Low moments of the four-loop splitting functions in QCD

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Abstract

We have computed the four lowest even-$N$ moments of all four splitting functions for the evolution of flavour-singlet parton densities of hadrons at the fourth order in the strong coupling constant $\alpha_s$. The perturbative expansion of these moments, and hence of the splitting functions for momentum fractions $x \gtrsim 0.1$, is found to be well behaved with relative $\alpha_s$-coefficients of order one and sub-percent effects on the scale derivatives of the quark and gluon distributions at $\alpha_s \lesssim 0.2$. More intricate computations, including other approaches such as the operator-product expansion, are required to cover the full $x$-range relevant to LHC analyses. Our results are presented analytically for a general gauge group for detailed checks and validations of such future calculations.
Fully consistent analyses of hard processes with initial-state hadrons at the \((n+1)\)-leading order \((N^n\text{LO})\) of renormalization-group improved perturbative QCD require parton distributions functions (PDFs) evolved with the \((n+1)\)-loop splitting functions. Over the past years, \(N^2\text{LO} = \text{NNLO}\) has become the standard approximation for many processes. Following pioneering computations of their lowest integer-\(N\) Mellin moments in refs. [1,2], the corresponding 3-loop splitting functions were obtained in refs. [3,4].

For certain benchmark cases, in particular Higgs-boson production at the LHC [5], \(N^2\text{LO}\) calculations are not sufficiently accurate, hence the 4-loop splitting functions need to be calculated. These have been determined for the flavour non-singlet quark-quark case in ref. [6] – analytically in the limit of a large number of colours \(n_c\), and numerically for the remaining contributions – and for the (next-to-)leading contributions for a large number of flavours \(n_f\) in ref. [7].

Here we present, as a first significant step towards at least approximate expressions for the 4-loop singlet splitting functions for use in phenomenological analyses, their lowest four even \(N\) moments in the standard \(\overline{\text{MS}}\) scheme, thus extending the computations of ref. [2] by one order in the strong coupling \(\alpha_s\). Following the approach of refs. [2,4] our calculations are performed via physical quantities in deep-inelastic scattering, i.e., instead of working with 4-loop off-shell flavour-singlet operator matrix elements (OMEs) which, at this point, is still theoretically challenging. Our present results, obtained analytically for a general gauge group, should also be useful for checking and validating future OME computations of these quantities.

The evolution equations for the flavour-singlet quark and gluon PDFs of hadrons,

\[
q_s(x, \mu_f^2) = \sum_{i=1}^{n_f} \left[ q_i(x, \mu_f^2) + \bar{q}_i(x, \mu_f^2) \right] \quad \text{and} \quad g(x, \mu_f^2),
\]

are

\[
\frac{d}{d \ln \mu_f^2} \left( \begin{array}{c} q_s \\ g \end{array} \right) = \left( \begin{array}{cc} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{array} \right) \otimes \left( \begin{array}{c} q_s \\ g \end{array} \right).
\]

Here \(\otimes\) represents the Mellin convolution in the momentum variable, and \(\mu_f\) is the factorization scale. For the determination of the splitting functions \(P_{ik}\), the renormalization scale can be identified with \(\mu_f\) without loss of information. The even-\(N\) moments of splitting functions in eq. (2) are identical to the anomalous dimensions of twist-2 spin-\(N\) operators up to a conventional sign,

\[
\gamma_{ik}(N, \alpha_s) = -\int_0^1 dx x^{N-1} P_{ik}(x, \alpha_s).
\]

Their perturbative expansions can be written as

\[
\gamma_{ik}(N, \alpha_s) = \sum_{n=0} a_s^{n+1} \gamma_{ik}^{(n)}(x) \quad \text{with} \quad a_s \equiv \frac{\alpha_s(\mu_f^2)}{4\pi}.
\]

The quark-quark entry in eq. (3) can be expressed as \(\gamma_{qq} = \gamma_{ns}^+ + \gamma_{ps}\) in terms of the non-singlet anomalous dimension \(\gamma_{ns}^+\) for quark-antiquark sums addressed at four loops in ref. [6] and a pure-singlet contribution \(\gamma_{ps}\) which is suppressed at \(N \gg 1\). At asymptotically large \(N\) the diagonal \(\overline{\text{MS}}\) entries \(\gamma_{kk}(N)\) in eq. (2) are governed by the (lightlike) cusp anomalous dimensions \(A_k\) [8], viz \(\gamma_{kk}(N) = A_k \ln N + O(1)\), which are now fully known at four loops [9,10].
The 4-loop contributions to the pure-singlet anomalous dimensions in eq. (4) at $N = 2, 4, 6$ are

$$
\gamma_{\text{ps}}^{(3)} (N=2) = n_f C_F^3 \left( \frac{227938}{2187} + \frac{1952}{81} \xi_3 + \frac{256}{9} \xi_4 - \frac{640}{3} \xi_5 \right) 
+ n_f C_A C_F^2 \left( -\frac{162658}{6561} + \frac{8048}{27} \xi_3 - \frac{1664}{9} \xi_4 + \frac{320}{9} \xi_5 \right) 
+ n_f C_A^2 C_F \left( -\frac{410299}{6561} - \frac{26896}{81} \xi_3 + \frac{1408}{9} \xi_4 + \frac{4480}{27} \xi_5 \right) 
+ n_f \frac{d_{R}^{abcd} d_{R}^{abcd}}{n_c} \left( \frac{1024}{9} + \frac{256}{9} \xi_3 - \frac{2560}{9} \xi_5 \right) - n_f^2 C_F^2 \left( \frac{73772}{6561} + \frac{5248}{81} \xi_3 - \frac{320}{9} \xi_4 \right) 
+ n_f^2 C_A C_F \left( \frac{160648}{6561} + \frac{48}{9} \xi_3 - \frac{320}{9} \xi_4 \right) + n_f^3 C_F \left( -\frac{1712}{729} + \frac{128}{27} \xi_3 \right),
$$

(5)

$$
\gamma_{\text{ps}}^{(3)} (N=4) = n_f C_F^3 \left( \frac{1995890620891}{52488000000} - \frac{897403}{202500} \xi_3 + \frac{18997}{2250} \xi_4 - \frac{484}{15} \xi_5 \right) 
+ n_f C_A C_F^2 \left( \frac{209865827521}{26244000000} + \frac{6743539}{202500} \xi_3 - \frac{29161}{750} \xi_4 + \frac{242}{45} \xi_5 \right) 
+ n_f C_A^2 C_F \left( -\frac{55187654921}{3280500000} + \frac{3104267}{67500} \xi_3 + \frac{34243}{1125} \xi_4 + \frac{3164}{135} \xi_5 \right) 
+ n_f \frac{d_{R}^{abcd} d_{R}^{abcd}}{n_c} \left( \frac{172231}{675} - \frac{5368}{25} \xi_3 - \frac{3728}{45} \xi_5 \right) - n_f^2 C_F^2 \left( \frac{141522185707}{26244000000} \right) 
+ n_f^2 C_A C_F \left( \frac{9398360351}{16402500000} + \frac{57877}{10125} \xi_3 - \frac{242}{45} \xi_4 \right) 
+ n_f^3 C_F \left( -\frac{46099151}{729000000} + \frac{484}{675} \xi_3 \right),
$$

(6)

$$
\gamma_{\text{ps}}^{(3)} (N=6) = n_f C_F^3 \left( \frac{140565274663259489}{5403265623000000} - \frac{62727544}{24310125} \xi_3 + \frac{343156}{77175} \xi_4 - \frac{1936}{147} \xi_5 \right) 
+ n_f C_A C_F^2 \left( \frac{336481838777617}{36021770820000} + \frac{2111992}{324135} \xi_3 - \frac{1398906}{77175} \xi_4 + \frac{968}{441} \xi_5 \right) 
+ n_f C_A^2 C_F \left( -\frac{6194882229735067}{8645224969680000} + \frac{2396237}{165375} \xi_3 + \frac{41866}{3087} \xi_4 + \frac{9544}{1323} \xi_5 \right) 
+ n_f \frac{d_{R}^{abcd} d_{R}^{abcd}}{n_c} \left( \frac{64697569}{330750} - \frac{426976}{3675} \xi_3 - \frac{39808}{441} \xi_5 \right) 
- n_f^2 C_F^2 \left( \frac{812984663253277}{270163281150000} + \frac{2594876}{694575} \xi_3 - \frac{968}{441} \xi_4 \right) + n_f^2 C_A C_F \left( \frac{3092531515013}{964868861250} \right) 
+ \frac{217432}{99225} \xi_3 - \frac{968}{441} \xi_4 \right) + n_f^3 C_F \left( -\frac{19597073837}{61261515000} + \frac{1936}{6615} \xi_3 \right),
$$

(7)

The complete $qq$ entries are obtained by adding the non-singlet contributions in app. B of ref. [6].
The corresponding results for the off-diagonal splitting functions are given by

\[
\begin{align*}
\gamma_{dq}^{(3)}(N=2) &= n_f C_F^3 \left( \frac{16489}{729} + \frac{736}{81} \zeta_3 + \frac{256}{9} \zeta_4 - \frac{320}{3} \zeta_5 \right) \\
&+ n_f C_A^3 \left( -\frac{88769}{729} + \frac{31112}{81} \zeta_3 - 132 \zeta_4 - \frac{3560}{27} \zeta_5 \right) - n_f C_A C_F^2 \left( \frac{1153727}{13122} - \frac{7108}{81} \zeta_3 \right) \\
&+ \frac{1136}{9} \zeta_4 - \frac{2000}{9} \zeta_5 \right) + n_f C_A^2 C_F \left( \frac{763868}{6561} - \frac{12808}{27} \zeta_3 + \frac{2068}{9} \zeta_4 + \frac{40}{9} \zeta_5 \right) \\
&+ n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left( \frac{368}{9} - \frac{992}{9} \zeta_3 - \frac{2560}{9} \zeta_5 \right) - n_f^2 C_F^2 \left( \frac{110714}{6561} + \frac{272}{9} \zeta_3 - \frac{224}{9} \zeta_4 \right) \\
&+ n_f^2 C_A C_F \left( \frac{249310}{6561} + \frac{5632}{81} \zeta_3 - \frac{440}{9} \zeta_4 \right) + n_f^2 C_A^2 \left( \frac{48625}{2187} - \frac{3572}{81} \zeta_3 \right) \\
&+ 24 \zeta_4 + \frac{160}{27} \zeta_5 \right) + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left( \frac{928}{9} - \frac{640}{9} \zeta_3 + \frac{2560}{9} \zeta_5 \right) \\
&+ \frac{8744}{2187} + \frac{128}{27} \zeta_3 \right) + n_f^3 C_F \left( \frac{3385}{2187} - \frac{176}{81} \zeta_3 \right),
\end{align*}
\tag{8}
\]

\[
\begin{align*}
\gamma_{dq}^{(3)}(N=4) &= n_f C_F^3 \left( -\frac{8103828487201}{104976000000} + \frac{5100751}{81000} \zeta_3 + \frac{154589}{4500} \zeta_4 - \frac{3158}{45} \zeta_5 \right) \\
&+ n_f C_A C_F^2 \left( \frac{5121012352507}{26244000000} - \frac{48971263}{405000} \zeta_3 - \frac{143489}{750} \zeta_4 + \frac{951}{5} \zeta_5 \right) \\
&+ n_f C_A^2 C_F \left( -\frac{314629497013}{1312200000} - \frac{2024593}{9000} \zeta_3 + \frac{1674889}{4500} \zeta_4 + \frac{1237}{45} \zeta_5 \right) \\
&+ n_f C_A^3 \left( \frac{143199094853}{1458000000} + \frac{11938031}{45000} \zeta_3 - \frac{26904}{125} \zeta_4 - \frac{17917}{135} \zeta_5 \right) \\
&+ n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left( \frac{12196}{135} - \frac{81008}{225} \zeta_3 + \frac{15976}{45} \zeta_5 \right) \\
&+ n_f^2 C_F^2 \left( -\frac{37295583467}{26244000000} + \frac{1400864}{50625} \zeta_3 + \frac{707}{45} \zeta_4 \right) + n_f^2 C_A C_F \left( \frac{217239001681}{13122000000} \right) \\
&+ \frac{4497112}{50625} \zeta_3 - \frac{103669}{225} \zeta_4 \right) + n_f^2 C_A^2 \left( -\frac{7131194093}{43740000000} - \frac{12599759}{202500} \zeta_3 \right) \\
&+ \frac{7591}{250} \zeta_4 + \frac{664}{135} \zeta_5 \right) + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left( -\frac{112424}{675} - \frac{2336}{75} \zeta_3 + \frac{10624}{45} \zeta_5 \right) \\
&+ \frac{312015851}{364500000} + \frac{6644}{3375} \zeta_3 \right) + n_f^3 C_A \left( \frac{33846151}{437400000} - \frac{5192}{2025} \zeta_3 \right),
\end{align*}
\tag{9}
\]

\[
\begin{align*}
\gamma_{dq}^{(3)}(N=6) &= n_f C_A^3 \left( \frac{49981299563940869}{345808999872000} + \frac{2383601783}{12965400} \zeta_3 - \frac{689907}{3430} \zeta_4 - \frac{159724}{1323} \zeta_5 \right) \\
&+ n_f C_A C_F^2 \left( \frac{324177529264517279}{960580555200000} - \frac{1154450237}{9724050} \zeta_3 - \frac{28952417}{154350} \zeta_4 + \frac{9832}{441} \zeta_5 \right)
\end{align*}
\]
\begin{align*}
&+ n_f C_A^2 C_F \left( \frac{-627686002393628869 - 6170262713 \zeta_3 + 1096679 \zeta_4 + 47774 \zeta_5}{1729044999360000 \cdot 48620250} \right) \\
&+ n_f C_F^3 \left( \frac{-2912197809548779709 - 1026604067 \zeta_3 + 2582141 \zeta_4 + 1328 \zeta_5}{21613062492000000 \cdot 24310125} \right) \\
&+ n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left( \frac{-23820479 - 11627738 \zeta_3 + 28624 \zeta_5}{264600 \cdot 33075} \right) \\
&+ n_f^2 C_F^2 \left( \frac{1942638296203817 - 113578219 \zeta_3 + 28724 \zeta_4}{5403265623000000 \cdot 4862025} \right) \\
&+ n_f^2 C_A C_F \left( \frac{3261418656515051 - 122909317 \zeta_3 - 600626 \zeta_4}{21613062492000000 \cdot 1620675} \right) \\
&+ n_f^2 C_A^2 \left( \frac{-55264268415947 - 38177677 \zeta_3 + 133186 \zeta_4 + 5360 \zeta_5}{6175160712000 \cdot 720300} \right) \\
&+ n_f^2 \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left( \frac{665983 - 192736 \zeta_3 + 85760 \zeta_5}{4725 \cdot 6615} \right) \\
&+ n_f^3 C_F \left( \frac{-1262351231147 + 15268 \zeta_3}{2572983630000 \cdot 9261} \right) + n_f^3 C_A \left( \frac{34431246007 - 8866 \zeta_3}{55135363500 \cdot 3969} \right).
\end{align*}

and

\begin{align}
\gamma_{gq}^{(3)} (N = 2) &= -\gamma_{qq}^{(3)} (N = 2), \\
\gamma_{gq}^{(3)} (N = 4) &= C_F^4 \left( \frac{-1438431824489 - 21061493 \zeta_3 + 259 \zeta_4 + 14408 \zeta_5}{17496000000 \cdot 101250} \right) \\
&+ C_A C_F^3 \left( \frac{270563159561 + 6105179 \zeta_3 + 5917 \zeta_4 - 17488 \zeta_5}{8748000000 \cdot 101250} \right) \\
&+ C_A^2 C_F^2 \left( \frac{1259255579057 + 16267093 \zeta_3 - 25621 \zeta_4 + 1484 \zeta_5}{4374000000 \cdot 67500} \right) \\
&+ C_A^3 C_F \left( \frac{-632341192829 \zeta_3 + 16048 \zeta_4 + 8782 \zeta_5}{21870000000 \cdot 8100} \right) \\
&+ \frac{d_R^{abcd} d_A^{abcd}}{n_c} \left( \frac{-12196 - 81008 \zeta_3 + 15976 \zeta_4 + 45 \zeta_5}{135 \cdot 225} \right) + n_f C_F^3 \left( \frac{-316818132031 - 32805000000}{32805000000} \right) \\
&+ \frac{411629 \zeta_3 - 4582 \zeta_4 + 352 \zeta_5}{16875 \cdot 225} + n_f C_A C_F^2 \left( \frac{569679966383 - 13919446 \zeta_3}{6561000000 \cdot 1350625} \right) \\
&+ \frac{12501 \zeta_4 - 176 \zeta_5}{125 \cdot 9} + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_c} \left( \frac{-112424 - 2336 \zeta_3 + 10624 \zeta_5}{675 \cdot 75} \right) \\
&+ n_f C_A^2 C_F \left( \frac{2203719743 + 2857549 \zeta_3 - 89599 \zeta_4 - 11872 \zeta_5}{52488000 \cdot 11250} \right) \\
&+ n_f^2 C_F^2 \left( \frac{9798304643 + 17096 \zeta_3 - 704 \zeta_4 + 45 \zeta_5}{32805000000 \cdot 675} \right) + n_f^2 C_A C_F \left( \frac{-1608863899 - 32805000000}{32805000000} \right) \\
&- \frac{39416 \zeta_3 + 704 \zeta_4}{2025 \cdot 45} + n_f^3 C_F \left( \frac{3990397 - 704 \zeta_3}{2733750 \cdot 405 \zeta_5} \right).
\end{align}
Finally the lowest three even moments \( \gamma \) of the four-loop gluon-gluon splitting function read

\[
\begin{align*}
\gamma_{\bar{g}g}^{(3)}(N=6) &= C_F^4 \left( -\frac{27548846012571077}{225136067625000} - \frac{28516720088}{121550625} \zeta_3 + \frac{6416}{105} \zeta_4 + \frac{260192}{735} \zeta_5 \right) \\
&+ C_A C_F^3 \left( \frac{15370144370986843}{900544270500000} + \frac{23472335174}{121550625} \zeta_3 - \frac{1023364}{25725} \zeta_4 - \frac{370016}{735} \zeta_5 \right) \\
&+ C_A^2 C_F^2 \left( \frac{58564721355491371}{720435416400000} + \frac{10781187328}{121550625} \zeta_3 - \frac{1215814}{25725} \zeta_4 + \frac{373832}{2205} \zeta_5 \right) \\
&+ C_A^3 C_F \left( -\frac{133292466369681947}{864522499680000} - \frac{226736591}{2701125} \zeta_3 + \frac{667258}{25725} \zeta_4 + \frac{67288}{6615} \zeta_5 \right) \\
&+ \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left( -\frac{23820479}{330750} - \frac{46510952}{165375} \zeta_3 + \frac{114496}{315} \xi_5 \right) \\
&+ n_f C_F^2 \left( -\frac{75665018489451691}{1350816405750000} + \frac{187225352}{24310125} \zeta_3 - \frac{36352}{2205} \zeta_4 + \frac{1408}{21} \xi_5 \right) \\
&+ n_f C_A C_F \left( \frac{331099053590779}{60036284700000} - \frac{3771301108}{24310125} \zeta_3 + \frac{4877248}{77175} \zeta_4 - \frac{704}{63} \xi_5 \right) \\
&+ n_f C_A^2 C_F \left( \frac{40511222207957}{3430644840000} + \frac{3610221368}{24310125} \zeta_3 - \frac{3604928}{77175} \zeta_4 - \frac{65344}{1323} \xi_5 \right) \\
&+ \frac{n_f d_R^{abcd} d_R^{abcd}}{n_c} \left( -\frac{2663932}{23625} - \frac{770944}{33075} \zeta_3 + \frac{68608}{441} \xi_5 \right) \\
&+ n_f^2 C_F^2 \left( \frac{275627366563631}{9648688612500} + \frac{266912}{19845} \zeta_3 - \frac{2816}{315} \zeta_4 \right) - n_f^2 C_A C_F \left( \frac{301286343367}{110270727000} - \frac{21623}{150} \zeta_3 - \frac{504}{5} \xi_5 \right) \\
&+ \frac{944432}{99225} \zeta_3 - \frac{2816}{315} \xi_4 \right) + n_f^2 C_F \left( \frac{3574461862}{3281866875} - \frac{2816}{2835} \zeta_3 \right). \quad (13)
\end{align*}
\]

Finally the lowest three even moments \( \gamma \) of the four-loop gluon-gluon splitting function read

\[
\gamma_{\bar{g}g}^{(3)}(N=2) = -\gamma_{\bar{g}g}^{(3)}(N=2), \quad (14)
\]

\[
\gamma_{\bar{g}g}^{(3)}(N=4) = C_A^4 \left( \frac{1502628149}{337500} + \frac{1146397}{1125} \zeta_3 - \frac{504}{5} \xi_5 \right) + \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left( \frac{160091}{675} + \frac{80072}{225} \zeta_3 - \frac{48016}{45} \xi_5 \right) \\
+ n_f C_A^3 \left( -\frac{20580892841}{72900000} - \frac{12550223}{22500} \zeta_3 + \frac{8613}{25} \zeta_4 + \frac{4316}{27} \xi_5 \right) \\
+ n_f C_A^2 C_F \left( \frac{4212122951}{41006250} + \frac{1170784}{5625} \zeta_3 - \frac{418198}{1125} \zeta_4 + \frac{17636}{45} \xi_5 \right) \\
+ n_f C_A C_F^2 \left( \frac{1913110089023}{26244000000} + \frac{3931783}{101250} \zeta_3 + \frac{26741}{750} \zeta_4 - \frac{3082}{5} \xi_5 \right) \\
+ n_f C_F^3 \left( \frac{34764568601}{2099520000} - \frac{958343}{40500} \zeta_3 - \frac{18997}{2250} \zeta_4 + \frac{908}{45} \xi_5 \right) \\
+ n_f^2 C_F^2 \left( \frac{3250393649}{218700000} + \frac{2969291}{20250} \zeta_3 - \frac{1566}{25} \zeta_4 - \frac{1276}{135} \xi_5 \right)
\]
For brevity, we here write down the results at $N = 6$ only numerically for the case of QCD:

\[
\gamma_{gs}^{(3)} (N=6) = C_A^4 \left( \frac{14796034088334539}{23053933324800} + \frac{198201877}{777924} \xi_3 - \frac{118210}{441} \xi_5 \right) + \frac{d_A^{abcd} d_A^{abcd}}{n_a} \left( \frac{1255552}{2205} + \frac{2997592}{1323} \xi_3 - \frac{472840}{147} \xi_5 \right) + n_f C_A^3 \left( \frac{51836938615212157}{360217708200000} + \frac{459844342}{972405} \xi_3 + \frac{1338986}{77175} \xi_4 - \frac{334352}{441} \xi_5 \right) + n_f C_A^2 C_F \left( \frac{10457671535671561}{540326562300000} - \frac{10055124}{8103375} \xi_3 - \frac{343156}{77175} \xi_4 + \frac{992}{147} \xi_5 \right) + n_f C_A^2 d_R^{abcd} d_R^{abcd} n_a \left( \frac{9661697}{22050} + \frac{22351528}{33075} \xi_3 - \frac{726848}{441} \xi_5 \right) + n_f^2 C_A^2 \left( \frac{2273514775943}{294055272000} + \frac{126516356}{694575} \xi_3 - \frac{18792}{245} \xi_4 - \frac{2200}{1323} \xi_5 \right) + n_f^2 C_A C_F \left( \frac{122395144706959}{2205414540000} - \frac{133661648}{694575} \xi_3 + \frac{173704}{2205} \xi_4 \right) + n_f^2 C_F^2 \left( \frac{61017705026527}{54032656230000} + \frac{2171164}{99225} \xi_3 - \frac{4576}{2205} \xi_4 \right) + n_f^2 d_R^{abcd} d_R^{abcd} n_a \left( \frac{788419}{3675} + \frac{180272}{441} \xi_3 - \frac{35200}{441} \xi_5 \right) + n_f^3 C_A \left( \frac{5226936307}{5250987000} + \frac{3224}{567} \xi_3 \right) + n_f^3 C_F \left( -\frac{9085701773}{30630757500} - \frac{1936}{6615} \xi_3 \right) .
\]

For brevity, we here write down the results at $N = 8$ only numerically for the case of QCD:

\[
\gamma_{gs}^{(3)} (N=8) = -24.014550n_f + 3.2351935n_f^2 - 0.0078892n_f^3 ,
\]

\[
\gamma_{gg}^{(3)} (N=8) = 294.58768n_f - 135.37676n_f^2 - 3.6097756n_f^3 ,
\]

\[
\gamma_{gg}^{(3)} (N=8) = -2803.6441 + 436.393057n_f + 1.8149462n_f^2 + 0.0735886n_f^3 ,
\]

\[
\gamma_{gg}^{(3)} (N=8) = 62279.744 - 17150.6967n_f + 785.88061n_f^2 + 1.8933103n_f^3 .
\]
In eqs. (5) – (16) $C_F$ and $C_A$ are the standard colour factors with $C_F = 4/3$ and $C_A = n_c = 3$ in QCD. The terms with the quartic group invariants $d_A^{abcd} d_A^{abcd}$, $d_R^{abcd} d_R^{abcd}$ and $d_R^{abcd} d_R^{abcd}$ agree with the results of ref. [11] where these particular contributions were obtained to much higher values of $N$ using OME calculations. All coefficients of the Riemann-$\xi$ value $\xi_4 = \pi^4/90$ agree with the all-$N$ predictions in eqs. (9) – (12) of ref. [12] based on the `no-$\pi^2$' conjecture of ref. [13]. The $n_f^3$ contributions to all four anomalous dimension are known for all $N$ [7], see also refs. [14].

The above results lead to the numerical expansions

\[
\begin{align*}
\gamma_{qq}(2,3) &= 0.282942 \alpha_s \left( 1 + 0.736828 \alpha_s + 0.517255 \alpha_s^2 + 0.756972 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(2,4) &= 0.282942 \alpha_s \left( 1 + 0.621883 \alpha_s + 0.146133 \alpha_s^2 + 0.362201 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(4,3) &= 0.555274 \alpha_s \left( 1 + 0.756202 \alpha_s + 0.672283 \alpha_s^2 + 0.701628 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(4,4) &= 0.555274 \alpha_s \left( 1 + 0.680253 \alpha_s + 0.427783 \alpha_s^2 + 0.345861 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(6,3) &= 0.716450 \alpha_s \left( 1 + 0.725387 \alpha_s + 0.685289 \alpha_s^2 + 0.663440 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(6,4) &= 0.716450 \alpha_s \left( 1 + 0.648931 \alpha_s + 0.426442 \alpha_s^2 + 0.324781 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(8,3) &= 0.832237 \alpha_s \left( 1 + 0.710075 \alpha_s + 0.650750 \alpha_s^2 + 0.643336 \alpha_s^3 + \ldots \right), \\
\gamma_{qq}(8,4) &= 0.832237 \alpha_s \left( 1 + 0.632824 \alpha_s + 0.423498 \alpha_s^2 + 0.312139 \alpha_s^3 + \ldots \right)
\end{align*}
\]

and

\[
\begin{align*}
\gamma_{gq}(2,3) &= -0.159155 \alpha_s \left( 1 + 0.900404 \alpha_s + 0.012215 \alpha_s^2 - 0.055970 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(2,4) &= -0.212207 \alpha_s \left( 1 + 0.900404 \alpha_s - 0.102840 \alpha_s^2 - 0.236731 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(4,3) &= -0.087535 \alpha_s \left( 1 - 0.280121 \alpha_s - 0.893969 \alpha_s^2 - 0.022754 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(4,4) &= -0.116714 \alpha_s \left( 1 - 0.280121 \alpha_s - 0.998634 \alpha_s^2 + 0.129659 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(6,3) &= -0.062525 \alpha_s \left( 1 - 0.838938 \alpha_s - 1.064575 \alpha_s^2 + 0.145572 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(6,4) &= -0.083367 \alpha_s \left( 1 - 0.838938 \alpha_s - 1.150113 \alpha_s^2 + 0.441744 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(8,3) &= -0.049728 \alpha_s \left( 1 - 1.255845 \alpha_s - 1.091729 \alpha_s^2 + 0.353099 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(8,4) &= -0.065430 \alpha_s \left( 1 - 1.255845 \alpha_s - 1.160288 \alpha_s^2 + 0.746929 \alpha_s^3 + \ldots \right)
\end{align*}
\]

for the upper row of the anomalous-dimension matrix, where the arguments of $\gamma_{ik}$ are $N$ and $n_f$; the values for $n_f = 5$ have been suppressed for brevity. The independent lower-row expansions – the values at $N = 2$ are fixed by the momentum sum-rule relations [11] and [14] – are given by

\[
\begin{align*}
\gamma_{gq}(4,3) &= -0.077809 \alpha_s \left( 1 + 1.165483 \alpha_s + 1.163066 \alpha_s^2 + 1.474368 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(4,4) &= -0.077809 \alpha_s \left( 1 + 1.115164 \alpha_s + 0.823447 \alpha_s^2 + 0.883269 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(6,3) &= -0.044462 \alpha_s \left( 1 + 1.314556 \alpha_s + 1.360970 \alpha_s^2 + 1.726679 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(6,4) &= -0.044462 \alpha_s \left( 1 + 1.301901 \alpha_s + 1.051619 \alpha_s^2 + 1.126955 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(8,3) &= -0.031157 \alpha_s \left( 1 + 1.416509 \alpha_s + 1.468523 \alpha_s^2 + 1.899893 \alpha_s^3 + \ldots \right), \\
\gamma_{gq}(8,4) &= -0.031157 \alpha_s \left( 1 + 1.430863 \alpha_s + 1.183046 \alpha_s^2 + 1.318370 \alpha_s^3 + \ldots \right)
\end{align*}
\]
\[ \gamma_{gg}(4,3) = 1.161831 \, \alpha_s \left( 1 + 0.475446 \alpha_s + 0.333272 \alpha_s^2 + 0.478025 \alpha_s^3 + \ldots \right), \]
\[ \gamma_{gg}(4,4) = 1.214882 \, \alpha_s \left( 1 + 0.383536 \alpha_s + 0.121966 \alpha_s^2 + 0.240469 \alpha_s^3 + \ldots \right), \]
\[ \gamma_{gg}(6,3) = 1.574497 \, \alpha_s \left( 1 + 0.489287 \alpha_s + 0.380902 \alpha_s^2 + 0.429696 \alpha_s^3 + \ldots \right), \]
\[ \gamma_{gg}(6,4) = 1.627549 \, \alpha_s \left( 1 + 0.393705 \alpha_s + 0.169676 \alpha_s^2 + 0.190156 \alpha_s^3 + \ldots \right), \]
\[ \gamma_{gg}(8,3) = 1.851503 \, \alpha_s \left( 1 + 0.497734 \alpha_s + 0.404644 \alpha_s^2 + 0.398779 \alpha_s^3 + \ldots \right), \]
\[ \gamma_{gg}(8,4) = 1.904554 \, \alpha_s \left( 1 + 0.401746 \alpha_s + 0.194306 \alpha_s^2 + 0.157133 \alpha_s^3 + \ldots \right). \] (32)

Except for \( \gamma_{gg} \) and \( \gamma_{gg} \) at \( N = 8 \) these numerical expansions have been presented before in ref. [15].

The results for the \( qq \) and \( gg \) cases at asymptotically (and unphysically) large values of \( N \) read
\[ \gamma_{qq}(N,3) = a_s \, \gamma_{qq}^{(0)}(N,3) \left( 1 + 0.726574 \alpha_s + 0.734054 \alpha_s^2 + 0.664730 \alpha_s^3 \right), \]
\[ \gamma_{qq}(N,4) = a_s \, \gamma_{qq}^{(0)}(N,4) \left( 1 + 0.638154 \alpha_s + 0.509978 \alpha_s^2 + 0.316848 \alpha_s^3 \right) \] (35)

and
\[ \gamma_{gg}(N,3) = a_s \, \gamma_{gg}^{(0)}(N,3) \left( 1 + 0.726574 \alpha_s + 0.734054 \alpha_s^2 + 0.415609 \alpha_s^3 \right), \]
\[ \gamma_{gg}(N,4) = a_s \, \gamma_{gg}^{(0)}(N,4) \left( 1 + 0.638154 \alpha_s + 0.509978 \alpha_s^2 + 0.064476 \alpha_s^3 \right) \] (36)

due to their relation to the cusp anomalous dimensions \( A_k \). The quark and gluon results are identical up to the ‘Casimir scaling’ of the prefactors, \( \gamma_{qq}^{(0)}(N,n_f) = 4C_F \) and \( \gamma_{gg}^{(0)}(N,n_f) = 4C_A \), to three loops and are related by a generalized (not numerical, except in the large-\( n_c \) limit, due to the presence of the quartic group invariants) Casimir scaling [11,16] at four loops.

The relative size of the \( N^2 \)LO and \( N^3 \)LO contributions in eqs. (21) – (36) is illustrated in fig. 1 for \( n_f = 4 \) at \( \alpha_s = 0.2 \): The \( N^3 \)LO corrections amount to less than 1%, and less than 0.5% of the NLO results except for \( P_{gg} \), the quantity with the lowest leading-order values, at \( N \geq 4 \). Unlike in the quark case, see also ref. [17] where also a first estimate of the five-loop contribution to \( A_q \) has been obtained, the \( N^2 \)LO and \( N^3 \)LO large-\( N \) limits in the gluon case do not, in general, roughly agree with values in the range \( 4 \leq N \leq 8 \) normalized as in eqs. (31) – (34).

The resulting low-\( N \) expansion for the evolution (2) of the singlet quark and gluon PDFs is illustrated in fig. 2 for the schematic but sufficiently realistic order-independent model input [4]
\[ xq_s(x,\mu_0^2) = 0.6 \, x^{-0.3} (1-x)^{3.5} \left( 1 + 5.0 \, x^{0.8} \right), \]
\[ xg(x,\mu_0^2) = 1.6 \, x^{-0.3} (1-x)^{4.5} \left( 1 - 0.6 \, x^{0.3} \right) \] (37)

with \( \alpha_s(\mu_0^2) = 0.2 \) and \( n_f = 4 \). The \( N^3 \)LO corrections are very small at the standard choice \( \mu_r = \mu_f = \mu_0 \) of the renormalization scale. They lead to a reduction of the scale dependence to about 1% (full width) at \( N \geq 4 \) for the conventional range \( \frac{1}{2} \mu_r^2 \leq \mu_f^2 \leq 4 \mu_r^2 \).

To summarize, we have employed the theoretical framework of refs. [1-4] together with an optimized in-house version of the FORM [18] program FORCER for 4-loop propagator integrals [19] to compute the moments \( N = 2, 4, 6, \) and 8 of all \( N^3 \)LO flavour-singlet splitting functions. The numerical effect of these contributions is small, but more work is needed to arrive at sufficient ‘data’ for a \( N^3 \)LO analogue of the earlier approximate \( N^2 \)LO splitting functions of ref. [20].
Figure 1: Moments of the splitting functions (2) at NNLO (lines) and $N^3$LO (even-$N$ points) at $\alpha_s = 0.2$ and $n_f = 4$, normalized to the NLO results. Also shown are the $qq$ and $gg$ large-$N$ limits.

Figure 2: The dependence of the logarithmic factorization-scale derivatives of the singlet PDFs on the renormalization scale $\mu_r$ at $N = 2$ (where the very small scaling violations of $q_s$ and $g$ are related by the momentum sum rule), $N = 4$ and $N = 6$ for the initial distributions (37).
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