Continuous controlled generalized fusion frames in Hilbert spaces

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Abstract

We introduce the notion of continuous controlled $g$-fusion frame in Hilbert space which is the generalization of discrete controlled $g$-fusion frame and give an example. Some characterizations of continuous controlled $g$-fusion frame have been presented. We define the frame operator and multiplier of continuous controlled $g$-fusion Bessel families in Hilbert spaces. Continuous resolution of the identity operator on a Hilbert space using the theory of continuous controlled $g$-fusion frame is being considered. Finally, we discuss perturbation results of continuous controlled $g$-fusion frame.

Keywords: Frame, $g$-fusion frame, continuous $g$-fusion frame, controlled frame, controlled $g$-fusion frame.

2010 Mathematics Subject Classification: 42C15; 42C40; 46C07.

1 Introduction

In 1952, Duffin and Schaeffer [10] introduced frame for Hilbert space to study some fundamental problems in non-harmonic Fourier series. Later on, after some decades, frame theory was popularized by Daubechies et al. [8].

Frame for Hilbert space was defined as a sequence of basis-like elements in Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is called a frame for a separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$, if there exist positive constants $0 < A \leq B < \infty$ such that

$$A \| f \|^2 \leq \sum_{i=1}^{\infty} | \langle f, f_i \rangle |^2 \leq B \| f \|^2 \text{ for all } f \in H.$$ 

For the past few years many other types of frames were proposed such as $K$-frame [13], fusion frame [5], $g$-frame [27], $g$-fusion frame [16, 25] and $K$-$g$-fusion frame [1] etc. P. Ghosh and T. K. Samanta [15] have discussed generalized atomic subspaces for operators in Hilbert spaces.
Controlled frame is one of the newest generalization of frame. P. Balaz et al. [4] introduced controlled frame to improve the numerical efficiency of interactive algorithms for inverting the frame operator. In recent times, several generalizations of controlled frame namely, controlled $K$-frame [22], controlled $g$-frame [23], controlled fusion frame [19], controlled $g$-fusion frame [26], controlled $K$-$g$-fusion frame [24] etc. have been appeared. Continuous frames were proposed by Kaiser [17] and it was independently studied by Ali et al. [2]. At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on.

In this paper, continuous controlled $g$-fusion frame in Hilbert space is presented and some of their properties are going to be established. We will see that any continuous controlled $g$-fusion frame is a continuous $g$-fusion frame and converse part is also true under some sufficient conditions. We consider the frame operator for a pair of continuous controlled $g$-fusion Bessel families. Multiplier of continuous controlled $g$-fusion Bessel families in Hilbert spaces is also discussed. Some useful results about continuous resolution of the identity operator on a Hilbert space using the theory of continuous controlled $g$-fusion frame is constructed. At the end, we study some perturbation results of continuous controlled $g$-fusion frame.

Throughout this paper, $H$ is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\mathbb{H}$ is the collection of all closed subspace of $H$. $(X, \mu)$ denotes abstract measure space with positive measure $\mu$. $I_H$ is the identity operator on $H$. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from $H_1$ to $H_2$. In particular $\mathcal{B}(H)$ denotes the space of all bounded linear operators on $H$. For $S \in \mathcal{B}(H)$, we denote $\mathcal{N}(S)$ and $\mathcal{R}(S)$ for null space and range of $S$, respectively. Also, $P_M \in \mathcal{B}(H)$ is the orthonormal projection onto a closed subspace $M \subset H$. $\mathcal{G}\mathcal{B}(H)$ denotes the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{G}\mathcal{B}(H)$, then $R^*, R^{-1}$ and $SR$ are also belongs to $\mathcal{G}\mathcal{B}(H)$. $\mathcal{G}\mathcal{B}^+(H)$ is the set of all positive operators in $\mathcal{G}\mathcal{B}(H)$ and $T, U$ are invertible operators in $\mathcal{G}\mathcal{B}(H)$.

2 Preliminaries

In this section, we recall some necessary definitions and theorems.

**Theorem 2.1.** (Douglas’ factorization theorem) [9] Let $S, V \in \mathcal{B}(H)$. Then the following conditions are equivalent:

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(V)$.

(ii) $SS^* \leq \lambda^2VV^*$ for some $\lambda > 0$.

(iii) $S = VW$ for some bounded linear operator $W$ on $H$.

**Theorem 2.2.** [7] The set $\mathcal{S}(H)$ of all self-adjoint operators on $H$ is a partially ordered set with respect to the partial order $\leq$ which is defined as for $R, S \in \mathcal{S}(H)$

$$R \leq S \iff \langle Rf, f \rangle \leq \langle Sf, f \rangle \quad \forall f \in H.$$
Definition 2.3. [20] A self-adjoint operator $U : H \to H$ is called positive if $\langle Ux, x \rangle \geq 0$ for all $x \in H$. In notation, we can write $U \geq 0$. A self-adjoint operator $V : H \to H$ is called a square root of $U$ if $V^2 = U$. If, in addition $V \geq 0$, then $V$ is called positive square root of $U$ and is denoted by $V = U^{1/2}$.

Theorem 2.4. [20] The positive square root $V : H \to H$ of an arbitrary positive self-adjoint operator $U : H \to H$ exists and is unique. Further, the operator $V$ commutes with every bounded linear operator on $H$ which commutes with $U$.

In a complex Hilbert space, every bounded positive operator is self-adjoint and any two bounded positive operators can be commute with each other.

Theorem 2.5. [12] Let $M \subset H$ be a closed subspace and $T \in B(H)$. Then $P_M T^* = P_M T^* P_M$. If $T$ is an unitary operator (i.e. $T^* T = I_H$), then $P_M T = T P_M$.

Definition 2.6. [20] Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a collection of positive weights, $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces and let $\Lambda_j \in B(H, H_j)$ for each $j \in J$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a generalized fusion frame or a g-fusion frame for $H$ respect to $\{H_j\}_{j \in J}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \|\Lambda_j P_{W_j}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H. \quad (1)$$

The constants $A$ and $B$ are called the lower and upper bounds of g-fusion frame, respectively. If $A = B$ then $\Lambda$ is called tight g-fusion frame and if $A = B = 1$ then we say $\Lambda$ is a Parseval g-fusion frame. If $\Lambda$ satisfies only the right inequality of (2) it is called a g-fusion Bessel sequence in $H$ with bound $B$.

Define the space

$$l^2(\{H_j\}_{j \in J}) = \left\{ \{f_j\}_{j \in J} : f_j \in H_j, \sum_{j \in J} \|f_j\|^2 < \infty \right\}$$

with inner product is given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{H_j}.$$ 

Clearly $l^2(\{H_j\}_{j \in J})$ is a Hilbert space with the pointwise operations $\| \|$.

Definition 2.7. [26] Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of $H$ and $\{v_j\}_{j \in J}$ be a collection of positive weights. Let $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces, $T, U \in GB(H)$ and $\Lambda_j \in B(H, H_j)$ for each $j \in J$. Then the family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a $(T, U)$-controlled g-fusion frame for $H$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \langle \Lambda_j P_{W_j}(T f), \Lambda_j P_{W_j}(U f) \rangle \leq B \|f\|^2 \quad \forall f \in H. \quad (2)$$
If $A = B$ then $\Lambda_{TU}$ is called $(T, U)$-controlled tight g-fusion frame and if $A = B = 1$ then we say $\Lambda_{TU}$ is a $(T, U)$-controlled Parseval g-fusion frame. If $\Lambda_{TU}$ satisfies only the right inequality of (2) it is called a $(T, U)$-controlled g-fusion Bessel sequence in $H$.

**Definition 2.8.** [20] Let $\Lambda_{TU}$ be a $(T, U)$-controlled g-fusion Bessel sequence in $H$ with a bound $B$. The synthesis operator $T_C : K_{\Lambda_j} \rightarrow H$ is defined as

$$T_C \left( \left\{ v_j \left( T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} \right) = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f,$$

for all $f \in H$ and the analysis operator $T_C^* : H \rightarrow K_{\Lambda_j}$ is given by

$$T_C^* f = \left\{ v_j \left( T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} \forall f \in H,$$

where

$$K_{\Lambda_j} = \left\{ \left\{ v_j \left( T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} : f \in H \right\} \subset l^2(\{H_j\}_{j \in J}).$$

The frame operator $S_C : H \rightarrow H$ is defined as follows:

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f \forall f \in H$$

and it is easy to verify that

$$\langle S_C f, f \rangle = \sum_{j \in J} v_j^2 \langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \rangle \forall f \in H.$$

Furthermore, if $\Lambda_{TU}$ is a $(T, U)$-controlled g-fusion frame with bounds $A$ and $B$ then $A I_H \leq S_C \leq B I_H$. Hence, $S_C$ is bounded, invertible, self-adjoint and positive linear operator. It is easy to verify that $B^{-1} I_H \leq S_C^{-1} \leq A^{-1} I_H$.

**Definition 2.9.** [21] Let $F : X \rightarrow \mathbb{H}$ be such that for each $h \in H$, the mapping $x \rightarrow P_{F(x)}(h)$ is measurable (i.e. is weakly measurable), $v : X \rightarrow \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in B(F(x), K_x)$. Then $\Lambda_F = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a generalized continuous fusion frame or a gc-fusion frame for $H$ with respect to $(X, \mu)$ and $v$, if there exists $0 < A \leq B < \infty$ such that

$$A \| h \|^2 \leq \int_X v^2(x) \| \Lambda_x P_{F(x)}(h) \|^2 \, d\mu \leq B \| h \|^2 \forall h \in H,$$

where $P_{F(x)}$ is the orthogonal projection onto the subspace $F(x)$. $\Lambda_F$ is called a tight gc-fusion frame for $H$ if $A = B$ and Parseval if $A = B = 1$. If we have only the upper bound, we call $\Lambda_F$ a Bessel gc-fusion mapping for $H$. 


Let $K = \oplus_{x \in X} K_x$ and $L^2(X, K)$ be a collection of all measurable functions $\varphi : X \to K$ such that for each $x \in X$, $\varphi(x) \in K_x$ and $\int_X \|\varphi(x)\|^2 d\mu < \infty$.

It can be verified that $L^2(X, K)$ is a Hilbert space with inner product given by

$$\langle \phi, \varphi \rangle = \int_X \langle \phi(x), \varphi(x) \rangle d\mu$$

for $\phi, \varphi \in L^2(X, K)$.

**Definition 2.10.** [11] Let $\Lambda_F = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ be a Bessel gc-fusion mapping for $H$. Then the gc-fusion pre-frame operator or synthesis operator $T_{gF}^* : L^2(X, K) \to H$ is defined by

$$\langle T_{gF}(\varphi), h \rangle = \int_X v(x) \langle P_F(x) \Lambda_x^* (\varphi(x)), h \rangle d\mu$$

where $\varphi \in L^2(X, K)$ and $h \in H$. $T_{gF}$ is a bounded linear mapping and its adjoint operator is given by

$$T_{gF}^* : H \to L^2(X, K), T_{gF}^*(h) = \{v(x) \Lambda_x P_F(x)(h)\}_{x \in X}, h \in H$$

and $S_{gF} = T_{gF} T_{gF}^*$ is called gc-fusion frame operator.

For each $f, h \in H$,

$$\langle S_{gF}(f), h \rangle = \int_X v^2(x) \langle P_F(x) \Lambda_x^* \Lambda_x P_F(x)f, h \rangle d\mu.$$

The operator $S_{gF}$ is bounded, self-adjoint, positive and invertible operator on $H$.

**Definition 2.11.** [13] A sequence $\{T_x : H \to H : x \in X\}$ is said to be a continuous resolution of the identity operator on $H$ if for each $f, g \in H$, the following are hold:

1. $x \to \langle T_x f, g \rangle$ is measurable functional on $X$.
2. $\langle f, g \rangle = \int_X \langle T_x f, g \rangle d\mu(x)$

## 3 Continuous controlled $g$-fusion frame

In this section, we give the continuous version of controlled $g$-fusion frame for $H$. Some of the recent results of controlled $g$-fusion frame are extended to continuous controlled $g$-fusion frame.

**Definition 3.1.** Let $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{GB}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a continuous $(T, U)$-controlled generalized fusion frame or continuous $(T, U)$-controlled $g$-fusion frame for $H$ with respect to $(X, \mu)$ and $v$, if
(i) For each \( f \in H \), the mapping \( x \rightarrow P_{F(x)}(f) \) is measurable (i.e. is weakly measurable).

(ii) There exist constants \( 0 < A \leq B < \infty \) such that

\[
A \| f \|^2 \leq \int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \rangle \, d\mu_x \leq B \| f \|^2, \tag{3}
\]

for all \( f \in H \), where \( P_{F(x)} \) is the orthogonal projection onto the subspace \( F(x) \). The constants \( A, B \) are called the frame bounds.

Now, we consider the following cases:

(I) If only the right inequality of \( \Box \) holds then \( \Lambda_{TU} \) is called a continuous \((T, U)\)-controlled \( g \)-fusion Bessel family for \( H \).

(II) If \( U = I_H \) then \( \Lambda_{TU} \) is called a continuous \((T, I_H)\)-controlled \( g \)-fusion frame for \( H \).

(III) If \( T = U = I_H \) then \( \Lambda_{TU} \) is called a continuous \( g \)-fusion frame for \( H \).

Remark 3.2. If the measure space \( X = \mathbb{N} \) and \( \mu \) is the counting measure then a continuous \((T, U)\)-controlled \( g \)-fusion frame will be the discrete \((T, U)\)-controlled \( g \)-fusion frame.

3.0.1 Example

Let \( H = \mathbb{R}^3 \) and \( \{ e_1, e_2, e_3 \} \) be an standard orthonormal basis for \( H \). Consider

\[
B = \{ x \in \mathbb{R}^3 : \| x \| \leq 1 \}.
\]

Then it is a measure space equipped with the Lebesgue measure \( \mu \). Suppose \( \{ B_1, B_2, B_3 \} \) is a partition of \( B \) where \( \mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 0 \). Let \( \mathbb{H} = \{ W_1, W_2, W_3 \} \), where \( W_1 = \text{span} \{ e_2, e_3 \} \), \( W_2 = \text{span} \{ e_1, e_3 \} \) and \( W_3 = \text{span} \{ e_1, e_2 \} \). Define

\[
F : B \rightarrow \mathbb{H} \quad \text{by} \quad F(x) = \begin{cases} W_1 & \text{if } x \in B_1 \\ W_2 & \text{if } x \in B_2 \\ W_3 & \text{if } x \in B_3 \end{cases}
\]

and

\[
v : B \rightarrow [0, \infty) \quad \text{by} \quad v(x) = \begin{cases} 1 & \text{if } x \in B_1 \\ 2 & \text{if } x \in B_2 \\ -1 & \text{if } x \in B_3. \end{cases}
\]

It is easy to verify that \( F \) and \( v \) are measurable functions. For each \( x \in B \), define the operator

\[
\Lambda_x(f) = \frac{1}{\sqrt{\mu(B_k)}} \langle f, e_k \rangle e_k, \quad f \in H,
\]
where $k$ is such that $x \in B_k$. Let $T \(f_1, f_2, f_3\) = (2f_1, 3f_2, 5f_3)$ and $U \(f_1, f_2, f_3\) = \left(\frac{f_1}{2}, \frac{f_2}{3}, \frac{f_3}{4}\right)$ be two operators on $H$. Then it is easy to verify that $T, U \in \mathcal{GB}^+(H)$ and $TU = UT$. Now, for any $f = (f_1, f_2, f_3) \in H$, we have

\[
\begin{aligned}
\int_B v^2(x) \langle \Lambda_x P_F(x)U f, \Lambda_x P_F(x)T f \rangle \, d\mu_x \\
= \sum_{i=1}^3 \int_{B_i} v^2(x) \langle \Lambda_x P_F(x)U f, \Lambda_x P_F(x)T f \rangle \, d\mu_x \\
= f_1^2 + 4f_2^2 + \frac{5f_3^2}{4}.
\end{aligned}
\]

$\Rightarrow \|f\|^2 \leq \int_B v^2(x) \langle \Lambda_x P_F(x)U f, \Lambda_x P_F(x)T f \rangle \, d\mu_x \leq 4\|f\|^2$.

Thus, $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $\mathbb{R}^3$ with bounds 1 and 4.

**Proposition 3.3.** Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion Bessel family for $H$ with bound $B$. Then there exists a unique bounded linear operator $S_C : H \to H$ such that

\[
\langle S_C f, g \rangle = \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x)U f, g \rangle \, d\mu_x \quad \forall \ f, g \in H.
\]

Furthermore, if $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ then $AI_H \leq S_C \leq BI_H$.

**Proof.** Define the mapping $\Psi : H \times H \to \mathbb{C}$ by

\[
\Psi(f, g) = \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x)U f, g \rangle \, d\mu_x \quad \forall \ f, g \in H.
\]
Then $\Psi$ is a sesquilinear functional. Now, by Cauchy-Schwarz inequality, we have

$$|\Psi(f, g)| = \left| \int_X v^2(x) \left\langle \left( T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \right)^{1/2} f, \left( T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \right)^{1/2} g \right\rangle \, d\mu_x \right|$$

$$\leq \left( \int_X v^2(x) \left\| \left( T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \right)^{1/2} f \right\|^2 \, d\mu_x \right)^{1/2} \times \left( \int_X v^2(x) \left\| \left( T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \right)^{1/2} g \right\|^2 \, d\mu_x \right)^{1/2}$$

$$= \left( \int_X v^2(x) \left\langle \Lambda_x P_F(x) U f, \Lambda_x P_F(x) T f \right\rangle \, d\mu_x \right)^{1/2} \times \left( \int_X v^2(x) \left\langle \Lambda_x P_F(x) U g, \Lambda_x P_F(x) T g \right\rangle \, d\mu_x \right)^{1/2}$$

$$\leq B \left\| f \right\| \left\| g \right\| .$$

Thus, $\Psi$ is a bounded sesquilinear functional with $\left\| \Psi \right\| \leq B$. Therefore, by Theorem 2.3.6 in [21], there exists a unique operator $S_C : H \to H$ such that $\Psi(f, g) = \langle S_C f, g \rangle$ and $\left\| \Psi \right\| = \left\| S_C \right\|$. Thus, for each $f, g \in H$, we have

$$\langle S_C f, g \rangle = \int_X v^2(x) \left\langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U f, g \right\rangle \, d\mu_x.$$

Now, for each $f \in H$, we have

$$\langle S_C f, f \rangle = \int_X v^2(x) \left\langle \Lambda_x P_F(x) U f, \Lambda_x P_F(x) T f \right\rangle \, d\mu_x$$

$$= \int_X v^2(x) \left\| \left( T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \right)^{1/2} f \right\|^2 \, d\mu_x.$$

This verifies that $S_C$ is a positive operator. Also, it is easy to verify that $S_C$ is a self-adjoint. Furthermore, if $A_{T,U}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ then by (3) it is easy to verify that $A_{I_H} \leq S_C \leq B_{I_H}$. 

**Theorem 3.4.** Let $A_{T,U}$ be a continuous $(T, U)$-controlled $g$-fusion Bessel family for $H$ with bound $B$. Then the mapping $T_C : L^2(X, K) \to H$ defined by

$$\langle T_C \Phi, g \rangle = \int_X v^2(x) \left\langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U f, g \right\rangle \, d\mu_x,$$
where for all \( f \in H \), \( \Phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} f \right\}_{x \in X} \) and \( g \in H \), is a linear and bounded operator with \( \| T_C \| \leq \sqrt{B}. \) Furthermore, for each \( g \in H \), we have

\[
T_C^* g = \left\{ v(x) \left(T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} g \right\}_{x \in X}.
\]

**Proof.** For \( \Phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} f \right\}_{x \in X} \in L^2(X, K), \)

\[
\| T_C \Phi \| = \sup_{\| g \| = 1} |\langle T_C \Phi, g \rangle | = \sup_{\| g \| = 1} \left| \int_X v^2(x) \left( T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \right) d\mu_x \right|^{1/2} \leq \sup_{\| g \| = 1} \left( \int_X v^2(x) \left\| \left( T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} g \right\|^2 d\mu_x \right)^{1/2} \| \Phi \|_2 \leq \sqrt{B} \| \Phi \|_2.
\]

This shows that \( T_C \) is a bounded linear operator with \( \| T_C \| \leq \sqrt{B}. \) Now, for each \( g \in H \) and \( \Phi \in L^2(X, K) \), we have

\[
\langle \Phi, T_C^* g \rangle = \langle T_C \Phi, g \rangle = \int_X v^2(x) \left( \left( T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} f, \left( T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} g \right) d\mu_x \leq \sup_{\| g \| = 1} \left( \int_X v^2(x) \left\{ v(x) \left( T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U\right)^{1/2} g \right\}_{x \in X} \right) \| \Phi \|_2.
\]

This completes the proof. \( \square \)

The operators \( T_C \) and \( T_C^* \) are called the synthesis operator and analysis operator of \( \Lambda_{TU} \), respectively.

In the following proposition, we will see that it is enough to check the continuous controlled \( g \)-fusion frame condition on a dense subset \( M \) of \( H \).

**Proposition 3.5.** Suppose that \( (X, \mu) \) is a measure space with \( \mu \) is \( \sigma \)-finite and \( \Lambda_{TU} \) is a continuous \((T, U)\)-controlled \( g \)-fusion frame for a dense subset \( M \) of \( H \) having bounds \( A \) and \( B \). Then \( \Lambda_{TU} \) is a continuous \((T, U)\)-controlled \( g \)-fusion frame for \( H \) with same bounds.
Proof. Let \( \{ X_n \}_{n=1}^{\infty} \) be a sequence of disjoint measurable subsets of \( X \) such that
\[
X = \bigcup_{n=1}^{\infty} X_n \quad \text{with} \quad \mu(X_n) < \infty \quad \text{for each} \quad n \in \mathbb{N}.
\]
Let
\[
\Omega_m = \{ x \in X : m \leq \| \phi(x) \| < m+1, \forall \phi \in L^2(X,K) \}, \quad m \geq 0.
\]
It is easy to verify that for each \( m \geq 0, \Omega_m \) is a measurable set and
\[
X = \bigcup_{m=0}^{\infty} \bigcap_{n=1}^{\infty} (X_n \cap \Omega_m).
\]
If possible suppose that \( \Lambda_{T,U} \) is not a continuous \((T,U)\)-controlled \(g\)-fusion Bessel mapping for \( H \). Then there exists \( f \in H \) such that
\[
\int_X v^2(x) \langle \Lambda_x P_{F(x)}U f, \Lambda_x P_{F(x)}T f \rangle \, d\mu_x > B \| f \|^2.
\]
It follows that there exist finite subsets \( I, J \) such that
\[
\sum_{m \in I} \sum_{n \in J} \int_{X_n \cap \Omega_m} v^2(x) \langle \Lambda_x P_{F(x)}U f, \Lambda_x P_{F(x)}T f \rangle \, d\mu_x > B \| f \|^2. \tag{4}
\]
Let \( \{ f_k \} \) be a sequence in \( M \) such that \( f_k \to f \) as \( k \to \infty \). Then, we have
\[
\sum_{m \in I} \sum_{n \in J} \int_{X_n \cap \Omega_m} v^2(x) \langle \Lambda_x P_{F(x)}U f_k, \Lambda_x P_{F(x)}T f_k \rangle \, d\mu_x \leq B \| f_k \|^2,
\]
and therefore by Lebesgue’s Dominated Convergence Theorem, it is a contradiction of (4). Hence, \( \Lambda_{T,U} \) is a continuous \((T,U)\)-controlled \(g\)-fusion Bessel mapping for \( H \). So, the analysis operator \( T^*_C \) is well-defined for \( H \). Let \( f \in H \) be arbitrary and \( \{ f_k \} \) be a sequence in \( M \) such that \( f_k \to f \) as \( k \to \infty \). Then
\[
A \| f_k \|^2 \leq \| T^*_C f_k \|^2.
\]
Taking \( k \to \infty \), we get
\[
A \| f \|^2 \leq \| T^*_C f \|^2
\]
\[
= \int_X v^2(x) \langle \Lambda_x P_{F(x)}U f, \Lambda_x P_{F(x)}T f \rangle \, d\mu_x.
\]
This completes the proof.

Next we will see that continuous controlled \(g\)-fusion Bessel families for \( H \) becomes continuous controlled \(g\)-fusion frames for \( H \) under some sufficient conditions. Consider \( G : X \to \mathbb{H} \) be such that for each \( h \in H \), the mapping \( x \to P_{G(x)}(h) \) is measurable and \( w : X \to \mathbb{R}^+ \) be a measurable function.

**Theorem 3.6.** Let the families \( \Lambda_{T,U} = \{(F(x), \Lambda_x, v(x))\}_{x \in X} \) and \( \Gamma_{T,U} = \{(G(x), \Gamma_x, w(x))\}_{x \in X} \) be two continuous \((T,U)\)-controlled \(g\)-fusion Bessel families for \( H \) with bounds \( B \) and \( D \), respectively. Suppose that \( T_C \) and \( T^*_C \) be their synthesis operators such that \( T_C, T^*_C = I_H \). Then \( \Lambda_{T,U} \) and \( \Gamma_{T,U} \) are continuous \((T,U)\)-controlled \(g\)-fusion frame for \( H \).
Thus, $\Gamma_{TU}$ bounds 1. This shows that $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with bounds $1/D$ and $B$. Similarly, it can be shown that $\Gamma_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$.

In the next result, we construct continuous controlled $g$-fusion frame by using bounded linear operator.

**Theorem 3.7.** Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with bounds $A, B$ and $V \in B(H)$ be an invertible operator on $H$ such that $V^* \text{ commutes with } T, U$. Then $\Gamma_{TU} = \left\{ (V F(x), \Lambda_x P_{F(x)} V^*, v(x)) \right\}_{x \in X}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$.

**Proof.** Since $P_{F(x)} V^* = P_{F(x)} V^* P_{VF(x)}$ for all $x \in X$, the mapping $x \to P_{VF(x)}$ is weakly measurable. Now, for each $f \in H$, using Theorem 2.5, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{VF(x)} U f, \Lambda_x P_{F(x)} V^* P_{VF(x)} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* U f, \Lambda_x P_{F(x)} V^* T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U V^* f, \Lambda_x P_{F(x)} T V^* f \right\rangle d\mu_x$$

$$\leq B \| V^* f \| ^2 \leq B \| V \| ^2 \| f \| ^2 .$$

(5)

On the other hand, from (5), we get

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{VF(x)} U f, \Lambda_x P_{F(x)} V^* P_{VF(x)} T f \right\rangle d\mu_x$$

$$\geq A \| V^* f \| ^2 \geq A \| V^{-1} \| ^{-2} \| f \| ^2 \forall f \in H.$$ 

Thus, $\Gamma_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$. 


Furthermore, for each \( f \in H \), using (5), we have
\[
\int_X v^2(x) \left< \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right> d\mu_x
= \int_X v^2(x) \left< \Lambda_x P_{F(x)} U V^* f, \Lambda_x P_{F(x)} T V^* f \right> d\mu_x
= \left< S_C V^* f, V^* f \right> = \left< V S_C V^* f, f \right>,
\]
where \( S_C \) is the corresponding frame operator for \( \Lambda T U \).

In particular, if \( V = S_C^{-1} \) then by the Theorem 3.7, the family \( \Lambda T U = \{ (S_C^{-1} F(x), \Lambda_x P_{F(x)} S_C^{-1}, v(x)) \}_{x \in X} \) is also a continuous \((T, U)\)-controlled \( g \)-fusion frame for \( H \). The family \( \Lambda^0 T U \) is called the canonical dual continuous controlled \( g \)-fusion frame of \( \Lambda T U \). It is easy to verify that the corresponding frame operator for \( \Lambda^0 T U \) is \( S_C^{-1} \).

A characterization of a continuous controlled \( g \)-fusion frame is given by in the next theorem.

**Theorem 3.8.** The family \( \Lambda T U \) is a continuous \((T, U)\)-controlled \( g \)-fusion frame for \( H \) if and only if \( \Lambda T U \) is a continuous \((T U, I_H)\)-controlled \( g \)-fusion frame for \( H \).

**Proof.** For each \( f \in H \), we have
\[
\int_X v^2(x) \left< \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right> d\mu_x = \left< T S_g F U f, f \right> = \left< S_g F T U f, f \right>
= \int_X v^2(x) \left< \Lambda_x P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} T U f, f \right> d\mu_x
= \int_X v^2(x) \left< \Lambda_x P_{F(x)} T U f, \Lambda_x P_{F(x)} f \right> d\mu_x,
\]
where
\[
\left< S_g F f, f \right> = \int_X v^2(x) \left< P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} f, f \right> d\mu_x.
\]
Hence, \( \Lambda T U \) is continuous \((T, U)\)-controlled \( g \)-fusion frame for \( H \) with bounds \( A \) and \( B \) is equivalent to:
\[
A \| f \|^2 \leq \int_X v^2(x) \left< \Lambda_x P_{W_x} T U f, \Lambda_x P_{W_x} f \right> d\mu_x \leq B \| f \|^2 \quad \forall f \in H.
\]
Thus, \( \Lambda T U \) is a continuous \((T U, I_H)\)-controlled \( g \)-fusion frame for \( H \) with bounds \( A \) and \( B \).
Corollary 3.9. The family $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ if and only if $\Lambda_{TU}$ is a continuous $((TU)^{1/2}, (TU)^{1/2})$-controlled $g$-fusion frame for $H$.

The following theorem shows that any continuous controlled $g$-fusion frame is a continuous $g$-fusion frame and conversely any continuous $g$-fusion frame is a continuous controlled $g$-fusion frame under some conditions.

Theorem 3.10. Let $T, U \in \mathcal{G} \mathcal{B}^+(H)$ and $S_{gF}T = TS_{gF}$. Then $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ if and only if $\Lambda_{TU}$ is a continuous $g$-fusion frame for $H$, where $S_{gF}$ is the continuous $g$-fusion frame operator defined by

$$\langle S_{gF}f, f \rangle = \int_X v^2(x) \langle P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} f, f \rangle d\mu_x, \ f \in H.$$ 

Proof. First we suppose that $\Lambda_{TU}$ is a continuous $g$-fusion frame for $H$ with bounds $A$ and $B$. Then for each $f \in H$, we have

$$A \| f \|^2 \leq \int_X v^2(x) \| \Lambda_x P_{F(x)} f \|^2 d\mu_x \leq B \| f \|^2.$$ 

Now according to the Lemma 3.10 of [3], we can deduced that

$$m m' A I_H \leq TS_{gF}U \leq M M' B I_H,$$

where $m, m'$ and $M, M'$ are positive constants. Then for each $f \in H$, we have

$$m m' A \| f \|^2 \leq \int_X v^2(x) \langle T P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \rangle d\mu_x \leq M M' B \| f \|^2$$

$$\Rightarrow m m' A \| f \|^2 \leq \int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \rangle d\mu_x \leq M M' B \| f \|^2.$$

Hence, $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$.

Conversely, suppose that $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame
for $H$ with bounds $A$ and $B$. Now, for each $f \in H$, we have

$$A \| f \|^2 = A \left\| (TU)^{1/2} (TU)^{-1/2} f \right\|^2$$

$$\leq \left\| (TU)^{1/2} \right\|^2 \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U (TU)^{-1/2} f, \Lambda_x P_{F(x)} T (TU)^{-1/2} f \right\rangle d\mu_x$$

$$= \left\| (TU)^{1/2} \right\|^2 \int_X v^2(x) \left\langle U^{1/2} T^{-1/2} P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U^{1/2} T^{-1/2} f, f \right\rangle d\mu_x$$

$$= \left\| (TU)^{1/2} \right\|^2 \int_X v^2(x) \left\langle U^{-1/2} T^{1/2} S_{gF} U^{1/2} T^{-1/2} f, f \right\rangle = \left\| (TU)^{1/2} \right\|^2 \left\langle S_{gF} f, f \right\rangle$$

$$= \left\| (TU)^{1/2} \right\|^2 \int_X v^2(x) \left\langle P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} f, f \right\rangle d\mu_x$$

$$\Rightarrow \frac{A}{\left\| (TU)^{1/2} \right\|^2} \| f \|^2 \leq \int_X v^2(x) \| \Lambda_x P_{F(x)} f \|^2 d\mu_x.$$

On the other hand, it is easy to verify that

$$\int_X v^2(x) \| \Lambda_x P_{F(x)} f \|^2 d\mu_x = \left\langle (TU)^{-1/2} (TU)^{1/2} S_{gF} f, f \right\rangle$$

$$= \left\langle (TU)^{1/2} S_{gF} f, (TU)^{-1/2} f \right\rangle = \left\langle S_{gF} (TU) (TU)^{-1/2} f, (TU)^{-1/2} f \right\rangle$$

$$= \left\langle TS_{gF} U (TU)^{-1/2} f, (TU)^{-1/2} f \right\rangle = \left\langle S_C (TU)^{-1/2} f, (TU)^{-1/2} f \right\rangle$$

$$\leq B \left\| (TU)^{-1/2} \right\|^2 \| f \|^2.$$

Thus, $\Lambda_{TU}$ is a continuous $g$-fusion frame for $H$. This completes the proof. \qed

4 Frame operator for a pair of continuous controlled $g$-fusion Bessel families

In this section, the frame operator for a pair of continuous controlled $g$-fusion Bessel families in $H$ is considered and some properties are going to be established. Also, we present multiplier of continuous controlled $g$-fusion Bessel families in $H$. We start this section by giving continuous resolution of the identity operator on $H$.

Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with the
corresponding frame operator $S_C$. Then for each $f, g \in H$, we have
\[
\langle f, g \rangle = \langle S_C S_C^{-1} f, g \rangle = \langle S_C^{-1} S_C f, g \rangle
\]
\[
= \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U S_C^{-1} f, g \rangle \, d\mu_x
\]
\[
= \int_X v^2(x) \langle S_C^{-1} T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U f, g \rangle \, d\mu_x.
\]
Thus, the families of bounded operators $\{ T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U S_C^{-1} \}_{x \in X}$ and $\{ S_C^{-1} T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U \}_{x \in X}$ are continuous resolution of the identity operator on $H$.

**Theorem 4.1.** Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with frame bounds $A, B$ and $S_C$ be its corresponding frame operator. Assume that $S_C^{-1}$ commutes with $T$ and $U$. Then $\{ v^2(x) T^* P_F(x) \Lambda_x^* T_x U \}_{x \in X}$ is a continuous resolution of the identity operator on $H$, where $T_x = \Lambda_x P_F(x) S_C^{-1}$, $x \in X$. Furthermore, for each $f \in H$, we have
\[
\frac{A}{B^2} \| f \|^2 \leq \int_X v^2(x) \langle T_x U f, T_x T f \rangle \, d\mu_x \leq \frac{B}{A^2} \| f \|^2.
\]
**Proof.** For $f, g \in H$, we have the reconstruction formula for $\Lambda_{TU}$:
\[
\langle f, g \rangle = \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U S_C^{-1} f, g \rangle \, d\mu_x
\]
\[
= \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) S_C^{-1} U f, g \rangle \, d\mu_x
\]
\[
= \int_X v^2(x) \langle T^* P_F(x) \Lambda_x^* T_x U f, g \rangle \, d\mu_x.
\]
Thus, $\{ v^2(x) T^* P_F(x) \Lambda_x^* T_x U \}_{x \in X}$ is a continuous resolution of the identity operator on $H$. Since $\Lambda_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with frame bounds $A$ and $B$, for each $f \in H$, we have
\[
\int_X v^2(x) \langle T_x U f, T_x T f \rangle \, d\mu_x
\]
\[
= \int_X v^2(x) \langle \Lambda_x P_F(x) S_C^{-1} U f, \Lambda_x P_F(x) S_C^{-1} T f \rangle \, d\mu_x
\]
\[
= \int_X v^2(x) \langle \Lambda_x P_F(x) U S_C^{-1} f, \Lambda_x P_F(x) T S_C^{-1} f \rangle \, d\mu_x
\]
\[
\leq B \| S_C^{-1} f \|^2 \leq B \| S_C^{-1} \|^2 \| f \|^2 \leq \frac{B}{A^2} \| f \|^2.
\]
On the other hand, for each \( f \in H \), we have

\[
\int_X v^2(x) \langle T_x U f, T_x T f \rangle \, d\mu_x \geq A \left\| S^{-1}_C f \right\|^2 \geq \frac{A}{B^2} \| f \|^2.
\]

This completes the proof. \( \square \)

Next we will see that a continuous controlled \( g \)-fusion Bessel family becomes a continuous controlled \( g \)-fusion frame by using a continuous resolution of the identity operator on \( H \).

**Theorem 4.2.** Let \( \Lambda_{TT} \) be a continuous \((T, T)\)-controlled \( g \)-fusion Bessel family in \( H \) with bound \( B \). Then \( \Lambda_{TT} \) is a continuous \((U, U)\)-controlled \( g \)-fusion frame for \( H \) provided \( \{ v^2(x) T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U \}_{x \in X} \) is a continuous resolution of the identity operator on \( H \).

**Proof.** Since \( \{ v^2(x) T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U \}_{x \in X} \) is a continuous resolution of the identity operator on \( H \), for \( f, g \in H \), we have

\[
\langle f, g \rangle = \int_X v^2(x) \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \rangle \, d\mu_x.
\]

By Cauchy-Schwartz inequality, for each \( f \in H \), we have

\[
\| f \|^4 = (\langle f, f \rangle)^2 = \left( \int_X v^2(x) \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \rangle \, d\mu_x \right)^2
\]

\[
= \left( \int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \rangle \, d\mu_x \right)^2
\]

\[
\leq \int_X v^2(x) \| \Lambda_x P_{F(x)} U f \|^2 \, d\mu_x \int_X v^2(x) \| \Lambda_x P_{F(x)} T f \|^2 \, d\mu_x
\]

\[
\leq B \| f \|^2 \int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} U f \rangle \, d\mu_x
\]

\[
\Rightarrow \frac{1}{B} \| f \|^2 \leq \int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} U f \rangle \, d\mu_x.
\]

On the other hand, for each \( f \in H \), we have

\[
\int_X v^2(x) \langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} U f \rangle \, d\mu_x
\]

\[
= \int_X v^2(x) \langle \Lambda_x P_{F(x)} TT^{-1} U f, \Lambda_x P_{F(x)} TT^{-1} U f \rangle \, d\mu_x
\]

\[
\leq B \| T^{-1} U f \|^2 \leq B \| T^{-1} \|^2 \| U \|^2 \| f \|^2.
\]
Thus, $\Lambda_{TT}$ is a continuous $(U, U)$-controlled $g$-fusion frame for $H$. Similarly, it can be shown that if $\Lambda_{TT}$ is a continuous $(U, U)$-controlled $g$-fusion Bessel family in $H$ then $\Lambda_{TT}$ is also a continuous $(T, T)$-controlled $g$-fusion frame for $H$. \[\square\]

Suppose $G : X \to \mathbb{H}$ be a weakly measurable function, $w : X \to \mathbb{R}^+$ be a measurable function and for each $x \in X$, $\Gamma_x \in B(G(x), K_x)$ and $\Gamma_{UU}$ denotes the family $\{(G(x), \Gamma_x, w(x))\}_{x \in X}$. Now, we present the frame operator for a pair of continuous controlled $g$-fusion Bessel families.

**Definition 4.3.** Let $\Lambda_{TT}$ and $\Gamma_{UU}$ be continuous $(T, T)$-controlled and $(U, U)$-controlled $g$-fusion Bessel families for $H$ with bounds $B$ and $D$, respectively. Then the operator $S_{\Lambda T \Gamma U} : H \to H$ defined by

$$
\langle S_{\Lambda T \Gamma U} f, g \rangle = \int_X v(x)w(x) \langle U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x,
$$

is called the frame operator for the pair of continuous controlled $g$-fusion Bessel families $\Lambda_{TT}$ and $\Gamma_{UU}$.

**Theorem 4.4.** Let $S_{\Lambda T \Gamma U}$ be the frame operator for the pair of continuous $(T, T)$-controlled and $(U, U)$-controlled $g$-fusion Bessel families $\Lambda_{TT}$ and $\Gamma_{UU}$ with bounds $B$ and $D$, respectively. Then $S_{\Lambda T \Gamma U}$ is well-defined and bounded operator with $\|S_{\Lambda T \Gamma U}\| \leq \sqrt{BD}$.

Proof. Let $f, g \in H$. Then by Cauchy-Schwartz inequality, we have

$$
\| \langle S_{\Lambda T \Gamma U} f, g \rangle \| = \left| \int_X v(x)w(x) \langle U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x \right|
$$

$$
\leq \int_X v(x)w(x) \left| \langle \Lambda_x P_{F(x)} T f, \Gamma_x P_{G(x)} U g \rangle \right| \, d\mu_x
$$

$$
\leq \int_X v(x)w(x) \| \Lambda_x P_{F(x)} T f \| \| \Gamma_x P_{G(x)} U g \| \, d\mu_x
$$

$$
\leq \left( \int_X v^2(x) \| \Lambda_x P_{F(x)} T f \|^2 \, d\mu_x \right)^{1/2} \left( \int_X w^2(x) \| \Gamma_x P_{G(x)} U g \|^2 \, d\mu_x \right)^{1/2}
$$

$$
\leq \sqrt{BD} \| f \| \| g \|.
$$

Thus, $S_{\Lambda T \Gamma U}$ is a well-defined and bounded operator with $\|S_{\Lambda T \Gamma U}\| \leq \sqrt{BD}$. \[\square\]

In particular, for $T = U = I_H$, the operator $S_{\Lambda \Gamma} : H \to H$ defined by

$$
\langle S_{\Lambda \Gamma} f, g \rangle = \int_X v(x)w(x) \langle P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} f, g \rangle \, d\mu_x,
$$

$f, g \in H$.
is well-defined bounded operator. Also, for each \( f, g \in H \), we have

\[
\langle S_{ATGU} f, g \rangle = \int_X v(x) w(x) \langle U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x
\]

\[
= \int_X v(x) w(x) \langle f, T P_{F(x)} \Lambda_x^* \Gamma_x P_{G(x)} U g \rangle \, d\mu_x = \langle f, S_{TUAT} g \rangle
\]

and hence \( S_{TATU}^* = S_{TUAT} \).

**Theorem 4.5.** Let \( S_{ATGU} \) be the frame operator for the pair of continuous \((T, T)\)-controlled and \((U, U)\)-controlled \(g\)-fusion families \( \Lambda_{TT} \) and \( \Gamma_{UU} \) with bounds \( B \) and \( D \), respectively. Then the following statements are equivalent:

(i) \( S_{ATGU} \) is bounded below.

(ii) There exists \( K \in B(H) \) such that \( \{T_x\}_{x \in X} \) is a continuous resolution of the identity operator on \( H \), where \( T_x = v(x) w(x) K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T \), \( x \in X \).

If one of the given conditions hold, then \( \Lambda_{TT} \) is a continuous \((T, T)\)-controlled \(g\)-fusion frame for \( H \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( S_{ATGU} \) is bounded below. Then for each \( f \in H \), there exists \( A > 0 \) such that

\[
\|f\|^2 \leq A \|S_{ATGU} f\|^2 \Rightarrow \langle I_H f, f \rangle \leq A \langle S_{ATGU}^* S_{ATGU} f, f \rangle
\]

\[
\Rightarrow I_H^* I_H \leq A S_{ATGU}^* S_{ATGU}.
\]

So, by Theorem 2.1, there exists \( K \in B(H) \) such that \( K S_{ATGU} = I_H \).

Therefore, for each \( f, g \in H \), we have

\[
\langle f, g \rangle = \langle K S_{ATGU} f, g \rangle = \int_X v(x) w(x) \langle K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x.
\]

Thus, \( \{T_x\}_{x \in X} \) is a continuous resolution of the identity operator on \( H \), where \( T_x = v(x) w(x) K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T \), \( x \in X \).

(ii) \( \Rightarrow \) (i) Since \( \{T_x\}_{x \in X} \) is a continuous resolution of the identity operator on \( H \), for each \( f, g \in H \), we have

\[
\langle f, g \rangle = \int_X v(x) w(x) \langle K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x = \langle K S_{ATGU} f, g \rangle.
\]

Thus, \( I_H = K S_{ATGU} \). So, by Theorem 2.1, there exists some \( \alpha > 0 \) such that \( I_H I_H^* \leq \alpha S_{ATGU}^* S_{ATGU} \) and hence \( S_{ATGU} \) is bounded below.
Last part: First we suppose that \( S_{\Lambda T \Gamma U} \) is bounded below. Then for all \( f \in H \), there exists \( M > 0 \) such that \( \| S_{\Lambda T \Gamma U} f \| \geq M \| f \| \) and therefore by (\ref{eq:inequality}), we have

\[
M^2 \| f \|^2 \leq \| S_{\Lambda T \Gamma U} f \|^2 \leq D \left( \int_X v^2(x) \| \Lambda_x P_{F(x)} T f \|^2 \, d\mu_x \right)^{1/2}
\]

\[
\Rightarrow \frac{M^2}{D} \| f \|^2 \leq \int_X v^2(x) \langle \Lambda_x P_{F(x)} T f, \Lambda_x P_{F(x)} T f \rangle \, d\mu_x
\]

Hence, \( \Lambda_{TT} \) is a continuous \((T, T)\)-controlled \(g\)-fusion frame for \( H \) with bounds \( M^2 / D \) and \( B \). Similarly, it can be shown that \( \Gamma_{UU} \) is a continuous \((U, U)\)-controlled \(g\)-fusion frame for \( H \) with bounds \( M^2 / B \) and \( D \).

Next, we suppose that the given condition (\textit{ii}) holds. Then for each \( f, g \in H \), we have

\[
\langle f, g \rangle = \int_X v(x) w(x) \langle K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, g \rangle \, d\mu_x, \ K \in \mathcal{B}(H).
\]

By Cauchy-Schwarz inequality, for each \( f \in H \), we have

\[
\| f \|^2 = \langle f, f \rangle = \int_X v(x) w(x) \langle K U P_{G(x)} \Gamma_x^* \Lambda_x P_{F(x)} T f, f \rangle \, d\mu_x
\]

\[
= \int_X v(x) w(x) \langle \Lambda_x P_{F(x)} T f, \Gamma_x P_{G(x)} U K^* f \rangle \, d\mu_x
\]

\[
\leq \left( \int_X v^2(x) \| \Lambda_x P_{F(x)} T f \|^2 \, d\mu_x \right)^{1/2} \left( \int_X w^2(x) \| \Gamma_x P_{G(x)} U K^* f \|^2 \, d\mu_x \right)^{1/2}
\]

\[
\leq \sqrt{D} \| K^* f \| \left( \int_X v^2(x) \| \Lambda_x P_{F(x)} T f \|^2 \, d\mu_x \right)^{1/2}
\]

\[
\Rightarrow \frac{1}{D \| K \|^2} \| f \|^2 \leq \int_X v^2(x) \langle \Lambda_x P_{F(x)} T f, \Lambda_x P_{F(x)} T f \rangle \, d\mu_x
\]

Therefore, in this case \( \Lambda_{TT} \) is also a continuous \((T, T)\)-controlled \(g\)-fusion frame for \( H \).

\[\blacksquare\]

**Theorem 4.6.** Let \( \Lambda_{TT} \) and \( \Gamma_{UU} \) be continuous \((T, T)\)-controlled and \((U, U)\)-controlled \(g\)-fusion frames for \( H \) and \( T, U, S_{\Lambda} + S_{\Gamma} \in \mathcal{G}\mathcal{B}^+(H) \) such that they are commutes with each others. Then \( S_{\Lambda_{TT}U} + S_{\Gamma_{UU}T} \) is a positive operator.
Proof. For each $f, g \in H$, we have
\[
\langle (S_{T \Gamma U} + S_{\Gamma U T}) f, g \rangle \\
= \int_X v^2(x) \langle U P_G(x) \Gamma_x^* \Lambda_x P_F(x) T f, g \rangle d\mu_x + \int_X v^2(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle d\mu_x \\
= \langle U S_{\Gamma A} T f, g \rangle + \langle T S_{\Gamma A} U f, g \rangle = \langle U S_{\Gamma A} T f, g \rangle + \langle U S_{\Gamma A} T f, g \rangle \\
= \langle U (S_{\Gamma A} + S_{\Gamma A}) T f, g \rangle.
\]
This shows that $S_{T \Gamma U} + S_{\Gamma U T} = U (S_{\Gamma A} + S_{\Gamma A}) T$. Since $T, U$ and $S_{\Gamma A} + S_{\Gamma A}$ are positive and commutes with each other. Therefore, $S_{T \Gamma U} + S_{\Gamma U T}$ is a positive operator. \qed

**Theorem 4.7.** Let $\Lambda_T$ and $\Gamma_U$ be continuous $(T, T)$-controlled and $(U, U)$-controlled $g$-fusion Bessel families for $H$ with bounds $B$ and $D$, respectively. Let $m \in L^\infty(X, \mu)$. Then the operator $M_{m, \Lambda_T, \Gamma_U} : H \to H$ defined by
\[
\langle M_{m, \Lambda_T, \Gamma_U} f, g \rangle = \int_X m(x) v(x) w(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle d\mu_x,
\]
for $f, g \in H$, is well-defined and bounded operator.

**Proof.** For each $f, g \in H$, we have
\[
|\langle M_{m, \Lambda_T, \Gamma_U} f, g \rangle| = \left| \int_X m(x) v(x) w(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle d\mu_x \right| \\
\leq \int_X |m(x)| |v(x)| |w(x)| \| \Lambda_x P_F(x) T g \| \| \Gamma_x P_G(x) U f \| d\mu_x \\
\leq \| m \|_\infty \left( \int_X v^2(x) \| \Lambda_x P_F(x) T g \|^2 d\mu_x \right)^{1/2} \left( \int_X w^2(x) \| \Gamma_x P_G(x) U f \|^2 d\mu_x \right)^{1/2} \\
\leq \| m \|_\infty \sqrt{B D} \| f \| \| g \|.
\]
Thus, $M_{m, \Lambda_T, \Gamma_U}$ is a well-defined and bounded operator with $\| M_{m, \Lambda_T, \Gamma_U} \| \leq \| m \|_\infty \sqrt{B D}$. \qed

Now, multiplier of continuous controlled $g$-fusion Bessel families in Hilbert spaces is presented.

**Definition 4.8.** Let $\Lambda_T$ and $\Gamma_U$ be continuous $(T, T)$-controlled and $(U, U)$-controlled $g$-fusion Bessel families for $H$ with bounds $B$ and $D$, respectively. Let $m \in L^\infty(X, \mu)$. Then the operator $M_{m, \Lambda_T, \Gamma_U} : H \to H$ defined by
\[
\langle M_{m, \Lambda_T, \Gamma_U} f, g \rangle = \int_X m(x) v(x) w(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle d\mu_x,
\]
for $f, g \in H$, is called the continuous $(T, U)$-controlled $g$-fusion Bessel multiplier of $\Lambda_{TT}$, $\Gamma_{UU}$ and $m$.

For each $f, g \in H$, we have

$$
\langle M_{m, \Lambda T, \Gamma U} f, g \rangle = \int_X m(x) v(x) w(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle \, d\mu_x
$$

and hence $M_{m, \Lambda T, \Gamma U}^* = M_{m, \Gamma U, \Lambda T}$.

**Theorem 4.9.** Let $M_{m, \Lambda T, \Gamma U}$ be the continuous $(T, U)$-controlled $g$-fusion Bessel multiplier of $\Lambda_{TT}$, $\Gamma_{UU}$ and $m$. Assume $\lambda \in (0, 1)$ such that

$$
\| f - M_{m, \Lambda T, \Gamma U} f \| \leq \lambda \| f \| \quad \forall f \in H.
$$

Then $\Lambda_{TT}$ and $\Gamma_{UU}$ are continuous $(T, T)$-controlled and $(U, U)$-controlled $g$-fusion frame for $H$.

**Proof.** For each $f \in H$, we have

$$(1 - \lambda) \| f \| \leq \| M_{m, \Lambda T, \Gamma U} f \| = \sup_{\| g \| = 1} \langle M_{m, \Lambda T, \Gamma U} f, g \rangle
$$

$$
= \sup_{\| g \| = 1} \int_X m(x) v(x) w(x) \langle T P_F(x) \Lambda_x^* \Gamma_x P_G(x) U f, g \rangle \, d\mu_x
$$

$$
\leq \sup_{\| g \| = 1} \| m \|_\infty \left( \int_X v^2(x) \| \Lambda_x P_F(x) T g \|^2 \, d\mu_x \right)^{1/2} \times
$$

$$
\left( \int_X w^2(x) \| \Gamma_x P_G(x) U f \|^2 \, d\mu_x \right)^{1/2}
$$

$$
\leq \| m \|_\infty \sqrt{B} \left( \int_X w^2(x) \| \Gamma_x P_G(x) U f \|^2 \, d\mu_x \right)^{1/2}
$$

$$
= \frac{(1 - \lambda)^2}{B \| m \|_\infty^2 \| f \|^2} \leq \int_X w^2(x) \langle \Gamma_x P_G(x) U f, \Gamma_x P_G(x) U f \rangle \, d\mu_x.
$$

Thus, $\Gamma_{UU}$ is a continuous $(U, U)$-controlled $g$-fusion frame for $H$. Similarly, it can be shown that $\Lambda_{TT}$ is a continuous $(T, T)$-controlled $g$-fusion frame for $H$. 

\[ \square \]
5 Perturbation of continuous controlled $g$-fusion frame

In frame theory, one of the most important problems is the stability of frame under some perturbation. P. Casazza and Christensen [6] have generalized the Paley-Wiener perturbation theorem to perturbation of frame in Hilbert space. P. Ghosh and T. K. Samanta [14] discussed stability of dual $g$-fusion frame. In frame theory, one of the most important problems is the stability of frame under some perturbation. P. Casazza and Christensen [6] have generalized the Paley-Wiener perturbation theorem to perturbation of frame in Hilbert space. P. Ghosh and T. K. Samanta [14] discussed stability of dual $g$-fusion frame.

Theorem 5.1. Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with bounds $A, B$ and $\Gamma_{TU} = \{(G(x), \Gamma_x, v(x)) \}_x \in X$. If there exist constants $\lambda_1, \lambda_2, \mu$ with

$$0 \leq \lambda_1, \lambda_2 < 1, \ A (1 - \lambda_1) - \mu \int_X v^2(x) d\mu_x > 0$$

such that for each $f \in H$,

$$0 \leq \langle T^* (P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} - P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)}) U f, f \rangle$$

$$\leq \lambda_1 \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \rangle + \lambda_2 \langle T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \rangle + \mu \|f\|^2$$

then $\Gamma_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$.

Proof. For each $f \in H$, we have

$$\int_X v^2(x) \langle T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \rangle d\mu_x$$

$$= \int_X v^2(x) \langle T^* (P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} - P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)}) U f, f \rangle d\mu_x +$$

$$+ \int_X v^2(x) \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \rangle d\mu_x$$

$$\leq (1 + \lambda_1) \int_X v^2(x) \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \rangle d\mu_x + \mu \|f\|^2 \int_X v^2(x) d\mu_x$$

$$+ \lambda_2 \int_X v^2(x) \langle T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \rangle d\mu_x$$

$$\Rightarrow (1 - \lambda_2) \int_X v^2(x) \langle T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \rangle d\mu_x$$

$$\leq (1 + \lambda_1) \int_X v^2(x) \langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \rangle d\mu_x + \mu \|f\|^2 \int_X v^2(x) d\mu_x.$$

$$\Rightarrow \int_X v^2(x) \langle \Gamma_x P_{G(x)} U f, \Gamma_x P_{G(x)} T f \rangle d\mu_x \leq \left[ \frac{(1 + \lambda_1) B + \mu \int_X v^2(x) d\mu_x}{(1 - \lambda_2)} \right] \|f\|^2.$$
On the other hand, for each $f \in H$, we have

$$\int_X v^2(x) \left< T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \right> \, d\mu_x \geq \int_X v^2(x) \left< T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \right> \, d\mu_x - \int_X v^2(x) \left< T^* \left( P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} - P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} \right) U f, f \right> \, d\mu_x.$$ 

$$\Rightarrow (1 + \lambda_2) \int_X v^2(x) \left< T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \right> \, d\mu_x \geq (1 - \lambda_1) \int_X v^2(x) \left< T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \right> \, d\mu_x - \mu \| f \|^2 \int_X v^2(x) \, d\mu_x.$$ 

$$\Rightarrow \int_X v^2(x) \left< \Gamma_x P_{G(x)} U f, \Gamma_x P_{G(x)} T f \right> \, d\mu_x \geq \frac{(1 - \lambda_1) A - \mu \int_X v^2(x) \, d\mu_x}{(1 + \lambda_2)} \| f \|^2.$$ 

Thus, $\Gamma_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$. \hfill \Box

**Corollary 5.2.** Let $\Lambda_{TU}$ be a continuous $(T, U)$-controlled $g$-fusion frame for $H$ with bounds $A, B$ and $\Gamma_{TU} = \{(G(x), \Gamma_x, v(x))\}_{x \in X}$. If there exists constant $0 < D \int_X v^2(x) \, d\mu_x < A$ such that for each $f \in H$, 

$$0 \leq \left< T^* \left( P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} - P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} \right) U f, f \right> \leq D \| f \|^2$$

then $\Gamma_{TU}$ is a continuous $(T, U)$-controlled $g$-fusion frame for $H$.

**Proof.** For each $f \in H$, we have

$$\int_X v^2(x) \left< \Gamma_x P_{G(x)} U f, \Gamma_x P_{G(x)} T f \right> \, d\mu_x = \int_X v^2(x) \left< T^* P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} U f, f \right> \, d\mu_x$$

$$= \int_X v^2(x) \left< T^* \left( P_{G(x)} \Gamma_x^* \Gamma_x P_{G(x)} - P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} \right) U f, f \right> \, d\mu_x + \int_X v^2(x) \left< T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, f \right> \, d\mu_x$$

$$\leq \left( B + D \int_X v^2(x) \, d\mu_x \right) \| f \|^2.$$
On the other hand,

\[
\int_X v^2(x) \left\langle T^* P_G(x) \Gamma_x^* \Gamma_x P_G(x) U f, f \right\rangle \, d\mu_x \\
\geq \int_X v^2(x) \left\langle T^* P_F(x) \Lambda_x^* \Lambda_x P_F(x) U f, f \right\rangle \, d\mu_x - \\
- \int_X v^2(x) \left\langle T^* \left( P_G(x) \Gamma_x^* \Gamma_x P_G(x) - P_F(x) \Lambda_x^* \Lambda_x P_F(x) \right) U f, f \right\rangle \, d\mu_x \\
\geq \left( A - D \int_X v^2(x) \, d\mu_x \right) \| f \|^2 \quad \forall f \in H.
\]

This completes the proof.

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