On a stack of surfaces obtained from the $\mathbb{C}P^{N-1}$ sigma models

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Abstract

Under the assumption that the $\mathbb{C}P^{N-1}$ sigma model is defined on the Riemann sphere and its action functional is finite, we derive surfaces induced by surfaces and we prove that the stacked surfaces coincide with each other, which demonstrates the idempotency of the recurrent procedure. In the process of finding the solutions of the $\mathbb{C}P^{N-1}$ model equations we prove that the Euler–Lagrange equations for the projectors admit larger classes of solutions than the ones corresponding to rank-1 projectors.

Keywords: sigma models, projector analysis, soliton surfaces, integrable systems

1. Introduction

Soliton surfaces associated with integrable models and with the $\mathbb{C}P^{N-1}$ sigma model in particular have been shown to play an essential role in many problems with physical applications (see e.g. [4, 13, 21, 23, 24, 26, 27]). The possibility of using a linear spectral problem (LSP) to describe a moving frame (whose equations are the Gauss–Weingarten equations) on a surface has yielded many new results concerning the intrinsic geometry of such surfaces [1, 19, 20]. It has recently proved fruitful to extend this characterization of soliton surfaces through the functions immersing them in Lie algebras. Representing integrable equations as conditions for the immersion of soliton surfaces in the appropriate spaces has for a long time been a useful tool for the visualisation of their integration. This visualisation often reveals new interesting properties of the equations and their solutions [2, 3, 28].

The construction of soliton surfaces related to the $\mathbb{C}P^{N-1}$ sigma model has been accomplished by representing the Euler–Lagrange (E–L) equations as conservation laws and
expressing them in terms of the rank-1 projector formalism. This allows us to define closed differential 1-forms for surfaces which can be explicitly integrated [18, 22]. This determination has led to a new way of constructing and investigating 2D surfaces in Lie algebras, Lie groups and homogeneous spaces [7, 8, 11, 15]. The algebraic-geometric approach based on this formalism applied to the $\mathbb{C}P^{N-1}$ sigma model equations and associated surfaces has proved to be a suitable tool for investigating the links between successive projection operators, wavefunctions of the LSP and immersion functions of surfaces in the $\mathfrak{su}(N)$ algebra [9]. The main advantages of this approach are that this formulation preserves the conformal and scaling invariance of these quantities. It allows us to construct a regular algorithm for finding certain classes of solutions having a finite action functional [11, 14]. A broad review of recent developments in this subject can be found in, e.g. [20, 29, 30].

In this paper we explore the fact that the soliton surfaces corresponding to the $\mathbb{C}P^{N-1}$ models are interesting objects in themselves. Namely, we make use of the fact that the immersion functions for the surfaces satisfy the same E–L equations as the projectors which were used to generate the surfaces, although with different constraints [12]. The possibility of further generation: surfaces over surfaces and so on, to the whole stack of surfaces, is analysed in the hope that it may give insight into the properties of surface generation. The result is astonishing: the procedure proves to already be idempotent in its second step. This way, the E–L equations (12) and (13) with the constraints satisfied by the first-step surfaces (14) and (15) make a unique example of a nonlinear model whose soliton surfaces are identical with the solutions of the equations.

We next investigate the E–L equations in more detail. It has been implicitly assumed in the literature that the constraint of the projective property imposed on their solutions infers that the solutions are rank-1 projectors of the $\mathbb{C}P^{N-1}$ model. We prove that the class of solutions which these equations admit is larger, even within the class of projectors.

In the next section we briefly recall the basic properties of $\mathbb{C}P^{N-1}$ models. Section 3 contains the main results on the construction of surfaces over surfaces. The theorem on the classes of projector solutions (not necessarily of rank 1) of the E–L equations is formulated and proven in section 4.

2. Preliminaries on the $\mathbb{C}P^{N-1}$ models

This section reviews the main notions to be used hereafter, certain theorems dealing with rank-1 Hermitian projector analysis and some techniques for obtaining soliton surfaces via the $\mathbb{C}P^{N-1}$ sigma models.

In our previous work [9, 11, 12] we discussed in detail the algebraic and analytic properties of 2D-soliton surfaces with the immersion functions $X_k, k = 0, 1, ..., N - 1$, which take values in the $\mathfrak{su}(N)$ algebra and are induced by rank-1 Hermitian projectors $P_k$ of $\mathbb{C}P^{N-1}$ sigma models. The dynamics of the $\mathbb{C}P^{N-1}$ sigma model defined on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ are determined by stationary points of the action functional [6–9]

$$A(P_k) = \int_{S^2} \text{tr}(\partial P_k \bar{\partial} P_k) d\xi d\bar{\xi}, \quad 0 \leq k \leq N - 1,$$

where $\partial$ and $\bar{\partial}$ denote the derivatives with respect to $\xi$ and $\bar{\xi}$, respectively

$$\partial = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} - i \frac{\partial}{\partial \xi^2} \right), \quad \bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial \bar{\xi}^1} + i \frac{\partial}{\partial \bar{\xi}^2} \right), \quad \xi = \xi^1 + i\xi^2.$$

The E–L equations within the constraint $P_k = P_k$ have the form

$$\dot{\xi} = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} - i \frac{\partial}{\partial \xi^2} \right), \quad \dot{\bar{\xi}} = \frac{1}{2} \left( \frac{\partial}{\partial \bar{\xi}^1} + i \frac{\partial}{\partial \bar{\xi}^2} \right).$$
\[ \left[ \partial \bar{\partial} P_k, P_k \right] = 0. \]  

(3)

The target space of a projector \( P_k \) is determined by a complex line in \( \mathbb{C}^N \), i.e. by a one-dimensional vector

\[ f_k(\xi, \bar{\xi}) = (f_k^0(\xi, \bar{\xi}), \ldots, f_k^{N-1}(\xi, \bar{\xi})) \in \mathbb{C}^N \setminus \{0\} \]  

related to \( P_k \) by

\[ P_k = \frac{f_k \otimes f_k^\dagger}{f_k^0 f_k}, \]  

(5)

where \( \otimes \) denotes the tensor product. In terms of the vector functions \( f_k \), the E–L equations corresponding to the action functional (1) take the form \[30\]

\[ \left( I - \frac{f_k \otimes f_k^\dagger}{f_k^0 f_k} \right) \left[ \partial \bar{\partial} f_k - \frac{1}{f_k^0 f_k} \left( (f_k^0 \bar{\partial} f_k) \partial f_k + (f_k^0 \partial f_k) \bar{\partial} f_k \right) \right] = 0. \]  

(6)

where \( I \) is the \( N \times N \) identity matrix. Equation (5) provides an isomorphism between the equivalence classes of the \( \mathbb{CP}^{N-1} \) model and the set of rank-1 Hermitian projectors \( P_k \). An entire class of solutions of (6) is obtained by acting on a holomorphic solution \( f_0 \) (or antiholomorphic solution \( f_{N-1} \)) with raising and lowering operators \[5, 30\].

The raising and lowering operators (7) have their counterparts in the corresponding operators acting on the projectors \( P_k \), namely \[9\]

\[ P_{k+1} = \Pi_+(P_k) = \frac{\partial P_k \partial P_k}{\text{tr}(\partial P_k \partial P_k)}, \quad P_{k-1} = \Pi_-(P_k) = \frac{\bar{\partial} P_k \bar{\partial} P_k}{\text{tr}(\bar{\partial} P_k \bar{\partial} P_k)}. \]  

(8)

We have proven \[10\] that if \( P_k \) satisfies the E–L equation (3) then \( P_{k+1} \) and \( P_{k-1} \) are also solutions of those equations and the projective property \( P_k^2 = P_k \) is preserved by the operations (8).

The elements of the set of \( N \) rank-1 projectors \( \{P_0, P_1, \ldots, P_{N-1}\} \) satisfy the orthogonality and completeness relations

\[ P_i P_j = \delta_{ij} P_k \]  

(no summations), \[ \sum_{j=0}^{N-1} P_j = I \]  

(9)

In the papers \[9, 11, 12, 17, 25\] it was shown that with each of these solutions we can associate a conformally parametrized surface in the Lie algebra \( \mathfrak{su}(N) \simeq \mathbb{R}^{N^2-1} \). That is, the E–L equation (3) can be written as the conservation law

\[ \partial [\partial P_k, P_k] + \bar{\partial} [\bar{\partial} P_k, P_k] = 0. \]  

(10)

This implies that there exist \( \mathfrak{su}(N) \) matrix-valued differential 1-forms

\[ dX_k = i \left( -[\partial P_k, P_k] d\xi + [\bar{\partial} P_k, P_k] d\bar{\xi} \right), \]  

(11)

which are closed. For the surfaces corresponding to the rank-1 projectors \( P_k \), the integration of (11) is performed explicitly \[18\]
\[ X_k = i \int_{\gamma_k} \left[ \partial P_k, P_k \right] d\xi + \left[ \bar{\partial} P_k, P_k \right] d\bar{\xi} \]

\[ = -i \left( P_k + 2 \sum_{j=0}^{k-1} P_j \right) + ic_k \mathbb{I} \in \mathfrak{su}(N), \quad c_k = \frac{1 + 2k}{N}, \]

where \( \gamma_k \) is a curve which is locally independent of the trajectory in \( \mathbb{C} \). The \( \mathfrak{su}(N) \) immersion functions \( X_k \) satisfy the cubic matrix equations (the minimal polynomial identity) \[ (X_k - ic_k \mathbb{I})(X_k - i(c_k - 1)\mathbb{I})(X_k - i(c_k - 2)\mathbb{I}) = 0, \quad 0 < k < N - 1 \]

for any mixed solution of the E–L equation (3). For holomorphic \((k = 0)\) or antiholomorphic \((k = N - 1)\) solutions of the E–L equation (3), the minimal polynomial for the immersion functions \( X_0 \) and \( X_{N-1} \) is quadratic

\[ (X_0 - ic_0 \mathbb{I})(X_0 - i(c_0 - 1)\mathbb{I}) = 0, \quad (X_{N-1} + ic_0 \mathbb{I})(X_{N-1} + i(c_0 - 1)\mathbb{I}) = 0, \]

(14)

where \( c_0 + c_{N-1} = 2 \) and the immersion functions \( X_k \) are linearly dependent

\[ \sum_{k=0}^{N-1} (-1)^k X_k = 0. \]

(16)

One of the results obtained in [12] was the derivation of the E–L equations satisfied by these surfaces, which were identical to the equations satisfied by the original projectors \( P_k \), namely

\[ [\partial \bar{\partial} X_k, X_k] = 0, \quad X_k^* = -X_k \in \mathfrak{su}(N), \]

(17)

subject to the constraint (14), which encompasses the constraints in (15). Since the E–L equation (3) for the projectors \( P_k \) constitute a basis for the construction of the set of surfaces \( X_k \) satisfying the same E–L equations, a natural question arises as to whether this technique can be further exploited to construct surfaces over surfaces and possibly a whole stack of surfaces in a similar way as the surfaces \( X_k \) were built from the projectors \( P_k \). The main goal of this paper is to analyse whether such a construction is possible. An unexpected result of our analysis is the fact that these surfaces over surfaces are identical to the original surfaces up to a multiplicative constant.

3. Stack of conformally parametrised surfaces

In this section we prove that the immersion functions of 2D-surfaces \( Y_k \) over the surfaces \( X_k \), defined analogously to \( X_k \) over the projectors \( P_k \), i.e.

\[ Y_k(\zeta, \bar{\zeta}) = i \int_{\gamma_k} -[\partial X_k, \bar{X}_k] d\zeta + [\bar{\partial} X_k, X_k] d\bar{\zeta} \in \mathfrak{su}(N) \]

(18)

are identical to the surfaces \( X_k \) up to a multiplication factor of \((-1)\) if we require that the \( Y_k \)’s be elements of the \( \mathfrak{su}(N) \) algebra. The E–L equation (3) written in terms of the \( \mathfrak{su}(N) \)-valued immersion functions \( X_k \) can be written equivalently as the conservation laws (CLs)

\[ \partial[\partial X_k, X_k] + \bar{\partial}[\bar{\partial} X_k, X_k] = 0. \]

(19)

Let us define the \( N \times N \) matrix functions
\[ M_k := [\bar{\partial}X_k, X_k], \quad \text{tr} M_k = 0. \]  

Then, we can write the CLs (19) as
\[ \partial M_k - \bar{\partial} M_k = 0. \]  

If the CLs (21) hold, then there exist matrix-valued differential 1-forms
\[ dY_k := i \left( M_k^\dagger d\xi + M_k d\bar{\xi} \right), \]  

which are closed $d(dY_k) = 0$ (the factor $i$ was introduced in order to make the $Y$’s belong to the Lie algebra $\mathfrak{su}(N)$). From the closure of the 1-forms (22) it follows that the integrated forms of the 2D surfaces (18) locally depend only on the end points of the curves $\gamma_k$ (i.e. they are locally independent of the trajectory in the complex plane $\mathbb{C}$). The integrals define mappings $Y_k : \Omega \ni (\xi, \bar{\xi}) \mapsto Y_k(\xi, \bar{\xi}) \in \mathfrak{su}(N) \simeq \mathbb{R}^N$. From the closure of the 1-forms (22) it follows that the integrated forms of the 2D surfaces (18) locally depend only on the end points of the curves $\gamma_k$ (i.e. they are locally independent of the trajectory in the complex plane $\mathbb{C}$). The integrals define mappings $Y_k : \Omega \ni (\xi, \bar{\xi}) \mapsto Y_k(\xi, \bar{\xi}) \in \mathfrak{su}(N) \simeq \mathbb{R}^N$. From the closure of the 1-forms (22) it follows that the integrated forms of the 2D surfaces (18) locally depend only on the end points of the curves $\gamma_k$ (i.e. they are locally independent of the trajectory in the complex plane $\mathbb{C}$). The integrals define mappings $Y_k : \Omega \ni (\xi, \bar{\xi}) \mapsto Y_k(\xi, \bar{\xi}) \in \mathfrak{su}(N) \simeq \mathbb{R}^N$.

**Lemma 1.** Let $P_k = P_k(\xi, \bar{\xi}) : \mathbb{C} \to GL(N, \mathbb{C})$ be a rank-1 Hermitian projector determined by a complex line in $\mathbb{C}^N$
\[ P_k = \frac{f_k \otimes f_k^\dagger}{f_k^\dagger f_k}, \quad k \in \{0, 1, \ldots, N-1\} \]  

where $f$ is a mapping $\mathbb{C} \supset \Omega \ni \xi \mapsto f = (f_0, f_1, \ldots, f_{N-1}) \in \mathbb{C}^N \setminus \{0\}$. Then the following relations hold for $k \leq N - 2$
\[ P_{k+1} \partial P_{k+1} = -\partial P_k P_k, \quad \bar{\partial} P_{k+1} P_{k+1} = -P_k \bar{\partial} P_k. \]  

For the proof see [11] (equation (25)).

We complete the lemma with two outermost cases, $k = -1$ and $k = N - 1$, namely
\[ P_0 \partial P_0 = 0, \quad \text{and} \quad P_{N-1} \bar{\partial} P_{N-1} = 0. \]  

Proof for the outermost cases:
For any projector $P_k$, the projective property $P_k^2 = P_k$ implies
\[ P_k \partial P_k = \partial P_k (\mathbb{I} - P_k), \quad \bar{\partial} P_k P_k = (\mathbb{I} - P_k) \bar{\partial} P_k. \]  

The same holds for the $\bar{\partial}$ derivative. We have
\[ P_0 \partial P_0 = \partial P_0 (\mathbb{I} - P_0) = \partial \left( \frac{f_0 \otimes f_0^\dagger}{f_0^\dagger f_0} \right) (\mathbb{I} - P_0) \]
\[ = \partial \left( \frac{f_0}{f_0^\dagger f_0} \otimes f_0^\dagger (\mathbb{I} - P_0) + \frac{f_0^\dagger}{f_0^\dagger f_0} \otimes (\bar{\partial} f_0^\dagger) (\mathbb{I} - P_0). \]  

For a holomorphic function $f_0$, its Hermitian conjugate $f_0^\dagger$ is antiholomorphic, which means that $\partial f_0^\dagger = 0$. On this basis the second term in (28) vanishes, while the first term vanishes due to the orthogonality of $(\mathbb{I} - P_0)$ to $f_0^\dagger$.
The proof of the second part of (26) is analogous (where holomorphic and antiholomorphic are interchanged).

Taking the Hermitian conjugates of (26), we get also
\[ \overline{\partial P_0 P_0} = 0, \quad \overline{\partial P_{N-1} P_{N-1}} = 0. \] (29)

We first demonstrate the usefulness of lemma 1 by proving equation (13) through straightforward induction (previously proven in [18] by a different method).

**Proposition 1.** Let the functions \( X_k \) be \( \mathfrak{su}(N) \)-valued immersion functions defined by the differential 1-form (11) or equivalently the integral (12). Then the immersion functions \( X_k \) can be explicitly written as (13).

**Proof.** For \( k = 0 \) and from equation (11) we have
\[ \partial X_0 = -i[\partial P_0, P_0] = -i(\partial P_0 P_0 - P_0 \partial P_0) = -i(I - 2P_0)\partial P_0 = -i\partial P_0, \] (30)
as \( P_0 \partial P_0 = 0 \), whence by Hermitian conjugate (which changes the sign of \( X_k \)) we have
\[ \overline{\partial X_0} = -i\overline{\partial P_0}. \]

Hence, the integrated form of the surface is \( X_0 = -iP_0 + \frac{i}{N} \mathbb{I} \) since the integration constant \( \frac{i}{N} \mathbb{I} \) is unique if we require that the matrix \( X_0 \) be traceless. Assume now that
\[ \partial X_k = -i[\partial P_k, P_k] = -i\partial P_k - 2i \sum_{j=0}^{k-1} \partial P_j, \] (31)
\[ \overline{\partial X_k} = i[\overline{\partial P_k}, P_k] = -i\overline{\partial P_k} - 2i \sum_{j=0}^{k-1} \overline{\partial P_j}, \] (32)
are satisfied for \( k = m, \quad m \in \{0, ..., N-2\} \) (the induction hypothesis). We will prove that these equation (31) hold for \( k = m + 1 \). We have
\[
[\partial P_{m+1}, P_{m+1}] = -P_{m+1} \partial P_{m+1} + \partial P_{m+1} P_{m+1} \\
= -P_{m+1} \partial P_{m+1} + (I - P_{m+1}) \partial P_{m+1} \\
= -2P_{m+1} \partial P_{m+1} + \partial P_{m+1} \\
= 2\partial P_{m+1} P_m + \partial P_{m+1},
\] (33)
where we have used lemma 1 in the form (25) to get the last line. Furthermore, from (25), expression (33) is equal
\[ \partial P_{m+1} + \partial P_m P_m + (I - P_m) \partial P_m = \partial P_{m+1} + [\partial P_m, P_m] + \partial P_m. \] (34)

Now we replace \([\partial P_m, P_m]\) using the induction hypothesis ((31) for \( k = m \)). Hence we obtain
\[
[\partial P_{m+1}, P_{m+1}] = \partial P_{m+1} + \partial P_m + \partial P_m + 2 \sum_{j=0}^{m-1} \partial P_j \\
= \partial P_{m+1} + 2 \sum_{j=0}^{m} \partial P_j,
\] (35)
Equation (35) constitutes the induction conclusion. The induction implies that
\[
[\partial P_k, P_k] = \partial P_k + 2 \sum_{j=0}^{k-1} \partial P_j \quad \text{for all} \quad 0 \leq k \leq N - 1. \tag{36}
\]

By Hermitian conjugation (which reverses the order of multiplication and thus changes the sign of the commutator) we obtain
\[
[\bar{\partial} P_k, P_k] = -\bar{\partial} P_k - 2 \sum_{j=0}^{k-1} \bar{\partial} P_j. \tag{37}
\]

Integration of (36) and (37) over the path \(\gamma_k\) yields (13) if we bear in mind that the constant of integration is unique to ensure tracelessness.

**Corollary 1.** For each of the complex tangent vectors \(\partial X_k\) and \(\bar{\partial} X_k\), we equate the two expressions given in (31) and obtain
\[
\partial P_k + 2 \sum_{j=0}^{k-1} \partial P_j = \partial P_k P_k = \partial P_k - 2 \partial P_k, \tag{38}
\]
where we have used \(\partial P_k P_k = P_k (1 - P_k)\) from (27). Thus, we get
\[
\sum_{j=0}^{k-1} \partial P_j = -P_k \partial P_k, \tag{39}
\]
and its respective Hermitian conjugate
\[
\sum_{j=0}^{k-1} \bar{\partial} P_j = -\bar{\partial} P_k P_k. \tag{40}
\]

**Corollary 2.** From the orthogonality property of the projectors (9), we get
\[
\partial P_k \sum_{j=0}^{k-1} P_j = -P_k \sum_{j=0}^{k-1} \partial P_j = -P_k (-P_k \partial P_k) = P_k \partial P_k, \tag{41}
\]
and
\[
\left( \sum_{j=0}^{k-1} P_j \right) \partial P_k = - \left( \sum_{j=0}^{k-1} \partial P_j \right) P_k = P_k \partial P_k P_k = 0, \tag{42}
\]
together with their respective Hermitian conjugate equations
\[
\left( \sum_{j=0}^{k-1} P_j \right) \bar{\partial} P_k = \bar{\partial} P_k P_k, \quad \bar{\partial} P_k \sum_{j=0}^{k-1} P_j = 0. \tag{43}
\]

Under these circumstances we have the following result.
Proposition 2. Let the $\mathbb{CP}^{N-1}$ model be defined on the Riemann sphere and have a finite action functional. Then the surfaces over surfaces defined by (18) are identical to the initial surfaces (13) from which they were derived, up to a factor of $(-1)$.

Proof. The proof is obtained by direct calculation from (18)

$$\partial Y_k = -i[\partial X_k, X_k] = (-i)^3 \left[ [\partial P_k, P_k], P_k + 2 \sum_{j=0}^{k-1} P_j \right]$$

$$= i \left( \partial P_k P_k P_k + 2 \partial P_k \sum_{j=0}^{k-1} P_j - P_k \partial P_k P_k - 2 P_k \partial P_k \sum_{j=0}^{k-1} P_j \right)$$

$$= -P_k \partial P_k P_k - 2 \sum_{j=0}^{k-1} P_j \partial P_k P_k + P_k \partial P_k + 2 \sum_{j=0}^{k-1} P_j P_k \partial P_k$$

$$= i (\partial P_k P_k + 0 - 0 - 2 P_k \partial P_k + 0 + 0 + P_k \partial P_k + 0)$$

$$= i (\partial P_k P_k - P_k \partial P_k) = i [\partial P_k, P_k] = -\partial X_k, \quad (44)$$

where we have used equations (41) and (42) from corollary 2. The Hermitian conjugate equation is

$$-\bar{\partial} Y_k = -(-\partial X_k) = \partial X_k. \quad (45)$$

Next, integrating (44) and (45) we obtain the vanishing of the expressions

$$X_k + Y_k = 0, \quad k = 0, 1, ..., N - 1, \quad (46)$$

where the constant of integration is chosen to be $-c_k$ in order to ensure the tracelessness of the immersion functions $Y_k$, which completes the proof.

To summarise, we have provided an explicit expression for 2D conformally parametrised surfaces induced by surfaces and demonstrated that these surfaces coincide with the original surfaces for any recurrence index $k$. This proof demonstrates the uniqueness of soliton surfaces obtained from the $\mathbb{CP}^{N-1}$ sigma models. In this way, our attempt to build the stack in which each next step is a surface over the previous step, becomes idempotent. This somewhat unexpected result provides important information on the structure of soliton surface constructions in the $\mathfrak{su}(N)$ algebra.

4. Higher-rank projectors as solutions of the Euler–Lagrange equations

It is interesting that the E–L equation (3) with the projective property $P^2 = P$ admit a larger class of solutions than the rank-1 Hermitian projectors $P_k$.

Proposition 3. Let $P$ be a linear combination of rank-1 orthogonal projectors which have been obtained from the projector $P_0$ (5) by the raising operators (8) and thus satisfy the E–L equation (3)

$$P = \sum_{i=0}^{N-1} \lambda_i P_i, \quad \lambda_i \in \mathbb{C}, \quad [\partial \bar{\partial} P_i, P_i] = 0, \quad (47)$$
where not all $\lambda_i$ are zero. Then $P$ also satisfies the E–L equation (3). If, in addition, for all $i \in \{0, \ldots, N - 1\}$ we have $\lambda_i = 0$ or $\lambda_i = 1$, then $P$ satisfies $P^2 = P$.

**Proof.** We first show that if $P_k$ satisfies the E–L equation (3) then its second mixed derivative can be represented as a combination of at most three rank-1 neighbouring projectors, namely

$$\partial \partial P_k = \alpha_k P_{k-1} - (\alpha_k + \bar{\alpha}_k)P_k + \bar{\alpha}_k P_{k+1},$$  \hspace{1cm} (48)

where

$$\alpha_k = \text{tr}(P_k \partial P_k \partial P_k), \quad \bar{\alpha}_k = \text{tr}(P_k \partial P_k \partial P_k),$$  \hspace{1cm} (49)

where $\alpha_k$ has been defined separately as it is a symbol complementary to $\bar{\alpha}_k$ rather than its complex conjugate (both $\alpha_k$ and $\bar{\alpha}_k$ are real).

For $k = 0$ the first component of (48) vanishes; for $k = N - 1$ the last one does.

We have

$$(\partial \partial P_k) = \partial(P_k \partial P_k) + \partial[(\mathbb{I} - P_k) \partial P_k] = \partial P_k \partial P_k + P_k \partial \partial P_k + \partial(\partial P_k P_k)$$

$$= \partial P_k \partial P_k + \bar{\partial} P_k \partial P_k + 2 \text{tr}(\partial P_k \partial P_k) P_k$$  \hspace{1cm} (50)

where we have used the E–L equation (3), property (27) and the obvious fact that for any rank-1 projector $P$ and any square matrix $A$ of the same dimension, we have $PAP = \text{tr}(PA)P$ see [10].

Bearing in mind that $\text{tr}(P_k \partial P_k) = 0$, we may write the last component of (50) in terms of $\alpha$ and $\bar{\alpha}$

$$2 \text{tr}(P_k \partial \partial P_k) P_k = 0 - 2 \text{tr}(\partial P_k \partial P_k) P_k$$

$$= 2[\text{tr}(P_k \partial P_k \partial P_k) - \text{tr}(\mathbb{I} - P_k) \partial P_k \partial P_k] P_k = -2(\alpha_k + \bar{\alpha}_k)P_k,$$  \hspace{1cm} (51)

as the projector $(\mathbb{I} - P_k)$ turns into $P_k$ when it passes $\partial P_k$, and the argument of the trace may be cyclically permuted.

The first component of (50) can be transformed with the use of the same property (27) of rank-1 projectors and with the recurrence formula (8)

$$\partial P_k \partial P_k = \partial P_k P_k \partial P_k + \partial P_k (\mathbb{I} - P_k) \partial P_k$$

$$= \text{tr}(\partial P_k P_k \partial P_k) P_k + \text{tr}(P_k \partial P_k \partial P_k) P_k = \bar{\alpha}_k P_{k+1} + \alpha_k P_k$$  \hspace{1cm} (52)

for $k = 0, \ldots, N - 2$, while $\partial P_{N-1} P_{N-1} = 0$ and (52) reduces to $\alpha_{N-1} P_{N-1}$ for $k = N - 1$.

Similarly, the second component of (50) is given by

$$\bar{\partial} P_k \partial P_k = \bar{\alpha}_k P_{k-1} + \alpha_k P_k,$$  \hspace{1cm} (53)

for $k = 1, \ldots, N - 1$, while $P_0 \partial P_0 = 0$ and (52) reduces to $\bar{\alpha}_0 P_0$ for $k = 0$.

Summing up the three components, we get the required decomposition of $\partial \partial P_k$ (48).

It follows from (48) that any linear combination of rank-1 orthogonal projectors $P$ satisfies the E–L equation.
\[
\left[ \sum_{i=0}^{N-1} \lambda_i P_i, \sum_{j=0}^{N-1} \lambda_j P_j \right] = \sum_{i=0}^{N-1} \lambda_i \lambda_j \left( \alpha_i P_{i-1}, P_j \right) + \bar{\alpha}_i \left( P_{i+1}, P_j \right) - (\alpha_i + \bar{\alpha}_i) \left( P_i, P_j \right) = 0,
\]
(54)
since the projectors \( P_i \) are mutually orthogonal (9). The idempotency condition for the projector \( P \) requires that
\[
\sum_{i=0}^{N-1} \lambda_i P_i = P = \left( \sum_{i=0}^{N-1} \lambda_i P_i \right)^2
\]
(55)
\[
= \sum_{i,j=0}^{N-1} \lambda_i \lambda_j P_i P_j = \sum_{i,j=0}^{N-1} \lambda_i \lambda_j \delta_{ij} P_i = \sum_{i=0}^{N-1} \lambda_i^2 P_i,
\]
(56)
In the last equality of (56) we have again used the orthogonality property (9). Hence \( \lambda_i^2 = \lambda_i \), which implies that \( \lambda_i = 0 \) or \( \lambda_i = 1 \) for all \( i \in \{0,...,N-1\} \).

Thus proposition 3 proves that the E–L equations have a much larger class of solutions possessing the projective property than the class of rank-1 projectors \( P_k \). This increases the range of projectors \( P \) solvable by the technique described in this paper.

The inverse theorem is not true. Namely there exist partitions of rank \( r > 1 \) projectors, which satisfy the E–L equations, into rank-1 projectors which do not satisfy them. A trivial example in \( CP^1 \) is a partition of the unit matrix (a rank-2 projector obviously satisfying the E–L equations) into any rank-1 projector function \( P \) of \((\xi, \bar{\xi})\) which does not satisfy the equations and a rank-1 projector \( 1 - P \) (which obviously does not satisfy them either).

A similar proposition holds for the immersion function \( X \).

**Proposition 4.** Let a function \( X \in \mathfrak{su}(N) \) be a linear combination of immersion functions \( X_k \) of 2D-soliton surfaces in the \( \mathfrak{su}(N) \) algebra

\[
X = \sum_{k=0}^{N-1} \lambda_k X_k, \quad \lambda_k \in \mathbb{C},
\]
(57)
where not all \( \lambda_k \) are zero, and the \( X_k \) satisfy the E–L equation (17). Then

\[
[\partial \bar{\partial} X, X] = 0
\]
(58)
holds. If all \( \lambda_k \) are real, then the immersion function of the multileaf surface \( X \) is also an element of the \( \mathfrak{su}(N) \) algebra.

**Proof.** Each immersion function \( X_k \) is a linear combination, with constant coefficients, of the projectors \( P_j \) and the unit matrix (13), while, for each \( j \), \( \partial \bar{\partial} P_j \) is a linear combination of projectors, given by (48). Therefore, the mixed second derivative of \( X_k \) is also a linear combination of orthogonal projectors, namely

\[
\partial \bar{\partial} X_k = i[\alpha_k P_{k-1} + (\bar{\alpha}_k - \alpha_k) P_k - \bar{\alpha}_k P_{k+1}].
\]
(59)
Hence \([\partial \bar{\partial} X, X]\) is a linear combination of commutators, either between projectors or between projectors and the unit matrix, i.e. all commutators are equal to zero.
If in addition all the $\lambda_k$ are real, this ensures the anti-Hermitian property of the matrix $X$, while its tracelessness follows from the fact that it is a linear combination of traceless matrices. These properties make $X$ an element of $\mathfrak{su}(N)$.

A question arises: does the idempotency stated in proposition 2 hold for the surfaces corresponding to higher-rank projectors? That is, let a projector $P$ of rank $r > 1$ satisfy the E–L equation (3), have the projective property $P^2 = P$, let $X$ be a surface obtained from $P$ by the contour integration (12) and $Y$ the ‘surface over surface’, i.e. the result of the same contour integration performed on the surface $X$ (18). Will this $Y$ prove to be the same immersion function as $X$ up to a constant factor?

The answer is negative, even if the projector is a linear combination of rank-1 orthogonal projectors (47) and the coefficients are 0 or 1. Direct calculation shows that there is a nonzero remainder. Namely, the $\xi$-derivative of the contour integral may be written as

$$
\partial Y = \sum_{k=0}^{N-1} \lambda_k^2 \partial X_k + 2i \sum_{l=0}^{N-1} \sum_{k=0}^{l-1} \lambda_k \lambda_l [\partial P_k \partial P_l - P_l \partial P_l] \\
+ \left[ (\partial P_l + \partial P_k) \sum_{j=k}^{l-1} P_j + \sum_{j=k}^{l-1} P_j (\partial P_l + \partial P_k) \right].
$$

(60)

The first sum is indeed $-X$, provided that all $\lambda_k$ are zero or one, but the remainder will generally be nonzero. The same holds for $-\overline{\partial Y}$, which is the Hermitian conjugate of (60).

The sum of the coefficients $\alpha_k$ has physical and geometric interpretations. Namely $\alpha_k + \overline{\alpha_k} = \text{tr}(\partial P_k \partial P_k)$ is the Lagrangian density in the action functional (1). Moreover, we have shown in [12] that $\text{tr}(\partial X_k \partial X_k) = -\text{tr}(\partial P_k \partial P_k)$, which makes this quantity also the Lagrangian density for the surface immersion functions. It is also the non-diagonal element $g_{12} = g_{21}$ of the metric tensor on the surface $X_k$, while the diagonal elements of the metric tensor are zero [9]. In this way $\alpha_k + \overline{\alpha_k}$ determines the metric properties of the surfaces $X_k$ (and obviously all the surfaces of the stack).

To conclude, we have shown that the E–L equation (58) may also describe multileaf surfaces in addition to the surfaces generated by rank-1 projectors, which were the subjects of earlier work [14]. This makes soliton surfaces associated with $\mathbb{C}P^{N-1}$ models a rather special and interesting subject to study. The interesting task of identifying the general solution of the E–L equations remains open and will be the subject of further studies. In this context it would also be pertinent to further develop the $\mathbb{C}P^{N-1}$ sigma models via the coherent states approach as performed in [16].

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