Statistical models of hadron production – simple models for complicated processes

Ludwik Turko

Institute of Theoretical Physics, University of Wroclaw, Pl. Maks. Borna 9, 50-204 Wroclaw, Poland
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Thermal statistical models are simple and effective tool to describe particle production in high energy heavy ion collision. It is shown that for higher moments finite volume corrections become important observable quantities. They make possible to differentiate between different statistical ensembles even in the thermodynamic limit.

I. INTRODUCTION

Particle production yields are nicely reproduced by thermal models, based on the assumption of noninteracting gas of hadronic resonances [1]. This simplicity can be however misleading as particle yields and particle ratios are not very sensitive to the underlying model. The main ingredient of the statistical models are probability densities, which allow to extract the whole physical information. The only way to reproduce those probability distributions is by means of higher and higher probability moments. These moments are in fact the only quantities which are phenomenologically available and can be used for the verification of theoretical predictions.

Particle yields in heavy ion collision are the first moments, so they lead to rather crude comparisons with the model. Fluctuations and correlations are second moments so they allow for the better understanding of physical processes in the thermal equilibrium.

Fluctuations and correlations measured in heavy ion collision processes give better insight into dynamical and kinematical properties of the dense hadronic medium created in ultrarelativistic heavy ion collisions. Systems under considerations are in fact so close to the thermodynamic limit that final volume effects seem to be unimportant — at least when productions yields are considered.

The aim of the paper is to show that finite volume effects become more and more important when higher moments, e.g. correlations and fluctuations are considered.

A preliminary analysis of the increasing volume effects was given in [2, 3]. It has been rigorously shown an influence of \( O(1/V) \) terms for a new class physical observables — semi-intensive quantities [3]. Those results completely explained also ambiguities noted in [4], related to "spurious non-equivalence" of different statistical ensembles used in the description of heavy ion collision processes.

We start from the simple example of the standard statistical physics. Is is shown that even in that case a notion of semi–intensive quantities is relevant for the physical situation. In the next step we consider an abelian symmetry corresponding to one conserved charge.

II. CHOICE OF VARIABLES

In the thermodynamical limit the relevant probabilities distributions are those related to densities. These distributions are expressed by moments calculated for densities — not for particles. In the practice, however, we measure particles — not densities as we do not know related volumes. Fortunately, volumes can be omitted by taking corresponding ratios.

Let us consider e.g. the density variance \( \Delta n^2 \). This can be written as

\[
\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = \frac{\langle N^2 \rangle - \langle N \rangle^2}{V^2}.
\]
By taking the relative variance
\[ \frac{\Delta n^2}{\langle n \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2}, \]
volume-dependence vanishes.

A. Semi-intensive variables

A special care should be taken for calculations of ratios of particles moments. Although moments are extensive variables their ratios can be finite in the thermodynamic limit. These ratios are examples of semi-intensive variables. They are finite in the thermodynamic limit but those limits depend on volume terms in density probability distributions. One can say that semi-intensive variables "keep memory" where the thermodynamic limit is realized from.

Let consider as an example the scaled particle variance
\[ \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = V \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle}. \]

The term
\[ \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle}, \]
tends to zero in the thermodynamic limit as \( O(V^{-1}) \). So a behavior of the scaled particle variance depends on the \( O(V^{-1}) \) term in the scaled density variance. A more detailed analysis of semi-intensive variables is given in [3].

To clarify this approach let us consider a well known classical problem of Poisson distribution but taken in the thermodynamic limit.

III. GRAND CANONICAL AND CANONICAL ENSEMBLES

A. Poisson distribution in the thermodynamic limit

Let us consider the grand canonical ensemble of noninteracting gas. A corresponding statistical operator is
\[ \hat{D} = \frac{e^{-\beta H + \gamma N}}{\text{Tr} e^{-\beta H + \gamma N}}. \]

This leads to the partition function
\[ Z(V, T, \gamma) = e^z e^\gamma. \]

where \( z \) is one-particle partition function
\[ z(T, V) = \frac{V}{(2\pi)^{3/2}} \int d^3p \ e^{-\beta E(p)} = V z_0(T), \]

A \( \gamma \) parameter (\( = \beta \mu \)) is such to provide the given value of the average particle number \( \langle N \rangle = V \langle n \rangle \). This means that
\[ e^\gamma = \frac{\langle n \rangle}{z_0}. \]

Particle moments can be written as
\[ \langle N^k \rangle = \left( \frac{1}{Z} \frac{\partial}{\partial \gamma} \right)^k Z. \]
The parameter $\gamma$ is taken in final formulae as a function $\gamma((n), z_0)$ from Eq (4). The resulting probability distribution to obtain $N$ particles under condition that the average number of particles is $\langle N \rangle$ is equal to Poisson distribution

$$P_{(N)}(N) = \frac{(N)^N}{N!} e^{-\langle N \rangle}.$$  

We introduce corresponding probability distribution $P$ for the particle number density $n = N/V$

$$P_{(n)}(n; V) = VP_{(n)}(Vn) = V \frac{(V(n))^V}{\Gamma(Vn + 1)} e^{-V(n)}.$$  

(6)

For large $Vn$ we are using an asymptotic form of Gamma function

$$\Gamma(Vn + 1) \sim \sqrt{2\pi(Vn)^{Vn-1/2}} e^{-Vn} \left\{ 1 + \frac{1}{12Vn} + O(V^{-2}) \right\}.$$  

This gives

$$P_{(n)}(n; V) \sim V^{1/2} \frac{1}{\sqrt{2\pi n}} \left( \frac{n}{V} \right)^{Vn} e^{V(n-n)} \left\{ 1 - \frac{1}{12Vn} + O(V^{-2}) \right\}.$$  

(7)

This expression is singular in the $V \to \infty$ limit. To estimate a large volume behavior of the probability distribution (6) one should take into account a generalized function limit. So we are going to calculate an expression

$$\langle G \rangle_V = \int dn \ G(n) P_{(n)}(n; V),$$  

where $P_{(n)}(n; V)$ is replaced by the asymptotic form from Eq (7). In the next to leading order in $1/V$ one should calculate

$$V^{1/2} \frac{1}{\sqrt{2\pi}} \int dn \ \frac{G(n)}{n^{1/2}} e^{V S(n)} - V^{-1/2} \frac{1}{12V^{1/2}} \int dn \ \frac{G(n)}{n^{3/2}} e^{V S(n)}.$$  

(8)

where

$$S(n) = n \ln(n) - n \ln n + n - \langle n \rangle.$$  

An asymptotic expansion of the function $\langle G \rangle_V$ is given by the classical Watson-Laplace theorem

**Theorem 1** Let $I = [a, b]$ be the finite interval such that

1. $\max S(x)$ is reached in the single point $x = x_0$, $a < x_0 < b$.

2. $f(x), S(x) \in C(I)$.

3. $f(x), S(x) \in C^\infty$ in the vicinity of $x_0$, and $S''(x_0) \neq 0$.

Then, for $\lambda \to \infty$, $\lambda \in S_\varepsilon$, there is an asymptotic expansion

$$F[\lambda] \sim e^{\lambda S(x_0)} \sum_{k=0}^\infty c_k \lambda^{-k-1/2},$$  

(9a)

$$c_k = \frac{\Gamma(k + 1/2)}{(2k)!} \left. \left( \frac{d}{dx} \right)^{2k} f(x) \left( \frac{S(x) - S(x_0)}{(x - x_0)^2} \right)^{-k-1/2} \right|_{x=x_0}.$$  

(9b)

$S_\varepsilon$ is here a segment $|\arg z| \leq \frac{\pi}{2} - \varepsilon < \frac{\pi}{2}$ in the complex z-plane.
To obtain $O(1/V)$ formula the first term in (8) should be calculated till the next to leading order term in $1/V$. For the second term it is enough to perform calculations in the leading order only.

The first term gives the contribution
\begin{equation}
V^{1/2} \frac{1}{\sqrt{2\pi}} \int dn \frac{G(n)}{n^{3/2}} e^{VS(n)} = G(\langle n \rangle) + \frac{1}{12\langle n \rangle V} G(\langle n \rangle) + \frac{\langle n \rangle}{2V} G''(\langle n \rangle),
\end{equation}
and the second term gives
\begin{equation}
V^{-1/2} \frac{1}{12\sqrt{2\pi}} \int dn \frac{G(n)}{n^{3/2}} e^{VS(n)} = \frac{1}{12\langle n \rangle V} G(\langle n \rangle),
\end{equation}
So we have eventually
\begin{equation}
\langle G \rangle_V = G(\langle n \rangle) + \frac{\langle n \rangle}{2V} G''(\langle n \rangle) + O(V^{-2}),
\end{equation}
for any function $G$.

This gives us the exact expression for the density distribution (10) in the large volume limit
\begin{equation}
P(\langle n \rangle; V) \sim \delta(n - \langle n \rangle) + \frac{\langle n \rangle}{2V} \delta''(n - \langle n \rangle) + O(V^{-2}).
\end{equation}
We are now able to obtain arbitrary density moments up to $O(V^{-2})$ terms.
\begin{equation}
\langle n^k \rangle_V = \int dn n^k P(\langle n \rangle; V) = \langle n \rangle^k + \frac{k(k-1)}{2V} \langle n \rangle^{k-1} + O(V^{-2}).
\end{equation}
We have for the second moment (intensive variable!)
\begin{equation}
\langle n^2 \rangle_V = \langle n \rangle^2 + \frac{\langle n \rangle}{V} + O(V^{-2}).
\end{equation}
This means
\begin{equation}
\Delta n^2 = \frac{\langle n \rangle}{V} \to 0.
\end{equation}
as expected in the thermodynamic limit.

The particle number and its density are fixed in the canonical ensemble so corresponding variances are always equal to zero. The result (14) can be seen as an example of the equivalence of the canonical and grand canonical distribution in the thermodynamic limit. This equivalence is clearly visible from the Eq (12) where the delta function in the first term can be considered as the particle number density distribution in the canonical ensemble.

A more involved situation appears for particle number moments (extensive variable!). Eq (13) translated to the particle number gives
\begin{equation}
\langle N^k \rangle = V^k \langle n \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle n \rangle^{k-1} + O(V^{k-2}),
\end{equation}
One gets for the scaled variance (semi-intensive variable!)
\begin{equation}
\frac{\Delta N^2}{\langle N \rangle} = 1,
\end{equation}
what should be compared with zero obtained for the canonical distribution.

The mechanism for such a seemingly unexpected behavior is quite obvious. The grand canonical and the canonical density probability distributions tend to the same thermodynamic limit. There are different however for any finite volume. Semi-intensive variables depend on coefficients at those finite volume terms so they are different also in the thermodynamic limit.
B. Energy distribution

It is interesting to perform similar calculation for the energy distribution in both ensembles. Energy moments and an average energy density can be written as

\[ \langle E^k \rangle = (-1)^k \frac{1}{Z} \frac{\partial^k Z}{\partial \beta^k} ; \quad \langle \epsilon \rangle = -\frac{dz_0}{d\beta} e^\gamma. \]  

(17)

One gets from Eq (17)

\[ \langle E^k \rangle = V^k \langle \epsilon \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle \epsilon \rangle^{k-2} n \frac{d^2 z_0}{dz_0 d\beta^2} + O(V^{k-2}). \]  

(18)

The grand canonical energy density distribution follows

\[ P(\epsilon | n, \langle \epsilon \rangle) = \delta (\epsilon - \langle \epsilon \rangle) + \frac{n}{2V} R^{GC} \left( \frac{\langle \epsilon \rangle}{n} \right) \delta'' (\epsilon - \langle \epsilon \rangle) + O(V^{-2}). \]  

(19)

\[ R^{GC} \] is given here as

\[ R^{GC} \left( \frac{\langle \epsilon \rangle}{n} \right) = \frac{1}{z_0} \frac{d^2 z_0}{dz_0 d\beta^2} \bigg|_{\beta = \beta(\langle \epsilon \rangle / n)}. \]

For the canonical distribution a corresponding statistical operator is

\[ \hat{D} = \frac{e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}} \]  

(20)

This leads to the partition function

\[ Z(V, T) = \frac{z^N}{N!} = \frac{e^{V n \log z}}{N!}. \]  

(21)

Internal energy moments are given by Eq (17). In particular

\[ \langle \epsilon \rangle = -n \frac{dz_0}{z_0 d\beta}. \]  

(22)

For the energy moments one gets now

\[ \langle E^k \rangle = V^k \langle \epsilon \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle \epsilon \rangle^{k-2} n \frac{d^2 z_0}{dz_0 d\beta^2} \left( \frac{1}{z_0} \frac{\partial z_0}{\partial \beta} \right) + O(V^{k-2}). \]  

(23)

A corresponding probability distribution is

\[ P(\epsilon | n, \langle \epsilon \rangle) = \delta (\epsilon - \langle \epsilon \rangle) + \frac{n}{2V} R^C \left( \frac{\langle \epsilon \rangle}{n} \right) \delta'' (\epsilon - \langle \epsilon \rangle) + O(V^{-2}), \]  

(24)

where \( R^C \) is given here as

\[ R^C \left( \frac{\langle \epsilon \rangle}{n} \right) = \frac{\partial}{\partial \beta} \left( \frac{1}{z_0} \frac{\partial z_0}{\partial \beta} \right) \bigg|_{\beta = \beta(\langle \epsilon \rangle / n)}. \]

IV. HIGH ENERGY STATISTICAL PHYSICS

Although the spirit and the philosophy of the statistical approach remains the same, ingredients of statistical models used in high energy problems are different. The main difference is that a number of particles is no longer conserved so we have no chemical potentials related to that quantity. The only nontrivial chemical potentials are those related to conserved charges, so the role of internal symmetries is a crucial one. For abelian charges, as electric charge or baryonic charge
an introduction of corresponding potential is rather obvious – it is just a Lagrange multiplier at a generator of the $U(1)$ symmetry. For non-abelian symmetries, as e.g. for the isotopic $SU(2)$ symmetry, problem is more involved. It appears [3] that in such a case the only relevant chemical potentials are those related to so called Cartan subgroup – a maximal abelian subgroup of the given non-abelian group of internal symmetry.

So for the hot hadronic gas a well approximated internal symmetry is $SU(3)$ or $SU(4)$ with a charm taken into account. On the top of it we have the exact $U(1)$ baryon number conservation. $SU(3)$ symmetry leads to two chemical potentials, related to the third isospin component and to the hypercharge. Those two chemical potentials are supplemented by the baryonic potential. One uses for practical reasons another set of chemical potentials, which are linear combinations of the basic set. These are the electric charge and the strangeness chemical potentials together with the unchanged baryonic potential.

Then, for the simplest case of an ideal hadron gas in thermal and chemical equilibrium, which consists of $l$ species of particles, energy density $\epsilon$, baryon number density $n_B$, strangeness density $n_S$ and electric charge density $n_Q$ read ($\hbar = c = 1$ always)

\[
\epsilon = \frac{1}{2\pi^2} \sum_{i=1}^{l} (2s_i + 1) \int_{0}^{\infty} dp \frac{p^2 E_i}{\exp \left( \frac{E_i - \mu_i}{T} \right) + g_i}, \tag{25a}
\]

\[
n_B = \frac{1}{2\pi^2} \sum_{i=1}^{l} (2s_i + 1) \int_{0}^{\infty} dp \frac{p^2 B_i}{\exp \left( \frac{E_i - \mu_i}{T} \right) + g_i}, \tag{25b}
\]

\[
n_S = \frac{1}{2\pi^2} \sum_{i=1}^{l} (2s_i + 1) \int_{0}^{\infty} dp \frac{p^2 S_i}{\exp \left( \frac{E_i - \mu_i}{T} \right) + g_i}, \tag{25c}
\]

\[
n_Q = \frac{1}{2\pi^2} \sum_{i=1}^{l} (2s_i + 1) \int_{0}^{\infty} dp \frac{p^2 Q_i}{\exp \left( \frac{E_i - \mu_i}{T} \right) + g_i}. \tag{25d}
\]

where $E_i = (m_i^2 + p^2)^{1/2}$ and $m_i$, $B_i$, $S_i$, $\mu_i$, $s_i$ and $g_i$ are the mass, baryon number, strangeness, chemical potential, spin and a statistical factor of specie $i$ respectively (we treat an antiparticle as a different specie). And $\mu_i = B_i \mu_B + S_i \mu_S + Q_i \mu_Q$, where $\mu_B$, $\mu_S$, and $\mu_Q$ are overall baryon number and strangeness chemical potentials respectively.

To get particle yields one should consider also entropy density $s$

\[
s = \frac{1}{6\pi^2 T^2} \sum_{i=1}^{l} (2s_i + 1) \int_{0}^{\infty} dp \frac{p^4 (E_i - \mu_i) \exp \left( \frac{E_i - \mu_i}{T} \right)}{\left( \exp \left( \frac{E_i - \mu_i}{T} \right) + g_i \right)^2}. \tag{26}
\]

To obtain the time dependence of temperature and baryon number and strangeness chemical potentials one has to solve numerically equations (25) with $s$, $n_B$, $n_Q$, and $n_S$ given as time dependent quantities. For $s(t)$, $n_B(t)$, and $n_Q(t)$ one obtains expressions form hydrodynamical calculations and $n_S = 0$ since we put the overall strangeness equal to zero during all the evolution.

These equations, enriched by unstable particles effects, form a basic for successful calculations of relativistic heavy ion production processes concerning particle yields and rates. All calculated observables are here the first moments of related probability distributions. If we are going to get correlations and fluctuations predictions, we have to calculate second moments. This gives quite new effects, statistical ensemble dependent, as was shown in previous sections devoted to standard statistical physics approach.

### A. Statistical ensembles of high energy physics

To make our considerations the simplest possible we consider the statistical model of a non–interacting gas constrained by the conservation of the abelian charge $Q$. The thermodynamic system of volume $V$ and temperature $T$ is considered to be composed of charged particles and their antiparticles carrying charge $\pm 1$ respectively. The requirement of charge conservation in the
system is imposed on the grand canonical or canonical level. The canonical level means the global charge conservation while the grand canonical level means the charge conservation on the average. The partition function of the above canonical and grand canonical statistical system is found to be

\begin{align}
Z_Q^C(V,T) &= \text{Tr}_Q e^{-\beta \hat{H}} = \sum_{N_+,N_-=0}^{\infty} \frac{z^{N_+} N_+!}{N_-! N_+!} I_Q(2z), \tag{27a} \\
Z_{GC}(V,T) &= \text{Tr} e^{-\beta (\hat{H} - \mu \hat{Q})} = \exp \left( 2z \cosh \frac{\mu}{T} \right). \tag{27b}
\end{align}

where \( z \) is the sum over all one-particle partition functions

\[ z(T) = \frac{V}{(2\pi)^3} \sum_i g_i \int d^3 p e^{-\beta \sqrt{p^2 + m_i^2}} = \frac{V}{2\pi^2} T \sum_i g_i m_i^2 K_2 \left( \frac{m_i}{T} \right) \equiv V z_0(T), \tag{28} \]

and \( g_i \) is the spin degeneracy factor. The sum is taken over all charged particles and resonances of mass \( m_i \) carrying the charge \( \pm 1 \). The functions \( I_Q \) and \( K_2 \) are modified Bessel functions. The chemical potential \( \mu \) determines the average charge in the grand canonical ensemble

\[ \langle Q \rangle = T \frac{\partial}{\partial \mu} \ln Z_{GC}. \]

This allows to eliminate the chemical potential from further formulae for the grand canonical probabilities distributions

\[ \frac{\mu}{T} = \text{arcsinh} \frac{\langle Q \rangle}{2z} = \ln \left( \frac{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4z^2}}{2z} \right). \tag{29} \]

In the canonical ensemble we have a system of volume \( V \) and total charge \( Q \). In the grand canonical ensemble we have a system with volume \( V \) and average charge \( \langle Q \rangle \). Number of particles is not conserved in both ensembles. Number \( N_- \) of negative charged particles shall be extracted from the relevant probability distributions. The probability distribution \( P_Q^C(N_-,V) \) to have \( N_- \) negatively and \( N_+ = N_- + Q \) positively charged particles is obtained \([2, 3]\) from the partition function \([27a]\) as

\[ P_Q^C(N_-,V) = \frac{z^{2N_- + Q}}{N_-!(N_- + Q)!} I_Q(2z). \tag{30} \]

On the other hand in the GC ensemble with volume \( V \) and average charge \( \langle Q \rangle \) the probability distribution \( P_{Q}^{GC}(N_-,Q,V) \) to find a system with a given charge \( Q \) and a given number of negatively charged particles \( N_- \) is expressed \([2, 3]\) as the product

\[ P_{Q}^{GC}(N_-,Q,V) = P_Q^C(N_-,V) P_{Q}^{GC}(Q,V), \tag{31} \]

of the canonical particle number distribution \( P_Q^C(N_-,V) \) from Eq. \([30]\) and the grand canonical probability distribution

\[ P_{Q}^{GC}(Q,V) = I_Q(2z) \left[ \frac{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4z^2}}{2z} \right]^Q \frac{e^{-\sqrt{\langle Q \rangle^2 + 4z^2}}}{2z} \] \tag{32} \]

to find the total charge \( Q \) in the system with the average charge \( \langle Q \rangle \).

**B. The thermodynamic limit**

The thermodynamic limit is understood as a limit \( V \to \infty \) such that densities of the system remain constant. So we have for the canonical ensemble

\[ Q \to \infty, \quad N_- \to \infty; \quad \frac{Q}{V} = q; \quad \frac{N_-}{V} = n_- \]
and

\[ \langle Q \rangle \to \infty, \ N_- \to \infty; \quad \frac{\langle Q \rangle}{V} = \langle q \rangle; \quad \frac{N_-}{V} = n_- \]

for the grand canonical ensemble.

To formulate correctly the thermodynamic limit of quantities involving densities, one defines the following probabilities

\[ \mathbf{P}_q^C(n_-, V) := V \mathbf{P}_q^C(V n_-, V), \quad \text{(33a)} \]
\[ \mathbf{P}_{(q)}^{GC}(n_-, q, V) := V^2 \mathbf{P}_{(q)}^{GC}(V n_-, V q, V), \quad \text{(33b)} \]
\[ \mathbf{P}_{(q)}^{GC}(q, V) := V \mathbf{P}_{(q)}^{GC}(V q, V). \quad \text{(33c)} \]

We are going to proceed now in a similar way as in the former section. In a large volume limit one gets

\[ \mathbf{P}_q^C(n_-, V) = \mathbf{P}_q^{\infty}(n_-) + \frac{1}{V} \mathbf{R}_q^C(n_-) + \mathcal{O}(V^{-2}), \quad \text{(34)} \]
\[ \mathbf{P}_{(q)}^{GC}(n_-, q, V) = \mathbf{P}_{(q)}^{\infty}(n_- q) + \frac{1}{V} \mathbf{R}_{(q)}^{GC}(n_- q) + \mathcal{O}(V^{-2}), \quad \text{(35a)} \]
\[ \mathbf{P}_{(q)}^{GC}(q, V) = \mathbf{P}_{(q)}^{\infty}(q) + \frac{1}{V} \mathbf{R}_{(q)}^{GC}(q) + \mathcal{O}(V^{-2}). \quad \text{(35b)} \]

All functional coefficients can be obtained here using Laplace-Watson theorem. From the careful analysis one gets \[ \text{(33)} \] for the probability distribution \[ \text{(34)} \] of the canonical ensemble

\[ \mathbf{P}_q^n(n_-; V) = \delta(n_- - \langle n_- \rangle) + \frac{1}{V} \frac{z_0^2}{q^2 + 4z_0} \delta'(n_- - \langle n_- \rangle) \]
\[ + \frac{1}{2V} \frac{z_0^2}{2q^2 + 4z_0} \delta''(n_- - \langle n_- \rangle) + \mathcal{O}(1/V^2), \quad \text{(36)} \]

and for the probabilities distributions \[ \text{(35a)} \] of the grand canonical distribution

\[ \mathbf{P}_{(q)}^{GC}(q, n_-; V) = \delta(n_- - \langle n_- \rangle) \delta(q - \langle q \rangle) \]
\[ + \frac{1}{2V} \frac{z_0^2}{\langle q \rangle^2 + 4z_0} \delta''(n_- - \langle n_- \rangle) + \mathcal{O}(1/V^2), \quad \text{(37a)} \]
\[ \mathbf{P}_{(q)}^{GC}(q, V) = \delta(q - \langle q \rangle) + \frac{\sqrt{\langle q \rangle^2 + 4z_0^2}}{2V} \delta''(q - \langle q \rangle) + \mathcal{O}(1/V^2), \quad \text{(37b)} \]
\[ \mathbf{P}_{(q)}^{GC}(n_, V) = \delta(n_- - \langle n_- \rangle) + \frac{1}{2V} \frac{z_0^2}{\langle n_- \rangle^2} \delta''(n_- - \langle n_- \rangle) + \mathcal{O}(1/V^2). \quad \text{(37c)} \]

An average limiting density of charged particles

\[ \langle n_\pm \rangle = \frac{\sqrt{q^2 + 4z_0} \pm q}{2} \quad \text{(38)} \]

is used in above formulae.

**C. Particle moments**

Probability distributions \[ \text{(36)} \] and \[ \text{(37)} \] allow to write compact expressions for for particle and charge distribution density moments of any order up to \( \mathcal{O}(1/V^2) \) terms. For particle moments one gets

\[ \langle n_\pm^k \rangle \simeq \langle n_\pm \rangle^k - \frac{k}{V} \frac{z_0^2}{q^2 + 4z_0} \langle n_\pm \rangle^{k-1} + \frac{k(k-1)}{2V} \frac{z_0^2}{\sqrt{q^2 + 4z_0}} (\langle n_\pm \rangle)^{k-2}, \quad \text{(39)} \]
for the canonical ensemble moments and
\[ \langle n_{\pm}^k \rangle^{GC} \simeq \langle n_{\pm} \rangle^k + \frac{k(k-1)}{2V} \langle n_{\pm} \rangle^{k-1}. \] (40)
for the grand canonical ensemble moments.

Although those moments are density moments they can be expressed by directly observable variables. Using Eq (38) one gets moments as functions of \( q/z_0 \) ratio. This ratio is observable as it can be written as a function of the ratio of charged particles
\[ \frac{q^2}{z_0^2} = \frac{\langle N_+ \rangle_{\infty}}{\langle N_- \rangle_{\infty}} + \frac{\langle N_- \rangle_{\infty}}{\langle N_+ \rangle_{\infty}} - 2. \]

D. Semi-intensive variables

Now we are in position to create some semi-intensive variables. They are finite in T-limit and have different values dependently on how the charge conservation is implemented in the description of the system. There is actually a broad class of variables. We take an an example
\[ S_k = \frac{\langle N_{\pm}^k \rangle - \langle N \rangle^k}{\langle N \rangle^{k-1}}. \] (41)

Indeed from (39) and (37) one gets canonical and grand canonical values for positive(negative) particles in the thermodynamic limit (denoted as T-limit in the subsequent formulae)
\[ T\text{-}lim S_k^{GC} = \frac{k(k-1)}{2} \left( \frac{q^2 + 4z_0^2}{\sqrt{q^2 + 4z_0^2}} \right), \]
for the canonical ensemble, while in the grand canonical ensemble
\[ T\text{-}lim S_k^{GC} = \frac{k(k-1)}{2}. \] (42b)
The scaled variance is just a special case of \( S_k \) corresponding to \( k = 2 \).

Another examples are classes of variables closely related to cumulant or factorial cumulant moments \[3\] or susceptibility ratios. Let define \( p \)-th order susceptibility
\[ \kappa_p = \frac{\partial^p \ln Z}{\partial \mu^p}. \]
One can easily check that the ratios
\[ K_{p,r} = \frac{\kappa_p}{\kappa_r}, \] (43)
are semi-intensive quantities.

One can also construct more involved semi-inclusive variables having a finite T-limit behavior which are determined by higher order asymptotic terms of the corresponding probability distributions.

V. CONCLUSION

We have discussed the differences in the asymptotic properties of the probability functions for a system with an exact, that is canonical, and with an average, that is grand canonical, implementation of charge conservation. We have shown that in the thermodynamic limit the corresponding probability distributions in the grand canonical and canonical ensembles coincide and are described as generalized functions. This property is a direct consequence of the grand canonical and canonical ensemble equivalence in the thermodynamic limit. However, the first finite volume corrections to the asymptotic value differ for both ensembles.
Finally, using the results of the probability functions we have derived the asymptotic behavior of the charged particle moments and established the differences in the grand canonical and canonical formulation. We have also applied these results to find the thermodynamic limit of a class of semi-intensive quantities. It was shown that in systems with exact and average charge conservation such quantities should naturally converge to different values in the thermodynamic limit. This is because the behavior of the semi-intensive quantities in the near vicinity to the thermodynamic limit are determined by the subleading, finite volume, corrections to the probability distributions which are specific to a given statistical ensemble.

It is important that first moments are the same in the canonical and grand canonical ensemble. This means that particle yields in heavy ion collision and equation of state of dense hadronic medium are insensitive to the statistical ensemble in the thermodynamic limit. This is not the case, however, for fluctuations and higher moments. Finite volume effect are more and more relevant for higher moments. Such a situation appears when comparing the statistical model with lattice gauge theory results obtained on a small lattice.

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