Dirichlet Polynomials form a Topos

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Abstract

One can think of power series or polynomials in one variable, such as \( P(y) = 2y^3 + y + 5 \), as functors from the category \( \text{Set} \) of sets to itself; these are known as polynomial functors. Denote by \( \text{Poly}_{\text{Set}} \) the category of polynomial functors on \( \text{Set} \) and natural transformations between them. The constants 0, 1 and operations +, \( \times \) that occur in \( P(y) \) are actually the initial and terminal objects and the coproduct and product in \( \text{Poly}_{\text{Set}} \).

Just as the polynomial functors on \( \text{Set} \) are the copresheaves that can be written as sums of representables, one can express any Dirichlet series, e.g. \( \sum_{n=0}^{\infty} n^\nu \), as a coproduct of representable presheaves. A Dirichlet polynomial is a finite Dirichlet series, that is, a finite sum of representables \( n^\nu \). We discuss how both polynomial functors and their Dirichlet analogues can be understood in terms of bundles, and go on to prove that the category of Dirichlet polynomials is an elementary topos.

1 Introduction

Polynomials \( P(y) \) and finite Dirichlet series \( D(y) \) in one variable \( y \), with natural number coefficients \( a_i \in \mathbb{N} \), are respectively functions of the form

\[
P(y) = a_n y^n + \cdots + a_2 y^2 + a_1 y + a_0 y^0, \\
D(y) = a_n n^\nu + \cdots + a_2 2^\nu + a_1 1^\nu + a_0 0^\nu. \tag{1}
\]

The first thing we should emphasize is that the algebraic expressions in (1) can in fact be regarded as objects in a category, in fact two categories: \( \text{Poly} \) and \( \text{Dir} \). We will explain the morphisms later, but for now we note that in \( \text{Poly} \), \( y^2 = y \times y \) is a product and \( 2y = y + y \) is a coproduct, and similarly for \( \text{Dir} \). The operators—in both the polynomial and the Dirichlet case—are not just algebraic, they are category-theoretic. Moreover, these categories have a rich structure.

The category \( \text{Poly} \) is well studied (see [GK12]). In particular, the following are equivalent:

**Theorem 1.1.** [GK12] For a functor \( P : \text{Fin} \to \text{Fin} \), the following are equivalent:

1. \( P \) is polynomial.
2. \( P \) is a sum of representables.
3. \( P \) preserves connected limits – or equivalently, wide pullbacks.

In Theorem 4.9 we prove an analogous result characterizing Dirichlet polynomials:

**Theorem 1.2.** For a functor \( D : \text{Fin}^{\text{op}} \to \text{Fin} \), the following are equivalent:
1. \( D \) is a Dirichlet polynomial.
2. \( D \) is a sum of representables.
3. \( D \) sends connected colimits to limits – or equivalently, \( D \) preserves wide pushouts.

We will also show that \( \text{Dir} \) is equivalent to the arrow category of finite sets,
\[
\text{Dir} \cong \text{Fin}^{\to},
\]
and in particular that \( \text{Dir} \) is an elementary topos.

If one allows arbitrary sums of functors represented by finite sets, one gets analytic functors in the covariant case—first defined by Joyal in his seminal paper on combinatorial species [Joy81]—and Dirichlet functors in the contravariant case—first defined by Baez and Dolan and appearing in Baez’s This Week’s Finds blog [BD]. Baez and Dolan also drop the traditional negative sign in the exponent (that is, they use \( n^s \) where \( n^{-s} \) usually appears), but also find a nice way to bring it back by moving to groupoids. Here, we drop the negative sign and work with finite sets to keep things as simple as possible. Similar considerations hold with little extra work for infinite Dirichlet series or power series, and even more generally, by replacing \( \text{Fin} \) with \( \text{Set} \).

2 Polynomial and Dirichlet functors

Recall that a co-representable functor \( \text{Fin} \to \text{Fin} \) is one of the form \( \text{Fin}(k, -) \) for a finite set
\[
k = \{1', 2', \ldots, k'\}.
\]
We denote this functor by \( y^k \) and say it is represented by \( k \in \text{Fin} \). Similarly, a (contra-)representable functor \( \text{Fin}^{\text{op}} \to \text{Fin} \) is contravariant functor of the form \( \text{Fin}(-, k) \); we denote this functor by \( k^y \). The functors \( y^{-} \) and \( -^y \) are the contravariant and covariant Yoneda embeddings,
\[
y^k := \text{Fin}(k, -) \quad \text{and} \quad k^y := \text{Fin}(-, k).
\]
For example \( y^3(2) \equiv 8 \) and \( 3^y(2) \equiv 9 \).

Note that the functor \( 0^y \equiv 0 \) is not the initial object in \( \text{Dir} \); it is given by
\[
0^y(s) = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1. \end{cases}
\]

The coefficient \( a_0 \) of \( 1 = y^0 \) in a polynomial \( P \) is called its constant term. We refer to the coefficient \( D_{zc} := a_0 \) of \( 0^y \) in a Dirichlet series \( D \) as its zero-content term. Rather than having no content, the content of the functor \( D_{zc} \cdot 0^y \) becomes significant exactly when it is applied to zero.
Example 2.1. The reader can determine which Dirichlet polynomial \( D(y) \in \text{Dir} \) as in Eq. (1) has the following terms

| \( y \) | \( \cdots \) | 5 | 4 | 3 | 2 | 1 | 0 |
| \( D(y) \) | \( \cdots \) | 96 | 48 | 24 | 12 | 6 | 7 |

Hint: its zero-content term is \( D_{zc} = 4 \).

The set \( P(1) \) (resp. the set \( D(0) \)) has particular importance; it is the set of pure-power terms \( y^k \) in \( P \) (resp. the pure-exponential terms \( k^y \) in \( D \)). For example if \( P = y^2 + 4y + 4 \) and \( D = 2^y + 4 + 4 \cdot 0^y \) then \( P(1) = D(0) = 9 \). We will later have reason to consider the inclusion

\[
D_{zc} \rightarrow D(0)
\]

of the zero-content terms into the set of all pure-exponential terms.

Definition 2.2. A polynomial functor is a functor \( P : \text{Fin} \rightarrow \text{Fin} \) that can be expressed as a sum of co-representable functors. Similarly, we define a Dirichlet functor to be a functor \( D : \text{Fin}^{\text{op}} \rightarrow \text{Fin} \) that can be expressed as a sum of representable presheaves (contravariant functors):

\[
P = \sum_{i=1}^{P(1)} y^{p_i} \quad \text{and} \quad D = \sum_{i=1}^{D(0)} (d_i)^y.
\]

That is, \( P(X) = \sum_{i=1}^{P(1)} \text{Fin}(p_i, X) \) and \( D(X) = \sum_{i=1}^{D(0)} \text{Fin}(X, d_i) \) as functors applied to \( X \in \text{Fin} \).

See Theorem 1.1 above for well-known equivalent conditions in Poly and Theorem 4.9 below for a Dirichlet analogue.

3 The categories Poly and Dir

For any small category \( C \), let \( \text{Fin}^C \) denote the category whose objects are the functors \( C \rightarrow \text{Fin} \) and whose morphisms are the natural transformations between them.

Definition 3.1. The category of polynomial functors, denoted Poly, is the full subcategory of \( \text{Fin}^{\text{Fin}} \) spanned by sums \( P \) of representable functors. The category of Dirichlet functors, denoted Dir, is the full subcategory of \( \text{Fin}^{\text{Fin}^{\text{op}}} \) spanned by the sums \( D \) of representable presheaves.

While we will not pursue it here, one can take \( \text{Poly}_{\text{Set}} \) to be the full subcategory of functors \( \text{Set} \rightarrow \text{Set} \) spanned by small coproducts of representables, and similarly for \( \text{Dir}_{\text{Set}} \).

Lemma 3.2. The set of polynomial maps \( P \rightarrow Q \) and Dirichlet maps \( D \rightarrow E \) are given by the following formulas:

\[
\text{Poly}(P, Q) := \prod_{i \in P(1)} Q(p_i) \quad \text{and} \quad \text{Dir}(D, E) := \prod_{i \in D(0)} E(d_i).
\]
Example 3.3. Let $P = 2y^2$, $Q = y + 1$, and let $D = 2 \cdot 2^y$ and $E = 1 + 0^y$. Then there are nine (9) polynomial morphisms $P \to Q$, zero (0) polynomial morphisms $Q \to P$, one (1) Dirichlet morphism $D \to E$, and eight (8) Dirichlet morphisms $E \to D$.

Remark 3.4. Sums and products of polynomials in the usual algebraic sense agree exactly with sums and products in the categorical sense: if $P$ and $Q$ are polynomials, i.e. objects in Poly, then their coproduct is the usual algebraic sum $P + Q$ of polynomials, and similarly their product is the usual algebraic product $PQ$ of polynomials. The same is true for Dir: sums and products of Dirichlet polynomials in the usual algebraic sense agree exactly with sums and products in the categorical sense.

**Formal structures.** We review some formal structures of the categories Poly and Dir; all are straightforward to prove. There is an adjoint quadruple and adjoint sextuple as follows, labeled by where they send objects $n \in \text{Fin}, P \in \text{Poly}, D \in \text{Dir}$:

\[
\begin{array}{ccc}
\text{Fin} & \xrightarrow{\mathbb{T}} & \text{Poly} \\
\text{Fin} & \xleftarrow{\mathbb{T}} & \text{Dir}
\end{array}
\]

All six of the displayed functors out of Fin are fully faithful.

For each $n : \text{Fin}$ the functors $P \mapsto P(n)$ and $D \mapsto D(n)$ have left adjoints, namely $n \mapsto ny^k$ and $n \mapsto nk^n$ respectively. These are functorial in $k$ and in fact extend to two-variable adjunctions $\text{Fin} \times \text{Poly} \to \text{Poly}$ and $\text{Fin} \times \text{Dir} \to \text{Dir}$. Indeed, for $n \in \text{Fin}$ and $P, Q \in \text{Poly}$ (respectively $D, E \in \text{Dir}$), we have

\[
\begin{align*}
\text{Poly}(nP, Q) & \cong \text{Poly}(P, Q^n) \cong \text{Fin}(n, \text{Poly}(P, Q)), \\
\text{Dir}(nD, E) & \cong \text{Dir}(D, E^n) \cong \text{Fin}(n, \text{Dir}(D, E)),
\end{align*}
\]

where $nP$ and $nD$ denote $n$-fold coproducts and $P^n$ and $D^n$ denote $n$-fold products.

The inclusion $D_{zc} \to D(0)$ from Eq. (2) is natural and induces three other natural transformations on Fin and Dir via the adjunctions in Eq. (4):

\[
D_{zc} \to D(0), \quad n \cdot 0^y \to n, \quad D(1) \xrightarrow{\pi_D} D(0), \quad n \to n^y.
\]

The one labeled $\pi_D$ is also $D(0)!$, where $0!: 0 \to 1$ is the unique function of that type.

The composite of two polynomial functors Fin $\to$ Fin is again polynomial, $(P \circ Q)(n) \coloneqq P(Q(n))$; this gives a nonsymmetric monoidal structure on Poly. The monoidal unit is $y$.

Day convolution for the cartesian product monoidal structure provides symmetric monoidal structure $\otimes: \text{Poly} \times \text{Poly} \to \text{Poly}$, for which the monoidal unit is $y$. This monoidal structure—like the Cartesian monoidal structure—distributes over $+$. We can write an explicit formula for $P \otimes Q$, with $P, Q$ as in Eq. (3):

\[
P \otimes Q = \sum_{i=1}^{P(1)} \sum_{j=1}^{Q(1)} y^{P(i)Q(j)}
\]
We call this the \textit{Dirichlet product} of polynomials, for reasons we will see in Remark \ref{remark:dirichlet-product}. The Dirichlet monoidal structure is closed; that is, for any $A, Q : \text{Poly}$ we define

\[
[A, Q] := \prod_{i : A(1)} Q \circ (a_i y), \quad (7)
\]

for example $[n y, y] \cong y^n$ and $[y^n, y] \cong n y$. For any polynomial $A$ there is an $(- \otimes A) \dashv [A, -]$ adjunction

\[
\text{Poly}(P \otimes A, Q) \cong \text{Poly}(P, [A, Q]). \quad (8)
\]

In particular we recover Lemma \ref{lemma:closed-monoidal} using Eqs. (4) and (7). The cartesian monoidal structure on $\text{Poly}$ is also closed, $\text{Poly}(P \times A, Q) \cong \text{Poly}(P, Q^A)$, and the formula for $Q^A$ is similar to Eq. (7):

\[
Q^A := \prod_{i : A(1)} Q \circ (a_i + y).
\]

If we define the \textit{global sections} functor $\Gamma : \text{Poly} \to \text{Fin}^{\text{op}}$ by $\Gamma P := \text{Poly}(P, y)$, or explicitly $\Gamma P = [P, y](1) = \prod_i p_i$, we find that it is left adjoint to the Yoneda embedding

\[
\text{Fin}^{\text{op}} \xleftarrow{\text{maps from}} \text{Poly}.
\]

Each of the categories $\text{Poly}$ and $\text{Dir}$ has pullbacks, which we denote using “fiber product notation” $A \times_C B$. We can use pullbacks in combination with monad units $\eta_P : P \to P(1)$ and $\eta_D : D \to D(0)$ arising from Eq. (4) to recover Eq. (3):

\[
P = \sum_{i=1}^{P(1)} P \times_{P(1)} 'i' \quad \text{and} \quad D = \sum_{i=1}^{D(0)} D \times_{D(0)} 'i'.
\]

Remark 3.5. By a result of Rosebrugh and Wood \cite{RW94}, the category of finite sets is characterized amongst locally finite categories by the existence of the five left adjoints to its Yoneda embedding $k \mapsto y^k : \text{Fin} \to \text{Fin}^{\text{op}}$. The adjoint sextuple displayed in (4) is just the observation that these six functors restrict to the subcategory $\text{Dir}$.

\section{Poly and Dir in terms of bundles}

There is a bijection between the respective object-sets of these two categories

\[
\text{Ob}(\text{Poly}) \xrightarrow{\cong} \text{Ob}(\text{Dir}) \quad \sum_{i=1}^{n} y^{k_i} \leftrightarrow \sum_{i=1}^{n} (k_i)^y.
\]

We call this mapping the \textit{Dirichlet transform} and denote it using an over-line $P \mapsto \overline{P}$. We will see in Theorem \ref{theorem:equivalence} that this bijection extends to an equivalence $\text{Poly}_{\text{cart}} \cong \text{Dir}_{\text{cart}}$ between the subcategories of cartesian maps.
Remark 4.1. With the Dirichlet transform in hand, we see why \( P \otimes Q \) can be called the Dirichlet product, e.g. in Eq. (6). Namely, the Dirichlet transform is strong monoidal with respect to \( \otimes \) and the cartesian monoidal structure \( \times \) in Dir:

\[
\overline{P \otimes Q} = \overline{P} \times \overline{Q}.
\]

**Proposition 4.2.** There is a one-to-one correspondence between the set of polynomials in one variable, the set of Dirichlet polynomials, and the set of (isomorphism classes of) functions \( \pi: s \to t \) between finite sets.

**Proof.** We already established a bijection \( P \leftrightarrow \overline{P} \) between polynomials and finite Dirichlet series in Eq. (9).

Given a finite Dirichlet series \( D \), we have a function \( \pi_D: D(1) \to D(0) \) as in Eq. (5). And given a function \( \pi: s \to t \), define \( D_{\pi} := \sum_{i=1}^{t} (d_i)^\nu \), where \( d_i := \pi^{-1}(i) \) for each \( 1 \leq i \leq t \). (N.B. Rather than constructing \( D_{\pi} \) from \( \pi \) by hand, one could instead use a certain orthogonal factorization system on Dir.)

It is easy to see that the roundtrip on Dirichlet series is identity, and that the round-trip for functions is a natural isomorphism. \( \square \)

We will upgrade Proposition 4.2 to an equivalence \( \text{Poly}_{\text{cart}} \simeq \text{Dir}_{\text{cart}} \) between certain subcategories of Poly and Dir in Theorem 4.7.

**Example 4.3.** Under the identification from Proposition 4.2, both the polynomial \( 2y^3 + y^2 + 3 \) and the Dirichlet series \( 2\cdot 3^\nu + 1\cdot 2^\nu + 3\cdot 0^\nu \) correspond to the function

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(1,1) & (2,2) & (3,1) & & & \\
(1,2) & (2,3) & & & & \\
(1,3) & & & & & \\
\end{array}
\]

We can think of a function \( \pi: s \to t \), e.g. that shown in (10), as a *bundle* of fibers \( \pi^{-1}(i') \), one for each element \( i' \in t \). In Definition 4.4 we define two different notions of morphism between bundles. We will see in Theorem 4.7 that they correspond to morphisms in the categories Poly and Dir.

For any function \( \pi': s' \to t' \) and function \( f: t \to t' \), denote by \( f^*(\pi') \) the pullback function as shown

\[
\begin{array}{ccc}
s \times_{t'} s' & \rightarrow & s' \\
\downarrow_{f^*(\pi')} & & \downarrow \pi' \\
t & \rightarrow & t' \\
f & & f
\end{array}
\]

**Definition 4.4.** Let \( \pi: s \to t \) and \( \pi': s' \to t' \) be functions between finite sets.

- a *bundle morphism* consists of a pair \((f, f')\) where \( f: t \to t' \) is a function and \( f': \pi \to f^*(\pi') \) is a morphism in the slice category over \( t \);
Figure 1: The categories Bun and Cont have the same objects, functions $\pi: s \to t$. Here a morphism $(f, f^\sharp): \pi \to \pi'$ in Bun and a morphism $(f, f^\sharp): \pi \to \pi'$ in Cont are shown.

- a container morphism consists of a pair $(f, f^\sharp)$ where $f: t \to t'$ is a function and $f^\sharp: f^*(\pi') \to \pi$ is a morphism in the slice category over $t$.

We say a bundle morphism $(f, f^\sharp)$ (resp. a container morphism $(f, f^\sharp)$) is cartesian if $f^\sharp$ (resp. $f^\sharp$) is an isomorphism.

Define Bun (resp. Cont) to be the category for which an object is a function between finite sets and a morphism is a bundle morphism (resp. container morphism); see Fig. 1. Denote by Bun$_\text{cart}$ (resp. Cont$_\text{cart}$) the subcategory of cartesian bundle morphisms.

One may note that Bun is the Grothendieck construction of the self-indexing $\text{Fin}/(-): \text{Fin}^{\text{op}} \to \text{Cat}$, while Cont is the Grothendieck construction of its point-wise opposite $(\text{Fin}/(-))^{\text{op}}: \text{Fin}^{\text{op}} \to \text{Cat}$.

The name container comes from the work of Abbot, Altenkirch, and Ghani [AAG03; AAG05; Abb03] (see Remark 2.18 in [GK12] for a discussion of the precise relationship between the notion of container and the notion of polynomial and polynomial functor).

**Remark 4.5.** By the universal property of pullbacks, Bun $\simeq \text{Fin}^\rightarrow$ is equivalent (in fact isomorphic) to the category of morphisms and commuting squares in Fin. Furthermore, Bun$_\text{cart}$ is equivalent to the category of morphisms and pullback squares in Fin, and Bun$_\text{cart} \simeq$ Cont$_\text{cart}$ (as in both cases a cartesian morphism $(f, f^\sharp)$ or $(f, f^\sharp)$ is determined by $f$ alone).

**Remark 4.6.** We can think of a function between finite sets $\pi: E \to B$ as the categorification of a Young diagram. A Young diagram consists of $k$ natural numbers $n_k \geq n_{k-1} \geq \cdots \geq n_1 > 0$; the number $k$ is the number of rows and $n_i$ is the number of boxes in row $i$. Here is a Young diagram corresponding to Eq. (10) (but ignoring the constant term, which is the number of empty rows):

If we allow for empty rows, then we can read a function $\pi: E \to B$ as a Young diagram in the following way:
- $B$ is the set of rows.
- $E$ is the set of pairs of a row and a box in that row.
- $\pi: E \to B$ is the projection which sends each pair of a box and a row to that row.
Thinking of maps $\pi: E \to B$ in this way, we can see $\text{Bun}$ as the category of Young diagrams with functions covariant in the rows and boxes, and $\text{Cont}$ as the category of Young diagrams with functions covariant in the rows but contravariant in the boxes. The category $\text{Bun}_{\text{cart}}$ (equivalently $\text{Cont}_{\text{cart}}$) is the category of functions of rows that preserve the number of boxes in each row (though it may permute the boxes within a row).

Next we show that $\text{Bun} \simeq \text{Dir}$ is also equivalent to the category of Dirichlet functors, from Definition 3.1. Recall that a natural transformation is called cartesian if its naturality squares are pullbacks.

**Theorem 4.7.** We have equivalences of categories

$$\text{Poly} \simeq \text{Cont} \quad \text{and} \quad \text{Dir} \simeq \text{Bun}.$$ 

In particular, this gives an equivalence $\text{Poly}_{\text{cart}} \simeq \text{Dir}_{\text{cart}}$ between the category of polynomial functors and cartesian natural transformations and the category of Dirichlet functors and cartesian natural transformations.

**Proof.** The functors $P_-$: $\text{Cont} \to \text{Poly}$ and $D_- : \text{Bun} \to \text{Dir}$ are defined on each object, i.e. function $\pi: s \to t$, by the formula $\pi \mapsto P_\pi$ and $\pi \mapsto D_\pi := P_\pi$ as in Proposition 4.2. For each $1 \leq i \leq t$, denote the fiber of $\pi$ over $i$ by $k_i := \pi^{-1}(i)$.

For any finite set $X$, consider the unique map $X! : X \to 1$. Applying $P_-$ and $D_-$ to it, we obtain the corresponding representable: $P_{X!} \simeq y^X$ and $D_{X!} \simeq X^u$. We next check that there are natural isomorphisms

$$\text{Poly}(P_{X!}, P_{\pi}) \simeq P_{\pi}(X) = \sum X^{k_i} \simeq \text{Cont}(X!, \pi),$$

$$\text{Dir}(D_{X!}, D_{\pi}) \simeq D_{\pi}(X) = \sum_{i=1}^{t} (k_i)^X \simeq \text{Bun}(X!, \pi).$$

(11)

In both lines, the first isomorphism is the Yoneda lemma and the second is a computation using Definition 4.4 (see Fig. 1). Thus we define $P_-$ on morphisms by sending $f: \pi \to \pi'$ in $\text{Cont}$ to the “compose-with-$f$” natural transformation, i.e. having $X$-component $\text{Cont}(X!, f): \text{Cont}(X!, \pi) \to \text{Cont}(X!, \pi')$, which is clearly natural in $X$. We define $D_-$ on morphisms similarly: for $f$ in $\text{Bun}$, use the natural transformation $\text{Bun}(-!, f)$.

By definition, every object in $\text{Poly}$ and $\text{Dir}$ is a coproduct of representables, so to prove that we have the desired equivalences, one first checks that coproducts in $\text{Cont}$ and $\text{Bun}$ are taken pointwise:

$$(\pi: s \to t) + (\pi': s' \to t') \simeq (\pi + \pi'): (s + s') \to (t + t'),$$

and then that $P_{\pi + \pi'} = P_{\pi} + P_{\pi'}$ and $D_{\pi + \pi'} = D_{\pi} + D_{\pi'}$; see Remark 3.4.

By Remark 4.5, we know that $\text{Bun}_{\text{cart}} \simeq \text{Cont}_{\text{cart}}$, and we have just established the equivalences $\text{Poly} \simeq \text{Cont}$ and $\text{Dir} \simeq \text{Bun}$. It thus remains to check that the latter equivalences identify cartesian natural transformations in $\text{Poly}$ with cartesian...
morphisms in $\text{Cont}$, and similarly for $\text{Dir}$ and $\text{Bun}$. For polynomial functors, we may refer to [GK12, Section 2].

Turning to Dirichlet functors, we want to show that for any $f : D \to D'$ the square

\[
\begin{array}{c}
 D(1) \xrightarrow{f_1} D'(1) \\
 \downarrow \pi \downarrow \downarrow \pi' \\
 D(0) \xrightarrow{f_0} D'(0)
\end{array}
\]

is a pullback in $\text{Set}$ iff for all functions $g : X \to X'$, the naturality square

\[
\begin{array}{c}
 D(X') \xrightarrow{f_{x'}} D'(X') \\
 D(g) \downarrow \downarrow D'(g) \\
 D(X) \xrightarrow{f_x} D'(X)
\end{array}
\]

is a pullback in $\text{Set}$; we will freely use the natural isomorphism $D_\pi(X) \cong \text{Bun}(X!, \pi)$ from Eq. (11). The square in Eq. (12) is a special case of that in Eq. (13), namely for $g \equiv 0$! the unique function $0 \to 1$; this establishes the only-if direction.

To complete the proof, suppose that Eq. (12) is a pullback, take an arbitrary $g : X \to X'$, and suppose given a commutative solid-arrow diagram as shown:

We can interpret the statement that Eq. (13) is a pullback as saying that there are unique dotted arrows making the diagram commute, since $DX \cong \text{Bun}(X!, D0!)$ and similarly for the other corners of the square in Eq. (13). So, we need to show that if the front face is a pullback, then there are unique diagonal dotted arrows as shown, making the diagram commute. This follows quickly from the universal property of the pullback. □

**Corollary 4.8.** $\text{Dir}$ is an elementary topos.

**Proof.** For any finite category $C$, the functor category $\text{Fin}^C$ is an elementary topos. The result now follows from Remark 4.5 and Theorem 4.7, noting that $\text{Dir} \cong \text{Fin}^{\to}$. □

As we mentioned in the introduction, this all goes through smoothly when one drops all finiteness conditions. The general topos of Dirichlet functors is the category of (arbitrary) sums of representables $\text{Set}^{\text{op}} \to \text{Set}$, and this is equivalent to the arrow category $\text{Set}^{\to}$ and so is itself a topos.

We conclude with the equivalence promised in Section 1.
Theorem 4.9. A functor $D: \text{Fin}^{\text{op}} \to \text{Fin}$ is a Dirichlet polynomial if and only if it preserves connected limits, or equivalently wide pullbacks.

Proof. Let $D(y) = \sum_{i:D(0)} (d_i)^y$, and suppose that $J$ is any connected category. Then for any diagram $X: J \to \text{Fin}$, we have

$$D(\text{colim} X_j) = \sum_{i:D(0)} (d_i)^\text{colim} X_i$$

$$\approx \sum_{i:D(0)} \text{lim}(d_i)^{X_i}$$

$$\approx \text{lim} \sum_{i:D(0)} (d_i)^{X_i}$$

$$= \text{lim} D(X_j)$$

since connected limits commute with sums in any topos (in particular $\text{Set}$).

Now suppose $D: \text{Fin}^{\text{op}} \to \text{Fin}$ is any functor that preserves connected limits; in particular, it sends wide pushouts to wide pullbacks. Every finite set $X$ can be expressed as the wide pushout

$\includegraphics[scale=0.5]{diagram}$

of its elements. Therefore, we have the following limit diagram:

$\includegraphics[scale=0.5]{diagram2}$

That is, an element of $D(X)$ is a family of elements $a_x \in D(1)$, one for each $x \in X$, such that the $D(0!)(a_x)$ are all equal in $D(0)$. But this is just a bundle map, i.e.

$$D(X) \equiv \text{Bun}(X!, D(0!))$$

where $X!: X \to 1$ and $D(0!): D(1) \to D(0)$. Thus by Theorem 4.7, the functor $D$ is the Dirichlet polynomial associated to the bundle $D(0!)$. \qed
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