NILPOTENCY AND DIMENSION SERIES FOR LOOPS

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Abstract. We take a step towards the development of a nilpotency theory for loops based on the commutator-associator filtration instead of the lower central series. This nilpotency theory shares many essential features with the associative case. In particular, we show that the isolator of the $n$th commutator-associator subloop coincides with the $n$th dimension subloop over a field of characteristic zero.

The lower central series for groups can be defined in two essentially different ways. Namely, the lower central series of a group $G$ is a descending filtration of $G$ by normal subgroups

$$G = G_1 \supseteq G_2 \supseteq \ldots$$

defined inductively by setting either:

$$G_i = [G, G_{i-1}]$$

where $[G, H]$ is the largest of all subgroups $K$ of $G$ with the property that $K/H$ is contained in the centre of $G/H$;

or, $G_i$ to be generated by all commutators $[x, y]$ with $x \in G_p$ and $y \in G_q$ with $p + q \geq i$.

These two definitions are equivalent for groups. However, in the non-associative case they give rise to rather different objects. The first definition, with “groups” replaced by “loops”, produces Bruck’s lower central series, see \[1\]. An analog of the second definition for loops was introduced in \[6\] under the name of “commutator-associator filtration”. The terms of the commutator-associator filtration contain, but do not necessarily coincide with the corresponding terms of the lower central series.

The main advantage of the commutator-associator filtration is the existence of a rich algebraic structure on the associated graded abelian group, consisting of an infinite number of multilinear operations. It can be seen that two of the operations, namely those induced by the loop commutator and the loop associator, satisfy the Akivis identity. However, the complete identification of this algebraic structure is a non-trivial problem.

In this paper we set up a nilpotency theory for loops based on the commutator-associator filtration. In this theory the standard techniques of the theory of nilpotent groups can be applied and various results valid for groups can be extended to loops. In particular, we shall prove that for an arbitrary loop the isolators of the terms of the commutator-associator filtration coincide with the dimension series. As a corollary, we identify the algebraic structure on the graded $\mathbb{Q}$-vector space associated to the commutator-associator filtration: it turns out to be a Sabinin algebra.

Throughout the text we make the emphasis on the similarities, rather than differences, between nilpotency theories for groups and for general loops. This should not leave the impression that extending the nilpotency theory from groups to loops is a straightforward task. In particular, the residual nilpotency of the free loop, established for the lower central series by Higman \[3\], remains an open question for the commutator-associator filtration. We did not strive for completeness; many relevant topics (such as applications to particular classes of loops, relation to the nilpotency of the multiplication group of the loop et cetera) have remained outside the scope of this paper.

Acknowledgments. I would like to thank Liudmila Sabinina and José María Pérez Izquierdo for discussions. This work was supported by the CONACyT grant CO2-44100.

2000 Mathematics Subject Classification. 20N05, 17D99.
1. \( N \)-sequences

1.1. The commutator-associator filtration. The commutator-associator filtration on a loop \( L \) is defined in terms of commutators, associators and associator deviations \([6]\).

The \textit{commutator} of two elements \( a, b \) of \( L \) is
\[
[a, b] = (ba) \setminus (ab)
\]
and the \textit{associator} of \( a, b \) and \( c \) is defined by
\[
(a, b, c) = (a(bc)) \setminus ((ab)c).
\]

There is an infinite number of \textit{associator deviations}. These are functions \( L^{l+3} \to L \) characterised by a non-negative number \( l \), called \textit{level} of the deviation, and \( l \) indices \( \alpha_1, \ldots, \alpha_l \) with \( 0 < \alpha_i \leq i + 2 \). The deviations of level one are
\[
(a, b, c, d)_1 = ((a, c, d)(b, c, d)) \setminus (ab, c, d),
\]
\[
(a, b, c, d)_2 = ((a, b, d)(a, c, d)) \setminus (a, bc, d),
\]
\[
(a, b, c, d)_3 = ((a, b, c)(a, b, d)) \setminus (a, b, cd).
\]

By definition, the deviation \((a_1, \ldots, a_{l+3})\alpha_1,\ldots,\alpha_l\) of level \( l \) is equal to
\[
(A(a_{\alpha_1})A(a_{\alpha_1+1})) \setminus A(a_{\alpha_1}a_{\alpha_1+1})
\]
where \( A(x) \) stands for the deviation \((a_1, \ldots, a_{\alpha_1-1}, x, a_{\alpha_1+2}, \ldots, a_{l+3})\alpha_1,\ldots,\alpha_{l+1}\) of level \( l - 1 \).

The subloop \( \gamma_nL \) is called the \textit{nth commutator-associator subloop} of \( L \).

\textbf{Lemma 1.} \([6]\) For an arbitrary loop \( L \) the commutator, the associator and the associator deviations induce multilinear operations on the graded abelian group \( \oplus \gamma_iL/\gamma_{i+1}L \); these operations respect the grading.

For the associator and the deviations the statement of the lemma follows straight from the definition of the commutator-associator subloops. As for the commutators, we shall now see that for arbitrary \( a \in \gamma_pL, b \in \gamma_qL \) and \( c \in \gamma_rL \), the commutator \([ab, c]\) is equal to \([a, c][b, c]\) modulo \( \gamma_{p+q+r}L \).

Indeed, modulo \( \gamma_{p+q+r}L \)
\[
(ca)b \cdot [a, c] \equiv ca \cdot (b[a, c]) \equiv ca \cdot ([a, c]b) \equiv (ca)[a, c] \cdot b = (ac)b \equiv a(cb).
\]

Hence, modulo \( \gamma_{p+q+r}L \)
\[
c(ab) \cdot ([a, c][b, c]) \equiv (ca)b \cdot ([a, c][b, c]) \equiv ((ca)b \cdot [a, c])[b, c] \equiv a(cb) \cdot [b, c] \equiv a \cdot (cb)[b, c] = a(bc) \equiv (ab)c,
\]
and therefore \( ([a, c][b, c]) \setminus [ab, c] \) is in \( \gamma_{p+q+r}L \). Similarly one proves that \( ([a, b][a, c]) \setminus [a, bc] \) belongs to \( \gamma_{p+q+r}L \). This implies that the commutator induces a bilinear operation on \( \oplus \gamma_iL/\gamma_{i+1}L \).

At this point it is convenient to introduce a notion that allows to speak of commutators, associators and deviations at the same time. A \textit{bracket of weight} \( n \) is an expression in \( n \) indeterminates formed by repeatedly applying commutators, associators and deviations, and in which every indeterminate appears only once. In particular, the commutator is a bracket of weight 2 and a deviation of level \( l \) is a bracket of weight \( l + 3 \).

\textbf{Lemma 2.} For an arbitrary finitely generated loop \( L \) the abelian groups \( \gamma_iL/\gamma_{i+1}L \) are finitely generated.
We say that $L$ is *nilpotent* if there exists $n$ such that $\gamma_{n+1}L = 1$. The minimal such $n$ is called the nilpotency class of $L$.

**Remark.** It can be seen that for $L$ nilpotent, the word “normal” can be omitted in the definition of the commutator-associator subloops.

1.2. **More on deviations.** The definition of the associator deviations given above does not use any specific property of the associator. In fact, the deviations can be constructed for any function $\phi(x) : L \to L$. We define the *deviation* $\phi(x_1, x_2)$ derived from $\phi(x)$ by setting

$$\phi(x_1, x_2) = (\phi(x_1)\phi(x_2))\phi(x_1, x_2).$$

If $\phi$ is a function $L^k \to L$ with $k > 1$ one can consider deviations with respect to each variable. The deviations derived from the associator are the usual associator deviations of level one.

Now, let $w$ be a word in the free loop on $k$ generators $F_k$. For any loop $L$ it induces a function $\phi_w(x_1, \ldots, x_k)$ from $L^k$ to $L$.

**Proposition 3.** Suppose that $\phi_w(x_1, \ldots, x_k)$ respects the commutator-associator filtration on any loop, that is, $x_i \in \gamma_n L$ implies $\phi_w(x_1, \ldots, x_k) \in \gamma_{n+\ldots+n_k} L$ for any $L$. Then a deviation derived from $\phi_w$ with respect to any variable also respects the commutator-associator filtration.

The rest of this subsection is dedicated to the proof of this statement.

Let $\phi : L^m \to L$ be a function which respects the commutator-associator filtration. We shall say that $\phi$ is *regular* if all deviations derived from $\phi$ also respect the commutator-associator filtration. Thus, Proposition 3 says that any function of the form $\phi_w$ that respects the commutator-associator filtration, is regular.

**Lemma 4.** Let $\phi : L^m \to L$ be a regular function. For each $1 \leq i \leq m$ let $X^i$ be a non-empty finite set of letters and let $\psi : L^{|X^i|} \to L$ be a regular function in the variables from the set $X^i$. (We do not assume that the sets $X^i$ are disjoint.) Then the composition $\phi(\psi_1, \ldots, \psi_m)$ is also regular.

Assume that $\phi$ is a function with two arguments and that $\psi_1$ and $\psi_2$ are functions of the same single variable. Then for $x \in \gamma_p L$ and $y \in \gamma_q L$ we have that, modulo $\gamma_{p+q} L$

$$\phi(\psi_1(xy), \psi_2(xy)) = \phi(\psi_1(x)\psi_1(y) \cdot \psi_1(x, y), \psi_2(x)\psi_2(y) \cdot \psi_2(x, y))$$

$$= \phi(\psi_1(x)\psi_1(y), \psi_2(x)\psi_2(y))$$

$$= \phi(\phi(\psi_1(x), \psi_2(x))\phi(\psi_1(x), \psi_2(y)))\phi((\psi_1(y), \psi_2(y)))$$

$$\equiv \phi(\psi_1(x), \psi_2(x)) \cdot \phi(\psi_1(y), \psi_2(y))$$

and, therefore, the deviation derived from $\phi(\psi_1, \psi_2)$ belongs to $\gamma_{p+q} L$.

The general case is entirely similar except for the complexity of notation; we omit the proof.

**Lemma 5.** Let $\psi_1, \psi_2 : L^m \to L$ be regular functions in the same set of variables. Then the functions $\psi_1 \psi_2$, $\psi_1/\psi_2$ and $\psi_1 \psi_2$ are also regular.

For the sake of simplicity assume that $\psi_1$ and $\psi_2$ are functions in one variable. Take $x \in \gamma_p L$ and $y \in \gamma_q L$. Then, modulo $\gamma_{p+q} L$

$$\psi_1(xy)\psi_2(xy) = (\psi_1(x)\psi_1(y) \cdot \psi_1(x, y))(\psi_2(x)\psi_2(y)\psi_2(x, y))$$

$$\equiv \psi_1(x)\psi_1(y) \cdot \psi_2(x)\psi_2(y)$$

$$\equiv \psi_1(x)\psi_2(x) \cdot \psi_1(y)\psi_2(y)$$

and it follows that $\psi_1 \psi_2$ is regular.

The regularity of $\psi_1/\psi_2$ and $\psi_1 \psi_2$ is proved in the same manner. The case of several variables is entirely similar.
In order to establish the truth of Proposition $\mathbf{3}$ it is sufficient to prove that the deviations derived from $\phi_w$ respect the commutator-associator filtration for nilpotent loops. Let $L$ be of nilpotency class $N - 1$. Then $\phi_w$ only depends on the image of $w$ in $F_k/\gamma_N F_k$.

It follows from the proof of Lemma $\mathbf{2}$ that for an arbitrary positive integer $N$, the word $w$ can be written, modulo $\gamma_N F_k$, as a word in brackets of weight at least $k$ whose arguments are the generators of $F_k$.

It can be assumed that each of these brackets contains every generator $x_1, \ldots, x_k$ of $F_k$ at least once. Indeed, the word $w$ can be written as

$$w = w_k = u_k v_k : w_{k+1}$$

where $w_{k+1}$ is a word in brackets of weight at least $k + 1$, $u_k$ is a word in brackets of weight $k$ among whose arguments the generator $x_i$ is present, and $v_k$ is a word in brackets of weight $k$ among whose arguments the generator $x_i$ is missing. Since $\phi_w(x_1, \ldots, x_k)$ respects the commutator-associator filtration on $F_k/\gamma_N F_k$, it follows that replacing $x_i$ by 1 in $w$ we obtain a word representing the identity in $F_k/\gamma_N F_k$. By definition, brackets of all weights respect the commutator-associator filtration. In particular, replacing $x_i$ by 1 in $u_k$ we also get the identity. Since $v_k$ does not change under replacing $x_i$ by 1, it follows that $v_k \in \gamma_{k+1} F_k/\gamma_N F_k$ and, hence, can be taken to be equal to the identity.

This shows that $u_k$ is a word in brackets of weight $k$ which contain all the $x_i$ among their arguments. Lemma $\mathbf{4}$ implies that $u_k$ gives rise to a regular function, and therefore, by Lemma $\mathbf{5}$ $u_k \backslash u_k$ also does. Now, writing $u_k \backslash w_k$ as

$$u_k \backslash w_k = u_{k+1} v_{k+1} : w_{k+2}$$

we can repeat the argument to show that $v_{k+1}$ can be taken to be the identity etc.

Finally, since $w = u_k(u_{k+1}(\ldots u_{N-1}))$ with all the $u_i$ giving rise to regular functions, it follows from Lemma $\mathbf{5}$ that $\phi_w$ is also regular.

1.3. Isolators. Let $F$ be the free loop on the single generator $x$ and $\delta : F \to \mathbb{Z}$ — the homomorphism that sends $x$ to 1. If $w(x)$ is a non-associative word in $x$ its degree is defined to be the integer $\delta(w)$.

Let $L$ be a loop and $K \subseteq L$ — a normal subloop. The isolator of $K$ in $L$, denoted by $\sqrt{K}$, is the minimal normal subloop of $L$ containing all such $x \in L$ that $w(x) \in K$ for some word $w$ of non-zero degree. An element of $L$ is called periodic if it belongs to the isolator of the identity. A loop is torsion-free if it has no periodic elements.

1.4. The dimension filtration. Let us now recall the definition of the dimension filtration for loops $\mathbf{7}$. Let $R$ be a commutative unital ring and $L$ — an arbitrary loop. The augmentation ideal $I \subseteq RL$ is the kernel of the $R$-linear map of the loop algebra $RL$ to $R$ that sends every element of $L$ to 1. The $n$th power of $I$ is the linear span of all products of at least $n$ elements of $I$. The loop $L$ sits inside $RL$ and its intersection with $1 + I^n$ is a normal subloop of $L$, called the $n$th dimension subloop over $R$ and denoted by $D_n(L, R)$. In what follows we shall only consider dimension subloops over a field $k$ of characteristic 0 and write $D_n L$ for $D_n(L, k)$.

Lemma 6. For any loop $L$ $\sqrt{\gamma_n L} \subseteq D_n L$.

For any loop $L$ the brackets of all weights respect the dimension filtration. In other words, if $x \in D_p L$ and $y \in D_q L$, the commutator $[x, y]$ belongs to $D_{p+q} L$ and similarly for the associator and the associator deviations $\mathbf{7}$. Since $\gamma_1 L = D_1 L = L$ it follows that $\gamma_n L \subseteq D_n L$ for all $n$.

Now, let $x$ be an element of $L$ that does not belong to $D_n L$. Then $u = 1 - x$ belongs to the augmentation ideal $I$ but not to $I^n$. Suppose $u$ belongs to $I^k$ but not to $I^{k+1}$. Here $1 \leq k < n$. For any word $w$ in $x$ of degree $m \neq 0$ we have

$$w(x) = 1 - mu + \ldots$$

and the omitted terms are integer multiples of monomials $u$ of degrees at least 2. Hence, $1 - w(x) \equiv mu \mod I^{k+1}$ and, since $k$ has characteristic 0, it follows that $w(x) \notin D_n L$. However, as $\gamma_n L$ is contained in $D_n L$ for all $n$, $w(x)$ cannot be contained in $\gamma_n L$. 


1.5. The isolators of $\gamma_iL$ as an N-sequence. A filtration of a loop $L$ by normal subloops $L = L_1 \supset L_2 \supset \ldots$ is said to be an N-sequence if for all $n$ any bracket of weight $n$ evaluated on arbitrary elements $x_i \in L_{p_i}, (0 < i \leq n)$ gives an element of $L_{p_1+\ldots+p_n}$. Both the commutator-associator filtration and the dimension filtration are N-sequences \cite{7}. As in Lemma 3 the brackets of weight $n$ induce $n$-linear operations on the graded group associated to an N-sequence.

**Proposition 7.** The filtration of any loop $L$ by $\sqrt[n]{\gamma_iL}$ is an N-sequence.

The proof of Proposition 3 is based on the following result.

Let $\theta$ be a non-associative word on $k$ letters and let $x_1, \ldots, x_k$ be elements of $L$. Define $\theta_{x_1, \ldots, x_k}$ to be the subloop of $L$ normally generated by all elements of the form $\theta(w_1(x_1), \ldots, w_k(x_k))$ where $w_i$ are words on one letter.

Let $W_i$, where $0 < i \leq k$, be words on one letter, each of non-zero degree.

**Lemma 8.** If $L$ is nilpotent, the quotient $\theta_{x_1, \ldots, x_k}/\theta_{W_1(x_1), \ldots, W_k(x_k)}$ is finite.

It is sufficient to prove Lemma 3 for the free class-$n$ nilpotent loop $F_k[n]$ on $k$ generators $x_1, \ldots, x_k$, that is, for the quotient of the free loop on the $x_i$ by the $n + 1$ term of its commutator-associator filtration.

The proof goes by induction on the nilpotency class. The lemma is obvious for abelian groups. Assume it is true for free loops of nilpotency class at most $n - 1$. The kernel of the homomorphism $F_k[n] \rightarrow F_k[n - 1]$ is the commutative group $\gamma_n F_k[n]$. It is enough to prove that $\gamma_n F_k[n] \cap \theta_{W_1(x_1), \ldots, W_k(x_k)}$ is of finite index in $\gamma_n F_k[n] \cap \theta_{x_1, \ldots, x_k}$.

The group $\gamma_n F_k[n]$ is generated by the brackets of weight $n$ evaluated on the $x_i$. In particular, we can choose a basis that consists of such brackets for the $\mathbb{Q}$-vector space $\gamma_n F_k[n] \otimes \mathbb{Q}$. The linearity of the brackets implies that the homomorphism $\omega : F_k[n] \rightarrow F_k[n]$ defined by sending $x_i$ to $W_i(x_i)$ induces a transformation of $\gamma_n F_k[n] \otimes \mathbb{Q}$ given by a diagonal matrix with non-zero diagonal entries. Hence, $\omega$ induces an isomorphism of $(\gamma_n F_k[n] \cap \theta_{x_1, \ldots, x_k}) \otimes \mathbb{Q}$ into itself. On the other hand, since the image of $\theta_{x_1, \ldots, x_k}$ under $\omega$ is contained in $\theta_{W_1(x_1), \ldots, W_k(x_k)}$, it follows that $\gamma_n F_k[n] \cap \theta_{W_1(x_1), \ldots, W_k(x_k)}$ is of finite index in $\gamma_n F_k[n] \cap \theta_{x_1, \ldots, x_k}$.

Now we are in the position to prove Proposition 7. It is sufficient to verify it for nilpotent loops; the general case can be reduced to the case of nilpotent loops by replacing $L$ with $L/\sqrt[n]{\gamma_iL}$ with sufficiently large $N$.

Assume that $L$ is nilpotent. Let $x \in \sqrt[p]{\gamma_iL}$ and $y \in \sqrt[q]{\gamma_iL}$ so that there exist words $w_1$ and $w_2$ on one letter and of non-zero degree such that $[w_1(x), w_2(y)] \in \gamma_{p+q}L$. Set $\theta = [x, y]$; applying Lemma 3 we see that $\theta_{w_1(x), w_2(y)}$ is of finite index in $\theta_{x, y}$ and, hence, there exists a word $w$ on one letter and of non-zero degree such that $w([x, y]) \in \theta_{w_1(x), w_2(y)} \subseteq \gamma_{p+q}L$. Therefore, $[\sqrt[p]{\gamma_iL}, \sqrt[q]{\gamma_iL}] \subseteq \sqrt[p+q]{\gamma_iL}$. Similarly one proves that $([\sqrt[p_1L], \ldots, \sqrt[p_{k+1}L]]_{\alpha_1, \ldots, \alpha_k} \subseteq \sqrt[p_1+\ldots+p_{k+1}L]$.

2. The Jennings theorem

Now we can state our main result.

**Theorem 9.** For any field $k$ of characteristic 0 and for any loop $L$, the isolator $\sqrt[n]{\gamma_iL}$ of $\gamma_iL$ in $L$ coincides with the dimension subloop $D_n(L, k)$.

The associative version of this theorem is due to Jennings \cite{4}. Our proof follows the argument given in Chapter 7 of \cite{2}, see also \cite{9} and \cite{8}.

Theorem 4 implies that after tensoring with a field of characteristic zero, the graded groups associated to the dimension and the commutator-associator filtrations become isomorphic. The group $D_nL/D_{n+1}L \otimes \mathbb{Q}$ has the structure of a Sabinin algebra.

Recall that Sabinin algebras are related to Lie algebras in the same way as loops are related to groups. They were initially introduced by Mikheev and Sabinin as tangent structures to general affine connections, see \cite{11} \cite{5}. Later, it was proved that primitive elements in a non-associative bialgebra form a Sabinin algebra \cite{12}, and that every Sabinin algebra arises this way \cite{10}.
It is known from [7] that $\bigoplus D_nL/D_{n+1}L \otimes \mathbb{Q}$ is the Sabinin algebra of primitive elements of the algebra $\bigoplus I^n/I^{n+1}$, the primitive operations of Shestakov-Umirbaev [12] being induced by associator deviations. Therefore, we have

**Corollary 10.** The graded group $\bigoplus \gamma_nL/\gamma_{n+1}L \otimes \mathbb{Q}$ is a Sabinin algebra with the commutator and the Shestakov-Umirbaev operations induced by the commutator and the associator deviations on $L$ respectively.

2.1. The outline of the proof. We have already seen that $\sqrt{\gamma_nL/\gamma_{n+1}L}$ is contained $D_nL$. So, just like in the associative situation, it is enough to prove that if $\sqrt{\gamma_nL} = 1$, then $D_nL$ is also trivial. We can assume that $L$ is finitely generated, since any element of $D_nL$ belongs to $D_nL'$ where $L' \subseteq L$ is some finitely generated subloop.

Let us fix some notation. For $a \in L$ denote the corresponding element of the left multiplication group of $L$ by $\lambda_a$. Similarly, for any $v$ in the loop algebra $kL$ we write $\lambda_v$ for the left multiplication by $v$ in $kL$. Writing a product $a_1a_2...a_m$ without parentheses we mean $a_1(a_2(...)a_m)) = \lambda_{a_1}\lambda_{a_2}...\lambda_{a_m}(1)$. The expression $a^m$ will stand for $\lambda_a^m(1)$.

All the quotients $\sqrt{\gamma_nL}/\sqrt{\gamma_{n+1}L}$ are torsion-free; let $M$ be the sum of the ranks of these quotients. There are $x_i \in L$ with $1 \leq i \leq M$ and integers $c_j$ with $1 \leq j \leq N$ such that $c_1 = 1$, $c_j \leq c_{j+1}$, $c_N = M + 1$ and $\sqrt{\gamma_nL} = \{\sqrt{\gamma_{n+1}L}, x_{c_n}, x_{c_{n+1}}, ..., x_{c_{n+1}-1}\}$.

Then each element of $L$ can be uniquely written as $A_i^{\lambda_1i}A_2^{\lambda_2i}...A_M^{\lambda_Mi}(1)$ with $r_i$ integers.

Let $x$ be a generator of the infinite cyclic group and set $u = 1 - x$. By Lemma 7.2 of [2], the group ring of the infinite cyclic group has the basis consisting of $1, u, u^2,...$ together with $u^m x^{-1}, u^m x^{-2},...$ where $m$ is any positive integer. Let $u_i = 1 - x_i \in kL$, then $\lambda_{u_i} = 1 - \lambda_{x_i}$. Therefore, we have the following

**Lemma 11.** The loop algebra $kL$ has a basis consisting of all elements of the form $A_1A_2...A_M(1)$

where $A_i$ is equal to either $\lambda_{x_i}$ or $\lambda_{\gamma_xi}^{-s_i}\lambda_{\gamma_{x_i}}^{N_i}$ with $r_i$ a non-negative and $s_i$ — a positive integer.

Define $\mu(u_i)$ to be the largest number $k$ such that $x_i \in \sqrt{\gamma_kL}$. For any basis element $v$ of the form described in Lemma 11 we define $\mu(v) \geq N$ if at least one of the $A_i$ in $v$ has the form $\lambda_{x_i}^{-s_i}\lambda_{\gamma_{x_i}}^{N_i}$; otherwise $v$ is of the form $v = \lambda_{u_i}^{N_i}...\lambda_{A_M}^{N_M}A_M$ and we set $\mu(v) = \sum \mu(u_i)r_i$.

For $k \leq N$ denote by $E_k$ the vector space over $k$ spanned by those basis elements $v$ with $\mu(v) \geq k$; for $k > N$ set $E_k = E_N$. It is clear that $E_k$ is contained in the $k$th power $I_k$ of the augmentation ideal $I$.

**Lemma 12.** $E_kE_l \subseteq E_{k+l}$.

The proof of this lemma will be given in the next section. Now, assuming the the truth of Lemma 12 let us finish the proof of Theorem 9.

The augmentation ideal $I$ is the same thing as $E_1$. Since $E_k \subseteq I_k$, it follows from Lemma 12 that $I_k$ coincides with $E_k$. Lemma 11 implies that the elements $u_i = \lambda_{u_i}(1)$ with $\mu(u_i) = k$ (these are the $u_i$ with $c_k \leq i < c_{k+1}$) are linearly independent modulo $E_{k+1}$, and hence, modulo $I^{k+1}$.

Now, any $y \in \sqrt{\gamma_kL} - \sqrt{\gamma_{k+1}L}$ is of the form $\lambda_{x_j}^{r_0}\lambda_{x_{j+1}}^{r_1}...\lambda_{x_{j+l}}^{r_l}(z)$ with $j = c_k$, $l = c_{k+1} - c_k - 1$, $z \in \sqrt{\gamma_{k+1}L}$ and not all $r_i$ equal to zero. Since $1 - z = I^{k+1}$, we have

$$1 - y \equiv \sum r_iu_{j+i} \mod I^{k+1}$$

where the sum is over all $0 \leq i \leq l$. Therefore, $1 - y$ does not belong to $I^{k+1}$ and, hence, $y \notin D_{k+1}L$. In particular, if $y \neq 1$, then $y \notin D_NL$. 

3. Proof of Lemma 12

Given a set $X$ of elements of $L$, an elementary bracket with respect to $X$ is a bracket whose arguments belong to $X$. Take $X = \{a, b_1, \ldots, b_p, c_1, \ldots, c_q\}$. Then, using the definition of associator deviations, one can decompose the loop associator $(a, b_1 \ldots b_p, c_1 \ldots c_q)$ as a product of elementary brackets. For every pair of non-empty subsets $I = \{i_1, \ldots, i_{|I|}\} \subseteq \{1, 2, \ldots, p\}$ and $J = \{j_1, \ldots, j_{|J|}\} \subseteq \{1, 2, \ldots, q\}$ this product contains precisely one deviation of the form $(a, b_{i_1}, \ldots, b_{i_{|I|}}, c_{j_1}, \ldots, c_{j_{|J|}})$, $\ldots$. We shall fix once and for all such a decomposition and call it $\Pi$.

Let $S$ be some subset of the set of all elementary brackets that form the product $\Pi$. One can then form a product of elementary brackets $\Pi_S$ by deleting from $\Pi$ all the brackets that do not belong to $S$. Now, replace in the product $\Pi_S$ each elementary bracket $w$ by $w - 1$; the resulting element of $kL$ is denoted by $P_S$. If $S$ is empty, then $\Pi_S = 1$ and $P_S = 0$.

Write $A, B_i$ and $C_j$ for $1 - a$, $1 - b_i$ and $1 - c_j$ respectively. Denote by $B_I$ and $C_J$ the products $B_{i_1}B_{i_2} \ldots B_{i_{|I|}}$ and $C_{j_1}C_{j_2} \ldots C_{j_{|J|}}$ respectively, where $I = \{i_1, \ldots, i_{|I|}\}$ and $J = \{j_1, \ldots, j_{|J|}\}$, in a similar way we define products $b_I$ and $c_J$. If $I$ (or $J$) is empty, then $B_I = 1$ ($C_J = 1$, respectively).

Then the following formula holds:

1. $A \cdot (B_1B_2 \ldots B_p \cdot C_1C_2 \ldots C_q) - (AB_1B_2 \ldots B_p) \cdot C_1C_2 \ldots C_q = (-1)^{p+q} \sum_S \left( \sum_{I,J} (-1)^{|I|+|J|}a \cdot B_IC_J \right) \cdot P_S.$

Here the sum inside the brackets on the right-hand side is taken over all subsets $I \subseteq \{1, \ldots, p\}$ and $J \subseteq \{1, \ldots, q\}$ with the property that if no bracket in $S$ contains $b_i$ (or $c_j$) as an argument, then $i \in I$ (or $j \in J$, respectively).

In order to prove (1), notice that

$$ab_I \cdot c_J = (a \cdot b_Ic_J)(1 + \sum_{S \subseteq S_{I,J}} P_S),$$

where $S_{I,J}$ is the subset consisting of all brackets from $\Pi$ which contain only the variables $a, b_I$ and $c_J$. Also,

$$aB_I \cdot C_J = \sum_{I' \subseteq I, J' \subseteq J} (-1)^{|I'|+|J'|}aB_I \cdot c_{J'},$$

$$= \sum_{I' \subseteq I, J' \subseteq J} (-1)^{|I'|+|J'|}(a \cdot b_{I'}c_{J'}) \left(1 + \sum_{S \subseteq S_{I',J'}} P_S \right),$$

$$= a \cdot B_IC_J + \sum_{I' \subseteq I, J' \subseteq J} (-1)^{|I'|+|J'|}(a \cdot b_{I'}c_{J'}) \sum_{S \subseteq S_{I',J'}} P_S.$$

Now, $aB_I \cdot C_J - a \cdot B_IC_J = A \cdot B_IC_J - AB_I \cdot C_J$. It remains to calculate the coefficient at $P_S$ for given $S$:

$$\sum_{I' \subseteq I, J' \subseteq J} (-1)^{|I'|+|J'|}(a \cdot b_{I'}c_{J'}) = \sum_{I' \subseteq I, J' \subseteq J} (-1)^{|I'|+|J'|}(a \cdot B_{I'}C_{J'}).$$

Now, setting $I = \{1, \ldots, p\}$ and $J = \{1, \ldots, q\}$ and writing $I, J$ instead of $I', J'$ we get (1).

We shall need two other formulae similar to (1). Consider the anti-associator

$$(a, b, c)' = ((ab)c) \setminus (a(bc)).$$

Mimicking the construction of deviations for the associator, we can build the hierarchy of deviations of all levels derived from the anti-associator. Then we have the following formula:

2. $$(A_1A_2 \ldots A_p \cdot B) \cdot C_1C_2 \ldots C_q - A_1A_2 \ldots A_p \cdot BC_1C_2 \ldots C_q = (-1)^{p+q} \sum_S \left( \sum_{I,J} (-1)^{|I|+|J|}A_Ib \cdot C_J \right) \cdot Q_S.$$
Here $Q_S$ is defined exactly as $P_S$ but with anti-associators and the deviations derived from them instead of associators and associator deviations. All other symbols have the same meaning as in (1). In the particular case when $p = 1$, the formulae (2) and (11) give

$$AB \cdot C_1 C_2 \cdots C_q - A \cdot B C_1 C_2 \cdots C_q = (-1)^{q+1} \sum_S \left( \sum_J (-1)^{|J|} ab \cdot C_J \right) \cdot Q_S$$

and

$$A_1 A_2 \cdots A_p \cdot B - B \cdot A_1 A_2 \cdots A_p = (-1)^{p+1} \sum_S \left( \sum_I (-1)^{|I|} b \cdot A_I \right) \cdot P_S$$

where $A$ and $B$ stand for $A_1$ and $B_1$ respectively, and $a$ and $b$ — for $a_1$ and $b_1$.

The same construction can also be performed for the commutator. The resulting formula reads

$$A_1 A_2 \cdots A_p \cdot B - B \cdot A_1 A_2 \cdots A_p = (-1)^{p+1} \sum_S \left( \sum_I (-1)^{|I|} b \cdot A_I \right) \cdot R_S.$$  

**Remark.** The anti-associator is readily seen to respect the commutator-associator filtration. Proposition (8) then implies that the deviations of all levels derived from the anti-associator also respect the commutator-associator filtration. The same thing can be said about the commutator and the deviations of all levels derived from it.

Set $v_i = 1 - x_i \setminus 1$ and consider products of the form

$$\Lambda_1^t \Lambda_2^t \cdots \Lambda_M^t(1)$$

where $\Lambda_i$ is equal to $\lambda_{d_i}^t \lambda_{r_i}^t$ with $d_i, r_i \geq 0$. For every such product $u$ set $\mu'(u) = \sum \mu(u_i)r_i$ and $l(u) = \sum (d_i + r_i)$.

Let $E_k$ be the subspace of $kL$ spanned by the products $\Lambda_1^t \cdots \Lambda_M^t(1)$ with $\mu' \geq k$. For $1 \leq s \leq M$, let $E_{k,s}$ be the subspace of $E_k$ spanned by the products that have $\Lambda_1^t = \Lambda_2^t = \cdots = \Lambda_{s-1}^t = 1$. In particular, $E_{k,1} = E_k$.

**Lemma 13.** If $y \in E_{k,s}^t$, $z \in E_{t,s}^t$ and $x = \lambda_{v_{s-1}}^t \lambda_{u_{s-1}}^t(y)$, then

$$xz = \sum_i \alpha_i \lambda_{v_{t-1}}^t \lambda_{u_{t-1}}^t(w_i)$$

where $w_i \in E_{m_i,s}^t$, $\alpha_i$ are integers, $m_i = k + l + \mu(u_{s-1})(r_i - i)$, $\delta_i \leq d$ and $p_i \leq r$.

In particular, for $y \in E_{k,s}^t$ and $z \in E_{t,s}^t$ the product $yz$ is contained in $E_{k+t,s}^t$.

**Proof.** Let $z \in E_{t,s}^t$ where $t \geq s$; we shall use descending induction on $t$. For $t = M$ the statement of the lemma is obvious since all elements of $E_{t,M}^t$ commute and associate with everything. Suppose that for $t > q$ and all $y, z$ as above the lemma has been established.

Consider first the case $z = u_q$. Set $p = l(x)$ and use induction on $p$.

For $p = 1$ we have that $x$ is equal to either $u_1$ or $v_j$ for some $j \geq s - 1$, and there are two possibilities. If $j \leq q$, the condition (7) is satisfied automatically. If $j > q$ we have

$$u_j u_q - u_q u_j = (1 - u_q)(1 - u_j) \left( [x_q, x_j] - 1 \right)$$

and

$$v_j u_q - u_q v_j = (1 - u_q)(1 - u_j) \left( [x_q, x_j \setminus 1] - 1 \right).$$

(These formulae are particular cases of (11).) Since $[x_q, x_j] - 1$ $[x_q, x_j \setminus 1] - 1$ belong to $E_{\mu'(u_q) + \mu'(u_j), q+j}$ the first induction assumption implies (11) for $p = 1$.

Assume that (7) holds for $z = u_q$ and all $p < p_0$.

Take $y \in E_{k,s}^t$ with $l(\lambda_{v_{s-1}}^t \lambda_{u_{s-1}}^t(y)) = p_0$. If $d > 0$, write $y = v_{s-1} \tilde{y}$. Then $v_s \tilde{y} \cdot u_q - v_s \cdot \tilde{y} u_q$ can be re-written with the help of (11). The right-hand side of (11) has the form

$$\pm \sum_{S,t} (-1)^{|l|} \left( [x_q \setminus 1] \cdot \tilde{y}l x_q \right) \cdot P_S.$$
By the second induction assumption, the product $\tilde{y}tu_q$ satisfies (7), and $P_S \in E'_q$ with $q' > q$. Therefore, applying the first induction assumption we see that the product $v_q\tilde{y} \cdot u_q$ also satisfies (7), so we see that (7) holds for $z = u_q$ and $p = p_0$ as well. In the situation where $d = 0$ but $r \neq 0$ the argument is completely analogous.

If $d = 0$ and $r = 0$, we only have to consider the case when $q = s$. Applying (5) and the induction assumptions we see again that (7) is fulfilled, so the lemma holds whenever $z = u_q$.

In a similar fashion one verifies the lemma for $z = v_q$.

Let us now pass to the case of arbitrary $z \in E'_{q}$. If the condition (7) fails for some $z = x_u\lambda^c_v(z')$ (where $z \in E'_{l+\omega'}$) and some $d$, $r$, and $y$, choose the counterexample with the smallest possible $c$. Then, on one hand, $x_u \cdot \lambda^c_u(z')$ satisfies (7). On the other hand, $x_u \cdot \lambda^c_u(z') - x \cdot \lambda^c_u(z')$ can be re-written using (4). However, using the induction assumption, we see that the right-hand side of (4) is a linear combination of products of the form $\lambda^{d'}_{q-1} \lambda^{d}_{q-1}(w_i)$ with $d_i \leq d$ and $\rho_i \leq r$ and the $w_i$ belonging to the “correct” terms of the filtration $E'_{q}$. Therefore, no such counterexample can exist.

Finally, it may happen that (7) fails for some $z = x_u \lambda^c_u(z')$. Then the argument of the previous paragraph carries over to this situation without modifications. This completes the induction step.

\[ \text{Lemma 14.} \text{ If } u \in E'_{k,s} \text{ and } v \in E'_{l,t}, \text{ then} \]
\[ (x_u u) v = \sum_i \alpha_i x_s w_i \]
\[ \text{where } w_i \in E'_{k+i,t} \text{ and } \alpha_i \text{ are integers.} \]

\[ \text{Proof.} \text{ Assume that } v \in E'_{l,t} \text{ and use descending induction on } t. \text{ For } t = M \text{ there is nothing to prove since } x_M \text{ commutes and associates with everything. Suppose that the lemma is established for } t > q. \]

We have
\[ (x_u u) v - x_s u v) = u_s (u v) - (u_s u) v \]
which, by formula (4), is equal to \pm \sum_s \left( \sum_{i,j} (-1)^{|i|} \cdot x_s u v_j \right) \cdot P_S. \text{ It follows from Lemma 13 that } u v \in E'_{k+l,s}. \text{ Moreover, in each term } u v_j \in E'_{k+l,s} \text{ and } P_S \in E'_{k',s'} \text{ with } k' + k'' = k + l \text{ and } s' > q \text{ and, hence, by the induction assumption } u_s (u v) - (u_s u) v \in E'_{k+l,s}. \]

For $1 \leq s \leq M$, let $E_{k,s}$ be the subspace of $E_k$ spanned by the basis elements $\Lambda_1 \Lambda_2 \cdots \Lambda_M(1)$ that have $\Lambda_1 = \Lambda_2 = \ldots = \Lambda_{s-1} = 1$. We have

\[ \text{Lemma 15. } E_{k,s} = E'_{k,s}. \]

Together with Lemma 13, this establishes Lemma 12 since $E_k = E_{k,1}$.

\[ \text{Proof.} \text{ Let us prove that every product of the form } \Lambda'_1 \Lambda'_{s+1} \cdots \Lambda'_M(1) \text{ with } \mu' \geq k \text{ is in } E_{k,s}. \text{ The proof uses descending induction on } s. \text{ For } s = M \text{ there is nothing to prove.} \]

Assume that this statement is true for $s = q + 1$. If it does not hold for $s = q$, the set of all $d$ such that $\lambda^d_u \lambda^e_v(v) \notin E_{k,q}$ for some $r \geq 0$ and some $v \in E_{k-r\mu(u_q),q+1}$ is non-empty. Take the smallest such $d$; clearly, $d \geq 1$. Choose $w = \lambda^{d-1}_u \lambda^e_u(v) \in E_{k,q}$ such that $v w$ is not contained in $E_{k,q}$. We have
\[ w - x_q((x_q \cdot 1)w) = (x_q \cdot x_q \cdot 1)w - x_q((x_q \cdot 1)w) = u_q v_q \cdot w - u_q \cdot v_q w. \]
Since $w$ can be written as a product $w_1 \ldots w_m$ with $w_i$ of the form $u_j$ or $v_j$, the formula (5) can be applied to the expression on the right-hand side of (8); it is equal to
\[ \pm \sum_s \left( \sum_{j} (-1)^{j} w_j \right) \cdot Q_S. \]
This lies in $E_{k,q}$ by Lemma \[ and the induction assumption; so does $w$. Therefore $x_q((x_q \setminus 1)w)$ also is in $E_{k,q}$, and, hence, $x_q(v_qw) \in E_{k,q}$.

The operator $\lambda_q^{-1}$ of left division by $x_q$ can be written as a linear combination of operators of the form $\lambda_q^t N$ and $\lambda_q^t \Lambda_q^k$ with $t \geq 0$, so $\lambda_q^{-1}(x_q(v_qw)) = v_qw$ is also in $E_{k,q}$, which contradicts our choice of $w$. Therefore, for $s = q$ all products of the form $\Lambda_q^r \Lambda_q^k \cdots \Lambda_q^1(1)$ with $\mu' \geq k$ belong to $E_{k,q}$.

It remains to see that $E_{k,s} \subseteq E_{k,s}'$. Assume that we have this established for $s > q$. Consider the set of all $d$ such that $\lambda_q^{-d} \Lambda_q^r(v) \in v_qw' \in E_{k,q}'$ for some $v \in E_{k',q+1}'$ and some $r \geq 0$ with $k' + r \geq k$. If this set is empty we are done. If not, take the smallest such $d$; we have $d \geq 1$. Choose $w = \lambda_q^{-d+1} \Lambda_q^r(v) \in E_{k,q}$ such that $x_q \setminus w$ is not contained in $E_{k,q}'$.

By construction $w \in E_{k,q}'$. Applying (1) to the expression on the right-hand side of (2) gives

$$u_q v_q\cdot w - u_q \cdot v_qw = \pm \sum_{S}\sum_j (-1)^{|J|} \left( x_q w_j - x_q \cdot v_q w_j \right) \cdot P_S$$

It follows then from Lemma \[ that

$$w - x_q((x_q \setminus 1)w) = \sum \alpha_i x_q \tilde{w}_i,$$

where each $\tilde{w}_i$ is in $E_{k,q}'$ and the $\alpha_i$ are some integers. Therefore, $x_q \setminus w$ is also in $E_{k,q}'$. □

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