About Synchronisation of Clocks in Free Fall Around a Central Body

F. Goy

Dipartimento di Fisica
Università di Bari
Via G. Amendola, 173
I-70126 Bari, Italy
E-mail: goy@axpba1.ba.infn.it

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The conventional nature of synchronisation is discussed in inertial frames, where it is found that theories using different synchronisations are experimentally equivalent to special relativity. In contrary, in accelerated systems only a theory maintaining an absolute simultaneity is consistent with the natural behaviour of clocks. The principle of equivalence is discussed, and it is found that any synchronisation can be used locally in a freely falling frame. Whatever the choosen synchronisation, the first derivatives of the metric tensor disappear and a geodesic is locally a straight line. But it is shown that only a synchronisation maintaining an absolute simultaneity allows to define time consistently on circular orbits of a Schwarzschild metric.

Key words: special and general relativity, synchronisation, one-way velocity of light, ether, principle of equivalence.

1 Introduction

Since a few decades there have been a revival of so-called “relativistic ether theories”. This revival is partly due to the parametrised test theory of special relativity of Mansouri and Sexl [1], which in contrary to the test theory of Robertson [2], takes explicitly the problem of synchronisation of distant clocks within an inertial frame into account. Thought its essential importance for the definition of time in special relativity, most modern textbooks of relativity treat very shortly the question of synchronisation of clocks or do not even mention it. The problem of synchronisation of distant clocks arose at the end of the 19th century from the fall of Newtonian mechanics, in which time was absolute and was defined without any reference to experiences, and in particular to procedures of synchronisation of clocks. The nature of Newtonian time, transcending any experimental definition was strongly critisized by Mach. On the other side, one had to take into account for the synchronisation procedure that no instantaneous action at distance exists in nature. In his 1905 [3] article founding the theory of relativity, Einstein influenced by the epistemological conceptions of Mach gave an operational definition of time: “It might appear possible to overcome all the difficulties attending the definition of “time” by substituing “the position of the small hand of my watch” for “time”. And in fact such a definition is satisfactory when we are concerned with defining a time exclusively for the place where the watch is located; but is no longer satisfactory when we have to connect in time series of events occurring at different places, or–what comes to the same thing– to evaluate the times of events occurring at places remote from the watch.”
Further he wrote: "If at the point A of space there is a clock, an observer at A can determine the time values of events in the immediate proximity of A by finding the positions of the hands which are simultaneous with these events. If there is at the point B of space another clock in all respects resembling the one at A, it is possible for an observer at B to determine the time values of events in the immediate neighbourhood of B. But it is not possible without further assumption to compare, in respect of time, an event at A with an event at B. We have so far defined only an “A time” and a “B time”. We have not defined a common “time” for A and B, for the latter cannot be defined at all unless we establish by definition that the “time” required by light to travel from A to B equals the “time” it requires to travel from B to A. Let a ray of light start at the “A time” \( t_A \) from A towards B, let it at the “B time” \( t_B \) be reflected at B in the direction of A, and arrive again at A at the “A time” \( t'_A \).

In accordance with definition the two clocks synchronize if

\[
t_B - t_A = t'_A - t_B \quad (1)
\]

We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:

1. If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.
2. If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other."

As Einstein underlines it himself, this is by definition that the time required by light to travel from A to B and from B to A is equal. It means that the one-way velocity of light is given by a convention and not by experience. What is known with a great precision is the (mean) two-way velocity of light, which obviously can be measured with only one clock and a mirror. This last is known with a precision of \( \Delta c/c = 10^{-9} \) and has always been found to be constant in any direction during the whole year despite of the motion of the earth. The one-way velocity of light, in contrary, cannot be determined experimentally. Let us imagine that someone would try to measure it: he might send a light ray from a clock located at A to a clock located at B, at a distance \( d \) from A, and would obtain the one-way velocity of light from A to B by dividing the distance \( d \) by the difference between the time of arrival in B and the time of departure from A. But in order to compute this time difference, he first needs clocks which are synchronised, by means of light rays whose one-way velocity is postulated. Thus the concepts of simultaneity and one-way velocity of light are bound logically in a circular way.

One can of course asks himself, if other conventions which are not in contradiction with experiments are possible. First we rewrite equation (1) such that the “B time” is defined in fonction of the “A time”. That is:

\[
t_B = t_A + \frac{1}{2}(t'_A - t_A) \quad (2)
\]

Reichenbach commented: "This definition is essential for the special theory of relativity, but is not epistemologically necessary. If we were to follow an arbitrary rule restricted only to the form

\[
t_B = t_A + \varepsilon(t'_A - t_A) \quad 0 < \varepsilon < 1 \quad (3)
\]

it would likewise be adequate and could not be called false. If the special theory of relativity prefers the first definition. i.e., sets \( \varepsilon \) equal to 1/2, it does so on the ground that this definition
leads to simpler relations.” Among the “conventionalists”, who agree that one can choose freely $\varepsilon$, are Winnie [6], Grünbaum [7], Jammer [8], Mansouri and Sexl [1], Sjödin [9], Cavalleri and Bernasconi [10], Ungar [11], Vetharaniam and Stedman [12], Anderson and Stedman [13]. Clearly, different values of $\varepsilon$ correspond to different values of the one way-speed of light.

A slightly different position was developed in the parametric test theory of special relativity of Mansouri and Sexl [1]. Following these authors, we assume that there is at least one inertial frame in which light behaves isotropically. We call it the privileged frame $\Sigma$ and denote space and time coordinates in this frame by the letter $s$: $(x_0, y_0, z_0, t_0)$. In $\Sigma$, clocks are synchronised with Einstein’s procedure. We consider also an other system $S$ moving with uniform velocity $v < c$ along the $x_0$-axis in the positive direction. In $S$, the coordinates are written with lower case letters ($x, y, z, t$). Under rather general assumptions on initial and symmetry conditions on the two systems ($S$ and $\Sigma$ are endowed with orthonormal axes, which coincide at time $t_0 = 0$, ... [1, 14]) the assumption that the two-way velocity of light is $c$ and furthermore that the time dilation factor has its relativistic value, one can derive the following transformation:

$$
\begin{align*}
  x &= \frac{1}{\sqrt{1 - \beta^2}} (x_0 - vt_0) \\
  y &= y_0 \\
  z &= z_0 \\
  t &= s (x_0 - vt_0) + \sqrt{1 - \beta^2} t_0 ,
\end{align*}
$$

(4)

where $\beta = v/c$. The parameter $s$, which characterizes the synchronisation in the $S$ frame remains unknown. Einstein’s synchronisation in $S$ involves: $s = -v/c^2 \sqrt{1 - \beta^2}$ and (4) becomes a Lorentz boost. For a general $s$, the inverse one-way velocity of light is given by [12]:

$$
\frac{1}{c_{\rightarrow}\left(\Theta\right)} = \frac{1}{c} + \left(\frac{\beta}{c} + s \sqrt{1 - \beta^2}\right) \cos \Theta ,
$$

(5)

where $\Theta$ is the angle between the $x$-axis and the light ray in $S$. $c_{\rightarrow}\left(\Theta\right)$ is in general dependent on the direction. A simple case is $s = 0$. This means from (4), that at $t_0 = 0$ of $\Sigma$ we set all clocks of $S$ at $t = 0$ (external synchronisation), or that we synchronise the clocks by means of light rays with velocity $c_{\rightarrow}\left(\Theta\right) = c/1 + \beta \cos \Theta$ (internal synchronisation). We obtain the transformation:

$$
\begin{align*}
  x &= \frac{1}{\sqrt{1 - \beta^2}} (x_0 - vt_0) \\
  y &= y_0 \\
  z &= z_0 \\
  t &= \sqrt{1 - \beta^2} t_0 ,
\end{align*}
$$

(6)

This transformation maintains an absolute simultaneity (the one of $\Sigma$) between all the inertial frames. It should be stressed that, unlike to the parameters of length contraction and time dilation, the parameter $s$ cannot be tested, but its value must be assigned in accordance with the synchronisation choosen in the experimental setup. It means, as regards experimental results, that theories using different $s$ are equivalent. Of course, they may predict different values of physical quantities (for example the one-way speed of light). This difference resides not in nature itself but in the convention used for the synchronisation of clocks. With
other words two transformations (4) with different \( s \) represent the same transformation but relative to different time coordinates. For a recent and comprehensive discussion of this subject, see [16]. A striking consequence of (4) is that the negative result of the Michelson-Morley experiment does not rule out an ether. Only an ether with galilean transformations is excluded, because the galilean transformations do not lead to an invariant two-way velocity of light in a moving system.

Strictly speaking, the conventionality of clock synchronisation was only shown to hold in inertial frames. The derivation of equation (4) is done in inertial frames and is based on the assumption that the two-way velocity of light is constant in all directions. This last assumption is no longer true in accelerated systems. But special relativity is not only used in inertial frames. A lot of textbooks bring examples of calculations done in accelerated systems, using infinitesimal Lorentz transformations. Such calculations use an additional assumption: the so-called Clock Hypothesis, which states that seen from an inertial frame, the rate of an accelerated ideal clock is identical to that of the instantaneously comoving inertial frame. With other words the rate of a clock is not influenced by acceleration per se. This hypothesis first used implicitly by Einstein in his article of 1905 was superbly confirmed in the famous timedecay experiment of muons in the CERN, where the muons had an acceleration of \( 10^{18} g \), but where their timedecay was only due to their velocity [17]. We stress here the logical independence of this assumption from the structure of special relativity as well as from the assumptions necessary to derive (4). The opinion of the author is that the Clock Hypothesis, added to special relativity in order to extend it to accelerated systems leads to logical contradictions when the question of synchronisation is brought up. This idea was also expressed by Selleri [18]. The following example (see [19]) shows it: imagine that two distant clocks are screwed on an inertial frame (say a train at rest) and synchronised with an Einstein’s synchronisation. We call this rest frame \( \Sigma \). The train accelerates during a certain time. After that, the acceleration stops and the train has again an inertial motion (system \( S \)). During acceleration, the clocks are submitted exactly to the same influences, so they have at all time the same rate, so they remain synchronous respective to \( \Sigma \). Because of the relativity of simultaneity in special relativity, where an Einstein’s procedure is applied to the synchronisation of clocks in all inertial frames, they are no more Einstein synchronous in \( S \). So the Clock Hypothesis is inconsistent with the clock setting of relativity. On the other hand, the Clock Hypothesis is tested with a high degree of accuracy [20] and cannot be rejected, so one has to reject the clock setting of special relativity. The only theory which is consistent with the Clock Hypothesis is based on transformations (4) with \( s = 0 \).

This is an ether theory. The fact that only an ether theory is consistent with accelerated motion gives strong evidences that an ether exist, but does not involve inevitably that our velocity relative to the ether is measurable. The opinion of the author is that it cannot be measured because (4) represents another coordinatisation of the Lorentz transformation (obtained by clock resynchronisation). This last avoid as a principle any detection of an uniform motion through the ether. By changing the coordinate system, one cannot obtain a physics in which new physical phenomenons appear. But we can obtain a more consistent description of these phenomenons.

In all the considerations above, space-time was flat and no gravitational forces were present. In the following we want to treat the question of synchronisation of clocks in the framework of general relativity were special relativity is only valid locally. In section 2, we calculate the equations of motion for circular orbits in a Schwarzschild metric. In section 3, we treat the problem of synchronisation of clocks on these orbits, and discuss the compatibility
of different synchronisations with the principle of equivalence.

## 2 Circular orbits in a Schwarzschild metric

In a system of reference $\mathcal{R}$ with coordinates $S, (x^0, x^1, x^2, x^3) = (ct, r, \varphi, \theta)$ ($\theta$ is the azimuthal angle) the spherical symmetric solution of Einstein’s equations in vacuum, with the boundary condition that the metric becomes Minkowskian at infinity is the Schwarzschild metric:

$$ds^2 = -(1 - \frac{\alpha}{r})(dx^0)^2 + (1 - \frac{\alpha}{r})^{-1}dr^2 + r^2(sin^2\theta d\varphi^2 + d\theta^2),$$  \hspace{1cm} (7)

were $\alpha = 2GM/c^2$ is the Schwarzschild radius of the field of total energy $Mc^2$ and $G$ the gravitational constant. We will consider in the following only geodesics of test particles of mass $m$ with $r > \alpha$ so that we are not concerned here with the breakdown of the coordinate system at $r = \alpha$. A Lagrangian function can be written as:

$$\mathcal{L} = -mc\sqrt{-g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}$$  \hspace{1cm} (8)

and the Lagrange equations by

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial (\frac{dx^i}{d\tau})} \hspace{1cm} i = 0, \ldots, 3.$$  \hspace{1cm} (9)

The variables $x^0, \theta, \phi$ are cyclic and their conjuged momentum are conserved. We can take without lost of generality: $\theta = \pi/2$, that is equatorial orbits only. The energy $E$ and angular momentum $L$ per unit of mass are conserved quantities:

$$L = r^2\frac{d\varphi}{d\tau}$$

$$E = c\frac{dx^0}{d\tau}(1 - \frac{\alpha}{r})$$  \hspace{1cm} (10)

From (8) and (10) the equation for the variable $r$ can be written

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - (1 - \frac{\alpha}{r})(c^2 + \frac{L^2}{r^2}) = \frac{1}{c^2}(E^2 - V^2(r))$$  \hspace{1cm} (11)

were $V(r)$ is an effective potential. This effective potential has a local minimum, thus we have stable circular orbits. From (10), we then find for these circular orbits:

$$\frac{dr}{d\tau} = 0 \Rightarrow r = \text{cst}$$

$$\frac{d\varphi}{d\tau} = \frac{L}{r^2} \Rightarrow \varphi(\tau) = \varphi(\tau = 0) + \text{cst}_1\tau$$

$$\frac{dt}{d\tau} = \frac{E}{c^2(1 - \frac{\alpha}{r})} \Rightarrow \tau(t) = \tau(t = 0) + \text{cst}_2t,$$  \hspace{1cm} (12)

where $\text{cst}_1 = \frac{c}{r}\sqrt{\frac{\alpha}{2r-3\alpha}}$ and $\text{cst}_2 = \sqrt{2-3\alpha/r}$. 

5
3 Two clocks in orbit

We now consider a clock $\mathcal{A}$ in event-point $A(x^0_A, r_A, \varphi_A)$ and make now all calculations in $1+2$ dimensional space-time since we treat equatorial orbits only. On a circular orbit, its velocity is given by $U = (c, 0, \omega)/\sqrt{1 - \alpha/r_A - r^2_A\omega^2/c^2}$. We have $U^iU^jg_{ij} = -c^2$ and $\omega = \frac{d\varphi}{dt}$ and is given by the Kepler law $\omega^2 = \frac{GM}{r_A^3}$ for circular orbits [21].

The principle of equivalence ensure us that we can find a system of reference $\tilde{\mathcal{O}}$, with a coordinate system $\tilde{S}$ such that at event-point $A$, $g^o_{ij}(A) = \eta_{ij}$ and $\frac{\partial g^o_{ij}}{\partial x^k}(A) = 0$, where $\eta_{ij} = \text{diag}(-1, 1, 1)$. In particular, it is possible to choose a set of three mutually orthogonal unit vectors $e^i_{(a)}$ such that $e^i_{(0)} = U^i/c$ and $e^i_{(1)}$ and $e^i_{(2)}$ fulfill the orthonormality conditions: $g_{ik}e^i_{(a)}e^k_{(b)} = \eta_{ab}$. Indices without parenthesis of $e^i$ are raised with $\eta^{ab}$, while indices with parenthesis are raised with $\eta^a_{~b}$. We can choose $e^i_{(1)}$ radial and $e^i_{(2)}$ tangential to the orbit:

$$e_{(1)} = (0, \sqrt{1 - \alpha/r_A}, 0)$$
$$e_{(2)} = \frac{1}{\sqrt{1 - \alpha/r_A - r^2_A\omega^2/c^2}} \left( \frac{r_A\omega}{c\sqrt{1 - \alpha/r_A}} - 0, \frac{\sqrt{1 - \alpha/r_A}}{r_A} \right)$$  \hspace{1cm} (13)

The following transformation from coordinate system $S$ to $\mathcal{O}$, is such that the metric tensor in the new coordinates is minkowskian and his first derivatives disapear at point $A$ [22, §9.6]:

$$\tilde{x}^i = e^i_r(x^r - x^r_A) + \frac{1}{2} e^i_r \Gamma^r_{st}(A)(x^s - x^s_A)(x^t - x^t_A) \quad i = 0, 1, 2.$$  \hspace{1cm} (14)

In the case of (7) the Christoffell’s symbols $\Gamma$ at $A$ are given by:

$$\Gamma^1_{00} = \frac{1}{2} \frac{\alpha(1 - \alpha/r_A)}{r^2_A}, \quad \Gamma^0_{01} = \frac{1}{2} \frac{\alpha}{(1 - \alpha/r_A)r^2_A}, \quad \Gamma^2_{12} = \frac{1}{r_A}$$
$$\Gamma^1_{11} = -\frac{1}{2} \frac{\alpha}{r^2_A(1 - \alpha/r_A)}, \quad \Gamma^3_{22} = -r(1 - \alpha/r_A)$$  \hspace{1cm} (15)

We obtain for the transformation between $S$ and $\mathcal{O}$:

$$\begin{align*}
\tilde{x}^0 & = \frac{1}{\sqrt{1 - \alpha/r_A - r^2_A\omega^2/c^2}} \left[ (1 - \alpha/r_A)(x^0 - x^0_A) - \frac{\omega r_A^2}{c} (\varphi - \varphi_A) \\
& \quad + \frac{1}{2} \frac{\alpha}{r^2_A} (x^0 - x^0_A)(r - r_A) - \frac{\omega r_A}{c} (\varphi - \varphi_A)(r - r_A) \right] \\
\tilde{x}^1 & = \frac{1}{\sqrt{1 - \alpha/r_A}} (r - r_A) + \frac{1}{4} \frac{\alpha}{r^2_A} (x^0 - x^0_A)^2 \\
& \quad - \frac{1}{4} \frac{\alpha}{r^2_A(1 - \alpha/r_A)} (r - r_A)^2 - \frac{1}{2} \frac{r_A}{r^2_A} \frac{1}{\sqrt{1 - \alpha/r_A}} (\varphi - \varphi_A)^2 \\
\tilde{x}^2 & = \frac{\sqrt{1 - \alpha/r_A}}{\sqrt{1 - \alpha/r_A - r^2_A\omega^2/c^2}} \left[ -\frac{\omega r_A^2}{c} (x^0 - x^0_A) + r_A (\varphi - \varphi_A) \\
& \quad - \frac{1}{2} \frac{\omega}{cr_A(1 - \alpha/r_A)} (x^0 - x^0_A)(r - r_A) + (r - r_A)(\varphi - \varphi_A) \right] \quad (16)
\end{align*}$$
This transformation looks like Lorentz transformation at first order, in particular, two distant events which are simultaneous in \( O_S \) are not simultaneous in \( S \). We now imagine that a clock \( B \) is located at \( B (x^0_A + dx^0, r_A, \varphi_A + d\varphi) \) and we want to synchronise it with \( A \) at \( A \) using an Einstein’s procedure. Since the metric is Minkowskian in \( O_S \), the velocity of light is \( c \) in this (local) frame. The two clocks will be Einstein synchronised when: \( \dot{x}^0_A = \dot{x}^0_B = 0 \). Using (16) we obtain that the infinitesimal time difference in \( S \) \( dx^0 \) between these events is given by:

\[
dx^0 = \frac{\omega r^2_A d\varphi}{c(1 - \alpha/r_A)}
\]

(17)

We generalise this synchronisation procedure all along the circular orbit. It means that we synchronise \( A \) in \((r_A, \varphi_A)\), with \( B \) in \((r_A, \varphi_B = \varphi_A + d\varphi)\), and then \( B \) with \( C \) located at \((r_A, \varphi_C = \varphi_B + d\varphi), \) etc. If we do a whole round trip, we find a time lag \( \Delta x^0 \) given by:

\[
\Delta x^0 = \oint \frac{\omega r^2_A d\varphi}{c(1 - \alpha/r_A)} = \frac{2\pi \omega r^2_A}{c(1 - \alpha/r_A)}
\]

(18)

It means that \( A \) is not synchronisable with itself, when we extend spatially the synchronisation procedure out of a local domain; this is clearly absurd. The problem occurs because \( dx^0 \) is not a total differential in \( r \) and \( \varphi \), thus the synchronisation procedure is path dependent. And in general one can say that if \( A \) synchronise with \( B \) then in general \( B \) does not synchronise with \( A \). The same remark is valid for the transitivity of the relation “is synchronised with” in the case of three clocks \( A, B \) and \( C \).

According to Einstein in the citation quoted here above, it means that the definition of synchronism given by (1) which is free of contradictions in the case of inertial frames in flat space is no more free of contradiction when we want to define time globally in a curved space. One could think that this difficulty is insuperable and that it is not possible to:

1. Find a local inertial system such that the equivalence principle is respected

2. Defining time in this system in such a way that the extension out of a local domain of the synchronisation procedure is self consistent: “is synchronised with” is an equivalence relation.

A similar problem occurs in the case of a rotating disk in flat space. It has been shown that only the transformation (6) allows a consistent definition of time on the rim of a rotating disk, while an Einstein’s synchronisation leads to the impossibility of defining time without contradictions on the rim of this disk [23].

Guided by the experimental equivalence of relativistic ether theories and special relativity, we are looking for an other synchronisation of clocks in \( \hat{O}_S \) such that the conditions 1 and 2 above are fulfilled. The spatial part of transformation (16) is not changed by a resynchronisation of clocks, and we can again choose the vectors \( e^{(1)} \), and \( e^{(2)} \) as they can be read out from (16). We are looking for a transformation from coordinate system \( S \) to local coordinate system \( \hat{S} \) such that the time transformation is not depending on the space variables at first order. It means that \( e^{(0)} \) is of the type \( e^{(0)} = (y,0,0) \). In order to find \( y \), we impone that the synchronisation only is different in \( \hat{S} \) and \( S \). That is the rate of a clock at rest at the origin of \( \hat{S} \) and \( O_S \) is the same when seen from \( S \). From (16) we calculate...
easily that: \( \delta \dot{x}^0 = \sqrt{1 - \alpha/r_A - \omega^2 r_A^2/c^2} \delta x^0 \), where \( \delta \dot{x}^0 \) is the coordinate time difference between two ticks of the clock in \( \hat{S} \) and \( \delta x^0 \) is the same quantity in \( S \). We find that \( y = \sqrt{1 - \alpha/r_A - \omega^2 r_A^2/c^2} \). Thus the transformation of the time coordinate from \( S \) to \( \hat{S} \) is now given by:

\[
\ddot{x}^0 = \sqrt{1 - \alpha/r_A - \omega^2 r_A^2/c^2} (x^0 - x_A^0) \quad (19)
\]

1. Are we sure that \( \hat{S} \) is a local inertial system of coordinates? Yes. The proof is indeed the same as it would be for \( \hat{S} \). From (14) and using the fact that \( e^{(r)}_i e_j^{(r)} = \delta_{ij} \), we have:

\[
e^{(r)}_i \ddot{x}^i = (x^i - x_A^i) + \frac{1}{2} \Gamma_{st}(A)(x^s - x_A^s)(x^t - x_A^t) \quad i = 0, 1, 2. \quad (20)
\]

Differentiating two times with respect to \( \dot{x}^k \) and \( \dot{x}^l \) gives:

\[
0 = \frac{\partial^2 x^i}{\partial \dot{x}^k \partial \dot{x}^k} + \Gamma^i_{st}(A) \left[ \frac{\partial^2 x^s}{\partial \dot{x}^i \partial \dot{x}^k}(x^t - x_A^t) + \frac{\partial x^s}{\partial \dot{x}^k} \frac{\partial x^t}{\partial \dot{x}^i} \right] \quad (21)
\]

Thus at point A:

\[
0 = \frac{\partial^2 x^i}{\partial \dot{x}^k \partial \dot{x}^k} + \Gamma^i_{st}(A) \left[ \frac{\partial x^s}{\partial \dot{x}^i} \frac{\partial x^t}{\partial \dot{x}^k} \right] \quad (22)
\]

Because of the law of transformation of Christoffel's symbols, this mean that: \( \ddot{\Gamma}_{kl}^i(A) = 0 \). So in \( \hat{S} \) at A, a geodesic becomes a straight line:

\[
\frac{d^2 \dot{x}^k}{d\lambda^2} + \ddot{\Gamma}_{kl}^i \frac{d\dot{x}^l}{d\lambda} \frac{d\dot{x}^i}{d\lambda} = \frac{d^2 \dot{x}^k}{d\lambda^2} = 0 \quad (23)
\]

2. Can time be defined consistently on the whole circular orbit? Yes. We treat again the problem of synchronising a clock \( \mathcal{A} \) at \( A(x_A^0, r_A, \varphi_A) \) and a clock \( \mathcal{B} \) at \( B(x_B^0 + dx^0, r_A, \varphi_A + d\varphi) \). The two clocks are synchronised in the system of coordinates \( \hat{S} \) if \( \dot{x}_A^0 = \dot{x}_B^0 \). Then the time difference \( dx^0 \) between these events in \( S \) calculated with (19) gives: \( dx^0 = 0 \). A similar calculation as in (18) shows that \( \Delta x^0 = 0 \) for a whole round trip. Thus the time can be defined consistently on the orbit with such a synchronisation.

The metric in system \( \hat{S} \) at \( A \) is given by \( e^{(r)}_i g_{ij} e^{(r)}_j = \hat{n}_{ab} \). We find

\[
\hat{n}_{ab} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -r_A^2/c^2(1-\alpha/r_A) \\
r_A^2/c^2(1-\alpha/r_A) & 0 & 0 & 1
\end{pmatrix} \quad (24)
\]

In the case where the vector potential \( \vec{n}_{ab} \): \( \alpha = 1, 2 \) is different from zero, the spatial part of the metric is not only given by the space-space coefficients of the metric but by \( \hat{\gamma}_{ab} = \hat{n}_{ab} - \hat{n}_{ab} \hat{n}_{ab} \). In our case we have \( \hat{\gamma}_{ab} = \delta_{ab} \). Thus the spatial system of coordinates is orthonormal. The velocity of light \( c(\Theta) \) is found by solving the equation \( ds^2 = \hat{n}_{ab} d\dot{x}^a d\dot{x}^b = 0 \). We find that:

\[
c(\Theta) = \frac{c}{1 + \frac{r_A\dot{\omega}}{c^2(1-\alpha/r_A)}} \quad (25)
\]

where \( \Theta \) is the angle between the light ray and the \( \dot{x}^2 \)-axis.
4 Remarks

1. The transformation of the time variable can easily be generalised to all synchronisations with a parameter $s$ like in (4):

$$x^0(s) = \sqrt{1 - \alpha/r_A - \omega^2 r_A^2/c^2} (x^0 + x_A^0) + s \left[ r_A (\varphi - \varphi_A) - \frac{r_A \omega}{c} (x^0 - x_A^0) \right] + O(x^i - x_A^i)^2$$

The transformation (19) is given by $s = 0$ and (16) by $s = -\frac{\omega r_A}{c \sqrt{1 - \alpha/r_A - \omega^2 r_A^2/c^2}}$. A similar argument as in section 3 shows that only $s = 0$ lead to $\Delta x^0 = 0$ for a whole round trip of synchronisation around the orbit.

2. The inertial coordinate systems $\hat{O}$ and $\hat{S}$ are different coordinatisations of the same reference frame $\hat{R}$. The transformation from $\hat{O}$ to $\hat{S}$ does not involve time in the transformation of space variables and thus is what Møller [22, p. 267, 316] calls a linear gauge transformation.

3. If a clock $A$ at $A(x_A^0, r_A, \varphi_A)$ and a clock $B$ at $B(x_A^0 + dx^0, r_A, \varphi_A + d\varphi)$ are Einstein’s synchronised in the system $\hat{S}$ of section 3 (i.e $dx^0$ is given by (17)), they remain Einstein’s synchronised during their trip around the orbit. From the equation of motion (12) one sees that they will be at a later time at point $\hat{A}$ and $\hat{B}$ with coordinates in $\hat{S}$: $(\hat{x}^0_A, r_A, \varphi_A)$ and $(\hat{x}^0_B + dx^0, r_A, \varphi_A + d\varphi)$. We can take a local inertial system at $\hat{A}$ and from (14) one sees that: $\hat{x}^0_A = \hat{x}^0_B = 0$.

5 Conclusion

In flat space, a whole set of theories equivalent to special relativity can be constructed. These theories are obtained by adopting an other convention on the synchronisation of clocks. In accelerated systems, only the theory maintaining an absolute simultaneity is logically consistent with the natural behaviour of clocks.

In general relativity, the principle of equivalence tells us that at every space-time point one can choose a local coordinate system such that the metric is minkowskian and its first derivatives disappear. Thus the laws of special relativity are locally valid in general relativity. In this local frame, we can choose an other synchronisation of clocks than the Einstein’s one. The frame is the same but the coordinatisation is different. All these coordinatisations are locally equivalent. The transformation between them is a linear gauge transformation. The spatial part of the metric is orthonormal and the derivates of the space time metric disappear at point in question. Thus a freely falling body has an uniform motion in straight line, and theses local coordinate systems are locally inertial.

An Einstein’s synchronisation lead to a contradictory definition of time when extended out of a local domain. It was shown in this article that in the case of circular orbits only a transformation maintaining an absolute simultaneity ales to define time globally and consistently on the orbit. An observer moving around a central body, who does not want to adopt a contradictory definition of time (when extended spatially out of his local domain) must then conclude that the velocity of light is not constant.
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