A LINEAR, DECOUPLED AND POSITIVITY-PRESERVING
NUMERICAL SCHEME FOR AN EPIDEMIC MODEL WITH
ADVECTION AND DIFFUSION

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Abstract. In this paper, we propose an efficient numerical method for a comprehensive infection model that is formulated by a system of nonlinear coupling advection-diffusion-reaction equations. Using some subtle mixed explicit-implicit treatments, we construct a linearized and decoupled discrete scheme. Moreover, the proposed scheme is capable of preserving the positivity of variables, which is an essential requirement of the model under consideration. The proposed scheme uses the cell-centered finite difference method for the spatial discretization, and thus, it is easy to implement. The diffusion terms are treated implicitly to improve the robustness of the scheme. A semi-implicit upwind approach is proposed to discretize the advection terms, and a distinctive feature of the resulting scheme is to preserve the positivity of variables without any restriction on the spatial mesh size and time step size. We rigorously prove the unique existence of discrete solutions and positivity-preserving property of the proposed scheme without requirements for the mesh size and time step size. It is worthwhile to note that these properties are proved using the discrete variational principles rather than the conventional approaches of matrix analysis. Numerical results are also provided to assess the performance of the proposed scheme.

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1. Introduction. Mathematical models have become an important and effective tool to describe dynamical behaviors of infectious diseases and assess the effectiveness of intervention and control measures [2, 9]. In their pioneering work [11], Kermack and McKendrick proposed a compartmental epidemic model to describe the transmission of communicable diseases. The total population is divided into three classes: susceptible, infected, and recovered (removed), which are labeled by S, I, and R respectively, and thus, the model is called the SIR model. This model is formulated as a system of ordinary differential equations based on the assumptions that the independent variable is the time only and the number of individuals in each compartment is a differentiable function of time. Since individuals often move and migrate in space, an infectious disease that happens first at one location may spread rapidly to the other areas. To describe the epidemic spreading in space accurately, the diffusive SIR models have been attracted more attentions of researchers in recent years [6,15,24,28–31]. Diffusion can produce traveling waves, the existence of which has been proved rigorously in [6,15,24,28–31]. In 2021, Ramaswamy, Oberai and Yortsos [18] proposed a comprehensive temporal-spatial infection model that takes into consideration the influence of advection on the spread of the infectious disease in addition to diffusion.

We assume that the total population density is constant in space and time for simplicity, which is also approximately true in the time frame of infectious diseases spreading. We use \( S(x,t) \), \( I(x,t) \) and \( R(x,t) \) to denote the fractions of susceptible, infected, and recovered individuals at space \( x \) and time \( t \). This paper is concerned with numerical methods for the following advection-diffusion-reaction equations [18]

\[
\begin{align*}
\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S - \nabla \cdot D \nabla S &= -\beta SI, \\
\frac{\partial I}{\partial t} + \mathbf{u} \cdot \nabla I - \nabla \cdot D \nabla I &= \beta SI - \gamma I, \\
\frac{\partial R}{\partial t} + \mathbf{u} \cdot \nabla R - \nabla \cdot D \nabla R &= \gamma I,
\end{align*}
\]

where \( \mathbf{u} \) is the (known) velocity, \( D \) denotes the diffusion rate, \( \beta \) is the transmission coefficient, and \( \gamma \) is the recovery rate.

Different from the classical SIR model and diffusive SIR models, the model (1.1) accounts for human movement through not only diffusion but also advection: diffusion occurs due to random walks, while advection arises from human travel or migration motivated by economical and social activities that are fairly common at present and actually speed up the epidemic spreading in the modern world. The parameter \( \beta \) represents the rate of new infections produced as a result of the interaction between susceptible and infected humans, while \( \gamma \) indicates the rate of recovered population from infected population that usually depends on specific diseases and medical treatment. Based on their meanings, we assume that \( \beta \) and \( \gamma \) are positive real numbers. Since the total population consists of susceptible, infected, and recovered individuals, we always have \( S + I + R = 1 \), and adding the three equations of (1.1) together, we obtain

\[
\frac{\partial (S + I + R)}{\partial t} = -\mathbf{u} \cdot \nabla (S + I + R) + \nabla \cdot D \nabla (S + I + R) = 0,
\]

which means that the property \( S + I + R = 1 \) is invariant with time. The diffusion rate \( D \) is assumed to be the same to three species such that the summation of three diffusion terms vanishes, and otherwise, (1.2) would never hold. In addition, we
assume that $D$ is a positive real variable that may vary in space but independent of time.

The infection model (1.1) has two essential characters: multivariable nonlinear coupling and positivity of unknown variables. As a result, an efficient numerical method is expected to be linear, decoupled and positivity-preserving. A linear and decoupled scheme is easy to implement, and also requires less computational resources than nonlinear coupling schemes. Positivity-preserving schemes not only satisfy the intrinsic requirement of the model, but also can eliminate spurious numerical solutions to dramatically improve the accuracy and stability of the long-time simulation. Actually, the preservation of positivity or boundedness is desirable for numerical methods of numerous scientific and engineering problems, for instance, [3,4,14,20,22].

The nonstandard finite difference (NSFD) method [7,16,17,27] is a popularly used method to construct numerical schemes for differential equations arising from the infection models. It has an advantage that the resulting schemes are able to preserve the positivity, but in order to obtain the explicit expressions of numerical solutions, it usually treats the advection and diffusion terms explicitly [16,17], thereby giving rise to critical constraints on the spatial mesh size and time step size to guarantee the positivity. Numerical methods for the epidemic models with advection and diffusion are even more scarce since such model are newly-developed and more complicated compared to the common models formulated by the ordinary differential equations. In [27], a NSFD scheme was developed for a viral infection model with a diffusion equation in one-dimensional space, and it deals with the diffusion term implicitly, but the advection is never considered in the model.

To our best knowledge, there is no numerical scheme developed for the comprehensive advection-diffusion infection model given in (1.1). In this paper, we for the first time develop an efficient, linear, decoupled and positivity-preserving numerical scheme for solving the model (1.1). The proposed scheme employs the cell-centered finite difference (CCFD) method [21] as the spatial discretization method, which can be equivalent to a special mixed finite element method with quadrature rules [1]. Due to its appealing property of local conservation and ease of use, the CCFD method has been widely applied to the discretization of various problems, for instance [5,8,10,12–14,19,22,23,25,26].

In the proposed scheme, motivated by the NSFD method, we use some subtle mixed explicit-implicit treatments for the variables $S$, $I$ and $R$ in their respective equations to construct a linearized and decoupled discrete system. For the advection terms, we propose a semi-implicit upwind treatment, which can remove the well-known CFL restriction to guarantee the positivity. The diffusion terms are discretized by the fully implicit approach, and thus, the positivity condition for diffusion can be unconditionally satisfied. We rigorously prove the unique existence of discrete solutions and positivity-preserving property of the proposed scheme without restrictions on the spatial mesh size and time step size. It is worthwhile to note that our proofs are carried out using the discrete variational principles rather than the conventional approaches of matrix analysis.

The rest of this paper is organized as follows. In Section 2, we propose a linear, decoupled numerical scheme for solving the model (1.1) based on the discrete difference operators. In Section 3, we prove the unique existence of discrete solutions and positivity-preserving property using the discrete variational principles. In Section
4. numerical results are provided to validate the proposed scheme. Finally, we give some concluding remarks in Section 5.

2. Discrete scheme. We consider a two-dimensional spatial domain $\Omega = [a_1, b_1] \times [a_2, b_2]$, where $a_1 < b_1$ and $a_2 < b_2$. Without loss of generality, we use a uniform division of the domain $\Omega$ and denote the mesh size by $h = \frac{a_1-a_2}{N-1} = \frac{b_2-a_2}{M-1}$, where $N$ and $M$ are positive integers. The grid-cell vertexes are denoted by $x_i = a_1 + i h$ and $y_j = a_2 + j h$, where $0 \leq i \leq N$ and $0 \leq j \leq M$, and furthermore, the cell centers are denoted by $(x_i + \frac{1}{2} h, y_j + \frac{1}{2} h)$, where $0 \leq i \leq N-1$ and $0 \leq j \leq M-1$.

We introduce the discrete operators and discrete variational principles [5, 8, 10, 12–14, 21–23, 25, 26], which are based on the cell-centered finite difference (CCFD) method [1, 21]. We use the discrete variable $\phi_{i+\frac{1}{2}, j+\frac{1}{2}}$ to approximate $\phi(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$, and furthermore, introduce the discrete function space as

$$C_h = \{ \phi : (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \mapsto \mathbb{R}, \ 0 \leq i \leq N-1, \ 0 \leq j \leq M-1 \}.$$  

For $\phi \in C_h$, the discrete gradient operator, denoted by $\nabla_h = [\nabla_{h,x}, \nabla_{h,y}]^T$, on the interfaces of interior cells is defined as follows

$$\nabla_{h,x} \phi_{i,j+\frac{1}{2}} = \frac{\phi_{i+\frac{1}{2}, j+\frac{1}{2}} - \phi_{i-\frac{1}{2}, j+\frac{1}{2}}}{h}, \quad (2.1a)$$

$$\nabla_{h,y} \phi_{i+\frac{1}{2}, j} = \frac{\phi_{i+\frac{1}{2}, j+\frac{1}{2}} - \phi_{i+\frac{1}{2}, j-\frac{1}{2}}}{h}. \quad (2.1b)$$

We consider the homogeneous Neumann boundary condition for diffusion and express the relevant discrete boundary conditions as

$$\nabla_{h,x} \phi_{0,j+\frac{1}{2}} = \nabla_{h,x} \phi_{N,j+\frac{1}{2}} = 0, \quad (2.2)$$

$$\nabla_{h,y} \phi_{i+\frac{1}{2},0} = \nabla_{h,y} \phi_{i+\frac{1}{2},M} = 0. \quad (2.3)$$

For $\phi \in C_h$, the discrete diffusion operator is defined as

$$\nabla_h \cdot D \nabla_h \phi_{i+\frac{1}{2}, j+\frac{1}{2}} = \frac{1}{h} \left( D_{i+1,j+\frac{1}{2}} \nabla_{h,x} \phi_{i+1,j+\frac{1}{2}} - D_{i,j+\frac{1}{2}} \nabla_{h,x} \phi_{i,j+\frac{1}{2}} \right)$$

$$+ \frac{1}{h} \left( D_{i+\frac{1}{2},j+1} \nabla_{h,y} \phi_{i+\frac{1}{2},j+1} - D_{i+\frac{1}{2},j} \nabla_{h,y} \phi_{i+\frac{1}{2},j} \right), \quad (2.4)$$

where the diffusion rate $D$ on a cell edge generally takes the harmonic mean of the data of two cells that share this edge. For $\phi, \varphi \in C_h$, we define the following discrete inner-products:

$$(\phi, \varphi)_h = h^2 \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \phi_{i,j} \varphi_{i,j},$$

$$(\nabla_h \phi, \nabla_h \varphi)_h = h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} \nabla_{h,x} \phi_{i,j} \nabla_{h,x} \varphi_{i,j}$$

$$+ h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{M-1} \nabla_{h,y} \phi_{i,j} \nabla_{h,y} \varphi_{i,j}.$$  

Correspondingly, we define the discrete norms for $\phi \in C_h$ as

$$\|\phi\|_h = (\phi, \phi)^{1/2}_h, \quad \|\nabla_h \phi\|_h = (\nabla_h \phi, \nabla_h \phi)^{1/2}_h.$$
By direct calculations, we can obtain the following discrete variational formula [5, 13, 14, 22, 23, 25, 26]

$$-(\nabla_h \cdot D \nabla_h \phi, \phi)_h = (D \nabla_h \phi, \nabla_h \phi)_h, \quad \phi, \phi \in \mathcal{C}_h. \quad (2.5)$$

The total time interval $[0, T]$ is uniformly divided into $K$ time steps as $0 = t_0 < t_1 < \cdots < t_K = T$, and the time step size is denoted by $\tau = t_{i+1} - t_i$. We use the the superscript to denote the variable $\phi$ at the time $t_n$ by $\phi^n$. We are now ready to introduce the discrete scheme for the model (1.1). We first propose a semi-implicit upwind approach to discretize the advection terms. Let $u^{n+1} = [u^{n+1}, v^{n+1}]^T$, then for $\phi = S, I, R$, we define

$$u^{n+1} \cdot \nabla_h \phi^{n+\frac{1}{2}} = u^{n+1} \nabla_h \phi^{n+\frac{1}{2}} + u^{n+1} \nabla_h \phi^{n+\frac{1}{2}}, \quad (2.6)$$

where the right-hand side terms have the component forms as follows

$$(u^{n+1} \nabla_h \phi^{n+\frac{1}{2}})_{i+\frac{1}{2}, j+\frac{1}{2}} = \begin{cases} u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \frac{\phi_i^{n+1} - \phi_{i+\frac{1}{2}, j+\frac{1}{2}}}{h}, & \quad u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \geq 0, \\ u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \frac{\phi_i^{n+1} + \phi_{i+\frac{1}{2}, j+\frac{1}{2}}}{h}, & \quad u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} < 0, \end{cases} \quad (2.7a)$$

$$(v^{n+1} \nabla_h \phi^{n+\frac{1}{2}})_{i+\frac{1}{2}, j+\frac{1}{2}} = \begin{cases} v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \frac{\phi_i^{n+1} - \phi_{i+\frac{1}{2}, j+\frac{1}{2}}}{h}, & \quad v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \geq 0, \\ v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \frac{\phi_i^{n+1} + \phi_{i+\frac{1}{2}, j+\frac{1}{2}}}{h}, & \quad v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} < 0. \end{cases} \quad (2.7b)$$

In (2.7), we use the approach of boundary extension to handle the boundary conditions; more precisely, the values of $\phi$ beyond the actual boundaries, i.e., $\phi_{-\frac{1}{2}, j+\frac{1}{2}}$, $\phi_{N+\frac{1}{2}, j+\frac{1}{2}}$, $\phi_{i+\frac{1}{2}, -\frac{1}{2}}$ and $\phi_{i+\frac{1}{2}, M+\frac{1}{2}}$ are provided through instantaneous boundary conditions.

The semi-implicit fully discrete scheme for the model (1.1) is stated as: given $S^n, I^n, R^n \in \mathcal{C}_h$, find $S^{n+1}, I^{n+1}, R^{n+1} \in \mathcal{C}_h$ such that

$$\frac{S^{n+1} - S^n}{\tau} + u^{n+1} \cdot \nabla_h S^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h S^{n+1} = -\beta I^n S^{n+1}, \quad (2.8a)$$

$$\frac{I^{n+1} - I^n}{\tau} + u^{n+1} \cdot \nabla_h I^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h I^{n+1} = \beta I^n S^{n+1} - \gamma I^{n+1}, \quad (2.8b)$$

$$\frac{R^{n+1} - R^n}{\tau} + u^{n+1} \cdot \nabla_h R^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h R^{n+1} = \gamma I^{n+1}, \quad (2.8c)$$

where $\tau$ is the time step size and $u^{n+1}$ is the velocity at the time $t_{n+1}$ that is provided but not an unknown.

We now elaborate upon the ideas for construction of the discrete scheme (2.8). The continuous model (1.1) has two prominent properties: one is multivariable nonlinear coupling, and the other is that adding three reaction terms together yields zero. A numerical scheme is highly preferable if it leads to a linearized and decoupled discrete system and ensures that the summation of three reaction terms remains zero. For the term $\beta IS$ in (1.1a), $I$ needs to be treated explicitly to decouple the relation between (1.1a) and (1.1b), while an implicit treatment should be applied to $S$ for guaranteeing the stability and positivity. Moreover, $\beta IS$ in (1.1b) should be discretized using the same treatments as that in (1.1a) to ensure that the summation of discrete reaction terms vanishes. The implicit treatment of $\gamma I$ can guarantee the stability and positivity. All diffusion terms are implicitly discretized to avoid any restriction on mesh and time step sizes. With the above subtle semi-implicit treatments, the discrete scheme (2.8) decouples the nonlinear and coupled relations.
between three unknown variables at the continuous level, and as a result, we can compute $S^{n+1}$, $I^{n+1}$ and $R^{n+1}$ by solving (2.8a)-(2.8c) sequently in their orders. We will prove that the proposed scheme can guarantee not only the positivity of three variables but also the summation constraint $S^{n+1} + I^{n+1} + R^{n+1} = 1$. Since $R$ is never involved in the equations (2.8a) and (2.8b), we have the other way to solve the discrete system (2.8): first solve (2.8a) and (2.8b) to obtain $S^{n+1}$ and $I^{n+1}$, and then update $R^{n+1}$ by $R^{n+1} = 1 - S^{n+1} - I^{n+1}$.

3. Theoretical analysis. In this section, we prove the unique existence of the discrete solutions and the positivity-preserving property using the discrete variational principles.

3.1. Unique existence of the discrete solutions. We first prove that the discrete scheme (2.8) admits a unique solution pair $(S^{n+1}, I^{n+1}, R^{n+1})$.

**Theorem 3.1.** Suppose that $I^n \geq 0$. For any given $\tau > 0$, $S^{n+1}$, $I^{n+1}$ and $R^{n+1}$ exist uniquely to solve (2.8) in $C_h$.

**Proof.** It suffices to prove the uniqueness of discrete solutions in $C_h$. Suppose that there exist another solutions $\tilde{S}^{n+1}, \tilde{I}^{n+1}, \tilde{R}^{n+1}$ in $C_h$ such that

$$\frac{\tilde{S}^{n+1} - S^n}{\tau} + u^{n+1} \cdot \nabla_h \tilde{S}^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h \tilde{S}^{n+1} = -\beta I^n \tilde{S}^{n+1}, \quad (3.1a)$$

$$\frac{\tilde{I}^{n+1} - I^n}{\tau} + u^{n+1} \cdot \nabla_h \tilde{I}^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h \tilde{I}^{n+1} = \beta I^n \tilde{S}^{n+1} - \gamma \tilde{I}^{n+1}, \quad (3.1b)$$

$$\frac{\tilde{R}^{n+1} - R^n}{\tau} + u^{n+1} \cdot \nabla_h \tilde{R}^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h \tilde{R}^{n+1} = \gamma \tilde{I}^{n+1}. \quad (3.1c)$$

The advection terms appearing in (3.1) have the form

$$u^{n+1} \cdot \nabla_h \tilde{S}^{n+\frac{1}{2}} = u^{n+1} \nabla_{h,x} \tilde{S}^{n+\frac{1}{2}} + u^{n+1} \nabla_{h,y} \tilde{S}^{n+\frac{1}{2}}, \quad (3.2)$$

where $\tilde{S} = \tilde{S}, \tilde{I}, \tilde{R}$, and the right-hand side terms are expressed as

$$\left(u^{n+1} \nabla_{h,x} \tilde{S}^{n+\frac{1}{2}}\right)_{i+\frac{1}{2},j+\frac{1}{2}} = \begin{cases} u^{n+1}_{i+1,j+\frac{1}{2}} - \frac{\tilde{S}^{n+\frac{1}{2}}}{h} - \frac{\phi^{n+\frac{1}{2}}}{h}, & u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} \geq 0, \\ u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{\tilde{S}^{n+\frac{1}{2}}}{h} - \frac{\phi^{n+\frac{1}{2}}}{h}, & u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} < 0, \end{cases} \quad (3.3a)$$

$$\left(u^{n+1} \nabla_{h,y} \tilde{S}^{n+\frac{1}{2}}\right)_{i+\frac{1}{2},j+\frac{1}{2}} = \begin{cases} u^{n+1}_{i+\frac{1}{2},j+1} - \frac{\tilde{S}^{n+\frac{1}{2}}}{h} - \frac{\phi^{n+\frac{1}{2}}}{h}, & u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} \geq 0, \\ u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{\tilde{S}^{n+\frac{1}{2}}}{h} - \frac{\phi^{n+\frac{1}{2}}}{h}, & u^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} < 0, \end{cases} \quad (3.3b)$$

Let $E_S^{n+1} = S^{n+1} - \tilde{S}^{n+1}, E_I^{n+1} = I^{n+1} - \tilde{I}^{n+1}$ and $E_R^{n+1} = R^{n+1} - \tilde{R}^{n+1}$. Subtracting (3.1) from (2.8) yields

$$\frac{1}{\tau} E_S^{n+1} + u^{n+1} \cdot \nabla_h E_S^{n+1} - \nabla_h \cdot D \nabla_h E_S^{n+1} = -\beta I^n E_S^{n+1}, \quad (3.4a)$$

$$\frac{1}{\tau} E_I^{n+1} + u^{n+1} \cdot \nabla_h E_I^{n+1} - \nabla_h \cdot D \nabla_h E_I^{n+1} = \beta I^n E_S^{n+1} - \gamma E_I^{n+1}, \quad (3.4b)$$

$$\frac{1}{\tau} E_R^{n+1} + u^{n+1} \cdot \nabla_h E_R^{n+1} - \nabla_h \cdot D \nabla_h E_R^{n+1} = \gamma E_I^{n+1}, \quad (3.4c)$$
where the advection terms are obtained subtracting (3.3) from (2.7) and have the following forms

\[
\begin{align*}
\mathbf{u}^{n+1} \cdot \nabla_h E_S^{n+1} &= \frac{1}{h} (|u^{n+1}| + |v^{n+1}|) E_S^{n+1}, \quad (3.5a) \\
\mathbf{u}^{n+1} \cdot \nabla_h E_I^{n+1} &= \frac{1}{h} (|u^{n+1}| + |v^{n+1}|) E_I^{n+1}, \quad (3.5b) \\
\mathbf{u}^{n+1} \cdot \nabla_h E_R^{n+1} &= \frac{1}{h} (|u^{n+1}| + |v^{n+1}|) E_R^{n+1}. \quad (3.5c)
\end{align*}
\]

In the derivations of (3.5), all boundary terms vanish since they exactly match boundary conditions. Taking the discrete inner product of (3.4) with $E_S^{n+1}$, $E_I^{n+1}$ and $E_R^{n+1}$ respectively, and then applying the discrete variational principles, we deduce that

\[
\begin{align*}
\frac{1}{\tau} \|E_S^{n+1}\|_h^2 + (\mathbf{u}^{n+1} \cdot \nabla_h E_S^{n+1}, E_S^{n+1})_h &= - (\beta I^n E_S^{n+1}, E_S^{n+1})_h, \quad (3.6a) \\
\frac{1}{\tau} \|E_I^{n+1}\|_h^2 + (\mathbf{u}^{n+1} \cdot \nabla_h E_I^{n+1}, E_I^{n+1})_h &= - (\gamma E_I^{n+1}, E_I^{n+1})_h, \quad (3.6b) \\
\frac{1}{\tau} \|E_R^{n+1}\|_h^2 + (\mathbf{u}^{n+1} \cdot \nabla_h E_R^{n+1}, E_R^{n+1})_h &= - (\gamma E_I^{n+1}, E_R^{n+1})_h. \quad (3.6c)
\end{align*}
\]

We first consider (3.6a) to estimate $E_S^{n+1}$. Using (3.5a), we have

\[
(\mathbf{u}^{n+1} \cdot \nabla_h E_S^{n+1}, E_S^{n+1})_h = \frac{1}{h} (|u^{n+1}| + |v^{n+1}|) E_S^{n+1} \geq 0. \quad (3.7)
\]

Thanks to $\beta > 0$ and $I^n \geq 0$, the right-hand side of (3.6a) is estimated as

\[
- (\beta I^n E_S^{n+1}, E_S^{n+1})_h \leq 0. \quad (3.8)
\]

Thus, from (3.7) and (3.8), we estimate (3.6a) as

\[
\frac{1}{\tau} \|E_S^{n+1}\|_h^2 + \|D^{1/2} \nabla_h E_S^{n+1}\|_h^2 \leq 0, \quad (3.9)
\]

which implies that $E_S^{n+1} \equiv 0$.

We next consider (3.6b). The two terms on the right-hand side of (3.6b) can be estimated taking into account $E_S^{n+1} \equiv 0$ and $\gamma > 0$ as

\[
(\beta I^n E_S^{n+1}, E_I^{n+1})_h = 0, \quad (3.10)
\]

\[
- (\gamma E_I^{n+1}, E_I^{n+1})_h \leq 0. \quad (3.11)
\]

The second term on the left-hand side of (3.6b) is non-negative similarly to (3.7), and thus, (3.6b) can be simplified as

\[
\frac{1}{\tau} \|E_I^{n+1}\|_h^2 + \|D^{1/2} \nabla_h E_I^{n+1}\|_h^2 \leq 0, \quad (3.12)
\]

which yields $E_I^{n+1} \equiv 0$. Using the similar routines, we can obtain $E_R^{n+1} \equiv 0$. Consequently, $S^{n+1}$, $I^{n+1}$ and $R^{n+1}$ exist uniquely in $C_h$. $\Box$
3.2. Positivity-preserving property. We now prove that the discrete scheme (2.8) preserves the positivity and the summation constraint $S + I + R = 1$. The proofs will be carried out using a variational approach, which has been developed to prove the discrete maximum principles in [14, 22, 23]. We define an auxiliary variable $\vec{\phi} = \min(\phi, 0)$ for $\phi \in C_h$.

**Lemma 3.2.** Assume that $D > 0$. For $\phi \in C_h$, we have

$$\vec{\phi} \leq 0, \quad \phi \vec{\phi} = \vec{\phi}^2,$$

(3.13)

$$- (\nabla_h \cdot D \nabla_h \phi, \vec{\phi})_h \geq \|D^{1/2} \nabla_h \vec{\phi}\|_h^2.$$  

(3.14)

**Proof.** The property (3.13) of $\vec{\phi}$ is apparent from its definition. The inequality (3.14) can be proved similarly to the proof of Lemma 5.1 in [14].

**Lemma 3.3.** Assume that $\phi^n \geq 0$ and boundary values are nonnegative. For $\phi^{n+1} \in C_h$, the semi-implicit upwind discrete approach given in (2.6)-(2.7) has a nonnegative lower bound as

$$(u^{n+1} \cdot \nabla_h \phi^{n+\frac{1}{2}}, \vec{\phi}^{n+1})_h \geq \frac{1}{h} \left( (|u^{n+1}| + |v^{n+1}|) \vec{\phi}^{n+1}, \vec{\phi}^{n+1} \right)_h \geq 0,$$

(3.15)

where $\vec{\phi}^{n+1} = \min(\phi^{n+1}, 0)$ and $u^{n+1} = [u^{n+1}, v^{n+1}]^T$.

**Proof.** First, we note that

$$(u^{n+1} \cdot \nabla_h \phi^{n+\frac{1}{2}}, \vec{\phi}^{n+1})_h = \left( u^{n+1} \nabla_h \phi^{n+\frac{1}{2}}, \vec{\phi}^{n+1} \right)_h + \left( v^{n+1} \nabla_h \phi^{n+\frac{1}{2}}, \vec{\phi}^{n+1} \right)_h.$$

(3.16)

We first estimate the first term on the right-hand side of (3.16). For the case $u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \geq 0$, taking into account $\phi^n \geq 0$, we deduce that

$$\vec{\phi}^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} = \frac{1}{h} \left( \phi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} - \phi^{n+1}_{i-\frac{1}{2}, j+\frac{1}{2}} \right) \vec{\phi}^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}}$$

$$\geq \frac{1}{h} |u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}}| |\vec{\phi}^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}}|^2,$$

(3.17)

where we have used the property

$$\vec{\phi}^{n+1}_{i-\frac{1}{2}, j+\frac{1}{2}} \leq 0.$$  

(3.18)
Using (3.13) and (3.16), the first term on the left-hand side of (3.24) is estimated as
\[
\frac{1}{\tau} \left[ u_{i,j}^{n+1} \cdot \nabla_h S_{i,j}^{n+\frac{1}{2}}, \mathbf{S}_n^{n+1} \right]_h \geq \frac{1}{\tau} \left[ u_{i,j}^{n+1} \cdot \nabla_h S_{i,j}^{n+\frac{1}{2}}, \mathbf{S}_n \right]_h + \frac{1}{\tau} \left[ u_{i,j}^{n+1} \cdot \nabla_h S_{i,j}^{n+\frac{1}{2}}, \mathbf{S}_n^{n+1} \right]_h^2. \tag{3.19}
\]
For the boundary cells, we can also obtain (3.17) and (3.19) since nonnegative boundary values have been assigned. The second term of the right-hand side of (3.16) can be estimated using the similar routines.

\[\square\]

**Theorem 3.4.** Assume that
\[
S^n \geq 0, \quad I^n \geq 0, \quad R^n \geq 0, \tag{3.20}
\]
\[
S^n + I^n + R^n = 1. \tag{3.21}
\]
Moreover, the boundary values always satisfy (3.20) and (3.21). The discrete solutions generated by the scheme (2.8) satisfy
\[
S^{n+1} \geq 0, \quad I^{n+1} \geq 0, \quad R^{n+1} \geq 0, \tag{3.22}
\]
\[
S^{n+1} + I^{n+1} + R^{n+1} = 1. \tag{3.23}
\]

**Proof.** Let \( \mathbf{S}^{n+1} = \min(S^{n+1}, 0), \) \( \mathbf{T}^{n+1} = \min(I^{n+1}, 0) \) and \( \mathbf{R}^{n+1} = \min(R^{n+1}, 0) \) respectively. Taking the discrete inner products of three equations in (2.8) with \( \mathbf{S}^{n+1}, \mathbf{T}^{n+1} \) and \( \mathbf{R}^{n+1} \) respectively, we obtain
\[
\frac{1}{\tau} (S^{n+1} - S^n, \mathbf{S}^{n+1})_h + \left( u^{n+1} \cdot \nabla_h S^{n+\frac{1}{2}}, \mathbf{S}^{n+1} \right)_h - \left( \nabla_h \cdot D \nabla_h S^{n+1}, \mathbf{S}^{n+1} \right)_h
= - \left( \beta I^n S^{n+1}, \mathbf{S}^{n+1} \right)_h, \tag{3.24}
\]
\[
\frac{1}{\tau} (I^{n+1} - I^n, \mathbf{T}^{n+1})_h + \left( u^{n+1} \cdot \nabla_h I^{n+\frac{1}{2}}, \mathbf{T}^{n+1} \right)_h - \left( \nabla_h \cdot D \nabla_h I^{n+1}, \mathbf{T}^{n+1} \right)_h
= \left( \beta I^n S^{n+1}, \mathbf{T}^{n+1} \right)_h - \left( \gamma I^{n+1}, \mathbf{T}^{n+1} \right)_h, \tag{3.25}
\]
\[
\frac{1}{\tau} (R^{n+1} - R^n, \mathbf{R}^{n+1})_h + \left( u^{n+1} \cdot \nabla_h R^{n+\frac{1}{2}}, \mathbf{R}^{n+1} \right)_h - \left( \nabla_h \cdot D \nabla_h R^{n+1}, \mathbf{R}^{n+1} \right)_h
= \left( \gamma I^{n+1}, \mathbf{R}^{n+1} \right)_h. \tag{3.26}
\]

We first consider (3.24) to prove \( S^{n+1} \geq 0. \) Thanks to \( S^n \geq 0, \) it is obtained using (3.13) that
\[
(S^{n+1}, \mathbf{S}^{n+1})_h = \| \mathbf{S}^{n+1} \|_h^2, \quad (S^n, \mathbf{S}^{n+1})_h \leq 0. \tag{3.27}
\]
Thus, the first term on the left-hand side of (3.24) is estimated as
\[
\frac{1}{\tau} (S^{n+1} - S^n, \mathbf{S}^{n+1})_h \geq \frac{1}{\tau} \| \mathbf{S}^{n+1} \|_h^2. \tag{3.28}
\]
The second term on the left-hand side of (3.24) is estimated applying Lemma 3.3
\[
\left( u^{n+1} \cdot \nabla_h S^{n+\frac{1}{2}}, \mathbf{S}^{n+1} \right)_h \geq 0. \tag{3.29}
\]
Using the discrete variational principle (3.14), we estimate the third term on the left-hand side of (3.24) as
\[
- \left( \nabla_h \cdot D \nabla_h S^{n+1}, \mathbf{S}^{n+1} \right)_h \geq \| D^{1/2} \nabla_h \mathbf{S}^{n+1} \|_h^2. \tag{3.30}
\]
Using (3.13) and \( I^n \geq 0, \) we estimate the right-hand side of (3.24) as
\[
- \left( \beta I^n S^{n+1}, \mathbf{S}^{n+1} \right)_h = - \left( \beta I^n \mathbf{S}^{n+1}, \mathbf{S}^{n+1} \right)_h \leq 0. \tag{3.31}
\]
Combining (3.28)-(3.31) yields
\[ \frac{1}{\tau} \| S^{n+1} \|_{h}^2 + \| D^{1/2} \nabla_h S^{n+1} \|_{h}^2 \leq 0, \] (3.32)
which implies that \( S^{n+1} \equiv 0 \), and consequently, \( S^{n+1} \geq 0 \).

We next prove \( I^{n+1} \geq 0 \) by considering (3.25). Since \( S^{n+1} \geq 0 \) has been already proved, we obtain
\[ \left( \beta I^n S^{n+1}, T^{n+1} \right)_h \leq 0. \] (3.33)
For the second term on the right-hand side of (3.25), we deduce that
\[ - \left( \gamma I^n, T^{n+1} \right)_h = - \gamma \| T^{n+1} \|_{h}^2 \leq 0. \] (3.34)
The terms on the left-hand side of (3.25) can be estimated by the similar routines used in the proof of \( S^{n+1} \geq 0 \), and the following inequality can be reached
\[ \frac{1}{\tau} \| T^{n+1} \|_{h}^2 + \| D^{1/2} \nabla_h T^{n+1} \|_{h}^2 \leq 0. \] (3.35)
Therefore, we deduce that \( I^{n+1} \geq 0 \). Using the similar routines, it can be proved that \( R^{n+1} \geq 0 \).

We now turn to prove that \( S^{n+1} + I^{n+1} + R^{n+1} = 1 \). Let us introduce the notation \( \xi^n = S^n + I^n + R^n \). Since \( S^n + I^n + R^n = 1 \) and the summation of the right-hand sides of (2.8a)-(2.8c) is zero, we add the three equations of the discrete scheme (2.8) together and obtain
\[ \frac{\xi^{n+1} - 1}{\tau} + u^{n+1} \cdot \nabla_h \xi^{n+\frac{1}{2}} - \nabla_h \cdot D \nabla_h \xi^{n+1} = 0. \] (3.36)
Since \( \xi^n = S^n + I^n + R^n = 1 \), the second term of (3.36) can be expressed as
\[ u^{n+1} \cdot \nabla_h \xi^{n+\frac{1}{2}} = u^{n+1} \nabla_{h,x} \xi^{n+\frac{1}{2}} + v^{n+1} \nabla_{h,y} \xi^{n+\frac{1}{2}}, \] (3.37)
which has the following component forms
\[ (u^{n+1} \nabla_{h,x} \xi^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}} = \begin{cases} \frac{1}{h} u^{n+1}_{i+1, j+1} (\xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} - 1), & u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \geq 0, \\ \frac{1}{h} u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} (1 - \xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}}), & u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} < 0, \end{cases} \] (3.38)
\[ (v^{n+1} \nabla_{h,y} \xi^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}} = \begin{cases} \frac{1}{h} v^{n+1}_{i+\frac{1}{2}, j+1} (\xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} - 1), & v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} \geq 0, \\ \frac{1}{h} v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} (1 - \xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}}), & v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} < 0. \end{cases} \] (3.39)
It is apparent that (3.38) and (3.39) hold for boundary cells since the summation constraint holds exactly for boundary values. We further simplify (3.38) and (3.39) as
\[ (u^{n+1} \nabla_{h,x} \xi^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{h} u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} (\xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} - 1), \] (3.40)
\[ (v^{n+1} \nabla_{h,y} \xi^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{h} v^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} (\xi^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}} - 1). \] (3.41)
Taking the discrete inner product of (3.36) with \( \xi^{n+1} \) yields
\[ \frac{1}{\tau} \| \xi^{n+1} - 1 \|_{h}^2 + \left( u^{n+1} \cdot \nabla_h \xi^{n+\frac{1}{2}}, \xi^{n+1} - 1 \right)_h \\
- \left( \nabla_h \cdot D \nabla_h \xi^{n+1}, \xi^{n+1} - 1 \right)_h = 0. \] (3.42)
It is deduced from (3.37), (3.40) and (3.41) that

\[
(u^{n+1} \cdot \nabla_h s^{n+1}, \xi^{n+1} - 1)_h = \frac{1}{h} ((|u^{n+1}| + |v^{n+1}|)(\xi^{n+1} - 1), \xi^{n+1} - 1)_h \geq 0.
\]  

(3.43)

The discrete variational principle gives

\[
- (\nabla_h \cdot D \nabla_h \xi^{n+1}, \xi^{n+1} - 1) = \|D^{1/2} \nabla_h (\xi^{n+1} - 1)\|_h^2.
\]  

(3.44)

Using (3.43) and (3.44), we can deduce from (3.42) that

\[
\frac{1}{\tau} \|\xi^{n+1} - 1\|_h^2 + \|D^{1/2} \nabla_h (\xi^{n+1} - 1)\|_h^2 \leq 0.
\]  

(3.45)

Therefore, we deduce that \(\xi^{n+1} \equiv 1\), i.e., \(S^{n+1} + I^{n+1} + R^{n+1} \equiv 1\).

By induction, Theorems 3.1 and 3.4 lead to the following theorem.

**Theorem 3.5.** Assume that the initial conditions for \(S, I\) and \(R\) are provided to satisfy

\[
S^0 \geq 0, \ I^0 \geq 0, \ R^0 \geq 0,
\]  

(3.46)

\[
S^0 + I^0 + R^0 = 1.
\]  

(3.47)

The boundary values are also assumed to be nonnegative and satisfy the summation constraint at any time. Then for any \(n \geq 1\), the solutions of the discrete scheme (2.8) exist uniquely and satisfy

\[
S^n \geq 0, \ I^n \geq 0, \ R^n \geq 0,
\]  

(3.48)

\[
S^n + I^n + R^n = 1.
\]  

(3.49)

It is worthwhile to note that the proposed scheme has a distinct and appealing feature that it is able to preserve the positivity without any condition on mesh and time step sizes against the classical schemes that suffer from the Courant–Friedrichs–Lewy (CFL) condition.

4. **Numerical results.** In this section, the proposed scheme is employed for numerical simulations of the epidemic spreading problems to validate its capability and effectiveness. In all numerical tests, the spatial domain is \(\Omega = [0, 100]^2\), which is divided by a uniform mesh with 100 \(\times\) 100 elements. For parameters, we take the diffusion coefficient \(D = 0.11\), the transmission coefficient \(\beta = 0.7\) and the recovery rate \(\gamma = 0.06\). In numerical simulations presented here, we solve (2.8a)-(2.8c) to get \(S^{n+1}, I^{n+1}\) and \(R^{n+1}\), and do not use the summation constraint to update \(R^{n+1}\) from \(S^{n+1}\) and \(I^{n+1}\). This enables us to verify whether the summation constraint can be preserved numerically.

4.1. **Example 1.** In this example, we consider a diffusive SIR problem in the absence of advection. The time step size is taken as \(\tau = 1\). The initial conditions for the susceptible (\(S\)) and infected (\(I\)) fractions in space are illustrated in Figure 1. Initially, there is only a tiny infected fraction \((7.5e - 6)\) in the central area of the domain, while the rest of the domain is free of infections and the recovered fraction is zero in the entire domain. In Figures 2, 3 and 4, we illustrate the spatial distributions of susceptible, infected and recovered fractions at different times computed by the proposed scheme.

Figure 5 depicts the temporal evolution of susceptible, infected and recovered fractions locating at the center of the domain and showcases a typical epidemic behavior. At the initial time, only a tiny population at this location is infected,
while the remainder is susceptible. The infection fraction is rising rapidly in the initial epidemic spread period. A maximum point is reached until a certain quantity of the population is infected or recovered, and subsequently the infection fraction begins to decline.

The traveling wave solutions to the diffusive epidemic models have been theoretically analyzed in [6, 15, 24, 28–31]. In Figure 6, we illustrate the traveling wave profiles of susceptible, infected and recovered fractions along the central horizontal line of the domain at different times. The disease propagation is advanced from the infected area to the neighbor areas with the susceptible, but initially disease-free population due to the influence of diffusion.

We compare numerical results with the real data about the COVID-19 infections of USA from March 1 2020 to June 9 2020, which can be found at the website covid.cdc.gov. In Figure 7, the presented data is normalized through dividing the original data by $10^7$ to match the format of numerical solutions, while numerical results are the spatially-averaged infected fractions. Figure 7 demonstrates that numerical solutions quite agree with the real data on the growing tendency of infections.

In Figure 8, we show that the proposed scheme is capable of guaranteeing the positivity of three variables, i.e., susceptible ($S$), infected ($I$), and recovered ($R$) fractions. The summation constraint is quantified by $\|S + I + R - 1\|_{\infty}$ that remains the errors on the order of $O(10^{-14})$ due to the roundoff errors. In general, numerical results are in full agreement with theoretical analysis.
Figure 3: Infected fraction profiles at different times in Example 1.

Figure 4: Recovered fraction profiles at different times in Example 1.

Figure 5: Temporal evolution of susceptible, infected and recovered fractions located at the center of the domain.

Figure 6: Traveling waves of (a) susceptible, (b) infected and (c) recovered fractions along the central horizontal line of the domain at different times.
4.2. Example 2. In this example, we consider the advection-diffusion epidemic spreading problem to explore the performance of the proposed scheme. The velocity field is taken as \( \mathbf{u} = [1, 1]^T \), which is constant in space and time. The boundary conditions for advection are given as \( S = 1 \) and \( I = R = 0 \) on all inflow boundaries.

As shown in Figure 9, at the initial time, a tiny infected fraction is placed in the southwest region of the domain, the infection does not take place in the rest of the domain and the recovered fraction is also zero in the entire domain. Figures 10, 11 and 12 depict the temporal-spatial distributions of susceptible, infected and recovered fractions computed by the proposed scheme with the time step size \( \tau = 0.8 \).

In the presence of velocity, it differs from the case of diffusion only under consideration that the epidemic epicenter is moving from the initial infected area to the rest areas along the population flow directions. Meanwhile, the epidemic is spreading towards the neighbor regions in the effect of diffusion, and the traveling waves of susceptible, infected and recovered fractions can also be observed from Figures 10, 11 and 12.

The velocity actually produces more complicated epidemic spreading process, thereby giving rise to a severe test for numerical schemes. We now consider the effect of the Courant–Friedrichs–Lewy (CFL) number, which is defined as

\[
\mathcal{C} = \frac{(|u| + |v|)\tau}{h}.
\]

Taking into account \( \mathbf{u} = [1, 1]^T \) and \( h = 1 \) in this example, we have

\[
\mathcal{C} = 2\tau.
\]
The CFL condition usually requires that \( C \leq 1 \) for stability and positivity preservation of the classical explicit methods. For the proposed scheme, we have tested two time step sizes, i.e., \( \tau = 0.8, 2 \), and correspondingly, \( C = 1.6, 4 \), both of which are against the CFL condition. Figure 13 demonstrates that the positivity of susceptible (\( S \)), infected (\( I \)), and recovered (\( R \)) fractions, as well as the summation constraint, can be preserved by the proposed scheme regardless of the CFL numbers.

We now compare the proposed scheme with the nonstandard finite difference (NSFD) method \([7, 16, 17, 27]\) that has been extensively employed for numerical approximations of infection models. The standard NSFD scheme for the model (1.1) can be formulated as

\[
\begin{align*}
S^{n+1} &- S^n \frac{\tau}{\tau} + \mathbf{u}^{n+1} \cdot \nabla^*_h S^n - \nabla_h \cdot D \nabla_h S^n = -\beta I^n S^{n+1}, \\
I^{n+1} - I^n \frac{\tau}{\tau} + \mathbf{u}^{n+1} \cdot \nabla^*_h I^n - \nabla_h \cdot D \nabla_h I^n = \beta I^n S^{n+1} - \gamma I^{n+1}, \\
R^{n+1} - R^n \frac{\tau}{\tau} + \mathbf{u}^{n+1} \cdot \nabla^*_h R^n - \nabla_h \cdot D \nabla_h R^n = \gamma I^{n+1}.
\end{align*}
\]

The scheme (4.1) is easy to implement due to the explicit treatment for both advection and diffusion, but it theoretically requires a critical positivity condition

\[
\mathcal{F} = \left( |u| + |v| \right) \frac{\tau}{\tau} + \frac{D \tau}{h^2} < 1,
\]

which can be simplified using the parameters of this example as

\[
\mathcal{F} = 2.11 \tau < 1.
\]

We employ the scheme (4.1) to simulate this example. The time step sizes are taken as \( \tau = 0.4 \) and \( \tau = 0.5 \) respectively, and correspondingly, \( \mathcal{F} = 0.844 < 1 \) and \( \mathcal{F} = 1.055 > 1 \). Figure 14 depicts the values of \( \|S + I + R - 1\|_\infty \) and minima of \( S, I \) and \( R \) computed by the scheme (4.1). We observe that the scheme (4.1) can ensure the positivity and summation constraint only if the condition (4.2) holds.

5. **Conclusions.** An efficient numerical scheme has been developed for simulating an epidemic model with advection and diffusion. The model is formulated by a system of nonlinear coupling advection-diffusion-reaction equations. More importantly, the preservation of positivity of variables is essential for the model. Consequently, linear, decoupled and positivity-preserving numerical approximations are more preferred, and the proposed scheme is able to preserve these properties. We employ
Figure 10: Susceptible fraction profiles at different times in Example 2.

Figure 11: Infected fraction profiles at different times in Example 2.

Figure 12: Recovered fraction profiles at different times in Example 2.

Figure 13: Preservation of the positivity and summation constraint of the proposed scheme (2.8) in Example 2.
the cell-centered finite difference method for the spatial discretization, and thus, the scheme is easy to implement. As a key ingredient of the proposed scheme, we propose a semi-implicit upwind discretization for the advection terms, which has a distinctive feature that the positivity of variables is preserved free of any restriction on the spatial mesh size and time step size. The diffusion terms are discretized implicitly to improve the robustness of the scheme. Using the discrete variational principles, we rigorously prove the unique existence of discrete solutions and positivity-preserving property of the proposed scheme without requirements for the mesh size and time step size. Numerical experiments demonstrate that the proposed scheme is capable of preserving the positivity of variables.

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