A Novel Geometric Modeling and Calculation Method for Forward Displacement Analysis of 6-3 Stewart Platforms

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Abstract: A novel geometric modeling and calculation method for forward displacement analysis of the 6-3 Stewart platforms is proposed by using the conformal geometric algebra (CGA) framework. Firstly, two formulas between 2-blade and 1-blade are formulated. Secondly, the expressions for two spherical joints of the moving platform are given via CGA operation. Thirdly, a coordinate-invariant geometric constraint equation is deduced. Fourthly, a 16-degree univariate polynomial equation without algebraic elimination by using the Euler angle substitution is presented. Fifthly, the coordinates of three spherical joints on the moving platform are calculated without judging the radical symbols. Finally, two numerical examples are used to verify the method. The highlight of this paper is that a new geometric modeling and calculation method without algebraic elimination is obtained by using the determinant form of the CGA inner product algorithm, which provides a new idea to solve a more complex spatial parallel mechanism in the future.

Keywords: 6-3 Stewart platform; conformal geometric algebra; forward displacement analysis

1. Introduction

The parallel mechanism (PM) [1–4] is composed of a moving platform and a fixed platform, which is connected by at least two independent kinematic chains and has two or more degrees of freedom (DOF). Compared with the traditional series mechanism, the spatial parallel mechanism has the advantages of higher rigidity, precision, and strong bearing capacity, etc. At present, the most widely studied parallel mechanism is the 6-DOF parallel mechanisms first proposed by Gough [1] and widely used by Stewart [2]. The classification of all self-motions of the original Stewart–Gough platform was achieved by Luces and Benhabib [3]. The redundant parallel kinematic mecha-nisms are introduced by Karger and Husty [4]. Traditionally, Eduard Study [5] put forward a novel method to define a rigid body displacement in 3-dimension space, and mapped each position of a rigid body onto a point on a quadric, now called the Study quadric. In this paper, the motion of the rigid body (here, the moving platform) is described by the position coordinates of three spherical joints on the moving platform. The forward displacement analysis of the parallel mechanism is to calculate the position and attitude of the moving platform relative to the static platform via the driving rod length of the parallel mechanism. This problem is the theoretical basis of the velocity, acceleration, error analysis, workspace analysis, singular position analysis, dynamic analysis, and other problems of the mechanism. For the elimination of equations, the Newton–Raphson method [6] and polynomial constraints solver [7] are commonly used.

In this discourse, we will revisit the forward displacement analysis of the 6-3 Stewart platforms (6-3SPS mechanisms), which has been researched in [8–14]. Griffis et al. [8] managed to present a closed-form solution to the platform mechanisms of the 6-3SPS. Nanua et al. [9] published the 16th-order input-output equation. Innocenti et al. [10] revealed the solution of a system of three second-order nonlinear equations. Merlet et al. [11] put forward a method for the forward displacement analysis of 6-3 Stewart platforms by
finding a polynomial in one variable whose roots determine the posture of the mobile plate. Zhang et al. [12] and Wei et al. [13] committed to an investigation into forward displacement analysis of the 6-3 Stewart platforms using the resultant method and the CGA theory. Zhang et al. [14] settled the forward displacement analysis for the 6-3 Stewart platforms only using the CGA method.

Conformal geometric algebra (CGA) [14–27] is a new geometric representation and calculation framework, which can uniformly represent point, line, plane, circle, and other geometric elements. CGA is committed to present direct algebraic operations on these geometric entities. These operations usually produce formulas that are independent of the coordinate system and make complex symbolic geometry computation possible. The above features are two important intrinsic attributes of CGA. Therefore, it is useful and suitable for the geometric modeling and calculation of mechanism kinematics. In recent decades, CGA had been used to deal with the inverse kinematics problem of the serial mechanism through CGA operation of geometric entities [19–21]. Additionally, Song et al. [22] employed the CGA theory in topology synthesis of mechanisms. Furthermore, Hrdina et al. [23–25] committed to an investigation into robotic modeling and control by CGA theory. Zhang et al. [14,26] and Huang et al. [27] performed exploration on addressing the forward displacement analysis of the serial mechanism in virtue of CGA theory.

In this paper, a new geometric modeling and calculation method for closed-form solution of the 6-3 Stewart platforms is proposed by using the CGA framework which contributes to a 16-degree univariate input-output polynomial equation without algebraic elimination and superfluous roots. Although the forward displacement problems of the 6-3 Stewart platforms have been addressed by Zhang et al. [12,14] and Wei et al. [13] using CGA, the proposed method based on the determinant operation formula (Equation (8)) of the CGA inner product provides a new idea to solve more a complex spatial parallel mechanism in the future, which is not available in Refs. [12–14]. In addition, compared with [14] in the back substitution procedure, the proposed method does not need to determine the radical symbols. Moreover, two formulas between 2-blade and 1-blade are first formulated.

The remaining of the paper is formulated as follows. In Section 2, the foundations of CGA will be presented. In Section 3, the derivation of two formulas between 2-blade and 1-blade under the CGA framework is elaborated. In Section 4, the geometric modeling and calculation process of the 6-3 Stewart platforms will be revealed. In Section 5, two numerical examples of 6-3 Stewart platforms will be given. Finally, the conclusions will be built in Section 6.

2. Conformal Geometric Algebra

2.1. Foundations of CGA

The 5-dimension (5D) CGA $\mathbb{C}^{4,1}$ consists of 3D Euclidean space $\mathbb{C}^3$ and a 2D Minkowski vector space $\mathbb{C}^{1,1}$. CGA has five orthonormal basis vectors introduced by $\{e_1, e_2, e_3, e_+, e_−\}$ with the following properties:

\[
\begin{align*}
\epsilon_1^2 &= \epsilon_2^2 = \epsilon_3^2 = \epsilon_+^2 = \epsilon_-^2 = 1 \\
\epsilon_i \cdot \epsilon_j &= 0 (i \neq j, i, j = 1, 2, 3, +, −) \\
\epsilon_i \wedge \epsilon_j &= -\epsilon_j \wedge \epsilon_i (i, j = 1, 2, 3, +, −)
\end{align*}
\]

where $\{e_1, e_2, e_3\}$ are three orthonormal basis vectors in the Euclidean space and $\{e_+, e_-\}$ are two orthogonal basis vectors in the Minkowski space.

In addition, two null bases can be presented by the vectors:

\[
e_0 = \frac{1}{2}(e_- - e_+) \quad e_\infty = e_+ + e_-
\]

with the properties of:

\[
\epsilon_0^2 = \epsilon_\infty^2 = 0, e_\infty \cdot e_0 = -1
\]
where $e_0$ is the conformal origin and $e_\infty$ is the conformal infinity.

Blade is the basic calculation element and basic geometric entity of geometric algebra. The 5D CGA consists of 0, 1, 2, 3, 4, and 5 blades. The blade with the largest grade, namely 5-blades, are called pseudo-scalars and defined by $I_C(e_\infty0123, I_C^2 = -1)$. A linear combination of blades with different grades is called a multi-vector.

The dual $X^*$ of a multi-vector $X$ is denoted by:

$$X^* = X I_C^{-1} = -XI_C$$

where $I_C^{-1}$ is the inverse of $I_C$.

Conformal geometric algebra has three algebraic operators: the inner product $(u \cdot v)$, the outer product $(u \wedge v)$, and the geometric product $(uv = u \cdot v + u \wedge v)$. Inner product is used to solve geometric scalars. The outer product is applied to build basic geometry or the geometry intersection. The inner and outer products of two vectors $u$ and $v$ can also be built as:

$$\begin{align*}
\{ \quad u \cdot v &= \frac{1}{2}(uv + vu) \\
&= \frac{1}{2}(uv - vu) \quad , \\
&= \frac{1}{2}(uv) \\
\}
\end{align*}$$

and:

$$\begin{align*}
\{ \quad u \wedge v &= v \cdot u \\
&= -v \cdot u \\
\}
\end{align*}$$

The inner product of between an r-blade $u_1 \wedge \cdots \wedge u_r$ and an s-blade $v_1 \wedge \cdots \wedge v_s$ can be defined by:

$$\begin{align*}
\{ \quad (u_1 \wedge \cdots \wedge u_r) \cdot (v_1 \wedge \cdots \wedge v_s) &= \left\{ \begin{array}{ll}
(u_1 \wedge \cdots \wedge u_r) \cdot (v_1 \wedge \cdots \wedge v_s) & \text{if } r \geq s \\
(u_1 \wedge \cdots \wedge u_{r-1}) \cdot (u_r \cdot (v_1 \wedge \cdots \wedge v_s)) & \text{if } r < s
\end{array} \right. \\
\}
\end{align*}$$

with:

$$
(u_1 \wedge \cdots \wedge u_r) \cdot v_1 = \sum_{i=1}^{r} (-1)^{r-i} u_1 \wedge \cdots \wedge u_{r-i} \wedge (u_{r-i} \cdot v_1) \wedge u_{r+1} \wedge \cdots \wedge u_r,
$$

$$u_r \cdot (v_1 \wedge \cdots \wedge v_s) = \sum_{i=1}^{s} (-1)^{s-i} v_1 \wedge \cdots \wedge v_{s-i} \wedge (u_r \cdot v_i) \wedge v_{i+1} \wedge \cdots \wedge v_s.
$$

As a supplement, if $r$ is equal to $s$, the inner product of between an r-blade $u_1 \wedge \cdots \wedge u_r$ and an s-blade $v_1 \wedge \cdots \wedge v_s$ can also be obtained by:

$$\begin{align*}
(u_1 \wedge \cdots \wedge u_r) \cdot (v_1 \wedge \cdots \wedge v_s) &= \begin{vmatrix}
& u_r \cdot v_1 & \cdots & u_r \cdot v_s \\
& \vdots & \ddots & \vdots \\
& u_1 \cdot v_1 & \cdots & u_1 \cdot v_s
\end{vmatrix}, \\
\end{align*}$$

For two blades $A_{[r]}$ and $B_{[s]}$ with non-zero grades $r$ and $s$, the inner and outer products can be denoted as:

$$\begin{align*}
\{ \quad A_{[r]} \cdot B_{[s]} &= \langle AB \rangle_{r-s} \\
A_{[r]} \wedge B_{[s]} &= \langle AB \rangle_{r+s}
\}
\end{align*}$$

2.2. Conformal Geometric Entities

CGA has a visual representation of basic geometric entities. The representations of geometric entities relative to inner product null space (IPNS) and outer product null space (OPNS) are listed in Table 1. IPNS refers to the geometric entity achieved by the intersection of geometric entities, and OPNS refers to the geometric entity described by the points belonging to the geometric entity. The two representations can be transformed by dual operators. Detailed information is given by references [15–18]. In Table 1, small italics and
bold characters represent points or vectors in Euclidean space. The character ρ is the radius of sphere S. The character n is a normal vector of plane Π. The symbol d represents the distance from the origin of the coordinate system to the plane Π.

### Table 1. The geometric element of CGA.

| Entity       | IPNS                  | Grade | OPNS                  | Grade |
|--------------|-----------------------|-------|-----------------------|-------|
| Point        | P = p + \(1/2p^2e_\infty + e_0\) | 1     | P* = S_1 \(\wedge\) S_2 \(\wedge\) S_3 \(\wedge\) S_4 | 4     |
| Sphere       | S = p + \((p^2 - \rho^2)e_\infty + e_0\) | 1     | S* = P_1 \(\wedge\) P_2 \(\wedge\) P_3 \(\wedge\) P_4 | 4     |
| Plane        | Π = n + de_\infty   | 2     | Π* = e_\infty \(\wedge\) P_1 \(\wedge\) P_2 \(\wedge\) P_3 | 4     |
| Line         | L = Π_1 \(\wedge\) Π_2 | 2     | L* = e_\infty \(\wedge\) P_1 \(\wedge\) P_2 | 3     |
| Circle       | C = S_1 \(\wedge\) S_2 | 2     | C* = P_1 \(\wedge\) P_2 \(\wedge\) P_3 | 3     |
| Point pair   | P_p = S_1 \(\wedge\) S_2 \(\wedge\) S_3 | 3     | P_p* = P_1 \(\wedge\) P_2 | 2     |

The inner product between two points \(P_1\) and \(P_2\) is summarized as:

\[
P_1 \cdot P_2 = \left( p_1 + \frac{1}{2}p_1^2e_\infty + e_0 \right) \cdot \left( p_2 + \frac{1}{2}p_2^2e_\infty + e_0 \right) = -\frac{1}{2} (p_1 - p_2)^2.
\]

### 3. Two Formulas under the Framework of CGA

In this section, the derivation of two formulas between 2-blade and 1-blade under the CGA framework is elaborated. In order to facilitate the derivation of the formula, the following formula is first given using Equation (7) under the framework of CGA, i.e.,

\[
P_p^* Q_q^* = (P_1 \wedge P_2) \cdot (Q_1 \wedge Q_2) = (P_2 \cdot Q_1)(P_1 \cdot Q_2) - (P_1 \cdot Q_1)(P_2 \cdot Q_2),
\]

\[
\left( Q_q^* X_1 \right) \wedge \left( Q_q^* X_2 \right) + \left( Q_q^* \left( e_\infty \wedge X_1 \right) \right) Q_q^* = 0,
\]

\[
Q_q^* (X_1 \wedge X_2) = \left( Q_q^* X_1 \right) X_2,
\]

\[
Q_q^* (e_\infty \wedge X_1) = Q_e \cdot X_i (i = 1, 2),
\]

\[
Q_q^* \left( Q_e \wedge X_2 \right) = \left( Q_q^* Q_e \right) X_2,
\]

\[
\left( Q_q^* X_1 \right) Q_e = -\left( Q_q^* Q_e \right) \cdot X_i (i = 1, 2),
\]

where \(P_p\) and \(Q_q\) are both 2-blades, \(P_p = P_1 \wedge P_2\), \(Q_q = Q_1 \wedge Q_2\) and \(Q_e = Q_q^* e_\infty\). \(P_1\), \(P_2\), \(Q_1\), \(Q_2\), \(Q_e\), \(X_1\) and \(X_2\) are all 1-blades. The detailed derived procedure of Equation (12) is given in the Appendix A.

#### 3.1. Derivation of the First Formula

According to Equation (12), we can obtain:

\[
\left( Q_q^* e_\infty \right) \wedge \left( Q_q^* X_1 \right) + \left( Q_q^* \left( e_\infty \wedge X_1 \right) \right) Q_q^* = 0.
\]

By multiplying both sides of Equation (17) with \(Q_q^* e_\infty\) \(\wedge\) \(X_2\), and simplifying it, we obtain:

\[
\left( Q_e \cdot X_2 \right) \left( Q_e \cdot \left( Q_q^* X_1 \right) \right) - \left( Q_e \cdot Q_e \right) \left( X_2 \cdot \left( Q_q^* X_1 \right) \right) + \left( Q_q^* \left( Q_e \wedge X_2 \right) \right) \cdot \left( Q_q^* \left( e_\infty \wedge X_1 \right) \right) = 0.
\]

Using Equations (13)–(16), we obtain:

\[
\left( Q_e \cdot X_1 \right) \cdot \left( Q_q^* \left( Q_e \cdot X_2 \right) \right) - \left( Q_e \cdot X_2 \right) \cdot \left( Q_q^* \left( Q_e \cdot X_1 \right) \right) = \left( Q_e \cdot Q_e \right) \left( Q_q^* \left( X_1 \wedge X_2 \right) \right).
\]
3.2. Derivation of the Second Formula

According to Equation (14), we can obtain:

\[
(Q_q^* Q_q^\ast) ((Q_{c1} X_1) \cdot (Q_{c2} X_2)) = 
\left( \left( Q_q^\ast (e_\infty \land X_1) \right) Q_q^j \right) \cdot \left( \left( Q_q^\ast (e_\infty \land X_2) \right) Q_q^j \right).
\]  

(20)

According to Equation (12), Equation (20) is simplified as:

\[
(Q_q^* Q_q^\ast) ((Q_{c1} X_1) \cdot (Q_{c2} X_2)) = 
\left( - \left( Q_q^\ast e_\infty \right) \land \left( Q_q^j X_1 \right) \right) \cdot \left( - \left( Q_q^\ast e_\infty \right) \land \left( Q_q^j X_2 \right) \right).
\]  

(21)

Expanding the right side of Equation (21) using Equation (11), we can obtain:

\[
(Q_q^* Q_q^\ast) ((Q_{c1} X_1) \cdot (Q_{c2} X_2)) = 
\left( Q_q^\ast X_1 \right) \cdot \left( Q_q^\ast e_\infty \right) \cdot \left( Q_q^j X_1 \right) \cdot \left( Q_q^\ast e_\infty \right) \cdot \left( Q_q^j X_2 \right).
\]  

(22)

Using Equation (15), Equation (22) is simplified as:

\[
(Q_q^* Q_q^\ast) ((Q_{c1} X_1) \cdot (Q_{c2} X_2)) = (Q_{c1} X_1) \cdot (Q_{c2} X_2).
\]  

(23)

Equations (19) and (23) are derived from Equation (12) and are frequently used in this paper.

4. CGA-Based Geometric Modeling and Calculation Procedure

In this section, the structure and coordinate system of the 6-3 Stewart platform is first presented. Then, the expressions for two point pairs \( B_{i2}^1 \) and \( B_{i3}^1 \) are given. Next, a coordinate-invariant geometric constraint equation is deduced. Finally, the coordinates of three spherical joints on the moving platform will be calculated. In this paper, three spherical joints on the moving platform are equivalent to three points.

4.1. The Structure and Coordinate System of the 6-3 Stewart Platforms

The 6-3 Stewart platforms structure is shown in Figure 1. The 6-3 Stewart platforms are connected by a moving platform \( \Delta B_1 B_2 B_3 \), a static platform \( \Delta A_1 A_2 A_3 \) and six SPS kinematic chains. The SPS kinematic chain is composed of revolute joint \( R \), prismatic joint \( P \), and spherical joint \( S \), where prismatic joint \( P \) is the actuated joint. The coordinate system \( O \) – \( X Y Z \) is the geodetic coordinate system. The coordinate systems \( O_1 \) – \( X_1 Y_1 Z_1 \) are attached to the static platform \( \Delta A_1 A_2 A_3 \). The origin of the coordinate system \( O \) – \( X Y Z \) is considered to coincide with the origin of the coordinate system \( O_1 \) – \( X_1 Y_1 Z_1 \). The geometric center coordinates of the spherical joints on the static platform and the spherical joints on moving platform are represented by \( a_i (i = 1, 2, \cdots , 6) \) and \( b_j (j = 1, 2, 3) \). The distance between the three spherical joints on the moving platform is denoted by \( r_j (j = 1, 2, 3) \). \( l_i (i = 1, 2, \cdots , 6) \) are the input of the six SPS kinematic chains and \( b_j (j = 1, 2, 3) \) are unknown.

The coordinate system \( O \) – \( X Y Z \) is rotated by \( \beta_1 \) about the \( Z_1 \)-axis, and the new \( X_1 \)-axis is rotated by \( a_1 \). The unit vector of the \( Z_1 \)-axis is defined as \( \left( z_{1x}, z_{1y}, z_{1z} \right)^T \). As a consequence, the coordinate system \( O \) – \( X Y Z \) coincides with the coordinate system \( O_1 \) – \( X_1 Y_1 Z_1 \), and the expressions of \( a_1 \) and \( \beta_1 \) are denoted as:

\[
\begin{cases}
\alpha_1 = \text{sgn}(-z_{1y}) \cos^{-1}(z_{1z}), \beta_1 = \tan^{-1}(-z_{1x}/z_{1y}), & z_{1y} \neq 0 \\
\alpha_1 = \text{sgn}(z_{1x}) \cos^{-1}(z_{1z}), \beta_1 = \pi/2, & z_{1y} = 0
\end{cases}
\]  

(24)
In addition, the matrix $M_r$ is represented as:

$$M_r = \begin{bmatrix}
c_{\beta_1} & -s_{\beta_1} & 0 \\
s_{\beta_1} & c_{\beta_1} & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{\alpha_1} & -s_{\alpha_1} \\
0 & s_{\alpha_1} & c_{\alpha_1}
\end{bmatrix}, \quad (25)$$

where $s_{\beta_1} = \sin(\beta_1), c_{\beta_1} = \cos(\beta_1), s_{\alpha_1} = \sin(\alpha_1),$ and $c_{\alpha_1} = \cos(\alpha_1)$. 

4.2. The Expressions for Two Point Pairs $B^*_2$ and $B^*_3$. 

According to Figure 1, point $B_2$ is situated on sphere $S_{A_3B_2}$, whose center is point $A_3$ and radius is $l_3$, and sphere $S_{A_4B_2}$ whose center is point $A_4$ and radius is $l_4$. In addition, point $B_2$ is also located on sphere $S_{B_1B_2}$ whose center is point $B_1$ and radius is $r_3$. Considering Table 1, the point pair $B^*_2$ can be denoted as:

$$B^*_2 = S_{B_1B_2} \wedge S_{A_3B_2} \wedge S_{A_4B_2}, \quad (26)$$

where $S_{B_1B_2}, S_{A_3B_2}$ and $S_{A_4B_2}$ can be formulated as:

$$S_{B_1B_2} = B_1 - \frac{1}{2}r_3^2 e_\infty, \quad S_{A_3B_2} = A_3 - \frac{1}{2}l_3^2 e_\infty, \quad S_{A_4B_2} = A_4 - \frac{1}{2}l_4^2 e_\infty$$

Using Equation (4), the dual of point pair $B^*_2$ can be represented as:

$$B'^*_2 = -(S_{B_1B_2} \wedge S_{A_3B_2} \wedge S_{A_4B_2}) I_C. \quad (27)$$

Homogeneously, point $B_3$ is located on sphere $S_{A_5B_3}$ whose center is point $A_5$ and radius is $l_5$, sphere $S_{A_6B_3}$ whose center is point $A_6$ and radius is $l_6$, and sphere $S_{B_1B_3}$ whose center is point $B_1$ and radius is $r_2$. Hence, the point pair $B^*_3$ can be formulated as:

$$B^*_3 = S_{B_1B_3} \wedge S_{A_5B_3} \wedge S_{A_6B_3}, \quad (28)$$

where $S_{B_1B_3}, S_{A_5B_3}$ and $S_{A_6B_3}$ can be expressed as:

$$S_{B_1B_3} = B_1 - \frac{1}{2}r_2^2 e_\infty, \quad S_{A_5B_3} = A_5 - \frac{1}{2}l_5^2 e_\infty, \quad S_{A_6B_3} = A_6 - \frac{1}{2}l_6^2 e_\infty$$
4.3. The Coordinate-Invariant Geometric Constraint Equation

According to Table 1 and Equation (10), we can obtain:
\[
B_3^*B_3^* = (S_{B_1B_1} \wedge S_{A_1B_1} \wedge S_{A_2B_1} \wedge S_{B_2B_3}) \cdot (S_{B_1B_3} \wedge S_{A_2B_3} \wedge S_{A_1B_3} \wedge S_{B_2B_3}) = 0
\]
\[
\Rightarrow (B_{i3} \wedge S_{B_2B_3}) \cdot (B_{i3} \wedge S_{B_2B_3}) = 0
\]  
(29)

According to the formula to dissect a point from the point pair \([14,27]\), we have:
\[
B_2 = \pm \sqrt{B_{i2}^*B_{i2}^* B_{2e}^{-1} + B_{i2}^*B_{2e}^{-1}},
\]
where \(B_{2e} = B_{i2}^* e_\infty\), and \(B_{2e}^{-1} = (B_{i2}^* e_\infty)^{-1} = \frac{B_{i2}^*}{B_{i2}^* - B_{2e}^*}\). B_{2e} and B_{2e}^{-1} are both 4-blade.

According to Table 1, sphere \(S_{B_2B_3}\) is represented as
\[
S_{B_2B_3} = B_2 - Re_\infty,
\]
where \(R = \frac{1}{2} r_1^2\).

By substituting Equation (30) into Equation (31), we obtain:
\[
S_{B_2B_3} = \pm \sqrt{B_{i2}^*B_{i2}^* B_{2e}^{-1} + B_{i2}^*B_{2e}^{-1} - Re_\infty}.
\]
(32)

By substituting Equation (32) into Equation (29) and simplifying it, we have:
\[
H_2 + H_1 + H_0 = 0,
\]
(33)

where:
\[
H_2 = (B_{i2}^*B_{i2}^*) \left( (B_{i3} \wedge B_{2}^{-1}) \cdot (B_{i3} \wedge B_{2}^{-1}) \right),
\]
\[
H_1 = \pm 2 \sqrt{B_{i2}^*B_{i2}^* (B_{i3} \wedge B_{2e}^{-1}) \cdot (B_{i3} \wedge (B_{i2}^* B_{2e}^{-1}) - R B_{3e})},
\]
\[
H_0 = (B_{i3} \wedge (B_{i2}^* B_{2e}^{-1} - R B_{3e}) \cdot (B_{i3} \wedge (B_{i2}^* B_{2e}^{-1} - R B_{3e}),
\]
\[
B_{3e} = B_{i3} \wedge e_\infty, B_{3e} is 4-blade.
\]

According to Equation (33), we derive:
\[
H_1^2 - (H_2 + H_0)^2 = 0,
\]
(34)

and expanding Equation (34), we have:
\[
C_4 R^4 + C_3 R^3 + C_2 R^2 + C_1 R + C_0 = 0,
\]
(35)

where:
\[
C_4 = -(B_{3e} \cdot B_{3e})^2,
\]
\[
C_3 = 4 (B_{3e} \cdot U_2)(B_{3e} \cdot B_{3e}),
\]
\[
C_2 = 4 (B_{i2e}^* B_{i2e}^*) (B_{3e} \cdot U_1)^2 - 4 (B_{3e} \cdot U_2)^2 - 2 (B_{3e} \cdot B_{3e}) ((B_{i2e}^* B_{i2e}^*) (U_1 \cdot U_1) + (U_2 \cdot U_2)),
\]
\[
C_1 = 4 (B_{i2e} \cdot U_2)((B_{i2e}^* B_{i2e}^*) (U_1 \cdot U_1) + (U_2 \cdot U_2)) - 8 (B_{i2e} B_{i2e}^*) (B_{3e} \cdot (U_1 \cdot U_2) + (U_2 \cdot U_1)),
\]
\[
C_0 = 4 (B_{i2e}^* B_{i2e}^*) (U_1 \cdot U_2)^2 - ((B_{i2e} B_{i2e}^*) (U_1 \cdot U_1) + (U_2 \cdot U_2))^2,
\]
\[
U_1 = (B_{i3} \wedge B_{2e}^{-1}), U_2 = (B_{i3} \wedge (B_{i2e} B_{2e}^{-1})). C_i (i = 0, 1, \ldots, 4) are all scalars. U_1 \text{ and } U_2 \text{ are both 4-blade.}
\]

By multiplying both sides of Equation (35) with \((B_{2e} B_{2e})(B_{2e} B_{2e})\) and simplifying it, we have:
\[
D_4 R^4 + D_3 R^3 + D_2 R^2 + D_1 R + D_0 = 0,
\]
(36)

where \(D_1 = (B_{2e} B_{2e})(B_{2e} B_{2e}) C_i (i = 0, 1, \ldots, 4)\).
Simplifying the coefficients \( D_4, D_3, D_2, D_1, \) and \( D_0 \) by using Equations (1)–(10), we have:

\[
D_4 = -(B_{2z} \cdot B_{2z})^2 (B_{3c} \cdot B_{3c})^2 \tag{37}
\]

\[
D_3 = 4(B_{2z} \cdot B_{2z}) (B_{3c} \cdot B_{3c}) (B_{3c} \cdot U_4), \tag{38}
\]

\[
D_2 = 4(B_{1z}^* B_{1z}^*) (B_{3c} \cdot U_3)^2 - 2(B_{3c} \cdot U_4)^2 + 2(B_{3c} \cdot B_{3c}) (B_{1z}^* B_{1z}^*) (U_3 \cdot U_3) + (U_4 \cdot U_4), \tag{39}
\]

\[
D_1 = 4(-V_7 M_1 M_6 + V_6 M_1 M_6 + V_6 M_2 M_6 - V_6 M_2 M_6) + 4(V_5 (V_4 T_1 + V_2 R_1 T_3) + V_1 M_4 (-V_6 T_2 - V_5 R_1 T_3) + V_2 M_6 (V_4 T_1 + V_2 R_1 T_3) + V_5 M_6 (-V_6 T_4 - V_4 R_1 T_3)) + 4(V_5 (-V_1 M_3 + V_3 M_5 + V_5 M_5 - V_6 M_5) M_6 + (V_6 M_1 + V_4 M_2) (-R_1 M_7 + M_5)) \tag{40}
\]

\[
D_0 = 4V_6 V_7 T_1 T_2 - V_7 T_2^2 + 4V_6 V_7 R_1 T_2 T_3 - 4V_6 V_7 R_1^2 T_2^2 + 4V_6 V_7 T_1 T_4 - V_7 T_2^2 + 8V_6 V_7 R_1^2 T_3 T_5 - 4V_6 V_7 R_1 T_4 T_6, \tag{41}
\]

where the coordinates of point \( U \) are only the design parameters, the inputs, and the coordinates of point \( B_1 \). In addition, the detailed simplification procedure of \( D_0 \) is given in the Appendix B.

For the 6-3 Stewart platforms, the geometric constraint equation Equation (36) depends only on the design parameters, the inputs, and the coordinates of point \( B_1 \).

4.4. The Univariate Polynomial Equation for Forward Displacement Analysis

According to Table 1, point \( B_1 \) can be denoted in CGA as:

\[
B_1 = b_1 + \frac{1}{2} b_1^2 e_{\infty} + e_0, \tag{42}
\]

where the coordinates of point \( B_1 \) in coordinate system \( O-XYZ \) can be expressed as:

\[
b_1 = a_1 + M_1 \cdot \left( (0, l_1 \sin(\theta_1), 0)^T + (l_1 \cos(\theta_1), 0, 0)^T \right). \tag{43}
\]

According to Equations (36) and (42), a 16th-degree polynomial equation about \( x \) is formulated by using the Euler angle substitution \( (\cos \theta_1 = (x + 1/x)/2, \sin \theta_1 = (x - 1/x)/(2i), i = \sqrt{-1} \) and \( x = e^{i\theta} \).
4.5. Back Substitution

After the values of $x$ are given, the coordinates of point $B_1$ can be solved by Equation (42). Two point pairs $B_{12}^e$ and $B_{12}^s$ can be calculated using Equations (27) and (28). According to Table 1, we can obtain:

$$B_1^* = S_{B_1} \wedge S_{A_5B_3} \wedge S_{A_6B_3} \wedge S_{B_2B_3} = B_{12} \wedge S_{B_2B_3}$$  \hspace{1cm} (44)

where the coordinates of point $B_2$ are unknown.

Since two values for point $B_2$ can be solved by Equation (30), aiming to receive the unique value of point $B_2$, a novel objective function $g_p$ for point $B_2$ is formulated as:

$$g_p = ( (B_{13} \wedge S_{p1}) \cdot (B_{13} \wedge S_{p1}) ) B_{22} + ( (B_{13} \wedge S_{p2}) \cdot (B_{13} \wedge S_{p2}) ) B_{21},$$  \hspace{1cm} (45)

where $S_{p1} = B_{21} - R_{e\infty}$, $S_{p2} = B_{22} - R_{e\infty}$, $B_{21} = \sqrt{B_{12}^e B_{12}^s B_{21}^e + B_{12}^e B_{21}^e}$,

$$B_{22} = -\sqrt{B_{12}^e B_{12}^s B_{22}^e + B_{12}^e B_{22}^e}, B_{21} \wedge B_{22} = B_{12}^e.$$  

By expanding Equation (45), we readily obtain:

$$g_p = F_2 R^2 + F_1 R + F_0,$$  \hspace{1cm} (46)

where $F_2 = (B_{3e} B_{3e})(B_{21} + B_{22})$, $F_1 = -2(2(3e \cdot (B_{13} \wedge B_{21})) B_{22} + (3e \cdot (B_{13} \wedge B_{22})) B_{21})$, $F_0 = ( (B_{13} \wedge B_{21}) \cdot (B_{13} \wedge B_{21}) ) B_{22} + ( (B_{13} \wedge B_{22}) \cdot (B_{13} \wedge B_{22}) ) B_{21}$. $F_0$, $F_1$, and $F_2$ are all 1-blade.

By simplifying the 1-blades $F_2$, $F_1$, and $F_0$, we obtain:

$$F_2 = 2(3e \cdot B_{3e}) (B_{12}^e B_{22}) / V_0,$$

$$F_1 = -4(-V_5(3e \cdot (B_{13} \wedge B_{2e})) B_{2e} + (3e \cdot (B_{13} \wedge (B_{12}^e B_{2e}))) (B_{12}^e B_{2e}) ) / V_0^2,$$

$$F_0 = 2 \left[ V_5(3e \wedge B_{2e})^2 + (B_{13} \wedge (B_{12}^e B_{2e}))^2 \right] (B_{12}^e B_{2e}) / V_0^2$$

$$-4V_5((B_{13} \wedge B_{2e}) \cdot (B_{13} \wedge (B_{12}^e B_{2e}))) B_{2e} / V_0^3.$$

The expression of point $B_2$ is the expression for a point $(B_{21}$ or $B_{22})$ of point pair $B_{12}^e$. Analyzing carefully Equation (45), on the one hand, when $(B_{13} \wedge S_{p1}) \cdot (B_{13} \wedge S_{p1})$ is equal to zero, we can have $B_{21} = g_p / ((B_{13} \wedge S_{p2}) \cdot (B_{13} \wedge S_{p2}))$. On the other hand, when $(B_{13} \wedge S_{p2}) \cdot (B_{13} \wedge S_{p2})$ is equal to zero, we can also acquire $B_{22} = g_p / ((B_{13} \wedge S_{p1}) \cdot (B_{13} \wedge S_{p1})). (B_{13} \wedge S_{p2}) \cdot (B_{13} \wedge S_{p2})$ and $(B_{13} \wedge S_{p1}) \cdot (B_{13} \wedge S_{p1})$ are both 4-blade. Therefore, we can obtain the coordinates of point $B_2$ by the normalization of a point in CGA, i.e.,:

$$B_2 = -g_p / (g_p e_{\infty}).$$  \hspace{1cm} (47)

By substituting Equation (47) into Equation (44), the coordinates of point $B_3$ are equal to the normalization of 4-blade $B_3^e$, i.e.,:

$$B_3 = -B_3^e / (B_3^e e_{\infty}).$$  \hspace{1cm} (48)

4.6. Two Comparisons

Compared with the traditional algebra method for forward displacement analysis of 6-3 Stewart platforms, the method proposed in this paper has the following characteristics:

1. The derivation process of the proposed method is geometrically intuitive due to the intuitiveness of CGA.

2. The univariate symbolic polynomial equation (Equation (36)) is derived directly by CGA operation without algebraic elimination.

Compared with the existing geometric algebra method to solve 6-3 Stewart platforms, the characteristics of this method proposed in this paper are as follows:

1. In the back substitution procedure, the proposed method does not need to determine the radical symbols.

2. Two formulas between 2-blade and 1-blade are first formulated.
(3) The proposed method based on the determinant operation formula (Equation (8)) of the CGA inner product provides a new idea to solve a more complex spatial parallel mechanism in the future, which is not available in Refs. [12–14].

5. Numerical Example

In order to verify the proposed modeling and calculation method using the CGA framework, two numerical examples of the 6-3 Stewart platforms are given.

5.1. Example 1

The structural parameters and inputs of the 6-3 Stewart platforms are as follows:

\[ a_1 = (50, 0, 100)^T, a_2 = (-25, -2, 40)^T, a_3 = (80, 20, 50)^T, a_4 = (-50, -20, 70)^T, a_5 = (48, 15, 68)^T, \]
\[ a_6 = (36, 31, -93)^T, \]
\[ l_1 = 76, l_2 = 160, l_3 = 139, l_4 = 55, l_5 = 128, l_6 = 217, r_1 = 135, r_2 = 190, r_3 = 141. \]

Substituting the link parameters and inputs into Equation (36), the polynomial of degree sixteen is formulated as:

\[
\begin{align*}
(0.349811 - 0.93682i)x + (2631.72 + 1824.52i)x^2 + (31170.7 - 13317.2i)x^3 \\
+ (78681.7 - 165524i)x^4 + (147210 + 486737i)x^5 - (347339 + 300463i)x^6 \\
+ (398116 - 230184i)x^7 + (994070 - 689922i)x^8 + (354906 - 292442i)x^9 \\
+ (159977 + 430499i)x^{10} + (404489 + 308175i)x^{11} + (182590 - 15808.4i)x^{12} \\
+ (23379.7 - 24542.8i)x^{13} - (788.664 + 3103.68i)x^{14} - (94.4066 - 87.4728i)x^{15} + x^{16} = 0.
\end{align*}
\]

(49)

The sixteen sets of solutions received by solving the polynomial equation Equation (49) agree with those given in [13,14]. The four sets of real solutions are shown in Table 2 and Figure 2.
Figure 2. Four assembly configurations (a, b, c, d) of 6-3 Stewart platforms.

Table 2. Four sets of real solutions of point $B_i (i=1, 2, 3)$.

| $i$ | $x$ | Coordinates | $B_1$ | $B_2$ | $B_3$ |
|-----|-----|-------------|------|------|------|
| 1   | 0.8920 + 0.4520i | $X$ 79.5345, $Y$ -45.8809, $Z$ 152.9020 | -26.0943 | -68.9458 | 54.3105 |
| 2   | -0.9997 + 0.0231i | $X$ 90.9017, $Y$ 53.3945, $Z$ 135.3850 | -40.7764 | 6.28516 | -108.359 |
| 3   | -0.9597 + 0.2809i | $X$ 82.5389, $Y$ 51.0783, $Z$ 145.9150 | -48.8262 | 24.7277 | -100.5170 |
| 4   | 0.6405 + 0.7680i | $X$ 68.8676, $Y$ -33.0062, $Z$ 165.8070 | -47.0212 | 21.0887 | 137.061 |

5.2. Example 2

The link parameters and inputs of the planar 6-3 Stewart platforms are shown in Table 3. According to the steps mentioned above, substituting the link parameters and inputs into Equation (36), we can obtain sixteen solutions. The sixteen sets of solutions are consistent with those given in [14]. The four sets of real solutions are shown in Table 4 and Figure 3.
Table 3. Link parameters and inputs for example 2.

| O-XYZ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $a_{ix}$ | 0 | 0 | 4.0803 | 1.4802 | 5.7676 | 6.2278 |
| $a_{iy}$ | -0.6749 | 3.2366 | -2.3929 | -3.5971 | 3.8962 | 0.7516 |
| $a_{iz}$ | 0 | 0 | 0 | 0 | 0 | 0 |

$r_1 = 2.0, r_2 = 2.0, r_3 = 3.0$

$l_1 = 5.0, l_2 = 4.5, l_3 = 5.0, l_4 = 5.5, l_5 = 5.5, l_6 = 5.7$

Table 4. Four sets of real solutions of point $B_i (i = 1, 2, 3)$.

| $x$ | Coordinates | $B_1$ | $B_2$ | $B_3$ |
|-----|--------------|-------|-------|-------|
| 1   | $0.0366 - 0.9993i$ | X 0.1570 Y 1.8880 Z 4.2903 | 1.6474 1.6311 1.6994 | 2.0248 2.0987 3.6070 |
| 2   | $0.0366 + 0.9993i$ | X 0.1570 Y 1.8880 Z -4.2903 | 1.6474 -1.6994 | 2.0248 2.0987 -3.6070 |
| 3   | $0.7239 - 0.6899i$ | X 3.1079 Y 1.8880 Z 2.9618 | 1.5287 1.8873 0.4111 | 1.1961 1.9774 2.3812 |
| 4   | $0.7239 + 0.6899i$ | X 3.1079 Y 1.8880 Z -2.9618 | 1.5287 -0.4111 | 1.1961 1.9774 -2.3812 |

Figure 3. Cont.
6. Conclusions

In this paper, a novel geometric modeling and calculation method for closed-form solution of 6-3 Stewart platforms based on the framework of conformal geometric algebra is presented. Two formulas between 2-blade and 1-blade are first obtained. A 16-degree and coordinate-invariant polynomial equation by using the Euler angle substitution was derived without algebraic elimination. The final numerical results show that the proposed geometric modeling and calculation method is effective. Since the result obtained by the proposed method is a closed symbolic solution, the method is suitable to write the computer program. Compared with the existing methods for the forward displacement analysis of the 6-3 Stewart platforms, the highlight of this paper is that a new geometric modeling and calculation method without algebraic elimination is obtained by using the determinant form of CGA inner product algorithm (Equation (8)), which can be used for the forward displacement analysis of other complex parallel mechanisms. In addition, compared with [14] in the back substitution procedure, the proposed method does not need to determine the radical symbols.

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Appendix A. Derivation of Equation (12)
\[
(Q_i^+ X_1) \land (Q_j^+ X_2) = ((Q_2-X_1)Q_1 - (Q_1-X_2)Q_2) \land ((Q_2-X_2)Q_2 - (Q_1-X_1)Q_1)
\]
\[
= -(Q_2-X_1)(Q_1-X_2)(Q_1 \land Q_2) - (Q_1-X_1)(Q_2-X_2)(Q_2 \land Q_1)
\]
\[
= -(Q_1 \land Q_2)((Q_1-X_1)(Q_2-X_2)) - (Q_2-X_2)(Q_1-X_1)
\]
\[
= -((Q_1 \land Q_2)(Q_1 \land Q_2) \cdot (X_1 \land X_2))
\]

Appendix B. Derivation of Equations (37)–(41)
According to Equations (19) and (23), we can obtain:
\[
(B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3})) = (B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3}))
\]
\[
(B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3})) = (B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3}))
\]
\[
(B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3})) = (B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3}))
\]
\[
(B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3})) = (B_{12}^* (B_{22}^* S_{A_3 B_3})) \cdot (B_{12}^* (B_{22}^* S_{A_3 B_3}))
\]

According to Equation (8), the expansions of $U_3 \cdot U_4 \cdot U_3 \cdot U_4 \cdot U_4 \cdot B_{3e} \cdot U_5$ and $B_{3e} \cdot U_4$ are

denoted as:
\[
U_3 \cdot U_4 = -V_1 M_1 + V_2 M_2,
\]
\[
U_3 \cdot U_3 = -V_1 M_3 + V_2 M_4 - V_0 M_5,
\]
\[
U_4 \cdot U_4 = V_0 R_{1} M_6 - V_3 M_1 + V_4 M_2 + V_5 V_0 M_5,
\]
\[
B_{3e} \cdot U_3 = V_1 M_8 - V_2 M_9,
\]
\[
B_{3e} \cdot U_4 = -V_0 R_{1} M_7 + V_3 M_8 - V_4 M_9 + V_5 M_5.
\]

(1). The derivation of Equation (37):
\[
D_4 = (B_{22}^* B_{22}^*) (B_{22}^* B_{22}^*) C_4 = -(B_{22}^* B_{22}^*)^2 (B_{3e}^* B_{3e}^*)^2.
\]

(2). The derivation of Equation (38):
\[
D_3 = (B_{22}^* B_{22}^*) (B_{22}^* B_{22}^*) C_3 = 4(B_{22}^* B_{22}^*) (B_{2e}^* B_{2e}^*) (B_{3e}^* U_2)
\]
\[
= 4(B_{2e}^* B_{2e}^*) (B_{3e}^* B_{3e}^*) (B_{3e}^* U_4).
\]

(3). The derivation of Equation (39):
\[
D_2 = (B_{22}^* B_{22}^*) (B_{22}^* B_{22}^*) C_2
\]
\[
= 4(B_{2e}^* B_{2e}^*) (B_{22}^* B_{22}^*) (B_{3e}^* U_1)^2 - 4(B_{22}^* B_{2e}^*) (B_{3e}^* U_2)^2
\]
\[
= 2(B_{3e}^* B_{3e}^*) ((B_{12}^* B_{12}^*) (U_1 \cdot U_1) + (U_2 \cdot U_2))
\]
\[
= 4(B_{12}^* B_{12}^*) (B_{3e}^* U_3)^2 - 4(B_{3e}^* U_4)^2 - 2(B_{3e}^* B_{3e}^*) ((B_{12}^* B_{12}^*) (U_3 \cdot U_3) + (U_4 \cdot U_4)).
\]

(4). The derivation of Equation (40):
\[
D_1 = (B_{22}^* B_{22}^*) (B_{22}^* B_{22}^*) C_1
\]
\[
= (B_{2e}^* B_{2e}^*)^2 (4(B_{3e}^* U_2) ((B_{12}^* B_{12}^*) (U_1 \cdot U_1) + (U_2 \cdot U_2)) - 8(B_{12}^* B_{12}^*) (B_{3e}^* U_1) (U_1 \cdot U_2)
\]
\[
= 4((B_{3e}^* U_3) ((B_{12}^* B_{12}^*) (U_3 \cdot U_3) + (U_4 \cdot U_4)) - 2(B_{12}^* B_{12}^*) (B_{3e}^* U_4)) / (B_{22}^* B_{22}^*)
\]
\[
= 4(N_1 + V_3 N_2 + N_3) / V_0.
\]

where
\[ N_1 = -V_5(B_{5c} \cdot U_3)(U_3 \cdot U_3) + W_3 W_4, \]
\[ N_2 = (-B_{5c} \cdot U_3)(U_3 \cdot U_4) + W_3 W_4 = -V_5(-V_1 M_1 + V_2 M_2)(V_1 M_8 - V_2 M_9) + W_3 W_4, \]
\[ N_3 = V_0 V_5(\omega R_{r1} M_6 W_5 + V_6 R_{r1} M_8 W_4 + V_0 W_3 W_5), \]
\[ W_4 = -V_1 M_3 + V_2 M_4, W_5 = -V_3 M_1 + V_4 M_2, W_4 = V_5 M_3 - V_4 M_9, \]
\[ W_5 = -R_{r1} M_7 + M_5. \]

The expansions of \( N_1, N_2, N_3 \) are simplified as:
\[ N_1 = -V_5(B_{5c} \cdot U_3)(U_3 \cdot U_4) + W_3 W_4 = -V_5(-V_1 M_1 + V_2 M_2)(V_1 M_8 - V_2 M_9) + W_3 W_4, \]  
\[ = V_0(-V_7 M_1 M_8 + V_6 M_1 M_9 + V_0 M_2 M_8 - V_8 M_2 M_9) \]
\[ N_2 = (-B_{5c} \cdot U_3)(U_3 \cdot U_4) + W_3 W_4 = (-V_1 M_1 + V_2 M_2)(V_1 M_8 - V_2 M_9) + W_3 W_4 \]
\[ = V_0 V_5(V_1 M_8(V_1 T_1 + V_1 R_{r1} T_3) + V_1 M_8(-V_6 T_2 - V_2 R_{r1} T_3) + V_2 M_9(V_6 T_1 + V_2 R_{r1} T_6) + V_2 M_9(-V_6 T_4 - V_1 R_{r1} T_6)) \]
\[ N_3 = V_0 V_5 W_3 W_5 + V_3 R_{r1} M_6 W_5 + V_0 R_{r1} M_8 W_4 + V_0 W_3 W_5 \]
\[ = V_0(V_5 W_3 W_5 + (R_{r1} M_6)(V_0 W_5 + W_4) + W_3 W_5) \]
\[ = V_0(V_5(-V_1 M_3 + V_2 M_4) + (-R_{r1} V_0 M_7 + V_0 M_5 + V_3 M_8 - V_4 M_9) R_{r1} M_6 + (-V_3 M_1 + V_4 M_2)(-R_{r1} M_7 + M_5)). \] 

According to Equations (A14)–(A16), the numerator and the denominator of Equation (A13) are divided by \( V_0 \), we can obtain:
\[ D_1 = 4(N_1 + V_5 N_2 + N_3)/V_0 \]
\[ = 4(-V_7 M_1 M_8 + V_6 M_1 M_9 + V_0 M_2 M_8 - V_8 M_2 M_9) \]
\[ + 4(V_5(V_1 M_8(V_1 T_1 + V_1 R_{r1} T_3) + V_1 M_8(-V_6 T_2 - V_2 R_{r1} T_3) + V_2 M_9(V_6 T_1 + V_2 R_{r1} T_6) + V_2 M_9(-V_6 T_4 - V_1 R_{r1} T_6)) \]
\[ + 4(V_5(-V_1 M_3 + V_2 M_4) + (-R_{r1} V_0 M_7 + V_0 M_5 + V_3 M_8 - V_4 M_9) R_{r1} M_6 + (-V_3 M_1 + V_4 M_2)(-R_{r1} M_7 + M_5)) \] 

(5). The derivation of Equation (41):
\[ D_0 = (B_{2a} - B_{2a})(B_{2a} - B_{2a}) \]
\[ = (B_{2a} - B_{2a})(B_{2a} - B_{2a}) \]
\[ - (B_{1a}^2 B_{1a}^2)(U_1 \cdot U_1) + U_2 \cdot U_2 \]
\[ = (B_{2a}^2 B_{2a}^2)(U_1 \cdot U_1) - (B_{1a}^2 B_{1a}^2)(U_3 \cdot U_3) + U_4 \cdot U_4 \]
\[ = (N_4 + N_5 + N_6 + N_7 + N_8)/V_0 \]

where
\[ N_4 = V_5(U_3 \cdot U_3)^2 - V_2^2 W_2^2, N_5 = 2 V_6 \left((U_3 \cdot U_4)^2 - W_3 W_5\right), N_6 = V_5(U_3 \cdot U_4)^2 - W_3, \]
\[ N_7 = 2 V_6 M_0(-V_3 W_2 - V_3) R_{r1}, N_8 = -V_0^2 R_{r1}^2 M_0^2. \]

The expansion of \( N_4 \) is simplified as:
\[ N_4 = V_5(U_3 \cdot U_3)^2 - V_2^2 W_2^2 = V_5 \left((U_3 \cdot U_3)^2 - V_2^2 W_2^2\right) \]
\[ = V_5 V_2^2 V_0 R_{r1} T_3^2 + 2 R_{r1}(V_1 T_1 - V_3 T_2) T_2 - 2 V_4 T_1 T_2 + V_8 T_2^2 + V_7 T_2^2 \]
\[ + V_0 V_5 V_2^2 (V_0 R_{r1} T_3^2 + 2 R_{r1}(V_4 T_4 - V_5 T_3) T_3 - 2 V_4 T_4 T_3 + V_8 T_3^2 + V_7 T_3^2) \]
\[ - 2 V_0 V_5 V_1 V_2 V_3 (V_0 R_{r1} T_3^2 + 2 R_{r1}(V_4 T_4 - V_5 T_3) T_3 + (V_4 T_4 - V_5 T_3) T_3) - 2 V_0 V_5 T_4 T_3 + V_6 T_4 T_3 + V_2 V_7 T_3) \]
\[ = V_0 V_5 \left(V_2^2 T_3^2 + V_2 T_3^2 - 2 V_1 V_2 T_3 T_3 - V_2 T_3^2 + V_0 V_5 T_4 + V_2 V_3 T_3 + V_2 V_3 T_3 + V_2 V_7 T_3\right) \]
\[ + V_0 V_5 \left(V_6(V_1 T_1 - V_2 T_2)^2 + (V_1 T_2 - V_2 T_3)^2 + (2 V_1 V_2 T_4 + 2 V_2 V_4 T_4 - 2 V_2 V_7 T_3)\right) \]
\[ = V_0^2 V_5(V_1 T_3 - V_2 T_3)^2 R_{r1}^2 + N_4 R_{r1} + N_4 \]

where
\[ N_4 I = 2 V_5 V_0(V_1 T_3 - V_2 T_3)(V_4 T_4 - V_3 T_2 + V_2 V_4 T_4 + 2 V_2 V_3 T_3), \]
\[ N_4 = V_0 V_5 \left(V_6(V_1 T_1 - V_2 T_2)^2 + (V_1 T_2 - V_2 T_3)^2 + (2 V_1 V_2 T_4 + 2 V_2 V_4 T_4 - 2 V_2 V_7 T_3)\right). \]

The expansion of \( N_5 \) is simplified as:
\[ N_5 = 2V_5 \left( \mathbf{u}_3 \cdot \mathbf{u}_4 \right)^2 - W_2 W_3 \]
\[ = 2V_0 V_1 V_5 (V_1 (V_0 R_0^2 T_2^2 + R_1 V_4 T_1 T_3 - R_2 V_3 T_2 T_3) + R_1 V_0 V_6 T_1 T_3 + V_4 V_5 T_2^2 - V_5 V_4 T_1 T_2) \]
\[ + 2V_0 V_2 V_5 (V_2 (V_0 R_0^2 T_2^2 + R_1 V_4 T_1 T_3 - R_2 V_3 T_2 T_3) + R_1 V_0 V_6 T_1 T_3 + V_4 V_5 T_2^2 + V_5 V_4 T_1 T_2) \]
\[ + 2V_0 V_3 V_5 (V_3 (V_0 R_0^2 T_2^2 + R_1 V_4 T_1 T_3 - R_2 V_3 T_2 T_3) + R_1 V_0 V_6 T_1 T_3 + V_4 V_5 T_2^2 + V_5 V_4 T_1 T_2) \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
\[ = 2V_0^2 V_5 (V_1 T_3 - V_2 T_2)^2 R_0^2 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_1 + 2V_0^2 V_5 (V_0 V_1 T_3 - V_2 T_2) R_2 \]
\[ + 2V_0 V_5 (V_4 (V_1 T_3 - V_2 T_2)) R_3 + 2V_0 (V_1 (T_2^2 - T_3^2)) R_4 \]
According to the expressions of $M_i (i = 1 \cdots 9)$ and Equations (A25)–(A28), the numerator and the denominator of Equation (A26) are divided by $V_0^2$, we can obtain:

\[
D_0 = 4V_2V_4T_1T_2 - V_2^2T_2^2 + 2V_2V_7V_R T_1 T_2 - V_4 V_5 V_R^2 T_2^2 + 4V_4 V_6 T_1 T_4 - V_2^2 T_2^2 - 8V_0 V_6 R_1 T_3 T_6 - 4V_4 V_6 R_1 T_4 T_6 - 4V_4 V_6 R_1 T_2^2 + 2V_2 V_8 - V_5 V_2^2 T_2^2 - 2(V_2^2 - V_4 V_5) T_2^2 - 2(V_2^2 - V_4 V_5) T_2^2 - 2(V_4 V_5) T_2^2 \tag{A29}
\]

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