Second post-Newtonian radiative evolution of the relative orientations of angular momenta in spinning compact binaries

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The radiative evolution of the relative orientations of the spin and orbital angular momentum vectors $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{L}$, characterizing a binary system on eccentric orbit is studied up to the second post-Newtonian order. As an intermediate result, all Burke-Thorne type instantaneous radiative changes in the spins are shown to average out over a radial period. It is proved that spin-orbit and spin-spin terms contribute to the radiative angular evolution equations, while Newtonian, first and second post-Newtonian terms together with the leading order tail terms do not. In complement to the spin-orbit contribution, given earlier, the spin-spin contribution is computed and split into two-body and self-interaction parts. The latter provide the second post-Newtonian order corrections to the 3/2 order Lense-Thirring description.

I. INTRODUCTION

Ongoing theoretical studies of the gravitational radiation emitted by compact binaries and of the radiation reaction on the orbit are strongly motivated by the desire to detect gravitational waves using the LIGO [2] / VIRGO [3] interferometers. In order to extract the signal from the experimental data one needs templates of sufficient accuracy. Based on the expectation that during the adiabatic regime the radiation reaction will circularize the orbit [3], templates for circular orbits are computed. If the orbits are eccentric, the detectability of the gravitational waves by using circular templates decreased [4]. Quinlan and Shapiro [5] and Hills and Bender [6] argue for a significant number of eccentric binaries in galactic nuclei for which the time before plunging is insufficient for circularization. A faithful description of the behavior of these binaries in the adiabatic regime requires not only high post-Newtonian orders, but also eccentric orbits. Such a generic treatment, valid up to the second post-Newtonian order was provided by Gopakumar and Iyer [7].

If the spins of the neutron stars/black holes which form the binary are equally considered, the number of kinematical variables increases significantly and the dynamics complicates. Currently a second post-Newtonian order accurate description of the motion of the spinning binary [8] - [11] is available. Spin-orbit and spin-spin type contributions appear at the 3/2 and second post-Newtonian orders, respectively. There is one notable exception over this rule: the spin-orbit and spin-spin terms in the spin precession equations

$$\dot{\mathbf{S}}_1 = \frac{G}{c^2 r^3} \left( \frac{4m_1 + 3m_2}{2m_1} \mathbf{L}_N - \mathbf{S}_2 + \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_2) \mathbf{r} \right) \times \mathbf{S}_1,$$

$$\dot{\mathbf{S}}_2 = \frac{G}{c^2 r^3} \left( \frac{4m_2 + 3m_1}{2m_2} \mathbf{L}_N - \mathbf{S}_1 + \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_1) \mathbf{r} \right) \times \mathbf{S}_2 \quad (1.1)$$

generate first and 3/2 post-Newtonian order changes in the angles between the angular momenta vectors [1]. Consequences of the spin precession equations (1.1) and of the conservation of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$, holding in the second post-Newtonian order are summarized in Table 1. In the present paper the angular momenta, $\dot{f}$ and $f$, are expressed in terms of a fiducial time $T$ and an angular frequency $\Omega$.
their directions, magnitudes and the angles, all enlisted in Table 1 will be needed to a precision at which precessional effects do not contribute.

All spin terms in the radiation reaction up to the 3/2 post-Newtonian order were evaluated in \([13]\), starting from the expressions of the radiated power and total angular momentum loss derived by Kidder [12] and from the Burke-Thorne potential \([13]\). The employed averaging method, based on a suitably introduced radial parameter \(\chi\) and on the residue theorem was described in detail in \([14]\).

The second-order spin contribution to the radiation reaction was considered recently in \([15]\). There we have developed the toolchest for computing secular radiative effects of spin-spin origin, essentially by proposing a description in terms of constants of the motion and the angular average \(\bar{L}\) of the magnitude of the orbital angular momentum \(L(\chi)\). The latter depends on the Keplerian true anomaly \(\chi\) through a spin-spin term:

\[
L(\chi) = \bar{L} - \frac{G\mu^2}{2c^2L^3} S_1 S_2 \sin\kappa_1 \sin\kappa_2 \{ 2\hat{A}\cos[\chi + 2(\psi_0 - \bar{\psi})] + (3Gm\mu + 2\hat{A}\cos\chi) \cos 2(\chi + \psi_0 - \bar{\psi}) \}. 
\]

(1.2)

The quantity \(\hat{A} = (G^2m^2\mu^2 + 2E\bar{L}^2/\mu)^{1/2}\) is the magnitude of the Laplace-Runge-Lenz vector for a Keplerian motion characterized by the energy \(E\) and magnitude of orbital angular momentum \(\bar{L}\). The angles \(\kappa_i = \cos^{-1}(\hat{S}_i \cdot \hat{L})\), \((i = 1, 2)\) together with \(\gamma = \cos^{-1}(\hat{S}_1 \cdot \hat{S}_2)\), characterize the relative orientation of the angular momentum vectors \(\hat{S}_1\), \(\hat{S}_2\) and \(\hat{L}\).

This description enabled us to introduce the generalized true anomaly parametrization \(r(\chi)\) for the radial motion: (Eq. (3.7) of Ref. \([13]\)). By employing \(\chi\) as radial variable, the average of a generic function \(f\) over a radial period \(T\)

\[
\langle f \rangle = \frac{1}{T} \int_0^{2\pi} f(\chi) \frac{1}{r} \frac{dr}{d\chi} d\chi, 
\]

(1.3)

is computed by Taylor expanding \(r^2\) taken from the radial equation, by substituting

\[
\frac{dr}{d\chi} = \frac{1}{2} \left( \frac{1}{r_{\text{min}}} - \frac{1}{r_{\text{max}}} \right) r^2 \sin \chi 
\]

(1.4)

\((r_{\text{min}}, r_{\text{max}} \) being the turning points) and finally, by switching to the complex parameter \(z = \exp(i\chi)\). Then the residue theorem is employed in the computation of the integral \([13]\), which in the majority of cases is given by the residues in the origin \([12]\).

As application, we have computed the secular losses of the energy \(E\) and of \(\bar{L}\). They depend on \(E\), \(\bar{L}\), on the angles \(\psi_i\) subtended by the node line with the projections of the spins in the plane of the orbit and on the angles \(\kappa_i\) and \(\gamma\). The enlisted five angles are constrained by two algebraic relations \([12]\):

\[
0 = S_1 \sin \kappa_1 \cos \psi_1 + S_2 \sin \kappa_2 \cos \psi_2, 
\]

\[
\cos \gamma = \cos \kappa_1 \cos \kappa_2 + \sin \kappa_1 \sin \kappa_2 \cos \Delta \psi, 
\]

(1.5)

(1.6)

where \(\Delta \psi = \psi_2 - \psi_1\). We also introduce here the notation \(\bar{\psi} = (\psi_1 + \psi_2)/2\). In order to have a closed system of differential equations, beside \(dE/dt\) and \(d\bar{L}/dt\) computed in \([13]\) the radiative change of the angles \(\kappa_i\) and \(\gamma\) is required. This was given to 3/2 post-Newtonian order accuracy in \([13]\). It is the purpose of the present paper to compute the radiative evolution of the angles \(\kappa_i\) and \(\gamma\) up to the second post-Newtonian order by the inclusion of the spin-spin terms.

### TABLE I. The effect of precessions on angular momenta.

| quantity \((f)\) | leading order change \(\mathcal{O}(df)\) | origin of the leading order change | constant at order |
|------------------|--------------------------------------|---------------------------------|------------------|
| \(S_i, \ S_i = S_i/S_i\), \(\gamma = \cos^{-1}(\hat{S}_1 \cdot \hat{S}_2)\) | \(\epsilon\) | spin-orbit | \(\epsilon^{1/2}\) |
| \(\bar{L}, \ \bar{L} = L/L\) | \(\epsilon^{3/2}\) | spin-orbit | \(\epsilon\) |
| \(\kappa_i = \cos^{-1}(\hat{S}_i \cdot \hat{L})\) | \(\epsilon^{3/2}\) | spin-spin | \(\epsilon\) |
| \(L\) | \(\epsilon^2\) | spin-spin | \(\epsilon^{3/2}\) |
| \(S_i\) | \(\geq \epsilon^2\) | higher order effects | \(\epsilon^{3/2}\) |
II. RADIATIVE LOSS IN THE SPINS

In [13] we have derived the radiative loss in the spin $\mathbf{S}_i$ of the $i^{th}$ axisymmetric body, following [15]. The integral over the volume of the body of the moment of the reaction force (the sign swapped gradient of the Burke-Thorne potential) gave:

$$\frac{d}{dt}(\mathbf{S}_i) = \frac{2Gm}{5c^5\Omega_i} \left( \frac{\Theta_i}{\Theta'_i} - 1 \right) \epsilon_{\mu\nu\rho} I_N^{(5)\nu\rho} S_i(\hat{S}_i)_{\mu} \hat{S}_i_{\sigma} \ . \quad (2.1)$$

(There is no summation over $i$.) Each body is characterized by its principal moments of inertia $\Theta_i$ and $\Theta'_i$ and by the angular velocity $\Omega_i = S_i/\Theta'_i$, while $I_N^{(5)\nu\rho}$ is the 5th time derivative of the system’s Newtonian symmetric trace-free mass quadrupole moment. As the loss in the spin vector described by Eq. (2.1) is of second post-Newtonian order (above the leading radiation term in the orbital angular momentum loss), Eq. (2.1) is still valid to this order for approximately axisymmetric bodies, with the deviation from axisymmetry being of any post-Newtonian order. We emphasize that Eq. (2.1) implies $d\mathbf{S}_i/dt = S_i \frac{d}{dt} \mathbf{S}_i$, therefore to this order the radiation reaction will change the orientation but not the magnitude of the spin vectors. We also stress that $d\mathbf{S}_i/dt$ generates 3/2 post-Newtonian order radiative changes $\delta_{\text{rad}} \mathbf{S}_i$ above the leading order radiation losses (in $\mathbf{L}$ for example).

In coordinates $(x, y, z) = r(\cos \psi, \sin \psi, 0)$ the spins are expressed as $\mathbf{S}_i = \mathbf{S}_i(x \sin \kappa_i \cos \psi_i, x \sin \kappa_i \sin \psi_i, x \cos \kappa_i)$. By employing the Keplerian equation of motion $a_N = -Gm/r^3$ and the radial equation $\dot{r}^2 = 2E/\mu + 2Gm/r - \bar{L}^2/\mu^2 r^2$ in Eq. (2.1), we obtain the instantaneous spin-loss equation:

$$\frac{1}{S_i} \frac{d}{dt}(\mathbf{S}_i) = \frac{2Gc^2 m \sin \kappa_i}{5c^5 \mu^2 r^7 \Omega_i} \left( \frac{\Theta_i}{\Theta'_i} - 1 \right) A_{i\mu}$$

with the coefficients

$$A_{i1} = \cos \kappa_i [a_1 \sin(2\psi - \psi_i) + a_2 \cos(2\psi - \psi_i) + a_3 \sin \psi_i]$$
$$A_{i2} = \cos \kappa_i [a_2 \sin(2\psi - \psi_i) - a_1 \cos(2\psi - \psi_i) - a_3 \cos \psi_i]$$
$$A_{i3} = -\sin \kappa_i [a_1 \sin(2\psi - \psi_i) + a_2 \cos(2\psi - \psi_i)] \ , \quad (2.2)$$

In order to obtain the secular radiative changes of the spins we insert the Newtonian expressions

$$\dot{r} = \frac{\bar{A}}{\bar{L}} \sin \chi \ , \quad r = \frac{L^2}{\mu(Gm + A \cos \chi)} \ , \quad \psi = \chi + \psi_0 \ . \quad (2.4)$$
in Eq. (2.2).

Then we average by the method described in the Introduction (see also [13]) employing the Newtonian expression \((1/r)(dr/d\chi) = dt/d\chi = \mu r^2/L\). We obtain the generic result that the radiative change in the spins averages to zero in the second post-Newtonian approximation:

\[
\left\langle \frac{dS_i}{dt} \right\rangle = 0 .
\]

(2.5)

Therefore the averaging-out property of some particular projections of the spin losses found in [12] and [18] also emerge.

As the first correction to both the acceleration and the Burke-Thorne potential is one post-Newtonian order higher than the respective leading terms [19], the averaging-out property of the radiative change in the spins derived above holds at 5/2 post-Newtonian orders as well.

III. RADIATIVE EVOLUTION OF THE ANGLES \(\kappa_i\) AND \(\gamma\)

We obtain the equations for the radiative evolution of \(\kappa_i = \cos^{-1}(\hat{S}_i \cdot \hat{L})\), \((i = 1, 2)\) and \(\gamma = \cos^{-1}(\hat{S}_1 \cdot \hat{S}_2)\) by differentiating the defining relations of these angles.

The simplest is the evolution of the angle \(\gamma\) spanned by the two spin vectors:

\[
\frac{d}{dt} \cos \gamma = \frac{d}{dt} \left( \hat{S}_1 \cdot \hat{S}_2 \right) = \hat{S}_1 \cdot \frac{d\hat{S}_2}{dt} + \hat{S}_2 \cdot \frac{d\hat{S}_1}{dt} .
\]

(3.1)

The terms of the right hand side were already evaluated in the framework of the spin-orbit contributions in [12]. Though they do not contain the spin explicitly, the angular velocities \(\Omega_i\) are present in their expression, Eq. (3.13) of [14]. An order of magnitude estimate shows that the ratio of the usual spin-orbit terms (taken for example from the losses of \(\kappa_i\), Eqs. (3.10)-(3.12) of [12]) and such terms is of the order unity for rapidly rotating objects. It was computed in [12], and one can check by simple inspection of Eq. (2.5) that \(\gamma\) will receive no secular radiative change.

The angles spanned by the spins with the orbital angular momentum evolve in a more complicated fashion:

\[
\frac{d}{dt} \cos \kappa_i = \frac{d}{dt} \left( \hat{S}_i \cdot \frac{L}{L(\chi)} \right) = \frac{1}{L(\chi)} \left[ \hat{S}_i \frac{d(J - S_1 - S_2)}{dt} - \frac{dL}{dt} \cos \kappa_i + L \cdot \frac{d\hat{S}_i}{dt} \right] .
\]

(3.2)

In the forthcoming expressions we replace \(L(\chi) \rightarrow \bar{L}\) in all post-Newtonian terms. This is possible in the required second order accuracy as \(L(\chi)\) differs from the constant \(\bar{L}\) in spin-spin terms (see Eq. (1.2)).

By inserting the following expression for the loss of the magnitude of orbital angular momentum

\[
\frac{dL}{dt} = \bar{L} \cdot \frac{dL}{dt} = \bar{L} \cdot \frac{dJ}{dt} - \bar{L} \cdot \frac{dS_1}{dt} - \bar{L} \cdot \frac{dS_2}{dt}
\]

(3.3)

into Eq. (3.2), and from \(\hat{S}_i \cdot d\hat{S}_j/dt = 0\) (no summation) we obtain:

\[
\frac{d}{dt} \cos \kappa_i = \frac{1}{L(\chi)} \left( \hat{S}_i - \bar{L} \cos \kappa_i \right) \frac{dJ}{dt} + \frac{1}{L} \left[ -\hat{S}_i \cdot \frac{dS_j}{dt} + \bar{L} \left( \frac{dS_1}{dt} + \frac{dS_2}{dt} \right) \cos \kappa_i + L \cdot \frac{d\hat{S}_i}{dt} \right] ,
\]

(3.4)

where \(j \neq i\). Remarkably all terms in the square bracket average out due to Eq. (2.3). Therefore we will not consider them further.

For the evaluation of the first term of Eq. (3.4) we need the loss in the total angular momentum \(dJ/dt\). It has the following structure:

To avoid confusion in notation, we denote the post-Newtonian, 2nd post-Newtonian, spin-orbit and spin-spin terms from a decomposition of a quantity expressed in terms of \(r, v, S_i\) and \(\dot{r}\) by lower-case indices \(pn, 2pn, so\) and \(ss\), respectively. This notation applies to all terms of \(dJ/dt\) computed in [11] and [12]. After inserting the expressions for \(v^2, \dot{r}^2\) and \(\dot{r}\), Eqs. (2.32), (2.33) of [13] and (2.4), the terms of the new decomposition will be denoted by the respective capital letters.
where the coefficients $\Gamma_0$ to the averaged tail contribution, given by Eq. (84) of [20]. To see this we note that all coefficients (denoted there by $\langle S \rangle$) of the spectral decomposition of the angular momentum loss (denoted there by $\langle S \rangle$) have only $z-$components, thus:

$$\langle \frac{dJ}{dt} \rangle_{\text{tail}} = \Gamma_{\text{tail}} \cdot L_N .$$

The next essential remark is that from

$$L = L_N + L_{PN} + L_{2PN} + L_{SO} = (1 + \gamma_1 + \gamma_2)L_N + L_{SO}$$

we can express the orbital angular momentum as

$$L_N = (1 - \gamma_1 - \gamma_2 + \gamma_1^2)L - L_{SO} .$$

Here the coefficients $\gamma_1, 2$ are given by Eqs. (2.9.b,d) of [11] and they are of first and second post-Newtonian order, respectively. Therefore up to an accuracy of second post-Newtonian order we can write

$$\left( \frac{dJ}{dt} \right)_{N + pn + 2pn} = (\Gamma_0 + \eta_1 + \eta_2)L - \Gamma_0 \cdot L_{SO}$$

and

$$\langle \frac{dJ}{dt} \rangle_{\text{tail}} = \Gamma_{\text{tail}} \cdot L ,$$

where the coefficients of the expansion (3.10) are given by

$$\eta_1 = \Gamma_1 - \Gamma_0 \gamma_1$$
$$\eta_2 = \Gamma_2 - \Gamma_1 \gamma_1 - \Gamma_0 (\gamma_2 - \gamma_1^2) .$$

We will not need their explicit expressions. By inserting Eq. (3.10) in Eq. (3.3), then the resulting expression in Eq. (3.4), due to the remark

$$(\hat{S}_i - \hat{L} \cos \kappa_i) \cdot L = 0$$

we obtain for the first term of Eq. (3.4):

$$\frac{1}{L(x)} (\hat{S}_i - \hat{L} \cos \kappa_i) \cdot \left( \frac{dJ}{dt} \right) = \frac{1}{L} (\hat{S}_i - \hat{L} \cos \kappa_i) \cdot \left[ (\frac{dJ}{dt})_{\text{tail}} + \left( \frac{dJ}{dt} \right)_{so} - \Gamma_0 L_{SO} + \left( \frac{dJ}{dt} \right)_{ss} \right] .$$

The tail term averages out due to Eq. (3.11) and property (3.13).

As suggested by the notation, the second and third terms of Eq. (3.14) are those spin-orbit contributions to the radiative loss of the angles $\kappa_i$, which do not originate in the Burke-Thorne potential. Indeed, modulo Burke-Thorne type terms they can be put into the concise form $(\hat{S}_i \cdot \frac{dL}{dt})_{SO}$ which was computed previously (Eqs. (3.8)-(3.10) of [13]). The averaged expression for the spin-orbit type loss in $\kappa_1$ was also given by Eq. (4.4) of [12], and a similar expression can be found for the loss of $\kappa_2$ by interchanging the indices $1 \leftrightarrow 2$ and the ratios $\eta = m_2/m_1 \leftrightarrow \eta^{-1}$.

The last term of Eq. (3.14), modulo Burke-Thorne type terms, is the spin-spin part of the radiative evolution of the angles $\kappa_i$:

$$\left( \frac{d}{dt} \cos \kappa_i \right)_{SS} \simeq \frac{1}{L} (\hat{S}_i - \hat{L} \cos \kappa_i) \cdot \left( \frac{dJ}{dt} \right)_{ss} ,$$

which will be computed in the next section. (We have denoted by $\simeq$ the equality modulo Burke-Thorne type terms.)
IV. SECULAR RADIATIVE EVOLUTION EQUATIONS

We start from the expression of \( \langle d J / dt \rangle_{ss} \) given by Kidder [11] in terms of the time derivatives of the mass quadrupole and velocity quadrupole moments (see also Eq. (4.18) of [12] for the respective expression with \( c \neq 1 \neq G \)). The required SO part of the velocity quadrupole moment was computed first by Kidder [11] and later verified by several authors. (Rieth and Schäfer [20] presented a derivation based on a different spin supplementary condition while Owen, Tagoshi and Ohasha [21] have employed a \( \delta \)-function type energy-momentum tensor.)

After computing the expression \( \langle d J / dt \rangle_{ss} \) in detail, we rewrite the last term of Eq. (3.14) as function of the radial variables \( r(\chi) \) and \( \chi \). The procedure we followed was described in detail in [13]. As a result both self-interaction and two-body spin-spin terms emerge:

\[
\left[ \frac{1}{L} (\vec{S}_i - \vec{L} \cos \kappa_i) \cdot \left( \frac{dJ}{dt} \right) \right]_{ss \rightarrow \text{self}} = \frac{2G^2m^2\mu \sin \kappa_i}{5c^7r^6} \left[ \left( \frac{S_i}{m_i} \right)^2 \sin \kappa_i \cos \kappa_i + \left( \frac{S_j}{m_j} \right)^2 \sin \kappa_j \cos \kappa_j \cos \Delta \psi \right]
\]

\[
\left[ \frac{1}{L} (\vec{S}_i - \vec{L} \cos \kappa_i) \cdot \left( \frac{dJ}{dt} \right) \right]_{ss \rightarrow \text{SS}} = \frac{2G^2S_iS_2\sin \kappa_i}{5c^7L^2r^7} \left\{ u_1 [\sin \kappa_i \cos \kappa_j + \sin \kappa_j \cos \kappa_i \cos \Delta \psi] + u_2 [\sin \kappa_i \cos \kappa_j \cos 2(\chi + \psi_0 - \psi_i) + \sin \kappa_j \cos \kappa_i \cos 2(\chi + \psi_0 - \psi_i)] + u_3 [\sin \chi \sin \kappa_i \cos \psi_j - \sin \psi_j - \psi_i] + u_4 [\sin \kappa_i \cos \kappa_j \sin 2(\chi + \psi_0 - \psi_i) + \sin \kappa_j \cos \kappa_i \sin 2(\chi + \psi_0 - \psi_i)] \right\}.
\]

Here \( j \neq i \) and the coefficients are

\[
u_1 = -\dot{L}^2(12\mu Er^2 + 4Gm\mu^2 r - 15\dot{L}^2)
\]

\[
u_2 = 3\dot{L}^2(Gm\mu^2 r + 3\dot{L}^2)
\]

\[
u_3 = 3\mu \dot{A} r(-2\mu Er^2 + 5\dot{L}^2)
\]

\[
u_4 = 3\mu \dot{A} r(-2\mu Er^2 + 3\dot{L}^2)
\]

The averaging procedure based on the parametrization \( r(\chi) \) and on the residue theorem yields the self-interaction and two-body spin-spin terms in the secular radiative loss of the angles \( \kappa_i \) and \( \gamma \). We enlist them together with the spin-orbit contributions:

\[
\langle \frac{d\gamma}{dt} \rangle = 0
\]

\[
\langle \frac{d\kappa_i}{dt} \rangle_{SO} = \langle \frac{d\kappa_i}{dt} \rangle_{SO} + \langle \frac{d\kappa_i}{dt} \rangle_{SS \rightarrow \text{self}} + \langle \frac{d\kappa_i}{dt} \rangle_{SS}
\]

\[
\langle \frac{d\kappa_i}{dt} \rangle_{SO} \text{ given by Eq. (4.4) of [12]}
\]

\[
\langle \frac{d\kappa_i}{dt} \rangle_{SS \rightarrow \text{self}} = -\frac{G^2\mu(2\mu E)^{3/2}}{20c^7L^9} V_1 \left[ \left( \frac{S_i}{m_i} \right)^2 \sin \kappa_i \cos \kappa_i + \left( \frac{S_j}{m_j} \right)^2 \sin \kappa_j \cos \kappa_j \cos \Delta \psi \right]
\]

\[
\langle \frac{d\kappa_i}{dt} \rangle_{s_1 s_2} = -\frac{G^2(2\mu E)^{3/2}S_iS_j}{20c^7L^9 \sin \kappa_i} \left\{ V_2 (\sin \kappa_i \cos \kappa_j + \sin \kappa_j \cos \kappa_i \cos \Delta \psi) + V_3 [\sin \kappa_i \cos \kappa_j \cos 2(\psi_0 - \psi_i) + \sin \kappa_j \cos \kappa_i \cos 2(\psi_0 - \psi_i)] \right\},
\]

where \( j \neq i \) and the coefficients \( V_{1-3} \) are:

\[
V_1 = 12E^2\dot{L}^4 + 60G^2m^2\mu^3E\dot{L}^2 + 35G^4m^4\mu^6
\]

\[
V_2 = 56E^2\dot{L}^4 + 1620G^2m^2\mu^3E\dot{L}^2 + 805G^4m^4\mu^6
\]

\[
V_3 = 60(8E^2\dot{L}^4 + 18G^2m^2\mu^3E\dot{L}^2 + 7G^4m^4\mu^6)
\]

Note that \( V_1 = D_1 \), because both arose from the average of \( r^{-6} \). (The coefficient \( D_1 \), given by Eq. (4.32) of [15], governs the self-interaction term in the radiative loss of \( L \)). In order the 3/2 post-Newtonian order-accurate spin-orbit contribution to hold at 2PN, the replacements \( L \rightarrow \dot{L} \) and \( A_0 \rightarrow \dot{A} \) should be carried on in Eq. (4.4) of [12].

We stress that Eqs. (4.1)-(4.8) contain all radiative terms in the angular evolution up to the second post-Newtonian order above the leading radiative effects.
V. CONCLUDING REMARKS

By proving the 5/2 post-Newtonian accurate result that axisymmetric objects do not radiate away any fraction of their initial spins, the computation of the secular angular evolutions induced by radiation reaction became simpler. Then from the analysis of $d\mathbf{J}/dt$ we could conclude that there are no $N$, $PN$, $2PN$ and tail contributions to the secular radiative angular evolutions. The $SO$ part of the equations was given in [12]. We have derived the $SS$ terms here.

No spin-spin terms appear in the radiative evolution of the angle $\gamma$. As the spin-orbit terms average out [2], we found the remarkable result that the angle spanned by the spins receives no radiative secular change up to the second post-Newtonian order. (However the angle $\gamma$, together with all other angles is subjected to precessional - both instantaneous and secular - evolution. Although the second post-Newtonian order precessional (nonradiative) evolution of the angles is not developed in the paper, inspection of Table 1 shows that the precessions do not contribute to the 2PN radiative evolution of the angles $\kappa_i$ and $\gamma$, which contain only 3/2 PN and 2PN parts.)

The angles $\kappa_i$ evolve under the influence of both spin-orbit and spin-spin terms. The radiative angular evolution equations (4.4)-(4.8) together with the algebraic constraints (1.5)-(1.6) and the expressions for $dE/dt$ and $dL/dt$ derived in [15] form a closed system of first order differential equations.

The angle $\psi_0$ appearing in this system is an integration constant. It is interpreted as the angle subtended by the node line with $r$ at $\chi = 0$ (with the periastron line). Each precession modifies the value of $\psi_0$ by a small amount. According to the arguments of Ryan in [23], where this angle first appeared, the terms containing periodic functions of $\psi_0$ average to zero whenever the precession time scale is short compared to the radiation reaction time scale.

As it happened with the loss of $E$ and $L$ [1], the spin-spin terms of the radiative $\kappa_i$-evolution could be decomposed into two-body and self-interaction terms. In the one-spin limit $S_2 = 0$ the terms proportional to $S_1^2$ from Eq.(1.7) represent the second post-Newtonian correction to the radiative evolution of the angle $\kappa_1$ derived earlier in the Lense-Thirring approximation [24].

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