Isocurvature constraints on gravitationally produced superheavy dark matter

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We show that the isocurvature perturbations imply that the gravitationally produced superheavy dark matter must have masses larger than few times the Hubble expansion rate at the end of inflation. This together with the bound on tensor to scalar contribution to the CMB induces a lower bound on the reheating temperature for superheavy dark matter to be about $10^7$ GeV. Hence, if the superheavy dark matter scenario is embedded in supergravity models with gravity mediated SUSY breaking, the gravitino bound will squeeze this scenario. Furthermore, the CMB constraint strengthens the statement that gravitationally produced superheavy dark matter scenario prefers a relatively large tensor mode amplitude if the reheating temperature must be less than $10^9$ GeV.

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I. INTRODUCTION

The primordial density perturbations which provide initial conditions for structure formation [1] can be classified as two types: curvature and isocurvature. Isocurvature perturbations are defined to be fluctuations in the composition of the energy density. Hence, the curvature remains fixed while the relative contributions of various fluid elements composing the total energy density changes. More explicitly, the isocurvature perturbation due to a cold dark matter particle species $X$ can be written as

$$\delta_S \equiv \frac{\delta n_X}{n_X} - \frac{\delta n_{\gamma}}{n_{\gamma}},$$

where $n_{\gamma}$ is the density of photons and $n_X$ is the density of the dark matter particles. As can be seen in the appendix, generically, isocurvature perturbations are almost always synonymous with entropy perturbations. Although the isocurvature perturbations do not give rise to potential energy during radiation domination, after matter domination occurs, the isocurvature perturbations also become source for gravitational potential energy leaving an imprint on the CMB.

In the late 80’s and early 90’s, isocurvature and curvature perturbation models were seen to be competing models for the theory of initial conditions for structure formation. Today, from the precise measurements of the CMB acoustic peaks and initial measurements of the TE correlations [2, 3], most people agree that the dominant contribution to the primordial density fluctuations arise from curvature perturbations. Nonetheless, as demonstrated by many recent works [2, 4, 5, 6, 7, 8, 9, 10], an order 10% contribution to the CMB power spectrum from isocurvature is possible. Perhaps more importantly, we generically expect from theoretical considerations a nonvanishing contribution to the...
CMB from isocurvature perturbations. (In other words, whenever there is more than one energy component in the universe, there is always some amount of isocurvature perturbations.)

In the most popular WIMP CDM (cold dark matter) picture in inflationary cosmology, the isocurvature perturbations are expected to be small because CDM abundance today is determined from a freeze out process starting from chemical equilibrium initial conditions. Since the chemical equilibrium is with radiation, which, by definition, has overdensities defined by curvature fluctuations during radiation domination, CDM in this case will naturally have suppressed isocurvature component since

$$\frac{\delta n_X}{n_X} = \frac{\delta n_\gamma}{n_\gamma} = \frac{3\delta T}{T},$$

(2)

where $X$ denotes the dark matter density, $\gamma$ denotes radiation, and $T$ denotes the equilibrium temperature.

However, for superheavy dark matter which never equilibrates, the isocurvature perturbations are generically not negligible. Furthermore, because, for gravitational particle production scenario, the superheavy dark matter homogenous density is determined by the same inflationary dynamics as the isocurvature perturbations themselves, one obtains a correlated constraint for the superheavy dark matter abundance and the isocurvature perturbation amplitude. In this paper, we compute the generic feature of this correlated constraint.

The gravitational production of superheavy dark matter has been well studied in the past [11, 12, 13, 14, 15]. In this scenario, the nonadiabatic change in the particle dispersion relationship due to a relatively sudden transition from a quasi-de Sitter phase of inflation to a small power law expansion phase of FRW cosmology causes particle production. The only non-generic feature of this dark matter scenario is the existence of stable particles with heavy mass. For $10^{11} - 10^{13}$ GeV particles, up to dimension 8 decay operators need to be absent to have the particle be long lived enough to be CDM. Nonetheless, there are many particle physics and string theory motivated candidates for superheavy dark matter (see for example [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]). Hence, what makes this scenario exciting is that if we can measure the signatures of superheavy dark matter, we would have a probe of the very early universe where the physics far beyond the standard model of particle physics was important. (Indeed, it may even probe string theory.) This paper serves as one step towards identifying the signature of superheavy dark matter on the CMB.

In particular, we investigate how much superheavy dark matter isocurvature perturbations are produced at around 50 efolds before the end of inflation and how those isocurvature perturbations will show up on the CMB. We find that the mass of the superheavy dark matter cannot be too small if one is to avoid overproducing isocurvature perturbations inconsistent with the CMB data. More specifically, the constraint is

$$\frac{m_X}{H_e} \gtrsim O(5),$$

(3)

where $m_X$ is the mass of the superheavy dark matter and $H_e$ is the Hubble expansion rate at the end of inflation. One implication of this is to strengthen the preference of large tensor perturbations for gravitationally produced superheavy dark matter scenario. More specifically, if the reheating temperature is less than $10^9$ GeV, then the tensor to scalar power ratio is bounded from below by

$$\frac{P_T}{P_R} \gtrsim 10^{-3}.$$  

(4)

Another implication of Eq. (3) is that there is a lower bound on the reheating temperature for superheavy dark matter scenario to be viable since the number density of particles produced falls off as $m_X/H_e$ and increasing the reheating temperature increases the final number density. Putting this together with the bound on the tensor perturbations from inflation, we find the approximate requirement

$$T_{RH} \gtrsim 10^7 \left(\frac{0.2}{r_S}\right) \text{ GeV}$$

(5)

in order to have the superheavy dark matter be the CDM while not contributing too much to isocurvature perturbations where $r_S$ represents the bound on the ratio of tensor to scalar perturbation power. Hence, we have a severe restriction on this scenario coming from the gravitino bound (see for example [27, 28, 29, 30, 31, 32, 33, 34] if this scenario is embedded in gravity mediate SUSY breaking scenario (for an introductory review of status of low energy supersymmetry, see for example, [35]). Note that given that both the superheavy dark matter and the gravity waves are produced by inflationary energy density, it is natural that there is a bound of the form Eq. (3). What is not obvious about Eq. (5) is how the bound on isocurvature perturbations make Eq. (3) independent of the mass of the superheavy dark matter. The explanation of this is one of the main results of this paper.
In the present work, we will consider superheavy dark matter model of massive scalar field $X$ minimally coupled to gravity without any other interaction for $X$. The results are not likely to change drastically with fermionic superheavy dark matter since none of the physics relies upon the existence of a vacuum expectation value of the dark matter field nor spin statistics. Furthermore, we restrict ourselves to minimal coupling in this paper. For any fixed dark matter mass, nonminimal coupling will change the total amount of dark matter produced, the spectral distribution of the dark matter, and the dark matter’s stress tensor correlator. The most important effect is likely to come from the dark matter’s stress tensor correlator. Because a coupling of the form $\xi \phi^2 R$ (where $R$ is the Ricci scalar) effectively shifts the mass of the superheavy dark matter as $m_X^2 \rightarrow m_X^2 + 12 \xi H^2$ (where $H$ is the expansion rate during inflation), a positive $\xi$ results will relax the mass bound shown in Eq. (8), possibly to an extent where there will be no constraint from isocurvature perturbations. On the other hand, a negative value for the nonminimal coupling will make the bound more stringent.

The order of presentation is as follows. In Section 2, we make general estimates to establish the results associated with Eqs. (3), (4), and (5). Following that, we review the basic physics of CMB and its relationship to isocurvature perturbations. We then compute the isocurvature power spectrum (essentially stress tensor correlation function) in Section 4. In Section 5, we present numerical results with a special emphasis on a particular inflationary model as a check of our general argument. Finally, we conclude in Section 6. In the appendix A, we give an elementary thermodynamic discussion of isocurvature perturbations to remind the reader of its physics. The technical details of the particle production computation for the toy model used in Section 5 is presented in Appendix B.

II. GENERAL ARGUMENTS

It is possible to semi-quantitatively describe the mass bound coming from isocurvature perturbations as follows. Field fluctuations are typically characterized by its $N$-point moments, and the most elementary nontrivial moment for perturbative theories is typically the two point function. The power spectrum of any two point function in a translationally invariant space is defined as

$$P_X(k) = \frac{k^3}{2\pi^2} \int d^3 r e^{-i\vec{k} \cdot \vec{r}} \langle X(\vec{x})X(\vec{y}) \rangle,$$

where $\vec{r}$ is defined to be $\vec{r} = \vec{x} - \vec{y}$. As will discuss below, the isocurvature perturbations that we are concerned with are characterized by the power spectrum

$$P_{\delta_X}(k) = \frac{k^3}{2\pi^2} \int d^3 r e^{-i\vec{k} \cdot \vec{r}} \langle \delta_X(\vec{x})\delta_X(\vec{y}) \rangle,$$

where $\delta_X \equiv \delta \rho_X / \rho_X$ is the CDM (superheavy dark matter in our case) energy overdensity.

As an order of magnitude estimate, we can represent the energy overdensity field as $\delta \rho_X \sim m_X^2 X^2$, which gives

$$P_{\delta_X}(k) \sim \frac{k^3 m_X^4}{2\pi^2 \rho_X^2} \int d^3 r e^{-i\vec{k} \cdot \vec{r}} \langle X^2(\vec{x})X^2(\vec{y}) \rangle \sim \frac{k^3 m_X^4}{2\pi^2 \rho_X^2} \int d^3 r e^{-i\vec{k} \cdot \vec{r}} \langle X(\vec{x})X(\vec{y}) \rangle^2.$$

Hence, in the long wavelength limit, we can approximate this as

$$P_{\delta_X}(k) \sim \frac{m_X^2 P_X^2(k)}{\rho_X^2}.$$

For scalar fields $X$ in FRW spacetimes in general, if $V''(X) \gg H^2$ (where $V$ is the scalar field potential and $H$ is the expansion rate), we know that the correlation function will behave as in flat Minkowski space. In that case, we know from Minkowski space field theory that the two point correlation functions receive most of its power from short distances. We also know that for long wavelengths ($k \rightarrow 0$) in de Sitter (dS) space, when $V'' \ll H^2$, that

$$P_X \sim \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^j,$$

where $(k/aH)^j$ represents the factor that is breaking scale invariance due to the fact that $V''(X) \neq 0$. 1 Hence, using the Minkowski intuition and the fact that there is generically large power contribution in the infrared for $V'' < 0$ (due

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1 It is important to emphasize that the form of Eq. (10) is only valid when $V''/H^2 \ll 9/4$ in the minimal coupling scenario. On the other hand, for nonminimal coupling, this form is valid only if $V''/H^2 \ll 9/4 - 12\xi$. 

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to instability), we can estimate the power index of Eq. (10) as
\[ j = \frac{Q}{H^2}, \tag{11} \]
where \( Q > 0 \) is a positive order unity dimensionless number. We thus find
\[ P_{\delta_X}(k) \sim \frac{m_X^4}{\rho_X(t_e)} \left( \frac{H_I}{2\pi} \right)^4 \left( \frac{k}{a_I H_I} \right)^{2QV''/H^2}. \tag{12} \]

Since inflation eventually ends and since the gravitational particle production does not really “occur” until the end of inflation, we have to assign the time at which we evaluate the time dependent quantities carefully. Since the isocurvature fluctuations are generated as the scales leave the horizon early in the inflationary epoch, we assign \( H \) to the expansion rate at about 50 efoldings before the end of inflation. On the other hand, the scaling of the wave vector continues until the end of inflation at which time particles are nonadiabatically produced. After the nonadiabatic particle production occurs, the power spectrum should remain approximately time independent in the long wavelength limit owing to the arguments of causality and adiabatic evolution. Hence, we write
\[ P_{\delta_X}(k) \sim \frac{m_X^4}{\rho_X(t_e)} \left( \frac{H_I}{2\pi} \right)^4 \left( \frac{k}{a_I H_I} \right)^{2QV''/H^2}, \tag{13} \]
where \( H_I \) represents the expansion rate at about 50 efoldings before the end of inflation and \( \rho_e \) represents the CDM energy density at the end of inflation. In Section 4 we will compute this more carefully and find that although this gives the right order of magnitude, there is comparable contribution coming from the kinetic terms neglected in this estimate.

The dark matter energy that appears in the denominator of Eq. (10) is not easy to estimate for minimal coupling because significant particle production contribution comes from the infrared which have ambiguous vacuum boundary conditions. As shown in the appendix, neglecting this infrared contribution, one finds
\[ 0.7 \times 10^{-3} m_X(m_\phi m_X)^{3/2} \exp \left( -\frac{5m_X}{m_\phi} \right) \leq \rho_X(t_e) \leq 1.6 \times 10^{-3} m_X(m_\phi m_X)^{3/2} \exp \left( -\frac{4.2m_X}{m_\phi} \right) \tag{14} \]
for \( V = \frac{1}{2} m_\phi^2 \phi^2 \) potential. More generally, we expect that
\[ \rho_e \sim 10^{-3} m_X(H_e m_X)^{3/2} \exp \left( -\frac{2m_X}{H_e} \right). \tag{15} \]
Hence, we generically find
\[ P_{\delta_X}(k) \sim 600 m_X \left( \frac{H_I}{H_e} \right)^3 \left( \frac{k}{a_e H_I} \right)^{2Q m_X^2/H_I^2} \exp \left( \frac{4m_X}{H_e} \right), \tag{16} \]
where we have assumed \( V'' \approx m_X^2 \) consistently with the particle production. Although naively, one would guess that the exponential would dominated on the right hand side of Eq. (16), in reality the \( k \)-dependent factor dominates since for large scales of interest for CMB, we have
\[ \ln \left( \frac{k}{a_e H_I} \right) \sim \ln \left( \frac{a_0 H_0}{a_e H_I} \right) \sim -55 + \ln \left( \frac{T_{RH}}{10^9 \text{GeV}} \right)^{-1/3} \left( \frac{V_e}{10^{15} \text{GeV}} \right)^{-1/6}, \tag{17} \]
which implies that the \( k \)-dependent term dominates:
\[ P_{\delta_X}(a_0 H_0) \sim 600 m_X \left( \frac{H_I}{H_e} \right)^3 \exp \left( \frac{4m_X}{H_e} - 110Q m_X^2/H_I^2 \right). \tag{18} \]
Hence, we see the inflationary model dependence of the power spectrum clearly in the ratio of \( H_I/H_e \) (ratio of the expansion rates at the beginning and end of inflation) and \( m_X/H_I \). Despite this model dependence, we would generically conclude \( m_X/H_I > O(1) \). More precisely, since the isocurvature contribution must not exceed the adiabatic contribution to \( C_I \), we have for small \( l \sim 2 \)
\[ C_{l=2}^{(X)} \sim 0.1 P_{\delta_X}(a_0 H_0) \lesssim 10^{-10}, \tag{19} \]
which gives a bound of
\[
\frac{m_X}{H_e} \gtrsim \frac{4}{\sqrt{Q_x}} \sqrt{\frac{1}{1 + \frac{0.05}{Q_x^2}}} + \frac{1}{Q_x^2},
\]
(20)
where \( x = 7H_e/H_I \) is close to unity. If we for example take \( Q = 2/3 \) and \( x = 1 \), we obtain \( m_X/H_e \gtrsim 6 \).

Since this is similar to the result of our more careful computation, we will use this bound as our example for the rest of this section. However, more generally, the bound is \( m_X/H_e \gtrsim O(5) \).

Now, according to our estimate of particle production shown in the appendix, the dark matter density today is given by
\[
\Omega_X h^2 \approx 4.31 \times 10^{-5} \frac{T_{RH}}{T_0} \frac{\rho_X(t_e)}{\rho(t_e)} \sim 10^4 \left( \frac{m_X}{10^{13}\text{GeV}} \right)^2 \frac{T_{RH}}{10^9\text{GeV}} \left( \frac{m_X}{H_e} \right)^{1/2} e^{-2m_X/H_e}.
\]
(21)
This implies that if \( m_X/H_e \gtrsim 6 \), we must increase the relic abundance by increasing \( m_X \) from \( 3 \times 10^{13}\text{GeV} \) while keeping the exponential fixed or increase \( T_{RH} \) or increase both \( m_X \) and \( T_{RH} \). Note that \( H_e \) dependence only comes through \( m_X/H_e \).

Keeping \( T_{RH} \) fixed, increasing \( m_X \) from \( 10^{13} \text{GeV} \) for a fixed ratio \( m_X/H_e \gtrsim 6 \) is restricted by a bound on the amplitude of the gravitational wave (limiting \( H_e \)). The CMB temperature fluctuation normalization \(^2\) fixes \( P_T \approx 2 \times 10^{-9} \) and since the tensor perturbation power spectrum is
\[
P_T = \left( \frac{H_I}{M_P(2\pi)} \right)^2,
\]
(22)
where \( M_P = 2 \times 10^{18}\text{GeV} \) is the reduced Planck mass, we have the tensor to scalar ratio
\[
\frac{P_T}{P_R} = 5 \times 10^8 \frac{H_I^2}{M_P^2(2\pi)^2} = 0.02 \left( \frac{H_I}{10^{14}\text{GeV}} \right)^2,
\]
(23)
which may be a detectable amplitude for \( H_I = 10^{14}\text{GeV} \) \(^3\) \(^4\) \(^5\). If the experimental bound on the tensor to scalar ratio is \( r_S \) (i.e., \( P_T/P_R < r_S \)), we have
\[
H_I < 7 \times 10^{14}\sqrt{r_S}\text{GeV},
\]
(24)
and consequently, we cannot raise \( H_e \) much above \( 10^{14} \text{GeV} \).\(^2\) Therefore, by Eq. (20) and recalling that we are for the moment focusing on fixing \( m_X/H_e \gtrsim 6 \) such that \( \Omega_X h^2 \sim 0.1 \), we find the bound on the superheavy dark matter mass of
\[
6 \lesssim \frac{m_X}{H_e} \lesssim 12 \left[ 1 + \frac{1}{24} \ln \left( \frac{H_e^2}{H_I^2} r_S \right) + \frac{1}{24} \ln \frac{T_{RH}}{10^9\text{GeV}} \right],
\]
(25)
where we have approximately solved a transcendental equation by iteration. This corresponds to an upper bound on the dark matter mass of
\[
m_X \lesssim 10^{15} \left( \frac{7H_e}{H_I} \right)^{1/2} \sqrt{r_S} \left[ 1 + \frac{1}{24} \ln \left( \frac{H_e^2}{H_I^2} r_S \right) + \frac{1}{24} \ln \frac{T_{RH}}{10^9\text{GeV}} \right].
\]
(26)

Now, suppose we want to find the minimum allowed \( T_{RH} \) while maintaining \( m_X/H_e \gtrsim 6 \). We can then solve for \( T_{RH} \) in Eq. (21) with \( \Omega_X h^2 = 0.1 \) and use Eq. (21) to obtain
\[
T_{RH} \gtrsim 10^7 \frac{0.2}{r_S} \left( \frac{H_I}{7H_e} \right)^2 \text{GeV},
\]
(27)
where the 0.2 factor would be much smaller if we were not restricted to \( m_X/H_e \gtrsim 6 \). This may be dangerous for the gravitino bound if the scenario is embedded in gravity mediated SUSY breaking scenario. In our explicit computations for \( V(\phi) = \frac{1}{2} m_S^2 \phi^2 \) model of slow roll inflation, we obtain a bound of \( T_{RH} \gtrsim 10^8 \text{GeV} \).

\(^2\) Eq. (20) with \( r_S = 1 \) is about the same bound one obtains in single field slow roll inflationary scenarios.
From Eqs. (26) and (27), we see that if we tighten the bound on the tensor amplitudes (by making $r_S$ smaller), the superheavy dark matter scenario can be severely restricted. In particular, the linear dependence of Eq. (27) on $1/r_S$ can change the reheating lower bound significantly.

Note if we require that $T_{RH} \lesssim 10^9$ GeV (for example, because we want to evade the gravitino bound), we have from Eq. (21) and Eq. (20), that

$$\Omega_X h^2 \lesssim 10^{-2} \left( \frac{m_X}{10^{13} \text{ GeV}} \right)^2.$$  \hspace{1cm} (28)

Since $\Omega_X h^2 \gtrsim 0.1$, we conclude $m_X \gtrsim 3 \times 10^{13}$ GeV, which implies

$$H_e \gtrsim 3 \times 10^{12} \text{ GeV},$$  \hspace{1cm} (29)

or according to Eq. (20) that

$$\frac{P_T}{P_K} \gtrsim 10^{-3} \left( \frac{H_I}{7H_e} \right)^2.$$  \hspace{1cm} (30)

Contrast this with the case in which we do not impose $m_X/H_e \gtrsim 6$. In that case, we can fix the value of the relic density, Eq. (21), to be $\Omega_X h^2 = 0.1$, impose $T_{RH} < 10^9$ GeV, and minimize the value of $H_e$ to obtain $H_e \gtrsim 3 \times 10^{11}$ GeV, where the minimization of $H_e$ occurs for $m_X/H_e = 5/4$. Since $H_e$ is one order of magnitude smaller, the gravitational wave signal in this case is bounded from below by a number that is 100 times smaller than Eq. (30). Hence, the isocurvature perturbation constraint strengthens the preference of large tensor perturbations for the gravitationally produced superheavy dark matter scenario.

### III. CMB TEMPERATURE FLUCTUATIONS AND ISOCURVATURE PERTURBATIONS

#### A. Dynamics in the tight-coupling regime

Here we follow the treatment of the Mukhanov [38] (for other good analytic treatments, see [39, 40]). The background FRW metric is taken with flat spatial sections and conformal time, whose time derivative is denoted with a prime ($'$). The scalar metric perturbation parameterization is chosen to be

$$ds^2 = a^2 \left[ (1 + 2\Phi) dt^2 - (1 - 2\Phi) dx^i dx^i \right].$$  \hspace{1cm} (31)

Assuming that the cold dark matter (denoted with subscript $CDM$) is decoupled from the baryon-photon plasma, the dark-matter stress-energy conservation equations lead to

$$(\delta_X - 3\Phi)' + au_{CDM,i}^{i} = 0 \hspace{1cm} (32)$$

$$[a(\delta_X - 3\Phi)'' - a\Delta \Phi] = 0.$$  \hspace{1cm} (33)

where $u_X^i$ is the velocity of the species $X$.

With the approximation of nonrelativistic baryons, the conservation equation $T_{b,\alpha}^\alpha = 0$ for baryons leads to

$$(\delta_b - 3\Phi) + au_{b,i}^{i} = 0,$$  \hspace{1cm} (34)

which corresponds to the conservation of baryon number. The photon energy conservation leads to

$$(\delta_\gamma - 4\Phi)' + \frac{4}{3} au_{\gamma,i}^{i} = 0.$$  \hspace{1cm} (35)

Assuming that the photons and baryons are tightly coupled, we set $u^i \equiv u_{b}^i = u_{\gamma}^i$, which from Eqs. (34) and (35) leads to

$$\frac{3}{4} \delta_\gamma - \delta_b = \text{constant}.$$  \hspace{1cm} (36)

With this tight-coupling approximation, the $T_{\gamma,\alpha}^\alpha = 0$ of the baryon-photon stress tensor leads to

$$\frac{1}{a^4} \left[ a^5 (\rho_b + \rho_\gamma + P_\gamma + P_b) u_{\gamma,i}^{i} \right]' - \frac{4}{3} \Delta u_{\gamma,i}^{i} + \Delta \delta(P_\gamma + P_b) + (\epsilon_\gamma + \epsilon_b + P_\gamma + P_b) \Delta \Phi = 0,$$  \hspace{1cm} (37)
where \( \eta \) is the viscosity:

\[
\eta = \frac{4}{15} \rho \gamma \tau \gamma,
\]

(38)

with \( \tau \gamma \) the mean free path of the photons. Assuming no isocurvature perturbations from the baryons, we use

\[
\delta_b = \frac{3}{4} \delta \gamma
\]

(39)

and arrive at

\[
\left( \frac{\delta' \gamma}{c_s^2} \right)' - \frac{3 \eta}{\rho \gamma a} \Delta \delta \gamma - \Delta \delta \gamma = \frac{4}{3} c_s^2 \Delta \Phi + \left( \frac{4 \Phi'}{c_s^2} \right)' - \frac{12 \eta}{\rho \gamma a} \Delta \Phi',
\]

(40)

with the sound speed

\[
c_s^2 = \frac{1}{3} \frac{1}{1 + 3 \rho_b/4 \rho \gamma}.
\]

(41)

Finally, the 00 component of the Einstein’s equations is

\[
\Delta \Phi - \frac{3 \alpha'}{a} \Phi' - 3 \left( \frac{\alpha'}{a} \right)^2 \Phi = \frac{4 \pi}{M_{pl}^2 a^2} \left( \rho_X \delta_X + \frac{\rho_b \delta \gamma}{3c_s^2} \right).
\]

(42)

The fields that need to be determined are \( \{ \delta_X, \delta \gamma, \Phi \} \) and the three independent equations are Eqs. (33), (40), and (42). (The only remaining equation simply determines \( u_{CDM} \).)

As with any differential equations, these need boundary conditions. The CDM fluctuation initial spectrum is determined by a quantum computation of \( \langle \delta_X \delta_X \rangle \) which is assumed to evolve collisionlessly at least until decoupling. Similarly, quantum computation of adiabatic perturbations \( \langle \zeta \zeta \rangle \) provide another set of boundary conditions. Finally, since the 00 component of Einstein’s equation, Eq. (42), is a parabolic equation for \( \Phi \), we only require one boundary condition for \( \Phi \). The fact there is a growing solution supported by a source obviates the need for a boundary condition for \( \Phi \) for the cases of our interest.

**B. Relationship between CMB temperature and isocurvature fluctuation**

Let the temperature of the CMB photons be a field \( T + \Delta T(\tau, \vec{x}, \hat{p}) \) where \( \hat{p} = p_i^c/|\vec{p}_c| \) corresponds to the direction of photon propagation (the subscript refers to coordinate momentum). We would like to find \( \Delta T(\tau_0, \vec{x}, \hat{p}) \), the temperature fluctuation of photons today, given \( \Delta T(\tau_i, \vec{x}, \hat{p}) \), the temperature fluctuations of photons at the time of last-scattering surface. We can define the relativistic phase space volume as

\[
|g_{ab} - n_a n_b| d^3x d^3p,
\]

(43)

where the absolute value signifies determinant and \( n_a \) are timelike vectors normal to the spacelike hypersurface. This is just a fancy way of writing physical momentum and space volume for a fixed-time slicing. The free-streaming Boltzmann equation for evolving \( \Delta T \) is given by

\[
p_c^\mu \partial_\mu f - p_c^\alpha p_c^\beta \Gamma^\mu_{\alpha \beta} \partial_\mu f = 0,
\]

(44)

with \( \Gamma^\mu_{\alpha \beta} \) calculated using the metric of Eq. (31). After standard manipulations, one finds the temperature field evolution to obey

\[
\left( \partial_\tau + \frac{\dot{c}_s}{|\vec{p}_c|} \partial_\tau \right) \left( \frac{\Delta T}{T} + \Phi \right) = 2 \partial_b \Phi.
\]

(45)

In integrating this equation for the temperature field today, the contribution of the right-hand side of this equation is called the integrated Sachs-Wolfe effect, and with the right-hand side neglected, one obtains what is simply referred to as the Sachs-Wolfe effect.
If $\Delta T/T + \Phi$ is not vanishing initially, to leading approximation we can neglect the integrated Sachs-Wolfe effect term $2\partial_0\Phi$, and the relevant Boltzmann equation is

$$
\left(\partial_0 + \frac{P^i}{|\vec{p}|}\partial_i\right)\left(\frac{\Delta T}{T} + \Phi\right) = 0. 
$$

(46)

Let us consider the initial condition

$$
\frac{\Delta T}{T} + \Phi = \frac{\Delta T}{T}|_i + \Phi|_i
$$

(47)

for this Boltzmann equation where we again remind the reader that $i$ corresponds to the last scattering surface time and $f$ corresponds to today. Inflationary computation gives us values for $\Delta T/T + \Phi$ during radiation domination when the modes of interest had wavelengths far outside of the horizon. We shall denote these initial conditions with a subscript $p$ (primordial) such that for example $\Phi|_p$ denotes the value during radiation domination. Starting from these primordial values, we would like to derive the initial condition Eq. (47) in terms of the “gauge invariant” curvature perturbation $\zeta$ and the isocurvature perturbation $\delta_S$ (see appendix for its definition) evaluated at the last scattering surface. To accomplish this, we need to express everything in terms of the set of field variables $\{\delta_\gamma, \Phi, \delta_X\}$ and then express the final result in terms of $\zeta$ and $\delta_S$. To start off, the temperature variable is easy to exchange in terms of the photon overdensity since by definition

$$
\Delta T/T|_i = \frac{1}{4} \delta_\gamma|_i.
$$

(48)

From now on, we will go to the Fourier space (spatially flat $e^{i\vec{k} \cdot \vec{x}}$ basis) and assume all of our variables now represent amplitudes in Fourier space. We have according to $T^\alpha_{\alpha,0} = 0$ (Eq. (35)) that

$$
C_1 \equiv \delta_\gamma(\tau, k) - 4\Phi(\tau, k)
$$

(49)

approximately a constant on large length scales. Because the curvature invariant $\zeta$ computed through the usual inflationary formalism is $\zeta|_p = -\Phi|_p + \delta_\gamma|_p/4$ where the subscript $p$ denotes the quantities are evaluated in the radiation dominated era, we can write

$$
\zeta|_p = \frac{C_1}{4}.
$$

(50)

Since $\zeta$ is an adiabatic invariant, $\zeta$ will continue to equal $C_1/4$ even after matter domination. By Eq. (49), we find

$$
\zeta = \frac{\delta_\gamma}{4} - \Phi
$$

(51)

for all time in the long-wavelength limit.

Inserting Eqs. 45 and 51 into Eq. 17, we arrive at

$$
\left.\frac{\Delta T}{T}\right|_i + \Phi|_i = \frac{1}{4} \delta_\gamma|_i + \Phi|_i = \zeta + 2\Phi|_i.
$$

(52)

Note that we have here assumed that the wavelengths of interest are sufficiently long such that radiation era quantities are valid at the last scattering surface which is assumed to be after matter domination.

Now, we want to reexpress $\Phi$ this in terms of the isocurvature perturbations. As discussed in the appendix, the isocurvature perturbation of interest is

$$
\delta_S \equiv \delta_X - \frac{3}{4} \delta_\gamma.
$$

(53)

Hence, we need to eliminate $\delta_X$ and $\delta_\gamma$ in terms of $\delta_s$ and $\zeta$.

---

3 The “gauge invariant” curvature perturbations are usually described by $\zeta$ or $R$ which are equivalent when the scales are far outside of the horizon. The relationship between the two are $\zeta \equiv -\Phi + \delta_\rho/3(\rho + P) = R + (k/aH)^2 \Phi 2\rho/9(\rho + P)$ and we will choose to work with $\zeta$. 

To relate $\delta X$ with $\Phi$, we can use the energy conservation equation (Eq. 33) in the long-wavelength limit, and restricting to the non decaying mode, to write
\[ \delta X - 3\Phi = C_2, \] (54)
where $C_2$ is a constant. Now, since during matter domination, the 00 component of Einstein’s equation (Eq. 42) in the long-wavelength limit is
\[ \Phi'|_i + \frac{a'}{a} \Phi|_i = -\frac{1}{2} \frac{a'}{a} \delta X|_i, \] (55)
we can use Eq. (54) to solve for the growing solution, which is
\[ \delta X|_i \approx -2 \Phi|_i. \] (56)
During matter domination, we thus find using Eqs. (56) and (51) that
\[ \delta S = -2 \Phi|_i - 3 (\zeta + \Phi|_i) = -5 \Phi|_i - 3\zeta. \] (57)
This is a remarkable feature of the isocurvature perturbations in which even if $\zeta = 0$, the isocurvature perturbations grow into potential perturbations during matter domination.
We say that the isocurvature perturbations grow into potential perturbations because during radiation domination, if $\zeta = 0$, then the gravitational perturbations approximately vanish. Let us see how this happens in detail. Solving the 00 part of the Einstein equation (Eq. 42), we find during radiation domination (RD) that
\[ \Phi|_p (\tau, k) = -\frac{C_1}{6}, \] (58)
where as before, the subscript $p$ denotes the radiation domination era when the wavelengths of interest are far outside the horizon. This and Eq. (50) imply that in the absence of curvature perturbations $\zeta$, the gravitational potential $\Phi$ is 0 during radiation domination. However, during matter domination, the potential $\Phi$ grows due to the existence of the isocurvature perturbations. One can also understand from this (the fact that during radiation domination $\Phi = \delta, = 0$ if $\zeta = 0$) that if matter domination never occurs then we would have in a hypothetical radiation dominated last scattering surface that
\[ \frac{\Delta T}{T}|_{\tau, \text{hypothetical}} + \Phi|_i \text{ hypothetical} = 0 \] (59)
which is in accord with the intuition that the superheavy dark matter perturbations are irrelevant if their density is too small.
Combining Eq. (57) with Eq. (52), we arrive at the desired expression
\[ \frac{\Delta T}{T}|_i + \Phi|_i = -\frac{1}{5} \zeta - \frac{2}{5} \delta S. \] (60)

C. $C_l$ characterization of CMB

Now consider the computation of $C_l$ which is defined as
\[ \langle a_{lm} a_{lm'}^* \rangle = \delta_{ll'} \delta_{mm'} C_l, \] (61)
where
\[ \frac{\Delta T(\tau_f, \vec{x}, \hat{p})}{T} = \sum_{l,m} a_{lm}(\tau_f, \vec{x}) Y_{lm}(\hat{p}) \] (62)
and $\hat{p}$ is the direction vector of the photon. Using
\[ a_{lm}(\vec{x}, \tau_f) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \int d\Omega Y_{lm}^*(\hat{p}) \frac{\Delta T(\tau_f, \vec{x}, \hat{p})}{T}, \] (63)
where \( \Omega \) is the solid angle for \( \hat{\rho} \), we write the ensemble average for \( C_l \) as
\[
\langle a_{lm} a_{l'm'}^* \rangle = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \int d\omega_1 \int d\omega_2 \left\langle \frac{\Delta T(\tau_f, \vec{k}_1, \hat{\rho}_1) \Delta T^*(\tau_f, \vec{k}_2, \hat{\rho}_2)}{T} \right\rangle e^{i\vec{k}_1 \cdot \vec{x}} e^{-i\vec{k}_2 \cdot \vec{x}} Y_{lm}(\hat{\rho}_1) Y_{l'm'}^*(\hat{\rho}_2),
\]
where we have gone to spatial Fourier space. Now, use the property that after the last scattering surface the Boltzmann equation Eq. (40) for long wavelengths is approximately
\[
(\partial_0 + \hat{\rho}^i \partial_i) \left( \frac{\Delta T}{T} + \Phi \right) \approx 0
\]
to write
\[
\frac{\Delta T(\tau, \vec{x}, \hat{\rho})}{T} + \Phi(\tau, \vec{x}) \approx \int \frac{d^3k}{(2\pi)^3} \left( \frac{\Delta T(\vec{k})}{T} \right) e^{i\vec{k} \cdot \vec{x}} e^{-i\hat{\rho} \cdot \vec{k}(\tau - \tau_i)}.
\]
Neglecting \( \Phi(\tau, \vec{x}) \) (today) gives
\[
\frac{\Delta T(\tau_f, \vec{k}, \hat{\rho}_1)}{T} \approx \left( \frac{\Delta T(\vec{k})}{T} \right) e^{-i\hat{\rho}_1 \cdot \vec{k}(\tau_f - \tau_i)}.
\]
Hence, we find
\[
\langle a_{lm} a_{l'm'}^* \rangle = \frac{1}{V} \int \frac{d^3k_1}{(2\pi)^3} \int d\omega_1 \int d\omega_2 \left\langle \left( \frac{\Delta T(\vec{k}_1)}{T} \right) e^{i\vec{k}_1 \cdot \vec{x}} e^{-i\hat{\rho}_1 \cdot \vec{k}_1(\tau_f - \tau_i)} \right\rangle \left( \frac{\Delta T^*(\vec{k}_2)}{T} \right) e^{-i\hat{\rho}_2 \cdot \vec{k}_2(\tau_f - \tau_i)}
\]
\[
\times e^{-i\hat{\rho}_1 \cdot \vec{k}_1(\tau_f - \tau_i)} e^{i\vec{k}_2 \cdot \vec{x}} e^{-i\hat{\rho}_2 \cdot \vec{k}_2} Y_{lm}(\hat{\rho}_1) Y_{l'm'}^*(\hat{\rho}_2).
\]
To leading order, the power temperature correlation function can be written as
\[
\left\langle \left( \frac{\Delta T(\vec{k}_1)}{T} \right) e^{i\vec{k}_1 \cdot \vec{x}} e^{-i\hat{\rho}_1 \cdot \vec{k}_1(\tau_f - \tau_i)} \right\rangle \left( \frac{\Delta T^*(\vec{k}_2)}{T} \right) e^{-i\hat{\rho}_2 \cdot \vec{k}_2(\tau_f - \tau_i)} = \frac{2\pi^2}{k_1^3} P(k_1)(2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2).
\]
Inserting
\[
e^{-i\hat{\rho}_1 \cdot \vec{k}(\tau_f - \tau_i)} = \sum_{l=0}^{\infty} i^l (2l + 1) j_{l}(\vec{k}(\tau_f - \tau_i)) P_l(-\hat{\rho}_1 \cdot \vec{k})
\]
where \( P_l \) are Legendre polynomials, we find
\[
\langle a_{lm} a_{l'm'}^* \rangle = \int \frac{d^3k}{(2\pi)^3} \int d\omega \int d\omega_2 \frac{2\pi^2}{k_1^3} P(k) \sum_{l=0}^{\infty} i^l (2l + 1) j_{l}(\vec{k}(\tau_f - \tau_i)) P_l(-\hat{\rho}_1 \cdot \vec{k}) Y_{lm}(\hat{\rho}_1)
\]
\[
\times \sum_{l=0}^{\infty} i^l (2l + 1) j_{l}(\vec{k}(\tau_f - \tau_i)) P_l(\hat{\rho}_2 \cdot \vec{k}) Y_{l'm'}(\hat{\rho}_2),
\]
Now using the identity
\[
P_l(-\hat{\rho}_1 \cdot \vec{k}) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} Y_{lm}(-\hat{\rho}_1) Y_{lm}^*(\vec{k}),
\]
where \( Y_{lm} \) are orthonormal spherical harmonics, we find
\[
\langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} Y_{lm} Y_{lm}^* d(k) j_l([\vec{k}(\tau_f - \tau_i)])^2.
\]
Thus, we see that \( C_l \) is essentially the power spectrum on long wavelengths up to angular projection effects of the Bessel function. Defining the photon travel distance from the last scattering surface
\[
L \equiv \tau_0 - \tau_{dec} \approx \frac{1}{H_0 a_0} \int_{z_{dec}}^{z_{dec}} \frac{dz}{\sqrt{\Omega_m(z + 1)^3 + \Omega_\Lambda}}.
\]

we arrive at the desired expression for $C_l$ as

$$C_l = 4\pi \int \frac{dk}{k} P(k) |j_l(kL)|^2.$$ (75)

To evaluate $C_l$ from this expression, we merely need to compute $P(k)$, which from Eqs. (69) and (70) is seen to be

$$P(k) = \frac{1}{25} P_\zeta(k)|_i + \frac{4}{25} P_S(k)|_i,$$ (76)

where the subscript $i$ denotes last scattering surface.

To compute $P_S$ at the last scattering surface, from Eqs. (49) and (54) we find

$$\delta_S = C_2 - \frac{3}{4} C_1.$$ (77)

In other words, isocurvature perturbations are approximately constant far outside of the horizon, whether the quantities are evaluated in matter or radiation domination era. Therefore, since we can express $\delta_\gamma$ in terms of $\zeta_p$ in radiation domination using Eqs. (49), (50), and (58) as

$$\zeta_p = \frac{3}{4} \delta_\gamma p,$$ (78)

it is convenient to compute $P_S$ in the radiation domination era using the relationship

$$\delta_S|_p = \delta_X|_p - \zeta|_p,$$ (79)

where one should understand the existence of nonzero $\delta_S|_p$ even when $\delta_X|_p = 0$ from the fact that the dark matter number density would then not trace the radiation number density. Hence, $P_\zeta(k)$ and $P_S(k)$ are

$$P_S(k)|_i = P_{\delta_X}|_p + P_{\zeta}|_p - P_{\zeta X}|_p - P_{X \zeta}|_p$$ (80)

$$P_{\delta_X} = \frac{k^3}{2\pi^2} \int d^3r \ e^{-i \mathbf{k} \cdot \mathbf{r}} \langle \delta_X(\mathbf{x}) \delta_X(\mathbf{y}) \rangle$$ (81)

$$P_{\zeta}(k) = \frac{k^3}{2\pi^2} \int d^3r \ e^{-i \mathbf{k} \cdot \mathbf{r}} \langle \zeta(\mathbf{x}) \zeta(\mathbf{y}) \rangle$$ (82)

$$P_{\zeta X}(k) = \frac{k^3}{2\pi^2} \int d^3r \ e^{-i \mathbf{k} \cdot \mathbf{r}} \langle \zeta(\mathbf{x}) \delta_X(\mathbf{y}) \rangle,$$ (83)

where all the functions with subscript $p$ are evaluated at the radiation-dominated era. For the usual thermal CDM scenario, we have

$$\delta_X = \frac{3}{4} \delta_\gamma,$$ (84)

which gives a zero power for the isocurvature as expected. For the isocurvature of superheavy dark matter, the cross correlation terms $P_{\zeta X}$ and $P_{X \zeta}$ vanish. The first term of Eq. (76) is the usual adiabatic contribution to the CMB which can be easily computed for slow roll inflationary scenarios in a standard way. The second term of Eq. (76) (or more specifically Eq. (81)) is what we are primarily concerned with in this paper, and we will turn to its computation in the next section.

Before beginning the computation of the matter correlation function, note that Eq. (78) allows us to rewrite the familiar Eq. (68) as

$$P(k) = \frac{1}{5} P_\zeta + \frac{4}{25} P_{\delta_X},$$ (85)

where we have dropped the subscript for brevity.

**IV. ISOCURVATURE POWER SPECTRUM**

In this section, we proceed to compute $P_{\delta_X}(k)$ needed in Eq. (85) for the computation of $C_l$ using Eq. (75). As we discussed in the introduction, we restrict ourselves to a massive scalar field $X$ minimally coupled to gravity without
any other interactions for $X$. The quantum fluctuations of the energy density during inflation is usually computed by using the curvature perturbation which remains constant far outside the horizon. However, the usual computation procedure which attempts to compute the quantum fluctuations induced about a classical field background does not apply to our scenario since the dark matter particles do not arise from a classical field background but from quantum particle production. Liddle and Mazumdar \cite{41} had a similar cosmological scenario as the one of interest in this paper, but instead of real particle production, their homogeneous dark matter density was implicitly from the vacuum energy contribution. This difference can be readily seen by computing the vacuum expectation value of the stress energy tensor for a massive scalar field $X$ with mass $m_X$ as

$$\int d^3x \langle T_{00}(x) \rangle = \int d^3 k \, a^3 \langle A^+_k A_k \rangle \left[ \left( |\beta_k|^2 + \frac{1}{2} \right) \left( |\dot{X}_k(t)|^2 + (k^2/a^2 + m_X^2)|X_k(t)|^2 \right) \right],$$

where the vacuum state $|0\rangle$ is defined such that $A^+_k|0\rangle = 0$ while the fields are defined by a Bogoliubov rotated boundary condition. The vacuum of Ref. \cite{41} corresponds to the case with $\beta_k = 0$, and their energy density came from the $\frac{1}{2} m^2 |X_k|^2$ term in the integral, which is part of the zero point energy term.

Given that the zero point energy must be taken care of by the cosmological constant problem solution, whether the zero point energy should be counted as an unambiguous production of particles is unclear. Indeed, in Minkowski space, we usually discard the zero point energy. One might argue that the zero point energy is what is contributing to the usual computation of generating density fluctuations in inflationary cosmology. However, this is not true. As we stated previously, the fluctuations that are being computed in the usual density perturbation computations are not the fluctuations about the classically zero energy state but about a quasi-de Sitter background of positive energy. Indeed, in computing the density fluctuations, the zero point energy of the inflaton is always subtracted. Hence, since treating the zero point energy as real particle production without specifying assumptions about the solution to the cosmological constant problem is speculative, we will in our computations throw away the zero point energy just as in Minkowski space. This is one of several important differences between Liddle and Mazumdar \cite{41} and our paper.

Assuming that the inflaton energy density dominates over the dark matter energy density during inflation, the curvature perturbations will be computed in the usual manner. This is an excellent assumption since the dark matter energy density to inflaton energy density during inflation needs to be much smaller than $10^{-10}$ to obtain the phenomenologically acceptable amount of dark matter today. Furthermore, because the superheavy dark matter field is assumed to be sitting at the minimum of its potential with zero vacuum energy, there is no vacuum energy contribution coming from the superheavy dark matter sector unlike the inflaton sector. In other words, the energy perturbation $\delta \rho$ coming from inflation is linear in the inflaton field fluctuation $\delta \phi$ while the energy perturbation coming from the superheavy dark matter is quadratic in $X$. Hence, to linear order, there is no mixing between the inflaton energy density and the superheavy dark matter energy density.

The dark matter energy density fluctuation is defined as

$$\delta \rho_X(x) =: T_{00}^{(X)}(x) : - \langle : T_{00}^{(X)} : \rangle,$$

where the normal ordering is with respect to the Bogoliubov rotated operators to eliminate the vacuum energy. Hence, we can write the correlation function as

$$\langle \delta \rho_X(x) \delta \rho_X(y) \rangle = \langle : T_{00}^{(X)}(x) : : T_{00}^{(X)}(y) : \rangle - \langle : T_{00}^{(X)} : \rangle^2.$$

Explicitly, the Bogoliubov rotation is given by

$$a_k = \alpha_k A_k + \beta_k A_k^\dagger,$$

and the normal ordered stress tensor vacuum expectation value is given as

$$\rho_X(t) = \langle : T_{00}^{(X)} : \rangle = \int \frac{d^3 k}{(2\pi)^3} \left[ \Re \left[ \alpha_k \beta_k^* \left( |\dot{X}_k(t)|^2 + w_k^2 |X_k(t)|^2 \right) \right] + |\dot{\beta}_k|^2 \left( |X_{k1}(t)|^2 + w_k^2 |X_{k1}(t)|^2 \right) \right],$$

where $dk = d^3k/(2\pi)^{3/2}$ and we have used the fact that $X_{k1} = X_{-k}$ and defined $w_k^2 = k^2/a^2 + m^2$. Note that this vanishes in the limit $\beta_k \to 0$ where there is no particle production consistent with our previous discussion of comparing our paper with \cite{41}. Hence, the dark matter correlation function is

$$\langle \delta_X(x) \delta_X(y) \rangle = \frac{\langle : T_{00}^{(X)}(x) : : T_{00}^{(X)}(y) : \rangle}{\rho_X^2} - 1.$$
With the mode mixing given by Eq. (69), the stress tensor correlator can be written down straightforwardly

\[
\left\langle : T_{00}^{(X)}(\vec{x}) : : T_{00}^{(X)}(\vec{y}) : \right\rangle = \frac{1}{2} \int [dk_1][dk_2] \left[ (\alpha_k \alpha_k \cdot k_1) \cdot k_2 \cdot k_2 \right] \left( \vec{X}_{k_1}(t) \vec{X}_{k_2}(t) - k_1 k_2 \right) X_{k_1} X_{k_2} + m^2 X_{k_1} X_{k_2} \right] e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{x}}
\]

+ \left[ (\beta_k \beta_k \cdot k_1) \cdot k_2 \cdot k_2 \right] \left( \vec{X}_{k_1}(t) \vec{X}_{k_2}(t) - k_1 k_2 \right) X_{k_1} X_{k_2} + m^2 X_{k_1} X_{k_2} \right] e^{-i(\vec{k}_1 + \vec{k}_2) \cdot \vec{y}}
\]

Although not rigorously justified, we can approximate to within an order of magnitude \(|\beta_k| \ll |\alpha_k|\). In the expansion with \(\beta_k \to 0\), the correlator is simply

\[
\left\langle : T_{00}^{(X)}(\vec{x}) : : T_{00}^{(X)}(\vec{y}) : \right\rangle = \frac{1}{2} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} |\alpha_{k_1}|^2 |\alpha_{k_2}|^2 \left[ X_{k_1} X_{k_2} - \frac{k_1 k_2}{a^2} X_{k_1} X_{k_2} + m^2 X_{k_1} X_{k_2} \right] e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{x}}
\]

+ O(\beta_k).

The power spectrum is

\[
P_{5X}(k) = \frac{k^3}{2\pi^2} \int d^3r e^{-i\vec{k} \cdot \vec{r}} \delta_X(x) \delta_X(y) = \frac{1}{k^3} \frac{1}{(2\pi)^3} \int d\cos \theta \left| \vec{X}_{k_1} \cdot \vec{X} \right| X_{k_1} \sqrt{k^2 + k_1^2 - 2k_1 k \cos \theta} + a^{-2} \left| \vec{k}_1 \right|^2 \left| \vec{k}_2 \right| \left| \vec{k}_2 \right| \cos \theta \right) X_{k_1} X_{k_2} \sqrt{k^2 + k_1^2 - 2k_1 k \cos \theta} + m^2 X_{k_1} X_{k_2} \sqrt{k^2 + k_1^2 - 2k_1 k \cos \theta})^2.
\]

Note that the negative unity in Eq. (91) disappears because

\[
- \frac{k^3}{2\pi^2} \int d^3r e^{-i\vec{k} \cdot \vec{r}} = 0.
\]

In the flat space limit, we find

\[
\left\langle : T_{00}^{(X)}(\vec{x}) : : T_{00}^{(X)}(\vec{x}) : \right\rangle = \frac{1}{2} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \frac{1}{(2E_{k_1})(2E_{k_2})} \left| - E_{k_1} E_{k_2} - \frac{k_1 k_2}{a^2} + m^2 \right|.
\]

If we neglect the kinetic terms, we find

\[
\left\langle : T_{00}^{(X)}(\vec{x}) : : T_{00}^{(X)}(\vec{x}) : \right\rangle = \frac{m^4}{2} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \frac{1}{(2E_{k_1})(2E_{k_2})} = \frac{m^4}{2} \left( X(x) X(x) \right)^2.
\]
which is what we expect since in that limit \( \rho \sim m^2 X^2 \) and
\[
\langle : X^2(x) : X^2(y) : \rangle = 2 \langle X(x) X(y) \rangle^2.
\] (98)

Note that in general, the kinetic part of the correlation function cannot be neglected.

To evaluate the correlation function in the curved spacetime of interest, we must compute the wave function, which solves the following Klein Gordon differential equation:
\[
X''_k(\tau) + 2 H X'_k(\tau) + \left( k^2 + a^2(\tau) m \right) X_k(\tau) = 0
\] (99)

Note that, since the Dark Matter field has no VEV, the equation of motion does not couple to the metric fluctuations at first order.

We write the evolution of the scale factor in conformal time \( \tau \) as
\[
\frac{1}{a} \frac{d}{d \tau} H = -\epsilon H^2,
\] (100)
where \( H = a^{-2} da/d\tau \), and \( \epsilon \) is the usual slow roll parameter (potential derivatives) which we take to be approximately constant (This is true at first order in the slow roll parameters). With the boundary condition that \( a(\tau_i) = a_i = -1/(H_I \tau_i) \), we find
\[
a = -\frac{1}{H_I} \left\{ 1 + \epsilon \left[ \left( 1 - \frac{\tau_i}{\tau} \right) - \ln \frac{\tau}{\tau_i} \right] \right\},
\] (101)
where one recognizes the usual dS scale factor \(-1/(H_I \tau)\) in the limit that \( \epsilon \to 0 \). However, using this would make an analytic solution impossible since \( a^2 \) will have \( 1/\tau^3 \) terms as well as \( \ln(\tau/\tau_i) \) terms while \( a''/a \) will have \( 1/\tau^3 \) terms. Hence, we will use for the dark matter correlation function in the dS approximation.

Still, we can approximately take into account the changing \( H \) for the isocurvature perturbations during slow roll inflation as follows. Since the amplitude of the perturbations are approximately frozen when the physical wavelength crosses the horizon, we will assume that \( H \) relevant for the density perturbation to be \( k \) dependent such that
\[
\frac{k}{a(t_k) H_k} = 1,
\] (102)

where \( t_k \) corresponds to the time at which \( k/a = H_k \). Now, solving the slow roll equation
\[
\frac{\dot{H}}{H^2} = -\epsilon,
\] (103)
we find
\[
-\frac{1}{H_k} + \frac{1}{H_I} = -\epsilon(t_k - t_I),
\] (104)
where \( H_I \) corresponds to the expansion rate at some initial time \( t_I \). Using approximate dS expansion
\[
(t_k - t_I) = \frac{1}{H_I} \ln \left( \frac{a_k}{a_I} \right),
\] (105)
we write
\[
-\frac{1}{H_k} + \frac{1}{H_I} = - \frac{\epsilon}{H_I} \ln \left( \frac{a_k}{a_I} \right),
\] (106)
or equivalently
\[
H_k \approx H_I \left( \frac{a_k}{a_I} \right)^{-\epsilon}
\] (107)
Using Eq. (102), we arrive at
\[
H_k = H_I \left( \frac{k}{a_I H_I} \right)^{-\epsilon} + O(\epsilon^2).
\] (108)
Hence, the mode function can be approximated as

\[
X_k \approx \sqrt{\frac{\pi}{2a^3 H}} e^{i\frac{\nu}{2}(\nu+1/2)} H^{(1)}_{\nu} \left( \frac{k}{aH_k} \right),
\]

where

\[
\nu \equiv \sqrt{\frac{9}{4} \left( \frac{m_X}{H_k} \right)^2}.
\]

The mode functions satisfy the usual normalization

\[
23 \left( \dot{X}_k X_k \right) = a^{-3}.
\]

In the long-wavelength limit, we find

\[
X_k \approx \frac{1}{\sqrt{\pi}} (-1)^{3/4} 2^{-1-\nu} e^{-i\nu\pi/2} e^{-H_k t(3/2-\nu)} H^{\nu-1/2}_{\nu} k^{-\nu} \left[ \frac{e^{-2\nu H_k t}}{H_k^\nu} k^{2\nu} \Gamma(-\nu) + 4^\nu e^{i\nu\pi} \Gamma(\nu) \right],
\]

where we have used \( a = e^{H_k t} \). When \( \nu \) is real and positive, this behaves as

\[
X_k \approx -\frac{1}{\sqrt{\pi}} (-1)^{3/4} 2^{-1-\nu} e^{i\nu\pi/2} \left( \frac{k}{aH_k} \right)^{-\nu} \frac{4^\nu}{\sqrt{H_k a^3}} \Gamma(\nu),
\]

which can become large when \( k/(aH_k) \ll 1 \) and \( \nu \neq 0 \). When \( \nu \) is imaginary and positive, the wave function behaves as

\[
X_k \approx -\frac{1}{\sqrt{\pi}} (-1)^{3/4} 2^{-1-|\nu|} e^{i|\nu|\pi/2} \left( \frac{k}{aH_k} \right)^{i|\nu|} \frac{1}{\sqrt{H_k a^3}} \Gamma(-i|\nu|),
\]

whose magnitude is essentially independent of \( k \), as it can be understood by noting that this limit corresponds to \( m \gg k \).

Let’s see how the mode function \( X_k \) scales after inflation ends. Assuming that the scale factor scales as

\[
a = a_i(t/t_i)^q,
\]

we find for the mode equation

\[
\ddot{f} + \left[ m^2 + \left( \frac{3}{2q} - \frac{9}{4} \right) H^2(t) + \left( \frac{k}{a(t)} \right)^2 \right] f = 0,
\]

where \( X_k(t) = a^{-3/2} f(t) \). Neglecting the \( k \) term in the long-wavelength limit and defining \( \xi_q \equiv 3/2q - 9/4 \), we find that in the limit \( m^2 \gg H^2 \),

\[
X_k \sim \frac{c}{a^{3/2}} e^{-i\nu t},
\]

while in the limit \( m^2 \ll H^2 \) that

\[
X_k \sim \frac{c}{a^{3/2}} e^{(1+\sqrt{1-4q^2\xi_q})/2}. \tag{118}
\]

Hence, we see that since \( H \sim t^{-1} \), eventually, \( m^2 \gg H^2 \) and \( X_k \) will fall like \( a^{3/2} \). Hence, for any quantity we compute involving \( X_k \) mode function in dS space, as far as the scaling with \( a \) is concerned in the absence of further interactions, we should freeze its value at the point when \( m = H \) and then scale it as \( a^{-3/2} \).

Because of the \( k \) suppression, the term \( a^{-2}(|\vec{k}_1| - |\vec{k}|) X_k X_{\sqrt{k^2+|\vec{k}_1|^2-2k\vec{k} \cos \theta}} \) in Eq. \( 92 \) should be negligible unless there is a cancellation of the other terms in the limit that \( |\vec{k}| \to 0 \). Quantitatively, the ratio

\[
\lim_{k,k_1 \to 0} \frac{\dot{X}_{k_1} X_{\sqrt{k^2+|\vec{k}_1|^2-2k\vec{k} \cos \theta}}}{m^2 X_{k} X_{\sqrt{k^2+|\vec{k}_1|^2-2k\vec{k} \cos \theta}}} = \frac{9H^2}{4m^2}.
\]

\[
\frac{1}{15}
\]

\[
\frac{1}{15}
\]
Hence, as long as \( m^2 \neq 9H^2/4 \), there should be no cancellation. Hence, the power spectrum Eq. (114) becomes

\[
P_{\delta_X} \approx \frac{1}{\rho_X^2} \frac{k^3}{2\pi} \frac{1}{(2\pi)^3} \int dk_1 k_1^2 \int_{-1}^{1} d\cos\theta \left| \dot{X}_{k_1} \dot{X} \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} + m^2 X_{k_1} X \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \right|^2
\]

\[
\approx \frac{1}{\rho_X^2} \frac{k^3}{2\pi} \frac{1}{(2\pi)^3} \int dk_1 k_1^2 \int_{-1}^{1} d\cos\theta \left\{ \left| \dot{X}_{k_1} \dot{X} \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \right|^2 + m^4 \left| X_{k_1} X \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \right|^2 \right\} + 2m^2 \text{Re} \left[ \dot{X}_{k_1} \dot{X} \right] \left[ X_{k_1} X \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \right]
\]

(120)

In the language of [41], we have

\[
P_X(k) = \frac{k^3}{2\pi^2} |X_k|^2
\]

(121)

which implies, keeping only the term proportional to \( m^4 \), that

\[
P_{\delta_X}(k) = \frac{m^4 k^3}{\rho_X^2} \frac{1}{2\pi} \frac{1}{(2\pi)^3} \int dk_1 k_1^2 \int_{-1}^{1} dx P_X(k_1) P_X(|\vec{k}_1 - \vec{k}|) \frac{2\pi^2}{k_1^4} \frac{2\pi^2}{|\vec{k}_1 - \vec{k}|^3} = \frac{m^4}{\rho_X} \int d^3 k_1 P_X(k_1) P_X(|\vec{k}_1 - \vec{k}|)
\]

(122)

which matches the result of Ref. [41]. However, unlike their analysis, we do not neglect terms involving \( \dot{X}_{k_1} \dot{X} \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \) which can be a priori of same order of magnitude as the \( m^2 X_{k_1} X \sqrt{k^2 + k_1^2 - 2k_1 k \cos\theta} \) term they accounted for. We will see the resulting correction is close to a factor of 2.

We can express the isocurvature power spectrum Eq. (120) in terms of the power spectrum of the dark matter field as follows:

\[
P_{\delta_X}(k) = \frac{k^3}{\rho_X^2} \frac{1}{(2\pi)^3} \int d^3 k_1 \frac{1}{|\vec{k}_1 - \vec{k}|^3} \left\{ \left( \dot{P}_X(k_1) \right)^2 + \frac{k_1^6}{a^6} \left( 2\pi^2 \right)^2 \left( \dot{P}_X(|\vec{k}_1 - \vec{k}|) \right)^2 + \frac{k^6}{a^6} \left( 2\pi^2 \right)^2 \right\}
\]

\[
+ m^4 P_X(k_1) P_X(|\vec{k}_1 - \vec{k}|) + \frac{m^2}{2} \left( \dot{P}_X(k_1) \dot{P}_X(|\vec{k} - \vec{k}_1|) - k_1^3 |\vec{k} - \vec{k}_1|^3 / a^6 (2\pi^2)^2 \right) \right\}
\]

(123)

Note the presence of additional terms with respect to Eq. (122) due to the kinetic contributions to the stress tensor appear as derivatives of the power spectrum.

At the end of inflation, the Hubble expansion rate is much smaller than the value at which the isocurvature perturbations are generated. For example, in the chaotic inflationary scenario that we consider, we have

\[
H_e \approx H_I / \sqrt{50},
\]

(124)

where \( H_I \) is the expansion rate during the time that the isocurvature perturbations are generated. Since gravitational particle production is determined by

\[
\frac{m_X}{H_e} \approx \frac{m_X \sqrt{50}}{H_I},
\]

(125)

and since to have enough particle production for dark matter with \( T_{RH} \lesssim 10^9 \) GeV,\(^4\) we must satisfy the condition

\[
\frac{m_X}{H_e} \lesssim 7,
\]

(126)

we must have

\[
\frac{m_X}{H_I} \lesssim 1.
\]

(127)

\(^4\) We will see later that the isocurvature bound of interest is within the regime where \( T_{RH} < 10^9 \) GeV.
This implies that \( \nu \) is real and positive, \(^5\) which means that the wave function will behave as

\[
X_k \approx -\frac{1}{\sqrt{\pi}} (-1)^{3/4} e^{-\nu \pi/2} \left( \frac{k}{aH_k} \right)^{-\nu} \frac{4^\nu}{\sqrt{H_k a^3}} \Gamma(\nu),
\]

and the spectrum

\[
P_X(k) \approx \frac{k^3}{2\pi^2} \left| \frac{-1}{\sqrt{\pi}} (-1)^{3/4} e^{\nu \pi/2} \left( \frac{k}{aH_k} \right)^{-\nu} \frac{4^\nu}{\sqrt{H_k a^3}} \Gamma(\nu) \right|^2 = A \frac{k^3}{H_k a^3} \left( \frac{k}{aH_k} \right)^{-2\nu}
\]

where

\[
A \equiv \frac{2^{2\nu-3}}{\pi^3} |\Gamma(\nu)|^2.
\]

When we take the time derivative of \( P_X(k) \), we will only account for the time dependence in \( a \) since that is the main time dependence information contained in the approximate Bessel function equation. Hence, we find

\[
\dot{P}_X \approx A \frac{k^3}{H_k a^3} \left( \frac{k}{aH_k} \right)^{-2\nu} (2\nu - 3) H_k
\]

where in the second equality, we have assumed approximate dS expansion again.

Finally, using Eq. (108), we find

\[
P_X(k) = A \frac{k^3}{H_1 a^3} \left( \frac{k}{aH_1} \right)^n \dot{P}_X = A \frac{k^3}{a^3} \left( \frac{k}{aH_1} \right)^{n-\epsilon} (2\nu - 3)
\]

where

\[
n = \epsilon - 2\nu(1 + \epsilon).
\]

Hence, we can write the simplified form of the power spectrum as

\[
P_3X(k) = \int \frac{k^6}{\rho_X^4} \frac{1}{4} \int_0^{\infty} du u^2 \left\{ -\frac{k}{256a^6 A^2 \pi^8 H_1} \left( \frac{k}{aH_1} \right)^{-(1+2n)} (u)^{-(1+n)} \left( 1 + 4A^2 \left( \frac{k}{aH_1} \right)^{2(n-\epsilon)} u^{2(n-\epsilon)} (3 - 2\nu)^2 \pi^4 \right) \right.
\]

\[
\times \left[ (1 - u)^{-n+2} \left( 1 + \frac{4A^2(3 - 2\nu)^2 \pi^4}{2(1 - \epsilon) + n} \left( \frac{k}{aH_1} \right)^{2(n-\epsilon)} (1 - u)^{2(n-\epsilon)} \right) \right.
\]

\[
+ |1 + u|^{-2(n-\epsilon)} \left( \frac{1}{2 - n} + \frac{4A^2(3 - 2\nu)^2 \pi^4}{2(1 - \epsilon) + n} \left( \frac{k}{aH_1} \right)^{2(n-\epsilon)} (1 + u)^{2(n-\epsilon)} \right) \right]
\]

\[
+ A^2 \left( \frac{k}{aH_1} \right)^{2n-1} u^{n-1} \frac{m^4 k}{a^7 H_1^3(n + 2)} [1 + u^{n+2} - |1 - u|^{n+2}]
\]

\[
+ \frac{m^2}{4a^6} \left( \frac{-1}{\pi^4} + \frac{2A^2(3 - 2\nu)^2}{u(2 - \epsilon + n)} \left( \frac{k}{aH_1} \right)^{2n-2\epsilon} |1 + u|^{-\epsilon} |1 - u|^{-\epsilon} u^{n-\epsilon} \right.
\]

\[
\times \left( (1 - u)^{n+2} |1 + u|^{\epsilon} - |1 - u|^{\epsilon} (1 + u)^{n+2} \right). \]

One can easily check that the term proportional to \( m^4 \) matches the integral expression of \([11]\). \(^6\)

\(^5\) With nonminimal coupling, \( \nu \) may easily be imaginary in which case Eq. \([11]\) will be relevant instead of Eq. \([13]\). In that case, the spectral index for the isocurvature perturbations will not be as sensitive to the mass \( m_X \) since \( |\Gamma(-ix)| \exp(x\pi/2) \) scales as \( 1/\sqrt{x} \) for real \( x > 1 \).

\(^6\)
TABLE I: Coefficients to the integrals governing the order of magnitude of the power spectrum is presented. The estimate in the third column includes the contribution from the integral multiplying the coefficients in the expansion of the power spectrum. As explained in the text, the divergent integrals were cut off at $k_{\text{max}} = a_e H_e$ and the $k$ value of the spectrum set at $k = a_i H_I y$ where $a_i$ is the scale factor at the “beginning” (near 50 efolds from the end) of the inflation and $a_e$ is the scale factor at the end of inflation. The degree of divergence in the integrals is given in the fourth column. It is clear that $a_i I_i$ contributions to the power spectrum dominate because these terms are not diluted by the enormous scale factor. For $c_2 I_2$ (where $I_2$ is the integral), the divergence is one degree smaller than the degree of divergence in each term in the integrand because of cancellation.

| coeff. | value                                                                 | order of magnitude contribution including integral | degree of divergence |
|--------|----------------------------------------------------------------------|-------------------------------------------------|----------------------|
| $c_1$  | $(256A^2\pi^8)^{-1}(k/aH_I)^{12-2n_i}$                               | $6 \times 10^{-4}(a_i/a)^{10}(a_i/a)^2(H_e/H_I)^{10}y^2$ | 10                   |
| $c_2$  | $(3-2\nu)^2\pi^4(n_i-1-e)^{-1}(256\pi^8)^{-2}(k/aH_I)^{6-2\nu}$   | $7 \times 10^{-5}(a_i/a)^3(a_i/a)^2(H_e/H_I)^3(m_X/H_I)^4y^3$ | 3                    |
| $c_3$  | $c_i/(5-n_i)$                                                          | $6 \times 10^{-4}(a_i/a)^{10}(a_i/a)^2(H_e/H_I)^{10}y^2$ | 10                   |
| $c_4$  | $(n_i-1-e)c_2$                                                        | $-7 \times 10^{-5}(a_i/a)^4(a_i/a)^2(H_e/H_I)^2(m_X/H_I)^4y^4$ | 2                    |
| $c_5$  | $-\left((3-2\nu)(4\pi^4)^{-1}(n_i-5)-1\right)(k/aH_I)^{6-2\nu}$    | $7 \times 10^{-3}(a_i/a)^2(H_e/H_I)^2(m_X/H_I)^4y^4$ | 2                    |
| $c_6$  | $(n_i-1-e)c_2$                                                        | $6 \times 10^{-4}(m_X/H_I)^4$                     | 4                    |
| $a_1$  | $A^2(3-2\nu)^416^{-1}(1+2e-n_i)^{-1}(k/aH_I)^{2n_i-4\nu}$            | $8 \times 10^{-6}(m_X/H_I)^8$                     | 8                    |
| $a_2$  | $A^2(m_X/H_I)^4(1-n_i)^{-1}(k/aH_I)^{2n_i}$                            | $6 \times 10^{-4}(m_X/H_I)^4$                     | 4                    |
| $a_3$  | $A^2(3-2\nu)^2(m_X/H_I)^22^{-1}(1-n_i+e)^{-1}(k/aH_I)^{2(n_i-\nu)}$ | $10^{-4}(m_X/H_I)^6$                             | 6                    |

Since we know from Eq. (133) that the power law index $n$ is generically a negative number close to $-3$ (since $m/H_I$ is expected to be much less than 1), we will define a new power law index

\[ n_l = n + 3. \]  

(135)

The $u$ integrals can be separated as

\[ P_X = \frac{H^8}{4\rho_X} \left( \sum_{j=1}^{3} a_j F_j + \sum_{i=1}^{6} c_i I_i \right), \]

(136)

where all the coefficients are shown in Table I and the integrands to the integrals are shown in Table II. The estimate in the third column is given with any divergent integrals cut off at $k_{\text{max}} = a_e H_e$ corresponding to $u_{\text{max}} = a_e H_e/k$ and $k = a_i H_I y$ where $y < 10^4$ is a scaling parameter relevant for the CMB: e.g.,

\[ I_2 = \int_0^{a_e H_e/k} du \left[ u^{4-n_i} \left( |1 + u|^{n_i-1-2\nu} - |1 - u|^{n_i-1-2\nu} \right) \right]. \]

(137)

This is a sensible cutoff since the power law form of the wave function and the decoherence of quantum fluctuations no longer holds when $k > aH_I$ and the shortest wavelength that gets stretched beyond the horizon during inflation is $k = a_e H_e$. It is clear that $a_i I_i$ contributions to the power spectrum dominate because these terms are not diluted by the enormous scale factor ($(a_e/a_i \gtrsim 10^{22}$). This means that the prediction is insensitive to the cutoff and the
FIG. 1: Plots of the functions important for the isocurvature spectrum. Here, $\epsilon = 0.01$ for $F_1$ and $F_3$.

prediction is robust. Note that in making the order of magnitude estimate in the the third column, we have assumed that $n_l \sim O(m_X^2/H_I^2)$.

The spectrum can thus be written simply as

$$P_{\delta_X}(k) \approx \frac{H_i^8}{\rho_X^2} \left[ \frac{A^2(3-2\nu)^4}{16(1+2\epsilon-n_l)} \left( \frac{k}{aH_i} \right)^{2n_l-4\epsilon} F_1(n_l) + \frac{A^2(m_X/H_I)^4}{1-n_l} \left( \frac{k}{aH_i} \right)^{2n_l} F_2(n_l) + \frac{A^2(3-2\nu)^2(m_X/H_I)^2}{2(1-n_l+\epsilon)} \left( \frac{k}{aH_i} \right)^{2(n_l-\epsilon)} F_3(n_l) \right].$$

(138)

Strictly speaking, this expression unfortunately cannot be used beyond the end of inflation since the wave function we have used to compute the spectrum is no longer valid even approximately. Indeed, as we discussed earlier, when the Hubble expansion rate falls below the mass of the dark matter, the mode function scales like $1/a^{3/2}$. However, in the spirit of the usual scalar density perturbations during inflation, we will simply treat the isocurvature perturbations as a classical fluid at the end of inflation. Owing to Eqs. (77) and (79), this means that for long wavelengths $P_{\delta_X}$ is approximately frozen at the end of inflation assuming that the final particle production density is simply scaled back.
to the end of inflation.\textsuperscript{6} Hence, our final expression for the spectrum is

\[
P_{\delta_X}(k) = \rho_{X}^{\Delta}(t_{e}) \frac{H_i^4}{4 a_e H_i} \left( \frac{k}{a_e H_i} \right)^{2n_l} \left\{ \frac{(3 - 2\nu)^4}{16(1 + 2\epsilon - n_l)} \left( \frac{k}{a_e H_i} \right)^{-4\epsilon} F_1(n_l) + \frac{(m_X/H_i)^4}{1 - n_l} F_2(n_l) + \frac{(3 - 2\nu)^2(m_X/H_i)^2}{2(1 - n_l + \epsilon)} \left( \frac{k}{a_e H_i} \right)^{-2\epsilon} F_3(n_l) \right\}
\]

The three function \(F_i(n_l)\) are shown in Figure\textsuperscript{11}

V. CMB SPECTRUM

In this section we compute numerical values for \(C_l\) contribution coming from the superheavy dark matter isocurvature perturbations within the context of a specific slow roll inflationary scenario, namely that of \(V(\phi) = \frac{1}{2}m_{\phi}^2\phi^2\) scenario. In particular, using Eq. \textsuperscript{85}, we define the isocurvature contribution to Eq. \textsuperscript{75} as

\[
C_l^{(X)} = \frac{16\pi}{25} \int \frac{dk}{k} P_{\delta_X}(k) |j_l(kL)|^2.
\]

For \(\Omega_\Lambda = 0, \Omega_m = 1, \) and \(z_{\text{dec}} = \infty, L\) defined by Eq. \textsuperscript{71} becomes the well known expression \(2/(a_0H_0)\). Numerically, we will instead take more realistic values of \(\Omega_\Lambda = 0.7, \Omega_m = 0.24, \) and \(z_{\text{dec}} = 1100, \) which yields

\[
L \approx \frac{3.5}{a_0H_0}.
\]

Now, we will use the approximation

\[
n_l \approx 3 + \epsilon - 2(1 + \epsilon)\nu,
\]

where \(\nu\) is defined as [c.f., Eq. \textsuperscript{110}]

\[
\nu = \sqrt{\frac{9}{4} - \left( \frac{m}{H_i} \right)^2}
\]

and the slow-roll parameter is evaluated at when the longest wavelengths of interest is leaving the horizon and \(H_i\) corresponds to the Hubble expansion rate at that same time. Using the well known formula

\[
\int \frac{dk}{k} \left( \frac{k}{a_e H_i} \right)^{q} |j_l(kL)|^2 = \frac{2^{q-3\pi}}{(a_e H_i L)^q} \frac{\Gamma(l + q/2)\Gamma(2 - q)}{\Gamma(l + 2 - q/2)\Gamma(3/2 - q/2)} \cdot \Gamma(l + n_l - 2\epsilon)\Gamma(2 - 2n_l + 4\epsilon)
\]

we find

\[
C_l^{(X)} = \frac{16\pi}{25} \rho_{X}^{\Delta}(t_{e}) \frac{H_i^4}{4 a_e H_i} \frac{1}{\pi^5} \Gamma(\nu)^4 \left[ \frac{(3 - 2\nu)^4}{16(1 + 2\epsilon - n_l)} F_1(n_l) + \frac{(m_X/H_i)^4}{1 - n_l} F_2(n_l) + \frac{(3 - 2\nu)^2(m_X/H_i)^2}{2(1 - n_l + \epsilon)} \left( \frac{k}{a_e H_i} \right)^{-2\epsilon} F_3(n_l) \right]
\]

Note that Eq. \textsuperscript{144} says that the effect of the Bessel function integral is to merely set \(k\) to \(1/L\) and multiply the \((k/(a_e H_i))^q|_{k=1/L}\) by a dimensionless number of order \(0.1.\)

\textsuperscript{6} As discussed before, note that we are also effectively neglecting Bogoliubov mixing effects for the computation of the two point function. This is also not rigorously justified in this paper but is consistent with the usual assumptions made in computing scalar perturbations in inflation.
FIG. 2: The region of parameter space excluded by the isocurvature constraint. The shaded area bounded by dot-dashed curve is ruled out due to overproduction of isocurvature perturbations. The bound on $C_l^{(X)}$ we took was $5 \times 10^{-10}$ corresponding to $C_l(l+1)/(2\pi)_{l=2} = 3532 (\mu K)^2$. The dashed curve indicates the uncertainty in the isocurvature computation due to the uncertainty in the particle production computation. Below the solid curve, there is not enough gravitational production of particles to have sufficient dark matter ($\Omega_X h^2 < 0.1$). Above the dotted curve, there is overproduction ($\Omega_X h^2 > 0.2$) of dark matter. Although this specific example is for $V = \frac{1}{2} m_\phi^2 \phi^2$ inflationary potential, as we argued in Section 2, the constraints are general in that superheavy dark matter scenario requires a reheating temperature of above $10^8$ GeV.

Finally, for the numerical analysis, we need an expression for $\rho_e$. As we explained in the appendix, the infrared contribution to this density is ambiguous. Although an accurate computation of the non-infrared contribution can only be computed on a model by model basis, the mass bound coming from the isocurvature contribution is not sensitive to the uncertainty because of the large number in the exponential. On the other hand, the uncertainty on the reheating temperature is the same as the one in the density of particles produced. Hence, the reheating temperature bound is uncertain by a factor of about 10.

For the slow-roll inflationary scenario of $V_\phi = \frac{1}{2} m_\phi^2 \phi^2$, one can compute the non-infrared contributions to the $\rho_X(t_e)$ to be bounded as shown in Eq. (14). For a more general class of slow roll inflationary models, the particle energy density can be estimated by Eq. (15). To obtain the numerical value of $a_e H_I L$ appearing in Eq. (145), we use the standard reheating relationships

$$a_e/a_{RH} = \left(\rho_{RH}/\rho_e\right)^{1/3} = (2.2 \times 10^{-62})^{1/3} \left(\frac{T_{RH}}{\text{GeV}}\right)^{4/3}$$

$$a_{RH}/a_0 = \left(\frac{g_* S(t_0)}{g_* S(t_{RH})}\right)^{1/3} \frac{T_0}{T_{RH}} = 8.1 \times 10^{-14} \frac{\text{GeV}}{T_{RH}}$$

(146)

In Fig. 2 we plot the parameter space allowed by the combined constraints of isocurvature fluctuations and $\Omega_{CDM}$ in the $V = \frac{1}{2} m_\phi^2 \phi^2$ slow roll inflationary model. The somewhat arbitrary bound on $C_l^{(X)}$ we took was $5 \times 10^{-10}$ corresponding to $C_l(l+1)/(2\pi)_{l=2} = 3532 (\mu K)^2$ in units of temperature. This rather large upper bound was taken to obtain a conservative bound on the mass of the superheavy dark matter. Because of the largeness of $a_e H_I L$ in Eq. (145), the mass bound is not sensitive to the exact bound of $C_l$ we use. As one can see, since the dark matter must
compose the CDM of the universe, we have a robust bound on the reheating temperature of

$$T_{RH} > 10^8 \text{GeV.}$$

and on the mass of the superheavy dark matter of about

$$\frac{m_X}{H_e} \gtrsim 6.$$  

As argued in Sec. 2, this bound on the mass and the reheating temperature should be robust for most slow roll inflationary models. The reheating temperature bound has an uncertainty of about a factor of 10 and the $m_X/H_e$ bound has an uncertainty of less than unity. Note that one cannot naively extrapolate using our results to situations to large $m_X/H_e$ (corresponding to small lower bound on reheating temperature) because there, the gravitational production is insufficient to compose the dark matter and the computations have assumed that the gravitationally produced superheavy dark matter dominates the matter density during matter domination. When the gravitationally produced contribution becomes negligible fraction of CDM, we expect the isocurvature bounds to disappear. 

Unlike in section 2, the tensor to scalar amplitude limit does not play much role here because the tensor to scalar ratio here is predicted in this specific inflationary scenario to be much less than unity (of order $10^{-6} \approx 0.1$) on long wavelengths, consistent with $r_S = 0.1$. It is important to note that all of the results in this section are in agreement with the more general discussion of Sec. 2.

One might worry that since $m_X/H_e$ considered in Fig. 4 corresponds to a number larger than 1, we may have violated our assumption that $\nu$ is real. In most inflationary scenarios, this constraint is not violated since $H_I \gtrsim 3H_e$ even when slowly rolling. (For example, $H_I > 7H_e$ for $V = \frac{1}{2} m_\phi^2 \phi^2$ type potential.)

Note also that numerically, the relative contributions to $C_i^{(X)}$ of the $\{F_1, F_2, F_3\}$ terms are

$$(\text{term } F_1) : (\text{term } F_2) : (\text{term } F_3) = 1 : 11 : 7$$

at $m_X/H_e = 6$ and $T_{RH} = 10^9$. These ratios are typical in the region of interest. Hence, the kinetic contribution gives almost a factor of 2 correction to the isocurvature perturbation computation.

VI. CONCLUSION

Isocurvature perturbations are generic prediction of inflationary cosmology. We have explored the isocurvature perturbation constraint on models of superheavy dark matter minimally coupled to gravity. Surprisingly, there is a robust lower bound on the reheating temperature for these models of about $10^7(0.2/r_S) \text{GeV}$ where $r_S$ is the bound on tensor to scalar. This means that, if the superheavy dark matter scenario is embedded in supergravity models with gravity mediated SUSY breaking, the gravitino bound, which is typically an upper bound on the reheating temperature of about $10^8 \text{GeV}$, strongly squeezes these models. If, for example, LHC data favors a gravity mediated SUSY breaking scenario in which the cascade of gravitino decay reactions favors an upper bound on the reheating temperature of $10^7 \text{GeV}$, then superheavy dark matter scenario can be ruled out. Note also that because of the presence of tensor to scalar power limit $r_S$, improved experimental bounds on this quantity can also squeeze this dark matter scenario.

There is also a corresponding $m_X/H_e > 5$ bound on the mass $m_X$ of the superheavy dark matter. This implies that the gravitational particle production is in the exponentially suppressed regime since the exponent of the suppression is of order $-2m_X/H_e$: i.e. noninfrared modes make up most of the dark matter. Moreover, as can be seen in Eq. (29), one cannot lower $H_e$ to less than $3 \times 10^{12} \text{GeV}$ while keeping $m_X/H_e$ fixed and $T_{RH} < 10^9 \text{GeV}$ to obtain sufficient abundance of dark matter because the overall dark matter density scales as $m_X^{5/2}$. Hence, we see that for the superheavy CDM scenario of gravitational particle production, CMB data forcing the isocurvature contribution to be small favors a relatively large gravitational wave amplitude.

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7 It is important to remember that this work has been done assuming that the superheavy dark matter makes up all of the dark matter.

8 Note that for $m_X < H_e$, this scaling law is expected to change to something closer to a power 2, but the present argument still applies.
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**APPENDIX A: MEANING OF ISOCURVATURE PERTURBATIONS**

Given $N$ particle species with their energy densities different from the average energy density, the total energy densities contribute to the curvature perturbations (because local energy density determines the gravitational potential). The $N-1$ remaining energy density field degrees of freedom can contribute to what is often called isocurvature perturbations since $N-1$ degrees of freedom can be assigned any value while keeping the the total energy density and hence the curvature constant.

To write down explicitly the usual condition for adiabatic perturbations, denote the energy density deviation from the average density of particle species $i$ as

$$\delta_i = \frac{\delta \rho_i}{\rho_i}. \quad (A1)$$

For perturbations to be adiabatic, we have

$$\delta(s_ia^3) = 0 \quad (A2)$$

where $s_i$ is the entropy density and $a$ is the scale factor. Since the photon entropy is conserved assuming adiabatic evolution of the photons, Eq. (A2) implies

$$\delta(s_i/s_\gamma) = 0. \quad (A3)$$

Since according to first law of thermodynamics, we have $s_\gamma = (\rho_\gamma + P_\gamma)/T = \frac{4}{3}\rho_\gamma/T$, we find

$$\frac{\delta(s_i/s_\gamma)}{s_i/s_\gamma} = \frac{\delta s_i}{s_i} \frac{s_\gamma}{s_i} = \delta s_i \frac{s_\gamma}{s_i} = 3 \frac{\delta T}{T} \frac{s_i}{s_i} \frac{3}{4} \delta \gamma = 0$$

as the condition for adiabatic perturbations.

Now, we can derive a useful relationship in terms of equation of state. First, the first law of thermodynamics gives

$$d(\rho_ia^3) = T_id(s_ia^3) - P_ida^3 + \mu_id(n_ia^3). \quad (A5)$$

Setting the entropy to zero gives

$$d\rho_ia^3 = -d(a^3)[\rho_i + P_i - \mu_i n_i] + \mu_i d(n_ia^3) \quad (A6)$$

or equivalently

$$\delta_i = 3\frac{\delta a}{a} \left[ \frac{\mu_i n_i}{\rho_i} \right] \left[ 1 + \frac{P_i}{\rho_i} \right] + \mu_i \frac{\delta n_i}{\rho_i}. \quad (A7)$$

The adiabatic condition Eq. (A2) gives

$$\frac{\delta s_i}{s_i} = -3\frac{\delta a}{a} = \frac{\delta_i - \mu_i \delta n_i/\rho_i}{1 + P_i/\rho_i - \mu_i n_i/\rho_i} \quad (A8)$$

where we have used Eq. (A7). Hence, using Eqs. (A4) and (A8), the adiabatic condition can be written as

$$\frac{\delta_i - \mu_i \delta n_i/\rho_i}{1 + w_i - \mu_i n_i/\rho_i} - 3\frac{3}{4} \delta \gamma = 0. \quad (A9)$$

Note that here all evolution was adiabatic. For the nonrelativistic particles, this expression simplifies further since

$$\frac{\delta_i - \mu_i \delta n_i/\rho_i}{1 + w_i - \mu_i n_i/\rho_i} - 3\frac{3}{4} \delta \gamma = \delta_i - \frac{3}{4} \delta \gamma = 0. \quad (A10)$$

which is the usual condition that we see in the literature.
It is interesting to rederive Eq. (A10) from a kinetic theory point of view. From a microcanonical ensemble construction, we can write the entropy density as

$$s_i = \int \frac{d^3p}{(2\pi)^3} a^{-3} f_i (1 - \ln f_i), \quad (A11)$$

where $p$ is the comoving momentum and $f$ is the phase space distribution function. This directly gives

$$\delta s_i = - \int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i \ln f_i - 3 \frac{\delta a}{a} s_i, \quad (A12)$$

On the other hand, the adiabaticity condition Eq. (A2) implies

$$\delta s_i = - 3 \frac{\delta a}{a} s_i. \quad (A13)$$

Hence, we are led to conclude

$$\int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i \ln f_i = 0. \quad (A14)$$

In the near equilibrium case without degeneracy, $\ln f_i < 0$. If in addition, if $\delta f_i = 0$, we can conclude $\delta n_i = -3a^{-1} \delta a \int d^3p (2\pi)^{-3} a^{-3} f_i = -3a^{-1} n_i \delta a$. However, generically, the constrain is only given by Eq. (A14).

If we assume equilibrium situation without degeneracy, we have

$$s_i = \int \frac{d^3p}{(2\pi)^3} a^{-3} f_i \left(1 + \frac{E_i - \mu_i}{T_i}\right) = n_i + \frac{\rho_i - \mu_i n_i}{T_i}. \quad (A15)$$

The constraint Eq. (A14) gives

$$\int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i \left(\frac{E_i - \mu_i}{T_i}\right) = 0. \quad (A16)$$

Since

$$\rho_i = \int \frac{d^3p}{(2\pi)^3} a^{-3} E_i f_i, \quad (A17)$$

we have

$$\delta \rho_i = -3 \frac{\delta a}{a} \rho_i + \int \frac{d^3p}{(2\pi)^3} a^{-3} E_i \delta f_i + \int \frac{d^3p}{(2\pi)^3} a^{-3} \delta E_i f_i$$

$$= -3 \frac{\delta a}{a} \rho_i + \mu_i \int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i - \frac{\delta a}{a} \int \frac{d^3p}{(2\pi)^3} a^{-3} |\vec{p}/a|^2 f_i$$

$$= -3 \frac{\delta a}{a} (\rho_i + P_i) + \mu_i \int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i \quad (A18)$$

where we have used the adiabaticity constraint. Also, we have

$$\delta n_i = \int \frac{d^3p}{(2\pi)^3} a^{-3} \delta f_i - 3 \frac{\delta a}{a} n_i. \quad (A19)$$

Hence, we write

$$\delta \rho_i = -3 \frac{\delta a}{a} (\rho_i + P_i) + \mu_i \left(\delta n_i + 3 \frac{\delta a}{a} n_i\right) = -3 \frac{\delta a}{a} (\rho_i + P_i - \mu_i n_i) + \mu_i \delta n_i, \quad (A20)$$

arriving at

$$\frac{\delta s_i}{s_i} = -3 \frac{\delta a}{a} = \frac{\delta \rho_i - \mu_i \delta n_i}{\rho_i + P_i - \mu_i n_i} = \frac{\delta s_i - \mu_i \delta n_i / \rho_i}{1 + w_i - \mu_i n_i / \rho_i}. \quad (A21)$$
This is precisely the formula that we derived with thermodynamics. Now, we can easily show that adiabatic perturbations usually imply nonvanishing curvature perturbations. According to Eq. \((\ref{eq:thermodynamics})\), we have

\[
\frac{\delta \rho}{\rho} = \frac{1}{\rho} \sum_i \rho_i \delta \rho_i = -3\frac{\delta a}{a} \sum_i \rho_i \left[ (1 + w_i) - \frac{\mu_i \rho_i}{\rho} \right] + \frac{1}{\rho} \sum_i \mu_i \delta \rho_i = -3\frac{\delta a}{a} \sum_i T_i s_i + \frac{1}{\rho} \sum_i \mu_i \delta \rho_i, \tag{A22}
\]

which is nonvanishing as long as \(\mu_i \delta \rho_i\) term does not cancel the entropy term. If the adiabatic perturbations conserve particle number (which would be true for adiabatic perturbations of nonrelativistic particles such as baryons which carry nonnegligible chemical potential), we can write

\[
\delta \rho/\rho \neq 0. \tag{A23}
\]

which would imply there being no cancellation and \(\delta \rho/\rho \neq 0\). On the other hand, it is clear that one can have energy density perturbed without conserving entropy locally. Hence, curvature perturbations do not imply adiabatic perturbations.

**APPENDIX B: PARTICLE PRODUCTION IN THE LARGE MASS LIMIT**

1. General consideration of the exponential suppression

In this appendix, using \((\ref{eq:thermodynamics})\), we give an approximation of the exponential damping of particle production in the regime \(m_X/H \rightarrow \infty\).

Consider the approximate \(|\beta|^2\) formula in \((\ref{eq:thermodynamics})\)

\[
|\beta_k|^2 \approx \exp \left( -\frac{4(k/\sqrt{C(r)})^2}{m_X \sqrt{C''(r)/(2C^2(r))}} - \frac{4m_X}{\sqrt{C''(r)/(2C^2(r))}} \right), \tag{B1}
\]

\[C(\eta) = a^2(\eta) \left[ 1 + \left( \frac{1}{6} - \frac{6a''/a^3}{m_X^2} \right) \right], \tag{B2}
\]

where \(r\) is the real-value solution to the equations

\[
\frac{\mu^2}{6} C'''(r) + C'(r) = 0 \tag{B3}
\]

\[
\frac{\mu^2}{2} C''(r) + \frac{k^2 + m_X^2 C(r)}{m_X^2} = 0 \tag{B4}
\]

with \(\mu\) a pure imaginary number (that also needs to be solved along with \(r\)). The primes are conformal time derivatives as usual. Note that this approximation is valid only if the conditions

\[
\left| \frac{C''''(r)}{C''(r)} \right| \left( \frac{k^2 + m_X^2 C(r)}{10m_X^2 C''(r)} \right) \ll 1 \tag{B5}
\]

\[
\left| \frac{C''''(r)}{C'(r)} \right| \left( \frac{k^2 + m_X^2 C(r)}{6m_X^2} \right) \ll 1 \tag{B6}
\]

\[
\mu^2 < 0 \tag{B7}
\]

are valid.

To use this formula, we need to solve for \(r\) given an inflationary model with potential \(V(\phi)\) which determines \(C\). Solving Eqs. \((\ref{eq:thermodynamics})\) and \((\ref{eq:thermodynamics})\) for \(r\) is equivalent to solving

\[
\frac{C'(r)C''(r)}{C(r)C'''(r)} = 1 + \frac{k^2}{C(r)m_X^2} \tag{B8}
\]

from which we see that \(r\) is a \(k\) dependent quantity in general. Now, consider the large mass expansion:

\[
C(\eta) \approx a^2(\eta) \left[ 1 + O(H^2/m_X^2) \right], \tag{B9}
\]
We then write Eq. \( \text{(B8)} \) as
\[
q = \frac{2a'(a^2 + aa'')}{a(3a'a'' + aa''')} = 1 + \frac{k^2}{a^2(r)m_X^2}
\]
where the primes are with respect to conformal time. Note that this equation gives \( r \) as a function of \( k \). Hence we will implicitly mean \( r = r_k \).

Using standard equation of motion relationships one can easily derive\(^9\)
\[
q = \frac{2H[-(\dot{\phi}^2 - V)/3 + 2H^2]}{H(-2\dot{\phi}^2/3 + 8V/3 + 4H^2) + V,\phi^2 \dot{\phi}}.
\]

Now, substituting \( \dot{\phi}^2 = 2[3H^2 - V] \), we find
\[
q = \left[ 2 + \frac{1}{2VH} \frac{d}{dt} V(\phi(t)) \right]^{-1},
\]
which is a remarkably simple formula.

Substituting for \( \dot{\phi} \) in terms of \( H \) and \( V \), we find
\[
q = \frac{2HV}{4HV + \text{sign}(\dot{\phi})V,\phi \sqrt{2[3H^2 - V]}}.
\]

Except for the Taylor expansion approximation and the saddle point approximation that led to Eq. \( \text{(B1)} \), we have only used one additional assumption of Eq. \( \text{(B9)} \) thus far. Now, we reparameterize \( r = r_k \) by writing
\[
\dot{\phi}^2(r) \approx \lambda V(\phi(r)),
\]
where \( \lambda \) is a numerical coefficient of \( O(1) \) [anticipating the fact that \( r \) will be near the end of inflation]. Again, since \( r = r_k \), we have \( \lambda = \lambda_k \). The utility of this parameterization is to bound \( \lambda \) thereby bounding the final Bogoliubov coefficient.

We can then write
\[
H^2 \approx \frac{1}{3} \left( \frac{\lambda}{2} + 1 \right) V.
\]

Using \( \text{sign}(\dot{\phi}) = -\text{sign}(V,\phi) \) with \( H > 0 \), we find
\[
q = \left( 2 - \frac{1}{2} \sqrt{\frac{3\lambda}{1 + \lambda/2}} \frac{|V,\phi|}{V} \right)^{-1}.
\]

Hence, we arrive at the condition determining \( r \) [Eq. \( \text{(B10)} \)] as
\[
\left( 2 - \frac{1}{2} \sqrt{\frac{3\lambda}{1 + \lambda/2}} \frac{|V,\phi|}{V} \right)^{-1} \approx 1 + \frac{k^2}{a^2(r)m_X^2},
\]
where the \( k \) dependence cannot be trusted for \( k/a \ll H \). However, this does not matter here since we are concerned with \( H \ll m_X \).

Now, let us evaluate
\[
W = \frac{C''}{2C^2} = 2H^2 + \frac{\ddot{a}}{a},
\]
\[\text{(B18)}\]
which appears in the denominator of the exponential in Eq. 21. Since Einstein’s equation with inflaton domination gives
\[ \ddot{a} = -\frac{a}{3}(\dot{\phi}^2 - V), \]  
we find \( W = V \). Hence, we find the approximate Bogoliubov coefficient amplitude to be given by Eq. 21 with \( C''/2C^2 = V \) and \( r \) given by the solution to Eq. 21. Explicitly, the Bogoliubov coefficient is approximated by
\[ |\beta_k|^2 \approx \exp \left( -\frac{4}{m_X} \frac{(k/a(r))^2}{\sqrt{V(\phi(r))}} - \frac{4m_X}{\sqrt{V(\phi(r))}} \right). \]  

2. Application

Consider the quadratic monomial inflaton potential:
\[ V = \frac{1}{2} m_\phi^2 \phi^2. \]  
We find
\[ |\beta_k|^2 \approx \exp \left( -\frac{4\sqrt{2}}{\phi/M_P} \left( \frac{(k/a(r))^2}{m_X m_\phi} + \frac{m_X}{m_\phi} \right) \right), \]  
where we have restored the reduced Planck mass. We use Eq. 21 and find
\[ \frac{\phi(r)}{M_P} \approx \sqrt{\frac{3\lambda}{1 + \lambda/2 k^2/a^2(r) + m_X^2}}. \]  
Hence, we arrive at the Bogoliubov coefficient
\[ |\beta_k|^2 \approx \exp \left[ -4 \frac{2 + \lambda_k}{3\lambda_k} \left( \frac{k^2/a^2(r_k) + m_X^2}{m_\phi m_X} \right) \right]. \]  
Note the nontrivial factor of 2 in front of the \( k^2/a^2 \).

Now, we have to solve for \( \lambda_k \) in Eq. 21. Let us look for the field value at which
\[ \dot{\phi}^2(r_k) = \lambda_k V(\phi) = \frac{\lambda}{2} m_\phi^2 \phi^2(r_k). \]  
There is no closed form solution to this. Suppose we take the standard slow-roll-like approximation except with \( H^2 \approx 2V/3 \) instead of \( H^2 \approx V/3 \). We find
\[ \phi = \sqrt{\frac{2}{3} \frac{M_P}{\sqrt{\lambda_k}}}. \]  
We would then write Eq. 21 as
\[ \sqrt{\frac{2}{3} \frac{1}{\sqrt{\lambda_k}}} \approx \sqrt{\frac{3\lambda_k}{1 + \lambda_k/2 k^2/a^2(r_k) + m_X^2}}. \]  
This can then solved for \( \lambda_k \) as
\[ \lambda_k = \frac{1 + \sqrt{1 + 72f^2}}{18f^2} \]  
where
\[ f = \frac{k^2/a^2(r_k) + m_X^2}{2k^2/a^2(r_k) + m_X^2}. \]
Now $a(r_k)$ still needs to be determined. Since $f$ is a function bounded by $1/2$ and $1$ (with the long-wavelength being $1$), we have

$$0.53 \lesssim \lambda_k \lesssim 1.2$$

(B30)

where the long-wavelength limit corresponding to $\lambda_k \approx 0.53$. For $m_X \gg m_\phi \sim H$, we will have mostly nonrelativistic contributing to the particle production. For $[k/a(r_k)]^2 = m_X^2$, we have $\lambda_k \approx 0.84$. Hence, a more realistic range for $\lambda_k$ is

$$0.53 \lesssim \lambda_k \lesssim 0.84.$$  

(B31)

From now on, we will take this to be the uncertainty range of this parameter.

This means that $a(r_k)$ is bounded as well. Since by Eq. (B26), we have

$$0.89 M_P \leq \phi \leq 1.1 M_P$$

(B32)

where the long-wavelength limit corresponds to $1.1 M_P$. Since the Friedmann equations give

$$\int \frac{da}{a} = \int \frac{d\phi}{M_P} \sqrt{\frac{1}{6} [\dot{\phi}^2 + m_\phi^2 \phi^2]}$$

(B33)

we can write

$$\frac{\Delta a}{a(r_k)} \approx - \frac{\Delta \phi}{M_P} \sqrt{\frac{1}{6} \frac{2}{\lambda_k}}$$

(B34)

where $\Delta a/a \equiv [a(r_k) - a(r_{k_1})]/a$ and $\Delta \phi/M_P \equiv [\phi(r_k) - \phi(r_{k_1})]/M_P$ are both assumed to be small quantities and $\lambda_{k_1} \approx 0.53$ while $\lambda_{k_2} \approx 0.84$. This results in

$$\frac{\Delta a}{a(r_{k_1})} \approx 0.21.$$  

(B35)

To find density of particles produced along with the uncertainty, we can integrate Eq. (B24) assuming both $\lambda = \lambda_{k_1}$ and $a(r) = a(r_k)$ are constants independent of $k$ and then evaluating the integral for two extreme values of $\lambda_{k_1}$. The integral is

$$n = \int \frac{dk k^2}{a^3(2\pi)^2} |\beta_k|^2 = \left(\frac{a(r_k)}{a(t)}\right)^3 \frac{\left(m_\phi m_X\right)^{3/2}}{128 \pi^2 (2 + \lambda_{k_1})/3 \lambda_{k_1}} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{4 m_X}{m_\phi} \frac{2 + \lambda_{k_1}}{3 \lambda_{k_1}}\right).$$

(B36)

Hence, we can bound the number density to be

$$0.7 \times 10^{-3} \left(\frac{a(r_k)}{a(t)}\right)^3 \left(m_\phi m_X\right)^{3/2} e^{-5 m_X/m_\phi} \lesssim n \lesssim 1.6 \times 10^{-3} \left(\frac{a(r_k)}{a(t)}\right)^3 \left(m_\phi m_X\right)^{3/2} e^{-4.2 m_X/m_\phi}.$$  

(B37)

Expectedly, note that the $m_X$ dependence for $m_X \to 0$ of the density here differs from that of $[11]$. The reason why this is expected is that Eq. (B39) no longer applies in that limit. We use Eq. (B37) in the main body of the paper (e.g., in Eq. (14)).

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