ON ASYMPTOTIC VANISHING BEHAVIOR OF LOCAL COHOMOLOGY

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Abstract. Let $R$ be a standard graded algebra over a field $k$, with irrelevant maximal ideal $m$, and $I$ a homogeneous $R$-ideal. We study the asymptotic vanishing behavior of the graded components of the local cohomology modules $\{H^i_m(R/I^n)\}_{n \in \mathbb{N}}$. If these modules are Noetherian for $n \gg 0$, we show that, when $\text{char } k = 0$, $R/I$ is Cohen-Macaulay, and $I$ is locally a complete intersection, their lowest degrees are bounded by a linear function whose slope is controlled by the largest generating degree of the dual of the conormal module of $I$. Our result is a direct consequence of a related bound for symmetric powers of locally free modules. If no assumptions are made on the ideal, we show that the complexity of the sequence of lowest degrees is at most polynomial.

1. Introduction

Let $R$ be a Noetherian standard graded algebra over a field $k = R_0$, $m = \bigoplus_{i>0} R_i$, and $I$ a homogeneous $R$-ideal. In this paper we study asymptotic behavior of the lowest degree of the local cohomology modules $\{H^i_m(R/I^n)\}_{n \in \mathbb{N}}$, provided that they are finite. As we make clear below, such behavior can be viewed as some “asymptotic Kodaira vanishing for thickenings” phenomenon, and have recently appeared in various works such as [1, 6, 14].

To describe our motivations and questions precisely, let us recall some notations. For a graded $R$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ one defines

$$\text{indeg } M = \min \{i \mid M_i \neq 0\}, \quad \text{topdeg } M = \max \{i \mid M_i \neq 0\};$$

If $M = 0$, we set $\text{indeg } M = \infty$ and $\text{topdeg } M = -\infty$. We also set

$$\beta(M) = \max \{i \mid (M/mM)_i \neq 0\},$$

i.e., $\beta(M)$ is the maximal degree of an element in a minimal set of homogeneous generators of $M$. The Castelnuovo-Mumford regularity of $M$ is defined as

$$\text{reg}(M) = \max \{\text{topdeg } H^i_m(M) + i\}.$$ 

It is known that $\text{reg}(R/I^n)$ agrees with a linear function for $n \gg 0$, this fact was proved independently in [4] and [11] when $R$ is a polynomial ring over a field, and extended in [18] for arbitrary standard graded rings.

The topdeg of $H^i_m(M)$ is always finite, however this is not case for indeg. In fact, since $H^i_m(M)$ is an Artinian module, we have that $\text{indeg } H^i_m(M) > -\infty$ if and only if $H^i_m(M)$ is Noetherian. Our work is guided by the following questions raised in [6]:

**Question 1.1.** Assume $H^i_m(R/I^n)$ is Noetherian for $n \gg 0$.

1. Does there exist $\alpha \in \mathbb{Z}$ such that $\text{indeg } H^i_m(R/I^n) > \alpha n$ for every $n \gg 0$? In other words, is $\liminf_{n \to \infty} \frac{\text{indeg } H^i_m(R/I^n)}{n}$ finite?

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(2) if so, does the limit \( \lim_{n \to \infty} \frac{\text{indeg } H^i_n(R/I^n)}{n} \) exist?

It follows from [1, 1.4] that when \( R \) is a polynomial ring over a field of characteristic 0, \( X := \text{Proj}(R/I) \) is locally a complete intersection (lc), and \( i \) is at most the codimension of the singular locus of \( X \), then \( \text{indeg } H^i_n(R/I^n) \geq 0 \) for all \( n > 0 \). As explained there, this can be viewed as a Kodaira Vanishing Theorem for thickenings of \( X \). When \( I \) is a determinantal ideal, more precise behavior of vanishing results, and other homological invariants of thickenings of \( I \) are available, see for instance [14] and the survey [15].

Our initial interest in the question came from [6] where we need an affirmative answer to part (1) of Question 1.1 to obtain efficient bounds on length of the local cohomology modules. Robert Lazarsfeld pointed out to us that this is indeed the case when \( X = \text{Proj}(R/I) \) is a l.c.i variety, \( k \) is of characteristic zero, and \( i \) is at most the dimension of \( X \). So Kodaira vanishing may not hold, but the lowest degrees of \( H^i_n(R/I^n) \) are still bounded below by a linear function.

In this work we provide further answers to Question 1.1 above, in the case when \( R \) is not necessarily a polynomial ring and \( I \) may not be prime, or even reduced. The first main general result of this article (Theorem 2.4), provides a linear lower bound for initial degrees of local cohomology of symmetric powers of a graded module that is locally free away from the graded maximal ideal. Our proof relies on a duality statement and the result on regularity by Trung and Wang in [18].

**Theorem 1.2.** Let \( (R, \mathfrak{m}) \) be a standard Cohen-Macaulay graded algebra over a field \( k \) of characteristic zero. Set \( d = \dim R \geq 2 \). Let \( E \) be a graded \( R \)-module which is free locally on \( \text{Spec } R \setminus \{ \mathfrak{m} \} \). Then there exists an integer \( \varepsilon \) such that

\[
\text{indeg } H^i_{\mathfrak{m}}(S^n(E)) \geq -\beta(E^*)n + \varepsilon
\]

for every \( n \gg 0 \) and \( 1 \leq i < d \).

We apply this Theorem to answer Question 1.1 (1) affirmatively and effectively when \( R \) is any standard graded algebra over a field of characteristic 0, \( R/I \) is Cohen-Macaulay, and \( I \) is a complete intersection locally on \( \text{Spec } R \setminus \{ \mathfrak{m} \} \). In this case, the \( \lim \inf_{n \to \infty} \frac{\text{indeg } H^i_{\mathfrak{m}}(R/I^n)}{n} \) is bounded below by \( -\max\{\beta(E^*), 0\} \) where \( E \) is the conormal module \( I/I^2 \) (see Corollary 2.6). This result can be seen as an algebraic version of [1, 1.4] and [6, 5.6], our proof via local cohomology of symmetric powers of conormal modules is inspired by the proofs of these results.

In general, if one only assumes that \( H^i_{\mathfrak{m}}(R/I^n) \) is Noetherian for \( n \gg 0 \), it is complicated to find bounds on its lowest degrees. However, we are able to prove that there is a polynomial lower bound, regardless of the characteristic of \( k \). The proof rests on a result by Chardin, Ha, and Hoa [3], and is provided in Section 3.

In Section 4 we study the situation when \( I \) is a monomial ideal in a polynomial ring \( R \). As expected, the extra combinatorial structure allows for better results. Assuming that \( H^i_{\mathfrak{m}}(R/I^n) \) is Noetherian for \( n \gg 0 \), one can show that either \( \text{indeg } H^i_{\mathfrak{m}}(R/I^n) = 0 \) for \( n \gg 0 \) or \( \lim \inf_{n \to \infty} \frac{\text{indeg } H^i_{\mathfrak{m}}(R/I^n)}{n} \geq 1 \), and the latter holds precisely when \( \tilde{H}_{i-1}(\Delta(I)) = 0 \), where \( \Delta(I) \) is the complex for which \( \sqrt{I} \) is the Stanley-Reisner ideal.

2. **Symmetric Powers of Locally Free Modules and Linear Lower Bound**

Let \( E \) be a Noetherian graded module and

\[
F_1 \xrightarrow{\phi} F_0 \xrightarrow{g} M \xrightarrow{0}
\]

2
a minimal presentation of $M$, where $\phi$ is an $u \times s$ matrix with entries in $m$ and $u = \mu(M) := \dim_k M/mM$. Let $T_1, \ldots, T_u$ be a set of variables and $\ell_1, \ldots, \ell_s$ the linear forms determined by

$$[\ell_1, \cdots, \ell_s] = [T_1, \cdots, T_u] \phi.$$  

The ring $\text{Sym}(E) := R[T_1, \ldots, T_u]/(\ell_1, \ldots, \ell_s)$ is the symmetric algebra of $E$. Let $d_1, \ldots, d_u$ be the degrees of a homogeneous minimal generating set of $E$. We can assign to $\text{Sym}(E)$ a bi-graded structure where $T_i$ has bi-degree $(d_i, 1)$ for every $i = 1, \ldots, u$. The $n$th-graded component of $\text{Sym}(E)$, $S^n(E) = \bigoplus_{a \in \mathbb{Z}} \text{Sym}(E)_{(a,n)}$, is the $n$th-symmetric power of $E$.

Let $M$ be any Noetherian graded $R$-module and $U \subseteq E$ a graded submodule. We say $U$ is an $M$-reduction of $E$, if $S^n(U \otimes_R M) = S^1(U)S^{n-1}(E) \otimes_R M$ for $n \gg 0$, where $S^1(U)$ is seen as a submodule of $S^1(E)$. Following [18], we define

$$\rho_M(E) := \min \{ \beta(U) \mid U \text{ is an } M\text{-reduction of } E \}.$$  

We note that $\rho_M(E) \leq \beta(E)$ for every $R$-module $M$. The following theorem is the module version of [18, 3.2] and the proofs of these results are identical, however we include some relevant details for the reader’s convenience. We remark that even though the algebras in [18] are positively graded, the proof of this result does not use this assumption.

**Theorem 2.1.** Let $R$ an standard graded algebra over a Noetherian ring $A$. Let $E$ and $M$ be finitely generated graded $R$-modules. Then

$$\text{reg}(S^n(E) \otimes_R M) = \rho_M(E)n + e$$  

for some integer $e \geq \text{indeg } M$ and every $n \gg 0$.

**Proof.** Let $U$ be an $M$-reduction of $E$ such that $d(U) = \rho_M(E)$. Let $\mathcal{M} = \text{Sym}(E) \otimes_R M = \bigoplus_{n \geq 0} S^n(E) \otimes_R M$ and notice $\mathcal{M}$ is a finitely generated graded $\text{Sym}(U)$-module. Let $s = \mu_A(R_1)$, $v = \mu(U)$, and $u_1, \ldots, u_v$ the degrees of a homogeneous minimal generating set of $U$, then $\text{Sym}(U)$ is a quotient ring of the bi-graded polynomial ring $A[x_1, \ldots, x_s, y_1, \ldots, y_v]$ where $x_i$ has degree $(1,0)$ for each $i$ and $y_j$ has degree $(u_j,1)$ for each $j$. Therefore, [18, 2.2] implies $\text{reg}(S^n(E) \otimes_R M)$ is a linear function $\rho m + e$ for some $e$ and $\rho \leq \rho_M(E)$. Finally, proceeding as in [18, 3.1] we obtain $\rho \geq \rho_M(E)$ and $e \geq \text{indeg } M$, finishing the proof. \hfill $\Box$

For the next result, we assume $R$ is a local ring or standard graded over a field. Let $m$ be the (irrelevant) maximal ideal of $R$ and $k = R/m$ and $E_R(k)$ the (graded) injective hull of $k = R$. For a (graded) $R$-module $M$ we set

$$M^\vee := \text{Hom}_R(M, E_R(k)).$$  

The following is a generalization of a duality result of Horrocks [10].

**Proposition 2.2.** Let $(R, m, k)$ be a Cohen-Macaulay local ring (or positively graded $k$-algebra). Set $d = \dim R \geq 2$ and $\omega$ a canonical module of $R$. Fix $1 \leq i \leq d-1$, then for a (graded) $R$-module $M$ of dimension $d$ that is $S_{i+1}$ locally on $\text{Spec } R \setminus \{m\}$ we have (graded) isomorphisms

$$H^i_m(M)^\vee \cong H^{d-i+1}_m(\text{Hom}_R(M, \omega)) \quad \text{if } i \geq 2,$$

and,

$$H^1_m(M)^\vee \cong \ker (H^d_m(\text{Hom}_R(M, \omega)) \to H^d_m(\text{Hom}_R(F_0, \omega))),$$

where $F_0 \to M$ is a free module.
Proof. Let $N$ be a (graded) $R$-module that is Maximal Cohen-Macaulay (MCM) locally on $\text{Spec } R \setminus \{m\}$ and $\cdots \to F_0 \to N \to 0$ a (graded) free resolution of $N$. Then, $\text{Ext}_R^1(N, \omega) \cong \ker \left( H_m^2(\text{Hom}_R(N, \omega)) \to H_m^2(\text{Hom}_R(F_0, \omega)) \right)$ if $d = 2$, and $\text{Ext}_R^1(N, \omega) \cong H_m^2(\text{Hom}_R(N, \omega))$ if $d \geq 3$.

In order to prove this claim, we consider the $R$-modules $K$ and $C$ that fit in the following two exact sequences

\begin{equation}
0 \to \text{Hom}_R(N, \omega) \to \text{Hom}_R(F_0, \omega) \to C \to 0 \quad \text{and} \quad (2.1)
\end{equation}

\begin{equation}
0 \to K \to \text{Hom}_R(F_1, \omega) \to \text{Hom}_R(F_2, \omega).
\end{equation}

By applying the depth lemma to (2.2) we obtain depth $K \geq 2$, therefore

\[ \text{Ext}_R^1(N, \omega) = H_m^0(\text{Ext}_R^1(N, \omega)) = H_m^0(K/C) \cong H_m^1(C) \]

where the first equality follows by the assumption on $N$. Hence, the proof of the claim follows from (2.1).

Now, notice that the result follows by the claim and local duality [2, 3.6.19] if $d = 2$, then we may assume $d \geq 3$. Let $\Omega^n M$ be the $n$th-syzygy module of $M$. Again by local duality and the claim we have

\[ H_m^i(M)^{\vee} \cong \text{Ext}_R^{d-i}(M, \omega) \cong \text{Ext}_R^1(\Omega^{d-i-1}M, \omega) \cong H_m^2(\text{Hom}_R(\Omega^{d-i-1}M, \omega)). \]

Let $0 \leq t \leq d - i - 2$, by assumption $\Omega^t M$ is MCM in codimension $i + t + 1$, which implies that $\dim \text{Ext}_R^1(\Omega^t M, \omega) < d - i - t - 1$. Let $\cdots \to F_0 \to N \to 0$ be a (graded) free resolution of $M$. From the exact sequence

\[ 0 \to \text{Hom}_R(\Omega^t M, \omega) \to \text{Hom}_R(F_t, \omega) \to \text{Hom}_R(\Omega^{t+1} M, \omega) \to \text{Ext}_R^1(\Omega^t M, \omega) \to 0 \]

we obtain

\[ H_m^{d-i-t}(\text{Hom}_R(\Omega^{t+1} M, \omega)) \cong H_m^{d-i-t+1}(\text{Hom}_R(\Omega^t M, \omega)) \]

if $i + t \geq 2$ and

\[ H_m^{d-1}(\text{Hom}_R(\Omega^1 M, \omega)) \cong \ker \left( H_m^d(\text{Hom}_R(M, \omega)) \to H_m^d(\text{Hom}_R(F_0, \omega)) \right), \]

the statement follows. \hfill \Box

Given an $R$-module $M$, we denote by $\Gamma(M)$ the divided powers algebra of $M$ [12]. We set $M^* := \text{Hom}_R(M, R)$ and for a graded $R$-algebra $S = \oplus_{n \geq 0} S_n$, we denote by $S^* := \oplus_{n \geq 0} S_n^*$ the graded dual of $S$.

We need the following technical lemma for the proof of our main result.

Lemma 2.3. Let $R$ be a commutative (graded) ring and $M$ a (graded) $R$-module. Then there exist natural (graded) maps

\[ \text{Sym}(M^*) \overset{\alpha}{\to} \Gamma(M^*) \overset{\beta}{\to} \text{Sym}(M)^*. \]

Moreover, $\alpha$ is an isomorphism if $R$ contains the field of rational numbers and $\beta$ is an isomorphism if $M$ is free.

Proof. For the construction and results on $\alpha$, see [16, Proposition III.3., p. 256]. See [12, (5.6), p. 5.4] for the corresponding information for $\beta$. \hfill \Box

The following is the main theorem of this section.
Theorem 2.4. Let \((R, m)\) be a standard Cohen-Macaulay graded algebra over a field \(k\) of characteristic zero. Let \(d = \dim R \geq 2\). Let \(E\) be a graded \(R\)-module which is free locally on \(\text{Spec} \ R \setminus \{m\}\). Then there exists an integer \(\varepsilon\) such that

\[
\text{indeg } \text{H}_m^i(S^n(E)) \geq -\beta(E^*)n + \varepsilon
\]

for every \(n \gg 0\) and \(1 \leq i < d\).

Proof. First, assume \(i \geq 2\). Let \(\omega\) be the canonical module of \(R\). By Hom-Tensor adjointness and the isomorphism \(R \cong \text{Hom}_R(\omega, \omega)\), we have \(S^n(E^*)^* \cong \text{Hom}_R(S^n(E^*) \otimes_R \omega, \omega)\).

By the assumption we have \(S^n(E^*) \otimes_R \omega\) is MCM locally on \(\text{Spec} \ R \setminus \{m\}\), therefore the natural inclusion

\[
S^n(E^*) \otimes_R \omega \hookrightarrow \text{Hom}_R(\text{Hom}_R(S^n(E^*) \otimes_R \omega, \omega), \omega)
\]

is an isomorphism locally on \(\text{Spec} \ R \setminus \{m\}\). Hence, by Proposition 2.2

\[
\text{H}_m^i(S^n(E^*)^*) \cong \text{H}_m^i(\text{Hom}_R(S^n(E^*) \otimes_R \omega, \omega)) \cong \text{H}_m^{d-i+1}(S^n(E^*) \otimes_R \omega)^\vee,
\]

By Theorem 2.1, we have

\[
\text{topdeg } \text{H}_m^{d-i+1}(S^n(E^*) \otimes_R \omega) \leq \beta(E^*)n - \varepsilon
\]

for some \(\varepsilon \in \mathbb{Z}\) and every \(n \gg 0\). Therefore \(\text{indeg } \text{H}_m^i(S^n(E^*)^*) \geq -\beta(E^*)n + \varepsilon\) for every \(n \gg 0\) and \(i \geq 2\). The map \(\text{Sym}(E^*) \xrightarrow{\beta \alpha} \text{Sym}(E)^*\) in Lemma 2.3 with \(M = E\) is an isomorphism locally on \(\text{Spec} \ R \setminus \{m\}\), hence

\[
S^n(E^*)^* \cong S^n(E)^{**}.
\]

The result now follows for \(i \geq 2\) by observing that \(\text{H}_m^i(S^n(E)) \cong \text{H}_m^i(S^n(E)^{**})\) as \(S^n(E)\) is reflexive locally on \(\text{Spec} \ R \setminus \{m\}\).

Now, we show the statement for \(i = 1\). Fix \(n \gg 0\) and consider the short exact sequence

\[
0 \to S^n(E) \xrightarrow{\varphi} S^n(E)^{**} \to C \to 0.
\]

Since \(\varphi\) is an isomorphism locally on \(\text{Spec} \ R\setminus \{m\}\), we have \(\dim C = 0\). Then \(\text{H}_m^1(S^n(E)) = C\), as depth \(S^n(E)^{**} \geq 2\). Therefore,

\[
\text{indeg } \text{H}_m^1(S^n(E)) = \text{indeg } C \geq \text{indeg } S^n(E)^{**} = \text{indeg } S^n(E^*)^*,
\]

where the last equality follows from (2.3). Let \(\oplus_{i=1}^n R(-a_i) \to S^n(E^*) \to 0\) be the first map of a minimal homogeneous resolution of \(S^n(E^*)\), where \(u = \mu(S^n(E^*))\). Then, \(S^n(E^*)^* \hookrightarrow \oplus_{i=1}^n R(a_i)\). We conclude

\[
\text{indeg } S^n(E^*)^* \geq -\max \{a_i\} \geq -\text{reg}(S^n(E^*)) \geq -\beta(E^*)n + \varepsilon,
\]

for some \(\varepsilon \in \mathbb{Z}\) and \(n \gg 0\) by Theorem 2.1. \hfill \Box

Assume \(E\) is a graded submodule of a free graded \(R\)-module \(F = \oplus_{i=1}^\gamma R(-d_i)\). We have the natural inclusion of symmetric algebras

\[
\text{Sym}(E) \to \text{Sym}(F) = R[T_1, \ldots, T_\gamma],
\]

where each \(T_i\) has bidegree \((d_i, 1)\). The image of this map is the bi-graded algebra

\[
\mathcal{R}[E] := \oplus_{n \geq 0} E^n \subset R[T_1, \ldots, T_\gamma].
\]

The ring \(\mathcal{R}[E]\) is called the Rees algebra of \(E\) with respect to the embedding \(E \subset F\). It is known that if \(E\) has a rank, i.e., \(E_P\) is free of constant rank for every \(P \in \text{Ass}(R)\), then \(\mathcal{R}[E]\) is isomorphic to \(\text{Sym}(E)/(R\text{-torsion})\) and hence it is independent of the graded embedding of \(E\) into a free module (cf. [7]).
Corollary 2.5. Let \((R, m, k)\) and \(E\) be as in Theorem 2.4. Assume that \(E\) has a rank, then
\[
\text{indeg} \ H^i_m(E^n) \geq -\beta(E^*)n + \varepsilon
\]
for some \(\varepsilon \in \mathbb{Z}\) and every \(n \gg 0, 1 \leq i < d\).

Proof. Since \(E \) and \(E^* \) have a rank, we have
\[
E^n \cong S^n(E)/(R\text{-torsion}) \quad \text{and} \quad (E^*)^n \cong S^n(E^*)/(R\text{-torsion}),
\]
therefore
\[
((E^*)^n)^* \cong S^n(E^*)^* \quad \text{and} \quad ((E^n)^*)^* \cong S^n(E)^*.
\]
The rest of the proof follows as in Theorem 2.4. \(\square\)

Corollary 2.6. Let \((R, m)\) be a standard graded algebra over a field \(k\) of characteristic zero. Let \(I \) be a homogeneous \(R\text{-ideal} \) such that \(S := R/I\) is Cohen-Macaulay. Assume \(I_\mathfrak{p}\) is generated by a regular sequence in \(R_\mathfrak{p}\) for every \(\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\) and that \(\dim S \geq 2\). Let \(E := I/I^2\) be the conormal module of \(I\) and \(E^* = \text{Hom}_R(E, S)\), then one of the following holds

1. if \(\beta(E^*) \leq 0\) then there exists \(C \in \mathbb{Z}\) such that
   \[
   \text{indeg} \ H^i_m(R/I^n) \geq C
   \]
   for every \(n \gg 0 \) and \(1 \leq i < \dim R/I\);
2. if \(\beta(E^*) > 0\), then there exists \(\varepsilon \in \mathbb{Z}\) such that
   \[
   \text{indeg} \ H^i_m(R/I^n) \geq -\beta(E^*)n + \varepsilon
   \]
   for every \(n \gg 0 \) and \(1 \leq i < \dim R/I\).

Proof. By assumption \(E\) is a \(R/I\)-module that is free locally on \(\text{Spec} R \setminus \{\mathfrak{m}\}\). Since the natural epimorphism \(S^n(E) \to I^n/I^{n+1}\) is an isomorphism locally on \(\text{Spec} R \setminus \{\mathfrak{m}\}\) we have \(H^i_m(S^n(E)) \cong H^i_m(I^n/I^{n+1})\). The conclusion now follows from Theorem 2.4 and by induction on \(n\) via the inequality
\[
\text{indeg} \ H^i_m(R/I^{n+1}) \geq \min\{\text{indeg} H^i_m(I^n/I^{n+1}), \text{indeg} H^i_m(R/I^n)\}
\]
for \(n \geq 1\). \(\square\)

Example 2.7. Let \(R = k[x, y, u, v] / (xv^t - yu^t)\) for some \(t \geq 1\) and \(\text{char} k = 0\). Let \(I = (x, y)\) and note that \(S := R/I\) and \(I\) satisfy the assumptions of Corollary 2.6. The graded free resolution of \(I/I^2\) is
\[
0 \to S(-1-t) \xrightarrow{egin{bmatrix} u^t \\ -v^t \\ \end{bmatrix}} S^2(-1) \to I/I^2 \to 0.
\]
Therefore, \((I/I^2)^*\) is isomorphic to the kernel of the map \(S^2(1) \xrightarrow{egin{bmatrix} u^t \\ v^t \\ \end{bmatrix}} S(1+t)\), which is generated by \(\begin{bmatrix} u^t \\ v^t \end{bmatrix}\). Therefore, \(\beta((I/I^2)^*) = t - 1\).

If \(t \geq 2\), then by Corollary 2.6 we have \(\text{indeg} H^1_m(R/I^n) \geq -(t-1)n + \varepsilon\) for some \(\varepsilon \in \mathbb{Z}\). On the other hand, computing \(H^1_m(R/I^n)\) via the Cech complex of the system of parameters \(\{u, v\}\) of the ring \(R/I^n\), we obtain that the class of \(\frac{u^{n-1}}{u}, \frac{v^{n-1}}{v}\) is nonzero and has degree \(-(t-1)n + (t-1)\). We conclude that
\[
-(t-1)n + (t-1) \geq \text{indeg} H^1_m(R/I^n) \geq -(t-1)n + \varepsilon
\]
for every \( n \geq 1 \). Therefore,
\[
\lim_{n \to \infty} \frac{\text{indeg } H^1_m(R/I^n)}{n} = -(t - 1).
\]

Now, if \( t = 1 \) we have \( \beta((I/I^2)^*) = 0 \) and hence \( \{\text{indeg } H^1_m(R/I^n)\}_{n \in \mathbb{N}} \) is bounded below by a constant.

We record below some values of \( \text{indeg } H^1_m(R/I^n) \) as \( t \) and \( n \) vary. These values were obtained with the help of Macaulay2 [8].

| \( t \) | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1     | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2     | -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | -9 | -10| -11| -12| -13| -14| -15|
| 3     | -2 | -4 | -6 | -8 | -10| -12| -14| -16| -18| -20| -22| -24| -26| -28| -30|
| 4     | -3 | -6 | -9 | -12| -15| -18| -21| -24| -27| -30| -33| -36| -39| -42| -45|
| 5     | -4 | -8 | -12| -16| -20| -24| -28| -32| -36| -40| -44| -48| -52| -56| -60|

**Example 2.8.** Let \( X \) be a 2 \( \times \) 3 generic matrix and \( R = k[X] \) with char \( k = 0 \). Let \( I = I_2(X) \) the ideal generated by the 2 \( \times \) 2 minors of \( X \), then \( R/I \) is Cohen-Macaulay of dimension 4, and \( \text{Proj } R/I \) is lci. Using Macaulay2 [8] we obtain \( \beta((I/I^2)^*) = -1 \), therefore by Corollary 2.6 \( \{\text{indeg } H^3_m(R/I^n)\}_{n \in \mathbb{N}} \) is bounded below by a constant. Indeed, \( \text{indeg } H^3_m(R/I^n) = 0 \) for every \( n \geq 2 \) [1, Example 5.1].

In the following example we demonstrate that the lower bound \( C \) in Corollary 2.6 (1) may be negative.

**Example 2.9.** Let \( R = k[x, y, z, u, v, w]/(x^2u^2 + y^2v^2 + z^2w^2) \) with char \( k = 0 \) and let \( I = (x, y, z) \). Computations Macaulay2 [8] show that \( \beta((I/I^2)^*) = -1 \) and suggest that \( \text{indeg } H^i_m(R/I^n) = -2 \) for every \( n \geq 3 \).

In the following example we observe that, even when the ring \( R \) is regular, the sequence \( \{\text{indeg } H^i_m(R/I^n)\}_{n \in \mathbb{N}} \) may have linear behavior with negative slope.

**Example 2.10.** Let \( R = k[x, y, u, v] \) with char \( k = 0 \) and \( I = (x^2u - y^2v, uv, v^2) \). Computations Macaulay2 [8] show that \( \beta((I/I^2)^*) = 2 \) and suggest that \( \text{indeg } H^i_m(R/I^n) = -2n + 1 \) for every \( n \geq 1 \).

A local ring \((S, \mathfrak{n})\) is said to be cohomologically full if for every surjection \( T \to S \) from a local ring \((T, \mathfrak{q})\), such that \( T_{\text{red}} = S_{\text{red}} \), the natural map \( H^i_q(T) \to H^i_n(S) \) is surjective for every \( i \). If \( R \) is a standard graded algebra over a field \( k \) and irrelevant maximal ideal \( \mathfrak{m} \), then we say \( R \) is cohomologically full if the local ring \( R_m \) is. For more information and examples of cohomologically full rings see [5].

The following result answers Question 1.1 in a particular case.

**Corollary 2.11.** Let \( R, I, \) and \( E \) be as in Corollary 2.6, and fix an integer \( 1 \leq i < \dim R/I \). Assume \( R/J \) is cohomologically full for some \( R \)-ideal \( J \) such that \( \sqrt{J} = \sqrt{I} \) and \( H^i_m(R/J) \neq 0 \). If \( \beta(E^*) < 0 \), then there exists an integer \( C \leq 0 \) such that \( \text{indeg } H^i_m(R/I^n) = C \) for every \( n \gg 0 \).
Proof. By assumption we have that the map $H^i_m(R/I^n) \to H^i_m(R/J)$ is surjective for $n \gg 0$. Therefore, $H^i_m(R/J)$ has finite length and

$$\text{indeg } H^i_m(R/I^n) \leq \text{indeg } H^i_m(R/J) = 0,$$

where the last equality follows from [5, 4.9]. Now, by Theorem 2.4 we have that $H^i_m(R/I^n)_{<0} = H^i_m(R/I^{n+1})_{<0}$ for $n \gg 0$, and the conclusion follows. \hfill \qed

Remark 2.12. In the setting of Corollary 2.11, let $X = \text{Proj } R/I$. If $i \leq \text{codim Sing } X$ it was proved in [1, 3.1] that $H^i_m(R/I^n)_{<0} = 0$ for every $n \geq 1$. Hence, if $I^n$ is cohomologically full for every $n \gg 0$, [5, 4.9] shows $\text{indeg } H^i_m(R/I^n) = 0$ for $n \gg 0$.

The following example shows that the characteristic assumption is necessary in Corollary 2.6 and hence in Theorem 2.4.

Example 2.13. Let $R$ and $I$ be as in Example 2.8 but assume instead that $R$ has characteristic $p > 0$. Moreover, assume the conclusion of Theorem 2.4 holds in positive characteristic. Computations by Macaulay2 [8] show $\beta((I/I^2)^*) = -1$ and then by Theorem 2.4 we have

$$H^3_m(R/I^{t+1})_0 \to H^3_m(R/I^t)_0$$

is an isomorphism for $t \gg 0$. Furthermore, by [1, 5.5], $H^3_m(R/I^t)_0 \neq 0$ for every $t \geq 2$. However, if $t' \gg t \gg 0$, there exists $e \in \mathbb{N}$ such that $I^t \subseteq I^{[p^e]} \subseteq I^{t'}$ and hence $H^3_m(R/I^t') \to H^3_m(R/I^t)$ is the zero map as $R/I^{[p^e]}$ is Cohen-Macaulay, which is a contradiction.

3. POLYNOMIAL BOUND FOR HOMOGENEOUS IDEALS

Let $I$ be a homogeneous ideal in a standard graded ring over a field. The purpose of this section is to prove that whenever the modules $H^i_m(R/I^n)$ are Noetherian for $n \gg 0$, the rate of growth of the sequence $\{\text{indeg } H^i_m(R/I^n)\}_{n \in \mathbb{N}}$ is at most polynomial. The results of this section apply in wide generality and without assumptions on the characteristic of the base field.

Let $M$ be a Noetherian $R$-module of dimension $d$. We denote by $e_0(M), \ldots, e_d(M)$ the Hilbert coefficients of $M$, i.e.,

$$\lambda(M/m^tM) = \sum_{i=0}^{r} (-1)^i e_i(M) \binom{t + d - i}{d - i} \quad \text{for } t \gg 0,$$

where $\lambda(N)$ denotes the length of the $R$-module $N$.

We say that a sequence of integers $\{a_n\}_{n \in \mathbb{N}}$ is $O(n^s)$ if $\limsup_{n \to \infty} \frac{|a_n|}{n^s} < \infty$.

Theorem 3.1. Let $R$ be a standard graded algebra over a field $k$ and with irrelevant maximal ideal $m$. Let $I$ be a homogeneous $R$-ideal and set $d = \dim R/I \geq 2$. Assume $H^i_m(R/I^n)$ is Noetherian for some $1 \leq i < d$ and $n \gg 0$. Then there exists $s \in \mathbb{N}$ such that

$$\text{indeg } H^i_m(R/I^n) = O(n^s).$$

Proof. Let $e = \dim_k R_1$. Consider an epimorphism $S := k[x_1, \ldots, x_e] \to R$ from a polynomial ring $S$ and let $n = (x_1, \ldots, x_e) \subset S$. By graded local duality [2, 3.6.19] we have a graded isomorphism

$$H^i_m(R/I^n) \cong H^n(S/I^n, S)^{\vee}(e)$$

for every $n \in \mathbb{N}$. Therefore, by assumption

$$\text{indeg } H^i_m(R/I^n) = \text{topdeg } H^{e-i}(S/I^n, S)^{\vee} - e = \text{reg}(H^{e-i}(S/I^n, S)) - e.$$
For an $R$-module $M$ of dimension $r$, we set $Q_M(n) = \sum_{i=0}^{r} |e_i(M)| \binom{n+r-i}{r-i}$. For an $R$-ideal $J$, set $J = (0 :_R n^\infty)$.

Let $r_n = \reg(R/I^n)$ and notice that from the regular sequence

$$0 \to \tilde{I}^n/I^n \to R/I^n \to R/\tilde{I}^n \to 0$$

we obtain $\reg(R/\tilde{I}^n) \leq r_n$. Therefore, by [3, 3.5], there exists $C \in \mathbb{N}$ such that

$$\reg(\Ext^n_S(R/I^n, S)) < C(Q_{R/I^n}(r_n))^{2^{d-2}}.$$ 

By [9, 1.1], the functions $e_i(R/I^n)$ agree with a polynomial of degree $\leq e-d-i$ for every $i$ and $n \gg 0$, therefore there exists a polynomial in two variables, $q(n,t) \in \mathbb{Z}[n,t]$ of degree at most $e$ in $n$ and of degree $d$ in $t$, such that $Q_{R/I^n}(t) \leq q(t,n)$ for $t,n \gg 0$. Since $r_n$ is eventually agrees with a linear function by [18, 3.2], it follows that $(Q_{R/I^n}(r_n))^{2^{d-2}} = O(n^{(e+d)(2^{d-2})})$.

The conclusion now follows from (3.1) and the fact that the sequence $\{\indeg H^n_m(R/I^n)\}_{n \in \mathbb{N}}$ is bounded above by a linear function. \hfill $\Box$

The previous result provides a polynomial bound for lengths of local cohomology modules of homogeneous ideals (cf. [6, 7.1])

**Corollary 3.2.** Let $I$ be a homogeneous $R$-ideal in a polynomial ring $R$. Assume $H^n_m(R/I^n)$ is Noetherian for $n \gg 0$, then there exists $s \in \mathbb{N}$ such that $\lambda(H^n_m(R/I^n)) = O(n^s)$.

**Proof.** By Theorem 3.1 the exists $C$ and $s$ such that $H^n_m(R/I^n)_{< Cn^s} = 0$. The proof now follows as in [6, 5.3]. \hfill $\Box$

4. **Monomial ideals**

The purpose of this section is to analyze the asymptotic behavior of $\{\indeg H^n_m(R/I^n)\}_{n \in \mathbb{N}}$ for monomial ideals. From now on we assume $R = k[x_1, \ldots, x_d]$, $m = (x_1, \ldots, x_d)$, and $I$ is a monomial ideal.

We now state Takayama’s formula which expresses the graded components of local cohomology of monomial ideals in terms of reduced homology of some associated simplicial complexes. Let $F$ be a subset of $[d] = \{1, \ldots, d\}$. We consider the map $\pi_F : R \rightarrow R$, defined by $\pi_F(x_i) = 1$ if $i \in F$, and $\pi_F(x_i) = x_i$ otherwise. We set $I_F := \pi_F(I)$. For $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$, we use the notation $x^a := x_1^{a_1} \cdots x_d^{a_d}$.

For $a \in \mathbb{Z}^d$ we also consider $G_a = \{i \mid a_i < 0\}$. We also define $a^+ = (a_1^+, \ldots, a_d^+)$, where $a_i^+ = a_i$ if $i \notin G_a$ and $a_i^+ = 0$ otherwise. We set $\Delta_a(I)$ to be the simplicial complex of all subsets $F$ of $[d] \setminus G_a$ such that $x^a \notin I_{F \cup G_a}$. We note that $\Delta_a(I)$ is a subcomplex of $\Delta(I)$, the simplicial complex for which $\sqrt{I}$ is the Stanley-Reisner ideal ([13, 1.3]).

The following is Takayama’s formula.

**Theorem 4.1** ([17, Theorem 1]). For every $a \in \mathbb{Z}^d$ and $i \geq 0$ we have

$$\dim_k H^n_m(R/I)_a = \dim_k \tilde{H}_{i-|G_a|-1}(\Delta_a(I),k)$$

The following is the main theorem of this section.

**Theorem 4.2.** Let $I$ be a monomial ideal and assume $H^n_m(R/I^n)$ is Noetherian for $n \gg 0$. Then one of the following holds

1. If $\tilde{H}_{i-1}(\Delta(I),k) \neq 0$, then $\indeg H^n_m(R/I^n) = 0$ for $n \gg 0$.
2. If $\tilde{H}_{i-1}(\Delta(I),k) = 0$ then $\lim_{n \to \infty} \frac{\indeg H^n_m(R/I^n)}{n} \geq 1$. 

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implies $\text{indeg } H_i$.

It suffices to show $(1)$ is automatically satisfied. Let $H_{i-1}(\Delta(I), k) \neq 0$, Theorem 4.1 implies $\text{indeg } H_m(R/I^n) = 0$ for every $n \gg 0$.

Now, assume $H_{i-1}(\Delta(I), k) = 0$. Fix $n \in \mathbb{N}$ and $a \in \mathbb{N}^d$ such that $|a| < n$. For every facet $F$ of $\Delta(I)$ we have $I_F \neq 1$, hence by degree reasons $x^a \not\in I_F$. It follows $\Delta_n(I^n) = \Delta(I)$ and then $H_m(R/I^n) = 0$ by Theorem 4.1. We conclude $\text{indeg } H_m(R/I^n) \geq n$, finishing the proof. □

The following example answers Question 1.1, (2) in the particular case that $\Delta(I)$ is a cycle graph $C_d$ for $d \geq 5$.

 Remark 4.3. The condition $H_{i-1}(\Delta(I), k) \neq 0$ in Theorem 4.2 (1) is automatically satisfied if $H_m(R/I^n)$ is Noetherian for some $n \in \mathbb{N}$ and $H_m(R/\sqrt{I}) \neq 0$ (see [5, 4.9]).

Example 4.4. Let $d \geq 5$ and $C_d$ the cycle graph of length $d$, i.e., the edges of $C_d$ are indexed by $\{i, i + 1\}$ for $1 \leq i \leq d$ where $\{d, d + 1\} = \{1, 1\}$. Let $I$ be the Stanley-Reisner ideal of $R$ associated to the complex $\Delta(I) = C_d$. Then

$$\lim_{n \to \infty} \frac{\text{indeg } H_m(R/I^n)}{n} = 1.$$ 

Since $\Delta(I)$ is connected, we have $H_0(\Delta(I)) = 0$. Then by Theorem 4.2 it suffices to show $$\limsup_{n \to \infty} \frac{\text{indeg } H_m(R/I^n)}{n} \leq 1.$$ For each $1 \leq i \leq d$, set $p_i = (\{x_j \mid j \neq i, i + 1\})$. Hence, $\text{Ass}(I) = \{p_1, \ldots, p_d\}$.

Note that

$$I_{(1)} = (x_3, \ldots, x_n) \cap (x_2, \ldots, x_{n-1}) = (x_2x_n, x_3, x_4, \ldots, x_{n-1})$$

which is a complete intersection. Likewise, $I_{(i)}$ is a complete intersection for every $1 \leq i \leq d$, therefore $\text{Proj } R/I$ is lci. Hence, $H^i_m(R/I^n)$ is Noetherian for every $n \in \mathbb{N}$ and we have

$$\hat{I}^n := (I^n : R m)^\infty = \cap_{i=1}^d p_i^n.$$ 

Fix $n \gg 0$ and let $a_n = (n - d + 4, 0, 1, \ldots, 1, 0) \in \mathbb{N}^d$, then one readily verifies $\Delta_n(\hat{I}^n)$ is the subcomplex of $\Delta(I)$ whose facets are $\{i, i + 1\}$ for $i \neq 2, n - 1$. Since $\Delta_n(\hat{I}^n)$ is disconnected, we have $\dim_k H^i_m(R/\hat{I}^n) = \dim_k H_0(\Delta_n, k) \neq 0$. We conclude

$$\text{indeg } H^i_m(R/\hat{I}^n) \leq |a_n| = n + 1.$$ 

Finally, the conclusion follows by noticing $H^i_m(R/I^n) \cong H^i_m(R/\hat{I}^n)$.

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