AN INVENTORY MODEL FOR ITEMS WITH IMPERFECT QUALITY AND QUANTITY DISCOUNTS UNDER ADJUSTED SCREENING RATE AND EARNED INTEREST

TIEN-YU LIN, MING-TE CHEN AND KUO-LUNG HOU

Abstract. In this paper we develop a new inventory model for items with imperfect quality and quantity discounts under adjusted screening rate and earned interest. Three highlights are included in this new model: (1) interest is earned by depositing the sales revenue from the perfect and imperfect items into an interest-bearing account (2) the screening rate may not be given but is a decision variable (3) the supplier offers quantity discounts to trigger the retailer into ordering greater lot sizes. This scenario has not been discussed in previous EOQ models with imperfect quality. Our model could determine two decision variables, order quantity and screening rate, to maximize retailer profit. The expected total profit function is derived with two special cases explored to validate the proposed model. An algorithm is developed to help the manager determine the optimal order quantity and screening rate. A numerical example is given to illustrate the proposed model and algorithm. Sensitivity analyses are carried out to investigate the model parameters effects on the optimal solution. Managerial insights are also included.

1. Introduction. The issue of economic order quantity with imperfect quality has received considerable attention in recent years from academicians and practitioners. Numerous studies have demonstrated that the production process may deteriorate and thus defective items may occur. Salameh and Jaber [30] recently developed an economic order quantity (EOQ) model with imperfect quality in which the defective items could be sold at a reduced price by the end of the screening process for each batch. The above model is more reasonable than the traditional EOQ model. Many extensions were developed for the above model. Cárdenas-Barrón [3] corrected an error appearing in the work of Salameh and Jaber [30]. Goyal and Cárdenas-Barrón [9] reconsidered the task done in Salameh and Jaber [30] and presented a simplified method to determine the optimal lot size. Chan et al. [5] proposed a non-shortage model similar to that in Salameh and Jaber [30], wherein products can be classified as good quality, good quality after reworking, imperfect quality and scrap. With...
respect to the inventory model proposed in Salameh and Jaber [30], Chang [6], and Mahata and Goswami [23] fuzzified the defective rate and the annual demand and then derived the corresponding optimal lot sizes. Huang [12] investigated a single-vendor, single-buyer integrated production-inventory problem, where the buyer’s inventory policy follows Salameh and Jaber’s model. Eroglu and Ozdemir [8], Wee et al. [36], and Chang and Ho [7] further extended Salameh and Jaber’s work to the case in which shortage backordering is permitted. Building upon the work of Salameh and Jaber [30], Maddah and Jaber [20] employed renewal theory to correct the flaw in their work and extended the analysis by allowing several batches of defectives to be consolidated and shipped in a single lot. Jaber et al. [13] assumed the percentage defective items in a shipment is reduced in conformance with a learning curve and thus developed two models subject to learning effects. Maddah et al. [21] developed two models for newsvendor and EOQ-type inventory systems under random yield and items of different quality. More recently, Wahab and Jaber [35] presented models based on Salameh and Jaber [30], Maddah and Jaber [20], and Jaber et al. [13] with different holding costs for good and imperfect items. Cárdenas-Barrón [4] considered a production system with a back-order policy in which defective items could be reworked. Khan et al. [17] extended Salameh and Jaber’s work to the EOQ model for items with imperfect quality with learning during inspection. Hsu and Yu [11] dealt with an EOQ model for products with imperfect quality under an announced price increase, where the defectives are screened out using a 100 % inspection process and which can be sold as a single batch at the end of the inspection process. Yu et al. [37] investigated an ”acceptable defective part” inventory model where the acceptable defective items can affect the consumption of perfect quality items under the condition that shortages are not allowed. Hsu and Hsu [10] developed an economic order quantity model with imperfect quality items, inspection errors, shortage backordering and sales returns. A closed form solution was obtained for the optimal order size, the maximum shortage level and the optimal order/reorder point. Jaber et al. [14] proposed an entropic version of an EOQ model with imperfect quality items in which an entropy cost term is added to the classical inventory cost. Lin and Hou [18] developed an imperfect quality inventory model with overlapped and advanced receiving in which the supplier provides a procurement cost discount rate to compensate the buyer for the additional holding cost and maintain a cooperative relationship. Vörös [34] revisited the economic order quantity models when a lot may randomly contain defective items in which he dropped the well-known assumption intended to avoid shortages with two lot sizing rules derived. A review of the modified EOQ model extensions for imperfect quality items can be found in Khan et al.’s [16] work.

Note that all of the above researchers assumed that no shortage occurred during the screening duration. However, this assumption discussed in Salameh and Jaber’s [30] work may not be true because defective items may be screened out under several consecutive screenings. Therefore, Papachristos and Kousstantaris [27] questioned the validity of this assumption but failed to provide a correction for this flaw. They merely pointed out that there is no easy solution that guarantees the validity of this scenario. More recently, Maddah et al. [22] developed a practical alternative, an order “overlapping” scheme that allows meeting the demand during the screening process from the ”previous order” to avoid shortages. Although Maddah et al.’s [22] work efficiently eliminates the uncertainty of shortages, they merely considered the holding cost for both good items and defective items to be the same. However,
in the real manufacturing environment (e.g. many semiconductor industries), the defective items are usually stored in a different warehouse from the good items to separate cost and quantity tracking for the defective items (Wahab and Jaber [35]; Paknejad et al. [26]). With this consideration, the holding costs for a unit of good items per period and a unit of defective items per period should be different. Note that all of the above papers treated the screening rate as a known parameter. From the practical point of view, semiconductor industry manufacturers often increase the screening rate to achieve the following benefits: (1) avoid the goods suffering from deterioration risk (2) removing imperfect quality items from the good ones and then storing them in another warehouse with lower holding cost (3) selling the imperfect quality items as soon as possible and then depositing the funds into an interest-bearing account. Simultaneously, an additional cost (e.g. an overtime wage) for upgrading the screening rate performance occurs. Thus, treating the screening rate as a decision variable is worth study.

Another common unrealistic assumption in the above models is that the direct product cost is irrelevant. In many cases quantity discounts can provide the buyer lower per-unit purchase cost, lower ordering costs and decreased likelihood of shortages (Burwell et al. [2]). It is recognized that quantity discounts can provide economic advantages for both the buyer and vendor (Burwell et al. [2]; Ji and Shao [15]). Therefore, in real life, the supplier may employ quantity discounts to stimulate the buyer into ordering more items. The first researcher focusing on an EOQ model with quantity discounts was Monahan [25] in which he assumed a lot-for-lot replenishment policy for a vendor and showed that a vendor could encourage the buyer to order larger quantities by offering a price discount. Ever since that time, many excellent extensions were made that showed that quantity discounts can provide the retailer with lower per-unit purchase cost, lower ordering cost and decreased likelihood of shortages. Benton and Park [1] further classified the literature on lot sizing determination under several types of discount schemes and discussed some of the significant literature in this area over two decades. Taking different paths, some researchers (Tsai [33]; Lin [19]) adopted quantity discounts as a tool to coordinate a two-echelon system with variation in demand. Some researchers (Mendoza and Ventura [24]; Qin et al. [28]) extended a previous EOQ model with modes of transportation by introducing all-units and incremental quantity discount structures into the analysis. It is worth noting that all of these works considered the items to be perfect and neglected to discuss screening process efficiency, which is very important for the EOQ model with imperfect quality.

Based on the above arguments, this study develops a profit maximization model with imperfect quality and different holding costs to explicitly obtain the optimal lot size and screening rate under quantity discount consideration. Specifically, this paper deals with the retailer’s inventory model for items with imperfect quality and different holding costs with three considerations included in this new model: (i) interest is earned by depositing the sales revenue for the perfect items sold in each cycle and for the imperfect items sold at the end of the screening process into an interest-bearing account (ii) the screening rate may not be given but is a decision variable when the screening rate can be adjusted (iii) the supplier may offer quantity discounts to trigger the retailer into ordering greater lot sizes.

The remainder of this paper is organized as follows. Section 2 presents the notations and assumptions used in this paper. Section 3 develops a new retailer’s inventory model for items with imperfect quality and quantity discounts. Section 4
develops the properties and an algorithm to find the optimal solution that maximizes the expected total profit. Section 5 provides a numerical example to illustrate the proposed model and the algorithm. A sensitivity analysis is also made to explore the effects of five important parameters on the optimal solution in this section. Conclusions are provided in Section 6.

2. Notations and assumptions.

2.1. Notations. The following notations are used throughout this paper to develop the mathematical model.

- \( Q \): order size
- \( D \): demand rate
- \( K \): ordering cost per order
- \( I \): interest rate / $/year
- \( p \): percentage rate of imperfect quality items in \( Q \)
- \( f(p) \): probability density function of \( p \)
- \( x \): screen rate, \( x > D \)
- \( v \): unit selling price of perfect items
- \( w \): unit salvage value of imperfect quality items, \( w < v \)
- \( c_j \): unit cost of \( j \)th level corresponding to the cost discount structure
- \( d \): unit screening cost
- \( g \): holding cost rate for a unit of good item per period, expressed as a fraction of dollar value and \( f_g > f_d \).
- \( d \): holding cost rate for a unit of defect item per period, expressed as a fraction of dollar value and \( f_d < f_g \).
- \( \theta \): unit cost of upgrading the screening rate performance
- \( t \): inspection duration
- \( T \): cycle length

2.2. Assumptions. In this paper, the following assumptions are considered:

(a) The demand rate for items is known and constant.
(b) The screening process and demand proceeds simultaneously, but the screening rate is greater than the demand rate, \( x > D \). This implies the condition \( \{\theta < (K/D)\} \) holds (See Appendix B).
(c) The holding cost for good items stored in a specific warehouse is higher than that for imperfect items stored in another warehouse.
(d) Shortages are not allowed and the supplement rate is infinite.
(e) Generated sales revenue from the perfect items sold in each cycle and from the imperfect items sold at the end of the screening process is deposited into an interest bearing account.
(f) This paper employs the policy of all-unit quantity discount. Let \( c_j \) be the unit price of \( j \)th level and \( Q_{j-1} \) be the \( j \)th lowest quantity \( (Q_{j-1} < Q < Q_j) \). If \( Q_{j-1} < Q < Q_j \), then the unit price is \( c_j \). The price discount schedule is shown in Table 1.

3. Mathematical model. Consider the inventory model with imperfect quality and different holding costs depicted in figure 1 in which \( T \) is the ordering cycle duration, defined as the time between the placements of two orders. The shaded area is represented as the average inventory quantities of defective items in each order. The 100% lot screening process is originally finished at time \( t \) which is
affected simultaneously by decision variables $Q$ and $x$ and thus somewhat different from Salameh and Jaber’s [30] work. This paper considers three real-life scenarios for the proposed model: (i) generated sales revenue from the imperfect items sold at the end of the screening process is deposited into an interest bearing account (ii) the screening rate may not be given but is a decision variable when the screening rate can be adjusted (iii) the supplier may offer quantity discounts to trigger the retailer into ordering greater lot sizes.

**Table 1. Price discount structure**

| $j$ | $Q_{j-1} \leq Q < Q_j$ | $c_j$ |
|-----|------------------------|------|
| 1   | $0 < Q < Q_1$          | $c_1$ |
| 2   | $Q_1 \leq Q < Q_2$     | $c_2$ |
|     |                        |      |
| ... |                        |      |
| $m$ | $Q_{m-1} \leq Q < Q_m$ | $c_m$ |

Observing Figure 1, the order will be placed when there are $Dt$ units left. This implies the overlapping policy suggested by Maddah et al.[22] is employed to avoid shortages during the screening process. The triangular $ZBC$ area in Figure 1 is the same as the triangular area of $DEF$ and $HIJ$. The retailer orders $Q$ units per cycle. The screening time is $t$ affected simultaneously by the order quantity (decision variable) and screening rate (decision variable). The imperfect quality is given by $pQ$. The cycle time $T$ is defined as the perfect items are exhausted and can be expressed as

$$T = (1 - p)Q/D$$

Let $TR(Q, x)$ and $TC_j(Q, x)$ be the total revenue and the total cost per cycle, respectively. $TR(Q, x)$ includes the revenues from perfect items, imperfect items
and the interest produced by an interest-bearing account from selling perfect items sold in each cycle and imperfect items sold at the end of the screening process. Therefore, $TR(Q, x)$ is expressed as

$$TR(Q, x) = vQ(1 - p) + wQp + wQpI(T - Q/x) + \frac{vI[Q(1 - p)]^2}{2D}$$  \hspace{1cm} (2)$$

$TC_j(Q, x)$ consists of the procurement cost per cycle ($c_jQ$), setup cost per cycle ($K$), screening cost per cycle ($dQ$), total cost of upgrading the screening rate ($x\theta$), and the holding cost of goods and imperfect quality items illustrated as follows. According to Assumption (3), we let the holding costs for defective items and good items per cycle be $HC_d(Q)$ and $HC_g(Q)$, respectively. The items per lot were initially stored in a good-quality warehouse. Each item was then screened and the defective items were removed to another warehouse. Therefore, $HC_d(Q)$ can be shown as the shaded area in Figure 1 and expressed as

$$HC_d(Q) = c_jf_d(Q^2p/2x)$$

$HC_g(Q)$ can be obtained based on the non-shaded area in Figure 1 and can be computed through a geometric argument as follows:

$$HC_g(Q) = c_jf_g[A(ZBC) + A(BGR) + A(GIJR) + A(RJF) - A(DEF)]$$

where $A(ZBC)$, $A(BGR)$, $A(RJF)$, and $A(DEF)$ are the areas of triangle ZBC, BGR, RJF, and DEF, and $A(GIJR)$ is the area of rectangle GIJR. Because $A(ZBC) = A(DEF)$, we have

$$HC_g(Q) = c_jf_g[A(BGR) + A(GIJR) + A(RJF)]$$

$$= c_jf_g \left[ \frac{Q^2p}{2x} + \frac{Q^2(1 - p)}{x} + \frac{[Q(1 - p)]^2}{2D} \right]$$

Synthesizing the costs described above we have the total cost per cycle as follows:

$$TC_j(Q, x) = c_jQ + K + dQ + c_j(f_d + f_g) \left[ \frac{pQ^2}{2x} \right] + c_jf_g \left[ \frac{Q^2(1 - p)}{x} + \frac{[Q(1 - p)]^2}{2D} \right] + x\theta$$

Therefore, the total profit per cycle, $TP_j(Q, x)$, can be expressed as

$$TP_j(Q, x) = TR(Q, x) - TC_j(Q, x)$$

$$= vQ(1 - p) + wQp + wQpIT - \left[ \frac{Q^2pIw}{x} - \frac{vI[Q(1 - p)]^2}{2D} \right]$$

$$- c_jQ - K - dQ - c_j(f_d + f_g) \left[ \frac{pQ^2}{2x} \right]$$

$$- c_jf_g \left[ \frac{Q^2(1 - p)}{x} + \frac{[Q(1 - p)]^2}{2D} \right] - x\theta$$

$$\hspace{1cm} (j = 1, 2, \ldots, m)$$

Since the replenishment cycle length is a variable, using the renewal reward theorem (Ross [29]), the expected total profit per unit time can be given as follows:

$$\lim_{t \to \infty} \frac{TPU_j}{t} = E[TPU_j(Q, x)] = \frac{E[TP_j(Q, x)]}{E[T]} \hspace{1cm} (j = 1, 2, \ldots, m)$$  \hspace{1cm} (5)$$

where

$$E[TP_j(Q, x)] = vQ(1 - E[p]) + wQE[p] + wQE[p]IE[T]$$
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\[ - \left[ \frac{Q^2 E[p]}{x} - \frac{v I [Q^2 E(1-p)^2]}{2D} \right] \]

\[-c_f Q - K - dQ - c_j (f_d + f_g) \left[ \frac{E[p] Q^2}{2x} \right] \]

\[-c_f f_g \left[ \frac{Q^2 (1 - E[p])}{x} + \frac{Q^2 E(1-p)^2}{2D} \right] - x \theta \]

and

\[ E[T] = \frac{Q(1 - E[p])}{D} \]

Thus, the expected profit per unit time is given from equation (5) as

\[ E[TPU_j(Q,x)] = \frac{D [v \alpha + w E[p] - d - c_f] - (KD + x D \theta) / Q}{1 - E[p]} \]

\[ + \frac{Q [w I E[(1-p)^2]/2]}{1 - E[p]} \]

\[ + \frac{Q [w I E[p] - D w I E[p]/x - D c_f f_g \alpha / x]}{1 - E[p]} \]

\[ - \frac{Q [D c_f (f_d + f_g) E[p]/(2x) + c_f f_g E[(1-p)^2]/2]}{1 - E[p]} \]

\[(j = 1, 2, \ldots, m), \quad (6)\]

where

\[ \alpha = 1 - E[p] \]

Our objective is to maximize the integrated total profit by determining the order quantity and the screening rate under a certain cost structure. To find the unique solution we need to show that the Hessian matrix of Eq. (6) is positively definite. However, it is not easy to determine the concavity of the Hessian matrix in Eq. (6). Thus, we employ an alternative procedure to obtain the profit function properties.

4. Solution procedure. The following property is needed to find the unique solution for the expected total profit \( E[TPU_j(Q,x)] \).

**Property.** For fixed \( Q \), \( E[TPU_j(Q,x)] \) is concave in \( x \) for all \( j \).

**Proof.** See Appendix A.

According to Property 1 the possible optimal value of \( x \) can be determined by letting \( \frac{\partial E[TPU_j(Q,x)]}{\partial x_j} = 0 \), which yields

\[ x_j(Q) = Q \sqrt{\frac{w I E[p] + c_j f_g (1 - E[p]) + c_j (f_d + f_g) E[p]/2}{\theta}}, \quad (j = 1, 2, \ldots, m) \]

(7)

Because \( x > D > Q \), we know the square root in Eq. (7) is greater than 1. Therefore, \( x \) has a positive correlation with \( Q \). Substituting Eq. (7) into Eq. (6), we have
the following expression corresponding to the total expected profit $E[TPU_j(Q)]$:

$$E[TPU_j(Q)] = \frac{D(v(1 - E[p]) + wE[p]) + Q_jwIE[p]\alpha}{1 - E[p]}$$

$$+ \frac{Q_j(vI - c_jf_g)E[(1 - p)^2]/2 - KD/Q_j - c_jD - dD}{1 - E[p]}$$

$$- \frac{2D\sqrt{(wIE[p] + c_jf_g\alpha + c_j(f_d + f_g)E[p]/2)\theta}}{1 - E[p]}$$

\[(j = 1, 2, \cdots, m)\] (8)

Equation (8) yields

$$\frac{\partial TPU_j(Q)}{\partial Q_j} = \frac{(wIE[p]\alpha + (vI - c_jf_g)E[(1 - p)^2]/2 + KD/(Q_j^2))}{1 - E[p]}$$

\[(j = 1, 2, \cdots, m)\] (9)

$$\frac{\partial^2 E[TPU_j(Q)]}{\partial (Q_j)^2} = \frac{-2KD}{(Q_j)^3(1 - E[p])} < 0, \quad (j = 1, 2, \cdots, m)$$ (10)

Therefore, the possible optimal value of $Q$ can be determined by letting $\frac{\partial E[TPU_j(Q)]}{\partial Q_j} = 0$, which yields

$$Q_j = \sqrt{\frac{KD}{(c_jf_g - vI)E[(1 - p)^2]/2 - wIE[p]\alpha}}, \quad (j = 1, 2, \cdots, m)$$ (11)

In practice the interest rate and expected defective rate are small. Furthermore, we have $\alpha < 1$. Thus, if the denominator in Eq. (11) is greater than zero Eq. (11) holds and the possible optimal order quantities do exist. The possible optimal order lot size therefore could be obtained from Eq. (11) and the unit cost for upgrading the screening rate performance should be less than $(K/D)$ (See Appendix B).

We cannot obtain the optimal lot size directly from Eq. (11). This is because the unit purchasing cost depends on the lot size order and thus has a different cost curve. Thus, $Q_j$ in Eq. (11) is the lowest point on each cost curve $c_j$. An algorithm is developed as follows to find the optimal ordering quantity $Q^*$ and screening rate $x^*$:

**Step 1.** Employ Eq. (11) to compute $Q_j$ for all $j$.

**Step 2.** Adjust the $Q_j$ values that are above to be the allowable discount range.

**Step 3.** Employ Eq. (7) to compute $x_j$ corresponding to each possible optimal order lot size obtained from Step 1 or Step 2.

**Step 4.** Employ Eq. (6) to obtain the expected total profit corresponding to each candidate optimal set $(Q_j, x_j)$.

**Step 5.** Select the order lot size and screening rate with expected total profit maximization. Thus, the optimal solution is obtained.

Note that if $c_jf_d = c_jf_g = h$, and $I = 0$ (i.e. the retailer did not deposit the sale income into an interest bearing account). The optimal order quantity in Eq. (11) then reduces to

$$Q^* = \sqrt{\frac{2KD}{hE[(1 - p)^2]}}$$ (12)
which is equivalent to the results in Silver [32] and Shih [31]. If no defective items are produced and the above conditions are held, that is \( p = 0, c_j f_d = c_j f_g = h, \) and \( I = 0, \) then the optimal order quantity in Eq. (11) reduces to

\[
Q^* = \sqrt{\frac{2KD}{h}}
\]

which is equivalent to the result in the traditional EOQ model. This helps for validating our model.

5. **Numerical example.** Referring to the example provided in Salameh and Jaber [30], the parameters needed for analyzing the models developed in this paper are given below. Demand rate \( D = 50000 \) units/year; Ordering cost \( K = $100/\)cycle; Screening cost \( d = $0.5/\)unit; Holding cost rate for a unit of good item (a fraction of dollar value) \( f_g = $0.15/\$/year.\) Holding cost rate for a unit of defect item (a fraction of dollar value) \( f_d = $0.05/\$/year.\) Selling price of good quality items \( v = $50/\)unit. Salvage value of defect items \( w = $25/\)unit. Unit cost of upgrading the screening rate performance \( \theta = $0.001/\)unit. Interest rate \( I = $0.01/\$/year.\) The supplier offers a price discount schedule as shown in Table 2. For illustrative purposes, assume that the percentage defective \( p \) can take any value in the range \([L, U]\) with \( L = 0 \) and \( U = 0.04; \) that is, \( p \) is uniformly distributed and its p.d.f. is given by

\[
f(p) = \begin{cases} 25 & 0 \leq p \leq 0.04 \\ 0 & \text{otherwise.} \end{cases}
\]

| \( j \) | \( Q_{j-1} \leq Q < Q_j \) | \( c_j \) |
|---|---|---|
| 1 | \( 0 < Q < 350 \) | \( c_1 = 35.08 \) |
| 2 | \( 350 \leq Q < 700 \) | \( c_2 = 35.06 \) |
| 3 | \( 700 \leq Q < 1400 \) | \( c_3 = 35.04 \) |
| 4 | \( 1400 \leq Q < 2800 \) | \( c_4 = 35.02 \) |
| 5 | \( Q \geq 2800 \) | \( c_5 = 35.00 \) |

Applying the preceding algorithm developed in Section 4, one has the optimal order quantity and screening rate which maximum expected total profit as follows:

**Step 1.** Employ Eq. (11) to compute \( Q_j, \) we have

\( Q_1 = 1480.18; \) \( Q_2 = 1480.65; \) \( Q_3 = 1481.12; \) \( Q_4 = 1481.58; \) \( Q_5 = 1482.05. \)

**Step 2.** Adjust the \( Q_j \) values that are above the allowable discount range. It is obvious that \( Q_5 \) is below the allowable range of 2800 to infinite and therefore it must be adjusted to 2800 (i.e. \( Q_5 = 2800. \) Alternatively, since \( Q_4 \) is between 1400 and 2800, it does not have to be adjusted.
Step 3. Employ Eq. (7) to compute \( x_j \) corresponding to \( Q_5 = 2800 \) and \( Q_4 = 1481.58 \), therefore, we have \( x_5 = 202299 \) and \( x_4 = 107074 \).

Step 4. Employ Eq. (6) to obtain the expected total profit corresponding to each candidate optimal set \((Q_j, x_j)\), we have \( ETPU_5[2800, 202299] = 673077 \) and \( ETPU_4[1481.58, 107074] = 673493 \).

Step 5. Select that order lot size and screening rate that has the maximization expected total profit in STEP 4. We obtain the optimal order quantity \( Q^* = 1481.58 \), the optimal screening rate \( x^* = 107074 \), and the expected total profit \( ETPU^* = 673493 \).

Parameters including \( D, K, w, I, f_d, f_g, c_j, \theta \), and \( p \) theoretically should be investigated to understand the model parameter effects on the optimal quantity and screening rate. However, a thorough investigation of the impact these parameters have on the optimal solution would be laborious computational work. We take only \( D, f_g, I, K \) and \( p \) into account to save computational labor. All of the five parameters are set at two levels and described as follows: \( D \) (i) 50000 (ii) 60000; \( f_g \) (i) 0.12 (ii) 0.15; \( I \) (i) 0.01 (ii) 0.012; \( K \) (i) 100 (ii) 120; \( p \) (i) Uniform distribution with \([0, 0.04]\) (ii) Uniform distribution with \([0, 0.048]\). The other parameters remain unchanged except for the above five parameters. Table 3 lists the optimal solution under 32 combinations of \( D, f_g, I, K, \) and \( p \). From this table we obtain the findings shown below:

1. In general, \( Q^* \), \( x^* \), and \( ETPU^* \) increase with \( D \). That is, the higher the demand rate, the higher the order quantities, screening rate and expected annual total profit are. Obviously, as the demand rate increases, the retailer needs greater quantity per order to satisfy the demand. At the same time, the retailer employs a higher screening rate to meet the demand and sell the defective items as the screening process ends. These results may increase the expected total profit. Note that the optimal order lot size \( (Q^*) \) may occur at the break point of ordering quantity corresponding to the lowest cost in the cost discount schedule \((i.e., c_5 = 35)\). Thus, \( Q^* \) and \( x^* \) remain unchanged with \( D \), while \( ETPU^* \) increases with \( D \). This fact illustrates the retailer enjoys the benefit of quantity discount and thus orders the lot size up to the break point of ordering quantity.

2. \( Q^* \), \( x^* \), and \( ETPU^* \) all decrease with \( f_g \). This is rather intuitive because a higher holding cost for items leads to higher storage expenditure for the retailer, which reduces order quantities. In the meantime, the screening rate for items could slow down to meet the smaller orders and thus the holding cost is reduced. Therefore, the expected total profit decreases with the holding cost for good items. This reveals the higher the holding cost rate for a unit of good or re-workable items, the lower the total expected profit. Note that if the optimal order lot size \( (Q^*) \) occurs at the break point of ordering quantity corresponding to the lowest cost in the cost discount schedule \((i.e., c_5 = 35)\), \( Q^* \) remains unchanged with \( f_g \), while \( x^* \) increases with \( f_g \). That is, the retailer enjoys the benefit of quantity discount and orders quantities up to the break point of ordering quantity. This leads the retailer to continue with the same orders as in \( h_g \) in our example.

3. As expected, \( Q^* \), \( x^* \), and \( ETPU^* \) all increase in \( I \). That is, the greater the interest rate, the greater the expected total profit. This is because as the interest rate increases, the retailer could obtain greater revenue caused by
Table 3. **The values of \( Q^* \), \( N^* \), and \( ETPU^* \) corresponding to 32 combinations of \( D, f_g, I, K, p \)**

| \( D \) | \( f_g \) | \( I \) | \( K \) | \( U(p) \) | \( Q^* \) | \( x^* \) | \( ETPU^* \) | \( c_j \) |
|--------|---------|-------|------|---------|-------|-------|-------|-------|
| 50000  | 0.01    | 100   | 0.04 | 2800    | 181039| 675293| 35    |
|        |         |       | 0.048| 2800    | 180955| 672984| 35    |
|        |         | 120   | 0.04 | 2800    | 181039| 674928| 35    |
|        |         |       | 0.048| 2800    | 180955| 672618| 35    |
|        | 0.12    | 100   | 0.04 | 2800    | 181061| 675432| 35    |
|        |         |       | 0.048| 2800    | 180981| 673123| 35    |
|        |         | 120   | 0.04 | 2800    | 181061| 675067| 35    |
|        |         |       | 0.048| 2800    | 180981| 673123| 35    |
|        | 0.012   | 100   | 0.04 | 2800    | 180961| 675432| 35    |
|        |         |       | 0.048| 2800    | 180981| 673123| 35    |
|        |         | 120   | 0.04 | 2800    | 181061| 675067| 35    |
|        |         |       | 0.048| 2800    | 180981| 673123| 35    |
| 100    | 0.04    | 1481.58 | 107074| 673493 | 35.02 |
|        |         |       | 0.048| 1487.94| 107472| 671164| 35.02 |
|        |         | 120   | 0.04 | 1622.99| 117294| 672836| 35.02 |
|        |         |       | 0.048| 1629.95| 117729| 670507| 35.02 |
|        | 0.15    | 100   | 0.04 | 1497.78| 108256| 673567| 35.02 |
|        |         |       | 0.048| 1504.28| 108655| 671238| 35.02 |
|        |         | 120   | 0.04 | 1640.74| 118588| 672917| 35.02 |
|        |         |       | 0.048| 1647.86| 119036| 670588| 35.02 |
|        | 0.01    | 100   | 0.04 | 2800    | 181039| 811364| 35    |
|        |         |       | 0.048| 2800    | 180955| 808589| 35    |
|        |         | 120   | 0.04 | 2800    | 181039| 810926| 35    |
|        |         |       | 0.048| 2800    | 180955| 808589| 35    |
|        | 0.12    | 100   | 0.04 | 2800    | 181061| 811503| 35    |
|        |         |       | 0.048| 2800    | 180981| 808728| 35    |
|        |         | 120   | 0.04 | 2800    | 181061| 811066| 35    |
|        |         |       | 0.048| 2800    | 180981| 808728| 35    |
| 60000  | 0.01    | 100   | 0.04 | 2800    | 202299| 808993| 35    |
|        |         |       | 0.048| 2800    | 202182| 806222| 35    |
|        |         | 120   | 0.04 | 2800    | 202299| 808556| 35    |
|        |         |       | 0.048| 2800    | 202182| 806222| 35    |
|        | 0.15    | 100   | 0.04 | 2800    | 202318| 809132| 35    |
|        |         |       | 0.048| 2800    | 202206| 806361| 35    |
|        |         | 120   | 0.04 | 2800    | 202318| 808695| 35    |
|        |         |       | 0.048| 2800    | 202206| 805922| 35    |

Depositing the revenue into an interest bearing account from selling imperfect quality items. In general, the retailer could speed up the screening rate and then sell the defective items to salvage as soon as possible. In the meantime the retailer may also order greater quantities to obtain greater revenue. These results illustrate \( Q^* \), \( x^* \), and \( ETPU^* \) all increase in \( I \). Note that if the optimal order lot size (\( Q^* \)) occurs at the break point of ordering quantity corresponding to the lowest cost in the cost discount schedule (i.e. \( c_5 = 35 \)), \( Q^* \) remains unchanged with \( I \), while \( x^* \) and \( ETPU^* \) still increase in \( I \). The reason is similar to that in the above illustration.

4. \( Q^* \) and \( x^* \) increase with \( K \), while \( ETPU^* \) decreases in \( K \). The increase in \( Q^* \) with \( K \) is similar to the traditional EOQ/EPQ model in which greater
sized quantities are ordered to reduce the number of orders and thus decrease the setup cost in each cycle. Because the lot size in each order increases with the setup cost, the retailer may speed up the screening rate to complete the screening process as soon as possible and then sell the defective items to salvage. The retailer could thereby obtain interest income produced by selling the defective items earlier than expected. This also reduces the holding cost for items. However, increasing the order cost is greater than the cost savings caused by increasing the screening rate. The expected total profit thereby decreases with K. Note that if the optimal order lot size (Q*) occurs at the ordering quantity break point corresponding to the lowest cost in the cost discount schedule (i.e., c5 = 35), Q* and x* remain unchanged with K, while ETPU* still decreases with K.

5. Q* and x* increase in p; while ETPU* decreases in p. That is, the higher the defective rate, the higher the order quantities and screening rate, and the less expected total profit. Note that as the defective rate increases, the retailer needs more quantity per shipment to satisfy the demand. This result is consistent with Salameh and Jabers [30] work. This increment in lot size promotes the retailer to speed up the screening rate to finish the screening process to save costs. It is rather intuitive that if the defective rate increases the number of defective items increases and thus revenue decreases. Therefore, Q* and x* increase with p; while ETPU* decreases in p. Note that if the optimal order lot size (Q*) occurs at the ordering quantity break point corresponding to the lowest cost in the cost discount schedule (i.e., c5 = 35), Q* remains unchanged with p, x* increasing, and ETPU* decreasing in p.

6. Table 3 shows that none of the second order interactions between D, f, g, I, K and p are significant. Thus, an additive model is appropriate to model the relationship between Q* (x* and ETPU*) and the five parameters (D, f, g, I, K and p).

6. Conclusions. This paper constructed a retailer’s inventory model for items with imperfect quality and adjusted screening rate. This model improved the observable fact that shortages may occur during the screening process. The proposed approach takes the adjusted screening rate and quantity discount into account, which have not been discussed in previous EOQ models. The expected total profit function was derived and a solution procedure associated with an algorithm was established to find the optimal solution. The works of Silver [32] and Shih [31] are two special cases used in our model associated with c1 f2 = c1 f2 = h, and I = 0. If no defective items are produced, our model reduces to the traditional EOQ model. An investigation of five important parameter effects (the annual demand, the holding cost rate for a unit of good or re-workable item, interest rate, the ordering cost, and the defective rate) on the optimal solution was also made. Numerical results showed that (1) the lowest unit purchasing cost in the cost discount schedule may not guarantee that the retailer could obtain maximum expected total profit because the extra quantity purchased may add additional costs (holding cost and adjusted screening rate cost) and thus reduce the expected total profit. (2) As the annual demand increases, the order lot size, the screening rate and the expected annual total profit also increase. However, if the optimal order lot size occurs at the ordering quantity break point corresponding to the least unit purchasing cost in the cost discount schedule, the optimal order quantity and screening rate then remain unchanged with the annual
demand. The expected total profit increases as the annual demand increases, (3) the lower the holding cost rate for a unit of good or re-workable item, the lower the optimal order quantity, screening rate and the expected annual total profit. However, if the optimal order quantity occurs at the ordering quantity break point corresponding to the least unit purchasing cost of the cost discount schedule, the optimal order quantity then remains unchanged with the holding cost rate for a unit of good or re-workable items, while the screening rate increases and the expected total profit decreases as the holding cost rate for a unit of good or re-workable items increases. (4) As the interest rate increases, the optimal order quantity, screening rate and the expected total profit increases. If the optimal order lot size occurs at the ordering quantity break point corresponding to the lowest cost in the cost discount schedule, the optimal lot size remains unchanged with the interest rate, while the optimal screening rate and expected total profit still increase with the interest rate, (5) the higher the setup cost, the higher the optimal lot size and screening, while the expected total profit decreases with the setup cost. However, if the optimal order lot size occurs at the ordering quantity break point corresponding to the lowest cost in the cost discount schedule, the optimal lot size and screening rate remain unchanged with the setup cost; while the expected total profit still decreases with the setup cost, (6) the higher the defective rate, the higher the order quantities and screening rate, and the less expected total profit. If the optimal order lot size occurs at the ordering quantity break point corresponding to the lowest cost in the cost discount schedule, the optimal lot size remains unchanged with the defective rate, while the optimal lot size remains unchanged, the screening rate increases and the expected total profit decreases with the defective rate.

Acknowledgements. This study was partially supported by the National Science Research Council of the ROC under Grant NSC 101-2410-H-240-001.

Appendix A
Proof of property. Taking the first and second partial derivatives of $E[TPU_j(Q,x)]$ with respect to $x$, we have

$$\frac{\partial E[TPU_j(Q,x)]}{\partial x} = \frac{DQ[wIE[p] + c_j f_g \alpha + c_j (f_d + f_g) E[p]/2]}{x^2} - \frac{D \theta}{Q},$$

$$\left( j = 1, 2, \cdots, m \right)$$

(A1)

$$\frac{\partial^2 E[TPU_j(Q,x)]}{\partial x^2} = \frac{-2DQ[wIE[p] + c_j f_g \alpha + c_j (f_d + f_g) E[p]/2]}{x^3} < 0,$$

$$\left( j = 1, 2, \cdots, m \right)$$

(A2)

Therefore, for fix $Q$, $E[TPU_j(Q,x)]$ is concave in $x$ for all $j$. This completes the proof of Property.

Appendix B
According to Assumption (2), we know $x > D$. This implies the following holds from Eq. (7):

$$Q \sqrt{wIE[p] + c_j f_g (1 - E[p]) + c_j (f_d + f_g) E[p]/2} / \theta > D$$

(B1)
After some manipulations for Eq. (B1), we have
\[
\frac{Q^2[wIE[p] + c_jf_g(1 - E[p]) + c_j(f_d + f_g)E[p]/2]}{D^2} > \theta
\]
(B2)

Substituting Eq. (11) into Eq. (B2), we have
\[
\theta < \frac{K[wIE[p] + c_jf_g(1 - E[p]) + c_j(f_d + f_g)E[p]/2]}{D[(c_jf_g - vI)E[(1 - p)^2]/2 - wIE[p]\alpha]}
\]
(B3)

Observing Eq. (B3), we have
\[
wIE[p] + c_jf_g(1 - E[p]) + c_j(f_d + f_g)E[p]/2 > (c_jf_g - vI)E[(1 - p)^2]/2 - wIE[p]\alpha
\]
(B4)

This implies that \( D < \frac{\alpha}{K} \) if \( \theta \leq \frac{\alpha}{K/D} \).

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Received October 2014; 1st revision April 2015; 2nd revision October 2015

E-mail address: admtyl@ocu.edu.tw
E-mail address: bmamtc@ccu.edu.tw
E-mail address: klhou@ocu.edu.tw