Saturation and Wilson Line Distribution

C.S. Lam\textsuperscript{a}, Gregory Mahlon\textsuperscript{b}, and Wei Zhu\textsuperscript{c}

\textsuperscript{a}Department of Physics, McGill University, 3600 University St., Montréal, QC Canada H3A 2T8
email: Lam@physics.mcgill.ca

\textsuperscript{b}Department of Physics, Penn State Mont Alto, 1 Campus Drive, Mont Alto, PA 17237, U.S.A.
email: gdm10@psu.edu

\textsuperscript{c}Department of Physics, East China Normal University, Shanghai 200062, China
email: Zhuweia@public8.sta.net.cn

Abstract

We introduce a Wilson line distribution function $\overline{W}_\tau(v)$ to study gluon saturation at small Feynman $x_F$, or large $\tau = \ln(1/x_F)$. This new distribution can be obtained from the distribution $W_\tau(\alpha)$ of the Color Glass Condensate model and the JIMWLK renormalization group equation. $\overline{W}_\tau(v)$ is physically more relevant, and mathematically simpler to deal with because of unitarity of the Wilson line $v$. A JIMWLK equation is derived for $\overline{W}_\tau(v)$; its properties are studied. These properties are used to complete Mueller’s derivation of the JIMWLK equation, though for $\overline{W}_\tau(v)$ and not $W_\tau(\alpha)$. They are used to derive a generalized Balitsky-Kovchegov equation for higher multipole amplitudes. They are also used to compute the unintegrated gluon distribution at $x_F = 0$, yielding a completely flat spectrum in transverse momentum squared $k^2$, with a known height. This is similar but not identical to the mean field result at small $k^2$.

I. INTRODUCTION

Soft gluons are produced by radiation from more energetic partons. Since the number of sources increases at small Feynman $x_F$, the soft gluon density $x_F G$ per unit rapidity interval increases with $\tau = \ln(1/x_F)$. In fact, both the DGLAP equation \cite{1} and the BFKL equation \cite{2} predict a growth so fast that the unitarity bound $\tau^2$ is violated. To restore unitarity, a new mechanism is required to slow down the growth towards $x_F = 0$ \cite{3}. The momentum $Q_s$ at which this mechanism sets in is known as the saturation momentum.

The phenomenological implications for the presence of a saturation momentum have been discussed in many papers \cite{4}, but it is not yet clear whether saturation has been observed experimentally. On the theoretical side, the important thing to note for our present
discussion is that soft gluons can be treated as a classical color potential \( \alpha(\vec{x}) \), because of its large density at small \( x_F \). In this background, energetic partons interact with soft gluons through their Wilson lines. It is this interaction that is responsible for the saturation process.

With the replacement of soft gluons by a classical Yang-Mills potential, the growth of soft gluon density is determined by the \( \tau \) dependence of the distribution function \( W_\tau(\alpha) \). This dependence is given by a renormalization group equation known as the JIMWLK equation \([5]\).

In this paper we introduce and study a distribution \( W_\tau(v) \) for the Wilson line \( v \). We will show that this distribution can be obtained from \( W_\tau(\alpha) \), and that it still satisfies the JIMWLK equation. Compared to \( W_\tau(\alpha) \), it has the advantage of being more directly physically relevant, because many physical processes can be described in terms of Wilson lines or dipole amplitudes. Moreover, the Wilson lines \( v \) are unitary matrices, living on the color group manifold which is compact. This compactness brings with it a number of mathematical advantages not shared by \( W_\tau(\alpha) \), whose argument \( \alpha \) lives in a non-compact linear space.

\( W_\tau(v) \) can be used to derive a number of properties not easily obtainable directly from \( W_\tau(\alpha) \). We will use these properties to complete Mueller’s proof of the JIMWLK equation \([6]\), though the proof is valid for \( W_\tau(v) \) and not for \( W_\tau(\alpha) \). They will be used to derive a generalized Balitsky-Kovchegov (BK) equation for multipole amplitudes. We find that once the non-linear BK equation for the dipole amplitude is solved, all subsequent multipole amplitudes can be obtained by solving only linear equations. This means saturation of the dipole amplitude automatically leads to saturation of higher multipole amplitudes.

\( W_\tau(v) \) will also be used to compute the asymptotic behavior of unintegrated gluon distribution at \( x_F = 0 \). We get a flat distribution in the gluon transverse momentum \( k^2 \), with a height given by eq. (6.7). This is to be contrasted with the mean field result (2.16) which yields a logarithmic dependence on \( k^2 \) with an undetermined normalization. Pre-asymptotic corrections will also be briefly discussed.

In the next section we start with a short review of saturation, the JIMWLK equation, the related BK equation, the BFKL equation, as well as some of their solutions. In Sec. 3, the Wilson line distribution \( W_\tau(v) \) is introduced, together with some mathematical preliminaries. The JIMWLK equation for \( W_\tau(v) \) is derived, and its properties studied. In Sec. 4, the missing steps of Mueller’s derivation of the JIMWLK equation are supplied. In Sec. 5, the infrared divergence encountered in the JIMWLK equation is discussed. It is shown that certain multipole functions which we call multipole traces are free of these divergences. A generalized BK equation is derived to describe the evolution of the multipole amplitudes. In Sec. 6, we discuss the property and implications of the asymptotic solution of \( W_\tau(v) \) at \( x_F = 0 \). In particular, the unintegrated momentum spectrum of the gluon density is derived. Correction to the asymptotic limit when \( x_F \neq 0 \) is briefly discussed. Certain mathematical details can be found in Appendices A and B at the end.

After this paper was submitted for publication, we were informed by the Referee of an interesting recent paper \([7]\) in which \( W_\tau(v) \) was also introduced and its JIMWLK equation derived. It went on to give a random-walk interpretation of the JIMWLK equations, with \( \tau \) playing the role of time, \( v \) the position in the group manifold, and \( \alpha \) the velocity.
II. A BRIEF REVIEW

The number of gluons emitted by a valence quark, per unit rapidity \( \tau = \ln(1/x_F) \) and per unit transverse momentum squared \( k^2 \), is given in perturbation theory by \( \alpha_s C_F/\pi k^2 \), where \( \alpha_s = g^2/4\pi \) is the QCD fine structure constant. \( C_F \) is the Casimir number in the fundamental representation, being \((N_c^2 - 1)/2N_c\) for the gauge group \( SU(N_c) \), and \( 1/2N_c \) for \( U(N_c) \). A nucleus with atomic number \( A \) has \( AN_c \) valence quarks, so its unintegrated gluon distribution is \( dN/d\tau dk^2 \equiv d(x_F G)/dk^2 = \alpha_s C_F (AN_c)/\pi k^2 \).

When integrated, this formula encounters an infrared divergence at small \( k^2 \), brought about by the long range gluon field of the unshielded valence quarks. However, quarks are confined inside color-singlet nucleons, so such a long range force is absent beyond the nucleon radius \( a \). Thus \( x_F G(x_F, Q^2) = (\alpha_s C_F (AN_c)/\pi) \ln(Q^2a^2) \).

In the central rapidity region where Feynman \( x_F \) is small, the gluon density is much larger than the amount given by the perturbation formula. This is so because soft gluons can be radiated also from energetic gluons and sea quarks, not just the valence quarks considered so far. According to the DGLAP \([1]\) equation, the soft gluon density grows like \( \exp(k\sqrt{\tau}) \) for some positive constant \( k \), and, according to the BFKL equation \([2]\), it grows like \( \exp(4\alpha_s \ln(2)N_c \tau/\pi) \). Both exceed the unitarity limit \( \tau^2 \), so a new mechanism must kick in to dampen the growth and restore unitarity at small \( x_F \) \([3]\). This effect is known as saturation.

Saturation is thought to arise from a non-linear mechanism which occurs when gluons are sufficiently dense to interact among themselves \([4]\). The number of gluons per unit rapidity interval is \( x_F G \). In a nucleus of radius \( R_A \), the transverse area per gluon is therefore \( \pi R_A^2/x_F G(x_F, Q^2) \). The average color-charge squared of a gluon is \( N_c/(N_c^2 - 1) \), their interaction strength is \( \sim \alpha_s/\pi \), and their natural size is \( \sim 1/Q \). Hence the cross section for two gluons to interact can be estimated to be \( (\alpha_s/\pi) (N_c/(N_c^2 - 1)) (\pi/Q^2) \). If the cross section is larger than the transverse area per gluon, then interaction will take place to set off the non-linear mechanism. The onset therefore occurs at a momentum scale \( Q_s \) such that

\[
\pi R_A^2/x_F G(x_F, Q_s^2) = c(\alpha_s/\pi) (N_c/(N_c^2 - 1)) (\pi/Q_s^2),
\]

or equivalently,

\[
Q_s^2 = \frac{\alpha_s N_c x_F G(x_F, Q_s^2)}{N_c^2 - 1} \pi R_A^2.
\]

A constant \( c \) has been inserted to account for the qualitative nature of this argument. Even when \( Q_s^2 \) is obtained from a detailed calculation, the constant \( c \) is still somewhat ambiguous because transition into saturation does not occur sharply. Thus one finds a number of \( c \)'s used in the literature. For example, \( c = 1 \) in \([10]\), \( c = \pi^2 \) in \([9]\) when estimated from the mean field approximation on the large \( k^2 \) side, and \( c = 16\pi^2c_1 \) when estimated on the small \( k^2 \) side, where \( c_1 \) is some unknown constant. And, \( c = \pi \) in \([11]\).

\( Q_s \) can also be defined through the unintegrated gluon density \( d(x_c G)/dk^2 \). When we reduce \( k^2 \) from infinity, this density increases until a point \( k^2 = Q_s^2 \) when gluons become sufficiently dense to set off the non-linear mechanism. From there on we enter a saturation region with a much slower growth. However, this definition is also ambiguous unless the
slowdown occurs fairly sharply, which turns out to be the case at \( x_F = 0 \). As we shall see in Sec. 6A, at \( x_F = 0 \), the saturation region is large and the unintegrated spectrum \( d(x_F G)/dk^2 \) in this region is absolutely flat in \( k^2 \). This then allows \( c \) to be determined unambiguously to be \( c = 8\pi^3 \).

Using the BFKL solution for \( x_F G \) as a qualitative estimate, and assuming that \( x_F G \) is proportional to \( A \), we see from (2.2) that \( Q_s^2 \) grows with a power of \( 1/x_F \) and \( A^{1/3} \), making it large for large nucleus or small \( x_F \). (2.2) also implies that the gluon number per unit transverse area at saturation is \( \sim Q_s^2(x)/\alpha_s \).

The large number of gluons present at saturation allows them to be treated as a classical Yang-Mills (YM) potential \( \alpha^a(\vec{x}) \). The superscript \( a \) is the color index, and \( \vec{x} = (x^-, \vec{x}) \) are the lightcone (LC) coordinates, defined for a hadron moving along the +z direction to be \( x^\pm = (t \pm z)/\sqrt{2} \) and \( \vec{x} = (x^1, x^2) \). It is also convenient to introduce the spacetime rapidity variable \( y = \ln(x^-P^+) \), where \( P^+ \) is the + component of the hadron momentum, and the gluon potential \( \alpha^a_y(x) = x^-\alpha(x^-, \vec{x}) \).

For a fast moving hadron (or nucleus), Lorentz contraction forces \( \alpha(\vec{x}) \) to be concentrated around \( x^- = 0 \), and time dilation makes it effectively (LC) time \((x^+)^{\text{independent}} \). The soft gluons are produced by partons within the hadron, so one can assume \( \alpha^a_y(x) = 0 \) for \( y > \tau \).

Energetic partons, whether in the same hadron or not, interact with the soft gluons through the Wilson-line factors:

\[
\begin{align*}
v^\dagger(x) &= P \exp \left( +ig \int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x}) t_a \right) = P \exp \left( +ig \int_{-\infty}^{\tau} dy \alpha^a_y(x) t_a \right), \\
v(x) &= \bar{P} \exp \left( -ig \int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x}) t_a \right) = \bar{P} \exp \left( -ig \int_{-\infty}^{\tau} dy \alpha^a_y(x) t_a \right), \\
V^\dagger(x) &= P \exp \left( +ig \int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x}) T_a \right) = P \exp \left( +ig \int_{-\infty}^{\tau} dy \alpha^a_y(x) T_a \right), \\
V(x) &= \bar{P} \exp \left( -ig \int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x}) T_a \right) = \bar{P} \exp \left( -ig \int_{-\infty}^{\tau} dy \alpha^a_y(x) T_a \right). \tag{2.3}
\end{align*}
\]

where \( P \) and \( \bar{P} \) indicate respectively path-ordering and anti path-ordering. The first two expressions describe the propagation of quarks and anti-quarks, respectively, through the dense background of the soft gluons, and the last two expressions describe the propagation of gluons. These Wilson lines play a central role in the rest of the paper.

---

1 We shall use upper case letters to denote the adjoint representation and lower case letters to denote the defining representation. In this notation, the generators in the defining representation will be denoted by \( t_a \), and they will be normalized to be \( \text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab} \). The generators in the adjoint representation are denoted by \( T_a \). They are related to the totally antisymmetric structure constants by \( (T_a)_{bc} = if_{bac} \). Hence \( (T_a)_{bc} \) is imaginary and totally antisymmetric in the three indices. Similarly, the quark and anti-quark Wilson lines will be denoted by the lower-case letters \( v^\dagger \) and \( v \), and the gluon Wilson line will be denoted by the upper-case letters \( V^\dagger \) and \( V \). A slight drawback of this convention is that we are forced to denote the dipole amplitude (2.8) by a lower-case letter \( s_r \), whereas the usual notation for it is \( S_r \).
In this representation of soft gluons by a classical background field, gluon distribution is determined by the distribution \( W_\tau(\alpha) \) of the YM potential. \( W_\tau \) depends on \( \tau \) because the number of sources available to emit soft gluons increases at small \( x_F \).

The resulting change of the distribution functional \( W_\tau(\alpha) \) can be shown to satisfy the JIMWLK renormalization group equation \[5\]

\[
\frac{\partial W_\tau(\alpha)}{\partial \tau} = -HW_\tau(\alpha),
\] (2.4)

where

\[
H = \frac{1}{\pi} \int d^2 z d^2 x d^2 y K(xy|z) O(xy|z),
\]

\[
O(xy|z) = \frac{\delta}{\delta \alpha^a_\tau(x)} \left[ V^\dagger(x) - V^\dagger(z) \right]_{ac} \left[ V(y) - V(z) \right]_{cb} \frac{\delta}{\delta \alpha^b_\tau(y)},
\] (2.5)

and

\[
K(xy|z) = \frac{1}{4\pi^3} \frac{(x-z) \cdot (y-z)}{(x-z)^2 (y-z)^2}.
\] (2.6)

A consequence of (2.4) and (2.5) is that the normalization \( \int D[\alpha] W_\tau(\alpha) \) is independent of \( \tau \). We will normalize it to be 1, so that the average of any functional of \( \alpha \) is given simply by \( \langle F \rangle_\tau = \int D[\alpha] F(\alpha) W_\tau(\alpha) \).

The functional derivatives \( \frac{\delta}{\delta \alpha^a_\tau(x)} \) of \( V^\dagger \) and \( V \) in (2.5), and similarly of \( v^\dagger \) and \( v \) that we will encounter later, are

\[
\frac{\delta V^\dagger(z)}{\delta \alpha^a_\tau(x)} = ig T_a V^\dagger(z) \delta(x-z),
\]

\[
\frac{\delta v^\dagger(z)}{\delta \alpha^a_\tau(x)} = ig t_a v^\dagger(z) \delta(x-z),
\]

\[
\frac{\delta V(z)}{\delta \alpha^a_\tau(x)} = -ig V(z) T_a \delta(x-z),
\]

\[
\frac{\delta v(z)}{\delta \alpha^a_\tau(x)} = -ig V(z) t_a \delta(x-z).
\] (2.7)

For calculational simplicity it is useful to note that \( V^\dagger(x) - V^\dagger(z) \) in (2.5) may be put in front of the operator \( \frac{\delta}{\delta \alpha^a_\tau(x)} \). This follows from (2.7) and the observation that \( (T_a)_{ac} = i f_{aac} = 0 \).

A particularly important physical quantity to study is the dipole amplitude \[12\]

\[
s_{\tau}(x, y) = \frac{1}{N_c} \langle \text{tr} \left[ v^\dagger(x)v(y) \right] \rangle_\tau.
\] (2.8)

At coincident points

\[
s_{\tau}(x, x) = 1
\] (2.9)
because of unitarity of $v$. It can be shown form (2.4) that the dipole amplitude satisfies the Balitsky equation \[ 13 \]
\[
\frac{\partial s_{\tau}(x, y)}{\partial \tau} = -\frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \cdot \left\{ s_{\tau}(x, y) - \frac{1}{N_c} \langle [v^\dagger(x)v(z)] tr [v^\dagger(z)v(y)] \rangle_{\tau} \right\}.
\] 
(2.10)

Note that the infrared divergence occuring at large $z$ in (2.5) is absent in the Balitsky equation.

For large $N_c$, the last term factorizes and we arrive at the Kovchegov equation \[ 14 \]
\[
\frac{\partial s_{\tau}(x, y)}{\partial \tau} = -\frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \cdot \left\{ s_{\tau}(x, y) - s_{\tau}(x, z)s_{\tau}(z, y) \right\}.
\] 
(2.11)

Since $s_{\tau}(x, x) = 1$, the quantity $t_{\tau}(x, y) \equiv 1 - s_{\tau}(x, y)$ is expected to be small when $x \sim y$. If we deal with soft gluons outside of the saturation region, when $\alpha(\vec{x})$ is small, the Wilson lines $v$ and $v^\dagger$ are close to 1 anyway, so we expect to be able to drop the quadratic term $t_{\tau}(x, z)t_{\tau}(z, y)$. In this way we get the dipole form of the BFKL equation \[ 2 \]
\[
\frac{\partial t_{\tau}(x, y)}{\partial \tau} = -\frac{\alpha_s N_c}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \cdot \left\{ t_{\tau}(x, y) - t_{\tau}(x, z) - t_{\tau}(z, y) \right\}.
\] 
(2.12)

For $|x - y|^2 \gg 1/Q_s^2$ and inside the saturation region, the strong classical YM potential $\alpha$ causes large and independent oscillations to both Wilson lines $v^\dagger$ and $v$. Consequently the dipole amplitude $s_{\tau}(x - y)$ is expected to be small, thus enabling the non-linear term in (2.11) to be dropped. The solution of the resulting linear equation can be shown to be \[ 8 \]
\[
s_{\tau}(x - y) = \exp \left[ -\frac{\alpha_s N_c}{\pi} \int_{\tau_0}^{\tau} dy \ln(Q_s^2(y)x^2) \right] s_{\tau_0}(x - y).
\] 
(2.13)

If $Q_s^2(y) = \exp[\alpha_s N_c(y - \tau_0)/\pi]Q_s^2(\tau_0)$ \[ 15 \], then
\[
s_{\tau}(x - y) = \exp \left[ -\frac{c}{2} \left( \frac{\alpha_s N_c}{\pi} \right)^2 (\tau - \tau_0)^2 \right] s_{\tau_0}(x - y),
\] 
(2.14)

provided $\tau_0$ is chosen so that $Q_s^2(\tau_0)(x - y)^2 = 1$. This condition implies $(x - y)^2 = Q_s^{-2}(\tau_0) \gg Q_s^{-2}(\tau)$, if $\tau \gg \tau_0$. The solution (2.14) then confirms the expectation that $s_{\tau}$ is small in that region.

To solve any of these evolution equations we need an initial condition at some $\tau = \tau_0$. For a large nucleus, and a $\tau_0$ where the source is dominated by the valence quarks, the
initial condition is provided by the McLerran-Venugopalan model [16], in which a Gaussian
distribution is assumed for $W_\tau(\alpha)$. For large $A$ and small $\alpha_s(Q_s^2)$, the Gaussian
distribution can be shown to be a good approximation [17]. Saturation is now provided by the valence
quarks alone, so $A$ has to be very large and the resulting $Q_s^2$ is relatively small. The detail
of confinement which affects the small $k^2$ region then becomes relatively important [18].

With the Gaussian distribution, the perturbative gluon distribution is modified by a
gluon dipole factor [9]

$$S_{x^-}(r) = \frac{1}{N_c^2 - 1} \langle \text{Tr} U^\dagger(x^-, r) U(x^-, 0) \rangle,$$  \hspace{1cm} (2.15)

where $U^\dagger$ is equal to $V^\dagger$ in (2.3) with $\tau$ replaced by $x^-$, and $r$ is the conjugate variable to the
transverse momentum $k$. This factor gives rise to the non-linear effect that is responsible
for saturation, with a saturation momentum $Q_s$ given by (2.2).

For $\tau > \tau_0$, the $W_\tau(\alpha)$ determined by (2.4) no longer has a Gaussian distribution. Nev-
ertheless, in a mean field approximation, the approximate solution is still Gaussian. The
gluon distribution for small $k^2$ is then given by [9]

$$\frac{d(x_F G)}{d^2k} \simeq c_2 \frac{N_c^2 - 1}{N_c} \frac{\pi R_A^2}{\alpha_s} \ln \frac{Q_s^2(\tau)}{k^2}, \quad (k^2 \ll Q_s^2(\tau))$$  \hspace{1cm} (2.16)

for some constant $c_2$.

### III. WILSON LINE DISTRIBUTION

We shall show later in this section that the distribution $W_\tau(\alpha)$ of YM potential $\alpha$ leads
to a distribution $\overline{W}_\tau(v)$ of the Wilson lines.

The notation might suggest that $\overline{W}_\tau(v)$ describes only the distribution of anti-quark
Wilson line $v$, but actually it provides a distribution for the Wilson lines of other partons
as well. Since $v$ is unitary, $v^\dagger = v^{-1}$, the variable $v^\dagger$ is a rational function of $v_{ij}$, so $\overline{W}_\tau(v)$
does provide the distribution for quark Wilson lines $v^\dagger$.

From the group-theoretical relation

$$t_a V_{ac} \dagger = v^\dagger t_c v = V_{cb} t_b,$$  \hspace{1cm} (3.1)

or equivalently, the relation

$$V_{ac} \dagger = 2\text{tr}(v^\dagger t_c v t_a) = V_{ca},$$  \hspace{1cm} (3.2)

gluon Wilson lines can be expressed as a quark anti-quark pair of Wilson lines, so we can
also compute the distribution of gluon Wilson lines from $\overline{W}_\tau(v)$.

Physical observables are often expressible in terms of Wilson lines, so it is clearly desirable
to know their distributions directly. Moreover, the Wilson line $v$ is unitary, which allows
the theory of representation of the unitary group to be used for computations. For example,
an orthonormal complete set of polynomials, given by the irreducible representations of the
unitary group, exists on the group manifold. Therefore a harmonic analysis of $\overline{W}_\tau(v)$ and
other physical quantities can be carried out to allow their integrals to be computed. In contrast, \( W_\tau(\alpha) \) is a function on a non-compact linear space of the Lie algebra, and the only functional that can be integrated in practice is the Gaussian.

We shall show that \( W_\tau(v) \) satisfies the same JIMWLK equation (2.4), but with \( H \) replaced by \( \overline{H} \). The latter is obtained from \( H \) simply by replacing the \( \alpha \) derivative by a differential operator in \( v \). Therefore, \( \overline{H} \) is still hermitean and positive semi-definite.

Since the group manifold is compact, the spectrum and eigenfunctions of \( H \) are more manageable. For example, \( W_\tau(v) = 1 \) is a normalized eigenfunction of \( H \) with zero eigenvalue, whereas \( W_\tau(\alpha) = 1 \) has a divergent integral in the \( \alpha \) space.

Before showing how \( W_\tau(v) \) is obtained from \( W_\tau(\alpha) \), let us first review some basic facts about integrations and orthonormal relations on a compact Lie group.

A. Inner products on the color group

For the sake of definiteness we shall assume the color group to be \( U(n) \), though a similar analysis can be carried out for \( SU(n) \). The \( n^2 \)-dimensional group \( U(n) \) will be parameterized by the \( n^2 \) matrix elements \( v_{ij} (1 \leq i, j \leq n) \) in the defining representation. Unitarity equates \( v^\dagger \) to \( v^{-1} \), so \( v^*_{ij} \) is to be regarded as a dependent variable, given as a rational function of \( v_{kl} \) via this relation.

The invariant volume element on \( U(n) \) will be denoted by \( d_H[v] \). This Haar measure is a left and right invariant \( n^2 \)-form, satisfying

\[
d_H[v] = d_H[v_0v] = d_H[vv_0]
\]

for any constant \( v_0 \in U(n) \). It is positive, and it shall be normalized to \( \int d_H[v] = 1 \).

The Haar measure \( d_H[v] \) is proportional but not equal to the product measure \( d[v] \), obtained by taking the exterior product of the \( n^2 \) 1-forms \( dv_{ij} \). They differ by a Jacobian \( J(v) \), so

\[
d_H[v] = J(v)d[v].
\]

To get an idea how this comes about, consider a change \( dv \) in the vicinity of a group element \( v \in U(n) \). Then \( v^{-1}dv \) constitutes a change around the identity, so it can be parametrized in the form \(-it_a(d\eta_a)\). The volume element at the identity is proportional to the exterior product of the \( n^2 \) \( d\eta_a \)'s, or equivalently, the exterior product of \( 2itr(v^{-1}dv) \). If we want the volume element to be left and right invariant, as in (3.3), this expression should be taken as the volume element at any point \( v \in U(n) \). The presence of \( v^{-1} \) in this expression is the origin of the Jacobian \( J(v) \) in (3.4). When \( n \) is odd, \( d_H[v] \) has a very simple analytical form, given by (A1) in Appendix A.

Let \( f(v) \) and \( g(v) \) be two functions on the group manifold. Their inner product is defined to be

\[
\langle f(v)|g(v) \rangle = \int d_H[v]f^*(v)g(v).
\]

It will be shown later that the operator \( \overline{D} = v_{ij}(\partial/\partial v_{ij}) \) is hermitean with respect to this inner product. Since \( \overline{D}v_{kl} = v_{kl} \), the eigenfunctions of \( \overline{D} \) are monomials of \( v \), whose eigenvalues
are the degrees of the monomials. The inner product of two eigenfunctions of a hermitean operator is zero if their eigenvalues are different. Hence if $M_k(v)$ denotes a monomial of degree $k$ and $M'_l(v)$ a monomial of degree $l \neq k$, then

$$\langle M_k(v)|M'_l(v) \rangle = 0 \quad (k \neq l). \quad (3.6)$$

This result can be stated in another way. An integral $\int d_H[v] B(v, v^\dagger)$ is non-zero only when the number of $v$’s in $B$ is equal to its number of $v^\dagger$’s. We shall refer to this later as the matching rule. It is one of the main tools for our later calculations.

To compute inner products when $k = l$, we resort to the theory of representation of the $U(n)$ group, which asserts that if $D^\lambda CD(\nu)$ is the matrix element of an irreducible representation $\lambda$, then

$$\int d_H[v] D^\lambda (v) D^\lambda(\nu) = \frac{1}{N(\lambda)} \delta_{\lambda \lambda'} \delta_{AC} \delta_{BD}, \quad (3.7)$$

where $N[\lambda]$ is the dimension of the irreducible representation $\lambda$.

For example, the defining representation $v$ is irreducible and has dimension $N_c$, hence

$$\langle v_{ij}|v_{kl} \rangle = \frac{1}{N_c} \delta_{ik} \delta_{jl}. \quad (3.8)$$

The orthonormal relation (3.7) can be used to compute inner products of any two monomials in the following way. First, apply Young’s idempotent operators of $k$ boxes to decompose $M_k(v)$ into a linear combination of irreducible representations $D^\lambda(v)$. Similarly $M'_l(v)$ is decomposed into a linear combination of $D^\lambda(v)$. Then (3.7) can be used to calculate $\langle M_k(v)|M'_l(v) \rangle$. Clearly it can also be used to calculate integrals $\int d_H[v] B(v, v^\dagger)$.

**B. Measures and functionals on $A$ and $U$**

To apply these properties to the physical problem on hand, we need to generalize them to the case when $v$ depends on the transverse position $x$. We shall denote the color group containing $v(x)$ as $U(n)_x$, and $U \equiv \prod_x U(n)_x$.

The product over $x$ is to be interpreted in the following way. Cover the transverse $x$-plane by a square lattice with a lattice constant $a$. The product $x$ is to be taken over all lattice points within the Lorentz-contracted nucleus of transverse radius $R_A$. The same convention will be applied to sums over $x$.

The measure on $U$ is defined to be

$$D_H[v] = \prod_x d_H[v(x)], \quad (3.9)$$

where $d_H[v(x)]$ is the Haar measure on $U(n)_x$. Using (3.4) and denoting $\prod_x J(v(x))$ by $\mathcal{J}(v)$, we get

$$D_H[v] = \mathcal{J}(v) D[v],$$

$$D[v] \equiv \prod_x d[v(x)]. \quad (3.10)$$
The measure $\mathcal{D}[\alpha]$ on the Lie algebra $\mathcal{A}$ of YM potentials can be defined in the following way. Divide the $y$ axis into intervals of size $\epsilon$. Since we are interested in $\alpha^a_y(x)$ only for $y \leq \tau$, the appropriate measure is

$$\mathcal{D}[\alpha] = \mathcal{D}_\tau[\alpha] \mathcal{D}_\prec[\alpha],$$

(3.11)

where

$$\mathcal{D}_\tau[\alpha] \equiv \prod_{a,x} d\alpha^a(x),$$

$$\mathcal{D}_\prec[\alpha] \equiv c_3 \prod_{y \leq \tau - \epsilon} \prod_{a,x} d\alpha^a_y(x),$$

(3.12)

with a normalization constant $c_3$ to be chosen later. In the same vein, the Wilson line $v(x)$ of (2.3) can be factorized into

$$v(x) = v_\prec(x) v_\tau(x),$$

(3.13)

with

$$v_\prec(x) = \tilde{P} \exp \left( -ig \int_{-\infty}^{\tau - \epsilon} dy \, \alpha^a_y(x) t_a \right),$$

$$v_\tau(x) = \exp \left( -iga \tau^a(x) t_a \right).$$

(3.14)

A change of $\alpha^a_y(x)$ of amount $d\alpha^a_y(x)$ causes a change of $v_\tau(x)$ of amount $dv_\tau(x) = -igev_\tau(x) t_a d\alpha^a_y(x)$, and hence a change in $v(x)$ by the amount $dv(x) = -igev(x) t_a d\alpha^a_y(x)$. Or, $v^{-1}dv(x) = -ige t_a d\alpha^a_y$. In light of the remark below (3.4), we can now choose the constant $c_3$ in (3.12) so that

$$\mathcal{D}_H[v] = \mathcal{D}_\tau[\alpha].$$

(3.15)

We are now ready to discuss how $\mathcal{W}_\tau(v)$ can be obtained from $W_\tau(\alpha)$.

A functional of $\alpha_y(x)$ for $y \leq \tau$ can be folded into a functional of $v(x)$ using the formula

$$\mathcal{F}(v) = \int F(\alpha) \delta(v - u) \frac{1}{\mathcal{J}(u)} \mathcal{D}[\alpha]$$

$$= \int F(\alpha) \delta(v - u) \mathcal{D}[u] \mathcal{D}_\prec[\alpha],$$

(3.16)

where

$$u(x) = \tilde{P} \exp \left( -ig \int_{-\infty}^{\tau} dy \, \alpha^a_y(x) t_a \right),$$

(3.17)

and

$$\delta(v - u) \equiv \prod_x \prod_{i,j=1}^n \delta[v(x)_{ij} - u(x)_{ij}],$$

(3.18)

so that $\int f(u) \delta(v - u) \mathcal{D}[u] = f(v)$ for any functional $f(u)$. The second equality of (3.16) comes from (3.10), (3.12), and (3.15).
It follows from (3.16) that
\[
\int F(v) \mathcal{D}_H[v] = \int \mathcal{D}_H[v] \mathcal{D}[\alpha] \delta(v - u) F[\alpha] / \mathcal{J}(u) \\
= \int F(\alpha) \mathcal{D}[\alpha],
\]
(3.19)
so \(W_\tau(v)\) is normalized if \(W_\tau(\alpha)\) is.

It also follows from (3.16) by an integration by parts that the transform of \(\delta F(\alpha) / \delta \alpha_i(x)\) is \(\hat{D}_a(x) F(v)\), where
\[
\hat{D}_a(x) \equiv igv_{ij}(x)(t_a)_{jk} \frac{\delta}{\delta v_{ik}(x)} = ig \text{tr} \left[ v(x) t_a \frac{\delta}{\delta v^T(x)} \right],
\]
(3.20)
and the functional derivative in \(v\) is defined so that
\[
\frac{\delta v_{pq}(y)}{\delta v_{ij}(x)} = \delta_{pi} \delta_{qj} \delta(x - y).
\]
(3.21)

The inner product between two functionals of \(v\) is defined to be
\[
\langle f(v) | g(v) \rangle = \int f^*(v) g(v) \mathcal{D}_H[v].
\]
(3.22)
It can be shown (see Appendix A) that \(\hat{D}_a(x)\) is anti-hermitean with respect to this inner product. In particular,
\[
\hat{D}_0(x) = igv_{ij}(x)[\delta / \delta v_{ij}(x)] / \sqrt{2N_c}
\]
is anti-hermitean, so the operator \(\mathcal{D}\) defined below eq. (3.10) is hermitean, as previously claimed.

Instead of \(\hat{D}_a(x)\) in (3.20), it is more convenient to deal with the matrix operators \(\mathcal{D}(x)\) and \(\mathcal{D}'(x)\), whose \((mn)\) matrix elements are defined to be
\[
\mathcal{D}_{mn}(x) \equiv \frac{2}{ig} (t_a)_{mn} \hat{D}_a(x) = v_{in}(x) \frac{\delta}{\delta v_{im}(x)},
\]
\[
\mathcal{D}'_{mn}(x) \equiv v_{mi}(x) \frac{\delta}{\delta v_{ni}(x)}.
\]
(3.23)
It can be checked that these two are related by
\[
v(x) \mathcal{D}(x) v^T(x) = \mathcal{D}'(x),
\]
(3.24)
where the subscript \(x\) in \(v^T(x)\) indicates that this \(v^T(x)\) should not be differentiated by the \(\delta / \delta v(x)\) in \(\mathcal{D}(x)\). In other words, in component forms, (3.24) reads \(v_{im} v_{nj}^\dagger \mathcal{D}_{mn} = \mathcal{D}'_{ij}\).

When operated on \(v_{pq}\) and \(v^T_{pq} = v_{pq}^{-1}\), they yield
\[
\mathcal{D}_{mn}(x) v_{pq}(y) = + \delta(x - y) \delta_{mq} v_{pn}(y),
\]
\[
\mathcal{D}_{mn}(x) v^T_{pq}(y) = - \delta(x - y) \delta_{np} v^T_{mq}(y),
\]
\[
\mathcal{D}'_{mn}(x) v_{pq}(y) = + \delta(x - y) \delta_{np} v_{mq}(y),
\]
\[
\mathcal{D}'_{mn}(x) v^T_{pq}(y) = - \delta(x - y) \delta_{mq} v^T_{pn}(y).
\]
(3.25)
These formulas give rise to the following formulas which are very useful in practical calculations. When $D$ or $D'$ operates on a $v$ or $v^\dagger$ in a trace, we have
\begin{align*}
D(x)\text{tr} [Av(y)] &= +\delta(x - y) [Av(y)], \\
D(x)\text{tr} [v^\dagger(y)A] &= -\delta(x - y) [v^\dagger(y)A], \\
D'(x)\text{tr} [v(y)A] &= +\delta(x - y) [v(y)A], \\
D'(x)\text{tr} [Av^\dagger(y)] &= -\delta(x - y) [Av^\dagger(y)].
\end{align*}
(3.26)

In other words, when the trace is written in a certain order, the operators $D(x)$ and $D'(x)$ simply remove the trace, and append to it the factor $\pm\delta(x - y)$. When $D$ or $D'$ operates on a $v$ or $v^\dagger$ in the same trace, we get
\begin{align*}
\text{tr} [AD(x)Bv(y)] &= +\delta(x - y)\text{tr} [A]\text{tr} [Bv(y)], \\
\text{tr} [AD(x)Bv^\dagger(y)] &= -\delta(x - y)\text{tr} [B]\text{tr} [v^\dagger(y)A], \\
\text{tr} [AD'(x)Bv(y)] &= +\delta(x - y)\text{tr} [B]\text{tr} [v(y)A], \\
\text{tr} [AD'(x)Bv^\dagger(y)] &= -\delta(x - y)\text{tr} [A]\text{tr} [Bv^\dagger(y)].
\end{align*}
(3.27)

In other words, the single trace is broken up into a product of two traces. The matrices $A, B$ in these equations are constant matrices.

C. $W_\tau(v)$ and its JIMWLK equation

Using (3.10), the distribution function $W_\tau(\alpha)$ can be folded into the distribution function $\overline{W}_\tau(v)$ of Wilson lines. Since $\delta/\delta\alpha_\tau(x)$ is transformed into $\widehat{D}_a(x)$ of (3.20), the JIMWLK equation (2.7) for $W_\tau(\alpha)$ is now changed into a JIMWLK equation for $\overline{W}_\tau(v)$:
\begin{equation}
\frac{\partial \overline{W}_\tau(v)}{\partial \tau} = -\overline{H} \overline{W}_\tau(v),
\end{equation}
(3.28)
where
\begin{align*}
\overline{H} &= \frac{1}{\pi} \int d^2z d^2xd^2y K(xy|z)\overline{\mathcal{O}}(xy|z), \\
\overline{\mathcal{O}}(xy|z) &= \widehat{D}_a(x) \left(V^\dagger(x) - V^\dagger(z)\right)_{ac} (V(y) - V(z))_{cb} \widehat{D}_b(y).
\end{align*}
(3.29)

Like $H$, $\overline{H}$ is also hermitean and positive semi-definite.

Using (3.23) and (3.24), the operator $\overline{\mathcal{O}}(xy|z)$ can be written in a form more convenient for practical calculations. Remember for this purpose the remark after (2.11), that the factor $[V^\dagger(x) - V^\dagger(z)]_{ac}$ in (2.3) can be written to the left of the differential operator $\delta/\delta\alpha_\tau(x)$. In terms of (3.29), this means to the left of $\widehat{D}_a(x)$. In what follows we shall use $\overline{\mathcal{O}}(xy|z)$ of (3.29) in this form.

Using (3.23), (3.24) and (3.2), and this remark above, we have
\[ V_{cb}(z)\bar{D}_b(y) = ig \text{tr} \left( v^\dagger(z)t_c v(z) \mathcal{D}(y) \right), \]
\[ \hat{D}_a(x)V^\dagger(z) = ig \text{tr} \left( v^\dagger(z)t_c v(z) \mathcal{D}(x) \right), \]
\[ V_{cb}(y)\bar{D}_b(y) = ig \text{tr} \left( v^\dagger(y)t_c v(y) \mathcal{D}(y) \right) = ig \text{tr} \left( t_c \mathcal{D}'(y) \right), \]
\[ \hat{D}_a(x)V^\dagger(x) = ig \text{tr} \left( v^\dagger(x)t_c v(x) \mathcal{D}(x) \right) = ig \text{tr} \left( t_c \mathcal{D}'(x) \right). \quad (3.30) \]

From these relations, and the identity \( \text{tr}(t_c A) \text{tr}(t_c B) = \frac{1}{2} \text{tr}(AB) \), we get
\[
\overline{\mathcal{O}}(xy|z) = \overline{\mathcal{O}}_{xy} + \overline{\mathcal{O}}_{xz} + \overline{\mathcal{O}}_{zy} + \overline{\mathcal{O}}_{zz},
\]
\[
\overline{\mathcal{O}}_{xy} = -\frac{1}{2} g^2 \text{tr} [\mathcal{D}'(x)\mathcal{D}'(y)],
\]
\[
\overline{\mathcal{O}}_{xz} = +\frac{1}{2} g^2 \text{tr} \left[ \mathcal{D}'(x)v(z)\mathcal{D}(y)v_\dagger(z) \right],
\]
\[
\overline{\mathcal{O}}_{zy} = +\frac{1}{2} g^2 \text{tr} \left[ v(z)\mathcal{D}(x)v_\dagger(z)\mathcal{D}'(y) \right],
\]
\[
\overline{\mathcal{O}}_{zz} = -\frac{1}{2} g^2 \text{tr} \left[ v(z)\mathcal{D}(x)v_\dagger(z)v(z)\mathcal{D}(y)v_\dagger(z) \right]
= -\frac{1}{2} g^2 \text{tr} [\mathcal{D}(x)\mathcal{D}(y)]. \quad (3.31)
\]

Assuming \( \overline{\mathcal{W}}_\tau(v) \) to be normalized, \( \int \mathcal{D}_H[v] \overline{\mathcal{W}}_\tau(v) = 1 \), the average of any functional \( B(v, v^\dagger) \) is equal to
\[
\langle B \rangle_\tau = \int \mathcal{D}_H[v] B(v, v^\dagger) \overline{\mathcal{W}}_\tau(v). \quad (3.32)
\]

If \( B \) and \( \overline{\mathcal{W}}_\tau \) are both monomial functionals of \( v \) and \( v^\dagger \), this functional integral factorizes into a product of integrals on the group \( U(n) \), each of which can be computed using (3.6) and (3.7). In particular, the functional integral is non-zero only when the number of \( v \)'s and \( v^\dagger \)'s in \( B \overline{\mathcal{W}}_\tau \) are the same for every transverse position \( x \). This is the functional form of the matching rule previously considered.

**IV. MUELLER’S DERIVATION OF THE JIMWLK EQUATION**

In a recent paper [6], Mueller proposed a simple derivation of the JIMWLK equation in the following way. He showed that if \( W_\tau(\alpha) \) is equal to the dipole functional \( v^\dagger(s)_{ij} v(t)_{kl} \), then the Feynman diagrams for \( \partial W_\tau(\alpha) / \partial \tau \) can be written in the form \( -HW_\tau(\alpha) \), with \( H \) given by (2.5). He then stated that the same is true if \( W_\tau(\alpha) \) is equal to any multipole functional \( v^\dagger(s_1)_{i_1j_1} v^\dagger(s_2)_{i_2j_2} \cdots v(t_1)_{k_1l_1} v(t_2)_{k_2l_2} \cdots \), and hence it is likely that the JIMWLK equation for \( W_\tau(\alpha) \) is also valid.

In this section we shall supply the missing steps of this proof. This consists of filling in the detailed arguments for the multiple functionals, and then showing that they lead to the JIMWLK equation for \( \overline{\mathcal{W}}_\tau(v) \). We do not know whether the JIMWLK equation for \( W_\tau(\alpha) \) follows or not. However, for all our applications, a JIMWLK equation for \( \overline{\mathcal{W}}_\tau(v) \) is sufficient, so it really does not matter whether the equation for \( W_\tau(\alpha) \) can be proven this way or not.
Instead of starting from the Feynman diagrams to derive the evolution equation, as is done in Ref. [6], we find it easier to do everything in reverse. That is, we start from the JIMWLK equation and show that they lead to the correct set of Feynman graphs. This inverse approach makes it more manageable to deal with the complicated multipole functionals. Actually, simplification already occurs at the dipole level: a necessary cancelation in the original derivation is avoided altogether in this way.

The Wilson line $v^\dagger(s)$ for a quark is drawn in Fig. 1 as a left-pointing arrow, and the Wilson line $v(t)$ for an anti-quark is drawn as a right-pointing arrow. Time $x^-$ is drawn to increase from right to left; multiplication of color matrices from left to right should be carried out against the arrow of the fermion lines.

The short vertical bars in the middle of the lines (labelled IP) indicate the lightcone longitudinal position $x^-=0$ where interaction between the multipole and the pancake nucleus takes place. Since $\alpha(\vec{x})$ is concentrated around $x^-=0$, we may regard $v$ and $v^\dagger$ to be located at the IP.

The operation of $\hat{D}_{a}^\dagger(x)$ or $\delta/\delta\alpha^b_{\tau}(x)$ on the Wilson lines is given in (2.7). This operation can be represented graphically by putting a vertex to the left (the larger $x^-$ side) of IP, both for the quark Wilson line $v^\dagger$ and the anti-quark Wilson line $v$. The vertex for a quark is $igt_{a}$, and the vertex for an anti-quark is $-igt_{a}$.

Using the remark following eq. (2.7), the operator $O(xyz)$ in (2.5) can be written in the form

$$O = \left[V^\dagger(x) - V^\dagger(z)\right]_{ac} \frac{\delta}{\delta\alpha^{a}_{\tau}(x)} \cdot \left[V(y) - V(z)\right]_{cb} \frac{\delta}{\delta\alpha^{b}_{\tau}(y)}$$

$$\equiv O_{xy} + O_{xz} + O_{zy} + O_{zz}.$$ (4.1)

Using (3.1), and the unitarity relation $vv^\dagger = v^\dagger v = 1$, we see that the effect of $[V(y)]_{cb} (\delta/\delta\alpha^{b}_{\tau}(y))$, operating on a Wilson line, is to place a vertex to the right (the small $x^-$ side) of the IP on the Wilson line. Similarly, the effect of $[V^\dagger(x)]_{ac} (\delta/\delta\alpha^{a}_{\tau}(x))$, operating on a Wilson line, is also to place a vertex to the right (the small $x^-$ side) of the IP on the Wilson line.

We are now ready to see what happens when

$$H = \int d^2zd^2xd^2y K(xy|z) (O_{xy} + O_{xz} + O_{zy} + O_{zz})$$

$$\equiv H_{xy} + H_{xz} + H_{zy} + H_{zz}.$$ (4.2)

operates on a multiple functional, i.e., a collection of $p$ quark and $q$ anti-quark Wilson lines. $H_{xy}$ puts a vertex $y$ to the right of the IP on a Wilson line, and vertex $x$ also to the right of the IP of the same or a different Wilson line. These two vertices are then connected by the ‘gluon propagator’ $\int d^2zK(xy|z)\delta_{ab}$, where $a,b$ are the color indices at the two vertices. The gluon propagator is shown in Fig. 1(a) with a dashed line. This operation is to be applied to every pair of Wilson lines, including the possibility of applying to the same line twice.

Similarly, since $V^\dagger(z)V(z) = 1$, the effect of $H_{zz}$ is to put a vertex $y$ to the left of the IP of a Wilson line, and another vertex $x$ to the left of the IP of the same or another Wilson
The two vertices are then linked by the same ‘gluon propagator’. This is shown in Fig. 1(b).

In both of these cases, the two vertices are both to the same side of the IP. This is not the case with the other two terms.

\( H_{xz} \) puts a vertex \( y \) with color \( b \) to the left of the IP, and a vertex \( x \) with color \( a \) to the right of the IP. These two vertices are linked by a ‘gluon propagator’ \( \int d^2z \; [V(z)]_{ab} K(xy|z) \). This is shown in Fig. 1(c).

There is potentially another contribution to \( H_{xz} \) when \( V^\dagger_{ac}(x) \delta/\delta \alpha_a(x) \) operates on \( V_{ab}(z) \). However, this term is proportional to \( \delta(x - z) V_{ac}(x) V_{cd}(z) (T_a)_{ab} \), which is proportional to \( (T_a)_{ab} = 0 \), so that term is actually absent.

Similarly, \( H_{zy} \) puts a vertex \( y \) of color \( b \) to the right of the IP, and a vertex \( x \) with color \( a \) to the left of the IP. These two vertices are linked by the ‘gluon propagator’ \( \int d^2z \; \left[V^\dagger(z)\right]_{ab} K(xy|z) \). This is shown in Fig. 1(d).

Appropriate signs and Dirac \( \delta \)-functions on the transverse coordinates must also be inserted.

One might be bothered that the gluon propagators in the four diagrams appear to be different. Fortunately this is only superficial. To see why they are actually the same, first note that \( T_a \) is imaginary so \( V \) is real. Hence the gluon Wilson line \( [V^\dagger(z)]_{ab} \) in the propagator in Fig. 1(d) can be written as \( [V(z)]_{ab} \). If we compare this with that of Fig. 1(c), we see that these two are the same, both equal to \( [V(z)]_{a_1a_2} \), where \( a_1 \) is the color index before the interaction point IP, and \( a_2 \) is the color index after. Since \( \alpha^a(x^-, z) \) is concentrated near \( x^- = 0 \), we may write, in both cases, the gluon Wilson line to be

\[
[V(z)]_{a_1a_2} = \hat{P} \exp \left[ -ig \int_{x^-_1}^{x^-_2} dx^- \alpha^a(x^-, z) T_a \right],
\]

where \((a_1, x^-_1)\) is the interaction vertex to the right of IP, and \((a_2, x^-_2)\) is the one to the left of IP. For the first two diagrams, we may replace \( \delta_{ab} \) by the same expression \((4.3)\), because in that case both \( x^-_1 \) and \( x^-_2 \) are to the same side of IP, hence \( \alpha^a(x^-, z) = 0 \) throughout the integration interval, so \( [V(z)]_{ab} = \delta_{ab} \).

These four types of Feynman graphs are precisely those needed for the development \( \partial W_\tau(\alpha)/\partial \tau \) \([9]\). Therefore the JIMWLK equation is satisfied whenever \( W_\tau(\alpha) \) is given by a multipole function, namely, a monomial functional of \( v \) and \( v^\dagger \). Since polynomial functionals on the group manifold form an orthonormal complete set, the JIMWLK equation \((3.28)\) must be true in general. This completes Mueller’s proof for \( W_\tau(v) \). However, since \( W_\tau(\alpha) \) depends on many more variables \( \alpha_a(x^-, x) \) than \( v(x) \), it does not necessarily follow from this argument that \((2.4)\) for \( W_\tau(\alpha) \) is true.

V. INFRARED DIVERGENCE AND THE GENERALIZED BK EQUATION

The kernel \( K(xy|z) \) of the JIMWLK equation, given in \((2.6)\), goes like 1/\( z^2 \) for large \( |z| \). This causes a log divergence in \( HW_\tau(\alpha) \) of \((2.3)\), and \( \overline{H} W_\tau(v) \) of \((3.29)\). On the other hand, the dipole amplitude \((6.1)\) satisfies the Balitsky equation \((2.10)\), whose kernel goes like 1/(\( z^2 \))^2 for large \( |z| \), so infrared divergence is absent in that case. In this section, we
shall use the Feynman diagrams derived in the last section to show that the multipole traces defined below are equally free of infrared divergence.

*Multipole traces* are defined by

\[
m(s_1t_1 \cdots s_k t_k) \equiv \frac{1}{N_c} \text{tr} \left[ v^\dagger(s_1)v(t_1) \cdots v^\dagger(s_k)v(t_k) \right].
\]

(5.1)

We will call the functional average of a multiple trace, \( \langle m_k(s_1t_1 \cdots s_k t_k) \rangle \), a *multiple amplitude*. When \( k = 1 \), this reduces to the dipole amplitude \( (3.28) \).

The diagram for a multiple trace is shown in Fig. 2. As far as color-matrix multiplication is concerned, these \( 2k \) lines should be considered to be joined together at their ends to form a single big loop with the arrows all pointing the same way. The joints are indicated by dotted lines in the figure. Note that if the amplitude did not have the form displayed in (5.1), with \( v^\dagger \) and \( v \) occuring alternately inside a trace, such a big loop cannot be formed and the argument below will not be valid.

Consider the two gluon propagators shown in Fig. 2. The color structures are identical, but the two terms differ by a sign because the vertex B is on a quark line in one diagram and an anti-quark line in another diagram. Their gluon propagators may therefore be combined into \( K(xx|z) - K(xy|z) \). For large \(|z|\), this is proportional to \( \mathbf{z} \cdot (\mathbf{x} - \mathbf{y})/(z^2)^2 = O(1/|z|^3) \), hence the infrared divergence is absent. As a matter of fact, we can also combine diagrams with vertex A similarly shifted. The four \( K \) thus combined actually dies down like \( 1/(z^2)^2 \) for large \(|z|\).

It is clear that all the diagrams for the multipole function can be paired up in a way similar to Fig. 2, thus eliminating all infrared divergences. Moreover, since the combination is obtained by combining two diagrams with vertex B in different positions, but the same vertex A, finiteness persists for products of multipole traces.

In the rest of this section, we shall derive a generalization of the Balitsky-Kovchegov (BK) equation to multipole amplitudes. We will see that the kernel of the equation actually behaves like \( 1/(z^2)^2 \) for large \(|z|\), just like the kernel for the Balitsky equation.

Since \( \Pi \) is hermitean, the JIMWLK equation satisfied by \( \Pi_r(v) \) is also satisfied by the multipole trace \( (5.1) \). We may now use \( (3.20) \), \( (3.27) \), \( (3.28) \) and \( (3.31) \) to simplify \( \Pi m(s_1t_1 \cdots s_k t_k) \). The resulting equation is

\[
\frac{\partial m(s_1t_1 \cdots s_k t_k)}{\partial \tau} = - \left[ H_{xy} + H_{zx} + H_{xz} + H_{yz} \right] m(s_1t_1 \cdots s_k t_k),
\]

(5.2)

where

\[
H_{xy}m(s_1 \cdots t_k) = + \frac{1}{2} g^2 N_c \sum_{i,j=1}^{k} \int d^2z I_{ij}(z)m^{aa}_{ij}(s_1 \cdots t_k)
\]

\[
H_{zx}m(s_1 \cdots t_k) = + \frac{1}{2} g^2 N_c \sum_{i,j=1}^{k} \int d^2z I_{ij}(z)m^{bb}_{ij}(s_1 \cdots t_k)
\]

\[
H_{xz}m(s_1 \cdots t_k) = - \frac{1}{2} g^2 N_c \sum_{i,j=1}^{k} \int d^2z I_{ij}(z)m^{ab}_{ij}(s_1 \cdots t_k)
\]

\[
H_{yz}m(s_1 \cdots t_k) = - \frac{1}{2} g^2 N_c \sum_{i,j=1}^{k} \int d^2z I_{ij}(z)m^{ba}_{ij}(s_1 \cdots t_k).
\]

(5.3)
The kernel in these equations is

\[ I_{ij}(z) = K(s_i s_j|z) - K(s_i t_j|z) - K(s_j t_i|z) + K(t_i t_j|z), \quad (5.4) \]

which is \( O(1/(z^2)^2) \) for large \( |z| \). It is also symmetric in \( i \) and \( j \): \( I_{ij}(z) = I_{ji}(z) \). The meaning of the multipole traces in (5.2) will now be explained.

The argument \( (s_1 \cdots t_k) \) of the multiple trace in (5.1) is circular, because the trace is. We can consider the argument \( s_1 \) to be behind the argument \( t_k \), forming a circle. For example,

\[ m(s_1 t_1 \cdots s_k t_k) = m(t_1 s_2 \cdots t_k s_1) = \frac{1}{N_c} \text{tr} \left[ v(t_1) v^\dagger(s_2) \cdots v(t_k) v^\dagger(s_1) \right]. \quad (5.5) \]

In these formulas, \( v \) is always associated with a \( t_i \) and \( v^\dagger \) is always associated with a \( s_i \).

The quantity \( m_{ij}^{ab}(s_1 \cdots t_k) \) appearing in the \( H_{xy} \) term in (5.3) is defined as follows. Put a vertical bar after \( s_i \), and another vertical bar after \( s_j \) in the circular argument \( (s_1 \cdots t_k) \). This pair of vertical bars separates the circular argument into two circular arguments. \( m_{ij}^{ab}(s_1 \cdots t_k) \) is defined to be the product of two multiple traces with these two circular arguments. For example,

\[ m_{13}^{ab}(s_1 t_1 s_2 t_2 s_3 t_3) \equiv m(s_1 t_1 s_2 t_2 s_3 t_3) = m(s_1 s_2 s_3) m(t_1 t_2 t_3). \quad (5.6) \]

The superscript \( a \) stands for 'after'. The superscript \( b \) to be found in the other three terms of (5.3) stands for 'before'. Each superscript together with its corresponding subscript pair up to tell us where the vertical bar is put. The pair \( (ai) \) means to put a vertical bar after \( s_i \). The pair \( (bj) \) tells us to put a vertical bar before \( s_j \). With this understanding the quantity \( m_{ij}^{ab}(s_1 \cdots t_k) \) can be defined similarly. For example,

\[ m_{13}^{ab}(s_1 t_1 s_2 t_2 s_3 t_3) \equiv m(s_1 t_1 s_2 t_2 s_3 t_3) = m(s_1 t_1 s_2 t_2) m(s_3 t_3). \quad (5.7) \]

If we apply these recipes literally to the last two equations in (5.3) we will end up with something non-sensical, in that the numbers of \( v \)'s and \( v^\dagger \)'s within each trace is not identical, thus neither is a multipole trace. What we should do in these two cases is to insert a \( v(z) \) into one trace and a \( v^\dagger(z) \) into another so that both become multiple functionals. For example,

\[ m_{13}^{ab}(s_1 t_1 s_2 t_2 s_3 t_3) \equiv m(s_1 t_1 s_2 t_2 s_3 t_3) = m(t_1 t_2 t_3) m(z_3 s_3 s_1). \quad (5.8) \]

The \( z \) in the first factor is a \( v^\dagger(z) \), and that in the second factor is a \( v(z) \).

This completes the explanation of the symbols in (5.3), except for one last remark. If the two vertical bars are side by side, then the corresponding multiple trace \( m \) should be interpreted as \( \text{tr}(1)/N_c = 1 \).

When we take expectation values on both sides of (5.2), we get an equation for the multiple amplitudes. For large \( N_c \), the expectation of products of traces factorizes into products of expectations of a trace. In this form (5.2) and (5.3) generalize the Balitsky-Kovchegov (BK) equation to higher multiple amplitudes. We should interpret \( m \) in these equations as the average \( \langle m \rangle \), and products of two \( m \)'s as the product of the averages.

In the special case of a dipole, \( k = 1 \), hence \( i = j = 1 \) in (5.3). In that case,
\[ n_{11}^a = m(s_1|t_1) = m(s_1t_1), \]
\[ m_{11}^b = m(|s_1t_1) = m(s_1t_1), \]
\[ m_{11}^a = m_{11}^b = m(|s_1t_1) = m(s_1z)m(zt_1), \]
\[ I_{11}(z) = \frac{1}{4\pi^3} \frac{(s_1 - t_1)^2}{(s_1 - z)^2(t_1 - z)^2}. \] (5.9)

Equation (5.2) coincides with the BK equation (2.11), as it should.

A very interesting fact emerges from these generalized BK equations for multipole amplitudes. If the multipole amplitudes for \( k = 1, 2, \ldots, \ell - 1 \) are known, then the evolution equation determining the \( \ell \)th multipole is a linear inhomogeneous equation. Thus the only non-linear equation one has to solve is the original BK equation for the dipole amplitude. It is well known that the non-linearity of this equation leads to saturation. The linearity of the higher multipole amplitudes therefore means saturation of the dipole amplitude automatically drives saturation of all higher multipole amplitudes.

If we require the solution of the JIMWLK equation (3.28) to be free of infrared singularities, then presumably it will be made out of the multiple traces (5.1) and their products. In that case the difficult functional differential JIMWLK equation can be replaced by the more manageable set of generalized BK equations (5.2) and (5.3).

**VI. SOLUTION OF THE JIMWLK EQUATION**

Suppose we decompose \( \overline{W}_\tau(v) \) into a linear combination of eigenfunctions \( \overline{\phi}_\lambda(v) \) of \( \overline{H} \). The operator \( \overline{H} \) is hermitean and positive semi-definite, hence the eigenvalues \( \lambda \) are real and non-negative. From (3.28), the \( \tau \) dependence of \( \overline{W}_\tau(v) \) is thus given by a linear combination of \( \exp(-\lambda \tau) \). In the asymptotic limit \( \tau \to \infty \), the lowest eigenvalue of \( \overline{H} \) dominates.

The lowest eigenvalue is \( \lambda = 0 \), and its normalized eigenfunction is \( \overline{\phi}_0(v) = 1 \). This eigenfunction is normalized because \( \int \mathcal{D}[v] = \prod_x \int \mathcal{D}[v(x)] = 1 \).

In contrast, \( \phi_0(\alpha) = 1 \) is also an eigenfunction of \( H \) with \( \lambda = 0 \), but the integral of this eigenfunction is divergent because the linear space \( \mathcal{A} \) is non-compact.

In the next subsection, we will discuss what happens in the asymptotic limit \( \tau \to \infty \) when \( \lambda = 0 \) dominates. In the subsequent subsection, we will look at \( \overline{W}_\tau(v) \) and its applications for a smaller \( \tau \).

**A. Asymptotic limit**

When \( \tau = \infty \), only \( \overline{\phi}_0(v) \) contributes, so we can set \( \overline{W}_\infty(v) = \overline{\phi}_0(v) = 1 \). Averages are then given by the integral \( \langle B(v, v^\dagger) \rangle = \int \mathcal{D}[v]B(v, v^\dagger) \). From the matching rule, this integral is non-zero only when the number of \( v \)'s in \( B \) exactly matches the number of \( v^\dagger \)'s at every transverse position \( x \).

For example, the dipole amplitude (2.8) is

\[ s_{\tau}(x, y) = \frac{1}{N_c} \langle \text{tr} [v^\dagger(x)v(y)] \rangle = \delta_{x,y} \] (6.1)
because \( v_x^t v_x = 1 \) and \( \text{tr}(1) = N_c \). This is consistent with \((2.3)\), and also \((2.14)\) at \( \tau = \infty \). In other words, whenever the dipole has a finite size, the dense gluon will have such a strong absorption that the dipole amplitude always become zero.

A similar statement can be made about multipole amplitudes.

Let us now compute the gluon spectrum at \( \tau = \infty \) by using the formula \[ (6.2) \]

\[
\frac{d(x_{\rho} G)}{d^2 k} = \frac{1}{4\pi^3} \langle F_{a}^{\pm i}(\vec{k}) F_{a}^{\pm i}(-\vec{k}) \rangle = \frac{1}{4\pi^3 N_c} \langle \text{Tr} \left[ F_{a}^{\pm i}(\vec{k}) F_{a}^{\pm i}(-\vec{k}) \right] \rangle,
\]

where \( \vec{k} = (k^+, \mathbf{k}) \), and

\[
F_{a}^{\pm i}(-\vec{k}) = \int d^3 x \exp(ik \cdot x) F_{a}^{\pm i}(\vec{x})
\]

is the color electric field in the lightcone gauge. In the approximation \( F_{a}^{\pm i}(\vec{x}) \simeq (i/g) \delta(x^-) V(x) \partial^i V^+(x) \) which is supposed to be valid for a Lorentz-contracted pancake nucleon, the formula becomes

\[
\frac{d(x_{\rho} G)}{d^2 k} = -\frac{1}{4\pi^3 g^2 N_c} \int d^2 x d^2 y \exp[i(k \cdot (x - y))] \langle C(x, y) \rangle,
\]

\[
C(x, y) = \text{Tr} \left[ V(x) V_{i}^{\dagger}(x) V(y) V_{i}^{\dagger}(y) \right],
\]

where the subscript \( i \) represents a differentiation, namely, \( V_{i}^{\dagger} \equiv \partial_{i} V^{\dagger} \).

To compute the average \( \langle C(x, y) \rangle = \int D_{H}[v] B(x, y) \) we need to use \((3.2)\) to convert the gluon Wilson lines \( V \) and \( V^{\dagger} \) to the quark and anti-quark Wilson lines \( v^{\dagger} \) and \( v \). The result is

\[
C(x, y) = 2C_{\rho} \text{tr} \left[ v(y) v_{i}^{\dagger}(y) v(x) v_{i}^{\dagger}(x) + v_{i}(y) v_{i}^{\dagger}(y) v_{i}(x) v_{i}^{\dagger}(x) \right]
\]

\[
+ \text{tr} \left[ v_{i}(y) v_{i}^{\dagger}(y) \right] \text{tr} \left[ v(x) v_{i}^{\dagger}(x) \right]
\]

\[
+ \text{tr} \left[ v(y) v_{i}^{\dagger}(y) \right] \text{tr} \left[ v_{i}(x) v_{i}^{\dagger}(x) \right].
\]

The functional integral \( \langle C \rangle = \int D_{H}[v] C \) is computed in Appendix B. The result is

\[
\langle C(x, y) \rangle = -2N_{c}^{2} - 1 \delta_{x, y}.
\]

Changing the integral in \((6.4)\) into a finite sum, and letting \( r = x - y \), we get

\[
\frac{d(x_{\rho} G)}{d^2 k} = \frac{\pi R_{A}^{2}}{4\pi^3 g^2 N_c} \sum_{r} a^2 \exp(i k \cdot r) \frac{2(N_{c}^{2} - 1)}{a^2} \delta_{r,0}
\]

\[
= \frac{2\pi R_{A}^{2}(N_{c}^{2} - 1)}{16\pi^3 \alpha_{s} N_c}.
\]

The unintegrated spectrum \( d(x_{\rho} G)/d^2 k^2 \) is therefore absolutely flat, up to the saturation momentum \( Q_{s}^{2} \) which is in this case infinite. That is not unexpected at \( \tau = \infty \).

The integrated cross section is then

\[
x_{\rho} G = Q_{s}^{2} \pi R_{A}^{2}(N_{c}^{2} - 1) \frac{1}{8\pi^3 \alpha_{s} N_c}.
\]

The spectrum in \((6.7)\) differs from the mean field prediction \((2.9)\), in that \((6.7)\) is flat and \((2.14)\) has a logarithmic dependence on \( k^2 \). The integrated density \( x_{\rho} G \) is however quite similar to the estimate given in \((2.2)\); the only difference is a factor \( c = 8\pi^3 \).
B. Below the asymptotic limit

It is much more difficult to solve the JIMWLK equation for finite $\tau$, because we know nothing about the other eigenfunctions and eigenvalues of $\mathcal{H}$. In this section, we will discuss an approximate correction to $\mathcal{W}_{\tau}(v)$ below the asymptotic limit, in a region where the Wilson lines are far apart.

To avoid the infrared divergence, the distribution functional $\mathcal{W}_{\tau}(v)$ will be assumed to depend on $v$ and $v^\dagger$ only through the multipole traces (5.1), or products of them. From results of the last subsection, and discussions in Sec. 2, we know that for large $\tau$, the average of multipole traces (i.e., multipole amplitudes) are small if the Wilson lines in the multipoles are far apart. In fact, the higher the order of the multipole is, the smaller the amplitude is expected to be. Therefore it is reasonable to include only quadratic dependences of $v$ and $v^\dagger$ in a first correction to the asymptotic limit, at least in the region when the Wilson lines are far apart. We will therefore assume

$$\mathcal{W}_{\tau}(v) = 1 + \sum_{s,t} \text{tr} \left[ v^\dagger(s)v(t) \right] b_{\tau}(t,s). \quad (6.9)$$

The contribution $\text{tr}[v^\dagger(s)v(s)]b_{\tau}(s,s) = N_cb_{\tau}(s,s)$ may be absorbed into the $v$-independent term. We may therefore assume $b_{\tau}(s,s) = 0$ without any loss of generality. In that case, using the matching rule and unitarity of $v$ to do the functional integral, we see that $\mathcal{W}_{\tau}(v)$ is still normalized:

$$\int D_H[v] \mathcal{W}_{\tau}(v) = 1. \quad (6.10)$$

To compute the dipole amplitude (2.8), we need the following integration formula, which can be obtained from the matching rule, eq. (3.8), and the unitarity of $v$:

$$\int D_H[v] \text{tr} \left[ v^\dagger(s)v(t) \right] \left[ v^\dagger(x)v(y) \right] = N_c^2 \delta_{s,t} \delta_{x,y} + \delta_{s,y} \delta_{t,x} - \delta_{s,y,t,x}. \quad (6.11)$$

The last Kronecker $\delta$ is by definition non-zero only when the four arguments in its subscript are all equal.

We may now compute the dipole amplitude from (2.8) to be

$$s_{\tau}(x,y) = \frac{1}{N_c} \int D_H[v] \mathcal{W}_{\tau}(v) \text{tr} \left[ v^\dagger(x)v(y) \right] = \left[ \delta_{x,y} + \frac{1}{N_c} b_{\tau}(x,y) \right]. \quad (6.12)$$

Since $b_{\tau}(x,x) = 0$, we get $s_{\tau}(x,x) = 1$, as it should be. If we stay away from $x = y$, then $b_{\tau}(x,y)/N_c$ is just the dipole amplitude $s_{\tau}(x,y)$. As such it should satisfy the BK equation. On the other hand, we should be able to get the equation of $b_{\tau}(x,y)$ directly from the JIMWLK equation (3.28) by using (6.9), (3.31), and (3.25). The left hand side is proportional to $\text{tr} \left[ v^\dagger(s)v(t) \right]$, but the right hand side has two terms, one is proportional to

20
tr \[v^\dagger(s)v(t)\] , and the other it proportional to \(\text{tr}\left[v^\dagger(z)v(t)\right]\text{tr}\left[v^\dagger(s)v(z)\right]\). If we drop this quartic term because it involves a higher order multipole function which is expected to be small, then \(b_\tau(x,y)/N_c = s_\tau(x,y)\) simply satisfies the BK equation (2.11) with the quadratic term of \(s_\tau\) dropped. This is justified when the dipole amplitude is small, which is the case when the two Wilson lines are far apart, as expected. The solution is given by (2.13).

We may use (6.9) to calculate higher-order amplitude. The result is a sum of \(b_\tau/N_c = s_\tau\), one for each dipole pairs inside the multipole.

\textbf{VII. CONCLUSION}

Density of soft gluons is determined by the distribution \(W_\tau(\alpha)\) of the classical Yang-Mills potential \(\alpha(\vec{x})\). Interaction of fast partons with such a background is given by their Wilson lines. In this paper, we introduced the distribution \(\overline{W}_\tau(v)\) of Wilson lines. It can be obtained from \(W_\tau(\alpha)\), and it also satisfies the JIMWLK equation. We completed Mueller’s derivation of the JIMWLK equation, though for \(\overline{W}_\tau(v)\) and not for \(W_\tau(\alpha)\). We derived a generalized BK equation for multipole amplitudes. We also used the normalizable property of \(\overline{W}_\tau(v)\) to compute the properties of physical observables at \(x_F = 0\). We obtained in this way an unintegrated gluon spectrum \(d(x_F G)/dk^2 = \pi R_A^2(N_c^2 - 1)/8\pi^3\alpha_s N_c\), independent of its transverse momentum \(k^2\) of the gluon. Correction to this asymptotic behavior was briefly discussed.

\textbf{ACKNOWLEDGMENTS}

This work of CSL is supported partly by the Natural Sciences and Engineering Research Council of Canada, and the Fonds de recherche sur la nature et les technologies of Québec. The work of WZ is supported by the National Natural Science Foundation of China, Grant Nos. 10075020 and 90103013. He wants to express his gratitude to the hospitality of McGill University where this work is carried out.

\textbf{APPENDIX A: MATHEMATICAL DETAILS}

The Haar measure of \(U(n)\) for odd \(n\) is given by

\[d_H[v] = c_4 \text{tr} \left[v^{-1}dv \wedge v^{-1}dv \wedge \cdots \wedge v^{-1}dv\right],\]  

(A1)

where \(c_4\) is a normalization constant determined by \(\int d_H[v] = 1\), and the argument of \(\text{tr}\) consists of the exterior product of \(n^2\) factors of \(v^{-1}dv\). From the cyclic nature of the trace, and the antisymmetry character of the exterior product, it can be seen that \(d_H[v]\) defined this way is 0 when \(n\) is even. Hence we must confine ourselves to an odd \(n\) of we want to use the expression (A1).

It is easy to check that (3.3) is satisfied for (A1).

We will show in two ways that \(\hat{D}_\alpha(x)\) defined in (3.21) is anti-hermitean. First, define the inner product of two functionals \(\alpha\) by

\[\alpha^\dagger = c_4 \text{tr} \left[\alpha^{-1}d\alpha \wedge \cdots \wedge \alpha^{-1}d\alpha\right].\]  

21
\[ \langle f\vert g \rangle_\alpha = \int \mathcal{D}[\alpha] \mathcal{D}[\alpha'] f^*(\alpha) g(\alpha') \delta (u - u') / \mathcal{J}(u), \]  

(A2)

where \( u \) is given by (3.17) and \( u' \) is given similarly in terms of \( \alpha' \). From (3.10) and (3.13), we see that

\[ \delta(u - u') / \mathcal{J}(u) = \prod_{a,x} \delta(\alpha_\tau(x) - \alpha'_\tau(x)). \]  

(A3)

Using integration by parts and assuming the resulting surface term to be zero, it is easy to see that \( \delta/\delta \alpha_\tau^2(\tau, y) \) is anti-hermitean with respect to this inner product.

If \( \overline{f}(v) \) and \( \overline{g}(v) \) are obtained respectively from \( f(\alpha) \) and \( g(\alpha) \) by (3.10), then the inner product between \( \overline{f}(v) \) and \( \overline{g}(v) \) defined by

\[ \langle \overline{f} \vert \overline{g} \rangle_v \equiv \int \mathcal{D}_H[v] \overline{f}(v) g(v) \]  

is equal to the inner product \( \langle f \vert g \rangle_\alpha \) defined in (A2). This is so because

\[ \langle \overline{f} \vert \overline{g} \rangle_v = \int \mathcal{D}_H[v] \mathcal{D}[\alpha] \mathcal{D}[\alpha'] f^*(\alpha) \delta(v - u) g(\alpha) \delta(v - u') / \mathcal{J}(u) \mathcal{J}(u') \]

\[ = \int \mathcal{D}[\alpha] \mathcal{D}[\alpha'] \delta(u - u') f^*(\alpha) g(\alpha') / \mathcal{J}(u) = \langle f \vert g \rangle_\alpha. \]  

(A5)

We know that the transform of \( \delta f(\alpha) / \delta \alpha_\tau(\tau, y) \) is \( \hat{D}_a(y) \). Since \( \langle \overline{f} \vert \overline{g} \rangle_v = \langle f \vert g \rangle_\alpha \) and since \( \delta / \delta \alpha_\tau^2(y) \) is anti-hermitean with respect to \( \langle f \vert g \rangle_\alpha \), it follows that \( \hat{D}_a(y) \) must also be anti-hermitean with respect to \( \langle \overline{f} \vert \overline{g} \rangle_v \).

The second proof of the anti-hermiticity of \( \hat{D}_a(y) \) makes use of the explicit formula (A1). It proceeds as follows.

Using the explicit formula (3.20) for \( \hat{D}_a(y) \), we must show that \( \overline{D}_a(y) \equiv v_{ij}(y) (t_a)_{jk} \delta / \delta v_{ik}(y) \) is hermitean with respect to the inner product (A4).

Let us first show that \( \overline{D}_a(y) \) is imaginary. Since \( v^t = v^{-1} \), it follows that

\[ \frac{\delta}{\delta v_{ik}} = -v_{is} v_{rk} \frac{\delta}{\delta v_{sr}}. \]  

(A6)

Using also the fact that \( t_a \) is hermitean, then

\[ \overline{D}_a = v_{ij}^* (t_a^*)_{jk} \frac{\delta}{\delta v_{ik}} = -(v t_a)_{rs} \frac{\delta}{\delta v_{rs}} = -\overline{D}_a. \]  

(A7)

Now we use integration by parts to show that \( \overline{D}_a \) is antisymmetric with respect to the inner product \( \langle f \vert g \rangle_v \). Remembering (3.10), integration by parts changes \( \overline{D}_a \) into

\[ -\frac{1}{\mathcal{J}(v)} \frac{\delta}{\delta v_{ik}} (vt_a)_{ik} \mathcal{J}(v). \]  

(A8)

This would indeed be equal to \( -\overline{D}_a \) if

\[ \frac{\delta}{\delta v_{ik}} [(vt_a)_{ik} \mathcal{J}(v)] = 0. \]  

(A9)
Since $\delta (d[v]) / \delta v_{ik} = d [\delta v / \delta v_{ik}] = 0$, (A3) is true if

$$\frac{\delta [(v_t)_ik D_H[v]]}{\delta v_{ik}} = 0,$$

which in turn is true if

$$n tr(t_a) d_H[v] + (v_t)_ik \frac{\partial d_H[v]}{\partial v_{ik}} = 0.$$

(A11)

Using (A1), we get

$$(v_t)_ik \frac{\partial d_H[v]}{\partial v_{ik}} = -n^2 tr \left[ t_a v^{-1} dv \wedge \cdots \wedge v^{-1} dv \right],$$

namely, it is equal to $-n^2$ times a $t_a$ inserted into the measure in (A1). If $a = 0$, then $t_0$ is proportional to the identity matrix, so indeed (A11) is true. For $a > 0$, $tr(t_a) = 0$. Since $v^{-1} dv$ is a linear combination of the $U(n)$ generators $t_b$, we conclude that (A11) is valid if

$$\epsilon_{b_1b_2\cdots b_n} tr \left[ t_a t_{b_1} \cdots t_{b_n} \right] = 0.$$

(A13)

This is indeed the case because $tr(t_a) = 0$ and $s \equiv \epsilon_{b_1b_2\cdots b_n} t_a t_{b_1} \cdots t_{b_n}$ is proportional to the identity matrix. This last statement can be proven as follows.

Let $v_0 \in U(n)$. Then $v_0^{-1} t_a v_0 = (V_0)_{ab} t_b$, where $V_0$ is the adjoint representation of $v_0$. Hence $v_0^{-1} sv_0 = det(V_0) s$. Since the adjoint generator $T_a$ is has no diagonal matrix elements, it is traceless, so $det(V_0) = 1$. Therefore $s$ commutes with every element $v_0$ of $U(n)$, so by Schur’s lemma, it must be proportional to the identity matrix.

We have thus shown that $D_a(y)$ is antisymmetric and imaginary. Hence it is hermitean.

**APPENDIX B: SATURATION CALCULATION**

We want to compute $\langle C(xy) \rangle = \int D_H[v] C(x,y)$ for the function $C(xy)$ given in (B.3). This function contains four terms. We will label them consecutively as $C_1, C_2, C_3,$ and $C_4$.

We interpret the derivative $\partial_i v(x) = v_i(x)$ on the lattice to be

$$v_i(x) = \frac{1}{2a} [v(x + a_i) - v(x - a_i)],$$

(B1)

where $a_i$ is a lattice vector along the $i$th direction. $v_i(x)$ is defined similarly. The following computations make use of (B.6), $C_F = N_c/2$ for $U(n)$, the matching rule discussed at the end of Sec. 3, and unitarity of $v$. There are also extra factors of 2 obtained by summing the two $a_i$ directions. In this way we get

$$\langle C_1 \rangle = \frac{2C_F}{4a^2} \langle \text{tr} \left[ v(y) \left( v^\dagger(y + a_i) - v^\dagger(y - a_i) \right) \right] \rangle v(x) \left( v^\dagger(x + a_i) - v^\dagger(x - a_i) \right)$$

$$\langle C_2 \rangle = \frac{2C_F}{4a^2} \langle \text{tr} \left[ v(y) \left( v^\dagger(y + a_i) - v^\dagger(y - a_i) \right) \right] \rangle v(x) \left( v^\dagger(x - a_i) - v^\dagger(x + a_i) \right)$$

$$\langle C_3 \rangle = \frac{2C_F}{4a^2} \langle \text{tr} \left[ v(y) \left( v^\dagger(y + a_i) - v^\dagger(y - a_i) \right) \right] \rangle v(x) \left( v^\dagger(x - a_i) - v^\dagger(x + a_i) \right)$$

$$\langle C_4 \rangle = \frac{2C_F}{4a^2} \langle \text{tr} \left[ v(y) \left( v^\dagger(y + a_i) - v^\dagger(y - a_i) \right) \right] \rangle v(x) \left( v^\dagger(x + a_i) - v^\dagger(x - a_i) \right)$$

23
\[
\begin{align*}
&= -\frac{2C_FN_c}{4a^2} [\delta_{x,y-a_i} + \delta_{x,y+a_i}] \simeq -\frac{2C_FN_c}{a^2} \delta_{x,y} \\
&= -\frac{N_c^2}{a^2} \delta_{x,y}, \\
\langle C_2 \rangle &= \frac{2C_F}{4a^2} \langle \text{tr} \left[ (v(y + a_i) - v(y - a_i)) v^\dagger(y) \right. \\
&\left. \left( v(x + a_i) - v(x - a_i) \right) v^\dagger(x) \right] \rangle \\
&= \langle C_1 \rangle, \\
\langle C_3 \rangle &= \frac{1}{4a^2} \text{tr} \left[ (v(y + a_i) - v(y - a_i)) v^\dagger(y) \right] \\
&\text{tr} \left[ v(x) \left( v^\dagger(x + a_i) - v^\dagger(x - a_i) \right) \right] \\
&= \frac{1}{a^2} \delta_{x,y}, \\
\langle C_4 \rangle &= \frac{1}{4a^2} \text{tr} \left[ v(y) \left( v^\dagger(y + a_i) - v^\dagger(y - a_i) \right) \right] \\
&\text{tr} \left[ (v(x + a_i) - v(x - a_i)) v^\dagger(x) \right] \\
&= \frac{1}{a^2} \delta_{x,y}. \\
\end{align*}
\]

Thus

\[
\langle C(x, y) \rangle = \langle C_1 + C_2 + C_3 + C_4 \rangle = -2\frac{N_c^2 - 1}{a^2} \delta_{x,y}.
\]
REFERENCES

[1] G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977); V.N. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15, 438, 675 (1972); Yu.L. Dokshitzer, Sov. Phys. JETP. 46, 641 (1977).
[2] V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett. B60, 50 (1975); E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov. Phys. JETP 44, 443 (1976); ibid. 45, 199 (1977); Ya.Ya. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978).
[3] L.V. Gribov, E.M. Levin and M.G. Ryskin, Phys. Rep. 100, 1 (1983).
[4] M.A. Betemps, M.B. Gay Ducati and M.V.T. Machado, hep-ph/0206218; J. Raufeisen, J.C. Peng and G.C. Nayak, hep-ph/0204093; J. Schaffner-Bielich, D. Kharzeev, L. McLerran and R. Venugopalan, hep-ph/0108043; K. Golec-Biernat and M. Wüsthoff, Phys. Rev. D60, 114023 (1999) (hep/ph 9903358); K. Golec-Biernat and M. Wüsthoff, Phys. Rev. D59, 014017 (1998); F. Gelis and A. Peshier, Nucl. Phys. A697, 879 (2002) (hep-ph/0107142); F. Gelis and J. Jalilian-Marian, hep-ph/0205037; A. Dumitru and J. Jalilian-Marian, hep-ph/0204025; R. Baier, A.H. Mueller, D. Schiff and D.T. Son, hep-ph/0204211; K.J. Eskola, K. Kajantie, P.V. Ruuskanen and K. Tuominen, hep-ph/0204034. J. Jalilian-Marian and S. Jeon, Phys. Rev. C65, 055201 (2002) (hep-ph/0110417); S. Munier, A.M. Stasto and A.H. Mueller, Nucl. Phys. B603, 427 (2001) (hep-ph/0102291); L. McLerran and J. Schaffner-Bielich, Phys. Lett. B514, 29 (2001) (hep-ph/0101133); A.H. Mueller and G.P. Salam, Nucl. Phys. B475, 293 (1996).
[5] J. Jalilian-Marian, A. Kovner, L. McLerran and H. Weigert, Phys. Rev. D55, 5414 (1997); J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Nucl. Phys. 503, 415; Phys. Rev. D59, 014014; H. Weigert, hep-ph/0004044; E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. A692, 583 (2001); Phys. Lett. B510, 133 (2001).
[6] A.H. Mueller, Phys. Lett. B525, 243 (2001). (hep-ph/0110169)
[7] J-P. Blaizot, E. Iancu, and H. Weigert, hep-ph/0206279.
[8] For a review, see A.H. Mueller, hep-ph/0111244, and references therein, as well as
[9] E. Iancu, A. Leonidov and L. McLerran, hep-ph/0202270, and references therein.
[10] E. Iancu, hep-ph/0206073.
[11] E. Fereiro, E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. A703, 489 (2002) (hep-ph/0109113).
[12] A.H. Mueller, Nucl. Phys. B335, 115 (1990); Yu. V. Kovchegov and A.H. Mueller, Nucl. Phys. B529, 451 (1998); A.H. Mueller, Nucl. Phys. A654, 37c (1999) (hep-ph/9911289).
[13] I. Balitsky, Nucl. Phys. B463, 99 (1996); hep-ph/0101042; I.I. Balitsky and A.V. Balitsky, Nucl. Phys. B629, 290 (2002) (hep-ph/0110158).
[14] Yu. Kovchegov, Phys. Rev. D60, 034008 (1999); Phys. Rev. D 61 074018 (2000).
[15] A. Mueller, Nucl. Phys. 558, 285 (1999) (hep-ph/9904404); E. Iancu and L. McLerran, Phys. Lett. B510, 145 (2001).
[16] L. McLerran and R. Venugopalan, Phys. Rev. D49, 2233 (1994); Phys. Rev. D49, 3352 (1994); Phys. Rev. D50, 2225 (1994).
[17] C.S. Lam and G. Mahlon, Phys. Rev. D64, 016004 (2001) (hep-ph/0102337).
[18] C.S. Lam and G. Mahlon, Phys. Rev. D61, 014005 (2000) (hep-ph/9907281); Phys. Rev. D62, 114023 (2001) (hep-ph/0007138).
FIG. 1. Diagrams representing the result of $H = H_{xy} + H_{zz} + H_{xz} + H_{zy}$, or $\mathcal{H} = \mathcal{H}_{xy} + \mathcal{H}_{zz} + \mathcal{H}_{xz} + \mathcal{H}_{zy}$, operating on a multipole functional with 2 anti-quark Wilson lines $v^\dagger$ (left-pointing arrows) and three quark Wilson lines $v$ (right-pointing arrows). The dashed lines are the ‘gluon propagators’. See the text in Sec. 4. for further explanation.
FIG. 2. A multiple trace with $k = 3$. The two diagrams have identical color structure, so their two ‘gluon propagators’ can be combined into an infrared finite expression.