi_L) + \frac{1}{2\pi i} \int_{|t| = \frac{1}{2}L^{3/4+3\epsilon}} \frac{E(X, t\phi_L)}{t^4(t-1)} dt \]  
\tag{145}

Note that since \(\phi_L \in L^{-3/4-3\epsilon}\mathcal{R}_k\) on the circle \(|t| = \frac{1}{2}L^{3/4+3\epsilon}\) we have \(t\phi_L \in \frac{1}{2}\mathcal{R}_k\). It follows that the second term in \((145)\) is bounded by \(O(1)L^{-3-12\epsilon}\|E(X)\|_k\) which suffices.

For the first term in \((145)\) we pick a point \(x_0 \in X\) and insert into \(E_2(X, 0; \phi_L, \phi_L)\) the expansion
\[ \phi_L(x) = \phi_L(x_0) + (x - x_0) \cdot \partial \phi_L(x_0) + \Delta \phi_L(x, x_0) \]  
\tag{146}

where
\[ \Delta \phi_L(x, x_0) = \int_{x_0}^x (\partial \phi_L(y) - \partial \phi_L(x_0)) \cdot dy \]  
\tag{147}

Then
\[ E_2(X, 0; \phi_L, \phi_L) = E_2(X, 0; \phi_L(x_0), \phi_L(x_0)) + 2E_2(X, 0; \phi_L(x_0), (x - x_0) \cdot \partial \phi_L(x_0)) + E_2(X, 0; (x - x_0) \cdot \partial \phi_L(x_0), (x - x_0) \cdot \partial \phi_L(x_0)) + 2E_2(X, 0; (x - x_0) \cdot \partial \phi_L(x_0), \Delta \phi_L) + E_2(X, 0; \Delta \phi_L, \Delta \phi_L) + 2E_2(X, 0; \phi_L(x_0), \Delta \phi_L) \]  
\tag{148}

The first and second terms vanish due to our normalization conditions.

The remaining terms will be estimated by Cauchy inequalities. In general if \(f_1 \in a_1\mathcal{R}_k, f_2 \in a_2\mathcal{R}_k\) we can write
\[ E_2(X, 0; f_1, f_2) = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \left[ E(X, t_1 f_1 + t_2 f_2) \right]_{t_1 = t_2 = 0} = \frac{1}{(2\pi)^2} \int_{|t_1| = (2a_1)^{-1}, |t_2| = (2a_2)^{-1}} \frac{dt_1}{t_1^2} \frac{dt_2}{t_2^2} E(X, t_1 f_1 + t_2 f_2) \]  
\tag{149}

This gives the estimate
\[ |E_2(X, 0; f_1, f_2)| \leq O(1)a_1a_2\|E(X)\|_k \]  
\tag{150}

Now we claim that if \(\phi \in \mathcal{R}_{k+1}\) and \(x, x_0 \in X\) then
\[ \phi_L(x_0) \in L^{-3/4-3\epsilon}\mathcal{R}_k \]
\[ (x - x_0) \cdot \partial \phi_L(x_0) \in L^{-7/4-2\epsilon}\mathcal{R}_k \]
\[ \Delta \phi_L \in L^{-7/4-\alpha-\epsilon}\mathcal{R}_k \]  
\tag{151}
Then the third term in \([148]\) has a factor \(L^{-7/2-4\epsilon}\), the fourth term has a factor \(L^{-7/2-\alpha-3\epsilon}\), the fifth term has a factor \(L^{-7/2-2\alpha-2\epsilon}\), and the sixth term has a factor \(L^{-5/2-\alpha-3\epsilon}\). This easily gives

\[
|E_2(X, 0; \phi_L, \phi_L)| \leq O(1)L^{-3-\epsilon}\|E(X)\|_k
\]

and completes the proof of \([142]\).

For the first inclusion in \([151]\) we already have by \([126]\), \(|\phi_L(x_0)| \leq L^{-3/4-3\epsilon}\lambda_k^{-1/4-3\epsilon}\). The derivatives vanish so this establishes \(\phi_L(x_0) \in L^{-3/4-\epsilon}R_k\).

For the second inclusion in \([151]\) note that if \(X\) is small then \(|X|_M \leq 9(d_M(X) + 1) \leq 9(L + 1)\) so the largest distance between points in \(X\) is \(9M(L + 1)\). Then by \([126]\) for \(\lambda_k\) sufficiently small

\[
|(x - x_0) \cdot \partial \phi_L(x_0)| \leq 9M(L + 1)L^{-7/4-2\epsilon}\lambda_k^{-1/4-2\epsilon}
\]

\[
\leq (9M(L + 1)\lambda_k^{-1/4-3\epsilon})L^{-7/4-2\epsilon}\lambda_k^{-1/4-3\epsilon} \leq L^{-7/4-2\epsilon}\lambda_k^{-1/4-3\epsilon}
\]

(153)

Furthermore the derivative is the constant

\[
|\partial (x - x_0) \cdot \partial \phi_L(x_0)| = |\partial \phi_L(x_0)| \leq L^{-7/4-2\epsilon}\lambda_k^{-1/4-2\epsilon}
\]

(154)

The difference of derivatives is zero, so \((x - x_0) \cdot \partial \phi_L(x_0) \in L^{-7/4-2\epsilon}R_k\).

For the third inclusion in \([151]\) we estimate by \([126]\)

\[
|\Delta_{\phi_L}(x)| = |\int_{x_0}^x (\partial \phi_L(y) - \partial \phi_L(x_0)) \cdot dy|
\]

\[
\leq L^{-7/4-\alpha-\epsilon}\lambda_k^{-1/4-\epsilon} \int_{x_0}^x d(y, x_0)^\alpha |dy|
\]

\[
\leq L^{-7/4-\alpha-\epsilon}\lambda_k^{-1/4-\epsilon} (9M(L + 1))^{1+\alpha}
\]

\[
\leq L^{-7/4-\epsilon}\lambda_k^{-1/4-3\epsilon}
\]

and also

\[
|\partial \Delta_{\phi_L}(x)| = |\partial \phi_L(x) - \partial \phi_L(x_0)|
\]

\[
\leq L^{-7/4-\alpha-\epsilon}\lambda_k^{-1/4-\epsilon} d(x, x_0)^\alpha
\]

\[
\leq L^{-7/4-\alpha-\epsilon}\lambda_k^{-1/4-\epsilon} (9M(L + 1))^{\alpha}
\]

\[
\leq L^{-7/4-\epsilon}\lambda_k^{-1/4-2\epsilon}
\]

(156)

Similarly

\[
|\partial \Delta_{\phi_L}(x) - \partial \Delta_{\phi_L}(y)| = |\partial \phi_L(x) - \partial \phi_L(y)| \leq L^{-7/4-\alpha-\epsilon}\lambda_k^{-1/4-\epsilon} d(x, y)^\alpha
\]

(157)

The last three bounds imply \(\Delta_{\phi_L} \in L^{-7/4-\alpha-\epsilon}R_k\). This completes the proof of \([151]\) and the theorem.

We also explain how to arrange the normalization for small polymers. Given \(E \in K_k\) we define \(RE \in K_k\) as follows. If \(X\) is small \((X \in S)\) then \(RE(X)\) is defined by

\[
E(X, \phi) = \alpha_0(E, X)\text{Vol}(X) + \alpha_2(E, X) \int_X \phi^2 + \sum_\mu \alpha_2,\mu(E, X) \int_X \phi \partial_\mu \phi + RE(X, \phi)
\]

(158)

where

\[
\alpha_0(E, X) = \frac{1}{\text{Vol}(X)} E(X, 0) \\
\alpha_2(E, X) = \frac{1}{2 \text{Vol}(X)} E_2(X, 0; 1, 1) \\
\alpha_2,\mu(E, X) = \frac{1}{\text{Vol}(X)} \left( E_2(X, 0; 1, x_\mu - x_\mu^0) - \frac{1}{\text{Vol}(X)} E_2(X, 0; 1, 1) \int_X x_\mu - x_\mu^0 \right)
\]

(159)
The last is independent of the base point \( x^0 \), which we take to be in \( X \). Then it is straightforward to check that \( RE \) is normalized for small polymers. If \( X \) is large then \( RE(X) = E(X) \).

**Lemma 12.** For \( E \in \mathcal{K}_k \) and \( \lambda_k \) sufficiently small

\[
\|RE(X)\|_k \leq O(1)\|E(X)\|_k
\]  

(160)

**Proof.** It suffices to check for small polymers \( X \). We check that every other term in (158) satisfies such a bound. This is immediate for \( E(X, \phi) \) and \( \alpha_0(E, X) \text{Vol}(X) = E(X,0) \).

For the next term note that since \( 1 \in \lambda_k^{1/4+3\epsilon} R_k \) we have by (150)

\[
|E_2(X, 0; 1, 1)| \leq O(1)\lambda_k^{1/2+6\epsilon} \|E(X)\|_k
\]

(161)

Also for \( \phi \in R_k \) we have \( \int_X \phi^2 \leq \text{Vol}(X)\lambda_k^{-1/2-6\epsilon} \). Therefore

\[
|\alpha_2(E, X) \int_X \phi^2| \leq O(1)\|E(X)\|_k
\]

(162)

For the next term note that

\[
|\partial_{\nu}(x_\mu - x_\mu^0)| \leq N M\epsilon \leq (9M(L + 1))^{1/4+2\epsilon} \lambda_k^{-1/4-3\epsilon} \leq [\lambda_k^{1/4+2\epsilon}] \lambda_k^{-1/4-3\epsilon}
\]

(163)

Therefore \( x_\mu - x_\mu^0 \in \lambda_k^{1/4+2\epsilon} R_k \) and so

\[
|E_2(X, 0; 1, 1)| \leq O(1)\lambda_k^{1/2+5\epsilon} \|E(X)\|_k
\]

(164)

Also \( \int_X |x_\mu - x_\mu^0| \leq \text{Vol}(X)\lambda_k^{-\epsilon} \) and for \( \phi \in R_k \) we have \( \int_X \phi \partial_{\mu} \phi | \leq \text{Vol}(X)\lambda_k^{-1/2-5\epsilon} \). These combine to give

\[
|\alpha_{2,\mu}(E, X) \int_X \phi \partial_{\mu} \phi| \leq O(1)\|E(X)\|_k
\]

(165)

This completes the proof.

Inserting (158) into \( E = \sum_X E(X) \) and defining \( RE = \sum_X RE(X) \) we find we have extracted energy and mass terms:

\[
E = -\varepsilon(E)\text{Vol}(\mathbb{T}_{M+N-k}) - \frac{1}{2}\mu(E)\|\phi\|^2 + RE
\]

(166)

Here

\[
\varepsilon(E) = - \sum_{X \supseteq \Box, X \in S} \alpha_0(E, X)
\]

\[
\frac{1}{2}\mu(E) = - \sum_{X \supseteq \Box, X \in S} \alpha_2(E, X)
\]

(167)

are independent of \( \Box \) by translation invariance. We have also used

\[
\sum_{X \supseteq \Box, X \in S} \alpha_{2,\mu}(E, X) = 0
\]

(168)

which follows by choosing \( x^0 \) in the center of \( \Box \) and using \( \alpha_{2,\mu}(E, r_\mu X) = -\alpha_{2,\mu}(E, X) \) where \( r_\mu \) is reflection thru the plane \( x_\mu = x_\mu^0 \).
Lemma 13.

\[ |\epsilon(E)| \leq O(1)\|E\|_{k,\kappa} \]
\[ |\mu(E)| \leq O(1)\lambda_k^{1/2 + 6\epsilon} \|E\|_{k,\kappa} \]  \hspace{1cm} (169)

**Proof.** For the first bound we have \( \epsilon(E) \leq \sum_{X \supset \Box} \|E(X)\|_k \leq K_0\|E\|_{k,\kappa} \) as in (124). The second bound uses (161) and follows in the same way.

4 The RG transformation with small fields

4.1 the theorem

Now we study the RG transformation with the potential, but modified with a small field assumption. The starting point is still the density

\[ \rho_0(\Phi_0) = \exp \left( - S_0(\Phi_0) - V_0(\Phi_0) \right) \]  \hspace{1cm} (170)

where \( \Phi_0 : T_{M+N}^0 \to \mathbb{R} \) and

\[ S_0(\Phi_0) = \frac{1}{2} \|\partial \Phi_0\|^2 + \frac{1}{2} \mu_0 \|\Phi_0\|^2 \]
\[ V_0(\Phi_0) = \varepsilon_0 \text{Vol}(T_{M+N}^0) + \frac{1}{2} \mu_0 \|\Phi_0\|^2 + \frac{1}{4} \lambda_0 \sum_x \Phi_0(x)^4 \]  \hspace{1cm} (171)

But now instead of (26) we add some characteristic functions and define \( \rho_k \) recursively as follows. For \( \Phi_k : T_{M+N-k}^0 \to \mathbb{R} \) and \( \Phi_{k+1} : T_{M+N-k-1}^1 \to \mathbb{R} \) let

\[ \tilde{\rho}_{k+1}(\Phi_{k+1}) \]
\[ = N_{aL,T_{M+N-k}}^{-1} \int \exp \left( - \frac{a}{2L^2} \|\Phi_{k+1} - Q\Phi_k\|^2 \right) \chi_{W_k}^{\chi_k} \left( C_k^{-1/2}(\Phi_k - \Psi_k) \right) \chi_k(\Phi_k) \rho_k(\Phi_k) \ d\Phi_k \]  \hspace{1cm} (172)

and as before for \( \Phi_{k+1} : T_{M+N-k-1}^1 \to \mathbb{R} \)

\[ \rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_k(\Phi_{k+1}, L) L^{-|T_{M+N-k-1}|/2} \]  \hspace{1cm} (173)

Here the characteristic functions are

\[ \chi_k^{\chi_k}(W) = \chi(|W| \leq p_{0,k}) \]
\[ \chi_k(\Phi_k) = \chi(\Phi_k \in S_k) \]  \hspace{1cm} (174)

With the free minimizer \( \Psi_k = \Psi_k(\Phi_{k+1}, \phi_{k+1}^0) \) defined in (59) the function \( \chi_k^{\chi_k} \left( C_k^{-1/2}(\Phi_k - \Psi_k) \right) \) enforces that the fluctuation field \( \Phi_k - \Psi_k \) be small. The size is determined by

\[ p_{0,k} = p_0(\lambda_k) = (-\log \lambda_k)^{p_0} \]  \hspace{1cm} (175)

This has the same form as \( p_k \) but with a smaller integer exponent \( p_0 < p \). The function \( \chi_k(\Phi_k) \) is the small field restriction. Actually we only consider \( \Phi_{k+1} \in S_{k+1} \) in which case this restriction is unnecessary as we will see.

Adding the characteristic functions gives us the leading term in an expansion of the full integral into various blocks of field values. This is developed in paper II.
We are going to assert that after \( k \) steps the modified density can be written in the following local form. For \( \Phi_k \in S_k \)
\[
\rho_k(\Phi_k) = Z_k \exp \left( -S_k(\Phi_k, \phi_k) - V_k(\phi_k) + E_k(\phi_k) \right)
\]  
(176)
where \( \phi_k : T_{M+N-k}^{-1} \rightarrow \mathbb{R} \) is \( \phi_k = a_k G_k Q_k^T \Phi_k \) and
\[
S_k(\Phi_k, \phi_k) = \frac{1}{2} a_k \| \Phi_k - Q_k \phi_k \|^2 + \frac{1}{2} \| \partial \phi_k \|^2 + \frac{1}{2} \mu_k \| \phi_k \|^2
\]
\[
V_k(\phi_k) = \varepsilon_k \text{Vol}(T_{M+N-k}) + \frac{1}{2} \mu_k \| \phi_k \|^2 + \frac{1}{4} \lambda_k \int \phi_k^2(x) dx
\]
(177)
We further assert that the functional \( E_k(\phi) \) is defined and analytic in the larger set \( \phi \in R_k \) and can be written
\[
E_k(\phi) = \sum_X E_k(X, \phi)
\]
(178)
where \( E_k(X, \phi) \in Re(\mathcal{K}_k^{\text{norm}}) \) and so is normalized for small polymers.

**Theorem 14.** Let \( L, M \) be sufficiently large, let \( \lambda_k \) be sufficiently small (depending on \( L, M \)). Suppose \( \rho_k(\Phi_k) \) has the representation \([176]-[178]\) for \( \Phi_k \in S_k \) and
\[
|\mu_k| \leq \lambda_k^{1/2} \quad \|E_k\|_{k, \kappa} \leq 1
\]
(179)
Then \( \rho_{k+1}(\Phi_{k+1}) \) has a representation of the same form for \( \Phi_{k+1} \in S_{k+1} \). The bounds are not the same but we do have
\[
\varepsilon_{k+1} = L^3 \varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^*(\lambda_k, \mu_k, E_k)
\]
\[
\mu_{k+1} = L^2 \mu_k + \mathcal{L}_2 E_k + \mu_k^*(\lambda_k, \mu_k, E_k)
\]
\[
\lambda_{k+1} = L \lambda_k
\]
\[
E_{k+1} = \mathcal{L}_3 E_k + \mu_k^*(\lambda_k, \mu_k, E_k)
\]
(180)
where the \( \mathcal{L}_i \) are linear operators which satisfy
\[
|\mathcal{L}_1 E_k| \leq \mathcal{O}(1) L^{-\tau} \|E_k\|_{k, \kappa}
\]
\[
|\mathcal{L}_2 E_k| \leq \mathcal{O}(1) L^{-\tau} \lambda_k^{1/2+\epsilon} \|E_k\|_{k, \kappa}
\]
\[
\|\mathcal{L}_3 E_k\|_{k+1, \kappa} \leq \mathcal{O}(1) L^{-\tau} \|E_k\|_{k, \kappa}
\]
(181)
and where
\[
|\varepsilon_k^*| \leq \mathcal{O}(1) L^3 \lambda_k^{1/4-10\epsilon}
\]
\[
|\mu_k^*| \leq \mathcal{O}(1) L^3 \lambda_k^{3/4-4\epsilon}
\]
\[
\|E_k^*\|_{k+1, \kappa} \leq \mathcal{O}(1) L^3 \lambda_k^{1/4-10\epsilon}
\]
(182)

**Remarks.** The unstarred terms represent scalings and rearrangements of the existing terms, but not the effects of the fluctuation integral. The starred terms are the effect of the fluctuation integral. To put another way the unstarred terms are zeroth order perturbation theory, and the starred terms are all higher order contributions. The starred terms are not necessarily smaller than the unstarred terms, although they do have better bounds.

This flow shows strong growth, but it is tolerable due to the ultraviolet origin of the problem: we start with very small coupling constants. Our concern will be that the growth is not too rapid. We want to finish at a good place.

The main idea is that we have removed mass and energy terms from \( E_k \) by normalizing, and included them in corrections to \( \varepsilon_k, \mu_k \). These are the fastest growing terms and in this form they will be susceptible to analysis.
4.2 start of the proof

We study \( \hat{\rho}_{k+1}(\Phi_{k+1}) \) for \( \Phi_{k+1} \in S^0_{k+1} \) defined as all \( \Phi_{k+1} : T^1_{M+N-k} \to \mathbb{R} \) such that

\[
|\Phi_{k+1} - Q_k \phi^0_{k+1}| \leq L^{-1/2} p_{k+1} \\
|\partial \phi^0_{k+1}| \leq L^{-3/2} p_{k+1} \\
|\phi^0_{k+1}| \leq L^{-1/2} \delta_{k+1}^1 p_{k+1}
\]

where \( \phi^0_{k+1} = \phi^0(\Phi_{k+1}). \) This is the appropriate choice since we eventually want \( \hat{\rho}_k(\Phi_{k+1,L}) \) for \( \Phi_{k+1} : T^1_{M+N-k} \to \mathbb{R} \) and \( \Phi_{k+1} \in S^0_{k+1} \). These conditions imply \( \Phi_{k+1} \in S^0_{k+1} \) as can be demonstrated using \( \phi^0_{k+1}(\Phi_{k+1,L}) = [\phi^0_{k+1}(\Phi_{k+1})]_L \).

Returning to \( \Phi_{k+1} \in S^0_{k+1} \) we note that this condition implies \( \phi^0_{k+1}(\Phi_{k+1}) \in L^{-\delta_2^{-r}} \mathcal{R}_k. \) (Proof: Then \( \Phi_{k+1,L-1} \in S^0 \) which implies \( \phi^0_{k+1}(\Phi_{k+1,L-1}) \in \mathcal{R}_k \) by lemma 8.) But \( \phi_{k+1}(\Phi_{k+1,L-1}) = [\phi^0_{k+1}(\Phi_{k+1})]_L \) so \( \phi^0_{k+1}(\Phi_{k+1}) \in L^{-\delta_2^{-r}} \mathcal{R}_k \) by lemma 9.

Now for \( \Phi_{k+1} \in S^0_{k+1} \) we have

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = Z_k N^{-1}_{aL_1 T_{M+N-k}} \int \exp \left( -\frac{a}{2L^2} \| \Phi_{k+1} - Q_k \phi \|_2^2 - S_k(\Phi_k, \phi) - V_k(\phi) + E_k(\phi) \right) \chi_k(\Phi_k - \Psi_k) d\Phi_k
\]

We expand in \( \Phi_k \) around the minimizer \( \Psi_k \) for the first two terms in the exponential by writing \( \Phi_k = \Psi_k + Z. \) As in (70) this generates \( S^0_{k+1}(\Phi_{k+1}, \phi^0_{k+1}) + \frac{a}{2} \left( Z, (\Delta_k + aL^{-2}Q^TQ)Z \right). \) We also have \( \phi_k(\Psi_k + Z) = \phi^0_{k+1} + Z \) where \( Z = a_kG_kQ^TZ \) as in (65). Changing the integration variable from \( \Phi_k \) to \( Z \) yields

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = Z_k N^{-1}_{aL_1 T_{M+N-k}} \int \exp \left( -S^0_{k+1}(\Phi_{k+1}, \phi^0_{k+1}) \right) \int \exp \left( E_k^+(\phi_{k+1}^0 + Z) - \frac{1}{2} < Z, (\Delta_k + aL^{-2}Q^TQ)Z > \right) \chi_k(\Psi_k + Z) \chi_k^w(C_k^{-1/2}Z) dZ
\]

Here we have introduced

\[
E_k^+(\phi) = E_k(\phi) - V_k(\phi)
\]

If we define

\[
V_k(\square, \phi) = \varepsilon_k \text{Vol}(\square) + \frac{1}{2} \mu_k \| \phi \|_2^2 + \frac{1}{4} \lambda_k \int \phi_k(x) d\phi_k(x)
\]

then \( V_k(\square, \phi) = 0 \) for \( |X|_M \geq 2 \) then \( V_k(\phi) = \sum X V_k(X, \phi). \) Together with (178) this gives a local expansion

\[
E_k^+(\phi) = \sum_X E_k^+(X, \phi)
\]

Recall also that \( C_k = (\Delta_k + aL^{-2}Q^TQ)^{-1} \) and let \( \mu_{C_k} \) be the Gaussian measure with covariance \( C_k. \) Then (185) can be written

\[
\hat{\rho}_{k+1}(\Phi_{k+1}) = Z_k N^{-1}_{aL_1 T_{M+N-k}} (2\pi)^{1/2} |det C_k|^{1/2} \exp \left( -S^0_{k+1}(\Phi_{k+1}, \phi^0_{k+1}) \right) \int \exp \left( E_k^+(\phi_{k+1}^0 + Z) \right) \chi_k(\Psi_k + Z) \chi_k^w(C_k^{-1/2}Z) d\mu_{C_k}(Z)
\]

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If we multiply by $L^{-|T_{M+N-k}|/2}$ we can identify the constant in front as $Z_{k+1}$ by (74). We make one further adjustment in the integral by changing from Gaussian $Z : T_{M+N-k}^0 \to \mathbb{R}$ with covariance $C_k$ to $Z = C_k^W$ where Gaussian $W : T_{M+N-k}^0 \to \mathbb{R}$ has identity covariance. Thus we have

$$\tilde{\rho}_{k+1}(\Phi_{k+1})L^{-|T_{M+N-k}|/2} = Z_{k+1} \exp \left(-S^0_{k+1}(\Phi_{k+1}, \phi^0_{k+1})\right)$$

$$\int \exp \left(E_{k}^{+}(\phi_{k+1}^{0} + W_{k})\right) \chi_{k}(\Psi_{k} + C_{k}^W) \chi_{k}^W(W) d\mu_{L}(W)$$

(190)

where $W_{k} : T_{M+N-k}^k \to \mathbb{R}$ is given by

$$W_{k} = \phi_k(C_{k}^W) = a_k G_k Q_k^T C_{k}^W$$

(191)

We will need a more explicit representation of $C_k^W$. For $\lambda > 0$

$$\lambda^{-1/2} = \frac{1}{\pi} \int_0^{\infty} \frac{dr}{\sqrt{r}(\lambda + r)^{-1}}$$

(192)

Hence we have the operator identity.

$$C_k^W = \frac{1}{\pi} \int_0^{\infty} \frac{dr}{\sqrt{r}} C_{k,r}$$

$$C_{k,r} = \left(\Delta_k + \frac{a}{L^2}Q_k^T Q + r\right)^{-1}$$

(193)

In appendix C we establish

$$C_{k,r} = A_{k,r} + a_k^2 Q_k^T A_{k,r} Q_k$$

(194)

where

$$A_{k,r} = \frac{1}{a_k + r} (I - Q_k^T Q) + \frac{1}{a_k + aL^{-2} + r} Q_k^T Q$$

$$G_{k,r} = \left(-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k - a_k^2 Q_k^T A_{k,r} Q_k\right)^{-1}$$

(195)

An alternative expression for $G_{k,r}$ is

$$G_{k,r} = \left(-\Delta + \bar{\mu}_k + \frac{a_k r}{a_k + r} Q_k^T Q_k + \frac{a_k^2 aL^{-2}}{(a_k + r)(a_k + aL^{-2} + r)} Q_{k+1}^T Q_{k+1}\right)^{-1}$$

(196)

This shows that we are inverting a positive operator. Note that $G_{k,r}$ interpolates between $G_{k,0} = G_0^0$ (use (20)) and $G_{k,\infty} = G_k^W$.

## 4.3 a simplification

The next lemma shows that we can drop $\chi_k(\Psi_k + C_k^W)$ from the expression (190).

**Lemma 15.** For $\Phi_{k+1} \in S_{k+1}^0$ and $|W| \leq p_{0,k}$ we have $\Psi_k + C_k^W \in S_k$ and hence

$$\chi_k(\Psi_k + C_k^W) = 1$$

(197)

---

If $k = 0$ this is $\cal{W}_0 = (C_0)^W = (-\Delta + aL^{-2}Q_k^T Q)^{-1/2}W$
Proof. We must show
\[
|\Psi_k + C_k^\frac{1}{3} W - Q_k \phi_k (\Psi_k + C_k^\frac{1}{3} W)| \leq p_k
\]
\[
|\partial \phi_k (\Psi_k + C_k^\frac{1}{3} W)| \leq p_k
\]
\[
|\phi_k (\Psi_k + C_k^\frac{1}{3} W)| \leq \lambda_k^{-\frac{1}{4}} p_k
\]
(198)

We give separate bounds on the terms involving \(\Psi_k\) and \(W\).

For the \(\Psi_k\) terms we identify \(\phi_k (\Psi_k) = \phi_k^{0}_{k+1}\) and show
\[
|\Psi_k - Q_k \phi_k^{0}_{k+1}| \leq \frac{1}{2} p_k
\]
\[
|\partial \phi_k^{0}_{k+1}| \leq \frac{1}{2} p_k
\]
\[
|\phi_k^{0}_{k+1}| \leq \frac{1}{2} \lambda_k^{-\frac{1}{4}} p_k
\]
(199)

These follow from (183). The last follows from \(|\phi_k^{0}_{k+1}| \leq L^{-1/2} \lambda_k^{-1/4} p_{k+1} \leq L^{-3/4} \lambda_k^{-1/4} p_k\). The second follows by \(|\partial \phi_k^{0}_{k+1}| \leq L^{-3/2} p_{k+1} \leq L^{-3/2} p_k\). The first follows by
\[
|\Psi_k - Q_k \phi_k^{0}_{k+1}| \leq |Q^T (\Phi_{k+1} - Q_{k+1} \phi_k^{0}_{k+1})| \leq \|\Phi_{k+1} - Q_{k+1} \phi_k^{0}_{k+1}\|_\infty \leq L^{-1/2} p_{k+1} \leq L^{-1/2} p_k
\]
(200)

Here we have used the explicit expression (59) for \(\Psi_k\).

For the \(W\) terms we need
\[
|C_k^\frac{1}{3} W - Q_k W_k| \leq \frac{1}{2} p_k
\]
\[
|\partial W_k| \leq \frac{1}{2} p_k
\]
\[
|W_k| \leq \frac{1}{2} \lambda_k^{-1/4} p_k
\]
(201)

In the next lemma we show that \(|C_k^\frac{1}{3} W|, |W_k|, |\partial W_k|\) are all bounded by a constant times \(p_{0,k}\) Then if \(\lambda_k\) is sufficiently small we have \(p_{0,k}/p_k = (-\log \lambda_k)^{p_0 - p}\) as small as we like since \(p_0 < p\). Hence these functions are bounded by say \(\frac{1}{4} p_k\) which suffices to prove (201).

Lemma 16. If \(|W| \leq p_{0,k}\) then
\[
|C_k^\frac{1}{3} W|, |W_k|, |\partial W_k|, |\delta a \partial W_k| \leq C p_{0,k}
\]
(202)

Furthermore \(W_k \in \lambda_k^{1/4} R_k\)

Proof. We use the representation (193), (194), (195) of \(C_k^\frac{1}{3}\). These express \(C_k^\frac{1}{3}\) in terms of \(D_{k,r} = Q_{k} G_{k,r} Q_{r}^T\). In appendix B we establish a random walk expansion for \(G_{k,r}\) which leads to \(L^2\) bounds. For the kernel \(D_{k,r}(y, y') = \langle Q_k^\frac{1}{3} \delta_y, G_{k,r} Q_r^T \delta_{y'} \rangle >\) these say
\[
|D_{k,r}(y, y')| \leq C e^{-\gamma_0 d(y,y')} \|Q_k^\frac{1}{3} \delta_y\|_2 \|Q_r^T \delta_{y'}\|_2 \leq C e^{-\gamma_0 d(y,y')}
\]
(203)

This gives the \(L^\infty\) bound \(|D_{k,r} W| \leq C \|W\|_\infty\). We also have \(|A_{k,r} W| \leq O(1) (1 + r)^{-1} \|W\|_\infty\). Hence \(C_{k,r} = A_{k,r} + a_k^2 A_{k,r} D_{k,r} A_{k,r}\) satisfies \(C_{k,r} W \leq C (1 + r)^{-1} \|W\|_\infty\) and so \(|C_k^\frac{1}{3} W| \leq C \|W\|_\infty \leq C p_{0,k}\) as announced.

The other bounds follow by (100). For example
\[
|W_k| = |a_k G_{k} Q_k^T C_k^\frac{1}{3} W| \leq C \|C_k^\frac{1}{3} W\|_\infty \leq C p_{0,k} \leq p_k \leq \lambda_k^{-\epsilon} \leq \lambda_k^{-3\epsilon}
\]
(204)

The last bound is the one needed for \(W_k \in \lambda_k^{1/4} R_k\). The bounds on \(\partial W_k\) and \(\delta a \partial W_k\) are similar.
4.4 fluctuation integral

With the characteristic function gone we now have

\[ \tilde{\rho}_{k+1}(\Phi_{k+1})L^{-|T_{M+N-k}|/2} = Z_{k+1} \exp \left( -S_{k+1}^0(\Phi_{k+1}, \phi_{k+1}^0) \right) \Xi_k(\phi_{k+1}) \]

where

\[ \Xi_k(\phi) = \int \exp \left( E_k^+(\phi + \omega_k) \right) \chi_k^w(W) d\mu_l(W) \]

This is the fluctuation integral. We are going to study it for \( \phi \in \frac{1}{2}R_k \). It is well defined with this restriction since \( E^+ \) is defined on \( R_k \) and \( \omega_k \in \chi_k^{1/4}R_k \subset \frac{1}{2}R_k \). Note also that the point of interest \( \phi_{k+1}^0 \in L^{-3/4 - 3r}R_k \) is included in \( \frac{1}{2}R_k \).

We make a couple of adjustments in \( \Xi_k \). First change to the probability measure

\[ d\mu_k^*(W) = N_{X,k}^{-1} \chi_k^w(W) d\mu_l(W) \]

Here the normalizing factor is

\[ N_{X,k} = \int \chi_k^w(W) d\mu_l(W) \]

\[ = \prod_{x \in T_{M+N-k}^0} \int \chi_k^w(W(x)) d\mu_l(W(x)) \]

\[ = \prod_{x \in T_{M+N-k}^0} \exp(-\varepsilon_k^0) = \exp \left( -\varepsilon_k^0 \text{Vol}(T_{M+N-k}) \right) \]

where \( \varepsilon_k^0 > 0 \) is defined by

\[ \varepsilon_k^0 = -\log \left( \int \chi_k^w(W(x)) d\mu_l(W(x)) \right) \]

It is straightforward to show

\[ \left| \int \chi_k^w(W(x)) d\mu_l(W(x)) - 1 \right| \leq O(e^{-R_0,k/2}) \]

and hence \( \varepsilon_k^0 \leq O(e^{-R_0,k/2}) \) as well. It is very small.

Secondly define \( \delta E_k^+(\phi, \omega_k) \) by

\[ E_k^+(\phi + \omega_k) = E_k^+(\phi) + \delta E_k^+(\phi, \omega_k) \]

There is also a local decomposition inherited from \( E_k^+ \). The term \( E_k^+(\phi) \) is pulled out of the integral. It is not necessarily small and would make subsequent estimates awkward.

Now we have

\[ \Xi_k(\phi) = \exp \left( -\varepsilon_k^0 \text{Vol}(T_{M+N-k}^0) + E_k^+(\phi) \right) \Xi_k(\phi) \]

\[ \Xi_k(\phi) = \int \exp \left( \delta E_k^+(\phi, \omega_k) \right) d\mu_k^*(W) \]

To analyze \( \Xi_k(\phi) \) we start with a general bound on the local pieces \( \delta E_k^+(X, \phi, \omega_k) \) which are small.

Lemma 17. For \( \phi \in \frac{1}{2}R_k \) and \( |W| \leq p_{0,k} \).

\[ |\delta E_k^+(X, \phi, \omega_k)| \leq O(1)\lambda_k^{1/4 - 10\varepsilon} e^{-\varepsilon_{ld}k} \]

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Proof. We have $\delta E_k^+ = \delta E_k - \delta V_k$ and we first consider $\delta V_k$. We have

$$\delta V_k(\Box) = \frac{1}{4\lambda_k} \int_{\Box} \left[ (\phi + W_k)^4 - \phi^4 \right] + \frac{1}{2\mu_k} \int_{\Box} \left[ (\phi + W_k)^2 - \phi^2 \right]$$ (214)

Now $|\phi| \leq \frac{1}{2}\lambda_k^{-1/4-3\epsilon}$ and $|W_k| \leq p_k$ so the first term has a contribution

$$|2\lambda_k \int_{\Box} \phi^3 W_k| \leq 2M^3 \cdot \lambda_k^{1/4-9\epsilon} p_k \leq \lambda_k^{1/4-10\epsilon}$$ (215)

The other contributions to the first term are smaller. Similarly the second term has a contribution

$$\mu_k \int_{\Box} \phi W_k \leq \lambda_k^{1/2} M^\epsilon \cdot \lambda_k^{-1/4-3\epsilon} p_k \leq \lambda_k^{1/4-4\epsilon}$$ (216)

The other term is smaller. Overall then $|\delta V_k(\Box)| \leq O(1)\lambda_k^{1/4-10\epsilon}$.

By lemma [16] we have $W_k \in \lambda_k^{1/4} R_k$. So if $|t| \leq \lambda_k^{1/4}$ then $tW_k \in 1/4 R_k$ and $\phi + tW_k \in 3/4 R_k$. The function $t \to E_k(\phi + tW_k)$ is analytic in this domain and so

$$\delta E_k(X, \phi, W_k) = \frac{1}{2\pi i} \int_{|t| = \lambda_k^{-1/4}/4} \frac{dt}{t(t-1)} E_k(X, \phi + tW_k)$$ (217)

Since

$$|E_k(X, \phi + tW_k)| \leq \|E_k\|_{k, \epsilon} e^{-\kappa d_M(X)} \leq e^{-\kappa d_M(X)}$$ (218)

this gives the bound

$$|\delta E_k(X, \phi, W_k)| \leq O(1)\lambda_k^{1/4} e^{-\kappa d_M(X)}$$ (219)

which is sufficient.

Remark. It is convenient at this point to reblock to $LM$ cubes defining $(\delta E_k^+)'(Y) = (B\delta E_k^+)'(Y)$. Then we have by (213) and lemma [10] for $Y \in \mathcal{D}_{k+1}^n$

$$|(\delta E_k^+)'(Y, \phi, W_k)| \leq O(1)L^3 \lambda_k^{1/4-10\epsilon} e^{-\kappa' d_{LM}(Y)}$$ (220)

and now $\delta E_k^+ = \sum_Y (\delta E_k^+)'(Y)$.

4.5 localization

In preparation for the cluster expansion we localize the dependence of $(\delta E_k^+)'(Y, \phi, W_k)$ in $W$. Consider the random walk expansion $G_k = \sum_{\omega} G_{k,\omega}$ of section 2.3. However to match the $LM$ polymers in $(\delta E_k^+)'(Y)$ we take an expansion based on $LM$ cubes rather than $M$ cubes. As explained in section 2.3 we introduce a variable $s = \{s_{\Box}\}$ with $0 \leq s_{\Box} \leq 1$ for every $LM$-cube $\Box$. In the random walk expansion $G_k = \sum_{\omega} G_{k,\omega}$ we weaken the coupling through $\Box$ by introducing

$$G_k(s) = \sum_{\omega} s_{\omega} G_{k,\omega}$$ (221)

We can also give a weakened form for $C_k^{1/2}$ as follows. We use the representation (193), (194), (195) for $C_k^{1/2}$ in terms of $G_{k,r}, A_{k,r}$. Now $G_{k,r}$ also has a random walk expansion $G_{k,r} = \sum_{\omega} G_{k,r,\omega}$ as explained in appendix [12] but now also taken based on $LM$ cubes. Hence it has a weakened version

$$G_{k,r}(s) = \sum_{\omega} s_{\omega} G_{k,r,\omega}$$ (222)

28
Lemma 19.

For other connected components of $Z$ connected paths such that on the component of $X$ $\square \subset \omega$ is true of $Y$

\[ C_{k/2}(s) = \frac{1}{\pi} \int_0^\infty \frac{dr}{r} C_{k,r}(s) \]
\[ C_{k,r}(s) = A_{k,r} + a_k^2 A_{k,r} Q_k G_{k,r}(s) Q_k^T A_{k,r} \]

Combining these we get a weakened form for $W_k = a_k G_k Q_k^T C_{k/2} W$ which is

\[ W_k(s) = a_k G_k(s) Q_k^T C_{k/2}(s) W \]

(224)

The term $(\delta E_k^+)'(Y, \phi, W_k)$ is local in $\phi, W_k$, but not in $W$ because $W_k$ at any point depends on $W$ at every point. We remedy this with the following localization expansion. Break the coupling outside of $Y$ by interpolating with $\delta E_k^+(Y, \phi, W_k(s))$. Use the identity

\[ f(s_\square = 1) = f(s_\square = 0) + \int_0^1 ds_\square \frac{\partial f}{\partial s_\square} \]

successively in each variable in $s_Y = \{ s_\square \} \in Y$ and obtain

\[ (\delta E_k^+)'(Y) = \sum_{Z \supset Y} \delta E_k^+(Y, Z) \]
\[ (\delta E_k^+)(Y, Z; \phi, W) = \int ds_{Z-Y} \frac{\partial}{\partial s_{Z-Y}} [(\delta E_k^+)'(Y, \phi, W_k(s))]_{s_{Z-Y} = 0, s_Y = 1} \]

Now we write

\[ \delta E_k^+ = \sum_Y (\delta E_k^+)'(Y) = \sum_Y \sum_{Z \supset Y} \delta E_k^+(Y, Z) = \sum_Z (\delta E_k^+)_{\text{loc}}(Z) \]

(227)

where

\[ (\delta E_k^+)_{\text{loc}}(Z) = \sum_{Y \subset Z} \delta E_k^+(Y, Z) \]

(228)

Here $Y$ is connected but $Z$ may not be. However we have:

**Lemma 18.** In the expansion $\delta E_k^+ = \sum_Z (\delta E_k^+)_{\text{loc}}(Z)$ we can restrict to connected $Z$, i.e. $Z \in D_{k+1}$. Furthermore $(\delta E_k^+)_{\text{loc}}(Z) = (\delta E_k^+)_{\text{loc}}(Z, \phi, W)$ only depends on $\phi, W$ on $Z$.

**Proof.** Consider the random walk expansion (221) for $G_k(s)|_{s_Z=0}$ which occurs in $W_k(s)|_{s_Z=0}$. If $\square \subset Z$ then $s_\square = 0$ and so then $s_\omega = 0$ for any path $\omega$ such that $X_\omega \supset Q_k$. Thus in $G_k(s)|_{s_Z=0}$ paths such that $X_\omega$ intersect $Z$ do not occur, and we must have $X_\omega \subset Z$. But $X_\omega$ is connected so only paths such that $X_\omega$ is in a single connected component of $Z$ contribute. This means that $G_k(s)|_{s_Z=0}$ preserves the subspaces of functions on the various connected components of $Z$. The same is true of $C_{k/2}(s)|_{s_Z=0}$ and $M_k(s)|_{s_Z=0} \equiv a_k G_k(s) Q_k^T C_{k/2}(s)|_{s_Z=0}$. But we are interested in $W_k(s)|_{s_Z=0} = M_k(s) \|_{s_Z=0}$ on $Y$ which means that only $M_k(s)|_{s_Z=0}$ restricted to functions on the component of $Z$ containing $Y$ contributes. Therefore derivatives in $\partial/\partial s_{Z-Y}$ for cubes in other connected components of $Z$ give zero. Hence in (226) we can restrict the sum over $Z \supset Y$ to connected $Z$ which proves the first statement. Furthermore we see that $\delta E_k^+(Y, Z; \phi, W)$ and hence $(\delta E_k^+)_{\text{loc}}(Z, \phi, W)$ only depends on $\phi, W$ in $Z$. This completes the proof.

**Lemma 19.** For $\phi \in T R_k$ and $|W| \leq p_{0,k}$

\[ |(\delta E_k^+)_{\text{loc}}(Z, \phi, W)| \leq O(1)L^3 \lambda_k^{1/4-10\epsilon} e^{-L(\kappa-2\epsilon_0-2)} d_{LM}(Z) \]

(229)
Proof. For \( \square \subset Z - \mathbf{Y} \) we consider \( s_{\square} \) complex and satisfying \( |s_{\square}| \leq M^{1/2} \). As explained in section 2.3 the operator \( G_k(s) \) satisfies bounds of the same form as \( G_k \). In the same way \( C_k^{1/2}(s) \) satisfies bounds of the same form as \( C_k^{1/2} \). Hence \( \mathcal{W}_k(s) \) satisfies bounds of the same form as \( \mathcal{W}_k \). Therefore \( (\delta E^+_k)(Y, \phi, \mathcal{W}_k(s)) \) is analytic in \( |s_{\square}| \leq M^{1/2} \) and satisfies there

\[
| (\delta E^+_k)^i(Y, \phi, \mathcal{W}_k(s)) | \leq O(1) L^3 3^{1/4 - 10e} e^{-\kappa' d_{LM}(Y)}
\] (230)

just as in (220). If we let \( \kappa_1 = 1/2 \log M \) we can write the condition as \( |s_{\square}| \leq e^{\kappa_1} \). Now if we restrict to \( |s_{\square}| \leq 1 \) we get Cauchy bounds on the derivatives:

\[
\left| \frac{\partial}{\partial s_{Z-Y}} [(\delta E^+_k)^i(Y, \phi, \mathcal{W}_k(s))] \right| \leq O(1) L^3 3^{1/4 - 10e} e^{-\kappa' |Z-Y|_{LM} e^{-\kappa' d_{LM}(Y)}} (231)
\]

We can assume \( \kappa_1 - 1 \geq \kappa' \). Using this, integrating over \( s_{Z-Y} \) and summing over \( Y \subset Z \) yields

\[
| (\delta E^+_k)^{loc}(Z, \phi, W) | \leq O(1) L^3 3^{1/4 - 10e} \sum_{Y \subset Z} e^{\kappa' |Z-Y|_{LM} - \kappa' d_{LM}(Y)} (232)
\]

We show below that \( |Z - Y|_{LM} + d_{LM}(Y) \geq d_{LM}(Z) \). Then we can extract a factor \( e^{-(\kappa' - \kappa_0) d_{LM}(Z)} \) and obtain

\[
| (\delta E^+_k)^{loc}(Z, \phi, W) | \leq O(1) L^3 3^{1/4 - 10e} e^{-(\kappa' - \kappa_0) d_{LM}(Z)} \sum_{Y \subset Z} e^{-\kappa_0 d_{LM}(Y)} (233)
\]

But the sum is bounded by \( O(1)|Z|_{LM} \) by 234 in the appendix. Furthermore by 122 \( |Z|_{LM} \leq O(1) e^{d_{LM}(Z)} \). Since \( \kappa' - \kappa_0 - 1 \geq L(\kappa - 2\kappa_0 - 2) \) this gives the result.

Lemma 20. For \( X, Y \in \mathcal{D}_k \) and \( X \subset Y \):

\[
M d_M(Y) \leq M|Y - X|_M + M d_M(X) (234)
\]

Proof. Let \( \tau \) be a minimal tree on the \( M \)-cubes in \( X \) of length \( M d_M(X) \). Let \( (Y - X)_i \), be the connected components of \( Y - X \). Every component \( (Y - X)_i \), has a cube \( \square_i \) adjacent to a cube in \( \square' \subset X \) across a 2-dimensional face. Let \( x'_i \) be the point in \( \square_i \) which is a vertex of \( \tau \). Now extend the tree \( \tau \) by taking a line from \( x'_i \) to the translated point \( x_i \) in \( \square_i \). Then extend it to all of \( (Y - X)_i \) by taking lines across two dimensional faces joining translates of \( x_i \). For each \( i \) this adds a length \( M|Y - X|_i|_{LM} \). Thus we have constructed a tree joining all the blocks of \( Y \) of length \( M d_M(X) + \sum_i M|Y - X|_i|_{M} = M d_M(X) + M|Y - X|_{M} \). This must be greater than the length of a minimal tree \( M d_M(Y) \).

4.6 cluster expansion

The fluctuation integral is now

\[
\Xi_k(\phi) = \int \exp \left( \sum_{Y \in \mathcal{D}_{k+1}} (\delta E^+_k)^{loc}(Y, \phi, W) \right) d\mu_k(W) (235)
\]

The cluster expansion gives this a local structure. The result is:

Lemma 21. (cluster expansion) Let \( \lambda_k \) be sufficiently small. For \( \phi \in \frac{1}{2} \mathcal{R}_k \)

\[
\Xi_k(\phi) = \exp \left( \sum_{Y \in \mathcal{D}_{k+1}} E^\phi_k(Y, \phi) \right) (236)
\]

where

\[
|E^\phi_k(Y, \phi)| \leq O(1) L^3 3^{1/4 - 10e} e^{-L(\kappa - 5\kappa_0 - 5) d_{LM}(Y)} (237)
\]
For the standard proof see appendix [13]. Here it is applied with \( LM \) cubes. The bound (237) follows from the bound \( \phi \). The latter is small enough to fall within the range of validity of the cluster expansion if \( \mathcal{O}(1) L^3 \lambda_k^{1/4-10\epsilon} \leq \phi_0 \).

Inserting this result into (212) and defining \( E^\#(\phi) = \sum_Y E^\#(Y, \phi) \) we have

\[
\Xi_k(\phi) = \exp \left( -\varepsilon_k^2 \text{Vol}(T_{M+N-k}^0) + E_k^+(\phi) + E_k^\#(\phi) \right)
\]  

Insert this into (205) and obtain

\[
\tilde{\rho}_{k+1}(\Phi_{k+1}) L^{-\lceil T_{M+N-k}^0 / 2 \rceil} = Z_{k+1} \exp \left( -S_{k+1}^0(\Phi_{k+1}, \phi_{k+1}) - \varepsilon_k^0 \text{Vol}(T_{M+N-k}^0) + E_k^+(\phi_{k+1}) + E_k^\#(\phi_{k+1}) \right)
\]  

### 4.7 scaling

From the last expression we form \( \rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_{k}(\Phi_{k+1, L}) L^{-\lceil T_{M+N-k}^0 / 2 \rceil} \).

We have seen in lemma [3] that \( \phi_{k+1}^0(\Phi_{k+1, L}) = \phi_{k+1, L} \) and that \( S_{k+1}^0(\Phi_{k+1}, \phi_{k+1}^0) \) scales to \( S_{k+1}(\Phi_{k+1}, \phi_{k+1}) \). We also have \( \varepsilon_k^0 \text{Vol}(T_{M+N-k}^0) = L^3 \varepsilon_k^0 \text{Vol}(T_{M+N-k}^1) \).

In \( E_k^+ = E_k - V_k \) we have

\[
V_k(\phi_{k+1, L}) = L^3 \varepsilon_k \text{Vol}(T_{M+N-k}^1) + \frac{1}{2} L^2 \mu_k \phi_{k+1}^2 + \frac{1}{4} L \lambda_k \int \phi_{k+1}^2
\]

For \( E_k \) we reblock before scaling, and have for \( \phi \in \mathcal{R}_{k+1} \) that \( E_k(\phi_L) = (BE_k)_{L-1}(\phi) \). Since \( E_k \) is normalized for small polymers lemma [11] says

\[
\| (BE_k)_{L-1} \|_{k+1, \kappa} \leq \mathcal{O}(1) L^{-\epsilon} \| E_k \|_{k, \kappa}
\]

The function \( E_k^\# \) is already reblocked. We have

\[
E_k^\#(\phi_L) = \sum_{Y \in \mathcal{D}_{k+1}} E_k^\#(Y, \phi_L) = \sum_{X \in \mathcal{D}_{k+1}} E_k^\#(LX, \phi_L) = \sum_{X \in \mathcal{D}_{k+1}} E_{k, L-1}^\#(X, \phi) \equiv E_{k, L-1}^\#(\phi)
\]

For \( \phi \in \mathcal{R}_{k+1} \) we have \( \phi_L \in \frac{1}{2} \mathcal{R}_k \) and so by (237)

\[
|E_{k, L-1}^\#(X, \phi)| \leq \mathcal{O}(1) L^3 \lambda_k^{1/4-10\epsilon} e^{-L(\kappa - 5\kappa_0 - 5)d_{LM}(X)}
\]

We need \( L(\kappa - 5\kappa_0 - 5) \geq \kappa \) or equivalently \( \kappa \geq 5L(L-1)^{-1}(\kappa_0 + 1) \). Since \( L \geq 2 \) it suffices that \( \kappa \geq 10(\kappa_0 + 1) \) which we assume. Then

\[
\|E_{k, L-1}^\#\|_{k+1, \kappa} \leq \mathcal{O}(1) L^3 \lambda_k^{1/4-10\epsilon}
\]

Altogether then

\[
\rho_{k+1}(\Phi_{k+1}) = Z_{k+1} \exp \left( -S_{k+1}(\Phi_{k+1}, \phi_{k+1}) - L^3 (\varepsilon_k + \varepsilon_k^0) \text{Vol}(T_{M+N-k}^1) 
- \frac{1}{2} L^2 \mu_k \phi_{k+1}^2 - \frac{1}{4} L \lambda_k \int \phi_{k+1}^2 + (BE_k)_{L-1}(\phi_{k+1}) + E_{k, L-1}^\#(\phi_{k+1}) \right)
\]
4.8 completion of the proof

Neither \((BE_k)_{L-1}\) nor \(E^k_{L-1}\) are normalized for small polymers, and we need this feature to complete
the induction. By \(166\) we remove energy and mass terms to normalize them.

\[
(BE_k)_{L-1}(\phi_{k+1}) = -L_1E_k\text{Vol}(T_{N+M-k-1}) - \frac{1}{2}L_2E_k\|\phi_{k+1}^2\| + (L_3E_k)(\phi_{k+1})
\]  

(246)

where

\[
\begin{align*}
L_1E_k &= \varepsilon((BE_k)_{L-1}) \\
L_2E_k &= \mu((BE_k)_{L-1}) \\
L_3E_k &= R((BE_k)_{L-1})
\end{align*}
\]  

(247)

From the bound (241) and lemma 12 and lemma 13 we have that \(|L_1E_k|\) and \(L_3E_k\|_{k+1,\kappa}\) are bounded
by \(O(L^{-\varepsilon})\|E_k\|_{k,\kappa}\) and that \(L_2E_k\) is bounded by \(O(L^{-\varepsilon})\lambda_{k}^{3/2+6\varepsilon}\|E_k\|_{k,\kappa}\). These are the required
bounds

We also apply (166) to \(E^k_{L-1}\) but now tack on the extra term \(\varepsilon^0_k\) We have

\[
E^k_{L-1}(\phi_{k+1}) = -L_1\varepsilon_k^k\text{Vol}(T_{N+M-k-1}) - \frac{1}{2}\mu_k^k\|\phi_{k+1}^2\| + E_k^*(\phi_{k+1})
\]  

(248)

where

\[
\begin{align*}
\varepsilon^*_k &= L_1\varepsilon^0_k + \varepsilon(E^k_{L-1}) \\
\mu^*_k &= \mu(E^k_{L-1}) \\
E^*_k &= R(E^k_{L-1})
\end{align*}
\]  

(249)

From the bound (241) and lemma 12 and lemma 13 \(|\varepsilon^*_k|\) and \(E^*_k\|_{k+1,\kappa}\) are bounded by \(O(1)L^3\lambda_{k}^{3/4-10\varepsilon}\)
and \(|\mu^*_k|\) is bounded by \(O(1)L^3\lambda_{k}^{3/4-4\varepsilon}\). These are the required bounds.

Insert these expansions into (245) and obtain the final form

\[
\rho_{k+1}(\Phi_{k+1}) = Z_{k+1} \exp \left(-S_{k+1}(\Phi_{k+1}, \phi_{k+1}) - \varepsilon_{k+1}\text{Vol}(T_{M+N-k-1})
- \frac{1}{2}\mu_{k+1}\|\phi_{k+1}\|^2 - \frac{1}{4}\lambda_{k+1}\int\phi_{k+1}^4 + E_{k+1}(\phi_{k+1}) \right)
\]  

(250)

where \(\varepsilon_{k+1}, \mu_{k+1}, E_{k+1}\) are given by (180). This completes the proof of theorem 14.

4.9 derivatives

The previous proof was carried out under the assumption that \(\lambda_k\) is small and \(\mu_k \in \mathbb{R}, E_k \in \text{Re}(K_{k}^{\text{norm}})\)
satisfy \(|\mu_k| \leq \lambda_{k}^{1/2}\) and \(\|E_k\|_{k,\kappa} \leq 1\). In this domain \(\mu_k^* = \mu_k^*(\lambda_k, \mu_k, E_k)\) and \(E_k^* = E_k^*(\lambda_k, \mu_k, E_k)\)
satisfy the bounds

\[
\begin{align*}
|\mu_k^*| &\leq O(1)L^3\lambda_{k}^{3/4-4\varepsilon} \\
\|E_k^*\|_{k,\kappa} &\leq O(1)L^3\lambda_{k}^{3/4-10\varepsilon}
\end{align*}
\]  

(251)

However the proof works as well for \(\mu_k \in \mathbb{C}, E_k \in K_{k}^{\text{norm}}\) with exactly the same bounds, and one can
show that \(\mu_k^*, E_k^*\) are analytic functions of \(\mu_k, E_k\) on this domain. This means we can use Cauchy
bounds to get estimates on partial derivatives in a slightly smaller region.

\footnote{If we had normalized \(E_k^*(X)\) for all polymers, not just small polymers, then \((BE_k)_{L-1}(X)\) would still be normalized
and \(L_1, L_2\) would not appear below. This strategy is possible, but presents other difficulties.}
Lemma 22. In the region \(|\mu_k| \leq \frac{1}{2} \lambda_k^{1/2}\) and \(\|E_k\|_k \leq \frac{1}{2}\) we have

\[
\frac{\partial \mu_k^\ast}{\partial \mu_k} \leq O(1) L^3 \lambda_k^{1/4-4\epsilon} \quad \left\| \frac{\mu_k^\ast}{E_k} \right\| \leq O(1) L^3 \lambda_k^{3/4-4\epsilon} \quad (252)
\]

\[
\frac{\partial E_k^\ast}{\partial \mu_k} \leq O(1) L^3 \lambda_k^{-1/4-10\epsilon} \quad \left\| \frac{E_k^\ast}{E_k} \right\| \leq O(1) L^3 \lambda_k^{1/4-10\epsilon} \quad (253)
\]

Proof. We have

\[
\frac{\partial \mu_k^\ast}{\partial \mu_k} = \frac{d}{dt} \left[ \mu_k^\ast(\lambda_k, \mu_k + t, E_k) \right]_{t=0} = \frac{1}{2\pi i} \int_{|t|=\frac{1}{2} \lambda_k^{1/2}} \frac{1}{t^2} \mu_k^\ast(\lambda_k, \mu_k + t, E_k) dt
\]

whence

\[
\left\| \frac{\partial \mu_k^\ast}{\partial \mu_k} \right\| \leq O(1) \lambda_k^{-1/2} (L^3 \lambda_k^{3/4-4\epsilon}) \leq O(1) L^3 \lambda_k^{1/4-4\epsilon}
\]

We also have for \(\|\dot{E}\|_{k,k} \leq 1\)

\[
< \frac{\partial \mu_k^\ast}{\partial E_k}, \dot{E} > = \frac{d}{dt} \left[ \mu_k^\ast(\lambda_k, \mu_k, E_k + t\dot{E}) \right]_{t=0} = \frac{1}{2\pi i} \int_{|t|=\frac{1}{2} \lambda_k^{1/2}} \frac{1}{t^2} \mu_k^\ast(\lambda_k, \mu_k + t\dot{E}) dt
\]

whence

\[
| < \frac{\partial \mu_k^\ast}{\partial E_k}, \dot{E} > | \leq O(1) L^3 \lambda_k^{3/4-4\epsilon}
\]

Then

\[
\left\| \frac{\partial \mu_k^\ast}{\partial E_k} \right\| = \sup_{\|\dot{E}\|_{k,k} \leq 1} | < \frac{\partial \mu_k^\ast}{\partial E_k}, \dot{E} > | \leq O(1) L^3 \lambda_k^{3/4-4\epsilon}
\]

The estimates on the derivatives of \(E^\ast_k\) are similar.

5 the flow

We seek well-behaved solutions of the RG equations \(180\). We continue to treat \(\lambda_k\) as a parameter, not a dynamical variable. Thus the equations of interest are

\[
\begin{align*}
\varepsilon_{k+1} &= L^3 \varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^\ast \\
\mu_{k+1} &= L^2 \mu_k + \mathcal{L}_2 E_k + \mu_k^\ast \\
E_{k+1} &= \mathcal{L}_3 E_k + E_k^\ast
\end{align*}
\]

(259)

Keep in mind that the quantities \(\varepsilon_k, \mu_k, \lambda_k, E_k\) determine a density \(\rho_k\) on the lattice \(\mathbb{T}_N^{M+N-K}\) as given by \(179\) \(177\).

The transformation is defined as long as \(\lambda_k\) is sufficiently small and \(|\mu_k| \leq \lambda_k^{1/2}\) and \(\|E_k\|_k \leq 1\). We make no restriction on the size of the bare coupling \(\lambda\) but the initial values \(\lambda_0 = \lambda_0^N = L^{-N}\lambda\) will be small enough for \(N\) sufficiently large, and we assume the other conditions are satisfied initially. We iterate it as long as the conditions are satisfied. Our goal is to show that for any \(N\) we can choose the initial point so that the solution exists for \(k = 0, 1, \ldots, K\) with \(K = N - \Delta\) and \(\Delta \geq 0\) independent of \(N\). Then at \(k = K\) we are on the lattice \(\mathbb{T}_N^{M+N-K} = \mathbb{T}_M^{\Delta}\) and can make estimates on \(\rho_K\) uniformly in \(N\) (for small fields).

\*This section is not particularly due to Balaban. We study the RG flow by a discrete dynamical systems approach. Somewhat similar methods can be found in \(32\), \(30\). However those papers are concerned with infrared problems, not ultraviolet problems of the type considered here.
To accomplish this tuning we do not at first specify the initial values for $\varepsilon_k, \mu_k$ but instead specify final values for these quantities which for simplicity we take to be zero. Thus we look for solutions $\varepsilon_k, \mu_k, E_k$ for $k = 0, 1, 2, \ldots, K$ satisfying

$$
\varepsilon_K = 0 \quad \mu_K = 0 \quad E_0 = 0
$$

This is non-perturbative renormalization - the initial values for $\varepsilon, \mu$ will depend on $K$ and hence $N$. Note that the total mass at level $K$ is then $\mu_K + \mu_K = \bar{\mu}_K$.

At this point we temporarily drop $\varepsilon_k$ as a variable since it does not affect the others. Then we rewrite the flow equation as

$$
\mu_k = L - 2(\mu_k + 1 - L \frac{E_k}{2} - \mu_k^*)
$$

$$
E_k = L_3 E_{k-1} + E_k^{*}
$$

The first equation is for $k = 0, 1, 2, \ldots, K - 1$ with value at $K$ given by $\mu_K = 0$. The last equation for $k = 1, 2, \ldots, K$ with $E_0 = 0$. These equations have the same solutions as (259), but are contractive and hence more tractable. We analyze them as a fixed point problem.

Let $\xi_k = (\mu_k, E_k)$ be an element of the real Banach space $\mathbb{R} \times \text{Re}(K^\text{form}_k)$ and consider sequences

$$
\xi = (\xi_0, \ldots, \xi_K)
$$

Pick a fixed $\beta$ satisfying

$$
0 < \beta < \frac{1}{4} - 10 \epsilon
$$

and let $B$ be the Banach space of all such sequences with norm

$$
||\xi|| = \sup_{0 \leq k \leq K} \{\lambda_k^{-\frac{1}{2} - \beta} |\mu_k|, \lambda_k^{-\beta} ||E_k||_{k, \kappa}\}
$$

This anticipates the kind of growth we can establish for solutions. Let $B_0$ be the subset of all sequences satisfying the boundary conditions. Thus

$$
B_0 = \{\xi \in B : \mu_K = 0, E_0 = 0\}
$$

This is a complete metric space with distance $||\xi - \xi'||$. Finally let

$$
B_1 = B_0 \cap \{\xi \in B : ||\xi|| < 1\}
$$

Next define an operator $\xi' = T\xi$ by

$$
\mu_k' = L - 2(\mu_{k+1} - L_2 E_k - \mu_k^*)
$$

$$
E_k' = L_3 E_{k-1} + E_k^{*}
$$

Then $\xi$ is a solution of (260), (261) iff it is a fixed point for $T$ on $B_0$. We look for such fixed points in $B_1$.

We proceed under the assumption that

$$
\lambda_K = \lambda_N^N = L^{-(N-K)} \lambda = L^{\Delta \lambda}
$$

is sufficiently small. This can be arranged either by taking $\lambda$ small (in which case we can take $\Delta = 0$ and $K = N$), or more generally by taking $\Delta$ large. If $\lambda_K = L^{\Delta \lambda}$ is sufficiently small then $T$ is defined on $B_1$. This follows since we have $\lambda_k \leq \lambda_K$ small and

$$
|\mu_k| \lambda_k^{-1/2} \leq \lambda_k^\beta \quad ||E_k||_{k, \kappa} \leq \lambda_k^\beta
$$

which is well within the allowed region $|\mu_k| \lambda_k^{-1/2} \leq 1, ||E_k||_{k, \kappa} \leq 1$. 34
Lemma 23. Let $\lambda_K = L^{-\Delta} \lambda$ be sufficiently small. Then for all $N \geq \Delta$ and $K = N - \Delta$

1. The transformation $T$ maps the set $B_1$ to itself.

2. There is a unique fixed point $T \xi = \xi$ in this set.

Proof. (1.) We use the bounds of theorem [13] for $L_2, L_3$ (replacing $O(1)L^{-\epsilon}$ by 1 ) and for $\mu^*_k, E_k^*$. To show the the map sends $B_1$ to itself we estimate

\[
\lambda_k^{-\frac{1}{2} - \beta} |\mu_k' - \mu_k^*| \leq \lambda_k^{-\frac{1}{2} - \beta} L^{-2} \left( |\mu_{k+1}| + \lambda_k^{1/2+\epsilon} \|E_k\|_{k,k} + O(1)L^3 \lambda_k^{3/4-\epsilon} \right)
\]

\[
\leq L^{-\frac{5}{4}} \left[ \lambda_k^{-\frac{1}{2} - \beta} |\mu_{k+1}| \right] + L^{-2} \lambda_k^{1/4+\epsilon} \left[ \lambda_k^{-\beta} \|E_k\|_{k,k} \right] + O(1)L \lambda_k^{\frac{1}{2} - \epsilon} (270)
\]

\[
\leq \frac{1}{2} (||\xi|| + 1) \leq 1
\]

Here we use that $L^{-\beta-3/2} \leq 1/4$ for $L$ large, that $\lambda_k^{1/2} |\mu_{k+1}| \leq ||\xi||$, that $L^{-2} \lambda_k^{6\epsilon} \leq 1/4$, that

\[
\lambda_k^{-\beta} \|E_k\|_{k,k} \leq ||\xi||
\]

and that $O(1)L \lambda_k^{-\beta} \leq 1/2$ for $\lambda_k$ small (depending on $L$). We also have for $L$ sufficiently large

\[
\lambda_k^{-\beta} \|E_k\|_{k,k} \leq \lambda_k^{-\beta} \left( \|E_{k-1}\|_{k-1,k} + O(1)L^3 \lambda_k^{1/4-10\epsilon} \right)
\]

\[
\leq L^{-\frac{5}{4}} \left[ \lambda_k^{-\frac{1}{2} - \beta} |E_{k-1}|_{k-1,k} \right] + O(1) L^3 \lambda_k^{-\frac{1}{4} - 10\epsilon} (271)
\]

\[
\leq \frac{1}{2} (||\xi|| + 1) \leq 1
\]

Combining this with (270) yields $\|T(\xi)\| \leq 1$ as required.

(2.) By the standard fixed point theorem in a complete metric space it suffices to show that the mapping is a contraction. We show that under our assumptions

\[
\|T(\xi_1) - T(\xi_2)\| \leq \frac{1}{2} \|\xi_1 - \xi_2\| (272)
\]

First for the $\mu$ terms we have (suppressing the dependence of $\mu_k^*$ on $\lambda_k$)

\[
\mu_{1,k}^* - \mu_{2,k}^* = L^{-2} \left( (\mu_{1,k} + \mu_{2,k+1}) - L_2(E_{1,k} - E_{2,k}) - (\mu_{1,k}^*(\mu_{1,k}, E_{1,k}) - \mu_{2,k}^*(\mu_{1,k}, E_{1,k})) \right) (273)
\]

Then

\[
\lambda_k^{-\frac{1}{2} - \beta} |\mu_{1,k}^* - \mu_{2,k}^*| \leq L^{-2} \lambda_k^{-\frac{1}{2} - \beta} |\mu_{1,k+1} - \mu_{2,k+1}| + L^{-2} \lambda_k^{-\beta+\epsilon} \|E_{1,k} - E_{2,k}\|_{k,k} + L^{-2} \lambda_k^{-\beta} |\mu_{1,k}^*(\mu_{1,k}, E_{1,k}) - \mu_{2,k}^*(\mu_{1,k}, E_{1,k})| (274)
\]

The first term is

\[
L^{-\frac{5}{4}} \left[ \lambda_k^{-\frac{1}{2} - \beta} |\mu_{1,k+1} - \mu_{2,k+1}| \right] \leq L^{-\frac{5}{4}} \|\xi_1 - \xi_2\| (275)
\]

The second term is

\[
L^{-2} \lambda_k^{1/4+\epsilon} \left[ \lambda_k^{-\beta} \|E_{1,k} - E_{2,k}\|_{k,k} \right] \leq L^{-2} \lambda_k^{1/4+\epsilon} \|\xi_1 - \xi_2\| (276)
\]

For the last term we write with $\mu(t) = t \mu_{1,k} + (1-t) \mu_{2,k}$ and $E(t) = t E_{1,k} + (1-t) E_{2,k}$ and

\[
\begin{align*}
\mu^*_k(\mu(t), E_{1,k})(\mu_{1,k} - \mu_{2,k}) & = \mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{1,k}) + \mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{2,k}) \\
& = \int_0^1 \frac{\partial \mu_k^*}{\partial \mu_k}(\mu(t), E_{1,k})(\mu_{1,k} - \mu_{2,k})dt + \int_0^1 \left( \frac{\partial \mu_k^*}{\partial E_k}(\mu_{2,k}, E(t), E_{1,k} - E_{2,k}) \right)
\end{align*}
\]
We use the bounds $|\partial \mu_k^*/\partial \mu_k| \leq \mathcal{O}(1)L^3 \lambda_k^{1/4-4\epsilon}$ and $\|\partial \mu_k^*/\partial E_k\|_k \leq \mathcal{O}(1)L^3 \lambda_k^{3/4-4\epsilon}$ from lemma \[\text{22}\].

Thus we have
\[
\lambda_k^{-\frac{1}{2}-\beta}|\mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{2,k})| \\
\leq \lambda_k^{-\frac{1}{2}-\beta} \left( \mathcal{O}(1)L^3 \lambda_k^{1/4-4\epsilon}|\mu_{1,k} - \mu_{2,k}| + \mathcal{O}(1)L^3 \lambda_k^{3/4-4\epsilon} \|E_{1,k} - E_{2,k}\|_{k,\kappa} \right) \\
\leq \mathcal{O}(1)L^3 \lambda_k^{1/4-4\epsilon} \left[ \lambda_k^{-\frac{1}{2}-\beta}|\mu_{1,k} - \mu_{2,k}| \right] + \mathcal{O}(1)L^3 \lambda_k^{1/4-4\epsilon} \left[ \lambda_k^{-\beta} \|E_{1,k} - E_{2,k}\|_{k,\kappa} \right] \\
\leq \mathcal{O}(1)L^3 \lambda_k^{1/4-4\epsilon} \|\xi_k - \xi_2\| \tag{278}
\]

Altogether then for $L$ large and $\lambda_k$ small $\lambda_k^{-\frac{1}{2}-\beta}|\mu_{1,k} - \mu_{2,k}| \leq 1/2\|\xi_k - \xi_2\|$ as required.

Now consider the $E$ terms. We have
\[
E_{1,k}^* - E_{2,k}^* = L_3(E_{1,k-1} - E_{2,k-1}) + (E_{1,k-1}^*(\mu_{1,k-1}, E_{1,k-1}) - E_{2,k-1}^*(\mu_{2,k-1}, E_{2,k-1})) \tag{279}
\]

Then
\[
\lambda_k^{-\beta} \|E_{1,k}^* - E_{2,k}^*\|_{k,\kappa} \leq L^{-\beta} \lambda_k^{-\frac{1}{2}} \left( \|E_{1,k-1} - E_{2,k-1}\|_{k-1,\kappa} + \|E_{1,k-1}^* - E_{2,k-1}^*\|_{k-1,\kappa} \right) \tag{280}
\]

The first term is bounded by $L^{-\beta}\|\xi_k - \xi_2\|$. For the second term let $\mu(t) = t \mu_{1,k-1} + (1 - t)\mu_{2,k-1}$ and $E(t) = tE_{1,k-1} + (1 - t)E_{2,k-1}$ and write
\[
E_{k-1}^*(\mu_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(\mu_{2,k-1}, E_{2,k-1}) \\
= \int_0^1 \frac{\partial E_{k-1}^*}{\partial \mu_{k-1}}(\mu(t), E_{k-1})(\mu_{1,k-1} - \mu_{2,k-1})dt - \int_0^1 \frac{\partial E_{k-1}^*}{\partial E_{k-1}}(\mu_{2,k-1}, E(t), E_{1,k-1} - E_{2,k-1}) \tag{281}
\]

We use the bounds $|\partial E_k^*/\partial \mu_k| \leq \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon}$ and $\|\partial E_k^*/\partial E_k\|_k \leq \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon}$ from \[\text{22}\]. Then we have
\[
L^{-\beta} \lambda_k^{-\frac{1}{2}} \|E_{1,k-1}^* - E_{2,k-1}^*\|_{k-1,\kappa} \\
\leq L^{-\beta} \lambda_k^{-\frac{1}{2}} \left( \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon}|\mu_{1,k-1} - \mu_{2,k-1}| + \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon} \|E_{1,k-1} - E_{2,k-1}\|_{k-1,\kappa} \right) \\
\leq \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon} \left[ \lambda_k^{-\frac{1}{2}-\beta}|\mu_{1,k-1} - \mu_{2,k-1}| \right] + \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon} \left( \lambda_k^{-\beta} \|E_{1,k-1} - E_{2,k-1}\|_{k-1,\kappa} \right) \\
\leq \mathcal{O}(1)L^3 \lambda_k^{1/4-10\epsilon} \|\xi_k - \xi_2\| \tag{282}
\]

Altogether then for $L$ large and $\lambda_k$ small we have $\lambda_k^{-\beta} \|E_{1,k}^* - E_{2,k}^*\|_{k,\kappa} \leq \frac{1}{2}\|\xi_k - \xi_2\|$ which completes the proof.

Now we can state:

**Theorem 24.** Let $\lambda_K = L^{-\Delta} \lambda$ be sufficiently small. Then for $N \geq \Delta$ there is a unique sequence $\xi_k, \mu_k, E_k$ for $k = 0, 1, 2, \ldots, K = N - \Delta$ satisfying the dynamical equation \[\text{259}\], the boundary conditions \[\text{207}\], and
\[
|\mu_k| \leq \lambda_k^{\frac{1}{2}+\beta} \quad \|E_k\|_{k,\kappa} \leq \lambda_k^\beta \tag{283}
\]

Furthermore
\[
|\xi_k| \leq \mathcal{O}(1)\lambda_k^\beta \tag{284}
\]

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Proof. This solution is the fixed point from the previous lemma and the bounds (283) are a consequence.

To complete the proof we check the estimate on the vacuum energy. Once $\mu_k, E_k$ are fixed $\varepsilon_k$ is determined by $\varepsilon_{k+1} = L^3 \varepsilon_k + L_1 (E_k) + \varepsilon_1^*(\mu_k, \lambda_k, E_k)$ or by

$$
\varepsilon_k = L^{-3}(\varepsilon_{k+1} - L_1 (E_k) - \varepsilon_k^*)
$$

(285)

starting with $\varepsilon_K = 0$. We have $|L_1 (E_k)| \leq \|E_k\|_{k,\kappa} \leq \lambda_k^{\beta}$ and $|L^{-3}\varepsilon_k^*| \leq O(1) \lambda_k^{\frac{1}{4} - 10\epsilon} \leq O(1) \lambda_k^{\beta}$. Therefore for some constant $b = O(1)$

$$
|\varepsilon_k| \leq L^{-3}|\varepsilon_{k+1}| + b\lambda_k^{\beta}
$$

(286)

At $k = K - 1$ it says $|\varepsilon_{K-1}| \leq b\lambda_{K-1}^{\beta}$. This gives an inequality for $|\varepsilon_{K-2}|$ and we repeat this process. We claim that in general

$$
|\varepsilon_{K-n}| \leq b\left(\sum_{j=0}^{n-1} L^{(\beta-3)j}\right)\lambda_{K-n}^{\beta}
$$

(287)

Suppose it is true for $K - n$. Then

$$
|\varepsilon_{K-n-1}| \leq L^{-3}b\left(\sum_{j=0}^{n-1} L^{(\beta-3)j}\right)\lambda_{K-n}^{\beta} + b\lambda_{K-n-1}^{\beta}
$$

(288)

Here we used $\lambda_{K-n}^{\beta} = L^{\beta} \lambda_{K-n-1}^{\beta}$. Thus (287) is true for $K - n - 1$, hence (287) is established, and this implies the result (284) since the series converges.

Remarks.

1. Our method is efficient, but the estimates are not very sharp. For example we get $\varepsilon_k = O(\lambda_k^{\beta})$ and $\mu_k = O(\lambda_k^{\frac{1}{4} + \beta})$ for $\beta < \frac{1}{4}$, whereas perturbation theory suggests that both are $O(\lambda_k)$.

2. In the second paper we analyze the renormalization group transformations without the small field assumptions in this paper. This is accomplished with by splitting the fluctuation integrals into large and small field region at each step. The result is an expansion with terms labeled by decreasing sequence of small field regions. The leading term in this expansion is the case where each small field region is the whole torus - the case considered in this paper. The bounds of this paper will also be useful in estimating the other terms in the expansion, leading to a proof of theorem [1].
A estimates

Let $X$ be an $M$-polymer as defined in section 3.1, although not necessarily in dimension $d = 3$. So $X$ is a connected union of $M$-blocks $\square$ centered on lattice points. Let $|X_M|$ be the number of $M$-blocks in $X$ and let $Md_M(X)$ be the length of a minimal tree connecting the blocks in $X$.

Lemma 25.

1. There are constants $a, b$ such that for any $\square$

$$\sum_{X: \square \subseteq X} \exp(-a|X_M|) \leq b$$ (289)

2. There are constants $\kappa_0, K_0$ such that for any $\square$

$$\sum_{X: \square \subseteq X} \exp(-\kappa_0 d_M(X)) \leq K_0$$ (290)

Remark. The sums are independent of $M$ so it suffices to prove it for $M = 1$. In this case we drop the subscript $M$. The constants depend only on the dimension.

A version of (290) holds even if one drops the condition that $X$ is connected. This is presented in paper II.

Proof. [37]

1. We have

$$\sum_{X: \square \subseteq X} \exp(-a|X|) = \sum_{n \geq 1} e^{-an} |\{X : \square \subseteq X : |X| = n}\}|$$ (291)

Thus we have to estimate the number of polymers $X$ with $|X| = n$ containing $\square$. For each such $X$ consider the connected graph with lines joining the centers of adjacent cubes. Delete lines until you have a tree. The tree will connect all the cubes in $X$ and have $n - 1$ lines of unit length. The tree can be traversed with a path starting at $\square$ that goes over each line twice and has length $2(n - 1)$. Distinct polymers give distinct paths so the number of polymers is bounded by the number of paths of length $2(n - 1)$. But the latter can be estimated by $(2^d)^{2(n-1)}$. Thus

$$\sum_{X: \square \subseteq X} \exp(-a|X|) \leq \sum_{n \geq 1} e^{-an} (2^d)^{2(n-1)} = 2^{-2d} \sum_{n \geq 1} \exp((-a + 2d \log 2)n) \leq b$$ (292)

for suitable $b$ provided $a > 2d \log 2$.

2. We use the inequality $d(X) \geq 3^{-d}|X| - 1$ quoted in (122). Therefore

$$\sum_{X: \square \subseteq X} \exp(-\kappa_0 d(X)) \leq e^{\kappa_0} \sum_{X: \square \subseteq X} \exp(-3^{-d} \kappa_0 |X|) \leq K_0$$ (293)

provided $\kappa_0 \geq 3^d a$ and $K_0 \geq e^{\kappa_0} b$.

Corollary 26.

$$\sum_{X: X \cap Y \neq \emptyset} e^{-a|X_M|} \leq b|Y_M|$$ (294)

$$\sum_{X: X \cap Y \neq \emptyset} e^{-\kappa_0 d_M(X)} \leq K_0|Y_M|$$
Proof. Again it suffices to take $M = 1$. The first follows by
\[
\sum_{X : X \cap Y \neq \emptyset} e^{-a|X|} \leq \sum_{\square \subset Y} \sum_{X \supset \square} e^{-a|X|} \leq b|Y|
\] (295)
The second is similar.

B. cluster expansion

We give a treatment of the standard cluster expansion adapted to our circumstances. General references are [33], [40], [29], [37]. We present an ultralocal version favored by Balaban.

Consider fields $\Phi$ and $M$—polymers $X$ on a $d$-dimensional unit toroidal lattice. We are given localized functionals $H(X, \Phi)$ depending on $\Phi$ only in $X$ and integrals of the form
\[
\Xi = \int \exp \left( \sum_{X} H(X, \Phi) \right) d\mu(\Phi)
\] (296)
where $d\mu(\Phi) = \prod_x d\mu(\Phi(x))$ is an ultralocal probability measure. These do not occur naturally in quantum field theory, but can be arranged as we have seen in the text. Our goal is to give a local structure to this integral. This is particularly important if there are other spectator fields which for which we want to localize the dependence.

Theorem 27. (cluster expansion) There is a constant $c_0$ depending only on the dimension such that if $H(X, \Phi)$ satisfies
\[
|H(X, \Phi)| \leq H_0 e^{-\kappa d_M(X)}
\] (297)
on the support of $\mu$ with $\kappa > 3\kappa_0 + 3$ and $H_0 \leq c_0$ then
\[
\Xi = \exp \left( \sum_{Y} H^\#(Y) \right)
\] (298)
where $H^\#(Y)$ only depends on $H(X)$ for $X \subset Y$ and
\[
|H^\#(Y)| \leq O(1)H_0 e^{-3\kappa_0 - 3) d_M(Y)}
\] (299)
The constant $O(1)$ depends only on the dimension.

Proof. step 1: Start with a Mayer expansion which yields
\[
\exp \left( \sum_{X} H(X, \Phi) \right) = \prod_{X} \left( (e^{H(X, \Phi)} - 1) + 1 \right)
\]
\[
= \sum_{\{X_i\}} \prod_{i} (e^{H(X_i, \Phi)} - 1)
\]
\[
= \sum_{\{Y_j\}} \prod_{j} K(Y_j, \Phi)
\] (300)
Here the product over $X$ is written as a sum over collections of distinct polymers $\{X_i\}$. Then terms in this sum are grouped together into collections of disjoint polymers $\{Y_j\}$ (possibly empty), defining for connected $Y$
\[
K(Y, \Phi) = \sum_{\{X_i\} \cup X_i = Y} \prod_{i} (e^{H(X_i, \Phi)} - 1)
\] (301)

In this sum we require that the \( \{X_i\} \) cannot be divided into two disjoint sets. Instead of unordered \( \{X_i\} \) we can write this as a sum over ordered sets \( (X_1, \ldots, X_n) \) by

\[
K(Y, \Phi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1, \ldots, X_n) \cup_{i=1} Y = \emptyset} \prod_{i=1}^{n} (e^{H(X_i, \Phi)} - 1)
\]  

(302)

still with the same conditions on the \( X_i \).

If \( H_0 \leq \log 2 \) then on the support of \( \mu \)

\[
|e^{H(X, \Phi)} - 1| \leq 2H(X, \Phi) \leq 2H_0 e^{-\kappa_0 d_M(X)}
\]  

(303)

and so

\[
|K(Y, \Phi)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1, \ldots, X_n) \cup_{i=1} Y = \emptyset} \prod_{i=1}^{n} 2H_0 e^{-\kappa_0 d_M(X_i)}
\]  

(304)

Next we claim that if \( \cup_{i=1}^{n} X_i = Y \) as above, then

\[
M d_M(Y) \leq \sum_{i=1}^{n} M d_M(X_i) + M(n-1)
\]  

(305)

Indeed let \( \tau_i \) be a minimal tree on \( X_i \) of length \( M d_M(X_i) \). Also consider the connected graph whose edges are pairs \( \{X_i, X_j\} \) such that \( X_i \cap X_j \neq \emptyset \). Take a tree which is a subgraph with \( (n-1) \) edges. Each pair \( \{X_i, X_j\} \) in this tree will have a block \( \emptyset \) in common. For each pair add a line in \( \emptyset \) joining the point in \( \tau_i \) to the point in \( \tau_j \). This line has length at most \( M \). The tree graph consisting of the \( \tau_i \) and the \( (n-1) \) extra lines now joins all the blocks in \( Y \) and has length less than \( \sum_{i=1}^{n} M d_M(X_i) + M(n-1) \). The minimal tree must have shorter length which is the claim.

We use this to extract a factor \( \exp\left( - (\kappa - \kappa_0) (d_M(Y) - (n-1)) \right) \). Dropping all conditions on the \( X_i \) except \( X_i \subset Y \) we have

\[
|K(Y, \Phi)| \leq e^{-(\kappa - \kappa_0) d_M(Y)} \sum_{n=1}^{\infty} \frac{1}{n!} e^{(\kappa_0 - \kappa_0) n} \sum_{(X_1, \ldots, X_n) \cup_{i=1} Y = \emptyset} \prod_{i=1}^{n} 2H_0 e^{-\kappa_0 d_M(X_i)}
\]  

\[
\leq e^{-(\kappa - \kappa_0) d_M(Y)} \sum_{n=1}^{\infty} \frac{1}{n!} \left( e^{\kappa - \kappa_0} \sum_{X \subset Y} 2H_0 e^{-\kappa_0 d_M(X)} \right)^n
\]  

\[
\leq e^{-(\kappa - \kappa_0) d_M(Y)} \sum_{n=1}^{\infty} \frac{1}{n!} \left( 2H_0 K_0 e^{\kappa - \kappa_0} |Y|_M \right)^n
\]  

\[
\leq e^{-(\kappa - \kappa_0) d_M(Y)} 2H_0 K_0 e^{\kappa - \kappa_0} |Y|_M \exp\left( 2H_0 K_0 e^{\kappa - \kappa_0} |Y|_M \right)
\]  

Now \( |Y|_M \leq 3^d (1 + d_M(Y)) \leq \kappa_0 (1 + d_M(Y)) \). Furthermore we assume \( c_0 \) is small enough so that \( 2c_0 K_0 \kappa_0 e^{\kappa - \kappa_0} \leq 1 \). (So \( c_0 \) does depend on \( \kappa \).) Then the exponent is bounded by \( O(1) e^{d_M(Y)} \) and downstairs \( 2H_0 K_0 e^{\kappa - \kappa_0} |Y|_M \) is bounded by \( O(1) H_0 e^{d_M(Y)} \). Altogether then on the support of \( \mu \)

\[
|K(Y, \Phi)| \leq O(1) H_0 e^{-(\kappa - \kappa_0 - 2) d_M(Y)}
\]  

(307)

step 2: Now because the \( Y_j \) are disjoint and because fields at different sites are independent random variables

\[
\int \left( \sum_{\{Y_j\}} \prod_{j} K(Y_j, \Phi) \right) d\mu(\Phi) = \sum_{\{Y_j\}} \prod_{j} K^\#(Y_j)
\]  

(308)
where
\[
K^\#(Y) = \int K(Y, \Phi) d\mu(\Phi)
\] (309)
satisfies the same bound
\[
|K^\#(Y)| \leq \mathcal{O}(1) H_0 e^{-(k-\kappa_0 - 2) d_M(Y)}
\] (310)

**step 3:** Next we claim that
\[
\sum_{\{Y_i\}} \prod_i K^\#(Y_i) = \exp \left( \sum_Y H^\#(Y) \right)
\] (311)

where
\[
H^\#(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n): Y_1 = Y} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i)
\] (312)
and \(\rho^T(Y_1, \ldots, Y_n)\) vanishes if the \(Y_j\) can be divided into disjoint sets. At first we demonstrate the identity as formal series. Afterwards we demonstrate convergence.

Start by writing
\[
\sum_{\{Y_i\}} \prod_i K^\#(Y_i) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n): Y_1 = Y} \prod_i K^\#(Y_i)
\] (313)

where the sum is now over ordered \(n\)-tuples of polymers. Next let \(\zeta(X, Y) = 1\) if \(X \cap Y = \emptyset\) and \(\zeta(X, Y) = 0\) if if \(X \cap Y \neq \emptyset\). Then this can be written
\[
1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n)} \prod_{i=1}^{n} K^\#(Y_i) \prod_{\{i,j\} \subset \{1, \ldots, n\}} \zeta(Y_i, Y_j)
\] (314)

Next write
\[
\prod_{\{i,j\}} \zeta(Y_i, Y_j) = \prod_{\{i,j\}} [1 + (\zeta(Y_i, Y_j) - 1)] = \sum_G \prod_{\{i,j\} \in G} (\zeta(Y_i, Y_j) - 1)
\] (315)

Here in the second step we expand out the product and identify the sum with a sum over collections of pairs \(\{i, j\}\) from \(\{1, \ldots, n\}\), that is with graphs \(G\) on \(\{1, \ldots, n\}\). Each graph determines a partition \(\{I_1, \ldots, I_K\}\) of \(\{1, \ldots, n\}\) and we group together terms which give the same partition. Then we have
\[
\prod_{\{i,j\}} \zeta(Y_i, Y_j) = \sum_{K=1}^{n} \sum_{\{I_1, \ldots, I_K\} \in \pi_n, K} \prod_{k=1}^{K} \rho^T(Y_{I_k})
\] (316)

where \(\pi_n, K\) is the partitions of \(\{1, \ldots, n\}\) into \(K\) subsets. We have defined \(\rho^T(Y) = 1\) and for \(n \geq 2\)
\[
\rho^T(Y_1, \ldots, Y_n) = \sum_G \prod_{\{i,j\} \in G} (\zeta(Y_i, Y_j) - 1)
\] (317)

where the sum is now over connected graphs \(G\) on \(\{1, \ldots, n\}\). We do have \(\rho^T(Y_1, \ldots, Y_n) = 0\) if the \(Y_j\) can be divided into disjoint sets.
The result now follows from \( \sum_{\{Y_i\}} \prod_{i} K^\#(Y_i) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{I_1,\ldots,I_K} \prod_{k=1}^{K} K^\#(Y_i) \prod_{i \in I_k} K^\#(Y_i) \prod_{i=1}^{n} f(T(Y_i)) \)

\[= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{K=1}^{n} \prod_{k=1}^{K} \rho^T(Y_{i_k}) \prod_{i \in I_k} K^\#(Y_i) \prod_{i=1}^{n} f(|I_k|) \]  

(318)

In the last step we defined for \( N \geq 1 \)

\[f(N) = \sum_{\{Y_1,\ldots,Y_N\}} \prod_{i=1}^{N} K^\#(Y_i) \]  

(319)

Replace the sum over partitions \( \{I_1,\ldots,I_K\} \) by a sum over ordered partitions \( (I_1,\ldots,I_K) \). The summand only depends on the number of elements \( N_k = |I_k| \) in each set. For each \( (N_1,\ldots,N_K) \) with \( N_k \geq 1 \) the number of partitions with these numbers is \( n!/N_1! \ldots N_K! \). Thus we have

\[\sum_{\{I_1,\ldots,I_K\} \in \pi_{n,K}} \prod_{k=1}^{K} f(|I_k|) = \frac{1}{K!} \sum_{(I_1,\ldots,I_K) \in \pi_{n,K}} \prod_{k=1}^{K} f(|I_k|) \]

(320)

\[= \frac{1}{K!} \sum_{(N_1,\ldots,N_K)} \frac{n!}{N_1! \ldots N_K!} \prod_{k=1}^{K} f(N_k) \]

Insert this into (318) and change the order of summations

\[\sum_{\{Y_i\}} \prod_{i} K^\#(Y_i) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{(N_1,\ldots,N_K)} \frac{n!}{N_1! \ldots N_K!} \prod_{k=1}^{K} f(N_k) \]

(321)

\[= 1 + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{N_1,\ldots,N_K} \frac{1}{N_1! \ldots N_K!} \prod_{k=1}^{K} f(N_k) \]

\[= 1 + \sum_{K=1}^{\infty} \frac{1}{K!} \left( \sum_{N=1}^{\infty} \frac{1}{N!} f(N) \right)^K = \exp \left( \sum_{N=1}^{\infty} \frac{1}{N!} f(N) \right) \]

The result now follows from

\[\sum_{N=1}^{\infty} \frac{1}{N!} f(N) = \sum_{Y} H^\#(Y) \]

(322)

**step 4:** We now demonstrate that under our assumptions the series (312) defining \( H^\# \) converges.

Each indivisible \( n \)-tuple \( (Y_1,\ldots,Y_n) \) determines a connected graph \( g \) on \( (1,\ldots,n) \): a pair \( \{i,j\} \in g \) if \( Y_i \cap Y_j \neq \emptyset \). We write \( (Y_1,\ldots,Y_n) \to g \) the expression \( \rho^T(Y_1,\ldots,Y_n) \) only depends on the graph (it is a certain sum over subgraphs) and one can show that

\[|\rho^T(Y_1,\ldots,Y_n)| \leq \text{number of tree graphs contained in } g \]  

(323)
Now fix \(n\) and \(Y\) and let us restrict to sums over \((Y_1, \ldots, Y_n)\) such that \(\cup_i Y_i = Y\).

\[
\sum_{(Y_1, \ldots, Y_n)} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i) \leq \sum_g \sum_{(Y_1, \ldots, Y_n) \rightarrow g} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i)
\]

\[
\leq \sum_g \sum_{(Y_1, \ldots, Y_n) \rightarrow g} \sum_{\tau \subseteq g} \prod_i K^\#(Y_i)
\]

\[
= \sum \sum_{\tau \supseteq g} \sum_{(Y_1, \ldots, Y_n) \rightarrow g} \prod_i K^\#(Y_i)
\]

\[
= \sum \sum_{(Y_1, \ldots, Y_n) \rightarrow g : g \supseteq \tau} \prod_i K^\#(Y_i)
\]

(324)

If \(\kappa_2 = \kappa - \kappa_0 - 1\) we have by (310) the bound

\[
\left| \sum_{(Y_1, \ldots, Y_n)} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i) \right| \leq (\mathcal{O}(1) H_0)^n \sum_{\tau} \sum_{(Y_1, \ldots, Y_n) \rightarrow g : g \supseteq \tau} \prod_i e^{-\kappa_2 d_M(Y_i)}
\]

(325)

Next use the inequality (305) to bound this by

\[
e^{-\kappa_2 \mathcal{O}(1) d_M(Y)} (\mathcal{O}(1) H_0)^n \sum_{\tau} \sum_{(Y_1, \ldots, Y_n) \rightarrow g : g \supseteq \tau} \prod_i e^{-\kappa_0 d_M(Y_i)}
\]

(326)

After relabeling a tree graph \(\tau\) can be thought of as a map \(\tau\) from \((1, \ldots, n)\) to itself such that \(\tau(j) < j\). The restrictions in the sum over \((Y_1, \ldots, Y_n)\) are then that \(Y_{\tau(j)} \cap Y_{\tau(j)} \neq \emptyset\). For the sum over \(Y_n\) we have by (294)

\[
\sum_{Y_n \cap Y_{\tau(n)} \neq \emptyset} e^{-\kappa_0 d_M(Y_n)} \leq K_0|Y_{\tau(n)}| M
\]

(327)

Continue summing over \(Y_{n-1}, Y_{n-2}, \ldots\). By the time we get to the \(j^{th}\) vertex we will have accumulated a factor \(|Y_j|^{d_{j-1}} = |Y_j| |d_{j-1}|\) where \(d_j\) is the incidence number at \(j\) for \(\tau\). Then we estimate

\[
\sum_{Y_{\tau(j)} \neq \emptyset} e^{-\kappa_0 d_M(Y_j)} |Y_j|^{d_{j-1}} \leq (d_j - 1)! \sum_{Y_{\tau(j)} \neq \emptyset} e^{-\kappa_0 d_M(Y_j)} e|Y_j| M
\]

(328)

Here we used again \(|Y_j| M \leq \kappa_0(1 + d_M(Y_j))\) The last step is

\[
\sum_{Y_{j} \subseteq Y} e^{-\kappa_0 d_M(Y_j)} |Y_j|^{d_{j-1}} \leq (d_j - 1)! \sum_{Y_{j} \subseteq Y} e^{-\kappa_0 d_M(Y_j)} e|Y_j| M
\]

(329)

Combining the above yields

\[
\left| \sum_{(Y_1, \ldots, Y_n)} \rho^T(Y_1, \ldots, Y_n) \prod_i K^\#(Y_i) \right| \leq e^{-\kappa_2 \mathcal{O}(1) d_M(Y)} (\mathcal{O}(1) H_0)^n \sum_{\tau} \prod_{j=1}^{n} (d_j - 1)!
\]

(330)

By Cayley’s theorem the number of trees with incidence numbers \(d_j\) is \((n - 2)! / \prod_{j=1}^{n} (d_j - 1)!\) so we have

\[
\sum_{\tau} \prod_{j=1}^{n} (d_j - 1)! = \sum_{d_1, \ldots, d_n} \left( \prod_{j=1}^{n} (d_j - 1)! \right) \sum_{\tau \text{ with } d_j} 1 \leq \sum_{d_1, \ldots, d_n} (n - 2)! \leq (n - 2)! 4^{n-1}
\]

(331)
In the last step we used that a tree graph has \( n - 1 \) lines so
\[
\sum_{d_1, \ldots, d_n: \sum_j (d_j - 1) = n - 2} 1 \leq 2^{n-2} \sum_{(d_1, \ldots, d_n)} 2^{-\sum_j (d_j - 1)} \leq 2^{n-2} 2^n \leq 4^{n-1}
\] (332)

Now use (331) in (330), divide by \( n! \) and sum over \( n \) to get a bound on \( H^\#(Y) \). We have
\[
\left| H^\#(Y) \right| \leq e^{-((\kappa_2 - 2\kappa_0 - 1))d_M(Y)} \sum_{n=1}^{\infty} (O(1)H_0)^n \leq O(1)H_0e^{-(\kappa_2 - 2\kappa_0 - 1)d_M(Y)}
\] (333)

provided \( H_0 \leq c_0 \) and \( c_0 \) is sufficiently small. Since \( \kappa_2 - 2\kappa_0 - 1 = \kappa - 3\kappa_0 - 3 \) this completes the proof.

C  an identity

We seek an expression for \( C_{k, r} = \left( \Delta_k + \frac{a_k^2}{Q_k} Q^T Q + r \right)^{-1} \)

Lemma 28.
\[
C_{k, r} = A_{k, r} + a_k^2 A_{k, r} Q_k G_{k, r} Q_k^T A_{k, r}
\] (334)

where
\[
A_{k, r} = \frac{1}{a_k + r} (I - Q_k^T Q) + \frac{1}{a_k + aL^{-2} + r} Q^T Q
\]
\[
G_{k, r} = \left( -\Delta + \bar{\mu}_k + a_k Q_k^T Q_k - a_k^2 Q_k^T A_{k, r} Q_k \right)^{-1}
\] (335)

Proof. Start with
\[
\exp(\frac{1}{2} \langle f, C_{k, r} f \rangle) = \text{const} \int d\Phi \exp \left( < \Phi, f > - \frac{a_k}{2L^2} \| Q\Phi \|^2 - \frac{r}{2} \| \Phi \|^2 - \frac{1}{2} < \Phi, \Delta_k \Phi > \right)
\] (336)

and from section 2.2
\[
\exp \left( -\frac{1}{2} < \Phi, \Delta_k \Phi > \right) = \text{const} \int \exp \left( -\frac{a_k}{2} \| \Phi - Q_k \phi \|^2 - \frac{1}{2} < \phi, (-\Delta + \bar{\mu}_k) \phi > \right) d\phi
\] (337)

Insert the second into the first and do the integral over \( \Phi \) which is
\[
\int d\Phi \exp \left( < \Phi, f > - \frac{a_k}{2L^2} \| Q\Phi \|^2 - \frac{r}{2} \| \Phi \|^2 - \frac{a_k}{2} \| \Phi - Q_k \phi \|^2 \right)
\]
\[
= \int d\Phi \exp \left( < \Phi, f + a_k Q_k \phi > - \frac{1}{2} < \Phi, (a_k + r + aL^{-2} Q_k^T Q_k) \Phi > - \frac{a_k}{2} \| Q_k \phi \|^2 \right)
\]
\[
= \text{const} \exp \left( \frac{1}{2} \left( \langle f + a_k Q_k \phi, A_{k, r} (f + a_k Q_k \phi) \rangle - \frac{a_k}{2} \| Q_k \phi \|^2 \right) \right)
\] (338)

Here we used \( (a_k + r + aL^{-2} Q_k^T Q_k)^{-1} = A_{k, r} \) which follows since \( Q_k^T Q_k \) is a projection. Hence
\[
\exp(\frac{1}{2} \langle f, C_{k, r} f \rangle) = \text{const} \int \exp \left( \frac{1}{2} \left( \langle f + a_k Q_k \phi, A_{k, r} (f + a_k Q_k \phi) \rangle - \frac{1}{2} < \phi, (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k) \phi > \right) d\phi \right)
\]
\[
= \text{const} \exp \left( \frac{1}{2} \left( \langle f, A_{k, r} f \rangle \right) \left( \langle \phi, a_k Q_k^T A_{k, r} f \rangle - \frac{1}{2} < \phi, G_{k, r}^{-1} A_{k, r} f > \right) \right)
\]
\[
= \text{const} \exp \left( \frac{1}{2} \langle f, A_{k, r} f \rangle + \frac{a_k^2}{2} < f, A_{k, r} Q_k G_{k, r} Q_k^T A_{k, r} f > \right)
\]
44
Estimate on $G_k(\tilde{\Box})$

As in the text let $\Box$ be a $M = L^m$-cube in a partition of $T_{M+N-k}$, let $\tilde{\Box}$ be a $3M$ enlargement of $\Box$, and let $G_k(\tilde{\Box}) = [-\Delta + \bar{\mu}_k + a_kQ_k^TQ_k]_\Omega^{-1}$ be the Green’s functions on $\tilde{\Box}$ with Neumann boundary conditions. We sketch a proof of the bounds (80), (81) which say for $x, x' \in \Delta_y, \text{supp} f \subset \Delta_y'$

\[
|\langle (G_k(\tilde{\Box})f)(x) \rangle| \leq C e^{-\gamma_0 d(y,y')} \|f\|_\infty
\]

\[
|\langle \partial G_k(\tilde{\Box})f)(x) \rangle| \leq C e^{-\gamma_0 d(y,y')} \|f\|_\infty
\]

\[
|\langle \delta_\alpha \partial G_k(\tilde{\Box})f)(x) \rangle| \leq C e^{-\gamma_0 d(y,y')} \|f\|_\infty
\]

The proof follows [4] with improvements suggested by [19].

We consider the more general case

\[
G_k(\Omega) = [-\Delta + \bar{\mu} + a_kQ_k^TQ_k]^{-1}_\Omega
\]

where $\Omega \subset T_{M+N-k}$ is a union of $M$ cubes, and we impose Neumann boundary conditions. Eventually we want $\Omega$ to be rectangular, but for the first results it is any union of $M$ cubes. Another restriction is that $\Omega$ should be small enough so that it can be identified with a subset of $(L^{-k}Z)^3$ with the same distances. Then we can study $G_k(\Omega)$ as an operator on functions on $\Omega \subset (L^{-k}Z)^3$. We want pointwise bounds, but start with $L^2(\Omega)$ bounds.

**Lemma 29.** The following holds for a constant $c_0 = O(1)$.

1. For a unit cube $\Delta$, as operators on $L^2(\Delta)$

\[
\left[ -\Delta + \bar{\mu}_k + a_kQ_k^TQ_k \right]_\Delta \geq c_0(-\Delta + I)
\]

2. For $\Omega$ a union of $M$ cubes, as operators on $L^2(\Omega)$

\[
\left[ -\Delta + \bar{\mu}_k + a_kQ_k^TQ_k \right]_\Omega \geq c_0(-\Delta + I)
\]

**Remark.** The idea is that the averaging operator $a_kQ_k^TQ_k$ supplies an effective mass. The parameter $\bar{\mu}_k = L^{-2(N-k)}\bar{\mu}$ is generally tiny and cannot help with this uniform bound. However if $k$ is a bounded distance from $N$ then $\bar{\mu}_k$ is not tiny. Then the $a_kQ_k^TQ_k$ is unnecessary and the $\bar{\mu}_k$ is sufficient for a lower bound.

**Proof.** If $f \in L^2(\Delta)$ is constant we have for a constant $c_0 = O(1)$

\[
< f, a_k|Q_k^TQ_k|f > = a_k\|f\|^2 = O(1)\|f\|^2
\]

If $f \in L^2(\Delta)$ is orthogonal to constants, then since the lowest non-zero eigenvalue of $-\Delta$ is $O(1)$ we have

\[
< f, [-\Delta]f > \geq O(1)\|f\|^2
\]

These combine to give

\[
< f, \left[ -\frac{1}{2}\Delta + a_kQ_k^TQ_k \right]f > \geq O(1)\|f\|^2
\]

which suffices to prove the first inequality.
For the second inequality we have for $f \in L^2(\Omega)$
\[
\left\langle f, \left[-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k\right] f \right\rangle \geq c_0 \sum_{\Delta \subset \Omega} \| f_{\Delta} \|^2 \geq c_0 \| f \|^2
\] (347)

Here in the first inequality we take advantage of the Neumann boundary conditions and drop bonds connecting adjacent unit squares. This completes the proof.

Now we consider $G_k(\Omega) = [-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k]^{-1}$. The lemma implies that
\[
\| G_k(\Omega) f \|_2, \| \partial G_k(\Omega) f \|_2 \leq O(1) \| f \|_2
\] (348)

The next result improves this.

**Lemma 30.** Let $\text{supp} f \subset \Delta_1, \text{supp} f' \subset \Delta_2$ with $\Delta_1, \Delta_2 \subset \Omega$. Then with $\delta_0 = O(1)$ we have
\[
| < f, G_k(\Omega) f' > | \leq O(1) e^{-\delta_0 |y-y'|} \| f \|_2 \| f' \|_2
\] (349)

**Remark.** This result also holds for Dirichlet or mixed boundary conditions.

**Proof.** (1.) For $q \in \mathbb{R}^3$ let $e_q$ be the exponential function $e_q(x) = e^{ix}$. For $|q| \leq 1$ we consider the operator
\[
D_q \equiv e_{-q} \left[-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k\right] e_q
\] (350)

We claim that there is a constant $c_1 = O(1)$ such that for $f \in L^2(\Omega)$
\[
| < f, [D_q - D_0] f > | \leq c_1 \| f \|_2 \| (-\Delta + I) f > |
\] (351)

There are two terms to consider. One is $< f, [e^{-q} Q_k^T Q_k e^q - Q_k^T Q_k] f >$. If we define for $y \in \Omega \cap \mathbb{Z}^3$
\[
(Q_{k-q})(y) = \int_{|x-y| < \frac{1}{2}} e^{i(x-y)} f(x) dx
\] (352)

then the identities $Q_k e_q = e_0 Q_{k-q}$ and $e_{-q} Q_k^T = Q_k^T e_{-q}$ hold. Then this term can be written $< f, [Q_{k-q}^T Q_k - Q_k^T Q_k] f >$. It is straightforward to establish $\| Q_{k-q} f - Q_k f \|_2 \leq O(1) |q| \| f \|_2$ and it follows that
\[
| < f, [Q_{k-q}^T Q_k - Q_k^T Q_k] f > | \leq O(1) |q| \| f \|_2^2
\] (353)

which is sufficient.

The other term is $< f, [e^{-q} (-\Delta) e^q - (-\Delta)] f >$, also written as $< \partial e^{-q} f, \partial e^q f > - < \partial f, \partial f >$. If we define $\partial_q = e^{-q} \partial e^q$ then it is $< \partial_{-q} f, \partial_q f > - < \partial f, \partial f >$. It is straightforward to show
\[
\| (\partial_q - \partial) f \|_2 \leq O(1) |q| \| f \|_2
\] (354)

Then (351) is established by
\[
| < \partial_{-q} f, \partial_q f > - < \partial f, \partial f > | \leq \| (\partial_{-q} - \partial) f \|_2 + O(1) |q| \| f \|_2 + \| \partial f \|_2 \| (\partial_q - \partial) f \|_2 \leq O(1) |q| \| f \|_2 = O(1) |q| < f, (-\Delta + I) f >
\] (355)
(2.) Now we write
\[< f, D_q f > = < f, D_0 f > + < f, [D_q - D_0] f > \] (356)

By the previous lemma \( < f, D_q f > \geq c_0 < f, (-\Delta + I) f > \). Combining this with \[551\] we conclude that for \( |q| \leq \frac{1}{2} c_1^{-1} c_0 \)
\[| < f, D_q f > | \geq \frac{1}{2} c_0 < f, (-\Delta + I) f > \geq \frac{1}{2} c_0 \| f \|_2^2 \] (357)

Now substitute \( f = D_q^{-1} h \) and get
\[| < h, D_q^{-1} h > | \geq \frac{1}{2} c_0 \| D_q^{-1} h \|_2^2 \] which implies \( \| D_q^{-1} h \|_2 \leq 2 c_0^{-1} \| h \| \).
Since \( D_q^{-1} = e_q G_k(\Omega) e_q \) this reads
\[\| e_q G_k(\Omega) e_q h \|_2 \leq O(1) \| h \|_2 \] (358)

Now let \( \delta_0 = \min \left\{ \frac{1}{2} c_1^{-1} c_0, 1 \right\} \). Then for \( |q| \leq \delta_0 \) and \( \text{supp} f \subset \Delta_y, \text{supp} f' \subset \Delta_{y'} \)
\[| < f, G_k(\Omega) f' > | = | < e_q f, e_q G_k(\Omega) e_q f' > | \leq O(1) \| e_q f \|_2 \| e_q f' \| \leq O(1) e^{\gamma (y - y')} \] (359)

Here we used that \( \| e_q f \|_2 \leq O(1) e^{\gamma y \| y - y' \|} \). Take \( q = \delta_0 \| - (y - y') / |y - y'| \| \) and get the bound
\( O(1) e^{-\delta_0 |y - y'|} \). This completes the proof.

We continue to assume \( \Omega \) be a union of \( M \) cubes in \( (L^{-k} \mathbb{Z})^3 \), and consider the operator
\[\Delta_k(\Omega) = a_k - a_k^2 Q_k G_k(\Omega) Q_k^T \] (360)
defined on \( \Omega \cap \mathbb{Z}^3 \). This is a local version of the global operator \( \Delta_k = a_k - a_k^2 Q_k G_k Q_k^T \) considered in the text. We study the inverse
\[C_k(\Omega) = \left[ \Delta_k(\Omega) + \frac{a}{L^2} Q^T Q \right]^{-1} \] (361)

By a variation of the identity \[334\] at \( r = 0 \) and with everything restricted to \( \Omega \) we have
\[C_k(\Omega) = A_k + a_k^2 A_k Q_k G_{k+1}^0(\Omega) Q_k^T A_k \] (362)
where
\[G_{k+1}^0(\Omega) = \left[ -\Delta + \mu_k + \frac{a}{L^2} Q_k^T Q_{k+1} \right]^{-1} \] \( \Omega \)
\[A_k(\Omega) = \left[ a_k + \frac{a}{L^2} Q_k^T Q \right]^{-1} \] \( \Omega \)

For \( \text{supp} f \subset \Delta_y, \text{supp} f' \subset \Delta_{y'} \)
\[| < f, G_{k+1}^0(\Omega) f' > | \leq O(1) L^2 e^{-\delta_0 L^{-2} d(y, y')} \| f \|_2 \| f' \|_2 \] (364)

This follows by scaling up \[349\] for \( G_{k+1}(L^{-1} \Omega) \). Since \( A_k, Q_k \) are local operators, it follows that the kernel \( C_k(\Omega; y, y') = < \delta_{y'}, C_k(\Omega) \delta_y > \) satisfies
\[| C_k(\Omega; y, y') | \leq O(1) L^2 e^{-\delta_0 L^{-2} d(y, y')} \] (365)

Now let \( \Omega \) be a rectangular box in \( (L^{-k} \mathbb{Z})^3 \) which is a union of \( M \) cubes. Consider the operator \( \mathcal{H}_k(\Omega) \) from functions on \( \Omega \cap \mathbb{Z}^3 \) to functions on \( \Omega \) defined by
\[\mathcal{H}_k(\Omega) = a_k G_k(\Omega) Q_k^T \] (366)
Lemma 31. The kernel $H_k(\Omega; x, y) = \left( H_k(\Omega)\delta_y \right)(x)$ satisfies for $\delta_1 = O(1)$

$$|H_k(\Omega; x, y)| \leq O(1)e^{-\delta_1 d(x, y)}$$
$$|\partial H_k(\Omega; x, y)| \leq O(1)e^{-\delta_1 d(x, y)}$$
$$|\delta_1 \partial H_k(\Omega; x, x', y)| \leq O(1)e^{-\delta_1 d(x, x', y)}$$  \hspace{1cm} (367)

Remark. $H_k(\Omega)$ is easier to treat than $G_k(\Omega)$ since it has no short distance singularity.

Proof. [4]. First establish the result by Fourier series on the whole lattice. Then extend the result to $\Omega$ by multiple reflections.

Lemma 32. For $\Omega$ a rectangular union of $M$ cubes define

$$C_k'(\Omega) = H_k(\Omega)C_k(\Omega)H_k^T(\Omega)$$  \hspace{1cm} (368)

Then with $\gamma_0 \equiv \frac{1}{2}\delta_0L^{-2} < \delta_1$

$$|(C_k'(\Omega)f)(x)| \leq Ce^{-\gamma_0 d(x, \text{supp } f)}\|f\|_{\infty}$$  \hspace{1cm} (369)

with the same bound for $\partial C_k'(\Omega)$ and $\delta_1 \partial C_k'(\Omega)$.

Proof. We have

$$(C_k'(\Omega)f)(x) = \sum_{y, y'} H_k(\Omega)(x, y)C_k(\Omega; y, y')(H_k^T(\Omega)f)(y')$$  \hspace{1cm} (370)

By (365) and (367)

$$|(C_k'(\Omega)f)(x)| \leq O(1)L^2 \sum_{y, y'} e^{-\delta_1 d(x, y)}e^{-\delta_0L^{-1}d(y, y')}e^{-\delta_1 d(y', \text{supp } f)}\|f\|_{\infty}$$
$$\leq O(1)L^2 e^{-\gamma_0 d(y, \text{supp } f)} \sum_{y, y'} e^{-\frac{1}{2}\delta_1 d(x, y)}e^{-\frac{1}{2}\delta_0L^{-2}d(y, y')}\|f\|_{\infty}$$
$$\leq Ce^{-\gamma_0 d(y, \text{supp } f)}\|f\|_{\infty}$$ \hspace{1cm} (371)

Now our main result is:

Lemma 33. Let $\Omega$ be a rectangular union of $M$ cubes. Then with $\gamma_0 = O(L^{-2})$

$$|(G_k(\Omega)f)(x)| \leq Ce^{-\gamma_0 d(x, \text{supp } f)}\|f\|_{\infty}$$
$$|(\partial G_k(\Omega)f)(x)| \leq Ce^{-\gamma_0 d(x, \text{supp } f)}\|f\|_{\infty}$$
$$|\delta_1 \partial G_k(\Omega)f(x, x')| \leq Ce^{-\gamma_0 d(x, x', \text{supp } f)}\|f\|_{\infty}$$  \hspace{1cm} (372)

Proof. Let $f_L(x) = f(x/L)$ in this proof only. The proof is based on the identity (see [4] or [34])

$$(G_k(\Omega)f)(x) = \sum_{j=0}^{k-1} L^{-2(k-j)} \left( C_j'(L^{k-j}\Omega)f_{L^{k-j}} \right)(L^{k-j}x)$$ \hspace{1cm} (373)
From lemma 32
\[ |\left( C_j^\ast (L^{k-j} \Omega) f_{L^{k-j}} \right)(L^{k-j} x) | \leq C e^{-\gamma_0 L^{-1} d(L^{k-j} x, \text{supp} f_{L^{k-j}})} \| f \|_\infty \]
\[ = C e^{-\gamma_0 L^{k-1-j} d(x, \text{supp} f)} \| f \|_\infty \] (374)

Therefore
\[ |(G_k(\Omega) f)(x)| \leq C \sum_{j=0}^{k-1} L^{-2(k-j)} e^{-\gamma_0 L^{k-1-j} d(x, \text{supp} f)} \| f \|_\infty \leq C e^{-\gamma_0 d(x, \text{supp} f)} \| f \|_\infty \] (375)

The other bounds are similar.

E Random walk expansion for $G_{k,r}$

We want a random walk expansion for $G_{k,r}$ on $\mathbb{T}^{k}_{M+n-k}$ as defined in (196). For this operator $L^2$ bounds are sufficient.

To begin we get a local result and consider for $\Omega \subset (L^{-k} \mathbb{Z})^3$, a union of $M$ cubes, the operator
\[ G_{k,r}(\Omega) = \left[ -\Delta + \tilde{\mu} + \frac{akr}{a_k + r} Q_k^T Q_k + \frac{a_k^2 aL^{-2}}{(a_k + r)(a_k + aL^{-2} + r)} Q^T_{k+1} Q_{k+1} \right]^{-1} \] (376)

**Lemma 34.** For unit cubes $\Delta_y, \Delta_{y'} \subset \Omega$ and $r \geq 0$
\[ \| 1_{\Delta_y} G_{k,r}(\Omega) 1_{\Delta_{y'}} f \|_2 \leq C e^{-\delta_0 L^{-2} d(y, y')} \| f \|_2 \]
\[ \| 1_{\Delta_y} \partial G_{k,r}(\Omega) 1_{\Delta_{y'}} f \|_2 \leq C e^{-\delta_0 L^{-2} d(y, y')} \| f \|_2 \] (377)

**Proof.** We follow the proofs of lemma 29, lemma 30. First we claim that there is a constant, again called $c_0$, such that as operators on $L^2(\Omega)$
\[ \left[ -\Delta + \tilde{\mu} + \frac{akr}{a_k + r} Q_k^T Q_k + \frac{a_k^2 aL^{-2}}{(a_k + r)(a_k + aL^{-2} + r)} Q^T_{k+1} Q_{k+1} \right] \geq c_0 (-\Delta + L^{-2}) \] (378)

If $r \geq 1$ just drop the second and fourth terms and get the lower bound $c_0 (-\Delta + I)$ from lemma 29. If $0 \leq r \leq 1$ drop the second and third terms and look for a lower bound on $-\Delta + a_k^2 L^{-2} Q^T_{k+1} Q_{k+1}$. Argue just as in lemma 29 but now using $L \Delta$ cubes instead of unit cubes $\Delta$, and get the lower bound $O(1)(-\Delta + L^{-2})$.

It follows that
\[ \| G_{k,r}(\Omega) f \|_2, \| \partial G_{k,r}(\Omega) f \|_2 \leq O(1) L^2 \| f \|_2 \] (379)

Continuing as in lemma 30, $D_q$ is replaced by
\[ D_{q,r} = e^{-\theta} \left[ -\Delta + \tilde{\mu} + \frac{akr}{a_k + r} Q_k^T Q_k + \frac{a_k^2 aL^{-2}}{(a_k + r)(a_k + aL^{-2} + r)} Q^T_{k+1} Q_{k+1} \right] e^\theta \] (380)

As in (351) there is a constant, again called $c_1$, such that for $|q| < L^{-1}$
\[ |<f, [D_{q,r} - D_{0,r}] f>| \leq c_1 |q| |<f, (-\Delta + I) f>| \] (381)

Then for $|q| \leq \frac{1}{2} c_1 c_0 L^{-2} = \delta_0 L^{-2}$ we have $|q| < L^{-1}$ and
\[ |<f, D_{q,r} f>| \geq \frac{1}{2} c_0 |<f, (-\Delta + L^{-2}) f>| \geq \frac{1}{2} c_0 L^{-2} \| f \|_2^2 \] (382)
As in Lemma 30, the last bound implies that $D_{q,r}^{-1} = e^{-q}G_{k,r}^{-1}$ satisfies
\[
\|D_{q,r}^{-1}f\|_2 = \|q^{-q}G_{k,r}^{-1}f\|_2 \leq C\|f\|_2
\] (383)
This leads to a bound of the form (339). But here we formulate it a little differently and write
\[
\|1_{\Delta^y}G_{k,r}(\Omega)1_{\Delta^y'}f\|_2 \leq O(1)\|e^{-q}G_{k,r}(\Omega)e^{-q}1_{\Delta^y}e^{-q}f\|_2 \leq Ce^{q\gamma(y-y')}\|1_{\Delta^y}e^{-q}f\|_2 \leq Ce^{q\gamma(y-y')} (384)
\]
With the choice $q = \delta_0 L^{-2} |(y-y')/(y-y')|$ we obtain the first result in (377).
For the second result start with
\[
\|\partial f\|_2 = |< f, (-\Delta) f > - O(1) | < f, D_{q,r}f > | (385)
\]
from (382). Let $f = D_{q,r}^{-1}h$ and get $\|\partial D_{q,r}^{-1}h\|_2^2 \leq O(1) |h, D_{q,r}^{-1}h| \leq C\|h\|_2^2$. We also have by (383) and (388) that $\|\partial f - \partial h\|_2^2 \leq C\|h\|_2$. The last two combine to give $\|\partial f, D_{q,r}^{-1}h\|_2 \leq C\|h\|_2$ or
\[
\|e^{-q}(\partial G_{k,r}) e^{-q}h\|_2 \leq C\|h\|_2 (386)
\]
Now argue as in (384) to get the second result in (377).

**Lemma 35.** $G_{k,r}$ on $\mathbb{T}^k_{M+N-k}$ has a random walk expansion of the form $G_{k,r} = \sum_{\omega} G_{k,r,\omega}$ which converges in $L^2$ norm for $M$ sufficiently large. It yields the bounds for $r \geq 0$:
\[
\|1_{\Delta^y}G_{k,r,1_{\Delta^y'}}f\|_2 \leq Ce^{-\gamma_0 d(y,y')}\|f\|_2
\] (387)

**Proof.** This follows by a random walk expansion similar to lemma 6. As in (344) the random walk expansion has the form
\[
G_{k,r} = \sum_{n=0}^{\infty} \sum_{\omega_0,\omega_1,\ldots,\omega_n} \left(h_{\omega_0}G_{k,r,\omega_0}(\Box_{\omega_0}h_{\omega_0})\left(K_{r,\omega_1}G_{k,r,\omega_1}(\Box_{\omega_1}h_{\omega_1})\right)\ldots\left(K_{r,\omega_n}G_{k,r,\omega_n}(\Box_{\omega_n}h_{\omega_n})\right) (388)
\]
where $\Box$ is still an $M$-cube, and $\Box$ is an enlargement to a $3M$-cube. The operator $K_{r,z}$ is
\[
K_{r,z} = -\left[\left(-\Delta + \frac{a_k T}{a_k + r}Q_k T + \frac{a_k^2 a L^{-2}}{(a_k + r)(a_k + a L^{-2} + r)}Q_k T Q_k + T Q_k T Q_k T\right), h_z\right] (389)
\]
Estimates on this operator, together with the bounds (364), yield
\[
\|1_{\Delta^y}K_{r,z}G_{k,r,\omega_0}(\Box_{\omega_0}h_{\omega_0})1_{\Delta^y'}f\|_2 \leq CM^{-1}e^{-\gamma_0 d(y,y')}\|f\|_2 (390)
\]
This is sufficient to give convergence of the expansion in the $L^2$ norm for $M$ large, and the decay $e^{-\gamma_0 d(y,y')}$ with $\gamma_0 = \frac{1}{2}\delta_0 L^{-2}$.

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