Effective Theory of Wilson Lines and Deconfinement

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To study the deconfining phase transition at nonzero temperature, I outline the perturbative construction of an effective theory for straight, thermal Wilson lines. Certain large, time dependent gauge transformations play a central role. They imply the existence of interfaces, which can be used to determine the form of the effective theory as a gauged, nonlinear sigma model of adjoint matrices. Especially near the transition, the Wilson line may undergo a Higgs effect. As an adjoint field, this can generate eigenvalue repulsion in the effective theory.

Recent results at the Relativistic Heavy Ion Collider (RHIC) demonstrate qualitatively new behavior for the collisions of heavy ions at high energies [1]. RHIC appears to have entered a region above $T_c$, the temperature for deconfinement, reaching up to temperatures a few times $T_c$. The experimental results cannot be explained if the transition is directly from a confined phase to a perturbative Quark-Gluon Plasma (QGP). Instead, RHIC seems to probe a novel region, which has been dubbed the “sQGP” [2].

In this paper I sketch how to develop an effective theory for the sQGP. Classically, the model is a familiar spin system, a gauged principal chiral field [3]; beyond leading order, it is more general. A mean field approximation of perturbation theory. As such, it applies only when fluctuations in $A_0$ are small. Computing the pressure to four loop order, $\sim \alpha^4$, the results are complete up to one undetermined constant [10]. Even with the most favorable choice for this constant, however, the pressure does not agree with that from numerical simulations on the lattice below temperatures of $\sim 3T_c$ [6, 8].

These computations are done in imaginary time, where the “energies” are multiples of $2\pi T$. Thus the coupling constant $\alpha_s(T)$ runs with a scale which is of order $\sim 2\pi T$ [5]. Computations to two loop order show that even better, this mass scale is $\sim 9T$ in QCD [8]. For $T_c \sim 175$ MeV, this is $\sim 1.6$ GeV; at $3T_c$, it is $\sim 4.7$ GeV. While these mass scales are not asymptotic, neither are they obviously in a non-perturbative regime: e.g., $\alpha_s(1.6$ GeV) $\sim 0.28$ [8]. Hence the question becomes: why does this effective theory fail between $T_c$ and $\sim 3T_c$, if the coupling is not that large?

To see how this might occur, consider a straight, thermal Wilson line in the fundamental representation:

$$L(x, \tau) = P \int_0^\tau A_0(x, \tau') d\tau'$$

P denotes path ordering, $x$ is the spatial position, and $\tau$, the imaginary time, runs from 0 to $1/T$. A closed loop is formed by wrapping all of the way around in imaginary time, $L(x, 1/T)$. As this quantity arises frequently, I denote it by $L(x)$.

The Wilson line is a matrix in color space, and so is not directly gauge invariant: under a gauge transformation $U(x, \tau), L(x) \rightarrow U(x, 1/T) L(x) U(x, 0)$. The trace of the Wilson line is gauge invariant, and is the Polyakov loop in the fundamental representation. Normalizing so that this loop is one when $A_0 = 0$, then its expectation value should be near one if $gA_0/(2\pi T)$ is small. Numerical simulations of a lattice $SU(3)$ gauge theory show that while the expectation value of the renormalized triplet loop is near one at $3T_c$, this is not so when $T < 3T_c$.

Without dynamical quarks, it drops to a value of $\approx 0.45$ at $T_c$ [1, 12, 13, 14]; its value with dynamical quarks is similar [15].
Since the triplet loop is significantly less than one between $T_c$ and $\sim 3T_c$, in this region it is necessary to extend the program of [5, 6, 8, 9, 10] to construct an effective, three dimensional theory for arbitrary values of $gA_0/(2\pi T)$. While $A_0$ can be large, as it applies only for distances $\gg 1/T$, we can assume that all spatial momenta are small relative to $2\pi T$ [16, 17, 18, 19, 20, 21]. This is like chiral perturbation theory, with temperature playing the role of the pion decay constant.

Certainly the effective theory must be invariant under static gauge transformations, $U(x, \tau) = U(x)$. In addition, and somewhat unexpectedly for a theory in three dimensions, certain time dependent gauge transformations matter. For a $SU(N)$ gauge group, consider

$$U_c(\tau) = e^{2\pi i \tau T_3} t_N, \quad t_N = \begin{pmatrix} 1 & 0 \\ 0 & -(N-1) \end{pmatrix}; \quad (3)$$

$1_{N-1}$ is the unit matrix. This is spatially constant and strictly periodic in $\tau$, $U_c(1/T) = e^{2\pi i} 1_N = U_c(0)$, and so appears to be rather trivial. Instead, it turns out to be essential in constraining the form of the effective Lagrangian at large $A_0$. Since they don’t alter the boundary conditions in imaginary time, similar gauge transformations exist for any gauge group, coupled to matter fields in arbitrary representations.

The problem cannot be ignored at large $A_0$, and arose previously [20, 21]. The $N$th root of $U_c$ is an aperiodic gauge transformation, $e^{2\pi i/N} 1_N$ at $\tau = 1/T$. If there are no dynamical quarks present, this is an allowed gauge transformation, and reflects the $Z(N)$ center symmetry of a $SU(N)$ gauge group [22]. Ref. [20] computed in the presence of nonzero, background fields for both $A_0$ and $A_i$, allowing $A_0$ to be large. They found that if the effective Lagrangian is formed from terms such as $D_i A_0$, then the $Z(N)$ center symmetry appears to be violated at one loop order. The argument above, applied to $U_c^{1/N}$, shows that even classically, $E_i = D_i A_0$ is not consistent with the requisite $Z(N)$ symmetry.

The significance of these large gauge transformations can be understood by looking at the Wilson line. Since $L$ is a $SU(N)$ matrix, $L^\dagger(x) L(x) = 1_N$, it can be diagonalized by a unitary transformation [23],

$$L(x) = \Omega(x)^\dagger e^{i\lambda(x)} \Omega(x). \quad (4)$$

$\lambda(x)$ is a diagonal matrix, with elements $\lambda_a, a = 1 \ldots N$. As det$(L) = 1$, $\text{tr} \lambda(x) = 0$, modulo $2\pi$. Under static gauge transformations, $U(x) = U$, the adjoint covariant derivative and the Wilson line transform similarly, $D_i \rightarrow U^\dagger D_i U$ and $L \rightarrow U^\dagger L U$. Hence the $\lambda_a$ do not change, while $\Omega$ is gauge dependent, $\Omega \rightarrow \Omega U$ [24].

The $\lambda_a$ can change under time dependent gauge transformations: under (3), $\lambda \rightarrow \lambda + 2\pi t_N$, so each $\lambda_a$ shifts by an integral multiple of $2\pi$. Hence gauge transformations such as (3) ensure that the $\lambda_a$ are periodic. Of course this is obvious from the definition of the Wilson line, since its eigenvalues are just $e^{i\lambda_a}$.

This periodicity is present for an abelian gauge group, where the Wilson line is merely a phase, $L = e^{i\lambda}$. Shifting $\lambda \rightarrow \lambda + 2\pi$ is an Aharonov-Bohm effect, where the Wilson line, in imaginary time, wraps around a patch of magnetic flux in a fictitious fifth dimension. This illustrates elementary topology [25]. At nonzero temperature, imaginary time is isomorphic to a sphere in one dimension, $S^1$. Topologically nontrivial windings are given by mappings from $S^1$ into $U(1)$, and are classified by the first homotopy group, $\pi_1(U(1)) = Z$, where $Z$ is the group of the integers.

The result for a nonabelian group is an exercise in abelian projection [23]. The $r$ diagonal generators in the Cartan subalgebra define the maximal torus, which is an abelian subgroup of $U(1)^r$, the direct product of $r$ $U(1)$’s. Nontrivial windings are then given by $\pi_1(U(1)^r) = Z^r$. In $SU(N)$, $r = N - 1$, and $t_N$ is one of these diagonal generators.

The effective Lagrangian must respect the periodicity of the $\lambda_a$’s. This is automatic if it is constructed from the Wilson line. What is then obscure is the form of the effective electric field, $E_i$. Consider

$$E_i(x) = \frac{T}{ig} L^\dagger(x) D_i L(x). \quad (5)$$

Like the original electric field, this is gauge covariant, $E_i \rightarrow U^\dagger E_i U$. It is also hermitean, and so is not $\sim D_i L$. If the gauge group has a center symmetry, then $E_i$ is trivially center symmetric. In accord with the conclusions of [22], though, the presence of a center symmetry is really secondary for what follows.

For small $A_0$, and static $A_i \neq 0$, this reduces to the expected form, $E_i = D_i A_0$, as in (2). This rules out using an $E_i$ constructed entirely from the eigenvalues of $L$. The simplest example is $E_i \sim \partial_\tau \lambda$, with an infinity of other terms, such as $|\text{tr} L|^2$ times this, etc.

There is one last limit which is essential in establishing (5), although its origin will only be clear after the discussion of interfaces below. I require that when $A_i = 0$, and $A_0$ is static and diagonal — but of arbitrary magnitude — that it reduces to the abelian form, $E_i = \partial_t A_0$. This forbids an infinity of terms, formed by taking various combinations of traces of $L$ times (5), such as $|\text{tr} L|^2$, $|\text{tr} L^2|^2$, etc. (Equivalently, one can write these terms times (8), as in (9)-(13) of [26],.) To leading order, these conditions uniquely determine $E_i$. In mathematics, (5) is known as the left invariant one form of $L$ [27].

Using the properties of path ordering, the effective electric field can be written as

$$E_i/T = \frac{1}{T} \int_0^{1/T} d\tau L(\tau) \frac{1}{i} \partial_t A_0(\tau) L(\tau) - L^\dagger [A_i, L]; \quad (6)$$

$L = L(1/T)$, and the $x$ dependence is suppressed. Up to the various Wilson lines — which are, after all, elements in the gauge group — this is a plausible form for a gauge covariant electric field formed by averaging over $\tau$. 
With this $E_i$, the effective Lagrangian is that of a
gauged, nonlinear sigma model [3, 28]:

$$\mathcal{L}_{\text{eff, classical}}(A_i, L) = \frac{1}{2} \text{tr} \, G_{ij}^2 + \frac{T^2}{g^2} \text{tr} \, |L^i D_i L|^2 .$$  \hspace{1cm} (7)

Using the decomposition of the Wilson line in (4), the
electric field term is proportional to

$$\text{tr} \, |D_i L|^2 = \text{tr} \, (\partial \lambda)^2 + \text{tr} \, \left[ \Omega \, D_i \Omega^\dagger , e^{i \lambda} \right]^2 .$$  \hspace{1cm} (8)

The first term is that of an abelian theory, while the sec-
ond couples the nonabelian electric and magnetic sectors
together. Since $e^{i \lambda}$ is invariant under static gauge trans-
formations, so is the combination $\Omega \, D_i \Omega^\dagger$ [24].

On the lattice, the analogy of (8) is well known from
Banks and Ukawa [29]. I suggested (8) previously [19],
but only by expanding in small $A_0$. This does not suffice
for computations on a lattice with no quark mass.

The effective Lagrangian of (7) is not renormalizable in
three dimensions (it is in two [26], but this is a standard
feature of effective theories [5]. It is also common that the
effective fields are only indirectly related to those in
the original theory, although it is especially striking here.

As an aside, I remark that the instanton number in four
dimensions carries over directly to the effective the-
ory. Start with a smooth, strictly periodic classical field,
$A_0(x, \tau)$, and then transform to $A_0 = 0$ gauge. The
gauge transformation which does this is just $U(x, \tau)$, (2).
The instanton number is then a difference of Chern-
Simons terms between $\tau = 1/T$ and 0 [16]. One can show
that the instanton number equals the winding number of the
Wilson line:

$$\frac{1}{24 \pi^2} \int d^3x \, \epsilon^{ijk} \text{tr} \, (C_i C_j C_k) \; ; \; C_i = L^i \partial_i L ,$$  \hspace{1cm} (9)

which is an integer. This suggests an analogy to the color
Skyrmions of [27]: these have nonzero winding number,
but are not instantons.

To establish (7), it is necessary to show that it gives
the same physics as the original theory, especially at large
$A_0$. One possibility is to use the interfaces which exist
because the $\lambda_\alpha$’s are periodic.

This is most familiar for the $Z(N)$ interface of a $SU(N)$
gauge theory without quarks [18, 30]. This is given by
taking a box, long in one spatial direction, with $L = 1_N$
at one end of the box, and $L = e^{2 \pi i / N} 1_N$ at the other.
A $Z(N)$ interface is related to the disorder parameter of
’t Hooft [31, 32].

The interface which corresponds to $U_c$, (3), is given by
taking $A_0 = 0$ at one end of the box, and $A_0 = 2 \pi T \tau / g$
at the other [33]. This is a $U(1)$ interface: while $L = 1_N$
at both ends of the box, the change in $A_0$ is nontrivial,
and cannot be undone [25]. It is not related to a disorder
parameter. Without quarks, there is a row of $N$, distinct
$Z(N)$ interfaces. With quarks, these coalesce into one
$U(1)$ interface; like the expectation value of $L$, it exists
in both phases.

To leading order in $g^2$, it is easy to use interfaces to
match the effective theory to the original. In the original
theory, compute for constant $L$ to one loop order. For a
$SU(N)$ gauge theory without quarks, this gives [16]

$$\mathcal{L}_{\text{eff, loop}}(L) = -\frac{2 T^4}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^4} \left| \text{tr} \, L^m \right|^2 + \frac{\pi^2 T^4}{45} .$$  \hspace{1cm} (10)

At leading order, the effective Lagrangian is the sum of
(7) and (10) [18]. Because $E_i = \partial_i A_0$ when $A_0 = 0$
and $A_0$ is static, diagonal, and of arbitrary magnitude,
trivially a $Z(N)$ interface is the same in both theories.
Dynamical quarks add new terms to the potential, which
lift the $Z(N)$ symmetry, and so remove $Z(N)$ interfaces.
$U(1)$ interfaces remain, and are analyzed similarly, with
the same result for $E_i$.

At higher order, matching between the original and
effective theories is much more involved. The effective
Lagrangian is constructed from $L$ and $G_{ij}$ in a derivative
expansion, with terms for constant $L$ [4, 12, 13], two
derivatives [12, 18, 19, 20, 21, 26], four [20], and so on.
At higher order, matching will involve computing both the
interface tension, $\sim T^4 / \sqrt{\alpha_s}$, and expectation values of
gauge invariant operators in the presence of an interface.

Further, my discussion of (2) was incomplete: it is merely
the first step of three, with the others integrating out the
electric and magnetic sectors [5, 6, 8, 9, 10]. While $A_0$ is large at the center of an interface, it is small
at the ends, and so there the electric sector must be
treated more carefully. For the pressure, it should be possible
to isolate that piece which is $L$ dependent, after
subtracting the vacuum energy of the static magnetic
sector for $L = 0$ [10].

While meaningful statements can only be made after
computation at next to leading order, when the scale of
$\alpha_s$ is set, qualitatively much of the physics can be under-
stood from (10). The perturbative vacuum, $\langle \bar{L} L \rangle = 1_N$,
gives minus the pressure of an ideal $SU(N)$ gas, $p_{\text{ideal}} =
-\mathcal{L}_{\text{eff, loop}}(1_N) = + (N^2 - 1) \pi^2 T^4 / 45$.
This is the absolute minimum at leading order, and it is at least metastable,
order by order in $\alpha_s$.

In a $SU(N)$ gauge theory without quarks, deconfinement is related to the breaking of a global $Z(N)$ sym-
metry: under a $Z(N)$ transformation, $L \rightarrow zL$, where
$z = e^{2 \pi i / N}$. Consider the diagonal $SU(N)$ matrix

$$L_c = \text{diag} \left( 1, z, z^2, \ldots, z^{N-1} \right)$$  \hspace{1cm} (11)

Of the loops constructed from $L_c$, only those which are
$Z(N)$ neutral are nonzero: if $m$ is an integer, $\text{tr} \, (L_c)^m = 0$
when $m$ is not a multiple of $N$, and $= N$ when it is.
Hence $L_c$ might represent the $Z(N)$ symmetric, confined vacuum [17, 34, 35]. However, at leading order, (10),
$p(L_c) = -\mathcal{L}_{\text{eff, loop}}(L_c) = -(1 - 1/N^2) \pi^2 T^4 / 45$. Thus for
any finite $N$, $L_c$ has negative pressure, and is not a phys-
ic state.

At infinite $N$, however, $L_c$ does represent the confined vacuum. While its pressure is negative, this is $\sim 1$, and is
negligible relative to that $\sim N^2$ in the deconfined phase [4, 34, 35]. While (10) is only valid at leading order, since any trace of $L_c$ vanishes at $N = \infty$, the pressure for $L_c$ remains $\sim 1$ to all orders in $\alpha_s N$ [35].

At infinite $N$, $L_c$ is familiar from random matrix models: there is complete eigenvalue repulsion, and a flat eigenvalue density [3, 4, 34]. Numerical simulations suggest that in the confined phase, the eigenvalue density for small $N$ is like that of $N = \infty$. By factorization, in the confined phase the expectation value of the renormalized adjoint loop is $\sim 1/N^2$ [12]. For $N = 3$, though, numerically this is found to be not $\sim 10\%$, but only $\sim 1$% [12, 14]. That the expectation value of a $Z(N)$ neutral loop is so small indicates that the functional integral is close to an integral over the group measure; i.e., that the eigenvalue density is nearly flat.

In perturbation theory, though, there is no sign of any eigenvalue repulsion which might produce a flat distribution. As in (10), and seen to three loop order in [4], the perturbative potential for constant $L$ only involves sums of eigenvalues, and not differences. Thus eigenvalue repulsion, and so confinement, must be generated by fluctuations in the effective theory.

It is known how this happens for $SU(\infty)$ on a very small sphere [4]. The effective Lagrangian is a single integral for the constant mode of $L$: as a random matrix model, the Vandermonde determinant in the measure generates eigenvalue repulsion and drives the transition [3, 4, 13]. In infinite volume, though, terms in the measure depend upon the regularization; e.g., they vanish with dimensional regularization.

To represent the non-perturbative effects which might drive the transition in infinite volume, consider adding to the effective Lagrangian

$$L_{\text{non-pert.}}^{\text{eff}} \sim + B_f T^2 |\text{tr} L|^2 .$$

While motivated by precise results from numerical simulations [36], this term is only meant to illustrate what is possible near $T_c$ [37]. It shifts the minimum in the loop potential from the perturbative value, $\langle L \rangle = 1_N$, to some $\langle L \rangle \neq 1_N$. This is interesting because it produces a Higgs effect for $A_i$. As an adjoint field, in perturbation theory the mass magnetic gluons acquire from $\langle L \rangle \neq 1_N$ involves differences of eigenvalues, as diagonal gluons remain massless, and off diagonal gluons develop a mass [23]. Integrating out fluctuations in $A_i$ and $L$ to one loop order (which is easiest in unitary gauge), there is a qualitatively new term in the effective Lagrangian,

$$\Delta L^{\text{eff}} \sim - \sum_{a,b=1}^N (g^2 |e_i \lambda_a - e_i \lambda_b|^2)^{3/2} .$$

The mass dimensions are made up by $T$ and $B_f$. The sign is physical, and corresponds to eigenvalue repulsion. Once (13) is included, $\langle L \rangle$ is given by a distribution of eigenvalues.

These calculations are only suggestive. It is not obvious how to characterize, gauge invariantly, such an adjoint Higgs phase for strongly coupled gluons in three dimensions. Qualitatively, a Higgs phase should increase the mixing between the Wilson line and magnetic glueballs [30, 38], which is usually very small.

The effective theory, as determined perturbatively, can be studied in various ways. Since the ultraviolet cutoff is physical, it is reasonable to start with mean field theory [12, 13]. To do better, the theory could be simulated numerically on a lattice, to directly measure quantities such as the eigenvalue distribution, glueball masses, etc. At large $N$ [39], analytic approximations may help [40].

The usual justification for an effective Lagrangian is the presence of a small mass scale, but generically, there is none here. If the effective coupling is small at $T_c$ [8], though, then with care nothing is lost by going to an effective Lagrangian. Presumably, this is worthwhile while $(2)$ fails: from $\sim 3T_c$, down to some point below $T_c$ [41]. For constant $L$, the effective potential shows no signs of a transition to a confining phase: at $N = \infty$, and perhaps even for small $N$, this must involve eigenvalue repulsion. In infinite volume, this arises dynamically, especially from fluctuations in the angular variables, $\Omega$, and the gauge fields, $A_i$. This, then, is why the effective theory is of interest: we can use it to uniquely isolate the dynamic origin of the transition, as eigenvalue repulsion. It thus provides a notable example of a field theory of (not so) random matrices [3].

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In a $SU(3)$ gauge theory without quarks, using the lattice data of [36] one can show that $(e - 3p)/T^4$, times $T^2$, is nearly constant from $\sim 1.1T_c$ to $4.0T_c$, where $e$ is the energy density and $p$ the pressure. (As the deconfining transition for $N = 3$ is nearly second order, this quantity rises sharply from near zero at $T_c$.) This, and $p(T_c) \approx 0$, implies that

$$p(T) \approx f_{\text{pert.}} \left( T^4 - T^2 T^2 \right), \quad 1.1T_c < T < 4.0T_c;$$

for constant $f_{\text{pert.}}$. To extrapolate to $T > 4.0T_c$, let $f_{\text{pert.}}$ include all perturbative contributions to the pressure, so it becomes a slowly varying function of temperature. This also suggests a form for the pressure in the deconfined phase of QCD, with dynamical quarks:

$$p_{\text{QCD}}(T) \approx f_{\text{pert.}} T^4 - B_f T^2 - B_{\text{MIT}} + \ldots .$$

While this includes the usual MIT bag constant, $B_{\text{MIT}}$, the leading non-perturbative contribution to the pressure is given by a temperature dependent, or “fuzzy” bag constant, $B_f$. This applies at temperatures above the maximum in $(e - 3p)/T^4$, see R. D. Pisarski, [arXiv:hep-ph/0612191]. For the present analysis, the
moral I draw is that non-perturbative terms in the effective Lagrangian begin $\sim T^2$, as in (12); that its coefficient is denoted $B_f$ is only meant to be suggestive.

[38] To see changes in mixing, it may be useful to measure magnetic masses from spatial plaquettes split in $\tau$: take two spatial links at $\tau = 0$, and two at $\tau = 1/T$, tied together by two thermal Wilson lines.

[39] B. Lucini, M. Teper and U. Wenger, J. High Energy Phys. 033 (2005) 0502 [arXiv:hep-lat/0502003]; B. Bringoltz and M. Teper, Phys. Lett. B 628, 113 (2005) [arXiv:hep-lat/0506034].

[40] L. G. Yaffe, Rev. Mod. Phys. 54, 407 (1982).

[41] At low temperature, dimensional reduction is not of use, and it cannot suffice to include only straight, thermal Wilson lines; those which oscillate in time also contribute. At zero temperature, for instance, only closed loops matter.