ACTIONS OF GALOIS GROUPS ON INVARIANTS OF NUMBER FIELDS

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Abstract. In this paper we investigate the connection between relations among various invariants of number fields $L^H$ corresponding to subgroups $H$ acting on $L$ and of linear relations among norm idempotents.

1. Introduction

Let $C$ be an algebraic curve defined over an algebraically closed field of arbitrary characteristic and let $G \subset \text{Aut}(C)$ be a subgroup of the automorphism group $G$, acting on $C$. For a subgroup $H$ of $G$, let $C^H$ be the quotient group and let $g_H$ and $\gamma_H$ be the the genus and the $p$-rank of the Jacobian of $C^H$. In the group algebra $k[G]$ the norm idempotents $\varepsilon_H$ are defined by

$$\varepsilon_H = \frac{1}{|H|} \sum_{h \in H} h.$$ 

E. Kani and M. Rosen [3], [4], studied the action of automorphisms on the Jacobian variety of the curve, and they proved that every linear relation among the norm idempotents coming from subgroups $H$ of $G$ implies the same relations for $g_H, \gamma_H$. This is a generalization of results proved by R. Accola [1].

They also have proved that this linear relation imply the same relations for the zeta functions of the corresponding fields $L^H$ where $L$ is the function field of an algebraic curve or a number field [4, prop. 1.2], i.e.,

$$\sum r_H \varepsilon_H = 0 \Rightarrow \prod \zeta_H(s)^{r_H} = 1.$$ 

The rings of integers of number fields have a theory similar to that of non singular algebraic curves, in the sense that the ring of integers are Dedekind so they give rise to one-dimensional affine schemes, that can be completed with the aid of infinite primes. For number fields there is a notion of genus, and the analogs for Jacobian varieties and Tate modules can be defined.

It is known that a lot of information concerning a number field, can be found in the corresponding zeta function. Let $L^H$ be the number field corresponding to the subgroup $H$, of the Galois group $G$. Using the characterization of the residues of the zeta functions for number fields at $s = 1$, Kani and Rosen arrived at a formula, involving the class number $h_{L^H}$, the regulator $\text{Reg}(L^H)$, and the number $w_H$ of roots of unity in $L^H$:

$$\prod (h_{L^H}\text{Reg}(L^H))^{r_H} = \prod w_H^{r_H},$$

Date: August 28, 2018.
where $H$ runs over the subgroups of $G$. The last equality was also proved by R. Brauer in 1950.

In this paper we study the dependence of several group invariants of the subfields $L^H$, corresponding to the Galois subgroups $H$, in terms of the linear relations among norm idempotents defined by the subgroup $H$.

In order to do so we give a generalization of the notion of Tate modules for the “Jacobian” of a Number Field, and we consider the action of the Galois group on it. In our study, problems arise that are similar to those stemming from wild ramification of the action of a group on a curve defined over a field of positive characteristic.

Consider a number field $L$. Fix a subfield $K$ such that the extension $L/K$ is Galois with Galois group $G$. For every subgroup $H$ of $G$ we define, as usual, the fixed field $L^H$. The following functions from the set of subgroups $G$ to $\mathbb{Z}$ are defined:

1. Let $r_H, 2s_H$ be the number of real and imaginary embeddings of $L^H$ to $\mathbb{Q}$. We set 
   $$\lambda_H = r_H + s_H - 1.$$  

2. Consider the class group $Cl(L^H)$. It is a finite Abelian group, hence it can be written as 
   $$Cl(L^H) = \bigoplus_{p|Cl(L^H)} A(H, p),$$  
   where $A(H, p)$ is the $p$-part of the abelian group $Cl(L^H)$ and $A(H, p)$ in turn can be expressed as a finite direct sum of abelian groups $A(H, p, n)$ that are sums of $\lambda_{H, p, n}$ summands of cyclic groups of order $p^n$, i.e., 
   $$A(H, p) = \bigoplus_{n=1}^{\infty} A(H, p, n),$$  
   $$A(H, p, n) = \bigoplus_{\mu=1}^{\lambda_{H, p, n}} \mathbb{Z}/p^n\mathbb{Z}.$$  
   Notice that in the above formulas $\lambda_{H, p, n} = 0$ for all but finite integers $n$, and that $A(H, p, n)$ are free $\mathbb{Z}/p^n\mathbb{Z}$-modules of rank $\lambda_{H, p, n}$.

3. Consider the group $\mu(L^H)$ of units of finite order in the field $L^H$. It is a cyclic group and can be written as 
   $$\mu(L^H) = \bigoplus_{p|\mu(L^H)} \mathbb{Z}/p^\nu(p, H)\mathbb{Z}.$$  
   We will show that the above functions $\lambda_H, \lambda_{H, p, n}$, behave like the $p$-ranks of the Jacobians of algebraic curves. Namely, we prove that every linear relation among norm idempotents of the subgroup implies the same relations for the $\lambda$ functions:

**Theorem 1.1.** Let $L/K$ be a Galois extension with Galois group $G$ of order $n$. Every relation 

$$\sum_{H \subseteq H} r_H \varepsilon_H = 0$$  

We will show that the above functions $\lambda_H, \lambda_{H, p, n}$, behave like the $p$-ranks of the Jacobians of algebraic curves. Namely, we prove that every linear relation among norm idempotents of the subgroup implies the same relations for the $\lambda$ functions:
among norm idempotents implies relations

\[ \sum r_H \lambda_H = 0 \]
\[ \sum r_H \lambda_{H,p,n} = 0 \]
\[ \sum r_H \nu(H,p) = 0 \text{ for every } p \nmid n. \]

On a number field \( L \) we can define the notion of the Arakelov genus \( g_L \), so it is interesting to ask whether a relation among norm idempotents implies the same relation among Arakelov genera. The answer is yes provided we have “tame ramification” in the group of units that are contained in \( L \), i.e.,

**Proposition 1.2.** Let \( L/K \) be a Galois extension with Galois group \( G \). Let \( w_L \) be the order of the group of units contained in \( L \). Consider the set \( S \) of subgroups \( H < G \), such that \( (|H|, w_L) = 1 \). Every linear relation \( \sum_{H \in S} r_H \varepsilon_H = 0 \) among norm idempotents corresponding to subgroups \( H \in S \), implies the same relation among the Arakelov genera \( g_{L,H} \), of the fixed fields \( L^H \). In particular, if the order of the Galois group is prime to \( w_L \), then \( \sum r_H \varepsilon_H = 0 \) implies \( \sum r_H g_{L,H} = 0 \).

In [6] G. Van der Geer and R. Schoof introduced the notion of effectivity of an Arakelov divisor, a notion that is close to the definition of the effectivity of a divisor on an algebraic curve. This notion gives rise to a new notion of \( H^0(D) \), for Arakelov divisors \( D \) and introduces naturally a new invariant \( \eta_L \) for the number field \( L \):

\[ \eta_L := \left( \sum_{x \in \mathcal{O}_L} e^{-\pi ||x||^2_{L,0}} \right), \]

where \( || \cdot ||_{L,0} \) is the metric on the Minkowski space of the number field \( L \) defined by

\[ ||x||^2_{L,0} = \sum |\sigma(x)|^2. \]

Given a relation \( \sum n_H \varepsilon_H = 0 \), we will prove a formula for the \( \eta \)-invariants corresponding to subfields \( L^H \) of \( L \). In order to do so we have to change the model at the infinite primes by considering a different metric \( || \cdot ||_{L,A} \) on the Minkowski vector space. This metric is defined in [7]. We introduce the invariants

\[ \eta_A(L) := \left( \sum_{x \in \mathcal{O}_L} e^{-\pi ||x||^2_{L,A}} \right), \]

for every divisor \( A \) supported at infinite primes. We will prove the following:

**Proposition 1.3.** Let \( \sum n_H \varepsilon_H = 0 \) be a linear relation among norm idempotents. If \( \mathbb{P}(L^H, \mathbb{R}) \) (resp. \( \mathbb{P}(L^H, \mathbb{C}) \)) denotes the real (resp. complex) infinite primes and

\[ B(H) = -\frac{\log(|H|)}{2\pi} \sum_{\sigma \in \mathbb{P}(L^H, \mathbb{R})} \frac{\sigma - \log(|H|/2)}{\pi} \sum_{\sigma \in \mathbb{P}(L^H, \mathbb{C})} \sigma, \]

is a divisor supported on infinite primes of the field \( L^H \), then the following formula holds

\[ 0 = \sum_H \lambda_H \eta_B(H)(L^H). \]

Aknowledgement: The author wishes to thank professor G. Van der Geer for his remarks and commends.
2. Notations

Let $K$ be a number field with ring of algebraic integers $\mathcal{O}_K$. We will follow the notation of the book of J. Neukirch [5].

An Arakelov divisor of $K$, is a formal sum

$$D = \sum \nu_p p,$$

where $p$ runs over the finite and infinite primes of $K$, and $\nu_p \in \mathbb{Z}$ if $p$ is a finite prime and $\nu_p \in \mathbb{R}$ if $p$ is an infinite prime. We will denote by

$$\text{Div}(\overline{\mathcal{O}}_K) \cong \text{Div}(\mathcal{O}_K) \times \bigoplus_{|\mathbb{P}|}\mathbb{R}$$

the set of Arakelov divisors on $K$. There is a canonical homomorphism

$$\text{div} : K^* \to \text{Div}(\overline{\mathcal{O}}_K)$$

sending $f \in K^*$ to $\sum \nu_p p(f)$, where $\nu_p(f)$ is the normalized $p$-adic valuation of $f$ if $p$ is a finite prime, and $\nu_p(f) = -\log|\tau(f)|$, where $\tau \in \text{Hom}_{\mathbb{Q}}(K, \overline{\mathcal{O}}_K)$ is the monomorphism corresponding to the infinite prime $p$. The Arakelov class group

$$\text{CH}^1(\overline{\mathcal{O}}_K) = \text{Div}(\overline{\mathcal{O}}_K) / \text{div}(K^*)$$

and it is equipped with the quotient topology. Since $\prod_p |f_p| = 1$ we can define on $\text{CH}^1(\overline{\mathcal{O}}_K)$ a continuous function

$$\text{deg} : \text{CH}^1(\overline{\mathcal{O}}_K) \to \mathbb{R}$$

sending $D = \sum \nu_p p$ to $\sum \nu_p \log(N(P))$, where $N(P)$ denotes the norm of $P$. The kernel of the degree map is a compact group denoted by $\text{CH}^1(\overline{\mathcal{O}}_K)^0 =: J_K$. It can be proved [5, Satz 1.11] that $J_K$ is given by the short exact sequence:

$$1 \to H/\Gamma \to J_K \to \text{Cl}(K) \to 1,$$

where $H/\Gamma$ is homeomorphic to a torus of dimension $r + s - 1$ and $\text{Cl}(K)$ is the ordinary class group of the number field $K$.

Following the theory of Jacobian varieties on a curve we set

$$J_{K,p^n} := \{ \text{Elements in } J_K \text{ of order } p^n. \}$$

where $p$ is a prime number of $\mathbb{Z}$. For the $p$-part of $J_K$ we have the following short exact sequence:

$$1 \to (\mathbb{Z}/p^n\mathbb{Z})^{r+s-1} \to J_{K,p^n} \to \text{Cl}(K)_{p^n} \to 1,$$

where $\text{Cl}(K)_{p^n} \cong \oplus_{i=1}^{\lambda_{K,n}} \mathbb{Z}/p^n\mathbb{Z}$ is the subgroup killed by multiplication by $p^n$. Using the classification theorem of finite Abelian groups we can write

$$J_{K,p^n} \cong \bigoplus_{i=1}^{r+s-1+\lambda_{K,n}} \mathbb{Z}/p^n\mathbb{Z}.$$

The groups $J_{K,p^n}$ form an inverse system and we can define the inverse limit forming the Tate module of $J_K$ at $p$. Namely we set

$$T_p(J_K) = \lim_{\leftarrow} J_{K,p^n}.$$

The Tate module is a free $\mathbb{Z}_p$-submodule of rank $s + r - 1$. Since the order of the ordinary class group $\text{Cl}(K)$ is finite $\lambda_{K,n} = 0$, for large $n$, and this implies that the
information of the \( p \)-part of the class group is lost after taking the inverse limit. We will study the \( p \)-part \( \text{Cl}(K)_p \) of the class group separately.

The action of \( G \) on the primes of \( L \) induces a representation

\[
\rho : G \to \text{End}(J_L),
\]

and since endomorphisms of \( J_L \) preserve the orders of the elements in the class group we can define representations:

\[
\rho_p : G \to \text{End}(J_{L,p}).
\]

Every \( \rho_p \) gives rise to a representation

\[
\hat{\rho}_p : \mathbb{Q}_p[G] \to \text{End}^0(T_p(J_L)) := \text{End}(T_p(J_L)) \otimes_{\mathbb{Z}} \mathbb{Q}_p
\]

and to a representation

\[
\tilde{\rho}_p : \mathbb{Z}_{(p)}[G] \to \text{End}(\text{Cl}(L)_p) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)},
\]

where \( \mathbb{Z}_{(p)} \) denotes the localization of the integer ring with respect to the prime ideal \( p \). We define the \( \mathbb{Q}_p \) vector space \( V_p(J_L) := T_p(J_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), so \( \text{End}^0(T_p(J_L)) \cong \text{End}(V_p) \). Since there is \( p \)-torsion on the \( \mathbb{Z} \)-module \( \text{Cl}(L)_p \) we cannot tensor by a field, without trivializing. The closest structure to vector space we can obtain without trivializing, is by tensoring with the localization \( \mathbb{Z}_{(p)} \). So we define the \( \mathbb{Z}_{(p)} \)-module \( V_p(\text{Cl}(L)) \) by \( V_p(\text{Cl}(L)) := \text{Cl}(L)_p \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \). The \( p \)-part of the class group can be factored, by the classification theorem of Abelian groups, as follows:

\[
\text{Cl}(L)_p = \bigoplus_{\nu = 1}^{\infty} \bigoplus_{\mu = 1}^{\lambda_{(1),p,\nu}} \mathbb{Z}/p^\nu \mathbb{Z},
\]

and

\[
V_p(\text{Cl}(L)) = \bigoplus_{\nu = 1}^{\infty} \bigoplus_{\mu = 1}^{\lambda_{(1),p,\nu}} \mathbb{Z}_{(p)}/p^\nu \mathbb{Z}_{(p)}.
\]

Since endomorphisms that came from \( \mathbb{Z}_{(p)}[G] \) preserve the order of the group, the representation \( \tilde{\rho}_p \) can be factored as a sum of matrix representations:

\[
\tilde{\rho}_{p,\nu} : \mathbb{Z}_{(p)}[G] \to M_{\lambda_{(1),p,\nu}}(\mathbb{Z}_{(p)}/p^\nu \mathbb{Z}_{(p)}),
\]

where \( M_r(R) \) denotes the \( r \times r \) matrices with coefficients from the ring \( R \). We define the trace of \( \tilde{\rho}_p \) to be the sequence \( \text{tr}(\tilde{\rho}_p) := (\text{tr}(\tilde{\rho}_{p,\nu}))_\nu \). Obviously, \( \text{tr}(\tilde{\rho}_{p,\nu}) = 0 \) for all but finite \( \nu \).

3. Field extensions

Let \( K, L \) be two number fields and let

\[
\tau : K \to L
\]

be a homomorphism. For an Arakelov divisor \( D = \sum_p \nu_p P \) of \( L \) we define:

\[
\tau_*(D) := \sum_p \left( \sum_{P | P} \nu_p f_{P/P} \right) p \in \text{Div}(\mathcal{O}_K),
\]

5
where \( f_{P/p} \) denotes the inertia degree of \( P \) over \( \tau K \) and \( P \mid \tau K \). Conversely for an Arakelov divisor \( D = \sum_p \nu_p p \) of \( K \) we define
\[
\tau^*(D) = \sum_p \sum_{P \mid \tau K} \nu_p e_{P/p} P \in \text{Div}(\mathcal{O}_L),
\]
where \( e_{P/p} \) denotes the ramification index of \( P \) over \( \tau K \). The maps \( \tau_* \), \( \tau^* \) induce maps
\[
\tau_* : \text{CH}^1(\mathcal{O}_L) \to \text{CH}^1(\mathcal{O}_K)
\]
and
\[
\tau^* : \text{CH}^1(\mathcal{O}_K) \to \text{CH}^1(\mathcal{O}_L)
\]
such that \( \tau_* \circ \tau^* = [L : K] \) and \( \deg(\tau_* D) = \deg(D) \), \( \deg(\tau^* D) = [L : K] \deg(D) \) \footnote{p. 204}. By the above formulas for the degree, we have that there are well defined homomorphisms
(2) \( \tau_* : J_L \to J_K \),
and
(3) \( \tau^* : J_K \to J_L \).

**Definition 3.1.** Let \( \mathbb{Z}_{(p)} \) denote the localization of the ring of integers with respect to a prime ideal. We will denote by \( R \) either a field of characteristic zero or \( \mathbb{Z}_{(p)} \).

**Lemma 3.2.** Let \( V, W \) be two finitely generated \( R \)-modules. Suppose that there are two \( R \)-module homomorphisms \( f_{V,W} : V \to W \), \( f_{W,V} : W \to V \), such that
\[
f_{V,W} \circ f_{W,V} = n \text{Id}_W
\]
and with \((n, p) = 1 \) if \( R = \mathbb{Z}_{(p)} \). Then there is a map:
\[
\phi : \text{End}(W) \to \text{End}(V),
\]
such that \( \text{tr}(a) = \text{tr}(\phi(a)) \). In particular, if \( a = \text{Id}_W \), then \( \phi(a) \in \text{End}(V) \), has trace equal to \( \text{rank}(W) \)

**Proof.** For every \( a \in \text{End}(W) \) we define \( \phi(a) \in \text{End}(V) \) by
\[
\phi(a) := \frac{1}{n} f_{W,V} \circ a \circ f_{V,W}.
\]
Since \( n \) is an invertible element in \( R \) the map \( f_{V,W} \) is onto. We consider the following short exact sequence of \( R \)-modules:
\[
0 \longrightarrow \ker f_{V,W} \longrightarrow V \xrightarrow{f_{V,W}} W \longrightarrow 0
\]

By construction, \( \phi(a) \) is zero on \( \ker f_{V,W} \), and \( \text{tr}(a) = \text{tr}(\phi(a)) \). In particular, for \( a = \text{Id}_W \) we have that \( \text{tr}(\text{Id}_W) = \text{rank}(W) \), hence \( \text{tr}(\phi(a)) = \text{rank}(W) \) \( \square \)

**Lemma 3.3.** For a given group \( G \), let \( S \) be the following set of subgroups of \( G \):
\[
S := \begin{cases} 
\text{all subgroups of } G & \text{if } R \text{ is a field} \\
H < G, p \nmid |H| & \text{if } R = \mathbb{Z}_{(p)}
\end{cases}
\]
Let \( V \) be a free \( R \)-module. To every \( H \in S \) we attach a free \( R \)-module \( V(H) \) and two \( R \)-module homomorphisms \( f^H : V(H) \to V \) and \( f_H : V \to V(H) \), such that
\[
f_H \circ f^H = |H| \cdot \text{Id}_{V(H)}
\]
and 
\[ f^H \circ f_H = \sum_{h \in H} h. \]

Moreover, there is a map \( \phi : \text{End}(V(H)) \to \text{End}(V) \), such that \( \phi(\text{Id}_{V(H)}) = \epsilon_H \) and 
\[ \text{tr}(\epsilon_H) = \text{rank}_RV(H). \]

Under the above assumptions, if \( \sum_{H \in S} n_H \epsilon_H = 0 \) then \( \sum_{H \in S} n_H \text{rank}_RV(H) = 0. \)

**Proof.** We apply lemma (3.2) for \( \epsilon \) not divisible by \( p \) and Proposition 3.4. Let \( V \) be Galois with Galois group \( H \). For every Galois extension \( L/L^H \), with Galois group \( H \), we have 
\[ \text{tr}(\hat{\rho}_p(\epsilon_H)) = \lambda_H. \]

For all \( p \) that do not divide the order of \( H \) we have 
\[ \text{tr}(\hat{\rho}_p(\epsilon_H)) = \lambda_{H,p}. \]
Proof. This result is clear for $H = \{\text{Id}\}$. By the above two propositions it is also clear for the general $H$, since

$$\text{tr}(\tilde{\rho}_p(\varepsilon_H)) = \text{tr}(\phi(\text{Id}_{T_p(J_K)})) = \lambda_H$$

and

$$\text{tr}(\tilde{\rho}_p(\varepsilon_H)) = \text{tr}(\phi_2(\text{Id}_{V_p(C_LK)})) = \lambda_{H,p,p}.$$  

□

We will now study the action of the Galois group on the group $\mu(L), \mu(L^H)$ of units contained in the fields $L, L^H$ respectively.

**Proposition 3.7.** Let $L/L^G$ be a Galois extension with Galois group $G$. Let $w_L$ be the order of the group $\mu(L)$ of units of finite order. For a fixed prime $p \mid w_L$ we define the set $S_p$ of subgroups $H$ of $G$ with the property

$$(\forall H \in S_p) \text{ } p \nmid |H|.$$  

Let $\nu(H,p)$ be the valuation at $p$ of the order $w_H$ of the unit group of $L^H$. Every linear relation of the form

$$\sum_{H \in S_p} n_H \varepsilon_H = 0$$

among norm idempotents of groups $H \in S_p$, implies the same relation

$$\sum_{H \in S_p} n_H \nu(H,p) = 0.$$  

Proof. Consider the norm

$$N_{L/L^H} : \mu(L) \to \mu(L^H),$$

and the inclusion function $i_{L^H,L} : \mu(L^H) \to \mu(L)$. We have

$$N_{L/L^H} \circ i_{L^H,L} = |H| \cdot \text{Id}_{\mu(L^H)}$$

and

$$i_{L^H,L} \circ N_{L/L^H} = \sum_{h \in H} h.$$  

The unit group $\mu(L)$ is a cyclic group of order $m$, and $\mu(L^H)$ is a subgroup of the cyclic group $\mu(L)$. The group $\mu(L)$ can be considered as a direct sum

$$\mu(L) = \bigoplus_{i=1}^r \frac{\mathbb{Z}}{p_i^{\nu(1,p_i)}} = \bigoplus_{i=1}^r \mu_i(L),$$

where $p_i$ are the different prime divisors of $m$. Each direct summand $\mu_i(L)$, gives rise to a $\mathbb{Z}(p_i)$-module $\mu_i(L) \otimes \mathbb{Z}(p_i)$. We can have a similar decomposition of $\mu(L^H)$ as a direct sum of $\mathbb{Z}(p_i)$-modules:

$$\mu(L^H) = \bigoplus_{i=1}^r \frac{\mathbb{Z}}{p_i^{\nu(H,p_i)}} = \bigoplus_{i=1}^r \mu_i(L^H),$$

The $N_{L/L^H}$ and $i_{L^H,L}$ group homomorphism give rise to $\mathbb{Z}(p_i)$-module homomorphisms from $\mu_i(L) \otimes \mathbb{Z}(p_i)$ to $\mu_i(L^H) \otimes \mathbb{Z}(p_i)$. The desired result follows by lemma □

4. Analytic methods

In this section we give an analytic proof of proposition 3.2. Consider the zeta function of an algebraic number field $L$,

$$\zeta_L(s) := \sum_{A \in \mathcal{I}_L} \frac{1}{N(A)^s}, \quad Re(s) > 1,$$
where $A$ runs over the integral ideals $I_L$ of the ring of integers of $L$. It is known that $\zeta_L(s)$ admits a meromorphic extension in $\mathbb{C} \setminus \{1\}$ with only one pole at $s = 1$. Moreover the residue at $s = 1$ can be computed \[5\text{ Satz VII 5.11}\]

\[
\text{Res}_{s=1} \zeta_L(s) = \lim_{s \to 1^+} (s - 1) \zeta_L(s) = \frac{2^r (2\pi)^s \text{Reg}(L)}{|\mu(L)| \sqrt{|D_L|}} h_L.
\]

The Arakelov genus $g_L$ of the number field $L$ is defined by

\[
g_L = \log \frac{|\mu(L)| \sqrt{|D_L|}}{2^r (2\pi)^s},
\]

therefore

\[
\text{Res}_{s=1} \zeta_L(s) = e^{-g_L \text{Reg}(L)} h_L.
\]

Let $\sum n_H \varepsilon_H$ be a norm idempotent relation. The product formula \[11\] implies that

\[
\lim_{s \to 1^+} \sum n_H \log((s - 1) \zeta_{L,H}(s)) = \sum n_H \lim_{s \to 1^+} \log(s - 1).
\]

The left hand side is finite (the regulator of every number field is not zero), therefore $\sum n_H = 0$ and moreover

\[
\sum n_H (-g_{L,H} + \log(\text{Reg}(L^H) h_{L,H})) = 0 \Rightarrow
\]

\[
\sum n_H g_{L,H} = \sum n_H \log(\text{Reg}(L^H) h_{L,H}).
\]

**Remark:** The relation $\sum n_H = 0$ can also be proved by applying the character of the trivial representation on the sum $\sum n_H \varepsilon_H$.

On the other hand using the analytic continuation of the $\zeta_L(s)$ we can prove that

\[
\lim_{s \to 0} \frac{\zeta_L(s)}{s^{r+s-1}} = \frac{h_L \text{Reg}(L^H)}{|\mu(L)|},
\]

therefore

\[
\sum n_H \log(h_{L,H} \text{Reg}(L^H)) = \sum n_H \log |\mu(L^H)|.
\]

Combining \[11\] and \[11\], we arrive at

\[
\sum n_H g_{L,H} = \sum n_H \log(\mu(L^H)).
\]

For every $H$, such that $|H|, |\mu(L)| = 1$, we write $|\mu(H)| = \prod_{i=1}^r p_i^{\nu_{H,p_i}}$. Therefore the right hand side of \[11\] is written

\[
\sum_H n_H \log(|\mu(L^H)|) = \sum_H n_H \log(\prod_{i=1}^r p_i^{\nu_{H,p_i}}) = \sum_{i=1}^r \log(p_i) \sum_H n_H \nu_{H,p_i} = 0
\]

by proposition \[11\] and the proof of proposition \[11\] is now complete.
5. The $\eta$ invariant

In order to apply the proof given in previous sections we would like to realize the function
\[ \sum_{x \in \mathcal{O}_L} e^{-\pi ||x||_L^2} \]
as the trace of a suitable linear operator.

If $L$ is a number field we will denote by $r_L$ the number of real embeddings and by $s_L$ the number of complex nonequivalent embeddings. The Minkowski space is defined by
\[ M(L) = \mathbb{R}^{r_L} \times \mathbb{C}^{s_L}. \]

We will define the set of real infinite primes of $L$ by $\mathbb{P}(L, \mathbb{R})$ and by $\mathbb{P}(L, \mathbb{C})$ the set of complex infinite primes. The field $L$ can be embedded on the Minkowski space by the map
\[ i_L : L \to M(L), \]
x $\mapsto (\sigma_1(x), \ldots, \sigma_{r_L}(x), \sigma_{r_L+1}(x), \ldots, \sigma_{r_L+s_L}(x))$.

Every divisor
\[ D = \sum_{\sigma \in \mathbb{P}(L, \mathbb{R})} a_\sigma \sigma + \sum_{\sigma \in \mathbb{P}(L, \mathbb{C})} a_\sigma \sigma, \]
supported on the set of infinite primes gives rise to the metric
\[ ||x||^2_{L,D} = \sum_{\sigma \in \mathbb{P}(L, \mathbb{R})} |x_{\sigma}|^2 e^{-2a_\sigma} + \sum_{\sigma \in \mathbb{P}(L, \mathbb{C})} |x_{\sigma}|^2 e^{-a_\sigma}, \]
where $x = (x_\sigma)$ is an element of the Minkowski space $M(L)$.

Let $L/K$ be a Galois extension of number fields with Galois group $Gal(L/K) = H$. An infinite complex prime $\sigma$ of $K$ is extended to $|H|$ infinite primes of $L$. Moreover,
\[ \sigma = \sum_{i=1}^{\frac{|H|}{2}} \sigma_i \]
on the other hand an infinite real prime $\tau$ of $K$ gives rise to $a(\tau)$ real infinite primes $\{\sigma_1, \ldots, \sigma_a\}$ of $L$ and $b(\tau)$ pairs $\{\sigma_{a(\tau)+1}, \ldots, \sigma_{a(\tau)+b(\tau)}\}$ complex infinite primes of $L$, where $a(\tau) + 2b(\tau) = |H|$. So the real infinite prime $\sigma$ is decomposed in $L$ as follows:
\[ \sigma = \sum_{i=1}^{a(\tau)} \sigma_i + \sum_{j=1}^{\frac{b(\tau)}{2}} \sigma_{a(\tau)+j}^2. \]

Lemma 5.1. Let $L/K$ be a Galois extension of number fields, with Galois group $H$. Consider the set $\mathbb{P}(L, \infty)$ (resp. $\mathbb{P}(K, \infty)$) of infinite primes of $L$ (resp. $K$) and let $r_L$, $s_L$ (resp. $r_K$, $s_K$) denote the number of real and complex embeddings of $K$ (resp. $L$). Let $D$ be a divisor supported at the infinite primes of $L$, such that $D$ is $H$-invariant i.e.,
\[ D = \sum_{\sigma \in \mathbb{P}(L, \infty)} a_\sigma \sigma = \sum_{\tau \in \mathbb{P}(K, \infty)} a_{\tau} \sum_{\sigma | \tau} \sigma. \]
Let us denote by \( D^H \) the divisor

\[
D^H = \sum_{\tau \in \mathbb{P}(K,\infty)} a_\tau \tau.
\]

If \( || \cdot ||_{L,D} \) is the metric on the Minkowski space \( \mathbb{R}^{r_L} \times \mathbb{C}^{s_L} \) introduced by \( D \) and \( || \cdot ||_{K,D^H} \) is the metric on the Minkowski space \( \mathbb{R}^{r_K} \times \mathbb{C}^{s_K} \) introduced by \( D^H \), then for every \( x \in K \subset L \) considered as an element on the spaces \( \mathbb{R}^{r_L} \times \mathbb{C}^{s_L} \) and \( \mathbb{R}^{r_K} \times \mathbb{C}^{s_K} \) we have:

\[
(10) \quad ||i_L(x)||_{L,D}^2 = ||H||i_K(x)||_{K,D^H}^2.
\]

Proof. Let \( x \in K \subset L \). We compute:

\[
||i_K(x)||_{K,D^H}^2 = \sum_{\tau \in \mathbb{P}(K,\mathbb{R})} |\tau(x)|^2 e^{-2a_\tau} + \sum_{\tau \in \mathbb{P}(K,\mathbb{C})} |\tau(x)|^2 2e^{-a_\tau}.
\]

If \( \tau \in \mathbb{P}(K,\mathbb{R}) \) then the contribution to the norm of the infinite primes \( a_\tau \sum_{\sigma|\tau} \sigma \) above \( \tau \) is according to (9):

\[
|\tau(x)|^2 \left( a(\tau)e^{-2a_\tau} + b(\tau)2e^{-2a_\tau} \right) = |\tau(x)|^2 |H|e^{-2a_\tau}.
\]

On the other hand if \( \tau \) is a complex prime, then the contribution to the norm of the infinite primes above \( \tau \) is according to (8)

\[
|\tau(x)|^2 |H|2e^{-a_\tau}.
\]

The desired result follows by adding all the contributions of primes of \( L \) above each infinite prime \( \tau \) of \( K \).

To every number field \( L \) we attach the Hilbert space \( V_L \) consisting of functions \( f : \mathcal{O}_L \to \mathbb{R} \), such that \( \sum_{y \in \mathcal{O}_L} |f(y)|^2 < \infty \).

Let \( H \) be a group acting on the number field \( L \). The space \( V_L \) is acted on by \( H \) as follows:

\[
f^h(x) = f(hx), \quad \text{for } h \in H.
\]

Let \( V_{L,H} \) be the Hilbert space of functions \( f : \mathcal{O}_{L,H} \to \mathbb{R} \) such that \( \sum_{y \in \mathcal{O}_{L,H}} |f(y)|^2 < \infty \).

The norm idempotent \( \epsilon_H \) induces a map \( \epsilon^*_H : V_{L,H} \to V_L \), sending the function \( f : \mathcal{O}_{L,H} \to \mathbb{R} \) to the function \( f \circ \epsilon_H : \mathcal{O}_L \to \mathbb{R} \). Moreover we will consider the restriction map \( \text{rest} : V_{\mathcal{O}_L} \to V_{\mathcal{O}_{L,H}} \) sending a function \( f : \mathcal{O}_L \to \mathbb{R} \), to the restriction on \( \mathcal{O}_{L,H} \).

Since the vector spaces we treat are of infinite dimension we can not use the trace of the identity map. Instead, we consider the diagonal linear operator

\[
T_D : V_{\mathcal{O}_L} \to V_{\mathcal{O}_L},
\]

sending a function

\[
\mathcal{O}_L \ni x \mapsto f(x) \quad \text{to} \quad \mathcal{O}_L \ni x \mapsto T_D f(x) = e^{-\pi ||x||_{L,D}^2} f(x).
\]

Let \( \delta_x(\cdot) \) denote the basis functions

\[
\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}
\]

We observe that the trace of the linear operator \( T \circ \epsilon \) is given by

\[
(11) \quad \text{tr}(T \circ \epsilon_H) = \sum_{x \in \mathcal{O}_L} \langle T \circ \epsilon^*_H(\delta_x), \delta_x \rangle = \sum_{x \in \mathcal{O}_{L,H}} e^{-\pi ||x||_{L,D}^2}.
\]
Indeed, if \( x \in \mathcal{O}_L \) is an element of \( \mathcal{O}_L \mathcal{H} \), then \( \epsilon_H(x) = x \), and if \( x \in \mathcal{O}_L \setminus \mathcal{O}_L \mathcal{H} \) then \( \epsilon_H(x) \neq x \) since \( \epsilon_H(x) \in \mathcal{O}_L \mathcal{H} \). We compute

\[
\langle \epsilon_H^*(\delta_x), \delta_x \rangle = \begin{cases} 
1 & \text{if } x \in \mathcal{O}_L \mathcal{H} \\
0 & \text{if } x \in \mathcal{O}_L \setminus \mathcal{O}_L \mathcal{H} \end{cases},
\]

and the formula (10) follows. Suppose now that \( D \) is an \( H \)-invariant divisor. Then equation (10) together with (11) gives that

\[
\text{tr}(T \circ \epsilon_H) = \sum_{x \in \mathcal{O}_L \mathcal{H}} e^{-\pi |x|^2_{K,D\mathcal{H}|H}},
\]

Proposition 5.2. Given a number field \( K \) and a divisor \( A \) supported on infinite primes, define the numbers

\[
\eta_A(K) = \sum_{x \in \mathcal{O}_K} e^{-\pi |x|^2_{K,A}}.
\]

If \( \sum_H \lambda_H \epsilon_H \) is a linear relation among norm idempotents, and \( D \) is an \( H \)-invariant divisor supported on infinite primes of \( L \) then the following relation holds

\[
0 = \sum_H \lambda_H \eta_{D+B(H)}(L^H),
\]

where

\[
B(H) = -\frac{\log(|H|)}{2\pi} \sum_{\sigma \in \mathfrak{F}(L^H,\mathbb{R})} \sigma - \frac{\log(|H|/2)}{\pi} \sum_{\sigma \in \mathfrak{F}(L^H,\mathbb{C})} \sigma.
\]

In particular if \( D = 0 \) then

\[
0 = \sum_H \lambda_H \eta_{B(H)}(L^H).
\]

Proof. The desired result follows by linearity of the trace map composed by \( T \), the relation \( \sum_H \lambda_H \epsilon_H \) and equation (12). \( \square \)

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