Analysis on Path Spaces over Riemannian Manifolds with Boundary

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Abstract

By using Hsu’s multiplicative functional for the Neumann heat equation, a natural damped gradient operator is defined for the reflecting Brownian motion on compact manifolds with boundary. This operator is linked to quasi-invariant flows in terms of a integration by parts formula, which leads to the standard log-Sobolev inequality for the associated Dirichlet form on the path space.

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1 Introduction

Stochastic analysis on the path space over a complete Riemannian manifold without boundary has been well developed since 1992 when B. K. Driver [3] proved the quasi-invariance theorem for the Brownian motion on compact Riemannian manifolds. A key point of the study is to first establish an integration by parts formula for the associated gradient operator induced by the quasi-invariant flow, then prove functional inequalities for the corresponding Dirichlet form (see e.g. [1][2] and references within). Moreover,

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some efforts have been made for the study of geometry and topology on Riemannian path or loop spaces (see e.g. [4] and references within).

On the other hand, however, the analysis on the path space over a manifold with boundary is still very open. To see this, let us mention [10] where an integration by parts formula was established on the path space of the one-dimensional reflecting Brownian motion. Let e.g. $X_t = |b_t|$, where $b_t$ is the one-dimensional Brownian motion. For $h \in C([0, T]; \mathbb{R})$ with $h_0 = 0$ and $\int_0^T |\dot{h}_t|^2 dt < \infty$, let $\partial_h$ be the derivative operator induced by the flow $X + \varepsilon h$, i.e.

$$\partial_h F = \sum_{i=1}^n h_{t_i} \nabla_i f(X_{t_1}, \ldots, X_{t_n}),$$

where $n \in \mathbb{N}, 0 < t_1 < \cdots < t_n \leq T$ and $F(X) = f(X_{t_1}, \ldots, X_{t_n})$ for some $f \in C^\infty(M^n)$.

As the main result of [10], when $h \in C^2_0(0, T)$, [10, Theorem 2.3] provides an integration by parts formula for $\partial_h$ by using an infinite-dimensional generalized functional in the sense of Schwartz. Since for non-trivial $h$ the flow is not quasi-invariant, this integration by parts formula can not be formulated by using the distribution of $X$ with a density function, and the induced gradient operator does not provide a Dirichlet form on the $L^2$-space of the distribution of $X$.

In this paper, we shall define quasi-invariant flows on a $d$-dimensional Riemannian manifolds with boundary for all $h \in \mathbb{H}$ in an intrinsic way, where

$$\mathbb{H} := \left\{ h \in C([0, T]; \mathbb{R}^d) : h_0 = 0, \int_0^T |\dot{h}_t|^2 dt < \infty \right\}$$

is the Cameron-Martin space. When $M$ is a half-space of $\mathbb{R}^d$, which essentially reduces to the one-dimensional setting, quasi-invariant flows has been constructed in [1, §4(a)] by solving SDEs with reflecting boundary. We shall modify the idea to the reflecting Brownian motion on a manifold with boundary. By establishing integration by parts formula, these flows will be linked to a damped gradient operator defined by using Hsu’s multiplicative functionals constructed in [9]. From this we will derive the Gross log-Sobolev inequality for the associated Dirichlet form.

To explain the idea of the study in a simple way, we first consider the one-dimensional situation. Let $l_t$ be the local time of $X_t := |b_t|$ at point 0. We have

$$dX_t = db_t + dl_t, \quad X_0 = 0.$$  

Now, for any $h \in \mathbb{H}$ and $\varepsilon > 0$, let $X_{\varepsilon,h}^t$ and its local time $l_t^{\varepsilon,h}$ at 0 solve the equation

$$dX_{\varepsilon,h}^t = db_t + \varepsilon \dot{h}_t dt + dl_t^{\varepsilon,h}, \quad X_{\varepsilon,h}^0 = 0.$$  

By the Girsanov theorem $\{b_t + \varepsilon \dot{h}_t : 0 \leq t \leq T\}$ is a Brownian motion under the probability $R_\varepsilon P$, where
\[ R_\varepsilon = \exp \left[ -\varepsilon \int_0^T \dot{h}_t \, db_t - \frac{\varepsilon^2}{2} \int_0^T |\dot{h}_t|^2 \, dt \right] \]

is a functional of \( X \) since \( db_t = dX_t - dt \). Thus, the distribution of \( X^{\varepsilon,h} \) under \( R_\varepsilon \mathbb{P} \) coincides with that of \( X \) under \( \mathbb{P} \). Therefore, the flow \( X^{\varepsilon,h} \) is quasi-invariant. Moreover, it is easy to see that \( X^{\varepsilon,h}_t = |b_t + \varepsilon h_t| \). So, for a cylindrical function \( \gamma \mapsto F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n}) \), where \( n \geq 1, 0 < t_1 < \cdots < t_n \leq T, f \in C_0^\infty(\mathbb{R}^n_+) \) and \( \gamma \in C([0,1];[0,\infty)) \) with \( \gamma_0 = 0 \), one has

\[
\lim_{\varepsilon \downarrow 0} \frac{F(X^{\varepsilon,h}) - F(X)}{\varepsilon} = \sum_{i=1}^n \text{sgn}(b_{t_i}) h_{t_i} \partial_i f(X_{t_1}, \cdots, X_{t_n})
\]

\[
= \sum_{i=1}^n \text{sgn}(X_{t_i} - L_{t_i}) h_{t_i} \partial_i f(X_{t_1}, \cdots, X_{t_n}),
\]

which is a functional of \( X \). Let \( \tilde{f}(x_1, \cdots, x_n) = f(|x_1|, \cdots, |x_n|) \). We have

\[
D^0_h F := \lim_{\varepsilon \downarrow 0} \frac{F(X^{\varepsilon,h}) - F(X)}{\varepsilon} = \sum_{i=1}^n h_{t_i} \partial_i \tilde{f}(b_{t_1}, \cdots, b_{t_n}).
\]

Combining this with the known integration by parts formula for the Brownian motion, we obtain

\[
\mathbb{E} D^0_h F = \mathbb{E} \left\{ F(X) \int_0^T \dot{h}_t \, db_t \right\}.
\]

Furthermore, let the gradient of \( F \) be fixed as an \( \mathbb{H} \)-valued random variable such that \( \langle D^0 F, h \rangle_{\mathbb{H}} = D^0_h F, h \in \mathbb{H} \). So,

\[
D^0 F = \sum_{i=1}^n (t_i \wedge t) \text{sgn}(X_{t_i} - L_{t_i}) h_{t_i} \partial_i f(X_{t_1}, \cdots, X_{t_n}).
\]

Let \( \mu \) be the distribution of \( X \). By (1.2), the form

\[
\mathcal{E}(F, G) = \mathbb{E}(D^0 F, D^0 G)_{\mathbb{H}}
\]

defined for cylindrical functions \( F \) and \( G \) is closable in \( L^2(\mu) \), and the closure \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \)

is a conservative Dirichlet form. Finally, by the known log-Sobolev inequality on the path space of the Brownian motion, this Dirichlet form satisfies the log-Sobolev inequality

\[
\mu(F^2 \log F^2) \leq 2 \mathcal{E}(F, F), \quad F \in \mathcal{D}(\mathcal{E}).
\]
The main purpose of this paper is to realize the above idea on the path space of the reflecting Brownian motion on a Riemannian manifold with boundary. In this case we no longer have explicit expression of $D^0$. But in Section 2 we shall present an integration by parts formula, which identifies the adapted projection of $D^0$ and that of the damped gradient operator induced by Hsu’s multiplicative functional constructed in [9]. This integration by parts formula will be proved in Section 3. Finally, using the resulting integration by parts formula, the standard log-Sobolev inequality will be addressed in Section 4.

2 Damped Gradient and Integration by Parts

Let $M$ be a $d$-dimensional compact connected Riemannian manifold with boundary $\partial M$. Let $o \in M$ and $T > 0$ be fixed. Then the path space for the reflecting Brownian motion on $M$ starting at $o$ is

$$W = \{ \gamma \in C([0,T]; M) : \gamma_0 = o \}.$$

Let $B_t$ be the $d$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. For any $x \in M$, let $O_x M$ be the set of all orthonormal bases for the tangent space $T_x M$ at point $x$, and let $O(M) := \bigcup_{x \in M} O_x (M)$ be the frame bundle. Then for any $X_0 \in M$, the reflecting Brownian motion can be constructed by solving the SDE

$$dX_t = u_t \circ dB_t + N(X_t)dl_t,$$

where $u_t \in O_{X_t}(M)$ is the horizontal lift of $X_t$ on the frame bundle $O(M)$, and $l_t$ is the local time of $X_t$ on the boundary $\partial M$. Let $\mu$ be the distribution of $X := \{X_t : 0 \leq t \leq T\}$ for $X_0 = o$. Then $\mu$ is a probability measure on the path space $W$.

To define the damped gradient operator, let us introduce the multiplicative functional constructed in [9]. To this end, we need to introduce some $\mathbb{R}^d \otimes \mathbb{R}^d$-valued functionals on the frame bundle. Let Ric be the Ricci curvature on $M$ and $\mathbb{I}$ the second fundamental form on $\partial M$. For any $u \in O(M)$, let

$$R_u(a,b) = \text{Ric}(au, bu), \quad a, b \in \mathbb{R}^d.$$

Let $\pi_\partial : TM \to T\partial M$ be the orthogonal projection at points on $\partial M$, and let $\pi : O(M) \to M$ is the canonical projection. For any $u \in O(M)$ with $\pi u \in \partial M$, let

$$\mathbb{I}_u(a,b) = \langle \pi_\partial au, \pi_\partial bu \rangle, \quad a, b \in \mathbb{R}^d.$$

Finally, let $N$ be the inward unit normal vector field on $\partial M$. For $\pi u \in \partial M$, let

$$P_u(a,b) = \langle ua, N \rangle \langle ub, N \rangle, \quad a, b \in \mathbb{R}^d.$$
For any $u_0 \in O(M)$, let $X_t$ be the reflecting Brownian motion on $M$ with horizontal lift $u_t$. For any $\varepsilon > 0$, let $Q^\varepsilon_t$ solve the following SDE on $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$
(2.2) \quad dQ^\varepsilon_t = -Q^\varepsilon_t \left\{ \frac{1}{2} R_{u_t} dt + (\varepsilon^{-1} P_{u_t} + I_{u_t}) dt \right\}, \quad Q^\varepsilon_0 = I.
$$

According to \cite{9, Theorem 3.4}, when $\varepsilon \downarrow 0$ the process $Q^\varepsilon_t$ converges in $L^2$ to an adapted right-continuous process $Q_t$ with left limit, such that $Q_t P_{u_t} = 0$ if $X_t = \pi u_t \in \partial M$. Consequently, if $\text{Ric} \geq -K$ and $\mathbb{I} \geq -\sigma$ for some continuous functions $K$ and $\sigma$ on $M$, then

$$
\|Q_t\| \leq \exp \left\{ \frac{1}{2} \int_0^t K(X_s) ds + \int_0^t \sigma(X_s) dl_s \right\}, \quad t \geq 0,
$$

where $\| \cdot \|$ is the operator norm on $\mathbb{R}^d$. In particular, $\mathbb{E}\|Q_t\|^p < \infty$ holds for any $p > 1$. For $f \in C(M)$, let

$$
P_t f(x) = \mathbb{E} f(X^x_t), \quad x \in M,
$$

where and in the sequel, $X^x_t$ denotes the solution to \eqref{2.1} for $X_0 = x$. Then $P_t$ is the Neumann semigroup. By \cite{9, Theorem 4.2} (see also the last display in the proof of \cite{9, Theorem 5.1}), $s \mapsto Q_s u_s^{-1} \nabla P_{t-s} f(X_s)$ is a martingale. So,

$$
(2.3) \quad u_0^{-1} \nabla P_t f(x) = \mathbb{E} \{ Q^x_t (u^x_t)^{-1} \nabla f(X^x_t) \}, \quad x \in M, u_0 \in O_x(M),
$$

where $Q^x_t$ and $u^x_t$ are the multiplicative functional and horizontal lift of $X^x_t$.

In general, for $s \geq 0$, let $(Q_{s,t+s})_{t \geq 0}$ be the associated multiplicative functional for the process $(X_{t+s})_{t \geq 0}$. We have

$$
(2.4) \quad Q_{s,t} Q_{t,r} = Q_{s,r}, \quad 0 \leq s \leq t \leq r.
$$

We shall use these multiplicative functionals to define the damped gradient operator (see \cite{7} for the damped gradient operator for manifolds without boundary).

Let

$$
\mathcal{F} C^\infty = \left\{ W \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : n \geq 1, 0 < t_1 < \cdots < t_n \leq T, f \in C^\infty(M) \right\}
$$

be the class of smooth cylindrical functions on $W$. For any $F \in \mathcal{F} C^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define the damped gradient $D F$ as an $H$-valued random variable by setting $(D F)_0 = 0$ and

$$
\frac{d}{dt} (D F)_t = \sum_{i=1}^n 1_{\{t < t_i\}} Q_{t,t_i} u_{t_i}^{-1} \nabla_i f(X_{t_1}, \ldots, X_{t_n}), \quad t \in [0, T],
$$

where $\nabla_i f(X_{t_1}, \ldots, X_{t_n})$ denotes the $i$-th partial derivative of $f$ with respect to $X_{t_i}$. In particular,

$$
\mathbb{E}(D F)_t = 0, \quad t \geq 0.
$$

We shall use these multiplicative functionals to define the damped gradient operator (see \cite{7} for the damped gradient operator for manifolds without boundary).
where $\nabla_i$ denotes the gradient operator w.r.t. the $i$-th component. Then, for any $\mathbb{H}$-valued random variable $h$, we have

$$Dh F := \langle DF, h \rangle_{\mathbb{H}} = \sum_{i=1}^{n} \int_{0}^{t_i} \langle u_{t_i}^{-1} \nabla_i f(X_{t_1}, \ldots, X_{t_n}), Q_{t,t_i} \dot{h}_t \rangle \, dt. \tag{2.5}$$

Next, let $\tilde{\mathbb{H}}$ denote the set of all square-integrable $\mathbb{H}$-valued adapted random variables, i.e.

$$\tilde{\mathbb{H}} = \{ h \in L^2(\Omega \rightarrow \mathbb{H}; \mathbb{P}) : h_t \text{ is } \mathcal{F}_t \text{-measurable, } t \in [0, T] \}. \tag{2.6}$$

Then $\tilde{\mathbb{H}}$ is a Hilbert space with inner product

$$\langle h, \tilde{h} \rangle_{\tilde{\mathbb{H}}} := \mathbb{E} \int_{0}^{T} \langle \dot{h}_t, \dot{\tilde{h}}_t \rangle \, dt = \mathbb{E} \langle h, \tilde{h} \rangle_{\mathbb{H}}, \quad h, \tilde{h} \in \tilde{\mathbb{H}}.$$

To describe $DF$ by using a quasi-invariant flow, for $h \in \tilde{\mathbb{H}}$ and $\varepsilon > 0$ let $X_{t}^{\varepsilon,h}$ solve the SDE

$$dX_{t}^{\varepsilon,h} = u_{t}^{\varepsilon,h} \circ dB_{t} + N(X_{t}^{\varepsilon,h}) d\hat{l}_{t}^{\varepsilon,h} + \varepsilon \dot{h}_t u_{t}^{\varepsilon,h} \, dt, \quad X_{0}^{\varepsilon,h} = X_{0} = \pi u_{0}, \tag{2.7}$$

where $\hat{l}_{t}^{\varepsilon,h}$ and $u_{t}^{\varepsilon,h}$ are, respectively, the local time on $\partial M$ and the horizontal lift on $O(M)$ for $X_{t}^{\varepsilon,h}$. To see that $\{X_{t}^{\varepsilon,h}\}_{\varepsilon \geq 0}$ has the flow property, let

$$\Theta : W_{0} := \{ \omega \in C([0, T]) : \omega_0 = 0 \} \rightarrow W$$

be measurable such that $X = \Theta(B)$. For any $\varepsilon > 0$ and a function $\Phi : W_{0} \rightarrow W$, let $(\dot{\theta}_{\varepsilon}^{h} \Phi)(\omega) = \Phi(\omega + \varepsilon h)$. Then $X_{t}^{\varepsilon,h} = (\dot{\theta}_{\varepsilon}^{h} \Theta)(B), \varepsilon \geq 0$. Hence,

$$X_{t}^{\varepsilon_{1} + \varepsilon_{2},h} = \dot{\theta}_{\varepsilon_{1}}^{h} X_{t}^{\varepsilon_{2},h}, \quad \varepsilon_{1}, \varepsilon_{2} \geq 0.$$

We shall try to link the multiplicative functional $Q$ to the vector field (if exists) generating the flow $X_{t}^{\varepsilon,h}$.

First of all, let us explain that the flow $X_{t}^{\varepsilon,h}$ is quasi-invariant. Let

$$R_{\varepsilon}^{h} = \exp \left[ \varepsilon \int_{0}^{T} \langle \dot{h}_t, dB_t \rangle - \frac{\varepsilon^2}{2} \int_{0}^{T} |\dot{h}_t|^2 \, dt \right].$$

By the Girsanov theorem,

$$B_{t}^{\varepsilon,h} := B_{t} - \varepsilon h_{t}$$

is the $d$-dimensional Brownian motion under the probability $R_{\varepsilon}^{h} \mathbb{P}$. Thus, the distribution of $X$ under $R_{\varepsilon}^{h} \mathbb{P}$ coincides with that of $X_{t}^{\varepsilon,h}$ under $\mathbb{P}$. Therefore, $X_{t}^{\varepsilon,h}$ is quasi-invariant.
Theorem 2.1. For any $u_0 \in O(M)$ and $F \in \mathcal{F}C^\infty$,

$$\mathbb{E}\{ D_h F \} = \lim_{\varepsilon \downarrow 0} \mathbb{E}\left\{ \frac{F(X_{\varepsilon,h}) - F(X)}{\varepsilon} \right\} = \mathbb{E}\left\{ F(X) \int_0^T \langle h_t, dB_t \rangle \right\}$$

holds for all $h \in \bar{H}_b$, the set of all elements in $\bar{H}$ with bounded $\|h\|_\bar{H}$.

Remark 2.1. Since $\bar{H}_b$ is dense in $\bar{H}$, the above result implies that the projection of $D$ onto $\bar{H}$ can be determined by the flows $X_{\varepsilon,h}$, $h \in \bar{H}_b$. But in general

$$(2.7) \quad D_h F = \lim_{\varepsilon \downarrow 0} \frac{F(X_{\varepsilon,h}) - F(X)}{\varepsilon}, \quad h \in \bar{H}$$

does not hold, so that the flow $\{X_{\varepsilon,h}\}$ is not generated by the vector field

$$W \ni \gamma \mapsto \left\{ u_t(\gamma) \int_0^t Q_{s,t}(\gamma) h_s ds \in T_{T_t}M : 0 \leq t \leq T \right\},$$

where $u_s(\gamma)$ and $Q_s(\gamma)$ are the horizontal lift and the multiplicative functional of $\gamma$ respectively.

To disprove (2.7), let us consider $M = [0, 1] \subset \mathbb{R}$ and $X_0 = 0$. Let $h_t = t$ and $F(\gamma) = gg_1$. By (1.1) we have

$$(2.8) \quad \lim_{\varepsilon \downarrow 0} \frac{F(X_{\varepsilon,h}) - F(X)}{\varepsilon} = f'(X_1) \text{sgn}(X_1 - l_1) = \text{sgn}(X_1 - l_1),$$

provided

$$\tau_1 := \inf\{t > 0 : X_t = 1\} > 1.$$ 

On the other hand, for the one-dimensional case we have $R_u = I_u = 0$ and $P_u = 1$. Then by (2.2)

$$Q_{s,t}^\varepsilon = \exp \left[ -\varepsilon^{-1}(l_t - l_s) \right] > 0, \quad s \leq t.$$ 

This implies that $Q_{t,1} \geq 0$ for all $t \in [0, 1]$. Combining this with (2.6) we obtain

$$(2.9) \quad D_h F = f'(X_1) \int_0^1 Q_{t,1} dt = \int_0^1 Q_{t,1} dt \geq 0.$$ 

Since

$$\mathbb{P}(X_1 - l_1 < 0, \tau_1 > 1) = \mathbb{P}\left( \inf_{s \in [0,1]} B_s < 0, \sup_{s \in [0,1]} B_s < 1 \right) > 0,$$
where \(B_s\) is now the one-dimensional Brownian motion. Combining this with (2.8) and (2.9), we see that (2.7) does not hold.

To prove Theorem 2.1, we need some preparations. In particular, we shall use (2.3) and a conducting argument as in [8] for the case without boundary.

### 3 Proof of Theorem 2.1

**Lemma 3.1.** Let \(u_0 \in O(M)\) and \(F \in \mathcal{F} \mathcal{C}^\infty\). Then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \int_0^T \langle \dot{h}_t, dB_t \rangle \}
\]

holds for all \(h \in \tilde{H}_b\).

**Proof.** Let \(B_t^{\varepsilon,h} = B_t - \varepsilon h_t\), which is the \(d\)-dimensional Brownian motion under \(R^{\varepsilon,h} \mathbb{P}\). Reformulating (2.1) as

\[
dX_t = u_t \circ dB_t^{\varepsilon,h} + N(X_t)dt + \varepsilon \dot{h}_t u_t dt, \quad X_0 = \pi u_0.
\]

By the weak uniqueness of (2.6), we conclude that the distribution of \(X\) under \(R^{\varepsilon,h} \mathbb{P}\) coincides with that of \(X_{\varepsilon,h}\) under \(\mathbb{P}\). In particular, \(\mathbb{E} F(X_{\varepsilon,h}) = \mathbb{E} [R^{\varepsilon,h} F(X)]\). Thus,

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \int_0^T \langle \dot{h}_t, dB_t \rangle \}
\]

where the last step is due to the dominated convergence theorem since \(\{R^{\varepsilon,h}\}_{\varepsilon \in [0,1]}\) is uniformly integrable for \(h \in \tilde{H}_b\).

**Lemma 3.2.** For any \(n \geq 1, 0 < t_1 < \cdots < t_n \leq T\), and \(f \in \mathcal{C}^\infty(M^n)\),

\[
u_0^{-1} \nabla_x \mathbb{E} f(X_{t_1}, \cdots, X_{t_n}) = \sum_{i=1}^n \mathbb{E} \{ M_{t_i}^x(u_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_n}^x) \}
\]

holds for all \(x \in M\) and \(u_0 \in O_x(M)\), where \(\nabla_x\) denotes the gradient w.r.t. \(x\).

**Proof.** By (2.3), the desired assertion holds for \(n = 1\). Assume that it holds for \(n = k\) for some natural number \(k \geq 1\). It remains to prove the assertion for \(n = k + 1\). To this end, set

\[
g(x) = \mathbb{E} f(x, X_{t_2-t_1}, \cdots, X_{t_{k+1}-t_1}), \quad x \in M.
\]
By the assumption for \( n = k \) we have

\[
    u_0^{-1} \nabla g(x) = \sum_{i=1}^{k+1} \mathbb{E}\left\{ M_{t_i-t_{i-1}}^x (u_0^{-1} \nabla_i f(x, X_{t_{i-1}-t_1}, \ldots, X_{t_{k+1}-t_1}) \right\}
\]

for all \( x \in M, u_0 \in O_x(M) \). Combining this with the assertion for \( k = 1 \) and using the Markov property, we obtain

\[
    u_0^{-1} \nabla_x \mathbb{E} f(X_{t_1}, \ldots, X_{t_{k+1}}) = u_0^{-1} \nabla_x \mathbb{E} g(X_{t_1}^x) = \sum_{i=1}^{k+1} \mathbb{E}\left\{ Q_x^{t_1} (u_0^{-1} \nabla_if(X_{t_1}^x)) \right\}.
\]

\[\Box\]

**Lemma 3.3.** Let \( f \in C^\infty(M) \). Then for any \( u_0 \in O(M) \) and \( t > 0 \),

\[
    \mathbb{E}\left\{ f(X_t) \int_0^t \langle \dot{h}_s, dB_s \rangle \right\} = \mathbb{E} \int_0^t \langle u_0^{-1} \nabla f(X_t), Q_s^x \dot{h}_s \rangle ds, \quad h \in \mathbb{H}, \ t_1 \in [0, T].
\]

**Proof.** Noting that

\[
    \frac{d}{ds} P_s f = \frac{1}{2} \Delta P_s f, \quad NP_s f|_{\partial M} = 0, \quad s > 0,
\]

by (2.1) and the Itô formula we obtain

\[
    d(P_{t-s} f)(X_s) = \langle \nabla P_{t-s} f(X_s), u_s dB_s \rangle, \quad s \in [0, t).
\]

This implies

\[
    f(X_t) = P_t f(X_0) + \int_0^t \langle u_0^{-1} \nabla P_{t-s} f(X_s), dB_s \rangle, \quad s \in [0, t].
\]

Therefore,

\[
    (3.1) \quad \mathbb{E}\left\{ f(X_t) \int_0^t \langle \dot{h}_s, dB_s \rangle \right\} = \mathbb{E} \int_0^t \langle u_0^{-1} \nabla P_{t-s} f(X_s), \dot{h}_s \rangle ds.
\]

By (2.3) and the Markov property we have

\[
    u_0^{-1} \nabla P_{t-s} f(X_s) = \mathbb{E}\left\{ Q_{s,t} u_0^{-1} \nabla f(X_t) | \mathcal{F}_s \right\}.
\]

So, the desired formula follows from (3.1) since \( \dot{h}_s \) is \( \mathcal{F}_s \)-measurable. \[\Box\]

As a consequence of (2.3) and Lemma 3.3, we have the following Bismut formula.
Corollary 3.4. Let $P_t f(x) = \mathbb{E}^x f(X_t)$, $t \geq 0, x \in M, f \in C(M)$. Then for any $v \in T_x M$ and any $h \in \mathcal{H}$ with $h_t = u_0^{-1}v$,

$$
\langle v, \nabla P_t f(x) \rangle = \mathbb{E}^x \left\{ f(X_t) \int_0^t \langle Q_s^* \dot{h}_s, dB_s \rangle \right\}.
$$

Proof. By (2.4) and applying Lemma 3.3 to $\tilde{h} \in \mathcal{H}$ in place of $h$, where $\tilde{h}_s = Q_s^* \dot{h}_s$, we obtain

$$
\mathbb{E} \left\{ f(X_t) \int_0^t \langle Q_s^* \dot{h}_s, dB_s \rangle \right\} = \mathbb{E} \int_0^t \langle u_t^{-1} \nabla f(X_t), Q_t^* \dot{h}_s \rangle ds = \mathbb{E} \langle Q_t u_t^{-1} \nabla f(X_t), u_0^{-1}v \rangle.
$$

Then the proof is completed by combining this with (2.3).

Proof of Theorem 2.7. By Lemma 3.1 it suffices to prove

$$
(3.2) \quad \mathbb{E} \{ D_h F \} = \mathbb{E} \left\{ F(X) \int_0^T \langle \dot{h}_t, dB_t \rangle \right\}, \quad h \in \mathcal{H}
$$

for $F = f(X_{t_1}, \cdots, X_{t_n})$ with $f \in C^\infty(M^n)$, where $n \geq 1, 0 < t_1 < \cdots < t_n \leq T$. According to Lemma 3.3, (3.2) holds for $n = 1$. Assuming (3.2) holds for $n = k$ for some $k \geq 1$, we aim to prove it for $n = k + 1$. To this end, let

$$
g(x) = \mathbb{E} f(x, X_{t_2-t_1}, \cdots, X_{t_{k+1}-t_1}), \quad x \in M.
$$

By the result for $n = 1$ and the Markov property,

$$
\begin{align*}
\int_0^{t_1} \mathbb{E} \langle u_t^{-1} \nabla g(X_{t_1}), Q_{t_1}^* \dot{h}_t \rangle dt &= \mathbb{E} \left\{ \mathbb{E} \langle F(X) | \mathcal{F}_{t_1} \rangle \int_0^{t_1} \langle \dot{h}_t, dB_t \rangle \right\} \\
&= \mathbb{E} \left\{ F(X) \int_0^{t_1} \langle \dot{h}_t, dB_t \rangle \right\}.
\end{align*}
$$

On the other hand, by (2.4), Lemma 3.2 and the Markov property,

$$
\begin{align*}
\int_0^{t_1} \mathbb{E} \langle u_t^{-1} \nabla g(X_{t_1}), Q_{t_1}^* \dot{h}_t \rangle dt \\
&= \int_0^{t_1} \mathbb{E} \left\langle \mathbb{E} \left( \sum_{i=1}^{k+1} Q_{t_1}^* u_t^{-1} \nabla_i f(X_{t_1}, \cdots, X_{t_{k+1}}) | \mathcal{F}_{t_1} \right), Q_{t_1}^* \dot{h}_t \right\rangle dt \\
&= \sum_{i=1}^{k+1} \int_0^{t_1} \langle u_t^{-1} \nabla_i f(X_{t_1}, \cdots, X_{t_{k+1}}), Q_{t_1}^* \dot{h}_t \rangle dt.
\end{align*}
$$
Combining this with (2.5) and (3.3) we obtain

\[
\mathbb{E}\{D_t F\} = \mathbb{E}\left\{F(X) \int_0^{t_1} \langle \dot{h}_t, dB_t \rangle \right\} \\
+ \mathbb{E} \sum_{i=2}^{k+1} \int_{t_1}^{t_{i+1}} \langle u_{t_i}^{-1} \nabla f(X_{t_i}, \cdots, X_{t_{i+1}}), Q^*_{t,t_i} \dot{h}_t \rangle \, dt.
\]

By the Markov property and the assumption for \(n = k\), we have

\[
\sum_{i=2}^{k+1} \mathbb{E} \int_{t_1}^{t_{i+1}} \langle u_{t_i}^{-1} \nabla f(X_{t_i}, \cdots, X_{t_{i+1}}), Q^*_{t,t_i} \dot{h}_t \rangle \, dt = \mathbb{E}\left\{F(X) \int_{t_1}^{T} \langle \dot{h}_t, dB_t \rangle \right\}.
\]

Combining this with (3.4) we complete the proof. \(\square\)

4 The Log-Sobolev Inequality

Let \(\mu\) be the distribution of \(X\) with \(X_0 = o\), and let

\[
\mathcal{E}(F,G) = \mathbb{E}\langle DF, DG \rangle_H, \quad F, G \in \mathcal{F}C^\infty.
\]

Since both \(DF\) and \(DG\) are functionals of \(X\), \((\mathcal{E}, \mathcal{F}C^\infty)\) is a positive bilinear form on \(L^2(W; \mu)\). It is standard that the integration by parts formula (3.2) implies the closability of the form (see Lemma 4.1). We shall use \((\mathcal{E}, \mathcal{R}(\mathcal{E}))\) to denote the closure of \((\mathcal{E}, \mathcal{F}C^\infty)\). Moreover, (3.2) also implies the Clark-Ocone type martingale representation formula (see Lemma 4.2), which leads to the standard Gross [6] log-Sobolev inequality. It is well known that the log-Sobolev inequality implies that the associated Markov semigroup is hypercontractive and converges exponentially to \(\mu\) in the sense of relative entropy.

**Lemma 4.1**. \((\mathcal{E}, \mathcal{F}C^\infty)\) is closable in \(L^2(W; \mu)\).

**Proof**. Although the proof is standard by using the integration by parts formula, we include it here for completeness. Let \(\{F_n\}_{n \geq 1} \subset \mathcal{F}C^\infty\) such that \(\mathcal{E}(F_n, F_n) \leq 1\) for all \(n \geq 0\) and \(\mu(F_n^2) + \mathcal{E}(F_n, F_n - F_m) \to 0\) as \(n, m \to \infty\). We aim to prove that \(\mathcal{E}(F_n, F_n) \to 0\) as \(n \to \infty\). Since

\[
\mathcal{E}(F_n, F_n) = \mathcal{E}(F_n, F_n - F_m) + \mathcal{E}(F_n, F_m) \leq \sqrt{\mathcal{E}(F_n - F_m, F_n - F_m)} + \mathcal{E}(F_n, F_m),
\]

it suffices to show that for any \(G \in \mathcal{F}C^\infty\), one has \(\mathcal{E}(F_n, G) \to 0\) as \(n \to \infty\). To this end, let \(\{h^i\}_{i \geq 1}\) be an ONB on \(H\). For any \(\varepsilon > 0\) there exists \(k \geq 1\) such that

\[
\left| \mathcal{E}(F_n, G) - \sum_{i=1}^k \mathbb{E}(D_{h^i} F_n)(D_{h^i} G) \right| < \varepsilon,
\]

11
where \( D_h F := \langle DF, h \rangle \) for \( F \in \mathcal{F}C^\infty \) and \( h \in \mathbb{H} \). Since \( \mathcal{F}C^\infty \) is dense in \( L^2(W; \mu) \), there exists \( G_i \in \mathcal{F}C^\infty \) such that

\[
\mathbb{E}|D_h G - G_i|^2 < \varepsilon, \quad 1 \leq i \leq k.
\]

Therefore,

\[
|\mathcal{E}(F_n, G)| \leq 2\varepsilon + \sum_{i=1}^{k} |\mathbb{E}(G_i DF_n, h_i)_{\mathbb{H}}|.
\]

Noting that \( G_i DF_n = D(F_n G_i) - F_n DG_i \), by (3.2) we obtain

\[
|\mathcal{E}(F_n, G)| \leq 2\varepsilon + \sum_{i=1}^{k} \left| \mathbb{E} \left[ F_n(X) \left\{ G_i(X) \int_0^T \langle h_i, dB_t \rangle - D_h G_i \right\} \right] \right|.
\]

Since \( \mu(F_n^2) \to 0 \) as \( n \to \infty \), by letting first \( n \to \infty \) then \( \varepsilon \to 0 \) we complete the proof.

**Lemma 4.2.** For any \( F \in \mathcal{F}C^\infty \), let \( \tilde{D}F \) be the projection of \( DF \) on \( \tilde{H} \), i.e.

\[
\frac{d}{dt}(\tilde{D}F)_t = \mathbb{E} \left( \frac{d}{dt}(DF)_t \bigg| \mathcal{F}_t \right), \quad t \in [0, T], (\tilde{D}F)_0 = 0.
\]

Then

\[
F(X) = \mathbb{E}F(X) + \int_0^T \left\langle \frac{d}{dt}(\tilde{D}F)_t, dB_t \right\rangle.
\]

**Proof.** By Theorem 2.1, we have

\[
(4.1) \quad \mathbb{E}\langle h, \tilde{D}F \rangle_{\mathbb{H}} = \mathbb{E} \left\{ F(X) \int_0^T \langle h_t, dB_t \rangle \right\}, \quad h \in \tilde{H}.
\]

On the other hand, by the martingale representation, there exists a predictable process \( \beta_t \) such that

\[
(4.2) \quad \mathbb{E}(F(X)|\mathcal{F}_t) = \mathbb{E}F(X) + \int_0^t \langle \beta_s, dB_s \rangle, \quad t \in [0, T].
\]

Let

\[
\varphi_t = \int_0^t \beta_s ds, \quad t \in [0, T].
\]

We have \( \varphi \in \tilde{H} \) and by (4.2),
\[ E(h, \varphi)_{\tilde{H}} = E \int_0^T \langle \dot{h}_t, \beta_t \rangle dt = E \left\{ F(X) \int_0^T \langle \dot{h}_t, dB_t \rangle \right\} \]

holds for all \( h \in \tilde{H} \). Combining this with (4.1) we conclude that \( \tilde{D}F = \varphi \). Therefore, the desired formula follows from (4.2).

\[ \square \]

**Theorem 4.3.** For any \( T > 0 \) and \( o \in M, (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) satisfies the following log-Sobolev inequality

\[ \mu(F^2 \log F^2) \leq 2\mathcal{E}(F, F), \quad F \in \mathcal{D}(\mathcal{E}), \quad \mu(F^2) = 1. \]

**Proof.** It suffices to prove for \( F \in \mathcal{F}C^\infty \). Let \( m_t = E(F(X)^2|\mathcal{F}_t), \ t \in [0, T] \). By Lemma [4.2] and the Itô formula,

\[ dm_t \log m_t = (1 + \log m_t)dm_t + \frac{|\frac{d}{dt}(\tilde{D}F^2)_t|^2}{2m_t} dt. \]

Thus,

\[ \mu(F^2 \log F^2) = Em_T \log m_T = \int_0^T 2E(F(X) \frac{d}{dt}(DF)_t|\mathcal{F}_t)^2 dt \]

\[ \leq 2 \int_0^T E \left| \frac{d}{dt}(DF)_t \right|^2 dt = 2E\|DF\|^2_{\tilde{H}} = 2\mathcal{E}(F, F). \]

\[ \square \]

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