A DECENTRALIZED PROXIMAL-GRADIENT METHOD WITH NETWORK INDEPENDENT STEP-SIZES AND SEPARATED CONVERGENCE RATES

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Abstract. This paper considers the problem of decentralized optimization with a composite objective containing smooth and non-smooth terms. To solve the problem, a proximal-gradient scheme is studied. Specifically, the smooth and nonsmooth terms are dealt with by gradient update and proximal update, respectively. The studied algorithm is closely related to a previous decentralized optimization algorithm, PG-EXTRA [37], but has a few advantages. First of all, in our new scheme, agents use uncoordinated step-sizes and the stable upper bounds on step-sizes are independent from network topologies. The step-sizes depend on local objective functions, and they can be as large as that of the gradient descent. Secondly, for the special case without non-smooth terms, linear convergence can be achieved under the strong convexity assumption. The dependence of the convergence rate on the objective functions and the network are separated, and the convergence rate of our new scheme is as good as one of the two convergence rates that match the typical rates for the general gradient descent and the consensus averaging. We also provide some numerical experiments to demonstrate the efficacy of the introduced algorithms and validate our theoretical discoveries.

Key words. decentralized optimization, proximal-gradient, convergence rates, network independent

1. Introduction. This paper focuses on the following decentralized optimization problem:

\begin{equation}
\min_{x \in \mathbb{R}^p} \bar{f}(x) := \frac{1}{n} \sum_{i=1}^{n} (s_i(x) + r_i(x)),
\end{equation}

where $s_i : \mathbb{R}^p \to \mathbb{R}$ and $r_i : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ are two lower semi-continuous proper convex functions held privately by agent $i$ to encode the agent’s objective function. We assume that one function (e.g., without loss of generality, $s_i$) is differentiable with a Lipschitz continuous gradient with parameter $L > 0$, and the other function $r_i$ is proximable, i.e., its proximal mapping

\[ \text{prox}_{\lambda r_i}(x) = \arg \min_{x \in \mathbb{R}^p} \lambda r_i(x) + \frac{1}{2} \|x - y\|^2, \]

has a closed-from solution or can be computed easily. Examples of $s_i$ include linear functions, quadratic functions, and logistic functions, while $r_i$ could be the $\ell_1$ norm, total variation, or indicator functions of convex constraints. In addition, we assume that the agents are connected through a fixed bi-directional communication network. Every agent in the network wants to obtain an optimal solution of (1) while it can only receive/send messages that are not sensitive from/to its immediate neighbors.

Specific problems of form (1) that require such a decentralized computing architecture have appeared in various areas including networked multi-vehicle coordination, distributed information processing, and decision making in sensor networks. Some examples include distributed average consensus [6, 28, 45], distributed spectrum sensing [1], information control [27, 34], power systems control [13, 32], statistical inference and learning [12, 23, 30]. In general, decentralized optimization fits the scenarios where the data is collected and/or stored in a distributed network. A fusion center is either inapplicable or unaffordable, and/or computing is required to be performed in a distributed but collaborative manner by multiple agents or network designers.

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1We believe that agent $i$’s instantaneous estimation on the optimal solution is not a piece of sensitive information but the functions $s_i$ and $r_i$ are.

1
1.1. Literature Review. The study on distributed algorithms dates back to early 1980s [2,41]. Since then, due to the emergence of large-scale networks, decentralized (optimization) algorithms, as a special type of distributed algorithms for solving problem (1), have received significant attentions. Many efforts have been made on star networks with one master agent and multiple slave agents [5,7]. This scheme is “centralized” due to the use of a “master” agent and may suffer a single point of failure and violate the privacy requirement in certain applications. In this paper, we focus on solving (1) in a decentralized fashion, where there is no “master” agent that is connected to all other agents.

The incremental algorithms [3,17–19,31,42] can solve (1) without the need of a “master” but requires that the network contains a ring structure. To handle general (possibly time-varying) networks, the distributed sub-gradient algorithm was proposed in [24]. It and its variants [16,33] are intuitive and simple but usually slow due to the diminishing step-size that is needed to obtain a consensual and optimal solution, even if the objective functions are differentiable and strongly convex. With a fixed step-size, these distributed methods can be fast, but they only converge to a neighborhood of the solution set. This phenomenon creates an exactness-speed dilemma.

A class of distributed approaches that bypasses this dilemma is based on introducing Lagrangian dual. The resulting algorithms include distributed dual decomposition [40] and decentralized alternating direction method of multipliers (ADMM) [4]. The decentralized ADMM and its proximal-gradient variant can employ a fixed step-size to achieve $O(1/k)$ rate under general convexity assumptions [8,14,43]. Under the strong convexity assumption, the decentralized ADMM has been shown to have linear convergence for time-invariant undirected graphs [38]. There exist some other distributed methods that do not (explicitly) use dual variables but can still converge to an exact consensual solution with fixed step-sizes. In particular, works in [9,15] employ multi-consensus inner loops, Nesterov’s acceleration, and/or the adapt-then-combine (ATC) strategy. Under the assumption that the objectives have bounded and Lipschtiz gradients\(^2\), the algorithm proposed in [15] is shown to have $O(\ln(k)/k^2)$ rate. References [36,37] use a difference structure to cancel the steady state error in decentralized gradient descent [24,47], thereby developing the algorithm EXTRA and its proximal-gradient variant PG-EXTRA. It converges at an $O(1/k)$ rate when the objective function in (1) is convex, and it has a linear convergence rate when the objective function is strongly convex.

A number of recent works employed the so-called gradient tracking technique [51] in distributed optimization to conquer different issues [11,21,25,29,46]. To be specific, works [21,46] relax the step-size rule to allow uncoordinated step-sizes across agents. Paper [11] solves non-convex optimization problems. Paper [25] aims at achieving geometric convergence over time-varying graphs. Work [29] improves the convergence rate over EXTRA, and its formulation is the same as that in [21].

Another topic of interest is decentralized optimization over directed graphs [20,25,39,44,50], which is beyond the scope of this paper.

1.2. Proposed Algorithm. By making a simple modification over PG-EXTRA [37], our proposed algorithm brings a big improvement in the speed and the dependency of convergence over networks. To better expose this simple modification, let us compare our proposed algorithm with the EXTRA algorithm for the smooth case, i.e., $r_i(x) = 0$.

\[
\begin{align*}
(\text{EXTRA}) \quad x^{k+2} &= (I + W)x^{k+1} - \frac{1+W}{2}x^k - \alpha \nabla s(x^{k+1}) + \alpha \nabla s(x^k), \\
(\text{Proposed NIDS}) \quad x^{k+2} &= (I + W)x^{k+1} - \frac{1+W}{2}(x^k + \alpha \nabla s(x^{k+1}) - \alpha \nabla s(x^k)).
\end{align*}
\]

Here in both algorithms\(^3\), the doubly stochastic matrix $W$ is the mixing matrix that is used to represent information exchange between neighboring agents. See Assumption 1 for more details about setting this matrix. There is only a small difference between EXTRA and the proposed algorithm.

\(^2\)This means that the nonsmooth terms $r_i$’s are absent. Such assumption is much stronger than the one used for achieving the $O(1/k^2)$ rate in Nesterov’s optimal gradient method [26].

\(^3\)In the original EXTRA, two mixing matrices $W$ and $\bar{W}$ are used. For simplicity, we take $\bar{W} = \frac{I + W}{2}$ here.
on what to communicate between the agents. In fact, the ATC strategy is applied on the successive difference of gradients. Because of this small modification, the proposed algorithm can have 
Network Independent Step-sizes. Therefore we call our proposed algorithm NIDS in the following. For the nonsmooth case, more detailed comparison between PG-EXTRA and NIDS will be followed in Section 2.

The proposed protocol works as long as each agent can estimate their local functional parameters. There is no need for any agent to have any sense of the global situation including the number of agents in the whole network. In other words, an agent in the network only needs to agree on the optimization framework and participate computation and information exchange at certain time/frequency with pure local knowledge. There is no need for any agent to negotiate any system level parameters thus it is less likely to compromise privacy.

Large step-size of NIDS: All of the works mentioned above either employ pessimistic step-sizes or have network dependent bounds on step-sizes. Furthermore, the step-sizes for the strongly convex case are more conservative. For example, the linear convergence rate has been achieved by EXTRA in [36] with the step-size being in the order of $O(\mu/L^2)$, where $\mu$ is the strong convexity constant of the function $s(x) = \sum_n s_i(x)$. This step-size has a loss of $\mu/L$ compared to $O(1/L)$ for the centralized gradient descent. Though faster convergence is achieved in the ATC variant of DIGing [21], its step-size is still very conservative compared to $O(1/L)$. We will show that the step-size of NIDS can has the same order as that of the (centralized) gradient descent. The achievable step-sizes of NIDS for $O(1/k)$ rate in the general convex case and the linear convergence rate in the strongly convex case are at the order of $O(1/L)$. Specifically, we can choose the step-size for agent $i$ as large as $2/L_i$ on any connected network.

Sublinear convergence rate of NIDS for the general case: Under the general convexity assumption, we show that NIDS has a convergence rate of $o(1/k)$, which is slightly better than the $O(1/k)$ rate of PG-EXTRA. Because the step-size of NIDS does not depend on the network topology and is much larger than that of PG-EXTRA, NIDS can be much faster than PG-EXTRA, as shown in the numerical experiments.

Linear convergence rate of NIDS for the strongly convex case: For the case where the non-smooth terms are null and the functions $\{s_i\}^n_{i=1}$ are strongly convex, we show that NIDS achieves a linear convergence rate whose dependencies on the functions $\{s_i\}^n_{i=1}$ and the network topology are decoupled. To be specific, to reach $\epsilon$-accuracy, the number of iterations needed for NIDS is

$$O \left( \max \left( \frac{L}{\mu}, \frac{1 - \lambda_n(W)}{1 - \lambda_2(W)} \right) \right) \log \frac{1}{\epsilon}.$$ 

Both $\frac{L}{\mu}$ and $\frac{1 - \lambda_n(W)}{1 - \lambda_2(W)}$ are typical in the literatures of optimization and average consensus, respectively. The scaling factor $\frac{L}{\mu}$, also called the condition number of the objective function, is aligned with the scalability for the standard gradient descent [26]. The scaling factor $\frac{1 - \lambda_n(W)}{1 - \lambda_2(W)} = O(n^2)$, could also be understood as the condition number of the network, is aligned with the scalability of the simplest linear iterations for distributed averaging [22].

Separating the condition numbers of the objective function and the network provides a way to determine the bottleneck of NIDS for a specific problem and a given network. Therefore, the system designer might be able to smartly apply preconditioning on $\{s_i\}^n_{i=1}$ or improve the connectivity of the network to cost-effectively obtain a better convergence.

Finally, we note that, references [48, 49], appearing simultaneously with this work, also proposed (2b) to enlarge the step-size and use column stochastic matrices rather than symmetric doubly stochastic matrices. However, their algorithm only works for smooth problems, and their analysis seems to be restrictive and requires twice differentiability and strong convexity of $\{s_i\}^n_{i=1}$.

\[\text{When we heuristically choose } W = I - \tau L \text{ where } L \text{ is the Laplacian of the underlying graph and } \tau \text{ is a positive tunable constant, we have } \frac{1 - \lambda_n(W)}{1 - \lambda_2(W)} = \frac{\lambda_n(L)}{\lambda_{n-1}(L)} \text{ which is the condition number of } L. \text{ Note that } \lambda_n(L) = 0.\]
1.3. Future Works. The capability of our algorithm using purely locally determined parameters increases its potential to be extended to working over dynamic networks with time-varying number of nodes. Given such flexibility, we may use similar scheme to solve the decentralized empirical risk minimization problems. Furthermore, it also enhances the privacy of the agents through allowing each to perform their own optimization procedure without negotiation on any parameters.

By using Nesterov’s acceleration technique, reference [28] shows that the scaling factor of a new average consensus protocol can be improved to $O(n)$; When the nonsmooth terms $r_i$’s are absent, reference [35] shows that the scaling factor of a new dual based accelerated distributed gradient method can be improved to $O(\sqrt{L/\mu})$. One of our future work would be exploring the convergence rates/scalability of the Nesterov’s accelerated version of our algorithm.

1.4. Paper Organization. The rest of this paper is organized as follows. To facilitate the description of the technical ideas, the algorithms, and the analysis, we introduce the notation in Subsection 1.5. In Section 2, we introduce our algorithm NIDS and discuss its relation to some other existing algorithms. In Section 3, we first show that NIDS can be understood as an iterative algorithm for seeking a fixed point. Following this, we establish that NIDS converges at an $o(1/k)$ rate for the general case and a linear rate for the strongly convex case. Then, numerical simulations are given in Section 4 to corroborate our theoretical claims. Final remarks are given in Section 5.

1.5. Notation. Agent $i$ holds a local variable $x_i \in \mathbb{R}^p$, whose value at the $k$th iteration is denoted by $x_i^k$. We introduce an objective function that sums all the local ones as

$$f(x) := \frac{1}{n} \sum_{i=1}^{n} (s_i(x_i) + r_i(x_i))$$

where

$$x := \begin{pmatrix} -x_1^\top & \cdots & -x_n^\top \end{pmatrix} \in \mathbb{R}^{n \times p}.$$  

If all local variables are identical, i.e., $x_1 = \cdots = x_n$, we call that $x$ is consensual. In addition, we define

$$s(x) := \frac{1}{n} \sum_{i=1}^{n} s_i(x_i), \quad r(x) := \frac{1}{n} \sum_{i=1}^{n} r_i(x_i).$$

Therefore, $f(x) = s(x) + r(x)$. The gradient of $s$ at $x$ is given in the same way as $x$ in (4) by

$$\nabla s(x) := \begin{pmatrix} - (\nabla s_1(x_1))^\top \\ - (\nabla s_2(x_2))^\top \\ \vdots \\ - (\nabla s_n(x_n))^\top \end{pmatrix} \in \mathbb{R}^{n \times p}.$$ 

We use bold upper-case letters such as $W$ to define matrix in $\mathbb{R}^{n \times n}$ and bold lower-case letter such as $x$ and $z$ to define matrix in $\mathbb{R}^{n \times p}$. Let $1$ and $0$ be matrices with all ones and zeros, respectively, and their dimensions are provided when necessary. For matrices $x, y \in \mathbb{R}^{n \times p}$, we define the inner product of $x$ and $y$ as $\langle x, y \rangle = \text{tr}(x^\top y)$ and the induced norm as $\|x\| = \sqrt{\langle x, x \rangle}$. Additionally, by an abuse of notation, we define $\langle x, y \rangle_Q = \text{tr}(x^\top Q y)$ and $\|x\|_Q = \langle x, x \rangle_Q$ for any given symmetric matrix $Q \in \mathbb{R}^{n \times n}$. Note that $\langle , \rangle_Q$ is an inner product defined in $\mathbb{R}^{n \times p}$ if and only if $Q$ is positive definite. However, when $Q$ is not positive definite, $\langle x, y \rangle_Q$ can still be an inner product defined in a subspace of

1.5.
We define the range of $\mathbf{A} \in \mathbb{R}^{n \times n}$ by $\text{range}(\mathbf{A}) := \{ \mathbf{x} \in \mathbb{R}^{n \times p} : \mathbf{x} = \mathbf{Ay}, \mathbf{y} \in \mathbb{R}^{n \times p} \}$. Let $\lambda_i(\mathbf{A})$ define the $i$th largest eigenvalue of a symmetric matrix $\mathbf{A}$. The largest eigenvalue of a symmetric matrix $\mathbf{A}$ is also denoted as $\lambda_{\text{max}}(\mathbf{A})$. For two symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, $\mathbf{A} \succeq \mathbf{B}$ (or $\mathbf{A} \succ \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is positive definite (or positive semidefinite).

### 2. Proposed Algorithm NIDS.

In this section, we describe our proposed NIDS in Algorithm 1 for solving (1) in more detail and explain the connections between other related methods.

#### Algorithm 1 NIDS

Set mixing matrix: $\mathbf{W} \in \mathbb{R}^{n \times n}$.

Choose parameters: $\{\alpha_i\}_{i=1}^n$, $c$, and $\mathbf{W} := [\mathbf{w}_{ij}] \in \mathbb{R}^{n \times n}$;

all agents $i = 1, \ldots, n$ pick arbitrary initial $x_0^i \in \mathbb{R}^p$ and do

$$z_i^1 = x_0^i - \alpha_i \nabla s_i(x_0^i),$$

$$x_i^1 = \arg\min_{x \in \mathbb{R}^p} \alpha_i r_i(x) + \frac{1}{2} \|x - z_i^1\|^2.$$

for $k = 1, 2, 3 \ldots$

all agents $i = 1, \ldots, n$ do

$$z_i^{k+1} = z_i^k - x_i^k + \sum_{j=1}^n \mathbf{w}_{ij} (2x_j^k - x_j^{k-1} - \alpha_j \nabla s_j(x_j^k) + \alpha_j \nabla s_j(x_j^{k-1})), $$

$$x_i^{k+1} = \arg\min_{x \in \mathbb{R}^p} \alpha_i r_i(x) + \frac{1}{2} \|x - z_i^{k+1}\|^2.$$

end for

The mixing matrix satisfies the following assumption, which comes from [36, 37].

**Assumption 1** (Mixing matrix). *The connected network $G = \{\mathcal{V}, \mathcal{E}\}$ consists of a set of agents $\mathcal{V} = \{1, 2, \ldots, n\}$ and a set of undirected edges $\mathcal{E}$. An undirected edge $(i, j) \in \mathcal{E}$ means that there is a connection between agents $i$ and $j$ and both agents can exchange data. The mixing matrix $\mathbf{W} = [\mathbf{w}_{ij}] \in \mathbb{R}^{n \times n}$ satisfies:*

1. (Decentralized property). If $i \neq j$ and $(i, j) \notin \mathcal{E}$, then $w_{ij} = 0$;
2. (Symmetry). $\mathbf{W} = \mathbf{W}^T$;
3. (Null space property). $\text{Null}(\mathbf{I} - \mathbf{W}) = \text{span}(\mathbf{1}_{1 \times n})$;
4. (Spectral property). $2\mathbf{I} \succeq \mathbf{W} + \mathbf{I} > \mathbf{0}_{n \times n}$.

**Remark 1.** Assumption 1 implies that the eigenvalues of $\mathbf{W}$ lie in $(−1, 1]$ and the multiplicity of eigenvalue 1 is one, i.e., $1 = \lambda_1(\mathbf{W}) > \lambda_2(\mathbf{W}) \geq \cdots \geq \lambda_n(\mathbf{W}) > −1$. Item 3 of Assumption 1 shows that $(\mathbf{I} - \mathbf{W})\mathbf{1}_{1 \times n} = \mathbf{0}$ and the orthogonal complement of $\text{span}(\mathbf{1}_{1 \times n})$ is the row space of $\mathbf{I} - \mathbf{W}$, which is also the column space of $\mathbf{I} - \mathbf{W}$ because of the symmetry of $\mathbf{W}$.

The functions $\{s_i\}_{i=1}^n$ and $\{r_i\}_{i=1}^n$ satisfy the following assumption.

**Assumption 2.** Functions $\{s_i(x)\}_{i=1}^n$ and $\{r_i(x)\}_{i=1}^n$ are lower semi-continuous proper convex, and $\{s_i(x)\}_{i=1}^n$ have Lipschitz continuous gradients with constants $\{L_i\}_{i=1}^n$, respectively. Thus, we have

$$\langle x - y, \nabla s(x) - \nabla s(y) \rangle \geq \|\nabla s(x) - \nabla s(y)\|_{L^{-1}},$$

where $L = \text{Diag}(L_1, \ldots, L_n)$ is the diagonal matrix with the Lipschitz constants [26].

Instead of using the same step-size for all the agents, we allow agent $i$ to choose its own step-size
\(\alpha_i\) and let \(\Lambda = \text{Diag}(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^{n \times n}\). Then NIDS can be expressed as

\begin{align}
\text{(8a)} & \quad z^{k+1} = z^k - x^k + \tilde{W}(2x^k - x^{k-1} - \Lambda \nabla s(x^k) + \Lambda \nabla s(x^{k-1})), \\
\text{(8b)} & \quad x^{k+1} = \arg \min_{x \in \mathbb{R}^{n \times p}} r(x) + \frac{1}{2\alpha} \|x - z^{k+1}\|_A^2,
\end{align}

where \(\tilde{W} = I - c\Lambda(I - W)\) and \(c\) is chosen such that \(\Lambda^{-1/2}\tilde{W}\Lambda^{1/2} = I - c\Lambda^{1/2}(I - W)\Lambda^{1/2} = 0\).

If all agents choose the same step-size, i.e., \(\Lambda = \alpha I\). We can choose \(c = 1/(2\alpha)\) from Assumption 1, and (8) becomes

\begin{align}
\text{(9a)} & \quad z^{k+1} = z^k - x^k + \frac{1 + W}{2}(2x^k - x^{k-1} - \alpha \nabla s(x^k) + \alpha \nabla s(x^{k-1})), \\
\text{(9b)} & \quad x^{k+1} = \arg \min_{x \in \mathbb{R}^{n \times p}} r(x) + \frac{1}{2\alpha} \|x - z^{k+1}\|^2.
\end{align}

**Remark 2.** When the mixing matrix \(\tilde{W}\) in the PG-EXTRA algorithm is typically chosen as \(\tilde{W} = \frac{1 + W}{2}\), the updates of PG-EXTRA algorithm are equivalent to

\begin{align}
\text{(10a)} & \quad z^{k+1} = z^k - x^k + \frac{1 + W}{2}(2x^k - x^{k-1} - \alpha \nabla s(x^k) + \alpha \nabla s(x^{k-1})), \\
\text{(10b)} & \quad x^{k+1} = \arg \min_{x \in \mathbb{R}^{n \times p}} r(x) + \frac{1}{2\alpha} \|x - z^{k+1}\|^2.
\end{align}

The only difference between NIDS and PG-EXTRA is that the mixing operation is further applied to the successive difference of the gradients \(-\alpha \nabla s(x^k) + \alpha \nabla s(x^{k-1})\) in NIDS.

When there is no nonsmooth function \(r(x)\), (8) becomes

\begin{align}
\text{(11)} & \quad x^{k+1} = \tilde{W}(2x^k - x^{k-1} - \Lambda \nabla s(x^k) + \Lambda \nabla s(x^{k-1})),
\end{align}

and it reduces to (2b) when \(\Lambda = \alpha I\) and \(c = 1/(2\alpha)\). Note that, though (2b) appears in [48, 49], its convergence still needs a small step-size that also depends on the network topology and the strongly convex constant. In Theorem 1 of [49], the upper bound for the step-size is also \(O(\mu/L^2)\), which is the same as that of PG-EXTRA.

**3. Convergence Analysis of NIDS.** In order to show the convergence of NIDS, we also need the following assumption.

**Assumption 3 (Solution existence).** Problem (1) has at least one solution.

To simplify the analysis, we introduce a new sequence \(\{d^k\}_{k \geq 0}\) which is defined as

\begin{align}
\text{(12)} & \quad d^k := \Lambda^{-1}(x^{k-1} - z^k) - \nabla s(x^{k-1}).
\end{align}

By using the sequence \(\{z^k\}_{k \geq 0}\), we obtain a recursive (update) relation for \(\{d^k\}_{k \geq 0}\):

\begin{align*}
d^{k+1} = & \quad \Lambda^{-1}(x^k - z^{k+1}) - \nabla s(x^k) \\
= & \quad \Lambda^{-1}(x^k - z^k + x^k) - \nabla s(x^k) \\
& \quad - \Lambda^{-1}\tilde{W}(2x^k - x^{k-1} - \Lambda \nabla s(x^k) + \Lambda \nabla s(x^{k-1})) \\
= & \quad \Lambda^{-1}(x^k - z^k + x^k - 2x^k + x^{k-1}) - \nabla s(x^k) + \nabla s(x^k) - \nabla s(x^{k-1}) \\
& \quad + c(I - W)(2x^k - x^{k-1} - \Lambda \nabla s(x^k) + \Lambda \nabla s(x^{k-1})) \\
= & \quad d^k + c(I - W)(2x^k - z^k - \Lambda \nabla s(x^k) - \Lambda d^k),
\end{align*}
where the second equality comes from the update of $z^{k+1}$ in (8a) and the last one holds because of the definition of $d^k$ in (12). Therefore, the iteration (8) is equivalent to, with the update order $(x, d, z)$,

$$
(13a) \quad x^k = \arg\min_{x \in \mathbb{R}^{n \times p}} r(x) + \frac{1}{2}\|x - z^k\|_{\Lambda^{-1}}^2,
$$

$$
(13b) \quad d^{k+1} = d^k + c(I - W) (2x^k - z^k - \Lambda\nabla s(x^k) - \Lambda d^k),
$$

$$
(13c) \quad z^{k+1} = x^k - \Lambda\nabla s(x^k) - \Lambda d^{k+1},
$$
in the sense that both (8) and (13) generate the same $\{x^k, z^k\}_{k>0}$ sequence.

Because $x^k$ is determined by $z^k$ only and can be eliminated from the iteration, iteration (13) is essentially an operator for $(d, z)$. Note that we have $d^1 = \Lambda^{-1}(x^0 - z^1) - \nabla s(x^0) = 0$ from Algorithm 1. Therefore, from the update of $d^{k+1}$ in (13b), $d^k \in \text{range}(I - W)$ for all $k$. In fact, any $z^1$ such that $d^1 \in \text{range}(I - W)$ works for NIDS. The following two lemmas show the relation between fixed points of (13) and optimal solutions of (1).

**Lemma 1** (Fixed point of (13)). $(d^*, z^*)$ is a fixed point of (13) if and only if there exists a subgradient $q^* \in \partial r(x^*)$ such that:

$$
(14a) \quad d^* + \nabla s(x^*) + q^* = 0,
$$

$$
(14b) \quad (I - W)x^* = 0.
$$

and $z^* = x^* + \Lambda q^*$.

**Proof.** $\Rightarrow$ If $(d^*, z^*)$ is a fixed point of (13), we have

$$
0 = c(I - W)(2x^* - z^* - \Lambda\nabla s(x^*) - \Lambda d^*) = c(I - W)x^*,
$$

where the two equalities come from (13b) and (13c), respectively. Combining (13c) and (13a) gives

$$
0 = z^* - x^* + \Lambda\nabla s(x^*) + \Lambda d^* = \Lambda(q^* + \nabla s(x^*) + d^*),
$$

where $q^* \in \partial r(x^*)$.

$\Leftarrow$ In order to show that $(d^*, z^*)$ is a fixed point of iteration (13), we just need to verify that $(d^{k+1}, z^{k+1}) = (d^*, z^*)$ if $(d^k, z^k) = (d^*, z^*)$. From (13a), we have $x^k = x^*$, then

$$
d^{k+1} = d^* + c(I - W)(2x^* - z^* - \Lambda\nabla s(x^*) - \Lambda d^*) = d^* + c(I - W)x^* = d^*,
$$

$$
z^{k+1} = x^* - \Lambda\nabla s(x^*) - \Lambda d^* = x^* + \Lambda q^* = z^*.
$$

Therefore, we have $(d^*, z^*)$ being a fixed point of iteration (13). \qed

**Lemma 2** (Optimality condition). $x^*$ is consensual with $x^n_1 = x^n_2 = \cdots = x^n_n = x^*$ being an optimal solution of problem (1) if and only if there exists $p^*$ and a subgradient $q^* \in \partial r(x^*)$ such that:

$$
(15a) \quad (I - W)p^* + \nabla s(x^*) + q^* = 0,
$$

$$
(15b) \quad (I - W)x^* = 0.
$$

In addition, $(d^* = (I - W)p^*, z^* = x^* + \Lambda q^*)$ is a fixed point of iteration (13).

**Proof.** $\Rightarrow$ Because $x^* = 1_{n \times 1}(x^*)^\top$, we have $(I - W)x^* = (I - W)1_{n \times 1}(x^*)^\top = 0_{n \times 1}(x^*)^\top = 0$. The fact that $x^*$ is an optimal solution of problem (1) means there exists $q^* \in \partial r(x^*)$ such that $(\nabla s(x^*) + q^*)^\top 1_{n \times 1} = 0$. That is to say all columns of $\nabla s(x^*) + q^*$ are orthogonal to $1_{n \times 1}$. Therefore, Remark 1 shows the existence of $p^*$ such that $(I - W)p^* + \nabla s(x^*) + q^* = 0$.

$\Leftarrow$ Equation (15b) shows that $x^*$ is consensual because of item 3 of Assumption 1, i.e., $x^* = 1_{n \times 1}(x^*)^\top$ for some $x^*$. From (15a), we have $0 = ((I - W)p^* + \nabla s(x^*) + q^*)^\top 1_{n \times 1} = (p^*)^\top (I - W)1_{n \times 1} + (\nabla s(x^*) + q^*)^\top 1_{n \times 1} = (\nabla s(x^*) + q^*)^\top 1_{n \times 1}$. Thus, $0 \in \sum_{i=1}^n(\nabla s_i(x^*) + \partial r_i(x^*))$ because $x^*$ is consensual. This completes the proof for the equivalence.

Lemma 1 shows that $(d^* = (I - W)p^*, z^* = x^* + \Lambda q^*)$ is a fixed point of iteration (13). \qed
Lemma 2 shows that we can find a fixed point of iteration (13) to obtain an optimal solution of problem (1). In addition, Lemma 2 tells us that we need \( d^* \in \text{range}(I - W) \) to get the optimal solution of problem (1). Therefore, we need \( d^1 \in \text{range}(I - W) \).

**Lemma 3 (Norm over range space).** For any symmetric positive semidefinite matrix \( A \in \mathbb{R}^{n \times n} \) with rank \( r \) (\( r \leq n \)), let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \) be its \( r \) eigenvalues. Then \( \text{range}(A) \) is a \( r \)-dimensional subspace in \( \mathbb{R}^{n \times p} \) and has a norm defined by \( \| x \|_{A^1}^2 := \langle x, A^1 x \rangle \), where \( A^1 \) is the pseudo inverse of \( A \). In addition, \( \lambda_1^{-1} \| x \|^2 \leq \| x \|_{A^1}^2 \leq \lambda_r^{-1} \| x \|^2 \) for all \( x \in \text{range}(A) \).

**Proof.** Let \( A = USU^T \), where \( \Sigma = \text{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_r) \) and the columns of \( U \) are orthonormal eigenvectors for corresponding eigenvalues, i.e., \( U \in \mathbb{R}^{n \times r} \) and \( U^T U = I_{r \times r} \). Then \( A^1 = U \Sigma^{-1} U^T \), where \( \Sigma^{-1} = \text{Diag}(\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_r^{-1}) \).

Letting \( x = Ay \), we have \( \| x \|^2 = \langle UΣU^T y, UΣU^T y \rangle = \langle ΣU^Ty, ΣU^Ty \rangle = \| ΣU^Ty \|^2 \). In addition,

\[
\langle x, A^1 x \rangle = \langle Ay, A^1 Ay \rangle = \langle UΣU^Ty, UΣU^Ty \rangle = \langle ΣU^Ty, ΣU^Ty \rangle = \| ΣU^Ty \|^2.
\]

Therefore,

\[
\lambda_1^{-1} \| x \|^2 = \lambda_1^{-1} \| ΣU^Ty \|^2 \leq \langle x, A^1 x \rangle \leq \lambda_r^{-1} \| ΣU^Ty \|^2 = \lambda_r^{-1} \| x \|^2,
\]

which means that \( \| : : A^1 \| = \langle \cdot, A^1 \cdot \rangle \) is a norm for \( \text{range}(A) \).

Let \( M = c^{-1}(I - W)^\dagger - \Lambda \) and we have the following proposition.

**Proposition 4.** \( \| : : M \| \) is a norm defined for \( \text{range}(I - W) \).

**Proof.** Rewrite the matrix \( M \) as

\[
M = c^{-1}(I - W)^\dagger - \Lambda = \Lambda^{1/2}(c^{-1}\Lambda^{-1/2}(I - W)^\dagger \Lambda^{-1/2} - I)\Lambda^{1/2}
\]

\[
= \Lambda^{1/2}((c\Lambda^{1/2}(I - W)\Lambda^{1/2})^\dagger - I)\Lambda^{1/2}.
\]

Then, the proposition holds by using Lemma 3 and the condition \( \Lambda^{-1/2} W \Lambda^{1/2} \) being positive definite, which implies \( I > c\Lambda^{1/2}(I - W)\Lambda^{1/2} > 0 \).

The following lemma compares the distance to a fixed point of (13) for two consecutive iterates.

**Lemma 5 (Fundamental inequality).** Let \( (d^*, z^*) \) be a fixed point of iteration (13) and \( d^* \in \text{range}(I - W) \), then

\[
\| z^{k+1} - z^* \|^2_{\Lambda^{-1}} + \| d^{k+1} - d^* \|^2_M 
\leq \| z^k - z^* \|^2_{\Lambda^{-1}} + \| d^k - d^* \|^2_M - \| z^k - z^{k+1} \|^2_{\Lambda^{-1}} - \| d^k - d^{k+1} \|^2_M
\]

\[
+ 2\langle \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle + 2\langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle.
\]

**Proof.** From the update of \( z^{k+1} \) in (13c), we have

\[
(d^{k+1} - d^*, z^{k+1} - z^k + x^k - x^*)
\]

\[
= (d^{k+1} - d^*, 2x^k - z^k - \Lambda \nabla s(x^k) - \Lambda d^{k+1} - x^*)
\]

\[
= (d^{k+1} - d^*, c^{-1}(I - W)^\dagger (d^{k+1} - d^k) + \Lambda d^k - \Lambda d^{k+1})
\]

\[
= (d^{k+1} - d^*, d^{k+1} - d^k)_M,
\]

where the second equality comes from (13b), (15b), and \( d^{k+1} - d^* \in \text{range}(I - W) \). From (13a), we have that

\[
\langle x^k - x^*, z^k - z^k - z^* + x^* \rangle \Lambda^{-1} \geq 0.
\]
Therefore, we have
\[
\langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle \\
\leq \langle x^k - x^*, \Lambda^{-1}(z^k - x^k - z^* + x^* + \nabla s(x^k) - \nabla s(x^*)) \rangle \\
= \langle x^k - x^*, \Lambda^{-1}(z^k - z^{k+1}) - d^{k+1} + d^* \rangle \\
= \langle x^k - x^*, d^* - d^{k+1} \rangle_{\Lambda^{-1}} + \langle x^k - x^*, d^{k+1} \rangle \\
= \langle x^k - x^*, z^k - z^{k+1} \rangle_{\Lambda^{-1}} + \langle d^{k+1} - d^*, z^{k+1} - z^k \rangle - \langle d^{k+1} - d^*, d^{k+1} - d^* \rangle_{\Lambda^{-1}} \\
= \langle \Lambda^{-1}(x^k - x^*) - d^{k+1} + d^*, z^k - z^{k+1} \rangle - \langle d^{k+1} - d^*, d^{k+1} - d^* \rangle_{\Lambda^{-1}} \\
= \langle \Lambda^{-1}(z^{k+1} - z^*) + \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle - \langle d^{k+1} - d^*, d^{k+1} - d^* \rangle_{\Lambda^{-1}} \\
= \langle z^{k+1} - z^*, z^k - z^{k+1} \rangle_{\Lambda^{-1}} + \langle \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle \\
+ \langle d^{k+1} - d^*, d^{k+1} - d^* \rangle_{\Lambda^{-1}}.
\]

The inequality and the third equality comes from (19) and (18), respectively. The first and fifth equalities hold because of the update of $z^{k+1}$ in (13c). Using $2\langle a, b \rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2$ and rearranging the previous inequality give us that
\[
\begin{align*}
2\langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle - 2\langle \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle \\
\leq 2\langle z^{k+1} - z^*, z^k - z^{k+1} \rangle_{\Lambda^{-1}} + 2\langle d^{k+1} - d^*, d^k - d^{k+1} \rangle_{\Lambda^{-1}} + 2\langle \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle_{\Lambda^{-1}} \\
+ \|d^k - d^*\|^2_{\Lambda^{-1}} - \|d^{k+1} - d^*\|^2_{\Lambda^{-1}} \leq \langle z^{k+1} - z^*, d^k - d^{k+1} \rangle_{\Lambda^{-1}} + \|d^k - d^*\|^2_{\Lambda^{-1}} - \|d^{k+1} - d^*\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}}.
\end{align*}
\]

Therefore, (17) is obtained. \[\square\]

3.1. Sublinear convergence of NIDS. Using Assumption 2, we are able to show in the following lemma that the distance to a fixed point of (13) is decreasing.

Lemma 6 (A key inequality of descent). Let $(d^*, z^*)$ be a fixed point of (13) and $d^* \in \text{range}(I - W)$, then
\[
\|z^{k+1} - z^*\|^2_{\Lambda^{-1}} + \|d^{k+1} - d^*\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}} \\
\leq \|z^k - z^*\|^2_{\Lambda^{-1}} + \|d^k - d^*\|^2_{\Lambda^{-1}} - (1 - \max_i \frac{\alpha_i L_i}{2})(\|z^k - z^{k+1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}}).
\]

Proof. Young’s inequality and (7) give us
\[
\begin{align*}
2\langle \nabla s(x^k) - \nabla s(x^*), z^k - z^{k+1} \rangle - 2\langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle \\
\leq \frac{1}{2}\|z^k - z^{k+1}\|^2_2 + 2\|\nabla s(x^k) - \nabla s(x^*)\|_{\Lambda^{-1}}^2 - 2\|\nabla s(x^k) - \nabla s(x^*)\|_{\Lambda^{-1}}^2 \\
= \frac{1}{2}\|z^k - z^{k+1}\|^2_2.
\end{align*}
\]

Therefore, from (17), we have
\[
\begin{align*}
\|z^{k+1} - z^*\|^2_{\Lambda^{-1}} + \|d^{k+1} - d^*\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}} \\
\leq \|z^k - z^*\|^2_{\Lambda^{-1}} + \|d^k - d^*\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}} - (1 - \max_i \frac{\alpha_i L_i}{2})(\|z^k - z^{k+1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}}) \\
\leq \|z^k - z^*\|^2_{\Lambda^{-1}} + \|d^k - d^*\|^2_{\Lambda^{-1}} - (1 - \max_i \frac{\alpha_i L_i}{2})(\|z^k - z^{k+1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_{\Lambda^{-1}}).
\end{align*}
\]

This completes the proof. \[\square\]
Lemma 7 (Monotonicity of successive difference). If \( \alpha_i < 2/L_i \) for all \( i \), the sequence \( \{\|z^{k+1} - z^k\|_{\beta}^2 + \|d^{k+1} - d^k\|_M^2\}_{k \geq 0} \) is monotonically nonincreasing.

Proof. Similar to the proof for Lemma 5, we can show that

\[
\begin{align*}
(21) & \quad \langle d^{k+1} - d^k, z^{k+1} - z^k + x^k \rangle = \langle d^{k+1} - d^k, d^{k+1} - d^k \rangle_M, \\
(22) & \quad \langle d^{k+1} - d^k, z^k - z^{k-1} + x^{k-1} \rangle = \langle d^{k+1} - d^k, d^k - d^{k-1} \rangle_M, \\
(23) & \quad \langle x^k - x^{k-1}, z^k - x^k - z^{k-1} + x^{k-1} \rangle_{\Lambda} \geq 0.
\end{align*}
\]

Subtracting (22) from (21) on both sides, we have

\[
\begin{align*}
\langle d^{k+1} - d^k, x^k - x^{k-1} \rangle & = \|d^{k+1} - d^k\|_M^2 - \langle d^{k+1} - d^k, d^k - d^{k-1} \rangle_M + \langle d^{k+1} - d^k, 2z^k - z^{k-1} - z^{k+1} \rangle \\
& \geq \|d^{k+1} - d^k\|_M^2 - \frac{1}{2}\|d^{k+1} - d^k\|_M^2 - \frac{1}{2}\|d^k - d^{k-1}\|_M^2 \\
& \quad + \langle d^{k+1} - d^k, 2z^k - z^{k-1} - z^{k+1} \rangle \\
& = \frac{1}{2}\|d^{k+1} - d^k\|_M^2 - \frac{1}{2}\|d^k - d^{k-1}\|_M^2 + \langle d^{k+1} - d^k, 2z^k - z^{k-1} - z^{k+1} \rangle,
\end{align*}
\]

where the inequality comes from the Cauchy-Schwarz inequality. Then, the previous inequality, together with (23) and the Cauchy-Schwarz inequality, gives

\[
\begin{align*}
\langle x^k - x^{k-1}, \nabla s(x^k) - \nabla s(x^{k-1}) \rangle & \leq \langle x^k - x^{k-1}, \Lambda^{-1}(z^k - x^k - z^{k-1} + x^{k-1}) + \nabla s(x^k) - \nabla s(x^{k-1}) \rangle \\
& = \langle x^k - x^{k-1}, \Lambda^{-1}(z^k - z^{k+1} - z^{k-1} + z^k) - d^{k+1} + d^k \rangle \\
& \leq \langle x^k - x^{k-1}, z^k - z^{k+1} - z^{k-1} + z^k \rangle_{\Lambda} + \langle d^{k+1} - d^k, 2z^k - z^{k-1} - z^{k+1} \rangle \\
& \quad - \frac{1}{2}\|d^{k+1} - d^k\|_M^2 + \frac{1}{2}\|d^k - d^{k-1}\|_M^2 \\
& = \langle \Lambda^{-1}(z^{k+1} - z^k) + \nabla s(x^k) - \nabla s(x^{k-1}), z^k - z^{k+1} - z^{k-1} + z^k \rangle \\
& \quad - \frac{1}{2}\|d^{k+1} - d^k\|_M^2 + \frac{1}{2}\|d^k - d^{k-1}\|_M^2 \\
& \leq \langle z^{k+1} - z^k, z^k - z^{k+1} - z^{k-1} + z^k \rangle_{\Lambda} + \frac{1}{2}\|z^k - z^{k+1} - z^{k-1} + z^k\|_{\Lambda}^2 \\
& \quad + \frac{1}{2}\|\nabla s(x^k) - \nabla s(x^{k-1})\|_{\Lambda}^2 - \frac{1}{2}\|d^{k+1} - d^k\|_M^2 + \frac{1}{2}\|d^k - d^{k-1}\|_M^2 \\
& = \frac{1}{2}\|z^k - z^{k-1}\|_{\Lambda}^2 - \frac{1}{2}\|z^{k+1} - z^k\|_{\Lambda}^2 + \frac{1}{2}\|\nabla s(x^k) - \nabla s(x^{k-1})\|_{\Lambda}^2 \\
& \quad - \frac{1}{2}\|d^{k+1} - d^k\|_M^2 + \frac{1}{2}\|d^k - d^{k-1}\|_M^2.
\end{align*}
\]

The three inequalities hold because of (23), (24), and the Cauchy-Schwarz inequality, respectively.
The first and third equalities come from (13c). Rearranging the previous inequality, we obtain
\[
\|z^{k+1} - z^k\|^2_{\Lambda^{-1}} + \|d^{k+1} - d^k\|^2_M \\
\leq \|z^k - z^{k-1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k-1}\|^2_M + \frac{1}{2}\|\nabla s(x^k) - \nabla s(x^{k-1})\|^2_{\Lambda} \\
- \langle x^k - x^{k-1}, \nabla s(x^k) - \nabla s(x^{k-1}) \rangle \\
\leq \|z^k - z^{k-1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k-1}\|^2_M + \frac{1}{2}\|\nabla s(x^k) - \nabla s(x^{k-1})\|^2_{\Lambda-2L^{-1}} \\
\leq \|z^k - z^{k-1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k-1}\|^2_M.
\]
where the second and last inequalities come from (7) and $\Lambda < 2L^{-1}$, respectively. It completes the proof. 

**Theorem 8 (Sublinear rate).** If $\alpha_i < 2/L_i$ for all $i$, we have
\[
\|z^k - z^{k+1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_M \leq \frac{\|z^1 - z^\ast\|^2_{\Lambda^{-1}} + \|d^1 - d^\ast\|^2_M}{k(1 - \max_i \frac{\alpha_i}{2L_i})}
\]
Furthermore, $(d^k, z^k)$ converges to a fixed point $(\bar{d}, \bar{z})$ of iteration (13) and $\bar{d} \in \text{range}(I - W)$.

**Proof.** Lemma 7 shows that \( \{\|z^{k+1} - z^k\|^2_{\Lambda^{-1}} + \|d^{k+1} - d^k\|^2_M\}_{k \geq 0} \) is monotonically nonincreasing. Summing up (20) from 1 to $k$, we have
\[
\sum_{j=1}^{k} (\|z^j - z^{j+1}\|^2_{\Lambda^{-1}} + \|d^j - d^{j+1}\|^2_M) \\
\leq \frac{1}{1 - \max_i \frac{\alpha_i}{2L_i}} (\|z^1 - z^\ast\|^2_{\Lambda^{-1}} + \|d^1 - d^\ast\|^2_M - \|z^{k+1} - z^\ast\|^2_{\Lambda^{-1}} - \|d^{k+1} - d^\ast\|^2_M).
\]
Therefore, we have
\[
\|z^k - z^{k+1}\|^2_{\Lambda^{-1}} + \|d^k - d^{k+1}\|^2_M \leq \frac{1}{k} \sum_{j=1}^{k} (\|z^j - z^{j+1}\|^2_{\Lambda^{-1}} + \|d^j - d^{j+1}\|^2_M) \\
\leq \frac{1}{k(1 - \max_i \frac{\alpha_i}{2L_i})} (\|z^1 - z^\ast\|^2_{\Lambda^{-1}} + \|d^1 - d^\ast\|^2_M),
\]
and [10, Lemma 1] gives us (26).

Inequality (20) shows that the sequence $(d^k, z^k)$ is bounded, and there exists a convergent subsequence $(d^{k_i}, z^{k_i})$ such that $(d^{k_i}, z^{k_i}) \to (\bar{d}, \bar{z})$. Then (26) gives the convergence of $(d^{k_i+1}, z^{k_i+1})$. More specifically, $(d^{k_i+1}, z^{k_i+1}) \to (\bar{d}, \bar{z})$. Therefore $(\bar{d}, \bar{z})$ is a fixed point of iteration (13). In addition, because $d^k \in \text{range}(I - W)$ for all $k$, we have $\bar{d} \in \text{range}(I - W)$. Finally Lemma 6 implies the convergence of $(d^k, z^k)$ to $(\bar{d}, \bar{z})$. 

**Remark 3.** Recall that
\[
z^{k+1} - z^k = x^k - \Lambda \nabla s(x^k) - \Lambda d^{k+1} - z^k = -\Lambda(d^{k+1} + \nabla s(x^k) + q^k),
\]
where $q^k \in \partial r(x^k)$. Therefore, $\|z^{k+1} - z^k\|^2_{\Lambda^{-1}} \to 0$ implies the convergence in terms of $(15a)$.

Combining (13b) and (13c), we have
\[
d^{k+1} = d^k + c(I - W)(2x^k - z^k - \Lambda \nabla s(x^k) - \Lambda d^k) \\
= d^k + c(I - W)(2x^k - z^k + z^{k+1} - x^k + \Lambda d^{k+1} - \Lambda d^k) \\
= d^k + c(I - W)(x^k - z^k + z^{k+1}) + c(I - W)\Lambda(d^{k+1} - d^k).
\]
Rearranging it gives
\[(I - c(I - W)\Lambda) (d^{k+1} - d^k) = c(I - W) (x^k - z^k + z^{k+1}).\]

Then we have
\[
\|c(I - W) (x^k - z^k + z^{k+1})\|^2 = \|(I - c(I - W)\Lambda) (d^{k+1} - d^k)\|^2 \\
= \|c(I - W) M^{1/2} M^{1/2} (d^{k+1} - d^k)\|^2 \\
\leq \|c(I - W) M^{1/2}\|^2 \|d^{k+1} - d^k\|^2_\mathcal{M},
\]
where the second equality comes from \(d^{k+1} - d^k \in \text{range}(I - W)\). Thus \(\|x^{k+1} - z^k\|_\Lambda^{1/2} + \|d^{k+1} - d^k\|^2_\mathcal{M} \to 0\) implies the convergence in terms of (15b).

3.2. Linear convergence for special cases. In this subsection, we provide the linear convergence rate for the case when \(r(x) = 0\), i.e., \(z^k = x^k\) in NIDS. Theorem 9. If \(\{s_i(x)\}_{i=1}^n\) are strongly convex with parameters \(\{\mu_i\}_{i=1}^n\), then
\[
\langle x - y, \nabla s(x) - \nabla s(y) \rangle \geq \|x - y\|^2_S,
\]
where \(S = \text{Diag}(\mu_1, \ldots, \mu_n) \in \mathbb{R}^{n \times n}\). Let
\[
\rho = \max \left(1 - (2 - \max_i (\alpha_i L_i)) \min_i (\mu_i \alpha_i), 1 - \frac{c}{\lambda_{\max}(\Lambda^{-1/2}(I - W)^{1/2})} \right).
\]
Then we have
\[
\|x^{k+1} - x^*\|_\Lambda^{1/2} + \|d^{k+1} - d^*\|^2_\mathcal{M + \Lambda} \leq \rho \left(\|x^k - x^*\|^2_\Lambda^{1/2} + \|d^k - d^*\|^2_\mathcal{M + \Lambda}\right).
\]
Proof. From (17), we have
\[
\|x^{k+1} - x^*\|^2_\Lambda^{1/2} + \|d^{k+1} - d^*\|^2_\mathcal{M} \\
\leq \|x^k - x^*\|^2_\Lambda^{1/2} + \|d^k - d^*\|^2_\mathcal{M} - \|x^k - x^{k+1}\|^2_\Lambda^{1/2} - \|d^k - d^{k+1}\|^2_\mathcal{M} \\
+ 2 \langle \nabla s(x^k) - \nabla s(x^*), x^k - x^{k+1} \rangle - 2 \langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle.
\]
For the two inner product terms, we have
\[
2 \langle \nabla s(x^k) - \nabla s(x^*), x^k - x^{k+1} \rangle - 2 \langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle \\
= - \|x^k - x^{k+1} - \Lambda \nabla s(x^k) - \Lambda \nabla s(x^*)\|^2_\Lambda^{1/2} - \|x^k - x^{k+1}\|^2_\Lambda^{1/2} \\
+ \|\nabla s(x^k) - \nabla s(x^*)\|^2_\Lambda^{1/2} - 2 \langle x^k - x^*, \nabla s(x^k) - \nabla s(x^*) \rangle \\
\leq - \|d^{k+1} - d^*\|^2_\Lambda^{1/2} + \|x^k - x^{k+1}\|^2_\Lambda^{1/2} + \|\nabla s(x^k) - \nabla s(x^*)\|^2_\Lambda^{1/2} \\
- \max_i (\alpha_i L_i) \|\nabla s(x^k) - \nabla s(x^*), x^k - x^{k+1}\|_\Lambda^{1/2} - (2 - \max_i (\alpha_i L_i)) \|x^k - x^*\|^2_\Lambda^{1/2} \\
\leq - \|d^{k+1} - d^*\|^2_\Lambda^{1/2} + \|x^k - x^{k+1}\|^2_\Lambda^{1/2} \\
- (2 - \max_i (\alpha_i L_i)) \|x^k - x^*\|^2_\Lambda^{1/2}.
\]
The first inequality comes from \(x^{k+1} = x^k - \Lambda \nabla s(x^k) - d^{k+1}, \Lambda \nabla s(x^*) + d^* = 0\), (7) and (27). Combing (30) and (31), we have
\[
\|x^{k+1} - x^*\|^2_\Lambda^{1/2} + \|d^{k+1} - d^*\|^2_\mathcal{M} \leq \|x^k - x^*\|^2_\Lambda^{1/2} + \|d^k - d^*\|^2_\mathcal{M} - \|d^{k+1} - d^*\|^2_\Lambda^{1/2} \\
- (2 - \max_i (\alpha_i L_i)) \|x^k - x^*\|^2_\Lambda^{1/2}.
\]
Therefore,

\[ \| \mathbf{x}^{k+1} - \mathbf{x}^* \|^2_{\Lambda^{-1}} + \| d^{k+1} - d^* \|^2_{\Lambda + \Lambda^*} \leq (1 - (2 - \max_i (\alpha_i L_i)) \min_i (\mu_i, \alpha_i)) \| \mathbf{x}^k - \mathbf{x}^* \|^2_{\Lambda^{-1}} + \frac{\lambda_{\max}(\Lambda^{-1/2} \Lambda^* \Lambda^{-1/2})}{\lambda_{\max}(\Lambda^{-1/2} \Lambda^* \Lambda^{-1/2}) + 1} \| d^k - d^* \|^2_{\Lambda + \Lambda^*}. \]

Since \( \lambda_{\max}(\Lambda^{-1/2} \Lambda^* \Lambda^{-1/2}) = 1 - \lambda_{\max}(\Lambda^{-1/2} (\Lambda + \Lambda^*) \Lambda^{-1/2}) = 1 - \frac{1}{\lambda_{\max}(\Lambda^{-1/2} (I - W)^T \Lambda^{-1/2})} \), let \( \rho \) be defined as (28), then we have (29).

**Remark 4.** The condition \( \mathbf{I} > c \Lambda^{1/2} (I - W) \Lambda^{1/2} \) implies \( c < \lambda_{n-1}(\Lambda^{-1/2} (I - W)^T \Lambda^{-1/2}) \). When \( \Lambda = \alpha \mathbf{I} \), i.e., the stepsize for all agents are the same, we have

\[ \rho = \max \left( 1 - (2 - \alpha \max_i L_i) \min_i \mu_i, 1 - \frac{c \alpha}{\lambda_{\max}(\mathbf{I} - W)} \right). \]

Letting \( \alpha = \frac{1}{\max_i L_i} \) and \( c = \frac{1}{\alpha(1 - \lambda_{n-1}(W))} \), we have \( \rho = \max \left( 1 - \frac{\min_i \mu_i}{\max_i L_i}, 1 - \frac{\lambda_{n-1}(W)}{\lambda_{\max}(\mathbf{I} - W)} \right) \).

If we let \( \Lambda = L^{-1} \), then we have \( \rho = \max \left( 1 - \frac{\min_i \mu_i}{L_i}, 1 - \frac{\lambda_{n-1}(L^{-1/2} (I - W)^T \Lambda^{-1/2})}{\lambda_{\max}(L^{-1/2} (I - W)^T \Lambda^{-1/2})} \right) \).

**Remark 5.** Theorem 9 separates the dependence of linear convergence rate on the functions and the network structure. In our current scheme, all the agents perform information exchange and proximal-gradient step once at each iteration. If the proximal-gradient adaptation is expensive, this explicit rate formula can help us to decide whether the so-called multi-step consensus can help reducing the computational time.

For the sake of simplicity, let us assume for this moment that all the agents have the same strong convexity constant \( \mu \) and gradient Lipschitz constant \( L \). Suppose that the “t-step consensus” technique is employed, i.e., the mixing matrix \( W \) in our algorithm is replaced by \( W^t \), where \( t \) is a positive integer. Then to reach \( \epsilon \)-accuracy, the number of iterations needed is

\[ O \left( \max \left( \frac{L}{\mu}, \frac{1 - \lambda_n(W^t)}{1 - \lambda_2(W^t)} \right) \right) \log \frac{1}{\epsilon}. \]

When \( L/\mu = 1 \) and the step sizes are chosen as \( \Lambda = L^{-1} \), this formula says that we should let \( t \to +\infty \) if the network is not a complete graph. Such theoretical result is correct in intuition since in this case, the centralized gradient descent only needs one step to reach optimal while the only bottleneck in decentralized optimization is the graph.

Suppose \( t_{\max} \) is a reasonable upper bound on \( t \), which is set by the system designer. It is difficult to explicitly find an optimal \( t \). But with the above analysis as an evidence, we suggest that one choose \( t = \min \left( \left\lfloor \log_{\lambda_2(W)} (1 - \frac{\mu}{L}) \right\rfloor, t_{\max} \right) \) if \( 1 - \frac{\mu}{L} > \lambda_2(W) \); otherwise \( t = 1 \). Here the operation \( \lfloor \cdot \rfloor \) gives the nearest integer.

4. Numerical Experiments. In this section, we compare the performance of our proposed NIDS with several state-of-the-art algorithms for decentralized optimization. These methods are

- EXTRA/PG-EXTRA (10) for both the smooth and non-smooth cases.
- DIging-ATC [25] for the smooth case only. The DIging-ATC can be expressed as:

\[
\begin{align*}
x^{k+1} &= W(x^k - \alpha y), \\
y^{k+1} &= W(y^k + \nabla f(x^{k+1}) - \nabla f(x^k)).
\end{align*}
\]

Note there are two rounds of communication in each iteration of DIging-ATC while there are only one round of communication in EXTRA/PG-EXTRA and NIDS. For all the experiments, we first compute the exact solution \( \mathbf{x}^* \) for (1) using the centralized (proximal) gradient descent.

The experiments are carried in Matlab R2016b running on a laptop with Intel i7 CPU @ 2.60GHZ, RAM 16.0 GB, and Windows 10 operating system. The source code for reproducing the numerical results is available via https://github.com/mingyan08/NIDS.
4.1. The strongly convex case with \( r(x) = 0 \). Consider the decentralized sensing problem that solves for an unknown signal \( x \in \mathbb{R}^p \). Each agent \( i \in \{1, \ldots, n\} \) takes its own measurement via \( y_i = M_i x + e_i \), where \( y_i \in \mathbb{R}^{m_i} \) is the measurement vector, \( M_i \in \mathbb{R}^{m_i \times p} \) is the sensing matrix, and \( e_i \in \mathbb{R}^{m_i} \) is the independent and identically distributed noise. Then we apply the decentralized algorithms to solve

\[
\text{minimize } s(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \| M_i x_i - y_i \|^2 \text{ subject to } Wx = x.
\]

In order to ensure that each subfunction \( s_i(x) = \frac{1}{2} \| M_i x - y_i \|^2 \) is strongly convex, we choose \( m_i = 60 \) and \( p = 50 \) and set the number of nodes \( n = 40 \). For the first experiment, we chosen \( M_i \) such that the Lipschitz constant of \( \nabla s_i \) satisfies \( L_i = 1 \) and the strongly convex constant \( \mu_i = 0.5 \) for all \( i \). From Remark 4, we would choose \( \alpha = 1/(\max_i L_i) = 1 \) and \( c = 1/(1 - \lambda_n(W)) \) for NIDS. In addition, we choose \( c = 1/2 \) such that \( \bar{W} = \frac{1}{2}W \), which is the same as in EXTRA. Then, the stopping criteria for all decentralized algorithms is \( \| x^k - x^* \| < 10^{-11} \).

The comparison of these four methods (NIDS with \( c = 1/(1 - \lambda_n(W)) \), NIDS with \( c = 1/2 \), EXTRA, and DIGing-ATC) is shown in Figure 1. We choose the same step-size \( \alpha = 1 \) for all the four methods. The experiments presented in the left one in Figure 1, is conducted under a less connected network comparing with the right one in Figure 1. Figure 1 shows the better performance of both NIDS than that of EXTRA and DIGing-ATC. NIDS with \( c = 1/(1 - \lambda_n(W)) \) always takes less than the half number of steps used by EXTRA to converge. Also DIGing-ATC is sensitive to network since it communicates twice per iteration, so that under the better connected network on the right, DIGing-ATC can catch up with the NIDS with \( c = 1/2 \).

![Figure 1: Under two different networks, the plots of the relative error \( \| x^k - x^* \| / \| x^* \| \) with respect to the iteration number. All methods (two NIDS, EXTRA, and DIGing-ATC) use the same step-size \( 1/(\max_i L_i) \). The two NIDS have different \( c \) values: \( c = 1/(\alpha(1 - \lambda_n(W))) \) and \( c = 1/(2\alpha) \).](image)

Remark 4 tells that different nodes can choose different step-sizes based on its own local function \( s_i \). Therefore, in the next experiment, we change two \( M_i \) such that they have different \( L_i \) and \( \mu_i \) from the rest. More specifically, we adjust the two \( M_i \) in \( s_i \), so that \( L_i = 2 \) and \( \mu_i = 0.4 \), while we keep the rest of \( s_i \) to have \( L_i = 1 \) and \( \mu_i = 0.2 \). Then we compare the performance of two NIDS (NIDS with step-size \( 1/L_i \) for each node and NIDS with a constant step-size \( 1/(\max_i L_i) \) ) with that of EXTRA and DIGing-ATC in Figure 2. The \( c \) values in both NIDS are chosen based on Remark 4.

With the same step-size \( 1/(\max_i L_i) \), NIDS, EXTRA and DIGing-ATC have similar performance on this experiment. We found that DIGing-ATC converges more smoothly than EXTRA, but slower than NIDS. The ability of NIDS to choose different step-size over nodes improves the convergence in this experiment, as shown in Figure 2.

4.2. The case with nonsmooth function \( r(x) \). In this subsection, we compare the performance of NIDS with PG-EXTRA [37] because DIGing can not be applied to this nonsmooth case.
We consider a decentralized compressed sensing problem. Again, each agent $i \in \{1, \cdots, n\}$ takes its own measurement via $y_i = M_i x + e_i$, where $y_i \in \mathbb{R}^{m_i}$ is the measurement vector, $M_i \in \mathbb{R}^{m_i \times p}$ is the sensing matrix, and $e_i \in \mathbb{R}^{m_i}$ is the independent and identically distributed noise. Here, $x$ is a sparse signal.

$$\min_x s(x) + r(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \|M_i x_i - y_i\|^2 + \frac{1}{n} \sum_{i=1}^{n} \lambda_i \|x_i\|_1, \text{ subject to } Wx = x.$$  

We normalize the problem to make sure that the Lipschitz constant for each node satisfies $L_i = 1$ for each node, we choose $m_i = 3$ and $p = 200$ and set the number of nodes $n = 40$. Considering the size of problem, the stopping criteria is reset to $\|x^k - x^*\| < 10^{-7}$.

Fix the network $(W)$, in Figure 3, NIDS allows larger step-size which leads to faster convergent speed. With step-size 1, NIDS and PG-EXTRA converge at the same speed, but if we keep increasing the step-size, PG-EXTRA will diverge with step-size 1.4, but the step-size of NIDS can be increased to 1.9, which leads to better rate of convergence.
5. Conclusion. In this paper, we proposed a novel decentralized consensus algorithm NIDS whose step-size does not depend on the network structure. The connection between NIDS and PG-EXTRA is that NIDS applies the ATC strategy on the successive gradient. In NIDS, the step-size depends only on the objective function, and it can be as large as $2/L$, where $L$ is the Lipschitz constant of the gradient of the smooth function. We showed that NIDS converges at the $o(1/k)$ rate for the general convex case and at the linear rate for the strongly convex case. For the strongly convex case, we separated the linear convergence rate’s dependence on the objective function and the network. The separated convergence rates match the typical rates for the general gradient descent and the consensus averaging. Furthermore, every agent in the network can choose its own step-size independently by its own objective function. Numerical experiments validated the theoretical results and demonstrated better performance of NIDS over the state-of-the-art algorithms. Because the step-size of NIDS does not depend on the network structure, there are many possible future extensions. One extension is to apply NIDS on dynamic networks where nodes can join and drop off.

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