GENERALIZATION OF A THEOREM OF CLUNIE AND HAYMAN

MATTHEW BARRETT AND ALEXANDRE EREMENKO

(Communicated by Mario Bonk)

ABSTRACT. Clunie and Hayman proved that if the spherical derivative \( \| f' \| \) of an entire function satisfies \( \| f' \|(z) = O(|z|^\sigma) \), then \( T(r, f) = O(r^{\sigma + 1}) \).

We generalize this to holomorphic curves in projective space of dimension \( n \) omitting \( n \) hyperplanes in general position.

INTRODUCTION

We consider holomorphic curves \( f : \mathbb{C} \to \mathbb{P}^n \); for the general background on the subject we refer to [7]. The Fubini–Study derivative \( \| f' \| \) measures the length distortion from the Euclidean metric in \( \mathbb{C} \) to the Fubini–Study metric in \( \mathbb{P}^n \). The explicit expression is

\[
\| f' \|^2 = \| f \|^{-4} \sum_{i<j} |f_i' f_j - f_i f_j'|^2,
\]

where \( (f_0, \ldots, f_n) \) is a homogeneous representation of \( f \) (that is, the \( f_j \) are entire functions which never simultaneously vanish), and

\[
\| f \|^2 = \sum_{j=0}^n |f_j|^2.
\]

See [3] for a general discussion of the Fubini–Study derivative.

We recall that the Nevanlinna–Cartan characteristic is defined by

\[
T(r, f) = \int_0^r \frac{dt}{t} \left( \frac{1}{\pi} \int_{|z| \leq t} \| f' \|^2(z) dm(z) \right),
\]

where \( dm \) is the area element in \( \mathbb{C} \). So the condition

\[
\limsup_{z \to \infty} |z|^{-\sigma} \| f'(z) \| \leq K < \infty
\]

implies

\[
\limsup_{r \to \infty} \frac{T(r, f)}{r^{2\sigma + 2}} < \infty.
\]
Clunie and Hayman [4] found that for curves $C \rightarrow P^1$ omitting one point in $P^1$, a stronger conclusion follows from (1), namely

$$\limsup_{r \to \infty} \frac{T(r,f)}{r^{\sigma+1}} \leq KC(\sigma).$$

In the most important case of $\sigma = 0$, a different proof of this fact for $n = 1$ is due to Pommerenke [8]. Pommerenke’s method gives the exact constant $C(0)$. In this paper we prove that this phenomenon persists in all dimensions.

**Theorem.** For holomorphic curves $f : C \rightarrow P^n$ omitting $n$ hyperplanes in general position, condition (1) implies (3) with an explicit constant $C(n,\sigma)$.

In [6], the case $\sigma = 0$ was considered. There it was proved that holomorphic curves in $P^n$ with bounded spherical derivative and omitting $n$ hyperplanes in general position must satisfy $T(r,f) = O(r)$. With a stronger assumption that $f$ omits $n + 1$ hyperplanes this was earlier established by Berteloot and Duval [2] and by Tsukamoto [9]. The proof in [6] has two drawbacks: it does not extend to arbitrary $\sigma \geq 0$, and it is non-constructive; unlike Clunie–Hayman and Pommerenke’s proofs mentioned above, it does not give an explicit constant in (3).

It is shown in [6] that the condition that $n$ hyperplanes are omitted is exact: there are curves in any dimension $n$ satisfying (1), $T(r,f) \sim c r^{2\sigma+2}$ and omitting $n - 1$ hyperplanes.

**Preliminaries**

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations $\{w_j = 0\}$, $1 \leq j \leq n$. We fix a homogeneous representation $(f_0, \ldots, f_n)$ of our curve, where $f_j$ are entire functions and $f_n = 1$. Then

$$u = \log \sqrt{|f_0|^2 + \ldots + |f_n|^2}$$

is a positive subharmonic function, and Jensen’s formula gives

$$T(r,f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta})d\theta - u(0) = \int_0^r \frac{n(t)}{t} dt,$$

where $n(t) = \mu(\{z : |z| \leq t\})$ and $\mu = \mu_u$ is the Riesz measure of $u$, that is, the measure with the density

$$\frac{1}{2\pi} \Delta u = \frac{1}{\pi} \|f^\prime\|^2.
$$

This measure $\mu$ is also called Cartan’s measure of $f$. Positivity of $u$ and (2) imply that all $f_j$ are of order at most $2\sigma + 2$, normal type. As $f_j(z) \neq 0$, $1 \leq j \leq n$, we conclude that

$$f_j = e^{P_j}, \quad 1 \leq j \leq n,$$

where

$$P_j$$

are polynomials of degree at most $2\sigma + 2$.

We need two lemmas from potential theory.

**Lemma 1** ([6]). Let $v$ be a non-negative harmonic function in the closure of the disc $B(a,R)$, and assume that $v(z_1) = 0$ for some point $z_1 \in \partial B(a,R)$. Then

$$v(a) \leq 2R|\nabla v(z_1)|.$$
We include a proof, suggested by the referee, which is simpler than that given in [6]. Without loss of generality, assume that $a = 0, R = 1, z_1 = 1$. Then Harnack’s inequality gives
\[
\frac{v(0)}{1 + r} \leq \frac{v(r)}{1 - r} = \frac{v(r) - v(1)}{1 - r}.
\]
Passing to the limit as $r \to 1$, we obtain the result.

**Lemma 2.** Let $v$ be a non-negative superharmonic function in the closure of the disc $B(a, R)$ and suppose that $v(z_1) = 0$ for some $z_1 \in \partial B(a, R)$. Then
\[
|\mu_{v}(B(a, R/2))| \leq 3R \left| \frac{\partial v}{\partial n}(z_1) \right|.
\]

By $|\partial v/\partial n|$ we mean here $\liminf |v(rz_1)|/(R(1 - r))$ as $r \to 1-$. 

**Proof.** The function $v(a + Rz)$ satisfies the conditions of the lemma with $R = 1$. So it is enough to prove the lemma with $a = 0$ and $R = 1$. Let
\[
w(z) = \int_{|\zeta| \leq 1/2} G(z, \zeta) d\mu_{v}(\zeta)
\]
be the Green potential of the restriction of $\mu_{v}$ onto the disc $|\zeta| \leq 1/2$, that is,
\[
G(z, \zeta) = \log \left| \frac{1 - \overline{\zeta} z}{z - \zeta} \right|.
\]
Then $w \leq v$ and $w(z_1) = v(z_1) = 0$, which implies that
\[
\left| \frac{\partial v}{\partial n}(z_1) \right| \geq \left| \frac{\partial w}{\partial |z|}(z_1) \right|.
\]
Minimizing $|\partial G/\partial |z||$ over $|z| = 1$ and $|\zeta| = 1/2$, we obtain $1/3$, which proves the lemma. \hfill \Box

**Proof of the theorem**

We may assume without loss of generality that $f_0$ has infinitely many zeros. Indeed, we can compose $f$ with an automorphism of $\mathbb{P}^n$; for example replace $f_0$ by $f_0 + cf_1$, $c \in \mathbb{C}$, and leave all other $f_j$ unchanged. This transformation changes neither the $n$ omitted hyperplanes nor the rate of growth of $T(r, f)$ and multiplies the spherical derivative by a bounded factor.

Let $u_j = \log |f_j|$ and
\[
u^* = \max_{1 \leq j \leq n} u_j.
\]
Here and in what follows $\max$ denotes the pointwise maximum of subharmonic functions.

**Proposition 1.** Suppose that at some point $z_1$ we have
\[
u_m(z_1) = u_k(z_1) \geq u_j(z_1)
\]
for some $m \neq k$ and all $j$ where $m, k, j \in \{0, \ldots, n\}$. Then
\[
\|f^\prime(z_1)\| \geq (n + 1)^{-1} |\nabla \nu_m(z_1) - \nabla \nu_k(z_1)|.
\]
Proof. \[ \| f'(z_1) \| \geq \frac{|f'_m(z_1)f_k(z_1) - f_m(z_1)f'_k(z_1)|}{|f_0(z_1)|^2 + \ldots + |f_n(z_1)|^2} \geq (n + 1)^{-1} \frac{|f'_m(z_1)|}{|f_m(z_1)|} \cdot \]

and the conclusion of the proposition follows since \( |\nabla \log |f|| = |f'/f| \).

Proposition 2. For every \( \epsilon > 0 \), we have

\[ u(z) \leq u^*(z) + K(2 + \epsilon)^{\sigma + 1}(n + 1)|z|^\sigma + 1 \]

for all \( |z| > r_0(\epsilon) \).

Proof. If \( u_0(z) \leq u^*(z) \) for all sufficiently large \( |z| \), then there is nothing to prove. Suppose that \( u_0(a) > u^*(a) \), and consider the largest disc \( B(a, R) \) centered at \( a \) where the inequality \( u_0(z) > u^*(z) \) persists. If \( z_0 \) is the zero of the smallest modulus of \( f_0 \), then \( R \leq |a| + |z_0| < (1 + \epsilon)|a| \) when \( |a| \) is large enough.

There is a point \( z_1 \in \partial B(a, R) \) such that \( u_0(z_1) = u^*(z_1) \). This means that there is some \( k \in \{1, \ldots, n\} \) such that \( u_0(z_1) = u_k(z_1) \geq u_m(z_1) \) for all \( m \in \{1, \ldots, n\} \). Applying Proposition 1 we obtain

\[ \| \nabla u_k(z_1) - \nabla u_0(z_1) \| \leq (n + 1)||f'(z_1)||. \]

Now \( u_0(z) > u^*(z) \) for \( z \in B(a, R) \), so we can apply Lemma 1 to \( v = u_0 - u_k \) in the disc \( B(a, R) \). This gives

\[ u_0(a) - u_k(a) \leq 2R|\nabla u_k(z_1) - \nabla u_0(z_1)| \leq 2R(n + 1)|f'(z_1)|. \]

Now \( R < (1 + \epsilon)|a| \) and \( |z_1| \leq (2 + \epsilon)|a| \), so

\[ u_0(a) \leq u^*(a) + K(2 + \epsilon)^{\sigma + 1}(n + 1)|a|^\sigma + 1, \]

and the result follows because \( u = \max\{u_0, u^*\} + O(1) \).

Next we study the Riesz measure of the subharmonic function

\[ u^* = \max\{u_1, \ldots, u_n\}. \]

We begin with the maximum of two harmonic functions. Let \( u_1 \) and \( u_2 \) be two harmonic functions in \( \mathbb{C} \) of the form \( u_j = \Re P_j \) where \( P_j \neq 0 \) are polynomials. Suppose that \( u_1 \neq u_2 \). Then the set \( E = \{z \in \mathbb{C} : u_1(z) = u_2(z)\} \) is a proper real-algebraic subset of \( \mathbb{C} \) without isolated points. Apart from a finite set of ramification points, \( E \) consists of smooth curves. For every smooth point \( z \in E \), we denote by \( J(z) \) the jump of the normal (to \( E \)) derivative of the function \( w = \max\{u_1, u_2\} \) at the point \( z \). This jump is always positive and the Riesz measure \( \mu_w \) is given by the formula

\[ (7) \quad d\mu_w = \frac{J(z)}{2\pi}|dz|, \]

which means that \( \mu_w \) is supported by \( E \) and has a density \( J(z)/2\pi \) with respect to the length element \( |dz| \) on \( E \).

Now let \( E_{i,j} = \{z : u_i(z) = u_j(z) \geq u_k(z), 1 \leq k \leq n\} \), and let \( E = \bigcup E_{i,j} \) where the union is taken over all pairs \( 1 \leq i, j \leq n \) for which \( u_i \neq u_j \). Then \( E \) is a proper real semi-algebraic subset of \( \mathbb{C} \) and \( \infty \) is not an isolated point of \( E \). For the elementary properties of semi-algebraic sets that we use here, see, for example,
There exists $r_0 > 0$ such that $\Gamma = E \cap \{r_0 < |z| < \infty\}$ is a union of finitely many disjoint smooth simple curves,

$$\Gamma = \bigcup_{k=1}^{m} \Gamma_k.$$ 

This union coincides with the support of $\mu_{u^*}$ in $\{z : r_0 < |z| < \infty\}$.

Consider a point $z_0 \in \Gamma$. Then $z_0 \in \Gamma_k$ for some $k$. As $\Gamma_k$ is a smooth curve, there is a neighborhood $D$ of $z_0$ which does not contain other curves $\Gamma_j$, $j \neq k$, and which is divided by $\Gamma_k$ into two parts, $D_1$ and $D_2$. Then there exist $i$ and $j$ such that $u^*(z) = u_i(z)$, $z \in D_1$ and $u^*(z) = u_j(z)$, $z \in D_2$, and $u^*(z) = \max\{u_i(z), u_j(z)\}$, $z \in D$. So the restriction of the Riesz measure $\mu_{u^*}$ on $D$ is supported by $\Gamma_k \cap D$ and has density $J(z)/(2\pi)$ where

$$|J(z)| = |\partial u_i/\partial n - \partial u_j/\partial n|(z) = |\nabla(u_i - u_j)|(z)$$

and $\partial/\partial n$ is the derivation in the direction of a normal to $\Gamma_k$. Taking into account that $u_j = \text{Re}P_j$ where $P_j$ are polynomials, we conclude that there exist positive numbers $c_k$ and $b_k$ such that

$$(8) \quad J(z)/(2\pi) = (c_k + o(1))|z|^b, \quad z \to \infty, \quad z \in \Gamma_k.$$ 

Let $b = \max_k b_k$, and among those curves $\Gamma_k$ for which $b_k = b$ choose one with maximal $c_k$ (which we denote by $c_0$). We denote this chosen curve by $\Gamma_0$ and fix it for the rest of the proof.

**Proposition 3.** We have

$$b \leq \sigma \quad \text{and} \quad c_0 \leq 3 \cdot 4^\sigma K(n + 1).$$

**Proof.** We consider two cases.

**Case 1.** There is a sequence $z_n \to \infty$, $z_n \in \Gamma_0$, such that $u_0(z_n) \leq u^*(z_n)$. Then (1) and Proposition 1 imply that

$$J(z_n) \leq (n + 1)K|z_n|^\sigma,$$

and comparison with (8) shows that $b \leq \sigma$ and $c_0 \leq K(n + 1)/(2\pi)$.

**Case 2.** $u_0(z) > u^*(z)$ for all sufficiently large $z \in \Gamma_0$. Let $a$ be a point on $\Gamma_0$, $|a| > 3r_0$, and $u_0(a) > u^*(a)$. Let $B(a, R)$ be the largest open disc centered at $a$ in which the inequality $u_0(z) > u^*(z)$ holds. Then

$$(9) \quad R \leq |a| + O(1), \quad a \to \infty,$$

because we assume that $f_0$ has zeros, so $u_0(z_0) = -\infty$ for some $z_0$.

In $B(a, R)$ we consider the positive superharmonic function $v = u_0 - u^*$. Let us check that it satisfies the conditions of Lemma 2. The existence of a point $z_1 \in \partial B(a, R)$ with $v(z_1) = 0$ follows from the definition of $B(a, R)$. The Riesz measure of $\mu_v$ is estimated using (7), (8):

$$|\mu_v(B(a, R/2))| \geq |\mu_v(\Gamma_0 \cap B(a, R/2))| \geq c_0 R(|a| - R/2)^b.$$ 

Now Lemma 2 applied to $v$ in $B(a, R)$ implies that

$$(10) \quad |\nabla v(z_1)| \geq (c_0/3)(|a| - R/2)^b.$$ 

On the other hand (1) and Proposition 1 imply that

$$|\nabla v(z_1)| \leq K(n + 1)(|a| + R)^\sigma.$$
Combining these two inequalities and taking \( \sigma \) into account, we obtain \( b \leq \sigma \) and \( c_0 \leq 3 \cdot 4^\sigma K (n + 1) \), as required. \( \square \)

We denote
\[
T^*(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^*(re^{i\theta})d\theta - u^*(0).
\]
This is the characteristic of the “reduced curve” \( (f_1, \ldots, f_n) \).

**Proposition 4.**

\[
T^*(r) \leq 6 \cdot 4^\sigma K n(n + 1)^2 \sigma + 1 r^{\sigma + 1}.
\]

**Proof.** By Jensen’s formula,
\[
T^*(r) = \int_0^r \nu(t) \frac{dt}{t},
\]
where \( \nu(t) = \mu_{u^*}(\{z : |z| \leq t\}) \). The number of curves \( \Gamma_k \) supporting the Riesz measure of \( u^* \) is easily seen to be at most \( 2n(n - 1)(\sigma + 1) \). The density of the Riesz measure \( \mu_{u^*} \) on each curve \( \Gamma_k \) is given by \( \Box \), where \( c_k \leq c_0 \) and \( b_k \leq b \) and the parameters \( c_0 \) and \( b \) are estimated in Proposition 3. Combining all these data, we obtain the result. \( \square \)

It remains to combine Propositions 2 and 4 to obtain the final result.

**Acknowledgment**

The authors thank the referee for many valuable remarks and suggestions.

**References**

[1] R. Benedetti and J. Risler, Real algebraic and semi-algebraic sets, Hermann, Paris, 1990. MR1070358 (91j:14045)

[2] F. Berteloot and J. Duval, Sur l’hyperbolicit´e de certains compl´ementaires, Enseign. Math. (2) 47 (2001), no. 3-4, 253–267. MR1876928 (2002m:32042)

[3] W. Cherry and A. Eremenko, Landau’s theorem for holomorphic curves in projective space and the Kobayashi metric on hyperplane complement, Pure and Appl. Math. Quarterly 7 (2011), no. 1, 199–221.

[4] J. Clunie and W. Hayman, The spherical derivative of integral and meromorphic functions, Comment. Math. Helv. 40 (1966) 117–148. MR0192055 (33:282)

[5] M. Coste, An introduction to semialgebraic geometry, Inst. editoriali e poligrafici internazionali, Pisa, 2000.

[6] A. Eremenko, Brody curves omitting hyperplanes, Ann. Acad. Sci. Fenn. 35 (2010) 565–570. MR2731707

[7] S. Lang, Introduction to complex hyperbolic spaces, Springer-Verlag, New York, 1987. MR0886677 (88f:32065)

[8] Ch. Pommerenke, Estimates for normal meromorphic functions, Ann. Acad. Sci. Fenn. Ser. A I 476 (1970). MR0285710 (44:2928)

[9] M. Tsukamoto, On holomorphic curves in algebraic torus, J. Math. Kyoto Univ. 47 (2007), no. 4, 881–892. MR2413072 (2009a:32023)

**Department of Mathematics, Purdue University, West Lafayette, Indiana 47907**

**E-mail address:** eremenko@math.purdue.edu