Matrix representation of the generalized Moyal algebra

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Abstract

It is shown that the isomorphism between the generalized Moyal algebra and the matrix algebra follows in a natural manner from the generalized Weyl quantization rule and from the well known matrix representation of the destruction and creation operators.

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This short note is motivated by Merkulov’s paper “The Moyal product is the matrix product” [1], where the canonical isomorphism between the Moyal algebra and an infinite matrix algebra has been found.

Here we are going to show how the results of previous works [2,3,4] and the well known in quantum mechanics [5,6] representation of the position, \( \hat{x} \), and the momentum, \( \hat{p} \), operators lead to isomorphisms between various \(*\)-algebras and infinite matrix algebra.

First remind the basic theorems [3,4].

Let \( P\big[[x, p, \hbar]\big] \) be the \( \mathbb{C} \) linear space of all formal power series of \( x, p \) and \( \hbar \) where \( (x, p) \in \mathbb{R} \times \mathbb{R} \) are the coordinates of the phase space \( \Gamma = \mathbb{R} \times \mathbb{R} \) and \( \hbar \) is a real parameter (the deformation parameter). The phase space \( \Gamma = \mathbb{R} \times \mathbb{R} \) is endowed with usual symplectic form

\[
\omega = dp \wedge dq
\] (1)

Let also \( \hat{P}\big[[\hat{x}, \hat{p}, \hbar]\big] \) be an associative algebra over \( \mathbb{C} \) of the formal power series of \( \hat{x}, \hat{p}, \hbar \). The self-adjoint operators \( \hat{x} \) and \( \hat{p} \) act in a Hilbert space \( \mathcal{H} \) and satisfy the commutation relation

\[
[\hat{x}, \hat{p}] := \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\hat{1}.
\] (2)

As usual, \( \hat{1} \) denotes the unity operator.

\( \hat{P}\big[[\hat{x}, \hat{p}, \hbar]\big] \) is the enveloping algebra of the Heisenberg-Weyl algebra generated by \( \hat{x}, \hat{p}, \hbar \hat{1} \).

The following theorem holds [3,4]

Theorem 1 There exists a vector space isomorphism

\[
W_g : P[[x, p, \hbar]] \longrightarrow \hat{P}\big[[\hat{x}, \hat{p}, \hbar]\big]
\]

such that

(i) \( W_g(1) = \hat{1} \)

\[
W_g(p^m x^n) = \sum_{s=0}^{\min(m,n)} g(m, n, s) \hbar^s \hat{p}^{m-s} \hat{x}^{n-s}
\]

\( m, n \in \mathbb{N}, \quad m + n \neq 0, \quad g(m, n, s) \in \mathbb{C}, \quad g(m, n, 0) = 1 \)

(ii) \( i\hbar W_g \{x, A\}_P = [\hat{x}, W_g(A)] \)
\[ i\hbar W_g \{ p, A \}_P = [\hat{p}, W_g(A)] \]

for every \( A \in P[[x, p, h]] \), with \( \{ \cdot, \cdot \}_P \) denoting the Poisson bracket.

Moreover, every isomorphism \( W_g : P[[x, p, h]] \rightarrow \hat{P}[[\hat{x}, \hat{p}, h]] \) satisfies
the conditions (i) and (ii) iff

\[ g(m, n, s) = \frac{(-1)^s m! n!}{s!(m - s)!(n - s)!} \left( \frac{d^s f(y)}{dy^s} \right)_{y=0} \]

where \( f(y) = \sum_{k=0}^{\infty} f_k y^k \), \( f_0 = 1 \), is a formal series independent of \( \hbar \)

(Of course, one can easily recognize in the conditions (ii) of Theorem 1, the modified Dirac quantization rules).

Then, the second theorem reads [4]

**Theorem 2** Let \( W_g : P[[x, p, h]] \rightarrow \hat{P}[[\hat{x}, \hat{p}, h]] \) be the vector space isomorphism defined in Theorem 1.

Then for any \( A, B \in P[[x, p, h]] \)

\[ W_g(A) W_g(B) = W_g(A \ast_g B) \]

where

\[ A \ast_g B = \alpha^{-1} [(\hat{\alpha} A) \ast (\hat{\alpha} B)] \]

\[ \hat{\alpha} := \alpha \left( -\hbar \frac{\partial^2}{\partial x \partial p} \right) = f \left( -\hbar \frac{\partial^2}{\partial x \partial p} \right) \exp \left\{ \frac{i}{2} \left( -\hbar \frac{\partial^2}{\partial x \partial p} \right) \right\} \]

and “\( \ast \)” stands for the usual Moyal product

\[ A \ast B = A \exp \left\{ \frac{i\hbar}{2} \hat{P} \right\} B \]

\[ A \hat{\rightarrow} B := \{ A, B \}_P = \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} \]
It can be also shown that $W_g(A)$ is a symmetric operator for every real $A \in P[[x, p, h]]$ if and only if the formal series $\alpha = \alpha(y) = f(y) \exp \left\{ \frac{i}{2} y \right\}$ is real.

In terms of $\alpha$ we have

$$g(m, n, s) = \left( \frac{i}{2} \right)^s \frac{m! n!}{(m-s)! (n-s)!} \sum_{k=0}^{s} \frac{(2i)^k}{(s-k)!} \alpha_k$$  \hspace{1cm} (7)

where $\alpha_k$ are defined by

$$\alpha(y) = \sum_{k=0}^{\infty} \alpha_k y^k, \quad \alpha_0 = 1$$ \hspace{1cm} (8)

Now we introduce the well known in quantum mechanics operators, $\hat{a}$ (“the destruction operator”) and its hermitian conjugate $\hat{a}^\dagger$ (“the creation operator”) such that

$$\hat{x} = \frac{1}{2} (\hat{a}^\dagger + \hat{a}) \quad \hat{p} = i\hbar (\hat{a}^\dagger - \hat{a})$$  \hspace{1cm} (9)

$$[\hat{a}, \hat{a}^\dagger] = 1$$

It is an easy matter to show that

$$\hat{x} = \exp \left\{ \frac{1}{2} (\hat{a}^\dagger)^2 \right\} \exp \left\{ \frac{1}{4} \hat{a} \right\} \hat{a}^\dagger \exp \left\{ -\frac{1}{4} \hat{a}^2 \right\} \exp \left\{ -\frac{1}{2} (\hat{a}^\dagger)^2 \right\}$$  \hspace{1cm} (10)

$$\hat{p} = \exp \left\{ \frac{1}{2} (\hat{a}^\dagger)^2 \right\} \exp \left\{ \frac{1}{4} \hat{a}^2 \right\} (-i\hbar \hat{a}) \exp \left\{ -\frac{1}{4} \hat{a}^2 \right\} \exp \left\{ -\frac{1}{2} (\hat{a}^\dagger)^2 \right\}$$

Therefore one can define an algebra isomorphism

$$L := \hat{P}[[\hat{x}, \hat{p}, \hbar]] \longrightarrow \hat{P}[[\hat{a}^\dagger, -i\hbar \hat{a}, \hbar]]$$

by

$$L (\hat{x}) = \hat{a}^\dagger \quad \text{and} \quad L (\hat{p}) = -i\hbar \hat{a}.$$  \hspace{1cm} (11)

Consequently, by Theorems 1 and 2 we obtain the algebra isomorphism

$$L \circ W_g : P[[x, p, h]] \longrightarrow \hat{P}[[\hat{a}^\dagger, -i\hbar \hat{a}, \hbar]]$$
\[
L \circ W_g(p^m x^n) = \sum_{s=0}^{\min(m, n)} \frac{(-\hbar)^s m! n!}{s!(m-s)! (n-s)!} \left. \frac{d^s f(y)}{dy^s} \right|_{y=0} (-i\hbar)^{m-s} (\hat{a}^\dagger)^{n-s}
\]

\[
(L \circ W_g(A)) (L \circ W_g(B)) = L \circ W_g(A \ast_g B), \quad A, B \in P[[x, p, \hbar]]. \quad (12)
\]

Now, employing the standard matrix representation of \(\hat{a}\) and \(\hat{a}^\dagger\) [5,6]

\[
\hat{a} \mapsto a = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & 0 & \sqrt{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
\hat{a}^\dagger \mapsto a^\dagger = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
1 & 0 & 0 & \cdots \\
0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & \sqrt{3} & 0 \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\] \quad (13)

and substituting the matrices \(a\) and \(a^\dagger\) instead of \(\hat{a}\) and \(\hat{a}^\dagger\), respectively, into (12) one finds the algebra isomorphism \(\tilde{W}_g\) between the generalized Moyal algebra \((P[[x, p, \hbar]], \ast_g)\) and the matrix algebra \(P[[a^\dagger, -i\hbar a, \hbar]]\).

Denote \(F^{(m, n)} := (-i\hbar)^m (a^\dagger)^n\). Simple calculations lead to the following non-vanishing elements of the matrices \(F^{(m, n)} (m + n > 0)\):

\[
\left( F^{(m, 0)} \right)^{j, j+m} = (-i\hbar)^m \sqrt{j} (j+1) \ldots (j+m-1),
\]

\[
\left( F^{(0, n)} \right)^{j+n, j} = \sqrt{j} (j+1) \ldots (j+n-1),
\]

\[
\left( F^{(m, n)} \right)^{j, j+m-n} = (-i\hbar)^m (j + m - n) \ldots (j + m - 1) \sqrt{j} (j+1) \ldots (j+m-n-1),
\]

for \(m > n > 0\);

\[
\left( F^{(m, m)} \right)^{j, j} = (-i\hbar)^m j (j+1) \ldots (j+m-1),
\]

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\[
(F^{(m,n)})_{j+n-m,j} = (-i\hbar)^m (j + n - m) \ldots (j + n - 1) \sqrt{j} (j + 1) \ldots (j + n - m - 1),
\]

for \( n > m > 0 \). \quad (14)

Finally, we have
\[
\tilde{W}_g (p^m x^n) = \sum_{s=0}^{\min(m,n)} \frac{(-\hbar)^s m! n!}{s! (m - s)! (n - s)!} \frac{d^s f(y)}{dy^s} |_{y=0} F^{(m-s,n-s)} \]
\quad (15)

This formula corresponds to Merkulov’s result but in slightly another representation and in our case we deal with generalized Moyal products \(*_g\).

**Examples**

(1) *The Moyal \(*_g\)-algebra*

It is well known that this algebra is induced by the Weyl ordering of operators [2,3,4]. In this case the operator \( \hat{\alpha} = 1 \). Hence by (5)
\[
f(y) = \exp \left\{ -\frac{i}{2} y \right\} \Rightarrow \frac{d^s f(y)}{dy^s} |_{y=0} = (-\frac{i}{2})^s
\]

and we get now (the index “\( g \)” is omitted)
\[
\tilde{W} (p^m x^n) = \sum_{s=0}^{\min(m,n)} \frac{(i\hbar)^s m! n!}{2^s s! (m - s)! (n - s)!} F^{(m-s,n-s)} \]
\quad (16)

(compare with Merkulov’s result)

(2) *The \(*_{(st)}\)-algebra*

This algebra follows from the standard ordering
\[
p^m x^n \mapsto \hat{x}^n \hat{p}^m
\]
Here \( \alpha(y) = \exp \left\{ -\frac{i}{2} y \right\} \). Hence
\[
f(y) = \exp \left\{ -iy \right\} \Rightarrow \frac{d^s f(y)}{dy^s} |_{y=0} = (-i)^s
\]

Consequently
\[
\tilde{W}_{st} (p^m x^n) = \sum_{s=0}^{\min(m,n)} \frac{(i\hbar)^s m! n!}{s! (m - s)! (n - s)!} F^{(m-s,n-s)} \]
\quad (17)
(3) The $\ast_{(\text{ast})}$-algebra
This is the algebra which follows from the anti-standard ordering
\[ p^m x^n \mapsto \hat{p}^m \hat{x}^n \]
Now $\alpha(y) = \exp \{ \frac{1}{2} y \}$. Hence $f(y) = 1$ and it remains only one term with $s = 0$ in (15).
Hence
\[ \tilde{W}_{\text{ast}} (p^m x^n) = F^{(m,n)} \] (18)
(Compare with Merkulov’s paper [1]).

(4) The $\ast_{(\text{sym})}$-algebra
Here we deal with the algebra generated by the symmetric ordering. So one has $\alpha(y) = \cos \left( \frac{y}{2} \right)$. Therefore,
\[ f(y) = \frac{1}{2} (1 + \exp \{ -iy \}) \implies \left. \frac{d^s f(y)}{dy^s} \right|_{y=0} = \frac{1}{2} (\delta_{s,0} + (-i)^s). \]
Consequently:
\[ \tilde{W}_{\text{sym}} (p^m x^n) = F^{(m,n)} + \sum_{s=1}^{\min(m,n)} \frac{(ih)^s m! n!}{2(s!) (m-s)! (n-s)!} F^{(m-s,n-s)} \] (19)
Finally we consider

(5) The $\ast_{\text{BJ}}$-algebra
This algebra follows from the Born-Jordan ordering.
Now $\alpha(y) = \frac{\sin \left( \frac{y}{2} \right)}{\left( \frac{y}{2} \right)}$. Therefore
\[ f(y) = \frac{1}{iy} (1 - \exp \{ -iy \}) \implies \left. \frac{d^s f(y)}{dy^s} \right|_{y=0} = \frac{(-i)^s}{s+1} \]
Hence
\[ \tilde{W}_{\text{BJ}} (p^m x^n) = \sum_{s=0}^{\min(m,n)} \frac{(ih)^s m! n!}{(s+1)! (m-s)! (n-s)!} F^{(m-s,n-s)}. \]

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