Influence of Impurity on the Rate of Single Photon Superradiance in Disordered N Qubit Chain

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We investigate the rate of superradiant emission for a number of artificial atoms (qubits) embedded in a one-dimensional open waveguide. More specifically, we study the 1D (N+1)-qubit chain where N qubits are identical in respect to their excitation frequency $\Omega$ but have different rates of spontaneous emission $\Gamma_n$, and a single impurity qubit which is different from N qubits by its excitation frequency $\Omega_p$ and rate of spontaneous emission $\Gamma_p$. This system is shown to have two hybridized collective states which accumulates the widths of all qubits. The energy spectrum of these states and corresponding probabilities are investigated as the function of the frequency detuning between the impurity and other qubits in a chain. It is shown that the inclusion of impurity qubit alter the resonance widths of the system only in a narrow range of the frequency detuning between qubits and impurity, where the resonance widths experience a significant repulsion. The photon transmission through disordered N-qubit chain with impurity qubit is also considered. It is shown that a single photon transport through this system is described by a simple expression which predicts for specific photon frequency the existence of a complete transmission peak and transparency window between frequencies $\Omega$ and $\Omega_p$.

I. INTRODUCTION

The phenomenon of superradiance was first discovered by R. H. Dicke [1], who showed that the system of N identical two level excited atoms undergoes a spontaneous coherent transition to the ground state. This is accompanied by the emission of N photons, the intensity of which scales as $N^2$, and the decay rate of which is $N\gamma$, where $\gamma$ is the decay rate of an isolated atom. In [2] the possibility of a single photon superradiance was indicated, which is realized when a single photon is sent through a qubit chain of which is much less the photon wavelength. In this case a single photon Dicke state is formed when N identical two level atoms are in a symmetrical superposition of the states with one excited atom and N-1 atoms in the ground state. In this case, the decay rate of a single photon is also equal to $N\gamma$.

However, unlike the real atoms, qubits are intrinsically not identical due to technological scattering of their parameters. While the excitation energy of every qubit in a chain can, in principle, be adjusted by external circuit, the decay rate of every qubit is individual and cannot be controlled externally. Nevertheless, an array of many intrinsically disordered artificial atoms globally coupled to a single waveguide mode reveals the existence of collective quantum behavior corresponding to coherent oscillations of qubits [3, 4].

In this paper we consider a one dimensional $N+1$ qubit chain in an open waveguide, where N qubits have the same excitation frequency $\Omega$ and are disordered with respect to their rates of spontaneous emission $\Gamma_n$, while a single impurity qubit has different excitation frequency $\Omega_p$ and the rate of spontaneous emission $\Gamma_p$. The incident photon is directed along waveguide axis. The qubits are not coupled directly. They interact only via a common photon field in a waveguide.

The goal of our study is to investigate the influence of impurity qubit on the formation of superradiant state and its characteristics. The system under study is described by a non Hermitian Hamiltonian which accounts for the irreversible decay of the excited qubits to the continuum.

The paper is organized as follows. In the Section II we find the collective states for N disordered qubits in the chain. In spite of the fact that for N qubits all rates of spontaneous emission are different there exists a single collective state which accumulates all the qubits’ widths. Other N-1 states are non-decaying degenerate states with the same energy, $\hbar\Omega$. In Section III the Hilbert space is enlarged to include the impurity qubit in the system. The presence of the impurity qubit results in a hybridization of its state with the superradiant collective state of N qubit chain. We investigated the dependence of the energy spectrum of this enlarged system on the detuning $\Delta\Omega = \Omega - \Omega_p$. At the end of the section we calculated the contribution of N- qubit state and impurity wavefunction to the formation of hybridized states depending on the frequency detuning. In Section IV the photon transmission through disordered N- qubit chain with impurity qubit is considered. It is shown that a single photon transport through this system is described by a simple expression which predicts for specific photon frequency the existence of a complete transmission peak and transparency window between frequencies $\Omega$ and $\Omega_p$.

II. COLLECTIVE STATES OF DISORDERED N- QUBIT CHAIN

As a basis set of state vectors we take the states where one qubit is in the excited state $|e>$ and the other N-1 qubits are in the ground state $|g>$. Therefore, we have N vectors $| n > = | g_1, g_2, ..., g_{n-1}, e_n, g_{n+1}, ..., g_{N-1}, g_N >$. The spontaneous emission of the excited qubit results in a continuum states $| k > = | g_1, g_2, ..., g_{N-1}, g_N, k >$, where all qubits are in the ground state and there is one photon in a waveguide. The interaction of qubits with a photon field can be described
by a non-Hermitian Hamiltonian [†]:

\[ H = H_S - iW \] (1)

where

\[ H_S = \sum_{n=1}^{N} \frac{1}{2} \left( 1 + \sigma_z^{(n)} \right) \Omega_n \] (2)

is Hamiltonian of \( N \) noninteracting qubits, and \( W \) describes the interaction of qubits with the photon field. The matrix elements of Hamiltonian (1) in the \( |n> \) representation are:

\[ \langle m | H | n \rangle = \Omega_n \delta_{m,n} - i \langle m | W | n \rangle \; ; \; 1 \leq m, n \leq N \] (3)

If the distance between qubits along the direction of the photon scattering (\( z \)-axis) is much less than the photon wavelength, the matrix element on the right-hand side of Eq. (3) takes the form [5]:

\[ \langle m | W | n \rangle = \sqrt{\Gamma_n \Gamma_m} \] (4)

where \( \Gamma_n \) is the rate of spontaneous photon emission from a state where the \( n \)-th qubit is excited. Therefore, we get from (3) a non-Hermitian \( N \times N \) matrix, where the main diagonal elements and off-diagonal elements are \( \Omega_n - i\Gamma_n \) and \( -i\sqrt{\Gamma_n \Gamma_m} \), respectively. The incident photon, when absorbed, can excite any qubit. As we do not know which of the \( N \) qubits is excited, the wave functions of the system should be expressed as a superposition of the state vectors \( |n> \) :

\[ \Psi_i = \sum_{n=1}^{N} c_{n,i} |n> \] (5)

where \( i = 1, 2, \ldots, N \).

In general, it is hard to obtain for large \( N \) the energy spectrum of this system, however, the problem is much simplified if all qubits have the same excitation energy: \( \Omega_n = \Omega \). It is not difficult to show that in this case the solution of the Schrödinger equation \( H \Psi = E \Psi \) with \( H \) and \( \Psi \) from Eqs. (1) and (5), respectively, has the following properties.

1) There is a single non stationary state with the complex energy \( E_S = \Omega - i\Gamma_S \) where \( \Gamma_S = \sum_{n=1}^{N} \Gamma_n \). This resonance accumulates decay widths of all qubits. The wave function of this state is as follows:

\[ |\Psi_S> = A \sum_{n=1}^{N} \sqrt{\frac{\Gamma_n}{\Gamma}} |n> \] (6)

where \( A \) is a normalizing factor: \( A = \sqrt{\Gamma_1/\Gamma_S} \).

2) There are \( N-1 \) degenerate mutual orthogonal stationary states with the energy \( E = \Omega \):

\[ |\Psi_m> = \sum_{n=1}^{N} \sqrt{\frac{\Gamma_n}{\Gamma_1}} c_{m,n} |n> \; (m = 1, 2, \ldots, N-1) \] (7)

where the coefficients \( c_{m,n} \) in (7) satisfy \( N-1 \) conditions

\[ \sum_{n=1}^{N} \Gamma_n c_{m,n}^* = 0 \] (8)

which ensures the orthogonality of the functions (6) and (7):

\[ \langle \Psi_S | \Psi_m > = 0 \]

If we assume the same rate of spontaneous emission for all qubits: \( \langle m | W | n \rangle = \Gamma \), then we get the known result [5, 6]: there is a single resonant state \( E_S = \Omega - i\Omega \), the wave function of which is a symmetric coherent superposition of the state vectors \( |n> \), where all quantities \( c_{n,i} \) are the same:

\[ |\Psi_S> = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n> \] (9)

The \( N-1 \) degenerate mutual orthogonal stationary states with the energy \( E = \Omega \) are given by the wave functions (7) with \( N-1 \) conditions \( \sum_{n=1}^{N} c_{m,n} = 0 \).

The collective state (9), which we call a single photon Dicke state [2], is formed by a single photon, which propagates through the \( N \) qubit chain. Therefore, under this condition, the state (9) decays with a rate which is \( N \) times faster than the rate of the spontaneous emission of a single qubit.

III. HYBRIDIZED STATES OF DISORDERED N- QUBIT CHAIN AND THE IMPURITY

We consider \( N \) qubit chain, where all qubits have the same excitation frequency \( \Omega \) but different rates of spontaneous emission \( \Gamma_n \). The parameters of the impurity qubit we denote as \( \Omega_p \) and \( \Gamma_p \). The non Hermitian Hamiltonian for the whole system reads:

\[ H = H_S - iW + H_p - iW_p \] (10)

The first two terms in right hand side of (10) are just Hamiltonian (1) for \( N \)- qubit chain, \( H_S \) is a Hamiltonian of impurity qubit, \( W_p \) describes the decay of impurity qubit via its direct interaction with a photon field and indirectly via its photon mediated interactions with other qubits in a chain. The matrix elements of Eq. (10) in the \( |n> \) representation are:

\[ \langle m | H | n \rangle = \Omega \delta_{m,n} - i \langle m | W | n \rangle = \Omega \delta_{m,n} - i \sqrt{\Gamma_n \Gamma_m} \]

\[ \langle p | H | n \rangle = -i \langle p | W_p | n \rangle = -i \sqrt{\Gamma_p \Gamma_n} \]

\[ \langle p | H | p \rangle = \Omega_p - i \langle p | W_p | p \rangle = \Omega_p - i \Gamma_p \] (11)

where \( |p> \) is the state of impurity qubit.

[†] In what follows we set for convenience \( \hbar = 1 \), so that all energies are expressed in frequency units.
A. CALCULATION OF THE ENERGY SPECTRUM

It is convenient to write (11) in explicit form as a sum of two \((N + 1) \times (N + 1)\) matrices \(H = H_0 + V:\)

\[
H_0 = \begin{pmatrix}
\Omega_p - i\Gamma_p & 0 & \ldots & 0 \\
0 & \Omega - i\Gamma_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & -i\Gamma_{1N} & \ldots & -i\Gamma_{N}
\end{pmatrix} \\
V = \begin{pmatrix}
0 & -i\Gamma_{p1} & \ldots & -i\Gamma_{pN} \\
-\Gamma_{p1} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\Gamma_{pN} & 0 & \ldots & 0
\end{pmatrix}
\]

(12)

(13)

where \(\Gamma_{mn} = \Gamma_{nm} = \sqrt{\Gamma_m \Gamma_n}, \Gamma_{pn} = \Gamma_{np} = \sqrt{\Gamma_p \Gamma_n}.\)

The matrix \(V\) exhibits photon mediated interaction between impurity atom and other N qubits.

The basis wave functions for the system (N qubits + impurity atom) are the states (6) and (7), and the state of the impurity atom, which is denoted as \(|\varphi_p\rangle\). For subsequent calculations we express these states as the N+1-component column vectors:

\[
|\varphi_p\rangle = \begin{pmatrix} 1 \\
0 \\
\vdots \\
0 \end{pmatrix} \\
|\Psi_S\rangle = \frac{1}{\sqrt{\Gamma_1}} \begin{pmatrix} \sqrt{\Gamma_1} \\
\sqrt{\Gamma_2} \\
\vdots \\
\sqrt{\Gamma_{N-1}} \\
\sqrt{\Gamma_N} \end{pmatrix} \\
|\Psi_n\rangle = \frac{1}{\sqrt{\Gamma_1}} \begin{pmatrix} \sqrt{\Gamma_1}e^n \\
\sqrt{\Gamma_2}e^n \\
\vdots \\
\sqrt{\Gamma_{N-1}}e^n \\
\sqrt{\Gamma_N}e^n \end{pmatrix}
\]

(14)

It can be easily verified that the wave functions (14) are the eigenfunctions for Hamiltonian \(H_0\): \(H_0 |\varphi_p\rangle = (\Omega_p - i\Gamma_p) |\varphi_p\rangle\), \(H_0 |\Psi_S\rangle = \Omega |\Psi_S\rangle\), \(H_0 |\Psi_n\rangle = (\Omega - i\Gamma_S) |\Psi_n\rangle\).

The decomposition of \(H\) into \(H_0\) and \(V\) is justified by the fact that due to the conditions (8) the dark states \(|\Psi_n\rangle\) are unaffected by the interaction term \(V\. Hence, the interaction matrix \(V\) mixes up only the states \(|\Psi_S\rangle\) and \(|\varphi_p\rangle\). Therefore, we express the solution of Hamiltonian \(H\) as a hybridized state

\[
|\Psi\rangle = C_p |\varphi_p\rangle + C_S |\Psi_S\rangle
\]

(15)

The energies \(E\) and the superposition factors \(C_p, C_S\) can be found from Schrodinger equation \(H |\Psi\rangle = E |\Psi\rangle\) for \(H\) (12), (13) and \(|\varphi_p\rangle, |\Psi_S\rangle\) (14). Thus, we obtain for the coefficients \(C_p, C_S\) in (15) two coupled equations:

\[
(\Omega_p - i\Gamma_p - E) C_p - i\sqrt{\Gamma_p \Gamma_s} C_S = 0
\]

(16)

The energies \(E\) are found by equating determinant of this system to zero. The result is as follows:

\[
E_\pm = \Omega - \Delta \Omega \mp \frac{1}{2} (\Gamma_S + \Gamma_P)
\]

\[
\pm \frac{1}{2} \sqrt{[\Delta \Omega - \frac{1}{2} (\Gamma_S + \Gamma_P)]^2 - 4\Gamma_S \Gamma_P}
\]

(18)

where \(\Delta \Omega = \Omega - \Omega_p\).

In what follows we analyze the behavior of real and imaginary parts of complex resonances \(E_\pm\) of Eq. (18) as a function of the impurity detuning \(\Delta \Omega\). We show that the widths of two resonances experience the repulsion in a narrow range of \(\Delta \Omega\) near zero. The effect of repulsion of resonance widths in open systems is known for a long time [8, 9]. Due to this effect the impurity atom gives rise to the enhancement of the rate of superradiance emitted by assembly of N qubits.

First, using the method described in [8], we obtain from (18):

\[
(E_+ - E_-)^2 = (\Delta \Omega)^2 - (\Gamma_S + \Gamma_P)^2 - 2i\Delta \Omega (\Gamma_S - \Gamma_P)
\]

(19)

Second, we explicitly split \(E_\pm\) into real and imaginary parts

\[
E_+ = \varepsilon_+ - i\gamma_+; E_- = \varepsilon_- - i\gamma_-
\]

(20)

Summing up these two equations and using explicit expression (13) for \(E_\pm\) we obtain:

\[
\varepsilon_+ + \varepsilon_- = \Omega - \frac{\Delta \Omega}{2}; \gamma_+ + \gamma_- = \Gamma_S + \Gamma_P
\]

(21)

Second equation in (21) is of general nature: total width of all resonances has to be equal to the sum of all individual width (10).

Then, from (19) we obtain two coupled equations for \(\Delta \varepsilon = \varepsilon_+ - \varepsilon_-\) and \(\Delta \gamma = \gamma_+ - \gamma_-:\)

\[
(\Delta \varepsilon)^2 - (\Delta \gamma)^2 = (\Delta \Omega)^2 - (\Gamma_S + \Gamma_P)^2
\]

(22)

\[
\Delta \varepsilon \Delta \gamma = \Delta \Omega (\Gamma_S - \Gamma_P)
\]

(23)

We could solve these equations for \(\Delta \varepsilon\) and \(\Delta \gamma\), however, we prefer to analyze their properties and show the behavior of \(\varepsilon_\pm\) and \(\gamma_\pm\) on the graphs below. First, we notice from (22) that at the point \(\Delta \Omega = 0\), the only solution which is consistent with (22) is \(\Delta \varepsilon = 0, \Delta \gamma = \pm (\Gamma_S + \Gamma_P)\). If \(\Gamma_S = \Gamma_P\) there are two possibilities: 1) \(\Delta \gamma = 0, \Delta \varepsilon \neq 0\) for \(|\Delta \Omega| > 2\Gamma_P\), and 2) \(\Delta \gamma \neq 0, \Delta \varepsilon = 0\) for \(|\Delta \Omega| < 2\Gamma_P\).

For the case \(\Delta \Omega = 0\), we obtain from (18) the expected result: \(E_+ = \Omega, E_- = \Omega - i(\Gamma_S + \Gamma_P)\). If detuning is large \((\Delta \Omega \gg \Gamma_S)\) we obtain from (18): \(E_+ = \Omega - i\Gamma_S, E_- = \Omega + i\Gamma_P\). This result is also reasonable, since for large detuning two systems (the impurity qubit and N-qubit chain must be independent.

Below we study the dependence of the real and imaginary parts of energies (18) on the detuning \(\Delta \Omega\). For above quantities we take the values which are relevant for superconducting qubits which operate in a microwave domain. We take
\( \Omega/2\pi = 5 \) GHz, \( \Gamma_{av}/2\pi = 5 \) MHz, where we introduce the average rate \( \Gamma_{av} = \Gamma_{S}/N \). The dependence of real parts of the energies \( E_{\pm} \) on the detuning is shown in Fig. 1 for five qubits with the same excitation frequency and one impurity qubit with excitation frequency \( \Omega_p \).

Two lines in Fig[1] do not intersect but only touch each other in a single point \( \Delta \Omega = 0 \).

The dependence of normalized widths on the detuning is shown in Fig.[2] We note from this figure that the widths of two resonances experience the repulsion in a narrow range of \( \Delta \Omega \) near zero. It is seen that the transition from the width \( \Gamma_{S} + \Gamma_{P} \) (for the point \( \Delta \Omega = 0 \)) to the width \( \Gamma_{S} \), and from zero width to \( \Gamma_{P} \) for the impurity occurs in a narrow frequency range of frequency detuning not exceeding one percent from the qubit excitation frequency. Another interesting point is that the "phases" of two complex resonances (18) experience a jump when crossing zero point of detuning. The reason for this is that the quantity \( \Delta \epsilon \) does not change the sign when crossing the zero detuning point \( \Delta \Omega = 0 \) as is seen from Fig[1]. Then, from equation (23) it follows that the quantity \( \Delta \gamma \) must change its sign together with the sign of \( \Delta \Omega \).

Below we consider the behavior of complex energies (18) for \( \Gamma_{S} = \Gamma_{P} \). As is shown in Fig[3] the real parts of the complex energies are merged between the points A and B where \( |\Delta \Omega| < \Gamma_{P} \). However, in this region the resonance widths are repulsed reaching the maximum repulsion \( \Gamma_{S} + \Gamma_{P} \) at zero detuning \( \Delta \Omega = 0 \) (see Fig[4]). In the region where \( |\Delta \Omega| > \Gamma_{P} \), the picture is different: the real parts of energies are repulsed while the widths are merged.

**B. Calculation of probability factors**

Here we find the probability factors \( C_P \) and \( C_S \) in (15). The eigenenergies (18) correspond to two eigenfunctions

\[
|\Psi_{\pm}\rangle = C_{\pm}^{P} |\phi_{P}\rangle + C_{\pm}^{S} |\psi_{S}\rangle
\]

(24)

Since non-Hermitian Hamiltonian \( H \) is symmetric (see (12), (13)), \( H = H^{\dagger} \), we have \( H^{\dagger} = H^{\dagger} \) with the consequence that the dual basis states of \( H^{\dagger} \), \( |\Phi_{\pm}\rangle \) can be chosen as (11).
expression (18) for $C$ in the following form:

$$|\Phi_\pm\rangle = |\Psi_\pm^*\rangle = C_p^{\pm\ast}|\varphi_p\rangle + C_S^{\pm\ast}|\Psi_S\rangle$$  \(25\)

with the orthogonality conditions

$$\langle \Psi_\pm^*|\Psi_\pm\rangle = 1; \quad \langle \Psi_\pm^*|\Psi_\mp\rangle = 0$$  \(27\)

From (27) we obtain

$$(C_p^{\pm})^2 + (C_S^{\pm})^2 = 1$$  \(28\)

$$C_p^{\pm}C_p^{\mp} + C_S^{\pm}C_S^{\mp} = 0$$  \(29\)

Using the equation (16) in the form

$$(\Omega_p - i\Gamma_p - E_\pm)C_p^{\pm} - i\sqrt{\Gamma_p}\Gamma_s C_S^{\pm} = 0$$  \(30\)

and the conditions (28), (29), we find the following expressions for $C_p^{\pm}$, $C_S^{\pm}$:

$$C_p^{\pm} = \mp i\frac{\Omega - i\Gamma_s - E_\pm}{\sqrt{\Gamma_s}\Gamma_p - (\Omega - i\Gamma_S - E_\pm)^2}$$  \(31\)

$$C_S^{\pm} = \pm\frac{\sqrt{\Gamma_p}\Gamma_s}{\sqrt{\Gamma_s}\Gamma_p - (\Omega - i\Gamma_S - E_\pm)^2}$$  \(32\)

The orthogonality condition (29) can be written as

$$(\Omega - i\Gamma_s - E_\pm)(\Omega - i\Gamma_s - E_-) - \Gamma_S\Gamma_p = 0$$  \(33\)

which can be proved analytically with the aid of explicit expression (13) for $E_\pm$.

The equation (33) allows one to rewrite the factors $C_p^{\pm}$, $C_s^{\pm}$ in a following form:

$$C_p^{\pm} = \mp i\frac{\sqrt{\Omega - i\Gamma_s - E_\pm}}{\pm(\Delta\varepsilon - i\Delta\gamma)}$$  \(34\)

$$C_S^{\pm} = \pm\frac{1}{\sqrt{\pm(\Delta\varepsilon - i\Delta\gamma)}}\frac{\sqrt{\Gamma_p}\Gamma_s}{\sqrt{\Omega - i\Gamma_S - E_\pm}}$$  \(35\)

where $\Delta\varepsilon$ and $\Delta\gamma$ are defined in (20).

As is seen from (34) and (35), the quantities $|C_p^{\pm}|^2$, $|C_S^{\pm}|^2$ cannot be taken as a measure of the probabilities, since they can be greater than one, especially for the case $\Gamma_S = \Gamma_P$ in the vicinity of exceptional points $\Delta\Omega = \pm 2\Gamma_P$, where $\Delta\varepsilon = \Delta\gamma = 0$. The reason for this is that the wave function (24) of non Hermitian Hamiltonian (12), (13) cannot be normalized to unity (12). Therefore, we define the corresponding probabilities as follows:

$$P_p^{\pm} = \frac{|\langle \varphi_p|\Psi_\pm\rangle|^2}{|\Psi_\pm|^2} = \frac{|C_p^{\pm}|^2}{|C_p^{\pm}|^2 + |C_S^{\pm}|^2}$$  \(36\)

$$P_S^{\pm} = \frac{|\langle \Psi_S|\Psi_\pm\rangle|^2}{|\Psi_\pm|^2} = \frac{|C_S^{\pm}|^2}{|C_p^{\pm}|^2 + |C_S^{\pm}|^2}$$  \(37\)

The explicit forms for these probabilities read:

$$P_p^{\pm} = \frac{(\Omega - i\Gamma_S - E_\pm)(\Omega + i\Gamma_S - E_\pm)}{(\Omega - i\Gamma_S - E_\pm)(\Omega + i\Gamma_S - E_\pm) + \Gamma_S\Gamma_p}$$  \(38\)

$$P_S^{\pm} = \frac{\Gamma_S\Gamma_p}{(\Omega - i\Gamma_S - E_\pm)(\Omega + i\Gamma_S - E_\pm) + \Gamma_S\Gamma_p}$$  \(39\)

The detuning dependence of the probabilities $P_p^{\pm}$ and $P_S^{\pm}$ for hybridized state $|\Psi_+\rangle$ (24) is shown in Fig. 5. These plots were drawn for the same parameters as those in Fig. 1 and Fig. 2. We see that for relatively large detuning the contribution of the one of two wave functions $|\varphi_p\rangle$ or $|\Psi_S\rangle$ to the formation of hybridized state $|\Psi_+\rangle$ becomes dominating. For positive detuning the contribution of the state $|\Psi_S\rangle$ dominates the formation of hybridized state, while for negative detuning the domination is observed for the state $|\varphi_p\rangle$. For small detuning $(\Delta\Omega < (\Gamma_S + \Gamma_P))$ the probabilities $P_p^{\pm}$ and $P_S^{\pm}$ are weakly dependent on detuning. At zero detuning we observe the jump between $P_p^{\mp}$ and $P_S^{\mp}$, the origin of which is related to the phase jump shown in Fig. 2.

![FIG. 5: Color online. The dependence of probabilities $P_p^{\pm}$, $P_S^{\pm}$ for the state $|\Psi_\pm\rangle$ on the detuning. N=5, $\Gamma_p/\Gamma_{av} = 4$, $\Omega/2\pi = 5$ GHz, $\Gamma_{av}/2\pi = 5$ MHz. Blue (solid) line is for $P_p^{\pm}$, green (dashed) line is for $P_S^{\pm}$](image)
FIG. 5. Hence, all properties of the plots in Fig. 6 regarding to the contribution of the states (|φ⟩ and |Ψ⟩) to the formation of hybridized state |Ψ⟩ are the same (with account for mirror reflection) as those shown in Fig. 5.

FIG. 6: Color online. The dependence of probabilities |P⟩ and |P⟩ for the state |Ψ⟩ on the detuning. N=5, P_=/P_ave = 4, Ω/2π = 5 GHz, P_ave/2π = 5 MHz. Blue (solid) line is for |P⟩, green (dashed) line is for |P⟩.

Figs. 7, 8 show the same dependences as those from Figs. 5, 6 but for the case Γ_S = Γ_P ≡ Γ. The plots of real and imaginary parts of the complex energy for this case are shown in Figs. 3, 4. We see that in the central region of these plots, |ΔΩ| < 2Γ, the contributions of the wave functions |ϕ⟩ and |Ψ⟩ to the formation of hybridized states |Ψ⟩ are equal (|P_⟩ = |P_⟩ in Fig. 7 and |P_⟩ = |P_⟩ in Fig. 8). It can be shown explicitly, that for this case in the detuning range |ΔΩ| ≤ 2Γ the following relation holds

\[(Ω - iΓ - E_±)(Ω + iΓ - E_±^*) = Γ^2\]

Therefore, from (38), (39) we obtain |P_⟩ = |P_⟩ = 0.5 in this range.

FIG. 7: Color online. The dependence of probabilities |P_⟩ and |P_⟩ on the detuning. N=5, Γ_P/Γ_ave = 5, Ω/2π = 5 GHz, Γ_ave/2π = 5 MHz. Blue (solid) line is for |P_⟩, green (dashed) line is for |P_⟩.

FIG. 8: Color online. The dependence of probabilities |P_⟩ and |P_⟩ on the detuning. N=5, Γ_P/Γ_ave = 5, Ω/2π = 5 GHz, Γ_ave/2π = 5 MHz. Blue (solid) line is for |P_⟩, green (dashed) line is for |P_⟩.

IV. PHOTON TRANSPORT THROUGH A DISORDERED N-QUBIT CHAIN

There are many papers where the one-photon transmission and reflection spectra are studied for two or more qubits interacting with waveguide, plasmonic or resonator modes. It was reported that for some specific qubit arrays it is possible to form a reflection or transmission window in a broad wave range [13–16].

Here we show that for system we study, namely, N qubits with equal excitation frequency Ω and different rates of spontaneous emission Γ_n, and a single impurity qubit with excitation frequency Ω_P and rate of spontaneous emission Γ_P, the expressions for transmission and reflection spectra are greatly simplified: they are reduced to those for two qubits with different excitation frequencies.

In the single-photon approximation and long wavelength limit (ω_L/v_g ≪ 1, where L is the chain length, v_g group velocity of the waveguide mode) the general expression for the transmission coefficient for N-qubit chain is as follows [5]:

\[t_N = \frac{\prod_{n=1}^{N} (ω - Ω_n)}{\prod_{n=1}^{N} (ω - z_n)}\]  

(40)

where ω is the frequency of incident photon, z_n are the eigenenergies of non Hermitian Hamiltonian (1). For the system studied in this paper the expression (40) can be reduced to the form below:

\[t_N = \frac{(ω - Ω)(ω - Ω_P)}{(ω - E_+)(ω - E_-)}\]  

(41)

where the quantities E_± are given in (18).

Explicit expressions for E_± allow to rewrite (41) in the following form

\[t_N = \frac{x(x + ΔΩ)}{x[2Γ + iΓ_P] + ΔΩ[x + iΓ_S]}\]  

(42)
where $x = \omega - \Omega; \Delta \Omega = \Omega - \Omega_P; \Gamma_S = \sum_{n=1}^{N} \Gamma_n$.

Similar calculations give the expression for reflection amplitude:

$$r_N = -i \frac{x(\Gamma_S + \Gamma_P) + \Delta \Omega \Gamma_S}{x[x + i(\Gamma_S + \Gamma_P)] + \Delta \Omega[x + i\Gamma_S]}$$  \hspace{1cm} (43)

If $\Delta \Omega = 0$, the transmission (42) takes the form:

$$t_N = \frac{\omega - \Omega}{[\omega - \Omega + i(\Gamma_S + \Gamma_P)]}$$ \hspace{1cm} (44)

It follows from (44) that for N qubits identical in their excitation frequencies the width of the resonance dip accumulates the widths of all resonances in the system. It corresponds to the point $\Delta \Omega = 0$ at the plots shown in Figs. 2 and 3 where the resulting width of the resonance at this point becomes a sum of the widths of all N qubits, $\Gamma_S$ and the width of the impurity, $\Gamma_P$. As it follows from (44), the best way to reveal the existence of collective qubit state is to measure a phase of transmission signal the magnitude of which scales as the number of qubits which take part in the formation of this collective state [3, 4].

The expression (42) describes, in fact, two qubits with the parameters $\Omega, \Gamma_S$ and $\Omega_P, \Gamma_P$, respectively. The interesting feature of this expression is that there exists the photon frequency at which a complete transmission is observed. This frequency corresponds to the point where the imaginary part of denominator in (42) (or nominator in (43)) becomes zero:

$$x(\Gamma_S + \Gamma_P) + \Delta \Omega \Gamma_S = 0$$

Therefore, we see that at the point

$$\omega_c = \Omega - \Delta \Omega - \frac{\Gamma_S}{\Gamma_S + \Gamma_P} = \frac{\Omega \Gamma_P + \Omega_P \Gamma_S}{\Gamma_S + \Gamma_P}$$ \hspace{1cm} (45)$$

the transmission $t_N=1$. The frequency $\omega_c$ lies just in between two frequencies, $\Omega$ and $\Omega_P$.

This property was earlier reported in [5] for three qubits (Fig.14 in [5]). As was noted there the frequencies at which complete transmission is observed, correspond to the frequencies which provide zero to the reflection amplitude. They do not, in general, coincide with the frequencies of the resonances (real parts of the complex energies) found from the system Hamiltonian.

For our system it can be shown explicitly that $\omega_c$ can be expressed in terms of real and imaginary parts of complex energies $E_{\pm}$ as follows:

$$\omega_c = \frac{\varepsilon + \gamma_- - \varepsilon - \gamma_+}{\gamma_+ + \gamma_-}$$ \hspace{1cm} (46)$$

The equivalence of right hand sides of Eqs. (45) and (46) can be seen if in (46) we express $\varepsilon_{\pm}$ and $\gamma_{\pm}$ in terms of $\Delta \varepsilon$ and $\Delta \gamma$ and use the equation (23).

The form of the transmission plot depends mainly on the relation between two quantities $\Gamma_S(\Gamma_P)$ and the frequency detuning $\Delta \Omega$. As the spontaneous rates $\Gamma_S, \Gamma_P$ are directly proportional to the qubit - photon interaction, we expect that for $\Gamma_S, \Gamma_P \approx \Delta \Omega$ the plot will have the form of interference pattern as shown in Fig.9, where $\Delta \Omega = 50$ MHz, $\Gamma_S/2\pi = 20$ MHz, $\Gamma_P/2\pi = 15$ MHz. In the opposite case, $\Gamma_S, \Gamma_P \ll \Delta \Omega$, we obtain between the frequencies $\Omega$ and $\Omega_P$ a transparency window with steep walls as shown in Fig.10 where $\Delta \Omega = 50$ MHz, $\Gamma_S/2\pi = 2$ MHz, $\Gamma_P/2\pi = 1.5$ MHz.

The formation of transparency window here is more or less understandable, and was reported for similar structures [13, 14]. But what is not trivial is that for our system there always exist between the frequencies $\Omega$ and $\Omega_P$ the frequency $\omega_c$ (45) where the complete transmission is observed.

**V. CONCLUSION**

We considered 1D N qubit chain in an open waveguide. All qubits have equal excitation frequencies but disordered in respect to their rates of spontaneous emission. We showed that this system has a unique superradiant state with the width being the sum of all individual qubits’ widths. Other $N - 1$ states are dark in that they have not any widths at all and, hence, these states are not observable. The inclusion of impurity qubit with different excitation frequency and the rate of spontaneous emission results in the formation of hybridized states which alter significantly the resonance widths of the system only in a narrow range of the frequency detuning between qubits and impurity. We also calculated the contribu-
tion of N-qubit state and impurity wavefunction to the formation of hybridized states depending on the frequency detuning. We showed that a single photon transport through our system is described by a simple expression which predicts for specific photon frequency the existence of a complete transmission peak and transparency window between two frequencies $\Omega$ and $\Omega_P$.

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