Veli B. Shakhmurov  
Okan University, Department of Mechanical Engineering, Akfirat, Tuzla 34959  
Istanbul, Turkey, E-mail: veli.sahmurov@okan.edu.tr

Separability properties of singular degenerate abstract differential operators and applications

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Abstract

In this paper, we study the separability and spectral properties of singular degenerate elliptic equations in vector valued $L^p$ spaces. We prove that a realization operator by this equation with some boundary conditions is separable and Fredholm in $L^p$. The leading part of the associated differential operator is not self-adjoint. The sharp estimate of the resolvent, discreteness of spectrum and completeness of root elements of this operator is obtained. Moreover, we show that this operator is positive and generates a holomorphic $C_0$-semigroups on $L^p$. In application, we examine the regularity properties of degenerate elliptic problem with Wentzell–Robin boundary conditions and boundary value problem for system of degenerate elliptic equations of either finite or infinite number.

Key Words: Abstract function spaces, Separable differential operators; Spectral properties of differential operators; Degenerate differential equations; Differential-operator equations

1. Introduction, notations and background

In this work, boundary value problem (BVP) for singular degenerate abstract elliptic equations are considered. BVPs for abstract differential equations (ADEs) have been studied extensively by many researchers (see e.g. [1–3], [6–8], [10–19], [21–22] and the references therein). A comprehensive introduction to the ADEs and historical references may be found in [1] and [22]. The maximal regularity properties for differential operator equations have been investigated e.g. in [6–8], [14–18] and [21–22]. The main objective of the present paper is to discuss the BVP for the following singular degenerate DOE

$$
\sum_{k=1}^{n} -x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + \sum_{k=1}^{n} x_k^{\alpha_k} A_k(x) \frac{\partial u}{\partial x_k} + Au = f(x), \quad (1.1)
$$

where $A, A_k$ are linear operators in a Banach space $E$.

We derive $L_p$–separability properties and sharp resolvent estimates of the associated differential operator. Especially, we show that this differential operator is $R$-positive and also is a generator of an analytic semigroup.
By using separability properties of the elliptic problem (1.1) we derive spectral properties of differential operator $Q$ generated by (1.1). Namely, we prove that the operator $Q$ is Fredholm in $L_p$, the inverse $Q^{-1}$ belong to some Schatten class $\sigma_q(L_p)$ and the system of root functions of this operator is complete in $L_p$.

One of the most important aspects of this ADE considered here is that the degeneration in different directions is at different speeds, in general. Unlike the regular degenerate equations, because of the singularity of the degeneracy of the equation, the boundary conditions are only given on the lines without degeneracy.

In application, the BVP for infinity system of singular degenerate partial differential equations and Wentzell-Robin type BVP for singular degenerate partial differential equations on cylindrical domain are studied.

Since the Banach space $E$ is arbitrary and $A$ is a possible linear operator, by choosing $E$ and $A$ we can obtain numerous classis of degenerate elliptic and quasiequilibrium equations which have a different applications. Let we choose $E = L_2(0, 1)$ and $A$ to be differential operator providing the Wentzell-Robin boundary condition defined by

$$D(A) = \{ u \in W^2_2(0, 1), A(j)u(j) = 0, j = 0, 1 \},$$

$$Au = a(y)u^{(2)} + b(y)u^{(1)} \text{ for all } y \in (0, 1),$$

where $a$ is positive and $b$ is a real-valued functions on $(0, 1)$. By virtue of $L_p$-regularity properties of (1.1) (see Theorem 2.1) we obtain the separability properties of Wentzell-Robin type BVP for singular degenerate elliptic equation

$$\sum_{k=1}^n x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + a(y) \frac{\partial^2 u}{\partial y^2} + b(y) \frac{\partial u}{\partial y} = f(x, y), \quad x \in G, \quad y \in (0, 1), \quad (1.2)$$

$$L_ku = 0, \text{ for a.e. } y \in (0, 1),$$

$$A(j)u(j) = a(j)uyy(x, j) + b(j)uy(x, j) = 0, \quad (1.3)$$

$$j = 0, 1, \text{ for a.e. } x \in G,$$

in the mixed $L_p(\Omega)$ spaces, where $L_k$ are boundary conditions with respect $x \in G \subset R^n$ that will be definit in late and $L_p(\Omega)$ denotes the space of all $p$-summable complex-valued functions with mixed norm and

$$\Omega = G \times (0, 1), \quad p = (p, 2).$$

Note that, the regularity properties of Wentzell-Robin type problems for elliptic and parabolic equations were studied e.g. in [5, 9, 11] and the references therein.
Let $\gamma = \gamma (x)$ be a positive measurable function on a domain $\Omega \subset R^n$. Here, $L_{p,\gamma} (\Omega;E)$ denote the space of strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

$$\|f\|_{L_{p,\gamma}(\Omega;E)} = \left( \int \|f(x)\|_E^p \gamma (x) \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$ 

For $\gamma (x) \equiv 1$ the space $L_{p,\gamma} (\Omega;E)$ will be denoted by $L_p = L_p (\Omega;E)$.

The Banach space $E$ is called an UMD-space if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy$ is bounded in $L_p (R,E)$, $p \in (1, \infty)$ (see. e.g. [4]). UMD spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p, q \in (1, \infty)$.

Let $C$ be the set of the complex numbers and $S_\varphi = \{ \lambda; \lambda \in C, \arg \lambda \leq \varphi \} \cup \{ 0 \}, \ 0 \leq \varphi < \pi$.

Let $E_1$ and $E_2$ be two Banach spaces. $L (E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ into $E_2$. For $E_1 = E_2 = E$ it will be denoted by $L (E)$.

**Definition 1.** A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\|(A + \lambda I)^{-1}\|_{L(E)} \leq M(1 + |\lambda|)^{-\varphi}$ for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where $I$ is an identity operator in $E$. Sometimes $A + \lambda I$ will be denoted by $A + \lambda A$. It is known [20, §1.15.1] that a positive operator $A$ has well-defined fractional powers $A^\alpha$.

**Remark 1.1.** By virtue of [20, §1.13] if $A$ is $\varphi$-positive in $E$, then the operator $-A^\alpha$ generate an analytic semigroup $U_{A^\alpha} (t)$ for $0 < \alpha \leq 1, \varphi \geq \frac{\pi}{2}$ and for $\alpha \leq \frac{1}{2}, \varphi < \frac{\pi}{2}$. Moreover, there exists a positive constant $\omega$ such that the estimate holds

$$\|U_{A^\alpha} (t)\|_{L(E)} \leq Me^{-\omega t}.$$ 

Let $E (A^\theta)$ denote the space $D (A^\theta)$ equipped with the norm

$$\|u\|_{E(A^\theta)} = \left( \|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \theta < \infty.$$ 

Let $E_1$ and $E_2$ be two Banach spaces. Now $(E_1, E_2)_\theta, 0 < \theta < 1, 1 \leq p < \infty$ will denote interpolation spaces obtained from $(E_1, E_2)$ by the $K$ method [20, §1.3.1].

**Definition 2.** Let $\mathbb{N}$ denote the set of natural numbers and $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. A set $K \subset L (E_1, E_2)$ is called $R$-bounded if there is a positive constant $C$ such that for all $T_1, T_2, \ldots, T_m \in K$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j (y) T_j u_j \right\|_{E_2} \, dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j (y) u_j \right\|_{E_1} \, dy.$$ 

3
The smallest $C$ for which the above estimate holds is called a $R$-bound of the collection $K$ and denoted by $R(K)$.

**Definition 3.** The $\varphi$-positive operator $A$ is said to be $R$-positive in $E$ if the set \( \{ \lambda (A + \lambda I)^{-1} : \lambda \in S_\varphi \} \), $0 \leq \varphi < \pi$ is $R$-bounded.

Let $E_1$ and $E_2$ be two Banach spaces. $\sigma_\infty (E_1, E_2)$ denotes the space of all compact operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $\sigma_\infty (E)$.

$s_j (A)$ will denote approximation numbers of operator $A$ [20, § 1.16.1]. Let

\[
\sigma_q (E_1, E_2) = \left\{ A \in \sigma_\infty (E_1, E_2), \sum_{j=1}^{\infty} s_j^q (A) < \infty, \ 1 \leq q < \infty \right\}.
\]

Here, $\Omega$ is a domain in $\mathbb{R}^n$. Assume $E_0$ and $E$ are two Banach spaces so that $E_0$ is continuously and densely embedded into $E$. Let $\gamma_k = \gamma_k (x)$ be a positive measurable functions on $\Omega$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$. Consider, the Sobolev-Lions type space $W_{p,\gamma}^m (\Omega; E_0, E)$, i.e. the space consisting of all functions $u \in L_p (\Omega; E_0)$ that have generalized derivatives $D_k^m u = \frac{\partial^m u}{\partial x_k^m} \in L_p, \gamma_k (\Omega; E)$ equipped with the norm

\[
\| u \|_{W_{p,\gamma}^m (\Omega; E_0, E)} = \| u \|_{L_p (\Omega; E_0)} + \sum_{k=1}^{n} \left\| \gamma_k^m \frac{\partial^m u}{\partial x_k^m} \right\|_{L_p (\Omega; E)} < \infty.
\]

Let $\chi = \chi (t)$ be a positive measurable function on $(0, a)$ and

\[
u^i (t) = \left( \chi (t) \frac{d}{dt} \right)^i u (t).
\]

Consider the following $E-$valued weighted function spaces

\[
W_{p,\chi}^m (0, a; E_0, E) = \{ u; u \in L_p (0, a; E_0), u^m \in L_p (0, a; E) \},
\]

\[
\| u \|_{W_{p,\chi}^m} = \| u \|_{L_p (0, a; E_0)} + \| u^m \|_{L_p (0, a; E)} < \infty \}
\]

\[
W_{p,\chi}^m (0, a; E_0, E) = \{ u; u \in L_{p, \chi} (0, a; E_0), u^m \in L_{p, \chi} (0, a; E) \},
\]

\[
\| u \|_{W_{p,\chi}^m} = \| u \|_{L_{p, \chi} (0, a; E_0)} + \| u^m \|_{L_{p, \chi} (0, a; E)} < \infty \}
\]

Let

\[
\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ D^{[\alpha]} = D_1^{[\alpha_1]}D_2^{[\alpha_2]}...D_n^{[\alpha_n]}, \ D_k^{[\alpha]} = \left( \gamma_k (x) \frac{\partial}{\partial x_k} \right)^{[\alpha]}.
\]


Consider the space $W^{[m]}_{p,\gamma}(\Omega;E_0,E)$, consisting of all functions $u \in L_p(\Omega;E_0)$ that have generalized derivatives $D^m_k u \in L_p(\Omega;E)$ with the norm
\[
\|u\|_{W^{[m]}_{p,\gamma}(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^n \|D^m_k u\|_{L_p(\Omega;E)} < \infty.
\]

From [15, Theorem 1, Theorem 3] we obtain

**Theorem A.1.** Suppose the following conditions are satisfied:

1. $E$ is an UMD space and $A$ is an $R$-positive operator in $E$;
2. $\gamma_k(x) = x^{\nu_k}$, $\nu_k \in (1,p)$, $p \in (1,\infty)$, $m$ is an integer and $0 \leq \mu \leq 1 - \frac{|\alpha|}{m}$, $1 < p < \infty$;
3. $\Omega = \prod_{k=1}^n (0,a_k)$.

Then, the embedding $D^\alpha W^{m}_{p,\gamma}(\Omega;E(A),E) \subset L_p(\Omega;E(A^{1-|\alpha|-\mu}))$ is continuous. Moreover for all $h > 0$ with $h \leq h_0 < \infty$ and $u \in W^{m}_{p,\gamma}(\Omega;E(A),E)$ the following uniform estimate holds
\[
\|D^\alpha u\|_{L_p(\Omega;E(A^{1-|\alpha|-\mu}))} \leq h^\mu \|u\|_{W^{m}_{p,\gamma}(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega;E)},
\]

where
\[
\gamma_\alpha(x) = \prod_{k=1}^n x^{\alpha_k \nu_k k}.
\]

**Theorem A.2.** Assume the conditions of Theorem A.1 are satisfied. Moreover, suppose $a_k < \infty$ and $A^{-1}$ is a compact operator $E$. Then for $0 < \mu \leq 1 - \frac{|\alpha|}{m}$ the embedding $D^\alpha W^{m}_{p,\gamma}(\Omega;E(A),E) \subset L_p(\Omega;E(A^{1-\kappa-\mu}))$ is compact.

Let $I(E_0,E)$ denote the embedding operator from $E_0$ to $E$. By reasoning as in [14, Theorem 3.1] we have

**Theorem A.3.** Let $E$ be Banach spaces with base $\alpha_k \in (1,p)$ for $p \in (1,\infty)$ and $\alpha_k < m$. Suppose the embedding $E_0 \subset E$ is compact and $s_j(I(E_0,E)) \sim j^{-\frac{1}{\nu}}$, $\nu > 0$, $j = 1,2,\ldots,\infty$.

Then
\[
s_j(I(W^{m}_{p,\alpha}(G;E_0,E),L_p(G;E))) \sim j^{-\frac{1}{\nu m - \alpha_k}}, \quad \nu = \sum_{k=1}^n \frac{1}{m - \alpha_k}.
\]

Consider the BVP
\[
-u^{(2)}(t) + Au(t) = f(t),
\] (1.4)
\[
L_1 u = \sum_{i=0}^{m} \left[ \delta_i u^{(i)}(a) + \sum_{j=1}^{N} \nu_{ij} u^{(i)}(t_{ij}) \right] = 0,
\]
where \( m \in \{0, 1\} \); \( \delta, \nu_{ij} \), are complex numbers, \( t_{ij} \in (0, a) \) and \( A \) is a linear operator in \( E \).

**Condition 1.1.** Let the following conditions be satisfied:
1. \( E \) is a UMD space and \( A \) is a \( R \) positive operator in \( E \);
2. \( \delta_m \neq 0 \), \( \gamma(t) = t^\nu \), \( \nu \in (1, p) \) for \( p \in (1, \infty) \);
3. Here, \( t_0 = \min_j t_{ij}, U^{-1}A^{-1}(t_0) \in L(E) \) and
\[
\sum_{i=0}^{m} \left| \frac{\delta_i}{e^{-\omega(a-t_0)}} + \sum_{j=1}^{N} \frac{\nu_{ij}}{e^{-\omega(t_{ij}-t_0)}} \right| < |\nu_0|,
\]
where \( \omega \) is a positive constant defined in the Remark 1.1.

Let \( \gamma(t) = t^\nu \). In a similar way as in [16, Theorem 5.1] we obtain

**Theorem A.1.** Assume the Condition 1.1 are satisfied. Then, the problem (1.14) has a unique solution
\[
u \in W^{2,p,\gamma}(a, \infty; E(A), E)
\]
for all \( f \in L^{p,\gamma}(a, \infty; E), |\arg \lambda| \leq \varphi \) with sufficiently large \(|\lambda|\) and the uniform coercive estimate holds
\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{2}{p}} \left\| u^{(i)} \right\|_{L^{p,\gamma}(a, \infty; E)} + \left\| Au \right\|_{L^{p,\gamma}(a, \infty; E)} \leq C \left\| f \right\|_{L^{p,\gamma}(a, \infty; E)}.

2. **Singular degenerate abstract elliptic equations**

Consider the BVP for the following singular degenerate ADO
\[
\sum_{k=1}^{n} \left[ -x_{2k}^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + x_k^\alpha A_k(x) \frac{\partial u}{\partial x_k} \right] + Au + \lambda u = f(x), \quad x \in G, \quad \text{(2.1)}
\]
\[
L_k u = \sum_{i=0}^{m_k} \left[ \delta_{ki} u^{[i]}_{x_k}(a_k, x(k)) + \sum_{j=0}^{N_k} \nu_{kij} u^{[i]}_{x_k}(x_{kij}, x(k)) \right] = 0, \quad \text{(2.2)}
\]
where \( x(k) \in G_k, x_{kij} \in (0, a_k) \) and
\[
u^{[i]}_{x_k} = \left[ x_k^{\alpha_k} \frac{\partial}{\partial x_k} \right]^{i} u(x), \quad G = \prod_{k=1}^{n} (0, a_k), \quad G_k = \prod_{j \neq k} (0, a_j),
\]
\[
m_k \in \{0, 1\}, \quad x(k) = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad j, \quad k = 1, 2, \ldots, n.
\]
\( \delta_{ki}, \nu_{kij} \) are complex numbers, \( \lambda \) is a complex parameter, \( A \) and \( A_k(x) \) are linear operators in a Banach space \( E \).

Let we denote \( W^m_{p,\gamma}(\Omega;E(A),E) \) by \( W^m_{p,\gamma}(\Omega;E(A),E) \) for \( \gamma_k(x) = a_k^\nu \).

Condition 2.1. Assume the following conditions are satisfied:

(1) \( E \) is an UMD space and \( A \) is a \( R \)-positive operator in \( E \);
(2) \( \delta_{km} \neq 0, \alpha_k \in (1, p) \) for \( p \in (1, \infty) \) and \( k = 1, 2, \ldots, n \);
(3) Here, \( x_{k0} = \min_j x_{kij} \) and \( U^{-1}_A(x_{k0}) \in L(E) \). Moreover,

\[
\sum_{i=0}^m \left| \delta_{ki} \right| e^{-\omega(a_k-x_{k0})} + \sum_{j=1}^N \left| \nu_{kij} \right| e^{-\omega(x_{kij}-x_{k0})} \right| < \left| \nu_{k0} \right| ,
\]

where \( \omega \) is a positive constant defined in the Remark 1.1.

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \gamma_k(x) = a_k^\nu \). The main result is the following:

Theorem 2.1. Assume the Condition 2.1 are hold and for any \( \varepsilon > 0 \) there is a positive constant \( C(\varepsilon) \) such that

\[ \| A_k(x)u \| \leq \varepsilon \| u \|_{(E(A),E)\frac{1}{\varepsilon} \infty} + C(\varepsilon) \| u \| \text{ for } u \in (E(A),E)\frac{1}{\varepsilon} \infty. \]

Then, problem \((2.1) - (2.2) \) has a unique solution \( u \in W^2_{p,\alpha}(G;E(A),E) \) for \( f \in L_p(G;E) \) and sufficiently large \( |\lambda| \) with \( |\arg \lambda| \leq \varphi \) and the following uniform coercive estimate holds

\[
\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \left\| a_k^{\nu} \frac{\partial^i u}{\partial x_k^i} \right\|_{L_p(G,E)} + \| Au \|_{L_p(G,E)} \leq M \| f \|_{L_p(G,E)}. \tag{2.3}
\]

For proving the main theorem, consider at first the BVP for the singular degenerate ordinary DOE

\[
- u^{[2]}(t) + (A + \lambda) u(t) = f, \ t \in (0, a), \tag{2.4}
\]

\[
L_1 u = \sum_{i=0}^m \left[ \delta_i u^{[i]}(a) + \sum_{j=1}^N \nu_{ij} u^{[i]}(t_{ij}) \right] = 0,
\]

where \( u^{[i]} = (t^\nu d^{i} \overline{d})^i, \nu > 1, m \in \{0, 1\}, \delta_i, \nu_{ij}, \) are complex numbers, \( t_{ij} \in (0, a) \) and \( A \) is a linear operator in \( E \).

Let \( \gamma = \gamma(t) = t^\nu \).

Remark 2.1. Consider the following substitution

\[
\tau = - \int_a^t \gamma^{-1}(z) \, dz, \ t = t(\tau) = \left[ a^{1-\nu} - (\nu - 1) \tau \right]^{\frac{1}{\nu-1}}, \tag{2.5}
\]
Under the substitution (2.5) the spaces \( L_p (0, a; E), W_{p, \gamma}^{[2]} (0, a; E (A), E) \) are mapped isomorphically onto weighted spaces

\[
L_{p, \bar{\gamma}} (-\infty, 0; E), \ W_{p, \bar{\gamma}}^{2} (-\infty, 0; E (A), E),
\]

respectively, where

\[
\bar{\gamma} = \gamma (t (\tau)) = [a^{1-\nu} - (\nu - 1) \tau]^\frac{1}{\nu - 1}.
\]

Moreover, under the substitution (2.5) the problem (2.1) – (2.2) is transformed into the following undegenerate problem

\[
-u^{(2)} (\tau) + Au (\tau) = \tilde{f} (\tau),
\]

\[
L_1 u = \sum_{i=0}^{m} \left( \delta_i u^{(i)} (0) + \sum_{j=1}^{N} \nu_{ij} u^{(i)} (\tau_{ij}) \right) = 0
\]

considered in the weighted space \( L_{p, \bar{\gamma}} (-\infty; 0; E) \), where \( \tilde{f} (\tau) = f (t (\tau)) \) and \( \tau_{ij} \in (-\infty, 0) \).

By using Theorem A4 we have

**Proposition 2.1.** Assume the Condition 1.1 are satisfied with \( t_{1j} = \tau_{1j} \). Then, the problem (2.4) has a unique solution \( u \in W_{p, \gamma}^{[2]} (0, a; E (A), E) \)

for all \( f \in L_p (0, a; E), \) for \( |\arg \lambda| \leq \varphi \) with sufficiently large \( |\lambda| \) and the uniform coercive estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| u^{(i)} \right\|_{L_p (0, a; E)} + \|Au\|_{L_p (0, a; E)} \leq C \|f\|_{L_p (0, a; E)}.
\]

**Proof.** Consider the transformed problem (2.6). By the substitution

\[
y = a^{1-\nu} - (\nu - 1) \tau, \ \tau = \tau (y) = \frac{1}{\nu - 1} (a^{1-\nu} - y)
\]

the spaces \( L_{p, \bar{\gamma}} (-\infty, 0; E), \ W_{p, \bar{\gamma}}^{2} (-\infty, 0; E (A), E) \) are mapped isomorphically onto weighted spaces

\[
L_{p, \bar{\gamma}} (a^{1-\nu}, \infty; E), \ W_{p, \bar{\gamma}}^{2} (a^{1-\nu}, \infty; E (A), E),
\]

respectively, where

\[
\bar{\gamma} = \gamma (\tau (y)) = y^\frac{1}{1-\nu}.
\]

Moreover, under the substitution (2.7) the problem (2.6) is transformed into the following undegenerate problem

\[
-u^{(2)} (y) + Au (y) = \tilde{f} (\tau),
\]

\[
L_1 u = \sum_{i=0}^{m} \left( \delta_i u^{(i)} (0) + \sum_{j=1}^{N} \nu_{ij} u^{(i)} (\tau_{ij}) \right) = 0
\]
\[ L_1 u = \sum_{i=0}^{m} \left[ \delta_i u^{(i)} (a_1^{-\nu}) + \sum_{j=1}^{N} \gamma_{ij} u^{(i)} (y_{ij}) \right] = 0 \]

considered in the weighted space \( L_{\rho, \gamma} (a_1^{-\nu}; \infty; E) \), where \( \bar{f}(y) = f(\tau(y)) \) and \( y_{ij} \in (a_1^{-\nu}, \infty) \).

By Theorem A, we obtain that the problem (2.8) has a unique solution \( u \in W_{\rho, \gamma}^2 (a_1^{-\nu}, \infty; E(A), E) \) for all \( f \in L_{\rho, \gamma} (a_1^{-\nu}, \infty; E) \), \(|\arg \lambda| \leq \phi\) with sufficiently large \(|\lambda|\) and the uniform coercive estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \| u^{(i)} \|_{L_{\rho, \gamma}(a_1^{-\nu}; \infty; E)} + \| Au \|_{L_{\rho, \gamma}(a_1^{-\nu}; \infty; E)} \leq C \| f \|_{L_{\rho, \gamma}(a_1^{-\nu}; \infty; E)}.
\]

From the above estimate we obtain the assertion.

Consider the operator \( B \) generated by problem (2.4), i.e.

\[
D(B) = W_{\rho, \gamma}^2 (0, a; E(A), E, L_1), Bu = -u^{[2]} + Au.
\]

Result 2.1. From the Proposition 2.1 we obtain that the operator \( B \) is positive in \( L_p (0, a; E) \) and there is positive constants \( C_1 \) and \( C_2 \) that

\[
C_1 \| (B + d) u \|_{L_p(0, a; E)} \leq \| u \|_{W_{\rho, \gamma}^2 (0, a; E(A), E)} \leq C_2 \| (B + d) u \|_{L_p(0, a; E)}
\]

for sufficiently large \( d > 0 \) and \( u \in D(B) \).

In a similar way as in [16, Theorem 3.1] we obtain

**Proposition 2.2.** Assume the Condition 1.1 are satisfied with \( t_{1j} = \tau_{1j} \).

Then, the operator \( B \) is \( R \)-positive in \( L_p (0, a; E) \).

From [15, Theorem 1] and Remark 2.1 we obtain

**Theorem A.** Suppose the following conditions are satisfied:

1. \( E \) is an UMD space and \( A \) is an \( R \)-positive operator in \( E \);
2. \( \gamma_k(x) = x^\nu_k \), \( \nu_k \in (1, p) \), \( p \in (1, \infty) \) and \( m \) is an integer, \( \nu = \frac{|\nu|}{m} \leq 1 \), \( 1 < p < \infty \);
3. \( \Omega = \prod_{k=1}^{n} (0, a_k) \).

Then, the embedding

\[
D^\alpha W_{\rho, \gamma}^{[m]} (\Omega; E(A), E) \subset L_p \left( \Omega; E \left( A^{1-\nu} \right) \right)
\]

is continuous. Moreover for all \( h > 0 \) with \( h \leq h_0 < \infty \) and \( u \in W_{\rho, \gamma}^{[m]} (\Omega; E(A), E) \) the following uniform estimate holds

\[
\| D^{[\alpha]} u \|_{L_p(\Omega; E(A^{1-\nu}))} \leq h^\mu \| u \|_{W_{\rho, \gamma}^{[m]} (\Omega; E(A), E)} + h^{-(1-\mu)} \| u \|_{L_p(\Omega; E)}.
\]

Consider now, the following degenerate problem
\[- t^{2\nu} u^{(2)} (t) + (A + \lambda) u (t) = f, \ t \in (0, a), \quad (2.10)\]

\[L_1 u = \sum_{i=0}^{m} \delta_i u^{[i]} (a) + \sum_{j=1}^{N} \nu_{ij} u^{[i]} (t_{ij}) = 0,\]

where \(u^{[i]} = (t^\nu \frac{d}{dt})^i, \nu > 1, \ m \in \{0, 1\}, \ \delta_i, \nu_{ij}, \) are complex numbers, \(t_{ij} \in (0, a)\) and \(A\) is a linear operator in \(E.\)

Here, \(\gamma (t) = t^{2\nu}.\)

**Proposition 2.3.** Assume all conditions of the Proposition 2.1 are satisfied. Then, problem (2.10) has a unique solution \(u \in W^2_{p, \gamma} (0, a; E (A), E)\) for all \(f \in L_p (0, a; E), \ \arg \lambda \leq \varphi\) and sufficiently large \(|\lambda|\). Moreover, the uniform coercive estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-i} \left\| \nu^i u^{(i)} \right\|_{L_p (0, a; E)} + \|Au\|_{L_p (0, a; E)} \leq C \|f\|_{L_p (0, a; E)}. \quad (2.11)
\]

**Proof.** Since \(\nu > 1,\) by Theorem A5 for all \(\varepsilon > 0\) there is a continuous function \(C (\varepsilon)\) such that

\[
\left\| \nu^i u^{(i)} \right\|_{L_p (0, a; E)} \leq \varepsilon \left\| u \right\|_{W^2_{p, \gamma} (0, a; E (A), E)} + C (\varepsilon) \left\| u \right\|_{L_p (0, a; E)} \quad (2.12)
\]

for all \(u \in W^2_{p, \gamma} (0, a; E (A), E).\) By Result 2.1, the operator \(B\) is positive in \(L_p (0, a; E).\) Then, in view of (2.9), (2.12), by Proposition 2.1 and resolvent properties of positive operator (see Definition1) we have

\[
\left\| \nu^i u^{(i)} \right\|_{L_p (0, a; E)} \leq \varepsilon \left\| (B + \lambda) u \right\|_{L_p (0, a; E)} + C (\varepsilon) \left\| u \right\|_{L_p (0, a; E)} \leq
\]

\[
\varepsilon \left\| (B + \lambda) u \right\|_{L_p (0, a; E)} + \frac{C (\varepsilon)}{|\lambda|} \left\| (B + \lambda) u \right\|_{L_p (0, a; E)}
\]

for each \(u \in D (B).\) From the above estimate we obtain

\[
\left\| \nu^i u^{(i)} \right\|_{L_p (0, a; E)} < \delta \left\| (B + \lambda) u \right\|_{L_p (0, a; E)}, \quad (2.13)
\]

where \(\delta = \varepsilon + \frac{C (\varepsilon)}{|\lambda|} < 1\) for sufficiently large \(|\lambda| > 0.\) Since \(-x^{2\nu} u^{(2)} = -u^{[2]} + \nu x^{\nu-1} u^{[1]},\) the assertion is obtained from Proposition 2.1 and estimate (2.13).

Consider the operator \(S\) generated by problem (2.10), i.e.

\[D (S) = W^2_{p, \gamma} (0, a; E (A), E, L_k), \ Su = -x^{2\nu} u^{(2)} + Au.\]

**Result 2.2.** Suppose all conditions of Proposition 2.1 are satisfied. Then, the operator \(S\) is \(R\)-positive in \(L_p (0, a; E).\)
Proof. Indeed, by Proposition 2.2 the operator $B$ is $R$-positive in $L_p(0, a; E)$.

By definition of $R$ positive operators (see Definition 3)

$$
R \left\{ \lambda (B + \lambda)^{-1}, \lambda \in S_{\varphi} \right\} \leq M_1. \quad (2.14)
$$

Then by estimates (2.13), (2.14), definition of $R$-bounded sets (see Definition 2) and in view of the Kahane’s contraction principle and from the product properties of the collection of $R$-bounded operators (see e.g. [4] Lemma 3.5, Proposition 3.4) we obtain

$$
R \left\{ \lambda (S + \lambda)^{-1}, \lambda \in S_{\varphi} \right\} \leq M_2. \quad (2.15)
$$

Consider now the leading part of the problem (2.1) – (2.2), i.e.

$$
- \sum_{k=1}^{n} x_k^2 \frac{\partial^2 u}{\partial x_k^2} + Au + \lambda u = f(x), \; L_k u = 0, \; k = 1, 2, \ldots, n. \quad (2.16)
$$

**Proposition 2.4.** Assume the Condition 2.1 are satisfied. Then problem (2.15) has a unique solution $u \in W^{2, p, \alpha}_{p, a_2}(G; E(A), E)$ for $f \in L_p(G; E)$ and sufficiently large $|\lambda|$ with $|\arg \lambda| \leq \varphi$. Moreover, the uniform coercive estimate holds

$$
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{p}} \left\| x_k^i \frac{\partial^i u}{\partial x_k^i} \right\|_{L_p(G; E)} + \|Au\|_{L_p(G; E)} \leq M \|f\|_{L_p(G; E)}. \quad (2.16)
$$

Proof. Consider first, the problem (2.15) for $n = 2$ i.e.

$$
- \sum_{k=1}^{2} x_k^2 \frac{\partial^2 u}{\partial x_k^2} + Au + \lambda u = f(x_1, x_2), \; L_k u = 0, \; k = 1, 2. \quad (2.17)
$$

Since

$$
L_p(0, a_2; L_p(0, a_1; E)) = L_p((0, a_1) (0, a_2) \times; E),
$$

then the BVP (2.17) can be expressed as

$$
- x_2^{2\alpha_2} \frac{\partial^2 u}{\partial x_2^2} + (S + \lambda) u(x_2) = f(x_2), \; L_2 u = 0. \quad (2.18)
$$

By virtue of [1, Theorem 4.5.2], $F = L_p(0, a_1; E) \in UMD$ provided $E \in UMD$ for $p \in (1, \infty)$. By Result 2.2 the operator $S$ is $R$-positive in $F$. Then by virtue of Proposition 2.3 we get that, for $f \in L_p(0, a_2; F)$ the problem (2.18), i.e. problem (2.17) for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$ has a unique solution $u \in W^{2, p, \alpha}_{p, a_2}(0, a_2; D(S), F)$ and the coercive uniform estimate (2.16) holds for solution of the problem (2.12). By continuing the above proses $n$ time, we obtain that problem (2.15) has a unique solution $u \in W^{2, p, \alpha}_{p, a_2}(G; E(A), E)$ for
\( f \in L_p (G; E), \, |\arg \lambda| \leq \varphi \) and sufficiently large \(|\lambda|\), moreover, the uniform estimate (2.16) holds.

**Proof of Theorem 2.1.** Let \( Q_0 \) denote the operator generated by problem (2.15) i.e.,

\[
D (Q_0) = \{ u \in W^2_{p, \alpha} (G; E (A), E), \, L_k u = 0, \, k = 1, 2, ..., n \},
\]

\[
Q_0 u = - \sum_{k=1}^{n} x^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + A u.
\]

The estimate (2.16) implies that the operator \( Q_0 \) has a bounded inverse from \( L_p (G; E) \) to \( W^2_{p, \alpha} (G; E (A), E) \), i.e. the following estimate holds

\[
\left\| (Q_0 + \lambda)^{-1} f \right\|_{W^2_{p, \alpha} (G; E (A), E)} \leq C \left\| f \right\|_{L_p (G; E)}
\]

for all \( f \in L_p (G; E), \, \lambda \in S (\varphi) \) with sufficiently large \(|\lambda|\). Moreover, by Theorem A1 and in view of assumption (3), for all \( \varepsilon > 0 \) there is a continuous function \( C (\varepsilon) \) such that

\[
\sum_{k=1}^{n} \left\| x^{\alpha_k} A_k u \right\|_{L_p (G; E)} \leq \varepsilon \left\| u \right\|_{W^2_{p, \alpha} (G; E (A), E)} + C (\varepsilon) \left\| u \right\|_{L_p (G; E)}.
\]

From the above estimates we obtain that there is a positive number \( \delta < 1 \) such that

\[
\left\| Q_1 u \right\|_{L_p (G; E)} < \delta \left\| (Q_0 + \lambda) u \right\|_{L_p (G; E)}
\]

for \( u \in W^2_{p, \alpha} (G; E (A), E) \), where

\[
Q_1 u = \sum_{k=1}^{n} x^{\alpha_k} A_k (x) \frac{\partial u}{\partial x_k}.
\]

Let \( Q \) denote differential operator generated by problem (2.1) – (2.2) for \( \lambda = 0 \). It is clear that

\[
(Q + \lambda) = \left[ I + Q_1 (Q_0 + \lambda)^{-1} \right] (Q_0 + \lambda).
\]

Therefore, we obtain that the operator \((Q + \lambda)^{-1}\) is bounded from \( L_p (G; E) \) to \( W^2_{p, \alpha} (G; E (A), E) \) and the estimate (2.16) is satisfied.

Let \( L = L (L_p (G; E)) \). We get the following result from Theorem 2.1:

**Result 2.3.** Theorem 2.1 implies that differential operator \( Q \) has a resolvent \((Q + \lambda)^{-1}\) for \(|\arg \lambda| \leq \varphi\) and the following estimate holds

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| x^{\alpha_k} \frac{\partial^i}{\partial x_k^i} (Q + \lambda)^{-1} \right\|_L + \left\| A (Q + \lambda)^{-1} \right\|_L \leq M.
\]

3. Spectral properties of singular degenerate elliptic operators
In this section, the spectral properties for singular degenerate abstract differential operators are derived. Note that, the leading part of this operator is non-self-adjoint. Consider the differential operator $Q$ generated by BVP (2.1) – (2.2) for $\lambda = 0$. Let

$$X = L_p(G; E) \text{ and } Y = W^2_{p,\alpha}(G; E(A), E).$$

The main results of this section are the following theorems:

**Theorem 3.1.** Assume the conditions of Theorem 2.1 are satisfied and $A^{-1}$ is compact in $E$. Then, problem (2.1) – (2.2) is Fredholm in $L_p(G; E)$ for $\lambda = 0$.

**Proof.** Theorem 2.1 implies that the operator $(Q + \lambda)^{-1}$ from $L_p(G; E)$ to $W^2_{p,\alpha}(G; E(A), E)$ for sufficiently large $|\lambda|$, that is the operator $Q + \lambda$ is Fredholm from $W^2_{p,\alpha}(G; E(A), E)$ into $L_p(G; E)$. Then by Theorem A2 and in view of perturbation theory of linear operators we obtain that the operator $Q$ is Fredholm from $W^2_{p,\alpha}(G; E(A), E)$ into $L_p(G; E)$.

**Theorem 3.2.** Suppose all conditions of Theorem 3.1 hold, $\alpha_k < 2$ and

$$s_j(I(E(A), E)) \sim j^{-\frac{1}{\nu}}, \quad j = 1, 2, \ldots, \infty, \quad \nu > 0.$$

Then:

(a) 

$$s_j((Q + \lambda)^{-1}L_p(G; E)) \sim j^{-\frac{1}{\nu}}, \quad (3.1)$$

where

$$\kappa = \sum_{k=1}^{n} \frac{1}{2 - \alpha_k}.$$

(b) the system of root functions of operator $Q$ is complete in $X$.

**Proof.** By virtue Theorem 4.1, there exists a resolvent operator $(Q + \lambda)^{-1}$ which is bounded from $X$ to $Y$. Moreover, by virtue of Theorem A3 the embedding operator $I(Y, X)$ is compact and

$$s_j(I(Y, X)) \sim j^{-\frac{1}{\nu + \kappa}}. \quad (3.2)$$

It is clear to see that

$$(Q + \lambda)^{-1}(X) = (Q + \lambda)^{-1}(X, Y) \times I(Y, X).$$

Hence, from the relation (3.1) and Theorem A3 we obtain (3.2). The Result 2.3 and the relation (3.2) implies that operator $Q + \lambda_0$ is positive in $X$ for sufficiently large $\lambda_0$ and

$$\lambda_0^{-1} \in \sigma_q(X), \quad q > \nu + \kappa. \quad (3.3)$$

Then in view of the Result 2.3, the relation (3.3) and by virtue of [2, Theorem 3.4.1] we obtain the assertion (b).

4. Singular degenerate boundary value problems for infinite systems of equations
Consider the infinite system of BVPs

\[
\sum_{k=1}^{n} -x_k^{2\alpha_k} \frac{\partial^2 u_m}{\partial x_k^2} + \sum_{k=1}^{n} \sum_{j=1}^{\infty} x_k^{\alpha_k} a_{kmj}(x, y) \frac{\partial u_j}{\partial x_k} = \lambda u_m, \tag{4.1}
\]

where \(\lambda\) is a complex parameter, \(L_k\) are defined by (2.2) and \(x \in G = \prod_{k=1}^{n} (0, a_k)\).

\[
L_1 u = 0, \quad L_2 u = 0, \tag{4.2}
\]

\[
D = \{d_m\}, \quad d_m > 0, \quad u = \{u_m\}, \quad Du = \{d_m u_m\}, \quad m = 1, 2, \ldots,
\]

\[
l_q(D) = \{u: u \in l_q, \|u\|_{l_q(D)} = \left( \sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty, q \in (1, \infty) \}.
\]

Let \(O\) denote the operator in \(L_p(G; l_q)\) generated by problem (4.1)–(4.2). Here,

\[
\alpha_k(x) = x_k^{2\alpha_k}, \quad \alpha = \alpha(x) = (x_1^{2\alpha_1}, x_2^{2\alpha_2}, \ldots, x_n^{2\alpha_n}).
\]

From Theorem 2.1, we obtain

**Theorem 4.1.** Assume \(\alpha_k \in (1, p)\) for \(p \in (1, \infty), \quad k = 1, 2, \ldots, n\) and \(a_{kmj} \in L_\infty(G)\). Moreover, for \(0 < \mu_k < \frac{1}{2}\) and for all \(x \in G\)

\[
\sup_{m} \sum_{j=1}^{\infty} a_{kmj}(x, y) d_j^{-\left(\frac{1}{2} - \mu_k\right)} < M_k.
\]

Then:

(a) for all \(f(x) = \{f_m(x)\}_{k=1}^{\infty} \in L_p(G; l_q)\), \(p, q \in (1, \infty), \quad |\arg \lambda| \leq \varphi, \quad 0 \leq \varphi < \pi\) and for sufficiently large \(|\lambda|\) problem (4.1)–(4.2) has a unique solution \(u = \{u_m(x)\}_{k=1}^{\infty}\) that belongs to \(W^2_{p, \alpha}(G, l_q(D), l_q)\) and

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1 - \frac{1}{2}} \left\| x_k^{\alpha_i} \frac{\partial^2 u}{\partial x_k^i} \right\|_{L_p(G; l_q)} + \|Du\|_{L_p(G; l_q)} \leq M \|f\|_{L_p(G; l_q)};
\]

(b) the operator \(O\) is Fredholm in \(L_p(G; l_q)\);

(c) the system of root functions of operator \(O\) is complete in \(L_p(G; l_q)\).

**Proof.** Let \(E = l_q\), \(A\) and \(A_n(x)\) be infinite matrices, such that

\[
A = [d_m \delta_{mj}], \quad A_k(x) = [d_{kmj}(x)], \quad m, j = 1, 2, \ldots, \infty.
\]

By [4], \(l_q\) is the UMD space. It is clear to see that the operator \(A\) is \(R\)-positive in \(l_q\). The problem (4.1) can be rewritten in the form of (2.1) – (2.2). From
Theorem 2.1 we obtain that problem (4.1) – (4.2) has a unique solution \( u \in W_{p,\alpha}^2(G;l_q(D), l_q) \) for all \( f \in L_p(G;l_q) \) and

\[
\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{p}} \left\| x_k^{\alpha_k} \frac{\partial^i u}{\partial x_k^i} \right\|_{L_p(G;l_q)} + \| Du \|_{L_p(G;l_q)} \leq M \| f \|_{L_p(G;l_q)}.
\]

From the above estimate we obtain the assertion (a). The assertions (b) and (c) are obtained from Theorems 3.1 and 3.2, respectively.

5. Wentzell-Robin type BVP for degenerate elliptic equation

Consider the problem

\[
\sum_{k=1}^n x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + \lambda u = f(x, y), \quad x \in G, \quad y \in (0, 1),
\]

\[
L_k u = 0, \quad \text{for a.e. } y \in (0, 1),
\]

where \( a = a(y), \quad b = b(y) \) are real-valued functions on \((0, 1), \lambda \) is a complex parameter and \( L_k \) are boundary condition in \( x \) defined by (2.2). For \( \Omega = G \times (0, 1), \quad p = (p, 2) \) and \( L_p(\Omega) \) will denote the space of all \( p \)-summable scalar-valued functions with mixed norm. Analogously, \( W_{p,\alpha}^2(\Omega) \) denotes the Sobolev space with corresponding mixed norm, i.e., \( W_{p,\alpha}^2(\Omega) \) denotes the space of all functions \( u \in L_p(\Omega) \) possessing the derivatives \( x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} \in L_p(\Omega) \) with the norm

\[
\| u \|_{W_{p,\alpha}^2(\Omega)} = \| u \|_{L_p(\Omega)} + \sum_{k=1}^n \left\| x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} \right\|_{L_p(\Omega)}.
\]

**Condition 5.1** Assume;

1. \( \alpha_k \in (1, p) \) for \( p \in (1, \infty) \) and \( k = 1, 2, ..., n; \)
2. \( a \) is positive, \( b \) is a real-valued functions on \((0, 1); \)
3. \( a(\cdot) \in C[0, 1] \) and

\[
\exp \left( - \int_{\frac{1}{2}}^{x} b(t) a^{-1}(t) \, dt \right) \in L_1(0, 1).
\]

Let \( H \) denote the elliptic operator in \( L_p(\Omega) \) generated by problem (5.1) – (5.3). In this section, we present the following result:
Theorem 5.1. Suppose the Condition 5.1 hold. Then:

(a) for \( f \in L^p(\Omega) \) problem (5.1) – (5.3) for \( \lambda \in S(\varphi) \) and sufficiently large \(|\lambda| > 0\) has a unique solution \( u \) belonging to \( W^{2,p}_{\alpha} (\Omega) \) and the following coercive uniform estimate holds

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| x_k^{i} \frac{\partial^i u}{\partial x_k^i} \right\|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)};
\]

(b) the problem (5.1) – (5.3) is Fredholm in \( L^p(\Omega) \) for \( \lambda = 0 \);

(c) the system of root functions of operator \( H \) is complete in \( L^p (G; l_q) \)

Proof. Let \( E = L^2 (0, 1) \). It is known [10] that \( L^2 (0, 1) \) is an UMD space. Consider the operator \( A \) defined by

\[
D(A) = W^2_2 (0, 1; A (j) u (j) = 0), \quad j = 0, 1, \quad Au = a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y}.
\]

Therefore, the problem (5.1) – (5.2) can be rewritten in the form of (2.1) – (2.2), where \( u(x) = u(x, .) \), \( f(x) = f(x, .) \) are functions with values in \( E = L_2 (0, 1) \). By virtue of [9, 10] the operator \( A \) generates analytic semigroup in \( L_2 (0, b) \). Then in view of Hill-Yosida theorem (see e.g. [20, § 1.13]) this operator is positive in \( L_2 (0, b) \). Since all uniform bounded set in Hilbert space is \( R \)-bounded (see [4] ), i.e. we get that the operator \( A \) is \( R \)-positive in \( L_2 (0, b) \). Then from Theorem 2.1 we obtain the assertion (a). Since the embedding \( W^2_2 (0, 1) \subset L_2 (0, 1) \) is compact, the assertions (b) and (c) are obtained from Theorems 3.1 and 3.2, respectively.

From Theorem 5.1 we obtain:

Result 5.1. Theorem 5.1 implies that operator \( H \) has a resolvent \((H + \lambda)^{-1}\) for \(|\arg \lambda| \leq \varphi\) and the following sharp coercive resolvent estimate holds

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| x_k^{i} \frac{\partial^i (H + \lambda)^{-1}}{\partial x_k^i} \right\|_{L^p(\Omega)} \leq M.
\]

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