HYPERBOLIC TESSELLATIONS ASSOCIATED TO BIANCHI GROUPS

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Abstract. Let $F/Q$ be number field. The space of positive definite binary Hermitian forms over $F$ form an open cone in a real vector space. There is a natural decomposition of this cone into subcones, which descend give rise to hyperbolic tessellations of 3-dimensional hyperbolic space by ideal polytopes. We compute the structure of these polytopes for a range of imaginary quadratic fields.

1. Introduction

Let $F/Q$ be a number field. The space of positive definite binary Hermitian forms over $F$ form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoï polyhedron [Gun99, Koe60, Ash77]. This has been computationally explored for real quadratic fields in [Ong86, GY08] and the cyclotomic field $Q(\zeta_5)$ in [Yas].

For $F$ an imaginary quadratic field, the polyhedral cones give rise to ideal polytopes in $\mathbb{H}_3$, 3-dimensional hyperbolic space. In work of Cremona and his students [Whi90, Byg98, Lin05, Cre94, CW94], analogous polytopes have already been computed for class number 1 fields imaginary quadratic fields as well as a few fields with class number 2 and 3 using different methods. The structure of the polytopes was used to compute Hecke operators on modular forms for the Bianchi groups over those fields. These polytopes were used by Goncharov [Gon08] in his study of Euler complexes on modular curves. The data of the polytope and stabilizer could also be used to give explicit presentations of $GL_2(\mathcal{O})$. We will examine this in a later project.

In this paper, we investigate the structure of these ideal polytopes for a much larger range of imaginary quadratic fields. Our approach and implementation works for general imaginary quadratic fields, but we restrict the range to ease the computation. Specifically, we compute the ideal polytope classes for all imaginary quadratic fields of class number 1 and 2, as well as some fields of higher class number with small discriminant. Specifically, we compute the ideal polytopes for the fields $Q(\sqrt{-d})$ for square-free $d$, where

$$-d \in \{1 - 100, 115, 123, 163, 187, 235, 267, 403, 427\}.$$ 

There is no theoretical obstruction to computing these tessellations for higher class number and higher discriminant.

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The structure of the paper is as follows. We set the notation for the quadratic fields and Hermitian forms in Section 2. The implementation is described in Section 3. Finally, in Section 4 we summarize some of the data collected so far.

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2. Notation and background

Let $F = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ be an imaginary quadratic number field. We always take $d < 0$ to be a square-free integer. Let $\mathcal{O} \subset F$ denote the ring of integers in $F$. Then $\mathcal{O}$ has a $\mathbb{Z}$-basis consisting of 1 and $\omega$, where

$$\omega = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \mod 4. \end{cases}$$

Then $[\text{Sta67, Bak71}]$ $F$ has class number $h_F = 1$ if $-d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ and $h_F = 2$ if $-d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$.

Let $\overline{\cdot}$ denote complex conjugation, the nontrivial Galois automorphism of $F$.

**Definition 2.1.** A binary Hermitian form over $F$ is a map $\phi : F^2 \rightarrow \mathbb{Q}$ of the form

$$\phi(x, y) = ax\overline{x} + bx\overline{y} + \overline{b}xy + cy\overline{y},$$

where $a, c \in \mathbb{Q}$ and $b \in F$ such that $\phi$ is positive definite.

By choosing a $\mathbb{Q}$-basis for $F$, $\phi$ can be viewed as a quadratic form over $\mathbb{Q}$. In particular, it follows that $\phi(\mathcal{O}^2)$ is discrete in $\mathbb{Q}$.

**Definition 2.2.** The minimum of $\phi$ is

$$m(\phi) = \inf_{v \in \mathcal{O}^2 \setminus \{0\}} \phi(v).$$

A vector $v \in \mathcal{O}^2$ is minimal vector for $\phi$ if $\phi(v) = m(\phi)$. The set of minimal vectors for $\phi$ is denoted $M(\phi)$.

**Definition 2.3.** A Hermitian form over $F$ is perfect if it is uniquely determined by $M(\phi)$ and $m(\phi)$.

3. Implementation

The space of positive definite binary Hermitian forms over $F$ form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoi polyhedron $\Pi$ [Gun99, Koe60, Ash77]. The top-dimensional cones of this decomposition correspond to perfect forms and descend to ideal polytopes in $\mathbb{H}_3$, 3-dimensional hyperbolic space.

There is an algorithm [Gun99] to compute the $\text{GL}_2(\mathcal{O})$-equivalence classes of perfect forms. The algorithm uses linear algebra and convex geometry, but requires
an initial input of a perfect form. To this end, we describe the method that was used to compute an initial perfect form.

A perfect form $\phi$ is uniquely determined by its minimum $m(\phi)$ and set of minimal vectors $M(\phi)$. By scaling, we can assume $m(\phi) = 1$. Since each minimal vector defines a linear equation in $V$, and $V$ is 4-dimensional, generically 4 minimal vectors will uniquely determine $\phi$. Note that this does not imply that $\#M(\phi) = 4$. Indeed in many examples $M(\phi) > 4$.

For each field $F = \mathbb{Q}(\sqrt{-d})$, we need only to find a single perfect form to begin the algorithm. Thus we limit our search to a particular family of quadratic forms. Specifically, let $S_0 \subset V$ be the subset of quadratic forms $\phi$ such that
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq M(\phi).
\]
For $\phi \in S_0$, the Hermitian matrix $A_\phi$ associated to $\phi$ must have the form
\[
A_\phi = \begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix}, \quad \text{where } \beta \in F \text{ with } \text{Re}(\beta) = -\frac{1}{2}.
\]
If $\phi \in S_0$ and $\phi$ has an additional minimal vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathcal{O}^2$, then
\[
(1) \quad \beta = -\frac{1}{2} + \left( \frac{1 - a_1^2 + a_2^2 b + a_1 b_1 - a_2 db_2 - b_1^2 + b_2^2 d}{2 da_1 b_2 - 2 da_2 b_1} \right) \sqrt{d},
\]
where $a = a_1 + a_2 \sqrt{d}$ and $b = b_1 + b_2 \sqrt{d}$. Since $\phi$ is positive definite, we must have $\beta \bar{\beta} < 1$. Combined with (1), this implies
\[
(2) \quad -\left( \frac{1 - a_1^2 + a_2^2 b + a_1 b_1 - a_2 db_2 - b_1^2 + b_2^2 d}{2 da_1 b_2 - 2 da_2 b_1} \right)^2 < \frac{3}{4}.
\]
By reduction theory, the values $N_{F/\mathbb{Q}}(a)$, $N_{F/\mathbb{Q}}(b)$, and $N_{F/\mathbb{Q}}(b-a)$ are bounded above by a constant depending upon $d$. Thus we implement a brute force search over $a, b \in \mathcal{O}$ beginning at 0 and moving out. When a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is found satisfying (2), we check that the corresponding form $\phi$ satisfies
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\} \subseteq M(\phi).
\]
This corresponds to an ideal polytope whose vertices contain $\{\infty, 0, 1, \frac{a}{b}\}$.

Once the initial form is found, we implement the algorithm of [Gun99] to find all the perfect forms and the full structure of the Voronoï polyhedron in Magma [BCP97].

4. Results

In this section we collect the results of the computations of the $\text{GL}_2(\mathcal{O})$-conjugacy classes of the ideal Voronoï polytopes.
4.1. **Example**: $d = -14$. Let $F = \mathbb{Q}(\sqrt{-14})$. Then $F$ has class number 4 and ring of integers $\mathcal{O} = \mathbb{Z}[\omega]$, where $\omega = \sqrt{-14}$. There are $9 \text{GL}_2(\mathcal{O})$-classes of polytopes which are of 3 combinatorial types. There are 3 triangular prisms with cuspidal vertices

$$P_1 = \left\{ \infty, 1, \frac{5 + 2\omega}{9}, \frac{2 + \omega}{4}, \frac{4 + 2\omega}{9}, 0 \right\},$$

$$P_2 = \left\{ \frac{11 + 4\omega}{23}, 1, \frac{5 + 2\omega}{9}, \frac{4 + 2\omega}{9}, \frac{12 + 4\omega}{23}, 0 \right\},$$

$$P_3 = \left\{ \frac{8 + 5\omega}{23}, \frac{2 + \omega}{5}, \frac{1 + \omega}{5}, \frac{2 + \omega}{6}, \frac{3 + 2\omega}{10}, \frac{7 + 4\omega}{21} \right\},$$

and 5 tetrahedra with cuspidal vertices

$$T_1 = \left\{ \frac{11 + 4\omega}{23}, \frac{2 + \omega}{5}, \frac{4 + 2\omega}{9}, 0 \right\},$$

$$T_2 = \left\{ 1, \frac{5 + 2\omega}{9}, \frac{3 + \omega}{5}, \frac{12 + 4\omega}{23} \right\},$$

$$T_3 = \left\{ \frac{11 + 4\omega}{23}, \frac{2 + \omega}{5}, \frac{2 + \omega}{6}, 0 \right\},$$

$$T_4 = \left\{ \frac{8 + 5\omega}{23}, \frac{2 + \omega}{5}, \frac{4 + 2\omega}{9}, 0 \right\},$$

$$T_5 = \left\{ \frac{4 + \omega}{6}, 1, \frac{3 + \omega}{5}, \frac{12 + 4\omega}{23} \right\},$$

and a square pyramid with cuspidal vertices

$$S = \left\{ \frac{8 + 5\omega}{23}, \frac{2 + \omega}{5}, \frac{1 + \omega}{5}, \frac{2 + \omega}{6}, 0 \right\}.$$

Given the cuspidal vertices, one can easily compute the stabilizers of each polytope. The stabilizers are all cyclic in this case. For each stabilizer, we compute a generator. The results are given in Table 1.

4.2. **Summary**. We compute the Voronoï polytopes for all imaginary quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ with class number 1 and 2 as well as higher class number for $d > -100$. Although there is no reason an arbitrary convex 3-dimensional polytope could not arise, in all of these cases only 8 combinatorial types show up. We give the names and $F$-vector $(|\text{#vertices}, \text{#edges}, \text{#faces}|)$ for each in Table 2. We also note that the triangular dipyramid shows up in this range much less frequently than the other polytopes.

In Table 3 we give the number of $\text{GL}_2(\mathcal{O})$-classes of each polytope type for $F$ with class number 1 or 2. In Table 4 we give the number of $\text{GL}_2(\mathcal{O})$-classes of each polytope type for the remaining imaginary quadratic fields with $d > -100$.

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Table 1. Stabilizer groups of Voronoï ideal polytopes for $\mathbb{Q}(\sqrt{-14})$

| Polytope | Stabilizer | Generator |
|----------|------------|-----------|
| $P_1$    | $C_6$      | $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ |
| $P_2$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $P_3$    | $C_4$      | $\begin{bmatrix} \omega + 1 & -\omega + 6 \\ 2 & -\omega - 1 \end{bmatrix}$ |
| $T_1$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $T_2$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $T_3$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $T_4$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $T_5$    | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $S$      | $C_2$      | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |

Table 2. Combinatorial types of ideal polytopes that occur in this range.

| Polytope       | F-vector | Picture |
|----------------|----------|---------|
| tetrahedron    | [4, 6, 4] | ![tetrahedron](image) |
| octahedron     | [6, 12, 8]| ![octahedron](image) |
| cuboctahedron  | [12, 24, 14]| ![cuboctahedron](image) |
| triangular prism | [6, 9, 5] | ![triangular prism](image) |
| hexagonal cap  | [9, 15, 8]| ![hexagonal cap](image) |
| square pyramid | [5, 8, 5] | ![square pyramid](image) |
| truncated tetrahedron | [12, 18, 8]| ![truncated tetrahedron](image) |
| triangular dipyramid | [5, 9, 6]| ![triangular dipyramid](image) |

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Table 3. $GL_2(O)$-classes of Voronoï ideal polytopes for class number 1 and 2.

| $h_F$ | $d$ | Polytope 1 | Polytope 2 | Polytope 3 | Polytope 4 | Polytope 5 | Polytope 6 | Polytope 7 | Polytope 8 |
|-------|-----|------------|------------|------------|------------|------------|------------|------------|------------|
| 1     | -1  | 0 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -2  | 0 0 1 0 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -3  | 1 0 0 0 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -7  | 0 0 0 1 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -11 | 0 0 0 0 0 0 0 1 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -19 | 0 0 1 1 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -43 | 0 0 0 2 1 0 1 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -67 | 0 1 0 2 1 2 1 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 1     | -163| 11 0 1 8 2 3 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -5  | 0 0 0 2 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -6  | 0 0 0 0 0 1 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -10 | 0 1 0 1 0 2 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -13 | 1 0 0 3 1 1 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -15 | 1 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -22 | 5 0 1 4 0 2 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -35 | 3 4 0 1 0 2 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -37 | 10 0 0 8 1 8 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -51 | 1 0 1 2 1 1 0 0 | 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -58 | 47 0 0 7 2 6 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -91 | 5 1 0 5 0 3 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -115| 3 1 0 5 2 1 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -123| 1 1 1 6 3 3 1 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -187| 18 1 1 4 1 9 1 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -235| 13 1 0 12 4 11 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -267| 24 1 1 13 5 10 1 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -403| 66 1 0 16 2 20 0 2| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |
| 2     | -427| 65 2 0 19 4 24 0 0| 0 1 0 0 0 0 0 0 |            |            |            |            |            |            |

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Table 4. GL$_2(\mathcal{O})$-classes of Voronoï ideal polytopes with $d > -100$.  

| $h_F$ | $d$ | $\triangle$ | $\square$ | $\pentagon$ | $\diamond$ | $\triangle$ | $\square$ | $\pentagon$ | $\diamond$ |
|-------|-----|-------------|-----------|-------------|-----------|-------------|-----------|-------------|-----------|
| 3     | -23 | 0           | 1         | 0           | 1         | 0           | 0         |               |           |
| 3     | -31 | 0           | 0         | 0           | 3         | 0           | 1         | 0           |           |
| 3     | -59 | 0           | 1         | 1           | 3         | 0           | 2         | 0           |           |
| 3     | -83 | 6           | 0         | 0           | 2         | 2           | 1         | 1           | 0         |
| 4     | -14 | 5           | 0         | 0           | 3         | 0           | 1         | 0           |           |
| 4     | -17 | 5           | 0         | 0           | 2         | 1           | 3         | 1           |           |
| 4     | -21 | 8           | 2         | 0           | 2         | 1           | 4         | 0           |           |
| 4     | -30 | 6           | 0         | 0           | 6         | 4           | 4         | 0           |           |
| 4     | -33 | 9           | 0         | 1           | 8         | 1           | 6         | 1           |           |
| 4     | -34 | 20          | 0         | 0           | 3         | 1           | 6         | 1           |           |
| 4     | -39 | 1           | 0         | 0           | 3         | 1           | 1         | 0           |           |
| 4     | -46 | 32          | 1         | 0           | 5         | 0           | 9         | 0           |           |
| 4     | -55 | 5           | 1         | 0           | 2         | 0           | 2         | 0           |           |
| 4     | -57 | 33          | 1         | 0           | 10        | 3           | 14        | 2           |           |
| 4     | -73 | 57          | 1         | 1           | 13        | 1           | 14        | 0           | 2         |
| 4     | -78 | 69          | 1         | 0           | 11        | 4           | 18        | 0           |           |
| 4     | -82 | 92          | 0         | 0           | 8         | 3           | 11        | 1           |           |
| 4     | -85 | 56          | 0         | 0           | 17        | 0           | 28        | 0           |           |
| 4     | -93 | 79          | 1         | 0           | 20        | 7           | 21        | 0           |           |
| 4     | -97 | 95          | 0         | 1           | 19        | 3           | 19        | 0           |           |
| 5     | -47 | 5           | 0         | 0           | 1         | 1           | 2         | 0           |           |
| 5     | -79 | 9           | 0         | 0           | 5         | 0           | 4         | 0           |           |
| 6     | -26 | 18          | 1         | 0           | 2         | 1           | 4         | 0           |           |
| 6     | -29 | 15          | 0         | 0           | 6         | 0           | 6         | 0           |           |
| 6     | -38 | 33          | 1         | 0           | 2         | 1           | 6         | 1           |           |
| 6     | -53 | 45          | 0         | 0           | 7         | 2           | 13        | 0           |           |
| 6     | -61 | 41          | 1         | 0           | 11        | 1           | 16        | 0           |           |
| 6     | -87 | 6           | 0         | 0           | 6         | 2           | 3         | 0           |           |
| 7     | -71 | 7           | 1         | 0           | 4         | 0           | 4         | 0           |           |
| 8     | -41 | 31          | 0         | 1           | 9         | 0           | 8         | 0           |           |
| 8     | -62 | 81          | 0         | 0           | 7         | 2           | 7         | 0           |           |
| 8     | -65 | 69          | 2         | 0           | 9         | 0           | 19        | 0           |           |
| 8     | -66 | 67          | 1         | 1           | 9         | 4           | 12        | 1           |           |
| 8     | -69 | 51          | 2         | 0           | 15        | 2           | 21        | 0           |           |
| 8     | -77 | 81          | 1         | 0           | 9         | 2           | 26        | 0           |           |
| 8     | -94 | 125         | 1         | 0           | 10        | 2           | 17        | 0           |           |
| 8     | -95 | 12          | 0         | 0           | 4         | 0           | 9         | 0           |           |
| 10    | -74 | 105         | 1         | 0           | 9         | 1           | 12        | 0           |           |
| 10    | -86 | 130         | 0         | 0           | 9         | 1           | 18        | 1           |           |
| 12    | -89 | 136         | 0         | 0           | 14        | 1           | 21        | 1           | 0         |
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