Asymptotic states in brane cosmology with a nonlocal anisotropic stress

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Abstract

We investigate the dynamics of a Bianchi I brane Universe in the presence of a nonlocal anisotropic stress $P_{\mu\nu}$ proportional to a "dark energy" $\mathcal{U}$. Using this ansatz for the case $\mathcal{U} > 0$ we prove that if a matter on a brane satisfies the equation of state $p = (\gamma - 1)\rho$ with $\gamma \leq 4/3$ then all such models isotropize. For $\gamma > 4/3$ anisotropic future asymptotic states are found. We also describe the past asymptotic regimes for this model.

1 Introduction

The idea that our space has more than three dimensions was attractive during many years, mostly in the scenario of compactification. Recently a new paradigm appeared. In this new approach the matter is trapped in three-dimensional brane embedded in a multi-dimensional space (bulk) [1]. In the case of one extra dimension it is shown that isotropic cosmological evolution being unusual in early stages of the brane Universe [2] tends to the standard cosmological scenario in the late time [3].

The field equations on the brane are [4, 5]

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - \varepsilon_{\mu\nu}.$$  \hspace{1cm} (1)

Here $\tilde{\kappa}$ is the fundamental 5-dimensional gravitational constant, $\kappa$ is the effective 4-dimensional gravitation constant on the brane. Bulk corrections to the Einstein equations on the brane are of two forms: there are quadratic energy-momentum corrections via the tensor $S_{\mu\nu}$ and nonlocal effects from the free gravitational field in the bulk $\varepsilon_{\mu\nu}$. The quadratic corrections are significant only in the early stages of cosmological evolution (see the next section). The nonlocal corrections can be decomposed into a scalar (nonlocal energy density), a vector (nonlocal energy flux) and a tensor (nonlocal anisotropic stress) parts [5]. On an isotropic brane there is only a scalar part $\mathcal{U}$ which decays during the expansion of the Universe as $\mathcal{U} \sim a^{-4}$ where $a$ is the scale factor.
However, if we remove the suggestion of isotropy (for example, in order to study the problem of isotropization of the brane Universe), then the tensor part of nonlocal contribution $P_{\mu\nu}$ should be taken into account. The main obstacle to do this is the absence of an evolution equation for the anisotropic stress $P_{\mu\nu}$ \([5, 6]\). Its correct form should be derived from a full 5-dimensional anisotropic metric which is still unknown. In this situation we have to choose some ansatz for $P_{\mu\nu}$. In first papers on Bianchi I brane either all nonlocal corrections \([6, 7]\) or its tensor part \([8, 9, 10]\) have been neglected. This approach gives some interesting results such as isotropic character of the initial singularity \([6, 7]\) and the possible existence of an intermediate anisotropic stage \([9]\). However, whether these results remain valid when the contribution from $P_{\mu\nu}$ becomes important is still an open problem.

Recently Barrow and Maartens have proposed an ansatz for the nonlocal stress which generalizes known anisotropic sources in General Relativity \([11]\). They have studied the dynamics of isotropization in the limit of small anisotropy. In the present paper we use this ansatz for describing the global properties of the dynamics. We use the techniques of dynamical systems and the expansion normalized variables (for the description of this formalism see \([12]\)) to find the past and the future attractors of our system. The transient behavior is also covered by studying the future asymptotic states for a pure nonstandard (i.e. with $S_{\mu\nu}$ and without $T_{\mu\nu}$ terms in (1)) brane dynamics. Of course, in this framework we can not find regimes originated from an interplay between standard (proportional to $T_{\mu\nu}$) and nonstandard (proportional to $S_{\mu\nu}$) source terms. Up to now several interesting regimes of this type have been found (for example, an analog of the Einstein static Universe \([13]\) and a stable oscillationary regime \([8]\)).

The paper is organized as follows: in Sec.2. we describe the dimensionless variables and use them to write down the system of differential equations in a convenient form. In Sec.3 we list stable points of this system neglecting the square energy-momentum correction and describe the future asymptotic states of the Bianchi I brane model. In Sec.4 the stable points of the system without linear energy-momentum contribution are listed and possible transient behavior is obtained. We also discuss the past asymptotic states of our model. Sec.5 provides a brief summary of the results obtained.

2 Expansion normalized variables and equation of motion

We will use an orthonormal frame of vector fields $e_a$ such that $e_0 = \partial_t$. In the Bianchi I case we have the following commutation relations for the vector basis (Latin indices run from 0 to 3 and Greek indices from 1 to 3):

\[
[e_a, e_b] = C_{ab}^c e_c
\]

where
\[ C^\alpha_{\alpha\beta} = -\Theta^\alpha_{\beta} - \epsilon^\alpha_{\alpha\beta} \Omega^\alpha, \quad C^\alpha_{\alpha\beta} = 0. \]

Here \( \Omega^\alpha \) is the angular velocity of the spatial triad with respect to a Fermi-propagated frame and \( \Theta_{\mu\nu} \) can be decomposed into a volume expansion rate \( \Theta \) and a shear \( \sigma_{\mu\nu} \):

\[ \Theta_{\mu\nu} = \frac{1}{3} \Theta \delta_{\mu\nu} + \sigma_{\mu\nu}. \]

If we introduce the metric for the Bianchi I model in the standard form

\[ ds^2 = -dt^2 + a_i^2(t)(dx^i)^2, \]

then the volume expansion rate in terms of the main scale factor \( a = (a_1 a_2 a_3)^{1/3} \) is \( \Theta = 3H = \frac{6\dot{a}}{a} \), where \( H \) is the main Hubble parameter.

For these variables we can write the Raychaudhuri equation

\[ \dot{\Theta} + \frac{1}{3} \Theta^2 + \sigma^{\alpha\nu} \sigma_{\alpha\nu} + \frac{1}{2} \kappa^2 (\rho + 3p) = -\frac{1}{2} \kappa^2 (2\rho + 3p) \frac{\rho}{\lambda} - \frac{6\dot{a}}{\kappa^2 \lambda}, \]

the Gauss-Codazzi equations

\[ \dot{\sigma}_{\mu\nu} + \Theta \sigma_{\mu\nu} = \frac{6}{\kappa^2 \lambda} P_{\mu\nu}, \]

\[ -\frac{2}{3} \Theta^2 + \sigma^{\mu\nu} \sigma_{\mu\nu} + 2\kappa^2 \rho = -\kappa^2 \frac{\rho^2}{\lambda} - \frac{12\dot{a}}{\kappa^2 \lambda} \]

and the conservation equations

\[ \dot{\rho} + \Theta (\rho + p) = 0, \]

\[ \dot{\mathcal{U}} + \frac{4}{3} \Theta \mathcal{U} + \sigma^{\mu\nu} P_{\mu\nu} = 0, \]

\[ D^\nu P_{\mu\nu} = 0. \]

Derivation of these equations from Eq. (1) and the explicit form of \( S_{\mu\nu} \) see in [5]. Here \( \lambda \) is the brane tension. Quadratic energy-momentum corrections (first terms in RHS of Eqs. (3) and (5)) become significant if a matter density \( \rho \) on the brane is greater than \( \lambda \). The current experimental limit on the brane tension is \( \lambda > 100(GeV)^4 \) [14].

Due to presence of the nonlocal stress \( P_{\mu\nu} \) this system of equations is not closed. Therefore, it is necessary to specify the form of \( P_{\mu\nu} \) using some additional suggestion. In the present paper we deal with the proposal of Barrow and Maartens [11]. They have suggested that the nonlocal stress is proportional to the nonlocal energy density

\[ P_{\mu\nu} = 2D_{\mu\nu} \mathcal{U}, \]

\[ D_{\mu\nu} = \text{diag}\{c_1, c_2, -(c_1 + c_2)\}, \]
where \( c_1, c_2 \) are some constant.

It is worth to notice that choosing the appropriate coordinate system we can diagonalize either \( \mathcal{P}_{\mu \nu} \) or \( \sigma_{\mu \nu} \). We choose to make \( \mathcal{P}_{\mu \nu} \) diagonal.

Nevertheless, our system of equations is not completely determined yet. We must fix the angular velocity \( \Omega^\alpha \) of the spatial triad associated with \( \mathcal{P}_{\mu \nu} \). After that we can write the derivatives of the shear tensor as

\[
\dot{\sigma}_{\mu \nu} = \partial_t \sigma_{\mu \nu} + 2\sigma_{(\mu, \epsilon)}^{\delta} \gamma^\delta \Omega^\gamma.
\]

In the GR description of cosmological anisotropic sources (for example, the homogeneous magnetic field [15]) the angular velocity components are constrained by the corresponding equations of motion of the anisotropic fluid (for example, the Maxwell equations in the magnetic field case). As we have no equation for \( \mathcal{P}_{\mu \nu} \), we should choose an ansatz for \( \Omega^\alpha \). In the present paper we consider the simplest possible form \( \Omega^\alpha = 0 \) (in the end of the next section one important statement not depending on \( \Omega^\alpha \) is proved).

After introduction of the deceleration parameter

\[
q = \frac{\dot{a}a}{a^2},
\]

and dimensionless variables

\[
\Sigma^+ = \frac{3}{2}(\sigma_{22} + \sigma_{33})/\Theta,
\]

\[
\Sigma^- = \frac{1}{2}\sqrt{3}(\sigma_{22} - \sigma_{33})/\Theta,
\]

\[
\Sigma_{12} = \sqrt{3}\sigma_{12}/\Theta,
\]

\[
\Sigma_{23} = \sqrt{3}\sigma_{23}/\Theta,
\]

\[
\Sigma_{13} = \sqrt{3}\sigma_{13}/\Theta,
\]

\[
U = \frac{18\mathcal{U}}{\kappa^2 \lambda \Theta^2},
\]

we may rewrite our system in a simpler form.

We have also auxiliary evolution equation for the matter density. There are two ways for description of the quadratic energy-momentum terms. It is possible either to incorporate them into a single evolution equation for the matter as it have been done in [10] or to consider separately as some kind of a particular effective matter [7, 8]. We choose the latter case and write two evolution equations - one for "standard" matter and the other for "nonstandard" matter. In order to do this we introduce dimensionless variables

\[
\Omega_\lambda = \frac{3\kappa^2 \rho^2}{2\lambda \Theta^2},
\]

\[
\Omega_\mu = \frac{3\kappa^2 \rho}{\Theta^2}.
\]
Note that these values are not independent, they connected to each other by $\Omega_\Lambda/\Omega_\mu = \rho/2\lambda$.

Finally, we introduce a new time variable $\tau$ such that $\frac{d}{d\tau} = \frac{3}{U}$. Now our system becomes (prime denotes a derivative with respect to $\tau$)

$$U' = 2U[(q - 1) - 3\Sigma_+(c_1 + c_2) - \sqrt{3}\Sigma_-(c_1 - c_2)],$$

$$\Sigma_+' = 3U(c_1 + c_2) + \Sigma_+(q - 2),$$

$$\Sigma_-' = \sqrt{3}U(c_1 - c_2) + \Sigma_-(q - 2),$$

$$\Sigma_{12}' = (q - 2)\Sigma_{12},$$

$$\Sigma_{13}' = (q - 2)\Sigma_{13},$$

$$\Sigma_{23}' = (q - 2)\Sigma_{23}$$

with the constraint

$$1 = \Sigma^2 + \Omega_\Lambda + \Omega_\mu + U,$$

the equation for $q$

$$q = 2\Sigma^2 + \frac{1}{2}(3\gamma - 2)\Omega_\mu + (3\gamma - 1)\Omega_\Lambda + U,$$

and the evolution equations for matter density

$$\Omega_\mu' = [2(1 + q) - 3\gamma]\Omega_\mu,$$

$$\Omega_\Lambda' = [2(1 + q) - 6\gamma]\Omega_\Lambda.$$

Here

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 + \Sigma_{12}^2 + \Sigma_{23}^2 + \Sigma_{13}^2.$$
\[ \Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23} \text{ varying in the range } [-1, 1] \text{ and } \Omega_\mu, \Omega_\lambda, U \text{ varying in the range } [0, 1]. \]

In the next section we start to analyze the system (13) – (17) neglecting the "nonstandard" matter. All the future attractors for this system are also the future attractors for the initial general problem, because for \( t \to \infty \) we can neglect a nonstandard matter on the brane in comparison with a standard one.

3 Equilibrium points and future evolution

Now we list equilibrium points for the system (13) – (17) in the case \( \Omega_\lambda = 0 \). It is convenient to use the notations \( c_+ = c_1 + c_2 \) and \( c_- = c_1 - c_2 \). We also have found the corresponding eigenvalues \( \lambda_i \) of the linearization of system (13) about the equilibrium points and have used the condition \( \lambda_i < 0 \) to study the stability of the equilibrium points in the future direction.

1. The isotropic Universe

\[ U = \Sigma_+ = \Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0 \]

\[ q = \frac{3}{2} \gamma - 1 \quad \Omega_\mu = 1 \quad 0 \leq \gamma \leq 2 \]

Stable for \( \gamma < 4/3 \) independently on \( c_+, c_- \).

2. Equilibrium point

\[ U = \frac{(2 - \gamma)(3\gamma - 4)}{12c_+^2 + 4c_-^2} \quad \Sigma_+ = \frac{c_+(3\gamma - 4)}{6c_+^2 + 2c_-^2} \]

\[ \Sigma_- = \frac{\sqrt{3}c_-(3\gamma - 4)}{18c_+^2 + 6c_-^2} \quad \Omega_\mu = 1 - \frac{3\gamma - 4}{18c_+^2 + 6c_-^2} \]

\( \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0 \)

\( \gamma \in (4/3, 2) \)

Stable for \( \gamma < 4/3 + 6c_+^2 + 2c_-^2 \).

3. Equilibrium point

\[ U = 1 - 9c_+^2 - 3c_-^2 \quad \Sigma_+ = 3c_+ \quad \Sigma_- = \sqrt{3}c_- \]

\( \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0 \)

\[ \Omega_\mu = 0 \quad 3c_+^2 + c_-^2 < 1/3 \]

Stable for \( \gamma > 4/3 + 6c_+^2 + 2c_-^2 \).
Figure 1: Stability zones of the equilibrium points for the case $\Omega_\lambda = 0$. Below the plane the point 1 is stable, above the plane outside the compact region the point 2 is stable, inside the compact region the point 3 is stable.

4. Equilibrium set (the Kasner Universe)

$$U = 0 \quad \Omega_\mu = 0 \quad q = 2$$

$$\Sigma_+^2 + \Sigma_-^2 + \Sigma_{12}^2 + \Sigma_{13}^2 + \Sigma_{23}^2 = 1$$

The results on stability of the equilibrium points are presented graphically in Fig.1. The space $(\gamma, c_+, c_-)$ is divided into several zones in which only one stable future attractor exists.

We can see that the only stable point for $\gamma < 4/3$ is the isotropic one, independently on $c_+$ and $c_-$. It is possible to strengthen this result and prove that all the solutions with $\Omega_\mu > 0$ and $\gamma \leq 4/3$ tend to the isotropic solution ($\Omega_\mu = 0$ is an invariant subset of our phase space, as it can be seen from Eq. (16)). We point out that a physically important boundary case $\gamma = 4/3$ is included in the conditions of this statement. The key point is the equation

$$\Omega'_\mu = [(6 - 3\gamma)\Sigma^2 + (4 - 3\gamma)U]\Omega_\mu$$

(19)

which can be obtained from Eqs. (14) – (16). Note, that Eq.(19) follows only from the Raychaudhuri and constraint equations and we do not use the evolution equations for the shear. This means that the consideration presented below does not depend on a particular choice of the angular velocity $\Omega^\alpha$. 
First, consider the case $\gamma < 4/3$. Then there is a positive number $\alpha$ such that
\[ \Omega'_\mu \geq \alpha[\Sigma^2 + U]\Omega_\mu. \]

From this inequality it follows that the quantity $\Omega_\mu$ is strictly increasing if $\Omega_\mu(\tau_0) > 0$. Using this fact and the constraint $\Sigma^2 + \Omega_\mu + U = 1$, one can easily understand that the sum $\Sigma^2 + U$, in turn, strictly decreases. As decreasing and bounded below by zero, this sum has some limit $\beta \geq 0$. Assume that $\beta$ is not equal to zero. Then, the quantity $\Omega_\mu$ goes to infinity:
\[ \ln \Omega_\mu(\tau) - \ln \Omega_\mu(\tau_0) > \alpha\beta(\tau - \tau_0) \]
which contradicts the inequality $\Omega_\mu(\tau) \leq 1$. Thus, for any initial conditions satisfying $\Omega_\mu(\tau_0) > 0$ the solution tends to the point $\Sigma^2 = \Sigma - \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = U = 0$.

In the boundary case $\gamma = 4/3$ it is necessary to be more careful. For $\gamma = 4/3$ we have
\[ \Omega'_\mu = 2\Sigma^2\Omega_\mu. \]  

So, $\Omega_\mu$ is monotone increasing and, then, tends to some limit value $\Omega_{lim}$. Assume that $\Omega_{lim} < 1$. Using the constraint (14) and the evolution equations for $U$ and $\Omega_\mu$, we obtain the evolution equation for $\Sigma^2$:
\[ \frac{\partial}{\partial \tau} \Sigma^2 = 6(c_1 + c_2)U\Sigma_\mu + 2\sqrt{3}(c_1 - c_2)U\Sigma - 2\Sigma^2(\Sigma^2 - 1). \]

In sequel, we shall use only the fact that the absolute value of the LHS of this equation is bounded above: $|\frac{\partial}{\partial \tau} \Sigma^2(\tau)| < C$ for all $\tau$. Now assume that $\Sigma^2$ does not tend to zero. Then there is a sequence $\tau_n$ going to infinity such that $\Sigma^2(\tau_n) > c$ for some $c > 0$. But the boundedness of the shear square derivative means that there is a positive $b$ which does not depend on $n$ such that $\Sigma^2(\tau) > c/2$ for $\tau \in (\tau_n - b, \tau_n + b)$ and all $n$. In this case from equation (20) it follows that the quantity $\Omega_\mu$ goes to infinity:
\[ \ln \Omega_\mu(\tau) - \ln \Omega_\mu(\tau_0) = \int_{\tau_0}^{\tau} 2\Sigma^2 dt > \sum_{n: \tau_n < \tau} 2cb. \]

This fact contradicts the inequality $\Omega_\mu(\tau) \leq 1$. Thus, $\Sigma^2$ tends to zero. By the constraint $1 = \Sigma^2 + \Omega_\mu + U$, this means that the quantity $U$ converges to $1 - \Omega_{lim} > 0$. But it is impossible because the point $\Sigma^2 = 0$ and $U = \text{const} > 0$ is not an equilibrium point of our system for nonzero $c_1$ and $c_2$. Thus, $\Omega_\mu$ goes to 1 and both $U$ and $\Sigma^2$ decrease to zero.

For the brane problem this means that if $P_{\mu\nu}$ satisfies the ansatz (9) with $U > 0$, then an arbitrary anisotropy of the whole 5-dimensional metric does not destroy the future isotropic regime unless $\gamma > 4/3$.

It is interesting that in the case of $\gamma = 4/3$ without an anisotropic stress we have a one-dimensional set of equilibria instead of the isotropic equilibrium point. Points of this set have $\Sigma^2 = 0$ and are marked by the constant ratio $U/\Omega_\mu$. A nonzero $P_{\mu\nu}$ makes all these points except for $U = 0, \Omega_\mu = 1$ unstable. As a
result, in the presence of a nonlocal anisotropic stress the "dark energy" \( U \) can be washed out during the cosmological evolution even in the radiation-dominated epoch.

We finish this section by one remark which is not directly connected with the brane cosmology. As we consider the case of \( \Omega_\lambda = 0 \), the system of equations (13) – (16) with Barrow-Maartens ansatz for \( P_{\mu\nu} \) has the same form (with appropriate rescaling of source terms) as the system describing the evolution of the Universe in the presence of an anisotropic matter in the ordinary General Relativity. Since our result on isotropisation does not depend on angular velocity \( \Omega^a \) (these values should be determined separately for each particular anisotropic source), we can claim that in the presence of an anisotropic matter with an anisotropic stress \( \pi_{\mu\nu} \) proportional to its energy density \( \epsilon \) and an average pressure \( \bar{p} \) equals to \( \epsilon/3 \) all the models isotropize if \( \gamma \leq 4/3 \). Moreover, we can consider an anisotropic matter with an average pressure \( \bar{p} = (\gamma_{\text{eff}} - 1)\epsilon \) and, using the same procedure, claim that all such models with \( \gamma \leq \gamma_{\text{eff}} \) tend to the isotropic attractor. This fact was previously discovered in the particular case of the homogeneous magnetic field [15] and for the general source in the limit of a small anisotropy [16] (the "criticality condition" of Ref. [16] is exactly \( \gamma = \gamma_{\text{eff}} \) in our notation).

4 A pure brane regime

In this section we neglect the standard matter on a brane. This case describes a purely nonstandard brane dynamics when the matter density on the brane is large in comparison with the brane tension. The past asymptotic states of these models correspond to a brane dynamics near a singularity, future asymptotic states describe some intermediate regime of the brane Universe when a nonstandard asymptotic behavior have already been established but the matter density is still larger than the brane tension.

1'. The isotropic Universe

\[
U = \Sigma_+ = \Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0
\]

\[
q = 3\gamma - 1 \quad \Omega_\lambda = 1 \quad 0 \leq \gamma \leq 2
\]

Stable in the future for \( \gamma < 2/3 \), stable in the past for \( \gamma > 1 \).
2'. Equilibrium point

\[ U = \frac{(1-\gamma)(3\gamma-2)}{3c_+^2 + c_-^2} \quad \Sigma_+ = \frac{c_+(3\gamma-2)}{3c_+^2 + c_-^2} \]

\[ \Sigma_- = \frac{\sqrt{3}c_-(3\gamma-2)}{9c_+^2 + 3c_-^2} \quad \Omega_\lambda = 1 - \frac{3\gamma-2}{9c_+^2 + 3c_-^2} \]

\[ \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0 \]

\[ \gamma \in (2/3, 1) \]

Stable in the future for \( \gamma < 2/3 + 3c_+^2 + c_-^2 \).

3'. Equilibrium point

\[ U = 1 - 9c_+^2 - 3c_-^2 \quad \Sigma_+ = 3c_+ \quad \Sigma_- = \sqrt{3}c_- \]

\[ \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0 \]

\[ \Omega_\lambda = 0 \quad 3c_+^2 + c_-^2 < 1/3 \]

Stable in the future for \( \gamma > 2/3 + 3c_+^2 + c_-^2 \).

4'. Equilibrium set (the Kasner Universe)

\[ U = 0 \quad \Omega_\lambda = 0 \quad q = 2 \]

\[ \Sigma_+^2 + \Sigma_-^2 + \Sigma_{12}^2 + \Sigma_{13}^2 + \Sigma_{23}^2 = 1 \]

Stable in the future for \( 3c_+^2 + c_-^2 > 1/3 \).

Zones of stability of these equilibrium points are plotted in Fig.2. For \( \gamma < 1 \) we can see a complete analogy with Fig.1 if \( \gamma \) is divided by 2. It is natural, because the nonstandard matter behaves effectively as a "normal" matter with \( \gamma_{NS} = 2\gamma \). Then, to receive expressions for equilibrium points and stability zones of the system (13) - (17) with \( \Omega_\mu = 0 \) it is enough to take the corresponding expressions for \( \Omega_\lambda = 0 \) and replace \( \gamma \) by \( 2\gamma \). In particular, the isotropic attractor is stable for \( \gamma < 2/3 \).

The part \( \gamma > 1 \) in Fig.2 is new (in Fig.1 this part would correspond to the unphysical region \( \gamma > 2 \)). In this part of the space \((\gamma, c_+, c_-)\) the stability does not depend on \( \gamma \). As a remarkable feature we can note appearance of the Kasner solution as a future attractor for \( 3c_+^2 + c_-^2 > 1/3 \). For \( 3c_+^2 + c_-^2 < 1/3 \) the point 3' is stable.

As in the previous section, using the expression

\[ \Omega'_\lambda = [(6 - 6\gamma)\Sigma^2 + 2(2 - 3\gamma)U]\Omega_\lambda \quad (21) \]
Figure 2: Stability zones of equilibrium points for the case $\Omega_p = 0$. Below the solid plane the point $1'$ is stable, between solid and dashed planes outside the compact region the point $2'$ is stable, inside the compact region the point $3'$ is stable, above the dashed plane outside the compact region the point $4'$ is stable.
we can conclude that all the models with $0 < \Omega_\lambda < 1$ and $\gamma \in (0, 2/3]$ isotropize.

Anisotropic brane attractors may naturally appear in the inflationary brane scenario if the transition from nonstandard dynamics to standard one occurs at some moment after the inflation. During the nonstandard brane phase the condition for inflation is $\gamma < 1/3$, after the end of the inflation the parameter $\gamma$ grows to $4/3$ in typical models. In this case the brane Universe evolves consecutively from one stable regime to another one according to a corresponding $c_+=$const, $c_-=$const line in Fig.2. When the pressure becomes positive ($\gamma > 1$), the Universe finally reaches one of the two possible attractors depending on the parameters $c_+, c_-$. and stays in this regime until the matter density on the brane drops below the brane tension and the transition to the standard regime occurs.

Near the initial singularity the nonstandard matter on the brane dominates and we can put the standard matter density $\Omega_\mu = 0$. As we now consider the backward time direction, the condition for the stability of the equilibrium points becomes $\lambda_i > 0$. The results are follows.

For the Kasner set two nonzero eigenvalues are $\lambda_1 = 6 - 6\gamma$ and $\lambda_2 = 1 - 6\Sigma_+ c_+ - 2\sqrt{3\Sigma_- c_-}$. Four other eigenvalues are equal to zero. This indicates that we have a four-dimensional set of equilibria. The second nonzero eigenvalue is positive on some subset of the Kasner set and this subset exists for arbitrary $(c_+, c_-)$. On the other hand, $\lambda_1$ is positive only for $\gamma < 1$. So, the condition for stability of the Kasner solution in the past is $\gamma < 1$ independently on $(c_+, c_-)$.

In the opposite case $\gamma > 1$ (a matter with a positive pressure) a stable solution in the past direction is the isotropic Universe. This result appears only in the brane scenario. In the case of $P_{\mu\nu} = 0$ the isotropic character of a brane cosmological singularity in the presence of matter with a positive pressure has been found earlier by several authors \cite{1, 2, 3}.

5 Conclusions

We have investigated the Bianchi I cosmological dynamics on a brane in the presence of a nonlocal anisotropic stress $P_{\mu\nu}$ satisfying the Barrow-Maartens ansatz (9) with $U > 0$. The dynamics appears to be more complicated in comparison with the previously known case of $P_{\mu\nu} = 0$. However, we can mention two important regimes not modified by the anisotropic stress of the form studied in this paper:

- If $0 < \gamma \leq 4/3$ the anisotropic stress does not prevent the Bianchi I model from isotropization. Using the existence of a monotonic function (19) we can prove that all such models isotropize.

- The anisotropic stress does not alter the past asymptotic state which is the isotropic Universe for $\gamma > 1$ and the Kasner solution for $\gamma < 1$.

We also describe the set of future asymptotic states in the case $\gamma > 4/3$ and possible transient brane dynamics which can exist before the linear energy-momentum contribution in the effective Einstein equations becomes important.
An important problem for the future development is to consider the $U < 0$ case.

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