The Study of Combined Effect of Delay Induced by Acid and Toxic Metal on Plant Population: A Modeling Approach

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Abstract: A mathematical framework is put forward to study the combined effect of acid and toxic metal on plant population growth with delay. It is supposed that the rate at which the nutrient is taken up by the plant from the soil is adversely affected in the excessive availability of acid and toxic metal. It is also observed that the concentration of nutrient and the plant population density decreases due to the presence of acid and toxic metal. This effect is shown by considering the delay in state variable: favourable resources. The stability of the system gets disturbed by the introduction of delay parameter. For the critical value of delay parameter, Hopf bifurcation is also seen. The sensitivity of model solutions for different values of model parameters is established using sensitivity analysis. MATLAB code is used for simulation.

Key Words: Concentration of acid, Concentration of toxic metal, Concentration of nutrients, Density of favourable resources, Plant population density, Delay, Equilibrium, Hopf bifurcation.

1. Introduction

Plant population density is directly proportional to the availability of nutrient pool in the soil. The entire nutrient pool consists of two kinds of nutrients: Macronutrients and Micronutrients. The nutrients such as Nitrogen, Phosphorus, Potassium, Calcium, Sulphur, Magnesium, Carbon, Hydrogen and Oxygen are macronutrients. These macronutrients act as favourable resources for the plant growth. Nearly 95 percent plant growth depends on these favourable resources or macronutrients. The micronutrients include Iron, Boron, Chlorine, Zinc, Copper, Nickel. Only small amount of these trace minerals or micronutrients is needed for plant growth. Over the period, number of statistical, empirical, exponential growth model, logistic growth model and various other mathematical models consisting of differential equations have been proposed for individual and overall plant population growth. The book written by Thornley [1] on mathematical models on plant physiology was the breakthrough and the first significant step towards application of ordinary differential equations in plant growth processes. It dealt with all the major plant physiological aspects namely light intercept by plants and crops, crop photosynthesis and a whole-plant model with structure-storage partitioning. Reynolds and Thornley [2] gave a model for partitioning of newly synthesized dry matter between roots and shoots in vegetative plants. Later Makela [3] proved that portioning given by Reynolds and Thornley is always not equal turn-over and well behaved and a more general derivation was given which considered their model as a special case. A two-compartment model by Thornley [4] is a good contribution to mathematical modelling of plant growth showing transport and chemical conversion as processes for allocating carbon and nitrogen. Deleuze and Houllier [5] gave a transport model for tree ring width. Shukla et al [6] studied the effect of primary and secondary toxicants on the biomass of resources such as forestry and agricultural crops. Overman [7] studied the effect of geographic location on crop growth and yield. He developed two mathematical models to couple plant biomass and mineral elements using analytical functions in contrast to numerical procedures. Misra and Kalra [8-9] gave a model which concluded that under the effect of toxicant, both the nutrient concentration and structural dry weight gets adversely affected. The decrease in plant biomass and oscillatory behaviour for a value of delay under the effect of toxicant was shown by Naresh et al [10]. Global stability with the help of non-linear delay diff. equs. was studied by Huang et al [11]. Zhang et al [12] gave a neural network model where the nature of the roots of a 5th degree exponential polynomial was discussed. Bharathi A et.al [13] gave model in which the coupled effect of...
acid and metal on survival of aquatic population was studied. Chaturvedi et al [14] studied the effect of pollutant and toxicants on fish population. Bocharov and Rihan [15] gave adjoint and direct methods for sensitivity analysis in numerical modelling in biosciences using delay differential equations. Rihan [16] did the Sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model are not only constants but also variables of time. Banks et al [17] presented theoretical foundations for traditional sensitivity and generalized sensitivity functions for a general class of nonlinear delay differential equations. They Included theoretical results for sensitivity with respect to the delays. Ingalls et al [18] developed A parametric sensitivity analysis for periodic solutions of delay differential equations. Kalra and Kumar [19] studied the role of time lag in plant growth dynamics using a two-compartment mathematical model. Sharma et al [20] used plastic track detectors for the measurement of concentration of toxic metals in soil samples collected from some villages of Kangra district, Himachal Pradesh, India. The concentration of uranium has been assessed in drinking water samples collected from different locations in Bathinda district, Punjab, India by Kumar et al [21]. Singh et al [22] concluded that Arsenic (As) is a deadly poison at high concentrations. It is mysterious in the sense that people are exposed to it most of the time through drinking groundwater, fortunately at much lower concentrations than the deadly levels, and usually without knowing it. Sharma et al [23] concluded that Salt-affected soils in arid and semi-arid tracts of the Indian Punjab are prone to deficiency of micronutrients. A field trial was conducted by Kumar et al [24] for two years in Hoshiarpur district of Punjab to know the major nutrient requirement of kinnon in a low fertility status soil.

Although a lot of work has been done on plant growth under the effect of toxicants, but the use of delay differential equations is rare in this field. In the consideration of this fact, a mathematical framework is put forward here for studying the combined effect of acid and metal on plant population by introducing delay parameter.

2. Mathematical Model

The following system of non-linear delay differential equations (1) - (5) governs the plant population growth having five state variables: Density of favourable resources $R$ including soil and surrounding environment, Plant population density $B$, Concentration of nutrients $N$ in the soil, Concentration of acid $T$ in the soil and Concentration of toxic metal $M$ in the soil.

\[
\frac{dR}{dt} = \beta_1 NR - \beta_2 R - \alpha_1 BR(t - \tau) \quad (1)
\]

\[
\frac{dB}{dt} = \alpha_1 BR(t - \tau) - \alpha_2 B \quad (2)
\]

\[
\frac{dN}{dt} = N_0 - \gamma_1 N - \beta_1 NR - K\delta_2 TN - K\varepsilon_2 MN + K\beta_2 R + K\alpha_2 B \quad (3)
\]

\[
\frac{dT}{dt} = T_0 - \delta_1 T - \delta_2 TN \quad (4)
\]

\[
\frac{dM}{dt} = M_0 - \varepsilon_1 M - \varepsilon_2 MN \quad (5)
\]

With the initial conditions: $R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0, M(0) > 0$ for all $t, R(t - \tau) = \text{constant for } t \in [-\tau, 0]$.

The parameters are defined as: $N_0 = \text{Constant nutrient input in soil}, T_0 = \text{Total input rate of acid}, M_0 = \text{Total input rate of metal}, \delta_1 = \text{Natural washout rate of acid}, \varepsilon_1 = \text{Natural washout rate of metal}, \delta_2 = \text{Depletion rate of nutrient due to acid}, \varepsilon_2 = \text{Depletion rate of nutrient due to metal}, \alpha_2 = \text{Natural decay rate of plant population density}, \beta_2 = \text{Natural decay rate of favourable resources}, \gamma_1 = \text{Nutrient leaching rate}, \alpha_1 = \text{Specific rate of utilization of favourable resources by plant population density}, \beta_1 = \text{Rate of}
interaction of nutrient and favourable resources. \( K (0 < K < 1) \) is proportional amount of resource and biomass which is recycled back to nutrient pool after degradation.

3. Boundedness

Boundedness means all the quantities considered in this model being real, their individual values as well as their total will always be finite and non-negative. The following lemma proves the boundedness of solutions of the model given by (1)-(5):

**Lemma 1.** All the solutions of the model (1)-(5) lie in the region: \( \Omega = \left[ (R, B, N, T, M) \in R_+^5 : 0 \leq T \leq \frac{T_0}{\delta_1}, 0 \leq M \leq \frac{M_0}{\varepsilon_1}, 0 \leq \frac{N_0 + T_0 + M_0}{\varphi} \leq R + B + N + T + M, 0 \leq R + B + N \leq \frac{N_0}{\varphi_1} \right] \), as \( t \to \infty \), for all +ve values \( \{ R(0), B(0), N(0), T(0), M(0), R(t - \tau) \} = \text{Constant for all } t \in [-\tau, 0] \in \Omega \subset R_+^5, \varphi_1 = \text{minimum} \left( (1 - K)\alpha_2, (1 - K)\beta_2, \gamma_1 \right) \) and \( \varphi = \max (\delta_1 + \varepsilon_1 + \frac{(1 + K)\delta_2 T_0}{\delta_1} + \frac{(1 + K)\varepsilon_2 M_0}{\varepsilon_1}, \alpha_2, \beta_2, \delta_1, \varepsilon_2) \).

**Proof.** From equation (4): \( \frac{dT}{dt} \leq T_0 - \delta_1 T \)

Applying the comparison theorem, we get as \( t \to \infty \): \( T \leq \frac{T_0}{\delta_1} \)

From equation (5): \( \frac{dM}{dt} \leq M_0 - \varepsilon_1 M \)

Applying the comparison theorem, we get as \( t \to \infty \): \( M \leq \frac{M_0}{\varepsilon_1} \)

Let \( W(t) = R(t) + B(t) + N(t) \)

\[
\frac{dW}{dt} = \frac{d(R + B + N)}{dt} = N_0 - \gamma_1 N - K\delta_2 T N - K\varepsilon_2 M N - (1 - K)\alpha_2 B - (1 - K)\beta_2 R
\]

Let \( \varphi_1 = \text{min.} \left( (1 - K)\alpha_2, (1 - K)\beta_2, \gamma_1 \right) \), then \( \frac{dW}{dt} \leq N_0 - \varphi_1 W \)

Applying the comparison theorem, we get as \( t \to \infty \): \( W \leq \frac{N_0}{\varphi_1} \)

\( R + B + N \leq \frac{N_0}{\varphi_1} \)

Again, let \( W_1(t) = W(t) + T(t) + M(t) \)

\[
\frac{dW_1}{dt} = (N_0 + T_0 + M_0) - \delta_1 T - \varepsilon_1 M - (1 + K)\delta_2 T N - (1 + K)\varepsilon_2 M N - \varphi_1 W
\]

Let \( \varphi = \max (\delta_1 + \varepsilon_1 + \frac{(1 + K)\delta_2 T_0}{\delta_1} + \frac{(1 + K)\varepsilon_2 M_0}{\varepsilon_1}, \alpha_2, \beta_2, \delta_1, \varepsilon_2) \),

then \( \frac{dW_1}{dt} \geq (N_0 + T_0 + M_0) - \varphi W_1 \)

Applying the comparison theorem, we get as \( t \to \infty \): \( W_1 \geq \frac{(N_0 + T_0 + M_0)}{\varphi} \)

Hence \( R + B + N + T + M \geq \frac{(N_0 + T_0 + M_0)}{\varphi} \)

4. Positivity of Solutions
For system to sustain, solution space must be positive. For positive solutions, we need to show that all solution \( (R(t), B(t), N(t), T(t), M(t)) \) of the model governed by Equations (1)–(5) stay positive for all time \( t > 0 \); \( R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0, M(0) > 0 \) for all \( t \) and \( R(t - \tau) = \text{constant for } t \in [-\tau, 0] \).

From equation (4): \[ \frac{dT}{dt} \geq - (\delta_1 + \delta_2 NT) T \] i.e. \[ \frac{dT}{dt} \geq - \left( \frac{\beta_1 N^* + \delta_2 N_0}{\varphi_1} \right) T \]

\[ T \geq c_1 e^{- \left( \frac{\beta_1 N^* + \delta_2 N_0}{\varphi_1} \right) t} \Rightarrow T > 0 \text{ for all } t. \]

From equation (5): \[ \frac{dM}{dt} \geq - (\varepsilon_1 + \varepsilon_2 N) M \] i.e. \[ \frac{dM}{dt} \geq - \left( \frac{\varepsilon_1 N^* + \varepsilon_2 N_0}{\varphi_1} \right) M \]

\[ M \geq c_2 e^{- \left( \frac{\varepsilon_1 N^* + \varepsilon_2 N_0}{\varphi_1} \right) t} \Rightarrow M > 0 \text{ for all } t. \]

Based on the same logic, \( R > 0, B > 0 \) and \( N > 0. \)

5. Equilibrium Points of the Model

The dynamical system governed by equations (1)–(5) has two non-negative equilibrium points \( E_2(i = 1, 2) \) out of which \( E_1 \) is a uniform equilibrium point and \( E_2 \) is the interior equilibrium point. At all points of equilibrium: \( R(t - \tau) = R(t) \)

(I) The first uniform equilibrium point \( E_1(\bar{R} \neq 0, \bar{B} \neq 0, \bar{N} \neq 0, \bar{T} = 0, \bar{M} = 0) \)

\[ \bar{R} = \frac{\alpha_2}{\alpha_1}, \bar{B} = \frac{\beta_1 N^* - \beta_2}{\alpha_1} \text{ if } \beta_1 N^* > \beta_2 \text{ and } \bar{N} = \frac{N_0 + K \beta_2 \bar{R} + K \alpha_2 \bar{B}}{\gamma_1 + \beta_1 \bar{R}} \]

(II) The second interior equilibrium point \( E_2(R^* \neq 0, B^* \neq 0, N^* \neq 0, T^* \neq 0, M^* \neq 0) \)

\[ T^* = \frac{T_0}{\delta_1 + \delta_2 N^*}, M^* = \frac{M_0}{\varepsilon_1 + \varepsilon_2 N^*}, R^* = \frac{\alpha_2}{\alpha_1}, B^* = \frac{\beta_1 N^* - \beta_2}{\alpha_1} \text{ if } \beta_1 N^* > \beta_2, \]

\[ N^* = \frac{N_0 + K \beta_2 R^* + K \alpha_2 B^*}{\gamma_1 + \beta_1 R^* + K \delta_2 T^* + K \varepsilon_2 M^*} \]

such that \( N^* = -h_1 + \frac{h_2^2 - 4h_1 h_3}{2h_1} > 0 \)

Where \( h_1 = \gamma_1 \alpha_1 \delta_2 + (1 - K) \alpha_2 \beta_1 \delta_2, \)

\( h_2 = \gamma_1 \alpha_1 \delta_2 + K \delta_2 T_0 \alpha_1 + K \varepsilon_2 M_0 - N_0 \delta_2 + (1 - K) \alpha_2 \beta_1 \delta_1, h_3 = -N_0 \delta_1 \alpha_1 \)

6. Study of Uniform Equilibrium \( E_1 \) and Local Stability

In the absence of acid and toxic metal \( (T = 0, \bar{M} = 0) \), there is no delay \( (\tau = 0) \). The system of equations governing the dynamics about equilibrium \( E_1(\bar{R} \neq 0, \bar{B} \neq 0, \bar{N} \neq 0, \bar{T} = 0, \bar{M} = 0) \) becomes:

\[ \frac{d\bar{R}}{dt} = \beta_1 \bar{N} \bar{R} - \beta_2 \bar{R} - \alpha_1 \bar{B} \bar{R}(t) \] \hspace{1cm} (6)

\[ \frac{d\bar{B}}{dt} = \alpha_1 \bar{B} \bar{R}(t) - \alpha_2 \bar{B} \hspace{1cm} (7) \]

\[ \frac{d\bar{N}}{dt} = N_0 - \gamma_1 \bar{N} - \beta_1 \bar{N} \bar{R} + K \beta_2 \bar{R} + K \alpha_2 \bar{B} \hspace{1cm} (8) \]

The characteristic equation associated with variational matrix about equilibrium \( E_1 \) is given by:

\[ \lambda^3 + \xi_1 \lambda^2 + \xi_2 \lambda + \xi_3 = 0 \] \hspace{1cm} (9)
Where $\xi_1 = \beta_2 + \alpha_2 + \gamma_1 + \alpha_1 \beta + \beta_1 \bar{R} - \beta_1 \bar{N} - \alpha_1 \bar{R}$

$\xi_2 = (\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B})(\alpha_1 \bar{R} - \alpha_2) - (\alpha_1 \bar{R} - \alpha_2)(\gamma_1 + \beta_1 \bar{R}) - (\gamma_1 + \beta_1 \bar{R})(\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B}) - \bar{R} (\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B}) - \alpha_1 \bar{R}^2 (\beta_2 - \beta_1 \bar{N})$

$\xi_3 = (\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B})(\alpha_1 \bar{R} - \alpha_2)(\gamma_1 + \beta_1 \bar{R}) + \alpha_1 \bar{R}^2 (\beta_2 - \beta_1 \bar{N})$

By Routh-Hurwitz’s criteria, the real parts of roots of the equation (9) will be negative iff $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_3 > 0$.

Hence, the uniform equilibrium $E_1$ will always be locally stable, provided $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$ and $\xi_1 \xi_2 - \xi_3 > 0$.

7. Study of Interior Equilibrium $E_2$ and Local Hopf Bifurcation

In the presence of acid and toxic metal ($T^* \neq 0, M^* \neq 0$), it is assumed that there will be delay ($\tau \neq 0$). The system of equations governing the dynamics about equilibrium $E_2(R^* \neq 0, B^* \neq 0, N^* \neq 0, T^* \neq 0, M^* \neq 0)$ becomes:

$$\frac{dR^*}{dt} = \beta_1 N^* R^* - \beta_2 R^* - \alpha_1 B^* R^*(t - \tau)$$

$$\frac{dB^*}{dt} = \alpha_1 B^* R^*(t - \tau) - \alpha_2 B^*$$

$$\frac{dN^*}{dt} = N_0 - \gamma_1 N^* - \beta_1 N^* R^* - K \delta_2 T^* N^* - K \varepsilon_2 M^* N^* + K \beta_2 R^* + K \alpha_2 B^*$$

$$\frac{dT^*}{dt} = T_0 - \delta_1 T^* - \delta_2 T^* N^*$$

$$\frac{dM^*}{dt} = M_0 - \varepsilon_1 M^* - \varepsilon_2 M^* N^*$$

The exponential characteristic equation about equilibrium $E_2$ is given by:

$$\mu^5 + \zeta_1 \mu^4 + \zeta_2 \mu^3 + \zeta_3 \mu^2 + \zeta_4 \mu + \zeta_5 + (\eta_1 \mu^4 + \eta_2 \mu^3 + \eta_3 \mu^2 + \eta_4 \mu + \eta_5)e^{-\mu \tau} = 0$$

Here $\zeta_1 = -(L_1 + L_7 + L_{13} + L_{19} + L_{25})$.

$\zeta_2 = (L_{15} L_{23} + L_1 L_7 + L_{13} L_{19} + L_{19} L_{25} + L_1 L_{25} + L_7 L_{19} + L_{25} L_7 + L_{13} L_{25} + L_1 L_{19} - L_{11} \beta_1 R^*)$.

$\zeta_3 = -(L_1 L_{15} L_{23} + L_7 L_{15} L_{23} + L_{15} L_{19} L_{23} + L_1 L_7 L_{13} + L_7 L_1 L_{19} + L_{13} L_{19} L_{25} + L_1 L_{19} L_{25} + L_1 L_7 L_{25} + L_1 L_{13} L_{19} + L_7 L_{19} L_{25} + L_1 L_{13} L_{25} + L_7 L_{13} L_{25} + L_1 L_{14} L_{18} + \beta_1 \gamma_1 L_6 L_{12} - \beta_1 R^* L_7 L_{11} - \beta_1 R^* L_1 L_{19} - \beta_1 R^* L_{11} L_{25})$.

$\zeta_4 = (L_1 L_7 L_{15} L_{25} + L_7 L_{15} L_{19} L_{23} + L_1 L_{15} L_{19} L_{23} + L_1 L_7 L_{13} L_{19} + L_7 L_1 L_{19} L_{25} + L_1 L_{13} L_{19} L_{25} + L_1 L_7 L_{19} L_{25} + L_1 L_{19} L_{25} + L_1 L_7 L_{13} L_{19} + L_7 L_{14} L_{18} + L_7 L_{14} L_{18} + \beta_1 R^* L_6 L_{12} L_{19} + \beta_1 R^* L_6 L_{12} L_{25} - \beta_1 R^* L_7 L_{11} L_{25} - \beta_1 R^* L_7 L_{11} L_{19} - \beta_1 R^* L_7 L_1 L_{25})$.

$\zeta_5 = -(L_1 L_2 L_{15} L_{19} L_{23} + L_1 L_2 L_{13} L_{19} L_{25} + L_1 L_2 L_{14} L_{18} - \beta_1 R^* L_7 L_{11} L_{25} - \beta_1 R^* L_7 L_{11} L_{19} - \beta_1 R^* L_7 L_1 L_{25})$.

$\eta_1 = \alpha_1 B^* \eta_2 = \alpha_1 B^*(L_7 + L_{13} + L_{19} + L_{25})$, $\eta_3 = \alpha_1 B^*(L_{15} L_{23} + L_7 L_{13} + 2L_{19} L_{25} + L_7 L_{25} + L_{13} L_{19} + L_{13} L_{25}), \eta_4 = \alpha_1 B^*(L_1 L_7 L_{13} + L_{12} L_{25} - L_{14} L_{18} - L_7 L_{15} L_{23} - L_{15} L_{19} L_{23} - L_{13} L_{19} - L_{19} L_{25} - L_7 L_{13} L_{25}), \eta_5 = \alpha_1 B^*(\beta_1 R^* L_1 L_2 L_{19} L_{25} - L_7 L_{15} L_{19} L_{23} - L_7 L_{13} L_{19} L_{25} - L_7 L_{14} L_{18})$.

Where $L_1 = -(\beta_2 - \beta_1 N^*), L_2 = 0, L_3 = \beta_1 R^* L_4 = 0, L_5 = 0 L_4 = \alpha_1 B^*, L_7 = -\alpha_2 L_8 = 0, L_9 = 0, L_{10} = 0, L_{11} = (\beta_2 K - \beta_1 N^*), L_{12} = \alpha_2 K, L_{13} = -(\gamma_1 + \beta_1 R^* + \delta_2 KT^* + \varepsilon_2 KM^*)$.
\(-\delta_2 KN^*, L_{15} = -\varepsilon_2 KN^*, L_{16} = 0, L_{17} = 0, L_{18} = -\delta_2 T^*, L_{19} = -(\delta_1 + \delta_2 N^*), L_{20} = 0, L_{21} = 0, L_{22} = 0, L_{23} = -\varepsilon_2 KM^*, L_{24} = 0, L_{25} = -(\varepsilon_1 + \varepsilon_2 N^*).\)

As \(\mu = io\) is a root of equation (15), So

\[(i\omega^5 + \zeta_1 \omega^4 - i\zeta_3 \omega^3 - \zeta_3 \omega^2 + i\zeta_4 \omega + \zeta_5) + (\eta_1 \omega^4 - i\eta_2 \omega^3 - \eta_3 \omega^2 + i\eta_4 \omega + \eta_5)(\cos \omega \tau - isin \omega \tau) = 0\]

Separating real and imaginary parts

\[(\omega^5 - \zeta_2 \omega^3 + \zeta_3 \omega^2) + (\eta_4 \omega - \eta_2 \omega^3 \cos \omega \tau - (\eta_1 \omega^4 - \eta_3 \omega^2 + \eta_5) \sin \omega \tau = 0 \quad (16)\]

\[(\zeta_1 \omega^4 - \zeta_3 \omega^2 + \zeta_5) + (\eta_4 \omega - \eta_3 \omega^2 + \eta_5) \cos \omega \tau + (\eta_4 \omega - \eta_2 \omega^3 \sin \omega \tau = 0 \quad (17)\]

Squaring and adding equation (15) and (16), we get

\[\omega^{10} + a\omega^8 + b\omega^6 + c\omega^4 + d\omega^2 + r = 0 \quad (18)\]

Where \(a = (\zeta_1^2 - \eta_1^2 - 2\zeta_2), b = (\zeta_1^2 - \eta_1^2 - 2\zeta_4 - 2\zeta_3\zeta_5 + 2\eta_1\eta_3), c = (\zeta_3^2 - \eta_3^2 - 2\zeta_2 \zeta_4 + 2\eta_2 \eta_4 - 2\eta_1 \eta_5), d = (\zeta_4^2 - \eta_4^2 - 2\eta_3 \eta_5 + 2\eta_3 \eta_5), r = (\zeta_5^2 - \eta_5^2)\]

Let \(\omega^2 = \chi\), then, equation (18) becomes:

\[\chi^5 + \alpha \chi^4 + \beta \chi^3 + \gamma \chi^2 + \delta \chi + \varepsilon = 0 \quad (19)\]

**Lemma 2.** If \(\varepsilon < 0\), Equation (19) has contains at least one +ve real root.

**Proof.** Let \(f(\chi) = \chi^5 + \alpha \chi^4 + \beta \chi^3 + \gamma \chi^2 + \delta \chi + \varepsilon\)

Here \(f(0) = \varepsilon < 0\), \(\lim_{\chi \to \infty} f(\chi) = \infty\). So, \(\exists \chi_0 \in (0, \infty)\) such that \(f(\chi_0) = 0\). Proof completed.

Also \(f'(\chi) = 5\chi^4 + 4\alpha \chi^3 + 3\beta \chi^2 + 2\gamma \chi + \delta\)

Let \(f'(\chi) = 0 \Rightarrow 5\chi^4 + 4\alpha \chi^3 + 3\beta \chi^2 + 2\gamma \chi + \delta = 0 \quad (20)\)

Which becomes \(u^4 + au^3 + bu + c = 0 \quad (21)\)

Where \(u = \chi + \frac{\alpha}{5}, a = \frac{3\beta}{5} - \frac{6\alpha^2}{25}, b = \frac{2\gamma}{5} + \frac{6\alpha \beta}{25} + \frac{8\alpha^3}{125}, c = \frac{\delta}{5} - \frac{2a \gamma}{25} + \frac{3\alpha^2 \beta}{125} + \frac{3\alpha^4}{625}\)

If \(b = 0\), then, roots of equ. (21) are:

\[u_1 = \sqrt{-\frac{a + \sqrt{\pi}}{2}}, u_2 = \sqrt{-\frac{a - \sqrt{\pi}}{2}}, u_3 = \sqrt{-\frac{a - \sqrt{\pi}}{2}}, u_4 = \sqrt{-\frac{a + \sqrt{\pi}}{2}}\]

Thus \(\chi_i = u_i - \frac{\alpha}{5}, \quad i = 1, 2, 3, 4\) are roots of equ. (19); \(H = a^2 - 4c\)

**Lemma 3.** Suppose \(\varepsilon \geq 0\) and \(b = 0\).

(I) If \(H < 0\), then equation (19) has no +ve real roots.

(II) If \(H \geq 0, a \geq 0, c \geq 0\), then equation (19) has no +ve real roots.

(III) If (I) and (II) are not satisfied, then equation (19) has +ve real roots iff \(\exists \) at least one \(\chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4)\) such that \(\chi^* > 0\) and \(f(\chi^*) \leq 0\).

**Proof.** (I) If \(H < 0\), then equation (19) has no +ve real roots. Since \(\lim_{\chi \to \infty} f(\chi) = \infty\), we have \(f'(\chi) > 0\) for \(\chi \in R\). Hence \(f(0) = \varepsilon \geq 0\) implies \(f(\chi)\) has no zero in \((0, \infty)\).
(II) Condition $H \geq 0$, $a \geq 0$, $c \geq 0$ imply that $f'(\chi)$ has no zero in $(-\infty, \infty)$. It is similar to (I) that $f(\chi)$ has no zero in $(0, \infty)$.

(III) The sufficiency is obvious. We need only to prove the necessity. If $H \geq 0$, we know that equation (21) has only four roots $u_1, u_2, u_3$ and $u_4$, that is equation (20) has only four roots $\chi_1, \chi_2, \chi_3$ and $\chi_4$ at least $\chi_1$ is a real root. Without loss of generality, we assume that $\chi_1, \chi_2, \chi_3$ and $\chi_4$ are all real. This implies that $f(\chi)$ has at most four stationary points $\chi_1, \chi_2, \chi_3$ and $\chi_4$. If it is not true, then we have that either $\chi_1 \leq 0$ or $\chi_1 > 0$ and $\min f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4 > 0$. If $\chi_1 \leq 0$, then $f'(\chi)$ has no zero in $(0, \infty)$. Since $f(0) = \epsilon \geq 0$ is the strict minimum of $f(\chi)$ for $\chi \geq 0$ which implies $f(\chi) > 0$ in $(0, \infty)$. If $\chi_1 > 0$ and $\min f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4 > 0$, since $f(\chi)$ is a derivable function and $\lim_{\chi \to \infty} f(\chi) = \infty$, then we have $\min_{\chi>0} f(\chi) = \min f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4 > 0$. The necessity is proved.

Next, if $b \neq 0$. Consider the resolvent of equ. (21)

$$b^2 - 4(v - a)(\frac{v^2}{4} - c) = 0$$

i.e. $v^3 - av^2 - 4cr + 4ac - b^2 = 0$

Then, equ. (22) has roots:

$$v_1 = \left(\frac{-b_1}{2} + \sqrt{H_1}\right)^{1/3} + \left(\frac{-b_1}{2} - \sqrt{H_1}\right)^{1/3} + \frac{a}{3}$$

$$v_2 = \sigma\left(\frac{-b_1}{2} + \sqrt{H_1}\right)^{1/3} + \sigma^2\left(\frac{-b_1}{2} - \sqrt{H_1}\right)^{1/3} + \frac{a}{3}$$

$$v_3 = \sigma^2\left(\frac{-b_1}{2} + \sqrt{H_1}\right)^{1/3} + \sigma\left(\frac{-b_1}{2} - \sqrt{H_1}\right)^{1/3} + \frac{a}{3}$$

Where $e_1 = -\frac{a}{3} - 4c$, $b_1 = \frac{-2a^3}{27} + \frac{8ac}{3} - b^2$, $H_1 = \frac{e_1^3}{27} + \frac{b_1^2}{4}, \sigma = \frac{1+\sqrt{3}i}{2}$

Let $v_* = v_1 \neq a$, then equation (22) becomes

$$u^4 + v_*u^2 + \frac{v_*^2}{4} - \left((v_* - a)u^2 - bu + \frac{v_*^2}{4} - c\right) = 0$$

(23)

If $v_*>a$, then equation (22) becomes

$$\left(u^2 + \frac{v_*}{2}\right)^2 - \left(\sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}}\right)^2 = 0$$

$$\Rightarrow u^2 + \sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}} + \frac{v_*}{2} \text{ and } u^2 - \sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}} + \frac{v_*}{2}$$

So, roots of equ. (21) are:

$$u_1 = -\sqrt{v_* - a} + \frac{\sqrt{H_2}}{2}, u_2 = \sqrt{v_* - a} - \frac{\sqrt{H_2}}{2}, u_3 = \frac{-\sqrt{v_* - a} + \sqrt{H_3}}{2}, u_4 = \frac{-\sqrt{v_* - a} - \sqrt{H_3}}{2}$$

Where $H_2 = -v_* - a + \frac{b}{2\sqrt{v_* - a}}$ and $H_3 = -v_* - a - \frac{b}{2\sqrt{v_* - a}}$

Then $\chi_i = u_i - \frac{a}{5}, i = 1, 2, 3, 4$ are roots of equ. (20).

Lemma 4. Suppose that $\epsilon \geq 0, b_1 \neq 0$ and $v_* > a$.

(I) If $H_2 < 0$ and $H_3 < 0$, then equation (19) has no +ve real roots.
(II) If (I) is not satisfied, then equation (19) has +ve real roots iff \( \exists \) at least one \( \chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4) \) such that \( \chi^* > 0 \) and \( f(\chi^*) \leq 0 \).

**Proof.** The proof is similar to lemma 2. We omit it. Finally, if \( v, < a \), then equation (23) becomes
\[
\left( u^2 + \frac{v_n}{2} \right)^2 - \left( \sqrt{a - v_n} - \frac{b}{2 \sqrt{a - v_n}} \right)^2 = 0
\]
Let \( \bar{x} = \frac{b}{2(a-v_n)} - \frac{a}{2} \).

**Lemma 5.** Suppose that \( \epsilon \geq 0, b_1 \neq 0 \) and \( v, < a \), then equation (22) has positive real roots iff \( \frac{b^2 - 4(a-v)^2}{4(a-v)^2} + \frac{v_n}{2} = 0 \) and \( \bar{x} > 0 \) and \( f(\bar{x}) \leq 0 \).

**Proof.** Assume equation (23) has a real root \( u_0 \) satisfying \( u_0 = \frac{b}{2(a-v_n)} \), \( u_0^2 = \frac{-v_n}{2} \) which implies that \( \frac{b^2 - 4(a-v)^2}{4(a-v)^2} + \frac{v_n}{2} = 0 \). Therefore, equation (23) has a real root \( u_0 \) if \( \frac{b^2 - 4(a-v)^2}{4(a-v)^2} + \frac{v_n}{2} = 0 \). The rest of the proof is similar to lemma 2. We omit it.

Suppose equation (20) possesses positive roots. In general, we suppose that it has 5 positive roots denoted by \( \chi^*_i, i = 1, 2, 3, 4, 5 \). Then equation (19) has 5 positive roots \( \omega_i = \sqrt{\chi^*_i}, i = 1, 2, 3, 4, 5 \).

We have
\[
\cos \omega \tau = \frac{\zeta_6}{(\eta_4 \omega - \eta_2 \omega^3)^2 + (\eta_4 \omega^4 - \eta_2 \omega^2 + \eta_3)^2}
\]
Which gives
\[
\tau = \frac{1}{\omega} \left[ \cos^{-1} \left( \frac{\zeta_6}{(\eta_4 \omega - \eta_2 \omega^3)^2 + (\eta_4 \omega^4 - \eta_2 \omega^2 + \eta_3)^2} \right) + 2j\pi \right]; j = 0, 1, 2, 3, - - -
\]
Where \( \zeta_6 = - \left( (\eta_1 \omega^4 - \eta_3 \omega^2 + \eta_5)(\zeta_1 \omega^4 - \zeta_3 \omega^2 + \zeta_5) + (\eta_4 \omega - \eta_2 \omega^3)(\omega^5 - \zeta_2 \omega^3 + \zeta_4 \omega) \right) \)

Let \( \tau_k(j) = \frac{1}{\omega_k} \left[ \cos^{-1} \left( \frac{\zeta_6}{(\eta_4 \omega - \eta_2 \omega^3)^2 + (\eta_4 \omega^4 - \eta_2 \omega^2 + \eta_3)^2} \right) + 2j\pi \right]; j = 1, 2, 3, 4, 5, - - -
\]
Then \( \mp i \omega \tau_k(j) \) is a pair of purely imaginary roots of equation (15)

Where \( \tau = \tau_k(j), k = 1, 2, 3, 4, 5, - - - \). We have \( \lim_{j \to \infty} \tau_k(j) = \infty, k = 1, 2, 3, 4, 5 \).

Thus, we can define: \( \tau_0 = \tau_{k_0(j)} = \min_{1 \leq k \leq 4, j \geq 1} \left\{ \tau_k(j) \right\} \), \( \omega_0 = \omega_{k_0}, \chi_0 = \chi_{k_0} \).

**Lemma 6.** Suppose that \( a_1 > 0, (a_1 a_2 - a_3) > 0, a_3 (a_1 a_2 - a_3) + a_1 (a_5 - a_4) > 0, (a_2 a_5 + a_3 a_2)(a_1 a_2 - a_3) + a_1 a_4 (a_5 - a_4) > 0, a_5 > 0 \).

Where \( a_1 = (\zeta_1 + \eta_1), a_2 = (\zeta_2 + \eta_2), a_3 = (\zeta_3 + \eta_3), a_4 = (\zeta_4 + \eta_4), a_5 = (\zeta_5 + \eta_5) \).

(I) If any one of the following condition holds: (i) \( \epsilon < 0 \) (ii) \( \epsilon \geq 0, b = 0, H = 0 \) and \( a < 0 \) or \( c \leq 0 \) and there exists a \( \chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4) \) such that \( \chi^* > 0 \) and \( f(\chi^*) \leq 0 \) (iii) \( \epsilon \geq 0, b \neq 0, \nu > a, H \geq 0 \) or \( H \geq 0 \) and there exists a \( \chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4) \) such that \( \chi^* > 0 \) and \( f(\chi^*) \leq 0 \) (iv) \( \epsilon \geq 0, b \neq 0, \nu > a, \frac{b^2}{4(a-v)^2} + \frac{v_n}{2} = 0, \chi > 0 \) and \( f(\chi) \leq 0 \), then negativ...
\[ \mu^4 + \zeta_2 \mu^3 + \zeta_3 \mu^2 + \zeta_4 \mu + \zeta_5 = 0 \]  
(26)

All roots of equation (26) have negative real parts iff supposition of lemma 6 holds (Routh-Hurwitz’s criteria).

From lemmas 1-4, we know that if conditions (i)-(iv) of (I) are not satisfied, then none of the roots of equation (26) will have zero real part for all \( \tau \geq 0 \).

If one of the conditions (i)-(iv) holds, when \( \tau \neq \tau_k^{(j)} \), \( k = 1,2,3,4,5; j \geq 1 \), then none of the roots of equation (26) will have zero real part and \( \tau_0 \) is the minimum value of \( \tau \) for which the roots of equation (26) are purely imaginary. This lemma is concluded by using Theorem 1.

Let \( \mu(\tau) = \psi(\tau) + i\omega(\tau) \)

be the roots of equation (26) satisfying: \( \psi(\tau_0) = 0, \omega(\tau_0) = \omega_0 \). Then we have the following lemma.

**Lemma 7.** Suppose \( h'(\chi_0) \neq 0 \). If \( \tau = \tau_0 \), then \( \mp i\omega_0 \) is a pair of simple purely imaginary roots of equation (26). Moreover, If the condition of lemma 6(I) are satisfied, then \( \frac{d}{d\tau} (Re\mu(\tau_0)) > 0 \).

**Proof.** Substituting \( \lambda(\tau) \) into equation (15) and differentiating both sides with respect to \( \tau \)

\[ (d\mu/d\tau)^{-1} = \frac{(5\mu^4 + 4\zeta_2 \mu^3 + 3\zeta_3 \mu^2 + 2\zeta_4 \mu + \zeta_5)e^{\mu \tau} + (4\eta_1 \mu^3 + 3\eta_2 \mu^2 + 2\eta_3 \mu + \eta_4)}{(\eta_4 \mu^3 + \eta_3 \mu^2 + \eta_4 \mu + \eta_5)} - \frac{\tau}{\mu} \]

By calculation, we have:

\[
\begin{align*}
(5\mu^4 + 4\zeta_2 \mu^3 + 3\zeta_3 \mu^2 + 2\zeta_4 \mu + \zeta_5)e^{\mu \tau} & = \zeta_7 \cos \omega_0 \tau + \zeta_8 \sin \omega_0 \tau + i(-\zeta_8 \cos \omega_0 \tau + \zeta_7 \sin \omega_0 \tau) \\
(4\eta_1 \mu^3 + 3\eta_2 \mu^2 + 2\eta_3 \mu + \eta_4) & = \eta_4 - 3\eta_2 \omega_0^2 + i\omega_0 (2\eta_3 - 4\eta_1 \omega_0^2) \\
(\eta_1 \mu^3 + \eta_2 \mu^2 + \eta_4 \mu + \eta_5) & = \omega_0^2 (\eta_2 \omega_0^2 - \eta_5) + i\omega_0 (\eta_5 - \eta_3 \omega_0^2 + \eta_1 \omega_0) \\
\end{align*}
\]

Where \( \zeta_7 = (5\omega_0^4 - 3\zeta_3 \omega_0^2 + \zeta_4), \zeta_8 = (4\zeta_1 \omega_0^3 - 2\zeta_5 \omega_0) \)

Then, we have

\[ (d\mu/d\tau)^{-1} = \frac{\zeta_9 h'(\chi_0)}{\zeta_6} \]

Where \( \zeta_9 = \omega_0^2 [(\eta_2 \omega_0^3 - \eta_5 \omega_0^2)^3 + (\eta_5 - \eta_3 \omega_0^2 + \eta_1 \omega_0)^4] \)

\[ \Rightarrow \text{sign} \left[ \frac{d\mu}{d\tau} \right] = \text{sign} \left[ \frac{d\mu}{d\tau} \right]^{-1} = \text{sign} \left[ \frac{\chi_6 h'(\chi_0)}{\zeta_6} \right] \]  
(29)

Where \( \zeta_6, \chi_6 > 0 \).

**8. Numerical Example**

To support the analytical result, numerical method has been used to solve equations (1)-(5) and simulation has been done with MATLAB. The following set of parametric values has been considered:

\[ N_0 = 3, M_0 = 1.9, T_0 = 1.9, K = 0.1, \alpha_1 = 0.13, \alpha_2 = 0.2, \beta_1 = 0.8, \beta_2 = 0.3, \gamma_1 = 0.1, \]
\[ \delta_1 = 0.4, \delta_2 = 0.1, \epsilon_1 = 0.45, \epsilon_2 = 0.1 \]

**Behaviour of the system about equilibrium **\( E_1: \)**

The uniform equilibrium point \( E_1: R = 1.5891, B = 2.8553, N = 0.8388 \).

With the initial values: \( R(0) = 1, B(0) = 1, N(0) = 1 \)
Figure 1. Trajectories of the model without acid and toxic metal with respect to time shows stable behaviour of the equilibrium $E_1(1.5891, 2.8553, 0.8388)$.

Figure 2. Trajectories showing adverse effect of acid $T$ and metal $M$ on nutrient concentration $N$ with respect to time.
Figure 3. Trajectories showing adverse effect of acid $T$ and metal $M$ on plant population density $B$ with respect to time $t$.

**Behaviour of the system about equilibrium $E_2$:**

The interior equilibrium point $E_2: R^* = 1.5891, B^* = 2.5130, N^* = 0.7929, T^* = 3.9177, M^* = 3.5513$.

with initial conditions: $R(0) = 1, B(0) = 1, N(0) = 1, T(0) = 1, M(0) = 1$.

Figure 4. The equilibrium points $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$ of the system is stable in the absence of delay i.e. $\tau = 0$. 
Figure 5. The equilibrium point $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$ is asymptotically stable with delay $\tau < 3.38$.

Figure 6. The equilibrium point $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$ breaks away from stability and Hopf-bifurcation is exhibited with delay $\tau \geq 3.38$.

9. Sensitivity Analysis

Estimation of the general sensitivity coefficients is done using the ‘Direct Method’ which assumes that all the parameters considered in the model are constants. Here, the sensitivity coefficients can be estimated by solving sensitivity equations simultaneously with the original system. For an instance, the partial derivatives of the solution $(R, B, N, T, M)$ with respect to $\beta_1$ (interaction rate between nutrient and resources) give the following set of sensitivity equations:
\[
\frac{dS_1}{dt} = (\beta_1 N - \beta_2 S_1 + \beta_1 R S_3 - \alpha_1 B S_1(t - \tau) + N R
\]
(30)

\[
\frac{dS_2}{dt} = -\alpha_1 S_2 + \alpha_1 B S_1(t - \tau)
\]
(31)

\[
\frac{dS_3}{dt} = (K \beta_2 - \beta_3 N) S_3 + K \alpha_2 S_3 - (\gamma_1 + \beta_1 N + K \delta_2 T + K \epsilon_2 M) S_3 - K \delta_2 N S_4 - K \epsilon_2 N S_5
\]
(32)

\[
\frac{dS_4}{dt} = -\delta_2 T S_3 - (\delta_1 + \delta_2 N) S_4
\]
(33)

\[
\frac{dS_5}{dt} = -\epsilon_2 M S_3 - (\epsilon_1 + \epsilon_2 N) S_5
\]
(34)

Where \(S_1 = \frac{\partial R}{\partial a_1}, S_2 = \frac{\partial B}{\partial a_1}, S_3 = \frac{\partial N}{\partial a_1}, S_4 = \frac{\partial T}{\partial a_1}, S_5 = \frac{\partial M}{\partial a_1}\)

**Figure 7.** Time series graph between partial changes in \(R\) (density of favourable resources) and different values of parameter \(\beta_1\) (interaction rate between nutrient and resources).
Figure 8. Time series graph between partial changes in $B$ (plant population density) and different values of parameter $\beta_1$ (interaction rate between nutrient and resources).

$\beta_1$ (interaction rate between nutrient and resources).

Figure 9. Time series graph between partial changes in $N$ (concentration of nutrients) and different values of parameter
**Figure 10.** Time series graph between partial changes in $T$ (concentration of acid in soil) and different values of parameter $\beta_1$ (interaction rate between nutrient and resources).

**Figure 11.** Time series graph between partial changes in $M$ (concentration of toxic metal in soil) and different values of parameter $\beta_1$ (interaction rate between nutrient and resources).
Figure 12. Time series graph between partial changes in $R$ (density of favourable resources) and different values of parameter $\alpha_1$ (specific rate of utilization of resources by biomass).

Figure 13. Time series graph between partial changes in $B$ (plant population density) and different values of parameter $\alpha_1$ (specific rate of utilization of resources by biomass).
Figure 14. Time series graph between partial changes in $N$ (concentration of nutrients) and different values of parameter $\alpha_1$ (specific rate of utilization of resources by biomass).

Figure 15. Time series graph between partial changes in $T$ (concentration of acid in soil) and different values of parameter $\alpha_1$ (specific rate of utilization of resources by biomass).
Conclusion

The role of delay on the plant population growth under the combined effect of acid and toxic metal is studied with the help of proposed mathematical model. In the absence of acid and toxic metal, the system shows stable behaviour as shown by the Figure 1. The introduction of the acid and toxic metal has adverse effect on plant growth. It is evident that the value of nutrient concentration decreases (From 0.8388 to 0.7929) under the combined effect of acid and toxic metal as shown by the Figure 2. It is also observed that the plant population density undergoes a decrease in its value (From 2.8553 to 2.5130) under the combined effect of acid and toxic metal as shown by the Figure 3.

The local stability of the uniform equilibrium $E_1$ is studied. It is shown that the equilibrium point $E_1(\bar{R} = 1.5891, \bar{B} = 2.8553, \bar{N} = 0.8388)$ is stable as shown by Figure 1 using Routh-Hurwitz’s criteria. The stability and Hopf- bifurcation about the interior equilibrium $E_2$ is also studied. Using lemma 6 (Routh-Hurwitz’s criteria), it is shown that interior equilibrium $E_2(R^* = 1.5891, B^* = 2.5130, N^* = 0.7929, T^* = 3.9177, M^* = 3.5513)$ is stable, in the absence of delay ($\tau = 0$) as shown by Figure 4. But the system is asymptotically stable for all values which are below the critical value of delay parameter ($\tau < 3.38$), keeping all the other parameters same (lemma 1-7) as shown by Figure 5. Once the critical value of the delay parameter is reached ($\tau \geq 3.38$), the system losses stability and becomes unstable (lemma 1-7) as shown by Figure 6. The system shows the periodic oscillation when it passes through that critical value that is Hopf bifurcation occurs.

The sensitivity of model solutions is established by taking different values of the parameters appearing in system. It improves the understanding of the role played by specific model parameters.

As we start increasing the rate of interaction of nutrient and resources, the entire system starts converging to stability. For $\beta_1 = 0.8$, the system i.e. the concentration of nutrients, the density of resources and plant population density, concentration of acid and concentration of toxic metal show Hopf bifurcation through periodic oscillations. But as we increase the value of $\beta_1$ from $\beta_1 = 0.8$ to $\beta_1 = 0.9$, the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further increase the value of $\beta_1$ from $\beta_1 = 0.9$ to $\beta_1 = 1$. It has also been observed that density of resources remain almost same throughout these increasing values of $\beta_1$, but concentration of nutrient keep on decreasing with increase in the value.
of $\beta_1$. On the contrary, plant population density, concentration of acid and concentration of toxic metal show similar kind of increase as we increase the values of $\beta_1$. This phenomenon is shown by the Figures 7-11.

As we start decreasing the specific rate of utilization of delayed resources by plant population density, the entire system starts converging to stability. For $\alpha_1 = .13$, the system i.e. the concentration of nutrients, the amount of resources and plant population density, concentration of acid and concentration of toxic metal show Hopf bifurcation through periodic oscillations. But as we decrease the value of $\alpha_1$ from $\alpha_1 = .13$ to $\alpha_1 = .125$, the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further decrease the value of $\alpha_1$ from $\alpha_1 = .125$ to $\alpha_1 = .11$. It has also been observed that density of resources, concentration of acid and concentration of toxic metal start increasing too with decrease in the value of $\alpha_1$, but this increase is more visible in case of resources as compared to concentration of acid and concentration of toxic metal. On the contrary, plant population density and concentration of nutrients show similar kind of decrease in their values with decrease in the value of $\alpha_1$. This phenomenon is graphically shown by Figures 12-16.

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