KÄHLER-RICCI FLOW FOR DEFORMED COMPLEX STRUCTURES

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Abstract. Let \((M, J_0)\) be a Fano manifold which admits a Kähler-Ricci soliton, we analyze the behavior of the Kähler-Ricci flow near this soliton as we deform the complex structure \(J_0\). First, we will establish an inequality of Lojasiewicz’s type for Perelman’s entropy along the Kähler-Ricci flow. Then we prove the convergence of Kähler-Ricci flow when the complex structure associated to the initial value lies in the kernel \(Z\) or negative part of the second variation operator of Perelman’s entropy. As applications, we solve the Yau-Tian-Donaldson conjecture for the existence of Kähler-Ricci solitons in the moduli space of complex structures near \(J_0\), and we show that the kernel \(Z\) corresponds to the local moduli space of Fano manifolds which are modified \(K\)-semistable. We also prove an uniqueness theorem for Kähler-Ricci solitons.

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0. Introduction

Let \((M, J_0)\) be a Fano manifold which admits a Kähler-Ricci, abbreviated as KR, soliton \(\omega_{KS} \in 2\pi c_1(M, J_0)\). It is known that for any initial metric \(\omega_0 \in 2\pi c_1(M, J_0)\), the KR flow will evolve \(\omega_0\) to a KR soliton \(\omega'_{KS}\) smoothly [39, 11]. Moreover, by the uniqueness of KR solitons [35, 36], \(\omega'_{KS} = \sigma^*\omega_{KS}\) for some \(\sigma \in \text{Aut}(M, J_0)\). Thus it is a natural question how to extend the above convergence result to Kähler manifolds with complex structures near \(J_0\). The question is closely related to the study of the moduli space of Fano manifolds.

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1We always denote a Kähler metric \(g\) by its Kähler form \(\omega_g\).
related to the existence problem of KR solitons for the local moduli of complex structures at \( J_0 \) as well as the local moduli space of KR solitons near \( \omega_{KS} \).

When \((M, J_0)\) admits a Kähler-Einstein (KE) metric, it has been proved that the KR flow is always convergent to a KE metric in the \( C^\infty \)-topology for any initial metric \( \omega_0 \in 2\pi c_1(M, J) \) so long as \( J \) is sufficiently close to \( J_0 \) \cite{15, 22}. As a consequence, the Mabuchi’s K-energy on \((M, J)\) is bounded from below \cite{15, Lemma 7.1} (also see \cite{11, 8}), so all Fano manifolds \((M, J)\) are \( K \)-semistable. We refer the reader to \cite{22, 20, 24, 10, Proposition 4.17} etc. for more general \( K \)-semistable Fano manifolds. However, for a KR soliton \((M, J_0, \omega_{KS})\) which is not KE metric, the flow on \((M, J)\) may not always converge to a KR soliton smoothly even if \( J \) is close to \( J_0 \), as explained in \cite{15, Remark 6.5} by using Pasquier’s example of horospherical variety which is a degeneration of the Grassmannian manifold \( Gr_2(2, 7) \) \cite{29}. Thus we are led to understanding those deformed complex structures for which the KR flow is \( C^\infty \)-convergent to a KR soliton.

According to the deformation theory \cite{14, 15}, the local moduli of complex structures at \((M, J_0)\) can be parameterized by using the \( C^\infty \)-smooth cohomology class \( H^1(M, J_0, \Theta) \), which is the infinitesimal deformation space of complex structures on \( M \). In our case of Fano manifolds, we introduce the \( h \)-harmonic space \( \mathcal{H}_h^0(M, T^{1,0}M) \) associated to \( H^1(M, J_0, \Theta) \), where \( h \) is a Ricci potential of Kähler metric \( \omega \in 2\pi c_1(M, J_0) \) (cf. Section 1). In particular, \( h \) is same as a potential function \( \theta \) of soliton vector field (VF) if \( \omega = \omega_{KS} \). Then the local deformation of Kähler metrics near \((M, J_0, \omega)\) can be parameterized by \( \mathcal{U} = B(\epsilon) \times C^\infty(M) \) (cf. Section 2), where \( B(\epsilon) \) denotes a small \( \epsilon \)-ball in \( \mathcal{H}_h^0(M, T^{1,0}M) \) centered at the origin. Thus the variation space \( \mathcal{U} \) of Kähler metrics becomes

\[
\mathcal{U} = \mathcal{H}_h^0(M, T^{1,0}M) \times C^\infty(M).
\]

By computing the second variation of Perelman’s entropy \( \lambda(\cdot) \) at \((M, J_0, \omega_{KS})\) via the parameter space \( \mathcal{U} \), we are able to get the following product formula of the second variation operator (cf. Proposition \( \ref{20} \)).

\[
\delta^2 \lambda(\omega_{KS}) = H(\psi, \chi) = H_1(\psi) \oplus H_2(\chi), \quad \forall (\psi, \chi) \in \mathcal{U},
\]

where \( H_2(\chi) \) is always non-positive \cite{40}. However, the sign of operator \( H_1(\psi) \) is in general not definite according to Pasquier’s example mentioned above. Thus the convergence problem of KR flow for deformed complex structure \( J_{\psi_\tau} \) associated to \( \psi_\tau \in B(\epsilon) \subseteq \mathcal{H}_h^0(M, T^{1,0}M) \) \( (\epsilon << 1) \) via the Kuranishi map will depend on the sign of \( H_1(\psi_\tau) \).

Recall the (normalized) KR flow on a Fano manifold \((M, J)\),

\[
\partial_t \omega(t) = -\text{Ric}(\tilde{\omega}(t)) + \tilde{\omega}_0 \in 2\pi c_1(M, J).
\]

By establishing an inequality of Lojasiewicz type for the entropy \( \lambda(\cdot) \) along the flow \( \ref{0.1} \), we will prove

**Theorem 0.1.** Let \((M, J_0)\) be a Fano manifold which admits a KR soliton \( \omega_{KS} \in 2\pi c_1(M, J_0) \) with respect to a holomorphic vector field (HVF) \( X \). Then there exists a small \( \epsilon \) such that for any \( \tau \in B(\epsilon) \) with \( \psi_\tau \in \text{Ker}(H_1) = Z \) the KR flow \( \ref{0.1} \) converges smoothly to a KR soliton \((M, J_\infty, \omega_\infty)\) for any initial metric \( \tilde{\omega}_0 \in 2\pi c_1(M, J_\psi) \). Moreover, \( X \) can be lifted to become a soliton VF of \((M, J_\infty, \omega_\infty)\) and

\[
\lambda(\omega_\infty) = \lambda(\omega_{KS}).
\]

Also, the convergence is of polynomial rate.

We note that \( Z = \mathcal{H}_h^0(M, T^{1,0}M) \) in case of \( \omega_{KS} = \omega_{KE} \) \cite{10}. Thus Theorem 0.1 generalizes the results in \cite{15, 22}. We also note that \( \psi_\tau \in Z \) if and only if \( X \) can be lifted to a HVF on \((M, J_\psi)\) (cf. Corollary \( \ref{15} \)). Moreover, as in the case of \( \omega_{KS} = \omega_{KE} \), the set of complex structures associated to the kernel \( Z \) corresponds to the local moduli space of Fano manifolds which are modified \( K \)-semistable \cite{18}. In particular, we prove the following existence theorem of KR solitons in the deformation space of complex structures.
Theorem 0.2. Let $(M, J_0)$ be a Fano manifold which admits a KR-soliton $\omega_{KS} \in 2\pi c_1(M, J_0)$. Then there exists a small $\epsilon$ such that $(M, J_{\psi_\epsilon})$ admits a KR soliton close to $\omega_{KS} \in 2\pi c_1(M)$ in the Cheeger-Gromov topology for $\tau \in B(\epsilon)$ if and only if the Fano manifold $(M, J_{\psi_\epsilon})$ is modified K-polystable.

Theorem 0.2 gives a confirmative answer to the Yau-Tian-Donaldson conjecture for the existence of KR solitons in the deformation space of complex structures (cf. [17, 10, 8, etc.]). We also mention that Theorem 0.2 was proved by Inoue for the equivariant deformation space of complex structures by the Hamilton-Tian conjecture [19] (also see Corollary 1.3 and Remark 1.4). Inoue’s result is a generalization of Székelyhidi’s for KE metrics [33].

Theorem 0.2 will first derive such an inequality for the restricted $\nu(\cdot)$ of $\lambda(\cdot)$ on the parameter space $U_\epsilon$ (cf. Proposition 5.1). The advantage of $\nu(\cdot)$ is that its gradient and the second variation $\delta^2 \nu(\cdot)$ of $\nu(\cdot)$ are both of maps to $U$ (cf. Definition 3.3, 3.8). In particular, we get an explicit formula for the kernel of $\delta^2 \nu(\cdot)$ at a KR soliton (cf. 3.9 and Remark 3.10). Then by the spectral theorem 3.11, we prove the Lojasiewicz inequality for the original $\lambda(\cdot)$ (cf. Lemma 3.6, Corollary 5.3, 6.11). We would like to mention that such an inequality for the space of Riemannian metrics along the Ricci flow has been studied by Sun-Wang [32].

Let us say a few of words about how to prove the Lojasiewicz inequality for $\lambda(\cdot)$. In fact, we will first derive such an inequality for the restricted $\nu(\cdot)$ of $\lambda(\cdot)$ on the parameter space $U_\epsilon$ (cf. Proposition 5.1). The advantage of $\nu(\cdot)$ is that its gradient and the second variation $\delta^2 \nu(\cdot)$ of $\nu(\cdot)$ are both of maps to $U$ (cf. Definition 3.3, 3.8). In particular, we get an explicit formula for the kernel of $\delta^2 \nu(\cdot)$ at a KR soliton (cf. 3.9 and Remark 3.10). Then by the spectral theorem 3.11, we prove the Lojasiewicz inequality for the original $\lambda(\cdot)$ (cf. Lemma 3.6, Corollary 5.3, 6.11). We would like to mention that such an inequality for the space of Riemannian metrics along the Ricci flow has been studied by Sun-Wang [32].

Theorem 0.2 will be generalized for Kähler manifolds $(M, J_{\psi_\epsilon})$ with $H_1(\psi_\tau) \leq 0$ as follows.

Theorem 0.3. Let $(M, J_0)$ be a Fano manifold which admits a KR soliton $\omega_{KS} \in 2\pi c_1(M, J_0)$. Suppose that

1. $(M, J_{\psi_\epsilon})$ admits a KR soliton for any $\psi_\epsilon \in B(\epsilon) \cap Z$;
2. $H_1(\cdot) \leq 0$, on $H^{0,1}_g (M, T^{1,0} M)$.

Then there exists a small $\epsilon$ such that for any $\tau \in B(\epsilon)$ the KR flow (0.1) converges smoothly to a KR soliton $(M, J_{\infty}, \omega_{\infty})$ for any initial metric $\tilde{\omega}_0$ in $2\pi c_1(M, J_{\psi_\epsilon})$. Moreover,

$$\lambda(\omega_{\infty}) = \lambda(\omega_{KS}),$$

and the convergence is of polynomial rate.

The condition (1) may not be necessary in view of Theorem 0.4. In fact, we have the following conjecture.

Conjecture 0.4. Let $(M, J_0)$ be a Fano manifold which admits a KR soliton $\omega_{KS} \in 2\pi c_1(M, J_0)$. Let $A_-$ be the linear subspace of $H^{0,1}_g (M, T^{1,0} M)$ associated to the negative eigenvalues of $H_1$. Then there exists a small $\epsilon$ such that the following is true:

1. For any $\tau \in B(\epsilon)$ with $\psi_\tau \in A_- \cup Z$ the KR flow (0.1) converges smoothly to a KR soliton $(M, J_{\infty}, \omega_{\infty})$ for any initial metric $\omega_0$ in $2\pi c_1(M, J_{\psi_\epsilon})$. Moreover, the convergence is of polynomial rate with $\lambda(\omega_{\infty}) = \lambda(\omega_{KS})$.
2. For any $\tau \in B(\epsilon)$ with $\psi_\tau \in H^{0,1}_g (M, T^{1,0} M) \setminus (A_- \cup Z)$, the KR flow (0.1) converges to a singular KR soliton $\omega_{\infty}$ for any initial metric $\omega_0$ in $2\pi c_1(M, J_{\psi_\epsilon})$ with $\lambda(\omega_{\infty})$ strictly bigger than $\lambda(\omega_{KS})$.

The above conjecture means that the set of complex structures, for which the KR flow (0.1) is stable, corresponds to the linear semistable subspace $A_- \cup Z$ of $H^{0,1}_g (M, T^{1,0} M)$ in the deformation space of Fano manifolds which admit a KR-soliton. Otherwise, the flow on $(M, J_{\psi_\epsilon})$ is unstable, but will in general converge to a singular KR soliton $\omega_{\infty}$ with $\lambda(\omega_{\infty})$ strictly bigger than $\lambda(\omega_{KS})$ by the Hamilton-Tian conjecture [53, 2, 17, 20].

2This also answers a question of Chen-Sun [8, Remark 5.5].
The organization of the paper is as follows. In Section 1, we introduce the weighted Hodge-Laplace operator $\Box_h$ and the $h$-harmonic space of $\mathcal{H}^{0,1}_h(M, T^{1,0}M)$ on a Fano manifold. In Section 2, we study the Kuranishi deformation theory of complex structures associated to $\Box_h$ and prove that the solution of modified Kuranishi’s equation is divergent-free (cf. Proposition 2.1)). In Section 3, we compute the first and second variations of $\nu(\cdot)$ on $\mathcal{U}_\epsilon$. In Section 4, we give a new version of the second variations of $\nu(\cdot)$ and prove the local maximality of $\lambda(\cdot)$ (cf. Theorem 4.4 and Proposition 4.11). Section 5 is devoted to prove the Lojasiewicz inequality (cf. Proposition 5.1). Theorem 0.1 is proved in Section 6 while both of Theorem 0.2 and Theorem 7.5 are proved in Section 7. In Section 8, we prove Theorem 0.3.

1. Weighted Hodge-Laplace on $A^{p,q}(M, T^{1,0}M)$

In this section, we introduce a weighted Hodge-Laplace operator on the space $A^{p,q+1}(M, T^{1,0}M)$ on a Fano manifold $(M, J_0)$ with a Kähler form $\omega \in 2\pi c_1(M)$. This is very similar with the weighted Laplace operator associated to the Bakry-Émery Ricci curvature studied in [44]. Let $h$ be a Ricci potential of $\omega$, which satisfies that

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h. \tag{1.1}$$

We denote $\delta_h = \partial^\ast h$ the dual operator of $\partial$ with respect to the inner product

$$\int_M<,>_e h,\omega^n,$$

where $<,>_e$ is the inner product on $\Lambda^{p,q} T^* M \otimes T^{1,0} M$ induced by $\omega$. Namely, for any $\varphi \in A^{p,q}(M, T^{1,0} M)$, $\varphi' \in A^{p,q+1}(M, T^{1,0} M)$, we have

$$\int_M \bar{\partial} \varphi, \varphi' > e^h \omega^n = \int_M \varphi, \delta_h \varphi' > e^h \omega^n.$$

Then weighted Hodge-Laplace operator $\Box_h$ on $A^{p,q+1}(M, T^{1,0} M)$ is defined by

$$\Box_h = \bar{\partial} \cdot \delta_h + \delta_h \cdot \bar{\partial}.$$

Thus, similarly with the Hodge-Laplace operator, we see that $\Box_h \varphi = 0$ if only if $\delta_h \varphi = 0$ and $\delta_h \varphi = 0$.

For simplicity, we denote $\mathcal{H}_h^{p,q}(M, T^{1,0} M) = \{ \varphi \in A^{p,q}(M, T^{1,0} M) \mid \Box_h \varphi = 0 \}$ the harmonic space of $\Box_h$ on $A^{p,q+1}(M, T^{1,0} M)$.

In this paper, we are interested in the space $A^{0,1}(M, T^{1,0} M)$. Then under local holomorphic coordinates, for any

$$\varphi = \varphi_i^j dz^j \otimes \frac{\partial}{\partial z^i} \in A^{0,1}(M, T^{1,0} M),$$

we have

$$\delta_h \varphi = -(\varphi^k_i + \varphi^h_i) g^{ij} \frac{\partial}{\partial z^i}. \tag{1.2}$$

Moreover,

$$\mathcal{H}_h^{0,1}(M, T^{1,0} M) \cong H^1(M, J, \Theta). \tag{1.3}$$

The latter is the Čech cohomology group associated to the infinitesimal deformation of complex structures on $(M, J_0)$ [14].

We define a $h$-divergence operator on $A^{p,q+1}(M, T^{1,0} M)$ by

$$\text{div}_h \varphi = e^{-h} \text{div}(e^h \varphi).$$
In particular, if
\[ \varphi = \varphi^a_j dz^j \otimes \frac{\partial}{\partial z^j} \in A^{0,1}(M, T^{1,0} M), \]
then
\[ (1.4) \quad \text{div}_{\varphi} \varphi = (\varphi^a_j)_t + \varphi^a_j h_i dz^j. \]

Also we introduce an inner product \( \varphi \cdot \mu \) with respect to
\[ \mu = \sqrt{-1} \varphi^1 \mu_1 dz^1 \wedge d\bar{z}^1 \in A^{1,1}(M) \]
by
\[ \varphi \cdot \mu = \sqrt{-1} \varphi^1 \mu_1 dz^1 \wedge d\bar{z}^1 \in A^{0,2}(M). \]

We list some identities for the above two operators in the following lemma.

**Lemma 1.1.**  \( (1) \quad \overline{\partial}(\varphi \cdot \mu) = \overline{\partial} \varphi \cdot \mu + \varphi \cdot \overline{\partial} \mu. \)

(2) \( \delta_h(\text{div}_{\varphi} \varphi) = \text{div}_{\varphi}(\delta_h \varphi). \)

(3) \( \delta_h(\varphi \cdot \omega) = -\delta_h(\varphi \cdot \omega) - \sqrt{-1} i \text{div}_{\varphi} \varphi. \)

(4) \( \overline{\partial} \text{div}_{\varphi} \varphi - \text{div}_{\varphi}(\overline{\partial} \varphi) = -\sqrt{-1} \varphi \cdot \omega. \)

(5) \( \text{div}_{\varphi} [\varphi, \psi] = \psi \cdot \partial \text{div}_{\varphi} \varphi + \varphi \cdot \partial \text{div}_{\varphi} \psi. \)

(6) \( [\varphi, \psi] \cdot \omega = \varphi \cdot \partial (\psi \cdot \omega) + \psi \cdot \partial (\varphi \cdot \omega). \)

**Proof.** All identities can be verified directly. (1) is simple. For (2), we note that
\[ \delta_h(\text{div}_{\varphi} \varphi) = -(\varphi^a_j)_t + \varphi^a_j h_i dz^j + (\varphi^a_j)_t + \varphi^a_j(h_i)_s) g^{ij} \]
and
\[ \text{div}_{\varphi}(\delta_h \varphi) = -(\varphi^a_j)_t + \varphi^a_j h_s + \varphi^a_j h_i dz^j + (\varphi^a_j)_t + \varphi^a_j(h_s)_i) g^{ij}. \]

Thus (2) is true.

Since \( \varphi \cdot \omega = \sqrt{-1} \varphi^a_j g^{ij} dz^i \wedge d\bar{z}^j \), we have
\[ \delta_h(\varphi \cdot \omega) = -\sqrt{-1} (\varphi^a_j g^{ij} dz^i + \varphi^a_j g^{sj} h_s) + \varphi^a_j g^{ij} dz^i \]
\[ = -\sqrt{-1} \varphi^a_j g^{ij} dz^i - \sqrt{-1} (\delta_h \varphi)^i g^{ij} dz^i \]
\[ = -\sqrt{-1} \text{Idiv}_{\varphi} - (\delta_h \varphi) \cdot \omega. \]

This is (3).

In order to prove (4), we choose a normal coordinate \((z^1, \ldots, z^n)\) around each \( p \in M \). By (1.4), we have
\[ \overline{\partial} \text{div}_{\varphi} \varphi = (\varphi^a_j)_t + (\varphi^a_j)_t h_s + \varphi^a_j h_i dz^j d\bar{z}^j. \]

Since
\[ \overline{\partial} \varphi = (\varphi^a_j)_t + (\varphi^a_j)_t + (\varphi^a_j)_t \wedge d\bar{z}^j \]
we see that at \( p \),
\[ \text{div}_{\varphi} \overline{\partial} \varphi = (\varphi^a_j)_t + (\varphi^a_j)_t + (\varphi^a_j)_t + (\varphi^a_j)_t) dz^j \wedge d\bar{z}^j. \]

Thus we get
\[ (1.5) \quad \overline{\partial} \text{div}_{\varphi} \varphi - \text{div}_{\varphi}(\overline{\partial} \varphi) = (\varphi^a_j)_t - \varphi^a_j h_s - (\varphi^a_j)_t - (\varphi^a_j)_t) dz^j \wedge d\bar{z}^j. \]

On the other hand,
\[ \varphi^a_j - \varphi^a_j = -R_{s_{\Gamma} t} g^{s_{\Gamma}_{ij}} - R_{s_{\Gamma} t} g^{s_{\Gamma}_{ij}_{s_{\Gamma}}} \]
\[ = -R_{s_{\Gamma} t} g^{s_{\Gamma}_{ij}} + \partial_h g^{s_{\Gamma}_{ij}}. \]

By (1.5) it follows that
\[ \overline{\partial} \text{div}_{\varphi} \varphi - \text{div}_{\varphi}(\overline{\partial} \varphi) = (-R_{s_{\Gamma} t} h_s) \varphi^a_j dz^j \wedge d\bar{z}^j. \]

We note that
\[ (-R_{s_{\Gamma} t} + h_s) \varphi^a_j dz^j \wedge d\bar{z}^j = \varphi_{\cdot} \text{Ric}(\omega) = \sqrt{-1} \varphi_{\cdot} \omega. \]
Together with the assumption $\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h$, we conclude that
\[
\partial \text{div}_h \varphi - \text{div}_h (\partial \varphi) = \varphi \cdot \frac{\text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} h}{\sqrt{-1}} = -\sqrt{-1} \varphi \omega,
\]
which proves (4).

For (5), we recall that for any $\varphi = \psi^{t \alpha}_{\beta} dz^\alpha \otimes \frac{\partial}{\partial z^\beta}$ and $\psi = \psi^{t \alpha}_{\beta} dz^\alpha \otimes \frac{\partial}{\partial z^\beta}$ it holds
\[
[\varphi, \psi] = \psi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \psi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j}) + \varphi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \varphi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j})
\]
\[
= \psi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \psi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j}) - \varphi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \varphi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j})
\]
\[
= (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
Then,
\[
\text{div}[\varphi, \psi] = (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
Hence by (1.4), we get
\[
\omega \varphi \cdot (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
(6)
\[
\text{On the other hand, by (1.3)} \quad \text{we also have}
\]
\[
\partial \text{div}_h \varphi = (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
It follows that
\[
\psi \varphi \partial \text{div}_h \varphi = (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
Similarly we have
\[
\varphi \psi \partial \text{div}_h \psi = (\psi^{t \alpha}_{\beta} h + \varphi^{t \alpha}_{\beta}) dz^i \wedge d \bar{z}^j \wedge \frac{\partial}{\partial z^j}.
\]
We note that $h_{ij} = h_{ji}$. Hence, combing the above (1.3)-(1.8) we will derive (5) immediately.

Finally we prove (6). From the proof of (5) we see that
\[
[\varphi, \psi] = \psi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \psi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j}) - \varphi^{t \alpha}_{\beta} \frac{\partial}{\partial z^\alpha} (h \bar{h} \varphi_{\beta}) \wedge (d \bar{z}^i \otimes \frac{\partial}{\partial z^j})
\]
Thus we have
\[
[\varphi, \psi] \omega = \sqrt{-1} \psi^{t \alpha}_{\beta} g_{ij} \bar{g} dz^i \wedge d \bar{z}^j \wedge d \bar{z}^i - \sqrt{-1} \varphi^{t \alpha}_{\beta} g_{ij} \bar{g} dz^i \wedge d \bar{z}^i \wedge d \bar{z}^j.
\]
On the other hand, we have
\[
\partial (\psi \omega) = \sqrt{-1} \psi^{t \alpha}_{\beta} g_{ij} \bar{g} dz^i \wedge d \bar{z}^i \wedge d \bar{z}^j.
\]
It follows that
\[
\psi \varphi \partial (\psi \omega) = \sqrt{-1} \psi^{t \alpha}_{\beta} g_{ij} \bar{g} dz^i \wedge d \bar{z}^i \wedge d \bar{z}^j.
\]
Similarly we have
\[
\varphi \psi \partial (\varphi \omega) = \sqrt{-1} \varphi^{t \alpha}_{\beta} g_{ij} \bar{g} dz^i \wedge d \bar{z}^i \wedge d \bar{z}^j.
\]
Hence, combining the above (1.9)-(1.11) we get (6).
Lemma 1.2. Suppose that
\[ \delta h \varphi = 0 \text{ and } \overline{\partial}(\varphi, \omega) = 0. \]
Then
\[ \Box h (\varphi, \omega) = -\sqrt{-1} (\text{div}_h \partial \varphi) - \varphi, \omega. \]

Proof. Using the assumption \( \overline{\partial}(\varphi, \omega) = 0 \), we have
\[ \Box h (\varphi, \omega) = \overline{\partial} \delta h (\varphi, \omega) + \delta h \overline{\partial}(\varphi, \omega) \]
\[ = \overline{\partial} \delta h (\varphi, \omega). \]
Then by Lemma 1.1-(3) and the assumption \( \delta h \varphi = 0 \), we get
\[ \Box h (\varphi, \omega) = \overline{\partial} (-\delta h \varphi, \omega - \sqrt{-1} \text{div}_h \varphi) \]
\[ = \overline{\partial} (-\sqrt{-1} \text{div}_h \varphi). \]
Hence, by Lemma 1.1-(4), we obtain (1.12). □

Corollary 1.3. Let \( \varphi \in H^{0,1}_h(M, T^{1,0} M) \). Then
\[ \Box h (\varphi, \omega) = -\varphi, \omega. \]

As a consequence,
\[ \varphi, \omega = 0. \]

Proof. Note that \( \overline{\partial} \varphi = 0 \). Then by Lemma 1.1-(1) we have \( \overline{\partial}(\varphi, \omega) = 0 \). On the other hand, since \( \delta h \varphi = 0 \), by Lemma 1.2 together with \( \overline{\partial} \varphi = 0 \), we see that
\[ \Box h (\varphi, \omega) = -\sqrt{-1} \text{div}_h (\overline{\partial} \varphi) - \varphi, \omega \]
\[ = -\varphi, \omega. \]
Thus
\[ 0 \leq (\varphi, \omega, \varphi, \omega)_h = -(\Box h (\varphi, \omega), \varphi, \omega)_h \]
\[ = -(\delta h (\varphi, \omega), \delta h (\varphi, \omega))_h - (\overline{\partial}(\varphi, \omega), \overline{\partial}(\varphi, \omega))_h \]
\[ \leq 0. \]
It follows that
\[ (\varphi, \omega, \varphi, \omega)_h = 0, \]
which implies (1.13). □

2. Kuranishi deformation theory for \( \Box h \)-operator

We choose a basis \( \{ e_i \}_{i=1, \ldots, l} \) of \( H^{0,1}_h(M, T^{1,0} M) \) and denote an \( \epsilon \)-ball by
\[ B(\epsilon) = \{ \tau = (\tau_1, \ldots, \tau_l) \mid \psi_\tau = \sum_\tau \tau_i e_i \in H^{0,1}_h(M, T^{1,0} M) \text{ with } \sum_\tau \tau_i^2 < \epsilon^2 \}. \]
Then by the Kodaira deformation theory for complex structures on \( (M, J_0) \), there is a map
\[ \Psi : B(\epsilon) \mapsto A^{0,1}(M, T^{1,0} M) \]
as long as \( \epsilon \) is small enough such that \( \Psi(t) \) is a solution of Cartan-Maurier equation,
\[ (2.1) \quad \overline{\partial} \Psi(\tau) = \frac{1}{2} [\Psi(\tau), \Psi(\tau)]. \]
More precisely, analogous to the Kodaira theory \[14\], we can reduce (2.1) to solving the following equation with gauge fixed,
\[ (2.2) \quad \left\{ \begin{array}{l} \delta h \varphi(\tau) = 0 \\ \overline{\partial} \varphi(\tau) = \frac{1}{2} [\varphi(\tau), \varphi(\tau)] \end{array} \right. \]
The above equation is equivalent to
\[
\varphi(\tau) = \sum_{i=1}^{m} \tau_{i}e_{i} + \frac{1}{2} \delta h G_{h}[\varphi(\tau), \varphi(\tau)],
\]
where \(G_{h}\) the Green operator associated to \(\Box_{h}\). Following the Kuranshi’s method by the implicity
function theorem \([13]\), there is a unique solution of (2.3) as long as \(\sum |\tau_{i}|^2 < \epsilon^2\). Thus there is a
map
\[
\Phi : B_{\epsilon} \mapsto A_{0,1}^1(M, T^{1,0}M)
\]
such that \(\Phi(\tau) = \varphi(\tau) = \varphi_{\tau}\). For simplicity, we call \(\Phi\) the Kuranishi map associated to \(\Box_{h}\)-
orator.

On the other hand, according to \([14]\), we may write \(\varphi(\tau)\) as a convergent expansion to solve
(2.1),
\[
\varphi(\tau) = \sum \tau_{i}e_{i} + \sum_{|I| \leq 2} \tau_{I} \varphi_{I}.
\]
(2.5)

By (2.5), we prove

**Proposition 2.1.** Let \(\varphi(\tau)\) be a Kuranishi solution of (2.3) as the form of (2.5). Then
\[
\varphi(\tau), \omega = 0 \text{ and } \text{div}_{h} \varphi(\tau) = 0.
\]
(2.6)

**Proof.** Let
\[
\psi_{k} = \sum_{|I| \leq k} \tau_{I} \varphi_{I}, k = 1, 2, \ldots.
\]
(2.7)

Then
\[
\delta_{h} \psi_{k} = 0, \text{ for each } k \geq 1.
\]
We need to prove (2.10) for each \(\psi_{k}\) by induction. Since
\[
\delta_{h} \psi_{1} = 0, \overline{\partial} \psi_{1} = 0,
\]
by Corollary \([13]\) we have
\[
\psi_{1}, \omega = 0.
\]
(2.8)

By Lemma \([1, 1]\)(3), it follows that
\[
\text{div}_{h} \psi_{1} = 0.
\]
Thus we may assume that
\[
\psi_{k}, \omega = 0 \text{ and } \text{div}_{h} \psi_{k} = 0, \text{ for any } k < l.
\]
(2.9)

By (2.2), we see that
\[
\overline{\partial} \psi_{l} = \frac{1}{2} \sum_{k=1}^{l-1} [\varphi_{k}, \varphi_{l-k}], \delta_{h} \psi_{l} = 0.
\]
(2.10)

Then by Lemma \([1, 1]\)(6) together with the first relation in (2.9), we get
\[
\overline{\partial} \psi_{l}, \omega = 0.
\]
(2.11)

By Lemma \([1, 1]\)(1), it follows that
\[
\overline{\partial} (\psi_{l}, \omega) = 0.
\]
(2.12)

Thus by Lemma \([1, 2]\) together with the second relation in (2.10), we derive
On the other hand,
\[
\text{div}_h \partial \psi_l = \text{div}_h \left( \frac{1}{2} \sum_{k=1}^{l-1} [\varphi_k, \varphi_{l-k}] \right) = \frac{1}{2} \sum_{k=1}^{l-1} \text{div}_h [\varphi_k, \varphi_{l-k}].
\]
By Lemma 1.1(5), we have
\[
\text{div}_h \partial \psi_l = 0.
\]
Thus by (2.12), we get
\[
\Box_h (\psi_l \omega) = -\psi_l \omega.
\]
Hence, as in the proof of (1.13), we obtain
\[
(\psi_l \omega = 0.
\]
This proves the first relation of (2.9) for \(k = l\).

By Lemma 1.1(3), we also have
\[
\delta_h (\psi_l \omega) = -\delta_h \psi_l \omega - \sqrt{-1} \text{div}_h \psi_l.
\]
Thus (2.13) implies the second relation of (2.9) for \(k = l\). The proposition is proved. □

By (2.6), for \(v, w \in T^{0,1}M\) we have
\[
\omega(v + \varphi_\tau(v), w + \varphi_\tau(w)) = 0.
\]
Since \(T^{0,1}_{\varphi,\tau} = (id + \varphi_\tau)(T^{0,1}M)\) and \(\omega\) is real, \(\omega\) is still a \((1, 1)\)-form with respect to the complex structure \(J_{\psi_\tau}\) defined by \(\varphi_\tau\). Namely, \(\varphi_\tau\) is compatible with the Kähler form \(\omega\). As a consequence, \((\omega, J_{\psi_\tau})\) defines a family of Kähler metrics by
\[
g_\tau = \omega(\cdot, J_{\psi_\tau} \cdot).
\]
(2.14)
Hence, we get

**Corollary 2.2.** For any \(\tau \in B(\epsilon)\), \(g_\tau\) defined in (2.14) is a family of Kähler metrics with the same Kähler form \(\omega\).

The following proposition gives a relationship between the first and second relations in (2.6), which means that the Kuranishi equation is equivalent to the Cartan-Maurer equation with the divergence free gauge.

**Proposition 2.3.** Suppose that \(\varphi \in A^{0,1}(M, T^{1,0}M)\) satisfies
\[
\overline{\partial} \varphi = \frac{1}{2} [\varphi, \varphi] \text{ with } \text{div}_h \varphi = 0.
\]
Then
\[
\varphi_\omega = 0 \text{ and } \delta_h \varphi = 0.
\]
**Proof.** By Lemma 1.1(5) and the assumption that \(\text{div}_h \varphi = 0\), we have
\[
\text{div}_h (\overline{\partial} \varphi) = \frac{1}{2} \text{div}_h [\varphi, \varphi] = \varphi_\omega \text{div}_h \varphi = 0.
\]
By Lemma 1.1(4) together with the assumption \(\text{div}_h \varphi = 0\), it follows that
\[
(2.15)
\]
On the other hand, by Lemma 1.1(3) and (2.15), we see that
\[
\delta_h \varphi_\omega = 0,
\]
which means that \(\delta_h \varphi = 0\). The proposition is proved. □
3. Restricted entropy \( \nu(\cdot) \) and its variations

From this section, we will always assume that \((M, J_0)\) is a Fano Kähler manifold which admits a KR soliton \((\omega_{KS}, X)\). Here the soliton \(VF_X\) can be regarded as an element in the center of Lie algebra \(\eta_r(M, J_0)\) of a maximal reductive subgroup \(\text{Aut}_r(M, J_0)\) of \(\text{Aut}(M, J_0)\) \[35\]. Namely \((\omega_{KS}, X)\) satisfies the soliton equation,

\[
\text{Ric}(\omega_{KS}) - \omega_{KS} = \mathcal{L}_X\omega_{KS},
\]

where \(\mathcal{L}_X\omega_{KS}\) is the Lie derivative of \(\omega_{KS}\) along \(X\). Then there is a real smooth function \(\theta = \theta_X(\omega_{KS})\) which satisfies

\[
i_X\omega_{KS} = \int_M e^\theta \omega_{KS}^n = \int_M \omega^n_{KS}.
\]

Thus by Corollary 2.2, there is a small ball \(B(\varepsilon) = \{\tau\mid \psi_\tau = \sum \tau_i e_i \in \mathcal{H}_{0,1}(M, T^{1,0}M), \sum_i \tau_i^2 < \varepsilon^2\}\), such that the Kuranishi map:

\[
\Phi : B(\varepsilon) \mapsto A^{0,1}(M, T^{1,0}M)
\]

induces a family of Kähler metrics \(g_\tau\) with the same Kähler form \(\omega_{KS}\). As a consequence, any Kähler metric \(g\) in \(2\pi c_1(M)\) with small perturbed integral complex structure of \(J_0\) can be parameterized by the following map:

\[
L : B(\varepsilon) \times C^\infty(M) \mapsto \text{Sym}^2(T^*M)
\]

with \(g_{\tau,\chi}(\cdot, \cdot) = L(\psi, \chi)(\cdot, \cdot)\) satisfying

\[
g_{\tau,\chi} = (\omega_{KS} + \sqrt{-1}\partial\bar{\partial}J_{\psi,\chi})(\cdot, J_{\psi,\cdot}).
\]

The purpose of this section is to compute the variation of Perelman’s entropy \(\lambda(\cdot)\) for Kähler metrics \(g_{\tau,\chi}\). Recall that the Perelman’s W-functional for Kähler metrics \(g\) in \(2\pi c_1(M)\) is defined for a pair \((g, f)\) by (cf. \[40\]),

\[
W(g, f) = (2\pi)^{-n} \int_M [R(g) + |\nabla f|^2 + f] e^{-f} \omega^n_g,
\]

where \(f\) is a real smooth function normalized by

\[
\int_M e^{-f} \omega^n_g = \int_M \omega^n_g.
\]

Then \(\lambda(g)\) is defined by

\[
\lambda(g) = \inf_f \{W(g, f)\mid (g, f) \text{ satisfies (3.4)}\}.
\]

The number \(\lambda(g)\) can be attained by some \(f\) (cf. \[30\]). In fact, such a \(f\) is a solution of the equation,

\[
2\triangle f + f - |DF|^2 + R = \lambda(g).
\]

In particular, \(f = \theta\) if \(\omega_g = \omega_{KS}\), so the minimizer of \(W(g, \cdot)\) is unique near a KR soliton \[38, 32\]. Thus by the relation (3.2), we get the restricted entropy \(\nu(\cdot)\) of \(\lambda(g)\) on \(\mathcal{U}_\varepsilon = B(\varepsilon) \times C^\infty(M)\) by

\[
\nu(\psi, \chi) = \lambda(L(\psi, \chi)),
\]

which is a smooth functional near \(\omega_{KS}\). In fact, \(\nu(\cdot)\) is analytic (cf. \[32\]).
3.1. Variation of Kähler metrics. We calculate the variation of Kähler metrics \( g_{\tau, \chi} \) at \( g = g_{0,0} \) with its Kähler form \( \omega = \sqrt{-1}g_{ij}dz^i \wedge dz^j \). Let \( \varphi \in \mathcal{A}^{0,1}(M, T^{1,0}M) \) with the almost complex structure \( J_\varphi \) associated with \( \varphi \). Under local coordinates \( (z^1, \ldots, z^n) \) on \( (M, J_0) \), \( \varphi \) is written as a Beltrami differential by

\[
\varphi = \varphi \frac{dz^i}{dz^j} \otimes \frac{\partial}{\partial z^j}.
\]

Decompose \( T_C M \) with respect to \( J_\varphi \) by

\[
T_C M = T_{J_\varphi}^1 M \oplus T_{J_\varphi}^{0,1} M.
\]

Then

\[
T_{J_\varphi}^{0,1} M = (id + \varphi) T_{J_\varphi}^{0,1} M,
\]

where \( \varphi \) is viewed as a map from \( T^{1,0}M \to T^{1,0}M \).

Let

\[
\left\{ \begin{array}{l}
J_\varphi \frac{\partial}{\partial z^i} = J^i_j \frac{\partial}{\partial z^j} + J^i_{\overline{j}} \frac{\partial}{\partial \overline{z}^j} \\
J_\varphi \frac{\partial}{\partial \overline{z}^i} = J^i_{\overline{j}} \frac{\partial}{\partial \overline{z}^j} + J^i_j \frac{\partial}{\partial z^j}
\end{array} \right.
\]

Since \( J_\varphi \) is real,

\[
J^i_i = J^i_{\overline{j}} = J^\overline{i}_i = J^\overline{i}_{\overline{j}}.
\]

Thus for a family of \( \varphi = \varphi(\tau) \in \mathcal{A}^{0,1}(T^{1,0}M) \) with \( \phi(0) = 0 \), we get the derivative of almost complex structure \( J_\varphi \) as follows,

\[
\left. \frac{dJ^i_j}{d\tau} \right|_{\tau=0} = -2\sqrt{-1} \frac{d\varphi^i_j}{d\tau},
\]

\[
\left. \frac{dJ^i_{\overline{j}}}{d\tau} \right|_{\tau=0} = 0.
\]

By Corollary 2.2, \( g_\tau \) given in (2.14) is a family of Kähler metrics with the fixed Kähler form \( \omega \). Locally, as a Riemannian tensor, \( g_\tau \) is of form,

\[
g_\tau = 2\text{Re}(-\sqrt{-1}g_{ij}J^i_j dz^i \otimes d\overline{z}^j - \sqrt{-1}g_{k\overline{l}}J^k_{\overline{l}} dz^k \otimes dz^\overline{l}).
\]

Then the derivative of \( L \) at \( (0,0) \) is given by

\[
\eta = DL_{(0,0)}(\psi, 0) = -4\text{Re}(g_{k\overline{l}} \psi^k_j dz^k \otimes d\overline{z}^j)
\]

and

\[
DL_{(0,0)}(0, \chi) = 2\text{Re}(\chi^i_j dz^i \otimes d\overline{z}^j),
\]

where \( \psi \in \mathcal{H}_{h}^{0,1}(M, T^{1,0}M) \) and \( \chi \in C^\infty(M) \). Clearly, \( \eta \) is anti-hermitian symmetric and \( DL_{(0,0)}(0, \chi) \) is hermitian symmetric.

We define the divergence for a \( (0,2) \)-type tensor \( \eta \) by

\[
\text{div}_h \eta = e^{-h} \text{div}(e^h \eta) = 2\text{Re}[\eta_{\overline{i}j}^k + \eta_{i\overline{j}}^k] g^{k\overline{l}} dz^i \otimes d\overline{z}^j).
\]

The following lemma shows that the tensor \( \eta \) is also divergence-free as \( \psi \).

**Lemma 3.1.**

(3.6) \( \text{div}_h \eta = 0 \) and \( \eta_{\overline{i}j}^k = \eta_{i\overline{j}}^k \).

**Proof.** By (1.4) we have

\[
(\text{div}_h \eta)^\tau = (\eta_{\overline{i}j}^k + \eta_{i\overline{j}}^k) g^{k\overline{l}}
\]

\[
= -4g_{i\overline{j}}(\psi^i_{\overline{k}} + \psi^i_k) g^{k\overline{l}}
\]

\[
= -4g_{i\overline{j}}(\text{div}_h \psi)^i = 0.
\]
The last equality comes from (2.7). Thus \( \nabla h. \eta = 0 \). On the other hand, since \( \bar{\partial} \varphi = 0 \), we have
\[
\begin{align*}
\eta & = \frac{1}{2} \sum_{i=1}^{l} (\nabla \varphi_i |_{\mathcal{T}}) \cdot \nabla \varphi_i |_{\mathcal{T}} \\
& = \frac{1}{2} \sum_{i=1}^{l} (\nabla \varphi_i |_{\mathcal{T}}) \cdot \nabla \varphi_i |_{\mathcal{T}} \\
& = \frac{1}{2} \sum_{i=1}^{l} (\nabla \varphi_i |_{\mathcal{T}}) \cdot \nabla \varphi_i |_{\mathcal{T}}.
\end{align*}
\]
The lemma is proved. \( \square \)

**Remark 3.2.** When \((M, \omega, J_0)\) is a KE manifold, (3.7) in Lemma 3.1 was verified by Koiso \([15]\).
By using the h-harmonic space in \((3.3)\), we can generalize Koiso’s result for any Fano manifold. The lemma will be used in the computation of the second variation of \( \nu \) below.

### 3.2. The first variation of \( \nu(\cdot) \)

As in \([10]\), we have the first variation of \( \lambda \) at \((M, \omega, J_0)\) as a Riemannian manifold,
\[
(3.7) \quad \delta \lambda = -\frac{1}{2} \int_M <\text{Ric}(g) - g + \text{Hess}_g(f), \delta g > - f^g \omega^n.
\]

Then
\[
(3.8) \quad \mathcal{N}(g) = \text{Ric}(g) - g + \text{Hess}_g(f)
\]
can be regarded as the derivative of \( \lambda \) which is a map from the space of 2-symmetric tensors to itself. But for the restricted entropy \( \nu(\cdot) \), we shall define its derivative from the product space
\( \mathcal{U} = \mathcal{H}_0^{0,1}(M, T^{1,0} M) \times C^\infty(M) \) to itself in the following.

Recall \( f_{\psi, \chi} \) be the minimizer of \( W \)-functional at \( g_{\tau, \chi} \) in \([30]\). We introduce a map \( \mathbf{R} : \mathcal{U} \to C^\infty(M) \) by
\[
\mathbf{R}(\psi, \chi) = \frac{1}{2} \sqrt{-1} (\bar{\partial} f_{\psi, \chi})^* \cdot (\bar{\partial} f_{\psi, \chi})^* \cdot (\text{Ric} \omega_{\tau, \chi}) - \omega_{\tau, \chi} + \sqrt{-1} \bar{\partial} J_{\psi, \chi} f_{\psi, \chi} e^{-f_{\psi, \chi}} \cdot \omega^n_{\tau, \chi},
\]
where \((\bar{\partial} f_{\psi, \chi})^* \) and \((\bar{\partial} f_{\psi, \chi})^* \) are the dual operators of \( \bar{\partial} f_{\psi, \chi} \) and \( \bar{\partial} f_{\psi, \chi} \), respectively as same as \( \delta_0 \) in Section 1 with respect to the following inner product,
\[
(\chi_1, \chi_2) = \int_M \chi_1 \chi_2 e^{-f_{\psi, \chi}} \omega^n_{\psi, \chi}, \quad \chi_1, \chi_2 \in L^2(M, e^{-f_{\psi, \chi}}).
\]

Choose an unitary orthogonal basis \( \{e_1, \ldots, e_l\} \) of \( \mathcal{H}_0^{0,1}(M, T^{1,0} M) \) with respect to the inner product
\[
(a, b)_0 = \int_M <a, b> \omega_{KS} e^g \omega^n_{KS}, \quad \forall \ a, b \in \mathcal{H}_0^{0,1}(M, T^{1,0} M).
\]
We define another map \( \mathbf{Q} : \mathcal{U} \to \mathcal{H}_0^{0,1}(M, T^{1,0} M) \) by
\[
\mathbf{Q}(\psi, \chi) = \sum_{i=1}^{l} (a_i(\psi, \chi) + \sqrt{-1} b_i(\psi, \chi)) e_i,
\]
where
\[
a_i(\psi, \chi) = \frac{1}{2} \int_M (\text{Ric}(g_{\tau, \chi}) - g_{\tau, \chi} + \text{Hess}_{g_{\tau, \chi}}(f_{\psi, \chi}), DL_{\psi, \chi}(e_i, 0)) e^{-f_{\psi, \chi}} \omega^n_{\tau, \chi}
\]
and
\[
b_i(\psi, \chi) = \frac{1}{2} \int_M (\text{Ric}(g_{\tau, \chi}) - g_{\tau, \chi} + \text{Hess}_{g_{\tau, \chi}}(f_{\psi, \chi}), DL_{\psi, \chi}(\sqrt{-1} e_i, 0)) e^{-f_{\psi, \chi}} \omega^n_{\tau, \chi}.
\]

**Definition 3.3.** We call the pair \((\mathbf{Q}, \mathbf{R})\) the gradient map \( \nabla \nu \) of \( \nu \) on \( \mathcal{U} \) with the inner product
\[
(3.10) \quad (\nabla \nu, (\psi, \chi))_0 = (\mathbf{Q}, \psi)_0 + \int_M \mathbf{R} e^g \omega^n_{KS},
\]
where \((\psi, \chi) \in \mathcal{U} \).
It is easy to verify that $\nabla \nu = (Q, R)$ is real analytic on $U$. The following lemma gives the first variation $\nu$ by $\nabla \nu$.

**Lemma 3.4.** Let $(\psi, \chi_0) \in U$ and $(\psi, \chi) \in \mathcal{H}_{\rho}^{0,1} (M, T^{1,0} M) \times C^\infty (M)$. Then

\begin{equation}
\frac{d}{ds} |_{s=0} \nu((\psi, \chi_0) + s(\psi, \chi)) = (\nabla \nu(\psi, \chi_0), (\psi, \chi))_0.
\end{equation}

**Proof.** For convenience, we let $g = L(\psi, \chi_0)$ and $f = f_g$ be the minimizer of $W$ functional at $g$. By (3.7), we have

\begin{align*}
\frac{d
u((\psi, \chi_0) + s(\psi, \chi))}{ds} |_{s=0} &= \frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(\psi, \chi) > e^{-f} \omega_{\psi, \chi_0}^n \\
&= \frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(\psi, 0) > e^{-f} \omega_{\psi, \chi_0}^n \\
&= \frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(0, \chi) > e^{-f} \omega_{\psi, \chi_0}^n.
\end{align*}

It is easy to see that

\begin{align*}
I &= -\frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(\psi, 0) > e^{-f} \omega_{\psi, \chi_0}^n \\
&= (Q(\psi, \chi_0), \psi)_0
\end{align*}

and

\begin{align*}
II &= -\frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(0, \chi) > e^{-f} \omega_{\psi, \chi_0}^n \\
&= \frac{d\lambda((\psi, \chi_0) + s(0, \chi))}{ds} |_{s=0} \\
&= -\frac{1}{2} \int_M < -\partial J_{\psi_0} J_{\eta_0} \chi, \text{Ric}(\omega_{\psi_0, \chi_0}) - \omega_{\psi_0, \chi_0} + \sqrt{-1} \partial J_{\psi_0} J_{\eta_0} f > e^{-f} \omega_{\psi_0, \chi_0}^n \\
&= \int_M R(\psi_0, \chi_0) \chi e^{-f} \omega_{\psi_0, \chi_0}^n.
\end{align*}

Thus (3.11) is true. \hfill \square

**Remark 3.5.** From the above, we actually have

\begin{align*}
\frac{d}{ds} |_{s=0} \nu((\psi, \chi_0) + s(\psi, \chi)) &= \nu((\psi, \chi), \nabla \nu(\psi, \chi_0))_0 \\
&= -\frac{1}{2} \int_M < \text{Ric}(g) - g + \text{Hess}_{g}(f), DL_{(\psi, \chi_0)}(\psi, \chi) > e^{-f} \omega_{\psi, \chi_0}^n.
\end{align*}

The $L^2$-norm of $R$ can be controlled by $\nabla \lambda(\cdot) = \mathcal{N}(\cdot)$ as follows.

**Lemma 3.6.** Let $k > 2$ be an integer and $\epsilon$ small enough. Then there exists a constant $C = C(\epsilon, k)$ such that for any $(\psi, \chi) \in U$ with $\|\chi\|_{C^{4,\gamma}} \leq \epsilon$ it holds

\begin{equation}
\|R(\psi, \chi)\|_{L^2} \leq C(\int_M |\text{Ric}(\omega_{\psi, \chi}) - \omega_{\psi, \chi} + \sqrt{-1} \partial \overline{\partial} f_{\psi, \chi} e^{-f_{\psi, \chi}} \omega_{\psi, \chi}^n|)_{L^2}.
\end{equation}

**Proof.** Let

\begin{equation}
\text{Ric}(\omega_{\psi, \chi}) - \omega_{\psi, \chi} + \sqrt{-1} \partial \overline{\partial} f_{\psi, \chi} = \sqrt{-1} \partial J_{\psi} J_{\eta} F.
\end{equation}
We extend $R(\psi, \chi)$ to a fourth-order operator on the Hilbert space $W^2_0(M, e^{-f_{o.x}})$ with respect to the inner product $(3.9)$ by
\begin{equation}
S = -(\partial_{J^o_o} \overline{J^o_o})^* \partial_{J^o_o} \overline{J^o_o} = -\overline{J^o_o} \cdot \partial_{J^o_o} \overline{J^o_o}.
\end{equation}
We show that $S$ is elliptic and self-adjoint like the Lichnerowicz operator \[4\]. Let $\eta, \phi \in C^\infty(M)$ and $\Delta \eta = \eta_{ij} g^{ij}$ associated to the metric $g = L(\psi, \chi)$. Then in local coordinates on $(M, J_o)$, we compute
\[
(S\eta, \phi) = \int_M \eta_{ij} \overline{g^{ij}} g^{kl} g^{lj} e^{-f_{o.x} \omega^n_{\psi, \chi}}
- \int_M \eta_{ij} \overline{g^{ij}} g^{kl} g^{lj} e^{-f_{o.x} \omega^n_{\psi, \chi}} + \int_M \eta_{ij} \overline{g^{ij}} g^{kl} g^{lj} e^{-f_{o.x} \omega^n_{\psi, \chi}}
= \int_M [(\Delta^2 \eta) - (\Delta \eta) (f_{\psi, \chi}) g^{ij}] \phi e^{-f_{o.x} \omega^n_{\psi, \chi}}
\]
It follows that
\[
(S\eta, \phi) = \int_M [(\Delta^2 \eta) - (\Delta \eta) (f_{\psi, \chi}) g^{ij}] \phi e^{-f_{o.x} \omega^n_{\psi, \chi}}
= \int_M [(\Delta^2 \eta) - (\Delta \eta) (f_{\psi, \chi}) g^{ij}] \phi e^{-f_{o.x} \omega^n_{\psi, \chi}}
\]
Thus we derive
\[
S\eta = \Delta^2 \eta - (\Delta \eta) (f_{\psi, \chi}) g^{ij} - (\Delta \eta) (f_{\psi, \chi}) g^{ij} - \eta_{ij} (f_{\psi, \chi}) g^{ik} g^{lj} + \eta_{ij} (f_{\psi, \chi}) g^{ik} g^{lj} + \eta_{ij} (f_{\psi, \chi}) g^{ik} g^{lj}.
\]
Hence, $S$ is an elliptic operator. By the regularity theorem of elliptic operator, $S$ is a self-adjoint operator on the domain $D(S) = W^2_0(M, e^{-f_{o.x}})$. Since $S$ is non-negative, by the spectral theorem we see that for integer $k > 0$, $x \in D(S)$ it holds
\begin{equation}
(S^k x, x) = \int_0^\infty \lambda^k dE_{x,x}(\lambda),
\end{equation}
where $E_{x,x}(\lambda)$ is the spectral decomposition defined by $S$ \[31\]. By the Hölder inequality and \[3.16\] it follows that
\begin{equation}
(S^2 x, x) \leq (Sx, x) \cdot (S^k x, x) \leq \int_0^\infty \lambda^2 dE_{x,x}(\lambda).
\end{equation}
Thus by (3.14), we get

\[ \|R(\psi, \chi)\|_{L^2} \leq C((\sqrt{-1}\partial\bar{\partial}\psi_0)F, (\sqrt{-1}\partial\bar{\partial}\chi_0)F) = C(SF, SF) = C(S^2 F, F) \leq C(SF, F)^{\frac{k-2}{k}} (S^k F, F)^{\frac{1}{k-2}}. \]

(3.18)

Note that

\[ SF = (\sqrt{-1}\partial\bar{\partial}\psi_0)F, (\sqrt{-1}\partial\bar{\partial}\chi_0)F = (S) = (S^k F, F) = (S^{k-2} SF, SF) = (S^{k-2}) (Ric(\omega_{\psi, \chi}) - \omega_{\psi, \chi} + \sqrt{-1}\partial\bar{\partial}\psi_0 F, \chi_0 F). \]

Then

\[ (S^k F, F) = (S^{k-2} SF, SF) = (S^{k-2}) (Ric(\omega_{\psi, \chi}) - \omega_{\psi, \chi} + \sqrt{-1}\partial\bar{\partial}\psi_0 F, \chi_0 F). \]

Hence

\[ (S^k F, F) C(\epsilon) < C(\epsilon) \to 0, \text{ as } \epsilon \to 0. \]

Therefore, we get (3.13) immediately by (3.18).

3.3. The second variation of \( \nu(\cdot) \). We will calculate the second valuation of \( \nu \) at \( (0, 0) \in \mathcal{U}_c \). Let \( (g, f) \) be a pair of \( W \)-functional in (3.3) while \( f \) is a smooth solution of (3.3). Denote \( \eta = \delta g, k = \delta f \) to be the variations of \( g \) and \( f \) respectively. Then it has been shown in [5, 40] that

\[ \delta (\text{Ric}(g) - g + \text{Hess}(f)) = -\Delta f - \text{Rm}(\eta, \cdot) - \text{div}_f \text{div}_f - \frac{1}{2} \text{Hess}(v_\eta), \]

where the function \( v_\eta = \text{tr} \eta - 2k \). Moreover, by differentiating (3.3), we get (cf. [17]),

\[ \Delta f (v_\eta) + \frac{v_\eta}{2} = -\frac{1}{2} \text{div}_f \text{div}_f (v_\eta). \]

For convenience, we write the right term in (3.20) as

\[ N(\eta) = -\Delta f - \text{Rm}(\eta, \cdot) - \text{div}_f \text{div}_f (\eta) - \frac{1}{2} \text{Hess}(v_\eta). \]

In case that \( -f = \theta \), namely \( g \) is a KR soliton, if

\[ \text{div}_f \eta = 0, \]

then

\[ \Delta_f (v_\eta) + \frac{v_\eta}{2} = 0. \]

Since the first non-zero eigenvalue of \( -\Delta_f \) is not less than one [10], we get

\[ v_\eta = 0. \]

Thus

\[ N(\eta) = -\Delta_f (\eta) - \text{Rm}(\eta, \cdot), \]

which is an anti-hermitian symmetric 2-tensor.

Now we restrict the Kähler metric \( g = g_{\tau, \chi} \) in \( \mathcal{U}_c \). Let

\[ (\psi, \chi, \psi', \chi') \in \mathcal{U} = \mathcal{H}^{0,1}_0(M, T^{1,0} M) \times C^\infty(M). \]
Then by (3.12), we have

\[
\frac{d}{ds}\big|_{s=0}(\nabla \nu(s(\psi, \chi)), (\psi', \chi'))_\theta
= -\frac{1}{2} \int_M \left\langle \text{Ric}(g_{\psi, \chi}) + \text{Hess}(f_{\psi, \chi}) - g_{\psi, \chi}, DL_{(0,0)}(\psi', \chi') \right\rangle > e^\theta \omega^n_{KS}
\]

(3.24)

Lemma 3.7.

\[
\int_M <N(DL_{(0,0)}(\psi, \chi), DL_{(0,0)}(\psi', \chi') > e^\theta \omega^n_{KS} = 0.
\]

Proof. We note that \( DL_{(0,0)}(\psi, 0) \) is anti-hermitian symmetric and (3.22) holds by Lemma 3.1. Thus \( N(DL_{(0,0)}(\psi, 0)) \) is an anti-hermitian symmetric 2-tensor by (3.23). On the other hand, \( DL_{(0,0)}(0, \chi') \) is hermitian symmetric. Hence the lemma is true. \( \Box \)

Analogous to the \( Q \)-map in Section 3.1, we introduce a map

\[
H_1 : \mathcal{H}^{0,1}_\theta(M, T^{1,0}M) \rightarrow \mathcal{H}^{0,1}_\theta(M, T^{1,0}M)
\]

by

\[
H_1(\psi) = \sum_{i=1}^l (c_i(\psi) + \sqrt{-1}d_i(\psi))e_i,
\]

where

\[
c_i(\psi) = -\frac{1}{2} \int_M <N(DL_{(0,0)}(\psi, 0), DL_{(0,0)}(\delta_i, 0) > e^\theta \omega^n_{KS}, i = 1, 2, \ldots, l,
\]

and

\[
d_i(\psi) = -\frac{1}{2} \int_M <N(DL_{(0,0)}(\psi, 0), DL_{(0,0)}(\delta_i, 0) > e^\theta \omega^n_{KS}, i = 1, 2, \ldots, l.
\]

Recall a fourth order non-positively elliptic operator \( H_2 \) on \( C^\infty(M) \) introduced in [40] by

\[
H_2 = [P_0^{-1}(\overline{T}_iL_i)(\overline{T}_iL_i)],
\]

where

\[
P_0\chi = 2\Delta \theta \chi + \chi - (X + \overline{X})(\chi),
\]

\[
L_1 \chi = \Delta \theta \chi + \psi - X(\chi)
\]

and

\[
L'_1 \chi = \Delta \theta \chi - X(\chi).
\]

Definition 3.8. Define a map \( H_{\psi, \chi} : \mathcal{U} \rightarrow \mathcal{U} \) by

\[
H_{\psi, \chi} = D_{\psi, \chi} \nabla \nu
\]

and write \( H = H_{0,0} \) for convenience. We call \( H \) the second variation operator of \( \nu(\cdot) \) at \( \omega_{KS} \).

By the above definition and (3.24), we see that for any \((\psi, \chi) \in \mathcal{U}, \) it holds

\[
\delta^2 \nu_{(0,0)}((\psi, \chi), (\psi, \chi)) = \int_M <H(\psi, \chi), (\psi, \chi) > e^\theta \omega^n_{KS}
\]

(3.25)

Moreover, by Lemma 3.7 we get

Proposition 3.9. \( H = H_1 \oplus H_2, \) namely, \( H(\psi, \chi) = H_1(\psi) + H_2(\chi). \)

By a result in [43] (also see [18]), we have the following explicit formula for \( H_1. \)
Lemma 3.10.

\begin{equation}
(H_1(\psi), \psi)_\theta = \int_M |\psi|^2 \tilde{\theta} e^\theta \omega^n_{KS}, \ \forall \ \psi \in \mathcal{H}^{0,1}_\theta(M, T^{1,0}M),
\end{equation}

where \( \tilde{\theta} \) differs from \( \theta \) with a constant such that

\begin{equation}
\int_M \tilde{\theta} e^\theta \omega^n_{KS} = 0.
\end{equation}

Proof. By Lemma [\ref{lem:delta_nu}], \( \eta = DL(\psi, 0) \) satisfies that

\[ \eta_\sigma = 0, \text{div}_\sigma \eta = 0 \text{ and } \eta_{\bar{k}, \bar{j}} = \eta_{k,j}. \]

For such a variation of Kähler metrics, \( \delta^2 \lambda \) has been computed by the following formula [\ref{ref:delta_nu}],

\[ \delta^2 \nu((\psi, 0), (\psi, 0)) = \delta^2 \lambda(\eta, \eta) = \frac{1}{2} \int_M |\eta|^2 \nu(\omega_{KS}) - 2m + \theta) e^\theta \omega^n_{KS}. \]

Notice that \(|\eta|^2 = 2|\psi|^2\) and

\[ \int_M (\nu(\omega_{KS}) - 2m + \theta)e^\theta \omega^n_{KS} = 0. \]

This implies that [\ref{eq:delta_nu}] holds.

\[ \square \]

4. A NEW VERSION OF \( \delta^2 \nu \)

In this section, we give a new version of \( H_1 \) and then describe the geometry of the kernel space \( Z = \ker(H_1) \). As an application, we are able to prove the local maximality of \( \lambda(\cdot) \) on the space of Kähler metrics associated to the complex structures determined by \( Z \).

4.1. \textbf{Kernel space} \( Z \) of \( H_1 \). We begin with the following technical lemma.

\textbf{Lemma 4.1.} Suppose that \( \varphi, \psi \in A^{0,1}(M, T^{1,0}M) \) satisfy

\[ \begin{cases} \bar{\partial} \varphi = \bar{\partial} \psi = 0 \\ \varphi, \omega \psi = \psi, \omega = 0. \end{cases} \]

Let \( v \) be a \((1, 0)\)-VF. Then

\[ v \varphi, \psi > = - < [v, \varphi], \psi > = \text{div}_\theta(v, \varphi, \omega) - (v, \omega) \text{ div}_\theta \varphi. \]

Proof. Let \( v = v^i \frac{\partial}{\partial z^i} \), \( \varphi = \varphi^i_{\bar{k}} \frac{\partial}{\partial \bar{z}^i} \otimes \frac{\partial}{\partial z^k} \), \( \psi = \psi^i_{\bar{k}} \frac{\partial}{\partial \bar{z}^i} \otimes \frac{\partial}{\partial z^k} \).

Then

\[ [v, \varphi] = (v^i \frac{\partial \varphi^j_k}{\partial z^i} - \varphi^j_k \frac{\partial v^i}{\partial \bar{z}^j}) \frac{\partial}{\partial \bar{z}^k} \otimes \frac{\partial}{\partial z^j} = (v^i \varphi^j_k - \varphi^j_k v^i) dz^j \otimes \frac{\partial}{\partial \bar{z}^k}. \]

and

\[ < [v, \varphi], \psi > = (v^i \varphi^j_k - \varphi^j_k v^i) \overline{\partial \varphi}^j g_{\bar{k}, \bar{i}} g^\bar{k}. \]

Similarly,

\[ < \varphi, \psi > = \varphi^i_{\bar{k}} \overline{\partial \varphi}^j g_{\bar{k}, \bar{i}} g^\bar{k} \]

and

\[ v \varphi, \psi > = v^i \varphi^j_k \overline{\partial \varphi}^j g_{\bar{k}, \bar{i}} g^\bar{k} + v^i \varphi^j_k \overline{\partial \varphi}^j g^\bar{k}. \]

On the other hand, by the condition,

\[ \varphi, \omega \psi = \psi, \omega = 0, \]

we have

\[ \varphi^j_k \overline{\partial \varphi}^j g_{\bar{k}, \bar{i}} g^\bar{k} = \psi^i_{\bar{k}} \overline{\partial \psi}^j g_{\bar{k}, \bar{i}} g^\bar{k} \]

and by the condition,

\[ \overline{\partial} \psi = 0. \]
we have
\[ \psi^s_{T_T} = \psi^s_{T_T}. \]
Thus
\[ v < \varphi, \psi > = \left< v, \varphi \right> - \left< \left[ v, \varphi \right], \psi \right> = v_i \varphi^i \psi^l \]
\[ = v_i \varphi^i \psi^l + v_i \partial^i \theta \]
\[ = (v_i \varphi^i \psi^l)_i - v_i (\varphi^i \psi^l \theta)_i \]
\[ = \text{div}_g (v, \varphi, \psi) - (v, \varphi, \psi)_\omega. \]
Theorem 4.4. Let $\xi = \im(X)$, where $X$ is a soliton HVF of $(M, J_0, \omega_{KS})$. Then

\begin{equation}
H_1(\psi) = 2\sqrt{-1}\mathcal{L}_\xi \psi, \forall \psi \in \mathcal{H}_\theta^{0,1}(M, T^{1,0}M).
\end{equation}

Proof. Note that

\[ X = \text{grad}^{1,0}\tilde{\theta} \]

and

\[ g^{\tilde{\theta}\bar{\tilde{\theta}}} + g^{\tilde{\theta}\tilde{\theta}} = -\tilde{\theta}. \]

Then by Lemma 4.2 for any $\psi' \in \mathcal{H}_\theta^{0,1}(M, T^{1,0}M)$, we have

\[ \int_M <\mathcal{L}_X \psi, \psi'> > e^\theta \omega_{KS}^n = \int_M \tilde{\theta} <\psi, \psi'> > e^\theta \omega_{KS}^n. \]

It follows that

\begin{equation}
\int_M <\mathcal{L}_X \psi, \psi'> > e^\theta \omega_{KS}^n = \int_M \tilde{\theta} <\psi, \psi'> > e^\theta \omega_{KS}^n.
\end{equation}

Let $(z^1, \ldots, z^n)$ be holomorphic local coordinates on $(M, J_0)$. Write

\[ \psi = \psi_k d\bar{z}^k \otimes \frac{\partial}{\partial z^k}, \quad X = X^i \frac{\partial}{\partial z^i}. \]

Then

\[ \mathcal{L}_X \psi = \left( \frac{\partial X^k}{\partial z^i} \psi_k + X_k \frac{\partial \psi_k}{\partial z^i} \right) d\bar{z}^j \otimes \frac{\partial}{\partial z^i}. \]

Since

\[ \overline{\partial} \psi = 0, \quad \frac{\partial \psi_k}{\partial z^i} = \frac{\partial \psi_k}{\partial z^i}, \]

Thus

\[ \overline{\partial}(X \psi) = \frac{\partial X^k}{\partial z^i} \psi_k d\bar{z}^j \otimes \frac{\partial}{\partial z^i} = \left( \frac{\partial X^k}{\partial z^i} \psi_k + X_k \frac{\partial \psi_k}{\partial z^i} \right) d\bar{z}^j \otimes \frac{\partial}{\partial z^i} = \left( \frac{\partial X^k}{\partial z^i} \psi_k + X_k \frac{\partial \psi_k}{\partial z^i} \right) d\bar{z}^j \otimes \frac{\partial}{\partial z^i} = \mathcal{L}_X \psi. \]

Hence, by the fact that

\[ \delta_\theta \psi = 0, \]

we derive

\begin{equation}
\int_M <\mathcal{L}_X \psi, \psi'> > e^\theta \omega_{KS}^n = \int_M <\bar{X} \psi, \delta_\theta \psi'> > e^\theta \omega_{KS}^n = 0.
\end{equation}

By (4.5) and (4.7), we have

\[ \int_M <\mathcal{L}_\xi \psi, \psi'> > e^\theta \omega_{KS}^n = -\frac{\sqrt{-1}}{2} \int_M \tilde{\theta} <\psi, \psi'> > e^\theta \omega_{KS}^n. \]

By Lemma 4.10 it follows that

\begin{equation}
\int_M <\mathcal{L}_\xi \psi, \psi'> > e^\theta \omega_{KS}^n = -\frac{\sqrt{-1}}{2} \int_M <H_1(\psi), \psi'> > e^\theta \omega_{KS}^n.
\end{equation}

On the other hand, for any $\sigma \in K_X$, where $K_X$ is the compact 1-ps of holomorphic transformations generated by $\xi$, $\sigma$ preserves the Kähler form $\omega_{KS}$ and the function $\theta$. Then $\sigma$ maps $\mathcal{H}_\theta^{0,1}(M, T^{1,0}M)$ to itself. As a consequence, $\mathcal{L}_\xi$ maps $\mathcal{H}_\theta^{0,1}(M, T^{1,0}M)$ to itself. Thus we prove (4.4) by (4.7) since $\psi'$ is an arbitrary element in $\mathcal{H}_\theta^{0,1}(M, T^{1,0}M)$.
By Theorem 4.4, we have

\[ \text{Ker}(H_1) = Z = \{ \psi \in H_0^{0,1}(M, T^{1,0}M) | L_\xi \psi = 0 \}. \]

Moreover, we have the following character for the kernel \( Z \).

**Corollary 4.5.** \( \psi \in Z \) if only if \( \xi J_{J_0(\tau \psi)} = J_{J_0(\tau \psi)} \xi + \sqrt{-1} \xi \) is a HVF on \( (M, J_\Phi(\tau \psi)) \) for some small \( \tau \).

**Proof.** Recall that \( K_X = e^{s\xi} \) is the 1-ps of holomorphic transformations on \( (M, J_0) \) generated by \( \xi \). Then \( \xi \) is holomorphic on \( (M, J_\Phi(\tau \psi)) \) if and only if

\[ (e^{s\xi})^* J_{J_0(\tau \psi)} = J_{J_0(\tau \psi)} \xi, \quad \forall s \in \mathbb{R}, \]

which is equivalent to

\[ J_{(e^{s\xi})^* J_{J_0(\tau \psi)}} = J_{J_0(\tau \psi)}, \quad \forall s \in \mathbb{R}. \]

The latter is also equivalent to

\[ (e^{s\xi})^* \Phi(\tau \psi) = \Phi(\tau \psi), \quad \forall s \in \mathbb{R}. \]  \hspace{1cm} (4.10)

Note that \( e^{s\xi} \) preserves the Kähler form \( \omega_{KS} \) and \( \theta \). Then

\[ (e^{s\xi})^* \psi \in H_0^{0,1}(M, T^{1,0}M). \]

Moreover, for any \((1,1)\)-form \( \varphi_k \) \( (k \geq 2) \) defined as in (2.7) for \( \tau \psi \), we have

\[ \delta_\theta ((e^{s\xi})^* \varphi_k) = 0, \]

Thus by the uniqueness of Kuranishi’s solutions we get

\[ (e^{s\xi})^* \Phi(\tau \psi) = \Phi(\tau (e^{s\xi})^* \psi). \]

Since the Kuranishi map is injective, by (4.10), we derive

\[ (e^{s\xi})^* \psi = \psi \forall s \in \mathbb{R}. \]  \hspace{1cm} (4.11)

It follows that

\[ \mathcal{L}_\xi \psi = 0. \]

By Theorem 4.4, we prove that \( \psi \in Z \) from (4.9). The inverse is also true. In fact, if \( \psi \in Z \), then (4.11) holds for any small \( \tau \). Hence, \( \psi \in Z \) if only if \( e^{s\xi} \) is a family of holomorphisms on \( (M, J_\Phi(\tau \psi)) \). The corollary is proved.

\[ \square \]

**Remark 4.6.** It has been shown [40] that the kernel of \( H_2 \) is finitely dimensional, which is isomorphic to the linear space generalized by the real and imaginary parts on HVFs of \( (M, J_0) \). Thus by Proposition 3.7,

\[ \text{Ker}(H) = \text{Ker}(H_1) \oplus \text{Ker}(H_2) \]

is also finitely dimensional.
4.2. Index of $H_1$. Since $H_2$ is always non-positively elliptic operator, the index (the number of positive eigenvalues) of $H$ depends only on the Lie operator $\mathcal{L}_\xi$ on $\mathcal{H}^{0,1}(M,T^{1,0}M)$ by Theorem 4.4. In the following, we will show that it just depends on the cohomology group $H^1(M,J_0,\Theta)$.

Suppose that $e_1,\ldots,e_l$ are the eigenvectors of $\mathcal{L}_\xi$. Namely there exists $\lambda_i, i=1,2,\ldots,l$ such that

$$\mathcal{L}_\xi e_i = \lambda_i e_i.$$ 

By (4.4) we have

$$H_1 e_i = 2\sqrt{-1}\lambda_i e_i.$$ 

Thus $2\sqrt{-1}\lambda_i$ is real. Denote $\lambda_i = -\sqrt{-1}\delta_i, \delta_i \in \mathbb{R}$ and let

$$\theta_i = e_i, \theta_{t+i} = \sqrt{-1}e_i, i=1,2,\ldots,l.$$ 

It follows that

$$H_1 \theta_i = 2\delta_i \theta_i, H_1 \theta_{t+i} = 2\delta_i \theta_{t+i}.$$ 

Hence, we need to show that the number $\delta_i$ is independent of choice of representation of $[\theta_i]$. The following lemma can be found in the book of Kodaira [14].

Lemma 4.7. If $\phi \in A^{0,p}(M,T^{1,0}(M)), \psi \in A^{0,q}(M,T^{1,0}(M))$, we have

$$\bar{\partial}[\phi,\psi] = [\bar{\partial}\phi,\psi] + (-1)^p[\phi,\bar{\partial}\psi].$$ 

In particular, if $Y$ is a HVF, and $\phi \in A^{0,p}(M,T^{1,0}(M))$ then

$$\bar{\partial}\mathcal{L}_Y(\phi) = \mathcal{L}_Y \bar{\partial}(\phi).$$ 

Lemma 4.8. Let $\phi \in A^{0,1}(M,T^{1,0}(M))$. Then the following is true:

$$\bar{\partial}\mathcal{L}_\xi \phi = 0,$$ 

if $\bar{\partial}\phi = 0$;

$$\mathcal{L}_\xi \phi = \frac{\bar{\partial}(\mathcal{L}_X Y - \mathcal{L}_Y X \phi)}{2\sqrt{-1}},$$ 

if $\phi = \bar{\partial}Y, Y \in T^{1,0}M$.

Proof. By (4.6), we have

$$\bar{\partial}\mathcal{L}_X \phi = 0, \forall \phi \in H^1(M,J_0,\Theta).$$ 

By (4.13), we also have

$$\bar{\partial}\mathcal{L}_X \phi = 0.$$ 

Thus, $\mathcal{L}_\xi \phi = 0$.

By (4.13) we have

$$\mathcal{L}_X \bar{\partial}Y = \bar{\partial}\mathcal{L}_X Y.$$ 

Together with (4.6) we see that

$$\mathcal{L}_\xi \bar{\partial}Y = \frac{\mathcal{L}_X \bar{\partial}Y - \mathcal{L}_Y \bar{\partial}Y}{2\sqrt{-1}},$$ 

$$= \frac{\bar{\partial}(\mathcal{L}_X Y - \mathcal{L}_Y X \phi)}{2\sqrt{-1}}.$$ 

Hence (4.14) is true. 

By (4.14), we see that for any $\phi = \bar{\partial}Y, Y \in T^{1,0}M$, it holds

$$[\mathcal{L}_\xi \phi] = 0.$$ 

Thus

$$\mathcal{L}_\xi [\phi] = [\mathcal{L}_\xi \phi].$$ 

This means that the eigenvalue $\theta_i$ depends only on the operator $\mathcal{L}_\xi$ on $H^1(M,J_0,\Theta)$. Hence, we prove
Proposition 4.9. The index of $H$ depends only on the operator $L_\xi$ on $H^1(M, J_0, \Theta)$.

4.3. Maximality of $\lambda(\cdot)$ associated to $Z$. First we recall a formula computed for the $W$-functional on a Fano manifold $(M, J)$ in [39]. Let $Y$ be any HVF with $\text{Im}(Y)$ generating a compact 1-ps $K_Y$ of holomorphic transformations on $(M, J)$. Then for any $K_Y$-invariant Kähler form $\omega_g \in 2\pi c_1(M, J)$, the potential $\theta_Y(\omega_g)$ of $Y$ in [3.1] associated to $\omega_g$ is real. Define an invariant for $Y$ by

$$N_Y(c_1(M, J)) = \int_M \theta_Y(\omega_g) e^{\theta_Y(\omega_g)} \omega_g^n,$$

which is independent of choice of $K_Y$-invariant $\omega_g \in 2\pi c_1(M, J)$ [39]. Moreover, for any $K_Y$-invariant $\omega_g \in 2\pi c_1(M, J)$, we have the following formula,

$$W(g, -\theta_Y(\omega_g)) = (2\pi)^{-n} (nV - N_Y(c_1(M, J)) - F^J_Y(Y)),$$

where $F^J_Y(\cdot)$ is the modified Futaki-invariant on $(M, J)$ introduced in [39]. $N_Y(c_1(M, J)) + F^J_Y(Y)$ is also called the $H(Y)$-invariant on $\eta(M, J)$ [39].

By Corollary 4.5, we know that $X = \xi_{J_0} = J_0 \xi + \sqrt{-1} \xi$ is a HVF on $(M, J_{\psi_0})$ for any $\tau << 1$, where $\psi_\tau = \tau \psi \in B(\epsilon) \cap Z$ and $J_{\psi_0}$ is associated to $\varphi_\tau = \Phi(\psi_\tau)$. Thus by (4.18), for any $\omega_g = \omega_{g_{\tau, x}}$ with $\xi(\chi) = 0$ in (4.2), it holds

$$W(g, -\theta_X(\omega_g)) = (2\pi)^{-n} (nV - N_X(c_1(M, J_{\psi_0})) - F^J_{X_{\psi_\tau}}(X)).$$

Notice that $\theta = \theta_X(\omega_{K\Sigma})$ is independent of $\psi_\tau$. Thus

$$N_X(c_1(M, J_{\psi_0})) = \int_M \theta_X(\omega_{K\Sigma}) e^{\theta_X(\omega_{K\Sigma})} \omega^n_{K\Sigma}$$

$$= \int_M \theta e^{\theta_X(\omega_{K\Sigma})} \omega^n_{K\Sigma}$$

$$= N_X(c_1(M, J_0))$$

is independent of $\psi_\tau$.

Next we show that $F^J_{X_{\psi_\tau}}(X)$ is also independent of $\psi_\tau$. In fact, we prove

Lemma 4.10.

$$F^J_{X_{\psi_\tau}}(X) = F^J_{X_{\psi_0}}(X) = 0, \ \forall \ \psi_\tau \in B(\epsilon) \cap Z.$$

Proof. We use an argument in [19] to prove the lemma. Let $J_{K\Sigma}(M, \omega_{K\Sigma})$ be a set of almost $K\Sigma$-invariant complex structures which are compatible with $\omega_{K\Sigma}$. Then for any $J \in J_{K\Sigma}(M, \omega_{K\Sigma})$ it induces a Hermitian metric $g_J$. As in [19], we define a modified Hermitian scalar curvature function on $J_{K\Sigma}(M, \omega_{K\Sigma})$ by

$$s_J(\xi) = s(J) - n + 2\Box_J \theta - X(\theta) - \theta, \ \forall \ J \in J_{K\Sigma}(M, \omega_{K\Sigma})$$

where $s(J)$ is the Hermitian scalar curvature of $g_J$ (cf. [12]) and $\Box_J$ is the Lapalace operator induced by the Chern connection associated to $g_J$. Now we consider a family of $J_{\psi_0} \in J_{K\Sigma}(M, \omega_{K\Sigma})$. By [19] Proposition 3.1], we have the formula,

$$\frac{d}{ds} \int_M 2s_J(\psi_0, \xi) e^{\theta_X(\omega_{K\Sigma})} \omega^n_{K\Sigma} = \Omega_\xi(L_\xi J_{\psi_0}, J_{\psi_0}).$$

where $\Omega_\xi$ is a non-degenerate 2-form on $J_T(M, \omega)$ (cf. [12] [19]).

By Corollary 4.5, we have

$$L_\xi J_{\psi_0} = 0$$

and so we get

$$\Omega_\xi(L_\xi J_{\psi_0}, J_{\psi_0}) = 0.$$
Proof. By (4.18) and (4.21), we have
\[ \int_M 2s_\xi(J_{\psi_0}) \bar{\partial} e^\phi \omega_{K_S}^n = -F_X^{J_{\psi_0}}(X). \]
Thus by (4.20), we get
\[ \int_M s_\xi(J_{\psi_0}) \bar{\partial} e^\phi \omega_{K_S}^n = 0. \]
As a consequence, we get
\[ F_X^{J_{\psi_0}}(X) = 0. \]

Proposition 4.11. Let \( \psi_0 \in B(\epsilon) \cap Z \). Then for any \( \chi \in C^\infty(M) \) with \( \xi(\chi) = 0 \) it holds
\[ \lambda(g_{r,\chi}) \leq \lambda(g_{K_S}). \]

Proof. By (4.18) and (4.21), we have
\[ \lambda(g_{r,\chi}) \leq W(g_{r,\chi}, -\theta(\omega_{r,\chi})) \]
\[ = (2\pi)^{-n}(nV - N_X(c_1(M, J_{\psi_0}))). \]
Thus by (4.20), we get
\[ \lambda(g_{r,\chi}) \leq (2\pi)^{-n}(nV - N_X(c_1(M, J_0))) = \lambda(g_{K_S}). \]

Proposition 4.11 can be also proved by using the equivalent formula for the modified Futaki-invariant in [47] as follows.

Another proof of Proposition 4.11. According to the above proof of Proposition 4.11 we need to show that (4.21) holds for \( J_\psi \). In fact, as in the proof of [21, Lemma 2.1], by the partial \( C^0 \)-estimate, there is a large integer \( L_0 \) such that for any integer \( k \) the family of Fano manifolds \( (M, g_{r,0}) \) \( (r \leq \epsilon) \) can be embed into an ambient projection space \( CP^{N_k} \) by normal orthogonal bases of \( H^0(M, kK_M^{L_0} \cdot \cdot \cdot g_{r,0}) \). Then by introducing two equivariant Riemann-Roch formulas of \( S_1 \) and \( S_2 \) with \( G = (S^1)^2 \)-action by
\[ S_1 = k \frac{\partial}{\partial t} (\text{trace}(e^{\frac{\partial}{\partial s} + itk})), \quad S_2 = \frac{1}{2} \frac{\partial}{\partial s} \frac{\partial}{\partial t} (\text{trace}(e^{\frac{\partial}{\partial s} + itk})), \]
where \( X^k = (X^k_\alpha) \) is the induced HVF as an element of Lie algebra \( sl(N_k + 1, \mathbb{C}) \), \( F_X^{J_{\psi_0}}(X) \) is the leading term \( F_0 \) (which is a multiple of \( F_1 \) for a smooth Fano variety) in the following expansion
\[ F_1 = F_0 + F_1 k^{-1} + O(k^{-2}). \]
Here
\[ \text{trace}(e^{\frac{\partial}{\partial s} + itk}) = \int_M \text{ch}^G(kL) \text{Td}^G(M), \]
and \( \text{ch}^G(kL) \) is the G-equivalent Chern character of multiple line bundle \( kL = kK_{M_0}^{L_0} \) and \( \text{Td}^G(M) \) is the G-equivariant Chern character of \( M \) [7]. Since \( K_{M_0}^{-1} \) is the restriction of \( \frac{1}{\text{vol}(M_0)} \mathcal{O}(-1) \), where
5. Lojasiewicz inequality on the space of Kähler metrics

In this section we prove an inequality of Lojasiewicz type for the functional \( \nu(\cdot) \) on \( \mathcal{U} \). We note that \( \nu(\cdot) \) can be defined on the \( C^{k+4,\gamma} \) continuous space,

\[
W^{k+4,\gamma}(M) = \mathcal{H}^{0,1}_\theta(M, T^{1,0}M) \times C^{k+4,\gamma}(M),
\]

where \( k \geq 0 \) is any integer. Clearly, there is a natural \( C^{k+4,\gamma} \)-norm on \( W^{k+4,\gamma}(M) \) by

\[
\| (\psi, \chi) \|_{C^{k+4,\gamma}} = \| \psi \|_\theta + \| \chi \|_{C^{k+4,\gamma}}.
\]

Let

\[
W^{k+4,\gamma}_\epsilon(M) = \{ (\psi, \chi) \in W^{k+4,\gamma}(M) | \psi \in B(\epsilon) \}. \]

Set an \( \epsilon \)-neighborhood of \((0, 0)\) in \( W^{k+4,\gamma}_\epsilon(M) \) by

\[
\mathcal{V}_\epsilon = \{ (\psi, \chi) \in W^{k+4,\gamma}_\epsilon(M) | \psi \in B(\epsilon), \| \chi \|_{C^{k+4,\gamma}} < \epsilon \}.
\]

We prove

**Proposition 5.1.** There are \( \epsilon > 0 \) and \( \alpha \in [\frac{1}{2}, 1) \) such that for any \( a = (\varphi, \chi) \in \mathcal{V}_\epsilon \) it holds

\[
\| \nabla \nu(a) \|_{L^2} \geq c_0|\nu(a) - \nu(0)|^\alpha,
\]

where \( \| \cdot \|_{L^2} \) is taken for \( \nabla \nu = (Q, R) \in W^{k,\gamma}(M) \) and \( c_0 > 0 \) is some small constant.

**Proof.** We will follow the argument in [32]. By Remark 4.6, \( W_0 = \ker(H) \) is finitely dimensional. Then both of \( W_0 = \ker(H) \) and

\[
W^\perp = \{ a \in W^{k+4,\gamma}(M) | (a, b)_\theta = 0, \quad \forall b \in W_0 \}
\]

are closed sets of \( W^{k+4,\gamma}(M) \). Since \( H_2 \) is elliptic, there is a constant \( C > 0 \) such that

\[
\| H(\eta) \|_{L^2} \geq C \| \eta \|_{L^2}, \quad \forall \eta \in W^\perp.
\]

Consider the project map,

\[
\Phi = \text{pr}_{W^\perp} \nabla \nu : \mathcal{U}_\epsilon \to W^\perp.
\]

Then \( \Phi \) is analysis and it satisfies that

\[
\Phi(0, 0) = 0, D(0, 0) \Phi = \text{pr}_{W^\perp} \circ H.
\]

Thus

\[
D(0, 0) \Phi(\cdot) : 0 \oplus W^\perp \to 0 \oplus W^\perp
\]

is an isomorphic. By the implicity function theorem, there is a neighborhood \( U \) of 0 in \( W_0 \) and a map \( G(x) : U \to W^\perp \) such that

\[
\nabla \nu(x + G(x)) \in W_0, \quad \forall x \in U.
\]

Define a functional on \( U \) by

\[
F(x) = \nu(x + G(x)).
\]

Then for any \( x \in U, \ z \in W_0 \), we have

\[
\frac{dG(x + tz)}{dt}|_{t=0} \in W^\perp.
\]

It follows that

\[
\frac{dF(x + tz + G(x + tz))}{dt}|_{t=0} = (z + \frac{dG(x + tz)}{dt}|_{t=0}, \nabla \nu(x + G(x)))_\theta
\]

\[
= (z, \nabla \nu(x + G(x)))_\theta
\]
Thus \( \nabla F(x) = \nabla \nu(x + G(x)) \in W_0, \forall x \in U \).

Since \( F(x) \) is analytic, by the classic Lojasiewicz inequality on \( W_0 \) (cf. \( \eqref{5.3} \)), there is an \( \alpha \in [\frac{1}{2},1) \) such that

\[
|\nabla \nu(x + G(x))|_{L^2} \geq c_1|\nabla \nu(x + G(x))| \\
\geq c_2|\nu(x + G(x)) - \nu(0)|^\alpha, \forall |x| < < 1.
\]

By Definition \( \eqref{5.8} \) we have

\[
\|H_{(\psi,\chi)}\|_{C^{k,\gamma}} \leq C, \forall (\psi,\chi) \in V_c.
\]

Moreover, \( \|H - H_{(\psi,\chi)}\|_{C^{k,\gamma}} < < 1 \), as long as \( \|H(\psi,\chi)\|_{C^{k+4,\gamma}} < < 1 \).

For any \( a = (\psi,\chi) \in V_c \), we write \( a = x + G(x) + y \) for some \( y \in W^\perp \). Thus there exists a small \( \epsilon \) such that for any \( a = (\psi,\chi) \in V_c \) it holds

\[
\nabla \nu(a) = \nabla \nu(x + G(x) + y) = \nabla \nu(x + G(x)) + \int_0^1 \delta_y \nabla \nu(x + G(x) + sy)ds
= \nabla F(x) + \delta_y \nabla \nu(0) + \int_0^1 \delta_y \nabla \nu(x + G(x) + sy) - \delta_y \nabla \nu(x + G(x))ds + o(\|y\|_{W^2})
\]

Since \( \nabla F(x) \) is perpendicular to \( H(y) \) with respect to \( (\cdot,\cdot)_\theta \), and \( H \) is nondegenerate on \( W^\perp \) by \( \eqref{5.2} \), we get

\[
\|\nabla \nu(a)\|^2_{L^2} \geq \|\nabla F(x)\|^2_{L^2} + c_3\|y\|^2_{W^2}
\]

where \( c_3 > 0 \) is some small constant.

Similarly, by \( \eqref{5.4} \), we have

\[
\nu(a) = \nu(x + G(x) + y)
= \nu(x + G(x)) + \int_0^1 < \nabla \nu(x + G(x) + sy), y > ds
= \nu(x + G(x)) + \frac{1}{2} < \delta_y \nabla \nu(x + G(x) + y), y > \\
+ \int_0^1 \int_0^1 s < \delta_y \nabla \nu(x + G(x) + sy) - \delta_y \nabla \nu(x + G(x) + y), y > dtds
= F(x) + \frac{1}{2} < H(y), y > \\
+ \int_0^1 \int_0^1 s < \delta_y \nabla \nu(x + G(x) + sy) - H(y), y > dtds + o(\|y\|^2_{W^2})
\]

Notice that

\[
| < H(y), y > | \leq \|y\|_{L^2} \|H(y)\|_{L^2}
\leq C_1 \|y\|_{L^2} \|y\|_{C^{k+4,\gamma}}
\leq C_2 \|y\|_{L^2}
\leq C \|y\|_{W^2}.
\]
As a consequence, we have

\[ |\nu(a) - \nu(0)| \leq |F(x) - F(0)| + C \| y \|_{H^2}^2. \]

Hence combining \(5.6\) and \(5.8\) together with \(5.3\), we obtain

\[ \| \nabla \nu(a) \|_{L^2}^2 \geq c_2 |F(x) - F(0)|^{2\alpha} + c_3 \| y \|_{W^2_a}^2 \]

\[ \geq c_4 (|F(x) - F(0)| + \| y \|_{W^2_a}^2)^{2\alpha} \]

\[ \geq C_0 |\nu(a) - \nu(0)|^{2\alpha}. \]

This proves \(5.1\).

\[ \square \]

Recall the operator \(N(\cdot)\) in \(3.8\) from Sym\(^2(T^*M)\) to itself. Then we can rewrite \(3.12\) as

\[ (\nabla \nu(a_0), (\psi, \chi)_\theta) = -\frac{1}{2} (N(L(a_0)), DL_{a_0}((\psi, \chi)))_{L^2(L(a_0))}, \forall a_0 \in \mathcal{U}. \]

Thus there is a dual operator \((DL_{a_0})^*\) of \(DL_{a_0}\) with respect to the inner product \((\cdot, \cdot)_\theta\) such that

\[ ((DL_{a_0})^* N(L(a_0)), (\psi, \chi)_\theta) = (N(L(a_0)), DL_{a_0}((\psi, \chi)))_{L^2(L(a_0))}. \]

As a consequence, we have

\[ \nabla \nu(a_0) = -\frac{1}{2} (DL_{a_0})^* N(L(a_0)). \]

Hence,

\[ (\nabla \nu(a_0), \nabla \nu(a_0))_\theta = \frac{1}{4} (N(L(a_0)), DL_{a_0}((DL_{a_0})^* N(L(a_0))))_{L^2(L(a_0))}. \]

The following is a generalization of Lemma 3.6.

**Lemma 5.2.** Let \(k > 2\) be an integer, and \(\epsilon\) a small constant. Then there exists a constant \(C = C(\epsilon) > 0\) such that

\[ ((DL_a)^* N(L(a)), (DL_a)^* N(L(a)))_\theta \leq C(N(L(a), N(L(a))))_{L^2(L(a))}, \forall a \in \mathcal{V}_c. \]

**Proof.** Let \(S = (DL_a)^*(DL_a)\) be an operator on the Hilbert space

\[ \mathcal{H} = \mathcal{H}_{\theta}^{0,1}(M, T^{1,0}M) \times L^2(M), \]

whose domain \(D(S)\) contains \(\mathcal{H}_{\theta}^{0,1}(M, T^{1,0}M) \times C^\infty(M)\). As same as the operator in \(3.13\), \(S\) is a self-adjoint fourth-order non-negative elliptic operator. Thus by the spectral theorem, for integer \(k > 0, x \in D(S)\), it holds

\[ (S^k x, x)_\theta = \int_0^\infty \lambda^k dE_{x,x}(\lambda). \]

By the H"older inequality and \(5.11\), we get

\[ (S^2 x, x)_\theta \leq (Sx, x)^{\frac{1}{2}} (S^k x, x)^{\frac{1}{2}}. \]

For any \(a \in \mathcal{V}_c\), we decompose \(N(L(a))\) into

\[ N(L(a)) = DL_a x + y, \]

where \(x \in \mathcal{H}^{0,1}(M, T^{1,0}M) \times C^\infty(M)\) and \(y \in \text{Im}(DL_a)^\perp\). Since

\[ (z, (DL_a)^*(y))_\theta = (DL_a(z), y)_{L^2(L(a))} = 0, \forall z \in \mathcal{H}^{0,1}(M, T^{1,0}M) \times C^\infty(M), \]

\[ (DL_a)^*(y) = 0. \]
Thus by [5.12] and the fact that \( S \) is self-adjoint, we have
\[
((DL_a)^*N(L(a)), (DL_a)^*N(L(a)))_\theta \\
= ((DL_a)^*DL_a x, (DL_a)^*DL_a x)_\theta \\
= (Sx, Sx)_\theta \\
= (S^2 x, x)_\theta \\
\leq (Sx, x)_\theta^2 (S^k x, x)_\theta^{1-\gamma} \\
= (DL_a x, DL_a x)^{1/2}_{L^2(L(a))} (S^k x, x)_\theta^{1-\gamma}
\]
(5.14)
\[
\leq \|N(L(a))\|_{L^2(L(a))} (S^k x, x)_\theta^{1-\gamma}.
\]

On the other hand,
\[
(S^k x, x)_\theta = (S^{k-1} x, Sx)_\theta \\
= (S^{k-2}(DL_a)^*DL_a x, (DL_a)^*DL_a x)_\theta \\
= (S^{k-2}(DL_a)^*N(L(a)), (DL_a)^*N(L(a)))_\theta.
\]
Then
(5.15)
\[
(S^k x, x)_\theta^{1-\gamma} \leq C,
\]
as long as \( a \in \mathcal{V}_r \). Hence, combining (5.14) and (5.15), we derive (5.10). \( \square \)

**Remark 5.3.** In order to apply the spectral theorem, the fourth-order operator \( S \) should be self-adjoint as in [3.17]. In the other words, we shall define the domain \( D(S) \) of \( S \). By decomposing \( S \) into
\[
S(\varphi, \chi) = (S_{11}\varphi + S_{12}\chi, S_{21}\varphi + S_{22}\chi),
\]
it suffices to consider the domains \( D(S_{12}) \) and \( D(S_{22}) \). On the other hand, by (5.12),
\[
S_{22} = (\sqrt{-1}\partial_{J_2}\bar{\partial}_{J_2})^* (\sqrt{-1}\partial_{J_2}\bar{\partial}_{J_2})
\]
is a fourth-order elliptic self-adjoint operator. Thus \( D(S_{22}) = W^4_2(M) \). Note that \( S_{12} \) is a second order operator. Hence,
\[
D(S) = H^{0,1}_\theta (M, T^{1,0} M) \times W^4_2(M)
\]
and so \( S \) is self-adjoint on \( D(S) \).

**Corollary 5.4.** There are constant \( c_0 > 0 \) and number \( \alpha' \in (\frac{1}{2}, 1) \) such that
(5.16)
\[
\|N(L(a))\|_{L^2(L(a))} \geq c_0 \|\lambda(L(a)) - \lambda(L(0))\|^\alpha', \forall a \in \mathcal{V}_r.
\]

**Proof.** By [5.9], Lemma 5.2 and Proposition 5.1 we have
\[
\|N(L(a))\|_{L^2(L(a))} \geq C\|\nabla \nu(a)\|_{L^2}^{\frac{1}{\gamma}} \\
\geq C\|\nu(a) - \nu(0)\|_{L^2}^{\frac{2}{\gamma}} \\
= C\|\lambda(L(a)) - \lambda(L(0))\|_{L^2}^{\frac{2}{\gamma}}.
\]
Set \( \alpha' = \frac{2}{\gamma} \). Then \( \alpha' \in (\frac{1}{2}, 1) \) when \( k >> 1 \). Hence, (5.16) is true. \( \square \)

6. **Proof of Theorem 0.1**

In this section, we prove Theorem 0.1. Let \( \tilde{f}(t) \) be the minimizer of \( W \)-functional as in (5.3) for the solution \( \tilde{g}(t) \) of KR flow (1.1). Let \( X(t) \) be a family of gradient vector fields defined by
\[
X(t) = \frac{1}{2} \text{grad}_{\tilde{g}(t)} \tilde{f}(t),
\]
which generates a family of differential transformations $F(t) \subseteq \text{Diff}(M)$. Then $g(t) = F(t)^* \bar{g}(t)$ is a solution of the following modified Ricci flow,

$$\frac{\partial g(t)}{\partial t} = -\text{Ric}(g(t)) + g(t) + \text{Hess}_f(f(t)),$$

where $f(t) = F(t)^* \bar{f}(t)$ is the minimizer of $W$-functional for $g(t)$. It follows that

$$\|\dot{g}(t)\|_{L^2(t)} = \|\dot{N}(t)\|_{L^2(t)}.$$

Here $\dot{N}(t) = \text{Ric}(g(t)) - g(t) + \text{Hess}_f(f(t))$ just as one in (6.3).

Let $\omega(t) = F(t)^* \omega(t)$. Then $\omega(t)$ is just the Kähler form of $g(t)$. Thus by (6.1), we get

$$\frac{d\omega(t)}{dt} = -\text{Ric}(\omega(t)) + \omega(t) + \sqrt{-1} \partial J(t) \bar{\partial} J(t)(f(t))$$

and

$$\omega(t)(\cdot, \frac{dJ(t)}{dt}) = D_t f(t),$$

where $D_t f$ denotes the anti-Hermitian part of $\text{Hess}(f)$. Hence,

$$\|\dot{J}(t)\|_{L^2(t)} = \|D_t f(t)\|_{L^2(t)}$$

$$\leq \|\dot{N}(t)\|_{L^2(t)}$$

$$= \|\dot{g}(t)\|_{L^2(t)}.$$

We first prove the following convergence theorem.

**Theorem 6.1.** There exists a small $\epsilon$ such that for any $(\tau, \chi) \in V_\epsilon$ with $\psi_\tau = \sum \tau_i e_i \in Z$ and $\xi(\chi) = 0$ the flow (6.4) with the initial metric $g = L(\tau, \chi)$ converges smoothly to a KR soliton $(M, J_\infty, g_\infty)$. Moreover, the convergence is fast in the polynomial rate.

**Proof.** We need to prove that the normalized flow $g(t)$ of (6.1) is uniformly bounded in $C^{k+2, \gamma}$-norm. Fix a small number $\delta_0$ we consider

$$T = T_{\tau, \chi} = \sup \left\{ t \mid \|g(t) - g_{KS}\|_{C^{k+2, \gamma}} < \delta_0, \|J(t) - J_0\|_{C^{k+2, \gamma}} < \delta_0 \right\}.$$

Then it suffices to show that $T = \infty$. By the stability of Ricci flow for the short time, $T_{\tau, \chi} \geq T_0 > 0$. In fact, for the KR flow (6.1), $T_0$ can be made any large as long as $\tau$ and $\chi$ are small enough.

By the Kuranishi theorem for the completeness of deformation space of complex structures $[25]$, for any $t < T$, there is a $K(t) \in \text{Diff}(M)$ closed to the identity such that

$$K(t)^* J(t) = J_{\psi_t}, \ \bar{\psi}_t = \Phi(t) = \Phi(\psi_t)$$

for some $\psi_t \in H^0_g(M, T^1, 0 M)$ with $\|\bar{\psi}_t\| < \epsilon_0$, where $\epsilon_0 = \epsilon$ as chosen in $[25]$ and [5.16] in Corollary 5.1. In fact, by the construction of $K(t)$ in $[25]$ Theorem 3.1, we have the estimate

$$\|\bar{\psi}_t\|_{L^2(M)} = O(\|\bar{\phi}_t\|_{L^2(M)}) \leq \epsilon_0,$$

$$K(t)^* \omega(t) = (1 + O(\|\bar{\phi}_t\|_{L^2(M)})) \omega(t).$$

Thus there is a smooth function $\chi_t$ such that

$$\omega_{K(t)^* g(t)} = K(t)^* \omega(t) = \omega_{KS} + \sqrt{-1} \partial J_{\psi_t} \bar{\partial} J_{\psi_t} \chi_t.$$

Here $\chi_t$ can be normalized by

$$\int_M \chi_t \omega_{KS}^n = 0.$$

It follows that

$$\|\Delta_{gKS} \chi_t\|_{C^{k+2, \gamma}} \leq 2 \|\Delta_{g_{\psi_t}, 0} \chi_t\|_{C^{k+2, \gamma}}$$

$$= 2 \|\text{trace}_{g_{KS}}(K(t)^* \omega(t)) - n\|_{C^{k+2, \gamma}}$$

$$\leq 2 \|(1 + \delta_1) \text{trace}_{g(t)}(g(t)) - n\|_{C^{k+2, \gamma}} \leq \delta_2(\delta_0) < 1.$$
Similarly, we have
\[
\|\chi_t\|_{C^{\alpha+\gamma}(M)} \leq \epsilon_0
\]
as long as \(\delta_2\) is chosen small enough. Note that \(\|\mathcal{N}(g)\|_{L^2(g)}\) is invariant under the action of \(\text{Diff}(M)\). Therefore, by Corollary 5.3, we obtain the following Lojasiewicz inequality,
\[
\|\mathcal{N}(g(t))\|_{L^2(g(t))} = \|\mathcal{N}(L(\psi_t, \chi_t))\|_{L^2(L(\psi_t, \chi_t))} \geq c_0\|\lambda(g(t)) - \lambda(g_{K_S})\|^\alpha,
\]
where \(\alpha \in \left(\frac{1}{2}, 1\right)\).

Note that \(\tilde{g}(t)\) are all \(K_X\)-invariant for any \(t \geq 0\). Then by Proposition 4.11 we have
\[
\lambda(\tilde{g}(t)) \leq \lambda(g_{K_S}), \quad t > 0.
\]
and so,
\[
\|\lambda(g(t))\|_{L^2(g)} \leq \lambda(g_{K_S}), \quad t > 0.
\]
Thus by (6.11), for any \(\beta > 2 - \frac{1}{\alpha}\), we get
\[
\frac{d}{dt}[\lambda(g_{K_S}) - \lambda(g(t))]^{1-(2-\beta)\alpha} = -(1 - (2 - \beta)\alpha)[\lambda(g_{K_S}) - \lambda(g(t))]^{-(2-\beta)\alpha} \frac{d}{dt}\lambda(g(t))
\]
\[
= -(1 - (2 - \beta)\alpha)\lambda(g_{K_S}) - \lambda(g(t)))^{-(2-\beta)\alpha} \int_M <\mathcal{N}(t), \mathcal{N}(t) > e^{-f_1}\omega^n_t.
\]
By (6.11), it follows that
\[
\frac{d}{dt}[\lambda(g_{K_S}) - \lambda(g(t))]^{1-(2-\beta)\alpha} \leq -C\int_M <\mathcal{N}(t), \mathcal{N}(t) > e^{-f_1}\omega^n_t \leq \delta_3(\epsilon).
\]
As a consequence,
\[
\int_{t_1}^{t_2} \|\tilde{g}(t)\|_{L^2(g(t))}^{\beta} \, dt \leq C(\beta)[\lambda(g_{K_S}) - \lambda(g(t_1))]^{1-(2-\beta)\alpha}
\]
Hence, for any \((\tau, \chi) \in \mathcal{V}_\epsilon\), we prove
\[
\int_1^T \|\tilde{g}(t)\|_{L^2(g(t))}^{\beta} \, dt \leq C(\beta)[\lambda(g_{K_S}) - \lambda(g(0))]^{1-(2-\beta)\alpha} \leq \delta_3(\epsilon).
\]
Similarly, it also holds
\[
\int_1^T \|\tilde{j}(t)\|_{L^2(g(t))}^{\beta} \, dt \leq \delta_4(\epsilon).
\]
By (6.13) and (6.11), we see that
\[
\frac{d}{dt}[\lambda(g_{K_S}) - \lambda(g(t))]^{1-2\alpha} \leq -C'(\lambda(g_{K_S}) - \lambda(g(t)))^{\alpha_\beta}.
\]
Then
\[
\frac{d}{dt}[\lambda(g_{K_S}) - \lambda(g(t))]^{1-2\alpha} \geq C > 0.
\]
It follows that
\[
\lambda(g_{K_S}) - \lambda(g(t)) \leq C''(t + 1)^{-\frac{1}{\alpha_\beta}}.
\]
On the other hand, by the interpolation inequalities for tensors, for \(\beta \in (2 - \frac{1}{\alpha}, 1)\) and any integer \(p \geq 1\), there exists \(N(p)\) which is independent of \(t\), such that
\[
\|\tilde{g}(t)\|_{L^p_{K_S}(t)} \leq C(p)\|\tilde{g}(t)\|_{L^2(g(t))}^{\beta}\|\mathcal{N}(t)\|_{L^2_{K_S}(t)}^{1-\beta}
\]
\[
\leq C(p)\|\tilde{g}(t)\|_{L^2(g(t))}^{\beta}, \quad t < T.
\]
Similarly, we have
\[
\|\tilde{j}(t)\|_{L^p_{K_S}(t)} \leq C(p)\|\tilde{j}(t)\|_{L^2(g(t))}^{\beta}, \quad t < T.
\]
Hence, by (6.14) and (6.15), we derive
\[
\| g(t_1) - g(t_2) \|_{C^{\kappa+2,\gamma}} \leq \int_{t_1}^{t_2} \| \dot{g}(t) \|_{L_{\kappa}^2(t)}
\leq C \int_{t_1}^{t_2} \| \dot{g}(t) \|_{L_{\kappa}^2(t)}
\leq C \int_{t_1}^{t_2} \| \dot{g}(t) \|_{L_{\kappa}^2(t)}^2
\leq C(\beta) [\lambda(g_{KS}) - \lambda(g(t_1))]^{1-(2-\beta)\alpha}
\leq C(\beta)(t_1 + 1)^{\frac{1-(2-\beta)\alpha}{2\alpha}}, \forall t_2 > t_1.
\]
(6.17)
Therefore, we can choose a large \( T_0 \) such that
\[
\| g(t) \|_{C^{\kappa+2,\gamma}} < \delta_0, \forall t \geq T_0.
\]
Similarly we can use (6.14) and (6.15) to prove that
\[
\| J(t) \|_{C^{\kappa+2,\gamma}} < \delta_0, \forall t \geq T_0.
\]
As a consequence, we prove that \( T = \infty \). Furthermore, we can show that the limit of \( g(t) \) is a KR soliton by (6.2) (cf. [37, 58]). The convergence speed comes from (6.17).

\[\square\]

**Remark 6.2.** By (6.15) and a result of Dervan-Székelyhidi [11] (also see [45]), we know that
\[
\sup_{\omega_{\psi} \in 2\pi C_1(M,J_{\psi^*})} \lambda(g) = \lim_{t \to 1} \lambda(g(t)) = \lambda(g_{KS}).
\]
(6.19)
In particular, (6.12) holds for KR flow (0.7) with any initial metric in \( 2\pi C_1(M,J_{\psi^*}) \). Thus the condition \( \xi(\chi) = 0 \) can be removed in Theorem 6.1 according to the above proof. \[\square\]

We also remark that the limit complex structure \( J_\infty \) in Theorem 6.1 may be different with the original one \( J_{\psi^*} \). But the soliton VF of \( (M_\infty, J_\infty) \) must be conjugate to \( X \). In fact, we have the following analogy of [46] Proposition 5.10 for the uniqueness of soliton VFs.

**Lemma 6.3.** There is a small \( \delta_0 \) such that for any KR soliton \((M,J',g_{KS}')\) with
\[
\text{dist}_{CG,C^1}((M',g_{KS}'),(M,g_{KS})) \leq \delta_0,
\]
\( X \) can be lifted to a HVF on \((M,J')\) so that it is a soliton VF of \((M,J',g_{KS}')\). Moreover,
\[
\lambda(g_{KS}') = \lambda(g_{KS}),
\]
\[\text{Proof.} \] As in the second proof of Proposition 4.11, by the partial \( C^0 \)-estimate, one can embed any KR soliton \((M,J';\omega_{KS}',X')\) with satisfying (6.20) into an ambient projection space \( CP^N \). Then the lemma turns to prove that there is some \( \sigma \in U(N + 1, \mathbb{C}) \) such that
\[
X' = \sigma \cdot X \cdot \sigma^{-1}.
\]
Thus one can use the argument in [46] Proposition 5.10 to get the compactness of soliton VFs in \( sl(N + 1, \mathbb{C}) \) and prove the lemma by the uniqueness result of soliton VFs [46] Proposition 2.2.

On the other hand, by (6.20), there is a \( F \in \text{Diff}(M) \) such that
\[
\| F^*J' - J_0 \|_{C^1} \leq \delta_0',
\]
Then by the Kuranishi’s theorem, there are \( K \in \text{Diff}(M) \) and \( \tau' \in B(\varepsilon) \) such that
\[
K^*(F^*J') = J_{\psi^*,\tau'}.
\]

\[\text{The convergence part also comes from the Hamilton-Tian conjecture and the uniqueness result in [46].}\]
Thus, we may assume that \((M,\omega'_KS)\) is a KR soliton with respect to \(J' = J_{\psi',t}\). Since both of \(\theta_X(\omega'_KS)\) and \(\theta_X(\omega_{KS})\) are minimizers of the W-functional respect to \(\omega'_KS\) and \(\omega_{KS}\), by (1.19) in Section 4, we obtain

\[
\lambda(g'_KS) = W(g'_KS, -\theta_X(\omega'_KS)) = W(g_{KS}, -\theta_X(\omega_{KS})) = \lambda(g_{KS}).
\]

By Lemma 6.3, we can finish the proof of Theorem 0.1.

Proof of Theorem 0.1. By the uniqueness of limits of KR flow \([16, 45]\), we may assume that \(\bar{\omega}_0 = g = L(\tau, \chi)\) with \((\tau, \chi) \in U_\epsilon\) and \(\xi(\chi) = 0\). Thus the convergence part of theorem comes from Theorem 6.1. Moreover, by Lemma 6.3, \(X\) can be lifted to a HVF on \((M, J_{\infty})\) so that it is a soliton VF of \((M, J_{\infty}, g_{\infty})\) and (0.2) is satisfied. (0.2) also comes from (6.15) as well as the convergence speed of \(g(t)\) with the polynomial rate comes from (6.17).

7. Applications of Theorem 0.1

In this section, we first prove Theorem 0.2. Recall that a special degeneration on a Fano manifold \(M\) is a normal variety \(\mathcal{M}\) with a \(\mathbb{C}^*\)-action which consists of three ingredients \([34]\):

1. an flat \(\mathbb{C}^*\)-equivarant map \(\pi : \mathcal{M} \to \mathbb{C}\) such that \(\pi^{-1}(t)\) is biholomorphic to \(M\) for any \(t \neq 0\);
2. an holomorphic line bundle \(\mathcal{L}\) on \(\mathcal{M}\) such that \(\mathcal{L}|_{\pi^{-1}(t)}\) is isomorphic to \(K_M^{-r}\) for some integer \(r > 0\) and any \(t \neq 0\);
3. a center \(M_0 = \pi^{-1}(0)\) which is a Q-Fano variety.

The following definition can be found in \([47]\) (also see \([42, 3, 10]\), etc.).

Definition 7.1. A Fano manifold \(M\) is modified K-semistable with respect to a HVF \(X\) in \(\eta(M)\) if the modified Futaki-invariant \(F(\cdot) \geq 0\) for any special degeneration \(\mathcal{M}\) associated to a 1-ps \(\sigma_\tau\) induced by a \(\mathbb{C}^*\)-action, which communicates with the one-parameter subgroup \(\sigma^X_\tau\) associated to the lifting of \(X\) on \(\mathcal{M}\). In addition that \(F(\cdot) = 0\) if only if \(\mathcal{M} \cong \mathbb{C} \times \mathbb{C}\), \(M\) is called modified K-polystable with respect to \(X\).

For a Fano manifold \((M, J_{\psi_0})\) in the deformation space in Theorem 0.1, we introduce

Definition 7.2. A Fano manifold \((M, J_{\psi_0})\) is called modified K-semistable (modified K-polystable) in the deformation space of complex structures on \((M, J_0)\) which admits a KR soliton \((g_{KS}, X)\) if \(X\) can be lifted to a HVF on \((M, J_{\psi_0})\) and \((M, J_{\psi_0})\) is modified K-semistable (modified K-polystable) with respect to \(X\).

Proof of Theorem 0.2. By Lemma 6.3, the soliton VF on \((M, J_{\psi_0})\) is conjugate to \(X\). Thus we need to prove the sufficient part since the modified K-polystability is a necessary condition for the existence of KR solitons (cf. \([33]\)).

By Theorem 6.1 and Lemma 6.3, the KR flow \(g(t)\) converges smoothly to a KR soliton \((M, J_{\infty}, g_{\infty})\) with respect to \(X\) for the initial Kähler metric \(g_{\tau,0}\) on \((M, J_{\psi_0})\). Then as in the proof of \([24]\, Lemma 2.1\), by the partial \(C^0\)-estimate, there is a large integer \(L_0\) such that the family of Kähler manifolds \((M, g(t))\) and the limit manifold \((M, J_{\infty}, g_{\infty})\) can be embed into an ambient projection space \(\mathbb{C}P^N\) by normal orthogonal bases of \(H^0(M, K_{M}^{-L_0}, g(t))\) and the images \(\tilde{M}_t\) of \((M, g(t))\) converges smoothly to the image \(\tilde{M}_\infty\) of \((M, J_{\infty}, g_{\infty})\) in \(\mathbb{C}P^N\). Thus there is a family of group \(\sigma_t \in SL(N+1, \mathbb{C})\) such that

\[
\sigma_t(\tilde{M}_t) = \tilde{M}_t, \forall t \geq t_0.
\]

Moreover, we may assume that

\[
\sigma_t \cdot \sigma^X_s = \sigma^X_s \cdot \sigma_t,
\]

by modifying the base of \(H^0(M, K_{M}^{-L_0}, g(t))\) after a transformation in \(SL(N+1, \mathbb{C})\) (cf. \([10]\)). Here \(X\) can be lifted to a HVF in \(\mathbb{C}P^N\) such that it is tangent to each \(\tilde{M}_t\).
Set a reductive subgroup of \( SL(N + 1, \mathbb{C}) \) by
\[
G = \{ \sigma \in SL(N + 1, \mathbb{C}) | \sigma \cdot \sigma_s^X = \sigma_s^X \cdot \sigma \}.
\]
Then
\[
G_c = \{ \sigma \in G | \sigma(M_\infty) = \tilde{M}_\infty \} \subset \text{Aut}(\tilde{M}_\infty),
\]
where \( \text{Aut}_c(\tilde{M}_\infty) \) is a reductive subgroup of \( \text{Aut}(\tilde{M}_\infty) \) which contains \( \sigma_s^X \). Note that \( M_\infty \) admits a KR soliton and \( X \) is a center of \( \eta_c(M_\infty) \) by [30]. Thus \( G_c = \text{Aut}_c(\tilde{M}_\infty) \) is a reductive subgroup of \( G \). By GIT, there is a \( \mathbb{C}^* \)-action in \( G \) which induces a smooth degeneration \( M \) on the Fano manifold \( \tilde{M}_0 \) with the center \( \tilde{M}_\infty \). Clearly, this \( \mathbb{C}^* \)-action communicates with \( \sigma_s^X \). It follows that the corresponding modified Futaki-invariant \( F(\cdot) = 0 \) by the fact \( \tilde{M}_\infty \) admitting a KR soliton. Hence \( M \) must be a trivial degeneration and we get \( M \cong M_\infty \). This proves that \( M \) admits a KR soliton.

Let \( T \) be a torus subgroup of \( \text{Aut}_c(M, J_0) \) which contains the soliton VF \( X \). Then by Proposition [19] and Corollary [15] we see that any \( T \)-equivariant subspace \( H^1_{\text{KS}}(M, J_0, \Theta) \subset H^1_2(M, J_0, \Theta) \) is included in \( Z \), which is invariant under \( \text{Aut}_c(M, J_0) \). Thus by Theorem 0.2 we actually prove the following existence result for KR solitons in the \( T \)-equivariant deformation space.

**Corollary 7.3.** Let \( (M, J_0, \omega) \) be a Fano manifold which admits a KR soliton \( (\omega_{KS}, X) \). Then there exists a small \( \varepsilon \)-ball \( B_\varepsilon(0) \subset H^1_{\text{KS}}(M, J_0, \Theta) \) such that for any \( \psi_\varepsilon \in B_\varepsilon(0) \) the Fano manifold \( (M, J_\psi) \) admits a KR soliton if and only if \( (M, J_\psi) \) is modified \( K \)-polystable.

**Remark 7.4.** By using the deformation theory, Inoue proved the sufficient part of Corollary 7.3 in sense of the GIT-polystability via the group \( \text{Aut}(M, J_0) \) [19] Proposition 3.8. But it is still unknown whether the necessary part is true in sense of the GIT-polystability [19] Postscript Remark 1.

Next we prove the following uniqueness result for KR-solitons in the closure of the orbit by diffeomorphisms.

**Theorem 7.5.** Let \( \{\omega_1^i\} \) and \( \{\omega_2^i\} \) be two sequences of Kähler metrics in \( 2\pi c_1(M, J) \) which converge to KR-solitons \( (M_\infty^1, \omega_{KS}^1) \) and \( (M_\infty^2, \omega_{KS}^2) \) in sense of Cheeger-Gromov, respectively. Suppose that
\[
\lambda(\omega_{KS}^1) = \lambda(\omega_{KS}^2) = \sup \{ \lambda(\omega_{g_i}) | \omega_{g_i} \in 2\pi c_1(M, J) \}.
\]
Then \( M_\infty^1 \) is biholomorphic to \( M_\infty^2 \) and \( \omega_{KS}^1 \) is isometric to \( \omega_{KS}^2 \).

Theorem 7.5 generalizes the uniqueness result of Tian-Zhu for KR-solitons [35] as well as a recent result of Wang-Zhu [45] Theorem 0.4 in sense of diffeomorphisms orbit where both of \( \text{Aut}_0(M_\infty^1) \) and \( \text{Aut}_0(M_\infty^2) \) are assumed to be reductive[3] and it is also a generalization of uniqueness result of Chen-Sun for KE-metrics orbit [8] (also see [25]). We also note that the assumption (7.1) is necessary according to Pasquier’s counter-example of horospherical variety [45] Remark 6.5.

Recall [40]

**Definition 7.6.** Let \( (M, J) \) be a Fano manifold. A complex manifold \( (M', J') \) is called a canonical smooth deformation of \( (M, J) \) if there are a sequence of Kähler metrics \( \omega_i \) in \( 2\pi c_1(M, J) \) and diffeomorphisms \( \Psi_i : M' \to M \) such that
\[
\Psi_i^* \omega_i \overset{C^\infty}{\to} \omega', \quad \Psi_i^* J \overset{C^\infty}{\to} J', \quad \text{on } M'.
\]
In addition that \( J' \) is not conjugate to \( J, J' \) is called a jump of \( J \).

Theorem 7.5 is a direct corollary of following convergence result of KR flow together with the uniqueness result of Han-Li [16] (cf. [15]).

\[4\text{In fact, we expect to generalize Theorem 7.5 for the singular KR-solitons of } (M_\infty^1, \omega_{KR}^1) \text{ and } (M_\infty^2, \omega_{KR}^2) \text{ as in [15] Theorem 6.7.}\]
Proposition 7.7. Let \((M', J')\) be a canonical smooth jump of a Fano manifold \((M, J)\). Suppose that \((M', J')\) admits a KR soliton \(\omega_{KS}\) such that
\[
\lambda(\omega_{KS}) = \sup\{|\lambda(g)| : g \in 2\pi c_1(M, J)\}.
\]
Then for any initial metric \(\tilde{\omega}_0 \in 2\pi c_1(M, J)\) the flow \((M, J, \tilde{\omega}(t))\) of (7.4) is uniformly \(C^\infty\)-convergent to \((M', J', \omega_{KS})\).

Proof. By the assumption, there is a sequence of Kähler metrics \(\omega_i\) in \(2\pi c_1(M, J)\) such that
\[
\lim \text{dist}_{CG}((M, \omega_i), (M', \omega')) = 0,
\]
where \(\omega' \in 2\pi c_1(M', J')\). Note that the KR flow of (0.1) with the initial \(\omega'\) converges to the KR soliton \(\omega_{KS}\) \([29, 11]\). Then by the stability of (0.1) for the finite time, there is a sequence of \(\tilde{\omega}_i\), such that
\[
\tilde{\omega}_i = \omega_{KS} + \sqrt{-1} \partial \bar{\partial} \chi_i
\]
and
\[
\lim \text{dist}_{CG, C^\infty}((M, \tilde{\omega}_i), (M', \omega_{KS})) = 0,
\]
where \(\tilde{\omega}_i\) is the solution of flow (0.1) with the initial metric \(\omega_i\). It follows that there are diffeomorphisms \(\Phi_i\) such that
\[
\lim \|\Phi_i J - J'\|_{C^1, \gamma} = 0.
\]
Thus there are diffeomorphisms \(K_i\) (cf. (6.6)) such that
\[
J_{\phi_i} = K_i^*(\Phi_i^* J) \quad \text{and} \quad \|\phi_i\|_{C^{1, \gamma}(M)} << 1, \quad \text{as } i >> 1.
\]
Hence, as in the proof of Theorem \([6, 11]\), we conclude that the sequence \(\{g_i(t)\}\) of modified flows of (6.1) uniformly converges to the soliton flow generated by \(\omega_{KS}\). Since the limit of \(g_i(t)\) is independent of the initial metrics \(\tilde{\omega}_i\) \([16, 45]\), each \(g_i(t)\) must converge to \(\omega_{KS}\). Again by the uniqueness result in \([16, 45]\), we prove the proposition.

By Proposition 7.7, we in particular prove the following uniqueness result for the centers of smooth degenerations with admitting KR solitons.

Corollary 7.8. Let \((M, J)\) be a Fano manifold. Suppose there is a smooth degeneration on \((M, J)\) with its center \((M', J')\) admitting a KR soliton \(g_{KS}\) which satisfies
\[
\lambda(g_{KS}) = \sup\{|\lambda(g)| : g \in 2\pi c_1(M, J)\}.
\]
Then \((M', J')\) is unique.

Proof. By Corollary 2.2 and Kuranishi’s completeness theorem \([25]\), there is a smooth path \(\tau(s) \in B(e)\) \((s << 1)\) such that \(g_0 = g_{KS}\). Then
\[
g_{\tau(s)} = \omega(\cdot, J_{\tau(s)}')
\]
induces a smooth jump of \((M, J)\) to \((M', J')\). Here \(\omega\) is the Kähler form of \(g_{KS}\). Thus the corollary comes from Proposition 7.7 immediately.

8. Generalization of Theorem 0.1

In this section, we generalize the argument in the proof of Theorem 0.1 to prove Theorem 0.3. We will use a different way to obtain the entropy estimate (6.12) in the proof of Theorem 0.1. We first consider the special case of \(H_1 < 0\), i.e., the stable case, and prove the jump of KR flow (0.1) to the KR soliton \((M, J_0, \omega_{KS})\).
8.1. Linearly stable case.

**Theorem 8.1.** Let \((M, J_0)\) be a Fano manifold which admits a KR soliton \(\omega_{KS} \in 2\pi c_1(M, J_0)\). Suppose that for any \(\psi \neq 0\) in \(\mathcal{H}^{0,1}_0(M, T^{1,0}M)\) it holds

\[
H_1(\psi) < 0.
\]

Then there exists a small \(\epsilon\) such that for any \(\tau \in B(\epsilon)\) the flow \([6.7]\) with an initial metric \(g = L(\tau, \chi)\) converges smoothly to the KR soliton \((M, J_0, \omega_{KS})\). Furthermore, the convergence is fast in the polynomial rate.

**Proof.** By the Hamilton-Tian conjecture and the uniqueness result in \([16, 45]\), we need to prove the convergence of flow \(g(t)\) of \([6.1]\) with the initial metric \(g_0 = L(\tau, 0)\) for any \(\tau \in B(\epsilon)\). Moreover by \([8.1]\), we may assume that there is \(\psi \in \mathcal{H}^{0,1}_0(M, T^{1,0}M)\) such that

\[
\|\psi\| = 1, \psi_\tau = \sum \tau_i \epsilon_i = \tau \psi = \tau (\psi_1 + \psi')
\]

with the property:

\[
\|\psi_1\| \geq \frac{1}{\tau} > 0, H_1(\psi_1) = -\delta_1 \psi_1;
\]

\[
< H_1(\psi'), \psi' > \leq 0, \|\psi'\| \leq 1.
\]

Here \(\delta_1 \geq 0\). Without loss of generality, \(-\delta_1\) may be regarded as the largest negative eigenvalue of \(H_1\). Thus for any \(\chi\) with \(\|\chi\|_{C^{4,\gamma}} \leq \tau N_0\), where \(N_0\) is fixed, the following is true:

\[
(\tau, \chi) = \tau(\psi, \chi'), \|\chi'\|_{C^{4,\gamma}} \leq N_0,
\]

\[
\lambda(g_\tau, \chi) \leq \tau \lambda(\psi_1, \chi) - \tau^2 \left( \frac{\delta_1}{\tau^2} + o(1) \right) < \lambda(\omega_{KS}).
\]

Let \(T = T_{\tau, 0}\) be the set defined as in \([6.5]\) with \(\delta_0 = O(\tau)\). Let \(\tilde{\psi} = \psi_t \in \mathcal{H}^{0,1}_0(M, T^{1,0}M)\) with \(\|\tilde{\psi}\| < \epsilon_0 = O(\tau)\) defined as in \([6.6]\) such that

\[
K(t)^* J(t) = J_\tilde{\psi}, \tilde{\psi} = \Phi(\tilde{\psi}).
\]

Then as in \([8.2]\) and \([8.3]\), there is \(\tilde{\psi}_1 \in \mathcal{H}^{0,1}_0(M, T^{1,0}M)\) with \(\|\tilde{\psi}_1\| \geq \frac{1}{\tau}\) such that

\[
\|\tilde{\psi}\| = \tau' < \epsilon_0;
\]

\[
\tilde{\psi} = \tau'(\tilde{\psi}_1 + \tilde{\psi}'), \|\tilde{\psi}_1 + \tilde{\psi}'\| = 1;
\]

\[
H_1(\tilde{\psi}_1) = -\delta_1' \tilde{\psi}_1 \text{ and } < H_1(\tilde{\psi}'), \tilde{\psi}' > \leq 0,
\]

where \(\delta_1' \geq \delta_1\).

On the other hand, we may assume that \(\omega_{g(t)}\) is same as \(\omega_{KS}\). Otherwise, by Moser’s theorem, there is a diffeomorphism \(\tilde{K}(t) \in \text{Diff}(M)\) such that \(\omega_{\tilde{g}(t)} = \omega_{KS}\), where \(\tilde{g}(t) = \tilde{K}(t)^* g(t)\). Then we can replace \(g(t)\) by \(\tilde{g}(t)\). Thus, there are \(K(t) \in \text{Diff}(M)\) and \(\tilde{\chi} \in C^\infty(M)\) as in \([6.5]\) such that

\[
\omega_{K(t)^* g(t)} = K(t)^* \omega_{KS} + \sqrt{-1} \partial_{\tilde{J}_2} \bar{\partial}_{\tilde{J}_2} \tilde{\chi}, \int_M \tilde{\chi} \omega_{KS}^n = 0.
\]

Moreover, as in \([6.9]\), we have

\[
K(t)^* \omega_{KS} = (1 + O(||\phi||_{L^2(M)})) \omega_{KS}.
\]

It follows that (cf. \([6.7]\))

\[
\|\Delta \tilde{\chi}\|_{C^{4,\gamma}} \leq \tau' N_0.
\]

Consequently, by \([8.4]\), we derive

\[
\lambda(g(t)) = \lambda(L \tilde{\psi}, \tilde{\chi}) - (\tau')^2 \left( \frac{\delta_1}{\tau^2} + o(1) \right) \leq \lambda(g_{KS}).
\]

Hence, we can continue the argument in the proof of Theorem \([6.1]\) to show that the flow \([6.1]\) converges smoothly to a KR soliton \((M, J_\infty, \omega_\infty)\) such that

\[
\lambda(g_\infty) = \lambda(g_{KS}).
\]
Moreover, by (8.7), \( \tau' \) must go to zero. Therefore, \( \bar{\psi} \) goes to zero and \( g_\infty \) is same as \( g_{KS} \) by the uniqueness of KR solitons [35].

For a general initial metric in \( 2\pi\epsilon_1(M, J_\tau) \), we only remark how to get the decay estimate. In fact, we can choose a large \( T_0 \) such that
\[
\text{dist}_{CG, C^\epsilon}((M, g(T_0)), (M, g_\infty)) \leq \epsilon
\]
and there is an auto-diffeomorphism \( \sigma \) such that
\[
\|\sigma^* J(T_0) - J_0 \|_{C^\epsilon} \leq \epsilon'.
\]
Then as in (8.8), there is an auto-diffeomorphism \( K(T_0) \) and \( \psi \in \mathcal{H}^{0,1}_\varphi (M, T^{1,0}M) \) with \( \varphi = \Phi'(\psi) \) for the metric \( \sigma^* g(T_0) \) such that
\[
\|\psi\| = \tau << 1 \text{ and } \|\Delta \chi\|_{C^{1,\gamma}} \leq \tau N_0.
\]
Hence, by the above argument, the KR flow with the initial metric \( g(T_0) \) converges to \( g_\infty \) fast in the polynomial rate. The proof is complete.

Theorem 8.1 means that any KR flow with an initial metric in \( 2\pi\epsilon_1(M, J_\tau) \) with \( \tau << 1 \) under the condition (7.6) will jump up to the original KR soliton \((M, J_0, \omega_{KS})\) and also show that the KR soliton \((M, J_0, \omega_{KS})\) is isolated in the deformation space of complex structures in this case. But up to now, we do not know whether there is such an example of Fano manifold with admitting an isolated KR soliton around her deformation space of complex structures.

### 8.2. Linearly semistable case

In this subsection, we generalize Theorem 8.1 with the existence of non-trivial kernel \( Z \) of operator \( H_1 \) to prove Theorem 0.3. Let \( K \) be a maximal compact subgroup of \( \text{Aut}_r(M, J_0) \) which contains \( K_X \). Then \((\omega_{KS}, J_0)\) is \( K \)-invariant [35], and so \( \sigma \) maps \( \mathcal{H}^{0,1}_\varphi (M, T^{1,0}M) \) to itself for any \( \varphi \in K \). The latter means that \( \sigma(\psi) \) is still \( \varphi \)-harmonic whenever \( \psi \) is \( \varphi \)-harmonic. Thus we can extend the action\( \text{Aut}_r(M, J_0) \) on \( \mathcal{H}^{0,1}_\varphi (M, T^{1,0}M) \) so that \( \mathcal{H}^{0,1}_\varphi (M, T^{1,0}M) \) is an \( \text{Aut}_r(M, J_0) \)-invariant space. Since
\[
K_X \cdot \sigma = \sigma \cdot K_X, \forall \sigma \in \text{Aut}_r(M, J_0),
\]
we have
\[
\mathcal{L}_\xi \sigma^* \psi = 0, \forall \psi \in Z.
\]
Hence, \( Z \) is also an \( \text{Aut}_r(M, J_0) \)-invariant subspace of \( \mathcal{H}^{0,1}_\varphi (M, T^{1,0}M) \). Furthermore, we prove

**Lemma 8.2.** Suppose that \((M, J_\psi)\) admits a KR soliton for any \( \psi \in B(\epsilon) \cap Z \). Then
\[
(8.8) \quad \sigma^* \psi = \psi, \forall \sigma \in \text{Aut}_r(M, J_0), \psi \in Z.
\]

**Proof.** Since the union of maximal tori is dense in a reductive complex group, we need to prove (8.8) for any torus 1-ps \( \sigma_1 \) and \( \psi \in Z \). On the contrary, there are \( \sigma \in \text{Aut}_r(M, J_0) \) and \( \psi \in Z \) such that \( \sigma \psi = \sigma^{-1} \cdot \rho_t \cdot \sigma \) with some nontrivial weights for \( \psi \), where
\[
\rho_t : \mathbb{C}^* \mapsto \text{Aut}_r(M, J_0)
\]
is a diagonal torus action on \( Z \). Since \( Z \) is \( \text{Aut}_r(M, J_0) \)-invariant, there is a basis of \( Z \), \( \{e_1, \ldots, e_s\} \) \((s = \dim Z)\) such that \( \sigma(\psi) = \sum_{i \leq s} \epsilon_i \) and
\[
\sigma_t(\psi) = \sum_i \epsilon_i \sigma_t^{-1}(e_i), \quad t \in \mathbb{C}^*,
\]
where \( \epsilon_i \) are weights of \( \sigma_t \) with some \( \epsilon_i \neq 0 \). Without loss of generality, we may assume \( \epsilon_1 > 0 \).

Now we consider a degeneration of \( \rho_t(e_1) \) \((|t| < |e_1| << 1)\) in \( Z \). Note that
\[
J_{\rho_t^* \Phi(e_1)} = \rho_t^* J_{\Phi(e_1)}.
\]
Then by the Kuranishi theorem, there is a \( K_t \in \text{Diff}(M) \) such that
\[
J_{\Phi(\rho_t(e_1))} = K_t^* (J_{\Phi(e_1)}).
\]
Thus, for any small $t_1$ and $t_2$ it holds
\[(M, J_{\Phi(t_1)}) \cong (M, J_{\Phi(t_2)})\].
Hence, the family of $(M, J_{\Phi(t)})$ (\(|t| < \epsilon\)) is generated by a $C^0$-action on $(M, J_{\Phi(0)})$, which degenerates to $(M, J_0)$. Moreover, it communicates with 1-ps $\sigma_{\delta}^X$ and its modified Futaki-invariant is zero. Therefore, we get a contradiction with the modified K-polystability of $(M, J_{\Phi(0)})$ which admits a KR soliton. The proof is finished.

\[\Box\]

**Remark 8.3.** In case of $\omega_{KS} = \omega_{KE}$, $Z = \mathcal{H}^{0,1}(M, T^{1,0}M)$. Then the property \[^{[6, Theorem 3.1]}\] turns to hold for any $\sigma \in \text{Aut}(M, J_0)$, $\psi \in \mathcal{H}^{0,1}(M, T^{1,0}M)$. Thus Lemma \[^{[5]}\] is actually a generalization of \[^{[6]}\] Theorem 3.1 in the case of $KE$ metrics.

By Lemma \[^{[5,2]}\] we prove

**Lemma 8.4.** Under the assumption in Lemma \[^{[5,2]}\] there exists $\delta > 0$ such that for any $(\tau, \chi) \in \mathcal{V}_\delta$ it holds
\[\lambda(g_{\tau, \chi}) \leq \lambda(g_{KS}),\]
if $H_1(\cdot) \leq 0$ on $\mathcal{H}^{0,1}_0(M, T^{1,0}M)$.

**Proof.** Let $\mathfrak{t}$ be the Lie algebra of $K$, and $\eta_\nu$ the nilpotent part of $\eta(M, J_0; \mathbb{R})$. Then
\[\eta(M, J_0; \mathbb{R}) = \eta_\nu \oplus \eta_u = \mathfrak{t} \oplus J_0\mathfrak{t} \oplus \eta_u.\]
Fix a small ball $U(\epsilon_0) \subseteq \mathfrak{t} \oplus \eta_u$ centered at the origin. By the Kuranishi theorem there exists $\tau^* \in B(\epsilon)$ and $\gamma \in \text{Diff}(M)$ for any $\tau \in B(\epsilon), (v_1, v_2) \in U(\epsilon_0)$ such that
\[\left( e^{J_0^*v_1 + v_2} \right)^* J_{\tau^*} = \gamma^* J_{\tau^*}, \]
where $J_{\tau}$ and $J_{\tau^*}$ are complex structures associated to $\psi_{\tau}$ and $\psi_{\tau^*}$, respectively. We define a smooth map $\Phi : U(\epsilon) \times \mathcal{V}_\epsilon \mapsto \mathcal{V}_\epsilon$ by
\[\Phi((v_1, v_2), (\tau, \chi)) = (\tau^*, \chi'),\]
where the Kähler potential $\chi'$ is determined by
\[\left( \gamma^{-1} \right)^* (e^{J_0^*v_1 + v_2})^* (\omega + \sqrt{-1} \partial J_0 \bar{\partial} J_0 \chi) = \omega + \sqrt{-1} \partial J_0 \bar{\partial} J_0 \chi' \text{ and } \int_M \chi' \omega_{KS} = 0.\]
Let $\Psi : U(\epsilon) \times \mathcal{V}_\epsilon \mapsto \text{ker}(H_2)$ be a projection of $\Phi$ such that
\[\Psi = \text{Pr}_{\text{ker}(H_2)} \circ \text{Pr}_2 \circ \Phi.\]
Namely,
\[\Psi((v_1, v_2)(\tau, \chi)) = \text{Pr}_{\text{ker}(H_2)}(\chi').\]
A direct calculation shows that
\[D_{(0,0)}\Psi((v_1, v_2), 0) = \theta_Y + \bar{\theta}_Y,\]
where $\theta_Y$ is a potential of HVF,
\[Y = \frac{1}{2} \sqrt{-1}(v_1 - \sqrt{-1}J_0v_1) + \frac{1}{2}(v_2 - \sqrt{-1}J_0v_2).\]
Thus by \[^{[40]}\], we see that
\[D_{(0,0)}\Psi(\cdot, 0) : \mathfrak{t} \oplus \eta_u \mapsto \text{ker}(H_2)\]
is an isomorphism. By the implicit function theorem, there are $\delta < \epsilon$ and map $G : \mathcal{V}_\delta \mapsto U(\epsilon_0)$ such that
\[\text{Pr}_2 \circ \Phi(G(\tau, \chi), (\tau, \chi)) \in \text{ker}(H_2)\]
It follows that
\[(\tau^*, \chi') = \Phi(G(\tau, \chi), (\tau, \chi)) \in \mathcal{V}_\epsilon.\]
where $\chi' \in \ker(H_2)^\perp$ and $\delta'(\delta) \ll 1$.

Let

$$\tilde{\Psi} = \{ (\tau_0, \tilde{\chi}) \in \Psi_c | \tau_0 \in Z, \xi(\tilde{\chi}) = 0 \}.$$ 

Then the restricted map $\tilde{\Psi}$ of $\Psi$ on $\mathfrak{t} \times \tilde{\mathfrak{v}}$ is also smooth and

$$D_{(0,0)} \tilde{\Psi}(\cdot, 0) : \mathfrak{t} \mapsto \{ \theta_Y + \overline{\theta}_Y \}$$

is an isomorphic map. Here $Y = \frac{1}{2} \sqrt{-1}(v - \sqrt{-1}J_0 v)$, $v \in \mathfrak{k}$. We claim:

(8.12) $\ker(H_2) \cap \{ \chi | \xi(\chi) = 0 \} = \{ \theta_Y + \overline{\theta}_Y \}$.

On contrary, there is a nontrivial $v' \in \eta_0$ such that

$$\xi(\theta_Y + \overline{\theta}_Y) = 0,$$

where $Y' = v' - \sqrt{-1}J_0 v'$. Then for any $\sigma \in K_X$,

$$\sigma^*(\theta_Y + \overline{\theta}_Y) = \theta_Y + \overline{\theta}_Y.$$

It follows that

$$\sigma^* Y' = Y', \forall \sigma \in K_X.$$

This implies $Y' \in \eta_c(M, J_0)$, which is a contradiction!

By (8.12),

$$D_{(0,0)} \tilde{\Psi}(\cdot, 0) : \mathfrak{t} \mapsto \ker(H_2) \cap \{ \chi | \xi(\chi) = 0 \}$$

is an isomorphism. On the other hand, by Lemma 8.4

$$(e^{J_0 v})^* J_{\tau_0} = J_{\tau_0}, \forall \psi_{\tau_0} \in Z, v \in \mathfrak{k}.$$

Hence, applying the implicit function theorem to $\tilde{\Psi}$ at $(\tau_0, \tilde{\chi}) \in \tilde{\mathfrak{v}}$, there are $v \in \mathfrak{k}$ and Kähler potential $\chi_0 \in \ker(H_2)^\perp$ such that $(\tau_0, \chi_0) \in \tilde{\mathfrak{v}}'$ and

$$(e^{J_0 v})^*(\omega + \sqrt{-1}\partial\bar{\partial}\tau_0 \tilde{\chi}) = \omega + \sqrt{-1}\partial\bar{\partial}\chi_0.$$

Note that a Kähler potential $\tilde{\chi}$ of KR soliton with respect to $X$ on $(M,J_{\tau_0})$ satisfies $\xi(\tilde{\chi}) = 0$.

Therefore, there is a KR soliton $\omega + \sqrt{-1}\partial\bar{\partial}\chi_0$ on $(M,J_{\tau_0})$ such that

(8.13) $\chi_0 \in \ker(H_2)^\perp$ and $(\tau_0, \chi_0) \in \tilde{\mathfrak{v}}'$.

Decompose $\chi$ in (8.11) as

(8.14) $\chi = \chi_0 + \chi_1, \psi_{\tau_0} \in Z, \psi_{\tau_1} \in Z^\perp$.

Then by (8.13), there are Kähler potentials $\chi_0$ and $\chi_1 \in \ker(H_2)^\perp$ such that

(8.15) $\chi' = \chi_0 + \chi_1$,

where $\omega + \sqrt{-1}\partial\bar{\partial}\chi_0$ is a KR soliton. Thus, we get the following expansion for the reduced entropy $\nu$,

(8.16) $\nu(\tau', \chi') = \nu(\tau_0, \chi_0) + \frac{1}{2} D_2^2(\tau_{0,\chi_0}, \zeta(\tau_{1,\chi_1}))((\tau_1, \chi_1), (\tau_1, \chi_1)).$

Here $\zeta \in (0, 1)$. Since $D_2^2(0, \cdot)$ is strictly negative on $\ker(H_2)^\perp$, by the assumption in Lemma 8.4 it is strictly negative on $\ker(H_1)^\perp \times \ker(H_2)^\perp$. It follows that $D_2^2(\tau_{0,\chi_0}, \zeta(\tau_{1,\chi_1}) \nu)$ is strictly negative on $\ker(H_1)^\perp \times \ker(H_2)^\perp$. Hence, we derive

(8.17) $\nu(\tau', \chi') \leq \nu(\tau_0, \chi_0)$.

As a consequence, we prove

$$\lambda(g_{\tau, \chi}) \leq \lambda(g_{\tau_0, \chi_0}) = \lambda(g_{KS}).$$

$$\square$$

By Lemma 8.4 we can use the argument in the proof of Theorem 6.1 (or Theorem 8.1) to finish the proof of Theorem 0.3. We leave it to the reader.

We end this subsection by the following two remarks to Theorem 0.3.
Remark 8.5. In Theorem 0.3 we can actually prove that the convergence is of polynomial rate of any order. This is because we can improve the Lojasiewicz inequality of (8.1) for $\nu(\cdot)$ in Proposition 5.1 as follows: For any $a = (\varphi, \chi) \in V$, it holds
\begin{equation}
\|\nabla \nu(a')\|_{L^2} \geq c_0 |\nu(a') - \nu(0)|^\frac{1}{2},
\end{equation}
where $a' = (\tau', \chi') \in V$. It follows that the Lojasiewicz inequality of (6.1) for $\lambda(\cdot)$ along the flow holds for any $a' > \frac{1}{2}$. Hence, by (6.1), the convergence is of polynomial rate of any order. (8.18) can be proved together with (5.5) and (5.7) since the functional $F$ there can vanish by the decompositions (8.14) and (8.15).

Remark 8.6. We call the KR flow $\tilde{\omega}(t)$ of (7.1) is stable on $(M, J_\psi)$ if it converges smoothly to a KR soliton $(M_\infty, J_\infty, \omega_\infty)$ with $\lambda(\omega_\infty) = \lambda(\omega_{KS})$. In Theorem 0.3 we show that the flow (7.1) is always stable for any complex structure in the deformation space under the conditions of (1) and (2). By the above argument (also see Remark 6.3), it is easy to see that the flow $\tilde{\omega}(t)$ is stable on $(M, J_\psi)$ if and only if the energy level $L([M, J_\psi])$ of flow is equal to $\lambda(\omega_{KS})$. Here
\begin{equation}
L([M, J_\psi]) = \lim_{t \to \infty} \lambda(\tilde{\omega}(t)),
\end{equation}
which is independent of flow $\tilde{\omega}(t)$ with the initial metric $\tilde{\omega}_0 \in 2\pi c_1(M, J_\psi)$

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