Abstract

We develop a family of techniques to align word embeddings which are derived from different source datasets or created using different mechanisms (e.g., GloVe or word2vec). Our methods are simple and have a closed form to optimally rotate, translate, and scale to minimize root mean squared errors or maximize the average cosine similarity between two embeddings of the same vocabulary into the same dimensional space. Our methods extend approaches known as Absolute Orientation, which are popular for aligning objects in three-dimensions, and generalize an approach by Smith et al (ICLR 2017). We prove new results for optimal scaling and for maximizing cosine similarity. Then we demonstrate how to evaluate the similarity of embeddings from different sources or mechanisms, and that certain properties like synonyms and analogies are preserved across the embeddings and can be enhanced by simply aligning and averaging ensembles of embeddings.

1 Introduction

Embedding complex data objects into a high-dimensional, but easy to work with, feature space has been a popular paradigm in data mining and machine learning for more than a decade [27, 28, 34, 38]. This has been especially prevalent recently as a tool to understand language, with the popularization through word2vec [23, 21] and GloVe [25]. These approaches take as input a large corpus of text, and map each word which appears in the text to a vector representation in a high-dimensional space (typically $d = 300$ dimensions).

These word vector representations began as attempts to estimate similarity between words based on the context of their nearby text, or to predict the likelihood of seeing words in the context of another. Other more powerful properties were discovered. Consider each word gets mapped to a vector $v_{\text{word}} \in \mathbb{R}^d$. Synonym similarity: Two synonyms (e.g., $v_{\text{car}}$ and $v_{\text{automobile}}$) tend to have small Euclidean distances and large inner products, and are often nearest neighbors. Linear relationships: For instance, the vector subtraction between countries and capitals (e.g., $v_{\text{Spain}} - v_{\text{Madrid}}$, $v_{\text{France}} - v_{\text{Paris}}$, $v_{\text{Germany}} - v_{\text{Berlin}}$) are similar. Similar vectors encode gender (e.g., $v_{\text{man}} - v_{\text{woman}}$), tense ($v_{\text{eat}} - v_{\text{ate}}$), and degree ($v_{\text{big}} - v_{\text{bigger}}$). Analogies: The above linear relationships could be transferred from one setting to another. For instance the gender vector $v_{\text{man}} - v_{\text{woman}}$(going from a female object to a male object) can be transferred to another more specific female object, say $v_{\text{queen}}$. Then the result of this vector operation is $v_{\text{queen}} + (v_{\text{man}} - v_{\text{woman}})$ is close to the vector $v_{\text{king}}$ for the word “king.” This provides a mechanism to answer analogy questions such as “woman:man::queen:?”. Classification: More classically [27, 28, 34, 35], one can build linear classifiers or regressors to quantify or identify properties like sentiment.

At least in the case of GloVe, these linear substructures are not accidental; the embedding aims to preserve inner product relationships. Moreover, these properties all enforce the idea that these embeddings are useful to think of inheriting a Euclidean structure, i.e., its safe to represent them in $\mathbb{R}^d$ and use Euclidean distance.

However, there is nothing extrinsic about any of these properties. A rotation or scaling of the entire dataset will not affect synonyms (nearest neighbors), linear substructures (dot products), analogies, or linear classifiers. A translation will not affect distance, analogies, or classifiers, but will affect inner products since it effectively changes the origin. These substructures (i.e., metric balls, vectors, halfspaces) can be transformed...
in unison with the embedded data. Indeed Euclidean distance is the only metric on $d$-dimensional vectors that is rotation invariant.

The intrinsic nature of these embeddings and their properties adds flexibility that can also be a hinderance. In particular, we can embed the same dataset into $\mathbb{R}^d$ using two approaches, and these structures cannot be used across datasets. Or two data sets can both be embedded into $\mathbb{R}^d$ by the same embedding mechanism, but again the substructures do not transfer over. That is, the same notions of similarity or linear substructures may live in both embeddings, but have different meaning with respect to the coordinates and geometry. This makes it difficult to compare approaches; the typical way is to just measure a series of accuracy scores, for instance in recovering synonyms [17, 23]. However, these single performance scores do not allow deeper structural comparisons.

Another issue is that it becomes challenging (or at least messier) to build ensembles structures or embeddings. For instance, some groups have built word vector embeddings for enormous datasets (e.g., GloVe embedding using 840 billion tokens from Common Crawl, or the word2vec embedding using 100 billion tokens of Google News), which costs at least tens of thousands of dollars in cloud processing time. Given several such embeddings, how can these be combined to build a new single better embedding without revisiting that processing expense? How can a new (say smaller or specialized) data set from a slightly different domain use a larger high-accuracy embedding?

**Our approach.** In this paper we provide a simple closed form method to optimally align two embeddings. These methods find optimal rotation (technically an orthogonal transformation) of one dataset onto another, and can also solve for the optimal scaling and translation. They are optimal in the sense that they minimize the sum of squared errors under the natural Euclidean distance between all pairs of common data points, or they can maximize the average cosine similarity.

The methods we consider are easy to implement, and are based on 3-dimensional shape alignment techniques common in robotics and computer vision called “absolute orientation.” We observe that these approaches extend to arbitrary dimensions $d$; the same solution for the optimal orthogonal transformation was also recently re-derived by Smith et al. [35].

In this paper, we also show that an approach to choose the optimal scaling of one dataset onto another [16] does not affect the optimal choice of rotation. Hence, the choice of translation, rotation, and scaling can all be derived with simple closed form operations.

We then apply these methods to align various types of word embeddings, providing new ways to compare, translate, and build ensembles of them. We start by aligning data sets to themselves with various types of understandable noise; this provides a method to calibrate the error scores reported in other settings. We also demonstrate how these aligned embeddings perform on various synonym and analogy tests, whereas without alignment the performance is very poor. The results with scaling, translation, and weighting all consistently improve upon the results for only translation as advocated by Smith et al. [35].

Moreover, we show that we can boost embeddings, showing improved resulted when aligning various embeddings, and taking simple averages of the embedded words from different data sets. The results from these boosted embeddings provide the best known results for various analogy and synonym tests. It also hints that many other domains may benefit from directly boosting their embeddings with word embeddings found from very large text corpuses.

**Other Applications.** We highlight a few other applications we may be served by this alignment, comparison, and boosting mechanisms that we have designed and demonstrate the effectiveness.

1. Common Crawl is one of the largest textual data sources available. It consistently gets updated to include the ever increasing data on the internet. Each of these datasets is over 800B tokened and extracting embeddings from these can be computationally expensive. However, extracting embeddings from the additional data not included in the previous update of Common Crawl should be significantly less expensive. Aligning an embedding from just the new data, and performing a weighted average with the older larger one may work as well or better than the embedding made from scratch.

2. This approach may also help with specialized data. Consider data from scientific journals only, or from a dump of bio medical terms. These embeddings would be very specific and each words would have
a specific word sense based on the domain. Orienting these along the gigantic corpus can enrich the specific domain related regions on the larger embedding.

3. Tags and phrases in English can be single words or a string of words. Orienting an embedding of tags/phrases along say Common Crawl using an intersection of the single words in the two datasets can help place multi-worded tags or phrases around words related to them. This can help derive meaning from random or unknown phrases. Images too come with a set of tags. So orienting a set of tags can help orient images meaningfully among words.

4. Using this method we can orient heterogenous embeddings derived from a variety of methods e.g. for graphs including node2vec [12] or DeepWalk [20], and others [5, 11, 7], images [13, 2], and for kernel methods [27, 28]. For instance, RDF data can contain shorthand query phrases like ‘president children spouse’ which answers the question ‘who are the spouses and children of presidents?’ Using point 3, we see how we can beneficially orient these along word embeddings from Common Crawl. All heterogenous networks have a mixture of node types. If there is an intersection of some nodes (and node types) between any two embeddings (heterogenous or homogenous), we can orient them using AbsoluteOrientationmeaningfully.

5. Customer data from different years and iterations of data collection are embedded using different features (such as income bracket, credit score, location etc depending on the company). Using common customers over the year, new users can be added meaningfully to the embedding and inferred about, without having to embed all of the data. Embedding the same users from different years and orienting them can also help deduce the change in their features over time.

In all, there are many applications where as data is updated, specialized, or diversified, the family of techniques we propose can be used to analyze, enhance, and combine the power of these embeddings.

2 Absolute Orientation and Relatives

In many classic computer vision and shape analysis problems, a common problem is the alignment of two (often 3-dimensional) shapes. The most clean form of this problem starts with two points sets $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, each of size $n$, where each $a_i$ corresponds with $b_i$ (for all $i \in 1, 2, \ldots, n$). Generically we can say each $a_i, b_i \in \mathbb{R}^d$ (without restricting $d$), but as mentioned the focus of this work was typically restricted to $d = 3$ or $d = 2$. Then the standard goal was to find a rigid transformation — a translation $t \in \mathbb{R}^d$ and rotation $R \in SO(d)$ – to minimize the root mean squared error (RMSE). An equivalent formulation is to solve for the sum of squared errors as

$$
(R^*, t^*) = \arg\min_{t \in \mathbb{R}^d, R \in SO(d)} \sum_{i=1}^{n} \|a_i - (b_i R + t)\|^2.
$$

For instance, this is one of the two critical steps in the well-known iterative closest point (ICP) algorithm [3, 6].

In the 1980s, several closed form solutions to this problem were discovered; their solutions were referred to as solving absolute orientation. The most famous approach by Horn [16] uses unit quaternions. However, this approach seems to have been known earlier [9], and other techniques using rotation matrices and the SVD [13, 1], rotation matrices and an eigen-decomposition [33, 32], and dual number quaternions [37], have also been discovered. In 2 or 3 dimensions, all of these approaches take linear (i.e., $O(n)$) time, and in practice have roughly the same run time [8].

In this document, we focus on the SVD-based approach of Hanson and Norris [13], since it is clear, has an easy analysis, and unlike the quaternion-based approaches which only work for $d = 3$, generalizes to any dimension $d$. This approach, described in Algorithm 2.1 decouples the translation from the rotation; they can be solved independently. In particular, this approach first finds the means $\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i$ and $\bar{b} = \frac{1}{n} \sum_{i=1}^{n} b_i$ of each data set. Then it creates centered versions of those data sets $\hat{A} \leftarrow (A, \bar{a})$ and $\hat{B} \leftarrow (B, \bar{b})$. Next we need to compute the RMSE-minimizing rotation (all rotations are then considered around the origin) on centered data sets $\hat{A}$ and $\hat{B}$. First compute the sum of outer products $H = \sum_{i=1}^{n} \hat{b}_i \hat{a}_i$, which is a $d \times d$ matrix. We emphasize $\hat{a}_i$ and $\hat{b}_i$ are row vectors, so this is an outer product, not an inner
product. Next take the singular value decomposition of $H$ so $[U, S, V^T] = \text{svd}(H)$, and the ultimate rotation is $R = UV^T$. We can create the rotated version of $B$ as $\hat{B} = BR$ so we rotate each point as $\hat{b}_i = \hat{b}_i R$.

Technically, this may allow $R$ to include mirror flips, in addition to rotations. These can be detected (if the last singular value is negative) and factored out by multiplying by a near-identity matrix $R = U_{L} V^T$ where $I_{L}$ is identity, except the last term is changed to $-1$. We ignore this issue in this paper, and henceforth consider orthogonal matrices $R \in O(d)$ (all $d \times d$ orthogonal matrices, which includes mirror flips) instead of just rotations $R \in SO(d)$ (the $d \times d$ special orthogonal group). For simpler nomenclature, we still refer to $R$ as a “rotation.”

**Algorithm 2.1 AbsoluteOrientation($A, B$) [13]**

1. Compute $\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i$ and $\bar{b} = \frac{1}{n} \sum_{i=1}^{n} b_i$
2. Center $\hat{A} \leftarrow (A, \bar{a})$ so each $\hat{a}_i = a_i - \bar{a}$, and similarly $\hat{B} \leftarrow (B, \bar{b})$
3. Compute the sum of outer products $H = \sum_{i=1}^{n} \hat{b}_i^T \hat{a}_i$
4. Decompose $[U, S, V^T] = \text{svd}(H)$
5. Build rotation $R = UV^T$
6. Rotate $\hat{B} = \hat{B}R$ so each $\hat{b}_i = \hat{b}_i R$
7. Translate $B^* \leftarrow (\hat{B}, -\bar{a})$ so each $b_i^* = \hat{b}_i + \bar{a}$
8. **return** $B^*$

A large focus of this paper will be evaluating the variant outlined in Algorithm 2.2 that only aligns with the optimal “rotation” over $O(d)$ as

$$R^* = \arg \min_{R \in O(d)} \sum_{i=1}^{n} ||a_i - b_i R||^2.$$ (2)

This restricted form of Hanson and Norris’s Absolute Orientation algorithm [13] was recently rediscovered by Smith et al. [24] for aligning word embeddings for different languages. We typically do not advocate the translation step for two reasons. First, this effectively changes the origin and hence the inner products. Second, we observe the effect of translation is usually small, and does not consistently improve performance.

**Algorithm 2.2 AO-Rotation($A, B$)**

1. Compute the sum of outer products $H = \sum_{i=1}^{n} \hat{b}_i^T a_i$
2. Decompose $[U, S, V^T] = \text{svd}(H)$
3. Build rotation $R = UV^T$
4. **return** $\hat{B} = BR$ so each $\hat{b}_i = b_i R$

We discuss here a few other variants of this algorithm which take into account translation and scaling between $A$ and $B$. Note that the rotation $R$ and translation $t = -\bar{b} + \bar{a}$ derived within this Algorithm 2.1 are not exactly the optimal $(R^*, t^*)$ desired in formulation (1). This is because the order these are applied, and the point that the data set is rotated around is different. In formulation (1) the rotation is about the origin, but the dataset is not centered there, as it is in Algorithm 2.2.

**Translations.** To compare with the use of also optimizing for the choice of translations in the transformation, we formally describe this procedure here. In particular, we can decouple rotations and translations, so to clarify the discrepancy between Algorithm 2.1 and equation (1), we use a modified version of the above procedure. In particular, we first center all data sets, $\hat{A} \leftarrow A$ and $\hat{B} \leftarrow B$, and henceforth can know that they are already aligned by the optimal translation. Then, once they are both centered, we can then call AO-Rotation($\hat{A}, \hat{B}$). This is written explicitly and self-contained in Algorithm 2.3.

**Scaling.** In some settings, it makes sense to align data sets by scaling one of them to fit better with the other, formulated as

$$(R^*, t^*, s^*) = \arg \min_{s \in \mathbb{R}, t \in \mathbb{R}^d, R \in O(d)} \sum_{i=1}^{n} ||a_i - s(b_i - t) R||^2.$$ (3)
The sketch for Absolute Orientation with scaling, is in Algorithm 2.4.

Proof. We show that the optimal rotation $B$ to change the order of multiplication operations of $A,B$, i.e. re-solving for the optimal rotation. Define $\hat{B} = s^* B$, so each $\hat{b}_i = s^* b_i$. Now to complete the proof, we show that the optimal rotation $\hat{R}$ derived from $A$ and $\hat{B}$ is the same as was derived from $A$ and $B$. In addition to the choices of translation and rotation, the optimal choice of scaling can also be decoupled.

Horn et al. [10] introduced two mechanisms for solving for a scaling that minimizes RMSE. Assuming the optimal rotation $R^*$ has already been applied to obtain $\tilde{B}$, then a closed form solution for scaling is

$$s^* = \frac{1}{n} \sum_{i=1}^{n} \langle \hat{a}_i, \hat{b}_i \rangle / \| \tilde{B} \|_F^2.$$ 

The sketch for Absolute Orientation with scaling, is in Algorithm 2.4.

Algorithm 2.4 AO-Scaling($A,B$)

\[
\tilde{B} \leftarrow \text{AO-Rotation}(A,B)
\]

Compute scaling $s = \sum_{i=1}^{n} \langle \hat{a}_i, \hat{b}_i \rangle / \| \tilde{B} \|_F^2$.

return $\tilde{B}$ as $\tilde{B} \leftarrow s \tilde{B}$ so for each $\tilde{b}_i = s \tilde{b}_i$.

The steps of rotation, scaling and translation fit together to give us Algorithm 2.5.

Algorithm 2.5 AO-Centered+Scaling($A,B$)

\[
\hat{a} = \frac{1}{n} \sum_{i=1}^{n} a_i, \quad \hat{\bar{b}} = \frac{1}{n} \sum_{i=1}^{n} b_i
\]

Center $A \leftarrow (A, \hat{a})$ so each $\hat{a}_i = a_i - \hat{a}$, and similarly $B \leftarrow (B, \hat{\bar{b}})$

Compute the sum of outer products $H = \sum_{i=1}^{n} \hat{b}_i^T \hat{a}_i$

Decompose $[U,S,V^T] = \text{svd}(H)$

Build rotation $R = UV^T$

Rotate $\tilde{B} = \tilde{B}R$ so each $\tilde{b}_i = \tilde{b}_i R$

Compute scaling $s = \sum_{i=1}^{n} \langle \hat{a}_i, \hat{b}_i \rangle / \| \tilde{B} \|_F^2$

Scale $\tilde{B}$ as $\tilde{B} \leftarrow s \tilde{B}$ so for each $\tilde{b}_i = s \tilde{b}_i$.

return $\tilde{A}, \tilde{B}$

Horn et al. [10] presented an alternative closed form choice of scaling $s$ which minimizes RMSE, but under a slightly different situation. In this alternate formulation, $A$ must be scaled by $1/s$ and $B$ by $s$, so the new scaling is somewhere in the (geometric) middle of that for $A$ and $B$. We found this formulation less intuitive, since the RMSE is dependent on the scale of the data, and in this setting the new scale is aligned with neither of the data sets. However, Horn et al. [16] only showed that the choice of optimal scaling is invariant from the rotation in the second (less intuitive) variant. We present a proof that this rotation invariance also holds for the first variant. The proof uses the structure of the SVD-based solution for optimal rotation, with which Horn et al. may not have been familiar.

Lemma 1. Consider two points sets $A$ and $B$ in $\mathbb{R}^d$. After the rotation and scaling in Algorithm 2.4, no further rotation about the origin of $\tilde{B}$ can reduce the RMSE.

Proof. We analyze the SVD-based approach we use to solve for the new optimal rotation. Since we can change the order of multiplication operations of $s_b R$, i.e. scale then rotate, we can consider first applying $s^*$ to $B$, and then re-solving for the optimal rotation. Define $\hat{B} = s^* B$, so each $\hat{b}_i = s^* b_i$. Now to complete the proof, we show that the optimal rotation $\hat{R}$ derived from $A$ and $\hat{B}$ is the same as was derived from $A$ and $B$.以下是对文档内容的自然语言阅读。
Computing the outer product sum $H = \sum_{i=1}^{n} b_i^T a_i = \sum_{i=1}^{n} (s^* b_i)^T a_i = s^* \sum_{i=1}^{n} b_i^T a_i = s^* H$, is just the old outer product sum $H$ scaled by $s^*$. Then its SVD is $\text{svd}(H) \rightarrow [U, S, V^T] = [U, s^* S, V^T]$, since all of the scaling is factored into the $S$ matrix. Then since the two orthogonal matrices $U = \tilde{U}$ and $V = \tilde{V}$ are unchanged, we have that the resulting rotation $R = \tilde{U} V^T = U V^T = R$ is also unchanged.

**Preserving Inner Products.** While Euclidean distance is a natural measure to preserve under a set of transformations, many word vector embeddings are evaluated or accessed by Euclidean inner product operations. It is natural to ask if our transformations also maximizes the sum of inner products of the aligned vectors. Or, does it minimize the sum of cosine similarity: the sum of inner products of these vectors more in the sum, and this leads to a more stable method. Hence, we observe that $\text{AO-Rotation}(A, B)$ results in a rotation $\tilde{R} = \arg\max_{R \in O(d)} \sum_{i=1}^{n} \langle a_i, b_i R \rangle$.

**Lemma 2.** $\text{AO-Rotation}(A, B)$ rotates $B$ to $\tilde{B}$ to maximize $\sum_{i=1}^{n} \langle a_i, \tilde{b}_i \rangle$. If $a_i \in A$ and $b_i \in B$ are normalized $\|a_i\| = \|b_i\| = 1$, then the rotation maximizes the sum of cosine similarities $\sum_{i=1}^{n} \langle \frac{a_i}{\|a_i\|}, \frac{b_i}{\|b_i\|} \rangle$.

**Proof.** From Hanson and Norris [13] we know $\text{AO-Rotation}(B)$ finds a rotation $R^*$ so

$$R^* = \arg\min_{R \in O(d)} \sum_{i=1}^{n} ||a_i - (b_i R)||^2.$$ 

Expanding this equation we find

$$R^* = \arg\min_{R \in O(d)} \left( \sum_{i=1}^{n} ||a_i||^2 - \sum_{i=1}^{n} 2\langle a_i, b_i R \rangle + \sum_{i=1}^{n} ||b_i R||^2 \right).$$

Since $||a_i||^2$ and $||b_i R||^2 = ||b_i||^2$ are properties of the dataset and do not depend on the choice of $R$ and as desired

$$R^* = \arg\max_{R \in O(d)} \sum_{i=1}^{n} \langle a_i, b_i R \rangle.$$ 

If all $a_i, b_i$ are normalized, then $R$ does not change the norm $||b_i|| = ||b_i R|| = ||b_i|| = 1$. So for $\tilde{b}_i = b_i R$, each $\langle a_i, \tilde{b}_i \rangle = \langle \frac{a_i}{||a_i||}, \frac{b_i}{||b_i||} \rangle$ and hence, as desired,

$$R^* = \arg\max_{R \in O(d)} \sum_{i=1}^{n} \langle \frac{a_i}{||a_i||}, \frac{b_i}{||b_i||} R \rangle.$$ 

Several evaluations of word vector embeddings focus on cosine similarity, so it suggests first normalizing all vectors $a_i \in A$ and $b_i \in B$ before performing $\text{AO-Rotation}(A, B)$, as in Algorithm 2.6. However, we found this does not empirically work as well. The rational is that vectors with larger norm tend to have less noise and are supported by more data. So the unnormalized alignment effectively weights the importance of aligning the inner products of these vectors more in the sum, and this leads to a more stable method. Hence, in general, we do not recommend this normalization preprocessing.

**Algorithm 2.6 AO-NORMALIZED$(A, B)$**

Set $a_i = \frac{a_i}{||a_i||}$ and $b_i = \frac{b_i}{||b_i||}$

Compute the sum of outer products $H = \sum_{i=1}^{n} b_i^T a_i$

Decompose $[U, S, V^T] = \text{svd}(H)$

Build rotation $R = U V^T$

return $\tilde{B} = BR$ so each $\tilde{b}_i = b_i R$

But the relative instability of the normalized variant leads us to discuss another variant of $\text{AO-Rotation}$, called $\text{AO-WEIGHTED}$, where we scale each vector up by its norm when solving for the optimal rotation (as opposed to scaling down when normalizing). This is based on the idea that the stability of a vector should influence its effect on the optimal rotation. This stability is reflected in the number of times a word appears in the text and consequentially, in the norm of the vector.

Alternately, we also scale up the centered version of $\text{AO-CENTERED+SCALING}$ to compare the effects of weighting versus translation and scaling.
Algorithm 2.7 AO+Weighted(A, B)

Compute the scaled sum of outer products $H = \sum_{i=1}^{n} ||a_i|| ||b_i|| b_i^T a_i$
Decompose $[U, S, V^T] = \text{svd}(H)$
Build rotation $R = UV^T$
return $\tilde{B} = BR$ so each $\tilde{b}_i = b_i R$

Algorithm 2.8 AO+Centered+Weighted(A, B)

Compute $\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i$ and $\bar{b} = \frac{1}{n} \sum_{i=1}^{n} b_i$
Center $\hat{A} \leftarrow (A, \bar{a})$ so each $\hat{a}_i = a_i - \bar{a}$, and similarly $\hat{B} \leftarrow (B, \bar{b})$
Compute the scaled sum of outer products $H = \sum_{i=1}^{n} ||\hat{a}_i|| ||\hat{b}_i|| \hat{b}_i^T \hat{a}_i$
Decompose $[U, S, V^T] = \text{svd}(H)$
Build rotation $R = UV^T$
Rotate $\hat{B} = \hat{B} R$ so each $\tilde{b}_i = \hat{b}_i R$
Compute scaling $s = \sum_{i=1}^{n} \langle a_i, b_i \rangle / \|B\|_F^2$
Scale $\tilde{B}$ as $\hat{B} \leftarrow s\hat{B}$ so for each $\hat{b}_i = sb_i$.
return $\tilde{A}, \tilde{B}$

2.1 Related Approaches

As mentioned Smith et al. [35] use Algorithm 2.2 to align word2vec word embeddings on English and Italian corpuses, and show that this simple approach is effective in translation. Our work can be seen as building on this, in that we show how to interpret the intrinsic accuracy of such an alignment, how to align word vector corpuses created by different mechanisms, and when to use which variant of the closed form solutions. Additionally, we confirm some of their translation results and show that it extends to when the embedding mechanisms for the different language corpuses are not the same (e.g., one by word2vec and one by GloVe).

There are several other methods in the literature which attempt to jointly compute embeddings of datasets so that they are aligned, for instance in jointly embedding corpuses in multiple languages [14, 22]. The goal of the approaches we study is to circumvent these more complex joint alignments.

A couple of very recent papers propose methods to align embeddings after their construction, but focus on affine transformations, as opposed to the more restrictive but distance preserving rotations of our method. Bollegala et al. [4] uses gradient descent, for parameter $\gamma$, to directly optimize

$$\arg\min_{M \in \mathbb{R}^{d \times d}} \sum_{i=1}^{n} \|a_i - b_i M\|^2 + \gamma \|M\|^2_F.$$ 

Another approach, by Sahin et al. [31], is Low Rank Alignment (LRA) which is an extension of aligning manifolds using LLE [36]. This approach has a 2-step but closed form solution to find an affine transformation applied to both embeddings simultaneously so as to project the two embeddings to a joint manifold space. The closed form solution is produced using the eigenvectors of the sum of the Laplacian and Gram matrix of the two datasets. So, if $A$ and $B$ are the two embeddings, and $C$ is the correspondence matrix between $A$ and $B$, where,

$$C_{ij} = \begin{cases} 1, & \text{if } a_i \text{ corresponds with } b_j \\ 0, & \text{otherwise} \end{cases}.$$ 

Then the two datasets are embedded into a joint d-dimensional space and represented by $F = \begin{bmatrix} F_A \\ F_B \end{bmatrix}$ where $i$th row of $F_A$ is the embedding of $a_i$ and the $j$th row of $F_B$ is the embedding of $b_j$ [36].

The matrix $F$ is obtained by a 2 step algorithm, where first, for each dataset $A$ and $B$, a reconstruction
matrix $M_A$ and $M_B$ is obtained by minimizing the loss function, for an appropriate choice of $\lambda$,

$$
M_A = \arg\min_{M \in \mathbb{R}^{n \times n}} \frac{1}{2} \| A - AM \|^2_F + \lambda \| M \|.
$$

$$
M_B = \arg\min_{M \in \mathbb{R}^{n \times n}} \frac{1}{2} \| B - BM \|^2_F + \lambda \| M \|.
$$

These have a closed form solution. Then, with block matrix $M = \begin{bmatrix} M_A & 0 \\ 0 & M_B \end{bmatrix}$.

Now let $\mu$ be an appropriate hyperparameter, the embedding $F$ is obtained by minimizing

$$
F^* = \arg\min_{F} (1 - \mu) \| F - FM \|^2_F + \mu \sum_{i=1}^{\lfloor |A| \rfloor} \sum_{j=1}^{\lfloor |B| \rfloor} \| f_i - f_{|A|+j} \| C_{i,j},
$$

where $f_t$ is the $t$th row of $F$, and this formulation also has a closed form.

This approach is heuristic and also finds an affine transformation (encoded in $M$), not restricted to distance-preserving rotations as our approach. Moreover, it is not clear how to apply this transformation to a new point not in the original sets $A$ and $B$, for instance if we split the data into a train and test set from the union of $A$ and $B$. Also, it requires an eigendecomposition of a dense $n \times n$ matrix (intractable when $n = 100,000$ or in some cases millions), where ours only requires $O(nd)$ preprocessing and a decomposition of a $d \times d$ matrix, where $d = 300$ or so, making LRA far less scalable.

Neither approach directly optimizes for the optimal transformation, and requires regularization parameters; this implies if embeddings start far apart, they remain further apart than if they start closer. Moreover, both find affine transformations $M$ over $\mathbb{R}^{d \times d}$, not a rotation over the non-convex $O(d)$ as does our approach. This changes the Euclidean distance found in the original embedding to a Mahalanobis distance that will change the order of nearest neighbors under Euclidian and cosine distance.

In contrast, just allowing the optimization to choose a scaling also changes the distance, but retains the ordering. The ordering of inner products also is unchanged. Hence, similarly embedded words are still embedded similarly under such a transformation: the $k$-nearest neighbors are retained, and the ratios between distances remains the same with a single uniform scaling.

## 3 Word Embedding Techniques and Data Sets

We first discuss the data and the different embedding mechanisms used to extract the embeddings for various experiments.

### 3.1 Different Word Embeddings

Given some textual data, there are several ways of extracting a distributed vector representation or an embedding for the words in the data. Our default approach will be GloVe [4], which is prediction based. We also consider word2vec [23, 21], the other most common prediction-based approach, using the Gensim [29] implementation. Third, we use the counting-based embedding RAW, based on a normalized continuous bag of words model.

**GloVe**: [4] The GloVe model is a log-bilinear model based on ratios of word-word co-occurrence frequencies. The training objective is for the dot product of the vectors learned for words to equal the logarithm of their co-occurrence frequency. It regresses over a weighted least squares problem to get the optimal vector representation of words. We used the source code available online at [https://nlp.stanford.edu/projects/glove/](https://nlp.stanford.edu/projects/glove/).

**word2vec**: [23, 21] This skip-gram model framework uses cooccurrences of words to predict for any word $w$, what words are likely to be in its context i.e., what words are likely to be used nearby $w$ in some text. The training objective is to maximize

$$
\frac{1}{T} \sum_{t=1}^{T} \sum_{-c \leq j \leq c, j \neq 0} \log p(w_{t+j} \mid w_t)
$$
where \( c \) is the size of the training context and the window is centered at the word \( w_t \) and \( T \) is the total number of training words. We implemented this using the Gensim framework.

**RAW:** Each target word \( w_i \) here is expressed using all other words \( w_j \) it occurs within some specified context window. For a vocabulary of size \( n \), each word \( w_i \) is represented by a vector \( a_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n}) \in \mathbb{R}^d \) where \( a_{i,j} \) is the number of times word \( w_i \) co-occurs with word \( w_j \), divided by the total number of times word \( w_i \) occurs in the text.

For all three embeddings, we used a window size of 10 words (symmetric context with 5 on each side of target word) for the context. The cut-off minimum frequency for words to be used for the embedding was kept at 5.

### 3.2 Data Sets

For our experiments, the default dataset used is the Latest Wikipedia dump of about 4.57 billion tokens and with a vocabulary of 243K tokens (dumps.wikimedia.org/enwiki/latest/enwiki-latest-pages-articles.xml.bz2). The embedding created of this by GloVe is labeled G(W).

We also compare against

- G(W1): a GloVe embedding for the first billion characters of Wikipedia (129M tokens, 849K vocab),
- G(WG): the GloVe embedding for Wikipedia + Gigaword (6B tokens, 400K vocab),
- G(CC42): the GloVe embedding of Common Crawl (42B tokens, 1.9M vocab),
- G(CC840): the GloVe embedding of Common Crawl (840B tokens, 2.2M vocab), and
- W(GN): the word2vec embedding of Google News (100B tokens, 3M vocab)

Embeddings G(W), G(CC42), and G(CC840) are found on GloVe’s homepage (https://nlp.stanford.edu/projects/glove/), embedding W(GN) is found on https://code.google.com/archive/p/word2vec/ and we use these directly. Further, for embeddings in other languages, we use the French and Spanish word embeddings by FastText (https://github.com/facebookresearch/fastText) which uses a version of word2vec.

**Default settings.** Unless otherwise stated, the default embedding mechanism is GloVe, and the default dataset is the 4.57 billion tokens of Wikipedia.

In each embedding, we always consider a consistent vocabulary of \( n = 100,000 \) words. To decide on this set, we found the \( n \) most frequent words used in the default Wikipedia dataset and that were embedded by GloVe. In one case, we compare against smaller datasets and then only work with a small vocabulary of size \( n' = 10,000 \) found the same way.

For each embedding we represent each word as a vector of dimension \( d = 300 \). Note that RAW originally uses an \( n \)-dimensional vector. We reduce this to \( d \)-dimensions by taking its SVD, as advocated by Levy et al. [18].

### 4 Experiments

In this section we evaluate the effectiveness of our methods on a variety of scenarios, typically using the Root Mean Square Error: \( \text{RMSE}(A,B) = \sqrt{\frac{1}{|A|} \sum_{i=1}^{|A|} ||a_i - b_i||^2} \). When aligning GloVe embeddings to other GloVe embeddings, we use AO-ROTATION. When aligning embeddings from different sources we use AO+SCALING. We justify these choices in Section 4.4.

#### 4.1 Calibrating RMSE

In order to make sense of the meaning of an RMSE score, we calibrate it to the effect of some easier to understand distortions. To start, we make a copy of \( A \) (the default G(W) embedding – we use this notation to signify a GloVe embedding G(·) or the default Wikipedia corpus W) and apply an arbitrary rotation, translation, and scaling of it to obtain a new embedding \( B \). Invoking \( A, B \leftarrow \text{AO-CENTERED+SCALING}(A,B) \), we expect that \( \text{RMSE}(\hat{A}, \hat{B}) = 0 \); we observe RMSE values on the order of \( 10^{-14} \), indeed almost 0 withstanding numerical rounding.
We observe the noise is linear, and achieves an RMSE of 100M, 1B, 2.5B or 4.57B tokens. For each dataset we create an entry on last row (ie, along the direction of arrow) when a mental aligned to larger datasets equivalent to the one the larger dataset would produce. This however comes at a higher cost of computation and time. If after a certain point, adding data does not really affect the model too much, it might be a good trade off to make better predictions and calibrate itself more acutely. As a model sees more data, it is able to make better predictions and calibrate itself more acutely. This however comes at a higher cost of computation and time. If after a certain point, adding data does not really affect the model too much, it might be a good trade off to use a smaller dataset to make an embedding that is almost equivalent to the one the larger dataset would produce.

We evaluate this relationship using the RMSE values when a GloVe embedding from a smaller dataset B is incrementally aligned to larger datasets A using AO-Rotation. We do this by starting off with a dataset of the first 1 million tokens of Wikipedia (1M). We then add data sequentially to it, to create datasets of sizes of 100M, 1B, 2.5B or 4.57B tokens. For each dataset we create GloVe embeddings. Then we align each dataset.

Table 1: RMSE after alignment for Incremental Data. The alignment done from each entry in first column to an entry on last row (ie, along the direction of arrow)

| 2.5 B | 3.067 | 10K vocab | 2.5 B | 3.902 | 100K vocab |
|-------|-------|-----------|-------|-------|-----------|
| 1 B   | 4.051 | 4.579     | 1 B   | 4.179 | 4.313     |
| 100 M | 4.202 | 4.217 | 4.322 | 100 M | 4.374 | 4.413 |
| 1 M   | 5.076 | 5.185 | 5.127 | 1 M   | 4.642 | 4.655 |
|       | 4.7 B | 2.5 B | 1 B   |       | 4.7 B | 2.5 B | 1 B | 100 M |

Figure 1: RMSE error after noise and then AO-Rotation alignment. Left: We perturb 10%, 50% or all points with Gaussian noise. Right: We add structured and unstructured noise before embedding.

Gaussian Noise. Next we add Gaussian noise directly to the embedding. That is we define an embedding $B$ so that each $b_i = a_i + g_i$ where $g_i \sim N_d(0, \Sigma)$, where $N_d(\mu, \Sigma)$ is a $d$-dimensional Gaussian distribution, and $\sigma$ is a standard deviation parameter. Then we measure $\text{RMSE}(A, B)$ from $A, B \leftarrow \text{AO-Rotation}(A, B)$.

Figure 1(left) shows the effects for various $\sigma$ values, and also when only added to 10% and 50% of the points. We observe the noise is linear, and achieves an RMSE of 2 to 5 with $\sigma \in [0.1, 0.3]$.

Noise before embedding. Next, we append noisy, unstructured text into the Wikipedia dataset with 1 billion tokens. We specifically do this by generating random sequences of $m$ tokens, drawn uniformly from the $n = 10K$ most frequent words; we use $m = \{0.01, 0.1, 0.5, 1, 2.5\}$ billion. We then extract embeddings for the same vocabulary of $n = 100K$ words as before, from both datasets, and use AO-Rotation to linearly transform the noisy one to the one without noise. As observed in Figure 1(right), this only changes from about 0.7 to 1.6 RMSE. The embeddings seem rather resilient to this sort of noise, even when we add more tokens than the original data.

We perform a similar experiment of adding structured text; we repeat a sequence made of $s = \{100, 1000, 10,000\}$ tokens of medium frequency so the total added is again $m = \{10M, 100M, 500M, 1B, 2.5B\}$. Again in Figure 1(right), perhaps surprisingly, this only increases the noise slightly, when compared to the unstructured setting. This can be explained since only a small percentage of the vocabulary is by affected this noise, and by comparing to the Gaussian noise, when only added to 10% of the data, it has about a third of the RMSE as when added to all data.

Incremental Data. As a model sees more data, it is able to make better predictions and calibrate itself more acutely. This however comes at a higher cost of computation and time. If after a certain point, adding data does not really affect the model too much, it might be a good trade off to use a smaller dataset to make an embedding that is almost equivalent to the one the larger dataset would produce.

We evaluate this relationship using the RMSE values when a GloVe embedding from a smaller dataset B is incrementally aligned to larger datasets A using AO-Rotation. We do this by starting off with a dataset of the first 1 million tokens of Wikipedia (1M). We then add data sequentially to it, to create datasets of sizes of 100M, 1B, 2.5B or 4.57B tokens. For each dataset we create GloVe embeddings. Then we align each dataset.

Figure 2: Aligned RMSE on Incremental Data.
Table 2: Variance of RMSE after alignment onto 4.57 Wikipedia data set from embeddings created by subsets of the full data set of various size. And alignment onto subset datasets of the same size.

| Original data size | 4.57 B | Similar sized |
|--------------------|--------|---------------|
| 1M                 | 0.0248 | 0.2583        |
| 100M               | 0.02038| 0.2189        |
| 1B                 | 0.0187 | 0.1259        |

using AO-ROTATION\((A,B)\) where \(A\) (the target) is always the larger of the two data set, and \(B\) (the source) is rotated and is the smaller of the two.

Figure 2 and Table 1 show the result using a vocabulary of \(n = 100K\) and \(n' = 10K\). The small \(n'\) is also used since for smaller datasets, many of the top 100K words are not seen. We observe that even this change in data set size, decreasing from 4.57B tokens to 2.5B still results in substantial RMSE. However aligning with fewer but better represented words starts to show better results, which indicates that (in future work) it might be better to try other weightings in the optimization function.

Variance. The changes in the RMSE scores are fairly small in some cases, but still meaningful. To quantify this we also show the variance of these RMSE scores found with AO-ROTATION. We align each data set (of size 1M, 100M, and 1B) to both themselves (labeled “Similar sized”) and to the full data set of size 4.57B. For each comparison, we create 5 datasets of the same size, embed them with GloVe, and align them to each other and to the 4.57B one, with AO-ROTATION. The results, in Table 2, show that the variance of the RMSE is small and also decreases for both cases as we increase the dataset size.

4.2 Dependence on Datasets and Embeddings

Now with a sense of how to calibrate the meaning of RMSE, we can investigate the effect of changing the dataset entirely or changing the embedding mechanism.

Dependence of Datasets. Table 3(left) shows the RMSE when the 4 GloVe embeddings are aligned with AO-ROTATION, either as a target or source. The alignment of G(W) and G(WG) has less error than either to G(CC42) and G(CC840), likely because they have substantial overlap in the source data (both draw from Wikipedia). In all cases, the error is roughly on the scale of adding Gaussian noise with \(\sigma \in [0.25, 0.35]\) to the embeddings, or reducing the dataset to 10M to 100M tokens. This is much more alignment error than in other experiments, indicating that the change in the source data set (and likely its size) has a much larger effect than the embedding mechanism.

Dependence on Embedding Mechanism. We now fix the data set (the default 4.57B Wikipedia dataset \(W\)), and observe the effect of changing the embedding mechanism: using GloVe, word2vec, and RAW. We now use AO+SCALING instead of AO-ROTATION, since the different mechanisms tend to align vectors at drastically different scales.

Table 3(right) shows the RMSE error of the alignments; the columns show the target (\(A\)) and the rows show the source dataset (\(B\)). This difference in target and source is significant because the scale inherent in these alignments change, and with it, so does the RMSE. Also as shown, the scale parameter \(s^*\) from GloVe to word2vec in AO+SCALING is approximately 3 (and non-symmetrically about 0.25 in the other direction from word2vec to GloVe). This means for the same alignment, we expect the RMSE to be between 3 to 4 (\(\approx 1/0.25\)) times larger as well.

However, with each column, with the same target scale, we can compare alignment RMSE. We observe the differences are not too large, all roughly equivalent to Gaussian noise with \(\sigma = 0.25\) or using only 1B to 2.5B tokens in the dataset. Interestingly, this is less error that changing the source dataset; consider the GloVe column for a fair comparison. This corroborates that the embeddings find some common structure, capturing the same linear structures, analogies, and similarities. And changing the datasets is a more significant effect.

In Table 4 we see how each step of AO+SCALING affects the RMSE value when we go between datasets keeping the embedding mechanism constant versus when we change the embedding mechanism keeping the
Table 3: RMSE after alignment for embeddings. Left: Created from different datasets. Right: Created by different embeddings; uses AO+SCALING mapping rows onto columns and changing scale.

|       | G(W) | G(WG) | G(CC42) | G(CC840) |
|-------|------|-------|---------|----------|
| G(W)  | -    | 4.56  | 5.167   | 6.148    |
| G(WG) | 4.56 | -     | 5.986   | 6.876    |
| G(CC42)| 5.167 | 5.986 | -       | 5.507    |
| G(CC840)| 6.148 | 6.876 | 5.507   | -        |

Table 4: Change in RMSE with each step of AO+SCALING(rotation, translation and scaling)

|       | rotation | +translation | +scaling |
|-------|----------|--------------|----------|
| G(W) - G(W) | 17.417 | 17.087 | 3.68 |
| G(W) - G(CC42) | 5.167 | 5.113 | 5.110 |

Table 5 shows the scores on just the GloVe and word2vec embeddings, and then across these aligned datasets.

To understand how the variants of AbsoluteOrientation compare, we compute the scores after each of the various optimal transformation types are applied: rotation, then scaling, then translation, and finally we consider if we normalize all vectors before alignment to maximize cosine similarities. Before transformation (“untransformed”) the across-dataset comparison is very poor, close to 0; that is, extrinsically there is very little information carried over. However, alignment with just AO-Rotation achieves scores nearly as good as, and sometimes better than on the original datasets. word2vec scores higher than GloVe, and the across-dataset scores are typically between these two scores. Adding scaling with AO+SCALING has no affect on the scores on the similarity test because they are measured with cosine similarity. However also applying the optimal translation does increase the scores even though it optimizes Euclidean distance and not cosine distance. Perhaps surprisingly, applying rotation along with translation and scaling improves more than just applying rotation and translation. This method applies scaling after the dataset is centered, so this then alters the inner products, and in a useful way.

We perform the same experiments on 2 Google analogy datasets [23]: SEM has 8869 analogies and SYN has 10675 analogies. These are of the form “A:B::C:D” (e.g., “man:woman::king:queen”), and we evaluate across data sets by measuring if vector \( v_D \) is among the nearest neighbors in data set \( A \) of vector \( v_C + (v_B - v_A) \) in data set \( B \). The results are similar to the synonym tests, where AO-ROTATION alignment across-datasets performs similar to within either embedding, and scaling and rotation provided small further improvement. In this case, performing rotation and scaling improves upon just rotation. This is because the analogies are accessing something more complicated about the embedding, and so adjusting the scale more aligns the
Table 5: Spearman coefficient scores for synonym and analogy tests between the aligned \texttt{word2vec} to \texttt{GloVe} embeddings, \texttt{GloVe} transformed to \texttt{word2vec} and between \texttt{GloVe} embeddings of Wikipedia and CC42 dataset; r, s and t stand for the functions of optimal rotation, scaling and translation respectively and w() is the weighted version of that function.

| Test Sets | GloVe | \texttt{word2vec} | \texttt{word2vec} to GloVe | \texttt{GloVe} to \texttt{word2vec} | \texttt{G(W)} to \texttt{G(CC42)} |
|-----------|-------|------------------|---------------------------|-----------------------------------|----------------------------------|
|           |       | untransformed    | r            | r + s | r + t | r + s + t | normalized | w(r)   | w(r+s+t) | untransformed | r            | r + s | r + t | r + s + t | normalized | w(r)   | w(r+s+t) | untransformed | r            | r + s | r + t | r + s + t | normalized | w(r)   | w(r+s+t) |
| RG        | 0.614 | 0.696 | 0.041 | 0.584 | 0.584 | 0.570 | 0.594 | 0.553 | 0.592 | 0.597 |
| WSim      | 0.623 | 0.659 | 0.064 | 0.624 | 0.624 | 0.611 | 0.652 | 0.604 | 0.657 | 0.664 |
| MC        | 0.669 | 0.818 | 0.013 | 0.868 | 0.868 | 0.843 | 0.873 | 0.743 | 0.878 | 0.882 |
| SimLex    | 0.296 | 0.342 | 0.012 | 0.278 | 0.278 | 0.269 | 0.314 | 0.261 | 0.314 | 0.316 |
| SYN       | 0.587 | 0.582 | 0.000 | 0.501 | 0.525 | 0.517 | 0.528 | 0.493 | 0.535 | 0.539 |
| SEM       | 0.691 | 0.722 | 0.0001 | 0.624 | 0.656 | 0.633 | 0.697 | 0.604 | 0.702 | 0.712 |
| RG        | 0.614 | 0.696 | 0.091 | 0.633 | 0.633 | 0.602 | 0.641 | 0.607 | 0.639 | 0.645 |
| WSim      | 0.623 | 0.659 | 0.008 | 0.585 | 0.585 | 0.569 | 0.588 | 0.511 | 0.593 | 0.596 |
| MC        | 0.669 | 0.818 | 0.006 | 0.712 | 0.712 | 0.707 | 0.716 | 0.697 | 0.721 | 0.727 |
| SimLex    | 0.296 | 0.342 | 0.009 | 0.321 | 0.321 | 0.317 | 0.323 | 0.308 | 0.322 | 0.328 |
| SYN       | 0.587 | 0.582 | 0.000 | 0.528 | 0.542 | 0.528 | 0.557 | 0.501 | 0.554 | 0.554 |
| SEM       | 0.691 | 0.722 | 0.0001 | 0.641 | 0.646 | 0.641 | 0.646 | 0.602 | 0.663 | 0.682 |
| RG        | 0.614 | 0.817 | 0.363 | 0.818 | 0.818 | 0.811 | 0.821 | 0.815 | 0.818 | 0.825 |
| WSim      | 0.623 | 0.63 | 0.017 | 0.618 | 0.618 | 0.615 | 0.618 | 0.601 | 0.616 | 0.637 |
| MC        | 0.669 | 0.786 | 0.259 | 0.766 | 0.766 | 0.732 | 0.768 | 0.705 | 0.771 | 0.774 |
| SimLex    | 0.296 | 0.372 | 0.035 | 0.343 | 0.343 | 0.339 | 0.346 | 0.296 | 0.346 | 0.346 |
| SYN       | 0.587 | 0.625 | 0.00 | 0.566 | 0.576 | 0.572 | 0.576 | 0.502 | 0.576 | 0.576 |
| SEM       | 0.691 | 0.741 | 0.00 | 0.676 | 0.684 | 0.676 | 0.688 | 0.565 | 0.690 | 0.695 |
Table 6: Generalizing the optimal rotation $R$ learned from the top 10K most frequent words to all 100K words to test for performance on similarity and analogy tests.

|      | AO-Rotation(100K) | AO-Rotation(10K) | AO-Normalized(10K) |
|------|------------------|------------------|--------------------|
| RG   | 0.584            | 0.576            | 0.588              |
| WSIM | 0.624            | 0.612            | 0.643              |
| MC   | 0.868            | 0.817            | 0.851              |
| SIMLEX | 0.278      | 0.292            | 0.308              |
| SYN  | 0.501            | 0.505            | 0.511              |
| SEM  | 0.624            | 0.616            | 0.616              |

Euclidean distance and hence the vector structure needed to succeed in analogies.

The right part of the table then considers first normalizing all vectors $a_i \in A$ and $b_i \in B$ so $\|a_i\| = \|b_i\| = 1$ before alignment; recall this explicitly optimizes the average cosine similarity. This “normalized” score, perhaps surprisingly, performs slightly worse. We suspect this is probably because it deemphasizes the alignment of more stably embedded (and hence longer) vectors. However, when we add extra weight to the longer vectors, then the alignment performs even better. Adding translation and scaling, further improves the scores across both similarity and analogy tests.

We reverse the setup and align GloVe onto word2vec and perform the same tests and observe similar results as before, as illustrated in Table 5.

We also align G(W) to G(CC42), to observe the effect of only changing the dataset. The G(CC42) dataset performs better itself; it uses more data. The small similarity tests (RG, MC) show some extrinsic information is captured without any alignment, but otherwise across-embedding scores have a similar pattern to across-dataset scores.

Next in Table 6 we further investigate the effect of various weighting (or normalizing) before alignment. In these test we show the effect on AO-Rotation with three types of weighting. As before we simply apply AO-Rotation on all 100K words. But we also find the optimal $R$ on only the most frequent 10K words using AO-Rotation, and then again using AO-Normalized on just these 10K words. The rotation and evaluation is still on all 100K words needed for the tests. Surprisingly AO-Normalized(10K) performs better than AO-Rotation(10K), and comparably to AO-Rotation(100K). This indicates that similarity optimization is useful when the words all have sufficient data to embed them properly.

4.3.1 Comparison to baselines

Next, we do similarity tests to compare against alignment implementations of methods by Sahin et al. [31] (LRA) and Bollegela et al. [4] (Affine Transformations). We reimplemented their algorithms, but did not spend significant time to optimize the parameters; recall our method requires no hyperparameters. We only used the top $n^\prime = 10K$ words for these transformations because these other methods were much more time and memory intensive. We only computed similarities among pairs in the top 10K words for fairness (about two-thirds of the word pairs evaluated, so the scores do not match other tables), and did not perform analogy tests since fewer than one-third of analogies fully showed up in the top 10K. Table 7 shows results for aligning the G(W) and G(CC42) embeddings with these approaches. Our AbsoluteOrientation-based approach does significantly better than Bollega et al. [4]’s Affine Transformations and generally better than Sahin et al. [31]’s LRA. Our advantage over LRA increases when aligning all $n = 100K$ words; by comparison LRA ran out of memory since it requires an $n \times n$ dense matrix decomposition.

4.3.2 Aligning Embeddings across Languages and Embeddings

Word embeddings have been used to place word vectors from multiple languages in the same space [14, 22]. These either do not perform that well in monolingual semantic tasks as noted in Luong et al. [20] or use learned affine transformations [22], which distort distances and do not have closed form solutions. Smith et al. [35] use the equivalent of AO-Rotation to translate between word embeddings from different languages that have been extracted using the same method. We extend that here to verify that no matter the embedding mechanism, we can translate using a variant of AbsoluteOrientation. We use the ability to choose the right variant of Absolute Orientation as per Section 4.4 to orient different embeddings onto each
other coherently. We use the default English GloVe embedding from Wikipedia and the FastText [2], using jointly learned affine transformations).

We perform a similar experiment between English and French, and see similar results. We first obtain 300 dimensional embeddings for English wikipedia dump using GloVe, and for French words from the FastText embeddings [https://github.com/facebookresearch/fastText]. Then, we extract the embeddings for the most frequent 10,000 words from the default Wikipedia dataset (that have translations in French) and their translations in French and align them using AO+CENTERED+WEIGHTED. We test before and after alignment, for each of these 10,000 words, if their translation is among their nearest 1, 5, and 10 neighbors. Before alignment, the fraction of words with its translation among its closest 1, 5 and 10 nearest neighbors is 0.00, 0.160, and 0.160 respectively, while after alignment it is 0.372, 0.623 and 0.726, respectively. Table 8 lists some examples before and after translation.

We perform a cross-validation experiment to see how this alignment applies to new words not explicitly aligned. On learning the rotation matrix above, we apply it to a set of 1000 new ‘test’ Spanish words (the translations of the next 1000 most frequent English words) and bring it into the same space as that of English words as before. We test these 2000 new words in the embedded and aligned space of 12,000 words (now 6,000 from each language). Before alignment, the fraction of times their translations are among the closest 1, 5 and 10 neighbors are 0.00, 0.00, and 0.00, respectively. After alignment it is 0.311, 0.689, and 0.747, respectively (comparable to results and setup in Mikolov et al. [22], using jointly learned affine transformations).

We perform a cross-validation experiment to see how this alignment applies to new words not explicitly aligned. On learning the rotation matrix above, we apply it to a set of 1000 new ‘test’ French words (the translations of the next 1000 most frequent English words in the default dataset) and bring it into the same space as that of English words as before. We test in this space of 22,000 words now, if their translations are among the closest 1, 5 and 10 nearest neighbors of the 2000 new words (1000 French and their translations in English). Before alignment, the fraction of times their translations are among the closest 1, 5 and 10

Table 8: The 5 closest neighbors of a word before and after alignment by AO-ROTATION(between English - Spanish). Target word (translation) in bold.

| Word    | Neighbors before alignment     | Neighbors after alignment          |
|---------|-------------------------------|-----------------------------------|
| woman   | her, young, man, girl, mother | her, girl, **mujer**, mother, man |
| week    | month, year, monday, time     | days, **semana**, year, day, month|
| casa    | apartamento, casas, palacio, residencia, habitaci | casas, **home**, homes, habitaci, apartamento |
| caballo | caballos, caballer, jinete, jinetes, equitaci | caballos, **horse**, horses, caballos, jinete |
| sol     | sombra, luna, solar, amanecer, cielo | **sun**, moon, luna, solar, sombra |

Table 7: Similarity tests after alignment by Affine Transformation, LRA, and AO-ROTATION of Wikipedia and CC42 GloVe embeddings.

| Test Sets | LRA (10K) | Affine Transformations (10K) | AO-ROTATION (10K) |
|-----------|-----------|-----------------------------|-------------------|
| RG        | 0.701     | 0.301                       | 0.728             |
| WSM       | 0.616     | 0.269                       | 0.612             |
| MC        | 0.719     | 0.412                       | 0.722             |
| SIMLEX    | 0.327     | 0.126                       | 0.340             |

We perform a cross-validation experiment to see how this alignment applies to new words not explicitly aligned. On learning the rotation matrix above, we apply it to a set of 1000 new 'test' Spanish words (the translations of the next 1000 most frequent English words) and bring it into the same space as that of English words as before. We test before and after alignment, for each of these 10,000 words, if their translation is among their nearest 1, 5, and 10 neighbors. Before alignment, the fraction of words with its translation among its closest 1, 5 and 10 nearest neighbors is 0.00, 0.00, and 0.00, respectively. After alignment it is 0.372, 0.623 and 0.726, respectively. Table 8 lists some examples before and after translation.

We perform a similar experiment between English and French, and see similar results. We first obtain 300 dimensional embeddings for English wikipedia dump using GloVe, and for French words from the FastText embeddings [https://github.com/facebookresearch/fastText]. Then, we extract the embeddings for the most frequent 10,000 words from the default Wikipedia dataset (that have translations in French) and their translations in French and align them using AO+CENTERED+WEIGHTED. We test before and after alignment, for each of these 10,000 words, if their translation is among their nearest 1, 5, and 10 neighbors. Before alignment, the fraction of words with its translation among its closest 1, 5 and 10 nearest neighbors is 0.00, 0.00, and 0.00, respectively. After alignment it is 0.311, 0.689, and 0.747, respectively (comparable to results and setup in Mikolov et al. [22], using jointly learned affine transformations).
Table 9: The 5 closest neighbors of a word before and after alignment by AbsoluteOrientation (between English - French). Target word (translation) in bold.

| Word   | Neighbors before alignment | Neighbors after alignment |
|--------|---------------------------|---------------------------|
| woman  | her, young, man, girl, mother | her, young, man, femme, la |
| week   | month, day, year, monday, time | month, day, year, semaine, start |
| heureux| amoureux, plaisir, rire, gens, vivre | happy, plaisir, loving, amoureux, rire |
| cheval | chein, petit, bateau, pied, jeu | horse, dog, chien, red, petit |
| daughter| father, mother, son, her, husband | mother, fille, husband, mere, her |

neighbors are 0.00, 0.00, and 0.00, respectively. After alignment it is 0.307, 0.513, and 0.698, respectively.

Thus, even with embeddings derived from different languages and different embedding mechanisms, we can successfully translate words by AO+CENTERED+WEIGHTED.

4.4 Choosing the Right Variant of AbsoluteOrientation

With all the experiments in this section so far, we observed several pivotal points.

Most of the gain using AbsoluteOrientation is achieved by just finding the optimal rotation \( R \) with AO-Rotation. However, consistent improvement can be found by weighting the large points more using AO+Weighted and by applying translation or scaling, and slightly more by applying both.

When different datasets are aligned using the same mechanism (e.g., both with GloVe or both with word2vec), then it is debatable whether scaling and translation is necessary, since scaling does not affect cosine similarity, and translation changes intrinsic inner product properties. However, using a weighting to put more weight on more longer (and implied more robustly embedded) words does not alter any intrinsic properties, and only seems to create better alignments.

On the other hand, when datasets are embedded with different mechanisms (e.g., one with word2vec and one with GloVe) then they are not scaled properly with respect to each other. In this case, it is important to find the optimal scaling to put them in a consistent interpretable scale, and to ensure analogy relations are optimized. So we strongly recommend using scaling in this setting.

5 Boosting via Ensembles

A direct application of combining different embeddings can be to increase its robustness. In this section we show that we can use ensembles of pre-computed word embeddings found via different mechanisms and on different datasets to boost the performance on the similarity and analogy tests beyond that of any single mechanism or dataset. The boosting strategy we use here is just simple averaging of the corresponding words after the embeddings have been aligned; we leave more nuanced strategies to future work.

Table 10 shows the performance of these combined embedding in three experiments. The first set shows the default Wikipedia data set under GloVe (\( G(W) \)), under word2vec (\( W(W) \)), and combined (\( [G(W) \odot W(W)] \)). The second set shows word2vec embedding of GoogleNews (\( W(GN) \)), and combined (\( [G(W) \odot W(GN)] \)) with \( G(W) \). The third set shows GloVe embedding of CommonCrawl (840B) (\( G(CC840) \)) and then combined with \( W(GN) \) as \( [G(CC840) \odot W(GN)] \). Combining two embeddings consistently boosts the performance on similarity and analogy tests. The best score on each experiment is in bold, and in 5 out of 6 cases, it is from a combined embedding. Moreover, except for this one case, the combined embedding is always performs better on all tests that both of the individual embeddings, and in this one case, \( G(CC804) \odot W(GN) \) still outperforms \( W(GN) \) on SEM analogies. For instance, remarkably, \( G(W) \odot W(W) \) which only uses the default 4.57B token Wikipedia dataset, performs better or nearly as well as \( W(GN) \) which uses 100B tokens. Moreover, in some cases the improvement is significant; on the large similarities test SIMLEX, the \( [G(CC840) \odot W(GN)] \) score is 0.443 or 0.446 with weights, whereas the best score without boosting is only 0.408 using \( G(CC840) \).
Table 10: Similarity and Analogy tests before and after alignment and combining embeddings derived from different techniques and datasets by AO+SCALING. Best scores in bold.

| TestSets | G(W) | W(W) | [G(W)⊙W(W)] | W(GN) | [G(W)⊙W(GN)] | G(CC840) | [G(CC840)⊙W(GN)] |
|----------|------|------|--------------|-------|--------------|----------|-------------------|
| RG       | 0.614 | 0.696 | 0.715        | 0.760 | **0.837**     | 0.768    | 0.810             |
| WSim     | 0.623 | 0.659 | 0.697        | 0.678 | 0.703         | 0.722    | **0.742**         |
| MC       | 0.669 | 0.818 | **0.865**    | 0.800 | 0.814         | 0.798    | 0.846             |
| Simlex   | 0.296 | 0.342 | 0.366        | 0.367 | 0.391         | 0.408    | **0.443**         |
| SYN      | 0.587 | 0.582 | 0.594        | 0.595 | 0.602         | **0.618**| 0.609             |
| SEM      | 0.691 | 0.722 | **0.757**    | 0.713 | 0.733         | 0.729    | 0.734             |

Table 11: Similarity and Analogy tests before and after alignment and combining embeddings derived from different techniques and datasets by AO+CENTERED+WEIGHTED. Best scores in bold.

| TestSets | G(W) | W(W) | [G(W)⊙W(W)] | W(GN) | [G(W)⊙W(GN)] | G(CC840) | [G(CC840)⊙W(GN)] |
|----------|------|------|--------------|-------|--------------|----------|-------------------|
| RG       | 0.614 | 0.696 | 0.716        | 0.760 | **0.836**     | 0.768    | 0.810             |
| WSim     | 0.623 | 0.659 | 0.695        | 0.678 | 0.708         | 0.722    | **0.740**         |
| MC       | 0.669 | 0.818 | **0.869**    | 0.800 | 0.811         | 0.798    | 0.847             |
| Simlex   | 0.296 | 0.342 | 0.368        | 0.367 | 0.394         | 0.408    | **0.446**         |
| SYN      | 0.587 | 0.582 | 0.592        | 0.595 | 0.607         | **0.618**| 0.609             |
| SEM      | 0.691 | 0.722 | **0.759**    | 0.713 | 0.733         | 0.729    | 0.733             |

6 Discussion

We have provided simple, closed-form method to align word embeddings. It allows for transformations for any subset of translation, rotation, and scaling. These operations all preserve the intrinsic Euclidean structure which has been shown to give rise to linear structures which allows for learning tasks like analogies, synonyms, and classification. All of these operations also preserve the Euclidean distances, so it does not affect the tasks which are measured using this distance; note the scaling also scales this distance, but does not change its order. Our experiments indicate that the rotation is essential for a good alignment, and the scaling is used to compare embeddings generated by different mechanisms (e.g., GloVe and word2vec) and while helpful, not necessarily when the data set is changed. Also the translation provides minor but consistent improvement.

We also show how to explicitly optimize cosine similarity by first normalizing all words – however, this does not perform as well as instead optimizing Euclidean distance. Rather we propose to weight words in the alignment by their norms, and this further improves the alignment because it emphasizes the words which have more stable embeddings.

This alignment enables new ways that word embeddings can be compared. This has the potential to shed light on the differences and similarity between them. For instance, as observed in other ways, common linear substructures are present in both GloVe and word2vec, and these structures can be aligned and recovered, further indicating that it is a well-supported feature inherent to the underlying language (and dataset). We also show that changing the embedding mechanism has less of an effect than changing the data set, as long as that data set is meaningful. Unstructured noise added to the input data set appears not to have much effect, but changing from the 4.57B token Wikipedia data set to the 840B token Common Crawl data set has a large effect.

Additionally, we show that by aligning various embeddings, their characteristics as measured by standard analogy and synonym tests can be transferred from one embedding to another. We also demonstrate that cross-language alignment can aid in word translation even when coming from completely different embedding mechanisms, even in a cross-validation setting. This cross embedding-mechanism alignment opens the door for many other types of alignment word embeddings with embeddings generated from graphs, images, or any other data set which has some useful word labels.

Finally, we showed that we can “boost” embeddings without revisiting the (sometimes quite enormous) raw data. This is surprisingly effective in improving scores on similarity and analogy test, results in the best known scores from embeddings on these tests. For instance, on the SimLEX analogy test we improve upon the best known scores by almost 10% in the Spearman correlation coefficient.
References

[1] K. S. Arun, T. S. Huang, and S. D. Blostein. Least-squares fitting of two 3-d points sets. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 9:698–700, 1987.

[2] Herbet Bay, Tinne Tuytelaars, and Luc Van Gool. SURF: Speeded up robust features. In *ECCV*, 2006.

[3] Paul J. Besl and Neil D. McKay. A method for registration of 3-d shapes. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(2):239 – 256, 1992.

[4] Danushka Bollegala, Kohei Hayashi, and Ken ichi Kawarabayash. Learning linear transformations between counting-based and prediction-based word embeddings. *PloS ONE*, 12(9):3370–3374, 2017.

[5] Hongyun Cai, Vincent W. Zheng, and Kevin Chen-Chuan Chang. A comprehensive survey of graph embedding: Problems, techniques and applications. Technical report, arXiv:1709.07604, 2017.

[6] Y. Chen and G. Medioni. Object modelling by registration of multiple range images. *Image and Vision Computing*, 10:145–155, 1992.

[7] Yuxiao Dong, Nitesh V. Chawla, and Ananthram Swami. metapath2vec: Scalable representation learning for heterogeneous networks. In *KDD*, 2017.

[8] D.W. Eggert, A. Lorusso, and R.B. Fisher. Estimating 3-d rigid body transformations: A comparison of four major algorithms. *Machine Vision and Applications*, 9:272–290, 1997.

[9] O. D. Faugeras and M. Hebert. A 3-d recognition and positioning algorithm using geometric matching between primitive surfaces. In *Proceedings International Joint Conference on Artificial Intelligence*, volume 8, pages 996–1002, August 1983.

[10] Lev Finkelstein, Evgeniy Gabrilovich, Yossi Matias, Elad Rivlin, Zach Solan, Gadi Wolfman, and Eytan Ruppin. Placing search in context : The concept revisited. *ACM Transactions of Information Systems*, 20:116–131, 2002.

[11] Palash Goyal and Emilio Ferrara. Graph embedding techniques, applications, and performance: A survey. Technical report, arXiv:1705.02801, 2017.

[12] Aditya Grover and Jure Leskovec. node2vec: Scalable feature learning for networks. In *KDD*, 2016.

[13] Richard J. Hanson and Michael J. Norris. Analysis of measurements based on the singular value decomposition. *SIAM Journal of Scientific and Statistical Computing*, 27(3):363–373, 1981.

[14] K. M. Hermann and P. Blunsom. Multilingual Distributed Representations without Word Alignment. *ArXiv e-prints*, December 2013.

[15] F Hill, R Reichart, and A Korhonen. Simlex-999: Evaluating semantic models with (genuine) similarity estimation. *Computational Linguistics*, 41:665–695, 2015.

[16] Berthold K. P. Horn. Closed-form solution of absolute orientation using unit quaternions. *Journal of the Optical Society of America A*, 4, April 1987.

[17] Omer Levy and Yoav Goldberg. Linguistic regularities of sparse and explicit word representations. In *CoNLL*, 2014.

[18] Omer Levy and Yoav Goldberg. Neural word embedding as implicit matrix factorization. In *NIPS*, 2014.

[19] David G. Lowe. Distinctive image features from scale-invariant keypoints. *International Journal of Computer Vision*, 2004.

[20] Minh-Thang Luong, Hieu Pham, and Christopher D. Manning.

[21] Tomas Mikolov, Kai Chen, Greg Corrado, and Jeffrey Dean. Efficient estimation of word representations in vector space. Technical report, arXiv:1301.3781, 2013.
[22] Tomas Mikolov, Q. V. Le, and Ilya Sutskever. Exploiting Similarities among Languages for Machine Translation. *ArXiv e-prints*, September 2013.

[23] Tomas Mikolov, Ilya Sutskever, Kai Chen, Greg Corrado, and Jeffrey Dean. Distributed representations of words and phrases and their compositionality. In *NIPS*, 2013.

[24] G Miller and W Charles. Contextual correlates of semantic similarity. In *Language and Cognitive Processes*, volume 6, pages 1–28, 1998.

[25] Jeffrey Pennington, Richard Socher, and Christopher D. Manning. Glove: Global vectors for word representation. In *EMNLP*, 2014.

[26] Bryan Perozzi, Rami Al-Rfou, and Steven Skiena. Deepwalk: Online learning of social representations. In *KDD*, 2014.

[27] Ali Rahimi and Ben Recht. Random features for large-scale kernel machines. In *NIPS*, 2007.

[28] Ali Rahimi and Ben Recht. Weighted sums of random kitchen sinks: Replacing minimization with randomization in learning. In *NIPS*, 2008.

[29] Radim Řehůřek and Petr Sojka. Software Framework for Topic Modelling with Large Corpora. In *Proceedings of the LREC 2010 Workshop on New Challenges for NLP Frameworks*, pages 45–50, Valletta, Malta, May 2010. ELRA. [http://is.muni.cz/publication/884893/en](http://is.muni.cz/publication/884893/en).

[30] H Rubenstein and JB Goodenough. Contextual correlates of synonymy. In *Communications of the ACM*, volume 8, pages 627–633, 1965.

[31] Cem Safak Sahin, Rajmonda S. Caceres, Brandon Oselio, and William M. Campbell. Consistent Alignment of Word Embedding Models. *ArXiv e-prints*, February 2017.

[32] Peter H. Schönemann. A generalized solution to the orthogonal procrustes problem. *Psychometrika*, 31(1):1–10, March 1966.

[33] Jacob T. Schwartz and Micha Sharir. Identification of partially obscured objects in two and three dimensions by matching noisy characteristic curves. *International Journal on Robotics Research*, 6(2), Summer 1987.

[34] Qinfeng Shi, James Patterson, Gideon Dror, John Langford, Alex Smola, and SVN Vishwanathan. Hash kernels for structured data. *JMLR*, 10:2615–2637, 2009.

[35] Samuel L. Smith, David H. P. Turban, Steven Hamblin, and Nils Y. Hammerla. Offline bilingual word vectors, orthogonal transformations and the inverted softmax. *CoRR*, abs/1702.03859, 2017.

[36] Sridhar Mahadevan Thomas Boucher, CJ Carey and M. Darby Dyar. Aligning mixed manifolds, year = 2014. In *AAAI*.

[37] Michael W. Walker, Lejun Shao, and Richard A. Volz. Estimating 3-D location parameters using dual number quaternions. *CVGIP: Image Understanding*, 54:358–367, 1991.

[38] Kilian Weinberger, Anirban Dasgupta, John Attenberg, John Langford, and Alex Smola. Feature hashing for large scale multitask learning. In *ICML 2009*. 