Diophantine approximation with smooth numbers

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In memory of Richard Askey

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Abstract
Let \( \theta \) be an irrational number and \( \varphi \) a real number. Let \( C > 2 \) and \( \varepsilon > 0 \). There are infinitely many positive integers \( n \) free of prime factors > \((\log n)^C\), such that
\[
\|\theta n + \varphi\| < n^{-(\frac{1}{3} - \frac{2}{3C}) + \varepsilon}.
\]
Here \( \|y\| \) is the distance from \( y \) to \( \mathbb{Z} \).

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1 Introduction

Let \( \theta \) be an irrational number and \( \varphi \) a real number. There are many results in the literature concerning small distances \( \|\theta n + \varphi\| \) of \( \theta n + \varphi \) from the nearest integer suggested by the classical inequality.

\[
\|\theta n\| < 1/n
\]

for infinitely many \( n \in \mathbb{N} \). We mention five results here. Harman [3] showed that
\[
\|\theta p + \varphi\| < p^{-7/22}
\]
for infinitely many primes \( p \), and Matomaki [9] improved the exponent to \(-1/3 + \varepsilon\) in the case \( \varphi = 0 \). The exponent in [9] is the limit of current methods; however, Irving [8] showed that
\[
\|\theta n\| < n^{-8/23+\varepsilon}
\]

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for infinitely many $n$ that are products of two primes. (As usual, $\varepsilon$ denotes an arbitrary positive number.)

For square-free integers one can do much better. Heath-Brown [6] proved that

$$\|\theta n\| < n^{-2/3+\varepsilon}$$

for infinitely many square-free integers $n$.

It is natural to ask what can be done for smooth integers $n$. Yau [10] has shown (among other results) that

$$\|\theta n + \varphi\| < n^{-1/3+\varepsilon}$$

(1.1)

for infinitely many integers $n$ free of prime factors greater than $n^\varepsilon$. (Thanks are due to Glyn Harman for pointing out this paper to me.)

In the present paper we consider stronger smoothness conditions on $n$.

**Theorem** Let $\theta$ be an irrational number and $\varphi$ a real number. Let $C > 2$. There are infinitely many $n \in \mathbb{N}$ free of prime factors greater than $(\log n)^C$ for which

$$\|\theta n + \varphi\| < n^{-\left(\frac{1}{3} - \frac{2}{C}\right)+\varepsilon}. \tag{1.2}$$

Exponential sum bounds for the $y$-smooth numbers up to $x$ given by Fouvry and Tenenbaum [2] and Harper [5] appear to deliver weaker bounds than (1.2). Our exponential sum, by contrast, is tailored to this particular application.

We conclude this section with lemmas that will be used in the proof of the theorem. Let $\Psi(x, y)$ denote the number of $n \leq x$ that are free of prime factors $> y$. Let $u = \log x / \log y$.

**Lemma 1** For $x \geq y \geq 2$, denote by $\alpha = \alpha(x, y)$ the unique solution of

$$\sum_{p \leq y} \log p \cdot \frac{p^\alpha - 1}{p^\alpha} = \log x.$$

For $1 \leq c \leq y$, we have

$$\Psi(cx, y) = \Psi(x, y)\cdot c^{\alpha(x, y)}\left(1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right)\right).$$

**Proof** This is Corollary 1.7 of [7]. ⊓⊔

**Lemma 2** For sufficiently large $x$ and $y = (\log x)^C$, $C > 1$, we have, with $\alpha = \alpha(x, y)$,

$$\left|\alpha - \left(1 - \frac{1}{C}\right)\right| < \varepsilon \tag{1.3}$$

and

$$x^{1-\frac{1}{C}-\varepsilon} \ll \Psi(x, y) \ll x^{1-\frac{1}{C}+\varepsilon}. \tag{1.4}$$

Implied constants will depend at most on $\varepsilon$. 

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Proof In [7] it is shown that
\[ \alpha = 1 - \frac{\log(u \log(u + 1))}{\log y} + O \left( \frac{1}{\log y} \right). \]
The inequality (1.3) follows on substituting \( y = (\log x)^C \).

The bounds in (1.4) are obtained by combining (1.7) of [7] with Theorem 2 of [7]. \( \square \)

Lemma 3 Let \( x_1, \ldots, x_N \) be a real sequence with \( \|x_n\| \geq M^{-1} \) \((1 \leq n \leq N)\) where \( M \) is a positive integer. Then
\[ \sum_{h=1}^{M} \left| \sum_{n=1}^{N} e(hx_n) \right| \geq N/6. \]

Proof See e.g. [1]. \( \square \)

We write ‘\( m \sim M \)’ to indicate \( M \leq m < 2M \).

Lemma 4 Let \( M > 1, N > 1 \). Let \( a_m (m \sim M), b_n (n \sim N) \) be complex numbers with \( |a_m| \leq 1, |b_n| \leq 1 \). For a real number \( \theta \) and a fraction \( a/q \) with \( |\theta - a/q| < 1/q^2 \), \( (a, q) = 1 \), we have
\[ \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(mn\theta) \ll \left( \sum_{m \sim M} |a_m|^2 \sum_{n \sim N} |b_n|^2 \right)^{1/2} \left( \frac{MN}{q} + M + N + q \right)^{1/2} \left( \log 2MN \right)^{1/2}. \]

Proof See p. 23 of [4] \( \square \)

Let \( S_c(x, y) \) denote the set of integers in \([x, cx)\) free of prime factors \( > y \).

2 Proof of the Theorem

Let \( a/q \) be a convergent to the continued fraction of \( \theta \) with \( q \) sufficiently large. Thus
\[ \left| \theta - \frac{a}{q} \right| < \frac{1}{q^2}, \ (a, q) = 1. \] (2.1)

We may suppose that \( \varepsilon \) is sufficiently small, so that
\[ \gamma := \frac{1}{3} - \frac{2}{3C} - \frac{5\varepsilon}{6} \]
is positive.
Define \( x \) by
\[
x^{\frac{1+y}{2}} = q.
\] (2.2)

Suppose that \( n \in \mathbb{N} \) satisfies
\[
\left\| \frac{an}{q} + \varphi \right\| \leq x^{-\gamma}, \; n \in S_4(x, y).
\] (2.3)

Here and below, \( y = (\log x)^C \). Then \( n \) is free of prime factors \( > (\log n)^C \). Moreover,
\[
\left\| \theta n + \varphi \right\| \leq x^{-\gamma} + \left| \theta - \frac{a}{q} \right| \cdot 4x
\leq x^{-\gamma} + \frac{4x}{x^{1+y}} < x^{-\gamma} + \frac{4x}{x^{1+y}} < n^{-\left(\frac{1}{2} - \frac{2}{3C}\right) + \varepsilon},
\]
from (2.1)–(2.3). Hence it will suffice to show that (2.3) has a solution \( n \).

Let \( S = S_2(x^{1+y}, y), \; J = S_2(x^{-1+y}, y) \). By Lemmas 1 and 2,
\[
x^{1+y} \left(1 - \frac{1}{C}\right)^{-\varepsilon} \ll |S| \ll x^{1+y} \left(1 - \frac{1}{C}\right)^{+\varepsilon},
\] (2.4)
\[
x^{-1+y} \left(1 - \frac{1}{C}\right)^{-\varepsilon} \ll |J| \ll x^{-1+y} \left(1 - \frac{1}{C}\right)^{+\varepsilon},
\] (2.5)

Here \( |\cdots| \) indicates cardinality. We shall show that there are solutions of (2.3) with
\[
n = uv, \; u \in S, \; v \in J.
\]

Suppose there are no such solutions of (2.3). Let
\[
S_h = \sum_{u \in S} \sum_{v \in J} e\left(\frac{hauv}{q}\right).
\]

Let \( H = [x^y] + 1 \). We deduce from Lemma 3 and (2.4), (2.5) that
\[
\sum_{h=1}^{H} |S_h| \gg x^{1-y} \frac{1}{C} - \frac{\varepsilon}{8}.
\] (2.6)

We choose complex numbers \( c_h, \; |c_h| \leq 1 \), such that
\[
\sum_{h=1}^{H} |S_h| = \sum_{h=1}^{H} c_h \sum_{u \in S} \sum_{v \in J} e\left(\frac{hauv}{q}\right)
= \sum_{u \leq 2x^{(1+y)/2}} \sum_{w \leq 2x^{(1-y)/2}} a_u b_w e\left(\frac{auw}{q}\right).
\] (2.7)

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Here

\[ a_u = \begin{cases} 
1 & \text{if } u \in S, \\
0 & \text{otherwise}, 
\end{cases} \]

\[ b_w = \sum_{\substack{hv = w \\ h \leq H, v \in J}} c_h. \]

Thus, recalling (2.4), (2.5),

\[ \sum_{u \leq 2x^{(1+\gamma)/2}} a_u^2 \ll x^{\frac{1+\gamma}{2}}(1-\frac{\gamma}{2}) + \frac{\varepsilon}{2}, \]

while \( b_w \ll x^{\varepsilon/8} \) and

\[ \sum_{w \leq 2x^{(1-\gamma)/2}} b_w^2 \ll x^{\varepsilon/8} \sum_{w \leq 2x^{(1-\gamma)/2}} b_w \ll x^{\left(\frac{1-\gamma}{2}\right)(1-\frac{1}{C}) + \gamma + \frac{\varepsilon}{4}}. \]

Lemma 4 can be applied to the double sum in (2.7) (after a dyadic dissection of the ranges of \( u, w \)) to give the bound

\[ \sum_{h=1}^{H} |S_h| \ll \left(\frac{x^{\frac{1+\gamma}{2}}(1-\frac{1}{C}) + \frac{1-\gamma}{2}(1-\frac{1}{C}) + \gamma + \frac{\varepsilon}{2}}{q} + x^{\frac{1+\gamma}{2}} + x^{\frac{1-\gamma}{2}} H + q\right)^{1/2} \]

\[ \ll \left(x^{1-\frac{1}{C} + \gamma + \frac{\varepsilon}{2}}\right)^{\frac{1}{2}} \left(x^{\frac{1+\gamma}{2}}\right)^{\frac{1}{2}}. \quad (2.8) \]

Combining (2.6), (2.8), we have

\[ x^{1-\frac{1}{C} - \frac{\varepsilon}{8}} \ll x^{\frac{1}{2}(1-\frac{1}{C}) + \frac{3\gamma}{4} + \frac{1}{2} + \frac{\varepsilon}{4}}, \]

\[ \frac{3\gamma}{4} \geq \frac{1}{2} \left(1 - \frac{1}{C}\right) - \frac{1}{4} - \frac{2\varepsilon}{5}, \]

\[ \gamma \geq \frac{1}{3} - \frac{2}{3C} - \frac{4\varepsilon}{5}. \]

This contradicts the definition of \( \gamma \). We conclude that solutions of (2.3) exist. This completes the proof of the theorem.

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