Optical Supersymmetry in the Time Domain

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Abstract

Originally emerged within the context of string and quantum field theory, and later fruitfully extrapolated to photonics, the algebraic transformations of quantum-mechanical supersymmetry were conceived in the space realm. Here, we introduce a paradigm shift, demonstrating that Maxwell’s equations also possess an underlying supersymmetry in the time domain. As a result, we obtain a simple analytic relation between the scattering coefficients of a large variety of time-varying optical systems and uncover a wide new class of reflectionless, three-dimensional, all-dielectric, isotropic, omnidirectional, polarization-independent, non-complex media. Temporal supersymmetry is also shown to arise in dispersive media supporting temporal bound states, which allows engineering their momentum spectra and dispersive properties. These unprecedented features define a promising design platform for free-space and integrated photonics, enabling the creation of a number of novel reconfigurable reflectionless devices, such as frequency-selective, polarization-independent and omnidirectional invisible materials, compact frequency-independent phase shifters, broadband isolators, and versatile pulse-shape transformers.
Introduction

Supersymmetry (SUSY) emerged within the context of string and quantum field theory as a mathematical framework for the unification of all physical interactions of the universe \[1\]–\[5\]. Attempting to solve fundamental questions about SUSY, scientists subsequently created the field of supersymmetric quantum mechanics (SUSYQM), a non-relativistic model for testing the aforementioned theories \[6\]. Essentially, the one-dimensional (1D) version of SUSYQM considers two different quantum-mechanical systems described by the eigenvalue problems:

\[ \hat{H}_{1,2} \psi^{(1,2)} (x) = \Omega^{(1,2)} \psi^{(1,2)} (x), \]

with Hamiltonians \( \hat{H}_{1,2} = -\alpha d^2/dx^2 + V_{1,2} (x) \) and \( \alpha \in \mathbb{R}^+ \). The fundamental idea is to factorize \( \hat{H}_1 \) as \( \hat{H}_1 = \hat{A}^+ \hat{A} \) (to be referred to as the SUSY operators and \( W \) as the superpotential), and then construct the second (supersymmetric) Hamiltonian as \( \hat{H}_2 = \hat{A}^- \hat{A}^+ \). The power of SUSYQM lies in the fact that \( \hat{H}_2 \) yields the same eigenvalue spectrum and scattering properties as \( \hat{H}_1 \). For this reason, while evidence of SUSY in nature remains elusive \[7\], SUSYQM became of great interest in itself, leading to the discovery of new analytically-solvable potentials, explaining intriguing aspects of QM (such as the energy spectrum equality shared by very different systems or the existence of non-trivial reflectionless potentials), and offering a unique way to generate new families of isospectral and reflectionless systems \[6\].

Interestingly, under specific circumstances, equation (1) also describes the dynamics of electromagnetic waves (and of any physical phenomenon governed by Helmholtz’s equation), enabling a straightforward application of SUSYQM theory in optics \[8\]. As a result, notions of this formalism have been very recently utilized to design ground-breaking photonic devices \[9\]–\[13\].

Being a 1D theory, we asked ourselves whether a temporal supersymmetry might exist for time-varying potentials. Nevertheless, to our knowledge, the SUSYQM formalism has never been applied in the time domain, whether in QM, optics, or any other field (SUSY quantum field theory is a multidimensional spacetime theory, but the formalism is considerably different and more complex than that of SUSYQM). This is probably due to the fact that the vast majority of 1D SUSY work has been developed within the realm of QM, and the time derivative in Schrödinger’s equation is of first order, preventing a similar decomposition to that of equation (1) in the time domain (time-dependent potentials have been considered in SUSYQM, but also using SUSY operators based on first-order spatial derivatives \[14\]–\[15\], making it impossible to exploit the potential of the standard spatial SUSY (S-SUSY) factorization in the time domain. In fact, none of the results we will derive here could be obtained with such operators). On the other hand, only a few works deal with optical SUSY, all focused on S-SUSY. Remarkably, the fact that the temporal derivative in the electromagnetic wave equation is of second order may enable a temporal optical version of SUSYQM, which has been overlooked so far. This paradigm shift would extend the foundations and unique properties of SUSYQM to the time domain, adding an unprecedented degree of understanding and control over time-varying photonic systems and opening the door to a myriad of new applications. Actually, time-varying optical systems are becoming crucial in a broad range of scenarios, including optical modulation \[16\], isolation and non-reciprocity \[17\]–\[18\], all-optical signal processing \[19\]–\[20\], quantum information \[21\], and reconfigurable photonics \[22\]–\[23\].

Here, we show that Maxwell’s equations indeed possess an underlying time-domain supersymmetry (T-SUSY) for any non-dispersive optical system characterized by a refractive index of the form:

\[ n (r, t) = n_S (r) n_T (t) = \sqrt{\varepsilon_S (r)} \mu_S (r) \sqrt{\varepsilon_T (t)} \mu_T (t), \]

where \( \varepsilon_r (r, t) = \varepsilon_S (r) \varepsilon_T (t) \) is the medium relative permittivity and \( \mu_r (r, t) = \mu_S (r) \mu_T (t) \) its relative permeability (T-SUSY can also be found in dispersive systems, as discussed below, and in anisotropic and nonlocal media, as discussed in Supplementary Section 1). In the following, we develop this idea, illustrating its potential through different applications, sketched in Fig. 1.
Results

Consider a linear, isotropic, heterogeneous, time-varying non-dispersive medium with \( n_{T}^{2}(t) = \varepsilon_{T}(t) \). Applying separation of variables in the electric flux density \( D(r, t) = \phi(r) \psi(t) \) of Maxwell’s equations, we find that \( \psi(t) \) exactly obeys the Helmholtz’s equation:

\[
\left( \frac{d^{2}}{dt^{2}} + \omega_{0}^{2}N^{2}(t) \right) \psi(t) = 0,
\]

where \( N^{2}(t) := n_{-}/n_{T}(t) \), \( n_{-} := n_{T}(t \rightarrow -\infty) \), and \( \omega_{0} \) is the angular frequency (central angular frequency) at \( t \rightarrow -\infty \) in the monochromatic (non-monochromatic) regime. The same equation is obtained for \( n_{S}^{2}(t) = \mu_{T}(t) \) and even for general materials with \( n_{ST}^{2}(t) = \varepsilon_{T}(t) \mu_{T}(t) \) [in which case, \( \varepsilon_{T}(t) \) and/or \( \mu_{T}(t) \) must vary slowly in time], see Supplementary Section 1. Equation (3) exactly matches equation (1) by taking \( \alpha = 1 \), relabelling \( x \rightarrow t \), and identifying \( \Omega - V(t) \equiv \omega_{0}^{2}N^{2}(t) \).

Using the eigenvalue \( \Omega \) as a degree of freedom, this will allow us to apply 1D SUSY in the time domain, with two fundamental noteworthy features: 1) T-SUSY is exact for both all-dielectric and all-magnetic indices \( n_{T} \); 2) T-SUSY is completely uncoupled from space. Hence, it is valid for all polarizations, all propagation directions and any 3D spatial medium dependence \( n_{S}^{2}(r) = \varepsilon_{S}(r) \mu_{S}(r) \). This means that we can generate T-SUSY partners of devices such as waveguides or structures with any desired 3D scattering response while keeping the spatial properties of interest (e.g., ability of guiding or reflect/refract the fields in a specific way). Contrarily, 1D SUSYQM is, by definition, only valid for 1D spatial variations, and only for a specific polarization in the optical case \[9–11\]. Moreover, T-SUSY can be used to study temporal scattering in systems with continuous spectra, as well as time-varying systems supporting discrete-spectrum bound states. In both cases, its application is not as straightforward as that of S-SUSY.

First, unlike in S-SUSY, to develop T-SUSY for wave scattering, the concept of negative frequencies is essential. This comes from the differences between spatial and temporal scattering, exemplified in Fig. 2 with a simple model having one spatial dimension. As is well known, when a plane wave traverses a localized spatial variation in a time-invariant medium, there appear reflected and transmitted waves of the same frequency (photon energy), with the wave number (photon momentum) of the incident \( (k_{-}) \), reflected \( (k_{R}) \) and transmitted \( (k_{+}) \) waves fulfilling the Snell’s relations: \( k_{R} = -k_{-} \) and \( k_{+}/n_{-} = k_{+}/n_{+} \), resulting from spatial symmetry breaking, Fig. 2(a). Less known is the fact that, when a wave propagates through a homogeneous medium, reflections also appear under a localized time variation [Fig. 2(c)]. In this case, since only time symmetry is broken, momentum is conserved and photon energy changes, with the frequency of the incident \( (\omega_{-} = \omega_{0}) \), reflected \( (\omega_{R}) \) and transmitted \( (\omega_{+}) \) waves obeying the relations \( \omega_{R} = -\omega_{+} \) and \( n_{+}\omega_{+} = n_{-}\omega_{-} \[23\]. That is, light can exchange energy with the medium. Notably, the frequencies of the reflected and transmitted waves have opposite signs.

Although the physical meaning of negative-frequency waves is striking and controversial \[25–26\], mathematically, the Hermiticity of the fields in \( k-\omega \) space allows reinterpreting a negative-frequency wave as a counter-propagating positive-frequency one, leading to the standard use of only-positive-frequency frequencies. However, the introduction of negative frequencies in this work is not a mere convention. It is a mathematical tool that enables the analysis of temporal scattering and, more importantly, a necessary ingredient to relate the reflection and transmission coefficients of T-SUSY index profiles, which otherwise cannot be decoupled (Supplementary Section 2). Concretely, for a given system with a temporal index \( n_{T} \), T-SUSY provides a systematic way of generating a superpartner, whose index is (Supplementary Section 2):

\[
n_{T}^{2}(t) = \frac{n_{2,-}}{\sqrt{n_{-}^{2}/n_{T}^{2}(t)} - \frac{2}{\omega_{0}^{2}}W'(t)}.
\]
where \( n_{1,2,\pm} := n_{T1,2} (t \to \pm \infty) \) is assumed to be constant. In this case, equation (3) admits asymptotic solutions for \( n_{T1,2} \) in the form of the following incident, reflected and transmitted plane waves:
\[
\psi_1^{(1,2)} (t \to -\infty) = e^{i\omega t}, \quad \psi_R^{(1,2)} (t \to \infty) = R_{1,2} e^{-iN_n\omega t}, \quad \psi_T^{(1,2)} (t \to \infty) = T_{1,2} e^{iN_n\omega t},
\]
where \( N_+ := n_{1,-}/n_{1,+} = n_{2,-}/n_{2,+} \). The combined use of negative frequencies and T-SUSY then relates the reflection and transmission coefficients of both media as (Supplementary Section 2):
\[
R_1 = \frac{W_+ + iN_+ \omega_0}{W_- - i\omega_0} R_2, \quad T_1 = \frac{W_+ - iN_+ \omega_0}{W_- - i\omega_0} T_2,
\]
where \( W_\pm := W (t \to \pm \infty) \), \( W \) follows from the Riccati equation \( V_{1,2} (t) = W^2 (t) = W' (t) \), and \( V_{1,2} (t) = \Omega - \omega_0^2 N_1^2 (t) \). The first important consequence of equation (5) is that it can be employed to obtain the reflection and transmission coefficients of a large number of non-trivial time-varying optical media without solving Maxwell’s equations (specifically, of any supersymmetric partner of a medium with a known response). As a second important consequence, since \( n_{T1} \) and \( n_{T2} \) share the same eigenvalue \( \Omega \), \( |R_1| = |R_2| \) and \( |T_1| = |T_2| \). This allows us to readily generate families of temporal index profiles exhibiting the same scattering intensity as another medium (which can have any spatial variation), synthesizable over the same \((n_{1,-} = n_{2,-})\) or different \((n_{1,-} \neq n_{2,-})\) background materials, opening up a variety of applications.

As an example, consider the simplest case: a constant refractive index \( n_1 (r,t) = n_{1,-} \). Its SUSY partner is \( n_2 (r,t) = n_{T2} (t) = n_{2,-} [-1 + 2(\Omega/\omega_0^2 - 1) \text{sech}^2 (\sqrt{\Omega - \omega_0^2} t)]^{-1/2} \) [see Fig. 2(b)]. The free parameters \( n_{2,-} \) and \( \Omega \) allow tailoring the asymptotic value of \( n_{T2} \), as well as its temporal width \( \Delta t \) and maximal excursion \( \Delta n \) (Supplementary Section 3). Since \( n_{T1} \) is constant, \( R_1 = 0 \). Therefore, \( n_{T2} \) will also be reflectionless as demonstrated in Fig. 2(d) for a quasi-monochromatic optical pulse. From our previous discussion, \( n_{T2} \) represents a new class of all-dielectric (all-magnetic), omnidirectional, isotropic, polarization-independent, and transparent 3D media with real positive \((> 1)\) permittivity (permeability). No known spatially-varying material possesses all these features, including transformation media \([27]\), complex-parameter materials \([28]\), and S-SUSY media \([10]\). The only previously reported time-varying reflectionless media required simultaneously time-varying permittivity and permeability values \([29]\). Our T-SUSY proposal is completely different, since it is valid for all-magnetic \([n_2^2 = \mu_S (r) \mu_T (t)]\) and all-dielectric materials \([n_2^2 = \varepsilon_S (r) \varepsilon_T (t)]\), see Supplementary Section 1. The latter are particularly important, as implementing temporal permittivity variations is extremely easier than implementing permeability ones. Another general feature of T-SUSY is that it is only exact for the design frequency \( \omega = \omega_0 \). Therefore, \( n_2 \) will be invisible \((R_2 = 0, |T_2| = 1)\) for all directions and polarizations at \( \omega_0 \), while a reflected wave will appear for \( \omega \neq \omega_0 \). Moreover, the spectral span for which \( n_2 \) is almost invisible \((R_2 \simeq 0 \text{ and } |T_2| \simeq 1)\) can also be tailored via \( \Omega \) (being considerably wide around \( \Omega = \omega_0^2 \)), enabling us to generate custom-made transparent temporal windows within \( n_2 \) only for desired bands [Figs.1(a) and 2]. Notably, out of the invisible band, all waves are (partially) retroreflected along the input path, in contrast to spatial retroreflectors, in which the reflected path is parallel to, but different from, the input one [30].

Additionally, \( n_2 \) presents an outstanding unexpected property: the phases \( \Phi_{R_2} \) and \( \Phi_{T_2} \) of the reflected and transmitted waves \((R_2 = |R_2| e^{i\Phi_{R_2}}, T_2 = |T_2| e^{i\Phi_{T_2}})\) show a frequency-independent response, also tuneable through \( \Omega \) [Fig. 2(e), Fig.S3.2]. This makes \( n_2 \) a perfect dynamically-reconfigurable phase shifter: polarization-independent, flat-frequency, reflectionless, and requiring a short time variation \((< 10\pi/\omega_0)\) to generate any phase shift \( \phi \in [0, \pi]\) (allowing a dramatically reduced device length), paving the way to ideal ultra-compact optical modulators [see Fig. 1(b) and Supplementary Section 3]. Contrariwise, optical-path-based phase shifters demand slow index variations (and thus long devices) to be reflectionless, and are inherently frequency-dependent \([31][32]\). In contrast, the T-SUSY device can simultaneously introduce the same phase shift over many spectral channels, which could be useful in, e.g., wavelength-division multiplexing and frequency combs. Furthermore, the non-linear behaviour of \( \Phi_{T_2} \) may be employed to implement pulse shaping operations (Supplementary Section 3). Additional media can also be connected to a constant index via T-SUSY (being therefore reflectionless), such as the hyperbolic Rosen-Morse II (HRMII) potential [Figs.3(a), S3.7 and S3.8], which has the
advantage of allowing an independent design control over $\Delta n$ for a fixed $\Delta t \sim 40 \pi/\omega_0$, enabling a technology-oriented adjustment of the index modulation.

The previous results can be extended via different T-SUSY variants. Firstly, isospectral T-SUSY deformations provide a root to obtain $m$-parameter index families $\tilde{n}_T(t; \eta_1, \ldots, \eta_m)$ with exactly the same scattering properties in module and phase as another medium $n_T$ (Supplementary Section 2). As an example, Fig. 3(a) shows a reflectionless two-parameter family of the HRMII index. Secondly, for some $n_{T1}$ profiles, the shape invariance (SI) SUSY strategy allows us to construct T-SUSY index deformations $\{n_{T1}(t; a_1), n_{T2}(t; a_1), \ldots, n_{TM}(t; a_1)\}$ satisfying the relations $n_{TM}(t; a_1) \propto n_{T1}(t; a_m)$, $R_m(a_1) = R_1(a_m)$, $T_m(a_1) = T_1(a_m)$ and $a_m = f(a_{m-1}) = (f \circ f)(a_{m-2}) = (f \circ f \circ \ldots \circ f)(a_1)$, with $f$ a real function. Therefore, we can straightforwardly analyse or design the temporal scattering properties of a large number of time varying media. To illustrate the benefits of SI, consider the following variation of the HRMII index ($\alpha$ is a real parameter):

$$n_{T1}(t; a_1) = \frac{n_{1,-}(a_1)}{\sqrt{1 - \frac{2B}{\omega_0^2} + \frac{a_1(a_1 + \alpha)}{\omega_0^2} \sech^2(\alpha t) - \frac{2B}{\omega_0} \tanh(\alpha t)}, \quad (6)$$

which is also reflectionless in a wide spectral band [Figs. 3(c) and S3.10]. Since $n_{1,-} \neq n_{1,+}$, the system performs a frequency down-conversion with $\omega_+ = (n_{1,-}/n_{1,+})\omega_-$, where $n_{1,-}/n_{1,+}$ can be engineered via the design parameters $\omega_0$ and $B$. A device exhibiting all these properties has many potential applications. Unfortunately, the exotic shape (reaching values below $n_{1,-}$) and large maximal excursion of $n_{T1}(t; a_1)$ hampers its experimental implementation. T-SUSY can overcome this drawback by using SI. Specifically, equation (6) satisfies the SI condition with $a_m = a_1 - (m - 1)\alpha$, allowing us to generate different index profiles with the same reflectionless band as $n_{T1}(t; a_1)$ [Fig. 3(b,c)]. Taking $m = 6$, we find an index $n_{T6}(t; a_1) = (n_{6,-}(a_1)/n_{1,-}(a_6))n_{T1}(t; a_6)$ with a considerably smoother time variation and a significantly lower $\Delta n$. As illustrated in Fig. 1(c), $n_{T6}(t; a_1)$ can be used to build a polarization-independent optical isolator with an ideally unlimited bandwidth (unlike previous time- and spacetime-modulated isolators and frequency converters [17][33][34], which, in addition, usually involve complicated non-omnidirectional implementations), difficult to achieve by other means (Supplementary Section 3 includes more details on this device and additional SI examples). Reflectionless frequency converters can also be designed via transformation optics, but their implementation requires extremely complex spacetime-varying bianisotropic materials [35].

Let us now discuss the discrete-spectrum case. This scenario naturally arises in optical S-SUSY. Specifically, when applying 1D SUSYQM to a spatial dimension normal to the propagation direction, the propagation constant enters the wave equation as an effective energy, which is quantized by the eigenvalue problem [9][10]. However, since time is unidimensional, there is no possible quantity playing the role of an energy in T-SUSY, leading to free-particle systems. Outstandingly, a discrete-spectrum version of T-SUSY can be developed for time-varying dispersive media. To this end, we require the concept of temporal waveguide (TWG): two adjacent temporal index boundaries (allowed to move at a speed $v_B$) defining a (position-dependent) temporal index window, which can confine and carry optical pulses by temporal total internal reflection [36][38]. TWGs can be created by inducing a perturbation $\Delta n_{\text{eff}}(t - z/v_B)$ of the effective index $n_{\text{eff}}$ of a given mode in a spatial waveguide. In a co-moving reference frame, the complex envelope of the electric field associated with a TWG can be written as $A(z, \tau) = \sum_n \psi_n(\tau) e^{i(\Delta n_{\text{eff}}/2\beta_2)\tau} e^{iK_n z} \equiv \Psi_{\text{eff}}(z, \tau)$. It is then shown that a TWG supports temporal bound states $\psi_n (n \in \{0, 1, 2, \ldots\})$ fulfilling the discrete-spectrum eigenvalue equation:

$$\left(-\frac{d^2}{d\tau^2} + 2\frac{\beta_B(\tau)}{\beta_2}\right)\psi_n(\tau) = \left(\frac{K_n}{\beta_2} + \frac{\Delta n_{\text{eff}}^2}{\beta_2^2}\right)\psi_n(\tau), \quad (7)$$

where $\tau := t - z/v_B$, $\beta_1$ and $\beta_2$ are the inverse group velocity and group-velocity dispersion constant of the perturbed spatial mode, $\Delta \beta_1 := \beta_1 - 1/v_B$, $\beta_B(\tau) = k_0\Delta n_{\text{eff}}(\tau)$, $k_0 = \omega_0/c_0$, and $\omega_0$ is the optical carrier angular frequency. We can apply T-SUSY to equation (7), as it matches equation (1) for $\alpha = 1$, $x \rightarrow \tau$, $V(\tau) = 2\beta_B(\tau)/\beta_2$ and $\Omega_n = 2K_n/\beta_2 + \Delta n_{\text{eff}}^2/\beta_2^2$, expanding the TWG landscape and its potential applications.
As an example, consider an analytically-solvable TWG with a step temporal perturbation \( \beta_{B1} \). Its unbroken T-SUSY partner is \( \beta_{B2} (\tau) = \beta_{B1} (\tau) - \beta_2 (\ln \psi_0^{(1)} (\tau))' \), where \( \psi_0^{(1)} \) is the ground state (fundamental mode) of \( \beta_{B1} \) [Fig. 4(a), Supplementary Section 4]. From T-SUSY theory, both TWGs have the same energy spectrum. Moreover, \( \hat{A}^- \) maps each state \( \psi_1^{(1)} \) (\( \psi_2^{(2)} \)) of \( \beta_{B1} (\beta_{B2}) \) into a state of \( \beta_{B2} (\beta_{B1}) \) having the same eigenvalue \( \Omega_n \), with the exception of the ground state \( \psi_0^{(1)} \), which is annihilated by \( \hat{A}^- (\hat{A}^+ = 0) \) and thus has no equal-energy counterpart in \( \beta_{B2} \). Particularly, \( \psi_1^{(2)} \propto \hat{A}^- \psi_{n+1} \), where \( \hat{A}^- = \frac{d}{d\tau} - (\ln \psi_0^{(1)} (\tau))' \). In T-SUSY, energy is related to phase constant, implying that \( \psi_0^{(1)} \) is not phase-matched with any \( \psi_1^{(2)} \) and that \( \psi_{n+1} \) and \( \psi_n^{(2)} \) are perfectly phase-matched. This occurs in an extremely large optical bandwidth \( \Delta \nu \sim 0.5 (\nu \propto 1/\beta_2 \) is the normalized frequency), provided that both TWGs are built on a dispersion-flattened spatial waveguide \( \Delta \beta_2 / \Delta n_{\text{eff}} \). See Fig. 4(b). Furthermore, Fig. 4(b) reveals that \( \beta_{B2} \) is less dispersive than \( \beta_{B1} \), i.e., T-SUSY enables us to engineer the dispersion properties of TWGs.

To further unfold the potential of discrete-spectrum T-SUSY, we propose the concept of temporal photonic lantern (TPL): close-packed serial T-SUSY TWGs moving at the same speed and supporting linear combinations of degenerate temporal bound states (supermodes) [39]. To verify the TPL concept, we have developed the first version of coupled-mode theory (CMT) for serial TWGs (Supplementary Section 4). Fig. 4(c) shows an example of a TPL supermode. Remarkably, TWGs can carry soliton-like (shape-invariant) optical pulses in a dispersive medium, with the advantage of allowing arbitrary pulse amplitude and duration, as well as a tuneable propagation speed [37,38]. TPLs extend this ability to a serial combination of modes (each with arbitrary length, amplitude and node number), yielding solitonic supermodes with almost any desired shape. Achieving the required perfect phase-matching between modes of different order in serial TWGs without T-SUSY typically demands neighbouring TWGs of different width, whilst T-SUSY permits an independent control over this parameter and generally presents a much larger normalized phase-matching bandwidth [12], inherently implying a higher tolerance to fluctuations in \( T_B \) and \( \Delta n_{\text{eff}} \). More advanced functionalities emerge by noting that if only one of the TWGs of the TPL is excited, a periodic energy transfer between adjacent TWGs occurs [Supplementary Section 4, Fig. 4(d)]. Using two coupled spatial waveguides (WG1, WG2), this effect enables the construction of a pulse shape transformer with unprecedented versatility and reconfigurable capability [Fig. 1(d)]. Particularly, the final shape of each pulse propagating along WG1 can be dynamically chosen among a large gamut by launching the appropriate TPL over WG2. The proposed T-SUSY TPLs could also find application in optical wavelet transforms, coherent laser control of physicochemical and QM processes, spectrally-selective nonlinear microscopy, and mathematical computing [40,43]. T-SUSY TWG theory can be extended via temporal analogues of SI, broken SUSY and isospectral constructions [6].

Discussion

Overall, these results generalize the foundations of SUSYQM to the time domain, unveiling the temporal supersymmetric nature of Maxwell’s equations (which, unlike S-SUSY, has no previous direct analogue) and, consequently, leading to the emergence of an entire field of research within optics, as well as to a new photonic design toolbox. Compared to S-SUSY, T-SUSY relaxes the need for controlling the polarization state of light and the medium spatial index variation, which usually involves complex fabrication steps [9,11]. A possible T-SUSY technological difficulty might arise if high temporal index excursions are desired (e.g. to achieve a large phase variation), which can be circumvented via strong nonlinear media such as indium tin oxide [44], or multi-level phase-change materials [45], such as germanium-antimony-tellurium, indium antimonide and vanadium dioxide [46,48]. Note that extremely low index perturbations suffice to create T-SUSY TWGs, technologically feasible in standard optical fibres and waveguides via traveling-wave electro-optic phase modulators or the cross-phase modulation effect [26,37,38]. Finally, it is worth mentioning that the sound pressure satisfies a temporal Helmholtz equation formally equal to equation (3) (Supplementary Section 6). Hence, T-SUSY can be directly transferred to acoustics.
Methods

Numerical simulations of the temporal scattering problem (Figs. 2 and 3) have been performed by solving Eq. (S1.8) of the Supplementary Section 1 with COMSOL Multiphysics, taking $\omega_0 = 38 \text{ rad/s}$ and $c_0 = 1 \text{ m/s}$ to guarantee a low computational time. However, the conclusions derived from Figs. 2 and 3 are valid for any value of $\omega_0$ and $c_0$ (see pages 24 and 31 of the Supplementary Information for more details). On the other hand, the modal analysis of the TWGs and the TPL (Fig. 4) has been calculated with CST Microwave Studio and MATLAB by using the analogy reported in [37] between a dielectric slab waveguide and a TWG. Finally, the numerical simulation of Fig. 4(d), based on the CMT derived in Supplementary Section 4 for serial TWGs, was performed in MATLAB. In Supplementary Section 5 we include a detailed discussion about the numerical methods employed in Fig. 4.

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Author contributions

C.G.-M. conceived the idea of temporal SUSY. C.G.-M. and A.M.O. developed the theory, performed the numerical simulations and analysed the data. R.L.S. supervised the work. All authors contributed to write the manuscript.
References

[1] Gol’fand, Y. A. & Likhtman, E. P. Extension of the algebra of Poincare group generators and violation of P invariance. JETP Lett. 13, 452 (1971).

[2] Ramond, P. Dual theory for free fermions. Phys. Rev. D 3, 2415 (1971).

[3] Neveu, A. & Schwarz, J. H. Factorizable dual model of pions. Nucl. Phys. B 31, 86 (1971).

[4] Wess, J. & Zumino, B. Supergauge transformations in four dimensions. Nucl. Phys. B 70, 39 (1974).

[5] Freedman, D. Z., van Nieuwenhuizen, P. & Ferrara, S. Progress toward a theory of supergravity. Phys. Rev. D 13, 3214 (1976).

[6] Cooper, F., Khare, A. & Sukhatme, U. Supersymmetry and quantum mechanics. Phys. Rep. 251, 267 (1995).

[7] Ulmer, K. A. Supersymmetry: experimental status. Preprint at https://arxiv.org/abs/1601.03774 (2016).

[8] Chumakov, S. M. & Wolf, K. B. Supersymmetry in Helmholtz optics. Phys. Lett. A 193, 51 (1994).

[9] Miri, M.-A., Heinrich, M., El-Ganainy, R. & Christodoulides, D. N. Supersymmetric optical structures. Phys. Rev. Lett. 110, 233902 (2013).

[10] Miri, M.-A., Heinrich, M. & Christodoulides, D. N. SUSY-inspired one-dimensional transformation optics. Optica 1, 89-95 (2014).

[11] Heinrich, M. et al. Supersymmetric mode converters. Nature Commun. 5, 3698 (2014).

[12] Macho, A., Llorente, R. & García-Meca, C. Supersymmetric transformations in optical fibers. Phys. Rev. Appl. 9, 014024 (2018).

[13] Hokmabadi, M. P., Nye, N. S., El-Ganainy, R., Christodoulides, D. N. & Khajavikhan, M. Supersymmetric laser arrays. Science 363, 623 (2019).

[14] Baggrov V. G. & Samsonov, B. F. Supersymmetry of a nonstationary Schrödinger equation. Phys. Lett. A 210, 60 (1996).

[15] Schulze-Halberg, A. & Jimenez, J. M. C. Supersymmetry of generalized linear Schrödinger equations in (1+1) dimensions. Symmetry 1, 115-144 (2009).

[16] Yanik, M. F. & Fan, S. Time reversal of light with linear optics and modulators. Phys. Rev. Lett. 93, 173903 (2004).

[17] Sounas, D. L. & Alà, A. Non-reciprocal photonics based on time modulation. Nature Photon. 11, 774-783 (2017).

[18] Koutserimpas, T. T. & Fleury, R. Nonreciprocal gain in non-hermitian time-Floquet systems. Phys. Rev. Lett. 120, 087401 (2018).

[19] Vezzoli, S. et al. Optical time reversal from time-dependent epsilon-near-zero media. Phys. Rev. Lett. 120, 043902 (2018).

[20] Lustig, E., Sharabi, Y. & Segev, M. Topological aspects of photonic time crystals. Optica 5, 1390-1395 (2018).
[21] Law, C. K. Effective Hamiltonian for the radiation in a cavity with a moving mirror and a time-varying dielectric medium. *Phys. Rev. A* **49**, 433 (1994).

[22] Kord, A., Sounas, D. L. & Ali, A. Magnet-less circulators based on spatiotemporal modulation of bandstop filters in a delta topology. *IEEE Transactions on Microwave Theory and Techniques* **66**, 911-926 (2018).

[23] Zhang, L. et al. Space-time-coding digital metasurfaces. *Nature Commun.* **9**, 4334 (2018).

[24] Mendonça J. T. & Shukla, P. K. Time refraction and time reflection: two basic concepts. *Physica Scripta* **65**, 160-163 (2002).

[25] Horsley, S. A. R. & Bugler-Lamb, S. Negative frequencies in wave propagation: a microscopic model. *Phys. Rev. A* **93**, 063828 (2016).

[26] Philbin, T. G. et al. Fiber-optical analog of the Event Horizon. *Science* **319**, 1367 (2008).

[27] Leonhardt, U. & Philbin, T. G. *Geometry and Light: The Science of Invisibility*. (Dover Publications, New York, 2010).

[28] Horsley, S. A. R., Artoni, M. & La Rocca, G. C. Spatial Kramers-Kronig relations and the reflection of waves. *Nature Photon.* **9**, 436-439 (2015).

[29] Xiao, Y., Maywar, D. N. & Agrawal, G. P. Reflection and transmission of electromagnetic waves at a temporal boundary. *Opt. Lett.* **39**, 574-577 (2014).

[30] Ma, Y. G., Ong, C. K., Tyc, T. & Leonhardt, U. An omnidirectional retroreflector based on the transmutation of dielectric singularities. *Nature Materials* **8**, 639-642 (2009).

[31] Wang, C. et al. Integrated lithium niobate electro-optic modulators operating at CMOS-compatible voltages. *Nature* **562**, 101-104 (2018).

[32] Haffner, C. et al. All-plasmonic Mach-Zehnder modulator enabling optical high-speed communication at the microscale. *Nature Photon.* **9**, 525-528 (2015).

[33] Shi, Y., Han, S. & Fan, S. Optical circulation and isolation based on indirect photonic transitions of guided resonance modes. *ACS Photonics* **4**, 1639-1645 (2017).

[34] Lee, K. et al. Linear frequency conversion via sudden merging of meta-atoms in time-variant metasurfaces. *Nature Photon.* **12**, 765-773 (2018).

[35] Cummer, S. A. & Thompson, R. T. Frequency conversion by exploiting time in transformation optics. *Journal of Optics* **13**, 024007 (2010).

[36] Plansinis, B. W., Donaldson, W. R. & Agrawal, G. P. What is the temporal analog of reflection and refraction of optical beams? *Phys. Rev. Lett.* **115**, 183901 (2015).

[37] Plansinis, B. W., Donaldson, W. R. & Agrawal, G. P. Temporal waveguides for optical pulses. *J. Opt. Soc. Am. B* **33**, 1112-1119 (2016).

[38] Zhou, J., Zheng, G. & Wu, J. Comprehensive study on the concept of temporal optical waveguides. *Phys. Rev. A* **93**, 063847 (2016).

[39] Birks, T. A., Gris-Sánchez, I., Yerolatsitis, S., Leon-Saval, S. G. & Thomson, R. R. The photonic lantern. *Advances in Optics and Photonics* **7**, 107-167 (2015).

[40] Vázquez, J. M., Mazilu, M., Miller, A. & Galbraith, I. Wavelet transforms for optical pulse analysis. *J. Opt. Soc. Am. A* **22**, 2890-2899 (2005).
[41] Dantus, M. & Lozovoy, V. V. Experimental coherent laser control of physicochemical processes. *Chem. Rev.* **104**, 1813-1860 (2004).

[42] Weiner, A. Ultrafast optical pulse shaping: a tutorial review. *Optics Commun.* **284**, 3669-3692 (2011).

[43] Silva, A. et al. Performing mathematical operations with metamaterials. *Science* **343**, 6167 (2014).

[44] Alam, M. Z., De Leon, I. & Boyd, R. W. Large optical nonlinearity of indium tin oxide in its epsilon-near-zero region. *Science* **352**, 6287 (2016).

[45] Wang, Q. et al. Optically reconfigurable metasurfaces and photonic devices based on phase change materials. *Nature Photon.* **10**, 60-65 (2016).

[46] Michel, A.-K. U. et al. Using low-loss phase-change materials for mid-infrared antenna resonance tuning. *Nano Lett.* **13**, 3470-3475 (2013).

[47] Yang, Z. & Ramanathan, S. Breakthroughs in photonics 2014: phase change materials for photonics. *IEEE Photonics Journal* **7**, 070030 (2015).

[48] Wang, W. J. et al. Fast phase transitions induced by picosecond electrical pulses on phase change memory cells. *Appl. Phys. Lett.* **93**, 043121 (2008).
Figure 1. Potential T-SUSY applications. (a) Omnidirectional, isotropic, polarization-independent, all-dielectric (all-magnetic), and 3D time-varying material that is invisible in a given (reconfigurable) spectral region, allowing the generation of frequency-selective transparent temporal windows. (b) Perfect phase shifter: a short region in a waveguide (lighter grey) with a fast T-SUSY time-varying index inducing a dynamically-reconfigurable frequency-independent reflectionless phase shift $\Phi$ over an optical pulse. (c) Optical isolator: another T-SUSY time-varying index shifts the frequency of a right-propagating pulse (blue to yellow here), which can traverse the optical filters OF1 and OF2. Any left-propagating pulse is reflected at OF1 or OF2, protecting a left-side source from external reflections. (d) Reconfigurable pulse-shape transformer: an input pulse propagating along waveguide WG1 is spatially coupled to a T-SUSY temporal photonic lantern (TPL, a moving index perturbation) running along waveguide WG2. The pulse excites another TPL mode with a different desired shape, coupled back to WG1.
Figure 2. Reflectionless all-dielectric (all-magnetic) T-SUSY time-varying optical media. 
(a) Example of spatial reflection for an optical beam propagating through a time-invariant spatial step-index medium. (b) Index profiles of homogeneous media characterized by a temporal step-index $n_{T\,\text{STEP}}(t)$ (reflective), a constant index $n_{T1}(t)$ (non-reflective), and its T-SUSY partner $n_{T2}(t)$ (also non-reflective), with $n_{2,-} = 2$ and $\Omega = 2\omega_0^2 (T_0 = 2\pi/\omega_0)$. (c,d) Pulse propagation evolution at $\omega = \omega_0$ through the media with index profiles $n_{T\,\text{STEP}}(t)$ in (c) and $n_{T2}(t)$ in (d). Here, $\lambda = \lambda_0/n_{2,-}$ and $\lambda_0 = 2\pi c_0/\omega_0$. In contrast to (c), the result in (d) demonstrates the reflectionless nature of $n_{T2}$. 
(e) Scattering coefficients $T_2 = |T_2|\exp(i\Phi_{T2})$ and $R_2 = |R_2|\exp(i\Phi_{R2})$ of $n_{T2}$ as a function of $\omega/\omega_0$ and $\Omega/\omega_0^2$. 


Figure 3. T-SUSY reflectionless isospectral media and frequency converter. (a) Two members of the 2-parameter isospectral family \( \tilde{n}_T (t; \eta_1, \eta_2) \) of the HRMII index \( n_T (t) \). Numerical calculations show that, in all cases, \( R (\eta_1, \eta_2) = 0 \) and \( T (\eta_1, \eta_2) = e^{i0.23} \) (see Supplementary Section 3). (b) Supersymmetric refractive index profile \( n_{T1} (t; a_1) \) of equation (6) with \( a_1 = 40, B = a_1^2 / 10, \alpha = 10, \omega_0 = 38 \) rad/s and \( n_{1, -} (a_1) = 2 \). The 6-th order index of its corresponding SI chain \( n_{T6} (t; a_1) \) is also depicted. Both media induce a reflectionless frequency down-conversion in any incident optical signal. (c,d) Module (c) and phase (d) of the scattering coefficients of \( n_{T1} (t; a_1) \) and \( n_{T6} (t; a_1) \) as a function of frequency. The phase of \( R_1 \) and \( R_6 \) cannot be estimated due to the non-reflecting behaviour of \( n_{T1} (t; a_1) \) and \( n_{T6} (t; a_1) \) in an extremely large optical bandwidth. No reflected wave is observed in the numerical simulation when propagating wide-band optical pulses through these time-varying media.
Figure 4. Supersymmetric TWGs and TPLs. (a) Temporal index perturbations $\Delta n_{\text{eff}1}$ and $\Delta n_{\text{eff}2}$ of a step-index TWG ($2T_B = 660$ ps, $|\Delta \beta_1| = 10^{-3}$ ps/m and $\beta_2 = 0.06$ ps$^2$/m) and its T-SUSY partner. (b) Normalized dispersion diagram $b \cdot \nu$ of the temporal bound states $\psi_n^{(1)}$ and $\psi_n^{(2)}$ of both TWGs, where $b$ is the normalized phase constant, defined for each $n$-th order mode as $b_n := 1 - K_n/\Delta \beta - \Delta \beta_1^2/(2\beta_2 \Delta \beta)$, and $\nu$ is the normalized frequency, with $\nu^2 := 2T_B^2 \Delta \beta/\beta_2$ (here, $\nu = 6$). $\psi_n^{(2)}$ exhibits a lower slope (is less dispersive) than $\psi_n^{(1)}$. The grey area is the phase-matching bandwidth, the interval $\Delta \nu$ where $\Delta b \leq 0.2$ between SUSY bound states. (c) Index perturbation of a TPL (blue) constructed from two T-SUSY TWGs with a time separation of $T_B/4$ ($2T_B = 660$ ps), and its temporal supermode (red), generated from the perfect phase-matching between the states $\psi_0^{(2)}$ and $\psi_1^{(1)}$. (d) Pulse shape transformation resulting from the energy transfer between $\psi_0^{(2)}$ and $\psi_1^{(1)}$ in the TPL.
Supplementary Information:
Optical Supersymmetry in the Time Domain

Abstract
In this supplementary information we explain the theory of temporal supersymmetry in detail, we include additional numerical examples and we provided further information on the methods employed in this work. Equations and figures are denoted with a prefix “S” to distinguish them from the ones in the main text.

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1 Temporal scattering: optical wave equation

In general, the theory of temporal supersymmetry (T-SUSY) applied to temporal scattering is valid for dielectric (or magnetic), linear, anisotropic, heterogeneous, time-varying, and temporally non-dispersive media. As demonstrated in this section, these scenarios (including the particular situation in which a refractive index of the form given by Eq. (2) is assumed) lead to the temporal Helmholtz Eq. (3) of the paper.

Let us start by considering the all-dielectric case \[ \mu_r (r, t) = I_3 \]. For such media, by combining Faraday’s and Ampère’s laws (applying the curl operator in Faraday’s law and taking the time derivative of Ampère’s law) it is straightforward to demonstrate that the exact time-domain vector wave equation for the electric flux density \( D \) takes the form:

\[
- \nabla \times \nabla \times \left[ \varepsilon_r^{-1} (r, t) D (r, t) \right] = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} D (r, t), \tag{S1.1}
\]

where \( \varepsilon_r (r, t) \) is the medium relative permittivity tensor. Let us now assume that the relative permittivity can be expressed as:

\[
\varepsilon_r (r, t) = \varepsilon_T (t) \varepsilon_S (r), \tag{S1.2}
\]

with \( \varepsilon_S (r) \) being a tensor and \( \varepsilon_T (t) \) a scalar. Then it follows that:

\[
- \nabla \times \nabla \times \left( \varepsilon_S^{-1} (r) D (r, t) \right) = \frac{1}{c_0^2} \varepsilon_T (t) \frac{\partial^2}{\partial t^2} D (r, t). \tag{S1.3}
\]

The same wave equation applies to all-magnetic media \[ \varepsilon_r (r, t) = I_3 \] if \( D \) is replaced by the magnetic flux density \( B \) and \( \varepsilon_r (r, t) \) by the relative permeability tensor \( \mu_r (r, t) = \mu_T (t) \mu_S (r) \).

Applying separation of variables in the electromagnetic field under analysis \( F \in \{ D, B \} \):

\[
F (r, t) = \psi (t) \Phi (r), \tag{S1.4}
\]

Eq. (S1.3) becomes:

\[
- \nabla \times \nabla \times \left( \varepsilon_S^{-1} (r) \Phi (r) \right) = \frac{\varepsilon_T (t) \ddot{\psi} (t)}{c_0^2} \Phi (r), \tag{S1.5}
\]

with \( \ddot{\psi} (t) \) the second-order time derivative of \( \psi(t) \). Therefore, we must have:

\[
\varepsilon_T (t) \frac{\ddot{\psi} (t)}{\psi (t)} = C, \tag{S1.6}
\]

where \( C \) is a constant. Defining \( n_T^2 (t) := \varepsilon_T (t) \) or \( n_T^2 (t) := \mu_T (t) \) in the all-magnetic case and assuming that \( n_{\infty} := n_T (t \to -\infty) \) is also a constant, we obtain \( C = -\omega_0^2 n_{\infty}^2 \) for a wave with a frequency \( \omega_0 \) at \( t \to -\infty \), yielding Eq. (3) of the main text. Note that, in the particular case of isotropic all-dielectric media, Eq. (S1.1) can be recast as (using \( \nabla \cdot D = 0 \)):

\[
\Delta \left( \frac{1}{n^2 (r, t)} D (r, t) \right) - \nabla \left( \nabla \left( \frac{1}{n^2 (r, t)} \right) \cdot D (r, t) \right) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} D (r, t) = 0, \tag{S1.7}
\]

with \( n^2 (r, t) = \varepsilon_r (r, t) \). The same equation applies to isotropic all-magnetic media with \( n^2 (r, t) = \mu_r (r, t) \) and replacing \( D \) by \( B \). Obviously, proceeding as in the general case by applying separation of variables to Eq. (S1.7), Eq. (3) is again recovered. In the case of simultaneously dielectric and magnetic materials \[ n^2 (r, t) = \varepsilon_r (r, t) \mu_r (r, t) \], it is also possible to obtain Eq. (3) for the electric (magnetic) flux density if we assume a slowly-varying spatial and temporal evolution in \( \mu_r (\varepsilon_r) \). In the isotropic homogeneous case \[ n (r, t) = n_T (t) \], Eq. (S1.7) reduces to the familiar form of the wave equation:

\[\text{The slowly-varying temporal evolution in a constitutive parameter, e.g. } \mu_r \text{, requires to assume that } |\delta_t \mu_r| \ll |\mu_r (t)| \text{ in } \delta t \sim 2\pi/\omega_0, \text{ where } \delta_t \mu_r := \mu_r (t + \delta t) - \mu_r (t). \text{ In a similar way, the slowly-varying spatial evolution requires to assume that } |\delta_r \mu_r| \ll |\mu_r (r, t)| \text{ in } |\delta r| \sim \lambda_0, \text{ where } \delta_r \mu_r := \mu_r (r + \delta r, t) - \mu_r (r, t) \text{ and } \lambda_0 \text{ is the maximum wavelength of the problem.} \]
\[
\left( \Delta - \frac{n^2_T(t)}{c^2} \frac{\partial^2}{\partial t^2} \right) D(r, t) = 0. \tag{S1.8}
\]

As a final remark, although it is out of the scope of this work, it can also be demonstrated that Eq. (3) also follows in the case of nonlocal media by applying separation of variables in the electromagnetic fields, provided that the nonlocal and time-varying nature of the constitutive parameters can be decoupled.

Boundary conditions and polarization dependence

The natural boundary conditions of Eq. (3) arise by noticing that \( \psi \) must be twice differentiable, i.e., such conditions are the continuity of \( \psi \) and \( \psi' \). These conditions are also decoupled from the spatial part of the problem, governed by the wave equation (in the isotropic case):

\[
\Delta \left( \frac{1}{n_S^2(r)} \Phi(r) \right) + k_0^2 n_S^2 \Phi(r) - \nabla \left( \nabla \left( \frac{1}{n_S^2(r)} \right) \cdot \Phi(r) \right) = 0. \tag{S1.9}
\]

Under the previous assumptions, the temporal and spatial wave equations (as well as their corresponding boundary conditions) are completely uncoupled. Since, in addition, the orientation of the fields is fully determined by the spatial wave equation (as well as the response of the medium as a function of the polarization state), the temporal wave equation has no influence on the polarization dependence of the optical system. That is, if a system is polarization independent (or dependent), so will be its T-SUSY counterpart.

Finally, it is worth mentioning that T-SUSY systems share the same spatial solution \( \Phi^{(2)} = \Phi^{(1)} \), but possess a different temporal evolution \( \psi^{(2)} \neq \psi^{(1)} \). Consequently, the eigenvalue degeneracy and the scattering properties between both optical systems are preserved if and only if both \( \psi^{(1)} \) and \( \psi^{(2)} \) fulfil the aforementioned temporal boundary conditions.
2 Temporal scattering: T-SUSY theory

In all the aforementioned media, the temporal evolution of $D$ ($B$), encoded by the $\psi$ function, obeys Eq. (3) of the main text, reproduced here for clarity:

$$\left(\frac{d^2}{dt^2} + \omega_0^2 N^2(t)\right) \psi(t) = 0,$$

(S2.1)

where $N^2(t) := n^2_+ / n^2_+(t)$, $n_- := n_T(t \to -\infty)$, and $\omega_0$ is the angular frequency (central angular frequency) of the electromagnetic fields at $t \to -\infty$ in the monochromatic (non-monochromatic) regime.

As discussed in the main text, Eq. (S2.1) matches the 1D time-independent Schrödinger equation [Eq. (1)] taking $\alpha = 1$, performing the relabelling $x \to t$, and identifying:

$$\Omega - V(t) \equiv \omega_0^2 N^2(t).$$

(S2.2)

In this way, assuming $\Omega$ [the eigenvalue in Eq. (1)] as a degree of freedom of the problem, we will be able to use the algebraic transformations of 1D SUSYQM in the time domain, which will give rise to

time-varying refractive index profiles $n_{T1,2}(t)$ with similar scattering properties, provided that we use real $V_{1,2}$ potentials \([1,2]\).

In order to have a well-defined temporal scattering problem in both superpartners, defined on the full line ($t \in \mathbb{R}$), we require that\(^2\)

$$\psi^{(1,2)}(t \to \pm\infty) \neq 0.$$  

(S2.3)

A sufficient condition to satisfy Eq. (S2.3) is to consider $V_{1,2}(t \to \pm\infty) < \infty$, which is fulfilled by assuming: (i) $W_\pm := W(t \to \pm\infty)$ exists and is finite, and (ii) $W'$ is uniformly continuous on the full line. Thus, from Barbalat’s lemma \([3]\) and Riccati’s equation ($V_{1,2} = W^2 \mp W'$), we can infer that $W_\pm' = 0$ and $V_{1,2} = V_{2,2} = W_\pm^2 < \infty$. To summarize, we will have a well-defined scattering problem in both superpartners with:

$$|W_\pm| < \infty \quad \text{W' \ unif. cont.} \quad \Rightarrow \quad V_{1,2} = V_{2,2} = W_\pm^2 < \infty \Rightarrow \psi^{(1,2)}(t \to \pm\infty) \neq 0.$$  

(S2.4)

Hence, combining Eqs. (S2.2) and (S2.4), keeping in mind that both superpartners share the same eigenvalue $\Omega$ \([1]\), we infer that:

$$N^2_1(t \to \pm\infty) = N^2_2(t \to \pm\infty) \equiv N^2_\pm,$$

(S2.5)

with $N_- = 1$ by definition. From the above equation, the following remarks are in order:

- In spite of the fact that $N^2_{1,1} = N^2_{2,1}$, note that $n^2_{1,1} \neq n^2_{2,1}$ when $n^2_{1,-} \neq n^2_{2,-}$. This can be observed when the SUSY systems are implemented over different background materials.

- If we assume positive-real constitutive parameters, $N_{1,\pm} = N_{2,\pm} \equiv N_\pm$, and then:

$$\frac{n_{1,-}}{n_{1,+}} = \frac{n_{2,-}}{n_{2,+}}.$$  

(S2.6)

In order to connect the temporal scattering problem of both superpartners, consider a plane wave in each system at $t \to -\infty$. In this way, the asymptotic behaviour of $\psi^{(1,2)}$ at $t \to -\infty$ is equivalent to:

$$\psi^{(1,2)}(t) \xrightarrow{t \to -\infty} \psi^{(1,2)}(t) = \exp(iN_- \omega_0 t).$$  

(S2.7)

\(^2\)A spatial or temporal scattering problem is well-defined when we can observe an incident wave and at least one reflected or transmitted wave. In the temporal scattering case, the incident wave is always defined at $t \to -\infty$ and the reflected and transmitted waves are found at $t \to \infty$. Thus, we require a non-vanishing wave function at $t \to \pm\infty$. 

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Next, after the interaction with the refractive index variations \( n_{T1,2} (t) \), the asymptotic behaviour of \( \psi^{(1,2)} \) at \( t \to \infty \) will be found to be equivalent to:

\[
\psi^{(1,2)} (t) \sim \psi^{(1,2)} (t) = R_{1,2} \exp (-iN_+ \omega_0 t) + T_{1,2} \exp (iN_+ \omega_0 t),
\]

\( R_{1,2} \) and \( T_{1,2} \) being respectively the reflection and transmission coefficients, and \( \omega_0 N_\pm = \sqrt{\Omega - W_\pm^2} \).

In such a scenario, using the SUSY relation \( \psi^{(1)} = \xi \tilde{\psi}^{(2)} \) \( \tilde{\psi}^{(2)} \), where \( \xi \in \mathbb{C} \), \( \tilde{\psi}^{(2)} : = \mp d/dt + W (t) \) are the SUSY operators and \( W \) is the superpotential (a real-valued function in our case), we can relate the asymptotic behaviours as:

\[
\psi^{(1)}_\pm (t) = \xi \left( -\frac{d}{dt} + W_\pm \right) \psi^{(2)}_\pm (t).
\]

Thus, we have at \( t \to -\infty \):

\[
\exp (i\omega_0 t) = \xi ( -i\omega_0 + W_- ) \exp (i\omega_0 t),
\]

and at \( t \to \infty \):

\[
R_1 \exp (-iN_+ \omega_0 t) + T_1 \exp (iN_+ \omega_0 t) = \xi R_2 (iN_+ \omega_0 + W_+) \exp (-iN_+ \omega_0 t)
+ \xi T_2 (-iN_+ \omega_0 + W_+) \exp (iN_+ \omega_0 t).
\]

Finally, equating terms with the same exponent in Eqs. (S2.10) and (S2.11), we find:

\[
\frac{R_1}{R_2} = \frac{W_+ + iN_+ \omega_0}{W_- - i\omega_0}; \quad \frac{T_1}{T_2} = \frac{W_+ - iN_+ \omega_0}{W_- - i\omega_0}.
\]

Let us take a closer look at the above equations. In particular, it should be remarked that:

1. The SUSY refractive index profiles have identical intensity scattering behaviour:

\[
|R_1|^2 = |R_2|^2; \quad |T_1|^2 = |T_2|^2,
\]

as a direct consequence of the fact that both systems share the same eigenvalue \( \Omega = \omega_0^2 N_\pm^2 + W_\pm^2 \).

2. Although we consider complex-valued wave functions, the superpotential and the potentials must be real-valued functions to guarantee Eq. (S2.13). Otherwise, the intensity scattering behaviour could be different in each system.

3. Eq. (S2.12) does not depend on \( n_{1,2,-} \). Consequently, we can engineer time-varying refractive index profiles exhibiting the same intensity scattering behaviour using the same \( (n_{1,-} = n_{2,-}) \) or different \( (n_{1,-} \neq n_{2,-}) \) background materials.

4. Bearing in mind that \( N_- = 1 \), we find that \( \Omega = \omega_0^2 + W_-^2 \). Therefore, we infer that \( \Omega \geq \omega_0^2 \).

5. The use of negative frequencies in Eq. (S2.11) allows us, not only to describe adequately the temporal scattering problem (as discussed in the main text), but also to decouple the ratios \( R_1/R_2 \) and \( T_1/T_2 \). If we used only positive frequencies and negative wave numbers, we would find a scattering relation of the form:

\[
\frac{R_1 + T_1}{R_2 + T_2} = \frac{W_+ - iN_+ \omega_0}{W_- - i\omega_0},
\]

that is, without the possibility of decoupling the reflected and transmitted amplitudes.

6. The ratios of the scattering coefficients \( R_i \) and \( T_i \) of the electric (magnetic) field strength are analogous to those in Eq. (S2.12), since \( R_i = N_2^2 R_i \) and \( T_i = N_2^2 T_i \). Hence, we find that \( R_1/R_2 = R_1/R_2 \) and \( T_1/T_2 = T_1/T_2 \).

\[\text{The scattering coefficients } R_{1,2} \text{ and } T_{1,2} \text{ are equal to the complex amplitudes of the reflected and transmitted waves when the amplitude of the incident wave is set to } 1.\]
7. Note that $|W_+| = |W_-| \iff N_+ = N_- = 1 \iff n_{i,+} = n_{i,-}$. Such a situation takes place:
   - If $W_+ = -W_-$, in which case SUSY is said to be unbroken. Here, we observe that $R_1 = -R_2$.
   - The reflected wave has an extra phase shift of $\pi$ rad in the SUSY system.
   - If $W_+ = W_-$, in which case SUSY is said to be broken. Here, we find that $T_1 = T_2$.

8. In a SUSY refractive index chain $\{n_{Ts}(t)\}_{s=1}^m$ where the scattering problem is well-defined, the value of the potentials $\{V_s\}_{s=1}^m$ and superpotentials $\{W_s\}_{s=1}^{m-1}$ at $t \to \pm \infty$ is the same. That is:
   \[
   V_{1,\pm} = V_{2,\pm} = \ldots = V_{m,\pm} = W_{1,\pm},
   \]
   and $W_{1,\pm} = W_{2,\pm} = \ldots = W_{m-1,\pm} = W_{\pm}$. Consequently, the scattering coefficients of $n_{Tm}$ and $n_{T1}$ are related by the following expressions:
   \[
   \frac{R_1}{R_m} = \left( \frac{W_+ + iN_+ \omega_0}{W_- - i\omega_0} \right)^{m-1}; \quad \frac{T_1}{T_m} = \left( \frac{W_+ - iN_+ \omega_0}{W_- - i\omega_0} \right)^{m-1}.
   \]
   On the other hand, given a refractive index $n_{T1}(t) = n_{1,-}/N_1(t)$, its SUSY profile $n_{T2}(t) = n_{2,-}/N_2(t)$ can be directly found by combining Eq. (S2.2) and Riccati’s equation ($V_{1,2} = W^2 \mp W'$):
   \[
   n_{T2}(t; \Omega, n_{2,-}) = \frac{n_{2,-}}{\sqrt{n_{T1}(t)} - \frac{2}{\omega_0} W'(t; \Omega)},
   \]
   where the semicolon symbol is used to separate explicitly the system parameters (≡degrees of freedom) from the time variable. As mentioned above, $n_{2,-}$ allows us to change the background material of $n_{T2}(t)$ while preserving the same intensity scattering properties as the original modulation $n_{T1}(t)$, and the $\Omega$ parameter can be employed to tailor different features of $n_{T2}(t)$, such as its maximal excursion (see Section 3).

**Shape Invariant Potentials (SIP)**

Shape invariant potentials (SIP) are of great interest in quantum mechanics to find new analytically solvable potentials. In the framework of SUSY quantum mechanics, SIP allows us to: (i) calculate the spectrum of a given potential and its (unbroken or broken) SUSY Hamiltonian chain in a simple and elegant way, and (ii) analyse and design the scattering properties of a large number of potentials. In particular, we are interested in this second feature.

In general, we will say that two SUSY partner potentials $V_{1,2}$ are *shape invariant* if they obey the relation:
\[
V_2(t; a_1) = V_1(t; a_2) + M(a_1),
\]
where $(a_1, a_2) \in \mathbb{R}^p \times \mathbb{R}^p$ are a set of parameters related by a multivariate function $f \in \mathcal{F}(\mathbb{R}^p, \mathbb{R}^p)$ of the form $a_2 = f(a_1)$, and $M \in \mathcal{F}(\mathbb{R}^p, \mathbb{R})$. In our numerical examples (see Section 3), we use $p = 1$.

![Figure S2.1. Eigenvalue relation between SIP superpartners supporting continuous spectra.](image)
From the above equation, we can infer the following properties of the temporal scattering problem, with the QM superpartners supporting a continuous spectrum:

- Eq. (S2.18) establishes an eigenvalue relation between superpartners of the form (see Fig. S2.1):
  \[ \Omega \left( a_1 \right) = \Omega \left( a_2 \right) + M \left( a_1 \right), \]  
  \[ (S2.19) \]
  where \( \Omega \left( a_i \right) \in I_i \subset \mathbb{R} \) and \( I_i \) is the eigenvalue spectrum of the SUSY partners \( V_{1,2} \left( a_i \right) \).

- Since \( V_{i,\pm} \left( a_j \right) = W_{\pm}^2 \left( a_j \right) \ \forall \ (i, j) \in \{1,2\}^2 \), then:
  \[ W_{\pm}^2 \left( a_1 \right) = W_{\pm}^2 \left( a_2 \right) + M \left( a_1 \right). \]  
  \[ (S2.20) \]
  Consequently, the superpotential depends on the SIP parameters at \( t \to \pm \infty \).

- As demonstrated in \( \Pi \), the wave functions are connected as:
  \[ \psi^{(2)} \left( t; a_1 \right) = \psi^{(1)} \left( t; a_2 \right). \]  
  \[ (S2.21) \]

- Combining Eqs. (S2.18) and (S2.19) in \( \Omega \left( a_1 \right) - V_2 \left( t; a_1 \right) = \omega_0^2 N_2^2 \left( t; a_1 \right) \) we find that \( N_2 \left( t; a_1 \right) = N_1 \left( t; a_2 \right) \), and therefore:
  \[ n_{T2} \left( t; a_1 \right) = \frac{n_{2,-} \left( a_1 \right)}{n_{1,-} \left( a_2 \right)} n_{T1} \left( t; a_2 \right), \]  
  \[ (S2.22) \]
  with \( n_{2,-} \left( a_1 \right) \) a degree of freedom of the problem.

- Interestingly, \( N_{\pm} \) does not depend on the SIP parameters at \( t \to \pm \infty \). Using Eqs. (S2.19) and (S2.20):
  \[ \omega_0^2 N_{\pm}^2 \left( a_1 \right) = \Omega \left( a_1 \right) - W_{\pm}^2 \left( a_1 \right) = \Omega \left( a_2 \right) - W_{\pm}^2 \left( a_2 \right) - M \left( a_1 \right) \]
  \[ = \Omega \left( a_2 \right) - W_{\pm}^2 \left( a_2 \right) = \omega_0^2 N_{\pm}^2 \left( a_2 \right), \]  
  \[ (S2.23) \]
  we verify that \( N_{\pm} \left( a_1 \right) = N_{\pm} \left( a_2 \right) \).

- From Eq. (S2.21), we can infer that \( R_2 \left( a_1 \right) = R_1 \left( a_2 \right) \) and \( T_2 \left( a_1 \right) = T_1 \left( a_2 \right) \). As a result, Eq. (S2.12) can be restated as:
  \[ \frac{R_1 \left( a_1 \right)}{R_1 \left( a_2 \right)} = \frac{W_+ \left( a_1 \right) + iN_+ \omega_0}{W_- \left( a_1 \right) - i\omega_0}; \]
  \[ \frac{T_1 \left( a_1 \right)}{T_1 \left( a_2 \right)} = \frac{W_+ \left( a_1 \right) - iN_+ \omega_0}{W_- \left( a_1 \right) - i\omega_0}. \]
  \[ (S2.24) \]

- In a SUSY refractive index chain with \( m \) shape invariant potentials in the continuum, Eq. (S2.18) can be generalized to:
  \[ V_m \left( t; a_1 \right) = V_1 \left( t; a_m \right) + \sum_{i=1}^{m-1} M \left( a_i \right), \]  
  \[ (S2.25) \]
  with \( a_i = \left( \hat{f} \right)^{i-1} \left( a_1 \right) \) [e.g., \( a_3 = \left( \hat{f} \right)^2 \left( a_1 \right) = \left( \hat{f} \circ \hat{f} \right) \left( a_1 \right) = \hat{f} \left( \hat{f} \left( a_1 \right) \right) \)]. Hence, Eqs. (S2.19) - (S2.22) become:
  \[ \Omega \left( a_1 \right) = \Omega \left( a_m \right) + \sum_{i=1}^{m-1} M \left( a_i \right); \]  
  \[ (S2.26) \]
  \[ W_{\pm}^2 \left( a_1 \right) = W_{\pm}^2 \left( a_m \right) + \sum_{i=1}^{m-1} M \left( a_i \right); \]  
  \[ (S2.27) \]
  \[ \psi^{(m)} \left( t; a_1 \right) = \psi^{(1)} \left( t; a_m \right); \]  
  \[ (S2.28) \]
  \[ n_{T_m} \left( t; a_1 \right) = \frac{n_{2,-} \left( a_1 \right)}{n_{1,-} \left( a_m \right)} n_{T1} \left( t; a_m \right). \]  
  \[ (S2.29) \]
Thus, $R_m(a_1) = R_1(a_m)$, $T_m(a_1) = T_1(a_m)$ and the scattering relations given by Eq. (S2.16) can be recast as:

$$\frac{R_1(a_1)}{R_1(a_m)} = \left( \frac{W_+(a_1) + iN_+\omega_0}{W_-(a_1) - i\omega_0} \right)^{m-1}; \quad \frac{T_1(a_1)}{T_1(a_m)} = \left( \frac{W_+(a_1) - iN_+\omega_0}{W_-(a_1) - i\omega_0} \right)^{m-1}. \quad (S2.30)$$

### Isospectral Transformations

In the next lines, we will discuss the possibility of using SUSY transformations in the time domain to construct, from a given refractive index $n_{T_1}(t)$, an $m$-parameter family of isospectral refractive index profiles $\tilde{n}_{T_1}(t;\eta_1,\ldots,\eta_m)$, that is, time-varying optical systems with exactly the same scattering properties in module and phase as the original one.

The *one-parameter* isospectral family $V_1(x;\eta_1)$ of a given potential $V_1(x)$ can be calculated as indicated in Section 7.1 of [1]. In this vein, using our quantum-optical analogy [Eq. (S2.2)], we find the one-parameter isospectral family $\tilde{n}_{T_1}(t;\eta_1)$ of the refractive index $n_{T_1}(t)$ as:

$$\tilde{n}_{T_1}(t;\eta_1) = \frac{\tilde{n}_{1,-}(\eta_1)}{\sqrt{n_{T_1}^2(\tau) + \frac{2\pi}{\xi} d^2 \ln \left[ \eta_1 + \int_{t}^{t} \exp \left( -2 \int_{t}^{\alpha} W(\beta) d\beta \right) d\alpha \right]},} \quad (S2.31)$$

with $\eta_1$ and $\tilde{n}_{1,-}(\eta_1)$ degrees of freedom of the problem. Interestingly, the above family has exactly the same scattering coefficients as the original modulation, provided that we use a nonsingular superpotential family, which is fulfilled by taking $\eta_1 > 0$. It is straightforward to prove this statement. Let us denote the scattering coefficients of $\tilde{n}_{T_1}(t;\eta_1)$ as $R_1(\eta_1)$ and $T_1(\eta_1)$. The ratios $R_1(\eta_1)/R_2$ and $T_1(\eta_1)/T_2$ can be expressed in the same form as Eq. (S2.12), but replacing $W_\pm$ by $W_\pm(\eta_1)$ and $N_\pm$ by $\tilde{N}_\pm(\eta_1)$, where $\tilde{N}_\pm(\eta_1) := \tilde{N}_1(t \rightarrow \pm\infty;\eta_1) = \tilde{n}_{1,-}(\eta_1)/\tilde{n}_{T_1}(t \rightarrow \pm\infty;\eta_1)$ and:

$$\tilde{W}_\pm(\eta_1) := \tilde{W}(t \rightarrow \pm\infty;\eta_1)$$

$$\quad = \lim_{t \rightarrow \pm \infty} \left\{ W(t) + \frac{d}{dt} \ln \left[ \eta_1 + \int_{t}^{t} \exp \left( -2 \int_{t}^{\alpha} W(\beta) d\beta \right) d\alpha \right] \right\}. \quad (S2.32)$$

Concretely, $\tilde{W}(t;\eta_1)$ is the family of superpotentials connecting $\tilde{V}_1(t;\eta_1)$ and $V_2(t)$ via the Riccati equation. In order to preserve the scattering properties between both superpartners, the superpotential family must be nonsingular (sufficient condition but not necessary) [1]. To this end, we set $\eta_1 > 0$. Finally, taking into account that $\tilde{W}_\pm(\eta_1) = W_\pm$ and $\tilde{N}_\pm(\eta_1) = N_\pm$, we demonstrate that $R_1(\eta_1) = R_1$ and $T_1(\eta_1) = T_1$.

In addition, it is worth highlighting the possibility of combining the one-parameter isospectral transformation along with SIP. More specifically, given two potentials $V_i(t;a_i)$ and $V_{i+1}(t;a_i)$ of a SUSY chain calculated respectively with SIP from $V_1(t;a_1)$ and $V_1(t;a_{i+1})$ by using Eq. (S2.25), we can obtain the superpotential $\tilde{W}_i(t;a_1)$ from:

$$W_i(t;a_1) = W_-(a_1) + \frac{1}{2} \int_{-\infty}^{t} [V_1(\tau;a_{i+1}) - V_1(\tau;a_i) + M(a_i)] d\tau, \quad (S2.33)$$

and later calculate the family $\tilde{W}_i(t;a_1,\eta_1)$ to construct the isospectral family $\tilde{n}_{T_1}(t;a_1,\eta_1)$ with the same scattering properties in module and phase as $n_{T_1}(t;a_1)$. As seen, the temporal scattering properties of a large number of time-varying optical systems can be analysed and designed by combining both strategies.

On the other hand, the *multi-parameter* isospectral family $\tilde{n}_{T_1}(t;\eta_1,\ldots,\eta_m)$ can be calculated from the multi-parameter Darboux procedure detailed in Section 7.2 of [1]. Here, we only describe the different steps of this procedure applied to time-varying optical systems:
1. We start from a given refractive index profile \( n_{T1} (t) \) associated with a QM potential \( V_1 (t \to x) \) via Eq. [S2.2]. This potential must support bound states and must satisfy the sufficient conditions detailed at the beginning of this section to guarantee that the temporal scattering problem is well-defined [Eq. (S2.4)].

2. Next, we generate the family \( \tilde{V}_1 (x; \eta_1, \ldots, \eta_m) \) from \( V_1 (x) \) by using the multi-parameter Darboux procedure.

3. Finally, performing the relabelling \( \tilde{V}_1 (x \to t; \eta_1, \ldots, \eta_m) \) we obtain the sought family:

\[
\tilde{n}_{T1} (t; \eta_1, \ldots, \eta_m) = \frac{\tilde{n}_{1 \to \infty} (\eta_1, \ldots, \eta_m)}{\sqrt{1 + \frac{1}{\omega_0^2} [V_{1 \to \infty} - \tilde{V}_1 (t; \eta_1, \ldots, \eta_m)]}},
\]

where \( \{\eta_1, \ldots, \eta_m\} \subset \mathbb{R}^+ \) and \( \tilde{n}_{1 \to \infty} (\eta_1, \ldots, \eta_m) \) are degrees of freedom in the above equation.
3 Temporal scattering: numerical examples

In this section, we include additional numerical examples of the temporal scattering problem in T-SUSY optical systems.

Transparent hyperbolic secant modulation

In the main text, we analysed the SUSY refractive index profile \( n_2 (r, t) = n_{T2} (t) \) of a constant and, therefore, transparent refractive index \( n_1 (r, t) = n_{1-} \). In such a case, using Eq. (S2.2), we infer that the original potential is:

\[
V_1 (t) = \Omega - \omega_0^2,
\]

which leads to a superpotential of the form:

\[
W (t) = -\sqrt{\Omega - \omega_0^2} \tanh \left( \sqrt{\Omega - \omega_0^2} t \right).
\]

Finally, using Eq. (S2.17), we find the SUSY refractive index:

\[
n_2 (r, t) = n_{T2} (t) = \frac{n_{2-}}{\sqrt{1 + \frac{\omega_0^2}{\Omega}} (\sqrt{\Omega - \omega_0^2}) \sech^2 \left( \sqrt{\Omega - \omega_0^2} t \right)}.
\]

with \( \Omega \geq \omega_0^2 \), as discussed on page 19. In the following, we will first discuss the case \( \Omega > \omega_0^2 \), and later, we will analyse the case \( \Omega = \omega_0^2 \).

Considering \( \Omega > \omega_0^2 \), we observe two degrees of freedom in Eq. (S3.3): \( n_{2-} \) and \( \Omega \). The former allows us to implement the above refractive index modulation over different background materials, and the latter can be employed to tailor its maximal excursion (\( \Delta n \)) and its temporal width (\( \Delta t \)). Figure S3.1 shows the refractive index \( n_{T2} (t) \) for different values of the ratio \( \Omega/\omega_0^2 \). As seen, the higher the value of \( \Omega/\omega_0^2 \) is, the higher \( \Delta n \) and the lower \( \Delta t \) are. Along this line, it can be noted that the value of \( \Omega/\omega_0^2 \) also allows us to select the phase shifting performed by \( n_2 \). Specifically, bearing in mind that \( R_1 = 0 \) and \( T_1 = 1 \), we find from Eq. (S2.12) that \( R_2 = 0 \) and:

\[
T_2 = |T_2| \exp (i\Phi_{T_2}) = \exp \left[ -i \left( \pi + 2 \arctan \frac{1}{\sqrt{\Omega/\omega_0^2} - 1} \right) \right].
\]

Fig. S3.2 compares Eq. (S3.4) with the numerical results calculated by solving Eq. (S1.8) with COMSOL taking \( n_{2-} = 2 \), \( \omega_0 = 38 \) rad/s and \( c_0 = 1 \) m/s to guarantee a low computational time of the numerical simulations.\(^4\) We can note that the transmitted amplitude \( T_2 \) calculated with T-SUSY is in good agreement with the numerical results of the wave equation.

These graphics, along with Fig. 2 of the paper, could be of great interest to design and synthesize a perfect omnidirectional, polarization-independent, transparent and reconfigurable phase shifter using, e.g., electro-optic modulators at microwave frequencies (\( T_0 \in [3, 3000] \) ps), where the required index modulation speed (\( \Delta t \geq T_0 \)) may be implementable with current electro-optic technology.\(^5\) From the selected value \( \Omega/\omega_0^2 \), we can directly estimate the performed phase shifting (Fig. S3.2), the required \( \Delta n \) and \( \Delta t \) (Fig. S3.1), and the spectral band of transparency (Fig. 2). Remarkably, these results can be implemented in all-dielectric, all-magnetic materials, or a combination of both. In the latter case, despite the fact that we must assume a slowly-varying temporal evolution in one of the constitutive parameters (see Section IV), \( n_{T2} \) may also present rapidly-varying temporal fluctuations. Moreover, note that these results are independent of the value of \( \omega_0 \) and only depend on the ratio \( \Omega/\omega_0^2 \). Accordingly, they can be directly extrapolated to a different angular frequency.

\(^4\) The parameter \( \Delta n \) is defined as \( \Delta n := n_{2-} - \min \{n_{T2} (t)\} \) and the parameter \( \Delta t \) is defined as the full-width at 1/(2\( e \)) maximum of the temporal profile \( n_{2-} - n_{T2} (t) \).

\(^5\) The conclusions detailed below and in the next numerical examples are found to be valid for any value of \( \omega_0 \) and \( c_0 \).
Figure S3.1. Hyperbolic secant modulation Eq. (S3.3). (a) Normalized refractive index profile \( n_{T_2}(t)/n_{2,-} \), (b) normalized maximal excursion \( \Delta n/n_{2,-} \), and (c) normalized temporal width \( \Delta t/T_0 \), with \( T_0 = 2\pi/\omega_0 \). All graphics have been normalized to guarantee the same results for any value of \( \omega_0 \) and \( n_{2,-} \).

Figure S3.2. Scattering coefficient \( T_2 \) given by Eq. (S3.4) (blue line) and calculated numerically from Eq. (S1.8) using COMSOL Multiphysics (dots). (a) Module and (b) phase as a function of the ratio \( \Omega/\omega_0^2 \).

In addition, as seen in Fig. 2(e), \( \Phi_{T_2} \) shows a flat frequency response. Consequently, in our phase shifter, the reconfigurable and frequency-independent response of the phase could be of extreme utility for wavelength-division multiplexing (WDM) transmissions of narrow-band signals to generate the same phase shifting in each WDM channel. In such a scenario, if we use electro-optic modulators with a reduced \( \Delta n \) excursion (\( \Delta n \sim 10^{-3} \)), we must operate in the range \( \Omega/\omega_0^2 < 1.01 \), where \( \Phi_{T_2} < 1 \) rad. In these circumstances, it would be of interest to us to concatenate a chain of non-reflecting hyperbolic secant modulations to increase the phase shifting induced by a single hyperbolic secant. Concretely, the hyperbolic secant chain (HSC) can be described as:

\[
n_{HSC}(r, t) = \sum_{k=0}^{N_{HSC}-1} n_{T_2} \left( t + \left( \frac{N_{HSC} - 1}{2} - k \right) T_{HSC} \right) - (N_{HSC} - 1) n_{2,-}, \tag{S3.5}
\]

where \( T_{HSC} \) is the fundamental period of the chain and \( N_{HSC} \) is the number of fundamental periods [see Fig. S3.3(a)]. In order to analyse the scattering behaviour of \( n_{HSC} \), we numerically calculate the scattering coefficients \( R_{HSC} \) and \( T_{HSC} \) as a function of the ratio \( T_{HSC}/T_P \), where \( T_P \) is the full-width at 1/(2e) of the peak power of the incident pulse. Figure S3.3 shows the numerical results of \( R_{HSC} \) and \( T_{HSC} \) taking: \( \Omega/\omega_0^2 = 1.01 \) [Fig. S3.3(b)] and \( \Omega/\omega_0^2 = 6 \) [Fig. S3.3(c)] in Eq. (S3.5), \( N_{HSC} \in \{2, 3, 4\} \), \( T_{HSC}/T_P \in (0, 2] \), and \( T_P = 1 \) s. We can observe that the module and phase of \( R_{HSC} \) and \( T_{HSC} \) have a flat behaviour as a function of \( T_{HSC}/T_P \), even if \( T_{HSC} < T_P \). Hence, as expected, we can use an HSC to increase the phase shifting of the original hyperbolic secant \( \Phi_{T_{HSC}}(N_{HSC}) = N_{HSC}\Phi_{T_2} \) maintaining its non-reflecting nature.
Figure S3.3. Chain of non-reflecting hyperbolic secant refractive index profiles. (a) Refractive index profile $n_{HSC}$. Scattering coefficients $T_{HSC}$ and $R_{HSC}$ calculated numerically as a function of the ratio $T_{HSC}/T_P$, where $T_P$ is the full-width at $1/(2e)$ of the peak power of the incident pulse for: (b) $\Omega/\omega_0^2 = 1.01$ and (c) $\Omega/\omega_0^2 = 6$. The number of fundamental periods $N_{HSC}$ of the chain ranges from 2 to 4. The legend of (b) also applies to (c). We omit the subscripts of $T_{HSC}$, $R_{HSC}$ and $N_{HSC}$ in the legend due to space constraints.

Figure S3.4. Pulse propagation evolution through the time-varying medium described by $n_{HSC}(t)$. Reflected (R) and transmitted (T) optical pulses generated from the interaction of an incident (I) optical pulse with an hyperbolic secant chain operating at $\omega = 1.4\omega_0$. The transmitted pulse has a higher peak power than the incident pulse because the energy is not conserved in a time-varying system (I $\neq$ R + T). The temporal axis is normalized as $t_N = t/T_0$ with $T_0 = 2\pi/\omega_0$. The $x$-axis is normalized as $x_N = x/\lambda$, with $\lambda = \lambda_0/n_2_{\text{sc}}$ and $\lambda_0 = 2\pi c_0/\omega_0$. 

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On the other hand, the reflecting behaviour of \( n_{T2}(t) \) at \( \omega \neq \omega_0 \) and the nonlinear nature of \( \Phi_{T2}(\omega) \) [see Fig. 2(c)] can be exploited to implement pulse shaping operations in reflection and transmission. Specifically, in reflection, we can build a flat-top optical pulse using the HSC. The chain generates \( N_{HSC} \) reflected pulses of temporal width \( T_P \) and separated \( T_{HSC} \) in time. Hence, selecting \( T_P \sim T_{HSC} \), we will obtain a flat-top optical pulse emerging from the superposition of all the reflected pulses. Figure S3.4 illustrates this basic idea. We build an HSC with \( \Omega/\omega_0^2 = 6 \), \( \omega = 1.4\omega_0 \), \( T_{HSC}/T_P = 1.2 \), \( T_P = 1 \) s and \( N_{HSC} = 4 \). As seen, a flat-top optical pulse can be observed in reflection at the end of the chain. Additionally, the tail of the transmitted pulse has been distorted due to the nonlinear nature of \( \Phi_{T2}(\omega) \). This can be further investigated in future works to generate transmitted optical pulses with an exotic shape, e.g., for optical wavelet transforms, coherent laser control of physicochemical processes, or spectrally selective nonlinear microscopy among other application areas \[6-8\]. On the contrary, the pulse distortion induced by the nonlinear frequency response of \( \Phi_{T2}(\omega) \) can be reduced if we operate with narrow-band incident pulses.

So far, we have extensively evaluated the SUSY refractive index of \( n_1(r,t) = n_{1-,} \) by considering \( \Omega > \omega_0^2 \). Now, we focus our attention on the case \( \Omega = \omega_0^2 \). In such circumstances, \( V_1(t) = 0 \), \( W(t) = 1/(C-t) \) and:

\[
n_2(r,t) = n_{T2}(t) = \frac{n_{2-}}{\sqrt{1 - \frac{2}{\omega_0^2(C-t)^2}}}, \tag{S3.6}
\]

where \( C \) is an integration constant arising from Riccati’s equation. Figure S3.5 illustrates the SUSY refractive index profile given by the above equation. It is fundamental to note that now we have a singular superpotential that may break the degeneracy of the eigenvalue spectra between both super-partners (\( \Omega^{(1)} \neq \Omega^{(2)} \)). More precisely, note that \( D^{(2)} \propto (\partial_t + W(t)) D^{(1)} \) is not a continuous function at \( t = C \). Therefore, \( D^{(2)} \) cannot be a solution of Maxwell’s equations. The boundary conditions of the temporal wave equation [Eq. (3)] are not satisfied (\( D^{(2)} \) and \( \partial_t D^{(2)} \) must be necessarily continuous functions in the temporal variable). As a result, we cannot guarantee the same intensity scattering properties for \( n_1 \) and \( n_2 \).

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The nonlinear frequency dependence of the phase \( \Phi_{T2}(\omega) \) implies that each spectral component of \( D(r,t) \) is phase shifted by a different quantity. Hence, the envelope of an incident wide-band pulse will be distorted in transmission.
Hyperbolic Rosen-Morse II potential: general expressions

In the next two numerical examples, we will analyse the SUSY refractive index profiles arising from the shape invariant hyperbolic Rosen-Morse II (HRMII) potentials, described by the following equations:

\[ W (t) = A \tanh (\alpha t) + B/A; \]  
\[ V_1 (t) = A^2 + B^2/A^2 - A (A + \alpha) \text{sech}^2 (\alpha t) + 2B \tanh (\alpha t); \]  
\[ V_2 (t) = A^2 + B^2/A^2 - A (A - \alpha) \text{sech}^2 (\alpha t) + 2B \tanh (\alpha t), \]  
with \( A, B \) and \( \alpha \) real parameters and \( B < A^2 \). From the above expressions, we can verify that the SIP condition [Eq. (S2.18)] is fulfilled with \( a_1 \equiv A, a_2 = f (a_1) = a_1 - \alpha \) and \( M (a_1) = a_1^2 - (a_1 - \alpha)^2 + B^2 [1/a_1^2 - 1/(a_1 - \alpha)^2] \). Taking \( \Omega (a_1) = \omega_0^2 + W_2^2 = \omega_0^2 + (B/a_1 - a_1)^2 \), the corresponding refractive index profiles are:

\[ n_{T1} (t; a_1) = \frac{\omega_0 n_{1,-} (a_1)}{\sqrt{\omega_0^2 - 2B + a_1 (a_1 + \alpha) \text{sech}^2 (\alpha t) - 2B \tanh (\alpha t)}}, \]  
\[ n_{T2} (t; a_1) = \frac{\omega_0 n_{2,-} (a_1)}{\sqrt{\omega_0^2 - 2B + a_1 (a_1 - \alpha) \text{sech}^2 (\alpha t) - 2B \tanh (\alpha t)}}. \]  
Equation (S3.10) is Eq. (6) of the paper. We also observe that \( |W_-| = |W_+| (|W_-| \neq |W_+|) \) if \( B = 0 \) \( (B \neq 0) \). Hence, we have the ability of constructing time-varying optical systems with \( \omega_{i,-} = \omega_{i,+} \) \( (\omega_{i,-} \neq \omega_{i,+}) \). In the case \( B \neq 0 \), we should take into account that:

- The scattering coefficients of the electric (magnetic) flux density \( R_i \) and \( T_i \) are found to be different from the scattering coefficients of the electric (magnetic) field strength \( R_i \) and \( T_i \). More precisely, \( R_i = N_+^2 R_i \) and \( T_i = N_+^2 T_i \), with \( N_+^2 = 1 - 4B/\omega_0^2 \in (0,1) \). Nonetheless, \( R_1/R_2 = R_1/R_2 \) and \( T_1/T_2 = T_1/T_2 \).
- From the previous point, we deduce that \( 0 < N_+ < 1, n_{i,-} < n_{i,+} \) and \( B \leq \omega_0^2/4 \).
- From the temporal version of Snell’s law, we infer that the frequency of the transmitted signal is lower than that of the incident signal, with \( \omega_{i,+} = N_+ \omega_{i,-} \).

In both cases \( (B = 0 \) and \( B \neq 0) \), the scattering relations [Eq. (S2.12)] become:

\[ \frac{R_1 (a_1)}{R_2 (a_1)} = \frac{R_1 (a_1)}{R_2 (a_2)} = \frac{a_1 + B/a_1 + i\sqrt{\omega_0^2 - 4B}}{-a_1 + B/a_1 - i\omega_0}; \]  
\[ \frac{T_1 (a_1)}{T_2 (a_1)} = \frac{T_1 (a_1)}{T_2 (a_2)} = \frac{a_1 + B/a_1 - i\sqrt{\omega_0^2 - 4B}}{-a_1 + B/a_1 + i\omega_0}. \]  
Figure S3.6 shows the above ratios as a function of \( \omega_0 \) and \( a_1 \) setting \( B = 0.4a_1^2 \). As seen, the module is equal to 1 when \( N_+ \in \mathbb{R} \) and the phase has a low dependence on \( \omega_0 \). Furthermore, the lower the value of \( a_1 \), the lower the frequency dependence of the phase and the area where \( N_+ \in \mathbb{C} \) are.
Figure S3.6. Ratios of the scattering coefficients in the HRMII potential. The dashed white line separates the allowed \((N_+ \in \mathbb{R}, \text{top})\) and forbidden \((N_+ \in \mathbb{C}, \text{bottom})\) regions.

Hyperbolic Rosen-Morse II potential: case \(B = 0\)

The case \(B = 0\) leads to reciprocal, omnidirectional, polarization-independent and transparent phase shifters such as the hyperbolic secant profile of the first numerical example. Specifically, taking \(\alpha = a_1\) in Eq. (S3.10) we retrieve the aforementioned refractive index profile [Eq. (S3.3)]. Nevertheless, setting \(\alpha = 1\) and \(a_1 = m \in \mathbb{Z}\) we will find new refractive index modulations with an extremely large transparent optical bandwidth. In such a scenario, Eqs. (S3.10) and (S3.11) are reduced to:

\[
\begin{align*}
    n_{T1}(t; m) &= \frac{\omega_0 n_{1, -}(m)}{\sqrt{\omega_0^2 + m(m+1) \text{sech}^2 t}}, \\
    n_{T2}(t; m) &= \frac{\omega_0 n_{2, -}(m)}{\sqrt{\omega_0^2 + m(m-1) \text{sech}^2 t}},
\end{align*}
\]

obeying the SIP relation Eq. (S2.22), which can be rewritten as:

\[
n_{T2}(t; m) = \frac{n_{2, -}(m)}{n_{1, -}(m-1)} n_{T1}(t; m-1).
\]

From this recurrence relation, we can infer the transparent behaviour of these systems taking into account that they are SUSY-connected with the potential of the free particle \((m = 1)\). Nevertheless, in contrast to the first numerical example [Eq. (S3.3)], we can note that the HRMII has the advantage of allowing independent design control over \(\Delta n \sim n_{1, -} \frac{1 - \omega_0/\sqrt{\omega_0^2 + m(m\pm1)}}{n_{1, -}(m-1)}\) for a fixed \(\Delta t \sim 20T_0\), enabling a technology-oriented adjustment of the index modulation contrast. Figure S3.7(a) shows the SUSY refractive index profiles for the case \(m = 30\), and Fig. S3.7(b) depicts their scattering coefficients as a function of the frequency. As commented above, we can observe an extremely large spectral band of transparency.
Figure S3.7. (a) SUSY refractive index profiles given by Eqs. (S3.14) and (S3.15) with $m = 30$. (b) Scattering coefficients calculated numerically as a function of the ratio $\omega/\omega_0$.

On the other hand, in this case we can report an analytic solution for $T_1(m)$, which can be calculated in the same way as the transmitted probability amplitude associated with the potential $V_1(x;m) = m^2 + \omega_0^2[1 - N_1^2(t \to x;m)]$ in quantum mechanics (see page 299 of [1]):

$$T_1(m) = \frac{\Gamma(-m-i\omega_0) \Gamma(m+1-i\omega_0)}{\Gamma(-i\omega_0) \Gamma(1-i\omega_0)},$$  \hspace{1cm} (S3.17)

where $\Gamma$ is the Gamma function. Figure S3.8 shows an excellent fitting between the theoretical and numerical transmission coefficient associated with $n_{T1}(t;m)$. Strikingly, Eq. (S3.17) can be combined with Eq. (S3.13) to solve straightforwardly the temporal scattering problem in this family of time-varying systems without using Maxwell’s equations. Furthermore, we can select the desired phase of the transmitted wave via the $m$ parameter.

Figure S3.8. Transmission coefficient associated with the refractive index profile given by Eq. (S3.14) as a function of $m$. [Solid line: Eq. (S3.17). Dots: numerical results].
**Isospectral two-parameter family.** Figure 3(a) of the paper depicts the two-parameter isospectral family $\tilde{n}_{T1}(t;\eta_1,\eta_2)$ of the HRMII index given by Eq. (S3.14). Here, we detail how we have calculated $\tilde{n}_{T1}(t;\eta_1,\eta_2)$. Taking $\Omega = \omega^2_0$ in Eq. (S2.2), the corresponding quantum potential is of the form (step 1 of the isospectral theory, see page 23):

$$V_1(t \rightarrow x) = \Omega - \omega_0^2 \frac{n_1^2 - n_{T1}^2(t \rightarrow x;m)}{n_{T1}^2(t \rightarrow x;m)} = -m(m+1) \text{sech}^2 x.$$  \hspace{1cm} (S3.18)

In particular, $V_1$ holds $m$ bound states and the scattering problem is well-defined ($V_{1, < \infty}$). For the case $m = 2$, the two-parameter isospectral family $\tilde{V}_1(x;\eta_1,\eta_2)$ can be calculated from $V_1(x)$ by using the multi-parameter Darboux procedure (step 2):

$$\tilde{V}_1(x;\eta_1,\eta_2) = -12 \frac{3 + 4 \cosh(2x - 2\Lambda_2) + \cosh(4x - 2\Lambda_1)}{[\cosh(3x - \Lambda_2 - \Lambda_1) + 3 \cosh(x + \Lambda_2 - \Lambda_1)]^2},$$  \hspace{1cm} (S3.19)

with $\Lambda_{1,2} = -0.5 \ln(1 + 1/\eta_{1,2})$ and $\eta_{1,2} > 0$. Finally, performing the relabelling $\tilde{V}_1(x \rightarrow t;\eta_1,\eta_2)$ and using Eq. (S2.34), we obtain the two-parameter isospectral family of time-varying optical systems (step 3):

$$\tilde{n}_{T1}(t;\eta_1,\eta_2) = \frac{\eta_{1,-}(\eta_1,\eta_2)}{\sqrt{1 - \frac{\omega_0}{\omega_1}} - V_1(t;\eta_1,\eta_2)}.$$  \hspace{1cm} (S3.20)

Figure 3(a) of the main text shows the refractive index profile of different optical systems of the family taking $\eta_{1,-}(\eta_1,\eta_2) = n_{1,-} = 2$. We have numerically calculated the scattering coefficients of these refractive index profiles and we have found the same reflected and transmitted coefficients in module and phase as those of the original modulation $n_{T1}(t; m = 2)$ at $\omega = \omega_0$: $R_1(\eta_1,\eta_2) = 0$ and $T_1(\eta_1,\eta_2) = \exp(10.23)$.

**Hyperbolic Rosen-Morse II potential: case $B \neq 0$ (optical isolator)**

The case $B \neq 0$ is of great interest to us given that it builds a bridge to design non-reciprocal optical systems using time-varying refractive index modulations, as commented above and in the main text. In particular, from the temporal SIP theory developed above, the value of $n_{T6}$ employed for the optical isolator demonstrated in the paper can be obtained by replacing $a_1$ by $a_6 = a_1 - 5\alpha$ in Eq. (S3.10). The frequency down-conversion between the incident and transmitted waves, with angular frequencies $\omega_0 \equiv \omega_{6,-}$ and $\omega_{6,+}$, respectively, can be calculated from the temporal version of Snell’s law $\omega_{6,-}n_{6,-} = \omega_{6,+}n_{6,+}$. Hence, the ratio $\omega_{6,+}/\omega_{6,-}$ obeys the relation:

$$\frac{\omega_{6,+}}{\omega_{6,-}} = \frac{n_{6,-}}{n_{6,+}} = N_+ = \sqrt{1 - 4B/\omega_0^2}.$$  \hspace{1cm} (S3.21)

Figure S3.9 depicts the ratio $\omega_{6,+}/\omega_{6,-}$ for different values of $\omega_0$ and $B$. The dashed line separates the allowed ($N_+ \in \mathbb{R}$) and forbidden ($N_+ \in \mathbb{C}$) areas. In our case, we operate at $\omega_{6,+}/\omega_{6,-} \simeq 0.7$ with $\omega_0 \equiv \omega_{6,-} = 38$ rad/s. If we are interested in synthesizing the optical isolator of the paper for a different angular frequency, we must select an adequate value of $B$ that preserves the same ratio in Eq. (S3.21).

On the other hand, the ratios $R_6(a_1)/R_6(a_1)$ and $T_1(a_1)/T_6(a_1)$ can be calculated by combining Eqs. (S2.30) with Eqs. (S3.12) and (S3.13). In particular, we find that:

$$\frac{T_1(a_1)}{T_6(a_1)} = \frac{T_1(a_1)}{T_6(a_1)} = \left(\frac{a_1 + B/a_1 - i\sqrt{\omega_0^2 - 4B}}{a_1 + B/a_1 - i\omega_0}\right)^5 = \exp(i2.55),$$  \hspace{1cm} (S3.22)

which is in good agreement with Fig. 3(c) at $\omega = \omega_0$. The ratio $R_1(a_1)/R_6(a_1)$ could not be numerically estimated because the non-reflecting behaviour of $n_{T1}(t; a_1)$ and $n_{T6}(t; a_1)$ has a flat frequency response in an extremely large optical bandwidth (see Fig. 3). We could not observe any reflected wave in the numerical simulation when propagating wide-band optical pulses through the above media.
Figure S3.9. Frequency down-conversion ratio $\omega_{6,+}/\omega_{6,-}$ as a function of $\omega_0$ and $B$. The dashed white line separates the allowed ($N_+ \in \mathbb{R}$) and forbidden ($N_+ \in \mathbb{C}$) areas. The hollow circle indicates the operation point of the optical isolator of the main text.

Figure S3.10. Scattering coefficients of the hyperbolic step-index profile [Eq. (S3.10) taking $\alpha = A = B^{1/2}$] as a function of $\omega_0$ for the case $\alpha = 5$. [Solid line: Eqs. (S3.23) and (S3.24). Dots: numerical results].

Interestingly, for the hyperbolic step-index profile with $\alpha = A = B^{1/2}$, we can find an analytic expression for the scattering coefficients (the mathematical derivation of these expressions is detailed below):

\[
R = -\frac{1}{N_+} \frac{\Gamma(i\omega_0/\alpha) \Gamma(1-i\omega_0N_+ /\alpha)}{\Gamma(i\omega_0/\alpha) \Gamma(1+i\omega_0(1-N_+)/2\alpha) \Gamma(1+i\omega_0(1-N_+)/2\alpha)}; \tag{S3.23}
\]

\[
T = \frac{1}{N_+} \frac{\Gamma(i\omega_0/\alpha) \Gamma(1+i\omega_0N_+ /\alpha)}{\Gamma(i\omega_0/\alpha) \Gamma(1-i\omega_0(1+N_+)/2\alpha) \Gamma(1-i\omega_0(1+N_+)/2\alpha)}. \tag{S3.24}
\]

Figure S3.10 demonstrates a perfect fitting between the above expressions and the numerical results for a particular value of $\alpha$. In this example, we can observe the non-reflecting behaviour of the hyperbolic step-index profile when $\omega_0 > 10$ rad/s. In conclusion, we can combine the theoretical tools provided by T-SUSY, SIP and Eqs. (S3.23) and (S3.24) to design broadband polarization-independent optical isolators.
Derivation of Eqs. (S3.23) and (S3.24). Following a similar strategy as in [9], our first goal is to analyse the asymptotic behaviour of the general solution of the time-independent Schrödinger equation when considering the hyperbolic step potential:

\[ V(x) = \frac{1}{2} V_0 \left( 1 + \tanh \left( \frac{x}{2\alpha} \right) \right), \]

(S3.25)

with \( V_0 > 0 \) and \( \alpha > 0 \). The general solution to the equation \( \psi''(x) + (E - V(x)) \psi(x) = 0 \) is given by Eq.(9) of [9], where \( k = \sqrt{E} \) and \( k' = \sqrt{E - V_0} \). Thus, using the asymptotic behaviour of the hypergeometric functions, we find that \([\psi(x) \xrightarrow{x \to -\infty} \tilde{\psi}_-(x) \text{ and } \psi(x) \xrightarrow{x \to \infty} \tilde{\psi}_+(x)]: \)

\[ \psi_-(x) = (C \Gamma_1(\mu, \nu) + D \Gamma_2(\mu, \nu)) \exp(i k x) + (C \Gamma_3(\mu, \nu) + D \Gamma_4(\mu, \nu)) \exp(-i k x) \]

\[ \equiv A \exp(i k x) + B \exp(-i k x); \]

(S3.26)

\[ \psi_+(x) = C \exp(i k' x) + D \exp(-i k' x), \]

(S3.27)

with \( C \) and \( D \) integration constants. The functions \( \Gamma_i(\mu, \nu) \) can be found by identifying our Eq. (S3.26) with Eq. (11) of [9].

Along these lines, it is important to note that the sign convention employed in [9] is the same as that in Cooper’s tutorial [1], and it is analogous to the sign convention employed in the temporal scattering problem for the forward and backward plane waves. In this way, one could expect that relabelling \( k \to \omega_0 \) and \( k' \to N_+ \omega_0 \) in the expressions of \( R \) and \( T \) of [9], we would be able to obtain the temporal scattering reflection and transmission coefficients. Unfortunately, this procedure does not allow us to derive closed-form expressions of \( R \) and \( T \) for the temporal scattering problem because the equations connecting \( R_1(T_1) \) and \( R_2(T_2) \) in S-SUSY and T-SUSY are not analogous. More specifically, note that the asymptotic behaviours of the supersymmetric wave functions in the spatial scattering problem are S-SUSY-connected in [1] as:

\[ \exp(i k x) + R_1 \exp(-i k x) = \xi \left( -\frac{d}{dx} + W_- \right) \left( \exp(i k x) + R_2 \exp(-i k x) \right); \]

(S3.28)

\[ T_1 \exp(i k' x) = \xi \left( -\frac{d}{dx} + W_+ \right) T_2 \exp(i k' x). \]

(S3.29)

In contrast, the asymptotic behaviours of the supersymmetric wave functions in the temporal scattering problem are T-SUSY-connected as [we reproduce Eqs. (S2.10) and (S2.11) for clarity]:

\[ \exp(i \omega_0 t) = \xi \left( -\frac{d}{dt} + W_- \right) \exp(i \omega_0 t); \]

(S3.30)

\[ R_1 \exp(-i N_+ \omega_0 t) + T_1 \exp(i N_+ \omega_0 t) = \xi \left( -\frac{d}{dt} + W_+ \right) \left( R_2 \exp(-i N_+ \omega_0 t) + T_2 \exp(i N_+ \omega_0 t) \right). \]

(S3.31)

If \( R_1 = R_2 = 0 \), both systems are analogous and we can perform the previous relabelling to find the temporal scattering reflection and transmission coefficients. For instance, we calculated Eq. (S3.17) using this procedure. Nevertheless, \( R_{1,2} \neq 0 \) in this case (see Fig.S3.10). Consequently, we should recalculate the scattering coefficients for the temporal problem from the beginning.

Considering that \( \psi_- (t) = \exp(i \omega_0 t) \) and \( \psi_+ (t) = R \exp(-i N_+ \omega_0 t) + T \exp(i N_+ \omega_0 t) \), we infer from Eqs. (S3.26) and (S3.27) that \( A = 1, B = 0, C = T \) and \( D = R \). Now, using Eq. (14) of [9], we find that \( C = t_{11} \) and \( D = t_{21} \), where the transfer matrix \( t_{ij} \) is given by Eq.(16) of the aforementioned reference. Therefore, performing the transformation of parameters \( \alpha = 1/2\tilde{\alpha} \) to match the definition of the hyperbolic step potential of [9] with our definition of the hyperbolic step-index profile [Eq. (S3.10) taking \( \alpha = A = B^{1/2} \)], we finally obtain Eqs. (S3.23) and (S3.24).
Supersymmetric time-reversal modulations

Interestingly, an even superpotential $W(t) = W(-t)$ allows us to construct T-SUSY time-reversal refractive index profiles of the form $n_{T2}(t) = n_{T1}(-t)$ with the same intensity scattering behaviour, certainly an unexpected result taking into account that, in general, $n_{T1}(-t)$ has different scattering properties from those of the original modulation $n_{T1}(t)$ (consider, e.g., the step-index case [10]).

Let us analyse this scenario in more detail. To this end, consider a given $W$ with even symmetry. In such a case, we can infer from Riccati’s equation that:

$$V_2(t) = W^2(t) + W'(t) = W^2(-t) - W'(-t) = V_1(-t), \quad (S3.32)$$

and then, we find that $n_{T2}(t) = n_{T1}(-t)$ by setting $n_{2,-} = n_{1,-}$. Now, rewriting Eq. (S3.32) as $V_2(a_1 t) = V_1(a_2 t)$ and comparing this expression with the SIP condition [Eq. (S2.18)], we find that $M = 0$ and the set of parameters $a_1$ and $a_2$ are real numbers related as $a_2 = -a_1 = -1$, i.e., we have a scaling SIP relation between superpartners. In this way, T-SUSY and SIP allow us to find time-reversal refractive index modulations with the same intensity scattering properties.

![Figure S3.11](image)

**Figure S3.11.** (a) SUSY time-reversal refractive index profiles associated with the superpotential of Eq. (S3.33) and (b) ratios of the corresponding scattering coefficients $R_1/R_2$ and $T_1/T_2$ as a function of the ratio $\omega/\omega_0$.

As an example, consider an even superpotential of the form:

$$W(t) = B \text{sech}(\alpha t), \quad (S3.33)$$

where $B$ and $\alpha$ are real parameters. The SUSY refractive index profiles $n_{T1,2}(t)$ can be calculated by combining Riccati’s equation and Eq. (S2.2) bearing in mind that $\Omega = \omega_0^2 + W_2^2 = \omega_0^2$. The corresponding profiles are shown in Fig.S3.11(a), and the behaviour of the scattering coefficients as a function of the frequency [numerically calculated from the wave equation Eq. (S1.8)] is depicted in

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This superpotential is a particular case of the hyperbolic Scarf II superpotential $W(t) = A \tanh(\alpha t) + B \text{sech}(\alpha t)$. 

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Fig. S3.11(b). Outstandingly, we can see that both time-reversal refractive index profiles have the same intensity scattering properties, not only at $\omega = \omega_0$, but also at $\omega \neq \omega_0$. In other words, the ratios $R_1/R_2$ and $T_1/T_2$ have a flat frequency response. In any case, T-SUSY only provides control at the design frequency.\textsuperscript{5}

\textsuperscript{5}We analysed other even superpotentials and we found that $R_1/R_2$ and $T_1/T_2$ also presented a flat frequency response. Nevertheless, we cannot extrapolate this singular feature as a general rule in SUSY time-reversal modulations. At least, the demonstration is not straightforward.
4 Temporal waveguides: theory

Here, we first detail the theory of unbroken SUSY transformations in temporal waveguides (TWGs) and, secondly, we derive a coupled-mode theory (CMT) for serial TWGs moving with the same speed and direction in a given spatial waveguide.

4.1 T-SUSY in temporal waveguides

The unbroken SUSY relation between two quantum-mechanical superpartners \( V_1, V_2 \) is given by the expression (we set \( \hbar^2/2m \equiv 1 \) for simplicity) [1]:

\[
V_2 (x) = V_1 (x) - 2d^2 \frac{d}{dx} \ln \psi_0 (1) (x),
\]

(S4.1)

where \( \psi_0 (1) \) is the ground state of \( V_1 \). As detailed in the main text, we can identify an effective potential in Eq. (7) of the form:

\[
V_i (\tau) \equiv \frac{\beta_i}{2} \beta_i \psi_0 (\tau).
\]

(S4.2)

Hence, combining Eqs. (S4.1) and (S4.2) we infer that:

\[
\beta_i (\tau) = \frac{\beta_i (\tau)}{\beta_i (\tau)} - \beta_i \frac{d^2}{d\tau^2} \ln \psi_0 (1) (\tau),
\]

(S4.3)

where \( \psi_0 (1) \) is the ground state (fundamental mode) of \( \beta_1 \) in this case.

4.2 Coupled-mode theory for serial temporal waveguides

Consider two serial TWGs \( a \) and \( b \) constructed from two different temporal perturbations \( \beta_{B,a} (t - z/v_B) \) and \( \beta_{B,b} (t - z/v_B) \) of temporal width \( 2T_{B,a} \) and \( 2T_{B,b} \). Furthermore, both TWGs are moving with the same speed \( v_B \) through the longitudinal \( z \)-axis of a given spatial waveguide and are separated \( T_{ab} \) in time and \( v_B T_{ab} \) in space. Figure S4.1 illustrates this scenario.

![Figure S4.1](image)

**Figure S4.1.** Serial temporal waveguides (TWGs) moving with the same speed \( v_B \) through the longitudinal axis of a given spatial waveguide (blue area).

For the sake of simplicity, let us first assume that both TWGs are operating in the single-mode regime. The fundamental mode of each TWG will be denoted with the subscript \( a \) or \( b \). In such a scenario, the complex envelope of the global electric field of this optical structure can be approximated using perturbation theory as \( (\tau := t - z/v_B) \):

\[
A (z, \tau) \simeq \sum_{m=a,b} A_m (z) \psi_m (\tau) \exp \left( \frac{\Delta \beta_i}{\beta_i^2} \tau \right) \exp (iK_m z),
\]

(S4.4)
where $A_m$ is the complex amplitude of each mode. In isolated conditions (i.e., when each TWG is uncoupled from the other one), we find that $dA_m/dz = 0$ and $\psi_m$ must fulfill the temporal Helmholtz equation [Eq. (7) of the paper]:

$$\left[ \frac{d^2}{dz^2} + 2 \frac{K_m}{\beta_2} + \left( \frac{\Delta \beta_1}{\beta_2} \right)^2 - 2 \frac{\beta_{B,m}}{\beta_2} \right] \psi_m (\tau) = 0. \quad (S4.5)$$

Nonetheless, if the two TWGs are in close proximity (serially coupled), the longitudinal dependence of $A_m$, accounts for the power exchange between modes $[\psi_m]$, also fulfills Eq. (S4.5), and the complex envelope $A$ given by Eq. (S4.4) must satisfy the time-domain equation [11]:

$$\left( \frac{\partial}{\partial z} + \Delta \beta_1 \frac{\partial}{\partial \tau} + \frac{1}{2} \beta_2 \frac{\partial^2}{\partial \tau^2} - i \beta_{B,m} (\tau) \right) A (z, \tau) = 0, \quad (S4.6)$$

where $\beta_{B,m} (\tau) = \beta_{B,a} (\tau) + \beta_{B,b} (\tau)$. Note that the above equation plays the same role as the wave equation in optical couplers based on parallel spatial waveguides. Hence, substituting Eq. (S4.4) into Eq. (S4.6) and using Eq. (S4.5), we find after some algebra (we omit the independent variables for simplicity):

$$\sum_{m=a,b} \frac{dA_m}{dz} \psi_m \exp (i K_m z) - i (\beta_{B} - \beta_{B,m}) A_m \psi_m \exp (i K_m z) = 0. \quad (S4.7)$$

From the above equation, we can find the coupled-mode equations describing the power exchange between modes of both serial TWGs. For instance, the coupled-mode equation governing the mode-coupling from mode $b$ to mode $a$ is found: (i) multiplying Eq. (S4.7) by $\psi_a \exp (-i K_a z)$, (ii) integrating in $\tau \in (-\infty, \infty)$, and (iii) writing the first-order derivative of $A_{a(b)}$ at the left-hand side. In this way, we obtain:

$$\frac{dA_a}{dz} = i c_a A_a + \exp (i \Delta K_{b,a} z) \left( i \kappa_{a,b} - \chi_{a,b} \frac{d}{dz} \right) A_b, \quad (S4.8)$$

where $\Delta K_{b,a} := K_b - K_a$. A similar coupled-mode equation describing the coupled power from mode $a$ to mode $b$ can be found by exchanging the subscripts in the above equation. The mode-coupling coefficients (MCCs), accounting for the mode overlapping between $\psi_a$ and $\psi_b$, are defined as:

$$\chi_{a,b} := \frac{1}{N_a} \int_{-\infty}^{\infty} \psi_b (\tau) \psi_a (\tau) d\tau; \quad (S4.9)$$

$$c_a := \frac{1}{N_a} \int_{-\infty}^{\infty} \beta_{B,b} (\tau) \psi_a^2 (\tau) d\tau = \frac{1}{N_a} \int_{(2T_{B,b})} \beta_{B,b} (\tau) \psi_a^2 (\tau) d\tau; \quad (S4.10)$$

$$\kappa_{a,b} := \frac{1}{N_a} \int_{-\infty}^{\infty} \beta_{B,a} (\tau) \psi_b (\tau) \psi_a (\tau) d\tau = \frac{1}{N_a} \int_{(2T_{B,a})} \beta_{B,a} (\tau) \psi_b (\tau) \psi_a (\tau) d\tau, \quad (S4.11)$$

with $N_a := \int_{-\infty}^{\infty} \psi_a^2 (\tau) d\tau$.

It is worthy to note the complete analogy between the CMT of parallel spatial waveguides (see e.g. Eq. (4.11) in [12] or Eq. (5) in [13]) and the CMT of serial TWGs [Eq. (S4.5)]. In these references, it is demonstrated that $\chi_{a,b}$ can be neglected. However, in serially-coupled TWGs, the MCC $\chi_{a,b}$ is generally higher than the MCCs $c_a$ and $\kappa_{a,b}$ and, therefore, all the MCCs should be retained to guarantee a complete description of the mode-coupling phenomenon. Along this line, note that Eq. (S4.8) can be rewritten as:

$$\frac{dA_a (z)}{dz} = i c_a^{(eq)} A_a (z) + i \kappa_{a,b}^{(eq)} \exp (i \Delta K_{b,a} z) A_b (z), \quad (S4.12)$$

where $c_a^{(eq)} := (c_a - \chi_{a,b} \kappa_{b,a})/(1 - \chi_{a,b} \kappa_{b,a})$ and $\kappa_{a,b}^{(eq)} := (\kappa_{a,b} - \chi_{a,b} c_a)/(1 - \chi_{a,b} \kappa_{b,a})$. A similar equation for $dA_b (z)/dz$ can be obtained by exchanging the subscripts $a$ and $b$ in Eq. (S4.12).
Finally, for completeness, the following considerations are in order:

- If we solve the CMT assuming that only the mode $a$ is excited at $z = 0$ [$\mathcal{A}_a (0) = 0$], we find that:

$$
\mathcal{A}_b (z) = \frac{iK_{b,a}^{(eq)}}{\eta} \exp \left( -i \frac{\Delta K_{b,a} }{2} z \right) \sin (\eta z) \mathcal{A}_a (0), \quad (S4.13)
$$

where:

$$
\eta = \sqrt{\kappa_{a,b}^{(eq)} \kappa_{b,a}^{(eq)} + \left( \frac{\Delta K_{b,a} + \psi_{\text{eq}} (b) - \psi_{\text{eq}} (a)}{4} \right)^2}. \quad (S4.14)
$$

Bearing in mind that $\psi_0^{(2)}$ and $\psi_1^{(1)}$ are degenerate modes in the TPL shown in Fig.4(c) of the paper ($\Delta K_{b,a} = 0$), we conclude from Eqs. (S4.13) and (S4.14) that they must exchange their optical power periodically along the $z$-axis with a coupling length$^{10}$ $L_C = \pi/2\eta$ and a coupling efficiency $|\mathcal{A}_b (z = L_C ) / \mathcal{A}_a (0)|^2 = \left| \kappa_{b,a}^{(eq)}/\eta \right|^2$.

- In the multi-mode regime, the modes $\{a_n\}_{n=1}^N$ of TWG $a$ exchange optical power with the modes $\{b_n\}_{n=1}^N$ of TWG $b$. In order to describe this situation, Eq. (S4.4) must be restated as:

$$
A (z, \tau) \simeq \sum_{m=a,b} \sum_{n=1}^N \mathcal{A}_{mn} (z) \psi_{mn} (\tau) \exp \left( i \frac{\Delta \beta_1}{\beta_2} \tau \right) \exp (iK_{mn} z), \quad (S4.15)
$$

and, consequently, the mode-coupling from the modes $\{b_n\}_{n=1}^N$ to mode $a_i$ is governed by the coupled-mode equation:

$$
\frac{d\mathcal{A}_{ai} (z)}{dz} = i\zeta_{ai} \mathcal{A}_{ai} (z) + \sum_{n=1}^N \exp (i\Delta K_{bn,ai} z) \left( i\kappa_{ai,bn} - \chi_{ai,bn} \frac{d}{dz} \right) \mathcal{A}_{bn} (z). \quad (S4.16)
$$

We cannot observe internal mode-coupling among the modes of a given TWG $m$ if we assume that its temporal perturbation profile $\beta_{B,m} (\tau)$ is invariant during the propagation of the TWG along the longitudinal axis of the spatial waveguide.$^{11}$ Nevertheless, in practice, the temporal profile $\beta_{B,m}$ may experience dispersion along the $z$-axis. The precise longitudinal evolution of $\beta_{B,m}$ depends on the exact physical mechanism used to generate the temporal perturbation in the spatial waveguide. For example, using the XPM with a pump-probe set-up, the shape of the pump pulse will generally be affected by dispersion during propagation $^{11}$. In our case, we could overcome this drawback by selecting the pump wavelength at the zero-dispersion wavelength of the spatial waveguide, provided that the higher-order dispersion terms are negligible. This scenario requires to use, e.g., microstructured optical fibres to tailor the material dispersion properties of the spatial waveguide.

- In serial TWGs with different speed or different propagation directions, the MCCs are found to be space-dependent. Nevertheless, this scenario is out of the scope of this work.

$^{10}$The coupling length $L_C$ is the length that maximizes the sinusoidal term of Eq. (S4.13).

$^{11}$An internal mode-coupling among the modes of a TWG requires a temporal perturbation with a varying shape along the $z$-axis. This can be modelled by a profile of the form $\beta_{B,m} (\tau; z)$. The $\tau$ variable describes the ideal temporal profile of the TWG, and the $z$ variable accounts for the fluctuation of its shape during the propagation of the TWG along the longitudinal axis of the spatial waveguide. This is analogous to an optical fibre with a refractive index profile $n(\tau; z)$ which fluctuates along the $z$-axis due to manufacturing imperfections, which induce mode-coupling between different fibre modes.
5 Temporal waveguides: numerical analysis

| Analogy                | Spatial slab (TE modes)                                      | Temporal waveguide                                      |
|------------------------|-------------------------------------------------------------|--------------------------------------------------------|
| Helmholtz equation     | \[ \frac{d^2}{dx^2} + \left( \frac{-\beta_n^2}{a^2} \right) \psi_n(x) = 0 \] | \[ \frac{d^2}{dt^2} + \frac{2}{\beta_n^2} K_n + \left( \frac{\Delta\beta_1}{\beta_n} \right)^2 - \frac{2}{\beta_2^2} \psi_n(t) = 0 \] |
| Waveguide profile      | \[ n(x) = \begin{cases} n_{cl} & |x| > a \\ n_{co} & |x| \leq a \end{cases} \] | \[ \beta_n(t) = \begin{cases} \beta_{cl} & |t| > T_B \\ \beta_{co} & |t| \leq T_B \end{cases} ; \sqrt{2}\beta_2(\beta_{cl} - \beta_{co}) > |\Delta\beta_1| \]
| Eigenmodes             | \[ \psi_n(x) = \begin{cases} B \exp \left( -\frac{w_n}{a} (|x| - a) \right) & |x| > a \\ A \cos \left( \frac{\mu_n}{a} x - \frac{n\pi}{2} \right) & |x| \leq a \end{cases} \] | \[ \psi_n(t) = \begin{cases} B \exp \left( -\frac{w_n}{T_B^2} (|t| - T_B) \right) & |t| > T_B \\ A \cos \left( \frac{\mu_n}{T_B^2} t - \frac{n\pi}{2} \right) & |t| \leq T_B \end{cases} \]
| Eigenvalue equation    | \[ \nu \sqrt{1 - b_n} = \frac{n\pi}{2} + \arctan \left( \frac{b_n}{1 - b_n} \right) ; \] \[ n = 0, 1, 2, ... \] | \[ \nu \sqrt{1 - b_n} = \frac{n\pi}{2} + \arctan \left( \frac{b_n}{1 - b_n} \right) ; \] \[ n = 0, 1, 2, ... \]
| Dispersion diagram     | \[ b_n = f(\nu) \] | \[ b_n = f(\nu) \] |

Table S1. Analogy between a step-index dielectric slab waveguide [12] and a step-index temporal waveguide [11]. The constants \( A \) and \( B \) of the eigenmodes \( \psi_n \) fulfil the relation \( B = A \cos (u_n - n\pi/2) \text{sign} (\tau)^{\nu} \), which arises from the continuity boundary condition of the eigenmodes.

Table S1 summarizes the analogy between a step-index spatial dielectric slab waveguide and a step-index TWG, reported in [11]. Concretely, the analogy only applies to the transversal electric (TE) modes of the slab. As seen, the spatial and temporal Helmholtz equations are analogous with an effective potential \( V \) and eigenvalue \( \Omega_n \) correspondence of the form indicated in Table S1. In addition, note that the modal analysis of both structures involves the same eigenvalue equation, where \( b_n \) and \( \nu \) are respectively the normalized phase constant and normalized frequency, given by the expressions:

\[
\nu^2 = u^2 + w^2 = \frac{\omega^2}{c^2} a^2 \left( n_{co}^2 - n_{cl}^2 \right) \equiv \frac{2}{\beta_2^2} T_B^2 (\beta_{cl} - \beta_{co}) ; \quad (S5.1)
\]
\[
u_n = \nu \sqrt{1 - b_n} , \quad w_n = \nu \sqrt{b_n} ; \quad (S5.2)
\]
\[
b_n = \frac{\beta_n^2 / (\omega^2 / c^2) - n_{cl}^2}{n_{co}^2 - n_{cl}^2} = \frac{\beta_{cl} - K_n - (\beta_2 / 2) (\Delta\beta_1 / \beta_2)^2}{\beta_{cl} - \beta_{co}} . \quad (S5.3)
\]

In the gradual-index case, the analogy is only preserved if the slowly-varying condition of \( n(x) \) is satisfied.\[12\] Otherwise, the TE solutions of the spatial slab, calculated from Maxwell’s equations, do not obey the spatial Helmholtz equation depicted in Table S1 and, therefore, the analogy is broken.

Hence, in step- and gradual-index TWGs, the eigenfunctions \( \psi_n \) and eigenvalues \( \Omega_n \) of the temporal Helmholtz equation can be calculated from the spatial Helmholtz equation of Table S1, provided that the slowly-varying condition of the analogous \( n(x) \) profile is satisfied (which is the case in all TWGs analysed in this work). It is important to note that the analogy should be established by selecting values of \( \omega \) and \( a \) that guarantee the same normalized frequency in the spatial slab and in the TWG [Equation (S5.1)]. In this work, we have solved the spatial Helmholtz equation of the analogous slab with CST Microwave Studio. This software allows us to calculate numerically the TE modes of interest directly from Maxwell’s equations, which coincide with those obtained from the spatial Helmholtz equation for slowly-varying index profiles \( n(x) \), as mentioned above. Once we calculated the normalized dispersion diagram \( b_n = f(\nu) \) of the TE modes with CST, we verified in MATLAB that the corresponding eigenfunctions and eigenvalues fulfil the temporal Helmholtz equation of the TWG under analysis.

\[12\] The slowly-varying condition of \( n(x) \) requires that \( |\delta_x n| \ll |n(x)| \) in \( \delta x \sim \lambda_0 / \pi \), where \( \delta_x n := n(x + \delta x) - n(x) \), \( \lambda_0 \) is the wavelength in vacuum and \( \pi \) is the average value of \( n(x) \) in \( \delta x = 2a \).
Temporal bound states of SUSY TWGs

To complete the information provided by Fig. 4(b) of the paper, Fig. S5.1 shows the spatio-temporal profile of the temporal bound states $\psi_{n}^{(1,2)}$ supported by both TWGs. As seen in Fig. 4(b) and Fig. S5.1, $\psi_{0}^{(1)}$ has no SUSY counterpart in the eigenvalue spectrum of $\beta_{B2}$, i.e., $\psi_{0}^{(1)}$ is not phase-matched with any temporal bound state $\psi_{n}^{(2)}$. Contrariwise, the temporal bound states $\psi_{n+1}^{(1)}$ are perfectly phase-matched with the temporal bound states $\psi_{n}^{(2)}$.

\[
\psi_{n}^{(1)} \Omega_{n}^{(1)} = 14.3 \cdot 10^{-5}
\]

\[
\psi_{n}^{(2)} \Omega_{n}^{(2)} = 14.3 \cdot 10^{-5}
\]

\[
\psi_{0}^{(1)} \Omega_{0}^{(1)} = 6.7 \cdot 10^{-5}
\]

\[
\psi_{0}^{(2)} \Omega_{0}^{(2)} = 6.7 \cdot 10^{-5}
\]

\[
\psi_{0}^{(1)} \Omega_{0}^{(1)} = 1.7 \cdot 10^{-5}
\]

\[
\psi_{0}^{(2)} \Omega_{0}^{(2)} = 1.7 \cdot 10^{-5}
\]

Figure S5.1. Spatio-temporal profile $\psi_{n}^{(1,2)}(t - z/v_B)$ and corresponding eigenvalue $\Omega_{n}^{(1,2)} = 2k_{n}^{(1,2)}/\beta_{2} + \Delta\beta_{1}^{2}/\beta_{2}^{2}$ in Eq. (7) of the paper. The temporal and spatial axes are normalized as $t/T_B$ and $z/L$, where $L$ is the length of the spatial waveguide over which the TWGs propagate. Note that $\nu_B$ and $L$ are arbitrary parameters in the numerical simulation. (Colorbar: normalized amplitude).

Note on the numerical analysis of the temporal photonic lantern using the CMT

The numerical simulation shown in Fig. 4(d) has been performed in MATLAB by using the CMT derived in Subsection 4.2. Aimed to facilitate the comprehension of the results, we have taken $L = L_C$ keeping in mind that $L$ (the length of the spatial waveguide over which the TPL propagates) is an arbitrary parameter. Additionally, the parameter $L_C$ and the coupling efficiency of the mode conversion can be calculated as indicated on page 38. In particular, the coupling efficiency is 0.96.
The temporal acoustic Helmholtz equation

In this section, a temporal Helmholtz equation formally equal to Eq. (3) of the main text is derived for acoustic systems characterized by space- and time-varying properties. Particularly, in the case of pressure acoustics, the spatiotemporal evolution of the acoustic pressure \( p(\mathbf{r},t) \) is governed by the following wave equation [14]:

\[
- \frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} + B(\mathbf{r},t) \nabla \cdot (\rho^{-1}(\mathbf{r},t) \nabla p(\mathbf{r},t)) = 0,
\]

(S6.1)

where the medium bulk modulus \( B(\mathbf{r},t) \) and mass density \( \rho(\mathbf{r},t) \) are, in general, functions of space and time. Taking a medium for which \( \rho \) depends only on time (a similar result could be obtained for a medium with a slowly-varying spatial dependence of \( \rho \)), it is possible to rewrite Eq. (S6.1) as:

\[
- \frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} + \frac{B(\mathbf{r},t)}{\rho(t)} \Delta p(\mathbf{r},t) = 0.
\]

(S6.2)

Furthermore, if the bulk modulus can be expressed as \( B(\mathbf{r},t) = B_S(\mathbf{r})B_T(t) \), and applying separation of variables to the pressure as \( p(\mathbf{r},t) = p_S(\mathbf{r})p_T(t) \), the previous equation can be recast as:

\[
\frac{\rho(t)}{B_T(t)p_T(t)} \frac{\partial p_T(t)}{\partial t} = B_S(\mathbf{r}) \frac{\Delta p_S(\mathbf{r})}{p_S(\mathbf{r})}.
\]

(S6.3)

Once again, this is satisfied if and only if both sides of the equation are equal to a constant. In analogy with the electromagnetic case, defining \( n_T^2(t) := \frac{\rho(t)}{B_T(t)} \) and assuming that \( n^- := n_T(t \to -\infty) \) is also a constant, the following temporal Helmholtz equation is readily obtained:

\[
\left( \frac{d^2}{dt^2} + \omega_0^2 \frac{n_T^2}{n_T^2(t)} \right) p_T(t) = 0,
\]

(S6.4)

that is, Eq. (3) of the main text.
References

[1] Cooper, F., Khare, A. & Sukhatme, U. Supersymmetry and quantum mechanics. Phys. Rep. 251, 267 (1995).

[2] Midya, B. Supersymmetry-generated one-way-invisible PT-symmetric optical crystals. Phys. Rev. A 89, 032116 (2014).

[3] Farkas, B. & Wegner, S.-A. Variations on Barbalat’s lemma. Preprint at arxiv.org/abs/1411.1611 (2014).

[4] Esutia, A. R. High-efficient electrodes for novel optoelectronic devices in silicon photonics, (Ph.D. Dissertation, 2018).

[5] Mercante, A. J. et al. Thin film lithium niobate electro-optic modulator with terahertz operating bandwidth. Opt. Express 26, 14810-14816 (2018).

[6] Vázquez, J. M., Mazilu, M., Miller, A. & Galbraith, I. Wavelet transforms for optical pulse analysis. J. Opt. Soc. Am. A 22, 2890-2899 (2005).

[7] Dantus, M. & Lozovoy, V. V. Experimental coherent laser control of physicochemical processes. Chem. Rev. 104, 1813-1860 (2004).

[8] Weiner, A. Ultrafast optical pulse shaping: a tutorial review. Optics Communications 284, 3669-3692 (2011).

[9] Gadella, M., Kurub, Ş. & Negroa, J. The hyperbolic step potential: anti-bound states, SUSY partners and Wigner time delays. Annals of Physics 379, 86-101 (2017).

[10] Mendonça, J. T. & Shukla, P. K. Time refraction and time reflection: two basic concepts. Physica Scripta 65, 160-163 (2002).

[11] Plansinis, B. W., Donaldson, W. R. & Agrawal, G. P. Temporal waveguides for optical pulses. J. Opt. Soc. Am. B 33, 1112-1119 (2016).

[12] Okamoto, K. Fundamentals of Optical Waveguides. 2nd ed. (Elsevier, Burlington, 2006).

[13] Macho, A., Morant, M. & Llorente, R. Unified model of linear and nonlinear crosstalk in multi-core fiber. J. Lightwave Technol. 34, 3035-3046 (2016).

[14] Landau, L. D. & Lifshitz, L. M. Fluids Mechanics: Course of Theoretical Physics. (Pergamon Press, Oxford, 1959).