Semi-Classical Field Theory as Decoherence Free Subspaces

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We formulate semi-classical field theory as an approximate decoherence-free-subspace of a finite-dimensional quantum-gravity Hilbert space. A complementarity construction can be realized as a unitary transformation which changes the decoherence-free-subspace. This can be translated to signify that field theory on a global slice, in certain space-times, is the simultaneous examination of two different superselected sectors of a field theory. We posit that a correct course graining procedure of quantum gravity should be WKB states propagating in a curved background in which particles exiting a horizon have imaginary components to their phases. The field theory appears non-unitary, but it is due to the existence of approximate decoherence free sub-spaces. Furthermore, the importance of operator spaces in the course-graining procedure is discussed. We also briefly touch on Firewalls.

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I. INTRODUCTION

The covariant entropy bound$^1$ states that the Desitter Hilbert space $\mathcal{H}_A$ of area $A$ has a finite dimension given by:

$$\dim \mathcal{H}_A = e^{A/4}. \quad (1)$$

Here $A$ is the area of the Desitter horizon. This finite dimensional Hilbert space is in stark contradiction with the Hilbert (Fock) spaces of quantum field theory which are always infinite dimensional. One then must tread cautiously when attempting to import field theory intuition to a finite dimensional quantum gravity (QG) theory. In our universe the Hilbert space dimension scales as $e^{5.1 \times 10^{121}}$. The number of states corresponding to semi-classical configurations parametrically behave as $e^{cA^n}$ where $n < 1$ and $c$ is some constant (most likely $n = 3/4$, see $^2$).

This paper will be focused on the structure of quantum-computers which simulate a field theory in curved space-time. Qubits which give rise to a classical gravitational field are treated as living in an environmental Hilbert space sector while the matter lives in a system Hilbert space (here the environment is also included as part of the quantum computation). In order for matter to exhibit unitary evolution, with its own system Hamiltonian, one needs to consider matter states in decoherence-free-subspaces.$^3$ Appendix I. reviews the properties of decoherence free subspaces.$^4$

We wish to consider quantum-computation simulations of field theory in curved space-time because quantum-computers are “UV-Complete”. Therefore, we can begin to model quantum gravity effects by 1.) modifying the system environment interaction and 2.) running the semi-classical simulation to the point where the Dfs approximation becomes corrupted. Quantum gravity effects can then be related to environmental errors induced after sufficiently long times. This work will be focused on the relationship between a semi-classical field theory (field theory in curved space) and a quantum-computer exhibiting decoherence-free-subspaces. The modeling of quantum gravity effects will be the subject of future work.

Most of the discussion here will rely on the theory of decoherence, quantum computing, and field theory in curved space-time. In order to make this paper more accessible, we present an intuitive description of the proposal in the conclusions.

II. CONSTRUCTION

We consider a large but finite dimensional Hilbert space. We claim that the Hilbert space within a causal patch is given as:

$$\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_{sc} = \mathcal{H}_E \otimes (\mathcal{H}_s^+ \oplus \mathcal{H}_s), \quad (2)$$

where $\mathcal{H}_{sc}$ is the system, and $\mathcal{H}_s$ is a decoherence free subspace of the quantum-gravity Hamiltonian $H$. Furthermore, we postulate that there is a procedure, which we denote by $S$, which approximates the dynamics of $\mathcal{H}$ with a semi-classical field theory on a fixed background. Namely:

$$S(H, \mathcal{H}) \rightarrow (g_{\mu \nu}, \mathcal{H}_{Fock}) \quad (3)$$

where $\mathcal{H}_{Fock}$ is the fock space associated with particles propagating in a restricted space-time (For example, the construction may not describe states propagating past a horizon. It is still not known why the space-time must be restricted. Further understanding of the gravitational sector is needed and is the subject of further research.).

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1 Throughout the rest of this paper we shall use Dfs to mean decoherence free subspace and DFS to mean the plural. Furthermore all DFS are only approximate unless otherwise specified.
Also, there is a correspondence:

$$H_s \rightarrow H_{Fock}.$$ \hspace{1cm} (4)

It is unclear how this correspondence arises in the finite dimensional Hilbert space $H_s$. This is related to the question of how particles are approximated in a finite dimensional Hilbert space. For this correspondence to work it is necessary for the operator algebra of $H_{sc}$ to be working with as well as their commutation relations. We are including the commutator of any operator with the Hamiltonian when we speak of this space.

For large $S$ the angular momentum operators are approximately bosonic creation/annihilation operators. How fermions may emerge is a more complicated story and string net condensate model\(^2\) may provide an answer.

### III. COMPLEMENTARITY

Complementarity\(^2\) was initially proposed to resolve a conflict between unitary black hole evaporation and semi-classical field theory. Several analyses on a global space-like slice, which extended through the event horizon, lead to the conclusion that black holes clone quantum states. Due to the fact that the cloning of arbitrary states is a violation of quantum mechanics, a complementary picture of black hole physics was formulated. In brief, complementarity states that black hole physics can be described in the exterior or the interior but not both at the same time. A “complementarity transformation” is a transformation of the degrees of freedom (operators and states). Initially the degrees of freedom describe physics in the exterior of a black hole. Once the transformation is applied, the degrees of freedom describe physics in the interior.

Here we propose that the complementarity transformation should change the properties of the DFSs. This can be realized as a unitary transformation of the Hamiltonian which necessarily changes the decoherence-free-subspace.

Applying the approximation $S$ on the new decoherence-free-subspace results in a new space-time $g_{\mu\nu}'$ and fock space $H_{Fock}'$. Field theory in curved space-time, on a global slice, can then be thought of the union of the two fock spaces and the gluing of the restricted space-times. The motivation to consider changes in DFSs as part of the complementarity transformation is described in the next section.

Before closing this section we comment on the relation of decoherence-free subspaces and super-selection sectors. For certain Hamiltonians, see Appendix 2, states in DFSs are grouped according to specific eigenvalues they possess (charge, for example). Any superposition of states within the DFSs will evolve unitarily under the system Hamiltonian. Expressed in a field theory language, this is a statement of how superselection sectors decohere. In other words, super-selection sectors arise through a systems interaction with the environment. In the case studied here the environment is the microscopic degrees of freedom which give rise to a gravitational field.

Field theory in curved space-time, on a global slice, can thus be seen as the simultaneous examination of many super-selected sectors. Thus, one is improperly doing field theory in the sense described in ref\(^6\). The author however, prefers to view the emergence of superselection sectors as a consequence of the DFSs formulation.

### IV. PHYSICAL DESCRIPTION OF STATES

There are many special cases which can be considered but here we focus on the case where the Hamiltonian has only an approximate DFSs. This implies that there are no time-constants which vanish in the DFSs (see Appendix 3). If all time constants are positive, then unitarity will appear to be violated for large enough time.

The mapping $S$ then takes the states in these classes of DFSs to WKB modes propagating in a curved background. Furthermore, all modes will have a positive imaginary component to their phases\(^3\). Such non-unitary field theories can be constructed on a curved background. However, such theories have been ruled out in the past due to a violation of unitarity. This apparent violation of unitarity is justified in the DFSs framework.

We conjecture that Hamiltonians which exhibit only approximate DFSs are mapped to those which have initial data on a particular Cauchy surface. This Cauchy surface is one which the congruence of future directed geodesics emanating from it become inextensible in finite affine parameter (a singularity). Hamiltonians with exact DFSs are in another extreme. We conjecture that these Hamiltonians describe space-times with no singularities\(^4\).

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\(^2\) Here we will denote the operator space (equivalently operator algebra) of a Hilbert space $H$ to be represented by the symbol $O(H)$. Physically, this space represents the set of operators we will be working with as well as their commutation relations. We are including the commutator of any operator with the Hamiltonian when we speak of this space.

\(^3\) The convention we use is such that a positive imaginary part of the phase will result in a decaying exponential.

\(^4\) The exterior picture in black hole evaporation has an interesting
V. COMMENT ON THE PROCEDURE S

So far, this discussion has relied on the mysterious “procedure” called S. Here we do not construct the procedure but we discuss how it must be viewed in quantum information theory terms.

A quantum computation can be viewed as a selection of the quintet:

\[ (\tau, H, \mathcal{H}, \mathcal{O}(\mathcal{H}), |\psi\rangle) , \]

where \( H \) is the Hamiltonian, \( \mathcal{H} \) is the Hilbert space, \( \mathcal{O}(\mathcal{H}) \) is the relevant operator algebra, \( |\psi\rangle \) is a state of the Hilbert space, and \( \tau \) is time. The correspondence of these objects to quantum information theory is as follows:

\( \tau \rightarrow \) Parameter of evolution gate  
\( H \rightarrow \) A generator for the evolution gate  
\( \mathcal{H} \rightarrow \) Qubit Hilbert Space  
\( \mathcal{O}(\mathcal{H}) \rightarrow \) Set of gates and/or generators of gates  
\( |\psi\rangle \rightarrow \) Initial Qubit Configuration.

By evolution gate we simply mean \( e^{iH\tau} \), the standard time evolution operator.  
Focusing on the Dfs, we can see that it too has such a quintet. Namely:

\[ (\tau, H_{sc}, \mathcal{H}_{sc}, \mathcal{O}(\mathcal{H}_{sc}), |\psi_{sc}\rangle) , \]

where \( H_{sc} \) is the Hamiltonian of the system, and with similar definitions for the other elements of the quintet.  

VI. DISCUSSION ON FIREWALLS

Here the Author does not claim to solve the firewall paradox. This paradox was first posed in ref[30], and later expanded upon in ref[31]. We shall attempt to formulate this paradox in an information theoretic way, keeping decoherence-free subspaces in mind.

The firewall problem can be recast into the statement: “Can a semi-classical observer extract a purification of a bit from the early Hawking radiation”\(^{10}\). By “semi-classical observer”, we mean an observer which only has control over gates which act on his/her Dfs. The number of independent “semi-classical gates” scale as \((\dim \mathcal{H}_{sc})^2\), which is far smaller than the total number of gates in the

- \(^{7}\) By simulation, we mean a circuit construction which takes the initial qubit configuration to a particular out-state. The out-state can be used to obtain expectation values of interest (N-point functions). See ref[13] for a simulation of \(^{8}\) theory.  
- \(^{8}\) Readers familiar with the ADM formalism[26, 27] will recognize the notation  
- \(^{9}\) This is a purely mathematical question. The existence (non-existence) of these algebraic structures will validate (invalidate) this work.  
- \(^{10}\) The Author would like to thank Raphael Bousso for making this clear to him.
 Several authors have discussed attaching physical significance to descriptions in ref\[4, 14\]. Further study of these theories with emergent Lorentz symmetry, such as those in finite dimensional Hilbert spaces may describe representations of the Lorentz group. Quantum computation be the fundamental group since there are no finite unitary dimensional Hilbert space then the Poncaire group can’t be the fundamental symmetry group of nature. If we are a finite living in an approximately error-free register of a quan- tational theory. Viewed computationally, matter are states in a decoherence-free-subspace in a full quantum gravita- tion. If this formalism describes nature then it implies that matter states are living very far away and then jumping into the black hole. Wether or not someone can see a firewall is a question on the time-scale restrictions on quantum computations. We will not discuss such thought experiments here. We will not discuss such thought experiments here. The author conjectures that there is no firewall. The reason being that one expects a black hole to be a quantum object which preforms operations, on infalling matter, which simulate general relativity. This is simply a conjecture and the question still needs to be answered. If firewalls do exist however, they could be described as regimes in the quantum computation in which a subsystem of interest no longer evolves unitarily under its own system Hamiltonian (how this is done and the physical interpretation of such a procedure is a matter of further work).

VII. CONCLUSIONS

Here we take a brief aside to discuss the philosophical implications of this formalism. If this formalism describes nature then it implies that matter states are living in a decoherence-free-subspace in a full quantum gravitational theory. Viewed computationally, matter are states living in an approximately error-free register of a quantum computer.

Furthermore, this framework makes a statement on the fundamental symmetry group of nature. If we are a finite dimensional Hilbert space then the Poncaire group can’t be the fundamental group since there are no finite unitary representations of the Lorentz group. Quantum computations in finite dimensional Hilbert spaces may describe theories with emergent Lorentz symmetry, such as those described in ref\[3, 14\]. Further study of these theories and their relationship with quantum computation would be interesting.

Finally, it should be mentioned that the role decoherence has to play in quantum gravity has been emphasized by many authors\[20, 22\]. With pioneering work done by Kiefer and Joos\[15\], and Anglin, Laflamme, Zurek, and Paz\[10\]. This work hopes to add to the above body of work in a small way.

VIII. ACKNOWLEDGMENTS

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Appendix 1: Decoherence Free Subspaces and Subsystems

In this appendix we review some aspects of decoherence free subspaces and subsystems. For a more thorough review the reader should consult\[17, 18\]. The Hilbert space is given by:

$$\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_{sc}$$

(9)

The Hamiltonian is:

$$H = H_E + H_{sc} + H_{int}$$

(10)

where $H_E$ acts only on $\mathcal{H}_E$, $H_{sc}$ acts only on $\mathcal{H}_{sc}$, and $H_{int}$ is an interaction Hamiltonian. A subspace $\mathcal{H}_s \subset \mathcal{H}_{sc}$ is a decoherence-free-subspace, with precision $\epsilon$, if any density matrix $\rho_s$ with support in $\mathcal{H}_s$ satisfies:

$$\rho_s(t) = e^{iH_{sc}t} (\rho_s) e^{-iH_{sc}t} + \mathcal{O}(\epsilon).$$

(11)

where:

$$\rho_s(t) = \text{tr}_E (e^{iHt} \rho_E \otimes \rho_s e^{-iHt})$$

(12)

for a large set of $\rho_E$. By $\mathcal{O}(\epsilon)$ we mean that any expectation value of an operator has only order $\epsilon$ corrections.

It is worth noting that the identification of the decoherence-free-subspace is highly dependent on the Hamiltonian. Furthermore, the existence of exact decoherence-free-subspaces does not necessarily imply that $H_{int} = 0$.

The above definition is equivalent to the standard way of defining decoherence-free-subspaces but it is usually not the most practical. The above definition is useful however, because it allows one to work backwards and construct decoherence-free-subspaces. Thus, it is independent on whether $\mathcal{H}_{sc}$ has many subsystems.
Constructing decoherence-free-subspaces, in situations in which the relevant Hilbert space has many subsystems, is done by partitioning states based on the evolution of the subsystems.

Appendix 2: Operator Spaces and DFS

There is a relationship between Lie Groups and DFS in special cases where the Hamiltonian possesses a certain structure. The discussion here follows closely Theorem 1 of ref [19]. Here we state the theorem without proof.

Let the interaction Hamiltonian be given by:

$$H_{int} = \sum_{\alpha} B_{\alpha} \otimes F_{\alpha},$$

where $B_{\alpha}$ act only on $H_E$ and $F_{\alpha}$, called error generators, act on $H_{sc}$. The decoherence-free-subspaces are those which possess degenerate eigenvalues with respect to all the error generators. In other words, the set $|i\rangle$ which satisfies $F_{\alpha}|i\rangle = c_{\alpha}|i\rangle$, for all $\alpha$, form a DFS.

If the $F_{\alpha}$ form an $M$ dimensional semi-simple Lie algebra in the $N$ dimensional matrix representation, where $N = \dim H_{sc}$, then the decoherence free subspaces live in one dimensional irreducible representations of the Lie group (singlet states).

Appendix 3: DFS and Hilbert Spaces with many subsystems

Here we discuss how to view a DFS when $H_{sc}$ contains many subsystems. If the Hilbert space is given as:

$$\mathcal{H} = H_E \otimes \bigotimes_i H_i$$

Then we can define:

$$\rho_j(t) = \text{tr}_{H_E} \left[ \text{tr}_{i \neq j} \left( e^{iHt} \rho e^{-iHt} \right) \right].$$

Here $H$ is the total Hamiltonian. In many cases the form of the density matrix is:

$$\rho_j(t) \propto e^{-\alpha_j t}$$

with some time constant $\alpha_j$. Here we are using a short-hand notation to denote the strength of the non-unitary behaviour of the density matrix evolution. More formally, one can think about the evolution of the expectation values of all physically relevant operators. If all expectation values have decay behaviour equal or worse than that described in eq (16) then the time constant can be defined. We will be using the time constant as a measure of the breakdown of unitarity. (We remind the reader that density matrix evolution need not be unitary when an environmental trace is preformed. Also, it should be noted that the above measure of non-unitarity holds only in a free theory where $i$ and $j$ do not interact heavily. A better definition in which the subsystems interact is needed.)

Decoherence free subspaces may be defined as the density matrices in which the maximum time constant of the set of all time constants is below some threshold. In other words $\text{Max} (|\alpha_i|) < \alpha_{cut}$.

For exact DFS spaces we expect $\text{Max} (|\alpha_i|) \to 0$. What I described above is only an approximate way of defining DFS in the product Hilbert space. Also, this requires understanding on the initial factorization of the environment. However, when discussing decoherence, some knowledge of the environment is always required.
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