Five-Dimensional Cosmological Theory of Unified Space, Time and Velocity

Moshe Carmeli

1. INTRODUCTION

In this lecture we introduce a five-dimensional cosmological theory of space, time and velocity. The added extra dimension of velocity to the usual four-dimensional spacetime will be evident in the sequel. Before introducing the theory we have to deal, as usual, with coordinate systems in cosmology. Other important basic issues will be dealt with later on.

We will use cosmic coordinate systems that fill up spacetime. Given one system \( x \), there is another one \( x' \) that differs from the original one by a Hubble transformation \( x' = x + t_1 v \), \( t_1 = \text{constant} \), where \( v \) is a velocity parameter, and \( y \) and \( z \) are kept unchanged. A third system will be given by another Hubble transformation, \( x'' = x' + t_2 v = x + (t_1 + t_2)v \).

The cosmic coordinate systems are similar to the inertial coordinate systems, but now the velocity parameter takes over the time parameter. The analogous Galileo transformation to the latter that relates inertial coordinate systems is given, as known, by \( x'' = x' + v t = x + (v_1 + v_2)t \).

The universe expansion is also given by a formula of the above kind: \( x' = x + \tau v \), where \( \tau = H_0^{-1} \) in the limit of zero distance. However, the universe expansion is apparently incompatible with the Hubble spacetime transformation, namely one cannot add them. Thus, if we have \( x'' = x' + tv, \quad x' = \tau v \), then \( x'' \neq (\tau + t) v \). Rather, it is always \( x'' = \tau v \).

This situation is like that we have with the propagation of light, \( x'' \neq (c + v) t \), but it is always \( x'' = ct \) in all inertial coordinate systems, and where \( c \) is the speed of light in vacuum.

The constancy of the speed of light and the validity of the laws of nature in inertial coordinate systems, though they are both experimentally valid, they are not compatible with each other. We have the same situation in cosmology; the constancy of the Hubble constant in the zero-distance limit, and the validity of the laws of nature in cosmic coordinate systems, though both are valid, they are incompatible with each other.

In the case of light propagation, one has to abandon the Galileo transformation in favor of the Lorentz transformation. In cosmology one has to give up the Hubble transformation for the cosmological transformation given by

\[
\begin{align*}
x' &= \frac{x - tv}{\sqrt{1 - t^2/\tau^2}}, & v' &= \frac{v - tx/\tau^2}{\sqrt{1 - t^2/\tau^2}}, \\
y' &= y, & z' &= z,
\end{align*}
\]

(1)

for the case with fixed \( y \) and \( z \).

As is well known, the flat spacetime line element in special relativity is given by \( ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) \). The cosmological flat spacetime line element is given, accordingly, by

\[
ds^2 = \tau^2 dv^2 - (dx^2 + dy^2 + dz^2).
\]

(2)

The special-relativistic line element is invariant under the Lorentz transformation. So is the cosmological line element: it is invariant under the Lorentz-like cosmological transformation. The first keeps invariant the propagation of light,
whereas the second keeps invariant the expansion of the universe. At small velocities with respect to the speed of light, \( v \ll c \), the Lorentz transformation goes over to the nonrelativistic Galileo transformation. So is the situation in cosmology: the Lorentz-like cosmological transformation goes over to the nonrelativistic Hubble transformation that is valid for cosmic times much smaller than the Hubble time, \( t \ll \tau \).

2. Universe with Gravitation

The universe is, of course, not flat but filled up with gravity. When gravitation is invoked, the above spaces become curved Riemannian with the line element \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu \), where \( \mu, \nu \) take the values 0, 1, 2, 3, 4. The coordinates are: \( x^0 = ct, x^1, x^2, x^3 \) are spatial coordinates and \( x^4 = \tau v \). The signature is \((+−−−+)\).

The metric tensor \( g_{\mu\nu} \) is symmetric and thus we have fifteen independent components. They will be a solution of the Einstein field equations in five dimensions.

The five-dimensional field equations will not explicitly include a cosmological constant, the latter is derivable from the theory. Our cosmological constant will be equal to \( \Lambda = 3/\tau^2 \approx 1.934 \times 10^{-35}s^{-2} \) (for \( H_0 = 70\text{km/s-Mpc} \)). This should be compared with results of the experiments recently done with the supernovae which suggest the value of \( \Lambda \approx 10^{-35}s^{-2} \). Our cosmological constant is derived from the theory itself which is part of the classification of the cosmological spaces to describe decelerating, constant or accelerating universe. We now discuss some basic questions that are encountered in going from four to five dimensions.

First, we have to iterate what do we mean by coordinates in general and how one measures them. The time coordinate is measured by clocks as was emphasized by Einstein repeatedly. So are the spatial coordinates: they are measured by meters, as was originally done in special relativity theory by Einstein, or by use of Bondi’s more modern version of k-calculus.

But how about the velocity as an independent coordinate? One might incline to think that if we know the spatial coordinates then the velocities are just the time-derivative of the coordinates and they are not independent coordinates. This is, indeed, the situation for a dynamical system when the coordinates are given as functions of the time. But in general the situation is different, especially in cosmology. Take, for instance, the Hubble law \( v = H_0x \). Obviously \( v \) and \( x \) are independent parameters and \( v \) is not the time derivative of \( x \). Basically one can measure \( v \) by instruments like those used by traffic police.

To finish this section we discuss the important concept of the energy density in cosmology. We use the Einstein field equations, in which the right-hand side includes the energy-momentum tensor. For fields other than gravitation, like the electromagnetic field, this is a straightforward expression that comes out as a generalization to curved spacetime of the same tensor appearing in special-relativistic electrodynamics. However, when dealing with matter one should construct the energy-momentum tensor according to the physical situation (see, for example, Fock, Ref. 16). Often a special expression for the mass density \( \rho \) is taken for the right-hand side of Einstein’s equations, which sometimes is expressed as a \( \delta \)-function.

In cosmology we also have the situation where the mass density is put on the right-hand side of the Einstein field equations. There is also the critical mass density \( \rho_c = 3/8\pi G\tau^2 \), the value of which is about \( 10^{-29}\text{g/cm}^3 \), just a few hydrogen atoms per cubic meter throughout the cosmos. If the universe average mass density \( \rho \) is equal to \( \rho_c \) then the universe will have a constant expansion. A deviation from this necessitates an increase or decrease from \( \rho_c \). That is to say that \( \rho_{eff} = \rho - \rho_c \) is the active or the effective mass density that causes the universe not to have a constant expansion. Accordingly, one should use \( \rho_{eff} \) in the right-hand side of the Einstein field equations. Indeed, we will use such a convention throughout this paper. The subtraction of \( \rho_c \) from \( \rho \) in not significant for celestial bodies and makes no difference.
3. The Accelerating Universe

In the last two sections we gave arguments to the fact that the universe should be presented in five dimensions, even though the standard cosmological theory is obtained from Einstein’s four-dimensional general relativity theory. The situation here is similar to that prevailed before the advent of ordinary special relativity. At that time the equations of electrodynamics, written in three dimensions, were well known to predict that the speed of light was constant. But that was not the end of the road. The abandon of the concept of absolute space along with the constancy of the speed of light led to the four-dimensional notion. In cosmology now, we have to give up the notion of absolute cosmic time. Then this with the constancy of the Hubble constant in the limit of zero distance leads us to a five-dimensional presentation of cosmology.

We recall that the field equations are those of Einstein in five dimensions, \( R^\mu_{\nu} - \frac{1}{2}g^\mu_{\nu}R = \kappa T^\mu_{\nu} \), where Greek letters \( \alpha, \beta, \ldots, \mu, \nu, \ldots = 0, 1, 2, 3, 4 \). The coordinates are \( x^0 = ct \), \( x^1, x^2 \) and \( x^3 \) are space-like coordinates, \( r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \), and \( x^4 = \tau v \). The metric used is given by \( g_{00} = 1 + \phi \), \( g_{kl} = -\delta_{kl} \), \( g_{44} = 1 + \psi \), other components are zero. We will keep only linear terms. The components of the Ricci tensor and the Ricci scalar are given by

\[
\begin{align*}
R^0_0 &= \frac{1}{2} (\nabla^2 \phi - \phi_{,44} - \psi_{,00}), \\
R^0_0 &= \frac{1}{2} \psi_{,00}, \quad R^0_0 = -\frac{1}{2} \psi_{,00}, \quad R^0_4 = R^4_0 = 0, \\
R^m_n &= \frac{1}{2} (\phi_{,mn} + \psi_{,mn}), \\
R^4_4 &= -\frac{1}{2} \phi_{,44}, \quad R^4_4 = \frac{1}{2} \phi_{,44}, \\
R &= \nabla^2 \phi + \nabla^2 \psi - \phi_{,44} - \psi_{,00}. 
\end{align*}
\]

In the above equations \( \nabla^2 \) is the ordinary three-dimensional Laplace operator.

The line element in five dimensions is given by

\[
ds^2 = (1 + \phi)dt^2 - dr^2 + (1 + \psi)dv^2, 
\]

where \( dr^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \), and where \( c \) and \( \tau \) were taken, for brevity, as equal to 1. The line element (5) represents a spherically symmetric universe.

The expansion of the universe (the Hubble expansion) is recorded at a definite fixed time and thus \( dt = 0 \). Accordingly, taking into account \( d\theta = d\phi = 0 \), Eq. (5) gives the following equation for the expansion of the universe at a certain moment,

\[
dr^2 + (1 + \psi)dv^2 = 0, 
\]

and thus

\[
(dr/dv)^2 = 1 + \psi. 
\]

To find \( \psi \) we solve the Einstein field equation (noting that \( T^0_0 = g_{00}T^{00} \approx T^{00} = \rho(dx^0/ds)^2 \approx c^2\rho \), or \( T^0_0 \approx \rho \) in units with \( c = 1 \):

\[
R^0_0 - \frac{1}{2}g^0_0R = 8\pi G\rho_{eff} = 8\pi G (\rho - \rho_c), 
\]

where \( \rho_c = 3/8\pi G r^2 \).

A simple calculation using Eqs. (3) and (4) then yields

\[
\nabla^2 \psi = 6(1 - \Omega), 
\]

where \( \Omega = \rho/\rho_c \).

The solution of the field equation (9) is given by

\[
\psi = (1 - \Omega)^2 + \psi_0, 
\]

where the first part on the right-hand side is a solution for the non-homogeneous Eq. (9), and \( \psi_0 \) represents a solution to its homogeneous part, i.e. \( \nabla^2 \psi_0 = 0 \). A solution for \( \psi_0 \) can be obtained as an infinite series in powers of \( r \). The only term that is left is of the form \( \psi_0 = -K_2/r \), where \( K_2 \) is a constant whose value can easily be shown to be the Schwartzschild radius, \( K_2 = 2GM \). We therefore have

\[
\psi = (1 - \Omega)^2 - 2GM/r. 
\]

The universe expansion is therefore given by

\[
(dr/dv)^2 = 1 + (1 - \Omega)^2 - \frac{2GM}{r}. 
\]

For large \( r \) the last term on the right-hand side of (12) can be neglected, and therefore

\[
(dr/dv)^2 = 1 + (1 - \Omega)r^2, 
\]
or
\[ \frac{dr}{dv} = \left[ 1 + (1 - \Omega) r^2 / c^2 \right]^{1/2}. \]  
(14)

Inserting now the constants \( c \) and \( \tau \) we finally obtain for the expansion of the universe
\[ \frac{dr}{dv} = \tau \left[ 1 + (1 - \Omega) r^2 / c^2 \right]^{1/2}. \]  
(15)

This result is exactly that obtained by Behar and Carmeli (BC) (Eq. 5.10) when the non-relativistic relation \( z = v/c \), where \( z \) is the redshift parameter, is inserted in the previous result.

The second term in the square bracket of (15) represents the deviation from constant expansion due to gravity. For without this term, Eq. (15) reduces to \( dr/dv = \tau \), thus \( r = \tau v + \text{const} \). The constant can be taken zero if one assumes, as usual, that at \( r = 0 \) the velocity should also vanish. Accordingly we have \( r = \tau v \) or \( v = \tau^{-1} r \). Hence when \( \Omega = 1 \), namely when \( \rho = \rho_c \), we have a constant expansion.

The equation of motion (15) can be integrated exactly. The results are:

For the \( \Omega > 1 \) case
\[ r(v) = (\tau \alpha) \sin (\alpha v/c); \quad \alpha = (\Omega - 1)^{1/2}. \]  
(16)

This is obviously a decelerating expansion.

For \( \Omega < 1 \),
\[ r(v) = (\tau \beta) \sinh (\beta v/c); \quad \beta = (1 - \Omega)^{1/2}. \]  
(17)

This is now an accelerating expansion.

For \( \Omega = 1 \) we have, from Eq. (15),
\[ \frac{d^2 r}{dv^2} = 0, \]  
(18)

whose solution is, of course,
\[ r(v) = \tau v, \]  
(19)

and this is a constant expansion. It will be noted that the last solution can also be obtained directly from the previous two cases for \( \Omega > 1 \) and \( \Omega < 1 \) by going to the limit \( v \to 0 \), using L'Hospital’s lemma, showing that our solutions are consistent.

It has been shown in BC that the constant expansion is just a transition stage between the decelerating and the accelerating expansions as the universe evolves toward its present situation.

This occurred at 8.5 Gyr ago at a time the cosmic radiation temperature was 143K [3].

In order to decide which of the three cases is the appropriate one at the present time, it will be convenient to write the solutions (16), (17) and (19) in the ordinary Hubble law form \( v = H_0 r \). Expanding Eqs. (16) and (17) and keeping the appropriate terms then yields
\[ r = \tau v \left( 1 - \frac{\alpha^2 v^2}{6c^2} \right), \]  
(20)
\[ r = \tau v \left( 1 + \frac{\beta^2 v^2}{6c^2} \right), \]  
(21)

for the \( \Omega > 1 \) and \( \Omega < 1 \) cases, respectively. Using now the expressions for \( \alpha \) and \( \beta \) in Eqs. (20) and (21), then both of the latter can be reduced into the single equation
\[ r = \tau v \left[ 1 + (1 - \Omega) \frac{v^2}{6c^2} \right]. \]  
(22)

Inverting now this equation by writing it in the form \( v = H_0 r \), we obtain in the lowest approximation for \( H_0 \)
\[ H_0 = h \left[ 1 - (1 - \Omega) \frac{v^2}{6c^2} \right], \]  
(23)

where \( h = 1/\tau \). Using \( v \approx r/\tau \), or \( z \approx v/c \), we also obtain
\[ H_0 = h \left[ 1 - (1 - \Omega) \frac{r^2}{6c^2 \tau^2} \right] = \frac{h}{\tau} \left[ 1 - (1 - \Omega) \frac{z^2}{6} \right]. \]  
(24)

As is seen \( H_0 \) depends on the distance, or equivalently, on the redshift. Consequently \( H_0 \) has meaning only in the limits \( r \to 0 \) and \( z \to 0 \), namely when measured locally, in which case it becomes the constant \( h \). This is similar to the situation with respect to the speed of light when measured globally in the presence of gravitational field as the ratio between distance and time, the result usually depends on these parameters. Only in the limit one obtains the constant speed of light in vacuum (\( c = 3 \times 10^{10} \text{cm/s} \)).

Accordingly, \( H_0 \) is intimately related to the sign of the factor \( (1 - \Omega) \). If measurements of \( H_0 \) indicate that it increases with the redshift parameter \( z \) then the sign of \( (1 - \Omega) \) is negative, namely \( \Omega > 1 \). If, however, \( H_0 \) decreases when \( z \) increases then the sign of \( (1 - \Omega) \) is positive, i.e. \( \Omega < 1 \). The possibility of \( H_0 \) not to depend on the redshift parameter indicates that \( \Omega = 1 \). In recent years different measurements were obtained
for $H_0$ with the so-called “short” and “long” distance scales, in which higher values of $H_0$ were obtained for the short distances and the lower values for $H_0$ corresponded to the long distances. Indications are that the longer the distance of measurement, the smaller the value of $H_0$. If one takes these experimental results seriously, then that is possible only for the case in which $\Omega < 1$, namely when the universe is at an accelerating expansion phase, and the universe is thus open.

### 4. The Cosmological Constant

First, a historical remark. In order to allow the existence of a static solution for the gravitational field equations, Einstein made a modification to his original equations by adding a cosmological term,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$  \hspace{0.5cm} (25)

where $\Lambda$ is the cosmological constant and $\kappa = 8\pi G$ ($c$ is taken as 1). For a homogeneous and isotropic universe with the line element $ds^2 = dt^2 - a^2(t) R_0^2 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]$, \hspace{0.5cm} (26)

where $k$ is the curvature parameter ($k = 1, 0, -1$) and $a(t) = R(t)/R_0$ is the scale factor, with the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu},$$  \hspace{0.5cm} (27)

Einstein’s equations (5.1) reduce to the two Friedmann equations

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \rho + \frac{\Lambda}{3} - \frac{k}{a^2 R_0^2},$$  \hspace{0.5cm} (28)

$$\frac{\dot{a}}{a} = -\frac{\kappa}{6} (\rho + 3p) + \frac{\Lambda}{3}. \hspace{0.5cm} (29)$$

These equations admit a static solution ($\dot{a} = 0$) with $k > 0$ and $\Lambda > 0$. After Hubble’s discovery that the universe is expanding, the role of the cosmological constant to allow static homogeneous solutions to Einstein’s equations in the presence of matter, seemed to be unnecessary. For a long time the cosmological term was considered to be of no physical interest in cosmological problems.

From the Friedmann equation (28), for any value of the Hubble parameter $H$ there is a critical value of the mass density such that the spatial geometry is flat ($k = 0$), $p_c = 3H_0^2/\kappa$. One usually measures the total mass density in terms of the critical density $\rho_c$ by means of the density parameter $\Omega = \rho/\rho_c$.

In general, the mass density $\rho$ includes contributions from various distinct components. From the point of view of cosmology, the relevant aspect of each component is how its energy density evolves as the universe expands. In general, a positive $\Lambda$ causes acceleration to the universe expansion, whereas a negative $\Lambda$ and ordinary matter tend to decelerate it. Moreover, the relative contributions of the components to the energy density change with time. For $\Omega_{\Lambda} < 0$, the universe will always recollapse to a Big Crunch. For $\Omega_{\Lambda} > 0$ the universe will expand forever unless there is sufficient matter to cause recollapse before $\Omega_{\Lambda}$ becomes dynamically important. For $\Omega_{\Lambda} = 0$ we have the familiar situation in which $\Omega_M \leq 1$ universes expand forever and $\Omega_M > 1$ universes recollapse. (For more details see the paper by Behar and Carmeli, Ref. 6.)

Recently two groups, the Supernova Cosmology Project Collaboration and the High-Z Supernova Team Collaboration, presented evidence that the expansion of the universe is accelerating. These teams have measured the distances to cosmological supernovae by using the fact that the intrinsic luminosity of Type Ia supernovae is closely correlated with their decline rate from maximum brightness, which can be independently measured. These measurements, combined with redshift data for the supernovae, led to the prediction of an accelerating universe. Both teams obtained

$$\Omega_M \approx 0.3, \hspace{0.5cm} \Omega_{\Lambda} \approx 0.7, \hspace{0.5cm} (30)$$

and strongly ruled out the traditional ($\Omega_M, \Omega_{\Lambda}$)=(1, 0) universe. This value of the density parameter $\Omega_{\Lambda}$ corresponds to a cosmological constant that is small but nonzero and positive,

$$\Lambda \approx 10^{-52} m^{-2} \approx 10^{-35} s^{-2}. \hspace{0.5cm} (31)$$
In the paper of Behar and Carmeli a four-dimensional cosmological relativity theory that unifies space and velocity was proposed that predicts the acceleration of the universe and hence it is equivalent to having a positive value for \( \Lambda \) in it. As is well known, in the traditional work of Friedmann when added to it a cosmological constant, the field equations obtained are highly complicated and no solutions have been obtained so far. Behar-Carmeli’s theory, on the other hand, yields exact solutions and describes the universe as having a three-phase evolution with a decelerating expansion followed by a constant and an accelerating expansion, and it predicts that the universe is now in the latter phase. In the framework of this theory the zero-zero component of Einstein’s equations is written as

\[
R_0^0 - \frac{1}{2} \delta_0^0 R = \kappa \rho_{eff} = \kappa (\rho - \rho_c),
\]  
(32)

where \( \rho_c = 3/\kappa r^2 \approx 3H^2/\kappa \) is the critical mass density. Comparing Eq. (32) with the zero-zero component of Eq. (25), one obtains the expression for the cosmological constant in the Behar-Carmeli theory,

\[
\Lambda = \kappa \rho_c = 3/r^2 \approx 3H^2.
\]  
(33)

Assuming that Hubble’s constant \( H = 70 \text{ km/s-Mpc} \), then \( \Lambda = 1.934 \times 10^{-35} \text{ s}^{-2} \). This result is in good agreement with the recent supernovae experimental results. The analyses presented in this paper for determining the value of \( \Lambda \) show that the same value for \( \Lambda \) is obtained here also, although the theory now is different.

REFERENCES
1. M. Carmeli, Cosmological Special Relativity: The Large-Scale Structure of Space, Time and Velocity, 2nd Edition, World Scientific, River Edge, NJ, 2002.
2. A. Einstein, Autobiographical Notes, P.A. Schilpp (ed.), Open Court Pub. Co., La Salle and Chicago, 1979.
3. A. Einstein, in: A. Einstein, H.A. Lorentz, H. Minkowski and H. Weyl, The Principle of Relativity, Dover Publications (1923) 35-65.
4. H. Bondi, in: Lectures on General Relativity, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1965.
5. H. Bondi, Relativity and Common Sense: A New Approach to Einstein, Dover, 1962.
6. S. Behar and M. Carmeli, Intern. J. Theor. Phys. 39 (2000) 1375.
7. W.L. Freedman, in: Seventeenth Texas Symposium on Relativistic Astrophysics and Cosmology, H. Böhringer et al. (eds.), Vol. 759, The New York Academy of Sciences, New York, 1995.
8. W.L. Freedman et al., Nature 371 (1994) 757.
9. A. Riess et al., Astrophys. J. 438 (1995) L17.
10. A. Sandage et al., Astrophys. J. 401 (1992) L7.
11. D. Branch, Astrophys. J. 392 (1992) 35.
12. B. Schmidt et al., Astrophys. J. 395 (1992) 366.
13. A. Saha et al., Astrophys. J. 438 (1995) 8.
14. P.J.E. Peebles, in: Texas/Pascos 92: Relativistic Astrophysics and Particle Cosmology, C.W. Akerlof and M.A. Srednicki (eds.), Vol. 688, The New York Academy of Sciences, New York, 1993.
15. M. Carmeli, Intern. J. Theor. Phys. 39 (2000) 1397.
16. V. Fock, The Theory of Space, Time and Gravitation, Pergamen Press, Oxford, 1959.
17. H.C. Ohanian and R. Ruffini, Gravitation and Spacetime, Second Edition, W.W. Norton, New York and London, 1994.
18. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields, Pergamon Press, Oxford, 1979.
19. P.M. Garnavich et al., Astrophys. J. 493 (1998) L53.
20. B.P. Schmidt et al., Astrophys. J. 507 (1998) 46.
21. A.G. Riess et al., Astronom. J. 116 (1998) 1009.
22. P.M. Garnavich et al., Astrophys. J. 509 (1998) 74.
23. S. Perlmutter et al., Astrophys. J. 483 (1997) 565.
24. S. Perlmutter et al., Nature 391 (1998) 51.
25. S. Perlmutter et al., Astrophys. J. 517 (1999) 565.