Explosive particle creation by instantaneous change of boundary condition

Umpei Miyamoto
RECCS, Akita Prefectural University, Akita 015-0055, Japan
umpei@akita-pu.ac.jp

Abstract

Particle creation by time-dependent boundary conditions is of special importance in variety of physics involving quantum fields. In this paper, we consider an idealized but fundamental system, in which the boundary condition in a finite or semi-infinite cavity for a 1 + 1 dimensional free massless scalar field instantaneously changes from Neumann to Dirichlet or reversely. Then, we estimate the vacuum expectation value of energy-momentum tensor for the quantized scalar field to obtain it in analytic form for every case considered. It is revealed that two kind of non-renormalizable diverging flux can emanate from the point where the boundary condition changes, one of which has been overlooked in past studies. Such a thunderbolt-like flux would destroy or destabilize the system if its back-reaction is taken into account. The result in this paper can be a guideline in that most models of particle creation by time-dependent boundary conditions should reproduce it when the models have an instantaneous limit.

Contents

1 Introduction 2

2 Finite cavity I 3
   2.1 Quantization of massless scalar field ................................................. 3
   2.2 Particle creation by instantaneous change of boundary condition ................. 5
      2.2.1 From Neumann to Dirichlet .................................................. 5
      2.2.2 From Dirichlet to Neumann .................................................. 9

3 Finite cavity II: Revisit Ishibashi-Hosoya [5] 11
   3.1 Quantization of massless scalar field ................................................. 12
   3.2 Particle creation by instantaneous change of boundary condition: From Neumann-Neumann to Dirichlet-Dirichlet .................................................. 13
   3.3 Origin of discrepancy .............................................................. 15

4 Semi-infinite cavity 16
   4.1 Quantization of massless scalar field ................................................. 16
   4.2 Particle creation by instantaneous change of boundary condition ................. 17
      4.2.1 From Neumann to Dirichlet .................................................. 17
      4.2.2 From Dirichlet to Neumann .................................................. 20

5 Conclusion 21
1 Introduction

Particle creation plays important roles in many contexts of gravitational and high-energy physics. For example, the particle creation during the gravitational collapse leading to black holes and the inflation of early universe play crucial roles in the evolution of spacetime, and completely changed our classical picture of spacetime [1]. Besides the particle creation by dynamical spacetimes, that by time-dependent boundary conditions and topology changes of spacetime or medium is also expected to play important roles. For example, the topology change of spacetime and of strings, on which quantum fields live, is expected to happen in quantum gravity and string theory, resulting in intensive particle creation [2, 3, 4, 5].

Although the above examples are ones in high-energy physics, it should be stressed that the particle creation by time-dependent boundary condition is expected to happen in low-energy physics [6], and indeed had been realized for the quantum electromagnetic field in a laboratory experiment [7].

No one will disagree that more radical change of boundary condition, more particles are excited from the vacuum generically. And, it is worth asking what happens in the extreme of rapid change of boundary condition, namely, what happens when the change of boundary condition occurs instantaneously. To ask so is not only interesting but also important in that most models of particle creation by the change of boundary conditions have to reproduce the result of instantaneous-change model in the limit of infinitely rapid change, whenever such a limit exists in the models.

Thus, in this paper, we suppose the extreme situation in which the boundary condition for a massless scalar field in a one-dimensional (1D) finite or semi-infinite cavity changes instantaneously, i.e., without an intermediate interval of time, from Neumann to Dirichlet or reversely. Under this assumption, we quantize the scalar field and estimate the vacuum expectation value of its energy-momentum tensor. Thanks to the idealization of physical situation, the analysis enjoys the use of various mathematical formulas mainly regarding summation and integral, which enables us to obtain the results in completely analytic form for every case. It is added that there is no adjustable parameter in our model.

As the results, we find that a thunderbolt-like null diverging flux emanates from the world point at which the boundary condition changes. Furthermore, we reveal that there are two components in the
diverging flux. Interestingly, there is a kind of asymmetry between the Neumann-to-Dirichlet (N-D) case and Dirichlet-to-Neumann (D-N) case. Namely, the diverging energy flux in the N-D case consists of two components, while the flux in the D-N case consists of only one. In any case, the emergence of such a diverging flux suggests that the back-reaction of the quantum field to the background spacetime and cavity cannot be ignored. The back-reaction can destroy the system or prohibit the change of boundary condition.

Here, we introduce briefly past related works. Anderson and DeWitt examined the particle creation by the fission of a 1 + 1 universe, in which the spatial topology changes from $S^1$ to $S^1 + S^1$ [3]. They argued that a diverging flux appears at the world point of fission, though detailed calculation was not presented (see also [4]). A work most relevant to the present work is that by Ishibashi and Hosoya [5], who estimated the particle creation due to the instantaneous change of boundary condition from Neumann to Dirichlet at the both ends of a finite cavity. Although their position that they regard the change of boundary condition as the appearance of a strong naked singularity [8] is different from ours, the essential part of their computation seems the same as ours. Nevertheless, there seems a discrepancy in result between [5] and the present work. Therefore, we will revisit the analysis of [5] in Sec. 3 to look for the origin of discrepancy. The particle creation by the rapid appearance and/or disappearance of a wall in a 1D finite cavity were studied in [9] and [10]. In particular, the system with the instantaneous appearance and disappearance of a Dirichlet wall studied in [10] is more complex than but similar to the system in Sec. 2 of the present paper.

The organization of this paper is as follows. In Sec. 2, we investigate the particle creation due to the instantaneous change of boundary condition in a finite 1D cavity, for the N-D case (Sec. 2.2.1) and the D-N case (Sec. 2.2.2). The origin of discrepancy between the result in Sec. 2 and Ref. [5] is clarified in Sec. 3. In Sec. 4, the case of semi-infinite cavity is analyzed. We conclude in Sec. 5. The proof of consistency between different quantizations, called the unitarity relations, and some integration formulas are presented in Appendices A and C, respectively. The result for the semi-infinite cavity in Sec. 4 is reproduced in Appendix D with the Green-function method, which naturally involves the regularization of the vacuum expectation value of energy-momentum tensor. We work in the natural units in which $c = h = 1$.

2 Finite cavity I

2.1 Quantization of massless scalar field

We consider a free massless scalar field in a 1D cavity of which length is $L$,

$$(-\partial_t^2 + \partial_x^2)\phi(t, x) = 0, \quad -\infty < t < \infty, \quad 0 < x < L.$$  \hspace{1cm} (1)

At the right boundary $x = L$, we assume the homogeneous Dirichlet boundary condition all the time,

$$\phi(t, L) = 0, \quad -\infty < t < \infty.$$  \hspace{1cm} (2)

At the left boundary $x = 0$, we consider two kinds of boundary conditions. One is the homogeneous Neumann boundary condition,

$$\partial_x \phi(t, 0) = 0.$$  \hspace{1cm} (3)
Another is the Dirichlet boundary condition,
\[ \phi(t, 0) = 0. \]  

During boundary conditions (2) and (3) are imposed, a natural set of positive-energy mode functions \( \{ f_n \} \) is given by
\[ f_n(t, x) = \sqrt{\frac{2}{n\pi}} e^{-ip_n t} \cos(p_n x), \quad p_n := \frac{n\pi}{2L}, \quad n = 1, 3, 5, \ldots. \]  

In the rest of this paper, we suppose that \( n \) and \( n' \) entirely denote odd natural numbers, otherwise denoted. The above mode functions satisfy the following orthonormal conditions,
\[ \langle f_n, f_{n'} \rangle = -\langle f_n^*, f_{n'}^* \rangle = \delta_{nn'}, \quad \langle f_n, f_{n'}^* \rangle = 0, \]  
where the asterisk denotes the complex conjugate and \( \langle , \rangle \) denotes the Klein-Gordon inner product [1],
\[ \langle \phi, \psi \rangle := i \int_0^L (\phi^* \partial_t \psi - \partial_t \phi^* \psi) dx. \]  

During boundary conditions (2) and (4) are imposed, a natural set of positive-energy mode functions \( \{ g_m \} \) is given by
\[ g_m(t, x) = \frac{1}{\sqrt{m\pi}} e^{-iq_m t} \sin(q_m x), \quad q_m := \frac{m\pi}{L}, \quad m = 1, 2, 3, \ldots. \]  

In the rest of this paper, we suppose that \( m \) and \( m' \) entirely denote natural numbers, otherwise denoted. The above mode functions satisfy the following orthonormal conditions,
\[ \langle g_m, g_{m'} \rangle = -\langle g_m^*, g_{m'}^* \rangle = \delta_{mm'}, \quad \langle g_m, g_{m'}^* \rangle = 0. \]  

Associated with the above two sets of mode function, \( \{ f_n \} \) and \( \{ g_m \} \), there are two ways to quantize the scalar field. One is to expand the scalar field by \( f_n \),
\[ \phi = \sum_{n=1}^{\infty} (a_n f_n + a_n^\dagger f_n^*), \]  
and impose the commutation relations,
\[ [a_n, a_{n'}^\dagger] = \delta_{nn'}, \quad [a_n^\dagger, a_{n'}] = 0. \]  
By imposing the above commutation relations, the following equal-time canonical commutation relation is realized,
\[ [\phi(t, x), \partial_t \phi(t, x')] = i \delta(x - x'). \]  
Then, \( a_n \) and \( a_n^\dagger \) are interpreted as the annihilation and creation operators, respectively. The vacuum state in which no particle corresponding to mode function \( f_n \) exists is defined by
\[ a_n |0_f \rangle = 0, \quad n = 1, 3, 5, \ldots, \quad \langle 0_f |0_f \rangle = 1. \]
Another is to expand the field by \( g_m \),
\[
\phi = \sum_{m=1}^{\infty} (b_m g_m + b_m^\dagger g_m^*) ,
\]
and impose the commutation relations,
\[
[b_m, b_{m'}^\dagger] = \delta_{mm'}, \quad [b_m, b_m^\dagger] = 0.
\]
The vacuum state in which no particle corresponding to \( g_m \) exists is defined by
\[
b_m |0_g\rangle = 0, \quad m = 1, 2, 3 \cdots , \quad \langle 0_g |0_g\rangle = 1.
\]
Later, we will estimate the vacuum expectation value of energy-momentum tensor for the scalar field. The energy-momentum tensor operator is written as
\[
T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial \phi)^2 ,
\]
where \( \eta_{\mu\nu} = \text{Diag.}(−1, 1) \) is the 1 + 1 dimensional flat metric. Introducing double null coordinates, non-zero components of this tensor are
\[
T_{\pm\pm} = (\partial_{\pm} \phi)^2 , \quad z_\pm := t \pm x.
\]
Note that the energy density and momentum density in the original Cartesian coordinates are \( T^{tt} = T_{--} + T_{++} \) and \( T^{tx} = T_{--} - T_{++} \), respectively.

2.2 Particle creation by instantaneous change of boundary condition

Given the above quantization schemes, we investigate how the vacuum is excited when the boundary condition at left boundary \( x = 0 \) is instantaneously, say at \( t = 0 \), changed from Neumann to Dirichlet (Sec. 2.2.1) and reversely (Sec. 2.2.2).

2.2.1 From Neumann to Dirichlet

First, we assume that the boundary condition at \( x = 0 \) is Neumann for \( t < 0 \) and Dirichlet for \( t > 0 \), and that the quantum field is in vacuum \( |0_f\rangle \) in the Heisenberg picture. See Fig. 1 for a schematic picture of this situation. Then, we investigate how the vacuum is excited due to the change of boundary condition by computing the spectrum and energy flux of created particles.

Let us expand \( f_n \) by \( g_m \),
\[
f_n = \sum_{m=1}^{\infty} (\alpha_{nm} g_m + \beta_{nm} g_m^*),
\]
where the expansion coefficients, called the Bogoliubov coefficients, are computed by
\[
\alpha_{nm} = \langle g_m, f_n \rangle , \quad \beta_{nm} = -\langle g_m^* , f_n \rangle .
\]
Using the explicit form of mode functions and , we obtain
\[
\alpha_{nm} = \frac{2}{(2m - n)\pi} \sqrt{\frac{2m}{n}} , \quad \beta_{nm} = \frac{2}{(2m + n)\pi} \sqrt{\frac{2m}{n}} .
\]
Substituting Eq. (18) into Eq. (10), and comparing it with Eq. (14), we obtain

\[ b_m = \sum_{n=1}^{\infty} \left( \alpha_{nm} a_n + \beta_{nm}^* a_n^\dagger \right). \]  

(21)

Substituting Eq. (21) into Eq. (15) and using Eq. (11), we obtain

\[ \sum_{n=1}^{\infty} \left( \alpha_{nm} \alpha_{nm'}^* - \beta_{nm}^* \beta_{nm'} \right) = \delta_{mm'}, \quad \sum_{n=1}^{\infty} \left( \alpha_{nm} \beta_{nm'}^* - \beta_{nm}^* \alpha_{nm'} \right) = 0, \]

(22)

which should be satisfied for the two quantizations, Eqs. (10) and (14), to be consistent. In Appendix A.1 these consistency conditions, which we call unitarity relations, are shown to be satisfied by Bogoliubov coefficients (20).

The spectrum of created particles is given by the vacuum expectation value of number operator \( b_n^\dagger b_m \),

\[ \langle 0_f | b_m^\dagger b_m | 0_f \rangle = \sum_{n=1}^{\infty} |\beta_{nm}|^2 = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{m}{n(n + 2m)^2}. \]

(23)

Note that this is finite but its summation over \( m \), the total number of created particles, is divergent. This implies that the Fock-space representation associated with \( a_n \) is unitarily inequivalent to that associated with \( b_m \) [11].

The vacuum expectation value of energy-momentum tensor before the change of boundary condition at \( t = 0 \) is computed by substituting Eq. (10) into Eq. (17), and using Eqs. (11), (13), and (5) as

\[ \langle 0_f | T_{\pm \pm} | 0_f \rangle_{t<0} = \sum_{n=1}^{\infty} |\partial_{\pm} f_n|^2 = \frac{\pi}{8L^2} \sum_{n=1}^{\infty} n. \]

(24)
This represents the Casimir energy density \cite{[12]}, which can be made finite with standard regularization schemes \cite{[1]}.

The most interesting quantity is the vacuum expectation value of energy-momentum tensor after \( t = 0 \). Substituting Eq. (14) into Eq. (17) and using Eq. (21), we obtain

\[
\langle 0 | T_{\pm \pm} | 0 \rangle_{t>0} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} ((\alpha_{nm} \beta_{nm'} + \alpha_{nm'} \beta_{nm}) \text{Re}(\partial_{\pm} g_m \partial_{\pm} g_{m'})) + (\alpha_{nm} \alpha_{nm'} + \beta_{nm} \beta_{nm'}) \text{Re}(\partial_{\pm} g_m \partial_{\pm} g_{m'})].
\]

(25)

To derive Eq. (25), we symmetrize it with respect to dummy indices \( m \) and \( m' \), and use the fact that \( \alpha_{nm} \) and \( \beta_{nm} \) are real. Using the explicit expressions of Bogoliubov coefficients (20) and mode function (8), we obtain

\[
\langle 0 | T_{\pm \pm} | 0 \rangle_{t>0} = \frac{1}{2\pi L^2} \sum_{n=1, n: \text{odd}}^{\infty} \left( \frac{1}{4n} \left[ 4 \sum_{m=1}^{\infty} \cos(q_m z_{\pm}) + n^2 \sum_{m=1}^{\infty} \frac{\cos(q_m z_{\pm})}{m^2 - (n/2)^2} \right]^2 + n \sum_{m=1}^{\infty} \frac{m \sin(q_m z_{\pm})}{m^2 - (n/2)^2} \right).
\]

(26)

This is an even function of \( z_{\pm} \) with period \( 2L \) since it is invariant under reflection \( z_{\pm} \rightarrow -z_{\pm} \) and translation \( z_{\pm} \rightarrow z_{\pm} + 2L \). Therefore, it is sufficient to calculate it in \( 0 \leq z_{\pm} < 2L \), and then generalize the obtained expression appropriately to one valid in the entire domain.

The first and second summations over \( m \) in Eq. (26) can be computed to give

\[
\langle 0 | T_{\pm \pm} | 0 \rangle_{t>0} = \frac{1}{2\pi L^2} \sum_{n=1, n: \text{odd}}^{\infty} \left( \frac{1}{4n} \left[ 16L^2 \delta^2(z_{\pm}) + n^2 \sum_{m=1}^{\infty} m \sin(q_m z_{\pm}) \right] + n \sum_{m=1}^{\infty} \frac{m \sin(q_m z_{\pm})}{m^2 - (n/2)^2} \right)
\]

(27)

which is valid in \( 0 \leq z_{\pm} < 2L \), using the following formulas,

\[
\sum_{k=1}^{\infty} \cos \left( \frac{2k \pi}{a} y \right) = -\frac{1}{2} + a \sum_{\ell=-\infty}^{\infty} \delta(y - \ell a), \quad (-\infty < y < \infty),
\]

(28)

\[
\sum_{k=1}^{\infty} \frac{\cos ky}{k^2 - a^2} = -\frac{\pi}{2a} \cos[a(\pi - y)] \csc(a\pi) + \frac{1}{2a^2}, \quad (0 \leq y \leq 2\pi).
\]

(29)

See Ref. \cite{[13], p. 730} for the second formula.

For \( z_{\pm} = 0 \), from Eq. (27), we have

\[
\langle 0 | T_{\pm \pm} | 0 \rangle_{t>0} = \frac{1}{\pi} \sum_{n=1, n: \text{odd}}^{\infty} \frac{\delta^2(0)}{n}, \quad (z_{\pm} = 0).
\]

(30)

For \( 0 < z_{\pm} < 2L \), the rest summation over \( m \) in Eq. (27) can be computed to give

\[
\langle 0 | T_{\pm \pm} | 0 \rangle_{t>0} = \frac{\pi}{8L^2} \sum_{n=1, n: \text{odd}}^{\infty} n, \quad (0 < z_{\pm} < 2L),
\]

(31)
Figure 2: Vacuum expectation values of energy density \( \langle 0_f | (T_{--} + T_{++}) | 0_f \rangle_{t > 0} \) (left) and momentum density \( \langle 0_f | (T_{--} - T_{++}) | 0_f \rangle_{t > 0} \) (right) with cutoff, from which the uniform Casimir contribution is subtracted. We set \( L = 1 \) and summation over modes in Eq. (26) is taken up to \( n = m = 13 \). The exact results without cutoff are given by Eq. (34).

Using the following formula [13, p. 730],

\[
\sum_{k=1}^{\infty} \frac{k \sin ky}{k^2 - a^2} = \frac{\pi}{2} \sin[a(\pi - y)] \csc(a\pi), \quad (0 < y < 2\pi).
\] (32)

Combining Eqs. (30) and (31), we obtain

\[
\langle 0_f | T_{\pm \pm} | 0_f \rangle_{t > 0} = 2 \frac{\pi}{L^2} \sum_{n=1}^{\infty} \frac{\delta^2 (z_{\pm})}{n} + \begin{cases} 
0 & (z_{\pm} = 0) \\
\frac{\pi}{8L^2} \sum_{n=1}^{\infty} n & (0 < z_{\pm} < 2L) 
\end{cases}.
\] (33)

This is the expression for \( 0 \leq z_{\pm} < 2L \), what we wanted to know. Extending the domain of Eq. (33), we obtain

\[
\langle 0_f | T_{\pm \pm} | 0_f \rangle_{t > 0} = 2 \frac{\pi}{L^2} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell = -\infty}^{\infty} \delta^2 (z_{\pm} - 2\ell L) + \begin{cases} 
0 & (z_{\pm} = 2\ell L, \ell \in \mathbb{Z}) \\
\frac{\pi}{8L^2} \sum_{n=1}^{\infty} n & (\text{otherwise}) 
\end{cases}.
\] (34)

Let us consider the meaning of two terms in Eq. (34). The first term, the delta function squared multiplied by the logarithmically divergent series, represents the diverging flux emanating from the origin \((t, x) = (0, 0)\) and localizing on the null lines (Fig. 2). The dependence of energy density on the delta function squared implies also the divergence of total energy emitted. This component of flux is similar to that predicted in the topology change of 1D universe [3] and the same as that predicted in the formation of a strong naked singularity [5].

The second term, at first glance, seems to represent the ambient Casimir energy just like Eq. (24), which is negative and finite after a regularization, and its vanishing on the null lines. As will be
explicitly shown in the semi-infinite cavity case (Sec. 4 and Appendix B), however, this is not the case. A regularization corresponds to subtracting the spatially uniform diverging energy density due to the zero-point oscillation. If one subtracts such a uniform diverging quantity from Eq. (34), leading to the regularization of ambient Casimir term, a divergence appears on the null lines $z_{\pm} = 2\ell L$ ($\ell \in \mathbb{Z}$).

As far as the present author knows, this kind of flux component was first found in the particle creation due to the instantaneous appearance of Dirichlet wall in a cavity [10]. It was confirmed in the same paper that such a divergence appears in the instantaneous limit of smooth formation of a Dirichlet wall in cavity analyzed in [9].

It is suspicious that the second kind of flux component does not appear in the analysis of Ishibashi and Hosoya [5], since their system is quite similar to the present one. Thus, we will revisit their analysis in Sec. 3 and see that the component was just overlooked.

2.2.2 From Dirichlet to Neumann

We assume that the boundary condition at $x = 0$ is Dirichlet (4) for $t < 0$ and Neumann (3) for $t > 0$, and that the quantum field is in vacuum $\mid 0_g \rangle$. See Fig. 3 for a schematic picture of the situation. Since this situation is a kind of time reversal of that in Sec. 2.2.1 most parts of calculation can be reused but the results are different.

Let us expand $g_m$ by $f_n$,

$$g_m = \sum_{n=1}^{\infty} (\rho_{mn} f_n + \sigma_{mn} f_n^*)$$

where the expansion coefficients are given by

$$\rho_{mn} = \langle f_n, g_m \rangle = \alpha_{nm}^*, \quad \sigma_{mn} = -\langle f_n^*, g_m \rangle = -\beta_{nm}.$$  

Here, $\alpha_{nm}$ and $\beta_{nm}$ are given by Eq. (20).
Substituting Eq. (35) into Eq. (14), and comparing it with Eq. (10), we obtain
\[ a_n = \sum_{m=1}^{\infty} (\rho_{mn} b_m + \sigma_{mn}^{*} b_m^\dagger). \] (37)

Substituting Eq. (37) into Eq. (11), and using Eq. (15), we obtain
\[ \sum_{m=1}^{\infty} (\rho_{mn} \rho_{mn'}^{*} - \sigma_{mn}^{*} \sigma_{mn'}) = \delta_{nn'}, \sum_{m=1}^{\infty} (\rho_{mn} \sigma_{mn} - \sigma_{mn}^{*} \rho_{mn'}) = 0, \] (38)

which should be satisfied again for the two quantization, Eqs. (10) and (14), to be consistent. It is shown in Appendix A.2 that the Bogoliubov coefficients given by Eq. (36) indeed satisfy unitarity relations (38).

The vacuum expectation value of number operator \( a_n^\dagger a_n \), representing the energy spectrum of created particles, is computed as
\[ \langle 0 | a_n^\dagger a_n | 0 \rangle = \sum_{m=1}^{\infty} |\sigma_{mn}|^2 = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{m}{n(n+2m)^2}. \] (39)

This and its summation over odd \( n \), i.e., the total number of created particles, are divergent. This implies that the Fock-space representation associated with \( b_m \) is unitarily inequivalent to that associated with \( a_n \).

The vacuum expectation value of energy-momentum tensor before the change of boundary condition at \( t = 0 \) is computed by substituting Eq. (14) into Eq. (17), and using the explicit expression of mode function (8),
\[ \langle 0 | T_{\pm\pm} | 0 \rangle_{t<0} = \sum_{m=1}^{\infty} |\partial_{\pm} g_m|^2 = \frac{\pi}{4L^2} \sum_{m=1}^{\infty} m. \] (40)

This represents the Casimir energy density, which can be made finite by standard renormalization procedures [1].

The vacuum expectation value of energy-momentum tensor after \( t = 0 \) is computed by substituting Eq. (10) into Eq. (17), and using Eq. (37), as
\[ \langle 0 | T_{\pm\pm} | 0 \rangle_{t>0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} [(\rho_{mn}\sigma_{mn'} + \rho_{mn'}\sigma_{mn}) \text{Re}(\partial_{\pm} f_n \partial_{\pm} f_{n'}) \\
+ (\rho_{mn}\rho_{mn'} + \sigma_{mn}\sigma_{mn'}) \text{Re}(\partial_{\pm} f_n \partial_{\pm} f_{n'}^*)], \] (41)

which we symmetrize with respect to dummy indices \( n \) and \( n' \), and use the fact that \( \rho_{mn} \) and \( \sigma_{mn} \) are real.

Using the explicit form of Bogoliubov coefficients and mode function, Eqs. (36), (20), and (5), we obtain
\[ \langle 0 | T_{\pm\pm} | 0 \rangle_{t>0} = \frac{4}{\pi L^2} \sum_{m=1}^{\infty} \left( 4m^3 \left( \sum_{n=1}^{\infty} \frac{\cos(p_n z_{\pm})}{n^2 - (2m)^2} \right)^2 + m \left( \sum_{n=1}^{\infty} \frac{n \sin(p_n z_{\pm})}{n^2 - (2m)^2} \right)^2 \right). \] (42)

10
This is an even function of $z_\pm$ with period $2L$, since it is invariant under reflection $z_\pm \rightarrow -z_\pm$ and translation $z_\pm \rightarrow z_\pm + 2L$. Therefore, it is sufficient to calculate it in $0 \leq z_\pm < 2L$, and generalize it appropriately to one valid in the entire domain.

The first summation over odd $n$ in Eq. (42) can be computed to give

$$\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0} = \frac{4}{\pi L^2} \sum_{m=1}^{\infty} \left( \frac{m \pi^2}{16} \sin^2(q_m z_\pm) + m \left[ \sum_{n=1 \atop n: \text{odd}}^{\infty} \frac{n \sin(p_n z_\pm)}{n^2 - (2m)^2} \right] \right),$$

which is valid in $0 \leq z_\pm < 2L$. Here, we have used the following formula [13, p. 733],

$$\sum_{k=0}^{\infty} \frac{\cos((2k+1)y)}{(2k+1)^2 - a^2} = \frac{\pi}{4a} \sin\left[ \frac{a}{2}(\pi - 2y) \right] \sec\left( \frac{a \pi}{2} \right), \quad (0 \leq y \leq \pi).$$

(44)

It is noted here that there are typos in Ref. [13, p. 733] about formulas (44) and (47) (see below).

For $z_\pm = 0$, from Eq. (43), we have

$$\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0} = 0, \quad (z_\pm = 0).$$

(45)

For $0 < z_\pm < 2L$, the rest summation over odd $n$ in Eq. (43) can be computed to give

$$\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0} = \frac{\pi}{4L^2} \sum_{m=1}^{\infty} m \quad (0 < z_\pm < 2L),$$

using the following formula [13, p. 733],

$$\sum_{k=0}^{\infty} \frac{(2k+1) \sin((2k+1)y)}{(2k+1)^2 - a^2} = \frac{\pi}{4} \cos\left[ \frac{a}{2}(\pi - 2y) \right] \sec\left( \frac{a \pi}{2} \right), \quad (0 < y < \pi).$$

(47)

Combining Eqs. (45) and (46), and extending the domain periodically into the entire domain, we have

$$\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0} = \begin{cases} 0 & \left( z_\pm = 2\ell L, \ \ell \in \mathbb{Z} \right) \\ \frac{\pi}{4L^2} \sum_{m=1}^{\infty} m & \text{(otherwise)} \end{cases}.$$ 

(48)

Comparing the above result with that in the N-D case [34], one sees that there is no flux component of delta function squared in this case. As will be explicitly shown in the semi-infinite cavity case (Sec. 4 and Appendix B), Eq. (48) represents the non-renormalizable diverging flux localized on the null lines $z_\pm = 2\ell L \ (\ell \in \mathbb{Z})$ and the ambient Casimir energy. Thus, the diverging flux emanates from origin $(t, x) = (0, 0)$ and propagates along the null lines in a similar way to Fig. 2.

### 3 Finite cavity II: Revisit Ishibashi-Hosoya [5]

As seen in Sec. 2, the vacuum expectation value of energy-momentum tensor has two components in the N-D case as Eq. (34), and one component in the D-N case as Eq. (48). The origin of such a
difference between the N-D and D-N cases will be discussed in Conclusion. Here, let us consider the consistency between these results and a relevant past work.

In Ref. [5], the authors considered the instantaneous change of boundary condition at the both sides of finite cavity. The boundary conditions for \( t < 0 \) are Neumann at the both sides and those for \( t > 0 \) are Dirichlet at the both sides, which we call the NN-DD case. Since this NN-DD case resembles the N-D case, one can expect the similar results. Namely, we expect that two diverging flux components appear in the NN-DD case. Reference [5], however, concludes the flux involves only the component of delta function squared. Therefore, we will reconsider here the system adopted in [5].

### 3.1 Quantization of massless scalar field

We consider the situation that the Neumann boundary condition is imposed at \( x = 0 \) and \( x = L \) for \( t < 0 \), while the Dirichlet boundary condition is imposed at \( x = 0 \) and \( x = L \) for \( t > 0 \) (see Fig. 4).

In this case, a normalized positive-energy mode function for \( t < 0 \) is given by

\[
h_k(t, x) = \frac{1}{\sqrt{k\pi}} e^{-ir_k t} \cos(r_k x), \quad r_k := \frac{k\pi}{L}, \quad k = 1, 2, 3, \ldots.
\]  

A normalized mode function for \( t > 0 \) is given by Eq. (8).

The scalar field is quantized by expanding it by set of mode functions \( \{h_k\} \) and an additional zero-mode function \( h_0 \), being spatially uniform, as

\[
\phi = h_0 + \sum_{k=1}^{\infty} (c_k h_k + c_k^\dagger h_k^*), \quad h_0 = \frac{1}{\sqrt{L}}(Q + tP).
\]  

Here, \( Q \) and \( P \) are Hermitian (\( Q^\dagger = Q, \ P^\dagger = P \)), and the following commutation relations are imposed

\[
[Q, P] = i, \quad [Q, c_k] = [P, c_k] = 0, \quad [c_k, c_{k'}^\dagger] = \delta_{kk'}.
\]

Note that zero-mode \( h_0 \), which exists because the boundary conditions are Neumann at the both ends, is indispensable to realize the equal-time commutation relation (12) using commutation relations (51).
3.2 Particle creation by instantaneous change of boundary condition: From Neumann-Neumann to Dirichlet-Dirichlet

Let us expand $h_0$ and $h_k$ by $g_m$,

$$h_0 = \sum_{m=1}^{\infty} (\xi_m g_m + \xi_m^\dagger g_m^*), \quad h_k = \sum_{m=1}^{\infty} (\xi_{km} g_m + \zeta_{km} g_m^*),$$

(52)

where the Bogoliubov coefficients are given by

$$\xi_m = \langle g_m, h_0 \rangle, \quad \xi_{km} = \langle g_m, h_k \rangle, \quad \zeta_{km} = -\langle g_m^*, h_k \rangle.$$  

(53)

Using the explicit form of mode functions (8) and (49), and Eq. (50), Bogoliubov coefficients (53) are computed as

$$\xi_m = \frac{2}{\sqrt{m\pi L}} \left( Q + i \frac{L}{m\pi} P \right) \delta_{m:odd},$$

(54)

$$\xi_{km} = -\frac{2}{(k-m)\pi} \sqrt{\frac{m}{k}} \delta_{k+m:odd}, \quad \zeta_{km} = \frac{2}{(k+m)\pi} \sqrt{\frac{m}{k}} \delta_{k+m:odd}.$$  

(55)

Here, we have introduced the following symbols,

$$\delta_{k:odd} := \frac{1 - (-1)^k}{2}, \quad \delta_{k:even} := \frac{1 + (-1)^k}{2}, \quad k \in \mathbb{Z}.$$  

(56)

Substituting Eq. (52) into Eq. (50), and comparing it with Eq. (14), we have

$$b_m = \xi_m + \sum_{k=1}^{\infty} (\xi_{km} c_k + \zeta_{km} c_k^*).$$

(57)

Substituting Eq. (57) into Eq. (15) and using Eq. (51), we obtain the unitarity relations,

$$[\xi_m, \xi_m^\dagger] + \sum_{k=1}^{\infty} (\xi_{km} c_{km'}^* - \zeta_{km} c_{km'}) = \delta_{mm'},$$

(58)

$$[\xi_m, \xi_{m'}] + \sum_{k=1}^{\infty} (\xi_{km} c_{km'}^* - \zeta_{km} c_{km'}) = 0.$$  

In Appendix A.3 we will show that the operators given in Eqs. (54) and (55) satisfy unitarity relations (58).

We define the vacuum in which particle corresponding to $h_0$ and $h_k$ does not exist,

$$P|0_h\rangle = c_k|0_h\rangle = 0, \quad k = 1, 2, 3, \cdots.$$  

(59)

Then, the spectrum of created particles are given by the expectation value of number operator $b_m^\dagger b_m$,

$$\langle 0_h| b_m^\dagger b_m |0_h\rangle = \langle 0_h| \xi_m \xi_m^\dagger |0_h\rangle + \sum_{k=1}^{\infty} |\zeta_{km}|^2$$

$$= \frac{4}{m^2 \pi^2} \left( \frac{m\pi}{L} \langle 0_h|Q^2|0_h\rangle - 1 \right) \delta_{m:odd} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{m}{k(k+m)^2} \delta_{k+m:odd}.$$  

(60)
The vacuum expectation value of energy-momentum tensor before the change of boundary conditions at \( t = 0 \) is computed by substituting Eq. (50) into Eq. (17), and using explicit form of mode function (49) as

\[
\langle 0_b | T_{\pm \pm} | 0_h \rangle_{t < 0} = \sum_{k=1}^{\infty} |\partial_{\pm} h_k|^2 = \frac{\pi}{4L^2} \sum_{k=1}^{\infty} k. \tag{61}
\]

This represents the Casimir energy density, which can be made finite by standard regularization schemes such as the \( \zeta \)-function regularization, the point-splitting regularization, and so on [1].

The vacuum expectation value of energy-momentum tensor after \( t = 0 \) is computed by substituting Eq. (14) into Eq. (17), and using Eq. (57),

\[
\langle 0_b | T_{\pm \pm} | 0_h \rangle_{t > 0} = \sum_{m=1}^{\infty} \sum_{m' = 1}^{\infty} \left[ \frac{8\langle 0_b | Q^2 | 0_h \rangle}{\pi L \sqrt{mm'}} \text{Re}(\partial_{\pm}g_m \partial_{\pm}g_{m'}) + \frac{4i}{\sqrt{\pi^2 m^3 m'^3}} \text{Im}((m + m')\partial_{\pm}g_m \partial_{\pm}g_{m'} - (m - m')\partial_{\pm}g_m \partial_{\pm}g_{m'}^*) \right] + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{m' = 1}^{\infty} \left[ (\xi_k \zeta_{km'} + \zeta_k \xi_{km'}) \text{Re}(\partial_{\pm}g_m \partial_{\pm}g_{m'}) + (\xi_k \zeta_{km'} + \zeta_k \xi_{km'}) \text{Re}(\partial_{\pm}g_m \partial_{\pm}g_{m'}) \right], \tag{62}
\]

which we symmetrize with respect to dummy indices \( m \) and \( m' \), and we have used the fact that \( \xi_k \) and \( \zeta_k \) are real.

Using explicit form of mode functions (3) and Bogoliubov coefficients (55), we obtain

\[
\langle 0_b | T_{\pm \pm} | 0_h \rangle_{t > 0} = \frac{4\langle 0_b | Q^2 | 0_h \rangle}{L^3} \left[ \sum_{m=1}^{\infty} \sum_{m' = 1}^{\infty} \left( \frac{4i}{\pi L^2} \sum_{m=1}^{\infty} \frac{\sin(qmz_\pm)}{m} \sum_{m=1}^{\infty} \cos(qmz_\pm) \right) + \frac{4i}{\pi L^2} \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{m=2}^{\infty} \frac{\cos(qmz_\pm)}{m^2 - k^2} + \frac{1}{k} \sum_{m=1}^{\infty} \frac{\cos(qmz_\pm)}{m^2 - k^2} \right) \right] \tag{63}
\]

The summations over odd \( m \) in the first two terms of Eq. (63), both of which are the contributions of the zero-mode, are computed using the following formulas,

\[
\sum_{k=1}^{\infty} \frac{1}{k} \sin\left( \frac{2k\pi}{a} y \right) = \frac{\pi}{4} \sum_{\ell = -\infty}^{\infty} (-1)^\ell \Pi_0^{a/2} (y - \frac{a}{2} \ell), \quad (-\infty < y < \infty), \tag{64}
\]
\[
\sum_{k=1}^{\infty} \cos\left( \frac{2k\pi}{a} y \right) = \frac{a}{4} \sum_{\ell = -\infty}^{\infty} (-1)^\ell \delta(y - \frac{a}{2} \ell), \quad (-\infty < y < \infty), \tag{65}
\]

where \( \Pi_0^a(x) \) is the rectangular function defined as

\[
\Pi_0^a(x) := \int_a^b \delta(x - y) dy = \begin{cases} 0 & (x < a, \ b < x) \\ \frac{1}{2} & (x = a, b) \\ 1 & (a < x < b) \end{cases}. \tag{66}
\]
The rest summations over odd and even \( m \) in Eq. (63) are computed using formulas (28), (29), (32), (41), and (47) in addition to the above formulas, to obtain

\[
\langle 0_h | T_{\pm \pm} | 0_h \rangle_{t>0} = \left( \frac{\langle 0_h | Q^2 | 0_h \rangle}{L} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \right) \sum_{\ell=-\infty}^{\infty} \delta^2 (z_+ - \ell L) + \begin{cases} 0 & (z_+ = \ell L, \ell \in \mathbb{Z}) \\ \frac{\pi}{4L^2} \sum_{k=1}^{\infty} & \text{(otherwise)} \end{cases}.
\]

(67)

After setting \( L = \pi \) and regularizing the diverging summation as \( \sum_{k=1}^{\infty} k = -\frac{1}{12} \) by the \( \zeta \)-function regularization, Eq. (67) should be equal to Eq. (31) of Ref. [5]. The vanishing of Casimir energy on the null lines in Eq. (67), however, has no counterpart in Eq. (31) of Ref. [5].

While we have derived Eq. (67) with keeping the parallelism with the other analyses in the present paper, it is unclear from where the discrepancy comes. In the next subsection, therefore, we will re-derive Eq. (67) with a method similar to one in Ref. [5].

3.3 Origin of discrepancy

Substituting Eq. (12) into Eq. (17), and using Eq. (57), the vacuum expectation value of the energy-momentum tensor after \( t = 0 \) is written as

\[
\langle 0_h | T_{\pm \pm} | 0_h \rangle_{t>0} = \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \left[ \langle 0_h | \xi_m \xi_{m'} | 0_h \rangle + \sum_{k=1}^{\infty} \xi_{km} \xi_{km'}^* \partial_\pm g_m \partial_\pm g_{m'} \\
+ \langle 0_h | \xi_m \xi_{m'}^\dagger | 0_h \rangle + \sum_{k=1}^{\infty} \xi_{km} \xi_{km'}^* \partial_\pm g_m \partial_\pm g_{m'}^* + \langle 0_h | \xi_m \xi_{m'}^\dagger | 0_h \rangle + \sum_{k=1}^{\infty} \xi_{km} \xi_{km'}^\dagger \partial_\pm g_m^* \partial_\pm g_{m'}^* \\
+ \langle 0_h | \xi_m \xi_{m'} | 0_h \rangle + \sum_{k=1}^{\infty} \xi_{km} \xi_{km'}^\dagger \partial_\pm g_m^* \partial_\pm g_{m'} \right].
\]

(68)

Using explicit form of Bogoliubov coefficients (54) and (55), and mode function (8), this quantity is rewritten in a compact form,

\[
\langle 0_h | T_{\pm \pm} | 0_h \rangle_{t>0} = \frac{1}{L^2} \sum_{m=-\infty}^{m=-\infty} \sum_{m'=m: \text{odd}}^{m':=m: \text{odd}} \left( \langle 0_h | Q^2 | 0_h \rangle + \frac{L}{m \pi} \right) e^{-\imath q m z} + \sum_{m'=-\infty}^{m'=-\infty} \sum_{m':m': \text{odd}}^{m':m': \text{odd}} e^{-\imath q m' z} \\
+ \frac{1}{\pi L^2} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=-\infty}^{m=-\infty} \sum_{m' = m: \text{odd}}^{m':m': \text{odd}} \frac{\delta_{m-k: \text{odd}}}{m-k} e^{-\imath (q m + a k) z} + \sum_{m'=-\infty}^{m'=-\infty} \sum_{m':m': \text{odd}}^{m':m': \text{odd}} \frac{\delta_{m'+k: \text{odd}}}{m'+k}.
\]

(69)

The summations over odd \( m \) and \( m' \) in Eq. (69) can be evaluated with the following formulas,

\[
\sum_{k=-\infty}^{\infty} \frac{1}{k} \exp(-\imath 2k \pi a) y = -\frac{i \pi}{2} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \Pi_0^{1/2} (y - \frac{a}{2} \ell),
\]

(70)

\[
\sum_{k=-\infty}^{\infty} \exp(-\imath 2k \pi a) y = \frac{a}{2} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \delta(y - \frac{a}{2} \ell),
\]

(71)
which are equivalent to Eqs. (64) and (65), respectively.

Finally, in order to obtain the final result, it is necessary to use the following relation,

\[
\sum_{\ell = -\infty}^{\infty} (-1)^{\ell} \Pi_0^{L}(z_{\pm} - \ell L) \sum_{\ell' = -\infty}^{\infty} (-1)^{\ell'} \Pi_0^{L}(z_{\pm} - \ell' L) = \begin{cases} 
0 & (z_{\pm} = \ell L, \ell \in \mathbb{Z}) \\
1 &\text{(otherwise)}
\end{cases}.
\]  

(72)

Then, we obtain Eq. (67). It seems that Ref. [5] overlooked the left-hand side of Eq. (72) to vanish on null lines \(z_- = 0\) and \(z_+ = L\). This would be the origin of the discrepancy between our result and their result.

4 Semi-infinite cavity

In the rest of this paper, we investigate the particle creation by the instantaneous change of boundary condition in a semi-infinite cavity, which correspond to the limit \(L \to +\infty\) of the finite-cavity model in Sec. 2. We will see that some simplifications happen in such a limit. Namely, one needs just some simple integral formulas rather than the non-trivial summation formulas in Sec. 2. The analysis in semi-infinite space \(x \in [0, +\infty)\) can be a footing to generalize the present analysis, for example, to higher-dimensional models by regarding the spatial coordinate \(x\) as a radial coordinate of higher-dimensional spaces (see [14] for a relevant higher-dimensional consideration). While the Bogoliubov transformation will be used in this section again in order to keep the parallelism with the previous sections, the results will be re-derived in Appendix B with an independent method using the Green functions, which naturally involves the point-splitting regularization of the vacuum expectation value of energy-momentum tensor.

4.1 Quantization of massless scalar field

We consider a free massless scalar field in the semi-infinite cavity, of which equation of motion is given by Eq. (1) with \(L \to +\infty\).

At left boundary \(x = 0\), we consider two kinds of boundary conditions. One is the Neumann boundary condition (3). Another is the Dirichlet boundary condition (4).

During Neumann boundary condition (3) is satisfied, a natural set of positive-energy mode functions \(\{f_p\}\), which is labeled by continuous parameter \(p\), is given by

\[
f_p(t, x) = \frac{1}{\sqrt{\pi p}} e^{-ipt} \cos(px), \quad p > 0.
\]

(73)

This mode function satisfies the following orthonormal conditions,

\[
\langle f_p, f_{p'} \rangle = -\langle f_p^*, f_{p'}^* \rangle = \delta(p - p'), \quad \langle f_p, f_{p'}^* \rangle = 0,
\]

(74)

where the integration range of Klein-Gordon inner product, Eq. (7), is from 0 to +\(\infty\).

During Dirichlet boundary condition (4) is satisfied, a natural set of positive-energy mode functions \(\{g_q\}\) is given by

\[
g_q(t, x) = \frac{1}{\sqrt{\pi q}} e^{-iqt} \sin(qx), \quad q > 0.
\]

(75)
This mode function satisfies the following orthonormal conditions,
\( \langle g_q, g_{q'} \rangle = -\langle g_q^*, g_{q'}^* \rangle = \delta(q - q'), \quad \langle g_q, g_{q'}^* \rangle = 0. \)  

(76)

Associated with the above two sets of mode functions, \( \{f_p\} \) and \( \{g_q\} \), there are two ways to quantize the scalar field. Namely, we can expand the scalar field by two sets of mode functions,
\[ \phi = \int_0^\infty dp (a_p f_p + a_p^\dagger f_p^*), \]
\[ \phi = \int_0^\infty dq (b_q g_q + b_q^\dagger g_q^*), \]

(77)

(78)

where the expansion coefficients are imposed the commutation relations,
\[ [a_p, a_{p'}^\dagger] = \delta(p - p'), \quad [a_p, a_{p'}] = 0, \]
\[ [b_q, b_{q'}^\dagger] = \delta(q - q'), \quad [b_q, b_{q'}] = 0. \]

(79)

(80)

Operators \( a_p \) and \( b_q \) (resp. \( a_p^\dagger \) and \( b_q^\dagger \)) are interpreted as annihilation (resp. creation) operators.

Accordingly, we can define two normalized vacuum states,
\[ a_p |0_f\rangle = 0, \quad \forall p > 0, \quad \langle 0_f |0_f \rangle = 1, \]  
\[ b_q |0_g\rangle = 0, \quad \forall q > 0, \quad \langle 0_g |0_g \rangle = 1. \]  

(81)

(82)

Then, \( |0_f\rangle \) (resp. \( |0_g\rangle \)) is the state where no particle corresponding to \( f_n \) (resp. \( g_m \)) exists.

### 4.2 Particle creation by instantaneous change of boundary condition

Given the above quantization of scalar field in the semi-infinite cavity, we investigate how the vacuum is excited when the boundary condition at \( x = 0 \) instantaneously changes from Neumann to Dirichlet (N-D) in Sec. 4.2.1 and reversely (D-N) in Sec. 4.2.2.

#### 4.2.1 From Neumann to Dirichlet

We assume that the boundary condition at \( x = 0 \) is Neumann \( \square \) for \( t < 0 \) and Dirichlet \( \square \) for \( t > 0 \), and that the quantum field is in vacuum \( |0_f\rangle \), defined by Eq. (81). See Fig. 5 for a schematic picture of the situation.

Let us expand \( f_p \) by \( g_q \) as,
\[ f_p = \int_0^\infty dq (\alpha_{pq} g_q + \beta_{pq} g_q^*), \]

(83)

where the expansion coefficients are given by
\[ \alpha_{pq} = \langle g_q, f_p \rangle, \quad \beta_{pq} = -\langle g_q^*, f_p \rangle. \]

(84)

Using Eqs. (73) and (75), we obtain
\[ \alpha_{pq} = -\frac{1}{(p - q)\pi} \sqrt{\frac{q}{p}}, \quad \beta_{pq} = \frac{1}{(p + q)\pi} \sqrt{\frac{q}{p}}. \]

(85)
Figure 5: The boundary condition at the left end of domain \((x = 0)\) instantaneous changes at \(t = 0\) from Neumann (dashed) to Dirichlet (solid). Spatial configurations of mode functions \(f_p\) and \(g_q\) are schematically depicted.

where we have used integral formula \(\int_{-\infty}^{\infty} e^{iax} dx = ia^{-1} (-\infty < a < \infty)\).

Substituting Eq. (83) into Eq. (77), and comparing it with Eq. (78), we obtain

\[
b_q = \int_{0}^{\infty} dp (\alpha_{pq} a_p + \beta_{pq}^* a_p^\dagger).
\] (86)

Substituting Eq. (86) into Eq. (80), and using Eq. (79), we obtain the unitarity relations,

\[
\int_{0}^{\infty} dp (\alpha_{pq} \alpha_{pq}^* - \beta_{pq} \beta_{pq}^*) = \delta(q - q'), \quad \int_{0}^{\infty} dp (\alpha_{pq} \beta_{pq}^* - \beta_{pq} \alpha_{pq}^*) = 0.
\] (87)

In Appendix A.4 we prove that Bogoliubov coefficients (85) satisfy Eq. (87).

The spectrum of created particles are computed as

\[
\langle 0_f | b_q^\dagger b_q | 0_f \rangle = \int_{0}^{\infty} dp |\beta_{pq}|^2 = \frac{1}{\pi^2} \int_{0}^{\infty} dp \frac{q}{p(p + q)^2}.
\] (88)

This and its integration over \(q\) are divergent due to the contribution from the infrared regime.

The vacuum expectation value of energy-momentum tensor before the change of boundary condition at \(t = 0\) is computed by substituting Eq. (77) into Eq. (17), and using Eqs. (79) and (73), as

\[
\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t < 0} = \int_{0}^{\infty} dp |\partial_{\pm} f_p|^2 = \frac{1}{4\pi} \int_{0}^{\infty} dp p.
\] (89)

Unlike the finite-cavity case, there is no Casimir energy in this semi-infinite case. The above result just represents the divergent energy density due to the zero-point oscillation. Thus, the renormalized vacuum expectation value obtained by subtracting such a zero-point contribution identically vanishes everywhere as Eq. (146).

The vacuum expectation value of energy-momentum tensor after \(t = 0\) is computed by substituting Eq. (78) into Eq. (17), and using Eq. (86), as

\[
\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t > 0} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} dp dq dq' [\alpha_{pq} \beta_{pq}^* + \alpha_{pq}^* \beta_{pq}] Re(\partial_{\pm} g_q \partial_{\pm} g_{q'})
\]
\[+ (\alpha_{pq} \alpha_{pq}^* + \beta_{pq} \beta_{pq}^*) Re(\partial_{\pm} g_q \partial_{\pm} g_{q'})].
\] (90)
To derive Eq. (90), we symmetrize it with respect to integration variables \( q \) and \( q' \), and use the fact that \( \alpha_{pq} \) and \( \beta_{pq} \) are real. Using explicit expressions of Bogoliubov coefficients (85) and mode function (75), we obtain

\[
\langle 0 | T_{\pm\pm} | 0 \rangle_{t>0} = \frac{1}{\pi} \int_0^\infty dp \left( \frac{1}{p} \int_0^\infty dq \cos(qz_{\pm}) + p^2 \int_0^\infty dq \frac{\cos(qz_{\pm})}{q^2 - p^2} \right)^2 + p \left( \int_0^\infty dq \frac{q \sin(qz_{\pm})}{q^2 - p^2} \right)^2.
\]

(91)

The integration over \( q \) in Eq. (91) can be computed to give

\[
\langle 0 | T_{\pm\pm} | 0 \rangle_{t>0} = \frac{\delta^2(z_{\pm})}{\pi} \int_0^\infty dp + \frac{\operatorname{sgn}^2(z_{\pm})}{4\pi} \int_0^\infty dpp,
\]

(92)

where \( \operatorname{sgn} \) denotes the sign function,

\[
\operatorname{sgn}(a) := \begin{cases} 
\pm 1 & (a \gtrless 0) \\
0 & (a = 0)
\end{cases}.
\]

(93)

Note that we have used the following integration formulas,

\[
\int_0^\infty \cos(ax) dx = \pi \delta(a), \quad (\infty < a < \infty),
\]

(94)

\[
\int_0^\infty \frac{\cos(ax)}{x^2 - b^2} dx = -\operatorname{sgn}(a) \frac{\pi}{2b} \sin(ab), \quad (\infty < a < \infty, b > 0),
\]

(95)

\[
\int_0^\infty \frac{x \sin(ax)}{x^2 - b^2} dx = \operatorname{sgn}(a) \frac{\pi}{2} \cos(ab), \quad (\infty < a < \infty, b > 0).
\]

(96)

See Appendix C for the derivation of the second and third formulas.

Let us consider the meaning of two terms in Eq. (92). The first term, the delta function squared multiplied by a divergent integral, represents the diverging flux emanating from the origin \((t, x) = (0, 0)\) and localizing on the null line \(z_- = 0\). The divergent factor involves the infrared divergence too since there in no infrared cutoff introduced by finite \(L\). The dependence of energy density on the delta function squared implies also the divergence of total energy emitted.

The second term, at first glance, seems to represent an ambient divergent energy density and its vanishing on the null line emanating from the origin (note that \( \operatorname{sgn}(0) = 0 \)). As will be seen below, however, this is not the case. Namely, the divergence at \( z_{\pm} \neq 0 \) just represents the energy due to the zero-point oscillation just like Eq. (89). Therefore, the regularized vacuum expectation value of energy-momentum tensor should be defined by subtracting such a diverging quantity distributing uniformly in space and time. As the result of such a subtraction, the divergence appears on the null line \( z_- = 0 \). Such a renormalized vacuum expectation value of energy-momentum tensor is computed in Appendix B with the Green-function method, which naturally involves the point-splitting regularization. The result is

\[
\langle 0_g | T_{\pm\pm} | 0_g \rangle_{t>0}^{\text{ren}} = \frac{\delta^2(z_{\pm})}{\pi} \int_0^\infty dp \left( \frac{1}{p} \int_0^\infty dq \frac{1}{4\pi(z_\pm - z'_\pm)^2} \right)^2 (z_{\pm} = 0) \quad (\text{otherwise}).
\]

(97)
Here, $z_\pm$ and $z'_\pm$ are the coordinates of two points on which the Green functions are evaluated. As explained above, the second term diverges on the null line and vanishes elsewhere. Thus, there remain the two components of diverging flux even after the renormalization to propagate along the null line $z_- = 0$.

### 4.2.2 From Dirichlet to Neumann

We assume that the boundary condition at $x = 0$ is Dirichlet for $t < 0$ and Neumann for $t > 0$, and that the quantum field is in vacuum $|0_g\rangle$, given by Eq. (82). See Fig. 6 for a schematic picture of the physical situation. Then, we investigate how the vacuum is excited by computing the spectrum and energy flux of created particles.

Let us expand $g_q$ by $f_p$ as,

$$ g_q = \int_0^\infty dp (\rho_{pq} f_p + \sigma_{pq} f_p^*) . \quad (98) $$

Here, the expansion coefficients are given by

$$ \rho_{pq} = \langle f_p, g_q \rangle = \alpha^*_{pq}, \quad \sigma_{pq} = -\langle f_p^*, g_q \rangle = -\beta_{pq}, \quad (99) $$

where $\alpha_{pq}$ and $\beta_{pq}$ are given by Eq. (85).

Substituting Eq. (98) into Eq. (78), and comparing it with Eq. (77), we obtain

$$ a_p = \int_0^\infty dq (\rho_{pq} b_q + \sigma_{pq}^* b_q^*) . \quad (100) $$

Substituting Eq. (100) into Eq. (79), and using Eq. (80), we obtain the unitarity relations,

$$ \int_0^\infty dq (\rho_{pq} \rho_{pq'}^* - \sigma_{pq}^* \sigma_{pq'}) = \delta(p - p'), \quad \int_0^\infty dq (\rho_{pq} \sigma_{pq'}^* - \sigma_{pq}^* \rho_{pq'}) = 0. \quad (101) $$

In Appendix A.3 it is shown that Bogoliubov coefficients (99) indeed satisfy Eq. (101).

The spectrum is computed as

$$ \langle 0_g | a_p^\dagger a_p | 0_g \rangle = \int_0^\infty dq |\sigma_{pq}|^2 = \frac{1}{\pi^2} \int_0^\infty dq \frac{q}{p(p + q)^2}, \quad (102) $$

which is divergent.

The expectation value of energy-momentum tensor before the change of boundary condition at $t = 0$ is computed by substituting Eq. (78) into Eq. (17), and using Eqs. (80) and (75), as

$$ \langle 0_g | T_{\pm \pm} | 0_g \rangle_{t < 0} = \int_0^\infty dq |\partial_{\pm} g_q|^2 = \frac{1}{4\pi} \int_0^\infty dq q. \quad (103) $$

This represents the divergence due to the zero-point oscillation, and the regularized value vanishes as given by Eq. (160).

The expectation value of energy-momentum tensor for $t > 0$ is computed by substituting Eq. (77) into Eq. (17), and using Eq. (100), as

$$ \langle 0_g | T_{\pm \pm} | 0_g \rangle_{t > 0} = \int_0^\infty \int_0^\infty \int_0^\infty dqdpd\rho [ (\rho_{pq} \rho_{pq'}) Re(\partial_{\pm} f_p \partial_{\pm} f_{p'}) + (\rho_{pq} \rho_{pq'} + \sigma_{pq} \sigma_{pq'}) Re(\partial_{\pm} f_p \partial_{\pm} f_{p'}^*) ], \quad (104) $$

20
where we symmetrize it with respect to integration variables \( p \) and \( p' \), and use the fact that \( \rho_{qp} \) and \( \sigma_{qp} \) are real. Substituting explicit form of the Bogoliubov coefficients, given by Eqs. (99) and (85), and mode function (75) into Eq. (104), we have

\[
\langle 0_g | T_{\pm\pm} | 0_g \rangle_{t>0} = \frac{1}{\pi^3} \int_0^\infty dq \left( q^3 \left[ \int_0^\infty dp \frac{\cos(pz_\pm)}{p^2-q^2}\right]^2 + q \left[ \int_0^\infty dp \frac{p \sin(pz_\pm)}{p^2-q^2}\right]^2 \right). \tag{105}
\]

The integrations over \( p \) in Eq. (105) are evaluated using formulas (95) and (96) to obtain

\[
\langle 0_g | T_{\pm\pm} | 0_g \rangle_{t>0} = \frac{\text{sgn}(z_\pm)}{4\pi} \int_0^\infty dq q. \tag{106}
\]

Again, result (106) seems to represent a diverging flux and its vanishing on the null line emanating from the origin. After subtracting the uniform contribution from the zero-point oscillation, however, the divergence appears on the null line. This is explicitly shown by adopting the Green-function method in Appendix B. The result is given by

\[
\langle 0_g | T_{\pm\pm} | 0_g \rangle_{t>0}^{\text{ren}} = \begin{cases} \lim_{z_+ \to z_\pm} \frac{1}{4\pi(z_\pm - z'_\pm)^2} & (z_\pm = 0) \\ 0 & (\text{otherwise}) \end{cases}. \tag{107}
\]

Here, \( z_\pm \) and \( z'_\pm \) are the coordinates of two points on which the Green functions are evaluated. The flux diverges on the null line and vanishes elsewhere. Thus, there remains only one component of diverging flux after the renormalization to propagate along the null line \( z_- = 0 \).

### 5 Conclusion

We have investigated the particle creation due to the instantaneous change of boundary condition in the one-dimensional (1D) finite cavity (Secs. 2 and 3) and semi-infinite cavity (Sec. 4) by computing the vacuum expectation value of energy-momentum tensor for the free massless Klein-Gordon scalar
field. The boundary condition changes from Neumann to Dirichlet (N-D) in Secs. 2 and 4, from Neumann-Neumann to Dirichlet-Dirichlet (NN-DD) in Sec. 3, and from Dirichlet to Neumann (D-N) in Secs. 2 and 4.

Although any actual change of boundary condition takes a finite interval of time, we believe that these models are capable of extracting the essence of phenomenon when the boundary condition changes rapidly enough compared to typical time scales in the system. In addition, the choice of Dirichlet and Neumann boundary conditions introduced no adjustable parameters into the system, which made the whole analysis simple to be a good starting point for succeeding considerations. Most models of the particle creation due to time-dependent boundary conditions would have to reproduce the results in this paper in the limit of infinitely rapid change.

Thanks to the above simplifications made in our model, we could obtain almost all the results in completely analytic form. For the finite cavity N-D (resp. D-N) case, the vacuum expectation value of energy-momentum tensor was obtained as Eq. (34) (resp. (48)). Our result that the flux in the N-D and D-N cases consist of two terms and only one term, respectively, seemed to contradict the result in Ref. [5], which analyses the NN-DD case. Therefore, we revisited the NN-DD case in Sec. 3 to obtain Eq. (67), which is consistent with the result in Sec. 2. The flux in the N-D and NN-DD cases consist of terms of $\delta^2(z_{\pm})$ and $1/(z_{\pm} - z'_{\pm})^2$, while the flux in the D-N case consists of only term of $1/(z_{\pm} - z'_{\pm})^2$. Although we cannot argue which term is stronger to dominate at this point, it will be the case that not only the flux but also the total energy radiated becomes large since the integration of flux cross $z_{\pm} = 0$ diverges.

While the results in the semi-infinite cavity for the N-D case (92) and D-N case (106) are quite similar to their respective counterparts in the finite cavity, the analysis for the infinite cavity is much simpler than the finite-cavity case in that non-trivial mathematical formulas such as summation formulas of Eqs. (29), (44), and so on, are not necessary. This is a technical but an important point for succeeding studies such as the generalizations of this work (future works will be mentioned later).

In addition, the vacuum expectation value of energy-momentum tensor in the semi-infinite cavity was re-derived by the Green-function method in Appendix [13]. This method not only naturally involves the point-splitting regularization but also involves only simpler calculations than the Bogoliubov method in the text. Again, this is a technical but an important point. Finally, the analysis for the semi-infinite cavity confirmed that the divergence of flux due to the change of boundary condition is nothing but an ultraviolet effect rather than an infrared one, and that the divergence of the flux has nothing to do with the Casimir effect, which exists only when $L$ is finite.

Let us discuss the origin of asymmetry between the N-D and D-N cases, of which related conjecture was already proposed in the previous paper of the present author and his collaborators [10]. The $\delta^2$-term seems to stem from a temporal discontinuity of mode function $f_n$ and $f_p$. For instance, in the finite-cavity N-D case, mode function $f_n$ is given by Eq. (5) for $t < 0$, having a non-zero value at $x = 0$, but given by Eq. (18) for $t > 0$, vanishing at $x = 0$. Therefore, $f_n(t, 0)$ is discontinuous as a function of time at $t = 0$. On the other hand, in the finite-cavity D-N case, mode function $g_m$ is given by Eq. (35) for $t < 0$ and Eq. (36) for $t > 0$, both of which vanish at $x = 0$. Therefore, $g_m(t, 0)$ is continuous as a function of time at $t = 0$. In a similar way, $h_k(t, 0)$ and $h_q(t, L)$ are discontinuous as functions of time at $t = 0$ in the NN-DD case, and $f_p(t, 0)$ (resp. $g_q(t, 0)$) is discontinuous (resp. continuous) at $t = 0$ in the semi-infinite N-D (resp. D-N) case. We conjecture that such a discontinuity, which would create a shock in the classical mechanics point of view, is the origin of the delta function squared.
Naively speaking, the results in this paper suggest that the back-reaction of created particles to the spacetime and/or the cavity cannot be ignored. However, the analysis is based on the test-field approximation, therefore, it is too early to assert such an implication of the results. As a next step, it is natural to investigate the back-reaction through, say, the semi-classical Einstein equation, where the right-hand side of Einstein equation is replaced by the regularized vacuum expectation value of energy-momentum tensor of quantized fields [1].

Given the results in this paper, there would be several directions to proceed besides investigating the back-reaction mentioned above. Firstly, it is natural to generalize the present analysis to higher-dimensional spacetime (see Ref. [14] for a highly relevant study). Secondly, it would be important to generalize the boundary condition in the present paper (i.e., Dirichlet and Neumann) to the Robin-type boundary condition (see, e.g., [15]), which takes the form of $\phi(t, x) - a \partial_x \phi(t, x)|_{x=0} = 0$. Taking different values of constant $a$ before and after $t = 0$, one can generalize the present analysis. By such a generalization, we would be able to verify the above conjecture about the origin of asymmetry between the N-D and D-N cases, and understand more deeply how the time-dependent boundaries make the quantum vacuum excite in general.

Acknowledgments

The author would like to thank T. Harada and S. Kinoshita for useful discussions. This work was partially supported by JSPS KAKENHI Grant Numbers 15K05086 and 18K03652.

A Proof of unitarity relations

A.1 Equation (22)

Using Eq. (20), the left-hand sides of Eq. (22) are written as

$$
\sum_{n=1 \atop n: \text{odd}}^{\infty} (\alpha_{nm} \alpha_{nm}^* - \beta_{nm} \beta_{nm}^*) = \frac{32 (m + m') \sqrt{mm'}}{\pi^2} U_{mm'},
$$

$$
\sum_{n=1 \atop n: \text{odd}}^{\infty} (\alpha_{nm} \beta_{nm}^* - \beta_{nm} \alpha_{nm}^*) = -\frac{32 (m - m') \sqrt{mm'}}{\pi^2} U_{mm'},
$$

where we define

$$
U_{mm'} := \sum_{n=1 \atop n: \text{odd}}^{\infty} \frac{1}{n^2 - (2m)^2 |n^2 - (2m')^2|}.
$$

The summation over odd $n$ in Eq. (110) can be computed to give

$$
U_{mm'} = \frac{\pi^2}{16(2m)^2} \delta_{mm'},
$$

(111)
using the following formulas [13, pp. 688–689],

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - a^2} = \frac{\pi}{4a} \tan\left(\frac{a\pi}{2}\right),
\]

(112)

\[
\sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 - a^2]^2} = -\frac{\pi}{8a^3} \tan\left(\frac{a\pi}{2}\right) + \frac{\pi^2}{16a^2} \sec^2\left(\frac{a\pi}{2}\right).
\]

(113)

Substituting Eq. (111) into Eqs. (108) and (109), we see Eq. (22) to hold.

A.2 Equation (38)

Using Eqs. (36) and (20), the left-hand sides of Eq. (38) are

\[
\sum_{m=1}^{\infty} \left( \rho_{mn} \rho_{mn}^* - \sigma_{mn} \sigma_{mn}^* \right) = 2\left(n + n'\right) \sqrt{n n'} V_{nn'},
\]

(114)

\[
\sum_{m=1}^{\infty} \left( \rho_{mn} \sigma_{mn}^* - \sigma_{mn} \rho_{mn}^* \right) = -2\left(n - n'\right) \sqrt{n n'} V_{nn'},
\]

(115)

where we define

\[
V_{nn'} := \sum_{m=1}^{\infty} \frac{m^2}{m^2 - \left(n/2\right)^2} \frac{1}{m^2 - \left(n'/2\right)^2}
\]

\[
= \sum_{m=1}^{\infty} \frac{1}{m^2 - \left(n'/2\right)^2} + \left(\frac{n}{2}\right)^2 \sum_{m=1}^{\infty} \frac{1}{m^2 - \left(n/2\right)^2} \frac{1}{m^2 - \left(n'/2\right)^2}.
\]

(116)

The summations over \(m\) in Eq. (116) can be computed to give

\[
V_{nn'} = \frac{\pi^2}{4} \delta_{nn'},
\]

(117)

using the following formulas [16, pp. 68–69],

\[
\sum_{k=1}^{\infty} \frac{1}{y^2 - k^2} = \frac{\pi}{2y} \cot(\pi y) - \frac{1}{2y^2},
\]

(118)

\[
\sum_{k=1}^{\infty} \frac{1}{[(ky)^2 - 1]^2} = \frac{\pi^2}{4y^2} \csc^2\left(\frac{\pi}{y}\right) + \frac{\pi}{4y} \cot\left(\frac{\pi}{y}\right) - \frac{1}{2}.
\]

(119)

Substituting Eq. (117) into Eqs. (114) and (115), we see Eq. (38) to hold.
A.3 Equation (58)

Using Eqs. (51), (54), and (55), the left-hand side of Eq. (58) is written as

\[
\{\xi_m, \xi^*_{m'}\} + \sum_{k=1}^{\infty} (\xi_{km}\xi^*_{km'} - \zeta_{km}\zeta^*_{km'})
= 8(m+m')\sqrt{mm'} \left[ \left( \frac{1}{2m^2m'^2} + W_{mm'} \right) \delta_{m:odd}\delta_{m':odd} + X_{mm'}\delta_{m:even}\delta_{m':even} \right],
\]

where

\[
W_{mm'} := \sum_{k=2, k:even}^{\infty} \frac{1}{(k^2 - m^2)(k^2 - m'^2)}, \quad X_{mm'} := \sum_{k=1, k:odd}^{\infty} \frac{1}{(k^2 - m^2)(k^2 - m'^2)}.
\]

Applying formulas (118) and (119) to \(W_{mm'}\), and formulas (112) and (113) to \(X_{mm'}\), we obtain

\[
W_{mm'} = -\frac{1}{2m^2m'^2} + \frac{\pi^2}{16m^2}\delta_{mm'}, \quad X_{mm'} = \frac{\pi^2}{16m^2}\delta_{mm'}. \tag{123}
\]

Substituting Eq. (123) into Eqs. (120) and (121), we see that unitarity relation (58) holds.

A.4 Equation (87)

Using Eq. (85), the left-hand side of Eq. (87) is written as

\[
\int_{0}^{\infty} dp (\alpha_{pq}\alpha^*_{pq'} - \beta_{pq}\beta^*_{pq'}) = \frac{2(q + q')\sqrt{qq'}}{\pi^2} U_{qq'},
\]

\[
\int_{0}^{\infty} dp (\alpha_{pq}\beta^*_{pq'} - \beta_{pq}\alpha^*_{pq'}) = -\frac{2(q - q')\sqrt{qq'}}{\pi^2} U_{qq'}, \tag{125}
\]

where we define

\[
U_{qq'} := \int_{0}^{\infty} \frac{dp}{(p^2 - q^2)(p^2 - q'^2)}. \tag{126}
\]

By simple algebra, this is rewritten as

\[
U_{qq'} = \frac{1}{4q(q + q')} \int_{-\infty}^{\infty} dp \left[ \frac{1}{(p - q)(p - q')} + \frac{1}{(p + q)(p + q')} - \frac{2}{p^2 - q^2} \right]. \tag{127}
\]

Adapting the following formula [17, p. 488] to the first and second terms of Eq. (127),

\[
\int_{-\infty}^{\infty} \frac{dx}{(x - a)(x - b)} = \pi^2 \delta(a - b), \quad (-\infty < a, b < \infty), \tag{128}
\]

and noting the third term vanishes from Eq. (125), we have

\[
U_{qq'} = \frac{\pi^2}{4q^2} \delta(q - q'). \tag{129}
\]

Substituting Eq. (129) into Eqs. (124) and (125), we see Eq. (87) to hold.
A.5 Equation (101)

Using Eqs. (99) and (85), the left-hand side of Eq. (101) is written as

\[ \int_0^\infty dq (\rho_{qp}\rho_{qp}^* - \sigma_{qp}^*\sigma_{qp}) = \frac{2(p + p')}{\pi^2 \sqrt{pp'}} V_{pp'} \quad (130) \]

\[ \int_0^\infty dq (\rho_{qp}\sigma_{qp}^* - \sigma_{qp}^*\rho_{qp}) = -\frac{2(p - p')}{\pi^2 \sqrt{pp'}} V_{pp'} \quad (131) \]

where we define

\[ V_{pp'} := \int_0^\infty dq \frac{q^2}{(q^2 - p^2)(q^2 - p'^2)} \quad (132) \]

This is computed as

\[ V_{pp'} = \int_0^\infty dq \frac{1}{q^2 - p^2} + p^2 \int_0^\infty dq \frac{1}{(q^2 - p^2)(q^2 - p'^2)} = \frac{\pi^2}{4} \delta(p - p') \quad (133) \]

where the first term vanishes from Eq. (55), and the technique to obtain Eq. (129) is used to compute the second term. Substituting Eq. (133) into Eqs. (130) and (131), we see Eq. (101) to hold.

B Green-function method for semi-infinite cavity

We re-analyze the vacuum excitation by the change of boundary condition for the semi-infinite cavity using the Green-function method [1, 10], which naturally incorporates the renormalization of zero-point energy.

B.1 Green functions

Two Hadamard elementary functions, \( F^{(1)} \) and \( G^{(1)} \), are defined by

\[ F^{(1)}(z, z') := \langle 0_f | \{ \phi(z), \phi(z') \} | 0_f \rangle, \quad G^{(1)}(z, z') := \langle 0_g | \{ \phi(z), \phi(z') \} | 0_g \rangle \quad (134) \]

where we have introduced a simplified notation \( z := (z_-, z_+) \) and \( z' := (z'_-, z'_+) \), and \( \{ \cdot, \cdot \} \) denotes the anti-commutator, \( \{ \phi, \psi \} := \phi \psi + \psi \phi \). Two Pauli-Jordan or Schwinger functions, \( F \) and \( G \), are defined by

\[ iF(z, z') := \langle 0_f | [\phi(z), \phi(z')] | 0_f \rangle, \quad iG(z, z') := \langle 0_g | [\phi(z), \phi(z')] | 0_g \rangle \quad (135) \]

Using the decompositions of field operator (77) and (78), the Hadamard elementary functions are represented as

\[ F^{(1)}(z, z') = \int_0^\infty dp [f_p(z)f_p^*(z') + \text{c.c.}] \quad (136) \]

\[ = \frac{1}{2\pi} \int_0^\infty dp \left[ \cos(p\Delta z_-) + \cos(p\Delta z_+) + \cos(p(z_- - z'_+)) + \cos(p(z_+ - z'_-)) \right], \quad (137) \]

\[ G^{(1)}(z, z') = \int_0^\infty dq [g_q(z)g_q^*(z') + \text{c.c.}] \quad (138) \]

\[ = \frac{1}{2\pi} \int_0^\infty dq \left[ \cos(q\Delta z_-) + \cos(q\Delta z_+) - \cos(q(z_- - z'_+)) + \cos(q(z_+ - z'_-)) \right], \quad (139) \]
where c.c. denotes the complex conjugate and $\Delta z_\pm := z_\pm - z'_\pm$. For the Pauli-Jordan functions, the momentum integration can be evaluated to give

$$iF(z, z') = -\frac{i}{4} [\text{sgn}(\Delta z_-) + \text{sgn}(\Delta z_+) + \text{sgn}(z_- - z'_+) + \text{sgn}(z_+ - z'_-)],$$

(140)

$$iG(z, z') = -\frac{i}{4} [\text{sgn}(\Delta z_-) + \text{sgn}(\Delta z_+) - \text{sgn}(z_- - z'_+) - \text{sgn}(z_+ - z'_-)],$$

(141)

where we have used $\int_0^\infty \frac{\sin(ax)}{x} \, dx = \pm \frac{\pi}{2a} \, (a \gtrless 0)$ \cite[p. 251]{18}.

**B.2 From Neumann to Dirichlet**

The vacuum expectation value of energy-momentum tensor before the change of boundary condition is obtained by differentiating the Hadamard elementary function $F^{(1)}$ with respect to two points $z$ and $z'$, and taking the same-point limit $z' \to z$,

$$\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t<0}^{\text{Green}} = \frac{1}{2} \lim_{z' \to z} \frac{\partial_\pm \partial'_\pm F^{(1)}(z, z')}{\Delta z_\pm}.$$  

(142)

From Eqs. (137) and (142), one obtains

$$\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t<0}^{\text{Green}} = \lim_{z' \to z} \frac{1}{4\pi} \int_0^\infty dp \cos(p \Delta z_\pm)$$

$$= \lim_{z' \to z} \frac{-1}{4\pi (\Delta z_\pm)^2}.$$  

(143)

One can see that Eq. (143) reproduces Eq. (89) if one takes limit $z' \to z$ before the $p$-integration. Equation (144) shows that $\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t<0}^{\text{Green}}$ contains the ultraviolet divergence $\sim 1/(\Delta z_\pm)^2$, which is the vacuum energy due to the zero-point oscillation always existing even in a free Minkowski spacetime. Therefore, the renormalized energy-momentum is defined by subtracting this ultraviolet divergence as

$$\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t<0}^{\text{ren}} := \langle 0_f | T_{\pm\pm} | 0_f \rangle_{t<0}^{\text{Green}} - \lim_{z' \to z} \frac{-1}{4\pi (\Delta z_\pm)^2}$$

$$= 0,$$  

(145)

(146)

which reasonably vanishes before changing the boundary condition.

The vacuum expectation value of energy-momentum tensor after the change the boundary condition has the same expression as Eq. (132). However, since the boundary condition is changed at $t = 0$, Hadamard elementary function $F^{(1)}$ before the change of boundary condition has to be propagated into $t > 0$ region using Pauli-Jordan function $iG$ \cite{10}. Thus, the energy-momentum is represented as

$$\langle 0_f | T_{\pm\pm} | 0_f \rangle_{t>0}^{\text{Green}} = \frac{1}{2} \lim_{A \to B} \partial_\pm \partial'_\pm [iG(A, C) iG(B, D) F^{(1)}(C, D)],$$

(147)

where $A := z$ and $B := z'$. Namely, in this abbreviated notation, let a capital Latin letters (except $G$ and $F$) denote a world point, e.g., $\phi(A, B) = \phi(z, z')$. In addition, let a pair of repeated capital Latin letter denote the Klein-Gordon inner product at $t = 0$, e.g., $\langle \phi, \psi \rangle|_{t=0} = \langle \phi, \psi \rangle|_{t=0}$. 

27
Substituting Eq. (136) into Eq. (137), one obtains
\[
\langle 0_f | T_{zz}^\pm | 0_f \rangle_{t>0}^{\text{Green}} = \frac{1}{2} \lim_{B \to A} \int_0^\infty dp \left( i \partial^\pm G(A, C) f_p(C) \langle i \partial^\pm G(B, D) f_p(D) \rangle^* + \text{c.c.} \right),
\]
where we have used the property of inner product \( \langle \phi, \psi^* \rangle = -\langle \phi^*, \psi \rangle^* \). The inner product in Eq. (148) can be written as
\[
i \partial^\pm G(A, B) f_p(B) = \int_0^\infty dx' \left[ \partial^\pm G(z, z') \partial_{t'} f_p(z') - \partial_{t'} \partial^\pm G(z, z') f_p(z') \right]_{t'=0}.
\]
Using Eq. (141), derivatives of \( G \) in Eq. (149) are computed as
\[
\partial^\pm G(z, z')_{t'=0} = -\frac{1}{2} \left[ \delta(x' \mp z_\pm) - \delta(x' \pm z_\pm) \right],
\]
\[
\partial_{t'} \partial^\pm G(z, z')_{t'=0} = \mp \frac{1}{2} \partial_{t'} \left[ \delta(x' \mp z_\pm) + \delta(x' \pm z_\pm) \right],
\]
where \( \text{sgn}'(x) = 2\delta(x) \) was used. Substituting Eqs. (150) and (151) into Eq. (149), one obtains
\[
i \partial^\pm G(A, B) f_p(B) = \pm \frac{i}{2} \sqrt{\frac{p}{\pi}} \text{sgn}(z_\pm) e^{-ipz_\pm} \mp \frac{\delta(z_\pm)}{\sqrt{\pi p}},
\]
where we have used
\[
\int_0^\infty \delta(x - a) f(x) dx = \theta(a) f(a),
\]
\[
\theta(\pm x) - \theta(\mp x) = \pm \text{sgn}(x).
\]
With Eq. (152), Eq. (148) yields
\[
\langle 0_f | T_{zz}^\pm | 0_f \rangle_{t>0}^{\text{Green}} = \frac{\delta^2(z_\pm)}{\pi} \int_0^\infty \frac{dp}{p} + \text{sgn}^2(z_\pm) \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_\pm)^2}.
\]
The last term in Eq. (155) shows that \( \langle 0_g | T_{zz}^\pm | 0_g \rangle_{t>0}^{\text{Green}} \) contains the ultraviolet divergence due to zero-point oscillation. Thus, the renormalized energy-momentum is defined in the same way as Eq. (145) by subtracting the zero-point energy,
\[
\langle 0_f | T_{zz}^\pm | 0_f \rangle_{t>0}^{\text{ren}} := \langle 0_f | T_{zz}^\pm | 0_f \rangle_{t>0}^{\text{Green}} - \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_\pm)^2} \quad (z_\pm = 0)
\]
\[
= \frac{\delta^2(z_\pm)}{\pi} \int_0^\infty \frac{dp}{p} + \begin{cases} \frac{1}{4\pi(\Delta z_\pm)^2} & (z_\pm = 0) \\ 0 & \text{(otherwise)} \end{cases},
\]
which is nothing but Eq. (97).

### B.3 From Dirichlet to Neumann

The vacuum expectation value of energy-momentum tensor before the change of boundary condition is given by
\[
\langle 0_g | T_{zz}^\pm | 0_g \rangle_{t<0}^{\text{Green}} = \frac{1}{2} \lim_{z' \to z} \partial_{z'} \partial_{z'}^\prime G^{(1)}(z, z').
\]
Substituting Eq. (139) into Eq. (158), one obtains
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t<0}^{\text{Green}} = \lim_{z' \to z} \frac{1}{4\pi} \int_0^\infty dq \cos(q\Delta z_{\pm}) = \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_{\pm})^2}.
\] (159)

This represents the ultraviolet divergence due to the zero-point oscillation. The normalized energy-momentum is defined by subtracting such a divergence,
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t<0}^{\text{ren}} := \langle 0_g | T_{\pm \pm} | 0_g \rangle_{t<0}^{\text{Green}} - \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_{\pm})^2} = 0,
\] (160)
which reasonably vanishes before the change of boundary condition.

The energy-momentum after the change of boundary condition is obtained by propagating $G^{(1)}$ by $iF$,
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0}^{\text{Green}} = \frac{1}{2} \lim_{B \to A} \partial_\perp \partial_{\perp}' [iF(A,C)iF(B,D)G^{(1)}(C,D)].
\] (161)

Substituting Eq. (138), this quantity is represented as
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0}^{\text{Green}} = \frac{1}{2} \lim_{B \to A} \int_0^\infty dq \left( i\partial_\perp F(A,C)g_q(C)[i\partial'_{\perp} F(B,D)g_q(D)]^* + c.c. \right).
\] (162)

The inner product in Eq. (162) is written as
\[
i\partial_{\perp} F(A,B)g_q(B) = \int_0^\infty dx' [\partial_{\perp} F(z',z')\partial_t g_q(z') - \partial_t' \partial_{\perp} F(z,z')g_q(z')]|_{t'=0}.
\] (163)

Using Eq. (130), derivatives of $F$ in Eq. (163) are computed as
\[
\partial_{\perp} F(z,z')|_{t'=0} = -\frac{1}{2} \left[ \delta(x' \pm z_{\pm}) + \delta(x' \mp z_{\pm}) \right],
\] (164)
\[
\partial_t \partial_{\perp} F(z,z')|_{t'=0} = \pm \frac{1}{2} \partial_{x'} [\delta(x' \mp z_{\pm}) - \delta(x' \pm z_{\pm})].
\] (165)

Substitution of Eqs. (164) and (165) into Eq. (163) yields
\[
i\partial_{\perp} F(A,B)g_q(B) = -\frac{1}{2} \sqrt{\frac{q}{\pi}} \text{sgn}(z_{\pm}) e^{-iqz_{\pm}},
\] (166)
where we have used formulas (153) and (154). The combination of Eqs. (166) and (162) gives
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0}^{\text{Green}} = \text{sgn}^2(z_{\pm}) \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_{\pm})^2}.
\] (167)

The renormalized energy-momentum is obtained by subtracting the zero-point energy (159) from Eq. (167),
\[
\langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0}^{\text{ren}} := \langle 0_g | T_{\pm \pm} | 0_g \rangle_{t>0}^{\text{Green}} - \lim_{z' \to z} \frac{-1}{4\pi(\Delta z_{\pm})^2} \begin{cases} 
\frac{1}{4\pi(\Delta z_{\pm})^2} & (z_{\pm} = 0) \\
0 & (\text{otherwise}) 
\end{cases},
\] (168)
which is nothing but Eq. (107).
Figure 7: Two closed contours $C_+$ and $C_-$ in the complex plane, each of which contains an infinitely large semicircle and two infinitesimal semicircles to avoid $-b$ and $+b$ on the real axis.

C Integral formulas (95) and (96)

Let us calculate the principal values of following integrals,

\[
I := \int_0^\infty \frac{\cos(ax)}{x^2 - b^2} \, dx, \quad J := \int_0^\infty \frac{x \sin(ax)}{x^2 - b^2} \, dx, \tag{169}
\]

where $-\infty < a < \infty$, $b > 0$. Note that we always consider only principal values for improper integrals. These are written as

\[
I = \frac{1}{4}(I_+ + I_-), \quad I_\pm := \int_{-\infty}^\infty \frac{e^{\pm iax}}{x^2 - b^2} \, dx, \tag{170}
\]

\[
J = \frac{1}{4i}(J_+ - J_-), \quad J_\pm := \int_{-\infty}^\infty \frac{xe^{\pm iax}}{x^2 - b^2} \, dx. \tag{171}
\]

We suppose two contours $C_+$ and $C_-$ drawn in Fig. 7 and use Cauchy’s integral theorem and the residue theorem.

For $a > 0$, taking contour $C_\pm$ for $I_\pm$ and $J_\pm$, we have

\[
0 = \int_{C_\pm} \frac{e^{\pm iaz}}{z^2 - b^2} \, dz = I_\pm \mp \frac{1}{2} \cdot 2\pi i \text{Res}[I_\pm, -b] \mp \frac{1}{2} \cdot 2\pi i \text{Res}[I_\pm, b], \tag{172}
\]

\[
0 = \int_{C_\pm} \frac{ze^{\pm iaz}}{z^2 - b^2} \, dz = J_\pm \mp \frac{1}{2} \cdot 2\pi i \text{Res}[J_\pm, -b] \mp \frac{1}{2} \cdot 2\pi i \text{Res}[J_\pm, b]. \tag{173}
\]

Here, $\text{Res}[X, z_0]$ denotes the residue of integrand of $X$ at $z = z_0$, and the contributions from the large semicircles vanish from Jordan’s lemma. Substituting the following values of residues,

\[
\text{Res}[I_\pm, -b] = -\frac{e^{\pm iab}}{2b}, \quad \text{Res}[I_\pm, b] = \frac{e^{\pm iab}}{2b}, \quad \text{Res}[J_\pm, -b] = \frac{e^{\pm iab}}{2}, \quad \text{Res}[J_\pm, b] = \frac{e^{\pm iab}}{2} \tag{174}
\]

into Eqs. (172) and (173), we have

\[
I_\pm = -\frac{\pi}{b} \sin(ab), \quad J_\pm = \pm i\pi \cos(ab), \quad (a > 0). \tag{175}
\]
For $a < 0$, taking contour $C_\mp$ for $I_\pm$ and $J_\pm$, we have

$$0 = \int_{C_\mp} \frac{e^{\pm iaz}}{z^2 - b^2} \, dz = I_\pm \mp \frac{1}{2} \cdot 2\pi i \text{Res}[I_\pm, -b] \pm \frac{1}{2} \cdot 2\pi i \text{Res}[I_\pm, b], \quad (176)$$

$$0 = \int_{C_\mp} \frac{ze^{\pm iaz}}{z^2 - b^2} \, dz = J_\pm \mp \frac{1}{2} \cdot 2\pi i \text{Res}[J_\pm, -b] \pm \frac{1}{2} \cdot 2\pi i \text{Res}[J_\pm, b]. \quad (177)$$

Using Eq. (174) again, we have

$$I_\pm = \frac{\pi}{b} \sin(ab), \quad J_\pm = \mp i \pi \cos(ab), \quad (a < 0). \quad (178)$$

For $a = 0$, taking either $C_+$ or $C_-$ for $I_\pm$, one can show that $I_\pm$ vanishes. In addition, $J$ obviously vanishes by definition (169). Thus, we have

$$I_\pm = J = 0, \quad (a = 0). \quad (179)$$

Combining Eqs. (170), (171), (175), (178), and (179), we see formulas (95) and (96) to hold.

References

[1] N. D. Birrell and P. C. W. Davies, “Quantum Fields in Curved Space,” Cambridge University Press, Cambridge, UK (1984).

[2] A. D. Shapere, F. Wilczek and Z. Xiong, “Models of Topology Change,” arXiv:1210.3545 [hep-th].

[3] A. Anderson and B. S. DeWitt, “Does the Topology of Space Fluctuate?,” Found. Phys. 16, 91 (1986).

[4] C. A. Manogue, E. Copeland, and T. Dray, “The trousers problem revisited,” Pramana 30, 4, 279 (1998).

[5] A. Ishibashi and A. Hosoya, “Naked singularity and thunderbolt,” Phys. Rev. D 66 (2002) 104016 [gr-qc/0207054].

[6] G. T. Moore, “Quantum Theory of the Electromagnetic Field in a Variable-Length One-Dimensional Cavity,” J. Math. Phys. 11, 2679 (1970).

[7] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simonen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, “Observation of the dynamical Casimir effect in a superconducting circuit,” Nature 479, 376 (2011) [arXiv:1105.4714 [quant-ph]].

[8] A. Ishibashi and A. Hosoya, “Who’s afraid of naked singularities? Probing timelike singularities with finite energy waves,” Phys. Rev. D 60 (1999) 104028 [gr-qc/9907009].

[9] E. G. Brown and J. Louko, “Smooth and sharp creation of a Dirichlet wall in 1+1 quantum field theory: how singular is the sharp creation limit?,” JHEP 1508, 061 (2015) [arXiv:1504.05269 [hep-th]].

31
[10] T. Harada, S. Kinoshita and U. Miyamoto, “Vacuum excitation by sudden appearance and disappearance of a Dirichlet wall in a cavity,” Phys. Rev. D 94 (2016) no.2, 025006 [arXiv:1601.01172 [hep-th]].

[11] R. M. Wald, “Quantum field theory in curved space-time and black hole thermodynamics,” Chicago University Press, USA (1994).

[12] H. B. G. Casimir, “On the attraction between two perfectly conducting plates,” Proc. K. Ned. Akad. Wet. 51, 793 (1948).

[13] Y. Otsuki and Y. Muroya, “Shin Sugaku Koshiki Shu,” in Japanese, Maruzen Co., Ltd., Tokyo, Japan (1991).

[14] L. J. Zhou, M. E. Carrington, G. Kunstatter and J. Louko, “Smooth and sharp creation of a pointlike source for a (3+1)-dimensional quantum field,” Phys. Rev. D 95 (2017) no.8, 085007 [arXiv:1610.08455 [hep-th]].

[15] D. F. Griffiths, J. W. Dold, and D. J. Slivester, “Essential Partial Differential Equations: Analytical and Computational Aspects,” Springer International Publishing, Switzerland (2015).

[16] S. Moriguchi, K. Udagawa, and S. Hitotsumatsu, “Sugaku Koshiki II,” in Japanese, Iwanami Shoten, Tokyo, Japan (1957).

[17] G. B. Arfken and H. J. Weber, “Mathematical Methods for Physicists (6th edition),” Elsevier Academic Press, California, USA (2005).

[18] S. Moriguchi, K. Udagawa, and S. Hitotsumatsu, “Sugaku Koshiki I,” in Japanese, Iwanami Shoten, Tokyo, Japan (1957).