PATH INTEGRAL DISCUSSION
OF TWO AND-THREE-DIMENSIONAL
$\delta$-FUNCTION PERTURBATIONS

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Abstract. The incorporation of two- and three-dimensional $\delta$-function perturbations into the path-integral formalism is discussed. In contrast to the one-dimensional case, a regularization procedure is needed due to the divergence of the Green-function $G^{(V)}(x, y; E)$, $(x, y \in \mathbb{R}^2, \mathbb{R}^3)$ for $x = y$, corresponding to a potential problem $V(x)$. The known procedure to define proper self-adjoint extensions for Hamiltonians with deficiency indices can be used to regularize the path integral, giving a perturbative approach for $\delta$-function perturbations in two and three dimensions in the context of path integrals. Several examples illustrate the formalism.

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1. Introduction

δ-function perturbations play an important rôle in quantum physics because they can serve as simple models for more complicated interactions and, more important, they are in many cases explicitly solvable [3]. They are also called point interactions (e.g. the Fermi point interaction [9]) in nuclear physics, solid state physics, and for the other fundamental interactions. The incorporation of a δ-function interaction in one-dimensional models causes no serious problems, and is usually discussed as a simple solvable quantum model in many textbooks in quantum mechanics (e.g. [53]).

The construction of the propagator can be done by a perturbative approach in the path integral. Let $W(x) = V(x) - \gamma \delta(x-a) \ (a, x \in \mathbb{R})$, and we consider the δ-function as a perturbation of the potential $V$. Then

$$G^{(\delta)}(x'', x'; E) = G^{(V)}(x'', x'; E) + (\Gamma^{(V)}(E))^{-1} G^{(V)}(x'', a; E) G^{(V)}(a, x'; E) \quad (1.1)$$

with $\Gamma^{(V)} = 1/\gamma - G^{(V)}(a, a; E)$, and $G^{(V)}(E)$ and $G^{(\delta)}(E)$ are the Green functions (resolvent kernel) of the unperturbed, respectively perturbed problem. The general structure of (1.1) is also known as Krein’s formula [3].

A comprehensive approach can be found in [32], including numerous examples, based on earlier work [7, 27, 51], and, of course, on the monograph of Albeverio et al. [3]. Considering the limit $\gamma \to -\infty$, i.e. making the strength of the δ-function perturbation infinitely repulsive, gives a boundary value problem with Dirichlet boundary-conditions at the boundary $x = a$, therefore explicitly incorporating boundary problems in the path integral [15, 16, 18, 36, 37]. Note that for a symmetrical model, i.e. $V(x) = V(-x)$, a δ-function perturbation at $x = 0$ yields in the limit $\gamma \to \infty$ a doubly generated energy level spectrum with one “spurious ground state” $\Psi_0$ with infinite negative energy and a corresponding wave-function concentrated at $x = 0$, i.e. $|\Psi_0(x)|^2 = \delta(x)$ (e.g. [4] and references therein).

One-dimensional examples, however, describe only quantum mechanical models on a line, respectively models which can be reduced to one-dimensional models, e.g. radial problems. More realistic ones must be at least two-, respectively three-dimensional. They can represent lattices of interaction centers in thin films (e.g. [57]) and solids, respectively, and can be chosen arbitrarily located with arbitrary strength.

Historically, point interactions were introduced in nuclear physics starting in the 1930s. But this type of interactions appear also in many other branches of physics, electromagnetic interactions with vector potential, etc. Also very important, they model lattices in solid state physics, yielding in the limit of infinitely many interaction centers an electronic band structure. Often the notion of a δ-function perturbation is replaced by the notion of particular boundary-conditions which the δ-function perturbations actually describe, c.f. [3] for a comprehensive bibliography on this subject.

The purpose of this paper is to incorporate such models into the path integral formalism. As we will see, a regularization procedure will be needed. The origin of the emerging divergence can be easily seen. A perturbation expansion consideration
yields an expression, where the Green function \( G^{(V)}(E) \) of the unperturbed problem, say for the free particle Green function, must be evaluated at both arguments being equal, i.e. “\( G^{(V)}(x, x; E) \)”. In two dimensions Green functions generally diverge logarithmically in this case, i.e. \( \lim_{x \to y} G(x, y; E) \propto \ln |x - y| \), whereas in three dimensions there is a simple pole, i.e. \( \lim_{x \to y} G(x, y; E) \propto |x - y|^{-1} \). Consequently a regularization procedure is necessary. It consists basically on a proper definition of the corresponding Friedrich extension, making the perturbed Hamiltonian self-adjoint.

What has to be done is the following: “In physical terms, the coupling constant \( \lambda \) in the heuristic expression \(-\Delta + \lambda \delta_y \) has to be ‘renormalized’ and turns out to be of the form \( \lambda = \eta + \alpha \eta^2 \), with \( \eta \) infinitesimal and \( \alpha \in (-\infty, \infty) \). [3, p.3]”, a topic which was first discussed by Berezin and Faddeev [8]. To put it into the language of approximating, the \( \delta \)-function perturbation is rewritten in terms of a Gaussian packet \( \psi \) with parameter \( \epsilon \), i.e. \( \psi_\epsilon(x) = e^{-(x-a)^2/\epsilon} / \sqrt{\pi \epsilon} \), and for \( D = 3 \), say, \( \lambda \) in the expression \( \lambda \psi(x) \psi(y) \psi(z) \) has to be explicitly dependent on \( \epsilon \), i.e. \( \lambda = \lambda(\epsilon) \). However, this is only necessary for \( D = 2, 3 \). For \( D = 1 \) no regularization is needed, and for \( D \geq 4 \), the corresponding Hamiltonian is already self-adjoint [58], c.f. the note in the summary.

This kind of regularization procedure is known from functional analysis [3] to define the corresponding proper self-adjoint Hamiltonian, and our purpose is to show that it can also be applied to regularize the corresponding path integral formulation. We will find that the entire Green function has to be taken into account in the regularization procedure, and not only part of it (compare [50]).

Such regularization procedures are well-known in other branches of mathematical physics, let alone high energy physics, respectively renormalization and quantum field theory. For instance, in quantum mechanics let us note the Selberg trace formula [35, 41, 62, 64], where for non-compact Riemann surfaces, the trace of a properly chosen operator-valued function for the Laplacian on the non-compact Riemann surface must be regulated by a subtraction of Eisenstein series representing the continuous spectrum. This regularization does not simply cancels the continuous part of the trace and naïvely gives zero-terms in the trace formula, but instead leads to inadmissible contributions, important in the theory of the Selberg trace formula, and related fields, e.g. in number theory.

The further content of this paper will be as follows. In the next section the theory of incorporating two- and three-dimensional \( \delta \)-function perturbations into the path integral will be presented. The regularization procedure will be applied for the perturbation expansion. In the third section, several examples illustrate the formalism, and will include the free particle, the harmonic oscillator, and the Coulomb potential \((1/r-)\) case on the one hand. On the other, some models for nuclear physics potentials will be given. The fourth sections concludes with a summary and a discussion of the results. In appendix 1 the computation of the Green function for the one-dimensional harmonic oscillator is sketched, and in appendix 2 the evaluation of the propagator for a free particle with a point interaction in two dimension will be given.
2. Time ordered perturbation expansion of the path integral and perturbation summation for $\delta$-function potentials

The general method for the time-ordered perturbation expansion is simple. We assume that we have a potential $W(x) = V(x) + \tilde{V}(x)$ ($x \in \mathbb{R}^D$) in the path integral, where it is assumed that $W$ is so complicated that a direct path integration is not possible. However, the path integral corresponding to $V(x)$ is assumed to be known, which we call $K^{(V)}(T)$. We expand the path integral containing $V(x)$ in a perturbation expansion about $\tilde{V}(x)$ in the following way. The initial kernel corresponding to $V$ propagates in $\epsilon$-time unperturbed, then it is interacting with $\tilde{V}$, propagates again in another $\epsilon$-time unperturbed, a.s.o., up to the final state. Let us denote the path integral (Feynman kernel, respectively) corresponding to the potential $V$ by $(T = t'' - t')$

$$K^{(V)}(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}^D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} .$$

(2.1)

We introduce the (energy-dependent) Green function (resolvent kernel)

$$G^{(V)}(x'', x'; E) = \frac{i}{\hbar} \int_0^\infty dT \ e^{iET/\hbar} K^{(V)}(x'', x'; T)$$

$$K^{(V)}(x'', x'; T) = \frac{1}{2\pi i} \int_{-\infty}^\infty G^{(V)}(x'', x'; E) e^{-iET/\hbar} dE .$$

(2.2)

This gives the series expansion (see also e.g. [7, 23, 27, 32, 51, 61])

$$K(x'', x'; T)$$

$$= \int_{x(t') = x'} \mathcal{D}^D x(t') \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) - \tilde{V}(x) \right] dt \right\}$$

$$= K^{(V)}(x'', x'; T) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \frac{1}{n!} \prod_{j=1}^{n} \int_{\mathbb{R}^D} d^D x_j \int_{t'}^{t''} dt_j$$

$$\times K^{(V)}(x_1, x'_1; t_1) \tilde{V}(x_1) K^{(V)}(x_2, x'_1; t_2 - t_1) \times \ldots$$

$$\ldots \times \tilde{V}(x_{n-1}) K^{(V)}(x_n, x_{n-1}; t_n - t_{n-1}) \tilde{V}(x_n) K^{(V)}(x'', x_n; T - t_n)$$

$$= K^{(V)}(x'', x'; T) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \prod_{j=1}^{n} \int_{t'}^{t_{j+1}} dt_j \int_{\mathbb{R}^D} d^D x_j$$

$$\times K^{(V)}(x_1, x'_1; t_1 - t') \tilde{V}(x_1) K^{(V)}(x_2, x'_1; t_2 - t_1) \times \ldots$$
\[ \cdots \times \tilde{V}(x_{n-1})K^{(V)}(x_n, x_{n-1}; t_n - t_{n-1})\tilde{V}(x_n)K^{(V)}(x', x_n; t' - t_n) \]. \quad (2.3) \\

In the second step we have ordered the time as \( t' = t_0 < t_1 < t_2 < \cdots < t_{n+1} = t'' \) and paid attention to the fact that \( K^{(V)}(t_j - t_{j-1}) \) is different from zero only if \( t_j > t_{j-1} \).

We consider now an arbitrary potential \( V(x) \) in one dimension with an additional \( \delta \)-function perturbation so that [32]

\[ W(x) = V(x) - \gamma \delta(x - a) \]. \quad (2.4) \\

The path integral for this potential problem reads

\[ K^{(W)}(x'', x'; T) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - W(x) \right] dt \right\}. \quad (2.5) \]

Introducing the Green function \( G(E) \) of the perturbed system similarly to (2.2), and, for the time being, we assume that \( G^{(V)}(a, a; E) \) exists. We then obtain due to the convolution theorem of the Laplace-Fourier transformation

\[ G^{(\delta)}(x'', x'; E) = G^{(V)}(x'', x'; E) - \frac{G^{(V)}(a', a; E)G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E) - 1/\gamma}. \quad (2.6) \]

Let us consider \( \delta \)-function perturbations in two and three dimensions. A formal series summation produces a formula similar to (2.6). However, a simple example, say the free particle Green function \( G^{(0)}(x'', x'; E) \), shows that the denominator for \( D > 1 \) does not exist. One must circumvent the arising divergence and must regularize properly. One chooses the coupling “\( \gamma \)” appropriately, say, according to

\[ \frac{1}{\gamma} \mapsto \alpha + G^{(V)}(a, b; 0), \quad (2.7) \]

and considers the expression \( G^{(V)}(a, b; 0) - G^{(V)}(a, b; E) \) in the limit \( b \rightarrow a \), say. We see that in some sense the coupling \( \gamma \) has to be zero in a “suitable way” in order to make the final expression well defined.

This formal reasoning can be put more rigorously [3]. As a simple example we start with the free Hamiltonian in \( \mathbb{R}^3 \). It is transformed via a momentum Fourier transformation into a multiplication operator, so that \( H = \hbar^2 |p|^2/2m \). A cut-off \( \omega \) is introduced, and the coupling \( \gamma \) is made explicitly dependent on the cut-off, i.e. \( \gamma = \gamma(\omega) \). All what remains is to choose \( \gamma(\omega) \) “in such a way that the [perturbed operator with cut-off \( \omega \)] \( \tilde{H}^\omega \) has a reasonable and nontrivial limit \( [\tilde{H}] \) as we remove the cut-off, i.e. as \( \omega \) tends to infinity.” [3, p.111]. For the free particle case we chose

\[ \frac{1}{\gamma(\omega)} = \alpha + \frac{2m}{\hbar^2 (2\pi)^3} \int_{|p| \leq \omega} \frac{d^3p}{|p|^2} = \alpha + \frac{m}{2\pi \hbar^2} \omega, \quad (2.8) \]
with $\alpha \in \mathbb{R}$. We introduce the eigen-functions $\phi_{\mathbf{y}}(\mathbf{p}) = (2\pi)^{-3/2} e^{-i \mathbf{p} \cdot \mathbf{y}}$ of the momentum operator, and the corresponding cut-off eigen-functions $\phi_{\mathbf{y}}^\omega = \chi_{\omega} \phi_{\mathbf{y}}$, where

$$
\chi_{\omega}(\mathbf{p}) = \begin{cases} 1 & |\mathbf{p}| \leq \omega, \\ 0 & |\mathbf{p}| > \omega. \end{cases} \quad (2.9)
$$

Then ($E = \hbar^2 |\mathbf{k}|^2 / 2m$)

$$
\frac{1}{\gamma(\omega)} - \frac{2m}{\hbar^2} \left( \phi_{\mathbf{y}}^\omega, \frac{1}{|\mathbf{p}|^2 - |\mathbf{k}|^2} \phi_{\mathbf{y}}^\omega \right) = \alpha + \frac{2m}{\hbar^2} \frac{1}{(2\pi)^3} \int_{|\mathbf{p}| \leq \omega} d^3 \mathbf{p} \left( \frac{1}{|\mathbf{p}|^2} - \frac{1}{|\mathbf{p}|^2 - |\mathbf{k}|^2} \right)
$$

$$(\omega \to \infty)
$$

$$
= \alpha + \frac{m}{2\pi \hbar^3} \sqrt{-2mE}. \quad (2.10)
$$

It follows that the perturbed cut-off operator $\hat{H}^\omega$ (in $\mathbf{p}$-space) converges in norm resolvent sense to a self-adjoint operator, respectively the perturbed cut-off operator $e^{-i \hat{H}^\omega T / \hbar}$ to a unitary operator, hence the path integral $K^{(\alpha)}(\mathbf{x}'', \mathbf{x}'; T) = \langle \mathbf{x}'' | e^{-i \hat{H}^\omega T / \hbar} | \mathbf{x}' \rangle$ exists and is well defined, with the corresponding resolvent kernel given by [3, Theorem 1.1.1, p.113]

$$
G^{(\alpha)}(\mathbf{x}'', \mathbf{x}'; E) = G^{(0)}_3(\mathbf{x}'', \mathbf{x}'; E) + G^{(0)}_3(\mathbf{x}'', \mathbf{a}; E)G^{(0)}_3(\mathbf{a}, \mathbf{x}'; E)
$$

$$
\times \lim_{\omega \to \infty} \sum_{n=1}^{\infty} \left( \frac{\gamma(\omega)}{\hbar} \right)^n \left[ \frac{2m}{\hbar^2} \phi_{\mathbf{y}}^\omega, \frac{1}{|\mathbf{p}|^2 - |\mathbf{k}|^2} \phi_{\mathbf{y}}^\omega \right]^{n-1}
$$

$$
= G^{(0)}_3(\mathbf{x}'', \mathbf{x}'; E) + \lim_{\omega \to \infty} \frac{G^{(0)}_3(\mathbf{x}'', \mathbf{a}; E)G^{(0)}_3(\mathbf{a}, \mathbf{x}'; E)}{\frac{1}{\gamma(\omega)} - \frac{2m}{\hbar^2} \phi_{\mathbf{y}}^\omega, \frac{1}{|\mathbf{p}|^2 - |\mathbf{k}|^2} \phi_{\mathbf{y}}^\omega}
$$

$$
= G^{(0)}_3(\mathbf{x}'', \mathbf{x}'; E) + \frac{G^{(0)}_3(\mathbf{x}'', \mathbf{a}; E)G^{(0)}_3(\mathbf{a}, \mathbf{x}'; E)}{\Gamma^{(0)}_{\alpha, \mathbf{a}}}, \quad (2.11a)
$$

$$
\Gamma^{(0)}_{\alpha, \mathbf{a}} = \alpha + \frac{m}{2\pi \hbar^3} \sqrt{-2mE}, \quad (2.11b)
$$

$$
G^{(0)}_3(\mathbf{x}'', \mathbf{x}'; E) = \frac{m}{4\pi \hbar^2 |\mathbf{x}'' - \mathbf{x}'|} \exp \left( -\frac{\sqrt{-2mE}}{\hbar} |\mathbf{x}'' - \mathbf{x}'| \right). \quad (2.11c)
$$

$G^{(0)}_3(\mathbf{x}''; E)$ denotes the free particle Green function in three dimensions. Therefore we have obtained the three-dimensional generalization of (2.6) for $V \equiv 0$ and we conclude that the perturbation expansion (2.3) together with the regularization (2.10) is well defined and (2.11) justifies our perturbation expansion approach, respectively its formal series summation (2.6) generalized to the three-dimensional case. This is in accordance from general considerations that the perturbed Green function for a
\(\delta\)-function perturbation located at \(x = a\) \((a \in \mathbb{R}^2, \mathbb{R}^3)\) must look like (2.11) (Krein’s formula [3], appendix A)

Let us note that the quantity \(-1/4\pi\alpha\) represents the scattering length of the perturbed operator.

However, an analogous line of reasoning is valid for the incorporation of potentials \(V(x)\). Instead of performing the usual Fourier transformation into \(p\)-space, transforming the free Hamiltonian into a multiplication operator, we perform a generalized Fourier transformation (decomposition) in terms of its spectral expansion, taking the corresponding eigen-functions \(\phi_{E_\lambda}\) with energy \(E_\lambda\) and momentum \(p_{\lambda}^2 = 2mE_\lambda/h^2\), yielding a generalization of (2.11), with \(G^{(0)}(E)\) replaced by \(G^{(V)}(E)\).

Another way to determine \(\Gamma^{(V)}(a, \gamma)(E)\), which also includes potential problems in a more constructive way, exploits that the Hilbert space of the perturbed problem must be properly defined, i.e. one has to find the correct self-adjoint extension of the heuristic operator ([2, 3, 12, 19] and references therein):

\[
H_\gamma = -\frac{\hbar^2}{2m}\Delta + V(x) + "\gamma \delta(x - a)" ,
\]

and \(\Delta\) denotes the Laplace-Beltrami operator in two, respectively three dimensions. Let \(g(r)\) be a solution of the corresponding minimal (reduced) radial s-wave Schrödinger operator \(\left(\frac{1}{2} \leq \lambda < \frac{3}{2}, \beta, \eta \in \mathbb{R}, 0 < a < 2\right)\)

\[
\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\lambda(\lambda - 1)}{r^2} + \frac{\eta}{r} + \beta r^{-\alpha} + \hat{V}(r), \quad D(\hat{H}) = C^\infty_0(\mathbb{R}^+) , \quad (2.13)
\]

where the notion \(\hat{H}\) denotes the incorporation of a \(\delta\)-function perturbation at \(r = 0\), say \(\hat{H}\) has deficiency indices \((1, 1)\), and \(\hat{V}(r) \in L^\infty(\mathbb{R}^+)\). We then can formulate the following theorem:

**Theorem [3, 12]** Let

\[
F^{(0)}_\lambda(r) = r^\lambda, \quad G^{(0)}_\lambda(r) = \begin{cases} -\frac{m}{\pi\hbar^2}\sqrt{r}\ln r & \lambda = \frac{1}{2}, \\ \frac{m}{2\pi\hbar^2} & \lambda = 1 \end{cases}, \quad (2.14)
\]

All self-adjoint extensions of the operator \(\hat{H}\) are given by

\[
H_\nu = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\lambda(\lambda - 1)}{r^2} + \frac{\eta}{r} + \beta r^{-\alpha} + \hat{V}(r) ,
\]

\[
D(H_\nu) = \left\{ g \in L^2(\mathbb{R}^+) \mid g, g' \in AC_{loc}(\mathbb{R}^+); \alpha g_{0,\lambda} = g_{1,\lambda}; H_\nu g \in L^2(\mathbb{R}^+) \right\} \quad (2.15a)
\]

\[
-\infty < \alpha \leq \infty, \quad \frac{1}{2} \leq \lambda < \frac{3}{2}, \quad \beta, \eta \in \mathbb{R}, \quad 0 < a < 2 \quad (2.15b)
\]

\[
\alpha g_{0,\lambda} = g_{1,\lambda}; H_\nu g \in L^2(\mathbb{R}^+) \quad (2.15c)
\]

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\[ \dot{V} \in L^\infty(\mathbb{R}^+) \] real valued, and \( \text{AC}_{\text{loc}}(\mathbb{M}) \) denotes the set of absolutely continuous functions on \( \mathbb{M} \) (here = \( \mathbb{R}^+ \)). The boundary values \( g_{0,\lambda}, g_{1,\lambda} \) are defined by

\[
g_{0,\lambda} = \lim_{r \to 0^+} \frac{g(r)}{G_{\lambda}^{(0)}(r)}, \quad g_{1,\lambda} = \lim_{r \to 0^+} \frac{g(r) - g_{0,\lambda} G_{\lambda}^B(r)}{F_{\lambda}^{(0)}(r)}, \tag{2.16}
\]

where \( G_{\lambda}^B(r) \) denotes the asymptotic expansion of the irregular solution \( G_{\lambda}(r) \) of (2.15) up to order \( r^t, t \leq 2\lambda - 1 \).

Generally one has \( G_{\lambda}^B(r) = G_{\lambda}^{(0)}(r) + \) additional terms. Two special cases of \( G_{\lambda}^B(r) \) can be e.g. stated for the \( \lambda = \frac{1}{2}, 1 \), i.e. for the Schrödinger operator in two and three dimensions, respectively, and all other potential terms in (2.15) equal to zero, which will be sufficient for our purposes. Then

\[
G_{\frac{1}{2}}^B(r) = G_{\frac{1}{2}}^{(0)}(r) = -\frac{m}{\pi \hbar^2} \sqrt{r \ln r}, \tag{2.17}
\]

\[
G_1^B(r) = \frac{m}{2\pi \hbar^2} \left( 1 - \frac{m\hbar}{\hbar^2} r - \frac{2m\eta}{\hbar^2} r \ln r \right). \tag{2.18}
\]

Take into account the \( \beta \neq 0 \) contributions considerably complicates the expressions and will not be stated here, c.f [12].

We consider \( \lambda = (D - 1)/2 \) and usually we identify \( F_\lambda \equiv F_D, G_\lambda \equiv G_D, \) etc. in the following. Note that the additional potential \( \dot{V}(r) \) does not play any further rôle in the regularization. The condition \( \frac{1}{2} \leq \lambda < \frac{3}{2} \) can be physically interpreted as follows: Set \( N = \lambda - \frac{1}{2} \), then \( \lambda(\lambda - 1) = N^2 - 1/4 \) and the functional weight in the radial path integral corresponding to the Hamiltonian (2.15) [38] is proportional to \( \mu_N |r^2| \). The number \( N \) corresponds to the magnetic flux inside an infinitely thin Aharonov-Bohm solenoid. Only fluxes less than unity are allowed and there exists exactly one bound state solution and the parameter of the self-adjoint extension describes the anomaly of the magnetic moment of the particle bound by the \( \delta \)-function-like solenoid [10].

Summarizing, we can state that the result of the regularization procedure (2.15) of the formal series summation (2.6) of the path integral perturbation expansion (2.3) with the heuristic expression "\( \gamma \delta(x - a) \)" in the Lagrangian in the path integral in two and three dimensions yields in an analogous way as the consideration in (2.11) the Green function

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{i ET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \left. D^D x(t) \right|_{1/\gamma - \lim_{x,y \to a} G^{(0)}(x,y;E) \mapsto \alpha_{g_{0,D}^a - g_{1,D}^a}} \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \gamma \delta(x - a) \right] dt \right\}
\]

\[
\equiv \frac{i}{\hbar} \int_0^\infty dT \ e^{i ET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \left. D^D_{1_{V,a}} x(t) \right| \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \gamma \delta(x - a) \right] dt \right\}
\]
\[ G^{(V)}(x'', x'; E) + \frac{G^{(V)}(x'', a; E)G^{(V)}(a, x'; E)}{\Gamma^{(V)}_{\alpha, a}} . \]  

(2.19)

with \( \Gamma^{(V)}_{\alpha, a} \) given by

\[ \Gamma^{(V)}_{\alpha, a} = \alpha g_{0,D} - g_{1,D} . \]  

(2.20)

and we define the heuristic point-interaction potential \( \gamma \delta(x - a) \) in the path integral by means of (2.19,2.20) with the regularization prescription (2.15).

A convenient way to find the function \( g(r) \) is to set \( g(r) = \Omega^{-1}(D)G_{t=0}^{(V),red.}(r, r; E) = \Omega^{-1}(D)r^\lambda G^{(V)}_{t=0}(r, r; E) \), where \( V \) includes all potential terms in (2.13). Here \( \Omega(D) \) denotes the volume of the unit sphere \( S^{(D-1)} \).

In three dimensions it is also possible to state the regularization rule as follows [3]

\[ \frac{i}{\hbar} \int_0^\infty dT \, e^{i ET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} D_{\Gamma^{(V)}_{\alpha, a}}^D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \gamma \delta(x - a) \right] dt \right\} = G^{(V)}(x'', x'; E) + \left( \Gamma^{(V)}_{\alpha, a}(E) \right)^{-1} G^{(V)}(x'', a; E)G^{(V)}(a, x'; E) , \]  

(2.21)

with \( \Gamma^{(V)}_{\alpha, a} \) given by

\[ \Gamma^{(V)}_{\alpha, a} = \alpha + \frac{\partial}{\partial r_{12}} G^{(V)}(x'', x'; E) \Big|_{r_{12} = |x'' - x'| = 0} , \]  

(2.22)

provided an explicit expression for \( G^{(V)}(x'', x'; E) \) exists.

3. Examples

3.1. One-dimensional examples

For completeness we refer some one-dimensional examples. This includes the free particle, the harmonic oscillator and the Coulomb potential. No new material is presented. The results will give an easy possibility to compare the results in one, two and three dimensions with each other, respectively to introduce some notation. We also introduce the Green functions for the free particle, the harmonic oscillator, and the Coulomb potential. Of course, no regularization procedure is needed.

Let us note that an implicit equation for the time-dependent propagator corresponding to (1.1) is due to Gaveau and Schulman [25]. They obtained

\[ K^{(\delta)}(x'', x'; T) = K^{(V)}(x'', x'; T) + i \frac{\gamma}{\hbar} \int_{t'}^{t''} K^{(V)}(x'', a; t)K^{(\delta)}(a, x'; T - t) dt . \]  

(3.1)
3.1.1. The free particle. Let us start with the free particle (labeled by “(0)”). The
path integral of the $D$-dimensional free particle reads as $(x \in \mathbb{R}^D)$

$$K^{(0)}(x'', x'; T) = \int_{x(t')=x'} D^D x(t) \exp \left( \frac{i m}{2\hbar} \int_{t'}^{t''} x^2 dt \right)$$

$$= \left( \frac{m}{2\pi i \hbar T} \right)^{D/2} \exp \left[ - \frac{m}{2i \hbar T} (x'' - x')^2 \right]$$

$$= \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} d^D p \exp \left[ i p \cdot (x'' - x') - i T \frac{\hbar |p|^2}{2m} \right]. \quad (3.2)$$

The energy dependent kernel is given by

$$G_D^{(0)}(x'', x'; E) = \frac{i}{\hbar} \int_0^\infty dT \ e^{i E T / \hbar} \int_{x(t')=x'} D^D x(t) \exp \left( \frac{i m}{2\hbar} \int_{t'}^{t''} x^2 dt \right)$$

$$= 2 \frac{i}{\hbar} \left( \frac{m}{2\pi i \hbar} \right)^{D/2} \left( \frac{m}{2E} |x'' - x'|^2 \right)^{\frac{1}{2}(1-D/2)} K_{1-D/2} \left( \frac{|x'' - x'|}{\hbar} \right)$$

$$= \frac{i}{\hbar} \left( \frac{m}{2\hbar} \right)^{D/2} \left( \frac{m(-\pi)^2}{2E} |x'' - x'|^2 \right)^{\frac{1}{2}(1-D/2)} H_{1-D/2} \left( i \frac{|x'' - x'|}{\hbar} \sqrt{2mE} \right). \quad (3.3)$$

For $D = 1$ one has

$$G_1^{(0)}(x'', x'; E) = \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \exp \left( - \frac{|x'' - x'|}{\hbar} \sqrt{2mE} \right). \quad (3.4)$$

From the general theory we now have [7, 27, 32, 51, 52]

$$\frac{i}{\hbar} \int_0^\infty dT \ e^{i E T / \hbar} \int_{x(t')=x'} D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \gamma \delta(x - a) \right] dt \right\}$$

$$= \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \exp \left( - \frac{\sqrt{-2mE}}{\hbar} |x'' - x'| \right)$$

$$\exp \left[ - \frac{\sqrt{-2mE}}{\hbar} (|x'' - a| + |a - x'|) \right]$$

$$+ \frac{m\gamma}{2\hbar^2} \sqrt{-E} \left( \frac{\gamma}{\hbar} \frac{m}{2} \right). \quad (3.6)$$

To determine the propagator from (3.6) the inverse Laplace-Fourier transformation
must be applied, which can be explicitly done with result [27]

$$\int_{x(t')=x'} D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \gamma \delta(x - a) \right] dt \right\}$$
\[\sqrt{\frac{m}{2\pi i\hbar T}} \exp \left[ \frac{i m}{2\hbar T}(x'' - x')^2 \right] + \frac{m\gamma}{2\hbar^2} \exp \left( -\frac{m\gamma}{\hbar^2}(|x''| + |x'| + \frac{i m\gamma^2}{\hbar}) \right) \times \text{erfc} \left( \sqrt{\frac{m}{2i\hbar T}}(1 + |x''| + |x'| - \frac{i \gamma}{\hbar}) \right). \quad (3.7)\]

### 3.1.2. The harmonic oscillator

The harmonic oscillator (labeled by “(ω)”) path integral is given by [23]

\[
x(t'') = x'' \\
\int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left[ \frac{i m}{2\hbar} \int_{t'}^{t''} (\dot{x}^2 - \omega^2 x^2)dt \right]
= \left( \frac{m\omega}{2\pi i\hbar \sin \omega T} \right)^{1/2} \exp \left\{ -\frac{m\omega}{2i\hbar} \left( x''^2 + x'^2 \cos \omega T - 2x'x'' \right) \right\}. \quad (3.8)
\]

We do not discuss the case of caustics etc. The energy dependent kernel can be evaluated to be given by (c.f. [5], and appendix 1)

\[
\frac{i}{\hbar} \int_0^\infty d\tau \ e^{i\tau E/\hbar} \int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left[ \frac{i m}{2\hbar} \int_{t'}^{t''} (\dot{x}^2 - \omega^2 x^2)dt \right]
= \sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar}} x' \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( -\sqrt{\frac{2m\omega}{\hbar}} x'' \right). \quad (3.9)
\]

The general theory then gives for a perturbed harmonic oscillator [32] \((x'' \geq a \geq x')\):

\[
\frac{i}{\hbar} \int_0^\infty d\tau \ e^{i\tau E/\hbar} \int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \ddot{x}^2 - \frac{m}{2} \omega^2 x^2 + \gamma \delta(x - a) \right] dt \right\}
= \sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( -\sqrt{\frac{2m\omega}{\hbar}} x'' \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar}} x' \right) \times \left[ 1 - \frac{\gamma}{\hbar} \sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) \right]^{-1}, \quad (3.10)
\]
(x'' ≥ x' ≥ a):

\[
i \frac{1}{\hbar} \int_0^\infty dT \ e^{iT E/\hbar} \int_{x(t') = x'}^{x(t'') = x''} D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{m}{2} \omega^2 x^2 + \gamma \delta(x - a) \right] dt \right\}
\]

\[= \sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x'} D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar} x''} \right) \right) \]

\[+ \frac{m \gamma}{\pi \omega^3} D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar} a} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x'} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x''} \right) \]

\[\times \left[ 1 - \frac{\gamma}{\hbar} \sqrt{\frac{m}{\pi \hbar \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) \right]^{-1}, \quad (3.11)\]

(x'' ≤ x' ≤ a):

\[
i \frac{1}{\hbar} \int_0^\infty dT \ e^{iT E/\hbar} \int_{x(t') = x'}^{x(t'') = x''} D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{m}{2} \omega^2 x^2 + \gamma \delta(x - a) \right] dt \right\}
\]

\[= \sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x'} D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar} x''} \right) \right) \]

\[+ \frac{m \gamma}{\pi \omega^3} D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar} a} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x'} \right) D_{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( - \sqrt{\frac{2m\omega}{\hbar} x''} \right) \]

\[\times \left[ 1 - \frac{\gamma}{\hbar} \sqrt{\frac{m}{\pi \hbar \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) \right]^{-1}. \quad (3.12)\]

3.1.3. The Coulomb potential. The Coulomb potential (labeled by “(C)” in one dimension is also know as the Kratzer potential. Its corresponding Green function via path integration can be done by considering the D-dimensional Coulomb potential, say, and then restricting to D = 1 [17]. Of course, the problem is closely related to a Kustaanheimo-Stiefel transformation in the in the path integral, hence to the space-time transformation technique in path integrals. This has been discussed by many authors, let us mention [17, 21, 33, 38, 42, 44, 47, 54, 63] and references therein. A comprehensive survey will be given in [40].

We now have (x > 0, κ = q_1 q_2 \sqrt{-m/2E/\hbar})

\[i \frac{1}{\hbar} \int_0^\infty dT \ e^{iT E/\hbar} \int_{x(t') = x'}^{x(t'') = x''} D x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{x}^2 + \frac{q_1 q_2}{|x|} \right) dt \right\} \]
\[
\frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \Gamma(1-\kappa) W_{\kappa,1/2} \left( \sqrt{-8mE \frac{x}{\hbar}} \right) M_{\kappa,1/2} \left( \sqrt{-8mE \frac{x}{\hbar}} \right).
\]

(3.13)

For a \( \delta \)-perturbed \( 1/r \)-potential we then obtain \( [32] \) \( (\kappa = q_1 q_2 \sqrt{-m/2E}/\hbar, x'' \geq a \geq x') \)

\[
\frac{i}{\hbar} \int_0^\infty dT e^{iT E/\hbar} \int_{x(t') = x'}^\infty Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \frac{q^2}{x} + \gamma \delta(x-a) \right] dt \right\}
\]

\[
= \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \Gamma(1-\kappa) W_{\kappa,1/2} \left( \frac{x''}{\hbar} \sqrt{-8mE} \right) M_{\kappa,1/2} \left( \frac{x'}{\hbar} \sqrt{-8mE} \right)
\]

\[
\times \left[ 1 - \frac{\gamma}{\hbar^2} \sqrt{-\frac{m}{2E}} \Gamma(1-\kappa) W_{\kappa,1/2} \left( \frac{a}{\hbar} \sqrt{-8mE} \right) M_{\kappa,1/2} \left( \frac{a}{\hbar} \sqrt{-8mE} \right) \right]^{-1}
\]

(3.14)

and similarly as in (3.11,3.12) for the other cases.

3.1.4. Multiple \( \delta \)-function perturbations. For completeness we also cite the case of a multiple \( \delta \)-function perturbation. Since only the Green function \( G^{(V)}(E) \) of the unperturbed problem is relevant, the incorporation of multiple \( \delta \)-function perturbations can be successively done with result \( [3, 28, 36] \)

\[
\frac{i}{\hbar} \int_0^\infty dT e^{iT E/\hbar} \int_{x(t') = x'}^\infty Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \sum_{j=1}^N \gamma_j \delta(x-a_j) \right] dt \right\}
\]

\[
= \frac{G^{(V)}(x'',x'; E) G^{(V)}(x'',a_1; E) \cdots G^{(V)}(x'',a_N; E)}{G^{(V)}(a_1,a_1; E) G^{(V)}(a_1,a_2; E) \cdots G^{(V)}(a_1,a_N; E) - 1/\gamma_1}
\]

\[
\vdots
\]

\[
= G^{(V)}(x'',x'; E) - \sum_{j,j'=1}^N \left( \Gamma^{(V)}_{\{\alpha\},\{a\}} \right)_{j,j'}^{-1} G^{(V)}(x'',a_j; E) G^{(V)}(a_{j'},x'; E)
\]

(3.16)

with the matrix \( \Gamma^{(V)}_{\{\alpha\},\{a\}} \) given by \( \{\alpha\} = \{\kappa\}_{k=1}^N, \{a\} = \{a_k\}_{k=1}^N \)

\[
\left( \Gamma^{(V)}_{\{\alpha\},\{a\}} \right)_{j,j'} = G^{(V)}(a_j,a_{j'}; E) - \frac{\delta_{jj'}}{\gamma_j}.
\]

(3.17)

The determinant and matrix representations are checked by induction [36].
3.2. Two-dimensional examples

3.2.1. The Free Particle. We keep the ordering schema of the first subsection and start with the free particle case. Form the general $D$-dimensional free particle Green function we consider $D = 2$ and obtain

$$G^{(0)}_2(x'',x';E) = \frac{m}{\pi \hbar^2} K_0 \left( \frac{\sqrt{-2mE}}{\hbar} |x'' - x'| \right),$$

which is logarithmically divergent for $|x'' - x'| \to 0$ due to $K_0(z) \propto -\ln(z/2) + \Psi(1)$ ($z \to 0$). Here $-\Psi(1) = 0.57721566490153286061\ldots$ denotes Euler’s constant. In order to regularize the problem we apply the theory of section 2. We set ($r > 0$)

$$g(r) = \frac{m}{\pi \hbar^2} \sqrt{r} K_0 \left( \frac{\sqrt{-2mE}}{\hbar} r \right),$$

from which follows

$$g_{0,2} = \frac{m}{\pi \hbar^2}, \quad g_{1,2} = \Psi(1) - \ln \frac{\sqrt{-2mE}}{2\hbar},$$

($x \in \mathbb{R}^2 \setminus \{0\}$). We therefore obtain (compare also [2, 3])

$$\frac{i}{\hbar} \int_0^\infty dT \ e^{iET/\hbar} \int_{x(t')=x'} \mathcal{D}_{\alpha,a}^2 x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \gamma \delta(x - a) \right] dt \right\}$$

$$= \frac{m}{\pi \hbar^2} K_0 \left( \frac{\sqrt{-2mE}}{\hbar} |x'' - x'| \right)$$

$$+ \left( \frac{m}{\pi \hbar^2} \right)^2 K_0 \left( \frac{\sqrt{-2mE}}{\hbar} |x'' - a| \right) K_0 \left( \frac{\sqrt{-2mE}}{\hbar} |a - x'| \right)$$

$$\alpha + \frac{m}{\pi \hbar^2} \ln \frac{\sqrt{-2mE}}{2\hbar} - \Psi(1)$$

$$= \frac{m}{\pi \hbar^2} K_0 \left( \frac{\sqrt{-2mE}}{\hbar} r_\ge \right) I_0 \left( \sqrt{-2mE} \frac{r_\le}{\hbar} \right)$$

$$+ \left( \frac{m}{\pi \hbar^2} \right)^2 K_0 \left( \frac{\sqrt{-2mE}}{\hbar} \frac{r'_\ge}{\hbar} \right) K_0 \left( \sqrt{-2mE} \frac{r'_\le}{\hbar} \right)$$

$$\alpha + \frac{m}{\pi \hbar^2} \ln \frac{\sqrt{-2mE}}{2\hbar} - \Psi(1)$$

$$+ \frac{m}{\pi \hbar^2} \sum_{l=1}^\infty e^{il(\phi'' - \phi')} K_l \left( \sqrt{-2mE} \frac{r_\ge}{\hbar} \right) I_l \left( \sqrt{-2mE} \frac{r_\le}{\hbar} \right).$$

(3.22)

and polar coordinates about $|x - a|$ have been used in the last line. The one bound state wave-function

$$\Psi^{(\alpha)} = \frac{1}{\sqrt{\pi}} K_0 \left( |x - a| e^{-2[\alpha \pi \hbar^2/2m - \Psi(1)]} \right)$$

(3.23)
has the energy
\[ E^{(\alpha)} = -\frac{2\hbar^2}{m} e^{-2[\alpha \pi \hbar^2/m - \Psi(1)]} \]  
\tag{3.24}

The normalization of the bound state wave-function follows from the integral representation [29, p.672]
\[
\int_0^\infty x K_\nu(ax) K_\nu(bx) dx = \begin{cases} 
\frac{\pi(ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2 \sin(\nu \pi) (a^2 - b^2)}, & \nu > 0, \\
\frac{1}{2}, & \nu = 0. 
\end{cases}
\tag{3.25}

3.2.2. The harmonic oscillator. In order to discuss the harmonic oscillator we need the radial Green function for the harmonic oscillator in \( D \) dimensions. We have [6, 26, 38, 55]
\[
(r' r'')^{1-D} \frac{i}{\hbar} \int_0^\infty d\mathcal{T} \ e^{iET/h} \\
\times \int_{r'(t')=r'}^{r(t')=r''} Dr(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} r'^2 - \hbar^2 \left( (l + \frac{D-2}{2})^2 - 1/4 \right) - \frac{m}{2} \omega^2 r'^2 \right) dt \right] \\
= \frac{\Gamma[\frac{1}{2}(l + \frac{D-2}{2}) - \frac{E}{\hbar \omega}]}{\hbar \omega (r' r'')^{D/2} \Gamma(l + D/2)} W_{\frac{m}{2} \omega^2, \frac{1}{2}(l + \frac{D-2}{2})} \left( \frac{m \omega}{\hbar} r^2 \right) M_{\frac{m}{2} \omega^2, \frac{1}{2}(l + \frac{D-2}{2})} \left( \frac{m \omega}{\hbar} r^2 \right). 
\tag{3.26}
\]

We consider \( D = 2 \). For the determination of the behaviour of the Whittaker functions for \( z \to 0 \) use has to been made of the representations [29, p.1059]
\[
M_{\lambda, \mu}(z) = z^{\mu+1/2} e^{-z/2} \ _1F_1 \left( \frac{1}{2} + \mu - \lambda; 2\mu + 1; z \right), 
\tag{3.27}
\]
\[
W_{\lambda, \mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \lambda)} M_{\lambda, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \lambda)} M_{\lambda, -\mu}(z) 
\tag{3.28}
\]
respectively [29, p.1063]
\[
W_{\lambda, \mu}(z) = \frac{(-1)^{2\mu} z^{\mu+1/2} e^{-z/2}}{\Gamma(\frac{1}{2} - \mu - \lambda) \Gamma(\frac{1}{2} + \mu - \lambda)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\mu + k - \lambda + \frac{1}{2})}{k!(2\mu + k)!} z^k \\
\times \left[ \Psi(k + 1) + \Psi(2\mu + k + 1) - \Psi(\mu + k - \lambda + \frac{1}{2}) - \ln z \right] \\
+ (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu - k) \Gamma(k - \mu - \lambda + \frac{1}{2})}{k!} (-z)^k \right\}. 
\tag{3.29}
\]

In the latter equation it is required that \( 2\mu \in \mathbb{N}_0 \), and the last term is understood to be ignored for \( \mu = 0 \). Putting the interactions center at the coordinate origin, we set
\[
g(r) = \frac{\Gamma(\frac{1}{2} - E/2\hbar \omega)}{2\pi \hbar \omega r} W_{E/2\hbar \omega, 0} \left( \frac{m \omega}{\hbar} r^2 \right) M_{E/2\hbar \omega, 0} \left( \frac{m \omega}{\hbar} r^2 \right), 
\tag{3.30}
\]

\[ 14 \]
and therefore we obtain due to the general theory

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{iE_T/\hbar} \int_{x(t')}^x D^2 \mathcal{G}_{r_{1,0}}(x(t)) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{M}{2} \omega^2 x^2 + \gamma \delta(x) \right] dt \right\}
\]

\[
= \frac{\Gamma(\frac{1}{2} - E/2\hbar\omega)}{2\pi^2 h^3 \omega r \Omega t'} W_E/2h\omega,0 \left( \frac{m\omega}{\hbar} r^> \right) W_E/2h\omega,0 \left( \frac{m\omega}{\hbar} r^< \right)
\]

\[
+ \frac{m\Gamma^2(\frac{1}{2} - E/2\hbar\omega)}{4\pi^2 h^3 \omega r \Omega t'} W_E/2h\omega,0 \left( \frac{m\omega}{\hbar} r^> \right) W_E/2h\omega,0 \left( \frac{m\omega}{\hbar} r^< \right)
\]

\[
\alpha + \frac{m}{2\pi h^2} \left[ \Psi \left( \frac{1}{2} - \frac{E}{2\hbar\omega} \right) + \ln \frac{m\omega}{\hbar} - 2\Psi(1) \right]
\]

\[
+ \sum_{l=1}^\infty e^{i(l(\phi'' - \phi') - \frac{l+1}{2} l^2 \hbar^2 \phi^2)} \frac{\Gamma\left( \frac{1}{2} \left( l + 1 - \frac{E}{2\hbar\omega} \right) \right)}{2\pi h\omega l r \Omega t'} W_E/2h\omega,1/2 \left( \frac{m\omega}{\hbar} r^> \right) W_E/2h\omega,1/2 \left( \frac{m\omega}{\hbar} r^< \right) \]

(3.31)

The bound state energy levels \( E_n^{(\alpha)} \) are determined by

\[
\alpha + \frac{m}{2\pi h^2} \left[ \Psi \left( \frac{1}{2} - \frac{E_n^{(\alpha)}}{2\hbar\omega} \right) + \ln \frac{m\omega}{\hbar} - 2\Psi(1) \right] = 0 \ . \quad (3.32)
\]

3.2.3. The Coulomb potential. In order to treat the two-dimensional Coulomb potential case, we must know the radial two-dimensional Coulomb Green’s function. We have (c.f. the references in the one-dimensional case)

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{iTE/\hbar} \int_{x(t')}^x D^D \mathcal{G}(x(t)) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \frac{q_1 q_2}{|x|} \right] dt \right\}
\]

\[
= \frac{1}{\hbar} \sum_{l=0}^\infty S^{(D)}_l(\Omega') S^{(D)}_l(\Omega')(r'^>) \left( \frac{r^<}{\hbar} \right)^{D-2} \sqrt{-\frac{m}{2E} \left( \Gamma(l + \frac{D-1}{2} - \kappa) \right)}
\]

\[
\times W_{\kappa,l+\frac{D-1}{2}} \left( \sqrt{-8mE} \frac{r^>}{\hbar} \right) M_{\kappa,l+\frac{D-1}{2}} \left( \sqrt{-8mE} \frac{r^<}{\hbar} \right) \quad (3.33)
\]

(\( \kappa = q_1 q_2 \sqrt{-m/2E/\hbar} \)) and the \( S^{(D)}_l(\Omega) \) are the (real) hyperspherical harmonics on the \( S^{(D-1)} \)-sphere taken at the unit vector \( \Omega \) on the sphere. We consider \( D = 2 \), and putting the interactions center at the coordinate origin, we obtain in the regularization procedure by means of (3.27,3.29)

\[
g(r) = \frac{1}{2\pi h} \sqrt{-\frac{m}{2E} \left( \frac{1}{2} - \kappa \right)} W_{\kappa,0} \left( \sqrt{-8mE} \frac{r^>}{\hbar} \right) M_{\kappa,0} \left( \sqrt{-8mE} \frac{r^<}{\hbar} \right) \ . \quad (3.34)
\]
Therefore we get for the δ-perturbed two-dimensional Coulomb potential problem

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{i E T / \hbar} \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}^2_{\Gamma_{\{\alpha\},\{a\}}} \mathbf{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \frac{q_1 q_2}{|x|} + \gamma \delta(x) \right] dt \right\}
\]

\[
= \sqrt{-\frac{m}{2E}} \frac{\Gamma(\frac{1}{2} - \kappa)}{2\pi \hbar \sqrt{r'' - r'}} \left( \sqrt{-8 m E \frac{r'}{\hbar}} \right) M_{\kappa,0} \left( \sqrt{-8 m E \frac{r}{\hbar}} \right)
\]

\[
+ \left( \frac{m}{\pi \hbar^2} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \kappa)}{\sqrt{r'' r'}} \left[ \alpha + \frac{m}{2\pi \hbar^2} \left[ \Psi \left( \frac{1}{2} - \kappa \right) + \ln \frac{-8 m E_n^{(\alpha)}}{\hbar} - 2\Psi(1) \right] \right.
\]

\[
+ \left. \frac{1}{2\pi \hbar} \sqrt{-\frac{m}{2E}} \sum_{l=1}^{\infty} e^{i l (\phi'' - \phi')} \frac{\Gamma(l + \frac{1}{2} - \kappa)}{(2l)! \sqrt{r'' r'}} W_{\kappa,l} \left( \sqrt{-8 m E \frac{r'}{\hbar}} \right) M_{\kappa,l} \left( \sqrt{-8 m E \frac{r}{\hbar}} \right) \right] = 0 . \tag{3.36}
\]

3.2.4. Multiple δ-function perturbations. Similarly as in the one-dimensional case, we can consider the case of multiple δ-function perturbations in two dimensions, therefore a lattice of δ-function perturbations. Repeating the regularization process for each additional δ-function we obtain by induction \([2, 3]\)

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{i E T / \hbar} \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}^2_{\Gamma_{\{\alpha\},\{a\}}} \mathbf{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \sum_{k=1}^{N} \gamma_k \delta(x - a_k) \right] dt \right\}
\]

\[
= \frac{m}{\pi \hbar^2} \left[ \alpha - \Psi(1) + \frac{m}{\pi \hbar^2} \ln \frac{-2 m E}{\hbar} \right] \delta_{jj'} - \tilde{G}_{2}^{(0)}(a_j, a_j'; E) . \tag{3.38}
\]
where \( G_2^{(0)}(x; 0) = -(m/\pi \hbar^2) \ln |x| \) (\( x \in \mathbb{R}^2 \setminus \{0\} \)), and \( \tilde{G}_2^{(0)}(x; E) = G_2^{(0)}(x; E) \) for \( x \neq 0 \), and \( \tilde{G}_2^{(0)}(x; E) = 0 \) otherwise. The bound states energy levels \( E_n^{(\alpha)} \) are determined by \( \det \left( T^{(0)}_{\{\alpha\}, \{a\}}(E_n^{(\alpha)}) \right) = 0 \).

3.3. Three-dimensional examples

3.3.1. The free particle. Again we start with the free particle. Form (3.4) we know the free particle Green function and we obtain according to the general theory

\[
g(r) = \frac{m}{2\pi \hbar^2} e^{-r\sqrt{-2mE}/\hbar},
\]

(\( r = |x|, x \in \mathbb{R}^3 \setminus \{0\} \)). Therefore we get (c.f. (2.11) and [3])

\[
\begin{align*}
\frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} & \int_{x(t')=x'}^x \mathcal{D}^{(3)}_{\rho^{(0)},\alpha} \rho(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \rho^2 + \gamma \delta(x - a) \right] dt \right\} \\
& = G_3^{(0)}(x'', x'; E) + \frac{G_3^{(0)}(x'', a; E)G_3^{(0)}(a, x'; E)}{\alpha + \frac{m}{2\pi \hbar^3 \sqrt{-2mE}}}.
\end{align*}
\]

Expanding into three-dimensional polar coordinates we obtain (compare also [3])

\[
\begin{align*}
\frac{1}{r' r''} \frac{1}{\hbar} & \int_0^\infty dT e^{iET/\hbar} \int_{r(t')=r'}^{r(t'')=r''} \mu_{l+\frac{1}{2}}[\rho^{(2)}] \mathcal{D}_{\rho^{(0)},\alpha} \rho(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \rho^2 + \gamma \delta(r) \right] dt \right\} \\
& = \frac{1}{2\pi \hbar^2 r' r''} \sqrt{-\frac{m}{2E}} \sinh \left( \sqrt{-2mE} \frac{r}{\hbar} \right) \exp \left( -\sqrt{-2mE} \frac{r'}{\hbar} \right) \\
& + \left( \frac{m}{2\pi \hbar^2} \right)^2 \frac{1}{r' r''} \frac{\exp \left( -\sqrt{-2mE} \frac{r' + r''}{\hbar} \right)}{\alpha + \frac{m}{2\pi \hbar^3 \sqrt{-2mE}}} \\
& + \sum_{l=1}^{\infty} \frac{m}{\hbar^2 \sqrt{r' r''}} I_{l+\frac{1}{2}} \left( \sqrt{-2mE} \frac{r}{\hbar} \right) K_{l+\frac{1}{2}} \left( \sqrt{-2mE} \frac{r'}{\hbar} \right). \tag{3.42}
\end{align*}
\]

Denoting the free three-dimensional propagator by

\[
K^{(0)}(x'', x'; T) = \left( \frac{m}{2\pi i \hbar T} \right)^{3/2} \exp \left( \frac{i m}{2\hbar T} |x'' - x'|^2 \right), \tag{3.43}
\]
it is possible to apply the inverse Laplace-Fourier transformation and one obtains for
the perturbed propagator [60]

\[
x(t'') = x'' \\
\int_{x(t') = x'} D_{x, x'}^3 \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \gamma \delta(x - a) \right] dt \right\}
\]
\[
= K_3^{(0)}(x'', x'; T) + \frac{1}{|a - x'| |x'' - a|} \int_0^\infty e^{-2\pi \alpha h u / m} (u + |a - x'| + |x'' - a|)
\times K_3^{(0)}(u + |a - x'| + |x'' - a|, 0; T) du ,
\]
\[
= K_3^{(0)}(x'', x'; T) + \frac{i \hbar T}{m|a - x'| |x'' - a|} K_3^{(0)}(|a - x'| + |x'' - a|, 0; T) ,
\]
\[
= K_3^{(0)}(x'', x'; T) + \Psi^{(\alpha)}(x') \Psi^{(\alpha)}(x'') e^{i E^{(\alpha)} T / \hbar}
\]
\[
+ \frac{1}{|a - x'| |x'' - a|} \int_0^\infty e^{-2\pi \alpha h u / m} (u - |a - x'| - |x'' - a|)
\times K_3^{(0)}(u - |a - x'| - |x'' - a|, 0; T) du ,
\]
for \( \alpha > 0 \), \( \alpha = 0 \) and \( \alpha < 0 \), respectively. For \( \alpha < 0 \) there is one bound state
wave-function

\[
\Psi^{(\alpha)}(x) = \sqrt{\frac{-\alpha h^2 e^{-2\pi \alpha h |x - a| / m}}{m |x - a|}},
\]
and the energy eigen-value \( E^{(\alpha)} \) has the form

\[
E^{(\alpha)} = -2\pi^2 \alpha^2 h^6 / m^3 .
\]

3.3.2. The harmonic oscillator. In order to discuss the harmonic oscillator we use
the representation (A.7). Applying the regularization procedure and by means of
(3.27,3.29) we set

\[
g(r) = \frac{\Gamma[\frac{3}{2} (3/2 - E / \hbar \omega)]}{4\pi \hbar \omega r^2} W_{E / 2 \hbar \omega, 1 / 4} \left( \frac{m \omega}{\hbar r^2} \right) M_{E / 2 \hbar \omega, 1 / 4} \left( \frac{m \omega}{\hbar r^2} \right) .
\]

We then obtain by denoting the three-dimensional harmonic oscillator Green function
by \( G^{(\omega)}(E) \) from (A.7) \( (\nu = -\frac{1}{2} + E / \hbar \omega, a = |a|, \tilde{a} = \sqrt{2m \omega} / \hbar a \), compare also [45]).

\[
\frac{i}{\hbar} \int_0^\infty dT e^{i E T / \hbar} \int_{x(t') = x'} D_{x, x'}^3 \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{m}{2} \omega^2 x^2 + \gamma \delta(x - a) \right] dt \right\}
\]
\[
= G^{(\omega)}(x'', x'; E) + \frac{G^{(\omega)}(x'', a; E) G^{(\omega)}(a, x'; E)}{\Gamma^{(\omega)}(E)} ,
\]
3.3.3. The Coulomb potential

For $E_a$ where

$$
\Gamma^{(\omega)}_{\alpha, a} = \alpha + \frac{m\Gamma(-\nu)}{2(2\pi)^{3/2}\hbar^2}\left\{ \frac{1}{\alpha} \left[ D_{\nu}(\tilde{a})D_{\nu}'(-\tilde{a}) - D_{\nu}'(\tilde{a})D_{\nu}(-\tilde{a}) \right] + \sqrt{\frac{2m\omega}{\hbar}} \left[ D_{\nu}''(\tilde{a})D_{\nu}(-\tilde{a}) + 2D_{\nu}'(\tilde{a})D_{\nu}'(-\tilde{a}) + D_{\nu}(\tilde{a})D_{\nu}''(-\tilde{a}) \right] \right\} .
$$

(The case $a = 0$ gives:

$$
\Gamma^{(\omega)}_{\alpha, 0} = \frac{\Gamma\left[\frac{1}{2}(3/2 - E/\hbar\omega)\right]}{4\pi\hbar\omega(r'r'')^{3/2}} W_{E/2\hbar\omega, 1/4} \left( \frac{m\omega}{\hbar} r_2 \right) M_{E/2\hbar\omega, 1/4} \left( \frac{m\omega}{\hbar} r_2 \right) + \frac{m^2\Gamma^2\left[\frac{1}{2}(3/2 - E/\hbar\omega)\right]}{4\pi^3 \hbar^4 \hbar'^4} D_{E/\hbar\omega - \frac{1}{2}} \left( \sqrt{\frac{2m\omega}{\hbar}} r'' \right) D_{E/\hbar\omega - \frac{1}{2}} \left( \sqrt{\frac{2m\omega}{\hbar}} r' \right) \alpha - \frac{m}{2\pi\hbar^2} \sqrt{\frac{m\omega}{\hbar}} \Gamma\left[\frac{1}{2}(3/2 - E/\hbar\omega)\right] + \frac{m}{2\pi\hbar^2} \sqrt{\frac{m\omega}{\hbar}} \Gamma\left[\frac{1}{2}(1/2 - E/\hbar\omega)\right] + \sum_{l=1}^{\infty} \frac{\Gamma\left[\frac{1}{2}(l + 3/2 - E/\hbar\omega)\right]}{\hbar\omega \Gamma(l + 3/2)(r'r'')^2} \sum_{n=-l}^{l} Y_l^n* (\theta', \phi') Y_l^n (\theta'', \phi'')

\times W_{E/2\hbar\omega, 1/2(l+1/2)} \left( \frac{m\omega}{\hbar} r'^2 \right) M_{E/2\hbar\omega, 1/2(l+1/2)} \left( \frac{m\omega}{\hbar} r'^2 \right) .
$$

For $a = 0$ use has been made of $D_{\nu}(0) = \sqrt{\pi} 2^{\nu/2}/\Gamma(1-\nu/2)$, and $D_{\nu}'(z) = \nu D_{\nu-1}(z) + z D_{\nu}(z)$, and $Y_l^n(\theta, \phi)$ are the spherical harmonics on the sphere. The bound states energy levels $E_{\alpha}^{(\omega)}$ are determined by $\Gamma^{(\omega)}_{\alpha, \{a\}} (E_{\alpha}^{(\omega)}) = 0$.

3.3.3. The Coulomb potential. Let us denote by $G^{(C)}(E)$ the three-dimensional Coulomb Green function as given by Hostler [43] ($\kappa = q_1 q_2 \sqrt{-m/2E}$):

$$
\int_{0}^{\infty} \frac{d\tau}{\hbar} e^{i\tau E/h} \int_{x(t') = x'} D^3x(t) \exp \left[ \frac{i}{\hbar} \int_{t''}^{t'} \left( \frac{m}{2} \frac{x^2}{|x|} + \frac{q_1 q_2}{x} \right) dt \right]
$$

$$
= -\frac{m \Gamma(1 - \kappa)}{2\pi \hbar^2 |x'' - x'|} \left[ W_{\kappa, 1/2} \left( \sqrt{-8mE} \frac{x_2}{h} \right) M_{\kappa, 1/2} \left( \sqrt{-8mE} \frac{x_2}{h} \right) \right]

\times \sum_{l=0}^{\infty} \sum_{n=-l}^{l} Y_l^n* (\theta', \phi') Y_l^n (\theta'', \phi'') \frac{1}{r''} \frac{1}{h} \sqrt{-\frac{m \Gamma(1 + l - \kappa)}{2E} \frac{1}{(2l + 1)!}}

\times W_{\kappa, l + \frac{1}{2}} \left( \sqrt{-8mE} \frac{r_2}{h} \right) M_{\kappa, l + \frac{1}{2}} \left( \sqrt{-8mE} \frac{r_2}{h} \right) .
$$

(3.50)
According to the general theory we set (c.f. (3.27,3.29) and [3])

\[ g(r) = \sqrt{-\frac{m}{2E}} \frac{\Gamma(1-\kappa)}{4\pi h r} W_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r}{\hbar} \right) M_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r}{\hbar} \right), \]  

(3.51)

Therefore we obtain (compare also [3, 49])

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{iET/\hbar} \int_{x(t')=x'} \mathcal{D}^3 \mathbf{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \frac{\dot{\mathbf{x}}^2}{|\mathbf{x}|} + \frac{q_1 q_2}{\hbar} + \gamma \delta(\mathbf{x} - \mathbf{a}) \right] dt \right\}
\]

\[ = G^{(C)}(\mathbf{x}''; \mathbf{x}'; E) + \frac{G^{(C)}(\mathbf{x}''; \mathbf{a}; E) G^{(C)}(\mathbf{a}; \mathbf{x}'; E)}{\Gamma^{(C)}_{\alpha,\mathbf{a}}(E)}, \quad (3.52a)\]

where

\[
\Gamma^{(C)}_{\alpha,\mathbf{a}}(E) = \alpha + \frac{m \Gamma(1-\kappa)}{2\pi h^2} \sqrt{-8mE} \times \left[ 2 W_{\kappa,\frac{1}{2}} (2\tilde{a}) M_{\kappa,\frac{1}{2}} (2\tilde{a}) - W_{\kappa,\frac{1}{2}} (2\tilde{a}) M''_{\kappa,\frac{1}{2}} (2\tilde{a}) - M_{\kappa,\frac{1}{2}} (2\tilde{a}) W''_{\kappa,\frac{1}{2}} (2\tilde{a}) \right]. \quad (3.52b)
\]

(The case \( \mathbf{a} = 0 \) gives:

\[
= \sqrt{-\frac{m}{2E}} \frac{\Gamma(1-\kappa)}{4\pi h r''} W_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r''}{\hbar} \right) M_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r''}{\hbar} \right)
\]

\[ + \left[ \frac{m}{2\pi h^2} \frac{\Gamma(1-\kappa)}{r''} \right]^2 W_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r'}{\hbar} \right) W_{\kappa,\frac{1}{2}} \left( \sqrt{-8mE} \frac{r''}{\hbar} \right) \]

\[ + \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \sum_{l=1}^\infty \frac{\Gamma(l+1-\kappa)}{(2l+1)!} \sum_{n=-l}^l Y_l^{n*}(\theta', \phi') Y_l^n(\theta'', \phi'') \]

\[ \times W_{\kappa, l+1/2} \left( \sqrt{-8mE} \frac{r'}{\hbar} \right) M_{\kappa, l+1/2} \left( \sqrt{-8mE} \frac{r''}{\hbar} \right), \]  

(3.53)

where three-dimensional polar coordinates have been used in the last line. The bound state energy levels \( E^{(C)}_n \) are determined by \( \Gamma^{(C)}_{\alpha,\mathbf{a}}(E^{(C)}_n) = 0 \).

3.3.4. Multiple \( \delta \)-function perturbations. As the last example in this sequel we again consider the case of multiple \( \delta \)-function perturbations. Similarly as before we get by induction [3]

\[
\frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'} \mathcal{D}^3 \mathbf{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \frac{\dot{\mathbf{x}}^2}{|\mathbf{x}|} + \sum_{k=1}^N \gamma_k \delta(\mathbf{x} - \mathbf{a}_k) \right] dt \right\}
\]
\[
G_3^{(0)}(x'', x'; E) + \sum_{j,j'=1}^N \left( \Gamma_{\{\alpha\},\{a\}}^{(0)} \right)^{-1}_{j,j'} G_3^{(0)}(x'', a_j; E) G_3^{(0)}(a_{j'}, x'; E).
\]

with the matrix \( \Gamma_{\{\alpha\},\{a\}}^{(0)} \) given by \( \{\alpha\} = \{\alpha_k\}_{k=1}^N, \{a\} = \{a_k\}_{k=1}^N \), note that \( G_3^{(0)}(x, y; E) = G_3^{(0)}(x - y, 0; E) \equiv G_3^{(0)}(x; y; E) \)

\[
(\Gamma_{\{\alpha\},\{a\}}^{(0)})_{j,j'} = \left( \alpha_j + \frac{m}{2\pi\hbar^2} \sqrt{-2mE} \right) \delta_{j,j'} - \tilde{G}_3^{(0)}(a_j, a_{j'}; E)
\]

\[
= \begin{cases} 
\alpha_j + \lim_{|x|\to0} [G_3^{(0)}(x; 0) - G_3^{(0)}(x; E)] & j = j', \\
- G_3^{(0)}(a_j, a_{j'}; E) & j \neq j',
\end{cases}
\]

where \( G_3^{(0)}(x; 0) = m/2\pi\hbar^2|x| (x \in \mathbb{R}^3 \setminus \{0\}) \), and \( \tilde{G}_3^{(0)}(x; E) = G_3^{(0)}(x; E) \) for \( x \neq 0 \), and \( \tilde{G}_3^{(0)}(x; E) = 0 \) otherwise. The bound states energy levels \( E_n^{(\alpha)} \) are determined by \( \det(\Gamma_{\{\alpha\},\{a\}}^{(0)}(E_n^{(\alpha)})) = 0 \).

### 3.4. Nonstandard three-dimensional potentials

In this subsection we want to discuss several potential problems important in nuclear physics. They are

i) the potential well,

ii) the Wood-Saxon potential,

iii) and the rotating Morse oscillator.

These potentials can serve as models for the potential strength in the vicinity of the nucleus more or less approximating the strong interaction force. It is assumed that they are radial symmetric, which allows an explicit solution for \( s \)-waves.

#### 3.4.1. The potential step

We first discuss the potential step. The Green function for the radial potential step potential \( V(r) = \Theta(b - r)V_0 \) can be deduced from the smooth-step potential (see below) and has the form \[37\] \( (k^2 = 2m(E + V_0)/\hbar^2, \chi^2 = -2mE/\hbar^2, l = 0) \)

\[
\frac{i}{4\pi\hbar r''} \int_0^\infty dT \int_{r''}^{r(\nu)''} \int_{r('')} \mathcal{D}r(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} r'^2 - [\Theta(r - b) - 1]V_0 \right) dt \right]
\]

\[
= \frac{\Theta(b - r')\Theta(b - r'')}{4\pi\hbar r'r''} \sqrt{-\frac{m}{2(E + V_0)}} \times e^{-ik(r-b)} \left( e^{ik(r>-b)} - \frac{\chi + ik}{\chi - ik} e^{-ik(r>-b)} \right)
\]

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smooth-step potential can be derived \cite{31}.

solution of the SU(1, 1) potential can be done via the path integral 3.4.2. The Wood-Saxon potential

\begin{align}
G(r) &= \frac{1}{4\pi r \hbar} \int_0^\infty dt e^{i E t / \hbar} \int_{r(t') = r'}^r \mathcal{D}r(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{r}^2 - [\Theta(r - b) - 1] V_0 \right) dt \right] \\
&= G^{(PS)}(r'', r'; E) + (\Gamma^{(PS)}_{\alpha, 0}(E))^{-1} G^{(PS)}(r'', 0; E) G^{(PS)}(0, r'; E),
\end{align}

where

\[ \Gamma^{(PS)}_{\alpha}(E) = \alpha + \frac{m}{\pi \hbar^3} \frac{\sqrt{-2m(E + V_0)} \sqrt{-E - \sqrt{-E - V_0}}}{\sqrt{-E + \sqrt{-E - V_0}}}, \]

and \( \Gamma^{(PS)}_{\alpha}(E^{(\alpha)}_n) = 0 \) determines the bound state energy levels.

\[ \frac{1}{4\pi r \hbar} \int_0^\infty dt e^{i E t / \hbar} \int_{r(t') = r'}^r \mathcal{D}r(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{r}^2 + \frac{V_0}{1 + e^{(r-b)/R}} \right) dt \right] = \]

\begin{align}
\frac{2mR}{\hbar^2} \frac{\Gamma(m_1)\Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \\
\times \left( 1 - \tanh \frac{r - b}{2R} \right)^{m_1 - m_2} \left( 1 + \tanh \frac{r - b}{2R} \right)^{m_1 + m_2}
\end{align}

alternatively we can write \( e^{2i \arctan(\chi / k)} = (\chi + i k) / (\chi - i k) \equiv \rho \). Let us denote this Green function by \( G^{(PS)}(r'', r'; E) \). According to the theory we set

\[ g(r) = \frac{1}{4\pi r \hbar} \int_0^\infty \frac{m}{2(E + V_0)} e^{-i k(r-b)} \left( e^{i k(r-b)} - \rho e^{-i k(r-b)} \right) \left( 1 - \frac{e^{2i k(r-b) - \rho}}{e^{-2i k b} - \rho} \right). \]

Therefore we obtain

\[ \frac{\Gamma^{(PS)}_{\alpha}(E)}{\alpha + \frac{m}{\pi \hbar^3} \frac{\sqrt{-2m(E + V_0)} \sqrt{-E - \sqrt{-E - V_0}}}{\sqrt{-E + \sqrt{-E - V_0}}}} = \]

\[ \frac{2mR}{\hbar^2} \frac{\Gamma(m_1)\Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \times \left( 1 - \tanh \frac{r - b}{2R} \right)^{m_1 - m_2} \left( 1 + \tanh \frac{r - b}{2R} \right)^{m_1 + m_2} \]
\[ \times \left( \frac{1 - \tanh \frac{r' - b}{2R}}{2} \right)^{m_1 - m_2 \over 2} \left( \frac{1 + \tanh \frac{r'' - b}{2R}}{2} \right)^{m_1 + m_2 \over 2} \]

\[ \times 2F_1 \left( m_1, m_1 + 1; m_1 - m_2 + 1; \frac{1 - \tanh \frac{r' - b}{2R}}{2} \right) \]

\[ \times 2F_1 \left( m_1, m_1 + 1; m_1 + m_2 + 1; \frac{1 + \tanh \frac{r'' - b}{2R}}{2} \right) \]

(3.61)

\[ m_{1,2} = \sqrt{2mR} \left( \sqrt{-E - V_0} \pm \sqrt{-E} \right) / \hbar \]

we then can state the Green function of the rotating radial Wood-Saxon oscillator which has for s-waves the following form [36]

\[ i \frac{1}{4\pi \hbar r''r''} \int_0^\infty dT \ e^{iET/\hbar} \int_{r'(t')=r'}^{r''(t'')=r''} \mathcal{D}(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} r^2 + \frac{V_0}{1 + e^{(r'' - b)/R}} \right) dt \right] \]

\[ = \frac{mR}{2\pi \hbar^2 r''r''} \frac{\Gamma(m_1)\Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \times \left( \frac{1 - \tanh \frac{r' - b}{2R}}{2} \right)^{m_1 - m_2 \over 2} \left( \frac{1 + \tanh \frac{r'' - b}{2R}}{2} \right)^{m_1 + m_2 \over 2} \times \left\{ 2F_1 \left( m_1, m_1 + 1; m_1 - m_2 + 1; \frac{1 - \tanh \frac{r' - b}{2R}}{2} \right) \right. \]

\[ \times 2F_1 \left( m_1, m_1 + 1; m_1 + m_2 + 1; \frac{1 + \tanh \frac{r'' - b}{2R}}{2} \right) \]

\[ \left. - 2F_1 \left( m_1, m_1 + 1; m_1 - m_2 + 1; \frac{1 + \tanh \frac{b}{2R}}{2} \right) \times 2F_1 \left( m_1, m_1 + 1; m_1 - m_2 + 1; \frac{1 - \tanh \frac{r' - b}{2R}}{2} \right) \times 2F_1 \left( m_1, m_1 + 1; m_1 - m_2 + 1; \frac{1 - \tanh \frac{r'' - b}{2R}}{2} \right) \right\}, \quad (3.62) \]

Let us denote this Green function by \( G_0^{(WS)}(r'', r', E) \). Following the general theory we set \( g(r) = rG_0^{(WS)}(r, r; E) \). Let us abbreviate

\[ f_{\pm}(r) = 2F_1 \left( m_1, m_1 + 1; m_1 \pm m_2 + 1; \frac{1 \pm \tanh \frac{r'' - b}{2R}}{2} \right). \quad (3.63) \]
Note the relations
\[
f_{\pm}(0) = 2F_1\left(m_1, m_1 + 1; m_1 \pm m_2 + 1; \frac{1 \mp \tanh \frac{b}{2R}}{2}\right),
\]
\[
f'_{\pm}(0) = \pm \frac{2F'_1(m_1, m_1 + 1; m_1 \pm m_2 + 1; \frac{1}{2}(1 \mp \tanh \frac{b}{2R}))}{4R \cosh^2 \frac{b}{2R}},
\]
\[
f''_{\pm}(0) = \frac{2F''_1(m_1, m_1 + 1; m_1 \pm m_2 + 1; \frac{1}{2}(1 \mp \tanh \frac{b}{2R}))}{16R^2 \cosh^4 \frac{b}{2R}}
\begin{align*}
&\mp \tanh \frac{b}{2R} \frac{2F'_1(m_1, m_1 + 1; m_1 \pm m_2 + 1; \frac{1}{2}(1 \mp \tanh \frac{b}{2R}))}{4R \cosh^2 \frac{b}{2R}}.
\end{align*}
\]

We now find
\[
g^{(WS)}_{0,3} = \frac{R \Gamma(m_1) \Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1) \Gamma(m_1 - m_2 + 1)}
\begin{align*}
&\times \left(\frac{1 \mp \tanh \frac{b}{2R}}{2}\right)^{m_1 - m_2} \left(1 \mp \tanh \frac{b}{2R}\right)^{m_1 + m_2}
\left[f_-(0)f'_+(0) - f'_-(0)f_+(0)\right]
\end{align*}
\]
\[
g^{(WS)}_{1,3} = \frac{R \Gamma(m_1) \Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1) \Gamma(m_1 - m_2 + 1)}
\begin{align*}
&\times \left(\frac{1 \mp \tanh \frac{b}{2R}}{2}\right)^{m_1 - m_2} \left(1 \mp \tanh \frac{b}{2R}\right)^{m_1 + m_2}
\times \left[\frac{1}{2}\left(f_-(0)f''_+(0) - f''_-(0)f_+(0)\right) + f'_-(0)f'_+(0) - \frac{f_+(0)}{f_-(0)}f'_-(0)\right].
\end{align*}
\]

Therefore we obtain for the $\delta$-perturbed Wood-Saxon potential
\[
\frac{i}{4\pi hr''r'''} \int_0^\infty dT e^{iET/h} \int_{r(t') = r''}^{r(t''') = r'''} \mathcal{D} \Gamma^{(WS)}_{\alpha,0} r(t) \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} r'^2 + \frac{V_0}{1 + e^{(r-b)/R}}\right) dt\right]
\]
\[= G^{(WS)}(r''', r''; E) + \left(\Gamma^{(WS)}_{\alpha,0}(E)\right)^{-1} G^{(WS)}(r'', 0; E)G^{(WS)}(0, r''; E),
\]
with $\Gamma^{(WS)}_{\alpha,0}(E)$ given by
\[
\Gamma^{(WS)}_{\alpha,0}(E) = \alpha g^{(WS)}_{0,3} - g^{(WS)}_{1,3},
\]
and the bound-state energy levels $E_n$ are determined by $\Gamma^{(WS)}_{\alpha,0}(E_n^{(\alpha)}) = 0$. 

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3.4.3. The rotating Morse oscillator. Another model for a nuclear potential is the so-called rotating Morse oscillator [59]. Its Green function can be derived by path integration via means of the Green function of the Morse potential [14, 20, 30, 54] and has for s-waves the following form [36]

\[
\frac{i}{4\pi \hbar r''r''} \int_0^\infty d\tau \ e^{iEt/\hbar} \int_{r(\tau')=r''}^r D\tau(t) \exp \left\{ \frac{i}{\hbar} \int_t^{t''} \left[ \frac{m}{2} \frac{\partial}{\partial \tau} - \frac{V_0^2 h^2}{2m} (e^{-2r - 2\alpha e^{-r}}) \right] d\tau \right\} \\
= \frac{m}{4\pi V_0^2 h^2 r''r''} \frac{\Gamma\left(\frac{1}{2} + \sqrt{-2mE}/\hbar - \alpha V_0\right)}{\Gamma(1 + \sqrt{-8mE}/\hbar) e^{(r' + r'')/2}} \times \left\{ W_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r'} < ) M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r''} \ ) \\
- \frac{M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0)}{W_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0)} M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r'} \ ) M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r''}) \right\}.
\]  

(3.71)

Let us denote this Green function by \( G^{(M)}_0(r'', r'; E) \).

\[
w_1(r) = M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r}) , \quad w_2(r) = W_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0 e^{-r}). \quad (3.72)
\]

Note the relations

\[
w_1(0) = M_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0) , \quad w_1'(0) = -2V_0 M_{\alpha V_0, \sqrt{-2mE}/\hbar}'(2V_0) , \\
w_1''(0) = 2V_0 M_{\alpha V_0, \sqrt{-2mE}/\hbar}'(2V_0) + 4V_0^2 M_{\alpha V_0, \sqrt{-2mE}/\hbar}''(2V_0). \quad (3.73)
\]

\[
w_1''(0) = 2V_0 M_{\alpha V_0, \sqrt{-2mE}/\hbar}'(2V_0) + 4V_0^2 M_{\alpha V_0, \sqrt{-2mE}/\hbar}''(2V_0). \quad (3.74)
\]

According to the theory we set \( g(r) = rG^{(M)}(r, r; E) \). Therefore we obtain

\[
g^{(M)}_{0, 3} = \frac{\Gamma\left(\frac{1}{2} + \sqrt{-2mE}/\hbar - \alpha V_0\right)}{2V_0 \Gamma(1 + \sqrt{-8mE}/\hbar)} \left[ w_1(0)w_2'(0) - w_1(0)w_2(0) \right] \quad (3.75)
\]

\[
g^{(M)}_{1, 3} = \frac{\Gamma\left(\frac{1}{2} + \sqrt{-2mE}/\hbar - \alpha V_0\right)}{\Gamma(1 + \sqrt{-8mE}/\hbar)} \times \left[ \frac{1}{2} \left( w_1(0)w_2''(0) - w_1''(0)w_2(0) \right) + w_1'(0)w_2'(0) - \frac{w_2(0)}{w_1(0)}w_1'(0)w_2'(0) \right]. \quad (3.76)
\]
Therefore we obtain for the $\delta$-perturbed rotating Morse oscillator

\[
\frac{i}{4\pi \hbar r' r''} \int_0^\infty dT \, e^{iET/\hbar} \times \int_{r(t')=r'}^{r(t'')=r''} D_{r^{(M)}(t)} r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} r'^2 - \frac{V_0^2 \hbar^2}{2m} (e^{-2r} - 2\alpha e^{-r}) \right] dt \right\} = G^{(M)}(r'', r'; E) + \left( \Gamma^{(M)}_{\alpha,0}(E) \right)^{-1} G^{(M)}(r''; 0; E) G^{(M)}(0, r'; E),
\]

(3.77)

with $\Gamma^{(M)}_{\alpha,0}(E)$ given by

\[
\Gamma^{(M)}_{\alpha,0}(E) = \alpha g^{(M)}_{0,3} - g^{(M)}_{1,3},
\]

(3.78)

and the bound-state energy levels $E_n$ are determined by $\Gamma^{(M)}_{\alpha,0}(E_n^{(\alpha)}) = 0$.

4. Summary and discussion

In this paper I have presented a path integral approach for the incorporation of two- and three dimensional $\delta$-function perturbations by means of a perturbation expansion. The motivation was to demonstrate the validity and applicability of the formalism in the context of path integrals, i.e. “to build up quantum mechanics from the point of view of fluctuating paths [21]”, hence to put the results of the operator and the path integral approach in quantum mechanics on an equal footing, and to supply steps towards a complete classification of solvable Feynman path integrals [39]. A regularization procedure was needed to treat appearing divergencies, but could be systematically discussed. As was shown, a regularization procedure known from functional analysis could be applied in the context of path integrals. In order to do this, the coupling $\gamma$ in the heuristic expression “$\gamma \delta(x - a)$” in the Lagrangian in the path integral had to be properly interpreted and it was shown that the formal series summation (2.6) of the perturbative path integral approach (2.3) corresponding to the Hamiltonian (2.15) could be regularized for two and three dimensional $\delta$-function perturbations, making the two and three dimensional analogue of (2.6) well-defined.

We succeeded in developing the theory intaking into account potential problems together with $\delta$-function perturbations in two and three dimensions. The general feature of the Green function of the perturbed problem with one point-interaction at $x = a (x', x'', a \in \mathbb{R}^2, \mathbb{R}^3)$ has the form

\[
G^{(\delta)}(x'', x'; E) = G^{(V)}(x'', x'; E) + \left( \Gamma^{(V)}_{\alpha,a}(E) \right)^{-1} G^{(V)}(x'', a; E) G^{(V)}(a, x'; E)
\]

(4.1)

with $G^{(V)}(E)$ the Green function of the unperturbed one. The energy eigenvalues $E_n$ are determined by

\[
\Gamma_{\alpha,a}(E_n^{(\alpha)}) = 0,
\]

(4.2)

giving in general a transcendental expression for the $E_n^{(\alpha)}$. Generally, $\alpha < 0$ generates bound states, whereas $\alpha > 0$ generates resonance states.
We demonstrated the formalism with a sample of two- and three-dimensional examples. They included the free particle, the harmonic oscillator, the Coulomb potential case, and some potential models familiar in nuclear and electronic shell physics. The latter were studied in three-dimensional space for s-waves only. We chose the potential step, the Wood-Saxon potential, and the rotating Morse oscillator. Of course, the Wood-Saxon potential is just but one model in the class of potentials related to the modified Pöschl-Teller potentials, other exactly solvable potentials can also be considered, e.g. the modified Pöschl-Teller potential itself, or more specifically the Hulthén or a Scarf-like potential. These latter potentials can be used to describe a screened Coulomb potential, where an additional \(\delta\)-function perturbation at the center would model a point interaction (of the nucleus) in the environment of an overall Coulomb-screening, say, of an electron shell, respectively, a point interaction (inside a nucleus) in the environment of an overall screening of the strong force in the nucleus itself, and many more.

One may ask, what happens in higher dimensions? Actually, point interactions do not make sense for \(D \geq 4\): “Any possible mathematical definition of a self-adjoint operator \(H\) of the heuristic form \(-\Delta + \lambda \delta_y\) in \(L^2(\mathbb{R}^d)\) should take into account the fact that, on the space \(C^\infty_0(\mathbb{R}^d \setminus \{0\})\) of smooth functions which vanish outside a compact subset on the complement of \(\{y\}\) in \(\mathbb{R}^d\), \(H\) should coincide with \(-\Delta\). For \(d \geq 4\) this already forces \(H\) to be equal to \(-\Delta\) on \(H^{2,2}(\mathbb{R}^d)\) since \(-\Delta|_{C^\infty_0(\mathbb{R}^d \setminus \{0\})}\) is essentially self-adjoint for \(d \geq 4\) [3, p.2]”, and c.f. [58, Theorem X.11, p.161]. What remains are \(\delta\)-function perturbations on planes and hyperplanes, respectively, along perpendicular lines and planes [36], systems which can model a Casimir effect, e.g. [11].

In one dimension, making the strength of the \(\delta\)-function perturbation infinitely repulsive, produces Dirichlet boundary-conditions at the location of the \(\delta\)-function perturbation. In two and three dimensions, in setting \(\alpha = 0\) some sort of defect is produced, which one may call a zero-radius hard core. Let us emphasize that again the entire Green function has to be taken into account and not only part of it (compare [50]). This shows once more that it is not enough just to throw away the continuous part of the spectrum (say, in the Coulomb case), but a proper regularization always takes into account the entire Green function yielding inadmissible contributions from both the discrete and continuous spectrum of the unperturbed problem. This concludes the discussion.

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Appendix 1. Green function for the harmonic oscillator

We want to demonstrate how to calculate the Green function of the harmonic oscillator. Of course, it is always possible to determine the Green function of a quantum mechanical problem by means of an operator approach exploiting the theory of differential equations. However, this is not the point of view here. The purpose here is to show how a Green function can be obtained by knowing the propagator form a path integral calculation.

We consider the propagator

\[ K(x''', x'; T) \] (3.8) of the harmonic oscillator. We need the integral representation ([13, p.86; 29, p.729], \( a_1 > a_2, \Re(\frac{1}{2} + \mu - \nu) > 0 \)):

\[
\int_0^\infty \coth^{2\nu} \frac{x}{2} \exp \left[ -\frac{a_1 + a_2}{2} t \cosh x \right] I_{2\mu}(t\sqrt{a_1a_2} \sinh x) dx = \frac{\Gamma(\frac{1}{2} + \mu - \nu)}{t\sqrt{a_1a_2} \Gamma(1 + 2\mu)} W_{\nu,\mu}(a_1t) M_{\nu,\mu}(a_2t) . \tag{A1.1}
\]

Here \( W_{\nu,\mu}(z) \) and \( M_{\nu,\mu}(z) \) denote Whittaker-functions. Furthermore we make use of the relations of the parabolic cylinder functions in terms of Whittaker-functions [13, pp.39]

\[ D_{\nu}(z) = 2^{\nu/2} \left( \frac{z^2}{2} \right)^{-1/4} W_{\nu/2+1/4,1/4} \left( \frac{z^2}{2} \right) \] \tag{A1.2a}

\[ E^{(0)}_{\nu}(z) = \sqrt{2} e^{-z^2/4} 1 F_1 \left( -\nu/2; \frac{1}{2}; \frac{z^2}{2} \right) = \sqrt{2\pi} \left( \frac{z^2}{2} \right)^{-1/4} M_{\nu/2+1/4,-1/4} \left( \frac{z^2}{2} \right) \] \tag{A1.2b}

\[ E^{(1)}_{\nu}(z) = \sqrt{2} e^{-z^2/4} 1 F_1 \left( \frac{1 - \nu}{2}; \frac{3}{2}; \frac{z^2}{2} \right) = \sqrt{2\pi} \left( \frac{z^2}{2} \right)^{-1/4} M_{\nu/2+1/4,1/4} \left( \frac{z^2}{2} \right) \] \tag{A1.2c}

We now obtain

\[ G(x''', x'; E) \]

\[
= \frac{i}{\hbar} \int_0^\infty dT \ e^{iET/\hbar} \left( \frac{m\omega}{2\pi i \hbar \sin \omega T} \right)^{1/2} \exp \left\{ \frac{i m\omega}{2\hbar} \left[ (x'^2 + x''^2) \cot \omega T - 2 \frac{x'x''}{\sin \omega T} \right] \right\}
\]

(expand the \( x'x'' \) term in the exponential into modified Bessel-functions)

\[
= \frac{m\omega \sqrt{x'x''}}{2\hbar^2} \int_0^\infty \frac{dT}{\sin \omega T} \exp \left[ -\frac{m\omega}{2i\hbar} (x'^2 + x''^2) \cot \omega T + \frac{iET}{\hbar} \right]
\times \left[ I_{1/2} \left( \frac{m\omega x'}{i\hbar \sin \omega T} \right) + I_{-1/2} \left( \frac{m\omega x''}{i\hbar \sin \omega T} \right) \right]
\]

(perform coordinate transformation and Wick-rotation)

\[
= \frac{m\sqrt{x'x''}}{2\hbar^2} \int_0^\infty dv \left( \coth \frac{v}{2} \right)^{E/\hbar \omega} e^{-m\omega(x'^2 + x''^2) \cosh v/2\hbar}
\]
\[ \times \left[ I_{1/2} \left( \frac{m\omega}{\hbar} x' x'' \sinh \nu \right) + I_{-1/2} \left( \frac{m\omega}{\hbar} x' x'' \sinh \nu \right) \right] \]

(applying (A1.1), set \( \nu = -\frac{1}{2} + \frac{E}{\hbar\omega} \))

\[ = \frac{1}{2\omega\hbar \sqrt{x'} x''} W_{\nu/2+1/4, 1/4} \left( \frac{m\omega}{\hbar} x_2^2 \right) \times \left[ \Gamma \left( \frac{1-\nu}{2} \right) \mathcal{M}_{\nu/2+1/4, 1/4} \left( \frac{m\omega}{\hbar} x_2^2 \right) + \Gamma \left( -\nu/2 \right) \mathcal{M}_{-\nu/2+1/4, -1/4} \left( \frac{m\omega}{\hbar} x_2^2 \right) \right] \]

(applying (A1.2))

\[ = \sqrt{\frac{m}{\pi h^3 \omega}} \Gamma \left( \frac{E}{\hbar\omega} - \frac{1}{2} \right) D_{-1/2+E/h\omega} \left( \sqrt{\frac{2m\omega}{h}} x_2 \right) D_{-1/2+E/h\omega} \left( -\sqrt{\frac{2m\omega}{h}} x_2 \right) , \]

(A1.3)

which is the result of (3.9). The particular structure of the propagator allows to derive the higher dimensional case from the lower one. Let \( x \in \mathbb{R}^D \) and define \( \mu = x' x'' + x' x'' \), \( \nu = x' \cdot x'' \), then the following relation for the Feynman kernel for the harmonic oscillator is valid

\[ K^{(D)}(x'', x'; T) = \frac{1}{2\pi} \frac{\partial}{\partial \nu} K^{(D-2)}(x'', x'; T) \]

\[ = \frac{1}{2\pi} e^{-i\omega T} \left( \frac{m\omega}{\hbar} - 2 \frac{\partial}{\partial \mu} \right) K^{(D-2)}(x'', x'; T) . \]

(A1.4)

Introducing \( \xi = \frac{1}{2}(|x' + x''| + |x'' - x'|) \), \( \eta = \frac{1}{2}(|x' + x''| - |x'' - x'|) \) we obtain for dimensions \( D = 1, 3, 5 \ldots \) for the Green function

\[ \frac{i}{h} \int_0^\infty dT \ e^{iTE/h} \int_{x(t') = x'} \mathcal{D}^{D} x(t) \exp \left[ \frac{i}{2h} \int_{t'}^{t''} (\dot{x}^2 - \omega^2 x^2) dt \right] \]

\[ = \sqrt{\frac{m}{\pi h^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar\omega} \right) \left( \frac{1}{2\pi} \right)^{D-1} \left[ \frac{1}{\eta^2 - \xi^2} \left( \eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta} \right) \right]^{D-1} \]

\[ \times D_{-\frac{1}{2} + \frac{\nu}{\pi^2}} \left( \sqrt{\frac{2m\omega}{h}} \xi \right) D_{-\frac{1}{2} + \frac{\nu}{\pi^2}} \left( -\sqrt{\frac{2m\omega}{h}} \eta \right) . \]

(A1.6)

Note the various sign conventions used in the literature [5].

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Appendix 2. Propagator in two dimensions

We consider (3.21). The first term is just the free particle propagator in two dimensions. To treat the second term we consider the following inverse Laplace transformation table [22]:

| $g(s)$ | $f(t)$ |
|--------|--------|
| $\int_0^\infty f(t) e^{-st} \, dt$ | $g(as)$ |
| $\int_0^t f_1(u)f_2(t-u) \, du$ | $g_1(s)g_2(s)$ |
| $\frac{1}{\ln s}$ | $\int_0^\infty \frac{du}{\Gamma(u)} u^{u-1}$ |
| $K_\nu(a\sqrt{s})K_\nu(b\sqrt{s})$ | $\frac{1}{2t}e^{-(a^2+b^2)/4t}K_\nu\left(\frac{ab}{2t}\right)$ |

Obviously, we will get a rather complicated expression, so that we keep the discussion short. We use imaginary time $\tau = iT$, which has the consequence that we must set $-E \to s$. Furthermore we introduce the abbreviations

$$a = \frac{\sqrt{2m}}{\hbar}|x'' - a|, \quad b = \frac{\sqrt{2m}}{\hbar}|x' - a|, \quad \beta = \sqrt{\frac{m}{2\hbar^2}}e^{\frac{\pi\alpha^2}{4m^2} - \Psi(1)}.$$  \hfill (A2.1)

Applying successively the above rules we obtain for the propagator of the free particle perturbed by a point interaction in $\mathbb{R}^2$ (compare also [1])

$$\begin{align*}
K^{(s)}(x'', x'; \tau) &= \frac{m}{2\pi^2\hbar^2\tau} \exp \left[ -\frac{m}{2\hbar\tau}(x'' - x')^2 \right] \\
&\quad + \frac{m}{\pi\beta^2\hbar^2} \left( \int_0^\infty dv \int_0^\tau du \left( \frac{\tau - u}{\beta^2} \right)^{v-1} e^{-(a^2+b^2)/4u}K_0\left(\frac{ab}{2u}\right) \right). 
\end{align*}$$  \hfill (A2.2)

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