INSTABILITY OF THE STANDING WAVES FOR A BENNEY-ROSKES/ZAKHAROV-RUBENCHIK SYSTEM AND BLOW-UP FOR THE ZAKHAROV EQUATIONS

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Abstract. In this paper we establish the nonlinear orbital instability of ground state standing waves for a Benney-Roskes/Zakharov-Rubenchik system that models the interaction of low amplitude high frequency waves, acoustic type waves in $N = 2$ and $N = 3$ spatial directions. For $N = 2$, we follow M. Weinstein’s approach used in the case of the Schrödinger equation, by establishing a virial identity that relates the second variation of a momentum type functional with the energy (Hamiltonian) on a class of solutions for the Benney-Roskes/Zakharov-Rubenchik system. From this identity, it is possible to show that solutions for the Benney-Roskes/Zakharov-Rubenchik system blow up in finite time, in the case that the energy (Hamiltonian) of the initial data is negative, indicating a possible blow-up result for non radial solutions to the Zakharov equations. For $N = 3$, we establish the instability by using a scaling argument and the existence of invariant regions under the flow due to a concavity argument.

1. Introduction. This work is related with the Benney-Roskes/Zakharov-Rubenchik system that describes the interaction of high-frequency and low-frequency waves in plasmas and magnetohydrodynamics given by

\[
\begin{align*}
\partial_t \psi + \epsilon \partial_z^2 \psi &= -\sigma_1 \Delta_\perp \psi + (\sigma_2 |\psi|^2 + W(\rho + D\partial_z \varphi)) \psi, \\
\partial_t \rho + \sigma_2 \partial_z \rho &= -\Delta_\perp \varphi - \partial_z^2 \varphi - D\partial_z (|\psi|^2), \\
\partial_t \varphi + \sigma_2 \partial_z \varphi &= -\frac{1}{M} \rho - |\psi|^2.
\end{align*}
\]

(BR-ZR)

The system is written in nondimensional form according to the parameters and rescaling used by T. Passot et al. [16], G. Ponce and J. C. Saut [18] and J. Cordero [5], after considering a reference frame moving at the group velocity. In the context of gravity waves, this system was deduced by D. Benney and G. Roskes in [1], and

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in other physical situations by A. Rubenchik and V. Zakharov [17]. A more precise description on this system for water waves can be found in the monographic work by D. Lannes in [12].

In this system, the function \( \psi = \psi(x, t) \in \mathbb{C} \) denotes the complex amplitude of the high frequency, \( \rho = \rho(x, t) \in \mathbb{R} \) denotes the density fluctuation and \( \varphi = \varphi(x, t) \in \mathbb{R} \) is the hydrodynamic potential, \( x \in \mathbb{R}^N \) for \( N = 2, 3 \), and \( \Delta = \Delta_\perp + \partial_x^2 \) with \( \Delta_\perp = \partial_y^2 \) in the case \( N = 2 \). The parameter \( \sigma \) measures the self-interaction of the carrying wave, \( D \) is a proportional constant to the Doppler shift \( \alpha \), \( \epsilon \) is a dispersion constant, \( W = \frac{\epsilon \alpha^2}{v_s^2} > 0 \), and \( M = \frac{|v_s|}{c_s} > 0 \) is the Mach number due to group velocity \( v_s \) (only in the direction of the \( z \)-axis) and the sound velocity \( c_s \). The constant \( \sigma_1, \sigma_2 \) are parameters depending on the group velocity, and \( \sigma \) is a factor (positive or negative) of self-interaction of \( \psi \). We refer to [11, 17] for more details on the physical background of this system.

We note that F. Oliveira in [14] in the one-dimensional case \( (N = 1) \) established the global well-posedness of this system in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \), and existence and orbital stability of the standing wave solutions using some conservation laws.

We want to point out that the system (BR-ZR) has an additional interest due to the fact that is possible to obtain the well known Zakharov system (1) and Davey-Stewartson system (3). For instance, in the supersonic regime \( M > 1 \), the Doopler shift \( \alpha \to 0 \) (resp. \( D \to 0 \)), J. Cordero in [5] showed under some constraints that solutions of the (BR-ZR) system converges to solutions of the Zakharov system

\[
\begin{cases}
  i\partial_t \psi + \epsilon \partial_x^2 \psi + \sigma_1 \Delta_\perp \psi - \hat{W} \rho \psi = 0, \\
  \partial_t \rho - \frac{1}{M^2} \Delta \rho = \Delta(|\psi|^2),
\end{cases}
\]

without any limit layer. In the subsonic regime \( M < 1 \), it is expected (see the work by E. Zakharov and V. Kuznetsov[11]) that those solutions can be approximated by solutions of the system

\[
\begin{cases}
  i\partial_t \psi + \epsilon \partial_x^2 \psi + \sigma_1 \Delta_\perp \psi = (\sigma |\psi|^2 + W(\rho + D \partial_x \varphi)) \psi, \\
  \sigma_1 \partial_t \varphi = -\Delta \varphi \partial_x^2 \psi - D \partial_x(|\psi|^2), \\
  \sigma_2 \partial_x \varphi = -\frac{1}{M^2} \rho - |\psi|^2.
\end{cases}
\]  

If we decouple the last two equations (with \( \sigma_2^2 = 1 \)), the system takes the form of a Davey-Stewartson system

\[
\begin{cases}
  i\partial_t \psi + \epsilon \partial_x^2 \psi + \sigma_1 \Delta_\perp \psi = (\sigma + W M^2) |\psi|^2 \psi + W(D - M^2 \sigma_2) \partial_x \varphi, \\
  \Delta \varphi = (1 - M^2) \partial_x^2 \varphi = -(D - M^2 \sigma_2) \partial_x(|\psi|^2),
\end{cases}
\]  

or

\[
\begin{cases}
  i\partial_t \psi + \epsilon \partial_x^2 \psi + \sigma_1 \Delta_\perp \psi = (\sigma + \beta) |\psi|^2 \psi + (c_1 + \beta^2) \rho \psi, \\
  \Delta \rho = M^2 \partial_x^2 \rho + c_2 \partial_x^2(|\psi|^2),
\end{cases}
\]  

with \( \beta \sim M \). In this case, if we make \( M \to 0 \), we obtain a Davey-Stewartson type system again

\[
\begin{cases}
  i\partial_t \psi + \epsilon \partial_x^2 \psi + \sigma_1 \Delta_\perp \psi = \sigma |\psi|^2 \psi + c_1 \rho \psi, \\
  \Delta \rho = c_2 \partial_x^2(|\psi|^2),
\end{cases}
\]

but now a mass function is the component associated with the complex wave function.

The reduction from (BR-ZR) to (2) does not have a rigorous mathematical justification yet. The subsonic limit of (4) to the system (5) depends strongly on non local operators defined by the Fourier transform. In this case, one gets a \( L^2 \) multiplier and the associated nonlocal operator is controlled, when \( M \to 0 \). We also
note that (4) is a Zakharov-Schulman type system whose global theory is known (see [8]). For more details about this simplification, we refer to [11] and [4].

On the other hand, it is known that if we have initial data with negative energy, then radial solutions \((\psi, \rho, \varphi)(t)\) associated to the Zakharov equations (12) blows up in finite time or blows up in infinite time in \(H^1\), in the sense that

\[
\lim_{t \to +\infty} ||(\psi, \rho, \varphi)(t)||_{H^1} = +\infty.
\]

But nevertheless it is still open the existence of a special class (non radial) of initial data such that solutions \((\psi, \rho, \varphi)\) of Zakharov equations blow up in finite time in \(H^1\), as in the case of the focusing Schrödinger equation. For more details of this subject, one could revise the work of Merle [13] and references therein.

In the one dimensional case, F. Oliveira in [15] obtained pointwise convergence of the magnetic field \(\psi\) in the Zakharov-Rubenchik system to a solution of the nonlinear Schrödinger equation, in the adiabatic limit to zero.

Regarding the classical Davey-Stewartson system with spatial dimension \(N=2\)

\[
\begin{cases}
    i\partial_t \psi + \delta \partial^2_x \psi + \partial^2_y \psi = \chi |\psi|^2 \psi + b \psi \partial_x \varphi, \\
    \partial^2_x \varphi + m \partial^2_y \varphi = \partial_x (|\psi|^2),
\end{cases}
\]  

(6)

where \(b, m \in \mathbb{R}, \delta \in \{-1, 1\}\) and \(\chi \in \{0, -1, 1\}\) (see [6], [2]), J. Ghidaglia and J. C. Saut in [7] addressed the well-posedness for the Cauchy problem, according to the different signs of \((\delta, m)\). In the elliptic-elliptic case \(m \delta = m > 0\) and negative energy, they obtained blow-up results, which are also relevant for us.

From the discussion above, it seems that the Benney-Roskes/Zakharov-Rubenchik system (BR-ZR) is richer than the Zakharov system (1) and Davey-Stewartson system (3). So, we expect to obtain some relevant information for the (BR-ZR) system from similar dynamics in the case of the Zakharov system and the hamiltonian Davey-Stewartson system (3), for instance.

Before we go further, we first remark that there exists a Hamiltonian structure which characterizes standing waves for the (BR-ZR) system as critical points of the action functional and also provides relevant information for the instability analysis. In this case, the Hamiltonian structure is given by

\[
\partial_t \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix} = \mathcal{J} \mathcal{H}' \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -\frac{2}{W} & 0 \end{pmatrix},
\]

(7)

where the Hamiltonian \(\mathcal{H}\) is defined as

\[
\mathcal{H} \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}^N} \left( \sigma_1 |\nabla \psi|^2 + \epsilon |\partial_z \psi|^2 + \frac{\sigma}{2} |\psi|^4 + W \rho |\psi|^2 \\
+W \sigma_2 \rho |\varphi|^2 + \frac{W}{2} |\nabla \varphi|^2 + \frac{W}{2M^2} |\rho|^2 + DW |\psi|^2 |\varphi| \right) dV.
\]

(8)

Due to the Hamiltonian structure, it is straightforward to see that there exists a functional \(Q\) (known as the charge) which is conserved in time for classical solutions and even mild solutions (Noether’s theorem). In our case, the charge \(Q\) is given by

\[
Q \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}^N} |\psi|^2 dV.
\]

(9)
J. C. Saut and G. Ponce in [18] established local well posedness results for the (BR-ZR) system in the space $H^s \times H^{s-1/2} \times H^{s+1/2}$ in the case $s > N/2$, using smoothing effects associated to the Schrödinger group. They also obtained existence of a global weak solution for initial data in $\Phi \in H^1 \times L^2 \times H^1$ when $\mathcal{H}(\Phi) > 0$, by using that the Hamiltonian $\mathcal{H}$ and the charge $Q$ are conserved quantities on solutions of the system (BR-ZR). Those results hold for any parameters $\sigma, \sigma_1, \sigma_2$.

We will see that standing waves of angular speed $\omega$ for the (BR-ZR) system corresponds to stationary solutions of the modulated system

$$\partial_t \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix} = J L_\omega \begin{pmatrix} \psi \\ \rho \\ \varphi \end{pmatrix},$$

where $L_\omega = \mathcal{H} + \omega Q$, meaning non trivial solutions of the system

$$\mathcal{H}' + \omega Q' = 0.$$

As it is well known, M. Grillakis, J. Shatah and W. Strauss in [10] established orbital stability/instability of standing and solitary waves for a class of abstract Hamiltonian systems. We will see in this case that standing waves of least energy $Y_\omega$ are actually minimums of the action functional $L_\omega$ and the instability analysis depends on the strict concavity of the real function

$$d_1(\omega) = \inf \{ L_\omega(Y) : Y \in \mathcal{M}_\omega \},$$

where $\mathcal{M}_\omega$ is a suitable set.

From previous discussion, in order to look for standing waves for the (BR-ZR) system, we need to seek for solutions of the form

$$\psi(x, t) = e^{i\omega t} u(x), \quad \rho(x, t) = v(x), \quad \varphi(x, t) = w(x),$$

where $\omega > 0$ and $(u, v, w) \in H^1 \times L^2 \times H^1$. This solutions satisfy the subsonic limit model (2), and in particular, $(u, v, w)$ verifies that

$$\begin{cases}
-\omega u &= -\epsilon \partial_x^2 u - \sigma_1 \Delta u + (\sigma |u|^2 + W(v + D \partial_x w)) u, \\
\sigma_2 \partial_x v &= -\Delta w - \partial_x^2 w - D \partial_x (|u|^2), \\
\sigma_2 \partial_x w &= -\frac{1}{M^2} v - |u|^2.
\end{cases} \quad (11)$$

We point out that in the elliptic case $\epsilon = \sigma_1 = 1$, with $\sigma_2 = 0$ and $\sigma, D \to 0$ (supersonic limit when $v_g \to \infty$), we obtain the standing wave for the Hamiltonian Zakharov system

$$\begin{cases}
i \partial_t \psi &= -\Delta \psi + W \rho \psi, \\
\partial_t \rho &= -\Delta \varphi, \\
\partial_t \varphi &= -\frac{1}{M^2} \rho - |\psi|^2, \quad (12)
\end{cases}$$

with $\nabla \varphi$ free divergence, and so on the standing wave for the focusing cubic non-linear Schrödinger equation

$$i \partial_t \psi = -\Delta \psi - \lambda |\psi| \psi, \quad (13)$$

for $\lambda = M^2 W$.

This paper is organized as follows. In section 2, we establish a local theory for the system (2). In section 3, we present some preliminaries related with the existence of standing waves (solitary wave solutions) for the (BR-ZR) system and the link between this standing waves and the Schrödinger equation. In section 4, we use a variational approach to analyze instability, which includes the analysis when the function $d$ for $\omega > 0$ is either convex or concave. In section 5, we show that standing waves are unstable by adapting the approach used by R. Cipollati in [3] in the case
of the Davey-Stewartson system. For \( N = 3 \), we establish the instability by using a scaling argument and the existence of invariant regions under the flow due to the concavity of \( d \). In the case \( N = 2 \), we use the M. Weinstein’s approach for the Schrödinger equation in [21] by using the following virial identity
\[
\frac{d^2}{dt^2} \int_{\mathbb{R}^2} (c x^2 + \sigma_1 y^2) |u(x, y, t)|^2 dx dy = 16c^2 \mathcal{H}(\Phi(t)),
\]
for any solution \( \Phi(x, y, t) = (u(x, y, t), v(x, y), w(x, y)) \) of the \((\text{BR-ZR})\) system in the regime \( |\epsilon| = |\sigma_1| \). Moreover, in the special regime \( \epsilon = \sigma_1 > 0 \), we show that any solution \( \Phi(x, y, t) = (u(x, y, t), v(x, y), w(x, y)) \) of the \((\text{BR-ZR})\) system with \( \mathcal{H}(\Phi(0)) < 0 \), necessarily blows up in finite time, so we give a positive response to two of the three questions proposed by F. Merle in [13].

2. Cauchy problem for the Hamiltonian Davey Stewartson type system.
It is important to emphasize that system (11) looks like the first Hamiltonian approximation (2) of the Davey-Stewartson system (3), in the case \( M < 1 \), for which one has
\[
\rho = -M^2(\sigma_2 \partial_x \varphi + |\psi|^2).
\]
The first remark is that the system (3) can be written as a single equation for \( \psi \), due to the fact that the second equation here is a Poisson-like equation, and
\[
\partial_x \varphi = (M^2 \sigma_2 - D)E(|\psi|^2),
\]
where \( E \) is the non local operator defined through the Fourier transform
\[
\widehat{E(u)}(\xi) := \Gamma_M(\xi)\hat{u}(\xi),
\]
with Fourier multiplier \( \Gamma_M \) given by
\[
\Gamma_M(\xi) := \frac{\xi_2^2}{(\xi_1^2 + \xi_2^2 + (1 - M^2)\xi_3^2)}.
\]
So, we reduce (2) to a nonlinear like Schrödinger equation
\[
i \partial_t \psi + \epsilon\partial_x^2 \psi + \sigma_1 \Delta_1 \psi = (\sigma - M^2 W)|\psi|^2 \psi - W(D - M^2 \sigma_2 E(|\psi|^2) \psi,
\]
where the linear operator \( \mathcal{L} := \epsilon\partial_x^2 + \sigma_1 \Delta_1 \) is non-necessarily elliptic. In the case \( N = 2 \), according to the coordinate system, we have
\[
i \partial_t \psi + \epsilon\partial_x^2 \psi + \sigma_1 \partial_y^2 \psi = (\sigma - M^2 W)|\psi|^2 \psi - W(D - M^2 \sigma_2 E(|\psi|^2) \psi.
\]
On the other hand, we see that the system (3) can be put in the form of the system (6), if we make the changes
\[
\mathbf{x} = \sqrt{\sigma_1} \tilde{\mathbf{x}}, \quad \psi(\mathbf{x}, t) = \frac{\sqrt{1 - M^2}}{|\sigma_1| \sqrt{M^2 \sigma_2 - D}} \tilde{\psi}(\tilde{\mathbf{x}}, t), \quad \varphi(\mathbf{x}, t) = \tilde{\varphi}(\tilde{\mathbf{x}}, t)
\]
with the restrictions
\[
D < \sigma_2 M^2,
\]
and
\[
\delta = \frac{\epsilon}{|\sigma_1|}, \quad b = \frac{W(D - M^2 \sigma_2)}{|\sigma_1|^2 \sqrt{M^2 \sigma_2 - D}} , \quad \chi = \frac{1 - M^2}{\sqrt{|\sigma_1|(D - M^2 \sigma_2)}} , \quad m = \frac{1}{1 - M^2}.
\]

Then, after dropping the tilde symbols, the equation (18) takes the form
\[
i \partial_t \psi + \mathcal{L}_1 \psi = F(\psi),
\]
with nonlinearity
\[ F(\psi) = \chi|\psi|^2\psi + bE(|\psi|^2)\psi \] (22)
and linear part
\[ L_1\psi = \delta\partial_x^2\psi + \Delta_\perp\psi. \] (23)
If we denote by \( S(t) \) the unitary group in \( H^s(\mathbb{R}^N) \) generated by the Schrödinger type operator
\[ P\psi = i\partial_t\psi + L_1\psi, \]
then we transform (21) in the integral equation
\[ \psi(t) = S(t)\psi_0 - i\Lambda f(\psi(t)). \] (24)
In order to address the well-posedness issue, it is important to note that the multiplier \( \Gamma_M \) is homogeneous of order 0 for \( 0 < M < 1 \), so we can use the Hormander-Mikhlin theorem to ensure that
\[ \|\partial_x\varphi\|_{L^p(\mathbb{R}^N)} \leq C_{p\varphi}\|\psi\|^2_{L^2(\mathbb{R}^N)}. \] (25)
On the other hand, the quadratic form associated with the operator \( L_1 \) is not degenerate, that is, the \( N \times N \) matrix
\[ A = \begin{bmatrix} I_{N-1} & 0 \\ 0 & \delta \end{bmatrix} \] (26)
is invertible for \( \epsilon \neq 0 \), where \( I_{N-1} \) denotes the \( (N - 1) \times (N - 1) \) identity matrix. Therefore, \( L^p - L^q \) estimates of Strichartz type on the free propagator \( S(t) \) hold (see [7]), as in the usual elliptic Schrödinger equation.

By using a contraction argument in dimensions \( N = 2, 3 \), in appropriate spaces, we have the following local result for the Cauchy problem associated to (2).

**Theorem 2.1.** Suppose that \( 0 < M < 1 \) (\( m > 0 \)).

1. Let \( \psi_0 \in L^2(\mathbb{R}^N) \). There exist \( T^* > 0 \) and a unique maximal solution \((\psi, \varphi, \rho)\) of (2) on \([0, T^*)\) such that \( \psi(0) = \psi_0, \|\psi(t)\| = \|\psi_0\| \) for \( 0 \leq t < T^* \) and
\[ \psi \in C([0, T^*); L^2(\mathbb{R}^N)) \cap L^4((0, t) \times \mathbb{R}^N), \quad \nabla\varphi, \rho \in L^2((0, t) \times \mathbb{R}^N). \]
2. If \( \psi_0 \) is sufficiently small in \( L^2(\mathbb{R}^N) \), then \( T^* = \infty \).
3. If \( \psi_0 \in H^1(\mathbb{R}^N) \), then the previous solution satisfies, for every \( t \in [0, T^*), p \in [2, \infty) \) and \( q \in [2, 4] \),
\[ \psi \in C([0, T^*); H^1(\mathbb{R}^N)) \cap C^1([0, T^*); H^{-1}(\mathbb{R}^N)) \]
\[ \nabla\psi \in L^4((0, t) \times \mathbb{R}^N), \quad \nabla\varphi \in C([0, T^*); H^1(\mathbb{R}^N)), \rho \in C([0, T^*); L^2(\mathbb{R}^N)) \]
\[ \nabla^2\varphi \in L^4((0, t) \times \mathbb{R}^N), \quad \nabla\rho \in L^4((0, t); L^2(\mathbb{R}^N)). \]
Moreover, \( \mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0) \).

This local and regularity result associated with the Cauchy Problem for (2) is obtained by following the classical ideas in the context of the Schrödinger equation. The proof can be extended mutatis-mutandis from the well-posedness results for the 2D-Davey-Stewarton system obtained by J. Ghidaglia and JC. Saut in [7]. The respective topology for the components \( \varphi \) and \( \rho \), are consequences of its dependence of \( \psi \), as established in the representation (15) and (14).
Hence, replacing this in the second equation, we find out that

\[ \text{From the last equation in system (11), we have that} \]

standing wave equation for the (BR-ZR) system in terms of the first component.

3. Existence of standing waves.

Sobolev spaces

standing waves, we require having an existence result of solutions in the weighted Sobolev

such that the momentum type functional

\[ \mathcal{M}(t) = \int_{\mathbb{R}^2} (ex^2 + \sigma_1 y^2)|\psi(x,y,t)|^2 \, dx \, dy, \]

makes sense. As done for J. Ghidaglia and J. C. Saut in the case of the Davey Stewartson systems in Theorem 3.1 and Lemma 3.1 in [7], the idea is to characterize the existence of solutions as a fixed point problem, whose principal ingredients are Strichartz type estimates and a conformal invariance property due to the symmetric non-singular matrix \( A \) of the linear operator \( \mathcal{L}_1 \). In this context, we have the following result,

**Theorem 2.2.**

If \( \psi_0 \in \Sigma \), then the solution given by Theorem 2.1 satisfies for \( 0 \leq t < T^* \) that

\[ \psi \in C([0,T^*); \Sigma), \quad \nabla \psi \in L^4((0,t) \times \mathbb{R}^2), \quad (x^2 + y^2)^{\frac{1}{2}} \psi \in L^4((0,t) \times \mathbb{R}^2). \]

3. Existence of standing waves. Now, we focus our attention in describing the standing wave equation for the (BR-ZR) system in terms of the first component.

From the last equation in system (11), we have that

\[ v = -M^2 \sigma_2 \partial_z w - M^2 |u|^2. \]

Hence, replacing this in the second equation, we find out that

\[ \sigma_2 \partial_z v = -M^2 (\sigma_2)^2 \partial_z^2 w - \sigma_2 M^2 \partial_z (|u|^2) = -\Delta_1 w - \partial_z^2 w - D \partial_z (|u|^2), \]

which implies that we can eliminate \( v \). Moreover, we also have that

\[ -\Delta_1 w - (1 - M^2) \partial_z^2 w = -(M^2 \sigma_2 - D) \partial_z (|u|^2) \quad \text{(with} \sigma^2_2 = 1). \]

Now, if we take formally Fourier transform, we are able to express \( w \) in term of \( u \) since,

\[ (\xi_1^2 + \xi_2^2 + (1 - M^2) \xi_3^2) \tilde{w} = -(M^2 \sigma_2 - D)i \xi_3 |\tilde{u}|^2. \]

In the case \( 0 < M < 1 \), we see that

\[ \partial_z \tilde{w} = (M^2 \sigma_2 - D) \Gamma_M(\xi)(|\tilde{u}|^2). \]

So, we conclude that

\[ \partial_z w = (M^2 \sigma_2 - D) E(|u|^2). \]

From this fact, we have that \( u \) satisfies the single equation

\[ -\omega u = -\epsilon \partial_z^2 u - \sigma_1 \Delta_1 u + (\sigma - M^2 W)|u|^2 u - W(D - M^2 \sigma_2)^2 E(|u|^2)u. \]

Therefore, if \( (\psi, \rho, \varphi) \) is a solution of the (BR-ZR) system of the form (10), then \( u \) must satisfy the following problem

\[ \begin{cases} 
\omega u - \epsilon \partial_z^2 u - \sigma_1 \Delta_1 u = (M^2 W - \sigma)|u|^2 u + W(D - M^2 \sigma_2)^2 E(|u|^2)u. \\
 u \in H^1(\mathbb{R}^N) \setminus \{0\}. 
\end{cases} \]

On the other hand, \( u \) is a solution of (31) if and only if \( u \) is a critical point of the functional

\[ \ell_\omega(u) = \frac{1}{2} I_\omega(u) + \frac{1}{4} G(u), \]
where $G$ and $I_\omega$ are defined by

$$I_\omega(u) = \int_{\mathbb{R}^N} (\omega|u|^2 + \sigma_1|\nabla u|^2 + \epsilon|\partial_\perp u|^2) \, dV,$$

$$G(u) = -\int_{\mathbb{R}^N} (W(D - M^2\sigma_2)^2E(|u|^2)|u|^2 + (M^2W - \sigma)|u|^4) \, dV.$$ 

As done for many related models, existence of standing and solitary wave solutions are obtained via the Lion’s Concentration-Compactness Principle and the existence of a local compact embedding result. We want to mention that R. Cipolletti in [2] established the existence of solutions (standing waves) for a more general equation than (31), with a similar nonlocal term but with a non-linearity of order $\alpha + 1 > 1$. Even though, the proof in the cubic non-linearity in the present work is easier due to the homogeneity of the nonlinear part, we include the details to make easier the reading and to have a self-contained work.

The strategy of this approach is to consider the following minimization problem

$$I_\omega := \inf \{ I_\omega(u) : u \in H^1(\mathbb{R}^N) \text{ with } G(u) = -1 \}. \quad (33)$$

We note for $\omega, \epsilon, \sigma_1 > 0$ that $\sqrt{I_\omega(u)} \sim \|u\|_{H^1}$ since we have that

$$C^{-1}\|u\|_{H^1}^2 \leq I_\omega(u) \leq C\|u\|_{H^1}^2. \quad (34)$$

On the other hand, we note that for $\|\xi\| = 1$

$$\xi_1^2 + \xi_2^2 + (1 - M^2)\xi_3^2 = \|\xi\|^2 - M^2\xi_3^2 = 1 - M^2\xi_3^2 \neq 0,$$

for $M$ small enough, then we also have that

$$\Gamma_M(\xi) = \frac{\xi_3^2}{|\xi|^2 - M^2\xi_3^2}$$

is of class $C^2$ on the unit sphere, for $M$ small enough. From the fact that $\Gamma_M$ is a homogeneous of degree 0, then the Hormander-Mikhlin theorem implies that $\Gamma_M$ is a multiplier for $L^p(\mathbb{R}^N)$, that is, $E$ is a bounded operator in $L^p(\mathbb{R}^N)$, $1 < p < \infty$ (see [19]). Then we conclude that

$$|G(u)| \leq C\left(\|u\|_{L^4(\mathbb{R}^N)}^4 + \|E(|u|^2)\|_{L^2(\mathbb{R}^N)}\|u\|_{L^4(\mathbb{R}^N)}^4\right) \leq C_2\|u\|_{L^4}^4,$$

$$\leq C_3\|\nabla u\|_{L^2(\mathbb{R}^N)}^2\|u\|_{L^4(\mathbb{R}^N)}^2 \leq C_4\|u\|_{H^1(\mathbb{R}^N)}^4, \quad (35)$$

because of the Gagliardo-Nirenberg inequality. From this fact, we have for any $u \in H^1(\mathbb{R}^N)$ such that $G(u) = -1$, that $\|u\|_{H^1(\mathbb{R}^N)} > C_1$, implying that $I_\omega$ is finite and positive. Moreover, if $u_0$ is a minimizer for problem (33), then $u = \sqrt{2\nu}u_0$ is a nontrivial solution of (31).

Before we go further, we set some definitions. For a given $u \in H^1(\mathbb{R}^N)$, we define the measure

$$\nu(A) = \int_A \mu(x) \, dV$$

on the Borelians in $\mathbb{R}^N$, where the density $\mu$ is given by

$$\mu = \omega|u|^2 + \epsilon|\partial_\perp u|^2 + \sigma_1|\nabla u|^2.$$

On the other hand, we use the notation $\langle \cdot, \cdot \rangle_2$ to represent the inner product in $L^2(\mathbb{R}^N)$. Now, for a differentiable function $F : X \to Y$, we use the following convention: for $f, g \in X$,

$$F'(f)(g) := \langle F'(f), g \rangle.$$
The aim of this section is to establish the existence of standing waves for the system (BR-ZR) of the form (10) and for \( M \ll 1 \), which follows from the following result.

**Theorem 3.1.** If \( \{u_m\}_m \) is a minimizing sequence for (33), then there is a subsequence (which we denote the same), a sequence of points \((x_m)_m \in \mathbb{R}^N \), and a minimizer \( u_0 \in H^1(\mathbb{R}^N) \) of (33), such that the translated functions \( \tilde{u}_m = u_m(\cdot + x_m) \) converge to \( u_0 \) strongly in \( H^1(\mathbb{R}^N) \).

The proof of this result is a consequence of the Lion’s Concentration-Compactness Principle. For a given minimizing sequence \( \{u_m\}_m \) of (33), we consider the sequence of measures \( \{\nu_m\}_m \) given by \( \nu_m = \mu_m \, dV \). So, we have that \( \nu_m \) is a positive measure with finite total variation, since \( \nu_m(\mathbb{R}^N) \to I_\omega \), as \( m \to \infty \).

**Lemma 3.2.** Vanishing is not possible.

**Proof.** Suppose that for any \( R > 0 \), we have that
\[
\lim_{m \to \infty} \left( \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} \mu(u_m) \, dV \right) = 0,
\]where \( B_R(x) \) is the ball of radius \( R \) centered at \( x \). As
\[
\int_{\mathbb{R}^N} E(|u|^2)|u|^2 \, dV \leq C_1 \int_{\mathbb{R}^N} |u|^4 \, dV,
\]we conclude that \( 1 = |G(u_m)| \leq C_2 \int_{\mathbb{R}^N} |u_m|^4 \, dV \). On the other hand, there is \( C > 0 \) such that
\[
\|u_m\|_{L^2(B_R(x))}^2 + \|
abla u_m\|_{L^2(B_R(x))}^2 + \|\nabla \perp u_m\|_{L^2(B_R(x))}^2 \leq C(\omega, \epsilon, \sigma) \int_{B_R(x)} d\nu_m.
\]Moreover, we also have that
\[
\int_{B_R(x)} |u_m|^4 \, dV \leq \left( \int_{B_R(x)} |u_m|^2 \, dV \right)^{1/2} \left( \int_{B_R(x)} |u_m|^6 \, dV \right)^{1/2}
\leq C\|u_m\|_{H^1(B_R(x))}^3 \left( \int_{B_R(x)} d\nu_m \right)^{1/2}
\]because of the Sobolev embedding \( H^1(B_R(x)) \hookrightarrow L^6(B_R(x)) \) for \( N = 2, 3 \).

Now, we know that \( \mathbb{R}^N \) can be covered with balls of radius \( R \) in such a way that each point of \( \mathbb{R}^N \) is contained in at most \( N + 1 \) balls. So, from this we conclude that
\[
\int_{\mathbb{R}^N} |u_m|^4 \, dV \leq (N + 1)C\|u_m\|_{H^1(\mathbb{R}^N)}^3 \left( \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\nu_m \right)^{1/2}
\leq (N + 1)C(I_\omega(u_m))^{3/2} \left( \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\nu_m \right)^{1/2}.
\]
As a consequence of this we see that
\[
1 = \lim_{m \to \infty} |G(u_m)| \leq C \lim_{m \to \infty} \int_{\mathbb{R}^N} |u_m|^4 \, dV = 0,
\]reaching then a contradiction. \( \square \)

**Lemma 3.3.** Dichotomy is not possible.
Proof. If we assume Dichotomy, then there is \(0 < \gamma < I_\omega\) such that for a given \(\eta > 0\), there exist \(R_0 > 0\), a sequence \((x_m)_m \subset \mathbb{R}^N\), \(R_N \uparrow +\infty\), and a bounded sequence \((u_{m,i})_m \subset H^1(\mathbb{R}^N)\) for \(i = 1, 2\) (all depending on \(\eta\)) such that
\[
\text{supp}(u_{m,1}) \subset B_{R_0}(x_m), \quad \text{supp}(u_{m,2}) \subset \mathbb{R}^N \setminus B_{R_m}(x_m) \quad (37)
\]
\[
\|u_m - u_{m,1} - u_{m,2}\|_{H^1(\mathbb{R}^N)} \leq \eta \quad (38)
\]
\[
\|u_m\|_{L^2(B_{R_m}(x))}^2 + \|\partial_z u_m\|^2_{L^2(B_{R_m}(x))} + \|\nabla_{\perp} u_m\|^2_{L^2(B_{R_m}(x))} \leq C(\omega, \epsilon, \sigma_1) \int_{B_{R_m}(x)} d\nu_m \quad (39)
\]
\[
\limsup_{m \to \infty} \left( |\gamma - \int_{\mathbb{R}^N} \mu(u_{m,1}) d\nu| + |(I_\omega - \gamma) - \int_{\mathbb{R}^N} \mu(u_{m,2}) d\nu| \right) \leq \eta. \quad (40)
\]
The first remark (passing to a subsequence) is that
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (\mu(u_m) - \mu(u_{m,1}) - \mu(u_{m,2})) d\nu \right) = 0,
\]
and we have that
\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^N} (\mu(u_m) - \mu(u_{m,1}) - \mu(u_{m,2})) d\nu \right| \leq \lim_{n \to \infty} \left| \gamma - \int_{\mathbb{R}^N} \mu(u_{m,1}) d\nu \right| + \left| (I_\omega - \gamma) - \int_{\mathbb{R}^N} \mu(u_{m,2}) d\nu \right|.
\]
Moreover, we also have that
\[
\lim_{n \to \infty} \|u_m - (u_{m,1} + u_{m,2})\|_{L^1(\mathbb{R}^N)} = 0, \quad (41)
\]
and for \(r = 2, 4\) that
\[
\lim_{n \to \infty} \|u_m\|_{r}^r - \|u_{m,1}\|_{r}^r - \|u_{m,2}\|_{r}^r = 0. \quad (42)
\]
If we set \(G_1(u) = \langle E(|u|^2), |u|^2 \rangle_2\), we claim that
\[
\lim_{n \to \infty} |G_1(u_{m,1}) - G_1(u_{m,1}) - G_1(u_{m,2})| = 0.
\]
Now, a direct computation shows that
\[
G_1(u_{m,1} + u_{m,2}) = G_1(u_{m,1}) + G_1(u_{m,2}) + 2 \langle E(|u_{m,1}|^2), |u_{m,2}|^2 \rangle_2
\]
On the other hand, we see that
\[
\langle E(|u_{m,1}|^2), |u_{m,2}|^2 \rangle_2 = \int_{\mathbb{R}^N \setminus B_{R_m}(x_m)} E(|u_{m,1}|^2), |u_{m,2}|^2 d\nu
\]
\[
= \int_{\mathbb{R}^N \setminus B_{R_m}(0)} E(|\tilde{u}_{m,1}|^2), |\tilde{u}_{m,2}|^2 d\nu
\]
where \(\tilde{u}_{m,j}(x) = u_{m,j}(x - x_m), \quad j = 1, 2\). Now, we know that \(H^1(B_{R_0}(0)) \hookrightarrow L^q(B_{R_0}(0))\) compactly for \(q \geq 1\), if \(N = 2\), and for \(2 \leq q \leq 6\), if \(N = 3\). Due to the fact that \((u_{m,1})_m\) is bounded in \(H^1(\mathbb{R}^N)\), we conclude (passing to a subsequence, if necessary) that there is \(u_{0,1} \in H^1(B_{R_0}(0))\) such that \(u_{m,1} \rightarrow u_{0,1}\) in \(L^q(B_{R_0}(0))\).
So, we have that
\[
E(|\tilde{u}_{m,1}|^2) \rightarrow E(|u_{0,1}|^2) \quad \text{in} \ L^q(B_{R_0}(0)).
\]
So, from this we easily get that
\[
\lim_{m \to \infty} \langle E(|u_{m,1}|^2), |u_{m,2}|^2 \rangle_2 = 0,
\]
meaning that
\[ \lim_{m \to \infty} |G_1(u_{m,1} + u_{m,2}) - G_1(u_{m,1}) - G_1(u_{m,2})| = 0. \]

On the other hand, we see that
\[ |G_1(u_m) - G_1(u_{m,1}) - G_1(u_{m,2})| \]
\[ \leq |G_1(u_m) - G_1(u_{m,1} + u_{m,2})| + |G_1(u_{m,1} + u_{m,2}) - G_1(u_{m,1}) - G_1(u_{m,2})|. \]

Now, since \( u_m - (u_{m,1} + u_{m,2}) \to 0 \) in \( L^r(\mathbb{R}^N) \) for \( r = 2, 4 \), and that \( E \in L_b(L^p(\mathbb{R}^N, \mathbb{R}^N)) \), then we conclude from (41) that
\[ |G_1(u_m) - G_1(u_{m,1} + u_{m,2})| \leq \langle E(||u_m|^2 - |u_{m,1} + u_{m,2}|^2), |u_m|^2 \rangle_2 \]
\[ + \langle E(|u_{m,1} + u_{m,2}|^2), ||u_m|^2 - |u_{m,1} + u_{m,2}|^2 \rangle_2 \to 0, \ m \to \infty. \]

So, as claimed above,
\[ \lim_{m \to \infty} |G_1(u_{m,1} + u_{m,2}) - G_1(u_{m,1}) - G_1(u_{m,2})| = 0. \]

In other words, we have shown that
\[ \lim_{m \to \infty} (I_\omega(u_m) - I_\omega(u_{m,1}) - I_\omega(u_{m,2})) = 0 \]
\[ \lim_{m \to \infty} (G(u_m) - G(u_{m,1}) - G(u_{m,2})) = 0. \]

Now, let \( \lambda_{m,i} = -G(u_{m,i}) \), for \( i = 1, 2 \). So, passing to a subsequence, if necessary, we have that \( \lambda_i \) exists. We claim that \( \lambda_i \neq 0 \). First we may assume that \( \lim_{m \to \infty} \lambda_{m,1} = 0 \). Then we have that \( \lim_{m \to \infty} \lambda_{m,2} = 1 \), implying that \( \lambda_{m,2} > 0 \), for \( m \) large enough. Now, we consider
\[ w_m = \lambda_{m,2}^{-\frac{1}{2}} u_{m,2}. \]
Then we have that, \( G(w_m) = -1 \). On the other hand,
\[ I_\omega = \lim_{m \to \infty} \left( I_\omega(u_{m,1}) + I_\omega(u_{m,2}) \right) \]
\[ = \lim_{m \to \infty} \left( I_\omega(u_{m,1}) + \lambda_{m,2}^\frac{1}{2} I_\omega(w_m) \right) \]
\[ \geq \lim_{m \to \infty} \left( I_\omega(u_{m,1}) + \lambda_{m,2}^\frac{1}{2} I_\omega \right) = \gamma + I_\omega. \]
In other words, \( |\lambda_{m,i}| > 0 \) for \( m \) large enough. Then we are allowed to define
\[ w_{m,i} = \lambda_{m,i}^{-\frac{1}{2}} u_{m,i}, \ i = 1, 2. \]
Then as above, we have that \( G(w_{m,i}) = -1 \). So that,
\[ I_\omega = \lim_{m \to \infty} \left( I_\omega(u_{m,1}^1) + I_\omega(u_{m,2}^2) \right) \]
\[ = \lim_{m \to \infty} \left( \lambda_{m,1}^\frac{1}{2} I_\omega(w_{m,1}) + \lambda_{m,2}^\frac{1}{2} I_\omega(w_{m,2}) \right) \]
\[ \geq \left( \lambda_1^\frac{1}{2} + \lambda_2^\frac{1}{2} \right) I_\omega. \]

Then
\[ 1 \geq \lambda_1^\frac{1}{2} + \lambda_2^\frac{1}{2} \geq (\lambda_1 + \lambda_2)^\frac{1}{2} = 1, \]
meaning that
\[ \lambda_1^\frac{1}{2} + \lambda_2^\frac{1}{2} = (\lambda_1 + \lambda_2)^\frac{1}{2}. \]
But previous equality led us to a contradiction due to the fact that the function \( f(t) = t^2 \) is strictly concave for \( t \in \mathbb{R}^+ \), and so
\[
f(t_1 + t_2) < f(t_1) + f(t_2), \quad \text{for } t_1, t_2 > 0.
\]
In other words, we have ruled out dichotomy.

**Proof of Theorem 3.1.** From previous Lemmas, we have compactness. So, there exists a sequence \((x_m)_m \subset \mathbb{R}^N\) such that for a given \( \eta > 0 \), there exists \( R > 0 \) with the following property,
\[
\int_{B_R(x_m)} \mu(u_m) \, dV \geq I_\omega - \eta, \quad \text{for all } m \in \mathbb{N}.
\]
(44)

Now, we localize the minimizing sequence \((u_m)_m\) around the origin by defining \( \tilde{u}_m(x) = u_m(x + x_m) \). Thus, we have the following localized inequality
\[
\int_{B_R(0)} \mu(\tilde{u}_m) \, dV = \int_{B_R(x_m)} \mu(u_m) \, dV \geq I_\omega - \eta, \quad \text{for all } m \in \mathbb{N},
\]
(45)
and also that
\[
G(\tilde{u}_m) = G(u_m) = -1, \quad \lim_{m \to \infty} I_\omega(\tilde{u}_m) = \lim_{m \to \infty} I_\omega(u_m) = I_\omega.
\]
(46)

Then by (34) we note that \((\tilde{u}_m)_m\) is a bounded sequence in \( X = H^1(\mathbb{R}^N) \). On the other hand, since \( \tilde{u}_m \in H^1(U) \) for any bounded open set \( U \subset \mathbb{R}^N \) and the embedding \( H^1(U) \hookrightarrow L^q(U) \) is compact for \( q \geq 1 \), if \( N = 2 \), and for \( 2 \leq q \leq 6 \), if \( N = 2 \). Then there exist a subsequence (denoted the same) of \((\tilde{u}_m)_m\) and \( u_0 \in H^1(\mathbb{R}^N)\) such that for \( i = 1, 2 \),
\[
\tilde{u}_m \to u_0 \quad \text{in } H^1(\mathbb{R}^N), \quad \tilde{u}_m \to u_0 \quad \text{in } L^2(\mathbb{R}^N)
\]
and we also have that
\[
\tilde{u}_m \to u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \quad \partial_i \tilde{u}_m \to \partial_i u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N).
\]
Moreover, we also have
\[
\tilde{u}_m \to u_0 \quad \text{a.e. in } \mathbb{R}^N, \quad \partial_i \tilde{u}_m \to \partial_i u_0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{for } i = 1, 2.
\]

We claim that
\[
\tilde{u}_m \to u_0, \quad \partial_i \tilde{u}_m \to \partial_i u_0 \quad \text{in } L^2(\mathbb{R}^N).
\]
(47)
From the definition of \( I_\omega \) and using (45), (46), we have that for \( \eta > 0 \), that there exists \( R > 0 \) such that for \( m \) large enough,
\[
\int_{B_R(0)} |\tilde{u}_m|^2 \, dV \geq \int_{\mathbb{R}^N} |\tilde{u}_m|^2 \, dV - 2\eta.
\]

Now, from Fatou Lemma we have that
\[
\int_{\mathbb{R}^N} |u_0|^2 \, dV \leq \liminf_{m \to \infty} \int_{\mathbb{R}^2} |\tilde{u}_m|^2 \, dV
\]
\[
\leq \liminf_{m \to \infty} \int_{B_R(0)} |\tilde{u}_m|^2 \, dV + 2\eta
\]
\[
= \int_{B_R(0)} |u_0|^2 \, dV + 2\eta
\]
\[
\leq \int_{\mathbb{R}^N} |u_0|^2 \, dV + 2\eta.
\]
which implies that
\[ \liminf_{m \to \infty} \int_{\mathbb{R}^N} |\tilde{u}_m|^2 dV = \int_{\mathbb{R}^N} |u_0|^2 dV. \]
In other words, we have that \( \tilde{u}_m \to u_0 \) in \( L^2(\mathbb{R}^N) \). Using a similar argument we prove the other part of (47). Moreover, we can also see that
\[ \partial_i \tilde{u}_m \to \partial_i u_0, \quad \partial_{ij} \tilde{v}_m \to \partial_{ij} v_0 \] in \( L^2(\mathbb{R}^N) \).\hfill (48)
Now, using the fact that the inclusion \( H^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N) \) is continuous, we have
\[ G(u_0) = \lim_{m \to \infty} G(\tilde{u}_m) = -1, \tag{49} \]
which allows us to conclude \( u_0 \neq 0 \). On the other hand, from (47)-(48), we conclude that
\[ \lim_{m \to \infty} I_\omega(\tilde{u}_m) = I_\omega(u_0) = I_\omega, \quad \lim_{m \to \infty} I_\omega(\tilde{u}_m - u_0) = 0. \]
Moreover, the sequence \( (\tilde{u}_m)_m \) converges to \( u_0 \) in \( H^1(\mathbb{R}^N) \), since
\[ \|\tilde{u}_m - u_0\|_{H^1(\mathbb{R}^N)} \leq C_1 I_\omega(u_m - u_0). \]
In other words, we have established that the sequence \( (\tilde{u}_m)_m \) converges to \( u_0 \) in \( H^1(\mathbb{R}^N) \) and \( u_0 \) is a nontrivial minimizer for \( I_\omega \).\hfill \( \square \)

4. Variational approach to analyze instability. Recall that the standing waves of the (BR-ZR) system of the form (10) are characterized as critical points of the functional defined on \( \mathcal{X} \) by
\[ \ell_\omega(u) = \frac{1}{2} I_\omega(u) + \frac{1}{4} G(u) = \frac{1}{2} I_0(u) + \frac{\omega}{2} B(u) + \frac{1}{4} G(u), \]
where \( B(u) = \|u\|_2^2 \). Now, we define the functional \( K_\omega \) by
\[ K_\omega(u) = \left( \frac{N - 2}{2} \right) I_0(u) + \left( \frac{N \omega}{2} \right) B(u) + \frac{N}{4} G(u) \tag{50} \]
\[ := \frac{1}{2} I_{\omega, N}(u) + \frac{N}{4} G(u), \tag{51} \]
and define the set \( \mathcal{M}_\omega \) by
\[ \mathcal{M}_\omega = \{ u \in H^1(\mathbb{R}^N) : K_\omega(u) = 0, \ u \neq 0 \}. \tag{52} \]
From this definition, we get for \( u \in \mathcal{M}_\omega \) that
\[ \ell_\omega(u) = \frac{1}{N} I_0(u). \tag{53} \]
Hereafter we assume that \( \omega, \epsilon, \sigma > 0 \).

**Lemma 4.1.** If \( \phi_\omega \in H^1(\mathbb{R}^N) \) is a solution of (31), then we have that \( K_\omega(\phi_\omega) = 0 \).

**Proof.** Let \( u = \phi_\omega \), and \( x = \beta y \). For \( u_\beta(x) = u(y) \), we see that
\[ \ell_\omega(u_\beta) = \frac{\beta^{N-2}}{2} I_0(u) + \frac{\beta^N \omega}{2} B(u) + \frac{\beta^N}{4} G(u). \]
Using that \( u \) is a solution of equation (31), we have that 
\[ \ell'_\omega(u) = \left( \frac{d\ell_{\omega}(u)}{d\beta} \right)_{\beta=1} = 0, \]
which means that 
\[ K_\omega(u) = \left( \frac{N-2}{2} \right) I_0(u) + \left( \frac{N\omega}{2} \right) B(u) + \frac{N}{4} G(u) = 0. \]

\[ \Box \]

Now, we note that the set \( \mathcal{M}_\omega \) is an “artificial constrain” for minimizing the functional \( \ell_\omega \) on \( H^1(\mathbb{R}^N) \). As we know, the analysis of the orbital stability/instability of ground states solutions depends upon some properties of the function \( d \) defined by
\[ d(\omega) = \inf \{ \ell_\omega(u) : u \in \mathcal{M}_\omega \}. \]
For us, a ground state solution is a standing wave which minimizes the action functional \( L_\omega \) among all the nonzero solutions of (31). Moreover, the set of ground state solutions
\[ \mathcal{G}_\omega = \{ u \in \mathcal{M}_\omega : d(\omega) = \ell_\omega(u) \} \]
can be characterized as
\[ \mathcal{G}_\omega = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : d(\omega) = \frac{1}{N} I_0(u) \right\} \subset \mathcal{M}_\omega. \]

In the next lemmas we present important variational properties of \( d(\omega) \).

**Lemma 4.2.** We have that
1) \( \mathcal{M}_\omega \) is a \( C^1 \) hypersurface in \( H^1(\mathbb{R}^N) \) bounded away from zero.
2) \( d(\omega) = \inf \left\{ \frac{1}{N} I_0(u) : K_\omega(u) \leq 0, u \neq 0 \right\} \).
3) \( d(\omega) = \inf \{ \ell_\omega(u) : I_0(u) = I_0(\phi_\omega) \} \).

**Proof.** 1. The first remark is that \( K_\omega \in C^1 \). Let \( u_0 \in \mathcal{M}_\omega \) such that \( \delta K_\omega(u_0) = 0 \). Then we know that \( u_0 \) satisfies the differential equation
\[ (N-2)\tilde{\Delta} u_0 - N\omega u_0 - \frac{N}{4} \delta G(u_0) = 0, \]
where \( \tilde{\Delta} = \sigma \Delta_{\perp} + c\partial^2_{zz} \). On the other hand, multiplying for \( u_0 \) and integrating, we have that
\[ -(N-2)I_0(u_0) - N\omega B(u_0) - NG(u_0) = 0. \]
Using that \( K_\omega(u_0) = 0 \), we conclude that
\[ (N-2)I_0(u_0) + N\omega B(u_0) = 0, \]
which implies that \( u_0 = 0 \). Moreover, for \( u_0 \in \mathcal{M}_\omega \), we have that \( \delta K_\omega(u_0) \neq 0 \) and that \( \mathcal{M}_\omega \) is a \( C^1 \) hypersurface. Now we will show that \( K_\omega \) is bounded away from zero. We need to recall inequality (35)
\[ |G(u)| \leq C \|u\|_{H^1(\mathbb{R}^N)}^4. \]

Then, we see for \( N > 2 \) that
\[ K_\omega(u) = \left( \frac{N-2}{2} \right) I_0(u) + \left( \frac{N\omega}{2} \right) B(u) + \frac{N}{4} G(u) \]
\[ \geq C_1(N,\omega) \|u\|_{H^1(\mathbb{R}^N)}^2 - C_2(N) \|u\|_{H^1(\mathbb{R}^N)}^4 \]
\[ \geq \|u\|_{H^1(\mathbb{R}^N)}^2 (C_1(N,\omega) - C_2(N) \|u\|_{H^1(\mathbb{R}^N)}^2). \]
So, for $\|u\|_{H_1(\mathbb{R}^N)}^2$ small enough, we have that $K_\omega > 0$, meaning that $\mathcal{M}_\omega$ is bounded away from zero. Now, for $N = 2$, we have that

$$\left(\frac{N\omega}{2}\right) B(u) = -\frac{N}{4} G(u) \leq C(N)\|u\|_4^4,$$

So, from the Galiardo-Nirenberg-Sobolev's inequality, we conclude that

$$\left(\frac{N\omega}{2}\right) \|u\|_2^2 = \left(\frac{N\omega}{2}\right) B(u) \leq C(N)\|u\|_4^4 \leq C_1(N)\|\nabla u\|_2^2,$$

meaning again that $\mathcal{M}_\omega$ is bounded away from zero.

2. Now, take $u \in H_1(\mathbb{R}^N)$ such that $K_\omega(u) \leq 0$. Then we have that $G(u) < 0$. So, we take $\alpha \in [0, 1)$ defined by

$$\alpha^2 = -\frac{2I_0(Nu)}{NG(u)} \leq 1.$$

From this choice, we have that $K_\omega(\alpha u) = 0$, implying that $\alpha u \in \mathcal{M}_\omega$. So, we have that

$$d(\omega) \leq \ell_\omega(\alpha u) = \frac{\alpha^2}{N} I_0(u) \leq \frac{1}{N} I_0(u).$$

Hence, we obtain that

$$d(\omega) \leq \inf \left\{ \frac{1}{N} I_0(u) : K_\omega(u) \leq 0 \right\}.$$

Now, if $u \in \mathcal{M}_\omega$, we see that $L_\omega(u) = \frac{1}{N} I_0(u)$ and also that

$$\inf \left\{ \frac{1}{N} I_0(u) : K_\omega(u) \leq 0, u \neq 0 \right\} \leq \inf \left\{ \ell_\omega(u) : u \in \mathcal{M}_\omega \right\} = d(\omega),$$

as desired.

3. Let $u \in H_1(\mathbb{R}^N)$ such that $I_0(u) = I_0(\phi_\omega)$. Then we may assume that $K_\omega(u) \geq 0$ (otherwise we get the conclusion directly). So, from this have that

$$\ell_\omega(u) = \frac{1}{N} K_\omega(u) + \frac{1}{N} I_0(u) \geq \frac{1}{N} I_0(\phi_\omega) = d(\omega),$$

as desired.

\begin{proof}
On the Convexity/Concavity of $d$
As pointed out above, the orbital stability/instability of the ground states depends on the convexity/concavity of the function $d$. We must recall that $d$ is convex at $\omega_0$ if and only if

$$d(\omega) - d(\omega_0) \geq (\omega - \omega_0)d'(\omega_0).$$

The first observation is that $d'$ can be computed from the charge $Q$.

Lemma 4.3. If $u \in \mathcal{G}_\omega$, then we have that

$$d'(\omega) = Q(\Phi_\omega).$$

Now, we show the characterization of the convexity for $d$.

Theorem 4.4. Let $\omega_0 > 0$ and $\Phi_{\omega_0}$ be fixed. We have that $d(\omega)$ is convex at $\omega_0$ if and only if the functional $\mathcal{H}$ restricted to the manifold

$$M_0 = \{U \in (H^1(\mathbb{R}^N))^3 : Q(U) = Q(\Phi_{\omega_0})\}$$

has a local minimum at $\Phi_{\omega_0}$.
Proof. Define the curve \( \omega \rightarrow \Phi_\omega = (u_\omega, v_\omega, w_\omega) \) and consider the family of functions

\[
\Psi_\omega(x) := \Phi_\omega(y), \quad x = \lambda(\omega)y, \quad \lambda(\omega) = \frac{\mathcal{Q}(\Phi_\omega)}{\mathcal{Q}(\Phi_0)}.
\]

By definition, we have that \( \mathcal{Q}(\Psi_\omega) = \mathcal{Q}(\Phi_\omega) \). Now, using (50), we see directly that

\[
\mathcal{H}(\Psi_\omega) = \frac{\lambda^{N-2}}{2} I_0(u_\omega) + \frac{\lambda^N}{4} G(u_\omega)
\]

\[
= \frac{\lambda^{N-2}}{2} I_0(u_\omega) - \lambda^N \left( \left( \frac{N-2}{2N} \right) I_0(u_\omega) + \frac{\omega}{2} B(u_\omega) \right)
\]

\[
= \left( \frac{\lambda^{N-2}}{2} - \lambda^N \left( \frac{1}{2} - \frac{1}{N} \right) \right) I_0(u_\omega) - \omega \mathcal{Q}(\Phi_\omega_0).
\]

It is straightforward to see for \( N \geq 2 \) that

\[
\left( \frac{\lambda^{N-2}}{2} - \lambda^N \left( \frac{1}{2} - \frac{1}{N} \right) \right) \leq \frac{1}{N},
\]

which implies for \( N \geq 2 \) that

\[
\mathcal{H}(\Psi_\omega) \leq \frac{1}{N} I_0(u_\omega) - \omega \mathcal{Q}(\Phi_\omega_0) = d(\omega) - \omega d'(\omega_0).
\] (57)

On the other hand,

\[
\mathcal{H}(\Psi_\omega) \geq \mathcal{H}(\Phi_\omega_0)
\]

\[
= \frac{1}{2} I_0(u_\omega) + \frac{1}{4} G(u_\omega)
\]

\[
= \frac{1}{N} I_0(u_\omega) - \frac{\omega_0}{2} B(u_\omega)
\]

\[
= d(\omega_0) - \omega_0 d'(\omega_0).
\]

So, from the inequality (57), we see that \( d \) is convex at \( \omega_0 \), since we have shown that \( d \) satisfies the condition (54).

Now, we assume that \( d \) is convex at \( \omega_0 \). Let \( U = (u, v, w) \in M_0 \). From Young’s inequality we see that

\[
\mathcal{H}(U) \geq \frac{1}{2} I_0(u) \geq \frac{1}{N} I_0(u) - \omega \mathcal{Q}(\Phi_\omega_0).
\] (58)

Now, we set the functional \( T(\omega, U) = I_0(u) - I_0(u_\omega) \). Thus, we have that \( T(\omega, \Phi_\omega) = 0 \) for all \( \omega \). On the other hand, we have that \( \partial_\omega T(\omega, \Phi_\omega) = Nd'(\omega) \neq 0 \) (see Lemma 4.2 (2)). Then for a given \( U \in M_0 \) in a small neighborhood of \( \Phi_\omega_0 \), we can get \( \omega \) such that \( T(\omega, U) = I_0(u) - I_0(u_\omega) = 0 \). Using this in (58) and the convexity of \( d \), we conclude that

\[
\mathcal{H}(U) \geq \frac{1}{N} I_0(u_\omega) - \omega \mathcal{Q}(\Phi_\omega_0) \geq d(\omega) - \omega d'(\omega_0) \geq d(\omega_0) - \omega_0 d'(\omega_0) = \mathcal{H}(\Phi_\omega_0),
\]

for \( U \in M_0 \) in a small neighborhood of \( \Phi_\omega_0 \). \( \square \)

As a consequence of this result, we have that

Corollary 1. Let \( \Psi_\omega \) be as defined by (56) in previous theorem. Then we have that,

a) if \( d(\omega) \) is strictly concave at \( \omega_0 \), then \( \mathcal{H}(\Psi_\omega) < \mathcal{H}(\Phi_\omega_0) \) for \( \omega \) near \( \omega_0 \) \( (\omega_0 \neq \omega). \)

b) \( \partial^2_{\omega} \mathcal{H}(\Psi_\omega)|_{\omega=\omega_0} = \left((\mathcal{H}''(\Phi_\omega_0) + \omega_0 Q''(\Phi_\omega_0)) Y_0, Y_0 \right), \quad Y_0 = \partial_\omega \Phi_\omega|_{\omega=\omega_0}. \)
Proof. a) From the estimate (57), and using that $d$

d) If $d''(\omega_0) < 0$, then we have that $\partial_\omega I_0(\Psi^{(1)}_0)|_{\omega=\omega_0} > 0$.

e) We need to recall that

For $\theta \in \mathbb{R}$, we define $R(\theta)$ and $R_0(\theta)$ by

$$R(\theta) = \begin{pmatrix} e^{i\theta} & 0 & 0 \\
0 & e^{i\theta} & 0 \\
0 & 0 & e^{i\theta} \end{pmatrix}, \quad R_0(\theta) = \begin{pmatrix} e^{i\theta} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}$$

For $X \subset (H^1(\mathbb{R}^N))^3$, we consider the set

$$\Omega_X = \{ R(\theta)V : \theta \in \mathbb{R}, \ V \in X \},$$

and we also define the orbit associated with $\Psi$ by

$$\Omega_\Psi = \{ R(\theta)\Psi : \theta \in \mathbb{R} \}.$$
For $X \subset (H^1(\mathbb{R}^N))^3$ and $\delta > 0$, we define the $\delta$-neighborhood of $X$ in $(H^1(\mathbb{R}^N))^3$ as

$$\mathcal{V}_\delta(X) = \bigcup_{V \in X} B_\delta(V), \quad B_\delta(V) = \{U \in (H^1(\mathbb{R}^N))^3 : \|U - V\|_1 < \delta\}.$$ 

The first remark is that for $Y \subset (H^1(\mathbb{R}^N))^3$, we have

$$\mathcal{V}_\delta(Y) = \mathcal{V}_{\delta(Y)}.$$ 

**Definition 5.1.** (X-Stability) We say that $X \subset (H^1(\mathbb{R}^N))^3$ is stable by the flow of $S(t)$, if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $U_0 \in \mathcal{V}_\delta(X)$ we have that the existence time $T^*(U_0) = \infty$ and $S(t)U_0 \in \mathcal{V}_\varepsilon(X)$ for $t \geq 0$.

**Lemma 5.2.** Let $\omega > 0$ and $\Phi_\omega$ be given. Then there exist $\varepsilon_0 > 0$ and a functional $\Lambda_0 : \mathcal{V}_{\varepsilon_0}(\Omega_{\Phi_\omega}) \to \mathbb{R}$ satisfying for $\theta \in \mathbb{R}$, and $V \in \mathcal{V}_{\varepsilon_0}(\Omega_{\Phi_\omega})$ that

$$\|V - R(\Lambda_0(V))\Phi_\omega\|_2 \leq \|V - W\|_2, \text{ for all } W \in \Omega_{\Phi_\omega},$$

$$\Lambda_0(R(\theta)V) = \Lambda_0(V) + \theta \mod (2\pi),$$

$$\Lambda_0'(V) = \frac{iV}{\|\Phi_\omega(1)\|_2}, \quad V \in \Omega_{\Phi_\omega},$$

where $Z^{(j)}$ denotes the $j$th component of $Z$.

**Proof.** For given $\varepsilon > 0$, we consider the function $F : B_\varepsilon(\Phi_\omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(V, \theta) = \frac{1}{2} \int_{\mathbb{R}^N} |R(\theta)\Phi_\omega - V|^2 \, dW.$$ 

A straightforward computation shows that

$$\partial_\theta F(V, \theta) = -(ie^{i\theta}\Phi_\omega^{(1)}, V^{(1)})_2, \quad \partial_V F(V, \theta) = -R(\theta)\Phi_\omega + V.$$ 

Moreover,

$$\partial^2_{\theta\theta} F(V, \theta) = (ie^{i\theta}\Phi_\omega^{(1)}, V^{(1)})_2, \quad \partial_\theta \partial_V F(V, \theta) = -iR_0(\theta)\Phi_\omega,$$

$$\partial^2_{VV} F(V, \theta) = I.$$ 

Now, we set the function $\Gamma$ defined on $B_\varepsilon(\Phi_\omega) \times \mathbb{R}$ by

$$\Gamma(V, \theta) = \nabla F(V, \theta) = (F_V, F_\theta).$$ 

We see that $\Gamma(\Phi_\omega, 0) = 0$ and $\partial_\theta \Gamma(\Phi_\omega, 0) = \|\Phi_\omega^{(1)}\|_2 \neq 0$. Thus, the Implicit Function Theorem guarantees that there exists $\varepsilon_0 > 0$, $\theta_0 > 0$ and $R > 0$, and $C^2$ function $\Lambda_0 : B_{\varepsilon_0}(\Phi_\omega) \to (-\theta_0, \theta_0)$ such that $\Lambda_0(\Phi_\omega) = 0$, and for $V \in B_{\varepsilon_0}(\Phi_\omega)$ we have that

$$\Gamma(V, \Lambda_0(V)) = 0 \iff (iR_0(\Lambda_0(V))\Phi_\omega, V)_2 = 0, \quad R(\Lambda_0(V))\Phi_\omega - V = 0,$$

and that the Hessian of $F$ at the point $(V, \Lambda_0(V))$, $HessF(V, \Lambda_0(V))$ is a strictly positive defined operator.

Moreover, given $V \in B_{\varepsilon_0}(\Phi_\omega)$ we have that $R(\Lambda_0(V))\Phi_\omega$ is the unique element in $B_{\varepsilon_0}(\Phi_\omega)$ such that for $W \in \Omega_{\Phi_\omega} \cap B_{\varepsilon_0}(\Phi_\omega)$

$$\|V - R(\Lambda_0(V))\Phi_\omega\|_2 \leq \|V - W\|_2.$$ 

On the other hand, since we have for any $\hat{\theta}, \theta \in \mathbb{R}$ that

$$F(R(\hat{\theta})V, \hat{\theta} + \theta) = F(V, \theta),$$
We consider Remark 1. Then previous remark allows us to extend \( \Lambda_0 \) to \( \mathcal{V}_{\epsilon_0}(\Omega_{\Phi_\omega}) \) in such a way that condition (60) holds for any \( \theta \in \mathbb{R} \), and \( V \in B_{\epsilon_0}(\Phi_\omega) \). Moreover, if we set
\[
\mathcal{N}(V) = R(\Lambda_0(V))\Phi_\omega, \quad \mathcal{N}_0(V) = R_0(\Lambda_0(V))\Phi_\omega
\]
then, from (63) we conclude for any \( V \in B_{\epsilon_0}(\Phi_\omega) \) that \( \mathcal{N}(V) \) is the unique element of \( B_{\epsilon_0}(\Phi_\omega) \) such that
\[
\|V - \mathcal{N}(V)\|_2 \leq \|V - W\|_2, \text{ for all } W \in \Omega_{\Phi_\omega} \cap B_{\epsilon_0}(\Phi_\omega).
\]
Moreover, \( \mathcal{N} : \mathcal{V}_{\epsilon_0}(\Omega_{\Phi_\omega}) \to L_b(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) \) is a \( C^2 \) function with derivative given by
\[
\mathcal{N}'(V) \Phi = \langle \Lambda'_0(V); \Phi \rangle i\mathcal{N}_0(V)
\]
(63)
On the other hand, for \( s \in \mathbb{R} \) small enough, we have that \( V + s\Psi_{\omega_0} \in B_{\epsilon_0}(V) \), and so, from (62) we have for any \( \Phi \in (H^1(\mathbb{R}))^3 \) that
\[
(i\mathcal{N}(V + s\Phi), V + s\Phi)_2 = 0.
\]
If we differentiate in \( s \) and evaluate at \( s = 0 \), we see for any \( \Phi \in (H^1(\mathbb{R}))^3 \) that
\[
\langle \mathcal{N}'(V) \Phi, iV \rangle_2 = (i\mathcal{N}(V), \Phi)_2.
\]
From this, the Riesz representation theorem and formula (63), we conclude for any \( V \in \mathcal{V}_{\epsilon_0}(\Omega_{\Phi_\omega}) \) that
\[
i\mathcal{N}(V) = (V, \mathcal{N}_0(V))_2 \Lambda'_0(V).
\]
Moreover, from (62) we have for any \( V \in \Omega_{\Phi_\omega} \) that \( \mathcal{N}(V) = V \), and so from previous formula we have that,
\[
iV = i\mathcal{N}(V) = (V, \mathcal{N}_0(V))_2 \Lambda'_0(V) = \left(V^{(1)}, V^{(1)}\right)_2 \Lambda'_0(V) \iff \Lambda'_0(V) = \frac{iV}{||V^{(1)}||_2}.
\]
Moreover, since we have for \( V \in \Omega_{\Phi_\omega} \), then \( V = R(\theta)\Phi_\omega \), then we conclude for any \( V \in \Omega_{\Phi_\omega} \) that
\[
\Lambda'_0(V) = \frac{iV}{||\Phi^{(1)}_{\omega}||_2}.
\]

\[\square\]

**Remark 1.** We consider \( \tilde{G} : \mathcal{V}_\delta(\Omega_{\Phi_\omega}) \to B_{\delta}(\Phi_{\omega_0}) \) defined as \( \tilde{G}(V) = R(-\Lambda_0(V))V \).

The first observation is that \( \tilde{G}(R(\theta)V) = \tilde{G}(V) \). In fact, since \( \Lambda_0(R(\theta)V)) = \Lambda_0(V) + \theta \), then
\[
\tilde{G}(R(\theta)V) = R(-\Lambda_0(R(\theta)V))R(\theta)V = R(-\Lambda_0 - \theta)R(\theta)V = R(-\Lambda_0)V = \tilde{G}(V).
\]

Now, we are allowed to define the function \( A : \mathcal{V}_\delta(\Omega_{\Phi_\omega}) \to \mathbb{R} \) by
\[
A(V) = -\left< \mathcal{J}Y_0, \tilde{G}(V) \right>.
\]

From the discussion above, we have that \( A(R(\theta)V) = A(V) \). On the other hand, a direct computation shows that \( R'(\theta) = i\mathcal{R}_0(\theta) \) and that
\[
\tilde{G}'(V) = R'(\Lambda_0(V))VA'_0(V) + R(\Lambda_0(V)) = i\mathcal{R}_0(\Lambda_0(V))VA'_0(V) + R(\Lambda_0(V)).
\]

As a consequence of previous formula, we have that
\[
A'(V) = -R(\Lambda_0(V))\mathcal{J}Y_0 - \left< \mathcal{J}Y_0, i\mathcal{R}_0(\Lambda_0(V))V \right> \Lambda'_0(V).
\]
In particular, using that \( \Lambda_0(\Phi_{\omega_0}) = 0 \) and \( R(0) = I \), we conclude at \( V = \Phi_{\omega_0} \) that
\[
A'(\Phi_{\omega_0}) = -\mathcal{J}Y_0 - \langle \mathcal{J}Y_0, iR(0)\Phi_{\omega_0} \rangle \Lambda'_0(\Phi_{\omega_0}) \\
= -\mathcal{J}Y_0 - \langle Y_0, i\mathcal{J}^*R(0)\Phi_{\omega_0} \rangle \Lambda'_0(\Phi_{\omega_0}) \\
= -\mathcal{J}Y_0 + \langle Y_0, \mathcal{Q}'(\Phi_{\omega_0}) \rangle \Lambda'_0(\Phi_{\omega_0}) \\
= -\mathcal{J}Y_0,
\]
due to the conclusion (d) in Corollary 1. In other words, we have established the following result.

**Lemma 5.3.** There exists \( A : \mathcal{V}_{\epsilon_0}(\Omega_{\Phi_{\omega_0}}) \to \mathbb{R} \) such that
1. \( A(R(\theta)V) = A(V) \).
2. \( A'(\Phi_{\omega_0}) = -\mathcal{J}Y_0 \).
3. \( \langle A'(V), \mathcal{J}\mathcal{Q}'(V) \rangle = 0 \).

**Proof.** From previous remark, we only need to check the third conclusion. Since \( A(R(\theta)V) = A(V) \), it follows that
\[
\frac{d}{d\theta} A(R(\theta)V)|_{\theta=0} = \langle A'(R(\theta)V), R'(\theta)V \rangle |_{\theta=0} \\
= \langle A'(R(\theta)V), iR(0)(\theta)V \rangle |_{\theta=0} \\
= \langle A'(V), \mathcal{J}\mathcal{Q}'(V) \rangle = 0.
\]
\[\square\]

**Lemma 5.4.** For \( \gamma < \epsilon_0 \), there are \( \varepsilon > 0 \) and a function
\[
S : (1 - \varepsilon, 1 + \varepsilon) \times \mathcal{V}_{\gamma}(\Omega_{\Phi_{\omega_0}}) \to \mathcal{V}_{\epsilon_0}(\Omega_{\Phi_{\omega_0}})
\]
such that
1. \( \mathcal{Q}(S(\lambda, V)) = \mathcal{Q}(V) \).
2. If we set \( V_\lambda = S(\lambda, V) \), then there is some \( \lambda = \lambda(V) \) such that
\[
I_0(V^{(1)}_\lambda) = I_0(u_{\omega_0}).
\]

**Proof.** For given \( V \in \mathcal{V}_{\gamma}(\Omega_{\Phi_{\omega_0}}) \), we consider the initial value problem
\[
\left\{
\begin{array}{l}
\frac{dV_\lambda}{d\lambda} = -\mathcal{J}A'(V_\lambda), \\
\frac{dV_1}{d\lambda} = V.
\end{array}
\right.
\tag{64}
\]
Since the mapping \( V \to -\mathcal{J}A'(V_\lambda) \) is well defined, we conclude that there is a \( \epsilon \)-neighborhood of \( \lambda = 1 \) in which (64) has solution. We define \( S(\lambda, V) = V_\lambda \) to be the flow associated with the initial value problem (64). We note that
\[
\frac{d}{d\lambda} S(1, V) = -\mathcal{J}A'(V), \quad S(1, V) = V.
\tag{65}
\]
A direct computation shows that
\[
\partial_\lambda \mathcal{Q}(S(\lambda, V)) = \langle \mathcal{Q}'(S(\lambda, V)), \partial_\lambda S(\lambda, V) \rangle = \langle \mathcal{Q}'(V_\lambda), -\mathcal{J}A'(V_\lambda) \rangle = 0,
\]
from conclusion (3) in Lemma (5.3). Now, from Corollary 1 (c), we conclude for \( \lambda = 1 \) and \( V = \Phi_{\omega_0} \) that \( \partial_\lambda I_0(u_{\omega_0}) \neq 0 \). Then using the Implicit Function Theorem, we have for \( V \) close to \( \Phi_{\omega_0} \) that the equation
\[
I_0(V^{(1)}_\lambda) = I_0(u_{\omega_0})
\]
has solution for some \( \lambda = \lambda(V) \), as desired. \[\square\]
Remark 2. Consider the curve $\lambda \to V_\lambda$ provided by previous result, then we see (see Corollary 1) that

$$
\partial_\lambda \mathcal{H}(V_\lambda)_{|\lambda=1} = \langle \mathcal{H}'(V_\lambda), \partial_\lambda V_\lambda \rangle_{|\lambda=1} = -\langle \mathcal{H}'(V), \mathcal{J} A'(V) \rangle := \mathcal{P}(V),
$$

$$
\partial_\lambda^2 \mathcal{H}(V_\lambda)_{|\lambda=1} = \langle \mathcal{H}''(V_\lambda) \partial_\lambda V_\lambda, \partial_\lambda V_\lambda \rangle_{|\lambda=1} + \langle \mathcal{H}'(V_\lambda), \partial_\lambda^2 V_\lambda \rangle_{|\lambda=1} = 0.
$$

Then we conclude from these formulas at $\lambda = 1$ and $V = \Phi_{\omega_0}$ that

$$
\partial_\lambda^2 \mathcal{H}(\Phi_{\omega_0}) = \langle (\mathcal{H}''(\Phi_{\omega_0}) + \mathcal{Q}''(\Phi_{\omega_0})) V_0, V_0 \rangle \leq d''(\omega_0),
$$

where we are using (c) of Corollary 1.

From this remark, we obtain the following result.

**Theorem 5.5.** Let $\omega_0 > 0$ and suppose that $d''(\omega_0) < 0$. Then,

a) There exists $\varepsilon > 0$ such that for $V \in \mathcal{V}_0(\Omega_{\Phi_{\omega_0}}) \setminus \Omega_{\Phi_{\omega_0}}$ such that $\mathcal{Q}(V) = \mathcal{Q}(\Phi_{\omega_0})$, there is $\lambda = \lambda(V) \in (1 - \varepsilon, 1 + \varepsilon)$ such that

$$
\mathcal{H}(\Phi_{\omega_0}) < \mathcal{H}(V) + (\lambda - 1)\mathcal{P}(V).
$$

b) $\mathcal{P}(\Psi_{\omega})$ changes sign, as $\omega$ passes $\omega_0$, where $\Psi_{\omega}$ is given by (56).

**Proof.** a) We set the function $Z(\lambda) = \mathcal{H}(V_\lambda)$. From Taylor expansion at $\lambda_0 = 1$, we conclude for $\lambda$ close to 1 and $V$ near $\Phi_{\omega_0}$ that

$$
Z(\lambda) = Z(1) + Z'(1)(\lambda - 1) + \frac{1}{2} Z''(\hat{\lambda})(\lambda - 1)^2
$$

for some $\hat{\lambda}$ between $\lambda$ and 1. Moreover, from (66), we also have for $\lambda$ close to 1 that

$$
\mathcal{H}(V_\lambda) < \mathcal{H}(V) + (\lambda - 1)\mathcal{P}(V),
$$

for $V$ close to $\Phi_\omega$, where we are using the estimate (67) and that $d''(\omega_0) < 0$. On the other hand, as done in (58), we have that

$$
\mathcal{H}(V_\lambda) \geq \frac{1}{N} I_{\omega_0}(V^{(1)}_\lambda) - \omega_0 \mathcal{Q}(V_\lambda).
$$

Note that we also have that $\mathcal{Q}(\Phi_{\omega_0}) = \mathcal{Q}(V_\lambda)$. Arguing as above, we can choose $\lambda = \lambda(V)$ such that $I_0(V^{(1)}_\lambda) = I_0(\Phi_{\omega_0})$, which implies that

$$
\mathcal{H}(V_\lambda) \geq \frac{1}{N} I_{\omega_0}(V^{(1)}_\lambda) - \omega_0 \mathcal{Q}(V_\lambda) = \frac{1}{N} I_{\omega_0}(u_{\omega_0}) - \omega_0 \mathcal{Q}(\Phi_{\omega_0}) = \mathcal{H}(\Phi_{\omega_0}).
$$

Combining the last inequality with (69), we conclude the desired estimate for $V$ close to $\Phi_{\omega_0}$. Finally, if $V$ is close to $R(\theta)\Phi_{\omega_0}$, then $R(-\theta)V$ is close to $\Phi_{\omega_0}$, so the same argument works in this case for $\bar{V} = R(-\theta)V$.

b) We first note from (69) that

$$
(1 - \lambda)\mathcal{P}(\Psi_{\omega}) < \mathcal{H}(\Psi_{\omega}) - \mathcal{H}(\Phi_{\omega_0}) < 0,
$$

where we are using Corollary 1 (a). On the other hand, we know from (e) in Corollary 1 that $\partial_{\lambda_1} I_0(\Psi_{\omega}) > 0$ at the point $\omega_0$. Now, if we set the function $g(\omega) = I_0(\Psi_{\omega}) - I_0(\Phi_{\omega_0})$. Then $g(\omega_0) = 0$ and $g'(\omega_0) > 0$. So, $g$ must change sign at $\omega_0$, and so does $1 - \lambda = 1 - \lambda(\Psi_{\omega})$. So, previous inequality (70) implies that $\mathcal{P}(\Psi_{\omega})$ changes sign at $\omega_0$. □
Invariant regions under the flow for system (7) For given \( \omega_0 > 0 \), we set the \( V_{\omega_0} = V_c(\Omega_{\Phi_{\omega_0}}) \setminus \Omega_{\Phi_{\omega_0}} \), and introduce the regions \( R_{\omega_0}^1 \), as
\[
R_{\omega_0}^1 = \{ U \in V_{\omega_0} : H(U) < H(\Phi_{\omega_0}), \ Q(U) = Q(\Phi_{\omega_0}), \ P(U) > 0 \},
\]
\[
R_{\omega_0}^2 = \{ U \in V_{\omega_0} : H(U) < H(\Phi_{\omega_0}), \ Q(U) = Q(\Phi_{\omega_0}), \ P(U) < 0 \}.
\]
The first remark is that from (a) of Corollary 1 and Theorem 5.5, the region \( R_{\omega_0}^1 \) contains points very close to \( \Phi_{\omega_0} \), in the case \( d''(\omega_0) < 0 \).

**Lemma 5.6.** If \( d''(\omega_0) < 0 \), then regions \( R_{\omega_0}^1 \) and \( R_{\omega_0}^2 \) are invariant under the flow associated with the Hamiltonian system (7).

**Proof.** We note that \( Q \) and \( H \) are conserved in time for solutions of the Hamiltonian system (7). On the other hand, suppose that \( U_0 \in R_{\omega_0}^1 \) and assume that \( U(t) \in R_{\omega_0}^1 \) (or \( U(t) \in V_{\omega_0} \)) for \( 0 \leq t \leq r \), then
\[
0 \leq H(\Phi_{\omega_0}) - H(U(t)) < (\lambda - 1)P(U(t)).
\]
So, we have that \( P(U(t)) \neq 0 \) for \( 0 \leq t \leq r \), meaning by continuity that we have \( P(U(t)) > 0 \) for \( 0 \leq t \leq r \), since \( P(U_0) > 0 \). The same argument works for the region \( R_{\omega_0}^2 \).

**Lemma 5.7.** Let \( U_0^i \in R_{\omega_0}^1 \) and \( U_i \) be the solution of the Cauchy problem associated with (7) with initial condition \( U_i(0) = U_0^i \) for \( i = 1, 2 \). Let \( T_0^i \) be the existence time of \( U_i \) given by
\[
T_{0,i} = \sup \{ t : U_i(r) \in R_{\omega_0}^1, \ 0 \leq r < t \} \leq \infty.
\]
Then there is a positive \( \alpha_0 \) such that \( P(U_1(t)) > \alpha_0 \) for \( t < T_{0,1} \) and \( P(U_2(t)) < -\alpha_0 \) for \( t < T_{0,2} \).

**Proof.** For \( U_0^1 \in R_{\omega_0}^1 \), we set \( \alpha_0 = \varepsilon(H(\Phi_{\omega_0}) - H(U_0^1)) > 0 \), where \( \varepsilon > 0 \) is obtained by (a) of Theorem 5.5. Moreover, \( |\lambda - 1| < \varepsilon \). Now, from (68), we have that for \( 0 \leq t < T_{\omega_0} \)
\[
H(\Phi_{\omega_0}) - H(U_1(t)) < (\lambda - 1)P(U_1(t)), \quad P(U_1(t)) > 0.
\]
From (a) of Theorem 5.5, we have that \( 0 < \lambda - 1 < \varepsilon \). So, we then conclude that \( \alpha_0 < P(U_1(t)) \). The second case is completely analogous.

**Instability: Case \( N = 3 \).** Now, we are in position to establish orbital instability of the \( \Omega_{\Phi_{\omega_0}} \) associated with the standing wave solutions \( \Phi_{\omega_0} \) in the case \( N = 3 \), by using a scaling argument and the concavity of \( d \).

**Theorem 5.8.** If \( N = 3 \), then the set \( \Omega_{\Phi_{\omega_0}} \) is unstable under the flow of (7) in the following sense: If \( U_0 \in R_{\omega_0}^1 \cup R_{\omega_0}^2 \) close to \( \Omega_{\Phi_{\omega_0}} \), then the solution \( U(t) \) with initial condition \( U(0) = U_0 \) must exit \( V_{\omega_0} \) in finite time \( T_0 < \infty \).

**Proof.** Let \( \Phi_{\omega_0} = (u_{\omega_0}, v_{\omega_0}, w_{\omega_0}) \) be a solution of the (BR-ZR) system of the form (10) for \( N = 3 \), then \( u_{\omega_0} \) satisfies the differential equation
\[
\omega_0 u - \epsilon \partial_x^2 u - \sigma_1 \Delta u = (M^2 W - \sigma)|u|^2 u + W(D - M^2 \sigma_2)^2 E(|u|^2) u.
\]
If we set \( u_{\omega_0}(x) = \sqrt{\omega_0} \phi_0(y) \) where \( y = \sqrt{\omega_0} x \), then we see that \( \phi_0 \) satisfies the differential equation
\[
u - \epsilon \partial_y^2 u - \sigma_1 \Delta u = (M^2 W - \sigma)|u|^2 u + W(D - M^2 \sigma_2)^2 E(|u|^2) u.
\]
We also have that
\[
d(\omega_0) = \frac{1}{N} I_0(u_{\omega_0}) = \frac{1}{N} \omega_0^{\frac{3-N}{2}} I_0(\phi_0).
\]
We conclude for \( N = 3 \) that

\[
d''(\omega_0) = \frac{(4 - N)(2 - N)\omega_0^N}{4N} I_0(\phi_0) < 0. \tag{71}
\]

Now, take \( U_0 \in \mathcal{R}_{\omega_0}^1 \cup \mathcal{R}_{\omega_0}^2 \) and let \( U(t) \) be the solution associated with initial value problem for (7). From (66), we see that

\[
\frac{dA(U(t))}{dt} = \left\langle A'(U(t)), \frac{U(t)}{dt} \right\rangle = -\langle JA'(U(t)), H'(U(t)) \rangle = \mathcal{P}(U(t)),
\]

where \( A \) was defined in Lemma 5.3. Integrating from zero to \( t \), we conclude from previous result that

\[
|A(U(t)) - A(U_0)| = \int_0^t |\mathcal{P}(U(s))| ds > \alpha_0 t.
\]

In fact, if \( U_0 \in \mathcal{R}_{\omega_0}^1 \), then we have that \( \mathcal{P}(U(s)) > \alpha_0 \), and so

\[
A(U(t)) - A(U_0) = \int_0^t \mathcal{P}(U(s)) ds > \alpha_0 t > 0.
\]

In the case \( U_0 \in \mathcal{R}_{\omega_0}^2 \), we conclude that

\[
A(U(t)) - A(U_0) = \int_0^t \mathcal{P}(U(s)) ds < -\alpha_0 t < 0.
\]

If \( T_0 = \infty \), then \( U(t) \) must leave the bounded set \( \Omega_{\Phi_{\omega_0}} \), but the function \( A \) is bounded on \( \Omega_{\Phi_{\omega_0}} \). So, we have that \( T_0 < \infty \).

**Instability: Case \( N = 2 \).** In order to analyze the instability in the case \( N = 2 \), we first establish a virial identity associated with time variations of a momentum functional \( \mathcal{M} \), which makes sense due to Theorem 2.2. For convenience we will move from the variables \((x, z)\) to the variables \((x, y)\), we change \( \partial_z \) by \( \partial_x \) and \( \Delta_\perp \) by \( \partial_y^2 \). We set the space

\[
\mathcal{X} := H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2).
\]

**Theorem 5.9.** Let \( N = 2 \) and \( \Phi(x, y, t) = (\psi(x, y, t), \rho(x, y), \varphi(x, y)) \in \mathcal{X} \) be the solution of the (2) system when \( \psi_0 \in \Sigma \). If \( \mathcal{M} \) is the momentum type functional

\[
\mathcal{M}(t) = \int_{\mathbb{R}^2} (\epsilon^2 + \sigma_1) \psi(x, y, t)^2 dx dy,
\]

then we have that

\[
\frac{d^2}{dt^2} \mathcal{M}(t) = 8 \int_{\mathbb{R}^3} \left( \epsilon^2 |\psi_x|^2 + \sigma_1^2 |\psi_y|^2 + \frac{(\epsilon^2 + \sigma_1^2)\sigma}{4} |\psi|^4 - \frac{(\epsilon^2 + \sigma_1^2)W}{4M^2} \rho^2 \right.
\]

\[
+ \left. \frac{(\epsilon^2 + \sigma_1^2)WD}{2} \varphi_x |\psi|^2 + \frac{(\epsilon^2 + \sigma_1^2)W}{4} \varphi_x^2 + \frac{(-\epsilon^2 + 3\sigma_1^2)W}{4} \varphi_y^2 \right) dV. \tag{72}
\]

Moreover, for \( |\epsilon| = |\sigma_1| > 0 \) we have the virial identity

\[
\frac{d^2}{dt^2} \mathcal{M}(t) = 16\epsilon^2 \mathcal{H} \begin{pmatrix} \psi \\ \rho \end{pmatrix}. \tag{73}
\]
Proof. A direct computation shows that \((\psi, \rho, \varphi)\) satisfies the system

\[
\begin{align*}
    i\partial_t \psi &= -\epsilon \partial_x^2 \psi - \sigma_1 \partial_y^2 \psi + (\sigma |\psi|^2 + W(\rho + D\partial_x \varphi)) \psi, \\
    \sigma_2 \partial_t \rho &= -2(\varphi_{xx} + \varphi_{yy}) - D\partial_x (|\psi|^2), \\
    \sigma_2 \partial_t \varphi &= -\frac{\epsilon}{M} \rho - |\psi|^2. \\
\end{align*}
\]

(74)

so taking the real and imaginary parts we conclude that

\[
\begin{align*}
    -2\Im(\psi_t \bar{\psi}) &= -2\Re(\epsilon \psi_{xx} \bar{\psi} + \sigma_1 \psi_{yy} \bar{\psi}) + 2\sigma |\psi|^4 + 2W(\rho + D\partial_x \varphi)|\psi|^2 \\
    (|\psi|^2)_t &= -2\Im(\epsilon \psi_{xx} \bar{\psi} + \sigma_1 \psi_{yy} \bar{\psi}) = -2\Im(\epsilon \psi_{x} \bar{\psi}_x + \sigma_1(\psi_y \bar{\psi}_y)) \\
\end{align*}
\]

(75)

(76)

where we are using that \(2\Re(\psi_t \bar{\psi}) = (|\psi|^2)_t\). On the other hand, we also have that

\[
\begin{align*}
    i\partial_t \psi_x &= -\epsilon \psi_{xx} \bar{\psi}_x - \sigma_1 \psi_{yy} \bar{\psi}_x + \sigma |\psi|^2 \psi_x \bar{\psi}_x + W(\rho + D\partial_x \varphi) \psi_x \\
    i\partial_t \psi_y &= -\epsilon \psi_{xx} \bar{\psi}_y - \sigma_1 \psi_{yy} \bar{\psi}_y + \sigma |\psi|^2 \psi_y \bar{\psi}_y + W(\rho + D\partial_x \varphi) \psi_y. \\
\end{align*}
\]

(77)

(78)

Using (76), we have that

\[
\begin{align*}
    \frac{d}{dt} M(t) &= -2\Im \left( \int_{\mathbb{R}^2} ((\epsilon x^2 + \sigma_1 y^2)\epsilon(\psi_x \bar{\psi}_x) + \sigma_1(\psi_y \bar{\psi}_y)) \, dx \, dy \right) \\
    &= 4\Im \left( \int_{\mathbb{R}^2} (\epsilon^2 x \bar{\psi}_x \psi_x + \sigma_1^2 y \bar{\psi}_y \psi_y) \, dx \, dy \right),
\end{align*}
\]

and

\[
\begin{align*}
    \frac{d^2}{dt^2} M(t) &= 4\Im \left( \int_{\mathbb{R}^2} (\epsilon^2 x \bar{\psi}_x \psi_x + \sigma_1^2 y \bar{\psi}_y \psi_y + (\epsilon^2 + \sigma_1^2) \psi_x \bar{\psi}_x + \psi_y \bar{\psi}_y) \, dx \, dy \right) \\
    &= 8 \left( \epsilon^2 I_1 + \sigma_1^2 I_2 - \frac{(\epsilon^2 + \sigma_1^2)}{2} \Im \left( \int_{\mathbb{R}^2} \bar{\psi}_x \psi_x \, dx \, dy \right) \right),
\end{align*}
\]

(79)

where \(I_1\) and \(I_2\) are defined by

\[
\begin{align*}
    I_1 &= \Im \left( \int_{\mathbb{R}^2} x \bar{\psi}_x \psi_x \, dx \, dy \right), \\
    I_2 &= \Im \left( \int_{\mathbb{R}^2} y \bar{\psi}_y \psi_y \, dx \, dy \right).
\end{align*}
\]

A direct computation using equation (74) (a) shows that

\[
\begin{align*}
    I_1 &= \frac{1}{2} \int_{\mathbb{R}^2} \left( \epsilon |\psi_x|^2 - \sigma_1 |\psi_x|^2 - \frac{\sigma}{2} |\psi|^4 + W(Dx \varphi_x + x \rho) \left( |\psi|^2 \right)_x \right) \, dx \, dy, \\
    I_2 &= \frac{1}{2} \int_{\mathbb{R}^2} \left( -\epsilon |\psi_x|^2 + \sigma_1 |\psi_x|^2 - \frac{\sigma}{2} |\psi|^4 + W(Dy \varphi_x + y \rho) \left( |\psi|^2 \right)_y \right) \, dx \, dy.
\end{align*}
\]

(80)

(81)
Now, we require using the equation for \( \rho \) and \( \varphi \) in (74) to estimate the last terms in previous equations. Then we see that

\[
\int_{\mathbb{R}^2} x |\psi|^2 \, dx dy = \frac{1}{2D} \int_{\mathbb{R}^2} (\varphi_x^2 - \varphi_y^2 - \frac{1}{M} \rho^2 - 2|\psi|^2 \rho - 2x \rho |\psi|^2) \, dx dy
\]

\[
\int_{\mathbb{R}^2} y |\psi|^2 \, dx dy = \frac{1}{2D} \int_{\mathbb{R}^2} (\varphi_x^2 + 3\varphi_y^2 - \frac{1}{M} \rho^2 - 2|\psi|^2 \rho - 2y \rho |\psi|^2) \, dx dy,
\]

which implies that

\[
\int_{\mathbb{R}^2} (Dx \varphi_x + x \rho) (|\psi|^2) \, dx dy = \frac{1}{2} \int_{\mathbb{R}^2} (\varphi_x^2 - \varphi_y^2 - \frac{1}{M} \rho^2 - 2|\psi|^2 \rho) \, dx dy
\]

Putting these estimates together, we have that

\[
\mathcal{E} \left( \int_{\mathbb{R}^2} x \tilde{\psi}_i \psi_\alpha \, dx \right) = \frac{1}{2} \int_{\mathbb{R}^2} \left( \epsilon |\psi_x|^2 - \sigma_1 |\psi_y|^2 - \frac{1}{2} \sigma |\psi|^4 + \frac{W}{2} \left( \varphi_x^2 - \varphi_y^2 - \frac{1}{M} \rho^2 - 2|\psi|^2 \rho \right) \right) \, dx dy
\]

and also that

\[
\mathcal{E} \left( \int_{\mathbb{R}^2} y \tilde{\psi}_i \psi_\alpha \, dx \right) = \frac{1}{2} \int_{\mathbb{R}^2} \left( -\epsilon |\psi_x|^2 + \sigma_1 |\psi_y|^2 - \frac{1}{2} \sigma |\psi|^4 + \frac{W}{2} \left( \varphi_x^2 + 3\varphi_y^2 - \frac{1}{M} \rho^2 - 2|\psi|^2 \rho \right) \right) \, dx dy
\]

Thus, we conclude that

\[
e^2 I_1 + \sigma_1^2 I_2 = \frac{1}{2} \int_{\mathbb{R}^2} \left( (e^3 - e\sigma_1^2) |\psi_x|^2 + (\sigma_1^3 - \sigma_1 e^2) |\psi_y|^2 - \frac{(e^2 + \sigma_1^2)}{2} \sigma |\psi|^4 - \frac{(e^2 + \sigma_1^2)}{2} W \rho |\psi|^2 - \frac{(e^2 + \sigma_1^2)}{2} W \rho^2 \right) \, dx dy
\]

Finally, we have that

\[
\mathcal{E}( \int_{\mathbb{R}^2} \psi_i \bar{\psi}_j dV) = - \int_{\mathbb{R}^2} \left( \epsilon |\psi_x|^2 + \sigma_1 |\psi_y|^2 + \sigma |\psi|^4 + W \rho |\psi|^2 + WD \varphi_x |\psi|^2 \right) \, dx dy.
\]

Putting together these estimates, we conclude that

\[
\frac{d^2}{dt^2} \mathcal{M}(t) = 8 \int_{\mathbb{R}^2} \left( e^3 |\psi_x|^2 + \sigma_1^3 |\psi_y|^2 + \frac{(e^2 + \sigma_1^2)}{4} \sigma |\psi|^4 - \frac{(e^2 + \sigma_1^2)}{4} W \rho^2 - \frac{(e^2 + \sigma_1^2)}{2} W \varphi_x |\psi|^2 - \frac{(e^2 + \sigma_1^2)}{4} \varphi_y^2 \right) \, dx dy,
\]

which completes the proof of the estimate (72).
Finally, to get the estimate (73), we note, using the third equation in (74), that the energy can be written as

\[
\mathcal{H} \left( \frac{\psi}{\rho} \right) = \frac{1}{2} \int_{\mathbb{R}^2} \left( \epsilon |\psi_x|^2 + \sigma_1 |\psi_y|^2 + \frac{\sigma}{2} |\psi|^4 - \frac{W}{2M^2} \rho^2 + WD \frac{\varphi_x}{\rho} |\psi|^2 \right) \, dx \, dy,
\]

So, comparing the coefficients of \( \mathcal{H} \) and \( \mathcal{M}' \) and using that \( |\sigma_1| = |\epsilon| \), then we have that

\[
\frac{\epsilon^2 + \sigma_1^2}{4\epsilon^2} = \frac{1}{2}, \quad \frac{3\sigma_1^2 - \epsilon^2}{4\epsilon^2} = \frac{1}{2}, \quad \frac{\sigma_1^3}{\epsilon^2} = \sigma_1,
\]

which leads to the desired estimate

\[
\frac{d^2}{dt^2} M(t) = 16\epsilon^2 \mathcal{H} \left( \frac{\psi}{\rho} \right).
\]

\[\square\]

As a direct consequence of the virial identity, we establish that solutions for the Cauchy problem associated with the (2) system necessarily blow up in finite time, in the case of initial data having negative energy.

**Theorem 5.10.** Let \( N = 2 \), \( \epsilon = \sigma_1 > 0 \), \( \Phi_0 = (\psi_0, \rho, \varphi) \in \mathcal{X} \) and \( \psi_0 \in \Sigma \). If \( \mathcal{H}(\Phi_0) < 0 \), then the maximal existence time \( T_* > 0 \) for the unique solution \( \Phi = (\psi(x, y, t), \rho(x, y), \varphi(x, y)) \) of the (2) system with initial data \( \Phi(\cdot, 0) = \Phi_0 \) is finite. More exactly, \( T_* > 0 \) is such that

\[
\lim_{t \uparrow T_*} \| \nabla \psi(\cdot, t) \|_{L^2(\mathbb{R}^2)} = +\infty.
\]

**Proof.** The first observation is that if \( \psi(\cdot, t) \in H^1(\mathbb{R}^2) \), then there is a time \( T_* > 0 \) such that

\[
\lim_{t \uparrow T_*} \int_{\mathbb{R}^2} (x^2 + y^2) |\psi(x, y, t)|^2 \, dx \, dy = 0.
\]

In fact, using that \( \mathcal{H}(\Phi(\cdot, t)) = \mathcal{H}(\Phi(0, \cdot)) \) and the virial identity, we have that

\[
e \int_{\mathbb{R}^2} (x^2 + y^2) |\psi(x, y, t)|^2 \, dx \, dy = \mathcal{M}(t) = \mathcal{M}(0) + \frac{\sigma_1}{2} M(t) + 8 \epsilon^2 \mathcal{H}(\Phi_0) t^2.
\]

Due to the fact that \( \mathcal{H}(\Phi_0) < 0 \), we conclude that there is a time \( T_* > 0 \) such that

\[
\lim_{t \uparrow T_*} \int_{\mathbb{R}^2} (x^2 + y^2) |\psi(x, y, t)|^2 \, dx \, dy = 0.
\]

On the other hand, we have the Weyl-Heisenberg’s inequality (see also R. Glassey [9] and M. Tsutsumi [20] and M. Weinstein [21])

\[
\|f\|_2 \leq ||\nabla f||_2 ||z||_2,
\]

whenever \( |z| f \in L^2(\mathbb{R}^2) \) and \( |\nabla f| \in L^2(\mathbb{R}^2) \).

From the fact that \( ||\psi(\cdot, t)||_2 = ||\psi_0||_2 \) as long as the solution exists, we conclude that

\[
||\psi_0||_2 \leq ||\nabla \psi(\cdot, t)||_2 \left( \int_{\mathbb{R}^2} (x^2 + y^2) |\psi(x, y, t)|^2 \, dx \, dy \right)^{\frac{1}{2}},
\]
which implies that
\[ \lim_{t \uparrow T^*} \int_{\mathbb{R}^2} |\nabla \psi(x, y, t)|^2 \, dx \, dy = +\infty. \]

Now, we are able to establish that standing wave solutions are unstable in the case \( N = 2 \) with \( |\epsilon| = |\sigma_1| > 0 \).

**Theorem 5.11.** Let \( N = 2 \), \( |\epsilon| = |\sigma_1| > 0 \) and \( \Phi = (u, v, w) \in \mathcal{X} \) be a solution of the system (11), then \( \Phi(t, x) = (e^{i\omega t} u(x), v(x), w(x)) \) is a global solution of the Benney-Roskes/Zakharov-Rubenchik system (BR-ZR), which is unstable in the following sense: there is \( (\Phi_n)_n \subset \mathcal{X} \) such that \( \Phi_n \to \Phi \) in \( \mathcal{X} \) as \( n \to \infty \), and the solution \( U_n(t) \) with initial condition \( U_n(0) = \Phi_n \) blows up in finite.

**Proof.** Let \( \Phi = (u, v, w) \in \mathcal{X} \) be a solution of Benney-Roskes/Zakharov-Rubenchik system (BR-ZR). The first remark is that \( |\cdot| \in L^2(\mathbb{R}^2) \), as done for R. Cipolatti in the case of the Davey-Stewartson system (see estimate (2.25) in Theorem 2.4 in [2]). From the fact that \( \Phi = (u, v, w) \in \mathcal{X} \) is a solution of Benney-Roskes/Zakharov-Rubenchik system (BR-ZR), we observe that
\[ M(t) = \int_{\mathbb{R}^2} (\epsilon x^2 + \sigma_1 y^2)|e^{i\omega t} u(x, y)|^2 \, dx \, dy = \int_{\mathbb{R}^2} (\epsilon x^2 + \sigma_1 y^2)|u(x, y)|^2 \, dx \, dy, \]
and so we conclude form virial identity (73) that \( \mathcal{H}(\Phi) = 0 \) since \( M \) is a constant function. Moreover, we see directly for \( \lambda > 1 \) that \( \Phi_\lambda = (\lambda u, \lambda^2 v, \lambda^2 w) \) satisfies
\[ \mathcal{H}(\Phi_\lambda) = \frac{1}{2}(\lambda^2 - \lambda^4) I_0(\psi) + \lambda^4 \mathcal{H}(\Phi) = \frac{1}{2}(\lambda^2 - \lambda^4) I_0(\psi) < 0. \]

Now, we set \( \Phi_n = \Phi_{\lambda_n} \) for \( \lambda_n \downarrow 1 \). Then, if \( U_n(t) = (u_n, v_n, w_n) \) is the solution with initial condition \( U_n(0) = \Phi_n \), and we define the function
\[ M_n(t) = \int_{\mathbb{R}^2} (\epsilon x^2 + \sigma_1 y^2)|u_n(x, y, t)|^2 \, dx \, dy, \]
from the virial identity (73), we have that
\[ M_n''(t) = 16\epsilon^2 \mathcal{H}(U_n(t)) = 16\epsilon^2 \mathcal{H}(U_n(0)) = 16\epsilon^2 \mathcal{H}(\Phi_n) < 0, \]
as long as the solution exists. So, integrating twice from 0 to \( t \), we conclude that
\[ 0 \leq M_n(t) = M_n(0) + M_n'(0) t + 8\epsilon^2 \mathcal{H}(\Phi_n) t^2, \]
which implies necessarily that \( U(t) \) exists only for a finite time \( T_* < \infty \). \( \square \)

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