Vacuum fluctuations and balanced homodyne detection through ideal multi-mode photon number or power counting detectors

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Abstract
The balanced homodyne detection as a readout scheme of gravitational-wave detectors is carefully examined, which specifies the directly measured quantum operator in the detection. This specification is necessary to apply the quantum measurement theory to gravitational-wave detections. We clarify the contribution of vacuum fluctuations to the noise spectral density without using the two-photon formulation. We found that the noise spectral density in the two-photon formulation includes vacuum fluctuations from the main interferometer but does not include those from the local oscillator which depends on the directly measured operators.

Keywords: quantum measurement theory, vacuum fluctuations, balanced homodyne detection, gravitational-wave detectors

1. Introduction
One of the motivations of the recent quantum measurement theory [1] is gravitational-wave detection. However, the actual application of this theory to the gravitational-wave detection requires its extension to the quantum field theory. Furthermore, in quantum measurement theory, we have to specify the directly measured quantum operator. In interferometric gravitational-wave detectors, we may regard that the directly measured operator is specified at their “readout scheme” in the detectors. The readout scheme in gravitational-wave detectors is an optical system to specify the optical fields which are detected at the photodetectors through the signal output from the multi-interferometer and the reference field which injected from the “local oscillator.” The research on this readout scheme is important for the development and application of the quantum measurement theory to gravitational-wave detections.

Current gravitational-wave detectors use the DC readout scheme, in which the output photon power is directly measured. On the other hand, “homodyne detections” are regarded as one of candidates of the readout scheme in the near future [2]. In Ref. [3], it is written that the output quadrature \( \hat{b}_\theta \) defined by

\[
\hat{b}_\theta := \cos \theta \hat{b}_1 + \sin \theta \hat{b}_2 \tag{1}
\]

is measured by the “balanced homodyne detection.” The output operator \( \hat{b}_\theta \) includes gravitational-wave signal \( h(\Omega) \) as

\[
\hat{b}_\theta = R(\Omega, \theta) \left( \hat{h}_\Omega(\Omega, \theta) + h(\Omega) \right), \tag{2}
\]

where \( \theta \) is the homodyne angle, \( \hat{h}_\Omega(\Omega, \theta) \) is the noise operator which is given by the linear combination of the annihilation and creation operators of photons injected to the main interferometer. However, it does not seem that there is clear description on the actual measurement processes of the operator (1) or directly measured quantum operators in these processes.

For this motivation, in Refs. [4], we examined the case where the directly measured operators are the number operator:

\[
\hat{n}(\omega) = \hat{a}^\dagger(\omega)\hat{a}(\omega) \tag{3}
\]

using the annihilation \([\hat{a}(\omega)]\) and creation \([\hat{a}^\dagger(\omega)]\) operators of the electric field. From the usual commutation relation

\[
[\hat{a}(\omega), \hat{a}(\omega')] = \hat{a}^\dagger(\omega), \hat{a}^\dagger(\omega') = 0, \tag{4}
\]

\[
\left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = 2\pi \delta(\omega - \omega'), \tag{5}
\]

eigenvalue of the operator \( \hat{n}(\omega) \) becomes countable number. This countable number give rise to the notion of “photon” and we can count this number. As a result of this examination, we reached to a conclusion that we cannot measure the expectation value of the operator (1) by the balanced homodyne detection [5]. However, the operator (3) is not appropriate as a directly measured operator in gravitational-wave detectors, since multi-mode detections are essential in these detectors.

As a directly measured operators in multi-mode detections, Glauber’s photon number operator

\[
\hat{N}_a(t) := \frac{\kappa_a}{2\pi \hbar} A \hat{E}_a^{\dagger}(t)\hat{E}_a(t). \tag{6}
\]

is often used in many literatures. The factor \( \frac{\kappa_a}{2\pi} A \) in Eq. (6) is chosen for our convenience and the coefficients \( \kappa_a \) is a phenomenological parameter whose dimension is [time] which includes “quantum efficiency”. This factor is not important within our discussions. On the other hand, the output electric field \( \hat{E}_a \) are separated into their positive- and negative-frequency part as

\[
\hat{E}_a(t) = \hat{E}_a^{(+)}(t) + \hat{E}_a^{(-)}(t), \quad \hat{E}_a^{(-)}(t) = \left[\hat{E}_a^{(+)}(t)\right]^{\dagger}, \tag{7}
\]

\[
\hat{E}_a^{(+)}(t) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi \kappa_a}{A}} \hat{a}(\omega) e^{-i\omega t}. \tag{8}
\]
Here, $\mathcal{A}$ in Eqs. (5) and (5) is the sectional area of the laser beam. The operator (5) is an extension of the operator (5) to multi-mode cases within the Maxwell theory (5).

In quantum measurement theories of optical fields, there was a long controversy on which variable is the directly measured by photodetectors in multi-mode cases [5, 6, 7]. Some insisted that the directly measured operators at photodetectors is the above Glauber’s photon number operator (5), and some insisted that the direct observable of photodetectors is the power of the optical field. Within this history, Kimble and Mandel [8] pointed out that we cannot say neither, in general. The current consensus of this issue will be that the directly measured operator at the photodetectors is Glauber’s number operator (5). In Sec. 2.2, we discuss the expectation value under the premise that the directly measured operator at the photodetectors is the power operator (9).

Furthermore, we estimate the quantum noise in these two cases. In many literatures in which the photo-detection is treated as a classical process, in which the detection probability is proportional to the expectation value of the power operator (9).

In this Letter, we examine these two cases where the directly measured operators at the photodetector is Glauber’s number operator (5) and that is the power operator (9). Furthermore, we estimate the quantum noise in these two cases. In many literatures of the gravitational-wave detection, it is written that the single sideband noise spectral density $S^{(0)}_A(\omega)$ for an arbitrary operator $\hat{A}(\omega)$ with the vanishing expectation value in the “stationary” system is given by

$$\frac{1}{2}\pi\delta(\omega-\omega')S^{(0)}_A(\omega) := \frac{1}{2}(\hat{A}(\omega)\hat{A}^\dagger(\omega') + \hat{A}^\dagger(\omega')\hat{A}(\omega)).$$  

This noise spectral density is introduced by Kimble et al. in Ref. [5] in the context of the two-photon formulation [10]. This two-photon formulation is commonly used in the community of gravitational-wave detection. However, in this Letter, we do not use the two-photon formulation, though some final formulae are written in terms of the two-photon formulation. We also examine the original meaning of Kimble’s noise spectral density [10] and derive the deviation from this noise formula [10].

2. Balanced Homodyne Detections by multi-mode detectors

Here, we examine the expectation value of the balanced homodyne detection. In Sec. 2.1 this expectation value is evaluated under the premise that the directly measured operator at the photodetector is Glauber’s number operator (5). In Sec. 2.2 we discuss the expectation value under the premise that the directly measured operator is the power operator (5).

2.1. Balanced Homodyne Detections by Glauber’s Photon-Number Counting Detectors

Here, we review the balanced homodyne detection depicted in Fig. 1. Throughout this Letter, we want to measure the signal field $\hat{E}_b(\omega)$. The electric field from the local oscillator $\hat{E}_l(\omega)$ is in the coherent state $|\gamma\rangle$ with $\hat{b}(\omega)|\gamma\rangle_l = \gamma(\omega)|\gamma\rangle_l$. In the time domain, the state $|\gamma\rangle$ satisfies

$$\hat{E}_{l_+}(\omega)|\gamma\rangle_l = \sqrt{\frac{2\pi\hbar}{\mathcal{A}_c}}\gamma(t)|\gamma\rangle_l, \quad \gamma(t) := \int_0^\infty d\omega \sqrt{|\omega|} e^{-i\omega t}.$$  

The output signal field $\hat{E}_d(\omega)$ and the field $\hat{E}_e(\omega)$ from the local oscillator is mixed through the beam splitter with the transmissivity 1/2.

![Figure 1: Configuration of the interferometer for the balanced homodyne detection. Notations of the quadrature for the electric fields in the main text is also described.](image)

At the beam splitter in Fig. 1 the fields $\hat{E}_d(\omega)$ and $\hat{E}_e(\omega)$ are transformed to the fields $\hat{E}_c(\omega)$ and $\hat{E}_d(\omega)$ as

$$\hat{E}_c(\omega) = \frac{1}{\sqrt{2}}\hat{E}_d(\omega) + \frac{1}{\sqrt{2}}\hat{E}_e(\omega),$$  

$$\hat{E}_d(\omega) = -\frac{1}{\sqrt{2}}\hat{E}_d(\omega) + \frac{1}{\sqrt{2}}\hat{E}_e(\omega).$$  

The electric fields $\hat{E}_d$ and $\hat{E}_e$ are in their vacua

$$\hat{b}(\omega)|0\rangle_d = \hat{c}(\omega)|0\rangle_e = 0,$$

respectively, and the field $\hat{E}_d(\omega)$ is given by $\hat{E}_d$ and $\hat{E}_e$ through the beam splitter condition. In general, the state of the field $\hat{E}_b$ depends on the state of the input field $\hat{E}_d$ and the other optical fields which is injected to the main interferometer [5]. Furthermore, we consider the situation where the output electric field $\hat{E}_b$ includes the information of classical forces as in Eq. (2) and this information are measured through the expectation value of the operator $\hat{E}_b$. To evaluate the expectation value

$$\frac{\kappa p c}{4\pi \hbar} \mathcal{A} \left( \hat{E}_b(t) \right)^2$$  

of the measured optical field. The factor $\frac{\kappa p c}{4\pi \hbar} \mathcal{A}$ in Eq. (5) is chosen for our convention and the coefficient $\kappa p$ is a phenomenological parameter whose dimension is [time] which includes “quantum efficiency”. This is not important within our discussions as in the case of Eq. (5). It is also true that there are many literatures in which the photo-detection is treated as a classical stochastic process, in which the detection probability is proportional to the expectation value of the power operator (9).
of the signal field $\hat{E}_b$, we have to specify the state of the total system as

$$|\Psi\rangle = |\gamma\rangle_c \otimes |0\rangle_c \otimes |0\rangle_d \otimes |\psi\rangle_{\text{main}}.$$ (16)

Here, the state $|\psi\rangle_{\text{main}}$ is for the electric fields associated with the main interferometer, which is independent of the state $|\gamma\rangle_c$, $|0\rangle_c$, and $|0\rangle_d$. The expectation value of the field $\hat{E}_a$ means

$$\langle \hat{E}_a \rangle := \langle \Psi | \hat{E}_a | \Psi \rangle.$$ (17)

Here, we regard that Glauber’s photon number operators

$$\hat{N}_c(t) := \frac{k_{\text{c}} c}{2\pi \hbar} \hat{A}_b \hat{E}_c^{(+)}(t) \hat{E}_c^{(-)}(t),$$ (18)

$$\hat{N}_d(t) := \frac{k_{\text{c}} c}{2\pi \hbar} \hat{A}_b \hat{E}_d^{(+)}(t) \hat{E}_d^{(-)}(t)$$ (19)

are directly measured at the photodetector D1 and D2 in Fig. 1, respectively. Substituting Eq. (13) into Eq. (19), we obtain

$$\hat{N}_c(t) = \frac{1}{2} \hat{N}_b(t) + \frac{1}{2} \hat{N}_t(t)$$

$$+ \frac{1}{2} k_{\text{c}} c \frac{\hbar}{2\pi} \left( \hat{E}_c^{(-)}(t) \hat{E}_b^{(-)}(t) + \hat{E}_b^{(-)}(t) \hat{E}_c^{(-)}(t) \right),$$ (20)

while the substitution of Eq. (14) into Eq. (19) yields

$$\hat{N}_d(t) = \frac{1}{2} \hat{N}_t(t) + \frac{1}{2} \hat{N}_t(t)$$

$$- \frac{1}{2} k_{\text{c}} c \frac{\hbar}{2\pi} \left( \hat{E}_c^{(-)}(t) \hat{E}_b^{(-)}(t) + \hat{E}_b^{(-)}(t) \hat{E}_c^{(-)}(t) \right).$$ (21)

From the expectation values of Eqs. (20) and (21), we obtain

$$\frac{1}{k_{\text{c}}} \langle \hat{N}_c(t) - \hat{N}_d(t) \rangle = \frac{\hbar}{2\pi} \left( \gamma(t) \langle \hat{E}_c^{(+)}(t) \rangle + \gamma(t) \langle \hat{E}_b^{(+)}(t) \rangle \right),$$ (22)

and we define the signal operator $\hat{s}_\gamma(t)$ as

$$\hat{s}_\gamma(t) := \frac{1}{k_{\text{c}}} \left[ \hat{N}_c(t) - \hat{N}_d(t) \right]$$

$$= \frac{\hbar}{2\pi} \left[ \langle \hat{E}_c^{(-)}(t) \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \hat{E}_c^{(+)}(t) \rangle \right]$$ (23)

so that

$$\langle \hat{s}_\gamma(t) \rangle = \sqrt{\frac{\hbar}{2\pi}} \left[ \gamma(t) \langle \hat{E}_b^{(+)}(t) \rangle + \gamma(t) \langle \hat{E}_c^{(+)}(t) \rangle \right].$$ (24)

We note that $\hat{s}_\gamma(t)$ is a self-adjoint operator.

Here, we consider the monochromatic local oscillator case, in which the complex amplitude $\gamma(\omega)$ in Eq. (24) is given by

$$\gamma(\omega) = 2\pi \gamma_0 \delta(\omega - \omega_0), \quad \omega_0 > 0, \quad \gamma := |\gamma| e^{i\theta},$$ (26)

and consider the situation $\omega_0 \gg \omega > 0$. In this case, the Fourier transformation of the expectation value (25) is given by

$$\langle \hat{s}_\gamma(\omega) \rangle \sim \omega_0 |\gamma| \left( e^{-i\theta} \hat{b}(\omega_0 + \omega) + e^{i\theta} \hat{b}^\dagger(\omega_0 - \omega) \right).$$ (27)

We note that $\omega_0$ in Eq. (27) is just the central frequency of the local oscillator and have nothing to do with the central frequency of the signal field $\hat{E}_b(t)$. Therefore, Eq. (27) is still valid even in the case “heterodyne detection.”

Now, we choose $\omega_0$ so that this frequency coincides with the central frequency of the signal field $\hat{E}_b(t)$. This is the “homodyne detection”. Then, we may identify the quadratures $\hat{b}(\omega_0 + \omega)$ and $\hat{b}(\omega_0 - \omega)$ with the upper- and lower-sideband quadratures $\hat{b}_+(\omega)$ and $\hat{b}_-(\omega)$ in the two-photon formulation [14], respectively. We may introduce the amplitude quadrature $\hat{b}_1(\omega)$ and the phase quadrature $\hat{b}_2(\omega)$ by

$$\hat{b}_1 := \frac{1}{\sqrt{2}} \left( \hat{b}_+ + \hat{b}_- \right), \quad \hat{b}_2 := \frac{1}{\sqrt{2}} \left( \hat{b}_- - \hat{b}_+ \right).$$ (28)

In terms of these quadratures $\hat{b}_{1,2}(\omega)$ Eq. (27) is given by

$$\frac{1}{\sqrt{2\omega_0}} \langle \hat{s}_\gamma(\omega) \rangle \sim \langle \hat{b}_\gamma(\omega) \rangle.$$ (29)

Thus, when Glauber’s photon number operator is the directly measured operator at the photodetectors, we can measure expectation value of the operator $\hat{b}_\gamma$.

2.2. Balanced Homodyne Detections by Power Counting Detectors

Through the conditions [13] and [14] at the beam splitter and the definition (9) of the operator $\hat{P}_{\gamma}(t)$, the power operators $\hat{P}_{\gamma_c}(t)$ at the D1 and $\hat{P}_{\gamma_d}(t)$ at D2 are given by

$$\hat{P}_{\gamma_c}(t) = \frac{1}{2} \hat{P}_{\gamma}(t) + \frac{1}{2} \hat{P}_{\gamma_b}(t)$$

$$+ \frac{1}{2} k_{\text{c}} c \frac{\hbar}{2\pi} \left( \hat{E}_b(t) \hat{E}_b(t) + \hat{E}_b(t) \hat{E}_b(t) \right),$$ (30)

$$\hat{P}_{\gamma_d}(t) = \frac{1}{2} \hat{P}_{\gamma_b}(t) + \frac{1}{2} \hat{P}_{\gamma_b}(t)$$

$$- \frac{1}{2} k_{\text{c}} c \frac{\hbar}{2\pi} \left( \hat{E}_b(t) \hat{E}_b(t) + \hat{E}_b(t) \hat{E}_b(t) \right).$$ (31)

As Sec. 2.1 we define the signal operator $\hat{s}_{\gamma}(t)$ by

$$\hat{s}_{\gamma}(t) := \frac{1}{2k_{\text{c}}} \left[ \hat{P}_{\gamma_c}(t) - \hat{P}_{\gamma_d}(t) \right]$$

$$= \frac{\hbar}{4\pi} \left[ \hat{E}_b(t) \hat{E}_b(t) + \hat{E}_b(t) \hat{E}_b(t) \right].$$ (32)

To evaluate the expectation value of the signal operator $\hat{s}_{\gamma}(t)$, we assume the commutation relation

$$\left[ \hat{E}_b(t), \hat{E}_b(t) \right] = 0,$$ (34)

which is justified in Ref. [11]. Then, we obtain

$$\langle \hat{s}_{\gamma}(t) \rangle = \sqrt{\frac{\hbar}{2\pi}} \left( \gamma(t) + \gamma^*(t) \right) \langle \hat{E}_b(t) \rangle.$$ (35)

In the case of the monochromatic local oscillator [26] and the situation $\omega_0 \gg \omega > 0$, the Fourier transformation of Eq. (35) is

$$\langle \hat{s}_{\gamma}(\omega) \rangle = \int_{-\infty}^{+\infty} dt e^{it\omega} \langle \hat{s}_{\gamma}(t) \rangle \sim \sqrt{2\omega_0} |\gamma| \langle \hat{b}_\gamma(\omega) \rangle.$$ (36)

This is the same result as Eq. (29).
3. Noise Spectral Densities

In the two-photon formulation, sideband fluctuations in the frequency $\omega_0 \pm \omega$ with the central frequency $\omega_0$ is considered and Kimble’s single-sideband noise-spectral density is commonly used. The “single sideband” means the evaluation of noises only in the frequency range $\omega > 0$ of the positive- and negative-sideband $\omega_0 \pm \omega$. The frequencies $\omega$ and $\omega'$ in Eq. (10) is the sideband frequencies in the two-photon formulation. If we consider the noise in both sideband $\omega \geq 0$, the noise spectral density $S_A^{(d)}(\omega)$ is called “double sideband”

$$2\pi \delta(\omega - \omega') \tilde{S}_A^{(d)}(\omega) := \frac{1}{2} \langle \hat{A}(\omega) \hat{A}^\dagger(\omega') + \hat{A}^\dagger(\omega') \hat{A}(\omega) \rangle.$$  (37)

Furthermore, (double sideband) “correlation spectral density” of operators $\tilde{A}(\omega)$ and $\tilde{B}(\omega)$ with $\langle \hat{A}(\omega) \rangle = \langle \hat{B}(\omega) \rangle = 0$ is

$$2\pi \delta(\omega - \omega') \tilde{S}_{AB}^{(d)}(\omega) := \frac{1}{2} \langle \hat{A}(\omega) \hat{B}^\dagger(\omega') + \hat{B}^\dagger(\omega') \hat{A}(\omega) \rangle.$$  (38)

To examine the meaning of the correlation function, we consider the time-domain expression of this formulae for the correlation spectral density through the Fourier transformation. Introducing the time-domain variables $\hat{A}(t)$ and $\hat{B}(t)$ as

$$\hat{A}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \Theta(\omega) \hat{A}(\omega) + \Theta(-\omega) \hat{A}^\dagger(-\omega) \right) e^{-i\omega t},$$

$$\hat{B}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \Theta(\omega) \hat{B}(\omega) + \Theta(-\omega) \hat{B}^\dagger(-\omega) \right) e^{-i\omega t}.$$  (39)

From Eq. (38) yields

$$C_{AB}(\tau) = \frac{1}{2} \langle \hat{A}(t + \tau) \hat{B}(t) + \hat{B}(t) \hat{A}(t + \tau) \rangle.$$  (41)

Note that the left-hand side of Eq. (41) depends only on “$\tau$”, while the right-hand side may depend both on “$t$” and “$\tau$”, in general. This dependence implies the “stationarity” of the correlation. If we consider the non-stationary cases, the correlation function may depend on “$t$” as

$$C_{AB}(t, \tau) = \frac{1}{2} \langle \hat{A}(t + \tau) \hat{B}(t) + \hat{B}(t) \hat{A}(t + \tau) \rangle.$$  (42)

From Eq. (42), we estimate the correlation function for the stationary noise by

$$C^{(\omega)}(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt C_{AB}(t, \tau).$$  (43)

We use Eq. (43) of the correlation function for the stationary noise, instead of Eq. (41). When $\hat{A}(t) = \hat{B}(t)$, the autocorrelation function $C_{AA}(\tau)$ for “stationary noise” is given by

$$C_{AA}(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt C_{AA}(t, \tau).$$  (44)

When the operator $\hat{A}(t)$ has a non-trivial expectation value $\langle \hat{A}(t) \rangle$, we consider the noise operator $\tilde{A}_n(t)$ defined by

$$\tilde{A}_n(t) := \tilde{A}_n(t) + \langle \hat{A}(t) \rangle.$$  (45)

and evaluate the noise correlation function by

$$C^{(\omega)}(\tau) := C_{AA}(\tau) - C^{(\omega)}_{AB}(\tau).$$  (46)

where $C_{(\omega,c)A}(\tau)$ is the classical correlation defined by

$$C_{(\omega,c)A}(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \langle \hat{A}(t + \tau) \rangle \langle \hat{A}(t) \rangle.$$  (47)

The noise spectral density $S_A(\omega)$ is given by the Fourier transformation of $C_{AA}(\tau)$ as

$$S_A(\omega) := \int_{-\infty}^{\infty} dt C_{(\omega,c)A}(\tau) e^{i\omega \tau}.$$  (48)

In this Letter, we evaluate the quantum noise through the noise spectral density instead of Eq. (37).

4. Estimation of Quantum noise

In this section, we evaluate quantum noise in the case that Glauber’s number operator is the directly measured operator (Sec. 4.1) and that in the case the power operator is the directly measured operator (Sec. 4.2).

In the noise estimation, we carefully treat the two types of vacuum fluctuations from the signal field $\hat{E}_s$ and the field $\hat{E}_l$ from the local oscillator. From Eqs. (4.1) and (5), the commutation relations of the electric field $\hat{E}(t)$ are given by

$$[\hat{E}^{(s)}(t), \hat{E}^{(s)}(t')] = \frac{2\pi \hbar}{\mathcal{A}} \int_{0}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} = 0,$$

$$[\hat{E}^{(+)}(t), \hat{E}^{(-)}(t')] = \frac{2\pi \hbar}{\mathcal{A}} \Delta_{\omega}(t-t').$$  (50)

The subscription “$s$” of the function $\Delta_{\omega}(t-t')$ indicates the vacuum fluctuations from the electric field $\hat{E}_s$ with the quadrature $\hat{A}(\omega)$. Although the function $\Delta_{\omega}(t-t')$ formally diverges, we regard that the integration range over $\omega$ in Eq. (50) as $[\omega_{\text{min}}, \omega_{\text{max}}]$ instead of $[0, +\infty]$. In the actual measurement of a time sequence of a variable, we have the minimal time bin which gives the maximal frequency $\omega_{\text{max}}$ and the finite whole observation time which gives the minimum frequency $\omega_{\text{min}}$.

4.1. Glauber’s Photon-Number Detectors Case

Here, we evaluate the noise spectral density $S_{\chi_{\text{SN}}}(\omega)$ for the noise operator $\chi_{\text{SN}}(t)$ defined by

$$\hat{\chi}_{\text{SN}}(t) := \hat{\chi}(t) - \langle \hat{\chi}(t) \rangle.$$  (51)

The evaluation is carried out step by step. First, we evaluate the normal-ordered noise-spectral density $S_{\chi_{\text{SN}}}^{(\text{normal})}(\omega)$, in which all vacuum fluctuations are neglected. Second, we evaluate the contribution from the vacuum fluctuations of the signal field $\hat{E}_s(t)$. Finally, we include the contribution from the vacuum fluctuations from the local oscillator $\hat{E}_l(t)$.

4.1.1. Normal ordered noise spectral density

Here, we consider the normal-ordered noise spectral density $S_{\chi_{\text{SN}}}^{(\text{normal})}(\omega)$ through the normal-ordered correlation function

$$C^{(\text{normal})}_{\chi_{\text{SN}}}(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \langle \hat{\chi}_{\text{SN}}(t + \tau) \hat{\chi}_{\text{SN}}(t) \rangle.$$  (52)
Through Eqs. (24), (51), (52), its noise spectral density is

\[ S_{\text{res}}^{(\text{normal})}(\omega) = \frac{3\Lambda c}{2\pi N}\langle |b(\omega)|^2 \rangle \int_{-\infty}^{\infty} dt e^{i\omega t} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left( e^{-i\omega t} \langle \hat{E}_{\text{res}}^{(+)}(t) \hat{E}_{\text{res}}^{(-)}(t) \rangle + e^{i\omega t} \langle \hat{E}_{\text{res}}^{(-)}(t) \hat{E}_{\text{res}}^{(+)}(t) \rangle + e^{-i\omega (t+\tau)} \langle \hat{E}_{\text{res}}^{(+)}(t+\tau) \hat{E}_{\text{res}}^{(-)}(t) \rangle + e^{i\omega (t+\tau)} \langle \hat{E}_{\text{res}}^{(-)}(t+\tau) \hat{E}_{\text{res}}^{(+)}(t) \rangle \right). \] (53)

Here, we introduce the Fourier transformed expression of the field operator \( \hat{E}_{\text{res}}^{(+)}(t) \) with the noise quadrature \( \hat{b}_n(\omega) := b(\omega) - \langle \hat{b}(\omega) \rangle \) as Eq. (8). Furthermore, to evaluate Eq. (53), we use the fact that the measure of the function

\[ f(a) := \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-i\omega t} = \begin{cases} 0 & \text{for } a \neq 0, \\ 1 & \text{for } a = 0. \end{cases} \] (54)

in the integration over \( a \) is zero, except for the case where the delta-function \( \delta(a) \) is included in the integration [11].

If we consider the Michelson interferometer as an explicit example of the main interferometer, each term in Eq. (53) gives the finite value due to the appearance of the delta-function in the expectation value of the product of the noise quadrature \( \hat{b}_n \) and the average-integral by \( t \). These finite results of each term in Eq. (53) also implies that if we omit the integration by the frequency and specify the frequency so that the exponent in the averaged function vanishes, by hand, the factor of the delta-function appears in Eq. (53). This is the imposition of the stationarity to the noise spectral density. In other words, for stationary noise, the correct result is obtained if we regard the expression of Eq. (53) as

\[ 2\pi\delta(\omega - \omega') S_{\text{res}}^{(\text{normal})}(\omega) \approx \omega_0^2 |\langle b(\omega) \rangle|^2 \left( e^{-2i\omega \hat{b}_n(\omega) + \hat{b}_n^\dagger(\omega') \hat{b}_n(\omega) - \omega'} + \hat{b}_n^\dagger(\omega') \hat{b}_n(\omega) - \omega' \rangle + e^{-i(\omega + \omega') \hat{b}_n(\omega) + \hat{b}_n^\dagger(\omega') \hat{b}_n^\dagger(\omega')} \right). \] (55)

Here, we used the situation \( \omega_0 \gg \omega > 0 \). Note that \( \omega_0 \) is the central frequency of local oscillator and may not coincide with the central frequency of the signal field \( \hat{E}_b(t) \).

Here, we regard that the central frequency \( \omega_0 \) of the local oscillator coincides with the central frequency from the main interferometer. This is the “homodyne detection.” Then, we may use the sideband picture \( \hat{b}_n(\omega) := \hat{b}(\omega) \pm \omega \) and introduce the noise quadratures \( \hat{b}_n, \hat{b}_{2n}, \) and \( \hat{b}_0 \) as Eqs. (11) and (28)

\[ 2\pi\delta(\omega - \omega') S_{\text{res}}^{(\text{normal})}(\omega) \approx \omega_0^2 |\langle b(\omega) \rangle|^2 \left( \hat{b}_n(\omega') \hat{b}_n(\omega) + \hat{b}_n(\omega) \hat{b}_n^\dagger(\omega') \right) - 2\pi\delta(\omega - \omega'). \] (56)

Here, we note that \( \langle \hat{b}_n(\omega) \hat{b}_n(\omega') \rangle = 0 \). Since we only consider the positive frequency \( \omega \) by the definition (8), the first term in the right-hand side of Eq. (56) coincides with Kimble’s single-sideband noise spectral density (10):

\[ S_{\text{res}}^{(\text{normal})}(\omega) \approx \omega_0^2 |\langle b(\omega) \rangle|^2 \left[ S_{\text{0}}(\omega) - 1 \right]. \] (57)

4.1.2. Vacuum fluctuations from the main interferometer

Here, we clarify the contribution of the vacuum fluctuations from the signal field \( \hat{E}_b(t) \) through ignoring the vacuum fluctuations of the local oscillator \( \hat{E}_l(t) \). To carry out this, we evaluate \( \langle \hat{S}_{\text{res},t}(\tau) \hat{S}_{\text{res},t}(\tau + t) \hat{S}_{\text{res},t}(\tau) \hat{S}_{\text{res},t}(\tau + t + \tau) \rangle \) under the premises

\[ \hat{E}_b^{(+)}(t), \hat{E}_b^{(-)}(t) = \frac{2 \pi n}{\gamma c} \Delta_b(t-t') \neq 0, \] (58)

\[ \hat{E}_l^{(+)}(t), \hat{E}_l^{(-)}(t) = \frac{2 \pi n}{\gamma c} \Delta_l(t-t') \neq 0. \] (59)

Eq. (59) is not consistent with quantum field theory, but we dare to use these premises (58) and (59) to distinguish the contribution of vacuum fluctuations from \( \hat{E}_b \) and \( \hat{E}_l \). Then, the time-averaged correlation function is given by

\[ C_{\text{res}}^{(\text{normal}+\text{sign vac})}(\tau) = C_{\text{res}}^{(\text{normal})(\tau)} + C_{\text{res}}^{(\text{sign vac})(\tau)}, \] (60)

where

\[ C_{\text{res}}^{(\text{sign vac})(\tau)} := \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i\omega t} \gamma(t)\gamma(t+\tau) \Delta_b(t) \Delta_b(-t) \right). \] (61)

In the monochromatic local oscillator case (26), we obtain

\[ C_{\text{res}}^{(\text{sign vac})(\tau)} = \frac{1}{\pi} \omega_0^2 \gamma(t) \left[ e^{i\omega_0 t} \Delta_b(t) + e^{-i\omega_0 t} \Delta_b(-t) \right], \] (62)

\[ S_{\text{res}}^{(\text{sign vac})(\omega)} = \omega_0^2 |\langle b(\omega) \rangle|^2. \] (63)

in the situation \( \omega_0 \gg \omega > 0 \). Together with Eq. (57), we obtain

\[ S_{\text{res}}^{(\text{normal}+\text{sign vac})(\omega)} = \omega_0^2 \gamma(t) S_{\text{0}}(\omega). \] (64)

Thus, apart from the overall factor, this result coincides with Kimble’s single-sideband noise spectral density (10). This means that the Kimble’s single-sideband noise spectral density is already included the contribution of the vacuum fluctuations from the signal field \( \hat{E}_b \).

4.1.3. Vacuum fluctuations from the local oscillator

Here, we consider the vacuum fluctuations from the local oscillator \( \hat{E}_l(t) \) through the premise

\[ \hat{E}_l^{(+)}(t), \hat{E}_l^{(-)}(t) \neq 0 \] (65)

instead of the premise (59). Then, we obtain

\[ C_{\text{res}}^{(\text{loc vac})(\tau)} = C_{\text{res}}^{(\text{normal}+\text{sign vac})(\tau)} + C_{\text{res}}^{(\text{loc vac})(\tau)}, \] (66)

and

\[ S_{\text{res}}^{(\text{loc vac})(\omega)} := \int_{-\infty}^{+\infty} dt e^{i\omega t} C_{\text{res}}^{(\text{loc vac})(\tau)}(t) \]

\[ = \frac{1}{2} \int_{-\infty}^{+\infty} d\omega_1 \frac{d\omega_1}{2\pi} \int_{-\infty}^{+\infty} d\omega_2 \frac{d\omega_2}{2\pi} \sqrt{\Delta_l^2 \Delta_b^2} \left( \hat{b}_n(\omega_1) \hat{b}_n(\omega_2) \right) \]

\[ \times \left[ \Theta(\omega_1 - \omega) \langle \omega_1 - \omega \rangle + \Theta(\omega_1 + \omega) \langle \omega_1 + \omega \rangle \right] \]

\[ \times \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i\omega(t-\omega)\tau}. \] (67)
Here, we used the mode expansion expression of $\hat{E}_{\text{b}}^{(s)}(t)$ and “complete dark port condition” which means the absence of the delta-function $\delta(\omega - \omega_0)$ in the expectation value $\langle \hat{b}(\omega) \rangle$.

Here, we introduce a new noise-spectral density $S_{\text{b}}(\omega)$ by

$$2\pi \delta(\omega_1 - \omega_2)S_{\text{b}}(\omega_1) := \langle \hat{b}_n(\omega_1)\hat{b}^*_n(\omega_2) + \hat{b}_n^*(\omega_2)\hat{b}_n(\omega_1) \rangle.$$

(68)

This definition of $S_{\text{b}}(\omega)$ has the same form of the Kimble single-sideband noise spectral density \[10\]. However, the noise-spectral density $S_{\text{b}}(\omega)$ has nothing to do with the two-photon formulation. The frequencies $\omega_1$ and $\omega_2$ in Eq. 68 is not sideband frequencies, but the frequency $\omega$ in Eq. (8).

Through Eq. (68) and $\omega > 0$, we obtain

$$S_{\text{b}}(\omega) := S_{\text{b}}^{(\text{normal-sig.vac})} + S_{\text{b}}^{(\text{loc.vac})}$$

$$\sim \omega_0^2 |\gamma|^2 S_{\text{b}}^{(s)}(\omega) + \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \langle \omega_1^2 (S_{\text{b}}(\omega_1) - 1) \rangle$$

$$- \frac{1}{4} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1 (\omega - \omega_1) (S_{\text{b}}(\omega_1) - 1). \quad (69)$$

Here, we note that Kimble’s noise spectral density is realized when $|\gamma|^2$ is sufficiently large so that the second- and third lines in the right-hand side of Eq. (69) are negligible, while the effects of these lines appears when $|\gamma|^2$ is sufficiently small. These lines are contribution from the vacuum fluctuations from the local oscillator and the third line has the frequency dependence.

4.2. Power Counting Detectors Case

To evaluate the noise spectral density $S_{\text{b}}(\omega)$ of the operator $\hat{S}_{\text{p}} := \hat{S}_{\text{p}}(\omega) - \hat{S}_{\text{p}}$, we consider the noise correlation function

$$C_{\text{b}}(\omega_1, \omega_2) := \frac{1}{2} \langle \hat{S}_{\text{b}}(\omega_2)\hat{S}_{\text{b}}(\omega_1) + \hat{S}_{\text{b}}(\omega_1)\hat{S}_{\text{b}}(\omega_2) \rangle,$$

(71)

$$\sim \frac{1}{2} \langle \gamma^2 (\hat{S}_{\text{b}} + \hat{S}_{\text{b}}^*) \rangle + \frac{1}{2} \langle \gamma^2 (\hat{S}_{\text{b}} + \hat{S}_{\text{b}}^*) \rangle \times$$

$$\langle \sqrt{\frac{\mathcal{A}_c}{2\pi h}} \hat{E}_{\text{b}}(t + \tau) + \sqrt{\frac{\mathcal{A}_c}{2\pi h}} \hat{E}_{\text{b}}^*(t) \rangle$$

$$+ \frac{1}{2} \mathcal{A}_c \langle \hat{S}_{\text{b}}(t + \tau) \rangle + \frac{1}{2} \mathcal{A}_c \langle \hat{S}_{\text{b}}(t) \rangle.$$

(72)

The averaged noise correlation function $C_{\text{b}}(\omega_1, \omega_2)$ and the noise spectral density $S_{\text{b}}(\omega)$ are given by the similar manner to Glauber’s photon-number case \[10\] through Eqs. (44) and (45). Then, we have

$$S_{\text{b}}(\omega) = \omega_0^2 |\gamma|^2 S_{\text{b}}^{(s)}(\omega)$$

$$+ \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1^2 (S_{\text{b}}(\omega_1) - 1)$$

$$+ \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1 (\omega - \omega_1) S_{\text{b}}(\omega_1). \quad (73)$$

in the situation $\omega_0 \gg \omega > 0$. We also note that even in the noise spectral density \[73\] Kimble’s noise spectral density is realized when $|\gamma|^2$ is sufficiently large so that the second- and third lines in the right-hand side of Eq. (73) is negligible, while these effects appears when $|\gamma|^2$ is sufficiently small. These lines are contribution from the vacuum fluctuations from the local oscillator. We note that the third line in Eq. (73), which is the frequency dependent term, is different from that in Eq. (69).

5. Summary

In summary, we showed our estimation of quantum noise in the balanced homodyne detections which measure $\hat{b}_b(\omega)$ as the expectation value. We consider both cases in which the directly measured operators at the photodetectors are Glauber’s photon-number operator \[6\] and the power operator \[9\], respectively. In our estimation, we did not use the two-photon formulation which is widely used in the gravitational-wave community. We concentrate on the stationary noise of the system through the time-average procedure. We also carefully treat vacuum fluctuations in our noise estimation. In both cases, we have derived the deviations from the Kimble’s noise spectral density \[10\], which are beyond the two-photon formulation.

In general, such deviations from the Kimble’s noise spectral density may also occur due to the physical properties of the photodetectors such as the band structure of photodiodes. However, the noise spectral densities derived here are for ideal Glauber’s photon-number counting detectors or ideal power counting detectors. The obtained deviations are due to the vacuum fluctuations from the local oscillator. The derived noise spectral densities \[69\] and \[73\] yield the Kimble noise spectral density when the amplitude $|\gamma|$ of the coherent state from the local oscillator is sufficiently large. On the other hand, when the amplitude of the coherent state from the local oscillator is sufficiently small, the difference of these noise spectral appears. This will be able to use for the characterization of the ideal multi-mode photon number detectors or ideal multi-mode photon power detectors.

In the case where the directly measured operator of the photodetector is the number operator of each frequency modes in Refs. \[44\], we reached to the conclusion that the measurement of the expectation value of the operator $\hat{b}_b(\omega)$ by the balanced homodyne detection is impossible. Therefore, we had to consider the eight-port homodyne detection which enable us to measure the expectation value of the operator $\hat{b}_b(\omega)$. Together with the ingredients of this Letter, this indicates that the choice of the directly measured operator at the photodetector affects the result not only of the noise properties but also the output signal expectation values themselves. Therefore, we conclude that the specification of the directly measured operator is crucial in the development of quantum measurement theory. Details of the derivation of our formulae will be seen in elsewhere \[11\].

Acknowledgments

The author deeply acknowledges to Prof. Masa-Katsu Fujimoto for valuable comments and his continuous encouragement. The author also acknowledges to Prof. Takayuki
Tomaru, and Prof. Tomotada Akutsu, Prof. Shinji Miyoki (ICRR, Tokyo Univ.), and Prof. Osamu Miyakawa (ICRR, Tokyo Univ.), and the other members of the GWSP in NAOJ for their continuous encouragement and discussions to this research.

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