Degeneration of torsors over families of del Pezzo surfaces

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Abstract
Let $S$ be a split family of del Pezzo surfaces over a discrete valuation ring such that the general fiber is smooth and the special fiber has ADE-singularities. Let $G$ be the reductive group given by the root system of these singularities. We construct a $G$-torsor over $S$ whose restriction to the generic fiber is the extension of structure group of the universal torsor. This extends a construction of Friedman and Morgan for individual singular del Pezzo surfaces. In case of very good residue characteristic, this torsor is unique and infinitesimally rigid.

KEYWORDS
degeneration, del Pezzo surface, universal torsor

MSC (2010)
14D06, 11E57, 14J26, 14L30

1 | INTRODUCTION

Cubic surfaces over $\mathbb{C}$ have been studied since the 19th century by Cayley, Clebsch, Schläfli, Segre, Manin, and many others. In particular, the 27 lines on smooth cubic surfaces have an interesting combinatorial structure: their symmetry group is the Weyl group of type $E_6$. Schläfli classified cubic surfaces with ADE-singularities: the worst has type $E_6$. In this paper, we explore a geometric connection between smooth cubic surfaces, singular cubic surfaces, and an algebraic group of type $E_6$.

More generally, the lines on a split smooth del Pezzo surface $S$ of degree $d = 1, 2, 3, 4, 5, 6, 7$ have a symmetry group that is a Weyl group of type $E_8, E_7, E_6, D_5, A_4, A_2 + A_1, A_1$, respectively. The underlying root system

$$\Phi_0 = \{ \alpha \in \Lambda \mid (\alpha, \alpha) = -2, (\alpha, -K_S) = 0 \}$$

is the set of $(-2)$-classes in the Picard group $\Lambda = \text{Pic}(S)$, where $-K_S$ is the anticanonical class [36].

If $S$ is a del Pezzo surface with ADE-singularities, then its minimal desingularization $\tilde{S}$ is a weak del Pezzo surface [16], [19, § 8]. If $\tilde{S}$ is split of degree $d \leq 7$, then the $(-2)$-classes in $\Lambda = \text{Pic}(\tilde{S})$ again form a root system $\Phi_0$ of the same type as above. It contains a subsystem $\Phi$ corresponding to the singularities of $S$, whose set $\Delta$ of simple roots consists of the $(-2)$-curves on $\tilde{S}$.

These combinatorial data also correspond to algebraic groups. Let $T := \text{Hom}(\Lambda, G_m)$ be the torus with character group $\Lambda$. Let $G \subset G_0$ be the split reductive groups with maximal torus $T$ and root systems $\Phi \subset \Phi_0$, respectively. Let $B \subset G$ be the Borel subgroup containing $T$ given by $\Delta$. Choose a Borel subgroup $B_0 \subset G_0$ containing $B$. 
A fundamental tool in the arithmetic study of weak del Pezzo surfaces $\overline{S}$ are the universal torsors introduced by Colliot-Thélène and Sansuc [8–11]. These are certain $T$-torsors over $\overline{S}$. They have been used, for example, to study the Hasse principle and weak approximation (e.g., in [12, 13]) and the Manin conjecture (e.g., in [4, 5]) for certain $\overline{S}$.

However, universal torsors $\mathcal{T}$ over $\overline{S}$ never descend to $S$. This observation combined with physical considerations led Friedman and Morgan [21] to a geometric connection between singular del Pezzo surfaces and algebraic groups: They show over $\mathbb{C}$ that it is possible to lift $\mathcal{T}$ (along the canonical projection $B_0 \to \mathcal{T}$) to a $B_0$-torsor over $\overline{S}$ such that the induced $G_0$-torsor descends to $S$ [21, Thm. 3.1]. Their construction is based on their work, partly with Witten, on principal bundles over elliptic curves [22–26].

A different geometric connection between smooth del Pezzo surfaces and algebraic groups was conjectured by Batyrev and proved in [18, 37, 40, 41]: Every universal torsor $\mathcal{T}$ over a split smooth del Pezzo surface $S$ of degree $\leq 5$ has a natural $T$-equivariant embedding into the affine cone over a flag variety $G_0/P_0$, where $P_0 \subset G_0$ is a certain maximal parabolic subgroup.

Singular del Pezzo surfaces appear naturally as degenerations of smooth del Pezzo surfaces. For modern accounts of such degenerations, see Corti [14] and Hacking–Keel–Tevelev [29]. We consider flat families $\tilde{S}$ of split del Pezzo surfaces of arbitrary degree over a discrete valuation ring $R$ with residue field $k$ such that the generic fiber of $\tilde{S}$ is smooth and the special fiber of $S$ has at most $\text{ADE}$-singularities. In Section 2, we describe the precise setup and discuss the geometry in more detail. In particular, we have a desingularization $\overline{S} \to S$ that is minimal in the special fibers and an isomorphism in the generic fibers.

Our main result, proved in Section 3, provides a geometric connection between smooth del Pezzo surfaces, singular del Pezzo surfaces, and algebraic groups:

**Theorem.** Every universal torsor $\mathcal{T}$ over $\overline{S}$ can be lifted to a $B$-torsor $B$ over $\overline{S}$ such that the induced $G$-torsor $\mathcal{G}$ descends to a $G$-torsor $\mathcal{G}'$ over $S$. If $k$ has very good characteristic for the root system $\Phi$, then $B$, $\mathcal{G}$, and $\mathcal{G}'$ are all unique up to isomorphisms, and infinitesimally rigid.

As $G \subset G_0$, our $G$-torsor $\mathcal{G}'$ induces a $G_0$-torsor over $S$. Since every individual singular del Pezzo surface over $\mathbb{C}$ can be extended to such a degenerating family of del Pezzo surfaces over $R = \mathbb{C}[[t]]$, our result extends that of Friedman and Morgan. On the other hand, we see no direct relation to the work [18, 37, 40, 41] on Batyrev’s conjecture.

We view the uniqueness as evidence that the $G$-torsor $\mathcal{G}'$ is naturally associated with the family $\tilde{S}$. See (3.13) for the notion of very good characteristic, and Proposition 3.18 for the precise uniqueness statement. The torsor $\mathcal{G}'$ is called infinitesimally rigid if $H^1(S, \text{ad}(\mathcal{G}')) = 0$ for its adjoint vector bundle $\text{ad}(\mathcal{G}')$, and similarly for the other torsors. In Section 4, we give an example of a family of cubic surfaces with a $D_4$ singularity over a residue field $k$ of characteristic 2 for which $\mathcal{G}'$ is not infinitesimally rigid.

The work of Friedman and Morgan is generalized to other rational surfaces over $\mathbb{C}$ with $\text{ADE}$-singularities in [7]. For a physically motivated related construction over families of such surfaces with an emphasis on the case $A_n$, using a Fourier–Mukai transform, see [20].

## 2 Degenerating Del Pezzo Surfaces

Let $R$ be a discrete valuation ring with quotient field $K$, maximal ideal $m \subset R$ and residue field $k = R/m$. Recall that every split smooth del Pezzo surface has degree $d \in \{1, \ldots, 9\}$, and is

(i) either a blow-up of $\mathbb{P}^2$ in $9 - d$ points $x_1, \ldots, x_{9-d} \in \mathbb{P}^2(K)$ in general position (i.e., no three on a line, no six on a conic, and no eight on a cubic with one of them on a singularity),

(ii) or $\mathbb{P}^1 \times \mathbb{P}^1$ for $d = 8$.

As a degeneration of case (i), we consider the chain of blow-ups

\[
\overline{S} = \overline{S}_{9-d} \xrightarrow{p_{9-d}} \overline{S}_{8-d} \to \ldots \to \overline{S}_2 \xrightarrow{p_2} \overline{S}_1 \xrightarrow{p_1} \overline{S}_0 = \mathbb{P}^2_R
\]

where $p_i : \overline{S}_i \to \overline{S}_{i-1}$ is the blow-up in the closure $\overline{x}_i \in \overline{S}_{i-1}(R)$ of the preimage of $x_i$ in $\overline{S}_{i-1}(K)$. The generic fiber $\overline{S}_K$ is the blow-up of $\mathbb{P}^2_K$ in $x_1, \ldots, x_{9-d}$, and therefore a del Pezzo surface of degree $d$ over $K$.
Here, we assume that the images of $\mathbf{x}_i$ in $\bar{S}_{i-1}(k)$ are in almost general position, by which we mean that the image of $\mathbf{x}_i$ does not lie on a $(-2)$-curve in $\bar{S}_{i-1,k}$.

As degenerations of case (ii), we consider $\mathbb{P}^1$-bundles 
$$
\bar{S} \to \mathbb{P}^1_R
$$
whose restriction to the generic fiber $\mathbb{P}^1_k$ is the trivial bundle 
$$
\mathbb{P}^1_k \times_k \mathbb{P}^1_k = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k} \right) \to \mathbb{P}^1_k
$$
and whose restriction to the special fiber is either trivial or the Hirzebruch surface 
$$
F_2 = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^1_k}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(1) \right) \to \mathbb{P}^1_k.
$$

In both cases, the special fiber $\bar{S}_k$ is a weak del Pezzo surface over $k$ [16], [19, §8], i.e., a smooth rational surface whose anticanonical class is nef and big. In fact, every split weak del Pezzo surface appears as such a blow-up of $\mathbb{P}^2$ or such a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

**Lemma 2.1.** The canonical bundle $\omega_{\bar{S}_k}$ of the special fiber $\bar{S}_k$ is isomorphic to the restriction of the canonical bundle $\omega_{\bar{S}}$ of the total space $\bar{S}$.

**Proof.** The two differ by the normal bundle of $\bar{S}_k$ in $\bar{S}$, which is the pullback of the normal bundle of $\text{Spec}(k)$ in $\text{Spec}(R)$, and therefore trivial. \hfill \square

**Lemma 2.2.** The $R$-module $H^0 \left( \bar{S}, \omega_{\bar{S}}^{-m} \right)$ is free, and the natural map 
$$
H^0 \left( \bar{S}, \omega_{\bar{S}}^{-m} \right) \otimes_R k \to H^0 \left( \bar{S}_k, \omega_{\bar{S}_k}^{-m} \right)
$$
is an isomorphism, for each integer $m \geq 0$.

**Proof.** We carry some arguments from [34, §III.3] over to the weak del Pezzo surface $\bar{S}_k$. We have 
$$
H^1 \left( \bar{S}_k, \mathcal{O}_{\bar{S}_k} \right) = 0
$$
since this is a birational invariant [30, Prop. V.3.4]. Let $D$ be a general member of the anticanonical linear system on $\bar{S}_k$. Then $D$ does not contain any $(-2)$-curve on $\bar{S}_k$, since $\omega_{\bar{S}_k}^{-1}$ is globally generated. Therefore, 
$$
H^0 \left( D, \omega_{\bar{S}_k}^m \otimes \mathcal{O}_D \right) = 0
$$
for $m \geq 1$. Being a local complete intersection, $D$ has dualizing sheaf 
$$
\omega_D = \text{det} \left( I_{D \cap \bar{S}_k} / I_{D \cap \bar{S}_k}^2 \right)^{\vee} \otimes \omega_{\bar{S}_k} \cong \omega_{\bar{S}_k}(-D)_{|D} \otimes \mathcal{O}_{\bar{S}_k}(-D) = \mathcal{O}_D
$$
according to [35, Def. 6.4.7]. Therefore, Serre duality on $D$ implies 
$$
H^1 \left( D, \omega_{\bar{S}_k}^{-m} \otimes \mathcal{O}_D \right) = 0
$$
for $m \geq 1$. By means of the exact sequence 
$$
H^1 \left( \bar{S}_k, \omega_{\bar{S}_k}^{-(m-1)} \right) \to H^1 \left( \bar{S}_k, \omega_{\bar{S}_k}^{-m} \right) \to H^1 \left( D, \omega_{\bar{S}_k}^{-m} \otimes \mathcal{O}_D \right) = 0,
$$
and induction over $m$, we conclude that
\[ H^1(\overline{S}_k, \omega_{\overline{S}_k}^{-m}) = 0 \]
for $m \geq 0$. Using Cohomology and Base Change [30, Thm. III.12.11] together with Lemma 2.1, the claim follows.

Choosing a sufficiently large integer $m$ and a basis of $H^0(\overline{S}, \omega_{\overline{S}}^{-m})$, we get an anticanonical map
\[ \phi : \overline{S} \to S \subset \mathbb{P}^N. \]

Up to isomorphism over $R$, the scheme $S$ does not depend in the choices made. As $S$ is integral and $R$ is a discrete valuation ring, $S$ is flat over $R$ by [30, Prop. III.9.7]. Lemma 2.2 implies that the special fiber $S_k$ of $S$ is the anticanonical image of the weak del Pezzo surface $\overline{S}_k$.

In particular, $S_k$ is a del Pezzo surface with at most $\text{ADE}$-singularities, and $\phi$ contracts precisely the $(-2)$-curves on $\overline{S}_k$.

**Proposition 2.3.** We have $\phi^*\mathcal{O}_S = \mathcal{O}_S$, and $R^i\phi^*\mathcal{O}_S = 0$ for all $i > 0$.

**Proof.** Since $R^i\phi^*$ commutes with flat base change, and the completion of $R$ is flat over $R$, we may assume without loss of generality that $R$ is complete.

We show by induction that $\phi^*\mathcal{O}_{n\overline{S}_k} = \mathcal{O}_{n\overline{S}_k}$ and $R^i\phi^*\mathcal{O}_{n\overline{S}_k} = 0$ for all $i > 0$. For $n = 1$, this holds by [16, Th. V.2]. The induction step follows from the short exact sequence
\[ 0 \to \mathcal{O}_{(n-1)\overline{S}_k} \to \mathcal{O}_{n\overline{S}_k} \to \mathcal{O}_{\overline{S}_k} \to 0, \]
where $\pi \in R$ is a generator of $m$, and its analog for $S_k$.

Using the Theorem on Formal Functions [30, Thm. III.11.1], the claim follows. \qed

**Lemma 2.4.** Let $\Phi$ be a simply laced, irreducible root system, with simple roots $\Delta$ and positive roots $\Phi^+$. Let $\beta, \gamma \in \Phi^+$ such that $\gamma - \beta = \sum_{i=1}^t \alpha_i$, with all $\alpha_i \in \Delta$. Then there exist positive roots $\beta = \beta_0, \beta_1, \ldots, \beta_t = \gamma$ such that $\beta_{i+1} - \beta_i \in \Delta$ for all $i$.

**Proof.** We argue by induction on $t$; the cases $t = 0$ and $t = 1$ are clear. For $t \geq 2$, we note that
\[ -2 = (\gamma, \gamma) = (\gamma, \beta) + \sum_{i=1}^t (\gamma, \alpha_i). \]
The roots $\beta$ and $\gamma$ are not proportional since $\Phi$ is simply laced and since they are both positive, but not equal. Hence $(\gamma, \beta) \in \{0, 1, -1\}$ [3, Prop.IV.1.8]. Therefore, $(\gamma, \alpha_i) < 0$ for at least one $i$. Since both are positive, but not equal, we have $(\gamma, \alpha_i) = -1$. We define $\beta_{t-1} := \gamma - \alpha_i$, which is a root since
\[ (\beta_{t-1}, \beta_{t-1}) = (\gamma, \gamma) - 2(\gamma, \alpha_i) + (\alpha_i, \alpha_i) = -2 + 2 - 2 = -2, \]
and which is positive since it is the sum of $\beta$ and all $\alpha_j$ with $j \neq i$. By induction, we find a sequence $\beta = \beta_0, \ldots, \beta_{t-1}$ as required. \qed

For the rest of this section, we fix one singular point $x$ on $S_k$. Let $D_1, \ldots, D_r$ be the $(-2)$-curves on $\overline{S}_k$ that map to $x$. Let
\[ Z = n_1D_1 + \cdots + n_rD_r \]
with $n_1, \ldots, n_r \geq 1$ denote the fundamental cycle on $\overline{S}_k$ over $x$ (see [1]). It has the property that $(Z, D_i) \leq 0$ for all $i = 1, \ldots, r$, and is minimal with this property. Here $\langle \cdot, \cdot \rangle$ denotes the intersection number of divisors on $\overline{S}_k$. Put $N := n_1 + \cdots + n_r$, and $Z^{\text{red}} := D_1 + \cdots + D_r$. 
Lemma 2.5. There is a sequence of effective divisors

\[ 0 = Z_0 < Z_1 < Z_2 < \cdots < Z_r = Z_{\text{red}} < \cdots < Z_N = Z \quad (2.1) \]

on \( \widetilde{S}_k \) such that

- \( Z_j - Z_{j-1} \) is a \((-2)\)-curve \( D_{ij} \) for all \( j = 1, \ldots, N \), and
- \( (Z_j, D_i) \leq 1 \) for all \( i = 1, \ldots, r \) and \( j = 1, \ldots, N \).

Proof. The classes of \( D_1, \ldots, D_r \) are the simple roots of an irreducible root system \( \Phi \) of type \( \mathbf{A}_r \) (\( r \geq 1 \)) or \( \mathbf{D}_r \) (\( r \geq 4 \)) or \( \mathbf{E}_r \) (\( r = 6, 7, 8 \)). By [15, Rem. 0.2.1], the fundamental cycle \( Z \) is its maximal root. The reduced fundamental cycle \( Z_{\text{red}} \) is a positive root because

\[
(Z_{\text{red}}, Z_{\text{red}}) = \sum_{i=1}^{r} (D_i, D_i) + 2 \sum_{1 \leq i < j \leq r} (D_i, D_j) = r \cdot (-2) + 2 \cdots (r-1) \cdot 1 = -2
\]

and \( (Z_{\text{red}}, \omega^{-1}_{\widetilde{S}_k}) = 0 \).

Applying Lemma 2.4 first to \( D_1 \) and \( Z_{\text{red}} \) and then to \( Z_{\text{red}} \) and \( Z \) gives a sequence of effective divisors as in (2.1) such that \( Z_j - Z_{j-1} \) is a simple root and therefore a \((-2)\)-curve. Since \( \Phi \) is simply laced, each of the positive roots \( Z_j \) has intersection number \( \leq 1 \) with each of the simple roots \( D_i \). \( \square \)

Lemma 2.6. Let \( Z_j \subset \widetilde{S}_k \) be the closed subschemes given by Lemma 2.5.

(i) \( H^1(D_{ij}, I_{Z_{j-1} \subset Z_j}) = 0 \) for \( j = 1, \ldots, N \).

(ii) \( H^1(Z, I^n_{Z \subset \widetilde{S}_k} / I^{n+1}_{Z \subset \widetilde{S}_k}) = 0 \) for \( n \geq 0 \).

Proof. The ideal sheaf of the effective divisor \( Z \) on the smooth projective surface \( \widetilde{S}_k \) is the line bundle \( \mathcal{O}(-Z) := \mathcal{O}_{\widetilde{S}_k}(-Z) \). Therefore, we have

\[
\frac{T^n_{Z \subset \widetilde{S}_k}}{T^{n+1}_{Z \subset \widetilde{S}_k}} \cong \frac{\mathcal{O}(-nZ)}{\mathcal{O}(-(n+1)Z)} \cong \frac{\mathcal{O}_{\widetilde{S}_k}(-nZ)}{\mathcal{O}_{\widetilde{S}_k}} \otimes \mathcal{O}(-nZ) \cong \mathcal{O}_Z \otimes \mathcal{O}(-nZ). \quad (2.2)
\]

Since \( Z_j - Z_{j-1} = D_{ij} \) according to Lemma 2.5, we similarly have

\[
I_{Z_{j-1} \subset Z_j} \cong \frac{\mathcal{O}(-Z_{j-1})}{\mathcal{O}(-Z)} \cong \frac{\mathcal{O}_{\widetilde{S}_k}(-Z_{j-1})}{\mathcal{O}_{D_{ij} \subset \widetilde{S}_k}} \otimes \mathcal{O}(-Z_{j-1}) \cong \mathcal{O}_{D_{ij}} \otimes \mathcal{O}(-Z_{j-1}), \quad (2.3)
\]

which is a line bundle of degree \( -Z_{j-1}, D_{ij} \) \( \geq -1 \) on \( D_{ij} \cong \mathbb{P}^1 \). But the first cohomology of any such line bundle vanishes. This proves part (i).

Twisting the isomorphism (2.3) by the line bundle \( \mathcal{O}(-nZ) \) on \( \widetilde{S}_k \), we get

\[
I_{Z_{j-1} \subset Z_j} \otimes \mathcal{O}(-nZ) \cong \mathcal{O}_{D_{ij}} \otimes \mathcal{O}(-Z_{j-1} - nZ),
\]

which is now a line bundle of degree \( -Z_{j-1} - nZ, D_{ij} \) on \( D_{ij} \cong \mathbb{P}^1 \). But this degree is still \( \geq -1 \), because the fundamental cycle \( Z \) satisfies \( (Z, D_{ij}) \leq 0 \) by definition, and \( n \geq 0 \) by assumption. Hence we have more generally

\[
H^1(D_{ij}, I_{Z_{j-1} \subset Z_j} \otimes \mathcal{O}(-nZ)) = 0.
\]
Using induction over \( j \), and the short exact sequences
\[
0 \to I_{Z_{j-1} \cap Z_j} \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0
\]
twisted by the line bundle \( \mathcal{O}(-nZ) \) on \( \bar{S}_k \), we conclude that
\[
H^1(Z, \mathcal{O}_Z \otimes \mathcal{O}(-nZ)) = 0.
\]
Because of the isomorphism \((2.2)\), this proves part (ii) of the lemma. \( \square \)

**Proposition 2.7.** \( H^1 \left( Z, I^n_{Z \cap \bar{S}} / I^{n+1}_{Z \cap \bar{S}} \right) = 0 \) for \( n \geq 0 \).

**Proof.** Let \( \pi \in R \) be a generator of \( \mathfrak{m} \). We first claim that the inclusion
\[
\pi I^n_{Z \cap \bar{S}} \subset I^{n+1}_{Z \cap \bar{S}} \cap \pi \mathcal{O}_{\bar{S}}
\]
is an equality. It suffices to check this over the local ring \( \mathcal{O}_{\bar{S}, z} \) of each point \( z \in Z \). We choose a local function \( f \in \mathcal{O}_{\bar{S}, z} \) whose residue class
\[
\bar{f} \in \mathcal{O}_{\bar{S}, z} / \pi \mathcal{O}_{\bar{S}, z} = \mathcal{O}_{\bar{S}_e, z}
\]
is a local equation for the divisor \( Z \subset \bar{S}_k \). Then \( \pi \) and \( f \) generate \( I_{Z \cap \bar{S}} \) in \( z \). Hence \( \pi I^n_{Z \cap \bar{S}} \) and \( f^{n+1} \) generate \( I^{n+1}_{Z \cap \bar{S}} \) in \( z \).

Suppose that
\[
f^{n+1}g \in \pi \mathcal{O}_{\bar{S}_e, z}
\]
for some \( g \in \mathcal{O}_{\bar{S}_e, z} \). Then its residue class \( \bar{g} \in \mathcal{O}_{\bar{S}_k, z} \) satisfies
\[
\bar{f}^{n+1} \bar{g} = 0 \in \mathcal{O}_{\bar{S}_k, z}.
\]
Since \( \bar{S}_k \) is integral and \( \bar{f} \neq 0 \), this implies \( \bar{g} = 0 \), and hence \( g \in \pi \mathcal{O}_{\bar{S}_e, z} \). In particular, \( f^{n+1} \) lies in \( \pi I^n_{Z \cap \bar{S}} \). Therefore, \((2.4)\) is indeed an equality.

Because of the natural short exact sequence
\[
0 \to \mathcal{O}_{\bar{S}} \xrightarrow{\cdot \pi} I_{Z \cap \bar{S}} \to I_{Z \cap \bar{S}_k} \to 0,
\]
the induced map \( I^n_{Z \cap \bar{S}} / I^{n+1}_{Z \cap \bar{S}} \to I^n_{Z \cap \bar{S}_k} / I^{n+1}_{Z \cap \bar{S}_k} \) is surjective with kernel
\[
\left( I^n_{Z \cap \bar{S}} \cap \pi \mathcal{O}_{\bar{S}} \right) / \left( I^{n+1}_{Z \cap \bar{S}} \cap \pi \mathcal{O}_{\bar{S}} \right).
\]
As \((2.4)\) is an equality, this kernel is \( \pi I^{n-1}_{Z \cap \bar{S}} / \pi I^n_{Z \cap \bar{S}} \). Thus the sequence
\[
0 \to I^{n-1}_{Z \cap \bar{S}} / I^n_{Z \cap \bar{S}} \xrightarrow{\cdot \pi} I^n_{Z \cap \bar{S}} / I^{n+1}_{Z \cap \bar{S}} \to I^n_{Z \cap \bar{S}_k} / I^{n+1}_{Z \cap \bar{S}_k} \to 0
\]
is exact. The proposition follows from this by induction over \( n \), using part (ii) of Lemma 2.6 for the case \( n = 0 \) and for the induction step. \( \square \)
We continue in the setting of Section 2 and construct certain algebraic groups naturally associated to the Picard group of \( \bar{S}_k \). Since \( \bar{S}, \bar{S}_K \) and \( \bar{S}_k \) are obtained by the same sequence of blow-ups of a \( \mathbb{P}^2 \), or all as \( \mathbb{P}^1 \)-bundles over \( \mathbb{P}^1 \), the canonical restriction maps

\[
\text{Pic}(\bar{S}_K) \leftarrow \text{Pic}(\bar{S}) \rightarrow \text{Pic}(\bar{S}_k)
\]

are isomorphisms; we denote this abelian group by \( \Lambda \). The canonical bundles of \( \bar{S}, \bar{S}_K \) and \( \bar{S}_k \) define the same class in \( \Lambda \) due to Lemma 2.1; we denote it by \( K_{\bar{S}} \in \Lambda \).

The intersection forms on \( \bar{S}_K \) and on \( \bar{S}_k \) define the same bilinear form \( (\cdot, \cdot) \) on \( \Lambda \). Let \( \Lambda^\vee \) be the dual of \( \Lambda \), and denote the canonical pairing between \( \Lambda^\vee \) and \( \Lambda \) by \( \langle \cdot, \cdot \rangle \). The root system of the smooth del Pezzo surface \( \bar{S}_K \) is the set

\[
\Phi_0 := \{ \alpha \in \Lambda \mid (\alpha, \alpha) = -2, \ (\alpha, -K_{\bar{S}}) = 0 \}
\]

(3.1)
of \((-2)\)-classes in \( \Lambda \). It has type \( \mathbf{E}_8, \mathbf{E}_7, \mathbf{E}_6, \mathbf{D}_5, \mathbf{A}_4, \mathbf{A}_2+\mathbf{A}_1, \mathbf{A}_1 \) for \( d = 1, 2, 3, 4, 5, 6, 7 \), respectively, and type \( \mathbf{A}_1 \) for \( d = 8 \) in the case (ii) of \( \mathbb{P}^1 \)-bundles. Otherwise, \( \Phi_0 = \emptyset \).

Let \( \Phi \subset \Phi_0 \) be the set of \((-2)\)-classes that are effective or anti-effective on \( \bar{S}_k \). Put

\[
\Phi^\vee := \{ \alpha^\vee \in \Lambda^\vee \mid \alpha \in \Phi \},
\]

where \( \alpha^\vee \in \Lambda^\vee \) is defined by \( \langle \alpha^\vee, x \rangle := -(\alpha, x) \). Then a simple computation shows that \( (\Lambda, \Phi, \Lambda^\vee, \Phi^\vee) \) is a reduced root datum in the sense of [17, Exposé XXI, Déf. 1.1.1, 2.1.3]. Let \( G \) be the associated split reductive group over \( R \) [17, Exposé XXV, Cor. 1.2]. Then the commutator subgroup \( [G, G] \) is a semisimple group over \( R \) whose Dynkin diagram has the same type as the singularities of \( S_k \). The dimension of the maximal torus quotient \( G/[G, G] \) is \( 10 - d \) minus the rank of this Dynkin diagram.

Let \( g \) be the Lie algebra of \( G \), with root spaces \( g_\alpha \subset g \) for \( \alpha \in \Phi \). The maximal torus \( T \) of \( G \) has character group \( \Lambda \). Therefore, \( T, T_K, \) and \( T_k \) are the Néron–Severi tori of \( \bar{S}, \bar{S}_K \), and \( \bar{S}_k \), respectively. Let \( B \) be the Borel subgroup of \( G \) containing \( T \) such that the associated set \( \Delta \) of simple roots in \( \Phi \) is the set of classes of the \((-2)\)-curves on \( \bar{S}_k \). Let \( t \) and \( b \) be the Lie algebras of \( T \) and of \( B \), respectively. The corresponding set \( \Phi^+ \) of positive roots consists precisely of the effective \((-2)\)-classes on \( \bar{S}_k \).

By a universal torsor over \( \bar{S} \), we mean a \( T \)-torsor \( T \) such that the \( \mathbb{G}_m \)-torsor \( \lambda \cdot T \) over \( \bar{S} \) obtained by extension of structure group along every character \( \lambda \in \Lambda = \text{Hom}(T, \mathbb{G}_m) \) has class \( \lambda \in \Lambda = \text{Pic}(\bar{S}) \). Such universal torsors exist and are unique up to isomorphism because \( T \cong G_{10-d}^\text{in} \).

Remark 3.1. The notion of universal torsor has been defined by Colliot-Thélène and Sansuc [11, (2.0.4)] over base fields and, more generally, by Salberger [38, Def. 5.14] over Noetherian base schemes. Our definition is a special case of Salberger’s.

Let \( T \) be a universal torsor over \( \bar{S} \). Then the line bundle \( L_\alpha := T \times^T g_\alpha \) on \( \bar{S} \) has class \( \alpha \in \Lambda = \text{Pic}(\bar{S}) \) for each \( \alpha \in \Phi \).

Lemma 3.2. For \( \alpha \in \Phi^+ \), the \( R \)-module \( H^1(\bar{S}, L_\alpha) \) is non-zero, cyclic and torsion (hence isomorphic to \( R/m^{n_\alpha} \) for some \( n_\alpha \geq 1 \)), the canonical map

\[
H^1(\bar{S}, L_\alpha) \otimes_R k \rightarrow H^1(\bar{S}_k, L_{\alpha,k})
\]

(3.2)
is an isomorphism, and \( H^0(\bar{S}, L_\alpha) = H^2(\bar{S}, L_\alpha) = 0 \).

Proof. Since \( \alpha \) is effective on \( \bar{S}_k \), we know that

\[
\dim H^i(\bar{S}_k, L_{\alpha,k}) = \begin{cases} 1 & \text{for } i = 0, 1, \\ 0 & \text{for } i = 2. \end{cases}
\]

(3.3)
Indeed, \( p_d(\bar{S}_k) = p_d(P^2) = 0 \) since the arithmetic genus is a birational invariant, and hence the Riemann–Roch formula gives

\[
\chi(L_{\alpha, k}) = 0.
\]

The class \( K_{\bar{S}} - \alpha \) has intersection number \(-d < 0\) with the nef class \(-K_{\bar{S}}\), and is therefore not effective. Consequently, Serre duality gives

\[
H^2(\bar{S}_k, L_{\alpha, k}) = 0.
\]

Since the anticanonical morphism \( \bar{S}_k \to S_k \) is birational, there are only finitely many curves on \( \bar{S}_k \) whose intersection number with \(-K_{\bar{S}}\) is 0. But every curve of class \( \alpha \) has this property, which implies

\[
\dim H^0(\bar{S}_k, L_{\alpha, k}) \leq 1.
\]

As \( \alpha \) is effective on \( \bar{S}_k \), we get \( H^0(\bar{S}_k, L_{\alpha, k}) \cong k \), and hence also

\[
H^1(\bar{S}_k, L_{\alpha, k}) \cong k. \quad (3.4)
\]

Over \( K \) instead of \( k \), the same arguments apply, but \( \alpha \) is not effective over \( \bar{S}_K \), and therefore

\[
H^i(\bar{S}_K, L_{\alpha, K}) = 0 \quad (3.5)
\]

for \( i = 0, 1, 2 \).

This implies that \( H^1(\bar{S}, L_{\alpha}) \) is torsion, and \( H^2(\bar{S}, L_{\alpha}) = 0 \) by Grauert’s Theorem [30, Cor. III.12.9]. Each section of the line bundle \( L_{\alpha} \) vanishes on the generic fiber \( \bar{S}_K \), and hence on \( \bar{S} \). Therefore, \( H^0(\bar{S}, L_{\alpha}) = 0 \).

Applying Cohomology and Base Change, we consider the natural maps

\[
\varphi^i : H^i(\bar{S}, L_{\alpha}) \otimes_R k \to H^i(\bar{S}_k, L_{\alpha, k}).
\]

For \( i = 2 \), both sides vanish. Using [30, Thm. III.12.11] twice, we conclude first that \( \varphi^1 \) is surjective, and then that \( \varphi^1 \) is an isomorphism. Due to (3.4), this implies that \( H^1(\bar{S}, L_{\alpha}) \) is non-zero and cyclic. \( \square \)

For \( \alpha, \beta \in \Phi^+ \) with \( \alpha + \beta \in \Phi^+ \), the Lie bracket \([\_\_\_] : \mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}\) induces a morphism

\[
[\_\_\_] : L_{\alpha} \otimes L_{\beta} \to L_{\alpha+\beta}. \quad (3.6)
\]

**Lemma 3.3.** Let \( \alpha \in \Delta \) and \( \beta \in \Phi^+ \) such that \( \alpha + \beta \in \Phi^+ \). Then the cup product

\[
k \otimes_k k \cong H^0(\bar{S}_k, L_{\alpha, k}) \otimes_k H^1(\bar{S}_k, L_{\beta, k}) \to H^i(\bar{S}_k, L_{\alpha+\beta, k}) \cong k
\]

induced by (3.6) is non-zero for \( i = 0, 1 \).

**Proof.** Choose a non-zero section

\[
s \in H^0(\bar{S}_k, L_{\alpha, k}).
\]

Then \( s \) vanishes precisely on a \((-2)\)-curve \( D \subset \bar{S}_k \), and the sequence

\[
0 \to L_{\beta, k} \xrightarrow{[s, \_]} L_{\alpha+\beta, k} \to L_{\alpha+\beta} | D \to 0
\]

(3.7)
of coherent sheaves on $\tilde{S}_k$ is exact. The line bundle $L_{\alpha + \beta}|_D$ on $D \cong \mathbb{P}^1_k$ has degree $(\alpha, \alpha + \beta) = -2 + 1 = -1$ since $(\alpha, \beta) = 1$; consequently,

$$H^i(D, L_{\alpha + \beta}|_D) \cong H^i\left(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-1)\right) = 0$$

for $i = 0, 1$. In the long exact cohomology sequence resulting from (3.7),

$$H^i\left(\tilde{S}_k, L_{\alpha + \beta,k}\right) \longmapsto H^i\left(\tilde{S}_k, L_{\alpha,k}\right) \cong H^i(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-1)) = 0$$

is therefore an isomorphism for $i = 0, 1$.

The next step is to lift our universal torsor $T$ to a $B$-torsor $B$ over $\tilde{S}$. We construct $B$ as follows.

For $\alpha \in \Phi^+$, let $U_\alpha \cong G_{a,R}$ be the associated root group in $B$. Let $U_{\geq 2}$ be the subgroup of $B$ generated by all $U_\alpha$ with $\alpha \not\in \Delta$, and put $B_{\leq 1} := B/U_{\geq 2}$. We have the exact sequence

$$0 \to U_{\leq 1} := \bigoplus_{\alpha \in \Delta} U_\alpha \to B_{\leq 1} \to T \to 0. \tag{3.8}$$

Here $T$ acts on $U_{\leq 1}$ by conjugation. Associated to the $T$-torsor $T$ over $\tilde{S}$, we thus obtain a fibration over $\tilde{S}$ with fiber $U_{\leq 1}$. This group scheme over $\tilde{S}$ is by construction the underlying additive group scheme of $\bigoplus_{\alpha \in \Delta} L_{\alpha}$.

We will first lift $T$ to a $B_{\leq 1}$-torsor $B_{\leq 1}$ over $\tilde{S}$. This is possible because (3.8) comes with a splitting $T \to B_{\leq 1}$, and the lifts $B_{\leq 1}$ are parameterized by

$$H^1\left(\tilde{S}, \bigoplus_{\alpha \in \Delta} L_{\alpha}\right).$$

To make this precise, we consider the commutative diagram

$$\begin{array}{cccccc}
0 & \to & \bigoplus_{\alpha \in \Delta} g_\alpha & \to & b_{\leq 1} = t \oplus \bigoplus_{\alpha \in \Delta} g_\alpha & \to & t & \to & 0 \\
& & \downarrow \text{pr}_\alpha & & \downarrow \text{ad}_t \oplus \text{pr}_\alpha & & \downarrow \text{ad}_t \\
0 & \to & g_\alpha & \to & \text{End}(g_\alpha) \oplus g_\alpha & \to & \text{End}(g_\alpha) & \to & 0
\end{array}$$

of $B_{\leq 1}$-modules, where the upper exact sequence consists of the Lie algebras of (3.8), and $\text{ad}_t$ sends $t \in t$ to $[t, -] : g_\alpha \to g_\alpha$. Given one lift $B_{\leq 1}$, we obtain an associated commutative diagram

$$\begin{array}{cccccc}
0 & \to & \bigoplus_{\alpha \in \Delta} L_{\alpha} & \to & \text{ad}(B_{\leq 1}) & \to & \Lambda^\vee \otimes_{\mathbb{Z}} \mathcal{O}_{\tilde{S}} & \to & 0 \\
& & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
0 & \to & L_{\alpha} & \to & B_{\leq 1} \times_{B_{\leq 1}} \text{End}(g_\alpha) \oplus g_\alpha & \to & \mathcal{O}_{\tilde{S}} & \to & 0
\end{array} \tag{3.9}$$

of vector bundles over $\tilde{S}$. We denote the extension class of the lower exact sequence by

$$c_\alpha \in H^1\left(\tilde{S}, L_{\alpha}\right). \tag{3.10}$$

The classes $c_\alpha$ for $\alpha \in \Delta$ classify the lift $B_{\leq 1}$ (see [32, Prop. 3.1.ii], for example).

We choose a particular lift $B_{\leq 1}$ such that, for each $\alpha \in \Delta$, the component $c_\alpha$ of the class of $B_{\leq 1}$ generates $H^1\left(\tilde{S}, L_{\alpha}\right)$ as an $R$-module. This is possible by Lemma 3.2.
Lemma 3.4. Let $B'_{\leq 1}$ be another lift of $T$ such that $H^1(\bar{S}, L_\alpha)$ is also generated by the component $c'_\alpha$ of the class of $B'_{\leq 1}$ for each $\alpha \in \Delta$. Then there is an automorphism $\sigma$ of $G$ with $\sigma|_T = \text{id}_T$ such that the extension of structure group of $B_{\leq 1}$ along $\sigma|_{B_{\leq 1}} : B_{\leq 1} \rightarrow B_{\leq 1}$ is isomorphic to $B'_{\leq 1}$ as a lift of $T$.

Proof. Since $c_\alpha$ and $c'_\alpha$ both generate $H^1(\bar{S}, L_\alpha)$, we have $c'_\alpha = \lambda_\alpha c_\alpha$ with $\lambda_\alpha \in R^\times$ for each $\alpha \in \Delta$. According to [17, Exposé XXIII, Th. 4.1], there is a unique automorphism $\sigma$ of $G$ with $\sigma|_T = \text{id}_T$ such that $\sigma$ acts on $\mathfrak{g}_\alpha$ as multiplication by $\lambda_\alpha$ for each $\alpha \in \Delta$. This implies

$$(\sigma|_{B_{\leq 1}})_{\|} B_{\leq 1} \cong B'_{\leq 1}$$

as lifts of $T$. \qed

Remark 3.5. This automorphism $\sigma$ of $G$ can be described as follows. The action of $G$ on itself by conjugation descends to an action of $G^\text{ad} := G/Z$ on $G$, where $Z \subset G$ is the (scheme-theoretic) center. The subgroup $T^\text{ad} := T/Z$ of $G^\text{ad}$ acts trivially on $T$. Since $\Delta$ is a basis of the lattice $\text{Hom}(T^\text{ad}, \mathbb{G}_m)$, there is a unique point $t \in T^\text{ad}(R)$ such that $\alpha(t) = \lambda_\alpha$ for each $\alpha \in \Delta$. Conjugation by this point $t$ is the required automorphism $\sigma$ of $G$.

Lemma 3.6. The $B_{\leq 1}$-torsor $B_{\leq 1}$ can be lifted to a $B$-torsor $B$ over $\bar{S}$.

Proof. Let $\Phi^+_n (\text{resp. } \Phi^+_{\geq n})$ be the set of all $\alpha \in \Phi^+$ that are sums of precisely (resp. at least) $n$ not necessarily distinct simple roots. Generalizing the above notation, we let $U_{\geq n}$ be the subgroup of $B$ generated by all $U_\alpha$ with $\alpha \in \Phi^+_{\geq n}$, and put $B_{\leq n} := B/U_{\geq n+1}$. We have the exact sequences

$$0 \rightarrow U := n := \bigoplus_{\alpha \in \Phi^+_{= n}} U_\alpha \rightarrow B_{\leq n} \rightarrow B_{\leq n-1} \rightarrow 0,$$

in which $B_{\leq n} = B/U_{\geq n+1}$ acts on $U := n$ by conjugation. Here $U_{\geq 1}/U_{\geq n+1}$ acts trivially, because $[U_{\geq 1}, U_{\geq n}] \subset U_{\geq n+1}$. Therefore, the action descends to an action of $B/U_{\geq 1} = T$ on $U := n$. Associated to the $T$-torsor $T$ over $\bar{S}$, we thus obtain a fibration over $\bar{S}$ with fiber $U := n$. This group scheme over $\bar{S}$ is by construction the underlying additive group scheme of $\bigoplus_{\alpha \in \Phi^+_{= n}} L_\alpha$.

Using induction, we assume that $B_{\leq 1}$ can be lifted to a $B_{\leq n-1}$-torsor $B_{\leq n-1}$ for some $n \geq 2$. We try to lift $B_{\leq n-1}$ to a $B_{\leq n}$-torsor $B_{\leq n}$ along the exact sequence (3.11). The obstruction against such a lift is an element in

$$H^2(\bar{S}, \bigoplus_{\alpha \in \Phi^+_{= n}} L_\alpha)$$

(see [32, Prop. 3.1.i]). This cohomology vanishes by Lemma 3.2.

For sufficiently large $n$, we have $B_{\geq n} = B$, and $B := B_{\geq n}$ is the required lift of $B_{\leq 1}$. \qed

According to [42, 12.12], we can choose a nonzero element $x_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi^+$ such that

$$[x_\alpha, x_\beta] = \epsilon_{\alpha, \beta} x_\gamma$$

with $\epsilon_{\alpha, \beta} \in \{-1, 1\}$ for all $\alpha, \beta, \gamma \in \Phi^+$ with $\alpha + \beta = \gamma$.

Lemma 3.7. There are classes $e_\alpha \in H^0(\bar{S}, L_\alpha)$ for $\alpha \in \Phi^+$ and $f_\alpha \in H^1(\bar{S}, L_\alpha)$ for $\alpha \in \Phi^+$ such that

(i) $[e_\alpha, e_\beta] = \epsilon_{\alpha, \beta} e_\gamma$ for all $\alpha, \beta, \gamma \in \Phi^+$ with $\alpha + \beta = \gamma$,

(ii) $[e_\alpha, f_\beta] = \epsilon_{\alpha, \beta} f_\gamma$ for all $\alpha, \beta, \gamma \in \Phi^+$ with $\alpha + \beta = \gamma$, and

(iii) $f_\alpha = e_{\alpha, k}$ is the restriction of the class $e_\alpha$ from (3.10) for all $\alpha \in \Delta$. 

Proof. We choose a point \( p \in \tilde{S}(k) \) outside the \((-2)\)-curves, and a point \( \tau \in \mathcal{T}(k) \) above \( p \). The trivialization \( \tau \) of \( \mathcal{T} \) above \( p \) induces an isomorphism \( L_{\alpha, p} \rightarrow q_\alpha \) for each \( \alpha \in \Phi^+ \). We define \( e_\alpha \in H^0(\tilde{S}_k, L_{\alpha, k}) \) as the unique section whose value at \( p \) maps to \( x_\alpha \) under this isomorphism. Then (i) holds by construction because of (3.12).

For each irreducible component of \( \Phi \), consider its highest root \( \delta \), and choose a nonzero \( f_\delta \in H^1(\tilde{S}_k, L_{\delta, k}) \). Let \( \alpha \) be a positive root in the same component. Since the anticanonical morphism \( \tilde{S}_k \rightarrow S_k \) is birational, there are only finitely many curves on \( \tilde{S}_k \) whose intersection number with \( -K_{\tilde{S}} \) is 0. But every curve of class \( \delta - \alpha \) has this property, which implies

\[
\dim \text{Hom}(L_{\alpha, k}, L_{\delta, k}) \leq 1.
\]

On the other hand, the divisor class \( \delta - \alpha \) contains a sum of \((-2)\)-curves. Hence there is a unique morphism \( \phi_\alpha : L_{\alpha, k} \rightarrow L_{\delta, k} \) whose restriction to \( p \) sends \( x_\alpha \) to \( x_\delta \). The induced map

\[
H^1(\tilde{S}_k, L_{\alpha, k}) \rightarrow H^1(\tilde{S}_k, L_{\delta, k})
\]

is bijective because of Lemma 2.4 and Lemma 3.3. We define

\[
f_\alpha \in H^1(\tilde{S}_k, L_{\alpha, k})
\]

as the inverse image of \( f_\delta \).

Assume that \( \alpha + \beta = \gamma \) with \( \beta, \gamma \in \Phi^+ \). Then \( \beta \) and \( \gamma \) lie in the same irreducible component of \( \Phi \) as \( \alpha \) and \( \delta \). The diagram

\[
\begin{array}{ccc}
L_\beta & \overset{e_\alpha^{-1}[e_\alpha \cdots]}{\longrightarrow} & L_\gamma \\
\phi_\beta \downarrow & & \phi_\gamma \\
L_\delta & &
\end{array}
\]

commutes, as can be seen by evaluating at \( p \) and using \( e_\alpha^{-1}[x_\alpha, x_\beta] = x_\gamma \). Therefore, \( e_\alpha^{-1}[e_\alpha, f_\beta] = f_\gamma \), which proves (ii).

We have \( c_{\alpha, k} = \lambda_\alpha f_\alpha \) with \( \lambda_\alpha \in k^\times \) for each \( \alpha \in \Delta \) by construction. For \( \alpha = \sum_i \alpha_i \) with all \( \alpha_i \in \Delta \), we put \( \lambda_\alpha : = \prod_i \lambda_{\alpha_i} \in k^\times \). Replacing \( e_\alpha \) by \( \lambda_\alpha e_\alpha \) and \( f_\alpha \) by \( \lambda_\alpha f_\alpha \) for \( \alpha \in \Phi^+ \) preserves (i) and (ii) and ensures (iii). \( \square \)

Recall from [43, I.4] that a prime \( p \) is good for an irreducible root system if \( p \) does not occur as a coefficient of the highest root. A good prime \( p \) is very good if moreover \( p \) does not divide the determinant of the Cartan matrix. For the simply laced root systems, the very good primes are as follows.

\[
\begin{align*}
A_r &: \text{all } p \mid r + 1, \quad D_r &: \text{all } p \neq 2, \quad E_6, E_7 &: \text{all } p \neq 2, 3, \quad E_8 &: \text{all } p \neq 2, 3, 5.
\end{align*}
\]

The field \( k \) has very good characteristic for \( \Phi \) if \( \text{char}(k) = 0 \) or a very good prime for every irreducible component of \( \Phi \).

Lemma 3.8. Given \( \beta \in \Phi^+_{n-1} \) and \( \gamma \in \Phi^+_{n} \), we consider the map

\[
H^0(\tilde{S}_k, L_{\beta, k}) \rightarrow H^1(\tilde{S}_k, L_{\gamma, k})
\]

(3.14)

given by \( \lfloor \cdot, c_{\alpha, k} \rfloor \) if \( \gamma - \beta \) is a simple root \( \alpha \), and by 0 otherwise. If \( k \) has very good characteristic for \( \Phi \), then the sum

\[
\bigoplus_{\beta \in \Phi^+_{n-1}} H^0(\tilde{S}_k, L_{\beta, k}) \rightarrow \bigoplus_{\gamma \in \Phi^+_{n}} H^1(\tilde{S}_k, L_{\gamma, k})
\]

(3.15)

of all these maps is surjective for \( n \geq 2 \).
Proof. Choose $e_\alpha, f_\alpha$ as in Lemma 3.7. Given $\beta \in \Phi_{=n-1}^+$ and $\gamma \in \Phi_{=n}^+$ with $\gamma - \beta = \alpha \in \Delta$, the map (3.14) is given by the $(1 \times 1)$-matrix $(e_\beta, \alpha)$ with respect to the bases $\{e_\beta\}$ and $\{f_\gamma\}$ by Lemma 3.7(ii).

Therefore, the matrix of (3.15) with respect to the bases $\{e_\beta \mid \beta \in \Phi_{=n-1}^+\}$ and $\{f_\gamma \mid \gamma \in \Phi_{=n}^+\}$ has entries $e_{\beta, \alpha}$ whenever $\gamma - \beta = \alpha \in \Delta$, and 0 otherwise. If $\Phi$ is reducible, then these matrices are block diagonal. Computing the ranks of all these matrices for all possible irreducible root systems shows that the maps in question are surjective in very good characteristic.

□

Proposition 3.9. Assume that $k$ has very good characteristic for $\Phi$. Let the $B$-torsor $B$ over $\widetilde{S}$ be an arbitrary lift of $B_{\leq 1}$.

(i) Up to isomorphism of torsors, the restriction $B_k := B|_{\tilde{S}_k}$ does not depend on the choice of $B$.

(ii) The adjoint vector bundle $\text{ad}(B_k) \to \tilde{S}_k$ satisfies $H^1(\tilde{S}_k, \text{ad}(B_k)) = 0$.

Proof. Considering the restricted $B_{\leq n}$-torsor $B_{\leq n, k} := B_{\leq n}|_{\tilde{S}_k}$ over $\tilde{S}_k$, we argue by induction over $n$.

For the proof of (ii), the construction of $B_{\leq 1}$ provides us with an exact sequence

$$0 \to \bigoplus_{\alpha \in \Delta} L_{\alpha, k} \to \text{ad}(B_{\leq 1, k}) \to \Lambda^\vee \otimes \mathbb{Z} \tilde{S}_k \to 0$$

of vector bundles over $\tilde{S}_k$. Since $H^0(\tilde{S}_k, \mathcal{O}_{\tilde{S}_k}) = k$ and $H^1(\tilde{S}_k, \mathcal{O}_{\tilde{S}_k}) = 0$, the resulting long exact cohomology sequence reads

$$\Lambda^\vee \otimes \mathbb{Z} k \delta \bigoplus_{\alpha \in \Delta} H^1(\tilde{S}_k, L_{\alpha, k}) \to H^1(\tilde{S}_k, \text{ad}(B_{\leq 1, k})) \to 0. \quad (3.16)$$

The connecting homomorphism $\delta$ is the sum over $\alpha \in \Delta$ of the compositions

$$\Lambda^\vee \otimes \mathbb{Z} k \xrightarrow{\alpha \otimes \text{id}_k} k \xrightarrow{c_{\alpha, k}} H^1(\tilde{S}_k, L_{\alpha, k}).$$

This composition is surjective, because $c_{\alpha, k}$ generates $H^1(\tilde{S}_k, L_{\alpha, k})$ for each $\alpha \in \Delta$ and

$$(\alpha \otimes \text{id}_k)_{\alpha \in \Delta} : \Lambda^\vee \otimes \mathbb{Z} \to \bigoplus_{\alpha \in \Delta} k$$

is surjective (since its restriction to the span of the coroots in $\Lambda^\vee$ is given by the Cartan matrix, which is invertible in $k$ by assumption). Consequently,

$$H^1(\tilde{S}_k, \text{ad}(B_{\leq 1, k})) = 0$$

according to the exact sequence (3.16).

By induction, we may assume the same for $B_{\leq n-1, k}$. We again have an exact sequence

$$0 \to \bigoplus_{\gamma \in \Phi_{=n}^+} L_{\gamma, k} \to \text{ad}(B_{\leq n, k}) \to \text{ad}(B_{\leq n-1, k}) \to 0. \quad (3.17)$$
of vector bundles over $\overline{S}_k$. Using the resulting long exact cohomology sequence, it remains to prove that its connecting homomorphism

$$H^0(\overline{S}_k, \text{ad}(B_{\leq n-1,k})) \to \bigoplus_{\gamma \in \Phi^+_n} H^1(\overline{S}_k, L_{\gamma,k})$$

(3.18)

is surjective. We consider its restriction

$$\delta : \bigoplus_{\beta \in \Phi^+_{n-1}} H^0(\overline{S}_k, L_{\beta,k}) \to \bigoplus_{\gamma \in \Phi^+_n} H^1(\overline{S}_k, L_{\gamma,k})$$

(3.19)

to the subbundle

$$\bigoplus_{\beta \in \Phi^+_{n-1}} L_{\beta,k} \subset \text{ad}(B_{\leq n-1,k}).$$

Choose $\beta \in \Phi^+_{n-1}$ and $\gamma \in \Phi^+_n$ such that $\gamma - \beta = \alpha \in \Delta$. Then the component

$$H^0(\overline{S}_k, L_{\beta,k}) \to H^1(\overline{S}_k, L_{\gamma,k})$$

(3.20)

of $\delta$ is the connecting homomorphism of the exact sequence

$$0 \to L_{\gamma} \to B_{\leq 1} \times^{B_{\leq 1}} (g_{\beta} \oplus g_{\gamma}) \to L_{\beta} \to 0$$

(3.21)

of vector bundles over $\overline{S}$ associated with the exact sequence

$$0 \to g_{\beta} \oplus g_{\gamma} \to g_{\beta} \oplus g_{\gamma} \to g_{\beta} \to 0$$

of $B_{\leq 1}$-modules. Let $[-, -] : g_{\beta} \otimes g_{\alpha} \to g_{\gamma}$ be the linear map that sends $x_{\beta} \otimes x_{\alpha}$ to $-[x_{\beta}, x_{\alpha}] = [x_{\alpha}, x_{\beta}]$. Then the isomorphism

of short exact sequences is $B_{\leq 1}$-equivariant. Therefore, it induces the isomorphism

$$0 \to L_{\beta} \otimes L_{\alpha} \to B_{\leq 1} \times^{B_{\leq 1}} (g_{\beta} \oplus (g_{\beta} \otimes g_{\alpha})) \to L_{\beta} \to 0$$

from the second exact sequence in (3.9) tensored with $L_{\beta}$ to (3.21). Comparing the classes of these exact sequences, we see that

$$-[\_ , \_] : L_{\beta} \otimes L_{\alpha} \to L_{\gamma}$$
sends \( c_\alpha \in H^1(\widetilde{S}, L_\alpha) = \text{Ext}^1(L_\beta, L_\beta \otimes L_\alpha) \) to the class of (3.21) in \( \text{Ext}^1(L_\beta, L_\gamma) \). This shows that the component \((3.20)\) of \( \delta \) in question is given by

\[ -[\ldots, c_\alpha, k] : H^0(\widetilde{S}_k, L_\beta, k) \rightarrow H^1(\widetilde{S}_k, L_\gamma, k). \]

Therefore, \( \delta \) is surjective according to Lemma 3.8. Hence (3.18) is also surjective. This proves (ii).

For the proof of (i), we may assume by induction that \( B_{\leq n-1,k} \) does not depend on the choice of \( B \). Since

\[ [U_{\geq n-1}, U_{\geq 1}] \subset U_{\geq n}, \]

the subgroup \( U_{n-1} \subset B_{\leq n-1} \) is normal, and the conjugation action of \( B_{\leq n-1} \) on \( U_{n-1} \) factors through the action of \( T \). This implies

\[ \text{Aut}(B_{\leq n-1,k}) \supset \bigoplus_{\beta \in \Phi_{n-1}^+} H^0(\widetilde{S}_k, L_\beta, k). \]

(3.22)

The set of lifts of \( B_{\leq n-1,k} \) to a \( B_{\leq n} \)-torsor is a torsor under the group

\[ \bigoplus_{\gamma \in \Phi_n^+} H^1(\widetilde{S}_k, L_\gamma, k). \]

(3.23)

This set comes with an action of \( \text{Aut}(B_{\leq n-1,k}) \), whose restriction to the subgroup in (3.22) is the homomorphism \( \delta \) in (3.19). As we have seen, \( \delta \) is surjective by Lemma 3.8. Hence \( \text{Aut}(B_{\leq n-1,k}) \) acts transitively on the set of lifts \( B_{\leq n,k} \). Thus \( B_{\leq n,k} \) does not depend on the choice of \( B \).

Since \( B_{\leq n} = B \) for sufficiently large \( n \), this proves part (i).

Remark 3.10. If \( \Phi \) has type \( D_4 \) and \( k \) has characteristic 2, then Lemma 3.8 is not true, and hence our proof of Proposition 3.9 does not work in this case.

In Section 4, we will give an example of a family \( S \) of cubic surfaces over a discrete valuation ring \( R \) with residue field \( k \) of characteristic 2 that has a \( D_4 \) singularity in the special fiber and for which Proposition 3.9 (ii) is false.

Lemma 3.11. The canonical group homomorphism

\[ \text{Aut}(B) \rightarrow \text{Aut}(T) = T(R) \cong (R^\times)^{10-d} \]

is injective, and its image contains the subgroup

\[ (1 + \mathfrak{m}^j)^{10-d} \subset (R^\times)^{10-d} \]

for any sufficiently large integer \( j \).

Proof. By construction of the torsors \( B_{\leq n} \), we have exact sequences

\[ 0 \rightarrow \bigoplus_{\alpha \in \Phi_n^+} H^0(\widetilde{S}, L_\alpha) \rightarrow \text{Aut}(B_{\leq n}) \rightarrow \text{Aut}(B_{\leq n-1}) \]

(3.24)

for \( n \geq 1 \), with \( B_{\leq 0} := T \). Here \( H^0(\widetilde{S}, L_\alpha) = 0 \) according to Lemma 3.2. Therefore, the canonical group homomorphisms

\[ \text{Aut}(B) = \text{Aut}(B_{\leq N}) \rightarrow \text{Aut}(B_{\leq N-1}) \rightarrow \cdots \rightarrow \text{Aut}(B_{\leq 0}) = \text{Aut}(T) \]
are all injective. The obstruction against lifting an automorphism of $B_{\leq n-1}$ to an automorphism of $B_{\leq n}$ is an element in

$$\bigoplus_{\alpha \in \Phi^*_n} H^1(S, L_{\alpha}).$$

Since this $R$-module has finite length by Lemma 3.2, automorphisms of $T$ that are congruent to the identity modulo $m^j$ for sufficiently large $j$ can be lifted step by step to automorphisms of each $B_{\leq n}$ and of $B$.

\[\square\]

**Proposition 3.12.** If $k$ has very good characteristic for $\Phi$, then all lifts $B$ of $B_{\leq 1}$ are isomorphic as $B$-torsors, and satisfy

$$H^1(S, \text{ad}(B)) = 0.$$

**Proof.** We compare the $B$-torsor $B$ chosen above to another lift $B'$ of $B_{\leq 1}$. Let $B'_{\leq n}$ and $T'$ denote the $B_{\leq n}$-torsor and the $T$-torsor induced by $B'$, respectively. Here $T$ and $T'$ are isomorphic, but we will use various isomorphisms between them.

Part (i) of Proposition 3.9 allows us to choose an isomorphism

$$\phi_1 : B_k = B_{|\bar{S}_k} \to B'_{|\bar{S}_k} = B'_k.$$

We claim that $\phi_1$ can be lifted to a compatible system of isomorphisms

$$\phi_i : B_{|\bar{S}_k} \to B'_{|\bar{S}_k}$$

for $i \geq 1$. Indeed, the obstruction against lifting $\phi_{i-1}$ to $\phi_i$ is an element in

$$H^1(\bar{S}_k, \text{ad}(B_k))$$

due to [33, Th. VII.2.4.4], since the normal bundle of $\bar{S}_k$ in $\bar{S}$ is trivial. Therefore, part (ii) of Proposition 3.9 allows us to lift $\phi_{i-1}$ to $\phi_i$. Let

$$\phi_{i, < 0} : T_{|\bar{S}_k} \to T'_{|\bar{S}_k} \quad \text{and} \quad \phi_{i, \leq n} : B_{|\bar{S}_k} \to B'_{\leq n, |\bar{S}_k}$$

denote the isomorphisms of torsors induced by $\phi_i$.

The choice of an isomorphism $T \to T'$ induces bijections

$$\text{Isom}(T, T') \cong T(R) \quad \text{and} \quad \text{Isom}\left(T_{|\bar{S}_k}, T'_{|\bar{S}_k}\right) \cong T(R/m^i)$$

for $i \geq 1$. Therefore, the restriction map

$$\text{Isom}(T, T') \to \text{Isom}\left(T_{|\bar{S}_k}, T'_{|\bar{S}_k}\right)$$

is surjective. We choose an integer $j$ which is sufficiently large in the sense of Lemma 3.11, and lift $\phi_{j, < 0}$ to an isomorphism

$$\psi_{\leq 0} : T \to T'. $$

Its restrictions $\psi_{\leq 0, i}$ to $i\bar{S}_k \subset \bar{S}$ satisfy by construction

$$\psi_{\leq 0, i} = \phi_{i, \leq 0} \quad \text{(3.25)}$$

for $i \leq j$. For $i > j$, these differ by an automorphism of $T$, which can be lifted to an automorphism of $B$ due to Lemma 3.11. Modifying $\phi_{j+1}, \phi_{j+2}, \ldots$ by these automorphisms of $B$, we can achieve (3.25) for all $i \geq 1$. 

We show by induction over $n$ that $\psi_{\leq 0}$ can be lifted to an isomorphism

$$\psi_{\leq n} : B_{\leq n} \to B'_{\leq n}$$

whose restrictions $\psi_{\leq n,i}$ to $i\tilde{S}_k \subset \tilde{S}$ satisfy

$$\psi_{\leq n,i} = \phi_{i,\leq n}.$$  \hfill (3.26)

The obstruction against lifting $\psi_{\leq n-1}$ to an isomorphism $\psi_{\leq n}$ lies in

$$\bigoplus_{\alpha \in \Phi^+} H^1(\tilde{S}, L_{\alpha}).$$

The restriction of this class to $i\tilde{S}_k \subset \tilde{S}$ vanishes, because $\psi_{\leq n-1,i} = \phi_{i,\leq n-1}$ admits the lift $\phi_{i,\leq n}$. But the canonical map

$$H^1(\tilde{S}, L_{\alpha}) \to \lim\longrightarrow H^1\left(i\tilde{S}_k, L_{\alpha}|_{i\tilde{S}_k}\right)$$

is bijective by the Theorem on Formal Functions [30, Thm. III.11.1] and Lemma 3.2. Therefore, we can lift $\psi_{\leq n-1}$ to an isomorphism $\psi_{\leq n}$. Its restrictions $\psi_{\leq n,i}$ differ from the isomorphisms $\phi_{i,\leq n}$ by an element of

$$\lim\longrightarrow \bigoplus_{\alpha \in \Phi^+} H^0\left(i\tilde{S}_k, L_{\alpha}|_{i\tilde{S}_k}\right).$$

But any such element vanishes, because

$$\lim\longrightarrow H^0\left(i\tilde{S}_k, L_{\alpha}|_{i\tilde{S}_k}\right) \cong H^0(\tilde{S}, L_{\alpha}) = 0$$

according to the Theorem on Formal Functions [30, Thm. III.11.1] and Lemma 3.2 again. This proves that the chosen lift $\psi_{\leq n}$ automatically satisfies (3.26), which completes the induction. Taking $n$ sufficiently large, $\psi_{\leq n}$ is the required isomorphism from $B_{\leq n} = B$ to $B'_{\leq n} = B'$.

Infinitesimal rigidity follows from Proposition 3.9 (ii) using the Semicontinuity Theorem [30, Thm. III.12.8] and Grauert’s Theorem [30, Cor. III.12.9].

We still assume that the $B$-torsor $B$ over $S$ is a lift of $B_{\leq 1}$. Extending the structure group of $B$ to $G$, we obtain a $G$-torsor $G = B \times^B G$ over $S$.

**Corollary 3.13.** If $k$ has very good characteristic for $\Phi$, then $H^1(\tilde{S}, \text{ad}(G)) = 0$.

**Proof.** The vector bundle $\text{ad}(B) = B \times^B b$ is associated with the $B$-torsor $B$ and the $B$-module $b$. Similarly, $\text{ad}(G) = G \times^G g = B \times^B g$ is associated with $B$ and the $B$-module $g$. The $B$-module $g/b$ has a composition series with composition factors $g_{-\alpha}$ for $\alpha \in \Phi^+$. Therefore, the associated vector bundle $\text{ad}(G)/\text{ad}(B)$ has a composition series with composition factors $B \times^B g_{-\alpha} \cong L_{-\alpha}$. Using induction over this composition series and

$$H^0(\tilde{S}, L_{-\alpha}) = H^1(\tilde{S}, L_{-\alpha}) = 0$$

for $\alpha \in \Phi^+$, we conclude that the natural map

$$H^1(\tilde{S}, \text{ad}(B)) \to H^1(\tilde{S}, \text{ad}(G))$$  \hfill (3.27)

is an isomorphism. \hfill \Box

**Proposition 3.14.** The $G$-torsor $G$ is trivial on every $(-2)$-curve $D \subset \tilde{S}_k$. 


Proof. Let \( \alpha \in \Delta \) be the class of \( D \). The restriction \( L_\alpha |D \) is a line bundle of degree \( (\alpha, \alpha) = -2 \) on \( D \cong \mathbb{P}^1_k \), which implies

\[
H^1(D, L_\alpha |D) \cong H^1\left( \mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2) \right) \cong k. \tag{3.28}
\]

Tensoring the short exact sequence

\[
0 \to \mathcal{O}_{\tilde{S}_k}(-D) \to \mathcal{O}_{\tilde{S}_k} \to \mathcal{O}_D \to 0
\]

with the line bundle \( L_\alpha, k \cong \mathcal{O}_{\tilde{S}_k}(D) \), we get a short exact sequence

\[
0 \to \mathcal{O}_{\tilde{S}_k} \to L_\alpha, k \to L_\alpha |D \to 0
\]

of coherent sheaves on \( \tilde{S}_k \). Since \( H^i(\tilde{S}_k, \mathcal{O}_{\tilde{S}_k}) \) vanishes for \( i = 1, 2 \) by their birational invariance [30, Prop. V.3.4], the associated long exact cohomology sequence shows that the restriction homomorphism

\[
H^1(\tilde{S}_k, L_\alpha, k) \to H^1(D, L_\alpha |D)
\]

is bijective. For \( \beta \in \Phi^+ \) with \( \beta \neq \alpha \), the degree of \( L_\beta |D \) on \( D \cong \mathbb{P}^1_k \) is

\[
(\alpha, \beta) = -\langle \beta^\vee, \alpha \rangle =: n \in \{-1, 0, 1\},
\]

because \( \alpha \neq \beta \) are roots in the simply laced root system \( \Phi \). This implies

\[
H^1(D, L_\beta |D) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) = 0. \tag{3.30}
\]

Let \( G_\alpha \subset G \) be the split reductive subgroup with the same maximal torus \( T \) and only the two roots \( \pm \alpha \). Then \( B_\alpha := B \cap G_\alpha \) sits in an exact sequence

\[
0 \to U_\alpha \to B_\alpha \to T \to 0.
\]

Let the \( B_\alpha \)-torsor \( B_{\alpha} \) on \( \tilde{S} \) be the lift of the \( T \)-torsor \( T \) corresponding to the class \( c_\alpha \) chosen in (3.10). Let \( G_\alpha \) be the \( G_\alpha \)-torsor over \( \tilde{S} \) induced by \( B_{\alpha} \).

The \( B \)-torsor induced by \( B_{\alpha} \) becomes isomorphic to \( B \) when both are restricted to \( D \), because there the lifting over each \( U_\beta \) with \( \beta \in \Phi^+ \setminus \{\alpha\} \) is unique by (3.30). Hence it suffices to prove that \( G_\alpha \) is trivial on \( D \).

In the case (i) of blow-ups of \( \mathbb{P}^2 \), [16, II.2(6)] shows that \( \alpha = e_1 - e_2 \) for two classes \( e_i \in \Lambda \) satisfying \( (e_i, e_j) = -1 \) and \( (e_1, e_2) = 0 \). Since their intersection matrix

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]

is invertible over \( \mathbb{Z} \), we can extend \( e_1, e_2 \) to a basis \( e_1, \ldots, e_{10-d} \) of \( \Lambda \) with \( (e_i, e_j) = (e_j, e_i) = 0 \) for all \( i \geq 3 \).

In the case (ii) of \( P^1 \)-bundles over \( \mathbb{P}^1 \), with \( d = 8 \), we have \( \alpha = e_1 - e_2 \), where \( e_1 \) and \( e_2 \) in \( \Lambda = \text{Pic}(\tilde{S}) \) restrict to the classes of fiber and constant section in \( \tilde{S}_k = \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k \), respectively. Here, \( (e_i, e_j) = 0 \) and \( (e_1, e_2) = 1 \). We note that \( e_1, e_2 \) is a basis of \( \Lambda \).

In both cases (i) and (ii), we have \( \alpha = e_1 - e_2 \) and

\[
\langle \alpha^\vee, e_i \rangle = (-\alpha, e_i) = \begin{cases} 1 & \text{for } i = 1, \\ -1 & \text{for } i = 2, \\ 0 & \text{for } i \geq 3. \end{cases}
\]

These descriptions of \( \alpha \) and \( \alpha^\vee \) allow us to extend the decomposition \( T \cong G_{10-d} \) given by \( e_1, \ldots, e_{10-d} \) to a decomposition

\[
G_\alpha \cong \text{GL}_2 \times G_{8-d}.
\]

This also induces a decomposition of \( B_\alpha \).
Let $L_{e_i}$ be a line bundle on $\tilde{S}$ of class $e_i \in \text{Pic}(\tilde{S}) = \Lambda$. Under the above decompositions, the $B_\alpha$-torsor $B_\alpha$ corresponds to the $10-d$ line bundles $L_{e_i}$ over $\tilde{S}$ and the vector bundle extension

$$0 \to L_{e_1} \to E \to L_{e_2} \to 0$$

of class $c_\alpha \in \text{Ext}^1(L_{e_2}, L_{e_1}) \cong H^1(\tilde{S}, L_\alpha)$, and the $G_\alpha$-torsor $G_\alpha$ corresponds to the vector bundle $E$ and the line bundles $L_{e_1}, \ldots, L_{e_{10-d}}$ over $\tilde{S}$.

The restriction of $L_{e_i}$ to $D \cong \mathbb{P}^1_k$ is a line bundle of degree $(\alpha, e_i)$. For $i \geq 3$, we have $(\alpha, e_i) = 0$, and therefore $L_{e_i}|_D$ is trivial. Since $(\alpha, e_1) = 1$ and $(\alpha, e_2) = 1$ in both cases (i) and (ii), the restriction of $E$ to $D \cong \mathbb{P}^1_k$ is given as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-1) \to E|_D \to \mathcal{O}_{\mathbb{P}^1_k}(1) \to 0,$$

whose class in $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) \cong k$ corresponds to the restricted class

$$c_{\alpha|D} \in H^1(D, L_{\alpha|D})$$

under the isomorphism in (3.28). This class is nontrivial since the class

$$c_{\alpha|\tilde{S}_k} = c_{\alpha,k} \in H^1(\tilde{S}_k, L_{\alpha,k})$$

is nontrivial by the choice of $c_\alpha$ in (3.10) together with Lemma 3.2, and the restriction map from $\tilde{S}_k$ to $D$ in (3.29) is bijective. Therefore, the extension (3.31) does not split. This implies that the vector bundle $E|_D$ over $D \cong \mathbb{P}^1_k$ is trivial. Hence the $G_\alpha$-torsor $G_{\alpha|D}$ over $D$ is also trivial, as required.

**Corollary 3.15.** Let $x$ be a singular point on $S_k$. The $G$-torsor $G$ over $\tilde{S}$ constructed above becomes trivial over the following fiber product $\tilde{S}_x$:

$$\tilde{S} \xrightarrow{\phi} S \xrightarrow{\phi^*} \text{Spec} \left( \mathcal{O}_{\tilde{S},x} \right) \xrightarrow{} S$$

**Proof.** We work with the sequence of effective divisors on $\tilde{S}_k$

$$0 = Z_0 < Z_1 < \cdots < Z_r = Z^{\text{red}} < \cdots < Z_N = Z$$

from Lemma 2.5, where $Z$ is still the fundamental cycle on $\tilde{S}_k$ over $x$.

First, we show that $G$ is trivial on $Z_j$ for $j = 1, \ldots, r$. Indeed, by induction, we can find a trivialization of $G$ on $Z_{j-1}$. Then

$$Z_j = Z_{j-1} \cup D_{ij}$$

where $D_{ij}$ meets $Z_{j-1}$ in at most one point. Proposition 3.14 states that $G$ is trivial on $D_{ij}$. We can trivialize on $D_{ij}$ in such a way that both trivializations agree on $Z_{j-1} \cap D_{ij}$. Then they define a trivialization of $G$ on $Z_j$.

Next, we show by induction that $G$ is trivial on $Z_j$ for all $j = r+1, \ldots, N$. Since $G$ is trivial on $D_{ij}$ by Proposition 3.14, its adjoint vector bundle

$$\text{ad}(G) \to \tilde{S}$$
is also trivial on \(D_{ij}\). Therefore, Lemma 2.6 implies that

\[
H^1\left( D_{ij}, I_{Z_{j-1} \subset Z_j} \otimes \text{ad}(G)_{\mathcal{P}_{ij}} \right) = 0. \tag{3.33}
\]

Assuming by induction that \(G\) is trivial on \(Z_{j-1}\), the vanishing of (3.33) means that \(G\) is also trivial on \(Z_j\) [33, Th. VII.2.4.4]. In particular, \(G\) is trivial on \(Z\). Therefore, Proposition 2.7 implies that

\[
H^1\left( Z, I^n_{Z \subset S}/I^{n+1}_{Z \subset S} \otimes \text{ad}(G)_{\mathcal{P}} \right) = 0. \tag{3.34}
\]

Let \(m_x \subset \mathcal{O}_S\) denote the ideal sheaf of \(x\). We have

\[
I_{Z \subset \bar{S}} = \phi^*(m_x)
\]

according to [1, Thm. 4], and therefore

\[
I^n_{Z \subset \bar{S}} = \phi^*(m^n_x).
\]

Let \(Z^{(n)}\) denote the closed subscheme in \(\bar{S}\) with this ideal sheaf. Assuming by induction that we have a section of \(G\) over \(Z^{(n)}\), the vanishing of (3.34) means that this section can be extended to a section of \(G\) over \(Z^{(n+1)}\).

These compatible sections induce a section of \(G\) over \(\bar{S}_x\) by Grothendieck’s Existence Theorem [27, Scholie 5.4.2], since \(\bar{S}\) is proper over \(S\).

Recall that we have lifted a universal torsor \(T\) over \(\bar{S}\) nontrivially to a \(B_{<1}\)-torsor \(B_{<1}\), and further to a \(B\)-torsor \(B\); see Lemma 3.4 and Lemma 3.6.

**Theorem 3.16.** Let \(G\) still be the \(G\)-torsor over \(\bar{S}\) induced by \(B\).

(i) There is a unique \(G\)-torsor \(G'\) over \(S\) such that \(\phi^* G' \cong G\).

(ii) If \(k\) has very good characteristic for \(\Phi\), then \(H^1(S, \text{ad}(G')) = 0\).

**Proof.** Since \(G\) is an affine scheme over \(\bar{S}\), we have

\[
G \cong \text{Spec}_{\mathcal{O}_{\bar{S}}}(A)
\]

for some quasicoherent \(\mathcal{O}_{\bar{S}}\)-algebra \(A\). We define

\[
G' := \text{Spec}_{\mathcal{O}_{\bar{S}}}(\phi_* A).
\]

The adjunction morphism \(\phi^* \phi_* A \to A\) induces a natural map

\[
G \to G' \times_{\bar{S}} \bar{S}. \tag{3.35}
\]

Assume that \(G\) is the spectrum of the \(R\)-algebra \(A\). The group action

\[
G \times_R G \to G
\]

induces a morphism \(A \to A \otimes_R A\) of \(\mathcal{O}_{\bar{S}}\)-algebras, and hence a morphism

\[
\phi_* A \to \phi_* (A \otimes_R A) = A \otimes_R (\phi_* A)
\]
of $\mathcal{O}_S$-algebras. Here, the last equality holds because $G$, and hence also $A$, is flat over $R$. This morphism of $\mathcal{O}_S$-algebras induces a morphism

$$G \times_R \mathcal{G}' \rightarrow \mathcal{G}'$$  \hspace{1cm} (3.36)$$

over $S$. We claim that the following statements hold, which imply (i):

- The morphism (3.36) is a group action of $G$ on $\mathcal{G}'$ over $S$.
- This group action turns $\mathcal{G}'$ into a $G$-torsor over $S$.
- The natural map (3.35) is an isomorphism of $G$-torsors.

According to [28, Prop. 2.5.1 and 2.7.1], all this can be tested locally in the fpqc-topology on $S$. We use the fpqc-covering

$$\left(S \setminus S_k^{\text{sing}}\right) \sqcup \coprod_{x \in S_k^{\text{sing}}} \text{Spec}(\hat{\mathcal{O}}_{S,x}) \rightarrow S$$

where $S_k^{\text{sing}} \subset S_k \subset S$ denotes the singular locus of $S_k$.

All our claims hold over $S \setminus S_k^{\text{sing}}$ because $\phi$ is an isomorphism there. They also hold over each $\text{Spec}(\hat{\mathcal{O}}_{S,x})$ because $\mathcal{G}$ is trivial there, and

$$\left(\phi_x\right)_* \mathcal{O}_{S,x} = \mathcal{O}_{\text{Spec}(\hat{\mathcal{O}}_{S,x})}$$

by Proposition 2.3 and flat base change in the diagram (3.32).

Uniqueness of $\mathcal{G}'$ also follows from Proposition 2.3.

For the proof of (ii), we note that $\text{ad}(\mathcal{G}) = \phi^* \text{ad}(\mathcal{G}')$ by construction. Therefore, we have $R^i\phi_*(\text{ad} \mathcal{G}) = 0$ for all $i > 0$ since this can be tested Zariski locally on $S$, where it holds by Proposition 2.3. Using the Leray spectral sequence, we conclude that

$$H^i(S, \text{ad}(\mathcal{G}')) \cong H^i(\tilde{S}, \text{ad}(\mathcal{G})).$$  \hspace{1cm} (3.37)$$

Hence (ii) follows from Corollary 3.13.

Remark 3.17. The restriction of $\mathcal{G}$ to the generic fiber $S_K$ is induced by the $T$-torsor $T$. But over the special fiber $S_k$, the restriction of $\mathcal{G}$ does not come from a $T$-torsor. The universal $T$-torsor over the desingularization $\tilde{S}_k$ is nontrivial on the $(-2)$-curves, and therefore does not descend to $S_k$.

Proposition 3.18. Let the $B$-torsor $\bar{B}$ over $\tilde{S}$ be an arbitrary lift of $T$. Let $\bar{G}$ be the $G$-torsor over $\tilde{S}$ induced by $\bar{B}$. Suppose that $\bar{G}$ descends to a $G$-torsor $\bar{G}'$ over $S$. If $k$ has very good characteristic for $\Phi$, then there is an automorphism $\sigma$ of $G$ with $\sigma|_T = \text{id}_T$ such that

$$\bar{B} \cong \left(\sigma|_B\right)_* B, \quad \bar{G} \cong \sigma_* \mathcal{G} \quad \text{and} \quad \bar{G}' \cong \sigma_* \mathcal{G}'.$$

Proof. Let $\bar{B}_{\xi_1} = \bar{B} \times^T B_{\xi_1}$. Let $\bar{c}_\alpha \in H^1(\tilde{S}, L_\alpha)$ for $\alpha \in \Delta$ be the extension classes corresponding to $\bar{B}_{\xi_1}$ as in (3.10).

Suppose that each $\bar{c}_\alpha$ generates $H^1(\tilde{S}, L_\alpha)$. Then $\bar{B}_{\xi_1} \cong \left(\sigma|_{B_{\xi_1}}\right)_* B_{\xi_1}$ for some such automorphism $\sigma$ of $G$ by Lemma 3.4, and hence $\bar{B} \cong \left(\sigma|_B\right)_* B$ by Proposition 3.12. Therefore, $\bar{G} \cong \sigma_* \mathcal{G}$ by the uniqueness in Theorem 3.16 (i), this implies $\bar{G}' \cong \sigma_* \mathcal{G}'$.

Now suppose that $\bar{c}_\alpha$ does not generate $H^1(\tilde{S}, L_\alpha)$ for one $\alpha \in \Delta$. Let $D$ be the corresponding $(-2)$-curve on $\tilde{S}_k$. The $B$-torsor $T \times^T B$ becomes isomorphic to $\tilde{B}$ when both are restricted to $D$, because there $\bar{c}_\alpha$ vanishes and the lifting over each $U_\beta$ with $\beta \in \Phi^\perp \setminus \{\alpha\}$ is unique by (3.30). Hence $\bar{G}|_D \cong T|_D \times^T G$, and therefore

$$\text{ad}(\bar{G})|_D \cong T|_D \times^T \mathfrak{g} \cong \mathcal{O}_{\mathfrak{g}}^{10-d} \oplus \bigoplus_{\beta \in \Phi} \mathcal{O}_{\mathfrak{g}}^1((\beta, \alpha)).$$
The integer \((\beta, \alpha)\) is nonzero at least for \(\beta = \alpha\), so the vector bundle \(\text{ad}(\overline{G})_{|D}\) is nontrivial by the Krull–Remak–Schmidt theorem. Hence \(\overline{G}_{|D}\) is nontrivial, contradicting the assumption that \(\overline{G}\) descends to \(S\).

\[\square\]

The main theorem in the introduction follows from these results: The descent statement is contained in Theorem 3.16 (i), and the uniqueness in Proposition 3.18. The claims about infinitesimal rigidity follow from Proposition 3.12, Corollary 3.13 and Theorem 3.16 (ii).

## 4

### INFinitesimal Rigidity in One Bad Characteristic

In the setting of Section 2, we assume that the residue field \(k\) of \(R\) is of characteristic 2, and that \(S\) is a family of cubic surfaces over \(R\) whose special fiber \(S_k\) has one singularity, which is of type \(D_4\).

For the geometry of cubic surfaces with a \(D_4\)-singularity, which were already studied by Schlaffi [39], see [31, §4], for example. Up to the action of the Weyl groups, a root system of type \(D_4\) admits only one embedding into one of type \(E_6\). Choosing one particular embedding allows us to describe \(S_k\) as follows. We may assume that its minimal desingularization

\[\overline{S}_k \rightarrow S_k\]

is obtained from \(\mathbb{P}^2_k\) by blowing up three points \(x_1, x_2, x_3\) on a line and then three points \(x_4, x_5, x_6\), where \(x_{i+3}\) lies on the \(i\)-th exceptional divisor, for \(i = 1, 2, 3\). Let \(h \in \Lambda\) be the pullback of \([\mathbb{P}^2_k(1)]\), and let \(e_i \in \Lambda\) be the class of (the total transform of) the \(i\)-th exceptional divisor \(E_i\), for \(i = 1, \ldots, 6\). Then the classes of the \((-2)\)-curves are

\[\alpha_i = e_i - e_{i+3} \quad (i = 1, 2, 3), \quad \alpha_4 = h - e_1 - e_2 - e_3\]

with the following Dynkin diagram:

\[
\begin{array}{c}
\alpha_1 \quad \alpha_4 \quad \alpha_2 \\
\downarrow \\
\alpha_3
\end{array}
\]

In particular, \(\Phi\) is indeed a root system of type \(D_4\). The surface \(S_k\) contains six lines. Three of them, namely the images of \(E_4, E_5\), and \(E_6\), meet in the singularity. The other three lines \(\ell_i\) for \(i = 1, 2, 3\) are the images of curves in \(\overline{S}_k\) of class \(h - e_i - e_{i+3}\); they may or may not meet in one point on \(S_k\).

Let \(B\) be a lift as in Lemma 3.6 of a universal torsor \(T\), and let \(B_k := B_{|\overline{S}_k}\). In this situation, \(B_k\) may or may not be infinitesimally rigid:

**Proposition 4.1.** We have

\[h^1(\overline{S}_k, \text{ad}(B_k)) = \begin{cases} 0 & \text{if } \ell_1 \cap \ell_2 \cap \ell_3 = \emptyset \text{ on } S_k, \\ 1 & \text{otherwise.} \end{cases}\]

To prove this, we follow the strategy of Lemma 3.8 and Proposition 3.9 (ii). These and what follows take place only in the special fiber \(\overline{S}_k\). Replacing \(k\) by its algebraic closure and \(B_k\) by its base change, we may assume that \(k\) is algebraically closed. By Bertini’s theorem [30, Rem. II.8.18.1], we can intersect an anticanonical embedding \(S_k \subset \mathbb{P}^3_k\) with a suitable plane in \(\mathbb{P}^3_k\) to obtain a smooth curve of degree 3 in that plane, not containing the \(D_4\) singularity. Its preimage \(C \subset \overline{S}_k\) is an elliptic curve.

Since \(k\) is algebraically closed, \(S_k\) is isomorphic to the surface defined by

\[x_0(x_1 + x_2 + x_3)^2 - x_1x_2x_3 = 0 \quad (4.1)\]
if \( \ell_1 \cap \ell_2 \cap \ell_3 = \emptyset \), and to the surface defined by

\[
x_0 \left( x_1 + x_2 + x_3 \right)^2 + x_1 x_2 \left( x_1 + x_2 \right) = 0
\]  

(4.2)

if \( \ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset \); see [6, Lem. 4] and [31, Rem. 4.1]. In both cases, the singularity is \( (1:0:0:0) \), hence a plane not containing it is defined by \( x_0 = a_1 x_1 + a_2 x_2 + a_3 x_3 \) for some \( a_1, a_2, a_3 \in k \). Whenever its intersection with \( S_k \) is an elliptic curve \( C \), [30, Prop. IV.4.21] shows that \( C \) is ordinary in case (4.1) and supersingular in case (4.2).

**Lemma 4.2.** Let \( \alpha \in \Phi^+ \) be a positive root. Then the restriction maps \( H^i(S_k, L_{\alpha,k}) \to H^i(C, L_{\alpha,k}|_C) \) are isomorphisms.

**Proof.** As in [21, Lem. 3.3], the long exact sequence arising from

\[
0 \to \mathcal{O}_{S_k}(-C) \otimes L_{\alpha,k} \to L_{\alpha,k} \to L_{\alpha,k}|_C \to 0
\]

together with the Serre duality isomorphisms

\[
H^i(S_k, \mathcal{O}_{S_k}(-C) \otimes L_{\alpha,k}) \to H^{2-i}(S_k, L_{-\alpha,k}) = 0
\]

give the result. \( \square \)

Let \( U_{\leq n} \) be the kernel of the canonical projection \( B_{\leq n} \to T \). The resulting short exact sequence

\[
0 \to U_{\leq n} \to B_{\leq n} \to T \to 1
\]

induces the short exact sequence of Lie algebras

\[
0 \to u_{\leq n} \to b_{\leq n} \to t \to 0.
\]  

(4.3)

Let \( \text{ad}_n(B_{\leq n,k}) \) be the vector bundle associated with the \( B_{\leq n,k} \)-torsor \( B_{\leq n,k} \) via the \( B_{\leq n,k} \)-module \( u_{\leq n,k} \). Using (4.3), we obtain the short exact sequence

\[
0 \to \text{ad}_n(B_{\leq n,k}) \to \text{ad}(B_{\leq n,k}) \to \Lambda^\vee \otimes Z \mathcal{O}_{S_k} \to 0.
\]  

(4.4)

**Lemma 4.3.** The restriction maps

\[
H^i(S_k, \text{ad}_n(B_{\leq n,k})) \to H^i(C, \text{ad}_n(B_{\leq n,k})|_C)
\]

are isomorphisms.

**Proof.** Since \( \text{ad}_n(B_{\leq n,k}) \) has a composition series with composition factors \( L_\alpha \) for some \( \alpha \in \Phi^+ \), this follows from Lemma 4.2 and the five lemma. \( \square \)

For every \( \alpha \in \Phi \), we denote by \( \exp_\alpha \) the exponential map from the underlying additive group of \( g_\alpha \) onto \( U_\alpha \subset G \).

**Lemma 4.4.** Let \( \alpha, \beta \in \Phi \) be nonproportional roots. Then the adjoint action of \( U_\alpha \) on \( g \) satisfies

\[
\exp_\alpha(x) \cdot y = y + [x, y]
\]

for all \( x \in g_\alpha \) and \( y \in g_\beta \), where \( [x, y] \in g_{\alpha+\beta} \) is 0 if \( \alpha + \beta \notin \Phi \).

**Proof.** Let \( G' \subset G \) be the centralizer of the reduced identity component of \( \ker(\alpha) \cap \ker(\beta) \subset T \). Then \( G' \) is a reductive group with maximal torus \( T \) and root datum \( \Phi \cap \langle \alpha, \beta \rangle \subset \Lambda \) of semisimple rank 2. We have \( U_\alpha \subset G' \).
Let $G'' = G' / Z'$, where $Z'$ is the center of $G'$. Since the exponential maps $\exp_\alpha$ for $G, G', G''$ agree, it suffices to prove the claim in $G''$. Since $G''$ is isomorphic to $\text{PGL}_3$ or $(\text{PGL}_2)^2$, this is an easy computation. □

By [2, Thm. 5(i)], there is an indecomposable vector bundle on $C$ of rank $r$ and degree 0, unique up to isomorphism, which we denote by $F_r$.

**Proposition 4.5.** We have $\text{ad}_n(B_{\leq 3,k})|_C \cong F_3^2 \oplus F_2 \oplus \text{Frob}^* F_2$, where $\text{Frob} : C \to C$ is the (absolute) Frobenius morphism.

**Proof.** Let $B^{\text{ad}}$ be the quotient of $B$ modulo the center $Z$ of $G$. Note that $\text{Hom}(B^{\text{ad}}, \mathbb{G}_m) = \langle \Phi \rangle \subset \Lambda$. Let $B^{\text{ad}}_{\leq n}$ denote the quotient of $B_{\leq n}$ modulo the image of $Z$. Let $B^{\text{ad}}_{\leq n}$ denote the torsor induced from $B_{\leq n}$ by extension of structure group along the projection $B_{\leq n} \to B^{\text{ad}}_{\leq n}$.

The action of $B_{\leq 3}$ on $\mathfrak{u}_{\leq 3}$ factors through the quotient $B^{\text{ad}}_{\leq 2}$ of $B_{\leq 3}$. Let $B_{\text{PGL}_3}$ be the standard Borel subgroup of classes of upper triangular matrices in $\text{PGL}_3$ over $R$. The three embeddings of the Dynkin diagram $\mathbf{A}_2$ into $\mathbf{D}_4$ yield three group homomorphisms $p_i : B \to B_{\text{PGL}_3}$ as follows.

The Lie algebra of $B$ has the root space decomposition

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

For $i = 1, \ldots, 4$, we choose nonzero $x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$. For $i = 1, 2, 3$, we define

$$x_{\alpha_i + \alpha_4} := [x_{\alpha_4}, x_{\alpha_i}] \in \mathfrak{g}_{\alpha_i + \alpha_4}.$$

For $i \neq j \in \{1, 2, 3\}$, we define

$$x_{\alpha_i + \alpha_j + \alpha_4} := [x_{\alpha_j + \alpha_4}, x_{\alpha_i}] = [x_{\alpha_i + \alpha_4}, x_{\alpha_j}] \in \mathfrak{g}_{\alpha_i + \alpha_j + \alpha_4}.$$

As in (3.12), we have followed the sign convention from [42, 12.14] here, i.e.,

$$[x_\alpha, x_\beta] = \epsilon_{\alpha, \beta} x_{\alpha + \beta}$$

whenever $\alpha, \beta, \alpha + \beta \in \Phi^+$, where in this case $\epsilon_{\alpha, \beta} = (-1)^f(\alpha, \beta)$ for the bilinear form $f$ defined by

$$f(\alpha_i, \alpha_j) = \begin{cases} (\alpha_i, \alpha_j), & i < j, \\ \frac{1}{2}(\alpha_i, \alpha_i), & i = j, \\ 0, & i > j, \end{cases}$$

for $i, j \in \{1, \ldots, 4\}$. Note that this turns out to be independent of the ordering of $\alpha_1, \alpha_2, \alpha_3$.

For $i = 1, 2, 3$, let $p_i : B \to B_{\text{PGL}_3}$ be the surjective homomorphism that vanishes on $U_\alpha$ for all $\alpha \in \Phi^+ \setminus \{\alpha_i, \alpha_4, \alpha_i + \alpha_4\}$ and on $\ker(\alpha_i) \cap \ker(\alpha_4) \subset T$. Note that $\ker(\alpha_i) \cap \ker(\alpha_4)$ is central modulo these $U_\alpha$, so the subgroup generated by these $U_\alpha$ and $\ker(\alpha_i) \cap \ker(\alpha_4)$ is normal in $B$. More precisely, $p_i$ corresponds to the Lie algebra homomorphism $\mathfrak{b} \to \mathfrak{b}_{\text{PGL}_3}$ defined by

$$t = \Lambda^\vee \otimes R \to \mathfrak{b}_{\text{PGL}_3},$$

$$\rho \otimes 1 \mapsto \begin{pmatrix} \langle \rho, \alpha_i + \alpha_4 \rangle & 0 & 0 \\ 0 & \langle \rho, \alpha_i \rangle & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
and

\[
\begin{align*}
\mathfrak{g}_{\alpha_i} & \rightarrow \mathfrak{b}_{PGL_3}, \\
x_{\alpha_i} & \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
x_{\alpha_4} & \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
x_{\alpha_i + \alpha_4} & \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

vanishing on all other \(\mathfrak{g}_\alpha\) with \(\alpha \in \Phi^+\).

By construction, the product \((p_1, p_2, p_3) : B \rightarrow B^3_{PGL_3}\) factors through a homomorphism

\[
B^3_{\leq 2} \rightarrow B^3_{PGL_3}.
\]

Let \(p : B_{PGL_3} \rightarrow B_{PGL_2}\) be the projection \((a_{r,s})_{1 \leq r,s \leq 3} \mapsto (a_{r,s})_{1 \leq r,s \leq 2}\) onto the upper left \((2 \times 2)\)-minor. Then (4.5) induces an isomorphism

\[
B^3_{\leq 2} \cong \left\{ (b_1, b_2, b_3) \in B^3_{PGL_3} \mid p(b_1) = p(b_2) = p(b_3) \right\}.
\]

(4.6)

Note that \(U_{\leq 2}\) can be identified with the unipotent radical of \(B^3_{\leq 2}\). Identifying also the unipotent radical \(U_{GL_3}\) of the standard Borel subgroup \(B_{GL_3} \subset GL_3\) with the unipotent radical of \(B_{PGL_3}\), we obtain an isomorphism

\[
U_{\leq 2} \cong \left\{ (u_1, u_2, u_3) \in U^3_{GL_3} \mid p(u_1) = p(u_2) = p(u_3) \right\}.
\]

We choose \(e_\alpha\) and \(f_\alpha\) for \(\alpha \in \Phi^+\) as in Lemma 3.7. The sections \(e_\alpha\) define isomorphisms \(L_{\alpha | C} \cong \mathcal{O}_C\) since \(C\) does not intersect the vanishing locus of \(e_\alpha\), which consists of \((-2)\)-curves. This gives a reduction of structure group of \(B^3_{\leq 2, | C}\) to a \(U_{\leq 2 | C}\)-torsor \(V_{\leq 2}\), which we have only on \(C\).

Let \(V\) be the vector bundle of rank 2 over \(C\) associated with the \(U_{\leq 2 | C}\)-torsor \(V_{\leq 2}\) via the common composition

\[
p \circ p_1 = p \circ p_2 = p \circ p_3 : U_{\leq 2} \rightarrow U_{GL_3} \rightarrow U_{GL_2} \subset GL_2.
\]

Using the above identifications, \(V\) is by construction an extension

\[
0 \rightarrow \mathcal{O}_C \rightarrow V \rightarrow \mathcal{O}_C \rightarrow 0
\]

whose class \(c_4 \in H^1(C, \mathcal{O}_C)\) is given by

\[
e_\alpha_{| C} \cdot c_4 = e_\alpha_{| C} = f_\alpha_{| C}.
\]

Since \(c_{\alpha_{| C}}\) is nontrivial, we can identify the extension \(V\) with the Atiyah bundle \(F_2\). The composition \(F_2 \cong V \rightarrow \mathcal{O}_C\) induces an isomorphism \(H^1(C, F_2) \rightarrow H^1(C, \mathcal{O}_C)\).

For \(i = 1, 2, 3\), let \(V_i\) be the vector bundle of rank 3 over \(C\) associated with \(U_{\leq 2}\) via

\[
p_i : U_{\leq 2} \rightarrow U_{GL_3} \subset GL_3.
\]

Using the above identifications, \(V_i\) is by construction an extension

\[
0 \rightarrow F_2 \rightarrow V_i \rightarrow \mathcal{O}_C \rightarrow 0
\]

whose class \(c_{\alpha_{| C}} \in H^1(C, \mathcal{O}_C) = H^1(C, F_2) = \text{Ext}^1(\mathcal{O}_C, F_2)\) is given by

\[
e_\alpha_{| C} \cdot c_1 = e_\alpha_{| C} = f_\alpha_{| C}.
\]
Applying $[e_{x, j + \alpha_4}, -]$ to this equation, and using that

\[
[e_{x, j + \alpha_4}, e_{\alpha_4}] = e_{\alpha_4 + \alpha_4} = e_{\alpha_4},
\]

\[
[e_{x, j + \alpha_4}, f_{\alpha_4}] = f_{\alpha_4 + \alpha_4} = e_{\alpha_4},
\]

by Lemma 3.7, we conclude that

\[
c_1 = c_2 = c_3 \in H^1(C, O_C).
\] (4.7)

This allows us to identify $V_1, V_2, V_3$ as extensions of $O_C$ by $P_2$. This identification reduces $U_{\leq 2}$ to a torsor $U_{GL_3}$ under the diagonally embedded subgroup

\[
U_{GL_3} \subset U_{\leq 2}.
\] (4.8)

Since the class in (4.7) is nontrivial, we note that $V_1 \cong F_3$.

Similarly, $c_3 = c_4 \in H^1(C, O_C)$ because

\[
[e_{\alpha_4}, e_{\alpha_3}] = e_{\alpha_3 + \alpha_4} = -[e_{\alpha_3}, e_{\alpha_4}],
\]

\[
[e_{\alpha_4}, f_{\alpha_3}] = f_{\alpha_3 + \alpha_4} = -[e_{\alpha_3}, f_{\alpha_4}].
\]

This allows us to identify the subbundle $P_2 \cong V \subset V_3$ with the quotient $V_3/O_C \cong F_2$. This identification reduces $U_{GL_3}$ to a torsor $U'$ under the subgroup

\[
U' := \left\{ \begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \middle| \lambda, \mu \in k \right\} \subset U_{GL_3}.
\]

The next step is to study $u_{\leq 3}$ as a ten-dimensional representation of these subgroups $U' \subset U_{GL_3} \subset U_{\leq 2}$. We have

\[
u_{\leq 3} = \bigoplus_{\alpha \in \Phi^+_{\leq 3}} g_{\alpha},
\]

with basis $(x_\alpha)$ as introduced above.

For $\lambda \in k$, consider

\[
u_{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \in U_{GL_3}, \quad v_{\lambda} = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_{GL_3}.
\]

Its images under (4.8) are

\[
\exp_{\alpha_4}(\lambda x_{\alpha_4}) \exp_{\alpha_2}(\lambda x_{\alpha_2}) \exp_{\alpha_3}(\lambda x_{\alpha_3}) \in U_{\leq 2}, \quad \exp_{\alpha_4}(\lambda x_{\alpha_4}) \in U_{\leq 2};
\]

for the image of $\nu_{\lambda}$, the ordering does not matter since $[x_{\alpha_i}, x_{\alpha_j}] = 0$ for all $i, j \leq 3$. For every $\alpha \in \Phi^+_{\leq 3}$ and $y \in g_{\alpha}$, Lemma 4.4 then gives

\[
u_{\lambda} \cdot y = y + \sum_{i=1}^{3} \lambda x_{\alpha_i}, y + \sum_{1 \leq i < j \leq 3} [\lambda x_{\alpha_i}, [\lambda x_{\alpha_j}, y]],
\]

\[
u_{\lambda} \cdot y = y + [\lambda x_{\alpha_4}, y].
\]

Note that the last sum in the expression for $\nu_{\lambda} \cdot y$ vanishes unless $\alpha = \alpha_4$. 
In particular, using the notation
\[
\begin{align*}
    z_1 &:= \sum_{i=1}^{3} x_{\alpha_i}, \\
    z_2 &:= \sum_{i=1}^{3} x_{\alpha_i + \alpha_4}, \\
    z_3 &:= \sum_{1 \leq i < j \leq 3} x_{\alpha_i + \alpha_j + \alpha_4},
\end{align*}
\]
we have
\[
\begin{align*}
    u_\lambda \cdot x_{\alpha_4} &= x_{\alpha_4} - \lambda z_2 + \lambda^2 z_3, \\
    u_\lambda \cdot x_{\alpha_j + \alpha_4} &= x_{\alpha_j + \alpha_4} - \lambda \sum_{i \in \{1,2,3\} \setminus \{j\}} x_{\alpha_i + \alpha_j + \alpha_4} \quad (j = 1, 2, 3), \\
    v_\lambda \cdot x_{\alpha_j} &= x_{\alpha_j} + \lambda x_{\alpha_j + \alpha_4} \quad (j = 1, 2, 3),
\end{align*}
\]
while \( u_\lambda \) and \( v_\lambda \) acts as the identity on all other \( x_{\alpha_i} \). These imply that
\[
\begin{align*}
    v_\lambda \cdot z_1 &= z_1 + \lambda z_2, \\
    u_\lambda \cdot z_2 &= z_2 - 2\lambda z_3 = z_2 \quad \text{in characteristic 2},
\end{align*}
\]
while \( u_\lambda \) and \( v_\lambda \) act as the identity on all other \( z_i \). We observe that \( u_{\leq 3} \) decomposes as \( U_{GL_3} \)-module into the direct sum of the three vector spaces
\[
\begin{align*}
    u_{\leq 3}^1 &:= \langle x_{\alpha_1}, x_{\alpha_2 + \alpha_4}, x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_1 + \alpha_3 + \alpha_4} \rangle, \\
    u_{\leq 3}^2 &:= \langle x_{\alpha_2}, x_{\alpha_2 + \alpha_4}, x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_2 + \alpha_1 + \alpha_4} \rangle, \\
    u_{\leq 3}^' &:= \langle x_{\alpha_4}, z_1, z_2, z_3 \rangle.
\end{align*}
\]
The vector bundle \( \mathcal{V}^1 \) of rank 3 over \( C \) associated with the \( U_{GL_3} \)-torsor \( U_{GL_3} \) via the representation \( u_{\leq 3}^1 \) is isomorphic to \( F_3 \). Indeed, \( u_{\leq 3}^1 \) has the composition series
\[
0 \subset \langle x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_1 + \alpha_3 + \alpha_4} \rangle \subset \langle x_{\alpha_1 + \alpha_4}, x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_1 + \alpha_3 + \alpha_4} \rangle \subset u_{\leq 3}^1.
\]
The composition factors are the trivial one-dimensional representations. Therefore, \( \mathcal{V}^1 \) is a double extension of \( \mathcal{O}_C \) by \( \mathcal{O}_C \) by \( \mathcal{O}_C \). Here, both extensions of \( \mathcal{O}_C \) by \( \mathcal{O}_C \) are nontrivial because the corresponding representations
\[
\begin{align*}
    \langle x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_1 + \alpha_3 + \alpha_4} \rangle \text{ and } u_{\leq 3}^1 / \langle x_{\alpha_1 + \alpha_2 + \alpha_4} + x_{\alpha_1 + \alpha_3 + \alpha_4} \rangle
\end{align*}
\]
are the two two-dimensional standard representations of \( U_{GL_3} \). A similar argument shows that the vector bundle \( \mathcal{V}^2 \) of rank 3 over \( C \) associated with the \( U_{GL_3} \)-torsor \( U_{GL_3} \) via the representation \( u_{\leq 3}^2 \) is isomorphic to \( F_3 \).

The subgroup \( U' \subset U_{GL_3} \) is generated by
\[
\begin{align*}
    u_\lambda v_\mu &= \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \in U', \\
    [v_1, u_\mu] &= \begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U',
\end{align*}
\]
for \( \lambda, \mu \in k \). We have
\[
\begin{align*}
    u_\lambda v_\mu \cdot x_{\alpha_4} &= x_{\alpha_4} - \lambda z_2 + \lambda^2 z_3, \\
    u_\lambda v_\mu \cdot z_1 &= z_1 + \lambda z_2, \\
    u_\lambda v_\mu \cdot z_2 &= z_2, \\
    u_\lambda v_\mu \cdot z_3 &= z_3,
\end{align*}
\]
while \([v_1, u_m]\) acts as the identity on \(u'_\leq 3\). Hence \(u'_\leq 3\) decomposes as \(U'\)-module into the direct sum of the two vector spaces

\[
\begin{align*}
    u'_\leq 3 & := \langle z_1, z_2 \rangle, \\
    u'_\leq 3 & := \langle x_{\alpha_4} + z_1, z_3 \rangle.
\end{align*}
\]

The vector bundle \(V^3\) of rank 2 over \(C\) associated with the \(U'\)-torsor \(U'\) via \(u'_\leq 3\) is isomorphic to \(P_2\) because \(u'_\leq 3\) is isomorphic to the two-dimensional standard representation of \(U'\). The vector bundle \(V^4\) of rank 2 over \(C\) associated with the \(U'\)-torsor \(U'\) via \(u'_\leq 3\) is isomorphic to the Frobenius pullback of \(P_2\). Indeed, \(u'_\leq 3\) is isomorphic to the Frobenius pullback of the two-dimensional standard representation of \(U'\) since \(u_\lambda v_\lambda\) acts on \(u'_\leq 3\) as the matrix \(\begin{pmatrix} 1 & 0 \\ \lambda^2 & 1 \end{pmatrix}\).

\[\square\]

**Corollary 4.6.** For \(i = 0, 1\), we have

\[
h^i(C, \text{ad}_n(B_k))_{|C} = \begin{cases} 
4 & \text{if } C \text{ is ordinary}, \\
5 & \text{if } C \text{ is supersingular}.
\end{cases}
\]

**Proof.** If \(C\) is ordinary, then the endomorphism of \(H^1(C, \mathcal{O}_C)\) induced by Frobenius is nonzero, and therefore, \(\text{Frob}^*P_2 \cong P_2\). Otherwise, \(C\) is supersingular, so \(\text{Frob}^*P_2 \cong \mathcal{O}_C^2\). Using Proposition 4.5 and \(h^i(C, T_r) = 1\) for \(i = 0, 1\) and all \(r \geq 1\), we conclude that

\[
h^i(C, \text{ad}_n(B_{\leq 3,k})_{|C}) = \begin{cases} 
4 & \text{if } C \text{ is ordinary}, \\
5 & \text{if } C \text{ is supersingular}.
\end{cases}
\]  

(4.9)

Now consider the long exact cohomology sequence associated with (3.17) for \(n \geq 4\). Its connecting homomorphism (3.18) is surjective since Lemma 3.8 is again valid for \(n \geq 4\) in the \(D_4\)-case in characteristic 2. Therefore,

\[
h^1(C, \text{ad}_n(B_{\leq n,k})_{|C}) = h^1(C, \text{ad}_n(B_{\leq n-1,k})_{|C}).
\]

By induction starting with (4.9), we obtain the result for \(i = 1\). The result for \(i = 0\) follows since the Euler characteristic of \(\text{ad}_n(B_k)_{|C}\) vanishes.  

\[\square\]

**Proof of Proposition 4.1.** By Corollary 4.6, Lemma 4.3, and the discussion of elliptic curves on the two isomorphism classes of singular cubic surfaces, we have

\[
h^i(\widetilde{S}_k, \text{ad}_n(B_k)) = \begin{cases} 
4 & \text{for } S_k \text{ as in (4.1)}, \\
5 & \text{for } S_k \text{ as in (4.2)},
\end{cases}
\]  

for \(i = 0, 1\). By (4.4) for sufficiently large \(n\), it suffices to prove that the connecting homomorphism

\[
H^0(\tilde{S}_k, \Lambda^r \otimes \mathbb{Z} \mathcal{O}_{\tilde{S}_k}) \to H^1(\tilde{S}_k, \text{ad}_n(B_k))
\]  

(4.10)

has rank 4. Indeed, this rank is at least 4 since the composition of (4.10) with the natural map

\[
H^1(\tilde{S}_k, \text{ad}_n(B_k)) \to H^1(\tilde{S}_k, \text{ad}_n(B_{\leq 1,k})) \cong \bigoplus_{\alpha \in \Delta} H^1(\tilde{S}_k, L_{\alpha,k}) \cong \bigoplus_{\alpha \in \Delta} k \cong k^4
\]
is the surjective connecting homomorphism $\delta$ from (3.16). On the other hand, the rank of (4.10) is at most 4 since this map factors through the projection

$$H^0 \left( \bar{S}_k, \Lambda^\vee \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{S}_k} \right) \to \bigoplus_{\alpha \in \Delta} H^0 \left( \bar{S}_k, \mathcal{O}_{\bar{S}_k} \right) \cong \bigoplus_{\alpha \in \Delta} k \cong k^4$$

given by the simple roots $\alpha$.

\[\square\]

**Corollary 4.7.** The $R$-modules $H^1 \left( \bar{S}, \text{ad}(B) \right)$, $H^1(\bar{S}, \text{ad}(G))$, $H^1(S, \text{ad}(G'))$ are all zero if $\ell_1 \cap \ell_2 \cap \ell_3 = \emptyset$ on $S_k$, and are all nonzero otherwise.

**Proof.** Since $H^2 \left( \bar{S}_k, \text{ad}(B_k) \right) = 0$ because of (3.3), Cohomology and Base Change [30, Thm. III.12.11] implies that the natural map

$$H^1 \left( \bar{S}_k, \text{ad}(B_k) \right) \otimes_R k \to H^1 \left( \bar{S}_k, \text{ad}(B_k) \right)$$

is an isomorphism. Using Proposition 4.1, we conclude that $H^1(\bar{S}, \text{ad}(B))$ vanishes if and only if $\ell_1 \cap \ell_2 \cap \ell_3 = \emptyset$. The isomorphisms (3.27) and (3.37) give the remaining statements for $H^1(\bar{S}, \text{ad}(G))$ and $H^1(S, \text{ad}(G'))$.

\[\square\]

**ACKNOWLEDGMENT**

We thank Yuri Tschinkel for introducing us to these questions. The first author was supported by grant DE 1646/3-1 of the Deutsche Forschungsgemeinschaft. The second author was supported by Mary Immaculate College Limerick through the PLOA sabbatical program, and by the Riemann Center for Geometry and Physics of Leibniz Universität Hannover.

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How to cite this article: Derenthal U, Hoffmann N. Degeneration of torsors over families of del Pezzo surfaces. *Mathematische Nachrichten*. 2020;293:2306–2334. https://doi.org/10.1002/mana.201900546