Uniformization of Riemann surfaces revisited

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Abstract
We give an elementary and self-contained proof of the uniformization theorem for noncompact simply connected Riemann surfaces.

Keywords Riemann surfaces · Perron’s method · Green functions · Uniformization theorem

1 Introduction

Paul Koebe and shortly thereafter Henri Poincaré are credited with having proved in 1907 the famous uniformization theorem for Riemann surfaces, arguably the single most important result in the whole theory of analytic functions of one complex variable. This theorem generated connections between different areas and lead to the development of new fields of mathematics. After Koebe, many proofs of the uniformization theorem were proposed, all of them relying on a large body of topological and analytical prerequisites. Modern authors [7, 8] use sheaf cohomology, the Runge approximation theorem, elliptic regularity for the Laplacian, and rather strong results about the vanishing of the first cohomology group of noncompact surfaces. A more recent proof with analytic flavor appears in Donaldson [5], again relying on many strong results, including the Riemann–Roch theorem, the topological classification of compact surfaces, Dolbeault cohomology and the Hodge decomposition. In fact, one can hardly find in the literature a self-contained proof of the uniformization theorem of reasonable length and complexity. Our goal here is to give such a minimalistic proof.

Recall that a Riemann surface is a connected complex manifold of dimension 1, i.e., a connected Hausdorff topological space locally homeomorphic to $\mathbb{C}$, endowed with a holomorphic atlas.

Uniformization theorem (Koebe [10], Poincaré [16]) Any simply connected Riemann surface is biholomorphic to either the complex plane $\mathbb{C}$, the open unit disk $\mathbb{D}$, or the Riemann sphere $\hat{\mathbb{C}}$. 

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A weaker result for domains in $\mathbb{C}$ appeared for the first time in Riemann’s thesis [18]:

**Riemann mapping theorem** A simply connected domain strictly contained in $\mathbb{C}$ is biholomorphic to the unit disk.

Riemann’s proof for his mapping theorem was notoriously imprecise, as none of the necessary tools were yet fully developed at the time; Osgood [13] is credited with completing the argument, building on the work of many other mathematicians.

What we show below is the following statement:

**Theorem 1** Every noncompact simply connected Riemann surface is biholomorphic to a domain in $\mathbb{C}$.

Combined with the Riemann mapping theorem, Theorem 1 is clearly equivalent to the noncompact case of the uniformization theorem.

The initial proofs of the uniformization theorem implicitly assumed the surface to have a countable basis of topology, but this hypothesis was shown to be unnecessary some 20 years later [17]. Most known proofs start from Riemann’s idea of searching for the so-called Green functions, i.e., harmonic functions with a prescribed logarithmic singularity on subsurfaces with boundary of a given Riemann surface. Such a Green function will turn out to be the log-norm of a holomorphic map. By exhausting the surface with compact subsurfaces with boundary, one can find in principle a sequence of embeddings, and compactness results should yield a global embedding into $\mathbb{C}$. Technical complications arise while carrying out this program, and our task is to solve them with the minimum amount of machinery.

Our paper starts with a self-contained treatment of a few necessary results from Perron’s theory [15] of subharmonic functions, deducing from there Radó’s theorem on the second countability of Riemann surfaces. We follow closely the line of proof of [9] of constructing an exhaustion by compact surfaces with boundary by means of Sard’s lemma. Hubbard [9] proves the *a priori* stronger statement that every noncompact Riemann surface with trivial first Betti number is biholomorphic to either $\mathbb{C}$ or the disk. One technical issue in [9] is constructing the harmonic dual of the Green function and showing its continuity to the boundary, a particular case of the Osgood-Carathéodory theorem [3], which he uses without proof. He must also rely on Koebe’s $1/4$ theorem, further complicating the argument.

The novelty in our approach is to go around these problems by yet another application of Sard’s lemma, together with the standard Riemann mapping theorem. We also give a direct proof of the removal of singularities property for bounded harmonic functions, as we were unable to find a clear reference in the literature. Besides these new arguments, we also give full proofs of the necessary auxiliary results in the most economical way possible.

Our prerequisites are exclusively contained in the standard undergraduate curriculum:

- The Schwarz lemma, Montel’s theorem on normal families of holomorphic functions, and Riemann’s mapping theorem for simply connected plane domains; we refer to [19] or [4];
- The Poisson integral formula on the unit disk (see [8, Ch. 22]);
- Partitions of unity associated to open covers of second-countable manifolds, integration of 1-forms along a smooth path, and Stokes formula in the plane; see [20];
- Sard’s lemma; see for instance [12];
- The Tietze extension theorem, the universal cover and the fundamental group $\pi_1$; see [11].

What we do not use here includes: the Riemann–Roch theorem, the $\partial \bar{\partial}$ lemma, sheaves and their cohomology, singular and de Rham cohomology and their isomorphism, Runge theory, Weierstrass’ results on the existence of meromorphic functions, Koebe-Bieberbach’s
1/4 theorem, the existence of triangulations on surfaces, or the topological classification of compact surfaces. We hope that in this way, the uniformization theorem becomes accessible to a general audience, including not only students but also working mathematicians wishing to gain access to an elementary, short, and self-contained proof.

The proof of the uniformization theorem is useful for the study of moduli spaces of hyperbolic metrics of families of noncompact, non simply connected Riemann surfaces with prescribed geometry near infinity [1, 2]. We believe that our approach can be similarly applied to the Schottky uniformization of compact Riemann surfaces, and this will be carried out elsewhere.

The reader is referred to [14] for a comprehensive overview of problems in Riemann surface theory leading to the uniformization theorem, and to the collective work [6] for a historical account of the theorem and an outline of the proof.

2 Local behavior of holomorphic maps

We will repeatedly use the local description of holomorphic maps as monomials in suitable charts. Let \( f : X \rightarrow Y \) be a nonconstant holomorphic map between Riemann surfaces. Let \( x_0 \in X \). Choose charts \((\eta, U)\), \((\psi, V)\) centered at \( x_0 \) and \( y_0 = f(x_0) \), respectively. Then \( \psi \circ f \circ \eta^{-1}(z) = z^k u(z) \) for some \( k \in \mathbb{N} \) and some holomorphic function \( u \), with \( u(0) = c \neq 0 \). Choose \( z_0 \neq 0 \). Let \( v(z) = e^{\frac{k}{2} \int_{z_0}^z \frac{du}{u}} \). Then \( v(z)^k = \frac{u(z)}{u(z_0)} \). In the new chart on \( X \) defined by \( \phi(p) = \eta(p) v(\eta(p)) \), we have \( \psi \circ f \circ \phi^{-1}(z) = cz^k \). In particular, it follows that nonconstant holomorphic maps are open: they map open sets in \( X \) to open sets in \( Y \).

3 Perron’s method

A disk centered at \( x \) in a Riemann surface \( X \) is the preimage of the unit disk through a holomorphic chart \( \varphi : U \rightarrow \mathbb{C} \) mapping \( x \) to 0, with \( \overline{D} \subset \varphi(U) \).

Definition 2 Let \( X \) be a Riemann surface. A continuous function \( u : X \rightarrow \mathbb{R} \) is called subharmonic, respectively harmonic, if for every disk in \( X \), the real number:

\[
\frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt - u(0)
\]

is nonnegative, respectively, zero.

Poisson integral formula Every continuous function \( f : S^1 \rightarrow \mathbb{R} \) admits a unique continuous extension \( u : \overline{D} \rightarrow \mathbb{R} \) which is harmonic on \( D \). Moreover, on \( D \), \( u \) is given by the formula:

\[
u(z) = \frac{1}{2\pi} \Re \left( \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta.
\]

The proof is elementary (see for instance [8, Theorem 22.3]). Since \( \frac{e^{i\theta} + z}{e^{i\theta} - z} \) is holomorphic in \( z \), every harmonic function is the real part of a holomorphic function, hence it is real analytic and satisfies the Laplace equation \( \Delta u = 0 \). Conversely, by Cauchy’s integral formula, the real part of any holomorphic function is harmonic, so in particular \( \log |\xi| = \Re (\log \xi) \) is harmonic.
Proposition 3  Let $u$ be a continuous function on a Riemann surface $X$. The following definitions are equivalent:

$D1$: $u$ is subharmonic;

$D2$: (The maximum principle) For any harmonic function $h$ on any open connected subset $U \subset X$, either $u + h$ is constant on $U$, or it does not attain its supremum;

$D3$: For every disk $D \subset X$, one has $u \leq u^{(D)}$, where $u^{(D)}$ is the unique continuous function defined by modifying $u$ inside the disk by using the Poisson integral formula:

$$u^{(D)}(x) = \begin{cases} u(x) & x \in X \setminus D; \\ \text{harmonic} & x \in D. \end{cases}$$  \hspace{1cm} (1)

We leave to the reader the (immediate) proof of this proposition. Remark that by the maximum principle, a nonconstant subharmonic function cannot attain its supremum. Thus, the restriction of a subharmonic function to a compact set attains its maximum on the boundary.

Remark 4  Definition $D2$ is local: a function which is subharmonic in the sense of $D2$ on a neighborhood of every point is globally subharmonic. We deduce that the function $u^{(D)}$ constructed in (1) is subharmonic on $X$. Indeed, $u^{(D)}$ is clearly subharmonic on $D$ and on $X \setminus D$. Let $p$ be a point in $\partial D$. For every disk centered at $p$, we get:

$$u^{(D)}(p) = u(p) \leq \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u^{(D)}(e^{i\theta})d\theta.$$

Hence $u^{(D)}$ is subharmonic in a neighborhood of $p$ as well, therefore subharmonic on $X$.

Definition 5  Let $X$ be a Riemann surface. A set $\mathcal{F}$ of subharmonic functions on $X$ is called a Perron family if the following two properties are verified:

- If $f, g \in \mathcal{F}$, then $\max(f, g) \in \mathcal{F}$.
- If $f \in \mathcal{F}$, $D$ is a disk, and $f^{(D)}$ is the subharmonic function defined by (1), then $f^{(D)} \in \mathcal{F}$.

Theorem 6  Let $X$ be a (possibly not second countable) Riemann surface.

- (Perron’s principle) If $\mathcal{F}$ is a nonempty, locally bounded above Perron family on $X$, then $\sup_{f \in \mathcal{F}} f$ is harmonic.
- (Dirichlet’s principle) Let $Y \subset X$ be a submanifold with (possibly noncompact) boundary and $m, M \in \mathbb{R}$. If $f : \partial Y \to [m, M]$ is a bounded continuous function, then there exists a continuous function $\tilde{f} : Y \to [m, M]$ which extends $f$, and is harmonic on $\tilde{Y}$.

We prove this theorem in Appendix 1 and 1.A. As a corollary, we get:

Theorem 7  (Radó’s theorem) Every Riemann surface $X$ is second countable.

Proof  Let $\overline{D}, \overline{D}'$ be two disjoint closed disks in $X$. Consider $Y = X \setminus (D \cup D')$. By Dirichlet’s principle (which does not require second countability), there exists a continuous function $g : Y \to \mathbb{R}$, harmonic on $\tilde{Y}$ with $g_{|_{\partial D}} \equiv 0$, and $g_{|_{\partial D'}} \equiv 1$. Since $g$ is harmonic, the 1-form $\partial g = \frac{1}{2} \left( \frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} \right) dz$ is holomorphic. Fix $y_0 \in \tilde{Y}$. For every path $\gamma : [0, 1] \to \tilde{Y}$ with $\gamma(0) = y_0$, let $G(\gamma) = \int_{\gamma} \partial g$. Then $G$ is a well-defined, nonconstant, holomorphic function on the universal cover of $\tilde{Y}$, which by Appendix 1 must be second countable. This implies immediately that both $\tilde{Y}$ and $X$ are second countable. \hfill \Box
4 Green functions

Lemma 8 Every harmonic function $f : \mathbb{D}^* \to \mathbb{R}$ bounded near 0 extends continuously to a harmonic function on $\mathbb{D}$.

Proof Recall that from the Poisson integral formula $f$ is smooth, so we can define the 1-form $\beta = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx$. Since $f$ is harmonic, it follows that $\beta$ is closed. Choose $z_0 \in \mathbb{D}^*$. The multivalued map

$$g : \mathbb{D}^* \to \mathbb{R}; \quad g(z) = \int_{z_0}^z \beta,$$

defined by integration along a path from $z_0$ to $z$, depends only on the homotopy class of the chosen path. Let $\gamma$ be the circle centered at 0 passing through $z_0$. It is a generator of $\pi_1(\mathbb{D}^*)$, so the map $g$ is well-defined modulo $a \mathbb{Z}$, where $a = \int_\gamma \beta$. Hence $f + ig$ is well-defined modulo $ia \mathbb{Z}$. We claim that $f + ig$ satisfies the Cauchy–Riemann conditions. Indeed,

$$\frac{\partial g}{\partial y}(z) = \left. \frac{d}{dt} \right|_{t=0} \int_{z_0}^{z+it} \left( \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) = \left. \frac{d}{dt} \right|_{t=0} \int_{z_0}^t \frac{\partial f}{\partial x}(z + i\theta) d\theta = \frac{\partial f}{\partial x}(z).$$

The other condition is verified similarly. If $a = 0$, $e^{f+ig}$ is a well-defined holomorphic function on $\mathbb{D}^*$. Moreover, $e^{f+ig}$ is bounded near 0, thus its singularity in 0 is removable. Since $\mathbb{D}$ is simply connected and $|e^{f+ig}| = e^f > 0$, we deduce that $f = g(t \log e^{f+ig})$ extends harmonically on $\mathbb{D}$. Otherwise, if $a \neq 0$, we apply the same argument to the function $e^{\frac{\pi}{a}(f+ig)}$. □

Proposition 9 Let $K \subset X$ be a (not necessarily compact) subsurface with smooth boundary and fix $x_0 \in \hat{K}$. Then there exists a continuous function $G : K \setminus \{x_0\} \to [0, \infty)$ which is harmonic on the interior of $K \setminus \{x_0\}$, vanishes on $\partial K$, and has a logarithmic pole in $x_0$ (i.e., $G + \log |\xi|$ is harmonic on a neighborhood of 0, where $\xi$ is a local holomorphic coordinate centered at $x_0$).

Proof Consider a disk $D$ centered at $x_0$ with coordinate $\xi$, and let $D_{\frac{1}{2}} = \{|\xi| < \frac{1}{2}\} \subset D$. From Dirichlet's principle, we can define a continuous map $h_1 : K \setminus D_{\frac{1}{2}} \to [0, 1]$ by:

$$h_1(\xi) = \begin{cases} 1 & |\xi| = \frac{1}{2} \\ 0 & \xi \in \partial K \\ \text{harmonic} & \text{otherwise.} \end{cases}$$

By the maximum principle, $a = \sup_{\partial D} h_1 \in (0, 1)$ so we can take $A, B > 0$ such that:

$$Ba < A < B - \log 2.$$

From these inequalities, it follows that the map:

$$h : \overline{D} \setminus D_{\frac{1}{2}} \to \mathbb{R}, \quad h(\xi) = \max(-Bh_1(\xi), \log |\xi| - A)$$

can be continuously extended to a subharmonic function on $K \setminus \{x_0\}$, by setting it to be $-Bh_1$ on $K \setminus D$, and $\log |\xi| - A$ on $D_{\frac{1}{2}}$.

Let $F$ be the Perron family of those continuous functions $g : K \setminus \{x_0\} \to [0, \infty)$ which are subharmonic on the interior, restrict to 0 on $\partial K$, and satisfy $g + h \leq 0$. The family $F$ is bounded above (by $-h$) and contains the map $\alpha : D \setminus \{0\} \to \mathbb{R}, \alpha(\xi) = -\log |\xi|$ extended...
with 0 outside the unit disk. By Perron’s principle, the map \( G : K \setminus \{x_0\} \to [0, \infty), G = \sup_{g \in F} g \) is continuous, vanishes on \( \partial K \) and is harmonic on \( \bar{K} \setminus \{x_0\} \). Moreover \( \alpha \leq G \leq -h \), so \( -\log|\xi| \leq G \leq A - \log|\xi| \) on \( D_1^2 \) and therefore \( G + \log|\xi| \) is bounded near 0. Lemma 8 shows that \( G + \log|\xi| \) is actually harmonic near 0, ending the proof.

\[ \square \]

5 A holomorphic embedding in the unit disk

**Theorem 10** Let \( X \) be a Riemann surface and \( K \subset X \) a compact simply connected subsurface with precisely one (smooth) boundary component. Then there exists a biholomorphism \( \varphi : \hat{K} \to \mathbb{D} \).

**Proof** Let \( G \) be the Green function from Proposition 9 and consider the 1-form given in holomorphic coordinates by \( \omega = \frac{\partial G}{\partial x} dy - \frac{\partial G}{\partial y} dx \in \Omega^1(\hat{K}) \). One can easily check that it is invariant under holomorphic changes of variables (in fact, this form is just \( JdG \), where \( J \) is the almost complex structure). Fix \( z_0 \in \hat{K} \setminus \{x_0\} \) and consider the multivalued map

\[
F : \hat{K} \setminus \{x_0\} \to \mathbb{R}; \quad F(z) = \int_{z_0}^z \omega
\]

defined by integrating along any smooth path from \( z_0 \) to \( z \). As above, the value of \( F(z) \) depends only on the homotopy class of the chosen path. We claim that \( F(z) \) is well-defined modulo \( 2\pi \).

**Remark 11** One could try to prove that \( \pi_1(\hat{K} \setminus \{x_0\}) \) is cyclic as an application of Van Kampen’s theorem, since \( \hat{K} \setminus \{x_0\} \) is obtained from \( \hat{K} \) by removing a disk. In reality, all we can say is that \( \pi_1(\hat{K} \setminus \{x_0\}) \) equals the normalizer of a cyclic subgroup. We will avoid this problem by using the Sard lemma.

Take \( \gamma : S^1 \to \hat{K} \setminus \{x_0\} \) a smooth loop with \( \gamma(1) = z_0 \). Since \( \hat{K} \) is simply connected there exists \( h : \bar{D} \to \hat{K} \), a smooth extension of \( \gamma \) to the closed unit disk.

We can assume \( x_0 \) to be a regular value for \( h \). If not, consider a disk \( D \) centered at \( x_0 \) which does not intersect \( \gamma \). By Sard’s lemma, there exists a regular value \( y_0 \) for \( h \) inside \( D \). Pick \( \psi : \hat{K} \to \hat{K} \) a diffeomorphism which maps \( y_0 \) to \( x_0 \), and acts as the identity outside \( D \). Then \( \psi \circ h \) is a new homotopy for which \( x_0 \) is a regular value.

Hence \( h^{-1}(x_0) \) is a (possibly empty) finite set in the disk \( \mathbb{D} \). By pulling back and using Stokes’ theorem, we get:

\[
\int_\gamma \omega = \int_{S^1} h^* \omega = \sum_{q \in h^{-1}(x_0)} \int_{C_q} h^* \omega = \sum_{q \in h^{-1}(x_0)} \int_{h(C_q)} \omega,
\]

where the \( C_q \)’s are disjoint circles centered at \( q \), for each \( q \in h^{-1}(x_0) \), chosen such that their image through \( h \) lies inside a punctured disk \( D \setminus \{x_0\} \). Every circle \( C_q \) is homotopic in \( D \setminus \{x_0\} \) to the circle \( |\xi| = r \), where \( \xi \) is a local coordinate in a disk centered at \( x_0 \) and \( r < 1 \) is a positive number. Therefore the right-hand side is an integer multiple of \( \int_{|\xi|=r} \omega \). From Proposition 9, we know that \( H := G + \log|\xi| \) is harmonic on \( \{|\xi| < 1\} \), hence, using Stokes’ theorem:

\[
\int_{|\xi|=r} \frac{\partial G}{\partial x} dy - \frac{\partial G}{\partial y} dx = \int_{|\xi| \leq r} \left( \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) dx \wedge dy - \int_{|\xi|=r} \frac{xdy - ydx}{x^2 + y^2} = -2\pi.
\]
This proves the claim that $F$ is well-defined modulo $2\pi$; therefore, $\varphi = e^{-G-iF} : \hat{K} \setminus \{x_0\} \rightarrow \mathbb{D}$ is holomorphic and well defined. The singularity in 0 of such a function is removable because $|e^{-G}| = |x|e^H$ is bounded, so we can extend $\varphi : \hat{K} \rightarrow \mathbb{D}$ with $\varphi(x_0) = 0$. Clearly, $\varphi^{-1}(0) = \{x_0\}$.

Let us prove that $\varphi$ is injective. Let $A$ be the set of those points $x \in \hat{K}$ for which $d\varphi_x \neq 0$ and the preimage $\varphi^{-1}(\varphi(x))$ only contains $x$. First we prove that $A$ is open. Let $x \in A$. By continuity and using Sect. 2 we find a disk $D$ centered at $x$ where $d\varphi \neq 0$ and on which $\varphi$ is injective. Suppose there existed sequences $(x_j)_{j \in \mathbb{N}}, (y_j)_{j \in \mathbb{N}} \subset \hat{K}$ with $x_j \neq y_j, x_j \rightarrow x$ and $\varphi(x_j) = \varphi(y_j)$. Since $\varphi$ is injective on $D$, there exists a rank $j_0$ such that for any $j \geq j_0$, $x_j \in D$ and $y_j \notin D$. Since $(y_j)_{j \geq j_0} \subset K \setminus D$, there exist $y \in K \setminus D$ and a subsequence $(y_{jk})_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} y_{jk} = y$. If $y \in \hat{K}$ then we have $\varphi(x) = \varphi(y)$, contradicting $x \in A$. Otherwise, if $y \in \partial K$, we have $|\varphi(y_{jk})| = e^{-G(y) = 1}$ while $|\varphi(x_j)| \rightarrow |\varphi(x)| < 1$, contradiction.

Now we prove that $A$ is closed in $\hat{K}$. Let $(x_j)_{j \in \mathbb{N}} \subset A$, with $x_j \rightarrow p \in \hat{K}$. If $\varphi'(p) = 0$, then again by Sect. 2, in a suitable chart $U$ near $p$, the map $\varphi$ is given by $z \mapsto cz^k$, for some positive integer $k \geq 2$. Let $\zeta \neq 1$ be a k-th root of unity. Then, for $j$ large enough, we have that $\varphi(x_j) = \varphi(\zeta x_j)$, which is a contradiction. Otherwise, suppose there existed another point $q \in \hat{K}$ with $\varphi(p) = \varphi(q)$. Since $\varphi$ is an open map, we can find two disjoint neighborhoods $U \ni p$ and $V \ni q$ with $\varphi(U) = \varphi(V)$. But this means that, for $j$ large enough, $\varphi^{-1}(x_j) \cap V \neq \emptyset$, which is again a contradiction. Hence $A$ is closed in $\hat{K}$.

To conclude that $A = \hat{K}$, we are left to prove that $A \neq \emptyset$. Note that near $x_0$ we have $|\varphi(\xi)| = |e^{\log|\xi|^{-1}e^{-H(\xi)}}| = |\xi||e^{-H(\xi)}|$. Thus $\varphi'(x_0) \neq 0$, so $x_0 \in A$.

Since nonconstant holomorphic maps are open, the injection $\varphi$ is a biholomorphism onto its image $\varphi(\hat{K}) \subset \mathbb{D}$.

The Riemann mapping theorem allows us to conclude that $\hat{K}$ is biholomorphic to $\mathbb{D}$.

6 Exhaustion by compact simply connected submanifolds

Let $X$ be a simply connected, noncompact Riemann surface and $x_0 \in X$ a fixed point.

**Theorem 12** There exists a compact exhaustion of $X$

$$x_0 \in K_0 \subset K_1 \subset \ldots \subset X,
K_n \subset \hat{K}_{n+1}, \quad \bigcup_{n \geq 0} K_n = X$$

with simply connected surfaces with nonempty connected smooth boundary.

**Proof** By Radó’s theorem, we can choose a countable locally finite cover of $X$ by relatively compact sets $(U_m)_{m \in \mathbb{N}}$ and a subordinated partition of unity $(\phi_m)_{m \in \mathbb{N}}$. Define a smooth function

$$g : X \rightarrow [0, \infty), \quad g(x) = \sum_{m \in \mathbb{N}} m\phi_m(x).$$

For every integer $m$, $g^{-1}([0, m]) \subset \text{supp} \phi_1 \cup \ldots \cup \text{supp} \phi_m$ is compact, hence $g$ is proper and unbounded above. By Sard’s lemma, there exists an increasing sequence of regular values $(\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \rightarrow \infty$ and we may suppose that $a_0 > g(x_0)$. Consider $Y'_n = \{x \in X | g(x) \leq a_n\}$ and let $Y_n$ be the connected component of $Y'_n$ containing $x_0$. Obviously $Y'_n \subset Y'_{n+1}$, so $Y_n \subset Y_{n+1}$. Since $X$ is a connected manifold, it is path-connected so, for every $x \in X$, there...
exists a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. The set $g(\gamma[0, 1])$ is compact, so $g(\gamma[0, 1]) \subset [0, a_k]$ for some $k \in \mathbb{N}$. Then $\gamma([0, 1]) \subset Y'$ and, by connectedness, $\gamma([0, 1]) \subset Y_k$, hence $x \in Y_k$ so $\bigcup_n Y_n = X$.

The compact submanifolds $Y_n$ are not exactly what we are looking for because they might have ‘holes’, like an annulus in $\mathbb{C}$ (they are not holomorphically convex). The natural way to fill these holes is to add those connected components of $X \setminus Y_n$ with compact closure. Since any two components have distinct boundaries and the boundaries are subsets of $g^{-1}(a_n)$ (which is a compact 1-manifold), we can define the compact submanifold $K_n$ as the finite union of $Y_n$ with these connected components. We can easily convince ourselves that $\bigcup_{n \in \mathbb{N}} K_n = X$, $K_n \subset K_{n+1}$, and $K_n$ is connected (see picture).

To finish the proof we have to show that $\pi_1(K_n, x_0) = 0$. The idea is to find a retraction from $X$ to $K_n$. Let $Z$ be the closure of a connected component of $X \setminus K_n$. First, we need to show that $\partial Z$ is connected. Assume by contradiction that there existed two components $\gamma_1$ and $\gamma_2$ of $\partial Z$. We take a point on each and connect them by smooth simple paths $\delta_1$ in $Z$ and $\delta_2$ in $K_n$, both transversal to $\gamma_1$ and $\gamma_2$ (the existence of such paths linking any two points on a Riemann surface can be easily proved: the set of points which can be connected by a simple curve to a given point is both open and closed). Moreover, we can assume that $\delta = \delta_1 \cup \delta_2$ is smooth. Then $\delta$ is diffeomorphic to $S^1$ and we can consider a tubular neighborhood $S^1 \times (-\epsilon, \epsilon) \subset X$ with coordinates $(x, y)$, where $x$ is the coordinate on the circle. Choose $\eta : \mathbb{R} \rightarrow \mathbb{R}$ a test function such that supp $\eta \subset (-\epsilon, \epsilon)$ and $\int_{-\infty}^{\infty} \eta(y)dy = 1$. Define a closed 1-form $\omega \in \Omega^1(X)$ by $\omega = \eta(y)dy$, extended by 0 outside the tubular neighborhood. From the homotopy invariance of the integral, we can assume that the intersection of $\gamma_1$ with the tubular neighborhood is a segment of type $\{x = \text{constant}\}$, hence $\int_{\gamma_1} \omega = \pm 1$. However $X$ is simply connected, therefore the integral of any closed 1-form on a loop is 0, yielding the desired contradiction. Thus $\partial Z$ is connected, and we parametrize it by $\gamma : [0, 1] \rightarrow \partial Z \subset \partial K_n$, where $\gamma(0) = \gamma(1) = p$.

Let $\delta$ be a simple path in $Z$ starting at $p$ escaping from every compact subset of $Z$. We cut $Z$ along $\delta$ to obtain a topological surface $\tilde{Z}$ with connected boundary $\partial \tilde{Z} = \delta' \cup \{p', p''\} \cup \delta''$ (see picture).
Let $\mu : \partial \tilde{Z} \to [0, 1]$ be the continuous function

$$
\mu(x) = \begin{cases} 
0 & x \in \delta' \\
\gamma^{-1}(x) & x \in \gamma \setminus \{p\} \\
1 & x \in \delta''.
\end{cases}
$$

By the Tietze extension theorem, we can extend $\mu$ to a continuous function $\tilde{\mu} : \tilde{Z} \to [0, 1]$. Note that the composition $\gamma \circ \tilde{\mu}$ takes the same value (namely $p$) on $\delta'$ and $\delta''$; therefore, it descends to a well-defined map $\Psi_Z : Z \to \gamma([0, 1]) \subset X$:

$$
\tilde{Z} \xrightarrow{\tilde{\mu}} [0, 1] \xrightarrow{\gamma} X
$$

Now let $\rho_n : X \to K_n$ be the retraction which acts as the identity on $K_n$ and equals $\Psi_Z$ on every connected component $Z$ of $X \setminus K_n$. Since the composition $K_n \xrightarrow{\rho_n} X \xrightarrow{\rho_n} K_n$ is the identity map, it induces the identity on the fundamental groups. But $\pi_1(X) = 0$, thus $K_n$ is simply connected.

\section*{7 End of the proof of theorem 1}

For $n \geq 0$, let $\phi_n : \hat{K}_n \to r_n \mathbb{D}$ be the unique biholomorphism given by Theorem 10 with $\phi_n(x_0) = 0$ and $\phi_n'(x_0) = 1$, where $r_n$ is a positive number determined by $K_n$, and the derivative is computed in a fixed coordinate near $x_0$.

For every Riemann surface $S$, let $\mathcal{O}(S)$ be the topological vector space of holomorphic functions from $S$ to $\mathbb{C}$ with the topology defined by uniform convergence on compact subsets of $S$.

\begin{lemma}
Let $(f_n : \mathbb{D} \to \mathbb{C})_{n \in \mathbb{N}}$ be a sequence of injective holomorphic functions with $f_n(0) = 0$ and $f_n'(0) = 1$. Then there exists a convergent subsequence $f_{n_k} \to f \in \mathcal{O}(\mathbb{D})$.
\end{lemma}

We defer the proof of this lemma to Appendix 1. Consider the holomorphic map $g_n : \mathbb{D} \to \mathbb{C}$, $g_n(z) = \frac{1}{r_0} \phi_n \circ \phi_0^{-1}(r_0 z)$. Clearly $g_n$ is injective, $g_n(0) = 0$, and $g_n'(0) = 1$. By Lemma 13, there exists a convergent subsequence $(g_{n_{0k}})_{k \in \mathbb{N}}$. Therefore $(\phi_{g_{n_{0k}}})_{k \in \mathbb{N}}$ converges on $K_0$. Similarly, from the sequence $\left(\frac{1}{r_1} \phi_{g_{n_{0k}}} \circ \phi_1^{-1}(r_1 z)\right)_{k \in \mathbb{N}}$, extract a subsequence
which converges on \( \tilde{K}_1 \). Repeating this process, we obtain the sequences \((\phi_{n_k})_{k \in \mathbb{N}}\) which converges on \( \hat{K}_m \). Set \( \phi_{n_k} = \phi_{n_k} \). Now \((\phi_{n_k})_{k \in \mathbb{N}}\) converges on \( X \) to a holomorphic function \( \phi : X \rightarrow \mathbb{C} \), uniformly on every compact of \( X \). By continuity \( \phi'(0) = 1 \), so \( \phi \) is nonconstant. Let us show that \( \phi \) is injective: suppose there existed \( z_1 \neq z_2 \in X \) with \( \phi(z_1) = \phi(z_2) \). Since \( \phi \) is nonconstant, there exists a circle \( C \) centered at \( z_2 \) whose interior does not contain \( z_1 \), and such that the difference \( \phi - \phi(z_2) \) does not vanish on \( C \). Using the dominated convergence theorem, we get

\[
\lim_{n \to \infty} \frac{1}{2\pi i} \int_C (\phi_n(z) - \phi_n(z_1))'(z) \frac{dz}{z} = \frac{1}{2\pi i} \int_C (\phi - \phi(z_1))'(z) \frac{dz}{z},
\]

The first integral is zero, while the second one counts the number of zeros of the function \( \phi - \phi(z_1) \) in the disk bounded by \( C \), which is at least 1, yielding a contradiction. Thus, \( \phi \) is a biholomorphism onto its image, ending the proof of Theorem 1.

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**Data availability** Not applicable.

**Declarations**

**Conflicts of interest** Not applicable.

**Appendix A Proof of Perron’s principle**

Since the family \( \mathcal{F} \) is locally bounded above, we can define \( u : X \rightarrow \mathbb{R}, u = \sup_{f \in \mathcal{F}} f \). Our aim is to show that \( u \) is harmonic. Since harmonicity is a local property, it suffices to prove that \( u \) is harmonic on a disk \( D \subset X \). Let \( A = \{z_0, z_1, \ldots\} \subset D \) be a dense subset. For each \( i \), there exists a sequence \((v_{jk})_{k \in \mathbb{N}} \subset \mathcal{F} \) such that \( u(z_j) = \lim_{k \to \infty} v_{jk}(z_j) \). Since \( \mathcal{F} \) is a Perron family, the map \( h_1 = v_{1j} \) belongs to \( \mathcal{F} \), is harmonic on \( D \), and \( h_1 \geq v_{11} \). Suppose we constructed the functions \( h_1, \ldots, h_n \subset \mathcal{F} \) harmonic on \( D \) such that \( h_k \geq h_{k-1} \) and \( h_k \geq v_{ij} \) for all \( k \in \{1, \ldots, i\}, i, j \in \{1, \ldots, k\} \). Pick \( h_{n+1} = \max(v_{11}, v_{12}, \ldots, h_n) \) \( \in \mathcal{F} \) harmonic on \( D \), with \( h_{n+1} \geq h_n \) and \( h_{n+1} \geq v_{ij} \) for any \( i, j \in \{1, \ldots, n + 1\} \). Then \( h_n(z_j) \geq v_{jk}(z_j) \) for all \( n \geq k \geq j \). Letting \( k \to \infty \) in the previous inequality, we get:

\[
\lim_{n \to \infty} h_n(z_j) = u(z_j).
\]

Since \((h_n)_{n \in \mathbb{N}}\) is an increasing sequence, we can consider the Lebesgue measurable function

\[
w : D \rightarrow \mathbb{R}, \quad w = \lim_{n \to \infty} h_n \leq \sup_{f \in \mathcal{F}} g_1|_D = u|_D.
\]

By the dominated convergence theorem, for every disk \( D' \) centered at a point \( x \in D \) we have

\[
w(x) = \lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} \frac{1}{\pi} \int_{D'} h_n(z)dz = \frac{1}{\pi} \int_{D'} \lim_{n \to \infty} h_n(z)dz = \frac{1}{\pi} \int_{D'} w(z)dz.
\]
Here we used the mean property on disks for harmonic functions, which is an immediate
consequence of the definition. This implies easily that $w$ is $C^0$. Again by dominated
convergence,
\[ \frac{1}{2\pi} \int_0^{2\pi} w(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \lim_{n \to \infty} h_n(e^{it}) dt = \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} h_n(e^{it}) dt = \lim_{n \to \infty} h_n(x) = w(x) \]
thus showing that $w$ is harmonic. We claim that $w = u$.

Since $h_n \in \mathcal{F}$ for any $n \in \mathbb{N}$, then $h_n \leq u$, so $w \leq u$. But $w(z_j) = u(z_j)$ for all $j \in \mathbb{N}$, hence $w \geq v$ on the dense subset $A \subset D$ for every $v \in \mathcal{F}$. Since $\mathcal{F}$ is a family of continuous
functions, it follows that $w \geq v$ for any $v \in \mathcal{F}$, therefore $w \geq u$, which ends the proof.

Appendix B Proof of Dirichlet’s principle

If $f$ is constant, the conclusion is clear, otherwise let $m < M$ be the infimum, respectively,
the supremum of $f$ on $\partial Y$. Consider the family $\mathcal{F}$ of continuous functions $g : Y \to [m, M]$ which are subharmonic on the interior of $Y$ with $g|_{\partial Y} \leq f$. The first condition for $\mathcal{F}$ to be a Perron family is evident. Let $g \in \mathcal{F}$ and $D'$ a disk in $Y$. Using Remark 4, $g^{(D')}$ is subharmonic. By the maximum principle, $g^{(D')}$ still takes values in $[m, M]$. Thus $g^{(D')} \in \mathcal{F}$. Now $\mathcal{F} \neq \emptyset$
since it contains the constant function $m$. Let $F = \sup_{g \in \mathcal{F}} g$. By Perron’s principle, $F$ is
harmonic on $\hat{Y}$.

We must prove that for every $x \in \partial Y$, $\lim_{\xi \to x} \inf_{\xi} F(\xi) = f(x)$. Let us first show that $\lim \inf_{\xi \to x} F(\xi) \geq f(x)$. If $f(x) = m$, there is nothing to prove, hence let us suppose that $f(x) > m$. We work in a disk $D \subset X$ centered at $x$ with local coordinate $\xi$.

Fix $\epsilon > 0$ such that $f(0) - \epsilon > m$. There exists $1 > \delta > 0$ such that for every $\xi \in \partial Y$ with $|\xi| < \delta$ we have $f(0) - \epsilon < f(\xi)$. Consider the exterior normal to $\partial Y$ at $x = 0$. Take a circle centered at a point $p$ on this normal, of radius $r$ small enough such that it touches $Y$ only in $0$. Choose $t > 1$ such that $\log \frac{1}{r^t} + f(0) - \epsilon < m$ and set $R = rt$. By decreasing $r$
if needed, we can assume that $R < \delta$. Clearly, $|p| \leq |\xi - p|$ for every $\xi \in Y \cap D$. Define

$u : D \to [m, \infty), \quad u(\xi) = \max \left( m, \log \frac{|p|}{|\xi - p|} + f(0) - \epsilon \right).$

Notice that $u$ is the maximum of two harmonic functions, hence subharmonic. When $|\xi|$ is close to 0, $u(\xi) = \log \frac{|p|}{|\xi - p|} + f(0) - \epsilon$, while for $|\xi| \geq R$, $u(\xi) = m$. Hence $u$ can be continuously extended (by the constant value $m$) to $Y \cup D$, and is subharmonic on $\hat{Y}$. It is straightforward to check that $u|_{\partial Y}$ belongs to the family $\mathcal{F}$. Thus, for small $|\xi|$, we get:

$F(\xi) \geq u(\xi) = \log \frac{|p|}{|\xi - p|} + f(0) - \epsilon.$

Since $\epsilon$ was arbitrarily small, we obtain $\inf_{\xi \to 0} F(\xi) \geq f(0)$. We now prove that $\lim \sup_{\xi \to 0} F(\xi) \leq f(0)$. If $f(0) = M$, this is clear. Otherwise, take $\epsilon > 0$ such that $f(0) + \epsilon < M$. As above, consider $t > 1$ and $R = rt$ such that $-\log \frac{1}{r^t} + f(0) + \epsilon > M$, and $f(\xi) < f(0) + \epsilon$ for $|\xi| \leq R$. Define

$U : D \to (-\infty, M), \quad U(\xi) = \min \left( M, -\log \frac{|p|}{|\xi - p|} + f(0) + \epsilon \right).$

Notice that $U$ is the minimum of two harmonic functions, thus $-U$ is subharmonic. Also
$U(\xi) = M$ for $|\xi| > R$, $U(\xi) = -\log \frac{|p|}{|\xi - p|} + f(0) + \epsilon$ for small $|\xi|$, $U \geq m$, and
\( U(\xi) \geq f(\xi) \) on \( \partial Y \). For every \( g \in F \), the continuous function \( g - U \) is nonpositive on the boundary of the compact domain \( \{|\xi| \leq R\} \cap Y \) and subharmonic in the interior. By the maximum principle it follows that on this compact set \( g \leq U \), hence \( F \leq U \). Thus for small \( |\xi| \),

\[
F(\xi) \leq -\log \frac{|p|}{|z - p|} + f(0) + \epsilon.
\]

Therefore \( \limsup_{\xi \to 0} F(\xi) \leq f(0) \), showing that \( \lim_{\xi \to 0} F(\xi) \) exists and equals \( f(0) \).

**Appendix C Second countability in the presence of a holomorphic function**

Let us prove that if there exists a nonconstant holomorphic function \( f : X \to \mathbb{C} \), then the Riemann surface \( X \) is second countable.

Let \( B \) be a countable basis of topology for \( \mathbb{C} \). Let \( A \) be the set of those connected components of each \( f^{-1}(U) \), \( U \in B \), which are second countable. We claim that \( A \) is a basis of topology for \( X \). Indeed, let \( D \subset X \) be an open set and \( x \in D \). By the identity theorem for holomorphic functions, \( f^{-1}(f(x)) \) is discrete, hence there exists an open neighborhood \( W \) of \( x \), relatively compact in \( D \), such that \( W \cap f^{-1}(f(x)) = \{x\} \). We have that \( f(\partial W) \) is compact in \( \mathbb{C} \). Since \( f(x) \notin f(\partial W) \), there exists \( U \in B \) which contains \( f(x) \) such that \( U \cap f(\partial W) = \emptyset \). Let \( V \) be the connected component of \( f^{-1}(U) \) which contains \( x \). Since \( V \cap \partial W = \emptyset \), we get that \( V \subset W \), hence \( V \) is second countable. We found \( x \in V \subset D \), with \( V \in A \), therefore the claim is proved. We next prove that \( A \) is countable.

Each \( V \in A \) intersects countably many other elements in \( A \). Otherwise, there would exist \( U \in B \) such that \( V \) intersects uncountably many connected components of \( f^{-1}(U) \). It would follow that \( V \) contains uncountably many disjoint open subsets, which is a contradiction.

Consider \( V_0 \in A \). There exists a countable number of open sets from \( A \) which intersect \( V_0 \), denote their union with \( V_0 \) by \( V_1 \). For \( k \geq 1 \), define \( V_{k+1} \) as the union of \( V_k \) with those open sets in \( A \) which intersect \( V_k \). By induction, \( V_k \) is countable. It is clear that \( \cup_{k \geq 1} V_k \) is both open and closed, hence \( \cup_{k \geq 1} V_k = X \). Therefore all the open sets from \( A \) appear in this process, so \( A \) is a countable union of countable sets, hence countable.

**Appendix D Compactness of families of injective holomorphic functions**

**Proof of Lemma 13** Denote by \( r_n = \sup\{r \in \mathbb{R} : rD \subset f_n(D)\} \). Let \( a_n \in \partial(r_nD) \setminus f_n(D) \). From the Schwarz lemma for the function \( f_n^{-1} |_{r_nD} \), it follows that \( |a_n| = r_n \leq 1 \). By extracting a subsequence, we can assume that \( (a_n)_{n \in \mathbb{N}} \) is convergent. Consider the function \( g_n : \mathbb{D} \to \mathbb{C} \), \( g_n = a_n^{-1}f_n \). It is easy to see that \( \mathbb{D} \subset g_n(D) \) and also \( 1 \notin g_n(D) \). Since \( g_n \) is holomorphic and injective, it follows that \( g_n(D) \) is simply connected; hence, we can construct the square root

\[
\psi_n : g_n(D) \to \mathbb{C}^* \quad \psi_n(z) = ie^{\frac{1}{2}} \int_{0}^{z} \frac{dw}{w^{1/2}}.
\]

Then \( \psi_n(0) = i \) and \( \psi_n(z) = z - 1 \) for \( z \in g_n(D) \). Since \( \psi_n^2 \) is injective, the image of \( \psi_n \) cannot contain pairs of the form \((w, -w)\). Thus, there exists a disk \( D \) centered at \(-i\) disjoint from the image of \( \psi_n \) for every \( n \). Define \( h_n = \psi_n \circ g_n : D \to \mathbb{C} \setminus D \). There exists \( 0 < r < \infty \) so that the homography \( \alpha(z) = \frac{1}{z+1} \) maps \( \mathbb{C} \setminus D \) into \( rD \). Using Montel’s Theorem and relabeling the sequence, we can assume that \( \alpha \circ h_n \) converges to a holomorphic
map $h : \mathbb{D} \rightarrow r \mathbb{D}$. If $h$ is nonconstant, it is open by Sect. 2; hence, it maps $\mathbb{D}$ to $r \mathbb{D}$. Thus we can compose it with the inverse homography $\alpha^{-1}$. Otherwise, $h$ equals the constant $\frac{1}{2i}$. In both cases, $h_n$ converges to $\alpha^{-1} \circ h$, thus $f_n = a_n \left(1 + h_n^2\right)$ converges in $O(\mathbb{D})$. □

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