NOTE

Time step size limitation introduced by the BSSN Gamma Driver

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Abstract
Many mesh refinement simulations currently performed in numerical relativity counteract instabilities near the outer boundary of the simulation domain either by changes to the mesh refinement scheme or by changes to the gauge condition. We point out that the BSSN Gamma Driver gauge condition introduces a time step size limitation in a similar manner as a Courant–Friedrichs–Lewy condition, but which is independent of the spatial resolution. We give a didactic explanation of this issue, show why, especially, mesh refinement simulations suffer from it, and point to a simple remedy.

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1. Introduction

We assume that the reader is familiar with the concept of a Courant–Friedrichs–Lewy (CFL) condition [1]. Loosely speaking, the CFL condition states that when a partial differential equation, for example the wave equation

\[ \partial_t^2 u = c^2 \Delta u, \]

is integrated numerically, the time step size \( \delta t \) is limited by the spatial resolution \( \delta x \) and the maximum propagation speed \( c \) by

\[ \delta t < Q\frac{\delta x}{c}. \]

Here \( Q \) is a constant of order 1 that depends on the time integration method (and details of the spatial discretization). Choosing a time step size larger than this is unstable and must therefore be avoided. (There are time integration methods that do not have such a stability limit, but these are expensive and not commonly used in numerical relativity, so we will ignore them here.)

1 http://www.cct.lsu.edu/~eschnett/
2. Example: exponential decay

In real-world equations, there are also other restrictions which limit the time step size, and which may be independent of the spatial resolution. One simple example for this is the exponential decay

\[ \partial_t u = -\lambda u, \]  

(3)

where \( \lambda > 0 \) is the decay constant. Note that this equation is an ordinary differential equation, as there are no spatial derivatives. The solutions of (3) are given by

\[ u(t) = A \exp\{-\lambda t\} \]  

(4)

with an amplitude \( A \).

The decay constant \( \lambda \) has the dimension \( 1/T \). The time step size is limited by

\[ \delta t < \frac{1}{\lambda} \]  

(5)

where \( Q' \) is a constant of order 1 that depends on the time integration method. Choosing a time step size larger than this is unstable and must therefore be avoided. (As with the CFL criterion, there are time integration methods that do not have such a stability limit.)

As an example, let us consider the forward Euler scheme with a step size \( \delta t \). This leads to the discrete time evolution equation

\[ \frac{u^{n+1} - u^n}{\delta t} = -\lambda u^n \]  

(6)

or

\[ u^{n+1} = (1 - \delta t \lambda) u^n. \]  

(7)

This system is unstable e.g. if \( |u^{n+1}| > |u^n| \) (there are also other definitions of stability), or if

\[ |1 - \delta t \lambda| > 1, \]  

(8)

which is the case for \( \delta t > 2/\lambda \) (and also for \( \delta t < 0 \)). In this case, the solution oscillates between positive and negative values with an exponentially growing amplitude.

3. Gamma Driver

The BSSN [2] Gamma Driver condition is a time evolution equation for the shift vector \( \beta^i \), given by (see e.g. (43) in [3])

\[ \partial_t^2 \beta^i = F \partial_i \tilde{\Gamma}^i - \eta \partial_i \beta^i. \]  

(9)

There exist variations of the Gamma Driver condition, but the fundamental form of the equation remains the same. The term \( F \partial_i \tilde{\Gamma}^i \) contains second spatial derivatives of the shift \( \beta^i \) and renders this a hyperbolic, wave-type equation for the shift. The parameter \( \eta > 0 \) is a damping parameter, very similar to \( \lambda \) in (3) above. It drives \( \partial_t \beta^i \) to zero, so that the shift \( \beta^i \) will tend to a constant in stationary spacetimes. (This makes it a symmetry-seeking gauge condition, since \( \partial_t \) will then tend to the corresponding Killing vector.)

Let us now consider a simple spacetime which is spatially homogeneous, i.e. where all spatial derivatives vanish. In this case (see e.g. (40) in [3]), \( \partial_i \tilde{\Gamma}^i = 0 \), and only the damped oscillator equation

\[ \partial_t^2 \beta^i = -\eta \partial_i \beta^i \]  

(10)

remains. As we have seen above, solving this equation numerically still imposes a time step size limit, even though there is no length scale introduced by the spatial discretization.
so the spatial resolution can be chosen to be arbitrarily large; there is therefore no CFL limit. This demonstrates that the damping time scale set by the parameter $\eta$ introduces a resolution-independent time step size limit.

This instability was e.g. reported in [4], below (13) there, without explaining its cause. The authors state that the choice $\eta = 2$ is unstable near the outer boundary, and they therefore choose $\eta = 1$ instead. Decreasing $\eta$ by a factor of 2 increases the time step size limit correspondingly.

The explanation presented above was first brought forth by Gundlach [5] and Hawke [6]. To our knowledge, it has not yet been discussed in the literature elsewhere.

Harmonic formulations of the Einstein equations have driver parameters similar to the BSSN Gamma Driver parameter $\eta$. Spatially varying parameters were introduced in harmonic formulations to simplify the gauge dynamics in the wave extraction zone far away from the origin (see e.g. (8) in [7]). Reference [8] describes a harmonic formulation with mesh refinement, and describes using this spatial dependence also to avoid time stepping instabilities (see (45) there).

4. Mesh refinement

When using mesh refinement to study compact objects, such as black holes, neutron stars or binary systems of these, one generally uses a grid structure that has a fine resolution near the centre and successively coarser resolutions further away from the centre. With full Berger–Oliger AMR that uses sub-cycling in time, the CFL factors on all refinement levels are the same, and thus the time step sizes increase as one moves away from the centre. This makes it possible that the time step sizes on the coarsest grids no longer satisfy the stability condition for the Gamma Driver damping parameter $\eta$.

One solution to this problem is to omit sub-cycling in time for the coarsest grids by choosing the same time step size for some of the coarsest grids. This was first advocated by [9], although it was introduced there to allow large shift vectors near the outer boundary as necessary for a co-rotating coordinate system. It was later used in [10] (see section IV there) to avoid an instability near the outer boundary, although the instability there is not attributed to the Gamma Driver. Omitting sub-cycling in time on the coarsest grids often increases the computational cost only marginally, since most of the computation time is spent on the finest levels.

Another solution is to choose a spatially varying parameter $\eta$, e.g., based on the coordinate radius and mimicking the temporal resolution of the grid structure, which may grow linearly with the radius. This follows the interpretation of $\eta$ setting the damping timescale, which must not be larger than the timescale set by the time discretization.

One possible spatially varying definition for $\eta$ could be

$$\eta(r) := \eta^* \frac{R^2}{r^2 + R^2}, \quad (11)$$

where $r$ is the coordinate distance from the centre of the black hole. The parameter $R$ defines a transition radius between an inner region, where $\eta$ is approximately equal to $\eta^*$, and an outer region, where $\eta$ gradually decreases to zero. This definition is simple, smooth and differentiable, and mimics a ‘typical’ mesh refinement setup, where the resolution $h$ grows approximately linearly with the radius $r$.

Another, simpler definition for $\eta$ (which is not smooth—but smoothness is not necessary; $\eta$ could even be discontinuous) is
\[ \eta(r) := \eta^{*} \begin{cases} \frac{1}{r} & \text{for } r \leq R \text{ (near the origin)} \\ \frac{1}{r} & \text{for } r \geq R \text{ (far away)} \end{cases}, \tag{12} \]

which is e.g. implemented in the McLachlan code [11].

If there are multiple black holes, possibly with differing resolution requirements, then prescriptions such as (11) or (12) need to be suitably generalized, e.g., via

\[ \frac{1}{\eta(r)} := \frac{1}{\eta_1(r_1)} + \frac{1}{\eta_2(r_2)}, \tag{13} \]

where \(\eta_1\) and \(\eta_2\) are the contributions from the individual black holes, with \(r_1\) and \(r_2\) being the distances to their centres. This form of (13) is motivated by the dimension of \(\eta\), which is \(1/M\), so that two superposed black holes of masses \(m_1\) and \(m_2\) lead to the same definition of \(\eta\) as a single black hole with mass \(m_1 + m_2\).

Another prescription for a spatially varying \(\eta\) has been suggested in [12]. In this prescription, \(\eta\) depends on the determinant of the three-metric, and it thus takes the masses of the black hole(s) automatically into account. This prescription is motivated by binary systems of black holes with unequal masses, where \(\eta\) near the individual black holes should be adapted to the individual black holes’ masses, and it may be more suitable to use this instead of (13).

There can be other limitations of the time step size near the outer boundary, coming e.g. from the boundary condition itself. In particular, radiative boundary conditions impose a CFL limit that may be stricter than the CFL condition from the time evolution equations in the interior.

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