Statistical Ensembles with Fluctuating Extensive Quantities

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We suggest an extension of the standard concept of statistical ensembles. Namely, we introduce a class of ensembles with extensive quantities fluctuating according to an externally given distribution. As an example the influence of energy fluctuations on multiplicity fluctuations in limited segments of momentum space for a classical ultra-relativistic gas is considered.

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Successful application of the statistical model to hadron production in high energy collisions (see, e.g., recent papers \(^1\) and references therein) has stimulated investigations of properties of statistical ensembles of relativistic hadronic gases. Whenever possible, one prefers to use the grand canonical ensemble (GCE) due to its mathematical convenience. The canonical ensemble (CE) \(^2\) should be applied when the number of carriers of a conserved charges is small (of the order of 1), such as strange hadrons \(^3\), antibaryons \(^4\), or charmed hadrons \(^5\), in otherwise large systems. The micro-canonical ensemble (MCE) \(^6\) has been used to describe small systems (with total number of produced particles less than or equal to 10) with additionally fixed energy, and momentum, e.g. elementary particle collisions or annihilation. In these cases, calculations performed in different statistical ensembles yield different results. Hence, ensembles are not equivalent, and systems are ‘far away’ from the thermodynamic limit (TL).

Measurement of hadron multiplicity distributions \(P(N)\) in relativistic nucleus-nucleus collisions opens another interesting field of investigations. The average number of produced hadrons ranges from \(10^2\) to \(10^4\), and mean multiplicities (of light hadrons) obtained within GCE, CE, and MCE approach each other. One refers here to the thermodynamical equivalence of statistical ensembles and uses the GCE for fitting experimentally measured mean multiplicities. However, the number of particles fluctuates event-by-event. These fluctuations are usually quantified by the ratio of variance to mean value of a multiplicity distribution \(P(N)\), the scaled variance, and are a subject of current experimental activities. In statistical models there is a qualitative difference in the properties of mean multiplicity and scaled variance of multiplicity distributions. It was recently found \(^6\) that even in the TL corresponding results for the scaled variance are different in different ensembles. Hence the equivalence of ensembles holds for mean values in the TL, but does not extend to fluctuations.

Statistical mechanics is usually formulated through the following steps. Firstly, one fixes the system’s extensive quantities: volume \(V\), energy \(E\), momentum \(\vec{P}\), and conserved charges \(\{Q_i\}\). Secondly, one postulates that all microstates have equal probability of being realized. This defines the MCE. The CE introduces temperature \(T\). Each set of microstates with fixed energy \(E\) (a macrostate) is weighted with the Boltzmann factor \(e^{-E/T}\). The probability to find a macrostate with energy \(E\) in the CE is then proportional to the number of all microstates with energy \(E\) times the Boltzmann factor \(e^{-E/T}\). To define the GCE, one makes a similar construction for conserved charges \(\{Q_i\}\) and introduces chemical potentials \(\{\mu_i\}\). Canonical or grand canonical observables can then be obtained by averaging over the CE energy distribution or the GCE (joint-) distribution of both energy and conserved charges.

Fluctuations in statistical systems, e.g., multiplicity distributions \(P(N)\) in relativistic gases \(^6\), are sensitive to conservation laws obeyed by the system, and therefore to fluctuations of extensive quantities. For calculation of multiplicity distributions, the choice of statistical ensemble is then not a matter of convenience, but a physical question. Fluctuations of extensive quantities \(\tilde{A} \equiv \langle V, E, \vec{P}, \{Q_i\} \rangle\) around their average values depend not on the system’s physical properties, but rather on external conditions. One can imagine a huge variety of these conditions, thus, MCE, CE, GCE, or pressure ensembles \(^10\) are only some special examples.

A more general statistical ensemble can be defined by an externally given distribution of extensive quan-
tities, $P_\alpha(\hat{A})$. All microstates with a fixed set $\hat{A}$ are taken to be equiprobable. Thus, the probability $P_{mce}(N; \hat{A})$ of finding the system in a macrostate with fixed $\hat{A}$ and additionally fixed multiplicity $N$ is given by the ratio of the number of microstates with fixed $N$ and $\hat{A}$ to the number of microstates with fixed $\hat{A}$. The construction of multiplicity distributions in such an ensemble proceeds in two steps. Firstly, the MCE multiplicity distribution, $P_{mce}(N; \hat{A})$, at fixed values of the extensive quantities $\hat{A}$ is calculated. Secondly, this result is averaged over the external distribution $P_\alpha(\hat{A})$,

$$P_\alpha(N) = \int d\hat{A} P_\alpha(\hat{A}) P_{mce}(N; \hat{A}).$$ \hspace{1cm} (1)

The ensemble defined by Eq. (1), the $\alpha$-ensemble, includes the standard statistical ensembles as particular cases.

Let us illustrate above statements for a simple system of non-interacting massless particles, neglecting the effects of quantum statistics (Boltzmann approximation). For a particular realization of Eq. (1) we choose:

$$P_\alpha(N) = \int dE P_\alpha(E) P_{mce}(N; E)$$ \hspace{1cm} (2)

to calculate the multiplicity distribution $P_\alpha(N)$ in the presence of an energy distribution $P_\alpha(E)$. We will firstly discuss the MCE multiplicity distribution $P_{mce}(N; E)$ and then solve the integral (2) for a particular choice of $P_\alpha(E)$ in the large volume limit.

The MCE multiplicity distribution is given by [8],

$$P_{mce}(N; E) = \frac{1}{Z_{mce}(V, E)} \frac{E^{-1} x^N}{(3N-1)! N!},$$ \hspace{1cm} (3)

where $x = g V E^3/\pi^2$, and $g$ is the particle’s degeneracy factor. The MCE partition function $Z_{mce}(V, E)$ is defined by the normalization condition, $\sum_{N=1}^{\infty} P_{mce}(N; E) = 1$. In the TL, $x \gg 1$ in Eq. (3), one finds from the raw moments $\langle N^k \rangle \equiv \sum_{N=1}^{\infty} N^k P_{mce}(N; E)$:

$$\langle N \rangle \approx \left(\frac{x}{27}\right)^{1/4}, \quad \omega_{mce} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} \approx \frac{1}{4}. \hspace{1cm} (4)$$

Additionally it follows from the equivalence of statistical ensemble that GCE and MCE values for average multiplicity are equal to each other, $\langle N \rangle_{mce} \equiv \overline{N} = g V T^3/\pi^2$, where the temperature $T$ of the GCE is found from $E = \overline{E} = 3g V T^4/\pi^2$. Momentum spectra in the MCE also converge to GCE Boltzmann spectra under this limit. Multiplicity fluctuations, expressed by the scaled variances, are however different, $\omega_{gce} = 1$ and $\omega_{mce} = 1/4$. In the TL the MCE distribution (4) converges to a Gaussian,

$$P_{mce}(N; E) \approx \frac{1}{\sqrt{2\pi \omega_{mce}(N)}} \exp \left[ -\frac{(N - \langle N \rangle)^2}{2\omega_{mce}(N)} \right].$$ \hspace{1cm} (5)

The Normal form of distributions is a general feature of all statistical ensembles in the TL [11].

The GCE energy distribution is equal to:

$$P_{gce}(E) = \frac{1}{Z_{gce}(V, T)} \exp \left( -\frac{E}{T} \right) Z_{mce}(V, E),$$ \hspace{1cm} (6)

where the GCE partition function $Z_{gce}(V, T)$ is defined by the normalization condition, $\int dE P_{gce}(E) = 1$. We consider a system with fixed large volume $V$. As an illustrative example we will use the following asymptotic form of energy distribution,

$$P_\alpha(E) = \frac{1}{\sqrt{2\pi \omega_E \alpha^2 T}} \exp \left( -\frac{E - \overline{E}}{2\omega_E \alpha^2 T} \right),$$ \hspace{1cm} (7)

where $\overline{E} = 3g V T^4/\pi^2$ is the GCE energy expectation value and $\omega_E = (\overline{E}^2 - \overline{\delta})/\overline{E} = 4T$ is the scaled variance of GCE energy fluctuations. This choice for $P_\alpha(E)$ results in a simple correspondence to the GCE and MCE in the large volume limit. In Eq. (7), $\alpha$ is a dimensionless tuneable parameter for the width of the distribution. In the MCE limit $\alpha \rightarrow 0$, Eq. (7) becomes a Dirac $\delta$-function, $\delta(E - \overline{E})$. For $\alpha = 1$, Eq. (7) results in the GCE energy fluctuations [9] in the TL [11]. The physical interpretation of the GCE energy fluctuations corresponds to an ‘infinite heat bath’. The values, $0 < \alpha < 1$, would correspond to a ‘large, but finite’ heat bath. The case of $\alpha > 1$ we would like to denote as ‘strong’ energy fluctuations.

Lastly we note that the MCE distribution $P_{mce}(N; E)$ can be conveniently presented in terms of the GCE distributions [11], $P_{mce}(N; E) \equiv P_{gce}(N, E)/P_{gce}(E)$. At fixed volume $V$, the GCE distribution $P_{gce}(E)$ is given by Eq. (7) with $\alpha = 1$, while the joint GCE energy and multiplicity distribution $P_{gce}(N, E)$ is given by a bivariate normal distribution in the large volume limit [11],

$$P_{gce}(N, E) \approx \frac{1}{2\pi V \sqrt{\sigma_E \sigma_N}} \exp \left[ -\frac{1}{2V(1 - \delta^2)} \times \left( \frac{(\overline{E})^2 - 2\Delta E \Delta N}{\sigma_E^2} + \frac{(\overline{N})^2}{\sigma_N^2} \right) \right],$$ \hspace{1cm} (8)

where $\Delta E = E - \overline{E}$, $\Delta N = N - \overline{N}$, $\overline{E} = V \kappa_1^E$, $\overline{N} = V \kappa_1^N$, $\sigma_E^2 = \kappa_2^E \omega_E$, $\sigma_N^2 = \kappa_2^N \omega_{gce}$.
and the correlation coefficient \( \delta = \sigma_{EN}/(\sigma_E\sigma_N) \), with \( \sigma_{EN} = \kappa_2^{E,N} \). The relevant cumulants \( \kappa \), for both \( 4\pi \)-integrated particle yields and for yields in limited momentum windows \( \Delta p \) for the ultra-relativistic Boltzmann gas are given in the Appendix. One finds then for the multiplicity distribution:

\[
P_\alpha(N) = \int dE \, P_\alpha(E) \frac{P_{gce}(N,E)}{P_{gce}(E)} \\
\approx \frac{1}{(2\pi\omega_\alpha N)^{1/2}} \exp \left[ -\frac{(N - N^\alpha)^2}{2\omega_\alpha N} \right],
\]

where for our system \( \omega_{gce} = 1 \) and \( \omega_{mce} = 1/4 \). As it can be expected, \( \omega_\alpha = \omega_{mac} \) for \( \alpha = 0 \), and \( \omega_\alpha = \omega_{gce} \) for \( \alpha = 1 \). It also follows, \( \omega_{mce} < \omega_\alpha < \omega_{gce} \) for \( 0 < \alpha < 1 \), and \( \omega_\alpha > \omega_{gce} \) for \( \alpha > 1 \). The \( \alpha \)-ensemble defined by Eqs. (7) presents an extension of the GCE (\( \alpha = 1 \)) and MCE (\( \alpha = 0 \)) to a more general energy distribution. Different values of \( \alpha \) correspond, by construction, to the same expectation values of energy \( \bar{E} \), and multiplicity \( \bar{N} \). Hence in the TL all ensembles defined by Eq. (7) are thermodynamically equivalent. Energy and multiplicity fluctuations are however different.

As a next step we want to discuss the effect of energy fluctuations, Eq. (7), on multiplicity fluctuations in limited segments of momentum space for a classical ultra-relativistic gas in the TL. The results for the scaled variances, \( \omega_\alpha^{\Delta p} \equiv \langle (N_\Delta^{\alpha})^2 \rangle / \langle N^{\alpha} \rangle^2 \), in different momentum bins \( \Delta p \) are shown in Fig. 1 for several values of parameter \( \alpha \). As in Ref. [12], each momentum bin \( \Delta p = [p_1,p_2] \) contains the same fraction \( q \) of the total average multiplicity, \( q \equiv \langle N_\Delta^{\alpha} \rangle / \langle N^{\alpha} \rangle \). In the MCE (\( \alpha = 0 \)), multiplicity fluctuations are essentially suppressed with respect to the GCE (\( \alpha = 1 \)). Global energy conservation in the MCE introduces correlations in momentum space [12]. The larger the fraction of the total energy in a given momentum bin, the stronger is the MCE suppression effect (\( \alpha = 0 \) in Fig. 1). For \( 0 < \alpha < 1 \) the suppression effects become weaker. \( \alpha = 1 \) corresponds to the GCE results for multiplicity fluctuations: \( \omega_\alpha = 1 \), both in full momentum space, and in different momentum bins. For ‘strong’ energy fluctuations, \( \alpha > 1 \), we find an increase of multiplicity fluctuations with increasing bin momentum, as shown in Fig. 1.

In order to make a quantitative comparison with present and future data on multiplicity fluctuations in relativistic nucleus-nucleus collisions one needs to extend essentially the formulation considered in this letter. Please note that Eqs. (4-7) can be readily generalized to more complicated cases of non-zero particle masses, quantum statistics, many particle species, several conserved charges, and multiplicity fluctuations in limited segments of momentum space. In all these cases the expressions for \( P_{mce}(N;\vec{A}) \) and \( P_{gce}(\vec{A}) \) are already obtained [11, 12]. The MCE multiplicity distributions in systems with full hadron-resonance spectrum and quantum statistics effects were presented in Ref. [11]. The role of momentum conservation was discussed in Ref. [12]. The next step would be an extension of this formulation to an external distributions \( P_\alpha(\vec{A}) = P_\alpha(V,E,\vec{P},B,S,Q) \), e.g., taking into account fluctuations of energy \( E \), momentum \( \vec{P} \), and three Abelian charges - baryon number \( B \), strangeness \( S \), and electric charge \( Q \). This is to be done according to Eq. (1). Fluctuations and correlations of the extensive quantities presented by the distribution \( P_\alpha(V,E,\vec{P},B,S,Q) \) could then be con-
nected to measured fluctuations of hadron multiplicities in limited segments of momentum space. This will be a subject of future studies.

In this letter we suggested to extend the concepts of statistical ensembles and introduced the $\alpha$-ensemble defined by an external distribution of extensive quantities. The key assumption used for calculation of multiplicity distributions is the ‘equiprobability’ of all microstates with the same set of extensive quantities. We have discussed a simple example of a gas composed of classical massless particles to demonstrate the effects of energy fluctuations on multiplicity fluctuations in full momentum space and in limited segments of momentum space. Measurement of event-by-event multiplicity fluctuations in different momentum bins correspond to current experimental studies of hadron production in relativistic nucleus-nucleus collisions. We believe also that the concept of statistical ensembles with fluctuating extensive quantities may be appropriate in other situations too. In fact, in all cases when fluctuations of extensive quantities are a subject of interest and can be measured experimentally.

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Appendix

For the ideal Boltzmann gas of massless particles the momentum distribution is $f(p) = \exp(-p/T)$, and the cumulants are:

\[ \kappa_1^N = \frac{g}{2\pi^2} \int_0^\infty dp \ p^2 \ f(p) = \frac{gT^3}{\pi^2}, \]

\[ \kappa_1^E = \frac{g}{2\pi^2} \int_0^\infty dp \ p^3 \ f(p) = 3 \frac{gT^4}{\pi^2}, \]

\[ \kappa_2^{E,N} = \frac{g}{2\pi^2} \int_0^\infty dp \ p^4 \ f(p) = 12 \frac{gT^5}{\pi^2}. \]

Additionally, in Boltzmann approximation, $\kappa_2^{E,N} = \kappa_2^E$ and $\kappa_2^{N,N} = \kappa_1^N$. The cumulants in the momentum segments $\Delta p = [p_2, p_1]$ are:

\[ \left( \kappa_1^N \right)_{\Delta p} = \frac{g}{2\pi^2} \int_{p_1}^{p_2} dp \ p^2 \ f(p) = \frac{gT^3}{\pi^2}, \]

\[ \left( \kappa_1^E \right)_{\Delta p} = \frac{g}{2\pi^2} \int_{p_1}^{p_2} dp \ p^3 \ f(p) = 3 \frac{gT^4}{\pi^2}, \]

\[ \left( \kappa_2^{E,N} \right)_{\Delta p} = \frac{g}{2\pi^2} \int_{p_1}^{p_2} dp \ p^4 \ f(p) = 12 \frac{gT^5}{\pi^2}. \]

Additionally, in the Boltzmann approximation $(\kappa_2^{N,N})_{\Delta p} = (\kappa_1^N)_{\Delta p}$.

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