Geometrical simplification of the dipole–dipole interaction formula

Ladislav Kocbach and Suhail Lubbad

Department of Physics and Technology, University of Bergen, Norway
E-mail: ladislav.kocbach@ift.uib.no and suhail.lubbad@gmail.com

Abstract
Many students meet dipole–dipole potential energy quite early on when they are taught electrostatics or magnetostatics and it is also a very popular formula, featured in encyclopedias. We show that by a simple rewriting of the formula it becomes apparent that, for example, by reorienting the two dipoles, their attraction can become exactly twice as large. The physical facts are naturally known, but the transformation presented seems to underline the geometrical features in a rather unexpected way. The consequence of the features discussed is the so-called magic angle which appears in many applications. The present discussion contributes to an easier introduction of this feature. We also discuss the possibility of designing educational toys and try to suggest why this formula has not been written down frequently before this work. A similar transformation is also possible for the field of a single dipole. In this case we found one such formula on the Web, but we could not find any published detailed discussion for this case either.

Introduction
Most people who have had the chance to play with magnets are aware of the fact that the magnets attract each other when held in two different orientations, but very few will be able to establish that one of the forces of attraction is exactly twice as large as the one in the weaker configuration. This simple fact is hidden in the well known formula for the dipole–dipole interaction. Many students meet this dipole–dipole potential energy quite early on when they are taught electrostatics or magnetostatics (see e.g. Griffith [1] or Jackson [2]) and it is also a very popular formula, featured in encyclopedias (e.g. [3]). In our research this formula is also used for certain types of molecular interactions. The dipole–dipole formula is

\[ U(r_{12}) = C \frac{1}{r_{12}^3} \left( \mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{3(\mathbf{m}_1 \cdot \mathbf{r}_{12})(\mathbf{m}_2 \cdot \mathbf{r}_{12})}{r_{12}^2} \right) \]  

(1)

where \( r_{12} \) is the vector connecting the two dipoles. The constant \( C \) is very different for magnetic and electric dipoles and we shall not specify it at all in the present discussion. Although we discuss mainly magnetic dipoles because they are easily represented by small magnets, the geometrical features are exactly the same for the electric dipoles.

The formula looks rather uninviting, but obviously it contains the truth about the interaction, and when needed it does its job very well. Sometimes it appears as an especially ugly species which might even contain different inverse powers,
I. Kochbach and S Lubbad

such as $r^{-3}$ and $r^{-5}$. People who are familiar with the model of interatomic interaction, the Lennard-Jones potential, which also features the difference of two powers, might then wonder if there is something untold about the minimum or maximum. Naturally, one should remember that in the dipole–dipole case the two powers are only an artefact of a sort of economic notation. The idea of the discussed transformation has probably occurred to many people, but in our case a happy coincidence was the presence of two small magnets used for posting announcements on our newsboard, which gave the idea a practical test, as described below.

The formula describes the potential energy in general. However, details of the forces on the bodies carrying the dipoles and the torques on the dipoles depend on the physical arrangements and there are in general too many possibilities. In this article, we therefore consider only such situations where the directions of the two dipoles are kept fixed, which means that the forces on the two bodies are given only by the gradient of the radial part, which is indeed very simple. The torques on the dipoles and their effects may be spectacular and fascinating, but they are not a part of this discussion. This short paper is a result of our observations and we really think that the suggested new form can make it possible to present the interaction between the dipoles to students who prefer a more intuitive approach.

We present a toy to illustrate the formula. Further, an analogue of the trick shown here can be used to draw field lines of a dipole. Moreover, we think that this much more intuitive formula and the physical features which it so clearly displays, will be of use in explanations and discussions in the interdisciplinary field of molecular dynamics applications—and sometimes perhaps even in chemistry.

The standard form of the dipole–dipole interaction energy formula (repeated from equation (1)) is

$$U(r_{12}) = \frac{1}{r_{12}^3} \left[ \mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{3}{r_{12}^4} (\mathbf{m}_1 \cdot \mathbf{r}_{12})(\mathbf{m}_2 \cdot \mathbf{r}_{12}) \right],$$

where $\mathbf{r}_{12}$ is the vector connecting the two dipoles. If

$$\mathbf{r}_{12} = r_{12} \mathbf{e}_{12} \quad (2)$$

we can decompose both of the dipoles into two components and write

$$\mathbf{m}_1 = m_{1\parallel} \mathbf{e}_{12} + m_{1\perp} = m_{1\parallel} + m_{1\perp}. \quad (3)$$

With this notation the interaction energy of equation (1) can be written as

$$U = C \left( \frac{1}{r_{12}} \right)^3 \left[ m_{1\perp} \cdot m_{2\perp} - 2m_{1\parallel} \cdot m_{2\parallel} \right]. \quad (4)$$

Thus we see that the attraction can happen if the $m_{1\perp} \cdot m_{2\perp}$ is negative, i.e. pointing against each other, and then only with a factor one; or, as is the more well known fact, if the two moments are parallel in the same direction, i.e. the factor $-2m_{1\parallel} \cdot m_{2\parallel}$ dominates (see figure 1).

It is naturally true that the standard formula does not need any additional definition of the perpendicular components and, as before, should probably remain the standard notation when no geometrical analysis is involved.

**Experiments with small magnets**

The pictures below show the two small magnets which we have by chance found on a magnetic board. The magnets were made in an unexpected way, having the south–north axis not along the length, as expected, but in the direction shown in figures 2 and 3.

This makes it very attractive to explore the geometrical properties of the interaction. The attraction is too strong to be able to appreciate the difference between the ‘two terms’, but the repulsion can nicely be explored when the magnets are placed on paper and lightly manipulated by the fingers.

From the potential formula it follows that the forces go as an inverse fourth power of distance if the geometry is fixed. If the terms for the two configurations differ by a factor of two, they will be equally strong at distances related by the fourth root of 2, i.e. 1.1892, which makes the difference about 20% (18.92%). This can actually be experienced when we use the notepad paper, as shown on the photographs of figure 3. The distance at which our magnets ‘stop repelling’, i.e. when friction takes over, is about 2 cm, thus the 20% difference is easily seen. It should, however, be noted that the dipole formula itself might not be fully valid due to the physical extension of the
magnets as compared to the distances involved. It should be explained to the audience that it is used here only for the purpose of the demonstration, while precise measurements and analysis should be used in real studies of magnetic interactions.

The magic angle

The magic angle is a concept used often in connection with nuclear magnetic resonance applications (e.g. [4]), but we have also found a very nice application and illustrations in a master’s thesis on the interaction of ultracold atoms [5]. The magic angle is characteristic for the configuration where the interaction of two dipoles is weakened by vanishing terms. It can happen in the case where the two dipoles are parallel, one is placed at the origin and both point in the $z$-direction while their connecting vector’s polar angle $\theta$ defines the parallel and perpendicular components as

$$m_{1\perp} = m_1 \sin \theta \quad m_{1\parallel} = m_1 \cos \theta$$

and the same for the second dipole $m_2$ (see figure 4). Since the dipoles are parallel, the scalar products in parentheses reduce to

$$m_{1\perp} m_{2\perp} - 2m_{1\parallel} m_{2\parallel}$$

and this gives the condition for the magic angle

$$\sin^2 \theta - 2 \cos^2 \theta = 0 \quad \text{i.e.} \quad \tan \theta = \sqrt{2}.$$ 

This gives the magic angle 54.74°. In the literature this angle is usually derived from the original formula where number 3 appears, which leads simply to the condition $3 \cos^2 \theta = 1$ giving naturally the same magic angle.
The field lines of a dipole

The formula for the strength of magnetic field of the dipole $\mathbf{m}$ placed at the origin is

$$ B(r) = \frac{C}{r^3} \left( \frac{3}{r} \mathbf{r} \cdot \mathbf{r} - \mathbf{m} \right) $$

and it invites similar rewriting, as

$$ B(r) = \frac{C}{r^3} \left( 3 \mathbf{m} \cdot \hat{r} \hat{r} - \mathbf{m} \right) $$

(8)

This is again much simpler, and leads to the following suggestions for the visualization of the field. This is the prescription (figure 5):

- Draw the dipole vector. Make the line to a point in the plane.
- Transfer the dipole vector and decompose it in components along and perpendicular to the connecting line.
- Add together twice the projection along the position vector and add the reverse (negative) of the perpendicular part. This gives the tangential line.
- Move along this direction—and start the next point.

This makes quick construction in, for example, Matlab possible. (Note: if somebody has the same idea of rewriting as presented here, a natural place to look would probably be in the source files of simple graphics for field lines.)

Before writing this article, we tried to find out if similar simplifications had been published previously. The magnetic field of a dipole is also discussed in the physics part of Eric Weisstein’s science encyclopedia on the Wolfram Research pages [6], and it appears to be the only place on the Web where it does not contain the factor 3, but factor 2 as in our present discussed formula (but the entry is very brief with no discussion). The two formulae there read

$$ B(r) = \frac{3}{r^5} \left( \mathbf{m} \cdot \mathbf{r} - \mathbf{m} \right) $$

(10)

Here $\hat{r}$ and $\hat{\theta}$ are unit vectors along the position vector and perpendicular to it, respectively. Due to their sign convention the minus sign essential in our discussion is unfortunately turned into a plus sign and the entry is too brief to include any interpretation. Note also that the units used are not specified. Fortunately, when compiling the final version of this article we were given a reference to a published discussion in [7] which includes the correct units and a short discussion, but still without any interpretation along the lines given here.
Dipole–dipole geometry

(a)

(b)

Figure 3. The magnets repelling each other are kept in position with the fingers until the friction outweighs the repulsion (the dipole directions shown by arrows): (a) the parallel components are in opposite directions, the stronger repulsion is felt against the friction to a distance of about five marks on the paper; (b) the perpendicular or upright components are in the same direction, the weaker repulsion is felt against the friction to only about four marks on the paper.

Taylor expansion derivation of the electric dipole–dipole interaction

The purpose of this section is to demonstrate how a very straightforward treatment of the electric dipole–dipole interaction from the model of two pairs of charges shown in figure 6 leads directly to the standard dipole–dipole formula of equation (1). As is well known, the dipole formula is obtained in a limiting process, where the displacements are allowed to tend to zero while the product of charge and displacement, the dipole moment, is kept constant.

The Coulomb interaction between the two pairs of charges (see figure 6) can be written as (now we leave the units out altogether)

\[
\begin{align*}
\frac{q^2}{|\mathbf{r}_{12} + \mathbf{d}_2 - \mathbf{d}_1|} - \frac{q^2}{|\mathbf{r}_{12} - \mathbf{d}_2 + \mathbf{d}_1|} \\
\frac{q^2}{|\mathbf{r}_{12} + \mathbf{d}_2 + \mathbf{d}_1|} + \frac{q^2}{|\mathbf{r}_{12} - \mathbf{d}_2 - \mathbf{d}_1|} \\
\end{align*}
\]

(11)

To obtain the formula for dipole–dipole interaction we need to consider a case where \(\mathbf{r}_{12} \gg \mathbf{d}_1\) and \(\mathbf{r}_{12} \gg \mathbf{d}_1\). Up to the second term in the Taylor expansion for two variables:

\[
f(x + \Delta x, y + \Delta y) = f(x, y) + (f_x(x, y)\Delta x \\
+ f_y(x, y)\Delta y) + \cdots + \frac{1}{2!}[f_{xx}(x, y)(\Delta x)^2 \\
+ 2f_{xy}(x, y)\Delta x\Delta y + f_{yy}(x, y)(\Delta y)^2].
\]

In vector notation

\[
f(\mathbf{r} + \mathbf{d}) = f(\mathbf{r}) + (\mathbf{d} \cdot \nabla)f(\mathbf{r}) \\
+ \frac{1}{2!}[\mathbf{d} \cdot [(\mathbf{d} \cdot \nabla)(\nabla f(\mathbf{r})]).
\]

The first-order terms disappear due to the symmetry of the terms, thus we only need to consider the second-order terms.

The \(f(\mathbf{x})\) is

\[
\frac{1}{\mathbf{r}} = \frac{1}{\sqrt{x^2 + y^2}}
\]

thus we wish to obtain the vector formula for \(\mathbf{d} = (\Delta x, \Delta y)\) from the very simple non-vector
Figure 5. How to construct the field lines for the field of an electric or magnetic dipole. The unit vector in the dipole direction is decomposed into components parallel and perpendicular to the position vector. Then twice the parallel and once the negative of the perpendicular components are added to give the direction of the field, i.e. the tangential line to the field line in the given position.

From this formula it is then not difficult to show, by comparison with the vector form above, that the Taylor expansion of (11) leads directly to the starting formula (1). We add to the notation implied by figure 6, two direction unit vectors of the charge displacements in each dipole denoted by $n_1$ and $n_2$ by relations

$$d_1 = \frac{d}{2} n_1 \quad d_2 = \frac{d}{2} n_2.$$
Dipole–dipole geometry

process, meaning that while \( d \) approaches zero, the charges \( q \) must grow correspondingly

\[
\mathbf{m}_1 = (qd)\mathbf{n}_1 \quad \mathbf{m}_2 = (dq)\mathbf{n}_2
\]

the above derivation in equations (12) leads directly to the form of the starting formula (1).

\[
\frac{1}{r_{12}^3} \left[ \mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{3}{r_{12}^2} (\mathbf{m}_1 \cdot \mathbf{r}_{12})(\mathbf{m}_2 \cdot \mathbf{r}_{12}) \right].
\]

In particular, this shows that the first scalar product term appears as a consequence of the \( \delta_{ij} \) in the first term of the last line of (12), and that the second term containing the projections on \( r_{12} \) will have a factor 3 and opposite sign following the second term in (12). The above derivation possibly explains why the usual formula is a straightforward choice and there seems no need to transform the expression. This might have contributed to a general acceptance of the standard formula as the only reasonable choice. Also, the standard formula does not need any additional definition of the perpendicular components and appears as a most general coordinate system independent expression. Hopefully, this article has shown that there are indeed some advantages in transformations and rearrangements of the standard formula for dipole–dipole interaction as well as the related vector fields.

Conclusion

It has been shown that the dipole–dipole interaction formula can be transformed to a much more intuitive form. We have shown applications that are suitable for teaching and illustration. It appears, however, that in research applications the insight provided by this simple transformation of the standard formula might contribute to more intuitive presentations and discussions. The transformation itself is really very elementary and it is thus rather surprising that it has not been discussed earlier. In fact, while preparing the final version of this article, we by chance found a new textbook on quantum chemistry [8] showing a version similar to ours but in a component form. This form appears to the author to be unsafely coordinate system dependent and is thus immediately replaced by a ‘watertight’ form which is the standard version of equation (1).

Acknowledgment

We would like to thank Professor Lars Egil Helseth at the University of Bergen for very helpful and enlightening discussions and suggestions.

Received 1 October 2009, in final form 26 February 2010
doi:10.1088/0031-9120/45/4/003

References

[1] Griffith D J 1999 Introduction to Electrodynamics (Upper Saddle River, NJ: Prentice-Hall)
[2] Jackson J D 1997 Classical Electrodynamics (Reading, MA: Addison-Wesley)
[3] http://en.wikipedia.org/wiki/Magnetic_dipole-dipole_interaction wikipedia: Magnetic dipole–dipole interaction
[4] Mehring M and Waugh J S 1971 Phys. Rev. B 5 3459
[5] Gati R 2004 Interaction of ultracold atoms in periodical potentials Diploma Thesis University of Stuttgart, Germany
[6] http://scienceworld.wolfram.com/physics/MagneticDipole.html Eric Weisstein: Science World: Field of magnetic dipole
[7] Gauthier N 2001 Comment on ‘Field pattern of a magnetic dipole’ Am. J. Phys. 69 384
[8] Piela L 2007 Ideas of Quantum Chemistry (Amsterdam: Elsevier) p 701, section 13.6.4

Ladislav Kochbach is professor of physics at the University of Bergen, Norway. His field of research is theoretical atomic physics and he teaches topics in related fields. He also works with visualization and fundamental problems of physics.

Suhail Lubbad recently completed his PhD in physics at University of Bergen, Norway on the development of effective interactions for molecular dynamics. His interests also include computer-based visualization and the physical basis of molecular interactions.

July 2010

PHYSICS EDUCATION 351