Horizon Problem Remediation via Deformed Phase Space

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April 26, 2011

Abstract

We investigate the effects of a special kind of dynamical deformation between the momenta of the scalar field of the Brans–Dicke theory and the scale factor of the FRW metric. This special choice of deformation includes linearly a deformation parameter. We trace the deformation footprints in the cosmological equations of motion when the BD coupling parameter goes to infinity. One class of the solutions gives a constant scale factor in the late time that confirms the previous result obtained via another approach in the literature. This effect can be interpreted as a quantum gravity footprint in the coarse grained explanation. The another class of the solutions removes the big bang singularity, and the accelerating expansion region has an infinite temporal range which overcomes the horizon problem. After this epoch, there is a graceful exiting by which the universe enters in the radiation dominated era.

PACS number: 04.20.Fy; 04.50.Kd; 02.40.Gh; 98.80.Qc
Keywords: Deformed Phase Space; Brans–Dicke Theory; Noncommutative Phase Space; Quantum Cosmology.

1 Introduction

General relativity (GR) and quantum theory, as generally believed, are two prominent paradigms to illustrate the nature. The idea of GR was inspired by the general covariance, the principle of equivalence and also the Mach ideas. However after the formulation of GR established by Einstein, its full satisfaction with the Mach principle has been a matter of debate. That is, although the matter content of the universe affects the geometry, but there are still vacuum solutions for GR, contrary to the strong version of Mach idea that states if there is no matter then, there will be no geometry. This proposition has made further considerations for alternative gravitational theories to be more Machian. One approach to this purpose has been stated by the scalar–tensor theories which among them, the Brans–Dicke (BD) theory is the simplest one [1]. In this scope, a scalar field plays the role of Newtonian gravitational constant and makes the theory to be more Machian. Also in the BD theory, there exists an adjustable dimensionless parameter which, in principle, can be fixed by observations. On the other hand, with suitable boundary conditions, GR is deducible from the BD theory when making this parameter goes to infinity limit (though not always, see, e.g., Refs. [3, 4, 5, 6]). In this view, the BD theory presents a modified version of GR (at least for traceless energy–momentum tensor [6]). Hence in this sense, there may also exist some other features in the BD theory.
theory in addition to those that are usually in GR. Such a kind of features (if any) can be interpreted as internal symmetries (or structures) of GR which are somehow missed if one just starts purely with the Ricci scalar. This aspect can also be viewed as if GR is only an effective theory, and the BD theory when the $\omega$ goes to infinity is a more realistic one.

The quantum theory is mainly established to justify the behavior of very small scales. Thus, it should be considered for small scale behaviors of GR (which is an excellent theory for large scale structures) as well. For example, in the standard cosmology, the universe has been commenced by a big bang in the very early universe. It is generally believed that, in the big bang, the scale of the universe is almost zero, thus it is predictable that the quantum behaviors to be significant in this regime. Indeed, there have been considerable attempts to combine quantum theory and GR in order to achieve a quantum gravity theory [7, 8, 9, 10]. One of these approaches is deformation of the phase space structure [11] that introduces, if not all but at least, a part of the quantum effects. This mechanism has been employed in the context of cosmology [12, 13], in affecting the small and even large scale behaviors, by removing the singularity and by the coarse grained effects, respectively. In this approach, usually a (length) parameter, which can be interpreted as the Planck (length) constant, presents the quantum regime. However, it is crucial to recover the standard results by taking the appropriate limits of this parameter. The another significant aspect of this kind of mechanism is its correspondence to the other methods of quantization, which is not only interesting but also indispensable, see, e.g., Refs. [14, 15].

As the scalar–tensor gravitational theories involve more degrees of freedom, they can give more number of solutions than GR [16]. Hence, we purpose to investigate deformation of the phase space structure in this context by studying the flat FRW metric and employing noncommutativity between the momenta of the BD scalar field and the scale factor. Then, we obtain the BD dynamical equations, however for simplicity and to be able to compare the outcomes with the corresponding results in GR, we solve them when the BD coupling parameter goes to infinity. Though, it is important to note that such a procedure does not make a precise transition to GR when the trace of energy–momentum vanishes [6], but we should also emphasize that if one just starts the procedure by the GR formalism, one cannot achieve such a wide classes of solutions, as expected.

In the next section, we briefly describe the BD theory in the context of Hamiltonian formalism which is essential for introducing deformation (noncommutativity) in the phase space. In Section 3, we derive the BD equations of motion in the presence of a special kind of deformation, then deduce their cosmological implications when the BD coupling parameter goes to infinity. Then, we discuss the solutions for different signs of the deformation parameter and the integration constants of the solutions while highlighting their effects. Finally, we will end up the work by conclusions in the last section, while two appendixes have also been furnished.

2 Hamiltonian Formalism for Brans–Dicke Theory

In this section, we review the BD theory in the Hamiltonian formalism. However, in order to study the cosmological behavior, we consider the spatially flat FRW metric as the background geometry, namely

$$
\text{ds}^2 = -N^2(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right),
$$

where $N(t)$ is a lapse function and $a(t)$ is the scale factor.

The BD Lagrangian density in the Jordan frame [1, 17, 18] in vacuum is given by

$$
\mathcal{L} = \sqrt{-g}\left(\phi R - \frac{\omega}{\phi}g^{\mu\nu}\phi,_{\mu}\phi,_{\nu}\right),
$$

where the Greek indices run from zero to three. Also, the $\phi$ is the BD scalar field, $R$ is the Ricci scalar and $\omega$ is the BD coupling constant that is supposed to be bigger than $-3/2$ for non–ghost
scalar field situations [19, 20, 21, 22]. Replacing the Ricci scalar associated to metric (1) into the
above Lagrangian gives
\[ L = -6N^{-1}a^2\dot{a} - 6N^{-1}a^2\dot{a} + \omega N^{-1}a^3\dot{a} + \phi^2, \] (3)
where the dot represents derivative with respect to the time and a total time derivative term has
been neglected. Thus, the corresponding Hamiltonian is
\[ H_0 = \frac{N}{\chi} \left( -\frac{\omega}{12}a^{-1}\phi^2 + \frac{1}{2}a^{-3}\phi P_\phi^2 - \frac{1}{2}a^{-2}P_\phi P_\phi \right), \] (4)
where \( \chi \equiv 2\omega + 3 \). As the momentum conjugate to \( N(t) \) vanishes, one has to add it as a constraint
to the above Hamiltonian. Therefore, the Dirac Hamiltonian becomes
\[ H = H_0 + \lambda P_N, \] (5)
where \( \lambda \) is a Lagrange multiplier. Here, we consider the ordinary phase space structure described by
the usual ordinary Poisson brackets
\[ \{a, P_a\} = \{\phi, P_\phi\} = \{N, P_N\} = \{1, \} \] (6)
where the other brackets vanish. Therefore, the equations of motion with respect to the Hamiltonian (5) are
\[ \dot{a} = \{a, H\} = -\frac{N}{\chi} \left( \frac{\omega}{6}a^{-1}\phi^2 + \frac{1}{2}a^{-2}P_\phi \right), \] (7)
\[ \dot{P_a} = \{P_a, H\} = -\frac{N}{\chi} \left( \frac{\omega}{12}a^{-2}\phi^2 - \frac{3}{2}a^{-4}\phi P_\phi^2 + a^{-3}P_\phi \right), \] (8)
\[ \dot{\phi} = \{\phi, H\} = \frac{N}{\chi} \left( a^{-3}\phi^2 + \frac{1}{2}a^{-2}P_\phi \right), \] (9)
\[ \dot{P_\phi} = \{P_\phi, H\} = -\frac{N}{\chi} \left( \frac{\omega}{12}a^{-1}\phi^2 - \frac{1}{2}a^{-3}P_\phi ^2 \right), \] (10)
\[ \dot{N} = \{N, H\} = \lambda, \] (11)
\[ \dot{P_N} = \{P_N, H\} = \frac{1}{\chi} \left( \frac{\omega}{12}a^{-1}\phi^2 - \frac{1}{2}a^{-3}P_\phi ^2 + \frac{1}{2}a^{-2}P_\phi P_\phi \right). \] (12)

Let us work in the comoving gauge, that is we fix the gauge by \( N = 1 \). Also, to satisfy the
constraint \( P_N = 0 \) at all times, the secondary constraint \( \dot{P}_N = 0 \) should also be satisfied. Hence, by
Eq. (12), one obtains
\[ P_\phi = \frac{1}{2} \left( 1 \pm \sqrt{\frac{\chi}{3}} \right) a\phi^{-1}P_a. \] (13)
Now, differentiating Eq. (7) with respect to the time, while using Eqs. (8), (9) and (13), leads to
\[ \ddot{a} = -\left( \frac{2\chi}{\chi \pm V_3} \right) a^{-1}\dot{a}^2. \] (14)

In addition to the trivial static solution, one can get solutions as \( a(t) = C_1(t - t_{ini})^{q_\pm} \) where \( C_1 \) and
the initial time \( t_{ini} \) are integration constants, and \( q_\pm \) is
\[ q_\pm = \frac{2}{3\chi - 1} \left[ \frac{\chi}{2} \pm \sqrt{\frac{\chi}{3}} \right], \] (15)
when \( \chi \neq 1/3 \) (i.e. \( \omega \neq -4/3 \)). In what follows, we assume that \( t_{ini} = 0 \), and the constant \( C_1 \)
can be fixed by the scale of the universe at an appropriate definite time. Also, one can easily obtain
\[ \frac{\dot{\phi}}{\phi} = \pm \left( \frac{2\sqrt{3\chi}}{\chi \pm V_3} \right) \frac{\dot{a}}{a}, \] (16)
with solutions $\phi(t) = C_2 t^{s\pm}$, where $C_2$ is an integration constant and $s\pm$ is given by
\begin{equation}
    s\pm = \frac{2(1 \pm \sqrt{3\chi})}{3\chi - 1},
\end{equation}
when again $\chi \neq 1/3$. Through the Hamiltonian approach, we have actually rederived the O’Hanlon and Tupper solution [23, 24], as expected. This solution has a big bang singularity when $t$ tends to zero. Also, we should emphasize that the behavior of the scale factor and the scalar field depends on the BD coupling parameter. In particular, when $\omega$ goes to infinity, the non-trivial solution is not the same as the corresponding result of GR (i.e. the Minkowski space–time), for in this case one gets
\begin{equation}
    a(t) = C_1 t^{1/3} \quad \text{and} \quad \phi = \text{constant},
\end{equation}
where the scale factor has a decelerated expanding behavior.

In the case $\chi = 1/3$, the solutions for upper sign are $a = C_3(t - t'_0)^{2/3}$ and $\phi = C_4(t - t'_0)^{-1}$, and for lower sign are $a = a_0 \exp(t/t''_0)$ and $\phi = \phi_0 \exp(-3t/t''_0)$, where $C_3$, $C_4$, $a_0$, $\phi_0$, $t'_0$ and $t''_0$ are constants.

In the next section, we investigate a modified version of the above formalism by introducing a noncommutative model.

### 3 Deformed Phase Space Brans–Dicke Structure

As mentioned in the introduction, deformation of the phase space can present a sort of tracing quantum footprints in a given model [11]. Of course, a general deformation makes equations very difficult and even unsolvable. Hence, it is customary to pick a proper choice which not only makes calculations be possible, but also gives non-trivial results. Indeed, by appealing to the simplicity principle (or the Occam’s razor), simplifications are usually performed in most toy models, and even in real ones, before a consistent and complete theory is deduced. Thus in this work, we just consider a dynamical deformation between the conjugate momentum sector as
\begin{equation}
    \{P'_a, P'_\phi\} = l\phi'(t) \tag{19}
\end{equation}
and leave the other Poisson brackets among the primed parameters (corresponding to those appeared in relation (6)) unchanged. Hence, the Jacobi identity is still satisfied. In general, noncommutativity between the momenta is a kind of generalization of the usual noncommutativity between the spatial coordinates [25]. In addition, noncommutativity between the momenta, in effective, has similarity with the behavior of a charged particle in the presence of a magnetic field. In the scope of gravitational theories, this kind of noncommutativity can be interpreted as a gravitomagnetic field [26]. Also, one may find a clue in the string theory, especially in the flux compactification, as mentioned in Ref. [27]. The dynamical behavior of deformation as time dependence has also been employed in the literature, e.g. the $\kappa$–Minkowskian spacetime [28, 29] and the generalized uncertainty principle [30, 31], that are considered also in the cosmological phase space [32, 33]. In addition, the main reason for the peculiarity of the chosen term in the right hand side of (19) is the dimensionality analysis that is described in more details in the Appendix A.

The minimally deformed (noncommutative) version of Hamiltonian (4) is achieved by replacing the unprimed variables with the primed ones, namely
\begin{equation}
    H'_0 = \frac{N'}{\chi} \left(-\frac{\omega}{12} a'^{-1} \phi'^{-1} P'_a^2 + \frac{1}{2} \phi'^{-3} \phi'^2 P'_a^2 - \frac{1}{2} a'^{-2} P'_a P'_\phi \right). \tag{20}
\end{equation}

However, it is more convenient to re–introduce the new variables by applying the standard transformation [25, 34]
\begin{equation}
    P'_\phi = P_\phi - l a \phi, \tag{21}
\end{equation}
One could easily obtain the solution from Eq. (14) for this limit, however for the sake of completeness, we have firstly derived explicit solutions for any $\omega$ too.
where the other unprimed variables are equivalent to their primed counterparts. By considering
the above transformation, relation (19) is satisfied when the unprimed variables satisfy the ordinary
Poisson brackets. Then, substituting (33) into (31) yields
\[
\mathcal{H}_0^{nc} = \mathcal{H}_0 + \frac{N l}{\chi} \left( \frac{1}{2} a^{-1} \phi^3 - a^{-2} \phi^2 P_\phi + \frac{1}{2} a^{-1} \phi P_a \right),
\]
where we have also substituted \( \mathcal{H}_0 \), as a function of the primed variables, with \( \mathcal{H}_0^{nc} \), as a function
of the unprimed ones via the employed transformation. Once again, the noncommutative Dirac
Hamiltonian is
\[
\mathcal{H}^{nc} = \mathcal{H}_0^{nc} + \lambda P_N.
\]
Therefore, the equations of motion become
\[
\dot{a} = \{ a, \mathcal{H}^{nc} \} = \frac{N}{\chi} \left( -\frac{\omega}{6} a^{-1} \phi^{-1} P_a - \frac{1}{2} a^{-2} P_\phi + \frac{1}{2} a^{-1} \phi \right),
\]
\[
\dot{P}_a = \{ P_a, \mathcal{H}^{nc} \} = -\frac{N}{\chi} \left( \frac{\omega}{12} a^{-2} \phi^{-1} P_a^2 - \frac{3}{2} a^{-4} \phi P_\phi^2 + a^{-3} P_a P_\phi - \frac{l^2}{2} a^{-2} \phi^3 + 2 l a^{-3} \phi^2 P_\phi - \frac{l}{2} a^{-2} \phi P_a \right),
\]
\[
\dot{\phi} = \{ \phi, \mathcal{H}^{nc} \} = \frac{N}{\chi} \left( a^{-3} \phi P_\phi - \frac{1}{2} a^{-2} P_a - l a^{-2} \phi^2 \right),
\]
\[
\dot{P}_\phi = \{ P_\phi, \mathcal{H}^{nc} \} = -\frac{N}{\chi} \left( \frac{\omega}{12} a^{-1} \phi^{-2} P_a^2 + \frac{1}{2} a^{-3} P_\phi^2 + \frac{3}{2} l^2 a^{-2} \phi^2 - 2 l a^{-2} \phi P_\phi + \frac{l}{2} a^{-1} P_a \right),
\]
\[
\dot{N} = \{ N, \mathcal{H}^{nc} \} = \lambda,
\]
\[
\dot{P}_N = \{ P_N, \mathcal{H}^{nc} \} = \frac{1}{\chi} \left( \frac{\omega}{12} a^{-1} \phi^{-1} P_a^2 - \frac{1}{2} a^{-3} \phi P_\phi^2 + \frac{1}{2} a^{-2} P_a P_\phi - \frac{l^2}{2} a^{-2} \phi^3 + 2 l a^{-2} \phi^2 P_\phi - \frac{l}{2} a^{-1} \phi P_a \right).
\]

In the comoving gauge, i.e. \( N = 1 \), the secondary constraint \( \dot{P}_N = 0 \) gives
\[
P_\phi = a \left[ 2 l \phi + \left( 1 \pm \sqrt{\frac{1}{3}} \right) \phi^{-1} P_a \right].
\]
Employing Eqs. (24)–(27) and (30), and performing a little manipulation lead to
\[
\ddot{a} = - \left( \frac{2 \chi}{\chi \pm \sqrt{3} \chi} \right) a^{-1} a^2 \pm \frac{1}{\sqrt{12} \chi} l a^{-2} \phi \dot{a}
\]
and again
\[
\frac{\dot{\phi}}{\phi} = \mp \left( \frac{2 \sqrt{3} \chi}{\chi \pm \sqrt{3} \chi} \right) \frac{\dot{a}}{a}.
\]
Solution of Eq. (32) can be in the form
\[
\phi = \phi_0 a^\xi,
\]
where \( \phi_0 \) is a constant and \( \xi \) is
\[
\xi \equiv \pm \frac{2 \sqrt{3} \chi}{\chi \pm \sqrt{3} \chi}.
\]
Substituting (33) into (31) yields
\[
\ddot{a} = - \left( \frac{2 \chi}{\chi \pm \sqrt{3} \chi} \right) a^{-1} a^2 \pm \frac{\phi_0}{\sqrt{12} \chi} l a^{\xi-2} \dot{a}.
\]
Note that, when the deformation parameter \( l \) tends to zero, all noncommutative equations reduce to
their corresponding ones in the previous section.
Now, as proposed, we are interested to investigate effects when $\omega$ goes to infinity, though again it does not mean that it makes transition to the standard GR, as has been shown for the commutative case in the previous section. However, as it is obvious from Eq. (35), in order to obtain such effects, it crucially depends on how the constant $\phi_0$ is, or can be, related to the $\omega$. Actually for this purpose, the value of constant $\phi_0$ (which represents different initial conditions) proportional to $\sqrt{\chi}$ can be a reasonable one. A particular motivation for it, however, is the new term in Eq. (35) (in comparison to Eq. (14)) which should not vanish in the limit $\chi \to \infty$ on the one hand. Namely, if it would vanish, one would not be able to see any new effects in comparison to the original undeformed theory in the limit $\omega \to \infty$. On the other hand, the new term also should not become infinite. However, a consequence of this choice is that when $\omega$ tends to infinity, then $\phi_0$ goes to infinity as well. Though, this brings a technical problem, i.e. it makes some ambiguities in the behavior of $\phi$–field in solution (33) when $\phi_0 \to \infty$ and $\xi \to 0$. In this limit, $\phi$ is a time independent (constant) variable which is infinity. To study the inconvenience caused by this divergence of $\phi_0$, the renormalization argument may assist in the following manner.

It is well–known that the procedure of renormalization occurs in the quantum field theoretical level. However in our toy model, this may indicate itself naively only in the deformation parameter as the only presenter of quantum regime in this work. Hence, as the first option, the deformation parameter $l \equiv l_{\text{bare}}$ can be re–defined in an appropriate way that makes the transition from Eq. (35) to Eq. (36) being possible. That is, it can be re–defined as $l_{\text{renormalized}} \equiv \phi_0 l_{\text{bare}}/\sqrt{12\chi}$ (see below Eq. (36)) with a finite $\phi_0$. Then, when $\chi \to \infty$, the $l_{\text{bare}}$ deformation parameter goes to infinity such that the $l_{\text{renormalized}}$ deformation parameter becomes a finite constant. Also, there is an alternative approach which is considered in the Appendix B.

Therefore, by taking $\omega$ goes to infinity and choosing the minus sign$^3$ in Eq. (35), one gets

$$\ddot{a} = -2a^{-1} \dot{a}^2 - l a^{-2} \dot{a},$$

(36)

where $\phi_0 l = \sqrt{12\chi} l$, which fixes the new parameter $l$ with dimensionality $L^{-1}$. Substituting $a^2 \ddot{a} = (a^2 \dot{a}) - 2a \ddot{a}^2$ into Eq. (36) gives

$$(a^2 \dot{a}) = -l \dot{a},$$

(37)

that yields $a^2 \ddot{a} = -l \dot{a} + C$, where $C$ is an integration constant with dimensionality $[C] = L^{-1}$. Then, one easily obtains

$$\frac{1}{2} a^2 + \frac{C}{l} a + \frac{C^2}{l^2} \ln|a - \frac{C}{l}| = \ell(-t + t_0),$$

(38)

where $t_0$ is an integration constant too. Obviously, the above equation is invariant under the transformation $(\ell, C, t) \to (-\ell, -C, -t)$. This symmetry makes a counterpart relevant between the solutions and, consequently reduces the number of investigations for different cases by half$^4$. Thus, in the following categorization, we consider the two probable options (the Case I and Case II) of the logarithmic term in Eq. (38) only for interesting cases of different signs of the $\ell$ and $C$, without probing the counterpart solutions. Also, for the sake of completeness, we explicitly investigate the solutions when $\ell$ tends to zero in Case III.

### 3.1 Case I: $a - \frac{C}{\ell} > 0$

As mentioned, we investigate this case for different signs of the $\ell$ and $C$.

#### 3.1.1 Case Ia: Negative $\ell$ & $C$

For convenience, assume $\bar{\ell} \equiv -\ell > 0$ and $b \equiv -C > 0$, thus Eq. (38) reads

$$\frac{1}{2} a^2 + \vartheta a + \vartheta^2 \ln|a - \vartheta| = \bar{\ell}(t - t_0),$$

(39)

$^3$This choice is not restrictive, for in the following we will consider different signs for the $\ell$.

$^4$For example, the case $\ell < 0$ and $C < 0$ is the counterpart of the another case $\ell > 0$ and $C > 0$ when $t \to -t$. 

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where $\vartheta \equiv C/\ell = b/\tilde{\ell} > 0$ with valid domain\(^5\) $a > \vartheta$. This means that the initial value of the scale factor cannot be zero and indeed, the universe has been started with a non-vanishing size. Therefore, one may interpret that the existence of a deformation parameter, as an indicator which usually presents the quantum corrections to models, removes the big bang singularity. Actually, this result is a common expectation in the quantum cosmological models.

Then, by differentiating (39), for $a > \vartheta$, we obviously get

$$\dot{a} = \frac{\tilde{\ell}}{a^2}(a - \vartheta) > 0$$

and hence,

$$\ddot{a} = \frac{-\tilde{\ell}^2}{a^5}(a - \vartheta)(a - 2\vartheta).$$

Thus, the sign of $\ddot{a}$ depends on two different regions, $\vartheta < a < 2\vartheta$ and $a > 2\vartheta$, which we investigate in the following.

**Region $\vartheta < a < 2\vartheta$**

In this region, the expansion is an accelerated one in similar to the inflationary phase. Though, it is not exactly as the standard inflationary phase, but it can solve the horizon problem as will be discussed in the following. Usually, a successful candidate for the standard inflation should satisfy two essential properties among the other ones, namely the 60 e–fold duration and a graceful exit from this epoch. Our model naturally satisfies the latter requirement, for the scale factor transits to the next region, i.e. $a > 2\vartheta$, where it decelerates. However, at first glance, it looks that it does not satisfy the former condition as it has much less than 60 e–fold duration. Indeed, the number of e–fold definition, i.e. $N = a_{\text{final}}/a_{\text{initial}}$, for our model is $N = 2\vartheta/\vartheta = 2$. Nevertheless, its result is comparable to the standard inflation one by presenting a solution for the horizon problem, which we indicate it after a brief review on the successes of the standard inflation while clarifying the horizon problem.

It is well-known that the most important problem of the standard cosmology, which is solved by proposition of an inflationary scenario, is the horizon problem. Of course, the inflation also solves the relic particle abundances (or the monopoles) and the flatness problems. However, it is generally believed that among these problems, the horizon problem is the most important one, for at least there are alternative scenarios that can resolve the other two problems in the same manner as the inflation does [35, 36]. The horizon problem arises when the universe is observed to be isotropic and homogeneous in the large scale structure. This requires that the initial conditions must be in a way which give such a universe. The problem with the standard cosmology is that although the matter fluctuations have been inhomogeneous at the initial level, but these fluctuations did not have enough time for interactions and transforming information about their situations. Consequently, the inhomogeneous initial conditions should naturally result in inhomogeneous present large scale structure which is in contradiction to the observations. The inflationary idea solves this problem by taking a homogeneous part of the initial condition, and inflates it to an appropriate size for the beginning of the radiation dominated era. In our model, the horizon problem is solved in another way.

Actually, the accelerating phase occurs during $a_{\text{initial}} \longrightarrow \vartheta$ and $a_{\text{final}} = 2\vartheta$, which is from $t_{\text{initial}} \longrightarrow -\infty$ to a finite final time, that is $t_{\text{final}} = \vartheta^2(4 + \ln \vartheta)/\tilde{\ell}$ (assuming $t_0 = 0$). This means that the accelerating phase takes infinite time, $\Delta t = (t_{\text{final}} - t_{\text{initial}}) \longrightarrow \infty$, and during this phase, the matter fluctuations can interact with the other parts of initial conditions, exchange information about their local structures, and hence, approach to an equilibrium state which is presented by a homogeneous structure. Thus, although in our simple model, the accelerating phase cannot be interpreted as the standard inflationary era, but it can address the horizon problem of the standard cosmology.

\(^5\)We have neglected the equality $a = \vartheta$, for it makes $t$ becomes $-\infty$. 

Fig. 1: The solid line shows the behavior of the scale factor for Case Ia with $\ell = C = -0.1$. Below the dashed line (i.e. $\vartheta < a < 2\vartheta = 2C/\ell = 2$), one has an accelerating phase, and above it (i.e. $a > 2\vartheta = 2C/\ell = 2$), a decelerating phase. In the right figure, the dotted curve represents the $\ddot{a}(t)$ which is negative for $a > 2\vartheta = 2$. Note that, the $\ddot{a}(t)$ curve has been rescaled for a better clarification.

This quasi static\(^6\) accelerating phase is very similar to the Hagedorn phase of string gas cosmology [37].

**Region** $a > 2\vartheta$

In this case, the expansion of universe is decelerating, and when $a \rightarrow \infty$, the first term in Eq. (39) is the dominant one, hence in this limit, $a(t)$ tends to $t^{1/2}$ that behaves as the radiation era. This phase occurs exactly after the above accelerating phase, and can be interpreted as the radiation dominated phase after the usual inflationary epoch in the standard cosmology. Indeed, this result is completely in agreement with what is usually proposed for the universe in different cosmological models, the standard cosmology with or without an inflation.

The behavior of the scale factor is depicted in Fig. 1.

### 3.1.2 Case Ib: Negative $\ell$ & Positive $C$

This case is very similar to the previous one, except the sign change in the acceleration. Hence, for the entire valid region of the scale factor, i.e. $a > 0$, the $\dot{a} > 0$ and $\ddot{a} < 0$ give a decelerated expanding behavior. When $t \rightarrow \infty$ then $a(t)$ tends to $t^{1/2}$, exactly as the previous case, however here, there are not interesting properties. The behavior of the scale factor is depicted in Fig. 2(left).

### 3.2 Case II: $a - C/\ell < 0$

Once again, we investigate this case for different signs of the $\ell$ and $C$ too.

#### 3.2.1 Case IIa: Positive $\ell$ & $C$

To describe this case, let us repeat Eq. (38) as

$$\frac{1}{2}a^2 + \vartheta \dot{a} + \vartheta^2 \ln (\vartheta - a) = \ell (-t + t_0),$$

where the valid domain of the scale factor is $0 \leq a < \vartheta$, which the $a \geq 0$ is a physical constraint. That is, in this case, there is no regulator to prevent the big bang singularity, for $a = 0$ occurs at

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\(^6\)For it takes infinite time to double the value of the initial scale factor.
Fig. 2: The solid line in the left figure shows the behavior of the scale factor for Case Ib with \( \ell = -C = -0.1 \). As it is obvious, the scale factor asymptotically is proportional to \( t^{1/2} \) (the dotted line). The solid line in the right figure shows the behavior of the scale factor for Case IIa with \( \ell = C = 0.1 \). When \( t \to \infty \), the scale factor approaches \( \vartheta = C/\ell = 1 \).

\[
t = t_0 - (\vartheta^2 \ln \vartheta)/\ell.
\]

However, the upper bound of the scale factor approaches to a constant when \( t \) goes to infinity, i.e. for the late time, one gets

\[
a(t \to \infty) = \vartheta = \text{constant}. \tag{43}
\]

This approaching to a constant value is a direct consequence of the existence of a deformation parameter, and more interestingly, it shows itself far from the big bang. As mentioned before, since this deformation parameter may be interpreted as a consequence of the quantum effects, then this feature may also be viewed as a quantum gravity effect when the scale of the universe is significant. That is, this behavior can be a phenomenological property for the quantum gravity. However, these kind of modifications can be perceived as a semi–classical model or, as a model beyond the BD theory but still in the classical regime. The scale factor decelerates in this choice, and its diagram is plotted in Fig. 2(right). The graph shows that the behavior of the scale factor is in agreement with the results obtained by Ref. [32] in where a dynamical deformation between the lapse function and the scale factor has been employed.

### 3.2.2 Case IIb: Positive \( \ell \) & Negative \( C \)

The case \( l > 0 \) and \( C < 0 \) gives negative values for the scale factor which is not acceptable.

### 3.3 Case III: \( \ell \) Tends to Zero

As mentioned before, all equations reduce to their corresponding commutative ones when the deformation parameter vanishes. Now, let us explicitly investigate it for Eq. (38). Hence, taking the limit \( \ell \to 0 \) for the logarithmic term in Eq. (38) gives

\[
\frac{C^2}{\ell^2} \ln \left( \frac{C}{\ell} - a \right) = \frac{C^2}{\ell^2} \ln \frac{C}{\ell} - \frac{Ca}{\ell} - \frac{a^2}{2} - \frac{\ell a^3}{3C} - \cdots, \tag{44}
\]

where higher order terms in \( \ell \) can be neglected. And obviously, the second and the third terms in the above relation cancel the second and the first terms in Eq. (38), respectively. The first term in relation (44) is a constant and can be absorbed by re–definition of \( t_0 \). Consequently, the scale factor tends to \( t^{1/3} \) which recovers the commutative solution (18) when \( \omega \) goes to infinity, as expected.
4 Conclusions

We have introduced a deformation in the phase space structure of the two existing fields of the BD theory in the spatially flat FRW metric. Also, we have traced the quantum footprints in the cosmological equations of motion in the comoving gauge. All the noncommutative equations are shown that do reduce to their corresponding counterparts when the deformation parameter tends to zero, as expected. Then as proposed, we have investigated the effects when the BD coupling parameter goes to infinity. In this process, we have faced an integration constant that depends on the initial conditions, however in order to be able to trace the effects, we assume it to be proportional to the square root of the BD coupling parameter. We have also discussed our justifications for why we have to fix it in this way and to render other side effects.

Finally, different cosmological results have been deduced due to the different possible signs for the two arbitrary parameters of the solutions, namely the deformation parameter, $\ell$, and the another integration constant (i.e. the $C$ in Eq. (38)). For one class of the solutions, the result predicts a constant value for the scale factor in the late time that is in agreement with the results obtained in Ref. [32]. This feature may be interpreted as a quantum gravity footprint in the large scale.

A more interesting result is achieved by the another class of the solutions. In this case, it is shown that the existence of the deformation parameter (or equivalently, the quantum correction) removes the big bang singularity by preventing the scale factor tends to zero. This also has an infinite temporal range for an accelerating expansion region. However, this phase is not the standard inflationary phase, for its e-fold duration is a very small number, but it can appropriately overcome the horizon problem that is the main one in the standard cosmology. Indeed, due to the very long time duration of this phase, the matter fluctuations can transmit their information to the other parts of the universe, and consequently become homogeneous. Implicitly, after this epoch, there is a graceful exiting and then, the universe enters in a radiation dominated era which is naturally in agreement with the standard cosmology, with or without an inflation. Note that, these consequences are held just by introducing a constant parameter without considering any potential in the model. It should also be mentioned that the model just makes a (classical) background plausible to address the horizon problem similar to (classical) background of the inflationary models.

However, one of the major success of the standard inflation is its prediction of (quantum) fluctuations' behavior. The standard inflation anticipates a scale invariant spectral index which is in good agreement with the observations. For our model, considering the fluctuation of dynamics is important, for not only to compare with the observational data but also, to test the stability of the model. It should be checked whether inhomogeneous arbitrary initial conditions can have a significant effect in the late time behavior or not. To overcome such general questions, one needs to perform more investigations, perhaps employing the perturbative analysis which in our model is still more complicated than in the standard inflation, due to the existence of the BD scalar field as well as the deformation parameter. These are interesting investigations for further considerations, and are not in the scope of the current work.

In fact the above achievement in solving the horizon problem can be viewed as a natural consequence of the noncommutativity approaches. That is, it is well-known that the varying speed of light and noncommutative models are related to each other [38], where the first motivation for the former models has been raised in order to achieve an alternative approach to the standard inflation. Indeed, the coordinate noncommutativity has been employed in the cosmological context for the same purpose as well, see, e.g., Refs. [39, 40].

Appendix A: On Dynamical Deformation As Relation (19)

Let us first indicate the dimension of the Poisson bracket $\{P_a, P_\phi\}$. In this work, we have employed the units $\hbar = 1 = c$, therefore, from the Plank length, $l_P = \sqrt{\hbar G/c^3}$, the dimensions of $G$ and $\phi$ are $[G] = L^2$ and $[\phi] = L^{-2}$. The scale factor and the lapse function are dimensionless parameters,
the dimensions of coordinates and the BD Lagrangian are \([x^\mu] = L\) and \([\mathcal{L}] = L^{-4}\). Hence, one can conclude that \([P_a] = L^{-3}\) and \([P_\phi] = L^{-1}\), and consequently \([\{P_a, P_\phi\}] = L^{-1}\). On the other hand, the dimension of the deformation parameter is \([l] = L\). Therefore, from dimensionality aspects of view, the dynamical deformation (19) is a plausible choice. Besides, from simplicity point of view, with this choice, no other extra field has been introduced in the model.

Of course, one may also propose other choices that still can satisfy the dimensionality of \([P_a, P_\phi]\), but the suggested relation (19) is a first order (linear) term in the deformation parameter as well. This suggestion is also a length indicator that can present and trace the quantum behaviors, and if the length indicator vanishes, one will recover the standard (classical) counterpart relations. On the other hand, in cosmological models a length parameter (e.g. the Planck length that is a function of \(\hbar\)) is physically a more realistic and plausible choice. That is, among different choices that can be selected as a quantum indicator, a length scale is an appropriate one for cosmological models, that can compare different scales for the quantum or classical aspects as well.

**Appendix B: Discussion on Fixing Integration Constant \(\phi_0\)**

To fix the inconvenience behavior of the \(\phi_0\) when \(\chi \to \infty\), one may also apply the renormalization procedure alternatively to the matter field \(L_{\text{matter}}\). However in our model, there is no matter field and \(G\) (which is equal to \(1/\phi\) in the BD theory) does not appear in the equations of motion and makes its value non–effective, but the above procedure can be applied as if the matter field is turned on. Hence, in this case, the value of Newtonian gravitational constant can be regularized by re–definition of the matter field as well as the BD scalar field. To be more specific, in the presence of a matter field, the BD Lagrangian is

\[
\mathcal{L} = \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + L_{\text{bare matter}} \right). \tag{B.1}
\]

Now, as an overall constant has no role in the form of equations of motion, one can multiply the above BD Lagrangian by a dimensionless parameter \((\phi_0 G_{\text{bare}})^{-1}\) to get

\[
\frac{1}{\phi_0 G_{\text{bare}}} \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + L_{\text{bare matter}} \right) = \sqrt{-g} \left[ \frac{\phi}{\phi_0 G_{\text{bare}}} R - \omega \left( \frac{\phi}{\phi_0 G_{\text{bare}}} \right)^{-1} g^{\mu\nu} \left( \frac{\phi}{\phi_0 G_{\text{bare}}} \right)_{\mu\nu} + \frac{1}{\phi_0 G_{\text{bare}}} L_{\text{bare matter}} \right]. \tag{B.2}
\]

The \(\phi_0\), that goes to infinity when \(\omega \to \infty\), can be absorbed in re–definition of \(\phi\) and \(L_{\text{bare matter}}\) by a renormalization process such that

\[
\mathcal{L}_{\text{renormalized}} = \sqrt{-g} \left[ \frac{\phi}{\phi_0 G_{\text{bare}}} R - \omega \left( \frac{\phi}{\phi_0 G_{\text{bare}}} \right)^{-1} g^{\mu\nu} \left( \frac{\phi}{\phi_0 G_{\text{bare}}} \right)_{\mu\nu} + L_{\text{renormalized matter}} \right], \tag{B.3}
\]

where \(\phi_0 = G_{\text{renormalized}}^{-1} = \phi_{\text{renormalized}} \equiv \phi/(\phi_0 G_{\text{bare}})\) and \(L_{\text{renormalized matter}} \equiv L_{\text{bare matter}}/(\phi_0 G_{\text{bare}})\). Thus, the difficulty of infinite value of \(\phi_0\) can be solved, at least naively, by the renormalization procedure.

**Acknowledgement**

We would like to thank H. Firouzjahi and M.M. Sheikh–Jabbari for fruitful discussions.

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