We present results from a first study of $B$-mesons that is based on a transverse lattice formulation of light-front QCD. The shape of the Isgur-Wise form factor is in very good agreement with experimental data. However, the calculations yield rather large values for $f_B$ and $\Lambda$ compared to contemporary calculations based on other techniques.

1. Introduction

Parton distributions measured in deep inelastic scattering, as well as many other high-energy observables (e.g. the deeply virtual Compton scattering amplitudes), are dominated by correlations along the light-cone ($x^2 = 0$). This simple fact poses a big obstacle for non-perturbative calculations of these important observables. For example, this makes direct evaluations of parton distributions on a Euclidean lattice, where all distances are space-like, impossible and calculations performed in a Euclidean framework usually try to reconstruct parton distribution functions from their moments. Furthermore, in an equal time quantization scheme, deep inelastic structure functions are described by real time response functions which are not only very difficult to interpret but also to calculate.

Light-Front (LF) quantization seems a promising tool to describe the immense wealth of experimental information about structure functions for a variety of reasons:

• correlations along the light-cone become “static” observables in this approach [i.e. equal $x^+ \equiv (x^0 + x^3)/\sqrt{2}$ observables]

• structure functions are easy to evaluate from the LF wavefunctions

Further advantages of the LF formalism derive from the simplified vacuum structure (nontrivial vacuum effects can only appear in zero-mode degrees of freedom) which provides a physical basis for the description of hadrons that stays close to intuition [1–4].

2. The Transverse Lattice

Before one can apply the LF formalism to QCD one has to remove the divergences first (i.e. regularize and renormalize). Then one has to cast bound state problems into a form that can be solved numerically with a reasonable effort.

The basic idea of the transverse lattice is very simple: one keeps two directions (the time and the z-direction) continuous but discretizes the transverse space coordinates (Fig. [1]). The metric is Minkowskian. Two immediate advantages of this construction are

• manifest boost and translational invariance in the longitudinal direction — thus keeping parton distributions easily accessible

• a gauge invariant cutoff for divergences associated with large transverse momenta
Because of these features, the transverse lattice seems to be ideally suited for a light-cone formulation of QCD.

The gauge degrees of freedom on the transverse lattice are described by non-compact $A$-fields in the continuous longitudinal directions and by compact link fields $U$ in the transverse directions. The $A$ fields are defined on the sites of the lattice and the $U$s on the links. All degrees of freedom depend on two continuous and two discrete space-time variables.

For the canonical lattice approximation of $G_{\mu\nu}G^{\mu\nu}$ on the transverse lattice one needs to distinguish the following cases:

1. Both $\mu$ and $\nu$ are longitudinal. Here the lattice representation for $G_{\mu\nu}G^{\mu\nu}$ is formally identical to the continuum representation.

2. Both $\mu$ and $\nu$ are transverse. Here the lattice representation is just the plaquette interaction, which is familiar from Euclidean lattice gauge theory.

3. In the mixed case, i.e. for example when $\mu$ is longitudinal and $\nu$ is transverse, the lattice representation for $G_{\mu\nu}G^{\mu\nu}$ resembles the kinetic term of a gauged nonlinear sigma model

$$G_{\mu\nu}G^{\mu\nu} \rightarrow D_\mu U^\dagger D^\mu U. \quad (1)$$

The transverse lattice action was first introduced in Ref. [5] where it was already realized that a formulation with compact link-fields is not suitable for light-cone quantization. \(^2\) This has led to the color dielectric formulation of the transverse lattice where ‘macroscopic’ or smeared degrees of freedom are introduced to represent linearized link degrees of freedom on a coarse lattice. The effective action in the color dielectric formulation is obtained by making an ansatz which does not break the unbroken symmetries of the transverse lattice (e.g. gauge invariance, longitudinal boost invariance) and the coefficients in this ansatz are then obtained by seeking regions of enhanced Lorentz symmetry (e.g. where the static $Q\bar{Q}$ potential is ‘round’ and where light glueballs have a dispersion relation with the same transverse speed of light). In Ref. [9] such an ansatz which included terms up to $4^{\text{th}}$ order in the link fields yielded glueball masses that are consistent with Euclidean results.

3. Spectrum and structure of light mesons

We used the effective link-field interactions, as determined in previous pure glue calculations within the color-dielectric formulation of $\perp$ lattice QCD [9,10]. The fermion action was based on a $\perp$ lattice generalization of Wilson fermions [11]. In this formulation, there are two hopping terms for fermions (hopping with and without spin flip for the quarks), which represent also the most general hopping terms that are possible to the same order in the fields which are consistent with the residual symmetries of the $\perp$ lattice.

In addition to these hopping terms, one needs to introduce kinetic energy terms for the quarks as well as a coupling of the fermions to the longitudinal gauge degrees of freedom $A^-$. Because of gauge invariance, the quarks and $\perp$ link fields couple to the longitudinal gauge field with the same strength. Since gauge coupling for the $\perp$ link fields have already been determined in studies of glueball spectra as well the static $Q\bar{Q}$ potential, this coupling is no longer a free param-
eter and the only new parameters are the coefficients of the hopping terms as well as the (kinetic) quark masses. In the spirit of the color dielectric formulation, we determined these parameters by looking for regions in parameter space with enhanced Lorentz symmetry. The criteria that we used to test the violation of Lorentz symmetry were

- \( k_\perp \) dependence in the dispersion relations of \( \pi \) and \( \rho \) mesons: using the relation \( k_\perp = a p_\perp \) between the lattice and physical \( \perp \) momentum, one can extract the effective \( \perp \) lattice spacing \( a \) for each hadron individually by performing a Taylor series expansion of its numerically determined dispersion relation

\[
2p^+ p^- = m_n^2 + c_n^2 k_\perp^2 + O(k_\perp^4)
\]

The covariant dispersion relation \( 2p^+ p^- = m^2 + p_\perp^2 \) is satisfied if the \( \perp \) lattice spacing satisfies \( a_n c_n = 1 \). The non-perturbative renormalization condition that we used in this work was to demand that the respective lattice spacings for \( \pi \) and \( \rho \) mesons are the same and also agree with \( \perp \) lattice spacings determined within the pure glue sector.

- mass splitting within the \( \rho \) meson spin multiplet

In a first principle calculation, it would be sufficient to input the \( m_\pi \) to set all scales in the light quark sector. However, since the role of chiral symmetry and chiral symmetry breaking is still not very well understood within the LF framework, we needed to input the chiral symmetry breaking scale as one phenomenological parameter (on top of the string tension and \( m_n \)). We did this by using \( m_\rho \) as an additional input parameter.

In the numerical calculations \([4]\) we restricted ourselves to the \( N_C \to \infty \) limit, which limited not only the number of possible terms in \( P_{eff}^- \) but also simplified the classification of states. For the numerical calculations we restricted the number of link fields to the minimal number which allows \( \perp \) propagation of the entire hadron: \( q \bar{q} \) on the same \( \perp \) site as well as \( q \bar{q} \) separated by one link with a link field \( U \) connecting the two.

Typical results for the meson dispersion relations are displayed in Fig. 2. As one can read off from Fig. 2 it was not possible to restore full rotational invariance for the \( \rho \) multiplet, but we hope that future calculations including additional terms in \( P_{eff}^- \) as well as higher Fock components can improve this situation.

In the continuum limit the dispersion relations should all be parabolas with the same curvature at the bottom. Near \( k_\perp = 0 \) this is reasonably well satisfied, but of course there are larger violations of Lorentz invariance as the inverse momentum becomes comparable to the lattice spacing near the boundaries of the Brillouin zone. The level crossing is a remnant of species doubling since without \( r \) term species doubling manifests itself on the transverse lattice at the hadronic level by giving rise to \( \rho \) mesons at the boundary.
that are degenerate with π mesons in the center of the Brillouin zone (and vice versa).

Results for the π distribution amplitude are shown in Fig. 3. Although \( \phi_\pi(x) \) resembles the asymptotic distribution \( \phi_{\pi_{asy}}(x) = 6x(1-x) \), our numerically determined result is somewhat broader, but clearly does not exhibit any ‘double hump’ feature. The ρ meson distribution amplitude looks similar although it is slightly more peaked, which reflects the weaker binding of the quarks in the ρ.

For its normalization, i.e. for \( f_\pi \) we found a result that is about a factor 2 larger than the experimental value. This discrepancy is most likely caused by the Fock space truncation, since including more higher Fock components tends to decrease the probability to find a hadron in its lowest Fock component and therefore also the normalization of the distribution amplitude.

4. B mesons on the \( \perp \) lattice

In the limit where the \( b \) quark is infinitely heavy, it acts as a static color source to which the light quark is bound. The extension of our light meson calculations to such a heavy-light system is straightforward and since the static source does not propagate, no new parameters appear in the Hamiltonian for such a system [13].

For the decay constant, which also plays an important role in mixing phenomenology, we find \( f_B \approx 240 MeV \pm 20 MeV \), i.e. a value that is somewhat larger than those obtained using Euclidean lattice gauge theory or QCD sum rules, but here the discrepancy is much smaller than for \( f_\pi \). This result is consistent with our observation that the Fock expansion also seems to converge much more rapidly for \( B \) mesons.

For phenomenological applications [14] it is useful to have numerical estimates for the moments (normalization: \( \int_0^\infty dz\phi_\infty(z)dz = 1 \))

\[
\int_0^\infty dz z^2 \phi_\infty(z) = 1.51 \pm 0.1 GeV
\]

\[
\int_0^\infty dz z \phi_\infty(z) = 1.22 \pm 0.1 (GeV)^{-1}.
\] (3)

The numerical values for the moments indicate a larger momentum scale compared to other calculations. This result is consistent with a rather large value of the \( B \)-meson ‘binding energy’ \( \Lambda \approx \)

\[
\text{The error bars include only the estimated error from the numerical extrapolation to the heavy quark limit and the truncation of the Hilbert space, but not the systematic errors from the extrapolation in the Fock space and the transverse lattice spacing.}
\]
0.9 – 1.0 GeV obtained from the same transverse lattice eigenstates by calculating the expectation value of the $p^+$ momentum of all light degrees of freedom in the $B$ meson. We expect that including more Fock components will lead to a lowering of $\bar{\Lambda}$.

An important observable in $B$-physics is the Isgur-Wise (IW) form factor, because of its use in the extraction of the CKM matrix element $V_{bc}$ from decays like $B \to \bar{D}^* l\nu$. We work in the limit $m_c, m_b \to \infty$, where $\langle B' | \bar{b} \gamma^\mu b | B \rangle$, $\langle D^* | \bar{c} \gamma^\mu b | B \rangle$ are all described by the same universal form factor

$$\langle B' | \bar{b} \gamma^\mu b | B \rangle = m_B (\nu^\mu + \nu^\rho) F(v \cdot v')$$

For $m_b, m_c \ll \Lambda_{QCD}$, heavy quark pair creation is suppressed, i.e. relevant matrix element is diagonal in Fock space and an overlap representation exists for $F(v \cdot v')$

$$F(v \cdot v') = F^{(2)}(v \cdot v') + F^{(3)}(v \cdot v'),$$

where

$$F^{(2)}(x) = \frac{2}{2 - x} \sum_s \int_x^1 dz \psi_s(z) \psi_s^* \left( \frac{z - x}{1 - x} \right)$$

$$F^{(3)}(x) = \frac{2}{2 - x} \frac{1}{\sqrt{1 - x}} \sum_s \int_x^1 dz \int_0^{1 - z} dw \psi_s(z, w) \psi_s^* \left( \frac{z - x}{1 - x}, \frac{w}{1 - x} \right)$$

Here, $\psi_s(z)$ and $\psi_s(z, w)$ are the wave functions in the 2 and 3 particle Fock component and $s$ represents the spin/orientation labels.

Numerical results for the shape of the IW form factor, obtained from our numerically determined eigenstates on the $\perp$ lattice are consistent with experimental results.

**REFERENCES**

1. R.J. Perry, lecture notes, [hep-ph/9710175](http://arxiv.org/abs/hep-ph/9710175).
2. S.J. Brodsky, H.-C. Pauli and S.S. Pinsky, Phys. Rept. 301, 299 (1998); S.J. Brodsky et al., Part. World 3, 109 (1993).
3. K. G. Wilson et al., Phys. Rev. D49, 6720 (1994).
4. M. Burkardt, Adv. Nucl. Phys. 23, 1 (1996).
5. W. A. Bardeen and R. B. Pearson, Phys. Rev. D14, 547 (1976); W. A. Bardeen, R. B. Pearson and E. Rabinovici, Phys. Rev. D 21, 1037 (1980).
6. P. Griffin, Proc. to ‘Theory of Hadrons and Light-Front QCD’, Polona Zgorgelisko, August 1994, [hep-ph/9410243](http://arxiv.org/abs/hep-ph/9410243).
7. M. Burkardt, AIP Conf. Proc. 494, 239 (1999); [hep-th/9908195](http://arxiv.org/abs/hep-th/9908195).
8. M. Burkardt and S. Dalley, to appear in Prog. Part. Nucl. Phys.; [hep-ph/0112007](http://arxiv.org/abs/hep-ph/0112007).
9. S. Dalley and B. van de Sande, Phys. Rev. Lett. 82, 1088 (1999), Phys. Rev. D 59, 065008 (1999).
10. B. Klindworth and M. Burkardt, in “Confinement and the Hadron Spectrum III” Jefferson Lab., June 1998, [hep-ph/9809283](http://arxiv.org/abs/hep-ph/9809283).
11. M. Burkardt and H. El-Khozondar, Phys. Rev. D 60, 054504 (1999) or Phys. Rev. D 55, 6514 (1997).
12. M. Burkardt and B. Klindworth, Phys. Rev. D 55, 1001 (1997).
13. P. Griffin, Phys. Rev. D47, 1530 (1993).
14. M. Burkardt and S. Seal, to appear in Phys. Rev. D, [hep-ph/0102245](http://arxiv.org/abs/hep-ph/0102245).
15. M. Burkardt and S. Seal, Phys. Rev. D 64, 111501 (2001).
16. M. Beneke and Th. Feldmann, Nucl. Phys. B 592, 3(2001).
17. CLEO collaboration, hep-exp/0007052.