The existence of Hall polynomials for $x^2$-bounded invariant subspaces of nilpotent linear operators

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Abstract
We prove the existence of Hall polynomials for $x^2$-bounded invariant subspaces of nilpotent linear operators.

1. Introduction

Let $k$ be a field and let $k[x]$ be the algebra of polynomials in one variable $x$ and coefficients in the field $k$. We denote by $\mathcal{N} = \mathcal{N}(k)$ the category of all finitely generated nilpotent $k[x]$-modules, i.e. each object of $\mathcal{N}$ is isomorphic to a module of the form:

$$N_\alpha = N_\alpha(k) = k[x]/(x^{\alpha_1}) \oplus \ldots \oplus k[x]/(x^{\alpha_m}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a partition, that is a sequence of natural numbers satisfying $\alpha_1 \geq \ldots \geq \alpha_m$. The map $\alpha \mapsto N_\alpha(k)$ defines a bijection between the set of all partitions and the set of isomorphism classes of objects in $\mathcal{N}(k)$. We call the partition $\alpha$ the type of $N_\alpha(k)$ and denote it by $\text{type}(N_\alpha(k))$.

Let $\alpha, \beta, \gamma$ be a triple of partitions. It is well-known that there exists the Hall polynomial $\varphi^{\beta}_{\alpha, \gamma} \in \mathbb{Z}[t]$, i.e. a polynomial such that for any finite field $k$:

$$F_{N_\alpha(k), N_\gamma(k)} = \varphi^{\beta}_{\alpha, \gamma}(q),$$

where $q = |k|$, $F_{N_\alpha(k), N_\gamma(k)} = |\{U \subseteq N_\beta : U \text{ is a submodule of } N_\beta, \ U \simeq N_\alpha \text{ and } N_\beta/U \simeq N_\gamma\}|$

and given a set $X$ we denote by $|X|$ its cardinality.

The existence of Hall polynomials in the category $\mathcal{N} = S_0$ was proved by P. Hall, let us quote also I. G. Macdonald, T. Klein, J. A. Green, A. Zelevinski, see [11], where one can find the connections to symmetric functions. C.
M. Ringel adopted the theory of Hall polynomials and Hall algebras to the representations theory of finite dimensional algebras, see [14]. In [15] Ringel conjectured the existence of Hall polynomials for all representation-finite algebras.

The main aim of the paper is to prove Theorem 2.4, which states that there exist Hall polynomials for finite dimensional $\Lambda_{2,w}(k)$-modules for all $w \geq 2$, where

$$\Lambda_{r,w}(k) = \begin{bmatrix} k[x]/(x^r) & k[x]/(x^r) \\ 0 & k[x]/(x^w) \end{bmatrix}.$$

Investigation of $\Lambda_{r,w}$-modules is related to the Birkhoff problem of classification subgroups of finite abelian groups, see [2]. Categories of finite dimensional $\Lambda_{r,w}$-modules are wild in general, see [19], that means, for arbitrary $r, w$ there is no hope for a nice classification of $\Lambda_{r,w}$-modules. However there are many results describing properties of modules over these algebras: e.g. [1, 3, 7, 9, 16, 18, 19].

The paper is organized as follows.

- In Section 2 we present preliminaries on $\Lambda_{2,w}(k)$-modules and we formulate the main Theorem 2.4.
- Section 3 contains definitions and some elementary facts about Ringel-Hall algebras.
- In Section 4 certain properties of the Steinitz's classical Hall algebra are collected. Moreover applying properties of the Kostka numbers we prove the existence of some polynomials that are needed in the proof of the main result.
- The most computational part of the proof of the main result is placed in Section 5 where some relations in the Ringel-Hall algebra of $\Lambda_{2,w}(k)$ are collected.
- Finally, we finish in Section 6 the proof of Theorem 2.4.

2. Preliminaries and the main theorem

By $\soc N$ we denote the socle of a $k[x]$-module $N \in \mathcal{N}$, i.e. the sum of all simple submodules. The following fact seems to be well-known.
Lemma 2.1. Let $0 \neq v \in \text{soc} N$, where $N \in \mathcal{N}$ and let $v = x^{m-1}v'$ for
some $v' \in N$, where $m$ is the maximal number such that $x^{m-1}$ divides $v$. Then the $k[x]$-submodule $\langle v' \rangle$ of $N$ generated by $v'$ is a direct summand of $N$ isomorphic to $k[x]/(x^m)$.

Proof. Follows directly from the description of the finite dimensional nilpotent $k[x]$-modules.

In what follows we use the standard notation (see [11]): Given a partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ we set $|\alpha| = \sum_i \alpha_i$, $n(\alpha) = \sum(i-1)\alpha_i$ and we denote by $\alpha' = (\alpha'_1, \ldots, \alpha'_s)$ the dual partition defined by $\alpha'_i = \max\{j: \alpha_j \geq i\}$. We write $(1^r)$ for the partition $(1, \ldots, 1)$ ($r$ ones). We write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for every $i$.

We denote by $S = S(k)$ the category of all triples $(N', N'', f)$, where $f : N' \to N''$ is a monomorphism. The morphisms in $S$ are defined in the natural way. Note that an object $(N_\alpha, N_\beta, f)$ gives a short exact sequence

$$0 \to N_\alpha \xrightarrow{f} N_\beta \to N_\gamma \to 0,$$

where $\gamma = \text{type}(\text{Coker} f)$.

Let $S_r(k)$ denotes the full subcategory of $S(k)$ consisting of all triples isomorphic to $(N_\alpha, N_\beta, f)$ with $\alpha_1 \leq r$. Note that the category $S_0$ is equivalent to $N$. The categories $S_r$ for $r \geq 3$ have wild representation type, see [19]. It means that it is hopeless to find a nice parametrization of isomorphism classes of objects in $S_r$, for $r \geq 3$. In the paper we work in the category $S_2$ that has a discrete representation type, i.e. for any integer $m$ there is only finitely many (up to isomorphism) objects in $S_2$ with $k$-dimension $m$.

If the nilpotency degree of the ambient space is bounded by $w$, then an object of $S_r(k)$ ($r \leq w$) can be treated as a right module over the algebra

$$\Lambda_{r,w}(k) = \begin{bmatrix} k[x]/(x^r) & k[x]/(x^r) \\ 0 & k[x]/(x^w) \end{bmatrix}.$$ 

Lemma 2.2. The algebra $\Lambda_{2,w}(k)$ is representation finite (i.e. there is only finitely many isomorphism classes of indecomposable $\Lambda_{2,w}(k)$-modules) and every indecomposable $\Lambda_{2,v}(k)$-module is isomorphic to one from the following list.

(1) $P_0^m(k) = (0, N_{(m)}(k), 0)$, $1 \leq m \leq w$,
(2) $P^m_1(k) = (N_{(1)}(k), N_{(m)}(k), \iota), \ 1 \leq m \leq w, \ \iota(1) = x^{m-1}$,
(3) $P^m_2(k) = (N_{(2)}(k), N_{(m)}(k), \iota), \ 2 \leq m \leq w, \ \iota(1) = x^{m-2}$,
(4) $B_{m', m}^2(k) = (N_{(2)}(k), N_{(m')}_{(m)}(k) \oplus N_{(m)}(k), \iota), \ 3 \leq m + 2 \leq m' \leq w$
   \[ \iota(1) = \begin{bmatrix} x^{m'-2} \\ x^{m-1} \end{bmatrix}, \]
(5) $P^0_1(k) = (N_{(1)}(k), 0, 0)$,
(6) $P^0_2(k) = (N_{(2)}(k), 0, 0)$,
(7) $P^1_1(k) = (N_{(2)}(k), N_{(1)}(k), \iota), \ \iota(1) = 1$,
(8) $Z^m_2(k) = (N_{(2)}(k), N_{(m)}(k), \kappa), \ \kappa(1) = x^{m-1}, \ 1 \leq m \leq w$.

Proof. Let $V = (V', V'', f)$ be an indecomposable $\Lambda_{2,w}(k)$-module. Given an
element $w$ of $V'$ we denote by $\langle w \rangle$ the $k[x]/(x^2)$-submodule of $V'$ generated
by $w$. If $f$ is a monomorphism, then $V$ is isomorphic to a module of one of the
types (1)-(4) by [3]. Suppose otherwise and let $f(v) = 0$ for some $0 \neq v \in V'$. If $xv \neq 0$, then the submodule $\langle v \rangle$ of $V'$ is isomorphic to $k[x]/(x^2)$, thus by
Lemma 2.1 (Let us recall that $x$ is a direct summand of $V$). It follows that $P^0_2(k)$ is a direct summand of $V$, hence $V \cong P^0_2(k)$. If $xv = 0$ then $v \in \text{soc } V'$. If $v$ is not divisible by $x$, then the submodule $\langle v \rangle$ of $V'$ is a direct summand of $V'$ by Lemma 2.1 and it is isomorphic to $k[x]/(x)$. Then $V$ has a direct summand isomorphic to $P^0_1(k)$, hence $V$ is isomorphic to $P^0_1(k)$. Otherwise, let $xv' = v$; then the submodule $\langle v' \rangle$ of $V'$ is isomorphic to $k[x]/(x^2)$. Moreover, $f(v') \in \text{soc } V''$, thus, again Lemma 2.1 $f(v')$ is an element of the socle of a direct summand of $V''$ isomorphic to $k[x]/(x^m)$, where $m$ is the maximal number such that $x^{m-1}$ divides $f(v')$. Then $V$ has a direct summand isomorphic to $Z^m_2(k)$, hence $V \cong Z^m_2(k)$.

The objects $(N_{\alpha}, N_\beta, f)$ of $S_2(k)$ with $\beta_1 \leq n$ are the $\Lambda_{2,n}(k)$-modules
having no direct summands isomorphic to $P^0_1(k)$, $P^0_2(k)$, $P^1_1(k)$ or $Z^m_2(k)$
($m \leq n$).

Observe that $Z^1_2(k) = P^1_2(k)$.

It follows that the isomorphism classes of indecomposable $\Lambda_{2,n}(k)$-modules are in a 1-1 correspondence with a finite set

\[ I_{2,n} = \{ P^m_0 : 1 \leq m \leq n \} \cup \{ P^m_1 : 1 \leq m \leq n \} \cup \{ P^m_2 : 2 \leq m \leq n \} \cup \{ B^m_{2', m} : 3 \leq m + 2 \leq m' \leq n \} \cup \{ P^0_1, P^0_2 \} \cup \{ Z^m_2 : 1 \leq m \leq n \} \]

\[ (2.3) \]

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which is independent on the field $k$. We denote by $X(k)$ the module corresponding to $X \in I_{2,n}$. Thus, for any function $a : I_{2,n} \to \mathbb{N}_0$ we can associate the $\Lambda_{2,n}(k)$-module $\bigoplus_{X \in I_{2,n}} X(k)^{a(X)}$, which we shall denote by $a(k)$. It is convenient to write $\bigoplus_{X \in I_{2,n}} X^{a(X)}$ instead of $a$ and call it a $\Lambda_{2,n}$-module.

Given right modules $X, Y, Z$ over a finite dimensional algebra $A$ over a finite field $k$ we set

$$F_{X,Z}^Y = |\{U \subseteq Y : U \text{ is a submodule of } Y, \ U \simeq X \text{ and } Y/U \simeq Z\}|.$$

The main aim of the paper is the following theorem.

**Theorem 2.4.** Let $a, b, c : I_{2,n} \to \mathbb{N}_0$. There exists $\varphi_{b, a, c}^b \in \mathbb{Z}[t]$ such that for any finite field $k$

$$F_{X,b(a,k),X,c(k)}^{X_{b,(k)}} = \varphi_{b,a,c}^b(q),$$

where $q = |k|$.

The polynomials $\varphi_{b,a,c}^b$ are called **Hall polynomials**.

3. **Generalities on Ringel-Hall algebras**

Let $k$ be a finite field, $A$ be a finite dimensional $k$-algebra and let $X_1, \ldots, X_t$, $Y$ be (right) $A$-modules. Let $F_{X_1,\ldots,X_t}^Y$ be the number of filtrations

$$0 = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_{t-1} \subseteq U_t = Y$$

of $Y$ such that $U_i/U_{i-1} \simeq X_i$ for all $i = 1, \ldots, t$. Note that $F_{X,Z}^Y$ is the number of submodules $U$ in $Y$ with $U \simeq X$ and $Y/U \simeq Z$.

Following [14] we define the Ringel-Hall algebra $\mathcal{H}(A)$ to be the $\mathbb{C}$-vector space with basis $\{u_X\}_{[X]}$ indexed by the isomorphism classes $[X]$ of finite dimensional $A$-modules and with multiplication given by the following formula

$$u_X u_Z = \sum_{[Y]} F_{X,Z}^Y u_Y.$$

It is well-known that $\mathcal{H}(A)$ is an associative algebra with $1 = u_0$ and

$$(3.1) \quad u_{X_1} u_{X_2} u_{X_3} = \sum_{[T]} F_{X_1,X_2,X_3}^T u_T,$$

$^1$More precisely, we can define a $\mathbb{Z}$-algebra $\Lambda_{2,n}$ and $\Lambda_{2,n}$-modules $P_{1,m}^m$ etc. in such a way that $P_{1,m}(k) = P_{1,m}^m \otimes_{\mathbb{Z}} k$. 

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where $[T]$ runs through the set of the isomorphism classes of finite dimensional $A$-modules, see [14 Proposition 1].

Note that

\[(3.2) \sum_{[T]} F^Y_{X_1,T} F^T_{X_2,x_3} = F^Y_{X_1,x_2,x_3} = \sum_{[T]} F^T_{X_1,x_2} F^Y_{T,x_3},\]

see [13, page 442].

More generally,

\[(3.3) u_{X_1} \ldots u_{X_t} = \sum_{[T]} F^T_{X_1,\ldots,X_t} u_T\]

and

\[(3.4) F^T_{X_1,\ldots,X_t} = \sum_{[Y_1],\ldots,[Y_t-1]} F^Y_{X_1,Y_2} F^X_{X_2,Y_3} \ldots F^Y_{X_t-1,X_t},\]

where $[Y_1],\ldots,[Y_{t-1}]$ run through the set of the isomorphism classes of finite dimensional $A$-modules, see [13, Remark on page 4].

The following lemma is a version of [12, Lemma 2.1].

**Lemma 3.5.** Let $X, X_1, X_2, X_3, X_4, Y, Z$ be $A$-modules (possibly equal to 0), $a, b \in \mathbb{C}$ and assume that $u_X = au_{X_1} u_{X_2} + bu_{X_3} u_{X_4}$ in $\mathcal{H}(A)$ Then

$$F^Y_{X,Z} = aF^Y_{X_1,x_2,Z} + bF^Y_{X_3,x_4,Z}$$

**Proof.** Let $u_X = au_{X_1} u_{X_2} + bu_{X_3} u_{X_4}$. Then by (3.1)

$$\begin{align*}
(au_{X_1} u_{X_2} + bu_{X_3} u_{X_4}) u_Z &= au_{X_1} u_{X_2} u_Z + bu_{X_3} u_{X_4} u_Z = \sum_{[Y'] \neq [Y]} (aF^Y_{X_1,x_2,Z} + bF^Y_{X_3,x_4,Z}) u_{Y'}.
\end{align*}$$

Moreover

$$u_X u_Z = F^Y_{X,Z} u_Y + \sum_{[Y'] \neq [Y]} F^Y_{X,Z} u_{Y'}. $$

Comparing these formulae we get

$$F^Y_{X,Z} = aF^Y_{X_1,x_2,Z} + bF^Y_{X_3,x_4,Z},$$

because elements $\{u_Y\}_{[Y]}$ form a $\mathbb{C}$-basis of $\mathcal{H}(A)$.

\[\square\]
Let us denote by $\text{Mon}_A(X, Y)$ the set of the monomorphisms $X \to Y$.

**Lemma 3.6.** Assume that $X, Y, Z$ are $A$-modules and $X = X' \oplus I$, where $I$ is an injective module. Then $F^Y_{X,Z}$ is nonzero if and only if $Y \cong Y' \oplus I$ for some module $Y'$ and $F^{Y'}_{X',Z}$ is nonzero.

In this case

$$F^Y_{X,Z} = \frac{|\text{Mon}_A(I, Y)|}{|\text{Mon}_A(I, X)|} \cdot F^{Y'}_{X',Z}. $$

**Proof.** The first assertion follows easily, because $I$ is injective. For the proof of the formula note that by (3.2) applied to $X_1 = I$, $X_2 = X'$, $X_3 = Z$ we get

$$\sum_{[T]} F^Y_{I,T} F^T_{X',Z} = \sum_{[U]} F^U_{I,X'} F^Y_{U,Z}$$

and, as $I$ is injective, the equality reduces to

$$F^Y_{I,Y',I,X',Z} = F^X_{I,X'} F^Y_{X,Z}. $$

Since $F^I_{I,U} = \frac{|\text{Mon}_A(I \oplus U)|}{|\text{Aut}(I)|}$ for injective $I$ and arbitrary $U$, the lemma follows.

4. Some relations in the Steinitz’s classical Hall algebra

Given a finite field $k$ with $|k| = q$ we denote by $\mathcal{H}(q)$ the (classical) Hall algebra defined as the free $\mathbb{C}$-vector space with the basis $u_\lambda$, where $\lambda$ runs through the set of all partitions, equipped with the multiplication defined by the formula

$$u_\mu u_\nu = \sum_\lambda F^\lambda_{\mu,\nu}(q) u_\lambda,$$

where $F^\lambda_{\mu,\nu}(q)$ is the number of the submodules $U$ of $N(\lambda)$ which are isomorphic to $N(\mu)$ and $N(\lambda)/U \cong N(\nu)$. The classical results mentioned in the Introduction asserts that $F^\lambda_{\mu,\nu}(q)$ are polynomials in $q$. In what follows we shall use some more precise properties, which we list in the following proposition.

**Proposition 4.1.** (a) If $F^\lambda_{\mu,\nu}(q) \neq 0$ then $\mu, \nu \subseteq \lambda$, $|\lambda| = |\mu| + |\nu|$ and $F^\lambda_{\mu,\nu}$ is a polynomial in $q$ of degree $n(\lambda) - n(\mu) - n(\nu)$. 


(b) $\mathcal{H}(q)$ is generated as a $\mathbb{C}$-vector space by the elements $u_{(1^{l_1})} \ldots u_{(1^{l_r})}$, where $r \in \mathbb{N}$ and $l_1, \ldots, l_r \in \mathbb{N}$, and those elements are linearly independent.

Proof. This is Chapter II, (4.3) and (2.3) in [11].

For sake of simplicity of further formulations let us define $\mathcal{F}_0$ as the free associative $\mathbb{Z}[t]$-algebra generated by $u_\lambda$ as free commutative generators, where $\lambda$ run through all partitions. Given $R \in \mathcal{F}_0$ and a number $q$ we denote by $R(q)$ the element of the $\mathcal{H}(q)$ obtained by evaluating the coefficients of $R$ at $q$.

Let $\mathcal{P}_n$ denote the set of all partitions of the number $n$. Given a partition $\lambda$ we denote by $v_{\lambda'}$ the product

$$u_{(1^{\lambda'_1})} \ldots u_{(1^{\lambda'_s})},$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the partition dual to $\lambda$.

Recall [11, I.(6.4)] that given two partition $\lambda, \mu \in \mathcal{P}_n$ the Kostka number $K_{\lambda,\mu}$ is defined as the number of tableaux of shape $\lambda$ and weight $\mu$.

**Lemma 4.2.** The matrix $K = [K_{\lambda,\mu}]_{\lambda,\mu \in \mathcal{P}_n}$ is invertible and $(K^{-1})_{(n),(1^n)} = (-1)^{n-1}$.

Proof. Can be found in several places. See eg. [5, Theorem 3], also the proof of Lemma 5 there or [11, I.6], Example 4(d), formula (2).

The aim of this section is the following proposition.

**Proposition 4.3.** For every $n \in \mathbb{N}$ there exists $h_n \in \mathbb{Z}[t]$ with the leading coefficient $\pm 1$ and $R_n \in \mathcal{F}_0$ which is a linear combination of elements of the form $u_{(1^{l_1})} \ldots u_{(1^{l_r})}$, where $r > 1$ (consequently $l_i < n$ for every $i = 1, \ldots, r$) such that for every finite field $k$

$$\sum_{\mu \in \mathcal{P}_n} u_\mu = h_n(q)u_{(1^n)} + R_n(q)$$

in $\mathcal{H}(q)$.

Proof. Most of the proof is a compilation of well-known facts on the classical Hall algebras. Let $\lambda \in \mathcal{P}_n$. Then (see [11, Chapter II, (2.3)])

$$v_{\lambda'} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu} u_\mu,$$
where \( a_{\lambda,\mu} = 0 \) unless \( n(\mu) \geq n(\lambda) \) and \( a_{\lambda,\lambda} = 1 \). Moreover, it follows (see eg. [17, (2.5)]) that

\[
a_{\lambda,\mu} = \sum_{(\nu_1, \ldots, \nu_{s-1})} F^{\nu_s}_{(1^{\lambda_1})\nu_{s-1}} F^{\nu_{s-1}}_{(1^{\lambda_2})\nu_{s-2}} \cdots F^{\nu_2}_{(1^{\lambda_{s-1}})\nu_1},
\]

where \( \nu_1 = (1^{\lambda_1}), \nu_s = \mu \) and \( (\nu_2, \ldots, \nu_{s-1}) \) runs through all sequences of partitions. Clearly, the r.h.s sum is finite. Moreover, by [17, Example 2.4] and Proposition 4.1 every nonzero summand of this sum is a monic polynomial in \( q \) of degree

\[
\sum_{j=2}^{s} (n(\nu_j) - n(1^{\lambda_{j-1}})) = n(\nu_s) - n(\nu_1) - \sum_{j=2}^{s} n(1^{\lambda_{j-1}}) = n(\mu) - n(\lambda)
\]

and the number of nonzero summands is equal to the number of the tableaux of shape \( \mu' \) and weight \( \lambda' \) (see [11, I.1]), thus it is the Kostka number \( K_{\mu',\lambda'} \) ([11, I.(6.4)]\(^2\)). Let \( A \) be the matrix \( [a_{\lambda,\mu}]_{\lambda,\mu \in P_n} \).

We summarize our observations in the following Claim 1:

\( A \) is upper triangular (with respect to a linear ordering of the partitions \( \lambda \geq \mu \) implying \( n(\lambda) \geq n(\mu) \)), with 1 on the diagonal and the element \( a_{\lambda,\mu} \) is a polynomial in \( q \) of degree \( n(\mu) - n(\lambda) \) and the leading term \( K_{\mu',\lambda'} \).

The matrix \( A \) is invertible. Let \( A^{-1} = [a'_{\lambda,\mu}]_{\lambda,\mu \in P_n} \). The elements \( a'_{\lambda,\mu} \) are polynomials in \( q \).

Claim 2. \( \deg(a'_{\lambda,\mu}) \leq n(\mu) - n(\lambda) \). In order to prove it, note that \( a'_{\lambda,\mu} \) is a linear combination (with integral coefficients) of elements of the form

\[
a_{\sigma} = \prod_{\nu} a_{\nu,\sigma(\nu)},
\]

where \( \nu \) runs through the partitions in \( P_n \) different than \( \mu \) and \( \sigma : P_n \setminus \{\mu\} \to P_n \setminus \{\lambda\} \) is a bijection. Thus, if \( a_{\sigma} \neq 0 \),

\[
\deg(a_{\sigma}) = \sum_{\nu} \deg(a_{\nu,\sigma(\nu)}) = n(\mu) - n(\lambda),
\]

the last equality by Claim 1. Thus the Claim 2. follows.

\(^2\)It follows by [11, (AZ.4) page 200].
Claim 3. $\deg(a'_{(n), (1^n)}) = n(1^n) - n((n)) = \frac{n(n-1)}{2} > n(\mu) - n(\lambda)$ whenever $\lambda \neq (n)$ and the leading coefficient of $a'_{(n), (1^n)}$ equals $(-1)^{n-1}$. Indeed, if $\lambda = (n)$, $\mu = (1^n)$, then by Claim 2 every nonzero $a_\sigma$ has degree $n(1^n) - n((n))$ and the leading coefficient of $a'_{(n), (1^n)}$ is the element at the position $(n, (1^n))$ of the matrix inverse to $[K_{\mu, \lambda}]_{\lambda, \mu \in \mathcal{P}_n}$.

Observe that this element is equal to the element at the position $((n), (1^n))$ of the matrix inverse to $[K_{\lambda, \mu}]_{\lambda, \mu \in \mathcal{P}_n}$, hence by Lemma 4.2 equals to $(-1)^{n-1}$. The claim follows.

Now,

$$u_\mu = \sum_{\lambda} a'_{\mu, \lambda} v_{\lambda' \lambda} = a'_{\mu, (1^n)} v_{(1^n)'} + \sum_{\lambda : \lambda \neq (1^n)} a'_{\mu, \lambda} v_{\lambda'}$$

for every $\mu$. Observe that $v_{(1^n)'} = u_{(1^n)}$ and if $\lambda \neq (1^n)$, then $v_{\lambda'}$ is a product of at least two elements of the form $u_{(1)}$. Summing up the above equalities over $\mu \in \mathcal{P}_n$ we obtain what we want, since by the Claims 2. and 3. all polynomials $a'_{\mu, (1^n)}$, with $\lambda \neq (n)$ have degree strictly less than $n(1^n) - n((n))$, whereas the degree of $a'_{(n), (1^n)}$ is equal to $n(1^n) - n((n))$ and its leading coefficient equals $\pm 1$.

**Example 4.5.** The matrix $[a_{\lambda, \mu}]$ for $n = 4$ is the following (the rows and columns are indexed by the partitions of 4 ordered as follows: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$.)

$$
\begin{bmatrix}
1 & 3q + 1 & 2q^2 + 3q + 1 & 3q^3 + 5q^2 + 3q + 1 & q^5 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 3q + 1 \\
0 & 1 & q + 1 & 2q^2 + q + 1 & q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1 \\
0 & 0 & 1 & q + 1 & q^4 + q^3 + 2q^2 + q + 1 \\
0 & 0 & 0 & 1 & q^3 + q^2 + q + 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

and its inverse is the following:

$$
\begin{bmatrix}
1 & -3q + 1 & q^2 + q & 2q^3 - 2q^2 & -q^6 + 2q^5 + 3q^4 + 3q^3 + 4q^2 + q \\
0 & 1 & -q - 1 & -q^2 + q & q^5 - q \\
0 & 0 & 1 & -q - 1 & q^3 + q \\
0 & 0 & 0 & 1 & -q^3 - q^2 - q - 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

It follows that

$$\sum_{\mu \in \mathcal{P}_n} u_\mu = (-q^6 + 3q^5 + 3q^4 + 3q^3 + 3q^2)u_{(4)} + u_{(1^4)} + u_{(1)} - 3qu_{(1^2)}u_{(1)} + q^2u_{(1^2)} + (2q^3 - 3q^2 - 1)u_{(1^3)}u_{(1)}.$$
Lemma 4.6.

\[ \frac{q^n - 1}{q - 1} u_{(1^n)} = -u_{(2,1^{n-2})} + u_{(1^{n-1})} u_1. \]

Proof. The middle term of an extension of \( k[x]/(x) \) by \( (k[x]/(x))^{n-1} \) is isomorphic either to \( (k[x]/(x))^n \) or to \( k[x]/(x^2) \oplus (k[x]/(x))^{n-2} \). The number of submodules of \( (k[x]/(x))^n \) isomorphic to \( (k[x]/(x))^{n-1} \) equals \( \frac{q^n - 1}{q - 1} \) and there is unique submodule of \( k[x]/(x^2) \oplus (k[x]/(x))^{n-2} \) isomorphic to \( (k[x]/(x))^{n-1} \). The factor module is isomorphic to \( (k[x]/(x))^n \) in every case. It follows that

\[ u_{(1^{n-1})} u_1 = \frac{q^n - 1}{q - 1} u_{(1^{n-1})} + u_{(2,1^{n-2})}. \]

\[ \Box \]

5. Some relations in Hall algebras of \( \Lambda_{2,w}(k) \)

In this section we fix a finite field \( k \) and for any \( X \in I_{2,w} \) we write \( X = X(k) \).

Remark 5.1. The category \( \mathcal{S} \) (treated as a subcategory of the category of \( \Lambda_{2,w}(k) \)-modules) is closed under extensions.

Lemma 5.2. Let \( |k| = q \) and \( w \geq 2 \). The following equalities hold for \( \mathcal{H}(\Lambda_{2,w}(k)) \):

1. \( u_{p^n_1} = u_{p^n_0} u_{p^n_2} - u_{p^n_0} u_{p^n_2} u_{p^n_0} \) for \( w \geq n \geq 1 \);  
2. \( u_{p^n_2} = u_{p^n_0} u_{p^n_2} - u_{p^n_0} u_{p^n_2} \);  
3. \( u_{z_2^n} = u_{z_2^{n-1}} u_{p^n_0} - u_{p^n_0} u_{z_2^{n-1}} + u_{p^n_2} \) for \( w \geq n > 2 \) and \( u_{z_2^n} = u_{z_2^n} u_{p^n_0} - \frac{1}{q} u_{z_2^{n+1}} + \frac{1}{q} u_{z_2^n} \);  
4. \( u_{p^n_2} = u_{p^n_0} u_{p^n_2} - u_{p^n_0} u_{p^n_2} - u_{z_2^n} \) for \( w \geq n \geq 2 \);  
5. \( u_{p^n_2 \oplus p^n_1} = \frac{1}{q} (u_{p^n_2} u_{p^n_1} - u_{p^{n+1}_2}) \) for \( w > n \geq 3 \) and \( u_{p^n_2 \oplus p^n_1} = \frac{1}{q} u_{p^n_2} u_{p^n_1} \);  
6. \( u_{p^n_{2,1}} = \frac{1}{q} (u_{p^n_1} u_{p^n_1} - u_{p^n_1} u_{p^n_1} + u_{p^n_{2,1}}) \) for \( w > n \geq 3 \) and \( u_{p^n_{2,1}} = \frac{1}{q} (u_{p^n_1} u_{p^n_1} - u_{p^n_1} u_{p^n_1}) \);  
7. \( u_{p^n_{2,2,1}} = \frac{1}{q} u_{p^n_2} u_{p^n_1} - \sum_{s=1}^{\max(w-n,m-1)} \frac{q-1}{q^{w+1}} u_{p^n_{2,2,s}} \oplus u_{p^{n-s}_0} - \frac{1}{q} u_{p^n_{2,2}} \) and \( q^m u_{p^n_{2,2}} = u_{p^n_{2,1}} u_{p^n_{2,1}} - u_{p^n_{2,1}} u_{p^n_{2,1}} + q^{m-2} u_{p^n_{2,2,1,m-1}} + (q-1) q^{m-1} u_{p^n_{2,2,1}} \) for \( n > m \geq 2 \). We agree that \( P^0_0 = 0 \).
Proof. Let \( w \geq 2 \) and \( A = \Lambda_{2,w}(k) \).

1. Let \( n \geq 1 \). It is easy to see that \( F^{P^n_{0_1}}_{P^0_1} = 1 \),

\[
\text{Hom}_A(P^0_0, P^0_1) = \text{Hom}_A(P^n_1, P^0_1) = \text{Ext}^1_A(P^0_0, P^0_1) = 0,
\]
and the only extensions of \( P^0_0 \) by \( P^n_0 \) are: \( P^n_1 \), \( P^0_0 \oplus P^0_1 \). Therefore, \( u_{P^0_0}u_{P^1_0} = u_{P^n_1} + u_{P^0_0 \oplus P^0_1} \) and \( u_{P^0_0}u_{P^n_0} = u_{P^0_0 \oplus P^0_1} \). We are done.

2. The proof is similar to the proof of the statement 1.

3. Let \( n \geq 2 \). Note that there is no monomorphism \( Z^{n-1}_2 \to P^n_0 \) neither \( Z^{n-1}_2 \to B^{n-1,1}_2 \). Therefore the only extensions of \( P^1_0 \) by \( Z^{n-1}_2 \) are: \( Z^n_2 \) and \( P^1_0 \oplus Z^{n-1}_2 \). Therefore \( u_{Z^{n-1}_2}u_{P^1_0} = u_{Z^n_2} + qu_{P^1_0 \oplus Z^{n-1}_2} \) for \( n > 2 \) and

\[
u_{Z^{n-1}_2}u_{P^1_0} = u_{Z^n_2} + u_{P^1_0 \oplus Z^n_2}.
\]
On the other hand, \( F^{P^1_0 \oplus Z^{n-1}_2}_0 = q, F^{P^n_0 \oplus Z^{n-1}_2}_0 = 1 \) and there are no epimorphisms: \( Z^n_2 \to Z^{n-1}_2, B^{n-1,1}_2 \to Z^{n-1}_2 \). It follows that \( u_{P^1_0}u_{Z^{n-1}_2} = qu_{P^1_0 \oplus Z^{n-1}_2} + u_{P^n_0} \) for \( n \geq 2 \). We are done.

4. Let \( n \geq 2 \). It is easy to see that \( F^{P^n_0}_{P^0_0, P^0_2} = F^{Z^n_2}_{P^0_0, P^0_2} = 1 \),

\[
\text{Hom}_A(P^n_0, P^0_2) = \text{Hom}_A(P^0_0, P^0_2) = \text{Ext}^1_A(P^0_0, P^0_2) = 0,
\]
and the only extensions of \( P^0_2 \) by \( P^n_0 \) are: \( P^n_2 \), \( Z^n_2 \) and \( P^n_0 \oplus P^0_2 \). Therefore, \( u_{P^0_0}u_{P^0_2} = u_{P^n_2} + u_{P^n_0 \oplus P^0_2} \) and \( u_{P^0_2}u_{P^0_0} = u_{P^n_0 \oplus P^0_2} \). We are done.

5. Let \( n \geq 3 \). The only extensions of \( P^1_0 \) by \( P^n_2 \) are: \( P^{n+1}_2 \) (if \( n < w \)) and \( P^1_0 \oplus P^n_2 \), because \( P^n_2, P^1_0 \in S_2 \), the category \( S \) is closed under extensions and there is no monomorphism \( P^n_2 \to B^{n+1}_2 \). It is straightforward to check that \( F^{P^{n+1}_0}_{P^n_2, P^0_1} = 1 \) (if \( n < w \)) and \( F^{P^n_0 \oplus P^0_1}_{P^n_2, P^0_1} = q \). We are done.

6. Let \( n \geq 3 \). It is straightforward to check that \( F^{P^{n+1}_0}_{P^n_1, P^1_0} = F^{P^n_0 \oplus P^0_1}_{P^n_1, P^1_0} = F^{P^0_1 \oplus P^n_0}_{P^n_1, P^0_1} = q \) and \( F^{P^{n+1}_0}_{P^n_1, P^1_0} = 1 \). The only extensions of \( P^n_0 \) by \( P^n_1 \) are \( B^{n+1}_2 \) and \( P^n_1 \oplus P^n_0 \), because the category \( S \) is closed under extensions and there is no epimorphism \( P^{n+1}_2 \to P^n_1 \). Moreover, the only extensions of \( P^n_0 \) by \( P^n_1 \) are \( P^{n+1}_2 \) and \( P^n_1 \oplus P^n_0 \), because the category \( S \) is closed under extensions and there is no epimorphism \( B^{n+1}_2 \to P^n_1 \). Therefore,
7. Let \( n - 2 \geq m \geq 2 \) and \( m > s \geq 1 \). It is easy to check that \( F_{P_2}^{P_{n+m}} P_{n} = 1 \), \( F_{P_2}^{P_{n+m} \oplus P_{n}} P_{n} = q^{m} \), \( F_{P_2}^{P_{n+m} \oplus P_{n-s}} P_{n} = (q - 1)q^{m-s-1} \) and the only extensions of \( P_{0} \) by \( P_{2} \) are \( P_{2} \) and \( P_{2} \oplus P_{m-s} \) for all \( m > s \geq 1 \). It is because the category \( S \) is closed under extensions and there are no monomorphisms \( P_{0} \to B_{2}^{s+m-s} \), \( P_{2} \to P_{0} \oplus P_{m-s} \) for \( s \geq 0 \). The first formula follows.

We divide the proof of the second formula into several steps.

(a) For \( s = 1, \ldots, m - 1 \) we have \( F_{P_1}^{P_{n+m} \oplus P_{m-s}} P_{n} = F_{P_1}^{P_{n+m} \oplus P_{m-s}} P_{1} \). This follows since the Steinitz’s classical Hall algebra is commutative and there is 1-1 correspondence between exact sequences \( 0 \to P_{1} \to P_{1} \oplus P_{m-s} \to P_{1} \to 0 \) (resp. \( 0 \to P_{1} \to P_{1} \oplus P_{m-s} \to P_{1} \to 0 \)) and the exact sequences of the ambient spaces: \( 0 \to N_{(s)} \to N(n+s) \oplus N(m-s) \to N(m) \to 0 \) (resp. \( 0 \to N_{(m)} \to N(n+s) \oplus N(m-s) \to N(m) \to 0 \)).

(b) For \( s = 2, \ldots, m - 1 \) we have \( F_{P_1}^{P_{n+m} \oplus P_{m-s}} P_{n} = F_{P_1}^{P_{n+m} \oplus P_{m-s}} P_{1} = q^{m-s-1}(q - 1) \).

(c) We have \( F_{P_1}^{P_{n+1,m-1}} P_{1} = q^{m-2}(q - 2) \) and \( F_{P_1}^{P_{n+1,m-1}} P_{1} = q^{m-2}(q - 1) \).

(d) \( F_{P_1}^{P_{n,m}} P_{1} = q^{m} \) and \( F_{P_1}^{P_{n,m}} P_{1} = 0 \), because there is no epimorphism \( B_{2}^{m,m} \to P_{1} \).

(e) \( F_{P_1}^{P_{n+1,s} \oplus P_{m-s}} P_{1} = 0 \) for \( s \geq 0 \), because there is no epimorphisms \( P_{2} \oplus P_{n-s} \to P_{1} \).

(f) \( F_{P_1}^{P_{n+1,s} \oplus P_{m-s}} P_{1} = 0 \) for \( s = 0 \) and \( s > 1 \), because there is no epimorphisms \( P_{2} \oplus P_{m-s} \to P_{1} \).

(g) \( F_{P_1}^{P_{n+1} \oplus P_{m-s}} P_{1} = q^{m-1}(q - 1) \).

(h) straightforward calculations show that \( F_{P_1}^{P_{n+1} \oplus P_{m-s}} P_{1} = 0 = F_{P_1}^{P_{n+1} \oplus P_{m-s}} P_{1} \) for \( s = 1, \ldots, m - 1 \).

The fact that the category \( S \) is closed under etensions and the statements (a)-(h) imply the second formula.
This finishes the proof.

6. The proof of the main result

As in Section 4 let us define $F_0$ as the $\mathbb{Z}[t]$-algebra generated by $u_a$ as free associative algebra, where $a \in \mathbb{N}_0^{I_{2,w}}$. Given $R \in F$ and a number $q$ we denote by $R(q)$ the element of the $H(\Lambda_{2,w}(k))$, where $q = |k|$, obtained by evaluating the coefficients of $R$ at $q$.

The set of the isomorphism classes of finite-dimensional $\Lambda_{2,w}(k)$-modules is in a natural 1-1 correspondence with the set $\mathbb{N}_0^{I_{2,w}}$ of functions $a : I_{2,w} \to \mathbb{N}_0$. We say that $a \in \mathbb{N}_0^{I_{2,w}}$ is in the polynomial zone if for every $b, c \in \mathbb{N}_0^{I_{2,w}}$ there exists $\varphi_{b, c}^{a} \in \mathbb{Z}[t]$ such that for any finite field $k$, $F_{a(k), c(k)}^{b(k)} = \varphi_{a, c}^{b}(q)$, where $q = |k|$. Let us denote by $P_w$ the polynomial zone, that is, the set of the $a \in I_{2,w}$ which are in the polynomial zone. We need to prove that $P_w = \mathbb{N}_0^{I_{2,w}}$. In what follows we identify functions $a$ with the corresponding modules.

If $w \leq w'$ then $I_{2,w} \subset I_{2,w'}$ and we treat $\mathbb{N}_0^{I_{2,w}}$ as a subset of $\mathbb{N}_0^{I_{2,w'}}$ in an obvious way: we extend a function defined on $I_{2,w}$ to $I_{2,w'}$ by putting zeros outside $I_{2,w}$. Given a partition $\alpha = (\alpha_1, \ldots, \alpha_l)$ we denote by $P^\alpha_0$ the module $(0, N(\alpha), 0) = \bigoplus_{i=1}^{l} P_{0}^{\alpha_i}$.

**Lemma 6.1.** If $w \leq w'$ then $\mathbb{N}_0^{I_{2,w}} \cap P_{w'} \subseteq P_w$.

**Proof.** Obvious.

**Lemma 6.2.** For every $w \in \mathbb{N}$ the simple modules, that is $P^1_0$ and $P^0_1$ are in the polynomial zone $P_w$.

**Proof.** Let $b, c \in \mathbb{N}_0^{I_{2,w}}$ be such that $\dim_k b(k) - \dim_k c(k) = 1$. Note that kernels of all epimorphisms $b(k) \to c(k)$ are isomorphic to $P^1_0$ or kernels of all epimorphisms $b(k) \to c(k)$ are isomorphic to $P^0_1$. It follows that for $a \in \{P^1_0, P^0_1\}$ the number $F_{a(k), c(k)}^{b(k)}$ equals the number of all subobjects $U$ of $b(k)$ such that $b(k)/U \simeq c(k)$ and it is given by a polynomial by Ringel’s results in [14] (4) at page 441.

**Lemma 6.3.** Let $R \in F_0$ be a linear combination, with coefficients in $\mathbb{Z}[t]$, of products of elements $u_b$ with $b \in P_w$. Assume that $a \in \mathbb{N}_0^{I_{2,w}}$ and there exists $h \in \mathbb{Z}[t]$ with the leading coefficient equals to $\pm 1$ and such that

\[ h(q) \cdot u_a = R(q) \]

(6.4)
in the Hall algebra $\mathcal{H}(\Lambda_{2,w}(k))$ for every finite field $k$, where $q = |k|$. Then $a \in \mathbb{P}_w$.

**Proof.** Let $b, c \in I_{2,w}$. By multiplying the equation (6.4) by $u_c$ we obtain

$$h(q) \cdot u_a u_c = R(q) u_c.$$ 

Comparing the coefficients at the basic element $u_b$ we get

$$h(q) F_{a(k),c(k)} (q) = R_{b,c}(q)$$

where $R_{b,c}(q)$ depends polynomially on $q = |k|$ by the formula (3.4) thanks to the fact that $R$ is a linear combination of products of elements $u_b$ with $b \in \mathbb{P}_w$. Therefore, by [14, Lemma, p. 441] also $F_{a(k),c(k)} (q)$ depends polynomially on $q$, because the leading coefficient of $h$ is equal to ±1. We have proved that $a$ is in the polynomial zone.

**Lemma 6.5.** For every $m, w \in \mathbb{N}$, the module $(P_0^1)^m$ is in the polynomial zone $\mathbb{P}_w$.

**Proof.** We proceed by induction on $m$. For $m = 1$ the assertion follows by Lemma 6.2. Assume that $m > 1$ and $(P_0^1)^s$ is in polynomial zone for every $1 \leq s < m$. By Lemma 6.4 we can assume that $w \geq m$. Then, by Proposition 4.3

$$(6.6) \quad \sum_{\mu \in \mathbb{P}_m} u_\mu = h_m(q) u_{(P_0^1)^m} + R_m(q)$$

in $\mathcal{H}(\Lambda_{2,w}(k))$, where $h_m(q)$ is a nonzero polynomial in $q$ with the leading coefficient ±1 and $R_m \in \mathcal{F}$ is a linear combination of elements of the form $u_{(P_0^1)^{r_1}} \ldots u_{(P_0^1)^{r_r}}$ with $r \geq 2$.

Let $b, c \in \mathbb{N}_{I_{2,w}}$. By multiplying (6.6) by $u_c$ in the Hall algebra and comparing the coefficients at the basic element $u_b$ we get

$$(6.7) \quad \sum_{\lambda \in \mathbb{P}_m} F_{\lambda^b} (q) F_{a(k),c(k)} (q) = h_m(q) F_{(P_0^1)^m,c(k)} (q) + \tilde{R}_{m,b,c},$$

where $\tilde{R}_{m,b,c}$ is a linear combination (with scalar coefficients) of products of numbers of the form $F_{(P_0^1)^{l,c'}} (q)$ with $l < m$ and $c', b' \in \mathbb{N}_{I_{2,w}}$, to see this combine formulae (3.4) and (6.7). By the induction hypothesis, $\tilde{R}_{m,b,c}$
depends polynomially on \( q = |k| \). On the other hand, the l.h.s of (6.7) is the number of all subobjects \( U \) of \( b(k) \) such that \( b(k)/U \simeq c(k) \). It is known by [14] that this number depends polynomially on \( q \). Therefore, by [14, Lemma, p. 441] so does \( F^{b(k)}_{(P^1_0(k))^m,c(k)}(q) \), because the leading coefficient of \( h_m \) is equal to \( \pm 1 \). We have proved that \((P^0_1(k))_m \) is in the polynomial zone.

**Lemma 6.8.** For every \( m, w \in \mathbb{N} \), the module \((P^0_1)^m \) is in the polynomial zone \( \mathbb{P}_w \).

**Proof.** We proceed by induction on \( m \). For \( m = 1 \) the assertion follows by Lemma 6.2. Assume that \( m > 1 \) and \((P^0_1)^s \) is in polynomial zone for every \( 1 \leq s < m \). The \( \Lambda_{2,w}(k) \)-module \( P^0_2(k) \) is injective, thus by Lemma 3.6 \( P^0_2 \oplus (P^0_1)^{m-2} \) is in the polynomial zone\(^3\) as \((P^0_1)^{m-2} \) is in the polynomial zone by the induction hypothesis. Now \((P^0_1)^m \) is in the polynomial zone thanks to Lemma 6.3 and Lemma 4.6.

**Lemma 6.9.** The modules \( P^0_n \) are in the polynomial zone for every \( 1 \leq n \leq w \).

**Proof.** For \( \lambda = (n) \) the formula (4.4) yields

\[
(6.10) \quad u_{P^0_n} = \sum_{\lambda} a'_{(n),\lambda} u_{\lambda'} = a'_{(n),(1^n)} u_{((P^0_1)^n)} + \sum_{\lambda: \lambda \neq (1^n)} a'_{(n),\lambda} u_{((P^0_1)^{\lambda_1})} \cdots u_{((P^0_1)^{\lambda_s})},
\]

where we set \( \lambda' = (\lambda'_1, \ldots, \lambda'_s) \). It follows by Lemma 6.5 and Lemma 6.3 that \((P^0_n)^{(n)} \) is in the polynomial zone. We use the fact that the leading coefficient of the polynomial \( a'_{(n),(1^n)} \) is \( \pm 1 \) (Claim 3 in the proof of Proposition 4.3).

**Lemma 6.11.** Every indecomposable module is in polynomial zone.

**Proof.** Let \( N \) be one of the indecomposable modules listed in Lemma 2.2. We are going to prove that \( N \) is in the polynomial zone. If \( N \) is of the form \( P^0_n \) then this follows by Lemma 6.9. This is also true for \( N = P^0_1 \) by Lemma 6.2 and for \( N = P^0_2 \) by the injectivity and Lemma 3.6.

Now by applying Lemma 6.3 and Lemma 5.2 (1)-(3) we get the assertion for \( N \) of the form \( P^0_n \) for \( n \geq 1 \) and \( P^1_1 = Z^1_2 \) and \( Z^1_2 \) for \( n \geq 2 \).

Having this, we prove similarly by Lemma 5.2 (4) that \( P^0_n \) are in the polynomial zone for \( 2 \leq n \leq w \).

\(^3\)We apply the fact that the number \( |\text{Mon}(a(k), b(k))| \) depends polynomially on \( q = |k| \), see [14, p. 441].
In the next step, by Lemma 5.2 (6) we prove the assertion for $N = B_{2}^{n,1}$ for $n \geq 3$, again applying Lemma 6.3. At the same time we show by Lemma 5.2 (5) the decomposable modules $P_{2}^{n} \oplus P_{0}^{1}$, $n \geq 3$ are in the polynomial zone.

We shall prove that for every $n, m \leq w$ the modules $P_{2}^{n} \oplus P_{0}^{m}$ and $B_{2}^{n,m}$ are in the polynomial zone by the induction on $m$. For $m = 1$ the assertion is proved above. Assume that $m > 1$ and the modules $P_{2}^{n} \oplus P_{0}^{m-s}$ and $B_{2}^{n,m-s}$ are in the polynomial zone for every $m > s > 1$. Then our assertion follows by Lemma 5.2 (7) and Lemma 6.3.

Now Theorem 2.4 follows by [12, Theorem 2.9]:

**Theorem 6.12.** Let $A$ be a representation finite algebra over finite field $k$. Then $A$ has a Hall polynomials if and only if for each $X, Z \in \text{mod}A$ and $Y \in \text{ind}A$, Hall polynomial $\varphi_{X,Y}^{Z}$ exist.

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