CHARACTERIZATION OF PSEUDO-COLLARABLE MANIFOLDS WITH BOUNDARY

SHIJIE GU

ABSTRACT. In this paper we obtain a complete characterization of pseudo-collarable n-manifolds for n ≥ 6. This extends earlier work by Guilbault and Tinsley to allow for manifolds with noncompact boundary. In the same way that their work can be viewed as an extension of Siebenmann’s dissertation that can be applied to manifolds with non-stable fundamental group at infinity, our main theorem can also be viewed as an extension of the recent Gu-Guilbault characterization of completable n-manifolds in a manner that is applicable to manifolds whose fundamental group at infinity is not peripherally stable.

In 1965, Siebenmann’s PhD thesis [Sie65] provided necessary and sufficient conditions for an open manifold $M^m$ of dimension at least 6 to contain an open collar neighborhood of infinity, i.e., a manifold neighborhood of infinity $N$ such that $N \approx \partial N \times [0, 1)$. His collaring theorem can be easily extended to cases when $M^m$ is noncompact but with compact boundary. However, the situation becomes much subtler if $\partial M^m$ is noncompact. Instead of “collaring”, the term “completion” serves as an appropriate analog. An $m$-manifold $M^m$ with (possibly empty) boundary is completable if there exists a compact manifold $\hat{M}^m$ and a compactum $C \subseteq \partial \hat{M}^m$ such that $\hat{M}^m \setminus C$ is homeomorphic to $M^m$. In this case $\hat{M}^m$ is called a (manifold) completion of $M^m$. After Siebenmann did some initial work in such topic, O’Brien [O’B83] characterized completable $m$-manifolds ($m > 5$) in the case where $M^m$ and $\partial M^m$ are both 1-ended. But, in general, a completable manifold with (noncompact) boundary may have uncountably many non-isolated ends. For example, one can take any favorite compact manifold with boundary and remove a Cantor set from its boundary. In a very recent work, Guilbault and the author settled such case, hence, providing a complete characterization for high-dimensional manifolds (with boundary).

Theorem 0.1. [GG17] [Manifold Completion Theorem] An $m$-manifold $M^m$ ($m \geq 6$) is completable if and only if

1. $M^m$ is inward tame,
2. $M^m$ is peripherally $\pi_1$-stable at infinity,
3. $\sigma_\infty(M^m) \in \lim \left\{ \tilde{K}_0(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \right\}$ is zero, and
4. $\tau_\infty(M^m) \in \lim^1 \left\{ \text{Wh}(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \right\}$ is zero.

Key words and phrases. ends, inward tame, completable, homotopy collar, plus construction, pseudo-collar, semistable, $Z$-compacification, Wall finiteness obstruction, Whitehead torsion.
Although Condition \( (2) \) is necessary in order for such a completion to exist, such condition is too rigid to characterize many exotic examples related to current research trends in topology and geometric group theory. For instance, the exotic universal covering spaces produced by Mike Davis in [Dav83] are not collarable (because Condition \( (2) \) fails) yet their ends exhibit some nice geometric structure. Other examples such as (open) manifolds that satisfy Conditions \( (1), (3) \) and \( (4) \) but Condition \( (2) \) can be found in [GT03, Thm.1.3]. Define a manifold neighborhood of infinity \( N \) in a manifold \( M^m \) to be a homotopy collar provided \( \text{Fr} N \hookrightarrow N \) is a homotopy equivalence. A pseudo-collar is a homotopy collar which contains arbitrarily small homotopy collar neighborhoods of infinity. A manifold is pseudo-collarable if it contains a pseudo-collar neighborhood of infinity. When \( M^m \) is an open manifold (or more generally, a manifold with compact boundary), Guilbault [Gui00] initiated a program to produce a generalization of Siebenmann’s collaring theorem. The idea of pseudo-collars and a detailed motivation for the definition are nicely exposited in [Gui00]. Through a series of papers [Gui00, GT03, GT06], a complete characterization for pseudo-collarable manifolds with compact boundary was provided.

**Theorem 0.2.** [GT06] An \( m \)-manifold \( M^m \) \((m \geq 6)\) with compact boundary is pseudo-collarable iff each of the following conditions holds:

(i) \( M^m \) is inward tame

(ii) \( M^m \) is perfectly \( \pi_1 \)-semistable at infinity,

(iii) \( \sigma_\infty(M^m) \in \lim \left\{ K_0(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \right\} \) is zero.

Just as Theorem 0.1 is a natural generalization of Siebenmann’s dissertation to manifolds with noncompact boundaries, it is natural to extend the study of pseudo-collarability to manifolds with noncompact boundaries. Moreover, since all completable manifolds are pseudo-collarable (a key step in the proof of Theorem 0.1), a more general study of pseudo-collarability also generalizes Theorem 0.1 in the same way that Theorem 0.2 generalized [Sie65]. Our main result is the following characterization theorem.

**Theorem 0.3** (Pseudo-collarability characterization theorem). An \( m \)-manifold \( M^m \) \((m \geq 6)\) is pseudo-collarable iff each of the following conditions holds:

(a) \( M^m \) is inward tame

(b) \( M^m \) is peripherally perfectly \( \pi_1 \)-semistable at infinity,

(c) \( \sigma_\infty(M^m) \in \lim \left\{ K_0(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \right\} \) is zero.

**Remark 1.** It is worth noting that Condition \((b)\) of Theorem 0.3 is strictly weaker than Condition \((2)\) of Theorem 0.1. Furthermore, it reduces to Condition \((ii)\) of Theorem 0.2 when boundary \( \partial M^m \) is compact.

The strategy of our proof is heavily relying on techniques and results developed by several substantial and technical papers [Sie65, Gui00, GT03, GT06, GG17]. For a full understanding, the readers should be familiar with the Pseudo-collarability
Characterization Theorem in [GT06] and the Manifold Completion Theorem in [GG17]. We will not reprove all these results, but our goal is to take shortcuts afforded by both papers, hence, provide a proof of Theorem 0.3 efficiently.

About the organization of this paper: §1 contains basic definitions and notation. §2 provides background materials and technical set-up about neighborhoods of infinity, ends, peripheral perfect semistability condition, etc. §3 and §4 give illustrations for Conditions (a) and (c) respectively. §5 sets forth some crucial lemmas. In §6 and §7, we prove Theorem 0.3. In the final section of this paper, we discuss some related open questions.

1. Conventions, notation, and terminology

For convenience, all manifolds are assumed to be piecewise-linear (PL). Equivalent results in the smooth and topological categories may be obtained in the usual ways. For instance, some technical issues in smooth category requiring “rounding off corners” or “smoothing corners” have been nicely exposited in [Sie65] and [O’B83]. Unless stated otherwise, an $m$-manifold $M^m$ is permitted to have a boundary, denoted $\partial M^m$. We denote the manifold interior by $\text{int} M^m$. For $A \subseteq M^m$, the point-set interior will be denoted $\text{Int}_M^A$ and the frontier by $\text{Fr}_M^A$. A closed manifold is a compact boundaryless manifold, while an open manifold is a non-compact boundaryless manifold.

A $q$-dimensional submanifold $Q^q \subseteq M^m$ is properly embedded if it is a closed subset of $M^m$ and $Q^q \cap \partial M^m = \partial Q^q$; it is locally flat if each $p \in \text{int} Q^q$ has a neighborhood pair homeomorphic to $(\mathbb{R}^m, \mathbb{R}^q)$ and each $p \in \partial Q^q$ has a neighborhood pair homeomorphic to $(\mathbb{R}^m_+, \mathbb{R}^q_+)$. By this definition, the only properly embedded codimension 0 submanifolds of $M^m$ are unions of its connected components; a more useful variety of codimension 0 submanifolds are the following: a codimension 0 submanifold $Q^m \subseteq M^m$ is clean if it is a closed subset of $M^m$ and $\text{Fr}_M^M Q^m$ is a properly embedded locally flat (hence, bicollared) $(m-1)$-submanifold of $M^m$. In that case, $M^m \setminus Q^m$ is also clean, and $\text{Fr}_M^M Q^m$ is a clean codimension 0 submanifold of both $\partial Q^m$ and $\partial (M^m \setminus Q^m)$.

When the dimension of a manifold or submanifold is clear, we will often omit the superscript; for example, denoting a clean codimension 0 submanifold simply by $Q$. Similarly, when the ambient space is clear, we denote (point-set) interiors and frontiers by $\text{Int}_A$ and $\text{Fr}_A$

For any codimension 0 clean submanifold $Q \subseteq M^m$, let $\partial_M Q$ denote $Q \cap \partial M^m$ and $\text{int}_M Q = Q \cap \text{int} M^m$; alternatively, $\partial_M Q = \partial Q \setminus \text{int} (\text{Fr} Q)$ and $\text{int}_M Q = Q \setminus \partial M^m$. Note that $\text{int}_M Q$ is a $m$-manifold and $\partial (\text{int}_M Q) = \text{int} (\text{Fr} Q)$.

2. Ends, pro-$\pi_1$, and the peripherally perfectly semistable condition

Most of definitions and terminologies in this section are based on ones developed in [GG17], we give a brief review in terms of the topology of ends of manifolds and inverse sequences of groups.
2.1. **Neighborhoods of infinity, partial neighborhoods of infinity, and ends.**

Let $M^m$ be a connected manifold. A *clean neighborhood of infinity* in $M^m$ is a clean codimension 0 submanifold $N \subseteq M^m$ for which $M^m \setminus N$ is compact. Equivalently, a clean neighborhood of infinity is a set of the form $\overline{M^m \setminus C}$ where $C$ is a compact clean codimension 0 submanifold of $M^m$. A *clean compact exhaustion* of $M^m$ is a sequence $\{C_i\}_{i=1}^{\infty}$ of clean compact connected codimension 0 submanifolds with $C_i \subseteq \text{Int}_{M^m} C_{i+1}$ and $\cup C_i = M^m$. By letting $N_i = M^m \setminus C_i$ we obtain the corresponding cofinal sequence of clean neighborhoods of infinity. Each such $N_i$ has finitely many components $\{N^j_i\}_{j=1}^{k_i}$. By enlarging $C_i$ to include all of the compact components of $N_i$ we can arrange that each $N^j_i$ is noncompact; then, by drilling out regular neighborhoods of arcs connecting the various components of each Fr$_{M^m} N^j_i$ (thereby further enlargening $C_i$), we can arrange that each Fr$_{M^m} N^j_i$ is connected. An $N_i$ with these latter two properties is called a 0-neighborhood of infinity. For convenience, most constructions in this paper will begin with a clean compact exhaustion of $M^m$ with a corresponding cofinal sequence of clean 0-neighborhoods of infinity.

Assuming the above arrangement, an *end* of $M^m$ is determined by a nested sequence of components $\varepsilon = (N^k_i)_{i=1}^{\infty}$ of the $N_i$; each component is called a *neighborhood of* $\varepsilon$. In §3.3, we discuss components $\{N^j\}$ of a neighborhood of infinity $N$ without reference to a specific end of $M^m$. In that situation, we will refer to the $N^j$ as *partial neighborhoods of infinity* for $M^m$ (partial 0-neighborhoods if $N$ is a 0-neighborhood of infinity). Clearly every noncompact clean connected codimension 0 submanifold of $M^m$ with compact frontier is a partial neighborhood of infinity with respect to an appropriately chosen compact $C$; if its frontier is connected it is a partial 0-neighborhood of infinity.

2.2. **The fundamental group of an end.** For each end $\varepsilon$, we will define the *fundamental group at* $\varepsilon$; this is done using inverse sequences. Two inverse sequences of groups and homomorphisms $A_0 \leftarrow A_1 \leftarrow A_3 \leftarrow \cdots$ and $B_0 \leftarrow B_1 \leftarrow B_3 \leftarrow \cdots$ are pro-isomorphic if they contain subsequences that fit into a commutative diagram of the form

\[
\begin{array}{ccccccc}
G_{i_0} & \leftarrow & \lambda_{i_0+1,i_1} & G_{i_1} & \leftarrow & \lambda_{i_1+1,i_2} & G_{i_2} & \leftarrow & \lambda_{i_2+1,i_3} & G_{i_3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_{j_0} & \leftarrow & \mu_{j_0+1,j_1} & H_{j_1} & \leftarrow & \mu_{j_1+1,j_2} & H_{j_2} & \leftarrow & \mu_{j_2+1,j_3} & \cdots \\
\end{array}
\]

An inverse sequence is *stable* if it is pro-isomorphic to a constant sequence $C \leftarrow \cdots$. Clearly, an inverse sequence is pro-isomorphic to each of its subsequences; it is stable if and only if it contains a subsequence for which the images
stabilize in the following manner

\[
G_0 \leftarrow \frac{\lambda_1}{\implies} G_1 \leftarrow \frac{\lambda_2}{\implies} G_2 \leftarrow \frac{\lambda_3}{\implies} G_3 \leftarrow \cdots
\]

(2.2)

\[
\text{Im}(\lambda_1) \leftarrow \frac{\sim}{\implies} \text{Im}(\lambda_2) \leftarrow \frac{\sim}{\implies} \text{Im}(\lambda_3) \leftarrow \cdots
\]

where all unlabeled homomorphisms are restrictions or inclusions.

Given an end \(\varepsilon = (N^k_i)_{i=1}^{\infty}\), choose a ray \(r : [1, \infty) \to M^m\) such that \(r ([i, \infty)) \subseteq N^k_i\) for each integer \(i > 0\) and form the inverse sequence

\[
(2.3) \quad \pi_1 (N^k_1, r (1)) \leftarrow \frac{\lambda_2}{\implies} \pi_1 (N^k_2, r (2)) \leftarrow \frac{\lambda_3}{\implies} \pi_1 (N^k_3, r (3)) \leftarrow \frac{\lambda_4}{\implies} \cdots
\]

where each \(\lambda_i\) is an inclusion induced homomorphism composed with the change-of-basepoint isomorphism induced by the path \(r|_{[i-1,i]}\). We refer to \(r\) as the base ray and the sequence (2.3) as a representative of the “fundamental group at \(\varepsilon\) based at \(r\)”—denoted \(\text{pro-} \pi_1 (\varepsilon, r)\). We say the fundamental group at \(\varepsilon\) is stable if (2.3) is a stable sequence. A nontrivial (but standard) observation is that both semistability and stability of \(\varepsilon\) do not depend on the base ray (or the system of neighborhoods if infinity used to define it). See [Gui16] or [Geo08].

If \(\{H_i, \mu_i\}\) can be chosen so that each \(\mu_i\) is an epimorphism, we say that our inverse sequence is semistable (or Mittag-Leffler, or pro-epimorphic). In this case, it can be arranged that the restriction maps in the bottom row of (2.1) are epimorphisms. Similarly, if \(\{H_i, \mu_i\}\) can be chosen so that each \(\mu_i\) is a monomorphism, we say that our inverse sequence is pro-monomorphic; it can then be arranged that the restriction maps in the bottom row of (2.1) are monomorphisms. It is easy to see that an inverse sequence that is semistable and pro-monomorphic is stable.

Recall that a commutator element of a group \(H\) is an element of the form \(x^{-1}y^{-1}xy\) where \(x, y \in H\); and the commutator subgroup of \(H\); denoted \([H, H]\) or \(H^{(1)}\), is the subgroup generated by all of its commutators. The group \(H\) is perfect if \(H = [H, H]\). An inverse sequence of groups is perfectly semistable if it is pro-isomorphic to an inverse sequence.

\[
(2.4) \quad G_0 \leftarrow \frac{\lambda_1}{\implies} G_1 \leftarrow \frac{\lambda_2}{\implies} G_2 \leftarrow \frac{\lambda_3}{\implies} \cdots
\]

of finitely generated groups and surjections where each \(\ker(\lambda_i)\) perfect. The following shows that inverse sequences of this type behave well under passage to subsequences.

**Lemma 2.1.** A composition of surjective group homomorphisms, each having perfect kernels, has perfect kernel. Thus, if an inverse sequence of surjective group homomorphisms has the property that the kernel of each bonding map is perfect, then each of its subsequences also has this property.

**Proof.** See [Gui00, Lemma 1].

\(\Box\)
2.3. Relative connectedness, relatively perfectly semistability, and the peripheral perfect semistability condition. Let $Q$ be a manifold and $A \subseteq \partial Q$. We say that $Q$ is $A$-connected at infinity if $Q$ contains arbitrarily small neighborhoods of infinity $V$ for which $A \cup V$ is connected.

**Lemma 2.2.** [GG17, Lemma 4.1] Let $Q$ be a noncompact manifold and $A$ a clean codimension 0 submanifold of $\partial Q$. Then $Q$ is $A$-connected at infinity if and only if $Q \setminus A$ is 1-ended.

If $A \subseteq \partial Q$ and $Q$ is $A$-connected at infinity: let $\{V_i\}$ be a cofinal sequence of clean neighborhood of infinity for which each $A \cup V_i$ is connected; choose a ray $r : [1, \infty) \to \text{Int} Q$ such that $r([i, \infty)) \subseteq V_i$ for each $i > 0$; and form the inverse sequence

$$\pi_1(A \cup V_1, r(1)) \leftarrow^\mu_2 \pi_1(A \cup V_2, r(2)) \leftarrow^\mu_3 \pi_1(A \cup V_3, r(3)) \leftarrow^\mu_4 \cdots$$

where bonding homomorphisms are obtained as in (2.3). We say $Q$ is $A$-perfectly $\pi_1$-semistable at infinity (resp. $A$-$\pi_1$-stable at infinity) if (2.5) is perfectly semistable (resp. stable). Independence of this property from the choices of $\{V_i\}$ and $r$ follows from the traditional theory of ends by applying Lemmas 2.2 and 2.3. Because each boundary component of a manifold with boundary is collared, the following lemma is true because “throwing away” part of the boundary will preserve the homotopy type of the original manifold.

**Lemma 2.3.** [GG17, Lemma 4.2] Let $Q$ be a noncompact manifold and $A$ a clean codimension 0 submanifold of $\partial Q$ for which $Q$ is $A$-connected at infinity. Then, for any cofinal sequence of clean neighborhoods of infinity $\{V_i\}$ and ray $r : [1, \infty) \to Q$ as described above, the sequence (2.5) is pro-isomorphic to any sequence representing $\text{pro-}\pi_1(Q \setminus A, r)$.

**Remark 2.** In the above discussion, we allow for the possibility that $A = \emptyset$. In that case, $A$-connected at infinity reduces to 1-endedness and $A$-perfectly $\pi_1$-semistable (resp. $A$-$\pi_1$-stability) to ordinary perfectly semistable (resp. $\pi_1$-stability) at that end.

We can now formulate one of the key definitions of this paper.

**Definition 2.1.** Let $M^m$ be an manifold and $\varepsilon$ be an end of $M^m$

1. $M^m$ is peripherally locally connected at infinity if it contains arbitrarily small 0-neighborhoods of infinity $N$ with the property that each component $N^j$ is $\partial_M N^j$-connected at infinity.

2. $M^m$ is peripherally locally connected at $\varepsilon$ if $\varepsilon$ has arbitrarily small 0-neighborhoods $P$ that are $\partial_M P$-connected at infinity.

An $N$ with the property described in Condition (1) will be called a strong 0-neighborhood of infinity for $M^m$, and a $P$ with the property described in Condition (2) will be called a strong 0-neighborhood of $\varepsilon$. More generally, any connected partial 0-neighborhood of infinity $Q$ that is $\partial_M Q$-connected at infinity will be called a strong partial 0-neighborhood of infinity.
Lemma 2.4. [GG17, Lemma 4.4] $M^m$ is peripherally locally connected at infinity iff $M^m$ is peripherally locally connected at each of its ends.

In the next section, one will see that every inward tame manifold $M^m$ is peripherally locally connected at infinity. As a consequence, that condition plays less prominent role than the next definition.

Definition 2.2. Let $M^m$ be a manifold and $\varepsilon$ an end of $M^m$.

1. $M^m$ is peripherally perfectly $\pi_1$-semistable at infinity if it contains arbitrarily small strong 0-neighborhoods of infinity $N$ with the property that each component $N^j$ is $\partial_M N^j$-perfectly $\pi_1$-semistable at infinity.

2. $M^m$ is peripherally perfectly $\pi_1$-semistable at $\varepsilon$ if $\varepsilon$ has arbitrarily small strong 0-neighborhoods $P$ that are $\partial_M P$-perfectly $\pi_1$-semistable at infinity.

If $M^m$ contains arbitrarily small 0-neighborhoods of infinity $N$ with the property that each component $N^j$ is $\partial_M N^j$-perfectly semistable at infinity, then those components provide arbitrarily small neighborhoods of the ends satisfying the necessary perfectly semistable condition. Thus, it’s easy to see peripheral perfect $\pi_1$-semistability at infinity implies peripheral perfect $\pi_1$-semistability at each end.

3. Finite domination and inward tameness

A topological space $P$ is finitely dominated if there exists a finite polyhedron $K$ and maps $u : P \to K$ and $d : K \to P$ such that $d \circ u \simeq \text{id}_P$. If choices can be made so that $d \circ u \simeq \text{id}_P$ and $u \circ d \simeq \text{id}_K$, i.e., $P \simeq K$, we say that $P$ has finite homotopy type. For convenience we will restrict our attention to cases where $P$ is a locally finite polyhedron—a class that contains the (PL) manifolds and submanifolds (and certain subspaces of these) under consideration in this paper.

A locally finite polyhedron $P$ is inward tame if it contains arbitrarily small polyhedral neighborhoods of infinity that are finitely dominated. Equivalently, $P$ contains a cofinal sequence $\{N_i\}$ of closed polyhedral neighborhoods of infinity each admitting a “taming homotopy” $H : N_i \times [0, 1] \to N_i$ that pulls $N_i$ into a compact subset of itself. By an application of the Homotopy Extension Property, we may require taming homotopies to be fixed on $\text{Fr} \ N_i$. From there, it is easy to see that, in an inward tame polyhedron, every closed neighborhood of infinity admits a taming homotopy.

Lemma 3.1. [GG17] Lemma 5.3 Let $M^m$ be a manifold and $A$ a clean codimension 0 submanifold of $\partial M^m$. If $M^m$ is inward tame then so is $M^m \setminus A$.

It is easy to see that a finitely dominated space $P$ has finitely generated homology. The following result is vital to this paper.

Proposition 3.2. [GG17, Prop.5.4] If a noncompact connected manifold $M^m$ and its boundary each have finitely generated homology, then $M^m$ has finitely many ends. More specifically, the number of ends of $M^m$ is bounded above by $\dim H_{m-1}(M^m, \partial M^m; \mathbb{Z}_2) + 1$.

\[^1\]A discussion of the “inward tame” terminology can be found in [Gui16].
Corollary 3.3. [GG17, Cor.5.5] If $M^m$ is inward tame, then $M^m$ is peripherally locally connected at infinity.

4. Finite homotopy type and the $\sigma_\infty$-obstruction

Finitely generated projective left $\Lambda$-modules $P$ and $Q$ are stably equivalent if there exist finitely generated free $\Lambda$-modules $F_1$ and $F_2$ such that $P \oplus F_1 \cong Q \oplus F_2$. Under the operation of direct sum, the stable equivalence classes of finitely generated projective modules form a group $\tilde{K}_0(\Lambda)$, the reduced projective class group of $\Lambda$. In [Wal65], Wall associated to each path connected finitely dominated space $P$ a well-defined $\sigma(P) \in \tilde{K}_0(\mathbb{Z}[\pi_1(P)])$ which is trivial if and only if $P$ has finite homotopy type.

As one of his necessary and sufficient conditions for completability of a 1-ended inward tame open manifold $M^m$ ($m > 5$) with stable pro-$\pi_1$, Siebenmann defined the end obstruction $\sigma_\infty(M^m)$ to be (up to sign) the finiteness obstruction $\sigma(N)$ of an arbitrary clean neighborhood of infinity $N$ whose fundamental group “matches” pro-$\pi_1(\varepsilon(M^m))$.

In cases where $M^m$ is multi-ended or has non-stable pro-$\pi_1$ (or both), a more general definition of $\sigma_\infty(M^m)$, introduced in [CS76], is required. Here we inherit the definition discussed in [GG17]. For inward tame finitely dominated locally finite polyhedron $P$ (or more generally locally compact ANR), let $\{N_i\}_i$ be a nested cofinal sequence of closed polyhedral neighborhoods of infinity and define

$$\sigma_\infty(P) = (\sigma(N_1), \sigma(N_2), \sigma(N_3), \cdots) \in \lim_{\leftarrow} \left\{ \tilde{K}_0(\mathbb{Z}[\pi_1(N_j)]) \right\}$$

The bonding maps of the target inverse sequence

$$\tilde{K}_0(\mathbb{Z}[\pi_1(N_1)]) \leftarrow \tilde{K}_0(\mathbb{Z}[\pi_1(N_2)]) \leftarrow \tilde{K}_0(\mathbb{Z}[\pi_1(N_3)]) \leftarrow \cdots$$

are induced by inclusion maps, with the Sum Theorem for finiteness obstructions [Sie65, Th.6.5] assuring consistency. Clearly, $\sigma_\infty(P)$ vanishes if and only if each $N_i$ has finite homotopy type; by another application of the Sum Theorem, this happens if and only if every closed polyhedral neighborhood of infinity has finite homotopy type.

We close this section by quoting a result from [GG17, Lemma 6.1], which plays a key role in proving Theorem 0.3.

Lemma 4.1. Let $M^m$ be a manifold and $A$ a clean codimension 0 submanifold of $\partial M^m$. If $M^m$ is inward tame and $\sigma_\infty(M^m)$ vanishes, then $M^m\setminus A$ is inward tame and $\sigma_\infty(M^m\setminus A)$ also vanishes.

---

2Here $\mathbb{Z}[\pi_1(P)]$ denotes the integral group ring corresponding to the group $\pi_1(P)$. In the literature, $\tilde{K}_0(\mathbb{Z}[\pi_1(P)])$ is sometimes abbreviated to $\tilde{K}_0(G)$.

3Each $N_i$ can be non-path-connected. However, inward tameness assures that each $N_i$ has finitely many components — each finitely dominated. See [GG17] for the details about defining the functor $\tilde{K}_0$ and the finiteness obstruction in this situation.
5. Concatenation of one-sided $h$-precobordisms

Recall that an (absolute) cobordism is a triple $(W, A, B)$, where $W$ is a manifold with boundary and $A$ and $B$ are disjoint manifolds without boundary for which $A \cup B = \partial W$. The triple $(W, A, B)$ is a relative cobordism if $A$ and $B$ are disjoint codimension 0 clean submanifolds of $\partial W$. In that case, there is an associated absolute cobordism $(V, \partial A, \partial B)$ where $V = \partial W\setminus (\text{int } A \cup \text{int } B)$. We view absolute cobordisms as special cases of relative cobordisms where $V = \emptyset$. A relative cobordism $(W, A, B)$ is a one-sided $h$-cobordism if one of the pairs of inclusions $(A, \partial A) \hookrightarrow (W, V)$ or $(B, \partial B) \hookrightarrow (W, V)$ is a homotopy equivalence. A relative cobordism is nice if it is absolute or if $(V, \partial A, \partial B) \approx (\partial A \times [0, 1], \partial A \times 0, \partial A \times 1)$. Readers are referred to [RSS82] for more details of such topic.

**Remark 3.** A situation similar to a nice relative cobordism occurs when $\partial W = A \cup B'$, where $A$ and $B'$ are codimension 0 clean submanifolds of $\partial W$ with a common nonempty boundary $\partial A = \partial B'$. We call such cobordism a precobordism. By choosing a clean codimension 0 submanifold $B \subseteq B'$ with the property that $B'\setminus \text{int } B \approx \partial B \times [0, 1]$ we arrive at a nice relative cobordism $(W, A, B)$. When this procedure is applied, we will refer to $(W, A, B)$ as a corresponding nice relative cobordism. For notational consistency, we will always adjust the term $B'$ on the far right of the triple $(W, A, B')$, leaving $A$ alone. A precobordism is a one-sided $h$-precobordism if one of the pairs of inclusions $A \hookrightarrow W$ or $B' \hookrightarrow W$ is a homotopy equivalence.

The role played by one-sided $h$-precobordisms in the study of pseudo-collars is illustrated by the following easy proposition.

**Proposition 5.1.** Let $W_i$ be a disjoint union of finitely many relative one-sided $h$-cobordisms $\bigsqcup_j(W_i^j, A_i^j, B_i^j)$ with $A_i^j \hookrightarrow W_i^j$ a homotopy equivalence. Let $\bigsqcup_j A_i^j$ and $\bigsqcup_j B_i^j$ be $A_i$ and $B_i$ respectively. Suppose for each $i \geq 1$, there is a homeomorphism $h_i : B_i \rightarrow A_{i+1}$ identifying a clean codimension 0 submanifold $B_i^1 \subset B_i$ with a clean codimension 0 submanifold $A_{i+1}^1 \subset A_{i+1}$. Then the adjunction space

$$N = W_1 \cup_{h_1} W_2 \cup_{h_2} W_3 \cup_{h_3} \cdots$$

is a pseudo-collar. Conversely, every pseudo-collar may be expressed as a countable union of relative one-sided $h$-cobordisms in this manner.

**Proof.** For the forward implication, the definition of relative one-sided $h$-cobordism implies that Fr $N = A_1 \hookrightarrow W_1 \cup_{h_1} \cdots \cup_{h_{k-1}} W_k$ is a homotopy equivalence for any finite $k$. Then a direct limit argument shows that Fr $N \hookrightarrow N$ is a homotopy equivalence. Hence, $N$ is a homotopy collar. To see that $N_i$ is a pseudo-collar, we apply the same argument to the subset $N_i = W_{i+1} \cup_{h_{i+1}} W_{i+2} \cup_{h_{i+2}} W_{i+3} \cup_{h_{i+3}} \cdots$.

For the converse, assume $N$ is a pseudo-collar. Choose a homotopy collar $N_1 \subset \text{Int } N$ and let $W_1 = N \setminus \text{Int } N_1$. Then Fr $N \hookrightarrow W_1$ is a homotopy equivalence. So, $(W_1, \text{Fr } N, \text{Fr } N_1)$ is a relative one-sided $h$-cobordism. Denote a component of $N_1$ by $N_1^j$. Let $N_2$ be the disjoint union of homotopy collars in $N_1^j$ and $W_2 = N_1^j \setminus \text{Int } N_2$. Since Fr $N_1^j \hookrightarrow W_2$ is a homotopy equivalence, each $(W_2^j, \text{Fr } N_1^j, \text{Fr } N_2)$ is a relative
one-sided $h$-cobordism. Repeating the procedure concludes the argument. See Figure 1.

**Figure 1.** A concatenation of relative one-sided $h$-cobordisms.

By the cleanliness of Fr $N$ and Fr $N_i$’s, one can re-define relative one-sided $h$-cobordisms

$$(W_1, \text{Fr } N, \text{Fr } N_1), (W_2^j, \text{Fr } N_1^j, \text{Fr } N_2'), \ldots$$

as precobordisms

$$(W_1, \text{Fr } N, \text{Fr } N_1 \cup \partial_N W_1), (W_2^j, \text{Fr } N_1^j, \text{Fr } N_2' \cup \partial_{N_1^j} W_2^j), \ldots$$

Then it’s easy to see that those precobordisms are one-sided $h$-precobordisms. □

The following lemma proved by duality and standard covering space theory is crucial in this paper.

**Lemma 5.2.** Let $(W, A, B')$ be a one-sided $h$-precobordism with $A \hookrightarrow W$ a homotopy equivalence. Then the inclusion induced map

$$i_\#: \pi_1(B') \to \pi_1(W)$$

is surjective and has perfect kernel.

**Proof.** The proof is similar to the argument of Theorem 2.5 in [GT03]. Let $p : \tilde{W} \to W$ be the universal covering projection, $\tilde{A} = p^{-1}(A)$ and $\tilde{B'} = p^{-1}(B')$. By generalized Poincaré duality for non-compact manifolds [Hat02, Thm.3.35, P. 245],

$$H_k(\tilde{W}, \tilde{B'}; \mathbb{Z}) \cong H_c^{-k}(\tilde{W}, \tilde{A}; \mathbb{Z}),$$

where cohomology is with compact supports. Since $\tilde{A} \hookrightarrow \tilde{W}$ is a proper homotopy equivalence, all of these relative cohomology groups vanish, so $H_k(\tilde{W}, \tilde{B'}; \mathbb{Z}) = 0$ for
all $k$. It follows that $H_1(\tilde{W}, \tilde{B}'; \mathbb{Z})$ vanishes. Then by considering the long exact sequence for $(\tilde{W}, \tilde{B}')$, we have $H_0(\tilde{B}'; \mathbb{Z}) = \mathbb{Z}$. Thus, $\tilde{B}'$ is connected. By covering space theory, the components of $\tilde{B}'$ are 1-1 corresponding to the cosets of $i_#(\pi_1(\tilde{B}'))$ in $\pi_1(W)$. So, $i_#$ is surjective. To see the kernel of $i_#$ is perfect, we consider the long exact sequence for $(\tilde{W}, \tilde{B}')$ again. Using $H_2(\tilde{W}, \tilde{B}'; \mathbb{Z}) = 0$ together with the simple connectivity of $\tilde{W}$, $H_1(\tilde{B}'; \mathbb{Z})$ vanishes. Hence, $\pi_1(\tilde{B}')$ is perfect. By covering space theory, $\pi_1(\tilde{B}') \cong \ker i_#$ is perfect.

The following lemma plays an important role in the proof of Theorem 0.3. The proof follows easily from the Seifert-van Kampen Theorem.

**Lemma 5.3.** Let $X$ be a connected CW complex and $Y \subset X$ a connected subcomplex. Let $Y'$ be the resulting space obtained by attaching 2-cells to $Y$ along loops $\{l_i\}$ in $Y$. Then $\pi_1(Y') \cong \pi_1(Y)/N$, where $N$ is the normal closure in $\pi_1(Y)$ of $\{l_i\}$. Let $X' = X \cup Y'$. Suppose $i_# : \pi_1(Y) \to \pi_1(X)$ is the inclusion induced map. Then $\pi_1(X') \cong \pi_1(X)/N'$, where $N'$ is the normal closure in $\pi_1(X)$ of $i_#(N)$. Thus, if $N$ is perfect, so is $N'$ (since the image of a perfect group is perfect and the normal closure of a perfect group is perfect.)

**Lemma 5.4.** Let $P$ be a compact $(n-1)$-manifold with boundary and $\{A_i\}$ a finite collection of pairwise disjoint compact codimension 0 clean (and connected) submanifolds of $P$. Let $\{(W_i, A_i, B'_i)\}$ be a collection of one-sided $h$-precobordisms with $A_i \hookrightarrow W_i$ a homotopy equivalence. Assume each $W_i$ intersects $P$ along $A_i$. Let $R = P \cup (\cup_i W_i)$ and $Q = (P \setminus (\cup_i A_i)) \cup (\cup_i B'_i)$. Then $\pi_1(Q) \to \pi_1(R) \cong \pi_1(P)$ is surjective and has perfect kernel.

**Proof.** We begin with $Q = (P \setminus (\cup_i A_i)) \cup (\cup_i B'_i)$. Choose a finite collection of arcs in $P$ that connect up the $A_i$. By adding tubular neighborhoods of these arcs, we get a clean connected codimension 0 submanifold $A$ of $P$. Attaching $W_1$ along $B'_1$. See Figure 2.

By Lemma 5.2, the inclusion induced map $\lambda_1 : \pi_1(B'_1) \to \pi_1(W_1)$ is surjective and $\ker \lambda_1$ is perfect. Let $L$ be a wedge of loops in $B'_1$ which together generate $\ker \lambda_1$ and $Y'_1$ be the space obtained by attaching 2-cells to the interior $B'_1$ along these loops. Since $A_1 \hookrightarrow W_1$ is a homotopy equivalence, by Lemma 5.3

$$\pi_1(W_1) \cong \pi_1(A_1) \cong \pi_1(Y'_1) \cong \pi_1(B'_1)/N_1,$$

where $N_1 = \ker \lambda_1$ is the normal closure in $\pi_1(B'_1)$ of $L$. Note that $A_1 \cap B'_1 = \partial A_1 = \partial B'_1$. By Seifert-van Kampen,

$$\pi_1((Q\setminus B'_1) \cup A_1) \cong \pi_1(Q \cup W_1) \cong \pi_1(Q \cup Y'_1).$$

Let $i^*_1 : \pi_1(B'_1) \to \pi_1(Q)$ be the inclusion induced map. Then Lemma 5.3 implies $\pi_1(Q \cup Y'_1) \cong \pi_1(Q)/N'_1$, where $N'_1$ is the normal closure in $\pi_1(Q)$ of $i^*_1(N_1)$. Hence, $\phi_1 : \pi_1(Q) \to \pi_1(Q \cup W_1)$ is surjective and has perfect kernel.

Attaching $W_2$ along $B'_2$ in $Q \cup W_1$. Repeat the above argument, one can show that $\phi_2 : \pi_1(Q \cup W_1) \to \pi_1(Q \cup W_1 \cup W_2)$ is surjective and has perfect kernel. Assume
there are $k A_i$’s. By induction, we have the following sequence
\[
\pi_1(Q) \xrightarrow{\phi_1} \pi_1(Q \cup W_1) \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} \pi_1(Q \cup (\bigcup_{i=1}^k W_i))
\]
Since each $\ker \phi_i$ is perfect, by Lemma 5.2, the composition $\Phi = \phi_k \circ \cdots \circ \phi_2 \circ \phi_1$ yields a desired surjection $\pi_1(Q) \twoheadrightarrow \pi_1(R) \cong \pi_1(P)$ and $\ker \Phi$ is perfect.

\[\square\]

6. Proof of Theorem 0.3: necessity
The proof of the necessity of Conditions (a) and (c) of Theorem 0.3 follow readily by definition of pseudo-collar. Thus, it suffices to show that pseudo-collarability implies Condition (b).

Proof of Theorem 0.3 (necessity). Suppose $M^m$ is pseudo-collarable and $N$ is a homotopy collar. Then it’s easy to see that each component $N^j$ of $N$ is a homotopy collar. By the definition of pseudo-collarability, we choose a desired cofinal sequence of clean neighborhoods of infinity $\{N_i^l\}_{i=1}^{k_i}$ such that each $N_i^l$ is a homotopy collar contained in $N^j$. Proposition 3.2 guarantees that each $N_i^l \setminus \partial M^m$ is 1-ended — thus, each $N_i^l$ is $\partial_M N_i^l$-connected at infinity. Let $N_{i,s} = N_i^l \cap (\bigcup_{i+1}^k N_{i+1}^l)$ $(s = 1, 2, \ldots)$ is the disjoint union of finitely many components $N_{i,s}^l$ contained in $N_i^l$. By Proposition 5.1, $N^j (= N_1^l)$ can be subdivided into relative one-sided $h$-cobordisms. That is, each $W_i^l = N_i^l \setminus N_{i+1}^l$. By definition, we may consider the sequence
\[
\pi_1(\partial_M N_1^l \cup N_{1,2}^l) \leftarrow \pi_1(\partial_M N_1^l \cup N_{1,3}^l) \leftarrow \pi_1(\partial_M N_1^l \cup N_{1,4}^l) \leftarrow \cdots
\]
where base rays are suppressed and bonding homomorphisms are compositions of maps induced by inclusions and change-of-basepoint isomorphisms. Let $\partial_M N_i^l \setminus \partial_M N_{i,i+1}^l$
be \( D'_{i,i+1} \) \((i = 1, 2, 3, \ldots)\) and \( D'_{i,i+2} = D'_{i,i+1} \cup D'_{i+1,i+2} \). Consider the following diagram. Each bonding map in the top row is an inclusion.

\[
\begin{array}{c}
\partial M N_1 \cup N_1^1 & \leftarrow & \partial M N_1^1 \cup N_1^1 \rightarrow & \partial M N_1^1 \cup N_1^4 \rightarrow & \ldots \\
\uparrow \text{incl.} & & \uparrow \text{incl.} & & \uparrow \text{incl.} \\
D_{1,2} \cup \text{Fr} N_1^1 & & D_{1,3} \cup \text{Fr} N_1^1 & & D_{1,4} \cup \text{Fr} N_1^1 & & \ldots
\end{array}
\]

Since each \( \text{Fr} N_i^1 \hookrightarrow N_i^1 \) is a homotopy equivalence, all the vertical maps are homotopy equivalence. By \( \partial_3 \) in the proof of Proposition 5.1, \((W'_i, \text{Fr} N_i^1, \text{Fr} N_{i,i+1}^i \cup \partial M W_i')\) is a one-sided \( h \)-precobordism. Apply Lemma 5.4,

\[
\pi_1(D_{1,i+2}^1 \cup \text{Fr} N_{1,i+2}^1) \twoheadrightarrow \pi_1(D_{1,i+1}^1 \cup \text{Fr} N_{1,i+1}^1)
\]

is surjective and has perfect kernel.


\[\square\]

7. Proof of Theorem 0.3: sufficiency

We begin the proof of the “sufficiency argument” with three theorems that will be key ingredients in the proof. Each is a straightforward extension of an established result from the literature.

The following theorem is a modest generalization of the Pseudo-collarability Characterization Theorem in [GT06] to some manifolds with noncompact boundary in the same way the Siebenmann’s “Relativized Main Theorem 10.1” provided a mild extension of the Main Theorem of [Sie65] to some manifolds with noncompact boundary.

**Theorem 7.1** (Relativized Pseudo-collarability Characterization Theorem). Suppose \( M^m \) \((m \geq 6)\) is one-ended and \( \partial M^m \) is homeomorphic to the interior of a compact manifold. Then \( M^m \) is pseudo-collarable iff \( M^m \) is

1. inward tame,
2. \( \pi_1(\varepsilon(M^m)) \) is perfectly semistable,
3. \( \sigma_\infty(M^m) = 0. \)

Quillen’s famous “plus construction” [Qui71] or [FQ90, Section 11.1] provides a partial converse to Lemma 5.2.

**Theorem 7.2** (The Relativized Plus Construction). Let \( B \) be a compact \((n - 1)\)-manifold \((n \geq 6)\) and \( h : \pi_1(B) \twoheadrightarrow H \) a surjective homomorphism onto a finitely presented group such that ker(h) is perfect. There exists a compact \( n \)-dimensional nice relative cobordism \((W, A, B)\) such that ker(\( \pi_1(B) \rightarrow \pi_1(W) \)) = ker \( h \), and \( A \hookrightarrow W \) is a simple homotopy equivalence. These properties determine \( W \) uniquely up to homeomorphism rel \( B \).

**Remark 4.** For \( n = 5 \), the above theorem still holds as long as \( H \) is restricted to be “good” (see [FQ90, Th. 11.1A, P.195]). For \( n \geq 6 \), the proof is the same as the proof of Th. 11.1A in [FQ90, P.195] except that 2-spheres on which the 3-handles are attached embedded simply by general position. When \( n = 4 \), the theorem is false.
When a nice rel one-sided $h$-cobordism has trivial Whitehead torsion, i.e., when the corresponding homotopy equivalence is simple, we refer to it as a nice rel plus cobordism.

**Theorem 7.3** (Relativized Embedded Plus Construction). Let $R$ be a connected manifold of dimension at least 6; $B$ a compact codimension 0 submanifold of $\partial R$; and $G \subseteq \ker(\pi_1(B) \to \pi_1(R))$ a perfect group which is the normal closure in $\pi_1(B)$ of a finite set of elements. Then there exists a nice rel plus cobordism $(W, A, B)$ embedded in $R$ which is the identity on $B$ for which $\ker(\pi_1(B) \to \pi_1(W)) = G$.

**Proof.** The proof of Theorem 3.2 in [GT06] will work for our situation with simple replacement of plus construction by the relativized plus construction and duality by generalized Poincaré duality [Hat02, Thm. 3.35, P. 245] for noncompact manifolds.

**Proof of Theorem 7.1.** For a full understanding, the reader should be familiar with the proof of the Main Existence Theorem [Gui00]. To generalize all the arguments made in [Gui00], especially Theorem 5, Lemmas 13-15, one need use frontiers $\text{Fr}$ of generalized $k$-neighborhoods to replace boundaries $\partial$. All handle operations should be performed away from $\partial M$. This is doable for nearly the same reasons given by Siebenmann for [Sie65, Th. 10.1]; in particular, all handle moves in the proof [GT06, Th. 1.1] can be performed away from $\partial M$. More specifically, the above procedure will assure the end has generalized $(n-3)$-neighborhoods $\{U_i\}$. To modify $\{U_i\}$ to generalized $(n-2)$-neighborhoods, one has to replace Theorem 3.2 in [GT06, P. 554] by Theorem 7.3. Then imitate the argument in [GT06, P. 554-555] via replacing $\partial$ by $\text{Fr}$ and keeping the handle decompositions away from $\partial M$.

The proof of the sufficiency of Theorem 0.3 follows readily from the following result.

**Proposition 7.4.** If $M^m$ satisfies Conditions (a) - (c) of Theorem 0.3 then there exists a clean compact exhaustion $\{C_i\}$ so that, for the corresponding neighborhoods of infinity $\{N_i\}$, $\text{Fr} N_i \to N_i$ is a homotopy equivalence.

**Proof.** The proof is a variation on the argument of Proposition 10.2 in [GG17]. By Lemma 4.1 and the definition of peripheral perfect semistability at infinity, we can begin with a clean compact exhaustion $\{C_i\}^\infty$ of $M^m$ and a corresponding sequence of neighborhoods of infinity $\{N_i\}^\infty$, each with a finite set of connected components $\{N^j_i\}^k_i$, so that for all $i \geq 1$ and $1 \leq j \leq k_i$,

i) $N^j_i$ is inward tame,

ii) $N^j_i$ is $(\partial M N^j_i)$-connected and $(\partial M N^j_i)$-perfectly-semistable at infinity, and

iii) $\sigma_\infty(N^j_i) = 0$.

By Lemmas 4.1 and 2.3 we have

i’) $N^j_i \setminus \partial M^m$ is inward tame,

ii’) $N^j_i \setminus \partial M^m$ is 1-ended and has perfectly semistable fundamental group at infinity, and

...
iii') \[ \sigma_\infty(N^j_i \setminus \partial M^m) = 0. \]

These are precisely the hypotheses of Theorem 7.1. That means \( N^j_i \setminus \partial M^m \) contains a homotopy collar neighborhood of infinity \( V^j_i \), i.e., \( \partial V^j_i \to V^j_i \) is a homotopy equivalence. Following the proof of Theorem 7.1 one can further arrange \( \partial N^j_i \setminus \partial M^m \) (\( = \text{int}(\text{Fr} N^j_i) \)) and \( \partial V^j_i \) contain clean compact codimension 0 submanifolds \( A^j_i \) and \( B^j_i \), respectively, so that \( \text{int}(\text{Fr} N^j_i) \setminus \text{int} A^j_i = \partial V^j_i \setminus \text{int} B^j_i \approx \partial A^j_i \times [0, 1) \). See Figure 3.

![Figure 3. \( V^j_i \) is a homotopy collar.](image)

Note that \( K^j_i = N^j_i \setminus V^j_i \) is a clean codimension 0 submanifold of \( M^m \) intersecting \( C_i \) in \( A^j_i \). To save on notation, we replace \( C_i \) with \( C_i \cup (\cup K^j_i) \), which is still a clean compact codimension 0 submanifold of \( M^m \), but with the additional property that

\[
\text{int}(\text{Fr} N_i) \hookrightarrow N_i \setminus \partial M^m \text{ is a homotopy equivalence.}
\]

Since adding \( \partial_M N_i \) back in does not affect homotopy types, we have

\[
\text{Fr} N_i \hookrightarrow N_i \text{ is a homotopy equivalence.}
\]

Having enlarged the \( C_i \), if necessary, one can easily retain the property that \( C_i \subseteq \text{Int} C_{i+1} \) for all \( i \) by passing to a subsequence. Then \( N_i = M^m \setminus C_i \) gives a desired nested cofinal sequence of clean neighborhoods of infinity \( \{N_i\} \) with the property that each inclusion \( \text{Fr} N_i \hookrightarrow N_i \) is a homotopy equivalence, i.e., \( M^m \) is pseudo-collarable. □

8. Questions

The idea of pseudo-collarability is related to a term named \( Z \)-compactification. The motivation first came from the modification of manifold completion applied to Hilbert cube manifolds in [CS76]. A compactification \( \tilde{X} = X \sqcup Z \) of a space \( X \) is a \( Z \)-compactification if, for every open set \( U \subseteq \tilde{X} \), \( U \setminus Z \hookrightarrow U \) is a homotopy equivalence. This compactification has been proven to be useful in both geometric group theory
and manifold topology, for example, in attacks on the Borel and Novikov Conjectures. A major open problem is a characterization of $\mathbb{Z}$-compactifiable manifolds [CS76] [GT03] [GG17].

**Question 1.** Are Conditions (1), (3) and (4) of Theorem 0.1 sufficient for manifolds to be $\mathbb{Z}$-compactifiable?

Although it’s still not well-understood whether these conditions are sufficient, in [GG17], Guilbault and the author provided a best possible result.

**Theorem 8.1.** An $m$-manifold $M^m$ ($m \geq 5$) satisfies Conditions (1), (3) and (4) of Theorem 0.1 if and only if $M^m \times [0, 1]$ admits a $\mathbb{Z}$-compactification.

**Remark 5.** Conditions (1), (3) and (4) are precisely the conditions that characterize $\mathbb{Z}$-compactifiable Hilbert cube manifolds [CS76]. The early version of Question 1 was posed more generally in [CS76] for locally ANR’s, but in [Gui01] a counterexample was constructed.

Obviously, completable manifolds are both pseudo-collarable and $\mathbb{Z}$-compactifiable. Despite the fact that many manifolds such as Davis’ manifolds are both pseudo-collarable and $\mathbb{Z}$-compactifiable but not completable, the relationship between pseudo-collarable manifolds and $\mathbb{Z}$-compactifiable manifolds are not well-understood. There are several interesting questions around such topic.

**Question 2.** Are pseudo-collarability and Condition (4) of Theorem 0.1 sufficient for manifolds to be $\mathbb{Z}$-compactifiable?

**Question 3.** Are $\mathbb{Z}$-compactifiable manifolds pseudo-collarable?

We suspect the answer to Question 3 is negative. Crossing manifolds constructed in [KM62], [Ste77] and [Gu18] with half-open interval $[0, 1)$ might be potential counterexamples. However, the biggest obstacle is closely related to the following question in knot theory.

**Question 4.** Let $K$ be a trefoil knot and $WD(K)$ be a twisted Whitehead double of $K$. Is the knot group of $WD(K)$ hypoabelian?

**Definition 8.1.** A group $G$ is said to be hypoabelian if the following equivalent conditions are satisfied:

(1) $G$ contains no nontrivial perfect subgroup

(2) the transfinite derived series terminates at the identity. (Note that this is the transfinite derived series, where the successor of a given subgroup is its commutator subgroup and subgroups at limit ordinals are given by intersecting all previous subgroups.)

**Question 5.** Can a $\mathbb{Z}$-compactifiable open $n$-manifold fail to be pseudo-collarable?
Acknowledgements

The work presented here is part of the author’s PhD dissertation at University of Wisconsin-Milwaukee. I would like to express my sincere gratitude to my thesis advisor, Craig Guilbault, for his guidance, enthusiasm and encouragement in course of this work.

REFERENCES

[CS76] T. A. Chapman and L. C. Siebenmann, Finding a boundary for a Hilbert cube manifold, Acta Math. 137 (1976), no. 3-4, 171–208. MR 0425973

[Dav83] M. W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. Math. 117 (1983) 293–325.

[FQ90] Michael H. Freedman and Frank Quinn, Topology of 4-manifolds, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990. MR 1201584

[Geo08] R. Geoghegan, Topological methods in group theory, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008. MR 2365352

[GG17] Shijie Gu and Craig R. Guilbault, Compactifications of manifolds with boundary, preprint. arXiv: 1712.05995

[GT03] C. R. Guilbault and F. C. Tinsley, Manifolds with non-stable fundamental groups at infinity. II, Geom. Topol. 7 (2003), 255–286. MR 1988286

[GT06] ———, Manifolds with non-stable fundamental groups at infinity. III, Geom. Topol. 10 (2006), 541–556. MR 2224464

[Gu18] Shijie Gu, Contractible open manifolds which embed in no compact, locally connected and locally 1-connected metric space, in progress.

[Gui00] Craig R. Guilbault, Manifolds with non-stable fundamental groups at infinity, Geom. Topol. 4 (2000), 537–579. MR 1800296

[Gui01] Craig R. Guilbault, A non-$Z$-compactifiable polyhedron whose product with the Hilbert cube is $Z$-compactifiable, Fund. Math. 168 (2001), no. 2, 165–197. MR 1852740

[Gui16] ———, Ends, shapes, and boundaries in manifold topology and geometric group theory, Topology and geometric group theory, Springer Proc. Math. Stat., vol. 184, Springer, [Cham], 2016, pp. 45–125. MR 3598162

[Hat02] Allen Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. MR 1867354

[KM62] J. M. Kister and D. R. McMillan, Jr., Locally Euclidean factors of $E^4$ which cannot be embedded in $E^3$, Ann. of Math. 76 (1962), 541–546.

[O’B83] Gary O’Brien, The missing boundary problem for smooth manifolds of dimension greater than or equal to six, Topology Appl. 16 (1983), no. 3, 303–324. MR 722123

[Qui71] D. Quillen, Cohomology of groups, from: “Actes du Congrès International des Mathematiciens (Nice, 1970), Tome 2”, Gauthier-Villars, Paris (1971) 47–51. MR 0488054

[RS82] Colin Patrick Rourke and Brian Joseph Sanderson, Introduction to piecewise-linear topology, Springer Study Edition, Springer-Verlag, Berlin-New York, 1982, Reprint. MR 665919

[Sie65] Laurence Carl Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, ProQuest LLC, Ann Arbor, MI, 1965, Thesis (Ph.D.)Princeton University. MR 2615648

[Ste77] Robert William Sternfeld, A contractible open n-manifold that embeds in no compact n-manifold, ProQuest LLC, Ann Arbor, MI, 1977, Thesis (Ph.D.)-The University of Wisconsin - Madison. MR 2627427

[Wal65] C. T. C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. (2) 81 (1965), 56–69. MR 0171284
Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53211

E-mail address: shijiegu@uwm.edu