LOSSY ASYMPTOTIC EQUIPARTITION PROPERTY FOR NETWORKED DATA STRUCTURES

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Abstract. In this article we prove a Generalized Asymptotic Equipartition Property for Networked Data Structures modelled as coloured random graphs. The main techniques in this article remains large deviation principles for suitably defined empirical measures on coloured random graphs. We apply our main result to a concrete example from the field of Biology.

1. Introduction

Suppose we have a networked data structure $x = \{(x(u), x(v)) : uv \in e\}$ generated by a memoryless source $G$ with distribution $P(x)$ to be compressed with distortion no greater than $d \geq 0$, using a memoryless random codebook $\hat{G}$ with distribution $P(y)$. Then the compression performance can be determined by the “generalized asymptotic equipartition property” (AEP), which states that the probability of finding a $d-$ close match between $x$ and any given networked data structure (codeword) $y = \{(y(u), y(v)) : uv \in e\}$, is approximately $2^{-nR(P(x), P(y), d)}$. The rate function $R(P(x), P(y), d)$ can be expressed as an infimum of relative entropies. The main aim of this article is to extend the results that have appeared in the recent literature as [DA16] and the reference therein.

To be specific, in this article we develop a Lossy AEP for networked data structures modelled as coloured random graphs. We prove process large deviation principle (LDP) for the coloured random graph conditioned to have a given empirical colour measure and empirical pair measure, see Doku-Amponsah [DA06], using similar coupling arguments as in the article by Boucheron et. al [BGL02]. From this LDP and the techniques employed by Dembo and Kontoyiannis [DK02] for the random field on $\mathbb{Z}^2$, we obtain the proof of the Lossy AEP for the Networked Data Structures.

We apply our Lossy AEP to the following concrete examples from biology: Metabolic network; This is a graph of interactions forming a part of the energy generation and biosynthesis metabolism of the bacterium E.coli. Here, the units represent substrates and products, and links represent interactions. See Newman [13].

The article is organized as follows. Generalized AEP for Coloured Random Graph Model section contain the main result of the paper, Theorem 2.1. LDP for two-dimensional Coloured Random Graph Model section gives process level LDP’s, Theorem 3.1 and 3.2, which form the bases of the proof of the main result of the article. Proof of Theorem 2.1 3.1 and 3.2 section provides the proofs of all Process Level LDP’s for the paper and hence the main result of the article.

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2. Generalized AEP for Coloured Random Graph Process

2.1 Main Result

Consider two Coloured Random Graph processes \( X = \{(X(u), X(v)) : uv \in E\} \) and \( Y = \{(Y(u), Y(v)) : uv \in E\} \) which take values in \( \mathcal{G} = \mathcal{G}(\mathcal{X}) \) and \( \hat{\mathcal{G}} = \hat{\mathcal{G}}(\mathcal{X}) \), resp., the spaces of finite graphs on \( \mathcal{X} \). We equip \( \mathcal{G}(\mathcal{X}) \), \( \hat{\mathcal{G}}(\mathcal{X}) \) with their Borel fields \( \mathcal{F}(x) \) and \( \hat{\mathcal{F}}(x) \). Let \( \mathbb{P}^{(x)} \) and \( \mathbb{P}^{(y)} \) denote the probability measures of the entire processes \( X \) and \( Y \). For \( (\sigma, \pi) \) and \( (\sigma', \pi') \) we denote a finite alphabet and denote by \( \mathcal{N}(\mathcal{X}) \) the space of counting measure on \( \mathcal{X} \) equipped with the weak topology. By \( \mathcal{M}(\mathcal{X}) \) we denote the space of probability measures on \( \mathcal{X} \) equipped with the weak topology and \( \mathcal{M}_s(\mathcal{X}) \) the space of finite measures on \( \mathcal{X} \) equipped with the weak topology.

Throughout the rest of the article we will assume that \( X \) and \( Y \) are Coloured Random Graph processes, See [Pe98]. For \( n \geq 1 \), let \( P_n \) denote the marginal distribution of \( X \) on \( V = \{1, 2, 3, \ldots, n\} \) taking with respect to \( \mathbb{P}^{(y)} \) and \( Q_n^{(y)} \) denote the marginal distribution \( Y \) on \( V = \{1, 2, 3, \ldots, n\} \) with respect to \( \mathbb{P}^{(y)} \).

Let \( \rho : \mathcal{X} \times \mathcal{N}(\mathcal{X}) \times \mathcal{X} \times \mathcal{N}(\mathcal{X}) \to [0, \infty) \) be an arbitrary non-negative function and define a sequence of single-letter distortion measures \( \rho^{(n)} : \mathcal{G} \times \hat{\mathcal{G}} \to [0, \infty) \), \( n \geq 1 \) by

\[
\rho^{(n)}(x, y) = \frac{1}{n} \sum_{v \in V} \rho(B_x(v), B_y(v)),
\]

where \( B_x(v) = (x(v), L_x(v)) \) and \( B_y(v) = (y(v), L_y(v)) \). Given \( d \geq 0 \) and \( x \in \mathcal{G} \), we denote the distortion-ball of radius \( d \) by

\[
B(x, d) = \{y \in \hat{\mathcal{G}} : \rho^{(n)}(x, y) \leq d\}.
\]

For \( (\sigma, \pi) \in \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \), we write

\[
K_{(\sigma, \pi)}(a, l) = \sigma(a) \prod_{b \in \mathcal{X}} \frac{e^{-\pi(a,b)/\sigma(a)[\pi(a,b)/\sigma(a)]^\ell(b)}}{\ell(b)!}, \text{ for } \ell \in \mathcal{N}(\mathcal{X})
\]

and define the rate function \( I_1 : \mathcal{M}((\mathcal{X} \times \mathcal{N}(\mathcal{X}))^2) \to [0, \infty] \) by

\[
I_1(\nu) = \left\{ \begin{array}{ll}
H(\nu \| K_{(\sigma, \pi)} \otimes K_{(\sigma, \pi)}), & \text{if } \nu \text{ is consistent and } \nu_{1,1} = \nu_{1,2} = \sigma, \\noalign{\vspace{1em}}
\infty, & \text{otherwise,}
\end{array} \right.
\]

where

\[
K_{(\sigma, \pi)} \otimes K_{(\sigma, \pi)}((a_x, a_y), (l_x, l_y)) = K_{(\sigma, \pi)}(a_x, l_x)K_{(\sigma, \pi)}(a_y, l_y).
\]

By \( x \overset{D}{\sim} p \) we mean \( x \) has distribution \( p \). For \( (\sigma, \pi) \in \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \), we write

\[
d_{av}(\sigma, \pi) = \langle \log(e^{\rho(B_x, B_y)}), K_{(\sigma, \pi)} \rangle,
\]

for \( (\sigma, \pi) \in \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \).
Throughout the proof we may assume that $n > 1$. Assume that

$$d^{(n)}_{\text{min}}(\sigma, \pi) = \mathbb{E}_{P^{(n)}} \left[ \text{essinf}_{Y \in \mathcal{G}} Q_{(Y)}^{(n)}(X, Y) \right] \to d_{\text{min}}(\sigma, \pi).$$

For $n > 1$, we write

$$R_n(P^{(x)}_n, Q^{(y)}_n, d) := \inf_{V_n} \left\{ \frac{1}{n} H(V_n \| P^{(x)}_n \times Q^{(y)}_n) : V_n \in \mathcal{M}(\mathcal{G} \times \hat{\mathcal{G}}) \right\}$$

and

$$d^\infty_{\text{min}}(\sigma, \pi) := \inf \left\{ d \geq 0 : \sup_{n \geq 1} R_n(P^{(x)}_n, Q^{(y)}_n, d) < \infty \right\}.$$

**Theorem 2.1.** Suppose $X$ and $Y$ are coloured random graph. Assume $\rho$ are bounded function. Then,

1. With $\mathbb{P}^{(x)}$ - probability 1, conditional on the event $\{ \Phi(\mathcal{L}_{n,1}) = \Phi(\mathcal{L}_{n,2}) = \sigma, \pi \}$ the random variables $\{\rho^{(n)}(x, Y)\}$ satisfy an LDP with deterministic, convex rate-function

$$I_\rho(z) := \inf_\nu \left\{ I_1(\nu) : \langle \rho, \nu \rangle = z \right\}. $$

2. For all $d \in (d_{\text{min}}(\sigma, \pi), d_{\text{av}}(\sigma, \pi)]$, except possibly at $d = d^\infty_{\text{min}}(\sigma, \pi)$,

$$\lim_{n \to \infty} -\frac{1}{n} \log Q^{(x)}_n(B(X, D)) = R(\mathbb{P}^{(x)}(\sigma, \pi), \mathbb{P}^{(y)}(\sigma, \pi), d)$$

almost surely,

$$\text{(2.2)}$$

where $R(p, q, D) = \inf_\nu H(\nu \| p \times q)$.

**2.2 Application [DA10]**

**Metabolic network.** We consider a metabolic network of the energy and biosynthesis metabolism of the bacterium E.coli modelled as coloured random graph on $n$ nodes partition into $n\sigma_n(\text{substrate})$ block of substrates and $n\pi_n(\text{product})$ block of products, and $n\|\pi_n\| \text{ number of interactions divided into } n\pi_n(\text{substrate, product}), n\pi_n(\text{substrate, product}), n\pi_n(\text{substrate, substrate})/2, n\pi_n(\text{product, product})/2 \text{ different interactions, respectively. Assume } \sigma_n \text{ converges } \sigma \text{ and } \pi_n \text{ converges } \pi.$ If we take $\rho(s, r) = (s - r)^2$ then, by Theorem 2.1 we have the distortion-rate

$$R(P, Q, D) = \begin{cases} 0, & \text{if } D \geq 2\pi(\text{subs, prod}) + \pi(\text{subs, subs}) + \pi(\text{prod, prod}) + 2\pi(\text{subs, prod}), \\ \infty, & \text{otherwise}. \end{cases} \quad \text{(2.3)}$$

where $\text{subs} = \text{substrate}$ and $\text{prod} = \text{product}$.

**3. LDP for two-dimensional Coloured Random Graph process**

For any $n \in \mathbb{N}$ we define

$$\mathcal{M}_n(\mathcal{X}) := \{ \sigma \in \mathcal{M}(\mathcal{X}) : n\sigma(a) \in \mathbb{N} \text{ for all } a \in \mathcal{X} \},$$

$$\tilde{\mathcal{M}}_n(\mathcal{X} \times \mathcal{X}) := \{ \pi \in \tilde{\mathcal{M}}_n(\mathcal{X} \times \mathcal{X}) : \frac{n}{1+1_{\{a=b\}}} \pi(a, b) \in \mathbb{N} \text{ for all } a, b \in \mathcal{X} \}.$$
Assign colours to the vertices by sampling without replacement from the collection of $n$ colours, which contains any colour $(a_x, a_y) \in \mathcal{X}$ exactly $n\omega_n(a_x, a_y)$ times;

for every unordered pair $\{a, b\}$ of colours create exactly $m_n(a, b)$ edges by sampling without replacement from the pool of possible edges connecting vertices of colour $a$ and $b$, where

$$m_n(a, b) := \begin{cases} n \pi_n(a, b) & \text{if } a = a_x, b = b_x \text{ and } a_x \neq b_x \\ n \pi_n(a, b) & \text{if } a = a_y, b = b_y \text{ and } a_y \neq b_y \\ \frac{n}{2} \pi_n(a, b) & \text{if } a = a_x, b = b_x \text{ and } a_x = b_x \\ \frac{n}{2} \pi_n(a, b) & \text{if } a = a_y, b = b_y \text{ and } a_y = b_y. \end{cases} \quad (3.1)$$

We define the process-level empirical measure $\mathcal{L}_n$ induced by $X$ and $Y$ on $\mathcal{G} \times \hat{\mathcal{G}}$ by

$$\mathcal{L}_n(\beta_x; \beta_y) = \frac{1}{n} \sum_{v \in V} \delta(\mathcal{B}_X(v), \mathcal{B}_Y(v))(\beta_x, \beta_y), \quad \text{for } (\beta_x, \beta_y) \in \mathcal{M}[(\mathcal{X} \times \mathcal{X}^*)^2].$$

Note that we have

$$\mathcal{L}_n \otimes \phi^{-1}((x(v), y(v)), \ell_{x,y}(v)) = \frac{1}{n} \sum_{v \in V} \delta(\mathcal{B}_X(v), \mathcal{B}_Y(v)) \left( \phi^{-1}(x(v), y(v)), \ell_{x,y}(v) \right)$$

$$= \frac{1}{n} \sum_{v \in V} \delta((X(v), Y(v)), L_{X,Y}(v)) \left( (x(v), y(v)), \ell_{x,y}(v) \right)$$

$$= \hat{\mathcal{L}}_n((x(v), y(v)), \ell_{x,y}(v)).$$

where $\phi(\beta_x, \beta_y) = ((x(v), y(v)), \ell_{x,y}(v)).$

The next Theorem which is the LDP for $\mathcal{L}_n$ of the process $X, Y$ is the main ingredient in the proof of the Lossy AEP.

**Theorem 3.1.** The sequence of empirical measures $\mathcal{L}_n$ satisfies a large deviation principle in the space of probability measures on $(\mathcal{X} \times \mathcal{N}(\mathcal{X}))^2$ equipped with the topology of weak convergence, with convex, good rate-function $I_1$.

The proof of Theorem 3.1 above is dependent on the LDP for $\tilde{\mathcal{L}}_n$ given below:

**Theorem 3.2.** The sequence of empirical measures $\tilde{\mathcal{L}}_n$ satisfies a large deviation principle in the space of probability measures on $\mathcal{X}^2 \times \mathcal{N}(\mathcal{X})^2$ equipped with the topology of weak convergence, with convex, good rate-function

$$I_2(\omega) = \begin{cases} H(\omega \parallel \mathcal{K}_{(\sigma, \pi)} \otimes \mathcal{K}_{(\sigma, \pi)}), & \text{if } \omega \text{ is consistent and } \omega_{1,1} = \omega_{1,2} = \sigma, \\ \infty, & \text{otherwise,} \end{cases} \quad (3.2)$$

where $\mathcal{K}_{(\sigma, \pi)} \otimes \mathcal{K}_{(\sigma, \pi)}((a_x, a_y), (l_x, l_y)) = \mathcal{K}_{(\sigma, \pi)}(a_x, l_x)\mathcal{K}_{(\sigma, \pi)}(a_y, l_y).$

We denote, for any bin $v \in \{1, \ldots, n\}$, by $(\tilde{X}(v), \tilde{Y}(v))$ its colours, and for $h = x, y$ by $l^h(b_h)$ the number of balls of colour $b_h \in \mathcal{X}$ it contains. Now define the empirical process-level occupancy measure of the constellation by

$$\tilde{\mathcal{L}}_n^+(a_x, a_y, \ell_{x,y}) = \frac{1}{n} \sum_{v \in V} \delta((\tilde{X}(v), \tilde{Y}(v)), \tilde{L}_{X,Y}(v))((a_x, a_y), \ell_{x,y}), \quad \text{for } (a_x, a_y, \ell_{x,y}) \in \mathcal{X}^2 \times \mathcal{N}^2(\mathcal{X}),$$

where $\tilde{L}_{X,Y}(v) = (l^x(b_x), l^y(b_y), (b_x, b_y) \in \mathcal{X} \times \mathcal{X})$ is the colour distribution in bin $v$. In our first theorem we establish exponential equivalence of the law of the empirical process-level measure $\tilde{\mathcal{L}}_n$ under $\mathbb{P}_{(\sigma_n, \omega_n)}$ the law of the coloured random graph conditioned to have colour law $\sigma_n$ and edge distribution...
\( \pi_n \), and the law of the empirical process-level occupancy measure \( \tilde{\mathcal{L}}_n^+ \) under the random allocation model \( \tilde{\mathbb{P}}(\pi_n, \pi_n) \). Recall the definition of exponential equivalence, see [DZ98, Definition 4.2.10].

**Lemma 3.3.** The law of \( \tilde{\mathcal{L}}_n^+ \) under \( \tilde{\mathbb{P}}(\pi_n, \pi_n) \) is exponentially equivalent to the law of \( \mathcal{L}_n \) under \( \mathbb{P}(\pi_n, \pi_n) \).

Define the metric \( d \) of total variation by
\[
d(\nu, \tilde{\nu}) = \frac{1}{2} \sum_{(a, y), (l_x, l_y) \in \mathcal{X} \times \mathcal{N}^2(\mathcal{X})} |\nu(\{a, y\}, (l_x, l_y)) - \tilde{\nu}(\{a, y\}, (l_x, l_y))|,
\]
for \( \nu, \tilde{\nu} \in \mathcal{M}(\mathcal{X} \times \mathcal{N}^2(\mathcal{X})) \).

As this metric generates the weak topology, the proof of Lemma 3.3 is equivalent to showing that for every \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\{d(\tilde{\mathcal{L}}_n^+, \mathcal{L}_n) \geq \varepsilon\} = -\infty,
\]
where \( \mathbb{P} \) indicates a suitable coupling between the random allocation model and the coloured graph.

To begin, denote by \( V(a) \) the collection of vertices (bins) which have colour \( a \in \mathcal{X} \) and observe that
\[
\#V(a) = n\omega_n(a).
\]

For \( h = x, y \) and every \( a_h, b_h \in \mathcal{X} \), begin: At each step \( k = 1, \ldots, m_n(a_h, b_h) \), we randomly pick two vertices \( V_1^k \in V(a_h) \) and \( V_2^k \in V(b_h) \). Drop one ball of colour \( b_h \) in bin \( V_1^k \) and one ball of colour \( a_h \) in \( V_2^k \), and link \( V_1^k \) to \( V_2^k \) by an edge unless \( V_1^k = V_2^k \) or the two vertices are already connected. If one of these two things happen, then we simply choose an edge randomly from the set of all possible edges connecting colours \( a_h \) and \( b_h \), which are not yet present in the graph. This completes the construction of a graph with \( \Phi(\tilde{\mathcal{L}}_n, 1) = \Phi(\tilde{\mathcal{L}}_n, 2) = (\omega_n, \pi_n) \) and
\[
d(\tilde{\mathcal{L}}_n^+, \mathcal{L}_n) \leq \frac{2}{n} \left( \sum_{a,b \in \mathcal{X}} B^n(a, b) + \sum_{a,b \in \mathcal{X}} B^n(a, b) \right),
\]
where \( B^n(a, b) \) is the total number of steps \( k \in \{1, \ldots, m_n(a, b)\} \) at which there is disparity between the vertices \( V_1^k, V_2^k \) drawn and the vertices which formed the \( k \)th edge connecting \( a \) and \( b \) in the random graph construction.

Given \( a, b \in \mathcal{X} \), the probability that \( V_1^k = V_2^k \) or the two vertices are already connected is equal to
\[
p[k|(a_h, b_h)] := \frac{1}{m_n(a_h, b_h)} \mathbb{I}_{\{a_h = b_h\}} + (1 - \frac{1}{m_n(a_h, b_h)} \mathbb{I}_{\{a_h = b_h\}} (1 - \frac{1}{m_n(a_h, b_h)}))^k.
\]

\( B^n(a_h, b_h) \) is a sum of independent Bernoulli random variables \( X^{(h)}_1, \ldots, X^{(h)}_{n\omega_n(a_h, b_h)/2} \) with ‘success’ probabilities equal to \( p[k|(a_h, b_h)] \). Note that \( \mathbb{E}[X_k] = p[k|(a_h, b_h)] \) and
\[
\text{Var}[X^{(h)}_k] = p[k|(a_h, b_h)](1 - p[k|(a_h, b_h)]).
\]

Now, we have
\[
\mathbb{E}B^n(a_h, b_h) = \sum_{k=1}^{n(a_h, b_h)} p[k|(a_h, b_h)] = \mathbb{I}_{\{a_h = b_h\}} + (1 - \mathbb{I}_{\{a_h = b_h\}} \frac{1}{m_n(a_h, b_h)} (1 - \frac{1}{m_n(a_h, b_h)})) \leq 1 + \mathbb{I}_{\{a_h = b_h\}}.
\]

We write
\[
\sigma^2_n(a_h, b_h) := \frac{1}{m_n(a_h, b_h)} \sum_{k=1}^{n(a_h, b_h)} \text{Var}[X^{(h)}_k]
\]
and observe that
\[
\lim_{n \to \infty} \mathbb{E}(B^n(a_h, b_h)) = \lim_{n \to \infty} \text{Var}(B^n(a_h, b_h)) = \lim_{n \to \infty} m_n(a_h, b_h) \sigma^2_n(a, b) = \mathbb{I}_{\{a_h = b_h\}} + 1.
\]
We Define $e(t) = (1 + t) \log(1 + t) - t$, for $t \geq 0$ and use Bennett’s inequality, see [Be62], to obtain, for sufficiently large $n$

$$
P\left\{ \frac{1}{n} \sum_{h=x,y} B^n(a_h, b_h) \geq \frac{\sum_{h=x,y} \mathbb{I}(a_h=b_h)+1}{n} + \delta_1 \right\} \leq \exp \left[ - \sum_{h=x,y} m_n(a_h, b_h) \sigma_n^2(a_h, b_h) e\left( \frac{n \delta_1}{\sum_{h=x,y} m_n(a_h, b_h) \sigma_n^2(a_h, b_h)} \right) \right],$$

for any $\delta_1 > 0$. Let $\varepsilon \geq 0$ and choose $\delta_1 = \frac{\varepsilon}{2m}$. Suppose that we have $B^n(a_h, b_h) \leq \delta_1$, for $h = x, y$. Then, by (3.4),

$$d(\tilde{L}, \nu_n) \leq 2\delta_1 m^2 = \varepsilon.$$ 

Hence,

$$
P \{d(\tilde{L}, \tilde{L}^+) > \varepsilon\} \leq \max_{h=x,y} \sum_{a_h, b_h \in \mathcal{X}} P \{B^n(a_h, b_h) \geq n\delta_1\}
$$

$$\leq m^2 \max_{h=x,y} \sup_{a_h, b_h \in \mathcal{X}} P \{B^n(a_h, b_h) \geq \mathbb{I}(a_h=b_h) + 1 + (n\delta_1)/2\}
$$

$$\leq m^2 \max_{h=x,y} \sup_{a_h, b_h \in \mathcal{X}} \exp \left[ - m_n(a_h, b_h) \sigma_n^2(a_h, b_h) e\left( \frac{n \delta_1}{m_n(a_h, b_h) \sigma_n^2(a_h, b_h)} \right) \right].$$

Let $0 \leq \delta_2 \leq 1$. The, for sufficiently large $n$ we have

$$
\frac{1}{n} \log P \{d(\tilde{L}, \tilde{L}^+) > \varepsilon\} \leq -(1-\delta_2) e\left( \frac{n \delta_1}{2(1+\delta_2)} \right)
$$

$$= -(\mathbb{I}(a-b) + 1 - \delta_2) \left[ \left( \frac{1}{2} + \frac{\delta_1}{2(\mathbb{I}(a-b)+1+\delta_2)} \right) \log(1 + \frac{n \delta_1}{2(\mathbb{I}(a-b)+1+\delta_2)}) - \frac{\delta_1}{2(\mathbb{I}(a-b)+1+\delta_2)} \right].$$

This completes the proof of the lemma.

4. PROOF OF THEOREM 2.1, 3.1 AND 3.2

4.1 Proof of Theorem 3.2. We write $\hat{\vartheta}_2^{(n)} := \vartheta_2^{(n)}(\varpi_n, \nu_n)$, $\vartheta_1^{(n)} := \vartheta_1^{(n)}(\varpi_n, \nu_n)$ and state the following Lemma. Denote by $\Sigma^{(n)}(\sigma_n, \pi_n)$ the space of all empirical neighbourhood measures with empirical colour measure $\sigma_n$ and empirical pair measure $\pi_n$.

Lemma 4.1 (Doku-Amponsah, 2014). For any process level empirical measure, $\nu_n$ with $\nu_{n,1}, \nu_{n,2} \in \Sigma^{(n)}(\sigma_n, \pi_n),$

$$e^{-n(H(\nu_{n,1} \parallel K(\sigma_n, \pi_n)) + H(\nu_{n,2} \parallel K(\sigma_n, \pi_n))) + \hat{\vartheta}_2^{(n)}} \leq \hat{P}(\sigma_n, \pi_n)(\tilde{L}_n^+ = \nu_n)
$$

$$\leq |\Sigma^{(n)}(\sigma_n, \pi_n)|^{-2} e^{-n(H(\nu_{n,1} \parallel K(\sigma_n, \pi_n)) + H(\nu_{n,2} \parallel K(\sigma_n, \pi_n))) + \vartheta_1^{(n)}},$$

where $K(\sigma_n, \pi_n)(a_h, l_h) = \sigma_n(a_h)K_{\pi_n}(l_h \mid a_h)$ and

$$K_{\pi_n}(l_h \mid a_h) = \prod_{b_h \in \mathcal{X}} e^{-\pi_n(a_h, b_h)/\sigma_n(a_h)} \frac{\pi_n(a_h, b_h)/\sigma_n(a_h)}{\ell(b_h)!},$$

for $\ell_h \in \mathcal{N}(\mathcal{X})$ and $h = x, y.$

$$\lim_{n \to \infty} \hat{\vartheta}_2^{(n)} = \lim_{n \to \infty} \vartheta_1^{(n)} = 0.$$
Proof. Note, by construction For any process level empirical measure, \( \nu_n \) with \( \nu_{n,1}, \nu_{n,2} \in \Sigma(\sigma_n, \pi_n) \), we have

\[
\hat{\mathbb{P}}_{(\sigma_n, \pi_n)}(\bar{\mathcal{L}}_n) = \nu_n = \hat{\mathbb{P}} \{ \mathcal{L}_n = \nu_n \mid \Phi(\bar{\mathcal{L}}_{n,1}) = \Phi(\bar{\mathcal{L}}_{n,2}) = (\sigma_n, \pi_n) \} = \prod_{h=x,y} \prod_{a_h \in \mathcal{X}} \left( n\nu_{n,\alpha(h)}(a_h, \ell_h), \ell_h \in \mathcal{N}(\mathcal{X}) \right) \prod_{a_h, b_h \in \mathcal{X}} \left( n\nu_n(a_h, b_h), j = 1, \ldots, n\nu_n(a_h) \right) \left( \frac{1}{n\nu_n(a_h)} \right) n\nu_n(a_h, b_h),
\]

while \( \hat{\mathbb{P}}_{(\sigma_n, \pi_n)}(\mathcal{L}_n) = 0 \) when \( \Phi(\bar{\mathcal{L}}_{n,1}) \neq (\sigma_n, \pi_n) \) or \( \Phi(\bar{\mathcal{L}}_{n,2}) \neq (\sigma_n, \pi_n) \) by convention. Therefore, by similar combinatoric computations as in the proof of [DAL14 Lemma 0.6] and the Sterling’s formula see, [Fe67] we have 4.1.

The proof of Theorem 3.2 follows from Lemma 4.1 and similar arguments as [DAL14 Page 13].

4.2 Proof of Theorem 3.1. Let \( \Gamma \in \mathcal{M}[(\mathcal{X} \times \mathcal{N}(\mathcal{X}))^2] \) and write \( \Gamma_\phi = \{ \omega \otimes \phi^{-1} : \omega \in \Gamma \} \). Note that if \( A \) is closed (open) then \( \Gamma_\phi \) is closed (open) since \( \phi \) is linear. Now suppose \( F \) is closed subset of \( \mathcal{M}[(\mathcal{X} \times \mathcal{N}(\mathcal{X}))^2] \) then by Theorem 3.2 we have

\[
- \inf_{\omega \in F} I_2(\omega \otimes \phi^{-1}) = - \inf_{\nu \in F_\phi} I_2(\nu) \leq \liminf_{n \to \infty} \frac{1}{n} \log \hat{\mathbb{P}} \{ \mathcal{L}_n \in F_\phi \} \leq \limsup_{n \to \infty} \frac{1}{n} \log \hat{\mathbb{P}} \{ \mathcal{L}_n \in F_\phi \} = - \inf_{\nu \in F} I_2(\nu) = - \inf_{\omega \in F} I_2(\omega \otimes \phi^{-1}).
\]

We obtain the form of the rate function in Theorem 3.1 if we solve the optimization problem

\[
\inf \left\{ I_2(\nu) : \omega \otimes \phi^{-1} = \nu \right\} = I_1(\omega).
\]

4.3 Proof of Theorem 2.1

We write \( \mathcal{M} := \mathcal{M}[(\mathcal{X} \times \mathcal{N}(\mathcal{X}))^2] \) and define the set \( \mathcal{C}^\varepsilon \) by

\[
\mathcal{C}^\varepsilon(\sigma, \pi) = \left\{ \nu \in \mathcal{M} : \sup_{\beta_x, \beta_y \in \mathcal{X} \times \mathcal{N}(\mathcal{X})} |\nu(\beta_x, \beta_y) - K(\sigma, \pi) \otimes K(\sigma, \pi)(\beta_x, \beta_y)| \geq \varepsilon \right\}.
\]

Lemma 4.2. Suppose the sequence of measures \((\sigma_n, \pi_n)\) converges to the pair of measures \((\sigma, \pi)\). For any \( \varepsilon > 0 \) we have \( \lim_{n \to \infty} \mathbb{P}_{(\sigma_n, \pi_n)}(\mathcal{C}^\varepsilon) = 0 \).

Proof. Observe that \( \mathcal{C}^\varepsilon \) defined above is a closed subset of \( \mathcal{M} \) and so by Theorem 3.1 we have that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{(\sigma_n, \pi_n)}(\mathcal{C}^\varepsilon) \leq - \inf_{\nu \in \mathcal{C}^\varepsilon} I_1(\nu).
\]

We use proof by contradiction to show that the right hand side of 3.1 is negative. Suppose that there exists sequence \( \nu_n \) in \( \mathcal{C}^\varepsilon \) such that \( I_1(\nu_n) \downarrow 0 \). Then, there is a limit point \( \nu \in F_1 \) with \( I(\nu) = 0 \). Note \( I \) is a good rate function and its level sets are compact, and the mapping \( \nu \mapsto I(\nu) \) lower semi-continuity. Now \( I_1(\nu) = 0 \) implies \( \nu(\beta_x, \beta_y) = K(\sigma, \pi) \otimes K(\sigma, \pi)(\beta_x, \beta_y) \), for all \( \beta_x, \beta_y \in \mathcal{X} \times \mathcal{N}(\mathcal{X}) \) which contradicts \( \nu \in \mathcal{C}^\varepsilon \).
(i) Notice \( \rho^{(n)}(X, Y) = \langle \rho, \mathcal{L}_n \rangle \) and if \( \Gamma \) is open (closed) subset of \( \mathcal{M} \) then
\[
\Gamma_{\rho} := \{ \nu : \langle \rho, \nu \rangle \in \Gamma \}
\]
is also open (closed) set since \( \rho \) is bounded function.

\[
- \inf_{z \in \text{Int}(\Gamma)} I_{\rho}(z) = - \inf_{\nu \in \text{Int}(\Gamma_{\rho})} I_1(\nu)
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left\{ \rho^{(n)}(X, Y) \in \Gamma | X = x, \Phi(\mathcal{L}_{n,1}) = \Phi(\mathcal{L}_{n,2}) = (\sigma_n, \pi_n) \right\}
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left\{ \rho^{(n)}(X, Y) \in \Gamma | X = x, \Phi(\mathcal{L}_{n,1}) = \Phi(\mathcal{L}_{n,2}) = (\sigma_n, \pi_n) \right\}
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left\{ \rho^{(n)}(X, Y) \in \Gamma | X = x, \Phi(\mathcal{L}_{n,1}) = \Phi(\mathcal{L}_{n,2}) = (\sigma_n, \pi_n) \right\}
\]
\[
\leq - \inf_{\nu \in \text{cl}(\Gamma_{\rho})} I_1(\nu) = - \inf_{z \in \text{cl}(\Gamma)} I_{\rho}(z).
\]

(ii) Observe that \( \rho \) are bounded, therefore by Varadhan’s Lemma and convex duality, we have
\[
R(\mathbb{P}^x, \mathbb{P}^y, d) = \sup_{t \in \mathbb{R}} [td - \Lambda_\infty(t)] = \Lambda^*_\infty(d)
\]
where
\[
\Lambda^*_\infty(t) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}Q_n \left( e^{nt\langle \rho, \mathcal{L}_n \rangle} dQ_n(y) \right)
\]
extists for \( \mathbb{P} \) almost everywhere \( x \). Using bounded convergence, we can show that
\[
\Lambda_\infty(t) = \lim_{n \to \infty} \Lambda_n(t) := \lim_{n \to \infty} \frac{1}{n} \int \left[ \log \int e^{nt\langle \rho, \mathcal{L}_n \rangle} dQ_n(y) \right] dP_n(x).
\]
Using Lemma 1.41 by boundedness of \( \rho \) we have that
\[
\frac{1}{n} \Lambda(nt) = \frac{1}{n} \sum_{j=1}^{n} \log \mathbb{E}Q_n \left( e^{t\rho(B_{s,j}, \mathcal{B}_{y}(j))} \right) \to (\log \langle e^{t\rho(B_X, \mathcal{B}_Y)} \rangle, \mathcal{K}(\sigma, \pi), \mathcal{K}(\sigma, \pi)) = d_{av}(\sigma, \pi).
\]
Also let
\[
D_{\text{min}}^{(n)} := \lim_{t \downarrow -\infty} \frac{\Lambda_n(t)}{t}
\]
so that \( \Lambda^*_n(d) = \infty \) for \( d < d_{\text{min}}^{(n)} \), while \( \Lambda^*_n(D) < \infty \) for \( d > d_{\text{min}}^{(n)} \). Observe that for \( n < \infty \) we have
\[
D_{\text{min}}^{(n)}(d) = \mathbb{E}_{P_n} \left[ \text{essinf}_{Y} \mathbb{P}Q_n \rho^{(n)}(X, Y) \right],
\]
which converges to \( d_{\text{min}} \). Using similar arguments as [DK02 Proposition 2] we obtain
\[
R_n(P_n, Q_n, d) = \sup_{t \in \mathbb{R}} \left( td - \Lambda_n(t) \right) := \Lambda^*_n(d)
\]
Now we observe from [DK02 Page 41] that the converge of \( \Lambda^*_n(\cdot) \to \Lambda_\infty(\cdot) \) is uniform on compact subsets of \( \mathbb{R} \). Moreover, \( \Lambda_n \) convex, continuous functions converge informally to \( \Lambda_\infty \) and hence we can invoke [See48 Theorem 5] to obtain
\[
\Lambda^*_n(d) = \lim_{\delta \to 0} \limsup_{n \to \infty} \inf_{|d - d| < \delta} \Lambda^*_n(\hat{d}).
\]
Using similar arguments as [DK02 Page 41] in the lines after equation (64) we have \( (2.3) \) which completes the proof.

**Conflict of Interest**
The author declares that he has no conflict of interest.

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REFERENCE

[BC13] C. Bordenave and P. Caputo. Large deviations of empirical neighborhood distribution in sparse random graphs. [arXiv:1308.5725] (2013).

[Be62] G. Bennett. Probability Inequalities for the Sum of Independent Random Variables Journal of the American Statistical Association 57 (297): 3345. doi:10.2307/2282438 (1962)

[BGL02] S. Boucheron, F. Gamboa and C. Leonard. Bins and balls: Large deviations of the empirical occupancy process. Ann. Appl. Probab. 12 607-636 (2002).

[CT91] T.M. Cover and J.A. Thomas. Elements of Information Theory. Wiley Series in Telecommunications, (1991).

[DA06] K. Doku-Amponsah. Large deviations and basic information theory for hierarchical and networked data structures. PhD Thesis, Bath (2006).

[DA10] K. Doku-Amponsah. Asymptotic equipartition properties for hierarchical and networked structures. ESAIM: PS 16 (2012): 114-138.DOI: 10.1051/ps/2010016.

[DA14] K. Doku-Amponsah. Exponential Approximation, Method of types for Empirical Neighbourhood Measures of Random graphs by Random Allocation. Int. Journal of Statistics and Probability,Vol 3, No.2,110-120 (2014).

[DA16] K. Doku-Amponsah. Large deviation Results for Critical Multitype Galton-Watson trees. https://arxiv.org/pdf/1009.3096.pdf

[DK02] A. Dembo and I. Kontoyiannis. Source Coding, Large deviations and Approximate Pattern. Invited paper in IEEE Transaction on information Theory, 48(6):1590-1615, June (2002).

[DMS03] A. Dembo, P. Mörters and S. Sheffield. Large deviations of Markov chains indexed by random trees. Ann. Inst. Henri Poincaré: Probab.et Stat.41, (2005) 971-996.

[DZ98] A. Dembo and O. Zeitouni. Large deviations techniques and applications. Springer, New York, (1998).

[Fe67] W. Feller. An introduction to probability theory and its applications. Vol. I, Wiley, New York. Third edition, (1967).

[13] M. E. Newman. Random graphs as models of networks. http://arxiv.org/abs/cond-mat/0202208

[Pe98] D.B. Penman. Random graphs with correlation structure. PhD Thesis, Sheffield 1998.

[Sce48] C.E. Shannon.(1948) A Mathematical Theory of Communication. Bell System Tech. J., 27:379-423,623-656.