TWO DIMENSIONAL ADELIC ANALYSIS AND CUSPIDAL AUTOMORPHIC REPRESENTATIONS OF $GL(2)$

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Abstract. Two dimensional adelic objects were introduced by I. Fesenko in his study of the Hasse zeta function associated to a regular model $\mathcal{E}$ of the elliptic curve $E$. The Hasse-Weil $L$-function $L(E, s)$ of $E$ appears in the denominator of the Hasse zeta function of $E$. The two dimensional adelic analysis predicts that the integrand $h$ of the boundary term of the two dimensional zeta integral attached to $\mathcal{E}$ is mean-periodic. The mean-periodicity of $h$ implies the meromorphic continuation and the functional equation of $L(E, s)$. On the other hand, if $E$ is modular, several nice analytic properties of $L(E, s)$, in particular the analytic continuation and the functional equation, are obtained by the theory of the cuspidal automorphic representation of $GL(2)$ over the ordinary ring of adele (one dimensional adelic object). In this article we try to relate the theory of two dimensional adelic object to the theory of cuspidal automorphic representation of $GL(2)$ over the one dimensional adelic object, under the assumption that $E$ is modular. Roughly speaking, they are dual each other.

1. Introduction

Let $X \to \text{Spec} \mathbb{Z}$ be a scheme separated and of finite type. The Hasse zeta function of $X$ is defined by the Euler product

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |\kappa(x)|^{-s})^{-1},$$

where $X_0$ is the set of all closed points $x$ of $X$ with residue field $\kappa(x)$ of cardinality $|\kappa(x)| < \infty$. For a number field $k$ with the ring of integers $\mathcal{O}_k$ the Hasse zeta function of the affine scheme $\text{Spec} \mathcal{O}_k$ is the Dedekind zeta function $\zeta_k(s) = \prod_{p \subset \mathcal{O}_k} (1 - |\mathcal{O}_k/p|^{-s})^{-1}$. It is conjectured that $\zeta_X(s)$ has several nice analytic properties such as a meromorphic continuation and a functional equation. However, the known result is very few when the dimension of $X$ is larger than one.

If the dimension of $X$ is one, the Hasse zeta function $\zeta_X(s)$ is essentially the Dedekind zeta function $\zeta_k(s)$. Due to the celebrated work of Iwasawa and Tate, the analytic properties of $\zeta_k(s)$ are obtained by the Fourier analysis on adele $\mathbb{A}_k$. The completed Dedekind zeta function $\hat{\zeta}_k(s)$ is defined by multiplying $\zeta_k(s)$ with a finite product of $\Gamma$-factors. It has the integral representation

$$\hat{\zeta}_k(s) = \int_{\mathbb{A}_k^\times} f(x)|x|^sd\mu_{\mathbb{A}_k^\times}(x) =: \zeta_k(f, s),$$

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where \( f \) is an appropriate Schwartz–Bruhat function on \( \mathbb{A}_k \) and \( | \cdot | \) is a module on the ideles \( \mathbb{A}_k^\times \) of \( k \). On the other hand, one has
\[
\zeta_k(f, s) = \xi(f, s) + \xi(\hat{f}, 1-s) + \omega(f, s)
\]
on \( \Re(s) > 1 \), where \( \hat{f} \) is the Fourier transform of \( f \) on \( \mathbb{A}_k \), \( \xi(f, s) \) is an entire function given by an integral which converges absolutely for any \( s \in \mathbb{C} \) and the boundary term
\[
\omega(f, s) = \int_0^1 h_f(x)x^s dx
\]
for some function \( h_f \) on \( (0, 1) \). The meromorphic continuation and the functional equation for \( \zeta_k(s) \) are equivalent to the meromorphic continuation and the functional equation for \( \omega(f, s) \). The properties of the function \( h_f(x) \) are crucial in order to have a better understanding of \( \omega(f, s) \). Fourier analysis and analytic duality on \( k \subset \mathbb{A}_k \) leads to
\[
h_f(x) = -\mu \left( \mathbb{A}_k^1/k^\times \right) \left( f(0) - x^{-1}\hat{f}(0) \right).
\]
As a consequence, \( \omega(f, s) \) is a rational function of \( s \) invariant with respect to \( f \mapsto \hat{f} \) and \( s \mapsto (1-s) \). Thus, \( \zeta_k(s) \) admits a meromorphic continuation to \( \mathbb{C} \) and satisfies a functional equation with respect to \( s \mapsto (1-s) \).

Let \( E \) an elliptic curve over \( k \) and let \( \mathcal{E} \to B = \text{Spec} \mathcal{O}_k \) be a regular model of \( E \) over \( k \). Then the description of geometry of models in [8, Thms 3.7, 4.35 in Ch. 9 and section 10.2.1 in Ch. 10] implies that
\[
\zeta_\mathcal{E}(s) = n_\mathcal{E}(s)\zeta_E(s) \quad \text{with} \quad \zeta_E(s) = \frac{\zeta_k(s)\zeta_k(s-1)}{L(E, s)} \tag{1.1}
\]
on \( \Re(s) > 2 \). Here \( n_\mathcal{E}(s) \) is the product of zeta functions of affine lines over finite extension \( \kappa(b_j) \) of the residue fields \( \kappa(b) \):
\[
n_\mathcal{E}(s) = \prod_{j=1}^J \left( 1 - |\kappa(b_j)|^{1-s} \right)^{-1} \tag{1.2}
\]
where \( J \) is the number of singular fibres of \( \mathcal{E} \to B \) (see [5, section 7.3]).

The modularity conjecture for \( E/k \) asserts that there exists a cuspidal automorphic representation \( \pi_E \) of \( \text{GL}_2(\mathbb{A}_k) \) such that
\[
L(E, s) = L(\pi_E, s - 1/2).
\]
Then the general theory of \( L \)-function \( L(\pi, s) \) of cuspidal automorphic representation \( \pi \) of \( \text{GL}_2(\mathbb{A}_k) \) leads to an analytic continuation and a functional equation of \( L(E, s) \) via \( L(\pi, s) \). The analytic properties of \( L(\pi, s) \) are obtained by extending the Iwasawa-Tate theory from the commutative group \( \text{GL}_1(\mathbb{A}_k) \) to the noncommutative group \( \text{GL}_2(\mathbb{A}_k) \). In this story, the theory of noncommutative group \( \text{GL}_2(\mathbb{A}_k) \) relates to \( \zeta_\mathcal{E}(s) \) via the modularity conjecture and the \( L \)-function \( L(E, s) \) of \( E \).

In contrast with the above story, I. Fesenko proposed another way to study \( \zeta_\mathcal{E}(s) \) in [3, 5, 4] by using a commutative group associated to two dimensional ideles. The ordinary ring of ideles \( \mathbb{A}_k \) is regarded as an one dimensional object in the sense that it is associated to the one dimensional scheme \( \text{Spec} \mathcal{O}_k \). He introduced the two dimensional adelic space \( \mathbb{A}_\mathcal{E} \) associated to the two dimensional scheme \( \mathcal{E} \) and established a theory of translation invariant measure and integrals on its subring \( \mathbb{A}_{\mathcal{E}, S} \subset \mathbb{A}_\mathcal{E} \), where \( S \) is a
defined by where is the space of annihilators of functions on . Similar to the Iwasawa-Tate theory, we have the boundary term theory of mean-periodic functions, see Kahane [7], Schwartz [10] or a reference of [12]. For the general theory of mean-periodic functions ([5, section 7], see also [12]). The zeta integral converges absolutely for . If the test function is well-chosen, the zeta integral equals

where is an extension of determined by each horizontal fiber in and is a positive real number determined by . On the other hand, similar to the Iwasawa-Tate theory, the zeta integral is decomposed as

on , where is an entire function and is the Fourier transform of on . Hence the meromorphic continuation of implies the meromorphic continuation of the Hasse zeta function . If we can prove the meromorphic continuation of using analysis and duality on two dimensional adelic space , it leads the meromorphic continuation of the -function without proving the modularity property!

One possible approach for the meromorphic continuation of is proposed via the theory of mean-periodic functions ([5, section 7], see also [12]). For the general theory of mean-periodic functions, see Kahane [7], Schwartz [10] or a reference of [12]. Similar to the Iwasawa-Tate theory, we have the boundary term

for some function on . So the boundary term is the Laplace transform of .

Let be a locally convex separated topological vector space consisting of complex valued functions on . It has a natural representation as for every . For we denote by the closure of with respect to the topology of . A function is called mean-periodic if . Using the representation the convolution for and is defined by

where . The mean-periodicity is equivalent that the space of annihilators of is trivial;

As a consequence of the general theory of mean-periodic function, if is mean-periodic, the Laplace transform of is continued meromorphically to the whole complex plane.
Now we suppose that \( h_{f_0} \in \mathcal{X} \). Then the conjectural mean-periodicity of \( h_{f_0} \) implies the meromorphic continuation of the Hasse zeta function \( \zeta_E(s) \). Hence it is important to understand the space of annihilators \( T(h_{f_0})^\perp \).

In this paper, in the case \( k = \mathbb{Q} \), we describe the space of annihilators \( T(h_{f_0})^\perp \) by using the cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}_\mathbb{Q}) \) whose existence follows from the modularity of \( E/Q \) (see Theorem 3.1, Theorem 3.2 for more detail). Such description of \( T(h_{f_0})^\perp \) suggests some duality between the commutative theory of two dimensional adeles \( \mathbb{A}_E \), \( \mathbb{A}_E, S \) and the noncommutative theory \( \text{GL}_2(\mathbb{A}_\mathbb{Q}) \) of one dimensional adele \( \mathbb{A}_Q \).

In section 2 we include several definitions, notations and already known properties. In section 3 we state the results, and we prove them in section 4.

2. Preliminaries

Let \( S(\mathbb{R}) \) be the Schwartz space on \( \mathbb{R} \) which consists of smooth functions on \( \mathbb{R} \) satisfying

\[
\|f\|_{m,n} = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty
\]

for all nonnegative integer \( m \) and \( n \). It is a Fréchet space over the complex numbers with the topology induced from the family of seminorms \( \| \|_{m,n} \). Let us define the Schwartz space \( S(\mathbb{R}_+^\times) \) on \( \mathbb{R}_+^\times \) and its topology via the homeomorphism

\[
S(\mathbb{R}) \to S(\mathbb{R}_+^\times); \quad f(t) \mapsto f(-\log t),
\]

where \( t = -\log x \). The strong Schwartz space \( S(\mathbb{R}_+^\times) \) ([9]) is defined by

\[
S(\mathbb{R}_+^\times) := \bigcap_{\beta \in \mathbb{R}} \{ f : \mathbb{R}_+^\times \to \mathbb{C}, [x \mapsto x^{-\beta} f(x)] \in S(\mathbb{R}_+^\times) \}. \tag{2.1}
\]

One of the family of seminorms on \( S(\mathbb{R}_+^\times) \) defining its topology is given by

\[
\|f\|_{m,n} = \sup_{x \in \mathbb{R}_+^\times} |x^m f^{(n)}(x)| \tag{2.2}
\]

for integer \( m \) and nonnegative integer \( n \). The strong Schwartz space \( S(\mathbb{R}_+^\times) \) is a Fréchet space over the complex numbers where the family of seminorms defining its topology is given in (2.2). This space is closed under the multiplication by a complex number and the pointwise addition and multiplication ([9]). Let \( S(\mathbb{R}_+^\times)^\ast \) be the dual space of \( S(\mathbb{R}_+^\times) \) with the weak \(*\)-topology. The pairing between \( S(\mathbb{R}_+^\times) \) and \( S(\mathbb{R}_+^\times)^\ast \) is denoted by \( \langle \cdot, \cdot \rangle \), namely \( \langle f, \varphi \rangle = \varphi(f) \) for \( f \in S(\mathbb{R}_+^\times) \) and \( \varphi \in S(\mathbb{R}_+^\times)^\ast \). The (multiplicative) representation \( \tau \) of \( \mathbb{R}_+^\times \) on \( S(\mathbb{R}_+^\times)^\ast \) is defined by

\[
\tau_x f(y) := f(y/x), \quad \forall x \in \mathbb{R}_+^\times
\]

and the (multiplicative) convolution \( f \ast \varphi \) of \( f \in S(\mathbb{R}_+^\times) \) and \( \varphi \in S(\mathbb{R}_+^\times)^\ast \) by

\[
(f \ast \varphi)(x) = \langle \tau_x f, \varphi \rangle, \quad \forall x \in \mathbb{R}_+^\times
\]

where \( \tilde{f}(x) := f(x^{-1}) \). The dual representation \( \tau^\ast \) on \( S(\mathbb{R}_+^\times)^\ast \) is defined by

\[
\langle f, \tau_x^\ast \varphi \rangle := \langle \tau_x f, \varphi \rangle.
\]
If \( V \) is a \( \mathbb{C} \)-vector space then the bidual space \( V^{**} \) (the dual space of \( V^* \) with respect to the weak *-topology on \( V^* \)) is identified with \( V \) in the following way. For a continuous linear functional \( F \) on \( V^* \) with respect to its weak *-topology, there exists \( v \in V \) such that \( F(v^*) = v^*(v) \) for every \( v^* \in V^* \). Therefore, we do not distinguish the pairing on \( V^{**} \times V^* \) from the pairing on \( V \times V^* \).

**Definition 2.1.** Let \( \mathfrak{X} = S(\mathbb{R}_+^\times)^* \). An element \( x \in \mathfrak{X} \) is said to be \( \mathfrak{X} \)-mean-periodic if there exists a non-trivial element \( x^* \) in \( \mathfrak{X}^* \) satisfying \( x \ast x^* = 0 \).

For \( x \in \mathfrak{X} = S(\mathbb{R}_+^\times)^* \), we denote by \( T(x) \) the closure of the \( \mathbb{C} \)-vector space spanned by \( \{ \tau_g(x), g \in \mathbb{R}_+^\times \} \). The Hahn-Banach theorem leads to another definition of \( \mathfrak{X} \)-mean-periodic functions.

**Proposition 2.1.** An element \( x \in \mathfrak{X} = S(\mathbb{R}_+^\times)^* \) is \( \mathfrak{X} \)-mean-periodic if and only if \( T(x) \neq \mathfrak{X} \).

Let \( L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \) be the space of locally integrable functions on \( \mathbb{R}_+^\times \) satisfying

\[
h(x) = \begin{cases} O(x^a) & \text{as } x \to +\infty, \\ O(x^{-a}) & \text{as } x \to 0^+ \end{cases}
\]

for some real number \( a \geq 0 \). Each \( h \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \) gives rise to a distribution \( \varphi_h \in S(\mathbb{R}_+^\times)^* \) defined by

\[
\langle f, \varphi_h \rangle = \int_0^{+\infty} f(x)h(x)\frac{dx}{x}, \quad \forall f \in S(\mathbb{R}_+^\times).
\]

If there is no confusion, we denote \( \varphi_h \) by \( h \) itself and use the notations \( \langle f, h \rangle = (f, \varphi_h) \) and \( h(x) \in S(\mathbb{R}_+^\times)^* \). Then

\[
x^\lambda \log^k(x) \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \subset S(\mathbb{R}_+^\times)^*
\]

for all \( k \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{C} \). Moreover, if \( h \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \) then the convolution \( f \ast \varphi_h \) coincides with the ordinary convolution on functions on \( \mathbb{R}_+^\times \) namely

\[
(f \ast h)(x) = \langle \tau_x f, h \rangle = \int_0^{+\infty} f(x/y)h(y)\frac{dy}{y} = \int_0^{+\infty} f(y)h(x/y)\frac{dy}{y}.
\]

For a \( h \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \) define \( h^+ \) and \( h^- \) by

\[
h^+(x) := \begin{cases} 0 & \text{if } x \geq 1, \\ h(x) & \text{otherwise} \end{cases} \quad h^-(x) := \begin{cases} h(x) & \text{if } x \geq 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Clearly, \( h^\pm \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \) for all \( h \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \).

**Lemma 2.1.** Let \( h \in L^1_{\text{loc.poly}}(\mathbb{R}_+^\times) \). If \( f \ast h = 0 \) for some non-trivial \( f \in S(\mathbb{R}_+^\times) \) then the Mellin transforms

\[
M(f \ast h^\pm)(s) = \int_0^{+\infty} (f \ast h^\pm)(x)x^s\frac{dx}{x}
\]

are entire functions on \( \mathbb{C} \).
Definition 2.2. Let \( h \in L^1_{\text{loc}, \text{poly}}(\mathbb{R}^+_+) \). If \( f * h = 0 \) for some non-trivial \( f \in S(\mathbb{R}^+_+) \) then the Mellin–Carleman transform \( \text{MC}(h)(s) \) of \( h(x) \) is defined by

\[
\text{MC}(h)(s) := \frac{\text{M}(f * h^+)(s)}{\text{M}(f)(s)} = -\frac{\text{M}(f * h^-)(s)}{\text{M}(f)(s)}.
\]

The Mellin–Carleman transform \( \text{MC}(h) \) does not depend on the particular choice of non-trivial \( f \) satisfying \( f * h = 0 \). By Lemma 2.1 we have

Proposition 2.2. Let \( h \in L^1_{\text{loc}, \text{poly}}(\mathbb{R}^+_+) \subset S(\mathbb{R}^+_+)^* \). If \( f * h = 0 \) for some non-trivial \( f \in S(\mathbb{R}^+_+) \), in other words, \( h \) is \( S(\mathbb{R}^+_+)^* \)-mean-periodic, then the Mellin–Carleman transform \( \text{MC}(h)(s) \) of \( h(x) \) is a meromorphic function on \( \mathbb{C} \).

The Mellin–Carleman transform \( \text{MC}(h)(s) \) of \( h(x) \) is not a generalization of the Mellin transform of \( h \) but is a generalization of the following integral, half Mellin transform,

\[
\int_{0}^{1} h(x)x^s \frac{dx}{x}.
\]

See also section 2 of [12] for more detail.

Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with conductor \( q_E \). Then the completed \( L \)-function \( \Lambda(E, s) \) is defined by

\[
\Lambda(E, s) := q^{s/2}_E (2\pi)^{-s} \Gamma(s) L(E, s).
\]

It is conjectured that \( \Lambda(E, s) \) is continued to an entire function and satisfies the functional equation \( \Lambda(E, s) = \omega_E \Lambda(E, 2 - s) \) for some sign \( \omega_E \in \{\pm 1\} \). By (1.1), the meromorphic continuation and the functional equation of \( \Lambda(E, s) \) implies the meromorphic continuation and the functional equation of \( \zeta_E(s) \). Moreover such nice analytic properties of \( \Lambda(E, s) \) lead to mean-periodicity of the \( \omega(f_0, s) \).

Theorem 2.1. Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( \mathcal{E} \to \text{Spec} \mathbb{Z} \) be its regular model. Assume that \( \Lambda(E, s) \) is continued meromorphically to \( \mathbb{C} \) with a finite poles and satisfies the functional equation

\[
\Lambda(E, s)^2 = \Lambda(E, 2 - s)^2.
\]

Then the function

\[
h_\mathcal{E}(x) := f_\mathcal{E}(x) - x^{-1} f_\mathcal{E}(x^{-1})
\]

with

\[
f_\mathcal{E}(x) = \frac{1}{2\pi i} \int_{(c)} \Lambda(s/2 + 1/4)^2 c^{-s-1/2}_\mathcal{E} \zeta_\mathcal{E}(s + 1/2)^2 x^{-s} ds \quad (c > 1)
\]

belongs to \( S(\mathbb{R}^+_+)^* \), where \( c_\mathcal{E} \) is a positive real constant determined by the singular fiber of \( \mathcal{E} \) ([5, section 5]). Moreover \( h_\mathcal{E} \) is \( S(\mathbb{R}^+_+)^* \)-mean-periodic and has the expansion

\[
h_\mathcal{E}(x) = \lim_{T \to \infty} \sum_{3(\lambda) \leq T} \sum_{m=1}^{m_\lambda} C_m(\lambda) \frac{(-1)^{m-1}}{(m-1)!} x^{-\lambda} (\log x)^{m-1}
\]

where \( \lambda \) are poles of \( \Lambda(s/2 + 1/4)^2 c^{-s-1/2}_\mathcal{E} \zeta_\mathcal{E}(s + 1/2)^2 \) of multiplicity \( m_\lambda \), \( C_m(\lambda) \) are constants determined by the principal part at \( s = \lambda \):

\[
\Lambda(s/2 + 1/4)^2 c^{-s-1/2}_\mathcal{E} \zeta_\mathcal{E}(s + 1/2)^2 = \sum_{m=1}^{m_\lambda} \frac{C_m(\lambda)}{(s - \lambda)^m} + O(1) \quad \text{when} \quad s \to \lambda,
\]

Let \( \text{MC}(h)(s) \) be the Mellin–Carleman transform of \( h \), then

\[
\text{MC}(h)(s) = \frac{\text{M}(f h^+)(s)}{\text{M}(f)(s)} = -\frac{\text{M}(f h^-)(s)}{\text{M}(f)(s)}.
\]
and the sum over $\lambda$ is converges uniformly on every compact subset of $\mathbb{R}_+^\times$.

Proof. See section 5 of [12, section 5]. \qed

So the mean-periodicity of $h_\mathcal{E}(x)$ and the meromorphic continuation of $\Lambda(E, s)^2$ are equivalent to each other in the first approximation.

**Remark 2.1.** Let $S$ be the set of fibres of $\mathcal{E} \to \text{Spec} \mathbb{Z}$ consisting of one horizontal curve which is the image of the zero section of $\mathcal{E} \to \text{Spec} \mathbb{Z}$ and all vertical fibres of $\mathcal{E} \to \text{Spec} \mathbb{Z}$. Then we have

$$
\zeta_{\mathcal{E}, S}(f_0, s) = \Lambda(s/2)^2 \zeta\mathcal{E}(s)^2
= \int_1^\infty x^{-1/2} f_\mathcal{E}(x) x^s \frac{dx}{x} + \int_1^\infty x^{-1/2} f_\mathcal{E}(x) x^{2-s} \frac{dx}{x} + \int_0^1 x^{-1/2} h_\mathcal{E}(x) x^s \frac{dx}{x}.
$$

Hence the function $h_{f_0}(x)$ in the introduction is $x^{-1/2} h_\mathcal{E}(x)$.

**Remark 2.2.** We hope to prove the mean-periodicity of $h_\mathcal{E}(x)$ without using the meromorphic continuation of $\Lambda(E, s)$.

### 3. Statement of Results

Throughout this section we denote by $\mathbb{A}$ the adele $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$. At first we settle the following basic assumption.

**Basic assumption.** Suppose that $E/\mathbb{Q}$ is modular. We denote by $(\pi, V_\pi)$ the corresponding cuspidal automorphic representation in $L^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), 1)$, where $1$ is the trivial central character.

Of course the modularity of $E/\mathbb{Q}$ is now a theorem by the famous work of Wiles et al. However it is not proved for a general number field $k$. We emphasize this assumption for the future study of this direction.

#### 3.1. Construction on the positive real line

In this part we construct the space of annihilators $T(h_\mathcal{E})^\perp$ of $h_\mathcal{E}$ associated to $\zeta\mathcal{E}(s)$ as in Theorem 2.1 by using $GL_2(\mathbb{A})$-theory of Soulé [11] which is an extension of the original theory of Connes [1].

Let $M = \text{Mat}_2$ and $G = GL_2$. Let $| | : G_\mathbb{A} \to \mathbb{R}_+^\times$ be the module map given by $|g| = |\det g|_\mathbb{A}$. Let $f_\pi$ be an admissible matrix coefficient of the cuspidal automorphic representation $(\pi, V_\pi)$ on $L^2(G_\mathbb{Q} \backslash G_\mathbb{A}, 1)$, and let $\phi$ be a Schwartz-Bruhat function on $M_\mathbb{A}$. For $x \in \mathbb{R}_+^\times$, we set $G_x = \{g \in G_\mathbb{A} | |g| = x\}$. Define a complex valued function $\mathcal{E}(\phi, f_\pi)$ on $\mathbb{R}_+^\times$ by

$$
\mathcal{E}(\phi, f_\pi)(x) = \int_{G_x} \phi(g) f_\pi(g) dg \quad (x \in \mathbb{R}_+^\times).
$$

Then

i) the integral (3.1) converges absolutely;

ii) for any integer $N > 0$, there exists a positive constant $C$ such that

$$
|\mathcal{E}(\phi, f_\pi)(x)| \leq C x^{-N}, \quad (x \in \mathbb{R}_+^\times),
$$

for all $x \in \mathbb{R}_+^\times$. 

iii) we have the functional equation
\[
\mathbf{E}(\phi, f_\pi)(x) = x^{-2}\mathbf{E}(\hat{\phi}, \hat{f_\pi})(x^{-1})
\]  
(3.3)

where \(\hat{\phi}\) is the Fourier transform of \(\phi\) and \(\hat{f_\pi}(g) = f_\pi(g^{-1})\).

Let
\[
S(\pi) = \{ (\phi, f_\pi) | \phi \in S(M(\mathcal{A})), f_\pi: \text{admissible coefficient of } \pi \}.
\]

Then (3.2) and (3.3) show that \(\mathbf{E}\) is a map from \(S(\pi)\) into \(S(\mathbb{R}_+^{\times})\):
\[
\mathbf{E} : S(\pi) \rightarrow S(\mathbb{R}_+^{\times}); (\phi, f_\pi) \mapsto \mathbf{E}(\phi, f_\pi). 
\]

We denote by \(\mathcal{V}_\pi \subset S(\mathbb{R}_+^{\times})\) the image of \(\mathbf{E}\). Using the function
\[
w_0(x) = \frac{1}{2\pi i} \int \Gamma(s/4)^2 \cdot \left(\frac{c_E}{q_E}\right)^s \cdot s^4(s-2)^4 \cdot (s-1)^2 \cdot x^{-s}ds,
\]
we define the space \(\mathcal{W}_\pi\) by
\[
\mathcal{W}_\pi = w_0 \ast \mathcal{V}_\pi \ast \mathcal{V}_\pi = \text{span}_\mathbb{C}\{ w_0 \ast v_1 \ast v_2 | v_i \in \mathcal{V}_\pi \}. 
\]
(3.5)

Then the space \(\mathcal{W}_\pi\) is a subspace of \(S(\mathbb{R}_+^{\times})\), since \(w_0 \in S(\mathbb{R}_+^{\times})\) and \(S(\mathbb{R}_+^{\times})\) is closed under the multiplicative convolution. For \(h \in S(\mathbb{R}_+^{\times})\), we define
\[
\mathcal{T}(h)^\perp = \{ g \in S(\mathbb{R}_+^{\times}) | g \ast \tau = 0, \forall \tau \in \mathcal{T}(h) \}
\]
and
\[
\mathcal{W}_\pi^\perp = \{ \varphi \in S(\mathbb{R}_+^{\times})^* | w \ast \varphi = 0, \forall w \in \mathcal{W}_\pi \}.
\]

**Theorem 3.1.** Let \(E\) be an elliptic curve over \(\mathbb{Q}\) and let \(\mathcal{E} \rightarrow \text{Spec } \mathbb{Z}\) be its regular model. Let \(h_E\) be the function on \(\mathbb{R}_+^{\times}\) associated to the Hasse zeta function \(\zeta_E(s)^2\) as in Theorem 2.1. Then we have
\[
\mathcal{T}(h_E)^\perp \supset \mathcal{W}_\pi \quad \text{and} \quad \mathcal{T}(h_E) \subset \mathcal{W}_\pi^\perp.
\]

Hence \(\mathcal{W}_\pi \neq \{0\}\) means \(S(\mathbb{R}_+^{\times})^*\)-mean-periodicity of \(h_E(x)\). Further the equality
\[
\mathcal{T}(h_E)^\perp = \mathcal{W}_\pi \quad \text{or} \quad \mathcal{T}(h_E) = \mathcal{W}_\pi^\perp
\]
implies the non-existence of cancelations of zeros between
\[
(s-1)\zeta(s/2)\zeta(s)\zeta(s-1) \quad \text{and} \quad n_E(s)^{-1}\Lambda_E(s).
\]

3.2. Adelic construction. In this part, we consider the adelic version of the previous one according to Deitmar [2]. Let \(S(M_0)\) be the space of all \(\phi \in S(M_\mathcal{A})\) such that \(\phi\) and \(\hat{\phi}\) send \(\{ g \in M_\mathcal{A} | \text{det}(g) = 0 \} = M_\mathcal{A} \subset G_\mathcal{A}\) to zero. For \(\phi \in S(M_0)\) we define functions \(\mathbf{E}(\phi)\) and \(\hat{\mathbf{E}}(\phi)\) on \(G_\mathcal{A}\) by
\[
\mathbf{E}(\phi)(g) = \sum_{\gamma \in M_\mathcal{Q}} \phi(\gamma g) = \sum_{\gamma \in G_\mathcal{Q}} \phi(\gamma g),
\]
\[
\hat{\mathbf{E}}(\phi)(g) = \sum_{\gamma \in M_\mathcal{Q}} \phi(g\gamma) = \sum_{\gamma \in G_\mathcal{Q}} \phi(g\gamma).
\]
(3.6)

Then for any \(\phi \in S(M_0)\), we have
i) the sums \(\mathbf{E}(\phi)\) and \(\hat{\mathbf{E}}(\phi)\) converge locally uniformly in \(g\) with all derivatives,

ii) for any \(N > 0\) there exists \(C > 0\) such that
\[
|\mathbf{E}(\phi)(g)|, |\hat{\mathbf{E}}(\phi)(g)| \leq C \min(|g|, |g|^{-1})^N,
\]
(3.7)
iii) for \( g \in G_A \) we have the functional equation
\[
\mathcal{E}(\phi)(g) = |g|^{-2} \mathcal{E}(\hat{\phi})(g^{-1}).
\] (3.8)
Hence \( \mathcal{E}(\phi) \) belongs to the strong Schwartz space
\[
\mathbf{S}(G_Q \backslash G_A) = \bigcap_{\beta \in \mathbb{R}} |\beta|^\beta \mathbf{S}(G_Q \backslash G_A).
\]
Let \( G_A^1 \) be the kernel of the module map \( g \mapsto |g| \). Fix a splitting \( \beta : \mathbb{R}_+^\times \rightarrow G_A \) of the exact sequence \( 1 \rightarrow G_A^1 \rightarrow G_A \rightarrow 1 \) such that \( (\text{id}, \beta) : G_A^1 \times \mathbb{R}_+^\times \rightarrow G_A \) is an isomorphism. We denote by \( R \) the image of splitting \( \beta \). Let \( \varphi_\pi \in V_\pi \subset L^2(G_Q \backslash G_A^1) \simeq L^2(\mathbb{R}G_Q \backslash G_A) \) be a vector \( \varphi_\pi = \otimes_v \varphi_{\pi,v} \) such that \( \varphi_{\pi,v} \) is a normalized class one vector for almost all places. Further we assume that \( \varphi_\pi \) is smooth and \( \varphi_\pi(1) \neq 0 \).

We define
\[
\mathcal{W}_\pi = \text{span}_C \{(w_0 \circ \beta) \ast (\mathcal{E}(\phi_1) \cdot \varphi_\pi) \ast (\mathcal{E}(\phi_2) \cdot \varphi_\pi) \mid \phi_i \in \mathcal{S}(M_A) \} \subset \mathbf{S}(G_Q \backslash G_A),
\]
where \( w_0 \) is the function in (3.4), \( \mathcal{E}(\phi) \cdot \varphi_\pi)(x) = \mathcal{E}(\phi)(x) \cdot \varphi_\pi(x) \) and \( \ast \) is the convolution on \( G_Q \backslash G_A \) via the right regular representation \( R \), and
\[
\mathcal{W}_\pi^\perp = \{ \eta \in \mathbf{S}(G_Q \backslash G_A)^* \mid w \ast \eta \equiv 0, \forall w \in \mathcal{W}_\pi \}.
\]
For \( \eta \in \mathbf{S}(G_Q \backslash G_A)^* \) we define
\[
T(\eta) = \text{span}_C \{ R^*(g) \eta \mid g \in G_A \},
\]
where \( R^* \) is the transpose of the right regular representation of \( G_A \) on \( \mathbf{S}(G_Q \backslash G_A) \) with respect to the pairing \( \langle , \rangle \) of \( \mathbf{S}(G_Q \backslash G_A) \) and \( \mathbf{S}(G_Q \backslash G_A)^* \),

**Theorem 3.2.** Let \( h_\varepsilon \) be the function on \( \mathbb{R}_+^\times \) associated to the Hasse zeta function \( \zeta_\varepsilon(s)^2 \) as in Theorem 2.1. Under the above notations, we have
\[
T(h_\varepsilon \circ \beta) \subset \mathcal{W}_\pi^\perp.
\]
The equality
\[
T(h_\varepsilon \circ \beta) = \mathcal{W}_\pi^\perp
\]
implies the non-existence of cancelation of the zeros between
\[
(s - 1)\hat{\zeta}(s/2)\hat{\zeta}(s)\hat{\zeta}(s - 1) \quad \text{and} \quad n_\varepsilon(s)^{-1} \Lambda_E(s).
\]

4. **Proof of Results**

4.1. **Proof of Theorem 3.1.** First we prove the implication \( \mathcal{W}_\pi \subset T(h_\varepsilon)^\perp \). It suffices to prove that \( w \ast h_\varepsilon \equiv 0 \) for any \( w \in \mathcal{W}_\pi \). By Theorem 2.1 the function \( h_\varepsilon \) is a series consisting of functions \( f_{\lambda,k}(x) = x^{-\lambda}(\log x)^k \). For \( w(x) \in \mathcal{W}_\pi \),
\[
w \ast f_{\lambda,k}(x) = \int_0^\infty w(y) f_{\lambda,k}(x/y) \frac{dx}{x} = \sum_{j=1}^k (-1)^j \binom{k}{j} x^{-\lambda}(\log x)^{k-j} \int_0^\infty w(y)y^\lambda(\log y)^j \frac{dy}{y}.
\]
Here
\[
\int_0^\infty w(y)y^\lambda(\log y)^j \frac{dy}{y} = \frac{d^j}{d\lambda^j} \int_0^\infty w(y)y^\lambda \frac{dy}{y}.
\]
By definition of $W_\pi$,
\[
\int_0^\infty w(y) y^{\lambda} \frac{dy}{y} = \int_0^\infty (w * E(\phi_1, f_\pi) * E(\phi_2, f'_\pi))(y) y^{\lambda} \frac{dy}{y} = \int_0^\infty w_0(y) y^{\lambda} \frac{dy}{y} \cdot \int_0^\infty E(\phi_1, f_\pi)(y) y^{\lambda} \frac{dy}{y} \cdot \int_0^\infty E(\phi_2, f'_\pi)(y) y^{\lambda} \frac{dy}{y}
\]
for some $(\phi_1, f_\pi), (\phi_2, f'_\pi) \in S(\pi)$. From the construction of $V_\pi$, we have
\[
\int_0^\infty E(\phi, f_\pi)(y) y^{\lambda} \frac{dy}{y} = F_{\phi, f_\pi}(\lambda) L(\pi, \lambda - 1/2) = F_{\phi, f_\pi}(\lambda) L(E, \lambda)
\]
where $F_{\phi, f_\pi}(\lambda)$ is an entire function determined by $(\phi, f_\pi)$ (see [6, Theorem 13.8] and [11, section 2.5]). The second equality is a consequence of modularity. Therefore
\[
\begin{aligned}
\int_0^\infty w(y) y^{\lambda} \frac{dy}{y} &= \Gamma(\lambda/4) 2^\lambda (\lambda - 2)^4 (\lambda - 1)^2 (c_\lambda/q_\lambda)^\lambda \\
&\times n_\lambda(\lambda)^{-2} L(E, \lambda)^2 F_{\phi_2, f_\pi}(\lambda) F_{\phi_2, f'_\pi}(\lambda),
\end{aligned}
\]
and $w * h_\lambda$ is a series consisting of (4.1) and its $j$-th derivative with $j \leq m_\lambda$. Because $E/\mathbb{Q}$ is modular, $\Lambda(E, s)$ is an entire function. Therefore the complex numbers $\lambda$ appearing in the expansion of $h_\lambda(x)$ is one of the followings:

1. $\lambda = 0$ or 2 and $m_\lambda = 4$,
2. $\lambda \neq 1$ is a zero of $\Lambda(E, s)$ with $n_\lambda(\lambda)^{-1} \neq 0$, and $0 \leq m_\lambda \leq$ the multiplicity of zero of $\Lambda(E, s)^2$ at $s = \lambda$,
3. $\lambda \neq 1$ is a common zero of $\Lambda(E, s)$ and $n_\lambda(s)^{-1}$, and $-2 \leq m_\lambda - 2 \leq$ the multiplicity of zero of $\Lambda(E, s)^2$ at $s = \lambda$,
4. $\lambda \neq 1$ is a zero of $n_\lambda(s)^{-1}$ with $\Lambda(E, \lambda) \neq 0$, and $m_\lambda = 2$
5. $\lambda = 1$ and $-2 - 2J \leq m_\lambda - 2 - 2J \leq$ the multiplicity of zero of $\Lambda(E, s)^2$ at $s = \lambda$, where $J$ is the number of singular fibers of $E$ (see (1.2)).

Hence $w * h_\lambda \equiv 0$. Because $w$ was arbitrary, we obtain $W_\pi \subset T(h_\lambda)^\perp$.

The other implication $T(h_\lambda) \subset W_\pi^\perp$ is proved by a similar way. The following fact is useful for this direction (see [6, section 13] and [11, section 2.5]); there exists finitely many $(\phi_\alpha, f_{\pi, \alpha}) \in S(\pi)$ such that
\[
\sum_\alpha \int_0^\infty E(\phi_\alpha, f_{\pi, \alpha})(x) x^s \frac{dx}{x} = L(\pi, s - 1/2).
\]

The final assertion for $T(h_\lambda)^\perp = W_\lambda$ is obvious from (4.1) and (1) \sim (5). For $T(h_\lambda) = W_\pi^\perp$ we note that $W_\pi^\perp$ consists of $f_{\lambda, k}$ such that $\lambda$ is a zero of $n_\lambda(s)^{-2} \Lambda(E, s)^2(s - 2)^4(s - 1)^2$ and $k \leq$ the multiplicity of $\lambda$ ([11]). If $f_{\lambda, k} \in T(h_\lambda)$ then $\lambda$ is a pole of order $\geq k$ of $\text{MC}(h_\lambda)$ by the general theory of mean-periodic function (e.g. [7, Theorem in lecture 4]). Hence the cancelation can not occur when $T(h_\lambda) = W_\pi^\perp$.

4.2. Proof of Theorem 3.2. This is proved similarly to the proof of Theorem 3.1. For $T(h_\lambda \circ \mathcal{B}) \subset W_\pi^\perp$, it is sufficient to prove that $(h_\lambda \circ \mathcal{B}) * w = 0$ for any $w \in W_\pi$. By the expansion of $h_\lambda(x)$ in Theorem 2.1, $h_\lambda \circ \mathcal{B}$ is a series consisting of $f_{\lambda, k} \circ \mathcal{B}$. We have
\[
(f_{\lambda, k} \circ \mathcal{B}) * w(y) = \sum_{j=0}^k (-1)^j \binom{k}{j} |y|^{-\lambda} (|y|)^{k-j} \int_{G_\lambda \setminus \Delta} w(x) |x|^\lambda (\log |x|)^j dx.
\]
Here
\[ \int_{G_{Q} \backslash G_{A}} w(x) |x|^{\lambda} (\log |x|)^{2} dx = \frac{d^{j}}{d\lambda^{j}} \int_{G_{Q} \backslash G_{A}} w(x) |x|^{\lambda} dx, \]
and
\[ \int_{G_{Q} \backslash G_{A}} w(x) |x|^{\lambda} dx = \int_{R_{+}^{n}} w_{0}(x) x^{\lambda} \frac{dx}{x} \]
\[ \times \int_{G_{Q} \backslash G_{A}} \mathcal{E}(\phi_{1})(x) \varphi_{\pi}(x) |x|^{\lambda} dx \int_{G_{Q} \backslash G_{A}} \mathcal{E}(\phi_{2})(x) \varphi_{\pi}(x) |x|^{\lambda} dx, \]
since \( |x|^{\lambda} \) is a multiplicative (quasi) character. By Lemma 3.5 of [2],
\[ \int_{G_{Q} \backslash G_{A}} \mathcal{E}(\phi)(x) \varphi_{\pi}(x) |x|^{\lambda} dx = L(\pi, s - 1/2) F_{\phi, \varphi_{\pi}}(s) = L(E, s) F_{\phi, \varphi_{\pi}}(s), \]
where \( F_{\phi, \varphi_{\pi}}(s) \) is an entire function. Therefore \( (f_{\lambda,k} \circ \beta) \ast w(y) = 0 \) for each \( \lambda, 1 \leq k \leq m_{\lambda} \) appearing in the expansion of \( h_{E} \), since \( \lambda \) is a zero of \( L(E, s) \) or a zero of \( \int_{R_{+}^{n}} w_{0}(x) x^{\lambda} \frac{dx}{x} \). Hence \( (h_{E} \circ \beta) \ast w = 0 \) for any \( w \in W_{\pi} \).

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