On the Monotonicity of the Copula Entropy

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Abstract

Understanding the way in which random entities interact is of key interest in numerous scientific fields. This can range from a full characterization of the joint distribution to single scalar summary statistics. In this work we identify a novel relationship between the ubiquitous Shannon’s mutual information measure and the central tool for capturing real-valued non-Gaussian distributions, namely the framework of copulas. Specifically, we establish a monotonic relationship between the mutual information and the copula dependence parameter, for a wide range of copula families. In addition to the theoretical novelty, our result gives rise to highly efficient proxy to the expected likelihood, which in turn allows for scalable model selection (e.g. when learning probabilistic graphical models).

1 Introduction

Understanding the joint behavior of random entities is of great importance in essentially all scientific fields ranging from computational biology and health care to economics and astronomy. Accordingly, the study of joint distributions is fundamental to all the data sciences and goes back at least to the seminal studies of Sir Francis Galton [Galton, 1888].

In multivariate modeling, our goal may range from the task of characterizing the full joint distribution which can be difficult, to specific summary statistics such as correlation measures, e.g. Pearson’s correlation or mutual information. In this work, we identify a novel and useful relationship between two central frameworks for these tasks, namely the frameworks of copulas and information theory.

In real-valued domains, the most prominent general purpose framework for going beyond the multivariate normal distribution is that of copulas [Joe, 1997, Nelsen, 2007] pioneered by Sklar [Sklar, 1959]. In a nutshell, copulas allow us to separate the modeling of the (possibly nonparametric) univariate marginals and that of the dependence function. Formally, given a set of univariate marginal cumulative distribution functions \( \{F_{X_i}\} \), a copula function \( C_{U}(u_1, \ldots, u_n) \) is a joint distribution over variables that are marginally uniform in the \([0, 1]\) range, so that

\[
F_{X}(x_1, \ldots, x_n) = C_{\{F_{X_i}\}}(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)),
\]

arXiv:1611.06714v1 [math.ST] 21 Nov 2016
is a valid joint distribution.

This separation between the marginal representation and the copula function that links them allows us, for example, to easily capture multi-modal or heavy-tailed distributions. Indeed, the popularity of copulas as a flexible tool for capturing dependence has grown substantially in recent years [Elidan, 2013].

Most if not all popular copula families are governed by a dependence parameter that spans the range (or part of it) between independence and full dependence. In fact, in the bivariate case, this is captured by a well known and fundamental relationship between the dependence parameter of the copula and rank-based correlation measures, such as Spearman’s rho or Kendall’s tau (see, for example, chapter 3 in Nelsen [2007]).

In the field of information theory, Shannon’s mutual information measures the reduction in entropy that the knowledge of one variable induces on another. Mutual information is a fundamental tool used to quantify the strength of dependence between random variables [Shannon, 2001], and is used throughout the exact sciences. Formally, Shannon’s mutual information is defined as:

$$MI(X; Y) = \int f_{X,Y}(x, y) \log \frac{f_{X,Y}(x, y)}{f_X(x)f_Y(y)} \, dx \, dy.$$ (1.1)

Whether in machine learning or physics, mutual information is often the de facto tool for measuring correlation and/or identifying independence [Cover and Thomas, 2012]. A natural question is thus how does this statistic relate to the framework of copulas.

In this work we establish a relationship between the mutual information of two variables and their corresponding copula dependence parameter. Concretely, we prove that the mutual information (equivalently the copula entropy) is monotonic in the bivariate copula dependence parameter for a wide range of copula families, covering the vast majority of copulas used in practice. We also extend our results for the popular class of Archimedean copulas to higher dimensions where other measures of dependence such as Spearman’s rho or Kendall tau are not well defined.

The monotonicity result is a theoretical one and an obvious question is whether it has practical merit in the statistical sense. In our simulations we show that it holds in practice using a modest number of samples. An immediate implication is that the mutual information between different pairs of variables can be ranked by evaluating simple statistics that are substantially simpler than the copula entropy, namely Spearman’s rho or Kendall’s tau.

Thus, for example, given gene expression data for a large number of genes, we can identify the two genes that have the highest mutual information without ever evaluating the mutual information or the copula density for all pairs of genes, a computationally formidable task. More broadly, our results facilitate highly efficient model selection in scenarios involving many interactions, e.g. when learning probabilistic graphical models. This practical implication is studied in depth in our earlier paper [Tenzer and Elidan, 2013], where the theoretical results were substantially more limited.
The rest of the paper is organized as follows. In Section 2 we briefly review the necessary background on copulas, TP2, super-modular functions and stochastic orders. In Section 3 we present our main theoretical result: broad-coverage sufficient conditions that guarantee the monotonicity of the mutual information (copula entropy) in the copula dependence parameter. We extend the results to the class of bivariate two-parameter families in Section 3.3. We then generalize the results for Archimedean copulas of any dimension in Section 4. In Section 6 we show the applicability of our theoretical results in the empirical finite-sample scenario. We finish with concluding remarks in Section 7.

2 Preliminaries

In this section we briefly review basic definitions and results related to copulas, super modular functions and stochastic orders that will be needed in the sequel.

2.1 Copulas

A copula is a multivariate joint distribution whose univariate marginals are uniformly distributed. Formally:

**Definition 2.1.** Let \( U_1, \ldots, U_n \) be random variables marginally uniformly distributed on \([0, 1] \). A copula function \( C : [0, 1]^n \rightarrow [0, 1] \) is a joint distribution

\[
C_\theta(u_1, \ldots, u_n) = P(U_1 \leq u_1, \ldots, U_n \leq u_n),
\]

where \( \theta \) are the parameters of the copula function.

Now consider an arbitrary set \( X = \{X_1, \ldots, X_n\} \) of real-valued random variables (typically not marginally uniformly distributed). Sklar’s seminal theorem states that for any joint distribution \( F_X(x) \), there exists a copula function \( C \) such that

\[
F_X(x) = C(F_1(x_1), \ldots, F_n(x_n)).
\]

When the univariate marginals are continuous, \( C \) is uniquely defined.

The constructive converse, which is of central interest from a modeling perspective, is also true. Since \( U_i \equiv F_i \) is itself a random variable that is always uniformly distributed in \([0, 1] \), any copula function taking any marginal distributions \( \{U_i\} \) as its arguments, defines a valid joint distribution with marginals \( \{U_i\} \). Thus, copulas are “distribution generating” functions that allow us to separate the choice of the univariate marginals and that of the dependence structure.

Deriving the joint density \( f(x) = \frac{\partial^n F(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n} \) from the copula construction, assuming \( F \) has n-order partial derivatives (true almost everywhere when \( F \) is...
continuous) is straightforward. Using the chain rule we can write
\[ f(x) = \frac{\partial^n C(F_1(x_1), \ldots, F_n(x_n))}{\partial F_1(x_1) \ldots \partial F_n(x_n)} \prod_i f_i(x_i) \]
where \( c(F_1(x_1), \ldots, F_n(x_n)) \) is called the \textit{copula density function}.

Copulas are intimately related to the fundamental concept of Mutual Information (MI) of random variables [Cover and Thomas, 2012]. In the bivariate case, the MI of two random variables \( X \) and \( Y \) is defined as in Equation (1.1). Denote \( U \equiv F_X \) and \( V \equiv F_Y \) so that \( c_{U,V} \) is the copula of the joint distribution of \( X \) and \( Y \) Applying Equation (2.1) we get
\[ MI(X;Y) = -\int c_{U,V}(u,v) \log c_{U,V}(u,v) dudv \]
In words, the MI between two random variables equals to the entropy of the corresponding copula. It is easy to see that this also applies to higher dimensions.

Copulas are also closely tied to other measures of association. The following relationship between the copula function and Spearman’s \( \rho_{X,Y} \) is well known in the bivariate case:
\[ \rho_{X,Y} = 12 \int C(F_X(x), F_Y(y)) dxdy - 3, \]
and a similar relationship is known for Kendall’s tau [Nelsen, 2007]

\[2.2\] TP2 and Super-Modular Functions

Total positive of order two functions (TP2) [Olkin and Marshall, 2016] play a central role in statistics and many common copula families have a TP2 density function. Below we define the TP2 concept and provide a simply connection to super-modular functions that will be useful for our developments. Formally, denote \( u \vee v = \min(u,v) \) and \( u \wedge v = \max(u,v) \).

\textbf{Definition 2.2.:} A function \( \Psi : \mathbb{R}^2 \Rightarrow \mathbb{R} \) is called TP2 if
\[ \forall \quad u, v \in \mathbb{R}^2 \quad \Psi(u \vee v) \cdot \Psi(u \wedge v) \geq \Psi(u) \cdot \Psi(v). \]
When the inequality is reversed, the function is reverse rule of order 2 (RR2).

\textbf{Definition 2.3.:} A function \( \Psi : \mathbb{R}^2 \Rightarrow \mathbb{R} \) is said to be super-modular if
\[ \forall \quad u, v \in \mathbb{R}^2, \Psi(u \vee v) + \Psi(u \wedge v) \geq \Psi(u) + \Psi(v) \]
Note that if the inequality is reversed the function is called sub-modular [Olkin and Marshall, 2016]. The following two simple results will be useful in the sequel.

**Lemma 2.4.** Let \( \Psi(u,v) \) be a positive TP2 (RR2) function. Then \( \Phi(u,v) = \log(\Psi(u,v)) \) is super-modular (sub-modular).

**Proof:** From the definition of a TP2 function \( \Psi(u \vee v) \cdot \Psi(u \wedge v) \geq \Psi(u) \cdot \Psi(v) \), we have

\[
\Phi(u \vee v) + \Phi(u \wedge v) = \log(\Psi(u \vee v)) + \log(\Psi(u \wedge v)) \\
= \log(\Psi(u \vee v)\Psi(u \wedge v)) \\
\geq \log(\Psi(u)\Psi(v)) \\
= \log(\Psi(u)) + \log(\Psi(v)) \\
= \Phi(u) + \Phi(v)
\]

**Lemma 2.5.** Let \( f_1(x,y), f_2(x,y) \) be two real non-negative TP2 (RR2) functions. Then \( \Psi(x,y) = f_1 f_2 \) is TP2 (RR2).

The proof is immediate and we omit the details.

### 2.3 PQD and Super-Modular Orderings

Stochastic orderings introduce the notion of partial orders between random variables. Perhaps the most well known is the standard stochastic order, or using the name more commonly used in the copula community, the positive quadrant dependent (PQD) order:

**Definition 2.6.** Let \( X, X' \) be bivariate random vectors and let \( F_X(u,v), F_{X'}(u,v) \) be the corresponding distribution functions. \( X' \) is said to be more PQD than \( X \) if:

\[
\forall (u,v) \in \mathbb{R}^2, F_X(u,v) \leq F_{X'}(u,v)
\]

Another stochastic ordering that will be useful in our development is the Super-modular (SM) ordering:

**Definition 2.7.** Let \( X, X' \) be bivariate random vectors, \( X' \) is said to be greater than \( X \) in the super-modular order, denoted by \( X \leq_{\text{SM}} X' \), if \( \forall \Psi \) such that \( \Psi \) is super modular:

\[
E_X(\Psi(x)) \leq E_{X'}(\Psi(x'))
\]

Note that since most bivariate families are PQD ordered by construction [Joe, 1997], an immediate result of Equation (2.3) is that Spearman’s rho is monotonic in the copula dependence parameter, within a specific copula family.
3 The Monotonicity of Mutual Information in the Dependence Parameter for Bivariate Copulas

We now present our central result and establish a novel and elegant connection between the copula function of the distribution of two random variables and the mutual information between these variables (equivalently the copula entropy). In Section 4 we generalize some of these results to higher dimensions.

**Theorem 3.1.** Let $C_\theta(u,v)$ be an absolutely continuous bivariate copula, and let $X, X'$ be two bivariate random vectors distributed according to the same copula family with two different parameterizations so that $X \sim C_{\theta_1}(u,v), X' \sim C_{\theta_2}(u,v)$, where $\theta_2 \geq \theta_1 \geq 0$. Then $-H(X) \leq -H(X')$ if one of the following condition holds:

(a) $C_\theta(u,v)$ is increasing (decreasing) in $<_{SM}$ and the copula density $c_\theta(u,v)$ is TP2 (RR2) (for all $\theta$).

(b) $C_\theta(u,v)$ is an Archimedean copula whose generator $\phi_\theta$ is completely monotone and satisfies the boundary condition $\phi_\theta(0) = 1$. In addition the copula is increasing in $<_{SM}$.

(c) $C_{\theta,\delta}(u,v)$ is a two-parameters Archimedean copula whose generator $\eta_{\theta,\delta}(s)$ is completely monotone and satisfies the boundary condition $\eta_{\theta,\delta}(0) = 1$. In addition the copula is increasing in $<_{SM}$, with respect to $\theta$, for a fixed $\delta$.

(d) $C_\theta(u,v)$ is an elliptical copula.

That is, the negative copula entropy defined in Equation (2.3) is monotonic increasing (decreasing) in $\theta$ if any of the above conditions hold for the copula family.

We note that in the bivariate case, SM ordering is equivalent to PQD ordering [Shaked and Shanthikumar, 2007] so that the above conditions (a)-(c) can equivalently be stated using a PQD order condition. We also note that condition (d) was proved in [Elidan, 2012] using an explicit formula of the copula entropy and is stated here for completeness.

An immediate consequence of Theorem 3.1 and the known monotonicity of $\rho_s$ in the dependence parameter $\theta$ for PQD ordered families [Nelsen 2007], is

**Corollary 3.2.** If any of the above conditions (a)-(d) hold for a copula family, then the magnitude of Spearman’s $\rho_s$ is monotonic in the copula entropy.

Importantly, one of the above conditions holds in most if not all commonly used bivariate copula families so that our result is widely applicable. In the sections below, we prove the result for each of the sufficient conditions (a)-(c). In Section 5 we survey some example families.
3.1 Proof for (a): TP2 Density and SM/PQD Order

To prove the result, we are going to show that the following holds:

\[ \int c_{\theta_1}(u,v) \log(c_{\theta_1}(u,v)) \partial u \partial v \leq \int c_{\theta_2}(u,v) \log(c_{\theta_2}(u,v)) \partial u \partial v. \]

Let \( \Psi(u,v) = \log(c_{\theta_1}(u,v)) \). Since \( c_{\theta_1}(u,v) \) is TP2, \( \Psi(u,v) \) is super modular. 

\[ X \leq_{SM} X'. \]

Thus, \( E(\Psi(x)) \leq E(\Psi(x')) \) for all super modular function \( f(u,v) \) and in particular for \( f = \Psi(u,v) \), according to Lemma 2.4. Thus,

\[ \int c_{\theta_1}(u,v) \Psi(u,v) \partial u \partial v \leq \int c_{\theta_2}(u,v) \Psi(u,v) \partial u \partial v. \]

Now, substituting the explicit form of \( \Psi \) we get the first inequality:

\[ \int c_{\theta_1}(u,v) \log(c_{\theta_1}(u,v)) \partial u \partial v \leq \int c_{\theta_2}(u,v) \log(c_{\theta_1}(u,v)) \partial u \partial v. \]

To prove the second inequality we observe that

\[ \int c_{\theta_2}(u,v) \log(c_{\theta_1}(u,v)) \partial u \partial v \leq \int c_{\theta_2}(u,v) \log(c_{\theta_2}(u,v)) \partial u \partial v \]

\[ \Leftrightarrow \int c_{\theta_2}(u,v) \log \left( \frac{c_{\theta_2}(u,v)}{c_{\theta_1}(u,v)} \right) \partial u \partial v \geq 0 \]

\[ \Leftrightarrow KL(c_{\theta_2}; c_{\theta_1}) \geq 0 \]

where \( KL(\cdot) \) is the Kullback-Leibler divergence [Cover and Thomas 2012], which is always non-negative. Putting these inequalities together we can conclude:

\[ -H(X) \equiv \int c_{\theta_1}(u,v) \log(c_{\theta_1}(u,v)) dudv \]

\[ \leq \int c_{\theta_2}(u,v) \log(c_{\theta_2}(u,v)) dudv \equiv -H(X') \]

3.2 Proof for (b): Bivariate Archimedean Copulas

Archimedean copulas are probably the most common non-elliptical copulas that allow for asymmetric or heavy-tail distributions.

Condition (a) in Theorem 3.1 is a fairly general one in the sense that it does not put any restrictions on the copula family type. In particular, it does not require the family to be elliptical or Archimedean. As we shall see below, Condition (a) actually implies condition (a) in the special case of Archimedean copulas with completely monotone generator. Among many examples of copula families for which this holds are the Ali-Mikhail (AMH), Clayton, Frank and the Gumbel families (see Section 5 for details).
Definition 3.3.: Let \( \psi_\theta(x) : [0, \infty) \to [0, \infty) \) be a strictly convex univariate function that is parametrized by \( \theta \in \mathbb{R} \). In addition assume that \( \psi_\theta(\infty) = 0 \). The Archimedean copula that is generated by \( \psi_\theta(x) \) is defined as:

\[
C_\theta(u, v) = \psi_\theta^{-1}(u_1) + \psi_\theta^{-1}(u_2). \tag{3.1}
\]

We say that \( \psi_\theta(x) \) is the copula generator of \( C_\theta(u, v) \) [Nelsen 2007]. We consider the subclass of copula generators:

\[
L_\infty = \{ \psi_\theta : (-1)^i \psi_\theta(x)^{(i)} \geq 0 \ \forall i = 0, 1, 2, \ldots, \psi_\theta(0) = 1 \}.
\]

This is the class of generators whose derivatives alternate signs (this property is widely known as completely monotonicity) and in addition these generators satisfy the boundary condition \( \psi_\theta(0) = 1 \).

The following lemma shows that an Archimedean copula whose generator \( \psi_\theta(x) \) is in \( L_\infty \), can be written as a mixture of two univariate CDFs. We will then use this to show that [3] implies [1].

Lemma 3.4.: Let \( C_\theta(u, v) \) be an Archimedean copula and let \( \psi_\theta \) be its generator such that \( \psi_\theta \in L_\infty \). Then there exists a CDF \( M(\alpha) \), of a positive random variable \( \alpha \), and unique CDFs \( G_1(u), G_2(v) \) such that:

\[
C_\theta(u, v) = \int_0^\infty G_1(u)^\alpha \cdot G_2(v)^\alpha dM(\alpha).
\]

For the sake of completeness, we give a simple proof here. Note that a different proof can be found in [Joe 1997].

Proof: From Bernstein’s theorem we have that each \( \psi_\theta \in L_\infty \) is a Laplace transform of some distribution function of a positive random variable \( \alpha \). That is:

\[
\psi_\theta(s) = \int_0^\infty e^{-sa} dM(\alpha), \quad s \geq 0.
\]

Thus

\[
C_\theta(u, v) = \psi_\theta^{-1}(u) + \psi_\theta^{-1}(v) = \int_0^\infty e^{-\alpha(\psi_\theta^{-1}(u) + \psi_\theta^{-1}(v))} dM(\alpha). \tag{3.2}
\]

In addition, for any arbitrary distribution function \( F \), and any positive random variable \( M(\alpha) \), there exists a unique distribution function \( G \) such that [Joe 1997, p.84]:

\[
F(x) = \int_0^\infty G(x)^\alpha dM(\alpha) = \int_0^\infty e^{-\alpha(-\ln G(x))} dM(\alpha) = \psi_\theta(-\ln G(x)),
\]

Thus, \( G(x) = e^{-\psi_\theta^{-1}(F(x))} \). In particular, if \( U, V \) are uniform, then \( F(u) = u, F(v) = v \) and therefore there exist \( G_1(u), G_2(v) \) such that \( G_1(u) = e^{-\psi_\theta^{-1}(u)} \) and \( G_2(v) = e^{-\psi_\theta^{-1}(v)} \). Substituting this into Equation (3.2), we get

\[
C_\theta(u, v) = \int_0^\infty G_1(u)^\alpha \cdot G_2(v)^\alpha dM(\alpha).
\]
Finally, a mixture representation implies a TP2 density \cite{Joe1997}. Therefore if in addition PQD/SM ordering holds for these copula families, then condition \(a\) is implied.

### 3.3 Proof for (d): Bivariate Two-parameters Archimedean Copulas

Two-parameter families are used to capture more than one type or aspect of dependence. For example, one parameter may control the strength of upper-tail dependence while the other indicates concordance or the strength of lower tail dependence. Bivariate Archimedean copulas generalize the one-parameter families. In particular, these families have the following form:

\[
C_{\theta,\delta}(u,v) = \psi_{\theta}(-\log K_\delta(e^{-\psi_{\theta}(u)}, e^{-\psi_{\theta}(v)})), \tag{3.3}
\]

where \(K_\delta\) is an Archimedean copula, parametrised by \(\delta\), as defined in Equation (3.1), and \(\psi_{\theta}\) is a Laplace transform. Let \(\phi_{\delta}\) be the generator of \(K_\delta\). The resulting copula is then also an Archimedean copula with generator \(\eta_{\theta,\delta}(s) = \psi_{\theta}(-\log \phi_{\delta}(s))\). For further details see \cite{Joe1997}.

For certain families, if we fix the \(\delta\) parameter, the copula is then increasing in SM order, with respect to \(\theta\). Therefore if the copula generator \(\eta_{\theta,\delta}(s)\), is completely monotone and in addition the boundary condition \(\eta_{\theta,\delta}(0) = 1\) holds, then we are back to the settings of condition \(c\) in Theorem 3.1. As a result we have the following corollary:

**Corollary 3.5.** Let \(C_{\theta,\delta}(u,v)\) be a two parameters bivariate copula of the above form specified in 3.3. Let \(\eta_{\theta,\delta}(s) = \psi_{\theta}(-\log \phi_{\delta}(s))\) be its associated generator. Then, if \(\eta_{\theta,\delta}(s)\) is completely monotone such that \(\eta_{\theta,\delta}(0) = 1\) and \(C_{\theta,\delta}(u,v)\) is increasing in SM order with respect to \(\theta\) for a fixed \(\delta\), then the copula entropy is increasing in \(\theta\).

The proof is immediate given our previous results and we omit the details. Examples of bivariate two-parameters families for which the conditions given in Corollary 3.5 hold are the BB1, BB2 and BB6 families \cite{Joe1997}. See Section 5 for details.

### 4 Multivariate Copulas

In this section we generalize some of our results for higher dimensions. Before presenting the main results of this section, let us introduce the following class of positive real univariate functions:

\[
L^*_\infty = \{ \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ | \phi(0) = 0, \phi(\infty) = \infty, (-1)^{j-1}\phi^{(j)}(0) \geq 0, j \geq 1 \}.
\]

As we shall see, this class plays a central role in our result regarding multivariate Archimedean copulas. We are now ready to state our main result:
Theorem 4.1.: Let $C_\theta(u_1, \ldots, u_n)$ be an absolutely continuous bivariate copula, and let $X, X'$ be two random $n$-dimensional vectors such that $X \sim C_{\theta_1}(u_1, \ldots, u_n), X' \sim C_{\theta_2}(u_1, \ldots, u_n)$, where $\theta_2 \geq \theta_1 \geq 0$. ($\theta_2 \leq \theta_1 \leq 0$). Then $-H(X) \leq -H(X')$ if one of the following condition holds:

(a) $C_\theta(u_1, \ldots, u_n)$ is increasing (decreasing) in $<_S M$ and the copula density $c_\theta(u_1, \ldots, u_n)$ is TP2 (RR2) (for all $\theta$).

(b) $C_\theta(u_1, \ldots, u_n)$ is an Archimedean copula whose generator, $\phi_\theta$, is completely monotone and satisfies the boundary condition $\phi_\theta(0) = 1$. In addition for all $\theta_1 \leq \theta_2$, $\phi_\theta^{-1}_1 \phi_\theta_2 \in L^\infty$.

It can be easily shown that condition [a] in Theorem 4.1 is sufficient in any dimension. We omit the proof since it is essentially identical to the bivariate case. Generalizing condition $[b]$ however, requires more work which we present in this section.

4.1 Proof for (b): Multivariate Archimedean Copulas

Recall that our proof for [b] in the bivariate case relies on the fact that a completely monotone generator that satisfies the boundary condition implies a mixture representation, which in turn implies a TP2 bivariate density. We now show that both properties also hold in the general case of Archimedean copulas with completely monotone generators for which the boundary condition holds. To complete the proof, we will then also need to show that the SM ordering holds under the composition condition $\phi_\theta^{-1}_1 \phi_\theta_2 \in L^\infty$ for $\theta_1 \leq \theta_2$.

The following lemma, proved in [Joe, 1997, p.85-89] substantiates the first property:

Lemma 4.2.: Let $C_\theta(u_1, \ldots, u_n)$ be an Archimedean copula and let $\psi_\theta$ be its generator such that $\psi_\theta$ is completely monotone and $\psi_\theta(0) = 1$. Then there exists a CDF $M(\alpha)$ of a positive r.v. and a unique CDF-s $G_1(u_1), \ldots, G_n(u_n)$ such that: $C_\theta(u_1, \ldots, u_n) = \int_0^\infty G_1(u_1)^\alpha \ldots G_n(u_n)^\alpha dM(\alpha)$

The following lemma shows that the second property also holds:

Lemma 4.3.: Let $C_\theta(u_1, \ldots, u_n)$ be a copula that can be represented as in Lemma 4.2. Then $C_\theta(u_1, \ldots, u_n)$ has a TP2 density.

Proof: We assume that $C_\theta(u_1, \ldots, u_n) = \int_0^\infty G_1(u_1)^\alpha \ldots G_n(u_n)^\alpha dM(\alpha)$. Taking the derivative with respect to each argument we then have that the copula density is:

$$c_\theta(u_1, \ldots, u_n) = \alpha^n \prod_i g_i(u_i) \int_0^\infty G_1(u_1)^\alpha \ldots G_n(u_n)^\alpha dM(\alpha).$$

Using $\phi(u_1, \ldots, u_{n-1}, \alpha) = \prod_{i=1}^{n-1} G_i(u_i)^\alpha$, we can write

$$c_\theta(u_1, \ldots, u_n) = \alpha^n \prod_i g_i(u_i) \int_0^\infty \phi(u_1, \ldots, u_{n-1}, \alpha) \cdot G_n(u_n)^\alpha dM(\alpha).$$
Now, for each $i$, $G_i(u_i)^{-1}$ is TP2 in $(u_i, \alpha)$. Using Observation 2.5 we then have that $\phi(u_1, \ldots, u_{n-1}, \alpha)$ is also TP2. From this and [Karlin and Rinott, 1980] we have that $\int_0^\infty \phi(u_1, \ldots, u_{n-1}, \alpha) \cdot G_n(u_n)^{-1} dM(\alpha)$ is TP2 (with respect to $u_1, \ldots, u_n$). As $\prod_i g_i(u_i)$ is trivially TP2 [Joe, 1997], using Observation 2.5 again we have that the resulting density is TP2.

We have shown that, as in the bivariate case, a completely monotone generator that satisfies the boundary condition implies a TP2 copula density. We now characterize the conditions that also ensure $SM$ ordering. Concretely $L^*_\infty$ provides us with the needed condition via the following theorem [Wei and Hu, 2002]:

**Theorem 4.4.** Let $C_1, C_2$ be two $n$-dimensional Archimedean copulas and let $\phi_1, \phi_2$ be their associated generators, respectively. If $\phi_1, \phi_2$ are two Laplace transforms (equivalently $\phi_1, \phi_2$ are completely monotone and satisfy the boundary condition $\phi_i(0) = 1$, $i = 1, 2$, [Joe, 1997]), such that $\phi_1 \phi_2^{-1} \in L^*_\infty$, then $C_1 \leq SM C_2$.

Putting this and Lemma 4.3 together we get the desired result.

The conditions of our theorem for multivariate copulas may seem somewhat obscure and therefore not of practical interest. However, we note that they actually apply to some of the most popular multivariate Archimedean copulas, namely the Clayton, Gumbel, Frank and Joe copulas. See Section 5 for details.

5 Examples

Below we demonstrate the wide applicability of our theory. We begin with condition a of Theorem 3.1 that provides the broadest coverage and then also provide examples for the others cases.

5.1 Examples Satisfying Condition a of Theorem 3.1

In all the following examples, the copulas under consideration are known to be increasing in the PQD order and have a TP2/RR2 density function [Joe, 1997]. Therefore the entropy for these families is monotonic in $\theta$ by Theorem 3.1:

- **Bivariate Normal:**
  $\theta \geq 0$, $C_{\theta}(u, v) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v))$,
  where $\Phi$ is the $N(0, 1)$ cdf, and $\Phi_\theta$ is the BVSN cdf with correlation $\theta$.

- **Bivariate Farlie-Gumbel-Morgenstern (FGM):**
  $\theta \leq 0$, $C_{\theta}(u, v) = uv + \theta uv(1-u)(1-v)$.
  Note that for this family the density is TP2 when $\theta \geq 0$ and RR2 when $\theta \leq 0$ [Joe, 1997]. Thus its negative entropy is monotonic increasing in $\theta$, when $\theta \in (0, 1]$ and monotonic decreasing when $\theta \in [-1, 0]$ [Joe].
Therefore overall the FGM copula negative entropy is monotonic increasing in $|\theta|$.

- **Bivariate Frank:**
  \[
  0 \leq \theta \leq \infty, \quad C_\theta(u,v) = -\theta^{-1}\log\left(\frac{\tau - (1 - e^{-\theta u})(1 - e^{-\theta v})}{\tau}\right),
  \]
  where $\tau = 1 - e^{-\theta}$.

- **Bivariate Gumbel:**
  \[
  1 \leq \theta \leq \infty, \quad C_\theta(u,v) = e^{-[(\hat{u})^\theta + (\hat{v})^\theta]^{1/\theta}},
  \]
  where $\hat{u} = -\log(u)$, $\hat{v} = -\log(v)$.

- **Bivariate Clayton:**
  \[
  0 \leq \theta \leq \infty, \quad C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{(-1/\theta)}.
  \]

### 5.2 Archimedean Copula Examples Satisfying condition b in Theorem 3.1

In the following we demonstrate condition b for several Archimedean families that are not covered by the previous section such as the Clayton/Frank/Gumbel copulas. These examples are slightly lesser known but still useful Archimedean copula families. In all the following examples, the copulas under consideration are known to be increasing in the PQD order [Joe, 1997]. In addition it can be easily verified that the boundary condition $\psi_\theta(0) = 1$ holds.

In order to establish the completely monotonicity (and hence also the total positivity of the corresponding density function) of the copula generator, we use the following sufficient conditions [Nelsen, 2007, Widder, 1942]:

**Lemma 5.1.** Let $f(x), g(x)$ be two real univariate functions and let $h_1(x) = f(x) \circ g(x)$, $h_2(x) = f(x)g(x)$. Then:

(i) If $g$ is completely monotonic and $f$ is absolutely monotonic, i.e., $\frac{\partial^k f(x)}{\partial x^k} \geq 0$ for $k = 0, 1, 2, \ldots$ then $h_1(x)$ is completely monotone.

(ii) If $f$ is completely monotonic and $g$ is a positive function with a completely monotone derivative, then $h_1(x)$ is completely monotone.

(iii) If $f$ and $g$ are completely monotone, then so is $h_2(x)$.

- **Bivariate Ali-Mikhail (AMH):**
  \[
  \theta \in [-1, 1], \quad C_\theta(u, v) = \frac{uv}{1 - \theta (1 - u)(1 - v)},
  \]
  The generator of this copula is given by:
  \[
  \psi_\theta(t) = \frac{1 - \theta}{\exp(t) - \theta}.
  \]
This generator is completely monotone for $\theta \in (0, 1]$ \cite{Jaworski et al. 2010}. When $\theta \in [-1, 0)$, it can easily been shown that the corresponding densities are RR2, using Lemma 2.5. Therefore overall the AMH copula negative entropy is monotonic increasing in $|\theta|$.

- **Bivariate Joe:**

  $\theta \in [1, \infty)$, \quad $C_\theta(u, v) = 1 - [(1 - u)^{\theta} + (1 - v)^{\theta} - (1 - u)^{\theta}(1 - v)^{\theta}]^{1/\theta}$.

  The generator of this copula is given by:

  $$\psi_\theta(t) = 1 - (1 - \exp(-t))^{1/\theta}.$$ 

  Taking $f(t) = 1 - t^{1/\theta}$ and $g(t) = 1 - \exp(-t)$, we see that $f$ is completely monotone and $g$ is a positive function whose first derivative completely monotone. Thus their composite is also completely monotone.

- **Family 4.14** \cite{Nelsen 2007}:

  $\theta \in [1, \infty)$, \quad $C_\theta(u, v) = \left(1 + \left(u^{-1/\theta} - 1\right)^{\theta} + \left(v^{-1/\theta} - 1\right)^{\theta}\right)^{-\theta}$.

  The generator of this copula is given by:

  $$\psi_\theta(t) = (t^{1/\theta} + 1)^{-\theta}.$$ 

  By taking $f = (1 + t)^{-\theta}$, $g = t^{1/\theta}$ and repeating the same arguments as in previous examples, we get that this generator is completely monotone.

- **Family 4.19** \cite{Nelsen 2007}:

  $\theta \in (0, \infty)$, \quad $C_\theta(u, v) = \frac{\theta}{\ln(e^{\theta/u} + e^{\theta/v} - e^{\theta})}$.

  The generator of this copula is given by:

  $$\psi_\theta(t) = \frac{\theta}{\ln(t + \exp(\theta))}.$$ 

  By taking $f = \theta/t$, $g = \ln(t + \exp(\theta))$ and repeating the same arguments as in previous examples, we get that this generator is completely monotone.

### 5.3 Examples of Bivariate Two-parameter Families

We now provide some examples of two-parameter copula families that satisfy condition 4 of Theorem 3.1. That is, they are all positively PQD/SM ordered with respect to $\theta$ and have a completely monotone generator that satisfies the boundary condition \cite{Joe 1997}. Therefore by Corollary 3.5 for fixed $\delta$, their entropy is monotonic in $\theta$.
Recall that bivariate two-parameter families have the following form:

\[ C_{\theta,\delta}(u,v) = \psi(-\log K_\delta(e^{-\psi_\theta(u)}, e^{-\psi_\theta(v)})), \]  

(5.1)

where \( K_\delta \) is an Archimedean copula, parametrised by \( \delta \), as defined in Equation (3.1), and \( \psi_\theta \) is a Laplace transform. Let \( \phi_\delta \) be the generator of \( K_\delta \). The resulting copula is then also an Archimedean copula with generator \( \eta_{\theta,\delta}(s) = \psi_\theta(-\log \phi_\delta(s)) \). For further details see [Joe, 1997].

- **Family BB1 [Joe, 1997]:** Taking \( K \) to be Gumbel copula and \( \psi_\theta(s) = (1 + s)^{-1/\theta}, \theta \geq 0 \) the resulting copula is:

\[ C_{\theta,\delta}(u,v) = \left(1 + ((u^{-\theta} - 1)^\delta + (v^{-\theta} - 1)^\delta)^{1/\delta}\right)^{-1/\theta}, \]

where \( \eta_{\theta,\delta}(s) = (1 + s^{1/\delta})^{-1/\theta}. \)

- **[Joe, 1997], bivariate BB2:** Taking \( K \) to be Clayton copula and \( \psi_\theta(s) = (1 + s)^{-1/\theta}, \theta \geq 0 \) the resulting copula is:

\[ C_{\theta,\delta}(u,v) = \left(1 + \delta^{-1}\log\left(e^{\delta(u^{-\theta} - 1)} + e^{\delta(v^{-\theta} - 1)}\right) - 1\right)^{-1/\theta}, \]

where \( \eta_{\theta,\delta}(s) = [1 + \delta^{-1}\log(1 + s)]^{-1/\theta}. \)

- **[Joe, 1997], bivariate BB6:** Taking \( K \) to be Gumbel copula and \( \psi_\theta(s) = 1 - (1 - e^{-s})^{1/\theta}, \theta \geq 1 \) the resulting copula is:

\[ C_{\theta,\delta}(u,v) = 1 - \left(1 - \exp\left(-((\log(1 - \hat{u}^\theta))^\delta + (\log(1 - \hat{v}^\theta))^\delta)^{1/\delta}\right)\right)^\delta, \]

where \( \hat{u} = 1 - u, \hat{v} = 1 - v, \eta_{\theta,\delta}(s) = 1 - [1 - \exp(-s^{1/\delta})]^{1/\theta}. \)

### 5.4 Examples of Multivariate Copulas

We finish with some multivariate examples that satisfy the conditions of Theorem [4.1]. That is, in each example the copula generator, \( \phi_\theta \), is completely monotone and in addition \( \phi_\theta^{-1} \phi_{\theta_2} \in L^*_\infty \) for \( \theta_1 \leq \theta_2 \) [Joe, 1997].

- **Multivariate Clayton:**

\[ \theta_1 \leq \theta_2 \in (0, \infty), \quad \phi_{\theta_1}^{-1} \phi_{\theta_2} = \frac{1}{\theta_1}(\theta_2 + 1)^{\theta_1/\theta_2} - \frac{1}{\theta_1}. \]

- **Multivariate Gumbel:**

\[ \theta_1 \leq \theta_2 \in (1, \infty), \quad \phi_{\theta_1}^{-1} \phi_{\theta_2} = \theta_1^{\theta_1/\theta_2}. \]
• Multivariate Frank:

\[ \theta_1 \leq \theta_2 \in (0, \infty), \quad \phi_{\theta_1}^{-1} \phi_{\theta_2} = -\ln \left( \frac{(e^{-t}(e^{-\theta_2} - 1) + 1)^{\theta_1/\theta_2} - 1}{e^{-\theta_1} - 1} \right). \]

• Multivariate Joe :

\[ \theta_1 \leq \theta_2 \in (0, 1), \quad \phi_{\theta_1}^{-1} \phi_{\theta_2} = -\ln \left( 1 - \left(1 - \exp(-t)\right)^{\theta_1/\theta_2} \right). \]

6 Simulations

The theory presented in this paper suggests that, in the case of an infinite number of samples generated from a given copula family, the entropy curve is monotonic in the copula dependence parameter. In practice, however, we almost always have access only to a finite number of samples. An obvious empirical question is thus whether monotonicity approximately holds given a reasonable number of samples, and what is the impact of the sample size on the entropy vs. dependence parameter curve.

To answer these questions we explore the finite-sample behaviour of the copula entropy monotonicity via a simulation study. For each of the different theoretical scenarios discussed in the previous section, we choose representative popular copula families. We then generate \( M = 1000 \) samples from the copula for different values of the dependence parameter. For each of these samples, we compute the entropy using the standard empirical estimator

\[ \hat{H}(c_\theta) = -\sum_{u \in \mathbb{D}} \log c_\theta(u), \]

where \( \mathbb{D} \) denotes the set of \( M \) samples generated from a copula \( C_\theta \).

For each value of the dependence parameter, we repeat the above 50 times, and report the mean empirical entropy along with a 95% confidence interval vs. the value of the dependence parameter. Figure 1, first two rows, show the results for single-parameter bivariate copula families. It is clear that with as little as 1000 samples, near-monotonicity consistently holds for all families evaluated including both elliptical ones (Gaussian, student-T) and Archmedian copula families (Clayton, Frank, Gumbel, Joe, AMH).

Next we turn to bivariate two parameters families. In this case we also need to test the monotonicity along the \( \delta \) axis, since as formalized in Corollary 3.5 monontonicity holds in \( \theta \) for each fixed value of \( \delta \). Results are shown in Figure 1, third row, for the BB1 and BB6 copula families, for two different values of \( \delta \). Results for other values of \( \delta \) as well as for the BB2 family were qualitatively similar. As before, near monotonicity is evident.

To evaluate the finite-sample monotonicity in the multivariate case, we repeat the same evaluation for several copula families of dimension 5. Results are shown in the last row of Figure 1, and are qualitatively similar for higher dimensions. As before, the empirical monotonicity is quiet impressive.
Figure 1: Simulation study of the monotonicity of the empirical entropy in the copula dependence parameter. Shown is the mean (red line) and 95% range (black lines) of the empirical entropy over 50 random computations using from $M = 1000$ samples for each value of the dependence parameter (x-axis). We consider several popular bivariate ($d=2$) single parameter copula families (first and second rows), bivariate two parameters families (third row), as well as multivariate ($d=5$) copula families (fourth row).
Finally, we evaluate the impact of the sample size on the extent to which the empirical entropy is monotonic in the copula dependence parameter. We repeat the same evaluation procedure for a wide range of sample sizes (from 500 to 10000). To measure empirical monotonicity, for a given sample size, we measure the fraction of consecutive parameter values for which monotonicity holds. That is, we measure the empirical rate at which the following inequality holds:

\[
\theta_1 \leq \theta_2 \Rightarrow \frac{\sum_{u \in D} \log c_{\theta_1}(u)}{M} \leq \frac{\sum_{u \in D} \log c_{\theta_2}(u)}{M},
\]

for two consecutive values of \(\theta_1\) and \(\theta_2\). We repeat this process 50 times and report the average. Results for the Clayton family are shown in Figure 2. As expected, with a greater sample size, monotonicity holds more frequently nearing 100% at just 5000 training instances. Appealingly, even at much smaller sample sizes, monotonicity is quite appealing.

7 Conclusion

In this work we establish a novel theoretical relationship between the main general purpose framework for capturing non-Gaussian real-valued distributions, namely copulas, and the ubiquitous Shannon’s mutual information measure that is used throughout the exact science to quantify the dependence between random variables.

Our main result is that the mutual information between two variables (equivalently the copula entropy) is monotonic in the copula dependence parameter for a (very) wide range of copula families. Concretely, we provide fairly general sufficient conditions for this monotonicity that cover the vast majority of com-
monly used bivariate copulas, as well as a wide class of multivariate Archimedean copulas.

As we demonstrated in our earlier work where the theory was substantially less developed [Tenzer and Elidan, 2013], the monotonicity result also has practical merits. Specifically, it allows us to rank the entropy (equivalently the expected log-likelihood) of copula-based probabilistic models by simple computation of association measures such as Spearman’s rho. This gives rise to highly efficient model selection in scenarios involving many interactions, e.g. when learning copula graphical models. The practical merit depends, of course, on the finite sample behavior of the entropy. Fortunately, as our simulations clearly show, near perfect monotonicity holds even with modest sample sizes.

Essentially all copula families are constructed so as to span some or all of the range between the independence and full dependence copula via the dependence parameter. We have not been able to identify a single family where the above monotonicity does not hold empirically. However, monotonicity does appear to hold for the Plackett copula family [Nelsen, 2007] and identifying further sufficient conditions remains a future challenge. Another direction if interesting is developing finite sample theory that will explicitly quantify the amount by which the empirical entropy can deviate from the expected monotonic behavior.

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