THE BLOCH-OKOUNKOV CORRELATION
FUNCTIONS, A CLASSICAL HALF-INTEGRAL CASE

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Abstract. Bloch and Okounkov’s correlation function on the infinite wedge space has connections to Gromov-Witten theory, Hilbert schemes, symmetric groups, and certain character functions of \( \hat{\mathfrak{gl}}_\infty \)-modules of level one. Recent works have calculated these character functions for higher levels for \( \mathfrak{gl}_\infty \) and its Lie subalgebras of classical type. Here we obtain these functions for the subalgebra of type \( D \) of half-integral levels and as a byproduct, obtain \( q \)-dimension formulas for integral modules of type \( D \) at half-integral level.

1. Introduction

Bloch and Okounkov [BO] introduced an \( n \)-point correlation function on the infinite wedge space and found an elegant closed formula in terms of theta functions. From a representation theoretic viewpoint, the Bloch-Okounkov \( n \)-point function can be also easily interpreted as correlation functions on integrable modules over Lie algebra \( \hat{\mathfrak{gl}}_\infty \) of level one (cf. [Ok, Mil, CW]). Along this line, Cheng and Wang [CW] formulated and calculated such \( n \)-point correlation functions on integrable \( \hat{\mathfrak{gl}}_\infty \)-modules of level \( l \) (\( l \in \mathbb{N} \)). These correlation functions proved to be very useful in many applications such as in Gromov-Witten theory, Hilbert schemes, and the study of the symmetric groups.

The author and Wang [TW] extended the formulation and computation of these correlation functions to the other classical subalgebras of \( \hat{\mathfrak{gl}}_\infty \); there we have calculated the \( n \)-point correlation functions for integrable modules of arbitrary positive level for the subalgebras classically identified as \( b_\infty \), \( c_\infty \), and \( d_\infty \). The author, along with Cheng and Wang [CTW], later further extended results to modules of negative level for \( \hat{\mathfrak{gl}}_\infty \) and its same subalgebras. For more history of this problem, we refer the reader to the introduction of [TW].

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\]

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It has been known since Hermann Weyl that representations of the orthogonal Lie algebras have certain annoying features due to two connected components of the orthogonal group; in particular, Weyl’s character formula gives a better result for the character of certain pairs of representations rather than the individual components. In [TW], we are forced to consider the direct sum of two irreducible $d_\infty$-modules for this reason. In this paper, we aim to examine the case for $d_\infty$-modules of positive half-integral level. Our main strategy, as in [TW], is to use a free-field realization [DJKM] and a Howe duality due to Wang [W1] between $d_\infty$ and the Lie group $O(2l+1)$. We develop an operator in $d_\infty$ that is able to distinguish between the two components of this direct sum and use this operator to help compute a formula for the $n$-point correlation functions on the irreducible $d_\infty$-modules.

The paper is organized as follows. In section 2 we review some of the preliminaries. First we review the definitions and notations we will use regarding $\hat{\mathfrak{gl}}_\infty$ and $d_\infty$. Then we give a brief review of the Lie group $O(2l+1)$ and conclude with a quick review of the problem in the $\hat{\mathfrak{gl}}_\infty$ case. This section also introduces some of the Fock space definitions as well as the original Bloch-Okounkov function which will appear in several of our formulas. Finally, in section 3 we present our main theorems with proof.

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2. The preliminaries

2.1. Classical Lie algebras of infinite dimension. In this subsection we review Lie algebras $\hat{\mathfrak{gl}} = \hat{\mathfrak{gl}}_\infty$ and Lie subalgebras of type $D$ (cf. [DJKM, Kac]).

2.1.1. Lie algebra $\hat{\mathfrak{gl}}$. Denote by $\mathfrak{gl}$ the Lie algebra of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ satisfying $a_{ij} = 0$ for $|i-j|$ sufficiently large. Denote by $E_{ij}$ the infinite matrix with 1 at $(i, j)$ place and 0 elsewhere and let the weight of $E_{ij}$ be $j - i$. This defines a $\mathbb{Z}$–principal gradation $\mathfrak{gl} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{gl}_j$. Denote by $\hat{\mathfrak{gl}} \equiv \hat{\mathfrak{gl}}_\infty = \mathfrak{gl} \oplus \mathbb{C}C$ the central extension given by the following 2–cocycle with values in $\mathbb{C}$:

$$C(A, B) = \text{tr} \left( [J, A]B \right)$$

(1)
where $J = \sum_{j \leq 0} E_{ii}$. The $\mathbb{Z}$-gradation of Lie algebra $\mathfrak{gl}$ extends to $\widehat{\mathfrak{gl}}$ by letting the weight of $C$ to be 0. This leads to a triangular decomposition

$$\widehat{\mathfrak{gl}} = \widehat{\mathfrak{gl}}_+ \oplus \widehat{\mathfrak{gl}}_0 \oplus \widehat{\mathfrak{gl}}_-$$

where $\widehat{\mathfrak{gl}}_\pm = \oplus_{j \in \mathbb{N}} \widehat{\mathfrak{gl}}_{\pm,j}$, $\widehat{\mathfrak{gl}}_0 = \mathfrak{gl}_0 \oplus \mathbb{C} C$. Let $H^a_i = E_{ii} - E_{i+1,i+1} + \delta_{i,0} C$ ($i \in \mathbb{Z}$).

Denote by $L(\widehat{\mathfrak{gl}}, \Lambda)$ the highest weight $\widehat{\mathfrak{gl}}$-module with highest weight $\Lambda \in \widehat{\mathfrak{gl}}^*$, where $C$ acts as a scalar which is called the level. Let $\Lambda^a_j \in \widehat{\mathfrak{gl}}_0^*$ be the fundamental weights, i.e. $\Lambda^a_j(H^a_i) = \delta_{ij}$.

2.1.2. Lie algebra $d_{\infty}$. Let

$$d_{\infty} = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -a_{1-j,1-i}\}$$

be a Lie subalgebra of $\mathfrak{gl}$ of type $D$. Denote by $d_{\infty} = d_{\infty}^\oplus \mathbb{C} C$ the central extension given by the 2-cocycle $[\Pi]$. Then $d_{\infty}$ has a natural triangular decomposition induced from $\widehat{\mathfrak{gl}}$ with Cartan subalgebra $d_{\infty}^\oplus = \mathfrak{gl}_0 \cap d_{\infty}$. Given $\Lambda \in d_{\infty}^\oplus$, we let

$$H^d_i = E_{ii} + E_{-i,-i} - E_{i+1,i+1} - E_{-i+1,-i+1} \quad (i \in \mathbb{N}),$$

$$H^d_0 = E_{0,0} + E_{-1,-1} - E_{2,2} - E_{1,1} + 2C.$$

Denote by $\Lambda^d_i$ the $i$-th fundamental weight of $d_{\infty}$, i.e. $\Lambda^d_i(H^d_j) = \delta_{ij}$.

2.2. Classical Lie group $O(2l+1)$.

2.2.1. $O(2l+1)$. Let $O(2l+1) = \{g \in GL(2l+1) \mid \, ^t gJg = J\}$, where

$$J = \begin{bmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Lie algebra $\mathfrak{so}(2l+1)$ is the Lie subalgebra of $\mathfrak{gl}(2l+1)$ consisting of $(2l+1) \times (2l+1)$ matrices of the form

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \gamma & -t\alpha & h \\ -t\delta & h & 0 \end{bmatrix}$$

(2)

where $\alpha, \beta, \gamma$ are $l \times l$ matrices and $\beta, \gamma$ skew-symmetric. The Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l+1))$ consists of matrices (2) by putting $\gamma, h, \delta$ to be 0 and $\alpha$ to be upper triangular. The Cartan subalgebra $\mathfrak{h}(\mathfrak{so}(2l+1))$ consists of diagonal matrices of the form $\text{diag}(t_1, \ldots, t_l; -t_1 \ldots -t_l; 0)$, $t_i \in \mathbb{C}$. An irreducible module of $SO(2l+1)$ is parameterized by its highest weight $(m_1, \ldots, m_l) \in \mathcal{P}^l$, where $\mathcal{P}^l$ denotes the set of partitions with at most $l$ non-zero parts.
It is well known that $O(2l+1)$ is isomorphic to the direct product $SO(2l+1) \times \mathbb{Z}_2$ by sending the minus identity matrix to $-1 \in \mathbb{Z}_2 = \{ \pm 1 \}$. Denote by $\det$ the non-trivial one-dimensional representation of $O(2l+1)$. An representation $\lambda$ of $SO(2l+1)$ extends to two different representations $\lambda$ and $\lambda \otimes \det$ of $O(2l+1)$. Then we can parameterize irreducible representations of $O(2l+1)$ by $(m_1, \ldots, m_l)$ and $(m_1, \ldots, m_l) \otimes \det$. We shall denote $\Sigma(B) = P^l \cup \{ \lambda \otimes \det \mid \lambda \in P^l \}$.

For more details regarding a parametrization of irreducible modules of various classical Lie groups including $O(2l+1)$, we refer the reader to [BtD].

2.3. The Fock space $\mathcal{F}^l$. Consider a pair of fermionic fields

$$
\psi^+(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \psi^+_n z^{-n - \frac{1}{2}}, \quad \psi^-(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \psi^-_n z^{-n - \frac{1}{2}},
$$

with the following anti-commutation relations

$$
[\psi^+_m, \psi^-_n]_+ = \delta_{m+n,0}, \quad [\psi^+_m, \psi^+_n]_+ = 0.
$$

Denote by $\mathcal{F}$ the Fock space of the fermionic fields $\psi^\pm(z)$ generated by a vacuum vector $|0\rangle$ which satisfies

$$
\psi^-_n |0\rangle = \psi^+_n |0\rangle = 0 \quad \text{for } n \in \frac{1}{2} + \mathbb{Z}_+.
$$

We have the standard charge decomposition (cf. [MJD])

$$
\mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}^{(k)}.
$$

Each $\mathcal{F}^{(k)}$ becomes an irreducible module over a certain Heisenberg Lie algebra. The shift operator $S : \mathcal{F}^{(k)} \to \mathcal{F}^{(k+1)}$ matches the highest weight vectors and commutes with the creation operators in the Heisenberg algebra.

Now we take $l$ pairs of fermionic fields, $\psi^\pm_p(z)$ ($p = 1, \ldots, l$) and denote the corresponding Fock space by $\mathcal{F}^l$. Introduce the following generating series

$$
E(z, w) \equiv \sum_{i,j \in \mathbb{Z}} E_{ij} z^i w^{-j} = \sum_{p=1}^{l} : \psi^+_p(z) \psi^-_p(w) :,
$$

where the normal ordering $::$ means that the operators annihilating $|0\rangle$ are moved to the right with a sign. It is well known that the operators $E_{ij}$ ($i, j \in \mathbb{Z}$) generate a representation in $\mathcal{F}^l$ of the Lie algebra $\hat{\mathfrak{gl}}$ with level $l$. 
Let
\[ e^-_{pq} = \sum_{r \in \mathbb{Z}} \psi_r^{-p} \psi_r^{-q}, \quad e^+_{pq} = \sum_{r \in \mathbb{Z}} \psi_r^{+p} \psi_r^{+q}; \quad p \neq q, \] (4)
and let
\[ e_{pq} = \sum_{r \in \mathbb{Z}} \psi_r^{+p} \psi_r^{-q}; + \delta_{pq} \epsilon. \] (5)
The operators \( e^+_{pq}, e_{pq}, e^-_{pq} \) (\( p, q = 1, \cdots, l \)) generate Lie algebra \( \mathfrak{so}(2l) \) (cf. \( \text{[FF, W1]} \)).

2.4. The main results of \([BO, CW]\). Recall that Bloch and Okounkov \([BO]\) introduced the following operators in \( \hat{\mathfrak{gl}} \):
\[ A(t) = \sum_{k \in \mathbb{Z}} t^{k-\frac{1}{2}} E_{k,k}, \quad A(t) = :A(t): + \frac{1}{t^\frac{1}{2} - t^{-\frac{1}{2}}} C. \]

Given \( \lambda = (m_1, \ldots, m_l) \in \Sigma(A) \), we denote by \( \Lambda(\lambda) \) the \( \hat{\mathfrak{gl}} \)-highest weight \( \Lambda^a_{m_1} + \cdots + \Lambda^a_{m_l} \). The energy operator \( L_0 \) on the \( \hat{\mathfrak{gl}} \)-module \( L(\hat{\mathfrak{gl}}; \Lambda(\lambda)) \) with highest weight vector \( v_{\Lambda(\lambda)} \) is characterized by
\[ L_0 \cdot v_{\Lambda(\lambda)} = \frac{1}{2} \| \lambda \|^2 \cdot v_{\Lambda(\lambda)}, \] (6)
\[ [L_0, E_{ij}] = (i-j) E_{ij}, \]
where
\[ \| \lambda \|^2 := \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2, \]
On \( \mathcal{F}^l \), we can realize \( L_0 \) as
\[ L_0 = \sum_{p=1}^l \sum_{k \in \mathbb{Z} + \frac{1}{2}} k : \psi_{-k}^{+p} \psi_{k}^{-p}. \]

The \( n \)-point \( \hat{\mathfrak{gl}} \)-correlation function of level \( l \) associated to \( \lambda \) is defined in \([BO]\) for \( l = 1 \) and in \([CW]\) for general \( l \) as
\[ \mathfrak{A}_l^t(q; t) \equiv \mathfrak{A}_l^t(q; t_1, \ldots, t_n) := \text{tr} \, L(\hat{\mathfrak{gl}};\Lambda(\lambda)) (q L_0 A(t_1) A(t_2) \cdots A(t_n)). \]

Here and below we denote \( t = (t_1, \ldots, t_n) \).

Let \( (a; q)_\infty := \prod_{r=0}^\infty (1 - aq^r) \). Define the theta function
\[ \Theta(t) := (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(q; q)_\infty^{-2} (qt; q)_\infty (qt^{-1}; q)_\infty \] (7)
\[ \Theta^{(k)}(t) := \left( t \frac{d}{dt} \right)^k \Theta(t), \quad \text{for } k \in \mathbb{Z}_+. \] (8)
Denote by $F_{bo}(q; t)$ or $F_{bo}(q; t_1, \ldots, t_n)$ the following expression
\begin{equation}
\frac{1}{(q; q)_\infty} \cdot \sum_{\sigma \in S_n} \det \left( \Theta(t_{\sigma(1)}) \Theta(t_{\sigma(1)}t_{\sigma(2)}) \cdots \Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots t_{\sigma(n)}) \right)_{i,j=1}^n \Theta(t_{\sigma(1)}) \Theta(t_{\sigma(2)}) \cdots \Theta(t_{\sigma(1)}t_{\sigma(2)} \cdots t_{\sigma(n)})^n.
\end{equation}

It is understood here that $1/(-k)! = 0$ for $k > 0$, and for $n = 1$, we have $F_{bo}(q; t) = (q; q)_\infty^{-1} \Theta(t)^{-1}$. The following summarizes the main results of Bloch-Okounkov [BO] for $l = 1$ and Cheng-Wang [CW] for general $l \geq 1$.

**Theorem 2.1.** Associated to $\lambda = (\lambda_1, \ldots, \lambda_l)$, where $\lambda_1 \geq \ldots \geq \lambda_l$ and $\lambda_i \in \mathbb{Z}$, the $n$-point $\hat{\mathfrak{gl}}$-function of level $l$ is given by

$$A_l^\lambda(q; t) = q^{\|\lambda\|^2} t^\lambda \cdot \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) \cdot F_{bo}(q; t)^l$$

where $|\lambda| := \lambda_1 + \cdots + \lambda_l$.

In the simplest case, i.e. $l = n = 1$, we have

$$A_1^\lambda(q; t) = q^{\frac{\lambda^2}{2}} t^\lambda \cdot F_{bo}(q; t) = \frac{q^{\frac{\lambda^2}{2}} t^\lambda}{(q; q)_\infty \Theta(t)}.$$  

**3. The Correlation Functions on $d_\infty$-Modules**

Let $t$ be an indeterminate and define the following operators in $d_\infty$:

$$:D(t): = \sum_{k \in \mathbb{N}} (t^{k-\frac{1}{2}} - t^{\frac{1}{2}-k})(E_{k,k} - E_{1-k,1-k}),$$

$$D(t) = :D(t): + \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C.$$  

**Definition 3.1.** The $n$-point $d_\infty$-correlation function of level $l + \frac{1}{2}$ associated to $\lambda \in \mathcal{P}^l \cup \mathcal{P}^l \otimes \det$, denoted by $D^{l+\frac{1}{2}}_\lambda(q, t)$ or also by $D^{l+\frac{1}{2}}_\lambda(q, t_1, \ldots, t_n)$, is

$$\text{tr}_{L(d_\infty; \Lambda(\lambda))} q^{L_0} D(t_1) \cdots D(t_n).$$

**Remark 3.1.** In [TW], the trace is taken over the direct sum of the modules $L(d_\infty; \Lambda(\lambda))$ and $L(d_\infty; \Lambda(\lambda \otimes \det))$ for $\lambda$ in $\mathcal{P}^l$ for technical reasons.
3.1. Fock space $\mathcal{F}^{l+\frac{1}{2}}$ and $D(t)$ realization. Consider the neutral fermion

$$\varphi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \varphi_n z^{-n-\frac{1}{2}}$$

which satisfies the commutation relation

$$[\varphi_m, \varphi_n]_+ = \delta_{m,-n}.$$ 

We denote by $\mathcal{F}^{l+\frac{1}{2}}$ the Fock space of one neutral fermion $\varphi(z)$ and $l$ pairs of complex fermions $\psi^{\pm, p}(z), 1 \leq p \leq l$, generated by a vacuum vector $|0\rangle$ which satisfies

$$\varphi_n |0\rangle = \psi^+_n |0\rangle = \psi^-_n |0\rangle = 0 \quad \text{for } n \in \frac{1}{2} + \mathbb{Z}.$$ 

Let

$$e^\pm_p = \sum_{r \in \mathbb{Z}} : \psi^\pm_p \varphi_{-r} : , \quad 1 \leq p \leq l.$$ 

It is known (cf. [FF, W1]) that the above operators $e^+_p, e^-_p$ together with $e^{+pq}_p, e^{-pq}_q$ ($p, q = 1, \ldots, l$) defined in [41, 51] generate Lie algebra $\mathfrak{so}(2l + 1)$.

When acting on $\mathcal{F}^{l+\frac{1}{2}}$, we may then rewrite $D(t)$ as

$$D(t) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k \left( \sum_{i=1}^l (\psi^+_k \psi^-_i + \psi^-_k \psi^+_i) + \varphi_{-k} \varphi_k \right).$$

For later use, we have the following lemma giving an isomorphism of Fock spaces.

**Lemma 3.1.** Given a pair of complex fermions $\psi^\pm(z)$, we let

$$\varphi_n := (\psi^+_n + \psi^-_n)/\sqrt{2}, \quad \varphi'_n := i(\psi^+_n - \psi^-_n)/\sqrt{2}.$$ 

Then, $\varphi_n$ and $\varphi'_n$ satisfy the anti-commutation relations:

$$[\varphi_n, \varphi_m]_+ = \delta_{n,-m}, \quad [\varphi'_n, \varphi'_m]_+ = \delta_{n,-m},$$

$$[\varphi_n, \varphi'_m]_+ = 0, \quad \text{for } m, n \in \mathbb{Z}.$$ 

Hence, there is an isomorphism of Fock spaces

$$\mathcal{F}^{\frac{1}{2}} \otimes \mathcal{F}^{\frac{1}{2}} \cong \mathcal{F}$$

**Proof.** Verified by a direct computation. □
3.2. The \( n \)-point \( d_\infty \)-correlation functions of level \( l + \frac{1}{2} \). Consider the \( d_\infty \) operator
\[
\alpha = \sum_{k > 0} \varphi_{-k} \varphi_k
\]
and set the following notation
\[
\mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) = \text{tr}_{L(d_\infty; \Lambda(\lambda))} q^{t_0} D(t_1) \cdots D(t_n)
\]
\[
\mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) = \text{tr}_{L(d_\infty; \Lambda(\lambda) \otimes \text{det})} (-1)^{\lambda} q^{t_0} D(t_1) \cdots D(t_n).
\]
Note that \( \mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \) was computed in [TW].

**Proposition 3.1. [TW Theorem 4.1]** The function \( \mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \) is equal to
\[
\mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \times \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{1}{2} \lambda + \rho - \sigma(\rho)} \prod_{a=1}^{l} \left( \sum_{\tilde{c}_a \in \{\pm 1\}^n} \left[ \tilde{c}_a \right] (\Pi t^\tilde{c}_a) k_a F_{bo}(q; t^\tilde{c}_a) \right)
\]
where \( k_a = (\lambda + \rho - \sigma(\rho), \varepsilon_a) \), \( W(B_l) \) is the Weyl group of type \( B \), \( \rho \) is the usual half-sum of positive roots for type \( B \), \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \), \([\varepsilon] = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n\), and \( \Pi t^\varepsilon = t_1^{\varepsilon_1} \cdots t_n^{\varepsilon_n} \). Also, \( \mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \) is given in [TW Proposition 4.2].

The following formula from [TW] will be used later.
\[
\text{tr}_{\mathbb{C}^n} q^{t_0} D(t_1) \cdots D(t_n) = \sum_{k \in \mathbb{Z}} \varphi_k \sum_{\varepsilon \in \{\pm 1\}^n} \left[ \varepsilon \right] \cdot (\Pi t^\varepsilon)^k F_{bo}(q; t^\varepsilon) \quad (10)
\]

The main results of this paper are the computation of the function \( \mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \) and Proposition 3.2 below. A recursive formula for \( \mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) \) can be obtained similar to [TW Proposition 4.2]. Note that [W2 Theorem 8] implicitly gives the 1-point version as
\[
\mathbb{D}_{l + \frac{1}{2}}^{\lambda}(q, t) = -(q^{\frac{1}{2}})_\infty \left( \sum_{n=1} q^{n \frac{1}{2}} \frac{(t^{n \frac{1}{2}} - t^{-n + \frac{1}{2}})}{1 - q^{n \frac{1}{2}}} \right) + \frac{t^{\frac{1}{2}}}{t - 1} (q^{\frac{1}{2}})_\infty.
\]

**Lemma 3.2.** We have
\[
[\alpha, \varphi_r] = \varphi_r, \quad [\alpha, \psi_r^\pm] = 0.
\]
Equivalently, \( \alpha \) acts on vectors of \( d_\infty \)-modules by counting the number of \( \varphi \)'s in the vector.
Proof. The lemma follows by direct computation using the anti-commutation relations amongst the $\varphi$s and $\psi$s. □

**Proposition 3.2.** For $\lambda \in \mathcal{P}$, the $n$-point $d_\infty$-correlation functions of level $l + \frac{1}{2}$ are given by

$$
\mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t) = \frac{\mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t) + \mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t)}{2}
$$

$$
\mathcal{D}^{l+\frac{1}{2}}_{\lambda \otimes \det}(q, t) = \frac{\mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t) - \mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t)}{2}.
$$

Proof. Using Lemma 3.2, the operator $\alpha$ acting on an element of either $L(d_\infty; \Lambda(\lambda))$ or $L(d_\infty; \Lambda(\lambda \otimes \det))$ counts the number of $\varphi$s in the vector. The structure of the highest weight vectors for these modules is well-known (cf. [W1, Theorem 4.1]) and elements of $L(d_\infty; \Lambda(\lambda))$ (respectively $L(d_\infty; \Lambda(\lambda \otimes \det))$) have an even (respectively odd) number of $\varphi$s; hence $(-1)^{\alpha}$ acts as 1 on $L(d_\infty; \Lambda(\lambda))$ and as $-1$ on $L(d_\infty; \Lambda(\lambda \otimes \det))$ and the result follows. □

We set

$$
\alpha' = \sum_{k>0} \left( \varphi_{-k} \varphi_k + \varphi'_{-k} \varphi'_k \right)
$$

and given a subset $I = (i_1, \ldots, i_s) \subseteq \{1, \ldots, n\}$, we denote by $I^c$ the complementary set to $I$, and $t_I = (t_{i_1}, \ldots, t_{i_s})$. By convention, we let

$$
\mathcal{D}^{l+\frac{1}{2}}_{\lambda}(0)(q, t_\emptyset) = \text{tr}_{g_{\lambda}'^{\alpha'}} (-1)^{\alpha'} q^{L_0} = (q^{\frac{1}{2}}; q)_{\infty}.
$$

(11)

**Proposition 3.3.** We have

$$
\text{tr}_{g_{\lambda}'^{\alpha'}} (-1)^{\alpha'} q^{L_0} \mathcal{D}(t_1) \cdots \mathcal{D}(t_n) = \sum_{I \subseteq \{1, \ldots, n\}} \mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t_I) \mathcal{D}^{l+\frac{1}{2}}_{\lambda \otimes \det}(q, t_{I^c}).
$$

(12)

Equivalently, we have

$$
\mathcal{D}^{l+\frac{1}{2}}_{\lambda}(q, t) = \frac{1}{2} (q^{\frac{1}{2}}; q)_{\infty}^{-1} \left( \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{k^2}{2}} \sum_{\bar{e} \in \{\pm 1\}^n} [\bar{e}] \cdot (\Pi t^\bar{e})^k F_{bo}(q; t^\bar{e}) 

- \sum_{\emptyset \subsetneq I \subseteq \{1, \ldots, n\}} \mathcal{D}_{\lambda}(q, t_I) \mathcal{D}_{\lambda \otimes \det}(q, t_{I^c}) \right).
$$

Proof. A simple calculation reveals that

$$
\psi_{-k}^+ \psi_k^- + \psi_k^- \psi_{-k}^+ = \varphi_{-k} \varphi_k + \varphi'_{-k} \varphi'_k.
$$
so under the isomorphism $\mathcal{F} \cong \mathcal{F}_1^{\frac{1}{2}} \otimes \mathcal{F}_2^{\frac{1}{2}}$, we may write $D(t) = D_1(t) + D_2(t)$ where $D_1(t) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} t^k \varphi_k \varphi_k'$ and $D_2(t) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} t^k \varphi_k' \varphi_k'$. Therefore, we have
\[
\text{tr}_{\mathcal{F}} q^{L_\alpha} D(t_1) \cdots D(t_n) = \text{tr}_{\mathcal{F}_1^{\frac{1}{2}} \otimes \mathcal{F}_2^{\frac{1}{2}}} q^{L_\alpha} (D_1(t_1) + D_2(t_1)) \cdots (D_1(t_n) + D_2(t_n))
\]
which is equivalent to the first formula in the proposition.

Observe that
\[
\alpha' = \sum_{k > 0} (\varphi_k - \varphi_k' \varphi_k')
\]
from the isomorphism of Fock spaces. Recalling that
\[
e_{11} = \sum_{k > 0} (\psi_k^+ \psi_k^- - \psi_k^- \psi_k^+)
\]
it follows that
\[
(-1)^{\alpha'} = (-1)^{e_{11}}.
\]

Thus we have
\[
\text{tr}_{\mathcal{F}} (-1)^{\alpha'} q^{L_\alpha} D(t_1) \cdots D(t_n) = \text{tr}_{\mathcal{F}} (-1)^{e_{11}} q^{L_\alpha} D(t_1) \cdots D(t_n);
\]
the proposition follows by noting that on the right-hand side of (12), there are exactly two terms equal to $\mathcal{D}_{(0)}^{\frac{1}{2}}(q; t)$ which come from $I = \emptyset$ and \{1, \ldots, n\}. Note that a formula for $\text{tr}_{\mathcal{F}} (-1)^{e_{11}} q^{L_\alpha} D(t_1) \cdots D(t_n)$ is given by (10) with $z = -1$.

We now present our main theorem.

**Theorem 3.1.** The function $\mathcal{D}_{\lambda}^{\ell + \frac{1}{2}}(q, t)$ is equal to
\[
\mathcal{D}_{(0)}^{\frac{1}{2}}(q; t) \times
\]
\[
\times \sum_{\sigma \in W(B_l)} (-1)^{t(\sigma)} q^{\frac{1}{2}(\lambda + \rho - \sigma(\rho))^2} \prod_{a=1}^{l} \left( \sum_{\tilde{e}_a \in \{\pm 1\}^n} [\tilde{e}_a](I\tilde{t}^{\tilde{e}_a})^{k_a} F_{\lambda}(q; t^{\tilde{e}_a}) \right)
\]
where $k_a = (\lambda + \rho - \sigma(\rho), \tilde{e}_a)$.

**Proof.** From [W1, Theorem 4.1], as $(O(2l + 1), d_{\infty})$-modules, $\mathcal{F}^{\ell + \frac{1}{2}} \cong \bigoplus_{\lambda \in \Sigma(B)} V_{\lambda}(O(2l + 1)) \otimes L(d_{\infty}, \Lambda(\lambda))$. (13)
Apply $\text{tr}_{g^{l+\frac{1}{2}}} (-1)^\alpha z_1^{\epsilon_1} \cdots z_l^{\epsilon_l} q^{L_0} D(t_1) \cdots D(t_n)$ to both sides of this Howe-duality decomposition. As $\alpha$ only acts on $F^{l+\frac{1}{2}}$, we obtain

$$\text{tr}_{g^{l+\frac{1}{2}}} (-1)^\alpha q^{L_0} D(t_1) \cdots D(t_n) \cdot \prod_{i=1}^l \text{tr}_{g^i} z_i^{\epsilon_i} q^{L_0} D(t_1) \cdots D(t_n) = \sum_{\lambda \in \Sigma(B)} (-1)^{\det +1} \text{ch}_\lambda^b(z_1, \ldots, z_l) D^{l+\frac{1}{2}_\lambda}(q, t),$$

where $(-1)^{\det +1}$ is equal to 1 if $\lambda \in \mathcal{P}^l$ and -1 otherwise.

For $\lambda \in \mathcal{P}^l$, the character of the irreducible $O(2l+1)$-module associated to $\lambda$ and $\lambda \otimes \det$ is the same, and is given as follows (cf. [FH, p. 408])

$$\text{ch}_\lambda(z_1, \ldots, z_l) = \left| \frac{z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})}}{z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})}} \right|,$$  \hspace{1cm} (14)

so we may rewrite the above as

$$\mathbb{D}^{l+\frac{1}{2}}_0(q; t) \prod_{i=1}^l \text{tr}_{g^i} z_i^{\epsilon_i} q^{L_0} D(t_1) \cdots D(t_n) = \sum_{\lambda \in \Sigma(B)} (-1)^{\det +1} \left| \frac{z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})}}{z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})}} \right| D^{l+\frac{1}{2}_\lambda}(q, t).$$}

The Weyl denominator of type $B_l$ reads that

$$\left| z_j^{l - i + \frac{1}{2}} + z_j^{-(l - i + \frac{1}{2})} \right| = \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} z^{\sigma(\rho)}.$$  \hspace{1cm} (15)

so by cross-multiplying terms, we may write

$$\sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} z^{\sigma(\rho)} \cdot \mathbb{D}^{l+\frac{1}{2}}_0(q; t) \prod_{i=1}^l \text{tr}_{g^i} z_i^{\epsilon_i} q^{L_0} D(t_1) \cdots D(t_n) = \sum_{\lambda \in \Sigma(B)} (-1)^{\det +1} \left| \frac{z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})}}{z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})}} \right| D^{l+\frac{1}{2}_\lambda}(q, t).$$

We may use (14) to expand each multiplicand on the left-hand side and compare coefficients of the dominant monomial $z^{l+\rho}$ on each side to finish the proof. \hfill $\square$

Remark 3.2. In the spirit of this paper, there are three more cases where one can consider the correlation functions on irreducible components of a direct summand. At the positive level, the integral case of $d_\infty$
remains. This case is technically more difficult and different than the case we consider here; we do not know of an operator in $d_{\infty}$ that is able to differentiate between the two modules of a direct summand, and this phenomenon only occurs for certain irreducible modules. A much different strategy may be required.

Also, at the negative level, both the integral and half-integral cases for $c_{\infty}$ are similar to $d_{\infty}$ for the positive levels. Given the already different nature of the negative level cases, again, a much different strategy may be required. The author plans to consider these in the future.

### 3.3. The $q$-dimension of a $d_{\infty}$-module of level $l + \frac{1}{2}$.

For a $d_{\infty}$-module $M$, we denote by 

$$\dim_q M = \text{tr}_M q^{L_0}$$

the $q$-dimension of the module $M$. Set

$$Q(q)^+ = \text{tr}_M (L(d_{\infty}; \Lambda(\lambda)) \oplus L(d_{\infty}; \Lambda(\lambda \otimes \det))) q^{L_0}$$

and

$$Q(q)^- = \text{tr}_M (L(d_{\infty}; \Lambda(\lambda)) \oplus L(d_{\infty}; \Lambda(\lambda \otimes \det))) (-1)^{\alpha} q^{L_0}.$$ 

The following proposition is a direct consequence of the above notation and the proof of Proposition 3.2.

**Proposition 3.4.** For $\lambda \in \mathcal{P}^l$, we have

$$\dim_q L(d_{\infty}, \Lambda(\lambda)) = \frac{Q(q)^+ + Q(q)^-}{2}$$

and

$$\dim_q L(d_{\infty}, \Lambda(\lambda \otimes \det)) = \frac{Q(q)^+ - Q(q)^-}{2}.$$ 

Note that $Q(q)^+ = \dim_q [L(d_{\infty}; \Lambda(\lambda)) \oplus L(d_{\infty}; \Lambda(\lambda \otimes \det))]$ which is equal to the following equivalent formulas (cf. [TW]):

$$Q(q)^+ = \frac{(-q^{\frac{1}{2}}; q)_{\infty}}{(q; q)_{\infty}} \sum_{\sigma \in \mathcal{W}(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|_2^2}{2}}$$

$$= \frac{(-q^{\frac{1}{2}}; q)_{\infty}}{(q; q)_{\infty}} q^{\frac{\|\lambda\|_2^2}{2}} \prod_{1 \leq i \leq l} (1 - q^{\lambda_i + l - i + 1/2}) \times$$

$$\times \prod_{1 \leq i < j \leq l} (1 - q^{\lambda_i - \lambda_j + j - i}) (1 - q^{\lambda_i + \lambda_j + 2l - i - j + 1}).$$

It remains to compute $Q(q)^-$. 
Proposition 3.5. We have

\[ Q(q)^{−} = \frac{(q^{− \frac{1}{2}}; q)_{\infty}}{(q; q)_{l}^{l}} \cdot \sum_{\sigma \in W(B_{l})} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho - \sigma(\rho)\|^{2}}{2}} \]

\[ = \frac{(q^{− \frac{1}{2}}; q)_{\infty}}{(q; q)_{l}^{l}} \cdot q^{\frac{\|\lambda\|^{2}}{2}} \prod_{1 \leq i \leq l} \left( 1 - q^{\lambda_{i} + l - i + 1/2} \right) \times \]

\[ \times \prod_{1 \leq i < j \leq l} \left( 1 - q^{\lambda_{i} - \lambda_{j} + j - i} \right) \left( 1 - q^{\lambda_{i} + \lambda_{j} + 2l - i - j + 1} \right). \]

Proof. In the proof of Theorem 3.1, we instead apply

\[ \text{tr}_{\mathcal{F}^{l+\frac{1}{2}} \left( -1 \right) ^{\alpha} z_{1}^{\alpha} \cdots z_{l}^{\alpha} q^{L_{0}}} \]

to both sides of the duality in (13). The same strategy applies, with the substitutions

\[ \text{tr}_{\mathcal{F}^{\frac{1}{2}}} \left( -1 \right) ^{\alpha} q^{L_{0}} = (q^{− \frac{1}{2}}; q)_{\infty} \]

and

\[ \text{tr}_{\mathcal{F} z_{i}^{\alpha} q^{L_{0}}} = \dim_{q} \mathcal{F}(0) \sum_{k \in \mathbb{Z}} z_{i}^{k} q^{k^{2}/2} = (q; q)_{\infty}^{-1} \sum_{k \in \mathbb{Z}} z_{i}^{k} q^{k^{2}/2}. \]

The equivalence of the two statements follows from above. \( \square \)

We note that the \( q \)-dimension formula was also obtained in an alternate form using a very different strategy in [KWy].

References

[BO] S. Bloch and A. Okounkov, The characters of the infinite wedge representation, Adv. in Math. 149 (2000), 1–60.

[BtD] T. Bröcker and T. tom Dieck, Representations of compact Lie groups, Springer-Verlag.

[CTW] S.-J. Cheng, D. Taylor and W. Wang, The Bloch-Okounkov correlation functions of negative levels, J. Algebra 319 (2008), 457-490.

[CW] S.-J. Cheng and W. Wang, The Bloch-Okounkov correlation functions at higher levels, Transformation Groups 9 (2004), 133–142.

[DJKM] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, A new hierarchy of soliton equations of KP-type. Transformation groups for soliton equations IV, Physics 4D (1982), 343–365.

[FF] A. Feingold and I. Frenkel, Classical affine algebras, Adv. in Math. 56 (1985), 117–172.

[FH] W. Fulton and J. Harris, Representation Theory: A First Course, Springer, 1991.

[Kac] V. Kac, Infinite dimensional Lie algebras, third edition, Cambridge Univ. Press, 1990.

[KWy] V. Kac, C.H. Yan, and W. Wang, Quasifinite representations of classical Lie subalgebras of \( \mathcal{W}_{1+\infty} \), Adv. in Math. 139 (1998), 56–140.
A. Milas, *Formal differential operators, vertex operator algebras and zeta-values, II*, J. Pure Appl. Algebra 183 (2003), 191–244.

T. Miwa, M. Jimbo and E. Date, *Solitons. Differential equations, symmetries and infinite dimensional algebras*, (originally published in Japanese 1993), Cambridge University Press, 2000.

A. Okounkov, *Infinite wedge and random partitions*, Select. Math., New Series 7 (2001), 1–25.

D. Taylor and W. Wang, *The Bloch-Okounkov correlation functions of classical type*, Commun. Math. Phys. 276 (2007), 473–508.

W. Wang, *Duality in infinite dimensional Fock representations*, Commun. Contemp. Math. 1 (1999), 155–199.

———, *Correlation functions of strict partitions and twisted Fock spaces*, Transformation Groups 9 (2004), 89–101.

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