Musical stylistic analysis: a study of intervallic transition graphs via persistent homology

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We develop a novel method to represent a weighted directed graph as a finite metric space and then use persistent homology to extract useful features. We apply this method to weighted directed graphs obtained from pitch transitions information of a given musical fragment and use these techniques to the quantitative study of stylistic trends. As a first illustration, we analyse a selection of string quartets by Haydn, Mozart and Beethoven and discuss possible implications of our results in terms of different approaches by these composers to stylistic exploration and variety. We observe that Haydn is stylistically the most conservative, followed by Mozart, while Beethoven is the most innovative. Finally we also compare the variability of different genres, namely minuets, allegros, prestos, and adagios, by a given composer and conclude that the minuet is the most stable form of the string quartet movements.

Keywords: Topological data analysis; persistent homology; string quartet; intervallic transitions; stylistic analysis

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1. Introduction

Several geometrical and topological representations of music have been proposed over the past decades in order to extract both qualitative and quantitative features from musical works and make precise comparisons between works by the same author, by different authors, among genres, etc. (see e.g. “The Topos of Music” by Mazzola 2012 as well as several papers in the more recent book “Computational Music Analysis” edited by Meredith 2016). In particular, once a piece of music has been represented as a topological object, we have at our disposal powerful methodological tools. Recently, Topological Data Analysis, TDA, has been applied to the study of musical works, in particular Persistent Homology, PH, a popular tool of TDA. For example in Bergomi (2015) and Bergomi and Baratè (2020) persistent homology is applied to deformations of the Tonnetz that vary over time. In Liu, Jeng, and Yang (2016) persistent homology is incorporated in a convolutional neural network for an automatic music tagging.

However, there are still fundamental questions that remain open. Specifically, identifying the stylistic character of a composer through some precise mathematical definition has been an elusive question, as it has been also finding adequate tools in order to quantitatively distinguish the evolution and richness of different genres.
In this work we start an investigation in these two directions by using a combination of tools from graph theory, TDA, and information theory. In particular, we begin by constructing a directed graph associated to any piece of music and then we propose a novel way of obtaining a simplicial complex from it. Once this is done, we compute persistent homology and summarize the results as barcodes. We then use statistical and information-theoretical quantities to analyse them. After developing the tools described above, we apply them to study stylistic features in a specific corpus, namely, a sample of movements from string quartets by Haydn, Mozart and Beethoven. We propose that the results obtained with our methods can be effectively used to support musicological claims about the different attitudes these three composers had in terms of the exploration of stylistic possibilities, confirming the traditional view of Haydn as the initiator of the form and Beethoven as a natural innovator. Finally, we compare the variability of different genres, namely minuets, allegros, prestos, and adagios, by a given composer and conclude that the minuet is the most stable form of the string quartet movements.

This paper is organized as follows: in Section 2 we review some basic concepts from algebraic topology and persistent homology. We also state the definitions of graph theory that we will use. In Section 3 we introduce a novel method to obtain information of weighted directed graphs by means of persistent homology. In Section 4 we first describe how to obtain a weighted directed graph from a musical work and then apply it to the techniques described in Section 3. We propose statistical and information-theoretical quantities to extract features from the data. In Section 5 we reach conclusions in two directions. One about stylistic differences between movements from three different authors and the other about stylistic differences between various types of composition of a given author. We end the paper with conclusions and further research perspectives in Section 6. We remark that all codes developed for this work are available at https://github.com/MartinMij/TDA-SQ.

2. Background

2.1. Simplicial complexes and homology

In this subsection we review some basic algebraic topology concepts. More details can be found in Munkres (2018) and Hatcher (2005).

Intuitively, a simplicial complex is a topological space composed of smaller pieces called simplices. These simplices are points, lines, triangles, tetrahedra, and their higher dimensional generalizations. A formal definition of a simplicial complex is the following (Munkres 2018, Section §1.3).

**Definition 2.1** An (abstract) simplicial complex is a set $K$ of finite subsets of a set $S$ in such a way that $\{v\} \in K$ for all $v \in S$ and if $\tau \subseteq \sigma \in K$ then $\tau \in K$. Elements $\{v\} \in K$ are the vertices of $K$ and $S$ is its vertex set. Any element in $K$ is a simplex of $K$. A face of $\sigma \in K$ is a simplex $\tau$ such that $\tau \subseteq \sigma$. We say that the dimension of a simplex $\sigma$ is $p$ or that $\sigma$ is a $p$-simplex if $|\sigma| = p + 1$ where $|\sigma|$ denotes the cardinality of the set $\sigma$. The dimension of the complex $K$ is defined as the largest dimension of any of its simplices or as infinite if there is no upper bound in such dimensions.

**Definition 2.2** A subcomplex $L$ of $K$ is a subset of $K$ that is a simplicial complex itself. In particular, for a given $i \in \mathbb{N}$, the $i$-skeleton of $K$ is the subcomplex composed by all the simplices of $K$ of dimension at most $i$. 
Figure 1. The Vietoris-Rips complex $VR(X, 3)$ for $X = \{x_0, x_1, x_2, x_3\}$ with the distance $d$ chosen to coincide with the corresponding Euclidean distance in $\mathbb{R}^2$. $VR(X, 3)$ consists of four 0-simplices, four 1-simplices and one 2-simplex.

**Definition 2.3** Given two simplicial complexes $K$ and $L$, a simplicial map $f : K \rightarrow L$ is a function that takes the vertices of $K$ into the vertices of $L$ and such that if $\sigma$ is a simplex of $K$ then $f(\sigma)$ is a simplex of $L$.

As we will see in Section 4, the musical data of interest yield finite metric spaces, that is, spaces with a finite number of points equipped with a distance function. Applying simplicial homology to these sets would only provide trivial information. It is therefore especially important to consider a class of simplicial complexes that can encapsulate the non-trivial topological information contained in these spaces. To do this, we can associate these finite metric spaces with a simple class of simplicial complexes called the Vietoris-Rips complexes (Zomorodian 2010). Intuitively, these complexes have the points of the metric space as their vertex set and higher-dimensional simplices are added as long as the vertices are within a specified threshold distance from one another. This construction allows us to capture topological information of the data (Attali, Lieutier, and Salinas 2013) by using precisely the closed balls of the metric space to recover the topology (see Figure 1 for a graphical illustration). Formally, we define them as follows.

**Definition 2.4** Let $(X, d)$ be a finite metric space and $r$ a non-negative real number. A Vietoris-Rips complex $VR(X, r)$ is a simplicial complex with vertex set $X$ and such that a $p$-simplex $\{x_0, x_1, \ldots, x_p\}$ is in $VR(X, r)$ if $d(x_i, x_j) \leq r$ for all $i, j \in \{0, 1, \ldots, p\}$, $i \neq j$, or equivalently if $B(x_i, r/2) \cap B(x_j, r/2) \neq \emptyset$, where $B(x_i, r/2)$ represents the closed ball with center $x_i$ and radius $r/2$. Observe that if $0 \leq r_1 \leq r_2$, then $VR(X, r_1)$ is a subcomplex of $VR(X, r_2)$.

We now define homology for simplicial complexes (Munkres 2018, Section §1.5).
Definition 2.5 Let $K$ be a simplicial complex and consider the field with two elements $\mathbb{Z}_2$. We indicate by $C_p(K)$ the $\mathbb{Z}_2$-vector space with basis the $p$-simplices of $K$. An element in $C_p(K)$ is called a $p$-chain.

Note that by definition a $p$-chain is a finite sum of the form

$$a_1\sigma_1 + a_2\sigma_2 + \cdots + a_n\sigma_n,$$

with $a_i \in \mathbb{Z}_2$ and $\sigma_i$ a $p$-simplex.

Definition 2.6 We define a boundary map as follows:

(i) for any $p \geq 1$

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$

is the map that acts on any basis element $\sigma = \{v_0, v_1, \ldots, v_n\}$ as

$$\partial_p(\sigma) = \sum_{i=0}^{p}\{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$$

where the hat above a vertex indicates that the corresponding vertex has been removed

(ii) for any $p < 0$, $C_p(K)$ and $\partial_p$ are defined as zero

(iii) $\partial_0 : C_0(K) \longrightarrow 0$ is the zero map.

$(C_p(K), \partial_p), p \in \mathbb{Z}$, is called a chain complex.

It is a basic and most important fact that $\partial_p \circ \partial_{p+1} = 0$ for all $p \in \mathbb{Z}$. We now define the homology groups of the simplicial complex $K$ as follows.

Definition 2.7 For a $p \in \mathbb{Z}$,

$$H_p(K) = \ker \partial_p / \text{im} \partial_{p+1},$$

is the $p$th homology group of $K$. For any element $c \in \ker \partial_p$, we indicate with $\bar{c} \in H_p(K)$ the corresponding homology class.

Note that all homology groups are trivial for $p < 0$. Moreover, they are all vector spaces, as they are quotients of vector spaces over $\mathbb{Z}_2$. Hence we can define the Betti numbers $\beta_p(K)$ as the dimension of the vector space $H_p(K)$.

Intuitively, $\beta_p(K)$ is the numbers of $p$-dimensional holes in $K$ (Hatcher 2005). For instance, $\beta_0(K)$ represents the number of connected components of $K$, $\beta_1(K)$ is the number of non-contractible loops and $\beta_2(K)$ is the number of voids or cavities of $K$.

As a final definition, we recall the following.

Definition 2.8 Given a simplicial map $f : K \longrightarrow L$, for all $p \in \mathbb{Z}$ we have the induced maps $(f_*)_p : H_p(K) \longrightarrow H_p(L)$ given by

$$(f_*)_p : \bar{c} \longmapsto \overline{f(c)},$$

where $\bar{c}$ and $\overline{f(c)}$ are the homology classes in $H_p(K)$ and $H_p(L)$ respectively and the subscript $p$ is used to denote the restriction of the map $f_*$ to the $p$th homology group.

To conclude, we provide a simple example that helps clarifying Definition 2.8.
Consider the simplicial complexes \( K \) and \( L \) defined as

\[
K = \{ \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\},
\]
\[
L = \{ \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}\}.
\]

A geometric representation of these complexes is shown in Figure 2.

We will not make the formal computations of the homology but we can use the intuition described above. In the case of \( K \), it is made up by one piece (that is, it has only one connected component) and has a non-contractible loop given by the 1-chain \( \{x_1, x_2\} + \{x_2, x_3\} + \{x_1, x_3\} \). In this way, we have the Betti numbers \( \beta_0(K) = 1 \), \( \beta_1(K) = 1 \), and \( \beta_p(K) = 0 \) for \( p > 1 \) since there are no simplices of dimension greater than 1. As these numbers are the ranks of the homology vector spaces, we have that \([x_1] \) and \([x_1, x_2] + [x_2, x_3] + [x_1, x_3] \) are basis elements of \( H_0(K) \) and \( H_1(K) \) respectively, which correspond respectively to any vertex in the unique connected component and to the non-contractible loop. In the case of \( L \), it has two connected components, it does not have any non-contractible loop and, although it has 2-dimensional simplices, it has no cavities. This implies that \( \beta_0(L) = 2 \) and \( \beta_p(L) = 0 \) for \( p > 0 \), or equivalently, \( H_0(L) = \mathbb{Z}_2^2 \) and \( H_p(L) = 0 \) for \( p > 0 \). As basis elements of \( H_0(L) \) we can take the homology classes of one vertex in each component, for instance, \([x_1]\) and \([x_4]\).

Note finally that \( K \) is a subcomplex of \( L \), so there is a simplicial map \( f : K \rightarrow L \) which is given by the inclusion. We compute the induced maps in homology as follows. In dimension 0, \( (f_0)_0 : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^2 \) is given in the basis element as \( (f_0)_0([x_1]) = f([x_1]) = \{x_1\} \), and since \([x_1]\) is a basis element in \( H_0(L) \), \( (f_0)_0 : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^2 \) can be seen as the inclusion of \( \mathbb{Z}_2 \) in the first component of \( \mathbb{Z}_2^2 \). For the 1-dimensional case, since \( H_1(L) = 0 \), \( (f_1)_1 : \mathbb{Z}_2 \rightarrow 0 \) is the zero map, which agrees with the fact that the basis element \([x_1, x_2] + [x_2, x_3] + [x_1, x_3] \) of \( H_1(K) \) is mapped to zero under \( (f_1) \) since the loop \([x_1, x_2] + [x_2, x_3] + [x_1, x_3] \) is contractible in \( L \).

### 2.2. Persistent homology

When dealing with data analysis we typically want a way to assess the importance of different aspects. For instance, we may want to identify noise from real features, or we may want to pay special attention to some features within a certain threshold. This is what persistent homology is useful for: homology alone captures the features of a “static” simplicial complex. Then the key idea is to take simplicial complexes at different scales to represent our data, and assess the importance of different features based on how long they persist when we move along all scales. We make this idea precise as follows (Otter et al. 2017; Carlsson 2009).

**Definition 2.10** A filtration of a simplicial complex \( K \) is a sequence of nested simplicial complexes \( K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq K \). Moreover, we set \( K_i = \emptyset \) for \( i < 0 \) and \( K_i = K \) for \( i > n \).
Example 2.11  Given a finite metric space \((X, d)\) (where \(d\) can also be allowed to be infinite for some pair of points), we can construct the Vietoris-Rips complex \(VR(X, r)\) for any given threshold \(r\), as in Definition 2.4. Given the great arbitrariness in fixing \(r\) (if any) is the best suited to capture the topological properties of the data, it turns out that it is more convenient to vary \(r\) at different scales, inducing a filtration of simplicial complexes. More precisely, given a sequence of non-negative real numbers \(\epsilon_0 \leq \epsilon_1 \leq \cdots \leq \epsilon_n\), we have a filtration of Vietoris-Rips complexes

\[
VR(X, \epsilon_0) \subseteq VR(X, \epsilon_1) \subseteq \cdots \subseteq VR(X, \epsilon_n).
\]

In this work we will consider filtrations of Vietoris-Rips complexes with \(\epsilon_0 = 0\) and \(\epsilon_n = \max\{d(x_j, x_i) \neq \infty | x_i, x_j \in X\}\).

Once we have a filtration of simplicial complexes, we can compute the simplicial homology of each of them. Moreover, we would like to track the homological changes that these spaces undergo throughout the filtration. This is where we use persistent homology, as described below.

As we saw in Definition 2.8 and Example 2.9, an inclusion \(\rho_{i, j} : K_i \hookrightarrow K_j, i \leq j\), induces a map

\[
(\rho_{i, j}^j)_p : H_p(K_i) \rightarrow H_p(K_j)
\]

for all \(p \in \mathbb{Z}\). To simplify the notation, from now on we will drop the subscript \(*\) and just denote these maps as \(\rho_{i, j}^j\). Given a class \(\bar{c}\) in \(H_p(K_i)\) (which can be considered as a \(p\)-dimensional hole) we can track its persistence as we move along filtration by means of the maps \(\rho_{i, j}^j\) according to the next definition.

Definition 2.12  Given a homology class \(\bar{c} \in H_p(K_i)\) we define its birth and death as in Edelsbrunner and Harer (2008), that is, we say that

(i) \(\bar{c}\) is born at \(K_i\) if \(\bar{c} \in H_p(K_i) \setminus \{0\}\) and \(\bar{c} \notin \text{im}(\rho_{i, i-1}^{i-1})\).

(ii) \(\bar{c}\) dies entering \(K_j\) if \(\rho_{i, j}^{j-1}(\bar{c}) \notin \text{im}(\rho_{i, j-1}^{j-1})\) but \(\rho_{i, j}^{j-1}(\bar{c}) \in \text{im}(\rho_{i, j-1}^{j-1})\).

Then the persistence of \(\bar{c}\) is the associated half-open interval \([b_{\bar{c}}, d_{\bar{c}})\), where \(b_{\bar{c}}\) represents the level in the filtration where \(\bar{c}\) was born and \(d_{\bar{c}}\) the level where it dies. If a class never dies we set \(d_{\bar{c}} = \infty\). The multiset given by the intervals of all the \(p\)-homology classes appearing in a filtration \(F\) is called the \(p\)-persistence barcode, denoted by \(bc_p(F)\).

In Figure 3 we provide an illustration of a filtration of simplicial complexes, together with the corresponding barcodes in dimension 0 and 1, while in Figure 4 we present a pictorial representation of a class that is born at \(K_i\) and dies entering \(K_j\).

2.3. Directed graphs

When studying objects with relations among them, a natural way of representing the information is through a graph. We start by setting the definitions that we will use (Berge 1973; Chartrand, Lesniak, and Zhang 2016; Diestel 2017).

Definition 2.13  An undirected graph, or simply a graph, is a pair \(G = (V, E)\) where \(V\) is the set of vertices and \(E\) is a set of 2-element subsets of \(V\). The elements of \(E\) are called the edges of \(G\).
Figure 3. A filtration of simplicial complexes and the corresponding barcodes in dimension 0 and 1. The number in the left-upper corner of each simplicial complex stands for its level in the filtration.

Figure 4. Given a filtration $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq K$, we represent the vector spaces $H_p(K_l)$ as ellipses whose lowest point is the zero element. Let the inner ellipses in $H_p(K_i), H_p(K_{i+1}), \ldots, H_p(K_j)$, denote the image of $H_p(K_{i-1})$ under the induced maps in homology. Then the blue line follows a point $\bar{c}$ which represents a homology class that is born at $K_i$ and dies entering $K_j$. 
Note that this definition does not allow for the existence of edges joining a vertex with itself, nor for multiple edges between a pair of vertices. Sometimes this graph is also referred to as a simple graph.

Definition 2.14 A directed graph (or digraph) is a pair \( G = (V, E) \) where \( V \) is the set of vertices and \( E \subseteq V \times V \) is the set of edges.

Note that this definition allows for the existence of loops, that is, edges joining a vertex with itself.

Definition 2.15 A weighted graph (resp. a weighted digraph) \( G^w = (V, E, w) \) is a graph (resp. digraph) equipped with a function \( w : E \rightarrow \mathbb{R}^+ \), that is, a positive number is associated to each edge. Intuitively, such number gives the weight of the corresponding edge.

Definition 2.16 Given a weighted graph or digraph \( G^w = (V, E, w) \) with finite vertex set \( V = \{v_1, \ldots, v_n\} \), its adjacency matrix is the \( n \times n \) matrix \( A = (a_{ij}) \) whose \((i, j)\) entry is given by

1. \( a_{ij} = w(v_i, v_j) \), if there is an edge between \( v_i \) and \( v_j \)
2. \( a_{ij} = 0 \), otherwise.

3. PH of weighted directed graphs

Given a directed graph \( G \), we would like to apply persistent homology to extract its main features. To do so we need a filtration of simplicial complexes.

One way to obtain a filtration is to first associate a simplicial complex to a directed or undirected graph by means of the so-called neighbourhood complex \( N(G) \) or the clique complex \( C(G) \) (Horak, Maletić, and Rajković 2009). Then a filtration \( \emptyset \subset K_0 \subset K_1 \subset \cdots \subset K_n = K \) is given by taking \( K_i \) as the \( i \)-skeleton of \( K \). A disadvantage of this approach is that it is hard to adapt it to consider weighted graphs (see e.g. Petri et al. 2013 for an extension to include weighted undirected graphs).

Here we propose a new way to obtain a filtration, which is naturally adapted to weighted directed graphs. The key idea is to transform a weighted directed graph into its associated undirected graph (Definition 3.2 below), then use the weights to endow it with a metric, and finally define the corresponding Vietoris-Rips complexes to obtain a filtration.

We start by showing how any weighted undirected graph can be thought of as a finite metric space (Otter et al. 2017). Intuitively, the weight of an edge can be considered as a similarity measure between the vertices that it joins. In this way, two vertices should be represented as close points in the metric space if and only if they are joined by edges (or paths) with strong weights. More precisely, we have the following definition.

Definition 3.1 Let \( G^w = (V, E, w) \) be an undirected graph with positive weights. For any pair of vertices \( u \) and \( v \) in \( V \) and any path \( \gamma(u, v) \) connecting them, define the length of \( \gamma(u, v) \), denoted \( \ell(\gamma(u, v)) \), as the sum of the inverses of the weights of the edges in \( \gamma(u, v) \). Then define the distance between \( u \) and \( v \), denoted \( d(u, v) \), as

1. \( d(u, v) = 0 \), if \( u = v \)
2. \( d(u, v) = +\infty \), if there is no path \( \gamma(u, v) \)
3. \( d(u, v) = \min_{\gamma(u,v)} \ell(\gamma(u,v)) \), otherwise.

It is straightforward to verify that \( d : V \times V \rightarrow [0, +\infty] \) is a (extended) metric on \( V \). We call this metric the weight metric of \( G^w \).
Clearly the former technique to obtain a metric space from an undirected graph does not work for directed graphs, since in this case the so-defined distance would not be a symmetric function. To overcome this issue, we propose the following strategy: given a directed graph, we first associate to it an undirected graph and then apply to this associated graph the above definition of a distance. Loosely speaking, we would like to preserve as much as possible of the information contained in the original directed graphs, so for instance if two vertices are joined by two edges with different weights, we cannot simply choose one of them in order to “symmetrize” the graph. Moreover, we would also like to preserve loops whenever they are present, as they might carry important information (as it is indeed the case for musical pieces). This line of reasoning leads us to introduce the following new definition.

**Definition 3.2** Let $G = (V, E)$ be a directed graph. We define its associated undirected graph as the graph $G' = (V', E')$, where

$$V' := V \cup \{v_{ab} | (a, b) \in E\} \quad \text{and} \quad E' = \{(a, v_{ab}) | (a, b) \in E\} \cup \{(b, v_{ab}) | (a, b) \in E\}.$$ 

Moreover, if the directed graph $G$ is weighted with weight function $w$, then we define the associated weight function $w'$ on $G'$ by

$$w'(\{x, v_{ab}\}) = \begin{cases} \frac{1}{2}w(a, b), & \text{if } a \neq b \text{ and } x = a \text{ or } x = b \\ w(a, a), & \text{if } x = a = b. \end{cases}$$

The idea behind this construction is to add a vertex $v_{ab}$ for each directed edge $(a, b)$ that works as a label. In this way we preserve the original “directional” information, at the cost of having to introduce more vertices (see Figure 5, where the weights $w(u, v)$ are denoted as $w_{uv}$).

We note in passing the property that if $w$ is a probability distribution over the edges of $G$, then $w'$ is a probability distribution over the edges of $G'$. This ensures that after the transformation the resulting undirected graphs can still be easily compared, as they are at the same scale. This is because a directed edge in $G$ which is not a loop splits into two edges in $G'$ whose weights add up to the weight of the original edge and a loop in $G$ becomes a regular edge in $G'$ with the same weight.

Now that we have an associated weighted undirected graph, the crucial step is completed: what follows is to apply the construction detailed in Definition 3.1 to transform it into a metric space, then obtain a filtration of Vietoris-Rips complexes as in Example 2.11, and finally use persistent homology to analyse the data.

Figure 5. A digraph and its associated undirected graph.
4. Stylistic analysis of musical works

Much research has been done trying to apply classification techniques to music. This ranges from basic statistical tools to sophisticated artificial intelligence methods. The results often have important musicological and analytical consequences related to authorship, chronology and genre identification. Besides their theoretical interest, these methodologies might have also practical implications, such as in legal disputes, e.g. plagiarism issues. In any case, the question of whether stylistic signatures can be obtained from data analysis has attracted much attention in the past few years. In this section we show how we can apply persistent homology to obtain features of musical works.

4.1. PH of musical works

In what follows we apply the techniques discussed above to a specific example, namely, the stylistic development and identification of the string quartets by Haydn, Mozart and Beethoven.

First of all, given a musical piece, we need to extract a graph from it. This is done following the approach recently put forward in Padilla et al. (2017) and Grant et al. (2022): given a score, the main idea is to analyse it by using its distribution of pitch class transitions weighted by note durations in seconds (indeed we will take durations modified according to Parncutt’s model, see e.g. Parncutt 1994) in order to convert it into a weighted directed graph.

To be precise, we have the following

**Definition 4.1** Let us enumerate the twelve pitch classes $C$, $C^\#$, $D$, $D^\#$, $E$, $F$, $F^\#$, $G$, $G^\#$, $A$, $A^\#$, $B$ from 1 to 12 respectively, and consider the set of pitch class transitions $X = \{(i,j)|i,j \in \{1,2,\ldots,12\}\}$ where the pair $(i,j)$ represents the transition from the pitch class $i$ to the pitch class $j$. In the musical piece, a transition $(i,j)$ can appear several times and in each case we consider a weighted duration of the transition as the product of the duration of the tone $i$ and the duration of the tone $j$. Finally, we let $p : X \rightarrow [0,1]$ be the probability function given by

$$p(i,j) = \frac{\text{total weighted durations of transitions } (i,j)}{\text{total weighted durations of all transitions}}.$$

This information can be organized in a $12 \times 12$-matrix $M = (m_{ij})$ where $m_{ij} = p(i,j)$. We use a built-in function of the Midi Toolbox (Eerola and Toiviainen 2004) implemented in MATLAB to calculate this matrix. Crucially, we can now see the matrix $M$ as the adjacency matrix of a weighted directed graph whose vertices are the pitch classes 1, 2, ..., 12, and if $m_{ij} \neq 0$, there is a weighted directed edge from $i$ to $j$ with weight $m_{ij}$. So, we have associated a weighted directed graph to the score we started with. We call this graph the intervallic transition graph.

**Remark 4.2** From previous analyses it has been shown that that standard classification procedures identify the voice structure (Knights et al. 2019; Knights, Rodríguez, and Padilla 2022). Therefore it seems to be more consistent, from a methodological perspective, to separate the voices from the very beginning. This is what we are going to do in this work. On the other hand, if harmonic features are to be incorporated it will be necessary to structure the data in a different way, such that the vertical (simultaneous) characteristics are taken into account. For instance, in Alcalá-Alvarez and Padilla-Longoria (2023), the vertices are constructed with rhythm and harmony information.

Now we are in the position to apply the ideas presented in Section 3 to obtain an extended metric space and from that we can use persistent homology over the filtration of Vietoris-Rips complexes to obtain the barcodes for each dimension. We illustrate this in the following example.
Figure 6. A score and its associated directed graph. (a) First measures from the score of the first violin in the second movement of the String quartet Op. 17 No. 1 by Haydn and (b) Induced intervallic transition graph.

Example 4.3 Consider the score of the first violin in the second movement of the String quartet Op. 17 No. 1 by Haydn. To this, we associate the intervallic transition graph according to Definition 4.1, see Figure 6. This weighted directed graph gives rise to an weighted undirected graph as described in Definition 3.2, which in turn, as mentioned in Definition 3.1, induces an extended metric space. Here the points represent pitch classes and the distance between a pair of such points reflects the strength of the transitions between them: the more interrelated, the closer they are.

Now that we have a metric space, we construct a sequence of Vietoris-Rips complexes indexed by a parameter, thought as time, where further points are joined as time progresses. We then compute its persistent homology and obtain its barcodes.

In the left panel of Figure 7 we show the 0-dimensional barcode. At time zero, the Vietoris-Rips complex thus constructed only consists of the vertices induced by the pitch classes, without any edges. Hence, since zero dimensional homology measures connected components and at time zero all points are their own connected component, resulting bars have zero as their birth time. As time goes on, points join together implying the death of some of the bars. In this sense, short bars correspond to highly interrelated pitch classes. Note that there are two infinite bars, which means that there are two connected components in the final Vietoris-Rips complex: the first component consisting of all the vertices appearing in at least one transition, and the second component consisting of a single vertex, which in turn represents a pitch class that is never played in the work.

In the right panel of Figure 7 we show the 1-dimensional barcode; here each bar represents 1-dimensional holes in the Vietoris-Rips complexes. This also captures some aspect of the topological structure intrinsic to the simplicial complexes thus constructed. However, this information is more difficult to interpret directly in musical terms, so we defer this analysis to future work.

4.2. Statistical measures of information

The main information in the persistent homology analysis is contained in the lengths of the bars. For this reason we now employ some standard statistical measures of information in order to extract quantitative and qualitative features of the distributions of such lengths. In particular, we will consider the mean, the variance and the entropy of the lengths, being defined as follows (see also Rucco et al. 2016).
Definition 4.4 Consider a filtration $F$ given by $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$, and the corresponding $p$-persistence barcode as in Definition 2.12,

$$bc_p(F) = \{[x_i, y_i]| 1 \leq i \leq r, r \in \mathbb{N} \},$$

where each interval corresponds to a bar starting at $x_i$ and ending at $y_i$. Define the length of a bar as $l_i = y_i - x_i$ and let $I \subseteq \{0, 1, \ldots, r\}$ be the subset of indices such that $y_i$ is finite. Then the $p$-persistent mean and the $p$-persistent standard deviation are calculated according to their standard statistical definitions as follows.

$$m_p(F) = \frac{1}{|I|} \sum_{i \in I} l_i \quad (2)$$

$$sd_p(F) = \sqrt{\frac{1}{|I| - 1} \sum_{i \in I} (l_i - m_p(F))^2} \quad (3)$$

Moreover, let $m = \max\{y_i| y_i \text{ is finite}\}$ and set $y'_i = y_i$ if $y_i$ is finite and $y'_i = m + 1$ otherwise. Define $l'_i = y'_i - x_i$ and let

$$p_i = \frac{l'_i}{\sum_{i=1}^r l'_i}$$

be the distribution of the finite bar lengths. Then the $p$-persistent entropy is defined as

$$e_p(F) = - \sum_{i=1}^r p_i \log(p_i). \quad (4)$$

Note that the definition of the persistent entropy requires a bound in the length of the bars. We have decided to take this bound as $m + 1$, motivated by the fact that it has been proved to be stable (Atienza, Gonzalez-Díaz, and Soriano-Trigueros 2020) and to perform well in applications (Merelli et al. 2016; Rucco et al. 2017).

In the next section we will use these tools in order to assign to each musical piece a unique topological footprint that we will use in order to address questions such as the stylistic exploration and variety of different authors with respect to a common genre and the dual perspective of comparing the richness of various genres for a given author.
Table 1. String quartets.

| Author   | Publication | Date | Title           | Key   | Movement number |
|----------|-------------|------|-----------------|-------|-----------------|
| Haydn    | Op. 17/2   | 1771 | Menuetto        | F major | 2               |
| Haydn    | Op. 20/1   | 1772 | Menuetto        | E♭ major | 2               |
| Haydn    | Op. 33/3   | 1781 | Scherzando      | C major | 2               |
| Haydn    | Op. 50/3   | 1787 | Menuetto        | E♭ major | 3               |
| Haydn    | Op. 64/1   | 1790 | Menuetto        | C major | 2               |
| Haydn    | Op. 71/1   | 1793 | Menuetto        | B♭ major | 3               |
| Haydn    | Op. 77/1   | 1799 | Menuetto        | G major | 3               |
| Mozart   | Quartet No. 5 | 1773 | Tempo di Minuetto | F major | 3               |
| Mozart   | Quartet No. 8 | 1773 | Menuetto        | F major | 3               |
| Mozart   | Quartet No. 13 | 1773 | Menuetto        | D minor | 3               |
| Mozart   | Quartet No. 14 | 1782 | Menuetto        | G major | 2               |
| Mozart   | Quartet No. 19 | 1785 | Menuetto        | C major | 3               |
| Mozart   | Quartet No. 23 | 1790 | Menuetto-Allegretto | F major | 3               |
| Beethoven | Op. 18/1  | 1798 | Scherzo: Allegro molto | F major | 3               |
| Beethoven | Op. 18/4  | 1798 | Menuetto: Allegretto | C major | 3               |
| Beethoven | Op. 18/6  | 1798 | Scherzo: Allegro | B♭ major | 3               |
| Beethoven | Op. 59/3  | 1805 | Menuetto: Grazioso | C major | 3               |
| Beethoven | Op. 127   | 1825 | Scherzando vivace – Presto | E♭ major | 3               |

5. Applications and results

In the previous section we have discussed an approach to apply persistent homology to musical works that we can summarize as the following pipeline: given a score, we generate the intervallic transition graph. This directed graph gives rise to a weighted undirected graph (as described in Section 3) which, in turn, induces a finite metric space. From this space, we derive a filtration of Vietoris-Rips complexes and compute persistent homology. Lastly, we analyse the resulting barcodes by means of some statistical descriptors, as described in the previous section.

It is worth noting that this approach is invariant under certain transformations. For instance, transposing the score leads to an isomorphic intervallic transition graph, that is, the same graph with possibly a relabelling of its vertices. Since the resulting graphs have the same topology, the analysis made through persistent homology yields the same results. Moreover, any transformation that leaves invariant the intervallic transition graph will not alter the results. In particular, these properties allow to compare musical works independently of the key in which they are written. On the other hand ornamentation, as its name suggests, is not considered in this first approach. Although it is clearly a significant stylistic feature, it should be clear that it can be neglected initially. Moreover, at least in the baroque period, ornaments were not consistently notated by authors.

In the following subsections we demonstrate the usefulness of our approach by applying it to several sets of string quartets.

5.1. Same genre, different authors

In this section we want to compare the musical style of different authors within a fixed genre using the techniques described above. To be precise, we consider the three most representative string quartet composers in classic style, namely Haydn, Mozart, and Beethoven, and compare some of their string quartets. For each author we choose a set of representative works of each stage of his musical compositional development. We summarize these works in Table 1.

Each of these works consists of four instruments (parts) and to each of them we may apply persistent homology as described above. For each musical work and each instrument we thus obtain the corresponding barcodes in dimension 0 and 1. We then calculate the mean, standard
deviation and entropy of their lengths according to Equations (2), (3), and (4), thus obtaining 6 statistical descriptors for each instrument, for a total of 24 descriptors for each work. Then each work is described by a point 

\[(m^1_j, sd^1_j, e^1_j, m^2_j, sd^2_j, \ldots, sd^4_j, e^4_j) \in \mathbb{R}^{24}\]

where \(m^i_j\), \(sd^i_j\), and \(e^i_j\), with \(i = 1, \ldots, 4\) and \(j = 0, 1\), stand for the mean, standard deviation, and entropy of the barcodes in dimension \(j\) of the \(i\)th instrument, respectively. Therefore for each composer we obtain a set of points in \(\mathbb{R}^{24}\).

Finally, in order to reduce the dimensionality of the problem and obtain a succinct but reliable visual description of each work, we plot all these points in \(\mathbb{R}^2\) using principal component analysis, PCA (Jolliffe 1986). In Figure 8 we illustrate the results of this analysis when considering only the first two principal components.

We now focus on the dispersion of the points for each author, meaning the deviation from their centre of mass.

At first sight it seems clear from Figure 8 that the dispersion is increasing in the order Haydn, Mozart, and Beethoven. Indeed, we can make this statement quantitative by computing the dispersion as follows.

**Definition 5.1** Let \(x_1, x_2, \ldots, x_l\) be the set of points in \(\mathbb{R}^{24}\) representing the works of a given composer and let \(\bar{x} = \frac{1}{l} \sum_{i=1}^{l} x_i\). Then the dispersion \(S\) is computed as

\[S := \sqrt{\frac{1}{l-1} \sum_{i=1}^{l} |x_i - \bar{x}|}.\]

The results for each author are plotted in Figure 9. We can clearly observe that there is an increasing trend when we move from Haydn to Mozart and from Mozart to Beethoven.
This trend has an interesting musicological interpretation. Indeed, it is natural to associate the dispersion of the works considered as a quantitative measure of the stylistic variability. In other words, the dispersion provides an indicator of the scope of stylistic exploration by each of these composers. Therefore our result coincides with the standard view in which Haydn is seen as the initiator of the string quartet and with a compositional style that remained relatively stable during his life. Correspondingly, Mozart is perceived as an innovator, but probably due to his early death, his stylistic range did not change as much as it could have, had he lived longer. Finally, the big variability shown by Beethoven’s works coincides with the conception of this author as the most innovative, expanding and modifying the string quartet as a musical form (Rosen 1997).

5.2. Same author, different genres

Contrary to the previous section, in this one we want to fix the author and compare works belonging to different genres.

Precisely, for every author (Haydn, Mozart, and Beethoven) we have selected a set of works belonging to different subgenres, such as minuets, allegros, adagios, etc. Namely, we chose the four most explored subgenres for each composer. Then we applied the same analysis as in the previous section, obtaining for each work first a point in \( \mathbb{R}^{24} \) and then the corresponding projection to \( \mathbb{R}^{2} \) given by PCA. The full list of the works considered for each author is available at https://github.com/MartinMij/TDA-SQ, where the codes used can also be found.

The results thus obtained are shown in the left panels of Figures 10, 11 and 12, while in the right panels we display the corresponding dispersions for each author.

As we can see, the dispersion of the different types of subgenre changes with each composer, but there are some interesting regularities: on the one hand, the minuets are in general the subgenre with the least dispersion, confirming the fact that the minuet has a more uniform formal and stylistic structure. On the other hand, the adagios have the greatest dispersion for all the authors, indicating that this subgenre is the most versatile in terms of stylistic exploration.
Figure 10. Analysis of several works of Haydn belonging to different subgenres. In the left panel we display the results of the PCA analysis. In the right panel we compute the dispersions of the points for each subgenre.

Figure 11. Analysis of several works of Mozart belonging to different subgenres. In the left panel we display the results of the PCA analysis (here two adagios that were outliers have been removed in order to better appreciate the dispersion). In the right panel we compute the dispersions of the points for each subgenre.

Figure 12. Analysis of several works of Beethoven belonging to different subgenres. In the left panel we display the results of the PCA analysis. In the right panel we compute the dispersions of the points for each subgenre.
5.3. What if we take rests?

Hitherto we have considered the pitch class distribution without taking into account the rests of a given score. A natural question is how much our previous results change if we either do not consider transitions between tones that are separated by a rest, or if we assign to them a smaller weight.

To test the robustness of our results under such changes, we modified the pitch class distribution in two ways: in the first case we removed completely the transitions that contain a rest, while in the second case we assigned to them a weight according to the following procedure: suppose we have a transition \((i,j)\) (i.e. from the pitch class \(i\) to \(j\)) where the first tone has a duration \(u\) and the second one has a duration \(v\), separated by a silence of duration \(s\). Instead of considering \(uv\) as the total duration of the transition \((i,j)\), as it was the case so far, now we assign to this transition the value \(uvf\) where \(0 \leq f \leq 1\) is a function of \(u\), \(v\) and \(s\) that weights \(uv\) so that transitions with long silences between shorts tones add little to the distribution, while transitions with short silences between long tones are more important for the distribution. Specifically, we take \(f\) to be

\[
f = \frac{1}{\frac{s}{uv} + 1}.
\]

Note in particular that for \(s = 0\) we obtain \(f = 1\) and the duration of the transition is unchanged, as it should be.

For the purpose of comparison, we apply persistent homology to the first violin of Haydn’s string quartets minuets shown in Table 1. As an illustration, in Figure 13 we show the 0-persistence means for each of the seven works. As we can see, only the 5th work exhibits a major change. If we look at the score (in Figure 14 we show the first measures) we realize why this happens. Indeed, many of the rests are very short, namely, sixteenth note rests, and are more to explicitly notate a desired articulation, rather than structural rests.

6. Conclusions and future work

We have developed a method to obtain quantitative measures to compare composition styles. Specifically, this method takes into account the duration of the notes involved in the played transitions of the work and results in a metric space where closer points correspond to more related pitch class tones. Persistent homology allows us to summarize this information in barcodes and through statistics and information-theoretical tools, we assign a point in a Euclidean space to each musical work. Then we can see how stylistically different two works are by comparing their associated points.
Our results from the analysis of the string quartets of Haydn, Mozart, and Beethoven are consistent with the standard view of Haydn as the initiator of the genre and Beethoven as the most innovative author among the three. Moreover, we were also able to make quantitative statements about the stylistic variety among different genres by the same author, and found out that minuets are the most uniform structure, while adagios are the most versatile. The results are robust with respect to different ways to consider rests in the transitions. Although these findings confirm well-known facts about the three authors, up to our knowledge, this is the first time that they have been recovered by using computational methods. This may open a way to use our approach in a quantitative way in order to, for example, establish or confirm authorship or chronologies.

In future research we will consider two important aspects. First, while the methods developed here apply to any set of works as long as all of them use the same instruments, it will be interesting to extend our techniques and adapt them to works having different instruments. As a second point, our method depends on the use of a metric space. The distance is obtained from a weighting function on the edges of a graph. If the graph is directed, the weighting function is not symmetric in general and neither is the distance. This is the reason why we had to induce an undirected graph from a directed one. But what if we could work with a non-symmetric “distance” and still be able to obtain filtrations and then compute persistent homology? A perspective in this direction is given in Edelsbrunner and Wagner (2018). We expect to develop this approach in future work.

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