Analysis of Eigenvalue and Eigenfunction of Klein Gordon Equation Using Asymptotic Iteration Method for Separable Non-central Cylindrical Potential

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Abstract. Analysis of relativistic energy and wave function for zero spin particles using Klein Gordon equation was influenced by separable noncentral cylindrical potential was solved by asymptotic iteration method (AIM). By using cylindrical coordinates, the Klein Gordon equation for the case of symmetry spin was reduced to three one-dimensional Schrodinger like equations that were solvable using variable separation method. The relativistic energy was calculated numerically with Matlab software, and the general unnormalized wave function was expressed in hypergeometric terms.

1. Introduction
Klein Gordon equation is used to describe the characteristic of spin-zero microscopic particles in many physics fields, relativistic quantum mechanics, in particular[1,2]. Klein Gordon equation for various potentials has been solved using different methods such as Kratzer potential [3] and Trigonometric Poschl-Teller potential using Asymptotic Iteration Method (AIM) [4]. Also, Klein Gordon equation is solved by using Nikiforov Uvarof for Hulten Potential [5] and using factorization method for Makarov Potential [6].

In this paper, Klein Gordon equation for separable non-central cylindrical potential in the spin symmetry case reduces into three one-dimensional Schrodinger like equations with shape invariant potential in the centrifugal approximation scheme. The three-dimensional separable non-central cylindrical potential [7] consists of a radial part potential was using Hyperbolic Manning Rosen potential [8], a angular part potential is using Hyperbolic Scarf type II potential [7] and a axial part (z part) is using trigonometric Scarf potential [9], and each potential part is invariant shape potential. The Hyperbolic Manning Rosen potential has been used to study quark-gluon dynamic [5,10]. Scarf trigonometric and hyperbolic type II potentials have been used to study the electrodynamics [11] and crystal model in solid state [12]. The Schrodinger like an equation with invariant shape potential is reduced to the generalized hypergeometric differential equation that can be solved by AIM [13-15].

The paper is organized as follows. Klein Gordon equation and the separable non-central cylindrical potential are discussed in section 2 and AIM is briefly discussed in section 3. Results and discussion are presented in section 4 and conclusion in section 5.

2. Klein Gordon Equation with Separable Non–central Cylindrical Potential
The separable non-central cylindrical potential is a potential that its variable can be separated completely and it is given as [7].
where \( v, q, a, b \) and \( \omega \) are positive constants. This potential is applied to cylindrical coordinates which its Laplacian is expressed,

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2}{\partial z^2}
\]

with \( 0 < r \leq \infty, 0 \leq \theta \leq 2\pi \), and \( 0 < z < \infty \). The Klein Gordon equation is given by [4,16],

\[
(E - V(\hat{r})) \psi(r) = \left[ \hat{p}^2 \psi + (M_c^2 + S(\hat{r})) \right]
\]

where \( V(\hat{r}) \) and \( S(\hat{r}) \) are vector and scalar potentials, \( E \) and \( M_c \) are relativistic energy and rest mass, respectively. For spin symmetric case \( (S(\hat{r}) = V(\hat{r})) \), and by taking the vector potential is as half of separable non-central cylindrical potential, then by using equations (1-3), the associated Klein Gordon equation as follows,

\[
\psi(r, \theta, z) = \left( k^2 - M_c^2 \right) \psi(r, \theta, z)
\]

By using variable separation method equation (4) reduces to three one-dimensional Schrodinger like equations that are solvable by AIM.

3. Asymptotic Iteration Method

Asymptotic Iteration Method is applied to solve the second order differential equation which is expressed as [13-15],

\[
y_n^I(x) = \lambda_n(x) y_n(x) + s_n(x) y_n(x)
\]

where \( \lambda_n(x) \neq 0 \) and \( s_n(x) \) are coefficient of a differential equation. The parameter \( n \) is the quantum number. By writing the \((k+1)^{th}\) derivative of \( y \) which is obtained from equation (5) given as,

\[
y_{n+1}^{k+1}(x) = \lambda_n(x) y_n(x) + s_n(x) y_n(x)
\]

where \( \lambda_n(x) = \lambda_{n+1}(x) + s_{n+1}(x) + \lambda_{n+1}(x) \lambda_n(x) ; s_n(x) = s_{n+1}(x) + s_n(x) \lambda_n(x) ; k = 1,2,3... \). The Eigenvalue is obtained from the quantization condition given by

\[
\Delta_n(x) = \lambda_n(x) s_{n+1}(x) - \lambda_{n+1}(x) s_n(x) = 0
\]

The one-dimensional Schrodinger like an equation is reduced to hypergeometric type differential equation which given as,

\[
y_n^I(x) = 2\frac{x^m + 1}{1 - bx^{m+2}} \int_0^1 \frac{t + 1}{x} y_n(t) - \frac{ax^n}{1 - bx^{m+2}} y_n(x)
\]

The Eigenfunction which is obtained from equation (8) is expressed,

\[
y_n(x) = (-1)^{n} C \left( N + 1 \right) (\sigma - n - 1, n + 1; \alpha \beta) F_1 \left( -n, \rho + n; \sigma \beta \right)
\]

where \( (\sigma)_{n} = \frac{\Gamma(\sigma + n)}{\Gamma(\sigma)} \); \( \sigma = \frac{2t + N + 3}{N + 2} \); \( \rho = \frac{(2t + 1)b + 2a}{(N + 2)b} \); \( C \) is normalization constant and \( F_1 \) is a hypergeometric function. The wave function of Klein Gordon equation can be obtained by using equation (7-9) [13-15].

4. Result and Discussion

The equation (4) was solved by using variable separation method by setting wave function \( \psi(r, \theta, z) = R(r) P(\theta) Z(z) \), so we had three part of second order differential equations, axial, angular, and radial parts as,
with \( \lambda_1 \) and \( \lambda_2 \) is variable separation constants. Equations (10-12) were solvable using AIM since all potentials in these three equations were shape invariant potentials.

### 4.1. Solution of The Axial Part of Klein Gordon Equation

By setting transformation variable of \( \sinh \omega_z = i(1-2u_z) \) and the new wave function of \( Z = u_z^\omega(1-u_z)^\nu f(u_z) \) in equation (10) which was reduced to AIM type equation by dividing it with \( u_z(1-u_z) \), so we obtained,

\[
f'(u_z) + \left[ \frac{2\alpha_z + \frac{1}{2}}{u_z(1-u_z)} \right] f(u_z) + \left[ \frac{\lambda_1 - (\alpha_z + \beta_z)^2}{u_z(1-u_z)} \right] f(u_z) = 0
\]

where,

\[
\eta_x = \frac{(E - M_x)}{\omega_z^2} ; \quad \lambda_x = \frac{\lambda_1}{\omega_z^2} ; \quad \alpha_z = \frac{1}{2} \sqrt{\eta_x \left[ (b - i(a + \frac{1}{2}))^2 + \frac{1}{4} \right] + \frac{1}{4}} ; \quad \beta_z = \frac{1}{2} \sqrt{-\eta_x \left[ (b + i(a + \frac{1}{2}))^2 + \frac{1}{4} \right] + \frac{1}{4}}
\]

By comparing equation (5) and equation (13), yields,

\[
\lambda_x = -\frac{2\alpha_z + \frac{1}{2}}{u_z} + \frac{2\beta_z + \frac{1}{2}}{1-u_z} ; \quad \frac{(\alpha_z + \beta_z)^2 - \lambda_x}{\alpha_z} = \frac{(\alpha_z + \beta_z)^2 - \lambda_x}{(1-u_z)}
\]

Using equations (6-7) and (13), we obtained Eigenvalue function, was given as

\[
\lambda_x = \omega_x^2 (\alpha_x + \beta_x + n_x)^2
\]

The \( n_x \) is an axial quantum number. The axial parts of wave functions were obtained with equations (7-9), (13), \( \sinh \omega_z = i(1-2u_z) \) and \( Z = u_z^\omega(1-u_z)^\nu f(u_z) \), we got wave function for \( n_x=0 \) and \( n_x=1 \) as follows,

\[
Z_x(z) = C \left( \frac{1-i\sinh \omega_z}{2} \right)^n \left( \frac{1+i\sinh \omega_z}{2} \right)^n
\]

\[
Z_x(z) = -C \left( \frac{1-i\sinh \omega_z}{2} \right)^n \left( \frac{1+i\sinh \omega_z}{2} \right)^n \left( \frac{2\alpha_z + \frac{1}{2}}{2} \right) \left( \frac{(2\alpha_z - 2\beta_z - 1)(i\sinh \omega_z - 1)}{2} \right)
\]

### 4.2. Solution of The Angular Part of Klein Gordon Equation

By applying \( \cos \omega_\theta = (1-2u_\theta) \) and \( P(\theta) = u_\theta^\nu(1-u_\theta)^\nu f(u_\theta) \) in equation (11), we got an equation that had similar form with equations (5) and (8), was given as,

\[
f'(u_\theta) + \left[ \frac{2\alpha_\theta + \frac{1}{2}}{u_\theta(1-u_\theta)} \right] f(u_\theta) + \left[ \frac{\lambda_1 - (\alpha_\theta + \beta_\theta)^2}{u_\theta(1-u_\theta)} \right] f(u_\theta) = 0
\]

where,
\[ \eta_s = \frac{(E - M_s)}{\alpha_s^2}; \beta_s = \frac{\lambda_s}{\alpha_s^2}; \alpha_s = \frac{1}{2} \sqrt{\eta_s \left( \frac{1}{4} - \frac{\lambda_s}{\alpha_s^2} \right)} + \frac{1}{8}; \beta_s = \frac{1}{2} \sqrt{\eta_s \left( \frac{1}{4} - \frac{\lambda_s}{\alpha_s^2} \right)} + \frac{1}{4} \] (20)

By comparing equation (5) and equation (19), we obtained,
\[ \lambda_s = \frac{-2\alpha_s + 2\beta_s}{\eta_s}; \beta_s = \frac{(\alpha_s + \beta_s)}{\eta_s} - \lambda_s \] (21)

Using equation (6-7) and (19), we obtained the Eigenvalue, as follows,
\[ \lambda_{z_s} = \omega_s^2 \theta_s + \beta_s + n\theta^2 \] (22)

where, \( n\theta \) is an angular quantum number. By comparing equations (8) and (19), equations (9-10) and also it was substituted into \( P(\theta) = u_{\theta^0} \frac{\alpha_s}{(1 - u_{\theta^0})} \frac{f(u_{\theta^0})}{\cos \omega_{\theta^0} (1 - 2u_{\theta^0})} \), we had the wave functions of the angular part for \( n_s = 0 \) and \( n_s = 1 \) as,
\[ P_1(\theta) = c \left( \frac{1 - \cos \omega_{\theta^0}}{2} \right)^{\lambda_{z_s}} \left( \frac{1 + \cos \omega_{\theta^0}}{2} \right)^{\beta_s} \] (23)
\[ P_\lambda(\theta) = -c \left( \frac{1 - \cos \omega_{\theta^0}}{2} \right)^{\lambda_{z_s}} \left( \frac{1 + \cos \omega_{\theta^0}}{2} \right)^{\beta_s} \left( 2\alpha_s + 1 \right) \] (24)

4.3. Solving The Radial Part of Klein Gordon Equation

By inserting \( R(r) = Q(r)/\sqrt{r} \), \( \frac{1}{r^\alpha} = \frac{\omega_s^2}{(\alpha_s)^2} \) \[14\] and \( \cosh \omega\omega = 1 - 2u_{\omega} \) in equation (14), then it must be reduced to AIM type equation by setting \( Q(r) = u_{\omega^0} \frac{\alpha_s}{(1 - u_{\omega^0})} \frac{f(u_{\omega^0})}{\cos \omega_{\omega^0} (1 - 2u_{\omega^0})} \) and by dividing it with \( u_{\omega^0} (1 - u_{\omega^0}) \), then we obtained,
\[ f'(u_{\omega^0}) + \left( \frac{2\alpha_s + 1 - (2\alpha_s + 2\beta_s)\omega_{\omega^0}}{u_{\omega^0} (1 - u_{\omega^0})} \right) f(u_{\omega^0}) + \left( \frac{\frac{E - (\alpha_s + \beta_s \omega_{\omega^0})}{u_{\omega^0} (1 - u_{\omega^0})}}{\beta_s} \right) = 0 \] (25)

where,
\[ \lambda_{s_r} = (E + M_{s_r})/(v - 1); \eta_s = (E + M_{s_r})/(v - 1); \] \[ \lambda_{s_r} = (E + M_{s_r})/(v - 1); \eta_s = (E + M_{s_r})/(v - 1) \] (26)

By comparing equation (5) and equation (19), we got
\[ \lambda_s = \frac{-2\alpha_s + 2\beta_s}{\eta_s}; \beta_s = \frac{(\alpha_s + \beta_s)}{\eta_s} - \lambda_s \] (27)

Using equation (6-7) and equation (19), we obtained the Eigenvalue, yields,
\[ \lambda_{s_1} = \alpha_s + \beta_s + n_s \] (28)

From the equation (29), we obtained the relativistic energy equation by using equations (26) and (28). The relativistic energy equation of Klein Gordon equation for separable non-central cylindrical potential was,
\[ (E + M_{r^2}) = \alpha_s \] (29)

The \( n_r \) is an axial quantum number. The relativistic energy of equation (29) could not calculate analytically. Therefore, to get the equation (29), we had to apply Matlab software to calculate numerically. The result was shown in Table 1.
Table 1. The relativistic energy for \( n_z = n_r = 1 \) and \( n_q = 0 \) with various parameters of a separable non-central cylindrical potential.

| Value of constant \( a, b, q, \nu \) | \( E (a) \) | \( E (b) \) | \( E (q) \) | \( E (\nu) \) |
|--------------------------------------|--------|--------|--------|--------|
| 0                                   | -0.9898 | -0.9896 | -0.9981 | -0.9895 |
| 1                                   | -0.9895 | -0.9895 | -0.9985 | -0.9895 |
| 2                                   | -0.9887 | -0.9895 | -0.9984 | -0.9887 |
| 3                                   | -0.9870 | -0.9872 | -0.9993 | -0.9869 |
| 4                                   | -0.9864 | -0.9868 | -0.9996 | -0.9600 |
| 5                                   | -0.9852 | -0.9861 | -0.9997 | -0.9812 |

Table 1 shows that the relativistic energy increased due to the increase a function of \( a, b, \) and \( \nu \). The relativistic energy decreased by the increase a function of \( q \). The radial wave functions were obtained with equation (8-10) and (19), \( Q(r) = u_r^n (1 - u_r)^{\beta} f(u_r) \) and together with \( \coth \omega r = (1 - 2u_r) \), we had wave functions for \( n_z = 0 \) and \( n_r = 1 \), was given as

\[
Q_r(r) = C \left( \frac{1 - \coth \alpha r}{2} \right)^n \left( \frac{1 + \coth \alpha r}{2} \right)^{\beta} \left( \frac{1 - \coth \beta r}{2} \right) \left( \frac{1 + \coth \beta r}{2} \right) \left( \frac{1 - \coth \gamma r}{2} \right) \left( \frac{1 + \coth \gamma r}{2} \right)
\]

The total wave function was obtained by multiplying equations for ground state wave function (18), (23), (30) and multiplication of equations (19), (24), (31) for the first excited state wave function.

5. Conclusion

Asymptotic iteration method (AIM) was used to analyze the solution Three-dimensional Klein Gordon equation with separable non-central cylindrical potential that was reduced to radial, angular, and axial of Klein Gordon equation by the use of cylindrical coordinates. The relativistic energy equation was obtained from the solution of radial Klein Gordon equation that was calculated numerically by using Matlab software. The relativistic energy increased with the increase of \( a, b \) and \( \nu \), but it decreased by the increase of \( q \). The wave function was expressed by hypergeometric terms.

Acknowledgement

This research was partly supported by Hibah PUT-MRG 2017 Sebelas Maret University.

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