CLOSED-FORM EXACT INVERSES OF THE WEAKLY SINGULAR AND HYPERSINGULAR OPERATORS ON DISKS

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Abstract. We introduce new boundary integral operators which are the exact inverses of the weakly singular and hypersingular operators for $-\Delta$ on flat disks. Moreover, we provide explicit closed forms for them and prove the continuity and ellipticity of their corresponding bilinear forms in the natural Sobolev trace spaces. This permits us to derive new Calderón-type identities that can provide the foundation for optimal operator preconditioning in Galerkin boundary element methods.

1. Introduction. We study the weakly singular and hypersingular Boundary Integral Operators (BIOs) arising when solving screen problems in $\mathbb{R}^3$ via Boundary Integral Equations (BIEs). In particular, we consider the following singular BIEs on the flat disk $D_a$ of radius $a > 0$:

\begin{align}
(V \sigma)(y) &:= \frac{1}{4\pi} \int_{D_a} \frac{\sigma(x)}{\|x-y\|} d\sigma_a(x) = g(y), \\
(W u)(y) &:= \frac{1}{4\pi} \int_{D_a} u(x) \frac{\partial^2}{\partial n_x \partial n_y} \frac{1}{\|x-y\|} d\sigma_a(x) = \mu(y),
\end{align}

for $y \in D_a$ and functions $\sigma, g, u, \mu$ in suitable trace Sobolev spaces that will be made explicit later on. Here $\|\cdot\|$ stands for the standard Euclidean norm and $\partial/\partial n$ for the normal derivative in a direction perpendicular to the disk $D_a$.

Equations (1.1) and (1.2) are connected with the following exterior boundary value problem for the Laplacian $-\Delta$ [16, 22, 23]: find $U \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D}_a)$ such that

\begin{align}
\begin{cases}
-\Delta U = 0 & \text{in } \Omega_a := \mathbb{R}^3 \setminus \overline{D}_a, \\
U = g & \text{or } \frac{\partial U}{\partial n} = \mu \quad \text{on } D_a, \\
U(x) = O(\|x^{-1}\|) & \text{as } \|x\| \to \infty,
\end{cases}
\end{align}

The derivation of BIEs relies on a fundamental solution and Green’s third identity, which yields the so-called integral representation. The latter allows to reconstruct the solution $U$ over the entire domain $\Omega_a$ from boundary data $(U|_{D_a}, \frac{\partial U}{\partial n}|_{D_a})$ via single and double layer potentials. When taking traces, we arrive at BIEs for the trace jumps across the disk, (1.1) for $\sigma := [\partial U/\partial n]|_{D_a}$, and (1.2) for $u := [U]|_{D_a}$, in the case of the exterior Dirichlet or Neumann problem respectively. Usually (1.1) is referred to as weakly singular BIE, while (1.2) is called hypersingular.

This remains true when replacing $D_a$ with a connected orientable Lipschitz manifold with boundary of co-dimension one, a so-called screen $\Gamma \subset \mathbb{R}^3$. A comprehensive theory of the arising BIOs in the framework of Sobolev spaces on screens is available [2][22][24][25].

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In (1.1) and (1.2), \( D_a \) may also be replaced with the unit 2-sphere \( S \). Then solutions \( \sigma \) and \( u \) will supply the missing boundary data for Dirichlet and Neumann boundary value problems for \(-\Delta\) in the exterior of the unit ball. Moreover, a remarkable Calderón identity will hold: On \( S \) the BIoS occurring in (1.1) and (1.2) regarded as linear mappings between the trace spaces \( H^{-1/2}(S) \) and \( H^{1/2}(S) \), (after scaling) turn out to be inverses of each other up to a compact perturbation. More precisely, on the 2-sphere we have \( V W = W V = \frac{1}{4}(V^2 - \text{Id}) \) [17, Eq. (3.2.33)]. Moreover, Dirichlet and Neumann trace spaces, \( H^{1/2}(S) \) and \( H^{-1/2}(S) \), are dual to each other.

However, the situation on screens differs from the case of closed surfaces. Indeed, instead of working with the standard Sobolev trace spaces as described above, one must use the spaces \( \tilde{H}^{1/2}(\Gamma) \) and \( \tilde{H}^{-1/2}(\Gamma) \) –also denoted as \( H^{\pm1/2}_0(\Gamma) \) [14]. In this setting, Calderón identities break down in the case of screens since the mapping properties of the weakly singular and hypersingular operators degenerate, and the double layer operator and its adjoint vanish [24]. The very same situation is encountered in two dimensions where the role of \( D_a \) is played by a straight line segment. In 2D, one of the authors jointly with J.-C. Nédélec managed to find the exact variational inverses of the weakly singular and hypersingular BIO established in [11] for the line segment. Their applicability as explicit Calderón-type preconditioners for open arcs was later shown in [7]. This breakthrough in 2D was the starting point to find the closed form of the BIoS that satisfy relations analogous to the ones established in [11] and find Calderón-type identities over the unit disk.

The key contribution of this article is to finally state these identities and to provide an explicit construction of the exact inverses of the hypersingular and the weakly singular BIOs on the unit disk. We call these inverse BIOs modified weakly singular and modified hypersingular operators, respectively, because their kernels feature the same structure as those of the standards BIOs but incorporate a smooth cut-off function that depends on the distance to the boundary of the disk \( \partial D_a \).

In addition, the modified hypersingular operator is given both in terms of a finite part integral operator with a special kernel and in terms of a variational (weak) form in suitable trace spaces. It turns out that this variational form is related to the modified weakly singular operator (introduced in Section 3.1 and 3) in exactly the same way as the weak form of the standard hypersingular operator can be expressed through that for the weakly singular operator [23, Thm. 6.17].

Instrumental in the derivation of this variational form of the inverse of the weakly singular operator have been recent yet unpublished results by J.-C. Nédélec [18,19] also elaborated in the PhD thesis of P. Ramaciotti [21] and in [19]. They use so-called projected spherical harmonics in order to state series expansion for the kernels of the boundary integral operators and their inverses on the disk. We make use of such relations and prove them in a different way as will be shown in Section 4.1.

2. Preliminaries.

2.1. Notation. Let \( d = 1, 2, 3 \). For a bounded domain \( K \subseteq \mathbb{R}^d \), \( C^m(K), \ m \in \mathbb{N}_0 \), denotes the space of \( m \)-times differentiable scalar functions on \( K \), and, similarly, for the space of infinitely differentiable, scalar continuous functions we write \( C^\infty(K) \). Let \( L^p(K) \) designate the class of \( p \)-integrable functions over \( K \). Dual spaces are defined in standard fashion with duality products denoted by angular brackets \( \langle \cdot, \cdot \rangle_K \).

Let \( \mathcal{O} \in \mathbb{R}^d, \ d = 2, 3 \) be open and \( s \in \mathbb{R} \). We denote standard Sobolev spaces by \( H^s(\mathcal{O}) \). For positive \( s \) and \( \mathcal{O} \) Lipschitz such that \( \partial \mathcal{O} = \overline{\mathcal{O}} \) for \( \mathcal{O} \in \mathbb{R}^d \) closed, let \( \tilde{H}^s(\mathcal{O}) \) be the space of functions whose extension by zero to \( \mathcal{O} \) belongs to \( H^s(\mathcal{O}) \), as
in [14]. In particular, the following duality relations hold
\[ \tilde{H}^{-1/2}(\Omega) \equiv \left( H^{1/2}(\Omega) \right)^\prime \quad \text{and} \quad H^{-1/2}(\Omega) \equiv \left( \tilde{H}^{1/2}(\Omega) \right)^\prime. \] (2.1)

Finite part integrals with distributional meaning as in [10] are labeled with a dash as in \( \tilde{f} \).

2.2. Geometry. We focus on the circular disk \( \mathbb{D}_a \) with radius \( a > 0 \), defined as \( \mathbb{D}_a := \{ x \in \mathbb{R}^3 : x_3 = 0 \text{ and } \| x \| < a \} \). Thus, the volume complement domain becomes \( \Omega_a := \mathbb{R}^3 \setminus \mathbb{D}_a \). Often, we will omit the third coordinate and use the following short polar coordinate notation: \( x = (r_x \cos \theta_x, r_x \sin \theta_x) \in \mathbb{D}_a \).

2.3. Variational BIEs on the Disk \( \mathbb{D}_a \).

2.3.1. Weakly Singular Integral Equation. As in [14], we consider the following singular integral equation: for \( g \in H^{1/2}(\mathbb{D}_a) \), we seek a function \( \sigma \in \tilde{H}^{-1/2}(\mathbb{D}_a) \) such that
\[ (V \sigma)(y) := \frac{1}{4\pi} \int_{\mathbb{D}_a} \frac{\sigma(x)}{\|x - y\|} d\mathbb{D}_a(x) = g(y), \quad y \in \mathbb{D}_a. \] (2.2)
The measure \( d\mathbb{D}_a(x) \) denotes the surface element in terms of \( x \in \mathbb{D}_a \), equal to \( r_x dr_x d\theta_x \), and the unknown \( \sigma \) is the jump of the Neumann trace of the solution \( U \) of the exterior Dirichlet problem in \([13]\).

The symmetric variational formulation for (2.2) is: find \( \sigma \in \tilde{H}^{-1/2}(\mathbb{D}_a) \) such that for \( g \in H^{1/2}(\mathbb{D}_a) \), it holds
\[ (V \sigma, \psi)_{\mathbb{D}_a} = \frac{1}{4\pi} \int_{\mathbb{D}_a} \int_{\mathbb{D}_a} \frac{\sigma(x)\psi(y)}{\|x - y\|} d\mathbb{D}_a(x)d\mathbb{D}_a(y) = \langle g, \psi \rangle_{\mathbb{D}_a}, \] (2.3)
for all \( \psi \in \tilde{H}^{-1/2}(\mathbb{D}_a) \) [22, Sect. 3.5.3].

2.3.2. Hypersingular Integral Equation. As in [14], we consider the following singular integral equation: for \( \mu \in \tilde{H}^{-1/2}(\mathbb{D}_a) \), we seek a function \( u \in \tilde{H}^{1/2}(\mathbb{D}_a) \) such that
\[ (W u)(y) = \frac{1}{4\pi} \int_{\mathbb{D}_a} u(x) \frac{\partial^2}{\partial n_x \partial n_y} \frac{1}{\|x - y\|} d\mathbb{D}_a(x) = \mu(y), \quad y \in \mathbb{D}_a, \] (2.4)
where the unknown \( u \) is the jump of the Dirichlet trace of the solution \( U \) of the exterior Neumann problem [13] [22, Sect. 3.5.3].

Let \( v \) be a continuously differentiable function over \( \mathbb{D}_a \), and let \( \tilde{v} \) be an appropriate smooth extension of \( v \) into a three-dimensional neighborhood of \( \mathbb{D}_a \). Let us introduce the vectorial surface curl operator [23, p.133] as
\[ \text{curl}_{\mathbb{D}_a} v(x) := \mathbf{n}(x) \times \nabla \tilde{v}(x), \] (2.5)
with \( \mathbf{n}(x) \) being the outer normal of \( \mathbb{D}_a \) in \( x \in \mathbb{D}_a \), and \( \nabla \) denoting the standard gradient.

**Proposition 2.1.** A symmetric variational formulation for (2.4) is given by: given \( \mu \in \tilde{H}^{-1/2}(\mathbb{D}_a) \), seek \( u \in \tilde{H}^{1/2}(\mathbb{D}_a) \) such that
\[ (W u, v)_{\mathbb{D}_a} := \frac{1}{4\pi} \int_{\mathbb{D}_a} \int_{\mathbb{D}_a} \frac{\text{curl}_{\mathbb{D}_a} u(y) \cdot \text{curl}_{\mathbb{D}_a} v(x)}{\|x - y\|} d\mathbb{D}_a(x)d\mathbb{D}_a(y) = \langle \mu, v \rangle_{\mathbb{D}_a} \] (2.6)
for all \( v \in \tilde{H}^{1/2}(\mathbb{D}_a) \).

**Proof.** The proof follows the same steps as [23, Thm. 6.17] for closed surfaces. Since \( u, v \in \tilde{H}^{1/2}(\mathbb{D}_a) \), when integrating by parts, the boundary term vanishes and [23, Lemma 6.16] still holds.

Existence and uniqueness of solution of problems (2.3) and (2.6) was proved by Stephan in [24, Thm. 2.7]. Moreover, for a screen \( \Gamma \), the bilinear form in (2.3) and (2.6) are continuous and elliptic in \( \tilde{H}^{-1/2}(\Gamma) \) and \( \tilde{H}^{1/2}(\Gamma) \), respectively (cf. [22, Thm. 3.5.9]). One can show that in these cases and for sufficiently smooth screens \( \Gamma \), when approaching the edges \( \partial \Gamma \), the solutions decay like the square-root of the distance to \( \partial \Gamma \) [3].

### 3. New Boundary Integral Operators.

#### 3.1. Modified Weakly Singular Integral Operator.

We define the modified weakly singular operator as the improper integral

\[
(\overline{V}v)(x) := -\frac{2}{\pi^2} \int_{\mathbb{D}_a} v(y) \frac{S_a(x, y)}{\|x - y\|} d\mathbb{D}_a(y), \quad x \in \mathbb{D}_a,
\]

with the bounded function on \( \mathbb{D}_a \times \mathbb{D}_a \)

\[
S_a(x, y) := \tan^{-1}\left(\frac{\sqrt{a^2 - r_x^2} - \sqrt{a^2 - r_y^2}}{a \|x - y\|}\right), \quad x \neq y.
\]

We remark that the standard weakly singular BIO (as defined in (2.2)) is given by (3.1) without \( S_a(x, y) \) and scaled by \( \frac{\pi}{2} \). Furthermore, since \( \lim_{x \to y} S_a(x, y) = \frac{\pi}{2} \) when \( x, y \in \mathbb{D}_a \), the kernels of \( \overline{V} \) and \( V \) have the same weakly singular behavior in the interior of \( \mathbb{D}_a \). Also note that \( S_a(x, y) = 0 \) if \( |x| = a \) or \( |y| = a \). As a consequence, \( S_a \), though bounded, will be discontinuous on \( \partial \mathbb{D}_a \times \partial \mathbb{D}_a \).

#### 3.2. Modified Hypersingular Singular Integral Operator.

We define the modified hypersingular operator \( \overline{W} \) through the finite part integral

\[
(\overline{W}g)(x) := -\frac{2}{\pi^2} \int_{\mathbb{D}_a} g(y) K_{\overline{W}}(x, y) d\mathbb{D}_a(y), \quad x \in \mathbb{D}_a,
\]

with

\[
K_{\overline{W}}(x, y) := \frac{a}{\|x - y\|^2 \sqrt{a^2 - r_x^2} \sqrt{a^2 - r_y^2}} + \frac{S_a(x, y)}{\|x - y\|^3}, \quad x \neq y.
\]

Since \( \lim_{x \to y} S_a(x, y) = \frac{\pi}{2} \) when \( x, y \in \mathbb{D}_a \), the kernel of the standard hypersingular operator \( W \) (defined as in (2.4)) and the second term in (3.4) have the same \( O(|x - y|^{-3}) \)-type singularity in the interior of \( \mathbb{D}_a \). On the other hand, the first term in (3.4) represents a strongly singular kernel in the interior of \( \mathbb{D}_a \) and features a hypersingular behavior when \( x = y \in \partial \mathbb{D}_a \). From these observations we point out that \( \overline{W} \) has a truly hypersingular kernel.

#### 3.3. Calderón-type Identities.

The next fundamental result reveals why we are interested in these exotic looking integral operators (cf. [11, Prop. 3.6] or [7, Thm. 2.2]).
\textbf{Theorem 3.1.} The following identities hold:

\begin{align}
\bar{W}V &= \text{Id}_{\overline{H}^{-1/2}(D_a)}, & \mathcal{V}\mathcal{W} &= \text{Id}_{H^{1/2}(D_a)}, \\
\nabla W &= \text{Id}_{\overline{H}^{1/2}(D_a)}, & \mathcal{W}\nabla &= \text{Id}_{H^{-1/2}(D_a)}.
\end{align}

Key tools for the proof of Theorem 3.1 are some auxiliary results by Li and Rong [13]. First, define the function $p(\rho, \theta)$ as

$$p(\rho, \theta) := \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \rho^{|n|} e^{in\theta} = \frac{1}{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos \theta}, \quad \forall \ |\rho| < 1,$$

with $\theta \in [0, 2\pi]$ (cf. [13] Chap. 1.1).

This function allows us to rewrite the kernel of $\mathcal{V}$ as will be shown in the following Lemma.

\textbf{Lemma 3.1 (Lemma 1 [13]).} Let us consider points $x$, $y$ on the disk $D_a$, satisfying $x \neq y$, whose polar coordinates are given by $x = (r_x \cos \theta_x, r_x \sin \theta_x) \in D_a$ and equivalently for $y$. Then, for a parameter $\alpha \in (0, 4)$, such that $\alpha \neq 2$, it holds

$$\frac{1}{4\pi} \frac{1}{||x-y||^\alpha} = \frac{1}{\pi} \sin \frac{\alpha \pi}{2} \int_{\min(r_x, r_y)}^{\max(r_x, r_y)} s^{\alpha-1} p \left( \frac{s^2}{r_x r_y}, \frac{\theta_x - \theta_y}{r_x^2 - r_y^2} \right) ds$$

$$= \frac{1}{\pi} \sin \frac{\alpha \pi}{2} \int_{\min(r_x, r_y)}^{\max(r_x, r_y)} s^{\alpha-1} p \left( \frac{r_x r_y}{s^2}, \frac{\theta_x - \theta_y}{r_x^2 - r_y^2} \right) ds.
$$

Here the integrals above are understood in the sense of finite-part integrals if $\alpha > 2$.

We can now follow Fabrikant [5] Chap. 1.1 and introduce

$$L(\rho)u(r, \theta) := \int_0^{2\pi} p(\rho, \theta - \theta_0) u(r, \theta_0) d\theta_0$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \rho^{|n|} e^{in\theta} \int_0^{2\pi} e^{-in\theta_0} u(r, \theta_0) d\theta_0,$$

with $\theta \in [0, 2\pi]$, and $r \in [0, a]$. This integral operator is sometimes called Poisson integral over the disk [5] Chap. 1.1. The properties of $L(\rho)u(r, \theta)$ combined with the formulas from Lemma 3.1 lead to a complete separation of variables. This fact plays a key role as the resulting expressions for $V$ and $W$ will be iterative systems of Abel integral equations, whose solutions are given in the next Theorem.

\textbf{Theorem 3.2 (Thm.1-2 [13]).} Let $\alpha \in \{1, 3\}$ and $f \in C^1(D_a)$. Then, the solution of

$$\frac{1}{4\pi} \int_{D_a} \frac{\varphi(x)}{||x-y||^\alpha} dD_a(x) = f(x), \quad x \in D_a,$$

is given by

$$\varphi(x) = \frac{1}{\pi} \int_{D_a} \frac{f(y)}{R_{D_a}^2 - a} dD_a(y),$$

where $R_{D_a} = \max(r, a)$.
By definition of \( \eta \) following primitive \( R \) from \( \sigma \) be written as with which leads to

\[ \int \frac{s^2 p \left( \frac{r_x r_y}{s^2}, \theta_x - \theta_y \right)}{(s^2 - r_x^2)(s^2 - r_y^2)^{3/2}} ds = -\frac{1}{2\pi} \left( \sqrt{s^2 - r_x^2} \sqrt{\frac{s^2 - r_y^2}{s}} \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{s \|x - y\|^2} \right) + \frac{s}{\|x - y\|^3} \right). \] \hspace{1cm} (3.13)

**Remark 1.** From (3.3), we notice that \( R_\sigma(x, y) \) is a scaled restriction of \( \|x - y\| \) from \( \mathbb{R}^3 \) to \( D_a \). Moreover, for \( a = \infty \), Theorem 3.2 implies \( \frac{1}{\pi R_\sigma(x, y)} = \frac{1}{\|x - y\|} \).

**Lemma 3.2.** Let \( a > 0 \) and \( x, y \in D_a \). If \( a \geq s \geq \max(r_x, r_y) \), we find the following primitive

\[ \int \frac{s^2 p \left( \frac{r_x r_y}{s^2}, \theta_x - \theta_y \right)}{(s^2 - r_x^2)(s^2 - r_y^2)^{3/2}} ds = -\frac{1}{2\pi} \left( \frac{s}{\|x - y\|^3} \right). \] \hspace{1cm} (3.12)

**Proof.** This can be shown by direct calculation using the following change of variable [5]:

\[ \eta := \sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2} \frac{s}{s}, \quad \frac{d\eta}{ds} = \frac{s^4 - r_x^2 r_y^2}{\eta s^3}, \]

which leads to

\[ \int \frac{s^2}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} \frac{1 - r_x^2 r_y^2}{s^2 + r_x r_y - 2r_x r_y \cos(\theta_x - \theta_y)} ds = \int \frac{\eta^{-2}}{\|x - y\|^2 + \eta^2} d\eta, \]

where

\[ \int \frac{\eta^{-2}}{\|x - y\|^2 + \eta^2} d\eta = -\frac{1}{\eta \|x - y\|^2} - \tan^{-1} \left( \frac{\eta}{\|x - y\|} \right). \] \hspace{1cm} (3.14)

By definition of \( \eta \) the result follows. \( \square \)

Combining the above elements we can prove the next result.

**Proposition 3.3.** The solution of the weakly singular integral equation (2.2) can be written as \( \sigma(x) = (Wg)(x) \), for all \( x \in D_a \), if \( g \) is continuously differentiable.

**Proof.** Applying Theorem 3.2 we get that the solution to (2.2) can be written as (3.11). Moreover, when \( a < \infty \), we may use Lemma 3.2 and write

\[ \frac{1}{\pi R_\sigma^2(x, y)} = \frac{1}{\pi} \int_{\max(r_x, r_y)}^{\infty} \frac{s^2}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} \frac{p \left( \frac{r_x r_y}{s^2}, \theta_x - \theta_y \right)}{s^2} ds \]

\[ = \frac{2}{\pi^2} f_p \left( \sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2} \right) \left( \frac{s}{\|x - y\|^2} \right) \left( \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{s \|x - y\|^2} \right) \right)^{\max(r_x, r_y)} \]
where the finite part (fp) of the last expression needs to be considered. This means that we drop the term corresponding to evaluating our primitive (3.13) on the lower bound $\max(r_x, r_y)$, as it becomes infinite.

Hence, we obtain

$$
\frac{1}{\pi} R_\Omega(x, y) = - \frac{2}{\pi^2} \left( \frac{a}{\|x - y\|^2 \sqrt{a^2 - r_x^2 \sqrt{a^2 - r_y^2}}^2} + S_a(x, y) \right) = - \frac{2}{\pi^2} K_W(x, y),
$$

with $K_W$ from (3.4) as stated.

**Lemma 3.3** (Eq. 1.2.14 in [5]). Let $a > 0$ and $x, y \in \mathbb{D}_a$. If $a \geq s \geq \max(r_x, r_y)$, we find the following primitive

$$
\int \frac{1}{(s^2 - r_x^2)^{1/2} (s^2 - r_y^2)^{1/2}} P \left( \frac{r_x r_y}{s^2}, \theta_x - \theta_y \right) ds = \frac{1}{2\pi \|x - y\|} \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{s \|x - y\|} \right). \tag{3.15}
$$

**Proof.** This can be shown by using the same change of variables in (3.3) and direct calculation.

**Proposition 3.4.** When $\mu$ is continuously differentiable, the solution of the hypersingular integral equation (2.4) can be written as $u(x) = (\nabla \mu)(x)$, for all $x \in \mathbb{D}_a$.

**Proof.** Using Theorem 3.2, we get that the solution to (2.4) can be written as (3.11). Moreover, when $a < \infty$, we may use Lemma 3.3 to write

$$
\frac{1}{\pi} R_\Omega(x, y) = - \frac{4}{\pi} \int_{\max(r_x, r_y)}^a \frac{1}{(s^2 - r_x^2)^{1/2} (s^2 - r_y^2)^{1/2}} P \left( \frac{r_x r_y}{s^2}, \theta_x - \theta_y \right) ds
$$

as stated with $S_a$ from (3.2).

With the above elements, we can now prove our Calderón-type Identities:

**Proof of Theorem 3.1.** Both identities in (3.5) follow from Proposition 3.3 combined with the density of $C^\infty(\mathbb{D}_a)$ in $H^{-1/2}(\mathbb{D}_a)$. Analogously, the relations in (3.6) are consequences of Proposition 3.4 and the density of $C^\infty(\mathbb{D}_a)$ in $H^1(\mathbb{D}_a)$. 

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Proposition 3.5. The operators

\[ \nabla : H^{-1/2}(\mathbb{D}_a) \to H^{1/2}(\mathbb{D}_a) \quad \text{and} \quad \overline{W} : H^{1/2}(\mathbb{D}_a) \to H^{-1/2}(\mathbb{D}_a) \]  

are continuous.

Proof. We begin our proof with \( \nabla : H^{-1/2}(\mathbb{D}_a) \to H^{1/2}(\mathbb{D}_a) \). Let us assume that \( \nabla \) is not a bounded operator. Then, by virtue of density, there exists a sequence \( (\mu_n)_n \in C^\infty(\mathbb{D}_a) \) such that

\[ \|\mu_n\|_{H^{-1/2}(\mathbb{D}_a)} = 1, \quad \|\nabla \mu_n\|_{H^{1/2}(\mathbb{D}_a)} \to \infty, \quad \text{as} \quad n \to \infty. \]  

(3.17)

Since \( \overline{W} : H^{1/2}(\mathbb{D}_a) \to H^{-1/2}(\mathbb{D}_a) \) is an isomorphism (cf. [24, Thm. 2.7]), it holds

\[ \|\nabla \mu_n\|_{H^{1/2}(\mathbb{D}_a)} \leq C \|\overline{W} \nabla \mu_n\|_{H^{-1/2}(\mathbb{D}_a)} \to C \|\mu_n\|_{H^{-1/2}(\mathbb{D}_a)}, \]  

from where we get a contradiction. The proof for \( \overline{W} \) is analogous. \( \square \)

Corollary 3.6. The bilinear forms

\[ (\partial, \mu) \mapsto \langle \nabla \partial, \mu \rangle_{\mathbb{D}_a}, \quad \partial, \mu \in H^{-1/2}(\mathbb{D}_a), \]  

(3.19)

\[ (u, g) \mapsto \langle \overline{W} u, g \rangle_{\mathbb{D}_a}, \quad u, g \in H^{1/2}(\mathbb{D}_a), \]  

(3.20)

are elliptic and continuous in \( H^{-1/2}(\mathbb{D}_a) \) and \( H^{1/2}(\mathbb{D}_a) \), respectively.

Proof. Follows from continuity and ellipticity of the standard BIOs \( W \) and \( V \) combined with Theorem [3.3]. \( \square \)

4. Bilinear Form for the Modified Hypersingular Integral Operator over \( \mathbb{D}_a \). We note that formula [3.3] is not practical when implementing a Galerkin BEM discretization. On \( S \) and, more generally, on every closed surface \( \partial \Omega \), we have

\[ \langle W u, v \rangle_{\partial \Omega} = \langle V \text{curl}_{\partial \Omega} u, \text{curl}_{\partial \Omega} v \rangle_{\partial \Omega}, \]  

(4.1)

where, abusing notations, we wrote \( V \) and \( W \) for the associated weakly singular and hypersingular BIOs (cf. [22,23]).

In this section we establish an analogous relation between \( \overline{W} \) and \( \nabla \). Let us begin by considering the modified weakly singular operator \( \nabla \) over the unitary disk given by [3.3] and denote \( \omega(x) := \sqrt{1 - r^2} \), \( x \in \mathbb{D}_1 \).

Proposition 4.1. The bilinear form associated to the modified hypersingular operator \( \overline{W} \) over \( \mathbb{D}_1 \) satisfies

\[ \langle \overline{W} u, v \rangle_{\mathbb{D}_1} = \frac{2}{\pi^2} \int_{\mathbb{D}_1} \int_{\mathbb{D}_1} \frac{S_1(x,y)}{||x-y||} \text{curl}_{\mathbb{D}_1, x} u(x) \cdot \text{curl}_{\mathbb{D}_1, y} v(y) d\mathbb{D}_1(x) d\mathbb{D}_1(y), \]  

(4.2)

for all \( u, v \in H^{1/2}(\mathbb{D}_1) \) : \( \{ v \in H^{1/2}(\mathbb{D}_1) : \langle v, \omega^{-1} \rangle_{\mathbb{D}_1} = 0 \} \).

This result was first reported by Nedélec and Ramacioti in their spectral study of the BIOs over \( \mathbb{D}_1 \) and their variational inverses [21]. For the sake of completeness, we introduce the key tools they derived and provide an alternative simpler proof of this proposition in Section [4.1]. A proof by means of formal integration by parts remains elusive, as it encounters difficulties due to the finite part integrals involved in the definition of \( \overline{W} \) and its kernel introduced in [3.3].
We also emphasize that the space $H^{1/2}(D_1)$ corresponds to $H^{1/2}(D_1)/\mathbb{R}$ (see end of Section 4.1 for further details). It is important to observe that the right-hand side of (4.2) maps constants to zero and thus has a non-trivial kernel if considered in the whole $H^{1/2}(D_1)$ space. In other words, the bilinear form in (1.2) is $H^{1/2}(D_1)$-elliptic but does not have this property on $H^{1/2}(D_1)$. For this reason, its extension to $H^{1/2}(D_1)$ does not actually match the bilinear form of $\overline{W}$ there, which is $H^{1/2}(D_1)$-elliptic. In order to remedy this situation, we add an appropriate regularizing term coming from the definition of $H^{1/2}(D_1)$ and the following result.

**Proposition 4.2.** The following identity holds:

$$\overline{(W1)}(y) = \frac{4}{\pi} \omega^{-1}(y), \quad y \in D_1. \quad (4.3)$$

**Proof.** Let us rewrite the modified hypersingular BIO acting on the constant function equal to one as

$$\overline{(W1)}(y) = \frac{2}{\pi^2} \int_{D_1} k_{\overline{W}}(x, y)d\mathcal{D}_1(x) = \frac{4}{\pi} \int_{r_y}^{1} \int_{0}^{s} \frac{r_x}{(s^2 - r^2_y)^{3/2}} ds \frac{d}{dr} \left( \frac{r_x}{s^2 - r^2_y} \right) dr_x ds, \quad (4.4)$$

where in the second line [13, Eq. (39)] was applied, with an appropriate scaling for (2.2). Based on Hadamard’s finite part integration, we can derive the following two formulas [13, Eq. (34)] [6, Eq. (50)]:

$$\frac{d}{ds} \int_{0}^{s} \frac{r_x}{\sqrt{s^2 - r^2_y}} dr_x = -s \int_{0}^{s} \frac{r_x}{(s^2 - r^2_y)^{3/2}} dr_x, \quad (4.5)$$

and

$$\frac{d}{dy} \int_{r_y}^{1} \sqrt{s^2 - r^2_y} \frac{sf(s)}{ds} ds = r_y \left[ \frac{sf(1)}{\sqrt{1 - r^2_y}} + r_y \int_{r_y}^{1} \frac{1}{\sqrt{s^2 - r^2_y}} ds \right] f(s) ds. \quad (4.6)$$

Using these derivatives in (4.4), we obtain

$$\overline{(W1)}(y) = -\frac{4}{\pi r_y} \frac{d}{dy} \int_{r_y}^{1} \frac{s}{\sqrt{s^2 - r^2_y}} ds \int_{0}^{s} \frac{r_x}{\sqrt{s^2 - r^2_y}} dr_x ds.$$

We integrate the inner integral and get

$$\int_{0}^{s} \frac{r_x}{\sqrt{s^2 - r^2_y}} dr_x = \left( -\sqrt{s^2 - r^2_y} \right)_{0}^{s} = s.$$

Our expression then becomes

$$\overline{(W1)}(y) = -\frac{4}{\pi r_y} \frac{d}{dy} \int_{r_y}^{1} \frac{s}{\sqrt{s^2 - r^2_y}} ds.$$
Since
\[
\int_{r_y}^{1} \frac{s}{\sqrt{s^2 - r_y^2}} ds = \sqrt{1 - r_y^2},
\]
it holds,
\[
(\overline{W})_1(y) = \frac{4}{\pi} \frac{1}{\sqrt{1 - r_y^2}},
\]
as stated. \(\square\)

As expected, this result is consistent with known solutions of \(\frac{\partial}{\partial r} f(r, \theta) = 0\) when the right-hand side is \(g = 1\) \[15\]. From this, we see that for \(u_c\) constant, \((\overline{W}u_c)(y)\) is equivalent to
\[
(\overline{W}u_c)(y) = \frac{2}{\pi^2} \int_{\mathcal{D}_1} u_c \omega^{-1}(x) \omega^{-1}(y) d\mathcal{D}_1(x), \quad y \in \mathcal{D}_1,
\]
(4.7)
since \((1, \omega^{-1})_{\mathcal{D}_1} = 2\pi\). Therefore, the following result holds:

**Proposition 4.3.** The symmetric bilinear form associated to the modified hypersingular operator \(\overline{W} : H^{1/2}(\mathcal{D}_1) \to H^{-1/2}(\mathcal{D}_1)\) can be written as
\[
(\overline{Wu}, v)_{\mathcal{D}_1} = \frac{2}{\pi^2} \int_{\mathcal{D}_1} \int_{\mathcal{D}_1} \frac{S_1(x, y)}{|x - y|} \frac{\text{curl}_{\mathcal{D}_1,x} u(x) \cdot \text{curl}_{\mathcal{D}_1,y} v(y)}{\omega(x) \omega(y)} d\mathcal{D}_1(x) d\mathcal{D}_1(y) + \frac{2}{\pi^2} \int_{\mathcal{D}_1} \int_{\mathcal{D}_1} \frac{u(x)v(y)}{\omega(x) \omega(y)} d\mathcal{D}_1(x) d\mathcal{D}_1(y), \quad \forall u, v \in H^{1/2}(\mathcal{D}_1).
\]
(4.8)

**Proof.** Note that we have added the required regularization to \(4.2\) such that \(4.3\) is preserved. This guarantees that our bilinear form defined in \(4.8\) is by construction equivalent to the bilinear form arising from our modified hypersingular operator \(\overline{W}\) on \(H^{1/2}(\mathcal{D}_1)\). Thus, it is \(H^{1/2}(\mathcal{D}_1)\)-continuous and elliptic. \(\square\)

Proposition 4.3 gives us a variational form for \(\overline{W}\) that can be easily implemented. Observe that the chosen regularization to extend the bilinear form \(4.2\) from \(H^{1/2}(\mathcal{D}_1)\) to \(H^{1/2}(\mathcal{D}_1)\) is analogous to the one needed for the modified hypersingular operator on a segment \(\square\) Eq. (2.11)].

In addition, by linearity of the scaling map \(\Psi_a : \mathcal{D}_1 \to \mathcal{D}_a\), we get the corresponding relationship on \(\mathcal{D}_a\).

**Corollary 4.4.** For \(a > 0\), the symmetric bilinear form associated to the modified hypersingular operator \(\overline{W} : H^{1/2}(\mathcal{D}_a) \to H^{-1/2}(\mathcal{D}_a)\) can be written as
\[
(\overline{W}u, v)_{\mathcal{D}_a} = \frac{2}{\pi^2} \int_{\mathcal{D}_a} \int_{\mathcal{D}_a} \frac{S_a(x, y)}{|x - y|} \frac{\text{curl}_{\mathcal{D}_a,x} u(x) \cdot \text{curl}_{\mathcal{D}_a,y} v(y)}{\omega_a(x) \omega_a(y)} d\mathcal{D}_a(x) d\mathcal{D}_a(y) + \frac{2}{a\pi^2} \int_{\mathcal{D}_a} \int_{\mathcal{D}_a} \frac{u(x)v(y)}{\omega_a(x) \omega_a(y)} d\mathcal{D}_a(x) d\mathcal{D}_a(y), \quad \forall u, v \in H^{1/2}(\mathcal{D}_a),
\]
(4.9)
where \(\omega_a(x) := \sqrt{a^2 - r_x^2}, \ x \in \mathcal{D}_a\).
For the standard weakly singular and hypersingular operators, the following relation between their kernels holds:

\[
\frac{1}{||x - y||^3} = \Delta \frac{1}{||x - y||} \quad x \neq y, \, x, y \in \mathbb{R}^3,
\]

while their modified versions do not satisfy this relation. Actually, one has for \(x \neq y, \, x, y \in \mathbb{R}^3\) [6, Eq. (23)]

\[
\Delta K_V(x, y) := \frac{(2\pi a) p \left( \frac{r_x r_y}{a^2 - r_x^2} \right) \theta_x - \theta_y}{(a^2 - r_x^2)^{3/2} \sqrt{a^2 - r_x^2}} + \frac{a}{||x - y||^2 \sqrt{a^2 - r_x^2} \sqrt{a^2 - r_y^2}} + S_n(x, y)
\]

\[
= \frac{(2\pi a) p \left( \frac{r_x r_y}{a} \theta_x - \theta_y \right)}{(a^2 - r_y^2)^{3/2} \sqrt{a^2 - r_y^2}} + K_W(x, y) \neq K_W(x, y),
\]

where the first term is singular on \(\partial \mathbb{D}_a\) and is surprisingly not symmetric. The interpretation of this term need further investigation.

**Remark 2.** The representation established in Proposition 4.3 is essential for the use of \(\overline{W}\) in the context of Galerkin boundary element methods, because the direct discretization of hypersingular boundary integral operators is not possible without some prior regularization.

### 4.1. Diagonalization of BIOs on the unit disk \(\mathbb{D}_1\). We begin by introducing key definitions and results on the unit disk \(\mathbb{D}_1\). When needed, we provide a short proof for the results reported in [21, Chap. 2].

**Definition 4.1 (Nedélec’s Projected Spherical Harmonics (PSHs) over \(\mathbb{D}_1\))** [18, Eq. (75)]. For \(l, m \in \mathbb{N}_0\) such that \(|m| \leq l\) and \(x = (r_x, \theta_x)\), we introduce the functions:

\[
y^m_l(x) := \gamma^m_l e^{im\theta_x} P^m_l(\sqrt{1 - r_x^2}), \quad \gamma^m_l := (-1)^m \sqrt{(2l + 1) (l - m)! \over 4\pi (l + m)!}.
\]

The PSH over \(\mathbb{D}_1\) satisfy the following orthogonality identity [18, eq. (79)]

\[
\int_{\mathbb{D}_1} \frac{y^m_l(y) y^{m_2}_{l_2}(y)}{\omega(y)} d\mathbb{D}_1(y) = {1 \over 2} \delta^{l_1 l_2} \delta^{m_1 m_2},
\]

where \(\delta^{l_1}_{l_2}\) is the Kronecker symbol and \(\omega(y) = \sqrt{1 - r_y^2}\).

**Proposition 4.5 ([26]).** The PSHs solve a generalized eigenvalue problem for \(V\) over \(\mathbb{D}_1\) in the sense that

\[
(V y^m_l) / \omega(x) = {1 \over 4} \lambda^m_l y^m_l(x), \quad l + m \text{ even},
\]

with \(\lambda^m_l := {\Gamma \left( \frac{l + m + 1}{2} \right) \Gamma \left( \frac{l - m + 1}{2} \right) \Gamma \left( \frac{l - m + 2}{2} \right) \over \Gamma \left( \frac{l + m + 2}{2} \right) \Gamma \left( \frac{l - m + 2}{2} \right) \Gamma \left( \frac{l - m + 3}{2} \right)}\), and \(\Gamma\) being the Gamma function.

**Proposition 4.6 ([12]).** The PSHs solve a generalized eigenvalue problem for \(W\) over \(\mathbb{D}_1\) in the sense that

\[
(W y^m_l)(x) = {1 \over \lambda^m_l} y^m_l(x) / \omega(x), \quad l + m \text{ odd},
\]
with $\lambda^m$ as in Proposition 4.2.

Remark 3. Propositions 4.3 and 4.6 nicely show how, in the case of a disk, the usual $V$ and $W$ have reciprocal symbols but $W V \neq \frac{1}{4} \operatorname{Id}$ due to their mapping properties. This is characterized here by the parity of the PSHs involved.

Proposition 4.7 ([21 Sect. 2.7]). The kernels of $V, W, \nabla, \mathcal{W}$ have the following symmetric series expansions on $\mathbb{D}_1$:

$$K_v(x, y) := \frac{1}{4\pi} \frac{1}{\|x - y\|^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \lambda_l^m \left( y_l^m(x) y_l^m(y) + \bar{y}_l^m(x) \bar{y}_l^m(y) \right).$$

$$K_\nabla(x, y) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \lambda_l^m \left( y_l^m(x) y_l^m(y) + \bar{y}_l^m(x) \bar{y}_l^m(y) \right).$$

$$K_w(x, y) := \frac{1}{4\pi} \frac{1}{\|x - y\|^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \lambda_l^m \left( y_l^m(x) y_l^m(y) + \bar{y}_l^m(x) \bar{y}_l^m(y) \right).$$

$$K_\mathcal{W}(x, y) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4}{\lambda_l^m} \left( y_l^m(x) y_l^m(y) + \bar{y}_l^m(x) \bar{y}_l^m(y) \right).$$

Proof. The proof for the standard BISOs kernels follows from Propositions 4.5 and 4.6 combined with the orthogonality of the PSHs (4.11). From (4.12), we can additionally find a key relationship fulfilled by its unique inverse operator $W = V^{-1}$ over $\mathbb{D}_1$

$$K_\mathcal{W}(x, y) := \frac{4}{\lambda_l^m} \frac{y_l^m(x)}{\omega(x)},$$

and use again the orthogonality property to get (4.17). The result for $K_\nabla$ can be obtained analogously from (4.18).

Definition 4.2 (Weighted $L^2$-spaces [18 sl. 36–37] [21 Def.2.7.8–2.7.9]). Let us define the spaces $L^2_{1/\omega}(\mathbb{D}_1)$ and $L^2_\omega(\mathbb{D}_1)$ as the $L^2$-spaces induced by the weighted inner products

$$(u, v)_{1/\omega} = \int_{\mathbb{D}_1} u(x) \overline{v(x)} \omega(x)^{-1} d\mathbb{D}_1(x),$$

and

$$(u, v)_{\omega} = \int_{\mathbb{D}_1} u(x) \overline{v(x)} \omega(x) d\mathbb{D}_1(x),$$

respectively.

Proposition 4.8 ([18 sl. 36–37] [21 Prop 2.7.8–2.7.9]). We have that

- $\{y_l^m : l + m \text{ odd}\}$ and $\{y_l^m : l + m \text{ even}\}$ are both orthogonal bases for $L^2_{1/\omega}$.

- $\{\omega^{-1}y_l^m : l + m \text{ odd}\}$ and $\{\omega^{-1}y_l^m : l + m \text{ even}\}$ are both orthogonal bases for $L^2_\omega$.

Proposition 4.9 ([21 Sect. 2.7.4]).

(i) $u \in \tilde{H}^{1/2}(\mathbb{D}_1)$ can be expanded in the basis $\{y_l^m : l + m \text{ odd}\}$ of $L^2_{1/\omega}$:

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_l^m y_l^m(x), \quad u_l^m = (u, y_l^m)_{1/\omega}, \quad l + m \text{ odd}. $$

(4.21)
(ii) \( g \in H^{1/2}(\mathbb{D}_1) \) can be expanded in the basis \( \{ y_l^m : l + m \text{ even} \} \) of \( L^2_{1/\omega} \):

\[
g(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l^m y_l^m(x), \quad g_l^m = (g, y_l^m)_{1/\omega}, \quad l + m \text{ even.} \quad (4.22)
\]

(iii) \( v \in H^{-1/2}(\mathbb{D}_1) \) can be expanded in the basis \( \{ y_l^m \omega^{-1} : l + m \text{ odd} \} \) of \( L^2_{\omega} \):

\[
v(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_l^m y_l^m(x) \frac{\omega(x)}{\omega}, \quad v_l^m = (v, y_l^m \omega^{-1})_{\omega}, \quad l + m \text{ odd.} \quad (4.23)
\]

(iv) \( \sigma \in \tilde{H}^{-1/2}(\mathbb{D}_1) \) can be expanded in the basis \( \{ y_l^m \omega^{-1} : l + m \text{ even} \} \) of \( L^2_{\omega} \):

\[
\sigma(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_l^m y_l^m(x) \frac{\omega(x)}{\omega}, \quad \sigma_l^m = (\sigma, y_l^m \omega^{-1})_{\omega}, \quad l + m \text{ even.} \quad (4.24)
\]

\[\text{Proof.} \] Since the bilinear form associated to \( W \) is symmetric, elliptic and continuous, it induces an energy inner product on \( \tilde{H}^{1/2}(\mathbb{D}_1) \). Then, the proof of (i) boils down to showing that for \( u \in \tilde{H}^{1/2}(\mathbb{D}_1) \):

\[
\langle Wu, y_l^m \rangle_{\mathbb{D}_1} = 0, \quad \forall l + m \text{ odd} \iff u \equiv 0.
\]

Using the symmetry of the bilinear form and (4.13), we derive

\[
\langle Wu, y_l^m \rangle_{\mathbb{D}_1} = \frac{1}{\lambda^m} \left( u, \frac{y_l^m}{\omega} \right)_{\mathbb{D}_1} = \frac{1}{\lambda^m} (u, y_{l1/\omega}),
\]

which is zero if and only if \( u \equiv 0 \) because \( \{ y_l^m : l + m \text{ odd} \} \) is an orthogonal basis for \( L^2_{1/\omega} \). The remaining three cases follow by analogy. \[\square\]

One can prove that \textbf{odd} \( \text{PSHs can be factored as} \)

\[
y_l^m(x) = e^{im\theta} \omega(x) \Psi(x), \quad l + m \text{ odd},
\]

where \( \Psi \) is a polynomial function (Combine [14, Eq. 14.3.21], [1] Eq. 3.2(7), and [1] Eq. 10.9(22))). However, this is not true when \( l + m \) is even. In that case, the radial part of \( y_l^m \) is already a polynomial (since [1] 10.9(21)] holds instead of [1] 10.9(22)]). This property confirms that the basis functions of our four fractional Sobolev spaces have the correct behaviour. Namely, when \( l + m \) is odd, \( y_l^m \sim \omega \) near the boundary and belongs to \( \tilde{H}^{1/2}(\mathbb{D}_1) \); when \( l + m \) is even, \( y_l^m \omega^{-1} \sim \omega^{-1} \) near \( \partial \mathbb{D}_1 \) and lies in \( \tilde{H}^{-1/2}(\mathbb{D}_1) \); while the basis functions of \( H^{1/2}(\mathbb{D}_1) \) and \( H^{-1/2}(\mathbb{D}_1) \) have no singular behaviour.

**Definition 4.3 (Kinetic moments on \( \mathbb{D}_1 \) [13, Eq. (51)])**. Define the operators \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) of derivation over \( \mathbb{D}_1 \) as

\[
\mathcal{L}_\pm u := e^{\pm i\theta} \left( \pm \frac{\partial u}{\partial r} + i \frac{1}{r} \frac{\partial u}{\partial \theta} \right). \quad (4.25)
\]

**Proposition 4.10 (Properties of the kinetic moments over \( \mathbb{D}_1 \) [13 sl. 21, Eq. (80)], [21] Prop. 2.7.7, Cor. 2.7.2, Lemma 2.7.3)**. Let \( u, v \in C^\infty(\mathbb{D}_1) \) and \( x, y \in \mathbb{D}_1 \). The kinetic moments satisfy over \( \mathbb{D}_1 \)

\[
\text{curl}_{\mathbb{D}_1,x} u(x) \cdot \text{curl}_{\mathbb{D}_1,y} v(y) = -\frac{1}{2} \left( \mathcal{L}_{+,x} u(x) \mathcal{L}_{-,y} v(y) + \mathcal{L}_{-,x} u(x) \mathcal{L}_{+,y} v(y) \right), \quad (4.26)
\]

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together with $\mathcal{L}_+ = \mathcal{L}_-$, and $\mathcal{L}_+ = \mathcal{L}_+$. Moreover, when applied to PSHs, we get

$$\mathcal{L}_+ y_l^m(x) = \sqrt{(l - m)(l + m + 1)} \frac{y_l^{m+1}(x)}{\omega(x)}, \quad (4.27)$$
$$\mathcal{L}_- y_l^m(x) = \sqrt{(l + m)(l - m + 1)} \frac{y_l^{m-1}(x)}{\omega(x)}. \quad (4.28)$$

Remark 4. Due to Proposition 4.9, $\mathcal{L}_\pm$ maps $H^{1/2}(\mathbb{D}_1)$ to $H^{-1/2}(\mathbb{D}_1)$, and $H^{1/2}(\mathbb{D}_1)$ to $H^{-1/2}(\mathbb{D}_1)$. 

Proof of 4.1. We begin our proof by introducing the following recursion formula

$$\frac{4}{\lambda_l^m} = \frac{1}{2} \left[ (l + m)(l - m + 1)\lambda_l^{m-1} + (l - m)(l + m + 1)\lambda_l^{m+1} \right], \quad (4.29)$$

which can be verified by direct computations using the multiplicative property of the Gamma function, i.e., $\Gamma(z + 1) = z\Gamma(z)$. Plugging this recursion formula into (4.28) gives

$$\left(\mathbf{W} y_l^m\right)(x) = \frac{4}{\lambda_l^m} \frac{y_l^m(x)}{\omega(x)}$$
$$= \frac{1}{2} \left[ (l + m)(l - m + 1)\lambda_l^{m-1} + (l - m)(l + m + 1)\lambda_l^{m+1} \right] \frac{y_l^m(x)}{\omega(x)}. \quad (4.30)$$

From (4.27) and (4.28), it is clear that

$$\mathcal{L}_+ y_l^{m-1}(x) = \sqrt{(l - m + 1)(l + m)} \frac{y_l^m(x)}{\omega(x)}, \quad (4.31)$$
$$\mathcal{L}_- y_l^{m+1}(x) = \sqrt{(l + m + 1)(l - m)} \frac{y_l^m(x)}{\omega(x)}. \quad (4.32)$$

Moreover, by unicity of $W^{-1} = \nabla$ (cf. Prop. 2.2), it must hold that

$$\left(\frac{\nabla y_l^{m \pm 1}}{\omega}\right)(x) = \lambda_l^{m \pm 1} y_l^{m \pm 1}(x), \quad l + m \pm 1 \text{ odd.} \quad (4.33)$$

Then, combining all these ingredients, it is clear that for $(l, m) \neq (0, 0)$, $l + m$ even\(^3\), our expression is equivalent to

$$\left(\mathbf{W} y_l^m\right)(x) = \frac{1}{2} \left( \mathcal{L}_+ \nabla \mathcal{L}_- y_l^m(x) + \mathcal{L}_- \nabla \mathcal{L}_+ y_l^m(x) \right).$$

It follows that the associated bilinear form is

$$\left(\mathbf{W} y_{l_1}^{m_1}, y_{l_2}^{m_2}\right)_{\mathbb{D}_1} = \frac{1}{2} \left( \left( \mathcal{L}_+ \nabla \mathcal{L}_- y_{l_1}^{m_1}, y_{l_2}^{m_2}\right)_{\mathbb{D}_1} + \left( \mathcal{L}_- \nabla \mathcal{L}_+ y_{l_1}^{m_1}, y_{l_2}^{m_2}\right)_{\mathbb{D}_1} \right)$$
$$= \frac{1}{2} \left( \left(\nabla \mathcal{L}_- y_{l_1}^{m_1}, \mathcal{L}_- y_{l_2}^{m_2}\right)_{\mathbb{D}_1} + \left(\nabla \mathcal{L}_+ y_{l_1}^{m_1}, \mathcal{L}_+ y_{l_2}^{m_2}\right)_{\mathbb{D}_1} \right), \quad (4.34)$$

\(^3\)It holds $\frac{4}{\lambda_0^m} = \frac{1}{2} \left( (0)(1)\lambda_0^{-1} + (0)(1)\lambda_0^1 \right) = \frac{4}{\pi} \Gamma(0)0$ where it is crucial that $\lim_{s \to 0} \Gamma(0)s = 1$.

In order to go from (4.29) to (4.30) one actually "splits this limit" and breaks the identity.
for \((l_1, m_1) \neq (0, 0)\) and \((l_2, m_2) \neq (0, 0)\).

Finally, (4.26) and

\[ \mathcal{L}_k = - \mathcal{L}_\tau, \quad (4.35) \]

imply that (4.34) can be rewritten as the desired formula. \(\square\)

It is worth noticing that the condition \((l, m) \neq (0, 0)\) only excludes the constants, characterized by \(y_0\). Due to the orthogonality (4.11), this space is defined by

\[ H^{1/2}(D_1) = \{ v \in H^{1/2}(D_1) : \langle v, \omega^{-1} \rangle_{D_1} = 0 \}, \]

as introduced in Proposition 4.1.

5. Conclusion. We have introduced new modified hypersingular and weakly singular integral operators that supply the exact inverses for the standard ones on disks. Moreover, we provide variational forms for these modified BIOs that are amenable to standard Galerkin boundary element discretization.

We consider the disk \(D_a\) to be the canonical shape for more general screens \(\Gamma\) and therefore an important starting point of our analysis. From this perspective, the new Calderón-type identities that we have shown over \(D_a\) are suitable for Calderón preconditioning on three-dimensional parametrized screens, in the same fashion as in 2D [7]. The applicability of these Calderón-type identities to construct \(h\)-independent preconditioners will be discussed in a upcoming article focusing on the numerical aspect of these new modified BIOs [8–10]. Though we developed our analysis only for the Laplace equation over disks with Dirichlet and Neumann boundary conditions, extensions to other elliptic equations, like the scalar Helmholtz equation, can be immediately achieved via perturbation arguments.

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