Finite sample Bernstein – von Mises theorems for functionals and spectral projectors of covariance matrix

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Abstract: We demonstrate that a prior influence on the posterior distribution of covariance matrix vanishes as sample size grows. The assumptions on a prior are explicit and mild. The results are valid for a finite sample and admit the dimension $p$ growing with the sample size $n$. We exploit the described fact to derive the finite sample Bernstein – von Mises theorem for functionals of covariance matrix (e.g. eigenvalues) and to find the posterior distribution of the Frobenius distance between spectral projector and empirical spectral projector. This can be useful for constructing sharp confidence sets for the true value of the functional or for the true spectral projector.

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1. Introduction

The prominent Bernstein – von Mises (BvM) phenomenon states some pivotal behaviour of the posterior distribution. It specifies conditions on a prior, under which the influence of the prior vanishes as the number of observations grows, and the posterior is asymptotically Gaussian. The main application of BvM is usage of Bayesian credible sets as frequentist confidence sets. It helps in situations when the frequentist uncertainty quantification does not allow to build confidence sets directly due to unknown parameters of the asymptotic distribution.

Classical BvM results for standard parametric setup are formulated in [21, 30]. More general semiparametric models were studied by [3]. In modern statistics main focus is on the growing parameter dimension, so the classical results should be reconsidered; see, e.g. [12, 16] for some examples in high dimensions. Moreover, many statisticians are focused on work with samples of limited size, however, only a few finite sample BvM results are available, e.g. [24]. We also mention BvM for linear functionals of the density derived in [26] and general theory for smooth functionals of the target parameter presented in [7], among other important works in the field.

This paper aims at deriving similar results for the following specific model.

Let the data $X^n = (X_1, \ldots, X_n)$ be independent identically distributed zero-mean random vectors in $\mathbb{R}^p$. Its covariance matrix is given by

$$\Sigma^* \overset{def}{=} \mathbb{E} (X_j X_j^T).$$

Natural estimate of the true unknown covariance is the sample covariance ma-
the spectral norm $\|\hat{\Sigma} - \Sigma^*\|_\infty$ arises in numerous problems and is well-examined; see, for instance, [19, 25, 28, 31, 1]. Functionals and spectral projectors of covariance matrix also appear in applications frequently, so this model is of special interest nowadays.

In this work we show that the posterior distribution of the covariance matrix $\Sigma$ stays approximately the same for different choices of prior distribution as soon as we have enough observations. In particular, we demonstrate that the posterior computed from some arbitrary prior deviates not a lot from the posterior that corresponds to one special class of priors – the Inverse Wishart priors. We do not impose any conditions on a prior; error of approximation of the corresponding posterior by the Inverse Wishart posterior is described in terms of two crucial concepts. One of these concepts is the posterior contraction, and the other is the flatness of the prior. So, our results makes sense if the prior is such that:

- the posterior concentrates in a relatively small vicinity of the true covariance $\Sigma^*$,
- the prior is “flat” enough in this vicinity, that is, can be well-approximated by a constant;

The described “posterior independence” enables the following strategy that allows to reduce the complexity of a problem at hand drastically. Instead of working with some complicated prior, one can consider the Inverse Wishart prior. Since this prior is conjugate to the multivariate Gaussian distribution, the posterior is again the Inverse Wishart, so we can study it directly. Moreover, in a wide range of situations nice properties of the Inverse Wishart distribution simplify the analysis significantly.

We apply the proposed strategy to the following important objects that are of interest in modern applications. First, we derive BvM theorem for approximately linear functionals of covariance matrix. The main focus here is on eigenvalues of covariance matrix, which are extensively studied object, see [22, 8, 15] and references therein. The asymptotic normality of the posterior measure of the functional was already shown in [32]. However, not only is their result for an
Infinite sample, it also imposes some non-trivial condition on a prior instead of our simple “flatness” assumption.

One more important object under consideration is a spectral projector of the covariance matrix. Let \( P_J^* \) be the projector onto some set \( J \) of eigenspaces of \( \Sigma^* \). Its sample version is given by \( \hat{P}_J \) based on the sample covariance \( \hat{\Sigma} \). For some recent results on the distribution of \( \| \hat{P}_J - P_J^* \|_2^2 \) we refer to [20, 23].

These objects are closely related to the Principal Component Analysis (PCA), probably the most famous dimension reduction method. Nowadays PCA-based methods are actively used in deep networking architecture [13] and finance [10], along with other applications. Recent developments in theoretical guarantees for sparse PCA in high dimensions engender attention to such methods, see [14, 4, 2, 5, 11]. Our approach is applied to the squared Frobenius distance \( \| P_J - \hat{P}_J \|_2^2 \) between the projector of \( \Sigma \) and the empirical projector from \( \hat{\Sigma} \). One remarkable fact is that while the posterior distribution of a functional is approximated by Gaussian, which is more or less usual for BvM, in the case of spectral projectors the limiting distribution is the distribution of Gaussian quadratic form.

Even though the assumptions that the eigenvalues of \( \Sigma^* \) are bounded from above and separated from zero, or that the spectral gaps (differences between consequent eigenvalues) are separated from zero, are pretty common, we avoid them. Our results admit growing spectral norm \( \| \Sigma^* \|_\infty \) and vanishing smallest eigenvalue and spectral gaps. The provided error bounds are explicit and allow to track what regimes of \( \Sigma^* \) still ensure convergence to the limiting distribution.

It is also worth mentioning that the presented approach does not rely on Gaussianity of the data. Even though we work with Gaussian likelihood, we allow model misspecification and formulate our results for quasi-posterior. As to the distribution of the data, we require only one property: the concentration of the sample covariance \( \hat{\Sigma} \) around the true covariance \( \Sigma^* \).

The main contributions of this paper are as follows.

- We establish a result stating that the prior influence on the posterior distribution of covariance matrix disappears as the sample size grows. The assumptions on a prior are mild and easy to verify. The data distribution can also be pretty general.
- We propose a novel strategy for analysing the posterior distribution for arbitrary prior. The strategy includes: first, approximation of our posterior...
at hand by the posterior based on the conjugate prior, and second, study of the latter posterior which has nice properties.

- The described strategy is applied to derive finite sample BvM theorems for functionals and spectral projectors of covariance matrix.

The rest of the paper is structured as follows. Some notations are introduced in Section 2.1. Section 2.2 explains our setup. Bayesian framework is described in Section 2.3. Section 3 is dedicated to the applications of the proposed strategy: functionals of covariance matrix are examined in Subsection 3.1, and spectral projectors are considered in Subsection 3.2. The proofs of main theorems are collected Section 4. Appendix A and Appendix B gather some auxiliary results from the literature and the rest of the proofs, respectively.

2. Setup and main result

This section explains our setup and states the main results.

2.1. Notations

We will use the following notations throughout the paper. The space of real-valued $p \times p$ matrices is denoted by $\mathbb{R}^{p \times p}$, while $S_+^p$ means the set of positive-semidefinite matrices. We write $I_d$ for the identity matrix of size $d \times d$, $r(A)$ and $\text{Tr}(B)$ stand for the rank of a matrix $A$ and the trace of a square matrix $B$. Further, $\|A\|_\infty$ stands for the spectral norm of a matrix $A$, while $\|A\|_1$ means the nuclear norm. The Frobenius scalar product of two matrices $A$ and $B$ of the same size is $\langle A, B \rangle_2 \overset{\text{def}}{=} \text{Tr}(A^\top B)$, while the Frobenius norm is denoted by $\|A\|_2$. When applied to a vector, $\|\cdot\|$ means just its Euclidean norm. The effective rank of a square matrix $B$ is defined by $\tilde{r}(B) \overset{\text{def}}{=} \text{Tr}(B) \|B\|_\infty$. The relation $a \lesssim b$ means that there exists an absolute constant $C$, different from line to line, such that $a \leq Cb$, while $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$. By $a \vee b$ and $a \wedge b$ we mean maximum and minimum of $a$ and $b$, respectively. In the sequel we will often be considering intersections of events of probability greater than $1 - \frac{1}{n}$. Without loss of generality, we will write that probability measure of such an intersection is $1 - \frac{1}{n}$, since it can be easily achieved by adjusting constants.

We write $\eta_n \overset{P}{\to} \eta$ for convergence in $\mathbb{P}$-probability of random elements $\eta_n$ to some random element $\eta$. If $\eta_n \overset{P}{\to} 0$, we also will use the notation $\eta_n = o_P(1)$. Throughout the paper we will assume that $p < n$. 

2.2. Setup

Without loss of generality, we can assume that \( \Sigma^* \in S^p_+ \) is invertible (otherwise one can easily transform data in such a way that the covariance matrix for the transformed data will be invertible). Therefore, when necessary, we can work in terms of precision matrix \( \Omega \) which is the inverse of a covariance matrix \( \Sigma \).

We do not need the assumption on Gaussianity of the data. The only condition that our main result require from the underlying distribution of the independent random vectors \( X^n = (X_1, \ldots, X_n) \) is the concentration of the sample covariance matrix \( \hat{\Sigma} \) around the true covariance \( \Sigma^* \):

\[
\| \hat{\Sigma} - \Sigma^* \|_\infty \leq \hat{\delta}_n \| \Sigma^* \|_\infty
\]

with probability \( 1 - 1/n \). Clearly, the bound \( \hat{\delta}_n \) from the condition can vary for different distributions of the data, but it allows to work with much wider classes of probability measures rather than just Gaussian or sub-Gaussian. For instance, in the Gaussian case one may take

\[
\hat{\delta}_n \asymp \sqrt{\frac{r(\Sigma^*)}{n}} \lor \sqrt{\frac{\log(n)}{n}}.
\]

Theorem A.1 from Appendix A provides a few more examples of possible distributions and the corresponding \( \hat{\delta}_n \) for them. So, throughout the rest of the paper we assume that the data satisfy condition (2.1).

2.3. Bayesian framework and main result

In Bayesian framework one imposes a prior distribution \( \Pi \) on the covariance matrix \( \Sigma \). Even though our data are not Gaussian, we can consider the Gaussian log-likelihood:

\[
l_n(\Sigma) = -\frac{n}{2} \log \det(\Sigma) - \frac{n}{2} \text{Tr}(\Sigma^{-1} \hat{\Sigma}) - \frac{np}{2} \log (2\pi),
\]

where further we will omit the additive constant term that does not affect the analysis. The posterior measure of a set \( A \subset S^p_+ \) can be expressed as

\[
\Pi (A \mid X^n) = \frac{\int_{A} \exp(l_n(\Sigma)) \, d\Pi(\Sigma)}{\int_{S^p_+} \exp(l_n(\Sigma)) \, d\Pi(\Sigma)}.
\]
As the Gaussian log-likelihood \( l_n(\Sigma) \) does not necessarily correspond to the true distribution of our data, we call the random measure \( \Pi (\cdot | X^n) \) a quasi-posterior. Once a prior is fixed, we can easily sample matrices \( \Sigma \) from this quasi-posterior distribution.

Our technique relies on two crucial concepts. The first one is \textit{posterior contraction}. Define the following \( \delta \)― vicinity of \( \Sigma^* \):

\[
B(\delta) \overset{\text{def}}{=} \{ \Sigma \in \mathbb{S}_+^p : \| \Sigma - \Sigma^* \|_\infty \leq \delta \| \Sigma^* \|_\infty \}.
\]

Then we can find a radius \( \delta_n \) such that the following posterior contraction condition is fulfilled:

\[
\Pi (B(\delta_n) | X^n) \geq 1 - \frac{1}{n}
\]

with probability \( 1 - 1/n \). The second concept that we introduce is “flatness” of the prior, defined as

\[
\rho(\delta) \overset{\text{def}}{=} \sup_{\Sigma \in B(\delta)} \left| \frac{\Pi(\Sigma)}{\Pi(\Sigma^*)} - 1 \right|.
\]

We will extensively use the conjugate prior to the multivariate Gaussian distribution, that is, the Inverse Wishart distribution \( \text{IW}_p(G, p + b - 1) \) with \( G \in \mathbb{S}_+^p \) and \( 0 < b \lesssim p \). Its density is given by

\[
\frac{d\Pi_W(\Sigma)}{d\Sigma} \propto \exp \left( -\frac{2p + b}{2} \log \det(\Sigma) - \frac{1}{2} \text{Tr}(G \Sigma^{-1}) \right).
\]

The following two lemmas state the posterior contraction and a bound for flatness of this special prior, respectively.

\textbf{Lemma 2.1.} Let the prior \( \Pi_W \) be given by the Inverse Wishart distribution \( \text{IW}_p(\Sigma^*, p + b - 1) \). Define

\[
\delta_n^W \overset{\text{def}}{=} \sqrt{\frac{\log(n) + p}{n}} + \hat{\delta}_n + \frac{n \| G \|_\infty}{n \| \Sigma^* \|_\infty}.
\]

Then

\[
\Pi_W (B(\delta_n^W) | X^n) \geq 1 - \frac{1}{n}
\]

with probability \( 1 - 1/n \).
Lemma 2.2. Let $\rho^W(\delta)$ be the flatness parameter of the prior $\Pi^W$ given by the Inverse Wishart distribution $\mathcal{IW}_p(\Sigma^*, p + b - 1)$. Then the following bound holds:

$$\rho^W(\delta) \lesssim \delta \left\{ p \| \Omega^* \|_1 + \| G \|_1 \| \Omega^* \|_\infty^2 \right\}.$$ 

The proofs are postponed to Appendix B.

The following result shows that the posterior distribution is approximately the same for sufficiently flat priors satisfying the posterior contraction condition.

Theorem 2.3. Assume the distribution of the data $\mathbf{X}^n = (X_1, \ldots, X_n)$ fulfills the sample covariance concentration property (2.1). Consider arbitrary prior $\Pi$ and the prior $\Pi^W$ given by the Inverse Wishart distribution $\mathcal{IW}_p(G, p + b - 1)$. Let $\rho, \rho^W$ be their flatnesses defined by (2.3). Define also

$$\delta_n \overset{\text{def}}{=} \delta_n \vee \delta_n^W$$

with $\delta_n$ from (2.2) and $\delta_n^W$ from (2.4). Then the following holds with probability $1 - 1/n$:

$$\sup_{A \subset \mathbb{S}_+^p} | \Pi(A \mid \mathbf{X}^n) - \Pi^W(A \mid \mathbf{X}^n) | \lesssim \hat{\rho}^*,$$

where

$$\hat{\rho}^* \overset{\text{def}}{=} \rho(\delta_n) + \rho^W(\delta_n) + \frac{1}{n}.$$ (2.5)

3. Applications

Before considering particular examples, let us introduce some additional notations concerning the true covariance $\Sigma^*$.

Let $\sigma_1^* \geq \ldots \geq \sigma_p^*$ be the ordered eigenvalues of $\Sigma^*$. Suppose that among them there are $q$ distinct eigenvalues $\mu_1^* > \ldots > \mu_q^*$. Introduce groups of indices $\Delta_r^* = \{ j : \mu_j^* = \sigma_r^* \}$ and denote by $m_r^*$ the multiplicity factor (dimension) $|\Delta_r^*|$ for all $r = 1, q$. The corresponding eigenvectors are denoted as $u_1^*, \ldots, u_p^*$. We will use the projector on the $r$-th eigenspace of dimension $m_r^*$:

$$P_r^* = \sum_{j \in \Delta_r^*} u_j^* u_j^{*\top}$$
and the eigendecomposition

\[ \Sigma^* = \sum_{j=1}^{p} \sigma_j^* u_j^* u_j^{*\top} = \sum_{r=1}^{q} \mu_r^* \left( \sum_{j \in \Delta_r^*} u_j^* u_j^{*\top} \right) = \sum_{r=1}^{q} \mu_r^* P_r. \]

We also introduce the spectral gaps \( g_r^* \):

\[
\begin{align*}
g_r^* &= \begin{cases} 
\mu_1^* - \mu_2^*, & r = 1, \\
(\mu_r^* - \mu_{r-1}^*) \wedge (\mu_r^* - \mu_{r+1}^*), & r \in [2, q-1], \\
\mu_{q-1}^* - \mu_q^*, & r = q.
\end{cases}
\end{align*}
\]

Similarly, suppose that \( \tilde{\Sigma} \) has \( p \) (distinct with probability one) eigenvalues \( \tilde{\sigma}_1 > \ldots > \tilde{\sigma}_p \). The corresponding eigenvectors are denoted as \( \tilde{u}_1, \ldots, \tilde{u}_p \). Suppose that \( \| \tilde{\Sigma} - \Sigma^* \|_\infty \leq \frac{1}{4} \min_{r \in [1, q]} g_r^* \). Then, as shown in [18], we can identify clusters of the eigenvalues of \( \tilde{\Sigma} \) corresponding to each eigenvalue of \( \Sigma^* \) and therefore determine \( \Delta_r^* \) and \( m_r^* \) for all \( r \in [1, q] \), so further we assume that they are known.

### 3.1. Functionals of covariance matrix

Represent a linear functional \( \phi(\cdot) \) as

\[
\phi(\tilde{\Sigma}) - \phi(\Sigma^*) = \text{Tr}[ (\tilde{\Sigma} - \Sigma^*) \Phi ] + \varepsilon(\tilde{\Sigma}, \Sigma^*),
\]

where we suppose there exists a symmetric matrix \( \Phi \in \mathbb{R}^{p \times p} \) such that the residual \( \varepsilon(\tilde{\Sigma}, \Sigma^*) \) is bounded in the following way:

\[
| \varepsilon(\tilde{\Sigma}, \Sigma^*) | \leq C_\phi(\Sigma^*) \| \tilde{\Sigma} - \Sigma^* \|_\infty^2,
\]

where the constant \( C_\phi(\Sigma^*) \) is different for different functionals and depends only on \( \Sigma^* \). So, in some sense \( \phi(\cdot) \) is "approximately linear" functional.

While entries \( \Sigma_{ij} \) or quadratic forms \( v^\top \Sigma v \) of covariance matrix are examples of linear functionals with \( C_\phi(\Sigma^*) = 0 \), one particular example of approximately linear functional is eigenvalue of covariance matrix. Define the functional as

\[
\phi(\tilde{\Sigma}) = \frac{1}{m_r^*} \sum_{j \in \Delta_r^*} \tilde{\sigma}_j.
\]
Clearly, \( \phi(\Sigma^*) = \mu_r^* \) and its natural estimate is
\[
\phi(\hat{\Sigma}) = \hat{\mu}_r = \frac{1}{m_r^*} \sum_{j \in \Delta_r} \hat{\sigma}_j.
\]

The next lemma shows that the functional defined in such a way is approximately linear functional of covariance matrix.

**Lemma 3.1.** Assume \( \| \bar{\Sigma} - \Sigma^* \| \leq \frac{g_r^*}{2e} \). Then the following bound for first-order approximation takes place:
\[
\left| \hat{\mu}_r - \mu_r^* - \text{Tr} \left( (\bar{\Sigma} - \Sigma^*) \frac{P_r^*}{m_r^*} \right) \right| \leq \frac{2e^2}{g_r^*} \| \bar{\Sigma} - \Sigma^* \|^2,
\]
or, in other terms, introducing \( \Phi = P_r^*/m_r^* \),
\[
|\varepsilon(\bar{\Sigma}, \Sigma^*)| = \left| \phi(\bar{\Sigma}) - \phi(\Sigma^*) - \text{Tr} \left( (\bar{\Sigma} - \Sigma^*) \Phi \right) \right| \leq \frac{2e^2}{g_r^*} \| \bar{\Sigma} - \Sigma^* \|^2.
\]

So, for eigenvalues the assumption (3.2) is fulfilled with \( C_\phi(\Sigma^*) = \frac{2e^2}{g_r^*} \).

We omit the proof of this result. We refer to [17] for the details on perturbation theory for eigenvalues. Let us continue with arbitrary functional \( \phi(\cdot) \) satisfying (3.1), (3.2), but keeping this example with eigenvalues in mind.

To derive the finite sample BvM theorem for the functionals of covariance matrix, we apply our general strategy described in the previous section. We first state the result for the Inverse Wishart prior.

**Theorem 3.2.** Assume the distribution of the data \( X^n = (X_1, \ldots, X_n) \) fulfills the sample covariance concentration property (2.1). Consider the prior \( \Pi^W \) given by the Inverse Wishart distribution \( IW_p(G, p + b - 1) \). Let \( \zeta \sim \mathcal{N}(0, 1) \).

Then with probability \( 1 - \frac{1}{n} \)
\[
\sup_{x \in \mathbb{R}} \left| \Pi^W \left( \sqrt{n} \left( \phi(\Sigma) - \phi(\bar{\Sigma}) \right) \right) \leq x \left| X^n \right) - \mathbb{P}(\zeta \leq x) \right| \lesssim \Diamond_\phi,
\]

where
\[
\Diamond_\phi \overset{\text{def}}{=} \Diamond_1 + \Diamond_2 + \Diamond_3 + \Diamond_4.
\]
The terms $\hat{\diamondsuit}_1$ through $\hat{\diamondsuit}_4$ can be described as

\[
\hat{\diamondsuit}_1 \overset{\text{def}}{=} \sqrt{n} \left( \delta_n^W + \hat{\delta}_n \right)^2 \frac{G_\phi(\Sigma^*) \| \Sigma^* \|_\infty^2}{\| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2},
\]

\[
\hat{\diamondsuit}_2 \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \cdot \frac{\| \Phi \|_1 \{ (\log(n) + p) \| \Sigma^* \|_\infty + \| G \|_\infty \}}{\| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2},
\]

\[
\hat{\diamondsuit}_3 \overset{\text{def}}{=} \frac{r^3(\Phi)}{\sqrt{n}},
\]

\[
\hat{\diamondsuit}_4 \overset{\text{def}}{=} \tilde{\delta}_n \frac{\| \Sigma^* \|_\infty \cdot \| \Phi \Sigma^* \Phi \|_1}{\| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2} + \frac{p}{n}
\]

with $\hat{\delta}_n$ and $\delta_n^W$ from (2.1) and (2.4), respectively.

Then we just exploit our main result Theorem 2.3 to derive the extended version of Theorem 3.2 for arbitrary prior.

**Theorem 3.3.** Assume the distribution of the data $X_n = (X_1, \ldots, X_n)$ fulfills the sample covariance concentration property (2.1). Let $\zeta \sim N(0, 1)$. Then with probability $1 - \frac{1}{n}$

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{E} \left( \frac{\sqrt{n} \left( \phi(\Sigma) - \phi(\hat{\Sigma}) \right)}{\sqrt{2 \| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2}} \right) - x \left| X_n \right) - \mathbb{P}(\zeta \leq x) \right| \lesssim \hat{\diamondsuit}_\phi + \hat{\diamondsuit}^*,
\]

where $\hat{\diamondsuit}_\phi$ and $\hat{\diamondsuit}^*$ are defined in (3.3) and (2.5), respectively.

Together with classical results on approximate normality of functionals of covariance matrix, these theorems provide a procedure for construction of credible sets with frequentist coverage guarantees for the true value of functional at $\Sigma^*$.

Next, we compare our theorem asymptotic functional BvM presented in [32]. In that work, a prior distribution is imposed on the precision matrix $\Omega = \Sigma^{-1}$.

Assume that there exist a set $A_n$ and a small value $\delta_n = o(1)$ such that

\[
A_n \subset \{ \| \Sigma - \Sigma^* \|_\infty \leq \delta_n \| \Sigma^* \|_\infty \},
\]

and let the functional $\phi(\cdot)$ be approximately linear on this set, that is, there exists a symmetric matrix $\Phi \in \mathbb{R}^{p \times p}$ such that

\[
\sup_{\Sigma \in A_n} \sqrt{n} \left| \phi(\Sigma) - \phi(\hat{\Sigma}) - \text{Tr} \left( (\Sigma - \hat{\Sigma}) \Phi \right) \right| = o_p(1). \quad (3.4)
\]

The following result takes place.
Theorem 3.4 ([32], Theorem 2.1). Under the assumptions of (3.4) and $\|\Sigma^*\| \vee \|\Omega^*\| = O(1)$, if for a given prior $\Pi$ the following two conditions are satisfied:

1. $\Pi \left( A_n \mid X^n \right) = 1 - o_p(1)$.
2. For any fixed $t \in \mathbb{R}$ holds

$$\frac{\int_{A_n} \exp \left( l_n(\Omega_t) \right) d\Pi(\Omega)}{\int_{A_n} \exp \left( l_n(\Omega) \right) d\Pi(\Omega)} = 1 + o_p(1)$$

for the perturbed precision matrix

$$\Omega_t = \Omega + \frac{\sqrt{2t}}{\sqrt{\|\Sigma^*\| \Phi \Sigma^* \|\Sigma^*\| \Phi}} \Phi.$$

Then

$$\sup_{x \in \mathbb{R}} \left| \Pi \left( \frac{\sqrt{n} \left( \phi(\Sigma) - \phi(\hat{\Sigma}) \right)}{\sqrt{2} \|\Sigma^*\| \Phi \Sigma^* \|\Sigma^*\| \Phi} \right) \leq x \mid X^n \right) - \mathbb{P}(\zeta \leq x) \right| = o_p(1),$$

where $\zeta \sim \mathcal{N}(0,1)$.

The proof of this result is based on the Laplace transform approach. Even though [32] assumes the data $X_1, \ldots, X_n$ to be Gaussian, the result can be extended in a straightforward way to non-Gaussian data. Nevertheless, this technique doesn’t allow to obtain a result for finite sample, since even if we know the rate of convergence of the exponential moments, it cannot provide any guarantee on the rate of convergence of distributions. Moreover, the second condition on a prior may not be easy to verify.

3.2. Spectral projectors of covariance matrix

First, define the sample projector on the $r$-th eigenspace of dimension $m_r^*$:

$$\hat{P}_r = \sum_{j \in \Delta_r^*} \hat{u}_j \hat{u}_j^\top.$$

More generally, pick a block of eigenspaces corresponding to an interval $\mathcal{J}$ in $\{1, \ldots, q\}$ from $r^-$ to $r^+$:

$$\mathcal{J} = \{ r^-, r^- + 1, \ldots, r^+ \}.$$
Define also the subset of indices

\[ \mathcal{I}_J \overset{\text{def}}{=} \{k: k \in \Delta^*_r, r \in J\}, \]

and introduce the projector onto the direct sum of the eigenspaces associated with \( P^*_r \) for all \( r \in J \):

\[ P^*_J \overset{\text{def}}{=} \sum_{r \in J} P^*_r = \sum_{k \in \mathcal{I}_J} u^*_k u^*_k \top. \]

Its empirical counterpart is given by

\[ \hat{P}_J \overset{\text{def}}{=} \sum_{r \in J} \hat{P}_r = \sum_{k \in \mathcal{I}_J} \hat{u}_k \hat{u}_k \top. \]

For instance, when \( J = \{1, \ldots, q_{\text{eff}}\} \) for some \( q_{\text{eff}} < q \), then \( \hat{P}_J \) is exactly what is recovered by PCA.

The projector dimension for \( J \) is given by \( m^*_J = \sum_{r \in J} m^*_r \). Its spectral gap can be defined as

\[
g^*_J \overset{\text{def}}{=} \begin{cases} 
\mu^*_r - \mu^*_r + 1, & \text{if } r^- = 1; \\
\mu^*_r - 1 - \mu^*_r, & \text{if } r^+ = q; \\
(\mu^*_r - 1 - \mu^*_r) \wedge (\mu^*_r + - \mu^*_r + 1), & \text{otherwise}.
\end{cases}
\]

Define also

\[ l^*_J = \mu^*_r - \mu^*_r. \]

For \( \Sigma \) generated by \( \Pi \) we can introduce \( P_J \) similarly. To describe the posterior distribution of \( \|P_J - \hat{P}_J\|_2^2 \), introduce the following matrix \( \Gamma^*_J \) of size \( m^*_J (p - m^*_J) \times m^*_J (p - m^*_J) \):

\[
\Gamma^*_J \overset{\text{def}}{=} \text{diag} \left( \Gamma^*_J \right)_{r \in J},
\]

\[
\Gamma^r_J \overset{\text{def}}{=} \text{diag} \left( \Gamma^r \right)_{s \notin J},
\]

\[
\Gamma^r,s \overset{\text{def}}{=} \frac{2\mu^*_r \mu^*_s}{(\mu^*_r - \mu^*_s)^2} \cdot I_{m^*_r m^*_s}, \quad r \in J, \ s \notin J.
\]

Before formulating the result for arbitrary prior, we consider the Inverse Wishart prior.
**Theorem 3.5** ([27], Theorem 2.1). Assume that the distribution of the data $X^n = (X_1, \ldots, X_n)$ fulfills the sample covariance concentration property (2.1). Consider the prior $\Pi^{W}$ given by the Inverse Wishart distribution $\mathcal{IW}_p(G, p + b - 1)$. Let $\xi \sim \mathcal{N}(0, \Gamma^*_J)$ with $\Gamma^*_J$ defined by (3.5). Then with probability $1 - \frac{1}{n}$

$$
\sup_{x \in \mathbb{R}} \left| \Pi^{W} \left( n \| P_J - \hat{P}_J \|_2^2 \leq x \mid X^n \right) - \mathbb{P}(\| \xi \|_2^2 \leq x) \right| \lesssim \Diamond_P,
$$

where

$$
\Diamond_P \overset{\text{def}}{=} \frac{\Diamond_1 + \Diamond_2 + \Diamond_3}{\sqrt{\lambda_1(\Gamma^*_J) \lambda_2(\Gamma^*_J)}} + \frac{1}{n},
$$

(3.6)

The terms $\Diamond_1$ through $\Diamond_3$ can be described as

$$
\Diamond_1 \overset{\text{def}}{=} \left\{ (\log(n) + p) \left( 1 + \frac{l_J^*}{g_J^*} \right) \sqrt{m_J^*} \frac{\| \Sigma^* \|_2^2}{g_J^*} \vee \| \Sigma^* \|_2 + \| G \|_2 \right\} \times \frac{\text{Tr}(\Sigma^*)}{g_J^*} \sqrt{\frac{\log(n) + p}{n}},
$$

$$
\Diamond_2 \overset{\text{def}}{=} \frac{\| \Sigma^* \|_\infty \left( m_J^* \| \Sigma^* \|_\infty^2 \land \text{Tr}(\Sigma^*)^{2} \right)}{g_J^*^3} \frac{p \left( \hat{\delta}_n + \frac{p}{n} \right)}{\sqrt{\frac{\log(n)}{n}}},
$$

$$
\Diamond_3 \overset{\text{def}}{=} \frac{m_J^*^{3/2} \| \Sigma^* \|_\infty \text{Tr}(\Sigma^*)}{g_J^*^2} \sqrt{\frac{\log(n)}{n}}
$$

with $\hat{\delta}_n$ from (2.1).

Then, from Theorem 3.5 and Theorem 2.3 we derive the following result.

**Theorem 3.6.** Assume the distribution of the data $X^n = (X_1, \ldots, X_n)$ fulfills the sample covariance concentration property (2.1). Let $\xi \sim \mathcal{N}(0, \Gamma^*_J)$ with $\Gamma^*_J$ defined by (3.5). Then with probability $1 - \frac{1}{n}$

$$
\sup_{x \in \mathbb{R}} \left| \Pi \left( n \| P_J - \hat{P}_J \|_2^2 \leq x \mid X^n \right) - \mathbb{P}(\| \xi \|_2^2 \leq x) \right| \lesssim \Diamond_P \overset{\text{def}}{=} \Diamond_P + \Diamond^*,
$$

where $\Diamond_P$ and $\Diamond^*$ are defined in (3.6) and (2.5), respectively.

As well as for functionals of covariance matrix, this result can be applied for building of sharp elliptic confidence sets for the true projector $P^*_J$. See again [27], Corollary 2.3 for the detailed description of the procedure.
4. Main proofs

4.1. Proof of Theorem 2.3

Step 1  “Localization”.
Fix arbitrary prior $\Pi$ (in particular, we may take $\Pi^W$). Posterior measure of a set $\mathcal{A} \subset \mathbb{S}_+^p$ is

$$\Pi(\mathcal{A}|X^n) = \frac{\int_{\mathcal{A}} \exp(l_n(\Sigma)) \Pi(\Sigma) \, d\Sigma}{\int_{\mathbb{S}_+^p} \exp(l_n(\Sigma)) \Pi(\Sigma) \, d\Sigma} = \frac{\tau(\mathcal{A})}{\tau(\mathbb{S}_+^p)},$$

where for shortness we introduce

$$\tau(\mathcal{A}) \overset{\text{def}}{=} \int_{\mathcal{A}} \exp(l_n(\Sigma)) \Pi(\Sigma) \, d\Sigma.$$

Observe that due to (2.2) since $\delta_n \geq \bar{\delta}_n$, we have

$$\frac{\tau(B(\bar{\delta}_n))}{\tau(\mathbb{S}_+^p)} \geq 1 - \frac{1}{n}. \tag{4.1}$$

Consider “localized” posterior defined by

$$\Pi_{loc}(\mathcal{A}|X^n) \overset{\text{def}}{=} \frac{\int_{\mathcal{A} \cap B(\bar{\delta}_n)} \exp(l_n(\Sigma)) \Pi(\Sigma) \, d\Sigma}{\int_{B(\bar{\delta}_n)} \exp(l_n(\Sigma)) \Pi(\Sigma) \, d\Sigma} = \frac{\tau(\mathcal{A} \cap B(\bar{\delta}_n))}{\tau(B(\bar{\delta}_n))}.$$ 

It is straightforwardly follows from (4.1) that

$$\sup_{\mathcal{A} \subset \mathbb{S}_+^p} |\Pi_{loc}(\mathcal{A}|X^n) - \Pi(\mathcal{A}|X^n)| \leq \frac{2}{n}. \tag{4.2}$$

with probability $1 - 1/n$. In particular, Lemma 2.1 and the fact that $\bar{\delta}_n \geq \delta_n^W$ imply similar bound for the Inverse Wishart prior:

$$\sup_{\mathcal{A} \subset \mathbb{S}_+^p} |\Pi_{loc}^W(\mathcal{A}|X^n) - \Pi^W(\mathcal{A}|X^n)| \leq \frac{2}{n}. \tag{4.3}$$

Step 2  “Flatness”.
Consider uniform prior $\Pi^U$ over $B(\bar{\delta}_n)$, which will be kind of a bridge between the priors $\Pi$ and $\Pi^W$. The corresponding posterior is given by

$$\Pi^U(\mathcal{A}|X^n) = \frac{\int_{\mathcal{A} \cap B(\bar{\delta}_n)} \exp(l_n(\Sigma)) \, d\Sigma}{\int_{B(\bar{\delta}_n)} \exp(l_n(\Sigma)) \, d\Sigma}.$$
Recalling the definition of flatness (2.3), it is easy to show that

$$\sup_{A \subset S_p^+} \left| \Pi_{loc}(A \mid X^n) - \Pi_U(A \mid X^n) \right| \leq 4 \rho(\delta_n)$$  (4.4)

with probability one. The same applies to the Inverse Wishart prior:

$$\sup_{A \subset S_p^+} \left| \Pi_{loc}^W(A \mid X^n) - \Pi_U(A \mid X^n) \right| \leq 4 \rho^W(\delta_n),$$  (4.5)

where the flatness of the Inverse Wishart prior is bounded in Lemma 2.2.

Applying the triangle inequality to (4.2), (4.3), (4.4) and (4.5), we derive the desired result.

4.2. Proof of Theorem 3.2

Write the representation (3.1) in two ways:

$$\phi(\Sigma) - \phi(\Sigma^*) = \text{Tr}[\Phi(\Sigma - \Sigma^*)] + \epsilon(\Sigma, \Sigma^*)$$

and

$$\phi(\hat{\Sigma}) - \phi(\Sigma^*) = \text{Tr}[\Phi(\hat{\Sigma} - \Sigma^*)] + \epsilon(\hat{\Sigma}, \Sigma^*).$$

Thus, subtracting the latter equality from the former,

$$\phi(\Sigma) - \phi(\hat{\Sigma}) = \text{Tr}[\Phi(\Sigma - \hat{\Sigma})] + \epsilon(\Sigma, \Sigma^*) - \epsilon(\hat{\Sigma}, \Sigma^*).$$

Define

$$\Delta_1 \overset{\text{def}}{=} \sqrt{n} \delta_n^W C_\phi(\Sigma^*) \|\Sigma^*\|_\infty^2,$$  (4.6)

$$\Delta_2 \overset{\text{def}}{=} \sqrt{n} \delta_n^W C_\phi(\Sigma^*) \|\Sigma^*\|_\infty^2$$  (4.7)

with $\hat{\delta}_n$ and $\delta_n^W$ from (2.1) and (2.4), respectively. Then the assumption (3.2) yields

$$\mathbb{P}(\sqrt{n} \mid \epsilon(\hat{\Sigma}, \Sigma^*) \geq \Delta_2) \leq \frac{1}{n}$$  (4.8)

and with probability $1 - \frac{1}{n}$

$$\Pi^W(\sqrt{n} \mid \epsilon(\Sigma, \Sigma^*) \geq \Delta_1 \mid X^n) \leq \frac{1}{n}.$$  (4.9)
Further, in order to work with the main linear part of the above display, we elaborate on $\Sigma$. Since $\Sigma \mid X^n \sim \mathcal{W}(G + n \hat{\Sigma}, n + p + b - 1)$ We make use of the following property of the Inverse Wishart distribution:

$$
\Sigma^{-1} \mid X^n \overset{d}{=} \sum_{j=1}^{n+p+b-1} W_j W_j^T,
$$

where $W_j \mid X^n \overset{i.i.d.}{\sim} \mathcal{N}(0, (G + n \hat{\Sigma})^{-1})$. Hence, introducing $n_p \overset{\text{def}}{=} n + p + b - 1$ and

$$
\Sigma_{n,p} = \frac{G + n \hat{\Sigma}}{n_p},
$$

we represent

$$
\Sigma^{-1} \mid X^n \overset{d}{=} \Sigma_{n,p}^{-1/2} \left[ \frac{1}{n_p} \sum_{j=1}^{n_p} Z_j Z_j^T \right] \Sigma_{n,p}^{-1/2} = \Sigma_{n,p}^{-1/2} (I_p + E_{n,p}) \Sigma_{n,p}^{-1/2},
$$

where $Z_j \mid X^n \overset{i.i.d.}{\sim} \mathcal{N}(0, I_p)$ and we also defined

$$
E_{n,p} \overset{\text{def}}{=} \frac{1}{n_p} \sum_{j=1}^{n_p} Z_j Z_j^T - I_p. \quad (4.10)
$$

We may think that in the posterior world all randomness comes from $E_{n,p}$. Moreover, due to Theorem A.1, (i), there is a random set $\Upsilon$ such that on this set

$$
\|E_{n,p}\|_{\infty} \lesssim \sqrt{\frac{\log(n_p) + p}{n_p}} \leq \sqrt{\frac{\log(n) + p}{n}} \quad (4.11)
$$

and its posterior measure

$$
\Pi(\Upsilon \mid X^n) \geq 1 - \frac{1}{n}. \quad (4.12)
$$

**Lemma 4.1.** The following decomposition holds:

$$
\text{Tr}[\Phi(\Sigma - \hat{\Sigma})] = - \text{Tr} \left( \Phi \Sigma^{1/2} E_{n,p} \Sigma^{1/2} \right) + \mathcal{R},
$$

and the remainder $\mathcal{R}$ satisfies on the random set $\Upsilon$

$$
\sqrt{n} |\mathcal{R}| \lesssim \hat{\Delta_3} \overset{\text{def}}{=} \frac{\|\Phi\|_1}{\sqrt{n}} \cdot \left\{ (\log(n) + p)\|\hat{\Sigma}\|_{\infty} + \|G\|_{\infty} \right\}.
$$
Proof. Define $R_{n,p}$ by

$$R_{n,p} \overset{\text{def}}{=} (I_p + E_{n,p})^{-1} - I_p + E_{n,p}.$$ 

Its spectral norm can be bounded as

$$\|R_{n,p}\|_\infty \lesssim \sum_{s=2}^\infty (-E_{n,p})^s \lesssim \sum_{s=2}^\infty \|E_{n,p}\|_\infty \leq \|E_{n,p}\|_\infty^2 \lesssim \|E_{n,p}\|_\infty^2.$$ 

So

$$\Sigma = \Sigma_{n,p}^{1/2} (E_{n,p} + I_p)^{-1} \Sigma_{n,p}^{1/2} = \Sigma_{n,p}^{1/2} (I_p - E_{n,p} + R_{n,p}) \Sigma_{n,p}^{1/2}.$$ 

Therefore for $\Sigma - \hat{\Sigma}$ we have

$$\Sigma - \hat{\Sigma} = \Sigma_{n,p}^{1/2} (I_p - E_{n,p} + R_{n,p}) \Sigma_{n,p}^{1/2} - \hat{\Sigma} = -\Sigma_{n,p}^{1/2} E_{n,p} \Sigma_{n,p}^{1/2} + \Sigma_{n,p}^{1/2} R_{n,p} \Sigma_{n,p}^{1/2} + \Sigma_{n,p} - \hat{\Sigma}. $$

From $\Sigma_{n,p}^{1/2} E_{n,p} \Sigma_{n,p}^{1/2}$ we pass to $\hat{\Sigma}_{n,p}^{1/2}$:

$$\Sigma - \hat{\Sigma} = -\hat{\Sigma}_{n,p}^{1/2} E_{n,p} \hat{\Sigma}_{n,p}^{1/2} + (\hat{\Sigma}_{n,p}^{1/2} E_{n,p} \hat{\Sigma}_{n,p}^{1/2} - \Sigma_{n,p}^{1/2} E_{n,p} \Sigma_{n,p}^{1/2})$$

$$+ \Sigma_{n,p}^{1/2} R_{n,p} \Sigma_{n,p}^{1/2} + \Sigma_{n,p} - \hat{\Sigma}$$

$$= -\hat{\Sigma}_{n,p}^{1/2} E_{n,p} \hat{\Sigma}_{n,p}^{1/2} + R_1 + R_2 + R_3,$$

where we introduce the remainder terms

$$R_1 \overset{\text{def}}{=} \hat{\Sigma}_{n,p}^{1/2} E_{n,p} \hat{\Sigma}_{n,p}^{1/2} - \Sigma_{n,p}^{1/2} E_{n,p} \Sigma_{n,p}^{1/2},$$

$$R_2 \overset{\text{def}}{=} \Sigma_{n,p}^{1/2} R_{n,p} \Sigma_{n,p}^{1/2},$$

$$R_3 \overset{\text{def}}{=} \Sigma_{n,p} - \hat{\Sigma}.$$ 

They can be bounded in Frobenius norm:

$$\|R_1\|_\infty \leq \|E_{n,p}\|_\infty \hat{\Sigma} - \Sigma_{n,p} \|_{\infty}^{1/2} \left( \|\Sigma_{n,p}\|_{\infty}^{1/2} + \|\hat{\Sigma}\|_{\infty}^{1/2} \right),$$

$$\|R_2\|_\infty \leq \|R_{n,p}\|_\infty \|\Sigma_{n,p}\|_\infty \lesssim \|E_{n,p}\|_\infty^2 \|\Sigma_{n,p}\|_\infty,$$

$$\|R_3\|_\infty \leq \frac{\|G\|_\infty + (n_p - n) \|\Sigma\|_\infty}{n_p}.$$
Hence, omitting higher order terms, on \( \Upsilon \) we have
\[
\| R_1 \|_\infty \lesssim \| \mathbf{\hat{\Sigma}} \|^{1/2} \left( \| G \|_\infty + p \| \mathbf{\hat{\Sigma}} \|_\infty \right)^{1/2} \frac{\sqrt{\log(n) + p}}{n},
\]
\[
\| R_2 \|_\infty \lesssim \| \mathbf{\hat{\Sigma}} \| \frac{\log(n) + p}{n},
\]
\[
\| R_3 \|_\infty \lesssim \frac{\| G \|_\infty + p \| \mathbf{\hat{\Sigma}} \|_\infty}{n} \sqrt{\log(n) + p}.\]
Therefore,
\[
\text{Tr}[\Phi(\mathbf{\Sigma} - \mathbf{\hat{\Sigma}})] = - \text{Tr} \left( \Phi \mathbf{\hat{\Sigma}}^{1/2} E_{n,p} \mathbf{\hat{\Sigma}}^{1/2} \right) + R_1 + R_2 + R_3,
\]
where \( R_i = \text{Tr}(\Phi R_i) \) and \( |R_i| \leq \| \Phi \|_1 \| R_i \|_\infty \) for all \( i = 1, 2, 3 \). Putting all the bound together, we obtain the desired statement.

The structure of \( E_{n,p} \) allows to apply Gaussian approximation and derive the following lemma.

**Lemma 4.2.** Let \( \mathbf{\hat{\zeta}} \sim \mathcal{N} \left( 0, \frac{2n}{n_p} \| \mathbf{\hat{\Sigma}} \|_2 \| \mathbf{\hat{\Sigma}} \|_2 \right) \). Then
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}^W \left( - \sqrt{n} \text{Tr} \left( \mathbf{\hat{\Sigma}}^{1/2} \Phi \mathbf{\hat{\Sigma}}^{1/2} E_{n,p} \right) \leq x \mid \mathbf{X}^n \right) - \mathbb{P} \left( \mathbf{\hat{\zeta}} \leq x \mid \mathbf{X}^n \right) \right| \lesssim \Delta_4 \overset{\text{def}}{=} \frac{\mathbb{V}_3(\mathbf{\Phi})}{\sqrt{n}}
\]
with probability one.

**Proof.** Recalling the definition of \( E_{n,p} \) from (4.10), we can represent
\[
- \sqrt{n} \text{Tr} \left( \mathbf{\hat{\Sigma}}^{1/2} \Phi \mathbf{\hat{\Sigma}}^{1/2} E_{n,p} \right) = \frac{1}{\sqrt{n_p}} \sum_{j=1}^{n_p} \left( \eta_j - \mathbb{E} \eta_j \right),
\]
where
\[
\eta_j \overset{\text{def}}{=} - \sqrt{\frac{n}{n_p}} Z_j^\top \mathbf{\hat{\Sigma}}^{1/2} \Phi \mathbf{\hat{\Sigma}}^{1/2} Z_j
\]
and \( Z_j \overset{i.i.d.}{\sim} \mathcal{N}(0, I_p) \). It is easy to verify that
\[
\mathbb{V} \left( \eta_j \right) = \frac{2n}{n_p} \| \mathbf{\hat{\Sigma}} \|_2 \| \mathbf{\hat{\Sigma}} \|_2 \| \mathbf{\hat{\Sigma}} \|_2 \| \mathbf{\hat{\Sigma}} \|_2.
\]
Besides, observe that
\[ E[|\eta_j - E\eta_j|^3] \lesssim r^3(\Phi) (\text{Var}(\eta_j))^{3/2}. \]

Now the classical Berry – Esseen theorem yield the result of the lemma. □

The next step is to compare two normal distributions with variances
\[ \frac{2n}{n_p} \| \hat{\Sigma}^{1/2} \Phi \hat{\Sigma}^{1/2} \|_2^2 \quad \text{and} \quad \| \Sigma^\ast^{1/2} \Phi \Sigma^\ast^{1/2} \|_2^2. \]

**Lemma 4.3.** Let
\[ \hat{\zeta} \sim \mathcal{N} \left( 0, \frac{2n}{n_p} \| \hat{\Sigma}^{1/2} \Phi \hat{\Sigma}^{1/2} \|_2^2 \right) \]
and
\[ \zeta \sim \mathcal{N} \left( 0, 2\| \Sigma^\ast^{1/2} \Phi \Sigma^\ast^{1/2} \|_2^2 \right). \]

Then, with probability one
\[ \sup_{x \in \mathbb{R}} | II \left( \hat{\zeta} \leq x \mid X^n \right) - P(\zeta \leq x) | \lesssim \widehat{\Delta}_5, \]
\[ \widehat{\Delta}_5 \overset{\text{def}}{=} \| \hat{\Sigma} - \Sigma^\ast \|_{\infty} \frac{\| \Phi(\hat{\Sigma} + \Sigma^\ast) \Phi \|_1}{2 \| \Sigma^\ast^{1/2} \Phi \Sigma^\ast^{1/2} \|_2^2} + \frac{p}{n}. \]

**Proof.** For shortness define
\[ \sigma_1^2 \overset{\text{def}}{=} \frac{2n}{n_p} \| \hat{\Sigma}^{1/2} \Phi \hat{\Sigma}^{1/2} \|_2^2, \]
\[ \sigma_2^2 \overset{\text{def}}{=} 2\| \Sigma^\ast^{1/2} \Phi \Sigma^\ast^{1/2} \|_2^2. \]

and let \( P_\sigma \) corresponds to one-dimensional normal distribution with zero mean and variance \( \sigma^2 \). By definition of total variation distance (\( d_{TV} \)) distance, we have
\[ \sup_{x \in \mathbb{R}} | II \left( \hat{\zeta} \leq x \mid X^n \right) - P(\zeta \leq x) | \leq d_{TV}(P_\sigma_1, P_\sigma_2) \]
in the posterior world for each \( X^n \). Due to the Pinsker’s lemma (see, e.g. [29], Lemma 2.5 (i)),
\[ d_{TV}(P_\sigma_1, P_\sigma_2) \leq \sqrt{d_{KL}(P_\sigma_1, P_\sigma_2)/2}, \]
\[ d_{TV}(P_\sigma_1, P_\sigma_2) = d_{TV}(P_\sigma_2, P_\sigma_1) \leq \sqrt{d_{KL}(P_\sigma_2, P_\sigma_1)/2}, \]
where \( d_{KL} \) is Kullback-Leibler divergence. Therefore,
\[
d_{TV}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) \leq \sqrt{\frac{d_{KL}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) \land d_{KL}(\mathbb{P}_{\sigma_2}, \mathbb{P}_{\sigma_1})}{2}}.
\]
Simple calculations show that
\[
d_{KL}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) = \log \left( \frac{\sigma_2}{\sigma_1} \right) + \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_2^2},
\]
\[
d_{KL}(\mathbb{P}_{\sigma_2}, \mathbb{P}_{\sigma_1}) = \log \left( \frac{\sigma_1}{\sigma_2} \right) + \frac{\sigma_2^2 - \sigma_1^2}{2\sigma_1^2},
\]
\[
d_{KL}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) \land d_{KL}(\mathbb{P}_{\sigma_2}, \mathbb{P}_{\sigma_1}) \leq \frac{d_{KL}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) + d_{KL}(\mathbb{P}_{\sigma_2}, \mathbb{P}_{\sigma_1})}{2}
\]
= \left( \frac{\sigma_2^2 - \sigma_1^2}{4\sigma_1^2\sigma_2^2} \right),
\]
which implies
\[
d_{TV}(\mathbb{P}_{\sigma_1}, \mathbb{P}_{\sigma_2}) \leq \frac{|\sigma_2^2 - \sigma_1^2|}{2\sigma_1\sigma_2}.
\]
Clearly, one has
\[
\frac{|\sigma_2^2 - \sigma_1^2|}{2} \leq \frac{n_p - n}{n_p} \| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2
\]
\[
+ \frac{n}{n_p} \| \Sigma^{1/2} \Phi \hat{\Sigma}^{1/2} \|_2^2 - \| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2
\]
\[
= \| \Sigma - \Sigma^* \|_\infty \| \Phi (\hat{\Sigma} + \Sigma^*) \Phi \|_1.
\]
For the second term we have
\[
\| \Sigma^{1/2} \Phi \hat{\Sigma}^{1/2} \|_2^2 - \| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2
\]
\[
\leq \| \Sigma - \Sigma^* \|_\infty \| \Phi (\hat{\Sigma} + \Sigma^*) \Phi \|_1.
\]
Then, assuming the right-hand side is small enough, we get
\[
\frac{|\sigma_2^2 - \sigma_1^2|}{2\sigma_1\sigma_2} \lesssim \frac{|\sigma_2^2 - \sigma_1^2|}{2\sigma_2^2} \lesssim \| \Sigma - \Sigma^* \|_\infty \| \Phi (\hat{\Sigma} + \Sigma^*) \Phi \|_1
\]
\[
\frac{n_p - n}{n_p} \| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2 + \frac{p}{n},
\]
that concludes the proof of the lemma. \(\square\)

Finally, we unite all the obtained bounds. Let us notice that for \( \hat{\Delta}_3, \hat{\Delta}_5 \) from Lemmas 4.1, 4.3 due to (2.1) holds
\[
\hat{\Delta}_3 \leq \Delta_3 \overset{\text{def}}{=} \frac{\| \Phi \|_1}{\sqrt{n}} \cdot \{(\log(n) + p)\| \Sigma^* \|_\infty + \| G \|_\infty \},
\]
\[
\hat{\Delta}_5 \leq \Delta_5 \overset{\text{def}}{=} \delta_n \cdot \frac{\| \Sigma^* \|_\infty \cdot \| \Phi \Sigma^* \Phi \|_1}{\| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2^2} + \frac{p}{n}.
\]
with probability $1 - 1/n$. So, for $\Delta_1, \Delta_2, \Delta_3$ from (4.6), (4.7), (4.13), respectively, we write

$$\Pi^W \left( \sqrt{n}(\phi(\Sigma) - \phi(\Sigma^*)) \leq x \right| X^n)$$

$$\leq \Pi^W \left( -\sqrt{n} \text{Tr} \left( \Phi \hat{\Sigma}^{1/2} E_{n,p} \hat{\Sigma}^{1/2} \right) \leq x + \Delta_1 + \Delta_2 + \Delta_3 \right| X^n)$$

$$+ \Pi^W \left( \sqrt{n} \varepsilon(\Sigma, \Sigma^*) \leq -\Delta_1 \right| X^n) + \Pi^W \left( -\sqrt{n} \varepsilon(\hat{\Sigma}, \Sigma^*) \leq -\Delta_2 \right| X^n)$$

$$+ \Pi^W \left( \sqrt{n} \mathcal{R} \leq -\Delta_3 \right| X^n).$$

The second term in the right-hand side is at most $1/n$ with probability $1 - 1/n$ due to (4.9). The third term is zero with probability $1 - 1/n$ according to (4.8). Lemma 4.1 and (4.13) imply that the fourth term does not exceed $1/n$ with probability $1 - 1/n$. Thus, with probability $1 - 3/n$ we have

$$\Pi^W \left( \sqrt{n}(\phi(\Sigma) - \phi(\Sigma^*)) \leq x \right| X^n)$$

$$\leq \Pi^W \left( -\sqrt{n} \text{Tr} \left( \Phi \hat{\Sigma}^{1/2} E_{n,p} \hat{\Sigma}^{1/2} \right) \leq x + \Delta_1 + \Delta_2 + \Delta_3 \right| X^n) + \frac{2}{n}.$$ 

Subtracting $\mathbb{P}(\zeta \leq x)$ with $\zeta \sim \mathcal{N}(0, 2\|\Sigma^{1/2}\Phi \Sigma^{1/2}\|_2^2)$ from the both sides and taking supremum over $x \in \mathbb{R}$, we obtain

$$\sup_{x \in \mathbb{R}} \left[ \Pi^W \left( \sqrt{n}(\phi(\Sigma) - \phi(\Sigma^*)) \leq x \right| X^n) - \mathbb{P}(\zeta \leq x) \right]$$

$$\leq \sup_{x \in \mathbb{R}} \left[ \Pi^W \left( -\sqrt{n} \text{Tr} \left( \Phi \hat{\Sigma}^{1/2} E_{n,p} \hat{\Sigma}^{1/2} \right) \leq x + \Delta_1 + \Delta_2 + \Delta_3 \right| X^n) \right.$$

$$\left. - \mathbb{P}(\zeta \leq x + \Delta_1 + \Delta_2 + \Delta_3) \right] + \sup_{x \in \mathbb{R}} \left[ \mathbb{P}(\zeta \leq x + \Delta_1 + \Delta_2 + \Delta_3) - \mathbb{P}(\zeta \leq x) \right] + \frac{2}{n}. $$

The first term in the right-hand side is bounded by $\Delta_4 + \Delta_5$ due to Lemma 4.2, Lemma 4.3 and (4.14). The second term is at most $\frac{\Delta_1 + \Delta_2 + \Delta_3}{\sqrt{2\pi} \sqrt{2\|\Sigma^{1/2}\Phi \Sigma^{1/2}\|_2}}$. 
Finally,
\[
\sup_{x \in \mathbb{R}} \left[ H^W \left( \sqrt{n} (\phi(\Sigma) - \phi(\Sigma^*)) \leq x \right| X^n \right) - P(\zeta \leq x) \right] 
\lesssim \frac{\Delta_1 + \Delta_2 + \Delta_3}{\| \Sigma^{1/2} \Phi \Sigma^{1/2} \|_2} + \Delta_4 + \Delta_5 + \frac{1}{n}
\]
with probability \(1 - 3/n\), which coincides with the claim of the theorem.

Appendix A: Auxiliary results

The following theorem gathers several crucial results on concentration of sample covariance.

**Theorem A.1.** Let \(X_1, \ldots, X_n\) be i.i.d. zero-mean random vectors in \(\mathbb{R}^p\). Denote the true covariance matrix as \(\Sigma^* \overset{\text{def}}{=} \mathbb{E} (X_j X_j^\top)\) and the sample covariance as \(\hat{\Sigma} \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^n X_j X_j^\top\). Suppose the data and are obtained from:

(i) Gaussian distribution \(\mathcal{N}(0, \Sigma^*)\). In this case, define \(\hat{\delta}_n\) as

\[
\hat{\delta}_n \asymp \sqrt{r(\Sigma^*)} n \lor \sqrt{\log(n)} n;
\]

(ii) Sub-Gaussian distribution. In this case, define \(\hat{\delta}_n\) as

\[
\hat{\delta}_n \asymp \sqrt{p} n \lor \sqrt{\log(n)} n;
\]

(iii) a distribution supported in some centered Euclidean ball of radius \(R\). In this case, define \(\hat{\delta}_n\) as

\[
\hat{\delta}_n \asymp \frac{R}{\sqrt{\| \Sigma^* \|}} \sqrt{\log(n)} n;
\]

(iv) log-concave probability measure. In this case, define \(\hat{\delta}_n\) as

\[
\hat{\delta}_n \asymp \sqrt{\log^6(n)} n p.
\]

Then in all the cases above the following concentration result for \(\hat{\Sigma}\) holds with the corresponding \(\hat{\delta}_n\):

\[
\| \hat{\Sigma} - \Sigma^* \|_\infty \leq \hat{\delta}_n \| \Sigma^* \|_\infty
\]

with probability at least \(1 - \frac{1}{n}\).
Proof. (i) See [19], Corollary 2. (ii) This is a well-known simple result presented in a range of papers and lecture notes. See, e.g. [25], Theorem 4.6. (iii) See [31], Corollary 5.52. Usually the radius \( R \) is taken such that \( R \sqrt{\|\Sigma\|} \lesssim \sqrt{\text{Tr}(\Sigma^{*))} \). (iv) See [1], Theorem 4.1.

Appendix B: Auxiliary proofs

B.1. Proof of Lemma 2.1

Using the definitions of \( n_p, \Sigma_{n,p}, E_{n,p} \) from the beginning of the proof of Theorem 3.2, we write

\[
\Sigma - \Sigma_{n,p} = \Sigma_{n,p}^{1/2} [(I_p + E_{n,p})^{-1} - I_p] \Sigma_{n,p}^{1/2}.
\]

Since

\[
\|(I_p + E_{n,p})^{-1} - I_p\|_{\infty} = \left\| \sum_{s=1}^{+\infty} (-1)^s E_{n,p}^s \right\|_{\infty} \leq \frac{\|E_{n,p}\|_{\infty}}{1 - \|E_{n,p}\|_{\infty}} \lesssim \|E_{n,p}\|_{\infty},
\]

then

\[
\|\Sigma - \Sigma_{n,p}\|_{\infty} \lesssim \|\Sigma_{n,p}\|_{\infty} \|E_{n,p}\|_{\infty}.
\]

Moreover,

\[
\|\Sigma_{n,p} - \hat{\Sigma}\|_{\infty} \leq \frac{G\|\Sigma\|_{\infty} + (n_p - n) \|\hat{\Sigma}\|_{\infty}}{n_p}.
\]

Besides, the covariance concentration condition (2.1) provides

\[
\|\hat{\Sigma} - \Sigma^*\|_{\infty} \leq \hat{\delta}_n \|\Sigma^*\|_{\infty}
\]

with probability \( 1 - 1/n \). Applying the triangle inequality, omitting higher-order terms and recalling (4.11) and (4.12), we deduce that the posterior measure of the event

\[
\|\Sigma - \Sigma^*\|_{\infty} \lesssim \|\Sigma^*\|_{\infty} \left( \sqrt{\frac{\log(n) + p}{n}} + \hat{\delta}_n + \frac{G\|\Sigma^*\|_{\infty}}{n_p\|\Sigma^*\|_{\infty}} \right)
\]

is at least \( 1 - 1/n \) with \( X^n \)– probability \( 1 - 1/n \), which concludes the proof of the lemma.
B.2. Proof of Lemma 2.2

The log-density of the Inverse Wishart prior is

\[
\log \Pi_W(\Sigma) = -\frac{2p + b}{2} \log \det(\Sigma) - \frac{1}{2} \text{Tr}(G\Sigma^{-1}) + C,
\]

where \(C\) is some universal constant related to normalization. For arbitrary \(\Sigma\) we have

\[
\left| \frac{\Pi_W(\Sigma)}{\Pi_W(\Sigma^*)} - 1 \right| = \left| \exp \left\{ \log \Pi_W(\Sigma) - \log \Pi_W(\Sigma^*) \right\} - 1 \right|
\]

\[
= \left| \exp \left\{ -\frac{2p + b}{2} (\log \det(\Sigma) - \log \det(\Sigma^*)) - \frac{1}{2} \text{Tr}(G(\Sigma^{-1} - \Sigma^*^{-1})) \right\} - 1 \right|
\]

\[
\lesssim \exp \left\{ p |(\log \det(\Sigma) - \log \det(\Sigma^*))| + \left| \text{Tr} \left[ G \left( \Sigma^{-1} - \Sigma^*^{-1} \right) \right] \right| \right\} - 1.
\]

Notice that

\[
|\log \det(\Sigma) - \log \det(\Sigma^*)| = \log \det(\Sigma^*^{-1/2} \Sigma \Sigma^*^{-1/2}) = \log \det(I_p + \Delta),
\]

where \(\Delta = \Omega^*1/2(\Sigma - \Sigma^*)\Omega^*1/2\). Then

\[
\log \det(I_p + \Delta) = \sum_{j=1}^p \log (\lambda_j(\Delta) + 1) = -\sum_{j=1}^p \sum_{s=1}^\infty \frac{\lambda_j(\Delta)}{s} = -\sum_{s=1}^\infty \frac{\text{Tr}(\Delta^s)}{s}.
\]

Hence, we get the bound

\[
|\log \det(I_p + \Delta)| \leq \sum_{s=1}^\infty \|\Omega^*\|_1 \|\Sigma - \Sigma^*\|_\infty = \frac{\|\Omega^*\|_1 \|\Sigma - \Sigma^*\|_\infty}{1 - \|\Omega^*\|_1 \|\Sigma - \Sigma^*\|_\infty}.
\]

Assuming \(\delta \|\Omega^*\|_1\) is small enough, we get

\[
|\log \det(I_p + \Delta)| \lesssim \delta \|\Omega^*\|_1
\]

whenever \(\Sigma \in B(\delta)\).

Besides, for \(\Sigma \in B(\delta)\) we have

\[
\|\Sigma^{-1} - \Sigma^{-1}\|_\infty = \|\Sigma^*^{-1}(\Sigma - \Sigma^*)\Sigma^{-1}\|_\infty \leq \delta \|\Sigma^*^{-1}\|_\infty \|\Sigma^{-1}\|_\infty
\]

\[
\leq \delta \|\Sigma^*^{-1}\|_\infty \|\Sigma^*^{-1}\|_\infty + \delta \|\Sigma^*^{-1}\|_\infty \|\Sigma^{-1} - \Sigma^*^{-1}\|_\infty.
\]
Therefore,
\[
\left\| \Sigma^{-1} - \Sigma^{-1}_{*} \right\|_{\infty} \leq \frac{\delta \left\| \Omega^{*} \right\|_{2}^{2}}{1 - \delta \left\| \Omega^{*} \right\|_{\infty}} \lesssim \delta \left\| \Omega^{*} \right\|_{\infty}^{2}
\]
whenever \( \delta \left\| \Omega^{*} \right\|_{\infty} \) is small enough. Hence,
\[
\left| \text{Tr} \left[ G \left( \Sigma^{-1} - \Sigma^{-1}_{*} \right) \right] \right| \leq \delta \left\| G \right\|_{1} \left\| \Omega^{*} \right\|_{\infty}^{2}.
\]
Finally,
\[
\rho^W(\delta) = \sup_{\Sigma \in \mathcal{B}_{\delta}} \left| \frac{\Pi^{W}(\Sigma)}{\Pi^{W}(\Sigma^{*})} - 1 \right| \lesssim \delta \left\{ p \left\| \Omega^{*} \right\|_{1} + \left\| G \right\|_{1} \left\| \Omega^{*} \right\|_{\infty}^{2} \right\}.
\]

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