Learning a high-dimensional classification rule using auxiliary outcomes

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Abstract

Correlated outcomes are common in many practical problems. Based on a decomposition of estimation bias into two types, within-subspace and against-subspace, we develop a robust approach to estimating the classification rule for the outcome of interest with the presence of auxiliary outcomes in high-dimensional settings. The proposed method includes a pooled estimation step using all outcomes to gain efficiency, and a subsequent calibration step using only the outcome of interest to correct both types of biases. We show that when the pooled estimator has a low estimation error and a sparse against-subspace bias, the calibrated estimator can achieve a lower estimation error than that when using only the single outcome of interest. An inference procedure for the calibrated estimator is also provided. Simulations and a real data analysis are conducted to justify the superiority of the proposed method.

Keywords: Auxiliary outcomes, Classification, Transfer learning, Multi-task learning, High-dimensional inference

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1 Introduction

Modern statistical learning addresses datasets that are increasingly massive in volume and diverse in variable category, and it is aimed at leveraging these datasets for knowledge discovery and decision support. In many problem settings, auxiliary outcomes are available to decision makers but one outcome is of primary interest. For instance, e-commerce retailers like Amazon own their customer profile data and purchase history associated with several types of products. These products can share similar features but only one product is of interest to the retailer whose goal is to decide whether to recommend this product to a particular customer or not. The purchase behavior associated with this specific product is considered as the target outcome, and that of the remaining products are auxiliary outcomes. In healthcare, many clinical outcomes are measured and recorded for individual patients. Some of them are closely relevant. For instance, cardiovascular diseases and diabetes share common risk factors; the events of a readmission or death within 30 days after a hospital discharge also share common predictors. In addition, some clinical outcomes are less accessible due to ethical or cost issues, whereas related biological outcomes as auxiliary outcomes can be relatively easily obtained. The prevalence of auxiliary outcomes in practical problems motivates a learning approach that transfers knowledge from these outcomes to improve the learning performance of the target problem.

One commonly used approach is transfer learning (Olivas et al. 2009). Transfer learning aims to improve the performance of target learners on target domains by transferring the knowledge contained in different but related source domains (Zhuang et al. 2019). Based on the consistency between domains, transfer learning can be broadly divided into two categories: homogeneous transfer learning, which handles the situation where the domains are of the same feature space; heterogeneous transfer learning, which refers to the situation where the domains own different feature spaces (Weiss et al. 2016). Our work belongs to the general category of homogeneous transfer learning as we mainly consider one dataset with multiple related outcomes. In this realm, the focus is typically on knowledge transfer at the level of parameters (or their supports), such as enforcing the models to share some common parameters (Zhuang et al. 2013), or requiring the parameters of the source and the target models to be similar. Another related topic is multi-task learning, which aims to model multiple outcomes simultaneously. Multi-task learning often assumes that some parameters are similar across tasks (i.e., knowledge at the parameter level) (Bakker and Heskes 2003; Yu et al. 2005; Ando and Zhang 2005; Zhang et al. 2008), or these tasks bear a shared sparsity structure (i.e., knowledge at the support level) (Obozinski et al. 2008; Lounici et al. 2009; Yang et al. 2009).

Recently, Li et al. (2020) addressed a heterogeneous transfer learning problem in high-dimensional linear regression. They assumed that the contrast between the parameters in the target model and those in the auxiliary models are sufficiently sparse. Nonetheless, the knowledge transfer at the parameter level or support level may not be optimal for classification problems. For example, let $Y_0$ be the target outcome, and $Y_1$ be an auxiliary outcome, and consider the decision rules of outcomes $Y_0$ and $Y_1$ following $d_j^*(X) = \text{sgn}\{X^\top \beta_j^*\}$, where $\beta_j^*$ is the coefficient vector for $j = 0$ and $1$, respectively. We aim to transfer the knowledge of $\beta_1^*$ to facilitate the estimation of $\beta_0^*$. Figure 1 illustrates a possible relationship between $\beta_0^*$ and $\beta_1^*$. In this context, the contrast $\beta_0^* - \beta_1^*$ represented by Arrow 1 is not sparse. Meanwhile, if we assume a shared support of $\beta_0^*$ and $\beta_1^*$, we still need to estimate all nonzero elements in $\beta_0^*$. For the knowledge transfer at the subspace level, the subspace spanned by $\beta_1^*$ is represented by the dotted line. Arrow 2 represents a possible contrast between $\beta_0^*$ and a vector within the subspace. Because the contrast represented by Arrow 2 is parallel to the axis, estimating such a contrast can be much easier than estimating $\beta_0^* - \beta_1^*$. Therefore, our work pursues the knowledge transfer at the subspace level to gain efficiency.

Added to the knowledge transfer challenge is the high-dimensional nature of modern applications. For example, in healthcare, the dimension of covariates in electronic health records (EHR) is typically very high compared to the sample size. With a low-dimensional setting, Cai and Wei (2019) studied the transfer learning for nonparametric classification using the posterior drift model; however, the
transfer learning approach for high-dimensional classification problems is highly demanded but less explored in the extant literature. The recent advancement in high-dimensional inference can be leveraged to fill this void. For example, van de Geer et al. (2014) proposed the de-biased lasso estimator for generalized linear models. Ning and Liu (2017) further proposed the de-correlated score to test the low-dimensional linear contrast under M-estimation. In the same vein, Dezeure et al. (2017) proposed a bootstrap-based procedure. To the best of our knowledge, these inference methods have not been applied to transfer learning, and their theoretical properties remain unknown.

To bridge the gap, we develop a robust transfer learning approach to estimating classification rules for the outcome of interest with the presence of auxiliary outcomes in high-dimensional settings. We introduce a novel bias decomposition that enables a knowledge transfer at the subspace level, and we propose a two-stage procedure. In the first stage, a pooled estimator is obtained using all outcomes to gain efficiency by borrowing relevant information from auxiliary outcomes. In the second stage, a calibration procedure is performed to reduce or correct the bias induced by pooling multiple outcomes to ensure the robustness of the estimator. To address the insufficiency of a parameter-level knowledge transfer, we add the pooled estimator as a new covariate to the targeted model in the calibration step. Further, we develop an inference procedure to identify the driving factors of the learned decision rule.

With this framework, we show in theory that the estimation error of the calibrated estimator has no worse a convergence rate than that using only the single outcome of interest if the sparsity level of the pooled estimator is relatively small, even though the auxiliary outcomes could have introduced biases. Moreover, when the pooled estimator has a low estimation error and the minimizer of pooled estimation lies in a subspace close to the true classification rule, the estimation error of the calibrated estimator can be lower than that using only the single outcome of interest. We also investigate the theoretical properties of the proposed inference procedure.

In addition to providing a theoretical guarantee of the performance of the proposed approach, we also identify the structural property of the construction of the convergence rate, which offers us a practical guideline to improving its performance. Our approach can be extended to an implementation that utilizes a large dataset with auxiliary outcomes in the pooled estimation step, regardless of whether the target outcome is contained or not. With sufficient samples in the large dataset, the estimation error of the calibrated estimator can be improved, and the conditions for a valid testing procedure can be less restrictive compared with those identified in Ning and Liu (2017).

The rest of the paper is organized as follows. Section 2 introduces the proposed robust transfer learning approach. In Section 3, we investigate the theoretical properties of the estimators. We present
the simulation studies in Section 4. In Section 5, we use the method to predict the risk of hospital readmissions by applying it to a dataset of a group of patients who were admitted to the University of Florida Health hospital due to a primary diagnosis of congestive heart failure. We present a discussion and concluding remarks in Section 6.

2 Robust learning with auxiliary outcomes

Let $X \in \mathbb{R}^p$ be a $p$-dimensional covariate and $Y_0 \in \{1, -1\}$ be a univariate target outcome. Learning a linear decision rule to predict the target outcome $Y_0$ using covariate vector $X$ entails a classification problem. Such problems are prevalent in medical, bioinformatics, transportation, and recommender systems, among others. Subsequently, the classification framework has been adopted for medical tests (Pepe et al. 2003), individualized clinical decision rules such as individualized treatment recommendations (Zhao et al. 2012), and many machine learning applications. These decision rules can be estimated through solving

$$
\min_{\beta_0, c_0} \mathbb{E} \left[ \phi \left( Y_0 (X^T \beta_0 - c_0) \right) \right],
$$

(1)

where $\phi(\cdot)$ is a surrogate loss. Let the minimizer of optimization problem (1) be $\beta_0^*$ and $c_0^*$, and define a decision rule, $d_0^*(X)$, with the form $d_0^*(X) = \text{sgn} \{ X^T \beta_0^* - c_0^* \}$. The decision rule $d_0^*(X)$ can be used for prediction purposes.

In these real-world problems, some auxiliary outcomes may be available along with the target outcome $Y_0$. The auxiliary outcomes chosen based on domain knowledge can be informative in terms of estimating $\beta_0^*$. To take the advantage of these auxiliary outcomes, we propose a pooling-and-calibrating procedure to improve the estimation of the decision rule. To facilitate the identification of the driving factors, we assume that the surrogate loss $\phi$ is strictly convex and has the second order derivative.

2.1 Pooled estimation step

Denote the available auxiliary binary outcomes as $Y_1, Y_2, \cdots, Y_J \in \{1, -1\}$, where $J$ is the number of available auxiliary outcomes. We first obtain a pooled estimator $\hat{\beta}_{\text{pool}}$ by solving

$$
\min_{\beta, \{c_j\}_{j=0}^J} \hat{\mathbb{E}}_n \left[ \frac{1}{J} \sum_{j=0}^J \phi \left( Y_j (X^T \beta - c_j) \right) \right] + \lambda_n \| \beta \|_1,
$$

(2)

where $\lambda_n$ is a tuning parameter and $\hat{\mathbb{E}}_n[\cdot]$ is the empirical expectation of training samples. In this pooled estimation, we estimate the $J + 1$ decision rules for $Y_0, Y_1, \cdots, Y_J$, simultaneously. These $J + 1$ decision rules are structured to learn a common parameter $\beta$, which is the direction shared by all the outcomes. In addition, the thresholds represented by $c_j$’s can be different for each outcome to accommodate possible heterogeneity.

Let $\left( \beta_{\text{pool}}^{*}, \{c_{\text{pool},j}^*\}_{j=0}^J \right)$ be the minimizer of $\min_{\beta, \{c_j\}_{j=0}^J} \mathbb{E} \left[ \frac{1}{J} \sum_{j=0}^J \phi \left( Y_j (X^T \beta - c_j) \right) \right]$. Leveraging information from auxiliary outcomes, the pooled estimator $\hat{\beta}_{\text{pool}}$ can approach $\beta_{\text{pool}}^*$ with a low estimation error. However, $\beta_{\text{pool}}^*$ may or may not lie in the subspace generated by $\beta_0^*$, and this will determine the theoretical performance of the proposed approach.

2.2 Calibration step

The pooled estimator $\hat{\beta}_{\text{pool}}$ may not be a good estimator of $\beta_0^*$ due to possible bias in $\beta_{\text{pool}}^*$ w.r.t. $\beta_0^*$. In order to obtain a consistent estimator of $\beta_0^*$, it is necessary to correct such a bias in $\hat{\beta}_{\text{pool}}$. To start with, we decompose the bias of $\beta_{\text{pool}}^*$ through the following. Given a coefficient vector $\beta$, the contrast $b := \beta - \beta_0^*$ can be generally decomposed as $b_w + b_a$, where $b_w$ refers to the within-subspace
bias, which lies in the subspace generated by \( \beta_0^* \), and \( b_a \) is the against-subspace bias, satisfying that \( b_a - \gamma \beta_0^* \neq 0 \) for any \( \gamma \). For example, if \( \beta = r \beta_0^* + e \) for some constant \( r \) and a vector \( e \) with \( e \perp \beta_0^* \), the bias \( b = \beta - \beta_0^* = (r-1)\beta_0^* + e \) can be decomposed as \( (r-1)\beta_0^* \) and \( e \). Then, \( (r-1)\beta_0^* \) lies in the subspace generated by \( \beta_0^* \); \( e \) is the against-subspace bias. In Figure 1, the bias along the blue dashed line is the within-subspace bias, and arrows 1 and 2 represent the against-subspace bias.

Motivated by this decomposition, we introduce an estimator \( \hat{\beta}_{cal} = \hat{\delta} + \hat{\gamma} \hat{\beta}_{pool} \), where \( \hat{\delta} \), \( \hat{\gamma} \) and \( \hat{c} \) are the minimizers of

\[
\min_{\delta, \gamma, c} \mathbb{E}_n \left[ \phi \left\{ Y_0 (X^T \delta + \gamma X^T \hat{\beta}_{pool} - c) \right\} \right] + \tilde{\lambda}_n \| \delta \|_1, \tag{3}
\]

and \( \tilde{\lambda}_n \) is a tuning parameter. Consequently, the estimated optimal decision rule becomes \( \hat{d}_0(X) = \text{sgn} \left\{ X^T \hat{\beta}_{cal} - \hat{c} \right\} \).

The loss function in (3) incorporates two adjustments to \( \hat{\beta}_{pool} \), which corresponds to the bias within the subspace and against the subspace. First, we calibrate the scaling parameter \( \gamma \) along the subspace generated by \( X^T \hat{\beta}_{pool} \). This calibration eliminates the bias within the subspace. For instance, if \( \beta_{pool}^* = 2\beta_0^* \), then, setting \( \gamma = 1/2 \) can eliminate such a bias. Second, we calibrate the subspace generated by \( \hat{\beta}_{pool} \) using \( \hat{\delta} \). This calibration reduces the bias against the subspace. If \( \beta_{pool}^* = \beta_0^* - e \), then setting \( \hat{\delta} = e \) can eliminate such a bias.

Nonetheless, the minimizers of the optimization problem (3) are not identifiable because the minimizer of \( \min_{\gamma} \| \beta_0^* - \gamma \beta_{pool}^* \|_q \) may not be unique, where \( q = 0 \) or 1. For example, let \( \beta_0^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \beta_{pool}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Then, any \( \gamma \in [0, 1] \) is a minimizer of \( \min_{\gamma} \| \beta_0^* - \gamma \beta_{pool}^* \|_1 \), and both \( \gamma = 1 \) and 0 are minimizers of \( \min_{\gamma} \| \beta_0^* - \gamma \beta_{pool}^* \|_0 \). Imposing \( \gamma \) to be a constant can resolve the identifiability issue. For example, the calibration procedure in Li et al. (2020) imposes \( \gamma = 1 \). However, these constraints may lead to a suboptimal estimation error as presented in Section 3.

Alternatively, we propose a different approach to resolve the identifiability issue. The idea is as follows: we first separate the space of \( \delta \) into several domains such that in each domain, the solution is unique; then, we select the best solution through validation on a pre-split dataset. Below we introduce how these domains can be defined, and show that in each domain, the solution is unique. We construct the following domains

\[
\Gamma_k = \left\{ \delta = (\delta_1, \delta_2, \cdots, \delta_p)^\top : \delta_k = 0 \right\}.
\]

Due to the strict convexity of \( \phi \) and the assumption that the coordinates of \( X \) are not linearly dependent, then, for any \( k \in S_{pool}^* \), there exists a unique \( \gamma \) such that \( \beta_0^* - \gamma \beta_{pool}^* \in \Gamma_k \), where \( S_{pool}^* \) is the set of indexes of the non-zero coefficients of \( \beta_{pool}^* \). This implies that the objective function (3) on each \( \Gamma_k \) has a unique minimizer, for any \( k \in S_{pool}^* \).

Motivated by this, we consider

\[
\min_{\delta \in \Gamma_k, \gamma, c} \mathbb{E}_n \left[ \phi \left\{ Y_0 (X^T \delta + \gamma X^T \hat{\beta}_{pool} - c) \right\} \right] + \tilde{\lambda}_n \| \delta \|_1 \tag{4}
\]

for \( k \in \hat{S}_{pool} \), where \( \hat{S}_{pool} \) is the set of the indexes of non-zero coefficients in \( \hat{\beta}_{pool} \). The optimization problem (4) has the same objective function as that of (3) but constrains the domain of \( \delta \), which has to be in \( \Gamma_k \). On each \( \Gamma_k \), the solution of optimization (4) is unique for \( k \in \hat{S}_{pool} \).

Although we have shown that on each domain, the minimizer is unique, it is unknown that whether the minimizer of optimization problem (3) will fall into one of these domains. As such, we provide a useful lemma, which shows that for every minimizer of optimization (3), there is a minimizer of optimization (4) with \( k \in \hat{S}_{pool} \) such that they correspond to the same \( \hat{\beta}_{cal} \). The proof is provided in
Lemma 1. Suppose that $\delta, \gamma$ and $c$ are the minimizer of the loss function in (3), there exists another minimizer $\hat{\delta}, \hat{\gamma}$ and $c$ of the loss function in (3), and it satisfies that 1) $\hat{\delta} \in \bigcup_{k \in \hat{S}_{\text{pool}}} \Gamma_k$, and 2) $\delta + \gamma \hat{\beta}_{\text{pool}} = \hat{\delta} + \hat{\gamma} \hat{\beta}_{\text{pool}}$.

Lemma 1 suggests that we can directly solve the optimization problem (4) instead of (3). First, we can calculate the minimizer of

$$
\min_{\delta \in \Gamma_k, \gamma, c} \tilde{E}_n \left[ \phi \left\{ Y_0 (X^\top \delta + \gamma X^\top \hat{\beta}_{\text{pool}} - c) \right\} \right] + \tilde{\lambda}_n \| \delta \|_1,
$$

for each $k \in \hat{S}_{\text{pool}}$, denoted as $\hat{\delta}(k), \hat{\gamma}(k), \hat{c}(k)$. Subsequently, we have all $\hat{\beta}_{\text{cal}}(k) = \hat{\delta}(k) + \hat{\gamma}(k) \hat{\beta}_{\text{pool}}$'s. Then, we can pick one of the $\hat{\beta}_{\text{cal}}(k)$'s and $\hat{c}(k)$'s based on their performance using a pre-split sample. The entire procedure can be referred to Algorithm 1.

**Algorithm 1: Estimation of the optimal decision rule using multiple outcomes**

**Input:** $n$ samples with variables $X$ and $Y_0, Y_1, \ldots, Y_J$.

**Output:** A decision rule $\hat{d}(X) = \text{sgn} \left\{ X^\top \hat{\beta}_0 - \hat{c}_0 \right\}$.

1. Randomly split the samples $I_1$ and $I_2$ with an equal size;
2. Using samples in $I_1$, we obtain the pooled estimation $\hat{\beta}_{\text{pool}, I_1}$ by solving

$$
\min_{\beta, (c_j)} \hat{E}_{n, I_1} \left[ \frac{1}{J} \sum_{j=0}^J \phi \left\{ Y_j (X^\top \beta - c_j) \right\} \right] + \lambda_n \| \beta \|_1,
$$

where $\hat{E}_{n, I_1}[\cdot]$ is the empirical expectation on $I_1$ and $\lambda_n$ is tuned by cross-validation;
3. Using samples in $I_1$, we obtain the calibrate estimator $\hat{\beta}_{\text{cal}, I_1}(k)$ and $\hat{c}_{I_1}(k)$ for each $k \in \hat{S}_{\text{pool}, I_1}$ by solving

$$
\min_{\delta \in \Gamma_k, \gamma, c} \hat{E}_{n, I_1} \left[ \phi \left\{ Y_0 (X^\top \delta + \gamma X^\top \hat{\beta}_{\text{pool}, I_1} - c) \right\} \right] + \hat{\lambda}_{n, 1} \| \delta \|_1,
$$

where $\hat{S}_{\text{pool}, I_1}$ is the index of the non-zero coefficients in $\hat{\beta}_{\text{pool}, I_1}$, and $\hat{\lambda}_{n, 1}$ is tuned by cross-validation;
4. Choose $\hat{\beta}_{\text{cal}, I_1} = \hat{\beta}_{\text{cal}, I_1}(k^*)$ and $\hat{c}_{I_1} = \hat{c}_{I_1}(k^*)$, where $k^*$ solves

$$
\min_{k} \hat{E}_{n, I_2} \left[ \phi \left\{ Y_0 (X^\top \hat{\beta}_{\text{cal}, I_1}(k) - \hat{c}_{I_1}(k)) \right\} \right],
$$

where $\hat{E}_{n, I_2}[\cdot]$ is the empirical expectation on $I_2$;
5. Replace $I_1$ by $I_2$ and $I_2$ by $I_1$, and repeat Steps 2 - 4. Obtain $\hat{\beta}_0$ and $\hat{c}_0$ by

$$
\hat{\beta}_0 = \left( \hat{\beta}_{\text{cal}, I_1} + \hat{\beta}_{\text{cal}, I_2} \right) / 2,
$$

and

$$
\hat{c}_0 = \left( \hat{c}_{I_1} + \hat{c}_{I_2} \right) / 2.
$$
2.3 A special case: auxiliary outcomes are related to a latent variable \( U \)

In this section, we focus on a special case where the different outcomes are related to a latent random variable \( U \). For instance, to diagnose whether a patient has a normal blood sugar level (\( Y_0 \)), or has pre-diabetes (\( Y_1 \)), or diabetes (\( Y_2 \)), the blood glucose Hemoglobin A1c (HbA1c) is measured as a continuous variable (\( U \)). Then, the optimal decision rule for \( Y_j \)'s is the same as the optimal decision rule for \( \{U > \tilde{c}_j \} \)'s, where \( \tilde{c}_j \)'s are some constants — in this case, different thresholds of the HbA1c value.

The existence of the latent random variable \( U \) guarantees that \( \beta^*_\text{pool} \) only has biases within the subspace of \( \beta^*_0 \); i.e., the bias \( b = \beta^*_\text{pool} - \beta^*_0 \) lies in the subspace generated by \( \beta^*_0 \). As such, the calibration step only needs to adjust for the within-subspace bias. Specifically, the parameter \( \hat{\gamma}(k) \) adjusts the within-subspace bias to obtain an unbiased \( c_0 \).

Given the existence of \( U \), Theorem 1 shows when the optimal decision rule of \( U \) can be reduced to a linear decision rule.

**Theorem 1.** Suppose that there exists a latent variable \( U \), a coefficient vector \( \beta \) and thresholds \( \tilde{c}_j \)'s such that \( P(Y_j = 1 | X) = P(U \leq \tilde{c}_j | X) = \Phi(X^\top \beta, \tilde{c}_j) \), where \( \Phi(\cdot, \cdot) \) is a non-increasing and continuous function for all \( j \in \{0,1,\cdots,J\} \). Then, there exists \( \beta_{\text{opt}} \propto \beta \) and \( c_{\text{opt}} \)'s such that

\[
\text{sgn} \left\{ P(Y_j = 1 | X) - P(Y_j = -1 | X) \right\} = \text{sgn} \left\{ X^\top \beta_{\text{opt}} - c_{\text{opt}} \right\}, \forall j.
\]

The condition regarding \( P(U \leq \tilde{c} | X) = \Phi(X^\top \beta, \tilde{c}) \) can be satisfied by a variety of models. For example, the cumulative link model (or ordinal regression) satisfies this condition (Winship and Mare 1984). The cumulative link model assumes that \( P(U \leq \tilde{c} | X) = \Phi(\tilde{c} - X^\top \beta) \), where \( \Phi \) is a distribution function. In addition, a model assuming that \( U = f(X^\top \beta, \epsilon) \) also satisfies this condition, where \( \epsilon \) is an independent error, and \( f(\cdot, \cdot) \) is a continuous bivariate function and a non-decreasing univariate function with respect to each argument. To see this, define \( h_\epsilon(\cdot) = \inf \{ f(\cdot, \epsilon) > \tilde{c} \} \), then \( P(U > \tilde{c} | X) = P(f(X^\top \beta, \epsilon) > \tilde{c} | X) = P(\epsilon > h_\epsilon(X^\top \beta | X) = 1 - \Phi_\epsilon(h_\epsilon(X^\top \beta)) \), where \( \Phi_\epsilon \) is the cumulative probability function of \( \epsilon \). By definition, \( h_\epsilon(\cdot) \) is non-increasing in \( X^\top \beta \). As such, this model assumption satisfies the condition in Theorem 1. In particular, \( f(X^\top \beta, \epsilon) = \tilde{f}(X^\top \beta) + \epsilon \) satisfies the condition, where \( \tilde{f} \) is non-decreasing and \( \epsilon \) is an independent error.

Theorem 1 provides a sufficient condition such that the optimal decision rule of \( Y_j \)'s can be reduced to a linear decision rule with a common \( \beta_{\text{opt}} \). Next, Theorem 2 and Corollary 1 further provide a sufficient condition such that the coefficient vector \( \beta^*_\text{pool} \) only has within-subspace bias w.r.t. \( \beta^*_0 \).

**Theorem 2.** We assume that 1) there exists \( \beta_{\text{opt}} \) and \( c_{\text{opt}} \)'s such that

\[
\text{sgn} \left\{ P(Y_j = 1 | X) - P(Y_j = -1 | X) \right\} = \text{sgn} \left\{ X^\top \beta_{\text{opt}} - c_{\text{opt}} \right\}, \forall j;
\]

2) \( f \) is strictly convex; 3) there exists a \( p \)-dimensional vector \( \beta^*_\text{opt} \) such that \( \mathbb{E}[X | X^\top \beta_{\text{opt}}] = PX^\top \beta_{\text{opt}} \). Then, the minimizer \( \beta^*_\text{pool} \) of the optimization problem

\[
\min_{\beta, \{c_j\}'_j} \mathbb{E} \left[ \sum_{j=0}^J \phi \left \{ Y_j(X^\top \beta - c_j) \right \} \right],
\]

satisfies that \( \beta^*_\text{pool} \propto \beta_{\text{opt}} \).

**Corollary 1.** We assume that 1) there exists \( \beta_{\text{opt}} \) and \( c_0, \text{opt} \) such that

\[
\text{sgn} \left\{ P(Y_0 = 1 | X) - P(Y_0 = -1 | X) \right\} = \text{sgn} \left\{ X^\top \beta_{\text{opt}} - c_0, \text{opt} \right\} ;
\]
2) $\phi$ is strictly convex; 3) there exists a $p$-dimensional vector $P$ such that $E[X|X^T \beta_{opt}] = PX^T \beta_{opt}$. Then, the minimizer $\beta_0^*$ of the optimization problem

$$\min_{\beta_0, c_0} E \left[ \phi \left( Y_0(X^T \beta_0 - c_0) \right) \right],$$

satisfies that $\beta_0^* \propto \beta_{opt}$, i.e., the minimizer of a surrogate loss can recover the minimizer of the 0-1 loss.

Condition 1) assumes that the optimal decision rule for $Y_j$’s are linear and share the same direction; this can be satisfied if conditions in Theorem 1 hold. Condition 2) can be satisfied by using a logistic loss as the surrogate loss. Condition 3) is a common condition in the dimension reduction literature (Li 1991; Lin et al. 2018, 2019) and inference of the high-dimensional single-index model (Eftekhari et al. 2019). It is satisfied if $E[b^T X | X^T \beta_{opt}]$ is linear in $X^T \beta_{opt}$ for any $b \in \mathbb{R}^p$. For example, when the distribution of $X$ is elliptically symmetric such as a Gaussian distribution, this condition is satisfied. When the distribution of $X$ is not elliptically symmetric, Diaconis and Freedman (1984) show that for most $b \in \mathbb{R}^p$, $E[b^T X | X^T \beta_{opt}]$ is approximately linear in $X^T \beta_{opt}$ in high-dimensional settings.

Theorems 2 provides a sufficient condition such that $\beta_{pool}^* \propto \beta_{opt}$. Following the proof of Theorem 2, we can similarly show that $\beta_0^* \propto \beta_{opt}$ (Corollary 1). As such, we have $\beta_{pool}^* \propto \beta_0^* \propto \beta_{opt}$. In this case, $\beta_{pool}^*$ only has the bias within the subspace spanned by $\beta_0^*$.

In practice, such a latent variable $U$ can be recognized in many ways. In healthcare, the disease status can be the unobservable latent variable $U$. A severe disease condition can lead to a hospital admission, and even worse, death. In the case of predicting hospital readmissions, the target outcome is an event of a readmission within 30 days of discharge, but whether the patient died or not can also be informative. It is worth noting that the latent variable can also be observable. As we have illustrated, the measure of HbA1c can be the latent variable which is used to diagnose diabetes.

3 Theoretical properties

To provide theoretical support for the effectiveness and applicability of the proposed method, we investigate the theoretical property of the calibrated estimators $\beta_{cal}(k)$’s. We first establish the estimation error of $\hat{\beta}_{pool}$ with respect to $\beta_{pool}^*$. This theoretical result allows the number of auxiliary outcomes $J$ to grow with the sample size. Then, we provide the convergence rate of the calibrated estimator, which can be structured as the sum of the estimation error of $\beta_{pool}$ and a term concerning the against-subspace bias between $\beta_0^*$ and $\beta_{pool}^*$. Lastly, we provide a valid testing procedure for examining the significance of coefficients in $\beta_0^*$. All the proofs of the theorems can be found in the supplementary material.

3.1 Estimation error of the calibrated estimator

In this section, we first provide a lemma that shows the convergence rate of $\hat{\beta}_{pool}$. We assume the following conditions for obtaining the theoretical results of the $\hat{\beta}_{pool}$:

a) There is a constant $R$ such that $\|X\|_{\infty}$, $\sup X |X^T \beta_{pool}^*|$, and $|c_{pool,j}^*|$’s are upper bounded by $R$.

b) Define $\tilde{X} = (X, -1)$. There is a constant $\lambda_{min}$ such that the smallest eigenvalue of $E[\tilde{X} \tilde{X}^T]$ is lower bounded by $\lambda_{min}$.

c) The number of the auxiliary outcomes $J = O(s_{pool}^*)$, where $s_{pool}^* = |S_{pool}^*|$, and $S_{pool}^*$ is the index set of non-zero coordinates in $\beta_{pool}^*$.

In Condition a), we assume that the design matrix is uniformly bounded for technical simplicity. Condition b) assumes that the covariance matrix of $\tilde{X}$ is positive-definite. This negates any design
matrix which has co-linearity. Notice that Condition b) does not request that the smallest eigenvalue of the Hessian matrix of the pooled estimation,

\[
\min_{\beta, \{c_j\}_{j=0}^{J}} \mathbb{E} \left[ \frac{1}{J} \sum_{j=0}^{J} \phi \left( Y_j (X^\top \beta - c_j) \right) \right],
\]

is bounded away from 0 because the smallest eigenvalue will approach 0 if \( J \to +\infty \). Condition c) allows the number of the auxiliary outcomes, \( J \), to increase to \(+\infty\) as \( n \to +\infty \). However, \( J \) cannot grow faster than the number of non-zero coordinates in \( \beta^*_\text{pool} \). In Algorithm 1, although we split the data into halves and obtain \( \hat{\beta}_{\text{pool}, I_1} \) and \( \hat{\beta}_{\text{pool}, I_2} \), separately, the following convergence rate is held for both. As such, we suppress the subscript and use \( \hat{\beta}_{\text{pool}} \) to refer to both.

**Lemma 2.** Under the Conditions a)–c), by choosing \( \lambda_n \asymp \sqrt{\frac{\log p}{n}} \), we have

\[
\left\| \hat{\beta}_{\text{pool}} - \beta^*_\text{pool} \right\|_1 = O_p \left( s^*_\text{pool} \sqrt{\frac{\log p}{n}} \right).
\]

In addition, we have \((\hat{\beta}_{\text{pool}} - \beta^*_\text{pool})^\top \hat{\mathbb{E}}_n [XX^\top] (\hat{\beta}_{\text{pool}} - \beta^*_\text{pool}) \lesssim \frac{s^*_\text{pool} \log p}{n}\).

**Remark.** The bound of \((\hat{\beta}_{\text{pool}} - \beta^*_\text{pool})^\top \hat{\mathbb{E}}_n [XX^\top] (\hat{\beta}_{\text{pool}} - \beta^*_\text{pool})\) holds for both \( \hat{\beta}_{\text{pool}, I_1} \) and \( \hat{\beta}_{\text{pool}, I_2} \). Specifically, we have \( (\hat{\beta}_{\text{pool}, I_1} - \beta^*_\text{pool})^\top \hat{\mathbb{E}}_n, I_{1, I_1} [XX^\top] (\hat{\beta}_{\text{pool}, I_1} - \beta^*_\text{pool}) \lesssim \frac{s^*_\text{pool} \log p}{n} \) and \((\hat{\beta}_{\text{pool}, I_2} - \beta^*_\text{pool})^\top \hat{\mathbb{E}}_n, I_{1, I_2} [XX^\top] (\hat{\beta}_{\text{pool}, I_2} - \beta^*_\text{pool}) \lesssim \frac{s^*_\text{pool} \log p}{n} \).

Lemma 2 shows that the convergence rate of \( \hat{\beta}_{\text{pool}} \) is \( O_p \left( s^*_\text{pool} \sqrt{\frac{\log p}{n}} \right) \) if \( J = O(s^*_\text{pool}) \). Nonetheless, the \( \hat{c}_{\text{pool}, j} \)’s may have different convergence rates. The proof of Lemma 2 shows that \( \sup_j |\hat{c}_{\text{pool}, j} - c^*_j| = O_p \left( s^*_\text{pool} \sqrt{\frac{\log p}{n}} \right) \) and \( \inf_j |\hat{c}_{\text{pool}, j} - c^*_j| = O_p \left( \sqrt{s^*_\text{pool} \log p} \right) \). The difference in the convergence rate of \( \beta_{\text{pool}} \) and \( \hat{c}_{\text{pool}, j} \)’s is a consequence of allowing \( J \to +\infty \).

We further investigate the estimation error of the calibrated estimator. In Algorithm 1, we set \( k \in \hat{S}_{\text{pool}, I_2} \), where \( a = 1 \) or \( 2 \). Without loss of generality, we consider any \( S \subset S^*_{\text{pool}} \), and set \( k \in S \). If external domain knowledge is available to determine \( S \), we can tailor \( S \) for practical needs. If there is no information on \( S \), we can use \( \hat{S}_{\text{pool}, I_2} \). When \( \hat{S}_{\text{pool}, I_2} \not\subset S^*_{\text{pool}} \), we may focus on those large coefficients in \( \hat{\beta}_{\text{pool}, I_2} \), for example, coefficients with an absolute value greater than a pre-specified threshold. If we assume that the selection consistency \( \hat{S}_{\text{pool}, I_a} = S^*_{\text{pool}} \) holds with probability \( 1 - o(1) \) for both \( a = 1 \) and \( 2 \), we can set \( S = \hat{S}_{\text{pool}, I_2} \).

In addition to Conditions a)–c), we need the following additional conditions:

d) There exists a constant \( \tilde{R} \) such that \( \sup_{X} |X^\top \beta^*_0| \leq \tilde{R} \), and \( |c^*_0| \leq \tilde{R} \). We also require that the minimizer of

\[
\min_{\beta^*_0} \|\beta^*_0 - \gamma \beta^*_{\text{pool}}\|_0 \text{ s.t. } \beta^*_0 - \gamma \beta^*_{\text{pool}} \in \bigcup_{k \in S_{\Gamma k}}
\]

is bounded by \( \tilde{R} \).

e) Let \( \beta_{\text{min}} = \min_{q \in S} \|\beta^*_{\text{pool}, q}\| \), where \( \beta^*_{\text{pool}, q} \) is the \( q \)-th coordinate in \( \beta^*_{\text{pool}} \). We require that \( \beta_{\text{min}} \gg s^*_\text{pool} \sqrt{\frac{\log p}{n}} \|\beta^*_{\text{pool}}\|_1 \).

f) Define \( \tilde{X}_j = (X^\top \beta^*_{\text{pool}}, X_{-j}, -1) \), where \( X_{-j} \) is the vector of covariates \( X \) excluding the \( j \)-th covariate. We assume that there is a constant \( \lambda_{\text{min}} \) such that the smallest eigenvalue of \( \mathbb{E}[\tilde{X}_j \tilde{X}_j^\top] \)
is lower bounded by $\tilde{\lambda}_{\min}$ for all $j \in S$.

Condition d) assumes a uniform bound by design for technical simplicity. Let $\gamma^*$ be the minimizer of $\min_{\gamma} \|\beta_0^* - \gamma \beta_{pool}^*\|_0$ s.t. $\beta_0^* - \gamma \beta_{pool}^* \in \cup_{k \in \mathbf{S}} \Gamma_k$. Condition d) requires that $|\gamma^*| \leq \tilde{R}$. This is satisfied with a sufficiently large $\tilde{R}$ when the non-zero coordinates in $\beta_{pool}^*$ are bounded away from 0 and $\|\beta_{pool}^*\|_\infty$ is bounded away from $+\infty$. Condition e) is required to control the estimation error in estimating $\gamma^*$. In Li et al. (2020), they fixed $\gamma = 1$, and such a condition is hence not needed. According to the proof of Theorem 3, the estimation error of $\gamma^*$ is amplified by a factor of $\|\beta_{pool}^* - \beta_{pool}^*\|_1 \|\beta_{pool}^*\|_1/\beta_{min}$.

As such, we require that $\beta_{min} \gg s^*_{pool}/\sqrt{\log p}/\|\beta_{pool}^*\|_1$, so that this factor is negligible. Condition f) assumes a uniform lower bound for the eigenvalues of the design matrix. When $S$ is a finite set, Condition f) is satisfied if $E[\tilde{X}_j \tilde{X}_j^\top]$ is positive-definite for all $j \in S$. Similar to Theorem 2, we suppress the subscription indicating $I_1$ or $I_2$ and use $\beta_{cal}(k)$ to replace $\beta_{cal,I_1}(k)$ and $\beta_{cal,I_2}(k)$.

**Theorem 3.** Under the Conditions a) - f), there exist constants $C_1$ and $C_2$ such that

$$\inf_{k \in \mathbf{S}} \|\hat{\beta}_{cal}(k) - \beta_0^*\|_1 \leq C_1 s^*_{pool} \sqrt{\log p} / n + C_2 \max \{\|\beta_0^* - \gamma^* \beta_{pool}^*\|_0, 1\} \sqrt{\log p} / n,$$

with probability at least $1 - 2s^*_{pool}/p$, where $\gamma^*$ is the minimizer of $\min_{\gamma} \|\beta_0^* - \gamma \beta_{pool}^*\|_0$ s.t. $\beta_0^* - \gamma \beta_{pool}^* \in \cup_{k \in \mathbf{S}} \Gamma_k$.

**Remark.** Let $k^*_S$ be the minimizer of $\inf_{k \in \mathbf{S}} \|\hat{\beta}_{cal}(k) - \beta_0^*\|_1$. We choose $k^*_S$ by executing Step 3 in Algorithm 1. Suppose that $k^*$ in Algorithm 1 equals to $k^*_S$, then we have

$$\left(\beta_0^* - \beta_{pool}^*\right)^\top \mathbb{E}_n \left[\tilde{X} \tilde{X}^\top\right] \left(\beta_0^* - \beta_{pool}^*\right) = O_p \left(s^*_{pool} \log p / n + \|\beta_0^* - \gamma^* \beta_{pool}^*\|_0 \log p / n\right).$$

The resultant rate in Theorem 3 is structured as the sum of two terms. The first term, $C_1 s^*_{pool} \sqrt{\log p} / n$, is related to the estimation error of $\hat{\beta}_{pool}$. When combining multiple outcomes, the estimation error of $\hat{\beta}_{pool}$ w.r.t. $\beta_{pool}^*$ can be small. The second term, $C_2 \max \{\|\beta_0^* - \gamma^* \beta_{pool}^*\|_0, 1\} \sqrt{\log p} / n$, is associated with the bias of $\beta_{pool}^*$ against the subspace spanned by $\beta_0^*$. It is at least of the same magnitude as $s^*_{pool} \sqrt{\log p} / n$ because $\|\beta_0^* - \gamma^* \beta_{pool}^*\|_0 \leq \|\beta_0^*\|_0$. Compared with Li et al. (2020) where $\gamma$ is fixed at 1, the second term also ensures a better convergence rate because $\|\beta_0^* - \gamma^* \beta_{pool}^*\|_0 \leq \|\beta_0^* - \beta_{pool}^*\|_0$. When $\beta_{pool}^*$ only has the bias within the subspace, the second term is negligible compared with the first term if $s^*_{pool} \rightarrow +\infty$. Collectively, Theorem 3 articulates that if $\beta_{pool}^*$ is minimum biased against the subspace spanned by $\beta_0^*$, and can be efficiently estimated through the pooled estimation procedure, an effective performance improvement regarding estimating $\beta_0^*$ can be achieved.

To achieve a low estimation error of $\hat{\beta}_{pool}$, the selection of auxiliary outcomes is critical. When the domain knowledge is known, the informative auxiliary outcome(s) can be selected accordingly. For example, death can be chosen as an auxiliary outcome of hospital readmission because the health condition of the patient is the latent variable that relates the two outcomes. On the other hand, if the domain knowledge is not explicitly known, but there exists an observable variable $U$ (see Section 2.3) that the target outcome is associated with, then, we can use quantiles or levels of $U$ to construct auxiliary outcomes. For example, let $U$ be the number of readmissions and let $Y_0$ be the target outcome indicating whether $U > 0$ or not. The auxiliary outcome can be chosen as indicators of $U > 1$ or not, $U > 2$ or not, etc. Lastly, when such a variable $U$ is unobserved and no domain knowledge is available, we need a guideline for selecting informative auxiliary outcome(s) among a set of candidate auxiliary outcomes. We propose the following selection strategy based on the F1-score, which is defined as the harmonic mean of the precision and recall:
Step 1. Randomly split the training data into two subsets.
Step 2. On the first subset, fit a classification rule using a logistic loss for each auxiliary outcome and the target outcome, correspondingly;
Step 3. On the second subset, calculate the F1-score for each outcome.
Step 4. Select the auxiliary outcome with the highest F1-score; if the F1-score of the target outcome exceeds the F1-scores of all auxiliary outcomes, then we do not use any auxiliary outcome.

The simulations using the levels of U as auxiliary outcomes when U is observable, and using the proposed selection procedure when U is unobserved can be found in the supplementary material.

3.2 Inference of $\beta_0^*$

To identify the driving factors of the classification problem, we are interested in testing which coordinates of $\beta_0^*$ are non-zero. For simplicity in exposition, we test the hypothesis, $\mathcal{H}_0 : \beta_{0,1}^* = 0$ versus $\mathcal{H}_0 : \beta_{0,1}^* \neq 0$, where $\beta_{0,1}^*$ is the first coordinate in $\beta_0^*$. Our approach can be easily extended to test $\mathcal{H}_0 : l^T \beta_0^*$, where $l$ is a $p$-dimensional vector. We utilize the de-correlated score proposed in Ning and Liu (2017) to conduct the testing.

Let $X_1$ be the first covariate and $X_{-1}$ be the rest of the covariate vector. We define $\bar{X}_{-1} = \begin{pmatrix} X_{-1} \\ -1 \end{pmatrix}$. Following Ning and Liu (2017), the inference procedure based on the estimates $(\beta_{\text{cal},I_1}, \hat{c}_{\text{cal},I_1})$ and $(\beta_{\text{cal},I_2}, \hat{c}_{\text{cal},I_2})$ is described in Algorithm 2.

**Algorithm 2: Inference of $\beta_{0,1}^*$.**

**Input:** $n$ samples with covariates $X$ and $Y_0$, and estimates $(\beta_{\text{cal},I_1}, \hat{c}_{\text{cal},I_1})$ and $(\beta_{\text{cal},I_2}, \hat{c}_{\text{cal},I_2})$ from the outputs of Algorithm 1.

**Output:** the p-value of the hypothesis $\mathcal{H}_0 : \beta_{0,1}^* = 0$.

1. Obtain $\hat{w}_1$ by solving
   \[
   \min_{\hat{w}_1} \hat{\mathbb{E}}_{n,I_1} \left[ \phi''(Y_0(X^T \hat{\beta}_{\text{cal},I_1} - \hat{c}_{\text{cal},I_1})) \left( X_1 - \bar{X}_{-1} \hat{w}_1 \right)^2 \right] + \tilde{\lambda}_{n,1} \| \hat{w}_1 \|_1,
   \]
   where $\tilde{\lambda}_{n,1}$ is the tuning parameter; Similarly, obtain $\hat{w}_2$;

2. Construct the test statistics $T$ as
   \[
   T = \frac{1}{2} \sum_{m=1}^{2} \hat{\mathbb{E}}_{n,I_m} \left[ Y_0 \phi' \left( Y_0(X^T \hat{\beta}_{\text{cal},I_m} - \hat{c}_{\text{cal},I_m}) \right) \left( X_1 - \bar{X}_{-1} \hat{w}_m \right) \right].
   \]
   Let the variance estimator be
   \[
   \hat{\sigma}^2 = \frac{1}{2} \sum_{m=1}^{2} \hat{\mathbb{E}}_{n,I_m} \left[ \phi''(Y_0(X^T \hat{\beta}_{\text{cal},I_m} - \hat{c}_{\text{cal},I_m})) \left( X_1 - \bar{X}_{-1} \hat{w}_m \right)^2 \right];
   \]

3. Obtain the p-value using $2(1 - \Phi(\sqrt{n}T/\hat{\sigma}))$, where $\Phi(\cdot)$ is the c.d.f. of standard Gaussian distribution.

Let $w^*$ be the minimizer of
   \[
   \min_w \mathbb{E} \left[ \phi''(Y_0(X^T \beta_0^* - c_0^*)) \left( X_1 - \bar{X}_{-1} w \right)^2 \right],
   \]
and $s_{w}^*$ be the number of the non-zero coefficients in $w^*$. Theorem 4 provides the theoretical property of the testing procedure in Algorithm 2.

**Theorem 4.** Assuming conditions in Theorem 3 and the following conditions
1. The calibrated estimator $\hat{\beta}_0$ satisfies that
\[
\left( \hat{\beta}_0 - \beta_0^* \right)^\top \hat{\Sigma} \left[ \hat{\Sigma} \right]^{-1} \left( \hat{\beta}_0 - \beta_0^* \right) = O_p \left( \frac{s_{\text{pool}}^* \log p}{n} + \frac{\|\beta_0^* - \gamma^* \beta_{\text{pool}}^*\|_0 \log p}{n} \right),
\]
and
\[
\left( s_{\text{pool}}^* + \max \{ \| \beta_0^* - \gamma^* \beta_{\text{pool}}^*\|_0, 1 \} \right) \log p / \sqrt{n} \to 0 \quad \text{and} \quad s_{\text{pool}}^* \log p / \sqrt{n} \to 0,
\]
are satisfied, then, under the null hypothesis, we have
\[
\sqrt{n}T \to N(0, \sigma^2),
\]
where $\sigma^2$ is defined in the supplementary material.

Condition 1 is the resultant convergence rate in Theorem 3 when $s_{\text{pool}}^*$ can be correctly selected through Step 4 in Algorithm 1. When $\beta_{\text{pool}}^* \propto \beta_0^*$, any $k \in S$ leads to the same estimation error in Condition 1. If $s_{\text{pool}}^*$ cannot be correctly selected, we can replace $\| \beta_{\text{pool}}^* - \gamma^* \beta_{\text{pool}}^*\|_0$ term in Conditions 1 and 2 by $\| \beta_0^*\|_0$. Condition 2 is equivalent to the conditions required in Ning and Liu (2017) if $s_{\text{pool}}^* = O(\| \beta_0^*\|_0)$.

### 3.3 Extension of pooled estimation on a separate dataset

Our approach can be extended to the setting where we have a small dataset including only the target outcome, and a large dataset including auxiliary outcomes. For example, when the goal is to decide whether to recommend a new product to a customer or not, the purchase behavior associated with this new product is limited in the database. However, the record of purchase behaviors associated with related former products can be abundant. In healthcare, when the goal is to decide whether to discharge a patient in a regional hospital, the patient database of the regional hospital may be limited. Nonetheless, the set of patient information from all state-wide hospitals is large.

In this setting, it is not required that the large dataset always includes the target outcome. Assume that outcomes $Y_1, \ldots, Y_J$ are included in the large dataset and $J = O(N)$, where $N$ is the sample size of the large dataset. We can obtain the pooled estimation on the large dataset with auxiliary outcomes and then calibrate it on the small dataset with the target outcome. In this case, the estimation procedure, Algorithm 1, needs to be modified to Algorithm 3. After obtaining an estimator for $\beta_0^*$, we can implement Algorithm 3 for the testing. Theorem 5 provides the theoretical properties regarding the estimation error and the validity of the testing in this setting.

**Theorem 5.** Assuming conditions in Theorem 3, then there exists $C_3$ and $C_4$ such that
\[
\inf_{k \in S} \left\| \hat{\beta}_{\text{cal}}(k) - \beta_0^* \right\|_1 \leq C_3 \left( s_{\text{pool}}^* \right)^2 \sqrt{\frac{\log p}{N}} \sqrt{\frac{n}{N}} + C_4 \max \left\{ \| \beta_0^* - \gamma^* \beta_{\text{pool}}^*\|_0, 1 \right\} \sqrt{\frac{\log p}{n}}.
\]
Further, if we assume
\[
1. \left\| \hat{\beta}_0 - \beta_0^* \right\|_1 \leq C_3 \left( s_{\text{pool}}^* \right)^2 \sqrt{\frac{\log p}{N}} \frac{\sqrt{n}}{N} + C_4 \max \left\{ \| \beta_0^* - \gamma^* \beta_{\text{pool}}^*\|_0, 1 \right\} \sqrt{\frac{\log p}{n}}, \quad \text{and}
\]
\[
2. \left( s_{\text{pool}}^* \right)^2 \frac{n}{N} + \max \left\{ \| \beta_0^* - \gamma^* \beta_{\text{pool}}^*\|_0, 1 \right\} \right) \log p / \sqrt{n} \to 0 \quad \text{and} \quad s_{\text{pool}}^* \log p / \sqrt{n} \to 0,
\]
then, under the null hypothesis, we have
\[
\sqrt{n}T \to N(0, \sigma^2),
\]
where $\sigma^2$ is defined in the supplementary material.
From Theorem 5, when $N \geq (s_{\text{pool}}^*)^2 n$, the convergence rate of $\inf_{k \in S} \left\| \hat{\beta}_{\text{cal}}(k) - \beta_0^* \right\|_1$ is dominated by

$$ \max \left\{ \| \beta_0^* - \gamma^* \beta_{\text{pool}}^* \|_0, 1 \right\} \sqrt{\frac{\log p}{n}}. $$

When $N \geq s_{\text{pool}}^* n$, the convergence rate of the calibrated estimator is faster than the resultant convergence rate in Theorem 3. In addition, the conditions required for a valid inference can be weaker than those in Theorem 4 and Ning and Liu (2017). Specifically, when $N \geq s_{\text{pool}}^* n$, Condition 2 required in Theorem 5 is weaker than the second condition in Theorem 4. When $N \geq (s_{\text{pool}}^*)^2 n$, Condition 2 required in Theorem 5 is also weaker than the condition required in Ning and Liu (2017). Although this extension is not the focus of our paper, but we would like to mention these results for their potential applications.

**Algorithm 3:** Estimation of the optimal decision rule using multiple outcomes based on two independent datasets.

**Input:** $n$ samples with covariates $\hat{X}$ and $Y_0$, for calibration and $N$ samples with covariates $\hat{X}$ and $Y_1, \ldots, Y_J$ for pooled estimation.

**Output:** A decision rule $\hat{d}(X) = \text{sgn} \left\{ \hat{X}^\top \beta_0 - \hat{c}_0 \right\}$.

1. Randomly split the (small) dataset for calibration into $\mathcal{I}_1$ and $\mathcal{I}_2$ with an equal size;
2. Using samples in the large dataset alone, we obtain the pooled estimation $\hat{\beta}_{\text{pool}}$ by solving

$$ \min_{\beta, \{c_j\}_{j=1}^J} \hat{E}_N \left[ \frac{1}{J} \sum_{j=1}^J \phi \left\{ Y_j (X^\top \beta - c_j) \right\} \right] + \lambda_N \| \beta \|_1, $$

where $\hat{E}_N[\cdot]$ is the empirical expectation on the large dataset and $\lambda_N$ is tuned by cross-validation;
3. Using samples in $\mathcal{I}_1$, we obtain the calibrated estimator $\hat{\beta}_{\text{cal}, \mathcal{I}_1}(k)$ and $\hat{c}_{\mathcal{I}_1}(k)$ for each $k \in \hat{S}_{\text{pool}, \mathcal{I}_1}$ by solving

$$ \min_{\delta \in \Gamma_k, \gamma, c} \hat{E}_{n, \mathcal{I}_1} \left[ \phi \left\{ Y_0 (X^\top \delta + \gamma X^\top \hat{\beta}_{\text{pool}, \mathcal{I}_1} - c) \right\} \right] + \tilde{\lambda}_n \| \delta \|_1, $$

where $\hat{S}_{\text{pool}, \mathcal{I}_1}$ is the set of indexes of the non-zero coefficients in $\hat{\beta}_{\text{pool}, \mathcal{I}_1}$ and $\tilde{\lambda}_n$ is tuned by cross-validation; $\hat{E}_{n, \mathcal{I}_1}[\cdot]$ is the empirical expectation on $\mathcal{I}_1$;
4. Choose $\hat{\beta}_{\text{cal}, \mathcal{I}_1} = \hat{\beta}_{\text{cal}, \mathcal{I}_1}(k^*)$ and $\hat{c}_{\mathcal{I}_1} = \hat{c}_{\mathcal{I}_1}(k^*)$, where $k^*$ solves

$$ \min_{k} \hat{E}_{n, \mathcal{I}_2} \left[ \phi \left\{ Y_0 (X^\top \hat{\beta}_{\text{cal}, \mathcal{I}_1}(k) - \hat{c}_{\mathcal{I}_1}(k)) \right\} \right], $$

where $\hat{E}_{n, \mathcal{I}_2}[\cdot]$ is the empirical expectation on $\mathcal{I}_2$;
5. Replace $\mathcal{I}_1$ by $\mathcal{I}_2$ and $\mathcal{I}_2$ by $\mathcal{I}_1$, and repeat Steps 2 – 4. Obtain $\hat{\beta}_0$ and $\hat{c}_0$ as

$$ \hat{\beta}_0 = \left( \hat{\beta}_{\text{cal}, \mathcal{I}_1} + \hat{\beta}_{\text{cal}, \mathcal{I}_2} \right) / 2, \text{ and } \hat{c}_0 = \left( \hat{c}_{\text{cal}, \mathcal{I}_1} + \hat{c}_{\text{cal}, \mathcal{I}_2} \right) / 2. $$

### 4 Simulations

To evaluate the performance of our proposed method, we compare it with four existing approaches using simulations. The baseline approach is the one using solely the target outcome. For the baseline
approach, we directly solve

\[
\min_{\beta, c} \mathbb{E}_n \left[ \phi \left\{ Y_0 (X^\top \beta - c) \right\} \right] + \mu_n \| \beta \|_1,
\]

where the logistic loss is chosen for \( \phi(\cdot) \) and \( \mu_n \) is a hyperparameter tuned by cross-validation. The other approaches for comparison include a direct transfer learning approach and two multi-task learning approaches. The direct transfer learning approach implements a modified Algorithm 1, where one fixes \( \gamma = 1 \) and does not split the samples. This modified algorithm can be considered an extension of the oracle Trans-Lasso Algorithm (TransferDirect) in Li et al. (2020). The multi-task learning approach 1 (MultiTask1) extends the algorithm proposed in Obozinski et al. (2008) using a logistic loss with a grouped lasso penalty. The multi-task learning approach 2 (MultiTask2) is the pooled estimation, which dictates that the decision rules for multiple tasks are in the same subspace. The TransferDirect approach aims to transfer the estimate of the pooled estimation; the MultiTask1 approach aims to transfer the sparsity structure shared by multiple tasks; the MultiTask2 approach aims to estimate a common decision rule for all tasks. The advantage of knowledge transfer at the subspace level using the proposed method can be demonstrated by the comparison with these methods.

Let \( \beta_{U_0} \) be the coefficients related to the latent variable \( U_0 \) for the target outcome. We generate experimental data using the following simulation scenarios:

(I) Set \( \beta_{U_0} = (1, -1, 1, -1, 0, \cdots, 0, 0.5, -0.5, 2, -2, 0.5, 0.5, 0, \cdots, 0)^\top; U_0 = 5G(X^\top \beta_{U_0}) + 0.2\epsilon_{U_0}, \)
where \( \epsilon_{U_0} \) follows a standard normal distribution. The function \( G(\cdot) \) is the cumulative distribution function of a standard normal distribution. Set \( \tilde{U} = 5G(X^\top \beta_{U_0}) + 0.2\epsilon_{\tilde{U}}, \)
where \( \epsilon_{\tilde{U}} \) follows a standard normal distribution, where the \( q \)-th coordinate of \( \beta_{U_0} \) satisfies \( \beta_{U_1,q} = \beta_{U_0,q} \) for \( q \neq 2, 4 \) and \( \beta_{U_2,q} = \beta_{U_4,q} = 1 \). The target outcome is generated by setting \( Y_0 = \text{sgn} \{ U_0 - u_{0,1/4} \}, \)
where \( u_{0,1/4} \) is the first quartile of \( U_0 \). We further introduce a weighting parameter \( \alpha \) and generate the auxiliary outcome \( Y_1 \) by setting \( Y_1 = \text{sgn} \{ U_1 - u_{1,3/4} \}, \)
where \( U_1 = (1 - \alpha)U_0 + \alpha \tilde{U} \), and \( u_{1,3/4} \) is the third quartile of \( U_1 \).

(II) Set \( \beta_{U_0} = (1, -1, 1, -1, 0, \cdots, 0)^\top \). We generate \( U_0 \) based on a binomial distribution \( B(4, G(X^\top \beta_{U_0})) \),
where the number of trials equals 4 and the success probability equals \( G(X^\top \beta_{U_0}) \). Then, we corrupt this \( U_0 \) if \( U_0 = 3 \) or 4. When \( U_0 = 3 \), we set \( U_1 = U_0 + 1 \) with probability \( \alpha \) and \( U_1 = U_0 \) with probability \( 1 - \alpha \); when \( U_0 = 4 \), we set \( U_1 = U_0 - 1 \) with probability \( \alpha \) and \( U_1 = U_0 \) with probability \( 1 - \alpha \). When \( U_0 \neq 3 \) and \( U_0 \neq 4 \), we set \( U_1 = U_0 \). The target outcome is set as \( Y_0 = \text{sgn} \{ U_0 - 0.5 \}; \)
the auxiliary outcome is set as \( Y_1 = \text{sgn} \{ U_1 - 3.5 \}. \)

For the choice of covariate vector \( X \), we have the following two designs. In Design I, the covariate vector \( X \) follows Gaussian distribution \( N(0, I_p) \). In Design II, we first generate a \( p \)-dimensional vector following \( N(0, \Sigma_p) \), where the \( (l, k) \)-th coordinate of \( \Sigma_p \) is \( 0.5|l-k| \); then, for \( l = 1, \cdots, \lfloor p/4 \rfloor \), we replace the \( 4l \)-th coordinates in the generated vector with a binary variable. This binary variable is 1 if and only if the generated coordinate is greater than 0. Compared with Design I, Design II has correlated covariates, and the covariates include discrete and continuous variables. We test our methods using both designs for Scenarios (I) and (II). In addition, both Scenarios (I) and (II) involve a parameter \( \alpha \). When \( \alpha = 0 \), the target outcome \( Y_0 \) and the auxiliary outcome \( Y_1 \) satisfy conditions in Theorem 2 because \( U_0 = U_1 \) in both settings. As such, the pooled estimation only has the within-subspace bias. With the increase of \( \alpha \), Scenario (I) involves more against-subspace bias. In Scenario (II), \( Y_1 \) may not have the against-subspace bias, but the signal-to-noise ratio decreases with the increase of \( \alpha \).

To compare the performance of different approaches, we generate a testing dataset with sample size \( n_{\text{test}} = 10^4 \) and calculate two scores. Let \( \mathbb{E}_{\text{test}}[\cdot] \) be the empirical expectation on the testing dataset. The first score is the accuracy. Given an estimated decision rule \( \hat{d}_0(X) = \text{sgn} \{ X^\top \beta_0 - \hat{c}_0 \}, \)
the accuracy is defined as \( \mathbb{E}_{\text{test}} \left[ 1 \left\{ Y_0 = \hat{d}_0(X) \right\} \right] \). The other score is the rank correlation. We calculate
the rank correlation between $X^\top \beta_{U_0}$ and $X^\top \hat{\beta}_0$ and use it as a proxy of the estimation error. We also report the Type I error and the power of the proposed inference procedure in the supplementary material.

In these simulations, we vary the sample size of the training dataset from $n = 200$, 350, to 500 and fix $p = 1000$. In Scenario (I), we change $\alpha$ from 0 to 1 with an increment of 0.25. In Scenario (II), we change $\alpha$ from 0, 0.1, 0.2 to 0.3. We repeat each simulation setting for 500 times.

Figures 2 and 3 illustrate how the performance metrics change with the increase of sample sizes and $\alpha$, for simulation Scenarios (I) and (II), respectively. In Scenario (I), in terms of the accuracy and the rank correlation, the proposed method outperforms the baseline approach regardless of the choice of sample sizes and $\alpha$. Compared with MultiTask1 and MultiTask2, our proposed method and TransferDirect are more robust w.r.t the change of $\alpha$; compared with TransferDirect, our proposed method shows great advantages in terms of prediction accuracy. In Scenario (II), the proposed method preforms better than the baseline approach. Compared with TransferDirect, the proposed method achieves better or similar accuracy and rank correlation. Compared with MultiTask1 and MultiTask2, in Design I, the proposed method preforms better when $\alpha < 0.2$. When $\alpha \geq 0.2$, the transfer learning approaches and MultiTask2 suffer from the noise introduced by $Y_1$ in Design I. When the design is complex (Design II), the performance of the proposed method is similar to that of MultiTask2, and better than other methods. In summary, the proposed method outperforms the baseline approach even when the auxiliary outcome introduces biases, but the auxiliary outcome cannot contain excessive noise. This can be avoided by constructing/selecting informative auxiliary outcomes as shown in the supplementary material.

5 Real data example

We apply our proposed method to a real world dataset that suits our setting. Reducing readmissions after a heart failure (HF) admission is an important quality improvement effort of hospitals. In this study, we aimed to develop a decision rule for predicting 90-day acute HF readmission using EHR and echocardiography records obtained from University of Florida Health. After cleaning the dataset and imputing the missing values in variables using the Multivariate Imputation by Chained Equations (MICE) method (Buuren and Groothuis-Oudshoorn 2010), we identified 3189 patients who underwent an index HF hospitalization, and considered 108 variables across various data categories, including 1) demographic and socioeconomic characteristics; 2) outpatient care characteristics; 3) social history, including lifestyle and noncompliance behavior; 4) medical history; 5) hospitalization characteristics; 6) laboratory values, including cardiac biomarkers; 7) vitals; 8) medications; and 9) comprehensive transthoracic echocardiographic findings. The target outcome $Y_0 = 1$, if the patient experienced an acute HF readmission within 90 days post his/her hospital discharge, and $Y_0 = -1$, otherwise. We treat death as an auxiliary outcome, since hospitalization for heart failure comes with a high risk of both short-term and long-term mortality (Rathore et al. 2005). If the patient was dead within 90 days post-discharge, regardless of being readmitted, $Y_1 = 1$, and $Y_1 = -1$, otherwise. Among 3178 patients, 654 (20.6%) patients experienced at least one HF readmission and 135 (4.2%) patients died within 90 days.

We randomly split the dataset into a training dataset (70% of the entire dataset) and a testing dataset (30% of the entire dataset) and repeat the experiment 500 times. With death as the auxiliary outcome, our method (mean 0.7761, se $5.06 \times 10^{-4}$) is significantly better than the other methods in terms of accuracy: TransferDirect (mean 0.7746, se $5.14 \times 10^{-4}$), MultiTask1 (mean 0.7749, se $5.16 \times 10^{-4}$), MultiTask2 (mean 0.7746, se $5.16 \times 10^{-4}$), and the baseline approach (mean 0.7737, se $5.12 \times 10^{-4}$). Because we do not know the true coefficients and there is no theoretical support to compute the rank correlation using the readmission as a binary label, we do not report the rank correlation for this analysis. We hereby report the significant covariates from the transfer learning approach with death as the auxiliary outcome. Nineteen covariates were identified with a p-value <
0.05. In addition to the risk factors found in previous studies, such as old age, a history of lung disease, peripheral arterial disease, and ventricular arrhythmia, as well as acute heart failure presentation, low albumin, and elevated BUN, among others (Keenan et al. 2008; Thavendiranathan et al. 2014; Han et al. 2018), there are unique predictors of increased readmission risk identified in this analysis. For instance, a lack of cardiology clinic visits and adherence to appointments (e.g., having many appointment no-shows) will increase risk, which suggests that post-discharge care is important to avoiding HF readmission. The full list of the identified risk factors can be found in the supplementary material.

6 Discussion

In this work, we develop a robust high-dimensional transfer learning approach to improving the estimation of classification rules using informative auxiliary outcomes. Our approach involves a two-step estimation procedure that takes advantage of the information provided by auxiliary outcomes and retains robustness against the bias introduced by auxiliary outcomes. Our novel bias decomposition enables knowledge transfer at the subspace level and achieves superior performance against existing approaches.

To apply our approach, we require that at least one candidate auxiliary outcome has a high signal-
Figure 3: Simulation results for Scenario II with the change of sample size and $\alpha$.

to-noise ratio. In future work, we would consider a situation where only a set of weakly associated auxiliary outcomes is available. A possible strategy is to combine multiple calibrated estimators, with each trained on a candidate set of multiple auxiliary outcomes. A similar strategy has been investigated in Li et al. (2020) for heterogeneous transfer learning in linear regression.

In addition to allowing the use of a separate and larger dataset in the pooled estimation step to boost the performance, the proposed method can also be modified to suit various practical needs. For example, we may consider a pooled estimation using a distributed learner (Duan et al. 2019) to overcome the communication barrier. This communication barrier comes from the fact that the datasets from different owners (e.g., hospitals) cannot be pooled on a single machine due to privacy regulations (e.g., HIPAA on sharing EHRs). Another possibility may be to extend the proposed method to learn an individualized treatment rule (ITR). In this setting, the pooled estimation can be implemented on observational data serving as “real-world evidence,” and the calibration can be done using clinical trial data. As such, a learned ITR can take advantage of volumes of observational data and retain robustness against possible confounding in observational studies.
Supplemental Materials

Proofs of all theorems and additional simulation results are contained in the online supplemental materials.

References

Ando, R. K. and Zhang, T. (2005). A framework for learning predictive structures from multiple tasks and unlabeled data. *Journal of Machine Learning Research*, 6(Nov):1817–1853.

Bakker, B. and Heskes, T. (2003). Task clustering and gating for bayesian multitask learning. *Journal of Machine Learning Research*, 4(May):83–99.

Buuren, S. v. and Groothuis-Oudshoorn, K. (2010). mice: Multivariate imputation by chained equations in r. *Journal of statistical software*, pages 1–68.

Cai, T. T. and Wei, H. (2019). Transfer Learning for Nonparametric Classification: Minimax Rate and Adaptive Classifier.

Dezeure, R., Bühlmann, P., and Zhang, C.-H. (2017). High-dimensional simultaneous inference with the bootstrap. *Test*, 26(4):685–719.

Diaconis, P. and Freedman, D. (1984). Asymptotics of graphical projection pursuit. *The annals of statistics*, pages 793–815.

Duan, R., Ning, Y., and Chen, Y. (2019). Heterogeneity-aware and communication-efficient distributed statistical inference.

Eftekhari, H., Banerjee, M., and Ritov, Y. (2019). Inference in high-dimensional single-index models under symmetric designs.

Han, J., Mauro, C. M., Kurlansky, P. A., Fukuhara, S., Yuzefpolskaya, M., Topkara, V. K., Garan, A. R., Colombo, P. C., Takayama, H., Naka, Y., et al. (2018). Impact of obesity on readmission in patients with left ventricular assist devices. *The Annals of thoracic surgery*, 105(4):1192–1198.

Keenan, P. S., Normand, S.-L. T., Lin, Z., Drye, E. E., Bhat, K. R., Ross, J. S., Schuur, J. D., Stauffer, B. D., Bernheim, S. M., Epstein, A. J., et al. (2008). An administrative claims measure suitable for profiling hospital performance on the basis of 30-day all-cause readmission rates among patients with heart failure. *Circulation: Cardiovascular Quality and Outcomes*, 1(1):29–37.

Li, K.-C. (1991). Sliced inverse regression for dimension reduction. *Journal of the American Statistical Association*, 86(414):316–327.

Li, S., Cai, T. T., and Li, H. (2020). Transfer Learning for High-dimensional Linear Regression: Prediction, Estimation, and Minimax Optimality.

Lin, Q., Zhao, Z., and Liu, J. S. (2018). On consistency and sparsity for sliced inverse regression in high dimensions. *The Annals of Statistics*, 46(2):580–610.

Lin, Q., Zhao, Z., and Liu, J. S. (2019). Sparse sliced inverse regression via lasso. *Journal of the American Statistical Association*, 114(528):1726–1739.

Lounici, K., Pontil, M., Tsybakov, A. B., and van de Geer, S. (2009). Taking Advantage of Sparsity in Multi-Task Learning.
Ning, Y. and Liu, H. (2017). A general theory of hypothesis tests and confidence regions for sparse high dimensional models. *Ann. Statist.*, 45(1):158–195.

Obozinski, G., Wainwright, M. J., and Jordan, M. I. (2008). High-dimensional union support recovery in multivariate regression. *Advances in Neural Information Processing Systems*, 21:3.

Olivas, E. S., Guerrero, J. D. M., Martinez-Sober, M., Magdalena-Benedito, J. R., Serrano, L., et al. (2009). *Handbook of Research on Machine Learning Applications and Trends: Algorithms, Methods, and Techniques: Algorithms, Methods, and Techniques*. IGI Global.

Pepe, M. S. et al. (2003). *The statistical evaluation of medical tests for classification and prediction*. Medicine.

Rathore, S. S., Foody, J. M., Wang, Y., Herrin, J., Masoudi, F. A., Havranek, E. P., Ordin, D. L., and Krumholz, H. M. (2005). Sex, quality of care, and outcomes of elderly patients hospitalized with heart failure: findings from the national heart failure project. *American heart journal*, 149(1):121–128.

Thavendiranathan, P., Yingchoncharoen, T., Grant, A., Seicean, S., Landers, S. H., Gorodeski, E. Z., and Marwick, T. H. (2014). Prediction of 30-day heart failure-specific readmission risk by echocardiographic parameters. *The American Journal of Cardiology*, 113(2):335–341.

van de Geer, S., Bühlmann, P., Ritov, Y., and Dezeure, R. (2014). On asymptotically optimal confidence regions and tests for high-dimensional models. *Ann. Statist.*, 42(3):1166–1202.

Weiss, K., Khoshgoftaar, T. M., and Wang, D. (2016). A survey of transfer learning. *Journal of Big data*, 3(1):9.

Winship, C. and Mare, R. D. (1984). Regression models with ordinal variables. *American sociological review*, pages 512–525.

Yang, X., Kim, S., and Xing, E. P. (2009). Heterogeneous multitask learning with joint sparsity constraints. In *Advances in neural information processing systems*, pages 2151–2159.

Yu, K., Tresp, V., and Schwaighofer, A. (2005). Learning gaussian processes from multiple tasks. In *Proceedings of the 22nd international conference on Machine learning*, pages 1012–1019.

Zhang, J., Ghahramani, Z., and Yang, Y. (2008). Flexible latent variable models for multi-task learning. *Machine Learning*, 73(3):221–242.

Zhao, Y., Zeng, D., Rush, A. J., and Kosorok, M. R. (2012). Estimating individualized treatment rules using outcome weighted learning. *Journal of the American Statistical Association*, 107(499):1106–1118. PMID: 23630406.

Zhuang, F., Luo, P., Du, C., He, Q., Shi, Z., and Xiong, H. (2013). Triplex transfer learning: Exploiting both shared and distinct concepts for text classification. *IEEE transactions on cybernetics*, 44(7):1191–1203.

Zhuang, F., Qi, Z., Duan, K., Xi, D., Zhu, Y., Zhu, H., Xiong, H., and He, Q. (2019). A comprehensive survey on transfer learning.