An analytic solution of the uncoupled multi-agent model and its benefit through optimal control system with attractor and repellant

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Abstract. This paper gives the analytic solution of the uncoupled multi-agent model. This solution describes the optimal path of each agent. In this exposition, the optimal control approach is used to model uncoupled multi-agent swarm with attractor and repellant. The special functional cost contains repellent cost functional is used to guarantee each agent never collides one to the others. The attractor term in the special functional cost makes each agent never move far away to the others. This paper also gives the benefit of the agents moves in the multi-agent system.

1. Introduction
In recent years, also as motivation in paper writing, multi-agent appears in many real cases. Multi-agent in the natural phenomenon is recognized as a swarm. Swarm is a natural phenomenon happening in groups of animals [1-3]. The swarming phenomenon is also happening in a group out of animals. For example, the phenomenon happens in unmanned aerial vehicles, robots, and airplanes.

Peoples modeled this phenomenon, for example, Breder [4] wrote a model of animal aggregation with repulsive force and attractive force. Warburton and Lazarus [5] studied the Tendency-distances model of social cohesion in animal groups. Mogilner and Keshet [6] described continuum models for swarming behavior based on non-local interactions. The interactions are assumed to influence the velocity of the organism. The model consists of integral-differential advection-diffusion equations. Keshet [7] proposed mathematical models of swarming and social aggregation. He surveyed some of the problems connected with the aggregation of the social organisms and indicate some mathematical and model challenges. In her paper, she modeled a swarm phenomenon, but she did not use a control approach. Gazi and Passino [8] provided a model of an aggregation swarm. They specify an "individual-based" continuous-time model for swarm in n-dimensional space and studied its stability properties. Topaz and Bertozzi [9] constructed a continuum for the motion of biological organisms experiencing social interactions and studied its pattern-forming behavior. The model took form of conservation law in two spatial dimensions. The social interactions are modeled in the velocity term, which is nonlocal in the population density and includes a parameter that controls the interaction length scale. Shi, Wang, and Chu [10] considered an anisotropic swarm model with an attraction/repulsion function and studied its aggregation properties. Pranoto, Tjahjana, and Muhammad [11-12] wrote a simulation of swarm modeling through optimal control and simulation of swarm modeling through bilinear optimal control.

The examples of multi-agent utilization, the reader canthe consult in [13] and [14] for the swarm of a flying robot. The other paper which covers multi-agent in cooperation among flying robots the reader who interesting can read [15] and [16]. The other researches in flying robot as multi-agent control can
be viewed in [17], [18] and [19]. If the reader wants to know the multi-agent system that the agents are quadrotor can reference [20] and [21]. The utilization of multi-agent in surveillance can be referenced in [22] and [23]. The research paper about the utilization of multi-agent in fuel-saving can consult in [24] and [25]. Research papers ([26]-[33]) are the newest papers that exposed multi-agent, but these papers did not consider the analytic solution of multi-agent.

Furthermore, since in some real cases the swarm leader does not exist, in this paper, we expose the model without swarm leader. The absence of the swarm leader is shown by the equity of the agents. Each agent cannot control the others. Since among the swarm agents, one and the others never collide, we put it in the cost functional. It penalizes all members if two or more members move too close to each other. We also put the attractor term to guarantee all members the away far from the others.

The main contribution of this paper, we propose an analytic solution a model of uncoupled multi-agent swarm system with attractor and repellant using an optimal control approach with special cost functional. The special cost functional which used in this paper is not standard cost functional as usually in optimal control course, but the cost functional contains repellant and attractor term. The analytic solution physically can be viewed as an optimal path for each agent, so the next section will explore the optimal path. The other main contribution of this paper is a theorem to show that the control used in multi-agent is lower than agents move in solo. Since an optimal solution can be viewed as an Optimal path then in the next section the optimal path will be discussed.

2. Optimal path

In general, the path of the i-th agent is governed by the vector function $x_i(t)$ in $\mathbb{R}^n$. Each agent controls itself through the piecewise-continuous control function $u_i(t)$. The initial formation of the swarm is described by the initial condition $x_i(0) = x^0$. The final formation of the swarm is described by the final condition $x_i(T) = x^1$. The cost is defined by $J = \int_0^T g(x_1, x_2, \ldots, x_m) + h(u_1, u_2, \ldots, u_m) \, dt$. We seek an optimal control $u_i \in \mathcal{U}$ such that the state of the system can be steered from $x^0$ to $x^1$ in time $T$ with the minimum value of $J$. The trajectory $x(t)$ related to the optimal control is the optimal path, and the corresponding value of $J$ is the optimal cost. We first define an extra state variable $x_0$ by state equation $\dot{x}_0 = f(x, u), x_0(0) = 0$. Thus, the cost is given by $J = x_0(T)$. We next introduce the extended state vector $\tilde{x}$ of dimension $nm + 1$, whose components are $x_0, x_1, x_2, \ldots, x_m$. If we define the extended vector $\tilde{f}$ similarly, the state equations can be written in the form $\dot{\tilde{x}} = \tilde{f}(x, u)$. The Hamiltonian $H(x, p, u)$ is defined by $H = \tilde{p}^T \dot{\tilde{x}} = \sum_{i=1}^n p_i f_i$, where $\tilde{p}$ is the extended co-state vector of dimension $m + 1$. Hamilton’s equation $\dot{\tilde{p}} = -\frac{\partial H}{\partial \tilde{x}}$ and $\dot{\tilde{x}} = \frac{\partial H}{\partial \tilde{p}}$. Since $H$ does not depend on $x_0$, these equations can be written in the form $\dot{p}_0 = 0, \dot{p}_i = -\frac{\partial H}{\partial \tilde{x}_i}, i = 1, 2, \ldots, m$. The Pontryagin Maximum Principle can now be stated as follows. Suppose that the problem has an optimal solution with optimal control $u^*$. Then the following conditions must hold

- $p_0 = 1$
- $u^*$ is the control function for which $H(x, p, u)$ reaches its infimum for all $u$
- The co-state equations have a solution $\tilde{p}^*$ and the state equation a solution $x^*$ which takes the value $x^0$ at $t = 0$ and $x^1$ at $t = T$
- The Hamiltonian function along the optimal trajectory and this constant is zero if the terminal is free, that is $H(x^*, \tilde{p}^*, u^*) = \text{constant}$ if $T$ is fixed and $H(x^*, \tilde{p}^*, u^*) = 0$ if $T$ is free and positive.

After the optimal path explored in this paper, the multi-agent model and its analytic solution will be considered in the next section.

3. Uncoupled multi-agent swarm with attractor and repellant

The controlling problem of multi-agent is brought to an optimal control problem. The group of agent’s dynamics is modeled by many decoupled control systems as follows

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t))$$
\[ \dot{x}_i(t) = f_i(x_i(t), u_i(t)) \]  
\[ \dot{x}_k(t) = f_k(x_k(t), u_k(t)). \]

Thus, the \( i \)-th agent is governed by \( \dot{x}_i(t) = f(x_i(t), u_i(t)) \). The initial and boundary conditions of the system are given by

\[ x_i(0) = s_i, \quad x_i(T) = q_i. \]

The common objective of the agents translated in a cost function defined as follows

\[ J = \int_0^T h(u_1, u_2, \ldots, u_n) + g_1(x_1, x_2, \ldots, x_n) + g_2(x_1, x_2, \ldots, x_n) dt. \]

The cost functional \( J \) consists of three terms, namely \( g_1, g_2 \) and \( h \). The term \( g_1 \) represents the fact that two agents or more cannot collide. In other words, the term \( g_1 \) contributes to increasing the cost if two agents or more move close to each other. The term \( g_2 \) is called the repellent term. The term \( g_2 \) represents the fact that two agents cannot move far from each other. The term \( g_2 \) is called the attractor term. The term \( h \) represents the total cost of the controls or energy used by the group of agents.

In this paper, we consider the model of the uncoupled multi-agent swarm with \( k \) members or agents as follows

\[ \dot{x}_i = A_i x_i + B_i u_i \]
\[ \dot{x}_2 = A_2 x_2 + B_2 u_2 \]
\[ \vdots \]
\[ \dot{x}_k = A_k x_k + B_k u_k, \]

for \( i = 1, 2, \ldots, k \). Model (4) is a special form from the model (1). In the model (4), the position of \( i \)-the agent is symbolized by \( x_i \), the symbol \( \dot{x}_i \) denote for differentiation \( x_i \) with respect to \( t \), \( A_i \) and \( B_i \) are contents. Control of the \( i \)-th agent is symbolized by \( u_i \). Since an agent just controls itself and cannot control the other agent, model (4) is an uncoupled multi-agent model. The initial and boundary conditions are given as

\[ x_i(0) = s_i, x_i(T) = q_i, i = 1, \ldots, k. \]

In (5), 0 is the initial time, so \( x_i(0) \) can be viewed as initial position the \( i \)-th agent, and \( T \) is fixed final time then \( x_i(T) \) can be viewed as final position the \( i \)-th agent. Consider the model (4) and conditions (5), \( x_i \) is the position of the \( i \)-th swarm agent, \( s_i \) is the initial position of the \( i \)-th swarm agent, and \( q_i \) is the final position of the \( i \)-th swarm agent. Since in this paper we will use optimal control approach and the most important things in optimal control is cost functional then the special cost functional which used in this paper is defined by

\[ J = -\frac{1}{2} \int_0^T \sum_{i=1}^{k} \delta u_i^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{\gamma}{||x_i-x_j||^2} + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \mu ||x_i-x_j||^2 dt. \]

The cost functional equation (6) is a special form from (3). In cost functional equation (6), the first summand represents the cost of the controls used and \( \delta \) is constant. The second summand represents the penalty if the members move too close to each other and \( \gamma \) is repellant constant. The third summand represents the attractor function between different two members and \( \mu \) is attractor constant. The solutions of the system of differential equations are \( x_i(t), \ i = 1, 2, \ldots, k \) and \( x_i(t) \) describes an equation of optimum trajectory or optimum path of the \( i \)-th agent. The main problem in this paper is with initial and boundary conditions (5) we want to minimize the cost function which satisfies the system (4). Since we are going to use the Pontryagin Maximum Principle, so the minimization of a cost function is equivalent to maximizing the negative of cost functional. It is the reason the cost functional (6) use the negative term. By the Pontryagin Maximum Principle and the repellant cost, the path of the agents never collide.
It is similar to the case of natural swarm phenomena where the agents never collide. Also, the fact that
an agent cannot control the others is described by the independent equation of \( x_i(t) \). The agent depends
on the other agents in collective duty. The independent equation of \( x_i(t) \) means \( x_i(t) \) is not influenced
by \( x_j(t) \). In most natural swarm phenomena, there is no leader. We translate the absence of the leader
by the similarity of \( k \) state equations.

**Theorem 3.1**
Consider the multi-agent model in (3.4), with initial and boundary values which given in (3.5) and also
maximize the cost functional which given in (3.6), then the control for the i-th agent is
\[
    u_i(t) = \frac{K_i e^{A_i(t)} B_{i|t}}{\delta p_0}, \quad i = 1, \ldots, k,
\]
and the analytic solution of the multi-agent system is
\[
    x_i(t) = -\frac{\left(-s \exp(-A_i T) + q \right) \exp(A_i T)}{\exp(\sqrt{A_i T}) + \exp(-\sqrt{A_i T})} + \frac{q_1 - \exp(A_i T) - \exp(\sqrt{A_i T})}{\exp(\sqrt{A_i T}) + \exp(-\sqrt{A_i T})}.
\]

**Proof:** The Hamiltonian function of the system is
\[
    H = \sum_{i=1}^{k} p_i (A_{i|t} x_i + B_{i|t} u_i) - \sum_{i=1}^{k} \frac{1}{2} \delta p_0 u_i^2 - \frac{1}{2} \gamma p_0 \sum_{i=1}^{k} \sum_{j>i}^{k} \frac{1}{|x_i - x_j|^2} - \frac{1}{2} \mu p_0 \sum_{i=1}^{k} \delta p_0 u_i^2 \sum_{j=1}^{k} \sum_{j>i}^{k} |x_i - x_j|^2.
\]
The Hamiltonian system is
\[
    \frac{\partial H}{\partial p_i} = \dot{x}_i = A_{i|t} x_i + B_{i|t} u_i, \quad i = 1, \ldots, k.
\]
\[
    \frac{\partial H}{\partial x_i} = -p_i = p_i A_{i|t} - \frac{1}{2} \gamma p_0 \frac{\partial}{\partial x_i} \left( \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{1}{|x_i - x_j|^2} \right) - \frac{1}{2} \mu p_0 \frac{\partial}{\partial x_i} \left( \sum_{i=1}^{k} \sum_{j=1}^{k} |x_i - x_j|^2 \right). \quad (8)
\]
By Pontryagin Maximum Principle[34], the necessary condition such that the system reaches its extremum is
\[
    \frac{\partial H}{\partial u_i} = 0 = p_i B_{i|t} - \delta p_0 u_i; \quad i = 1, \ldots, k
\]
Consider (8) we have
\[
    -\dot{p}_i - p_i A_{i|t} - p_2 - p_2 A_{2|t} - \cdots - p_{k-1} - p_{k-1} A_{k-1} = \dot{p}_k + p_k A_k.
\]
From (10) we get
\[
    -\dot{p}_1 - \cdots - p_k = p_1 A_1 + p_2 A_2 + \cdots + p_k A_k.
\]
Next, from equation (11), and we are going to find the analytic solution for \( p_i(t) \) we have equations as follows
\[
    \dot{p}_i = -p_i A_{i|t}
\]
for \( i = 1, 2, \ldots, k \). Then we conclude that
\[
    p_i(t) = K_i e^{A_i t}, \quad (13)
\]
with \( K_i; \quad i = 1, 2, \ldots, k \) are constants. From (9) we obtain
\[
    u_i(t) = \frac{p_i(t) B_{i|t}}{\delta p_0}, \quad i = 1, \ldots, k
\]
consider (13) then (14) can be written as
\[
    u_i(t) = \frac{K_i e^{A_i t} B_{i|t}}{\delta p_0}, \quad i = 1, \ldots, k.
\]
The equation (15) can be viewed as control used by the \( i \)-th agent. Consider the equation (7) and (15),

\[
\dot{x}_i = A_i x_i + B_i \frac{R_i(\exp(-A_it)B_i)}{\delta_{po}}, \quad i = 1, \ldots, k.
\]

(16)

The solution of the differential equation (16) with considering the initial and boundary conditions (5) is

\[
x_i(t) = \frac{(q_i - \exp(\sqrt{A_i}t)q_i) - \exp(\sqrt{A_i}t)}{-\exp(-\sqrt{A_i}t) + \exp(-\sqrt{A_i}t)}
\]

(17)

\[
+ \frac{\sum_{k=1}^{k} (u_i)^2}{2}.
\]

Theorem 3.1 above can be interpreted as an optimal trajectory determination for each agent. The result in (17) can be viewed as an optimal trajectory equation of the \( i \)-th agent.

4. The benefit of move as a multi-agent system

Consider the special cost function in equation (6), especially in the first term which represents the cost of the controls used. One of the benefits of the move as multi-agent is reduced control cost. The control cost if the agents move in multi-agent, follows (6) is

\[
\frac{1}{2} \delta \sum_{k=1}^{k} (u_i)^2.
\]

(18)

Follow (6), if the agents move in a solo then the control cost is

\[
\frac{1}{2} \delta (\sum_{k=1}^{k} u_i)^2.
\]

(19)

**Theorem 4.1**

*The control cost in move together is lower than move in solo*

**Proof**: Consider (18) and (19) above, through Cauchy-Schwarz inequality we get (18) less than or equal with (19). So in notation can be stated that \( \sum_{k=1}^{k} (u_i)^2 \leq (\sum_{k=1}^{k} u_i)^2 \), in other word is concluded that The control cost in move together is lower than move in solo.

5. Conclusion and future work

Multi-agent swarm modeling is interesting because through this process, a multi-agent swarm system can be controlled more effectively. Swarm multi-agent is a behavioral metaphor for solving distributed problems it is based on the principles underlying the behavior of natural systems of many agents. The abilities of such systems appear to transcend the abilities of the constituent individual agents; in all the biological cases studied so far, the emergence of high-level control has been found to be mediated by nothing more than a small set of swarm multi-agent low-level interactions among individuals, and between individuals and the environment. One way to analyze and understand the underlying common principles of swarm systems is to capture their dynamics at more abstract levels. Modeling is a means for saving time, enabling generalization to different platforms, and estimating optimal system parameters.

After the modeling process, the important step is to find the solution of the model. Mathematically, if the analytic solution from the model is founded, we can make many simulations for the model. An analytic solution is a general form equation which satisfies the given requirements. In this paper, the analytic solution of the model can be found successfully. The analytic solution of the uncoupled multi-agent swarm model with \( k \) members or agents described in (4) can be shown in (17). In future work, if possible, we will try to find an analytic solution model multi-agent swarm phenomenon through nonlinear optimal control with attractor and repellant. If the analytic solution for the nonlinear case is impossible to find, the solution with approximation will be used to get an optimal solution.

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