From non equilibrium quantum Brownian motion to impurity dynamics in one-dimensional quantum liquids

Julius Bonart and Leticia F. Cugliandolo

1Laboratoire de Physique Théorique et Hautes Energies,
Université Pierre et Marie Curie – Paris VI, 4 Place Jussieu, 75252 Paris Cedex 05, France.

Impurity motion in one dimensional ultra cold quantum liquids confined in an optical trap has attracted much interest recently. As a step towards its full understanding, we construct a generating functional from which we derive the position non equilibrium correlation function of a quantum Brownian particle with general Gaussian non-factorizing initial conditions. We investigate the slow dynamics of a particle confined in a harmonic potential after a position measurement; the rapid relaxation of a particle trapped in a harmonic potential after a quantum quench realized as a sudden change in the potential parameters; and the evolution of an impurity in contact with a one dimensional bosonic quantum gas. We argue that such an impurity-Luttinger liquid system, which has been recently realized experimentally, admits a simple modeling as quantum Brownian motion in a super Ohmic bath.

I. INTRODUCTION

Quantum Brownian motion has been the starting point for the understanding of more complex dissipative quantum systems [1]. Applications to quantum tunnel junctions [2], dissipative two–state systems [3] and reaction–rate theory [4] are just a few among many. In its simplest form, as proposed in the founding papers [5–7], the environment induced dissipation is modeled by an ensemble of quantum harmonic oscillators linearly coupled to the particle of interest. So far, in most studies of the dissipative dynamics of a harmonically confined [8, 9] or a free [10] quantum particle, the quantity of interest has been the reduced density matrix that is obtained by tracing away the bath degrees of freedom in the density matrix of the coupled system. For generic initial conditions this quantity has been obtained with the help of functional integral methods [9, 11]. An alternative simpler, though in general only approximate, description of the reduced density matrix is given by a master equation. For factorizing initial conditions [12] and thermalized initial conditions [13] an exact master equation can be obtained. However, it is also known that there cannot be a master equation – in the form of a partial differential equation local in time – for arbitrary initial conditions [13]. The alternative quantum Langevin approach [14] extensively used in quantum optics [15] is not sufficiently powerful either, for quite the same reason: only a few special initial conditions can be successfully treated within this approach and the quantum noise statistics are not tractable in the generic case.

Non equilibrium correlation functions have been recently observed in cold atoms experiments on the dynamics of an impurity atom moving in a one dimensional (1D) quantum liquid [16, 17]. Both the impurity and the quantum liquid are confined in an optical harmonic trap so that the impurity motion resembles the dynamics of a damped quantum harmonic oscillator. In [19] the authors attempted to describe the impurity dynamics within the Gross-Pitaevskii approach at zero temperature. In this paper we will follow an alternative way by applying quantum Brownian motion theory to the impurity problem. More specifically, we will generalize the path integral formalism in [8] to the language of generating functionals commonly used in quantum field theory. From the generating functional we will easily deduce the non-equilibrium correlation functions for generic non-factorizing Gaussian initial conditions (for a stochastic description of open quantum systems via a generating functional, see [20] that cannot be obtained within the density matrix formalism. We will use this technique to treat three problems: the relaxation dynamics of a particle confined in a harmonic potential after a position measurement performed at the “initial time”; the relaxation dynamics of a particle after an abrupt change in the parameters of the confining potential; the motion of an impurity [18, 21–24] in contact with a 1D quantum gas all confined in a harmonic potential. In the latter case, we will explicitly compare our theoretical findings to recent experimental results [16] on ultra cold quantum gases. We will argue that, although the impurity-Luttinger liquid system is described by the Fröhlich polaron Hamiltonian, many aspects of the impurity dynamics can be understood in the framework of quantum Brownian motion.

More precisely, the paper is organized as follows. In Sec. II we present the model and we review the main results obtained in [9]. In Sec. III we employ path integral methods to derive the generating functional of out of equilibrium correlations. Our results cover both factorized and non-factorized Gaussian initial conditions as well as the effects of an initial position measurement performed on the particle. In Sec. IV we study the equilibration processes after an initial position measurement and after a quench in the harmonic potential and we derive the equilibration times for low and high bath temperatures. In Sec. V we apply our formalism to impurity motion in a 1D quantum gas described by the Luttinger theory. To keep the discussion simple, we choose to use a simplified modeling of the experiment in which we neglect...
polaronic effects as well as the possible renormalization of the external potential. These subtle effects will be analyzed elsewhere. The Luttinger liquid is found to behave as an exotic quantum bath of harmonic oscillators with a highly non Ohmic spectral density and non-linearly coupled to the particle. This is shown to lead to the curious behavior that the oscillator frequency can increase upon increasing the coupling constant between the “bath” and the impurity in strong contrast to the behavior of an Ohmic damped oscillator. We further calculate the non equilibrium equal time correlation function (the variance of the position) and we compare our theoretical results to experimental evidence. In the last section, Sec. VI we conclude and we present further possible applications of our work.

II. THE MODEL

We study the evolution of a particle of mass $M$ evolving in a (possibly time-dependent) potential $V(\tilde{q}; t)$ where $\tilde{q}$ is the position operator. The Brownian motion stems from its interaction with a quantum heat bath which is usually modeled by an infinite set of harmonic oscillators linearly coupled to the position operator $\tilde{q}$. The full system is then described by the Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}$, with

$$\hat{H}_S[\tilde{q}, \tilde{p}] = \frac{\tilde{p}^2}{2M} + V(\tilde{q}; t) - H(t)\tilde{q},$$

$$\hat{H}_B[\{\tilde{x}_n, \tilde{p}_n\}] = \sum_{n=1}^{\infty} \frac{\tilde{p}_n^2}{2m_n} + \frac{m_n\omega_n^2}{2} \tilde{x}_n^2,$$

$$\hat{H}_{SB}[\{\tilde{x}_n\}, \tilde{q}] = -\dot{\tilde{q}} \sum_{n=1}^{\infty} c_n \tilde{x}_n + \tilde{q}^2 \sum_{n=1}^{\infty} \frac{c_n^2}{2m_n\omega_n^2}.$$

$\hat{p}$ is the momentum operator of the particle. The last term in Eq. (1) introduces a time-dependent source $H(t)$, a c-number, that couples linearly to the particle’s position $\tilde{q}$. $\tilde{x}_n$ and $\tilde{p}_n$ are the position and momentum operators of the $n$-th harmonic oscillator, with mass and frequency $m_n$ and $\omega_n$, respectively. $c_n$ is the coupling strength between the particle and the $n$-th oscillator’s position. The last term in Eq. (3) compensates for the bath-induced renormalization of the potential. Indeed, the sum of Eqs. (2) and (3) can be rewritten as

$$\hat{H}_B + \hat{H}_{SB} = \sum_{n=1}^{\infty} \frac{\tilde{p}_n^2}{2m_n} + \frac{m_n\omega_n^2}{2} \left[\tilde{x}_n - \frac{c_n}{m_n\omega_n^2} \tilde{q}\right]^2,$$

which shows the absence of any drift force induced by the bath and ensures that $V(\tilde{q}, t)$ corresponds to the physical potential right from the start. The model Hamiltonian Eqs. (2)-(3) has been widely used in the literature as a generic model for the dissipative dynamics of a quantum particle.

In the Heisenberg representation the time evolution of all possible observables $\hat{A}$ is governed by

$$\hat{A}(t) = \left[\mathcal{T} \exp \left(-\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right) \right]^\dagger \hat{A} \times \left[\mathcal{T} \exp \left(-\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right) \right],$$

with $\mathcal{T}$ the time-ordering operator. By introducing the density matrix of the initial state $\hat{\rho}_0$ the $N$-time average of a set $\{\hat{A}_i\}$ of $N$ operators, is

$$\langle \hat{A}_N(t_N)\hat{A}_{N-1}(t_{N-1})\cdots\hat{A}_1(t_1) \rangle = \text{Tr} \left[\hat{A}_N(t_N)\cdots\hat{A}_1(t_1)\hat{\rho}_0 \right],$$

where we took the product of the $\hat{A}_i$s to be time ordered (with $t_N \geq t_{N-1} \geq \cdots \geq t_1$) so that we can more easily make the connection between Eq. (6) and its path integral representation. We assumed that $\text{Tr} \hat{\rho}_0$ is normalized to one. Note that for a generic initial matrix $\hat{\rho}_0$ this, as well as any other, correlation function is not necessarily stationary, i.e., it may depend on the $N$ times explicitly.

In all cases the model has to be supplemented by information on the initial condition of the coupled system. These are incorporated in the initial density matrix $\hat{\rho}_0$. Equilibrium dynamics can be studied by choosing $\hat{\rho}_0$ to be the Boltzmann weight, that is

$$\hat{\rho}_0 = \exp(-\beta \hat{H}),$$

where $\hat{H}$ is the full coupled Hamiltonian and the normalization constant has been ignored. This truly equilibrium density matrix has to be distinguished from $\hat{\rho}_0 = e^{-\beta \hat{H}_S} \otimes e^{-\beta \hat{H}_B}$, a case in which each component of the “universe” (the whole particle–bath system) is in equilibrium on its own at the same temperature. This subtle point is often overlooked in the literature.

Non equilibrium dynamics can be studied whenever the initial density matrix is not of the form in Eq. (7). The simplest choice is an initial product state for which the initial density matrix factorizes into two contributions $\hat{\rho}_{S0} \otimes \hat{\rho}_{B0}$ which solely depend on particle and bath variables, respectively:

$$\hat{\rho}_0 = \hat{\rho}_{S0} \otimes \hat{\rho}_{B0}.$$

Brownian motion as well as the dynamics of more complex macroscopic systems with a factorized initial density matrix have been studied in a variety of physical situations. However, in many cases it is not appropriate to assume Eq. (8) since one has no command over the bath and it is impossible to “switch it on and off” at will. In addition, with recent developments in cold atom experiments, new classes of initial conditions become of relevance.

The first one covers all situations in which the particle is in equilibrium in a potential and either it is released or the potential is suddenly modified at $t = 0$. In this case
Eq. (7) holds with $\hat{H}$ replaced by $\hat{H}_0$ ($\neq \hat{H}$) describing the initial state.

The second class concerns all situations in which the position of the free particle is measured at $t = 0$. This procedure projects the initial density matrix onto the quantum states of the measurement outcome. We focus on the case where no quantum quench is performed in addition to the position measurement so that $\hat{H}_0 = \hat{H}$. If the position is exactly determined at $t = 0$ the initial density matrix is

$$\hat{\rho}_0 = \hat{\Pi}(q) e^{-\beta \hat{H}} \hat{\Pi}(q) \quad \text{with} \quad \hat{\Pi}(q) = |q\rangle \langle q| \quad (9)$$

the projection operator onto the state $|q\rangle$. If, instead, we take the measured position of the particle to be Gaussian distributed around $q_m$ the projection operator takes the form

$$\hat{\Pi}(q_m) = \int dq \, e^{-\frac{(q-q_m)^2}{4\Delta^2}} |q\rangle \langle q| , \quad (10)$$

where $\Delta$ measures the uncertainty of the particle’s position at $t = 0$. Once again we neglected the irrelevant normalization factor.

A third important class of initial conditions are the factorized density operators, see Eq. (8), in which the initial state is a pure state. Since any state can be expanded in terms of displaced Gaussians (or coherent states) it suffices to consider initial states of the form

$$\hat{\rho}_{S0} = |\psi\rangle \langle \varphi| , \quad (11)$$

where

$$\psi(q) = e^{-\frac{(q-q_0)^2}{4\delta^2}} \quad \text{and} \quad \varphi(q') = e^{-\frac{(q'-q'_0)^2}{4\delta^2}} , \quad (12)$$

to cover the whole class of initially factorized pure states.

In this article we derive a generating functional that allows us to obtain the $N$-time correlators for these types of initial conditions. We are mainly interested in the evolution and averages of the particle’s position observables for which $\hat{A} = A[\hat{q}]$ with some function $A$ depending on the position $\hat{q}$ of the particle. Note that due to the coupling of the bath to the particle’s position the momenta dynamics follow from the Heisenberg equation $M \partial_t \hat{q} = \hat{p}$. Therefore, by focusing on the particle position operator we simultaneously describe the dynamics of the particle’s momentum. While in $[9]$ the authors derived an explicit expression for the equilibrium correlation functions the generating functional will allow us to go beyond the equilibrium case.

III. THE GENERATING FUNCTIONAL

In this section we derive the generating functional of all (non-equilibrium) correlation functions. This goes beyond the analysis in $[9]$ where explicit expressions for equilibrium correlation functions were given. More precisely, we derive a functional of two time-dependent sources $\mathcal{J}[F,G]$ such that the two-time correlation is given by

$$\langle \hat{q}(t) \hat{q}(t') \rangle = \frac{e^{\mathcal{J}[F,G]}}{e^{\mathcal{J}[F,G]|_{F,G=0}}} \quad (13)$$

and similarly for higher order correlations.

We obtain the path integral formulation of the generating functional by making use of the coherent states of the bath variables $|\{\xi_{n,f}\}\rangle$ which are defined in App. $[\Delta]$. The ensuing functional integration includes paths over particle and bath variables. Since we are not interested in the degrees of freedom of the bath, we average over all bath variables to find a “reduced action” that only depends on the particle variables. In the special cases discussed below (e.g., harmonic potential) the remaining path-integrals can also be performed and the functional $\mathcal{J}$ can be fully determined. In this section we sketch all steps in the derivation. Further technical details are reported in App. $[\Delta]$ The reader who is not interested in these technical details can jump directly to Eq. (54) where its rather lengthy final expression is given.

A. The density matrix

In terms of the product states $|q, \{\xi_{n,f}\}\rangle$ between the particle and the bath eigenstates the matrix elements of the time evolution operator read

$$\mathcal{K}(q_f, \{\xi_{n,f}\}; q_i, \{\xi_{n,i}\}; t) \equiv \langle q_f, \{\xi_{n,f}\}|\hat{T} e^{\frac{-i}{\hbar} \int_0^t dt' \hat{H}(t')}|q_i, \{\xi_{n,i}\}\rangle \quad (14)$$

and of its Hermitian conjugate

$$\mathcal{K}^*(q'_f, \{\xi'_{n,f}\}; q'_i, \{\xi'_{n,i}\}; t) \equiv \langle q'_i, \{\xi'_{n,i}\}|\hat{T} e^{\frac{i}{\hbar} \int_0^t dt' \hat{H}(t')}|q'_f, \{\xi'_{n,f}\}\rangle \quad (15)$$

$\hat{T}$ is the anti-chronological time ordering operator. The elements of the time-dependent density matrix, $\hat{\rho}(t) \equiv \mathcal{K}_0(t) \mathcal{K}^*$, are given by

$$\mathcal{W}(q_f, \{\xi_{n,f}\}; q'_f, \{\xi'_{n,f}\}; t) \equiv \langle q_f, \{\xi_{n,f}\}|\hat{\rho}(t)\hat{\rho}(t)^\dagger|q'_f, \{\xi'_{n,f}\}\rangle \quad (16)$$

where the matrix elements of the initial density matrix have been denoted by

$$\mathcal{W}(q_i, \{\xi_{n,i}\}; q'_i, \{\xi'_{n,i}\}) = \langle q_i, \{\xi_{n,i}\}|\hat{\rho}_0\hat{\rho}_0^\dagger|q'_i, \{\xi'_{n,i}\}\rangle \quad (17)$$

and we used the short-hand notation

$$d\xi_i = \prod_{n=1}^\infty e^{-\xi_{n,i}^2} d\xi_{n,i} d\xi_{n,i}^* \quad (18)$$
The path integral representations of $K$ and $K^*$ are

$$K(q_f, \{\xi_{ni}\}; q_i, \{\xi_{ni}\}; t) = \int \mathcal{D}q^+ \mathcal{D}\xi^+ e^{\phi[S^+(\xi^+)]},$$

$$K^*(q'_{f}, \{\xi'_{ni}\}; q'_i, \{\xi'_{ni}\}; t) = \int \mathcal{D}q^- \mathcal{D}\xi^- e^{-\phi[S^-(\xi^-)]},$$

where we made clear with the superscripts $+$ and $-$ which paths belong to $K$ and $K^*$, respectively. The functional integration measures are defined in App. A.

### B. Reduced density matrix for a system initially coupled to an equilibrium bath

The time-dependent density matrix in Eq. (10) still contains information about the degrees of freedom of the bath which we are not interested in. Therefore, we average (trace) over all the bath variables to find a reduced density matrix that depends only on the particle variables and the external sources.

We are interested in a system that is initially coupled to an equilibrium bath. Therefore,

$$\hat{H}_{B0} = \hat{H}_B, \quad \hat{H}_{SB0} = \hat{H}_{SB}, \quad (19)$$

where all initial Hamiltonians are labeled with a subscript $0$. At this point it is not necessary to make $\hat{H}_{SB0}$ explicit since this term involves only particle variables that are not affected by the trace over the bath variables. The matrix element of the initial density operator Eq. (17) in Eq. (10) can be represented by an imaginary time path integral

$$W(q_i, \{\xi_{ni}\}; q'_i, \{\xi'_{ni}\}) = \int \mathcal{D}q^0 \mathcal{D}\xi^0 e^{-\frac{\beta}{2}S_0[q^0, (\xi^0)]},$$

where the initial action $S_0$ is in general different from $S$ reflecting the fact that $H_0 \neq H$. The reduced density matrix can now be recast as

$$W(q_f; q'_f; t) = \mathcal{D}q_d \mathcal{D}q'_d \mathcal{D}q^+ \mathcal{D}q^- \mathcal{D}q^0 \mathcal{D}\xi^0 e^{\frac{i}{\hbar}S_0[q^0, (\xi^0)]} \mathcal{F}[q^+, q^-, q^0],$$

where $\mathcal{F}[q^+, q^-, q^0]$ is the “influence functional” that depends only on the particle variables, as also do $S_S$ and $S_{S0}$. The path integral runs over all paths with

$$q^+(t) = q_f, \quad q^+(0) = q_i, \quad q^-(t) = q'_f, \quad q^-(0) = q'_i, \quad q^0(\beta \hbar) = q^0(0), \quad q^0(0) = q^0(0),$$

which is the reason for the name “closed-time path integral”. It is convenient to introduce the linear combinations

$$x = \frac{q^+ + q^-}{2} \quad \text{and} \quad \bar{x} = q^+ - q^-.$$  \hspace{1cm} (20)

The calculation of $\mathcal{F}$ can be found in App. A1 or in B; the result reads

$$\mathcal{F}[x, \bar{x}, q^0] = e^{\frac{i}{\hbar}\Phi[x, \bar{x}, q^0]},$$

with

$$\Phi[x, \bar{x}, q^0] = \frac{i}{2} \int_0^{\beta \hbar} d\tau \kappa(\tau - \sigma)q^0(\tau)q^0(\sigma)$$

$$+ \int_0^{\beta \hbar} d\tau \int_0^t ds K^*(s - i\tau)q^0(\tau)\bar{x}(s)$$

$$- \frac{i}{2} \int_0^t ds du \kappa_R(s - u)\bar{x}(s)\bar{x}(u)$$

$$- M \int_0^t ds \frac{d}{ds} \bar{x}(s) \int_0^s du \gamma(s - u)x(u). \quad (22)$$

The kernels $K(\theta)$, $\gamma(t)$ and $k(t)$ are defined in Eqs. (A31), (A32) and (A33), respectively. $K_R$ denotes the real part of $K$. Note that $\Phi[x, \bar{x}, q^0]$ depends on the fixed “end-points” $t$ and $\beta \hbar$ of the closed-time path.

Expected values evaluated at different times are now expressed in terms of a path-integral over $q^0$, $x$ and $\bar{x}$ with an effective action $\Sigma$,

$$\langle...\rangle = \int dx_f dx_i d\bar{x}_i \int Dx D\bar{x} Dq^0 ... e^{i\Sigma[x, \bar{x}, q^0]},$$

(23)

where $\Sigma[x, \bar{x}, q^0]$ is given by

$$\Sigma[x, \bar{x}, q^0] =$$

$$\Phi[x, \bar{x}, q^0] + iS_{S0}[q^0] + S_{S}[x + \bar{x}/2] - S_{S}[x - \bar{x}/2]$$

$$= \Phi[x, \bar{x}, q^0] + i \int_0^{\beta \hbar} d\tau \left[ \frac{M_0}{2} q^0(\tau)^2 + V_0(q^0) \right]$$

$$+ \int_0^t ds \left[ M \ddot{x} - V(x + \bar{x}/2) + V_H(x - \bar{x}/2) \right]. \quad (24)$$

We introduced the initial mass $M_0$ of the particle and the initial potential $V_0$ that are in general different from the “bulk” mass $M$ and potential $V$. This allows for quenches in these parameters. Note that the case in which the initial state is a pure state [e.g. Eq. (11)] can be easily recovered by setting $M_0 = 0$ and $V_0 = 0$ or, equivalently, by noting that the path $q^0$ shrinks identically to zero (since there is no initial Hamiltonian for this simple type of initial condition).

The superscripts in the path integral in Eq. (23) remind us of the constraint that the paths are subject to. One has

$$x(0) = x_i, \quad \bar{x}(0) = \bar{x}_i,$$

$$x(t) = x_f, \quad q^0(0) = q^0(0) = x_i + \frac{\bar{x}_i}{2}, \quad (25)$$

$$\bar{x}(t) = 0, \quad q^0(\beta \hbar) = q^0(0) = x_i - \frac{\bar{x}_i}{2}.$$  \hspace{1cm} (25)

Note that due to the periodic boundary conditions of the trace $\bar{x}_f = q^0(t) - q^0(t) = 0$.

### C. Generic Gaussian initial conditions

It is very easy to include the change of $\Phi[x, \bar{x}, q^0]$ induced by the initial position measurement in Eq. (9).
using the explicit Gaussian form of the projector $\hat{\Pi}(q_m)$ [see Eq. (11)] the dependence on the initial measurement can be simply incorporated in $\Sigma[x,\bar{x},q^0]$ by an additional term of the form

$$\frac{i\hbar}{4\Delta^2} \left[ (x_i + \bar{x}_i - q_m)^2 + (x_i - \bar{x}_i - q_m)^2 \right] = \frac{i\hbar}{2\Delta^2} \left[ (x_i - q_m)^2 + \frac{x_i^2}{4} \right].$$

(26)

In the limit of strong uncertainty $\Delta \to \infty$ the effect of the initial measurement is blurred.

In order to recover the case where the initial state of the system is pure and decouples from the environment [which corresponds to the factorized initial density matrix with $\hat{\rho}_{\text{in}}$ given by Eq. (11)] the action in Eq. (24) has to be supplemented by

$$\frac{i\hbar}{4\Delta^2} \left[ (x_i + \bar{x}_i - q_m)^2 + (x_i - \bar{x}_i - q_m)^2 \right] = \frac{i\hbar}{2\Delta^2} \left[ x_i^2 + \frac{x_i^2}{4} + x_m^2 + \frac{x_m^2}{4} - 2x_mx_i - \frac{1}{2} \bar{x}_m\bar{x}_i \right].$$

(27)

with the notation

$$x_m = (q_m + q_m')/2 \quad \text{and} \quad \bar{x}_m = (q_m - q_m').$$

Since Eq. (26) is a special case of Eq. (27) corresponding to $q_m = q_m'$ (or $x_m = q_m$ and $\bar{x}_m = 0$) we will work with the latter in the following. The relevant cases can then be selected by taking simple limits.

In the following expressions we will write only the terms that depend on $x_i$ or $\bar{x}_i$ since the ones depending on $x_m$ and $\bar{x}_m$ contribute only to an overall constant.

D. The sources

The source term appears as $\int dt' H(t')q^+(t')$ in $\mathcal{K}$ [see Eq. (14)] and as $-\int dt' H(t')q^-(t')$ in $\mathcal{K}^*$ [see Eq. (15)]. For convenience, we distinguished the function existing on the positive running branch of the closed time contour, which we still call $H(t)$, from the one existing on the negative running branch of the same contour, which we call $H'(t)$. This implies that the potentials in Eq. (24) are given by $V_H(y) = V(y) - H y$ and $V_{H'}(y) = V(y) - H' y$.

After the transformation of variables in Eq. (24) we obtain two external time-dependent sources $F(s) = [H(s) + H'(s)]/2$ and $G(s) = [H(s) - H'(s)]$ which couple linearly to the variables $\bar{x}(s)$ and $x(s)$, respectively. All correlation functions can be computed from the generating functional $\mathcal{J}[F,G]$ as derivatives of $\mathcal{J}$ with respect to $F$ or $G$ evaluated at $F = G = 0$. A physical force is represented by $H(s) = H'(s)$, that is by $F(s) \neq 0$ and $G(s) = 0$. Therefore, the linear response of the mean value Eq. (25) to an external force can be obtained for $F(s) \neq 0$.

The generating functional, that is to say, the trace over the reduced density matrix in the presence of the external sources reads

$$e^{\mathcal{J}[F,G]} \sim \int d\tau d\bar{x}_0 d\bar{x}_f \int' Dx D\bar{x} Dq^0 e^{\Sigma[x,\bar{x},q^0,x_0,\bar{x}_0,\bar{x}_f]} .$$

where the path integral is subject to the constraints in Eqs. (23). The overall normalization factor depends on $t$, $\beta$ and all parameters in the model but not on the fields. We can now write

$$\langle \ddot{q}(t) \rangle = \left. \frac{\hbar}{i} \frac{\delta}{\delta G(t)\delta G(t')} \right|_{F,G \equiv 0} e^{\mathcal{J}[F,G]}$$

(29)

and

$$\langle \ddot{q}(t)\ddot{q}(t') \rangle = \left. \frac{\hbar}{i} \frac{\delta}{\delta G(t)\delta G(t')} \right|_{F,G \equiv 0} e^{\mathcal{J}[F,G]}$$

(30)

and all other correlation functions can be obtained in a similar way by noting that $q^+(t) = q_f = x_f$ and $q^-(t') = x(t) + \bar{x}(t)/2$. At this point it has become obvious why two sources are needed in order to obtain all non-equilibrium correlation functions.

E. The harmonic case

To go further we restrict ourselves to the study of a quantum Brownian particle in a harmonic potential for which

$$-V(x + \bar{x}/2; s) + V(x - \bar{x}/2; s) = -M\omega^2 x \bar{x}$$

(31)

and

$$V_0(q^0) = \frac{1}{2} M\omega^2 q^0(q^0)^2 .$$

(32)

The choice of a quadratic potential renders the problem analytically solvable. The generating functional can be calculated by simply evaluating the action on its minimal action path (over the initial condition branch and the time-dependent branches) as Gaussian fluctuations yield only pre-factors that are independent of the sources and can be determined at the end of the calculation from the normalization of the density matrix. Note that, although both initial and bulk potentials are harmonic, they are not necessarily the same thus allowing for the study of quantum quenches.

F. Integration over the initial condition

We first treat the contribution of the initial condition path $q^0$ in Eq. (24). The equation of motion for $q^0$ can be easily obtained from Eq. (24):

$$Mq^0(\tau) = -\int_0^{\beta\hbar} d\sigma \ k(\tau - \sigma)q^0(\sigma) - M\omega^2 q^0(\tau)$$

$$\quad + i \int_0^t ds \ K^*(s - i\tau)\bar{x}(s) ,$$

(33)
with the fixed end-points \( q^0(0) = q^+ \) and \( q^0(\beta \hbar) = q^- \). As the \( q^0 \) path is part of the whole closed-time path it implicitly depends on the fixed end-time \( t \) as well. In \cite{9} one can find a detailed analysis of this equation of motion which uses a Fourier expansion of the path \( q^0(\tau) \) on the interval \([0, \beta \hbar]\). By using the results found therein we obtain

\[
\Sigma[x, \bar{x}, x_f, \bar{x}_i] = \frac{i}{2M_0} \int_0^t ds du \ R'(s, u) \bar{x}(s) \dot{x}(u) + iM_0 \left[ \frac{1}{2\Delta} q'' + \frac{\Omega}{2} q' \right] + \frac{i\hbar}{2\Delta^2} \left[ x'' + \frac{\bar{x}^2}{4} - 2x_{m} \bar{x}_i \right]
\]

\[
+ \int_0^t ds \ M \left[ \dot{x} \dot{\bar{x}} - \omega^2 \bar{x} x + \frac{1}{M} \dot{F}(s) \bar{x}(s) + \frac{1}{M} G(s) x(s) - \bar{x}(s) \frac{d}{ds} \int_0^s du \gamma(s - u) x(u) \right]. \tag{34}
\]

**G. Real time minimal action paths with external sources**

The equations of motion for \( x(s) \) and \( \bar{x}(s) \) read

\[
\ddot{x}(s) + \frac{1}{M} \int_0^s du \ \gamma(s - u) x(u) + \omega^2 x(s)
\]

\[
= \frac{\bar{F}(s)}{M} + \frac{i}{M M_0} \int_0^t du \ R'(s, u) \bar{x}(u), \tag{43}
\]

\[
\ddot{\bar{x}}(s) - \frac{d}{ds} \int_s^t du \ \gamma(u - s) \bar{x}(u) + \omega^2 \bar{x}(s) = \frac{G(s)}{M}. \tag{44}
\]

The action \( \Sigma \) evaluated along the minimal action paths can be determined by inserting the solutions to Eqs. \(43\) and \(44\) into Eq. \(34\). However, the authors of \cite{9} noted a simplification of the calculation which can be generalized to our case where the source \( G(s) \) is also present (in \cite{9} no external source for \( \bar{x} \) was used). The idea is the following. After a partial integration in the second line of Eq. \(34\) the action \( \Sigma \) takes the form

\[
\Sigma[x, \bar{x}, x_f, \bar{x}_i] = -M \bar{x} \dot{x}_f - \frac{i}{2M_0} \int_0^t ds du \ R'(s, u) \bar{x}(s) \dot{x}(u)
\]

\[
+ \int_0^t ds \ G(s) x(s) + \text{border}(x_i, x_f, \bar{x}_i) \tag{45}
\]

when evaluated along the minimal action paths determined by Eqs. \(43\) and \(44\), where we used the boundary condition \( \bar{x}_f = 0 \). Here, \( \text{border}(x_i, x_f, \bar{x}_i) \) stands for all border terms in Eq. \(34\).

On the other hand, one can split the force Eq. \(35\) into its real and imaginary parts \( \bar{F}(s) = \bar{F}_R(s) + i\bar{F}_I(s) \). Then, the minimal action path \( x(s) \) splits into \( x(s) = x_R(s) + i x_I(s) \), where \( x_I(s) \) satisfies the boundary conditions \( x_I(0) = x_I(t) = 0 \). The trick is to show now that one can simply focus on the real part \( x_R(s) \) of the minimal action path in order to obtain the complete stationary phase action. Indeed, if we evaluate the action

We introduced the complex “force”

\[
\bar{F}(s) = F(s) + x_i C_1(s) - i \bar{x}_i C_2(s), \tag{35}
\]

with the functions \( C_1 \) and \( C_2 \)

\[
C_1(s) = \frac{1}{\beta \hbar} \sum_{k=-\infty}^{\infty} u_k g_k(s),
\]

\[
C_2(s) = \frac{1}{\beta \hbar} \sum_{k=-\infty}^{\infty} u_k \nu_k f_k(s). \tag{36}
\]

The constants \( \Lambda \) and \( \Omega \) are given by

\[
\Lambda = \frac{1}{\beta \hbar} \sum_{k=-\infty}^{\infty} u_k \text{ and } \Omega = \frac{1}{\beta \hbar} \sum_{k=-\infty}^{\infty} u_k (\omega_0^2 + \zeta_k),
\]

\[
\text{with } u_k = (\omega_0^2 + \nu_k^2 + \zeta_k)^{-1}, \nu_k = 2 \pi k / \beta \hbar, \zeta_k = [M \gamma(t) - g_k(0)] / M_0 \text{ [for the definition of } \gamma(t) \text{ see Eq. \cite{12} below]}
\]

\[
g_k(s) = \int_{0}^{\infty} \frac{d\omega}{\pi} S(\omega) \omega^2 + \nu_k^2 \cos(\omega s), \tag{38}
\]

\[
f_k(s) = \int_{0}^{\infty} \frac{d\omega}{\pi} S(\omega) \frac{2 \nu_k}{\omega^2 + \nu_k^2} \sin(\omega s), \tag{39}
\]

where \( S(\omega) \) is the spectral density of the bath. The two-time function \( R'(s, u) \) reads

\[
R'(s, u) = R(s, u) + M_0 K_R(s - u),
\]

\[
R(s, u) = -\Delta C_1(s) C_1(u)
\]

\[
+ \frac{1}{\beta \hbar} \sum_{k=-\infty}^{\infty} u_k [g_k(s) g_k(u) - f_k(s) f_k(u)], \tag{40}
\]

with

\[
K_R(s - u) = \frac{1}{\beta \hbar} \sum_{k} g_k(s - u) \tag{41}
\]

the real part of the kernel \( K \). The time-dependent bath kernel \( \gamma(s) \) is given by [see Eq. \(\Delta12\)]

\[
\gamma(s) = \frac{2}{M} \int_{0}^{\infty} \frac{d\omega}{\pi} S(\omega) \cos(\omega s). \tag{42}
\]

The functions \( C_1 \) and \( C_2 \) as well as the kernel \( R(s, u) \) are not to be confused with the correlation functions and the linear response function that will be denoted by \( \mathcal{C} \) and \( \mathcal{R} \), respectively.
Eq. (53) only along the minimal \( x_R(s) \) and \( \bar{x}(s) \) we obtain

\[
\Sigma[x_R, \bar{x}, x_f, \bar{x}_f] = -M \bar{x} \dot{x}_{R,i} + \int_0^t ds \ G(s) x_R(s)
\]

\[
+ \int_0^t ds \ \bar{x}(s) \left[ \bar{F}_I(s) + \frac{i}{2M_0} \int_0^t du \ R'(s, u) \bar{x}(u) \right]
+ \text{border} (x_i, x_f, \bar{x}_i), \tag{46}
\]

where we used the fact that \( x_R(s) \) satisfies the real part of Eq. (43). We now want to show that Eqs. (46) and (45) coincide. It is sufficient to evaluate the right-hand-side (rhs) of Eq. (45) and the rhs of Eq. (46) coincide. The help of the imaginary part of Eq. (45) and the equation of motion (44) we can easily prove by integration by parts that

\[
\int_0^t ds \bar{x}(s) \left[ \bar{F}_I(s) + \frac{i}{M_0} \int_0^t du \ R'(s, u) \bar{x}(u) \right] = -M \bar{x} \dot{x}_{I,i} + \int_0^t ds \ G(s) x_I(s),
\]

and by using this identity in Eq. (46) we recover Eq. (45). Therefore, the right-hand-side (rhs) of Eq. (45) and the rhs of Eq. (46) coincide. It is sufficient to evaluate the action Eq. (46) along the real component \( x_R(s) \) that satisfies a much simpler equation than \( x(s) \).

In terms of the end points \( x_i, x_f \) and \( \bar{x}_i \), the solutions to the real parts of Eqs. (43) and (44) read

\[
x_R(s) = \frac{G_+(s)}{G_+(t)} x_f + \left( \frac{G_+(s)}{G_+(t)} - \frac{G_+(s)}{G_+(t)} \right) \bar{x}_i
+ \frac{1}{M} \int_0^s du \ G_+(s-u) \bar{F}_R(u)
- \frac{1}{M} \int_0^t du \ G_+(t-u) \bar{F}_R(u), \tag{47}
\]

and

\[
\bar{x}(s) = \frac{G_+(t-s)}{G_+(t)} \bar{x}_i + \frac{1}{M} \int_s^t du \ G_+(u-s) G(u)
- \frac{1}{M} \int_0^t du \ G_+(u) G(u), \tag{48}
\]

where \( G_+(t) \) is a propagator that in Laplace-transform reads

\[
\tilde{G}_+(\lambda) = \frac{1}{\lambda^2 + \lambda \gamma(\lambda) + \omega^2}. \tag{49}
\]

From Eq. (47) we immediately find, by using the boundary conditions \( G_+(0) = 0, \tilde{G}_+(0) = 1 \) and \( \tilde{G}_+(0) = 0, \)

\[
\dot{x}_{R,i} = \frac{1}{G_+(t)} x_f - \frac{\tilde{G}_+(t)}{G_+(t)} \bar{x}_i
- \frac{1}{G_+(t)} \frac{1}{M} \int_0^t du \ G_+(t-u) \bar{F}_R(u). \tag{50}
\]

Inserting the solutions to Eqs. (47) and (48), and Eq. (46) into Eq. (46) we find an effective action \( \Sigma[x_i, x_f, \bar{x}_i] \) that depends only on the end-points, \( x_i, x_f \) and \( \bar{x}_i \), and the external sources \( F \) and \( G \):

\[
\Sigma[x_i, x_f, \bar{x}_i] = -i M_0 \frac{\bar{x}}{\epsilon^2} x_i x_m - i M_0 \frac{\bar{x}}{4 \sqrt{2}} \bar{x}_i \bar{x}_m - M \bar{x} x_f \frac{1}{\tilde{G}_+(t)} + \frac{1}{\tilde{G}_+(t)} \int_0^t du \ G_+(t-u) \bar{F}_R(u)
- i \bar{x}_i \int_0^t ds C_2(s) \bar{x}(s) + i M_0 \frac{\bar{x}_i^2}{\lambda'} + \frac{1}{M} \int_0^t du \ G(s) \tilde{G}_+(t) G(s) \bar{F}_R(u)
+ \frac{1}{2} \int_0^t du \ G(s) \tilde{G}_+(t) G(s) \bar{F}_R(u)
- \frac{1}{M} \int_0^t du \ G(s) \tilde{G}_+(t-u) G(s) \bar{F}_R(u), \tag{51}
\]

with

\[
\frac{1}{\lambda'} = \frac{1}{\lambda} + \frac{1}{\epsilon}, \quad \Omega' = \Omega + \frac{1}{4 \epsilon^2},
\]

and

\[
\epsilon^2 = \frac{M_0 \lambda^2}{h}. \tag{52}
\]

For the sake of a clear presentation we have not replaced \( \bar{x}(s) \) and \( \bar{F}_R \) by their corresponding expressions in terms of the end-points yet.

### H. Integration over the end-points

In order to find the final expression for \( \mathcal{J}[F, G] \) we still have to integrate over the end-points \( x_i, x_f \) and \( \bar{x}_i \). Since the exponent \( \Sigma[x_i, x_f, \bar{x}_i] \) is linear in \( x_f \) the integration over this variable generates a \( \delta \)-function of the form

\[
\delta \left[ \bar{x}_i - \frac{1}{M} \int_0^t ds \ G_+(s) G(s) \right], \tag{53}
\]
up to a factor not depending on the end-points which, in combination with the integration over \( \bar{x}_i \), enforces a substitution of \( \bar{x}_i \) by \( \frac{1}{M} \int_0^t ds \, \mathcal{G}_+(s)G(s) \) and \( \bar{x}(s) \) by \( \frac{1}{M} \int_0^t du \, \mathcal{G}_+(u - s)G(u) \) [the first and the third terms of the rhs of Eq. \( \text{[15]} \) cancel]. Moreover, after the integration over \( \bar{x}_i \) the fifth and the last terms of the rhs of Eq. \( \text{[51]} \) cancel, too. Finally, the Gaussian integral over \( x_i \) yields

\[
\mathcal{J}[F,G] = -\frac{\Lambda'}{2\hbar M_0} \left( \int_0^t ds \, G(s)\hat{\mathcal{G}}_+(s) \right)^2 + \frac{M_0}{4\hbar e^2 M} \bar{x}_m \int_0^t ds \, \mathcal{G}_+(s)G(s) \\
- \frac{\Lambda'}{\hbar M M_0} \int_0^t ds \, G(s)\hat{\mathcal{G}}_+(s) \int_0^s ds' \int_0^t du' \mathcal{G}_+(s' - u')C_1(u')G(s') - \frac{\Omega M_0}{2\hbar M^2} \left( \int_0^t ds \, G(s)\hat{\mathcal{G}}_+(s) \right)^2 \\
+ \frac{i}{M} \int_0^t ds \, \mathcal{G}_+(s - u)G(s)F(u) + \frac{1}{\hbar M^2} \int_0^t ds \, \mathcal{G}_+(s)G(s) \int_0^t \int_0^s ds' \int_0^t du' \mathcal{G}_+(u' - s')C_2(s')G(u') \\
+ \frac{i\Lambda'}{\hbar e^2 x_m} \left[ \int_0^t ds \, G(s)\hat{\mathcal{G}}_+(s) + \frac{1}{M} \int_0^t ds \, \int_0^s du \, \mathcal{G}_+(s - u)G(s)C_1(u) \right] \\
- \frac{1}{2\hbar M^2 M_0} \int_0^t ds \, \int_0^t ds' \int_s^t du \, \mathcal{G}_+(u - s)R''(s, s')\mathcal{G}_+(u' - s')G(u)G(u') \tag{54}.
\]

Here we introduced the kernel

\[ R''(s, s') = \Lambda 'C_1(s)C_1(s') + R'(s, s') , \tag{55} \]

where we used Eq. \( \text{[A36]} \). Equation \( \text{[54]} \) is the central result of the first part of this paper. It allows us to derive all non-equilibrium correlation functions in a systematic way. Direct applications of this method will be presented in Secs. \( \text{IV} \) and \( \text{V} \).

### I. The correlation function

The two-time correlation function Eq. \( \text{[30]} \) has two contributions,

\[
\langle \hat{q}(t)\hat{q}(t') \rangle = \frac{1}{2} \langle [\hat{q}(t), \hat{q}(t')]_+ \rangle + \frac{1}{2} \langle [\hat{q}(t), \hat{q}(t')]_- \rangle \\
= \mathcal{C}(t, t') + i\mathcal{A}(t - t') \tag{56}.
\]

The first term, the average of the anti-commutator or symmetrized contribution, is real and the second one, the average of the commutator or anti-symmetrized contribution, is imaginary and proportional to the linear response function, \( R(t, t') \), as shown by the Kubo formula. For generic Gaussian initial conditions, Eq. \( \text{[11]} \), one finds

\[
\mathcal{C}(t, t') = \frac{\Lambda \hbar}{M_0} \left[ \hat{\mathcal{G}}_+(t)\hat{\mathcal{G}}_+(t') + \frac{\hat{\mathcal{G}}_+(t)}{M} \int_0^{t'} du \, \mathcal{G}_+(t' - u)C_1(u) + \frac{1}{M} \hat{\mathcal{G}}_+(t') \int_0^t du \, \mathcal{G}_+(t - u)C_1(u) \right] \\
+ \frac{\hbar \Omega M_0}{M^2} \mathcal{G}_+(t)\mathcal{G}_+(t') \left[ \mathcal{G}_+(t) \int_0^{t'} du \, \mathcal{G}_+(t' - u)C_2(u) + \mathcal{G}_+(t') \int_0^t du \, \mathcal{G}_+(t - u)C_2(u) \right] \\
+ \frac{\Lambda^2}{e^4} x_m^2 \left[ \hat{\mathcal{G}}_+(t) + \frac{1}{M} \int_0^t du \, \mathcal{G}_+(t - u)C_1(u) \right] \left[ \hat{\mathcal{G}}_+(t') + \frac{1}{M} \int_0^t du \, \mathcal{G}_+(t' - u)C_1(u) \right] \\
- \frac{M_0^2}{16e^4 M^2 x_m^2} \hat{\mathcal{G}}_+(t)\hat{\mathcal{G}}_+(t') + \frac{\hbar}{M^2 M_0} \int_0^t \int_0^{t'} du \, \mathcal{G}_+(t - u)R''(u, u')\mathcal{G}_+(t' - u') \tag{57}.
\]

To represent a pure initial condition that is initially decoupled from the bath, as in Eq. \( \text{[11]} \), we set \( M = M_0 \), \( \Lambda' = e^2 \) and \( \Omega' = 1/4e^2 \) as well as \( C_1(s) = 0 \), \( C_2(s) = 0 \) and \( R''(u, u') = MK(u - u') \). A non-factorized initial state with - say - an initial position measurement of "width" \( \epsilon \) is obtained with \( x_m = q_m \) and \( \bar{x}_m = 0 \). In the
classical case the non equilibrium correlator of a Brownian particle is given by Eq. (133).

IV. NON EQUILIBRIUM DYNAMICS AFTER QUANTUM QUENCHES.

Quenches from a high temperature initial state have been extensively studied in the literature. They correspond to the case of a factorizing density matrix as in Eq. (5). In this section we will study the non-equilibrium dynamics of a quantum Brownian particle after quenches from different initial states. As already mentioned in the Introduction, there are two experimental scenarios that are of interest to us. In the first one the initial position of a Brownian particle trapped by a harmonic potential is measured at $t = 0$. We will study this case in the first part of this section by focusing on the two-time correlation function. The second scenario consists in a quench of the trapping potential, which in the case of a harmonic potential corresponds to an abrupt change in the trapping frequency. Such a quench will be studied in the second part of this section. In both cases, we will derive the asymptotic equilibration behavior of the system in the presence of Ohmic dissipation.

A. A particle in a harmonic potential with an initial position measurement

The general results in Sec. III are here specialized to the case of a particle trapped in a harmonic potential, on which a position measurement is performed at $t = 0$. As we work with the same particle initially and subsequently, $M_0 = M$, while $\omega_0 = \omega > 0$. At $t = 0$ a measurement of the particle position is performed with outcome $q_m = 0$ and uncertainty $\Delta$. The initial density matrix is given by Eq. (7). Note that the particle is permanently coupled to the bath, hence the initial density matrix does not factorize. Thus, our starting point is the general expression Eq. (57) with $M = M_0$, $\omega = \omega_0$ and $x_m = \bar{x}_m = 0$. The Laplace transform of the correlator reads

$$\tilde{C}(\lambda, \kappa) = \frac{\hbar}{M} \tilde{G}_\perp(\lambda) \tilde{G}_\perp(\kappa) \left\{ \Lambda' \lambda \kappa + \frac{\Lambda'}{M} \lambda \tilde{C}_1(\kappa) + \frac{\Lambda'}{M} \kappa \tilde{C}_1(\lambda) + \frac{\Omega' M_0^2}{M^2} - \frac{1}{M} \tilde{C}_2(\lambda) - \frac{1}{M} \tilde{C}_2(\kappa) + \frac{1}{M^2} \tilde{R}(\lambda, \kappa) \right\} . \tag{58}$$

an expression that can be simplified by using the method explained in App. C.

Introducing the function $\tilde{G}_\perp(\lambda) = \tilde{g}(\lambda)/M + \lambda$, Eq. (58) can be written as

$$\tilde{C}(\lambda, \kappa) = \frac{\hbar}{M} \tilde{G}_\perp(\lambda) \tilde{G}_\perp(\kappa) \left( \frac{\Lambda' - \Lambda}{(\beta M)^2} \sum_{k,k'} u_k u_{k'} \tilde{h}_k(\lambda) \tilde{h}_{k'}(\kappa) \right) + \frac{\hbar}{4 M c^2} \tilde{G}_\perp(\lambda) \tilde{G}_\perp(\kappa) + \tilde{C}^{1\text{eq}}(\lambda) + \tilde{C}^{1\text{eq}}(\kappa), \tag{59}$$

with the equilibrium correlation function $\tilde{C}^{1\text{eq}}(\lambda)$ in the Laplace domain defined in Eq. (C12). In App. C it is shown that $\tilde{h}(\lambda)$ can be written in terms of $\tilde{G}_\perp^{-1}(\lambda)$ and $\tilde{G}_\perp^{-1}(\nu_k)$ [see Eq. (C11)]. For $\omega = \omega_0$ we use the fact that $u_k = \tilde{G}_\perp(\nu_k)$ and we find from the definition of the equilibrium correlator

$$\tilde{C}^{1\text{eq}}(\lambda) = \frac{1}{\beta M} \sum_k \frac{\lambda}{\nu_k - \omega} \left[ \tilde{G}_\perp(\lambda) - \tilde{G}_\perp(\nu_k) \right] , \tag{60}$$

which is derived in App. C [see Eq. (C12)], that the desired non equilibrium correlation function of a quantum Brownian particle with initial position measurement reads

$$\tilde{C}(\lambda, \kappa) = \frac{M}{\hbar} \frac{\Lambda' - \Lambda}{\Lambda^2} \tilde{C}^{1\text{eq}}(\lambda) \tilde{C}^{1\text{eq}}(\kappa) + \frac{\hbar}{4 M c^2} \tilde{G}_\perp(\lambda) \tilde{G}_\perp(\kappa) + \frac{\tilde{C}^{1\text{eq}}(\lambda) + \tilde{C}^{1\text{eq}}(\kappa)}{\lambda + \kappa} . \tag{61}$$

The classical correlator for an initial position measurement with outcome $q_0 = 0$ and uncertainty $\Delta$ can be obtained from Eq. (58) by replacing $(\tilde{g}(\lambda))$ by $\Delta^2$. The initial momentum is not measured and it is therefore distributed according to the Boltzmann–law with $(\nu_0^2) = (\beta M)^{-1}$. In the limit of a sharp position measurement ($\Delta \to 0$) the classical correlator of an equilibrium particle reads $\tilde{C}^{\text{eq}}(\lambda, \kappa) - B M \omega^2 \tilde{C}^{\text{eq}}(\lambda) \tilde{C}^{\text{eq}}(\kappa)$. As to the quantum correlator, we note that $\Lambda' = 0$ for $\Delta \to 0$ and the sum of the first and the third term in the rhs of Eq. (61) yields $\tilde{C}^{\text{eq}}(\lambda, \kappa) = M/(\hbar \Lambda) \tilde{C}^{\text{eq}}(\lambda) \tilde{C}^{\text{eq}}(\kappa)$ which already has the form of its classical counterpart. It is easy to show that in the high temperature limit $\beta \hbar \ll |\omega^2 - \gamma^2/4|$ the two expressions coincide exactly. The role of the second term in the rhs of Eq. (61) remains to be discussed: it describes the initial momentum due to Heisenberg’s uncertainty relation. Consequently, it diverges when the initial position measurement becomes sharp unless one considers that $\hbar/(M c^2) = \hbar^2/(M^2 \Delta^2) \to 0$ even though $\Delta \to 0$. More precisely, when $\beta \hbar |\omega^2 - \gamma^2/4|/12 \ll 1$ we have $\tilde{C}^{\text{eq}} \sim 1/(\beta M \omega^2)$ and $\tilde{G}_\perp \sim 1/\omega$. In order for the second term of the rhs of Eq. (61) to be small compared to the other terms the condition $\Delta \gg \lambda_T$ must hold, with $\lambda_T = \sqrt{\beta \hbar^2/(2 M \hbar)}$ the thermal de Broglie–wavelength of the particle. Only then can one speak of a classical particle: the condition $\beta \hbar |\omega^2 - \gamma^2/4|^{1/2} \ll 1$ that properly defines the “high temperature regime” is not sufficient. One also has to take the “macroscopic measurement limit” defined through $\Delta \gg \lambda_T$. 
In the real time domain the quantum correlation function is

$$C(t, t') = \frac{M \Lambda' - \Lambda}{\hbar \Lambda^2} C^{\text{eneq}}(t') C^{\text{eneq}}(t') + \frac{\hbar}{4 M c^2} G_{+}(t) G_{+}(t') + C^{\text{eneq}}(|t - t'|), \quad (62)$$

where $C^{\text{eneq}}(t)$ is the real time equilibrium correlation function. This result applies to any kind of spectral density of the bath. We will use the correlation function Eq. (62) in Sec. V to study the non equilibrium dynamics of an impurity in a Luttinger liquid bath for which a specific spectral density of the bath applies.

We are interested in the equilibration behavior of Eq. (62) in the presence of Ohmic dissipation for which the spectral density behaves as $S(\omega) \sim \gamma \omega$ for small $\omega$, with $\gamma$ playing the role of a friction coefficient [for more details see App. D]. It is of special interest to study the strong quantum regime $\beta \hbar \gg |\omega^2 - \gamma^2/4|^{-1/2}$. By using the long–time limits Eq. (65) and Eq. (68) presented in App. D valid for $t \gg \gamma^{-1}$ we find that the propagator $G_{+}(t)$ exponentially approaches zero whereas the equilibrium correlation function $C^{\text{eneq}}(t)$ relaxes with a power law $\sim t^{-2}$ for $t \to \infty$. In the long–time limit the correlator Eq. (62) thus relaxes as fast as the squared equilibrium correlation function. Therefore, for $\beta \hbar \gg |\omega^2 - \gamma^2/4|^{-1/2}$:

$$C(t, t') = C^{\text{eneq}}(|t - t'|) + O([tt'])^{-2} \quad (63)$$

when $t, t' \gg \gamma^{-1}$, and the equilibrium function $C^{\text{eneq}}(|t - t'|) = C^{\text{eq}}(t, t')$ is asymptotically approached during the algebraic relaxation of the non–equilibrium terms. Consequently, care has to be taken in experiments when an equilibrium system is desired at very low temperatures after an initial position measurement. The relaxation of the system is slow independently of the dissipation strength $\gamma$. At high temperatures $\beta \hbar \ll |\omega^2 - \gamma^2/4|^{-1/2}$ the relaxation is of order $O(e^{-\gamma t})$ [see the discussion in App. D] and therefore exponential as in the classical limit.

B. Quantum quench in the confining potential.

In this section we desire to gain insight into the equilibration process of a quantum Brownian particle after an abrupt change in the trapping frequency. At $t < 0$ the particle is confined in a harmonic potential with frequency $\omega_0 > 0$. At $t = 0$ the experimentalist abruptly changes the strength of the harmonic potential resulting in a higher or lower trapping frequency. We do not consider an initial position measurement since we assume that the particle is already localized by the initial harmonic trap. Hence we set $\Lambda = \Lambda'$, $\Omega = \Omega'$, $x_m = \bar{x}_m = 0$ and $c \to \infty$. By starting from Eq. (63) and by using Eq. (40) it is straightforward to show with the methods employed in App. C that the correlator in the Laplace domain reads

$$\tilde{C} (\lambda, \kappa) = \frac{\hbar}{M} \tilde{G}_{+}(\lambda) \tilde{G}_{+}(\kappa) \frac{1}{\lambda + \kappa} \left[ \delta^{\text{eneq}}(\lambda) + \delta^{\text{eneq}}(\kappa) \right], \quad (64)$$

where $\tilde{G}_{+}(\lambda) = 1/|\lambda^2 + \gamma(\lambda) + \omega_0^2|$ is the propagator with initial frequency $\omega_0$, and

$$\delta^{\text{eneq}}(\lambda) = \frac{1}{\beta M} \sum_k \frac{\lambda}{\nu_k^2 - \lambda^2} \left[ \tilde{G}_0^{\text{eneq}}(\lambda) - \tilde{G}_0^{\text{eneq}}(|\nu_k|) \right], \quad (65)$$

is the equilibrium correlation function of a particle in a harmonic potential with frequency $\omega_0$ [see Eq. C12]. The structure of Eq. (64) is very different from the classical counterpart Eq. (B7). Still, by using the high temperature approximation $\delta^{\text{eneq}}(\lambda) \approx - \left[ \tilde{G}_0^{\text{eneq}}(\lambda) - 1/\omega_0^2 \right]/\lambda$ [see the end of the discussion in App. D] one recovers the classical expression Eq. (B3).

The equilibration time for Ohmic dissipation in the strong quantum regime $\beta \hbar \gg |\omega^2 - \gamma^2/4|^{-1/2}$ can be found in the following way. Note first that when $\delta^{\text{eneq}}(\lambda)$ is multiplied by $\tilde{G}_{+}(\lambda)/\tilde{G}_0^{\text{eneq}}(\lambda)$ one obtains a new function that we call $C^{\text{eq}}$:

$$C^{\text{eq}}(\lambda) = \frac{1}{\beta M} \sum_k \frac{\lambda}{\nu_k^2 - \lambda^2} \left[ \tilde{G}_{+}(\lambda) - \tilde{G}_0^{\text{eneq}}(|\nu_k|) \tilde{G}_{+}(\nu_k) \right]. \quad (66)$$

Second, we observe that the Laplace transform of $(\partial_t + \partial_{t'}) C^{\text{eq}}(t, t')$ is equal to $(\lambda + \kappa) \tilde{f}(\lambda, \kappa) - \tilde{f}(t = 0; \kappa) - \tilde{f}(\lambda; t' = 0)$ where $\tilde{f}(\lambda, \kappa)$ is the Laplace transform of the generic function $f(t, t')$ with respect to both $t$ and $t'$ while $\tilde{f}(\lambda; t' = 0)$ [$\tilde{f}(t = 0; \kappa)$] is the Laplace transform of $f(t, t' = 0)$ [$f(t = 0, t')$] with respect to $t'$ ($t$). Now, by choosing $f(t, t') = C^{\text{eq}}(t, t')$ we find for the correlation function Eq. (64) in the time domain

$$(\partial_t + \partial_{t'}) C^{\text{eq}}(t, t') = C^{\text{eq}}(t) \left[ 1 + (\omega^2 - \omega_0^2) \tilde{G}_{+}(t') \right] + C^{\text{eq}}(t') \left[ 1 + (\omega^2 - \omega_0^2) \tilde{G}_{+}(t) \right] - C(0, t) - C(0, t'). \quad (67)$$

From Eq. (64) we easily find the expression of the Laplace transform of $C(t, t' = 0)$ by multiplying $\tilde{C}(\lambda, \kappa)$ by $\kappa$ and by taking the limit $k \to \infty$:

$$\tilde{C}(\lambda; t' = 0) = \tilde{C}_0^{\text{eneq}}(\lambda) \left[ 1 + (\omega^2 - \omega_0^2) \tilde{G}_{+}(\lambda) \right] = \tilde{C}^{\text{eq}}(\lambda). \quad (68)$$

Accordingly, Eq. (67) simplifies to

$$(\partial_t + \partial_{t'}) C^{\text{eq}}(t, t') = C^{\text{eq}}(t) (\omega^2 - \omega_0^2) \tilde{G}_{+}(t') + C^{\text{eq}}(t') (\omega^2 - \omega_0^2) \tilde{G}_{+}(t). \quad (69)$$

Finally, it is clear that $(\partial_t + \partial_{t'}) C^{\text{eq}}(t, t') = 0$ when $C(t, t') = C^{\text{eq}}(t, t') = C^{\text{eneq}}(|t - t'|)$ so that the terms of the rhs of
Eq. (69) can be understood as the derivative of the non equilibrium part of \( C(t,t') \). In combination with the results of the discussion in App. D the equilibration behavior of the correlator at low temperatures after a quench in the trapping potential can be summarized as follows [by noting that the asymptotic long–time behavior of \( C(t,t') \) and \( C_{\text{eq}}(t) \) are identical]:

\[
C(t,t') = C_{\text{eq}}(|t-t'|) + \mathcal{O} \left[ e^{-\gamma t'/2}/t' + e^{-\gamma t}/2t \right]
\]  

\[
(70)
\]

for \( t,t' \gg \gamma^{-1} \). Consequentially, the non equilibrium contributions are exponentially suppressed which leads to a faster equilibration than the one found after an initial position measurement.

At high temperatures the relaxation of \( C(t) \) and \( C_{\text{eq}}(t) \) are both exponential of \( \mathcal{O}(e^{-\gamma t/2}) \) so that in the high temperature regime \( C(t,t') = C_{\text{eq}}(|t-t'|) + \mathcal{O} \left[ e^{-\gamma(t+t')/2} \right] \).

V. EVOLUTION OF AN IMPURITY IN A 1D BOSE GAS

Many 1D systems are described by the Tomonaga-Luttinger theory [37 39]. The evolution of impurities in such a Luttinger liquid (LL) has attracted much attention in recent years [17, 19] since this problem can be explicitly realized with cold atom systems. In particular, modern techniques allow one to tune the interspecies interaction strength [10 42] so that it has become possible to study the diffusion of a minority species within an ensemble of majority atoms, as a function of the interaction and the trapping potential [16].

In this part of our work we apply the non equilibrium formalism developed in the first part of our article to such an impurity–LL system. In particular, we seek to mimic the experimental process described in [16] with our theoretical description. In this experiment the impurity atom is trapped in a 1D harmonic potential together with an ensemble of a different kind of atoms that form the LL. The impurity is initially localized at the center of the confining potential by a laser blade. When the whole impurity–LL system reaches equilibrium the impurity is released. The equal–times position correlation function of the impurity, \( C(t,t) \), then shows damped oscillations which strongly suggest that the impurity is de facto a quantum Brownian particle moving in a quantum liquid bath.

In the following we will present a precise description of the impurity motion in the LL from the quantum Brownian motion point of view. The LL itself will play the role of an exotic quantum bath that we here characterize.

A. The impurity model

The impurity and the atoms constituting the bath are all confined in a harmonic potential. We therefore take the Hamiltonian of the impurity, \( \hat{H}_S \), to be of the standard form Eq. (1) without external force \( (H = 0) \) and with the harmonic potential \( V(q) = \frac{1}{2} M \omega^2 q^2 \). We assume that the interaction Hamiltonian between the position operator \( \hat{q} \) of the impurity and the density of the boson liquid is of the form

\[
\hat{H}_{SB} = \int \text{d}x \text{d}y \, U(x-y) \rho(y) \delta(x-\hat{q}) ,
\]  

\[
(71)
\]

with the density operator \( \rho(x) \) of the LL approximately described by

\[
\rho(x) \approx \rho_0(x) - \frac{1}{\pi} \nabla \phi(x) ,
\]  

\[
(72)
\]

where \( \rho_0(x) \) is the unperturbed density of the fluid in the 1D trap and \( \phi(x) \) is the density variation. The Hamiltonian of the free LL reads

\[
\hat{H}_B = \frac{\hbar}{2\pi} \int \text{d}x \, \left[ \frac{u}{\hbar^2} (\pi \bar{\Pi}(x))^2 + \frac{u}{K_L} (\nabla \phi(x))^2 \right]
\]  

\[
(73)
\]

where \( \bar{\Pi}(x) \) and \( \phi(x) \) are conjugate operator fields. Equation (73) describes the low-energy properties of a Lieb-Liniger gas [42] with a contact interaction potential \( \hbar v_L \delta(x) \). The parameters \( u \) (with the dimension of a velocity) and \( K_L \) (dimensionless) have to be determined numerically for general \( v_L \). In order to reduce the complexity of the problem we will assume the background density \( \rho_0 \) to be constant in the following. Accordingly, we define

\[
\rho_0 \equiv \frac{1}{L} \int \text{d}x \, \rho_0(x) ,
\]  

\[
(74)
\]

where \( L \) is a length scale of the order of the length of the trap. Note that in this modeling we have not added the quadratic confining potential to the LL.

Since the Bose gas is confined in a space of length \( L \), the wave vectors are quantized with values \( k_n = \pi n/L \) with \( n \) an integer. The Fourier representation of Eq. (44) is

\[
\hat{H}_{SB} = \frac{1}{\sqrt{L}} \sum_n \hat{U}_n e^{ik_n \hat{q}} \left[ -\frac{i k_n}{\pi} \hat{\phi}(k_n) \right] ,
\]  

\[
(75)
\]

where we used \( \delta(x-x') = (1/L) \sum_n e^{ik_n(x-x')} \) and we neglected a constant contribution. We assume that the potential \( U \) has the form

\[
\hat{U}_k = \hbar v e^{-|k|/k_c} ,
\]  

\[
(76)
\]

with some finite cutoff \( k_c \) that depends on the microscopic properties of the interaction. The parameter \( v \) has the dimension of a velocity and it determines the strength of the impurity–bath interaction potential.

After redefining the fields according to

\[
\hat{\phi}(x) \mapsto \sqrt{(\pi K_L/\hbar)} \, \hat{\phi}(x)
\]  

\[
(77)
\]

and

\[
\bar{\Pi}(x) \mapsto \sqrt{(\hbar/\pi K_L)} \, \bar{\Pi}(x) ,
\]  

\[
(78)
\]
one introduces the bosonic ladder operators
\[ \hat{b}_k = \sqrt{|k|/2\hbar} \left( \hat{\phi}(k) + \frac{i}{|k|} \hat{\Pi}(k) \right) \] (79)
and
\[ \hat{b}_k^\dagger = \sqrt{|k|/2\hbar} \left( \hat{\phi}(k) - \frac{i}{|k|} \hat{\Pi}(k) \right), \] (80)

\[ \hat{\mathcal{H}} = \sum_{k \in \{k_n\}} \hbar u|k| \hat{b}_k^\dagger \hat{b}_k + \frac{\beta^2}{2M} q^2 - \frac{1}{\sqrt{L}} \sum_k \left( \frac{K_L}{2\pi|k|} \right)^{1/2} \tilde{U}_k \left[ (ik e^{i\tilde{q}}) \hat{b}_k + (ik e^{i\tilde{q}})^* \hat{b}_k \right] . \] (81)

For each \( k \) mode the coupling between the operator \( e^{ik\tilde{q}} \) and the bath operators \( \hat{b}_k^\dagger \) and \( \hat{b}_k \) is bilinear, so we can use the general results derived in Sec. [14] by consider-

\[ \Phi[q^+, q^-, q^0] = -\sum_k \left\{ \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\sigma \ K_k(-i\tau + i\sigma)e^{ikq^0(\tau)e^{-ikq(\sigma)}} + i \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} ds K^*_k(s - i\tau)e^{ikq^0(\tau)e^{-ikq(-s)}} \right. \\
- \left. \int_0^{\beta\hbar} ds \int_0^{\beta\hbar} du \left[ e^{ikq^+(-s)} - e^{ikq^-(s)} \right] \left[ K_{-k}(s - u)e^{-ikq^+(u)} - K^*_k(s - u)e^{-ikq^-(u)} \right] \right\} . \] (82)

Note that, since in the initial Hamiltonian there is no counter-term the counter-terms in Eq. (82) are absent in Eq. (83). The \( k \)-dependent kernel reads
\[ K_k(\theta) = \frac{1}{L} \frac{K_L|k|}{2\pi \hbar} \left| \tilde{U}_k \right|^2 \frac{\cosh[u|k|\beta\hbar/2 - i\theta u|k|]}{\sinh[u|k|\beta\hbar/2]} \] (83)

A well known feature of the Fröhlich polaron Hamiltonian \([81]\) is its polaronic mass shift which leads to an effective impurity mass \( M^* > M \) greater than the “bare mass” \( \frac{1}{2} \). Another process described by this Hamiltonian is that during a collision between the impurity and an LL atom the former loses momentum \( \hbar k \) by creating a density wave excitation \( \hat{b}_k^\dagger \) in the LL. However, the LL is itself confined in a harmonic potential and one may expect to have some momentum transfer absorbed (or provided) from the LL to the optical trap. We will here simply assume that the spring constant of the optical potential is renormalized by such a process in such a way that it balances the polaronic mass shift and we will henceforth work with the bare Hamiltonian \([81]\).

If the oscillations are small (if the impurity potential is sufficiently steep) we can expand the \( e^{ik\tilde{q}} \) in Eq. (82) to second order in \( k \). Note that the linear order vanishes in Eq. (82) due to the symmetry \( k \mapsto -k \). Ignoring the zero-th order in \( k \) we can make the replacements
\[ e^{ikq^0(\tau)e^{-ikq(\sigma)}} \rightarrow -\frac{k^2}{2} \left[ q^0(\tau) - q^0(\sigma) \right]^2 , \\
\[ e^{ikq^0(\tau)}e^{-ikq^0(\sigma)} \rightarrow -\frac{k^2}{2} \left[ q^0(\tau) - q^0(\sigma) \right]^2 , \\
k^2 \left[ q^+(s) - q^-(s) \right] \rightarrow \frac{k^2}{2} \left[ q^+(s)^2 - q^-(s)^2 \right] , \\
\left[ e^{ikq^+(s)} - e^{ikq^-(s)} \right] \times \left[ K_{-k}(s - u)e^{-ikq^+(u)} - K^*_k(s - u)e^{-ikq^-(u)} \right] \rightarrow \frac{k^2}{2} \left[ q^+(s)^2 - q^-(s)^2 \right] \left( K_k(s - u) - K^*_k(s - u) \right) . \] (84)
B. Steep potential: The Luttinger bath and the initial condition.

If the external potential is steep we expect the Gaussian approximation of the Fröhlich Hamiltonian [83] to be valid. Up to this order in \( k \) each \( k \)-mode plays the role of one bath harmonic oscillator. Accordingly, for \( L \to \infty \) we define the spectral density as

\[
S(\omega) = \frac{\pi}{L} \sum_{k \in \{k_n\}} \frac{K_L |k|^3}{2\pi \hbar} |\hat{U}_k|^2 \delta(\omega - u|k|)
\]

\[
\simeq \frac{K_L \omega^3 \hbar v^2}{\pi u^4} e^{-\omega/\omega_c} = \frac{\pi \mu}{4} \left( \frac{\omega}{\omega_c} \right)^3 e^{-\omega/\omega_c},
\]

(85)

The first line in Eq. (88) shows that only the non-local part of \( K(-i\tau + i\sigma) \) contributes. Hence we can replace the first line by \( \frac{1}{4} \int_0^{\beta \hbar} d\sigma \kappa(-i\tau + i\sigma)q^0(\tau)q^0(\sigma) \). Furthermore, we see that the last line in Eq. (88) exactly represents the counter-term proportional to \( \mu \). By rewriting Eq. (88) in terms of the kernels \( K_R \) and \( \gamma \) [see Eqs. (A33) and (A42)] and using the transformed variables \( x = (q^+ + q^-)/2 \) and \( \bar{x} = q^+ - q^- \) the action becomes

\[
\Sigma[\bar{x}, \bar{q}^0, q^0, x_i, f_i, \bar{f}_i] = i \int_0^{\beta \hbar} d\tau \left[ \frac{M}{2} \dot{q}^0(\tau)^2 + \frac{M \omega^2}{2} q^0(\tau)^2 + \frac{1}{2} \int_0^{\beta \hbar} d\sigma \kappa(\tau - \sigma)q^0(\tau)q^0(\sigma) \right]
\]

\[
+ \int_0^{\beta \hbar} d\tau \int_0^t ds K^*(s - i\tau)q^0(\tau)\bar{x}(s) + \int_0^t ds \left[ M \dot{x}(s)\dot{\bar{x}}(s) - M \omega^2 \bar{x}(s)x(s) - M \bar{x}(s) \frac{d}{ds} \int_0^s du \gamma(s - u)x(u) \right]
\]

\[
+ \frac{1}{2} \int_0^t ds \int_0^t du K_R(s - u)\bar{x}(s)x(u) + \frac{i \hbar}{2\Delta^2} \left[ x_i^2 + \frac{\bar{x}_i^2}{4} \right].
\]

(89)

We remind the reader of the main approximations used so far. First, we used the Gaussian approximation [84] of the Fröhlich Hamiltonian. Second, in order for the action [88] to make sense we assumed \( \omega \) to be large enough, so that non Gaussian effects are not too important. Note, that the mass and the potential renormalization can modify the oscillation amplitude and the final width of the impurity position. Finally, we interpreted the laser blade that initially localizes the impurity at the center of the quantum liquid as an initial position measurement with outcome \( q_m = 0 \) and uncertainty \( \Delta \). The effect of the initial position measurement is incorporated into the action via the last term of the rhs of Eq. (89). If the localization is performed itself by a very steep trapping potential with frequency \( \Omega_0 \) the particle is in its ground state (with respect to \( \Omega_0 \)) ground state at initial time. We then have

\[
\epsilon^{-2} = 2\Omega_0.
\]

(90)

This approximation is disputable since one could also consider the initial localization of the impurity as stemming from an initial trapping potential. The subsequent release of the impurity would then rather be described by a quench in the harmonic potential [see Sec. IV A]. However, in real experiments the “high temperature” regime \( \hbar \beta \omega \ll 1 \) prevails. From the discussion in Secs. IV A and IV B we know that in this case the difference between the particle motion after an initial position measurement and the one after an initial localization due to an initial trapping potential (followed by a quench in the potential)
is blurred. Since it is technically easier to deal with a position measurement we prefer this method to a quench in the trapping potential. Note that the “high temperature” regime is not equivalent to the classical regime, as we pointed out in Sec. IV A as it does not fulfill the “macroscopic measurement” condition $\Delta \gg \lambda T$ with $\lambda T$ the thermal de Broglie– wavelength of the impurity. For more details go back to the discussion in Sec. IV A.

C. Signature of a Luttinger liquid bath

A typical experimental scenario consists in holding the impurity in the center of the trap at $t < 0$ and switching off the localizing potential at $t = 0$ to let the impurity move in the residual harmonic potential. Let us for a moment forget about the potential renormalization and the mass shift. By using the results found in Sec. IV A we find in the limit $\epsilon \rightarrow 0$ (which corresponds to an almost perfect localization of the impurity at $t = 0$)

$$\tilde{C}(\lambda, \kappa) \approx \frac{\hbar}{4ME^2} \tilde{G}_+(\lambda) \tilde{G}_+(\kappa) - \frac{M}{\hbar \Lambda} \tilde{C}_{1eq}(\lambda) \tilde{C}_{1eq}(\kappa) + \frac{1}{\lambda + \kappa} \left[ \tilde{C}_{1eq}(\lambda) + \tilde{C}_{1eq}(\kappa) \right].$$

(91)

In the time domain the correlation function reads

$$C(t, t') \approx \frac{\hbar}{4ME^2} G_+(t) G_+(t') - \frac{M}{\hbar \Lambda} C_{1eq}(t) C_{1eq}(t') + C_{1eq}(|t - t'|).$$

(92)

For the moment, experimental measurements focus on the equal-time correlation (which corresponds to the time-dependent variance of the position)

$$C(t, t) \approx \frac{\hbar}{4ME^2} G_+^2(t) - \frac{M}{\hbar \Lambda} C_{1eq}(t)^2 + C_{1eq}(0).$$

(93)

The formula Eq. (93) is valid for all kinds of baths and for very small polaronic effects and potential renormalization.

In order to gain further insight into the dynamics of the impurity we need to understand the contribution of each of the three terms in Eq. (93).

We start from the analysis of the propagator $G_+(t)$. To determine its time-dependence we need the specific form of the spectral density of the LL bath $\tilde{S}$. In terms of the function

$$g(z) = \frac{1}{2} \int_0^\infty d\zeta \zeta^2 e^{-\zeta} \frac{z^2}{\zeta^2 + z^2} = \frac{z^2}{2} - \frac{z^4}{2} \int_0^\infty d\zeta \frac{e^{-\zeta}}{\zeta^2 + z^2}$$

(94)

the Laplace transform of the damping kernel $\tilde{g}(\lambda)$ [see Eqs. (85)] can be recast as

$$\lambda \tilde{g}(\lambda) = \frac{\mu}{M} g(\lambda/\omega_c)$$

(95)

and the propagator (which is proportional to the linear response function) reads

$$\tilde{G}_+(\lambda) = \frac{1}{\lambda^2 + \frac{1}{M^2} g(\lambda/\omega_c) + \omega^2}.$$  

(96)

Our objective is to find the oscillation frequency and the damping of the impurity motion in the small coupling limit. Thus, we need the inverse Laplace transform of Eq. (96) which can be expressed in terms of the Bromwich integral,

$$\tilde{G}_+(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda \ e^{\lambda t} \tilde{G}_+(\lambda),$$

(97)

where the real number $c$ is greater than the real part of all poles of $\tilde{G}_+(\lambda)$. The integral in Eq. (97) can be solved by displacing the complex contour towards the left and by evaluating the encountered residues. Hence, we seek for the complex points at which the denominator of Eq. (96) vanishes, that is $z^2 + \mu' g(z) + \omega^2/\omega_c^2 = 0$ with

$$z = i\sigma - \Gamma, \quad \Gamma = \frac{\Gamma}{\sigma} \ll 1 \quad \text{and} \quad \mu' = \frac{\mu}{M \omega_c^2}. $$

(98)

Note that $z$, $\sigma$, $\Gamma$ and $\mu'$ are dimensionless and $\sigma$ and $\Gamma$ real. From Eq. (97) it is clear that $\Gamma$ corresponds to the exponential damping (measured in units of $\omega_c$) of the propagator and hence $\Gamma$ has to be positive (whereas $\sigma$ is the oscillating frequency measured in units of $\omega_c$ and can be positive or negative). Thus, the first singularities encountered when displacing the contour towards the left are the imaginary axis for which $g(z)$ is obviously singular when $\text{Re} \ z = 0$. In other words, we have to account for the difference $g(i\sigma + 0^+) - g(i\sigma - 0^-)$ when displacing the integration contour past the imaginary axis. We will come back to this point in just a few lines. The integral in the second term of the rhs of Eq. (97) can be recast as

$$I(-\Gamma, \sigma) \equiv \int_0^\infty d\zeta \frac{e^{-\zeta}}{\zeta^2 + z^2}$$

$$\approx \frac{1}{|\sigma|} \int_0^\infty d\zeta \frac{e^{-|\sigma|\zeta}}{\zeta^2 - 1 - 2i\Gamma/\sigma}$$

$$\approx \frac{1}{|\sigma|} \left[ A(|\sigma|) + \text{sign} \left( \frac{\Gamma}{\sigma} \right) \frac{i\pi}{2} e^{-|\sigma|} \right]$$

$$+ \frac{\Gamma}{\sigma^2} \left[ \pi \frac{1}{2} (1 + |\sigma|) e^{-|\sigma|} + i \text{sign} \left( \frac{\Gamma}{\sigma} \right) B(|\sigma|) \right],$$

where we expanded the denominator of the integrand to first order in $\Gamma/\sigma$ and we defined

$$A(\sigma) = \sinh \sigma \ \text{Chi} \sigma - \cosh \sigma \ \text{Shi} \sigma,$$

$$B(\sigma) = \sigma [\cosh \sigma \ \text{Chi} \sigma - \sinh \sigma \ \text{Shi} \sigma] + \cosh \sigma \ \text{Shi} \sigma - \sinh \sigma \ \text{Chi} \sigma,$$

where Shi and Chi are the hyperbolic sine and cosine integrals, respectively. The contribution from the crossing of the imaginary axis is now easily obtained:
$\Delta I = I(0^+, \sigma) - I(-0^+, \sigma) = -i\pi e^{-|\sigma|}/\sigma$. After adding $-z^4 \Delta I/2$ to $g(z)$ we thus obtain for $\text{Re} \ z < 0$

$$g(z) \simeq -\sigma^2 \frac{\pi \Gamma}{4} e^{-|\sigma|} - \sigma^3 A(\sigma) - \pi \Gamma \sigma^2 e^{-|\sigma|}, \quad (99)$$

where we neglected all real terms of $O((\Gamma/\sigma)^2)$ and all imaginary terms of $O(\Gamma/\sigma)$.

At the poles, the real and the imaginary part of the denominator $z^2 + \mu' g(z) + (\omega/\omega_c)^2$ vanish simultaneously. By using the form of $g(z)$ just derived we see that this can be achieved with the choices

$$\Gamma = -\frac{\pi \mu'}{8} \sigma^2 e^{-|\sigma|}, \quad (100)$$

$$\frac{\omega^2}{\omega_c^2} = \sigma^2 + \frac{\mu'}{2} [\sigma^3 A(\sigma) + \sigma^2] + 2\Gamma^2(5 + |\sigma|). \quad (101)$$

Note the symmetry $\sigma \to -\sigma$ in Eq. (101). Equations (100) and (101) determine the oscillating behavior of $G_s(t)$: $\Gamma$ corresponds to the damping and $\sigma$ to the frequency of the oscillation (both quantities measured in units of $\omega_c$). We assumed the damping $\Gamma$ to be small compared to $\sigma$ such that $\sigma^2 \gg \Gamma$ (as in [16]).

The behavior of the oscillating frequency $\sigma$ (measured in units of $\omega_c$) strongly depends on the trapping frequency $\omega/\omega_c$. Figure 1 shows the dependence of $|\sigma|$ on $\mu' \equiv \mu/(M\omega_c^2)$ for the cases $\omega/\omega_c = 1.0$ and $\omega/\omega_c = 3.0$. When $h\beta\omega \ll 1$ (as in [16]) the properties of the oscillations of $G_s(t)$ coincide with the ones for $C^{eq}(t)$ [see App. B]. Then, by observing Eq. (101) one deduces that the actual oscillation frequency of $C(t,t)$ and its damping are rather $\sigma^2$ and $2\Gamma^2$, respectively. As can be seen for moderate to high trapping frequencies there is an increase of the oscillator frequency followed by a decrease induced by the bath [see Fig. 1]. Such a behavior is not observed for small frequencies (compared to $\omega_c$). The experimental results in [14] confirm such a “peak” although very few data-points are shown in this paper and the error bars are quite large so it is hard to draw firm conclusions on the actual experimental behavior at this stage. The “peak” in the curve of the oscillation frequency can be used in further experiments to determine whether the bath is actually described by a LL or not since it is a direct consequence of the non-Ohmic spectral density [35]. In Fig. 2 we show the correlator Eq. (93) for values of the impurity–bath coupling ($\mu' = 0.5$ and $\mu' = 0.2$) in the “high temperature” regime $\beta h\omega \ll 1$ typical for experiments. The parameters were chosen to be $1/(4e^2\omega_c) = 10$ in both images for the thick lines, $\omega/\omega_c = 1.0$ in the upper and $\omega/\omega_c = 3.0$ in the lower image. The thin line in the upper image has been obtained with $1/(4e^2\omega_c) = 50$. The curves qualitatively agree with the experimental data [16]. One clearly recognizes the increase of the oscillation frequency induced by an increase in the coupling $\mu'$ for $\omega/\omega_c = 3.0$. The typical oscillation width in experiments is about 15$\mu$m with a frequency of $\sigma \approx 550$ s$^{-1}$. The final position width of a (high temperature) quantum Brownian particle is given by $1/(\beta M\omega^2)$ [3]. With the mass $M$ of the $^{41}$K atoms used as an impurity and the bath temperature $T \approx 350$ nK in [16] this yields $\sqrt{C(t,t)} \approx 15\mu$m which is even quantitatively the right value.

VI. SUMMARY AND OUTLOOK

Motivated by recent experiments on impurity dynamics in quantum liquids we studied the non equilibrium dynamics of a quantum Brownian particle coupled to a quantum thermal bath of harmonic oscillators for generic Gaussian initial conditions. We derived a closed expression for the non equilibrium correlation function, which we showed to be easy to derive as variations of a generating functional Eq. (54). We used the analysis in [9] as a starting point to obtain this generating functional by employing path integral methods. We then showed that factorizing initial conditions (where the bath and the particle are initially uncoupled) are a special case of the non–factorizing initial conditions on which a position measurement has been performed. We demonstrated the correctness of our approach by deriving the equilibrium correlation function without imposing time–translational invariance (presented in App. C).

We applied this general formalism to the study of three physical situations. First, we studied the equilibration process of a trapped particle after an initial position measurement. In this case we considered Ohmic dissipation. While the classical (high temperature) correlator relaxes exponentially on a time scale $\gamma^{-1}$, the low temperature
correlator \( C(t, t') \) in the strongly quantum regime shows an algebraic relaxation of the form \( 1/(t't)^2 \) which is independent of the dissipation strength \( \gamma \). Therefore, the information that an initial measurement on the system has been performed persists for a very long time. We then showed that the equilibration process is different if, instead of a position measurement, a sudden quench in the trapping potential is performed at the initial time. We showed that in this case the relaxation is exponential in time even in the quantum regime with the slight difference that at very low temperatures the relaxation time is of order \( 2\gamma^{-1} \) rather than \( \gamma^{-1} \) for high temperatures. Accordingly, the relaxation due to quantum fluctuations is almost as effective as thermal relaxation in this case.

Third, we applied our approach to the motion of impurities in 1D ultra cold quantum gases confined in a very elongated harmonic trap. For low excitation energies the 1D gas is well described by the Luttinger-liquid theory which predicts a Gaussian theory. We modeled the impurity–gas interaction by a simple contact potential. The resulting Hamiltonian is also known as the Fröhlich polaron Hamiltonian which features a non-linear coupling of the polaron (here the impurity) to the bath (here the Luttinger liquid). It is well known that this Hamiltonian induces a mass shift on the impurity depending on the interaction strength. Although the mass shift is clearly observed, no significant change of the oscillation frequency is detected in [10]. Hence it is likely that the confining potential is renormalized as well in such a way that it counteracts the polaronic mass shift. We then proposed to use, as a first approximation to this problem, the Gaussian approximation of the polaron Hamiltonian (more subtle issues linked to polaron and harmonic trap renormalization together with a quantitative comparison between theory and experiment will be presented elsewhere). The spectral density of the resulting exotic “bath” turns out to be strongly super-Ohmic with the low frequency spectral density \( S(\omega) \sim \omega^3 \). With the help of the generating functional approach we were able to deduce the non equilibrium correlation function of the impurity position. Our formula correctly reproduces the oscillating behavior of the position variance (which is the equal time correlation function) as well as the initial momentum of the impurity due to its strong localization at the center of the quantum liquid which is a pure quantum effect. We showed that the typical impurity position width measured in [10] quantitatively matches the theoretical value for a quantum Brownian particle. This suggests that non equilibrium quantum Brownian motion theory can describe many aspects of such impurity dynamics. The super-Ohmic dissipation leads to a curious phenomenon: upon increasing the coupling between the impurity and “bath” the oscillation frequency increases before decreasing. This peculiar behavior is in strong contrast with that induced by Ohmic dissipation, for which the damped oscillator has a lower frequency than the free one and it can be used as a signature for Luttinger liquid baths.

To our knowledge our formalism is the only one that can fully access non equilibrium correlation functions in quantum Brownian motion in contrast to the density matrix approach in [3] that yields only one-time quantities. It proved effective for the description of impurity dynamics in Luttinger liquids. By using a quantum Langevin equation the authors of [10] have described the impurity motion with some success. However, the same authors pointed out that the Langevin approach failed to grasp all aspects of the experimental evidence. We are convinced that one can fit our theoretical results to experimental data as soon as more data become available in the future. The parameter \( \omega_c \) which depends on the actual impurity–quantum liquid coupling can then be experimentally determined within our framework. We hope to address this topic in a future study.

To conclude, our theoretical framework proved effective to deduce all kinds of non equilibrium correlation functions and we obtained promising results for its application to impurity motion in 1D systems. Further possible applications of our approach include various topics such as the study of decoherence problems, the impurity motion in other types of systems or non equilibrium...
quantum transport phenomena. The present work is expected to provide a further motivation for a thorough understanding of quantum Brownian motion.

Appendix A: Coherent state path integral formulation

In this Appendix we detail the derivation of a path-integral representation of the generating functional of the multi-time correlation functions of a quantum particle in contact with a generic quantum bath made of an ensemble of harmonic oscillators.

The terms in the Hamiltonian of the dissipative quantum Brownian particle that depend on the bath variables, Eqs. (2) and (3), can be rewritten in terms of creation and annihilation operators of the bath oscillators in view of a later use of coherent states. One defines the creator and the annihilator of the \( n \)-th oscillator mode by

\[
\hat{a}_n = \sqrt{\frac{m_n \omega_n}{2\hbar}} \left( \hat{x}_n - \frac{i}{m_n \omega_n} \hat{p}_n \right), \quad \hat{a}_n^\dagger = \sqrt{\frac{m_n \omega_n}{2\hbar}} \left( \hat{x}_n + \frac{i}{m_n \omega_n} \hat{p}_n \right).
\]

The operators \( \hat{a}_n \) and \( \hat{a}_n^\dagger \) satisfy the bosonic commutation relations \([\hat{a}_m^\dagger, \hat{a}_n] = \delta_{mn} \) and \([\hat{a}_n^\dagger, \hat{a}_m^\dagger] = [\hat{a}_n, \hat{a}_m] = 0\). Equations (2) and (3) read in terms of the \( \hat{a}_n^\dagger \) and \( \hat{a}_n \)

\[
\hat{H}_B = \sum_{n=1}^{\infty} \hbar \omega_n \hat{a}_n^\dagger \hat{a}_n , \tag{A3}
\]

\[
\hat{H}_SB = \sum_{n=1}^{\infty} g_n \hat{q} (\hat{a}_n^\dagger + \hat{a}_n) . \tag{A4}
\]

Here we introduced the notation \( g_n = \sqrt{\hbar \epsilon_n / 2m_n \omega_n} \).

We introduce the coherent states of the harmonic oscillators, which are particularly suitable when dealing with the bosonic ladder operators in Eq. (A1),

\[|\xi\rangle = e^{\xi \hat{a}^\dagger} |0\rangle, \quad \langle \xi| = (0| e^{-\xi \hat{a}} , \tag{A5}\]

where \( \xi \) is a complex number and \( \xi^* \) its complex conjugate. \( \hat{a}^\dagger \) and \( \hat{a} \) stand for the creation and annihilation operator of each harmonic oscillator. The coherent states are eigenstates of the annihilation operator, that is

\[\hat{a}|\xi\rangle = \xi|\xi\rangle , \quad \langle \xi|\hat{a}^\dagger = \langle \xi|\xi^* , \tag{A6}\]

with the properties

\[\langle \xi| \xi \rangle = e^{\xi^* \xi} \quad \text{and} \quad \mathbb{1}' = \int d\xi^* d\xi e^{-\xi^* \xi} \langle \xi| \langle \xi| . \tag{A7}\]

The trace of any observable \( \hat{B} \) that depends on the particle and the bath operators can be expressed as

\[\text{Tr} \hat{B} = \int dq \int \prod_n \left\{ d\xi_n^* d\xi_n e^{-\xi_n^* \xi_n} \right\} \langle q, \{\xi_n\}|\hat{B}|q, \{\xi_n\} \rangle . \tag{A9}\]

The generating functional can now be obtained by supplementing the potential \( V(q, t) \) in Eq. (10) by a linear term \(-H(s) \hat{q}\) where \( H(s) \) is a \( c \)-number function that plays the role of an external source. To be more explicit, we introduce two distinct sources \( H(s) \) and \( H'(s) \) in the potential \( V \) for the time evolution operator

\[\mathcal{K}(q_f, \{\xi_{n,f}\}; q_i, \{\xi_{n,i}\}; t) \equiv \langle q_f, \{\xi_{n,f}\}| \hat{T} e^{-\hat{H}(t')/\hbar} |q_i, \{\xi_{n,i}\} \rangle \tag{A10}\]

and its Hermitian conjugate

\[\mathcal{K}^*(q'_f, \{\xi_{n,f}'\}; q'_i, \{\xi_{n,i}'\}; t) \equiv \langle q'_i, \{\xi_{n,i}'\}| \hat{T}^\dagger e^{\hat{H}'(t')/\hbar} |q'_f, \{\xi_{n,f}'\} \rangle , \tag{A11}\]

respectively. \( \hat{T}^\dagger \) is the anti-chronological time ordering operator. In the following we will use the shorthand notation \( d\xi_n = \prod_{n=1}^{\infty} e^{-\xi_n^* \xi_n} d\xi_n^* d\xi_n \). All correlation functions of the position \( \hat{q} \) can be obtained by taking the corresponding variations of the trace of the time-dependent density matrix \( \hat{\rho}(t) \equiv \hat{K} \hat{\rho}_0 \hat{K}^* \), Eq. (16), with respect to \( H(s) \) and \( H'(s) \). The matrix elements of \( \hat{\rho}(t) \equiv \hat{K} \hat{\rho}_0 \hat{K}^* \) are given by

\[W(q_f, \{\xi_{n,f}\}; q'_f, \{\xi_{n,f}'\}; t) = \langle q_f, \{\xi_{n,f}\}| \hat{\rho}(t) |q'_f, \{\xi_{n,f}'\} \rangle , \tag{A12}\]

\[W(q_i, \{\xi_{n,i}\}; q'_i, \{\xi_{n,i}'\}) = \langle q_i, \{\xi_{n,i}\}| \hat{\rho}_0 |q'_i, \{\xi_{n,i}'\} \rangle . \tag{A13}\]

The path integral representation of \( \mathcal{K} \) and \( \mathcal{K}^* \) are found by using standard methods:

\[\mathcal{K}(q_f, \{\xi_{n,f}\}; q_i, \{\xi_{n,i}\}; t) = \int Dq^+ D\xi^+ \times \exp \left( \frac{i}{\hbar} S[q^+, \{\xi_{n}^+\}] \right) \tag{A14}\]

\[\mathcal{K}^*(q'_f, \{\xi_{n,f}'\}; q'_i, \{\xi_{n,i}'\}; t') = \int Dq^- D\xi^- \times \exp \left( -\frac{i}{\hbar} S^*[q^-, \{\xi_{n}^-\}] \right) , \tag{A15}\]

where we make clear by the superscripts \( ^+ \) and \( ^- \) which path belongs to \( \mathcal{K} \) and which to \( \mathcal{K}^* \), respectively. The real time interval \([0,t]\) has been discretized into \( T \in \mathbb{N} \) steps of length \( \Delta t \) with \( t = \Delta T T \). The functional integration measures are defined as \( Dq = \prod_{j=1}^{T-1} dq_j \) with
\[ q_j \equiv q(jT). \] The terms contributing to the total action, \( S[q, \xi_n] = S_S[q] + S_B[\xi_n] + S_{SB}[q, \xi_n] \) introduced in Eqs. (A14) and (A15), read in discretized form

\[
S_S[q] = \sum_{j=1}^{T} \Delta t \left[ \frac{M}{2} \left( \frac{q_j - q_{j-1}}{\Delta t} \right)^2 - V(q_j; j\Delta t) \right] + H(j\Delta t)q_j, \tag{A16}
\]

\[
S_B[\xi_n] + S_{SB}[q, \xi_n] = i\hbar \sum_{j=1}^{T-1} \xi_{n,j} (\xi_{n,j} - \xi_{n,j-1}) + \Delta t \sum_{j=1}^{T} \left[ \hbar \omega \xi^*_{n,j} \xi_{n,j-1} + g_n q_j (\xi^*_{n,j} + \xi_{n,j-1}) \right]. \tag{A17}
\]

The reduced density matrix depends only on the particle variables and the external sources,

\[
\mathcal{W}(q^0; q^0'; t) \equiv \iint dq_0 dq' \int dq dq' dq dq' dq \mathcal{K}(q^0, \{\xi_n^0, j \}; q, \{\xi_n; j \}; t) \times \mathcal{W}(q, \{\xi_n; j \}; q', \{\xi_n'; j \}) \mathcal{K}^*(q', \{\xi_n; j \}; q', \{\xi_n'; j \}; t). \tag{A18}
\]

In Eqs. (A14) and (A15) we omitted normalization factors that do not depend on the bath nor on the particle variables. Note that the integral runs only over bath and particle variables with an index between 1 and \( T - 1 \) since \( q_0 = q, \) \( q_T = q_T \) (and analogously for the bath variables) are fixed for \( \mathcal{K} \) and \( q_0 = q', \) \( q_T = q'_T \) (and analogously for the bath variables) are fixed for \( \mathcal{K}^* \).

The path integral description of \( \hat{\rho}_0 \) is obtained by dividing the imaginary time interval \([0, \beta\hbar]\) into \( T \) time steps. Consequently, by using \( \hat{\rho} = e^{-\beta\mathcal{H}_0} \) we find

\[
\mathcal{W}(q^0; \{\xi_n^0; j \}; q; \{\xi_n; j \}) = \mathcal{W}(q, \{\xi_n; j \}; q', \{\xi_n'; j \}) \mathcal{W}(q', \{\xi_n'; j \}; q', \{\xi_n; j \}) \mathcal{S}_S(q, \{\xi_n; j \}) \mathcal{S}_B[\xi] \mathcal{S}_{SB}[q, \xi] \mathcal{S}_0(q, \{\xi_n; j \}), \tag{A19}
\]

\[
\mathcal{S}_0[q, \xi_n] = \mathcal{S}_0[\xi_n] + \mathcal{S}_{0B}[\xi_n] + \mathcal{S}_{0SB}[q, \xi_n] \text{ with } \mathcal{S}_0[q, \xi_n] = \mathcal{S}_0[q] + \mathcal{S}_{0B}[\xi_n] + \mathcal{S}_{0SB}[q, \xi_n] \text{ with }
\]

\[
\mathcal{S}_0[\xi_n] + \mathcal{S}_{0B}[\xi_n] + \mathcal{S}_{0SB}[q, \xi_n] = \hbar T \sum_{j=1}^{T-1} \xi_{n,j} (\xi_{n,j} - \xi_{n,j-1}) \tag{A20}
+ \Delta t \sum_{j=1}^{T} \left[ \hbar \omega \xi^*_{n,j} \xi_{n,j-1} + g_n q_j (\xi^*_{n,j} + \xi_{n,j-1}) \right].
\]

We introduced the imaginary time path step \( \Delta t' = \beta\hbar/T.\)

1. Integration over the bath variables

The influence functional in discretized form reads

\[
\mathcal{F}(\{q_j\}) = \prod_{n} \prod_{j=1}^{3T} \int d\xi_{n,j} d\xi_{n,j}' e^{- \sum_{j,j'=1}^{3T} \xi_{n,j} \mathcal{K}^{-1}(j,j') \xi_{n,j}} \tag{A21}
\]

\[
\mathcal{K}^{-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & -k_1 \\
-k_2 & 1 & 0 & \cdots & 0 \\
0 & -k_3 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -k_4 & \cdots & 0 \\
k_{3T} & 0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}, \tag{A22}
\]

which depends on the correlation matrix

\[
\mathcal{K}^{-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & -k_1 \\
-k_2 & 1 & 0 & \cdots & 0 \\
0 & -k_3 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -k_4 & \cdots & \cdots \\
k_{3T} & 0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}, \tag{A23}
\]

Note that the exponential factors that stem from the bath integration measure Eq. (A13) exactly combine with the sums in the actions (A16) and (A19),

\[
\frac{i}{\hbar} \mathcal{S}_S(\{q_j\}) = \frac{i}{\hbar} \mathcal{S}_S(\{q_j\})^T \frac{i}{\hbar} \mathcal{S}_S(\{q_j\})^{2T} \frac{i}{\hbar} \mathcal{S}_S(\{q_j\})^{2T+1} \frac{i}{\hbar} \mathcal{S}_S(\{q_j\})^{T+1}.
\]

The Gaussian integral in Eq. (A21) is now readily done. Explicit expressions of Eq. (A23) in the continuum limit \( T \to \infty \) are easily obtained: for instance, when \( j, j' < T, \) \( K(j, j') \) couples to two \( q^+ \) fields and is given by

\[
K(j, j') = \begin{cases}
\frac{1}{1 - k_1} & \text{for } j = j' \\
\frac{1}{1 - k_1} \frac{1}{k_{j-j'} + 1} k_{j-j'} & \text{for } j > j' \\
\frac{1}{1 - e^{-\beta\hbar}} & \text{for } j < j' \leq T
\end{cases}
\tag{A24}
\]

The Gaussian integral in Eq. (A21) is now readily done. Explicit expressions of Eq. (A23) in the continuum limit \( T \to \infty \) are easily obtained: for instance, when \( j, j' < T, \) \( K(j, j') \) couples to two \( q^+ \) fields and is given by

\[
K(j, j') = \begin{cases}
e^{-\beta\hbar + i \omega(t-j')} & \text{for } j' < j \leq T \\
e^{-\beta\hbar + i \omega(t-j')} & \text{for } j' < j \leq T
\end{cases}
\tag{A25}
\]
$q^-$ and $q^0$ we find \[ q^\pm(q^+,q^-,q^0) = \exp \left( -\frac{1}{\hbar} \Phi[q^+,q^-,q^0] \right), \quad (A26) \]
where the exponent reads
\[
\Phi[q^+,q^-,q^0] = -\int_0^{\beta\hbar} d\tau \int_0^\tau d\sigma \ K(-i\tau + i\sigma)q^0(\tau)q^0(\sigma) + \int_0^{\beta\hbar} d\tau \frac{\mu}{2} q^{02}(\tau)
\]
\[
- i \int_0^{\beta\hbar} d\tau \int_0^t ds \ K^*(s-i\tau)q^0(\tau) \left[ q^+(s) - q^-(s) \right]
\]
\[
+ \int_0^t dt \int_0^s du \ [q^+(s) - q^-(s)] \left[ K(s-u)q^+(u) - K^*(s-u)q^-(u) \right]
\]
\[
+ i \int_0^{\beta\hbar} ds \frac{\mu}{2} \left[ q^+(s) - q^-(s) \right].
\]
The kernel $K$ reads for complex times $\theta = s - i\tau$, $0 \leq \tau \leq \beta\hbar$
\[
K(\theta) = \sum_n \frac{g_n^2}{\hbar} \frac{\cosh[\omega_n(\beta\hbar/2 + i\theta)]}{\sinh[|\omega_n\beta\hbar/2|]}, \quad (A28)
\]
and the constant $\mu$ is given by
\[
\mu = 2 \sum_n \frac{g_n^2}{\hbar \omega_n}. \quad (A29)
\]
Note that for a fermionic bath the only difference lies in the boundary conditions enforced by the trace operation: for fermions anti-periodic boundary conditions apply in contrast to periodic boundary conditions for bosons. This difference is incorporated by replacing $-k_1$ by $k_1$ in Eq. \[ A22 \]. The analysis for bosons can then be repeated, leading to a fermionic bath kernel where the cage and the sinh in Eq. \[ A28 \] interchange their positions.

The environment can be regarded as a proper heat bath only if the spectrum of the harmonic oscillators becomes quasi-continuous. Accordingly, we introduce the spectral density of the bath through
\[
S(\omega) = \pi \sum_n \frac{g_n^2}{\hbar} \delta(\omega - \omega_n) = \pi \sum_n \frac{c_n^2}{2m_n\omega_n} \delta(\omega - 4A30)
\]
Then the kernel $K(\theta)$ and the constant $\mu$ are rewritten in terms of the spectral density
\[
K(\theta) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{\cosh[\omega(\beta\hbar/2 + i\theta)]}{\sinh[|\omega\beta\hbar/2|]} \quad (A31)
\]
and
\[
\mu = 2 \int_0^\infty \frac{d\omega}{\omega} \frac{S(\omega)}{\pi}. \quad (A32)
\]
The real and the imaginary parts of the kernel $K(\theta) = K_R(\theta) + iK_I(\theta)$ are found to be
\[
K_R(s-i\tau) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{\cosh[\omega(\beta\hbar/2 - \tau)]}{\sinh[|\omega\beta\hbar/2|]} \cos(\omega s), \quad (A33)
\]
\[
K_I(s-i\tau) = -\int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{\sin[\omega(\beta\hbar/2 - \tau)]}{\sinh[|\omega\beta\hbar/2|]} \sin(\omega s). \quad (A34)
\]
The imaginary time argument $\tau$ varies in the interval $[0, \beta\hbar]$ so that it is convenient to introduce the Fourier series of $K(s-i\tau)$ with respect to $\tau$. Introducing the Matsubara frequencies
\[
\nu_k = \frac{2\pi k}{\beta\hbar} \quad (A35)
\]
we find
\[
K_R(s-i\tau) = \frac{1}{\beta\hbar} \sum_{k=-\infty}^{\infty} g_k(s) e^{i\nu_k \tau}, \quad (A36)
\]
and
\[
K_I(s-i\tau) = \frac{i}{\beta\hbar} \sum_{k=-\infty}^{\infty} f_k(s) e^{i\nu_k \tau}, \quad (A37)
\]
where the functions $g_k$ and $f_k$ are defined through
\[
g_k(s) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{2\omega}{\omega^2 + \nu_k^2} \cos(\omega s) \quad (A38)
\]
and
\[
f_k(s) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{2\nu_k}{\omega^2 + \nu_k^2} \sin(\omega s). \quad (A39)
\]
In the following we will express most quantities in terms of the functions $g_k$ and $f_k$. For real times the real and the imaginary parts of the kernel \[ A31 \] read [see Eqs. \[ A36 \] and \[ A37 \]]
\[
K_R(s) = \int_0^\infty \frac{d\omega}{\pi} S(\omega) \coth(\beta\hbar/2) \cos(\omega s), \quad (A40)
\]
and
\[
K_I(s) = -\int_0^\infty \frac{d\omega}{\pi} S(\omega) \sin(\omega s). \quad (A41)
\]
We now eliminate the local terms in Eq. \[ A27 \]. We define the two new kernels
\[
\gamma(s) = \frac{2}{M} \int_0^\infty \frac{d\omega}{\pi} S(\omega) \cos(\omega s) \quad (A42)
\]
and
\[
k(\tau) = \frac{M_0}{\beta\hbar} \sum_{k=-\infty}^{\infty} \zeta_k e^{i\nu_k \tau}, \quad (A43)
\]
where $\zeta_k$ is defined by
\[
\zeta_k = \frac{1}{M_0} \int_0^{\infty} \frac{d\omega}{\pi} \frac{S(\omega)}{\omega^2 + \nu_k^2}.
\]

The latter kernel is related to $K_R(-i\tau)$ via
\[
- \int_0^{h_\beta} d\tau \int_0^{t-\tau} d\sigma K_R(-i\tau + i\sigma)f(\tau, \sigma) = \frac{\mu}{2} \int_0^{h_\beta} d\tau f(\tau, \tau) + \int_0^{h_\beta} d\tau d\sigma K(\tau - \sigma)f(\tau, \sigma),
\]
with a generic function $f$. In terms of the kernels $g(\lambda), k(\tau), K^*(s-i\tau), K_R(s-u)$ and the linear combinations
\[
x = (q^+ + q^-)/2 \quad \text{and} \quad \bar{x} = q^+ - q^-.
\]
the exponent of the influence functional reads
\[
\Phi[x, \bar{x}, q^0] = \frac{1}{2} \int_0^{h_\beta} d\tau d\sigma K(\tau - \sigma)q^0(\tau)q^0(\sigma) - i \int_0^{h_\beta} d\tau \int_0^{t} d\sigma K^*(s-i\tau)q^0(\tau)\bar{x}(s)
\]
\[
+ \frac{1}{2} \int_0^{t} \int_0^{t} \int_0^{t} d\sigma d\tau d\sigma K_R(s-u)\bar{x}(s)\bar{x}(u)
\]
\[
+ iM \int_0^{t} \int_0^{t} \int_0^{t} d\sigma d\tau d\sigma K_R(s-u)\bar{x}(s)\bar{x}(u).
\]

Details of the derivation of Eq. (A47) can be found in the thorough analysis in [9].

**Appendix B: Classical Brownian particle in a harmonic potential: Initial position measurement and quenches in the trapping potential**

This part is meant to be a reminder on classical stochastic motion induced by generic baths. None of the results presented herein are new but they are useful to be confronted with the quantum results discussed in the body of the paper.

The classical Brownian motion of a particle confined in a harmonic potential can be described by the *Langevin equation*
\[
\ddot{q}(t) + \int_0^t ds \gamma(t - s)\dot{q}(s) + \omega^2 q(t) = \xi(t),
\]
where $\xi$ is a zero mean Gaussian noise with correlation $\langle \xi(t)\xi(s) \rangle = \frac{1}{2\tau} \gamma(|t - s|)$ and with $\gamma(t)$ given in Eq. (12). In the Laplace transform formulation, the solution to Eq. (151) reads
\[
\tilde{q}(\lambda) = \tilde{G}_+(\lambda) \left[ \tilde{\xi}(\lambda) + v_0 + (\lambda + \tilde{\gamma}(\lambda))q^0 \right],
\]
where we used $\tilde{G}_+$ defined in Eq. (19) and we introduced the initial conditions $q(0) = q^0$ and $\dot{q}(0) = v_0$. The correlation function is now easily computed and it reads
\[
\tilde{C}(\lambda, \kappa) = \langle \tilde{q}(\lambda)\tilde{q}(\kappa) \rangle = \frac{1}{\beta M} \frac{\tilde{\gamma}(\lambda) + \tilde{\gamma}(\kappa)}{\lambda + \kappa} \tilde{G}_+(\lambda)\tilde{G}_+(\kappa)
\]
\[
+ \tilde{G}_+(\lambda)\tilde{G}_+(\kappa) \left[ v_0^2 + (\lambda + \tilde{\gamma}(\lambda))(\kappa + \tilde{\gamma}(\kappa))q^0 \right] \tilde{G}_+(\lambda)\tilde{G}_+(\kappa) v_0 q^0 \left[ \lambda + \tilde{\gamma}(\lambda) + \kappa + \tilde{\gamma}(\kappa) \right],
\]
where we used the fact that the Laplace transform of $\gamma(t - s)$ with respect to $t$ and $s$ is given by $\langle \tilde{\gamma}(\lambda) + \tilde{\gamma}(\kappa) \rangle/(\lambda + \kappa)$. The initial values $q^0$ and $v_0$ can be sharp or drawn from a probability distribution which is typically of the Maxwell-Boltzmann type, that is
\[
P(q^0, v_0) = \frac{\beta M_0 \omega_0}{2\pi} \exp \left[ -\beta \left( \frac{M_0}{2} v_0^2 + \frac{M_0}{2} \omega_0^2 q^0 \right) \right],
\]
where $\omega_0$ is the frequency of the initial trapping potential and $M_0$ is the initial mass. From Eq. (B4) we easily derive
\[
\langle q^0 \rangle = (\beta M_0 \omega_0^2)^{-1} \quad \text{and} \quad \langle v_0^2 \rangle = (\beta M_0)^{-1}.
\]

As long as $\omega_0 = \omega$ and $M = M_0$ the correlation function can be rewritten as
\[
C(\lambda, \kappa) = \frac{\tilde{C}(\lambda) + \tilde{C}(\kappa)}{\lambda + \kappa} \tilde{G}_+(\lambda)\tilde{G}_+(\kappa)
\]
\[
+ \tilde{G}_+(\lambda)\tilde{G}_+(\kappa) \left[ \lambda + \tilde{\gamma}(\lambda) + \kappa + \tilde{\gamma}(\kappa) \right].
\]

which is the equilibrium correlation function.

The non equilibrium correlation can be recast in the form
\[
\tilde{C}(\lambda, \kappa) = \frac{\tilde{C}_1(\lambda) + \tilde{C}_1(\kappa)}{\lambda + \kappa} \tilde{G}_+(\lambda)\tilde{G}_+(\kappa)
\]
\[
+ \beta M \omega^2 \tilde{C}_1(\lambda)\tilde{C}_1(\kappa) \left[ \beta M \omega_2 q^0 \right] \tilde{G}_+(\lambda)\tilde{G}_+(\kappa) \left[ \lambda + \tilde{\gamma}(\lambda) + \kappa + \tilde{\gamma}(\kappa) \right].
\]

In many cases $q^0$ and $v_0$ are uncorrelated random variables. Then Eq. (B7) transforms into
\[
\tilde{C}(\lambda, \kappa) = \frac{\tilde{C}_1(\lambda) + \tilde{C}_1(\kappa)}{\lambda + \kappa} \tilde{G}_+(\lambda)\tilde{G}_+(\kappa)
\]
\[
+ \beta M \omega^2 \tilde{C}_1(\lambda)\tilde{C}_1(\kappa) \left[ \beta M \omega_2 q^0 \right] - \frac{1}{\beta M}.
\]

**Appendix C: The equilibrium initial condition**

In this Appendix we use Eq. (17) in the particular case of an equilibrium initial condition and a subsequent evolution still in equilibrium. We show how to derive the
equilibrium correlation function and we prove that the fluctuation-dissipation theorem (FDT) is satisfied without imposing time-translational invariance (TTI) as has been done before in the literature.

1. The fluctuation-dissipation theorem

The linear response is easily found by noting that the external source $F(s)$ represents a physical drift force. Therefore, by calculating

$$\langle \hat{q}(t) \rangle = \frac{i}{\hbar} \frac{\partial}{\partial G(t)} \exp(\mathcal{J}[F,G]) |_{G=0} = \int_0^t ds \mathcal{R}(t-s) F(s)$$

(C1)

for $F \neq 0$ one finds the response function $\mathcal{R}(t)$ with respect to the external force $F(t)$. We set $\epsilon \rightarrow \infty$ which corresponds to the absence of any initial measurement. By using Eq. (C4) we obtain

$$\mathcal{R}(t) = \frac{1}{M} \mathcal{G}_+(t) \quad \text{and} \quad \tilde{\mathcal{R}}(\lambda) = \frac{1}{M} \frac{1}{\omega^2 + \gamma(\lambda) + \lambda^2}$$

(C2)

in the time and Laplace transform domains, respectively. These expressions are independent of the initial condition. Therefore, the response function is the same in and out of equilibrium. Moreover, it is equal to the response function of a classical Brownian particle if it is coupled to a bath with the same friction kernel $\gamma$. We will confirm the validity of the FDT when the system is in equilibrium. We choose the initial Hamiltonian to be equal to the “bulk” one, that is

$$\omega = \omega_0 \quad \text{and} \quad M = M_0 ,$$

(C3)

so that the initial density matrix is equal to the Boltzmann weight $\exp(-\beta\mathcal{H})$ with the terms contributing to $\mathcal{H}$ given in Eqs. (1), (2) and (3). The initial state is not perturbed by any measurement, so we take $\epsilon \rightarrow \infty$ which implies $\Lambda' = \Lambda$ and $\Omega' = \Omega$. From Eq. (57) we find the equilibrium correlation function $C^{eq}(t, t')$ which in the Laplace transform version reads

$$\tilde{C}^{eq}(\lambda, \kappa) = \frac{\hbar}{M} \tilde{\mathcal{G}}_+(\lambda) \tilde{\mathcal{G}}_+^{*}(\kappa) \left\{ \Lambda \lambda \kappa + \frac{\Lambda}{M} \lambda \tilde{C}_1(\kappa) + \frac{\Lambda}{M} \kappa \tilde{C}_1(\lambda) + \Omega - \frac{1}{M} \tilde{C}_2(\lambda) - \frac{1}{M} \tilde{C}_2(\kappa) + \frac{1}{M^2} \tilde{R}(\nu, \lambda, \kappa) \right\} .$$

(C4)

This expression can be greatly simplified. We first note that from the definitions of $g_k$ and $f_k$ in Eqs. (A38) and (A39) it follows that

$$\tilde{f}_k(s) = \nu_k g_k(s) , \quad \tilde{g}_k(\lambda) = \frac{\nu_k}{\lambda} \tilde{g}_k(\lambda) ,$$

(C5)

where we used $f_k(0) = 0$. The Laplace transform of the kernel

$$R''(s, s') = \frac{1}{\beta h} \sum_k u_k [g_k(s)g_k(s')] - \frac{f_k(s)}{f_k(s')} + M_0 K_h(s - s')$$

(C6)

[see Eq. (55) and (49) for $\Lambda = \Lambda'$] can now be written as

$$\tilde{R}''(\lambda, \kappa) = \frac{1}{\beta h} \sum_k u_k \left( 1 - \frac{\nu_k^2}{\lambda \kappa} \right) \tilde{g}_k(\lambda) \tilde{g}_k(\kappa) + \frac{1}{\beta h} \sum_k \tilde{g}_k(\lambda) + \tilde{g}_k(\kappa)$$

(C7)

and by defining $\tilde{h}_k(\lambda) = \tilde{g}_k(\lambda)/M + \lambda$ we find that the expression in the curly brackets in the rhs of Eq. (C4) can be recast as

$$\frac{1}{\beta h} \sum_k u_k \left( 1 - \frac{\nu_k^2}{\lambda \kappa} \right) \tilde{h}_k(\lambda) \tilde{h}_k(\kappa) + \frac{1}{\beta h} \sum_k \tilde{h}_k(\lambda) + \tilde{h}_k(\kappa)$$

(C8)

where we used Eqs. (36) and (37). By combining the expression for the Laplace transform of the cosine $\int_0^\infty dt e^{-\lambda t} \cos(\omega t) = \lambda/(\lambda^2 + \omega^2)$ with Eqs. (A42) and (A38) we obtain

$$\frac{1}{M} \tilde{g}_k(\lambda) = \frac{\lambda}{\nu_k^2 - \lambda^2} \left[ |\nu_k| \tilde{\gamma}(|\nu_k|) - \lambda \tilde{\gamma}(\lambda) \right] .$$

(C9)

By using instead Eqs. (A42) and (A44) we derive

$$\zeta_k = |\nu_k| \tilde{\gamma}(|\nu_k|) .$$

(C10)

The kernel $\tilde{\gamma}$ can be eliminated in favor of $\tilde{\mathcal{G}}_+$ through Eq. (49) which yields

$$\tilde{h}_k(\lambda) = \lambda \nu_k^2 - \lambda^2 \left[ \tilde{\mathcal{G}}_+^{*}(|\nu_k|) - \tilde{\mathcal{G}}_+^{-1}(\lambda) \right] .$$

(C11)

This expression can now be inserted via Eq. (C8) into Eq. (C4) to find the equilibrium correlator. Note that, for $\omega_0 = \omega$ we have $\tilde{\mathcal{G}}_+^{*}(|\nu_k|) = u_k$. With the help of the one variable function

$$\tilde{C}^{eq}_1(\lambda) = \frac{1}{\beta M} \sum_k \frac{\lambda}{\nu_k^2 - \lambda^2} \left[ \tilde{\mathcal{G}}_+^{*}(\lambda) - \tilde{\mathcal{G}}_+^{*}(|\nu_k|) \right]$$

(C12)

the equilibrium correlation function becomes

$$\tilde{C}^{eq}(\lambda, \kappa) = \frac{C^{eq}_1(\lambda) + C^{eq}_1(\kappa)}{\lambda + \kappa} ,$$

(C13)

which clearly displays time translational invariance (TTI). Indeed, the Laplace transform with respect to $t$ and $t'$ of a generic function $\tilde{f}(t - t')$ that depends only on the time difference is equal to $[\tilde{f}(\lambda) + \tilde{f}(\kappa)]/(\lambda + \kappa)$, where $\tilde{f}(\lambda)$ denotes the Laplace transform of $f(t)$ with respect to $t$. Hence, we have $C^{eq}_1(t, t') = C^{eq}_1(|t - t'|)$ with the explicit Laplace representation of $C^{eq}_1$ in Eq. (C12).
The equilibrium correlation function is thus found without imposing TTI. By imposing TTI Eq. (C12) can be directly found from Eq. (17) by setting \( t' = 0 \) which simplifies the expression considerably. Remember that \( \hat{G}_+(t = 0) = 1 \) and \( G_+(t = 0) = 0 \). By taking the Laplace transform of the result with respect to \( t \) and using Eqs. (C2) and (C12) one easily recovers Eq. (C12).

It is now straightforward to establish the relation between \( C_1^{\text{eq}}(t) \) and \( R(t) \). Firstly, we note that since \( C_1^{\text{eq}}(t) \) is an even function of \( t \) its Fourier transform \( C_1^{\text{eq}}(\omega) \) is related to its Laplace transform through

\[
C_1^{\text{eq}}(\omega) = \hat{C}_1^{\text{eq}}(i\omega) + \hat{C}_1^{\text{eq}}(-i\omega).
\]

Thus, by using Eqs. (C2) and (C12) we have

\[
C_1^{\text{eq}}(\omega) = \frac{1}{\beta} \sum_k \frac{i\omega}{\omega^2 + \nu_k^2} \left[ \hat{R}(i\omega) - \hat{R}(-i\omega) \right].
\]

Now, since the Fourier transform of the response function, \( R(\omega) \), is related to its Laplace transform via \( R(\omega) = \mathcal{L}[R(\omega)] \) due to causality we obtain the quantum FDT in the form

\[
C_1^{\text{eq}}(\omega) = \hbar \coth[\omega/\beta\hbar/2] \mathcal{L}[R(\omega)],
\]

where we used the formula \( \sum_k \omega/(\omega^2 + \nu_k^2) = (\beta\hbar/2) \coth[\omega/\beta\hbar/2] \). This result is completely general, in the sense that it applies to any bath, as it should.

**Appendix D: Asymptotic behavior of \( G_+(t) \) and \( C_1^{\text{eq}}(t) \) for Ohmic dissipation**

In the case of *Ohmic dissipation* the spectral function has the form

\[
S(\omega) = \gamma \omega \quad \text{for} \quad \omega \to 0.
\]

For large frequencies one typically introduces a high frequency cutoff function (since the ultraviolet divergence is unphysical) that we choose to be of the Drude–type \( \omega_D^2/(\omega_D^2 + \omega^2) \) where \( \omega_D \gg \omega \) is the high frequency cutoff. The bath kernel

\[
\gamma(t) = \gamma \omega_D e^{-\omega_D t}
\]

then has a finite memory and a simple form in the Laplace domain, namely

\[
\hat{\gamma}(\lambda) = \gamma \frac{\omega_D}{\omega_D + \lambda}.
\]

We are interested in the equilibration behavior of the correlation function Eq. (D2) when quantum effects dominate. In order to find the long–time behavior of \( C_1^{\text{eq}}(t) \) and \( G_+(t) \) we study their small–\( \lambda \) behavior. In the very low temperature limit the sum over the Matsubara frequencies in Eq. (C12) can be replaced by an integral. For \( \lambda \to 0 \) one finds

\[
\hat{C}_1^{\text{eq}}(\lambda) \simeq \int_0^\infty \frac{d\nu}{\nu^2 - \lambda^2} \left[ \hat{G}_+(\lambda) - \hat{G}_+ (\nu) \right]
\]

\[
\simeq \frac{1}{\pi} \int_0^{\lambda^{-1}} \frac{d\nu}{\nu^2 - 1} \left[ \frac{\lambda^2(\nu^2 - 1) + \lambda(\nu\gamma(\nu) - \gamma)}{\omega^4} \right]
\]

\[
+ \frac{\lambda}{\pi} \int_0^\infty \frac{d\nu}{\nu^2} \frac{\nu^2 + \nu\gamma(\nu)}{\omega^2(\nu^2 + \nu\gamma(\nu) + \omega^2)} + \ldots,
\]

where the ellipsis stands for higher orders in \( \lambda \). Now, by observing the ultraviolet behavior of Eq. (D3) one easily argues that all the terms in the rhs of Eq. (D4) are of order \( \sim \lambda \). Therefore, the long time behavior of the Ohmic equilibrium correlation function at zero temperature is

\[
C_1^{\text{eq}} \sim \frac{1}{t^2} \quad \text{for} \quad t \to \infty \quad \text{and} \quad \beta\hbar \gg |\omega^2 - \gamma^2/4|^{-1}.
\]

It is straightforward to show by direct inversion of the Laplace transform that the propagator \( \mathcal{G}_+(t) \) is exponentially suppressed for large times (and for Ohmic dissipation) on a typical time scale \( \gamma/2 \), hence we have

\[
\mathcal{G}_+(t) \sim e^{-\gamma t/2} \quad \text{for} \quad t \to \infty.
\]

Equation (D6) holds for all temperatures. In the high temperature regime one has \( \nu_k \to \infty \) so that \( C_1^{\text{eq}}(\lambda) \simeq -\left[ \hat{G}_+(\lambda) - 1/\omega^2 \right]/(\beta M A) \). Translated into real time this states that \( \mathcal{G}_+(t) \) is proportional to the derivative of \( C_1^{\text{eq}}(t) \) which is nothing else than the classical FDT. Accordingly, we find

\[
C_1^{\text{eq}} \sim e^{-\gamma t/2} \quad \text{for} \quad t \to \infty \quad \text{and} \quad \beta\hbar \ll |\omega^2 - \gamma^2/4|^{-1}.
\]

**ACKNOWLEDGMENTS**

We thank T. Giamarchi for very useful discussions. This work was financially supported by ANR-BLAN-0346 (FAMOUS).
The underlying probability distribution is of the (Gaussian) Boltzmann-Gibbs type exp(−βH) with H the full coupled Hamiltonian of the particle–bath system. Equation \[ \text{[25]} \] thus describes the case where the harmonic oscillator bath and the particle are initially coupled as in the quantum case studied in the present work. This subtle point is often overlooked. For more details see p. 21-23 in \[ \text{[1]} \].

\[ \text{[25]} \] J. T. Devreese and A. S. Alexandrov, Reports on Progress in Physics \textbf{72}, 066501 (2009).
\[ \text{[26]} \] J. Tempere, W. Casteels, M. K. Oberthaler, S. Knoop, E. Timmermans, and J. T. Devreese, Physical Review B \textbf{80}, 184504 (2009).
\[ \text{[27]} \] P. Hänggi and G. Ingold, Chaos \textbf{15}, 026105 (2005).
\[ \text{[28]} \] L. F. Cugliandolo, T. Giamarchi, and P. L. Doussal, Phys. Rev. Lett. \textbf{96}, 217203 (2006).
\[ \text{[29]} \] L. F. Cugliandolo and G. S. Lozano, Phys. Rev. Lett. \textbf{80}, 4979 (1998).
\[ \text{[30]} \] L. F. Cugliandolo and G. S. Lozano, Phys. Rev. B \textbf{59}, 915 (1999).
\[ \text{[31]} \] L. F. Cugliandolo, D. R. Grempel, G. S. Lozano, H. Lozza, and C. A. da Silva Santos, Phys. Rev. B \textbf{66}, 014444 (2002).
\[ \text{[32]} \] M. P. Kennett and C. Chamon, Phys. Rev. Lett. \textbf{86}, 1622 (2001).
\[ \text{[33]} \] M. P. Kennett, C. Chamon, and Y. Ye, Phys. Rev. B \textbf{64}, 224408 (2001).
\[ \text{[34]} \] G. Biroli and O. Parcollet, Phys. Rev. B \textbf{65}, 094414 (2002).
\[ \text{[35]} \] C. Aron, G. Biroli, and L. F. Cugliandolo, J. Stat. Mech., P11018 (2010).
\[ \text{[36]} \] C. Aron, G. Biroli, and L. F. Cugliandolo, Phys. Rev. B \textbf{82}, 174203 (2010).
\[ \text{[37]} \] S. Tomonaga, Prog. in Theor. Phys. \textbf{5}, 544 (1950).
\[ \text{[38]} \] J. M. Luttinger, J. Math. Phys. \textbf{4}, 1154 (1963).
\[ \text{[39]} \] D. Mattis and E. Lieb, J. Math. Phys. \textbf{6}, 304 (1965).
\[ \text{[40]} \] G. Thalhammer, G. Barontini, L. D. Sarlo, J. Catani, F. Minardi, and M. Inguscio, Phys. Rev. Lett. \textbf{100}, 214002 (2008).
\[ \text{[41]} \] V. Peano, M. Thorwart, C. Mora, and R. Egger, New J. Phys. \textbf{7}, 192 (2005).
\[ \text{[42]} \] M. Olshanii, Phys. Rev. Lett. \textbf{81}, 938 (1998).
\[ \text{[43]} \] E. H. Lieb and W. Liniger, Phys. Rev. \textbf{130}, 16051616 (1963).
\[ \text{[44]} \] T. Giamarchi, Quantum Physics in One Dimension, Clarendon Press, Oxford, 2003.

The underlying probability distribution is of the (Gaussian) Boltzmann-Gibbs type exp(−βH) with H the full coupled Hamiltonian of the particle–bath system. Equation \[ \text{[25]} \] thus describes the case where the harmonic oscillator bath and the particle are initially coupled as in the quantum case studied in the present work. This subtle point is often overlooked. For more details see p. 21-23 in \[ \text{[1]} \].

\[ \text{[45]} \] J. Bonart, L. F. Cugliandolo, and A. Gambassi, J. Stat. Mech., P01014 (2012).