The Resultant of an Unmixed
Bivariate System

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Abstract

This paper gives an explicit method for computing the resultant of any sparse unmixed bivariate system with given support. We construct square matrices whose determinant is exactly the resultant. The matrices constructed are of hybrid Sylvester and Bézout type. The results extend those in [14] by giving a complete combinatorial description of the matrix. Previous work by D’Andrea [5] gave pure Sylvester type matrices (in any dimension). In the bivariate case, D’Andrea and Emiris [7] constructed hybrid matrices with one Bézout row. These matrices are only guaranteed to have determinant some multiple of the resultant. The main contribution of this paper is the addition of new Bézout terms allowing us to achieve exact formulas. We make use of the exterior algebra techniques of Eisenbud, Fløystad, and Schreyer [10, 9].

1. Introduction

Let \( f_1, \ldots, f_{n+1} \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) be Laurent polynomials in \( n \) variables with the same Newton polytope \( Q \subset \mathbb{R}^n \). Let \( A = Q \cap \mathbb{Z}^n \). So we can write:

\[
    f_i = \sum_{\alpha \in A} C_{i\alpha} x^\alpha.
\]

We will assume that \( Q \) is actually \( n \)-dimensional, and furthermore that \( A \) affinely spans \( \mathbb{Z}^n \).

Definition: The \( A \)-resultant \( \text{Res}_A(f_1, \ldots, f_{n+1}) \) is the irreducible polynomial in the \( C_{i\alpha} \), unique up to sign, which vanishes whenever \( f_1, \ldots, f_{n+1} \) have a common root in the algebraic torus \((\mathbb{C}^*)^n\).

The existence, uniqueness, and irreducibility of the \( A \)-resultant are proved in the book by Gelfand, Kapranov, and Zelevinsky [12]. The \( A \)-resultant, also

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called the sparse resultant, allows one to eliminate \( n \) variables from \( n+1 \) unmixed equations. Hence, resultants can be quite useful in solving systems of polynomial equations \([4]\). It is an important problem to find efficiently computable, explicit formulas for the resultant.

When \( n = 1 \), we are in the case of the classical resultant of two polynomials in one variable of the same degree. There are two formulas due to Sylvester and Bézout which represent the resultant as the determinant of an easily computable matrix. Sylvester’s matrix has entries that are either 0 or a coefficient of \( f_1 \) or \( f_2 \). The entries in Bézout’s matrix are linear in the coefficients of each of the \( f_i \), hence quadratic overall.

Our work deals with the case \( n = 2 \). We give a determinantal formula which is of hybrid Sylvester and Bézout type. A preliminary version of these results appeared in the ISSAC 2002 Proceedings \([4]\). This paper makes the formula completely explicit and provides complete proofs. Our approach follows work by Jouanolou \([13]\) and Dickenstein and D’Andrea \([6]\) who found formulas for the “dense” resultant, when the polytope \( Q \) is a coordinate simplex of some degree. We make heavy use of new techniques by Eisenbud, Floystad and Schreyer \([10, 9]\) relating resultants to complexes over an exterior algebra.

**Theorem 1.1:** The resultant of a system \((f_1, f_2, f_3) \in \mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}] \) with common Newton polygon \( Q \) is the determinant of the block matrix:

\[
\begin{pmatrix}
B & L \\
\tilde{L} & 0
\end{pmatrix}.
\]

The entries of \( L \) and \( \tilde{L} \) are linear forms, and the entries of \( B \) are cubic forms in the coefficients \( C_{iα} \).

The columns of \( B \) and \( \tilde{L} \) are indexed by the lattice points in \( Q \), the rows of \( B \) and \( L \) are indexed by the interior lattice points in \( 2 \cdot Q \), the matrix \( \tilde{L} \) has three rows indexed by \( \{f_1, f_2, f_3\} \), and the columns of the matrix \( L \) are indexed by pairs \((f_1, a)\) where \( i \in \{1, 2, 3\} \) and \( a \) runs over the interior lattice points of \( Q \). Each entry of \( L \) and \( \tilde{L} \) is either zero or is a coefficient of some \( f_i \) and is determined in the following straightforward manner. The entry of \( \tilde{L} \) in row \( f_i \) and column \( a \) is the coefficient of \( x^a \) in \( f_i \). The entry of \( L \) in row \( b \) and column \((f_i, a)\) is the coefficient of \( x^{b-a} \) in \( f_i \). The entries of the matrix \( B \) are linear forms in bracket variables. A bracket variable is defined as

\[
[abc] = \det \begin{bmatrix}
C_{1a} & C_{1b} & C_{1c} \\
C_{2a} & C_{2b} & C_{2c} \\
C_{3a} & C_{3b} & C_{3c}
\end{bmatrix},
\]

where \( C_{iα} \) is the coefficient of \( x^α \) in \( f_i \). An explicit formula for \( B \) is described in Section 3 below.
Example 1.2:

\[ f_1 = C_{11} + C_{12}x + C_{13}y + C_{14}xy + C_{15}x^2y + C_{16}xy^2 \]
\[ f_2 = C_{21} + C_{22}x + C_{23}y + C_{24}xy + C_{25}x^2y + C_{26}xy^2 \]
\[ f_3 = C_{31} + C_{32}x + C_{33}y + C_{34}xy + C_{35}x^2y + C_{36}xy^2 \]

The system above has Newton polygon as shown in Figure 1. We will show that the resultant of this system is the determinant of the matrix in Table 1.

| 0  | [124] | 0  | [126] − [234] | −[235] | −[236] | c_{11} | c_{21} | c_{31} |
|----|-------|----|---------------|-------|-------|--------|--------|--------|
| 0  | 0     | 0  | 0             | 0     | 0     | c_{12} | c_{22} | c_{32} |
| 0  | [126] − [135] | 0  | [146] − [236] | [156] + [345] | [346] | c_{13} | c_{23} | c_{33} |
| 0  | −[145] | 0  | [156] − [345] | [256] | [356] | c_{14} | c_{24} | c_{34} |
| 0  | 0     | 0  | 0             | 0     | 0     | c_{15} | c_{25} | c_{35} |
| 0  | [156] | 0  | [356]        | [456] | 0     | c_{16} | c_{26} | c_{36} |
| c_{11} | c_{12} | c_{13} | c_{14} | c_{15} | c_{16} | 0     | 0     | 0     |
| c_{21} | c_{22} | c_{23} | c_{24} | c_{25} | c_{26} | 0     | 0     | 0     |
| c_{31} | c_{32} | c_{33} | c_{34} | c_{35} | c_{36} | 0     | 0     | 0     |

Table 1: Resultant matrix for Example 1.2

In Section 2 we provide some preliminary results about toric varieties and their homogeneous coordinates which allow us to present our formula in Section 3. Section 4 describes the exterior algebra techniques of Eisenbud, Schreyer, and Fløystad. Section 5 applies these results to the toric setting, while Section 6 goes on to prove our formula. Finally Section 7 briefly discusses possible generalizations to more variables.
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2. Toric Varieties

Definition: Let $Q \subset \mathbb{R}^n$ be a lattice polytope of dimension $n$, and $A = Q \cap \mathbb{Z}^n = \{\alpha_1, \ldots, \alpha_N\}$. We assume that $A$ affinely spans $\mathbb{Z}^n$. The toric variety $X_A$ is the dimension $n$ variety defined as the Zariski closure of the following set in $\mathbb{P}^{N-1}$:

$$X_A = \{ (x^{\alpha_1}, \ldots, x^{\alpha_N}) : x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \}.$$

Now a polynomial system $(f_1, \ldots, f_{n+1})$ can be thought of as $n+1$ hyperplane sections of $X_A$ in $\mathbb{P}^{N-1}$. Generically, such a system defines a codimension $n+1$ plane.

For any $n$-dimensional irreducible projective variety $X$, it turns out that the condition on a linear subspace of codimension $n+1$ meeting $X$ is actually a closed condition of codimension 1 (see [12] for details). Therefore we can make the following definition.

Definition: If $X \subset \mathbb{P}^{N-1}$ is a variety of dimension $n$, the codimension $n+1$ planes meeting $X$ define a hypersurface in the Grassmannian $G(n+1, N)$. The equation of this hypersurface is called the Chow form of $X$.

In particular, the $A$-resultant is the Chow form of $X_A$. As a consequence we have the following strengthening of Definition 1.1.

Corollary 2.1: $\text{Res}_A(f_1, \ldots, f_{n+1}) = 0$ if and only if the $f_i$ have a common root on $X_A$.

Returning to the defining polytope $Q$, let $d_1, \ldots, d_s$ denote the facets of $Q$. Let $\eta_i$ be the first lattice vector along the inner normal to facet $d_i$. The normal fan of $Q$ is the set of cones, one for each vertex, spanned by the $\eta_i$ corresponding to facets incident to that vertex. The next proposition can be found in Fulton’s book [11].

Proposition 2.1: The $\eta_i$ are in 1-1 correspondence with the $T$-invariant prime Weil divisors on $X_A$. Let $D_i$ denote the divisor corresponding to $\eta_i$.

The polytope $Q$ can be characterized completely in terms of the rays in its normal fan as follows:

$$Q = \{ m \in \mathbb{R}^n : \langle m, \eta_i \rangle \geq -a_i, \ i = 1, \ldots, s \}.$$
The very ample divisor corresponding to the embedding of $X_A$ into $\mathbb{P}^{n-1}$ corresponding to $Q$ is just $D = \sum a_i D_i$. We can now define the homogeneous coordinate ring of $X_A$. This was introduced by Cox \cite{Cox} and the propositions below follow from this paper.

Let $S = \mathbb{C}[y_1, \ldots, y_s]$ be the polynomial ring with one variable for each $\eta_i$. Consider the short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\phi} \mathbb{Z}^s \xrightarrow{\pi} G \longrightarrow 0.$$ 

Here $\phi(m) = (\langle m, \eta_1 \rangle, \ldots, \langle m, \eta_s \rangle)$, and $G$ is the cokernel of $\phi$.

**Definition:** Define a $G$-grading on $S$ as follows. Given $y^\alpha \in S$, let $\deg(y^\alpha) = \pi(\alpha) \in G$.

Now we will identify the lattice points in $Q$ with a graded piece of $S$.

**Definition:** Let $\alpha \in Q \cap \mathbb{Z}^n$. Define $\alpha_i = \langle \alpha, \eta_i \rangle + a_i$ for $i = 1, \ldots, s$ and the $a_i$ are the defining data for $Q$ as above. The $Q$-homogenization of $x^\alpha$ is $\prod_{i=1}^s y_i^{a_i}$. We will write this as $y^\alpha$ and use the letter $\alpha$ to denote both a vector $\alpha \in \mathbb{Z}^n$ and its homogenization $(\alpha_1, \ldots, \alpha_s)$, where the meaning will be clear from the context.

**Proposition 2.2:** Let $a = (a_1, \ldots, a_s)$ be the defining data for $Q$. The monomials in the $\pi(a)$ graded piece of $S$ are in 1-1 correspondence with the lattice points in $Q$. Denote this graded piece by $S_Q$. Moreover, $H^0(X_A, \mathcal{O}(D)) \cong S_Q$.

There is a similar characterization of the interior lattice points of $Q$.

**Proposition 2.3:** Let $\omega_0 = (1, 1, \ldots, 1) \in \mathbb{Z}^s$. The monomials in the $\pi(a - \omega_0)$ graded piece of $S$ are in 1-1 correspondence with the interior lattice points of $Q$. Denote this graded piece $S_{\text{int}(Q)}$. We have $H^0(X, \mathcal{O}(D - \sum_{i=1}^s D_i)) \cong S_{\text{int}(Q)}$.

### 3. Formula for $B$.

We now return to case of two variables. So $(f_1, f_2, f_3) \in \mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}]$ have common Newton polygon $Q \subset \mathbb{R}^2$. The rays in the normal fan of $Q$ are $\{\eta_1, \ldots, \eta_s\}$, assumed to be in counterclockwise order. We pick out the distinguished cone spanned by $\{\eta_1, \eta_2\}$ and partition the vectors in the fan as follows:

$$R_1 = \{ i \mid \eta_i = c_1 \eta_1 + c_2 \eta_2 \text{ with } c_1 \geq 0 \text{ and } c_2 \leq 0 \}$$

$$R_2 = \{ i \mid \eta_i = c_1 \eta_1 + c_2 \eta_2 \text{ with } c_1 \leq 0 \text{ and } c_2 \geq 0 \}$$

$$R_3 = \{ i \mid \eta_i = c_1 \eta_1 + c_2 \eta_2 \text{ with } c_1 < 0 \text{ and } c_2 < 0 \}. \quad (1)$$

It is possible that $R_3$ as defined is empty. If that is the case we need to refine the fan, by adding in one new vector, say $\eta_{s+1} = -\eta_1 - \eta_2$. This new vector $\eta_{s+1}$
lies in the interior of some cone spanned by \( \eta_i \) and \( \eta_j \), hence can be written as 
\[ c_1 \eta_i + c_2 \eta_j \] 
for some positive \( c_1, c_2 \). Define \( a_{s+1} = c_1 a_i + c_2 a_j \). As above, given \( \alpha \in Q \) we denote by \( \alpha_{s+1} \) the quantity \( \langle \alpha, \eta_{s+1} \rangle + a_{s+1} \).

In fact, if there is a single fan vector \( \eta_i \) such that \( -\eta_i \) is not a ray in the fan, then we can choose our distinguished cone to be the one containing \( -\eta_i \), and \( R_3 \) is guaranteed not to be empty. However, for polytopes such that every edge has a corresponding parallel edge, this is not the case.

A good way to think about these sets is that we choose a distinguished vertex \( p \) of \( Q \) having normal cone spanned by \( \{ \eta_1, \eta_2 \} \). The set \( R_3 \) consists of all edges of \( Q \) such that the corresponding inner normals are maximized at \( v \). If there is no such edge, then our refinement adds in a “length 0” edge whose inner normal is maximized at \( p \). \( R_1 \) is the set of the remaining edges clockwise from \( v \), while \( R_2 \) is the set of remaining edges counterclockwise from \( v \).

This partition is illustrated in Figure 2 for Example 1.2 with the choice of the vertex \( p \). Edge 4 has the only normal maximized at \( p \), thus is the only element in \( R_3 \). The edges in \( R_1 \) and \( R_2 \) are \{1, 5\} and \{2, 3\} respectively.

We can now state an explicit formula for the matrix \( B \) appearing in the Theorem 1.1.

**Theorem 3.1:** The matrix \( B \) from Theorem 1.1 is the matrix of the linear map 
\( \Delta_Q : (S_Q)^* \to S_{\text{int}(2Q)} \) defined as follows:

\[
\Delta_Q((y^\alpha)^*) = \sum_{(u,v,w) \in F_\alpha \subset A^3} [uvw] y^{u+v+w-\alpha-\omega_0}.
\]

Here \( \omega_0 = (1, 1, \ldots, 1) \), and \( F_\alpha \) is the set of all triples \((u, v, w) \in A^3\) satisfying
the following Boolean combination of inequalities:

\[
\forall i \in R_1 \quad u_i + v_i + w_i > \alpha_i \\
\exists i \in R_1 \quad v_i + w_i \leq \alpha_i \\
\forall j \in R_2 \quad v_j + w_j > \alpha_j \\
\exists j \in R_2 \quad w_j \leq \alpha_j \\
\forall k \in R_3 \quad w_k > \alpha_k,
\]

where the \( R_i \) are as described in \( (1) \).

**Example 3.2:** Let’s see how this works for Example \( [12] \). Specifically, consider the point \( \alpha = (1, 1) \) corresponding to the monomial \( xy \). The homogenization is \( y_1y_2y_3y_4y_5 \). If the monomials are numbered \( 1, \ldots, 6 \) as in the equations, then the only solutions to the inequalities above are:

\[
(u, v, w) = \{(2, 6, 1), (4, 6, 1), (5, 6, 1), (2, 4, 3), (5, 4, 3), (2, 6, 3), (5, 6, 3)\}.
\]

It follows that

\[
\Delta_Q((y_1y_2y_3y_4y_5)^x) = ([261] + [243])y_3y_4^3y_5 + ([461] + [263])y_2y_3^3y_4^2 \\
+ ([561] + [543])y_1y_2y_3y_4y_5 + [563]y_1y_2y_3^2,
\]

which corresponds to the fourth column of the matrix in Table \( [4] \).

### 4. Tate Resolution

In this section we describe a complex used by Eisenbud and Schreyer \([10, 9]\) to compute Chow forms of projective varieties. This begins as a complex of free modules over an exterior algebra, however there is a functor which transforms it into a complex of vector bundles on the Grassmannian. The determinant of this new complex will be the Chow form.

Suppose \( X \subset \mathbb{P}^{N-1} \) is an irreducible variety of dimension \( n \). The ambient projective space \( \mathbb{P} = \mathbb{P}^{N-1} \) has the graded coordinate ring \( R = \mathbb{C}[X_1, \ldots, X_N] \). If we let \( W \) be the \( \mathbb{C} \) vector space spanned by the \( X_i \), identified with the degree 1 part of \( R \), then \( \mathbb{P} \) is the projectivization \( \mathbb{P}(W) \). The ring \( R \) can also be identified with the symmetric algebra \( \text{Sym}(W) \).

Now let \( V = W^* \), the dual vector space, with a corresponding dual basis \( e_1, \ldots, e_N \). We will consider the exterior algebra \( E = \wedge V \), which is also graded where the \( e_i \) have degree \( -1 \). We will use the standard notation \( E(k) \) to refer the rank 1 free \( E \)-module generated in degree \( -k \).

Now given any coherent sheaf \( \mathcal{F} \) on \( \mathbb{P} \), there is an associated exact complex of graded free \( E \)-modules, called the Tate resolution, denoted \( T(\mathcal{F}) \). The terms of \( T(\mathcal{F}) \) can be written in terms of the sheaf cohomology of twists of \( \mathcal{F} \). Namely, we have:

\[
T^e(\mathcal{F}) = \bigoplus_{j=0}^{N-1} [H^j(\mathcal{F}(e - j)) \otimes_{\mathbb{C}} E(j - e)]
\]

(3)
for all $e \in \mathbb{Z}$. See Eisenbud-Floystad-Schreyer [9].

Suppose further that $\mathcal{F}$ is chosen to be supported on $X$. Recall that the Chow form of $X$ is the defining equation of the set of codimension $n+1$-planes meeting $X$. Such a plane is specified by an $n+1$ dimensional subspace $W_f = \mathbb{C}\{f_1, \ldots, f_{n+1}\} \subset W$. Let $G_{n+1}$ be the Grassmannian of codimension $n+1$-planes on $P$. Let $\mathcal{T}$ be the tautological bundle on $G_{n+1}$, that is to say the fiber at the point corresponding to $f$ is just $W_f$.

The following proposition is a consequence of Theorem 0.1 in [9].

**Proposition 4.1:** There is an additive functor $U_{n+1}$ from graded free modules over $E$ to vector bundles on $G_{n+1}$, such that $U_{n+1}(E(p)) = \wedge^p \mathcal{T}$. Furthermore, if $\mathcal{F}$ is a sheaf of rank $k$ supported on a variety $X \subset \mathbb{P}(V)$ of dimension $n$, $U_{n+1}(T(\mathcal{F}))$ is a complex of vector bundles whose determinant is the $k$-th power of the Chow form of $X$.

The determinant of a complex of vector bundles on $G_{n+1}$ is a homogeneous polynomial function on $G_{n+1}$ whose value at a particular point is the corresponding determinant of the complex of vector spaces over that point. The determinant of a complex of vector spaces is defined in [12, Appendix A].

So, in particular if we could choose $\mathcal{F}$ so that enough cohomology vanishes, this new complex $U_{n+1}(T(\mathcal{F}))$ may have only two terms and a single non trivial map $\Psi_{\mathcal{F}}$. Such sheaves are called weakly Ulrich, see [10, Section 2]. In this case, to compute the Chow form we need only compute the determinant of $\Psi_{\mathcal{F}}$. This is exactly what we do in the next section. But first we need to describe the maps in the Tate resolution, and also how the functor $U_{n+1}$ acts.

The maps in the Tate resolution are composed of maps $H^j(\mathcal{F}(e-j)) \otimes E(j-e) \to H^k(\mathcal{F}(e+1-k)) \otimes E(k-e-1)$. All such maps for $k > j$ must be 0 by degree considerations. When $k = j$ we have a linear map $H^j(\mathcal{F}(e-j)) \otimes E(j-e) \to H^j(\mathcal{F}(e+1-j)) \otimes E(j-e-1)$ which is canonical and completely well understood. Explicitly we consider the graded $R$-module $M^j = \oplus_{l>0} H^j(\mathcal{F}(l))$. The Bernstein-Gel’fand-Gel’fand correspondence [9, Section 2] applied to $M^j$ results in a map $M^j_{e-j} \otimes E(j-e) \to M^j_{e-j+1} \otimes E(j-e-1)$ which is just multiplication by the element $m = \sum X_i \otimes e_i$. By [9, Theorem 4.1] these are exactly the linear maps in the Tate Resolution.

Much more mysterious are the nonlinear diagonal maps corresponding to $k < j$. Indeed one of the major contributions of this paper is an explicit formula for one of these diagonal maps in the case of a toric surface. Eisenbud and Schreyer [10] outline a general procedure for computing the Tate resolution, and therefore the diagonal maps, however it requires computing a free resolution and is not an explicit formulation.

Before moving on to the toric setting let us complete the description of the functor $U_{n+1}$ by describing how it acts on morphisms. The functoriality and other useful properties of the construction below are in Proposition 1.1 of [10].
Given a map $E(q) \to E(q - p)$ we need to construct a map $\bigwedge^q T \to \bigwedge^{q-p} T$. Any map $E(q) \to E(q - p)$ is defined by a single element $a \in \bigwedge^p V$. This also defines a map $\bigwedge^p W \to \mathbb{C}$. As $T$ is a subbundle of $W \otimes O_{G_{n+1}}$, there is an induced map $a : \bigwedge^p T \to O_{G_{n+1}}$. Finally, to construct the map $U_{n+1}(a) : \bigwedge^q T \to \bigwedge^{q-p} T \otimes \bigwedge^p T$ and compose with the map $1 \otimes a$.

We will need to use a more explicit description of the map, in terms of our chosen bases. Recall that a fiber of $T$ is a subspace $W_f = \mathbb{C}\{f_1, \ldots, f_{n+1}\}$. We can write the $f_i$ as:

$$f_i = \sum_{j=1}^N C_{ij} X_j.$$ 

The coefficients form a $(n + 1) \times N$ matrix $C$. Given ordered subsets $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n+1\}$ and $J = \{j_1, \ldots, j_p\} \subset \{1, \ldots, N\}$, of the same size $p$, let $C_{I,J}$ denote the determinant of the submatrix of $C$ with rows from $I$ and columns from $J$. We will also use the notation $f_I = (-1)^q \bigwedge_{i \in I} f_i$ and $e_J = \bigwedge_{j \in J} e_j$. Note the sign factor added to the $f$ part only in order to simplify the signs in the next proposition:

**Lemma 4.1:** Let $J \subset \{1, \ldots, N\}$ with $|J| = p$. We view $e_J$ as a map from $E(q)$ to $E(q - p)$. In that case for any $I \subset \{1, \ldots, n+1\}$ with $|I| = q$:

$$(U_{n+1}(e_J))(f_I) = \sum_{I_1 \subset I, \ |I_1| = p} C_{I_1,J} f_{I_1 \backslash I_1}.$$ 

**Proof:** This is a direct translation of the above description applied to our particular choice of bases. The diagonal map splits up $f_I$ into a sum of pieces corresponding to a choice of $I_1$ and its complement. The action of $e_J$ on the piece corresponding to $I_1$ is exactly the determinant of the specified minor. The only thing to check is that the sign works out. 

\[ \Box \]

## 5. Toric Tate Resolution

We return to the case in question, where $X_A$ is a toric surface with corresponding polytope $Q$. As we saw earlier, the sections of the corresponding very ample divisor are just the elements of the vector space $S_Q$. Therefore, we will apply the exterior algebra construction with $W = S_Q$ and $V = S_Q^*$. The corresponding projective space is $P = \mathbb{P}(W) \cong \mathbb{P}^{N-1}$, and the exterior algebra is $E = \bigwedge V$.

Any Weil divisor on the toric surface $X_A$ yields a rank one reflexive sheaf which can be extended to a sheaf on $P$ under the given embedding. We will consider the particular divisor corresponding to $\text{int}(2Q)$ i.e. $2D - \sum D_i$. Let $F$ be the corresponding sheaf $O_{X_A}(\text{int}(2Q))$ extended to a sheaf of $P$. 

Proposition 5.1:

\[ H^0(\mathcal{F}(k)) \cong S_{\text{int}(2+k)Q} \]  
\[ H^1(\mathcal{F}(k)) \cong 0 \]  
\[ H^2(\mathcal{F}(k)) \cong S_{(-2-k)Q}^{\ast} \]

for all \( k \in \mathbb{Z} \).

Proof: First of all, since all sheaves are supported on \( X_A \), it is equivalent to compute cohomology on \( X_A \). By construction, \( X_A \) is normal and thus Cohen Macaulay by Hochster’s theorem. The dualizing sheaf is \( \mathcal{O}(\omega) = \mathcal{O}(-\sum D_i) \).

Also, twisting by 1 on \( \mathcal{P} \) is the same as twisting by \( D \) on \( X_A \). Therefore, \( \mathcal{F}(k) = \mathcal{O}((k+2)D - \omega) \).

Now (4) follows from Proposition 2.3. For \( k > -2 \), \( \mathcal{F}(k) \) is an ample divisor minus the canonical divisor. Therefore, the higher cohomology, \( H^1 \) and \( H^2 \) must be zero by Mustata’s vanishing result, [15, Theorem 2.4 (ii)].

Furthermore, \( \mathcal{O}(D) \) is very ample, hence locally free, so Serre duality tells us \( H^i(\mathcal{O}((k+2)D - \omega)) \cong H^{2-i}(\mathcal{O}((-2-k)D))^\ast \). In particular, applying Proposition 2.2 to \( i = 2 \) gives us statement (6) in the proposition. For \( k \leq -2 \), \( \mathcal{O}((-2-k)D) \) is generated by its sections and so all higher cohomology, in particular \( H^1 \) vanishes, completing the proof of (5). \( \square \)

Corollary 5.1: The Tate resolution \( T(\mathcal{F}) \) has terms:

\[ T^e(\mathcal{F}) = S_{-eQ}^{\ast} \otimes E(2-e) \quad \text{for } e < -1 \]
\[ T^{-1}(\mathcal{F}) = S_{Q}^{\ast} \otimes E(3) \oplus S_{\text{int}(Q)} \otimes E(1) \]
\[ T^0(\mathcal{F}) = S_{0}^{\ast} \otimes E(2) \oplus S_{\text{int}(2Q)} \otimes E(0) \]
\[ T^e(\mathcal{F}) = S_{\text{int}(eQ)} \otimes E(-e) \quad \text{for } e > 0, \]

with maps as follows:

\[ \cdots \rightarrow i_m^! (S_{2Q})^\ast \otimes E(4) \rightarrow i_m^! (S_{Q})^\ast \otimes E(3) \rightarrow i_m^! (S_0)^\ast \otimes E(2) \rightarrow 0 \]
\[ \oplus \Delta_{2Q} \oplus \Delta_{Q} \oplus \Delta_0 \oplus \Delta_m \oplus \]
\[ 0 \rightarrow S_{\text{int}(Q)} \otimes E(1) \rightarrow^m S_{\text{int}(2Q)} \otimes E \rightarrow^m S_{\text{int}(3Q)} \otimes E(-1) \rightarrow^m \cdots \]

The horizontal maps \( \wedge m \) and \( i_m \) are all multiplication by the element \( m = \sum y^a \otimes e_\alpha \) where \( \alpha \) ranges over the lattice points in \( Q \), and \( e_\alpha \) is the corresponding dual vector in \( E \).

Proof: We simply plug in our known cohomology from 5.1 into (3) to obtain the terms. The horizontal maps are indeed multiplication by \( m \), as per our discussion in the previous section, noting only that the Serre duality respects the \( S \)-module structure of the cohomology. \( \square \)
Now we apply the functor $U_3$ to $T(F)$. Once again let $\mathcal{T}$ denote the tautological bundle on the Grassmannian of codimension 3 planes in $\mathbb{P}^{N-1}$. Note that $\bigwedge^p \mathcal{T} = 0$ for $p > 3$ or $p < 0$. Therefore $U_3(T(F))$ is the two term complex below:

$$
\begin{array}{c}
\bigwedge^3 \mathcal{T} \xrightarrow{i_m} (S_Q)^* \\
\bigwedge^2 \mathcal{T} \xrightarrow{\Delta_Q} \bigwedge^1 \mathcal{T} \\
\bigwedge^0 \mathcal{T}
\end{array}
$$

Since $\mathcal{F}$ is of rank 1, the resultant is up to a constant the determinant of the matrix of the nontrivial map $(\hat{i}_m + \hat{\Delta}_Q) \oplus \hat{\lambda}m$. However, we can of course normalize the maps in the Tate resolution so that we have the resultant up to sign. From here on we assume that such a normalization has been made.

All that is left to do is describe the maps $\hat{\lambda}m$, $\hat{\Delta}_Q$, and $\hat{i}_m$. It is enough to define these maps on each fiber, that is, for each choice of $(f_1, f_2, f_3)$.

To describe the maps $\hat{\lambda}m$ and $\hat{i}_m$, we introduce the Sylvester map $\Psi_t : S_t \otimes \mathbb{C}^3 \to S_{t+Q}$ which sends $(g_1, g_2, g_3)$ to $f_1g_1 + f_2g_2 + f_3g_3$.

**Proposition 5.2:** The map $\hat{\lambda}m$ is $\Psi_{\text{int}(Q)}$, and the map $\hat{i}_m$ is $(\Psi_0)^*$ on each fiber over the Grassmannian.

**Proof:** First consider $\lambda m$. We pass to $\bigwedge^1 \mathcal{T}$, which has a basis at each fiber indexed by $f_1, f_2, f_3$. By Lemma 4.1, on the factor corresponding to $f_i$ we must replace each $e_\alpha$ in $m$ by the corresponding coefficient $C_{i\alpha}$. So on the factor corresponding to $f_i$, multiplication by $m = \sum_{\alpha \in A} y^\alpha \otimes e_\alpha$ becomes multiplication by $\sum_{\alpha \in A} C_{i\alpha} y^\alpha = f_i$. This is exactly the Sylvester map.

On the other hand $i_m$ is the map sending $(y^\alpha)^*$ to $e_\alpha$. To apply the functor $U_3$ we pick the basis $(f_1 \wedge f_2 \wedge f_3)$ on $\bigwedge^3 \mathcal{T}$ and $(f_2 \wedge f_3, -f_1 \wedge f_3, f_1 \wedge f_2)$ on $\bigwedge^2 \mathcal{T}$. Another application of Lemma 4.1 shows that $e_\alpha$ is replaced by the vector $(C_{1\alpha}, C_{2\alpha}, C_{3\alpha})$ in terms of this second basis. This is exactly the dual Sylvester map $(\Psi_0)^*$. \qed

Computing $\hat{\Delta}_Q$ from $\Delta_Q$ is straightforward.

**Proposition 5.3:** Write

$$
\Delta_Q((y^\alpha)^*) = \sum_\beta \sum_{u,v,w} c_{uvw}(e_u \wedge e_v \wedge e_w)y^\beta;
$$

then for each fiber $(f_1, f_2, f_3)$ on the Grassmannian:

$$
\hat{\Delta}_Q((y^\alpha)^*) = \sum_\beta \sum_{u,v,w} c_{uvw}[uvw]y^\beta.
$$
Proof: Here, both $\bigwedge^3 T$ and $\bigwedge^0 T$ are 1 dimensional vector spaces. Lemma 4.1 tells us to replace $e_u \wedge e_v \wedge e_w$ by the determinant of the maximal minor with columns $u, v, w$ of the coefficient matrix of the $f_i$, i.e the bracket $[uvw]$. $\square$

Putting it all together we have a proof of Theorem 1.1.

Proof (Proof of Theorem 1.1): The Chow form is the determinant, up to sign, of the map $(\hat{\Delta} + \Delta Q) \oplus \wedge m$. However, the blocks of the matrix corresponding to $\wedge m$ and $i_m$ are just Sylvester maps, by Proposition 5.2, whose matrices are $L$ and $\tilde{L}$ respectively. The matrix of $\hat{\Delta} Q$ has entries which are linear forms in the bracket variables by Proposition 5.3 above. $\square$

As a corollary we note that the matrix must be square. That is, $3 + \#\text{int}(2Q) = 3 \cdot \#\text{int}(Q) + \#Q$. This identity also arises from the simple fact that the third difference of the quadratic Erhart polynomial of $Q$ is 0.

All that is left is to prove our formula for $\hat{\Delta} Q$ in Theorem 3.3, for which, by the above, we need to prove the corresponding formula for $\Delta Q$. It turns out that it is easy to compute $\Delta_0$, and we can verify a formula for $\Delta Q$ by making sure it lifts $\Delta_0$. This is described below.

6. The Map $\Delta Q$

The map $\Delta_0$ is closely related to the toric Jacobian $\sigma$. The toric Jacobian is usually constructed as the determinant of a matrix of partial derivatives. Cattani, Cox, and Dickenstein construct a different element, which they call $\Delta_\sigma$, referring to the choice $\sigma$ of a cone in the fan, which is a constant times the Jacobian modulo the ideal $I = (f_1, f_2, f_3)$. Moreover, while the Jacobian of three forms supported on $Q$ has toric residue equal to the normalized area of $Q$, this new element has residue 1. Therefore, we will call this the normalized Jacobian and it is unique modulo $I$.

Let $y_1, y_2$ be edge variables such that the corresponding edges meet at a vertex $p$. Let $y_3, \ldots, y_s$ be the remaining edge variables of the homogeneous coordinate ring $S$. A monomial $m$ in $S_Q$ is divisible by $y_i$ if and only if the corresponding lattice point in $Q$ is not on the corresponding edge.

Therefore, we can define a partition of the monomials in $S_Q$ into three sets $\mu_1, \mu_2, \mu_3$, where $\mu_1$ is defined to be the set of all monomials divisible by $y_1$, $\mu_2$ is the set of monomials divisible by $y_2$ but not divisible by $y_1$, and $\mu_3$ divisible by $y_3 \cdot \ldots \cdot y_s$ but not by either $y_1$ or $y_2$.

Note that $\mu_1$ corresponds to points not on edge 1, $\mu_2$ is the points on edge 1, but not edge 2, and $\mu_3$ is the unique point, the vertex $p$, on both edges 1 and 2.

Proposition 6.1: Set $M_i = \sum_{s \in \mu_i} s \otimes e_s \in S_Q \otimes E$. Define $J_0 = \frac{M_1 \wedge y_1}{y_2} \wedge \frac{M_3}{y_3 \cdot \ldots \cdot y_s}$, an element of $S_{\text{int}(3Q)} \otimes E_{-3}$. A choice for the map $\Delta_0 : (S_0)^* \otimes E(2) \rightarrow S_{\text{int}(3Q)} \otimes E(-1)$ is $1 \otimes 1 \mapsto -J_0$. 

The element $J_0$ is chosen so that $U_3(J_0)$ is the normalized toric Jacobian as constructed in [1].

**Proof:** First note that the map $\Delta_0$ is determined by the image of $1 \otimes 1 \in (S_0)^* \otimes E_0$. By abuse of notation we denote $\Delta_0(1 \otimes 1)$ by just $\Delta_0$. By exactness, $\Delta_0$ is in the kernel of $\wedge^m$ and not in the image of the previous map $\wedge^{m-1}$. Furthermore, $\Delta_0$ is unique with respect to this property, up to a constant and modulo the image of $\wedge^m$. Thus we need to check that our choice $J_0$ is also in the kernel of the horizontal map $\wedge^m$, but not in the image of the previous map $\wedge^{m-1}$. Finally, we argue that if we choose the constant $-1$, the determinant of the complex will be exactly the resultant (up to sign).

To start with we notice $m = M_1 + M_2 + M_3$, and so $J_0 \wedge e_\alpha = M_1 y_1 \wedge M_2 y_2 \wedge M_3 y_3 \wedge (M_1 + M_2 + M_3) = 0$. So $J_0$ is indeed in the kernel of $\wedge^m$.

To show that $J_0$ is not in the image of the previous map, we twist the whole Tate resolution by 1, so that the map $\Delta_0$ goes from $(S_0)^* \otimes E(3)$ to $S_{\text{int}(3Q)} \otimes E$, and then apply the functor $U_3$. This also gives a complex whose determinant is the resultant (Theorem 0.1, in [10]), in particular it is exact when the resultant is non-zero. In this situation the image of the lower map is just the int $(3Q)$ graded piece of the ideal $I = (f_1, f_2, f_3)$, and the normalized toric Jacobian is known to be a nonzero element modulo this ideal (see [1, 3]). Therefore, $J_0$, which specializes to the Jacobian, cannot be in the image of the map $\wedge^m$.

Finally, the specialized complex above, with the normalized toric Jacobian as the diagonal map, appears in [7] where the authors show that the determinant of the complex is exactly the resultant up to sign. Therefore, the map $1 \otimes 1 \mapsto -J_0$ above is a valid choice, up to sign, for the map $\Delta_0$ in Theorem 5.1. \hfill $\Box$

Now let’s take the degree $-3$ part of the Tate resolution to get:

$$
\begin{array}{cccccc}
\longrightarrow & \overset{\Delta_Q}{\longrightarrow} & \overset{\Delta_0}{\longrightarrow} & \overset{\Delta_0}{\longrightarrow} & \overset{\Delta_0}{\longrightarrow} & \overset{\Delta_0}{\longrightarrow} \\
0 & (S_Q)^* \otimes \Lambda^m \otimes \Lambda^m \otimes \Lambda^m \otimes \Lambda^m \otimes \Lambda^m & \longrightarrow & 0 \\
0 & S_{\text{int}(Q)} \otimes \Lambda^2 V \otimes S_{\text{int}(2Q)} \otimes \Lambda^3 V \otimes S_{\text{int}(3Q)} \otimes \Lambda^4 V. & \longrightarrow & 0
\end{array}
$$

Let $\{n_\alpha\}$ be the basis of $(S_Q)^*$ dual to the monomial basis $\{y^\alpha\}$ of $S_Q$. The map on the top row sends $n_\alpha$ to $e_\alpha$. Because these maps form a complex we have the relation $\Delta_Q(n_\alpha) \wedge m = -\Delta_0(e_\alpha) = J_0 \wedge e_\alpha$.

The map $\Delta_Q$ is not canonically defined, even after picking $\Delta_0$. In fact the next proposition shows that any map satisfying the above relation will do.

**Proposition 6.2:** Define $\Delta_Q(n_\alpha)$ to be any element $d_\alpha$, homogeneous of degree $-3$, such that $d_\alpha \wedge m = J_0 \wedge e_\alpha$. This defines a valid choice for $\Delta_Q$.

**Proof:** The map $i_m$ in the top row sending $n_\alpha$ to $e_\alpha$ for each $\alpha \in Q$ is clearly injective (in fact an isomorphism of vector spaces). We will use this to show that the bottom row is exact at the term $S_{\text{int}(2Q)} \otimes \Lambda^2 V$. So pick an element $k$
in the kernel of $\wedge m : S_{\text{int}(2Q)} \otimes \wedge^3 V \to S_{\text{int}(3Q)} \otimes \wedge^4 V$. Now $(0, k)$ is in the kernel of the whole complex. Therefore, by exactness there exists an element $(a, b) \in (S_Q)^* \oplus (S_{\text{int}(Q)} \otimes \wedge^2 V)$ mapping on to it. But now $i_m(a) = 0$, therefore $a = 0$. So $b \wedge m = k$ as desired.

Now suppose the Tate resolution is fixed with $\Delta_0$ defined as in Proposition 6.1. Let $\tilde{\Delta}_Q$ be any map satisfying the above relation. Therefore, for any $\alpha$, $\Delta_Q(n_\alpha) \wedge m = -\Delta_0(e_\alpha) = \tilde{\Delta}_Q(n_\alpha) \wedge m$. So, $\Delta_Q$ and $\tilde{\Delta}_Q$ differ by an element of the kernel of $\wedge m$. By the argument in the previous paragraph, this is the same as differing by an element of the image of the previous $\wedge m$. Therefore, replacing $\Delta_Q$ by $\tilde{\Delta}_Q$ does not change exactness at this step of the Tate resolution. As the Tate resolution is a minimal free resolution, this new choice can always be extended ad infinitum, and so $\tilde{\Delta}_Q$ is itself a valid map. □

So we need only find for every lattice point $\alpha$ in $Q$, an element $d_\alpha$ such that $d_\alpha \wedge m = J_0 \wedge e_\alpha$. In [14] it was shown how to reduce this to a problem in linear algebra. In this paper, we show instead that the explicit, combinatorial formula from Theorem 3.1 does the trick. We restate Theorem 3.1 below using the language of exterior algebras developed above. Recall the definitions of the sets $R_i$ from [1]. The fan has possibly been refined as described earlier to guarantee that $R_3$ is non-empty.

**Theorem 6.1:** The map $\Delta_Q : (S_Q)^* \otimes E \to S_{\text{int}(2Q)} \otimes E(-3)$ can be defined as follows:

$$\Delta_Q(n_\alpha) = \sum_{(u,v,w) \in F_\alpha \subseteq A^3} y^{u+u+w-\omega_0} \otimes e_u \wedge e_v \wedge e_w.$$

Here $\omega_0 = (1, 1, \ldots, 1)$, and $F_\alpha$ is the set of all triples $(u, v, w) \in A^3$ satisfying the Boolean combination of inequalities in (3).

The next lemma will rewrite $J_0 \wedge e_\alpha$ in a form more convenient for our purposes.

**Lemma 6.1:**

$$J_0 \wedge e_\alpha = \sum_{t,u,v,w} y^{t+u+v+w-\omega_0} \otimes e_u \wedge e_v \wedge e_w \wedge e_t,$$

where $t, u, v, w$ satisfy:

1. $\forall i \in R_1 \quad t_i + u_i + v_i + w_i > \alpha_i$ (7)
2. $\exists i \in R_1 \quad t_i + v_i + w_i \leq \alpha_i$ (8)
3. $\forall j \in R_2 \quad t_j + v_j + w_j > \alpha_j$ (9)
4. $\exists j \in R_2 \quad t_j + w_j \leq \alpha_j$ (10)
5. $\forall k \in R_3 \quad t_k + w_k > \alpha_k$ (11)
6. $\exists k \in R_3 \quad t_k \leq \alpha_k$. (12)
Proof: First note that if $\exists k \in R_3$ such that $w_k \leq \alpha_k$, then both $e_u \wedge e_v \wedge e_w \wedge e_t$ and $e_u \wedge e_v \wedge e_t \wedge e_w$, with the same power of $y$, appear in the sum and cancel out. So condition (I) can be replaced by the stronger condition

$$\forall k \in R_3 \quad w_k > \alpha_k. \quad (**)$$

We will show that every term in $J_0 \wedge e_\alpha$ satisfies these conditions, and conversely every tuple $(t, u, v, w)$ satisfying the conditions corresponds to a term in $\Delta_0 \wedge e_\alpha$.

The element $J_0$ can be rewritten as $y^{u+v+w-\omega_0} \otimes \sum e_u \wedge e_v \wedge e_w$ where $u_1 > 0$, $v_1 = 0$ but $v_2 > 0$, and $w_1 = w_2 = 0$. Wedge this with $e_\alpha$, and we show that the terms $e_u \wedge e_v \wedge e_w \wedge e_\alpha$ all appear on the right hand side. So choose $t = \alpha$ then $t_1 + v_1 + w_1 = \alpha_1$, $t_2 + w_2 = \alpha_2$ and $t_k = \alpha_k$ for all $k$, thus conditions (8), (10), and (**) are satisfied. On the other hand, $w_i > 0$ for all $i \neq 1, 2$. This, combined with $v_2 > 0$ implies condition (9), while $u_1 > 0$ implies condition (7). Now, the set $R_3$ is constructed so that $w$, the vertex where edges 1 and 2 meet, satisfies condition (**) for all $\alpha$ except when $\alpha = w$, in which case $J_0 \wedge e_\alpha = 0$. Thus all the terms in $J_0 \wedge e_\alpha$ appear in the desired sum.

Conversely, pick any tuple $(t, u, v, w)$ satisfying (7), (8), (9), (10), (12) , and the modified (**). Define $\gamma = \alpha - t$. So, in our notation, $\alpha_i - t_i = \langle \eta_i, \gamma \rangle$.

By conditions (8), (10), (12) there exists $i_0, j_0, k_0$ in $R_1, R_2, R_3$ respectively such that $\langle \eta_i, \gamma \rangle \geq 0$, $\langle \eta_j, \gamma \rangle \geq 0$ and $\langle \eta_k, \gamma \rangle \geq 0$. Since the region $R_3$ is between $R_1$ and $R_2$, we must either have $\eta_k$ a positive linear combination of $\eta_{i_0}$ and $\eta_{j_0}$, or $\langle \eta_k, \gamma \rangle = \langle \eta_{j_0}, \gamma \rangle = 0$.

However, we also have $w_{i_0} \leq \alpha_{i_0}$ and $w_{j_0} \leq \alpha_{j_0}$, but $w_{k_0} > \alpha_{k_0}$, which rules out the first case. Thus $t_{i_0} = \alpha_{i_0}$ and $t_{j_0} = \alpha_{j_0}$. By conditions (8) and (**) we must have $w_{i_0} = w_{j_0} = 0$. This is possible only if the facets corresponding to $\eta_{i_0}$ and $\eta_{j_0}$ meet at a vertex. The only vertex where the sets $R_1$ and $R_2$ meet is the vertex $p$ when $w_1 = w_2 = 0$. But now, $\gamma$ must be 0, since $\eta_1$ and $\eta_2$ are linearly independent. Thus $t = \alpha$. So, by condition (8), $v_1 = 0$, by condition (9) $v_2 > 0$, and by condition (7), $u_1 > 0$. Hence, every term in the right hand sum also appears in $J_0 \wedge e_\alpha$. 

Proof (Proof of Theorem (6.1))): We must show that if $\Delta_Q$ is defined as above, then $\Delta_Q(n_\alpha) \wedge m = J_0 \wedge e_\alpha$. The left hand side is the sum

$$\sum_{(u,v,w,t)} y^{u+v+w+t-\alpha-\omega_0} \otimes e_u \wedge e_v \wedge e_w \wedge e_t,$$

where $(u, v, w)$ satisfy (**) and $t$ is unconstrained.

On the other hand, by Lemma (5.4), the right hand side is

$$\sum_{t,u,v,w} y^{t+u+v+w-\alpha-\omega_0} \otimes e_u \wedge e_v \wedge e_w \wedge e_t,$$

where $(u, v, w, t)$ satisfy the inequalities (7)-(12).
So, it is enough to show for any fixed 4 tuple \((u, v, w, t)\) the sum of all signed permutations satisfying (2), is equal to the sum of all signed permutations satisfying (3)-(12).

We consider the poset corresponding to the power set of \(P = \{u, v, w, t\}\). This is a four-dimensional cube whose vertices are the 16 subsets of \(P\), and two subsets \(p\) and \(q\) are connected by a directed edge from \(p\) to \(q\) if \(p\) is the union of \(q\) with a single element of \(P\). A maximum oriented path (of length 5) in this poset corresponds to a permutation of \((u, v, w, t)\). Given a permutation \((u, v, w, t)\), the path starts at \(\emptyset\), has first vertex \(\{t\}\), second vertex \(\{w, t\}\) and so on. Define the sign of this path to be the sign of the corresponding permutation. We will consider formal sums of signed paths, remembering that if the same path occurs twice in the sum with opposite signs, then the contribution from that path is 0.

Let \(A_i\) be a condition on a vertex \(p\) which evaluates to true if \(\sum_{v \in p} v_k > \alpha_k\) holds for all indices \(k\) in \(R_i\). Note that if \(p\) satisfies \(A_i\) and \(q \supset p\) then \(q\) satisfies \(A_i\). Label a vertex \(B_i\) if it satisfies condition \(A_i, \ldots, A_3\) but fails to satisfy conditions \(A_1, \ldots, A_{i-1}\). With this notation the permutations \((u, v, w, t)\) satisfying (2) are oriented paths through the cube labeled \((B_1, B_3, B_2, B_1)\). The permutations, this time ordered \((t, u, v, w)\), satisfying (3)-(12) are paths of the form \((B_4, B_4, B_3, B_2, B_1)\). Note that this introduces a sign of \((-1)^3\) into our formula.

So, to complete the proof it is enough to show the following lemma that was proved by David Speyer in a personal communication.

**Lemma 6.2:** The sum of oriented paths in the cube of the form \((B_4, \ldots, B_i, B_i, B_{i-1}, \ldots, B_1)\) is \((-1)^{i-1}\) times the sum of paths of the form \((B_4, B_3, B_2, B_1, B_1)\).

In particular when \(i = 4\) we have our desired result.

**Proof:** By induction it is enough to show that the sum of paths of the form \((B_4, \ldots, B_i, B_i, B_{i-1}, \ldots, B_1)\) is negative the sum of paths of the form \((B_4, \ldots, B_{i-1}, B_i-1, \ldots, B_1)\). Let \(S_1\) denote the first sum and \(S_2\) the second.

For the moment, consider any two vertices \(p\) and \(q\) of the cube, labeled \(B_i\) and \(B_{i-1}\) respectively, joined by an oriented path of length 2. There are exactly two such paths passing through intermediate vertices \(a\) and \(b\) respectively. As \(a\) contains \(p\) and is contained in \(q\), by the definition of the labels \(a\) satisfies \(A_i, \ldots, A_3\) but fails to satisfy \(A_1, \ldots, A_{i-2}\). If \(a\) obeys \(A_{i-1}\) then it has label \(B_{i-1}\), otherwise it has label \(B_i\). The case for \(b\) is identical.

Returning to the claim consider two disjoint paths of vertices \(v_1, \ldots, v_i\) and \(v_{i-1}, \ldots v_1\) where \(v_j\) has label \(B_j\) and it is possible to join these paths by adding a single vertex between them. As above, there are two possibilities for this new vertex, \(a\) and \(b\), each of which has label \(B_i\) or \(B_{i-1}\). The permutations associated to the two ways of completing the path differ by a single exchange, hence have opposite signs. If \(a\) and \(b\) have the same label they cancel in the sum \(S_1\) or \(S_2\).
If they have opposite labels than one contributes positively to one of the sums, and the other contributes negatively to the other sum. Therefore, the two sums are negative of each other. □

7. Future Work

This paper is, in the author’s opinion, just the tip of the iceberg in the application of exterior algebra methods to sparse resultants. I am actively working on several more general results and have ideas on many more.

In this paper we investigated the sheaf $\mathcal{O}(\text{int}(2Q))$ on a toric surface. One of the important properties was the vanishing of all “middle” cohomology. Other sheaves also have this property and give rise to different formulas for the resultant of a surface. We can also consider sheaves that do have middle cohomology, although it seems more difficult to make the maps explicit. In the special case of products of projective spaces, this is hinted at in Section 6 of the paper by Dickenstein and Emiris [8].

It is of course of great interest to consider toric varieties of higher dimension, that is more than 3 equations. I know of a sheaf giving rise, via the Tate resolution, to a determinantal formula for the Chow form of any toric threefold. The sticking point is finding an explicit formula, analogous to Theorem 3.1. Hopefully, this will be worked out in a future publication.

For four dimensions or higher, it appears the best we can hope for is matrices whose determinant is a nontrivial multiple of the resultant. In this situation it should be possible to identify the extraneous factor with a minor of the matrix. See [8, 5].

An important generalization would be to mixed resultants, i.e. equations with different supports. Tate resolutions do not obviously apply, but there may be an appropriate extension.

Finally, returning to the specific formula presented here, there are several places where choice is involved. An interesting question would be to classify all possible formulas, for all the different choices. Another issue is to investigate the efficiency, both in theory and for an implementation. It may be possible to speed up the computation of the Bézout map $\Delta_Q$.

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