Rodrigues Formula for the Nonsymmetric Multivariable Hermite Polynomial

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Applying a method developed by Takamura and Takano for the nonsymmetric Jack polynomial, we present the Rodrigues formula for the nonsymmetric multivariable Hermite polynomial.

KEYWORDS: nonsymmetric multivariable Hermite polynomial, Calogero model, Dunkl-Cherednik operator formulation, Rodrigues formula

§1. Introduction

Explicit construction of the commuting conserved operators and identification of their simultaneous eigenfunctions are important fundamental problems in the study of quantum integrable systems. The orthogonal basis spanned by the simultaneous eigenfunctions of the conserved operators plays an essential role in detailed studies on the integrable systems such as the calculation of the correlation functions, the representation theory of the symmetry of the system and so on.

Among the integrable inverse-square interaction models in one dimension, the Calogero and the Sutherland models are considered to be the “twins” because the two models have their own Dunkl-Cherednik operator formulations that share the same structure of the commutator algebra. The Dunkl-Cherednik operator formulation provides an explicit construction of the commuting conserved operators of the two models. The celebrated Jack symmetric polynomials are the simultaneous eigenfunctions of the conserved operators made of the Cherednik operators of the Sutherland model. However, only a little had been known about the symmetric simultaneous eigenfunctions of the conserved operators made of the Cherednik operators of the Calogero model. Motivated by the Rodrigues formula for the Jack symmetric polynomial that was found by Lapointe and Vinet, we presented the Rodrigues formula for the Hi-Jack symmetric (multivariable Hermite) polynomial and identified it as the simultaneous eigenfunction of the conserved operators of the Calogero model. The multivariable Hermite polynomial is a one-parameter deformation of the Jack symmetric polynomial. They share many common properties, which reflect the same algebraic structure of the corresponding Dunkl-Cherednik operators.

To study the Calogero and Sutherland models including spin variables, we need the nonsymmetric simultaneous eigenfunctions of the Cherednik operators as the orthogonal basis of the orbital part of the wave function. Such a nonsymmetric simultaneous eigenfunction of the conserved operators of the Sutherland model is known to be the nonsymmetric Jack polynomial whose properties are extensively studied in mathematical context. On the other hand, the simultaneous eigenfunction of the Calogero model is identified as the nonsymmetric multivariable Hermite polynomial that is a one-parameter deformation of the nonsymmetric Jack polynomial. A recursive construction of the nonsymmetric Jack polynomial was invented by Knop and Sahi. They also found out a combinatorial formula of the nonsymmetric Jack polynomial which enables us to calculate the norm of the polynomial. Their results were translated to the theory of the nonsymmetric multivariable Hermite and Laguerre polynomials. However, a simple expression for an arbitrary nonsymmetric Jack polynomial and its variants that has a form of successive operations of the raising operators to the vacuum such as the Fock space of the quantum harmonic oscillator has not been given in their formulation.

Quite recently, Takamura and Takano presented a simplified version of an algebraic construction of the nonsymmetric Jack polynomial. Their intuitive formulation yields a single expression for an arbitrary nonsymmetric Jack polynomial, which is the advantage of Takamura and Takano’s formulation to Knop and Sahi’s. Applying Takamura and Takano’s results, we shall present the Rodrigues formula for the nonsymmetric multivariable Hermite polynomial. Recursion relations among norms and explicit forms of norms for some simple cases will be also presented.

The outline of the paper is as follows. In §2, we give a brief summary on the Dunkl-Cherednik operator formulation for the Calogero model and the nonsymmetric multivariable Hermite polynomial. In §3, the algebraic construction of the nonsymmetric multivariable Hermite polynomial is presented in detail. The final section is devoted to a summary.

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§2. Dunkl-Cherednik Operators and Nonsymmetric Multivariable Hermite Polynomials

We give a brief summary on the Dunkl-Cherednik operator formulation of the Calogero model and the nonsymmetric multivariable Hermite polynomial. The Calogero Hamiltonian is expressed as

\[ \hat{H}_C = \frac{1}{2} \sum_{j=1}^{N} (p_j^2 + \omega^2 x_j^2) + \frac{1}{2} \sum_{j<k}^{N} \frac{\alpha^2 - a K_{jk}}{(x_j - x_k)^2}, \]  

(2.1)

where \( p_j = -i \frac{\partial}{\partial x_j} \) and the coordinate exchange operator \( K_{jk} \) is defined as

\[ K_{jk}f(\cdots, x_j, \cdots, x_k, \cdots) = f(\cdots, x_k, \cdots, x_j, \cdots). \]

The ground state wave function is known to be the real Laughlin wave function,

\[ \phi_g(x) = \prod_{1 \leq j < k \leq N} |x_j - x_k|^a \exp\left(-\frac{1}{2} \omega \sum_{j=1}^{N} x_j^2\right), \]

where the ground state energy is

\[ E_g = \frac{1}{2} N \omega (N a + (1 - a)). \]

The excited state wave function of the Calogero model is expressed as the product of an inhomogeneous polynomial and the ground state wave function. To study the polynomial part of the wave functions, we introduce a transformed Hamiltonian whose eigenfunctions are polynomials,

\[ H_C = (\phi_g(x))^{-1} (\hat{H}_C - E_g) \phi_g(x). \]  

(2.2)

In the following, we call eq. (2.2) instead of eq. (2.1) the Calogero Hamiltonian. The commuting conserved operators for the Calogero model are known to be the Cherednik operators. To show this, we need to introduce the Dunkl operators and the creation-like and annihilation-like operators for the Calogero model,

\[ \nabla_j \equiv \frac{\partial}{\partial x_j} + a \sum_{k=1}^{N} \frac{1}{x_j - x_k} (1 - K_{jk}), \]

\[ \alpha^\dagger_l \equiv x_l - \frac{1}{2 \omega} \nabla_l, \quad \alpha_l \equiv \nabla_l. \]

Then the Cherednik operators are given by

\[ d_l \equiv \alpha^\dagger_l \alpha_l + a \sum_{k=1}^{N} (K_{lk} - 1) + a (N - l), \quad [d_l, d_m] = 0. \]  

(2.3)

In the above expression (2.3), we have adopted Knop and Sahi’s choice of the constant term in the Cherednik operator. The commutator algebra among the Dunkl-Cherednik operators is listed as follows,

\[ [\alpha_l, \alpha^\dagger_m] = 0, \quad \alpha_l, \alpha_m = 0, \]

\[ [\alpha^\dagger_l, \alpha^\dagger_m] = \delta_{lm} (1 + a \sum_{k=1}^{N} K_{lk}) - a (1 - \delta_{lm}) K_{lm}, \]

\[ [d_l, \alpha^\dagger_m] = \delta_{lm} (\alpha^\dagger_l + a \sum_{k=1}^{l-1} \alpha^\dagger_k K_{lk} + a \sum_{k=l+1}^{N} \alpha^\dagger_k K_{ik}) \]

\[ - a (\Theta(m - l) \alpha^\dagger_m K_{lm} + \Theta(l - m) \alpha^\dagger_l K_{lm}), \]

\[ [d_l, \alpha_m] = - \delta_{lm} (- a_l + a \sum_{k=1}^{l-1} \alpha_k K_{lk} + a \sum_{k=l+1}^{N} \alpha_k K_{ik}) \]

\[ + a (\Theta(m - l) \alpha_l K_{lm} + \Theta(l - m) \alpha_m K_{lm}), \]  

(2.4)

where \( \Theta(x) \) is the Heaviside function,

\[ \Theta(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \]

The Cherednik operators and the exchange operators satisfy

\[ d_l K_{l,l+1} - K_{l,l+1} d_l + 1 = a, \quad d_{l+1} K_{l,l+1} - K_{l,l+1} d_l = -a, \]

\[ [d_l, K_{m,m+1}] = 0, \quad (l \neq m, m + 1). \]  

(2.5)

In terms of the Cherednik operators, the Calogero Hamiltonian (2.2) can be expressed as

\[ H_C = \omega \sum_{l=1}^{N} \left( d_l - \frac{1}{2} a (N - 1) \right). \]

Thus the Cherednik operators \( \{d_l\} \) give a set of commuting conserved operators of the Calogero model.

Hereafter, all the wave functions are labeled by the symbol \( \lambda \) that denotes a sequence of \( N \) nonnegative integers (composition) which is defined by a partition \( \lambda \),

\[ \lambda \equiv \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}, \]

\[ \lambda_j, \quad j = 1, 2, \cdots, N, \]  

are nonnegative integers, and a distinct permutation \( \sigma \in S_N \). Distinct permutations \( \sigma \) and \( \tau \) for a partition must satisfy

\[ \lambda_{\sigma(j)} \neq \lambda_{\tau(j)} \]

for some \( j \in \{1, 2, \cdots, N\} \). For the definition of the non-symmetric multivariable Hermite polynomial, a partial order among compositions named the Bruhat order \( \prec \) is useful,

\[ \mu \ prec \lambda \quad \Leftrightarrow \quad \begin{cases} \mu \prec \lambda, & 1) \\ \mu = \lambda, & 2) \end{cases} \]

where \( \mu = \lambda \) then the first non-vanishing difference \( \tau(i) - \sigma(i) > 0 \).

Here, the symbol \( \prec \) denotes the dominance order among partitions

\[ \mu \prec \lambda \iff \mu \neq \lambda, \quad |\mu| = |\lambda| \quad \text{and} \quad \sum_{k=1}^{l} \mu_k \leq \sum_{k=1}^{l} \lambda_k, \]

for all \( l = 1, 2, \cdots, N \). A set of indistinct permutations
for a partition \( \lambda \),
\[
\{ \sigma \} \overset{\text{def}}{=} \{ \tau \in S_N| \lambda_\tau = \lambda_\sigma \text{ and } \lambda_\tau > \lambda_\sigma \text{ for } \tau \neq \sigma \},
\]
is represented by the permutation \( \sigma \) which gives the minimum in the sense of the Bruhat order \( <^B \).

The nonsymmetric multivariable Hermite polynomial is the nondegenerate simultaneous eigenfunction of the Cherednik operators \( \{ d_i \} \) with the coefficient of its top term \( x^{\lambda_\sigma} = x_1^{\lambda_\sigma(1)}x_2^{\lambda_\sigma(2)} \cdots x_N^{\lambda_\sigma(N)} \) conventionally taken to be unity,
\[
j_{\lambda_\sigma}(x; 1/a, \omega) = x^{\lambda_\sigma} + \sum_{\mu < \lambda_\sigma, |\mu| < |\lambda_\sigma|} w_{\lambda_\sigma \mu}(a, 1/2\omega)x^{\mu},
\]
d\(j_{\lambda_\sigma}(x; 1/a, \omega) = \tilde{\lambda}_{\sigma(i)}j_{\lambda_\sigma}(x; 1/a, \omega),
\]
where
\[
\tilde{\lambda}_{\sigma(i)} \overset{\text{def}}{=} \lambda_{\sigma(i)} - a\left( \# \{ 1 \leq j < l | \lambda_{\sigma(j)} < \lambda_{\sigma(i)} \} + \# \{ l < j \leq N | \lambda_{\sigma(j)} \leq \lambda_{\sigma(i)} \} \right).
\]
The energy eigenvalue of the eigenfunction \( j_{\lambda_\sigma} \) is given by
\[
H_{CJ_{\lambda_\sigma}} = \sum_{k=1}^{N} \lambda_{\sigma(k)}j_{\lambda_\sigma}.
\]
In the following section, we shall construct the simultaneous eigenfunction of the Cherednik operators.

§3. Rodrigues Formula
Following the Takamura-Takano version of the algebraic construction of the nonsymmetric Jack polynomials, we shall present the algebraic construction of the nonsymmetric multivariable Hermite polynomials.

Though the operator algebras of the Dunkl-Cherednik operators for the two nonsymmetric polynomials share the same structure, we note that some modifications, which will be explained later, are needed to apply Takamura and Takano’s approach to the nonsymmetric multivariable Hermite polynomials.

We introduce two types of operators to construct the nonsymmetric multivariable Hermite polynomials. The first type is the quantum number exchange operator, which is called the braid-exclusion operator in ref. [21].
\[
X_{l,l+1} \overset{\text{def}}{=} i[d_l, K_{l,l+1}] = -i[d_{l+1}, K_{l,l+1}].
\]
We note that the quantum number exchange operator is Hermitian. The quantum number exchange operators transpose the indices of the Cherednik operators,
\[
d_kX_{l,l+1} = X_{l,l+1}d_{l+1}, \quad d_{l+1}X_{l,l+1} = X_{l,l+1}d_l,
\]
\[
[d_k, X_{l,l+1}] = 0, \quad k \neq l, l + 1.
\]
Thus they play the role of permutations in compositions. By a straightforward calculation, we can verify the following relations for the quantum number exchange operators,
\[
X_{l,l+1}^2 = (d_l - d_{l+1})^2 - a^2,
\]
\[
X_{l,l+1}X_{l+1,l+2}X_{l+2,l+1} = X_{l+1,l+2}X_{l+2,l+1}X_{l,l+1}, \quad (|l - m| \geq 2).
\]
The second type is the raising operator that is given by
\[
e_{l}^\dagger = (X_{l,l+1} \cdots X_{N-1,N}e_1)^l, \quad l = 1, \cdots, N,
\]
where \( e_1^\dagger \) is a variant of the operator invented by Knop and Sahi,
\[
e_1^\dagger \overset{\text{def}}{=} K_{N,N-1} \cdots K_{2,1}a_1^\dagger.
\]
What we want to show in the following is the commutation relation between the Cherednik operator and the raising operator,
\[
d_{l},a_{m}^{\dagger} = (1 - \Theta(l - m))a_{m}^{\dagger},
\]
where \( \Theta \) is the Hermitian conjugate of the Knop-Sahi operator,
\[
es = (e_1^\dagger)^\dagger = \alpha_1K_{12} \cdots K_{N-1,N}.
\]
We note that the corresponding relations of the above expressions (3.7) for the nonsymmetric Jack polynomials are simply \( e_1^\dagger e = e^\dagger e = 1 \). Commutation relations related to the Knop-Sahi operator are straightforwardly calculated,
\[
d_{l}e_{l}^\dagger = e_{l+1}^\dagger, \quad l = 1, \cdots, N - 1,
\]
\[
d_{N}e_{1}^\dagger = e_{1}^\dagger,(d_{l} + 1),
\]
\[
X_{l,l+1}e_{l}^\dagger = e_{l}^\dagger X_{l+1,l+2}, \quad l = 1, \cdots, N - 2,
\]
\[
X_{N-1,N}(e_1^\dagger)^2 = (e_1^\dagger)^2X_{1,2}.
\]
We introduce
\[
b_{m}^\dagger \overset{\text{def}}{=} X_{l,l+1} \cdots X_{N-1,N}e_1^\dagger.
\]
Using eqs. (3.2), (3.3), (3.8) and (3.9), we easily verify
\[
d_{l}b_{m}^\dagger = \begin{cases} b_{m}^\dagger d_{l+1}, & 1 \leq l \leq m - 1, \\ b_{m}^\dagger(d_{l+1} + 1), & l = m, \\ b_{m}^\dagger d_{l}, & m + 1 \leq l \leq N, \end{cases}
\]
\[
X_{l,l+1}b_{m}^\dagger = \begin{cases} b_{m}^\dagger X_{l,l+1}, & 1 \leq l \leq m - 1, \\ b_{l+1}^\dagger((d_{l+1} - d_{l} - 1)^2 - a^2), & m = l, \\ b_{l}^\dagger, & m = l + 1, \\ b_{l}^\dagger X_{l+1,l+2}, & l + 2 \leq m \leq N, \end{cases}
\]
\[
b_{m}^\dagger b_{m}^\dagger = \begin{cases} b_{m}^\dagger X_{l,l+1}b_{m}, & 1 \leq m \leq l, \\ X_{m-1,m} \cdots X_{l+2,l}(b_{l})^2, & l \geq m. \end{cases}
\]
Now we are ready to prove the commutation relation (3.9). Using the definition of the raising operator (3.4), we have
\[
[d_{l},a_{m}^{\dagger}] = d_{l}(b_{m}^\dagger)^m - (b_{m}^\dagger)^m d_{l}.
\]
By use of eqs. (3.10), the first term of the above expres-
sion is cast into
\[ d_l (b_m^\dagger)^m = \begin{cases} (b_m^\dagger)^m d_l, & l > m, \\ (b_m^\dagger)^m (d_l + 1), & l \leq m. \end{cases} \]
Thus we obtain
\[ [d_l, a_m^\dagger] = \begin{cases} 0, & l > m, \\ a_m^\dagger, & l \leq m, \end{cases} \]
which proves eq. (3.6). We should note that the raising operators are commutative with each other,
\[ [a_l^\dagger, a_m^\dagger] = 0, \]
as is shown from eqs. (3.10).

The following relations are useful when we calculate the norms of the eigenstates,
\[ b_l^\dagger b_l = d_l \prod_{k=l+1}^{N} ((d_l - d_k)^2 - a^2), \]
\[ b_l b_l^\dagger = (d_l + 1) \prod_{k=l+1}^{N} ((d_l - d_k + 1)^2 - a^2), \]
\[ b_N^\dagger b_N = d_N, \quad b_N b_N^\dagger = d_l + 1. \]
Using eqs. (2.5), (3.10) and (3.11), we can rewrite the number-like operators in terms of the Cherednik operators,
\[ a_l^\dagger a_l = \prod_{m=1}^{l} \left( \prod_{k=l+1}^{N} (d_m - d_k)^2 - a^2 \right) d_m, \]
\[ a_l a_l^\dagger = \prod_{m=1}^{l} \left( \prod_{k=l+1}^{N} (d_m - d_k + 1)^2 - a^2 \right) (d_m + 1), \]
for \( l = 1, \ldots, N \). These relations are different from the corresponding formulas for the nonsymmetric Jack polynomials, which causes a slight modification of the norms.

We also need the following formulas in the calculation of the norms,
\[ X_{l,l-1} a_m^\dagger = a_m^\dagger X_{l,l-1}, \quad m \neq l, \]
\[ a_l^\dagger X_{l,l-1} a_l = ((d_l - d_{l+1} - 1)^2 - a^2) X_{l,l-1} a_l a_{l-1}^\dagger, \]
which can be verified by use of eqs. (3.10).

We are ready to write down the Rodrigues formula for the nonsymmetric multivariable Hermite polynomials up to normalization. For the composition \( \lambda_\sigma \) where the distinct permutation \( \sigma \) is expressed by the product of transpositions as
\[ \sigma = (k_1, k_1 + 1) \cdots (k_2, k_2 + 1)(k_1, k_1 + 1), \]
we can show by use of eqs. (3.2) and (3.6) that the nonsymmetric polynomial \( k_{\lambda_\sigma} \) generated by the Rodrigues formula,
\[ k_{\lambda_\sigma} \overset{\text{def}}{=} X_{k_1, k_1 + 1} X_{k_2, k_2 + 1} \cdots X_{k_1, k_1 + 1} \]
\[ (a_1^\dagger)^{\lambda_1 - \lambda_2} (a_2^\dagger)^{\lambda_2 - \lambda_3} \cdots (a_N^\dagger)^{\lambda_N \cdot 1}, \]
satisfies the definition of the nonsymmetric multivariable Hermite polynomials \( j_{\lambda_\sigma} \) except for the normalization,
\[ d_j k_{\lambda_\sigma} = \delta_{\sigma(j)} k_{\lambda_\sigma}. \]
Thus we conclude \( k_{\lambda_\sigma} \propto j_{\lambda_\sigma} \). Since the Cherednik operators \( d_l \) are Hermitian with respect to the following conventional inner product,
\[ \langle f, g \rangle \overset{\text{def}}{=} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j |\phi_j|^2 f^\dagger(x) g(x), \]
\[ |f|^2 \overset{\text{def}}{=} \langle f, f \rangle, \]
the nonsymmetric multivariable Hermite polynomials are orthogonal with respect to the inner product,
\[ \langle k_{\lambda_\sigma}, k_{\mu_\tau} \rangle = |k_{\lambda_\sigma}|^2 \delta_{\lambda_\sigma, \mu_\tau} \Leftrightarrow \langle j_{\lambda_\sigma}, j_{\mu_\tau} \rangle = |j_{\lambda_\sigma}|^2 \delta_{\lambda_\sigma, \mu_\tau}. \]
For the case \( \sigma = \text{id} \), the norm \( \langle k_{\lambda_\sigma}, k_{\lambda_\sigma} \rangle = |k_{\lambda_\sigma}|^2 \) is calculated in an algebraic manner,
\[ \langle k_{\lambda_\sigma}, k_{\lambda_\tau} \rangle (1, 1) = \prod_{k=1}^{N} \prod_{l=1}^{k} \prod_{m=1}^{l} \left( \lambda_k + a(N-l) - (m-1) \right) \]
\[ \prod_{n=k+1}^{N} \left( (\lambda_k - \lambda_n + a(n-l) - (m-1))^2 - a^2 \right), \]
where \( (1, 1) \) is the vacuum normalization.

\[ \langle 1, 1 \rangle = \frac{(2\pi)^{N}}{(2\omega)^{N(N+1-s)}} \prod_{1 \leq j \leq N} \Gamma(1 + ja) \Gamma(1 + a), \]
with \( \Gamma(z) \) being the gamma function. We note that the above formula for the ratio of the norms of \( k_{\lambda_\sigma} \) and the ground state is different from the corresponding formula in Takamura and Takano’s result. The difference comes from the definitions of the inner products that respectively make the nonsymmetric Jack and nonsymmetric multivariable Hermite polynomials orthogonal. For the general case, the norm \( \langle k_{\lambda_\sigma}, k_{\lambda_\tau} \rangle \) satisfies the following recursion relation,
\[ \frac{\langle X_{l,l+1} k_{\lambda_\sigma}, X_{l,l+1} k_{\lambda_\tau} \rangle}{\langle k_{\lambda_\sigma}, k_{\lambda_\tau} \rangle} = \left( (\lambda_\sigma(l) - \lambda_\sigma(l+1))^2 - a^2 \right), \]
which is equivalent to
\[ X_{l,l+1} k_{\lambda_\sigma} \frac{k_{\lambda_\sigma}}{|k_{\lambda_\sigma}|} = (\lambda_\sigma(l) - \lambda_\sigma(l+1))^2 - a^2 \right) \frac{k_{\lambda_\sigma(l,l+1)}}{|k_{\lambda_\sigma(l,l+1)}|}. \]
The coefficient of the r.h.s. of the above expression becomes zero when the two permutations \( \sigma \) and \( \sigma(l,l+1) \) are indistinct with respect to the Young diagram \( \lambda \), \( \lambda_\sigma(l) = \lambda_\sigma(l+1) \). Explicit form for \( \sigma = \text{id} \) and recursion relations of the norms can be straightforwardly derived from eqs. (3.3), (3.10) and (3.12).

§4. Summary

Generalizing the method developed by Takamura and Takano for the nonsymmetric Jack polynomials, we have presented an algebraic construction of the nonsymmetric multivariable Hermite polynomials which span the orthogonal basis of the orbital part of the wave function of the Calogero model including spin variables. Because of the difference in the definitions of the inner product,
the formulas of the norms of the two nonsymmetric orthogonal polynomials are different from each other.

Though the Calogero and the Sutherland models share a lot of properties of their algebraic structures and structures of their Hilbert spaces, calculations of correlation functions have been done only for the Sutherland model. The difficulty in the calculation toward the correlation functions of the Calogero model comes from the difference of the analytic properties and the inner product of the nonsymmetric multivariable Hermite polynomial and those of the nonsymmetric Jack polynomial. Thus any new approach and formulation for the Calogero and the Sutherland models should be tested if they will give a breakthrough in the calculation of correlation functions of the two models on an equal basis. We need further efforts toward such directions.

Takamura and Takano’s approach is intuitive enough to give a simple expression for arbitrary nonsymmetric multivariable Hermite polynomial. The coefficient of the top term \( x^{\lambda \sigma} \) of the polynomial \( k^{\lambda \sigma} \), however, has not been computed for general cases, which is a disadvantage to Knop and Sahi’s formulation. The same approach will be also applicable to the nonsymmetric multivariable Laguerre polynomial with even parity, which span the orthogonal basis of the \( B_N \) Calogero model. These problems are also left for future studies.

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