Dynamical Localization for the One-Dimensional Continuum Anderson Model in a Decaying Random Potential

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Abstract. We consider a one-dimensional continuum Anderson model where the potential decays in average like $|x|^{-\alpha}$, $\alpha > 0$. We show dynamical localization for $0 < \alpha < \frac{1}{2}$ and provide control on the decay of the eigenfunctions.

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1. Introduction

Disordered systems in material sciences have been the source of a plethora of interesting phenomena and many practical applications. The addition of impurities in otherwise fairly homogeneous materials is known to induce new behaviours such as Anderson localization where wave packets get trapped by the disorder and conductivity can be suppressed [1]. It is then natural to expect that accurate mathematical models for disordered media should display an interesting phase diagram.

As a model for the dynamics of an electron in a disordered medium, the Anderson model is expected to undergo a transition from a delocalized to a localized regime reflected at the spectral level by a transition from absolutely continuous to pure point spectrum. While the localized regime is well understood (see [5,39] and references therein), the existence of absolutely continuous spectrum remains a mystery (nonetheless, see [4,23,26,32]).

In order to understand how absolutely continuous spectrum survives in spite of the disorder, it has been proposed to modulate the random potential
by a decaying envelope \([20–22, 33, 34, 37]\), this is, to replace the usual i.i.d. random variables \(\{V(n) : n \in \mathbb{Z}^d\}\) by \(a_n V(n)\), where \((a_n)_n\) is a deterministic sequence satisfying \(a_n \sim |n|^{-\alpha}\) for some decay rate \(\alpha > 0\). For large values of \(\alpha\) and dimensions \(d \geq 3\), scattering methods can be applied, leading to the proof of absolutely continuous spectrum [33]. A wider range of values of \(\alpha\) was considered by Bourgain in dimension 2 [6] and higher [7]. Point spectrum was also showed to hold outside the essential spectrum of the free operator in [34].

It is well known that, in the i.i.d. case, the one-dimensional Anderson model always displays pure point spectrum \([10, 11, 15, 16, 21, 25, 28, 30, 31, 36]\), while the addition of a decaying envelope leads to a rich phase diagram as the value of \(\alpha\) varies. Transfer matrix analysis can be applied, leading to a complete understanding of the spectrum of the operator [21, 35] in the discrete and continuum setting (see also [37] for a related model). This time, absolutely continuous spectrum can still be observed for large values of \(\alpha\). As it is natural to expect, small values of \(\alpha\) lead to pure point spectrum. Interestingly, there is a critical value of \(\alpha\) for which a transition from pure point to singular continuous spectrum occurs as a function of the coupling constant. The three above regimes correspond to \(\alpha > \frac{1}{2}\), \(\alpha < \frac{1}{2}\) and \(\alpha = \frac{1}{2}\), respectively. A complete study of the spectral behaviour of the one-dimensional discrete and continuum models is given in [35] (see also [8]).

From the dynamical point of view, it is standard to show that the system propagates for \(\alpha > \frac{1}{2}\). For the critical case \(\alpha = \frac{1}{2}\), no transition occurs at the dynamical level, despite the spectral transition: there are non-trivial transport exponents for all values of the coupling constant [27] for both the discrete and continuum model (see also [8, 9] for elementary arguments showing delocalization). This provides yet another example of a model where spectral localization and transport coexist. Dynamical localization in the regime \(0 < \alpha < \frac{1}{2}\) for the discrete model was shown in [38].

In the present paper, we show dynamical localization for the continuum model in the sub-critical region \(0 < \alpha < \frac{1}{2}\). This was left as an open question in [17], where the authors develop a continuum version of the Kunz–Souillard method [36]. Instead, we have chosen to work within the framework of the continuum fractional moment method of [2] in the one-dimensional version of [30] (See [3] for this method in its original discrete context). In addition, our proof involves a fine-tuning of various auxiliary technical ingredients scattered in the literature, e.g. [12, 14, 35].

**Structure of the Article**

We present the model and the main results in Sect. 2. In Sect. 3 we recall the Prüfer transform formalism and give some preliminary bounds. The fractional moments of the Green’s function are studied in Sect. 4. In Sect. 5, we prove the main theorem on dynamical localization. Section 6 contains the proof of some consequences of our main result. Finally, the appendix contains some estimates used along the proofs.
2. Model and Main Results

Let \( \omega_n : n \in \mathbb{Z} \) be an i.i.d. family of bounded random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Assume that \(E[\omega_0] = 0\) and \(E[\omega_0^2] = 1\). We denote by \( \omega^+ \) and \( \omega^- \) the supremum and infimum of the support of the \( \omega_n \)'s. By choosing an appropriate probability space, we may assume that these random variables are bounded and not merely bounded with probability 1. We will always assume that the random variables \( \omega_n \) admit a bounded density \( \rho \) which is hence supported on \([\omega^-, \omega^+]\).

Let \( u \) be a non-negative bounded function with support in \((0, 1)\) and denote \( u_n(\cdot) = u(\cdot - n) \). This is often called the single-site potential. We assume that there exists a non-trivial interval \( J \subset (0, 1) \) and two constants \( c_u, C_u \in (0, \infty) \) such that
\[
c_u \chi_J \leq u \leq C_u \chi_{[0, 1]}.
\]
(2.1)

Finally, let \( \lambda \neq 0 \) and \( \alpha > 0 \), and let \((a_n)_n\) be a positive sequence such that \( \lim_{|n| \to \infty} a_n|n|^\alpha = 1 \).

We consider the random operator
\[
H_{\omega, \lambda} = -\Delta + \lambda V_\omega \text{ on } L^2(\mathbb{R}),
\]
(2.2)

where \( V_\omega \) is a multiplication operator by the function
\[
V_\omega(x) = \sum_{n \in \mathbb{Z}} a_n \omega_n u_n(x).
\]
(2.3)

Notice that \((H_{\omega, \lambda})_\omega\) is a non-ergodic family of random operators which are essentially self-adjoint on \(C_0^\infty(\mathbb{R})\), the space of infinitely differentiable compactly supported functions.

We start recalling the spectral results of [35] which give a complete characterization of the spectrum of \( H_{\omega, \lambda} \) for all the possible combinations of parameters. The following is proved for the model on the half-line.

**Theorem 2.1.** Under the hypothesis above, the essential spectrum of \( H_{\omega, \lambda} \) is \( \mathbb{P}\)-a.s. equal to \([0, \infty)\). Furthermore,

1. **Super-critical case.** If \( \alpha > \frac{1}{2} \), then for all \( \lambda \in \mathbb{R} \), the spectrum of \( H_{\omega, \lambda} \) is almost surely purely absolutely continuous in \((0, \infty)\).

2. **Critical case.** If \( \alpha = \frac{1}{2} \), then for all \( \lambda \neq 0 \), the a.c. spectrum of \( H_{\omega, \lambda} \) is almost surely empty. Furthermore, for each \( \lambda \neq 0 \), there exists \( E_0(\lambda) \geq 0 \) such that, almost surely, the spectrum of \( H_{\omega, \lambda} \) is pure point in \((0, E_0(\lambda))\) and purely singular continuous in \([E_0(\lambda), \infty)\).

3. **Sub-critical case.** If \( 0 < \alpha < \frac{1}{2} \), then for all \( \lambda \neq 0 \), the spectrum of \( H_{\omega, \lambda} \) is almost surely pure point in \((0, \infty)\).

Delocalization, or spreading of wave packets, for \( \alpha > \frac{1}{2} \) follows from the RAGE theorem [13]. The situation is particularly interesting for \( \alpha = \frac{1}{2} \): non-trivial transport occurs regardless of the precise nature of the spectrum. In particular, this provides an example of an operator displaying pure point spectrum but no dynamical localization. To describe the dynamics, we consider
the time-averaged random moment of order \( p \geq 0 \) for the evolution, initially spatially localized at the origin and localized in energy by a positive function \( f \in C_0^\infty(\mathbb{R}) \),

\[
M_\omega(p, f, T) := \frac{2}{T} \int_0^\infty e^{-\frac{2t}{T}} \| |X| e^{-itH_{\omega, \lambda}} f(H_{\omega, \lambda}) \chi_0 \|^2 dt,
\]

where \( |X| \) denotes the position operator and \( \chi_x \) denotes the characteristic function of the interval \([x, x+1]\). The following result is proved in [27] for the model defined on the half-line.

**Theorem 2.2.** Let \( \alpha = \frac{1}{2} \) and \( \lambda \in \mathbb{R} \). The following holds \( \mathbb{P} \)-almost surely: for all positive \( f \in C_0^\infty(\mathbb{R}) \) constantly equal to 1 on a compact interval \( J \subset (0, \infty) \), for any \( \nu > 0 \) and all \( p > 2\gamma_J + \nu \) where \( \gamma_J = \inf\{\lambda(8E)^{-1}|\hat{u}(\sqrt{E})| : E \in J\} \), there exists \( C_\omega(p, J, \nu) > 0 \) such that

\[
M_\omega(p, f, T) \geq C_\omega(p, J, \nu) T^{-2\gamma_J - \nu},
\]

for all sufficiently large values of \( T \) and where \( \hat{u} \) denotes the Fourier coefficient of \( u \).

We complete the dynamical study of the model by addressing the question of dynamical localization for \( 0 < \alpha < \frac{1}{2} \). The corresponding result for the discrete model was obtained in [38].

For an interval of energy \( I \), we define the correlator

\[
Q_{\omega, \lambda}(x, y; I) = \sup_{f \in \mathcal{C}_c(I)} \| \chi_x f(H_{\omega, \lambda}) \chi_y \|,
\]

where \( \mathcal{C}_c(I) \) denotes the space of bounded measurable functions compactly supported in \( I \). We now give the notion of dynamical localization we will use in this work.

**Definition 2.3.** We say that \( H_{\omega, \lambda} \) exhibits dynamical localization in an interval \( I \subset \mathbb{R} \) if we have

\[
\sum_{x \in \mathbb{Z}} \mathbb{E} \left[ Q_{\omega, \lambda}(x, y; I)^2 \right] < \infty,
\]

for all \( y \in \mathbb{Z} \).

Our main theorem is the following.

**Theorem 2.4.** Let \( 0 < \alpha < \frac{1}{2} \), \( \lambda \neq 0 \) and let \( I \in (0, \infty) \) be a compact interval. For each \( y \in \mathbb{Z} \), there exist constants \( c = c(I, y), C = C(I, y) \in (0, \infty) \) such that

\[
\mathbb{E} \left[ Q_{\omega, \lambda}(x, y; I) \right] \leq Ce^{-c|x|^{1-\alpha}},
\]

for all \( x \in \mathbb{R} \). In particular, dynamical localization in the sense of (2.5) holds in any compact subinterval of \((0, \infty)\).
Although the lack of ergodicity of the model induces the dependence of (2.5) on the base site $y$, it is standard to show that this bound still implies pure point spectrum and finiteness of the moments. We will recall the proof of pure point spectrum in “Appendix B” as some care is needed to overcome the non-uniform bounds. Respect to the finiteness of the moments, we will prove a stronger result in Theorem 2.6.

Our analysis provides a control on the eigenfunctions of the operator $H_{ω,λ}$. Let $φ_{ω,E}$ denote the eigenfunction of $H_{ω,λ}$ corresponding to the eigenvalue $E$. The analysis of [35, Theorem 8.6] can be adapted to the continuum setting to show that

$$
\lim_{n→∞} \frac{1}{n^{1-2\alpha}} \log \left( \sqrt{|φ_{ω,E}(n)|^2 + |φ'_{ω,E}(n)|^2} \right) = -\beta(λ, E), \quad P - a.s.,
$$

for almost every fixed $E ∈ (0, ∞)$, where $β(λ, E)$ is explicitly given in [35, Theorem 9.2] (see Proposition 3.1). In particular, this shows that for almost every $E ∈ (0, ∞)$, $P$-almost surely, there exists a finite constant $C_{ω,E}$ such that

$$
\sqrt{|φ_{ω,E}(x)|^2 + |φ'_{ω,E}(x)|^2} ≤ C_{ω,E} e^{-β(λ,E)}|x|^{1-2\alpha}.
$$

It is known that certain types of decay of eigenfunctions are closely related to dynamical localization [18,19,29]. Such criteria usually require a control on the localization centres of the eigenfunctions, uniformly in energy intervals. This information is missing in the above bound. We provide this uniform control in the next proposition.

**Theorem 2.5.** Let $0 < α < \frac{1}{2}$ and let $λ ≠ 0$. For all compact energy interval $I ⊂ (0, ∞)$, there exists two deterministic constants $c_1 = c_1(I)$, $c_2 = c_2(I)$ and almost surely finite positive random quantities $c_ω = c_ω(I)$, $C_ω = C_ω(I)$ such that

$$
c_ω e^{-c_1|x|^{1-2α}} ≤ \|χ_x φ_{ω,E}\| \leq C_ω e^{-c_2|x|^{1-2α}},
$$

for all $E ∈ I ∩ σ(H_{ω,λ})$ and all $x ∈ ℝ$.

The upper bound in (2.6) can be seen as a stretched form of the condition SULE where the localization centres are all equal to 0 [18,24].

We finally state our result on the moments.

**Theorem 2.6.** Let $0 < α < \frac{1}{2}$ and $λ ≠ 0$, and let $I ⊂ ℝ$ be a compact interval. Then,

$$
E\left(\sup_{t ∈ ℝ} \left\| e^{\frac{1}{2}|X|^α} e^{-itH_{ω,λ}} P_t(H_{ω,λ})ψ \right\|^2 \right) < ∞,
$$

for all $κ < 1 - 2α$ and $ψ ∈ L^2(ℝ)$ with bounded support, while

$$
\limsup_{t→∞} \left\| e^{\frac{1}{2}|X|^α} e^{-itH_{ω,λ}} ψ \right\|^2 = ∞, \quad P - a.s.,
$$

for all $κ > 1 - 2α$ and all $ψ ∈ RanP_t(H_{ω,λ})$. 
3. Asymptotics of Transfer Matrices and Prüfer Transform

Let \( \phi \) be a solution of the equation \( H_{\omega, \lambda} \phi = E \phi \) in some interval \([a, b]\). For \( x, y \in [a, b] \), we define the transfer matrices by the relation

\[
T_{\omega, \lambda}(y, x; E) \begin{pmatrix} \phi(x) \\ \phi'(x) \end{pmatrix} = \begin{pmatrix} \phi(y) \\ \phi'(y) \end{pmatrix}.
\]

(3.1)

**Proposition 3.1** ([35], Theorem 9.2). Let \( 0 < \alpha \leq \frac{1}{2} \). For all \( E \geq 0 \) such that \( 4\sqrt{E} \notin \pi \mathbb{Z} \), we have the almost sure limit

\[
\beta(\lambda, E) := \lim_{n \to \pm \infty} \frac{\log \| T_{\omega}(n, 0; E) \|}{\sum_{j=1}^{n} j^{-2\alpha}} = \frac{\lambda^2}{8E} \left( \int_{0}^{1} u(y) e^{i\sqrt{E}y} dy \right)^2.
\]

(3.2)

This result follows from the asymptotic analysis of the Prüfer transform associated with the system. Following [35], we denote \( k = \sqrt{E} \) and define the modified Prüfer coordinates \( R \) and \( \theta \) such that

\[
\phi(x) = kR(x) \cos \theta(x), \\
\phi'(x) = R(x) \sin \theta(x).
\]

(3.3)

Note that, if \( V = 0 \), then we have \( \theta(x) = \theta_0 + kx \). These satisfy the equations

\[
\frac{d}{dx} \theta(x) = k - \frac{V_{\omega}(x)}{k} \sin^2 \theta(x),
\]

(3.4)

\[
\frac{d}{dx} \log R(x) = \frac{1}{2k} V_{\omega}(x) \sin(2\theta(x)).
\]

(3.5)

Note that the functions \( R \) and \( \theta \) depend on the energy \( E \), which we removed from the notation as no confusion will arise. Nonetheless, we sometimes denote \( R_{x_0}(\cdot; \theta_0) \) and \( \theta_{x_0}(\cdot; \theta_0) \) to stress that the system is considered with initial conditions \( \phi(x_0) = \sin \theta_0 \) and \( \phi'(x_0) = \cos \theta_0 \).

We quote the following lemma from [35].

**Lemma 3.2** ([35], Lemma 2.1). For all compact energy interval \([a, b] \subset (0, \infty)\) and all \( \vartheta_1 \neq \vartheta_2 \), there exists positive deterministic constants \( C_1 = C_1(\vartheta_1, \vartheta_2, I) \) and \( C_2 = C_2(\vartheta_1, \vartheta_2, I) \) such that

\[
C_1 \max \{ R_x(y, \vartheta_1), R_x(y, \vartheta_2) \} \leq \| T_{\omega, \lambda}(E; y, x) \| \\
\leq C_2 \max \{ R_x(y, \vartheta_1), R_x(y, \vartheta_2) \},
\]

(3.6)

for all \( x, y \in \mathbb{R} \), \( E \in I \) and all \( \omega \).

This allows to reduce the asymptotics of transfer matrices (3.1) to the ones of the Prüfer radii (3.3). The analysis outlined in [35, Section 9] leads to

\[
\mathbb{E} \left[ \log \frac{R(n)}{R(m)} \right] = \frac{\lambda^2}{8k^2} \left( \int_{0}^{1} u(y) e^{2iky} dy \right)^2 \sum_{j=m}^{n} j^{-2\alpha} + K_{m,n},
\]

(3.7)

for \( m \leq n \), where \( |K_{m,n}| = o(\sum_{j=m}^{n} j^{-2\alpha}) \), uniformly on values of \( \sqrt{E} \notin \pi \mathbb{Z} \) ranging over compact energy intervals. The same estimate holds for \( n \leq m \leq 0 \). By (3.6), and the same asymptotics holds for the norms of the transfer matrices. We detail the above estimate in “Appendix B” and summarize it in
a form that will suit our purposes in the next lemma. For simplicity, we denote \( T_{\omega,n}(E) = T_{\omega,\lambda}(n + 1, n; E) \), dropping the dependence in \( \lambda \).

**Lemma 3.3.** Let \( I \subset (0, \infty) \) be a compact interval. Then, for all \( \beta' \) such that \( 0 < \beta' < \inf_{E \in I} \beta(\lambda, E) \), there exists \( n_0 = n_0(I, \beta') \geq 1 \) such that

\[
\mathbb{E} \left[ \log \| T_{\omega,ln_0}(E) \cdots T_{\omega,(l-1)n_0+1}(E) \psi_0 \| \right] \geq \beta' \sum_{j=(l-1)n_0+1}^{ln_0} \frac{1}{j^{2\alpha}} \tag{3.8}
\]

for all \( l \geq 1, \| \psi_0 \| = 1 \) and \( E \in I \).

Furthermore, there exists a constant \( C = C(I) \) such that

\[
\mathbb{E} \left[ \left( \log \| T_{\omega,ln_0}(E) \cdots T_{\omega,(l-1)n_0+1}(E) \psi_0 \| \right)^2 \right] \leq C \left( \sum_{j=(l-1)n_0+1}^{ln_0} \frac{1}{j^{2\alpha}} \right)^2, \tag{3.9}
\]

for all \( l \geq 1, \| \psi_0 \| = 1 \) and \( E \in I \).

**Proof.** From (3.6) and (3.7), we can find \( n_0 \) large enough such that

\[
\mathbb{E} \left[ \log \| T_{\omega,ln_0}(E) \cdots T_{\omega,(l-1)n_0+1}(E) \psi_0 \| \right] \geq \beta' \sum_{j=(l-1)n_0+1}^{ln_0} \frac{1}{j^{2\alpha}},
\]

for all \( l \geq 1, \| \psi_0 \| = 1 \) and all \( E \in I \) corresponding to values of \( k \notin \pi \mathbb{Z} \).

The bound for all energies in \( I \) then follows by continuity of the left-hand side above with respect to \( E \). This proves (3.8). The upper bound (3.9) follows from (3.6) and the estimate (B.3) from “Appendix B”.

## 4. Fractional Moments Estimates

For \( \Lambda \subset \mathbb{R} \), we denote by \( H_{\omega,\Lambda} \) the restriction of \( H_{\omega,\lambda} \) to \( L^2(\Lambda) \) and its resolvent by \( G_{\omega,\Lambda}(E) = (H_{\omega,\Lambda} - E)^{-1} \), where we hid the explicit dependence on \( \lambda \) to lighten the notation. The following is the main result of this section.

**Theorem 4.1.** Let \( 0 < \alpha < \frac{1}{2} \) and \( I \subset (0, \infty) \) be a compact interval. For each \( y \in \mathbb{Z} \), there exist constants \( c = c(I, y), C = C(I, y) \in (0, \infty) \) such that

\[
\mathbb{E} \left[ \| \chi_x G_{\omega,[a,b]}(E) \chi_y \|^{s} \right] \leq C(\lambda a_x)^{-1/2} e^{-c|x|^{1-2\alpha}},
\]

for all \( x \in \mathbb{R}, E \in I \) and all \( a < b \).

The proof is given at the end of Sect. 4.2. In Sect. 4.1, we relate the fractional moments of the Green’s function to negative fractional moments of the norm of transfer matrices which are then estimated in Sect. 4.2.
4.1. From Green’s Function to Transfer Matrices

The following analysis is a direct adaptation of [30, Section 3]. We provide the details for the sake of completeness and to carefully identify the dependence on the envelope \( a_x \).

Fix \( E \geq 0 \) and let \( I \subset \mathbb{R} \). For \( c \in [a, b] \) and \( \theta \in [0, 2\pi) \), we define \( \phi_c(\cdot; \theta) \) the solution of \( H_{\omega, [a, b]} \phi = E \phi \) such that \( \phi(c) = \sin \theta \) and \( \phi'(c) = \cos \theta \). This way, we can define Prüfer coordinates \( R_c(x; \theta) \) and \( \theta_c(x; \theta) \) with the convention that \( \theta_c(c; \theta) = \theta \) and imposing continuity. We will eliminate \( \theta \) from the notation whenever \( \theta = 0 \).

**Lemma 4.2.** Let \( I \subset (0, \infty) \) be a compact interval. For all \( s \in [0, \frac{1}{2}] \), there exists \( C = C(s, I) \in (0, \infty) \) such that, for all \( a < b \),

\[
\mathbb{E} \left[ \left\| \chi_x G_{\omega, [a, b]}(E) \psi_y \right\|^s \right] \leq C (\lambda a_x)^{-1/2} \mathbb{E} \left[ \left\| T_{\omega, \lambda}(E; x, y) \begin{pmatrix} \sin \theta_b(y) \\ \cos \theta_b(y) \end{pmatrix} \right\|^{-2s} \right]^{1/2},
\]

(4.1)

for all integers \( a \leq x < y \leq b \) and

\[
\mathbb{E} \left[ \left\| \chi_x G_{\omega, [a, b]}(E) \psi_y \right\|^s \right] \leq C (\lambda a_x)^{-1/2} \mathbb{E} \left[ \left\| T_{\omega, \lambda}(E; x, y) \begin{pmatrix} \sin \theta_a(y) \\ \cos \theta_a(y) \end{pmatrix} \right\|^{-2s} \right]^{1/2},
\]

(4.2)

for all integers \( a \leq y < x \leq b \).

**Proof.** We start from the identity

\[
G_{\omega, [a, b]}(s; t; E) = \frac{1}{W(\phi_a, \phi_b)} \begin{cases} \phi_a(s) \phi_b(t) & \text{if } s \leq t, \\ \phi_a(t) \phi_b(s) & \text{if } s > t, \end{cases}
\]

where \( W(f, g) = fg' - f'g \) is the Wronskian of the functions \( f \) and \( g \). We consider \( a \leq x < y \leq b \) as the opposite case follows by symmetry. Note that \( W(\phi_a, \phi_b)(x) = k R_a(x) R_b(x) \sin(\theta_a(x) - \theta_b(x)) \). Hence, by definition of the Prüfer transform, we can find a constant \( C = C(I) \in (0, \infty) \) such that

\[
\mathbb{E} \left[ \left\| \chi_x G_{\omega, [a, b]}(E) \psi_y \right\|^s \right] \leq C \mathbb{E} \left[ \frac{R_b(y)^s}{R_b(x)^s} \sin(\theta_a(x) - \theta_b(x)) \right]^{-s} \leq C \mathbb{E} \left[ \frac{R_b(y)^{2s}}{R_b(x)^{2s}} \mathbb{E} \left[ \sin(\theta_a(x) - \theta_b(x)) \right]^{-2s} \right]^{1/2}.
\]

From (2.1), we infer that \( R_b(x) = R_b(y) R_y(x; \theta_b(y)) \). Hence,

\[
\mathbb{E} \left[ \left\| \chi_x G_{\omega, [a, b]}(E) \psi_y \right\|^s \right] \leq C \mathbb{E} \left[ R_y(x, \theta_b(y))^{-2s} \right]^{1/2} \mathbb{E} \left[ \sin(\theta_a(x) - \theta_b(x)) \right]^{-2s} \right]^{1/2}.
\]

The first expected value above is bounded by the expected value on the right-hand side of (4.1). The bound on the second one is given in the next lemma. \( \square \)
Lemma 4.3. For all compact energy interval $I \in (0, \infty)$ and for all $s \in [0, \frac{1}{2})$, there exists $C = C(s, I) \in (0, \infty)$ such that
\[
\mathbb{E} \left[ | \sin(\theta_a(x) - \theta_b(x))|^{-2s} \right] \leq C(\lambda_{a_x})^{-1} \| \rho \|,
\]
for all $a < x < b$, $E \in I$.

Proof. We will prove that there exist $C = C(s, I) \in (0, \infty)$ such that
\[
\int_{\omega^-}^{\omega^+} \frac{\rho(\omega_x) d\omega_x}{|\sin(\theta_b(x) - \theta_a(x))|^{2s}} \leq C(\lambda_{a_x})^{-1} \| \rho \|,
\]
for all $a < x < b$, $E \in I$ and all realizations of $\{\omega_y : y \neq x\}$. The result then follows by averaging over $\{\omega_y : y \neq x\}$.

Observe that $\theta_a(x)$ is independent of $\omega_x$. We will change variables to $t = -\theta_b(x)$. Now, from (3.5), we see that, for $z \in [x, x + 1)$, we have
\[
\frac{d}{dz} \theta_b(z) = k - \frac{\lambda_{a_x} \omega_x}{k} u_x(s) \sin^2 \theta_b(z), \tag{4.3}
\]
\[
\frac{d}{dz} \log R_b(z) = \frac{\lambda_{a_x} \omega_x}{2k} u_x(z) \sin 2\theta_b(z). \tag{4.4}
\]
Hence, using (4.3) in the second equality,
\[
\frac{\partial}{\partial z} \left( R_b(z)^2 \frac{\partial}{\partial \omega_x} \theta_b(z) \right) = 2R_b(z) \frac{\partial}{\partial z} R_b(z) \frac{\partial}{\partial \omega_x} \theta_b(z) + R_b(z)^2 \frac{\partial^2}{\partial \omega_x \partial z} \theta_b(z)
\]
\[
= 2R_b(z)^2 \frac{\partial}{\partial z} \log R_b(z) \frac{\partial}{\partial \omega_x} \theta_b(z) - \frac{\lambda_{a_x}}{k} u_x(z) R_b(z)^2 \sin^2 \theta_b(z)
\]
\[
- R_b(z)^2 \frac{\lambda_{a_x} \omega_x}{k} u_x(z) \sin(2\theta_b(z)) \frac{\partial}{\partial \omega_x} \theta_b(z)
\]
\[
= -\frac{\lambda_{a_x}}{k^2} u_x(z) \phi_b(z)^2, \tag{4.5}
\]
where we used (4.4) and the identity $\phi_b(z) = R_b(z) \sin \theta_b(z)$ in the last step. Since $\theta_b(x + 1)$ is independent of $\omega_x$, (4.5) yields
\[
\frac{\partial}{\partial \omega_x} \theta_b(x) = -\frac{\lambda a - x}{k^2 R_b(x)^2} \int_x^{x+1} u_x(z) \phi_b(z)^2 dz.
\]
From (2.1), Lemma A.1 and A.2, we can then find two positive constants $C_1$ and $C_2$ such that
\[
C_1 \lambda a_x \leq \left| \frac{\partial}{\partial \omega_x} \theta_b(x) \right| \leq C_2 \lambda a_x,
\]
for all $x \in \mathbb{Z}$ and all $E \in I$. As anticipated, we will change variables to $t = -\theta_b(x)$. Let $[t_-(x), t_+(x)]$ be the range of $-\theta_b(x)$. Thanks to the last estimate above, $|t_+(x) - t_-(x)| \leq C_3 \lambda a_x (\omega^+ - \omega^-)$, for all $x \in \mathbb{Z}$ and $E \in I$, where $C_3 =$
max\{C_1, C_2\}. Hence there exists \( t^- < t^+ \) such that \([t^-(x), t^+(x)] \in [t^1, t^+] \) for all \( x \in \mathbb{Z} \) and \( E \in I \). Recalling that the density \( \rho \) is bounded,

\[
\int_{\omega^-}^{\omega^+} \frac{\rho(\omega_x)}{\sin(\theta_b(x) - \theta_a(x))} \, d\omega_x = \int_{\omega^-}^{\omega^+} \frac{\rho(\omega_x)}{\sin(\theta_b(x) - \theta_a(x))} \left| \frac{\partial}{\partial \omega_x} \theta_b(x) \right| \, d\omega_x \leq C_4 \frac{\|\rho\|}{\lambda a_x} \int_{t^-}^{t^+} \frac{dt}{\sin(t - \theta_a(x))}^{2s},
\]

for some \( C_4 > 0 \) and all \( x \in \mathbb{Z}, E \in I \). The last integral is finite if \( 2s < 1 \).

Since \( \theta_b(x + 1) \) is independent of \( \omega_x \), (4.5) yields

\[
\frac{\partial}{\partial \omega_x} \theta_a(x) = \frac{\lambda a_x}{k^2 R_a(x)^2} \int_x^{x+1} u_x(s) \phi_a(s)^2 \, ds. \tag{4.6}
\]

From (2.1), Lemmas A.1 and A.2, we can then find two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \lambda a_x \leq \left| \frac{\partial}{\partial \omega_x} \theta_a(x) \right| \leq C_2 \lambda a_x. \tag{4.7}
\]

This change of variables leads to the estimate. \( \square \)

### 4.2. Estimates on Transfer Matrices

We start with an a priori estimate on the norm of transfer matrices.

**Lemma 4.4.** For all compact interval \( I \subset (0, \infty) \) and for all \( [a, b] \subset \mathbb{R} \), there exists \( M = M(I, a, b) \in (0, \infty) \) such that

\[
M^{-1} \leq \|T_{\omega, \lambda}(x, y; E)\| \leq M,
\]

for all \( E \in I \) and \( x, y \in [a, b] \).

**Proof.** The estimates of Lemma A.1 from “Appendix B” imply that

\[
\exp \left( -\frac{1}{2} \int_a^b |1 + V(t) - E| \, dt \right) \leq \|T_{\omega, \lambda}(x, y; E)\| \leq \exp \left( \frac{1}{2} \int_a^b |1 + V(t) - E| \, dt \right).
\]

\( \square \)

The proofs of the next two lemmas are strongly inspired by [14], but we provide them in full detail as our non-ergodic situation requires finer estimates. For applications of this argument in the continuum ergodic setting, see [16,30].

Recall the notation \( T_{\omega, n}(E) = T_{\omega, \lambda}(n + 1, n; E) \).

**Lemma 4.5.** Let \( 0 < \alpha < \frac{1}{2} \) and \( \lambda \neq 0 \). For all compact interval \( I \subset (0, \infty) \), there exist \( n_0 = n_0(I) \geq 1 \), \( s_0 = s_0(I) \in (0, 1) \) and \( c = c(I) > 0 \) such that

\[
\mathbb{E} \left[ \|T_{\omega, n_0}(E) \cdots T_{\omega, (l-1)n_0+1}(E)\psi_0\|^{-s} \right] \leq 1 - \frac{c}{l^{2\alpha}},
\]

for all \( s \in (0, s_0], l \geq 1, \|\psi_0\| = 1 \) and \( E \in I \).
Proof. We drop the dependence on $E$ to lighten the notation. From Lemma 3.3, we obtain $n_0 = n_0(I) \geq 1$, $c_1 = c_1(I) > 0$ and $c_2 = c_2(I) > 0$ such that

$$
\mathbb{E} \left[ \log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \| \right] \geq c_1 \frac{n_0^{1-2\alpha}}{l^{2\alpha}},
$$

and

$$
\mathbb{E} \left[ (\log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|)^2 \right] \leq c_2 \frac{n_0^{2-4\alpha}}{l^{4\alpha}},
$$

for all $l \geq 1$, $\| \psi_0 \| = 1$ and $E \in I$. Now, we apply the inequality $e^y \leq 1 + y + y^2e^{|y|}$ to $y = -s \log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|$ with $s$ to be fixed later, so that

$$
\| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|^{-s} \leq 1 - s \log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|
$$

$$
+ s^2 \left( \log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \| \right)^2 e^{|s|\log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|}.
$$

Now,

$$
\log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \| \leq \sum_{j=(l-1)n_0+1}^{ln_0} \log \| T_{\omega,j} \|.
$$

On the other hand,

$$
1 = \| T_{\omega, (j-1)n_0+1} \cdots T_{\omega, ln_0} T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|
$$

$$
\leq \| T_{\omega, (l-1)n_0+1} \cdots T_{\omega, ln_0} \| \| T_{\omega, ln_0} \| \| T_{\omega, (l-1)n_0+1} \psi_0 \|,
$$

since $\| T_{\omega,j} \| = \| T_{\omega,j} \|$, so that we have

$$
\log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \| \geq - \sum_{j=(l-1)n_0+1}^{ln_0} \log \| T_{\omega,j} \|.
$$

Piecing these bounds together and remembering Lemma 4.4, we obtain

$$
\left| \log \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \| \right| \leq \sum_{j=(l-1)n_0+1}^{ln_0} \log \| T_{\omega,j} \| \leq c_3 n_0,
$$

for some $c_3 = c_3(I) > 0$. Hence,

$$
\mathbb{E} \left[ \left| \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|^{-s} \right| \right] \leq 1 - c_1 s \frac{n_0^{1-2\alpha}}{l^{2\alpha}} + c_2 s^2 e^{c_3 n_0} \frac{n_0^{2-4\alpha}}{l^{4\alpha}},
$$

for all $l \geq 1$, $\| \psi_0 \| = 1$ and $E \in I$. We can now find $s_0 = s_0(I) > 0$ small enough such that

$$
\mathbb{E} \left[ \left| \| T_{\omega, ln_0} \cdots T_{\omega, (l-1)n_0+1} \psi_0 \|^{-s} \right| \right] \leq 1 - \frac{c_4}{l^{2\alpha}},
$$

for some $c_4 > 0$, for all $s \in (0, s_0]$, $l \geq 1$, $\| \psi_0 \| = 1$ and $E \in I$. □

The following lemma contains the key estimate to Theorem 4.1.
Lemma 4.6. Let $0 < \alpha < \frac{1}{2}$ and $\lambda \neq 0$. For each $m \in \mathbb{Z}$ and each compact interval $I \subset (0, \infty)$, there exist $s_0 = s_0(m, I) \in (0, 1)$, $C = C(m, I)$ and $c = c(I) > 0$ such that
\[
E \left[ \| T_\omega(n, m; E) \psi_0 \|^{-s} \right] \leq Ce^{-c|n|^{1-2\alpha}}, \quad (4.8)
\]
for all $s \in (0, s_0]$, $\| \psi_0 \| = 1$ and $n \in \mathbb{Z}$.

Proof. Once again, we drop the dependence on $E$ to lighten the notation. We start proving the bound (4.8) for $m \geq 0$, $n \geq m + n_0$ and $s \in (0, s_0]$ where $n_0 = n_0(I) \geq 1$ and $s_0 = s_0(I) > 0$ are taken from the previous lemma. Write $m = l_1 n_0 - r_1$ and $n = l_2 n_0 + r_2$ with $0 \leq r_1, r_2 < n_0$. By Lemma 4.4,
\[
\| T_{\omega,l_2 n_0} \cdots T_{\omega,m} \psi_0 \| = \| T_{\omega,l_2 n_0+1}^{-1} T_{\omega,n} T_{\omega,n} \cdots T_{\omega,m} \psi_0 \|
\leq \prod_{j=l_2 n_0+1}^{n} \| T_{\omega,j} \| \cdot \| T_{\omega,n} \cdots T_{\omega,m} \psi_0 \|
\leq C_1 \| T_{\omega,n} \cdots T_{\omega,m} \psi_0 \|,
\]
for some $C_1 = C_1(I) > 0$. The rest of the proof is based on a careful conditioning that we now detail. Let
\[
\psi_1 = \frac{T_{l_1 n_0} \cdots T_m \psi_0}{\| T_{l_1 n_0} \cdots T_m \psi_0 \|},
\]
and observe that $\psi_{l-1}$ is measurable with respect to $F_{l-1} = \sigma(\omega_0, \cdots, \omega_{l-1})$. Hence, Lemma 4.5 can be applied to obtain
\[
E \left[ \| T_{\omega,l_2 n_0} \cdots T_{\omega,(l-1)n_0+1} \psi_{l-1} \|^{-s} \right| F_{l-1} \right] \leq 1 - \frac{c_4}{j^{2\alpha}},
\]
where $c_4 = c_4(I) > 0$ is the constant from Lemma 4.5. Hence,
\[
E \left[ \| T_{\omega,n} \cdots T_{\omega,m} \psi_0 \|^{-s} \right] \leq C_1 E \left[ \| T_{\omega,l_2 n_0} \cdots T_{\omega,m} \psi_0 \|^{-s} \right]
= C_1 E \left[ \| T_{\omega,(l_2-1)n_0} \cdots T_{\omega,m} \psi_0 \|^{-s} \| T_{\omega,l_2 n_0} \cdots T_{\omega,(l_2-1)n_0+1} \psi_{l_2-1} \|^{-s} \right]
= C_1 E \left[ \| T_{\omega,(l_2-1)n_0} \cdots T_{\omega,m} \psi_0 \|^{-s} \| T_{\omega,l_2 n_0} \cdots T_{\omega,(l_2-1)n_0+1} \psi_{l_2-1} \|^{-s} \left| F_{l_2-1} \right| \right]
= C_1 E \left[ \| T_{\omega,(l_2-1)n_0} \cdots T_{\omega,m} \psi_0 \|^{-s} E \left[ \| T_{\omega,l_2 n_0} \cdots T_{\omega,(l_2-1)n_0+1} \psi_{l_2-1} \|^{-s} \left| F_{l_2-1} \right| \right] \right]
\leq C_1 \left( 1 - \frac{c_4}{j^{2\alpha}} \right) E \left[ \| T_{\omega,(l_2-1)n_0} \cdots T_{\omega,m} \psi_0 \|^{-s} \right].
\]
Iterating, we get
\[
E \left[ \| T_{\omega,n} \cdots T_{\omega,m} \psi_0 \|^{-s} \right] \leq C_1 \prod_{j=l_1}^{l_2} \left( 1 - \frac{c_4}{j^{2\alpha}} \right) E \left[ \| T_{\omega,m+r_1} \cdots T_{\omega,m} \psi_0 \|^{-s} \right].
\]
Just as we did in the previous lemma, we have
\[
1 = \| T_{\omega,m}^{-1} \cdots T_{\omega,m+r_1}^{-1} T_{\omega,m+r_1} \cdots T_{\omega,m} \psi_0 \| \leq \prod_{j=m}^{m+r_1} \| T_{\omega,j} \| \cdot \| T_{\omega,m+r_1} \cdots T_{\omega,m} \psi_0 \|,
\]
so that, by Lemma 4.4,
\[
E \left[ \| T_{\omega,m+r_1} \cdots T_{\omega,m} \psi_0 \|^{-s} \right] \leq C_2,
\]
for some \( C_2 = C_2(I) > 0 \). Hence, using that \( 1 - z \leq e^{-z} \) for \( z \geq 0 \),

\[
E \left[ \| T_{\omega, n} \cdots T_{\omega, m} \psi_0 \|^{-s} \right] \leq C_3 \prod_{j=1}^{l_2} \left( 1 - \frac{c_4}{j^{2\alpha}} \right)
\leq C_3 \prod_{j=1}^{l_2} e^{-c_4 j^{-2\alpha}} \leq C_3 e^{c m^{1-2\alpha} - c\alpha m^{1-2\alpha}},
\]

for some suitable \( C_3 = C_3(I) > 0 \) and \( c = c(I) > 0 \).

The symmetric situation where \( m \leq 0 \) and \( n \leq m - n_0 \) is treated in the exact same way. If \( m \leq 0 \) and \( n \geq n_0 \), the analysis is essentially reduced to estimate \( T_{\omega, \lambda}(n, 0) \). Indeed, we notice that

\[
T_{\omega, \lambda}(n, 0) = T_{\omega, m}^{-1} \cdots T_{\omega, -1}^{-1} T_{\omega, \lambda}(n, m),
\]

so that, by Lemma 4.4,

\[
\| T_{\omega, \lambda}(n, m) \|^{-s} \leq C_4 \| T_{\omega, \lambda}(n, 0) \|^{-s},
\]

for some \( C_4 = C_4(I) > 0 \). The right-hand side is covered by our previous discussion. The case \( m \geq 0 \) and \( n \leq -n_0 \) is of course similar. In all the remaining cases, we simply use the a priori bound of Lemma 4.4 so that

\[
E \left[ \| T_{\omega, \lambda}(n, m) \psi_0 \|^{-s} \right] \leq E \left[ \| T_{\omega, \lambda}(n, m) \|^{s} \right] \leq M |n - m|,
\]

where \( M = M(I) \) was defined in Lemma 4.4.

\( \square \)

**Proof of Theorem 4.1.** Use Lemma 4.2 to bound the fractional moments of the Green’s function by the negative fractional moments of the norm of transfer matrices. These can be estimated by Lemma 4.6.

\( \square \)

5. **Proof of Dynamical Localization**

We outline the theory developed in [2, Section 2] to relate the fractional moments of the Green’s function to the correlator (2.4), with some one-dimensional adaptations from [30]. Since our potential is not ergodic, some care has to be taken to insure that the estimates remain uniform enough. We provide details when this is required. The proof of Theorem 2.4 is given at the end of the section.

To simplify the notation, we denote \( H_{\omega, L} = H_{\omega, [-L, L]} \) the restriction of \( H_{\omega, \lambda} \) to the box \([-L, L]\), and we let \( G_{\omega, L}(E) = (H_{\omega, L} - E)^{-1} \) its resolvent. For an energy interval \( I \), we consider the restricted correlator to the box \([-L, L]\) defined by

\[
Q_{\omega, L}(x, y; I) = \sup_{f \in C_{x}} \frac{\| \chi_x f(H_{\omega, L}) \chi_y \|}{\| f \|},
\]

where, once again, we dropped the dependence on \( \lambda \) to lighten the notation. The following is the main result of this section.
Theorem 5.1. Let $0 < v < s < 1$ and let $I \subset (0, \infty)$ be a compact interval. Then, there exists a constant $C = C(v, s, I)$ such that
\[
E |Q_{\omega,L}(x,m;I)| \leq C(\lambda a_m)^{\frac{v}{2}-1} \|\rho\|^{\frac{1}{2-v}} \|v\|^s \left(\int_I dE \|\chi_x G_{\omega,L}(E) \chi_m\|^s\right)^{\frac{s}{(2-v)s}},
\]
for all $m \in \mathbb{Z}$, $x \in \mathbb{R}$ and $L > 0$.

The proof, given at the end of the section, will use several reduction steps discussed below. We will need to work with the fractional eigenfunction correlator for which we introduce a family of perturbations of the finite volume operator $H_{\omega,L}$ so that
\[
H_{\omega,L}^{\mu,\xi} = H_{\omega,L} + \lambda a_m(\xi - \omega_m)u_m.
\]
This corresponds to setting the value of $\omega_m$ to $\xi$. By the general theory summarized in [2, Appendix B], we know that the eigenvalues $(E_n)_n$ of these operators and their corresponding normalized eigenfunctions $(\varphi_n)_n$ can be chosen analytically in the parameter $z = \lambda a_m(\xi - \omega_m)$. We will sometimes denote $E_n = E_n(z)$ and $\varphi_n(\cdot) = \varphi_n(z)(\cdot)$ to stress this dependence. We also call $\Gamma_n$ the inverse of the function $z \mapsto E_n(z)$ which is shown to be well defined [2]. Note that $|\Gamma_n(E)| \leq 2 \lambda a_M M$ where $M = \max\{|\omega^-|, |\omega^+|\}$.

For $v \in [0,2]$, we define the $v$-fractional eigenfunction correlator as
\[
Q_{\omega,L}(x,m;I,v) = \sum_{n:E_n \in I} \langle \chi_x \varphi_n, \varphi_n \rangle^{v/2} (u_m \varphi_n, \varphi_n)^{1-v/2},
\]
where $(E_n)_n$ and $(\varphi_n)_n$ are chosen with the conventions above. Note that
\[
Q_{\omega,L}(x,m;I,0) = \text{Tr} (u_m P_I(H_{\omega,L})),
\]
\[
Q_{\omega,L}(x,m;I,2) = \text{Tr} (\chi_x P_I(H_{\omega,L})).
\]
The next lemma allows us to control the correlator in (5.1) through the fractional correlator (5.4) with $v = 1$. We use the arguments of [30] to bypass a certain covering condition imposed on the single-site potential in [2].

Lemma 5.2. There exists a finite constant $c$ such that
\[
Q_{\omega,L}(x,m;I) \leq c Q_{\omega,L}(x,m;I,1),
\]
for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$.

Proof. Note that
\[
Q_{\omega,L}(x,m;I) \leq \sum_{n:E_n \in I} \|\chi_x P_{\varphi_n} \chi_m\| = \sum_{n:E_n \in I} \|\chi_x \varphi_n\| \|\varphi_n \chi_m\|,
\]
where $P_{\varphi_n} := |\varphi_n\rangle \langle \varphi_n|$ denotes the projector on the subspace spanned by $\varphi_n$. It follows from the hypothesis on the single-site potential together with Lemmas A.1 and A.2 that
\[
\|u_m^{1/2} \varphi_n\|^2 \geq c_u \|\chi_{J+m} \varphi_n\|^2 \geq c_1 (|\varphi_n(m)|^2 + |\varphi'_n(m)|^2) \geq c_2 \|\chi_m \varphi_n\|^2,
\]
for some constants $c_1$ and $c_2$ which are bounded away from 0, uniformly in $m$. Hence,
\[ Q_{\omega,L}(x, m; I) \leq c \sum_{n:E_n \in I} \|\chi_x \varphi_n\| \|u_m^{1/2} \varphi_n\| = c Q_{\omega,L}(x, m; I, 1), \]
for some finite $c$ and for all $m$ and $x$. \hfill \square

By the interpolation bound [2, Lemma 2.1], we have
\[ \mathbb{E}[Q_{\omega,L}(x, m; I, 1)] \leq \mathbb{E}[Q_{\omega,L}(x, m; I, v)]^{1/(2-v)} \mathbb{E}[Q_{\omega,L}(x, m; I, 2)]^{(1-v)/(2-v)}. \] (5.8)

We notice that the correlator $Q_{\omega,L}(x, m; I, 2)$ in (5.6) can be bounded deterministically. Indeed, if $I \subset (-\infty, E]$, then we have
\[ Q_{\omega,L}(x, m; I, 2) \leq \text{Tr} \left( \chi_x P_{(-\infty,E]}(H_{\omega,L}) \right) \]
\[ \leq (|E| + B)^p \text{Tr} \left( \chi_x (H_{\omega,L} + B)^{-p} \right), \] (5.10)
which is finite for $B$ and $p$ large enough, uniformly in $x$.

The following is the key identity to relate the correlator to the fractional moments of the resolvent.

**Theorem 5.3** ([2], Thm 2.1). If $\xi \neq \omega_m$ and $\sigma(H_{\omega,L}) \cap \sigma(H_{\omega,L}^{m,\xi}) \cap I = \emptyset$, then
\[ Q_{\omega,L}(x, m; I, v) = \sum_n \int_I dE \delta(\Gamma_n(E) + \lambda a_m(\xi - \omega_m))|\Gamma_n(E)|^v \]
\[ \times \|\chi_x (H_{\omega,L}^{m,\xi} - E)^{-1} u_m^{1/2} \psi_n(E)\|_v \|\psi_n(E)\|^{-v}, \] (5.11)
where
\[ \psi_n(E)(\cdot) = u_m^{1/2} \varphi_n(\Gamma_n(E))(\cdot). \] (5.12)
Furthermore, for any $E$ and $a < b$ such that $E$ is not an eigenvalue of $H_{\omega,L}^{m,a}$ or $H_{\omega,L}^{m,b}$,
\[ \int_a^b d\omega_m \sum_n \delta(\Gamma_n(E) + \lambda a_m(\xi - \omega_m)) = \lambda^{-1} a_m^{-1} \left( \text{Tr} P_E(H_{\omega,L}^{m,a}) - \text{Tr} P_E(H_{\omega,L}^{m,b}) \right), \] (5.14)
where $P_E$ is the projection on $(-\infty, E]$.

Note that, as we consider absolutely continuous environments, the hypothesis of the theorem is almost surely satisfied for each choice of the parameters (see, for instance, [2, Lemma B.2]). If we take $a = \omega^-$ and $b = \omega^+$ in the right-hand side of (5.14), we recover the spectral shift
\[ S_{\omega,m}(L, E) = \text{Tr} P_E(H_{\omega,L}^{m,\omega^+}) - \text{Tr} P_E(H_{\omega,L}^{m,\omega^-}). \] (5.15)
By a combination [12, Theorem 2.1 and Proposition 5.1], we know that $S_{\omega,m}(L, E)$ has finite moments of order $p \geq 1$ with respect to the Lebesgue measure, uniformly in $\omega$ and $m$. The uniformity in $m$ can be seen from [12, Formula (5.6)] where the dependence on the single-site potential is made explicit. We state this as a lemma.
Lemma 5.4. For all \( p \geq 1 \) and every compact interval \( I \subset (0, \infty) \), there exists a constant \( C = C(p, I) \) such that
\[
\int_I dE |S_{\omega,m}(L, E)|^p \leq C,
\]
for all \( L > 0, m \in \mathbb{Z} \) and \( \omega \in \Omega \).

The next lemma brings the relation between the fractional eigenfunction correlators and the fractional moments of the Green’s function. This finishes the proof of Theorem 5.1.

Lemma 5.5. For all \( v \in (0, 1) \), \( s \in (v, 1) \) and all compact interval \( I \subset (0, \infty) \), there exists a constant \( C = C(v, s, I) \) such that
\[
\mathbb{E} [Q_{\omega,L}(x; m, I, v)] \leq C(\lambda a_m)^{v-1} \| \rho \|_\infty \mathbb{E} \left[ \int_I dE \| \chi_x G_{\omega,L}(E) \chi_m \|^s \right]^{v/s},
\]
for all \( x \in \mathbb{R} \), \( m \in \mathbb{Z} \) and \( L > 0 \).

Proof. Let us denote by \( \mathbb{E}_m \) the expected value with respect to \( \omega_m \), which corresponds to integration against \( \rho(\omega_m) d\omega_m \) over the interval \( [\omega^-, \omega^+] \). Remember that \( H_{m,\xi}^m \) is independent of \( \omega_m \). Averaging (5.12) with respect to \( \mathbb{E}_m \), recalling that \( |\Gamma_n(E)| \leq 2\lambda a_m M \) with \( M = \max\{|\omega^-|, |\omega^+|\} \) and using (5.14),
\[
\mathbb{E}_m [Q_{\omega,L}(x; m, I, v)] \leq (2\lambda a_m M)^v \int_I dE \left( \int \omega_m \rho(\omega_m) \sum_n \delta(\Gamma_n(E) + \lambda a_m (\xi - \omega_m)) \right)
\times \| \chi_x (H_{m,\xi}^m - E)^{-1} u_m^{1/2} \|^v
\leq 2^v M^v (\lambda a_m)^{v-1} \| \rho \|_\infty \int_I dE S_m(L, E) \| \chi_x (H_{m,\xi}^m - E)^{-1} u_m^{1/2} \|^v.
\]
(5.16)

Integrating the inequality (5.16) with respect to \( \rho(\xi) d\xi \) and then with respect to \( \{\omega_n : n \neq m\} \), we obtain
\[
\mathbb{E} [Q_{\omega,L}(x; m, I, v)] \leq 2^v M^v (\lambda a_m)^{v-1} \| \rho \|_\infty \mathbb{E} \left[ \int_I dE S_m(L, E) \| \chi_x G_{\omega,L}(E) u_m^{1/2} \|^s \right]^{v/s}.
\]
(5.17)

Next, we apply H"older’s inequality to (5.17) with respect to \( \mathbb{P} \times dE \) to get
\[
\mathbb{E} [Q_{\omega,L}(x; m, I, v)] \leq 2^v M^v (\lambda a_m)^{v-1} \| \rho \|_\infty \mathbb{E} \left[ \int_I dE S_m(L, E)^{s/(s-v)} \right]^{(s-v)/s}
\times \mathbb{E} \left[ \int_I dE \| \chi_x G_{\omega,L}(E) u_m^{1/2} \|^s \right]^{v/s}
\leq C(\lambda a_m)^{v-1} \| \rho \|_\infty \mathbb{E} \left[ \int_I dE \| \chi_x G_{\omega,L}(E) \chi_m \|^s \right]^{v/s},
\]
where we used Lemma 5.4 in the last step. \( \square \)

We finish the proof of Theorem 5.1:
Proof of Theorem 5.1. The result follows from Lemma 5.2 together with the interpolation bound (5.8) and the uniform bound (5.10) on $Q_{\omega,L}(x,m;I,2)$, and Lemma 5.5.

Finally, we give the proof of our main result Theorem 2.4.

Proof of Theorem 2.4. Since $H_{\omega,L}$ converges to $H_{\omega,\lambda}$ in the strong resolvent sense as $L \to \infty$, we have

$$Q_{\omega,\lambda}(x,y;I) \leq \liminf_L Q_{\omega,L}(x,y;I).$$

By Fatou’s lemma and using that $Q_{\omega,\lambda}(m,n;I) \leq 1$, we have

$$\mathbb{E}[Q_{\omega,\lambda}(x,y;I)^2] \leq \liminf_L \mathbb{E}[Q_{\omega,L}(x,y;I)].$$

The result follows from Theorem 5.1 and the uniform bound of Theorem 4.1.

6. Proof of Theorem 2.5 and 2.6

Proof of Theorem 2.5. We will establish the lower bound for

$$\sqrt{\left|\phi_{\omega,E}(x)\right|^2 + \left|\phi'_{\omega,E}(x)\right|^2}$$

since the bound for $\|\chi_x \phi_{\omega,E}\|$ will then follow from Lemma A.2. Recall that we can reconstruct $\Psi_{\omega,E} = (\phi \phi')$ using the transfer matrices as $\Psi_{\omega,E}(x) = T_{\omega,\lambda}(x,0;E)\psi_0$ for some possibly random $\|\psi_0\| = 1$. This implies in particular that $\|\Psi_{\omega,E}(x)\| \geq \|T_{\omega,\lambda}(x,0;E)\|^{-1}$.

Assume $x > 0$, the opposite case being analogous. Using Lemma 3.2 with some $\vartheta_1 \neq \vartheta_2$,

$$\mathbb{P}\left[\|\Psi_{\omega,E}(x)\| \leq e^{-c_1|x|^{1-2\alpha}}\right] \leq \mathbb{P}\left[\|T_{\omega,\lambda}(x,0;E)\| \geq e^{c_1|x|^{1-2\alpha}}\right]$$

$$\leq e^{-2c_1|x|^{1-2\alpha}} \mathbb{E}\left[\|T_{\omega,\lambda}(x,0;E)\|^2\right]$$

$$\leq C_1(\vartheta_1,\vartheta_2)e^{-2c_1|x|^{1-2\alpha}} \{\mathbb{E}[R^2(x,\vartheta_1)] + \mathbb{E}[R^2(x,\vartheta_2)]\},$$

for some $C_1(\vartheta_1,\vartheta_2) > 0$. From the martingale decomposition (B.1), one has

$$R(x,\vartheta_1) = \prod_{j=1}^{[x]} \exp \left\{ \frac{\omega_j}{j^{\alpha}} A_j + \frac{\omega_j^2}{j^{2\alpha}} B_j + E_j \right\},$$

(6.1)

where $A_j$ and $B_j$ are independent of $\omega_j$ and bounded uniformly in $E \in I$, and $E_j = o(j^{-2\alpha})$, uniformly in $E \in I$. Hence, from standard estimates on the exponential moments of bounded centred random variables,

$$\mathbb{E}[R(x,\vartheta_1)^2] \leq \prod_{j=1}^{[x]} \left( 1 + \frac{C_2}{j^{2\alpha}} \mathbb{E}[\omega_j^2] + o(j^{-2\alpha}) \right) \leq e^{C_3x^{1-2\alpha}},$$

for some finite constants $C_2 = C_2(I)$ and $C_3 = C_3(I)$. The bound for $R^2(x,\vartheta_2)$ is of course similar. The result follows by Borel–Cantelli choosing $2c_1 > C_3$.
The upper bound is quite standard. Following, for instance, the proof of [13, Theorem 9.22], we obtain
\[ \|X_\omega \phi_{\omega,E}\| \leq C_\omega e^{-c_2 |x|^{1-2\alpha}}, \]
for some random almost surely finite $C_\omega > 0$ and deterministic $c_2 > 0$. We can use the lower bound we just proved to get a lower bound on $\|X_\omega \phi_{\omega,E}\|$ uniformly in $E \in I$.

**Proof of Theorem 2.6.** For the first statement, let $\kappa < 1 - 2\alpha$. Then,
\[
\mathbb{E} \left[ \sup_t \left\| e^{\frac{1}{2} |X|^\alpha} e^{-it H_{\omega,\lambda}} P_l(H_{\omega,\lambda}) \chi_m \right\|^2 \right] \\
= \mathbb{E} \left[ \sup_t \left\| e^{\frac{1}{2} |X|^\alpha} e^{-it H_{\omega,\lambda}} P_l(H_{\omega,\lambda}) \chi_m, e^{-it H_{\omega,\lambda}} \chi_m \right\| \right] \\
\leq \mathbb{E} \left[ \sup_t \sum_n \left| \left\langle e^{\frac{1}{2} |X|^\alpha} e^{-it H_{\omega,\lambda}} P_l(H_{\omega,\lambda}) \chi_m, \chi_n \right\rangle \right| \right] \\
\leq C \mathbb{E} \left[ \sup_t \sum_n e^{n^\kappa} \left| \left\langle \chi_n, e^{-it H_{\omega,\lambda}} P_l(H_{\omega,\lambda}) \chi_m \right\rangle \right|^2 \right] \\
= C \mathbb{E} \left[ \sum_n e^{n^\kappa} Q_{\omega,\lambda}(m,n;I)^2 \right],
\]
for some finite $C > 0$. The last sum is finite for each $m$ in virtue of (2.5).

For the second statement, let $c_\omega$ and $c_1$ be as in Theorem 2.5. Let $(\psi_l)_l$ be a basis of $\text{Ran} P_l(H_{\omega,\lambda})$ given by normalized eigenfunctions of the operator $H_{\omega,\lambda}$ with corresponding eigenvalues $(E_l)_l$. Let $N \geq 1$. Taking $\kappa > 1 - 2\alpha$ and applying the lower bound in Theorem 2.5,
\[
\left\| e^{\frac{1}{2} |X|^\alpha} \chi_{[0,N]} \psi_l \right\|^2 = \int_0^N \overline{\psi_l(x)} e^{\frac{1}{2} |X|^\alpha} \psi_l(x) \, dx \geq \sum_{n=0}^{N-1} e^{n^\kappa} \int_n^{n+1} |\psi_l(x)|^2 \, dx \\
\geq C \sum_{n=0}^{N-1} e^{n^\kappa} c_\omega e^{-c_1 n^{1-2\alpha}} \geq c_\omega' e^{\frac{1}{2} N^\kappa},
\]
for some $C > 0$ and some suitable random quantity $c_\omega' > 0$. Let $\psi \in \text{Ran} \ P_l(H_{\omega,\lambda})$ and write $\psi = \sum_l a_l \psi_l$ with $\sum_l |a_l|^2 = 1$. Then,
\[
\left\| e^{\frac{1}{2} |X|^\alpha} e^{-it H_{\omega,\lambda}} \psi \right\|^2 = \sum_{l,l'} a_l \overline{a}_{l'} e^{-it E_l - E_{l'}} \left\langle \psi_{l'}, e^{\frac{1}{2} |X|^\alpha} \psi_l \right\rangle. \tag{6.2}
\]
A careful application of the dominated convergence theorem to exchange sums and integrals yields
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left\| e^{\frac{1}{2} |X|^\alpha} e^{-it H_{\omega,\lambda}} \psi \right\|^2 \, dt = \sum_l |a_l|^2 \left\| e^{\frac{1}{2} |X|^\alpha} \psi_l \right\|^2 \geq c_\omega' e^{\frac{1}{2} N^\kappa}. \tag{6.3}
\]
Hence, there exists an diverging (random) sequence \((T_N)_N\) such that
\[
\frac{1}{T_N} \int_0^{T_N} \left\| e^{\frac{1}{2}|x|^\kappa} e^{-itH\omega,\lambda} \psi \right\|^2 dt \geq \frac{c'_\omega}{2} e^{\frac{1}{2}N^\kappa},
\] (6.4)
for all \(N \geq 1\). From here, we can find a diverging (random) sequence \((t_N)_N\) such that
\[
\left\| e^{\frac{1}{2}|x|^\kappa} e^{-it_N H\omega,\lambda} \psi \right\|^2 \geq \frac{c'_\omega}{4} e^{\frac{1}{2}N^\kappa},
\] (6.5)
for all \(N \geq 1\). This finishes the proof. □

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Appendix A. General Estimates

We quote two lemmas from [30] that were used repeatedly in the proofs.

Lemma A.1 ([30], Lemma A.1) For every \(q \in L^1_{\text{loc}}(\mathbb{R})\), every \(c < d\) and every solution of \(-\Delta \phi + q \phi = 0\) on \([c,d]\), we have
\[
\left( |\phi(c)|^2 + |\phi'(c)|^2 \right) \exp \left( - \int_c^d |1 + q(x)| \, dx \right) \leq \left( |\phi(d)|^2 + |\phi'(d)|^2 \right)
\]
\[
\leq \left( |\phi(c)|^2 + |\phi'(c)|^2 \right) \exp \left( \int_c^d |1 + q(x)| \, dx \right).
\]

Lemma A.2 ([30], Lemma A.2) For any positive numbers \(l\) and \(M\), there exists \(C > 0\) such that
\[
\int_c^{c+l} |\phi(t)|^2 \, dt \geq C \left( |\phi(c)|^2 + |\phi'(c)|^2 \right),
\] (A.1)
for all \(c \in \mathbb{R}\), all \(q \in L^1_{\text{loc}}(\mathbb{R})\) such that \(\int_c^{c+l} |1 + q(x)| \, dx \leq M\) and every solution of \(-\Delta \phi + q \phi = 0\) on \([c,c+l]\).

Appendix B. The Martingale Decomposition

We briefly describe the martingale analysis of [35, Section 9] needed in Lemma 3.3. Note that the analysis of [35] is almost sure. We only require a version in expectation.

Let \(I \subset \mathbb{R}\) be a compact interval and, for \(E \in I\), set \(k = \sqrt{E}\). Letting \(\tilde{\theta}_n(y) = \theta(n) + ky\), it can be showed that
\[
\log \frac{R(n)}{R(m)} = \sum_{j=m}^{n-1} \frac{\lambda \omega_j}{2k^\alpha} \int_0^1 u(y) \sin (2\tilde{\theta}_j(y)) \, dy
\]
\[
- \sum_{j=m}^{n-1} \frac{\lambda^2 \omega_j^2}{4k^2j^{2\alpha}} \int_0^1 u(y) \left( \int_0^y u(t) \, dt \right) \cos 2\tilde{\theta}_j(y) \, dy
\]
\[ + \sum_{j=m}^{n-1} \frac{\lambda^2 \omega_j^2}{8k^2 j^{2\alpha}} \left| \int_0^1 u(y)e^{2iky} \, dy \right|^2 \cos(4(\theta(j) - \nu_k)) \tag{B.1} \]

\[ + \sum_{j=m}^{n-1} \frac{\lambda^2 \omega_j^2}{8k^2 j^{2\alpha}} \left| \int_0^1 u(y)e^{2iky} \, dy \right|^2 + K_\omega(m, n), \]

for some \( \nu_k \in [0, 2\pi) \) and where

\[ |K_\omega(m, n)| = o \left( \sum_{j=m}^{n-1} j^{-2\alpha} \right), \tag{B.2} \]

uniformly in \( E \in I \) and \( \omega \). (The bound is indeed deterministic.) From (3.5), we can see that \( \{\bar{\theta}_j(y) : y \geq 0\} \) and \( \omega_j \) are independent. Recalling that the \( \omega_j \)'s are centred and satisfy \( \mathbb{E}[\omega_j^2] = 1 \), we can integrate (B.1),

\[
\mathbb{E} \left[ \log \frac{R(n)}{R(m)} \right] = -\frac{\lambda^2}{4k^2} \mathbb{E} \left[ \sum_{j=m}^{n-1} \frac{1}{j^{2\alpha}} \left| \int_0^1 u(y) \left( \int_0^y u(t) \, dt \right) \cos 2\bar{\theta}_j(y) \, dy \right|^2 \right] \\
+ \frac{\lambda^2}{8k^2} \mathbb{E} \left[ \sum_{j=m}^{n-1} \frac{1}{j^{2\alpha}} \left| \int_0^1 u(y)e^{2iky} \, dy \right|^2 \cos(4(\theta(j) - \nu_k)) \right] \\
+ \frac{\lambda^2}{8k^2} \sum_{j=m}^{n-1} \frac{1}{j^{2\alpha}} \left| \int_0^1 u(y)e^{2iky} \, dy \right|^2 + \mathbb{E} [\omega_\omega(m, n)].
\]

The deterministic analysis of [35, Section 9] allows us to control the first two sums on the right-hand side to show that they are \( o \left( \sum_{j=m}^{n-1} j^{-2\alpha} \right) \), uniformly in \( E \in I \) such that \( 4\sqrt{E} \notin \pi \mathbb{Z} \). This shows (3.7).

We now prove the second moment bound needed to complete the proof of the second statement in Lemma 3.3. From the decomposition (B.1), we can find a constant \( C_1 > 0 \) such that

\[
\mathbb{E} \left[ \left( \log \frac{R(n)}{R(m)} \right)^2 \right] \leq \mathbb{E} \left[ \left( \sum_{j=m}^{n-1} \frac{\lambda\omega_j}{2kj^\alpha} \int_0^1 u(y) \sin(2\bar{\theta}_j(y)) \, dy \right)^2 \right] + C_1 \left( \sum_{j=m}^{n-1} j^{-2\alpha} \right)^2,
\]

for all \( E \in I \). Using the fact that the random variables \( (\omega_j)_j \) are independent and centered, and that \( \{\bar{\theta}_j(y) : y \in [0, 1]\} \) is independend of \( (\omega_t)_{t \geq j} \), we obtain that

\[
\mathbb{E} \left[ \left( \sum_{j=m}^{n-1} \frac{\lambda\omega_j}{2kj^\alpha} \int_0^1 u(y) \sin(2\bar{\theta}_j(y)) \, dy \right)^2 \right] = \mathbb{E} \left[ \sum_{j=m}^{n-1} \frac{\lambda^2 \omega_j^2}{4k^2 j^{2\alpha}} \left( \int_0^1 u(y) \sin(2\bar{\theta}_j(y)) \, dy \right)^2 \right] \\
\leq C_2 \sum_{j=m}^{n-1} j^{-2\alpha},
\]
for some constant $C_2 > 0$ and all $E \in I$. These two last estimates together imply that
\[
E \left[ \left( \log \frac{R(n)}{R(m)} \right)^2 \right] \leq C_3 \left( \sum_{j=m}^{n-1} j^{-2\alpha} \right)^2,
\]
for some constant $C_3 > 0$ and all $E \in I$. This finishes the proof of Lemma 3.3.

Appendix C. Pure Point Spectrum

**Proposition C.1.** Assume dynamical localization for $H_{\omega,\lambda}$ holds in the sense of (2.5) in an energy interval $I \subset \mathbb{R}$. Then, the spectrum of $H_{\omega,\lambda}$ is almost surely pure point in $I$.

**Proof.** This is a consequence of the RAGE Theorem [13]. Suppose that (2.5) holds in an energy interval $I$ and consider $\chi_R$, the projector on the box $[-R, R]$. As $\chi_R$ converges strongly to the identity, it is enough to show that, $\mathbb{P}$-almost surely,
\[
\lim_{R \to \infty} \sup_t \left\| (1 - \chi_R) e^{-itH_{\omega,\lambda}} P_I(H_{\omega,\lambda}) \chi_m \right\|^2 = 0, \tag{C.1}
\]
for all integer $m$ as this implies that the range of $P_I(H_{\omega,\lambda})$ is almost surely included in the point spectrum of $H_{\omega,\lambda}$. Now,
\[
\left\| (1 - \chi_R) e^{-itH_{\omega,\lambda}} P_I(H_{\omega,\lambda}) \chi_m \right\|^2 \leq \sum_{|n| > R} \left| \langle \chi_n, e^{-itH_{\omega,\lambda}} P_I(H_{\omega,\lambda}) \chi_m \rangle \right|^2 = \sum_{|n| > R} \left| \langle P_I(H_{\omega,\lambda}) e^{itH_{\omega,\lambda}} \chi_n, \chi_m \rangle \right|^2 \leq \sum_{|n| > R} Q_\omega(m, n; I)^2,
\]
for all $t \in \mathbb{R}$. By Fatou’s lemma,
\[
E \left[ \lim_{R \to \infty} \sup_t \left\| (1 - \chi_R) e^{-itH_{\omega,\lambda}} P_I(H_{\omega,\lambda}) \chi_m \right\|^2 \right] \leq E \left[ \liminf_{R \to \infty} \sum_{|n| > R} Q_\omega,\lambda(m, n; I)^2 \right] \leq \liminf_{R \to \infty} E \left[ \sum_{|n| > R} Q_\omega,\lambda(m, n; I)^2 \right] = 0,
\]
which shows that (C.1) holds $\mathbb{P}$-almost surely for each $m$. This is, for each $m$, there exists $\Omega_m \subset \Omega$ with $\mathbb{P}[\Omega_m] = 1$ such that (C.1) holds for all $\omega \in \Omega_m$. Finally, the set $\tilde{\Omega} = \bigcap_m \Omega_m$ is such that $\mathbb{P}[\tilde{\Omega}] = 1$ and such that (C.1) holds for all $m$ simultaneously for all $\omega \in \tilde{\Omega}$. \[\square\]
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