\(\tau^- \rightarrow \nu_\tau M_1 M_2\), with \(M_1, M_2\) pseudoscalar or vector mesons

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Abstract. We perform a calculation of the \(\tau^- \rightarrow \nu_\tau M_1 M_2\), with \(M_1, M_2\) either pseudoscalar or vector mesons using the basic weak interaction and angular momentum algebra to relate the different processes. The formalism also leads to a different interpretation of the role played by \(G\)-parity in these decays. We also observe that, while \(PP\) \(p\)-wave production is compatible with chiral perturbation theory and experiment, \(VP\) and \(VV\) \(p\)-wave production is clearly incompatible with experiment and we develop the formalism also in this case, producing the \(VP\) or \(VV\) pairs in \(s\)-wave. We compare our results with experiment and other theoretical approaches for rates and invariant mass distributions and make predictions for unmeasured decays. We show the value of these reactions, particularly if the \(M_1 M_2\) mass distribution is measured, as a tool to learn about the meson-meson interaction and the nature of some resonances, coupling to two mesons, which are produced in such decays.

1 Introduction

Tau decays have been instrumental to learn about weak interaction as well as strong interaction affecting the hadrons produced on \(\tau^-\) hadronic decay [1–8,20]. \(\tau^-\) decays into \(\nu_\tau\) and a pair of mesons make up for a sizeable fraction of the \(\tau^-\) decay width [10]. Several modes are well measured, as \(\tau^- \rightarrow \nu_\tau K^0 K^- [11], \tau^- \rightarrow \nu_\tau \pi^0 \omega [12], \tau^- \rightarrow \nu_\tau K^0 K^- [15], \tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau K^0 K^- [15], \tau^- \rightarrow \nu_\tau \pi K^+ [16]\), \(\tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [17], \tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15], \tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15]\), \(\tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15]\), \(\tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15]\), \(\tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15]\), \(\tau^- \rightarrow \nu_\tau \pi K^+ [16], \tau^- \rightarrow \nu_\tau \pi K^+ [15]\). As we can see, there are modes with two pseudoscalar mesons and also modes with pseudoscalar-vector mesons using the basic weak interaction and angular momentum algebra to relate the different processes.

One may wonder whether there is some fundamental rearrangement of the quarks, it should be possible to relate the rates of decay for two pseudoscalar mesons and the related pseudoscalar-vector or vector modes, for instance, \(\tau^- \rightarrow \nu_\tau K^0 K^- [15], \nu_\tau K^0 K^- [12], \nu_\tau K^0 K^- [15]\), \(\tau^- \rightarrow \nu_\tau K^0 K^- [15], \nu_\tau K^0 K^- [12], \nu_\tau K^0 K^- [15]\), \(\tau^- \rightarrow \nu_\tau K^0 K^- [15], \nu_\tau K^0 K^- [12], \nu_\tau K^0 K^- [15]\).

Many of the works on \(\tau^-\) decay into two mesons rely upon vector meson dominance producing a vector meson that decays into two pseudoscalars, or a pseudoscalar and a vector [7,8]. One approach often used [20] relies upon production of a \(qq\) pair that posteriorly hadronizes into two hadrons, and the Nambu–Jona-Lasinio (NJL) model [9] is used for this purpose.

In the present work we use the basic dynamics of the weak interaction at the quark level, producing a primary \(qq\) pair. The hadronization of this pair into two mesons is done using the \(^3P_0\) model [22–24]. The method allows us to correlate many different processes. To make that possible we carry out an elaborate and detailed angular momentum algebra that allows us to write amplitudes analytically in a very simple form.

One interesting point concerning \(\tau^-\) mesonic decays is the issue of charge symmetry discussed in ref. [25] and the classification of the weak interaction into first and second class currents. The issue, with suggestions of experiments, is retaken in ref. [26]. One of the interesting reactions is the \(\tau^- \rightarrow \nu_\tau \pi^- \eta(q')\), which according to that classification is forbidden by \(G\)-parity.

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\(^1\) We mention explicitly the most recent experiments. The full information can be obtained in ref. [10].
The $G$-parity plays indeed an important role in these reactions and in this paper we offer a new perspective into this issue. We shall see that $G$-parity for the non strange mesons plays an important role and the rules are different for pseudoscalar-pseudoscalar ($PP$) pseudoscalar-vector ($PV$) or vector-vector ($VV$) production. Interestingly, an extension of these rules appears also in the strange sector for the $\tau^- \rightarrow \nu_\tau K^- \eta(\eta')$, $\nu_\tau K^{*-} \eta(\eta')$ reactions.

We make a thorough study of all possible Cabibbo-favored and Cabibbo-suppressed reactions and compare with present available data and with results of other theoretical approaches.

2 Formalism

The first step is to look at the $\tau^- \rightarrow \nu_\tau q\bar{q}$ decay depicted in fig. 1 for the Cabibbo-favored $d\bar{u}$ production. We obtain the Cabibbo-suppressed mode substituting the $d$ quark by an $s$ quark. However, we are interested in the production of two mesons, not just one, as it would come from the mechanism of fig. 1 when $q\bar{q}$ merge into a meson. The procedure to produce two mesons is hadronization by creating a new $q\bar{q}$ pair with the quantum numbers of the vacuum. This is depicted in fig. 2.

It is easy and important to see how two mesons appear, and in which order, to see the relevance of the $G$-parity in the reactions. For this purpose, and looking only at the flavor components, we proceed as follows [27–29]: we introduce the $SU(3)$ matrix $M$

$$M = \begin{pmatrix}
  w\bar{u} & u\bar{d} & u\bar{s} \\
  d\bar{u} & d\bar{d} & d\bar{s} \\
  s\bar{u} & s\bar{d} & s\bar{s}
\end{pmatrix},$$

and when we do the hadronization of $d\bar{u}$ we get

$$d\bar{u} \rightarrow 3 \sum_{i=1}^3 d\bar{q}_i \bar{u} = M_2 \bar{M}_{11} = (M \cdot M)_{21}.$$ 

And now we write the $M$ matrix in terms of pseudoscalar or vector mesons

$$P = \begin{pmatrix}
  \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}} & \pi^+ & K^+ \\
  -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}} & \pi^- & K^- \\
  K^- & K^0 & \phi
\end{pmatrix},$$

where the standard mixing of $\eta$ and $\eta'$ has been assumed [30].

Then $M^2$ becomes $PP$, $PV$, $VP$, $VV$ and it is important to keep the order of the mesons. Thus we get

$$\begin{align*}
(P \cdot P)_{21} &= \pi^- \left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}}\right) + \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}}\right) \pi^- + K^0 K^- \\
&= \left(\frac{\pi^0}{\sqrt{2}} - \frac{\pi^0}{\sqrt{2}} \pi^-\right) + \pi^- \left(\frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}}\right) + \left(\frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}}\right) \pi^- + K^0 K^-.
\end{align*}$$

(1)

We shall see later that it is precisely the combination of $\pi^- \eta(\eta')$ and $\eta(\eta')\pi^-$ that appears in eq. (1) what makes the $\tau^- \rightarrow \nu_\tau \pi^- \eta$ decay $G$-parity forbidden, while the $\pi^- \pi^-$, $\pi^0 \pi^-$ combination gets reinforced by the relative sign in eq. (1).

Similarly, we obtain

$$\begin{align*}
(V \cdot P)_{21} &= \rho^- \left(\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}}\right) + \left(-\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}}\right) \rho^- + K^0 K^- \\
&= \left(\frac{\rho^0}{\sqrt{2}} - \frac{\rho^0}{\sqrt{2}} \rho^-\right) + \rho^- \left(\frac{\omega}{\sqrt{2}}\right) + \left(\frac{\omega}{\sqrt{2}}\right) \rho^- + K^0 K^-.
\end{align*}$$

(2)

We see again that $\pi^- \rho^0$ appears as $\pi^- \rho^0$ or $-\rho^0 \pi^-$, $\pi^- \omega$ and $\omega \pi^-$ and $(\rho^0 + \omega^2 + \rho^0) \rho^-$ with $\rho^- (\rho^0 + \omega^2 + \rho^0)$. Once again, we shall see that the order matters in the $G$-parity conservation.
Thirdly, for the $VV$ combination we get

$$\langle V \cdot V \rangle_{31} = \left( \rho - \frac{\rho_0}{\sqrt{2}} - \frac{\rho_0}{\sqrt{2}} \rho^* \right) + \left( \rho - \frac{\omega}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \rho^* \right) + K^{*0} K^{-},$$  

with, again, relevant signs between the $\rho^- \rho^0$, $\rho^0 \rho^-$ and $\rho^- \omega$, $\omega \rho^-$ components.

Replacing the $d$ quark by an $s$ quark we get the Cabibbo-suppressed modes. The hadronization leads to

$$s \bar{u} \rightarrow \sum_{i=1}^{3} s \bar{q}_i q_i \bar{u} = M_{3i} M_{i1} = (M \cdot M)_{31},$$

with the results

$$(P \cdot P)_{31} = K^- \frac{\pi^0}{\sqrt{2}} + K^0 \pi^- + \left( K^- \eta \frac{\eta'}{\sqrt{3}} + \eta K^- \right)^*,$$  

$$(P \cdot V)_{31} = K^- \left( \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) + K^0 \rho^- + \left( - \eta \frac{\eta'}{\sqrt{3}} + \frac{2 \eta'}{\sqrt{6}} \right) K^*,$$  

$$(V \cdot P)_{31} = K^* \left( \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) + K^{*0} \pi^- + \phi K^-,$$  

$$(V \cdot V)_{31} = K^{*0} \left( \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) + K^{*0} \rho^- + \phi K^{*-}.$$  

Interestingly, even if here we do not have $G$-parity states, we have also some states appearing in different order, as $K^- \eta$, $\eta K^-$ and $K^- \eta'$ and $\eta K^-$ in $PP$ and $\eta K^{*-}$, $K^{*} \eta'$, $\eta' K^{*-}$, $K^{*} \eta'$ in $PV$, $VP$. This has also consequences, similar to those leading to $G$-parity selection rules, as we shall see.

### 2.1 Weak interaction

We shall not worry about the global normalization and concentrate only on the relationship of the different decay modes discussed before. Then the weak interaction is given by

$$H = C L^\mu Q_\mu,$$  

with $C$ containing weak interaction constants and radial matrix elements that we shall see later on, where $L^\mu$ is the leptonic current

$$L^\mu = \langle \bar{u}_\tau \gamma^\mu - \gamma^\mu \gamma_5 | u_\tau \rangle,$$  

and $Q_\mu$ the quark current

$$Q^\mu = \langle \bar{u} d | \gamma^\mu - \gamma^\mu \gamma_5 | v_u \rangle.$$  

As is usual in the evaluation of decay widths to three final particles, we evaluate the matrix elements in the frame where the two mesons system is at rest. For the evaluation of the matrix element $Q_\mu$, we assume that the quark spinors are at rest in that frame and we have in the Itzykson-Zuber normalization [31]

$$u_\tau = \left( \begin{array}{c} x_r \\ 0 \end{array} \right), \quad v_\tau = \left( \begin{array}{c} 0 \\ \chi_r \end{array} \right), \quad \chi_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \chi_2 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$  

with the $\gamma^\mu$ matrices,

$$\gamma^0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \gamma_5 = \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right), \quad \gamma^i = \left( \begin{array}{cc} 0 & \sigma^i \\ -\sigma^i & 0 \end{array} \right).$$  

For the spinors at rest we have

$$\gamma_5 v_\tau = u_\tau,$$

and then

$$Q^\mu = \langle \bar{u} | \gamma^\mu - \gamma^\mu \gamma_5 | v_\tau \rangle = \langle \bar{u} | \gamma^\mu + \gamma^\mu \gamma_5 | u_\tau \rangle = \langle \bar{u} | \gamma^\mu - \gamma^\mu \gamma_5 | u_\tau \rangle. \quad (14)$$

Thus, apart from a global sign we can work with the $u$ spinors all the time.

Next we must care about how to combine the spins of the quark-antiquark to states of given angular momentum. Indeed, in the $Wq\bar{q}$ vertex of fig. 1, we shall have the matrix element

$$ME = \langle m | Operator | m' \rangle,$$  

but we want to combine the spins to total angular momentum and for this we use for the antiparticles the rule of particle-hole conjugation [32], where the hole with $m'$ behaves as a particle state according to

$$| hole, m' \rangle \rightarrow (-1)^{\frac{1}{2} - m'} \left| \begin{array}{c} 1 \\ 2 - m' \end{array} \right). \quad (16)$$

We can include the minus sign of eq. (14) and then we will implement the rule

$$| hole, m' \rangle \rightarrow (-1)^{\frac{1}{2} + m'} \left| \begin{array}{c} 1 \\ 2 - m' \end{array} \right). \quad (17)$$

We shall, then, carry on the former phase and change the sign of $m'$ to combine spins in what follows.

The next step is to realize that for the spinors at rest and $\gamma^\mu$ matrices, eqs. (12) and eq. (13), $\bar{u} \gamma^0 u \rightarrow \langle \chi' | \chi \rangle$,

$$\bar{u} \gamma^0 u \rightarrow \langle \chi' | \chi \rangle,$$

which means, $\gamma^0$ becomes the operator 1 with bispinors, and

$$\bar{u} \gamma^i \gamma_5 u \rightarrow \langle \chi' | \sigma_i | \chi \rangle,$$

hence, replacing $\gamma^i \gamma_5$ by $\sigma_i$, the Pauli matrices, with bispinors. The rest of matrix elements are zero. Then

$$Q_0 = \langle \chi' | 1 | \chi \rangle \equiv M_0,$$  

$$Q_i = \langle \chi' | \sigma_i | \chi \rangle \equiv N_i.$$  

(18)
Denoting for simplicity,
\[ L^{\mu
u} = \sum \sum L^{\mu} L^{\nu}, \]
we can write
\[ \sum \sum L^{\mu} L^{\nu} Q_{\mu} Q_{\nu}^{*} = T^{00} M_{0} M_{0}^{*} + T^{01} M_{0} N_{1}^{*} + T^{10} N_{1} M_{0}^{*} + T^{11} N_{1} N_{1}^{*}, \]
where in \( M_{0} M_{0}^{*}, \) \( M_{0} N_{1}^{*}, \) \( N_{1} M_{0}^{*}, \) \( N_{1} N_{1}^{*} \) we shall sum over the final polarizations of the mesons produced. \( \sum \sum L^{\mu} L^{\nu} \) is easily evaluated and we have
\[ \sum \sum L^{\mu} L^{\nu} = \frac{1}{m_{\tau} m_{\nu}} (g^{\mu\nu} p^{\nu} + \bar{p}^{\nu} p^{\mu} - g^{\mu\nu} p^{\nu}, p \varepsilon^{\alpha\beta\gamma\delta} \bar{p}^\alpha \bar{p}^\beta \bar{p}^\gamma \bar{p}^\delta), \]
where \( p, p' \) are the momenta of the \( \tau \) and \( \nu \) respectively and we use the field normalization for fermions of ref. [33]. These techniques have also been used successfully in the evaluation of weak decays \( M_{1} \rightarrow M_{2} M_{3} \) [34] and semileptonic decays of \( B^{(*)}, D^{(*)} \) into \( \nu \ell \) and pseudoscalar or vector mesons [35].

Next we must evaluate \( M_{0} \) and \( N_{1} \) for the different \( PP, PV, VP \) and \( VV \) combinations. In order to implement the hadronization of fig. 2 we use the \( 3P_{0} \) model [22–24]. The \( 3P_{0} \) model has been widely used in the literature and recently it has been found very instrumental to address different problems in hadron physics. A few of the most recent examples include the study of selfenergies of baryons from the coupling to the meson-baryon continuum [36], or meson coupling to the meson-meson continuum [37], the unquenching of the quark model to account for sea quarks [38], the strong and electromagnetic decays of mesons [39], the strong decay of charmed states into open charm meson pairs [40], the weak decays of \( \Lambda_{b}, \Xi_{b} \) into open charm baryons and mesons [41,42], the strong decays of higher isovector scalar mesons [43], the strong decays of \( 1P \) and \( 2D \) double charmed states [44], or to mix \( qq \) bare meson states with meson-meson components [45]. The essence of this model is that a \( qq \) state is introduced with parity + and zero total angular momentum. Since \( \bar{q}q \) has negative parity we need \( L = 1 \) to restore parity, which forces the \( \bar{q}q \) to couple to spin \( S = 1 \) and then \( L, S \) couple to \( J = 0 \). We take the contribution of spin
\[ |1S_{3}\rangle = \sum_{s} C \left[ \frac{1}{2} \right]^{2} |1, 1; s, S_{3} - s\rangle \left[ \frac{1}{2} \right]^{2} |1, 2; \bar{S}_{3} - \bar{s}\rangle, \]
where \( \left[ \frac{1}{2} \right], \bar{s} \rangle \) corresponds to the antiparticle \( \bar{q} \) with sign and phase implicitly included and is considered as a normal particle state. This is now coupled to \( Y_{1, M_{3}} \) to give \( J = 0 \). Thus
\[ |00\rangle = \sum_{M_{3}} C(110; M_{3}, S_{3}) Y_{1, M_{3}}(\bar{r})|1, S_{3}\rangle, \]
\[ = \sum_{S_{3}} C(110; -S_{3}, S_{3}) Y_{1, -S_{3}}(\bar{r})|1, S_{3}\rangle. \]

Next we must look at the spatial matrix element. We assume that for this low energy problem all the quark states are in their ground state. This assumption leads naturally to the weak chiral Lagrangians [46,47]. Then we have
\[ ME(q) = \int d^{3}r \bar{\varphi}_{u}(r) \varphi_{d}(r) \varphi_{q}(r) e^{iq \cdot r} Y_{1, -S_{3}}(\bar{r}) \]
with \( q = p_{1} - p_{2} \), where \( p_{1}, p_{2} \) are the momenta of the mesons produced. By means of
\[ e^{iq \cdot r} = 4\pi \sum_{i} i\gamma_{i}(qr) \sum_{\mu} Y_{\mu}(q) Y_{\mu}^{*}(\bar{r}) \]
we obtain
\[ ME(q) = i4\pi q Y_{1, -S_{3}}(\bar{q}) \frac{1}{3} \int r^{2} d^{3}r \prod_{i} \varphi_{i}(r) \frac{3j_{1}(qr)}{qr} r \]
\[ \equiv q Y_{1, -S_{3}}(\bar{q}) F(q). \]

As we have commented, we do not wish to evaluate this matrix element which involves large uncertainties, but rather establish relationships between different decays based exclusively on the flavor-spin structure. However, due to the fact that \( j_{1}(qr) \) goes as \( qr \), hence \( q \), for low values of \( qr \), and the fact that \( q \) is very different for different decays, due to their different masses, the appropriate procedure is to write
\[ ME(q) = i4\pi q Y_{1, -S_{3}}(\bar{q}) \frac{1}{3} \int r^{2} d^{3}r \prod_{i} \varphi_{i}(r) \frac{3j_{1}(qr)}{qr} r \]
\[ \equiv q Y_{1, -S_{3}}(\bar{q}) F(q). \]

where in the evaluation of \( F(q) \) we use the factor \( \frac{3j_{1}(qr)}{qr} \) in the integrand which goes to \( 1 \) as \( qr \rightarrow 0 \) and is a smooth function over the range of \( \prod_{i} \varphi_{i}(r) \). This allows for a better comparison of rates for different decays, assuming \( F(q) \) the same for all of them since the quark wave functions refer to the ground state in all cases that we study. This factor \( q Y_{1, -S_{3}}(\bar{q}) = \sqrt{\frac{3}{4\pi}} q - S_{3} \) (in spherical basis) leads to the WPP coupling of chiral perturbation theory [46,47].

Once the integral over \( d^{3}r \) is done, and assuming \( F(q) \) the same in all the decays, the \( |00\rangle \) state of eq. (23) leads to
\[ |00\rangle = \sum_{S_{3}} (-1)^{1+S_{3}} \frac{1}{\sqrt{3}} Y_{1, -S_{3}}(\bar{q})|1S_{3}\rangle, \]
where we have permuted indices in \( C(110; -S_{3}, S_{3}) \) to obtain this Clebsch-Gordan coefficient (CGC) (we follow Rose conventions and formulas for all the coming Racah algebra in ref. [48]).

Next we must combine \( |00\rangle \) with the \( d, \bar{d} \) spins to obtain the final \( JM, J'M' \) angular momenta of the two mesons produced. This is accomplished by means of the
which requires $m = M - s, m' = S_3 - s - M'$, and combined with $|00⟩_q$ of eqs. (28) and (22) lead to the matrix elements

$$
ME = -\frac{1}{\sqrt{3}} \sum_{S_3} \sum_{s} \mathcal{C} \left( \frac{1}{2} \frac{1}{2} 1; s, S_3 - s, S_3 \right) \times (-1)^{j+s-M'q} Y_{J_1, -S_3} \left( \hat{q} \right) \times \mathcal{C} \left( \frac{1}{2} \frac{1}{2} J; M - s, s, M \right) \times \mathcal{C} \left( \frac{1}{2} \frac{1}{2} J'; S_3 - s, M' - S_3 + s, M' \right) \times \begin{cases} (m|\sigma|m') & \text{ii) } \\ (m|\sigma|m') & \text{i) } \end{cases}
$$

(30)

i) In the case of the operator 1 leading to $M_0$ of eq. (18) we have the extra constraint $m = m' = M - s$ and then $S_3 = M + M'$.

ii) We shall work in spherical basis and evaluate $(m|\sigma_m|m')(\mu = \pm 1, 0)$. We have $(m|\sigma_m|m') = \sqrt{3}\mathcal{C}(\frac{1}{2} \frac{1}{2}; m', \mu, m)$, which induces the constraint $m' + \mu = m, \mu = M - S_3 + M'$. We call $N_\mu$ the matrix element resulting from eq. (30) in this case.

In appendix A we evaluate these matrix elements explicitly for $PP, PV, VP, VV$ and we quote here the results. For convenience, in the final formulas we take as z axis of quantization the direction of the $\nu_\tau$ in the $\tau^- \rightarrow \nu_\tau M_1 M_2$ decay, which is the direction of $\tau^- \nu_\tau$ when we make a boost to have the $M_1 M_2$ system at rest.

i) $M_0$

a) $PP$: $J = 0, J' = 0$

$$M_0 = 0$$

(31)

b) $PV$: $J = 0, J' = 1$

$$M_0 = (-1)^{M-M'} \frac{1}{\sqrt{6}} q Y_{1,-(M+M')}(\hat{q}) \delta_{M0}$$

(32)

c) $VP$: $J = 1, J' = 0$

$$M_0 = (-1)^{M-M'} \frac{1}{\sqrt{6}} q Y_{1,-(M+M')}(\hat{q}) \delta_{M'0}$$

(33)

d) $VV$: $J = 1, J' = 1$

$$M_0 = (-1)^{M-M'} \frac{1}{\sqrt{3}} \mathcal{C}(111; M, M', M) \times q Y_{1,-(M+M')}(\hat{q})$$

(34)

Table 1. Signs resulting in the $M_0$ and $N_\mu$ amplitudes by permuting the order of the mesons.

|     | $PP$ | $PV$ | $VP$ | $VV$ |
|-----|------|------|------|------|
| $M_0$ | 0    | -    | -    | +    |
| $N_\mu$ | -    | +    | +    | -    |

ii) $N_\mu$

a) $PP$: $J = 0, J' = 0$

$$N_\mu = \frac{1}{\sqrt{6}} q Y_{1,\mu}(\hat{q}) \delta_{M0} \delta_{M'0}$$

(35)

b) $PV$: $J = 0, J' = 1$

$$N_\mu = (-1)^{1-M'} \frac{1}{\sqrt{3}} q Y_{1,\mu-M'}(\hat{q}) \times \mathcal{C}(111; M', -\mu, M' - \mu) \delta_{M0}$$

(36)

c) $VP$: $J = 1, J' = 0$

$$N_\mu = (-1)^{-M} \frac{1}{\sqrt{3}} q Y_{1,\mu-M}(\hat{q}) \times \mathcal{C}(111; M, -\mu, M - \mu) \delta_{M'0}$$

(37)

d) $VV$: $J = 1, J' = 1$

$$N_\mu = \frac{1}{\sqrt{6}} q Y_{1,\mu-M-M'}(\hat{q}) \times \mathcal{C}(111; M, -\mu, M - \mu) \delta_{M'0} \times \mathcal{C}(111; M', -M - M' + \mu, -M + \mu)$$

(38)

The formulas obtained allow us to exploit selection rules for G-parity. Let us see how it proceeds. By inspecting the change when we permute particle 1 and 2, taking into account that in this permutation $Y_{1,\mu}(\hat{q}) = Y_{1,\nu}((p_1 - p_2) \rightarrow p_1 - p_2)$ goes to $Y_{1,\mu}(p_2 - p_1) = (-1)^{1} Y_{1,\nu}(p_1 - p_2)$, we find the results of table 1.

In the signs of table 1 we have taken into account that when exchanging particle 1 and 2 in the PV case we go to the VP case. For the case of the $M_0$ amplitude there is no sign change (apart from $Y_{1,\mu-M'-M}(\hat{q})$) in the formula to go from PV to VP, but for the case of $VV$ we have $\mathcal{C}(111; M, M', M + M') = (-1)^{1+1-1} \mathcal{C}(111; M', M, M + M')$ and hence a change of sign. On the other hand, the situation in the $N_\mu$ amplitude is opposite. For PP there is no change of sign, apart from $Y_{1,\mu}(\hat{q})$. However, in the PV to VP exchange we see a change of sign from the phase of CGC, apart from $Y_{1,\mu-M}(\hat{q})$. Finally the case of $VV$ is more complicated but taking the z axis such that $\sigma_z$ becomes $\sigma_2$, only $\mu = 0$ contributes and one can explicitly see by inspection of all possible cases that the amplitude does not change by exchanging the two particles, except for the $Y_{1,\mu-M-M'}(\hat{q})$. Interestingly, in some cases the role of the first and second terms in eq. (38) are exchanged, but the sum remains the same.

Let us use the result of table 1 to see the contribution of the channels shown in eqs. (1), (2), (3) and
If we take the $\pi^-\pi^0$ channel it comes with the combination $\pi^-\pi^0 - \pi^0\pi^-$. As a consequence $N_\mu$ adds for the two terms and we have a weight $2\frac{1}{\sqrt{2}}$ for the $\pi^-\pi^0$ channel. On the other hand if we take $\pi^-\eta$, $\pi^-\eta'$ they come with the combinations $\pi^-\eta + \eta\pi^-$, $\pi^-\eta' + \eta'\pi^-$ and then the combination of the two terms cancels and we do not have $\pi^-\eta$, $\pi^-\eta'$ production. In the next subsection we shall see the relationship of this to G-parity. We can proceed like that for the $\pi^-\rho^0$, $\rho^0\pi^-$ where the two terms add in $M_0$ and cancel in $N_\mu$. The opposite happens to the $\pi^-\omega$ channel and so on. A consequence of that, although there is no G-parity in this case, is that the terms $K^-\eta$, $\eta K^-$ also add in $N_\mu$ to give a weight of $2\frac{1}{\sqrt{2}}$ for the $K^-\eta$ channel, and $K^-\eta'$, $\eta' K^-$ also lead to a weight $\frac{1}{\sqrt{3}}$ for $N_\mu$ for the $K^-\eta'$ channel. For the same reasons the contribution of $\eta K^+$ and $K^-\eta$ leads to a weight $\frac{1}{\sqrt{3}}$ in $M_0$ for $\eta K^+$ and zero in $N_\mu$, while $\eta' K^+$ and $K^-\eta'$ combine to give a weight $\frac{1}{\sqrt{2}}$ in $M_0$ and $\frac{1}{\sqrt{3}}$ in $N_\mu$ for $\eta' K^+$. Altogether we find the weight of $M_0$, $h_1$, and $N_\mu$, $\bar{h}_1$, for the different channels in table 2. Since we want to evaluate ratios, the Cabibbo suppressed modes go with $\frac{\sin\theta_c}{\cos\theta_c} = \tan\theta_c$ with respect to the allowed modes, with $\theta_c$ the Cabibbo angle, $\cos\theta_c = 0.97427$.

### 2.2 G-parity considerations

Taking into account the G-parity of the mesons, $\pi(-)$, $\eta(+)$, $\eta'(+)$, $\rho(+)$, $\omega(-)$, $\phi(-)$ we can associate a G-parity to all nonstrange $M_1 M_2$ pairs. On the other hand, the G-parity can already be established from the original $\bar{d}u$ pair and the operator producing them, $1$ or $\sigma_i$. We know that the G-parity for quarks belonging to the same isospin multiplet is given in ref. [24],

$$G = (-1)^{L+S+I},$$  \hspace{1cm} (39)$$

but here $L = 0$, $I = 1$ and $S = 0$ for the $1$ operator and $S = 1$ for the $\sigma_i$ operator. Thus we have G-parity negative for the $1$ operator and positive parity for the $\sigma_i$ operator. As a consequence we find the result of table 3 for the different channels.

We can see, comparing with table 2, that the G-parity rules of table 3 coincide with what we obtained in table 2 considering the order of the $M_1 M_2$ pairs in the hadronization and the explicit formulas for $M_0$ and $N_\mu$, with their properties under the exchange of $M_1$ and $M_2$. We can see that the matrix elements are all zero for $\pi^-\eta$, $\pi^-\eta'$ cases, which shows from a different perspective that it is the value of $M_0 = 0$ for $PP$ and G-parity what makes the matrix elements zero, in coincidence with results obtained through different methods [26]. Note, however, that the G-parity restrictions have clear repercussions on which of the $M_0$ or $N_\mu$ terms contribute to the process.

### 3 Evaluation of $\sum \sum |t|^2$ for the different processes

Following the nomenclature adopted in eq. (19) we must evaluate

$$\sum \sum |t|^2 = L^{00} M_0 N_0^0 + L^{0i} M_0 N_i^0 + L^{ij} N_i M_j^0 + L^{ji} N_j M_i^0$$  \hspace{1cm} (40)$$

and in this equation we must sum over $M,M'$ the spin third components of $J,J'$. This is done in appendix B and here we summarize the results.

1) $PP$, $J = 0$, $J' = 0$
Table 3. Contributions of the different non-strange $M_1M_2$ pairs. The cross indicates non zero contribution.

| Channels | G-parity | $M_0$ | $N_\mu$ |
|----------|----------|-------|----------|
| $\pi^-\pi^0$ | + | 0 | $\times$ |
| $\pi^-\eta$ | | 0 | 0 |
| $\pi^-\eta'$ | | 0 | 0 |
| $\pi^-\rho^0$ | | $\times$ | 0 |
| $\pi^-\omega$ | | 0 | $\times$ |
| $\pi^0\rho^0$ | | | 0 $\times$ |
| $\eta\rho^0$ | | | 0 $\times$ |
| $\rho^0\rho^0$ | | | 0 $\times$ |
| $\rho^-\omega$ | | | $\times$ |

Only the term $N_\mu$ contributes and we obtain

$$\sum |t|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{2\pi} \tilde{p}_1^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right),$$

which, as discussed previously, is evaluated in the frame where the system $M_1, M_2$ is at rest, $p$ is the momentum of the $\tau, \nu$ in that frame, given by

$$p = \frac{\lambda^{1/2}(m_\tau^2, m_\nu^2, M_{M_1M_2}^2)}{2 M_{M_1M_2}},$$

$E_{M_\tau} = \sqrt{m_\tau^2 + p^2}$, $E_{M_\nu}$ = $p$ and $\mathbf{L}_{\mu\nu}$ of eq. (21) is evaluated in this frame too. In eq. (41) $\tilde{p}_1$ is the momentum of the meson $M_1$ in the same frame where the system $M_1M_2$ is at rest,

$$\tilde{p}_1 = \frac{\lambda^{1/2}(M_{M_1M_2}^2, m_{M_1}^2, m_{M_2}^2)}{2 M_{M_1M_2}}.$$  

2) PV, $J = 0, J' = 1$; VP, $J = 1, J' = 0$

a) The $\mathbf{L}^{00}M_0N_0$ contribution, summed over $M, M'$ gives

$$\sum |t_a|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{2\pi} \tilde{p}_1^2 \left( E_\tau E_\nu + p^2 \right).$$

b) The $M_0N_1^*$ and $N_1^*M_0$ combinations give zero.

c) The $\mathbf{L}^{00}N_1 N_1^*$ term gives the result

$$\sum |t_c|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{\pi} \tilde{p}_1^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right).$$

3) VV, $J = 1, J' = 1$

a) The $\mathbf{L}^{00}M_0N_0$ term gives

$$\sum |t_a|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{2\pi} \tilde{p}_1^2 \left( E_\tau E_\nu + p^2 \right).$$

b) The $\mathbf{L}^{00}M_0N_1^*, \mathbf{L}^{00}_0 N_1^* N_0$ terms give zero.

c) The $\mathbf{L}^{00}N_1 N_1^*$ term gives the result

$$\sum |t_c|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{\pi} \tilde{p}_1^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right).$$

Taking into account the weights $h_i, \bar{h}_i$ of table 2, we get finally the following result

1) $PP, J = 0, J' = 0$

$$\sum |t|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{2\pi} \tilde{p}_1^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right).$$

2) $PV, J = 0, J' = 1; VP, J = 1, J' = 0$

$$\sum |t|^2 = \frac{1}{m_{\tau}m_\nu} \frac{1}{2\pi} \tilde{p}_1^2 \left[ h_i^2 \left( E_\tau E_\nu + p^2 \right) + \frac{7}{2} \bar{h}_i^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right) \right].$$

In the former equations the angle integrations are already done in a way that finally we must take into account the full phase space with the angle independent expressions obtained in the former equations and we obtain

$$\frac{d\Gamma}{dM_{M_1M_2}} = \frac{2m_{\tau}m_\nu}{(2\pi)^3} \frac{1}{4m_{\tau}^2} p_\nu \tilde{p}_1 \sum |t|^2,$$

where $p_\nu$ is the neutrino momentum in the $\tau$ rest frame

$$p_\nu = \frac{\lambda^{1/2}(m_{\tau}^2, m_\nu^2, M_{M_1M_2}^2)}{2 M_{\tau}},$$

and $\tilde{p}_1$ the momentum of $M_1$ in the $M_1M_2$ rest frame given in eq. (43). The mass distribution of eq. (51) is then integrated over the $M_1M_2$ invariant mass in order to obtain the width.

4 s-wave decays

In the previous sections we have assumed that the quarks $d, \bar{u}$ of fig. 1 are produced in their ground state, this leads to a negative parity $q\bar{q}$ state, which makes the pair of mesons after the hadronization to be produced in $p$-wave and this is in agreement with the results of chiral perturbation theory for $\tau^- \rightarrow$ decay into $\nu_\tau$ and a pair of pseudoscalar mesons.

We shall extrapolate the scheme to pseudoscalar-vector and vector-vector production, but we can anticipate that, since the masses of these mesons are larger, the resulting momenta for the mesons are much smaller and
the $p$-wave mechanism will lead to very small widths. Certainly, in this case, $s$-wave production shall be preferable. There is just one inconvenience. Two mesons with negative parity and $s$-wave have positive parity. This means that the $d\bar{u}$ must be produced in an $L' = 1$ state. This is accomplished creating one quark in $L' = 1$ state.

The formalism in this case proceeds in total analogy to what we have done before. There is only one difference. Since a $Y(L', M'_s, 1)$ is introduced, we have now two spherical harmonics: this one and the one from the $^3P_0$ model, and they must combine to a final $s$-wave. Hence

$$Y_{1M_s}Y_{1M'_s} = \sum_l \frac{3 \cdot 3}{4\pi(2l+1)} C(111; M_3, M'_3) \times C(111; 0,0,0) Y_l,M_3+M'_3$$

which can have $l = 0, 2$ for parity reasons and we then choose $l = 0$. Evaluating explicitly the CGC we obtain

$$Y_{1M_s}Y_{1M'_s} = \frac{1}{4\pi} (-1)^{M_s} \delta_{M_3,-M'_3}$$

and the rest of calculations proceed as in the case of $p$-wave, only the $Y_{1\mu}(q)$ does not appear. Also the form factor now implies $j_0(qr)$ in the integrand instead of $\frac{3j_1(qr)}{4r}$ and the factor $q$ outside the integral of eq. (27) does not appear now. We obtain the results

1) $M_0$
   a) $PP$, $J = 0$, $J' = 0$

$$M_0 = 0$$

(55)

b) $PV$, $J = 0$, $J' = 1$

$$M_0 = \frac{1}{\sqrt{6}} \frac{1}{4\pi}$$

(56)

c) $VP$, $J = 1$, $J' = 0$

$$M_0 = \frac{1}{\sqrt{6}} \frac{1}{4\pi} C(111; M, M', M + M')$$

(57)

d) $VV$, $J = 1$, $J' = 1$

$$M_0 = \frac{1}{\sqrt{3}} \frac{1}{4\pi} C(111; M, M', M + M')$$

(58)

2) $N_\mu$
   a) $PP$, $J = 0$, $J' = 0$

$$N_\mu = \frac{1}{\sqrt{6}} \frac{1}{4\pi} \delta_{M_0} \delta_{M'0} (-1)^{-\mu}$$

(59)

b) $PV$, $J = 0$, $J' = 1$

$$N_\mu = -(-1)^{-\mu} \frac{1}{\sqrt{3}} \frac{1}{4\pi} C(111; M', -\mu, M' - \mu) \delta_{M_0}$$

(60)

---

**Table 4. $h_i$ and $\overline{h}_i$ coefficient for different channels with the two final mesons in $s$-wave.**

| Channels          | $h_i$ (for $M_0$) | $\overline{h}_i$ (for $N_\mu$) |
|-------------------|-------------------|-------------------------------|
| $\pi^-\rho^0$     | 0                 | $\sqrt{2}$                    |
| $\pi^-\omega$     | $\sqrt{2}$        | 0                             |
| $\pi^0\rho^-$     | 0                 | $-\sqrt{2}$                   |
| $\eta\rho^0$      | $\frac{2}{\sqrt{2}}$ | 0                             |
| $\rho^-\rho^0$    | $\sqrt{2}$        | 0                             |
| $\rho^-\omega$    | 0                 | $\sqrt{2}$                    |
| $\eta K^{*0}$     | 0                 | $-\frac{3}{2\sqrt{3}} \tan \theta_c$ |
| $\eta' K^{*0}$    | $\frac{3}{\sqrt{2}} \tan \theta_c$ | $\frac{1}{2\sqrt{3}} \tan \theta_c$ |

---

c) $VP$, $J = 1$, $J' = 0$

$$N_\mu = (-1)^{-\mu} \frac{1}{\sqrt{3}} \frac{1}{4\pi} \delta_{M_0} + 2 (-1)^{-\mu - M'}$$

(61)

$$\times C(111; M, -\mu, M - \mu) \times C(111; M', -M - M' + \mu, -M + \mu)$$

(62)

In this case table 1 is changed and under the exchange of the two mesons we obtain signs opposite to those in this table because we do not have the $Y_{1\mu}(q)$ factor. As a consequence, the weights of some channels, particularly those of defined $G$-parity, are changed. Note that now the rule $(-1)^{L+1/2+S+I}$ for the $G$-parity implies positive $G$-parity for the operator “1” and negative $G$-parity for the operator $\sigma_i$. As a consequence, we get the results of table 4 for the new weights of the channels involved. The rest do not change. The final formulas for $\sum \sum |t|^2$, up to a global normalization, are the same for $p$-wave removing the factor $\frac{1}{2\pi}$, concretely:

1) $PP$, $J = 0$, $J' = 0$

$$\sum \sum |t|^2 = \frac{1}{m_\tau m_\nu} \left( \frac{1}{4\pi} \right)^2 \left( E_\tau E_\nu - \frac{p^2}{3} \right) \frac{1}{2} h_i^2$$

(63)

2) $PV$, $J = 0$, $J' = 1$; $VP$, $J = 1$, $J' = 0$

$$\sum \sum |t|^2 = \frac{1}{m_\tau m_\nu} \left( \frac{1}{4\pi} \right)^2 \left( E_\tau E_\nu + \frac{p^2}{3} \right) \frac{1}{2} \overline{h}_i^2$$

$$+ \left( E_\tau E_\nu - \frac{p^2}{3} \right) \frac{1}{2} \overline{h}_i^2$$

(64)
Table 5. Branching ratios for PP case in p-wave normalized by $\tau^−\rightarrow \nu_\tau K^− K^0$.

| Decay process | $BR$ (Theo.) | $BR$ (Exp.) |
|---------------|--------------|-------------|
| $1\tau^− \rightarrow \nu_\tau \pi^0\pi^−$ | $2.48 \times 10^{-2}$ | $(3.0 \pm 3.2) \times 10^{-3}$ |
| $1\tau^− \rightarrow \nu_\tau \eta\pi^−$ | 0 | $<9.9 \times 10^{-5}$ |
| $1\tau^− \rightarrow \nu_\tau \eta'\pi^−$ | 0 | $<4.0 \times 10^{-6}$ |
| $2\tau^− \rightarrow \nu_\tau \eta\pi^0K^−$ | $8.17 \times 10^{-5}$ | $(1.55 \pm 0.08) \times 10^{-4}$ |
| $2\tau^− \rightarrow \nu_\tau \eta'\pi^0K^−$ | $3.26 \times 10^{-7}$ | $<2.4 \times 10^{-6}$ |
| $1\tau^− \rightarrow \nu_\tau K^− K^0$ fit to the Exp. | $1.29 \times 10^{-4}$ | $(2.7 \pm 1.1) \times 10^{-4}$ |
| $2\tau^− \rightarrow \nu_\tau \pi^0K^0$ | $2.52 \times 10^{-4}$ | $(5.4 \pm 2.1) \times 10^{-4}$ |

$^a$ Means Cabibbo-allowed.

$^b$ Means Cabibbo-suppressed.

The PDG has only the whole contribution including $K^−K^0$ production. We evaluate the rates in two ways: $\frac{1}{2}$ of the rate of $\tau^− \rightarrow \nu_\tau \pi^0\pi^−$ (non-resonant) and taking the whole rate times the ratio of $BR(\tau^− \rightarrow \nu_\tau \pi^0K^−)$ (non-resonant) to $BR(\tau^− \rightarrow \nu_\tau \pi^0K^−)$ (whole). Both ways give the same result. The error is taken from $\tau^− \rightarrow \nu_\tau \pi^− K^0$ in the table.

3) $VV, J = 1, J^f = 1$

$\sum \sum |t|^2 = \frac{1}{m_1m_2} \left( \frac{1}{4\pi} \right)^2 \left[ (E_\tau E_\nu + p^2) h_1^2 + \frac{7}{2} \left( E_\tau E_\nu - \frac{p^2}{3} \right) t_1^2 \right]$. (65)

5 Results

In table 5 we show the results for the decays in table 2 assuming the mesons are in $p$-wave. We should be careful selecting the data because in some cases a strong resonance can appear. This is the case of $\tau^− \rightarrow \nu_\tau \pi^0\pi^0\pi^−$, where the $\rho^0$ (770) can be formed and decay to $\pi^0\pi^−$. We should note that the $\tau^− \rightarrow \nu_\tau \rho^0\pi^−$ decay does not require the hadronization because a $q\bar{q}$ can already produce the $\rho^0$ [49]. In this case the rate of $\rho^0$ production should be bigger than the non-resonant $\nu_\tau \pi^0\pi^−$ which is actually the case experimentally. We calculate only the non-resonant part of the decay, which involves the hadronization and we compare with the “non-resonant” results of the PDG [10]. The same can be said about the $\nu_\tau \pi^− K^0$ and $\nu_\tau \pi^0 K^−$. In fact, for $\nu_\tau \pi^− K^0$ the whole branching ratio is $8.4 \times 10^{-3}$ while the “non-resonant” part is $5.4 \times 10^{-4}$. In this case the resonant part comes from $\tau^− \rightarrow \nu_\tau K^0\pi^−$. For the $\nu_\tau \pi^0 K^−$ the PDG only quotes the whole branching ratio. We have estimated the non-resonant part as explained in the footnote of table 5.

If we look at table 5 for decay to two pseudoscalars, we find that fixing our normalization to $\nu_\tau K^− K^0$ the rates obtained in the other cases are close to experiment within a factor of two or less. The rates obtained for $\nu_\tau \eta\pi^− \pi^−$ and $\nu_\tau \eta'\pi^−$ are zero in our case, and experimentally the upper bounds are very small. For the case of $\nu_\tau \eta'\pi^− K^−$ we also get a value of the branching ratio which is smaller than the experimental upper bound. The exception to the rule is the $\tau^− \rightarrow \nu_\tau \pi^0\pi^−$ that in our case is about one order of magnitude bigger than experiment. This already indicates that the form factor of eq. (27), with $q$ quite big and $3q^2(q^2)$ in the integrand, which we have assumed equal for all decays, should be smaller in the case of $\nu_\tau \pi^0\pi^−$ production. We should also note that we are taking a pion as a simple $q\bar{q}$, but this light Goldstone boson should be more complicated. Our results, and the discrepancies found, could serve as a tool of comparison for theoretical models of this form factor.

There is another consideration that we must do. If we take $\tau^− \rightarrow \nu_\tau \pi^0\pi^−$, the $\pi^0\pi^−$ pair is in $I = 1$, but for identical spin zero particles within the same isospin multiple this implies $L = 1$, $p$-wave. However, in the case that we have different mesons in the final state this requirement does not hold and one could have some $s$-wave contribution in cases like $\tau^− \rightarrow \nu_\tau \pi^0K^−$, $\tau^− \rightarrow \nu_\tau K^− K^0$, etc. Note that we have related all the processes in table 5 using the $SU(3)$ results of eq. (5). Dynamically we have assumed that the diu pair (six pair) is produced in $L = 0$ in all these cases, but this is not necessarily the case when we have two different pseudoscalars in the final state. Since the form factors in these two cases are different, we cannot relate the two mechanisms and this brings an extra uncertainty to our non-resonant pseudoscalar pair production. Since the $s$-wave requires $L = 1$ primary quark pair production, the matrix element in eq. (26) will be somewhat suppressed relative to $L = 0$. The contribution from $s$-wave in $\tau^− \rightarrow \nu_\tau \pi K$ is found small in [50, 51], and is shown to contribute only at very low $\pi K$ invariant masses in the $\pi K$ spectrum of the experiment [12]. The integrated contribution is estimated in [51] to be of the order of $BR(\tau^− \rightarrow \nu_\tau (K\pi)_{s-wave}) = (3.88 \pm 0.19) \times 10^{-4}$, or about one half of the sum of the non-resonant $\tau^− \rightarrow \nu_\tau K^0\pi^0$ and $\tau^− \rightarrow \nu_\tau \pi^0 K^− \pi^0$ contributions in table 5.

The discrepancies with experiment that we find are in line with all the uncertainties that we have discussed. On the contrary the VP, PV and VV cases, which are much less studied both theoretically and experimentally, proceeding all in $s$-wave, can be better related and there is where our approach is more useful. Also, because these reactions also allow to study the VP and VV interaction where some dynamically generated resonance appear. We turn now to this sector.

In table 6, for PV, and VV decay, what we observe is that the assumption of $p$-wave in the mesons leads systematically to very small results compared to the experiment. There are two cases where the discrepancies are larger than in the other cases. This occurs for $\tau^− \rightarrow \nu_\tau K^− K^0$ and $\tau^− \rightarrow \nu_\tau K^0 K^−$. This has to be understood as a large contribution from the resonance $K_1(1270)$ decaying into $K\rho$, as found in [52], while we only calculate the non-resonant contributions. Yet, the findings of that work are illustrative because the $K_1(1270)$ couples to $K\rho$ in $s$-wave [53–55], which clearly indicate that $PV$ and $VV$ proceed via $s$-wave meson-meson production, not $p$-wave. We also take into account the mass distributions for the particles that have a width, but this leads to effects of the order of 10–
and $N$ is the normalization factor

$$N = \int_{(M_1+2\Gamma_1)^2} (M_1-2\Gamma_1)^2 \, dm_1^2 \left( -\frac{1}{\pi} \right) \text{Im} D_{M_1}(m_1).$$

(68)

For the case of two vectors we make a double convolution as

$$\Gamma_{\tau \to \nu \cdot M_1 M_2} = \frac{1}{N'} \int_{(M_1+2\Gamma_1)^2} (M_1-2\Gamma_1)^2 \int_{(M_2+2\Gamma_2)^2} (M_2-2\Gamma_2)^2 \, dm_1^2 \left( -\frac{1}{\pi} \right) \text{Im} D_{M_1}(m_1) \times \text{Im} D_{M_2}(m_2),$$

(69)

where

$$N' = \int_{(M_1+2\Gamma_1)^2} (M_1-2\Gamma_1)^2 \, dm_1^2 \left( -\frac{1}{\pi} \right) \text{Im} D_{M_1}(m_1) \times \int_{(M_2+2\Gamma_2)^2} (M_2-2\Gamma_2)^2 \, dm_2^2 \left( -\frac{1}{\pi} \right) \text{Im} D_{M_2}(m_2).$$

(70)
When performing the convolution, some of the decays forbidden in table 7, as \( \tau^- \to \nu_\tau \eta K^+ \) and \( \tau^- \to \nu_\tau K^{0*} \), are now allowed, and finite results arise in table 8, although with very small rates. By looking at table 8 and normalizing the results to the \( \tau^- \to \nu_\tau \eta K^+ \) branching ratio, we obtain fair results compared to experiment within a factor of about two, with two exceptions: \( \tau^- \to \nu_\tau K^- \rho^0 \) and \( \tau^- \to \nu_\tau K^0 \rho^- \). As discussed previously, these two decays have a large contribution from the \( K_1(1270) \) resonance [52] and thus, with the non-resonant part that we calculate we underestimate the experimental results by about a factor three or more. This can be used in an opposite direction: a gross underestimation of the rates that we have calculated compared with future experiments would be indicative of substantial resonance contribution, which can stimulate the research for such resonance in the mass distribution.

It is also worth mentioning that in the work of refs. [54, 55] two \( K_1(1270) \) resonances were found coupling mostly to \( \pi K^+ \) and \( K \rho \). The fair agreement with the data of \( \tau^- \to \nu_\tau K^{0*} \pi^- \) should be looked with caution, because we expect some overestimation due to the light pion mass, which indicates that there is room for a resonant contribution, in this case one of the two \( K_1(1270) \). Something similar could be said about the \( \tau^- \to \nu_\tau \pi^0 \rho^- \) and \( \tau^- \to \nu_\tau \pi^0 \rho^- \) decays. We should also expect an overestimation due to the small pion mass but we instead underestimate the data by about a factor of two. This again has to be looked with the perspective that the \( \pi \rho \) couples strongly to the \( h_1(1170) \) and \( a_1(1260) \) resonances [54].

An interesting and clean case is the decay \( \tau^- \to \nu_\tau \phi K^- \), where the agreement with experiment is fair. In this case the coupling of \( \phi K^- \) to the two \( K_1(1270) \) resonances found in ref. [54] is quite weak. Also in the \( \tau^- \to \nu_\tau \eta K^+ \) decay, taken as reference, the coupling of \( \eta K^+ \) to one \( K_1(1270) \) resonance is negligible and the coupling to the second \( K_1(1270) \) is also smaller than to the \( \rho K \) dominant channel [54]. This makes the comparison of these two modes fair.

For vector-vector there is also work leading to dynamically generated resonance from the \( VV \) interaction [56–58]. However we do not have data for \( \tau^- \) decay into \( \nu_\tau \) and \( VV \), something that could change in the future. In that case the comparison of the measured decay rates with our predictions would be of interest.

Finally, we should also mention that the formalism discussed here can be considered as a starting point to study the final state interaction of \( M_1 M_2 \), eventually leading to dynamically generated resonances. It would be most interesting to study experimentally in detail invariant mass distributions in the \( \tau^- \to \nu_\tau M_1 M_2 \) decays. One case that has deserved some attention from this perspective is the \( \tau^- \to \nu_\tau \pi \rho \) via the \( a_1(1260) \) [59]. In ref. [60] this decay is done via \( \tau^- \to \nu_\tau PV \), with \( PV \) coupled channels that generate the \( a_1(1260) \), which decays into \( \pi \rho \). In the approach [60] one would take the amplitudes evaluated here for \( \tau^- \to \nu_\tau M_1 M_2 \) with all possible coupled channels that lead to a given resonance, then propagate \( M_1 M_2 \) as they would do in scattering theory, and later these \( M_1 M_2 \) mesons would be coupled to \( M_1 M_2 \), which are the observed mesons. The transition of \( M_1 M_2 \) to \( M_1 M_2 \) is given by the \( M M \to M M \) matrix that contains information on the resonance [54, 55]. Work along these lines has already been done studying the \( \tau^- \) decay into a pseudoscalar and an axial-vector meson [61], or into \( \pi^- \) and the \( f_0(980) \), \( a_0(980) \) resonances [62].

We have used a novel formalism with the explicit \( ^3P_0 \) model for hadronization and the necessary angular momentum algebra to relate the different processes, and we have shown that only one form factor is needed for \( p \)-wave production and another one for \( s \)-wave production. It is interesting to relate our formalism with a more conventional one used in the related semileptonic \( M_1 \to \ell V M_2 \) process which involves several form factors. This comparison is done in sect. 5 of [63]. Indeed, taking the \( P \to \ell V \) reaction to compare, one can see in [64–66] that four form factors are needed. Equivalently, four other form factors are proposed in [67, 68], \( V(q^2), A_0(q^2), A_1(q^2), A_2(q^2) \). In [63] it was found that close to the end point, \( \omega = 1 (M_{\ell \nu}(\ell \nu) \) maximum) the two formalisms, the one used here and the one of [67, 68] gave practically identical results for ratios of rates. This limit corresponds to having the two mesons at rest, and this is the approximation that we have used in sect. 2.1, which allows us to use only one
form factor for this reaction. In [63] it was shown that at the end point, $M_{\text{inv}}(4\ell)$ maximum, only the form factor $A_1(q^2)$ contributes, which shows in a different way that only one form factor is needed in this limit. In [69] the application of the formalism of [67,68] to $\tau^-$ decay was done, showing that, indeed, the range of $M_{\text{inv}}$ of the mesons in $\tau^-$ decays falls well within the range where the single form factor $A_1(q^2)$ dominates. Our approach allows to extrapolate the results to the $\tau^- \rightarrow \nu_\tau VV$ case without extra form factors.

6 Discussion and perspective

The present paper brings some novelties and the results are useful for predictions of new decay modes, as well as a means to find out resonances that couple to certain meson-meson channels. We briefly discuss below the novelties of the present work:

1) Although the sums over CGC can be done numerically nowadays, the laborious but useful work reported in the appendix allowed us to get finally extremely simple matrix elements for all possible transitions to $PP$, $PV$, $VP$, $VV$. This allows a qualitative understanding of the different rates obtained.

2) Another useful output of these analytical calculations is the fact that we could see explicitly the role of $G$-parity in the different reactions, and how it affects different $PP$ or $PV$, $VV$ decay modes. This was done from a very different perspective to the conventional one, based on the order in which the $M_1M_2$ components are produced in the hadronization and the expressions of our amplitudes for $M_1M_2$ and $M_2M_1$ production. Also, it was interesting to find the extrapolation of these constraints to the strange sector, even if $G$-parity is not a good quantum number there, something that is pointed out for the first time.

3) The study done here also convinced us that the $VP$, $PV$ and $VV$ modes proceed in the s-wave meson-meson channel, unlike the $PP$ mode that requires p-wave, although for $PP$ corresponding to different particles a contribution from s-wave is also possible. Experimental data in some reactions support this finding.

4) For the case of $PP$ production we indicated that we did not consider the direct vector production followed by $V \rightarrow PP$. This is a part that has been calculated in many approaches and is simpler and more accurate to evaluate because the original $q\bar{q}$ produced can directly produce the vector meson [7,8,20]. Instead, we calculated the non-resonant part, which requires the hadronization of the initial $q\bar{q}$ pair and is subject to larger uncertainty, also in the experimental extraction. Our agreement with experiment for this non-resonant part within a factor of two is fair.

5) For the $PV$ and $VP$ decay modes there is no experimental separation between the resonant and non-resonant parts. In this case we turn the discrepancies found between our results and experiment into an advantage. Indeed, the results of [53–55] indicate that resonances in this sector stem from the interaction of $VP$ components in coupled channels. Hence, the discrepancies found indicate the need for final state interaction of coupled channels that couple to the final $VP$. In other words, these cases constitute clear examples where one can learn much about meson meson interaction and hadron dynamics. The same could be said in case of $VV$, which also produces dynamically generated resonances [56–58], but there are no data so far.

6) We also value that by means of the simple formulas that we obtained we could correlate many reactions and we did the calculations for eventually all possible $\tau^- \rightarrow \nu_\tau M_1M_2$ decays.

7) As a new path to continue with this problem, starting from the findings of the present work, we can suggest the test of quark models by implementing the form factor

\[
\left\{ \begin{array}{ll}
\int \! r^2 \! dr \! \frac{3j_i(qr)}{qr} \prod_i \phi_i(r), & \text{for } PP \text{ case;} \\
\int \! r^2 \! dr \! j_0(qr) \prod_i \phi_i(r), & \text{for } PV \text{ and } VV \text{ cases,}
\end{array} \right.
\]

and using them together with the amplitudes obtained here. Quark models like those in refs. [36,45] and [70] would be most suited for this work.

8) There is another benefit from the present formulation. We provide directly the amplitudes in terms of $JM, J'M'$. Recently the power of polarization measurements to provide information of models beyond the standard model in the related semileptonic decay of $B$ mesons has been stressed [71,72]. The present formalism provides accurate ratios of differential widths for different $M, M'$ since in these ratios the missing form factor cancels. Steps in this direction are given within the present formalism in [35] for semileptonic reactions. Similar steps could be done for $\tau$ decays into the different channels that we have studied. Steps in this direction are given in [69]. Actually, the polarization issue in $\tau$ decays has already been addressed, concretely in the $\tau^- \rightarrow \nu_\tau K^-\pi^0$ reaction, but only looking at the $\tau$ polarization [73].

9) Another path worth undertaking would be the consideration of final state interaction of the meson pair produced. As mentioned above, the rates evaluated by us correspond to non-resonant contribution of the meson pairs. In some cases of $PP$ there is a clear contribution from vector mesons, which does not come from final state interaction but from the direct production of $q\bar{q}$ that forms the vector mesons. Note that the dominance of the $q\bar{q}$ component for the vector mesons has been thoroughly checked [49]. Conversely, the meson-meson interaction in s-wave gives rise to many dynamically generated resonances, and the $M_1M_2$ invariant mass distribution would reveal this. Steps in this direction for other reactions have been done in refs. [41,42]. This is a promising source of information which should be accompanied by a parallel experimental study of
invariant mass distributions. Our work provides the $M_1 M_2$ mass distributions at tree level (non-resonant part). Any experimental diversion from these predictions is a signal of a resonance as a strong final state interaction of the pair of mesons. To help in this direction we plot three mass distributions to the $PP$, $PV$ and $VV$ cases, concretely the $\tau^- \to \nu_\tau K^- K^0$, $\tau^- \to \nu_\tau \eta K^{*+}$, and $\tau^- \to \nu_\tau \rho^- \omega$ in figs. 3, 4 and 5, respectively.

We have chosen cases where one expects very small final state interaction, according to the findings of refs. [54,55,57], and we expect them to agree well with experiments. It is rewarding that there are data for two of these reactions, the $\tau^- \to K^- K^0 \nu_\tau$ and the $\tau^- \to \eta K^{*+} \nu_\tau$.

We should note that the shapes of the $PP$ (fig. 3) and $PV$, $VV$ (figs. 4, 5) distributions are quite different.

In fig. 3 we show our predictions for the $K^- K^0$ mass distribution for the $\tau^- \to K^- K^0 \nu_\tau$ decay. We show experimental data from the experiments [74–76], the most recent one [76] providing a very precise spectrum. We can see that the agreement of our mass distribution with experiments is relatively good, particularly taking into account the very different shape of the mass distribution compared to the $VP$ and $VV$ cases. Our results and the data of [75] are normalized to the area of the integrated experimental $d\Gamma/dM_{inv}$ of the other two experiments.

In fig. 4 we also show our results compared to the experimental data for the $\eta K^{*+}$ invariant mass distribution from the $\tau^- \to \eta K^{*+} \nu_\tau$ decay [77]. The experimental mass distribution is for $\pi K \eta$, but in the same paper one can see that $\pi K$ comes from $K^*$. We see again that the agreement with experiment is good, and the shape of the distribution is very different to the one of $K^- K^0$ in fig. 3. This supports our conclusion that the $PV$ pair is produced in s-wave.

In fig. 5 we show our predictions for the $\rho^- \omega$ mass distribution in $\tau^- \to \nu_\tau \rho^- \omega$ decay. There are no data for this reaction, but we predict a shape similar to the one of fig. 4, which is tied to the s-wave character of the $VV$ pair produced.

It is interesting to mention that in [77] there is also a mass distribution for $\eta K$ coming from $\tau^- \to \nu_\tau \eta K$ decay. The shape is not symmetrical like the one of fig. 3, but more similar to the one of fig. 4. However, one should not conclude that this proceeds in $s$-wave, rather the reason for it is that the $\eta K$ channel couples very strongly to $K^*$ (stronger than $\pi K$ to $K^*$), and the $K^*$ mass is below the threshold of $\eta K$. As a consequence, the strength of the distribution tends to pile up close to threshold. One may wonder why this does not happen in fig. 3, since $KK$ also couples to the $\rho$. However, in studies of coupled channels with unitarity for the $\rho$ including $\pi \pi$ and $K \bar{K}$, the role of $KK$ is found small [78,79]. Yet, this could be responsible for
7 Comparison with other approaches

In this section we take advantage to compare our results with others obtained in different approaches. The example most studied is $\tau^- \to \pi^- \pi^0 \nu_\tau$ [3, 4, 7, 8, 80]. In refs. [7, 8] vector meson dominance is addressed in $\tau^- \to \nu_\tau \rho^-$ with the $\rho$ decaying into $\pi^- \pi^0$. In ref. [3] vector form factors are considered and the $\rho, \rho'$, contributions are evaluated. One model that has proved rather adequate to study these reactions is the NJL model, which is used in [80]. The model for this and related reactions is based on the mechanism of fig. 6(a) and (b).

In the diagram of fig. 6(a) the $W$ couples to a $q\bar{q}$ state and by means of a triangle mechanism, exchanging a quark $q'$, the $q\bar{q}'$ and $q'\bar{q}$ pairs give rise to two pions with the Lagrangians of the NJL model [9]. In the diagram of fig. 6(b) the $W$ also couples to $q\bar{q}$ but this $q\bar{q}$ couples later to a vector meson which again decays into $\pi^- \pi^0$, for which the NJL model is used again. Eventually all modes give fair results for this mode where the $\rho$ production is largely dominant.

As we discussed in the results section, for this mode we only calculated the non-resonant part and furthermore it relies on form factors implicitly evaluated from the $\tau^- \to \nu_\tau K^- K^0$ (our reference decay width). Since there are larger momentum transfers for the case of the $\pi$ production, the reduction should be bigger, and this is the reason why we overestimate the experimental rate in at least a factor of four (note also the large experimental errors). This is the largest disagreement of our results for $PP$ compared to the corresponding data. In all other $PP$ cases the agreement is much better, always within a factor of two, which we should accept as the intrinsic uncertainty for our model, trying to correlate such diverse data.

It is interesting to observe that although the NJL model and our way of hadronization using the $^3P_0$ model and quark wave functions look so different, they are not. Indeed, loops with intermediate particles (quarks or nucleons) are the alternative way in Field Theory to take into account wave functions of a conventional Quantum Mechanical approach [81, 82].

Another channel worth discussing is the $\tau^- \to \nu_\tau \eta K^-$ reaction. Our formalism, with the $^3P_0$ model for hadronization, produces amplitudes which clearly show why $G$-parity is conserved in these decays and other ones, but in $\tau^- \to \nu_\tau \eta$ ($\eta'$) $\tau^-$ there is $G$-parity violation and the amplitude vanishes in our approach. Certainly there are corrections when $G$-parity is broken, and again, within the NJL model, by taking different $u, d$ quark masses, a finite value for the width can be obtained [83], which is consistent with the present experimental bounds.

The Cabibbo-suppressed $\tau^- \to \nu_\tau K^- K^0$ reaction is also studied within the extended NJL model in [84] and a width of $1.45 \times 10^{-4}$ is obtained, which compares well with experiment. Our rate in table 6 is 0.56 times this value. As we have mentioned above, this kind of qualitative agreement is what we expect from our approach in which we have no free parameters (up to the fit to one particular rate).

The $\tau^- \to \nu_\tau \pi^0 K^-$ decay is also addressed in [85] with the extended NJL model. We cannot compare with the results of [85] because the full width is calculated which is dominated by $K^{*-}$ production. We compare with results of the experimental non-resonant contribution and we find a rate about one half smaller, like in the $\eta K^-$ case. The same occurs with the $\pi^- K^0$ non-resonant width, where we also get about one half the rate of the experiment. Actually, within the framework of [85] it is possible to calculate the non-resonant contribution, removing the $K^{*-}$ pole. Although some numbers are given in [85] for different options, it would be most interesting to address this individual problem from that perspective to the light of the existing experimental data.

One example worth discussing is the $\tau^- \to \nu_\tau K^- K^0$. We used this reaction to get the normalization for the rest of the reactions. So we cannot say anything about the width, but we can comment on the invariant mass distribution of fig. 3. There are some theoretical works that provide rates for this reaction. Models based on the vector meson dominance and form factors provide branching ratios of $27 \times 10^{-4}$ [7, 8], $12.5 \times 10^{-4}$ [86], $13.5 \times 10^{-4}$ [87], $16 \times 10^{-4}$ [88], and using the extended NJL model $12.7 \times 10^{-4}$ is obtained in [89]. Results are also available from [90] using vector meson dominance with a branching rate $\Gamma = 27 \times 10^{-4}$. In [89] the $K^- K^0$ invariant mass distribution is provided (see fig. 3 of [89]) and a large contribution is obtained from the $\rho'(1450)$, providing a big peak in this energy region, although uncertainties on this contribution are acknowledged. It also contains another peak at lower invariant masses. The recent accurate measure of this amplitude in [76] does not show any peak in the mass distribution. We plotted our results in fig. 3 and we find a fair agreement with experiment, relating the small shift to lower invariant mass of the experiment to the role of the moderate coupling of $K\bar{K}$ to $\rho$.

For the case of $\tau^- \to \nu_\tau PV$ decays we compare results with other ones for the cases of $\tau^- \to \nu_\tau \rho \eta, \tau^- \to \nu_\tau K^* \bar{K}$ and $\tau^- \to \nu_\tau \omega \pi^-$. 

![Fig. 6. The diagram of the $\tau^- \to \nu_\tau \pi^- \pi^0$ decay, (a) contact term; (b) vector meson production from ref. [80].](image-url)
For the reaction $\tau^- \rightarrow \nu_{\tau} \rho \eta$ a branching ratio of $1.44 \times 10^{-3}$ is obtained in [91] using the extended NJL model. We get for this ratio $4.5 \times 10^{-3}$ with no convolution of the $\rho$ and $5.34 \times 10^{-3}$ considering the $\rho$ convolution to account for the $\rho$ mass distribution. This is a rate which in our case is related to the similar $\tau^- \rightarrow \nu_{\tau} \eta K^-\pi^-$ reaction (used as reference, see tables 7, 8) and should be rather reliable. We do not expect divergences of more than a factor two from the experiment. Unfortunately there are no data for this reaction, but it is clear that its measurement will provide helpful information to get more insight into hadron physics from $\tau^-$ decay reactions.

The $\tau^- \rightarrow \nu_{\tau} K^+\bar{K}^-$ branching ratio is evaluated in [7,8], using vector meson dominance, with a result of $3.92 \times 10^{-3}$. Our result in table 8 is $4.87 \times 10^{-3}$, very close to the result of [7,8] and both of them within approximately a factor two from experiment.

In the case of $\tau^- \rightarrow \nu_{\tau} \omega \pi^-$ there are results in [7,8], with a branching ratio of $1.2 \times 10^{-2}$, compared to $1.95 \times 10^{-2}$ from experiment. Our result from table 8 exceeds the experiment by about a factor of two, following the usual trend. There are also results of $1.85 \times 10^{-2}$ using the extended NJL model in [92] and $(1.22 \pm 0.56) \times 10^{-2}$ from [90].

There are calculations from other models that we can compare with. From refs. [7,8] we find $BR[\tau^- \rightarrow \nu_{\tau}(K^0\rho^- + K^-\rho^0)] = 7.5 \times 10^{-4}$ versus $11.26 \times 10^{-4}$ from our results. Note however, that as mentioned earlier, this result would be enhanced in both approaches when final state interaction of the $K\rho$ is allowed. In [7,8] one finds another rate which is $BR[\tau^- \rightarrow \nu_{\tau} K^-\omega] = 2.5 \times 10^{-4}$ for which we obtain $3.1 \times 10^{-4}$ and the experiment is $(4.1 \pm 0.9) \times 10^{-4}$. Finally, in [7,8] the rate for $\tau^- \rightarrow \nu_{\tau} K^+\eta$ is also evaluated with the result $BR[\tau^- \rightarrow \nu_{\tau} K^+\eta] = 1.1 \times 10^{-4}$ versus $(1.38 \pm 0.15) \times 10^{-4}$ for the experiment. Here we cannot compare because this is the mode used by us for normalization of the rest of the results.

The discussion above served to put our results in perspective and see that, once we normalize to one datum, the branching ratios obtained for other reactions are in line with the results of other approaches, usually a bit less accurate, but this is the price to pay to obtain so many ratios with our approach with no parameters, while all theories with which we compare have a fair amount of parameters that are fitted to different observables. As discussed earlier, these results constitute a basis to begin with, using this nonstandard formalism, which can be improved with the use of form factors calculated for example from existing quarks models, and the implementation of final state interaction of the mesons, for which the use of the chiral unitary approach would be most suited.

8 Conclusions

We have performed a study of the $\tau^-$ decay into $\nu_{\tau}$ and two mesons, with the aim of establishing a relationship between production of two pseudoscalars, a pseudoscalar and a vector and two vectors. For this we have used the dynamics of the weak interaction and worked out all the angular momentum-spin algebra to relate these processes, provided the form factors stemming from the radial wave functions are the same in the different cases.

The calculations done allow us to present a new perspective of the role played by $G$-parity in these reactions, involving $u, d$ quarks. However, we also find that the selection rules of $G$-parity have repercussion in the matrix elements of $\tau^--\nu_{\tau} K^-\eta, \tau^--\nu_{\tau} K^-\eta', \tau^--\nu_{\tau} K^-\eta, \tau^--\nu_{\tau} K^+\eta'$, where $G$-parity does not apply.

We compare our results with experiment. For $\tau^-$ decays into $\nu_{\tau}$ and two pseudoscalars we assume that the two mesons are produced with $p$-wave. This is in agreement with the formalism of chiral perturbation theory. In our case the two mesons are produced from an initial $q\bar{q}$ formation by the $W$, followed by the hadronization of $q\bar{q}$ into two mesons, which is done using the $3\,P_0$ model. For the case that the two pseudoscalars are not identical particles, $s$-wave production is also possible, although somewhat suppressed since it involves producing the primary $d\bar{u}$ system in $L = 1$ rather than in its ground state.

However, we observe that assuming also $p$-wave for the pseudoscalar-vector and vector-vector production one obtains results clearly incompatible with experimental data. This fact and experimental evidence that in such case the mesons are produced in $s$-wave, leads us to redo the formalism for production of the two mesons in $s$-wave.

Comparison with the experimental results shows that our predictions are fair, in spite of the large differences in the rates for different cases. We also compare our results with other approaches and make predictions for unmeasured decays.

We have also compared our mass distributions with some available data and the agreement found is relatively good. We also see a very different shape, both theoretically and experimentally for $PP$ production or $PV, VV$ production.

Another point in the results is that sometimes there are larger discrepancies from the data in these cases in which we could identify the reason of the discrepancies to large resonance contribution, with the resonance decaying finally into the two mesons observed.

We also emphasize that our formalism can be directly used to take into account final state interaction of the mesons that in some cases lead to dynamically generated resonances.

Finally we also emphasize the value of these decays to study the meson-meson interaction and the nature of some resonances, which should stimulate experimentalists to measure the two-meson mass distributions with precision in these decays, as recently done in the $\tau^--\nu_{\tau} K^- K^0$ decay in ref. [76].

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Appendix A. Evaluation of the matrix elements for the operators “1” and $\sigma_i$

We start from eq. (30)

$$\text{ME} = -\frac{1}{\sqrt{3}} \sum_{i,j} \sum_s (-1)^{j-s-M'} C \left( \frac{1}{2} 1; s, S_3 - s, S_3 \right) \\
\times q Y_{1,-S_3}(q) C \left( \frac{1}{2} 1; J, M - s, s, M \right) \\
\times C \left( \frac{1}{2} 1; J', M - M' - s, s, M \right) \\
\times \left\{ \left( m | 1 | m' \right) \left( m | \sigma | m' \right) \right\} \quad (A.1)$$

where in the case of the “1” operator we have $m' = M - s$ and $S_3 = M + M'$, while in the case of $\sigma_i$, we have $m = M - s$, $m' = m, \mu = M - S_3 + M'$. In the case of the operator “1” we obtain $M_0$ and with $\sigma_i, N_\mu$.

1) $M_0$ case

$$\sqrt{3} \sum_s (-1)^{j-s-M'} q Y_{1,-(M+M')}(q)$$

$$\times C \left( \frac{1}{2} 1; J, M - s, s, M \right)$$

$$\times C \left( \frac{1}{2} 1; J', M - M' - s, s, M \right)$$

$$\times \frac{1}{\sqrt{3}} \sum_s \left( \left( m | 1 | m' \right) \left( m | \sigma | m' \right) \right) \quad (A.2)$$

Using the permutation relations in ref. [48]

$$C \left( \frac{1}{2} 1; s, M + M' - s, M + M' \right) =$$

$$(-1)^{j-s} \sqrt{3} C \left( \frac{1}{2} 1; M + M', -s, M + M' - s \right)$$

$$\times \left\{ \left( m | 1 | m' \right) \left( m | \sigma | m' \right) \right\} \quad (A.3a)$$

we obtain

$$M_0 = -\frac{1}{\sqrt{3}} (1) -M' \sqrt{3} q Y_{1,-(M+M')}(q)$$

$$\times \sum_s \left( \left( \frac{1}{2} 1; M + M', -s, M + M' - s \right) \left( \frac{1}{2} 1; J', M + M' - s, s, M \right) \left( \frac{1}{2} 1; J; -s, -M + s, -M \right) \right) \quad (A.4)$$

and summing over $s$, keeping $M$ fixed we obtain, using the formulas of ref. [48],

$$M_0 = -\frac{1}{\sqrt{3}} (1) -M' \sqrt{3} \frac{2}{J + 1} q Y_{1,-(M+M')}(q)$$

$$\times W \left( \frac{1}{2} 1; \frac{1}{2} 1; J \right) C \left( J J' 1; M, M', M + M' \right) \quad (A.5)$$

in terms of a Racah coefficient, $W(\cdots)$. We can write this in a more symmetrical way by taking

$$C \left( J J' 1; M + M', -M, M' \right) =$$

$$(-1)^j-M \frac{2J+1}{3} C \left( J J' 1; M, M', M + M' \right) \quad (A.6)$$

such that finally we obtain,

$$M_0 = -\frac{1}{\sqrt{3}} (1) -M' \frac{1}{\sqrt{3}} (J + 1)(2J + 1) \frac{2J+1}{3} \sqrt{3} (2J + 1) q Y_{1,-(M+M')}(q)$$

$$\times W \left( \frac{1}{2} 1; \frac{1}{2} 1; J \right) C \left( J J' 1; M, M', M + M' \right)$$

$$\times q Y_{1,-(M+M')}(q). \quad (A.7)$$

We apply it to the different $M_1 M_2$ cases:

a) $PP$: $J = 0, J' = 0$

The Clebsch-Gordan Coefficient (CGC) $C(001;\cdots)$ is zero, hence:

$$M_0 = 0. \quad (A.8)$$

b) $PV$: $J = 0, J' = 1$

Using the table in the appendix of ref. [48] we find

$$W \left( \frac{1}{2} 1; \frac{1}{2} 1; 0 \right) = -\frac{1}{\sqrt{6}} \quad (A.9)$$

and then, for any $M'$,

$$M_0 = (1) -M' \frac{1}{\sqrt{6}} \delta_{M0} q Y_{1,-(M+M')}(q) \quad (A.10)$$

c) $VP$: $J = 1, J' = 0$

Now

$$W \left( \frac{1}{2} 1; \frac{1}{2} 1; 0 \right) = \frac{1}{\sqrt{6}} \quad (A.11)$$

and thus we get, for any $M$,

$$M_0 = (1) -M' \frac{1}{\sqrt{6}} \delta_{M0} q Y_{1,-(M+M')}(q) \quad (A.12)$$
d) $VV$: $J = 1, J' = 1$

Now

$$W\left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}\right) = \frac{1}{3},$$

(A.13)

and thus we get

$$M_0 = (-1)^{-M-M'} \frac{1}{\sqrt{3}} \mathcal{C} (111; M, M', M + M') \times q Y_{1,-(M+M')}(\hat{q}).$$

(A.14)

2) $N_\mu$ case

$$N_\mu = \frac{1}{\sqrt{3}} \sum_s (-1)^{\frac{1}{2} - s} M' q Y_{1,\mu-\nu-(M+M')}(\hat{q})$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M - s, s, M \right)$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; s, M + M' - \mu - s, M + M' - \mu \right)$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J' , M + M' - \mu - s, -M + \mu + s, M' \right)$$

$$\times \sqrt{3} \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M - \mu - s, \mu, M - s \right).$$

(A.15)

Note that now the variable $s$ is in the four CGC and we cannot get directly a Racah coefficient. For this we use again formulas of ref. [48] to decompose two CGC into other two, one of which does not depend on $s$. First we use the permutations

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; s, M + M' - \mu - s, M + M' - \mu \right) = (-1)^{\frac{1}{2} - s}$$

$$\times \sqrt{\frac{3}{2}} \left(\frac{1}{2} \frac{1}{2} ; M + M' - \mu - s, -s, M + M' - \mu - s \right),$$

and

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J - s, -M + s, M \right) = \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J - s, -M + s, M \right),$$

and we find:

$$N_\mu = \frac{1}{\sqrt{3}} \sum_s (-1)^{-M'} \sqrt{6} q Y_{1,\mu-\nu-(M+M')}(\hat{q}) \sqrt{\frac{3}{2}}$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M + M' - \mu - s, M + M' - \mu - s \right)$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J' , M + M' - \mu - s, -M + \mu + s, M' \right)$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J - s, -M + s, M \right)$$

$$\times \sqrt{3} \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M - \mu - s, \mu, M - s \right).$$

(A.16)

We can write the first two CGC as

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M + M' - \mu - s, M + M' - \mu - s \right)$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J' , M + M' - \mu - s, -M + \mu + s, M' \right)$$

$$\times \sum_{j''} \sqrt{2(j'' + 1)} W \left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} j'' \right) j''$$

$$\times \mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J - s, -M + \mu + s, -M \mu + s, M' \right)$$

$$\times \mathcal{C} \left(1 j'' J' ; M + M' - \mu, -M + \mu, M' \right).$$

(A.17)

We use again CGC permutation relations:

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M - s - \mu, \mu, M - s \right) =$$

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; M - s - \mu, \mu, M - s \right),$$

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} ; J - s, -M + s, -M \right) =$$

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} J - s, -M + s, M \right),$$

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} J'' - s, M + \mu + s, -M + \mu \right) =$$

$$\mathcal{C} \left(\frac{1}{2} \frac{1}{2} J'' - s, M + \mu + s, -M + \mu \right).$$

(A.18)

Then summing over $M - s - \mu$, keeping $M - \mu$ fixed, we get for the sum of the three CGC to the right of eq. (A.18) in ref. [48]

$$\sqrt{2(j'' + 1)} W \left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} j'' \right) j'' \mathcal{C} \left(1 j'' J' ; M - \mu, -M + \mu \right).$$

(A.19)

So, finally we get

$$N_\mu = (-1)^{-M'} \sqrt{6} q Y_{1,\mu-\nu-(M+M')}(\hat{q})$$

$$\times \sum_{j''} \sqrt{2(j'' + 1)} W \left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} j'' \right) j''$$

$$\times \mathcal{C} \left(1 j'' J' ; M + M' - \mu, -M + \mu, M' \right).$$

(A.20)

We apply this equation to the different $M_1 M_2$ cases and find:

a) $PP$: $J = 0, J' = 0$

$$\mathcal{C} \left(1 j'' 0 ; \mu, -M + \mu, M - \mu \right) =$$

$$(-1)^{1-M'} \sqrt{\frac{1}{2j'' + 1}} \mathcal{C} \left(10 j'' ; \mu, -M + \mu, M - \mu \right)$$

(A.21)

which implies $M = 0$, and $j'' = 1$,

$$\mathcal{C} \left(1 j'' 0 ; M + M' - \mu, -M + \mu, M' \right) =$$

$$(-1)^{1-M' - M''} \mathcal{C} \left(10 j'' ; M + M' - \mu, -M + \mu, M' \right),$$

which also implies that $M' = 0$ and $j'' = 1$. The Racah coefficients are the same as in eq. (A.11) and we finally get

$$N_\mu = \frac{1}{\sqrt{6}} q Y_{1,\mu}(\hat{q}) \delta_{M'0} \delta_{M'0}.$$  

(A.23)

b) $PV$: $J = 0, J' = 1$

We use eq. (A.21) which implies $M = 0$ and $j'' = 1$ and then write

$$\mathcal{C} \left(111; M + M' - \mu, -M + \mu, M' \right) =$$

$$(-1)^{1-M' - M''} \mathcal{C} \left(111; M + M' - \mu, -M' + \mu, M - \mu \right).$$

(A.24)
We need the Racah coefficients of eqs. (A.11) and (A.13), and we get

\[ N_{\mu} = (-1)^{-M - \mu} \frac{1}{\sqrt{3}} q Y_{1,\mu-(M+M')} (\hat{q}) \delta_{M0} \]
\[ \times C (111; M' - \mu, -M' - \mu), \]  
(A.25)

and writing the CGC as \((-1)^{-M' - \mu} C(111; M', -\mu, M' - \mu)\) we get finally

\[ N_{\mu} = (-1)^{1-M-M'} \frac{1}{\sqrt{3}} q Y_{1,\mu-M'} (\hat{q}) \delta_{M0} \]
\[ \times C (111; M', -\mu, M' - \mu). \]  
(A.26)

c) \(VP; J = 1, J' = 0\) We use

\[ C (1j^0; M + M' - \mu, -M + \mu, M') = \]
\[ (-1)^{1-M-M'+\mu} \frac{1}{\sqrt{2}} \delta_{M0} + 1 \]
\[ \times C (10j; M + M' - \mu, -M', M - \mu), \]  
(A.27)

which implies \(M' = 0\) and \(j'' = 1\) and using

\[ C (111; M, -\mu, M - \mu) = C (111; M, -\mu, M - \mu) \]

we finally find

\[ N_{\mu} = (-1)^{-M} \frac{1}{\sqrt{3}} q Y_{1,\mu-M} (\hat{q}) \delta_{M0} \]
\[ \times C (111; M, -\mu, M - \mu). \]  
(A.28)
d) \(VV; J = 1, J' = 1\) We find now

\[ \mathcal{W}^2 \left( 1 \frac{1}{2} 1 \frac{1}{2} 1 \frac{1}{2} \right) = \frac{1}{36} (3 + j'')(2 - j''), \]  
(A.29)

which means that only \(j'' = 0, j'' = 1\) contribute and for \(j'' = 2\) the coefficient is zero.

i) if \(j'' = 0\) we get

\[ N_{\mu} \rightarrow (-1)^{-M'} \frac{1}{\sqrt{6}} \delta_{M,0} q Y_{1,\mu-(M+M')} (\hat{q}). \]  
(A.30)

ii) if \(j'' = 1\) we write

\[ C (111; M, M - \mu, M) = (-1)^{1-\mu} C (111; M, -\mu, M - \mu), \]
\[ C (111; M + M' - \mu, -M + \mu, M') = (-1)^{1-M-M'+\mu} \delta_{M,0} \]
\[ \times C (111; M', -M - M' + \mu - M + \mu), \]  
(A.31)

and then

\[ N_{\mu} \rightarrow (-1)^{-M} \sqrt{\frac{2}{3}} q Y_{1,\mu-(M+M')} (\hat{q}) \]
\[ \times C (111; M, -\mu, M - \mu) \]
\[ \times C (111; M', -M - M' + \mu - M + \mu), \]  
(A.32)

and for the sum of \(j'' = 0, j'' = 1\) we get the final result

\[ N_{\mu} = \frac{1}{\sqrt{6}} q Y_{1,\mu-(M+M')} (\hat{q}) \left\{ (-1)^{-M'} \delta_{M,0} \right\} \]
\[ + 2 (-1)^{-M} \times C (111; M, -\mu, M - \mu) \]
\[ \times C (111; M', -M - M' + \mu - M + \mu). \]  
(A.33)

**Appendix B. Evaluation of \(\sum \sum |t|^2\)**

Following the nomenclature \(\bar{L}^{\mu\nu} = \sum \sum L^{\mu \nu} L_{\mu \nu}\) adopted before for simplicity, we have for the leptonic sector

\[ \bar{L}^{\mu\nu} \equiv \sum \sum L^{\mu \nu} L_{\mu \nu} = \frac{1}{m_{M0}} \cdot \left\{ p(p'p' + p'^2) - g^{\mu\nu}(p' \cdot p) \right\} \]
\[ + i e \cos^2 \theta \omega_{\mu} p_{\nu} p_{\mu} \]  
(B.1)

Thus for the leptonic plus hadronic matrix elements we have

\[ \sum \sum |t|^2 = \bar{L}^{00} M_0 M_0' + \bar{L}^{01} M_0 N_0' + \bar{L}^{02} N_0 M_0' + \bar{L}^{11} N_1 N_1', \]  
(B.2)

We have to take the product of these hadronic components, sum over \(M, M'\) and contract with \(\bar{L}^{\mu\nu}\). We do that for the different \(M_1 M_2\) cases.

a) \(PP; J = 0, J' = 0\)

In this case \(M_0 = 0\) and we only have to calculate \(N_i N_i^*\).

We use eq. (A.23) and write

\[ q Y_{1, \mu} (\hat{q}) = \frac{3}{\sqrt{4\pi}} q_\mu = \frac{1}{\sqrt{4\pi}} (\hat{p}_1 - \hat{p}_2) = \frac{3}{\sqrt{4\pi}} 2 \hat{p}_1, \]  
(B.3)

since \(\hat{p}_1\) is evaluated in the rest frame of \(M_1 M_2\). This means that in cartesian coordinates we can write

\[ N_1 = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{\delta_{M0} \delta_{M'0} 2 \hat{p}_1}}, \]  
(B.4)

and then from eq. (B.2):

\[ \sum \sum |t|^2 = \sum_{M,M'} \frac{1}{m_{M0}} \frac{1}{2\pi} \left\{ 2 p_{i} p_{j} + \delta_{ij} (p \cdot p') \right\} \]
\[ \times \hat{p}_{i1} \hat{p}_{j1} \delta_{M0} \delta_{M'0}, \]  
(B.5)

with

\[ p_{i} p_{j} \hat{p}_{i1} \hat{p}_{j1} = (p \cdot \hat{p})^2 = (p_{\perp}^2) \cos^2 \theta - \frac{1}{3} (p_{\perp}^2), \]  
(B.6)

where the last step comes from the integral over \(\cos^2 \theta\). We replace \(\cos^2 \theta\) by \(1/3\) and put the whole phase space later independent on the angles. Then we get, including the weight \(h_i\) for the \(N_i\) term

\[ \sum \sum |t|^2 = \frac{1}{m_{M0}} \frac{1}{\pi} \left( E_{e} E_{e} - \frac{1}{3} p^2 \right) \hat{p}_{1}^2 h_{1}, \]  
(B.7)

with

\[ \hat{p}_1 = \frac{1/2 (M_{20}^2 (M_1 M_2), M_{21}^2, M_{22}^2)}{2 M_{20} (M_1 M_2)}, \]  
(B.8)

and \(p\) given in eq. (42),

\[ p = \frac{1/2 (m_2^2, m_2^2, M_{20}^2 (M_1 M_2))}{2 M_{20} (M_1 M_2)}. \]  
(B.9)
In eq. (B.7) and what follows $E_\tau$, $E_\nu$ are also calculated in the $M_1M_2$ rest frame, $E_\tau = \sqrt{m_\tau^2 + p^2}$, $E_\nu = p$.

The $\epsilon^{\alpha\beta\gamma} \vec{p}_i \cdot \vec{p}_j$ is zero. This term does not contribute in any case, but in some cases the cancellation comes from different terms in the sum over $M$, $M'$. The cancellation of this term when summing over polarizations in semileptonic decays was already found in [93,94] and we do not elaborate on it further here.

b) $PV$: $J = 0$, $J' = 1$

Now we have

$$M_\theta = (-1)^{-M} \frac{1}{\sqrt{6}} \delta_{M_0} q Y_{1,-M'}(\hat{q}),$$

$$N_\mu = (-1)^{1-M'} \frac{1}{\sqrt{3}} \delta_{M_0} q Y_{\mu,-M'}(\hat{q}) \times C (111; M', -\mu, M' - \mu).$$

First let us see that the $M_0N_\alpha$ components do not contribute. Indeed we find in the phase space integration

$$\sum_{M'} \int d\Omega Y_{1,-M'}^* Y_{1,-M'}(\hat{q}) = C (111; M', -\mu, M' - \mu) = 0.$$  

This is again the case also in $VP$ and $VV$ and we do not discuss it further.

Thus, we have contributions from:

i) $M_0M_0^*$

$$M_0M_0^* = \frac{1}{6} \delta_{M_0} Y_{1,-M'}(\hat{q}) Y_{1,-M'}(\hat{q}) q^2.$$  

In the phase space calculation we shall have

$$\int d\Omega Y_{1,-M'}(\hat{q}) Y_{1,-M'}(\hat{q}) = 1,$$

and then we replace $Y_1^*$ by $\frac{1}{\pi}$ evaluating later the phase space for an angle independent amplitude. Thus summing over $M'$ ($M = 0$) we get

$$\sum_{M'} M_0M_0^* = \frac{1}{6} \frac{1}{4\pi^3} q^2 = \frac{1}{2\pi^2} p_1^2,$$

which multiplied by $\tilde{L}^{00}$ gives

$$\sum \sum |t|^2 = \frac{1}{m_\nu m_\tau} (E_\tau E_\nu - \frac{1}{3} p^2).$$  

ii) $L^0N_iN_i^*$

For simplicity of the calculation we take $p$ in the $z$ direction. Then

$$L^0N_iN_i^* = \frac{1}{m_\nu m_\tau} (2p^2 \delta_\alpha \delta_\beta \delta_\gamma N_i N_i^* + (p \cdot p') N_i N_i^*),$$

but one can see that

$$\sum N_iN_i^* = \sum N_\mu N_\mu^*.$$  

Then, from eq. (A.26)

$$\sum \sum N_\mu N_\mu^*$$

$$= \sum \sum \sum \frac{1}{16\pi^3} q^2 \int Y_{1,-M'}(\hat{q}) Y_{1,-M'}(\hat{q}) d\Omega \times C (111; M', -\mu, M' - \mu)^2$$

$$= \sum \frac{1}{4\pi^3} C (111; M', -\mu, M' - \mu)^2$$

$$= \sum \frac{1}{4\pi^3} C (111; M', -\mu, M' - \mu)^2 4p_1^2$$

$$= \sum \frac{1}{\pi^2} p_1^2.$$  

On the other hand for $i = 3$, $N_i \equiv N_{\mu=0}$, and again

$$\sum \sum \sum \frac{1}{4\pi^3} q^2 \int Y_{1,-M'}(\hat{q}) Y_{1,-M'}(\hat{q}) d\Omega \times C (111; M', 0, M' - \mu)^2$$

$$= \frac{1}{4\pi^3} \sum_{M'} C (111; M', -\mu, 0)^2$$

$$= \frac{1}{\pi^2} p_1^2,$$

and we find for this term

$$\sum \sum |t|^2 = \frac{1}{m_\nu m_\tau} \frac{1}{\pi} p_1^2 \left( E_\tau E_\nu - \frac{1}{3} p^2 \right).$$  

Recalling that we have different weights for $M_0$ and $N_i$ in each channel we sum the two terms of eqs. (B.15) and (B.20) to give

$$\sum \sum |t|^2 = \frac{1}{m_\nu m_\tau} \frac{1}{2\pi^2} \left( \hat{h}_1^2 \left( E_\tau E_\nu - \frac{1}{3} p^2 \right) + \frac{1}{\pi^2} p_1^2 \right).$$

and (B.21)

c) $VP$: $J = 1$, $J' = 0$

The evaluation proceeds as before and we obtain the same result.

d) $VV$: $J = 1$, $J' = 1$

From eqs. (A.14) and (A.33) we have

$$M_0 = (-1)^{-M-M'} \frac{1}{\sqrt{3}} C (111; M, M', M + M') \times q Y_{1, -M'}(\hat{q})$$

$$N_\nu = \frac{1}{\sqrt{6}} q Y_{1, -M-M'}(\hat{q}) \left\{ (-1)^{-M} \delta_{M_0} + 2(-1)^{-M} C (111; M, -\mu, M + M) \right\}.$$  

(B.22b)
i) $M_0 M'_0$

\[ \frac{1}{4\pi} \int d\Omega Y_{0, -(M+M')}(\hat{q}) Y^*_{0, -(M+M')}(\hat{q}) = \frac{1}{4\pi}, \]

(B.23a)

\[ \sum_{M,M'} C_0(111; M, M', M + M')^2 = \sum_{M,M'} C(111; M, -M' - M, -M')^2 = \sum_{M'} 1 = 3. \]

(B.23b)

Then we get for this term

\[ \sum \sum |t|^2 = \frac{1}{m_{\mu} m_{\nu}} \frac{1}{\pi} (E_{\mu} E_{\nu} + p^2) \hat{p}_t^2. \]

(B.24)

ii) $L^{ij} N_i N_j^*$

We get again in the frame where $p$ is in the $z$ direction

\[ \frac{1}{m_{\mu} m_{\nu}} \left\{ 2p^2 \delta_{i\beta} \delta_{\beta j} + \delta_{ij} (p \cdot p') \right\} N_i N_j^* = \]

\[ \frac{1}{m_{\mu} m_{\nu}} \left\{ 2p^2 N_0 N_0^* + (E_{\mu} E_{\nu} - p^2) \sum_{\mu} N_{\mu} N_{\mu}^* \right\}, \]

(B.25)

\[ N_0 N_0^* - \frac{1}{6} \frac{q^2}{4\pi} \int d\Omega Y_{1, -(M+M')}(\hat{q}) Y^*_{1, -(M+M')}(\hat{q}) \]

\[ \times \left\{ \left( -1 \right)^{-M'} \delta_{M0} + 2 \left( -1 \right)^{-M} C(111; M, 0, M) \right\} \quad \}

\[ \times C(111; M', -M - M', -M) \}

\[ \times \left\{ \left( -1 \right)^{-M'} \delta_{M0} + 2 \left( -1 \right)^{-M} C(111; M, 0, M) \right\} \quad \}

\[ \times C(111; M', -M - M', -M). \]

(B.26)

Now for the $\delta_{M0} \delta_{M0}$ term we have

\[ \sum_{M'} \delta_{M0} \delta_{M0} = 3. \]

(B.27)

For the crossed term in eq. (B.26), $\delta_{M0} C(\cdots) C(\cdots)$, we have

\[ \delta_{M0} C(111; M, 0, M) = C(111; 0, 0, 0) = 0. \]

(B.28)

The last term in eq. (B.26) involves

\[ \sum_{M} C(111; M, 0, M)^2 \]

\[ \times \sum_{M'} C(111; M', -M - M', -M)^2 = 1. \]

(B.29)

Hence, altogether the $N_0 N_0^*$ contribution is

\[ \sum \sum |t|^2 = \frac{7}{6} \frac{1}{\pi} \hat{p}_t^2. \]

(B.30)

Next we must evaluate $\sum_{\mu} N_{\mu} N_{\mu}^*$

\[ \sum_{\mu} N_{\mu} N_{\mu}^* = \frac{1}{6} \frac{q^2}{4\pi} \int d\Omega Y_{1, \mu, -(M+M')}(\hat{q}) \]

\[ \times Y^*_{1, \mu, -(M+M')}(\hat{q}) \}

\[ \times \left\{ \left( -1 \right)^{-M'} \delta_{M\mu} + 2 \left( -1 \right)^{-M} C(111; M, -\mu, M - \mu) \right\} \quad \}

\[ \times C(111; M', -M - M' + \mu, -M + \mu) \}

\[ \times \left\{ \left( -1 \right)^{-M'} \delta_{M\mu} + 2 \left( -1 \right)^{-M} C(111; M, -\mu, M - \mu) \right\} \quad \}

\[ \times C(111; M', -M - M' + \mu, -M + \mu). \]

(B.31)

The first term involves

\[ \sum_{M} \delta_{M\mu} \delta_{M\mu} = \delta_{MM}, \]

(B.32a)

\[ \sum_{M} \delta_{MM} = 9. \]

(B.32b)

The crossed term involves

\[ \sum_{M'} \left( -1 \right)^{M'} C(111; M', -M', 0) = 0, \]

and vanishes, and the product of the second terms in eq. (B.31) gives

\[ \sum_{M,M'} \frac{1}{6} \frac{q^2}{4\pi} \int d\Omega Y_{1, \mu, -(M+M')}(\hat{q}) Y^*_{1, \mu, -(M+M')}(\hat{q}) \]

\[ \times C(111; M, -\mu, M - \mu) \]

\[ \times C(111; M', -M - M' + \mu, -M + \mu)^2. \]

(B.34)

Fixing $M - \mu$ the sum over $M$, and $M'$ of the CGC coefficients gives 1, and then

\[ \sum_{M - \mu} 1 = 3, \]

(B.35)

and we get altogether for $L^{ij} N_i N_j^*$,

\[ \sum \sum \sum \sum |t|^2 = \frac{1}{m_{\mu} m_{\nu}} \frac{7}{2} \hat{p}_t^2 \frac{1}{\pi}. \]

(B.36)

Summing the $N_0 N_0^*$ and $N_{\mu} N_{\mu}^*$ terms we find

\[ \sum \sum \sum \sum |t|^2 = \frac{1}{m_{\mu} m_{\nu}} \frac{1}{2} \hat{p}_t^2 \left\{ \frac{2p^2 7}{6} \left( (E_{\mu} E_{\nu} - p^2) \frac{7}{2} \right) \right\} \]

\[ = \frac{1}{m_{\mu} m_{\nu}} \frac{1}{2} \hat{p}_t^2 \left\{ \frac{2E_{\mu} E_{\nu} - \frac{1}{3} p^2 \frac{2}{3} \right\}. \]

(B.37)

and finally, considering the weights for the $M_0$ and $N_i$ parts we get

\[ \sum \sum \sum \sum |t|^2 = \frac{1}{m_{\mu} m_{\nu}} \frac{1}{2} \hat{p}_t^2 \left\{ \frac{7}{2} \left[ E_{\mu} E_{\nu} + p^2 \right] \right\}. \]

(B.38)
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