THE ARC SPACE OF A TORIC VARIETY

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ABSTRACT. The Nash problem on arc families is affirmatively answered for a toric variety by Ishii and Kollár’s paper which also shows the negative answer for general case. The Nash problem is one of questions about the relation between arc families and valuations. In this paper, the relation is described clearly for a toric variety. The arc space of a toric variety admits an action of the group scheme determined by the torus. Each orbit on the arc space corresponds to a lattice point in the cone and therefore corresponds to a toric valuation. The dominant relation among the orbits is described in terms of the lattice points. As a corollary we obtain the answer to the embedded version of the Nash problem for an invariant ideal on a toric variety.

Keywords: arc space, toric variety, Nash problem

1. Introduction

The concept of jet schemes and arc space over an algebraic variety or an analytic space was introduced by Nash in his preprint in 1968 which was later published as [12]. These schemes are considered as something to represent the nature of the singularities of the base space. In fact, papers [5], [10], [11] by Mustaţă, Ein and Yasuda show that geometric properties of the jet schemes determine certain properties of the singularities of the base space. Primarily the Nash problem posed in [12] is based on this idea. The Nash problem asks if the set of arc families through the singularities corresponds bijectively to the set of the essential components of resolutions of the singularities. Here an arc family through the singularities on $X$ is a good component of $\pi^{-1}(\text{Sing } X)$ (see 3.5 or [8] for the definition of a good component), where $\pi$ is the canonical projection from the arc space to $X$. The paper [8] proves that if $X$ is a toric variety, the answer to the Nash problem is “yes”, while the paper also shows the negative answer for general $X$. 

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In this paper, we study the structure of the arc space of a toric variety defined over an algebraically closed field \( k \) of arbitrary characteristic. We prove that each jet scheme or arc space admits a canonical action of the jet scheme or arc space of the torus. The arc space of a toric variety becomes an almost homogeneous space by this action, which means that the arc space is the closure of one orbit. A good component turns out to be the closure of a certain orbit and there is no non-good component in the arc space of a toric variety.

Each orbit of the arc space corresponds to a lattice point of the cone, therefore to a toric valuation, and the dominant relation of two orbits is translated to the order relation of the corresponding lattice points. As a corollary we show the answer to the embedded version of Nash problem posed by Ein, Lazarsfeld and Mustaţă in [6] for an invariant ideal on a toric variety.

This paper is organized as follows: In §2 we study some basic properties on jet schemes and arc spaces. The closed points in the arc spaces of varieties are discussed here. In §3 we introduce a stratification on the arc space of a toric variety according to the fan. Some basic properties of the arc space of a toric variety (non-existence of non-good components, irreducibility in any characteristic) are proved here. In §4 we study the orbits of the arc space of a toric variety by the action of the arc space of the torus. In §5 we give the answer to the embedded version of Nash problem for an invariant ideal on a toric variety.

Throughout this paper the base field \( k \) is an algebraically closed field of arbitrary characteristic unless otherwise stated.

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2. Basic properties of jet schemes and the arc space

**Definition 2.1.** Let \( X \) be a scheme of finite type over \( k \) and \( K \supset k \) a field extension. For \( m \in \mathbb{N} \) a morphism \( \text{Spec} K[t]/(t^{m+1}) \rightarrow X \) is called an \( m \)-jet of \( X \) and \( \text{Spec} K[[t]] \rightarrow X \) is called an arc of \( X \). We denote the closed point of \( \text{Spec} K[[t]] \) by \( 0 \) and the generic point by \( \eta \).

**2.2.** Let \( X \) be a scheme of finite type over \( k \). Let \( \mathcal{S} \text{ch}/k \) be the category of \( k \)-schemes and \( \text{Set} \) the category of sets. Define a contravariant functor \( F_m : \mathcal{S} \text{ch}/k \rightarrow \text{Set} \) by

\[
F_m(Y) = \text{Hom}_k(Y \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}), X).
\]
Then, \( F_m \) is representable by a scheme \( X_m \) of finite type over \( k \), that is

\[
\text{Hom}_k(Y, X_m) \simeq \text{Hom}_k(Y \times_{\text{Spec}\, k} \text{Spec}\, k[t]/(t^{m+1}), X).
\]

This \( X_m \) is called the \( m \)-jet scheme of \( X \). A \( K \)-valued point \( \alpha : \text{Spec}\, K \rightarrow X_m \) is regarded as an \( m \)-jet \( \alpha : \text{Spec}\, K[[t]]_{/t^m+1} \rightarrow X \).

Let \( X_\infty = \varprojlim_m X_m \) and call it the arc space of \( X \). \( X_\infty \) is a scheme which is not of finite type over \( k \), see [4]. Denote the canonical projection \( X_\infty \rightarrow X \) by \( \pi \). A \( K \)-valued point \( \alpha : \text{Spec}\, K \rightarrow X_\infty \) is regarded as an arc \( \alpha : \text{Spec}\, K[[t]]_{/t^m+1} \rightarrow X \).

Using the representability of \( F_m \) we obtain the following universal property of \( X_\infty \):

**Proposition 2.3.** Let \( X \) be a scheme of finite type over \( k \). Then

\[
\text{Hom}_k(Y, X_\infty) \simeq \text{Hom}_k(Y \widehat{\times}_{\text{Spec}\, k} \text{Spec}\, k[[t]], X)
\]

for an arbitrary \( k \)-scheme \( Y \), where \( Y \widehat{\times}_{\text{Spec}\, k} \text{Spec}\, k[[t]] \) means the formal completion of \( Y \times_{\text{Spec}\, k} \text{Spec}\, k[t] \) along the subscheme \( Y \times_{\text{Spec}\, k}\{0\} \).

**2.4.** A morphism \( \Phi : X \rightarrow Z \) of varieties over \( k \) induces a canonical morphism \( \Phi_m : X_m \rightarrow Z_m \) \((m \in \mathbb{N} \cup \{\infty\})\). Some properties of \( \Phi \) are inherited by \( \Phi_m \); for example, if \( \Phi \) is a closed immersion, an open immersion or étale, then \( \Phi_m \) is also a closed immersion, an open immersion or étale. But many properties of \( \Phi \) are not inherited by \( \Phi_m \); for example, properness, projectiveness, closedness, and so on.

Next we study the jet schemes and the arc space of a variety which admits an action of a group scheme.

**Proposition 2.5.** Let \( G \) be a group scheme of finite type over \( k \). Then \( G_m \) \((m \in \mathbb{N} \cup \{\infty\})\) is again a group scheme over \( k \). If \( G \) is irreducible, then \( G_m \) is also irreducible.

**Proof.** Let \( \mu : G \times G \rightarrow G \) be the multiplication of the group, let \( e \in G \) be the unit element of the group and let \( \iota : G \rightarrow G \) be the morphism defining the inverse elements. Then, \( G_m \) becomes a group scheme with \( \mu_m : G_m \times G_m \rightarrow G_m \) the multiplication of the group, where \( \mu_m \) is induced on \( (G \times G)_m \simeq G_m \times G_m \) from \( \mu \). The scheme \( \{e\}_m \) is a \( k \)-valued point of \( G_m \) and it is the unit element under this multiplication. The morphism \( \iota_m : G_m \rightarrow G_m \) induced from \( \iota \) gives the inverse elements. If \( G \) is irreducible, then it is a non-singular irreducible variety which yields that \( G_m \) is also non-singular and irreducible. \( \square \)

**Proposition 2.6.** Let \( G \) be a group scheme of finite type over \( k \) and \( X \) a variety admitting an action of \( G \). Then, for \( m \in \mathbb{N} \cup \{\infty\} \), \( X_m \) admits a canonical action of \( G_m \) induced from the action of \( G \) on \( X \).
Proof. Let \( \psi : G \times X \to X \) be the morphism defining the action of \( G \) on \( X \). Then the morphism \( \psi_m : G_m \times X_m \to (G \times X)_m \to X_m \) induced from \( \psi \) gives an action of \( G_m \) on \( X_m \).

**Example 2.7.** If \( G \) is an \( n \)-dimensional torus \( T^n \simeq (A^1_k \setminus \{0\})^n \), then \( G_m \simeq T^n \times A^m_k \). Let \( x = (x_1^{(0)}, \ldots, x_n^{(0)}, x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}) \) and \( y = (y_1^{(0)}, \ldots, y_n^{(0)}, y_1^{(1)}, \ldots, y_n^{(1)}, \ldots, y_1^{(m)}, \ldots, y_n^{(m)}) \) be two \( k \)-valued points of \( G_m \), where \((x_1^{(0)}, \ldots, x_n^{(0)}), (y_1^{(0)}, \ldots, y_n^{(0)}) \in T^n \). Then the multiplication \( x \cdot y \) of \( x \) and \( y \) is \( (x_1^{(0)} y_1^{(0)}, \ldots, x_n^{(0)} y_n^{(0)}, \sum_{i+j=1} x_1^i y_1^j, \ldots, \sum_{i+j=m} x_1^i y_1^j, \ldots, \sum_{i+j=m} x_n^i y_n^j) \). The unit element of \( G_m \) is
\[
(1, \ldots, 1, 0, \ldots, 0).
\]

**Example 2.8.** Let \( X \) be a toric variety with the torus \( T \). Then \( T_m \) acts on \( X_m \) for every \( m \in \mathbb{N} \cup \{\infty\} \).

**2.9.** As the \( m \)-jet scheme \( X_m \) of a variety \( X \) is of finite type over \( k \), a point of \( X_m \) is closed if and only if it is a \( k \)-valued point. But \( X_\infty \) is not of finite type and the equivalence above does not hold. First we will see the affirmative case under a condition on \( k \).

**Proposition 2.10.** Assume that the base field \( k \) is uncountable. Then, for every variety \( X \), a point of \( X_\infty \) is closed if and only if the point is a \( k \)-valued point.

Proof. As the problem is local, we may assume that \( X \) is affine. Therefore we have only to prove the assertion for the case \( X_\infty = \text{Spec} R \), \( R = k[x_1, x_2, \ldots, x_n, \ldots] \), where the variables \( x_1, x_2, \ldots, x_n, \ldots \) are countably infinite. For the assertion of the proposition, it is sufficient to prove that every prime ideal \( I \subset k[x_1, x_2, \ldots, x_n, \ldots] \) is contained in a maximal ideal \((x_1-a_1, x_2-a_2, \ldots, x_n-a_n, \ldots), a_1, a_2, \ldots, a_n, \ldots \in k \).

For every \( n \), let \( R_n \) be a subring \( k[x_1, \ldots, x_n] \) of \( R \) and \( I_n \) be the intersection \( I \cap R_n \). For \( m < n \) the inclusion \( R_m \hookrightarrow R_n \) induces the projection \( \text{Spec} R_n \to \text{Spec} R_m \) which induces a dominant map \( \psi_{n,m} : Z(I_n) \to Z(I_m) \), \((a_1, \ldots, a_m, \ldots) \mapsto (a_1, \ldots, a_m) \), where \( Z(I_n) \) is the set of closed points of the closed subscheme defined by \( I_n \). Fix \( r \geq 1 \). Since \( Z(I_r) \neq \emptyset \) for every \( n > r \), \( \text{Im} \psi_{n,r} \) is a non-empty constructible set and
\[
\text{Im} \psi_{r+1,r} \supset \text{Im} \psi_{r+2,r} \supset \ldots
\]
is a non-increasing sequence. As \( k \) is uncountable, the intersection \( \bigcap_{n \geq r} \text{Im} \psi_{n,r} \) is non-empty by [1, Proposition 6.5]. Take a point \( p_r \) from this set. In \( Z(I_{r+1}) \),
is a non-increasing sequence of non-empty constructible sets. Therefore, we can take a point $p_r \in (\bigcap_{n>r+1} \text{Im} \psi_{r+1,r}(p_r) \cap \text{Im} \psi_{r+1,r}(p_r) \cap \ldots$ is a non-increasing sequence of non-empty constructible sets. Therefore, we can take a point $p_{r+1} \in \psi_{r+1,r}(p_r) \cap (\bigcap_{n>r+1} \text{Im} \psi_{n,r+1})$. In the same way, we have points $p_{r+2} \in Z(I_{r+2}), p_{r+3} \in Z(I_{r+3}), \ldots$ such that $\psi_{n+1,n}(p_{n+1}) = p_n \in Z(I_n)$ for $n \geq r$. Therefore, there is a sequence $a_1, a_2, \ldots, a_n, \ldots \in k$ such that $p_n = (a_1, a_2, \ldots, a_n)$. Hence, $I_n \subset (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ for every $n$. Then, it follows $I = \lim_{\to} I_n \subset (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$. □

In the proposition above, the condition on $k$ is essential. In fact, we obtain the following:

**Proposition 2.11** (Watanabe, Yoshida). Let $k$ be a countable field. Then there is a closed point which is not a $k$-valued point in Spec $k[x_1, x_2, \ldots, x_n, \ldots]$.

**Proof.** Let $y$ be a transcendental element over $k$. As $k$ is countable, the extension field $k(y)$ is a countably generated $k$-algebra. Therefore there exists a surjective homomorphism $k[x_1, x_2, \ldots, x_n, \ldots] \rightarrow k(y)$. The kernel of this homomorphism is a maximal ideal which does not give a $k$-valued point. □

As we assume that the base field is an arbitrary algebraically closed field, a closed point of an arc space is not necessarily a $k$-valued point. In spite of such a difficulty we can see the structure of the arc space for a toric variety.

3. Basic properties of the arc space of a toric variety

3.1. We use the notation and terminology of [7]. Let $M$ be the free abelian group $\mathbb{Z}^n$ ($n \geq 1$) and $N$ its dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Z})$. We denote $M \otimes \mathbb{Z} \mathbb{R}$ and $N \otimes \mathbb{Z} \mathbb{R}$ by $M_\mathbb{R}$ and $N_\mathbb{R}$, respectively. The canonical pairing $\langle \ , \rangle : N \times M \rightarrow \mathbb{Z}$ extends to $\langle \ , \rangle : N_\mathbb{R} \times M_\mathbb{R} \rightarrow \mathbb{R}$. For a linear subspace $W \subset N_\mathbb{R}$, the induced pairing $(N_\mathbb{R}/W) \times W^\perp \rightarrow \mathbb{R}$ is also denoted by $\langle \ , \rangle$. Here, for $v \in N_\mathbb{R}, u \in W^\perp$ we have that $\langle v, u \rangle = \langle \rho(v), u \rangle$, where $\rho : N_\mathbb{R} \rightarrow N_\mathbb{R}/W$ is the projection.

For a finite fan $\Delta$ in $N$, the corresponding toric variety is denoted by $T_N(\Delta)$. If $\Delta$ is the fan consisting of all faces of a cone $\sigma$, then $T_N(\Delta)$ is affine and sometimes denoted by $T_N(\sigma)$.

For a cone $\tau \in \Delta$ we denote by $U_\tau$ the invariant affine open subset which contains $\text{orb} \tau$ as the unique closed orbit. The open set $U_\tau$ is isomorphic to $T_N(\tau)$. 
We can write \( k[M] \) as \( k[x^n]_{u \in M} \), where we use the shorthand \( x^n = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \) for \( u = (u_1, \ldots, u_n) \in M \). The torus \( \text{Spec} k[M] \) is denoted by \( T \). We also write \( T \) for the open orbit of the toric variety.

**Proposition 3.2.** Let \( X \) be a toric variety over \( k \) and \( f : Y \rightarrow X \) an equivariant resolution of the singularities. Then, the induced morphism \( f_\infty : Y_\infty \rightarrow X_\infty \) is surjective in a strong sense; i.e. for every extension field \( K \supset k \) the corresponding morphism \( Y_\infty(K) \rightarrow X_\infty(K) \) is surjective.

**Proof.** Let \( \alpha : \text{Spec} K[[t]] \rightarrow X \) be an arc of \( X \), then the generic point \( \eta \in \text{Spec} K[[t]] \) is mapped to \( \text{orb}\tau \) for some cone \( \tau \) in the defining fan of \( X \). As \( f \) is equivariant, \( f^{-1}(\text{orb}\tau) \) contains a subscheme isomorphic to \( \text{orb}\tau \times T^s \), where \( T^s \) is the torus of dimension \( 0 \leq s < n \). Hence the restriction \( \text{Spec} K((t)) \rightarrow X \) of \( \alpha \) can be lifted to \( Y \). Therefore, by the properness of \( f \), \( \alpha \) can be lifted to \( Y \). \( \Box \)

The irreducibility of the arc space of a variety is known for a base field of characteristic zero ([9]). In the positive characteristic case, [8, Example 2.13] gives an example of non-irreducible arc space. But for a toric variety, the characteristic is not a problem.

**Corollary 3.3.** The arc space of a toric variety \( X \) is irreducible.

**Corollary 3.4.** Since the arc space \( X_\infty \) of a toric variety contains \( T_\infty \) as an open orbit, \( X_\infty \) is an almost homogeneous space by the action of \( T_\infty \).

**Proof.** This follows immediately from the irreducibility of \( Y_\infty \) and Proposition 3.2. \( \Box \)

### 3.5.
An irreducible component of the fiber \( \pi^{-1}(\text{Sing} X) \) of the singular locus \( \text{Sing} X \subset X \) is called a **good component** if it contains an arc \( \alpha \) such that \( \alpha(\eta) \) is in the non-singular locus ([8]). If the characteristic of the base field is zero, then every component of \( \pi^{-1}(\text{Sing} X) \) is a good component, while there is a non-good component for a positive characteristic case ([8, Example 2.13]). The following shows that the characteristic does not affect on this problem for a toric variety.

**Proposition 3.6.** For a toric variety \( X \), every component of \( \pi^{-1}(\text{Sing} X) \) is a good component.

**Proof.** Let \( C \) be a non-good component of \( \pi^{-1}(\text{Sing} X) \). Let \( f : Y \rightarrow X \) be an equivariant resolution of the singularities and \( E_i \) (\( i = 1, 2, \ldots, r \)) be the irreducible components of \( f^{-1}(\text{Sing} X) \). Then, \( \pi_Y^{-1}(E_i) \)'s are the irreducible components of \( f_\infty^{-1}(\pi^{-1}(\text{Sing} X)) \), where \( \pi_Y : Y_\infty \rightarrow Y \) is
the canonical projection. By the surjectivity of $f_\infty$ proved in Proposition 3.2, there is a component $\pi_\infty^{-1}(E_i)$ mapped to $C$. However, $\pi_\infty^{-1}(E_i)$ contains an arc whose image of the generic point corresponds to a point in the non-singular locus on $X$, which is a contradiction. □

Now we are going to make a stratification of the arc space of a toric variety according to the fan. From now on we assume that a toric variety $X$ is defined by a fan $\Delta$. Let $X(\tau) \subset X$ be the closure $\overline{\text{orb}\tau}$ for the cone $\tau \in \Delta$. Then $X(\tau)$ is again a toric variety.

**Definition 3.7.** Let $X$ be a toric variety corresponding to a fan $\Delta$. We define $X(\tau)_\infty$ as follows:

$X(\tau)_\infty = \{ \alpha \in X_\infty | \alpha : \text{Spec } K[[t]] \rightarrow X \text{ factors through } X(\tau) \}$

but does not factor through $X(\gamma)$ for $\gamma \not< \tau$.

**Remark 3.8.**

(i) By definition, we have:

$X(\tau)_\infty = \{ \alpha \in X_\infty | \alpha(\eta) \in \text{orb}\tau \}$.

In particular,

$X(0)_\infty = \{ \alpha \in X_\infty | \alpha(\eta) \in T \}$.

(ii) $X(\tau)_\infty = X(\tau)_\infty(0)$, where $0$ is the cone consisting of the origin.

(iii) $X_\infty$ is the disjoint union:

$X_\infty = \bigsqcup_{\tau \in \Delta} X(\tau)_\infty$.

**Proposition 3.9.** Let $X$ be a toric variety defined by a fan $\Delta$, $T$ the torus acting on $X$ and $\tau$ a cone in $\Delta$. Then, the subset $X(\tau)_\infty$ is a locally closed subset which is invariant under the action of $T_\infty$.

**Proof.** As $X(\gamma)$ is closed in $X$ for every cone $\gamma \in \Delta$, $X(\gamma)_\infty$ is considered as a closed subscheme of $X_\infty$. By definition

(3.9.1) $X(\tau)_\infty = X(\tau)_\infty \setminus \left( \bigcup_{\gamma \not< \tau} X(\gamma)_\infty \right)$

as subsets in $X_\infty$, which shows that $X(\tau)_\infty$ is locally closed.

As $X(\gamma)$ is invariant under the action of $T$ for every $\gamma \in \Delta$, $X(\gamma)_\infty$ is invariant under the action of $T_\infty$. The description of $X(\tau)_\infty$ as above gives the assertion of the invariance. □

**Proposition 3.10.** Let $X$ be a toric variety defined by a fan $\Delta$ and $\tau$, $\gamma$ be cones in $\Delta$. Then, $\gamma < \tau$ if and only if $X(\gamma)_\infty \supset X(\tau)_\infty$. 

Proof. First note that $X(\gamma)_\infty$ and $X(\tau)_\infty$ are irreducible (Corollary 3.3) and closed in $X_\infty$. Then, the description (3.9.1) gives that $X_\infty(\gamma) = X(\gamma)_\infty$ and $X_\infty(\tau) = X(\tau)_\infty$. Therefore, the relation $X_\infty(\gamma) \supset X_\infty(\tau)$ holds if and only if $X(\gamma)_\infty \supset X(\tau)_\infty$ holds, which is equivalent to $X(\gamma) \supset X(\tau)$. It is well known that the last relation is equivalent to $\gamma < \tau$. □

4. ORBITS ON THE ARC SPACE OF A TORIC VARIETY

In this section we associate each $T_\infty$-orbit on $X_\infty$ to a lattice point, and describe the dominant relation of two orbits in terms of the corresponding lattice points.

Theorem 4.1. Let $X$ be a toric variety defined by a fan $\Delta$. Then,

(i) there is a surjective canonical map

$$
\Psi : X_\infty(0) \longrightarrow |\Delta| \cap N, \quad \alpha \mapsto v_\alpha,
$$

(ii) for every $v \in |\Delta| \cap N$ there exists a $k$-valued point $\alpha \in X_\infty(0)$ such that

$$
\Psi^{-1}(v) = T_\infty \cdot \alpha,
$$

where $T_\infty \cdot \alpha$ is the orbit of $\alpha$ by the action of $T_\infty$, and

(iii) for $v \in |\Delta| \cap N$, $\Psi^{-1}(v)$ is a locally closed subset of $X_\infty$.

Proof. For a $K$-valued point $\alpha \in X_\infty(0)$, take a cone $\sigma \in \Delta$ such that $\alpha(0) \in U_\sigma$. Then $\alpha$ is an arc of $U_\sigma$ with $\alpha(\eta) \in T$, therefore we have a commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\alpha} & K[[t]] \\
\cap & \cap & \\
k[M] & \xrightarrow{\alpha^*} & K((t))
\end{array}
$$

Let $v_\alpha : M \longrightarrow \mathbb{Z}$ be a map defined by $u \mapsto \text{ord} \alpha^*(x^u)$. Then $v_\alpha$ is a group homomorphism, therefore $v_\alpha \in N$ with the pairing $\langle v_\alpha, u \rangle = \text{ord} \alpha^*(x^u)$. For $u \in \sigma^\vee \cap M$, it follows $\langle v_\alpha, u \rangle = \text{ord} \alpha^*(x^u) \geq 0$, which implies that $v_\alpha \in \sigma$. Now we obtain a map $\Psi : X_\infty(0) \longrightarrow |\Delta| \cap N, \alpha \mapsto v_\alpha$. To show the surjectivity, take a point $v \in |\Delta| \cap N$. Let $\sigma$ be a cone containing $v$. Let $\alpha^* : k[M] \longrightarrow k((t))$ be a $k$-algebra homomorphism defined by $\alpha^*(x^u) = t^{\langle v, u \rangle}$ for $u \in M$. Then, $\alpha^*(k[\sigma^\vee \cap M]) \subset k[[t]]$, since $\langle v, u \rangle \geq 0$ for $u \in \sigma^\vee$. Hence, $\alpha^*$ gives a $k$-valued point $\alpha$ in $X_\infty(0)$.

For (ii), we prove the equality $\Psi^{-1}(v) = T_\infty \cdot \alpha$ for a $k$-valued point $\alpha \in \Psi^{-1}(v)$. For a $k$-valued point $\alpha \in X_\infty(0)$, take a cone $\sigma$ such that $\alpha \in (U_\sigma)_\infty$. Then $\alpha$ corresponds to a ring homomorphism $\alpha^* : k[\sigma^\vee \cap M] \longrightarrow k[[t]]$. On the other hand, a $K$-valued point $\gamma \in T_\infty$
corresponds to a ring homomorphism $\gamma^* : k[M] \to K[[t]]$. This homomorphism is equivalent to a ring homomorphism $\gamma^* : k[\sigma^\vee \cap M] \to K[[t]]$ such that the order of $\gamma^*(x^u)$ is zero for every $u \in \sigma^\vee \cap M$, because $\sigma^\vee \cap M$ generates $M$. Then, $\gamma \cdot \alpha$ corresponds to the homomorphism $k[\sigma^\vee \cap M] \to K[[t]]$ which maps $x^u$ to $\gamma^*(x^u)\alpha^*(x^u)$.

Now let $\alpha \in (U_\sigma)_\infty$ be the arc corresponding to $v$ which was constructed in (i). If $\beta \in T_\infty \cdot \alpha$, then there exists a $K$-valued point $\gamma \in T_\infty$ such that $\beta = \gamma \cdot \alpha$. Then, by the above remark, it follows that $\beta \in (U_\sigma)_\infty$ and $\beta$ corresponds to $\beta^* : k[\sigma^\vee \cap M] \to K[[t]]$ which maps $x^u$ to $\gamma^*(x^u)t^{(v,u)}$ whose order is $\langle v, u \rangle$. Therefore $\beta \in \Psi^{-1}(v)$. Conversely, suppose that $\beta \in \Psi^{-1}(v)$ and let $\sigma$ be a cone such that $\beta \in (U_\sigma)_\infty$. Then we can define $\gamma \in T_\infty$ by $\gamma^* : k[\sigma^\vee \cap M] \to K[[t]]$, $\gamma^*(x^u) = t^{-(v,u)}\beta^*(x^u)$. For this $\gamma$ we have that $\gamma \cdot \alpha = \beta$.

For the assertion (iii), take a cone $\sigma \in \Delta$ such that $T_\infty \cdot \alpha \subset (U_\sigma)_\infty$. It is sufficient to prove that $T_\infty \cdot \alpha$ is locally closed in $(U_\sigma)_\infty \cap X_\infty(0)$. Denote $(U_\sigma)_\infty$ by $Spec A$. Let $\Lambda : k[\sigma^\vee \cap M] \to A[[t]]$ be the ring homomorphism induced from the universal family of arcs on $(U_\sigma)_\infty$ (see Proposition 2.3). Let $\Lambda(x^u_j) = \sum_{i \geq 0} a_{j,i}t^i$ for generators $u_j$ ($j = 1, \ldots, r$) of the semigroup $\sigma^\vee \cap M$. Then
\[
T_\infty \cdot \alpha = \Psi^{-1}(v) = \{ \beta \in (U_\sigma)_\infty \cap X_\infty(0) \mid a_{j,i}(\beta) = 0 \text{ for } i < \langle v, u_j \rangle, \quad a_{j,i}(\beta) \neq 0 \text{ for } i = \langle v, u_j \rangle, \quad j = 1, \ldots, r \}.
\]
Hence, $T_\infty \cdot \alpha$ is locally closed in $(U_\sigma)_\infty \cap X_\infty(0)$. \hfill \square

4.2. For $\tau \in \Delta$, $X(\tau)$ is a toric variety $T_{N_\tau}(\Delta_\tau)$, where $\Delta_\tau$ consists of the cones $\sigma \subset N_\mathbb{R}/\tau \mathbb{R}$ which are the images of the cones $\sigma \in \Delta$ such that $\tau < \sigma$ and $N_\tau$ is the image of $N$ in $N_\mathbb{R}/\tau \mathbb{R}$. The affine open subset $U_\sigma \subset X(\tau)$ is $Spec k[\tau^\perp \cap \sigma^\vee \cap M]$.

Since $X_\infty(\tau) = X(\tau)_\infty(0)$ as is seen in Remark 3.8, we obtain the following from Theorem 4.1:

Corollary 4.3. Let $X$ be a toric variety defined by a fan $\Delta$ and $\tau \in \Delta$. Then,

(i) there is a surjective canonical map
\[
\Psi : X_\infty(\tau) \to |\Delta_\tau| \cap N_\tau, \quad \alpha \mapsto v_\alpha,
\]
(ii) for every $v \in |\Delta_\tau| \cap N_\tau$ there exists a $k$-valued point $\alpha \in X_\infty(\tau)$ such that
\[
\Psi^{-1}(v) = T_\infty \cdot \alpha,
\]
where $T_\infty \cdot \alpha$ is the orbit of $\alpha$ by the action of $T_\infty$, and
(iii) for $v \in |\Delta_\tau| \cap N_\tau$, $\Psi^{-1}(v)$ is a locally closed subset of $X_\infty$. 

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Corollary 4.4. \( \text{ (i) } \)
\[
X_{\infty} = \bigcup_{\alpha: \text{k-valued point of } X_{\infty}} T_{\infty} \cdot \alpha.
\]

\( \text{ (ii) For every cone } \tau, \text{ there is a bijection: } \)
\[
\{ T_{\infty} \cdot \alpha \mid \alpha \text{ is a k-valued point } \in X_{\infty}(\tau) \} \simeq |\Delta_{\tau}| \cap N_{\tau}.
\]

Definition 4.5. As an orbit of a k-valued point \( \alpha \) in \( X_{\infty}(\tau) \) is determined by the lattice point \( v = v_{\alpha} \in |\Delta_{\tau}| \), we sometimes denote the orbit \( T_{\infty} \cdot \alpha \) by \( T_{\infty}(v) \).

Definition 4.6. Let \( \sigma \) be a cone in \( N \) and \( v, v' \) two points in \( \sigma \). We denote \( v \leq_{\sigma} v' \) if \( v' \in v + \sigma \). It is clear that \( \leq_{\sigma} \) is an order in \( \sigma \).

Now we are going to study the dominant relation between orbits.

Proposition 4.7. Let \( X \) be a toric variety defined by a fan \( \Delta \). Let \( \alpha \in X_{\infty}(\tau) \) and \( \beta \in X_{\infty}(\gamma) \) be k-valued points for \( \tau, \gamma \in \Delta \). If \( T_{\infty} \cdot \alpha \supset T_{\infty} \cdot \beta \), then \( \tau < \gamma \) and there exists a cone \( \sigma \in \Delta \) containing \( \tau \) and \( \gamma \) such that \( \alpha, \beta \in (U_{\sigma})_{\infty} \).

Proof. By the condition of the proposition, it follows that \( \beta \in \overline{X_{\infty}(\tau)} = X(\tau)_{\infty} \). As \( \beta(\eta) \in \text{orb}_\gamma \), we have \( \text{orb}_\gamma \subset X(\tau) \), which implies \( \tau < \gamma \). To see the second assertion, take a cone \( \sigma \in \Delta \) such that \( \beta \in (U_{\sigma})_{\infty} \). Then \( \beta(\eta) \in \text{orb}(\gamma) \) implies \( \gamma < \sigma \). Since \( (U_{\sigma})_{\infty} \) is an open subset of \( X_{\infty} \) containing \( \beta \), there is an arc \( \alpha' \in T_{\infty} \cdot \alpha \cap (U_{\sigma})_{\infty} \). As \( (U_{\sigma})_{\infty} \) is \( T_{\infty} \)-invariant, it contains both \( T_{\infty} \cdot \alpha \) and \( T_{\infty} \cdot \beta \). \( \square \)

Hence, in order to interpret the condition of the domination \( \overline{T_{\infty} \cdot \alpha} \supset T_{\infty} \cdot \beta \) in terms of the corresponding lattice points, we may assume that \( X \) is an affine toric variety. If \( X \) is an affine toric variety defined by a cone \( \sigma \) and \( T_{\infty}(v) \subset X_{\infty}(\tau) \) for a face \( \tau < \sigma \), then \( v \in \sigma \cap N_{\tau} \) by Corollary 4.3, where \( \sigma \) is the image of \( \sigma \subset N_{\mathbb{R}} \) by the projection \( N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/\tau \mathbb{R} \).

Proposition 4.8. Let \( X \) be an affine toric variety defined by a cone \( \sigma \) in \( N \). Then, two orbits \( T_{\infty}(v) \) and \( T_{\infty}(v') \) in \( X_{\infty}(0) \) satisfy \( \overline{T_{\infty}(v)} \supset T_{\infty}(v') \) if and only if \( v \leq_{\sigma} v' \).

Proof. Assume \( \overline{T_{\infty}(v)} \supset T_{\infty}(v') \). If \( \langle v, u \rangle > \langle v', u \rangle \) for some \( u \in \sigma' \cap M \), then
\[
T_{\infty}(v) \subset \overline{T_{\infty}(v)} \cap \{ \alpha \in X_{\infty}(0) \mid \text{ord } \alpha^*(x^u) \geq \langle v', u \rangle + 1 \},
\]
where the right hand side is a proper closed subset of \( \overline{T_{\infty}(v)} \). This is a contradiction. Hence, \( v \leq_{\sigma} v' \).
Next, assume that \( v \leq_{\sigma} v' \) for \( v, v' \in \sigma \cap N \). To prove the converse, we divide the proof into two steps.

**Step 1.** The case \( X \) is non-singular.

Let \( e_1, e_2, \ldots, e_n \) be the basis of \( M \) such that \( e_1, \ldots, e_r, e_{r+1}^{\pm 1}, \ldots, e_n^{\pm 1} \) generate \( \sigma' \). Define a \( k \)-algebra homomorphism \( \Phi^* : k[\sigma'/M] \to k[[\lambda, t]] \) by

\[
\Phi^*(x^{e_i}) = t^{\langle v', e_i \rangle} + \lambda t^{\langle v, e_i \rangle}.
\]

Here, note that \( \Phi^*(x^{e_i}) = 1 + \lambda \) for \( i \geq r + 1 \), since \( \langle v, e_i \rangle = \langle v', e_i \rangle = 0 \) for these \( i \)'s. Then, we obtain a morphism \( \Phi : \text{Spec} \ k[[\lambda]] \to X_\infty(0) \) such that \( \Phi(0') \subset T_\infty(v') \) and \( \Phi(\eta') \subset T_\infty(v) \), where \( 0' \) is the closed point and \( \eta' \) is the generic point of \( \text{Spec} \ k[[\lambda]] \). This implies that \( T_\infty(v) \) contains a point of \( T_\infty(v') \). As \( T_\infty(v) \) is \( T_\infty \)-invariant, it follows that \( T_\infty(v) \supset T_\infty(v') \).

**Step 2.** The general case.

Define \( \sigma' \) as the cone generated by \( v \) and \( v' - v \). Then, note that \( \sigma' \subset \sigma \) and

\[
v \leq_{\sigma'} v'.
\]

Let \( N' \) be the subgroup of \( N \) generated by \( v, v' - v \) and \( v_1, v_2, \ldots, v_s \in N \), where their images \( \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s \in N/N \cap \sigma' \mathbb{R} \) are a basis of \( N/N \cap \sigma' \mathbb{R} \). Then, the toric variety \( Z = T_{N'}(\sigma') \) is non-singular and there is a canonical equivariant morphism

\[
\varphi : Z \to X
\]

with the surjective morphism \( T' \to T \) of the tori. By Step 1, \( \overline{T_\infty(v')} \supset T_\infty(v) \) follows from \( v \leq_{\sigma'} v' \). Take \( k \)-valued points \( \alpha, \beta \in Z_\infty(0) \) such that \( v_\alpha = v, v_\beta = v' \), then \( T_\infty \cdot \varphi_\infty(\alpha) = \varphi_\infty(T_\infty \cdot \alpha) \) and \( T_\infty \cdot \varphi_\infty(\beta) = \varphi_\infty(T_\infty \cdot \beta) \). Therefore \( T_\infty(v) \supset T_\infty(v') \) follows from \( v_\varphi_\infty(\alpha) = v, v_\varphi_\infty(\beta) = v' \).

As \( X_\infty(\tau) = X(\tau)_\infty(0) \) we obtain the following as a corollary of Proposition 4.8.

**Corollary 4.9.** Let \( X \) be an affine toric variety defined by a cone \( \sigma \) in \( N \). Then, for a face \( \tau < \sigma \), two orbits \( T_\infty(v) \) and \( T_\infty(v') \) in \( X_\infty(\tau) \) satisfy \( \overline{T_\infty(v)} \supset T_\infty(v') \) if and only if \( v \leq_{\sigma} v' \), where \( \overline{\sigma} \subset N_\mathbb{R}/\tau \mathbb{R} \) is the image of \( \sigma \).

Next we will see the relation of the orbits in mutually different strata.

To see this we need the following combinatorial lemma:

**Lemma 4.10.** Let \( \sigma \) be an \( n \)-dimensional cone in \( N \), where \( n = \dim N_\mathbb{R} \), and \( \tau \) an \( r \)-dimensional face of \( \sigma \). Then, there exist a non-singular \( n \)-dimensional cone \( \sigma_0 \) in \( N \) and its \( r \)-dimensional face \( \tau_0 \) such that \( \sigma_0 \subset \sigma \) and \( \tau_0 \subset \tau \).
Proposition 4.11. Let \( X \) be an affine toric variety defined by a cone \( \sigma \) in \( N \). Then, two orbits \( T_\infty(v) \subset X_\infty(\tau) \), \( T_\infty(v') \subset X_\infty(\gamma) \) satisfy the relation \( \overline{T_\infty(v)} \supset T_\infty(v') \) if and only if \( \tau < \gamma \) and \( \rho(v) \leq \rho(v') \), where \( \rho : N_\mathbb{R}/\tau \mathbb{R} \to N_\mathbb{R}/\gamma \mathbb{R} \) is the canonical projection and \( \rho \) is the image of \( \sigma \) in \( N_\mathbb{R}/\gamma \mathbb{R} \).

Proof. First assume that \( \overline{\rho(v)} > T_\infty(v') \). Then, we have \( \tau < \gamma \) by Proposition 4.7. By the assumption, there is a morphism \( \Phi : \text{Spec} k[[\lambda]] \to X_\infty(0) \) such that \( \beta := \Phi(0) \in T_\infty(v') \) and \( \alpha := \Phi(\eta) \in T_\infty(v) \), where \( 0 \) is the closed point and \( \eta \) is the generic point of \( \text{Spec} k[[\lambda]] \). As \( \alpha \in X_\infty(\tau) \), \( \Phi \) factors through \( X(\tau)_\infty \). This gives the \( k \)-algebra homomorphism:

\[
\Phi^* : k[\tau^\perp \cap \sigma^\vee \cap M] \to k[[\lambda, t]].
\]

By using \( \Phi^* \), we obtain \( \text{ord} \alpha^*(x^u) \leq \text{ord} \beta^*(x^u) \) for \( u \in \tau^\perp \cap \alpha^\vee \cap M \) in the same way as in the proof of Proposition 4.8. Therefore, for \( u \in \gamma^\perp \cap \sigma^\vee \cap M \subset \tau^\perp \cap \sigma^\vee \cap M \) the inequality \( \langle v, u \rangle = \langle \rho(v), u \rangle \leq \langle v', u \rangle \) holds. Hence, \( \rho(v) \leq \rho(v') \).

To prove the converse, assume \( \rho(v) \leq \rho(v') \). Then, it is sufficient to prove that \( \overline{T_\infty(v)} \supset T_\infty(\rho(v)) \), because \( T_\infty(\rho(v)) \supset T_\infty(v') \) follows from Corollary 4.9. To prove \( \overline{T_\infty(v)} \supset T_\infty(\rho(v)) \), we may assume that \( \gamma = \sigma \), since \( X_\infty(\gamma) = X(\gamma)_\infty(0) \). We also can assume that \( \dim \sigma = \dim N_\mathbb{R} \), because if \( \dim \sigma = s < n \), then \( T_\infty(v) = T_\infty^{n-s} \times T_\infty^s(\rho(v)) \), \( T_\infty(\rho(v)) = T_\infty^{n-s} \times T_\infty^s(\rho'(v)) \), where \( \rho' : \sigma \mathbb{R} \to \sigma \mathbb{R}/\tau \mathbb{R} \) is the projection and \( T^s \), \( T^{n-s} \) are \( s \) and \( (n-s) \)-dimensional tori, respectively. So the problem is reduced to proving that \( \overline{T_\infty^s(v)} \supset T_\infty^s(\rho'(v)) \).

Now, for \( \sigma \) and \( \tau \), let \( \sigma_0 \) and \( \tau_0 \) be as in Lemma 4.10. Let \( e_1, e_2, \ldots, e_n \) be a basis of \( M \) which generate \( \sigma_0^\vee \) and \( e_1, e_2, \ldots, e_r \) (\( r < n \)) generate \( \tau_0^\perp \cap \sigma_0^\vee \). Let

\[
\Lambda^* : k[\sigma_0^\vee \cap M] \to k[[\lambda]]((t))
\]

be a \( k \)-algebra homomorphism defined by

\[
\Lambda^*(x^{e_i}) = (\lambda + 1)^{(v, e_i)} \text{ for } i = 1, \ldots, r,
\]
\[
\Lambda^*(x^{e_i}) = \lambda t^{(v, e_i)} \text{ for } i = r + 1, \ldots, n.
\]

It is easy to check that \( \Lambda^*(x^u) \in k[[\lambda, t]] \) for every \( u \in \sigma^\vee \cap M \), since \( v \in \sigma \). Then, we obtain a morphism \( \Lambda : \text{Spec}[[\lambda]] \to X_\infty \). For every
Let Lemma 4.14. \( v : \sigma^\vee \cap M \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) be a homomorphism of semigroups. Then, there exists a face \( \tau < \sigma \) such that \( v^{-1}(\mathbb{Z}_{\geq 0}) = \tau^\perp \cap \sigma^\vee \cap M \).

**Proof.** Take the minimal face \( \gamma \) of \( \sigma \) containing \( C = v^{-1}(\mathbb{Z}_{\geq 0}) \). Then, \( C \) contains a relative interior point \( u \) of \( \gamma \). We will show that \( C = \gamma \cap M \). Assume that there exists a point \( u_0 \in \gamma \cap M \) such that \( v(u_0) = \infty \). Then, note that \( u_0 + \sigma^\vee \subset v^{-1}(\infty) \). Let \( \sigma^\vee \) be generated by
Then, there is a representation \( u = \sum_{i=1}^{r} a_i u_i \) with \( a_i > 0 \) for every \( i \) and \( u_0 = \sum_{i=1}^{r} b_i u_i \) with \( b_i \geq 0 \) for every \( i \). Then, in the equality:

\[
m u = \sum_{i} b_i u_i + \sum_{i} (ma_i - b_i) u_i,
\]

the second term of the right hand side is in \( \sigma' \) for \( m \gg 0 \). Hence, \( v(mu) = \infty \), but this contradicts to that \( v(mu) = mv(u) \in \mathbb{Z}_{\geq 0} \). Now, we obtain that \( C = \gamma \cap M \) and \( \gamma \) can be written as \( \tau^\perp \cap \sigma' \) for some \( \tau < \sigma \).

By Corollary 4.3 and Theorem 4.12, we obtain the following interpretation:

**Theorem 4.15.** Let \( X \) be a toric variety defined by a fan \( \Delta \), then we obtain the following:

(i) There is a bijective map:

\[
\{ T_\infty \cdot \alpha \mid \alpha : k\text{-valued point of } X \} \xrightarrow{\sim} \bigsqcup_{\sigma} \text{Hom}_{s.g.}(\sigma' \cap M, \mathbb{Z}_{\geq 0} \cup \{\infty\}),
\]

where \( \sigma \) varies the maximal cones in \( \Delta \). Via this map, each \( T_\infty \cdot \alpha \) can be written as \( T_\infty(v) \) for a suitable element \( v \) of the right hand side.

(ii) We have the relation \( T_\infty(v) \supset T_\infty(v') \) if and only if there is a maximal cone \( \sigma \) in \( \Delta \) such that \( v, v' \in \text{Hom}_{s.g.}(\sigma' \cap M, \mathbb{Z}_{\geq 0} \cup \{\infty\}) \) and \( v \leq v' \), where \( v \leq v' \) means that \( v(u) \leq v'(u) \) for every \( u \in \sigma' \cap M \).

5. **Contact loci of an invariant ideal**

In this section, we will give the answer to the embedded version of Nash problem for an invariant ideal of a toric variety.

**Definition 5.1.** Let \( X \) be a variety over an algebraically closed field \( k \) and \( k(X) \) the rational function field of \( X \). A divisorial valuation of \( k(X) \) is a positive integer times discrete valuation \( \text{val}_D \) associated to a prime divisor \( D \) on some normal variety \( X' \) which is birational to \( X \). Note that this definition is wider than the definition of “divisorial valuation” in [6].

**Definition 5.2.** Let \( X \) be an affine toric variety defined by a cone \( \sigma \) in \( N \). For every point \( v \in \sigma \cap N \) we can associate a valuation \( \text{val}_v \) on \( k(X) \) as follows:

Define \( \text{val}_v(f) := \min_{x \in f} \langle v, u \rangle \), for \( f \in k[\sigma' \cap M] \).
and extend it on \( k(X) \), the quotient field of \( k[\sigma^\vee \cap M] \). This valuation is called a toric valuation. Here \( x^u \in f \) means that the coefficient of the monomial \( x^u \) in \( f \) is not zero. Note that the toric valuation defined by a primitive element \( v \) is \( \text{val}_{D_\sigma} \), where \( D_\sigma \) is the irreducible invariant divisor \( \text{ord}(b(R_{\geq 0}v)) \) on some toric variety \( X \) which is birational to \( X \). Since every toric valuation is a positive integer times such a valuation, every toric valuation is a divisorial valuation.

5.3. For a variety \( X \) over an algebraically closed field \( k \), let \( \psi_m : X_\infty \rightarrow X_m \) (\( m \in \mathbb{Z}_{\geq 0} \)) be the truncation morphism. Note that \( \psi_0 = \pi \). Recall that a cylinder in \( X_\infty \) is a subset of the form \( \psi_m^{-1}(S) \), for some \( m \) and some constructible subset \( S \subset X_m \).

Example 5.4. Let \( X \) be a toric variety. Then an orbit \( T_\infty(v) \) of a \( k \)-valued point in \( X_\infty(0) \) is a cylinder. Indeed, we may assume that \( X \) is the affine toric variety defined by a cone \( \sigma \). The orbit is the subset of \( X_\infty \) consisting of arcs \( \alpha \) whose corresponding homomorphisms \( \alpha^* : k[\sigma^\vee \cap M] \rightarrow K[[t]] \) satisfy \( \text{ord} \alpha^*(x^u) = \langle v, u_i \rangle \) for generators \( u_1, \ldots, u_n \) of \( \sigma^\vee \cap M \). Let \( m \geq \max_{i=1,\ldots,n}(v, u_i) \) and \( S_m \subset X_m \) the subset consisting of \( m \)-jets \( \gamma \) whose corresponding homomorphisms \( \gamma^* : k[\sigma^\vee \cap M] \rightarrow \text{Spec } K[[t]]/(t^{m+1}) \) satisfy \( \text{ord} \gamma^*(x^u) = \langle v, u_i \rangle \). Then, \( S_m \) is a locally closed subset of \( X_m \) and \( T_\infty(v) = \psi_m^{-1}(S_m) \).

5.5. Let \( X \) be a non-singular variety over \( \mathbb{C} \) and \( C \) an irreducible cylinder in \( X_\infty \). In [6] a valuation \( \text{val}_C \) corresponding to \( C \) is defined as follows: Note first that if \( \alpha \in X_\infty \) is a \( \mathbb{C} \)-valued point, and if \( f \) is a rational function on \( X \) defined in a neighborhood of \( \pi(\alpha) \), then \( \text{ord} \alpha^*(f) \) is well defined, where \( \alpha^* : \mathcal{O}_X \rightarrow \mathbb{C}[[t]] \) is the ring homomorphism corresponding to \( \alpha \). If the domain of \( f \) intersects \( \pi(C) \), then \( \text{val}_C(f) := \text{ord} \alpha^*(f) \), for general \( \alpha \in C \). Then \( \text{val}_C(f) \) is well defined and can be extended to a valuation of the function field of \( X \).

Proposition 5.6 ([6]). Let \( X \) be a non-singular variety over \( \mathbb{C} \) and \( C \) an irreducible cylinder in \( X_\infty \) which does not dominate \( X \). Then \( \text{val}_C \) is equal with a divisorial valuation.

In the proof of Proposition 5.6, the condition that \( X \) is non-singular is used. Therefore, this proposition does not imply that for a cylinder \( C = T_\infty(v) \subset X_\infty(0) \) on a singular toric variety \( X \), the corresponding valuation \( \text{val}_C \) is a divisorial valuation. However, the following proposition shows that \( \text{val}_C \) is a divisorial valuation for \( C = T_\infty(v) \).

Proposition 5.7. Let \( X \) be a toric variety over an algebraically closed field \( k \) and \( C = T_\infty(v) \subset X_\infty(0) \), then, \( \text{val}_C = \text{val}_v \). In particular \( \text{val}_C \) is a divisorial valuation.
Proof. We may assume that $X$ is an affine toric variety defined by a cone $\sigma$. It is sufficient to prove that $val_C(f) = val_v(f)$ for every element $f \in k[\sigma^\vee \cap M]$. Note that $val_C(f) = \text{ord}\, \alpha^*(f)$ for the generic point $\alpha \in C$. If $f$ is a monomial $x^u$ ($u \in \sigma^\vee \cap M$), then by the definition of $C = T_\infty(v)$ we have

$$val_C(x^u) = \text{ord}\, \alpha^*(x^u) = \langle v, u \rangle = val_v(x^u).$$

For general $f$, we have

$$val_C(f) \geq \min_{x^u \in f} val_C(x^u) = \min_{x^u \in f} \langle v, u \rangle = val_v(f).$$

On the other hand, let $R_v$ is the discrete valuation ring of the divisorial valuation $val_v$. Then there is an indeterminate $t$ such that the composite

$$\beta^* : k[\sigma^\vee \cap M] \hookrightarrow R_v \hookrightarrow \hat{R}_v \simeq K[[t^e]] \hookrightarrow K[[t]]$$

satisfies $\text{ord}\, \beta^*(f) = val_v(f)$ for $f \in k[\sigma^\vee \cap M]$. Here, $K$ is the residue field of $R_v$ by the maximal ideal and $e$ is the positive integer such that $v = ev_0$ for a primitive element $v_0$. As the arc $\beta : \text{Spec} K[[t]] \rightarrow X$ corresponding to $\beta^*$ is a $K$-valued point of $C$, we obtain the following inequality by the upper semicontinuity

$$val_C(f) = \text{ord}\, \alpha^*(f) \leq \text{ord}\, \beta^*(f) = val_v(f).$$

Therefore, we obtain $val_C(f) = val_v(f)$.

Now we recall the definition of the contact locus of an ideal of a variety $X$. Let $X$ be an affine variety over an algebraically closed field $k$ with the coordinate ring $A$ and $a$ an ideal of $A$. Then, we define the $p$-th contact locus of $a$ by

$$\text{Cont}^p(a) = \{ \alpha \in X_\infty \mid \min_{f \in a} \text{ord}\, \alpha^*(f) = p \}.$$

It is clear that this is a cylinder. If $X$ is non-singular then the irreducible components are also cylinders. Therefore each irreducible component of the contact locus corresponds to a divisorial valuation. Now, we can state the embedded version of Nash problem posed in [6].

Problem 5.8. Which valuations correspond to the irreducible components of $\text{Cont}^p(a)$?

We consider this problem for an invariant ideal $a$ on a toric variety $X$. We should note that for a singular variety $X$, an irreducible component of a cylinder is not a cylinder in general, therefore an irreducible component does not necessarily correspond to a divisorial valuation. But in our toric case, an irreducible component of the contact locus corresponds to a divisorial valuation.
Lemma 5.9. Let $X$ be an affine toric variety and $\mathfrak{a}$ an invariant ideal on $X$. Then, for every integer $p > 0$, an orbit $T_\infty(v)$ is either contained in $\text{Cont}^p(\mathfrak{a})$ or disjoint from $\text{Cont}^p(\mathfrak{a})$.

Proof. Take an arc $\alpha \in T_\infty(v)$. Then $\alpha$ belongs to $\text{Cont}^p(\mathfrak{a})$ if and only if

$$p = \min \text{ord} \alpha^*(x^u) = \min_{x^u \in \mathfrak{a}} \langle v, u \rangle,$$

where we define $\langle v, u \rangle = \infty$ if $v \in N_{\mathbb{R}}/\tau \mathbb{R}$ and $u \not\in \tau^\perp$ for a cone $\tau$. The assertion of the lemma follows immediately from this. \hfill \Box

By this lemma it follows that $\text{Cont}^p(\mathfrak{a})$ is a union of $T_\infty(v)$’s.

Lemma 5.10. Let $X$ be an affine toric variety defined by a cone $\sigma$ in $N$ and $\mathfrak{a}$ an invariant ideal on $X$. If an orbit $T_\infty(v) \subset \text{Cont}^p(\mathfrak{a})$ is in $X_\infty(\tau)$ for $\tau \neq 0$, then there is an orbit $T_\infty(\tilde{v}) \subset X_\infty(0)$ such that $T_\infty(\tilde{v}) \subset \text{Cont}^p(\mathfrak{a})$ and $\overline{T_\infty(\tilde{v})} \supset T_\infty(v)$.

Proof. Let $\rho : N_{\mathbb{R}} \to N_{\mathbb{R}}/\tau \mathbb{R}$ be the projection. As $v$ is in the image $\rho(\sigma \cap N)$, we can take a point $v_0 \in \sigma \cap N$ such that $\rho(v_0) = v$. Then $\langle v, u \rangle = \langle v_0, u \rangle$ for $u \in \sigma^\vee \cap \tau^\perp$. We can naturally define $\langle v, u \rangle = \infty$ for $u \in \sigma^\vee \setminus \tau^\perp$. Let $v_1 \in \tau \cap N$ be in the relative interior of $\tau$. Then $\langle mv_1, u \rangle > p$ for every $u \in (\sigma^\vee \setminus \tau^\perp) \cap N$ and an integer $m > p$. Let $\tilde{v} = v_0 + mv_1$ ($m > p$). Then, for every $u \in \tau^\perp \cap \sigma^\vee \cap M$ it follows that $\langle \tilde{v}, u \rangle = \langle v_0, u \rangle = \langle v, u \rangle$, while for every $u \in (\sigma^\vee \setminus \tau^\perp) \cap M$ it follows that $\langle \tilde{v}, u \rangle > p$. Therefore

$$\min_{x^u \in \mathfrak{a}} \langle \tilde{v}, u \rangle = \min_{x^u \in \mathfrak{a}} \langle v, u \rangle = p.$$

Hence $T_\infty(\tilde{v}) \subset \text{Cont}^p(\mathfrak{a})$.

On the other hand, $\rho(\tilde{v}) = v$ yields $\overline{T_\infty(\tilde{v})} \supset T_\infty(v)$, by Proposition 4.11. \hfill \Box

By these lemmas, we obtain that an irreducible component of $\text{Cont}^p(\mathfrak{a})$ is the closure of $T_\infty(v)$ for some $v \in \sigma \cap N$ such that $\min_{x^u \in \mathfrak{a}} \langle v, u \rangle = p$. Here, by Proposition 4.8 and Proposition 5.7, we obtain the answer to the embedded version of Nash problem.

Theorem 5.11. Let $\mathfrak{a}$ be an invariant ideal on an affine toric variety $X$ defined by a cone $\sigma$. Then, an irreducible component of $\text{Cont}^p(\mathfrak{a})$ is the closure of $T_\infty(v)$ for an element $v$ minimal in $V(\mathfrak{a}, p) = \{ v' \in \sigma \cap N \mid \min_{x^u \in \mathfrak{a}} \langle v', u \rangle = p \}$ with respect to the order $\leq_{\sigma}$. Therefore the valuations $\{ \text{val}_v \mid v \in \sigma \cap N \text{ minimal in } V(\mathfrak{a}, p) \}$ correspond bijectively to the irreducible components of $\text{Cont}^p(\mathfrak{a})$. 

Remark 5.12. Let $G(a) \subset M_\mathbb{R}$ be the Newton polytope of $a$ as in Figure 1 and $\Delta(a)$ the dual fan of $G(a)$. The dual fan is the subdivision of $\sigma$. Then, the function $g(v) := \min_{u \in G(a)} \langle v, u \rangle$ ($v \in \sigma$) is a strongly convex piecewise linear function with respect to the fan $\Delta(a)$. Therefore the subset $g^{-1}(p) = \{ v \in \sigma \mid g(v) = p \}$ is the boundary of some convex polytope as in the Figure 2. The minimal elements of $V(a,p)$ are on this boundary. It is clear that this convex polytope is $pG(a)^{\circ}$, where $G(a)^{\circ} is the polar polytope defined as $\{ v \in \sigma \mid g(v) \geq 1 \}$.

We can see that a lattice point of a compact face of $g^{-1}(p)$ is always a minimal element of $V(a,p)$, therefore it gives a valuation corresponding to an irreducible component of $\text{Cont}^p(a)$. If $p$ is divisible enough so that every vertex of $pG(a)^{\circ}$ is in $N$, then the minimal elements in $V(a,p)$ coincide with the lattice points on the compact faces of $g^{-1}(p)$.

Remark 5.13. The referee kindly informed the following to the author: For $u \in \sigma^\vee \cap M$, the log canonical threshold $\text{lc}(X,V(a),V(x^u))$ turns out to be the maximal value $\lambda$ such that $x^u \not\in I(X,a^{\lambda})$ by [2], where $I(X,a^{\lambda})$ is a multiplier ideal for $a$. Some multiple of the primitive vector $v \in \sigma \cap N$ corresponding to a divisor which computes $\text{lc}(X,V(a),V(x^v))$ lies on a compact face of $g^{-1}(p)$ for some $p$. Conversely, for some multiple of a primitive vector $v \in \sigma \cap N$ on a compact face of $g^{-1}(p)$, there exists $u \in \sigma^\vee \cap M$ such that the divisor corresponding to $v$ computes the log canonical threshold $\text{lc}(X,V(a),V(x^u))$.

Example 5.14. Let $X$ be an affine toric variety defined by a cone $\sigma$. Then the components in $\pi^{-1}(\text{Sing} X)$ are $T_\infty(v)$'s, where $v$'s are the minimal elements in $\bigcup_{\tau < \sigma, \text{regular}} \tau^o \cap N$ with respect to the order $\leq_{\sigma}$. Here, $\tau^o$ is the relative interior of $\tau$. This is proved as follows: Let $\mathfrak{a}$ be the ideal of $\text{Sing} X$, then it is an invariant ideal. As $\pi^{-1}(\text{Sing} X) = \bigcup_{p \geq 1} \text{Cont}^p(a)$, it follows that an irreducible component of $\pi^{-1}(\text{Sing} X)$ is $T_\infty(v)$, where $v$ is minimal among $v$'s such that $v' \in \sigma \cap N$ and $\min_{x \in a}(v',u) \geq 1$ by Theorem 5.11. Here, $\min_{x \in a}(v',u) \geq 1$ if and only if $\alpha(0) \in \text{Sing} X$ for $\alpha$ with $v_\alpha = v'$, which is equivalent to the fact that $v' \in \tau^o$ for a singular face $\tau < \sigma$ by [8, Proposition 3.9].

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