SCATTERING AND RADIATION DAMPING IN
GYROSCOPIC LORENTZ ELECTRODYNAMICS

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Abstract
The nonlinear system of equations of relativistic Lorentz electrodynamics (LED) is stud-
ied in a “gyroscopic setup” in which the Lorentz electron is assumed to remain at rest,
leaving the electromagnetic fields and the particle spin as the only dynamical degrees of
freedom. The global existence and uniqueness of this gyroscopic spin-plus-field dynamics
in unbounded space is proven. It is further shown that for rotation-reflection symmetric
initial data any gyroscopic solution also satisfies the world-line equations consistent with a
non-moving Lorentz electron, thus furnishing a proper solution of the complete set of equa-
tions of LED. Rotation-reflection symmetric scattering is shown to occur for sufficiently
small ratio of electrostatic to (positive) bare rest mass, with deviations from the station-
ary spin state dying out exponentially fast through radiation damping. The previously
proven result that the renormalized spinning Lorentz electron evolves like a soliton in scat-
tering processes combined with the present results that scattering does occur establish the
solitonic character of the renormalized Lorentz electron.

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1 Introduction

In recent years the century-old Lorentz program of electrodynamics [10] has attracted the attention of mathematical physicists. Most of the rigorous results established so far belong to the semi-relativistic Abraham model and are surveyed in [14]. Recently [1] the authors presented the first properly renormalized approach to truly relativistic Lorentz electrodynamics (LED), picking up on the pioneering work [8]. While the dynamical initial value problem for the model in [8] is seriously singular, our Lorentz-covariant LED displays most of the features considered crucial for a realistic, consistent classical electrodynamics, namely:

- the Cauchy problem for the evolution of the physical state in “massive” LED with strictly positive bare rest mass and bare moment of inertia is regular;
- the pre- and post-scattering values of the renormalized electron rest mass and electron spin magnitude are identical, i.e. the Lorentz electron evolves as soliton in scattering processes;
- there exists a simple curve in the charge – bare rest mass – radius – gyration frequency — parameter space of the stationary bare particle on which the stationary renormalized particle data charge, magnetic moment, and mass match the empirical electron data without involving superluminal gyration speeds.

In [1] we also studied LED’s renormalization flow to vanishing bare rest mass with empirically matched data when the positive bare mass and charge are distributed on the surface of a sphere. The renormalized “purely electromagnetic” LED which emerges in the limit has the following additional characteristics:

- the renormalized purely electromagnetic LED constitutes a classical field theory equipped with an ultraviolet cutoff at about the physical electron’s Compton length;
- in the limit of vanishing bare rest mass the equatorial gyration speed reaches the speed of light and the bare gyrational mass converges to a “photonic” mass;
- in the same limit, the renormalized spin magnitude converges to $3\hbar/2$, up to corrections of order $\alpha$ (Sommerfeld’s fine structure constant).

In this paper we supply several rigorous results regarding scattering in LED conditioned on a straight particle world-line. For a straight particle world-line the set of Maxwell–Lorentz plus gyration equations decouples from the world-line equations, which in turn become constraint equations that impose on the admissible set of initial conditions for the dynamical equations for the spin and the electromagnetic fields. We prove that all physically reasonable Cauchy data for the fields and the spin launch unique global forward and backward evolutions of Maxwell–Lorentz plus gyration equations. For rotation-reflection symmetric data it is shown that these gyroscopic solutions satisfy the world-line equations for a non-moving Lorentz electron, too, thus furnishing solutions of the complete set of equations of LED. Rotation-reflection symmetric scattering is shown to occur if the ratio of electrostatic to bare rest mass is smaller than $\approx 1$. The previously proven result that the renormalized spinning Lorentz electron evolves like a soliton in scattering processes [1] combined with the present results that scattering does occur establish the solitonic character of the renormalized Lorentz electron. It is further shown that the rotation-reflection symmetric deviations from the soliton state die out exponentially fast through radiation damping. The results proven here are somewhat stronger and cover more general mass and charge densities than announced in [1].
2 Notation

We use the notation of \[\mathbb{I}\], which largely follows the conventions of \[\mathbb{I}\]. Abstract Minkowski space is identified with \(\mathbb{R}^{1,3}\), equipped with a Lorentzian metric of signature \(+2\). Thus, any orthonormal basis \(\{e_0, e_1, e_2, e_3\}\) satisfies the elementary inner product rules \(e_0 \cdot e_0 = -1\), \(e_{\mu} \cdot e_{\mu} = 1\) for \(\mu > 0\), and \(e_{\mu} \cdot e_{\nu} = 0\) for \(\mu \neq \nu\). A constant basis defines a Lorentz frame, denoted \(\mathcal{F}_L\). We use \(x, y, \) etc. to denote four-vectors representing events in \textit{spacetime}. With respect to \(\mathcal{F}_L\), we decompose \(x^\mu = (ct, x)\), where \(x = (x^1, x^2, x^3)\) is a “point in space,” and \(t = x^0/c\) an “instant of time,” where \(c\) is the speed of light in \textit{vacuo}. Henceforth we shall use units in which \(c = 1\). We call \(v\) spacelike, lightlike, or timelike according as \(v \cdot v > 0\), \(v \cdot v = 0\), or \(v \cdot v < 0\), respectively. We define \(\|v\|\) as the principal value of \((v \cdot v)^{1/2}\). The tensor product \(e_\mu \otimes e_\nu\) is defined by its inner-product action on four-vectors thus, \((e_\mu \otimes e_\nu) \cdot c \overset{\text{def}}{=} e_\mu (e_\nu \cdot c)\) and \(c \cdot (e_\mu \otimes e_\nu) \overset{\text{def}}{=} (e_\mu \cdot c)e_\nu\). In general a rank-two tensor reads \(T = T^\mu_\nu e_\mu \otimes e_\nu\), and if \(T^{\mu\nu} = \pm T^{\nu\mu}\) it is symmetric (+ sign), respectively \textit{anti-symmetric} (− sign). The \textit{metric tensor} \(g = g^{\mu\nu} e_\mu \otimes e_\nu\), with \(g^{\mu\nu} = e_\mu \cdot e_\nu\), is clearly symmetric and has the same components \(g^{\mu\nu}\) in all Lorentz frames. Notice that \(g\) acts as identity on four-vectors, i.e. \(g \cdot v = v\). A particular class of anti-symmetric tensors is given by the exterior product between two four-vectors, \(a \wedge b \overset{\text{def}}{=} a \otimes b - b \otimes a\). Finally, \([A, B]_\pm \overset{\text{def}}{=} A \cdot B \pm B \cdot A\) is the \textit{(anti-)commutator} of any two tensors of rank two \(A\) and \(B\).

For a differentiable function \(f(x)\) we denote by \(\nabla_g f\) its four-gradient w.r.t. \(g\). In time-plus-space decomposition, \(\nabla_g f(x) = (\partial_\mu f, \nabla f)\), where \(\nabla\) is the usual three-gradient. The four-curl of a differentiable four-vector function is defined in analogy with the conventional curl as the anti-symmetric four tensor function

\[
\nabla_g \wedge A(x) = \varepsilon^{\mu\nu\lambda\eta} e_\mu \otimes e_\nu (e_\lambda \cdot \nabla_g)(e_\eta \cdot A)
\]

where the \(\varepsilon^{\mu\nu\lambda\eta}\) are the entries of the rank-four Levi-Civita tensor. The four-Laplacian with respect to \(g\) is just the (negative) d’Alembertian, or wave operator, i.e. \(\Delta_g \overset{\text{def}}{=} \nabla_g \cdot \nabla_g = -\Box\).

3 Covariant massive LED with a straight particle world-line

In this section we present the manifestly covariant equations of massive LED \[\mathbb{I}\] for the special case that the particle’s world-line is straight.

3.1 Kinematical pre-requisites

We recall that the particle’s \textit{world-line} is a map \(\tau \mapsto x = q(\tau)\), where \(d\tau = \sqrt{-dx \cdot dx}\), with \(dx\) taken along the world-line, is the invariant \textit{proper-time} element. The map \(\tau \mapsto u = \dot{q}(\tau)\), where \(\dot{q}\) is the particle’s four-velocity, is the \textit{world hodograph}. The \textit{world gyrograph} of the particle is an anti-symmetric tensor-valued map \(\tau \mapsto \Omega_E(\tau)\) of space-space type with respect to \(u\) (i.e. \(\Omega_E \cdot u = 0\)) which describes the angular velocity of the inert gyrational motions of the Lorentz particle that may occur in excess of the inertia-free Thomas precession. Thus, \(\Omega_E \overset{\text{def}}{=} \Omega - \dot{u} \wedge u\), where \(\Omega\) is the angular velocity tensor of the particle’s co-rotating body frame, while \(\dot{u} \wedge u\) is the familiar angular velocity tensor of Fermi–Walker transport \[\mathbb{I}\].

For a straight world-line \(q(\tau) = u_0 \tau + q_0\) the particle’s four-velocity is a constant four-vector, \(u(\tau) = u_0\) for all \(\tau\). A constant four-velocity in turn implies that \(\Omega_E = \Omega\).
3.2 Field equations

The electromagnetic Maxwell–Lorentz fields are gathered into the anti-symmetric rank-two Faraday tensor field \( \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) \), which satisfies the manifestly covariant Maxwell–Lorentz equations

\[
\nabla_g \cdot \star \mathbf{F} = 0, \\
\nabla_g \cdot \mathbf{F} = 4\pi \mathbf{J},
\]

where \( \star \mathbf{F} \) is the (left) Hodge dual of \( \mathbf{F} \) and \( \mathbf{J} \) is the charge-current density four-vector field, given by Nodvik’s manifestly covariant expression

\[
\mathbf{J}(\mathbf{x}) = \int_{-\infty}^{+\infty} (\mathbf{u}_0 - \mathbf{\Omega}(\tau) \cdot \mathbf{x}) \, f_e(\|\mathbf{x} - \mathbf{q}(\tau)\|) \, \delta(\mathbf{u}_0 \cdot (\mathbf{x} - \mathbf{q}_0) + \tau) \, d\tau,
\]

where \( f_e : [0, R] \to \mathbb{R}^- \) is the \( SO(3) \) invariant charge “density” of the Lorentz particle, and \( 0 < R < \infty \) its radius. For a Lorentz electron, \( \int_{\mathbb{R}^3} f_e(|\mathbf{x}|) d^3x = -e \), where \( e > 0 \) the elementary charge.

Conditioned on the world-line \( \tau \mapsto \mathbf{q}(\tau) = \mathbf{u}_0\tau + \mathbf{q}_0 \) and gyrograph \( \tau \mapsto \mathbf{\Omega}_e(\tau) = \mathbf{\Omega}(\tau) \) being given, the Maxwell–Lorentz equations are linear equations for \( \mathbf{F} \).

3.3 World-gyrograph equations

The equations for the gyrograph are

\[
\frac{d}{d\tau} \mathbf{S}_b = \mathbf{t},
\]

where

\[
\mathbf{S}_b(\tau) = \int_{\mathbb{R}^{1,3}} (\mathbf{y} - \mathbf{u}_0\tau) \wedge \frac{-\mathbf{\Omega}_e(\tau) \cdot \mathbf{y}}{\sqrt{1 - \|\mathbf{\Omega}_e(\tau) \cdot \mathbf{y}\|^2}} f_m(\|\mathbf{y} - \mathbf{u}_0\tau\|) \, \delta(\mathbf{u}_0 \cdot \mathbf{y} + \tau) \, d^4\mathbf{y}
\]

is the anti-symmetric tensor of bare Minkowski spin \( \text{about} \ \mathbf{q}(\tau) = \mathbf{u}_0\tau + \mathbf{q}_0 \) associated with the gyrational motion of the \( SO(3) \) invariant bare rest mass “density” \( f_m : [0, R] \to \mathbb{R}^+ \) of the particle, while

\[
\mathbf{t}(\tau) = \int_{\mathbb{R}^{1,3}} (\mathbf{y} - \mathbf{u}_0\tau) \wedge (\mathbf{F}(\mathbf{y}) \cdot (\mathbf{u}_0 - \mathbf{\Omega}(\tau) \cdot \mathbf{y})) \cdot f_e(\|\mathbf{y} - \mathbf{u}_0\tau\|) \, \delta(\mathbf{u}_0 \cdot \mathbf{y} + \tau) \, d^4\mathbf{y}
\]

is the Abraham–Lorentz type Minkowski torque, with \( a^\perp \overset{\text{def}}{=} (\mathbf{g} + \mathbf{u}_0 \otimes \mathbf{u}_0) \cdot \mathbf{a} \).

3.4 World-line equations

The world-line equations are

\[
\frac{d}{d\tau} \mathbf{p} = \mathbf{f},
\]

where

\[
\mathbf{p}(\tau) = \mathbf{M}(\tau) \cdot \mathbf{u}_0
\]
is the *Minkowski momentum* four-vector of the particle, with $M = M_n + M_b g$ its symmetric *Minkowski tensor mass*, where

$$M_n(\tau) = - \int_{\mathbb{R}^{1,3}} [(y - u_0 \tau) \otimes (y - u_0 \tau), [F(y), \Omega_c(\tau)] + f_e(\|y - u_0 \tau\|) \delta(u_0 \cdot y + \tau) d^4 y$$

(3.9)

is the *Nodvik tensor mass* [1], extracted from the Minkowski momentum four-vector associated with electromagnetic spin-orbit coupling given in [8], and where

$$M_b(\tau) = \int_{\mathbb{R}^{1,3}} \left(1 - \|\Omega_c \cdot y\|^2\right)^{-\frac{1}{2}} f_m(\|y - u_0 \tau\|) \delta(u_0 \cdot y + \tau) d^4 y$$

(3.10)

is the *gyroational bare mass* [1]. Finally,

$$f(\tau) = \int_{\mathbb{R}^{1,3}} F(y) \cdot (u_0 - \Omega_c(\tau) \cdot y) f_e(\|y - u_0 \tau\|) \delta(u_0 \cdot y + \tau) d^4 y$$

(3.11)

is the Abraham–Lorentz type *Minkowski force* [8].

4 The Cauchy problem for the state in LED

We now choose a convenient Lorentz frame, called the “laboratory frame” $F_{lab}$, in which the space-plus-time decomposition of our manifestly covariant equations takes a simple form. In particular, since we consider only evolutions for which $u(\tau) = u_0$ for all $\tau$, we can work with the standard foliation of space-time in our frame $F_{lab}$. The standard foliation of $F_{lab}$ consists of the level sets $T_{F_{lab}}(x) = t$ of the function $T_{F_{lab}}(x) \equiv -e_0 \cdot x$, which has a constant timelike four-gradient $\nabla_x T_{F_{lab}}(x) = -e_0$. The space-plus-time decomposition of events in $F_{lab}$ written as $(t, x)$, is understood w.r.t. this standard foliation.

By a boost we can achieve that the timelike unit vector $e_0 = (1, 0, 0, 0)$ of $F_{lab}$ coincides with the four-velocity of the particle, i.e. $u_0 = e_0$. By at most a spacetime translation we can furthermore assume that $q(0) = 0$ in $F_{lab}$, so that the particle’s space position is at the origin of the space hypersurface of $F_{lab}$, and that laboratory time $t$ and particle proper-time $\tau$ coincide. Accordingly, from now on we will write $t$ in place of $\tau$. The world-line as seen in $F_{lab}$ is now simply given by $q(t) = (t, 0)$. As for the gyrograph, since $\Omega_c = \Omega$, we will henceforth simply omit the subscript $E$. In $F_{lab}$ we clearly have $\Omega(\tau) \cdot e_0 = 0$ for all $\tau$, so that $\Omega$ is dual to a spacelike four-vector $w(t)$ which satisfies $\Omega(t) \cdot w(t) = 0$ and $w(t) \cdot e_0 = 0$ for all $t$. Hence, in our $F_{lab}$, we have $w = (0, \omega)$, where $\omega(t)$ is the usual angular velocity three-vector, directed along the instantaneous (i.e., at time $t$) axis of body gyration in the space hypersurface of $F_{lab}$. Finally, the field tensor $F(x)$ at $x$ is decomposed as usual into its electric and magnetic Maxwell–Lorentz components w.r.t. the standard foliation of $F_{lab}$, here conveniently grouped together as a complex electromagnetic three-vector field,

$$G(x, t) \equiv E(x, t) + iB(x, t),$$

(4.1)

whose real and imaginary part are, respectively, the electric (i.e. time-space) and magnetic (i.e. space-space) components of the field tensor $F$ in $F_{lab}$. Since by hypothesis $q(t) = (t, 0)$ for all $t$, the state at time $t$ in LED is uniquely characterized by specifying $\omega(t)$ and $G(\cdot, t)$. 

5
4.1 Evolution equations

The covariant equations now decompose into a system of first-order evolution equations for the state variables of LED, plus a set of constraint equations. We begin with the evolution equations.

4.1.1 Field equations

Beginning with the covariant field equations, we note the space-plus-time decomposition of the current density four-vector as \( J(x) = (1, \omega(t) \times x)f_e(|x|) \). The space components of the covariant field equations combine into the Maxwell–Lorentz evolution equations for \( \mathcal{G} \),

\[
\partial_t \mathcal{G}(x,t) = -i \nabla \times \mathcal{G}(x,t) - 4\pi \omega(t) \times x f_e(|x|),
\]

where \( \partial_t \) means first-order partial derivative w.r.t. Lorentz time and \( \nabla \times \) is the standard curl operator.

4.1.2 Spin equations

Turning next to the gyrational equations, we recall that \( \Omega_e \) is dual to the space vector \( \omega \). In the same vein, the space projector \( g + u_0 \otimes u_0 \) under the integral in (3.6) guarantees the space-space character of \( S \) w.r.t. \( u_0 \), i.e. \( S \cdot u_0 = 0 \) for all \( \tau \), so that the bare spin Minkowski tensor (3.5) is dual to the space vector of bare spin,

\[
s_b(t) = \int_{\mathbb{R}^3} \frac{x \times (\omega(t) \times x)}{\sqrt{1 - |\omega(t) \times x|^2}} f_m(|x|) \, d^3x,
\]

and the Minkowski torque (3.6) is dual to the torque space vector

\[
t(t) = \int_{\mathbb{R}^3} x \times (\mathcal{E}(x,t) + (\omega(t) \times x) \times \mathcal{B}(x,t)) f_e(|x|) \, d^3x.
\]

Equation (3.4) together with (3.6) is therefore dual to the evolution equation

\[
\frac{d}{dt}s_b = t,
\]

for \( \omega(t) \).

4.2 Constraint equations

4.2.1 Divergence equations

The time components of the covariant field equations combine into the Maxwell–Lorentz divergence equation

\[
\nabla \cdot \mathcal{G}(x,t) = 4\pi f_e(|x|).
\]

Notice that (4.6) is merely a constraint on the set of initial data, for the (three-) divergence of (4.2) implies that a solution \( \mathcal{G}(x,t) \) of (4.2) for given \( \omega(t) \times x f_e(|x|) \) automatically satisfies (4.6) for all \( t > 0 \) if the initial data \( \mathcal{G}_0 \) for \( \omega(t) \times x f_e(|x|) \) satisfy the constraint (4.6) at time \( t = 0 \), i.e. if \( \nabla \cdot \mathcal{G}_0(x) = 4\pi f_e(|x|) \).
4.2.2 World-line equations

The four-momentum $p$ has the space-plus-time decomposition $p = (M_b, \mathbf{N}_e \cdot \omega)$, where

$$M_b(t) = \int_{\mathbb{R}^3} \frac{1}{\sqrt{1 - |\omega(t) \times \mathbf{x}|^2}} f_m(|\mathbf{x}|) \, d^3x$$ (4.7)

is the bare gyrational mass at time $t$, and where

$$\mathbf{N}_e(t) = \int_{\mathbb{R}^3} \mathbf{x} \otimes \left( \mathbf{x} \times \mathbf{E}(\mathbf{x}, t) \right) f_e(|\mathbf{x}|) \, d^3x$$ (4.8)

is a spin-orbit coupling tensor. Furthermore, the Abraham–Lorentz type Minkowski force now has the space-plus-time decomposition $f = (P, \mathbf{f})$, where

$$P(t) = \omega(t) \cdot \int_{\mathbb{R}^3} \left( \mathbf{x} \times \mathbf{E}(\mathbf{x}, t) \right) f_e(|\mathbf{x}|) \, d^3x$$ (4.9)

is the power delivered by the field to the particle, and where

$$\mathbf{f}(t) = \int_{\mathbb{R}^3} \left( \mathbf{E}(\mathbf{x}, t) + \left( \omega(t) \times \mathbf{x} \right) \times \mathbf{B}(\mathbf{x}, t) \right) f_e(|\mathbf{x}|) \, d^3x$$ (4.10)

is the Abraham–Lorentz force on the particle. The space-plus-time decomposition of the world-line equation then becomes

$$\frac{d}{dt} M_b = P$$ (4.11)

and

$$\frac{d}{dt} (\mathbf{N}_e \cdot \omega) = \mathbf{f}.$$ (4.12)

Despite their appearance, equations (4.11) and (4.12) are not evolution equations for the world-line; instead, they have to be satisfied by the active state variables $\omega(t)$ and $\mathcal{G}(., t)$ to ensure consistency with the constraint that the world-line is given by $q(t) = e_0 t$ in $\mathcal{F}_{\text{lab}}$. However, we shall show that (4.11) is automatically satisfied for all time by any solution of the evolution equations for spin and fields that obeys the divergence equations initially. This leaves (4.12) as the only true constraint equation coming from the world-line equation. While we will show that certain symmetric initial conditions launch a dynamics consistent with (4.12), it seems difficult to precisely characterize the complete set of initial conditions that will launch such a consistent dynamics.

4.3 Cauchy data

The field evolution equation (4.2) are supplemented by initial data consistent with the constraint equations (4.6) and satisfying the asymptotic condition that $\mathcal{G}(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$, the real part as $\mathcal{E}(\mathbf{x}, t) \sim -e \mathbf{x}/|\mathbf{x}|^3 + o(|\mathbf{x}|^{-2})$, the imaginary part satisfying $|\mathcal{B}| = O(|\mathbf{x}|^{-3})$.

Equation (4.3) is to be supplemented by initial data $\omega(0) = \omega_0$ satisfying the requirement of strict subluminality, $|\omega_0| R < 1$, or subluminality, $|\omega_0| R \leq 1$, depending on the choice of $f_m$. 

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Viewed from a dynamical systems perspective, Cauchy data may be prescribed in any consistent manner, and for our existence and uniqueness result of a strong solution in some weighted $L^1$ norm we only need that the cumulative time integral of the wave fields over the support of the particle stays bounded. However, the scope of LED as a theory, in the classical limit, of the dynamics of an electron coupled to the electromagnetic fields, its self-fields included, basically limits the physically sensible choices of initial data to a stationary electron well-separated from some localized radiation field that has compact support in space disjoint from the fixed support of the particle. To have a dynamically interesting scenario, the time-evolved support of the initial radiation fields should eventually overlap with the support of the electron.

5 Gyroscopic LED

We study first the subsystem of equations obtained by neglecting the world-line equations (4.11), (4.12) from the LED with a straight world-line. For obvious reasons, we will call this dynamical model the gyroscopic LED.

We have to solve the Maxwell–Lorentz equations (4.2), (4.6) for the field (4.1) together with the gyrograph equations (4.5), (4.4) for the bare spin (4.3). Our strategy is to solve first the Maxwell–Lorentz equations in terms of integral representations involving the unknown bare spin dynamics. Inserting this representation into the gyrograph equation, we rewrite the latter into a fixed point problem for $s_b(t)$. We then prove that the fixed point map is a Lipschitz map, from which the global well-posedness of the gyroscopic problem follows. Subsequently we will show that the gyroscopic problem conserves the energy, angular momentum and the canonical spin magnitude, but generally not the linear momentum. Energy conservation is coincidental with the fact that (4.11) is automatically satisfied by a gyroscopic solution.

5.1 Forward integration of the Maxwell-Lorentz equations

We recall that in virtue of the homogeneous Maxwell–Lorentz equations (3.1), there exists a (non-unique) four-vector field $A$ satisfying the Lorentz gauge $\nabla \cdot A = 0$ such that $F = \nabla \wedge A$. The inhomogeneous Maxwell–Lorentz equation (3.3) then becomes the inhomogeneous wave equation $\Box A(x) = 4\pi J(x)$. Recalling furthermore the time-plus-space decomposition for the current density four-vector, $J(x) = (1, \omega(t) \times x)f_e(|x|)$, and introducing the time-plus-space decomposition for the electromagnetic potential four-vector as $A(x) = (\phi(x,t), A(x,t))$, the equation $F = \nabla \wedge A$ becomes

$$G(x,t) = -\nabla \phi(x,t) - \partial_t A(x,t) + i \nabla \times A(x,t). \tag{5.1}$$

The Coulomb potential $\phi$ and vector potential $A$ satisfy the inhomogeneous wave equations

$$\Box \phi(x,t) = 4\pi f_e(|x|), \tag{5.2}$$

$$\Box A(x,t) = 4\pi f_e(|x|) \omega(t) \times x, \tag{5.3}$$

supplemented (i) by the asymptotic conditions $\phi(x,t) \sim -e|x|^{-1}$ and $A(x,t) \sim \mu_0 \times x |x|^{-3}$ as $|x| \to \infty$, for all $t \in \mathbb{R}$, where $\mu_0$ is the particle’s magnetic moment at $t = 0$,

$$\mu_0 = \frac{1}{2} \int_{\mathbb{R}^3} x \times (\omega_0 \times x)f_e(|x|)d^3x \tag{5.4}$$
with $\omega_0 = \omega(0)$, and (ii) by compatible Cauchy data at $t = 0$.

We first integrate the wave equations (5.2), (5.3) for the potentials $\phi$ and $A$. Clearly, (5.2) is solved by $\phi(x, t) = \phi_{\text{Coul}}(x) + \phi_{\text{wave}}(x, t)$, where

$$
\phi_{\text{Coul}}(x) = \int_{\mathbb{R}^3} \frac{1}{|x - y|} f_e(|y|) \, d^3 y
$$

(5.5)

is the static Coulomb potential for $f_e$ and $\phi_{\text{wave}}(x, t)$ is a solution of the homogeneous scalar wave equation $\Box \phi_{\text{wave}}(x, t) = 0$. After at most a gauge transformation, we may assume that $\phi_{\text{wave}} \equiv 0$. Next, (5.3) for $t > 0$ is solved by $A(x, t) = A_{\text{source}}(x, t) + A_{\text{wave}}(x, t)$, where

$$
A_{\text{source}}(x, t) = \int_{\mathbb{R}^3} \left( \omega_0 + \Theta(t - |x - y|)(\omega(t - |x - y|) - \omega_0) \right) \times \frac{y}{|x - y|} f_e(|y|) \, d^3 y
$$

(5.6)

solves the inhomogeneous vector wave equation (5.3) ($\Theta$ is the Heaviside function), and where $A_{\text{wave}}(x, t)$ solves the homogeneous vector wave equation $\Box A_{\text{wave}}(x, t) = 0$ for initial data $A_{\text{wave}}(x, 0) = A_0'(x)$ and $\partial_t A_{\text{wave}}(x, 0) = -E_0'(x)$, where $A_0'(x) = A_0(x) - A_{\text{source}}(x, 0)$, with $A_0(x)$ the initial magnetic vector potential, and where $E_0'(x) = E_0(x) + \nabla \phi_{\text{Coul}}(x)$, with $E_0(x)$ the initial electric field strength. Thus $A_{\text{wave}}$ is given by Kirchhoff’s formula

$$
A_{\text{wave}}(x, t) = -\frac{1}{t} \int_{\partial B_t(x)} E_0'(y) d\Omega_y + \frac{\partial}{\partial t} \left( \frac{1}{t} \int_{\partial B_t(x)} A_0'(y) d\Omega_y \right),
$$

(5.7)

where $d\Omega_y$ is the uniform surface measure on $\partial B_t(x)$ divided by $4\pi$.

### 5.2 Canonical form of the gyrograph equation

With the help of the potential representation of $G$ we now rewrite (4.3), (4.4) into the more accessible canonical format. Recalling that $E(x, t) = -\nabla \phi_{\text{Coul}}(x) - \partial_t A(x, t)$, with $\phi_{\text{Coul}}(x)$ given in (5.5), and with $B(x, t) = \nabla \times A(x, t)$, and noticing that $x \times \nabla \phi_{\text{Coul}}(x) = 0$, we find

$$
t(t) = \int_{\mathbb{R}^3} x \times \left( -\partial_t A(x, t) + (\omega(t) \times x) \times \nabla \times A(x, t) \right) f_e(|x|) \, d^3 x
$$

$$
= -\frac{1}{dt} \int_{\mathbb{R}^3} x \times A(x, t) f_e(|x|) \, d^3 x + \int_{\mathbb{R}^3} x \times \left( (\omega(t) \times x) \times \nabla \times A(x, t) \right) f_e(|x|) \, d^3 x.
$$

(5.8)

The last term in (5.8) can be rewritten as

$$
\int_{\mathbb{R}^3} x \times ((\omega(t) \times x) \times \nabla \times A(x, t)) f_e(|x|) \, d^3 x = \omega(t) \times \int_{\mathbb{R}^3} x \times A(x, t) f_e(|x|) \, d^3 x.
$$

(5.9)

To verify (5.9), first note that $x \times ((\omega(t) \times x) \times \nabla \times A(x, t)) = \omega(t) \times x \times (x \cdot \nabla \times A(x, t))$ (for $x \cdot (\omega \times x) = 0$) and pull $\omega(t)$ in front of the integral, next use $\nabla \times x = 0$ and another standard identity from vector analysis to rewrite $x \cdot \nabla \times A = x \cdot \nabla \times A - A \cdot \nabla \times x = \nabla \cdot (A \times x)$, then integrate by parts, use the identity $(x \times A(x, t)) \cdot \nabla f_e(|x|) = 0$ and get

$$
\int_{\mathbb{R}^3} x(x \cdot \nabla \times A(x, t)) f_e(|x|) \, d^3 x = \int_{\mathbb{R}^3} (x \times A(x, t)) \cdot \nabla (x f_e(|x|)) \, d^3 x
$$

$$
= \int_{\mathbb{R}^3} x \times A(x, t) f_e(|x|) \, d^3 x.
$$

(5.10)
as claimed. Defining now the electromagnetic field spin vector of the particle by

\[ s_{\text{f}}(t) = \int_{\mathbb{R}^3} x \times A(x, t) f_{\text{e}}(|x|) d^3x, \]  

(5.11)

and its canonical spin vector by \( s = s_{\text{b}} + s_{\text{f}} \), and finally recalling that \( \omega \times s_{\text{b}} = 0 \), we conclude that (4.13) can be recast into the canonical evolution equation for the spin (in \( \mathcal{F}_{\text{lab}} \)),

\[ \frac{d}{dt} s = \omega \times s. \]  

(5.12)

**Remark:** It follows directly from (5.12) that \( |s| \) is conserved during the evolution.

### 5.3 The bare spin / angular velocity relation

Inserting the explicit integral representation for \( A(x, t) \) into the canonical equation (5.12), and recalling that \( s_{\text{b}}(t) \) is given in terms of \( \omega(t) \) by (4.3), we see that (5.12) becomes a closed, non-autonomous, nonlinear first-order vector differential equation for \( \omega(t) \). However, it is advisable to eliminate \( \omega(t) \) in favor of \( s_{\text{b}}(t) \).

We rewrite (4.3) as

\[ s_{\text{b}}(t) = I_{\text{b}}(|\omega(t)|) \cdot \omega(t), \]

where

\[ I_{\text{b}}(|\omega|) = \int_{\mathbb{R}^3} \frac{|x|^2 1 - x \otimes x}{\sqrt{1 - |\omega \times x|^2}} f_m(|x|) d^3x, \]  

(5.13)

is the inertia tensor of the bare particle. Clearly, \( I_{\text{b}} \) acts as a number on \( \omega \), viz. \( I_{\text{b}} \cdot \omega = I_{\text{b}} \omega \).

Performing the angular integrations we are left with

\[ I_{\text{b}}(|\omega|) = 2\pi \frac{1}{|\omega|^4} \int_0^{|\omega| R} f_m\left(\frac{\xi}{|\omega|}\right) \left( (\xi^2 + 1) \text{artanh}(\xi) - \xi \right) \xi d\xi. \]  

(5.14)

By hypothesis, \( 0 < I_{\text{b}}(0) < \infty \). This implies that the map \( |\omega| \mapsto I_{\text{b}}(|\omega|) \) is strictly positive, increasing, and strictly convex for \( |\omega| \in [0, 1/R] \). Depending on the choice for \( f_m \), the bare spin magnitude \( |s_{\text{b}}| \) may or may not approach a finite limit \( s_{\text{b}}^* \) as \( |\omega| R \to 1 \). In any event, it follows that for \( |s_{\text{b}}| < s_{\text{b}}^* \), we can invert the map \( \omega \mapsto s_{\text{b}} = I_{\text{b}}(|\omega|) \omega \) to get the Euler angular velocity vector \( \omega \) uniquely in terms of the bare spin vector \( s_{\text{b}} \), viz. \( \omega = \mathcal{W}(s_{\text{b}}) \), where

\[ \mathcal{W}(s_{\text{b}}) = \frac{s_{\text{b}}}{|s_{\text{b}}|} (I_{\text{b}} \text{id})^{-1}(|s_{\text{b}}|) \quad \text{for} \quad |s_{\text{b}}| < s_{\text{b}}^* . \]  

(5.15)

Note that the map \( |s_{\text{b}}| \mapsto (I_{\text{b}} \text{id})^{-1}(|s_{\text{b}}|) \) is bounded, strictly increasing, and concave, hence it has its steepest slope when \( |s_{\text{b}}| \to 0^+ \). This slope at \( 0^+ \) is simply the reciprocal value of the slope of the map \( |\omega| \mapsto |\omega| I_{\text{b}}(|\omega|) \) at \( |\omega| \to 0^+ \), viz. slope of \( (I_{\text{b}} \text{id})^{-1} \leq I_{\text{b}}(0) \ (\ < \infty \), for \( I_{\text{b}}(0) > 0 \), by hypothesis). Finally, if \( s_{\text{b}}^* < \infty \), we extend \( \mathcal{W} \) continuously differentiably to \( \mathbb{R}^3 \) by setting

\[ \mathcal{W}(s_{\text{b}}) \overset{\text{def}}{=} \frac{1}{R |s_{\text{b}}|} s_{\text{b}} \quad \text{for} \quad |s_{\text{b}}| \geq s_{\text{b}}^*. \]  

(5.16)
5.4 Bare spin evolution as fixed point problem

Substituting $\mathcal{W}(s_b)$ for $\omega$ in (5.12) and integrating (5.12) w.r.t. $t$, supplementing the initial datum $s_b(0)$ (automatically compatible with the subliminality requirement $|\omega_0|R \leq 1$, and writing out dependencies on $\mathcal{A}$ explicitly, we arrive at the following integral equation for $s_b$,

$$s_b(t) = s_b(0) + \int_{\mathbb{R}^3} x \times \left( \mathcal{A}_0(x) - \mathcal{A}(x,t) \right) f_e(|x|) \, \mathrm{d}^3x$$

$$+ \int_0^t \mathcal{W}(s_b(\tilde{t})) \times \int_{\mathbb{R}^3} x \times \mathcal{A}(x,\tilde{t}) f_e(|x|) \, \mathrm{d}^3x \, \mathrm{d}\tilde{t},$$

where $\mathcal{A} = \mathcal{A}_{\text{wave}} + \mathcal{A}_{\text{source}}$ is given by the integral representations (5.7) and (5.6), and where $\omega(t) = \mathcal{W}(s_b(t))$ in (5.6), closing the chain. Substituting (5.6) for $\mathcal{A}_{\text{source}}$ in (5.17) and rearranging some integrations gives the explicit fixed-point equation for $s_b$,

$$s_b(t) = s_b(0) + \int_{\mathbb{R}^3} x \times \left( \mathcal{A}_{\text{wave}}(x,0) - \mathcal{A}_{\text{wave}}(x,t) \right) f_e(|x|) \, \mathrm{d}^3x$$

$$- \int_0^t \left( \mathcal{W}(s_b(\tilde{t})) - \omega_0 \right) K(t - \tilde{t}) \, \mathrm{d}\tilde{t}$$

$$+ \int_0^t \mathcal{W}(s_b(\tilde{t})) \times \int_{\mathbb{R}^3} x \times \mathcal{A}_{\text{wave}}(x,\tilde{t}) f_e(|x|) \, \mathrm{d}^3x \, \mathrm{d}\tilde{t}$$

$$- \omega_0 \times \int_0^t \mathcal{W}(s_b(\tilde{t})) \int_{\tilde{t}}^{2R} K(t') \, \mathrm{d}t' \, \mathrm{d}\tilde{t}$$

$$+ \int_0^t \mathcal{W}(s_b(\tilde{t})) \times \int_0^t \mathcal{W}(s_b(t')) K(\tilde{t} - t') \, \mathrm{d}t' \, \mathrm{d}\tilde{t},$$

where $K$ is the electron’s retarded self-interaction kernel,

$$K(t) = \frac{2}{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x \cdot y}{|x - y|} f_e(|x|) f_e(|y|) \delta(t - |x - y|) \, \mathrm{d}^3x \, \mathrm{d}^3y.$$ (5.19)

Notice that $K \in L^\infty(\mathbb{R})$, and that $\text{supp}(K) \subseteq [0, 2R]$. By the SO(3) invariance of $f_e$ we can carry out the angular integrations in (5.19), obtaining a double integral,

$$K(t) = \frac{8\pi^2}{3} \int_0^R \int_0^R \Theta(t - |r - s|) \Theta(r + s - t)(r^2 + s^2 - t^2) r s f_e(r) f_e(s) \, \mathrm{d}r \, \mathrm{d}s.$$ (5.20)

5.5 Lipschitz estimates

**Lemma 1**: The map $\mathcal{W} : \mathbb{R}^3 \to \mathbb{R}^3$ is Lipschitz continuous for the standard Euclidean norm, with Lipschitz constant $1/\mathcal{I}_b(0)$.

**Proof**. The sole action of $\mathcal{W}$ is to scale any input vector $u$ by the factor $|\mathcal{W}(u)|/|u|$, with $|\mathcal{W}(u)| = (\mathcal{I}_b \text{id})^{-1}(|u|)$ for $|u| < s_b^2$, and $|\mathcal{W}(u)| = 1/R$ for $|u| \geq s_b^2$. The map $|u| \mapsto (\mathcal{I}_b \text{id})^{-1}(|u|)$ is increasing and concave, vanishing with finite slope $1/\mathcal{I}_b(0)$ for $|u| \to 0^+$, and saturating for $|u| \to s_b^2$ to $(\mathcal{I}_b \text{id})^{-1}(|u|) \to 1/R$, with vanishing slope. Thus, $|\mathcal{W}(u)|/|u|$ is monotonic decreasing and bounded above by $\lim_{|u| \to 0^+}(|\mathcal{W}(u)|/|u|) = 1/\mathcal{I}_b(0)$. Hence, the map $\mathcal{I}_b(0)\mathcal{W}$ shrinks any input vector $u$ by a factor which is the smaller the longer $u$ is, but leaving its direction unchanged. It now follows right away that $|\mathcal{I}_b(0)\mathcal{W}(u_1) - \mathcal{I}_b(0)\mathcal{W}(u_2)| \leq |u_1 - u_2|$ for any two vectors $u_1$ and $u_2$. QED
Lemma 2: The two-point map $\mathbf{W}^{*2} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $(u, v) \mapsto \mathbf{W}(u) \times \mathbf{W}(v)$ is Lipschitz continuous with Lipschitz constant $(\mathcal{I}_b(0)R)^{-1}$.

Proof. By a simple identity, followed by the triangle inequality, followed by the upper bound $|\mathbf{W}| \leq 1/R$ and by Lemma 1, we find

$$\left|\mathbf{W}(u_1) \times \mathbf{W}(v_1) - \mathbf{W}(u_2) \times \mathbf{W}(v_2)\right|$$

$$= \left|\mathbf{W}(u_1) \times (\mathbf{W}(v_1) - \mathbf{W}(v_2)) + (\mathbf{W}(u_1) - \mathbf{W}(u_2)) \times \mathbf{W}(v_2)\right|$$

$$\leq |\mathbf{W}(u_1)|\mathbf{W}(v_1) - \mathbf{W}(v_2)| + |\mathbf{W}(u_1) - \mathbf{W}(u_2)|\mathbf{W}(v_2)|$$

$$\leq R^{-1}\left(|\mathbf{W}(v_1) - \mathbf{W}(v_2)| + |\mathbf{W}(u_1) - \mathbf{W}(u_2)|\right)$$

$$\leq (R\mathcal{I}_b(0))^{-1}(|u_1 - u_2| + |v_1 - v_2|) \quad \text{QED}$$

Writing $\mathbf{s}_b = \mathbf{F}(\mathbf{s}_b)$ defines a map $\mathbf{F}$ in the space $L^1_1(\mathbb{R}^+, \mathbb{R}^3)$ of $\mathbb{R}^3$-valued functions $u$ on $\mathbb{R}^+$, equipped with the weighted $L^1$ norm $\|u\|_{1,\lambda} = \int_0^\infty \exp(-\lambda t)|u(t)|dt$, $\lambda > 0$. Since by assumption the integral of the wave fields over the particle support is bounded for all $t$, there exist two constants $C_1$ and $C_2$, determined by the initial data alone, such that $|\mathbf{F}(u)| < C_1 + C_2 t$ for any $u \in L^1_1(\mathbb{R}^+, \mathbb{R}^3)$. Hence, $\mathbf{F}$ maps $L^1_1(\mathbb{R}^+, \mathbb{R}^3)$ into some ball $\{\|u\|_{1,\lambda} \leq C\} \subset L^1_1(\mathbb{R}^+, \mathbb{R}^3)$, where $C$ is determined by the initial data. This also implies that $\|\mathbf{s}_b\|_{1,\lambda}$ is well defined for any solution of $(5.18)$.

Proposition 1: The map $u \mapsto \mathbf{F}(u)$ is $\|u\|_{1,\lambda}$-Lipschitz continuous with Lipschitz constant

$$L = \frac{1}{\lambda \mathcal{I}_b(0)} \left(\|\mathbf{s}_{\text{wave}}\|_\infty + \left(1 + \frac{1}{\lambda R}\right)\|K\|_\infty + \frac{2}{R}\|K\|_1\right),\quad (5.21)$$

where $\|\mathbf{s}_{\text{wave}}\|_\infty = \sup_t \int_{\mathbb{R}^3} x \times \mathbf{A}_{\text{wave}}(x, t)f_e(\|x\|)d^3x < \infty$; $\|K\|_\infty = \sup_{t \in [0, 2R]} |K(t)| < \infty$, and $\|K\|_1 = \int_0^{2R} |K(t)| dt < \infty$.

Proof. By definition of $\mathbf{F}$,

$$\mathbf{F}(u)(t) - \mathbf{F}(v)(t) = -\int_0^t \left(\mathbf{W}(u(\tilde{t})) - \mathbf{W}(v(\tilde{t}))\right)K(t - \tilde{t})d\tilde{t}$$

$$+ \int_0^t \left(\mathbf{W}(u(\tilde{t})) - \mathbf{W}(v(\tilde{t}))\right) \times \int_{\mathbb{R}^3} x \times \mathbf{A}_{\text{wave}}(x, \tilde{t})f_e(\|x\|)d^3x d\tilde{t}$$

$$- \omega_0 \times \int_0^t \left(\mathbf{W}(u(\tilde{t})) - \mathbf{W}(v(\tilde{t}))\right) \int_{\tilde{t}'}^{2R} K(t')dt' d\tilde{t}$$

$$+ \int_0^t \int_0^{t'} \left(\mathbf{W}(u(\tilde{t})) - \mathbf{W}(v(\tilde{t}))\right) \times \mathbf{W}(v(t'))K(\tilde{t} - t')dt' d\tilde{t}.$$
Subadditivity of the norm gives
\[
\| \mathcal{F}(u) - \mathcal{F}(v) \|_{1, \lambda} \leq \left\| \int_0^t \left( \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right) K(t - \tilde{t}) \, d\tilde{t} \right\|_{1, \lambda} \\
+ \left\| \int_0^t \left( \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right) \times \int_{\mathbb{R}^3} x \times A_{\text{wave}}(x, \tilde{t}) f_e(|x|) \, dx \, d\tilde{t} \right\|_{1, \lambda} \\
+ \left\| \omega_0 \times \int_0^t \left( \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right) \int_{t}^{2R} K(t') dt' \, d\tilde{t} \right\|_{1, \lambda} \\
+ \left\| \int_0^t \int_0^t \left( \mathcal{W}(u(\tilde{t})) \times \mathcal{W}(u(t')) - \mathcal{W}(v(\tilde{t})) \times \mathcal{W}(v(t')) \right) K(\tilde{t} - t') dt' \, d\tilde{t} \right\|_{1, \lambda}.
\]

We now estimate one by one the terms on the right-hand side. For the first term we find
\[
\left\| \int_0^t \left( \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right) K(t - \tilde{t}) \, d\tilde{t} \right\|_{1, \lambda} \\
\leq \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right| |K(t - \tilde{t})| \, d\tilde{t} \, dt \\
\leq \|K\|_{\infty} \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right| \, d\tilde{t} \, dt \\
= \|K\|_{\infty} \lambda^{-1} \int_0^t e^{-\lambda t} \left| \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right| \, dt \\
\leq \|K\|_{\infty} (\lambda I_b(0))^{-1} \int_0^t e^{-\lambda t} |u(t) - v(t)| \, dt \\
= \|K\|_{\infty} (\lambda I_b(0))^{-1} \|u - v\|_{1, \lambda},
\]
where in the third step we used integration by parts together with \(|\mathcal{W}| < 1/R\) and with \(te^{-\lambda t} = 0\) for \(t = 0\) and \(t \to \infty\). The last estimate then is Lemma 1. Similarly, for the second term we find
\[
\left\| \int_0^t \left( \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right) \times \int_{\mathbb{R}^3} x \times A_{\text{wave}}(x, \tilde{t}) f_e(|x|) \, dx \, d\tilde{t} \right\|_{1, \lambda} \\
\leq \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right| \left\| \int_{\mathbb{R}^3} x \times A_{\text{wave}}(x, \tilde{t}) f_e(|x|) \, dx \right\| \, d\tilde{t} \, dt \\
\leq \|s_{\text{wave}}\|_{\infty} \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(\tilde{t})) - \mathcal{W}(v(\tilde{t})) \right| \, d\tilde{t} \, dt \\
= \|s_{\text{wave}}\|_{\infty} \lambda^{-1} \int_0^t e^{-\lambda t} \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| \, dt \\
\leq \|s_{\text{wave}}\|_{\infty} (\lambda I_b(0))^{-1} \int_0^t e^{-\lambda t} |u(t) - v(t)| \, dt \\
= \|s_{\text{wave}}\|_{\infty} (\lambda I_b(0))^{-1} \|u - v\|_{1, \lambda}.
\]
Proceeding analogously for the third term, we find
\[
\left\| \omega_0 \times \int_0^t \left( \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right) K(t') dt' \right\|_{1, \lambda} \\
\leq |\omega_0| \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| |K(t')| dt' dt \\
\leq |\omega_0| \|K\|_1 \lambda^{-1} \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| dt' dt \\
\leq |\omega_0| \|K\|_1 \lambda^{-1} \int_0^t e^{-\lambda t} \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| dt \\
= |\omega_0| \|K\|_1 \left( \lambda I_b(0) \right)^{-1} \int_0^t e^{-\lambda t} \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| dt.
\]
For the fourth term we need Lemma 2, otherwise we proceed along the same lines to find
\[
\left\| \int_0^t \int_0^t \left( \mathcal{W}(u(t')) - \mathcal{W}(v(t')) \right) K(t' - t') dt' dt \right\|_{1, \lambda} \\
\leq \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| |K(t' - t')| dt' dt \\
= \lambda^{-1} \int_0^t e^{-\lambda t} \int_0^t \left| \mathcal{W}(u(t)) - \mathcal{W}(v(t)) \right| |K(t' - t')| dt' dt \\
\leq (\lambda R I_b(0))^{-1} \int_0^t e^{-\lambda t} \left( |u(t) - v(t)| + |u(t') - v(t')| \right) |K(t' - t')| dt' dt \\
= (\lambda R I_b(0))^{-1} \int_0^t e^{-\lambda t} \left( |u(t) - v(t)| + |u(t') - v(t')| \right) |K(t' - t')| dt' dt \\
\leq (\lambda R I_b(0))^{-1} \left( \|K\|_1 \|u - v\|_{1, \lambda} + \lambda^{-1} \|K\|_\infty \|u - v\|_{1, \lambda} \right).
\]
Adding all estimates together and finally noting that |\omega_0| R \leq 1, we find that
\[
\| \mathcal{F}(u) - \mathcal{F}(v) \|_{1, \lambda} \leq L \|u - v\|_{1, \lambda}
\]
with L given in (5.21).

QED

5.6 Global well-posedness

The existence of a unique \| \cdot \|_{1, \lambda}-strong forward solution \( t \mapsto s_b(t), t \geq 0 \), of (5.12) now follows right away from the \| \cdot \|_{1, \lambda}-Lipschitz continuity of \( \mathcal{F} \) and the fact that \( \mathcal{F} \) maps \( L^1_{\lambda}(\mathbb{R}^+, \mathbb{R}^3) \) into some ball \( \| \cdot \|_{1, \lambda} \leq C \), with C determined by the initial data. Moreover, we can exchange \( t \to -t \) and the conclusions holds for the backward evolution as well. Furthermore, for any permissible incoming data (not necessarily scattering data) \( \mathcal{A}_{\text{wave}}(x, 0) \) we can find a \( \lambda_* \) such that \( L < 1 \) for all \( \lambda > \lambda_* \). We summarize these findings in the following theorem.

**Theorem 1:** There exists a unique \( \| \cdot \|_{1, \lambda}-strong \) solution \( t \mapsto s_b(t) \) of (5.12) globally in \( t \in \mathbb{R} \). Furthermore, for all \( \lambda > \lambda_* \) the map \( \mathcal{F} \) is a \( \| \cdot \|_{1, \lambda}-contraction \) mapping, and in these norms the simple iteration
\[
s_b^{(n+1)} = \mathcal{F}(s_b^{(n)}), \tag{5.22}
\]
starting with initial datum \( s_b^{(0)} \equiv s_b(0) \), converges \( \| \cdot \|_{1, \lambda}-strongly \) to the solution \( t \mapsto s_b(t) \).
Proposition 2: The following quantities are conserved during the evolution:

\[-e = \int_{\mathbb{R}^3} \rho \, d^3x \quad \text{(charge)},\]

\[W = \frac{1}{8\pi} \int_{\mathbb{R}^3} (|\mathcal{E}|^2 + |\mathcal{B}|^2) \, d^3x + \mathcal{M}_b(|\mathbf{\omega}|) \quad \text{(energy)},\]

\[\mathcal{L} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbf{x} \times (\mathcal{E} \times \mathcal{B}) \, d^3x + \mathbf{s}_b \quad \text{(angular momentum)},\]

\[\sigma = |\mathbf{s}_b + \mathbf{s}_l| \quad \text{(canonical spin magnitude)}.\]

Proof. We basically follow [4] where the conservation laws for the semi-relativistic theory are discussed.

As for charge conservation, by way of construction [8], LED honors the continuity equation

\[\partial_t \rho(x, t) + \nabla \cdot \mathbf{j}(x, t) = 0,\]

where \(\rho\) is the electric charge density and \(\mathbf{j}\) the vector of the electric current density, and this fact does not change by simply imposing the condition that the world-line be straight. Indeed, one directly verifies that for our \(\mathbf{j}(x, t) = f_e(|x|) \mathbf{\omega}(t) \times x\) we have \(\nabla \cdot (\mathbf{\omega}(t) \times x f_e(|x|)) = 0\), and of course \(\rho(x, t) = f_e(|x|)\) independent of \(t\), i.e. \(\partial_t \rho(x, t) = 0\). Hence, charge is conserved.

As for the energy conservation, taking the time derivative of the field energy gives us [3]

\[\frac{d}{dt} \left( \frac{1}{8\pi} \int_{\mathbb{R}^3} (|\mathcal{E}(x, t)|^2 + |\mathcal{B}(x, t)|^2) \, d^3x \right) = -\int_{\mathbb{R}^3} \mathcal{E}(x, t) \cdot \mathbf{j}(x, t) \, d^3x,\]

here with \(\mathbf{j}(x, t) = f_e(|x|) \mathbf{\omega}(t) \times x\). On the other hand, by direct calculation with (4.7) and (1.3) one readily verifies that

\[\frac{d}{dt}\mathcal{M}_b(|\mathbf{\omega}|) = \mathbf{\omega} \cdot \frac{d}{dt}\mathbf{s}_b.\]
Next, taking the Euclidean inner product with $\omega$ on both sides of the canonical evolution equation for the total spin, (5.12), we see that

$$\omega \cdot \frac{d}{dt}s_b = -\omega \cdot \frac{d}{dt}s_f.$$  \hfill (5.31)

Recalling now the definition of the electromagnetic field spin, (5.11), then using the cyclicity of $\omega \cdot (x \times \partial_t \mathcal{A})$, noting next that $-\partial_t \mathcal{A} = \mathcal{E} + \nabla \phi_{\text{Coul}}$ and that $(\omega \times x) \cdot \nabla \phi_{\text{Coul}}(|x|) = 0$, and at last recalling that $f_e(|x|) \omega(t) \times x = j(x,t)$ we find

$$-\omega(t) \cdot \frac{d}{dt}s_f(t) = -\omega(t) \cdot \int_{\mathbb{R}^3} x \times \partial_t \mathcal{A}(x,t) f_e(|x|) d^3x$$

$$= \int_{\mathbb{R}^3} (\omega(t) \times x) \cdot \mathcal{E}(x,t) f_e(|x|) d^3x = \int_{\mathbb{R}^3} \mathcal{E}(x,t) \cdot j(x,t) d^3x.$$  \hfill (5.32)

Hence, energy conservation is proved.

As for the angular momentum conservation, taking the time derivative of the field angular momentum gives the well-known formula \[3\]

$$\frac{d}{dt} \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} x \times (\mathcal{E}(x,t) \times \mathcal{B}(x,t)) d^3x \right) = -\int_{\mathbb{R}^3} x \times \left( \rho(x,t) \mathcal{E}(x,t) + j(x,t) \times \mathcal{B}(x,t) \right) d^3x.$$  \hfill (5.33)

Inserting our expressions $\rho(x,t) = f_e(|x|)$ and $j(x,t) = f_e(|x|) \omega(t) \times x$, we see that

$$\int_{\mathbb{R}^3} x \times \left( \rho(x,t) \mathcal{E}(x,t) + j(x,t) \times \mathcal{B}(x,t) \right) d^3x = \frac{d}{dt} s_b(t),$$  \hfill (5.34)

and conservation of angular momentum is proven.

Finally, we already remarked that (5.12) implies at once that $|s|$ is conserved. QED

The proof that the total energy is conserved has the following spin-off.

**Corollary 1**: The constraint equation (4.11) is automatically satisfied by any solution of gyroscopic LED.

As for the total linear momentum,

$$\mathcal{P} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathcal{E} \times \mathcal{B} d^3x + N_e \cdot \omega,$$  \hfill (5.35)

we remark that conservation of (5.35) is equivalent to (4.12). However, our assumption of a straight particle world-line is generally *not* compatible with (4.12), unless special symmetries prevail. An example is discussed in the next section.

### 6 Rotation-reflection symmetric scattering

Our Theorem 1 reduces the global existence and uniqueness problem for proper LED with a straight particle world-line to finding the class of non-stationary initial conditions for which momentum conservation holds with a non-moving particle. Such a class of initial conditions is given by the rotation-reflection symmetric field decorations of spacetime, with the particle’s axis of rotation necessarily identical to the axis of symmetry $a$, linearly superimposed on which
is a compactly supported, non-symmetric electromagnetic radiation field that is never going to interact with the particle. Since a non-interacting radiation field is evidently rather uninteresting, we confine our discussion to the rotation-reflection symmetric evolutions.

More precisely, let \((\zeta, \theta, z)\) denote cylindrical coordinates of \(x\), with origin in the particle center, axis unit vector \(a\), \(z = x \cdot a\), \(\theta\) the polar angle of \(x\) about \(a\), and \(\zeta = |x - za|\). The axis \(a\) is fixed during the evolution, and \(\omega = \omega a\), so that \(\omega\) is (assumed, and below verified to be) the only remaining dynamical degree of freedom of the particle. Aside from the non-dynamical and spherically symmetric Coulomb field (5.5), the remaining electromagnetic field is now determined by a vector potential of the form

\[
\mathbf{A}(x, t) = \psi(\zeta, z, t) \nabla \theta,
\]

satisfying the reflection symmetry \(\psi(\zeta, z, t) = \psi(\zeta, -z, t)\), and obviously rotation invariant. The inhomogeneous wave equation for \(\mathbf{A}\), (5.3), reduces to the inhomogeneous, scalar, generalized wave equation

\[
(\partial_{tt} - \partial_{\zeta \zeta} + \zeta^{-1} \partial_{\zeta} - \partial_{zz}) \psi(\zeta, z, t) = 4\pi \omega(t) \zeta^2 f_e(\sqrt{\zeta^2 + z^2}),
\]

(6.1)

with accordingly simplified scalar solution formulas for \(\psi\). An elementary calculation with \(\mathcal{E} = -\nabla \phi_{\text{Coul}} - \partial_t \psi \nabla \theta\) and \(\mathcal{B} = \nabla \psi \times \nabla \theta\) then shows that the torque \(\int x \times (\mathcal{E} \times \mathcal{B}) f_e d^3x \propto a\), establishing the consistency at the level of the gyroscopic problem, indeed.

6.1 Momentum balance

We already saw that the time component (4.11) of the covariant world-line equation is automatically satisfied, see section 5. We now show that for rotation-reflection symmetric solutions to the gyroscopic problem the space-part of the world-line constraint equation (4.12) is satisfied, too. Since the fulfillment of (4.12) is equivalent to the conservation of linear momentum (5.35), it suffices to show that (5.35) is a constant vector for all time.

By direct computation with \(\mathcal{E} = -\nabla \phi_{\text{Coul}} - \partial_t \psi \nabla \theta\) and \(\mathcal{B} = \nabla \psi \times \nabla \theta\) one verifies that

\[
\int_{\mathbb{R}^3} \mathcal{E}(x, t) \times \mathcal{B}(x, t) f_e(|x|) d^3x = -\int_{\mathbb{R}^3} \zeta^{-2} \partial_t \psi(\zeta, z, t) \nabla \psi(\zeta, z, t) d^3x = 0
\]

(6.2)

for our rotation-reflection symmetric fields. As for the spin-orbit coupling term, another direct calculation yields that rotation-reflection symmetry implies

\[
\mathbf{N}_e(t) \cdot \omega(t) = -\omega(t) \frac{d}{dt} \int_{\mathbb{R}^3} x \psi(\zeta, z, t) f_e(|x|) d^3x = 0,
\]

(6.3)

and the satisfaction of the world-line constraint equation (4.12) follows.

6.2 Exponential convergence to the soliton state

In [1] we proved that the conservation of \(\sigma = |s_b + s_f|\) together with the invertibility of the map \(\omega \mapsto s\) in stationary situations implies that any scattering process connects two boosted stationary particle states with identical values for the renormalized mass and the magnitudes of spin and magnetic moment. In short: the Lorentz electron scatters like a soliton. We now complement this result by proving that rotation-reflection symmetric scattering does occur, and that the soliton state is approached exponentially fast. For our proof we need to assume that the ratio of electrostatic to bare rest mass is small.
Proposition 3: Assume that the electromagnetic potential data are rotation-reflection symmetric in the sense explained above, and of class $C^1$. Assume furthermore that $\psi_{\text{wave}}(\zeta, z, 0)$ has compact support a finite distance away from $\text{supp}(f_e)$. Finally, assume that

$$\|K\|_1 < \mathcal{I}_b(0). \quad (6.4)$$

Then, as $t \to \infty$, the bare spin $s_b(t)$ converges exponentially fast to a stationary vector, $s_b(t) \to s_b^\infty$, and $s_b^\infty = s_b(0)$.

Proof. Clearly, since $\omega \propto a$ for all $t$, all terms $\omega \times \mathcal{W}(s_b)$ and $\mathcal{W}(s_b(t)) \times \mathcal{W}(s_b(t))$ vanish. Also, by direct calculation one verifies that $\int_{\mathbb{R}^3} x \times \mathcal{A}_{\text{wave}}(x, t) f_e(|x|) \, d^3x \propto a$ for all $t$, so that its cross product with $\mathcal{W}$ vanishes as well for all $t$. Furthermore, by hypothesis, the initial wave data don’t overlap with the support of the particle, hence $\int_{\mathbb{R}^3} x \times \mathcal{A}_{\text{wave}}(x, 0) f_e(|x|) \, d^3x = 0$. Finally, by the wave propagation, there exists a $T \geq 2R$ such that $\text{supp}(\mathcal{A}_{\text{wave}}(x, t) \cap \text{supp}(f_e(|x|))) = \emptyset$ for all $t > T$. Then, for $t > T$, we have

$$s_b(t) + \int_{t-2R}^t \mathcal{W}(s_b(\bar{t})) K(t - \bar{t}) \, d\bar{t} = s(0), \quad \text{for } t > T \quad (6.5)$$

where $s(0) = s_b(0) + \kappa \omega_0$, with $\kappa \overset{\text{def}}{=} \int_0^2 R K(t) \, dt$. Notice that (6.5) is effectively a scalar equation because all vectors are $\propto a$. We now define $s_b^\infty$ as the – unique – solution of

$$s_b^\infty + \kappa \mathcal{W}(s_b^\infty) = s(0). \quad (6.6)$$

Clearly, since $s(0) = s_b(0) + \kappa \omega_0$ and $\mathcal{W}(s_b(0)) = \omega_0$, (6.6) is solved by $s_b^\infty = s_b(0)$, and by uniqueness this is the only solution. We next rewrite (6.5) as

$$s_b(t) - s_b^\infty = -\int_{t-2R}^t \left( \mathcal{W}(s_b(\bar{t})) - \mathcal{W}(s_b^\infty) \right) K(t - \bar{t}) \, d\bar{t} \quad \text{for } t > T \quad (6.7)$$

and estimate

$$|s_b(t) - s_b^\infty| \leq \int_{t-2R}^t |\mathcal{W}(s_b(\bar{t})) - \mathcal{W}(s_b^\infty)| \|K(t - \bar{t})\| \, d\bar{t}$$

$$\leq \left( \mathcal{I}_b(0) \right)^{-1} \int_{t-2R}^t |s_b(\bar{t}) - s_b^\infty| \|K(t - \bar{t})\| \, d\bar{t}$$

$$\leq \|K\|_1 \max_{t \in [t-2R,t]} \left| s_b(\bar{t}) - s_b^\infty \right| \quad (6.8)$$

where we used the Lipschitz continuity of $\mathcal{W}$ (Lemma 1) and the continuity of $t \mapsto s_b(t)$. Now assume that $t \in [n2R, (n+1)2R]$, with $n$ big enough so that $n2R > T$. By (6.3) and the inclusion $[t - 2R, t] \subset [(n-1)2R, (n+1)2R]$ we have that

$$\max_{t \in [n2R,(n+1)2R]} |s_b(t) - s_b^\infty| \leq \|K\|_1 \left( \mathcal{I}_b(0) \right)^{-1} \max_{t \in [(n-1)2R,(n+1)2R]} |s_b(t) - s_b^\infty| \quad (6.9)$$

By our smallness condition (6.4) we conclude that $\max_{t \in [(n-1)2R,(n+1)2R]} |s_b(t) - s_b^\infty|$ cannot be attained in $[n2R, (n+1)2R]$, hence it is attained in $[(n-1)2R, n2R]$. By induction from one interval of length $2R$ to the next one we now get $|s_b(nT) - s_b^\infty| \leq C \exp(-n\Gamma)$, i.e. exponential convergence with rate $\Gamma = \ln \left( \mathcal{I}_b(0)/ \|K\|_1 \right)$. QED
The exponentially fast convergence \( s_b(t) \to s_b^\infty \) implies for all rotation-reflection symmetric initial conditions of the type discussed that the field-particle system in fact converges exponentially fast on families of nested compact sets to a stationary particle-field bound state, the soliton state, while a departing field of electromagnetic radiation escapes to spatial infinity. Put differently, our class of rotation-reflection states consists of scattering states, with the exception of the stationary bound state itself. For late times the evolution of the electromagnetic field thus satisfies the scattering formulas (* means complex conjugate)

\[
\mathcal{G}(x, t) \quad t \to +\infty \quad \mathcal{G}_{\text{out}}(x) + e^{-it \nabla \times} \mathcal{G}_{\text{rad}}^\text{out}(x),
\]

and

\[
\mathcal{G}^*(x, t) \quad t \to -\infty \quad \mathcal{G}_{\text{sol}}^\text{in}*(x) + e^{it \nabla \times} \mathcal{G}_{\text{rad}}^\text{in}*(x),
\]

where the soliton fields \( \mathcal{G}_{\text{sol}}^\text{in} \) and \( \mathcal{G}_{\text{sol}}^\text{out} \) coincide in this rotation-reflection symmetric setting.

### 7 Open problems

It is instructive to have some explicit numbers. As in [1], consider the example where \( f_e \) and \( f_m \) are given by the uniform surface measure on a sphere of radius \( R \), i.e. \( f_e(|x|) = -e(4\pi R^2)^{-1} \delta(|x| - R) \), and \( f_m(|x|) = m_b(4\pi R^2)^{-1} \delta(|x| - R) \), with \( m_b \) the strictly positive bare rest mass of the Lorentz electron. This gives \( \mathcal{I}_b(0) = (2/3)m_bR^2 \), and

\[
K(t) = e^2 \frac{1}{3} \left(1 - \frac{1}{2} \frac{t^2}{R^2}\right) \Theta(t) \Theta(2R - t),
\]

so that \( \|K\|_1 = e^2 R^2(2\sqrt{2} - 1)/9 \). Our smallness condition \( \|K\|_1 < \mathcal{I}_b(0) \) then becomes

\[
\frac{e^2}{m_bR} < \frac{3}{2\sqrt{2} - 1}.
\]

Roughly speaking, the particle’s electrostatic Coulomb energy must be less than the bare rest mass. (This conclusion holds with minor numerical differences also when \( f_m \) is uniform volume measure in \( B_R \).) The interesting question now is whether deviations from the soliton state decay exponentially fast also when the smallness condition (7.2) is violated, especially since one is interested in a renormalization flow limit \( m_b \to 0^+ \) where \( R \to 1.5R_c \) (with \( R_c \) the electron’s Compton length) [1]. Conceivably some long-lived resonances may emerge and render a more complicated picture. Nonlinear resonances have been studied rigorously in the simpler semi-relativistic model of a particle interacting with a scalar wave field [5]; see also [13] for certain nonlinear wave equations. A corresponding study for gyroscopic LED is in its infancy.

For general non-rotation-reflection symmetric initial data we proved global existence and uniqueness of gyroscopic solutions (which typically do not satisfy the world-line equations of LED), but we do not yet know that on families of nested compact sets the field-particle system converges to a stationary state. All we can show is that \( s_b(t) \) converges to some \( s_b^\infty \) as \( t \to \infty \) whenever the iterated integral \( \int_t^\infty \int_0^t \mathcal{W}(s_b(t')) \times \mathcal{W}(s_b(t')) K(t - t') dt' \ d\tilde{t} \) has a limit in \( \mathbb{R}^3 \) as \( t \to \infty \), but we have nothing to say about exponentially fast convergence, then. In case of a scattering scenario, i.e. with convergence to a soliton, the fields \( \mathcal{G}_{\text{sol}}^\text{in} \) and \( \mathcal{G}_{\text{sol}}^\text{out} \) are generally
not identical; however, they differ by at most a space rotation as a consequence of the soliton dynamics. The explicit characterization of the scattering operator from the “in” states to the “out” states has yet to be worked out.

Eventually we would like to be able to establish control over the problem of many-body scattering. While well developed in quantum theory [2, 9, 11, 12], very little is known rigorously for truly relativistic LED. Interestingly enough, the solution to this problem requires the construction of a self-consistent nontrivial foliation of space-time, injecting a technical element from general relativity into the analysis.

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