Article

A-Statistical Convergence Properties of Kantorovich Type λ-Bernstein Operators Via \((p, q)\)-Calculus

Liang Zeng 1, Qing-Bo Cai 2,* and Xiao-Wei Xu 3

1 School of Mathematical Sciences, Xiamen University, Xiamen 361005, China; zengliang0409@aliyun.com
2 School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China
3 School of Computer and Data Engineering, Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China; lampminket263.net
* Correspondence: qbcai@126.com; Tel.: +86-150-6079-5559

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Abstract: In the present paper, Kantorovich type \(\lambda\)-Bernstein operators via \((p, q)\)-calculus are constructed, and the first and second moments and central moments of these operators are estimated in order to achieve our main results. An \(A\)-statistical convergence theorem and the rate of \(A\)-statistical convergence theorems are obtained according to some analysis methods and the definitions of \(A\)-statistical convergence, the rate of \(A\)-statistical convergence and modulus of smoothness.

Keywords: \(A\)-statistical convergence; \(A\)-Bernstein operators; \((p, q)\)-calculus; modulus of continuity; rate of convergence

MSC: 41A10; 41A25; 41A36

1. Introduction

As we know, one of the simplest and most elegant ways to prove the famous Weierstrass Approximation Theorem was given by S. N. Bernstein [1] in 1912 by constructing a sequence of polynomials which were defined as follows,

\[ B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right) \]

for \( f \in C[0,1] \) and \( x \in [0,1] \). These polynomials are called Bernstein operators or Bernstein polynomials. Due to the fine properties of approximation, Bernstein operators play a significant role in Approximation Theory and Computer Aided Geometric Design (CAGD).

In 2016, Mursaleen et al. [2] defined the following \((p, q)\)-analogue of Bernstein operators:

\[ B_{n,p,q}(f; x) = \sum_{k=0}^{n} b_{n,k}(x; p, q) f \left( \frac{\lfloor k/p \rfloor}{p^{k-n} \lfloor n/p \rfloor} \right), \quad x \in [0,1], \] (1)

where \( b_{n,k}(x; p, q) \) \((k = 0, 1, \ldots, n)\) are \((p, q)\)-Bernstein basis functions and defined as

\[ b_{n,k}(x; p, q) = \frac{1}{p^{\lfloor (n-1)/2 \rfloor}} \binom{n}{k} p^{k-1} x^{\lfloor (n-1)/2 \rfloor} \prod_{s=0}^{\lfloor (n-1)/2 \rfloor} (p^s - q^s x), \quad x \in [0,1]. \] (2)

Then, there are many papers mention about the approximation properties of \((p, q)\)-type positive linear operators, such as [2–22].
Very recently, Cai et al. [23] proposed the following positive linear $\lambda$-Bernstein operators based on $(p, q)$-integers as

$$B_{n,p,q}^\lambda(f;x) = \sum_{k=0}^{n} b_{n,k}^\lambda(x;p,q)f \left( \frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \right), \quad x \in [0,1],$$

where

$$
\begin{cases}
  b_{n,0}^\lambda(x;p,q) = b_{0,0}(x;p,q) - \frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,1}(x;p,q), \\
  b_{n,k}^\lambda(x;p,q) = b_{n,k}(x;p,q) + \lambda \left( \frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{k-2n}[n]_{p,q}+1} b_{n+1,k}(x;p,q) \\ - \frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{k-2n}[n]_{p,q}+1} b_{n+1,k+1}(x;p,q) \right), \\
  b_{n,n}^\lambda(x;p,q) = b_{n,n}(x;p,q) - \frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,n}(x;p,q), 
\end{cases}
$$

$b_{n,k}(x;p,q)$ $(k = 0, 1, \ldots, n)$ are defined in Equation (2), $\lambda \in [-1,1]$, $n \geq 2$, $x \in [0,1]$ and $0 < q < p \leq 1$.

Inspired by the above research, based on Equation (3), we introduce Kantorovich type $A$-Bernstein operators via $(p, q)$-calculus as

$$K_{n,p,q}^A(f;x) = [n+1]_{p,q} \sum_{k=0}^{n} b_{n,k}^A(x;p,q) p^{-k} \int_{[k]_{p,q}}^{[k+1]_{p,q}} f \left( p^{-k} u \right) d_{p,q} u,$$

where $x \in [0,1]$, $0 < q < p \leq 1$ and $b_{n,k}^A(x;p,q)$ $(k = 0, 1, \ldots, n)$ are defined in Equation (4).

Firstly, we give some definitions of $(p, q)$-integers, which can be referred to [24–28]. For any fixed real number $p > 0$ and $q > 0$, $[n]_{p,q}$ are defined by $[n]_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{cases} [n]_{p,q} [n-1]_{p,q} \cdots [1]_{p,q}, & n = 1, 2, \ldots; \\ 1, & n = 0, \\ \frac{n!}{[n]_{p,q} ![n-k]_{p,q}}, & n \geq 1. \end{cases}$

The $(p, q)$-power basis $(x \oplus t)_p^n$ and $(x \ominus t)_p^n$ are defined by

$$(x \oplus t)^n_{p,q} = (x+t)(px+q^2t) (p^2x+q^4t) \cdots (p^{n-1}x+q^{2n-2}t)$$

and

$$(x \ominus t)^n_{p,q} = (x-t)(px-q^2t) (p^2x-q^4t) \cdots (p^{n-1}x-q^{2n-2}t).$$

We also give the fundamental theorem of $(p, q)$-calculus, say, if $F(x)$ is an anti-derivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, then $\int_a^b f(x) d_{p,q} x = F(b) - F(a)$ holds, where $0 \leq a < b \leq \infty$ and $F(x)$ is given by the formula

$$F(x) = (p-q)x \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} x \right) + F(0),$$

the infinite series here converges.

The main goal of the present work is to study the rate of $A$-statistical convergence of Kantorovich type $\lambda$-Bernstein operators based on $(p, q)$-integers by means of modulus of continuity. The rest of this
paper are mainly organized as follows: in Section 2, some moments and central moments of \( K_{\lambda,p,q}(f;x) \) are estimated; in Section 3, we prove \( K_{\lambda,p,q}(f;x) \) is \( A \)-statistically convergent to \( f(x) \) and investigate the rate of \( A \)-statistical convergence by means of the first and second modulus continuity.

2. Some Preliminary Results

In the sequel, consider sequences of functions \( e_i(x) = x^i \) \((i = 0, 1, 2)\), and \( \phi_j(t,x) = (t - x)^j \) \((j = 1, 2)\), \( x, t \in [0, 1] \). Before we give our main theorems, we need the following lemmas.

**Lemma 1.** The following statements are true:

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} d_{p,q}u = \frac{p^k}{n+1}_{p,q},
\]

(6)

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} p^{-k}u d_{p,q}u = 2q[k]_{p,q} + \frac{p^k}{2}_{p,q|n+1}_{p,q}^2,
\]

(7)

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} (p^{-k}u)^2 d_{p,q}u = \frac{3p^{-k}q^2[k]_{p,q}^2 + 3q[k]_{p,q} + p^k}{3}_{p,q|n+1}_{p,q}.
\]

(8)

**Proof.** By the fundamental theorem of \((p,q)\)-calculus given in Section 1, we have

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} d_{p,q}u = \frac{(p-q) \cdot k+1}{p_{n+1} q_{p,q}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} + (p-q) \frac{-q}{p_{n+1}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} = \frac{p^k}{n+1}_{p,q}.
\]

Similarly,

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} p^{-k}u d_{p,q}u = \frac{p^{-k}k+1}{p_{n+1} q_{p,q}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} + (p-q) \frac{-q}{p_{n+1}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} = \frac{2q[k]_{p,q} + p^k}{2}_{p,q|n+1}_{p,q}.
\]

Finally,

\[
\int_{[\lambda+1,p,q]}^{[k+1,p,q]} (p^{-k}u)^2 d_{p,q}u = \frac{p^{-2k}k+1}{p_{n+1} q_{p,q}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} + (p-q) \frac{-q}{p_{n+1}} \sum_{j=0}^{\infty} q^j \frac{-q}{p_{n+1}} = \frac{3p^{-k}q^2[k]_{p,q}^2 + 3q[k]_{p,q} + p^k}{3}_{p,q|n+1}_{p,q}.
\]

Lemma 1 is proved. □
Lemma 2. Let $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $0 < q < p \leq 1$, for the operators $K_{n,p,q}^{\lambda}(f; x)$, we have

\[ K_{n,p,q}^{\lambda}(e_1; x) = 1, \]

\[ K_{n,p,q}^{\lambda}(e_2; x) = \frac{2x}{[2]_{p,q} p^n} + \frac{1 - 2x}{[2]_{p,q}[n + 1]_{p,q}} + 2\lambda \left( \frac{2q}{[2]_{p,q}} \left( 1 - \frac{q}{p} \right)x^2 \left( 1 - x^{n-1} \right) - \frac{1 - \frac{q}{p}}{[2]_{p,q}} \left( p [n]_{p,q} + p^n \right) \right) - 2^{p^n - 1} \left( 0, 1 \right)^2 \left( p^n [n]_{p,q} \right) \] \[ - \frac{2^{p^n - 1} \left( 0, 1 \right)^2 \left( p^n [n]_{p,q} \right) \left( p^n [n]_{p,q} - p^n \right) - \left( 0, 1 \right)^2 \left( p^n [n]_{p,q} \right) \left( p^n [n]_{p,q} - p^n \right) \right) \] \[ \leq \frac{\Theta^\lambda_{p,q}(n, x)}{[2]_{p,q} p^n} \right) \] \[ \leq \frac{\Theta^\lambda_{p,q}(n, x)}{[2]_{p,q} p^n} \right). \]

Proof. We get Equation (9) easily by Equations (5) and (6) and [23]. From Equations (5), (7), (8) and (2), we have

\[ K_{n,p,q}^{\lambda}(e_1; x) = \left[ n + 1 \right]_{p,q} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{[n]_{p,q}} \right)^k \frac{1}{[n]_{p,q}^{k+1}} \frac{1}{[n]_{p,q}^{k+1}} \] \[ - \frac{x^k}{[n]_{p,q}^{k+1}} \frac{1}{[n]_{p,q}^{k+1}} \] \[ - \frac{1}{[n]_{p,q}^{k+1}} \frac{1}{[n]_{p,q}^{k+1}} \] \[ - \frac{x^k}{[n]_{p,q}^{k+1}} \frac{1}{[n]_{p,q}^{k+1}} \] \[ - \frac{1}{[n]_{p,q}^{k+1}} \frac{1}{[n]_{p,q}^{k+1}} \] \[ \leq \frac{\Theta^\lambda_{p,q}(n, x)}{[2]_{p,q} p^n} \right) \] \[ \leq \frac{\Theta^\lambda_{p,q}(n, x)}{[2]_{p,q} p^n} \right) \] \[ \leq \frac{\Theta^\lambda_{p,q}(n, x)}{[2]_{p,q} p^n} \right). \]

Therefore, Equations (11) and (12) can be obtained by Equations (12), (14), [23] and some tedious computations, here we omit those processes.

Lemma 3. Let $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $0 < q < p \leq 1$, we have the following inequalities for the operators $K_{n,p,q}^{\lambda}(f; x)$.

\[ K_{n,p,q}^{\lambda}(e_1; x) \leq \frac{2 - [2]_{p,q} p^n}{[2]_{p,q} p^n} + \frac{1}{[2]_{p,q}[n + 1]_{p,q}} + \frac{4}{[2]_{p,q} \left( p [n]_{p,q} + p^n \right)} + \frac{5}{[2]_{p,q}[n + 1]_{p,q} \left( p [n]_{p,q} + p^n \right)} \]
\[ \text{Mathematics} \ 2020, \ 8, \ 291 \]

By Equations (9) and (11), we can get

\[ \leq \Theta(p, q; n), \]

\[ K_{n,p,q}^\lambda (q_2(t, x); x) \leq \frac{3[2p_{n,q} - 4\cdot 3p_{n,q}P^n + 3p_{n,q}2P^{2n}]}{8} \]

\[ + \frac{10}{4} \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{16}{4} \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{16}{4} \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \]

\[ + \frac{6}{18} \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^{2n}) + \frac{3p_{n,q}P^{n-1} | n+1 | p_{n,q}]}{18} (p_{n,q}^n | n+1 | p_{n,q} - p^{2n}) \]

\[ \leq \Phi(p, q; n). \]

**Proof.** By Equations (9) and (11), we can get

\[ K_{n,p,q}^\lambda (q_1(t, x); x) \]

\[ = K_{n,p,q}^\lambda (e_1; x) - x \]

\[ = \left( \frac{2}{[2p_{n,q}P^n - 1]} \right) x + \frac{1 - 2x}{[2p_{n,q} | n+1 | p_{n,q}]} + \lambda \left\{ \begin{array}{ll}
\frac{4}{p} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-1}) - \frac{2}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
\frac{4}{p} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-1}) + \frac{2}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
\frac{4}{p} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-1}) - \frac{2}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
\end{array} \right. \]

\[ \leq \lambda \left\{ \begin{array}{ll}
2 - \frac{2}{[2p_{n,q}P^n] - 1} + \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{4}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{5}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
2 - \frac{2}{[2p_{n,q}P^n] - 1} + \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{4}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{5}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
2 - \frac{2}{[2p_{n,q}P^n] - 1} + \frac{1}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{4}{[2p_{n,q} | n+1 | p_{n,q}]} + \frac{5}{[2p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \\
\end{array} \right. \]

Similarly, by Lemma 2, we have

\[ K_{n,p,q}^\lambda (q_2(t, x); x) = K_{n,p,q}^\lambda (e_2; x) - 2xK_{n,p,q}^\lambda (e_1; x) + x^2K_{n,p,q}^\lambda (e_0; x) \]

\[ = \left( \frac{3}{[3p_{n,q}P^{2n}] + 1} \right) x^2 + \frac{3x - 6x^2 + 3\frac{4}{p} x(1 - x)}{[3p_{n,q}P^n | n+1 | p_{n,q}]} - 2x(1 - 2x) + \frac{1 - 3x(1 - x)}{[3p_{n,q} | n+1 | p_{n,q}]} + \frac{1 - 3x(1 - x)}{[3p_{n,q} | n+1 | p_{n,q}]} (p_{n,q}^n - p^n) \]

\[ + \lambda \left\{ \begin{array}{ll}
\frac{3}{3} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-2}) - \frac{1}{[3p_{n,q}P^n | n+1 | p_{n,q}]} \frac{8}{p} (1 - \frac{2}{p}) x^3 (1 - x^{n-1}) \\
\frac{3}{3} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-2}) - \frac{1}{[3p_{n,q}P^n | n+1 | p_{n,q}]} \frac{8}{p} (1 - \frac{2}{p}) x^3 (1 - x^{n-1}) \\
\frac{3}{3} \frac{1 - \frac{2}{p}}{2} x^2 (1 - x^{n-2}) - \frac{1}{[3p_{n,q}P^n | n+1 | p_{n,q}]} \frac{8}{p} (1 - \frac{2}{p}) x^3 (1 - x^{n-1}) \\
\end{array} \right. \]
\[ + 4 \left(1 - \frac{2}{3}\right) x^2 (1 - x^n) + 3 \left(\frac{3}{2} - 1\right) x (1 - x^n) + 8p^{n-1} q \left[x^2 (1 - x) + \frac{2}{3} x^3 (1 - x^{n-1})\right] \]
\[ - 4x (1 - x^{n+1}) \]
\[ + \frac{3}{2} x - 6x^2 + \frac{3y}{2} x (1 - x) \]
\[ + 2x (1 - 2x) \]
\[ - \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{2}{3} p \left[p^n [n]_{p,q} + p^n\right] \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ + \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]

Next, we will discuss in two cases:

Case 1: for \( \lambda \in [0, 1] \), we have

\[ K_{\lambda}^n (p^n [n]_{p,q} + p^n) + \frac{1}{4} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ = \frac{3}{2} x - 6x^2 + \frac{3y}{2} x (1 - x) + \frac{2}{3} p \left[p^n [n]_{p,q} + p^n\right] + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]

Case 2: for \( \lambda \in [-1, 0] \), we have

\[ K_{\lambda}^n (p^n [n]_{p,q} + p^n) + \frac{1}{4} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ = \frac{3}{2} x - 6x^2 + \frac{3y}{2} x (1 - x) + \frac{2}{3} p \left[p^n [n]_{p,q} + p^n\right] + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]

\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[ + \frac{1}{3} \left(1 + \frac{3}{p}\right) x (1 - x) \]
\[ - \frac{2}{3} p \left(p^n [n]_{p,q}^2 - p^{2n}\right) \]
\[
\begin{align*}
&\left\{ 16p^n x^n_\theta (1 - x^n) \right. \\
&\left. + \frac{4}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) \right\} \\
&\leq \frac{3}{3[p,q]p^{2n}} - \frac{4}{2[p,q]p^n} + 1 + \frac{4}{3[p,q]p^n[n+1][p,q]} + \frac{4}{2[p,q][n+1][p,q]} + \frac{1}{3[p,q][n+1]\sqrt{p}} \\
&+ \frac{3}{3[p,q]p^n} \left( p[n]_{p,q} + p^n \right) + \frac{2}{2[p,q]} \left( p[n]_{p,q} - p^n \right) + \frac{3}{3[p,q][n+1][p,q]} \left( p[n]_{p,q} - p^n \right) \\
&+ \frac{2}{2[p,q][n+1][p,q]} \left( p^{1-n}[n]_{p,q} - 1 \right) + \frac{3}{3[p,q][n+1][p,q]} \left( p[n]_{p,q} + p^n \right) + \frac{12}{3[p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) \\
&+ \frac{6}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) + \frac{12}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right).
\end{align*}
\]

Combining Equations (17) and (18), we obtain

\[
K_{\ell,\alpha}^3 (\phi_2(t,x); x)
\]

\[
\leq \frac{3}{3[p,q]p^{2n}} - \frac{4}{2[p,q]p^n} + 1 + \frac{4}{3[p,q]p^n[n+1][p,q]} + \frac{4}{2[p,q][n+1][p,q]} + \frac{1}{3[p,q][n+1]\sqrt{p}} \\
+ \frac{4}{2[p,q]p^n} \left( p[n]_{p,q} + p^n \right) + \frac{2}{2[p,q]} \left( p[n]_{p,q} - p^n \right) + \frac{4}{2[p,q][n+1][p,q]} \left( p[n]_{p,q} - p^n \right) \\
+ \frac{15}{3[p,q][n+1][p,q]} \left( p[n]_{p,q} + p^n \right) + \frac{3}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) \\
+ \frac{18}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) \\
\leq \frac{3}{3[p,q]p^{2n}} - \frac{4}{2[p,q]p^n} + 1 + \frac{10}{3[p,q]p^n[n+1][p,q]} + \frac{4}{2[p,q][n+1][p,q]} + \frac{16}{3[p,q][n+1]\sqrt{p}} \\
+ \frac{8}{2[p,q]} \left( p[n]_{p,q} - p^n \right) + \frac{4}{2[p,q]} \left( p^{n-1}[n]_{p,q} - 1 \right) + \frac{2}{2[p,q][n+1][p,q]} \left( p[n]_{p,q} - p^n \right) \\
+ \frac{6}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) + \frac{18}{3[p,q][n+1][p,q]} \left( p^2 [n]_{p,q}^2 - p^{2n} \right) \tag{17}
\]

with the fact that

\[
\frac{1}{[p]_{p,q} + p^n} \leq \frac{1}{[p]_{p,q} + p^n} = \frac{1}{[p]_{p,q} + 1}. \]}

Thus, Lemma 3 is proved. \(\square\)

3. A-Statistical Convergence Properties

Let \(C[0,1]\) be the space of all real-valued continuous bounded functions \(f\) on \([0,1]\), endowed with the norm \(\|f\|_{C[0,1]} = \sup_{x\in[0,1]} |f(x)|\). In this section, we will give some A-statistical convergence properties for positive linear operators \(K_{\alpha}^a(f; x)\) by the following definition of A-statistical convergence and the first and second modulus of continuity.

**Definition 1.** (See [29]) For a given non-negative infinite summability matrix \(A = (a_{nk}), n, k \in \mathbb{N}\), A-transform of \(x\) denoted by \(Ax := \{ (Ax)_n \}\) is defined as \((Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k\) provided the series converges for each \(n\). We say that \(A\) is regular if \(\lim_n (Ax)_n = L\) whenever \(\lim x = 1\). Assume that \(A\) is non-negative regular summability matrix, a sequence \(x = \{ x_k \}\) is called A-statistically convergent to \(L\) provided that for every \(\epsilon > 0\), \(\lim_n \sum_{k: |x_k - L| \geq \epsilon} a_{nk} = 0\). We denote this limit by \(st_A - \lim x = L\).
As we know, $A$-statistical convergence becomes ordinary statistical convergence when $A = (C_1)$, the Cesaro matrix of order one, and it becomes classical convergence when $A = I$, the identity matrix. There is also a conclusion, every convergent sequence is statistically convergent to the same limit but not conversely.

We need the following Korovkin theorem via the conception of $A$-statistical convergence to prove our main results.

**Theorem 2.** (See [29]) Let $A = (a_{nk})$ be a non-negative regular summability matrix and $L_n(f; x)$ be positive linear operators over $C[a, b]$. Then the following two statements are equivalent:

(i) $\lim_{n \to \infty} ||L_n(f) - f||_{C[a, b]} = 0$ for $f \in C[a, b]$;

(ii) $\lim_{n \to \infty} ||L_n(e_i) - e_i||_{C[a, b]} = 0$ for $i = 0, 1, 2$.

Consider sequences $p := \{p_n\}$, $q := \{q_n\}$ for $0 < q_n < p_n \leq 1$ satisfying the following conditions

$$\lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = \lim_{n \to \infty} p_n^2 = 1.$$  \hspace{1cm} \text{(18)}

We now give main results related to statistical convergence of operators in Equation (5).

**Theorem 3.** Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix for $n, k \in \mathbb{N}$, $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $0 < q < p \leq 1$. For any $f \in C[0, 1]$, we have

$$\lim_{n \to \infty} ||K_{n,p,q}^\lambda (f) - f||_{C[0,1]} = 0.$$  \hspace{1cm} \text{(19)}

**Proof.** According to Theorem 2, it is sufficient fo satisfy

$$\lim_{n \to \infty} ||K_{n,p,q}^\lambda (e_i) - e_i||_{C[0,1]} = 0, \; i = 0, 1, 2.$$  \hspace{1cm} \text{(20)}

From Lemma 2, it is clear that Equation (20) is true for $i = 0$. For $i = 1$, by Equation (15), we have

$$\left|\left|K_{n,p,q}^\lambda (e_1) - e_1\right|\right|_{C[0,1]} = \sup_{x \in [0,1]} |K_{n,p,q}^\lambda (\Phi (t, x); x)| \leq \Theta (p, q; n)$$

$$= \frac{2 - [2]_{p,q} p^n}{[2]_{p,q} p^n} + \frac{1}{[2]_{p,q} p^n} \left( [2]_{p,q} [n + 1]_{p,q} + \frac{4}{[2]_{p,q} (p[n]_{p,q} - p^n)} + \frac{5}{[2]_{p,q} [n + 1]_{p,q} (p[n]_{p,q} - p^n)} \right) + \frac{2}{[2]_{p,q} p^{n-1}} \left( {n-1} \right) [n + 1]_{p,q} \left( p[n]_{p,q} - p^n \right).$$

Given $\epsilon > 0$, we define the following sets:

$$D = \left\{ n : \left|\left|K_{n,p,q}^\lambda (e_1) - e_1\right|\right|_{C[0,1]} \geq \epsilon \right\}, \; D_1 = \left\{ n : \frac{2 - [2]_{p,q} p^n}{[2]_{p,q} p^n} \geq \frac{\epsilon}{5} \right\},$$

$$D_2 = \left\{ n : \frac{1}{[n + 1]_{p,q}} \geq \frac{[2]_{p,q} \epsilon}{5} \right\}, \; D_3 = \left\{ n : \frac{1}{p[n]_{p,q} - p^n} \geq \frac{[2]_{p,q} \epsilon}{20} \right\},$$

$$D_4 = \left\{ n : \frac{1}{[n + 1]_{p,q} (p[n]_{p,q} - p^n)} \geq \frac{[2]_{p,q} \epsilon}{25} \right\},$$

$$D_5 = \left\{ n : \frac{1}{p^{n-1} [n + 1]_{p,q} (p[n]_{p,q} - p^n)} \geq \frac{[2]_{p,q} \epsilon}{10} \right\}.$$
Then it is clear that $D \subseteq D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$. Hence, for every $l \in \mathbb{N}$, we have

$$
\sum_{n \in D} a_n \leq \sum_{n \in D_1} a_n + \sum_{n \in D_2} a_n + \sum_{n \in D_3} a_n + \sum_{n \in D_4} a_n + \sum_{n \in D_5} a_n.
$$

Letting $j \to \infty$, we get $s_{t_A} - \lim \frac{1}{n+1} [\sum_{n \in D} a_n] = 0$ with the help of Equation (18) and

$$
st_A - \lim_n \frac{1}{n+1} [\sum_{n \in D} a_n] = 0, \quad st_A - \lim_n \frac{1}{n} [\sum_{n \in D} a_n] = 0.
$$

For $i = 2$, by Lemma 2, we have

$$
\left| \left| K^\lambda_{n,q} e_2 - e_2 \right| \right|_{C([0,1])} = \sup_{x \in [0,1]} \left| \frac{3 - [3]_{p,q} p^{2n}}{[3]_{p,q} p^{2n}} x^2 + \frac{3x - 6x^2 + \frac{3n}{2n} x(1 - x)}{[3]_{p,q} p^{2n} p^{n}} + \frac{1 - \frac{4}{n}}{[3]_{p,q} p^{n+1}} x(1 - x) \right| + \frac{\left( \frac{q - 1}{p} \right) x(1 - x^n)}{[3]_{p,q} p^{n+1} p^n}
$$

$$
\leq \frac{3 - [3]_{p,q} p^{2n}}{[3]_{p,q} p^{2n}} + \frac{15}{2[3]_{p,q} p^{n+1} p^n} + \frac{6}{[3]_{p,q} p^{n+1} p^n} + \frac{7}{[3]_{p,q} p^{n+1} p^n} + \frac{2}{[3]_{p,q} p^{n+1} p^n}.
$$

For a given $\epsilon > 0$, let us define the following sets:

$$
U = \left\{ n : \left| K^\lambda_{n,q} e_2 - e_2 \right|_{C([0,1])} \geq \epsilon \right\}, \quad U_1 = \left\{ n : \frac{3 - [3]_{p,q} p^{2n}}{[3]_{p,q} p^{2n}} \geq \frac{\epsilon}{6} \right\},
$$

$$
U_2 = \left\{ n : \frac{1}{p^n [n + 1]_{p,q}} \geq \frac{[3]_{p,q} \epsilon}{45} \right\}, \quad U_3 = \left\{ n : \frac{1}{p^n [n + 1]_{p,q}} \geq \frac{[3]_{p,q} \epsilon}{36} \right\},
$$

$$
U_4 = \left\{ n : \frac{1}{[n + 1]_{p,q} (p^n [n + 1]_{p,q} - p^n)} \geq \frac{[3]_{p,q} \epsilon}{42} \right\},
$$

$$
U_5 = \left\{ n : \frac{1}{[n + 1]_{p,q} (p^n [n + 1]_{p,q} - p^n)} \geq \frac{[3]_{p,q} \epsilon}{36} \right\},
$$

$$
U_6 = \left\{ n : \frac{1}{p^n [n + 1]_{p,q} (p^n [n + 1]_{p,q} - p^n)} \geq \frac{[3]_{p,q} \epsilon}{12} \right\}.
$$
It is obvious that \( U \subseteq U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \), then we obtain
\[
\sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn} + \sum_{n \in U_3} a_{jn} + \sum_{n \in U_4} a_{jn} + \sum_{n \in U_5} a_{jn} + \sum_{n \in U_6} a_{jn}.
\] (21)

Letting \( j \to \infty \) in Equation (21), using Equation (18) and
\[
st_A - \lim_{n} \frac{1}{|n+1|_{p,q}} = 0, \quad st_A - \lim_{n} \frac{1}{p^{n} - p^n} = 0.
\]

We obtain \( st_A - \lim_{n} \left\| K_{n,p,q}^A(\varepsilon_2) - \varepsilon_2 \right\|_{C[0,1]} = 0 \). Therefore, Equation (20) is proved, which yields the result of Theorem 3. \( \square \)

We now need the following definitions to estimate the rate of \( A \)-statistical convergence of \( K_{n,p,q}^A(f; x) \).

**Definition 4.** (See [29]) Let \( A = (a_{nk}) \) be a non-negative regular summability matrix and let \( (u_n) \) be a positive non-increasing sequence. The sequence \( x = \{ x_k \} \) is \( A \)-statistically convergent to the number \( L \) with the rate of \( o(u_n) \) if for every \( \varepsilon > 0 \),
\[
\lim_{n} \frac{1}{u_n} \sum_{k: |x_k - L| \geq \varepsilon} a_{nk} = 0.
\]

In this case we write \( x_k - L = st_A - o(u_k) \) as \( k \to \infty \).

**Definition 5.** (See [30]) Let \( f \in C[0,1] \), Peetre’s K-functional is defined by
\[
K_2(f; \delta) = \inf_{g \in C^2[0,1]} \left\{ \| f - g \|_{C[0,1]} + \delta \| g'' \|_{C[0,1]} \right\},
\]
where \( \delta > 0 \) and \( C^2[0,1] = \{ g \in C[0,1] : g', g'' \in C[0,1] \} \). The second order modulus of smoothness of \( f \in C[0,1] \) is defined by
\[
\omega_2(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,1]} |f(x + 2h) - 2f(x + h) + f(x)|.
\]

There exist an absolute constant \( C > 0 \) such that \( K_2(f; \delta) \leq C \omega_2 \left( f; \sqrt{\delta} \right) \). We also denote the usual of modulus of continuity by
\[
\omega(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,1]} |f(x + h) - f(x)|.
\]

**Theorem 6.** Let \( A = (a_{jn}) \) be a non-negative regular summability matrix. Assume that \( \omega \left( f; \sqrt{\Phi(p,q,n)} \right) = st_A - o(u_n) \), where \( \Phi(p,q,n) \) is defined in Equation (16). Then for \( f \in C[0,1] \), we have
\[
\left\| K_{n,p,q}^A(f) \right\|_{C[0,1]} = st_A - o(u_n).
\]

**Proof.** Since
\[
|f(t) - f(x)| \leq \omega(f; |t - x|) \leq \left( 1 + \frac{|t - x|}{\delta} \right) \omega(f; \delta).
\]
Applying $K_{n,p,q}^λ(f; x)$ to both ends and using Cauchy–Schwarz inequality, we have
\[ \left| K_{n,p,q}^λ(f; x) - f(x) \right| \leq K_{n,p,q}^λ(|f(t) - f(x)|; x) \leq \left( 1 + \sqrt{K_{n,p,q}^λ(\phi_2(t,x))} \right) \omega(f; \delta) \]
\[ \leq \left( 1 + \sqrt{K_{n,p,q}^λ(\phi_2(t,x))} \right) \omega(f; \delta) \]

Letting $\delta = \sqrt{K_{n,p,q}^λ(\phi_2(t,x))}$, we get
\[ \left| K_{n,p,q}^λ(f; x) - f(x) \right| \leq 2\omega \left( f; \sqrt{K_{n,p,q}^λ(\phi_2(t,x))} \right) \leq 2\omega \left( f; \sqrt{\Phi(p,q;n)} \right) , \]
where $\Phi(p,q;n)$ is defined in Equation (16). Taking supremum over $[0,1]$ on both sides, we obtain
\[ \left\| K_{n,p,q}^λ(f; x) - f(x) \right\|_{C[0,1]} \leq 2\omega \left( f; \sqrt{\Phi(p,q;n)} \right) . \]

For a given $\epsilon > 0$, consider following sets
\[ V = \left\{ n : \left\| K_{n,p,q}^λ(f; x) - f(x) \right\|_{C[0,1]} \geq \epsilon \right\}, \quad V_1 = \left\{ n : \omega \left( f; \sqrt{\Phi(p,q;n)} \right) \geq \frac{\epsilon}{2} \right\} . \]

Obviously, we have $V \subseteq V_1$ and we also can obtain
\[ \frac{1}{u_j} \sum_{n \in V} a_{jn} \leq \frac{1}{u_j} \sum_{n \in V_1} a_{jn} . \]

Thus, let $j \to \infty$, by hypothesis we are led to the fact that $\left\| K_{n,p,q}^λ(f) - f \right\|_{C[0,1]} = st_A - o(u_n)$. Theorem 6 is proved. \[ \Box \]

**Theorem 7.** Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $\omega \left( f; \Theta(p,q;n) \right) = st_A - o(a_n), \quad \omega_2 \left( f; \sqrt{\Phi(p,q;n)} + \Theta(p,q;n)^2/2 \right) = st_A - o(b_n)$, where $\Theta(p,q;n)$ and $\Phi(p,q;n)$ are defined in Equations (15) and (16). Then for $f \in C[0,1]$, we have
\[ \left\| K_{n,p,q}^λ(f) - f \right\|_{C[0,1]} = st_A - o(\max\{a_n, b_n\}) . \]

**Proof.** Let’s define the following auxiliary operators
\[ \tilde{K}_{n,p,q}^λ(f; x) = K_{n,p,q}^λ(f; x) - f \left( \theta_{p,q}^λ(n,x) \right) + f(x) , \]
where $x \in [0,1], \theta_{p,q}^λ(n,x)$ is defined in Equation (11). Thus, we get
\[ \tilde{K}_{n,p,q}^λ(\phi_1(t,x); x) = 0 \]
by Lemma 2. Letting $g \in C^2_{[0,1]}, t \in [0,1]$, by Taylor’s expansion, we have
\[ g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du . \]
Applying $\hat{K}_{n,p,q}^\lambda$ on both sides for Equation (24) and using Equation (23), we obtain

$$\hat{K}_{n,p,q}^\lambda(g; x) = g(x) + \hat{K}_{n,p,q}^\lambda \left( \int_x^t (t-u)g''(u) du; x \right).$$

Therefore, by Equations (22), (15) and (16), we have

$$\left| \hat{K}_{n,p,q}^\lambda(g; x) - f(x) \right| \leq K_{n,p,q}^\lambda \left( \int_x^t (t-u)g''(u) du; x \right) + \int_x^f \theta_{p,q}^\lambda(n,x) \left( \theta_{p,q}^\lambda(n,x) - u \right) g''(u) du,$$

$$\leq K_{n,p,q}^\lambda \left( \int_x^t g''(u) du; x \right) + \int_x^f \theta_{p,q}^\lambda(n,x) \left( \theta_{p,q}^\lambda(n,x) - u \right) g''(u) du,$$

$$\leq \left( \Phi(p,q;n) + \Theta(p,q;n)^2 \right) \left\| g'' \right\|_{C[0,1]}.$$

(25)

Besides, by Equations (22) and (9), we get

$$\left| \hat{K}_{n,p,q}^\lambda(f; x) \right| \leq ||f||_{C[0,1]} K_{n,p,q}^\lambda(e_0;x) + 2 ||f||_{C[0,1]} = 3 ||f||_{C[0,1]}.$$

(26)

By Equations (22), (26) and (25), we have

$$\left| K_{n,p,q}^\lambda(f; x) - f(x) \right| = \left| K_{n,p,q}^\lambda(f - g; x) - (f - g)(x) + \hat{K}_{n,p,q}^\lambda(g; x) - g(x) + f \left( \theta_{p,q}^\lambda(n,x) \right) - f(x) \right|,$$

$$\leq \left| K_{n,p,q}^\lambda(f - g; x) - (f - g)(x) \right| + \left| \hat{K}_{n,p,q}^\lambda(g; x) - g(x) \right| + \left| f \left( \theta_{p,q}^\lambda(n,x) \right) - f(x) \right|,$$

$$\leq 4 ||f - g||_{C[0,1]} + \left( \Phi(p,q;n) + \Theta(p,q;n)^2 \right) \left\| g'' \right\|_{C[0,1]} + \omega(f; \Theta(p,q;n)).$$

Taking infimum on the right hand side over all $f \in C^2_{[0,1]}$, we obtain

$$\left| K_{n,p,q}^\lambda(f; x) - f(x) \right| \leq 4K_2 \left( f; \left( \Phi(p,q;n) + \Theta(p,q;n)^2 \right) / 4 \right) + \omega(f; \Theta(p,q;n)).$$

Hence, we get

$$\left| K_{n,p,q}^\lambda(f; x) - f(x) \right| \leq C\omega_2 \left( f; \sqrt{\Phi(p,q;n) + \Theta(p,q;n)^2} / 2 \right) + \omega(f; \Theta(p,q;n)).$$

Taking supremum over $[0,1]$ on both sides, we have

$$\left\| K_{n,p,q}^\lambda(f) - f \right\|_{C[0,1]} \leq C\omega_2 \left( f; \sqrt{\Phi(p,q;n) + \Theta(p,q;n)^2} / 2 \right) + \omega(f; \Theta(p,q;n)).$$

For a given $\epsilon > 0$, set

$$M = \left\{ n : \left\| K_{n,p,q}^\lambda(f) - f \right\|_{C[0,1]} \geq \epsilon \right\},$$

$$M_1 = \left\{ n : \omega_2 \left( f; \sqrt{\Phi(p,q;n) + \Theta(p,q;n)^2} / 2 \right) \geq \frac{\epsilon}{2C} \right\},$$

$$M_2 = \left\{ n : \omega(f; \Theta(p,q;n)) \geq \frac{\epsilon}{2} \right\}.$$
Then we have $M \subseteq M_1 \cup M_2$ and

$$\frac{1}{\max\{a_j, b_j\}} \sum_{j \in M} a_{jn} \leq \frac{1}{b_j} \sum_{j \in M_1} a_{jn} + \frac{1}{a_j} \sum_{j \in M_2} a_{jn}.$$ 

According to the assumptions of Theorem 7, we have

$$\lim_{j \to \infty} \frac{1}{b_j} \sum_{j \in M_1} a_{jn} + \lim_{j \to \infty} \frac{1}{a_j} \sum_{j \in M_2} a_{jn} = 0.$$ 

Thus,

$$\lim_{j \to \infty} \frac{1}{\max\{a_j, b_j\}} \sum_{j \in M} a_{jn} = 0.$$ 

Therefore, we get the desire result of Theorem 7.

4. Conclusions

In this paper, we introduced a kind of Kantorovich type $\lambda$-Bernstein operators $K_{n,p,q}^{\lambda}(f; x)$ via $(p, q)$-calculus, we estimated the moments and central moments and used these results to obtain an $A$-statistical convergence theorem and the rate of $A$-statistical convergence of $K_{n,p,q}^{\lambda}(f; x)$ to $f(x)$. In the future research work, we will continue to investigate some approximation properties of Durrmeyer type $\lambda$-Bernstein operators via $(p, q)$-calculus.

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