

LOCAL DIMENSION-FREE ESTIMATES FOR VOLUMES OF SUBLEVEL SETS OF ANALYTIC FUNCTIONS

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§1. The result

In what follows, we denote complex balls \( \{ z \in \mathbb{C}^n : |z - w| < r \} \) by \( B_c(w, r) \) and real balls \( \{ x \in \mathbb{R}^n : |x - u| < r \} \) by \( B(u, r) \). For any real ball \( B \), we denote by \( \text{Vol}_B \) the normalized volume \( \text{Vol}_B(E) = \frac{\text{Vol}(B \cap E)}{\text{Vol}(B)} \).

We shall prove

**Theorem**

Let \( F \) be a non-constant analytic function in \( B_c(0, 1) \subset \mathbb{C}^n \) such that \( \sup_{B_c(0, 1)} |F| \leq 1 \). Let \( \varepsilon \leq \frac{1}{4} \), let \( B \) be any real ball contained in \( B(0, 1 - \varepsilon) \), and let \( M_B(F) \) be a (unique) positive number such that

\[
\text{Vol}_B\{|F| \geq M_B(F)\} = \frac{1}{e}.
\]

Then, for every \( \lambda > 1 \),

\[
\text{Vol}_B\{|F| \leq (C\lambda)^{-\sigma} M_B(F)\} \leq \frac{1}{\lambda},
\]

and

\[
\text{Vol}_B\{|F| \geq (C\lambda)^{\sigma} M_B(F)\} \leq e^{-\lambda},
\]

where one can take \( C = 8 \) and \( \sigma = 48\varepsilon^{-3} \log \frac{1}{|F(0)|} \).

The main feature of the result is its dimensionless character. Dimension-dependent versions of the theorem were obtained by N. Garofalo and P. Garrett (see [GG]), and A. Brudnyi (see [Br1], [Br2]).
If needed, the reader can adjust the theorem to plurisubharmonic functions in the unit ball of $\mathbb{C}^n$ and to analytic functions with values in a Banach space. Without changing the proof, one can replace real balls $B$ by arbitrary convex bodies $V \subset B(0, 1 - \varepsilon)$ whose boundaries have sectional curvatures bounded from below by some fixed positive constant. This “curvature” restriction can, probably, be relaxed but cannot be removed completely: a simple example given in the end of this note shows that estimates (1.1) and (1.2) may fail for thin rectangles in $B(0, 1 - \varepsilon) \subset \mathbb{R}^2$.

Compiling the theorem with the technique from [NSV, §3], one can obtain an Offord-type statement about the distribution of zeroes of analytic functions in families that depend analytically on some parameters. Informally speaking, the result is that the portion of the family occupied by the functions whose distribution of zeroes deviates from the “average” one by some fixed amount, is about $\text{Const} \exp\{-\text{size of the deviation}\}$. This might be a possible embryo of a nonlinear and dimensionless value-distribution theory.

The theorem appeared as an attempt to “generalize” the similar statement for polynomials $P$ in $\mathbb{R}^n$. The main difference is that, for polynomials, the counterparts of (1.1) and (1.2) hold with $\sigma = \deg P$ and $C = 4$ in any convex body $V \subset \mathbb{R}^n$ (see [NSV]). The quantity $\log \frac{1}{|P(0)|}$ appears as a “natural analogue” of the degree of a polynomial just as it does in the classical Cartan lemma.

As to the history of “dimension-free estimates”, the pioneering dimensionless results are due to A. C. Offord [O], M. Gromov and V. Milman [GM1] (the case of linear functions), and J. Bourgain [B] (a somewhat cruder form of (1.2) for polynomials). For other developments, see A. Brudnyi [Theorem 1.11, Br2]), S. Bobkov [Bo], and Carbery and Wright [CW].

The proof of the theorem will be cooked from three ingredients.

A. The geometric Kannan-Lovász-Simonovits lemma:

A continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}_+$ is called logarithmically concave if

$$\Phi \left( \frac{x + y}{2} \right) \geq \sqrt{\Phi(x) \Phi(y)}$$

for all $x, y \in \mathbb{R}^n$.

**Lemma A**

Let $\Phi$ be a logarithmically concave function in $\mathbb{R}^n$. Let $S \subset \text{supp}(\Phi)$ be a convex compact, and let $E \subset S$ be a closed subset. For $\lambda > 1$, define

$$E_{\lambda, S} := \left\{ x \in E : \frac{|E \cap J|}{|J|} \geq \frac{\lambda - 1}{\lambda} \text{ for every interval } J \text{ such that } x \in J \subset S, \right\}.$$
Then

\[ \frac{\int_{E \times S} \Phi}{\int_S \Phi} \leq \left( \frac{\int_E \Phi}{\int_S \Phi} \right)^\lambda. \]

This lemma was proved in [NSV] using the needle decomposition technique developed by M. Gromov and V. Milman [GM2] and by L. Lovász and M. Simonovits [LS]. It can also be derived from a result of R. Kannan, L. Lovász, and M. Simonovits [Theorem 2.7, KLS].

B. One dimensional Remez property:

We shall use the following result (which, probably, should be called the Boutroux-Cartan-Remez property):

**Lemma B**

Let \( f \) be an analytic function in the unit disk \( \mathbb{D} \) such that \( \sup_{\mathbb{D}}|f| \leq 1 \), and let \( |f(a)| = |f(-a)| > 0 \) for some \( a \in (0,1) \). Then \( f \) has the Remez property on the interval \([-a,a]\), i.e., for every sub-interval \( I \subset [-a,a] \) and every set \( E \subset I \),

\[ \max_I |f| \leq \left( \frac{C|I|}{|E|} \right)^\sigma \sup_E |f| \]

with \( C = 8 \) and \( \sigma = \frac{3}{1-a} \log \frac{1}{|f(a)|} \).

For the sake of completeness, we provide the proof of Lemma B in §2.

C. A change of variable:

Let \( \delta \leq \frac{1}{8} \). Set \( A = 1 - \delta^3 \), \( a = \sqrt{A} \), \( \varphi(\zeta) = \frac{A-\zeta}{1-A\zeta} \), and consider the mapping \( T \) defined on the unit ball \( B_c(0,1) \subset \mathbb{C}^n \) by the formula

\[ T(z) := \varphi\left( \sum_{j=1}^n z_j^2 \right) z, \quad z = (z_1, \ldots, z_n) \in B_c(0,1) \subset \mathbb{C}^n . \]

In particular, \( T(x) = \varphi(|x|^2)x \) for \( x \in B(0,1) \).

**Lemma C**

Set \( R_0 = 1 - 3\delta - \delta^3 \), \( r_0 = \sqrt{R_0} \). Then the mapping \( T \) has the following properties:

1. \( TB_c(0,1) \subset B_c(0,1) \);
2. \( T \) maps the real sphere \(|x|=a\) to the origin;
3. \( T \) is one-to-one in the ball \( B(0,r_0) \);
4. \( TB(0,r_0) \) is a ball centered at the origin of radius greater than \( 1 - 2\delta \);
5. The Jacobian \( |\det D_xT| \) is a logarithmically concave function in \( B(0,r_0) \);
6. The (partial) pre-image \( B(0,r_0) \cap T^{-1}B \) of every (real) ball \( B \subset TB(0,r_0) \) is convex.
The first two properties are obvious; the others will be proved in §3.

**Proof of Theorem:**

Let $F$ be a non-constant analytic function in $B_c(0,1)$. We shall show that for every $c > 0$ and every $\lambda > 1$,

$$\text{Vol}_{B_c}\{|F| \geq (C\lambda)^\sigma c\} \leq \left(\text{Vol}_{B_c}\{|F| \geq c\}\right)^\lambda.$$  \hfill (1.3)

The rest is the same as in [NSV]: to get (1.2), we just set $c = M_B(F)$ in (1.3); to get (1.1), we rewrite (1.3) in the form

$$\text{Vol}_{B_c}\{|F| \geq c\} \leq \left(1 - \text{Vol}_{B_c}\{|F| < (C\lambda)^{-\sigma} c\}\right)^\lambda$$

and, taking $c = M_B(F)$, obtain

$$\text{Vol}_{B_c}\{|F| < (C\lambda)^{-\sigma} M_B(F)\} \leq 1 - e^{-1/\lambda} < \frac{1}{\lambda},$$

which is identical to (1.1) since Vol$^c_B\{|F| = \text{const}\} = 0$.

To prove (1.3), choose $\delta = \frac{\lambda}{2}$ and consider the composition $F_T(z) = (F \circ T)(z)$ of the function $F$ with the mapping $T$ defined above. The function $F_T$ is analytic in the complex unit ball and $\sup_{B_c(0,1)}|F_T| \leq 1$. The advantage we gain by this trick is that the new function $F_T$ has a lower bound on a massive set (the real sphere) instead of just one point (the origin): $F_T(u) = F(0)$ for every $u \in \mathbb{R}^n$ with $|u| = a$. Let $S = B(0,r_0) \cap T^{-1}B$. Due to Lemma C (property (6)), this is a convex compact subset of $B(0,r_0)$. We shall show that for every $c > 0$ and for every $\lambda > 1$,

$$\frac{\int_S\{|F_T| \geq (C\lambda)^\sigma c\} \det D_x T}{\int_S \det D_x T} \leq \left(\frac{\int_S\{|F_T| \geq c\} \det D_x T}{\int_S \det D_x T}\right)^\lambda$$ \hfill (1.4)

which is equivalent to (1.3).

Let $E = \{x \in S : |F_T(x)| \geq c\}$. To prove (1.4), we check that

$$S \cap \{|F_T| > (C\lambda)^\sigma c\} \subset E_{\lambda,S},$$ \hfill (1.5)

where the set $E_{\lambda,S}$ is defined in Lemma A. Since $F_T$ is a non-constant analytic function, the level set $\{|F_T| = (C\lambda)^\sigma c\}$ has zero volume. Then Lemma A with the
function $\Phi = |\det D_x T|$ (which is logarithmically concave due to property (5) in Lemma C) gives us (1.4).

Assume that $x \notin E_{\lambda,S}$, i.e., that there exists an interval $J \subset S$ containing the point $x$ and such that the length of the set $J \setminus E$ is at least $\lambda^{-1}|J|$. Extend this interval until the endpoints appear on the unit sphere $\partial B(0,1)$ and denote the extended interval by $J^*$. Let $\Delta$ be the one-dimensional complex disk with diameter $J^*$. Then $\Delta \subset B_c(0,1)$ and $|F_T(x)| = |F(0)|$ for $x \in J^* \cap \partial B(0,a)$. Further, $|J^* \cap B(0,a)| \leq a|J^*|$ and we can apply the one-dimensional Remez property (Lemma B) to the analytic function $F_T|_\Delta$, the interval $J$, and its subset $J \setminus E$. We get

$$|F_T(x)| \leq \max_J |F_T| \leq \left(\frac{C|J|}{|J \setminus E|}\right)^{\sigma} \sup_{J \setminus E} |F_T| \leq (C\lambda)^\sigma c,$$

with $C = 8$ and

$$\sigma = \frac{3}{1 - a} \log \frac{1}{|F_T(a)|} = \frac{3}{1 - \sqrt{1 - \delta^3}} \log \frac{1}{|F(0)|}$$

$$< \frac{6}{\delta^3} \log \frac{1}{|F(0)|} = \frac{48}{\varepsilon^3} \log \frac{1}{|F(0)|},$$

completing the proof of (1.5) and, thereby, of the theorem. □

§2. Proof of Lemma B

We shall use the standard factorization $f(z) = U(z)B(z)$ where $U(z)$ has no zeroes in the disk and $B(z)$ is the Blaschke product. Since for every $x \in [-a,a]$,

$$\log |U(x)| = -\int_T \frac{1 - x^2}{1 - x\zeta^2} d\mu(\zeta)$$

where $\mu$ is some positive measure on the unit circle $T$, and since

$$\frac{1}{|1 - x\zeta|^2} \leq \frac{1}{|1 - a\zeta|^2} + \frac{1}{|1 + a\zeta|^2}$$

for every $\zeta \in D$, $x \in [-a,a]$, we immediately conclude that

$$\log |U(x)| \geq -\frac{1 - x^2}{1 - a^2} \int_T \left(\frac{1 - a^2}{|1 - a\zeta|^2} + \frac{1 - a^2}{|1 + a\zeta|^2}\right) d\mu(\zeta) = \frac{1 - x^2}{1 - a^2} \log |U(a)U(-a)|$$

and, therefore,

$$\min_{[-a,a]} |U| \geq |U(-a)U(a)| \frac{1}{1-a^2}. \tag{2.1}$$
We shall split the Blaschke product
\[ B(z) = \prod_{\zeta} \frac{z - \zeta}{1 - z\overline{\zeta}} \]
into two factors: \( B_1(z) \), which is the product over all zeroes \( \zeta \) satisfying
\[ \frac{1 - |\zeta|^2}{|1 + a\overline{\zeta}|^2} + \frac{1 - |\zeta|^2}{|1 - a\zeta|^2} \leq \frac{2}{3}, \]
and \( B_2(z) \), which is the product over all zeroes \( \zeta \) for which the opposite inequality holds. Our next aim will be to show that for all \( x \in [-a, a] \),
\[ |B_1(x)| \geq |B_1(-a)B_1(a)|^{\frac{2(1-x^2)}{1-a^2}}, \]
which yields
\[ \min_{[-a,a]} |B_1| \geq |B_1(a)B_1(-a)|^{\frac{2}{1-a^2}}. \quad (2.2) \]
Clearly, it is enough to establish this inequality for every Blaschke factor in \( B_1(z) \). Using the inequality \( 1 - t \geq e^{-2t} \ (0 \leq t \leq \frac{2}{3}) \), we obtain
\[ \left| \frac{x - \zeta}{1 - x\zeta} \right|^2 = 1 - \frac{(1 - x^2)(1 - |\zeta|^2)}{|1 - x\zeta|^2} \]
\[ \geq 1 - \left[ \frac{(1 - x^2)(1 - |\zeta|^2)}{|1 + a\overline{\zeta}|^2} + \frac{(1 - x^2)(1 - |\zeta|^2)}{|1 - a\zeta|^2} \right] \]
\[ \geq \exp \left\{ - \frac{2(1 - x^2)}{1 - a^2} \left[ \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 + a\overline{\zeta}|^2} + \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 - a\zeta|^2} \right] \right\} \]
\[ \geq \left[ 1 - \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 + a\overline{\zeta}|^2} \right]^{\frac{2(1-x^2)}{1-a^2}} \cdot \left[ 1 - \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 - a\zeta|^2} \right]^{\frac{2(1-x^2)}{1-a^2}} \]
\[ = \left| \frac{-a - \zeta}{1 + a\overline{\zeta}} \cdot \frac{a - \zeta}{1 - a\zeta} \right|^{\frac{2(1-x^2)}{1-a^2}}, \]
proving the statement.

The next observation is that the number \( N \) of factors in \( B_2(z) \) satisfies the inequality
\[ N \leq \frac{3}{1 - a^2} \log \frac{1}{|B_2(-a)B_2(a)|^{\frac{6}{1-a^2}}} \quad (2.3) \]
Indeed, for every zero $\zeta$ in $B_2$, we have
\[
\left| \frac{-a - \zeta \cdot a - \zeta}{1 + a\zeta} \right|^2 = \left[ 1 - \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 + a\zeta|^2} \right] \cdot \left[ 1 - \frac{(1 - a^2)(1 - |\zeta|^2)}{|1 - a\zeta|^2} \right]
\leq \exp \left\{ -\left(1 - a^2\right) \left[ \frac{1 - |\zeta|^2}{|1 + a\zeta|^2} + \frac{1 - |\zeta|^2}{|1 - a\zeta|^2} \right] \right\}
\leq e^{-\frac{2(1-a^2)}{3}}.
\]

Thus,
\[
|B_2(a)B_2(-a)| \leq \exp \left[ -\frac{N(1 - a^2)}{3} \right],
\]
which is equivalent to (2.3).

Now write $B_2(z) = P(z)R(z)$ where $P(z) = \prod_{k=1}^{N}(z-\zeta_k)$, and $R(z) = \prod_{k=1}^{N} \frac{1}{1 - z\zeta_k}$.

We have
\[
\max_{[-a,a]} |R| \leq \left( \frac{1 + a}{1 - a} \right)^{N} \min_{[-a,a]} |R| \leq \left( \frac{2}{1 - a} \right)^{N} \min_{[-a,a]} |R|.
\] (2.4)

At last, according to the classical Remez inequality (see, for example, [DR] or [BG]), for any sub-interval $I \subset [-a, a]$ and any measurable subset $E \subset I$,
\[
\max_{I} |P| \leq \left( \frac{4|I|}{|E|} \right)^{N} \sup_{E} |P|.
\] (2.5)

Combining estimates (2.1)–(2.5), we get
\[
\max_{I} |f| \leq \max_{I} |B_2|
\leq \max_{I} |P| \cdot \max_{[-a,a]} |R|
\leq \left( \frac{4|I|}{|E|} \right)^{N} \sup_{E} |P| \cdot \left( \frac{2}{1 - a} \right)^{N} \min_{[-a,a]} |R|
\leq \left( \frac{8|I|}{|E|} \right)^{N} \cdot \left( \frac{1}{1 - a} \right)^{N} \sup_{E} |B_2|
\leq \left( \frac{8|I|}{|E|} \right)^{N} \cdot \left( \frac{1}{1 - a} \right)^{N} \cdot \max_{[-a,a]} \frac{1}{|U|} \cdot \max_{[-a,a]} \frac{1}{|B_1|} \cdot \sup_{E} |f|
\leq \left( \frac{8|I|}{|E|} \right)^{N} \cdot \left( \frac{1}{1 - a} \right)^{N} \cdot \left| \frac{1}{U(a)U(-a)B_1(a)B_1(-a)} \right| \frac{1}{1 - a^2} \sup_{E} |f|
\leq \left( \frac{8|I|}{|E|} \right)^{N} \cdot \sup_{E} |f|.
\]
where
\[
\sigma \leq \left(1 + \log \frac{1}{1-a}\right) N + \frac{2}{1-a^2} \log \frac{1}{|U(a)U(-a)B_1(a)B_1(-a)|} + \frac{1}{1-a} \log \frac{1}{|U(a)U(-a)B_1(a)B_1(-a)|} \leq 3 \left(1 + \log \frac{1}{1-a}\right) \log \frac{1}{|B_2(a)B_2(-a)|} + \frac{2}{1-a} \log \frac{1}{|U(a)U(-a)B_1(a)B_1(-a)|} \leq \frac{3}{1-a} \log \frac{1}{|f(a)f(-a)|},
\]
(in the last line we used the inequality \(1 + \log t \leq t\) when \(t \geq 1\)). Lemma B is proved. \(\square\)

§3. Proof of Lemma C

\(T\) is one-to-one in the ball \(B(0, r_0)\):

We show that the function \(r \mapsto r\varphi(r^2)\) where, as before, \(\varphi(\zeta) = \frac{A-\zeta}{1-A\zeta}\), is increasing on the interval \([0, r_0]\). Set \(R = r^2\). We have
\[
\frac{d}{dr}(r\varphi(r^2)) = \varphi(R) \left(1 + 2R'\varphi'(R)\right).
\]
Since \(0 \leq R \leq R_0 < A\), we have \(\varphi(R) > 0\). So, it will suffice to show that \(\frac{|\varphi'(R)|}{\varphi(R)} \leq \frac{1}{2}\). A direct computation yields
\[
\frac{|\varphi'(R)|}{\varphi(R)} \leq \frac{\varphi'(R)}{\varphi(R)^2} = \frac{1-A^2}{(A-R)(A-R_0)} \leq \frac{2(1-A)}{(A-R_0)^2} \leq \frac{2\delta}{9} < \frac{1}{30},
\]
since \(\delta \leq \frac{1}{8}\). \(\square\)

\(TB(0, r_0)\) is a ball centered at the origin with radius bigger than \(1-2\delta\):

It is clear now that \(TB(0, r_0) = B(0, r_0\varphi(R_0))\), so we need only to show that \(r_0\varphi(R_0) > 1 - 2\delta\). We have
\[
1 - AR_0 = 1 - (1 - \delta^3)(1 - 3\delta - \delta^3) = 3\delta + \delta^3 + \delta^3(1 - 3\delta - \delta^3) \leq 3\delta + 2\delta^3
\]
and, thereby,
\[
\varphi(R_0) = \frac{A - R_0}{1 - AR_0} \geq \frac{3\delta}{3\delta + 2\delta^3} > \frac{1}{1 + \delta^2}.
\]
Thus, to prove our inequality, we need to check that
\[
1 - 3\delta - \delta^3 \geq (1 - 2\delta)^2 (1 + \delta^2)^2.
\]

The right hand side does not exceed
\[
(1 - 4\delta + 4\delta^2)(1 + 3\delta^2) \leq 1 - 4\delta + 7\delta^2.
\]

Since \(\delta \leq \frac{1}{8}\), we have \(7\delta^2 + \delta^3 < 8\delta^2 \leq \delta\), finishing the proof. □

The Jacobian \(|\det D_x T|\) is a logarithmically concave function in \(B(0, r_0)\):

First, we compute the Jacobian. Let \(T_i(x) = \phi(|x|^2)x_i\). Then
\[
\frac{\partial T_i}{\partial x_j} = \begin{cases} 
\phi'(r^2)2x_ix_j, & i \neq j \\
\phi(r^2) + \phi'(r^2)2x_i^2, & i = j
\end{cases}
\]

whence \(\det D_x T = \det(\xi I + A)\), where \(\xi = \phi(r^2)\) and \(A_{ij} = 2\phi'(r^2)x_ix_j\). Since the rank of \(A\) is one, \(\det(\xi I + A) = \xi^n + \xi^{n-1}\tr(A)\), and
\[
|\det D_x T| = (\phi(r^2) + 2r^2\phi'(r^2)) \cdot (\phi(r^2))^{n-1}.
\]

(This result can also be obtained in a purely geometric way: just consider the image of a small domain containing \(x\) and bounded by two concentric spheres and a thin cone).

The Taylor expansions
\[
\phi(R) = A - (1 - A^2) \sum_{k=1}^{\infty} A^{k-1} R^k,
\]
and
\[
\phi(R) + 2R\phi'(R) = A - (1 - A^2) \sum_{k=1}^{\infty} (1 + 2k) A^{k-1} R^k
\]

immediately show that both \(\phi(r^2)\) and \(\phi(r^2) + 2r^2\phi'(r^2)\) are concave decreasing functions of \(r\) on the interval \([0, 1]\). Since they are also positive on \([0, r_0]\), they are logarithmically concave on that interval. Hence the function \(r \mapsto [\phi(r^2) + 2r^2\phi'(r^2)] [\phi(r^2)]^{n-1}\) is also logarithmically concave on the interval \([0, r_0]\). It remains to recall that if \(\Phi(r)\) is a decreasing logarithmically concave function on the
interval \([0, r_0]\), then \(x \mapsto \Phi(|x|)\) is logarithmically concave in the ball \(B(0, r_0) \subset \mathbb{R}^n\).

\(\square\)

**The pre-image \(T^{-1}B\) of every (real) ball \(B \subset TB(0, r_0)\) is convex:**

Since the pre-image \(T^{-1}B\) is a body of revolution around the axis containing both the origin and the center of the ball \(B\), it is enough to prove our statement on the plane \(\mathbb{R}^2\). In order to do so, we shall show that the curvature of the image of any straight line tangent to the boundary of \(T^{-1}B\) does not exceed the curvature of the boundary of \(B\) which is \(\frac{1}{\text{rad}(B)}\). It is going to be a simple but somewhat boring exercise in differential geometry.

Let \(rx\ (0 \leq r \leq r_0, x \in \mathbb{R}^2, |x| = 1)\) be a point on the boundary of \(T^{-1}B\) and let \(y(t) = rx + tv\ (v \in \mathbb{R}^2, |v| = 1, t \in \mathbb{R})\) be the corresponding tangent line. Let \(\alpha\) be the angle between the vectors \(x\) and \(v\). The image of our tangent line is the curve

\[\sigma(t) = \varphi(|y(t)|^2)y(t) = \varphi(r^2 + 2rt\cos\alpha + t^2)(rx + tv).\]

To estimate the curvature, we need to compute the first and second derivatives of \(\sigma\). Differentiation yields

\[
\sigma'(t) = \varphi(|y(t)|^2)v + 2\varphi'(|y(t)|^2)\langle y(t), v \rangle y(t),
\]

\[
\sigma''(t) = 4\varphi'(|y(t)|^2)\langle y(t), v \rangle v + 2\varphi'(|y(t)|^2)y(t) + 4\varphi''(|y(t)|^2)\langle y(t), v \rangle^2 y(t).
\]

Plugging in \(t = 0\) and denoting, as above, \(r^2 = R\), we obtain

\[
\sigma'(0) = \varphi(R)v + 2R\varphi'(R)(\cos\alpha) x,
\]

\[
\sigma''(0) = 4r\varphi'(R)(\cos\alpha) v + 2r\varphi'(R) x + 4rR\varphi''(R)(\cos^2\alpha) x.
\]

Now we are ready to estimate the curvature. We shall use the standard formula

\[
\text{curvature} = \frac{|\sigma'(0) \times \sigma''(0)|}{|\sigma'(0)|^3}.
\]

We have

\[
\frac{|\sigma'(0)|}{\varphi(R)} \geq 1 - 2R \frac{|\varphi'(R)|}{\varphi(R)} \cos\alpha \geq \frac{14}{15}
\]

(recall that \(R < 1\) and \(\frac{|\varphi'(R)|}{\varphi(R)} \leq \frac{1}{30}\)), and therefore \(|\sigma'(0)|^3 \geq \frac{4}{3}\varphi(R)^3\). Using this
estimates, we finally obtain

\[
\text{curvature} \leq \frac{\left| \sigma'(0) \times \sigma''(0) \right|}{\frac{4}{3} \varphi(R)^3} \\
= \frac{5}{2} \left| v \times x \right| \cdot \left| \frac{\varphi'(R)}{\varphi(R)^2} + 2R \left[ \frac{\varphi''(R)}{\varphi(R)^2} - \frac{2 \varphi'(R)^2}{\varphi(R)^3} \right] \cos^2 \alpha \right| \\
= \frac{5}{2} \left| \sin \alpha \right| \cdot \left| \frac{\varphi'(R)}{\varphi(R)^2} + 2R \left[ \frac{\varphi'(R)}{\varphi(R)^2} \right]' \cos^2 \alpha \right| \\
\leq \frac{5}{2} \left[ \frac{1 - A^2}{(A - R)^2} + 4R \frac{1 - A^2}{(A - R)^3} \right] \\
\leq \frac{10(1 - A^2)}{(A - R)^3} \cdot \left[ 1 + \frac{A - R}{4} \right] \\
\leq \frac{20(1 - A)}{(A - R)^3} \cdot \frac{5}{4} \\
= \frac{25}{27} < 1 < \frac{1}{\text{rad}(B)},
\]

completing the proof of Lemma C. □

§4. An Example

Let \( Q(z) \) be an arbitrary polynomial. Let \( \eta > 0 \) be so small that

\[
\eta \max_{|z| \leq 1} |Q(z)| < \frac{1}{8}.
\]

Consider the analytic function \( F \) in the unit ball \( B_c(0, 1) \subset \mathbb{C}^2 \) defined by

\[
F(z_1, z_2) = \frac{1}{2} \left[ 2 \eta Q(z_1) + z_2 + \frac{1}{2} \right]
\]

and take rectangles

\[
V_\delta = \left\{ 0 \leq x_1 \leq \frac{1}{4}, \ 0 \leq x_2 + \frac{1}{2} \leq \delta \right\} \subset B \left( 0, \frac{3}{4} \right), \quad 0 < \delta \leq \frac{1}{2}.
\]

It is easy to see that \( |F| \leq 1 \) in \( B_c(0, 1) \) and \( |F(0, 0)| \geq \frac{1}{4} \) regardless of the choice of \( Q \). Notice that for very small \( \delta > 0 \), the distribution of \( |F| \) in the rectangle \( V_\delta \) with respect to the normalized area \( \frac{1}{\text{Area}(V)} \text{dArea}(x) \) is practically indistinguishable
from the distribution of $\eta Q(t)$ on the interval $[0, \frac{1}{4}]$ with respect to the normalized Lebesgue measure $4dt$. If the estimates (1.1) and (1.2) of the theorem were true in every rectangle $V_\delta$, they would also hold for the measures of level sets of the polynomial $\eta Q(t)$ on the interval $[0, \frac{1}{4}]$. Since they are scale-invariant, they would also hold for the measures of level sets of the polynomial $Q$ on the interval $[0, \frac{1}{4}]$. But, since polynomials are dense in the space of continuous functions, this would imply that they hold for level sets of any continuous function $g(t)$ on the interval $[0, \frac{1}{4}]$, which is clearly false. □

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