5D EChS Cosmology with Perfect Fluid

Luis Avilés, Patricio Mella and Patricio Salgado
Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile.
E-mail: luisaviles@udec.cl; patriciomella@udec.cl; pasalgad@udec.cl

Abstract. We consider a Chern-Simons gravity theory in the cosmological context for a flat Friedmann-Robertson-Walker metric in five dimensions, chosen barotropic perfect fluids filled the Universe, that is, an equation of state with constant state parameter for the fluid representing the \( h \) field of Einstein-Chern-Simons cosmology and an equation of state with variable state parameter for the fluid representing the matter field.

1. Introduction
The Einstein-Chern-Simons (EChS) cosmology was introduced by [1] in order to have a dynamical description to a five dimensional Friedmann-Robertson-Walker (FRW) cosmology from Chern-Simons gravity theory for a certain Lie algebra B [2], which was obtained from the AdS algebra and a particular semigroup S by means of the S-expansion procedure development by [3, 4].

Using the extended Cartan’s homotopy formula as in Ref.[5], and integrating by parts, the five dimensional EChS Lagrangian for the B algebra can be written as

\[
L^{(5)}_{\text{EChS}} = \alpha_1 l^2 \varepsilon_{abcdef} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcdef} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2 \alpha_1 l^2 k_{ab} R^{cd} e^e + l^2 R^{ab} R^{cd} h^e \right)
\]

where \( \alpha_1, \alpha_3 \) are parameters of the theory, \( l \) is a coupling constant, \( T^a = de^a + \omega^a_b e^b \) is the torsion, \( R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} \) corresponds to the curvature 2-form in the first-order formalism related to the 1-form spin connection [6], [7], [8], and \( e^a, h^a \) and \( k_{ab} \) are others gauge fields presents in the theory [2].

We consider now the fields equations for the lagrangian given by

\[
L = L^{(5)}_{\text{ChS}} + \kappa L_M
\]

where \( L^{(5)}_{\text{ChS}} \) is the five-dimensional Chern-Simons Lagrangian given by (1), \( L_M = L_M (e^a, h^a, \omega^{ab}) \) is the matter Lagrangian and \( \kappa \) is a coupling constant related to the effective Newton’s constant. If \( T^a = 0 \) and \( k_{ab} = 0 \). The variation of the lagrangian (2), leads to the following field equations

\[
\varepsilon_{abcdef} \left( 2 \alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} \right) = \kappa \delta L_M \delta e^e,
\]

\[
\alpha_3 l^2 \varepsilon_{abcdef} R^{ab} R^{cd} = \kappa \delta L_M \delta h^e.
\]
\[
\frac{\delta L_{M}}{\delta k_{ab}} = 0, \quad \text{(5)}
\]
\[
2\alpha l^{2} \varepsilon_{abde} R^{cd} D_{\omega} h^{e} = \kappa \frac{\delta L_{M}}{\delta \omega_{ab}}, \quad \text{(6)}
\]

consider here \(\frac{\delta L_{M}}{\delta \omega_{ab}} = 0\), imposed for consistency with \(T^a = 0\).

Notice that from these fields equations we recover the odd-dimensional Einstein gravity theory when the curvature \(R_{ab}\) takes values not excessively large and the parameter \(l\) takes small values \((l \rightarrow 0)\) [2] and the constant \(\alpha\) takes values not excessively large, namely

\[
\varepsilon_{abde} R_{ab}^{e} e_{d} \approx 4\kappa_{5} \frac{\delta L_{M}}{\delta e^{e}}, \quad \text{(7)}
\]

where \(\kappa_{5} = \kappa / 8\alpha_{3}\).

If \(R_{ab}\) is not large then \(\frac{\delta L_{M}}{\delta e^{a}}\) is also not large. This means that General Relativity can be seen as a low energy limit of Einstein-Chern-Simons gravity. So that, in the range of validity of the General Relativity, the equations (3-6) are given by

\[
\varepsilon_{abde} R_{ab}^{e} e_{d} = 4\kappa_{5} \frac{\delta L_{M}}{\delta e^{e}}, \quad \text{(8)}
\]
\[
\varepsilon_{abde} R^{cd} D_{\omega} h^{e} = 0. \quad \text{(9)}
\]

On the other hand, if \(R_{ab}\) is large enough, so that when it is multiplied by \(l^{2}\) (which is very small) will have a non-negligible results, then we will find that \(\frac{\delta L_{M}}{\delta h^{a}}\) is not negligible. This means that, in this case, we must consider the entire system of equations (3-6).

Next we consider a flat five dimensional FRW metric and given an equation of state for the \(h\) field of the EChS theory like a barotropic perfect fluid with constant state parameter, this generates a variable state parameter in the equation of state chosen for the matter field. We think that is important because this no was explored in [1] and this open a new family of exact solutions to the EChS cosmology.

2. 5D-FRW-CHS field equations: Case \(k=0\).

In order to solve the EChS field equations, we consider the FRW metric of the universe in five dimensions given by

\[
d s^{2} = -dt^{2} + a^{2}(t) \left( \frac{dr^{2}}{1 - kr^{2}} + r^{2}(d\theta^{2} + sin^{2}\theta(d\phi^{2} + sin^{2}\phi d\psi^{2})) \right), \quad \text{(10)}
\]

with \(k = 0\) and the energy-momentum tensors

\[
T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} + Pg_{\mu\nu}, \quad \text{(11)}
\]
\[
T_{\mu\nu}^{(h)} = (\rho^{(h)} + P^{(h)}) u_{\mu} u_{\nu} + P^{(h)} g_{\mu\nu}, \quad \text{(12)}
\]

where \(\rho\) and \(P\) are the energy density and pressure of matter, and \(\rho^{(h)}\) and \(P^{(h)}\) are the energy density and pressure for \(h\) field, so the field equations are [1]

\[
6 H^{2} + \alpha l^{2} H^{4} = \kappa_{1} \rho, \quad \text{(13)}
\]
\[
\dot{\rho} + 4H(\rho + P) = 0, \quad \text{(14)}
\]
\[
l^{2} H^{4} = \kappa_{2} \rho^{(h)}, \quad \text{(15)}
\]
\[
\rho^{(h)} + 4H(\rho^{(h)} + P^{(h)}) = 0, \quad \text{(16)}
\]
\[
(g - f)H + \dot{g} = 0, \quad \text{(17)}
\]

where \(H = \frac{\dot{a}}{a}\).
2.1. Solutions to the barotropic equation of state

We further assume that the perfect fluid for the field $h$ obeys the barotropic equation of state

$$P^{(h)} = \omega^{(h)} \rho^{(h)}, \quad (18)$$

where $\omega^{(h)}$ is a constant.

In order to obtain solutions for the scale factor, energy density of the field $h$, field matter, we analyze the case when $\omega^{(h)} \neq -1$ and when $\omega^{(h)} = -1$ separately.

2.1.1. Case $\omega^{(h)} \neq -1$. Writing the balance equation for the field $h$ (16) in terms of the Hubble parameter $H$ and using the equation of state (18), we obtain

$$H(t) = \frac{1}{(\omega^{(h)} + 1) (t - t_0)}, \quad (19)$$

from this result we can find the scalar factor

$$a(t) = a_0 \left[ (\omega^{(h)} + 1) (t - t_0) \right]^{-\frac{1}{\omega^{(h)}+1}}. \quad (20)$$

Now introducing (20) in (15), we obtain

$$\rho^{(h)}(t) = \frac{l^2}{\kappa_2 [(\omega^{(h)} + 1) (t - t_0)]^4} \quad (21)$$

and introducing (20) in (13), we obtain

$$\rho(t) = \frac{6 \left[ (\omega^{(h)} + 1) (t - t_0) \right]^2 + \alpha l^2}{k_1 [(\omega^{(h)} + 1) (t - t_0)]^4}. \quad (22)$$

If we now propose an equation of state with variable state parameter $P = \omega(t) \rho$ for the field matter, we can find from (14) that is

$$\omega(t) = \frac{\alpha l^2 \omega^{(h)} + 3(\omega^{(h)} - 1)(\omega^{(h)} + 1)^2(t - t_0)^2}{\alpha l^2 + 6(\omega^{(h)} + 1)^2(t - t_0)^2} \quad (23)$$

this result is new and not reported on Ref.[1].

2.1.2. Case $\omega^{(h)} = -1$. Writing the balance equation for the field $h$ (16) in terms of the Hubble parameter $H$ and using the equation of state (18), we obtain

$$H(t) = H_0 \quad (24)$$

from this result we can find the scalar factor

$$a(t) = a_0 e^{tH_0}. \quad (25)$$

Now introducing (25) in (15), we obtain

$$\rho^{(h)}(t) = \frac{l^2 H_0^4}{\kappa_2} \quad (26)$$

and introducing (25) in (13), we obtain

$$\rho(t) = H_0^2 \left[ 6 + \frac{l^2 \alpha H_0^2}{\kappa_1} \right]. \quad (27)$$
If we now propose an equation of state with variable state parameter \( P = \omega(t) \rho \) for the field matter, we can find from (14) that

\[
\omega = -1
\]  

This result agrees with that found in Ref.[1].

3. Conclusions
In this paper we have considered a Chern-Simons gravity theory in the cosmological context for a flat FRW metric in five dimensions, considering an equation of state with \( \omega^{(h)} \) constant for the fluid that represents the \( h \) field of EchS theory and an equation of state with \( \omega(t) \) variable for the fluid representing the matter field. With this we have obtained an exact solution for our cosmological model, finding the Hubble parameter, the scalar factor, pressures and energy densities for the matter field and \( h \) field, and the variable parameter of state for the matter field, considering the cases \( \omega^{(h)} \neq -1 \) and \( \omega^{(h)} = -1 \).

Acknowledgments
This work was supported by Comisión Nacional de Ciencias y Tecnología through FONDECYT Grants 3130444 (PM).

References
[1] Gomez, F., Minning, P., Salgado, P.: Phys. Rev. D84, 063506 (2011)
[2] F. Izaurieta, P. Minning, A. Perez, E. Rodriguez, P. Salgado, Phys. Lett. B 678 (2009) 213
[3] F. Izaurieta, E. Rodriguez, P. Salgado, Jour. Math. Phys. 47 (2006) 123512.
[4] F. Izaurieta, A. Perez, E. Rodriguez, P. Salgado, Jour. Math. Phys. 50 (2009) 073511.
[5] F. Izaurieta, E. Rodriguez, P. Salgado, Lett. Math. Phys. 80 (2007) 127
[6] J. Zanelli, ”Lecture notes on Chern-Simons (super)gravities. Second edition (February 2008), 2005.
[7] A. H. Chamseddine, Phys. Lett. B 233 (1989) 291.
[8] A. H. Chamseddine, Nucl. Phys. B 346 (1990) 213.
[9] K. Andrew, B. Bolen, and C. Middleton, Gen. Relativ. Gravit. 39, 2061 (2007).
[10] N. Mohammedi, Phys. Rev. D 65, 104018 (2002).