Resonance Frequencies of a Slab with Subwavelength Slits: a Fourier-transformation Approach

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Abstract

This paper proposes a novel, rigorous and simple Fourier-transformation approach to study resonances in a perfectly conducting slab with finite number of subwavelength slits of width $h \ll 1$. Since regions outside the slits are variable separated, by Fourier transforming the governing equation, we could express field in the outer regions in terms of field derivatives on the aperture. Next, in each slit where variable separation is still available, wave field could be expressed as a Fourier series in terms of a countable basis functions with unknown Fourier coefficients. Finally, by matching field on the aperture, we establish a linear system of infinite number of equations governing the countable Fourier coefficients. By carefully asymptotic analysis of each entry of the coefficient matrix, we rigorously show that, by removing only a finite number of rows and columns, the resulting principle sub-matrix is diagonally dominant so that the infinite dimensional linear system can be reduced to a finite dimensional linear system. Resonance frequencies are exactly those frequencies making the linear system rank-deficient. This in turn provides a simple, asymptotic formula describing resonance frequencies with accuracy $O(h^3 \log h)$. We emphasize that such a formula is more accurate than all existing results and is the first accurate result especially for slits of number more than two to our best knowledge. Moreover, this asymptotic formula rigorously confirms a fact that the imaginary part of resonance frequencies is always $O(h)$ no matter how we place the slits as long as they are spaced by distances independent of width $h$.

1 Introduction

Electro-magnetic wave scattering problems for optical devices with subwavelength structures have been extensively studied in recent years [2, 8, 7, 22, 26, 27, 29]. Distinctive phenomena such as extraordinary optical transmission and local field enhancement have been experimentally observed: light can be localized and greatly enhanced near subwavelength apertures or holes. Such features are vastly demanded in many areas, such as biological sensing and imaging, microscopy, spectroscopy and communication [21, 15]. The underlying theory of field enhancement, arguably, is largely related to wave frequency matching some resonance frequency in a scattering problem. Roughly speaking, a resonance frequency refers to certain complex frequency, at which the scattering problem allows a nonzero wave field to survive under no external excitation.

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In the past decades, a number of subwavelength structures have been studied to quantitatively analyze the enhancement of wave field \[3, 4, 5, 6, 9, 10, 11, 12, 13, 16, 17, 18, 19, 24, 23\]. Overall, these works have either numerically illustrated or rigorously proved the fact that: when wave frequency coincides with real part of some resonance frequency, the wave field can be enhanced by a factor inversely proportional to certain power of imaginary part of the resonance frequency. Existing theories treating resonances can be roughly categorized into two approaches: boundary-integral approach and matched-asymptotics approach. A representative work of the first approach is \[5\] by Bonnetier and Triki. They studied wave scattering by a perfectly conducting half plane with a subwavelength cavity and proposed a novel integral-equation technique incorporated with an operator version of Rouché’s theorem to asymptotically describe resonance frequencies; Green functions of subregions were used to develop governing integral equations on the aperture of the cavity, which, by asymptotic analysis of the integral kernels, leads to asymptotic behavior of resonance frequencies. Following this boundary-integral-equation approach, Babadjian et al. in \[3\] studied resonances by two interacting subwavelength cavities; later on, Lin and Zhang simplified the analyzing procedure of \[5\] and studied resonances by a slab with a single slit \[17\] or periodic slits \[18, 19\]; Gao et al. \[9\] studied resonance frequencies by a rectangular cavity with different conducting boundaries; recently, Lin et al. \[16\] studied Fano resonances in a slab with a periodic array of two subwavelength slits, and proved that such a subwavelength structure could support real resonance frequencies, a.k.a, bound states in the continuum \[28\] or embedded eigenvalues \[4, 24, 23\]. Joly and Tordeux \[11, 12, 13\] and Clausel et al. \[6\] have used the second approach to study resonances of thin slots. It is worthwhile to mention a nice work of Holley and Schnitzer \[10\], who used matched asymptotic analysis to get a closed-form of leading term of resonance frequencies of a slab with a single slit; Brandão et al. \[1\] have recently extended this approach to study resonances in a slab of finite conductivity with a single slit or a periodic array of slits.

This paper aims to establish a novel, rigorous but much simpler theory to study resonances in a perfectly conducting slab with a finite number of subwavelength slits of width \(h \ll 1\). Unlike the boundary-integral-equation approach which relies on subregion Green functions, our theory does not make use of any Green function, but only relies on Fourier transformations. The underlying motivation is now that subregion Green functions are basically derived by Fourier transformations, it is certainly more straightforward to study resonances by such an approach.

Since regions outside the slits are variable separated, by Fourier transforming the governing equation, we express field in the outer regions in terms of field derivatives on the aperture. Next, in each slit where variable separation is still available, waves could be expressed as Fourier series in terms of a countable basis functions with unknown Fourier coefficients. Finally, by matching field on the aperture, we establish a linear system of infinite number of equations governing the countable unknown coefficients. We note that \[10\] has already used a similar approach to establish a linear system, which, however, presumed certain symmetry of wave field and served only as a numerical solver when normal incidences of real frequencies are specified. By carefully asymptotic analysis of each entry of the coefficient matrix and by retaining entries of the matrix up to only leading algebraic order of \(h\), we rigorously show that, by removing only a finite number of rows and columns, the resulting principle sub-matrix is diagonally dominant so that the infinite dimensional linear system can be reduced to a finite dimensional linear system. Resonance frequencies are exactly those frequencies making the linear system rank-deficient. This in turn provides an asymptotic formula of resonance frequencies that is accurate up to
the order of $O(h^3 \log h)$, which is more accurate than all existing results; to the best of our knowledge, this is the first accurate result especially for a slab with slits of any finite number more than two. Moreover, our asymptotic formula rigorously confirms a fact that imaginary parts of all resonance frequencies are always $O(h)$ no matter how we place the slits as long as they are spaced by distances independent of $h$. As no Green function is used in our formulation, we expect that such an approach could be more flexible to study resonances in more complicated and realistic structures, e.g., a slab of finite/infinite conductivities with single or periodic slits or with single or periodic holes, which we shall report elsewhere.

The rest of this paper is organized as follows. In section 2, we present the Fourier-transformation approach by studying a perfectly conducting slab with a single slit. In section 3, we extend the approach to study resonances of a slab with multiple slits. In section 4, we draw the conclusion and present some potential applications of the current method.

## 2 Single slit

To clarify the basic idea of our Fourier-transformation approach, we begin with studying resonances of a perfectly conducting slab with a single slit, which has been studied in [17, 10]. Suppose a perfectly conducting slab of thickness $l$ is perturbed by a slit of width $h \ll 1$, as shown in Fig. 1 (a). Then, a TM-polarized electro-magnetic wave is governed by

$$\Delta u + k^2 u = 0, \quad \text{on} \quad \mathbb{R}^2 \setminus \Omega_h^{\pm},$$

$$\partial_{\nu} u = 0, \quad \text{on} \quad \partial \Omega_h^{\pm} \cup \partial \Omega_h^{-}, \quad \text{(1)} \quad \text{eq:helm}$$

where the two-dimensional Laplace operator $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $k$ is the freespace wavenumber, $u$ denotes the $z$-component of magnetic field, $\Omega_h^{\pm} = \{(x_1, x_2) : \pm x_1 > h/2, x_2 \in (-l, 0)\}$, $\nu$ denotes the outer normal vector along the boundaries $\partial \Omega_h^{\pm}$. Mathematically, a resonance frequency $k$ refers to a certain value in $\mathbb{C}$, at which there exists a nonzero solution $u$ solving (1,2) and is purely outgoing at infinity; we shall refer to the nonzero $u$ as a resonance mode in the following. It is known that when $\text{Im}(k) > 0$, the medium becomes lossy so that the original problem could not support a nonzero solution. On the other hand, we expect that $\text{Im}(k)$ should not be far away from 0. Thus, throughout this paper, we shall restrict the searching region to a bounded domain $S = \{k \in \mathbb{C} : \text{Re}(k) > \epsilon_0 > 0, |k| < M, \arg(k) \in (-\frac{\pi}{4}, 0)\}$ for a sufficiently large constant $M$ and a sufficiently small constant $\epsilon_0$. By rescaling the variables $x_1$ and $x_2$, we could assume $l = 1$ in the following.

![Figure 1](image-url)
Due to the symmetry of the structure, we split $u$ as the sum of even mode $u^e$ and odd mode $u^o$ about the axis $x_2 = -l/2$ in the following.

2.1 Even mode

For simplicity, we suppress the superscript $e$ in this section. Clearly, $u^e$ solves

$$\Delta u + k^2 u = 0, \quad \text{on } \Omega_h,$$

$$\partial_{\nu} u = 0, \quad \text{on } \Gamma_h,$$

where $\Omega_h = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \cup B_h \cup \{(x_1, 0) : |x_1| < h/2\}$, $B_h = \{(x_1, x_2) : |x_1| < h/2, x_2 \in (-l/2, 0)\}$, and $\Gamma_h$ denotes the boundary of $\Omega_h$, as shown in Fig. 1(b).

In region $\mathbb{R}^2_+$, Fourier transforming $x_1-$variable in (3) and making use of (4), we get

$$\hat{u}(x_2; \xi) = \hat{f}(\xi) e^{i\mu x_2},$$

where $\mu = \sqrt{k^2 - \xi^2}$, and

$$\hat{u}(x_2; \xi) = \int_{-\infty}^{+\infty} u(x)e^{i\xi x_1} dx_1,$$

$$\hat{f}(\xi) = \int_{-h/2}^{h/2} u(x_2(x_1, 0)e^{i\xi x_1} dx_1.$$  \hfill (7)  \hfill \{eq:hf\}

Throughout this paper, unless otherwise specified, we always choose the negative real axis as the branch cut of $\sqrt{\cdot}$. Thus, by Fourier inverse transform, in $\mathbb{R}^2_+$,

$$u(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\mu x_2 - i\xi x_1} d\xi.$$  \hfill (8)  \hfill \{eq:outer\}

Inside the slit $B_h$, by method of variable separations and by (4), $u$ can be represented as

$$u(x) = \sum_{n=0}^{+\infty} b_n \phi_n(x_1) [e^{is_n(x_2 + l)} + e^{-is_n x_2}], x_2 \in (-l/2, 0].$$  \hfill (9)  \hfill \{eq:inner\}

Here $\{b_n\}$ are unknown scaled Fourier coefficients to be determined,

$$\phi_n = \begin{cases} \sqrt{\frac{\pi}{h}} \cos \frac{n \pi x_1}{h} & n \mid 2; \\ i\sqrt{\frac{\pi}{h}} \sin \frac{n \pi x_1}{h} & n \nmid 2, \end{cases}$$

$$s_n = \sqrt{k^2 - \left(\frac{\pi n}{h}\right)^2}.$$  

When $h \ll 1$, using the negative real axis as the branch cut of $\sqrt{\cdot}$ could make $s_n$ a discontinuous and certainly nonholomorphic function for $k \in \mathbb{S}$. To resolve this issue, we use the negative imaginary axis as the branch cut here, only to define $s_n$ \hfill \{24\}. Clearly, when $\text{Im}(k) < 0$, $s_0 = k$ still preserves negative imaginary part while $s_n$ for $n > 0$ has positive imaginary part.
In the above, \( \{ \phi_n \} \) form a complete and orthogonal basis in the space \( L^2(-h/2, h/2) \) equipped with the natural inner product \( \langle \cdot, \cdot \rangle_2 \) (they are not orthonormal unless we redefine \( \phi_0 = \sqrt{1/h} \)). On segment \( \Gamma_A := \{(x_1,0)|x_1| < h/2, x_2 = 0\} \), we introduce the standard Sobolev space \( H^{1/2}(\Gamma_A) \) equipped with the following norm
\[
\|f\|_{H^{1/2}(\Gamma_A)} = \left( \sum_{n=0}^{+\infty} (1 + n^2)^{1/2} |\hat{f}_n|^2 \right)^{1/2},
\] where \( \hat{f}_n = (f, \phi_n)_2 \), and \( H^{-1/2}(\Gamma_A) \) as the completion of \( L^2(\Gamma_A) \) w.r.t the following norm: for any \( f \in L^2(\Gamma_A) \),
\[
\|f\|_{H^{-1/2}(\Gamma_A)} = \left( \sum_{n=0}^{+\infty} (1 + n^2)^{-1/2} |\hat{f}_n|^2 \right)^{1/2} < +\infty.
\]
Furthermore, we define
\[
\widetilde{H}^{-1/2}(\Gamma_A) = \{ f \in H^{-1/2}(\Gamma_A) : \exists \tilde{f} \in H^{-1/2}(\mathbb{R}) \text{ such that } \tilde{f} = 0 \text{ on } \mathbb{R} \setminus \Gamma_A, \text{ and } \tilde{f}|\Gamma_A = f \}.
\]
Clearly, \( \widetilde{H}^{-1/2}(\Gamma_A) = (H^{1/2}(\Gamma_A))' \). Note that the above norms are respectively equivalent to their standard norms \([13, 20]\). Furthermore, let \( \ell^2 := \{(a_n)_{n=1}^{+\infty} \subset \mathbb{C} : \sum_{n=1}^{+\infty} |a_n|^2 \leq +\infty \} \) be equipped with its natural norm. We see clearly that \( f \in H^{1/2}(\Gamma_A) \) iff \( \{(1 + n^2)^{1/4} \hat{f}_n \} \in \ell^2 \) and \( f \in H^{-1/2}(\Gamma_A) \) iff \( \{(1 + n^2)^{-1/4} \hat{f}_n \} \in \ell^2 \). In the following, we seek a nonzero solution \( u \in H^{1, \text{loc}}(\Omega_h) = \{ u : u \in H^1(\Omega_h \cap D_R), D_R = \{ x : |x| < R \} \forall R > 0 \} \) solving \([3]\) in the distributional sense. Thus, we require \( u|_{\Gamma_A} \in H^{1/2}(\Gamma_A) \) so that the following sequence
\[
\left\{(1 + n^2)^{1/4} b_n (1 + e^{is_n t}) \right\} \in \ell^2,
\]
and \( b_0 e^{is_0 t/2} = (u|_{x_2 = -t/2}, \phi_0) \) with \( |b_0| < \infty \). This implies
\[
\{a_n := \sqrt{n b_n} \}_{n=1}^{+\infty} \in \ell^2.
\]
On the contrary, if we are given a sequence \( \{a_n\}_{n=1}^{+\infty} \in \ell^2 \) with \( b_n = \frac{a_n}{\sqrt{n}} \) and if \( |b_0| < \infty \), then equation \([6]\) defines a solution \( u^\text{in} \in H^1(B_h) \) with \( \partial_n u^\text{in} = 0 \) on \( \partial B_h \cap \Gamma_h \). Thus, for \( h \ll 1 \),
\[
u_i^{\text{in}}(x_1,0) = \sum_{n=0}^{+\infty} i s_n b_n [e^{is_n t} - 1] \phi_n(x_1) \in \widetilde{H}^{-1/2}(\Gamma_A),
\]
since
\[
|(1 + n^2)^{-1/4} is_n b_n (e^{is_n t} - 1)| \leq (1 + n^2)^{1/4} O(b_n (1 + e^{is_n t})).
\]
Such a Neumann data \( u_i^{\text{in}}(x_1,0) \) on the real axis defines a unique solution \( u^\text{ext} \in H^{1, \text{loc}}(\mathbb{R}_+^2) \) of the Helmholtz equation \([3]\) in \( \mathbb{R}_+^2 \) with \( u_i^{\text{ext}}(x_1,0) = u_i^{\text{in}}(x_1,0) \). The two solutions \( u^\text{ext} \) and \( u^\text{in} \) together form a solution \( u \in H^{1, \text{loc}}(\Omega_h) \) of \([3]\) and \([4]\) as long as they share the same Dirichlet data on \( \Gamma_A \), i.e.,
\[
u_i^{\text{in}}(x_1,0) = u^\text{ext}(x_1,0) = u(x_1,0), \quad |x_1| < h/2.
\]
Thus, by (7) and (9),
\[
\hat{f}(\xi) = \sum_{n=0}^{+\infty} i s_n b_n [e^{i s_n \xi} - 1] \int_{-h/2}^{h/2} \phi_n(x_1)e^{i \xi x_1} dx_1
\]
\[
= \sqrt{\frac{2}{h}} \sum_{n'=0}^{+\infty} i s_{2n'} b_{2n'} [e^{i s_{2n'} \xi} - 1] \frac{2 \xi \sin(\xi h/2 + n' \pi)}{\xi^2 - \pi^2 (2n')^2 h^2}
\]
\[
+ \sqrt{\frac{2}{h}} \sum_{n'=0}^{+\infty} i s_{2n'+1} b_{2n'+1} [e^{i s_{2n'+1} \xi} - 1] \frac{2 \xi \sin(\xi h/2 + (n' + 1/2) \pi)}{\xi^2 - \pi^2 (2n'+1)^2 h^2}
\]
\[
= \sqrt{\frac{2}{h}} \sum_{n=0}^{+\infty} i s_n [e^{i s_n \xi} - 1] \frac{2 \xi \sin(\xi h/2 + n \pi/2)}{\xi^2 - \pi^2 n^2 h^2} b_n,
\]
so that equation (10) implies
\[
u(x_1, 0) = \sum_{n=0}^{+\infty} \phi_n(x_1) b_n [e^{i s_n \xi} + 1] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{-i \xi x_1} d\xi
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{i \mu} e^{-i \xi x_1} \sqrt{\frac{2}{h}} \sum_{m=0}^{+\infty} i s_m [e^{i s_m \xi} - 1] \frac{2 \xi \sin(\xi h/2 + m \pi/2)}{\xi^2 - \pi^2 m^2 h^2} b_m d\xi
\]
\[
= \sum_{m=0}^{+\infty} b_m [e^{i s_m \xi} - 1] \psi_m(x_1),
\]
where
\[
\psi_m(x_1) = \frac{1}{2\pi} \sqrt{\frac{2}{h}} \int_{-\infty}^{+\infty} \frac{s_m 2 \xi \sin(\xi h/2 + m \pi/2)}{\mu} e^{-i \xi x_1} d\xi.
\]
Rewriting the above equation in terms of \(\{a_n\} \in \ell^2\),
\[
\phi_0(x_1) b_0 [e^{i s_0 \xi} + 1] + \sum_{n=1}^{+\infty} n^{-1/2} \phi_n(x_1) a_n [e^{i s_n \xi} + 1]
\]
\[
= b_0 [e^{i s_0 \xi} - 1] \psi_0(x_1) + \sum_{m=1}^{+\infty} m^{-1/2} a_m [e^{i s_m \xi} - 1] \psi_m(x_1).
\]
Consequently, the above arguments in fact imply the following equivalent relation:

**Finding a nonzero solution** \(u \in H^1_{\text{loc}}(\Omega_h)\) of (3) and (4)

\[\iff\] Finding a nonzero sequence \(\{a_n\}_{n=1}^{\infty} \in \ell^2\) and \(b_0\) solving (11).

Now, taking \(\ell^2\)-inner product of (11) and \(\phi_n\) for \(n = 0, \cdots\), we get the following linear equations of infinite dimensions:
\[
2 b_0 [e^{i s_0 \xi} + 1] = b_0 [e^{i s_0 \xi} - 1] c_00 + \sum_{m=1}^{+\infty} a_{2m} c_{2m,0},
\]
\[\text{(12) } \{\text{eq:b0}\}\]
\[
a_{2n} = b_0 [e^{i s_0 \xi} - 1] c_{0,2n} + \sum_{m=1}^{+\infty} a_{2m} c_{2m,2n}, \quad n = 1, \cdots,
\]
\[\text{(13) } \{\text{eq:bneven}\}\]
\[
a_{2n-1} = \sum_{m=1}^{+\infty} a_{2m-1} c_{2m-1,2n-1}, \quad n = 1, \cdots,
\]
\[\text{(14) } \{\text{eq:bnodd}\}\]

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where the sequence \( \{d_{mn}\}_{m,n=0}^\infty \) is defined as
\[
d_{mn} = \int_{-\infty}^{+\infty} \frac{1}{\mu h^2} \frac{\xi \sin(\xi h/2 + m\pi/2)}{\xi^2 - \frac{\pi^2 m^2}{h^2}} \frac{\xi \sin(\xi h/2 + n\pi/2)}{\xi^2 - \frac{\pi^2 n^2}{h^2}} d\xi,
\]
(15) \{eq:def:dmn\}
and for \( m, n \geq 1 \),
\[
c_{mn} = 4s_m h \sqrt{n(e^{is_m} - 1)} d_{mn},
\]
\[
c_{m0} = 4s_m h \sqrt{m(e^{is_m} - 1)} d_{m0},
\]
\[
c_{0n} = 4s_0 h \sqrt{n} d_{0n},
\]
and we have used the fact that \( c_{mn} \equiv 0 \) when \( m + n \not\equiv 2 \). The above definition of \( d_{mn} \) is well-defined for \( k \in \mathbb{R}^+ \). However, to ensure that \( d_{mn} \) is a holomorphic function of \( k \in \mathcal{S} \), we should redefine \( d_{mn} \) as follows
\[
d_{mn} = \int_{\mathcal{I}} \frac{1}{\mu h^2} \frac{\xi \sin(\xi h/2 + m\pi/2)}{\xi^2 - \frac{\pi^2 m^2}{h^2}} \frac{\xi \sin(\xi h/2 + n\pi/2)}{\xi^2 - \frac{\pi^2 n^2}{h^2}} d\xi,
\]
(16) \{eq:def:dmn\}
where \( \mathcal{I} \) indicates a Sommerfeld integral path such that \( \mathcal{S} \) lies above \( \mathcal{I} \) and its symmetry about origin lies below \( \mathcal{I} \); as shown in Figure 2.

Figure 2: Sommerfeld integral path \( \mathcal{I} \), the searching region \( \mathcal{S} \) in the fourth quadrant and its symmetry.

For \( h \ll 1 \) so that \( \epsilon = kh \ll 1 \) for \( k \in \mathcal{S} \), we have the following lemma accounting for the asymptotics of \( d_{mn} \); its proof is presented in the Appendix.

**Lemma 2.1.** For \( \epsilon \ll 1 \), the sequence \( \{d_{mn}\}_{m,n=0}^\infty \) asymptotically behaves as: for \( k \in \mathbb{Z} \),
\[
d_{mn} = \begin{cases} 0, & \text{when } m + n \not\equiv 2; \\ \frac{\pi}{2} + \frac{1}{2} (\gamma - \log 2 - \frac{3}{2}) + \frac{1}{2} \log \epsilon + \mathcal{O}(\epsilon^2 \log \epsilon) & \text{when } m = n = 0; \\ i(-1)^{m/2} C_0(\pi m) + m^{-2} \mathcal{O}(\epsilon^2 \log \epsilon) & 0 < m | 2, n = 0; \\ i(-1)^{n/2} C_0(\pi n) + n^{-2} \mathcal{O}(\epsilon^2 \log \epsilon) & m = 0, 0 < n | 2; \\ i(-1)^{(m-n)/2} C_\ominus(\pi m, \pi n) - i \frac{1}{4m} \delta_{mn} + \frac{\log m - \log n \log \epsilon}{m^2 - n^2} \mathcal{O}(\epsilon^2) & 0 < m, n | 2; \\ i(-1)^{(m-n)/2} C_\oplus(\pi m, \pi n) - i \frac{1}{4m} \delta_{mn} + \frac{\log m - \log n \log \epsilon}{m^2 - n^2} \mathcal{O}(\epsilon^2 \log \epsilon) & 0 < m, n \not\equiv 2, \\ \end{cases}
\]
(17) \{eq:dmn\}
where \( \delta_{mn} \) denotes the Kronecker delta function, for \( b, b' \geq \pi \),

\[
C_0(b) = \int_0^\infty \frac{1 - e^{-kt}}{t(k^2t^2 + b^2)} dt \leq \frac{\log b}{b^2}, \tag{18} \]

\[
C_0^-(b, b') = \int_0^\infty \frac{t(1 - e^{-t})}{(t^2 + b^2)(t^2 + b'^2)} dt \leq \frac{\log b - \log b'}{b^2 - b'^2}, \tag{19} \]

\[
C_0^+(b, b') = \int_0^\infty \frac{t(1 + e^{-t})}{(t^2 + b^2)(t^2 + b'^2)} dt \leq \frac{\log b - \log b'}{b^2 - b'^2} + \frac{1}{b^2b'^2}, \tag{20} \]

limit is considered when \( b = b' \), and the invisible constants in \( O \) notations are independent of \( m \) and \( n \).

Moreover, as \( h \to 0 \),

\[
s_m h = \pi m + O(e^m n^{-1}),
1 \pm e^{s_m l} = 1 + O(e^{-m \pi l/h}).
\]

We get the asymptotic expansions of \( c_{mn} \) as follows.

**Lemma 2.2.** For \( \epsilon \ll 1 \), the sequence \( \{c_{mn}\}_{m,n=0}^\infty \) asymptotically behaves as: for \( k \in \mathbb{Z} \),

\[
c_{mn} = \begin{cases} 0, & m + n \not\mid 2; \\
\epsilon + \frac{2}{n} (\gamma - \log 2 - \frac{3}{2}) \epsilon + \frac{2}{n} \pi \log \epsilon + O(\epsilon^2 \log \epsilon) & m = n = 0; \\
4\sqrt{m(-1)^{m/2}C_0(\pi m) + m^{-3/2}O(\epsilon^2 \log \epsilon)} & n = 0 < m \mid 2; \\
\frac{4 \pi}{n} (-1)^{m/2} C_0(\pi n) + m^{-3/2}O(\epsilon^2 \log \epsilon) & m = 0 < n \mid 2; \\
p_m^{(e)} - \delta_{mn} + m^{-2n}O(\epsilon^2) & 0 < m, n \mid 2; \\
p_m^{(o)} - \delta_{mn} + \sqrt{m^{-2n} - n^{2}}O(\epsilon^2 \log \epsilon) & 0 < m, n \mid 2, \end{cases} \tag{21} \]

where we have defined

\[
p_m^{(e)} = 4 \sqrt{m(-1)^{(m-n)/2}C_0^-(\pi m, \pi n)} \tag{22} \]

\[
p_m^{(o)} = 4 \sqrt{m(-1)^{(m-n)/2}C_0^+(\pi m, \pi n)}. \tag{23} \]

Now for any integer \( n \geq 1 \), let \( A_n^{(e)} \) and \( A_n^{(o)} \) be defined as: for any \( \{f_j\}_{j=1}^\infty \in \ell^2 \),

\[
A_n^{(e)}(f_j) = \{\chi_n(i) \sum_{j=1}^\infty (c_{2i, 2j} + \delta_{ij}) f_j \chi_n(j)\}_{i=1}^\infty, \tag{24} \]

\[
A_n^{(o)}(f_j) = \{\chi_n(i) \sum_{j=1}^\infty (c_{2i-1, 2j} - \delta_{ij}) f_j \chi_n(j)\}_{i=1}^\infty, \tag{25} \]

where

\[
\chi_n(i) = \begin{cases} 1 & i \leq n; \\
0 & \text{otherwise}. \end{cases}
\]

We have the following theorem.

**Theorem 2.1.** For \( \epsilon \ll 1 \), the operators \( \{A_n^{(l)}\}_{n=1}^\infty, l = e, o \) mapping from \( \ell^2 \) to \( \ell^2 \) are uniformly bounded, i.e.,

\[
\|A_n^{(l)}\| \leq \frac{1}{2} + \frac{4}{\pi^4} + O(\epsilon^2).
\]
As a consequence, there exists a bounded and contracting operator \( A^{(e)} : \ell^2 \to \ell^2 \) such that for any \( \{f_j\}_{j=1}^{\infty} \) in \( \ell^2 \),

\[
A^{(e)} \{f_j\} = \{ \sum_{j=1}^{\infty} c_{2i,2j} f_j \}_{i=1}^{\infty} \in \ell^2, \quad A^{(o)} \{f_j\} = \{ \sum_{j=1}^{\infty} c_{2i-1,2j-1} f_j \}_{i=1}^{\infty} \in \ell^2,
\]

and

\[
\|A^{(l)}\| \leq \frac{1}{2} + \frac{4}{\pi^4} + O(\epsilon^2), \tag{26} \eqref{eq: Ai:norm}
\]

for \( l = e, o \).

**Proof.** Here, we prove the property of \( A^{(e)} \) only, and shall suppress the superscript for simplicity. We first prove the contraction of \( A_n \). For \( \epsilon \ll 1 \), we have

\[
c_{2i,2j} + \delta_{ij} = 8i\sqrt{ij}d_{2i,2j} + i^{-3/2}j^{1/2}\log i - \log j - O(\epsilon^2),
\]

where limit is considered when \( i = j \). Note that we should not use the expansion \( \eqref{eq: Ai:norm} \) since the neglected part is not symmetric. Let \( P_n \) and \( Q_n \) be operators defined as \( A_n \) but with \( a_{2i,2j} \) replaced respectively by

\[
P_{ij} = 8i\sqrt{ij}d_{2i,2j} := P^{(1)}_{ij} + iP^{(2)}_{ij}, \quad Q_{ij} = \epsilon^{-2}(c_{2i,2j} + \delta_{ij} - 8i\sqrt{ij}d_{2i,2j}),
\]

for \( i, j \geq 1 \). Then

\[
\|(Q_{ij})_{n \times n}\|_{\text{FRO}}^2 \lesssim \sum_{i,j=1}^{n} i^{-3,5}(\log i - \log j)^2 / (i^2 - j^2)^2 < \infty,
\]

where \( \| \cdot \|_{\text{FRO}} \) is the Frobenius norm. Since \( \|(Q_{ij})_{n \times n}\|_{\text{FRO}} \) is strictly increasing w.r.t \( n \), and the 2-norm \( \| \cdot \|_2 \leq \| \cdot \|_{\text{FRO}} \), we see clearly that \( \{Q_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathcal{L}(\ell^2; \ell^2) \), converging to a bounded operator \( Q : \ell^2 \to \ell^2 \). As for the \( n \times n \) symmetric matrix \( (P^{(i)}_{ij})_{n \times n}, i = 1, 2, \) its 2-norm is exactly the magnitude of its largest eigenvalue.

Thus, we choose to estimate the eigenvalue of matrix \( (\sqrt{P^{(i)}_{ij}}/\sqrt{J^{-1}})_{n \times n} \), which is similar to \( (P^{(i)}_{ij})_{n \times n} \). By Lemma \ref{lem: 2.2}

\[
P^{(1)}_{ij} = p_{2i,2j} + \frac{\sqrt{ij}(\log i - \log j)}{i^2 - j^2}O(\epsilon^2) = \frac{\sqrt{ij}(\log i - \log j)}{i^2 - j^2}(1 + O(\epsilon^2)),
\]

\[
P^{(2)}_{ij} = \frac{\sqrt{ij}(\log i - \log j)}{i^2 - j^2}O(\epsilon^2),
\]

and

\[
\sum_{j=1}^{n} \sqrt{\frac{\sqrt{ij}(\log i - \log j)}{i^2 - j^2}J^{-1}} \leq 8 \sum_{j=1}^{n} \frac{i \log i - \log j}{4\pi^2(i^2 - j^2)} \leq \frac{2}{\pi^2} \int_{0}^{+\infty} \frac{i \log x - \log i}{x^2 - i^2} dx = \frac{2}{\pi^2} \int_{0}^{+\infty} \frac{\log x}{x^2 - 1} dx \leq \frac{1}{2},
\]

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where we notice that the terms in the summation is decreasing in $j$. Therefore, for all $n \in \mathbb{N}$,

$$
\|\text{Re}(P_n)\| = \|(P_{ij}^{(1)})_{n \times n}\|_2 \leq \|(t^{1/2}P_{ij}^{(1)}j^{-1/2})_{n \times n}\|_1 \leq \frac{1}{2}(1 + O(\epsilon^2)),
$$

for $\epsilon \ll 1$. One similarly obtains that $\|\text{Im}(P_n)\| = O(\epsilon^2)$ so that $\|P_n\| \leq 1/2 + O(\epsilon^2)$ for all $n$, where the invisible constant in the $O$-notation is independent of $n$. Suppose $\ell^2_{\text{comp}} = \{\{f_i\} \in \ell^2 : \exists N > 0, f_i \equiv 0 \text{ for } i \geq N\}$. Clearly, $\ell^2_{\text{comp}}$ is dense in $\ell^2$. Now, we define $P^{(e)} : \ell^2_{\text{comp}} \to \ell^2 = (\ell^2)'$ as follows: for any $\{f_i\}_i, \{g_i\}_i \in \ell^2_{\text{comp}},$

$$
< P^{(e)}\{f_i\}, \{g_i\} > := \sum_{i,j=1}^{\infty} g_j p_{2i,2j}^{(e)} f_i.
$$

Clearly, the above summation is finite since $g_i = f_i \equiv 0$ for $i \geq N$. Then, by

$$
| < P^{(e)}\{f_i\}, \{g_i\} > | \leq \|(P_{2i,2j}^{(e)})_{N \times N}\|_2 \|\{f_i\}\|_2 \|\{g_i\}\|_2 \leq \frac{1}{2}\|\{f_i\}\|_2 \|\{g_i\}\|_2,
$$

we get from continuous extension theorem that $P^{(e)}\{f_i\} \in (\ell^2)'$, and

$$
\|P^{(e)}\{f_i\}\|_2 = \|P_D\{f_i\}\|_{(\ell^2)'} \leq \frac{1}{2}\|\{f_i\}\|_2,
$$

which states that $P^{(e)}$ is bounded from $\ell^2_{\text{comp}} \to \ell^2$. By continuous extension theorem again, we see that $P^{(e)}$ can be uniquely extended as a bounded operator from $\ell^2$ to $\ell^2$ such that

$$
\|P^{(e)}\| \leq \frac{1}{2}.
$$

One similarly proves the existence of $P_e : \ell^2 \to \ell^2$ defined by the elements $\{P_{ij} - p_{2i,2j}^{(e)}\}$ with $\|P_e\| \leq O(\epsilon^2)$. and the proof is completed by observing that $A = P^{(e)} + P_e + Q\epsilon^2$. \hfill \Box

We are ready to solve the infinite dimensional linear system \eqref{eq:op1}--\eqref{eq:op3}, which can be restated as: Seek nonzero $\{a_j\}_{j=1}^{\infty} \in \ell^2$ and $b_0 < \infty$, such that

\begin{align*}
2b_0(e^{x_{10}} + 1) &= b_0(e^{x_{10}} - 1)c_{00} + \{a_{2m}\}, \{c_{2m,0}\} > \epsilon, \quad (28) \quad \text{\eqref{eq:op1}}
\end{align*}

\begin{align*}
2\{a_{2n}\} &= b_0(e^{x_{10}} - 1)\{c_{0,2n}\} + A^{(e)}\{a_{2n}\}, \quad (29) \quad \text{\eqref{eq:op2}}
\end{align*}

\begin{align*}
2\{a_{2n-1}\} &= A^{(o)}\{a_{2n-1}\}, \quad (30) \quad \text{\eqref{eq:op3}}
\end{align*}

where we have used the fact that $\{c_{2m,0}\}, \{c_{0,2m}\} \in \ell^2$. As $A^{(i)}, i = e, o,$ is contracting for $\epsilon \ll 1$, $2\text{Id} - A^{(i)}$ is invertible, where $\text{Id}$ stands for the identity operator. Consequently, we arrive at our first theorem.

**Theorem 2.2.** For $\h \ll 1$, the system \eqref{eq:op1}--\eqref{eq:op3} has a nonzero solution if and only if $k$ solves

$$
2(e^{k\h} + 1) = (e^{k\h} - 1) \left(c_{00} + (2\text{Id} - A^{(e)})^{-1}\{c_{0,2m}\}, \{c_{2m,0}\} > \epsilon\right). \quad (31) \quad \text{\eqref{eq:gov:k0:e}}
$$

In fact, the solutions (resonance frequencies) to \eqref{eq:gov:k0:e} are

$$
kl = k_{m,e} - \frac{1}{2} \frac{e}{\pi} \Delta(\epsilon, m, e) - \left[k_{m,e}^{-1} - 5\pi^{-1}k_{m,e}^{-1}\h \right] \Delta^2(\epsilon, m, e) + \left[k_{m,e}^{-2} - \frac{1}{12}\right] \Delta^3(\epsilon, m, e)
$$
where $k_{m,e} = (2m - 1)\pi$ are the Fabry-Pérot frequencies and $\epsilon_{m,e} = k_{m,e}h \ll 1$,

$$\Delta(\epsilon) = \epsilon + \frac{2i}{\pi}(\gamma - \log 2 - \frac{3}{2} + \pi \alpha)\epsilon + \frac{2i}{\pi} \epsilon \log \epsilon,$$

$$\alpha = \frac{32}{\pi} < (2\text{Id} - P(\epsilon))^{-1}\{\sqrt{m(-1)^mC_0(2\pi m)}\}, \sqrt{m(-1)^mC_0(2\pi m)} > \epsilon^2,$$

and we recall that $P(\epsilon)$ is defined in (27). The corresponding solutions to (28-30) are

$$b_0 = 1,$$

$$\{a_{2m-1}\} = \{0\},$$

$$\{a_{2n}\} = (e^{ikl} - 1)(2\text{Id} - A^{(\epsilon)})^{-1}\{c_{0,2n}\},$$

**Proof.** By Lemma 2.2 and by

$$||(2\text{Id} - A^{(\epsilon)})^{-1} - (2\text{Id} - P(\epsilon))^{-1}|| = O(\epsilon^2 \log \epsilon),$$

equation (31) becomes:

$$2(e^{ikl} + 1) = (e^{ikl} - 1)\Delta(\epsilon) + O(\epsilon^3 \log \epsilon),$$

which is equivalent to

$$e^{ikl} + 1 = -\frac{\Delta(\epsilon)}{1 - \Delta(\epsilon)/2} + O(\epsilon^3 \log \epsilon).$$

As the right-hand side approaches 0 as $\epsilon \to 0$, we see that the resonance frequencies must satisfy: for some $m = 1, \cdots,$

$$\delta_{m,e} := kl - k_{m,e} = o(1).$$

Thus,

$$\epsilon - \epsilon_{m,e} = h\delta_{m,e},$$

$$\epsilon \log \epsilon - \epsilon_{m,e} \log \epsilon_{m,e} = h\delta_{m,e} \log \epsilon + h^2 \delta_{m,e} + O(h^3 \delta_{m,e}^2 \epsilon^{-1}),$$

as $h \to 0^+$. Therefore, we have

$$e^{i\delta_{m,e}} - 1 = \frac{\Delta(\epsilon)}{1 - \Delta(\epsilon)/2} + O(\epsilon^3 \log \epsilon),$$

so that by Taylor’s expansion of $\log(1 + x)$ and $1/(1 - x)$ at $x = 0$,

$$\delta_{m,e} = -i \log \left[1 + \frac{\Delta(\epsilon)}{1 - \Delta(\epsilon)/2} + O(\epsilon^3 \log \epsilon) \right]$$

$$= -i \left[ \frac{\Delta(\epsilon)}{1 - \Delta(\epsilon)/2} - \frac{\Delta^2(\epsilon)}{2(1 - \Delta(\epsilon)/2)^2} + \frac{\Delta^3(\epsilon)}{3(1 - \Delta(\epsilon)/2)^3} \right] + O(\epsilon^3 \log \epsilon)$$

$$= -i \left[ \Delta(\epsilon) + \frac{1}{12} \Delta^3(\epsilon) \right] + O(\epsilon^3 \log \epsilon).$$

Thus, $\delta_{m,e} \approx \frac{2}{\pi} \epsilon_{m,e} \log \epsilon_{m,e}$ implies

$$\epsilon \log \epsilon - \epsilon_{m,e} \log \epsilon_{m,e}$$
\[
\begin{align*}
\Delta(\epsilon) - \Delta(\epsilon_{m,e}) &= k_{m,e}^{-1}\epsilon_{m,e}\log(\epsilon_{m,e}) + k_{m,e}^{-1}\epsilon_{m,e}\delta_{m,e}\log(1 + \delta_{m,e}/k_{m,e}) + \epsilon_{m,e}\log(1 + \delta_{m,e}/k_{m,e}) \\
&= k_{m,e}^{-1}\epsilon_{m,e}\delta_{m,e}\log(\epsilon_{m,e}) + k_{m,e}^{-1}\delta_{m,e}\epsilon_{m,e} + \frac{1}{2}k_{m,e}^{-2}\delta_{m,e}^2\epsilon_{m,e} + O(\epsilon_{m,e}^3 \log \epsilon_{m,e}).
\end{align*}
\]

Based on the definition of \(\Delta\), we get

\[
\Delta^3(\epsilon) - \Delta^3(\epsilon_{m,e}) = (\Delta(\epsilon) - \Delta(\epsilon_{m,e}))(\Delta^2(\epsilon) + \Delta(\epsilon)\Delta(\epsilon_{m,e}) + \Delta^2(\epsilon_{m,e})) = O(\epsilon_{m,e}^4 \log^4 \epsilon_{m,e}).
\]

Therefore,

\[
\delta_{m,e} = -i\Delta(\epsilon_{m,e}) - ik_{m,e}^{-1}\delta_{m,e}\Delta(\epsilon_{m,e}) + \frac{2}{\pi}k_{m,e}^{-1}\delta_{m,e}\epsilon_{m,e} + \frac{1}{2}k_{m,e}^{-2}\delta_{m,e}^2\epsilon_{m,e} + O(\epsilon_{m,e}^3 \log \epsilon_{m,e}),
\]

which is equivalent to

\[
A\delta_{m,e}^2 + B\delta_{m,e} + C = 0,
\]

where

\[
A = \pi^{-1}k_{m,e}^{-2}\epsilon_{m,e} = O(\epsilon_{m,e}),
\]
\[
B = \frac{2}{\pi}k_{m,e}^{-1}\epsilon_{m,e} - ik_{m,e}^{-1}\Delta(\epsilon_{m,e}) - 1 \approx -1,
\]
\[
C = -i\Delta(\epsilon_{m,e}) - \frac{i}{12}\Delta^3(\epsilon_{m,e}) + O(\epsilon_{m,e}^3 \log \epsilon_{m,e}) = O(\epsilon_{m,e} \log \epsilon_{m,e}).
\]

Solving this quadratic equation,

\[
\delta_{m,e} = -\frac{2C}{B + \sqrt{B^2 - 4AC}} = -\frac{2C}{\frac{B}{2} + \frac{AC}{2B^2}} + O(\epsilon_{m,e}^5 \log^3 \epsilon_{m,e})
\]

\[
= -\frac{C}{B} + \frac{AC^2}{B^3} + O(\epsilon_{m,e}^5 \log^3 \epsilon_{m,e})
\]

\[
= -\frac{i\Delta(\epsilon_{m,e}) - \frac{i}{12}\Delta^3(\epsilon_{m,e})}{\frac{2}{\pi}k_{m,e}^{-1}\epsilon_{m,e} - ik_{m,e}^{-1}\Delta(\epsilon_{m,e}) - 1} - \frac{\pi^{-1}k_{m,e}^{-2}\epsilon_{m,e}(-i\Delta(\epsilon_{m,e}))^2}{(\frac{2}{\pi}k_{m,e}^{-1}\epsilon_{m,e} - ik_{m,e}^{-1}\Delta(\epsilon_{m,e}) - 1)^3} + O(\epsilon_{m,e}^3 \log \epsilon_{m,e})
\]

\[
= -i\Delta(\epsilon_{m,e})\left[1 + \left(\frac{2}{\pi}k_{m,e}^{-1}\epsilon_{m,e} - ik_{m,e}^{-1}\Delta(\epsilon_{m,e})\right)^2 + \left(\frac{2}{\pi}k_{m,e}^{-1}\epsilon_{m,e} - ik_{m,e}^{-1}\Delta(\epsilon_{m,e})\right)^3\right]
\]

\[
- \frac{i}{12}\Delta^3(\epsilon_{m,e}) - \pi^{-1}k_{m,e}^{-2}\epsilon_{m,e}\Delta^2(\epsilon_{m,e}) + O(\epsilon_{m,e}^3 \log \epsilon_{m,e})
\]
behave as $\epsilon^{kl}$ for $\epsilon_{m,e} \ll 1$. As for the existence of such solutions, one just notices that when $kl$ lies in $D_h = \{ k \in \mathbb{C} : |kl - k_{m,e}| \leq h^{1/2} \} \subset \mathcal{S}$, then on the boundary of this disk

$$
2(e^{ikl} + 1) - (e^{ikl} - 1) \left[ c_{00} + (\text{Id} - A^{(e)})^{-1}\{c_{0,2m}, \{c_{2m,0}\} \geq \epsilon^3\} - 2i(kl - k_{m,e}) \right]
$$

is $O(h) \leq 2\sqrt{h} = |2i(kl - k_{m,e})|$.

Rouché’s theorem indicates that there exists a unique solution to (31) in $D_h$. □

**Remark 1.** In (32), $\Delta^3(\epsilon_{m,e})$ contains terms greater than the error term $O(\epsilon_{m,e}^3 \log \epsilon_{m,e})$; we keep it here to make the expansion more compact and easier to evaluate. On the other hand, by (32),

$$
kl = k_{m,e} - i \left[ 1 + \frac{2h}{\pi} \right] \Delta(\epsilon_{m,e}) + O(h^2 \log^2 h),
$$

coincides with the result in Proposition 4.5 of [17]. By retaining leading behaviors of $c_{ij}$ here, we obtain an asymptotic formula of accuracy $O(h^3 \log h)$ that is much more accurate.

**Remark 2.** Like [17], our asymptotic formula contains an undetermined constant $\alpha$ as well. In [10], the authors use a method of matched asymptotic expansions to solve the single-slit scattering problem when a normal incident wave is specified and exactly describes the leading behavior of a real frequency (see Eq. (35) therein) at which transmission efficiency reaches a peak. In fact, such real frequencies are exactly real parts of resonance frequencies. By comparing their formula and the real part of (32), we easily conclude that

$$
\alpha = 1/\pi - 2/\pi \log(\pi/2).
$$

### 2.2 Odd mode

Now, we consider the odd mode $u^o$. Since the theory is essentially the same as the even case, we show briefly the results. $u^o$ solves

\[
\begin{align*}
\Delta u + k^2 u &= 0, \quad \text{on} \quad \Omega_h, \quad \{\text{eq:helm:o}\} \\
\partial_n u &= 0, \quad \text{on} \quad \Gamma \backslash \{(x_1, x_2) : x_2 = -l/2, |x_1| \leq h/2\}, \quad \{\text{eq:cond:o1}\} \\
u &= 0, \quad \text{on} \quad \{(x_1, x_2) : x_2 = -l/2, |x_1| < h/2\}. \quad \{\text{eq:cond:o2}\}
\end{align*}
\]

In $B_h$, we could represent $u$ as the following form,

\[
u(x) = \sum_{n=0}^{\infty} b_n \phi_n(x_1) [e^{is_n(x_2 + l)} - e^{-is_nx_2}]. \quad \{\text{eq:o}\}
\]

Thus, for $|x_1| \leq h/2$, we get

\[
u_{x_2}(x_1, 0) = \sum_{n=0}^{\infty} is_n b_n [e^{is_n l} + 1] \phi_n(x_1),
\]

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In fact, the nonzero solutions (resonance frequencies) to (47) are

\[ k_l = k_{m,o} - i \left[ 1 + \frac{2h}{\pi} \Delta(\epsilon_{m,o}) - \left[ k_{m,o}^{-1} - 5\pi^{-1}k_{m,o}^{-1}h \right] \Delta^2(\epsilon_{m,o}) + i \left( k_{m,o}^{-2} - \frac{1}{12} \right) \Delta^3(\epsilon_{m,o}) \right] + O(\epsilon_{m,o}^3 \log \epsilon_{m,o}), \quad m = 1, 2, \ldots , \]

Consequently, we obtain the second theorem.

\textbf{Theorem 2.3.} For \( h \ll 1 \), the system (43-45) has a nonzero solution if and only if \( k \) solves

\[ 2(e^{ikl} - 1) = (e^{ikl} + 1) \left[ c_{00}^{(o)} + < (2\text{Id} - A^{(e)})^{-1} \{ c_{0,2m}^{(o)} \} + c_{2m,0}^{(o)} > \right] . \]

\[ \text{In fact, the nonzero solutions (resonance frequencies) to (47) are} \]

\[ k_l = k_{m,o} - i \left[ 1 + \frac{2h}{\pi} \Delta(\epsilon_{m,o}) - \left[ k_{m,o}^{-1} - 5\pi^{-1}k_{m,o}^{-1}h \right] \Delta^2(\epsilon_{m,o}) + i \left( k_{m,o}^{-2} - \frac{1}{12} \right) \Delta^3(\epsilon_{m,o}) \right] + O(\epsilon_{m,o}^3 \log \epsilon_{m,o}), \quad m = 1, 2, \ldots , \]
where $k_{m,o} = 2m\pi$ are the Fabry-Pérot frequencies and $\epsilon_{m,o} = k_{m,o} h \ll 1$. The corresponding solutions to (43-45) are

\begin{align*}
    b_0 &= 1, \\
    \{a_{2n-1}\} &= \{0\}, \\
    \{a_{2n}\} &= (e^{ikl} + 1)(2I - A^e)^{-1}\{c_{o,2n}^{(o)}\},
\end{align*}

(49) \hspace{1cm} (50) \hspace{1cm} (51)

**Proof.** By Lemma 2.2 and by

\begin{equation}
    \|(2I - A^e)^{-1} - (2I - P^e)^{-1}\| = O(\epsilon^2 \log \epsilon),
\end{equation}

(52)

equation (47) becomes:

\begin{equation}
    2(e^{ikl} - 1) = (e^{ikl} + 1)\Delta(\epsilon) + O(\epsilon^3 \log \epsilon),
\end{equation}

which is equivalent to

\begin{equation}
    e^{ikl} - 1 = \frac{\Delta(\epsilon)}{1 - \Delta(\epsilon)/2} + O(\epsilon^3 \log \epsilon).
\end{equation}

As the right-hand side approaches 0 as $\epsilon \to 0$, we see that the resonance frequencies must satisfy: for some $m = 1, \cdots$,

\begin{equation}
    \delta_{m,o} := kl - k_{m,o} = o(1).
\end{equation}

The proof follows from similar arguments of Theorem 2.2. We omit the details. \hfill \Box

### 3 Multiple slits

We study the resonance frequencies for a slab with two slits first. As we shall see, the formula of resonance frequencies of a two-slit slab can be easily extended to a slab with any finite number of slits. Suppose the slab has two slits of the same width $h$ spaced by $D$ independent of $h$, as illustrated in Figure 3.

![Figure 3: A perfectly conducting slab with two slits of width $h$ spaced by $D$.](image)

We consider even modes first, i.e. $u$ satisfies $u(x_1, x_2 + l/2) = u(x_1, -x_2 + l/2)$. Thus, we have inside the two slits,

\begin{equation}
    u(x) = \sum_{n=0}^{\infty} b_n \phi_n(x_1) [e^{ik_n(x_2 + l)} + e^{-ik_n x_2}], \quad |x_1| < h/2;
\end{equation}
Then, we get
\[
u(x) = \sum_{n=0}^{\infty} b_n' \phi_n(x_1 - D)[e^{i\pi n(x_2 + l)} + e^{-i\pi n x_2}], |x_1 - D| < h/2.
\]

Thus, we obtain
\[
\sum_{n=0}^{\infty} \phi_n(x_1) b_n[e^{i\pi n}] + 1 = \sum_{m=0}^{\infty} (e^{i\pi n} - 1) \left[ \psi_m(x_1) b_m + \psi_m(x_1 - D) b'_m \right], |x_1| < h/2,
\]
\[
\sum_{n=0}^{\infty} \phi_n(x_1 - D) b'_n[e^{i\pi n}] + 1 = \sum_{m=0}^{\infty} (e^{i\pi n} - 1) \left[ \psi_m(x_1) b_m + \psi_m(x_1 - D) b'_m \right], |x_1 - D| < h/2.
\]

Taking inner product with \( \phi_n \), we obtain the following linear system of infinite dimensions,
\[
2b_0(e^{i\pi n} + 1) = b_0(e^{i\pi n} - 1)c_{00} + b_0'(e^{i\pi n} - 1)c_{00}(D)
\] (53) \{eq:op1:2holes\}
\[+ <a_{m,0}, c_{m,0}(D) > \epsilon_2 \]
\[
2b_0'(e^{i\pi n} + 1) = b_0'(e^{i\pi n} - 1)c_{00} + b_0(e^{i\pi n} - 1)c_{00}(-D)
\] (54) \{eq:op2:2holes\}
\[+ <a_{m,0}, c_{m,0}(-D) > \epsilon_2 \]
\[
2 \left[ \begin{array}{c}
\{a_{2n}\} \\
\{a_{2n-1}\}
\end{array} \right] = b_0(e^{i\pi n} - 1) \left[ \begin{array}{c}
\{c_{0,2n}\} \\
\{0\}
\end{array} \right] + b_0'(e^{i\pi n} - 1) \left[ \begin{array}{c}
\{c_{0,2n-1}(D)\}
\{c_{0,2n}(D)\}
\end{array} \right]
\] (55) \{eq:op1D\}
\[
+ \left[ \begin{array}{c}
A(e) \\
0
\end{array} \right] \left[ \begin{array}{c}
\{a_{2n}\} \\
\{a_{2n-1}\}
\end{array} \right] + A(D) \{a_n'\}
\]
\[
2 \left[ \begin{array}{c}
\{a_{2n}'\} \\
\{a_{2n-1}'\}
\end{array} \right] = b_0'(e^{i\pi n} - 1) \left[ \begin{array}{c}
\{c_{0,2n}\} \\
\{0\}
\end{array} \right] + b_0(e^{i\pi n} - 1) \left[ \begin{array}{c}
\{c_{0,2n-1}(-D)\}
\{c_{0,2n}(D)\}
\end{array} \right]
\] (56) \{eq:op2-D\}
\[
+ \left[ \begin{array}{c}
A(e) \\
0
\end{array} \right] \left[ \begin{array}{c}
\{a_{2n}'\} \\
\{a_{2n-1}'\}
\end{array} \right] + A(-D) \{a_n\}.
\]

In the above, we have defined two operators \( A(\pm D) : \ell^2 \rightarrow \ell^2 \) such that for any \( \{f_j\}_{j=1}^{\infty} \in \ell^2 \),
\[
A(\pm D) \{f_j\} = \left[ \begin{array}{c}
\sum_{j=1}^{\infty} c_{2j, j}(\pm D) f_j \\
\sum_{j=1}^{\infty} c_{2j-1, j}(\pm D) f_j
\end{array} \right]_{j=1}^{\infty} \in \ell^2,
\]
and \( c_{00}(\pm D) = \frac{4s_{D}d_{D}(\pm D)}{\pi} \) and for \( m, n \geq 1 \),
\[
c_{mn}(\pm D) = \frac{4s_m h}{\pi} \sqrt{n} (e^{i\pi n} - 1) d_{mn}(\pm D),
\]
\[
c_{m0}(\pm D) = \frac{4s_m h}{\pi} e^{i\pi n} d_{m0}(\pm D),
\]

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\[
c_{0n}(\pm D) = \frac{4s_0 h \sqrt{n}}{\pi} d_{0n}(\pm D),
\]
where
\[
d_{mn}(\pm D) = \int_{-\infty}^{+\infty} \frac{1}{\mu h^2} \frac{\xi \sin(\xi h/2 + m\pi/2)}{\xi^2 - \frac{\pi \nu^2}{h^2}} e^{\pm \xi D} d\xi.
\]  (57)

The boundedness of \(A(\pm D) : \ell^2 \rightarrow \ell^2\) can be seen from the following lemma.

**Lemma 3.1.** As \(h \to 0^+\), the sequence \(\{d_{mn}(\pm D)\}_{m,n=0}^{\infty}\) asymptotically behaves as following:

\[
d_{mn}(\pm D) = \begin{cases}
\frac{\pi}{4} H_0^{(1)}(\pm kD) + O(\epsilon^2) & m = n = 0; \\
-n^{-2}O(\epsilon^2) & m = 0, n > 0; \\
n^{-2}O(\epsilon^2) & m = 0, n > 0; \\
-m^{-2}n^{-2}O(\epsilon^2) & m, n > 0,
\end{cases}
\]  (58)  \(\text{eq:dmn:2holes}\)

where \(H_0^{(1)}\) is the first kind Hankel function of order 0, and the invisible constants in the \(O\)-notations are independent of \(m\) and \(n\). Thus, \(\{c_{mn}(\pm D)\}_{m,n=0}^{\infty}\) asymptotically behaves as following:

\[
c_{mn}(\pm D) = \begin{cases}
H_0^{(1)}(\pm kD)\epsilon + O(\epsilon^3) & m = n = 0; \\
-m^{-3/2}O(\epsilon^2) & n = 0, 0 < m; \\
m^{-3/2}O(\epsilon^3) & m = 0, 0 < n; \\
m^{-3/2}n^{-3/2}O(\epsilon^2) & m, n > 0,
\end{cases}
\]  (59)  \(\text{eq:cmn:2holes}\)

*Proof.* Here, we prove only the case \(m = n = 0\) as the other cases are much easier to justify. We have

\[
d_{00}(D) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{k^2 - \xi^2}} \frac{\sin^2(\xi h/2)}{\xi^2 h^2} e^{\xi D} d\xi
\]
\[
= \epsilon^2 \int_{+\infty+i \to 0}^{+\infty-i \to 0} \frac{1}{\sqrt{1 - \xi^2}} \frac{1 - \cos(\xi \epsilon)}{\xi^2} e^{\xi k D} d\xi =: \epsilon^{-2} I(\epsilon; D).
\]

Clearly,

\[
I''(\epsilon; D) = \int_{+\infty+i \to 0}^{+\infty-i \to 0} \frac{1}{\sqrt{1 - \xi^2}} \cos(\xi \epsilon)e^{\xi k D} d\xi
\]
\[
= \frac{\pi}{4} \left[ H_0^{(1)}(kD + \epsilon) + H_0^{(1)}(kD - \epsilon) \right]
\]
\[
= \frac{\pi}{2} H_0^{(1)}(kD) + O(\epsilon^2).
\]

Consequently, we get

\[
d_{00}(D) = \frac{\pi}{4} H_0^{(1)}(kD) + O(\epsilon^2).
\]

As a corollary, we obtain the following property of \(A(\pm D)\).

**Corollary 3.1.** For \(\epsilon \ll 1\), \(A(\pm D)\) is bounded from \(\ell^2\) to \(\ell^2\) with

\[
\|A(\pm D)\| = O(\epsilon^2).
\]
Thus, Eqs. (53–56) indicate that

\[
\begin{bmatrix}
\{a_{2n}\} \\
\{a_{2n-1}\} \\
\{a'_{2n}\} \\
\{a'_{2n-1}\}
\end{bmatrix} = (e^{is\theta} - 1)\epsilon 
\begin{bmatrix}
b_0(2Id - P^{(1)})^{-1}\{-\frac{4\sqrt{2n}}{\pi}i(-1)^nC_0(2n\pi)\} \\
b_0'(2Id - P^{(1)})^{-1}\{-\frac{4\sqrt{2n}}{\pi}i(-1)^nC_0(2n\pi)\} \\
0 \\
0
\end{bmatrix}
\]

\[
+ (e^{is\theta} - 1)b_0O(\epsilon^3 \log \epsilon)\{n^{-3/2} \log n\}
\]

\[
+ (e^{is\theta} - 1)b_0'O(\epsilon^3 \log \epsilon)\{n^{-3/2} \log n\}. \tag{60}
\]

Consequently, (53) and (54) are reduced to the following linear system

\[
\begin{bmatrix}
2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(\epsilon) \\
e \epsilon(Dk - 1)S_2(k, D)
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_0'
\end{bmatrix}
\]

\[
= (e^{ikl} - 1)E_2(\epsilon, D) \begin{bmatrix}
b_0 \\
b_0'
\end{bmatrix}, \tag{61} \{\text{eq:2holes}\}
\]

where \(Id_2\) denotes the 2 \times 2 identity matrix, all four entries of the 2 \times 2 matrix \(E_2(\epsilon, D)\) are \(O(\epsilon^3 \log \epsilon)\) and

\[
S_2(k, D) = \begin{bmatrix}
0 & H_0^{(1)}(-kD) \\
H_0^{(1)}(kD) & 0
\end{bmatrix}.
\]

We have the following theorem.

**Theorem 3.1.** For \(h \ll 1\), the resonance frequencies of even modes of the two-slit slab are

\[
kl = k_{m,e} - i\Delta_{1,j,m,e} - k_{m,e}^{-1}\Delta_{2,j,m,e} + O(\epsilon^2 \log \epsilon_{m,e}), j = 1, 2; m = 1, 2, \ldots, \tag{62} \{\text{eq:kl:e2}\}
\]

or

\[
kl = k_{m,e} - i \left[1 + \frac{2}{\pi}h\right] \Delta_{2,j,m,e} - \left[k_{m,e}^{-1} - 5\pi^{-1}k_{m,e}h\right] \Delta_{2,j,m,e}^2 
+ i \left[k_{m,e}^{-2} - \frac{1}{12}\right] \Delta_{2,j,m,e}^3 + O(\epsilon^3 \log \epsilon_{m,e}), j = 1, 2; m = 1, 2, \ldots, \tag{63} \{\text{eq:kl:e3}\}
\]

where

\[
\Delta_{1,j,m,e} = \Delta(\epsilon_{m,e}) + \epsilon_{m,e}\lambda_j(S_2(k_{m,e}, D))
\]

\[
\delta_{1,j,m,e} = -i\Delta_{1,j,m,e}(\epsilon_{m,e}) - k_{m,e}^{-1}\Delta_{1,j,m,e}^2,
\]

\[
\Delta_{2,j,m,e} = \Delta(\epsilon_{m,e}) + \epsilon_{m,e}\lambda_j(S_2(k_{m,e} + \delta_{1,j,m,e}, D)),
\]

and \(\lambda_j(S_2)\) indicates the \(j\)-th eigenvalue (in descending order of magnitude) of \(S_2\) for \(j = 1, 2\).

**Proof.** Clearly, [61] has a nonzero solution \([b_0, b_0']^T\) if and only if

\[
\begin{bmatrix}
2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(\epsilon) \\
\epsilon(Dk - 1)S_2(k, D) - (e^{ikl} - 1)E_2(\epsilon, D)
\end{bmatrix}
\]

has a zero eigenvalue or zero determinant. Since \(\|\epsilon(e^{ikl} - 1)S_2(k, D) - (e^{ikl} - 1)E_2(\epsilon, D)\|_2 = O(\epsilon)\), the resonance frequency \(k\) must satisfy

\[
2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(\epsilon) = O(\epsilon),
\]

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so that
\[ e^{ikl} + 1 = \mathcal{O}(\epsilon \log \epsilon). \]
Thus, as in Theorem 2.2
\[ k = k_{m,e} + o(1), \quad h \to 0, \]
for some \( m = 1, 2, \cdots \). Obviously, \( \epsilon \approx \epsilon_{m,e} \) and \( \log \epsilon \approx \log \epsilon_{m,e} \) so that
\[ \delta_{m,e} = k - k_{m,e} \approx (-i) \left[ e^{ik(k-k_{m,e})} - 1 \right] = \mathcal{O}(\epsilon \log \epsilon) = \mathcal{O}(\epsilon_{m,e} \log \epsilon_{m,e}). \]
Thus,
\[ 2(e^{ikl} + 1) - (e^{ikl} - 1) \Delta(\epsilon) - \epsilon(e^{ikl} - 1)\lambda_j(S_2(k_{m,e}, D)) = \mathcal{O}(\epsilon^2 \log \epsilon), \quad j = 1, 2, \]
where \( \lambda_j(S_2(k_{m,e}, D)) \) denotes the \( j \)-th eigenvalue of \( S_2(k_{m,e}, D) \) for \( j = 1, 2 \). By the same procedures in Theorem 2.2 we get
\[ \delta_{m,e} = \delta_{j,m,e} + \mathcal{O}(\epsilon_{m,e}^2 \log \epsilon_{m,e}), \]
where
\[ \delta_{j,m,e} = -i\Delta_{j,m,e} - k_{m,e}^{-1}\Delta_{j,m,e}^2, \]
\[ \Delta_{j,m,e} = \Delta(\epsilon_{m,e}) + \epsilon_{m,e}\lambda_j(S_2(k_{m,e}, D)). \]
Now, we have
\[ 2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(\epsilon) - \epsilon(e^{ikl} - 1)\lambda_j(S_2(k_{m,e} + \delta_{j,m,e}, D)) = \mathcal{O}(\epsilon^3 \log \epsilon), \]
where we note that the \( j \)-th eigenvalue \( \lambda_j \) of \( S_2(k_{m,e} + \delta_{j,m,e}, D) \) should be the one close to \( \lambda_j \) with distance \( O(\delta_{j,m,e}) \). Then, following similar arguments in Theorem 2.2 again, we get
\[ \delta_{m,e} = -i\left[ 1 + \frac{2}{\pi} h \right] \Delta_{j,m,e} - \left[ k_{m,e}^{-1} - 5\pi^{-1}k_{m,e}^{-1}h \right] \Delta_{j,m,e}^2 + i\left[ k_{m,e}^{-2} - \frac{1}{12} \right] \Delta_{j,m,e}^3 + \mathcal{O}(\epsilon_{m,e}^3 \log \epsilon_{m,e}). \]
We now prove the existence of those solutions. Since \( S_2(k, D) \) is skew-Hermitian, one could find a unitary matrix \( Q \), s.t., \( Q^*S_2(k, D)Q = \text{diag}\{\lambda_1(S_2(k, D)), \lambda_2(S_2(k, D))\} \). Then, Eq. (61) becomes
\[ \left( 2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(\epsilon) \right) \text{Id}_2 - \epsilon(e^{ikl} - 1)\text{diag}\{\lambda_1(S_2(k, D)), \lambda_2(S_2(k, D))\}) \left[ \begin{array}{c} b_0 \\ b_0' \end{array} \right] = (e^{ikl} - 1)Q^*[E_2(\epsilon, D) + \epsilon(S_2(k, D) - S_2(k, D))]Q \left[ \begin{array}{c} b_0 \\ b_0' \end{array} \right], \]
where
\[ \left[ \begin{array}{c} b_0 \\ b_0' \end{array} \right] = Q^* \left[ \begin{array}{c} b_0 \\ b_0' \end{array} \right]. \]
Assume that \( k \) lies in the disk \( D_{h} = \{ k \in \mathbb{C} : |k - k_{m,e}| \leq h^{1/2} \} \). Then, on the boundary of \( D_{h} \), all entries of
\[ Q^*[E_2(\epsilon, D) + \epsilon(S_2(k, D) - S_2(k, D))]Q, \]
are $\mathcal{O}(h^{3/2})$, so that by the linearity of determinant,

$$|\text{Det}_1 - \text{Det}_2| = \mathcal{O}(h^2) \leq \mathcal{O}(h) = |\text{Det}_2|,$$

where

\[
\begin{align*}
\text{Det}_1 &= \left| \text{diag} \left\{ 2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(e) - \epsilon(e^{ikl} - 1)\lambda_j(S_2(k_m, D)) \right\} \right|_{j=1}^2 \\
&\quad - (e^{ikl} - 1)Q^*[E_2(\epsilon, D) + \epsilon(S_2(k, D) - S_2(k_m, D))]Q, \\
\text{Det}_2 &= \left| \text{diag} \left\{ 2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(e) - \epsilon(e^{ikl} - 1)\lambda_j(S_2(k_m, D)) \right\} \right|_{j=1}^2.
\end{align*}
\]

For either $j = 1, 2$, it is clear that on the boundary of $D_h$,

$$\left| 2(e^{ikl} + 1) - (e^{ikl} - 1)\Delta(e) - \epsilon(e^{ikl} - 1)\lambda_j(S_2(k_m, D)) - 2i(kl - k_{m,e}) \right| = \mathcal{O}(h) \leq |2i(kl - k_{m,e})|.$$

The above two inequalities and Rouché’s theorem indicate that there are exactly two solutions in $D_h$.

The following theorem characterizes resonance frequencies of odd modes, i.e., when the wave field $u$ satisfies $u(x_1, -x_2 + l/2) = -u(x_1, x_2 + l/2)$.

**Theorem 3.2.** For $h \ll 1$, the resonance frequencies of odd modes of the two-slit slab are

$$kl = k_{m,o} - i\Delta_{1,j,m,o} - k_{m,o}^{-1}\Delta_{1,j,m,o}^2 + \mathcal{O}(\epsilon_{m,o}^2 \log \epsilon_{m,o}), j = 1, 2; m = 1, 2, \ldots, \tag{64} \{\text{eq:kl:o2}\}$$

or

$$kl = k_{m,o} - i\left[ 1 + \frac{2}{\pi} h \right] \Delta_{2,j,m,o} - \left[ k_{m,o}^{-1} - 5\pi^{-1}k_{m,o}^{-1}h \right] \Delta_{2,j,m,o}^2 + i\left[ k_{m,o}^{-2} - \frac{1}{12} \right] \Delta_{2,j,m,o}^3 + \mathcal{O}(\epsilon_{m,o}^3 \log \epsilon_{m,o}), j = 1, 2; m = 1, 2, \ldots, \tag{65} \{\text{eq:kl:o3}\}$$

where

$$\Delta_{1,j,m,o} = \Delta(\epsilon_{m,o}) + \epsilon_{m,o}\lambda_j(S_2(k_{m,o}, D))$$

$$\delta_{1,j,m,o} = -i\Delta_{1,j,m,o} - k_{m,o}^{-1}\Delta_{1,j,m,o}^2$$

$$\Delta_{2,j,m,o} = \Delta(\epsilon_{m,o}) + \epsilon_{m,o}\lambda_j(S_2(k_{m,o} + \delta_{1,j,m,o}, D)),$$

and $\lambda_j(S_2)$ indicates the $j$-th eigenvalue (in descending order of magnitude) of $S_2$ for $j = 1, 2$.

**Proof.** The proof follows from similar arguments as in Theorem 3.1.

The above results can be readily extended to a slab with three or more slits. Specifically, suppose now the slab has $N$ slits of the same width $h$ and thickness $l$, centered at

$$(D_1, -l/2), (D_2, -l/2), \ldots, (D_N, -l/2),$$

respectively. We state our main result in the following.
Theorem 3.3. For $h \ll 1$, the resonance frequencies of the $N$-slit slab are

$$ kl = k_m - i\Delta_{1,j,m} - k_m^{-1} \Delta_{1,j,m}^2 + \mathcal{O}(\epsilon_m^2 \log \epsilon_m), \quad j = 1, \ldots, N; m = 1, 2, \ldots, \quad \text{(66)} \{\text{eq:k1:N2}\} $$

or

$$ kl = k_m - i \left[ 1 + \frac{2}{\pi} h \right] \Delta_{2,j,m} - \left[ k_m^{-1} - 5\pi^{-1}k_m^{-1}h \right] \Delta_{2,j,m}^2 + i \left[ k_m^{-2} - \frac{1}{12} \right] \Delta_{2,j,m}^3 + \mathcal{O}(\epsilon_m^3 \log \epsilon_m), \quad j = 1, \ldots, N; m = 1, 2, \ldots, \quad \text{(67)} \{\text{eq:k1:N3}\} $$

where $k_m = m\pi$ are the Fabry-Pérot frequencies, $\epsilon_m = k_m h \ll 1$,

$$ \Delta_{1,j,m} = \Delta(\epsilon_m) + \epsilon_m \lambda_j(S_N(k_m, \{D_j\}_{j=1}^N)) $$

$$ \delta_{1,j,m} = -i\Delta_{1,j,m}(\epsilon_m) - k_m^{-1} \Delta_{1,j,m}^2, $$

$$ \Delta_{2,j,m} = \Delta(\epsilon_m) + \epsilon_m \lambda_j(S_N(k_m + \delta_{1,j,m}, \{D_j\}_{j=1}^N)), $$

and $\lambda_j(S_N(k, \{D_j\}_{j=1}^N))$ indicates the $j$-th eigenvalue (in descending order of magnitude) of

$$ S_N(k, \{D_j\}_{j=1}^N) = \left[ \begin{array}{cccc} 0 & H_0^{(1)}(kD_{12}) & \cdots & H_0^{(1)}(kD_{1N-1}) & H_0^{(1)}(kD_{1N}) \\ H_0^{(1)}(kD_{21}) & 0 & \cdots & H_0^{(1)}(kD_{2N-1}) & H_0^{(1)}(kD_{2N}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_0^{(1)}(kD_{N1}) & H_0^{(1)}(kD_{N2}) & \cdots & H_0^{(1)}(kD_{NN-1}) & 0 \end{array} \right], \quad \text{(68)} $$

with $D_{ij} = D_j - D_i$ for $i, j = 1, \ldots, N$.

Proof. The proof is analogous to that of Theorems 3.1 and 3.2 \square

Remark 3. Since $S_N(k_m, \{D_j\}_{j=1}^N)$ is skew-Hermitian, its eigenvalues are all pure imaginary, so that we see from (66) in Theorem 3.3 that the imaginary part of any resonance frequency is asymptotically leaded by $-k_m h$, and can never attain $\mathcal{O}(h^2)$ however those slits are placed as long as $D_{ij} \gg h$ in matrix $S_N(k, \{D_j\}_{j=1}^N)$. In other words, to make $\text{Im}(k) \ll h$, at least one of $D_{ij}$ should be comparable to $h$, as was illustrated by Babadjian et al. [3], where the structure contains two slits spaced by $\mathcal{O}(h)$.

4 Conclusion

We have proposed a quite simple Fourier-transformation approach to study resonances in a perfectly conducting slab with finite number of subwavelength slits of width $h \ll 1$. Outside the slits, we Fourier transformed the governing equation and expressed wave field in terms of field derivatives on the aperture. Inside the slit, wave field was expressed as Fourier series in terms of a countable basis functions with unknown Fourier coefficients. By matching field on the aperture, we established a linear system of infinite number of equations governing the countable Fourier coefficients. By asymptotic analysis of each entry of the coefficient matrix, we have rigorously shown that, by removing only a finite number of rows and columns, the resulting principle sub-matrix is diagonally dominant so that the infinite dimensional linear system is reduced to a finite dimensional linear system. This in turn provided a simple, asymptotic formula of resonance frequencies of accuracy $\mathcal{O}(h^3 \log h)$. This asymptotic formula rigorously confirms a fact that the imaginary part
of resonance frequencies is always $O(h)$ no matter how we place the slits as long as they are spaced by distances independent of $h$.

As no subregion Green functions are required, we could see immediate advantages in analyzing more complicated structures. For a slab with impedance boundary condition, the background Green function involves Sommerfeld integrals which are not easy to analyze. For a slab with periodic slits, our theory does not need to evaluate the quasi-periodic Green function. Furthermore, for an ideal PEC slab with a circular or rectangular hole, the Green function of the region of the hole could be quite challenging to derive or analyze. Therefore, we expect that our method could serve as an efficient approach in analyzing resonances in such structures. We shall report the results in a future work.

**Appendix**

To study the asymptotic behavior of $d_{mn}$ for $\epsilon = kh \ll 1$ where $k \in S$, we need the following technical lemmas.

**Lemma 4.1.** Let $b > 1$ and

$$I(\epsilon; b) = \int_1^{+\infty} \frac{1 - e^{-\epsilon t}}{\sqrt{1 + t^2}(\epsilon^2 t^2 + b^2)} \, dt. \tag{69} \{\text{eq:int:1}\}$$

As $h \to 0^+$, 

$$I(\epsilon; b) = C_0(b) - \sqrt{2}b^{-2} + b^{-2}O(\epsilon^2 \log \epsilon),$$

where the invisible constants in the $O$-notation are independent of $b$ and $k$, and we recall that $C_0(b)$ is defined in (18).

**Proof.** We make the rescaling $\epsilon = \frac{k}{|k|}(|k|h)$ so that one could assume $|k| = 1$ in the following. First,

$$I(\epsilon; b) = \int_h^{+\infty} \frac{1 - e^{-kt}}{\sqrt{h^2 + t^2(k^2 t^2 + b^2)}} \, dt$$

$$= \int_1^{+\infty} \frac{1 - e^{-kt}}{\sqrt{h^2 + t^2(k^2 t^2 + b^2)}} \, dt$$

$$= I_1(\epsilon; b) + I_2(\epsilon; b).$$

Here,

$$I_2(\epsilon; b) = \int_1^{+\infty} \frac{1 - e^{-kt}}{t(k^2 t^2 + b^2)} \, dt$$

$$= \int_1^{+\infty} \frac{1 - e^{-kt}}{t(k^2 t^2 + b^2)} \, dt - \int_1^{+\infty} \frac{1 - e^{-kt}}{(k^2 t^2 + b^2)} \left[ \frac{h^2}{\sqrt{h^2 + t^2} \left( \sqrt{h^2 + t^2} + t \right)} \right] \, dt$$

$$= \int_1^{+\infty} \frac{1 - e^{-kt}}{t(k^2 t^2 + b^2)} \, dt + b^{-2}O(h^2).$$

On the other hand,

$$I_1(\epsilon; b) = \int_h^1 \frac{1 - e^{-\epsilon t}}{\sqrt{h^2 + t^2(k^2 t^2 + b^2)}} \, dt.$$
Combining the above yields the asymptotic behavior of $I(\epsilon; b)$.

Lemma 4.2. Let $b \geq b' > e$ and

$$K^\pm(\epsilon; b, b') = \int_1^\infty \frac{e^{2t^2}(1 \pm e^{-t})}{\sqrt{1 + t^2(e^{2t^2} + b^2)(e^{2t^2} + b'^2)}} dt, \quad (70)$$

then

$$K^-(\epsilon; b, b') = C_0^-(b, b') + \frac{\log b - \log b'}{b^2 - b'^2} O(\epsilon^2),$$

$$K^+(\epsilon; b, b') = C_0^+(b, b') + \frac{\log b - \log b'}{b^2 - b'^2} O(\epsilon^2 \log \epsilon),$$

where the invisible constants in the big-O notation are independent of $b$ and $b'$, and $C_0^\pm(b, b')$ are defined in (19) and (20), respectively.

Proof. As in the previous lemma, we could assume $|k| = 1$ and $k = k_1 - ik_2$ with $k_1 \geq k_2 > 0$. We have

$$K^-(\epsilon; b, b') = \int_h^1 + \int_1^\infty \frac{k^2t^2(1 - e^{-kt})}{\sqrt{h^2 + t^2(k^2t^2 + b^2)(k^2t^2 + b'^2)}} dt$$

$$= K_1^-(\epsilon; b, b') + K_2^- (\epsilon; b, b').$$

Thus,

$$\left| K_2^- (\epsilon; b, b') - \int_1^\infty \frac{k^2t^2(1 - e^{-kt})}{t(k^2t^2 + b^2)(k^2t^2 + b'^2)} dt \right|$$
Moreover,

\[
\left. \frac{\sqrt{k^2 t^2 + b^2}}{k^2 t^2 + b^2} \right|_{t=1}^{\infty} = 1
\]

Proof of Lemma 2.1.

for \( \epsilon \) which concludes the proof.

\[
K = k
\]

we have for \( d \mid k \)

\[
1 - \epsilon \log(b^2 + (k_1^2 - k_2^2)) + h^2 \log(k_1^2 - k_2^2)
\]

\[
= \frac{\log b - \log b'}{b^2 - b'^2} \mathcal{O}(h^2).
\]

Moreover,

\[
\left| K^- (\epsilon; b, b') - \int_0^1 \frac{k^2 t^2 (1 - e^{-kt})}{(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt \right|
\]

\[
= \left| \int_0^h \frac{k^2 t^2 (1 - e^{-kt})}{(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt - \int_0^1 \frac{k^2 t^2 (1 - e^{-kt})}{(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} \left[ \frac{h^2}{t^2 + \sqrt{h^2 + t^2}} \right] dt \right|
\]

\[
= b^2 - b'^2 \mathcal{O}(h^2),
\]

which yields the desired results for \( K^- \). Similarly, one obtains

\[
K^+ (\epsilon; b, b') = K^- (\epsilon; b, b') + \int_0^\infty \frac{2k^2 t^2 e^{-kt}}{\sqrt{h^2 + t^2}(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt
\]

\[
= K^- (\epsilon; b, b') + \int_0^1 \frac{2k^2 t^2 e^{-kt}}{t(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt + \int_0^h \frac{2k^2 t^2 e^{-kt}}{h^2} dt + \int_h^1 \frac{2k^2 t^2 e^{-kt}}{h^2} dt + \int_1^\infty \frac{2k^2 t^2 e^{-kt}}{(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt + \log b - \log b' \mathcal{O}(h^2)
\]

\[
= C_0 (b, b') + \int_0^1 \frac{2k^2 t^2 e^{-kt}}{t(k^2 t^2 + b^2)(k^2 t^2 + b'^2)} dt + b^2 - b'^2 \mathcal{O}(h^2) + \log b - \log b' \mathcal{O}(h^2)
\]

\[
= \int_0^\infty \frac{\log b - \log b'}{b^2 - b'^2} \mathcal{O}(h^2) dt + \log b - \log b' \mathcal{O}(h^2) + \log b - \log b' \mathcal{O}(h^2)
\]

which concludes the proof.

By the above two lemmas, we are ready to analyze the asymptotic behavior of \( d_{mn} \) for \( \epsilon \ll 1 \).

**Proof of Lemma 2.1.** It is clear that \( d_{mn} = 0 \) when \( m + n \mid 2 \). For the case \( m + n \mid 2 \), we have for \( k \in \mathbb{R}^+ \) that

\[
d_{mn} = \int_{-\infty}^{\infty} \frac{\xi \sin(\xi h/2 + m \pi/2) \xi \sin(\xi h/2 + n \pi/2)}{\mu_2^2 \xi^2 - \pi^2 + \pi^2 \xi^2 - \pi^2 m^2 / k^2} d\xi
\]

\[
= (-1)^{(m-n)/2} \int_{-\infty}^{\infty} \frac{\xi^2 \xi^2 e^{-\xi^2} (1 - (-1)^m \cos(\xi \epsilon)))}{\sqrt{1 - \xi^2} e^{-\xi^2}} d\xi.
\]

We consider case \( m = n = 0 \) first. Then, we have by Cauchy’s theorem that

\[
d_{00} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 - \cos(\xi \epsilon)}{\sqrt{1 - \xi^2 e^{2\xi^2}}} d\xi = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 - e^{i\xi}}{\sqrt{1 - \xi^2 e^{2\xi^2}}} \xi^2 d\xi = \int_{\sqrt{\xi^2 e^{2\xi^2}}}^{\infty} \frac{1 - e^{i\xi} + i\xi}{\sqrt{1 - \xi^2 e^{2\xi^2}}} d\xi
\]

\[
= 0.24
\]
where \( \gamma \) is the Euler’s constant. Thus,

\[
I_2(\epsilon) = \frac{\pi}{4} + \frac{i}{6} + O(\epsilon^2).
\]

On the other hand,

\[
I_2''(\epsilon) = -i \int_0^{+\infty} \frac{e^{-\epsilon t}}{\sqrt{1 + \epsilon^2 t^2}} dt
= -i \frac{\gamma}{2} Y_0(\epsilon) - i \int_0^1 (1 - t^2)^{-1/2} \sin(\epsilon t) dt
= i(\log(\epsilon/2) + \gamma) - i\epsilon + O(\epsilon^2 \log(\epsilon)),
\]

where \( \gamma \) is the Euler’s constant. Thus,

\[
I_2(\epsilon) = \frac{i}{2} \gamma - \log 2 - \frac{3}{2} + i \frac{\epsilon^3}{6} + i \left[ \frac{\epsilon^2 \log \epsilon}{2} - \frac{3\epsilon^2}{4} \right] + O(\epsilon^4 \log \epsilon).
\]

Consequently, we get

\[
d_{00} = \frac{\pi}{4} + \frac{i}{2} (\gamma - \log 2 - \frac{3}{2}) + \frac{i}{2} \log \epsilon + O(\epsilon^2 \log \epsilon).
\]

Now, consider the case when \( m = 0 \) and \( 0 \neq n \mid 2 \). We have

\[
d_{n0} = d_{0n} = (-1)^{n/2} \left[ \int_0^1 \frac{1 - e^{i\epsilon \xi}}{\sqrt{1 - \xi^2 (\epsilon^2 t^2 + \pi^2 n^2)}} dt + i \int_0^{+\infty} \frac{1 - e^{-\epsilon t}}{\sqrt{1 + \epsilon^2 t^2 (\epsilon^2 t^2 + \pi^2 n^2)}} dt \right]
= (-1)^{n/2} \left[ I_3(\epsilon) + i I_4(\epsilon) \right].
\]

Clearly,

\[
I_3(\epsilon) = \frac{i\epsilon}{\pi^2 n^2} + n^{-2} O(\epsilon^2).
\]

On the other hand, according to Lemma 4.1

\[
I_4(\epsilon) = \int_0^1 \frac{1 - e^{-\epsilon t}}{\sqrt{1 + \epsilon^2 t^2 (\epsilon^2 t^2 + \pi^2 n^2)}} dt + I(\epsilon; \pi n)
= \frac{\epsilon}{\pi^2 n^2} (\sqrt{2} - 1) + n^{-2} O(\epsilon^3) + C_0(\pi n) - \frac{\sqrt{2}\epsilon}{\pi^2 n^2} + n^{-2} O(\epsilon^2 \log \epsilon)
= C_0(\pi n) - \frac{\epsilon}{\pi^2 n^2} + n^{-2} O(\epsilon^2 \log \epsilon).
\]

Consequently, we get

\[
d_{n0} = d_{0n} = i (-1)^{n/2} C_0(\pi n) + n^{-2} O(\epsilon^2 \log \epsilon).
\]

When \( mn \neq 0 \) and \( m \neq n \), we have by Cauchy’s theorem that

\[
d_{mn} = \frac{1}{2} (1 - \epsilon)^{(m-n)/2} \int_{-\infty}^{+\infty} \frac{e^{2\epsilon^2 (1 - (-1)^m e^{i\epsilon})}}{\sqrt{1 - \xi^2 (\epsilon^2 \xi^2 - \pi^2 m^2)(\epsilon^2 \xi^2 - \pi^2 n^2)}} d\xi
\]
\[ (-1)^{(m-n)/2} \left[ \int_0^1 \frac{e^{2\xi^2}(1 - (-1)^m e^{i\xi})}{\sqrt{1 - \xi^2(\xi^2 m^2 - \pi^2 n^2)} \xi^2} \, d\xi + i \int_0^\infty \frac{e^{2\xi^2}(1 - (-1)^m e^{-\xi^2})}{\sqrt{1 + \xi^2(\xi^2 m^2 + \pi^2 n^2)} \xi^2} \, d\xi \right] = (-1)^{(m-n)/2} [I_5(\epsilon) + iI_6(\epsilon)]. \]

Clearly, we have
\[ I_5(\epsilon) = \frac{\epsilon^2}{\pi^2 m^2 n^2} \left[ (1 - (-1)^m \frac{\pi}{4} - (-1)^m \frac{2}{3} i) \right] + m^{-2n-2} \mathcal{O}(\epsilon^4). \]

If \( m \mid 2 \), we have
\[ I_6(\epsilon) = \int_0^1 \frac{e^{2\xi^2}(1 - e^{-\xi^2})}{\sqrt{1 + \xi^2(\xi^2 m^2 + \pi^2 n^2)} \xi^2} \, d\xi + K^- (\epsilon; \pi m, \pi n) \]
\[ = m^{-2n-2} \mathcal{O}(\epsilon^3) + C^-_0 (\pi m, \pi n) + \frac{\log m - \log n}{m^2 - n^2} \mathcal{O}(\epsilon^2). \]

Consequently,
\[ d_{mn} = i(-1)^{(m-n)/2} C^-_0 (\pi m, \pi n) + \frac{\log m - \log n}{m^2 - n^2} \mathcal{O}(\epsilon^2). \]

If \( m \nmid 2 \), we similarly have
\[ d_{mn} = i(-1)^{(m-n)/2} C^+_0 (\pi m, \pi n) + \frac{\log m - \log n}{m^2 - n^2} \mathcal{O}(\epsilon^2 \log \epsilon). \]

If \( 0 \neq n = m \mid 2 \), we have by Cauchy’s theorem that
\[
\begin{align*}
    d_{mn} &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{2\xi^2}(1 - \cos(\epsilon \xi))}{\sqrt{1 - \xi^2(\xi^2 m^2 - \pi^2 n^2)} \xi^2} \, d\xi \\
    &= \int_0^1 \frac{(1 - e^{i\xi} + i(\epsilon \xi - \pi m))e^{\xi^2}}{\sqrt{1 - \xi^2(\xi^2 m^2 - \pi^2 n^2)^2}} \, d\xi + \left( \int_0^1 + \int_1^{+\infty} \frac{(1 - e^{-\xi^2} - \epsilon t)ie^{\xi^2}}{\sqrt{1 + \xi^2(\xi^2 m^2 + \pi^2 n^2)^2}} \, d\xi \right) \\
    &= m^{-3} \mathcal{O}(\epsilon^2) + m^{-4} \mathcal{O}(\epsilon^4) + iK^- (\epsilon; \pi m, \pi m) - i \int_1^{+\infty} \frac{e^{3t^3}}{\sqrt{1 + t^2(\epsilon^2 t^2 + \pi^2 m^2)^2}} \, dt \\
    &= iK^- (\epsilon; \pi m, \pi m) - iI_7(\epsilon) + m^{-3} \mathcal{O}(\epsilon^2).
\end{align*}
\]

where
\[ I_7(\epsilon) := \int_1^{+\infty} \frac{e^{3t^3}}{\sqrt{1 + t^2(\epsilon^2 t^2 + \pi^2 m^2)^2}} \, dt \]
\[ = m^{-4} \mathcal{O}(\epsilon^3) + \int_2^{+\infty} \frac{e^{3t^3}}{t(\epsilon^2 t^2 + \pi^2 m^2)^2} \left\{ \sum_{i=0}^{\infty} \left( -\frac{1}{2} \right)^i t^{-2i} \right\} \, dt \]
\[ = \sum_{i=0}^{\infty} \left( -\frac{1}{2} \right)^i \int_2^{+\infty} \frac{e^{3t^2-2i}}{t(\epsilon^2 t^2 + \pi^2 m^2)^2} \, dt \]
\[ = 8 \int_1^{+\infty} \frac{t^2}{(4\epsilon^2 + \pi^2 m^2)^2} \, dt + \mathcal{O}(\epsilon^2)m^{-3}. \]
\[ = \frac{1}{4m} + \mathcal{O}(\epsilon^2)m^{-3}. \]
Consequently, we get
\[ d_{mm} = iC_0^-(\pi m, \pi m) - \frac{i}{4m} + O(\epsilon^2) m^{-3}. \]

Finally, when \( 0 \neq m = n \nmid 2 \), one similarly gets
\[
\begin{align*}
& d_{mm} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\epsilon^2 \xi^2 (1 + \cos(\epsilon \xi))}{\sqrt{1 - \xi^2 (\epsilon^2 \xi^2 - \pi^2 m^2)^2}} d\xi \\
& = iK^+(\epsilon; \pi m, \pi m) - iI_7(\epsilon) + m^{-3}O(\epsilon^2) \\
& = iC_0^+(\pi m, \pi m) - \frac{i}{4m} + m^{-3}O(\epsilon^2 \log \epsilon).
\end{align*}
\]

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