World-volume Solitons of the D3-brane in the Background of \((p, q)\) Five-branes

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ABSTRACT

We analyze the world-volume solitons of a D3-brane probe in the background of parallel \((p, q)\) five-branes. The D3-brane is embedded along the directions transverse to the five-branes of the background. By using the S-duality invariance of the D3-brane, we find a first-order differential equation whose solutions saturate an energy bound. The \(SO(3)\) invariant solutions of this equation are found analytically. They represent world-volume solitons which can be interpreted as formed by parallel \((-q, p)\) strings emanating from the D3-brane world-volume. It is shown that these configurations are 1/4 supersymmetric and provide a world-volume realization of the Hanany-Witten effect.
1 Introduction

The study of the solitons of the world-volume theories which describe the dynamics of branes has provided a lot of information about the different aspects of brane physics and of their interrelations [1]. An interesting aspect of these soliton solutions is that they have the spacetime interpretation of branes ending on the world-volume. In fact, different supergravity solutions, which represent spacetime configurations of intersecting branes, have been realized as solitons on the world-volume [2, 3, 4]. These results have allowed to perform several consistency checks of the different approaches to describe branes in string theory and of its connections to Yang-Mills theories.

Of particular interest are the world-volume solitons of brane probes which move in the supergravity background created by another brane. In some of these cases the background acts as a source for the world-volume gauge field and, thus, induces a soliton on the world-volume of the brane probe. The resulting configuration is a triple intersection of the background, probe and soliton branes.

An example of this last type of systems is the baryon vertex [5], in which a D5-brane moves in AdS$_5 \times S^5$, which is the near-horizon geometry of a stack of parallel D3-branes. A BPS equation for the embedding of the D3-brane in AdS$_5 \times S^5$ was proposed in ref. [6]. This equation was analyzed in ref. [7], where it was generalized to the full geometry of the D3-brane background. In ref. [8] the BPS condition was reobtained as a saturation condition of an energy bound. These results were generalized in ref. [9], where the system of a D(8 − $p$)-brane in the background of a D$p$-brane for $p \leq 6$ was studied and the corresponding BPS condition was obtained and integrated analytically. Moreover, in ref. [10] it was shown that the BPS condition obtained from the energy argument is precisely the one needed to preserve 1/4 supersymmetry. In all these systems the world-volume soliton is a spike which can be interpreted as a bundle of fundamental strings emanating from the brane probe.

In this paper we extend the results of refs. [3–10] to the case of a D3-brane probe in the background of a stack of parallel $(p, q)$ five-branes (the case of a D3-brane in the background of Neveu-Schwarz five-branes was studied in ref. [11]). The $(p, q)$ five-brane is a magnetically charged object under both the Neveu-Schwarz (NS) and Ramond (R) three-form field strengths of the type IIB supergravity, i.e. is the magnetic analogue of the $(p, q)$ strings studied by Schwarz [12]. It can be regarded as a bound state of $p$ NS5-branes and $q$ D5-branes. The complete form of this background was obtained in ref. [13] by using the $SL(2, \mathbb{Z})$ S-duality symmetry of the type IIB supergravity.

The S-duality symmetry will be very important in our approach. Indeed, it is well-known [11] that the D3-brane is invariant under this symmetry. At the level of the world-volume action, this invariance is reflected by the fact that one can convert the problem of a D3-brane in a $(p, q)$ five-brane background into the problem of a D3-brane in a D5-brane background. A BPS condition for the latter was found in ref. [11] and, thus, by inverting the S-duality transformation, one can obtain a BPS condition for the $(p, q)$ five-brane background. This condition can be integrated analytically and we will show that the corresponding world-volume soliton can be interpreted as a bundle of $(-q, p)$ strings. We will also prove that this configuration is 1/4 supersymmetric.

This paper is organized as follows. In section 2 we will describe the $(p, q)$ five-brane
background. In section 3 we will analyze the action and equations of motion of the D3-brane probe and, in particular we will put its energy functional in a form in which the invariance under S-duality is manifest. A BPS condition minimizing the energy will be found in section 4, where we will also obtain the explicit form of the world-volume solitons. The analysis of the supersymmetries preserved by our solution will be the subject of section 5. In this section we will rederive the BPS condition as the one to be imposed if 1/4 supersymmetry is required. Finally, in section 6 we summarize our results and discuss some possible generalizations of our work.

2 The \((p, q)\) five-brane background

The massless bosonic fields of the type IIB superstring theory in the NSNS sector are the graviton \(G_{\mu\nu}\), the dilaton \(\phi\) and an antisymmetric Kalb-Ramond field \(B_{\mu\nu}\). In the RR sector the type IIB theory has an scalar \(\chi\), an antisymmetric tensor field \(C_{\mu\nu}\) and a four-form gauge field whose field strength is a self-dual five-form. When this five-form field strength is zero, the equations of motion of the type IIB supergravity \[14\] can be derived from an action whose bosonic part, written in the Einstein frame, is given by:

\[
S_{IIB} = \int d^{10}x \sqrt{-G} \left[ R - \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi - \frac{1}{2} e^{2\phi} \nabla_{\mu} \chi \nabla^{\mu} \chi - \frac{1}{12} \left( e^{-\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + e^{\phi} \left( G_{\mu\nu\lambda} - \chi H_{\mu\nu\lambda} \right) \left( G^{\mu\nu\lambda} - \chi H^{\mu\nu\lambda} \right) \right) \right].
\]

In this action \(H\) is the field strength of the NSNS \(B\) field:

\[H = dB,\]

whereas \(G\) is given by:

\[G = dB.\]

The action \((2.1)\), and the equations of motion derived from it, are invariant under an \(SL(2, \mathbb{R})\) group of transformations. This is the so-called S-duality of the type IIB effective action, which is a non-perturbative symmetry of the theory. The NSNS and RR gauge fields transform as a doublet of this \(SL(2, \mathbb{R})\) symmetry, whereas the dilaton and RR scalar are mixed under S-duality. At the quantum level, only the discrete subgroup \(SL(2, \mathbb{Z})\) of \(SL(2, \mathbb{R})\) survives. Actually, one can generate a multiplet of solutions by acting with the \(SL(2, \mathbb{Z})\) symmetry on a particular solution. This is the method used by Schwarz \[12\] to construct new string-like solutions of the type IIB string theory starting from the fundamental string, i.e., from a solution electrically charged under the NSNS gauge field. One gets in this way a series of extremal solutions labeled by two integers \(p\) and \(q\), which are usually referred to as \((p, q)\) strings.

One can apply the same procedure to the magnetically charged NSNS solution, which is nothing but the NS five-brane (NS5). This construction has been undertaken by Lu and Roy.
in ref. [13] and as a result one gets an infinite family of five-branes of the type IIB effective action. Each member of this family is labeled by two integers \( p \) and \( q \), which represent, respectively, the magnetic charges under the NSNS and RR gauge fields. The \( (p, q) = (1, 0) \) solution is the NS5-brane one starts with, the \( (p, q) = (0, 1) \) is the D5-brane and a general \( (p, q) \) five-brane can be regarded as an extremal bound state of \( p \) NS5-branes and \( q \) D5-branes. Since this is the type IIB background we are interested in, let us describe it in detail following ref. [13].

The metric for an stack of \( N \) parallel \( (p, q) \) five-branes is given by:

\[
d s^2 = \left[ H_{(p,q)}(r) \right] ^{-\frac{2}{3}} ( -dt^2 + d x_\parallel^2 ) + \left[ H_{(p,q)}(r) \right] ^{\frac{2}{3}} ( dr^2 + r^2 d\Omega_3^2 ) , \tag{2.4}
\]

where \( x^i_\parallel (i = 1, \cdots, 5) \) are the coordinates along which the five-brane is extended and we have used spherical coordinates to parametrize the directions transverse to the five-brane. \( d\Omega_3^2 \) represents the metric of the unit three sphere). In the metric written above, \( H_{(p,q)}(r) \) is an harmonic function:

\[
H_{(p,q)}(r) = 1 + \frac{R_{(p,q)}^2}{r^2} , \tag{2.5}
\]

where the “radius” \( R_{(p,q)} \) is given by:

\[
R_{(p,q)}^2 = N \left[ \rho_{(p,q)} \right] ^{\frac{2}{3}} \alpha' . \tag{2.6}
\]

In eq. (2.6) \( \alpha' \) is the Regge slope and \( \rho_{(p,q)} \) can be put in terms of the string coupling constant \( g_s \) and of the asymptotic value of the RR scalar \( \chi_0 \) as follows:

\[
\rho_{(p,q)} = g_s^{-1} p^2 + ( q + p\chi_0 )^2 g_s . \tag{2.7}
\]

Let us recall that the string coupling constant \( g_s \) is related to the asymptotic value of the dilaton \( \phi_0 = \phi(r = \infty) \) by means of the expression:

\[
g_s = e^{\phi_0} . \tag{2.8}
\]

The dilaton field for the stack of \( (p, q) \) five-branes at an arbitrary value of \( r \) is given by:

\[
e^{-\phi} = \frac{\rho_{(p,q)} e^{-\phi_0}}{p^2 e^{-\phi_0} \left[ H_{(p,q)}(r) \right] ^{\frac{2}{3}} + ( q + p\chi_0 )^2 e^{\phi_0} \left[ H_{(p,q)}(r) \right] ^{-\frac{2}{3}}} , \tag{2.9}
\]

whereas the RR scalar takes the form:

\[
\chi = \frac{\rho_{(p,q)} \chi_0 + pq e^{-\phi_0} \left( 1 - H_{(p,q)}(r) \right) }{ p^2 e^{-\phi_0} H_{(p,q)}(r) + ( q + p\chi_0 )^2 e^{\phi_0} } . \tag{2.10}
\]

In order to determine completely the solution, it remains to give the values of the NSNS and RR gauge fields. Their field strengths \( H \) and \( G \) can be written\footnote{Notice that the action of type IIB supergravity written in ref. [13] differs from the one above in the change \( G \rightarrow -G \). Therefore, in order to write the \( (p, q) \) five-brane solution with our conventions, this same change must be done in the solution of ref. [13].} in terms of the volume element \( \epsilon_3 \) of the three dimensional sphere as follows:

\[
H = 2 p N \alpha' \epsilon_3 , \quad G = -2 q N \alpha' \epsilon_3 . \tag{2.11}
\]
From the values of the field strengths displayed in eq. (2.11) it is clear that the object which generates the metric (2.4) and the values of the dilaton and RR scalar of eqs. (2.9) and (2.10) is magnetically charged under the NSNS and RR gauge fields, i.e. extended along the direction of the potentials of the Hodge duals of \( H \) and \( G \). Moreover, the \((p,q)\) five-brane solution is extremal in the sense that preserves \(1/2\) of the supersymmetries of the type IIB supergravity. The Killing spinors corresponding to this solution will be given in section 5.

3 The D3-brane probe

Let us now consider a D3-brane probe embedded along the transverse directions of the stack of \((p,q)\) five-branes described in sect. 2. The action of this D3-brane probe is the sum of the Dirac-Born-Infeld term and the Wess-Zumino term:

\[
S = S_{DBI} + S_{WZ},
\]

where the expressions of \( S_{DBI} \) and \( S_{WZ} \) in the Einstein frame are given by:

\[
S_{DBI} = -T_3 \int d^4 \xi \sqrt{-\det (g + e^{-\frac{\psi}{2}} F)},
\]

\[
S_{WZ} = T_3 \int d^4 \xi \left[ F \wedge C + \frac{1}{2} \chi F \wedge F \right].
\]

In eq. (3.2) \( g \) is the induced world-volume metric, \( C \) is the pullback of the RR gauge field and \( F \) is

\[
F = dA - B = F - B,
\]

where \( A \) is a \( U(1) \) world-volume gauge field, \( F \) is its field strength and \( B \) is the pullback of the Kalb-Ramond field of the NS sector of the superstring. Notice that in \( S_{WZ} \) the field \( C \) acts as a source for the world-volume electric charge, whereas the RR scalar \( \chi \) couples to the instanton number density of \( F \). The coefficient \( T_3 \) multiplying the two terms of the action is the tension of the D3-brane which, in terms of the Regge slope \( \alpha' \), can be written as:

\[
T_3 = \frac{1}{8\pi^3 (\alpha')^2}.
\]

Let us parametrize the transverse three-sphere by means of three angles \( \theta^1, \theta^2 \) and \( \theta^3 \), where \( \theta^3 \equiv \theta \) is the polar angle \( (0 \leq \theta \leq \pi) \). The line element \( d\Omega^2_3 \) in these coordinates is given by:

\[
d\Omega^2_3 = (d\theta) + (\sin \theta)^2 [(d\theta^2)^2 + (\sin \theta^2)^2 (d\theta^1)^2].
\]

Let us denote by \( \bar{g} \) the determinant of the metric (3.3). It is clear that:

\[
\sqrt{\bar{g}} = (\sin \theta)^2 \sin \theta^2,
\]

and, therefore, the volume element \( \epsilon_3 \) can be written in these coordinates as:

\[
\epsilon_3 = \sqrt{\bar{g}} \ d\theta^1 \wedge d\theta^2 \wedge d\theta.
\]
We shall take the angles $\theta^i$, together with the time variable $t$ as world-volume coordinates $\xi^\alpha$, i.e. $\xi^\alpha = (t, \theta^1, \theta^2, \theta^3)$. Actually, we are going to consider static (i.e. time independent) configurations of the D3-brane which correspond to a radial deformation of the probe. Accordingly, the only “active scalar” will be the coordinate $r$, which we will take to depend only on the polar angle $\theta$. These configurations are invariant under an $SO(3)$ symmetry, which is the maximal amount of symmetry that a radial deformation can have. As was mentioned above, the first term in $S_{WZ}$ in eq. (3.2) couples the world-volume gauge field to the RR background potential $C$. Actually, by using the expression (3.3) of $F$ and integrating by parts, one can easily verify that this term of $S_{WZ}$ couples the time component $A_0$ of the $U(1)$ world-volume field to the RR field strength $G$. Because of this coupling, we cannot take $A_0$ to vanish. In what follows we shall assume that $A_0$ depends only on the polar angle $\theta$, which again corresponds to the maximally symmetric solution. On the other hand, the NS field $B$ couples to the brane probe through $F$ (see eq. (3.3)). Moreover, it is possible to choose a gauge in which the only non-vanishing component of $B$ is $B_{\theta\theta^1}$.

Indeed, one can take $B$ as:

$$B = 2\rho N \alpha' \sin^2 b_\nu(\theta) \, d\theta^1 \wedge d\theta^2,$$

where $b_\nu(\theta)$ must verify:

$$\frac{d}{d\theta} b_\nu(\theta) = (\sin \theta)^2.$$

In what follows we shall adopt the gauge (3.8) and, accordingly, we shall assume that the only non-vanishing components of $F$ are $F_{0\theta} \equiv F_{0\theta}$ and $F_{\theta\theta^2} \equiv F_{12}$. The action $S$ for these configurations is:

$$S = T_3 T \int d^3 \theta \left\{ -\sqrt{r^4 \left[ H_{(p,q)}(r) \right]^\frac{3}{2} g + e^{-\phi} F_{12}^2} \times \right.$$  

$$\left. \times \sqrt{\left[ H_{(p,q)}(r) \right]^\frac{1}{2} (r^2 + r'^2) - e^{-\phi} F_{0\theta}^2 - 2qN \alpha' \sqrt{g} A_0 + \chi F_{12} F_{0\theta}} \right\},$$

where the prime denotes derivative with respect to the polar angle $\theta$ and $T = \int dt$. Let us now define the quantity:

$$\Pi \equiv \frac{1}{T_3 T} \frac{\partial S}{\partial F_{0\theta}}.$$

By using the form of $S$ given in eq. (3.10), one can find the expression of $\Pi$ in terms of $F_{0\theta}$ and $F_{12}$, namely:

$$\Pi = \frac{\sqrt{r^4 \left[ H_{(p,q)}(r) \right]^\frac{3}{2} g + e^{-\phi} F_{12}^2}}{\sqrt{\left[ H_{(p,q)}(r) \right]^\frac{1}{2} (r^2 + r'^2) - e^{-\phi} F_{0\theta}^2}} e^{-\phi} F_{0\theta} + \chi F_{12}.$$

Moreover, the Euler-Lagrange equation for $A_0$ can be written as the following equation for $\Pi$:

$$\partial_\theta \Pi = 2qN \alpha' \sqrt{g},$$

The label $\nu$ of the function $b_\nu(\theta)$ represents the constant of integration of equation (3.3). The explicit expression of $b_\nu(\theta)$ will be given in section 4.
which is nothing but the Gauss law constraint for the world-volume gauge field. Notice that
the right-hand side of eq. (3.13) only depends on the angles and, thus, the Gauss law can be
integrated and, as a result, one can obtain \( \Pi \) as a function of the \( \theta \)'s. Similarly, by computing
the exterior derivative of \( \mathcal{F} = dA - B \), we get the Bianchi identity for \( \mathcal{F} \):
\[
d \mathcal{F} = - H ,
\]
which implies that \( \mathcal{F}_{0\theta} \) is independent of the angles \( \theta^1 \) and \( \theta^2 \) (in agreement with our as-
sumption that \( A_0 \) only depends on \( \theta \)) and the following differential equation for \( \mathcal{F}_{12} \):
\[
\partial_\theta \mathcal{F}_{12} = -2pN\alpha' \sqrt{\bar{g}} .
\]
The similarity of eqs. (3.13) and (3.15) is manifest. It is clear that eq. (3.15) could also be
integrated and one could obtain \( \mathcal{F}_{12} \) as a function of the angles \( \theta \). We are going to postpone
the integration of these equations until the next section. However we shall proceed as if \( \Pi \)
and \( \mathcal{F}_{12} \) were known functions and we will try to put the action in terms of them. It is clear
from eq. (3.10) that one must eliminate \( \mathcal{F}_{0\theta} \) and \( A_0 \) in favor of \( \Pi \) and
\( \mathcal{F}_{12} \). This can be done
by inverting the relation between \( \Pi \) and \( \mathcal{F}_{0\theta} \). The result of this inversion is:
\[
\mathcal{F}_{0\theta} = \frac{\sqrt{H_{(p,q)}(r) \left[ r^2 + r'^2 \right]}}{\sqrt{r^4 \left[ H_{(p,q)}(r) \right]^2 \bar{g} + e^{-\phi} \mathcal{F}_{12}^2 + e^{\phi} (\Pi - \chi \mathcal{F}_{12})^2}} e^\phi (\Pi - \chi \mathcal{F}_{12}) .
\]
Moreover, by using the Gauss law (3.13) and integrating by parts, we can rewrite the WZ
term of the action as:
\[
S_{WZ} = -T_3 T \int d^3 \theta (\Pi - \chi \mathcal{F}_{12}) \mathcal{F}_{0\theta} .
\]
By means of eq. (3.16), and after representing the WZ term as in eq. (3.17), one can write
the total action \( S \) as:
\[
S = -TU ,
\]
where \( U \) is given by:
\[
U = T_3 \int d^3 \theta \sqrt{r^2 + r'^2} \times
\]
\[
\times \sqrt{\left[ \Sigma_{(p,q)} \right]^2 + \left[ H_{(p,q)}(r) \right]^2 \left( e^{-\phi} \mathcal{F}_{12}^2 + e^{\phi} (\Pi - \chi \mathcal{F}_{12})^2 \right)} .
\]
In eq. (3.19) \( \Sigma_{(p,q)} \) denotes:
\[
\Sigma_{(p,q)} \equiv r^2 H_{(p,q)}(r) \sqrt{\bar{g}} .
\]
It is clear from eq. (3.18) that \( S \) and \( U \) give rise to the same Euler-Lagrange equations. In
fact, one can regard \( U \) as the result of performing a Legendre transformation to \( S \) and, thus,
\( U \) can be considered as an energy functional for the D3-brane probe in the \( (p, q) \) five-brane
geometry. The equations of motion derived from \( U \) are simply the conditions required to
have minimal energy configurations.
An important point, which we shall exploit in the next section, is that $U$ can be written in a way which makes manifest its $SL(2, \mathbb{R})$ invariance. In order to recast $U$ as a function explicitly invariant under $S$-duality, let us define the matrix:

$$M = \begin{pmatrix} \chi^2 + e^{-2\phi} & \chi \\ \chi & 1 \end{pmatrix} e^{\phi},$$

and the vector

$$D = \begin{pmatrix} -F_{12} \\ \Pi \end{pmatrix}.$$

By inspecting eq. (3.19) and the definitions (3.21) and (3.22), one immediately verifies that $U$ can be put as:

$$U = T_3 \int d^3\theta \sqrt{r^2 + r'^2} \sqrt{\Sigma_{(p,q)}^2} + \left[ H_{(p,q)}(r) \right]^{\frac{1}{2}} D^T M D.$$  (3.23)

Let $\Lambda$ be an arbitrary $2 \times 2$ $SL(2, \mathbb{R})$ matrix of the form:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

where $a$, $b$, $c$ and $d$ are constant real numbers. It is now obvious from eq. (3.23) that $U$ is invariant under the following simultaneous transformation of $M$ and $D$:

$$M \rightarrow \Lambda M \Lambda^T, \quad D \rightarrow (\Lambda^{-1})^T D.$$  (3.25)

Notice that the matrix $M$ encodes the information of the scalar fields of the background. Actually, this same matrix $M$ appears when the background action (2.1) is written in an $SL(2, \mathbb{R})$-invariant form [1]. It is well-known that if one assembles the RR scalar $\chi$ and the dilaton $\phi$ into a complex scalar $\lambda$ as:

$$\lambda = \chi + i e^{-\phi},$$

then, the transformation law of $M$ (see eq. (3.23)) is equivalent to a Möbius bilinear transformation of the complex scalar $\lambda$:

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}.$$  (3.27)

It follows from eq. (3.25) that the $SL(2, \mathbb{R})$ transformation of the background must be accompanied by a transformation of $D$, which can be regarded as an electric-magnetic duality rotation of the world-volume gauge fields. Notice that under this symmetry the Gauss law and the Bianchi identity are mixed. This connection between world-volume electric-magnetic symmetry and $S$-duality of the background is well-known [15]-[20]. In the next section we shall use it to map our problem to a system in which the D3-brane is in the background of a D5-brane, i.e. we will reduce our problem for arbitrary values of the integers $p$ and $q$ to the case in which $(p, q) = (0, 1)$. This D5-D3 system was studied in ref. [9] and its equation of motion was integrated exactly by means of a BPS first-order condition. In section 4 we will be able to translate this BPS condition of the D5-D3 system into a condition for the $(p, q)$ five-brane case and, as a consequence, we will find exact solutions for the embedding of the D3-brane in the general $(p, q)$ background.
4 The BPS equation

In this section we will find a first-order (BPS) differential equation \[21\] whose solutions minimize the energy functional of the D3-brane \[22\]. Our strategy to find this BPS equation will be the following. First of all we shall perform an \(S\)-duality transformation which converts the background of eqs. \[2.4\], \[2.9\], \[2.10\] and \[2.11\] into a solution of IIB supergravity with line element given by eq. \[2.4\] in which only the dilaton \(\phi\) and the RR field strength \(G\) are non-vanishing. The problem of minimizing the energy in the dual variables has been solved in ref. \[9\]. After transforming back to our original fields, we will get the minimal energy configurations of our system.

The form of the \(SL(2, \mathbb{R})\) matrix with the properties required for our purposes is not difficult to find. Actually, the \((p, q)\) five-brane background was found in ref. \[13\] by means of an \(S\)-duality transformation of a solution in which only one of the two three-form field strengths is non-vanishing. By adapting the results of ref. \[13\], it is straightforward to conclude that the matrix \(\Lambda\) suited for our objectives has the following elements:

\[
\begin{align*}
a &= -e^{\phi_0} (q + p\chi_0) \left[ \rho_{(p,q)} \right]^{-\frac{1}{2}}, \\
b &= \left( e^{\phi_0} (q + p\chi_0) \chi_0 + e^{-\phi_0} p \right) \left[ \rho_{(p,q)} \right]^{-\frac{1}{2}}, \\
c &= -p \left[ \rho_{(p,q)} \right]^{-\frac{1}{2}}, \\
d &= -q \left[ \rho_{(p,q)} \right]^{-\frac{1}{2}}.
\end{align*}
\]  

(4.1)

Let us call \(\mathcal{M}^{(D)}\) to the transformed \(\mathcal{M}\) matrix, \(i.e.\):

\[
\mathcal{M}^{(D)} = \Lambda \mathcal{M} \Lambda^T.
\]  

(4.2)

After some algebra, one can check that, when the matrix elements of \(\Lambda\) are those displayed in eq. \[4.1\], the dual matrix \(\mathcal{M}^{(D)}\) has the simple form:

\[
\begin{pmatrix}
\frac{1}{2} H_{(p,q)}(r) & 0 \\
0 & \frac{1}{2} H_{(p,q)}(r)^{-\frac{1}{2}}
\end{pmatrix}.
\]  

(4.3)

This matrix corresponds to the following values of the dilaton and RR scalar:

\[
e^{-\phi^{(D)}} = \left[ H_{(p,q)}(r) \right]^\frac{1}{2}, \quad \chi^{(D)} = 0,
\]  

(4.4)

which are the values of \(\phi\) and \(\chi\) for a D5-brane solution of type IIB supergravity. Moreover, after the transformation, the vector \(\mathcal{D}\) takes the form:

\[
\mathcal{D}^{(D)} = (\Lambda^{-1})^T \mathcal{D} = \begin{pmatrix} -\mathcal{F}_{12}^{(D)} \\ \Pi^{(D)} \end{pmatrix},
\]  

(4.5)
where:
\[ F_{12}^{(D)} = d F_{12} + c \Pi, \quad \Pi^{(D)} = a \Pi + b F_{12}. \] (4.6)

In terms of the dual variables, the energy functional is simply given by:
\[ U = T_3 \int d^3\theta \sqrt{r^2 + r'^2} \sqrt{\Sigma^2_{(p,q)} + \left( \Pi^{(D)} \right)^2 + H_{(p,q)}(r) \left( F_{12}^{(D)} \right)^2}. \] (4.7)

From eq. (4.6) and the Gauss law and Bianchi identity for the original variables \( \Pi \) and \( F_{12} \) (eqs. (3.13) and (3.15)), it is immediate to obtain the equation that determines the dependence on \( \theta \) of \( F_{12}^{(D)} \), namely:
\[ \partial_\theta F_{12}^{(D)} = (q c - p d) 2 N \alpha' \sqrt{g}. \] (4.8)

By using the explicit values of the coefficients \( d \) and \( c \) from eq. (4.1), one can verify that the right-hand side of eq. (4.8) vanishes, i.e.:
\[ \partial_\theta F_{12}^{(D)} = 0. \] (4.9)

Eq. (4.9) is simply the Bianchi identity for the dual variables. As the dual background has vanishing NSNS gauge field, the result of eq. (4.9) was to be expected. Moreover, it follows from the analysis of ref. [9] that the D3-brane configurations which minimize the energy in a D5 background are such that the magnetic components of the world-volume field strength are zero. Thus, in order to solve the equations of motion derived from \( U \), we shall adopt the ansatz:
\[ F_{12}^{(D)} = 0. \] (4.10)

If this condition holds, it implies from (4.1) the following relation between \( F_{12} \) and \( \Pi \):
\[ F_{12} = -\frac{c}{d} \Pi = -\frac{p}{q} \Pi, \] (4.11)
which is, in principle, valid only when \( q \neq 0 \) (we will consider the \( q = 0 \) case below). Using the relation (4.11) in eq. (4.6), we can get \( \Pi^{(D)} \) in terms of \( \Pi \), namely:
\[ \Pi^{(D)} = -\left[ \frac{\rho_{(p,q)}}{q} \right]^{\frac{1}{2}} \Pi. \] (4.12)

Let us now determine \( \Pi \) by integrating the Gauss law constraint (3.13). Taking eq. (3.6) into account, one can solve eq. (3.13) as follows:
\[ \Pi = 2q N \alpha' \sin^2 \theta b_\nu(\theta), \] (4.13)
where \( b_\nu(\theta) \) satisfies the differential equation (3.3). It is immediate to verify that:
\[ b_\nu(\theta) = \frac{1}{2} \left[ \theta - \cos \theta \sin \theta - \pi \nu \right], \] (4.14)
is the general solution of eq. (3.9). In eq. (4.14) $\nu$ is an integration constant which, for example, determines the value of $b_\nu(\theta)$ at $\theta = \pi$. By plugging in eq. (4.12) the expression of $\Pi$ given by (4.13), one can obtain $\Pi^{(D)}$ as a function of the angles. One gets:

$$\Pi^{(D)} = -2 R_{(p,q)}^2 \sin \theta^2 b_\nu(\theta) .$$

Similarly, from eqs. (4.11) and (4.13) we can determine $F_{12}$:

$$F_{12} = -2 p N \alpha' \sin \theta^2 b_\nu(\theta).$$

It is immediate to verify that, indeed, the expression of $F_{12}$ given in eq. (4.16) solves the Bianchi identity (3.15). Actually, we could have obtained $F_{12}$ by direct integration of eq. (3.15). The advantage of our present derivation based on S-duality is that it fixes uniquely the integration constant of this equation. Moreover, if one chooses the gauge for $B$ as in eq. (3.8), i.e. with the same constant $\nu$ as is in eq. (4.16), one has:

$$F_{12} = -B_{\theta^1 \theta^2},$$

which implies that:

$$F_{g^1 g^2} = 0 .$$

Notice that eq. (4.18) implies that the spatial components $A_{g^i}$ of the world-volume gauge field vanish and, therefore, only $A_0$ is non-zero and depends on the polar angle $\theta$.

Let us now define the function $D_{(p,q)}(\theta)$ as:

$$D_{(p,q)}(\theta) \equiv -2 R_{(p,q)}^2 b_\nu(\theta).$$

The field strength $F_{12}$ and the dual momentum $\Pi^{(D)}$ can be rewritten in terms of $D_{(p,q)}(\theta)$. Indeed, by using eq. (2.6), one arrives at:

$$F_{12} = \frac{p}{|\rho_{(p,q)}|^2} \sin \theta^2 D_{(p,q)}(\theta),$$

while $\Pi^{(D)}$ is given by:

$$\Pi^{(D)} = \sin \theta^2 D_{(p,q)}(\theta).$$

Let us now consider the problem of finding the embeddings of the D3-brane with minimal energy $U$. First of all, we can substitute in the expression of $U$ written in eq. (1.7) the values of $F_{12}^{(D)}$ and $\Pi^{(D)}$ that we have obtained (eqs. (4.10) and (4.21)). As, by assumption, $r$ only depends on the polar angle $\theta$, it is possible to integrate over the other two angles $\theta^1$ and $\theta^2$, with the result:

$$U = 4\pi T_3 \int d\theta \sqrt{r^2 + r'^2} \sqrt{[\Delta_{(p,q)}(r,\theta)]^2 + [D_{(p,q)}(\theta)]^2},$$

where:

$$\Delta_{(p,q)}(r,\theta) \equiv r^2 H_{(p,q)}(r) (\sin \theta)^2 .$$

Eq. (4.22) will be the starting point in our determination of the minimal energy configurations. Before proceeding further, let us point out that eq. (4.22) is also valid when $q = 0,$
i.e. when the background is purely NS. Actually, although some of our intermediate equations are ill-defined when \( q = 0 \), the final expressions for \( \Pi \) and \( F_{12} \) (eqs. (4.13) and (4.16) respectively) are perfectly well-behaved in this case, and one can readily check that they solve the Gauss law and the Bianchi identity for \( q = 0 \). By substituting these \( q = 0 \) values of \( \Pi \) and \( F_{12} \) in eq. (3.19), one can easily conclude that the energy \( U \) is given by eq. (4.22) also when \( q = 0 \). Therefore, we will assume that this value is included in what follows.

The minimal energy embeddings of the D3-brane are characterized by a function \( r(\theta) \) which satisfies the Euler-Lagrange equation of motion derived from \( U \). This equation of motion is a rather complicated second-order differential equation which, in general, we will not be able to solve. Instead, we will find a first-order differential equation for \( r(\theta) \) whose solutions saturate an energy bound and, thus, also solve the Euler-Lagrange equation. In order to find this first-order equation we follow closely refs. [8, 9]. First of all, we define the function

\[
 f_{(p,q)}(r, \theta) \equiv \frac{\Delta_{(p,q)}(r, \theta) \sin \theta + D_{(p,q)}(\theta) \cos \theta}{\Delta_{(p,q)}(r, \theta) \cos \theta - D_{(p,q)}(\theta) \sin \theta} .
\]

(4.24)

In terms of \( f_{(p,q)}(r, \theta) \) it is easy to check that the square root in \( U \) can be written as a sum of squares. One has:

\[
 U = 4\pi T_3 \int d\theta \times \sqrt{Z^2 + r^2 \left[ \Delta_{(p,q)}(r, \theta) \cos \theta - D_{(p,q)}(\theta) \sin \theta \right]^2 \left[ \frac{r'}{r} - f_{(p,q)}(r, \theta) \right]^2} ,
\]

(4.25)

where \( Z \) is given by:

\[
 Z = r \left[ \Delta_{(p,q)}(r, \theta) \cos \theta - D_{(p,q)}(\theta) \sin \theta \right] \left[ 1 + \frac{r'}{r} f_{(p,q)}(r, \theta) \right] .
\]

(4.26)

From eq. (4.25) we obtain immediately that \( U \) satisfies the bound:

\[
 U \geq 4\pi T_3 \int d\theta \left| Z \right| ,
\]

(4.27)

which is saturated when \( r(\theta) \) satisfies the following first-order differential equation:

\[
 \frac{r'}{r} = f_{(p,q)}(r, \theta) .
\]

(4.28)

Eq. (4.28) is the first-order differential equation we were looking for. It is easy to prove that any function \( r(\theta) \) that satisfies (4.28) also solves the Euler-Lagrange equation. Moreover, \( Z \) can be written as a total derivative:

\[
 Z = \frac{d}{d\theta} \left[ D_{(p,q)}(\theta) r \cos \theta + \left( \frac{1}{3} + \frac{R_{(p,q)}^2}{r^2} \right) (r \sin \theta)^3 \right] .
\]

(4.29)

Eq. (4.29) holds for an arbitrary function \( r(\theta) \). In fact, in order to demonstrate it, one only has to use the value of the derivative of \( D_{(p,q)}(\theta) \), which can be easily obtained from
its explicit expression (eq. (4.19)). The main consequence of eq. (4.29) is that only the boundary values of $r(\theta)$ contribute to the energy bound of eq. (4.27) and, thus, the solutions of (4.28) certainly minimize the energy for given boundary conditions. We will refer to eq. (4.28) as the BPS condition. In section 5 we will reobtain this condition as the one needed for preservation of 1/4 of supersymmetry. In order to establish this connection with supersymmetry, it is interesting to recast the BPS condition as an ansatz for the electric components $F_{0\theta}$ of the world-volume field strength. Recall that $F_{0\theta}$, as a function of $\Pi$ and $F_{12}$ was given in eq. (3.16). Actually, by using the values of the dilaton and RR scalars in the $(p, q)$ five-brane background (eqs. (2.9) and (2.10)), together with the expressions of $\Pi$ and $F_{12}$ (eqs. (4.13) and (4.16)), one can prove, after some calculation, that:

$$e^\phi \left( \Pi - \chi \mathcal{F}_{12} \right) = - \left[ H_{(p,q)}(r) \right]^{\frac{1}{2}} \frac{q + p \chi_0}{\rho_{(p,q)}} e^{\phi_0} \sin \theta^2 \ D_{(p,q)}(\theta) .$$

(4.30)

By using eq. (4.30) to determine the right-hand side of eq. (3.16), it is not difficult to find the following expression for $F_{0\theta}$:

$$F_{0\theta} = - \frac{q + p \chi_0}{\rho_{(p,q)}} e^{\phi_0} \sqrt{\frac{\Delta_{(p,q)}(r, \theta)}{\Delta_{(p,q)}(r, \theta)}} D_{(p,q)}(\theta) .$$

(4.31)

Moreover, if $r(\theta)$ satisfies the BPS condition (4.28), one can readily check that:

$$\sqrt{\frac{\Delta_{(p,q)}(r, \theta)}{\Delta_{(p,q)}(r, \theta)}} D_{(p,q)}(\theta) = (r \cos \theta)' ,$$

(4.32)

which implies that, when the BPS condition is fulfilled, the electric field $F_{0\theta}$ takes the value:

$$F_{0\theta} = - \frac{q + p \chi_0}{\rho_{(p,q)}} e^{\phi_0} (r \cos \theta)' .$$

(4.33)

Notice that, as $F_{0\theta} = - \partial_\theta A_0$, the time component of the world-volume gauge field is related to $r(\theta)$ as:

$$A_0(\theta) = \frac{q + p \chi_0}{\rho_{(p,q)}} e^{\phi_0} r(\theta) \cos \theta .$$

(4.34)

It is also interesting to relate $\mathcal{F}_{12}$ to $r(\theta)$ and its derivative. This can be done by rewriting the BPS condition (4.28) as:

$$D_{(p,q)}(\theta) = \frac{(r \cos \theta)'}{(r \sin \theta)'} \Delta_{(p,q)}(r, \theta) .$$

(4.35)

After substituting $D_{(p,q)}(\theta)$, as given in eq. (4.33), on the right-hand side of eq. (4.20), one gets:

$$\mathcal{F}_{12} = \frac{p}{\rho_{(p,q)}} \sin \theta^2 \frac{(r \cos \theta)'}{(r \sin \theta)'} \Delta_{(p,q)}(r, \theta) .$$

(4.36)
We shall demonstrate in section 5 that, in order to preserve 1/4 supersymmetry, $F_{0\theta}$ and $F_{12}$ must be related to $r(\theta)$ as in eqs. (4.33) and (4.36). This result will clarify the nature of our first-order equation (4.28).

An amazing aspect of eq. (4.28) is the fact that, as was noticed in ref. [9], it can be integrated exactly. In fact, by using the expression of $D_{(p,q)}(\theta)$ (eq. (4.19)), one can rewrite eq. (4.28) as a total derivative:

$$
\frac{d}{d\theta} (r \cos \theta) = R^2_{(p,q)} \frac{\theta - \pi \nu}{r \sin \theta} .
$$

(4.37)

The integration of eq. (4.37) is immediate and the result can be written as:

$$
r \cos \theta = R^2_{(p,q)} \frac{\theta - \pi \nu}{r \sin \theta} - z_\infty ,
$$

(4.38)

where $z_\infty$ is a constant of integration. Notice that (4.38) determines $r(\theta)$ in implicit form. However, from eq. (4.38) one can easily obtain the explicit expression of $r(\theta)$. The result is:

$$
r(\theta) = -\frac{z_\infty}{2 \cos \theta} \pm \frac{1}{2 \cos \theta} \sqrt{z_\infty^2 + 4 R^2_{(p,q)} \frac{\theta - \pi \nu}{\tan \theta}} .
$$

(4.39)

In figure 1 we have represented the D3-brane embedding for different values of the constants $\nu$ and $z_\infty$. In order to choose the signs on the right-hand side of (4.39) one has to take into account that $r(\theta)$ must be real and non-negative. Notice that, in general, the function $r(\theta)$ is double-valued due to these two signs. This double-valueness can also be appreciated from the plots of figure 1. In order to have a global description of the D3-brane world-volume by means of a single-valued function, it is more convenient to work in a new set of variables $(z, \rho)$, related to $(r, \theta)$ as follows:

$$
z = -r \cos \theta , \quad \rho = r \sin \theta .
$$

(4.40)

Notice that $z$ and $\rho$ are nothing but cylindrical coordinates. Clearly, $z$ can take values on the interval $(-\infty, +\infty)$, whereas $0 \leq \rho < \infty$. In these coordinates the embedding of the D3-brane is determined by a single-valued function $z(\rho)$.

The solution (4.38) of the BPS differential equation was analyzed in detail in ref. [9]. Let us review here the main results obtained in [9]. First of all, in those regions which are far from the origin $r = 0$ (or $\rho \to \infty$ in cylindrical coordinates) the D3-brane world-volume can be described approximately by the equation $z = \text{constant} = z_\infty$. Thus, in these asymptotic regions, the shape of the D3-brane world-volume is just a plane, i.e. the brane probe is not bent by the action of the background. For some values of the integration constants the shape of the D3-brane near the polar axis (i.e. for $\theta = 0$ or $\theta = \pi$) resembles that of a tube (see figure 1). In this region, the D3-brane develops a world-volume soliton which is a spike that connects the D3-brane probe to the $r = 0$ region, where the $(p, q)$ five-branes of the background are located. In order to find out what this soliton represents, let us evaluate its energy. For this purpose it is more convenient to express the energy functional $U$ in cylindrical coordinates. By performing the change of variables (4.40) on the right-hand side of eq. (4.22), one gets the remarkably simple result (4.41):

$$
U = 4\pi T_3 \int d\rho \sqrt{\left( \Delta_{(p,q)} - D_{(p,q)} \frac{dz}{d\rho} \right)^2 + \left( D_{(p,q)} + \Delta_{(p,q)} \frac{dz}{d\rho} \right)^2} .
$$

(4.41)
Figure 1: Solutions of the BPS differential equation for several values of \( \nu \) and \( z_\infty \). The coordinates \((r, \theta)\) are the polar coordinates of the plane of the figure \( (\theta = \pi \) at the top). The horizontal and vertical axis correspond, respectively, to the coordinates \( \rho \) and \( z \). For every value of \( \nu \), the different curves plotted together correspond to different values of the constant \( z_\infty \).

Moreover, the BPS condition in the \((z, \rho)\) variables takes the form:

\[
\frac{dz}{d\rho} = -\frac{D_{(p,q)}}{\Delta_{(p,q)}}. \tag{4.42}
\]

Notice that, when \( z(\rho) \) satisfies eq. (4.42), the second term under the square root on the right-hand side of eq. (4.41) vanishes. Therefore, the energy of a BPS configuration is simply:

\[
U_{\text{BPS}} = -4\pi T_3 \int d\rho \left[ \frac{dz}{d\rho} + \left( \frac{dz}{d\rho} \right)^{-1} \right] D_{(p,q)}. \tag{4.43}
\]

Notice that the argument of the function \( D_{(p,q)} \) in (4.43) is \( \theta = -\arctan(-\rho/z) \). From eq. (4.43) it is very easy to evaluate the contribution of the soliton to the energy \( U \). Indeed, as the shape of the soliton is approximately a cylinder with \( \rho = \) constant, the derivative \( \frac{dz}{d\rho} \) is very large and, thus, one can neglect in (4.43) the term containing \( \left( \frac{dz}{d\rho} \right)^{-1} \). Moreover, the argument of the function \( D_{(p,q)} \) is almost constant and equal to 0 or \( \pi \) and, therefore, \( D_{(p,q)} \) can be taken out of the integral. Thus, for a tube at \( \theta = \pi \), the energy is given by:

\[
U_{\text{tube}} = 4\pi T_3 |D_{(p,q)}(\pi)| L_{\text{tube}}, \tag{4.44}
\]

where \( L_{\text{tube}} \) is the length of the tube defined as:

\[
L_{\text{tube}} = \int_{\text{tube}} d\rho \left| \frac{dz}{d\rho} \right|. \tag{4.45}
\]

It is clear that the world-volume soliton we have found is one-dimensional, \textit{i.e.} is a string-like object. The best way to identify it is by computing its tension and comparing the result with the strings of the type IIB theory. This tension is naturally defined in the string frame, where the metric \( g_{\text{string}} \) is related to the Einstein metric \( g \) we have been using as \( g_{\text{string}} = e^{\phi/2} g \). The world-volume action of any string has a Nambu-Goto term which
contains the square root of the determinant of the induced metric. This Nambu-Goto term is multiplied by a factor $e^{\phi/2}$ when one changes from the string frame to the Einstein frame. It is, thus, clear that this same factor, evaluated at infinity, appears in the relation of the string tensions in the two frames. Actually, the tension in the string frame is obtained by multiplying by $g_s^{-1/2}$ the tension evaluated in the Einstein frame. Taking these considerations into account, we obtain from eq. (4.44) that the tension of our soliton in the string frame is:

$$g_s^{-\frac{1}{2}} 4\pi T_3 | D_{(p,q)}(\pi) | .$$

(4.46)

The value of $D_{(p,q)}$ at $\theta = \pi$ can be obtained in an straightforward way from eqs. (4.19) and (4.14):

$$| D_{(p,q)}(\pi) | = \pi (1 - \nu) R_{(p,q)}^2 ,$$

(4.47)

where, for reasons which will become clear in a moment, we have taken $\nu$ in the interval $0 \leq \nu \leq 1$. By using the value of $R_{(p,q)}$, as given in eq. (2.6), one gets that the tension of the spike is:

$$(1 - \nu) N T_f \left[ \frac{\rho_{(p,q)}}{g_s} \right]^{\frac{1}{2}} .$$

(4.48)

Recalling [12] that the tension of a $(p, q)$ string is given by:

$$T_{(p,q)} = \sqrt{(p - q\chi_0)^2 + \frac{q^2}{g_s^2}} T_f = \left[ \frac{\rho_{(p,q)}}{g_s} \right]^{\frac{1}{2}} T_f ,$$

(4.49)

we conclude that the tension of our spike is:

$$(1 - \nu) N T_{(-q,p)} ,$$

(4.50)

which corresponds to a bundle constituted by $(1 - \nu) N$ parallel $(-q, p)$-strings. It is now clear that $\nu$ should be quantized in units of $1/N$ and restricted to take values in the interval $[0, 1]$.

The analysis we have done can also be carried out for world-volume solitons at $\theta = 0$. The conclusion, in this case, is that the spike can be interpreted as a bundle of $\nu N$ parallel $(-q, p)$-strings. Therefore, the configuration we have found involves three branes, background, probe and soliton, intersecting according to the array:

$$(p, q) \quad \text{five} - \text{brane} : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad - \quad - \quad - \quad - \quad - \quad \text{background}$$

$$D3 : \quad - \quad - \quad - \quad - \quad - \quad - \quad 6 \quad 7 \quad 8 \quad \text{probe}$$

$$(-q, p) \quad \text{string} : \quad - \quad - \quad - \quad - \quad - \quad - \quad 9 \quad \text{soliton}. \quad (4.51)$$

The 9-direction is the radial one.

The world-volume solitons we have found provide an explicit realization of the Hanany-Witten effect [23]. In order to clarify this point, let us consider, for concreteness, the $\nu = 0$ case. As can be seen from figure 1, when $z_\infty \rightarrow -\infty$, the brane is nearly flat. When $z_\infty$ is increased, the D3-brane gets more and more deformed near the $\theta = \pi$ axis, and, when $z_\infty \rightarrow +\infty$ the bundle of $(-q, p)$-strings is created. For $\nu = 1$ the soliton is created when $z_\infty \rightarrow -\infty$ and the brane is not bent when $z_\infty \rightarrow +\infty$. Notice that for $\nu = 0, 1$, the D3-brane does not reach $r = 0$ and only in those two cases the D3-brane wraps completely the sphere $S^3$. If, on the contrary, $\nu \neq 0, 1$, the $r = 0$ point is reached and the solution has the two types of soliton tubes (at $\theta = 0, \pi$), depending on the sign of $z_\infty$. 

15
5 Supersymmetry

In this section we will show that the world-volume configurations we have found preserve 1/4 of the background supersymmetry. As is well-known [24, 25], the number of supersymmetries preserved by a D3-brane moving in the background of a \((p, q)\) five-brane is the number of independent solutions of the equation:

\[
\Gamma_\kappa \epsilon_{(p,q)} = \epsilon_{(p,q)} , \tag{5.1}
\]

where \(\Gamma_\kappa\) is the matrix appearing in the \(\kappa\)-symmetry transformation of the D3-brane and \(\epsilon_{(p,q)}\) is a Killing spinor of the background. In general, the matrix \(\Gamma_\kappa\) depends on the background and on the type of brane. For a D3-brane in a type IIB background, \(\Gamma_\kappa\) is given by:

\[
\Gamma_\kappa = \frac{1}{\sqrt{-\det (g + e^{-\frac{\phi}{2}} F)}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \gamma_{\mu_1 \nu_1 \cdots \mu_n \nu_n} \times
\]

\[
x e^{-\frac{\phi}{2}} F_{\mu_1 \nu_1} \cdots e^{-\frac{\phi}{2}} F_{\mu_n \nu_n} J^{(n)} , \tag{5.2}
\]

where \(J^{(n)}\) is the following matrix:

\[
J^{(n)} = (-1)^n \sigma_3^n (i\sigma_2) \otimes \Sigma_0 , \tag{5.3}
\]

with \(\Sigma_0\) being:

\[
\Sigma_0 = \frac{1}{4!} \epsilon_{\mu_1 \cdots \mu_4} \gamma_{\mu_1 \cdots \mu_4} . \tag{5.4}
\]

Notice that the presence of the \(e^{-\frac{\phi}{2}}\) factors on the right-hand side of eq. (5.2) is due to the fact that \(g\) is the induced metric in the Einstein frame. Moreover, in eqs. (5.2) and (5.4) \(\gamma_{\mu_1 \cdots}\) are antisymmetrized products of the induced world-volume Dirac matrices which, in terms of the ten-dimensional constant gamma matrices \(\Gamma_m\) are given by:

\[
\gamma_\mu = \partial_\mu X^m E^m_m \Gamma_m , \tag{5.5}
\]

where \(E^m_m\) is the ten-dimensional vielbein.

The infinite sum appearing on the right-hand side of eq. (5.2) has, actually, only three terms due to the antisymmetry of the index structure of this equation, and, thus, \(\Gamma_\kappa\) can be written as:

\[
\Gamma_\kappa = \frac{1}{\sqrt{-\det (g + e^{-\frac{\phi}{2}} F)}} (i\sigma_2) \left[ \Sigma_0 - \frac{1}{4} \sigma_3 \epsilon^{\mu\nu\rho\sigma} e^{-\frac{\phi}{2}} F_{\mu\nu} \gamma_{\rho\sigma} + \right.
\]

\[
+ \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} e^{-\frac{\phi}{2}} F_{\mu\nu} e^{-\frac{\phi}{2}} F_{\rho\sigma} \right] . \tag{5.6}
\]
In eq. (5.6), and in what follows, we have suppressed the tensor product symbol. For a metric of the form (2.4), and if only the radial coordinate \( r \) is excited, the induced gamma matrices take the form:

\[
\gamma_0 = [H(p,q)(r)]^{-\frac{1}{2}} \Gamma_0, \\
\gamma_{\theta^i} = [H(p,q)(r)]^{\frac{3}{8}} \left[ r e^{\frac{1}{2}} \Gamma_{\theta^i} + \partial_{\theta^i} r \Gamma_0 \right], \tag{5.7}
\]

where \( e^i \) are the vielbeins on the sphere \( S^3 \). Moreover, when \( r \) only depends on \( \theta^3 \equiv \theta \), the matrix \( \Sigma_0 \) can be written as:

\[
\Sigma_0 = r^2 H(p,q)(r) \sqrt{g} [r' \Gamma_{\theta^3} - r \Gamma_0] \Gamma_{0\theta^1 \theta^2 \theta^3}. \tag{5.8}
\]

In agreement with our analysis of section 4, we shall assume from now on that the only non-vanishing components of \( F_{\mu\nu} \) are \( F_{0\theta} \) and \( F_{\theta^1 \theta^2} \). If we define:

\[
f_{0\theta} \equiv \frac{e^{-\frac{\phi}{2}} F_{0\theta}}{[H(p,q)(r)]^{\frac{3}{4}}}, \\
f_{\theta^1 \theta^2} \equiv \frac{e^{-\frac{\phi}{2}} F_{\theta^1 \theta^2}}{r^2 [H(p,q)(r)]^{\frac{3}{4}} \sqrt{g}}, \tag{5.9}
\]

then, after using eqs. (5.6)-(5.8), we can write eq. (5.1) as:

\[
\sqrt{(r^2 + r'^2 - f_{0\theta}^2)(1 + f_{\theta^1 \theta^2}^2)} \epsilon_{(p,q)} = \\
i\sigma_2 \left[ \sigma_3 (r' \Gamma_{\theta^3} + r \Gamma_0) - f_{0\theta} \right] \left[ \sigma_3 \Gamma_{\theta^1 \theta^2} - f_{\theta^1 \theta^2} \right] \epsilon_{(p,q)}. \tag{5.10}
\]

We shall regard (5.10) as an equation whose unknowns are both the spinors \( \epsilon_{(p,q)} \) and the world-volume field strength \( F \). In order to solve this equation we must know first the general form of the Killing spinors of the \((p,q)\) five-brane background. These spinors can be obtained from the supersymmetric variation of the gravitino and dilatino fields in the type IIB supergravity configurations of eqs. (2.4)-(2.11). In the coordinate system in which we are working, and in the Einstein frame, the \( \epsilon_{(p,q)} \)'s can be written as follows (20):

\[
\epsilon_{(p,q)} = \frac{1}{[H(p,q)(r)]^{\frac{3}{4}}} e^{-\frac{1}{4} \Phi(p,q)(r)} \sigma_2 e^{-\frac{\phi}{2} \Gamma_{\theta^3}} e^{-\frac{\phi}{2} \Gamma_{\theta^2 \theta^3}} e^{-\frac{\phi}{2} \Gamma_{\theta^1 \theta^2}} \epsilon_0, \tag{5.11}
\]

where \( \epsilon_0 \) is a constant spinor of positive ten-dimensional chirality (\( \Gamma_{11} \epsilon_0 = \epsilon_0 \)), satisfying the condition:

\[
\Gamma_{0} \Gamma_{12345} \sigma_1 \epsilon_0 = \epsilon_0, \tag{5.12}
\]

with \( \Gamma_{12345} \) being the product of the \( \Gamma \)-matrices along the parallel coordinates \( x^i \parallel (i = 1, \cdots, 5) \). We are taking the convention in which \( \Gamma_{11} = \Gamma_0 \Gamma_{12345} \Gamma_{\theta^1 \theta^2 \theta^3} \Gamma_7 \), in agreement
with the array \[4.51\]. The \(r\)-dependent angle \(\alpha_{(p,q)}(r)\) is given by:

\[
\sin \left( \alpha_{(p,q)}(r) \right) = \frac{p \left[ H_{(p,q)}(r) \right]^\frac{1}{2}}{\left[ p^2 H_{(p,q)}(r) + g_s^2 \left( q + p\chi_0 \right)^2 \right]^\frac{1}{2}},
\]

\[
\cos \left( \alpha_{(p,q)}(r) \right) = -\frac{g_s \left( q + p\chi_0 \right)}{\left[ p^2 H_{(p,q)}(r) + g_s^2 \left( q + p\chi_0 \right)^2 \right]^\frac{1}{2}}.
\]

\tag{5.13}

Notice that when \((p, q) = (1, 0), (0, 1)\) and \(\chi_0 = 0\) (i.e. for the NS5- and D5-brane, respectively), the angle defined by (5.13) is independent of \(r\) and, in those cases, the Killing spinors only depend on \(r\) through the power of the harmonic function in (5.11). In general, the dependence of \(\epsilon_{(p,q)}\) on \(r\) will be given by the first two terms on the right-hand side of eq. (5.11) and will not be proportional to the unit matrix. As we will verify in a moment, this fact will be crucial in our analysis. Actually, by commuting the matrix appearing on the left-hand side of eq. (5.12) with that of the right-hand side of (5.11), one can obtain the following condition for \(\epsilon_{(p,q)}\):

\[
\Gamma_0 \Gamma_{12345} \sigma_1 \epsilon_{(p,q)} = e^{i\alpha_{(p,q)}(r) \sigma_2} \epsilon_{(p,q)}.
\]

\tag{5.14}

By using the fact that \(\epsilon_{(p,q)}\) has positive chirality, it is easy to prove that eq. (5.14) is equivalent to:

\[
\Gamma_{0r} \epsilon_{(p,q)} = \sigma_{(p,q)}(r) \Gamma_{0\theta^1 \theta^2 \theta^3} (i\sigma_2) \epsilon_{(p,q)},
\]

\tag{5.15}

where \(\sigma_{(p,q)}(r)\) is the following \(r\)-dependent linear combination of the Pauli matrices \(\sigma_3\) and \(\sigma_1\):

\[
\sigma_{(p,q)}(r) \equiv \cos \left[ \alpha_{(p,q)}(r) \right] \sigma_3 + \sin \left[ \alpha_{(p,q)}(r) \right] \sigma_1.
\]

\tag{5.16}

In order to solve the supersymmetry preserving equation (5.10), we must impose some additional constraint to the spinor \(\epsilon_{(p,q)}\). This extra condition is responsible of the fact that our world-volume solitons preserve 1/4 supersymmetry, instead of the 1/2 supersymmetry of the background. The extra requirement we shall demand to \(\epsilon_{(p,q)}\) is the one which follows from the fact that we have a D3-brane placed on the \((p, q)\) five-brane background. This supersymmetry condition associated to the D3-brane probe has to be imposed locally \[10\], i.e. at a particular point of its world-volume. We shall choose this point to be the one with polar coordinate \(\theta^3 = \theta = 0\). Thus, we require that:

\[
\Gamma_{0\theta^1 \theta^2 \theta^3} (i\sigma_2) \epsilon_{(p,q)} \bigg|_{\theta=0} = \epsilon_{(p,q)} \bigg|_{\theta=0}.
\]

\tag{5.17}

Using this condition, it is a simple exercise to find, by using eq. (5.11), the result of acting with \(\Gamma_{0\theta^1 \theta^2 \theta^3} (i\sigma_2)\) on \(\epsilon_{(p,q)}\) at arbitrary angles:

\[
\Gamma_{0\theta^1 \theta^2 \theta^3} (i\sigma_2) \epsilon_{(p,q)} = e^{\theta \Gamma_{\theta^2 \theta^3}} \epsilon_{(p,q)} = \left( \cos \theta + \sin \theta \Gamma_{\theta^2 \theta^3} \right) \epsilon_{(p,q)}.
\]

\tag{5.18}
By using eq. (5.18) in eq. (5.19), one gets the result:

\[ \Gamma_{\theta r} \epsilon_{(p,q)} = \sigma_{(p,q)}(r) \left( \cos \theta + \sin \theta \Gamma_{\theta r} \right) \epsilon_{(p,q)}, \]  

(5.19)

which corresponds to the supersymmetry breaking condition required to a \((-q,p)\)-string in the radial direction at \(\theta = 0\). In order to check this fact, one can verify for the NS5- and D5-brane backgrounds that, indeed, eq. (5.19) implies that the radial spikes in these two cases can be interpreted as a D1-brane and an antifundamental string respectively. As a consequence of eqs. (5.18) and (5.19), one can obtain that:

\[ \Gamma_{\theta r} \epsilon_{(p,q)} = \sigma_{(p,q)}(r) \left( -\sin \theta + \cos \theta \Gamma_{\theta r} \right) \epsilon_{(p,q)}, \]

(5.20)

Eqs. (5.19) and (5.20) allow us to evaluate the right-hand side of eq. (5.10). After performing this evaluation, one gets two types of terms. First of all, one has terms which contain the matrix \(\Gamma_{\theta r}\) which should vanish. This condition leads to the equation:

\[ \sigma_3 \sigma_{(p,q)}(r) \left( r \sin \theta \right)' \Gamma_{\theta r} \sigma_2 - \sigma_1 \sigma_{(p,q)}(r) \Gamma_{\theta r} - i \sigma_2 \left( r \cos \theta \right)' = 0. \]  

(5.21)

We have, in addition, terms without \(\Gamma_{\theta r}\), which give rise to the equation:

\[ (r \sin \theta)' + \sigma_1 \sigma_{(p,q)}(r) (r \cos \theta)' \Gamma_{\theta r} \sigma_2 + i \sigma_2 f_{\theta r} = \]

(5.22)

Eq. (5.21) can be used to determine \(f_{\theta r}\) and \(f_{\theta r} \sigma_2\). Indeed, the left-hand side of this equation contains terms proportional to the Pauli matrix \(\sigma_2\), together with others which are multiple of the unit matrix. By requiring these two classes of terms to vanish independently, one gets two equations which can be solved for \(f_{\theta r}\) and \(f_{\theta r} \sigma_2\) with the result:

\[ f_{\theta r} = \cos[\alpha_{(p,q)}(r)] \left( r \cos \theta \right)' , \]

\[ f_{\theta r} \sigma_2 = \sin[\alpha_{(p,q)}(r)] \left( r \cos \theta \right)' \frac{\left( r \sin \theta \right)'}{r \sin \theta}. \]  

(5.23)

Quite remarkably, one can verify that the values of \(f_{\theta r}\) and \(f_{\theta r} \sigma_2\) given in eq. (5.23) also satisfy eq. (5.22) and, thus, we have succeeded to find the world-volume field strengths corresponding to the D3-brane supersymmetry condition (5.17). It remains to verify that the values we have just found are the same as those obtained in section 4 (eqs. (4.33) and (4.36)), by means of the energy bound argument. The first step in this verification will consist in rewriting the values of \(\sin[\alpha_{(p,q)}(r)]\) and \(\cos[\alpha_{(p,q)}(r)]\) in terms of the dilaton field \(\rho_{(p,q)}\) in the \((p,q)\) five-brane background. By combining eqs. (2.4) and (3.13), one arrives at:

\[ \sin[\alpha_{(p,q)}(r)] = \frac{p e^{-\frac{1}{2}} \left[ H_{(p,q)}(r) \right]^{\frac{1}{2}}}{\left[ \rho_{(p,q)} \right]^{\frac{1}{2}}}, \]

\[ \cos[\alpha_{(p,q)}(r)] = -g_s \left( q + p \chi_0 \right) e^{-\frac{1}{2}} \frac{\left[ H_{(p,q)}(r) \right]^{\frac{1}{2}}}{\left[ \rho_{(p,q)} \right]^{\frac{1}{2}}}. \]  

(5.24)
By substituting these results on the right-hand side of eq. (5.23) and, after using the definitions of \( f_{0\theta} \) and \( f_{0\theta'\theta} \) (eq. (5.9)), it is straightforward to check that the values of \( F_{0\theta} \) and \( J_{12} \) obtained from (5.23) coincide with the ones displayed in eqs. (4.33) and (4.36) (recall that we are adopting the gauge (2.8) for the NS field \( B \)). Therefore, we conclude that the BPS conditions we have used to minimize the energy of the D3-brane probe are, indeed, the ones needed to preserve 1/4 supersymmetry.

6 Concluding remarks

In this paper we have studied the world-volume solitons of a D3-brane probe in the background of a stack of parallel \((p, q)\) five-branes. We have used S-duality to find an energy bound, whose saturation is achieved when a certain first-order BPS equation is satisfied. By solving this BPS equation we have determined the D3-brane embeddings in the \((p, q)\) five-brane geometry. The resulting configurations of the brane probe contain spikes, which can be interpreted as a bundle of \((-q, p)\) strings emanating from the D3-brane world-volume.

An important point in our analysis is the rôle played by the background which, through its coupling to the D3-brane in the WZ term of the action \( S_{WZ} \), provides a source for the world-volume gauge field. For the background we studied two different terms in \( S_{WZ} \) are relevant. Moreover, we have verified that the configurations which saturate the energy bound are precisely those which are 1/4 supersymmetric. This result is very important since it provides a precise interpretation of the BPS equation obtained by the energy argument.

It is clear that the ansatz we have adopted to minimize the energy is not the most general one. We could take, for example, a radial deformation of the brane probe with less symmetry than the \( SO(3) \) invariant one we have considered here. In these new configurations the coordinate \( r \) would depend on the three angles \( \theta^i \) and, in principle, one could apply the methods of ref. [10] to study them. The resulting BPS equation for this case would be more complicated although, from the results of [10], one expects to find at least some of its particular solutions.

It would be also very interesting to apply the methodology followed here to other background and probes. In particular we could apply different chains of dualities (including T-duality, which was not used here) to generate new world-volume solitons from the known ones.

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