Tomographic probability representation in the problem of transitions between Landau levels

E D Zhebrak

Moscow Institute of Physics and Technology, Institutsky Lane 9, Dolgoprudny, Russia
E-mail: el1holstein@phystech.edu

Received 15 November 2012
Published 28 March 2013
Online at stacks.iop.org/PhysScr/T153/014063

Abstract
The problem of quantum motion of a charged particle in a magnetic field is considered in the framework of the tomographic probability representation. The coherent and Fock states of a charge moving in a time-dependent homogeneous magnetic field are studied in the tomographic probability representation. These states are expressed in terms of the quantum tomograms. The Fock state tomograms are given in the form of probability distributions described by multivariable Hermite polynomials with time-dependent arguments. These results are then generalized and applied to calculate the transition probabilities between the Landau levels for the case of transitions between stationary states of different magnetic fields, when the initially constant field becomes time dependent and then stays at some constant value again. The problem is studied in terms of wave functions and symplectic tomograms as well.

PACS numbers: 42.50.Ar, 42.50.Dv

1. Introduction

In [1] coherent states of a charge moving in a constant uniform magnetic field were constructed. These coherent states correspond to a Gaussian wave packet [2] that moves along the classical cyclotron trajectory with time-independent position dispersions. Such states were also studied in [3] and applied in [4–8]. A new formulation of quantum mechanics where the fair tomographic probability distribution was used as an alternative to a wave function and a density matrix has been suggested in [9]. The coherent states and Landau levels of a charge moving in a magnetic field were studied in the framework of the tomographic probability representation of quantum states in [10]. The charge moving in a time-dependent magnetic field was studied in the coherent state representation in [11, 12].

The problem of transitions between the Landau levels corresponding to a constant magnetic field is sufficiently trivial in contrast to the case when a constant magnetic field becomes time dependent, and when the varying component is ‘switched on’ the new constant magnetic field is set. The problem of transitions between the states of an initial and final constant field was studied in [11]. There the parametric excitation of the Landau levels for varying magnetic field was considered by using the coherent state method, and explicit expressions for the transition amplitudes were found in terms of classical polynomials. It was stated that the transition probability depends on the parameter having a physical meaning of reflection coefficient of a particle from the one-dimensional effective potential. This parameter cannot be obtained directly from the Schrödinger equation and its analytical value was not derived before. On the other hand, the excitation of the Landau levels can be reconsidered in the framework of a new formulation of quantum mechanics.

The aim of our work is to find the tomographic probability distributions called tomograms describing the charge coherent states and the Landau levels and their non-stationary analogues. We obtain new expressions for transition probabilities between the Landau levels expressed in terms of the tomograms of the charge quantum states. The paper is organized as follows. In section 2, we review the problem of the coherent states in both time-independent and time-dependent magnetic fields. In section 3, we construct quantum tomograms of the Landau energy levels and the Gaussian packets corresponding to the coherent
states. In section 4, we derive explicit expressions for the transition probabilities from the Landau levels to the ground Landau states in terms of integrals of the state tomogram products.

2. Coherent states of a charge moving in a magnetic field

The problem of a charge moving in a magnetic field was studied in [13] for the constant field case and in [10] for the time-dependent case.

Let us consider a charged particle with mass \( m = 1 \) and charge \( e = 1 \) moving in a magnetic field \( H = (0, 0, H) \) with the vector potential \( \vec{A} = \frac{1}{2} \left( \vec{H} \times \vec{r} \right) \).

The Hamiltonian of this quantum system reads (\( c = 1 \))

\[
H = \frac{1}{2} \left( (p_x - A_x)^2 + (p_y - A_y)^2 \right), \quad c = 1.
\]

Let us introduce a cyclotron frequency \( \omega (t) = H(t) \). For the constant case, it can be introduced as \( \omega = 1 \). Using the method of integrals of motion [1, 11] and introducing the operators

\[
\hat{A}^{\text{const}} = \frac{1}{\sqrt{2}} \left( p_x + ip_y \right),
\]

\[
\hat{B}^{\text{const}} = \frac{1}{\sqrt{2}} \left( p_x - ip_y \right),
\]

for a constant magnetic field and

\[
\hat{A}^{\text{var}}(t) = \exp \left( -\frac{i}{2} \int_0^t \omega (\tau) \, d\tau \right) \frac{\partial}{\partial x} - i \frac{\partial}{\partial y},
\]

\[
\hat{B}^{\text{var}}(t) = \exp \left( -\frac{i}{2} \int_0^t \omega (\tau) \, d\tau \right) \frac{\partial}{\partial y} + i \frac{\partial}{\partial x},
\]

with \( \omega (t) \) as was stated in [11] corresponds to both the equations \( \dot{\epsilon} (t) + \frac{1}{2} \epsilon (t)^2 \dot{\epsilon} = 0 \) and \( \frac{\partial}{\partial x} |\epsilon (t)| + \frac{1}{2} \omega (t)^2 |\epsilon (t)| - \frac{1}{it \omega} = 0 \) for varying field.

For an axially symmetric time-dependent magnetic field, we can obtain the quantum states corresponding to this motion

\[
\langle x, y | \alpha, \beta \rangle^{\text{const}} = \sqrt{\frac{1}{\pi \epsilon}} e^{-\frac{1}{4 \epsilon} \left| \alpha \right|^2 - \frac{1}{4 \epsilon} \left| \beta \right|^2} e^{\frac{i}{\sqrt{2 \epsilon}} \left( \left| \beta \right|^2 + \left| \alpha \right|^2 \right) + i \left( x \beta^* - y \alpha^* \right) - i \alpha \beta},
\]

\[
\langle x, y | \alpha, \beta \rangle^{\text{var}} = \frac{1}{\sqrt{2 \epsilon \pi}} \exp \left\{ \frac{i}{2 \epsilon} \left( x^2 + y^2 \right) - \frac{1}{2} \left( |\alpha|^2 + |\beta|^2 \right) \right\}

+ \frac{1}{|\epsilon|} \left( \beta \zeta e^{-i \gamma} + i \alpha \zeta^* e^{-i \gamma} - i \alpha \beta e^{-i \gamma} \right),
\]

where \( \alpha, \beta \) are real parameters of the coherent state, \( \zeta = x + iy \) and \( \gamma = \int_0^t \left[ \epsilon (r)^2 \pm H (r) \right] \, dr \).

In the following consideration of a time-dependent magnetic field, we will introduce for brevity \( \omega = \omega (t) \) and \( \epsilon = \epsilon (t) \).

The states (4) and (5) are called coherent states and are related to the Fock states as follows:

\[
|\alpha, \beta \rangle = \exp \left\{ -\frac{1}{2} \left( |\alpha|^2 + |\beta|^2 \right) \right\} \sum_{n_1, n_2 = 0}^{\infty} \frac{\alpha^{n_1} \beta^{n_2}}{\sqrt{n_1! n_2!}} |n_1, n_2 \rangle.
\]

The corresponding Fock states are eigenstates of the Hamiltonian operator \( H \) and the angular momentum operator \( L_z \):

\[
H |n_1, n_2 \rangle = (n_1 + \frac{1}{2}) |n_1, n_2 \rangle,
\]

\[
L_z |n_1, n_2 \rangle = (n_2 - n_1) |n_1, n_2 \rangle.
\]

As was shown in [1], the motion under consideration corresponds to a Gaussian wave packet with a center moving along the classical trajectory.

3. Tomographic representation of quantum states and energy levels of a charge moving in a magnetic field

The function called the symplectic tomogram was introduced in [9]. This function connected with the density matrix by the Radon transform can determine quantum states as well,

\[
\rho = \frac{1}{2 \pi} \int w (X, \mu, \nu) \, e^{i(X - \mu q - \nu p)} \, dX \, d\mu \, dv.
\]

where \( q \) and \( p \) are position and momentum operators, respectively, \( X, \mu, \nu \) are reals and \( X = \mu q + \nu p \). The tomogram is the non-negative probability distribution of a random variable \( X \) which is a position in the rotated and rescaled reference frame in the phase space. It is also a homogeneous normalized function. The inverse of (8) reads

\[
\rho = \frac{1}{2 \pi} \int w (X, \mu, \nu) \, e^{i(X - \mu q - \nu p)} \, dX \, d\mu \, dv.
\]

The symplectic tomogram of a pure state with the wave function \( \varphi (\gamma) \) is determined by the formula

\[
w (X, \mu, \nu) = \frac{1}{2 \pi |v|} \left| \int \varphi (\gamma) \, e^{i(X - \mu q - \nu p)} \, d\gamma \right|^2,
\]

which is related to the fractional Fourier transform of the wave function. Formulae (8)–(10) can be generalized for a system with several degrees of freedom. For two degrees of freedom, the symplectic tomogram \( w (X_1, \mu_1, v_1, X_2, \mu_2, v_2) \) is determined by the fractional Fourier transform of the wave function \( \varphi (y_1, y_2) \) and reads

\[
w (X_1, \mu_1, v_1, X_2, \mu_2, v_2) = \frac{1}{4 \pi^2 |v_1 v_2|} \left| \int \varphi (y_1, y_2) \, e^{i \frac{4 \pi^2}{v_1 v_2} y_1 \gamma_1 - \frac{4 \pi^2}{v_1 v_2} \gamma_2} \, dy_1 \, dy_2 \right|^2.
\]
For brevity, we will use \( w = w(X_1, \mu_1, v_1, X_2, \mu_2, v_2) \). Using (11), we can calculate symplectic tomograms directly from the wave functions. In such a manner there was yielded the symplectic tomogram \( u_{a,\beta}^{\text{const}} \) of the coherent state of a charged particle moving in a constant magnetic field in [13] was obtained:

\[
 u_{a,\beta}^{\text{const}} = \frac{e^{-|\alpha|^2 - |\beta|^2}}{2\pi^{\frac{3}{4}} \sqrt{\frac{\mu_1^2 + \mu_2^2}{\nu_1^2 + \nu_2^2}}} \times \exp \left( \frac{\frac{i\beta}{\nu_1} - \frac{i\alpha}{\nu_2}}{1 + \frac{2\mu_1}{\nu_1}} + \frac{\frac{i\alpha}{\nu_1} - \frac{i\beta}{\nu_2}}{1 + \frac{2\mu_2}{\nu_2}} - ia\beta \right)^2 .
\]  

(12)

The tomogram of a Fock state in the time-dependent field \( u_{n_1,n_2}^{\text{var}} \) is derived from (12) using the formula (6):

\[
 u_{n_1,n_2}^{\text{var}} = \frac{1}{n_1!n_2!} \frac{1}{2\pi^{\frac{3}{4}} \sqrt{\frac{\mu_1^2 + \mu_2^2}{\nu_1^2 + \nu_2^2}}} \times \exp \left( -\frac{X_1^2}{v_1 \left( 1 - \frac{2\mu_1}{\nu_1} \right)} - \frac{X_2^2}{v_2 \left( 1 - \frac{2\mu_2}{\nu_2} \right)} \right) \times H_{n_1, n_2}^{(S)} \left( \frac{\bar{k}}{\bar{\kappa}} \right)^2 ,
\]  

(13)

where \( H_{n_1, n_2}^{(S)} \left( \frac{\bar{k}}{\bar{\kappa}} \right) \) is a Hermite polynomial of two variables:

\[
 S = \begin{pmatrix}
 \bar{b} & -i \left( \sqrt{2} \bar{\alpha} - 1 \right) \\
 -i \left( \sqrt{2} \bar{\alpha} - 1 \right) & -\bar{b}
\end{pmatrix},
\]

\[
 \bar{k} = \sqrt{2} \begin{pmatrix}
 \frac{X_1}{v_1 - 2\mu_1} + \frac{iX_2}{v_2 - 2\mu_2} \\
 -\frac{iX_1}{v_1 - 2\mu_1} + \frac{X_2}{v_2 - 2\mu_2}
\end{pmatrix},
\]

\[
 \bar{\alpha} = \frac{1}{1 - \frac{2\mu_1}{v_1}} + \frac{1}{1 - \frac{2\mu_2}{v_2}}, \quad \bar{b} = \frac{1}{1 - \frac{2\mu_1}{v_1}} - \frac{1}{1 - \frac{2\mu_2}{v_2}}.
\]

Analogously for the time-dependent magnetic field

\[
 u_{a,\beta}^{\text{var}} = \frac{e^{-|\alpha|^2 - |\beta|^2}}{2\pi \left( \frac{\nu_1}{v_1} + \mu_1 \right) \left( \frac{\nu_2}{v_2} + \mu_2 \right)} \times \exp \left[ -\frac{X_1^2}{2v_1 \left( \frac{\nu_1}{v_1} + \frac{2\mu_1}{v_1} \right)} - \frac{X_2^2}{2v_2 \left( \frac{\nu_2}{v_2} + \frac{2\mu_2}{v_2} \right)} \right] \times e^{-\frac{1}{2} \Delta A^2 + \Delta D} \right]^2 ,
\]  

(14)

where

\[
 D = \begin{pmatrix}
 e^{-2i\nu_1} b & e^{-i(\nu_1 - \nu_2)} - (i - a) \\
 e^{-i(\nu_1 - \nu_2)} & e^{2i\nu_2} b
\end{pmatrix} ,
\]

\[
 \Delta A = \frac{1}{|\epsilon|^2} b - \frac{1}{|\epsilon|^2} (i - a) \\
 \Delta D = \frac{1}{|\epsilon|^2} (i - a) - \frac{e^{2i\nu_2}}{|\epsilon|^2} b.
\]

4. Transition probabilities between Landau levels

The problem of calculating transition probabilities between the Landau levels of any constant magnetic field with the help of wave functions is well studied. In this consideration, we suppose the magnetic field to be constant in the initial and final states, and between them there exists an arbitrary time dependence. As was shown in [11], in this case the transition probability \( P_{n_1, n_2}^{m_1, m_2} \) between the initial and final states with the wave functions \( \psi_{n_1, n_2} \) and \( \psi_{m_1, m_2} \) can be calculated by the formula

\[
 P_{n_1, n_2}^{m_1, m_2} = \left| \int \psi_{n_1, n_2}^*(x, y) \psi_{m_1, m_2}(x, y) \, dx \, dy \right|^2 .
\]  

(16)

These two states are non-orthogonal and coherent states can be expanded in them according to formula (6). The transition probability will have an explicit form [11]

\[
 P_{n_1, n_2}^{m_1, m_2} = \frac{m_2!n_1!}{n_1!n_2!} R_{m_1 - n_1}^{m_2 - n_2} \left( 1 - R \right)_{m_1 - n_1}^{m_2 - n_2} 
\]

\[
 \times \left| J_{m_1 - n_1}^{m_2 - n_2} \left( 1 - 2R \right) \right|^2 ,
\]  

(17)

where \( J_{m_1, m_2}^{m_1, m_2}(x) \) is the Jacobi polynomial and \( R \) can be treated as the reflection coefficient of a particle from the one-dimensional effective potential and cannot be calculated directly.

We consider a situation when the particle possessing a quantum state \( |n_1, n_2\rangle \) in a constant magnetic field transits to the ground state \( |0, 0\rangle \) when the time-dependent magnetic field is switched off. The tomographic approach also allows us to find \( P_{n_1, n_2}^{m_1, m_2} \) in terms of symplectic
Using the symplectic tomograms derived in section 3, we provide corresponding transition probabilities by formula (16):

\[
P^{00}_{n_1n_2} = \frac{1}{16\pi^4 n_1! n_2!} \int \frac{1}{\sqrt{\left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right) \left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right)}} \left| \left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right) \right| |k|^2 \exp \left[ - \frac{X_1^2}{v_1^2 \left(\frac{\nu_1}{\nu_2} + 2i\frac{\mu_1}{v_1} \right)} - \frac{X_2^2}{v_2^2 \left(\frac{\nu_1}{\nu_2} + 2i\frac{\mu_2}{v_2} \right)} \right] \left(\frac{Y_1^2}{v_1^2 \left(1 + 2i\frac{\mu_1}{v_1} \right)} - \frac{Y_2^2}{v_2^2 \left(1 + 2i\frac{\mu_2}{v_2} \right)} \right)^2 \times e^{i(X_1-Y_1+X_2-Y_2)} dX_1 dY_1 d\mu_1 d\nu_1 \times dX_2 dY_2 d\mu_2 d\nu_2.
\]

The integral (19) has such a form that the time-dependent function \(\varepsilon(t)\) disappears after integration. For the case of transition between the ground states, we have \(P^{00}_{0,0} = 1 - R\) [11]. Comparing with (19), we obtain an integral equation for reflection coefficient \(R\):

\[
R = 1 - \frac{1}{16\pi^4} \int \frac{1}{\sqrt{\left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right) \left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right)}} \left| \left(\frac{\nu_1}{\nu_2} + \mu_1 \right) \left(\frac{\nu_1}{\nu_2} + \mu_2 \right) \right| |k|^2 \exp \left[ - \frac{X_1^2}{v_1^2 \left(\frac{\nu_1}{\nu_2} + 2i\frac{\mu_1}{v_1} \right)} - \frac{X_2^2}{v_2^2 \left(\frac{\nu_1}{\nu_2} + 2i\frac{\mu_2}{v_2} \right)} \right] \left(\frac{Y_1^2}{v_1^2 \left(1 + 2i\frac{\mu_1}{v_1} \right)} - \frac{Y_2^2}{v_2^2 \left(1 + 2i\frac{\mu_2}{v_2} \right)} \right)^2 \times e^{i(X_1-Y_1+X_2-Y_2)} dX_1 dY_1 d\mu_1 d\nu_1 \times dX_2 dY_2 d\mu_2 d\nu_2.
\]

5. Conclusion

To resume, we point out the main results of our work. We studied the problem of finding the transition probabilities between the Landau levels induced by time dependence of the homogeneous magnetic field during some period of time using the symplectic probability description of the quantum states. The transition probability is expressed in terms of the non-local integral of the product of tomograms of the initial quantum state and the final quantum state with an exponential kernel. We have shown that this integral is expressed in terms of the Jacobi polynomial which corresponds to standard calculation of the transition probabilities by using the overlap integral of the initial and final wave functions. Comparing the results obtained by calculations of transition probabilities between the ground Landau levels of the initial and final states with the methods of wave functions and symplectic tomograms, we obtained the integral equation for the reflection coefficient of a particle from a one-dimensional effective potential. We generalize the obtained result to the case of the presence of a varying electric field in future work.

Acknowledgments

The author thanks the organizers of the Central European Workshop on Quantum Optics 2012 and the Moscow Institute of Physics and the Technology for partial support.

References

[1] Malkin I and Man’ko V I 1968 Zh. Ekspt. Teor. Fiz. 55 1014
[2] Kennard E H 1927 Z. Phys. 44 326
[3] Feldman A and Kahn A H 1970 Phys. Rev. B 1 4584
[4] Kuzmenko T, Kikoin K and Avishai Y 2006 Phys. Rev. B 73 235310
[5] Belov V V and Maslov V P 1990 Dokl. Akad. Nauk SSSR 311 849
[6] Rashba E I, Zhukov L E and Efros A L 1997 Phys. Rev. B 55 5306
[7] Bagrov V G, Gavrilov S P, Gitman D M and Gorska K 2012 J. Phys. A: Math. Theor. 45 244008
[8] Kowalskiy K, Rembielinski J and Papaloucas L C 1996 J. Phys. A: Math. Gen. 29 4149
[9] Mancini S, IMan’ko V and Tombesi P 1996 Phys. Lett. A 213 1
[10] Man’ko V I and Zhebrak E D 2012 Tomographic probability representation for states of charge moving in varying field arXiv:1204.3427v1 [quant-ph]
[11] Malkan I, Man’ko V I and Tritonov D A 1970 Phys. Rev. D 2 1371
[12] Dodonov V and Man’ko V 1988 Sov. Phys. Proc. FIAN 183 71
[13] Malko V I and Zhebrak E D 2011 Opt. Spectrosc. 111 666