TRACE OF ABELIAN VARIETIES OVER FUNCTION FIELDS AND THE GEOMETRIC BOGOMOLOV CONJECTURE

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ABSTRACT. We prove that the geometric Bogomolov conjecture for any abelian varieties is reduced to that for nowhere degenerate abelian varieties with trivial trace. In particular, the geometric Bogomolov conjecture holds for abelian varieties whose maximal nowhere degenerate abelian subvariety is isogenous to a constant abelian variety. To prove the results, we investigate closed subvarieties of abelian schemes over constant varieties, where constant varieties are varieties over a function field which can be defined over the constant field of the function field.

1. INTRODUCTION

In this paper, we contribute to the geometric Bogomolov conjecture for abelian varieties by investigating closed subvarieties of abelian schemes over constant varieties, where constant varieties are varieties over a function field which can be defined over the constant field of the function field.

1.1. Background. The geometric Bogomolov conjecture for abelian varieties is an analogue of the theorem established by Ullmo [17] and Zhang [25], called the (arithmetic) Bogomolov conjecture. To describe this theorem, let \( K \) be a number field. Let \( A \) be an abelian variety over \( K \). A line bundle on \( A \) is said to be even if it is preserved under the pull-back by the inverse \( a \mapsto -a \) of \( A \). To an even ample line bundle on \( A \), one associates the canonical height function, also called the Néron–Tate height, which is known to be a semi-positive definite quadratic form on the additive group \( A(K) \). For an \( x \in A(K) \), the value at \( x \) of the canonical height function is called the (canonical) height of \( x \). We say that a closed subvariety \( X \) of \( A \) has dense small points if, for any \( \epsilon > 0 \), the set of points of \( X(K) \) whose canonical heights are not greater than \( \epsilon \) is dense in \( X \). It is known that the density of small points does not depend on the choice of even ample line bundles on \( A \). Since the canonical height function is a quadratic form, it follows that a torsion subvariety has dense small points, where a torsion subvariety is the translate of an abelian subvariety by a torsion point. The (arithmetic) Bogomolov conjecture for \( A \), which is the theorem of Ullmo and Zhang, claims that the converse also holds, that is, any irreducible closed subvariety \( X \) of \( A \) with dense small points is a torsion subvariety.

The theorem of Ullmo and Zhang is generalized by Moriwaki in [15] to the case where \( K \) is a field finitely generated over \( \mathbb{Q} \). He considered the arithmetic heights associated to “big” polarizations (when the transcendence degree of \( K \) over \( \mathbb{Q} \) is positive) and established the Bogomolov conjecture with respect to the canonical height arising from a big polarization.
We remark that this kind of heights are different from the classical “geometric” heights over function fields; they are more arithmetic.

It is then quite natural to ask whether there is an analogue of the theorem of Ullmo and Zhang over a function field with respect to the classical geometric height. Now, let \( K \) be the function field of a (normal projective) variety over an algebraically closed field \( k \). In \([9]\), Gubler has proved that the same statement as the theorem holds for abelian varieties over \( \overline{K} \) which are totally degenerate at some place. However, this statement does not hold for any abelian variety in fact. For example, one sees that, if \( A \) is a constant abelian variety, that is, \( A = \tilde{A} \otimes_k \overline{K} \) for some abelian variety \( \tilde{A} \) over \( k \), then, for any closed subvariety \( \tilde{X} \) of \( \tilde{A} \), the subvariety \( X := \tilde{X} \otimes_k \overline{K} \) of \( A \) has dense small points. More generally, for any abelian variety \( A \) over \( \overline{K} \), let \( \big( \overline{A^{\text{K/k}}}, \text{Tr}_A \big) \) be the \( \overline{K}/k \)-trace of \( A \) (cf. §1.2). Then, one sees that, for any closed subvariety \( \tilde{Y} \) of \( \overline{A^{\text{K/k}}} \), the closed subvariety \( \text{Tr}_A \big( \tilde{Y} \otimes_k \overline{K} \big) \) of \( A \) has dense small points. This means that, over function fields, an abelian variety may in general have an irreducible closed subvariety that is not a torsion subvariety but has dense small points. It should be remarked that, in the case of Gubler, it follows from the assumption of total degeneracy that \( A \) has trivial \( \overline{K}/k \)-trace. Therefore, there are no such subvarieties, and actually, he succeeded in establishing the same statement for such abelian varieties.

Inspired by the work due to Gubler, we formulate in \([21]\) a Bogomolov conjecture for any abelian variety over a function field generalizing Gubler’s theorem, which we call the geometric Bogomolov conjecture (for abelian varieties) (cf. Conjecture 1.1). This conjecture claims that any irreducible closed subvariety with dense small points is a special subvariety, which has been defined as the sum of a torsion subvariety and a closed subvariety coming from the constant subvarieties of the \( \overline{K}/k \)-trace. In \([21]\), we established a partial answer to the conjecture, and this result generalizes the theorem of the totally degenerate case by Gubler in fact. Further development can be found in \([22]\), and more details of it are resumed in §1.3. We remark that the geometric Bogomolov conjecture is still an open problem.

We put a few comments on another version of the conjecture, the geometric Bogomolov conjecture for curves. It is a restricted version of the conjecture for abelian varieties, claiming that any projective curve of genus more than 1 embedded in its jacobian variety does not have dense small points. When \( \text{char}(k) = 0 \) and \( K \) has transcendence degree 1 over \( k \), this is proved by Cinkir in \([2]\). In positive characteristics, although one can find some partial answers in \([23, 13, 14, 19, 20]\), it has not yet been solved.

1.2. Notation and convention. Let \( k \) be an algebraically closed field, let \( \mathcal{B} \) be an irreducible normal projective variety of dimension \( b \geq 1 \) over \( k \), and let \( \mathcal{H} \) be an ample line bundle on \( \mathcal{B} \). Let \( K \) be the function field of \( \mathcal{B} \) and let \( \overline{K} \) be an algebraic closure of \( K \). All of them are fixed throughout this article. Any finite extension of \( K \) will be taken in \( \overline{K} \).

Let \( M_K \) be the set of points of \( \mathcal{B} \) of codimension 1. Each element of \( M_K \) is called a place of \( K \). For any \( v \in M_K \), the local ring \( \mathcal{O}_{\mathcal{B},v} \) is a discrete valuation ring with fractional field \( K \), and we let \( \text{ord}_v : K^\times \to \mathbb{Z} \) denote the order function. This gives rise to a non-archimedean value \( | \cdot |_{v,\mathcal{H}} \) on \( \overline{K} \) normalized in such a way that

\[
|x|_{v,\mathcal{H}} := e^{-\text{deg}_\mathcal{H}(\overline{v})\text{ord}_v(x)}
\]

In fact, its effective version is proved here.
for any \( x \in K^X \), where \( \deg_H(\mathcal{V}) \) denotes the degree with respect to \( \mathcal{H} \) of the closure \( \mathcal{V} \) of \( v \) in \( \mathfrak{B} \). It is well known that the set \( \{ \cdot | v, \mathcal{H} \} \in M_K \) of valuations satisfies the product formula, and hence the notion of (absolute logarithmic) heights with respect to this set of valuations is defined (cf. [12, Chapter 3 §3]). A place \( v \in M_K \) is identified with the corresponding valuation of \( \overline{K} \). We denote by \( K_v \) the completion of \( K \) with respect to \( v \).

Let \( F/k \) be a field extension and let \( X \) be a scheme over \( F \). For a field extension \( \mathfrak{g}/F \), we write \( X \otimes_F \mathfrak{g} := X \times_{Spec(F)} Spec(\mathfrak{g}) \). For a morphism \( \phi : X \to Y \) of schemes over \( F \), we write \( \phi \otimes_F \mathfrak{g} : X \otimes_F \mathfrak{g} \to Y \otimes_F \mathfrak{g} \) for the base-extension to \( \mathfrak{g} \). We call \( X \) a variety over \( F \) if \( X \) is a geometrically reduced algebraic scheme over \( F \).

A variety \( X \) over \( \overline{K} \) is called a constant variety if there exists a variety \( \tilde{X} \) over \( k \) with \( X = \tilde{X} \otimes_k \overline{K} \). Further, a subscheme \( Y \) of \( X \) is called a constant subscheme if \( Y = \tilde{Y} \otimes_k \overline{K} \) for some subscheme \( \tilde{Y} \) of \( \tilde{X} \). An abelian variety \( A \) over \( \overline{K} \) is called a constant abelian variety if there exists an abelian variety \( \tilde{A} \) over \( k \) with \( A = \tilde{A} \otimes_k \overline{K} \) as abelian varieties. Note that the group scheme structure of \( A \) is required to be defined over \( k \).

Let \( A \) be an abelian variety over \( \overline{K} \). It is well known that there exists a unique pair \((\overline{A}/k, \text{Tr}_A)\) consisting of an abelian variety \( \overline{A}/k \) over \( k \) and a homomorphism \( \text{Tr}_A : \overline{A}/k \otimes_{k} \overline{K} \to A \) of abelian varieties over \( \overline{K} \) characterized by the property that, for any abelian variety \( \tilde{B} \) over \( k \) and a homomorphism \( \phi : \tilde{B} \otimes_k \overline{K} \to A \), there exists a unique homomorphism \( \text{Tr}(\phi) : \tilde{B} \to \overline{A}/k \) such that \( \phi \) factors as \( \phi = \text{Tr}_A \circ (\text{Tr}(\phi) \otimes_k \overline{K}) \). This pair is called the \( \overline{K}/k \)-trace, or simply the trace, of \( A \). We sometimes call \( \overline{A}/k \) the \( \overline{K}/k \)-trace by abuse of words. See [11] for more details.

1.3. Geometric Bogomolov conjecture and known results. We review the geometric Bogomolov conjecture for abelian varieties and some known results. We begin by recalling the special subvarieties, introduced in [21] and used to formulate the geometric Bogomolov conjecture for abelian varieties. They are defined as irreducible closed subvarieties which are expressed as the sum of the image of a constant closed subvarieties in the trace and a torsion subvariety. To be precise, let \( A \) be an abelian variety over \( \overline{K} \) and let \((\overline{A}/k, \text{Tr}_A)\) be the trace of \( A \). An irreducible closed subvariety \( X \) of \( A \) is called a special subvariety if there exist a closed subvariety \( \tilde{Y} \) of \( \overline{A}/k \) and a torsion subvariety \( T \subset A \) such that \( X = T + \text{Tr}_A (\tilde{Y} \otimes_k \overline{K}) \).

In stating the conjecture, it is convenient to use the words of “density of small points”, which we are going to explain. Let \( L \) be an even ample line bundle on \( A \). Here \( L \) is said to be even if \([-1]^*(L) = L \), where \([-1] : A \to A \) is the homomorphism \( a \mapsto -a \). Let \( \hat{h}_L \) be the canonical height function over \( A(\overline{K}) \) associated to \( L \). It is a semi-positive quadratic form on the group \( A(\overline{K}) \). We set, for \( \epsilon > 0 \),

\[
X(\epsilon; L) := \left\{ x \in A(\overline{K}) \mid \hat{h}_L(x) \leq \epsilon \right\}.
\]

It follows from [21, Lemma 2.1] that whether or not \( X(\epsilon; L) \) is dense in \( X \) for any \( \epsilon > 0 \) does not depend on the choice of even ample \( L \). Therefore, it makes sense to say that \( X \) has

\footnote{In [16], there is a similar but different notion of special subvarieties due to Scanlon (cf. [22, Remark 7.3]).}
dense small points if $X(\epsilon; L)$ is dense in $X$ for any $\epsilon > 0$ and for some (and hence any) even ample line bundle $L$ on $A$ (cf. [21, Definition 2.2]).

A point $x \in A(K)$ is called a special point if $\{x\}$ is a special subvariety. It is classically known that a point is special if and only if it has height 0 (cf. [21, (2.5.4)]). This means that, for an irreducible subvariety of dimension 0, being special is the same thing as having dense small points. In the case of positive dimension also, it is verified that any special subvariety has dense small points (cf. [21, Corollary 2.8]). However, it is not known whether the converse holds true or not in general. The geometric Bogomolov conjecture for abelian varieties is stating the converse:

**Conjecture 1.1** (Geometric Bogomolov conjecture for abelian varieties). Let $A$ be an abelian variety over $\overline{K}$. Then any irreducible closed subvariety with dense small points should be a special subvariety.

Although Conjecture 1.1 is not solved in full generality, there are some results proved under some assumptions. In [9], Gubler proves that, if $A$ is totally degenerate at some place $v \in M_K$, then the conjecture holds true for $A$. Here, $A$ is said to be totally degenerate at $v$ if the base-change of $A$ to $\overline{K}_v$ can be uniformized by an algebraic torus in the category of non-archimedean analytic spaces. In [21], we generalize this result for abelian varieties allowing milder degenerations. In [22], we moreover prove that the conjecture holds for abelian varieties with “nowhere degeneracy rank” at most 1. (See Theorem 1.2 below.)

Let us recall here the notions of maximal nowhere degenerate abelian subvariety of an abelian variety and the nowhere degeneracy rank, defined in [22, Definition 7.10]. An abelian variety $A$ over $\overline{K}$ is said to be non-degenerate at a point $x \in M_K$ if $A \otimes \overline{K}_v$ is the generic fiber of an abelian scheme over the ring of integers of $\overline{K}_v$. We say that $A$ is nowhere degenerate if it is non-degenerate at any $v \in M_K$. We note the fact that there exists a unique maximal abelian subvariety $m$ of $A$ such that $m$ is nowhere degenerate, where “maximal” means “maximal with respect to inclusion”. This unique abelian subvariety is called the maximal nowhere degenerate abelian subvariety of $A$. Further, the nowhere-degeneracy rank of $A$ is defined to be $\text{nd-rk}(A) = \dim(m)$.

With the use of nowhere degeneracy rank, one of the main results of [22] is stated as follows.

**Theorem 1.2** (Theorem D in [22]). Let $A$ be an abelian variety over $\overline{K}$ with $\text{nd-rk}(A) \leq 1$. Then, any irreducible closed subvariety of $A$ with dense small points is a special subvariety.

In [22], we also pointed out that the conjecture for an abelian variety is equivalent to the conjecture for its maximal nowhere degenerate abelian subvariety:

**Theorem 1.3** (Theorem E in [22]). Let $A$ be an abelian variety over $\overline{K}$ with maximal nowhere degenerate abelian subvariety $m$. Then the geometric Bogomolov conjecture holds for $A$ if and only if it holds for $m$.

Since the geometric Bogomolov conjecture holds for elliptic curves, Theorem 1.2 follows from Theorem 1.3. By Theorem 1.3, the conjecture for any abelian variety is reduced to that for nowhere degenerate abelian varieties (cf. [22, Conjecture 7.22]).

1.4. Results and ideas. This paper includes two main results. One is the following theorem.
Theorem 1.4 (Theorem 3.5). Let $A$ be an abelian variety over $\overline{K}$ such that $\dim \left( \tilde{A}^{K/k} \right) = \text{nd-rk}(A)$. Then the geometric Bogomolov conjecture holds for $A$.

Theorem 1.4 generalizes Theorem 1.2 (see Remark 3.6 how it generalize Theorem 1.2). We notice that the proof of Theorem 1.4 uses Theorem 1.3.

The other theorem is the following, where note that $\text{Image}(\text{Tr}_A) \subset m$ by [22, Proposition 7.11].

Theorem 1.5 (Theorem 5.5). Let $A$ be an abelian variety over $K$ with maximal nowhere degenerate abelian subvariety $m$ and let $t := \text{Image}(\text{Tr}_A)$ be the image of the $\overline{K}/k$-trace homomorphism of $A$. Then the following statements are equivalent to each other.

(a) The geometric Bogomolov conjecture holds for $A$.
(b) The geometric Bogomolov conjecture holds for $m$.
(c) The geometric Bogomolov conjecture holds for $m/t$.

This theorem includes Theorem 1.3 and the new part is the equivalence between (c) and the others. Since (b) implies (c) by [22, Lemma 7.7], the essential part is that (c) implies (b), in fact. We remark that Theorem 1.3 leads us to Theorem 1.4 (See Remark 5.6).

Theorem 1.5 is interesting because it shows that Conjecture 1.1 can be reduced to the geometric Bogomolov conjecture for a quite special class of abelian varieties, that is, for nowhere degenerate abelian varieties with trivial trace (cf. Remark 5.7 and Conjecture 5.8).

We briefly outline the idea of the proofs. Let $A$ be an abelian variety over $\overline{K}$. The starting point of the proofs is the fact proved in [9] that a closed subvariety of $A$ has dense small points if and only if it has canonical height 0 (cf. Proposition 2.3). Using this fact together with the description of the canonical height in terms of intersection theory over a model (cf. Proposition 2.4 and Lemma 2.6), we show Proposition 3.2. This claims that, when $A$ is a constant abelian variety, a closed subvariety of canonical height 0 is a constant subvariety and hence a special subvariety. Then Theorem 1.4 follows from this proposition and Theorem 1.3.

The proof of Theorem 1.5 needs more arguments, which we are now going to explain. Since we may replace $A$ with an isogenous abelian variety in proving the geometric Bogomolov conjecture (cf. [22, Corollary 7.6]), we find that Theorem 1.5 is reduced to Theorem 5.3 which claims the following. Let $A$ be a nowhere degenerate abelian variety over $\overline{K}$ with trivial trace and let $B$ be a constant abelian variety. Let $X$ be a closed subvariety of $B \times A$ and let $Y$ and $T$ be the projections of $X$ to $B$ and $A$, respectively. Suppose that $X$ has dense small points and assume that the geometric Bogomolov conjecture holds for $A$. Then, $Y$ is a constant subvariety, $T$ is a torsion subvariety, and $X = Y \times T$.

We give the outline of the proof of Theorem 5.3. Let $A, B, X, Y$ and $T$ be as above. Suppose that $X$ has dense small points. Then $Y$ has dense small points by [22, Lemma 7.7] and hence has canonical height 0 by Proposition 2.3. Therefore by Proposition 3.2 $Y = \tilde{Y} \otimes_k \overline{K}$ for some variety $\tilde{Y}$ over $k$. It remains to show that $T$ is torsion and $X = Y \times T$. To do that, the “relative height” of $X \to Y$, which will be introduced in §1.2 plays a crucial role. It is a function which assigns to each point of a dense open subset of $Y$ the canonical height of the fiber of $X \to Y$ over the corresponding point of $Y$. In the proof of Theorem 5.3 we show in fact that, if $X$ has dense small points, then the relative height function vanishes.
over a dense open subset of \( \widetilde{Y} \). Then the geometric Bogomolov conjecture for \( A \) implies that the geometric generic fiber of \( X \rightarrow Y \) is a torsion subvariety, and arguments using Chow’s theorem (cf. Proposition 3.7) allow us to conclude that \( T \) is a torsion subvariety and \( X = Y \times T \).

1.5. Organization. This article consists of six sections including this section and an appendix. In \( \S 2 \) we recall canonical heights, and describe them in terms of intersection theory over a model when the abelian variety is nowhere degenerate. Using the results there, we show Theorem 1.4 in \( \S 3 \). In \( \S 4 \) we investigate families of closed subvarieties of an abelian variety parameterized by a constant variety, and we introduce the relative heights. In \( \S 5 \) applying the arguments in \( \S 4 \) to the setting of the geometric Bogomolov conjecture, we prove Theorem 1.5. In the appendix, we prove a lemma concerning the trace of an abelian variety and the base-change, which is used in \( \S 4 \).

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2. Canonical heights over nowhere degenerate abelian varieties

The purpose of this section is to describe the canonical height of closed subvarieties of nowhere degenerate abelian varieties in terms of intersection theory on models.

2.1. Canonical heights. We briefly recall properties of canonical heights of closed subvarieties and cycles of abelian varieties. We refer to \([3, 9, 10]\) for more details.

Let \( L \) be a line bundle on a variety \( W \) over \( \overline{K} \). A metric on \( L \) at \( v \) means a collection of \( \overline{K}_v \)-norms \( L(w) \rightarrow \mathbb{R} \) for all \( w \in W(\overline{K}_v) \), where \( L(w) := \omega^*(L) \) is the fiber of \( L \) at \( w \). A metric on \( L \) is a family \( || \cdot || = \{ || \cdot ||_v \}_{v \in M_K} \) of metrics on \( L \) at \( v \) for all places \( v \in M_K \). A line bundle \( L \) with a metric \( || \cdot || \) is called a metrized line bundle, denoted by \( \mathcal{T} = (L, || \cdot ||) \).

Example 2.1 (Algebraic metrics). Let \( K' \) be a finite extension of \( K \) and let \( \mathcal{B}' \) be the normalization of \( \mathcal{B} \) in \( K' \). Let \( f : \mathcal{W} \rightarrow \mathcal{B}' \) be a proper morphism with geometric generic fiber \( W \) and let \( \mathcal{L} \) be a line bundle on \( \mathcal{W} \) which equals \( L \) over \( W \). Then, it is known that an algebraic metric \( || \cdot ||_{\mathcal{L}} \) on \( L \) is defined (cf. \([10, 2.3]\)). Here, we do not recall what it is exactly but explain what it is like. For any \( v \in M_K \), let \( R_v \) be the ring of integers of \( \overline{K}_v \). Let \( \text{Spec} (\overline{K}_v) \rightarrow \mathcal{B}' \) be the morphism arising from the field extension \( \overline{K}_v/K' \). Since \( \mathcal{B}' \) is proper over \( k \), this morphism extends to a unique morphism \( \text{Spec}(R_v) \rightarrow \mathcal{B}' \) by the valuative criterion. Let \( \mathcal{W}_v \rightarrow \text{Spec}(R_v) \) be the base-change of \( f \) by this morphism. Take any \( w \in W(\overline{K}_v) \). Since \( f \) is proper, there exists a unique section \( \sigma_w : \text{Spec}(R_v) \rightarrow \mathcal{W}_v \) corresponding to \( w \). Note that \( \sigma_w^*(\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{W}_v}} R_v) \) a free \( R_v \)-module of rank 1 and \( \sigma_w^*(\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{W}_v}} R_v) \otimes_{R_v} \overline{K}_v = L(w) \). To a non-zero \( s(w) \in L(w) \), assign a non-negative number

\[
||s(w)||_{\mathcal{L}, v} := \inf \left\{ |a^{-1}|_{v, \mathcal{W}} \in \mathbb{R}_{>0} \mid a \in \overline{K}_v^\times, \ as(w) \in \sigma_w^*(\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{W}_v}} R_v) \right\}.
\]
This assignment defines a metric on $L$ at $v$ for each $v \in M_K$ and hence a metric $|| \cdot ||_{L} = \{|| \cdot ||_{L,v}\}_{v \in M_K}$ on $L$. This metric is called the algebraic metric associated to the model $(f, \mathcal{L})$.

**Example 2.2** (Canonical metrics). Let $A$ be an abelian variety over $\overline{K}$. For any $n \in \mathbb{Z}$, let $[n] : A \to A$ denote the endomorphism given by $a \mapsto na$. Let $L$ be a line bundle on $A$ and assume that $L$ is even, that is, $[-1]^*(L) \cong L$. Let $n$ be an integer with $n \geq 2$. It follows from the theorem of cube that there exists an isomorphism $\phi : [n]^*(L) \to L^{\otimes n^2}$. A metric $|| \cdot ||$ on $L$ is called a canonical metric if $\phi$ induces an isometry $[n]^*(L) \cong L^{\otimes n^2}$, where $\mathcal{T} := (L, || \cdot ||)$. It is known that, once the isomorphism $\phi$ is fixed, the canonical metric is determined uniquely, and that, for a different choice of isomorphisms, the canonical metric changes only by a non-zero constant multiple (cf. [24, Theorem 2.2] or [8, Theorem 10.9]).

To define the height with respect to metrized line bundles, we need to focus on suitable metrics, called admissible metrics, studied in [8]. We do not repeat the definition of them here but remark that any algebraic metric is known to be admissible. Further, we also remark that canonical metrics over abelian varieties are also admissible. Thus we may take the heights of closed subvarieties with respect to line bundles equipped with these metrics.

Here is a remark concerning the compatibility of metrics considered here with those developed in [8]. Metrics here are those on line bundles over an algebraic variety and are considered only at the closed points of the variety. On the other hand, metrics in [8] are those on line bundles over the analytic space associated to the algebraic variety and are considered at any point of the analytic space. As far as working with admissible metrics, however, we do not have to be serious about this difference. In fact, since the admissible metrics are continuous metrics on the analytic space and the set of closed points is dense in the analytic space, one can recover all information of metrics on line bundles over the analytic space from metrics over the closed points of the given algebraic variety.

Let $\mathcal{T}_0, \ldots, \mathcal{T}_d$ be admissibly metrized line bundles on a proper variety $W$ and let $Z$ be a $d$-dimensional cycle on a variety $W$. Then the height $h_{\mathcal{T}_0, \ldots, \mathcal{T}_d}(Z)$ of $Z$ with respect to $\mathcal{T}_0, \ldots, \mathcal{T}_d$ is defined in [10] Definition 3.6]. We do not recall the definition of heights because we do not need it in the following arguments. We note that the heights that will be used in the arguments are those with respect to line bundles with algebraic metric only, which can be described in terms of intersection theory (cf. [10] Theorem 3.5 (d)) or Lemma 2.6.

Now, we consider an abelian variety $A$ over $\overline{K}$. Let $L_0, \ldots, L_d$ be even line bundles over $A$. Fixing an isomorphism $[n]^*(L_i) \to L_i^{\otimes n^2}$ for each $i = 0, \ldots, d$, we obtain a canonically metrized line bundle $\mathcal{T}_i = (L_i, || \cdot ||_i)$ (cf. Example 2.2). Then the canonical height of a cycle $Z$ of dimension $d$ of $A$ with respect to $L_0, \ldots, L_d$ is defined to be

$$\widehat{h}_{L_0, \ldots, L_d}(Z) := h_{\mathcal{T}_0, \ldots, \mathcal{T}_d}(Z).$$

The canonical metrics on a line bundle depend on the choice of isomorphisms $[n]^*(L_i) \to L_i^{\otimes n^2}$, but it follows from the product formula that the canonical height does not. Thus the canonical height $\widehat{h}_{L_0, \ldots, L_d}(Z)$ is well-defined. If $L = L_0 = \cdots = L_d$ and there is no danger of confusion, we simply write $\widehat{h}_L(Z)$ for $\widehat{h}_{L_0, \ldots, L_d}(Z).$
For a closed subvariety $X$ of $A$ of pure dimension $d$, let $[X]$ denote the corresponding cycle. We write $\widehat{h}_{L_0,\ldots,L_d}(X) := \widehat{h}_{L_0,\ldots,L_d}([X])$, called the the canonical height of $X$ with respect to $L_0,\ldots,L_d$.

The following proposition can be found in [9].

**Proposition 2.3** (Corollary 4.4 in [9]). Let $A$ be an abelian variety over $\overline{K}$, let $L$ be an even ample line bundle on $A$, and let $X$ be an irreducible closed subvariety of $A$. Then $X$ has dense small points if and only if $\widehat{h}_L(X) = 0$.

By Proposition 2.3, the geometric Bogomolov conjecture for $A$ is equivalent to the statement that an irreducible closed subvariety $X$ of $A$ with $\widehat{h}_L(X) = 0$ for some even ample line bundle $L$ on $A$ is a special subvariety.

2.2. Models of nowhere degenerate abelian varieties. Let $W$ be a projective scheme over $\overline{K}$ and let $L$ be a line bundle on $W$. Let $\mathcal{U}$ be an open subset of $\mathcal{B}'$. A proper morphism $f : \mathcal{W} \to \mathcal{U}$ with geometric generic fiber $W$ is called a model of $W$ over $\mathcal{U}$. We note that, in this terminology, it is not required that $W$ is dense in $\mathcal{W}$. Let $\mathcal{L}$ be a line bundle on $\mathcal{W}$ such that the restriction of $\mathcal{L}$ to the geometric generic fiber equals $L$. Then the pair $(f, \mathcal{L})$ is called a model of $(W, L)$ over $\mathcal{U}$.

We construct a suitable model of nowhere degenerate abelian varieties.

**Proposition 2.4.** Let $A$ be a nowhere degenerate abelian variety over $\overline{K}$ and let $L$ be a line bundle on $A$. Then, there exist a finite extension $K'$ of $K$, a proper morphism $f : \mathcal{A} \to \mathcal{B}'$, where $\mathcal{B}'$ is the normalization of $\mathcal{B}$ in $K'$, and a line bundle $\mathcal{L}$ on $\mathcal{A}$ satisfying the following conditions.

(a) The pair $(f, \mathcal{L})$ is a model of $(A, L)$.

(b) There exists an open subset $\mathcal{U} \subset \mathcal{B}'$ with $\text{codim}(\mathcal{B}' \setminus \mathcal{U}, \mathcal{B}') \geq 2$ such that the restriction $f' : \mathcal{A}_\mathcal{U} := f^{-1}(\mathcal{U}) \to \mathcal{U}$ of $f$ is an abelian scheme.

(c) Let $0_{f'}$ be the zero-section of the abelian scheme $f'$ in (b). Then $0_{f'}^*(\mathcal{L}) \cong \mathcal{O}_\mathcal{U}$.

**Proof.** Let $K_0$ be a finite extension of $K$ such that $A$ and $L$ can be defined over $K_0$, that is, $A = A_0 \otimes_{K_0} \overline{K}$ for some abelian variety $A_0$ over $K_0$ and $L$ is the base-change of a line bundle on $A_0$. Let $\mathcal{B}_0$ be the normalization of $\mathcal{B}$ in $K_0$. Then there exist a dense open subset $\mathcal{U}_0 \subset \mathcal{B}_0$ and an abelian scheme $f_0 : \mathcal{A}_0 \to \mathcal{U}_0$ with zero-section $0_{f_0}$ having $A_0$ as its generic fiber.

There exist only a finite number of points of $\mathcal{B}_0$ of codimension 1 which are not contained in $\mathcal{U}_0$, so that let $v_1,\ldots,v_m$ be all such points. For each $i = 1,\ldots,m$, let $\mathcal{O}_i$ be the stalk at $v_i$ of the structure sheaf of $\mathcal{B}_0$. It is a discrete valuation ring with fractional field $K_0$. It follows from Grothendieck’s semistable reduction theorem (cf. [7, Exp IX, Théorème 3.6]) that there exists a finite extension $K'$ of $K_0$ such that, for any $i = 1,\ldots,m$, there exists a semistable model $\varphi_i$ of $A_0 \otimes_{K_0} K'$ over the integral closure $\mathcal{O}'_i$ of $\mathcal{O}_i$ in $K'$. By the assumption of nowhere degeneracy of $A$, these semistable models are abelian schemes.

Let $\nu : \mathcal{B}' \to \mathcal{B}_0$ be the normalization of $\mathcal{B}_0$ in $K'$. Note that $\text{Spec}(\mathcal{O}'_i) = \mathcal{B}' \times_{\mathcal{B}_0} \text{Spec}(\mathcal{O}_i)$. The abelian scheme $\varphi_i$ extends to an abelian scheme over a neighborhood of $\text{Spec}(\mathcal{O}'_i)$ in $\mathcal{B}'$. Since $\mathcal{O}'_i$ is finite over $\mathcal{O}_i$, it follows that, for each $i = 1,\ldots,m$, there exist an open neighborhood $\mathcal{U}_i$ of $v_i$ in $\mathcal{B}_0$ and an abelian scheme $f'_i : \mathcal{A}'_i \to \nu^{-1}(\mathcal{U}_i)$ with zero-section $0_{f'_i}$ such that the generic fiber of $f'_i : \mathcal{A}'_i \to \nu^{-1}(\mathcal{U}_i)$ equals $A_0 \otimes_{K_0} K'$. On the other hand,
taking the base-change by $\nu^{-1}(U_0) \to U_0$ of the abelian scheme $f_0 : \mathcal{A}_0 \to U_0$, we obtain an abelian scheme $f'_0 : \mathcal{A}'_0 \to \nu^{-1}(U_0)$ with zero-section $0'_f$, which has $A_0 \otimes_{K_0} K'$ as its generic fiber. Thus we have a family
\[
\Phi := \{ (f'_i : \mathcal{A}'_i \to \nu^{-1}(U_i), 0'_f) \}_{i=0}^m
\]
of abelian schemes with generic fiber $A_0 \otimes_{K_0} K'$.

It follows from the generalized Weil extension lemma (cf. [1 Proposition 1.3]) together with the valuative criterion of properness that the isomorphism between the generic fibers of the abelian schemes $(f'_i, 0'_f)$ and $(f'_j, 0'_f)$ extends to a unique isomorphism between the restrictions over $\nu^{-1}(U_i) \cap \nu^{-1}(U_j)$ of the abelian schemes, or in other words, the abelian schemes $(f'_i, 0'_f)$ and $(f'_j, 0'_f)$ coincide with each other over $\nu^{-1}(U_i) \cap \nu^{-1}(U_j)$. Thus the family $\Phi$ patches together to be an abelian scheme $f'_+ : \mathcal{A}'_+ \to \bigcup_{i=0}^m \nu^{-1}(U_i)$ with geometric generic fiber $A = A_0 \otimes_{K_0} K$.

Since $\nu (\bigcup_{i=0}^m \nu^{-1}(U_i)) = \bigcup_{i=0}^m U_i$ contains all the points of $\mathfrak{B}_0$ of codimension 1 and since $\nu$ is finite, $\bigcup_{i=0}^m \nu^{-1}(U_i)$ contains all the points of $\mathfrak{B}'$ of codimension 1, which means $\text{codim}(\mathfrak{B}' \setminus \bigcup_{i=0}^m \nu^{-1}(U_i), \mathfrak{B}') \geq 2$. Let $\mathfrak{U}$ be the regular locus of $\bigcup_{i=0}^m \nu^{-1}(U_i)$. Since $\mathfrak{B}'$ is regular in codimension 1, it follows that $\text{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$.

Let $f' : \mathfrak{A}' \to \mathfrak{U}$ be the restriction of $f'_+$ over $\mathfrak{U}$. Then $f'$ is an abelian scheme with geometric generic fiber $A$. Since $\mathfrak{U}$ is regular and $f'$ is smooth, $\mathfrak{A}'$ is regular. It follows that there exists a line bundle $\mathcal{L}'_1$ on $\mathfrak{A}'$ such that the pair $(f', \mathcal{L}'_1)$ is a model of $(A, L)$ over $\mathfrak{U}$. Put $\mathcal{L}' := \mathcal{L}'_1 \otimes (f')^* (0'_{f'} (\mathcal{L}'_1))^{-1}$, where $0'_{f'}$ is the zero-section of $f'$. Then we have a model $(f', \mathcal{L}')$ such that $0'_{f'} (\mathcal{L}')$ is trivial. Finally, by Nagata’s embedding theorem (see [18 Theorem 5.7] for a scheme-theoretic version), there exists a proper morphism $f : \mathfrak{A} \to \mathfrak{B}'$ such that $\mathfrak{A}'$ is an open dense subscheme of $\mathfrak{A}$ and that $f|_{\mathfrak{A}'} = f'$, and an invertible sheaf $\mathcal{L}$ on $\mathfrak{A}'$ such that $\mathcal{L}|_{\mathfrak{A}'} = \mathcal{L}'$. The pair $(f, \mathcal{L})$ satisfies the conditions (a) and (c), and also satisfies (b) with the above $\mathfrak{U}$. This proves the proposition. 

Remark 2.5. Let $B$ be a constant abelian variety over $\overline{K}$, and we take an abelian variety $\tilde{B}$ over $k$ with $B = \tilde{B} \otimes_k K$. Let $\tilde{M}$ be an even ample line bundle on $\tilde{B}$ and set $M := \tilde{M} \otimes_k K$. Let $\text{pr}_{\mathfrak{B}} : \tilde{B} \times_{\text{Spec}(k)} \mathfrak{B} \to \mathfrak{B}$ and $q : \tilde{B} \times_{\text{Spec}(k)} \mathfrak{B} \to \tilde{B}$ be the canonical projections. Then the pair $\left( \text{pr}_{\mathfrak{B}}, q^* \left( \tilde{M} \right) \right)$ is a model of $(B, M)$ satisfying the conditions in Proposition 2.4.

2.3. Height and intersection on a model. In this subsection, we describe the canonical heights of pure dimensional closed subschemes of a nowhere degenerate abelian variety in terms of intersection theory over models.

Let $A$ be an abelian variety over $\overline{K}$ with an even ample line bundle $L$. Assume that $A$ is nowhere degenerate, and we take a model $(f : \mathfrak{A} \to \mathfrak{B}', \mathcal{L})$ of $(A, L)$ satisfying the conditions in Proposition 2.4 where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in some finite extension $K'$ of $K$. Let $\mathfrak{U}$ be an open subset of $\mathfrak{B}'$ such as in (b) of Proposition 2.4. Then the restriction $f' : \mathfrak{A}_\mathfrak{U} \to \mathfrak{U}$, where $\mathfrak{A}_\mathfrak{U} := f^{-1}(\mathfrak{U})$, is an abelian scheme with zero-section $0'_f$ by (c).

It follows from (c) of Proposition 2.4 that, for any $n \in \mathbb{N}$, there exists an isomorphism

\[
[n]^* (\mathcal{L}|_{\mathfrak{A}_\mathfrak{U}}) \to (\mathcal{L}'|_{\mathfrak{A}_\mathfrak{U}})^{\otimes n^2},
\]
where \([n]: \mathcal{A}_t \to \mathcal{A}_t\) is the \(n\)-times endomorphism. Indeed, since \(L\) is even, there exists an isomorphism \([n]^* (L) \to L^\otimes n^2\), which extends to an isomorphism \([n]^* (\mathcal{L}|_U) \to (\mathcal{L}|_U)^\otimes n^2 \otimes f^*(\mathcal{M})\) for some line bundle \(\mathcal{M}\) on \(U\). Then we have an isomorphism

\[
0^*_f ([n]^* (\mathcal{L}|_U)) = 0^*_f ((\mathcal{L}|_U)^\otimes n^2 \otimes f^*(\mathcal{M})) = 0^*_f (\mathcal{L}|_U)^\otimes n^2 \otimes \mathcal{M}.
\]

By the condition (c), we have \(0^*_f ([n]^* (\mathcal{L}|_U)) = [n]^* (0^*_f (\mathcal{L}|_U)) \cong \mathcal{O}_U\) and \(0^*_f (\mathcal{L}|_U)^\otimes n^2 \cong \mathcal{O}_U\), which lead us to \(\mathcal{M} \cong \mathcal{O}_U\). This shows the existence of an isomorphism \((2.5.2)\).

Let \(|\cdot|_{\mathcal{L}}\) be the algebraic metric on \(L\) associated to the model \((f, \mathcal{L})\) (cf. Example 2.1). Noting \(\text{codim}(\mathcal{B}' \setminus \mathcal{A}, \mathcal{B}') \geq 2\), we see that the isomorphism \((2.5.2)\) gives rise to an isometry

\[
[n]^* (L, |\cdot|_{\mathcal{L},v}) \cong (L, |\cdot|_{\mathcal{L},v})^\otimes n^2
\]

for each \(v \in M_K\) (cf. Example 2.1). Thus the algebraic metric \(|\cdot|_{\mathcal{L}}\) is a canonical metric.

The heights of cycles with respect to algebraically metrized line bundles are described in terms of intersection theory, and so are the canonical heights. To be precise, let \(X\) be a closed subscheme of \(A\) of pure dimension \(d\). Replacing \(K'\) with a finite extension if necessary, we assume that \(X\) can be defined over \(K'\). Let \(\mathcal{X}\) be the closure of \(X\) in \(\mathcal{A}\). Note that \(X\) is the geometric generic fiber of \(\mathcal{X}\). Let \([\mathcal{X}']\) denote the cycle corresponding to \(\mathcal{X}\). Then we have, by \([10\) Theorem 3.5 (d)],

\[
(2.5.3) \quad \hat{h}_L(X) = \frac{\deg_{\mathcal{H}'} f_* (c_1(\mathcal{L})^{(d+1)} \cdot [\mathcal{X}'])}{[K' : K]},
\]

where \(\deg_{\mathcal{H}'}\) means the degree of the \((b-1)\)-dimensional cycle on \(\mathcal{B}'\) with respect to \(\mathcal{H}'\) and \([K' : K]\) is the extension degree.

In equality \((2.5.3)\), the model \(f|_{\mathcal{X}}: \mathcal{X} \to \mathcal{B}'\) is assumed to be the closure of \(X\), but this is too strong to use in the latter arguments. In fact, we verify equality \((2.5.3)\) under a milder assumption on \(\mathcal{X}\) as follows.

**Lemma 2.6.** Let \(A, L, X, f: \mathcal{A} \to \mathcal{B}', \mathcal{L}\), and \(\mathcal{H}'\) be as above. Let \(\mathcal{X}'\) be a closed subscheme of \(A\) such that the restriction \(f|_{\mathcal{X}'}: \mathcal{X}' \to \mathcal{B}'\) has geometric generic fiber \(X\) and is flat over any point of codimension 1 of \(\mathcal{B}'\). Then we have

\[
\hat{h}_L(X) = \frac{\deg_{\mathcal{H}'} f_* (c_1(\mathcal{L})^{(d+1)} \cdot [\mathcal{X}'])}{[K' : K]},
\]

where \(\deg_{\mathcal{H}'}\) means the degree of the \((b-1)\)-dimensional cycle on \(\mathcal{B}'\) with respect to \(\mathcal{H}'\).

**Proof.** Let \(\mathcal{X}\) be the closure of \(X\) in \(\mathcal{A}\). Then there exists an effective cycle \([\mathcal{Y}]\) on \(\mathcal{A}\) of dimension \(d + b\) such that \([\mathcal{X}'] = [\mathcal{X}] + [\mathcal{Y}]\). Since \(\mathcal{X}'\) is flat over any point of \(\mathcal{B}'\), \(\mathcal{X}'\) coincides with \(\mathcal{X}\) over any point of \(\mathcal{B}'\) of codimension 1. Therefore, any irreducible component of \(f(\mathcal{Y})\) has dimension not greater than \(b - 2\). Since \(c_1(\mathcal{L})^{(d+1)}[\mathcal{Y}]\) is a cycle class of dimension \(b - 1\) and can be represented by a cycle supported in \(\mathcal{Y}\), it follows that \(f_* (c_1(\mathcal{L})^{(d+1)} \cdot [\mathcal{Y}]) = 0\). This concludes \(f_* (c_1(\mathcal{L})^{(d+1)} \cdot [\mathcal{X}']) = f_* (c_1(\mathcal{L})^{(d+1)} \cdot [\mathcal{X}'])\), and thus, by \((2.5.3)\), the desired equality is obtained. \(\square\)
3. Constant abelian varieties and the geometric Bogomolov conjecture

3.1. Proof of Theorem 1.4. In this subsection, we show Proposition 3.2, which indicates that Conjecture 1.1 holds for abelian varieties which are isogenous to constant abelian varieties (cf. Remark 3.3). Using this proposition, we prove Theorem 1.4, which states that the geometric Bogomolov conjecture holds for any abelian variety whose nowhere degeneracy rank equals the dimension of the trace.

We begin with a preliminary argument. We will use the following lemma on intersection theory.

Lemma 3.1. Let \( X \) be an irreducible proper variety over \( k \), let \( Y \) be a closed subscheme of \( X \) of pure dimension \( d \), and let \( [Y] \) denote the corresponding cycle. Let \( \mathcal{L} \) be a line bundle and let \( \mathcal{M} \) be a sublinear system of the complete linear system associated to \( \mathcal{L} \). Suppose that \( \mathcal{M} \) is base-point free. Let \( e \) be a positive integer. Then, for general \( \mathcal{D}_1, \ldots, \mathcal{D}_e \in \mathcal{M} \), the cycle \( [\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_e \cap Y] \) represents the cycle class \( c_1(\mathcal{L})^e \cdot [Y] \).

Proof. Since \( \mathcal{M} \) is base-point free, a general \( \mathcal{D}_1 \in \mathcal{M} \) is away from any associated point of \( Y \). Then, at any point of the support of \( \mathcal{D}_1 \), a local equation of \( \mathcal{D}_1 \) is \( \mathcal{O}_Y \)-regular. Since \( \mathcal{M} \) is base-point free again, a general \( \mathcal{D}_2 \in \mathcal{M} \) is away from any associated point of \( \mathcal{D}_1 \cap Y \). Then a local equation of \( \mathcal{D}_2 \) is \( \mathcal{O}_{\mathcal{D}_1 \cap \mathcal{D}_2} \)-regular. Repeating this process, we obtain, by induction, general \( \mathcal{D}_1, \ldots, \mathcal{D}_e \) such that the sequence of their local equations is \( \mathcal{O}_Y \)-regular. By \([3, \text{Example 2.4.8}]\), therefore, the cycle of \( \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_e \cap Y \) represents the cycle class \( c_1(\mathcal{L})^e \cdot [Y] \).

Now, we show the key proposition to the proof of Theorem 1.4.

Proposition 3.2. Let \( \widetilde{B} \) be an abelian variety over \( k \) and let \( \widetilde{M} \) be an even ample line bundle on \( \widetilde{B} \). Set \( B := \widetilde{B} \otimes_k \overline{k} \) and \( M := \widetilde{M} \otimes_k \overline{k} \). Let \( Y \) be an irreducible closed subvariety of \( B \) of dimension \( d \). Suppose that \( \widetilde{h}_M(Y) = 0 \). Then \( Y \) is a constant subvariety.

Proof. Since the canonical height is multilinear on line bundles to which it is associated (cf. \([3, \text{Theorem 11.18 (a)]}\)), we have \( \widetilde{h}_M^n(Y) = n^{d+1}\widetilde{h}_M(Y) \) for any \( n \in \mathbb{N} \), so that we may and do assume that \( \widetilde{M} \) and hence \( M \) are very ample.

Let \( K' \) be a finite extension of \( K \) such that \( \overline{Y} \) be defined over \( K' \) and let \( \mathfrak{B}' \rightarrow \mathfrak{B} \) be the normalization of \( \mathfrak{B} \) in \( K' \). We set \( \mathfrak{B} := \widetilde{B} \times_{\text{Spec}(k)} \mathfrak{B}' \) and let \( f : \mathfrak{B} \rightarrow \mathfrak{B}' \) denote the second projection, which is an abelian scheme. Further, let \( \mathcal{M} \) be the pull-back of \( \widetilde{M} \) by the canonical projection \( \text{pr}_B : \mathfrak{B} \rightarrow \widetilde{B} \). Then the pair \((f, \mathcal{M})\) is a model of \((B, M)\) such that the conditions in Proposition 2.4 are satisfied (cf. Remark 2.5). Let \( \mathcal{V} \) be the closure of \( Y \) in \( \mathfrak{B} \). Then the geometric generic fiber of \( f|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathfrak{B}' \) equals \( Y \).

Set \( \mathcal{Y} := \text{pr}_B(\mathcal{V}) \). We are going to show that \( Y = \mathcal{Y} \otimes_k \overline{k} \). Let \( \mathfrak{U} \subset \mathfrak{B}' \) be a dense open subset such that \( f|_{\mathcal{Y}} \) is flat over \( \mathfrak{U} \). We put \( \mathcal{V} := f^{-1}(\mathfrak{U}) \cap \mathcal{Y} \). Note that \( f|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathfrak{U} \) is a proper flat morphism of relative dimension \( d \). Let \( [\widetilde{M}] \) be the complete linear system associated to \( \widetilde{M} \). Since \( \widetilde{M} \) is very ample, it follows that, for general members \( \widetilde{D}_1, \ldots, \widetilde{D}_{d+1} \in [\widetilde{M}] \), if we write \( \mathcal{D}_i := \text{pr}_B^{-1}(\widetilde{D}_i) \) for \( i = 1, \ldots, d+1 \), then the restriction \( \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_d \cap \mathcal{V} \rightarrow \mathfrak{U} \) of \( f \) is finite and surjective, and furthermore, by Lemma 3.1, the cycles of the closed
subspaces \( D_1 \cap \cdots \cap D_d \cap Y \) and \( D_1 \cap \cdots \cap D_d \cap Y \) represent the cycle classes \( c_1 (M)^d \cdot [Y] \) and \( c_1 (M)^{d+1} \cdot [Y] \) respectively.

For such \( D_1, \ldots, D_d \), we have

\[
(3.2.4) \quad D_1 \cap \cdots \cap D_d \cap Y = \emptyset.
\]

Indeed, let \( Z \) be the closure of \( D_1 \cap \cdots \cap D_d \cap Y \). Then \([Z]\) and \([D_1 \cap \cdots \cap D_d \cap Y] - [Z]\) are effective cycles\(^3\) of dimension \( b - 1 \), where we remark \( b = \dim (Y') \). Therefore we have

\[
\deg_{H'} f_* \left( c_1 (M)^{d+1} \cdot [Y] \right) = \deg_{H'} f_* (D_1 \cap \cdots \cap D_d \cap Y) \geq \deg_{H'} f_* [Z] \geq 0,
\]

where \( H' \) is the pull-back of \( H \) by the finite morphism \( Y' \to Y \), and \( \deg_{H'} \) means the degree of the \((b - 1)\)-dimensional cycle on \( Y' \) with respect to \( H' \). On the other hand, by \((2.5.3)\) or Lemma \(2.6\) we have

\[
\deg_{H'} f_* \left( c_1 (M)^{d+1} \cdot [Y] \right) = [K' : K] h_M (Y),
\]

and hence

\[
\deg_{H'} f_* \left( c_1 (M)^{d+1} \cdot [Y] \right) = 0
\]

by assumption. It follows that \( \deg_{H'} f_* [Z] = 0 \). Since \( f_* [Z] \) is an effective cycle, this implies that the support \( \text{Supp} (f_* [Z]) \) of \( f_* [Z] \) is empty. Since any irreducible component of \( \text{Supp} ([Z]) \) is generically finite over \( Y' \), we have \( \text{Supp} (f_* [Z]) = f (\text{Supp} ([Z])) \). Thus \( \text{Supp} ([Z]) = \emptyset \) follows, which concludes \( D_1 \cap \cdots \cap D_d \cap Y = \emptyset \).

Next, we claim that \( \dim \left( \tilde{Y} \right) = d \). Since \( \dim (Y) = d + b \) and the generic fiber \( Y \) of \( f|_Y : Y \to Y' \) has dimension \( d \), the dimension counting shows \( \dim \left( \tilde{Y} \right) = \dim (\text{pr}_{\tilde{B}} (Y')) \geq d \), so that it suffices to show \( \dim \left( \tilde{Y} \right) \leq d \). By \((3.2.4)\), we have

\[
(3.2.5) \quad \text{pr}_{\tilde{B}} (D_1 \cap \cdots \cap D_d \cap Y) = \emptyset
\]

for general \( D_1, \ldots, D_d \in \tilde{M} \), where we recall \( D_i := \text{pr}_{\tilde{B}}^{-1} (\tilde{D}_i) \) for \( i = 1, \ldots, d + 1 \). Since \( Y \) is a dense open subset of \( Y \) and \( \text{pr}_{\tilde{B}} : Y \to \tilde{B} \) is surjective, there exists a dense open subset \( \tilde{W} \) of \( \tilde{Y} \) with \( \tilde{W} \subset \text{pr}_{\tilde{B}} (Y') \). Then \((3.2.5)\) implies \( D_1 \cap \cdots \cap D_d \cap \tilde{W} = \emptyset \) for general \( D_1, \ldots, D_d \in \tilde{M} \). Since \( M \) is very ample and \( \tilde{W} \) is a dense open subset of \( \tilde{Y} \), it follows that \( \dim \left( \tilde{Y} \right) \leq d \). Thus, we find \( \dim \left( \tilde{Y} \right) = d \).

Since \( \tilde{Y} = \text{pr}_{\tilde{B}} (Y) \), we have \( Y \subset \tilde{Y} \times_{\text{Spec} (k)} Y' \). Since both \( Y \) and \( \tilde{Y} \times_{\text{Spec} (k)} Y' \) are irreducible varieties of dimension \( d + b \), we conclude that \( Y = \tilde{Y} \times_{\text{Spec} (k)} Y' \). This proves that \( Y = \tilde{Y} \otimes_k \overline{\kappa} \), and thus \( Y \) is a constant subvariety of \( B \).

\( \square \)

\textbf{Remark 3.3.} It follows from the definition of special subvarieties that any constant closed subvariety of a constant abelian variety is a special subvariety. Taking into account of Proposition \(2.3\) we then find that Proposition \(3.2\) verifies the geometric Bogomolov conjecture for constant abelian varieties.

Recall that, for an abelian variety \( A \) over \( \overline{K} \), \( (A^\ell/k, \text{Tr}_A) \) denotes the \( \overline{K}/k \)-trace of \( A \).
Remark 3.4. The trace homomorphism $\text{Tr}_A$ is a finite morphism (cf. [21, Lemma 1.4]). It is known that $\text{Tr}_A$ factors through $m$ by [22, Proposition 7.11], where $m$ is the maximal nowhere degenerate abelian subvariety of $A$. Thus $\dim \left( \overline{A}/k \right) \leq \text{nd-rk}(A)$ holds.

Now we establish Theorem 1.4 as a consequence of Proposition 3.2.

Theorem 3.5 (Theorem 1.4). Let $A$ be an abelian variety over $K$. Assume $\dim \left( \overline{A}/k \right) = \text{nd-rk}(A)$. Then the geometric Bogomolov conjecture holds for $A$.

Proof. Let $m$ be the maximal nowhere degenerate abelian subvariety of $A$. We see from Remark 3.4 that $\text{Tr}_A$ factors as a finite homomorphism $\overline{A}/k \otimes K \to m$, and it follows from the assumption that this homomorphism is an isogeny. By Remark 3.3, which is a reinterpretation of Proposition 3.2, the geometric Bogomolov conjecture holds for $\overline{A}/k \otimes K$, so that, by [22, Corollary 7.6], the conjecture holds also for $m$. By Theorem 1.3, the conjecture holds for $A$.

Remark 3.6. Theorem 1.4, which is Theorem 3.5 also, generalizes Theorem 1.2. Indeed, suppose that $\text{nd-rk}(A) \leq 1$. If $\text{nd-rk}(A) = 0$, then Remark 3.4 tells us $\dim \left( \overline{A}/k \right) = \text{nd-rk}(A) = 0$, so that Theorem 1.2 follows from Theorem 3.5 trivially. Suppose that $\text{nd-rk}(A) = 1$. This means that the maximal nowhere degenerate abelian subvariety $m$ of $A$ has dimension 1. We remark here that any nowhere degenerate abelian variety of dimension 1 is a constant abelian variety, which follows from the well-known fact that the moduli space of elliptic curves is an affine line. Therefore, $m$ is a constant abelian variety, and it follows from the universality of the trace that $\dim \left( \overline{A}/k \right) \geq \dim(m)$. By Remark 3.4 we then conclude that $\dim \left( \overline{A}/k \right) = \dim(m) = \text{nd-rk}(A)$. Thus Theorem 1.2 follows from Theorem 3.5.

3.2. Special subvarieties of constant abelian varieties. First, we note the following proposition, which is a slight generalization of Chow’s theorem [11, II, §1 Theorem 5] and will be significantly used later to prove Theorem 1.3.

Proposition 3.7. Let $F$ be an algebraically closed field and let $\mathcal{F}/F$ be a field extension with $\mathcal{F}$ algebraically closed. Let $A$ be an abelian variety over $F$ and let $Z$ be a torsion subvariety of $A \otimes_F \mathcal{F}$. Then there exists a torsion subvariety $T$ of $A$ such that $Z = T \otimes_F \mathcal{F}$.

Proof. Since $F$ is algebraically closed, the torsion points of $(A \otimes_F \mathcal{F}) (\mathcal{F})$ coincide with those of $A (F)$. Then the assertion follows from Chow’s theorem [11, II, §1 Theorem 5].

Proposition 3.7 is used in the following remark. This remark is not necessary in the sequel, but it is worth mentioning because it tells us a basic fact which characterizes special subvarieties of constant abelian varieties.

Remark 3.8. Let $B$ be a constant abelian variety over $K$ and let $Y$ be an irreducible closed subvariety of $B$. We then note that $Y$ is a special subvariety if and only if $Y$ is a constant subvariety. Indeed, the “if” part is noted in Remark 3.3. As for the “only if” part, we have two different proofs: one is the proof using Proposition 3.2 together with some basic facts on special subvarieties; the other is the proof just using the definition of special subvarieties and Proposition 3.7. The details are left to the reader.
4. Family of closed subvarieties over a constant variety

In this section, we investigate a family of closed subvarieties of an abelian variety which is parameterized by a constant variety.

4.1. Abelian subschemes and their translates. Let \( p : \mathcal{A} \to U \) be an abelian scheme over a noetherian scheme \( U \) with addition \( \alpha : \mathcal{A} \times_U \mathcal{A} \to \mathcal{A} \) and zero-section \( 0_p \). Let \( X \) be a closed subscheme of \( \mathcal{A} \) and let \( p|_X : X \to U \) be the restriction of \( p \). We call \( p|_X : X \to U \) a closed subgroup scheme of \( p \) if the group structure of \( p \) restricts to a group structure on \( p|_X \), that is, we have \( \alpha (X \times_U X) \subset X \), \( 0_p(U) \subset X \), and \(-X = X\), where \(-X\) is the image of \( X \) by the “minus” morphism of the abelian scheme \( p \).

The following lemma shows that the condition \(-X = X\) can be omitted for \( X \) to be a subgroup scheme.

Lemma 4.1. Let \( p : \mathcal{A} \to U \) be an abelian scheme with addition \( \alpha : \mathcal{A} \times_U \mathcal{A} \to \mathcal{A} \) and zero-section \( 0_p \). Let \( X \) be a closed subscheme of \( \mathcal{A} \) such that \( \alpha (X \times_U X) \subset X \) and \( 0_p(U) \subset X \). Then, \( p|_X : X \to U \) is a closed subgroup scheme of \( p \).

Proof. It suffices to show that \(-X = X\). Let \( S \) be any \( U \)-scheme and let \( s \in X(S) \) be any \( S \)-valued point of \( X \). Since the zero \( 0_{p,S} \) of the group \( \mathcal{A}(S) \) sits in \( X(S) \) by assumption, the addition \( X(S) \times X(S) \to X(S) \) is surjective. Therefore, there exists an \( s' \in X(S) \) such that \( s + s' = \alpha(s, s') = 0_{p,S} \), which shows that \(-s = s' \in X(S) \). Thus we have \(-X(S) \subset X(S) \). Since \( S \) is an arbitrary \( U \)-scheme, this means that \(-X \subset X \). We then have \( X = -(-X) \subset -X \), and thus \(-X = X\) as required.

Let \( p|_X : X \to U \) be a closed subgroup scheme of an abelian scheme \( p : \mathcal{A} \to U \). We call \( p|_X \) an abelian subscheme of \( p \) if it is a smooth morphism with geometrically connected fibers. Then \( p|_X \) itself is an abelian scheme in a natural way.

The following lemma shows that, under a certain condition, if there exists a geometric fiber of \( p|_X \) that is an abelian subvariety, then \( p|_X \) is an abelian subscheme.

Lemma 4.2. Let \( U \) be an integral noetherian scheme and let \( p : \mathcal{A} \to U \) be an abelian scheme with zero-section \( 0_p \). Let \( X \) be a subscheme of \( \mathcal{A} \) such that the restriction \( p|_X : X \to U \) is proper and smooth and such that \( 0_p \) factors through \( X \subset \mathcal{A} \). Suppose that there exists a point \( s \in U \) such that the geometric fiber \( X_s \) is an abelian subvariety of \( \mathcal{A}_s \). Then \( p|_X \) is an abelian subscheme of \( p \).

Proof. Since \( p|_X \) is proper and flat and \( X_s \) is connected, it follows from [5] Proposition 15.5.9 (ii)] that any fiber of \( p|_X \) is geometrically connected. Since \( p|_X \) is smooth, this means that any fiber of \( p|_X \) is geometrically integral.

Let \( \alpha : \mathcal{A} \times_U \mathcal{A} \to \mathcal{A} \) denote the addition of \( p \). Set \( Z := \alpha (X \times_U X) \), the scheme-theoretic image. Note that \( Z \) is an integral scheme. Indeed, by [4] Proposition 4.5.7, it follows that \( X \times_U X \) is irreducible. Further, since \( X \times_U X \to U \) is smooth and \( U \) is reduced, \( X \times_U X \) is reduced by [6] Proposition 17.5.7. It follows that \( X \times_U X \) is integral, and hence \( Z \) is integral.

It follows from \( 0_p(U) \subset X \) that \( Z \supset X \), and hence \( Z_u \supset X_u \) for any \( u \in U \). Further, we have \( Z_u = \alpha (X \times_U X)|_u = X_u + X_u \) as sets. For any \( u \in U \), since \( X_u \) is geometrically irreducible, it follows that \( Z_u \) is irreducible.
By Lemma 4.1, we only have to prove $Z \subset X$ as subschemes. Since $X_\pi$ is an abelian variety, we have $Z_\pi = X_\pi$ as sets. By the upper semicontinuity of the dimension of fibers (cf. [5, Corollaire 13.1.5]), there exists an open neighborhood $V \subset U$ of $s$ such that $\dim Z_u = \dim X_u$ for any $u \in V$. Since $Z_u$ is irreducible and $Z_u \supset X_u$, it follows that $Z_u = X_u$ as sets for $u \in V$. Therefore, $Z \times_U V = X \times_U V$ as sets. Since $Z$ is integral and $Z \supset X$, it follows that $Z \times_U V = X \times_U V$ as schemes in fact. Further, since $X$ is a closed subscheme of $\mathcal{A}$ and $Z$ is an integral scheme, taking the closures of the both sides of this equality concludes $Z \subset X$ as subschemes. Thus we obtain the lemma.

The following lemma will be used in the proof of Proposition 4.6.

**Lemma 4.3.** Let $Y$ be an integral noetherian scheme, let $p : \mathcal{A} \to Y$ be an abelian scheme, and let $X$ be a subscheme of $\mathcal{A}$. Assume that the restriction $p|_X : X \to Y$ is proper and surjective and that the geometric generic fiber $X_{\pi}$ is reduced, where $\eta$ is the generic point of $Y$. Suppose that there exists a dense subset $S$ of $Y$ such that, for any $s \in S$, the subscheme $(X_s)_{\text{red}} \subset \mathcal{A}_{\pi}$, the geometric fiber with its induced reduced subscheme structure, is a translate of an abelian subvariety. Then, $X_{\pi}$ is a translate of an abelian subvariety of $\mathcal{A}_{\pi}$.

**Proof.** We first claim that there exists a generically finite dominant morphism $\phi : U \to Y$ with $U$ integral such that, if $p' : U \times_Y \mathcal{A} \to U$ is the base-change of $p$ by $\phi$ and if $Z := U \times_Y X$, then the restriction $p'|_Z : Z \to U$ is a proper smooth morphism with a section $\sigma$. Indeed, let $X_{\eta}$ be the generic fiber of $p|_X$ and let $x \in X_{\eta}$ be a closed point. Further, let $V \subset X$ be the closure of $x$ in $X$. Then the restriction $V \to Y$ of $p|_X$ is a generically finite surjective morphism such that the base-change $V \times_Y X \to V$ of $p|_X$ has a section. Since $V \times_Y X \to V$ is generically flat, restricting $V \to Y$ to a dense open subset $U$ of $V$ gives us a generically finite morphism $\phi : U \to Y$ such that $p'|_Z : Z \to U$ is flat with a section $\sigma$. Since $p'|_Z$ has geometrically reduced generic fiber by assumption, [5, Théorème 12.2.4] allows us moreover to take $\phi$ so that any geometric fiber of $p'|_Z$ is reduced. It follows that, for any $u \in \phi^{-1}(S)$, we have $(p'|_Z)^{-1}(u) = ((p'|_Z)^{-1}(u))_{\text{red}}$, which is smooth by assumption. This implies that $p'|_Z$ is smooth over any $u \in \phi^{-1}(S)$ (cf. [5, Théorème 17.5.1]). Since $\phi^{-1}(S) \neq \emptyset$ by assumption, $p'|_Z$ is smooth over an open dense subset of $U$. Replacing $U$ with this open subset, we obtain $p'|_Z : Z \to U$ that is smooth (cf. [5, Théorème 12.2.4]). Thus, $\phi$ is taken so that $p'|_Z : Z \to U$ is proper and smooth.

Put $Z' := Z - \sigma(U)$, the translate of $Z$ by the section $-\sigma(U)$. Then $0_{p'}(U) \subset Z'$, where $0_{p'}$ is the zero-section of the abelian scheme $p'$. Since $p'|_Z : Z \to U$ is smooth and $U \supset \phi^{-1}(S) \neq \emptyset$, it follows from the assumption that there exits a geometric point $\overline{s}$ of $U$ such that $(p'|_Z)^{-1}(\overline{s})$ is an abelian subvariety. Now, applying Lemma 1.2 to these $p' : U \times_Y \mathcal{A} \to U$ and $Z'$, we find that $p'|_{Z'}$ is an abelian subscheme of $p'$. In particular, the geometric generic fiber of $p'|_{Z'}$ is an abelian subvariety, and hence the geometric generic fiber $Z_{\pi}$ of $p'|_Z$ is a translate of an abelian subvariety, where $\overline{\pi}$ denotes the geometric generic point of $U$. Since $X_{\pi} = Z_{\pi}$, this concludes that $X_{\pi}$ is a translate of an abelian subvariety of $\mathcal{A}_{\pi}$.

**4.2. Relative height.** We begin by fixing the notation for this subsection. Let $A$ be an abelian variety over $\overline{K}$ with an even ample line bundle $L$. Let $Y$ be an irreducible variety
over $k$ and put $Y := \widetilde{Y} \otimes_k \overline{K}$. Let $p : Y \times A \to Y$ be the first projection, which is an abelian scheme. Let $X$ be an integral closed subscheme of $Y \times A$ with $p(X) = Y$.

In this subsection, we are concerned with a sufficient condition for $X$ to be of form $Y \times T$ for some torsion subvariety $T$ of $A$. We will give such a condition in terms of a “relative height” function, which will be denoted by $h^r_{X/Y}$.

To define the relative height function, we assign to each point $\tilde{y} \in \widetilde{Y}$ a geometric point $\overline{y_K}$ of $Y$ in the following way. Taking an algebraic closure $\xi = \overline{k}(\tilde{y})$ of the residue field $k(\tilde{y})$ of $\tilde{y}$ gives rise to a geometric point $\overline{y} : \text{Spec} (\xi) \to \widetilde{Y}$. The base-change $\{\overline{y}\} \times_{\overline{Y}} (\widetilde{Y} \times \mathcal{B}) \cong \xi \otimes_k \mathcal{B}$, denoted by $\{\overline{y}\} \times \mathcal{B}$ simply, is a normal projective variety over $\xi$. Let $\overline{y}_K$ be the generic point of $\{\overline{y}\} \times \mathcal{B}$. We remark that the residue field $\xi(\overline{y}_K)$ is the function field of $\overline{y}_K$. Then, the field extensions $\xi(\overline{y}_K) / \xi$ and $\xi(\overline{y}_K) / K$ give rise to a morphism $\text{Spec} (\xi(\overline{y}_K)) \to \{\overline{y}\} \otimes_k K$ and hence to a geometric point

$$
\overline{y}_K : \text{Spec} (\xi(\overline{y}_K)) \to \widetilde{Y} \otimes_k \overline{K} = Y.
$$

That is the definition of $\overline{y}_K$.

**Remark 4.4.** If $\overline{y} = \overline{\eta}$ is the generic point of $\widetilde{Y}$, then the corresponding point $\overline{y_K}$ is the geometric generic point of $Y$.

Let $\mathcal{O}_Y \otimes_k \mathcal{H}$ be the pull-back of $\mathcal{H}$ by the natural projection $\widetilde{Y} \times_{\text{Spec}(k)} \mathcal{B} \to \mathcal{B}$. Since $\xi(\overline{y}_K)$ is the function field of the normal projective variety $\{\overline{y}\} \times \mathcal{B}$ equipped with an ample line bundle $\xi \otimes_k \mathcal{H} = (\mathcal{O}_Y \otimes_k \mathcal{H}) |_{\{\overline{y}\} \times \mathcal{B}}$, the notion of height over the base field $\xi(\overline{y}_K)$ makes sense. Indeed, we consider the abelian variety $\{\overline{y}_K\} \times_Y (Y \times A) \cong \xi(\overline{y}_K) \otimes_k A$ over $\xi(\overline{y}_K)$, which is denoted by $\{\overline{y}_K\} \times A$. This abelian variety is equipped with the even ample line bundle $\xi(\overline{y}_K) \otimes_k L = (\mathcal{O}_Y \otimes_k L) |_{\{\overline{y}_K\} \times A}$, where $\mathcal{O}_Y \otimes_k L$ is the pull-back of $L$ by the natural projection $Y \times A \to A$. Since the fiber $X_{\overline{y}_K}$ of $p|_X : X \to Y$ over $\overline{y}_K$ is a closed subscheme of $\{\overline{y}_K\} \times A$, we can therefore consider the canonical height of $X_{\overline{y}_K}$ with respect to this line bundle if $X_{\overline{y}_K}$ has pure dimension.

Now, varying $\overline{y}$ defines a function $h^r_{X/Y}$ which assigns to each $\overline{y}$ the height of $X_{\overline{y}_K}$ with respect to $\xi(\overline{y}_K) \otimes_k L = (\mathcal{O}_Y \otimes_k L) |_{\{\overline{y}_K\} \times A}$. To be precise, let $\widetilde{Y}_{pd}$ be a subset of $\widetilde{Y}$ given by

$$
\widetilde{Y}_{pd} := \left\{ \overline{y} \in \widetilde{Y} \mid X_{\overline{y}_K} \text{ has pure dimension dim}(X) - \text{dim}(Y) \right\}.
$$

Then we define a function $h^r_{X/Y}$ on $\widetilde{Y}_{pd}$ by

$$
(4.4.6) \quad h^r_{X/Y}(\overline{y}) := \widehat{h}_{\xi(\overline{y}_K) \otimes_k L} (X_{\overline{y}_K}) = \widehat{h}_{(\mathcal{O}_Y \otimes_k L)|_{\overline{y}_K} \times A} (X_{\overline{y}_K}).
$$

**Remark 4.5.** For any $\overline{y} \in \widetilde{Y}_{pd}$ and for any irreducible component $Z$ of $X_{\overline{y}_K}$, we have

$$
0 \leq \widehat{h}_{\xi(\overline{y}_K) \otimes_k L}(Z) \leq \widehat{h}_{\xi(\overline{y}_K) \otimes_k L}(X_{\overline{y}_K}) = h^r_{X/Y}(\overline{y}),
$$

where $\overline{y}_K$ is the generic point of $\overline{y}_K$. 


which follows from the non-negativity of the canonical height \([8, \text{ Theorem 11.18 (e)}]\) and the linearity of the canonical height on cycles \([10, \text{ Theorem 3.5 (a)}]\).

The generic point \(\tilde{\eta}\) of \(\tilde{Y}\) sits in \(\tilde{Y}_{pd}\). Indeed, since \(X\) is irreducible, the generic fiber of \(X \to Y\) is also irreducible and has dimension \(\dim(X) - \dim(Y)\). Therefore the geometric generic fiber of \(X \to Y\) has pure dimension \(\dim(X) - \dim(Y)\). Since the point \(\eta_{K}\) associated to \(\tilde{\eta}\) is the geometric generic point of \(Y\) (cf. Remark 4.4), we have \(\tilde{\eta} \in \tilde{Y}_{pd}\).

The following proposition gives us a condition in terms of the relative height for \(X\) to be the product of \(Y\) with a torsion subvariety of \(A\).

**Proposition 4.6.** Let \(A\) be an abelian variety over \(K\) with an even ample line bundle \(L\). Let \(\tilde{Y}\) be an irreducible variety over \(k\) and put \(Y := \tilde{Y} \otimes_k K\). Let \(p : Y \times A \to Y\) be the first projection. Let \(X\) be an integral closed subscheme of \(Y \times A\) with \(p(X) = Y\). Assume that \(A\) has trivial \(K/k\)-trace. Suppose that the following conditions are satisfied.

(a) We have \(h^L_{X/Y}(\tilde{\eta}) = 0\), where \(\tilde{\eta}\) is the generic point of \(\tilde{Y}\).

(b) There exists a dense subset \(S \subset Y\) such that, for any \(s \in S\), each irreducible component of the geometric fiber \(p_{|X}^{-1}(s)\) with its induced reduced subscheme structure is a torsion subvariety of \(\{s\} \times A\).

Then, there exists a torsion subvariety \(T \subset A\) such that \(X = Y \times T\).

**Proof.** We use the following notation and convention in this proof. For a morphism \(\phi : U \to Y\) and put \(X_U := U \times_Y X\) and let \((p|_X)_U : X_U \to U\) denote the base-change of \(p|_X\) by \(\phi\). When we say an irreducible component \(Z\) of a scheme, we regard \(Z\) as a closed subscheme with its induced reduced subscheme structure.

There exists a generically finite dominant morphism \(\phi : U \to Y\) with \(U\) integral and an irreducible component \(Z\) of \(X_U\) such that the restriction \(q : Z \to U\) of the morphism \((p|_X)_U : X_U \to U\) to \(Z\) is a proper flat morphism with geometrically integral fibers. Indeed, there exists a generically finite dominant morphism \(\phi : U \to Y\) with \(U\) integral such that, for any irreducible component \(Z\) of \(X_U\), the restriction \((p|_X)_U\) to \(Z\) is a surjective morphism with geometrically integral generic fiber. Shrinking \(U\) if necessary, we may take \(\phi\) so that \(Z \to U\) is flat, and by [5, Théorème 12.2.4], we may take it moreover so that any fiber of \(Z \to U\) is geometrically integral. The properness of \((p|_X)_U\) follows from the fact that \(X\) is proper over \(Y\).

Let \(K\) be an algebraic closure of the residue field at \(\tilde{\eta}\) and let \(\overline{\eta}_K : \text{Spec} \left( \mathfrak{F}(\overline{\eta}_K) \right) \to Y\) be the corresponding geometric point of \(Y\) defined at the beginning of this subsection. Let \(\overline{\xi}\) be the geometric generic point of \(U\). Then \(\phi(\overline{\xi})\) is the geometric generic point of \(Y\), and it follows from Remark 4.4 that \(\phi(\overline{\xi}) = \overline{\eta}_K\).

We apply Lemma 4.3 to \(U, U \times_{\text{Spec}(K)} A \to U\) and \(Z\) in place of \(Y, A \to Y\) and \(X\). To do that, we remark that, \(\phi^{-1}(S)\) is dense in \(U\) by the assumption (b). Further, we remark that for any \(u \in U\), \(Z_{\pi} := q^{-1}(\pi)\) is an integral subscheme that appears as an irreducible component of \((p|_X)_U^{-1}(\pi) = (p|_X)^{-1}(\phi(\pi))\). By assumption, it follows that, for any \(s \in \phi^{-1}(S), Z_{\pi}\) is a torsion subvariety. Then, Lemma 4.3 concludes that there exist a point \(\sigma \in A\left( \mathfrak{F}(\overline{\eta}_K) \right)\) and an abelian subvariety \(G' \subset \{\overline{\xi}\} \times A\) such that \(Z_{\overline{\xi}} = G' + \sigma\).
We prove that $Z_\tilde{\xi}$ is a torsion subvariety. By the assumption (a), we have
\[
\tilde{h}_{\tilde{\xi}(\eta_K)} \circ \tilde{\rho}^{-1} \left( (p|_X)^{-1} \left( \phi \left( \tilde{\xi} \right) \right) \right) = \tilde{h}_{\tilde{\xi}(\eta_K)} \left( X_{\tilde{\eta_K}} \right) = h^L_X(Y(\eta)) = 0.
\]
Since $Z_{\tilde{\xi}}$ is an irreducible component of $(p|_X)^{-1} \left( \phi \left( \tilde{\xi} \right) \right) = X_{\eta_K}$, we then see from Remark 4.5 that $\tilde{h}_{\tilde{\xi}(\eta_K)} \circ \tilde{\rho}^{-1} \left( Z_{\tilde{\xi}} \right) = 0$. By Proposition 2.3, it follows that $Z_{\tilde{\xi}}$ has dense small points, and hence $Z_{\tilde{\xi}}/G' \subset \{ \{ \xi \} \times A \} / G'$ has dense small points by [21] Lemma 2.1. This means that the image of $\sigma$ in $\{ \{ \xi \} \times A \} / G'$ has canonical height 0. Since $A$ has trivial $\mathcal{K}/k$-trace, $\{ \xi \} \times A$ has trivial $\mathcal{K}(\eta_K)/t$-trace by Lemma A.1 and will be proved in the appendix. Hence, by [21] Lemma 1.5], $\{ \{ \xi \} \times A \} / G'$ has trivial $\mathcal{K}(\eta_K)/t$-trace. By [12] Theorem 5.4, it follows that the image of $\sigma$ in $\{ \{ \xi \} \times A \} / G'$ is a torsion point. Since a surjective homomorphism between abelian varieties over an algebraically closed field induces a surjective homomorphism between the groups of torsion points, there exists a torsion point $\tau \in \{ \xi \} \times A$ having the same image as $\sigma$ in $\{ \{ \xi \} \times A \} / G'$. Then $Z_{\tilde{\xi}} = G' + \sigma = G' + \tau$, which shows that $Z_{\tilde{\xi}}$ is a torsion subvariety of $\{ \xi \} \times A$.

By Proposition 3.7, therefore, there exists a torsion subvariety $T$ of $A$ such that $Z_{\tilde{\xi}} = \{ \xi \} \times T$. We look at the closed subschemes $Y \times T$ and $X \times A$. We remark that, since $\phi \left( \tilde{\xi} \right) = \eta_K$, $Z_{\tilde{\xi}}$ equals the geometric generic fiber of $p|_{Y \times T} : Y \times T \to Y$. Since $Z_{\tilde{\xi}}$ is a closed subscheme of $X_{\eta_K}$, it follows that the generic fiber of $p|_{Y \times T}$ is a closed subscheme of the generic fiber of $p|_{X} : X \to Y$. Since $Y \times T$ is integral, this implies that $Y \times T \subset X$ as subschemes. Since $X$ is integral and $\dim(Y \times T) = \dim Z = \dim X$, we then conclude that $Y \times T = X$ as subschemes. Thus we obtain the proposition.

4.3. Generic constantness of relative heights. The purpose of this subsection is to show that the relative height $h^L_X(Y)$ is generically constant when $A$ is nowhere degenerate.

We begin with a technical lemma.

Lemma 4.7. Let $\tilde{Y}$ be a proper irreducible variety over $k$. Let $K'$ be a finite extension of $K$ and let $\mathcal{B}'$ be the normalization of $\mathcal{B}$ in $K'$. Let $h : \mathcal{X} \to \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$ be a proper surjective morphism with $\mathcal{X}$ integral. Then, there exists a dense open subset $\tilde{V} \subset \tilde{Y}$ satisfying the following conditions.

(a) Let $\text{pr}_Y : \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}' \to \tilde{Y}$ be the first projection. Then the restriction $\tilde{V} \times_Y \mathcal{X} \to \tilde{V}$ of $\text{pr}_Y \circ h : \mathcal{X} \to \tilde{Y}$ is flat.

(b) For any $\tilde{y} \in \tilde{V}$, the restriction $h|_{\{ \tilde{y} \} \times \mathcal{X}} : \{ \tilde{y} \} \times \mathcal{X} \to \{ \tilde{y} \} \times_{\text{Spec}(k)} \mathcal{B}'$ of $h$ is flat over any point of codimension 1 in $\{ \tilde{y} \} \times_{\text{Spec}(k)} \mathcal{B}'$.

Proof. First, since $\mathcal{X}$ is an integral scheme, the generic flatness gives us a dense open subset $\tilde{V}_1$ of $\tilde{Y}$ such that the restriction $\tilde{V}_1 \times \mathcal{X} \to \tilde{V}_1$ of $\text{pr}_Y \circ h : \mathcal{X} \to \tilde{Y}$ is flat.

Next, we construct $\tilde{V}$ which will suffice (b). Let $\tilde{Y}_{\text{reg}}$ be the subset of regular points of $\tilde{Y}$. It is a dense open subset. Since $\mathcal{X}$ is an integral scheme and $\tilde{Y}_{\text{reg}} \times_{\text{Spec}(k)} \mathcal{B}'$ is normal, there exists a closed subset $Z$ of $\tilde{Y}_{\text{reg}} \times_{\text{Spec}(k)} \mathcal{B}'$ with $\text{codim} \left( Z, \tilde{Y}_{\text{reg}} \times_{\text{Spec}(k)} \mathcal{B}' \right) \geq 2$ such that $h : \mathcal{X} \to \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$ is flat over $\left( \tilde{Y}_{\text{reg}} \times_{\text{Spec}(k)} \mathcal{B}' \right) \setminus Z$. Then, for any $\tilde{y} \in \tilde{Y}_{\text{reg}}$, $\{ \tilde{y} \} \times \mathcal{X} Z$
is a closed subset of \( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \), and the base-change \( \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{X} \to \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \) of \( h \) is flat over \( \left( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \right) \setminus \left( \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{Z} \right) \). Let \( \tilde{\eta} \) be the generic point of \( \tilde{Y} \). Then \( \tilde{\eta} \in \tilde{Y}_{\text{reg}} \), and \( \text{codim} \left( \{ \tilde{\eta} \} \times_{\tilde{Y}} \mathcal{Z}, \{ \tilde{\eta} \} \times_{\Spec(K)} \mathcal{B}' \right) \geq 2 \). It follows that there exists a dense open subset \( \tilde{V}_2 \subset \tilde{Y}_{\text{reg}} \) such that, for any \( \tilde{y} \in \tilde{V}_2 \), we have \( \text{codim} \left( \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{Z}, \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \right) \geq 2 \). This means that the condition (b) is satisfied for \( V = V_2 \).

Finally, we set \( Y := V_1 \cap V_2 \). Then it is a dense open subset of \( \tilde{Y} \) which satisfies both the conditions. Thus we conclude the lemma.

**Remark 4.8.** Under the setting of Lemma 4.7, let \( \tilde{V} \subset \tilde{Y} \) be an open subset as in the lemma. Then the above proof shows that there exists a closed subset \( \mathcal{Z}' \subset \tilde{V} \times_{\Spec(K)} \mathcal{B}' \) such that \( \text{codim} \left( \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{Z}', \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \right) \geq 2 \) for any \( \tilde{y} \in \tilde{V} \) and that \( h : \mathcal{X} \to \tilde{Y} \times_{\Spec(K)} \mathcal{B}' \) is flat over \( \left( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \right) \setminus \mathcal{Z}' \). Indeed, if \( \mathcal{Z} \) is the closed subset of \( \tilde{Y}_{\text{reg}} \times_{\Spec(K)} \mathcal{B}' \) as in the proof, then \( \mathcal{Z}' := \mathcal{Z} \cap \left( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \right) \) has those properties.

Now, we show the generic constantness of the relative height:

**Proposition 4.9.** Let \( A \) be an abelian variety over \( \overline{K} \) with an even ample line bundle \( L \). Let \( \tilde{Y} \) be an irreducible variety over \( k \) and put \( Y := {\tilde{Y} \otimes_k \overline{K}}. \) Let \( p : Y \times A \to Y \) be the first projection and let \( X \) be an integral closed subscheme of \( Y \times A \) with \( p(X) = Y \). Assume that \( A \) is nowhere degenerate. Then there exists a dense open subset \( \tilde{V} \) of \( \tilde{Y} \) contained in \( \tilde{Y}_{\text{pd}} \) such that \( h_{X/Y}^L \) is constant over \( \tilde{V} \).

**Proof.** Let \( K', \mathcal{B}', f : \mathcal{A} \to \mathcal{B}' \) and \( \mathcal{Z} \) be those as in Proposition 2.4 for \( A \) and \( L \). Taking a finite extension of \( K' \) if necessary, we assume that \( X \) can be defined over \( K' \). Let \( \mathcal{X} \) be the closure of \( X \) in \( \tilde{Y} \times_{\Spec(K)} \mathcal{A} \). Since \( X \) is integral, \( \mathcal{X} \) is also integral. Let \( h : \mathcal{X} \to \tilde{Y} \times_{\Spec(K)} \mathcal{B}' \) be the restriction to \( \mathcal{X} \) of the base-change \( f_{\tilde{Y}} : \tilde{Y} \times_{\Spec(K)} \mathcal{A} \to \tilde{Y} \times_{\Spec(K)} \mathcal{B}' \) of \( f \). Since \( f_{\tilde{Y}} \) is proper, \( h \) is also proper. The restriction of \( h \) to the geometric generic fiber over \( \mathcal{B}' \) equals \( p|_X \). Since \( p|_X \) is surjective and \( \mathcal{B}' \) is irreducible, it follows that \( h \) is surjective. By Lemma 4.7, we then take a dense open subset \( \tilde{V} \) satisfying the conditions (a) and (b) of Lemma 4.7.

Let us show that \( \tilde{V} \subset \tilde{Y}_{\text{pd}} \). Let \( \tilde{y} \in \tilde{V} \) be any point. Recall from \( \S 4.2 \) that \( \overline{y_K} \) is the geometric generic point of \( \{ \tilde{y} \} \times_{\Spec(k)} \mathcal{B}' \) and hence a geometric point of \( \tilde{Y} \times_{\Spec(K)} \mathcal{B}' \), and furthermore recall that \( X_{\overline{y_K}} \) is nothing but the fiber of \( h : \mathcal{X} \to \tilde{Y} \times_{\Spec(K)} \mathcal{B}' \) over \( \overline{y_K} \). Let \( \mathcal{Z}' \) be a closed subset of \( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \) as in Remark 4.8. Since \( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \) \( \setminus \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{Z}' \) is a dense open subset of \( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \) and \( \overline{y_K} \) is the geometric generic point of \( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \), \( \overline{y_K} \) is also the geometric generic point of \( \{ \tilde{y} \} \times_{\Spec(K)} \mathcal{B}' \) \( \setminus \{ \tilde{y} \} \times_{\tilde{Y}} \mathcal{Z}' \) and hence is a geometric point of \( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \) \( \setminus \mathcal{Z}' \). Since \( h \) is flat over \( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \) \( \setminus \mathcal{Z}' \) and \( \mathcal{X} \) is irreducible, it follows that the fiber \( X_{\overline{y_K}} \) of \( h \) over \( \overline{y_K} \) has pure dimension

\[
\dim(\mathcal{X}) - \dim \left( \tilde{V} \times_{\Spec(K)} \mathcal{B}' \right) = \dim(X) - \dim(Y).
\]

This shows \( \tilde{y} \in \tilde{Y}_{\text{pd}} \), and thus \( \tilde{V} \subset \tilde{Y}_{\text{pd}} \).
We are going to prove that $\mathbf{h}^L_{X/Y}$ is constant over $\tilde{V}$. To do that, we describe $\mathbf{h}^L_{X/Y}$ in terms of intersection product on models. For each $\tilde{y} \in \tilde{V}$, let $\tilde{y} : \text{Spec}(k(\tilde{y})) \to \tilde{Y}$ be the geometric point arising from $\tilde{y}$. Since the point $\overline{\tilde{y}_K}$ is the geometric generic point of $\{\tilde{y}\} \times \mathcal{B}'$, we note that $\{\overline{\tilde{y}_K}\} \times A$ is the geometric generic fiber of $f_{\tilde{y}} : \{\tilde{y}\} \times A \to \{\tilde{y}\} \times \mathcal{B}'$, where $f_{\tilde{y}}$ is the restriction of $f_{\tilde{Y}}$. Let $\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L}$ be the pull-back of $\mathcal{L}$ by the canonical projection $\tilde{Y} \times_{\text{Spec}(k)} A \to A$ and let $\mathcal{O}_{\tilde{Y}} \otimes \mathcal{L}$ be the pull-back of $\mathcal{L}$ by the canonical projection $Y \times A \to A$. Then we see that $(\{\tilde{y}\} \times A, (\mathcal{O}_{\tilde{Y}} \otimes \mathcal{L})|_{\{\tilde{y}\} \times A})$ is a model of $(\{\tilde{y}_K\} \times A, (\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L})|_{\{\tilde{y}_K\} \times A})$ satisfying the conditions of Proposition 2.4.

Let $\text{pr}_Y : \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}' \to \tilde{Y}$ be the projection. Put $\mathcal{H}_y := (\text{pr}_Y \circ h)^{-1}(\tilde{y})$, which is a closed subscheme of $\{\tilde{y}\} \times A$. We remark that $p|_X : X \to Y$ equals the restriction of $h : \mathcal{H} \to \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$ to the geometric generic fiber over $\mathcal{B}'$. Then we see that $X_{\tilde{y}_K} := (p|_X)^{-1}(\tilde{y}_K)$ is the geometric generic fiber of $h|_{\mathcal{H}} : \mathcal{H} \to \{\tilde{y}\} \times \mathcal{B}'$. By the condition (b) of Lemma 4.7 for $\tilde{V}$, the proper morphism $h|_{\mathcal{H}}$ is flat over any point of $\{\tilde{y}\} \times \mathcal{B}'$ of codimension 1. This means that it is a model of $X_{\tilde{y}_K}$ in our sense.

Recalling (4.4.6), we then apply Lemma 2.6 to obtain

$$
\mathbf{h}^L_{X/Y}(\tilde{y}) = \frac{\deg_{k(\tilde{y}) \otimes_k \mathcal{H}'}(f_{\tilde{y}}^*) \left( c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L})|_{\{\tilde{y}\} \times A} \right) \right)^{(\dim(X) - \dim(Y) + 1)} \cdot [\mathcal{H}_y]}{[K' : K]},
$$

where $\mathcal{H}'$ is the pull-back of $\mathcal{H}$ by the morphism $\mathcal{B}' \to \mathcal{B}$ and $k(\tilde{y}) \otimes_k \mathcal{H}'$ is the pull-back of $\mathcal{H}'$ by the natural morphism $\{\tilde{y}\} \times \mathcal{B}' \to \mathcal{B}'$. Using the projection formula, we see

$$
de\deg_{k(\tilde{y}) \otimes_k \mathcal{H}'}(f_{\tilde{y}}^*) \left( c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L})|_{\{\tilde{y}\} \times A} \right) \right)^{(\dim(X) - \dim(Y) + 1)} \cdot [\mathcal{H}_y]
$$

$$
= \deg \left( c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k f^*(\mathcal{H}'))|_{\{\tilde{y}\} \times A} \right)^{(b-1)} \cdot c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L})|_{\{\tilde{y}\} \times A} \right)^{(\dim(X) - \dim(Y) + 1)} \cdot [\mathcal{H}_y] \right),
$$

and hence

$$
\mathbf{h}^L_{X/Y}(\tilde{y}) = \frac{\deg \left( c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k f^*(\mathcal{H}'))|_{\{\tilde{y}\} \times A} \right)^{(b-1)} \cdot c_1 \left( (\mathcal{O}_{\tilde{Y}} \otimes_k \mathcal{L})|_{\{\tilde{y}\} \times A} \right)^{(\dim(X) - \dim(Y) + 1)} \cdot [\mathcal{H}_y] \right)}{[K' : K]}.
$$

By the condition (a) of Lemma 4.7, $\text{pr}_Y \circ h$ is flat over $\tilde{V}$. By [3] Theorem 10.2, it follows that the intersection number on the left-hand side in (4.9.8) is independent of $\tilde{y} \in \tilde{V}$, and hence $\mathbf{h}^L_{X/Y}$ is also independent of $\tilde{y} \in \tilde{V}$. Thus we conclude the proposition.

5. APPLICATION TO THE GEOMETRIC BOGOMOLOV CONJECTURE

In this section, we prove Theorem 1.5 and reduce the geometric Bogomolov conjecture for any abelian varieties to the conjecture for those abelian varieties which are nowhere
degenerate and have trivial trace. We begin with a proposition which will be the key to this goal.

**Proposition 5.1.** Let $A$ be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace and let $L$ be an even ample line bundle on $A$. Let $\tilde{B}$ be an abelian variety over $k$ and set $B := \tilde{B} \otimes_k \overline{K}$. Let $X$ be an irreducible closed subvariety of $B \times A$. Let $\text{pr}_B : B \times A \to B$ be the natural projection and set $Y := \text{pr}_B(X)$. Suppose that $X$ has dense small points. Then, there exists a closed subvariety $\tilde{Y} \subset \tilde{B}$ such that $Y = \tilde{Y} \otimes_k \overline{K}$. Furthermore, $h^L_{X/Y}(\tilde{y}) = 0$ holds for general $\tilde{y} \in \tilde{Y}(k)$.

**Proof.** Let $\tilde{M}$ be an even and very ample line bundle on $\tilde{B}$ and set $M := \tilde{M} \otimes_k \overline{K}$. Since $X$ has dense small points, so does $Y$ by [22, Lemma 7.7], and hence $\tilde{h}_M(Y) = 0$ by Proposition [23]. It follows from Proposition [3.2] that there exists a closed subvariety $\tilde{Y}$ of $\tilde{B}$ such that $Y = \tilde{Y} \otimes_k \overline{K}$. Thus we obtain the first part of the proposition.

Next, we discuss the second part of the proposition. For $A$ and $L$, let $K', f : \mathcal{A} \to \mathcal{B}'$, and a line bundle $\mathcal{L}$ on $\mathcal{A}$ be as in Proposition [2.4]. Taking a finite extension of $K'$ if necessary, we may assume that $X$ can be defined over $K'$. Let $\mathcal{H}'$ be the pull-back of $\mathcal{H}$ by the morphism $\mathcal{B}' \to \mathcal{B}$.

Set $\mathcal{B} := \tilde{B} \times_{\text{Spec}(k)} \mathcal{B}'$. Then the canonical projection $\mathcal{B} \to \mathcal{B}'$ is a model of $B$, and $\tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$ equals the closure of $Y$ in $\mathcal{B}$. Let $\text{pr}_{\mathcal{A}} : \mathcal{B} \times_{\mathcal{B}'} \mathcal{A} \to \mathcal{A}$ be the canonical projection and set $\varphi := f \circ \text{pr}_{\mathcal{A}} : \mathcal{B} \times_{\mathcal{B}'} \mathcal{A} \to \mathcal{B}'$. Then $\varphi$ is an abelian scheme and is a model of $B \times A$. Let $\mathcal{X}'$ be the closure of $X$ in the model $\mathcal{B} \times_{\mathcal{B}'} \mathcal{A}$. Since $X$ is integral, $\mathcal{X}'$ is also integral. Let $\text{pr}_{\mathcal{B}} : \mathcal{B} \times_{\mathcal{B}'} \mathcal{A} \to \mathcal{B}$ be the natural projection. Since $\text{pr}_B(X) = Y$, it follows that $\text{pr}_{\mathcal{B}}(\mathcal{X}') = \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$. Let $h : \mathcal{X} \to \tilde{Y} \times_{\text{Spec}(k)} \mathcal{B}'$ the morphism given from $\text{pr}_{\mathcal{B}}$ by restriction. Then $h$ is proper and surjective. Now, applying Lemma [4.7] to this $h$, we obtain a dense open subset $\tilde{V} \subset \tilde{Y}$ as in Lemma [4.7].

For any $\tilde{y} \in \tilde{V}(k)$, let $\tilde{y}_{K'}$ be the geometric point of $Y$ corresponding to $\tilde{y}$ (cf. §4.2). Note that $\{\tilde{y}\} \times \mathcal{B}' = \mathcal{B}'$ naturally and note that $\tilde{y}_{K'}$ is also regarded as the geometric generic point of $\{\tilde{y}\} \times \mathcal{B}' = \mathcal{B}'$ and is a $\overline{K}$-valued point. Let $f_{\tilde{y}} : \{\tilde{y}\} \times \mathcal{A} \to \{\tilde{y}\} \times \mathcal{B}'$ be the restriction of $\text{pr}_{\mathcal{B}} : \mathcal{B} \times_{\mathcal{B}'} \mathcal{A} \to \mathcal{B}$. We remark that, via natural identification $\{\tilde{y}\} \times \mathcal{A} = \mathcal{A}$ and $\{\tilde{y}\} \times \mathcal{B}' = \mathcal{B}'$, $f_{\tilde{y}}$ coincides with $f$. Further, we remark that the pair $(f_{\tilde{y}} : \{\tilde{y}\} \times \mathcal{A} \to \{\tilde{y}\} \times \mathcal{B}', \mathcal{L})$ is a model of $(\overline{\{y\}} \times A, L)$ as in Proposition [2.4] where $\mathcal{L}$ and $L$ are regarded as line bundles on $\{\overline{y}\} \times \mathcal{A}$ and $\{\overline{y}\} \times A$ via the natural isomorphism $\{\tilde{y}\} \times \mathcal{A} = \mathcal{A}$ and $\{\tilde{y}_{K}\} \times A = A$ respectively.

Let $q : \mathcal{B} = \tilde{B} \times_{\text{Spec}(k)} \mathcal{B}' \to \tilde{B}$ be the natural projection. For any $\tilde{y} \in \tilde{V}(k)$, set $\mathcal{X}_{\tilde{y}} := (q \circ h)^{-1}(\tilde{y})$, which is a closed subscheme of $\{\tilde{y}\} \times \mathcal{A}$. Let $p : Y \times A \to Y$ be the restriction of $\text{pr}_B : B \times A \to B$. Then we see that the geometric fiber $X_{\tilde{y}K}$ of $p|_X : X \to Y$ over $\overline{\{y\}}$ equals the geometric generic fiber of $h|_{\mathcal{X}_{\tilde{y}}} : \mathcal{X}_{\tilde{y}} \to \{\tilde{y}\} \times \mathcal{B}'$. By the condition (b) of Lemma [4.7], $h|_{\mathcal{X}_{\tilde{y}}} : \mathcal{X}_{\tilde{y}} \to \{\tilde{y}\} \times \mathcal{B}' = \mathcal{B}'$ is flat over any point of $\mathcal{B}'$ of codimension 1. Set $d := \dim(X)$ and $e := \dim(Y)$. Then, it follows from Lemma [2.6] and the definition (4.4.6) of $h^L_{X/Y}$ that, for any $\tilde{y} \in \tilde{V}(k)$, we have

\begin{equation}
(5.1.9) \quad [K' : K]h^L_{X/Y}(\tilde{y}) = \deg_{\mathcal{H}'}(f_{\tilde{y}})_* \left( c_1(\mathcal{L})^{d+1-e} \cdot [\mathcal{X}_{\tilde{y}}] \right),
\end{equation}
where \( \mathcal{H}' \) is naturally regarded as a line bundle on \( \{ \overline{y} \} \times \mathfrak{B}' \) via \( \{ \overline{y} \} \times \mathfrak{B}' = \mathfrak{B}' \). Equality (5.1.9) will be used later.

Set \( M \boxtimes L := \text{pr}_B^*(M) \otimes \text{pr}_A^*(L) \), where \( \text{pr}_A : B \times A \to A \) is the canonical projection. The multilinearity of the canonical height with respect to line bundles (cf. [8, Theorem 11.18 (a)]) gives us

\[
\hat{h}_{M \boxtimes L}(X) = \sum_{j=0}^{d+1} \binom{d+1}{j} \hat{h}_{\text{pr}_B^*(M), \ldots, \text{pr}_B^*(M), \text{pr}_A^*(L), \ldots, \text{pr}_A^*(L)}(X).
\]

Since \( X \) has dense small points, we have \( \hat{h}_{M \boxtimes L}(X) = 0 \) by Proposition 2.3. On the other hand, by [8, Theorem 11.18 (e)], we have

\[
\hat{h}_{\text{pr}_B^*(M), \ldots, \text{pr}_B^*(M), \text{pr}_A^*(L), \ldots, \text{pr}_A^*(L)}(X) \geq 0
\]

for each \( j \). It follows that

\[
\text{(5.1.10)} \quad \hat{h}_{\text{pr}_B^*(M), \ldots, \text{pr}_B^*(M), \text{pr}_A^*(L), \ldots, \text{pr}_A^*(L)}(X) = 0.
\]

Recall that \( \varphi : \mathfrak{B} \times_{\mathfrak{B}'} \mathfrak{A} \to \mathfrak{B}' \) is a model of \( B \times A \) and that \( q : \mathfrak{B} = \widetilde{B} \times \text{Spec}(k) \mathfrak{B}' \to \widetilde{B} \) is the canonical projection. Set \( \mathcal{M} := q^*(\overline{M}) \). Then \( (\varphi, \text{pr}_{\mathfrak{B}'}^*(\mathcal{M})) \) and \( (\varphi, \text{pr}_{\mathfrak{A}'}^*(\mathcal{L})) \) are models of \( (B \times A, \text{pr}_B^*(M)) \) and \( (B \times A, \text{pr}_A^*(L)) \) respectively, satisfying the conditions in Proposition 2.3 (cf. Remark 2.5). Via the description (2.5.3) or Lemma 2.6, equality (5.1.10) gives us

\[
\text{(5.1.11)} \quad \deg_{\mathcal{H}'} \varphi_* \left( c_1(\text{pr}_{\mathfrak{A}'}^*(\mathcal{L}))^{(d+1-e)} \cdot c_1(\text{pr}_{\mathfrak{B}'}^*(\mathcal{M}))^e \cdot [\mathcal{X}] \right) = 0.
\]

With the preparation so far, let us prove that \( h^L_{X/Y} \) vanishes generically on \( \overline{Y}(k) \). Let \( \overline{M} \) denote the complete linear system over \( \widetilde{B} \) associated to \( \overline{M} \). Since \( \widetilde{M} \) is very ample, the linear system \( (q \circ \text{pr}_{\mathfrak{B}'})^*\overline{M} \) over \( \mathfrak{B} \times_{\mathfrak{B}'} \mathfrak{A} \) is base-point free. By Lemma 3.1, it follows that, for a general \( (\overline{D}_1, \ldots, \overline{D}_e) \in \overline{M}^e \), the cycle

\[
\left( (q \circ \text{pr}_{\mathfrak{B}'})^{-1} (\overline{D}_1) \cap \cdots \cap (q \circ \text{pr}_{\mathfrak{B}'})^{-1} (\overline{D}_e) \cap \mathcal{X} \right) = \left( (q \circ \text{pr}_{\mathfrak{B}'})^{-1} (\overline{D}_1 \cap \cdots \cap \overline{D}_e) \cap \mathcal{X} \right)
\]

represents the cycle class \( c_1(\text{pr}_{\mathfrak{B}'}^*(\mathcal{M}))^e \cdot [\mathcal{X}] \). Furthermore, for a general \( (\overline{D}_1, \ldots, \overline{D}_e) \in \overline{M}^e \), the intersection \( \overline{D}_1 \cap \cdots \cap \overline{D}_e \) is a finite closed subscheme of the regular locus \( \overline{V}_{\text{reg}} \) of \( \overline{V} \subset \overline{Y} \), so that we write

\[
\overline{D}_1 \cap \cdots \cap \overline{D}_e = [\overline{y}_1] + \cdots + [\overline{y}_m]
\]

with \( \overline{y}_1, \ldots, \overline{y}_m \in \overline{V}_{\text{reg}}(k) \). Then we have

\[
\left( (q \circ \text{pr}_{\mathfrak{B}'})^{-1} (\overline{D}_1 \cap \cdots \cap \overline{D}_e) \cap \mathcal{X} \right) = \sum_{i=1}^m [(\{ \overline{y}_i \} \times \mathfrak{A}) \cap \mathcal{X}],
\]
and hence
\[(5.1.12) \quad \deg_{\mathcal{H}'} \varphi_*(c_1\left(\text{pr}_A^*(\mathcal{L})\right)^{(d+1-e)} \cdot c_1\left(\text{pr}_B^*(\mathcal{M})\right)^e \cdot [\mathcal{X}]) = \sum_{i=1}^m \deg_{\mathcal{H}'} \left(\varphi|_{(\tilde{y}_i) \times \mathcal{A}}\right)_* \left(c_1(\mathcal{L})^{(d+1-e)} \cdot \left(\{(\tilde{y}_i) \times \mathcal{A}\} \cap \mathcal{X}\right)\right),\]

where \(\mathcal{L}\) is naturally regarded as a line bundle on \(\{\tilde{y}_i\} \times \mathcal{A}\).

Since \((q \circ \text{pr}_B)|_{\mathcal{X}} = q \circ h\) by the definition of \(h\), we have \(\{(\tilde{y}_i) \times \mathcal{A}\} \cap \mathcal{X} = \mathcal{X}_{\tilde{y}_i}\). We remark that \(f_{\tilde{y}_i} = \varphi|_{(\tilde{y}_i) \times \mathcal{A}}\) via the natural isomorphism \(\{\tilde{y}_i\} \times \mathcal{B}' = \mathcal{B}'\). By \((5.1.11)\), it follows that the right-hand side of \((5.1.12)\) equals \(|K' : K| \sum_{i=1}^m h_{X/Y}(\tilde{y}_i)\). On the other hand, the left-hand side of \((5.1.12)\) equals 0 by \((5.1.11)\). By Remark \((4.5)\) we thus have \(h_{X/Y}(\tilde{y}_i) = 0\).

In summary, we have seen that there exists a dense open subset \(U \subset \tilde{M}\) such that, for any \((\tilde{D}_1, \ldots, \tilde{D}_e) \in U\), and for any \(\tilde{y} \in \tilde{D}_1 \cap \cdots \cap \tilde{D}_e\) we have
\[(5.1.13) \quad h_{X/Y}(\tilde{y}) = 0.\]

Since \(\tilde{M}\) is very ample, a general point \(\tilde{y} \in \tilde{Y}(k)\) is a point of the intersection \(\tilde{D}_1 \cap \cdots \cap \tilde{D}_e\) for some \((\tilde{D}_1, \ldots, \tilde{D}_e) \in U\). It follows from equality \((5.1.13)\) that \(h_{X/Y}(\tilde{y}) = 0\) for a general \(\tilde{y} \in \tilde{Y}(k)\). This is the second statement of the proposition, and thus we complete the proof. \(\square\)

**Proposition 5.2.** Let \(A\) be an abelian variety over \(\overline{K}\) with trivial \(\overline{K}/k\)-trace and let \(L\) be an even ample line bundle on \(A\). Let \(\tilde{Y}\) be an irreducible variety over \(k\) and set \(Y := \tilde{Y} \otimes_k \overline{K}\). Let \(p : Y \times A \to Y\) denote the canonical projection and let \(X\) be an irreducible closed subvariety of \(Y \times A\) such that the restriction \(p|_X : X \to Y\) is surjective. Suppose that there exists a dense open subset \(\tilde{U}\) of \(\tilde{Y}\) with \(\tilde{U} \subset \tilde{Y}_{pd}\) such that \(h_{X/Y}^L = 0\) over \(\tilde{U}\). Then, if the geometric Bogomolov conjecture holds for \(A\), then there exists a torsion subvariety \(T\) of \(A\) such that \(X = Y \times T\).

**Proof.** To use Proposition \((4.6)\) we check that the assumptions of it are satisfied. First, since the generic point \(\bar{\eta}\) of \(\tilde{Y}\) sits in \(\tilde{U}\), we have \(h_{X/Y}^L(\bar{\eta}) = 0\). Thus the condition (a) of Proposition \((4.6)\) is verified. To see the condition (b), let \(S\) be the preimage of \(\tilde{U}(k)\) by the natural morphism \(Y \to \tilde{Y}\). Note that \(S\) is a dense subset of \(Y\). For any \(\tilde{y} \in \tilde{U}(k)\), let \(\bar{y}_K\) be the corresponding geometric point of \(Y\). By definition, we have \(\bar{y}_K \in S \cap Y(\overline{K})\). Since \(h_{X/Y}^L(\tilde{y}) = 0\), it follows from Remark \((4.5)\) that any irreducible component of \(X_{\bar{y}_K}\) is a closed subvariety of \(A\) of canonical height zero. Suppose that the geometric Bogomolov conjecture holds for \(\{\tilde{y}\} \times A = A\). Then, it follows from Proposition \((2.3)\) that \(X_{\bar{y}_K}\) is a torsion subvariety. Thus, the condition (b) of Proposition \((4.6)\) is verified, and hence Proposition \((5.2)\) follows from Proposition \((4.6)\) \(\square\)

As a consequence, we obtain the following theorem.

**Theorem 5.3.** Let \(A\) be a nowhere degenerate abelian variety over \(\overline{K}\) with trivial \(\overline{K}/k\)-trace. Let \(B\) be an abelian variety over \(k\) and set \(B = B \otimes_k \overline{K}\). Let \(\text{pr}_A : B \times A \to A\) and
pr_{B} : B \times A \to B denote the canonical projections. Let X be an irreducible closed subvariety of \( B \times A \) and set \( Y := \text{pr}_{B}(X) \) and \( T := \text{pr}_{A}(X) \). Suppose that X has dense small points and assume that the geometric Bogomolov conjecture holds for A. Then, Y is a constant subvariety of B, T is a torsion subvariety of A, and \( X = Y \times T \) holds.

Proof. Let L be an even ample line bundle on A. It follows from Proposition 5.1 that Y is a constant subvariety and \( h^{L}_{X/Y}(y) = 0 \) holds for general \( y \in \tilde{Y}(k) \), where \( \tilde{Y} \) is a closed subvariety of \( \tilde{B} \) with \( Y = \tilde{Y} \otimes_{k} \overline{K} \). By Proposition 4.9, therefore, \( h^{L}_{X/Y} = 0 \) on some dense open subset of \( \tilde{Y} \) contained in \( \tilde{Y}_{pd} \). Then Proposition 5.2 concludes that T is a torsion subvariety, and \( X = Y \times T \) holds. \( \blacksquare \)

Let \( m \) be the maximal nowhere degenerate abelian subvariety of A. Let \( (\tilde{A}^{K/k}, \text{Tr}_{A}) \) be the \( \overline{K}/k \)-trace of A and let \( t \) be the image of \( \text{Tr}_{A} \). Then, by [22, Proposition 7.11], we have \( t \subset m \).

Remark 5.4. The quotient \( m/t \) is nowhere degenerate and has trivial \( \overline{K}/k \)-trace. Indeed, the nondegeneracy follows from [22, Lemma 7.8 (2)], and the triviality of the trace follows from the well-known fact that a surjective homomorphism between abelian varieties over \( \overline{K} \) induces a surjective homomorphism between their \( \overline{K}/k \)-traces (cf. [21, Lemma 1.5]).

Now, we establish the second main theorem of this paper, which includes Theorem 1.3.

Theorem 5.5 (Theorem 1.5). Let A be an abelian variety over \( \overline{K} \). Let \( m \) be the maximal nowhere degenerate abelian subvariety of A and let \( t \) be the image of the \( \overline{K}/k \)-trace homomorphism of A. Then the following statements are equivalent to each other.

(a) The geometric Bogomolov conjecture holds for A.
(b) The geometric Bogomolov conjecture holds for \( m \).
(c) The geometric Bogomolov conjecture holds for \( m/t \).

Proof. The equivalence between (a) and (b) is nothing but Theorem 1.3. It follows from [22, Lemma 7.7] that (b) implies (c).

To prove that (c) implies (b), suppose that the geometric Bogomolov conjecture holds for \( m/t \). We put B := \( \tilde{A}^{K/k} \otimes_{k} \overline{K} \). Since \( t \subset m \) and since \( \text{Tr}_{A} : B \to t \) is an isogeny, \( m \) is isogenous to \( B \times m/t \). By [22, Corollary 7.6], it suffices to show that the conjecture holds for this abelian variety. Let \( X \subset B \times m/t \) be an irreducible closed subvariety having dense small points. Let \( \text{pr}_{B} : B \times m/t \to B \) and \( \text{pr}_{m/t} : B \times m/t \to m/t \) be the natural projections. Since \( m/t \) is nowhere degenerate and has trivial \( \overline{K}/k \)-trace (cf. Remark 5.4), Theorem 5.3 then tells us that \( \text{pr}_{B}(X) \) is a constant subvariety, \( \text{pr}_{m/t}(X) \) is a torsion subvariety, and \( X = \text{pr}_{B}(X) \times \text{pr}_{m/t}(X) \). This shows that the geometric Bogomolov conjecture holds for \( B \times m/t \). Thus we conclude that (c) implies (b), which completes the proof the theorem. \( \blacksquare \)

Remark 5.6. (1) Theorem 1.4 follows from Theorem 5.5 (Theorem 1.5). Indeed, since the trace homomorphism is an isogeny, \( \dim(\tilde{A}^{K/k}) = \text{nd-rk}(A) \) implies \( m/t = 0 \). Then the geometric Bogomolov conjecture for A holds by Theorem 5.5 (Theorem 1.5).

(2) Since the geometric Bogomolov conjecture holds for abelian varieties of dimension not greater than 1, a direct application of Theorem 5.5 tells us that the conjecture holds...
for $A$ with $\dim(m/t) \leq 1$, or equivalently, with $\dim\left(\overline{A^{\mathbb{K}}/k}\right) \geq \text{nd-rk}(A) - 1$. This seems to give us a result stronger than Theorem 1.4. However, it is not stronger in fact because $\dim\left(\overline{A^{\mathbb{K}}/k}\right) \geq \text{nd-rk}(A) - 1$ leads us to $\dim\left(\overline{A^{\mathbb{K}}/k}\right) = \text{nd-rk}(A)$. Indeed, suppose $\dim\left(\overline{A^{\mathbb{K}}/k}\right) \geq \text{nd-rk}(A) - 1$, i.e., $\dim(m/t) \leq 1$. If we had $\dim(m/t) = 1$, then $m/t$ would be a constant variety (see the argument in Remark 3.6), but this is a contradiction by Remark 5.4. Thus $\dim\left(\overline{A^{\mathbb{K}}/k}\right) \geq \text{nd-rk}(A)$ implies $\dim(m/t) = 0$, i.e., $\dim\left(\overline{A^{\mathbb{K}}/k}\right) = \text{nd-rk}(A)$.

**Remark 5.7.** Since the abelian variety $m/t$ is a nowhere degenerate abelian variety over $K$ with trivial $K/k$-trace (cf. Remark 5.4), it follows from Theorem 5.5 that Conjecture 1.1 is reduced to the geometric Bogomolov conjecture for nowhere degenerate abelian varieties with trivial $K/k$-trace. Further, since any special subvariety of an abelian variety with trivial $K/k$-trace is a torsion subvariety, Conjecture 1.1 is in fact reduced to the following conjecture.

**Conjecture 5.8** (Geometric Bogomolov conjecture for nowhere abelian varieties with trivial trace). Let $A$ be a nowhere degenerate abelian variety over $K$ with trivial $K/k$-trace. Then any irreducible closed subvariety of $A$ with dense small points is a torsion subvariety.

**Appendix. Field extension and the trace**

In this appendix, we show the following lemma, which is used in the proof of Proposition 4.6.

**Lemma A.1.** Let $F/k$ be a field extension with $F$ algebraically closed and let $\mathfrak{t}/k$ be a field extension with $\mathfrak{t}$ algebraically closed such that $\mathfrak{t} \otimes_k F$ is an integral domain. Let $\mathfrak{g}$ be a field containing $\mathfrak{t} \otimes_k F$ as a subring. Let $A$ be an abelian variety over $F$ and suppose that $A$ has trivial $F/k$-trace. Then $A \times_{\text{Spec}(F)} \text{Spec}(\mathfrak{g})$ has trivial $\mathfrak{g}/\mathfrak{t}$-trace.

**Proof.** Let $B$ be an abelian variety over $\mathfrak{t}$. Then there exist a finitely generated $k$-subalgebra $R \subset \mathfrak{t}$ and an abelian scheme $B \rightarrow \text{Spec}(R)$ such that $B = B \times_{\text{Spec}(R)} \text{Spec}(\mathfrak{t})$. Let

$$\phi : B \times_{\text{Spec}(\mathfrak{t})} \text{Spec}(\mathfrak{g}) \rightarrow A \times_{\text{Spec}(F)} \text{Spec}(\mathfrak{g})$$

be a homomorphism. Then there exist a finitely generated $F$-algebra $S$ with $R \otimes_k F \subset S \subset \mathfrak{g}$ and a homomorphism

$$\Phi : B \times_{\text{Spec}(R)} \text{Spec}(S) \rightarrow A \times_{\text{Spec}(F)} \text{Spec}(S)$$

such that its base-change $\Phi_{\mathfrak{g}}$ to $\text{Spec}(\mathfrak{g})$ coincides with $\phi$.

We set $X' := \text{Spec}(R)$, $X := \text{Spec}(R \otimes_k F) = X' \times_{\text{Spec}(k)} \text{Spec}(F)$, and $Y := \text{Spec}(S)$. The morphism $f : Y \rightarrow X$ induced from the inclusion $R \otimes_k F \subset S$ is a morphism of varieties over $F$ and is dominant, so that there exists a dense open subset $U \subset X$ which is contained in the image of $f$. Since $X = X' \times_{\text{Spec}(k)} \text{Spec}(F)$, we have a natural injection $X'(k) \rightarrow X(F)$, and let $X(k)$ denote its image. Then $f^{-1}(X(k) \cap U)$ is dense in $Y$.\footnote{This should be compared with [22, Conjecture 7.22].}
Let \( g : X \to X' \) be the natural projection. For any \( x \in X(k) \cap U \), the fiber \( f^{-1}(x) \) is a closed subscheme of \( Y \) and is a scheme over \( X' \) via \( g \circ f \). We consider the restriction
\[
\Phi_{f^{-1}(x)} : \mathcal{B} \times_{X'} f^{-1}(x) \to A \times_{\text{Spec}(F)} f^{-1}(x)
\]
of \( \Phi \), which is a homomorphism of abelian schemes over \( f^{-1}(x) \). Let \( y \in f^{-1}(x) \) be any closed point, which is a \( F \)-valued point of \( Y \). Then the fiber \( \mathcal{B} \times_{X'} \{ y \} \) of the abelian scheme \( \mathcal{B} \times_{X'} f^{-1}(x) \to f^{-1}(x) \) coincides with
\[
\mathcal{B} \times_{X'} \{ y \} = (\mathcal{B} \times_{X'} \{ g(x) \}) \otimes_k F.
\]
Further, the fiber of \( A \times_{\text{Spec}(F)} f^{-1}(x) \to f^{-1}(x) \) over \( y \) equals \( A \). Since \( \mathcal{B} \times_{X'} \{ g(x) \} \) is an abelian variety over \( k \) and since \( A \) has trivial \( F/k \)-trace, it follows that the homomorphism \( \Phi_{f^{-1}(x)} \) is trivial over \( y \). This means that \( \Phi_{f^{-1}(x)} \) is trivial over any closed point of \( f^{-1}(x) \), and hence \( \Phi_{f^{-1}(x)} \) itself is the trivial homomorphism. Since \( x \) is any point of \( X(k) \cap U \) and \( f^{-1}(X(k) \cap U) \) is dense in \( Y \), it follows further that \( \Phi \) is the trivial homomorphism. Thus \( \phi \), which is the base-change of \( \Phi \) to \( \mathfrak{F} \), is also trivial.

Let \( F \), \( \mathfrak{t} \) and \( \mathfrak{F} \) be as in Lemma [A.1]. Let \( (A^F/k, \text{Tr}_A) \) be the \( F/k \)-trace of \( A \) and let \( (A^{\mathfrak{F}/\mathfrak{t}}, \text{Tr}_{A^{\mathfrak{F}}} ) \) be the \( \mathfrak{F}/\mathfrak{t} \)-trace of \( A^{\mathfrak{F}} := A \otimes_{F} \mathfrak{F} \). Let \( \phi : A^F/k \otimes_k \mathfrak{t} \to A^{\mathfrak{F}/\mathfrak{t}} \) be the homomorphism induced by the universality from the base-change \( \text{Tr}_A \otimes_F \mathfrak{F} : A^F/k \otimes_k \mathfrak{F} \to A^{\mathfrak{F}} \) of \( \text{Tr}_A \). We end with a remark that \( \phi \) is a purely inseparable isogeny. Since the trace homomorphism is purely inseparable to its image (cf. [11, VIII § 3 Corollary 2] or [21, Lemma 1.4]), the same holds for \( \phi \). Therefore, we only have to show that \( \dim (A^F/k \otimes_k \mathfrak{t}) \geq \dim (A^{\mathfrak{F}/\mathfrak{t}}) \). Let \( \mathfrak{t} \subset \mathfrak{F} \) be the image of \( \text{Tr}_A \) and let \( q : A \to A/\mathfrak{t} \) be the quotient. By the same argument as Remark [B.4] we note that \( A/\mathfrak{t} \) has trivial \( F/k \)-trace. Thus Lemma [A.1] tells us that \( (A/\mathfrak{t}) \otimes_F \mathfrak{F} \) has trivial \( \mathfrak{F}/\mathfrak{t} \)-trace. It follows that the composite
\[
A^{\mathfrak{F}/\mathfrak{t}} \otimes_{\mathfrak{t}} \mathfrak{F} \xrightarrow{\text{Tr}_{A^{\mathfrak{F}}}} A^{\mathfrak{F}} \xrightarrow{q \otimes_{\mathfrak{F}}} (A/\mathfrak{t}) \otimes_F \mathfrak{F}
\]
is trivial, which means that \( \text{Tr}_{A^{\mathfrak{F}}} (A^{\mathfrak{F}/\mathfrak{t}} \otimes_{\mathfrak{t}} \mathfrak{F}) \subset \mathfrak{t} \otimes_k \mathfrak{t} \). Since the trace homomorphisms are finite (cf. [21, Lemma 1.4]), we obtain
\[
\dim (A^F/k \otimes_k \mathfrak{t}) = \dim (\mathfrak{t} \otimes_k \mathfrak{t}) \geq \dim (\text{Tr}_{A^{\mathfrak{F}}} (A^{\mathfrak{F}/\mathfrak{t}} \otimes_{\mathfrak{t}} \mathfrak{F})) = \dim (A^{\mathfrak{F}/\mathfrak{t}} \otimes_{\mathfrak{t}} \mathfrak{F}) = \dim (A^{\mathfrak{F}/\mathfrak{t}}),
\]
as required.

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