A REMARK FOR AUTOMORPHIC FORMS ON PERIOD DOMAINS

MOHAMMAD REZA RAHMATI

ABSTRACT. In this article we mention the fact that the Hodge metric on Mumford-Tate domains is the same as the Peterson inner product of the automorphic forms on these spaces.

INTRODUCTION

The inter-action between Hodge theory and automorphic representations has been extensively studied during the last decades. From the Hodge literature point of view this goes back to P. Griffiths defining the automorphic cohomology of period domains in his famous treatise ‘Periods of Integrals ...‘. A variation of Hodge structure

\[ \Phi : S \to \Gamma \setminus D \]

naturally provides the Hodge domain \( \Gamma \setminus D \) and the space of functions on this space can be identified with the \( \Gamma \)-automorphic functions on \( D \). A basic example of this is a variation \( V = V^{1,0} \oplus V^{0,1} \) obtained from the middle cohomology of an elliptic fibration of curves. In this case \( D = H \) is the upper half plane, \( \Gamma = SL_2(\mathbb{Z}) \) and the functions on the Hodge domain \( SL_2(\mathbb{Z}) \setminus H \) are the usual modular forms. One can consider the Hodge bundles \( V^{n,0} = V^{1,0} \otimes^n \) on \( \Gamma \setminus D \) and correspondingly the space of automorphic functions with values in these bundles by only extending the coefficients.

We will consider automorphic forms with values in general Hodge bundles \( F^p \) of the Hodge structures on a vector space \( V \), and also the endomorphism bundle \( g = \text{End}(V) \) in general. The corresponding local systems on \( D \) are \( \mathcal{V} := \Gamma \setminus (D \times V) \) and \( \mathcal{G} := \Gamma \setminus (D \times g) \), respectively. Lets denote the hermitian metric by \( h \) to be any of the induced metric from that of the upper half plane \( H \). We use the hyperbolic metric descended from \( H \) to define Hodge \( * \)-operator on \( \mathcal{V} \) and \( \mathcal{G} \). Then the Laplace operator and the harmonic forms can be defined in the same fashion as in Hodge theory. In this way this question arises that what would be the relation between the Hodge inner product and some natural inner product on the automorphic forms.
We simply treat with this question in section 3 and identify these bilinear forms on Hodge domains.

The Peterson inner product between two modular forms on the upper half plane can be naturally stated for the vector valued automorphic forms. This may be regarded as a hermitian form induced from the hyperbolic metric on the upper half plane. The aforementioned identification of $V$ and $G$ identifies this inner product with the one induced by the Hodge inner product, defined by the Hodge star operator. If we add this fact from the mixed Hodge metric theorem that the polarization form on the Hodge structures are unique, it follows that the polarization form and the one induced from Peterson inner product must be the same up to a constant factor.

1. Moduli of Hodge structures

Let $h_{p,q}$ be the Hodge numbers of a Hodge structure

$$\phi : S(\mathbb{R}) \to Aut(V_{\mathbb{R}})$$

of weight $n$ and period domain $D$ with compact dual $\tilde{D}$. The group $G_{\mathbb{R}} = Aut(V_{\mathbb{R}}, Q)$ is a real simple Lie group that acts transitively on $D$. The isotropy group $H$ of a reference polarized Hodge structure $(V, Q, \phi)$ would be a compact subgroup of $G_{\mathbb{R}}$ that contains a compact maximal torus $T$. One has

$$D = \{ \phi : S^1 \to G_{\mathbb{R}} ; \phi = g^{-1}\phi_0 g \}$$

It follows that $H = Z_{\phi_0}(G_{\mathbb{R}})$, the centralizer of $\phi_0(S^1)$. An easy exercise in linear algebra shows

$$H \approx \begin{cases} 
U(h^{2m+1}) \times \ldots \times U(h^{m+1,m}) & n = 2m + 1 \\
U(h^{2m}) \times \ldots \times U(h^{m+1,m-1}) \times O(h^{m,m}) & n = 2m
\end{cases}$$

The group $G_{\mathbb{C}}$ is a complex simple Lie group that acts transitively on $\tilde{D}$. The subgroup $P$ that stabilizes a $F_{0}^{\bullet}$ is a parabolic subgroup with $H = G_{\mathbb{R}} \cap P$.

The case $n = 1$ is classical and one knows that $D = H_g$ the Siegel generalized upper half space $= \{ Z \in M_{g \times g} : Z =^t Z, \text{Im}(Z) > 0 \}$.
The Lie algebra $\mathfrak{g}$ of the simple Lie group $G_\mathbb{C}$ is a $\mathbb{Q}$-linear subspace of $End(V)$, and the form $Q$ induces on $\mathfrak{g}$ a non-degenerate symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

which up to scale is just the Cartan-Killing form $\text{tr}(\text{ad}(x)\text{ad}(y))$. For each point $\phi \in D$

$$\text{Ad}\phi : \mathbb{U}(\mathbb{R}) \to \text{Aut}(\mathfrak{g}_\mathbb{R}, B)$$

is a Hodge structure of weight 0 on $\mathfrak{g}$. This Hodge structure is polarized by $B$.

Associated to each nilpotent transformation $N \in \mathfrak{g}$ one defines a limit mixed Hodge structure. The local system $\mathfrak{g} \to \Delta^*$ is then equipped with the monodromy $T = e^{\text{ad}N}$ and Hodge filtration defined with respect to the multi-valued basis of $\mathfrak{g}$ by $e^{\log(t)\frac{N}{2\pi i}}F^\bullet$. It gives a limit MHS $(\mathfrak{g}, F^\bullet, W(N)_\bullet)$.

The polarizing form gives perfect pairings

$$B_k : Gr_{W(N)}^k \mathfrak{g} \times Gr_{-k}^W \mathfrak{g} \to \mathbb{Q}, \quad B_k(u, v) = B(v, N^k u)$$

defined via the hard Lefschetz isomorphism $N^k : Gr_{-k}^W \mathfrak{g} \cong Gr_{k}^W \mathfrak{g}$.

Let $D$ be a period domain for a PHS $(V, Q, \phi)$ of weight $n$ and set $\Gamma_Z = Aut(V_\mathbb{Z}, \mathbb{Q})$. In the tangent bundle $TD$ there is a homogeneous sub-bundle $W$ whose fiber at $\phi$ is $W_\phi = \mathfrak{g}_\phi^{-1,1} = \{ \psi \in T_\phi D : \psi(F^p_\phi) \subset F^{p+1}_\phi \}$ defined by, namely infinitesimal period relations (IPR).

A variation of Hodge structure (VHS) is given by a locally liftable holomorphic map

$$\tilde{S} \xrightarrow{\tilde{\phi}} D \xrightarrow{\phi} \Gamma_Z \setminus D$$
where the left vertical map is the universal covering, and the infinitesimal period relation says $\Phi_* : T\tilde{S} \to W$. $\Gamma := \Phi_*(\pi_1(S, s_0)) \subset G_Z$ is called the monodromy group.

2. Hodge bundles and Automorphic representations

On the compact dual $\tilde{D}$ there are $G_\mathbb{C}$-homogeneous vector bundles

\[(7) \quad F^p \to \tilde{D}\]

called Hodge bundles whose fiber at a given point $F^\bullet$ is $F^p$. Over $D \subset \tilde{D}$ we have

\[(8) \quad V^{p,q} = F^p / F^{p+1}\]

These are homogeneous vector bundles for the action of $G_\mathbb{R}$. They are hermitian vector bundles with $G_\mathbb{R}$-invariant Hermitian metric given in each fiber by the polarization form.

In order to explain automorphic forms on Hodge domains we proceed by an example. The general case is similar. Suppose for the moment $\text{dim}(V) = 2$ and $n = 1$. The equivalence classes of polarized Hodge structures of weight 1 can be identified with $Sl_2(\mathbb{Z}) \setminus H$, with $H$ to be the upper half plane. More generally for geometric reasons one wishes to consider congruence subgroups $\Gamma \subset Sl_2(\mathbb{Z})$ and the quotient spaces

\[(9) \quad M_\Gamma = \Gamma \setminus H\]

such that $M_\Gamma$ is a Riemann surfaces. It is not compact but has only cusps. Let $V^{n,0} := (V^{1,0})^{\otimes n}$ be the $n$-th tensor power of the space of holomorphic forms on the elliptic curve.

**Definition 2.1.** A holomorphic automorphic function of weight $n$ is given by a holomorphic section $\psi \in \Gamma(\Gamma \setminus H, V^{n,0})$ that is finite on the cusps.

Such an automorphic form can be written as

\[(10) \quad \psi(\tau) = f(\tau)d\tau^{n/2}\]
where $f$ is holomorphic on $H$ and satisfies

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^nf(\tau)$$

Around a cusp one sets $q = e^{2\pi i\tau}$ and expands in Laurent series

$$f(q) = \sum_n a_n q^n$$

The finiteness condition at the cusps are then $a_n = 0$ for $n < 0$. If we consider the Deligne extension $V_n,0$ the modular forms are sections of $V_n,0 \to \Gamma \backslash H$ that extend to $H_n,0 \to \Gamma \backslash \overline{H}$. A modular form is a cusp form if $a_0 = 0$, i.e it vanishes at the cusps. This condition is equivalent to $\int_{\Gamma \backslash \overline{H}} \| \psi \| d\mu < \infty$.

The aforementioned construction can be done in general over any period domain $\Gamma \backslash D$ as in (6). Then the $\Gamma_\mathbb{Z}$-invariant functions or forms are the automorphic forms with respect to $\Gamma$.

**Definition 2.2.** Let $\mathcal{V}$ be a Hodge bundle on the period domain $D$. A holomorphic automorphic function of weight $n$ is given by a holomorphic section $\psi \in \Gamma(\Gamma \backslash D, \mathcal{V})$ that is finite on the cusps.

A vector valued $\mathcal{A}^{p,q}$-automorphic form satisfies the following condition;

$$f(\gamma.z)\gamma'(z)^p\bar{\gamma}'(z)^q = Ad(\rho)f(z), \quad \gamma \in \Gamma$$

where $\rho : \pi_1 \to GL(V)$ is the representation of the monodromy on the parameter base space. $\mathcal{A}^{p,q}$ is vector space of smooth $C^\infty$-forms on $D$. In this way Hodge theory naturally inter-acts with Langlands program and the theory of automorphic representations. One may note that automorphic forms can be considered to take values in Hodge bundles by extending the coefficients. We investigate some relations with the polarization form of Hodge structure in this context.

### 3. Peterson Inner Product and Hodge Polarization

Let $f$, $g$ be two cusp forms of weight $n$ on $H$ the upper half plane. One proves easily that the measure
\[ (14) \quad \mu(f, g) = f(z) \overline{g(z)} y^{n-2} dx dy, \quad x = \text{Re}(z), \ y = \text{Im}(z) \]

is invariant by \( \Gamma = \text{PSl}_2(\mathbb{Z}) \) and it is a bounded measure on the space \( \Gamma \setminus H \). By putting

\[ (15) \quad \langle f, g \rangle = \int_{\Gamma \setminus H} \mu(f, g) \]

one obtains a hermitian scalar product on the space of modular forms which is positive and non-degenerate. One can check that

\[ (16) \quad \langle T(k)f, g \rangle = \langle f, T(k)g \rangle \]

where \( T(k) \) is the \( k \)-th Hecke operator defined by

\[ (17) \quad T(k).f = k^{n-1} \sum_{a \geq 0, \ 0 \leq b < d} d^{-n} f\left(\frac{az+b}{d}\right) \]

The hermitian metrics \( h_V \) and \( h_G \) define the Hodge star operators on the corresponding space of smooth \( (p, q) \)-forms. The Laplace operator is defined by \( \Delta = \overline{\partial}^* \partial^* \) where \( \overline{\partial}^* := - * \overline{\partial}^* \). It is a self adjoint operator on the space of \( L^2 \)-sections of the corresponding bundles. The identification \( V := \Gamma \setminus (H \times V) \) and \( G := \Gamma \setminus (H \times G) \) identify these Laplace operators with the Laplace operator on \( H \) acting on the space of \( \Gamma \)-automorphic functions.

When \( V \) or \( G \) are equipped with a monodromy representation \( \rho \). Then we are involved with the sections of \( A^{p,q}(H, M_\Gamma) \) and \( A^{p,q}(H, G) \), in which the automorphic forms are defined via

\[ (18) \quad f(\gamma, z) \gamma'(z)^p \gamma'(z)^q = \text{Ad}(\rho) f(z) \]

The Hodge inner product is

\[ (19) \quad \langle f, g \rangle = \int_H f(z) \wedge * g(z) dx dy = \int_{\Gamma \setminus H} \text{tr}(f(z) \wedge g(z)^*) y^{p+q-2} dx dy \]
where \( f^* = \overline{f^t} \). In the case of usual automorphic functions the right hand side is Peterson inner product (14). The vector valued case is similarly treated.

**Proposition 3.1.** The Hodge polarization of the Hodge bundles \( \mathcal{V} \) and \( \mathcal{G} \) is given by the Peterson Inner product. In the latter case the Cartan-Killing form on \( \mathfrak{g} \) is induced from the Peterson-inner product.

The proof goes trivially from the discussion in sections 1 and 2 with the Mixed Hodge Metric Theorem stating the uniqueness of polarization form for mixed Hodge structures.

**Corollary 3.2.** The Peterson inner product induces the Cartan-Killing form \( B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) on the fibers of \( \mathcal{G} \). Moreover there are graded forms \( \langle \cdot, \cdot \rangle_k : \text{PGr}_k^{W(N)} \mathfrak{g} \times \text{PGr}_{-k}^{W(N)} \mathfrak{g} \to \mathbb{Q} \), \( \langle \cdot, \cdot \rangle_k := \langle \mathfrak{l}^k u, v \rangle = \langle v, \mathfrak{l}^k v \rangle \) when the fibration is involved with the monodromy \( T = e^{	ext{ad}(N)} \), where \( \mathfrak{l} \) is induced from \( \text{ad}(N) \). Moreover the action of the Hecke operators induce self adjoint linear transformations on \( \mathfrak{g} \).

**Proof.** The corollary follows from the uniqueness of the polarization form of (mixed) Hodge structures and the explanations of (3), (4) and (5). The last part is a consequence of (16). \( \square \)

The vector space of complex modular (automorphic) forms on a Hodge domain is finite dimensional \( \mathbb{C} \)-vector space. This is general well-known fact in Langlands program. The above discussion and result says this vector space has a mixed Hodge structure which is polarized by the Peterson-inner product.

**Corollary 3.3.** The vector space of complex automorphic forms (representations) on a period domain (Mumford-Tate) \( S \) domain has a mixed Hodge structure which is polarized by the Peterson inner product.

The proof follows from 3.1,3.2 and the discussion in section 2.

**Corollary 3.4.** (Riemann-Hodge bilinear relations for automorphic forms) Let the primitive subspaces of \( S \) be defined by

\[ P_l : \ker(N^{l+1}) : \text{Gr}^{W}_l S \to \text{Gr}^{W}_{-l-2} S \]

Have a pure Hodge structure
of weight $k + l$ (for some $k > 0$) polarized by the bilinear form $\langle \cdot, \cdot \rangle_l : P_l \times P_l \to \mathbb{C}$ induced by Peterson inner product and satisfies

- $\langle \psi, \eta \rangle_l = 0$, $\psi \in S^{p,q}, \eta \in S^{r,s}$ unless $r = p, s = q$.
- $\ast \times \langle \psi, C_l \psi \rangle_l > 0$ for any $\psi \neq 0$ in $S^{p,q}$, where $C$ is the corresponding Weil operator and $\ast$ is a complex constant.

The proof is standard Hodge theory. The Riemann-Hodge bilinear relations give certain identities on automorphic forms. It would be interesting to investigate the arithmetic interpretation of different weights mentioned in the corollary 3.4.

4. Appendix 1: Unitarization of Representation

We generally work with a continuous representation of a locally compact Hausdorff group $G$ into a Banach space $H$, that is a homomorphism

\begin{equation}
\pi : G \rightarrow GL(H)
\end{equation}

Throughout the whole section $K \subset G$ would be a compact subgroup of Haar measure 1. We denote by $H(K)$ the algebraic space of $K$-finite vectors. This means the subspace of $H$ of elements $v$ such that the map $K \ni x \mapsto \pi(x)v$ has finite dimensional image. For $v \in H$ and $\lambda$ a functional on $H$ then

$$x \mapsto \lambda(\pi(x)v) = \pi_{v,\lambda}(x)$$

is called a coefficient (coordinate) function. In case $H$ is a Hilbert space one can of course represent $\lambda$ by an element $w \in H$, so the coefficient function becomes as

$$x \mapsto \langle \pi(x)v, w \rangle$$

Let $K$ be compact subgroup of $G$. On $L^2(K)$ lets consider the representation

\begin{equation}
T(y)f(x) = f(xy)
\end{equation}

With respect to the Haar measure, $T$ is obviously unitary. $T$ is also called the regular representation on $K$. It is well known that the regular representation of $K$
is completely reducible. Every irreducible representation of $K$ occurs in the regular representation on $L^2(K)$, i.e.

$$L^2(K) = \bigoplus m_\pi V_\pi$$

where hat means taking closure. In general such a decomposition is called admissible if the subspaces $V_\pi$ are finite dimensional.

**Example 4.1.** Let $X, Y$ be measured spaces, and $\phi$ a function on the product. Define

$$\Phi(f)(y) := \int_X f(x)\phi(x, y)\mu(x), \quad {}^t\Phi(g)(y) := \int_Y \phi(x, y)g(y)\mu(y)$$

Set $\Phi^* := {}^t\bar{\Phi}$. Then $\Phi^*$ is adjoint of $\Phi$ for the scalar products

$$\langle f_1, f_2 \rangle = \int_X f_1\overline{f_2}\mu(x), \quad \langle g_1, g_2 \rangle = \int_Y g_1\overline{g_2}\mu(y)$$

Then one has

$$\langle \Phi(f), g \rangle = \int_Y \int_X f\Phi(\overline{g}) = \int_X f \int_Y \overline{\phi}g = \langle f, \Phi^*(g) \rangle$$

It follows that $\Phi$ is unitary iff $\Phi^* = \Phi^{-1}$.

We say that an admissible representations $\pi_1$ and $\pi_2$ of a locally compact Hausdorff group $G$ into Banach spaces $H_1$ and $H_2$ are infinitesimally isomorphic if there exists a linear isomorphism $L : H_1(K) \to H_2(K)$ which commutes with the derived representations

$$L \circ d\pi_1 = d\pi_2 \circ L \quad \text{on } H_1(K)$$

where $K$ is a compact closed subgroup of $G$.

Let $E$ is a Banach space. We say that a representation $\pi : G \to Gl(E)$ is unitarizable if $\pi$ is infinitesimally isomorphic to a unitary representation on a Hilbert space $H$. We have the following fact,
Theorem 4.2. [4] Let $\pi$ be an irreducible admissible representation of $G$ in a Banach space $H$. Let $0 \neq \lambda \in H^\vee$ and $f_v(x) := \lambda(\pi(x)v)$ the corresponding coordinate function. Let $G$ operate by right translation on $L^2(G)$. If $f_u$ is in $L^2(G)$ for some $u \neq 0$ in $H(K)$, then $f_v \in L^2(G)$ for all $v \in H(K)$ and the map

$$v \rightarrow f_v$$

is a $(g = \text{Lie}(G))$-embedding of $H(K)$ into $L^2(G)$. In particular $\pi$ is unitarized by this embedding.

The theorem says that a representation on a Banach space is unitarizable if the $K$-coefficient functions are in $L^2(K)$.

Theorem 4.3. [4] Let $\pi$ and $\pi'$ be irreducible admissible representations of $G$ on Banach spaces $H, H'$. Assume there exists a positive integer $m$ such that $\pi$ and $\pi'$ have a lowest weight vector of weight $m$, say $u_m, u'_m$, respectively. If $H, H'$ are both either infinite dimensional or has the same finite dimensions then there exists an infinitesimal isomorphism

$$L : H(K) \rightarrow H'(K), \quad L(u_m) = u'_m$$

Similar theorem holds when the representations have common highest weight vectors of weight $-m$, for $m > 0$. Such infinite dimensional representations are called discrete series representations because they are infinitesimally isomorphic with irreducible subspaces of $L^2(G)$ on which $G$ acts by translations, [4].

5. Appendix 2: Parabolic bundles

Let $X$ be a compact connected Riemann surface of genus $g$ and $\emptyset \neq S \subset X$ a divisor. A parabolic bundle $E$ with parabolic divisor $S$ consists of the data of a filtration (quasi-parabolic condition)

$$E_s = E_{s,1} \supset \ldots \supset E_{s,t_s} \supset 0, \quad \forall s \in S$$

and rational numbers $0 \leq \alpha(s) < \ldots < \alpha_{t_s} < 1$ called parabolic weights. Fix a positive integer $r$. The parabolic degree of $E$ is defined by
\begin{equation}
\text{Par-deg}(E) = \deg(E) + \sum_{s \in S} \sum_{i} \alpha_{i}(s) \dim(E_{s,i}/E_{s,i+1})
\end{equation}

Then every sub-bundle of $E$ is parabolic in a natural way. We call a parabolic bundle semi-stable if the factor (par-deg/rank) is non-increasing when passing to sub-bundles. In case this number is decreasing for every proper sub-bundles we call the bundle stable.

A hermitian structure on a parabolic bundle $E$ is a hermitian structure on $E|_{X\setminus S}$ with the extra condition that around each $s \in S$ for any section $\sigma$ of $E$ defined around $s$, if $\sigma(s)$ is non-zero in $E_{s,i}$, then

\begin{equation}
\|s\| = f(z)|z|^{\alpha_{i}(s)}
\end{equation}

with positive real valued $f$.

**Theorem 5.1.** (Mehta-Seshadri) A parabolic bundle $E$ of rank $k$ and parabolic degree 0 is stable if and only if it is isomorphic to a bundle $E^{\rho}$, where $\rho : \Gamma \to U(k)$ is an irreducible unitary representation of the group $\Gamma$ admissible with respect to the weights and multiplicities of the parabolic structure of $E$. Moreover, parabolic bundles $E^{\rho_{1}}$ and $E^{\rho_{2}}$ are isomorphic if and only if the representations $\rho_{1}$ and $\rho_{2}$ are equivalent.

A unitary representation $\rho : \Gamma \to U(k)$ is called admissible with respect to a given set of weights and multiplicities at $S$ if for each $i$, we have $\rho(S_{i}) = U_{i}D_{i}U_{i}^{-1}$, where $S_{i}$ is a generator for the local monodromy at $P_{i} \in S$, with unitary $U_{i} \in U(k)$ and $D_{i} = \exp(2\pi i \text{diag}[\alpha_{i}^{1}, ..., \alpha_{i}^{l}])$, where each $\alpha_{i}^{l} = \alpha_{l}(P_{i})$ is repeated $k_{i}^{l} = k_{i}(P_{i})$ times.

Admissible matrices are parametrized by the flag varieties $F_{i} = U(k)/U(k_{1}) \times U(k_{2}) \times ... \times U(k_{r_{i}})$. The group $\Gamma$ acts on the trivial bundle $H \times \mathbb{C}^{k}$ by the rule $(z, \gamma) \mapsto \gamma z, \rho(\gamma)v)$. Take the sheaf of its bounded $(\Gamma, \rho)$-sections around the cusps. The direct image of this sheaf under $H \to X$ is a locally free sheaf of rank $k$. The parabolic structure at the image of cusps is defined by the matrices $\rho(S_{i})$. This gives a parabolic vector bundle on the Riemann-Surface $X$ of parabolic degree 0. Loosely speaking this bundle is the extension of the bundle $E^{\rho} = \Gamma \setminus (H \times \mathbb{C}^{k}) \to \Gamma \setminus H$.

The standard hermitian metric in $\mathbb{C}^{k}$ defines a $\Gamma$-invariant metric on the trivial vector bundle $H \times \mathbb{C}^{k} \to H$. It extends as a (pseudo)-metric to the bundle $E = E^{\rho}$.
Explicitly, we choose $\sigma_i \in SL_2(\mathbb{R})$ such that $\sigma_i(\infty) = x_i$ and $\sigma_i^{-1} S_i \sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, and make the change of coordinates $\zeta = e^{2\pi i \sigma_i^{-1} z}$ at $P_i \in \Gamma \setminus H$. Then the metric $h$ is given by a diagonal matrix $(\zeta^{2\alpha_i}, ..., \zeta^{2\alpha_i})$.

6. Appendix 3: Langlands Correspondence

Let $K$ be a number field with Galois group $G = Gal(K/\mathbb{Q})$, $p$ a prime in $\mathbb{Q}$ and $P|p$ a prime of $K$. there exists a unique element namely $Fr_P \in G$ which leaves the prime $P$ stable and induces the frobenius transformation on the residue field $\mathfrak{O}_K/P$. If the prime $p$ is unramified, the $Fr_P \in G$ are all in the same conjugacy class, and we may write $Fr_p$ instead. If $p$ is ramified, there exists a homomorphism

$$D_P = Stab_G(P) \to Gal(k(P)/k(p))$$

with kernel $I_P$, the inertia group at $P$. $D_P$ is the decomposition group of $P$. In this case $Fr_P$ is well-defined element of $D_P/I_P$.

If the extension $K$ is Galois and $\sigma : G \to \mathbb{C}^*$ a character of the Galois group, then there exists an integer $N_\sigma$ called the conductor of $\sigma$ and a primitive Dirichlet character $\chi_\sigma$ of $\mathbb{Z}$ of level $N_\sigma$ such that

$$(31) \quad \sigma(Fr_p) = \chi_\sigma(p), \quad \forall \ p \text{ unramified}$$

In this case one defines an $L$-function associated to the character as

$$(32) \quad L(s, \sigma) = \prod_{p \text{ unram.}} L_p(s, \sigma) = \prod_{p \text{ unram.}} \frac{1}{1 - \sigma(Fr_p)p^{-s}} = \sum_{n \geq 1} \chi_\sigma(n)n^{-s}$$

$L(s, \sigma)$ is entire and if $\chi \neq 1$ satisfies a functional equation.

R. Langlands consider a more general situation that $\sigma$ is a representation of $G$ on $\mathbb{C}^n$. Then the local $L$-factors $L_p(s, \sigma)$ are replaced by

$$(33) \quad L_p(s, \sigma) = \frac{1}{\det(1 - \sigma(Fr_p)p^{-s})}, \quad p \text{ unramified}$$
when \( p \) is ramified, the vector space \( V \) is replaced by \( V^{I_p} \) of invariant elements by the inertia group, and \( Fr_P \) acts on \( V^{I_p} \). The definition of the local \( L \)-factor does not depend on the choice of \( P \mid p \). Langlands formulated the conjecture that \( L(s, \sigma) \) is the \( L \)-function associated to an automorphic representation of \( \text{Gl}(n, \mathbb{A}) \), where \( \mathbb{A} \) is the ring of adeles of \( \mathbb{Q} \).

An automorphic cuspidal representation \( \pi \) of \( \text{Gl}(n, \mathbb{A}) \) is by definition an irreducible subrepresentation of \( \text{L}^2(\text{Gl}(n, \mathbb{A})/\text{Gl}(n, \mathbb{Q})Z_A) \), that satisfy certain cuspidal conditions. \( Z_A \) is the center of \( \text{Gl}(n, \mathbb{A}) \). Such representation is necessarily of the form

\[
(34) \quad \pi = \prod_p \pi_p
\]

when \( p \) is unramified, then by a theorem of Satake an irreducible representation \( \pi_p \) gives rise to a local \( L \)-factor. The Langlands conjecture then can be formulated as follows.

**Langlands Conjecture:** Let \( E \) be a finite extension of \( \mathbb{Q} \) with Galois group \( G \), and \( \sigma \) an irreducible representation of \( G \) in \( \mathbb{C}^n \). Then there exists an automorphic cuspidal representation \( \pi_\sigma \) of \( \text{Gl}(n, \mathbb{A}) \) such that

\[
(35) \quad L(s, \sigma) = L(s, \pi_\sigma)
\]

In fact the right group is the Weil group to make the Langlands correspondence,

\[
(36) \quad L : \text{Rep}_n(W_p) \cong \text{Irr}(\text{Gl}(n, \mathbb{Q}_p))
\]

such that the \( L \)-factors and the \( \epsilon \)-factors are equal. One may ask how much this correspondence is functorial with respect to the natural maps in the both sides. The case of finding the corresponding lift for \( G(\mathbb{Q}/E) \to G(\mathbb{Q}/\mathbb{Q}) \) on the automorphic side is called the base change. On the other hand the case of finding the the lift associated to a homomorphism \( \text{Gl}(n, \mathbb{C}) \to \text{Gl}(m, \mathbb{C}) \) is called functoriality, [7].

**Example 6.1.** Let \( E \) be a non-singular Elliptic curve. Its first \( l \)-adic cohomology group has dimension 2 over \( \overline{\mathbb{Q}}_l \) and the representation of rank 2 of \( G = G(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \) gives rise to the \( L \)-function attached by Hasse-Weil to \( E \). The \( L \)-factor is given by

\[
\frac{1}{\prod_{p \mid \text{ord}} \big(1 - a_p t_p \big)^{\frac{1}{2}}}
\]

where \( a_p = 1 - N(p) + p \) and \( N(p) = \#E(\mathbb{F}_p) \). The Shimura-Tanyama-Weil conjecture asserts that these \( L \)-functions are modular.
Appendix 4: Mumford-Tate groups

An abelian variety $\mathbb{C}$ is a quotient of $\mathbb{C}^n$ by a lattice such that can be embedded in a projective space. This means that it can be given by a homogeneous polynomial. One may be interested in a situation that the lattice degenerates in a family of these varieties. For instance a family of elliptic curves $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ parametrized by $\tau \in H$ the upper half plane is given by a homogeneous equation. Letting $\tau \to 0$ gives an example of the above degeneration. The degeneration may happen when considering an arithmetic deformation by replacing the parameter space $H$ by $\text{Spec}(\mathbb{Z})$ and study the reduction of the homogeneous equation modulo primes in $\text{Spec}(\mathbb{Z})$.

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. We denote by $k_p$ the residue field at $p$. Fix an embedding $i : K \hookrightarrow \mathbb{C}$, and let $A$ be an abelian variety over $K$. We say that $A$ has good reduction at $p$ if there exists a cartesian diagram as

\begin{equation}
A \longrightarrow A \\
\downarrow \quad \downarrow \\
\text{Spec}(K) \longrightarrow \text{Spec}(\mathcal{O}_{K,p})
\end{equation}

with $A$ an abelian group scheme. We say it has potentially good reduction at $p$ if such a situation happens for $p' | p$ after a finite extension.

Let $V = H_1(A_{\mathbb{C}}, \mathbb{Q})$. There exists an equivalence between the set of complex structures on $V$ and Hodge structures of weight 1, i.e. homomorphisms

\begin{equation}
h : \mathbb{C}^* \to \text{Gl}(V_{\mathbb{R}})
\end{equation}

such that $h(z)$ has eigenvalues $z, \bar{z}$ on $V_{\mathbb{C}}$. The Mumford-Tate group of $A$ denoted $MT(A)$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\text{Gl}(V)$ such that $h(\mathbb{C}^*) \subset H(\mathbb{R})$ In case of a generic elliptic curve $E$, one simply has $MT(E) = \text{Gl}_{2,\mathbb{Q}}$.

The assignment $A \mapsto MT(A)$, associates a Shimura variety $\Gamma \backslash MT(A)(\mathbb{R})/K_{\infty}$ to any abelian variety, where $\Gamma$ is congruence subgroup of $MT(\mathbb{Z})$ and $K_{\infty}$ is torus. It is a type of moduli for abelian varieties, with level structure of that of $\Gamma$. For elliptic curves this gives the Shimura variety $Sl_2(\mathbb{Z}) \backslash H$ and $K_{\infty} = \mathbb{C}^*$.

The representation
(39) \[ \rho : MT(A) \to Gl(H_1(A, \mathbb{Q})) \]

characterizes if \( A \) may have potentially good reductions at every place of \( K \). More specifically, \( A \) has potentially good reduction at every place if there exists a prime \( l \) of \( \mathbb{Q} \) such that

(40) \[ \rho_l : MT(A)(\mathbb{Q}_l) \to Gl(H_1(A, \mathbb{Q}))(\mathbb{Q}_l) \]

does not contain unipotents. By a theorem of Borel-Harisch Chandra this is equivalent to the fact that the associated Shimura variety of \( A \) is compact. In this case there exists a finite extension \( K'/K \) such that the action of \( Gal(\bar{K}/K') \) factors as

(41) \[ Gal(\bar{K}/K) \to MT(A)(\mathbb{Q}_l) \to Gl(H_1(A, \mathbb{Q}))(\mathbb{Q}_l) \]

\[ \blacksquare. \]

8. Hodge theory and Representation theory

Let \( g_\mathbb{C} = g_\mathbb{R} \otimes \mathbb{R} = \text{Lie}(G_\mathbb{R} \otimes \mathbb{C}) \) be a complex semi-simple Lie algebra, \( \mathfrak{h} = \mathfrak{t} \otimes \mathbb{C} \) a Cartan subalgebra, and \( K_\mathbb{C} \) a complex Lie group corresponding to the unique maximal compact subgroup \( K \subset G_\mathbb{R} \). We denote \( U(g_\mathbb{C}) \) to be the universal enveloping algebra of \( g_\mathbb{C} \). We assume the action of \( K_\mathbb{C} \) will be locally finite, and its differential agrees with the corresponding subspace of \( U(g_\mathbb{C}) \).

One may match these data with the case \( D = G_\mathbb{R}/H \) is a general Mumford-Tate domain siting in the diagram

(42) \[
\begin{array}{c}
G_\mathbb{R}/T \longrightarrow G_\mathbb{C}/B \\
\downarrow \quad \quad \downarrow \\
D = G_\mathbb{R}/H \longrightarrow G_\mathbb{C}/P = \check{D}
\end{array}
\]

with \( T \) a maximal torus, \( B \) a Borel subgroup, and horizontal arrows to be inclusions.

**Definition 8.1.** A Harish-Chandra (HC)-module \( M \) is a \( (g_\mathbb{C}, K_\mathbb{C}) \)-module that is finite as \( U(g_\mathbb{C}) \)-module with an admissible \( K_\mathbb{C} \)-action.
An example is given by $K$-finite vectors in a unitary representation or a discrete series. We propose to show when the automorphic cohomology $H^q(D, L_\mu)$ of a period domain gives a HC-module. Let $\mathcal{H} = \mathcal{U}(\mathfrak{h}_C)$. The Weyl group $W$ of $(g_C, h_C)$ acts on $\mathcal{H}$ and gives an isomorphism $Z(g_C) \cong \mathcal{H}^W$. For each $\zeta \in \mathfrak{h}_C^*$ the homomorphism

\[(43) \quad \chi_\zeta : Z(g_C) \to \mathbb{C}, \quad z \mapsto z(\zeta)\]

is called the infinitesimal character. A result of Harish-Chandra says that any character of $Z(g_C)$ is an infinitesimal character, and $\chi_\zeta = \chi_{\zeta'}$ iff $\zeta = w(\zeta')$ for some $w \in W$. Let's fix a set of positive roots $\Phi^+$ of $(g_C, h_C)$. Note that we have

\[(44) \quad g_C = t_C \oplus n^+ \oplus n^-, \quad n^+ = \oplus_{\alpha \in \Phi^+} g^\alpha\]

Any root $\mu \in \Phi^+$ defines an integrable almost complex structure on $G_\mathbb{R}/T$ as well as a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus n^-$ and a line bundle $L_\mu = G_\mathbb{C} \times_B \mathbb{C}$. The homogeneous vector bundle $H = G_\mathbb{C} \times_B \mathfrak{b}_C$ gives a $G_\mathbb{C}$-invariant connection on the principal bundle $G_\mathbb{C}/B$.

Let $E = G_\mathbb{R} \times_T E \to D$. The sections of $E$ are $E$-valued functions $f : G_\mathbb{R} \to E$ such that the right action of $T$ is given by $f(g.t) = r(t^{-1})f(g)$ where $r : T \to Aut(E)$ is the given representation of $T$. Now take

\[(45) \quad E = \Lambda^q T^{0,1} D \otimes L_\mu\]

Identifying $T^{0,1} = n$ obtain $A^{0,q}(D, L_\mu) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q n^* \otimes L_\mu)^T$. We abbreviate this by $(C^\infty(G_\mathbb{R}) \otimes \Lambda^q n^*)_{-\mu}$. Summarizing the identifications of the complexes of $G_\mathbb{R}$-modules we get

\[(46) \quad A^{0,q}((D, L_\mu), \bar{\partial}) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q n^*)_\mu, \delta), \quad H^q(D, L_\mu) = H^q(n, C^\infty(G_\mathbb{R}))_{-\mu}\]

These identity can be generalised as

\[H^q(D, E(\mu)) = \oplus_{\lambda \in \hat{K}} W^\lambda \otimes H^q(n_K, E \otimes W^{\lambda^*})_{-\mu}\]

for $V = \oplus_{\lambda \in \hat{K}} m_\lambda W^{\lambda^*}$ a Harish-Chandra module, [2].
References

[1] [P] F. Paugam, Galoisdarstellungen, Mumford-tate grouppe, und shöne Ermessigung von Abelschevarietäten, Lecture notes, Rensburg 2003

[2] [PG] P. Griffiths, Hodge theory and representation theory, Ten lectures given at TCU, June 18-22, 2012

[3] [KP] M. Kerr, G. Pearlstein, Boundary components of Mumford-Tate domains, arxiv preprint

[4] [L] S. Lang, $SL_2(\mathbb{R})$, Addison-Wesley Pub., 1975

[5] [TZ] L. A. Takhtajan, P. G. Zograf, The first chern form on moduli of parabolic bundles, arxiv preprint

[6] [S] J. P. Serre, A course in arithmetic, Springer-Verlag 7, 1973

[7] [V] M. Vergne, All what I wanted to know about Langlands program and was afraid to ask, preprint

E-mail address: mrahmati@cimat.mx