A Two-Level Fourth-Order Approach For Time-Fractional Convection-Diffusion-Reaction Equation With Variable Coefficients

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Abstract. This paper develops a two-level fourth-order scheme for solving time-fractional convection-diffusion-reaction equation with variable coefficients subjected to suitable initial and boundary conditions. The basis properties of the new approach are investigated and both stability and error estimates of the proposed numerical scheme are deeply analyzed in the $L^{\infty}(0,T;L^2)$-norm. The theory indicates that the method is unconditionally stable with convergence of order $O(k^{2-\frac{\lambda}{2}}+h^4)$, where $k$ and $h$ are time step and mesh size, respectively, and $\lambda \in (0,1)$. This result suggests that the two-level fourth-order technique is more efficient than a large class of numerical techniques widely studied in the literature for the considered problem. Some numerical evidences are provided to verify the unconditional stability and convergence rate of the proposed algorithm.

Keywords: time-fractional Caputo derivative, convection-diffusion-reaction equation with variable coefficients, two-level fourth-order approach, stability analysis, convergence rate.

AMS Subject Classification (MSC). 65M12, 65M06.

1 Introduction

In the last few decades, fractional calculus have played an important role in all areas in sciences and engineering \cite{20, 50, 47}. Most recently, the great potential of the fractional partial differential equations (FPDEs) has motivated the development of efficient numerical schemes for solving both stationary and evolutionary FPDEs describing nonlinear phenomena in medical, biological, physical, financial, and geological systems. For more details, the readers can consult \cite{5, 12, 48, 8}. Although the concepts and the calculus of fractional derivative are few centuries old, fractional advection-diffusion equations have received a great interest in recent years and have been used to model a broad range of problems in fluid flow, electrostatics, electricity, heat, electrodynamic and sound \cite{20, 43}. Owing to the increasing applications, a particular attention is given to the analytical and numerical solutions of FPDEs. In the literature the big challenge with such equations is the design of efficient and accurate numerical methods and computational cost is the main issue to be considered for any numerical scheme. For classical integer order ordinary or partial differential equations (ODEs/PDEs) such as: systems of ODEs, Navier-Stokes equations, mixed Stokes-Darcy model, shallow water equations, advection-diffusion equations, convection-diffusion-reaction equations, heat conduction \cite{22, 24, 52, 40, 6, 28, 30, 31, 14, 42, 33, 37, 10, 27}, a wide class of numerical approaches have been deeply analyzed: finite difference techniques, two-level MacCormack procedure, spectral methods, full implicit finite difference schemes, two-level factored approaches, compact ADI methods and multi-level finite difference formulations. For more details, we refer the readers to \cite{7, 11, 33, 32, 30, 49, 15, 38, 13, 38, 23, 44, 25, 25, 45, 33, 46, 29} and references therein. For fractional partial differential equations, a variety of numerical methods have been developed and their stability and accuracy...
have been widely discussed. Such techniques for solving FPSEs considered implicit meshless schemes based on radial basis functions, finite element formulations, spectral methods, finite difference procedures, meshless methods [17, 2, 16, 51, 13, 18, 41]. This work deals with a two-level fourth-order scheme applied to the time-fractional convection-diffusion-reaction equation with variable coefficients. Though the proposed approach is slightly more accurate (convergence order $O(k^{2-\frac{\lambda}{2}})$) than a large class of numerical methods widely studied in literature [9, 4, 46, 3], it is also less time computing. The main motivation of this paper are the following: (a) we introduce a new parameter $\alpha$ (where $\alpha = 1 - \lambda$) in the discrete time when approximating the time-fractional derivative $(cD_{0t}^\lambda u)$ at the grid point $(x_j, t_{i+1})$; (b) the considered problem has variable coefficients for convection, diffusion and reaction terms; (c) a new two-level method of order $O(k^{2-\frac{\lambda}{2}} + h^4)$ is developed and (d) both stability and convergence rate of the proposed algorithm are deeply analyzed in the $L^\infty(0, T; L^2)$-norm (also $L^2(0, T; L^2)$-norm for numerical examples) by introducing generalized sequences with positive increasing terms. The use of generalized sequences instead of the ordinary ones in the study of the stability and accuracy of a numerical method is an innovation since in our knowledge, there is not available works in the literature that use the generalized sequences in the analysis of both stability and convergence.

In this work, we propose a modified time discrete form combined with finite difference techniques for the time-fractional convection-diffusion-reaction equation involving Caputo fractional derivative and describing by the following initial-boundary value problem

$$cD_{0t}^\lambda u(x, t) - q(t)u_{xx} + p(t)u_x + g(x, t)u(x, t) = s(x, t),$$

where the coefficients $q$ and $p$ are functions depending on the time variable $t$, whereas $g$ and $s$ are functions that depend on both variables $x$ and $t$. Here $p(t) \geq 0$, $g(x, t) \geq 0$, and $q(t) \geq \gamma > 0$, where $\gamma$ is a constant. In [9], the Caputo fractional derivative $cD_{0t}^\lambda u$ $(0 < \lambda < 1)$ of the function $u$ is defined as

$$cD_{0t}^\lambda u(x, t) = \frac{1}{\Gamma(1-\lambda)} \int_0^t \frac{\partial_x u(x, \tau)}{(t-\tau)^\lambda} d\tau,$$

where $\partial_x u$ denotes $\frac{\partial u}{\partial x}$. The initial condition for equation (1) is given by

$$u(x, 0) = \psi_1(x), \quad \text{on} \quad (0, L_1),$$

and the boundary conditions are defined by

$$u(0, t) = \psi_2(t), \quad \text{and} \quad u(L_1, t) = \psi_2(t), \quad \text{on} \quad (0, T).$$

Furthermore, we assume that the exact solution of problem (1)-(3) is sufficiently smooth for the discretization and error estimates. We recall that the aim of this paper is to construct an efficient solution to the time-fractional equation (1) subjects to suitable initial and boundary conditions given by relations (2) and (3), respectively. More specifically, the attention is focused on the following three items:

(i1) detailed description of a two-level fourth-order approach for time-fractional convection-diffusion-reaction equation (1) with appropriate initial-boundary conditions (2)-(3),

(i2) analysis of the unconditional stability and error estimates of the proposed approach,

(i3) a broad range of numerical examples that confirm the theoretical study.

In the following we proceed as follows. Section 2 considers a full description of the two-level fourth scheme for solving the given initial-boundary value problem (1)-(3). In Section 3 we analyze both stability and error estimates of the new algorithm using the $L^\infty(0, T; L^2)$-norm. A wide set of numerical evidences that confirm the theoretical results are presented and discussed in Section 4. Finally, we draw in Section 5 the general conclusion and provide our future works.
2 Full description of a two-level time-fractional method

This section deals with a detailed description of the two-level time-fractional scheme for solving the initial-boundary value problem (1)-(3). The proposed algorithm is a two-step implicit method which approximates the time-fractional Caputo derivative using forward difference in each step and the convection and diffusion terms are approximated by the use of central difference. Since the aim of this section is to develop the method, without loss of generality we should use a constant time step \( k = \Delta t \) and space step \( h = \Delta x \). Let \( \lambda \) be a real number satisfying \( 0 < \lambda < 1 \), \( L_1 > 0 \) and \( T > 0 \) be the space interval length and time interval length, respectively. Suppose \( N \) and \( M \) be two positive integers. Set \( x_j = jh \), \( t_i = ik \) and let the superscript denoting the time level and space level of the approximation, where \( k = \frac{T}{N} \) and \( h = \frac{L_1}{M} \). Consider the uniform mesh space \( Y_{kh} = \{(x_j, t_i), 0 \leq i \leq N; 0 \leq j \leq M\} \). Furthermore, we introduce the positive parameter \( \alpha = 1 - \lambda \). For a function \( u \in C^0([0, T]; H^\alpha(0, L_1)) \), the Caputo fractional derivatives of order \( \alpha \) of the function \( u \) at the grid points \( (x_j, t_i, t_{i+\frac{1}{2}+\alpha}) \) and \( (x_j, t_{i+1+\alpha}) \) are given by

\[
cD_0^\lambda u(x_j, t_{i+\frac{1}{2}+\alpha}) = \frac{1}{\Gamma(1-\lambda)} \int_0^{t_{i+\frac{1}{2}+\alpha}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau
\]

and

\[
cD_0^\lambda u(x_j, t_{i+1+\alpha}) = \frac{1}{\Gamma(1-\lambda)} \int_0^{t_{i+1+\alpha}} u_{\tau,j}(\tau)(t_{i+1+\alpha} - \tau)^{-\lambda} d\tau.
\]

For the convenience of writing, we should set in the following the notations

\[
cD_0^\lambda u(x_j, t_{i+\frac{1}{2}+\alpha}) = cD_0^\lambda u_{j}^{i+\frac{1}{2}+\alpha} \quad \text{and} \quad cD_0^\lambda u(x_j, t_{i+1+\alpha}) = cD_0^\lambda u_{j}^{i+1+\alpha}, \quad \text{for } i \geq 0.
\]

Using this, equation (1) can be rewritten as

\[
cD_0^\lambda u_{j}^{i+\frac{1}{2}+\alpha} = \frac{1}{\Gamma(1-\lambda)} \int_0^{t_{i+\frac{1}{2}+\alpha}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau = \frac{1}{\Gamma(1-\lambda)} \sum_{l=0}^{i-1} \int_{t_l}^{t_{l+\frac{1}{2}}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau
\]

\[
+ \frac{1}{\Gamma(1-\lambda)} \left[ \int_0^{t_{i+\frac{1}{2}}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau + \int_{t_{i+\frac{1}{2}}}^{t_{i+\frac{1}{2}+\alpha}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau \right].
\]

Furthermore, let introduce the following operators

\[
\delta_u u_{j}^{i} = \frac{u_{j}^{i+1} - u_{j}^{i}}{h}; \quad \delta_x u_{j}^{i+\frac{1}{2}} = \frac{u_{j}^{i+1} - u_{j}^{i}}{h}; \quad \delta_x^2 u_{j}^{i} = \frac{u_{j}^{i+1} - 2u_{j}^{i} + u_{j}^{i-1}}{h^2}.
\]

We consider the following norms and inner product

\[
\|u\|_{L^2} = \left( \sum_{j=1}^M (u_{j}^{i})^2 \right)^{\frac{1}{2}}, \quad \|u\|_{C_{D}^{0,3}} = \max_{0 \leq s \leq 3} \max_{0 \leq \xi \leq 6} \|\partial_y^s \partial_x^\xi u(t_i)\|_{L^2} \quad \text{and} \quad (u^I, v^I) = h \sum_{j=1}^{M-1} u_{j}^I v_{j}^I.
\]

where \( D = [0, L_1] \times [0, T] \), \(|\cdot|\) denotes the \( C \)-norm. The spaces \( L^2(0, L_1) \) and \( C_{D}^{0,3} \) are endowed with the norms \( \|\cdot\|_{L^2} \) and \( \|\cdot\|_{C_{D}^{0,3}} \) respectively, whereas the Hilbert space \( L^2(0, L_1) \) is equipped with the inner product \((\cdot, \cdot)\).

Let \( P_{2}^{u_{j}^{i}, t} \) be the second-order polynomial interpolating \( u_{j}(t) \) at the points \( (t_{i+\frac{1}{2}}, u_{j}^{i+\frac{1}{2}}), (t_{i+1}, u_{j}^{i+1}) \) and \( (t_{i+\frac{1}{2}}, u_{j}^{i+\frac{1}{2}}) \). Simple calculations give

\[
P_{2}^{u_{j}^{i}, t}(t) = \frac{2}{k^2} \left[ (t - t_{i+1})(t - t_{i+\frac{1}{2}})u_{j}^{i+\frac{1}{2}} - 2(t - t_{i+\frac{1}{2}})(t - t_{i+1})u_{j}^{i+1} + (t - t_{i+\frac{1}{2}})(t - t_{i+1})u_{j}^{i+\frac{1}{2}} \right],
\]

and the corresponding error is defined as

\[
E_j^I(t) = u_{j}(t) - P_{2}^{u_{j}^{i}, t}(t) = \frac{1}{6} (t - t_{i+\frac{1}{2}})(t - t_{i+1})(t - t_{i+\frac{1}{2}})u_{3t,j}(t_i^I),
\]
where \( t_i^* \) is between the minimum and maximum of \( t_{i+\frac{1}{2}}, t_{i+1}, t_{i+\frac{3}{2}} \) and \( t \). The derivative of \( P_{l,l}^{\mu,j} \) with respect to the time-variable provides

\[
P_{l,l}^{\mu,j}(t) = \frac{2}{k^2} \left[ (u_j^{l+\frac{1}{2}} - 2u_j^{l+1} + u_j^{l+\frac{3}{2}}) + (2(t_{i+1} + t_{i+\frac{1}{2}})u_j^{l+\frac{1}{2}} + 2(t_{i+\frac{1}{2}} + t_{i+1})u_j^{l+1} - (t_{i+\frac{3}{2}} + t_{i+1})u_j^{l+\frac{3}{2}} \right].
\]

Setting

\[
I_i^{(\alpha)} = \frac{1}{\Gamma(1-\lambda)} \left[ \int_{t_{i+\frac{1}{2}}}^{t_{i+\frac{3}{2}}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau \right] + \frac{1}{\Gamma(1-\lambda)} \sum_{i=0}^{i-1} \int_{t_{i+\frac{1}{2}}}^{t_{i+\frac{3}{2}}} E_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau.
\]

Since \((1-\lambda)\Gamma(1-\lambda) = \Gamma(2-\lambda)\), plugging equations (9)-(12), direct computations yield

\[
cD_0^{\lambda}u_j^{l+\frac{1}{2}+\alpha} = \frac{2}{k^2\Gamma(2-\lambda)} \sum_{i=0}^{i-1} \left[ -2 \left( u_j^{l+\frac{1}{2}} - 2u_j^{l+1} + u_j^{l+\frac{3}{2}} \right) \left( \tau(t_{i+\frac{1}{2}+\alpha} - \tau)^{1-\lambda} + \frac{1}{2-\lambda}(t_{i+\frac{1}{2}+\alpha} - \tau)^{2-\lambda} \right) \right. \\
\left. + \left( (t_{i+1} + t_{i+\frac{1}{2}})u_j^{l+\frac{3}{2}} - 2(t_{i+\frac{1}{2}} + t_{i+1})u_j^{l+\frac{1}{2}} + (t_{i+\frac{1}{2}} + t_{i+1})u_j^{l+\frac{3}{2}} \right) \left( t_{i+\frac{1}{2}+\alpha} - \tau \right)^{1-\lambda} \right] t_{i+\frac{1}{2}}^{l+\frac{1}{2}} + I_i^{(\alpha)} \\
= \frac{2k^{1-\lambda}}{k^2\Gamma(2-\lambda)} \sum_{i=0}^{i-1} \left[ \left( (i+\alpha-l)1^{1-\lambda} - \frac{3}{2}(i+\alpha-l-1)^{1-\lambda} \right) \delta_t u_j^{l+\frac{1}{2}} + \frac{2}{2-\lambda} \left( (i+\alpha-l)^{2-\lambda} - (i+\alpha-l-1)^{2-\lambda} \right) \right] \delta_t u_j^{l+\frac{1}{2}} + I_i^{(\alpha)} \\
= \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{i=0}^{i-1} \left[ \tilde{f}_{i+\frac{1}{2},l}^{\lambda} \delta_t u_j^{l+1} + (\tilde{d}_{i+\frac{1}{2},l}^{\lambda} - \tilde{f}_{i+\frac{1}{2},l}^{\lambda}) \delta_t u_j^{l+\frac{1}{2}} \right] + I_i^{(\alpha)},
\]

where

\[
\tilde{d}_{i+\frac{1}{2},l}^{\lambda} = (i+\alpha-l)^{1-\lambda} - (i+\alpha-l-1)^{1-\lambda},
\]

and

\[
\tilde{f}_{i+\frac{1}{2},l}^{\lambda} = \frac{2}{2-\lambda} \left[ (i+\alpha-l)^{2-\lambda} - (i+\alpha-l-1)^{2-\lambda} \right] - \frac{3}{2} (i+\alpha-l-1)^{1-\lambda} + \frac{3}{2} (i+\alpha-l)^{1-\lambda} + 3(i+\alpha-l-1)^{1-\lambda},
\]

for \( 0 \leq l \leq i-1, i \geq 1 \). In addition, we should find an explicit expression of the term \( cD_0^{\lambda}u_j^{l+\frac{1}{2}+\alpha} \).

\[
cD_0^{\lambda}u_j^{l+\frac{1}{2}+\alpha} = \frac{1}{\Gamma(1-\lambda)} \int_0^{t_{i+\frac{3}{2}}} u_{\tau,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau \\
= \frac{1}{\Gamma(1-\lambda)} \left[ \int_0^{t_{i+\frac{3}{2}}} \delta_t u_j(t_0)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau \\
+ \int_0^{t_{i+\frac{3}{2}}} (u_{\tau,j}(\tau) - \delta_t u_j(t_0))(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau \right] = cD_0^{\lambda}u_j^{l+\frac{1}{2}+\alpha} + \frac{1}{\Gamma(1-\lambda)} \int_0^{t_{i+\frac{3}{2}}} (u_{\tau,j}(\tau) - P_{1,\tau}^{\mu,j,0})(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau,
\]
where

\[ c \Delta_0^u u_j^{t+1} = \frac{k^{1-\lambda}(\frac{1}{2} + \alpha)^{1-\lambda}}{\Gamma(2-\lambda)} \delta_i u_j^0. \]  

(17)

In a similar manner, the term \( \frac{1}{\Gamma(1-\lambda)} \int_{t_i+\frac{1}{2}}^{t_i+\frac{1}{2}+\alpha} u_{\tau,j}(\tau)(1+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau \) can be rewritten as

\[ \frac{1}{\Gamma(1-\lambda)} \int_{t_i+\frac{1}{2}}^{t_i+\frac{1}{2}+\alpha} u_{\tau,j}(\tau)(1+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau = \frac{k^{1-\lambda}(\frac{1}{2} + \alpha)^{1-\lambda}}{\Gamma(2-\lambda)} \delta_i u_j^i + \frac{1}{\Gamma(1-\lambda)} \int_{t_i+\frac{1}{2}}^{t_i+\frac{1}{2}+\alpha} u_{\tau,j}(\tau) - \delta_i u_j^i \frac{1}{(t_i+\frac{1}{2}+\alpha - \tau)^{\lambda}} d\tau. \]  

(18)

Furthermore, it is easy to observe that

\[ \frac{1}{\Gamma(1-\lambda)} \int_0^{t_i+\frac{1}{2}+\alpha} u_{\tau,j}(\tau)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau = \frac{1}{\Gamma(1-\lambda)} \int_0^{t_i+\frac{1}{2}+\alpha} (u_{\tau,j}(\tau) - P_{1,\tau,j}^u)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau, \]  

(19)

If \( i = 0 \),

\[ cD_0^u u_j^{t+1} = c\Delta_0^u u_j^{t+1} + \frac{1}{\Gamma(1-\lambda)} \int_0^{t_i+\frac{1}{2}+\alpha} (u_{\tau,j}(\tau) - P_{1,\tau,j}^u)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau = \]

\[ f_{i+\frac{1}{2}}^0 \delta_i u_j^0 + \frac{1}{\Gamma(1-\lambda)} \int_0^{t_i+\frac{1}{2}+\alpha} (u_{\tau,j}(\tau) - P_{1,\tau,j}^u)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau, \]  

(20)

and

\[ cD_0^u u_j^{t+1} = c\Delta_0^u u_j^{t+1} + \quad + \frac{1}{\Gamma(1-\lambda)} \left\{ \sum_{i=0}^{i-1} \int_{t_i+\frac{1}{2}}^{t_i+\frac{1}{2}+\alpha} E_{\tau,j}(\tau)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau + \right. \]

\[ \left. \int_{t_i+\frac{1}{2}}^{t_i+\frac{1}{2}+\alpha} (u_{\tau,j}(\tau) - \delta_i u_j^i)(t_i+\frac{1}{2}+\alpha - \tau)^{-\lambda} d\tau \right\}, \]  

(21)

for \( i \geq 1 \),

\[ c\Delta_0^u u_j^{t+1} = f_{i+\frac{1}{2},0} \delta_i u_j^i, \quad \text{if} \quad i = 0, \]  

(22)

\[ c\Delta_0^u u_j^{t+1} = f_{i+\frac{1}{2},0} \delta_i u_j^i + \sum_{i=0}^{i-1} \left[ f_{i+\frac{1}{2},i}^0 \delta_i u_j^i + (d_{i+\frac{1}{2},i} - f_{i+\frac{1}{2},i}^0) \delta_i u_j^i \right] + f_{i+\frac{1}{2},i}^0 \delta_i u_j^i, \quad \text{if} \quad i \geq 1, \]  

(23)

where

\[ f_{i+\frac{1}{2},0}^0 = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} f_{i+\frac{1}{2},0}; \quad f_{i+\frac{1}{2},0} = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} f_{i+\frac{1}{2},0}; \quad f_{i+\frac{1}{2},i} = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} f_{i+\frac{1}{2},i}; \quad f_{i+\frac{1}{2},i} = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} f_{i+\frac{1}{2},i}; \]  

(24)

\[ d_{i+\frac{1}{2},i} = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} d_{i+\frac{1}{2},i}; \quad \text{for} \quad 0 \leq l \leq i - 1, \]

with

\[ f_{i+\frac{1}{2},i} = 0 \]

(25)

\[ f_{i+\frac{1}{2},0} = 0; \quad f_{i+\frac{1}{2},0} = \frac{1}{\lambda}; \quad f_{i+\frac{1}{2},i} = (i + \frac{1}{2} + \alpha)^{1-\lambda} - (i + \alpha)^{1-\lambda}, \]

(26)

To construct the first-level of the proposed approach, we shall find a similar approximation for the function \( q(t)u_{2x} - p(t)u_x - g(x,t)u \) at the point \( (x_j,t_i+\frac{1}{2}+\alpha) \). Firstly,

\[ [q(t)u_{2x} - p(t)u_x - g(x,t)u](x_j,t_i+\frac{1}{2}+\alpha) = q_i^{i+\frac{1}{2}+\alpha} u_{2x,j}^{i+\frac{1}{2}+\alpha} - p_i^{i+\frac{1}{2}+\alpha} u_{x,j}^{i+\frac{1}{2}+\alpha} - g_i^{i+\frac{1}{2}+\alpha} u_j^{i+\frac{1}{2}+\alpha} + s_i^{i+\frac{1}{2}+\alpha}. \]  

(26)
Expanding the Taylor series for \( u \) about the point \((x_j, t_{i+\frac{1}{2}+\alpha})\) with step size \( h \) using both forward and backward representations provides

\[
u_{j+\frac{1}{2}+\alpha} = u_{j,2\alpha} + 2h u_{j+\frac{1}{2}+\alpha} + 2h^2 u_{2\alpha,j} + \frac{4h^3}{3} u_{3\alpha,j} + \frac{2h^4}{3} u_{4\alpha,j} + \frac{4h^5}{15} u_{5\alpha,j} + \frac{4h^6}{45} u_{6\alpha,j} (\varepsilon_j),
\]

(27)

\[
u_{j-\frac{1}{2}+\alpha} = u_{j,2\alpha} - 2h u_{j-\frac{1}{2}+\alpha} + 2h^2 u_{2\alpha,j} - \frac{4h^3}{3} u_{3\alpha,j} + \frac{2h^4}{3} u_{4\alpha,j} - \frac{4h^5}{15} u_{5\alpha,j} + \frac{4h^6}{45} u_{6\alpha,j} (\varepsilon_j),
\]

(28)

\[
u_{j+1/2+\alpha} = u_{j,2\alpha} + h u_{x,j} + \frac{h^2}{2} u_{2\alpha,j} + \frac{h^3}{6} u_{3\alpha,j} + \frac{h^4}{24} u_{4\alpha,j} + \frac{h^5}{120} u_{5\alpha,j} + \frac{h^6}{720} u_{6\alpha,j} (\varepsilon_j),
\]

(29)

\[
u_{j-1/2+\alpha} = u_{j,2\alpha} - h u_{x,j} + \frac{h^2}{2} u_{2\alpha,j} - \frac{h^3}{6} u_{3\alpha,j} + \frac{h^4}{24} u_{4\alpha,j} - \frac{h^5}{120} u_{5\alpha,j} + \frac{h^6}{720} u_{6\alpha,j} (\varepsilon_j),
\]

(30)

where \( \varepsilon_j \in (x_{j-1}, x_j), \ v_j^5 \in (x_j, x_{j+1}), \ v_j^6 \in (x_{j-2}, x_j), \ v_j^7 \in (x_j, x_{j+2}). \)

Combining equations (27) and (28), direct computations give

\[
u_{j+2}^{\frac{1}{2}+\alpha} + \nu_{j-2}^{\frac{1}{2}+\alpha} = 2u_{j,2\alpha} + 4h^2 u_{2\alpha,j} + \frac{4h^4}{3} u_{4\alpha,j} + \frac{4h^6}{45} [u_{i,6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j) + u_{i,6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^5)].
\]

(32)

In a similar manner, plugging equations (29) and (30) to get

\[
u_{j+1}^{\frac{1}{2}+\alpha} + \nu_{j-1}^{\frac{1}{2}+\alpha} = 2u_{j,2\alpha} + h^2 u_{2\alpha,j} + \frac{h^4}{12} u_{4\alpha,j} + \frac{h^6}{360} [u_{i,6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j) + u_{i,6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^5)].
\]

(33)

Multiplying both sides of equations (32) and (33) by 12 and \( \frac{\alpha}{6} \), respectively, and adding the resulting equations to obtain

\[12(u_{j+1}^{\frac{1}{2}+\alpha} + u_{j-1}^{\frac{1}{2}+\alpha}) - 3(u_{j+2}^{\frac{1}{2}+\alpha} + u_{j-2}^{\frac{1}{2}+\alpha}) = \frac{45}{2} u_{j,2\alpha} + 9h^2 u_{2\alpha,j} + \frac{h^4}{30} u_{4\alpha,j} + \frac{h^6}{30} u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^4) + u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^5) - 18u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^6) - 18u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^7).
\]

Solving this equation for \( u_{j,2\alpha} \), we obtain

\[
u_{j,2\alpha} = \frac{1}{12h^2} \left[-u_{j+2}^{\frac{1}{2}+\alpha} + 16u_{j+1}^{\frac{1}{2}+\alpha} - 30u_j^{\frac{1}{2}+\alpha} + 16u_{j-1}^{\frac{1}{2}+\alpha} - u_{j-2}^{\frac{1}{2}+\alpha} \right] + \frac{h^4}{270} \left[18u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^6) + 18u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^7) - u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^4) - u_{6\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^5) \right].
\]

(34)

In a similar way, one easily shows that

\[
u_{j,2\alpha} = \frac{1}{12h^2} \left[-u_{j+2}^{\frac{1}{2}+\alpha} + 8u_{j+1}^{\frac{1}{2}+\alpha} - 8u_j^{\frac{1}{2}+\alpha} + u_{j-2}^{\frac{1}{2}+\alpha} \right] + \frac{h^4}{180} \left[4u_{5\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^8) + 4u_{5\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^9) - u_{5\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^8) - u_{5\alpha}^{\frac{1}{2}+\alpha}(\varepsilon_j^9) \right],
\]

(35)

where \( \varepsilon_j^9 \in (x_{j-1}, x_j), \ v_j^8 \in (x_j, x_{j+1}), \ v_j^{11} \in (x_{j-2}, x_j), \ v_j^{10} \in (x_j, x_{j+2}). \)

On the other hand, the application of the Taylor series formulation for the function \( u \) about the point \((x_j, t_{i+\frac{1}{2}+\alpha})\) with time step \( k \) results in

\[
u_{j}^{\frac{1}{2}+\alpha} = u_j^{\frac{1}{2}+\alpha} - \alpha k u_j^{\frac{1}{2}+\alpha} + \frac{1}{2} \alpha^2 k^2 u_{2\alpha,j}(\varepsilon_j^4), \ u_j^{\frac{1}{2}+\alpha} = u_j^{\frac{1}{2}+\alpha} - (\frac{1}{2} + \alpha) k u_j^{\frac{1}{2}+\alpha} + \frac{1}{2} (\frac{1}{2} + \alpha)^2 k^2 u_{2\alpha,j}(\varepsilon_j^5),
\]

(37)

where \( \varepsilon_j^4 \in (t_{i+\frac{1}{2}}, t_{i+\frac{1}{2}+\alpha}) \) and \( \varepsilon_j^5 \in (t_i, t_{i+\frac{1}{2}+\alpha}). \) Combining both equations in (37), simple computations yield

\[
u_{j}^{\frac{1}{2}+\alpha} = (1 + 2\alpha) u_j^{\frac{1}{2}+\alpha} - 2\alpha u_j^\alpha - (\frac{1}{2} + \alpha) k^2 H_{1,j}^{\frac{1}{2}+\alpha},
\]

(38)
where

\[ H_{i,j}^1 = \alpha u_{2i,j}(e_i^1) - \left( \frac{1}{2} + \alpha \right) u_{2i,j}(e_i^2). \]  

(39)

Setting \( u_j^{\alpha_i} = (1 + 2\alpha)u_j^{i+\frac{1}{2}} - 2\alpha u_j^i \), equation (38) becomes

\[ u_j^{i+\frac{1}{2}+\alpha} = u_j^{\alpha_i} - \alpha \left( \frac{1}{2} + \alpha \right) k^2 H_{i,j}^1. \]

Utilizing this, equations (34) and (35) can be rewritten as

\[ \frac{1}{12h^2} \left[ -u_j^{\alpha_i} + 16u_j^{\alpha_i} - 30u_j^{\alpha_i} + 16u_j^{\alpha_i} \right] + \psi_{i,j}, \]

(40)

and

\[ \frac{1}{12h^2} \left[ -u_j^{\alpha_i} + 8u_j^{\alpha_i} - 8u_j^{\alpha_i} + u_j^{\alpha_i} \right] + \psi_{i,j}, \]

(41)

where

\[ \psi_{i,j} = \frac{h^4}{270} \left[ 18u_6^{i+\frac{1}{2}+\alpha}(e_6^1) + 18u_6^{i+\frac{1}{2}+\alpha}(e_6^2) - u_{6x}^{i+\frac{1}{2}+\alpha}(e_6^1) - u_{6x}^{i+\frac{1}{2}+\alpha}(e_6^2) \right] + \frac{\alpha (\frac{1}{2} + \alpha) k^2}{12h^2} [H_{i,j+2} - 16H_{i,j+1} + 30H_{i,j} - 16H_{i,j-1} + H_{i,j-2}], \]

(42)

and

\[ \psi_{i,j} = \frac{h^4}{180} \left[ 4u_5^{i+\frac{1}{2}+\alpha}(e_5^0) + 4u_5^{i+\frac{1}{2}+\alpha}(e_5^1) - u_{5x}^{i+\frac{1}{2}+\alpha}(e_5^0) - u_{5x}^{i+\frac{1}{2}+\alpha}(e_5^1) \right] + \frac{\alpha (\frac{1}{2} + \alpha) k^2}{12h^2} [H_{i,j+2} - 8H_{i,j+1} + 8H_{i,j-1} - H_{i,j-2}], \]

(43)

To develop the first-level of the proposed approach, we should combine equations (1), (21), (26) and (20-23). This provides

\[ c\Delta^\lambda_{i} u_j^{i+\frac{1}{2}+\alpha} + \frac{1}{\Gamma(1 - \lambda)} \left\{ \sum_{l=0}^{i-1} \int_{l+\frac{1}{2}}^{l+1} E_{i,j}(\tau)(t_{i+\frac{1}{2}+\alpha} - \tau)^{-\lambda} d\tau + \int_{0}^{i+\frac{1}{2}+\alpha} \frac{u_{x,j}(\tau) - \delta_i u_j^0}{(t_{i+\frac{1}{2}+\alpha} - \tau)^{\lambda}} d\tau \right\} = q^{i+\frac{1}{2}+\alpha} \left[ -u_j^{\alpha_i} + 16u_j^{\alpha_i} - 30u_j^{\alpha_i} + 16u_j^{\alpha_i} \right] - \frac{p^{i+\frac{1}{2}+\alpha}}{12h} \left[ -u_j^{\alpha_i} + 8u_j^{\alpha_i} - 8u_j^{\alpha_i} + u_j^{\alpha_i} \right] - g_j^{i+\frac{1}{2}+\alpha} u_j^{\alpha_i} + s_j^{i+\frac{1}{2}+\alpha} + q^{i+\frac{1}{2}+\alpha} \psi_{i,j} - \frac{p^{i+\frac{1}{2}+\alpha}}{12h} H_{i,j}. \]

(44)

Suppose \( U_j^i \) be the approximate solution at the grid point \((x_j, t_i)\). Tracking the error terms in both sides, (44) becomes

\[ c\Delta^\lambda_{i} U_j^{i+\frac{1}{2}+\alpha} = q^{i+\frac{1}{2}+\alpha} \left[ -U_j^{\alpha_i} + 16U_j^{\alpha_i} - 30U_j^{\alpha_i} + 16U_j^{\alpha_i} \right] - \frac{p^{i+\frac{1}{2}+\alpha}}{12h} \left[ -U_j^{\alpha_i} + 8U_j^{\alpha_i} - 8U_j^{\alpha_i} + U_j^{\alpha_i} \right] - g_j^{i+\frac{1}{2}+\alpha} U_j^{\alpha_i} + s_j^{i+\frac{1}{2}+\alpha}. \]

Using equation (23), expanding and rearranging terms, this approximation is equivalent to

\[ U_j^{i+\frac{1}{2}} = U_j^{i} - \frac{k}{2} \left( f_{\alpha_i}^{i+\frac{1}{2}} + f_{\alpha_i}^{i+\frac{1}{2}} \right)^{-1} \left\{ \left( \lambda_{\alpha_i}^{i+\frac{1}{2}} + (\delta_{\alpha_i}^{i+\frac{1}{2},-1} - f_{\alpha_i}^{i+\frac{1}{2},-1}) \right) u_j^{\alpha_i} + \sum_{l=0}^{i-2} \left[ f_{\alpha_i}^{i+\frac{1}{2},l} \delta_{\alpha_i}^{i+\frac{1}{2},l+1} \right] \right\} \]

\[ - \frac{q^{i+\frac{1}{2}+\alpha}}{12h^2} \left[ -U_j^{\alpha_i} + 16U_j^{\alpha_i} - 30U_j^{\alpha_i} + 16U_j^{\alpha_i} \right] + \]

\[ (f_{\alpha_i}^{i+\frac{1}{2}} - f_{\alpha_i}^{i+\frac{1}{2}}) \delta_{\alpha_i}^{i+\frac{1}{2}} \right\} - \frac{q^{i+\frac{1}{2}+\alpha}}{12h^2} \left[ -U_j^{\alpha_i} + 16U_j^{\alpha_i} - 30U_j^{\alpha_i} + 16U_j^{\alpha_i} \right]. \]
Here the sum equals zero is the upper summation index is less than the lower one.

For \( i = 0 \), combining equations \((1)\), \((24)\), \((25)\), and \((26)\) - \((28)\), it is not difficult to observe that

\[
f_f^\lambda_{x,0} \delta_i U_0^0 + \frac{1}{\Gamma(1 - \lambda)} \int_0^{t_i} \frac{u_{\tau,j}(\tau) - \delta_i U_0^0}{(t_i - \tau)^\lambda} d\tau = \frac{q^0_0 + \alpha}{12h} \left[ u_{\tau,j}^0 + 16u_{\tau,j+1}^0 - 30u_{\tau,j-2}^0 \right] - \frac{p^0_{x,0} q^0}{12h} \left[ u_{\tau,j}^0 + 16u_{\tau,j+1}^0 - 8u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + 8u_{\tau,j+1}^0 - 8u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right].
\]

Expanding and rearranging terms, it is easy to see that

\[
U_j^\lambda = U_j^0 + \frac{k}{2} \left( \frac{f_f^\lambda_{x,0}}{12h} \right)^{-1} \left[ u_{\tau,j}^0 + 16u_{\tau,j+1}^0 - 30u_{\tau,j-2}^0 \right] - \frac{p^0_{x,0} q^0}{12h} \left[ u_{\tau,j}^0 + 16u_{\tau,j+1}^0 - 8u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + 8u_{\tau,j+1}^0 - 8u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right] - \frac{q^0_{x,0}}{12h} \left[ u_{\tau,j}^0 + u_{\tau,j+1}^0 + u_{\tau,j-1}^0 \right].
\]

It worth noticing to mention that equation \((45)\) defines the first-level of the new method subject to the appropriate initial condition \((48)\).

Now, we should describe the second step of the two-level formulation for solving the considered time-fractional convection-diffusion-reaction equation.

Considering equation \((5)\), it is not hard to see that

\[
cD_0^\alpha u_j^{t+1,\lambda} = \frac{1}{\Gamma(1 - \lambda)} \int_0^{t_i} \frac{u_{\tau,j}(\tau)(t_i + 1 + \alpha - \tau)^{-\lambda} d\tau = \frac{1}{\Gamma(1 - \lambda)} \sum_{l=0}^{i} \int_{t_l}^{t_{l+1}} u_{\tau,j}(\tau)(t_i + 1 + \alpha - \tau)^{-\lambda} d\tau + \int_{t_{l+1}}^{t_{l+1} + \alpha} u_{\tau,j}(\tau)(t_i + 1 + \alpha - \tau)^{-\lambda} d\tau.\]

Let \( \hat{P}_{2,\lambda} u_{\tau,j} \) be the quadratic polynomial interpolating the function \( u_j(t) \) at the points \( (t_i, u_j^t), (t_i + 1, u_j^{t+1}) \) and \( (t_{i+1}, u_j^{t+1}) \). Direct computations result in

\[
\hat{P}_{2,\lambda} u_{\tau,j} = \frac{2}{k^2} \left[ (t - t_i + t_{i+1})(t - t_i)u_j^t - 2(t - t_i)(t - t_i + 1)u_j^{t+1} + (t - t_i)(t - t_i + 1)u_j^{t+1} \right].
\]

The corresponding error is given by

\[
\hat{E}_{2,\lambda} = u_j(t) - \hat{P}_{2,\lambda} u_{\tau,j} = \frac{1}{6} (t - t_i)(t - t_i + 1)(t_i + 1)u_{\tau,j}(t - t_i)(t - t_i + 1)(t_i + 1),
\]

where \( t_{i+1} \) is between the minimum and maximum of \( t_i, t_{i+1} \), \( t_{i+1} \) and \( t \). Furthermore

\[
\hat{P}_{2,\lambda} u_{\tau,j} = \frac{2}{k^2} \left[ 2(u_j^t - 2u_j^{t+1})t - (t_i + 1)(t_i + 1)u_j^{t+1} + 2(t_i + 1)u_j^{t+1} - (t_i + 1)u_j^{t+1} \right].
\]
Setting
\[ J_j^{(i, \alpha)} = \frac{1}{1 - \lambda} \left[ \int_{t_{i+1}}^{t_{i+1+\alpha}} u_{\tau,j}(\tau)(t_{i+1+\alpha} - \tau)^{-\lambda} d\tau + \sum_{l=0}^{i} \int_{t_l}^{t_{i+1}} \tilde{E}_{\tau,j}(\tau)(t_{i+1+\alpha} - \tau)^{-\lambda} d\tau \right]. \] (53)

Combining equations (49) - (53), simple calculations give
\[ cD_{t^i} u_j^{i+1+\alpha} = \frac{1}{1 - \lambda} \left[ \sum_{l=0}^{i} \int_{t_l}^{t_{i+1}} \tilde{P}_{2;\tau,l}^i(\tau)(t_{i+1+\alpha} - \tau)^{-\lambda} d\tau + \delta_t u_j^{i+\frac{1}{2}} \int_{t_{i+1}}^{t_{i+1+\alpha}} (t_{i+1+\alpha} - \tau)^{-\lambda} d\tau \right] + J_j^{(i, \alpha)} \]
\[ = \frac{k^{1-\lambda}}{1 - \lambda} \sum_{l=0}^{i} \left[ \hat{\delta}_l u_j^{i+\frac{1}{2}} - \delta_t u_j^{i+\frac{1}{2}} \right] \left( (i + l + \alpha - l)^{-\lambda} - (i + l + \alpha - l)^{-\lambda} + \frac{1}{2 - \lambda} \right) + \frac{1}{2 - \lambda} \left( (i + l + \alpha - l)^{-\lambda} - \left( (i + l + \alpha - l)^{-\lambda} \right) \right) + J_j^{(i, \alpha)} \]
\[ = \sum_{l=0}^{i} \left[ f_{j, i+j, i+1}^{\alpha} \delta_t u_j^{i+\frac{1}{2}} \delta_t u_j^{i+\frac{1}{2}} + (\hat{d}_{j, i+j, i+1}) \delta_t u_j^{i+\frac{1}{2}} \right] + f_{j, i+j, i+1}^{\alpha} \delta_t u_j^{i+\frac{1}{2}} + J_j^{(i, \alpha)} = cD_{t^i} u_j^{i+1+\alpha} + J_j^{(i, \alpha)}, \] (54)
where
\[ cD_{t^i} u_j^{i+1+\alpha} = \sum_{l=0}^{i} \left[ f_{j, i+j, i+1}^{\alpha} \delta_t u_j^{i+\frac{1}{2}} + (\hat{d}_{j, i+j, i+1}) \delta_t u_j^{i+\frac{1}{2}} \right] + f_{j, i+j, i+1}^{\alpha} \delta_t u_j^{i+\frac{1}{2}}, \] (55)
\[ d_{l, i+j, i+1}^{\alpha} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}} - \delta_t u_j^{i+\frac{1}{2}}, \] (56)
where
\[ \delta_t u_j^{i+\frac{1}{2}} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}}, \]
\[ \delta_t u_j^{i+\frac{1}{2}} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}}, \]
\[ \delta_t u_j^{i+\frac{1}{2}} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}}, \]
\[ \delta_t u_j^{i+\frac{1}{2}} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}}, \]
\[ \delta_t u_j^{i+\frac{1}{2}} = \frac{k^{1-\lambda}}{1 - \lambda} \delta_t u_j^{i+\frac{1}{2}}, \]
for \(0 \leq l \leq i, \) with \(i \geq 0.\) Applying the Taylor approximation, it is not difficult to show that
\[ u_j^{i+1+\alpha} = (1 + 2a)u_j^{i+\frac{1}{2}} - 2a u_j^{i+\frac{1}{2}} - a(\frac{1}{2} + \alpha)k^2 H_{2,j}^i, \] (58)
where
\[ H_{2,j}^i = au_{2,j}(c_{i}^j) - (\frac{1}{2} + \alpha)u_{2,j}(c_{i}^j), \] (59)
where \(c_{i}^j \in (t_{i+1}, t_{i+1+\alpha})\) and \(c_{i}^j \in (t_{i+\frac{1}{2}}, t_{i+1+\alpha}).\) Replacing \(t_{i+\frac{1}{2}}\) by \(t_{i+1+\alpha}\) in equation (49) - (51) to obtain
\[ u_{2x,j}^{i+1+\alpha} = \frac{1}{12h^2} \left[ -u_{j+2}^{i+\alpha} + 16u_{j+1}^{i+\alpha} - 30u_{j}^{i+\alpha} + 16u_{j-1}^{i+\alpha} - u_{j-2}^{i+\alpha} \right] + \psi_{3,j}^i, \] (60)
and
\[ u_{x,j}^{i+1+\alpha} = \frac{1}{12h^2} \left[ -u_{j+2}^{i+\alpha} + 8u_{j+1}^{i+\alpha} - 8u_{j-1}^{i+\alpha} + u_{j-2}^{i+\alpha} \right] + \psi_{4,j}^i, \] (61)
where we set
\[ u_{x,j}^{i+\alpha} = (1 + 2a)u_{j+1}^{i+\alpha} - 2a u_{j+\frac{1}{2}}^{i+\alpha}, \] (62)
\[ \psi_{3,j}^i = \frac{H^4}{270} \left[ 18u_{6x}^{i+1+\alpha}(c_{j}^{12}) + 18u_{6x}^{i+1+\alpha}(c_{j}^{13}) - u_{6x}^{i+1+\alpha}(c_{j}^{14}) - u_{6x}^{i+1+\alpha}(c_{j}^{15}) \right] + \frac{\alpha(\frac{1}{2} + \alpha)k^2}{12h^2} \left[ H_{2,j+2}^i - 16H_{2,j+1}^i + 30H_{2,j}^i - 16H_{2,j-1}^i + H_{2,j-2}^i \right], \] (63)
and
\[ \psi_{4,j}^i = \frac{H^4}{180} \left[ 18u_{6x}^{i+1+\alpha}(c_{j}^{18}) + 4u_{5x}^{i+1+\alpha}(c_{j}^{19}) - u_{5x}^{i+1+\alpha}(c_{j}^{16}) - u_{5x}^{i+1+\alpha}(c_{j}^{17}) \right] + \frac{\alpha(\frac{1}{2} + \alpha)k^2}{12h^2} \left[ H_{2,j+2}^i - 16H_{2,j+1}^i + 30H_{2,j}^i - 16H_{2,j-1}^i + H_{2,j-2}^i \right]. \]
\[
\frac{\alpha(\frac{1}{2} + \alpha)k^2}{12h} \left[ H_{2,j+2}^l - 8H_{2,j+1}^l + 8H_{2,j-1}^l - 8H_{2,j-2}^l \right],
\]

where \( \epsilon_j^{14}, \epsilon_j^{15}, \epsilon_j^{16} \in (x_j - x_{j+1}), \epsilon_j^{10}, \epsilon_j^{12}, \epsilon_j^{18} \in (x_j, x_{j+1}), \epsilon_j^{11}, \epsilon_j^{13}, \epsilon_j^{19} \in (x_j - x_{j+2}) \).

Here the terms \( H_{2,m}^l \) are defined as in equation (59).

Combining equations (1), (26), (54), (55), (60) and (61), direct computations result in

\[
c\Delta t^\lambda u_{j+1,i}^{\alpha} + J_{i}^{(\alpha)} = \frac{q^{i+1+\alpha}}{12h^2} \left[ -u_{j+2}^{\alpha} + 16u_{j+1}^{\alpha} - 30u_{j}^{\alpha} + 16u_{j-1}^{\alpha} - u_{j-2}^{\alpha} \right] - \frac{p^{i+1+\alpha}}{12h} \left[ -u_{j+2}^{\alpha} + 8u_{j+1}^{\alpha} - 8u_{j}^{\alpha} + 8u_{j-1}^{\alpha} - u_{j-2}^{\alpha} \right] - 8u_{j}^{\alpha} - u_{j-1}^{\alpha},
\]

where

\[
-8u_{j+1}^{\alpha}u_{j}^{\alpha} + s_{j+1+\alpha} + q^{i+1+\alpha}p_{i}^{\alpha} - p^{i+1+\alpha}p_{i}^{\alpha} + (1 + \alpha)k^2 g_{j}^{i+1+\alpha} H_{2,j}^l.
\]

Tracking the error terms \( J_{i}^{(\alpha)}, q^{i+1+\alpha}p_{i}^{\alpha} \) and \( \alpha(\frac{1}{2} + \alpha)k^2 g_{j}^{i+1+\alpha} H_{2,j}^l \), plugging equations (55) and (65) and rearranging terms, this yields

\[
U_{j+1}^{i+1} = U_{j}^{i+\frac{1}{2}} - \frac{k}{2} (f_{i+1,i}^{\alpha} + f_{i+1,i+1}^{\alpha})^{-1} \left\{ (d_{i+1,i}^{\alpha} - f_{i+1,i}^{\alpha})\delta_i U_{j}^0 + \sum_{l=0}^{i-1} \left[ f_{i+1,i}^{\alpha} \delta_i U_{j+1}^l + (d_{i+1,i}^{\alpha} - f_{i+1,i}^{\alpha})\delta_i U_{j}^l \right] - \frac{q^{i+1+\alpha}}{12h^2} \left[ -U_{j+2}^{\alpha} + 16U_{j+1}^{\alpha} - 30U_{j}^{\alpha} + 16U_{j-1}^{\alpha} - U_{j-2}^{\alpha} \right] + \frac{p^{i+1+\alpha}}{12h} \left[ -U_{j+2}^{\alpha} + 8U_{j+1}^{\alpha} - 8U_{j}^{\alpha} + 8U_{j-1}^{\alpha} - U_{j-2}^{\alpha} \right] - g_{j}^{i+1+\alpha} U_{j}^{\alpha} - s_{j+1}^{\alpha} \right\},
\]

for \( i \geq 1 \).

We recall that \( U_{m}^{\alpha} = (1 + 2\alpha)U_{m}^{i+1} - 2\alpha U_{m}^{i+\frac{1}{2}} \), for \( i \geq 0 \). Equation (66) denotes the second-level of the proposed procedure applied to the time-fractional parabolic equations (1) - (3).

Now, utilizing (66) and assuming that the sum equals zero when the upper summation index is less than the lower one, we obtain

\[
U_{j}^{i+1} = U_{j}^{i+\frac{1}{2}} - \frac{k}{2} (f_{i+1,i}^{\alpha} + f_{i+1,i+1}^{\alpha})^{-1} \left\{ (d_{i+1,i}^{\alpha} - f_{i+1,i}^{\alpha})\delta_i U_{j}^0 - \frac{q^{i+1+\alpha}}{12h^2} \left[ -U_{j+2}^{\alpha} + 16U_{j+1}^{\alpha} - 30U_{j}^{\alpha} + 16U_{j-1}^{\alpha} - U_{j-2}^{\alpha} \right] + \frac{p^{i+1+\alpha}}{12h} \left[ -U_{j+2}^{\alpha} + 8U_{j+1}^{\alpha} - 8U_{j}^{\alpha} + 8U_{j-1}^{\alpha} - U_{j-2}^{\alpha} \right] - g_{j}^{i+1+\alpha} U_{j}^{\alpha} - s_{j+1}^{\alpha} \right\},
\]

for \( i \geq 1 \).

An assembly of equations (48), (49), (67) and (66) after rearranging terms provides a two-level fourth-order approach for solving the initial-boundary value problem (1) - (3). That is, for \( i = 1, 2, ..., N - 1 \), and \( j = 2, 3, ..., M - 2 \),

\[
\frac{(1 + 2\alpha)k}{24h^2} \left[ \left( q^{i+\alpha} + hp^{i+\alpha} \right) U_{j+2}^{\alpha} - \left( 16q^{i+\alpha} + 8hp^{i+\alpha} \right) U_{j+1}^{\alpha} - \left( 16q^{i+\alpha} + 8hp^{i+\alpha} \right) U_{j}^{\alpha} + \left( q^{i+\alpha} - hp^{i+\alpha} \right) U_{j+2}^{\alpha} \right] + \left( f_{i+1,i+1}^{\alpha} + \frac{(1 + 2\alpha)k}{24h^2} \left( 30q^{i+\alpha} + 12h^2 g_{j}^{i+\alpha} \right) \right) U_{j}^{\alpha} = f_{i+1,i}^{\alpha} U_{j}^0 + \frac{(1 + 2\alpha)k}{2} s_{j}^{i+\alpha},
\]

where

\[
U_{j}^{\alpha} = (1 + 2\alpha)U_{j}^{i+\frac{1}{2}} - 2\alpha U_{j}^{i+1}, \text{ for } l = 2, 3, ..., M - 2,
\]

\[
\frac{(1 + 2\alpha)k}{24h^2} \left[ \left( q^{i+\alpha} + hp^{i+\alpha} \right) U_{j+2}^{\alpha} - \left( 16q^{i+\alpha} + 8hp^{i+\alpha} \right) U_{j+1}^{\alpha} - \left( 16q^{i+\alpha} + 8hp^{i+\alpha} \right) U_{j}^{\alpha} + \left( q^{i+\alpha} - hp^{i+\alpha} \right) U_{j+2}^{\alpha} \right] + \left( f_{i+1,i+1}^{\alpha} + \frac{(1 + 2\alpha)k}{24h^2} \left( 30q^{i+\alpha} + 12h^2 g_{j}^{i+\alpha} \right) \right) U_{j}^{\alpha} = \left( f_{i+1,i+1}^{\alpha} + f_{i+1,i}^{\alpha} \right) U_{j}^{i+1} - \frac{(1 + 2\alpha)k}{2} \left[ (d_{i+1,i+1}^{\alpha} - f_{i+1,i}^{\alpha})\delta_i U_{j}^0 - s_{j+1}^{\alpha} \right],
\]

(69)
where \( U^{\theta_0} = (1 + 2\alpha)U^1_j - 2\alpha U^2_j \), for \( r = 2, 3, ..., M - 2 \),
\[
\frac{(1 + 2\alpha)k}{24h^2} \left[ (q^{i+1/2,\alpha} + hp^{i+1/2,\alpha}) U^n_{j-2} - (16q^{i+1/2,\alpha} + 8hp^{i+1/2,\alpha}) U^n_{j-1} \right] + (16q^{i+1/2,\alpha} - 8hp^{i+1/2,\alpha}) U^n_{j+1} + (q^{i+1/2,\alpha} - hp^{i+1/2,\alpha}) U^n_{j+2} \right] + \left[ \left( f^{\lambda}_{i+1/2,j} + f^{\lambda}_{i+1/2,j+1} \right) \right] U^n_j - \left( \frac{1 + 2\alpha)k}{2} \right) \left( \frac{1 + 2\alpha)k}{2} \right) \{ f_{i+1/2,0} \delta_i U^0_j \} + (d^{\lambda}_{i+3/2,i-1} - f^{\lambda}_{i+3/2,j+1}) \delta_i U^{i+1/2}_j + \sum_{l=0}^{i-2} \left( f^{\lambda}_{i+3/2,i} \delta_i U^{i+1/2}_j + (d^{\lambda}_{i+3/2,i} - f^{\lambda}_{i+3/2,j}) \delta_i U^{i+1/2}_j \right) - s^{i+1/2}_j \right) \right),
\]
where \( U^{\alpha}_m = (1 + 2\alpha)U^1_m - 2\alpha U^2_m \), for \( m = 2, 3, ..., M - 2 \),
\[
\frac{(1 + 2\alpha)k}{24h^2} \left[ \left( f^{\lambda}_{i+1,i+1} + f^{\lambda}_{i+1,i+2} \right) \right] U^n_{j-2} - (16q^{i+1/2,\alpha} + 8hp^{i+1/2,\alpha}) U^n_{j-1} - (16q^{i+1/2,\alpha} - 8hp^{i+1/2,\alpha}) U^n_{j+1} \right] + \left( q^{i+1/2,\alpha} - hp^{i+1/2,\alpha} \right) U^n_{j+2} + \left( f^{\lambda}_{i+1,i+1} + f^{\lambda}_{i+1,i} \right) \right] U^n_j - \left( \frac{1 + 2\alpha)k}{2} \right) \left( \frac{1 + 2\alpha)k}{2} \right) \{ f_{i+1,1} \delta_i U^1_j \} + (d^{\lambda}_{i+1,i} - f^{\lambda}_{i+1,i}) \delta_i U^{i+1/2}_j + \sum_{l=0}^{i-1} \left( f^{\lambda}_{i+1,i} \delta_i U^{i+1/2}_j + (d^{\lambda}_{i+1,i} - f^{\lambda}_{i+1,i}) \delta_i U^{i+1/2}_j \right) - s^{i+1/2}_j \right) \right),
\]
\( U^0_j = \psi_{1,j}, j = 0, 1, ..., M; \text{ and } U^{\alpha}_{r} = \psi_{2,r} \), \( U^{\theta} = \psi_{2,r} \) for \( i = 0, 1, ..., N-1; r = 0, 1, M-1, M \). (72)
We recall that the sum equals zero if the upper summation index is less than the lower ones.

### 3 Stability analysis and convergence rate of the new algorithm (68)-(72)

This section considers the analysis of stability and convergence rate of the two-level fourth-order method applied to the time-fractional convection-diffusion-reaction equation subject to initial and boundary conditions. In this study, we assume that the parameters \( \lambda \) and \( \alpha \) satisfy \( 0 < \lambda < \frac{3}{2} \) and \( \alpha = 1 - \lambda \). This requirement plays a crucial role in the proof of Lemma (3.6) and Theorem (3.1). Firstly, it is worth noticing to mention that the stability analysis and convergence rate of the proposed approach requires some intermediate results (namely Lemmas 3.1, 3.9).

**Lemma 3.1.** For any \( \lambda \in (0, 1) \) and \( \alpha = 1 - \lambda \), suppose the function \( u(\cdot, \cdot) \in C^{6,3}(D) := C^{6,3}_D \) where \( D = [0, L_1] \times [0, T] \), so
\[
\max_{0 \leq i \leq N-1} \| cD^{\lambda}_{0}u^{i+r} - cD^{\lambda}_{0}u^{i+r}\| L^2 \leq C_r k^{2-\lambda}
\]
where \( r \in \{1, 2, 3\} \). \( cD^{\lambda}_{0}u^{i+r} \), (for \( i = 0 \) and \( 1 \leq i \leq N-1 \)) are defined by equations (20) and (21), respectively; \( cD^{\lambda}_{0}u^{i+1+r} \), (for \( i = 0 \) and \( 1 \leq i \leq N-1 \)) are defined by equations (22) and (23), respectively. \( cD^{\lambda}_{0}u^{i+r} \) and \( cD^{\lambda}_{0}u^{i+1+r} \), (for \( 0 \leq i \leq N-1 \)) are defined by equations (54) and (55), respectively. \( C_r \) are positive constants which do not depend on the time step \( k \) and space step \( h \).
Proof. A combination of equations (20) and (22) yields
\[ |cD_{0+}^\alpha u^i_j - c\Delta_{0+}^\lambda u^i_j| = \frac{1}{\Gamma(1 - \lambda)^2} \left[ \int_0^{t_+^i + \alpha} (u_{\tau, j}(\tau) - \delta_{i} u^0_j)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau \right]. \] (74)

The application of the Taylor series gives
\[ u_{\tau, j}(\tau) = u_{\tau, j}(t_+^{i}) + (\tau - t_+^{i})u_{2, \tau, j}(t_+^{i}) + \frac{1}{2}(\tau - t_+^{i})^2 u_{3, \tau, j}(\tau_1), \] (75)
where \( \tau_1 \) is between the minimum and maximum of \( \tau \) and \( t_+^{i} \). Substituting (75) into (74) to obtain
\[ |cD_{0+}^\alpha u^i_j - c\Delta_{0+}^\lambda u^i_j| = \frac{1}{\Gamma(1 - \lambda)^2} \left[ \int_0^{t_+^i + \alpha} (u_{\tau, j}(t_+^{i}) - \delta_{i} u^0_j) \int_0^{t_+^i + \alpha} (t_+^{i + \alpha} - \tau)^{-\lambda} d\tau + \int_0^{t_+^i + \alpha} (\tau - t_+^{i})^2 u_{3, \tau, j}(\tau)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau \right]. \] (76)
Performing simple calculations, it is easy to see that
\[ u^{i + \frac{1}{\lambda}} = u^{i + \frac{1}{\lambda}} + k u^{i + \frac{1}{\lambda}} + O(k^2), \] \[ u^i_j = u^i_j - \frac{k}{4} u^{i + \frac{1}{\lambda}} + \frac{k^2}{8} u^{i + \frac{1}{\lambda}} + O(k^3). \]
Using this, simple calculations result in
\[ u^{i + \frac{1}{\lambda}} - \delta_j u^i_j = O(k^2), \] for \( i \geq 0 \). (77)

We recall that \( \delta_j u^i_j = \frac{2}{k} (u^{i + \frac{1}{\lambda}} - u^i_j). \) So
\[ (u_{\tau, j}(t_+^{i}) - \delta_{i} u^0_j) \int_0^{t_+^{i + \alpha}} (t_+^{i + \alpha} - \tau)^{-\lambda} d\tau = \frac{-1}{1 - \lambda} (u_{\tau, j}(t_+^{i}) - \delta_{i} u^0_j) \int_0^{t_+^{i + \alpha}} (t_+^{i + \alpha} - \tau)^{-\lambda} d\tau = O(k^{3 - \lambda}). \] (78)

On the other hand
\[ u_{2, \tau, j}(t_+^{i}) \int_0^{t_+^{i + \alpha}} (\tau - t_+^{i})(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau = \frac{-u_{2, \tau, j}(t_+^{i})}{1 - \lambda} \int_0^{t_+^{i + \alpha}} (\tau - t_+^{i})(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau = \frac{k^{2 - \lambda}(\alpha + \lambda - \frac{3}{4})}{1 - \lambda} \left( \frac{1}{4} + \frac{\alpha}{2 - \lambda} \right) = O(k^{2 - \lambda}). \] (79)

Setting \( m(\tau) = (\tau - t_+^{i})^2, \) so \( \frac{d}{d\tau} m(\tau) = 0, \) implies \( \tau = t_+^{i}. \) So \( \max_{0 \leq \tau \leq t_+^{i + \alpha}} |m(\tau)| = (\frac{1}{4} + \alpha)^2 k^2. \) Thus
\[ \left| \int_0^{t_+^{i + \alpha}} (\tau - t_+^{i})^2 u_{3, \tau, j}(\tau)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau \right| \leq \frac{1}{2(1 - \lambda)} \int_0^{t_+^{i + \alpha}} (\tau - t_+^{i})^2 u_{3, \tau, j}(\tau)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau \leq \frac{1}{2(1 - \lambda)} \left( \frac{1}{4} + \alpha \right)^2 k^2 \max_{0 \leq \tau \leq t_+^{i + \alpha}} |u_{3, \tau, j}(\tau)| = O(k^{3 - \lambda}). \] (80)

Plugging estimates (76)–(80) to get \( |cD_{0+}^\alpha u^i_j - c\Delta_{0+}^\lambda u^i_j| = O(k^{2 - \lambda}). \) This fact, together with the definition of \( L^2 \)-norm give
\[ \|cD_{0+}^\alpha u^i_j - c\Delta_{0+}^\lambda u^i_j\|_{L^2} \leq C^2_0 k^{2 - \lambda}, \] (81)
where \( C^2_0 \) is a positive constant. Furthermore, for \( 1 \leq i \leq N - 1, \) a combination of equations (21) and (28) provides
\[ cD_{0+}^\alpha u^i_j - c\Delta_{0+}^\lambda u^i_j = \frac{1}{\Gamma(1 - \lambda)} \left( \sum_{l=0}^{i-1} \int_{t_+^{i + \alpha}}^{t_+^{i + \alpha}} E_{l, \tau, j}(\tau)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau + \int_0^{t_+^{i + \alpha}} (u_{\tau, j}(\tau) - \delta_{i} u^0_j)(t_+^{i + \alpha} - \tau)^{-\lambda} d\tau \right). \]
where \( \tilde{u}_{t,j} \) is given by (77). In addition, using equation (75), it is not difficult to prove that

\[
\left| \int_{t_{l+\frac{1}{2}}}^{t_{l+\frac{3}{2}}} u_{\tau,j}(\tau) - \tilde{u}_{t,j}(\tau) \, d\tau \right| \leq \frac{k}{2} |u_{\tau,j} - \tilde{u}_{t,j}|_{t_{l+\frac{1}{2}}} + \frac{k^2}{8} t_{\tau,j} + \frac{k^3}{128} \max_{0 \leq \tau \leq t_{l+\frac{1}{2}}} |u_{3\tau,j}(\tau)| = O(k^{2-\lambda}),
\]

which is (83). In addition, using equation (75), it is not difficult to prove that

\[
\left| \int_{t_{l+\frac{1}{2}}}^{t_{l+\frac{3}{2}}} u_{\tau,j}(\tau) - \tilde{u}_{t,j}(\tau) \, d\tau \right| \leq \frac{k}{2} |u_{\tau,j} - \tilde{u}_{t,j}|_{t_{l+\frac{1}{2}}} + \frac{k^2}{8} t_{\tau,j} + \frac{k^3}{128} \max_{0 \leq \tau \leq t_{l+\frac{1}{2}}} |u_{3\tau,j}(\tau)| = O(k^{2-\lambda}),
\]

since \( u_{t,j} - \tilde{u}_{t,j} = O(k^2) \) and \( t_{\tau,j} = k^{-\lambda} (i + \alpha - \lambda) \leq k^{-\lambda} \alpha^{-\lambda} \). Now, setting \( m_l(\tau) = \frac{1}{\alpha}(\tau - t_{l+\frac{1}{2}})(\tau - t_{l+\frac{3}{2}}) \), simple calculations result in

\[
\max_{t_{l+\frac{1}{2}} \leq \tau \leq t_{l+\frac{3}{2}}} |m_l(\tau)| = \frac{k^3}{72\sqrt{3}}.
\]

Combining of relations (10) and (55), direct computations provide

\[
\left| \int_{t_{l+\frac{1}{2}}}^{t_{l+\frac{3}{2}}} E_{\tau,j}(\tau)(t_{l+\frac{1}{2}} - \tau)^{\lambda} \, d\tau \right| \leq \frac{k^{3-\lambda}}{72\sqrt{3}} [t^{\lambda - (i + \alpha - l - 1)^{\lambda}] \max_{t_{l+\frac{1}{2}} \leq \tau \leq t_{l+\frac{3}{2}}} |u_{3\tau,j}(\tau)|].
\]

Since \( i \geq 1 \), summing this up from \( l = 0, 1, \ldots, i - 1 \), to obtain

\[
\sum_{l=0}^{i-1} \left| \int_{t_{l+\frac{1}{2}}}^{t_{l+\frac{3}{2}}} E_{\tau,j}(\tau)(t_{l+\frac{1}{2}} - \tau)^{\lambda} \, d\tau \right| \leq \frac{k^{3-\lambda}}{72\sqrt{3}} [t^{\lambda - (i + \alpha - l)^{\lambda}] \max_{0 \leq \tau \leq t_{i+\frac{1}{2}}} |u_{3\tau,j}(\tau)|] = O(k^{3-\lambda}).
\]

Plugging equation (82) and estimate (84), it is easy to see that

\[
\|cD_0^\lambda u_{l+\frac{1}{2}} - cD_0^\lambda \tilde{u}_{l+\frac{1}{2}}\|_{L^2} \leq C_3 k^{2-\lambda}, \quad \text{for } i \geq 1,
\]

where \( C_3 \) is a positive constant. In a similar manner, one easily proves the following inequality

\[
\|cD_0^\lambda u_{l+1} - cD_0^\lambda \tilde{u}_{l+1}\|_{L^2} \leq C_1 k^{2-\lambda}, \quad \text{for } i \geq 0,
\]

where \( C_1 > 0 \) is a constant. The proof of Lemma [3,1] is completed by taking the maximum over \( i (0 \leq i \leq N - 1) \) of estimates (31) and (87) (respectively, estimate (88)).

**Lemma 3.2.** Define the following linear operators

\[
L_h u_j = \frac{\bar{\gamma}_i}{12h^2} \left[ -u_{j+1}^i + 16u_{j+1}^i - 30u_j^i + 16u_{j-1}^i - u_{j-2}^i \right] -
\]

\[
\frac{p_i}{12h} \left[ -u_{j+1}^i + 8u_{j+1}^i - 8u_j^i + u_{j-2}^i \right] - g_j^i u_j^i,
\]

for \( j = 2, 3, \ldots, M - 2 \), and

\[
L^\alpha u_j = [g(t)u_{2x} - p(t)u_x - g(x,t)u]|_{(x, t_{\alpha})},
\]

where \( \bar{\gamma}_i \in \{ i + \frac{1}{2} + \alpha, i + 1 + \alpha \} \) and \( u_j^i = u_{\alpha}^i, \ u_j^i, \) for \( 0 \leq i \leq N \). So, it holds

\[
\|L_h u_j^i - L^\alpha u_j^i\|_{L^2} \leq A_9 \left[ 7\alpha(1 + 2\alpha)\left( \frac{1}{2} + 2\alpha \right) k^2 + \frac{53}{270} (12 + \alpha(\frac{1}{2} + \alpha)(\frac{1}{2} + 2\alpha) k^2)^3 \right] \|u\|_{c_3^2},
\]

for all \( i \geq 1 \).
where
\[ A_{pq}^\mu = \frac{\sqrt{3}}{12} \max\{ \|q\|_{C^0_{\partial}}, \|p\|_{C^0_{\partial}}, \|g\|_{C^{0,3}_{\partial}} \}, \]
where \( \Gamma = [0, T] \), \( \|v\|_{C^0_{\partial}} = \max_{0 \leq t \leq N} |v^t| \) and \( \| \cdot \|_{C^{0,3}_{\partial}} \) is defined in relation (3).

Proof. Considering relations (10)-(11), (58), (60)-(61) and (80)-(90), simple computations give
\[ Lu_{j} = \bar{L}_{j}u_{j} = \bar{q}^{\gamma_{j}} \psi_{l,j}^{\gamma_{j}} - p^{\gamma_{j}} \psi_{r,j}^{\gamma_{j}} + \alpha(\frac{1}{2} + \alpha)k^{2} \psi_{l,j}^{\gamma_{j}}H_{l,j}^{i}, \]
where \( (l, r, \gamma_{i}, m_{i}) \in \{ (1, 2, i + \frac{1}{2} + \alpha, 1), (3, 4, i + 1 + \alpha, 2) \} \), \( \psi_{l,j}^{\gamma_{j}} \) are given by equations (12) and (83), \( \psi_{r,j}^{\gamma_{j}} \) are given by equations (43) and (64). Working as in Section 2 to approximate the terms \( u_{2x}(x, t) \) and \( u_{x}(x, t) \), one easily shows that
\[ \partial_{2x}H_{l,j}^{i} = \frac{1}{12h^{2}} \left[ -H_{l,j}^{i} + 2H_{l,j}^{i} - 30H_{l,j}^{i} - 16H_{l,j}^{i} - H_{l,j}^{i} \right] + \]
\[ \frac{h^{4}}{270} \left[ 18\partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{20}) + 18\partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{12}) - \partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{23}) - \partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{21}) \right], \]
and
\[ \partial_{x}H_{r,j}^{i} = \frac{1}{12h} \left[ -H_{r,j}^{i} + 2H_{r,j}^{i} - 8H_{r,j}^{i} + H_{r,j}^{i} \right] + \]
\[ \frac{h^{4}}{180} \left[ 4\partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{20}) + 4\partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{21}) - \partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{22}) - \partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{23}) \right], \]
where \( \epsilon_{l,j}^{20}, \epsilon_{r,j}^{20} \in (x_{j-1}, x_{j}), \epsilon_{l,j}^{12}, \epsilon_{r,j}^{12} \in (x_{j}, x_{j+1}), \epsilon_{l,j}^{23}, \epsilon_{r,j}^{23} \in (x_{j-2}, x_{j}), \epsilon_{l,j}^{21}, \epsilon_{r,j}^{21} \in (x_{j-1}, x_{j+2}), \)
where \( \partial_{m}H_{s}^{i} \) for \( m = 1, 2 \) denote \( \partial_{m}H_{s}^{i} \). The functions \( H_{s}^{i} \) are defined by relations (39) and (59).

Plugging equations (12), (83) and (94), straightforward calculations provide
\[ \psi_{l,j}^{i} = -\frac{\alpha (\frac{1}{2} + \alpha)k^{2}}{12} \partial_{2x}H_{l,j}^{i} + \frac{h^{4}}{270} \left[ 18\partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{12}) + \frac{\alpha (\frac{1}{2} + \alpha)k^{2}}{12} \partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{20}) - \partial_{6x}H_{l}^{i}(\epsilon_{l,j}^{23}) \right], \]
where
\[ \epsilon_{l,j}^{1}, \epsilon_{r,j}^{1} \in \{ \epsilon_{l,j}^{1}, \epsilon_{r,j}^{1} \}, \epsilon_{l,j}^{2}, \epsilon_{r,j}^{2} \in \{ \epsilon_{l,j}^{2}, \epsilon_{r,j}^{2} \}, \epsilon_{l,j}^{3}, \epsilon_{r,j}^{3} \in \{ \epsilon_{l,j}^{3}, \epsilon_{r,j}^{3} \}, \epsilon_{l,j}^{4}, \epsilon_{r,j}^{4} \in \{ \epsilon_{l,j}^{4}, \epsilon_{r,j}^{4} \}, \epsilon_{l,j}^{5}, \epsilon_{r,j}^{5} \in \{ \epsilon_{l,j}^{5}, \epsilon_{r,j}^{5} \}. \]
Analogously, considering equations (43), (64) and (95), simple computations yield
\[ \psi_{r,j}^{i} = -\frac{\alpha (\frac{1}{2} + \alpha)k^{2}}{12} \partial_{2x}H_{r,j}^{i} + \frac{h^{4}}{180} \left[ 4\partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{12}) + \frac{\alpha (\frac{1}{2} + \alpha)k^{2}}{12} \partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{20}) + \partial_{6x}H_{r}^{i}(\epsilon_{r,j}^{23}) \right], \]
where
\[ \epsilon_{l,j}^{1}, \epsilon_{r,j}^{1} \in \{ \epsilon_{l,j}^{1}, \epsilon_{r,j}^{1} \}, \epsilon_{l,j}^{2}, \epsilon_{r,j}^{2} \in \{ \epsilon_{l,j}^{2}, \epsilon_{r,j}^{2} \}, \epsilon_{l,j}^{3}, \epsilon_{r,j}^{3} \in \{ \epsilon_{l,j}^{3}, \epsilon_{r,j}^{3} \}, \epsilon_{l,j}^{4}, \epsilon_{r,j}^{4} \in \{ \epsilon_{l,j}^{4}, \epsilon_{r,j}^{4} \}, \epsilon_{l,j}^{5}, \epsilon_{r,j}^{5} \in \{ \epsilon_{l,j}^{5}, \epsilon_{r,j}^{5} \}. \]
with \( (l, r, \gamma_{i}) \in \{ (1, 2, i + \frac{1}{2} + \alpha), (3, 4, i + 1 + \alpha) \} \).

Now, using equations (39) and (59), it is not hard to show that
\[ \| \partial_{m}H_{s}^{i} \|_{C^{0,3}_{\partial}} \leq \| \frac{\alpha (\frac{1}{2} + 2\alpha)}{12} \|_{C^{0,3}_{\partial}} \right. \]
Substituting equations (39) and (59) into equations (96) and (97), respectively, performing simple calculations and utilizing estimate equations (98), it is not difficult to observe that

$$\|\psi_j^l\|_{C^0_D}^2 \leq \frac{1}{12} \left( \alpha \left( \frac{1}{2} + \alpha \right) \frac{1}{2} + 2\alpha k^2 + \frac{19}{135} \left[ 12 + \alpha \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} + 2\alpha k^2 \right) h^4 \right] \right) \|u\|_{C^0_D}^2, \text{ for } l = 1, 3, \quad (99)$$

and

$$\|\psi_j^r\|_{C^0_D}^2 \leq \frac{1}{12} \left( \alpha \left( \frac{1}{2} + \alpha \right) \frac{1}{2} + 2\alpha k^2 + \frac{1}{18} \left[ 12 + \alpha \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} + 2\alpha k^2 \right) h^4 \right] \right) \|u\|_{C^0_D}^2, \text{ for } r = 2, 4. \quad (100)$$

Taking the square of both sides of equation (93) and applying the Cauchy-Schwarz inequality, it holds

$$[L_h u^{n_j} - L u^{n_j}]^2 = 3 \left( q^{\bar{g}_i} \psi_j^{t_i} \right)^2 + \left( p^{\bar{g}_i} \psi_j^{t_i} \right)^2 + \alpha^2 \left( \frac{1}{2} + \alpha \right) k^4 \left( g^{\bar{g}_i} H_{m_1,j}^2 \right)^2 \leq 3 \left[ \|q\|_{C^0_D}^2 \|\psi_j\|_{C^0_D}^2 + \|p\|_{C^0_D}^2 \|\psi_j\|_{C^0_D}^2 + \alpha^2 \left( \frac{1}{2} + \alpha \right) k^4 \|g\|_{C^0_D}^2 \|H_{m_1,j}\|_{C^0_D}^2 \right]^2.$$ 

Substituting estimates (98)-(99) into this inequality, summing up from $j = 1, 2, \ldots, M - 1$, and multiplying both sides of the obtained estimate by $h$, direct calculations result in

$$\|L_h u^{n_j} - L u^{n_j}\|_{L^2}^2 \leq \frac{3}{144} \max \{\|q\|_{C^0_D}^2, \|p\|_{C^0_D}^2, \|g\|_{C^0_D}^2\} \left[ \frac{7}{2} \alpha \left( \frac{1}{2} + \alpha \right) (1 + 2\alpha k^2) + \frac{53}{270} (12 + \alpha \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} + 2\alpha k^2 \right) h^4 \right] \|u\|_{C^0_D}^2.$$ 

Taking the square root of both sides of this estimate to get relation (91). This ends the proof of Lemma 3.2. \hfill \square

**Lemma 3.3.** Let $u \in C_{D}^{2,0}$, be a function defined on $D = [0, L_1] \times [0, T]$, such that $u(0, t) = u(L_1, t) = 0$, for every $t \in [0, T]$. Suppose $U(t) \in \mathcal{U}_{h}(t) = \{ U_j(t); j = 0, 1, \ldots, M \}$ be a grid function satisfying $U_j(t) = u(x_j, t)$, for $j = 0, 1, \ldots, M$. So, it holds

$$(-L u(t), u(t)) \geq \gamma \|u_j(t)\|_{L^2}^2 + \beta \|u(t)\|_{L^2}^2 \text{ and } (-L_h U(t), U(t)) \geq \gamma \|\delta_x U(t)\|_{L^2}^2 + \beta \|U(t)\|_{L^2}^2,$$

for any $t \in [0, T]$, where $\gamma = \inf_{t \in [0, T]} q(t) > 0$ and $\beta = \inf_{(x, t) \in D} g(x, t) \geq 0$.

**Proof.** In this proof, we should prove that the linear operator: $-Lu = -q(t) u_{2x} + p(t) u_x + g(x, t) u$, satisfies $(-Lu, u) \geq \gamma \|u\|_{L^2}^2 + \beta \|u\|_{L^2}^2$, and then use the definition of the discrete $L^2$-norm given in relation (8) to conclude. Since $\delta_x u_{j-\frac{1}{2}}(t) = \frac{u_{j}(t) - u_{j-1}(t)}{h}$ and $\delta_x^2 u_j(t) = \frac{u_{j+1}(t) - 2u_{j}(t) + u_{j-1}(t)}{h^2}$, applying the Taylor series expansion, it is easy to show that $u_{2x}(x, t) = \delta_x^2 u_j(t) + O(h^2)$ and $u_{2x}(x, t) = \delta_x^2 u_j(t) + O(h^2)$. Using the conditions $u_0(t) = u_M(t) = 0$, for every $t \in [0, T]$ together with the summation by parts and the equality $a(a-b) = \frac{1}{2}(a-b)^2 + a^2 - b^2$, for every real numbers $a$ and $b$, simple calculations provide

$$(-Lu(t), u(t)) = -h \sum_{j=1}^{M-1} \left[ q(t) \left( \delta_x^2 u_j(t) + O(h^2) \right) - p(t) \delta_x u_{j-\frac{1}{2}}(t) + O(h) \right] u_{j}(t) =$$

$$-hq(t) \sum_{j=1}^{M-1} \left( \delta_x^2 u_j(t) \right) u_j(t) + O(h^2) + hp(t) \sum_{j=1}^{M-1} \left( \delta_x u_{j-\frac{1}{2}}(t) \right) u_j(t) + O(h) + h \sum_{j=1}^{M-1} g_j(t) u_j(t)^2 =$$

$$-q(t) \left[ \left( \delta_x u_{M-\frac{1}{2}}(t) \right) u_{M-1}(t) - \left( \delta_x u_{\frac{1}{2}}(t) \right) u_1(t) - \sum_{j=1}^{M-2} \left( \delta_x u_{j+\frac{1}{2}}(t) \right)^2 \right] + O(h^2) + \frac{1}{2} p(t) \left[ h^2 \sum_{j=1}^{M-1} \left( \delta_x u_{j-\frac{1}{2}}(t) \right)^2 \right]$$. \hfill 15
Since $q(t) \geq 0$, $p(t) \geq 0$, for any $t \in [0, T]$ and $\beta = \inf_{(x,t) \in D} g(x, t) \geq 0$, using this and substituting approximation (102) into relation (101) to obtain

$$(-Lu(t), u(t)) \geq \gamma h \sum_{j=1}^{M-1} (u_{x,j}(t))^2 + \beta h \sum_{j=1}^{M-1} (u_j(t))^2 = \gamma \|u_x(t)\|_{L^2}^2 + \beta \|u(t)\|_{L^2}^2.$$ 

This ends the proof of the first estimate in Lemma 3.3. Furthermore, since $u_j(t) = U_j(t)$, for $j = 0, 1, ..., M$, for $h$ sufficiently small, neglecting the infinitesimal terms $O(h^2)$ and $O(h)$, equation (101) implies

$$(-Lu(t), u(t)) \geq \gamma h \sum_{j=1}^{M-1} (\delta_x U_{x,j}(t))^2 + \beta h \sum_{j=1}^{M-1} (U_j(t))^2 = \gamma \|\delta_x U(t)\|_{L^2}^2 + \beta \|U(t)\|_{L^2}^2.$$ 

The proof of Lemma 3.3 is completed thanks to the definition of the discrete $L^2$-norm given in relation (1).

The following Lemmas (namely Lemmas 3.4, 3.5 taken in [1]) help in proving Lemmas 3.6.

**Lemma 3.4.** [1] For every $i = 1, 2, 3, ..., $ and any $\lambda$ satisfying $0 < \lambda < 1$, the following inequalities are satisfied

$$\frac{1}{2} < Z_i < \frac{1}{2 - \lambda}$$

where

$$Z_i = \frac{(i + \alpha)^{2-\lambda} - (i - 1 + \alpha)^{2-\lambda} - (2 - \lambda)(i - 1 + \alpha)^{1-\lambda}}{(2 - \lambda)(i + \alpha)^{1-\lambda} - (i - 1 + \alpha)^{1-\lambda}}.$$ 

**Lemma 3.5.** [1] Suppose $\lambda$ be a positive number which is less than one. So, it holds

$$\frac{1}{2 - \lambda} \left[(l + \alpha)^{2-\lambda} - (l - 1 + \alpha)^{2-\lambda}\right] - \frac{1}{2} \left[(l + \alpha)^{1-\lambda} + (l - 1 + \alpha)^{1-\lambda}\right] > 0,$$

for every integer $l \geq 1$.

**Lemma 3.6.** Let $\lambda$ be a positive number such that $0 < \lambda < \frac{3}{2}$. For any positive integer $i$, we introduce the generalized sequences $(a_{i+\frac{1}{2},l})_{i}$ and $(\alpha_{l+\frac{1}{2},i})$ with step size equals $\frac{1}{2}$ (that is, $l = \frac{3}{2}, 1, \frac{5}{2}, 2, ...$) defined by:

$$a_{i+\frac{1}{2}} = f_{i+\frac{1}{2},0}, \text{ and for } i \geq 1, \quad a_{i+\frac{1}{2},i} = \bar{f}_{i+\frac{1}{2},0},$$

$$\alpha_{i+\frac{1}{2},l} = \begin{cases} \bar{f}_{i+\frac{1}{2},l-1} - \bar{f}_{i+\frac{1}{2},l-1}, & \text{if } l = 1, 2, 3, ..., i, \\ \bar{f}_{i+\frac{1}{2},l-\frac{1}{2}}, & \text{if } l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., i - \frac{1}{2}, \\ \bar{f}_{i+\frac{1}{2},i-\frac{1}{2}}, & \text{if } l = i + \frac{1}{2}. \end{cases}$$
where the terms \(\bar{f}_{i+\frac{1}{2},0}^{\lambda}, \bar{f}_{i+\frac{1}{2}}^{\lambda}, \bar{f}_{i+\frac{1}{2},s}^{\lambda}\) and \(\bar{d}^{\lambda}_{i+\frac{1}{2},s}\) (for \(s = 0, 1, 2, \ldots, i\)) are given by equations (115), (116) and (117). Furthermore,
\[
a_{i+1}^{\lambda} = \bar{d}_{i+1}^{\lambda} - \bar{f}_{i+1}^{\lambda} \quad \text{and} \quad a_{i+1}^{\lambda} = \bar{f}_{i+1}^{\lambda} + \bar{f}_{i+1}^{\lambda},
\]
(105)
and for \(i \geq 1\),
\[
a_{i+1,l+\frac{1}{2}}^{\lambda} = \begin{cases} 
\bar{d}_{i+1,l}^{\lambda} - \bar{f}_{i+1,l}^{\lambda}, & \text{if } l = 0, 1, 2, 3, \ldots, i, \\
\bar{f}_{i+1,l+\frac{1}{2}}^{\lambda}, & \text{if } l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, i - \frac{1}{2},
\end{cases}
\]
(106)
where the terms \(\bar{d}^{\lambda}_{i+1,s}\) and \(\bar{f}_{i+1,s}\) (for \(i \geq 0\) and \(s = 0, 1, 2, \ldots, i + 1\)) are defined by relation (118). Thus, for \(i \geq 1\) the following estimates are satisfied, for \(s = \frac{1}{2}, 1,\)
\[
\bar{f}_{i+1,0}^{\lambda}, \bar{f}_{i+1}^{\lambda}, \bar{d}_{i+1,0}^{\lambda}, \bar{f}_{i+1}^{\lambda}, \bar{a}_{i+1,s}^{\lambda}, \bar{d}_{i+1,s}^{\lambda} - \bar{h}_{i+1,s}^{\lambda} > 0, \quad j = 0, 1, 2, \ldots, i (\text{resp.}, \ i + 1),
\]
(107)
\[
\bar{f}_{i+1,s+1}^{\lambda} + \bar{f}_{i+1,s}^{\lambda} - \bar{d}_{i+1,s+1}^{\lambda} < 0, \quad j = 1, 2, \ldots, i (\text{resp.}, \ i + 1), \quad 2\bar{f}_{i+1,s}^{\lambda} - \bar{d}_{i+1,s}^{\lambda} > 0, \quad j = 0, 1, 2, \ldots, i (\text{resp.}, \ i + 1). \quad (108)
\]
Furthermore,
\[
a_{i+1,s}^{\lambda} < a_{i+1,s+1}^{\lambda}, \quad l = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, i (\text{resp.}, \ i + \frac{1}{2}),
\]
(109)
and
\[
a_{i+1,s}^{\lambda} > \frac{(2 - 3\lambda)(1 - \lambda)(i - \lambda)}{(2 - \lambda)}(i + \alpha - l)^{-\lambda}, \quad l = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, i (\text{resp.}, \ i + \frac{1}{2}).
\]
(110)

**Proof.** It follows from relations (115) and (116) that
\[
\bar{f}_{i+1,0}^{\lambda} = (\frac{1}{2} + \alpha)^{1-\lambda} > 0, \quad \bar{d}_{i+1,0}^{\lambda} = (\frac{1}{2} + \alpha)^{1-\lambda} - (i + \alpha)^{1-\lambda} > 0, \quad \bar{f}_{i+1,1}^{\lambda} = \bar{f}_{i+1,1}^{\lambda} = \alpha^{1-\lambda} > 0.
\]

In addition, it comes from equations (115), (116) and (117) that
\[
0 < \bar{d}_{i+1,0}^{\lambda} = (1 + \alpha)^{1-\lambda} - \alpha^{1-\lambda}, \quad 0 < \bar{d}_{i+1,l}^{\lambda} = (i + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda} = \bar{d}_{i+1,l+1}^{\lambda}, \quad l = 1, 2, 3, \ldots, i,
\]
(111)
\[
\bar{f}_{i+1,l} = \frac{2}{2 - \lambda}[(i + \alpha - l)^{2-\lambda} - (i + \alpha - l)^{2-\lambda}] - \frac{1}{2}[(i + \alpha - l)^{1-\lambda} + (i + \alpha - l)^{1-\lambda}] = \bar{f}_{i+1,l}^{\lambda}, \quad l = 1, 2, \ldots, i + 1,
\]
(112)
\[
\bar{d}_{i+1,0}^{\lambda} = (i + \alpha - l)^{1-\lambda} - (i + \alpha)^{1-\lambda} > 0.
\]

We should show that \(\bar{f}_{i+1,l}^{\lambda} > 0\) and \(\bar{d}_{i+1,l}^{\lambda} - \bar{f}_{i+1,l}^{\lambda} > 0\), for \(0 \leq l \leq i + 1\), and use this to end the proof of estimates given in relation (107).
\[
2Z_{i+1,l} - \frac{1}{2} = \frac{(i + \alpha + l) - (i + \alpha - l)^{2-\lambda} - (i + \alpha - l)^{2-\lambda} - (2 - \lambda)(i + \alpha - l)^{1-\lambda}}{(2 - \lambda)[(i + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda}]} - \frac{1}{2} =
\]
\[
\frac{2}{2 - \lambda}[(i + \alpha - l)^{2-\lambda} - (i + \alpha - l)^{2-\lambda}] - \frac{1}{2}[(i + \alpha - l)^{1-\lambda} + (i + \alpha - l)^{1-\lambda}] = \frac{(i + \alpha - l)^{1-\lambda} + (i + \alpha - l)^{1-\lambda}}{2[(i + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda}]} =
\]
\[
[(i + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda}]^{1-\lambda} - \bar{f}_{i+1,l}^{\lambda}.
\]
(113)
But it comes from Lemma (54) that \(Z_{i+1,l} > \frac{1}{2}\), this implies \(2Z_{i+1,l} - \frac{1}{2} > 0\). Thus, \(\bar{f}_{i+1,0}^{\lambda} > 0, \bar{f}_{i+1,1}^{\lambda} > 0, \bar{f}_{i+1,0}^{\lambda} > 0\) and \(\bar{f}_{i+1,l}^{\lambda} - \bar{f}_{i+1,l}^{\lambda} > 0\), for \(l = 1, 2, \ldots, i + 1\).
In a similar way, it is easy to show that

$$\frac{3}{2} - 2Z_{i+1-l} = [(i + 1 + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda}] - \frac{1}{2}[(i + 2 + \alpha - j)^{1-\lambda} - (i + 1 + \alpha - j)^{1-\lambda} + (i + \alpha - j)^{1-\lambda}].$$

(114)

Since \(\frac{3}{2} - \frac{2}{2} - \alpha = \frac{2-\frac{3}{2}}{2(2-\lambda)} > 0\), for \(0 < \lambda < \frac{2}{3}\). This fact together with from Lemma 3.4 provides \(\frac{3}{2} - 2Z_{i+1-l} > \frac{2}{2} - 2Z_{i+1-l} > 0\). Hence,

$$d_{i+1,0} - f_{i+1,0} > 0, \quad \text{and} \quad d_{i+1,l} - f_{i+1,l} = d_{i+1,l-1} - f_{i+1,l-1} > 0, \quad l = 1, 2, \ldots, i + 1.$$

This ends the proof of relation (107). Now, we should prove the inequalities in (108).

Firstly, it is not hard to observe that

$$-\lambda(1-\lambda) \int_0^1 \int_0^1 (i + y_1 + y_2 + \alpha - j)^{-1-\lambda} dy_1 dy_2 = (i + 2 + \alpha - j)^{-1-\lambda} - 2(i + 1 + \alpha - j)^{-1-\lambda} + (i + \alpha - j)^{-1-\lambda}.$$

Using this together with equation (107), simple calculations give

$$\bar{f}_{i+1,1,j-1} + \bar{f}_{i+1,1,j} - \bar{d}_{i+1,j} = \frac{2}{2-\lambda} [(i + 2 + \alpha - j)^{2-\lambda} - (i + \alpha - j)^{2-\lambda}] - \frac{1}{2} [(i + 2 + \alpha - j)^{-1-\lambda} - 2(i + 1 + \alpha - j)^{-1-\lambda} + (i + \alpha - j)^{-1-\lambda}] \leq 2(i + 1 + \alpha - j)^{-1-\lambda} - \frac{1}{4} (i + 1 + \alpha - j)^{-1-\lambda} = 0, \quad j = 0, 1, \ldots, i.$$

(115)

But, for \(j = 1, 2, \ldots, i\), plugging equation (114) and estimate (115) yields

$$\bar{f}_{i+1,1,j} + \bar{f}_{i+1,1,j} - \bar{d}_{i+1,j} < 0, \quad \text{for} \quad j = 0, 1, \ldots, i.$$

(116)

Furthermore, combining (57) and Lemma 3.5 we obtain

$$2\bar{f}_{i+1,j} - \bar{d}_{i+1,j} = \frac{4}{2-\lambda} [(i + 1 + \alpha - j)^{2-\lambda} - (i + \alpha - j)^{2-\lambda}] - 2[(i + 1 + \alpha - j)^{-1-\lambda} + (i + \alpha - j)^{-1-\lambda}] = 4 \left\{ \frac{1}{2-\lambda} [(i + 1 + \alpha - j)^{2-\lambda} - (i + \alpha - j)^{2-\lambda}] - \frac{1}{2} [(i + 1 + \alpha - j)^{-1-\lambda} + (i + \alpha - j)^{-1-\lambda}] \right\} > 0, \quad l = 0, 1, \ldots, i.$$

The last estimate comes from Lemma 3.5. Utilizing this inequality together with relation (111), it holds

$$2\bar{f}_{i+1,j} - \bar{d}_{i+1,j} > 0, \quad \text{for} \quad l = 0, 1, \ldots, i - 1.$$

This completes the proof of relation (108).
Finally, we should prove the first estimate in relation (109). The proof of the second one is similar.

Firstly, it comes from the expression of \( a_{i+\frac{2}{2}}^{\lambda_{y}} \) and \( a_{i+\frac{2}{2},1}^{\lambda_{y}} \), that

\[
a_{i+\frac{2}{2}}^{\lambda_{y}} - a_{i+\frac{2}{2},1}^{\lambda_{y}} = \tilde{a}_{i+\frac{2}{2}}^{\lambda_{y}} + \tilde{f}_{i+\frac{2}{2},0}^{\lambda_{y}} - \tilde{a}_{i+\frac{2}{2}}^{\lambda_{y}}.
\]

Utilizing equation (116) and (25), this becomes

\[
a_{i+\frac{2}{2}}^{\lambda_{y}} - a_{i+\frac{2}{2},1}^{\lambda_{y}} = -\frac{2}{2-\lambda} \left[(i+\alpha)^{2-\lambda} - (i+\alpha-1)^{2-\lambda}\right] + \frac{1}{2}(i+\alpha)^{1-\lambda} + 3(i+\alpha-1)^{1-\lambda} + (i+\frac{1}{2}+\alpha)^{1-\lambda} - (i+\alpha)^{1-\lambda}
\]

\[
= -\frac{2}{2-\lambda} \int_{0}^{1} (i + \alpha + y_1 - 1)^{1-\lambda} dy_1 - \frac{1}{2}(i+\alpha)^{1-\lambda} - 3(i+\alpha-1)^{1-\lambda} + (i+\frac{1}{2}+\alpha)^{1-\lambda}.
\]

Applying the integral mean value theorem, there exists \( y^\alpha \in (0,1) \) such that, \( \int_{0}^{1} (i + \alpha + y_1 - 1)^{1-\lambda} dy_1 = (i + \alpha + y^\alpha - 1)^{1-\lambda} \). Using this and assuming that \( 4y^\alpha \leq 3 \), it holds

\[
a_{i+\frac{2}{2}}^{\lambda_{y}} - a_{i+\frac{2}{2},1}^{\lambda_{y}} = -2(i + \alpha + y^\alpha - 1)^{1-\lambda} + \frac{3}{2}(i + \alpha - 1)^{1-\lambda} - \frac{1}{2}(i+\alpha)^{1-\lambda} + (i+\frac{1}{2}+\alpha)^{1-\lambda} =
\]

\[
(1 - \lambda) \left[ \int_{y^\alpha}^{\frac{1}{2}} (i + \alpha + y_1 - 1)^{-\lambda} dy_1 - \int_{0}^{y^\alpha} (i + \alpha + y_1 - 1)^{-\lambda} dy_1 - \frac{1}{2} \int_{0}^{1} (i + \alpha + y_1 - 1)^{-\lambda} dy_1 \right] \leq
\]

\[
(1 - \lambda) \left[ \left( \frac{3}{2} - y^\alpha \right) (i + \alpha + y^\alpha - 1)^{-\lambda} - y^\alpha (i + \alpha + y^\alpha - 1)^{-\lambda} - \frac{1}{2} (i + \alpha)^{-\lambda} \right] =
\]

\[
(1 - \lambda) \left[ \left( \frac{3}{2} - 2y^\alpha \right) (i + \alpha + y^\alpha - 1)^{-\lambda} - \frac{1}{2} (i + \alpha)^{-\lambda} \right] = \frac{1 - \lambda}{2(i + \alpha)^{\lambda}} \left[ (3 - 4y^\alpha) \left( \frac{i + \alpha}{i + \alpha + y^\alpha - 1} \right)^{\lambda} - 1 \right] =
\]

\[
\frac{1 - \lambda}{2(i + \alpha)^{\lambda}} \left[ (3 - 4y^\alpha) \left( 1 + \frac{1 - y^\alpha}{i + \alpha + y^\alpha - 1} \right)^{\lambda} - 1 \right] < \frac{1 - \lambda}{2(i + \alpha)^{\lambda}} \left[ (3 - 4y^\alpha) \left( 1 + \frac{\lambda(1 - y^\alpha)}{i + \alpha + (y^\alpha - 1)} \right) - 1 \right] =
\]

\[
-\frac{\alpha(\alpha + y^\alpha)}{2(i + \alpha)^{\lambda}} \left[ 4\alpha(y^\alpha)^2 + (5 - 3\alpha) y^\alpha - 3 + \alpha \right] \leq 0,
\]

for values of \( y^\alpha \) satisfying

\[
\max \left\{ 0, \frac{1}{8\alpha} \left[ -5 + 3\alpha + \sqrt{7\alpha^2 + 18\alpha + 25} \right] \right\} \leq y^\alpha \leq \frac{3}{4},
\]

(116)

where \( \alpha = 1 - \lambda \) and \( \lambda \in (0, \frac{1}{2}) \). In fact, since \( y^\alpha \in (0,1) \), without loss of this generality, we can assume that \( y^\alpha \) satisfies estimate (116). Thus

\[
a_{i+\frac{2}{2}}^{\lambda_{y}} < a_{i+\frac{2}{2},1}^{\lambda_{y}}.
\]

For \( l = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, i - \frac{1}{2}, \) if \( l \) is an integer, it follows from relation (104) that

\[
a_{i+\frac{2}{2}}^{\lambda_{y},l} = \tilde{d}_{i+\frac{2}{2},l-1}^{\lambda_{y}} - \tilde{f}_{i+\frac{2}{2},l-1}^{\lambda_{y}} \quad \text{and} \quad a_{i+\frac{2}{2},l+\frac{1}{2}}^{\lambda_{y}} = \tilde{a}_{i+\frac{2}{2},l-1}^{\lambda_{y}}.
\]

Utilizing the second inequality in (108), it is easy to see that

\[
a_{i+\frac{2}{2}}^{\lambda_{y},l} - a_{i+\frac{2}{2},l+\frac{1}{2}}^{\lambda_{y}} = \tilde{a}_{i+\frac{2}{2},l-1}^{\lambda_{y}} - 2\tilde{f}_{i+\frac{2}{2},l-1}^{\lambda_{y}} < 0.
\]

If \( l \) is not an integer, relation (104) provides

\[
a_{i+\frac{2}{2}}^{\lambda_{y},l} = \tilde{f}_{i+\frac{2}{2},l}^{\lambda_{y}} \quad \text{and} \quad a_{i+\frac{2}{2},l+\frac{1}{2}}^{\lambda_{y}} = \tilde{a}_{i+\frac{2}{2},l}^{\lambda_{y}} - \tilde{f}_{i+\frac{2}{2},l}^{\lambda_{y}}.
\]

Hence

\[
a_{i+\frac{2}{2}}^{\lambda_{y},l} - a_{i+\frac{2}{2},l+\frac{1}{2}}^{\lambda_{y}} = \tilde{f}_{i+\frac{2}{2},l}^{\lambda_{y}} + \tilde{f}_{i+\frac{2}{2},l}^{\lambda_{y}} - \tilde{a}_{i+\frac{2}{2},l}^{\lambda_{y}} < 0.
\]

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The last estimate comes from the first inequality in (108). Furthermore,
\[ a_{i+1}^{\lambda, m} - a_{i}^{\lambda, m} = \overline{f}_{i+1}^{\alpha, m} - 2\overline{f}_{i+1}^{\alpha, m-1} - f_{i+1}^{\alpha, m} < 0. \]
In a similar manner, Using relations (106) and (108), one easily shows that,
\[ a_{i+1, l}^{\lambda, m} - a_{i+1, l+\frac{1}{2}} < 0, \]
for \( l = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, i + \frac{1}{2} \). This completes the proof of relation (109). To end, we should prove the second inequality in relation (110). The proof of the first one is similar.

Combining equation (114) and the last equation in (107), it holds
\[ \left( \frac{3}{2} - 2Z_{i+1-l} \right) \left[ (i + 1 + \alpha - l)^{1-\lambda} - (i + \alpha - l)^{1-\lambda} \right] = \overline{a}_{i+1, l}^{\lambda, m} - \overline{f}_{i+1, l}^{\alpha, m} = a_{i+1, l+\frac{1}{2}}^{\lambda, m}, \]
if \( l \) is an integer satisfying \( 0 \leq l \leq i \). This fact together with Lemma 8.3 result in
\[ a_{i+1, l+\frac{1}{2}}^{\lambda, m} > \left( \frac{3}{2} - 2 \lambda \right) \left( 1 - \lambda \right) \int_0^1 (i + \alpha + y_1 - l)^{-\lambda} dy_1 = \frac{(2 - 3 \lambda)(1 - \lambda)}{2(2 - \lambda)} (i + 1 + \alpha - l)^{-\lambda}, \]
for \( l = 0, 1, \ldots, i \). Now, if \( l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, i + \frac{1}{2} \), plugging equation (113) and Lemma 8.4 it is not difficult to observe that
\[ \left( 2Z_{i+1-(l-\frac{1}{2})} - \frac{1}{2} \right) \left[ (i + 1 + \alpha -(l - \frac{1}{2}))^{1-\lambda} - (i + \alpha -(l - \frac{1}{2}))^{1-\lambda} \right] = \overline{f}_{i+1, l-\frac{1}{2}}^{\alpha, m} = a_{i+1, l+\frac{1}{2}}^{\lambda, m}. \]
Since \( 2Z_{i+1-(l-\frac{1}{2})} > 1 \), this implies
\[ a_{i+1, l+\frac{1}{2}}^{\lambda, m} > \frac{1 - \lambda}{2} \int_0^1 (i + \alpha + y_1 - (l - \frac{1}{2}))^{-\lambda} dy_1 \geq \frac{1 - \lambda}{2} (i + 1 + \alpha - (l - \frac{1}{2}))^{-\lambda} > \frac{(2 - 3 \lambda)(1 - \lambda)}{2(2 - \lambda)} (i + \frac{3}{2} + a - l)^{-\lambda}. \]
The last estimate follows from \( 1 > \frac{2 - 3 \lambda}{2} > 0 \), for any \( 0 < \lambda < \frac{3}{2} \). Thus,
\[ f_{i+1, l-\frac{1}{2}}^{\alpha, m} = \overline{f}_{i+1, l+\frac{1}{2}}^{\alpha, m} > \frac{(2 - 3 \lambda)(1 - \lambda)}{2(2 - \lambda)} (i + \frac{3}{2} + \alpha - l)^{-\lambda}, \]
for \( \lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, i + \frac{1}{2} \). Since \( \alpha = 1 - \lambda \) and \( 1 > \frac{2 - 3 \lambda}{2} > 0 \), for any \( 0 < \lambda < \frac{3}{2} \), a combination of (117), (106) and (118) gives
\[ a_{i+1, i+1}^{\lambda, m} = f_{i+1, i+1}^{\alpha, m} + f_{i+1, i+1}^{\alpha, m} > \frac{(2 - 3 \lambda)(1 - \lambda)}{2(2 - \lambda)} (1 + \alpha)^{-\lambda} + \alpha^{-\lambda} > \frac{1 - \lambda}{2} \alpha^{-\lambda} > \frac{(2 - 3 \lambda)(1 - \lambda)}{2(2 - \lambda)} \alpha^{-\lambda}. \]
An assembly of estimates (117)-(119) ends the proof of the second inequality in relation (110). In a similar way, one easily show the first estimate in (110). This completes the proof of Lemma 3.6.

The following Lemma (Lemma 3.7) plays a crucial role when proving Lemma 3.8.

**Lemma 3.7.** Consider the generalized sequences \((a_{i,l}^{\lambda, m})_l\) defined by relations (103)-106). For any mesh function \( w \) defined on the grid space \( Y_{kh} \), it holds
\[ \sum_{l=l_0}^{m} (a_{i,l}^{\lambda, m})^{-1}[(w^{l+\frac{1}{2}})^2 - (w^l)^2] = (a_{i,m}^{\lambda, m})^{-1}(w^m)^2 - (a_{i,m}^{\lambda, m})^{-1}(w^m)^2 + \sum_{l=l_0}^{m-\frac{1}{2}} [(a_{i,l}^{\lambda, m})^{-1} - (a_{i,l+1}^{\lambda, m})^{-1}] (w^{l+\frac{1}{2}})^2, \]
for \( m \in \{i,i+\frac{1}{2}\} \) and \( l = l_0, l_0 + \frac{1}{2}, l_0 + 1, l_0 + \frac{3}{2}, \ldots, m \), where \( l_0 \) is a nonnegative integer that satisfies \( l_0 \leq m \).
Proof. Expanding the left side of this equality and rearranging terms to obtain the result. \qed

Lemma 3.8. Given the generalized sequences \((a_{i,j}^{\alpha})_l\) defined by relations (103) - (106), respectively, for every grid function \(u(\cdot, \cdot)\) defined on the mesh space \(Y_{kh}\), the following estimates are satisfied

\[
u_j^{i+\frac{1}{2}}(c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}) = \frac{1}{2}c\Delta_{0t}^2(u_j^{i+\frac{1}{2}+\alpha})^2 + \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left\{ (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \left( \sum_{l=0}^{\frac{i}{2}} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_l u_j^l \right)^2 + \right.
\]

\[
\left[ a_{i+\frac{1}{2},i+\frac{1}{2}} - (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1}(a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] (\delta_l u_j^l)^2 + \sum_{l=\frac{i}{2}}^{i-\frac{1}{2}} \left[ (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] \left( \sum_{r=0}^{l} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_t u_j^r \right)^2 \right\},
\]

\[
u_j^{i} (c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}) = \frac{1}{2}c\Delta_{0t}^2(u_j^{i+\frac{1}{2}+\alpha})^2 - \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left\{ (a_{i+\frac{1}{2},i+\frac{1}{2}}) (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] \left( \sum_{l=0}^{\frac{i}{2}} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_l u_j^l \right)^2 + \right.
\]

\[
\left[ a_{i+\frac{1}{2},i+\frac{1}{2}} - (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1}(a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] (\delta_l u_j^l)^2 + \sum_{l=\frac{i}{2}}^{i-\frac{1}{2}} \left[ (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] \left( \sum_{r=0}^{l} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_t u_j^r \right)^2 \right\},
\]

Furthermore,

\[
u_j^{i+1} (c\Delta_{0t}u_j^{i+1+\alpha}) = \frac{1}{2}c\Delta_{0t}^2(u_j^{i+1+\alpha})^2 + \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left\{ (a_{i+1,i+1}^{\alpha})^{-1} \left( \sum_{l=0}^{\frac{i}{2}} a_{i+1,i+1}^{\alpha} \delta_l u_j^l \right)^2 + \right.
\]

\[
\left[ a_{i+1,i+1} - (a_{i+1,i+1}^{\alpha})^{-1}(a_{i+1,i+1}^{\alpha})^{-1} (a_{i+1,i+1}^{\alpha})^{-1} \right] (\delta_l u_j^l)^2 + \sum_{l=\frac{i}{2}}^{i-\frac{1}{2}} \left[ (a_{i+1,i+1}^{\alpha})^{-1} (a_{i+1,i+1}^{\alpha})^{-1} \right] \left( \sum_{r=0}^{l} a_{i+1,i+1}^{\alpha} \delta_t u_j^r \right)^2 \right\},
\]

\[
u_j^{i+\frac{1}{2}} (c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}) = \frac{1}{2}c\Delta_{0t}^2(u_j^{i+\frac{1}{2}+\alpha})^2 - \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left\{ (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] \left( \sum_{l=0}^{\frac{i}{2}} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_l u_j^l \right)^2 + \right.
\]

\[
\left[ a_{i+\frac{1}{2},i+\frac{1}{2}} - (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1}(a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] (\delta_l u_j^l)^2 + \sum_{l=\frac{i}{2}}^{i-\frac{1}{2}} \left[ (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} (a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha})^{-1} \right] \left( \sum_{r=0}^{l} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_t u_j^r \right)^2 \right\},
\]

where "*" denotes the usual multiplication in \(C\), \(c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}\) and \(c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}\) are defined by equations (230) and (55), respectively.

Proof. Firstly, combining equations (233), (234) and (104), it is not difficult to observe that

\[
u_j^{i+\frac{1}{2}}(c\Delta_{0t}u_j^{i+\frac{1}{2}+\alpha}) = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{l=0}^{i} a_{i+\frac{1}{2},i+\frac{1}{2}+\alpha} \delta_l u_j^l.
\]

Similarly, a combining equations (55), (56) and (106) gives

\[
u_j^{i+1} (c\Delta_{0t}u_j^{i+1+\alpha}) = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{l=0}^{l} a_{i+1,i+1}^{\alpha} \delta_l u_j^l,
\]

where the summation index \(l\) varies with a step size \(\frac{1}{2}\). Furthermore, using relation (124), we should prove only equations (120) and (121). The proof of relations (122) and (123) is similar thanks to equation (125).
Substituting (128) into equation (126) and using Lemma 3, we recall that the summation indices $a$.

Equality (127) is due to Lemma 3.

Let prove equation (121). Adding the term $\frac{1}{2}c\Delta_{\alpha}^\lambda (u_{j+\frac{1}{2}}^{\alpha})^2$ to $-u_j^\lambda (c\Delta_{\alpha}^\lambda u_j^{\alpha})$ and performing straightforward calculations, this yields

$$-u_j^\lambda (c\Delta_{\alpha}^\lambda u_j^{\alpha}) + \frac{1}{2}c\Delta_{\alpha}^\lambda (u_j^{\alpha})^2 = \frac{k^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{l=0}^{i} a_{l+\frac{1}{2},l+\frac{1}{2}} \delta_i u_j^l - \frac{1}{2} \sum_{l=0}^{i} a_{l+\frac{1}{2},l+\frac{1}{2}} \delta_i (u_j^l)^2 =$$
\[
\frac{k^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{l=0}^{i} a_{i+\frac{l}{2}, t+\frac{l}{2}} \delta t u_j^l \left( \frac{1}{2} (u_j^{r+\frac{l}{2}} - u_j^{r-\frac{l}{2}}) - \sum_{r=0}^{i} (u_j^{r+\frac{l}{2}} - u_j^{r-\frac{l}{2}}) \right) = \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \sum_{l=0}^{i} \frac{a_{i+\frac{l}{2}, t+\frac{l}{2}}}{(2-\lambda)} (\delta t u_j^{l})^2 - \\
\frac{2}{4\Gamma(2-\lambda)} \sum_{l=0}^{i} a_{i+\frac{l}{2}, t+\frac{l}{2}} \delta t u_j^l \sum_{l=0}^{i} \frac{a_{i+\frac{l}{2}, t+\frac{l}{2}}}{(2-\lambda)} (\delta t u_j^{l})^2 - 2\sum_{r=0}^{i} \delta t u_j^r \sum_{l=0}^{i} \frac{a_{i+\frac{l}{2}, t+\frac{l}{2}}}{(2-\lambda)} (\delta t u_j^{l})^2,
\]
since the sum equals zero if the lower summation index is less than the upper one. Using \((127) - (128)\) together with Lemma 3.7 this equation becomes

\[
-u_j^i (c \Delta \delta t u_j^{i+\frac{1}{2}+\alpha}) + \frac{1}{2} c \Delta \delta t (u_j^{i+\frac{1}{2}+\alpha})^2 = \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left[ -a_{i+\frac{l}{2}, t+\frac{l}{2}} (\delta t u_j^{l})^2 + \sum_{r=0}^{i} (a_{i+\frac{l}{2}, t+\frac{l}{2}} - W_i^{i+\frac{1}{2}, j} - W_i^{i+\frac{1}{2}, j})^2 \right] \\
+ a_{i+\frac{l}{2}, t+\frac{l}{2}} (\delta t u_j^{l})^2 - 2\sum_{r=0}^{i} (a_{i+\frac{l}{2}, r+\frac{1}{2}} (W_i^{i+\frac{1}{2}, j} - W_i^{i+\frac{1}{2}, j})) W_i^{i+\frac{1}{2}, j} \\
- \left[ (a_{i+\frac{l}{2}, r+\frac{1}{2}})^{-1} - (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} \right] (W_i^{i+\frac{1}{2}, j})^2 - \left[ (a_{i+\frac{l}{2}, r+\frac{1}{2}})^{-1} - (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} \right] (W_i^{i+\frac{1}{2}, j})^2.
\]

The proof of equation \((121)\) is completed thanks to equality \(W_i^{i+\frac{1}{2}, j} = a_{i+\frac{l}{2}, t+\frac{l}{2}} \delta t u_j^{l}\) and equation \((127)\). In a similar manner, one easily proves relations \((122)\) and \((123)\). This completes the proof of Lemma 3.8.\[\Box\]

**Lemma 3.9.** Let \((a_{i+\frac{l}{2}}^{\lambda})\) be the generalized sequences defined by equations \((103) - (106)\), respectively. For every grid function \(u_j(\cdot, \cdot)\) defined on the mesh space \(\mathcal{Y}_{kh}\), the following estimates hold

\[
(c \Delta \delta t u_j^{i+\frac{1}{2}+\alpha}, u_{0}) \geq \frac{h}{2} \sum_{j=1}^{M-1} c \Delta \delta t (u_j^{i+\frac{1}{2}+\alpha})^2,
\]
if \(\alpha = 1 - \lambda\) satisfies

\[
4\alpha^2 - (1 + 4\alpha) \left( (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} - 1 \right) \leq 0, \text{ for } i \geq 1.
\]

Furthermore,

\[
(c \Delta \delta t u_j^{i+1+\alpha}, u_{0}) \geq \frac{h}{2} \sum_{j=1}^{M-1} c \Delta \delta t (u_j^{i+1+\alpha})^2,
\]

whenever

\[
4\alpha^2 - (1 + 4\alpha) \left( (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} - 1 \right) \leq 0, \text{ for } i \geq 0.
\]

**Proof.** We should prove only estimate \((129)\), the proof of inequality \((131)\) is similar.

Multiplying both sides of relations \((120)\) and \((121)\) by \(1 + 2\alpha\) and \(-2\alpha\), respectively, using relation \((127)\) and summing, it is not hard to see that

\[
(1 + 2\alpha) u_j^{i+\frac{1}{2}} c \Delta \delta t (u_j^{i+\frac{1}{2}+\alpha} - 2\alpha u_j^c \Delta \delta t (u_j^{i+\frac{1}{2}+\alpha}) = \frac{1}{2} c \Delta \delta t (u_j^{i+\frac{1}{2}+\alpha})^2 + \frac{k^{2-\lambda}}{4\Gamma(2-\lambda)} \left( (1 + 2\alpha) (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} (W_i^{i+\frac{1}{2}, j})^2 + \right.
\]

\[
\left. + (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} - (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} \right) (\delta t u_j^{l})^2 + (1 + 2\alpha) \left[ (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} - (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1} \right] (W_i^{i+\frac{1}{2}, j})^2 - 2\alpha (a_{i+\frac{l}{2}, t+\frac{l}{2}})^{-1}
\]
In a similar way, one easily proves inequality (131). This completes the proof of Lemma 3.

**Remark 3.1.** Inequality (129) shows that the new method is convergent of order 2 − h in time and spatial fourth-order accurate.
Proof. (of Theorem 3.1) Setting \( I^{α,i}_j = cD_0^αu_j^{i+\frac{1}{α}+α} - cΔ_0^αu_j^{i+\frac{1}{α}+α} \), it comes from (3.2) that

\[
I^{α,i}_j = \frac{1}{Γ(1 - λ)} \left\{ \sum_{i=0}^{M-1} \int_{t_i+\frac{1}{α}+α}^{t_{i+1}+\frac{1}{α}+α} \frac{E_{τ,i}^j(τ)}{(t_i+\frac{1}{α}+α - τ)^λ} dτ + \int_0^{t_i+\frac{1}{α}+α} \frac{u_{τ,j}(τ) - δ_{i}u_0^j}{(t_i+\frac{1}{α}+α - τ)^λ} dτ + \int_{t_i+\frac{1}{α}+α}^{t_{i+1}+\frac{1}{α}+α} \frac{u_{τ,j}(τ) - δ_{i}u_0^j}{(t_i+\frac{1}{α}+α - τ)^λ} dτ \right\},
\]

for \( i \geq 1 \). Combining equations (141), (170) and equation (3.89) (case where \( γ_1 = α_1 \)), simple calculations give

\[
cΔ_0^αu_j^{i+\frac{1}{α}+α} + I^{α,i}_j - cΔ_0^αU_j^{i+\frac{1}{α}+α} = L_hu_α^j - L_hU_α^j + q_i^{i+\frac{1}{α}+α}ψ_1^{i,j} - p_i^{i+\frac{1}{α}+α}ψ_2^{i,j} + α(\frac{1}{2} + α)k^2σ_j^{i+\frac{1}{α}+α}H_{1,j},
\]

which is equivalent to

\[
cΔ_0^αe_j^{i+\frac{1}{α}+α} - L_he_α^j = -I^{α,i}_j + q_i^{i+\frac{1}{α}+α}ψ_1^{i,j} - p_i^{i+\frac{1}{α}+α}ψ_2^{i,j} + α(\frac{1}{2} + α)k^2σ_j^{i+\frac{1}{α}+α}H_{1,j},
\]

(137)

where \( H_{1,j}, ψ_1^{i,j}, ψ_2^{i,j} \) and \( I^{α,i}_j \) are defined by relations (3.89), (4.2), (4.3) and (3.30), respectively. Multiplying both sides of (137) by \( he_α^j \) and summing up the obtained equation from \( j = 1, 2, ..., M - 1 \), yields

\[
(cΔ_0^αe_j^{i+\frac{1}{α}+α}, e_α^j) + (-L_he_α^j, e_α^j) = -h \sum_{j=1}^{M-1} I^{α,i}_j e_α^j + h \sum_{j=1}^{M-1} (L_hu_α^{i+\frac{1}{α}+α} - L_hU_α^{i+\frac{1}{α}+α}) e_α^j.
\]

Utilizing equation (3.33), this becomes

\[
(cΔ_0^αe_j^{i+\frac{1}{α}+α}, e_α^j) + (-L_he_α^j, e_α^j) = -h \sum_{j=1}^{M-1} I^{α,i}_j e_α^j + h \sum_{j=1}^{M-1} (L_hu_α^{i+\frac{1}{α}+α} - L_hU_α^{i+\frac{1}{α}+α}) e_α^j.
\]

Applying the Hölder inequality to the right side of this equation and using Lemma 3.31 this implies

\[
(cΔ_0^αe_j^{i+\frac{1}{α}+α}, e_α^j) + γ∥Δ_0^αe_α^j∥^2_{L^2} + β∥e_α^j∥^2_{L^2} \leq \left[ \left( \sum_{j=1}^{M-1} |I^{α,i}_j|^2 \right)^{\frac{1}{2}} + \left( h \sum_{j=1}^{M-1} |L_hu_α^{i+\frac{1}{α}+α} - L_hU_α^{i+\frac{1}{α}+α}|^2 \right)^{\frac{1}{2}} \right] \cdot ||e_α^j||_{L^2}.
\]

By the Poincare-Friedrichs inequality, there is a positive parameter \( \tilde{C}_0 \) so that, \( ∥Δ_0^αe_α^j∥^2_{L^2} ≥ \tilde{C}_0∥e_α^j∥^2_{L^2} \). This fact, together with estimate (138) and Lemmas 3.6, 3.2 result in

\[
(cΔ_0^αe_j^{i+\frac{1}{α}+α}, e_α^j) + C_0∥e_α^j∥^2_{L^2} ≤ \left[ C_4 k^2 - λ + C_3 \left( k^2 + (1 + k^2)h^4 \right) ||u||^2_{C^0_D} \right] ||e_α^j||_{L^2},
\]

(139)

where \( C_0 = γ\tilde{C}_0 + β, C_4 \) is the constant given in estimate 3.1 and all the constants in estimate 3.1 are absorbed into a positive constant \( C_3 \).

Now, replacing \( u \) by \( e \) in estimate (129) of Lemma 3.9 and plugging the new estimate with (139) provides

\[
\frac{h}{2} \sum_{j=1}^{M-1} cΔ_0^α(e_j^{i+\frac{1}{α}+α})^2 + C_0∥e_α^j∥^2_{L^2} ≤ \left[ C_4 k^2 - λ + C_3 \left( k^2 + (1 + k^2)h^4 \right) ||u||^2_{C^0_D} \right] ||e_α^j||_{L^2}.
\]

Combining this together with relation (124), direct calculations give

\[
\frac{h}{2} \sum_{j=1}^{M-1} \sum_{j=0}^{M-1} a_{i+\frac{1}{α}+α} δ_τ(e_j^i)^2 + C_0∥e_α^j∥^2_{L^2} ≤ \left[ C_4 k^2 - λ + C_3 \left( k^2 + (1 + k^2)h^4 \right) ||u||^2_{C^0_D} \right] ||e_α^j||_{L^2}.
\]

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Performing simple computations, this yields

$$h \sum_{j=1}^{M-1} \sum_{l=0}^{i} \left( a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l+\frac{1}{2}})^2 - (e_{j}^{l})^2 \right) + \Gamma(2 - \lambda) C_0 k^\lambda \|e^{\alpha_i}\|_{L^2} \leq \Gamma(2 - \lambda) \left[ C_2 k^2 + C_3 (k^{2+\lambda} + (1 + k^2) k^\lambda h^4) \right] \|e^{\alpha_i}\|_{L^2}.$$ 

We recall that the summation index \( l \) varies in the range \( l = 0, \frac{1}{2}, 1, \frac{3}{2}, ..., i \). Using the summation by parts, this results in

$$h \sum_{j=1}^{M-1} \sum_{l=0}^{i} \left[ a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l+\frac{1}{2}})^2 - a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l})^2 - \sum_{l=0}^{i-\frac{1}{2}} \left[ a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l+\frac{1}{2}})^2 \right] \right] + \Gamma(2 - \lambda) C_0 k^\lambda \|e^{\alpha_i}\|_{L^2} \leq \Gamma(2 - \lambda) \left[ C_2 k^2 + C_3 (k^{2+\lambda} + (1 + k^2) k^\lambda h^4) \right] \|e^{\alpha_i}\|_{L^2}. \tag{140}$$

It follows from the initial condition given in (22) that \( e_{j}^0 = 0, \) for \( j = 0, 1, 2, ..., M. \) Using this, inequality (140) is equivalent to

$$a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} \|e^{l+\frac{1}{2}}\|_{L^2}^2 + \Gamma(2 - \lambda) C_0 k^\lambda \|e^{\alpha_i}\|_{L^2} \leq \sum_{l=0}^{i-\frac{1}{2}} \left[ a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l+\frac{1}{2}})^2 \right] \|e^{l+\frac{1}{2}}\|_{L^2} + \Gamma(2 - \lambda) F(k, h) \|e^{\alpha_i}\|_{L^2}, \tag{141}$$

where

$$F(k, h) = C_2 k^2 + C_3 \left( k^{2+\lambda} + (1 + k^2) k^\lambda h^4 \right) \|u\|_{C_D^0}. \tag{142}$$

But, it is not hard to observe that

$$F(k, h) \|e^{\alpha_i}\|_{L^2} = 2(\sqrt{C_0 k^\lambda} \|e^{\alpha_i}\|_{L^2}) \left( \frac{1}{2} \sqrt{C_0 k^\lambda} F(k, h) \right) \leq C_0 k^\lambda \|e^{\alpha_i}\|_{L^2} + \frac{1}{4C_0 k^\lambda} F(k, h)^2. \tag{143}$$

Substituting (143) into (141) and after simplifying provides

$$a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} \|e^{l+\frac{1}{2}}\|_{L^2}^2 \leq \sum_{l=0}^{i-\frac{1}{2}} \left[ a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} (e_{j}^{l+\frac{1}{2}})^2 \right] \|e^{l+\frac{1}{2}}\|_{L^2} + \frac{\Gamma(2 - \lambda)}{4C_0 k^\lambda} F(k, h)^2, \text{ for } i > 1. \tag{144}$$

Similarly, one easily shows that

$$a^{\lambda a}_{i+1, l+1} \|e^{l+1}\|_{L^2}^2 \leq \sum_{l=0}^{i} \left[ a^{\lambda a}_{i+1, l+1} (e_{j}^{l+1})^2 \right] \|e^{l+1}\|_{L^2} + \frac{\Gamma(2 - \lambda)}{4C_0 k^\lambda} F(k, h)^2, \text{ for } i \geq 0. \tag{145}$$

We should prove by mathematical induction that for \( i = 0, 1, 2, ..., \) the following estimates are satisfied

$$\|e^{l+\frac{1}{2}}\|_{L^2}, \|e^{l+1}\|_{L^2} \leq \beta_0 \gamma_0 \lambda^{\frac{1}{2}} \left[ 1 + \gamma_0^2 \left( a^{\lambda a}_{i+\frac{1}{2}, l+\frac{1}{2}} \right) \right] \frac{\gamma_0^2}{2} F(k, h), \tag{146}$$

where

$$\beta_0 = \frac{1}{2} \sqrt{\frac{\Gamma(2 - \lambda)}{C_0}} \text{ and } \gamma_0 = \left( \frac{2a^\lambda(2 - \lambda)}{(1 - \lambda)(2 - 3\lambda)} \right)^\frac{1}{2}. \tag{147}$$

Firstly, for \( i = 0, \) we should find a relation satisfied by \( a^{\lambda a}_{1,1} \|e^{\frac{1}{2}}\|_{L^2}^2, \) where \( a^{\lambda a}_{1,1} \) is given by (25). It comes from equation (23) that \( L_{ij} u^{\alpha_0} - L_{h} u^{\alpha_0} = q^\frac{1}{2} + \alpha \psi_{1, j} + p^\frac{1}{2} + \alpha \psi_{2, j} + \alpha \psi_{0, j} + \alpha \frac{1}{2} k^2 g^\frac{1}{2} + \alpha H_{1, j}. \) Setting \( I^{\alpha_0}_j = cD^\alpha_{ij} u^{\frac{1}{2} + \alpha} - c\Delta^\alpha_{ij} u^{\frac{1}{2} + \alpha}, \) using equation (21) and subtracting equation (17) from (46), it is easy to see that

$$I_j^{\alpha_0} \delta e_j^0 = L_{h} e^{\alpha_0} + L_{ij} u^{\frac{1}{2} + \alpha} - L_{h} u^{\alpha_0} + I^{\alpha_0}_j, \tag{148}$$
Utilizing Lemmas 3 where we absorbed all the constants in estimate (91) into a positive constant and \( \psi \).

Summing this from \( j = 1, 2, \ldots, M - 1 \), it is not difficult to observe that

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha_0} \|_{L_2}^2 = \frac{(1 + 2\alpha)k\Gamma(2 - \lambda)}{2} \left[ (L_h e^{\alpha_0}, e^{\alpha_0}) + \left( L u^{\frac{1}{2} + \alpha} - L_h u^{\alpha_0}, e^{\alpha_0} \right) + (I^{\alpha_0}, e^{\alpha_0}) \right].
\]

Applying Lemma 3.3 and the Poincare-Friedrichs inequality, this implies

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha_0} \|_{L_2}^2 \leq \frac{(1 + 2\alpha)k\Gamma(2 - \lambda)}{2} \left[ (L u^{\frac{1}{2} + \alpha} - L_h u^{\alpha_0}, e^{\alpha_0}) + (I^{\alpha_0}, e^{\alpha_0}) \right].
\]

where \( C_0 \) is the positive constant given in (139). But it holds

\[
(I^{\alpha_0}, e^{\alpha_0}) = 2 \left( \frac{1}{\sqrt{2}C_0} I^{\alpha_0}, \sqrt{\frac{C_0}{2}} e^{\alpha_0} \right) \leq \frac{1}{2C_0} \| I^{\alpha_0} \|_{L_2}^2 + \frac{C_0}{2} \| e^{\alpha_0} \|_{L_2}^2,
\]

and

\[
(L u^{\frac{1}{2} + \alpha} - L_h u^{\alpha_0}, e^{\alpha_0}) \leq \frac{1}{2C_0} \| L u^{\frac{1}{2} + \alpha} - L_h u^{\alpha_0} \|_{L_2}^2 + \frac{C_0}{2} \| e^{\alpha_0} \|_{L_2}^2.
\]

Substituting this into (149) and rearranging terms yields

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha_0} \|_{L_2}^2 \leq \frac{(1 + 2\alpha)k\Gamma(2 - \lambda)}{4C_0} \left[ \| I^{\alpha_0} \|_{L_2}^2 + \| L u^{\frac{1}{2} + \alpha} - L_h u^{\alpha_0} \|_{L_2}^2 \right].
\]

Utilizing Lemmas 3.1 and 3.2 we obtain

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha_0} \|_{L_2}^2 \leq \frac{(1 + 2\alpha)k\Gamma(2 - \lambda)}{4C_0} \left[ C_0^2 k^{4 - 2\lambda} + C_3^2 (k^2 + (1 + k^2)h^4)^2 \| u \|_{C^{\alpha_0}}^2 \right],
\]

where we absorbed all the constants in estimate (91) into a positive constant \( C_3 \). Since \( e^{\alpha_0} = (1 + 2\alpha)e^{\frac{x}{2}} \) and \( a^2 + b^2 \leq (a + b)^2 \), for any nonnegative real numbers \( a \) and \( b \), this becomes

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha_0} \|_{L_2}^2 \leq \frac{(1 + 2\alpha)^{-1}k\Gamma(2 - \lambda)}{4C_0} F(k, h)^2,
\]

\( F(k, h) \) is defined by (132). Using relation (110) and performing direct calculations to obtain

\[
\| e^{\alpha} \|_{L_2} \leq \left( \frac{\Gamma(2 - \lambda)}{4C_0} \right)^{\frac{1}{2}} \left( \frac{2\alpha \Gamma(2 - \lambda)}{(1 - \lambda)(2 - 3\lambda)} \right)^{\frac{1}{2}} k^{\lambda - \frac{\lambda}{4}} F(k, h) \leq \beta_0 \gamma_0 k^{\lambda - \frac{\lambda}{4}} \left[ 1 + \gamma_0^2 \left( a_{\frac{\alpha}{2}, \frac{3}{2}}^0 - a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \right) \right] F(k, h),
\]

since \((1 + 2\alpha)^{-1} < 1\). The first estimate in (150) together with inequality (145), for \( i = 0 \), result in

\[
a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \| e^{\alpha} \|_{L_2}^2 \leq \left( a_{\frac{\alpha}{2}, \frac{3}{2}}^0 - a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \right) \| e^{\alpha} \|_{L_2}^2 + \beta_0^2 k^{-\lambda} F(k, h)^2 \leq \beta_0^2 k^{-\lambda} \left[ 1 + \gamma_0^2 \left( a_{\frac{\alpha}{2}, \frac{3}{2}}^0 - a_{\frac{\alpha}{2}, \frac{3}{2}}^0 \right) \right] F(k, h)^2.
\]

But it comes from (110) that \( a_{\frac{\alpha}{2}, \frac{3}{2}}^0 > \gamma_0^{-2} \), where \( \gamma_0 \) is given in relation (147). Dividing each side of (151) by \( \gamma_0^{-2} \) and taking the square root to get the desired result.

Now, for any positive integer \( i \), we assume that estimates (146) holds for \( j = 0, 1, 2, \ldots, i - 1 \). Using the assumption, relation (144) implies

\[
a_{\frac{\alpha}{2}, \frac{3}{2}, \frac{i}{2}, \frac{3}{2}}^0 \| e^{i+\frac{1}{2}} \|_{L_2}^2 \leq \sum_{i=0}^{i-2} \left[ a_{\frac{\alpha}{2}, \frac{3}{2}, \frac{i+1}{2}, \frac{3}{2}} - a_{\frac{\alpha}{2}, \frac{3}{2}, \frac{i}{2}, \frac{3}{2}} \right] \| e^{i+\frac{1}{2}} \|_{L_2}^2 + \frac{\Gamma(2 - \lambda)}{4C_0k^{\lambda}} F(k, h)^2 \leq \beta_0^2 \gamma_0^{-2} k^{-\lambda} \left[ 1 + \gamma_0^2 \right] F(k, h)^2.
\]
Utilizing relation (110), it holds $a_{i+1,1}^α - a_{i+1,1}^α > γ_0^2$. So $-β_0^2γ_0^2k^{-λ} \left[ 1 + γ_0^2 \left( a_{i+1,1}^α - a_{i+1,1}^α \right) \right] a_{i+1,1}^α F(k, h)^2 < -β_0^2k^{-λ} \left[ 1 + γ_0^2 \left( a_{i+1,1}^α - a_{i+1,1}^α \right) \right] F(k, h)^2$. This fact, together with estimate (152) gives

$$a_{i+1,1}^α \| e^{i+\frac{1}{2}} \|_{L^2}^2 ≤ β_0γ_0k^{-\frac{1}{2}} \left[ 1 + γ_0^2 \left( a_{i+1,1}^α - a_{i+1,1}^α \right) \right] \frac{1}{2} F(k, h).$$

(153)

Similarly, using relation (153) together with the assumption and performing simple computations, estimate (145) becomes

$$a_{i+1,i+1}^α \| e^{i+1} \|_{L^2}^2 ≤ β_0γ_0k^{-\frac{1}{2}} \left[ 1 + γ_0^2 \left( a_{i+1,i+1}^α - a_{i+1,i+1}^α \right) \right] \frac{1}{2} F(k, h).$$

This ends the proof of estimates in (146). Substituting equation (142) into relation (146) and absorbing all the constants into a positive constant $C$ gives

$$\| e^{i+\frac{1}{2}} \|_{L^2}, \| e^{i+1} \|_{L^2} ≤ C \left[ k^{2\frac{1}{2}} + k^{2\frac{1}{2}} + (1 + k^2)k^{\frac{1}{2}}h^4 \right].$$

(154)

But $\| U^l \|_{L^2} ≤ \| u^l \|_{L^2}$ for $l = i + \frac{1}{2}, i + 1$. So,

$$\| U^l \|_{L^2} ≤ \| u^l \|_{L^2} + C \left[ k^{2\frac{1}{2}} + k^{2\frac{1}{2}} + (1 + k^2)k^{\frac{1}{2}}h^4 \right].$$

The proof of estimate (134) is completed by taking the maximum over $i$. Now, since $k < 1$ then, $(1 + k^2)k^{\frac{1}{2}} ≤ 2$ and $k^{2\frac{1}{2}} ≤ k^{2\frac{1}{2}}$, utilizing this, inequality (154) implies

$$\| e^{i+\frac{1}{2}} \|_{L^2}, \| e^{i+1} \|_{L^2} ≤ 2C \left[ k^{2\frac{1}{2}} + h^4 \right].$$

Taking the maximum over $i$, this completes the proof of Theorem 3.1.
for the initial-boundary value problem (1)-(3). We confirm the predicted convergence rate from the theory (see Section 3, Theorem 3.1). More precisely, Tables 1-4 and Figures 1-4 present the exact solution, the approximate ones and the errors between the computed solution and the analytical one with different values of time step $k$ and space step $h$ satisfying $k = h^{\frac{2}{3-\alpha}}$. In addition, we look at the error estimates of the proposed method for the parameters $\lambda \in \{6.6 \times 10^{-1}, 9 \times 10^{-1}\}$, $\alpha = 1 - \lambda$ and $T = L_1 = 1$.

Finally, to analyze the stability and convergence rate of our numerical scheme, we take the mesh size $h = 2^{-l}$, $l = 3, 4, 5, 6$ and time step $k \in \left\{2^{-\frac{2}{3-\alpha}}, l = 2, 3, ..., 40\right\}$, by a mid-point refinement. We set $k = h^{\frac{2}{3-\alpha}}$ and we compute the numerical solutions $\||U||_{L^2(0,T;L^2)}$, the exact ones $\|u\|_{L^2(0,T;L^2)}$, the error estimates $\|E(h)\|_{L^2(0,T;L^2)}$ related to the two-level scheme and the convergence rate using the formula $CR = \log_2(r(h))$, where $r(h) = \|E(2h)\|_{L^2(0,T;L^2)}/\|E(h)\|_{L^2(0,T;L^2)}$, to see that the new algorithm is unconditionally stable, convergent of order $2 - \frac{2}{3-\alpha}$ in time and spatial fourth-order accuracy. In addition, we plot the computed solutions, the analytical ones and the error versus $i$. We observe from this study that the proposed method is both efficient and effective than a broad range of numerical methods [3, 4, 5, 8, 10, 17] applied to the considered problem.

- **Example 1.** Let $\Omega = (0, 1)$ be the unit interval and $T = 1$ be the final time. The parameters $\lambda$ and $\alpha$ are given by 0.66, 0.9 and $\alpha = 1 - \lambda$. In [1], the functions $s, q, p, g$ are defined as: $s(x, t) = \Gamma(2 - \lambda)^{-1}t^{1-\lambda}\sin x + t\sin x + \cos x$, $q(t) = 1$, $p(t) = 1$, $g(x, t) = 0$ and the exact solution $u$ is given by $u(x, t) = t\sin(x)$.

The initial and boundary conditions are directly obtained from this analytical solution.

**Table 1.** Unconditional stability and convergence rate $O(h^4 + k^{2-\frac{2}{3-\alpha}})$ for the two-level fourth-order approach with $\log_2(r(h))$, varying spacing $h = \Delta x$ and time step $k = \Delta t$. In this test we take $\lambda = 0.9$, $\alpha = 1 - \lambda = 0.1$ and $h = h^{\frac{2}{3-\alpha}}$.

| $k$     | $\|u\|_{L^2}$ | $\||U||_{L^2}$ | $\|E(h)\|_{L^2}$ | RC   |
|---------|---------------|----------------|-------------------|------|
| $h^{-1}$| 2.4192 \times 10^{-1} | 2.4192 \times 10^{-1} | 3.2480 \times 10^{-2} | -    |
| $h^{-2}$| 1.3890 \times 10^{-1} | 1.3890 \times 10^{-1} | 2.0134 \times 10^{-3} | 4.0121 |
| $h^{-3}$| 7.1804 \times 10^{-2} | 7.1804 \times 10^{-2} | 1.2685 \times 10^{-4} | 3.9882 |
| $h^{-4}$| 3.6520 \times 10^{-2} | 3.6520 \times 10^{-2} | 7.9282 \times 10^{-6} | 4.0000 |
| $h^{-5}$| 1.8436 \times 10^{-2} | 1.8436 \times 10^{-2} | 4.9514 \times 10^{-7} | 4.0011 |

**Table 2.** Stability and Convergence rate $O(h^4 + k^{2-\frac{2}{3-\alpha}})$ of the new technique with $\log_2(r(h))$, varying spacing $h = \Delta x$ and time step $k = \Delta t$. Here we take $\lambda = 0.66$, $\alpha = 1 - \lambda = 0.34$ and $h = h^{\frac{2}{3-\alpha}}$.

| $k$     | $\|u\|_{L^2}$ | $\||U||_{L^2}$ | $\|E(h)\|_{L^2}$ | RC   |
|---------|---------------|----------------|-------------------|------|
| $h^{-1}$| 2.4211 \times 10^{-1} | 2.4212 \times 10^{-1} | 3.2389 \times 10^{-2} | -    |
| $h^{-2}$| 1.3894 \times 10^{-1} | 1.3893 \times 10^{-1} | 2.03254 \times 10^{-3} | 3.9982 |
| $h^{-3}$| 7.2046 \times 10^{-2} | 7.1923 \times 10^{-2} | 1.2665 \times 10^{-4} | 4.0044 |
| $h^{-4}$| 3.6541 \times 10^{-2} | 3.6541 \times 10^{-2} | 7.8599 \times 10^{-6} | 4.0102 |
| $h^{-5}$| 1.8458 \times 10^{-2} | 1.8458 \times 10^{-2} | 4.9075 \times 10^{-7} | 4.0016 |

- **Example 2.** Suppose $\Omega$ be the open interval $(0, 1)$ and $T$ be the final time, $T = 1$. We assume that the parameters $\lambda \in \{0.66, 0.9\}$ and $\alpha = 1 - \lambda$. We choose the function $g(t) = e^t$, $p(t) = 0$, $g(x, t) = 1 - \sin(2t)$ and $s(x, t) = [\pi^2t^2e^t + t^2(1 - \sin(2t)) + 2\Gamma(3-\lambda)^{-1}t^{2-\lambda}]\sin(\pi x)$ such that the analytical solution is given in [1] by $u(x, t) = t^2\sin(\pi x)$.

The initial and boundary conditions are determined from the exact solution $u$. 29
Tables 1-4

In this example we take $\lambda = 0.9$, $\alpha = 1 - \lambda = 0.1$ and $k = h^{\frac{1}{10}}$.

| $k$ | $\|u\|_{L^2}$ | $\|U\|_{L^2}$ | $\|\mathcal{E}(h)\|_{L^2}$ | RC |
|-----|---------------|---------------|-----------------|----|
| $h^{-1}$ | $9.2363 \times 10^{-1}$ | $9.2361 \times 10^{-1}$ | $4.4483 \times 10^{-3}$ | -- |
| $h^{-2}$ | $3.7291 \times 10^{-1}$ | $3.7292 \times 10^{-1}$ | $3.2732 \times 10^{-4}$ | 3.7645 |
| $h^{-3}$ | $1.1238 \times 10^{-1}$ | $1.1237 \times 10^{-1}$ | $2.3565 \times 10^{-5}$ | 3.7960 |
| $h^{-4}$ | $2.9527 \times 10^{-2}$ | $2.9527 \times 10^{-2}$ | $1.6832 \times 10^{-7}$ | 3.8074 |
| $h^{-5}$ | $7.5431 \times 10^{-3}$ | $7.5431 \times 10^{-3}$ | $1.1829 \times 10^{-9}$ | 3.8308 |

Tables 1-4

In this example we take $\lambda = 0.66$, $\alpha = 1 - \lambda = 0.34$ and $k = h^{\frac{1}{10}}$.

| $k$ | $\|u\|_{L^2}$ | $\|U\|_{L^2}$ | $\|\mathcal{E}(h)\|_{L^2}$ | RC |
|-----|---------------|---------------|-----------------|----|
| $h^{-1}$ | $8.7304 \times 10^{-1}$ | $8.7392 \times 10^{-1}$ | $1.0445 \times 10^{-2}$ | -- |
| $h^{-2}$ | $3.7291 \times 10^{-1}$ | $3.7292 \times 10^{-1}$ | $7.4499 \times 10^{-4}$ | 3.8094 |
| $h^{-3}$ | $1.1234 \times 10^{-1}$ | $1.1234 \times 10^{-1}$ | $5.2836 \times 10^{-5}$ | 3.8176 |
| $h^{-4}$ | $2.9519 \times 10^{-2}$ | $2.9519 \times 10^{-2}$ | $3.7499 \times 10^{-5}$ | 3.8166 |
| $h^{-5}$ | $7.5327 \times 10^{-3}$ | $7.5327 \times 10^{-3}$ | $2.6315 \times 10^{-7}$ | 3.8329 |

We observe from this table that the proposed method is temporal second order convergent and spatial fourth order accurate.

5 General conclusions and future works

In this paper we proposed a two-level fourth-order approach for solving the time-fractional convection-diffusion-reaction equation with variable coefficients and source terms. Both stability analysis and error estimates of the numerical scheme have been deeply analyzed. The theory has suggested that the proposed method is unconditionally stable, convergence with order $O(k^{2-\frac{1}{2}})$ in time and fourth accurate in space (see Theorem 3.1). The numerical tests are performed for values of the parameter $\lambda$ lying in the interval $(0, 1)$ and they confirmed the theoretical result provided in Section 3 (see Theorem 3.1 Tables 1-4 and Figures 11). Especially, the graphs (Figures 14) show that the new method is both unconditionally stable and convergent whereas Tables 1-4 indicate the convergence rate (accurate of order $O(k^{2-\frac{1}{2}})$ in time and fourth-order convergent in space) of the algorithm. Our future works will consider the numerical solution of the two-dimensional time-fractional convection-diffusion-reaction equation with variable coefficients using the new approach. Furthermore, the development of more efficient algorithms to reducing computational costs will be also subjected of other works.

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Stability and convergence rate of a two-level fourth-order approach for time-fractional advection-diffusion
with $\lambda = 0.9$ and $k = h^{\frac{3}{2}}$.

Figure 1: Exact solution (u: in green), Numerical solution (U: in blue) and Error (E: in red) for Problem 1
Analysis of stability and convergence rate of a two-level fourth-order method for time-fractional advection-diffusion with $\lambda = 0.66$ and $k = h^{\frac{4}{7}}$.

Figure 2: Exact solution (u: in green), Numerical solution (U: in blue) and Error (E: in red) for Problem 1
Stability and convergence rate of a two-level fourth-order numerical scheme for time-fractional advection-diffusion with $\lambda = 0.9$ and $k = h^{\frac{4}{5}}$.

Figure 3: Exact solution (u: in green), Numerical solution (U: in blue) and Error (E: in red) for Problem 2
Analysis of stability and convergence rate of a two-level fourth-order approach for time-fractional advection-diffusion with $\lambda = 0.66$ and $k = h^{2 - \frac{1}{2}}$.

Figure 4: Exact solution (u: in green), Numerical solution (U: in blue) and Error (E: in red) for Problem 2