QUANTUM COSMOLOGICAL PERFECT
FLUID MODEL
AND ITS CLASSICAL ANALOGUE

A.B. Batista†, J.C. Fabris‡, S.V.B. Gonçalves§ and J. Tossa¶

†Departamento de Física, Universidade Federal do Espírito Santo, 29060-900, Vitória, Espírito Santo, Brazil
‡IMSP - Université Nationaledu Bénin, Porto Novo, Bénin

Abstract

The quantization of gravity coupled to a perfect fluid model leads to a Schrödinger-like equation, where the matter variable plays the role of time. The wave function can be determined, in the flat case, for an arbitrary barotropic equation of state \( p = \alpha \rho \); solutions can also be found for the radiative non-flat case. The wave packets are constructed, from which the expectation value for the scale factor is determined. The quantum scenarios reveal a bouncing Universe, free from singularity. We show that such quantum cosmological perfect fluid models admit a universal classical analogue, represented by the addition, to the ordinary classical model, of a repulsive stiff matter fluid. The meaning of the existence of this universal classical analogue is discussed. The quantum cosmological perfect fluid model is, for a flat spatial section, formally equivalent to a free particle in ordinary quantum mechanics, for any value of \( \alpha \), while the radiative non-flat case is equivalent to the harmonic oscillator. The repulsive fluid needed to reproduce the quantum results is the same in both cases. PACS number(s): 04.20.Cv., 04.20.Me

1 Introduction

The standard cosmological model predicts that the Universe had an initial singular state, from which expansion followed. In spite of many success of this scenario, it is hard to believe that such initial state may have existed, since it is not possible to construct a physical scenario for a singular state. It is generally argued that quantum effects appear as the Universe approaches the initial singularity, the temperature mounting to levels comparable to the Planck temperature, leading to the avoidance of that singular initial state. The great problem with this mechanism is the absence, until today, of a consistent quantum theory of gravity. However, quantum cosmology permits, in principle, to

\[ \text{e-mail: brasil@cce.ufes.br} \]
\[ \text{e-mail: fabris@cce.ufes.br} \]
\[ \text{e-mail: sergio@cce.ufes.br} \]
\[ \text{e-mail: jtoss@syfed.bj.refer.org} \]
circumvent this difficulty, since it signs with the possibility that the entire Universe may admit a quantization procedure, from which a quantum scenario can be built up.

However, quantum cosmology faces many conceptual and technical problems [1, 2, 3, 4]. First, since it is based on the ADM decomposition, leading to the hamiltonian formulation of General Relativity, it can be applied only to space-times admitting foliation. From the cosmological point of view, this does not seem to be a serious restriction. However, the equation for the so-called wave function of the Universe, the Wheeler-DeWitt equation, is a functional equation defined in the superspace, the space of all possible spatial geometries, and no solution for it is known until now, unless an infinite number of degrees of freedom is frozen. Moreover, the gravity theory reveals to be a constrained system, and the time reparametrisation freedom of General Relativity implies that the superhamiltonian of gravity is zero. The consequence is that, in the process of quantization, time disappears.

In some situations, the notion of time can be recovered in quantum cosmology. One example is the case where gravity is coupled to a perfect fluid. Employing the Schutz’s formalism for the description of the perfect fluid [5, 6], based on some auxiliary potentials which represent the dynamical degrees of freedom of the fluid, we can obtain a hamiltonian in the minisuperspace, where the scale factor and the fluid variable are the only dynamical degrees of freedom. The conjugate momentum associated to the variables of the fluid appears linearly. After quantization, this implies a Schrödinger equation, where this conjugate momentum plays the role of time. The solution of this Schrödinger equation leads to eigenfunctions which are not square integrable. But finite wave functions can be constructed by superposing these eigenfunctions conveniently. The expectation value for the dynamical variable (in this case, the scale factor of the Universe) can be obtained, revealing a singularity-free Universe which exhibit a bounce and approaches asymptotically the classical solution.

The purpose of the present paper is to show that such quantum perfect fluid model can be mimetized, from the point of view of the behaviour of the scale factor, by a classical model where, besides the ordinary fluid characterized by the equation of state \( p = \alpha \rho \), a repulsive fluid with an equation of state \( p = \rho \) is introduced, which does not depend on the value of \( \alpha \) and, perhaps, on the spatial curvature \( k \). This generalizes the results obtained in [8]. In principle, a suitable classical model can reproduce certain aspects of a given quantum model. What is surprising here is that the classical model which reproduces the quantum cosmological perfect fluid model is obtained by introducing a universal repulsive fluid: the analogous classical model is valid for the flat case, which is formally equivalent to the free particle problem in ordinary quantum mechanics, and for the radiative curved case, which takes the form of a harmonic oscillator problem. We guess that other quantum cases, different from those quoted before, may also be reproduced by the analogous classical model.

The existence of such analogous classical model to the quantum cosmological perfect fluid model in so many different situations may rise the question if the quantization of a perfect fluid, at least in the mini-superspace, is a real quantization. We do not intend to answer this question here. But we notice that the important difference between the cases treated here and the equivalent cases in ordinary quantum mechanics (free
particle, harmonic oscillator) is connected to the fact that the dynamical variable in the present problem (the scale factor of the Universe) is positive definite, while in the quantum ordinary case the position variable is definite in all real axis. This restriction to half of the real axis is imposed by the fact that the system is coupled to gravity. So the classical analogue exhibited here does not occur in ordinary quantum mechanics, being specific of the quantum cosmological perfect fluid model.

This paper is organized as follows. In next section, we describe the quantum model, with the solutions for the flat case, in the minisuperspace. Wave packets are constructed, and the expectation value of the scale factor is computed. In section 3, the spreading of the wave packet is analyzed, revealing a behaviour very close to the free particle problem in ordinary quantum mechanics. The similarity of the flat quantum perfect fluid model with the free particle problem in ordinary quantum mechanics is discussed. In section 4, the classical analogue is presented. The problem with curvature in the spatial section is discussed in section 5, and it is shown the classical analogue can still be applied, at least for a radiative fluid. Section 6 contains our conclusions.

2 The quantum model

The action for a perfect fluid coupled to gravity in the Schutz’s formalism may be written as

$$S = \int_M d^4x \sqrt{-g} p - \int_M d^4x \sqrt{-gR} - \int_{\partial M} d^3x \sqrt{h} K,$$

where \( p \) is the pressure, \( K \) is the trace of the extrinsic curvature and \( h \) is the induced metric in the boundary of the manifold \( M \). The four velocity of the fluid is written with the aid of five potentials \( \phi, \epsilon, \beta, \theta \) and \( S \):

$$u_\nu = \frac{1}{\mu} (\epsilon_\nu + \phi \beta_\nu + \theta S_\nu)$$

where \( \mu \) is the specific enthalpy. The four velocity is subjected to the condition \( u^\nu u_\nu = 1 \). Introducing the Friedmann-Lemaître-Robertson-Walker metric in the action (1), it is possible to construct the super-hamiltonian [9]

$$H = -\frac{\Pi^2_a}{24a} - 6ka + \frac{p_{\epsilon}^{1+\alpha} e^{S}}{a^{3\alpha}},$$

where \( \alpha \) defines de equation of state of the fluid (\( p = \alpha \rho \)). Performing the canonical transformations

$$T = p_{\epsilon} e^{-S} p_{\epsilon}^{-(1+\alpha)} \quad \Pi_T = e^{S} p_{\epsilon}^{1+\alpha},$$

we finally obtain the following expression for the super-hamiltonian:

$$H = -\frac{\Pi^2_a}{24a} - 6ka + \frac{\Pi_T}{a^{3\alpha}},$$

where \( \Pi_T \) is the momentum associated with the matter variable. It appears linearly in the hamiltonian. The parameter \( k \) defines the curvature of the spatial section, taking the
values 0, 1, -1 for a flat, closed and open Universe, as usual. The Schutz’s formalism for the description of perfect fluids gives dynamical degrees of freedom to them. Hence, the hamiltonian constraint can not be used to eliminate all the degrees of freedom of the problem.

Imposing the quantization conditions and applying this hamiltonian to the wave function, we obtain the Wheeler-DeWitt equation in the minisuperspace:

\[
\frac{\partial^2 \Psi}{\partial a^2} - 144 a^2 - 24 a^{1-3\alpha} \frac{\partial \Psi}{\partial T} = 0 .
\] (6)

Due to the canonical transformations employed in the construction of the hamiltonian, the time coordinate \(T\) is connected with the cosmic time \(t\) by the relation \(dt = a^{3\alpha} dT\). Notice that the equation (6) is equivalent to the Schrödinger equation of ordinary quantum mechanics. Hence, all the formalism of quantum mechanics, like Hilbert space, observables represented by self-adjoint operators, can be applied to this problem. In order the hamiltonian operator to be self-adjoint, the scalar product between the wave functions \(\Phi\) and \(\Psi\) must take the form

\[
(\Phi, \Psi) = \int_0^\infty a^{1-3\alpha} \Phi^* \Psi da .
\] (7)

The non-conventional measure in the scalar product (7) implies that \(a\) must be positive definite in order to have a positive norm for the wave function, except for some specific equation of state, e.g., those for which \(\alpha = 1/3, -1/3, -1\). However, the physical requirement that \(a\) must specify the scale, force us to restrict it to positive values, even for those particular cases.

The Wheeler-DeWitt equation written above may be solved through the separation of variables method. Indeed, writing

\[
\Psi(a, T) = e^{iET} \xi(a)
\] (8)

we obtain

\[
\xi'' + \left( -144 a^2 + 24 E a^{1-3\alpha} \right) \xi = 0 ,
\] (9)

where the primes mean derivative with respect to \(a\). It is possible to show that the parameter \(E\) is positive, except when \(\alpha = 1\), a special case which will be discussed separately later. Notice that in principle the order ambiguity in (6) should be taken into account. But, it is possible to show that the final results do not depend on it.

When \(k = 0\), the equation (9) admits a solution under the form of Bessel functions, leading to the following final expression for the wave function:

\[
\Psi = e^{iET} \sqrt{a} \left[ c_1 J_{\frac{1}{2(1-\alpha)} \left( \frac{\sqrt{96E}}{3(1-\alpha)} a^{\frac{3(1-\alpha)}{2}} \right)} + c_2 J_{\frac{1}{2(1-\alpha)} \left( \frac{\sqrt{96E}}{3(1-\alpha)} a^{\frac{3(1-\alpha)}{2}} \right)} \right] .
\] (10)

These solutions are not valid for \(\alpha = 1\). In this particular case, equation (9) becomes an Euler’s type equation, and the final solution takes the form:

\[
\Psi = e^{iET} \sqrt{a} \left[ c_1 a^{\frac{1-3\alpha}{2}} + c_2 a^{-\frac{1-3\alpha}{2}} \right] .
\] (11)
The above solutions must obey convenient boundary conditions in order the original hamiltonian operator to be self-adjoint. These boundary conditions are

\[ \Psi(0, T) = 0 \quad \text{or} \quad \frac{\partial \Psi}{\partial a}(a, T)|_{a=0} = 0 \quad . \tag{12} \]

The first one amounts to impose \( c_2 = 0 \), while the second one implies \( c_1 = 0 \). For \( \alpha = 1 \) it is possible to satisfy the boundary conditions but the value of \( E \) must be bounded from above. This compromises the possibility to construct well-behaved wave packets. Hence this case seems to be pathological.

None of the above solutions is square integrable. Hence, wave packets must be constructed, by superposing those solutions, in order to obtain expressions with physical meaning. The general structure of these superpositions is

\[ \Psi(a, T) = \int_0^\infty A(E)\Psi_E(a, T)dE \quad . \tag{13} \]

We specialize the discussion from now on to the case \( \alpha < 1 \) and \( c_2 = 0 \). Nothing would have changed if we have choosed \( c_1 = 0 \) instead. Defining \( r = \frac{\sqrt{96E}}{3(1-\alpha)} \), we can obtain final closed expressions for the wave packet if we choose the function \( A(E) \) to be a quasi-gaussian superposition factor:

\[ \Psi(a, T) = \sqrt{a} \int_0^\infty r^{\nu+1}e^{-\gamma r^2 + i\frac{24}{32}(1-\alpha)^2r^2T} J_\nu(r a \frac{3(1-\alpha)}{2}) dr \quad , \tag{14} \]

where \( \nu = \frac{1}{3(1-\alpha)} \) and \( \gamma \) is a real parameter. The wave packet takes then the form

\[ \Psi(a, T) = a e^{-\frac{a^{3(1-\alpha)}}{4B}} \frac{4-4\alpha}{(2B)^{\frac{4(1-\alpha)}{3}}} \quad , \tag{15} \]

where \( B = \gamma - i\frac{24}{32}(1-\alpha)^2T \).

Now, we are interested in verifying which are the previsions of those quantum models for the behaviour of the scale factor of the Universe. In order to do this, we adopt the many worlds interpretation \[7\] and calculate the expectation value of the scale factor:

\[ < a >_T = \frac{\int_0^\infty a^{1-3\alpha}\Psi(a, T)^*a\Psi(a, T)da}{\int_0^\infty a^{1-3\alpha}\Psi(a, T)^*\Psi(a, T)da} \quad . \tag{16} \]

The integrals above are easily solved, leading to

\[ < a >_T = \left( \frac{\Gamma(\frac{5-3\alpha}{3(1-\alpha)})}{\Gamma(\frac{4-2\alpha}{3(1-\alpha)})} \right) (2\gamma)^{\frac{1}{3(1-\alpha)}} \left[ \frac{9(1-\alpha)^4}{(32)^2\gamma^2}T^2 + 1 \right]^{\frac{1}{3(1-\alpha)}} \quad . \tag{17} \]

These solutions represent a bouncing Universe, with no singularity, which goes asymptotically to the corresponding classical model when \( T \to \infty \). They were first written down in \[7\].
3 Spreading of the wave packet

In ordinary quantum mechanics, the width of the wave packet permits to have important informations about the classical limit of the quantum model. This width is defined as

$$\Delta^2 = \langle a^2 \rangle - \langle a \rangle^2 .$$

(18)

In the case of a free particle, for example, the distribution of the probability of finding the particle coincides asymptotically for large values of time with the distribution of trajectories of classical particles which had initially some spread in their initial conditions in the origin. Of course, this does not mean that the notion of classical trajectory is recovered asymptotically. As the gaussian factor goes to zero, the particle is completely localized and the notion of classical trajectory is recovered.

In quantum cosmology the situation is more subtle. At the present stage, it seems that there exist two main interpretation schemes that are specially useful in quantum cosmology: the many worlds interpretation and the Bohm-de Broglie interpretation. In the last one, the notion of trajectory is essential. In the former one, this question is less clear but it can be considered that it predicts also trajectories that bifurcate following the possible eigenvalues, leading to what is called consistent histories. Hence, it seems that it is very difficult to avoid the notion of trajectories in quantum cosmologies, due to the unicity of the Universe, even if in the many worlds interpretation we have many possible Universes that do not communicate among themselves.

Let us evaluate the spread of the wave packet determined before. Using the expression (15), we find

$$< a^2 > = \frac{\int_0^\infty a^{1-3\alpha}\Psi^*(a,T)a^{2}\Psi(a,T)da}{\int_0^\infty a^{1-3\alpha}\Psi^*(a,T)\Psi(a,T)da}$$

(19)

$$= \frac{\Gamma(2-\alpha)}{\Gamma[3(1-\alpha)]} \left\{ \frac{2\gamma^2 + \frac{18}{16\alpha}(1-\alpha)^4T^2}{\gamma} \right\} \frac{\gamma}{\gamma} .$$

(20)

The expression for the width of the wave packet takes then the form

$$\Delta^2 = \frac{\Gamma(2-\alpha)}{\Gamma[2(1-\alpha)]} - \frac{\Gamma[5-3\alpha]}{\Gamma[3(1-\alpha)]} \left\{ \frac{2\gamma^2 + \frac{18}{16\alpha}(1-\alpha)^4T^2}{\gamma} \right\} .$$

(21)

There is a striking similarity between the result for the spreading of the wave packet found here and the case of the free particle in ordinary quantum mechanics [10]. This similarity will be discussed in more details later in this section. Considering an ensemble of Universe with some spreading in their initial conditions, then the classical trajectories coincide with the quantum "trajectories" asymptotically for large values of time. The classical solution is recovered from the expectation value of the quantum solutions, for any value of time, when the gaussian parameter $\gamma$ goes to zero. Moreover, the wave packet has an initial small width that spreads as time evolves.

Even if technically the results of the quantum model for the Universe with a perfect fluid is similar to the result for a free particle, the question of interpretation makes all
the difference. In the case of a free particle in ordinary quantum mechanics, the notion of trajectory is not recovered asymptotically. However, in quantum cosmology it seems unavoidable that the Universe must follow a trajectory. Hence, if we adopt this point of view, the results exhibited above tell us that the ensemble of “quantum trajectories” initially differs from the possible classical trajectories, coinciding with them later. The initial discrepancy is due to the appearance of repulsive quantum effects as the existence of a classical analogue will reveal explicitly.

As it has already been remarked, the behaviour for the quantum perfect fluid model presented before has a great resemblance with what happens in the free particle problem in ordinary quantum mechanics. In this case, the solution of the Schrödinger equation leads to the wave function

$$\Psi(x, t) = A(k) \exp i(kx - \omega t) \quad ,$$  

with $\omega = \hbar k^2 / 2m$. This plane wave solution is not realistic, and the wave function for a free particle is obtained by constructing the wave packet

$$\Psi(x, t) = \frac{1}{(2\pi)^{3/2}} \int A(k) \exp i[kx - \omega(k)t] dk \quad .$$  

Even if a gaussian superposition does not lead to an expression for the expectation value of the position similar to (17), the wave packet spreads as it evolves in a manner similar to (21). This similarity in fact expresses a more deep connection between the quantum mechanical free particle and the flat quantum cosmological perfect fluid model.

In fact, let us consider again the super-hamiltonian (5). From it, we can define, for $k = 0$, the reduced hamiltonian

$$H_r = \frac{1}{24} a^{3\alpha - 1} p_a^2 \quad .$$  

It is this reduced hamiltonian that drives the evolution of the system, in the sense of ordinary quantum mechanics. The Wheeler-DeWitt equation in the minisuperspace can be written as a genuine Schrödinger equation,

$$H_r \Psi = i \frac{\partial \Psi}{\partial T} \quad .$$  

If a canonical transformation such that

$$p_x = \frac{1}{\sqrt{12}} a^{(3\alpha - 1)/2} p_a \quad , \quad x = \frac{4}{\sqrt{3}} a^{(1 - 3\alpha)/2} \frac{1}{1 - \alpha} \quad ,$$  

is performed in (24), then the reduced hamiltonian takes the form

$$H_r = \frac{1}{2} p_x^2 \quad$$  

which is equivalent to the free particle problem in ordinary quantum mechanics. In [11] such identification was made for $\alpha = 0$. Here, this reduction is valid for any value
of $\alpha$. To reduce the original problem to a free particle problem through a convenient canonical transformation does not mean that their physical content is the same. For example, the harmonic oscillator problem may also be expressed in terms of a free particle problem through a canonical transformation \[12\]. But, their physical contents are very different. The physical interpretation must be made in the original variables, which in the present case is the scale factor $a$. The quantum perfect fluid model is equivalent to a free particle, strictly speaking, only for $\alpha = 1/3$; however, for other values of $\alpha$ the free particle expression may be obtained through simple redefinitions, even if original problem is not exactly the free particle one. Another important difference is that, in the quantum mechanics ordinary case $-\infty < x < \infty$, while in the quantum cosmological case $0 \leq a < \infty$.

4 The classical analogous model

Now, a classical model which reproduces the results found before for the expectation value for the scale factor is worked out. This classical analogue is obtained by considering a Friedmann model with an ordinary perfect fluid model, with an equation of state $p = \alpha \rho$ (the same employed in the quantization in the previous sections), plus a repulsive fluid with an equation of state $p_q = \rho_q$, where the subscript $q$ was chosen in order to remember that we look for a term that may reproduce the quantum effects.

The Einstein’s field equations reduces then to

$$ \left(\frac{\dot{a}}{a}\right)^2 = \frac{C_1}{a^{3(1+\alpha)}} - \frac{C_2}{a^6} . \tag{28} $$

This equation may be solved by reparametrizing the time coordinate as

$$ dt = a^{3\alpha} dT , \tag{29} $$

leading to the expression

$$ \left(\frac{a'}{a}\right)^2 = C_1 a^{-3(1-\alpha)} - C_2 a^{-6(1-\alpha)} . \tag{30} $$

This equation can be easily solved, leading to the following expression for the scale factor:

$$ a(T) = \left(\frac{C_1}{C_2}\right)^{\frac{1}{2(1-\alpha)}} \left[ \frac{C_1^2 C_2}{36(1-\alpha)^2} T^2 + 1 \right]^{-\frac{1}{2(1-\alpha)}} . \tag{31} $$

The first thing to notice is that the time coordinate $T$ is the same obtained in the quantum model: this is due to the choice of the canonical variable in the quantum model. Consequently, the solution (31) is essentially the same as that obtained for the scale factor expectation value in the quantum model. In fact, both solutions coincide quantitatively if we fix the integration constants as

$$ C_1 = \left(\frac{\Gamma\left[\frac{5-3\alpha}{3(1-\alpha)}\right]}{\Gamma\left[\frac{2-3\alpha}{3(1-\alpha)}\right]}\right)^{1-\alpha} \frac{3}{8} \frac{3^{1/3}}{(1-\alpha)^2} \gamma , \quad C_2 = \left(\frac{\Gamma\left[\frac{5-3\alpha}{3(1-\alpha)}\right]}{\Gamma\left[\frac{2-3\alpha}{3(1-\alpha)}\right]}\right)^{-2(1-\alpha)} \frac{3}{4} 3^{1/3} (1-\alpha)^2 . \tag{32} $$
This remark is valid for any value of $\alpha \leq 1$, covering all the "free" particle problem (we remember that the free particle problem occurs, strictly speaking, only for $\alpha = 1/3$). For $\alpha = 1$, the solution for (28) depends on the relative values of $C_1$ and $C_2$: for $C_1 > C_2$, the traditional solution for a stiff matter is obtained with $a \propto t^{1/3}$; for $C_1 = C_2$, the only solution is the Minkowski space-time; for $C_1 < C_2$, there is no lorentzian solution. This must be compared with the fact, stressed before, that there is no consistent quantum solution for this case. The introduction of a order factor in (6) would just change the argument of the gamma functions.

The existence of a repulsive term implies that the energy conditions are violated as the singularity is approached, leading to its avoidance. In fact, let us consider the null energy condition
\begin{equation}
8\pi G(\rho + p) \geq 0.
\end{equation}
This condition establishes that a comoving observer measures a positive energy density; if this condition is violated, the co-moving observer will measure a negative energy density, and repulsive effects take place. Considering the $\rho_{eff}$ and $p_{eff}$ as the sum of the energy and pressure for both the attractive and repulsive fluids, we find
\begin{equation}
8\pi G(\rho_{eff} + p_{eff}) = \frac{2}{a^{6\alpha}} \left( -\frac{a''}{a} + (1 + 3\alpha)\frac{a'^2}{a^2} \right),
\end{equation}
with primes meaning derivative with respect to $T$. Inserting the solutions (31), with an unimportant absorption of integration constant in the definition of the time coordinate, we find
\begin{equation}
8\pi G(\rho_{eff} + p_{eff}) = \frac{4}{a^{6\alpha} \frac{1}{1 - \alpha}^2} \left[ \frac{(1 + \alpha)T^2 - (1 - \alpha)}{(T^2 + 1)^2} \right],
\end{equation}
which becomes negative for $T < \sqrt{\frac{1-\alpha}{1+\alpha}}$. Hence, for each value of $\alpha$ smaller than one, the dominant energy condition is violated around the bounce. For negative values of $\alpha$ the period of time during which the energy conditions are violated becomes larger, and in particular for $\alpha = -1$ the energy conditions are violated for any value of time.

5 The radiative case with curvature

If the curvature of the spatial section is taken into account, the integration of the Wheeler-DeWitt equation in the minisuperspace is not so easy, and perhaps there is no simple analytical solution for any value of $\alpha$. However, for some values of $\alpha$ we can integrate the equations, construct explicitly the wave packets and obtain the expectation value for the scale factor as before.

We will consider now the radiative case, for which such integration is possible, being also a very important particular case. For the radiative case the time coordinate $T$ becomes identical to the conformal time $\eta$. The Wheeler-DeWitt equation in the minisuperspace may be written as
\begin{equation}
\frac{\partial^2 \Psi}{\partial a^2} - 144k a^2 \Psi - i 24 \frac{\partial \Psi}{\partial \eta} = 0.
\end{equation}
Notice that the radiative case is equivalent to the quantum harmonic oscillator \[13\]. The analysis of this equation is more involved. In \[14, 15, 16\] Green’s function methods were employed, and gaussian superposition were constructed.

The equation (36) can be solved by writing

\[
\Psi(a, \eta) = \int_0^\infty G(a, a', \eta) \Psi_0(a') \, da'.
\]

The function \(\Psi_0(a)\) defines the initial configuration, which must satisfy the boundary conditions specified before. The propagator takes the form \[14, 15\]

\[
G(a, a', \eta) = \sqrt{\frac{6\sqrt{k}}{i\pi \sin(\sqrt{k}\eta)}} \exp\left\{\frac{6i\sqrt{k}}{\sin(\sqrt{k}\eta)} \left[(a^2 + a'^2)\cos(\sqrt{k}\eta) - 2aa'\right]\right\}.
\]

Choosing the initial configuration for the wave function as

\[
\Psi_0 = \left(\frac{8\sigma}{\pi}\right)^{1/4} \exp(-\beta a^2), \quad \beta = \sigma + ip,
\]

\(\sigma\) and \(p\) being real parameters, we obtain the following wave function for a curved radiative Universe:

\[
\Psi(a, \eta) = \left(\frac{8\sigma}{\pi}\right)^{1/4} \left\{\frac{6\sqrt{k}}{\cos(\sqrt{k}\eta)[\beta \tan(\sqrt{k}\eta) - 6i\sqrt{k}]} \right\}^{1/2} \times \exp\left\{\frac{6i\sqrt{k}}{\tan(\sqrt{k}\eta)} \left[1 + \frac{6i\sqrt{k}}{\cos^2(\sqrt{k}\eta)[\beta \tan(\sqrt{k}\eta) - 6i\sqrt{k}]}a^2\right]\right\}.
\]

Calculating the expectation value for the scale factor as before, we obtain

\[
\langle a \rangle_\eta = \begin{cases} \sqrt{\sigma^2 \sin^2 \eta + (6 - p \tan \eta)^2 \cos^2 \eta} & k = +1, \\ \sqrt{\sigma^2 \sinh^2 \eta + (6 - p \tanh \eta)^2 \cosh^2 \eta} & k = -1. \end{cases}
\]

The solutions (41) may be rewritten as

\[
\langle a \rangle_\eta = \begin{cases} \sqrt{A_1 \cos 2(\eta - \eta_{01}) + B_1} & k = +1, \\ \sqrt{A_2 \cosh 2(\eta - \eta_{02}) + B_2} & k = -1. \end{cases}
\]

The constants are given by

\[
(A_1)^2 = \left(18 - \frac{\sigma^2}{2} - \frac{p^2}{2}\right) + 36p^2, \quad (A_2)^2 = \left(18 + \frac{\sigma^2}{2} + \frac{p^2}{2}\right) - 36p^2, \quad \tan 2\eta_{01} = \frac{6p}{\frac{\sigma^2}{2} + \frac{p^2}{2} - 18}, \quad \tanh 2\eta_{02} = -\frac{6p}{\frac{\sigma^2}{2} + \frac{p^2}{2} + 18}, \\
B_1 = \frac{\sigma^2}{2} + \frac{p^2}{2} + 18, \quad B_2 = -\frac{\sigma^2}{2} - \frac{p^2}{2} + 18.
\]
Now, we will determine the classical analogue to this model. After the introduction of the stiff repulsive fluid, the equations of motion read, in the conformal time gauge,

\[
\left(\frac{a'}{a}\right)^2 = \frac{C_1}{a^2} - \frac{C_2}{a^4} - k
\]

which can be easily solved:

\[
a = \begin{cases} 
\sqrt{A'_1 \cos 2(\eta - \eta_{01}) + \frac{C_1}{2}} & k = +1 \\
\sqrt{A'_2 \cosh 2(\eta - \eta_{02}) - \frac{C_1}{2}} & k = -1 
\end{cases}
\]

where

\[
A'_1 = \sqrt{\frac{C_1^2}{4} - C_2}, \quad A'_2 = \sqrt{\frac{C_1^2}{4} + C_2}
\]

The solutions (47) represent non-singular Universe and have the same form as those obtained from the expectation value of the scale factor in the quantum model. Notice that the parameter \( p \), which leads to oscillations in the gaussian function, is directly connected with the time phase \( \eta_0 \). The classical analogue permit to give sense to some quantum parameters.

In spite of the fact that the Wheeler-DeWitt equation for the spatially curved radiative case is the same as the Schrödinger equation for the harmonic oscillator, the final solutions exhibited here are different from the equivalent problem in ordinary quantum mechanics. The reason for that relies again on the fact that the dynamical variable here, the scale factor, must be positive definite, a restriction which does not occur in the ordinary harmonic oscillator problem.

The universality of repulsive classical fluid needed to reproduce the quantum behaviour is more intriguing when we remark that it covers the radiative curved case. Hence, thinking in terms of ordinary quantum mechanics, both the free particle and the harmonic oscillator need the same repulsive classical term in order to reproduce the quantum behaviour. Perhaps, other curved cases (which are not reduced to a "free" particle or a harmonic oscillator problem) may be treated in the same lines. But, the lack of simple closed expressions for curved cases with \( \alpha \neq 1/3 \) in the quantum model makes this generalization a hard task.

6 Conclusions

Quantum perfect fluid models in minisuperspace exhibit a dynamical variable, connected with the matter degrees of freedom, which plays the role of time. Hence, the Wheeler-DeWitt equation can be reduced to a Schrödinger equation. For the flat case, solutions are easily obtained for any ordinary fluid with a barotropic equation of state \( p = \alpha \rho \). However, wave packets can be constructed only when \( \alpha < 1 \). For this case, the expectation value of the scale factor reveals a singularity-free Universe which exhibits a bounce. The spreading of the wave packet indicates a behaviour very similar to the free particle of ordinary quantum mechanics.
The main point of the present work is that these quantum perfect fluid models admit a universal classical analogue such that the quantum effects are reproduced by a repulsive fluid with an equation of state \( p = \rho \). This occurs also for the non-flat case with a radiative fluid. The existence of this universal classical analogue, in the sense that the nature of the repulsive fluid required to mimitize the quantum effects does not depend on \( \alpha \) and, perhaps, on \( k \) indicates that the true nature of the quantum perfect fluid model needs a deeper investigation.

Indeed, the flat quantum cosmological perfect fluid model can be reduced, through a canonical transformation, to the problem of a free particle in ordinary quantum mechanics. The variable \( x \) describing the free particle is connected to the scale factor by the relation \( x \propto a^{3(1-\alpha)} \). For \( \alpha = 1/3 \) we face a genuine free particle problem. But, for \( \alpha \neq 1/3 \), the flat case has some similarity with that of a free particle, as the canonical transformation employed in section 3 shows; there are also striking differences with respect to the free particle problem and the similarity may not hide these differences. The classical analogue is valid also for the curved radiative case, which is equivalent to a harmonic oscillator problem. We may guess that this analogue is valid to any value not only of \( \alpha \) but also of \( k \).

Hence, results for quite different quantum cosmological models can be reproduced classically using the same repulsive fluid in addition to the normal one which has been employed in the quantization. This situation does not seem to occur in ordinary quantum mechanics. In particular, it does not occur for the free particle and harmonic oscillator problem. Here, this situation was assured due to the restriction of the dynamical variable, the scale factor, to positive values, a restriction imposed by the fact that we are treating a gravitational system. Hence, this analogous model seems to be particular to the quantum cosmological model.

It can be argued that the question of equivalence between the quantum model and the classical analogue has an obvious negative answer. Quantum mechanics is a completely different framework compared with classical physics. The expectation value of an observable in quantum mechanics may, in principle, be reproduced by a suitable classical model, and in general this fact by itself has no deeper meaning. But here the situation is more involved since it seems that quantum cosmology needs the notion of trajectory and, moreover, the modification introduced in the classical model in order to reproduce the quantum one is universal, covering a large range of quite different models. In our point of view, the universality of the term added to the classical model in order to reproduce the results of the quantum model and the fact that this possibility is typical of the gravity system represent surprising features of the problem. For this reason, we think that the question of the formal equivalence between the quantum cosmological perfect fluid model and the classical model with a repulsive stiff matter fluid may hide a more profound meaning.

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