INVARIANT SUBSETS OF THE SPACE OF SUBGROUPS,
EQUATIONAL COMPACTNESS AND THE WEAK EQUIVALENCE OF
ACTIONS

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Abstract. Equationally compact subgroups of countable groups were introduced by
Banaschewski. For all known cases the orbit closure of such a subgroup is a countable
subset in the space of subgroups and has finite Cantor-Bendixson rank. We show that
there exists a finitely generated group $\Gamma$ such that for any countable ordinal $\alpha$
we have
an equationally compact subgroup $H \subset \Gamma$ for which the Cantor-Bendixson rank of the
orbit closure of $H$ equals to $\alpha + 2$. Then we give an explicit construction of continuum
many equationally compact subgroups of $\Gamma$ such that the associated ergodic Bernoulli shift
actions are pairwise weakly incomparable. We also answer two questions on equational
compactness posed by Prest and Rajani.

Keywords. equational compactness, space of subgroups, weak equivalence, Cantor-
Bendixson rank

1. Introduction

It is well-known that fields are equationally compact, that is, if we have a system
$E_i, i \in I$ of equations over a field $K$
$$\sum_j a_{ij}x_j = m_i$$

and for any finite subset of $I$ the equations $E_i$ can be solved simultaneously, then all the
equations can be solved simultaneously. Now let $\Gamma$ be a countable group acting on a set $X$
by permutations. Let $\text{Fix}(s)$ be the fixed point set of $s \in \Gamma$. Following Banaschewski [2],
we say that a $\Gamma$-action is equationally compact, if we have a subset $S$ of $\Gamma$ and for any finite
subset $T$ of $S$, $\cap_{s \in T} \text{Fix}(s)$ is non-empty, then $\cap_{s \in S} \text{Fix}(s)$ is non-empty. A subgroup $H$ of
$\Gamma$ is equationally compact (or PIP, [3]) if the left action on $\Gamma/H$ is equationally compact.
The following proposition is quite straightforward and is left for the reader.

Proposition 1.1. The subgroup $H \subset G$ is equationally compact if any of the following
three conditions hold:

- $H$ is a finite extension of a normal subgroup (in particular, if $H$ is finite or normal).
- The normalizer subgroup of $H$ has finite index in $G$.
- $H$ is malnormal.
On the other hand, Banaschewski proved ([2], Proposition 6.) that the free group of infinite generators has non-equationally compact subgroups.

Let $\Gamma$ be a countable group and $\{0, 1\}^\Gamma$ be the set of subsets of $\Gamma$ with the Tychonoff-topology. The set of all subgroups, $S(\Gamma)$ forms a closed, invariant (under the conjugate action) subspace of $\{0, 1\}^\Gamma$. If $\Gamma$ acts on a set $X$, then the set $\{\text{Stab}(x) : x \in X\} = M(\Gamma, X)$ is an invariant subspace of $S(\Gamma)$. The following proposition is easy to prove.

**Proposition 1.2.** Let $\Gamma$ act on the set $X$ and let $M(\Gamma, X) \subset S(\Gamma)$ be the corresponding invariant subspace of $S(\Gamma)$. Then, the action is equationally compact if and only if for any subgroup $K$ in the closure of $M(\Gamma, X)$, there exists an element of $L \in M(\Gamma, X)$ such that $K \subseteq L$. Particularly, $H \subset \Gamma$ is equationally compact if for any $K$ in the orbit closure of $H$, there exists a conjugate of $H$, $L = gHg^{-1}$ such that $K \subseteq L$.

The first goal of the paper is to answer two queries of Prest and Rajani [8] concerning equational compactness by proving the following two theorems.

**Theorem 1.** The only equationally compact subgroup of the finitary symmetric group $S^0_\infty$ on $\mathbb{N}$ are the finite subgroups and the group of even permutations $A^0_\infty$.

**Theorem 2.** There exists a countable group $\Gamma$ acting on a set $X$, such that the action is equationally compact, but for any $x \in X$, $\text{Stab}(x)$ is not an equationally compact subgroup of $\Gamma$.

Let $H$ be a subgroup of a countable group $\Gamma$. Then one can consider the Cantor-Bendixson rank of its orbit closure ([3]). In some sense the Cantor-Bendixson rank measures the complexity of a subgroup, how far they are from being normal. Notice that the orbit closures of all the subgroups described in Proposition 1.1 are countable sets and their Cantor-Bendixson ranks are finite. One of the two main results of our paper is the following theorem.

**Theorem 3.** Let $\Gamma = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ be the free product of three cyclic groups. Then, for any countable ordinal $\alpha$, there exists an equationally compact subgroup $H$ of $\Gamma$ such that orbit closure of $H$ is countable and its Cantor-Bendixson rank is $\alpha + 2$.

Finally, we apply our techniques for the construction of weakly incomparable essentially free and ergodic generalized Bernoulli actions of the group $\Gamma_5 = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ (see Section 7).

**Theorem 4.** There exist uncountable many equationally compact subgroups $\{H_\alpha\}_{\alpha \in \mathbb{T}}$ of $\Gamma_5$ such that the associated generalized Bernoulli actions $\Gamma_5 \curvearrowright ([0, 1], \lambda)_{\Gamma_5/H_\alpha}$ are pairwise weakly incomparable.
$A^0_\infty$. Before getting into the proof let us fix some notations. Let $S_{[1, \infty]}^0 \subset S^0_\infty$ be the subgroup of elements fixing the set $\{1, 2, \ldots, l-1\}$. Let $A_{[1, \infty]}^0 = S_{[1, \infty]}^0 \cap A^0_\infty$. For a permutation $\gamma \in S^0_\infty$, we define $s(\gamma)$ as the maximum of $k$'s for which $\gamma(k) \neq k$.

**Proposition 2.1.** Let $H$ be an equationally compact subgroup of $S^0_\infty$. Then one of the following two conditions are satisfied.

1. There exists $l \geq 0$ such that $H \cap S_{[l, \infty]}^0 = \{e\}$.
2. There exists $l \geq 0$ such that $A_{[1, \infty]}^0 \subset H$.

**Proof.** Let $H \subset S^0_\infty$ be a subgroup such that neither of the two conditions above are satisfied. Let $\kappa_1 H \kappa_1^{-1}, \kappa_2 H \kappa_2^{-1}, \ldots$ be an enumeration of the conjugates of $H$. Inductively, we will pick elements \{$\gamma_n\}_{n=1}^\infty \subset S^0_\infty$, \{$\delta_n\}_{n=1}^\infty \subset S^0_\infty$ such that

- $\{\delta_1, \delta_2, \ldots, \delta_n\} \subset \gamma_nH\gamma_n^{-1}$.
- $\delta_n \notin \kappa_n H \kappa_n^{-1}$.

Hence the subgroup $H$ cannot be equationally compact. Suppose that \{$\gamma_i\}_{i=1}^n$, \{$\delta_i\}_{i=1}^n$ has already been constructed and for any $1 \leq i \leq n$

- $\{\delta_1, \delta_2, \ldots, \delta_i\} \subset \gamma_iH\gamma_i^{-1}$.
- $\delta_i \notin \kappa_i H \kappa_i^{-1}$.

Let $l = \max(\max_{1 \leq i \leq n} s(\gamma_i), \max_{1 \leq i \leq n} s(\delta_i), \kappa_{n+1}) + 1$.

Since a conjugacy class always generates a normal subgroup, there exists a non-unit conjugacy class $C$ of $S^0_{[1, \infty]}$ such that $H \cap C$ is a proper subset of $C$. Let $\delta_{n+1} \in C \setminus H, \rho_{n+1} \in H \cap C$. Then, we have $\gamma \in S^0_{[1, \infty]}$ such that $\gamma \rho_{n+1} \gamma^{-1} = \delta_{n+1}$. Let $\gamma_{n+1} = \gamma \gamma_n$. By the definition of $l$, we have that $\gamma$ commutes with \{$\gamma_i\}_{i=1}^n$, \{$\delta_i\}_{i=1}^n$ and $\kappa_{n+1}$, hence

- $\delta_{n+1} \notin \kappa_{n+1} H \kappa_{n+1}^{-1}$.
- $\delta_i \notin \gamma_{n+1} H \gamma_{n+1}^{-1}$, whenever $1 \leq i \leq n$.

Therefore, $H$ is not equationally compact. \hfill $\square$

**Lemma 2.1.** If $H$ is equationally compact and contains $A_{[1, \infty]}^0$ for some $l > 0$, then either $H = S^0_\infty$ or $H = A^0_\infty$.

**Proof.** If $A_{[1, \infty]}^0 \subset H$, then for any $k \geq 1$ there exists a conjugate of $H$, $\gamma H \gamma^{-1}$ such that the subgroup $A_k$ is contained in $\gamma H \gamma^{-1}$. Hence, if $H$ is equationally compact, then it must contain the whole group $A^0_\infty = \cup_{k=1}^\infty A_k$. Therefore, $H = S^0_\infty$ or $H = A^0_\infty$. \hfill $\square$

The following proposition finishes to proof of Theorem 1.

**Proposition 2.2.** If there exists $l \geq 1$ such that $H \cap S_{[l, \infty]}^0 = \{e\}$, then $H$ is finite.

**Proof.** Suppose that $H$ is an infinite subgroup of $S^0_\infty$.

**Lemma 2.2.** There exists an infinite subset $\{\gamma_n\}_{n=1}^\infty \subset H$ such that for any $n \geq 1$, $\gamma_n(1) = 1$.
Proof. First, let us suppose that there exists \( k \geq 1 \) and an infinite subset \( \{\delta_n\}_{n=1}^{\infty} \subset H \) such that \( \delta_n(1) = k \). Let \( \gamma_n = \delta_n^{-1}\delta_1 \). Then for any \( n \geq 1 \), \( \gamma_n(1) = 1 \). If such \( k \) does not exist, then we have an increasing sequence of positive integers \( \{k_n\}_{n=1}^{\infty} \) and an infinite subset \( \{\gamma_n\}_{n=1}^{\infty} \subset H \) such that

- \( \gamma_n(1) = k_n \).
- \( k_n > s(\gamma_i) \), whenever \( 1 \leq i \leq n - 1 \).

Then for any \( n \geq 1 \) and \( 1 \leq i \leq n - 1 \), \( \gamma_n^{-1}\gamma_i\gamma_n(1) = 1 \), hence our lemma follows.

Now let \( s \) be a positive integer and \( \{\delta_n\}_{n=1}^{\infty} \subset H \) be an infinite set of permutations such that \( \delta_n(j) = j \) if \( 1 \leq j \leq s \).

Lemma 2.3. There exists an infinite subset \( \{\gamma_n\}_{n=1}^{\infty} \subset H \) such that \( \gamma_n(j) = j \), if \( 1 \leq j \leq s + 1 \).

Proof. Again, if there exists \( k \geq 1 \) and an infinite subset \( \{\rho_n\}_{n=1}^{\infty} \subset H \) such that

- \( \rho_n(j) = j \), if \( 1 \leq j \leq s \).
- \( \rho_n(s + 1) = k \).

then the set \( \{\gamma_n = \rho_i^{-1}\rho_1\}_{n=1}^{\infty} \) will satisfy the condition of our lemma. On the other hand, if such \( k \) does not exist then we have an increasing sequence of positive integers \( \{k_n\}_{n=1}^{\infty} \) and an infinite subset \( \{\delta_n\}_{n=1}^{\infty} \subset H \) such that

- \( \delta_n(j) = j \), if \( 1 \leq j \leq s \).
- \( \delta_n(s + 1) = k_n \).
- \( k_n \geq s(\delta_i) \) if \( 1 \leq i \leq n - 1 \).

Hence \( \delta_n^{-1}\delta_i\delta_n(j) = j \), if \( 1 \leq i \leq n \), \( 1 \leq j \leq s + 1 \). Thus our lemma follows.

By induction, we can construct infinitely many elements \( \{\gamma_n\}_{n=1}^{\infty} \subset H \) such that \( \gamma_n \in S^0_{[1,\infty]} \), in contradiction with the fact that \( H \cap S^0_{[1,\infty]} = \{e\} \).

3. Tree subgroups and equational compactness

Let \( T \) be a tree of vertex degrees at most three with edges properly colored by the letters \( a, b \) and \( c \), that is, adjacent edges are colored differently. From now on (until Section 5) all trees will be considered to be properly \( (a, b, c) \)-edge-colored. Trees are Schreier-graphs, the associated action of \( \Gamma = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \) on the vertex set \( V(T) \) is given the following way.

- If for \( x \in V(T) \) there exists an edge \( e(x, y) \) colored by \( a \), then \( ax = y \).
- If such edge does not exist, then \( a(x) = x \).

We define the action of the generators \( b \) and \( c \) in a similar fashion. Let \( w = p_n p_{n-1} p_{n-2} \ldots p_1 \), where \( p_i = a, b \) or \( c \) be a reduced word, \( x \in V(T) \). Then \( w \in \text{Stab}_T(x) \), if there exists a closed walk \( \{v_0, v_1, \ldots, v_n\} \) in \( T \) such that

- \( v_0 = v_n = x \).
- \( p_i(v_{i-1}) = v_i \).
If $x \in V(T)$, then $\text{Stab}_T(x)$ is called a tree subgroup of $\Gamma$. The orbit of $H = \text{Stab}_T(x)$ in the space of subgroups $S(\Gamma)$ is the set $\{\text{Stab}(y) \mid y \in V(T)\}$. Now we describe the orbit closure of $H$. First, let us recall the notion of convergence and limits for rooted trees. A rooted tree $(T, \rho)$ is an $(a, b, c)$-edge-colored tree with a distinguished vertex $\rho \in V(T)$. The distance of two rooted trees is defined the following way.

$$d_r((T_1, \rho_1), (T_2, \rho_2)) = 2^{-k}$$

if the balls $B_k(T_1, \rho_1)$ and $B_k(T_2, \rho_2)$ are isomorphic as rooted, colored graphs, but $B_{k+1}(T_1, \rho_1)$ and $B_{k+1}(T_2, \rho_2)$ are not isomorphic. The space $\mathcal{RT}$ of rooted trees is a totally disconnected compact space with respect to the metric $d_r$. We will denote by $T$ the set of all $(a, b, c)$-trees up to isomorphism.

**Lemma 3.1.** The sequence $\{T_n, \rho_n\}_{n=1}^{\infty}$ converges to $\{T, \rho\}_{n=1}^{\infty}$ if and only if $\text{Stab}_{T_n}(\rho_n)$ converges to $\text{Stab}_T(\rho)$ in the space of subgroups $S(\Gamma)$.

**Proof.** Let $\{T_n, \rho_n\}_{n=1}^{\infty}$ be a sequence of rooted trees converging to $(T, \rho)$. Let $w \in \Gamma$ be a reduced word of length $k$. By definition, if $n$ is large enough, then $B_k(T_n, \rho_n) \cong B_k(T, \rho)$, hence $w(\rho) = \rho$ if and only if $w(\rho_n) = \rho_n$ for large enough $n$. Therefore, $\text{Stab}_T(\rho)$ is the limit of the sequence $\text{Stab}_{T_n}(\rho_n)$ in the space $S(\Gamma)$. Conversely, let $H$ be the limit of the sequence $\text{Stab}_{T_n}(\rho_n)$ in $S(\Gamma)$. Let $W_k$ be the finite set of reduced words of length $k$ in $\Gamma$. Then there exists an integer $N > 0$ such that if $N \leq n$, then an element $w \in W_n$ fixes $\rho_n$ if and only if it fixes $\rho$. Hence, the Schreier-graph $\text{Sch}(\Gamma/H)$ must be a tree and $\{T_n, \rho_n\}_{n=1}^{\infty}$ converges to the rooted tree $\text{Sch}(\Gamma/H, H)$ in the space of rooted trees $\mathcal{RT}$. \qed

Before getting further, let us recall the notion of a branch. If $T$ is a tree and $x \in V(T)$, then $y, z \in V(T)\{x\}$ is called branch equivalent if the shortest paths connecting $y$ to $x$ resp. $z$ to $x$ have a joint edge. For a fixed $x$, the equivalence classes induce subtrees of $T$ and they are called branches. For any edge $e$ adjacent to $x$ there exists exactly one branch containing $e$. The vertex $x$ is called the root of the branch. Now let $T_3$ be the infinite tree of vertex degrees 3 with its unique $(a, b, c)$-edge coloring.

A decorated tree $T_3^M$ is constructed the following way. Let $M \subseteq E(T_3)$ be a set of edges of $T_3$ colored by the letter $a$. For each chosen edge $e(x, y) \in M$ we will have two extra vertices $x_e, y_e$ such that $x$ is connected to $x_e$ by an edge colored by $a$, $x_e$ is connected to $y_e$ by an edge colored by $b$ and $y_e$ is connected to $y$ by an edge colored by $a$ (and then we delete the original edge $e(x, y)$). It is easy to see that the resulting tree $T_3^M$ is properly $(a, b, c)$-colored, the “old” vertices have valency 3 and the “new” vertices have valency 2. The following lemma is trivial.

**Lemma 3.2.** For any decorated tree we have an $(a, b, c, D)$-coloring of the 3-tree such that edges adjacent to an edge colored by $D$ can be colored by either $b$ or $c$ (we call such edge-colorings good). Conversely, if we have a good $(a, b, c, D)$-coloring of the edges of the 3-tree, then there is an associated decorated tree $T_3^M$, where $M$ is the set of edges colored by $D$.

It is easy to see that if $\{(T_n, \rho_n)\}_{n=1}^{\infty}$ is a convergent sequence of good rooted $(a, b, c, D)$-trees, then its limit $(T, \rho)$ is a good tree as well. Let $\mathcal{RT}_D$ denotes the compact space of
Proposition 3.1. Let $S$ be an $(a, b, c, D)$-tree. Let $\{(T_\alpha, x_\alpha)\}_{\alpha \in I}$ be the set of all elements of the closure of the set $\{S, y\}_{y \in S}$ in $\mathcal{RT}_D$. Suppose that for any $\alpha \in I$, there exists $x_\alpha \in V(T_\alpha)$ and $y_\alpha \in V(S)$ such that $(S, y_\alpha)$ sequentially dominates $(T_\alpha, x_\alpha)$. Then for each $y \in V(S)$, $\text{Stab}_{\phi(S)}(y)$ is an equationally compact subgroup of $\Gamma$.
Proof. By Proposition 1.2, we only need to prove that if \( \{ (\phi(S), y_i) \}_{i=1}^{\infty} \) converges to \((Q, z)\) in \(RT\), then \( \text{Stab}_Q(z) \subset \text{Stab}_{\phi(S)}(y) \) for some \( y \in V(S) \). We can suppose that the vertices \( \{ y_i \} \) has degree 3. Indeed, if the degrees (for large \( i \)) are 2, then we can substitute \( y_i \) by an adjacent vertex \( y'_i \) having vertex degree 3. Then \( \{ (\phi(S), y'_i) \}_{i=1}^{\infty} \) will converge to \((Q, z')\), where \( z' \) is adjacent to \( z \). Now, if \( \text{Stab}_Q(z') \subset \text{Stab}_{\phi(S)}(y') \) for some \( y' \in V(S) \), then \( \text{Stab}_Q(z) \subset \text{Stab}_{\phi(S)}(y) \) for some \( y \in V(S) \). By Lemma 3.3 if \( \{ (\phi(S), y_i) \}_{i=1}^{\infty} \) converges to \((Q, z)\) in \(RT\), then \( \{ (S, y_i) \}_{i=1}^{\infty} \) converges to \((T, z)\) in \(RT_D\) and \( \phi(T) = Q \). By the condition of our proposition, there exists \( z'' \in V(T) \) such that the rooted tree \((T, z'')\) is sequentially dominated by \((S, y')\) for some \( y' \in V(S) \). Hence, by Proposition 3.4 \( \text{Stab}_{\phi(T)}(z'') \subset \text{Stab}_{\phi(S)}(y) \). That is,
\[
\text{Stab}_Q(z) = \text{Stab}_{\phi(T)}(z) \subset \text{Stab}_{\phi(S)}(y) \text{ for some } y \in V(S) \quad \square
\]

The following example is intended to illustrate the use of Proposition 3.1. We call a good \((a, b, c, D)\)-tree \( S \) sparse, if for any \( n \geq 1 \), there exists only finitely many pairs of edges \( E_D(S) \) having distance at most \( n \).

**Proposition 3.2.** Let \( S \) be a sparse tree. Then for any vertex \( y \in \phi(S) \), \( \text{Stab}_{\phi(S)}(y) \) is equationally compact.

**Proof.** Suppose that the sequence \( \{ (S, x_i) \}_{i=1}^{\infty} \) converges to \((T, x)\) in \(RT_D\). Recall, that it means that for any \( r \geq 1 \), the ball \( B_r(T, x) \) is rooted-colored-isomorphic to \( B_r(S, x_i) \) if \( i \) is large enough. We have three cases.

**Case 1.** The sequence \( \{ x_i \}_{i=1}^{\infty} \) is bounded. Then \((T, x)\) is isomorphic to \((S, y)\) for some vertex \( y \in V(S) \).

**Case 2.** There exists \( r \geq 1 \) such that \( B_r(S, x_i) \) contains exactly one \( D \)-colored edge. Then the limit tree \((T, x)\) has exactly one \( D \)-colored edge. Clearly, \((T, x)\) is sequentially dominated by \((S, y)\) for some \( y \in S \).

**Case 3.** For all \( r \geq 1 \), the balls \( B_r(S, x_i) \) does not contain \( D \)-colored edges if \( i \) is large enough. Then \((T, x)\) has no \( D \)-colored edges, so \((T, x)\) is sequentially dominated by \((S, y)\) for all \( y \in V(S) \) and \( \text{Stab}_{\phi(T)}(x) = \{ e \} \).

Hence, the conditions of Proposition 3.1 are satisfied. Therefore, for any vertex \( y \in \phi(S) \), \( \text{Stab}_{\phi(S)}(y) \) is equationally compact. \( \square \)

4. **The proof of Theorem 3**

Before getting into the proof of Theorem 3 let us introduce some definitions to make the construction of good \((a, b, c, D)\)-trees easier. A bi-infinite \((a, b)\)-path is a graph on the vertex set \( \{ x_n \}_{n \in \mathbb{Z}} \) such that \( x_n \) and \( x_m \) is adjacent if and only if \( |n - m| = 1 \) and the edge \( e(x_{2k}, x_{2k+1}) \) is colored by \( a \), the edge \( e(x_{2k}, x_{2k-1}) \) is colored by \( b \). An infinite \((c, b)\)-path is a graph on the vertex set \( \{ y_n \}_{n \geq 0} \) such that \( y_n \) and \( y_m \) is adjacent if and only if \( |n - m| = 1 \) and the edge \( e(y_{2k}, y_{2k+1}) \) is colored by \( c \), the edge \( e(y_{2k}, y_{2k-1}) \) is colored by \( b \). Now we build our \((a, b, c, D)\)-tree. We start with the standard infinite \((a, b, c)\)-colored tree \( T_3 \).
Step 1. We fix a bi-infinite \((a, b)\)-subpath in \(T_3\) on the vertices \(\{x_n\}_{n \in \mathbb{Z}}\).

Step 2. Let us consider the infinite \((c, b)\)-path starting from \(x_0\). The vertices of the path are \(\{y_n^0\}_{n=0}^\infty\), where \(y_0^0 = x_0\).

Step 3. Now for any \(k > 0\) we consider the infinite \((c, b)\)-path starting from \(x_{2^k}\). The vertices of this path are \(\{y_n^k\}_{n=0}^\infty\), where \(y_0^k = x_{2^k}\).

Step 4. For any \(i \geq 1\), we choose a subset \(L_i\) of the positive integers. If \(s \in L_i\), then we recolor the \(a\)-colored edge adjacent to the vertex \(y_{2i}^s\) by the color \(D\).

Step 5. We recolor the \(a\)-colored edges adjacent to the vertices \(\{y_{2^k}^0\}_{k=1}^\infty\) by \(D\).

**Proposition 4.1.** For all choices of the family \(\mathcal{L}\) of the sets \(\{L_i\}_{i=1}^\infty\) the resulting tree \(S_{\mathcal{L}}\) satisfies the conditions of Proposition 3.7 hence \(\text{Stab}_{\phi(S)}(z)\) is sequentially compact if \(z \in V(S)\).

**Proof.** Again, suppose that the sequence \(\{(S, z_i)\}_{i=1}^\infty\) converges to \((T, q)\) in \(\mathcal{RT}_D\). We need to show that \((T, q)\) is sequentially dominated by \((S, y)\) for some \(y \in V(S)\). The three possible cases described in Proposition 3.2 are already handled. The underlying limit trees are in these cases \(S, T_3\) and \(T_3\), where \(T_3\) is the tree with one single \(D\)-colored edge. Before considering the fourth case, let us define a specific tree. Let \(L \in \{0, 1\}^\mathbb{N}\) be representing a subset of the positive integers. Let \(p = p_0, p_1, p_2, \ldots\) be a \((c, b)\)-path in \(T_3\) starting from the vertex \(p\). Then recolor by \(D\) the \(a\)-colored edge adjacent to \(p_i\), if \(i = 2^j\), where \(j \in L\). The resulting tree is denoted by \(T_L\).

**Case 4.** For some \(r \geq 1\), the balls \(B_r(S, z_i)\) contains at least two \(D\)-colored edges provided that \(i\) is large enough.

Hence, there exists some integer \(d \geq 0\) such that \(\text{dist}_S(z_i, x_n) = d\) for some increasing sequence \(\{n_i\}_{i=1}^\infty\), where \(\text{dist}_S\) is the shortest path distance in \(S_{\mathcal{L}}\). First, let us suppose that \(z_i = x_{n_i}\) that is \(d = 0\). The following lemma is an easy consequence of the definition of rooted tree convergence.

**Lemma 4.1.** The sequence \(\{L_{n_i}\}_{i=1}^\infty \subset \{0, 1\}^\mathbb{N}\) is convergent in the Tychonoff-topology of pointwise convergence and \((T, q) \cong (T_L, p)\), where \(L\) is the limit of \(\{L_{n_i}\}_{i=1}^\infty\).

By our construction, \((T_L, p)\) is sequentially contained in \((S, x_0)\). Now, if \(d \neq 0\), then \((T, x) \cong (T_L, p')\), where \(p'\) is some vertex of \(T_L\). Hence, \((T, x)\) is still sequentially contained by \((S, x_0)\). This finishes the proof of Proposition 4.1. \(\square\)

Now we are ready to finish the proof of Theorem 3. Let \(\mathcal{N} = \{N_j\}_{j=1}^\infty \subset \{0, 1\}^\mathbb{N}\) be a countable closed set with Cantor-Bendixson rank \(\alpha\). It is well-known that such set exists. Choose the sets \(\{L_i\}_{i=1}^\infty\) in such a way that each \(L_i\) equals to some \(N_j\) and for each \(j\) we have infinitely many \(i\) so that \(L_i\) equals to \(N_j\). We denote by \(\text{Cl}(S_{\mathcal{L}})\) the closure of \(\{S_{\mathcal{L}}, y\}_{y \in S_{\mathcal{L}}}\) in \(\mathcal{RT}_D\). The following proposition finishes the proof of Theorem 3.
Proposition 4.2. The Cantor-Bendixson rank of $\text{Cl}(S_L)$ is $\alpha + 2$ if $\alpha$ is an infinite ordinal and $\alpha + 3$ if $\alpha$ is a finite ordinal.

Proof. As we have seen in the proof of Proposition 4.1, $\text{Cl}(S_L)$ is the union of the families $\{(S_L, y)\}_{y \in S_L}, \bigcup_{j=1}^{\infty} \{(T_{N_j}, z)\}_{z \in T_{N_j}}, \{(\hat{T}_3, x)\}_{x \in \hat{T}_3}$ and the element $(T_3, p)$. Let $\{M_k\}_{k=1}^{\infty} \subset \{0, 1\}^\mathbb{N}$ and $Q := \bigcup_{k=1}^{\infty} \{(T_{M_k}, z)\}_{z \in T_{M_k}}$. Again, by Proposition 4.1, $\text{Cl}(Q)$ is the union of the families $\bigcup_{L \in \bigcup_{k=1}^{\infty} M_k} \{(T_L, y)\}_{y \in T_L}, \{(\hat{T}_3, x)\}_{x \in \hat{T}_3}$ and the element $(T_3, p)$. Hence, the isolated points of $\text{Cl}(S_L)$ denoted by $\text{Cl}_0(S_L)$ are exactly the elements $\{(S_L, y)\}_{y \in \mathcal{L}}$. The isolated points of $\text{Cl}(S_L) \setminus \text{Cl}_0(S_L)$ denoted by $\text{Cl}_1(S_L)$ are $\bigcup_{N_j \in \mathcal{N}_0} \{(T_{N_j}, z)\}_{z \in T_{N_j}}$, where $\mathcal{N}_0$ is the set of isolated points in $\mathcal{N}$. For an ordinal $\gamma$, define $\mathcal{N}_\gamma$ to be the isolated points of $\mathcal{N} \setminus \bigcup_{\beta < \gamma} \mathcal{N}_\beta$. Similarly, define $\mathcal{N}_\gamma$ to be the isolated points of $\text{Cl}(S_L) \setminus \bigcup_{\beta < \gamma} \text{Cl}_\beta(S_L)$. By transfinite induction, we can see that for finite ordinals $\beta$

$$\text{Cl}_{\beta+1}(S_L) = \bigcup_{N_j \in \mathcal{N}_\beta} \{(T_{N_j}, z)\}_{z \in T_{N_j}},$$

and for infinite ordinals $\beta$

$$\text{Cl}_\beta(S_L) = \bigcup_{N_j \in \mathcal{N}_\beta} \{(T_{N_j}, z)\}_{z \in T_{N_j}}.$$

Therefore by the definition of the set $\mathcal{N}$, if $\alpha$ is a finite ordinal

$$\text{Cl}(S_L) \setminus \text{Cl}_{\alpha+1}(S_L) = \{(\hat{T}_3, x)\}_{x \in \hat{T}_3} \cup (T_3, p),$$

and if $\alpha$ is an infinite ordinal

$$\text{Cl}(S_L) \setminus \text{Cl}_\alpha(S_L) = \{(\hat{T}_3, x)\}_{x \in \hat{T}_3} \cup (T_3, p).$$

Hence for a finite ordinal $\text{Cl}_{\alpha+3}$ consists of one single element $(T_3, p)$. For an infinite ordinal $\text{Cl}_{\alpha+2}$ consists of one single element $(T_3, p)$. $\square$

5. Minimal systems

The goal of this section is to prove Theorem 2. First, let us recall the notion of minimal subshifts of the Bernoulli shift space. Let $\{a, b, c\}^\mathbb{Z}$ be the set of all $\{a, b, c\}$-valued functions $\sigma$ on the integers with the natural $\mathbb{Z}$-action

$$t_n(\sigma)(a) = \sigma(a - n).$$

A minimal subshift is a closed, invariant subspace $\Sigma \subset \{a, b, c\}^\mathbb{Z}$ such that the orbit closure of any $\sigma \in \Sigma$ is $\Sigma$ itself. Let $w = (q_k, q_{k-1}, \ldots, q_1) \in \Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and $\sigma \in \{a, b, c\}^\mathbb{Z}$. We say that $n \in \mathbb{Z}$ sees $w$ if

$$\sigma(n - i) = q_i \quad \text{for any} \quad 1 \leq i \leq k.$$  

It is easy to see (e.g. [11]) that the orbit closure of $\sigma$ is a minimal subshift if for any $w \in \Gamma$ that is seen by some integer $n$, there exists $m_w > 0$ such that the longest interval in $\mathbb{Z}$ without elements that see $w$ is shorter than $m_w$. We call such a $\sigma$ a minimal sequence. A good minimal sequence is a minimal sequence that does not contain the same letter consecutively. It is well-known that good minimal sequences exist for which the associated subshift has the cardinality of the continuum. Now, let us consider the line graph $L$ on $\mathbb{Z}$. That is, $a, b \in \mathbb{Z}$ is connected if and only if $|a - b| = 1$. Let $\sigma$ be a good minimal sequence.
Color the edge \((n, n + 1)\) of \(L\) by \(\sigma(n)\). Then we obtain the \(\{a, b, c\}\)-tree \(L_\sigma\). Let \(\Sigma\) be the orbit closure of \(\sigma\). By Lemma 3.3 and the definition of a good minimal sequence one can see immediately that if \(\tau \in \Sigma\) and \(n \in \mathbb{Z}\), then \(\text{Stab}_{L_\tau}(n)\) is in the orbit closure of \(\text{Stab}_{L_\sigma}(0)\) in the space of subgroups \(S(\Gamma)\). Conversely, any element of the orbit closure of \(\text{Stab}_{L_\sigma}(0)\) is in the form of \(\text{Stab}_{L_\tau}(n)\), for some \(\tau\) and \(n\). That is,
\[
\{\text{Stab}_{L_\tau}(n)\}_{\tau \in \Sigma, n \in \mathbb{Z}}
\]
is a minimal \(\Gamma\)-system in \(S(\Gamma)\). These systems are called a uniformly recurrent subgroup (URS) in \(\mathbb{Z}\) (we thank Miklós Abért to call our attention to this paper). Since there are continuum many minimal subshifts in \(\{a, b, c\}\mathbb{Z}\), in this way we obtain continuum many URS’s in \(S(\Gamma)\) (see Theorem 5.1 [5]). Note that if \(\Sigma\) and \(\Sigma'\) represents the same URS, then either \(\Sigma = \Sigma'\) or \(\Sigma' = \Sigma^{-1}\), where \(\sigma^{-1}(n) := \sigma(-n)\), for any \(\sigma \in \Sigma\). Clearly, an URS is an equationally compact set in \(S(\Gamma)\), therefore Theorem 2 follows from the lemma below.

**Lemma 5.1.** Let \(\sigma, \tau\) be good elements of a minimal subshift \(\Sigma\). Suppose that \(\tau\) is neither a \(\mathbb{Z}\)-translate of \(\sigma\) nor a \(\mathbb{Z}\)-translate of \(\sigma^{-1}\). Then for any \(n \geq 1\), \(\text{Stab}_{L_\sigma}(0) \not\subset \text{Stab}_{L_\tau}(n)\).

**Proof.** Let \(n \geq 1\) and let \(w = (q_k, q_{k-1}, \ldots, q_1)\) be the longest word such that one of the following two conditions hold.

1. \(\sigma(-i) = \tau(n - i)\) for any \(1 \leq i \leq k\).
2. \(\sigma(-i) = \tau(n + i)\) for any \(1 \leq i \leq k\).

Without a loss of generality we can suppose that the first condition holds, \(\sigma(-k) = c\), \(\sigma(-k - 1) = a\), \(\tau(n - k - 1) = b\). Then clearly, \(w^{-1}bw \in \text{Stab}_{L_\tau}(0)\). On the other hand, \(w^{-1}bw \not\in \text{Stab}_{L_\tau}(n)\). Indeed, \(bw(n) = n - k - 1\), hence \(w^{-1}bw(n) < n\). \(\square\)

### 6. Weak equivalence of actions I. (the basic construction)

In this section we construct a continuum of equationally compact subgroups of the group \(\Gamma_5 = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\). Let \(\{a, b, c, d, e\}\) be free generators of order two in \(\Gamma_5\). Let \(T_5\) be the infinite tree of vertex degrees five properly edge-colored by the symbols \(\{a, b, c, d, e\}\). We will consider good \(\{a, b, c, d, e, D\}\)-colorings of \(T_5\) and the associated \(\Gamma_5\)-actions, where a \(D\)-colored edge is modified exactly the same way as in Section 3 by adding two vertices \(p_x, p_y\) such that the edges \((x, p_x)\) and \((p_y, y)\) are colored by \(a\) and the edge \((p_x, p_y)\) is colored by \(b\). So, in the associated \(\{a, b, c, d, e\}\)-tree the degrees of the new vertices are two, the degrees of the old vertices are five.

**The master subtree** \(R\). In order to build our master subtree \(R\) in \(T_5\) we need some definitions. A \((2^n, b, a)\)-path in \(T_5\) is a path \((x_0, x_1, x_2, \ldots, x_{2^n})\) such that all edges \((x_{2i}, x_{2i+1})\) are colored by \(b\) and all edges \((x_{2i+1}, x_{2i+2})\) are colored by \(a\). Hence the path starts with a \(b\)-colored edge and ends with an \(a\)-colored edge. We define the \((2^n, c, a)\)-paths similarly. They start with a \(c\)-colored edge and end with an \(a\)-colored edge. Now, fix a vertex \(p \in V(T_5)\). Pick the \((2, b, a)\)-path and the \((2, c, a)\)-path starting from \(p\) to obtain two new endpoints. Then for each of the two endpoints pick the \((4, b, a)\)-path and the \((4, c, a)\)-path starting from them to obtain four new endpoints. In the \(n\)-th step we see \(2^{n-1}\)-endpoints.
and for each such endpoints \( x \) we pick the \((2^n, b, a)\)-path and the \((2^n, c, a)\)-path starting from \( x \) to obtain \( 2^n \) new endpoints. Inductively, we build the infinite subtree \( R \), as the union of all the chosen paths above.

**The codes.** A code \( C \) is an infinite sequence \( \{a_n\}_{n=1}^{\infty} \) such that if \( n \geq 1 \), then \( a_i = b \) or \( a_i = c \). Using our master tree \( R \), for each code \( C \) we build a good \((a, b, c, d, e, D)\)-tree \( T_C \) and the associated \((a, b, c, d, e)\)-tree \( S_C \). Note that \( T_C \) will be recoloring of \( T_5 \) and not a recoloring of the master subtree \( R \).

**Step 1.** We recolor by \( D \) the last edge of the \((2, a_1, a)\)-path starting from \( p = p_C \) we denote by \( x_1^C \) the endpoint of the other \( 2 \)-path \((a, b, a)\)-path or a \((2, c, a)\)-path) starting from \( p \).

**Step 2.** We recolor by \( D \) the last edge of the \((4, a_2, a)\)-path starting from \( x_1^C \). Then we denote the endpoint of the other \( 4 \)-path by \( x_2^C \).

**Step n.** We recolor by \( D \) the last edge of the \((2^n, a_n, a)\)-path starting from \( x_{n-1}^C \) and denote by \( x_n^C \) the endpoint of the other \( 2^n \)-path.

Inductively, we construct the good \((a, b, c, d, e, D)\)-tree \( T_C \).

Since the associated \((a, b, c, d, e)\)-tree \( S_C \) is sparse, by Proposition 3.2 \( \Gamma_C := \text{Stab}_{S_C}(p) \) is an equationally compact subgroup of \( \Gamma_5 \). The following technical proposition is crucial for the proof of Theorem 4.

**Proposition 6.1.** Let \( C = (a_1, a_2, \ldots) \) and \( C' = (a'_1, a'_2, \ldots) \) be different codes. Then there exists a finitely generated non-amenable subgroup \( H \subset \Gamma_C \) such that no finite subset of \( V(S_{C'}) \) is invariant under the action of \( H \).

**Proof.** We prove our propostion using two lemmas.

**Lemma 6.1.**

\[
\text{Stab}_{S_C}(p_C) \not\subset \text{Stab}_{S_{C'}}(p_{C'})
\]

**Proof.** Since \( C \neq C' \), there exist \( q, q' \in V(S_C), q' \in V(S_{C'}) \) and \( n \geq 1 \) such that

- The paths \((p_C, q)\) and \((p_{C'}, q')\) are colored-isomorphic paths of length \( n \).
- The degree of \( q \) is two and the degree of \( q' \) is five.

Let \( w \in \Gamma_5 \) be the word of length \( n \) such that \( w(p_C) = q \) and \( w(p_{C'}) = q' \). Then \( w^{-1}ew \in \text{Stab}_{S_C}(p_C) \) and \( w^{-1}ew \notin \text{Stab}_{S_{C'}}(p_{C'}) \). \( \square \)

Let \( \gamma = a_1aeaa_1, \delta = \overline{a_1}aa_2aa_2aeeaa_2aa_2a\overline{a_1}, \) where \( \overline{a_1} = b \) if \( a_1 = c \) and \( \overline{a_1} = c \) if \( a_1 = b \). Obviously, \( \gamma, \delta \in \text{Stab}_{S_C}(p_C) \).

**Lemma 6.2.** Let \( F \) be a finite subset of \( V(S_{C'}) \) containing at least one element that does not equal to \( p_{C'} \). Then \( F \) is not invariant under the group generated by \( \gamma \) and \( \delta \).

**Proof.** Suppose that \( q \in F \) and \( q \) is not a vertex of degree 2 or a vertex from the master subtree \( R \). Then either the \( a_1 \)-branch or the \( a_2 \)-branch of \( q \) contains only vertices of degree
five. Hence, either the set \( \{(\gamma \delta)^n(q)\}_{n=1}^\infty \) or the set \( \{(\delta \gamma)^n(q)\}_{n=1}^\infty \) is infinite. So, we can suppose that all the elements of \( F \) are either vertices of degree 2 or they are from the master subtree. Let \( s \) be one of the furthest elements from \( p_{C'} \) in \( F \). Then either \( \gamma(s) \) or \( \delta(s) \) are further from \( p_{C'} \) than \( s \). Hence \( F \) cannot be invariant under the group generated by \( \gamma \) and \( \delta \). □

Now, let \( H \subset \text{Stab}_{S_C}(p_C) \) be generated by the set
\[
\{\gamma, \delta, a_1adaa_1, a_1aeaa_1, a_1afaa_1, \alpha\},
\]
where \( \alpha \in \text{Stab}_{S_C}(p_C) \setminus \text{Stab}_{S_C'}(p_{C'}) \). Then \( H \) is finitely generated, non-amenable subgroup of \( \text{Stab}_{S_C}(p_C) \), without non-empty invariant subsets in \( V(S_{C'}) \). □

Now we recall the notion of amenable actions [7]. An action \( \alpha : \Gamma \actson X \) of a countable group \( \Gamma \) on a countable set \( X \) is called amenable if there exists a sequence of finite subsets in \( X \), the so-called Følner sets \( \{F_n\}_{n=1}^\infty \) such that for any \( g \in \Gamma \)
\[
\lim_{n \to \infty} \frac{|\alpha(g)(F_n) \cup F_n|}{|F_n|} = 1.
\]
Suppose that \( \Gamma \) is generated by the finite set \( \{g_1, g_2, \ldots, g_r\} \). Then the action \( \alpha \) is non-amenable if and only if there exists an \( \epsilon > 0 \) such that for any finite set \( F \subset X \)
\[
|\bigcup_{j=1}^r \alpha(g_j)(F) \cup F| \geq (1 + \epsilon)|F|.
\]
The left action of a group \( \Gamma \) on itself or on an overgroup of \( \Gamma \) is amenable if and only if the group \( \Gamma \) is amenable.

**Proposition 6.2.** Let \( C \neq C' \) be two codes and \( H \subset \text{Stab}_{S_C}(p_C) \) be the finitely generated non-amenable subgroup of \( \Gamma_5 \) as in Proposition 6.1, then the restricted action of \( H \) on the set \( V(S_{C'}) \) is non-amenable.

**Proof.** We start with a technical lemma.

**Lemma 6.3.** Let \( \{G_n\}_{n=1}^\infty \) be a sequence of connected, induced, finite subgraphs in \( V(S_{C'}) \). Then
\[
\liminf_{n \to \infty} \frac{|\bigcup_{j=1}^r \beta(h_j)(V(G_n)) \cup V(G_n)|}{|V(G_n)|} > 1.
\]
Here, \( \beta \) denotes the restricted action of \( H \).

**Proof.** For any \( n \geq 1 \), we consider the finite induced subgraph \( H_n \subset T_5 \) constructed the following way. If \( (p, q, r, s) \) is a path in \( S_{C'} \) such that the degrees of \( q \) and \( p \) are two in \( S_{C'} \) with respect to the standard generating system \( \{a, b, c, d, e\} \) (that is, this path was substituted for a \( D \)-colored edge) and the path is in \( G_n \), then let \( (p, s) \in E(H_n) \). If only \( (p, q) \) or \( (r, s) \) is an edge of \( G_n \), then \( (p, s) \notin E(H_n) \). On the other hand, if the degrees of \( p \) and \( q \) are five in \( S_{C'} \), then \( (p, q) \in H_n \) if and only if \( (p, q) \in G_n \). So we substitute
new paths with a single edge and cut off hanging edges starting from new vertices. Hence, there is a bijection between \( V_5(G_n) \) and \( V(H_n) \), where
\[
V_5(G_n) = \{ x \in V(G_n) \mid \deg_{S_{G'}}(x) = 5 \}.
\]
Since \( S_{G'} \) is sparse (that is for any \( n \geq 1 \) there are only finitely many pairs of degree 2 vertices of distance less than \( n \),
\[
\lim_{n \to \infty} \frac{|\bigcup_{j=1}^{r} \alpha(h_j)(V(H_n)) \cup V(H_n)|}{|\bigcup_{j=1}^{r} \beta(h_j)(V(G_n)) \cup V(G_n)|} = 1,
\]
where \( \alpha \) denotes the left action of \( H \) on the overgroup \( \Gamma \). By the non-amenability of \( \alpha \),
\[
\lim_{n \to \infty} \frac{|\bigcup_{j=1}^{r} \alpha(h_j)(V(H_n)) \cup V(H_n)|}{|V(H_n)|} > 1,
\]
hence by (1) our proposition follows.

**Corollary 6.1.** There exist \( \delta > 0 \) and \( C > 0 \) such that for any finite, connected induced subgraph \( L \subset G \) such that \( |V(L)| \geq C \)
\[
|\bigcup_{j=1}^{r} \beta(h_j)(V(L)) \cup V(L)| > (1 + \delta)|V(L)|.
\]

Now, we finish the proof of the proposition. Let \( M \subset V(S_{G'}) \) be a finite subset and \( L_M \) be the subgraph induced by \( M \). Let \( \{L_q\}_{q=1}^{s} \) be the connected components of \( L_M \). By Proposition 6.1 \( V(L_j) \) is not invariant under \( H \), hence for any \( 1 \leq q \leq s \)
\[
|\bigcup_{j=1}^{r} \beta(h_j)(V(L_q)) \cup V(L_q)| \geq \frac{C + 1}{C} |V(L_q)|,
\]
provided that \( |V(L_q)| \leq C \). Hence we have the following inequality. For any \( 1 \leq q \leq s \),
\[
|\bigcup_{j=1}^{r} \beta(h_j)(V(L_q)) \setminus V(L_q)| \geq m|V(L_q)|,
\]
where \( m = \min(\frac{1}{C}, \delta) \).

Observe that for any vertex \( x \in L_M \), there exist at most \( r \) values of \( q \) such that
\[
x \in \bigcup_{j=1}^{r} \beta(h_j)(V(L_q)) \setminus V(L_q),
\]
hence by (2), we have that
\[
|\bigcup_{j=1}^{r} \beta(h_j)(V(L_M)) \cup V(L_M)| \geq (1 + \frac{m}{r})|V(L_M)|.
\]

7. **Weak equivalence of actions II. (the proof of Theorem 4)**

The notion of weak equivalence was introduced by Kechris [6] and since then it has been studied extensively. Let \( \alpha : \Gamma \actson (M, \lambda), \beta : \Gamma \actson (M, \lambda) \) be measure preserving actions of a countable group \( \Gamma \) on a standard probability space \((M, \lambda)\). We say that \( \alpha \) weakly contains \( \beta \), \( \alpha \succeq \beta \) if for any finite measurable partition of \( M, M = \bigcup_{i=1}^{n} B_i, \) a finite subset \( \{g_j\}_{j=1}^{m} \subset \Gamma \) and a constant \( \epsilon > 0 \), there exists a partition \( M = \bigcup_{i=1}^{l} A_i \) such that for any \( 1 \leq j \leq m, 1 \leq k, l \leq n \)
\[
|\lambda(\alpha(g_j)(A_k) \cap A_l) - \lambda(\beta(g_j)(B_k) \cap B_l)| \leq \epsilon.
\]
The actions $\alpha, \beta$ are weakly incomparable if $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$. It was proved in [1] that for certain groups $\Gamma$ there exist uncountably many pairwise weakly incomparable free and ergodic probability measure preserving (p.m.p.) actions of $\Gamma$. We shall prove in Theorem 4 that for $\Gamma_5$ there exist uncountably many generalized shifts associated to subgroups of $\Gamma_5$ that are pairwise weakly incomparable. We will use a result of Kechris and Tsankov [Theorem 1.2] in a crucial way. Let $\alpha : \Gamma \curvearrowright (M, \lambda)$ be a p.m.p.action. A sequence of measurable subsets in $M$, $\{A_n\}_{n=1}^\infty$ are called asymptotically invariant with respect to the action $\alpha$ if for any $g \in \Gamma$

$$\lim_{n \to \infty} \lambda(\alpha(g)(A_n) \setminus A_n) = 0.$$ 

The action $\alpha$ is called strongly ergodic if for any asymptotically invariant sequence $\{A_n\}_{n=1}^\infty$

$$\lim_{n \to \infty} \lambda(A_n)\lambda(M \setminus A_n) = 0.$$ 

Clearly, a strongly ergodic action of $\Gamma$ cannot weakly contain a non-ergodic action of $\Gamma$. Now, let $\beta : \Gamma \curvearrowright X$ be an action of $\Gamma$ on a countable set by permutations. The associated generalized Bernoulli action $\hat{\beta}$ is defined on the product space $\prod_{x \in X}([0,1], \nu)$, where $\nu$ is the Lebesgue measure. The action is defined by

$$\hat{\beta}(g)(F)(x) = F(\beta(g^{-1})(x)),$$

where $F \in [0,1]^X, g \in \Gamma$. It is easy to see that $\hat{\beta}$ is essentially free if for any $e \neq g \in \Gamma$, there exist infinitely many $x \in X$ such that $\beta(g)(x) \neq x$. According to the theorem of Kechris and Tsankov, the associated generalized Bernoulli action $\hat{\beta} : \Gamma \curvearrowright \prod_{x \in X}([0,1], \nu)$ is strongly ergodic if and only if the action $\beta$ is non-amenable. Now we can finish the proof of Theorem 4. Let $\prod_{n=1}^\infty \{b, c\}$ be the set of all codes. For each code $C$ we constructed an $\{a, b, c, d, e\}$-colored tree $S_C$ and the associated $\Gamma_5$-action $\alpha_C : \Gamma_5 \curvearrowright V(S_C)$. Let $C \neq C'$ be two codes and $H \subset \Gamma_5$ be the subgroup of $\text{Stab}_{S_C}(p_C)$ as in Proposition 6.1. Then the associated generalized Bernoulli action restricted to $H$,

$$\hat{\alpha}_C |_H : H \curvearrowright \prod_{v \in V(S_C)}([0,1], \nu)$$

is clearly non-ergodic. On the other hand, by Proposition 6.2 and the aforementioned theorem of Kechris and Tsankov,

$$\hat{\alpha}_{C'} |_H : H \curvearrowright \prod_{v \in V(S_{C'})}([0,1], \nu)$$

is strongly ergodic, therefore $\hat{\alpha}_{C'} \not\leq \hat{\alpha}_C$. Hence Theorem 4 follows. $\square$

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