On Stationary Solutions of the 2D Doi-Onsager Model

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Abstract

We study the 2D Doi–Onsager models with general potential kernel, with special emphasis on the classical Onsager kernel. Through application of topological methods from nonlinear functional analysis, in particular the Leray–Schauder degree theory, we obtain the uniqueness of the trivial solution for low temperatures as well as the local bifurcation structure of the solutions.

1 Introduction

In 1949, Lars Onsager proposed a mathematical model for the phase transition of equilibria of dilute colloidal solutions of rod-like molecules between the isotropic and nematic phases ([Ons49]). As the fluid in both phases is homogeneous, that is the locations of the molecules do not matter, Onsager’s theory focuses on a probability density function \( f(r) \) over the unit sphere which models distribution of the directions of the rods. Although the original modeling is carried out in \( \mathbb{R}^3 \), the mathematical formulation can be generalized to \( \mathbb{R}^d \) for any dimension \( d \geq 2 \) in a straightforward manner. In the following we present this generalized version.

Denote by \( S^{d-1} \) the unit sphere in \( \mathbb{R}^d \). Let \( f(r) : S^{d-1} \to [0, \infty) \) be the probability density characterizing the directions of the rods, that is

\[
P \left( \text{the rod is along } r \in A \subseteq S^{d-1} \right) = \int_A f(r) d\sigma(r)
\]

where we denote by \( \sigma(r) \) the volume element on \( S^{d-1} \). As we are modeling “rod-like” molecules with no distinction between the two ends, we can further assume \( f(r) = f(-r) \). Consequently the constraints on \( f(r) \) are

\[
f(r) \geq 0, \quad f(r) = f(-r), \quad \int_{S^{d-1}} f(r) d\sigma(r) = 1.
\]

The equilibrium distributions correspond to the critical points of the following functional:

\[
E(f) := \int_{S^{d-1}} (\log f(r)) f(r) d\sigma(r) + \frac{1}{2} \int_{S^{d-1}} (U(f)(r)) f(r) d\sigma(r)
\]

which is derived in [Ons49] as the second order approximation of the free energy — neglecting interactions between three and more molecules.
The interaction potential $U(f)$ in (3) is given by

$$U(f)(r) := \lambda \int_{S^{d-1}} K(r, r') f(r') d\sigma(r')$$

(4)

where the parameter $\lambda > 0$ can be interpreted as either the concentration of the particles in the carrier fluid or the inverse of the absolute temperature. The interaction kernel $K(r, r')$ inherits the following symmetry properties.

$$K(-r, r') = K(r, r'); \quad K(r, r') = K(r', r); \quad K(r, r') = K(Tr, Tr') \quad \forall T \in O(3).$$

(5)

Note that when $d = 2$ we can use the natural parametrization of $S^1$ by the angle $\theta \in [0, 2\pi)$ and rewrite any kernel satisfying (5) as a convolution kernel $K(\theta - \theta')$ for some even function $K$ satisfying $K(\theta + \pi) = K(\theta)$. This reduces the right hand side of (4) to a convolution

$$U(f)(\theta) := \lambda \int_{0}^{2\pi} K(\theta - \theta') f(\theta') d\theta'.$$

(6)

The Euler–Lagrange equation for the system (2)–(4) can be easily written down as

$$f(r) = e^{-U(f)(r)} \int_{S^{d-1}} e^{-U(f)(r')} d\sigma, \quad f(r) = f(-r).$$

(7)

A moment’s inspection reveals that $f(r) \equiv \frac{1}{|S^{d-1}|}$ is always a solution. This constant solution corresponds to the uniform distribution of rod directions and therefore models the “isotropic” or “un-ordered” phase where all directions are equally likely to be taken by the molecules. On the other hand, it has been observed since 1888 (Rei88) that as the temperature $\lambda^{-1}$ decreases, the fluid may go through one or more phase transitions resulting in some order of the directions taken by the molecules. Such phase transition to the so-called nematic phases can be modeled by the bifurcation of the constant solution to non-constant solutions of (2)–(4), or equivalently of (7).

The original kernel proposed by Onsager is

$$K(r, r') = |\sin \theta| \quad (= |r \times r'| \text{ when } d = 3)$$

(8)

where $\theta$ is the angle between the unit vectors $r$ and $r'$. For (7) with this kernel, Onsager showed through asymptotic expansion in [Ons49] that when $\lambda$ is large enough, bifurcation to non-constant solutions occur.

More quantitative analysis of the system (7) with Onsager kernel turned out to be difficult. On the other hand there are kernels capturing the qualitative behavior of the solution while at the same time are more friendly to mathematical analysis. One such kernel, due to Maier and Saupe [MS58], reads

$$K(r, r') = \cos^2 \theta - \frac{1}{3} = (r \cdot r')^2 - \frac{1}{3}.$$  

(9)

The Maier–Saupe kernel is often simply written as $(r \cdot r')^2$ as (7) remains the same if we discard the constant $-\frac{1}{3}$.

The major difference between (7) with Maier–Saupe potential (9) and that with the Onsager potential (8) is that for the former the potential $U(f)$, given by (4), resides in a finite dimensional space, thus reducing the infinite dimensional problem (7) to a finite dimensional nonlinear system.
of equations. This reduced system, still highly nontrivial, is nevertheless more tractable than the original system. As a consequence, \( (7) \) with Maier–Saupe potential has been well understood through brilliant work of many researchers (see [CKT04], [FS05b], [LZZ05a], [ZWFW05], [Liu07], [ZWWF07] for the case \( d = 3 \), [CV05], [FS05a], [LZZ05b] for the case \( d = 2 \), and [WH08] for the general \( d \)-dimensional case.) Inspired by these works, \( (7) \) with other kernels enjoying similar “dimension-reduction” property has also been analyzed, see e.g. [CLW10].

With the Maier–Saupe model \( (7) \) understood, interest in the original Onsager model \( (7)–(8) \) was resurrected. Much progress has been made in the past few years in the case \( d = 2 \). In [CLW10], the axisymmetry of all possible solutions is proved, that is, for any solution \( f(\theta) \) to \( (7)–(8) \), there is \( \theta_0 \) such that \( f(\theta_0 - \theta) = f(\theta_0 + \theta) \). It is also proved in [CLW10] that for appropriate \( \lambda \), there are solutions of arbitrary periodicity. In [WZ08] the authors rewrite \( (7) \) into an infinite system of nonlinear equations for the Fourier coefficients of \( f(\theta) \) and calculated numerically the first few bifurcations. More recently, in [LV10] the authors study the case \( d = 2 \) through cutting-off \( (7)–(8) \) to a finite dimensional system of nonlinear equations, and obtain local bifurcation structure for this finite dimensional approximation.

In this article, we try to gain more understanding of the original infinite dimensional problem \( (7)–(8) \) in the case \( d = 2 \):

\[
\begin{align*}
  f(\theta) &= \frac{e^{-U(f)(\theta)}}{\int_0^{2\pi} e^{-U(f)(\theta')}d\theta'}, \\
  f(\theta) &= f(\pi + \theta), \\
  U(f)(\theta) &= \lambda \int_0^{2\pi} K(\theta - \theta')f(\theta')d\theta'.
\end{align*}
\]

(10)

We show that most of the results obtained in [LV10] for the finite dimensional truncated system of (10) can be generalized to the original infinite dimensional system (10) itself. More specifically, we have the following results.

Let \( k_m, m = 1, 2, 3, \ldots \) be defined through the Fourier expansion

\[
K(\theta) = \sum_{m=0}^{\infty} k_m \cos(2m\theta).
\]

(11)

- **(Theorem 1)** The problem has a unique solution, which must be the constant solution, when \( 0 < \lambda < \lambda_0 := (\sum_{m=1}^{\infty} |k_m|)^{-1} \). This generalizes Proposition 3.1 b) in [LV10].

- **(Theorem 2)** Two solutions bifurcate from the trivial solution at every \( \lambda_m := -\frac{2}{k_m} \). The bifurcation is supercritical if \( \frac{2k_m}{k_m^2} < 1 \) and subcritical if \( \frac{2k_m}{k_m^2} > 1 \). Furthermore, in the former case the first pair of bifurcated solutions are stable and the other bifurcated solutions are unstable, while in the latter case all bifurcated solutions are unstable. This generalizes Proposition 4.4 and Corollary 4.5 in [LV10].

Application of these results to the equation with Onsager’s kernel leads to the following conclusions.

- The problem has a unique (trivial) solution when \( 0 < \lambda < \frac{\pi}{2} \).

- Two solutions bifurcate from the trivial solution at \( \lambda_m = \frac{(4m^2-1)\pi}{2} \), \( m = 1, 2, 3, \ldots \). All bifurcations are supercritical.

- The pair of solutions bifurcating from \( \lambda_1 = \frac{3\pi}{2} \) is stable. All other bifurcated solutions are unstable.
Remark 1. Our method applies in principle to the general cases $d \geq 3$ as well. However some technical difficulties arise and many new measures need to be taken. We will report our effort in this direction in a forthcoming paper.

The remaining of the paper is organized as follows. In Section 2 we rewrite the problem into a new formulation better-suited for the application of topological methods, and carry out the calculation of the Jacobian matrix of the linearized operator. In Section 3 we prove that for all $0 < \lambda < \lambda_0$ the problem has a unique solution, which is trivial. In Section 4 we study the local bifurcation structure of the problem. To improve the readability of the paper, statements of classical results as well as some detailed calculations are delegated to Appendix A.

2 Preparations

2.1 Re-formulation of the Problem

Recall that we need to solve

$$f(r) = e^{-U(f)(r)} \int_{S^{d-1}} e^{-U(f)(r')} d\sigma(r'), \quad f(r) = f(-r). \quad (12)$$

with

$$U(f)(r) = \lambda \int_{S^{d-1}} K(r, r') f(r') d\sigma(r'). \quad (13)$$

Multiplying both sides of (12) by $\lambda K(r, r')$ and integrating over $S^{d-1}$, we cancel $f$ and reach an equation for the potential $U(r)$.

$$U(r) = \frac{\int_{S^{d-1}} \lambda K(r, r') e^{-U(r')} d\sigma(r')} {\int_{S^{d-1}} e^{-U(r)} d\sigma(r)}, \quad U(r) = U(-r). \quad (14)$$

Note that once (14) is solved, $f(r)$ can be recovered from

$$f(r) = \frac{e^{-U(r)} \int_{S^{d-1}} e^{-U(r)} d\sigma(r)} {\int_{S^{d-1}} e^{-U(r)} d\sigma(r)}. \quad (15)$$

Thus (14) is equivalent to the original problem (12)–(13).

From now on we restrict ourselves to the specific case $d = 2$. In this case we can apply the natural parametrization of $S^1$ and write $K(r, r')$ as a convolution kernel $K(\theta - \theta')$. This reduces (14) to

$$U(\theta) = \int_0^{2\pi} \frac{\lambda K(\theta - \theta') e^{-U(\theta')}} {\int_0^{2\pi} e^{-U(\theta')} d\theta'} d\theta', \quad U(\theta) = U(\theta + \pi). \quad (16)$$

Now we define $\overline{K} := \frac{1}{2\pi} \int_0^{2\pi} K(\theta) d\theta$ and denote

$$\tilde{K}(\theta) := K(\theta) - \overline{K}, \quad V(\theta) := U(\theta) - \lambda \overline{K}. \quad (17)$$

It is easy to see that (10) is equivalent to the following.

$$V(\theta) = \lambda \Gamma(V)(\theta) := \lambda \int_0^{2\pi} \frac{\tilde{K}(\theta - \theta') e^{-V(\theta')}} {\int_0^{2\pi} e^{-V(\theta')} d\theta'} d\theta', \quad \int_0^{2\pi} V(\theta) d\theta = 0, \quad V(\theta) = V(\theta + \pi). \quad (18)$$
As the kernel has rotational invariance and the solution is axisymmetric \cite{CLW10}, we can further require
\[ V(\theta) = V(2\pi - \theta). \]
We also assume that \( K(\theta) \in W^{1,\infty}([0,2\pi]) \). Note that this assumption is satisfied by all the kernels proposed in the literature. The natural function space we will be working in is
\[ H := \left\{ V(\theta) \in H^1([0,2\pi]); V(\theta) = V(\theta + \pi) a.e.; \int_0^{2\pi} V(\theta)d\theta = 0; V(\theta) = V(2\pi - \theta) a.e. \right\}. \quad (19) \]

To summarize, we will study the fixed-point problem
\[ V(\theta) = \lambda \Gamma(V)(\theta), \quad V(\theta) \in H, \quad (20) \]
where
\[ \Gamma(V)(\theta) := \frac{\int_0^{2\pi} \tilde{K}(\theta - \theta')e^{-V(\theta')}d\theta'}{\int_0^{2\pi} e^{-V(\theta')}d\theta}. \quad (21) \]

2.2 The Jacobian \( D\Gamma \)

We calculate the Jacobian matrix \( A := D\Gamma \).

Denote \( \phi_n := \frac{\cos(2n\theta)}{\sqrt{(4n^2 + 1)}} \) which with \( n = 1, 2, 3, \ldots \) form an orthonormal basis for \( H \). Then standard calculation gives
\[ D\Gamma(V)(U)(\theta) = \frac{\int_0^{2\pi} \tilde{K}(\theta - \theta') e^{-V(\theta')} d\theta'}{\int_0^{2\pi} e^{-V(\theta')} d\theta} \int_0^{2\pi} U(\theta) e^{-V(\theta')} d\theta - \int_0^{2\pi} \tilde{K}(\theta - \theta') U(\theta') e^{-V(\theta')} d\theta'. \quad (22) \]

If we define the probability measure
\[ d\mu_V := \left( \int_0^{2\pi} e^{-V(\theta')} d\theta' \right)^{-1} e^{-V(\theta)} d\theta \quad (23) \]
then we can simplify (22) to
\[ D\Gamma(V)(U) = \left[ \int_0^{2\pi} \tilde{K}(\theta - \theta') d\mu_V(\theta') \cdot \int_0^{2\pi} U(\theta') d\mu_V(\theta') - \int_0^{2\pi} \tilde{K}(\theta - \theta') U(\theta') d\mu_V(\theta') \right]. \quad (24) \]

which leads to
\[ A := (a_{mn}) := (D\Gamma(V)(\phi_n), \phi_m)_H \]
\[ = \int_0^{2\pi} (D\Gamma(V)(\phi_n))\phi_m d\theta + \int_0^{2\pi} (D\Gamma(V)(\phi_n))' \phi_m' d\theta \]
\[ = \frac{k_m A_{mn}(1 + 4mn)}{\sqrt{4m^2 + 1}\sqrt{4n^2 + 1}}. \quad (25) \]

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where \(k_m\) are the Fourier coefficients of \(\tilde{K}(\theta)\)
\[
\tilde{K}(\theta) := \sum_{m=1}^{\infty} k_m \cos(2m\theta)
\] (26)
and
\[
A_{mn} := \int_0^{2\pi} \cos(2m\theta) d\mu_V(\theta) \cdot \int_0^{2\pi} \cos(2n\theta) d\mu_V(\theta) - \int_0^{2\pi} \cos(2m\theta) \cos(2n\theta) d\mu_V(\theta),
\] (27)

An important property of the matrix \(A\) is \(|a_{mn}| \leq |k_m|\). To see this, we apply Lemma 3 (see Appendix A.3) to (27) to conclude \(|A_{mn}| \leq 1\), from which the conclusion immediately follows.

### 3 Uniqueness of the Trivial Solution

In this section we prove the following theorem.

**Theorem 1.** Assume \(K \in W^{1,\infty}([0, 2\pi])\). Let \(\lambda \sum_{m=1}^{\infty} |k_m| < 1\). Then \(V = 0\), that is the only solution for (7)–(8) is the constant solution. Here \(k_m\) is the \(m\)-th coefficient of the Fourier expansion of \(K(\theta)\).

\[
K(\theta) = \sum_{m=0}^{\infty} k_m \cos(2m\theta).
\] (28)

**Remark 2.** This is a direct generalization of Proposition 3.1 b) of [LV10] to the infinite dimensional case.

The proof applies the classical Leray–Schauder theory to the fixed point problem
\[
(I - \lambda \Gamma)(V) = 0, \quad V \in H
\] (29)
where \(\Gamma\) is defined in (18) and the space \(H\) is defined in (19). To do this we need \(H\) to be Hilbert and \(\Gamma\) to be compact, which are established by the following lemmas whose proofs are delegated to Appendix A.2.

**Lemma 1.** \(H\) is a Hilbert space. Furthermore \(\Gamma : H \to H\) if \(K \in W^{1,\infty}([0, 2\pi])\).

**Lemma 2.** Assume \(K(\theta) \in C([0, 2\pi])\). Then \(\Gamma : H \to H\) is compact.

**Remark 3.** We emphasize that since \(\Gamma\) is nonlinear, compactness here means (see e.g. [Nir01])

i. \(\Gamma\) is continuous;

ii. For every bounded closed \(\Omega \subset H\), \(\overline{\Gamma(\Omega)}\) is compact.

**Proof of Theorem 1.** As \(W^{1,\infty}([0, 2\pi]) \to C([0, 2\pi])\), we can apply Lemma 2 to conclude that \(\Gamma\) is compact.

The proof will now be carried out as follows. First we show the existence of a bounded open set \(\Omega \subset H\) such that there is no solution outside \(\Omega\). Next we show that the degree \(\text{deg}(I - \lambda \Gamma, \Omega, 0) = 1\). Finally we prove that any possible solution to \((I - \lambda \Gamma)(V) = 0\) is isolated with index 1. As in this case the degree is the sum of indices, we know that 0 is the only solution.
The existence of a bounded open set $\Omega \subset H$ such that $(I - \lambda \Gamma)(V) = 0$ has no solution outside $\Omega$. Let $R := \|K\|_{W^{1,\infty}} / \sum_{m=1}^{\infty} |k_m|$. Then it is easy to see that $\|\lambda \Gamma(V)\|_H \leq CR$ for all $\lambda$ satisfying the assumption of the theorem. Thus we can take $\Omega := B_{CR}$, the ball centered at the origin with radius $CR$.

$\deg(I - \lambda \Gamma, \Omega, 0) = 1$.

Introduce the homotopy $H(t) := I - t\lambda \Gamma$ with $t \in [0, 1]$. We easily verify that $H(t)(V) = 0$ has no solution on $\partial \Omega$ for all $t \in [0, 1]$. Consequently

$$\deg(I - \lambda \Gamma, \Omega, 0) = \deg(H(1), \Omega, 0) = \deg(H(0), \Omega, 0) = \deg(I, \Omega, 0) = 1.$$  \hfill (30)

The solutions are isolated.

The Fréchet differentiability of $\Gamma$ can be verified through straightforward calculation, taking advantage of the embedding $H \hookrightarrow L^\infty([0, 2\pi])$. The solutions are isolated if we can show that $I - \lambda D\Gamma$ is a homeomorphism. As $\lambda \Gamma$ is compact, so is the derivative $\lambda D\Gamma$. Applying standard Fredholm alternative (see e.g. [AP93]) we see that all we need to show is that $\ker(I - \lambda D\Gamma) = \{0\}$.

Take any $U \in \ker(I - \lambda D\Gamma)(V)$. We have, following (24) in Section 2.2:

$$U(\theta) = \lambda \left[ \int_0^{2\pi} \tilde{K}(\theta - \theta')d\mu_V(\theta') \cdot \int_0^{2\pi} U(\theta')d\mu_V(\theta') - \int_0^{2\pi} \tilde{K}(\theta - \theta')U(\theta')d\mu_V(\theta') \right] \tag{31}$$

where $d\mu_V$ is as defined in (23).

Application of Lemma 3 gives

$$|U(\theta)| \leq \lambda \|\tilde{K}\|_{L^\infty} \|U\|_{L^\infty}, \quad \forall \theta \in [0, 2\pi].$$  \hfill (32)

By assumption $\lambda \sum_{m=1}^{\infty} |k_m| < 1$ which leads to $\lambda \|\tilde{K}\|_{L^\infty} < 1$, consequently $U = 0$.

The index of any solution is 1.

Following the calculation in Section 2.2 we have $|a_{mn}| \leq |k_m|$ where $(a_{mn})$ is the infinite dimensional matrix representation of $D\Gamma$ with respect to the orthonormal basis $\left\{ \frac{\sqrt{4n^2+1}}{\pi} \cos(2n\theta) \right\}_{n=1}^{\infty}$ of $H$. By assumption $\sum_{m=1}^{\infty} |k_m| < 1$, therefore the eigenvalues of $I - \lambda D\Gamma$ are all bounded below by a positive constant. Consequently the index of the map $I - \lambda \Gamma$ is 1 everywhere.

Thus we see that the desired conclusion holds when $\lambda \|\tilde{K}\|_{L^\infty} < 1$ and $\sum_{m=1}^{\infty} \lambda |k_m| < 1$. As $\|\tilde{K}\|_{L^\infty} \leq \sum_{m=1}^{\infty} |k_m|$, Theorem 1 is proved.

Remark 4. For Onsager kernel we have $K(\theta) = |\sin \theta|$, $k_m = -\frac{4}{\pi(4n^2-1)}$. Theorems 1 then gives $\lambda_0 = \frac{\pi}{2}$.

## 4 Bifurcation Analysis

We study the criticality and stability of bifurcated solutions from the trivial solution. Our results generalize Proposition 4.4 and Corollary 4.5 in [LV10].
Theorem 2. Let $k_m < 0$ satisfies $-k_1 > -k_2 > \cdots > 0$. Then

a) **(Bifurcation points)** two solutions bifurcate from the trivial solution at every $\lambda_m := -\frac{2}{k_m}$.

b) **(Criticality)**

- if $\frac{2k_m}{k_m} < 1$, both bifurcated solutions from $\lambda_m$ are supercritical;
- if $\frac{2k_m}{k_m} > 1$, both bifurcated solutions from $\lambda_m$ are subcritical.

c) **(Stability)** the bifurcated solutions from $\lambda_m$, $m \geq 2$ are unstable. The bifurcated solutions from $\lambda_1$ are stable if $\frac{2k_1}{k_m} < 1$ and unstable if $\frac{2k_1}{k_m} > 1$.

**Proof.**

a) At the trivial solution $V = 0$ we have $d\mu_V = \frac{d\theta}{2\pi}$. Following the calculation in Section 2.2, the matrix $(a_{mn})$ for the Jacobian $D\Gamma$ is given by

$$a_{mn} = \begin{cases} -\lambda \frac{k_m}{2} & m = n \\ 0 & m \neq n \end{cases}$$

and is thus diagonal.

Denoting $\lambda_m = -\frac{2}{k_m}$ and $K = D\Gamma$, we see that $\lambda_m$ is a simple characteristic value of $K$, and the dimensions of $\ker(I - \lambda_m K)$ and $\text{Ran}(I - \lambda_m K)^\perp$ are both 1, which means $I - \lambda_m K$ is Fredholm with index zero.

Furthermore, as $K = D\Gamma(0)$ and $\Gamma(0) = 0$ we have

$$\Gamma(0) = \frac{\int_0^{2\pi} \tilde{K}(\theta - \theta')e^{-V(\theta')}d\theta'}{\int_0^{2\pi} e^{-V(\theta')}d\theta}.$$  \hspace{1cm} (35)

Expanding around the trivial solution we have

$$\Gamma(V) = T(V) + N(V)$$  \hspace{1cm} (36)

where

$$T(V) = -\frac{1}{2\pi} \int_0^{2\pi} \tilde{K}(\theta - \theta')V(\theta')d\theta';$$  \hspace{1cm} (37)

$$N(V) = -\frac{1}{2}T(V^2) + \frac{1}{6}T(V^3) - \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{V^2}{2^2} \right) T(V) + O(V^4);$$  \hspace{1cm} (38)

Writing the orthonormal basis as

$$\phi_n(\theta) = c_n \cos(2n\theta),$$  \hspace{1cm} (39)
where \( c_n = 1/\sqrt{4n^2 + 1} \), we have

\[
T(\phi_n) = \left(-\frac{k_n}{2}\right) \phi_n = \lambda_n^{-1} \phi_n. \tag{40}
\]

By Theorem 3 and Corollary 1 (see Appendix A.1) we can write the bifurcated solution from \( \lambda_n \) as

\[
V = t\phi_n + t^2 z \tag{41}
\]

where the \( H^1 \)-inner product \( \langle \phi_n, z \rangle = 0 \).

Next writing \( \lambda = \lambda_n + \mu \) we have

\[
t^2 z = \left(-\frac{2}{k_n}\right) t^2 T(z) + \mu t^2 T(z) + t \left(-\frac{k_n}{2}\right) \phi_n
\]

\[
+ (\lambda_n + \mu) \left[-\frac{t^2 \frac{c_n^2}{4}}{\phi_{2n}} \left(-\frac{k_{2n}}{2}\right) \phi_{2n} - t^3 T(\phi_n)z\right]
\]

\[
+ (\lambda_n + \mu) \left[-\frac{t^3 \frac{c_n^3}{24 \phi_{3n}} \left(-\frac{k_{3n}}{2}\right) \phi_{3n} - \frac{t^3}{8} \frac{c_n^2}{\phi_n} \left(-\frac{k_n}{2}\right) \phi_n\right] + O(t^4). \tag{42}
\]

Taking \( H^1 \)-inner product with \( \phi_n \) and using the facts that \( \langle \phi_n, z \rangle = 0 \), \( \langle T(z), \phi_n \rangle = 0 \) we reach

\[
\mu = t^2 \lambda_n \left(-\frac{k_n}{2}\right)^{-1} \left[ \langle T(\phi_n z), \phi_n \rangle + \left(-\frac{k_n}{2}\right) \frac{c_n^2}{8}\right] + O(t^3). \tag{43}
\]

Consequently \( \mu = O(t^2) \) and (42) can be simplified to

\[
t^2 z = \left(-\frac{2}{k_n}\right) t^2 T(z) + t \left(-\frac{k_n}{2}\right) \phi_n
\]

\[
+ \left(-\frac{2}{k_n}\right) \frac{t^2 \frac{c_n^2}{4}}{\phi_{2n}} \left(-\frac{k_{2n}}{2}\right) \phi_{2n} - t^3 T(\phi_n)z
\]

\[
+ \left(-\frac{2}{k_n}\right) \frac{t^3 \frac{c_n^3}{24 \phi_{3n}} \left(-\frac{k_{3n}}{2}\right) \phi_{3n} - \frac{t^3}{8} \frac{c_n^2}{\phi_n} \left(-\frac{k_n}{2}\right) \phi_n\right] + O(t^4). \tag{44}
\]

To obtain the sign of \( \mu \), we need to calculate \( \langle T(\phi_n z), \phi_n \rangle \). Writing

\[
z = \sum_{k=2}^{\infty} z_k \phi_k. \tag{45}
\]

We have

\[
\langle T(\phi_n z), \phi_n \rangle = \left(-\frac{k_n}{2}\right) z_{2n} \frac{c_{2n}}{2}. \tag{46}
\]

Now we calculate \( z_{2n} \). Taking \( H^1 \)-inner product of (44) with \( \phi_{2n} \), we finally reach

\[
z_{2n} = \frac{\gamma_n \frac{c_n^2}{\gamma_n - 1 \frac{c_{2n}}{4}}} \tag{47}
\]

where \( \gamma_n := \frac{k_{2n}}{k_n} \).
Putting things together, we have

\[
\mu = t^2 \lambda_n \left[ 2 \lambda_n \frac{C_{2n}}{2} + \frac{C_3}{8} \right] + O(t^3) = t^2 \lambda_n \frac{C_3}{8} 2 \gamma - 1 + O(t^3)
\]  
(48)

and consequently the bifurcation is super-critical if \((2 \gamma_n - 1)/(\gamma_n - 1) > 0\) and sub-critical if \((2 \gamma_n - 1)/(\gamma_n - 1) < 0\). The conclusion of the theorem thus follows.

c) It is clear that the trivial solution is stable for \(\lambda < \lambda_1\) and unstable for \(\lambda > \lambda_1\). Therefore the bifurcated solutions from \(\lambda_m\) with \(m \geq 2\) are unstable, independent of their criticality.

For the bifurcations from \(\lambda_1\), we check that the assumptions of Theorem 4 (see Appendix A.1) are all satisfied at \(\lambda_1\).

First recall that a bifurcation point \(\mu_0\) is “regular” if the linearized operator is invertible “for all \(\mu\) sufficiently close to \(\mu_0\) but \(\mu \neq \mu_0\)” ([Sat71]). As the linearized operator \(D \Gamma\) is diagonal, we easily see that all the bifurcation points under discussion are regular.

Next we check the smoothness conditions for Theorem 4. The nonlinear remainder term \(N(\lambda, V)\) is given by

\[
N(\lambda, V) = \lambda(D \Gamma(0) - \Gamma)(V) = -\lambda \left[ \frac{1}{2\pi} \int_0^{2\pi} \tilde{K}(\theta - \theta') V(\theta') d\theta' + \frac{\int_0^{2\pi} \tilde{K}(\theta - \theta') e^{-V(\theta')} d\theta'}{\int_0^{2\pi} e^{-V(\theta')} d\theta} \right].
\]  
(49)

It is easy to see that \(N(\lambda, V)\) is Fréchet differentiable as

\[
\delta V \to 0 \text{ in } H \implies \delta V \to 0 \text{ in } L^\infty \implies e^{-(V + \delta V)} \to e^{-V} \text{ uniformly}
\]  
(50)

thanks to the embedding \(H \hookrightarrow L^\infty([0, 2\pi])\). Similarly we can show that \(N(\lambda, V)\) is twice Fréchet differentiable.

Finally we define \(N_1(\lambda, V, \alpha) := \alpha^{-2} N(\lambda, \alpha V)\) and prove that it is Fréchet differentiable in \(\lambda, V\) and \(\alpha\). It is obvious that \(N_1\) is Fréchet differentiable in \(\lambda\) and \(V\). To see that it is also Fréchet differentiable in \(\alpha\), we write

\[
N_1(\lambda, V, \alpha) = -\lambda \int_0^{2\pi} \tilde{K}(\theta - \theta') R(\theta', \alpha) d\theta'
\]  
(51)

where

\[
R(\theta, \alpha) := \alpha^{-2} \left[ -1 + \alpha V(\theta) + \frac{e^{-\alpha V(\theta)}}{2\pi} \int_0^{2\pi} e^{-\alpha V(\theta')} d\theta' \right].
\]  
(52)

Using Taylor expansion with integral form of remainder, we have

\[
e^{-\alpha V(\theta)} = 1 - \alpha V(\theta) + \int_0^{\alpha V(\theta)} (\alpha V(\theta) - t) e^{-t} dt = 1 - \alpha V(\theta) + \alpha^2 \int_0^{V(\theta)} (V(\theta) - s) e^{-\alpha s} ds.
\]  
(53)
Substituting this into (52) we have
\[
R(\theta, \alpha) = \frac{(-1 + \alpha V(\theta)) \int_0^{2\pi} M(\theta, \alpha) d\theta + 2\pi M(\theta, \alpha)}{2\pi \int_0^{2\pi} e^{-\alpha V(\theta)} d\theta}
\]
(54)
where \( M(\theta, \alpha) := \int_0^V(\theta)(V(\theta) - s)e^{-\alpha s} ds \). We see that clearly \( R(\theta, \alpha) \) is differentiable in \( \alpha \) and so is \( N_1(\lambda, V, \alpha) \).

Thus we can apply Theorem 4 to immediately conclude:

- If \( \frac{d}{d\lambda} < 1 \), then the bifurcated solutions from \( \lambda_1 \) are stable;
- If \( \frac{d}{d\lambda} > 1 \), then the bifurcated solutions from \( \lambda_1 \) are unstable.

**Remark 5.** All the bifurcations from the trivial solution for the Onsager model are super-critical. For Onsager kernel the bifurcation values are \( \lambda_m = \frac{(4m^2 - 1)\pi}{2} \), \( m = 1, 2, 3, \ldots \). We see that the first bifurcation value is \( \frac{3\pi}{2} \). Thus there is a gap between it and the uniqueness region \( \lambda < \lambda_0 = \frac{5\pi}{2} \).

### A Auxiliary Lemmas and Known Theorems

#### A.1 Classical Results from Nonlinear Analysis

The following classical results in nonlinear analysis are crucial in our analysis.

**Theorem 3** ([Die85], Theorem 28.3). Let \( X \) be a real Banach space, \( K \in L(X) \), \( \Omega \subset \mathbb{R} \times X \) a neighborhood of \( (\lambda_0; 0) \) and \( G : \Omega \rightarrow X \) such that \( G_\lambda, G_z, G_{\lambda z} \) are continuous on \( \Omega \). Suppose also that

a) \( G(\lambda, x) = o(\|x\|) \) as \( x \rightarrow 0 \) uniformly in \( \lambda \) near \( \lambda_0 \);

b) \( I - \lambda_0 K \) is Fredholm of index zero and \( \lambda_0 \) is a simple characteristic value of \( K \).

Then \( (\lambda_0; 0) \) is a bifurcation point for \( F(\lambda, x) = x - \lambda K + G(\lambda, x) = 0 \) and there is a neighborhood \( U \) of \( (\lambda_0; 0) \) such that

\[
F^{-1}(0) \cap U = \{ (\lambda_0 + \mu(t), tv + tz(t)) : |t| < \delta \} \cup \{ (\lambda; 0) : (\lambda; 0) \in U \}
\]
(55)
for some \( \delta > 0 \), with continuous functions \( \mu(\cdot) \) and \( z(\cdot) \) such that \( \mu(0) = 0, z(0) = 0 \) and the range of \( z(\cdot) \) is contained in a complement of \( N(I - \lambda_0 K) = \text{span}\{v\} \).

**Corollary 1** ([Die85], Corollary 28.1). Let the hypotheses of Theorem 28.3 be fulfilled. If \( G \) is \( C^k \) near \( (\lambda_0; 0) \) for some \( k \geq 2 \) then the functions \( \mu(\cdot), z(\cdot) \), defining the branches of nontrivial zeros, are \( C^{k-1} \). If \( G \) is real (or complex) analytic then \( \mu(\cdot) \) and \( z(\cdot) \) are real (or complex) analytic.

We also made use of the following result by Sattinger.

**Theorem 4** ([Sat71], Theorem 4.2). Let \( (\mu_0, 0) \) be a regular bifurcation point of (3.1) and let \( N \) be twice continuously Fréchet differentiable, with \( N(\mu, ou) = \alpha^2 N_1(\mu; u; \alpha) \) where \( N_1 \) is Fréchet differentiable in \( \mu, u \) and \( \alpha \). Then the supercritical bifurcating solutions are stable and subcritical bifurcating solutions are unstable.
A.2 Properties of $H$ and $\Gamma$

Proof of Lemma 1. We first prove that $H$, as defined in (19),

$$H := \left\{ V(\theta) \in H^1([0, 2\pi]); V(\theta) = V(\theta + \pi) a.e.; \int_0^{2\pi} V(\theta) d\theta = 0; V(\theta) = V(2\pi - \theta) a.e. \right\}.$$  \hspace{1cm} (56)

is a Hilbert space.

Since $H$ is a subspace of the Hilbert space $H^1([0, 2\pi])$, all we need to show is that it is closed in the topology of $H^1$, which is trivial.

Next it is easy to check that $\Gamma(V)$ satisfies the 2nd, 3rd, and 4th requirements in (56). To show that it is in $H^1$, we calculate

$$\|\Gamma(V)\|_{L^2}^2 = \int_0^{2\pi} [\Gamma(V)(\theta)]^2 d\theta = \int_0^{2\pi} \left[ \int_0^{2\pi} \tilde{K}(\theta - \theta') e^{-V(\theta')} d\theta' \right]^2 d\theta \leq 2\pi \|\tilde{K}\|_{L^\infty}^2 = 2\pi \|\tilde{K}\|_{L^\infty}^2 < \infty.$$ \hspace{1cm} (57)

Similarly, we have $\|\frac{d}{d\theta} \Gamma(V)\|_{L^2}^2 \leq 2\pi \|K'\|_{L^\infty}^2 < \infty$. Thus ends the proof of Lemma 1.

Proof of Lemma 2. Next we prove the continuity and compactness of the operator $\Gamma$. Recall that $\Gamma$ is defined in (21) as

$$\Gamma(V)(\theta) = \frac{\int_0^{2\pi} \tilde{K}(\theta - \theta') e^{-V(\theta')} d\theta'}{\int_0^{2\pi} e^{-V(\theta')} d\theta}. \hspace{1cm} (58)$$

• Continuity. Let $\delta V \to 0$ in $H$. We first show that $\Gamma(V + \delta V)(\theta) \to \Gamma(V)(\theta)$ in $L^2$. Thanks to the embedding $H^1([0, 2\pi]) \hookrightarrow L^\infty([0, 2\pi])$, we have $\delta V \to 0$ in $L^\infty$. Consequently $e^{-(V + \delta V)} \to e^{-V}$ uniformly and it follows that $\Gamma(V + \delta V)(\theta) \to \Gamma(V)(\theta)$ uniformly and the conclusion follows.

Next we show that $\frac{d}{d\theta} \Gamma(V + \delta V)(\theta) \to \frac{d}{d\theta} \Gamma(V)(\theta)$ in $L^2$. We calculate

$$\frac{d}{d\theta} \Gamma(V)(\theta) = \frac{-\int_0^{2\pi} \tilde{K}(\theta - \theta') e^{-V(\theta')} V'(\theta') d\theta'}{\int_0^{2\pi} e^{-V(\theta')} d\theta}. \hspace{1cm} (59)$$

As $e^{-(V + \delta V)} \to e^{-V}$ uniformly, $e^{-(V + \delta V)}(V' + \delta V') \to e^{-V} V'$ in $L^2$ which together with $\tilde{K} \in L^\infty$ implies $\frac{d}{d\theta} \Gamma(V + \delta V)(\theta) \to \frac{d}{d\theta} \Gamma(V)(\theta)$ in $L^\infty$ and consequently also in $L^2$.

• Compactness. Assume $\Omega \subset B_R$ be a bounded closed subset of $H$, where $B_R$ denotes the ball with radius $R$ in $H$. It suffices to show that there are operators $\Gamma_n \to \Gamma$ whose ranges are finite dimensional (see e.g. [Nir01]). By the Weierstrass approximation theorem, for any $n \in \mathbb{N}$, there is $K_n(\theta) = \sum_{i=1}^{L_n} [a_{ni} \cos(m_n \theta) + b_{ni} \sin(m_n \theta)]$ such that

$$|\tilde{K}(\theta) - K_n(\theta)| < \frac{1}{n} \hspace{1cm} \forall \theta \in [0, 2\pi]. \hspace{1cm} (60)$$
Now we define
\[
\Gamma_n(V)(\theta) := \int_0^{2\pi} K_n(\theta - \theta') \frac{e^{-V(\theta')}}{e^{-V(\theta)}} d\theta'
\] (61)

It is easy to check that
\[
\Gamma_n(V)(\theta) \in \text{span}\{\cos(m_i\theta), \sin(m_i\theta)\}_{i=1}^{L_n}
\] (62)
for any \(V(\theta) \in H\) which means the range of \(\Gamma_n\) is finite dimensional.

Finally check
\[
\|\Gamma_n - \Gamma\|_{H \rightarrow H} \leq \sup_{\|V\|_{L^2} \leq R} \left\| \int_0^{2\pi} (\tilde{K}(\theta - \theta') - K_n(\theta - \theta')) e^{-V(\theta')} d\theta' \right\|_{H^1} \\
\leq \sup_{\|V\|_{L^2} \leq R} \left\| \int_0^{2\pi} (\tilde{K}(\theta - \theta') - K_n(\theta - \theta')) e^{-V(\theta')} d\theta' \right\|_{L^2} \\
+ \sup_{\|V\|_{H^1} \leq R} \left\| \frac{\int_0^{2\pi} (\tilde{K}(\theta - \theta') - K_n(\theta - \theta')) e^{-V(\theta')} V'(\theta') d\theta'}{\int_0^{2\pi} e^{-V(\theta')} d\theta} \right\|_{L^2} \\
\leq C(R) [||\tilde{K} - K_n||_{L^\infty} + ||\tilde{K} - K_n||_{L^2}] \leq \frac{C(R)}{n}. \quad (63)
\]

The calculation is similar to that in the proof of Lemma \(\square\) and is omitted here. The arbitrariness of \(n\) now gives the desired result.

### A.3 A Grüss type inequality

The following Grüss type inequality will play a crucial role in the proofs.

**Lemma 3.** Let \(\mu\) be a probability measure over a domain \(\Omega\). Let \(f, g \in L^\infty(\Omega)\) satisfy \(a \leq f \leq A, b \leq g \leq B\). Then
\[
|\int_{\Omega} f(x)g(x)d\mu - \left(\int_{\Omega} f(x)d\mu\right) \left(\int_{\Omega} g(x)d\mu\right)| \leq \frac{(A - a)(B - b)}{4}. \quad (64)
\]

in particular we have
\[
|\int_{\Omega} f(x)g(x)d\mu - \left(\int_{\Omega} f(x)d\mu\right) \left(\int_{\Omega} g(x)d\mu\right)| \leq \|f\|_{L^\infty} \|g\|_{L^\infty}. \quad (65)
\]

**Proof.** This is a simple generalization of the classical Grüss inequality. The proof is almost identical to that in [Dra00] and is therefore omitted.

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