C\textsuperscript{m_0}\textsuperscript{-Smoothness of Evaluation Maps

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In this note, we give a complete proof of the claim in [L2] that total l-fold evaluation map \( E: \Sigma^l \times M \to M^l \) given by \( E(x, f) = (f(x_1), \ldots, f(x_l)) \) is of class \( C^{m_0} \). Here \( \Sigma \) and \( M \) are two \( C^\infty \)-smooth Riemannian manifolds with \( \dim(\Sigma) = n \), \( M = M_{k,p} \) is the space of \( L^p_k \)-maps from \( \Sigma \) to \( M \) and \( m_0 \) is a positive integer such that \( m_0 + \gamma = k - \frac{n}{p} \) with \( 0 < \gamma < 1 \). Note that \( m_0 \) is the Sobolev differentiability of any generic elements in \( M \).

The proof is written only using elementary calculus on Banach spaces.

**Theorem 0.1** The total evaluation map \( E: \Sigma^l \times M \to M^l \) is of class \( C^{m_0} \).

**Proof:**

It is easy to see that \( E \) is linear, and hence, of \( C^\infty \) in \( M \)-direction. It is of \( C^{m_0} \) in \( \Sigma \)-direction by Sobolev embedding. The question is about those mixed partial derivatives as well as the continuity of all derivatives.

To this end, we make a few reductions. Clearly, this can be reduced to the case that \( l = 1 \), and it can be reduced further first to the case that \( M = R^k \) by using an embedding of \( M \) into \( R^k \), then to the case that \( M = R^1 \).

Since the computations for the partial derivatives are local in \( \Sigma \), by multiplying a fixed cut-off function on \( \Sigma \) supported near the point that we are interested, we may assume that \( \Sigma \) is either \( R^n \) or \( T^n \), the \( n \)-tours. In this setting, \( E \) becomes \( E: \Sigma \times L^p_k(\Sigma, R^1) \to R^1 \). We will denote \( L^p_k(\Sigma, R^1) \) by \( L^p_k \).

To compute these partial derivatives, let \( D \) be the space of all smooth function on \( \Sigma \) (with compact support if \( \Sigma \) being \( R^n \) with \( C^\infty \)-topology” in the sense of distribution theory, and \( D' \) is the collection of continuous linear functionals. Consider the collection of those elements of \( D' \) that can be extended to continuous linear functionals on \( L^p_k \). We denote it by \( (L^p_k)' \) with operator norm. Note that \( (L^p_k)' \) is usually denoted by \( L^q_{k,1} \) with \( 1/p + 1/q = 1 \). Its elements have more concrete expressions. But we will only consider \( (L^p_k)' \) as an abstract dual.

Let \( \delta: \Sigma \to D' \) defined by \( \delta(x) = \delta_x \in D' \) where \( \delta_x \) is the Dirac delta function at \( x \in \Sigma \). By our assumption, \( \delta(x) \) is in \( (L^p_k)' \). Therefore, we have \( \delta: \Sigma \to (L^p_k)' \), and \( E \) is the composition of \( \delta \times Id_{L^p_k}: \Sigma \times L^p_k \to (L^p_k)' \times L^p_k \) with the paring of \( (L^p_k)' \) and \( L^p_k \).
Now we list the following three elementary facts proved, for instance, in Lang’s book "Real Analysis":

(I) Any paring, as a bilinear continuous map $< -, - > : E_1 \times E_2 \to E_3$ between Banach spaces satisfying the condition that $\| < e_1, e_2 > \| \leq \| e_1 \| \cdot \| e_2 \|$, is of class $C^\infty$.

(II) A map $F = F_1 \oplus F_2 : E \to E_1 \oplus E_2$ between Banach spaces is $C^r$-smooth if and only if each $F_i$, $i = 1, 2$, is.

(III) The projection $p_i : E_1 \oplus E_2 \to E_i$, $i = 1, 2$, is linear, and hence $C^\infty$-smooth.

Note that in our case, for the paring $< -, - > : (L_k^p)' \times L_k^p \to \mathbb{R}$, we have

$$\| < \phi, \xi > \| = \| \phi \left( \frac{\xi}{\| \xi \|_{k,p}} \right) \| \cdot \| \xi \|_{k,p} \leq \sup_{\| \eta \|_{k,p} \leq 1} \| \phi (\eta) \| \cdot \| \xi \|_{k,p}$$

$$= \| \phi \|_{(L_k^p)'} \cdot \| \xi \|_{L_k^p}.$$

Using the above three facts, we only need to show that $\delta$ is of $C^{m_0}$. To this end, we observe that for each $x \in \Sigma$, $\delta_x$ extends to $L_{k-m_0}^p$ since by our assumption $L_{k-m_0}^p$ is in $C^\gamma$ with $0 < \gamma < 1$. In other words, the map $\delta_x$ is lifted as $\delta_x : \Sigma \to (L_{k-m_0}^p)' \subset (L_k^p)'$. The following fact will be used repeatedly: for any $\xi$ in $(L_{k-m_0+l}^p)'$ with $l \leq m_0$, $\| \xi \|_{(L_k^p)'} \leq \| \xi \|_{(L_{k-m_0+l}^p)'}$, which follows from the dual version of the inequality.

The result we are looking for follows from this observation. Roughly speaking, each time we take a partial derivative to $\delta$, we move it from the dual of $L_{k-l}^p$ to the dual of $L_{k+l}^p$ starting with $l = m_0$.

More precisely, we show this inductively by the following four steps:

- **Step I**: $\delta : \Sigma \to (L_{k-m_0}^p)'$ is continuous with respect to the operator norm on $(L_{k-m_0}^p)'$.

**Proof:**

$$\| \delta(x) - \delta(y) \|_{(L_{k-m_0}^p)'} = \sup_{\| \xi \|_{k-m_0,p} \leq 1} \| (\delta(x) - \delta(y))(\xi) \|$$

$$= \sup_{\| \xi \|_{k-m_0,p} \leq 1} \| \xi(x) - \xi(y) \|$$

$$\leq \sup_{\| \xi \|_{k-m_0,p} \leq 1} \| \xi \|_{C^{0,\gamma}} \| x - y \|^{\gamma}$$

$$\leq C_0 \cdot \sup_{\| \xi \|_{k-m_0,p} \leq 1} \| \xi \|_{k-m_0,p} \| x - y \|^{\gamma}$$

$$= C_0 \cdot \| x - y \|^{\gamma}$$

for some constant $C_0$.

Since $\| \delta(x) - \delta(y) \|_{(L_k^p)'} \leq \| \delta(x) - \delta(y) \|_{(L_{k-m_0}^p)'}$, this also proves that $\delta$ is continuous.
• Step II: (A) The value of the partial derivative of \( \delta \) at \( x \in \Sigma \), \( (\partial_x \delta)(x) \), is equal to the distribution derivative of \( \delta_x \), \( \partial_x (\delta_x) \). \( (B) \partial_x (\delta_x) \in (L_{k-m_0+1}^p)' \subset (L_k^p)' \). Therefore, \( \partial_x \delta : \Sigma \rightarrow (L_{k-m_0+1}^p)' \) defined by \( (\partial_x \delta)(x) = \partial_x (\delta_x) \).

Proof:

Since for any \( \xi \in L_{k-m_0+1}^p \),
\[
||\partial_j^{(\delta_x)}(\xi)|| = ||\delta_x(\partial_j(\xi))||
\]
\[
\leq ||\partial_j(\xi)||_{\infty} \leq C_1 \cdot ||\delta_x^{(\partial_j)}(\xi)||_{k-m_0, p}
\]
\[
= C_1 \cdot ||\xi||_{k-m_0+1, p}.
\]

This shows that \( (B) \) is true.

To prove \( (A) \), we compute
\[
||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}
\]
\[
= \sup_{||\xi||_{k-m_0+1, p} \leq 1} ||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}
\]
\[
= \sup_{||\xi||_{k-m_0+1, p} \leq 1} \frac{||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}}{||\xi||_{(L_{k-m_0+1}^p)'}}
\]
\[
= \sup_{||\xi||_{k-m_0+1, p} \leq 1} \frac{||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}}{||\xi||_{(L_{k-m_0+1}^p)'}}
\]
\[
= \sup_{||\xi||_{k-m_0+1, p} \leq 1} ||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}
\]
\[
\leq \sup_{||\xi||_{k-m_0+1, p} \leq 1} ||\partial_j \xi||_{\infty} \cdot ||he_j||_{\gamma}
\]
\[
\leq \sup_{||\xi||_{k-m_0+1, p} \leq 1} ||\xi||_{k-m_0+1, p} \cdot ||he_j||_{\gamma}
\]
\[
\leq C_1 \cdot ||\xi||_{k-m_0+1, p} \cdot ||he_j||_{\gamma}
\]
\[
\leq C_1 \cdot ||h||_{\gamma}.
\]

Here \( 0 < t < h \).

Therefore,
\[
||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}
\]
\[
\leq ||\delta( x + he_j ) - \delta( x ) ||_{(L_{k-m_0+1}^p)'}
\]
\[
\leq C_1 \cdot ||h||_{\gamma} \rightarrow 0
\]
as \( h \rightarrow 0 \). This proves that \( \partial_x \delta \) exists.

• Step III: Assume that \( \partial^\alpha \delta : \Sigma \rightarrow (L_{k-m_0+1}^p)' \subset (L_k^p)' \) for multi-indices \( \alpha = (\alpha_1, \cdots, \alpha_n) \) with \( |\alpha| = l \leq m_0 \), and \( \partial^\alpha \delta(x) = \partial^\alpha (\delta_x) \). Then \( \partial^\alpha \delta : \Sigma \rightarrow (L_{k-m_0+1}^p)' \subset (L_k^p)' \) is continuous.
Proof:

\[\|\partial^\alpha \delta(x) - \partial^\alpha \delta(y)\|_{(L^p_{k-m_0+1})'} \]

\[= \sup_{||\xi|| \leq 1} \|\partial^\alpha (\delta_x) - \partial^\alpha (\delta_y)\| \]

\[= \sup_{||\xi|| \leq 1} \|\partial^\alpha (\delta_x) - \partial^\alpha (\delta_y)\| \]

\[\leq \sup_{||\xi|| \leq 1} \|\partial^\alpha (\xi)\| \cdot \|x - y\|^\gamma \]

\[\leq C_l \cdot \sup_{||\xi|| \leq 1} \|\xi\| \cdot \|x - y\|^\gamma \]

\[= C_l \cdot \|x - y\|^\gamma \]

\[\square\]

**Step IV**: For multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(|\alpha| = l \leq m_0 - 1\), assume that \(\partial^\alpha \delta : \Sigma \to (L^p_{k-m_0+1})' \subset (L^p_k)'\) is continuous. Then (A) \(\partial \partial^\alpha \delta(x) = \partial_j(\partial^\alpha (\delta_x));\) and (B) \(\partial_j(\partial^\alpha (\delta_x))\) is in \((L^p_{k-m_0+1})' \subset (L^p_k)'.\)

**Proof:**

Since for any \(\xi \in L^p_{k-m_0+l+1},\)

\[\|\partial_j(\partial^\alpha (\delta_x))\| \]

\[= \|\partial_j(\partial^\alpha (\delta_x))\| \]

\[\leq C_{l+1} \cdot \|\partial_j(\partial^\alpha (\xi))\|_{k-m_0,p} \]

\[= C_{l+1} \cdot \|\xi\|_{k-m_0+l+1,p}.\]

This shows that (B) is true.

To prove (A), we compute

\[\|\partial^\alpha (x + h \epsilon_j) - \partial^\alpha (x)\|_{(L^p_{k-m_0+1})'} \]

\[= \sup_{||\xi|| \leq 1} \|\partial^\alpha (x + h \epsilon_j) - \partial^\alpha (x)\| \]

\[\leq \sup_{||\xi|| \leq 1} \|\partial_j(\partial^\alpha (\xi))\|_{k-m_0+p}.\]
\[
C_{t+1} \cdot \sup_{|k-m_0+t+1,p| \leq 1} \| \partial_j \partial^\gamma \xi \|_{k-m_0,p} t \| \xi \|_{k-m_0+t+1,p} \| t \|_{\gamma} \\
= C_{t+1} \cdot \sup_{|k-m_0+t+1,p| \leq 1} \| \xi \|_{k-m_0+t+1,p} \| t \|_{\gamma} \\
\leq C_{t+1} \cdot \| h \|_{\gamma}.
\]

Here \( 0 < t < h \).

Remark 0.1 In the above computation, we only prove that all partial derivatives of \( \delta : \Sigma \to (L^p_k)' \) exist and are continuous up to degree \( m_0 \). Since the domain \( \Sigma \) is of finite dimensional, this is equivalent to that \( \delta \) is of class \( C^{m_0} \) in the usual sense of the differential calculus in Banach space setting (see Lang’s “Real Analysis” for the proof of this equivalency). In particular, the proof here has nothing to do with the \( sc \)-smoothness in the usual polyfold theory.

• Note: Proposition 3.1 in [L2] is a corollary of the above theorem, which we state now.

Proposition 0.1 Let \( G \) be a Lie subgroup of the group of diffeomorphisms of \( \Sigma \). Fix an \( x \) in \( \Sigma \). Let \( \Phi_x : G \times L^p_k(\Sigma, M) \to L^p_k(\Sigma, M) \to M \) be the composition \( ev_x \circ \Psi \). Here \( ev_x : L^p_k(\Sigma, M) \to M \) is the evaluation map at \( x \) and \( \Psi : G \times L^p_k(\Sigma, M) \to L^p_k(\Sigma, M) \) is the total action map of \( G \) acting on \( L^p_k(\Sigma, M) \) as reparametrization group of \( \Sigma \). Then \( \Phi_x \) is of class \( C^{m_0} \).

Proof:
For the completeness, we include the argument in [L2] that reduces this proposition to the above theorem.

For any \( g \in G \) and \( \xi \in L^p_k(\Sigma, M) \), we have
\[
\Phi_x(g, \xi) = ev_x \circ \Psi(g, \xi) = ev_x(\xi \circ g) = \xi(g(x)) \\
= E(g(x), \xi) = E(\phi_x(g), \xi).
\]

Here \( \phi_x : G \to \Sigma \) is the orbit map of \( x \in \Sigma \) given by \( \phi_x(g) = g(x) \) which is \( C^\infty \)-smooth by our assumption that \( G \) acts on \( \Sigma \) smoothly. Therefore, \( \Phi_x = E \circ (\phi_x, Id) \). Here \( Id \) is the identity map on \( L^p_k(\Sigma, M) \), and \( (\phi_x, Id) : G \times L^p_k(\Sigma, M) \to \Sigma \times L^p_k(\Sigma, M) \) is of class \( C^\infty \).

As for the smoothness of \( E \), the proof in [L2] only establishes the trivial fact that \( E \) is of class \( C^\infty \) along \( M \)-direction and of class \( C^{m_0} \) along \( \Sigma \)-direction. Even the continuity of the first derivative is not proved in [L2]. The proof the Theorem 0.1 above is taken from [L1]. It is possible to give a more direct proof for the \( C^r \)-smoothness of \( E \) at least for small values of \( r \) starting with the continuity of the first derivative. However, the computation below shows
that similar considerations as above proof has to be used. In the following we
carry out this computation for \( C^1 \)-smoothness of \( E \). It also gives another way
to reduce the proof of the Theorem 0.1 to the above statement of the \( C^{m_0} \-
smoothness of the \( \delta \)-function.

- Continuity of \( E = E(k - m_0) : \Sigma \times L^p_{k-m_0} \rightarrow \mathbb{R}^1 \).

By Sobolev embedding, for any \( g \) in \( L^p_{k-m_0} \), we have \( \|g\|_{C^{0, \gamma}} \leq C\|g\|_{k-m_0, p} \)
for a positive constant \( C \).

\[
|E(x + v, f + \xi) - E(x, f)| = |(f(x + v) - f(x)) + \xi(x + v)| \\
\leq \sum_{i=0}^{n-1} |(f(x + v^{i+1}) - f(x + v^i)) + |\xi(x + v)| \\
\leq \sum_{i=0}^{n-1} \|f\|_{C^{0, \gamma}} |v_i|^{1+\gamma} + \|\xi\|_{C^0} \\
\leq C_1 \cdot (\|f\|_{k-m_0, p} \|v\|^{\gamma} + \|\xi\|_{k-m_0, p}).
\]

Here \( v = (v_1, v_2, \ldots, v_n) \) and \( v^i = (v_1, v_2, \ldots, v_i, 0, \ldots, 0) \). This proves the
continuity directly for \( E \) extended to \( \Sigma \times L^p_{k-m_0} \). It will be the starting point
for the induction below.

\[ DE_x, f(v, \xi) = Df_x(v) + \xi(x) = \sum_{i=1}^n (\partial_i f(x) \cdot v_i) + \xi(x). \]

Here \( v = (v_1, \ldots, v_n) \) is a tangent vector in \( T\Sigma \simeq \mathbb{R}^n \) and \( \xi \) is a tangent vector
in \( T^*_x \mathbb{L}^p_k \simeq \mathbb{L}^p_k \).

**Proof:**

\[
E(x + tv, f + t\xi) - E(x, f) \\
= (f(x + tv) - f(x)) + t\xi(x + tv) = (Df_x + htv)(v + \xi(x + tv))t.
\]

Here \( 0 < h < 1 \)

\[
\frac{\|E(x + tv, f + t\xi) - E(x, f) - (Df_x(v) + \xi(x))\|}{t} \\
\leq \|Df_x + htv - Df_x\| \cdot \|v\| + \|\xi(x + tv) - \xi(x)\| \\
\leq \|Df\|_{C^{0, \gamma}} \|ht\| \|v\|^{1+\gamma} + \|\xi\|_{C^{0, \gamma}} \|t\| \|v\|^{\gamma} \\
\leq \|Df\|_{k-m_0, p} \|ht\| \|v\|^{1+\gamma} + \|\xi\|_{k-m_0, p} \|t\| \|v\|^{\gamma}. \\
\leq \|f\|_{k, p} \|ht\| \|v\|^{1+\gamma} + \|\xi\|_{k, p} \|t\| \|v\|^{\gamma} \rightarrow 0
\]
as \( t \rightarrow 0 \). \qed

Therefore, \( DE : \Sigma \times L^p_k \rightarrow L(\mathbb{R}^n \oplus L^p_k, \mathbb{R}) \) is given by \( DE(x, f) = Df_x + \delta_x \).

Note that

\[
L(\mathbb{R}^n \oplus L^p_k, \mathbb{R}) \simeq L(\mathbb{R}^n, \mathbb{R}) \oplus L(L^p_k, \mathbb{R}) = (\mathbb{R}^n)' \oplus (L^p_k)' = L(\mathbb{R}^n \oplus L^p_k, \mathbb{R}).
\]

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• Degree of smoothness of $DE$:

Note that the second term of $DE$ is exactly the map $\delta: \Sigma \times L^p_k \to (L^p_k)' \to L(\mathbb{R}^n \oplus L^p_k, \mathbb{R})$ before. It is of class $m_0$.

Denote the first term of $DE$ by $E^1: \Sigma \times L^p_k \to (\mathbb{R}^n)' = ((\mathbb{R}^1)')^n$. Then $E^1$ can be identified with $(E^1_1, \cdots, E^1_n)$ with $E^1_j: \Sigma \times L^p_k \to (\mathbb{R}^1)' = \mathbb{R}$ defined by $E^1_j(x, f) = \partial_j f(x)$. Clearly $E^1_j = E(k-1) \circ (I_\Sigma \times P_j(k))$. Here $P_j(k): L^p_k \to L^p_{k-1}$ is the bounded linear map given by $P_j(k)(f) = \partial_j f$, which is smooth, and $E(k-1): \Sigma \times L^p_{k-1} \to \mathbb{R}$ is just the total evaluation map $E = E(k)$ with a shifting from level $k$ to $k-1$. We already proved that $E(k - m_0)$ is continuous. We are in the position to apply induction to conclude that $E(k-1)$ is of class $m_0 - 1$ provided that required smoothness for the delta function is already established. This implies that the first term of $DE$ is of class $m_0 - 1$, and $E$ is of class $m_0$. In particular, the above argument shows that for the proof of $C^1$-smoothness of $E$, only the continuity of $\delta$ is needed.

• Acknowledgement: In [L2], the author stated that:

(A) Proposition 3.1 in [L2] is weaker than the following statements that (i) the action map $\Psi: G \times \mathcal{M}_{k,p} \to \mathcal{M}_{k-1,p}$ is of class $C^1$; (ii) inductively $\Psi: G \times \mathcal{M}_{k,p} \to \mathcal{M}_{k-l,p}$ is of class $C^l$ with $l \leq m_0$.

(B) Above (i) and (ii) follows from the considerations in the theory of sc-smoothness.

I am grateful to McDuff for pointing out that there is a difference between the usual smoothness and the sc-smoothness in Polyfold theory, and the statement (B) needs to be clarified.

Indeed, the first derivative appeared in (A) (i) above is just the ordinary derivative even we use sc-type of computation. However, the continuity of the first derivative in the sc-smoothness is measured in a weaker topology on $L(E_1, E_2)$ (called strong topology in operator theory) rather than in norm topology.

The author’s intention for (A) is to give another proof for Proposition 3.1 in [L2]. Since our method for regularizing the moduli spaces of $J$-holomorphic curves in [L2] does not use sc-smoothness, we will not discuss the statement (A) further here. In [L3], we will prove that (A) above is true in the sense of usual calculus on Banach Manifolds in the case that $(k - l) - \frac{n}{p} > 0$. In other words, in the above situation, the sc-smoothness in the standard polyfold theory is not the optimal result for the purpose here despite of the fact that sc-smoothness using the weaker topology is the right choice for various other reasons in polyfold theory.

• A question:
The question still remains that whether or not Proposition 3.1 in [L2] is weaker than the two statements in (A) interpreted in the sense of $sc$-smoothness. I am expecting a positive answer to this question.

References

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