The Effect of Anisotropy on Vortex Lattice Structure
and Flux Flow in d-Wave Superconductors

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Abstract

We describe effects of anisotropy caused by the crystal lattice in d-wave superconductors using effective free energy approach in which only one order parameter, the d-wave order parameter field, is used. All the effects of rotational symmetry breaking, including that of the s-wave mixing, can be parametrized by a single four derivative term. We find solutions for single vortex and the vortex lattice. Extending the formalism to include the time dependence, effects of anisotropy on moving vortex structure are calculated. Both direct and Hall I-V curves as functions of the angle between the current and the crystal lattice orientation are obtained.

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It is widely believed that the superconductivity in layered high $T_c$ cuprates is largely due to the $d_{(x^2−y^2)}$ pairing [1]. It has been observed that this influences the vortex lattice structure [3,4]. There are also indications that even though the major bulk pairing mechanism is of $d$ wave nature, there is a small admixture of the $s$-wave pairs in the condensate. Ren et al [5], using a phenomenological microscopic model (in the weak-coupling limit), and Soininen et al [6], considering attractive nearest neighbors interaction, proposed an effective Ginzburg-Landau (GL) type theory using two fields. Two order parameters, $s$ and $d$, describe the gap functions in corresponding channels. The free energy was constructed to include the fourfold symmetry $D_{4h}$ [7]:

$$f = \alpha_s |s|^2 - \alpha_d |d|^2 + \beta_1 |s|^4 + \beta_2 |d|^4 + \beta_3 |s|^2 |d|^2 + \beta_4 (s^* d^2 + d^* s^2$$

$$+ \gamma_s |\Pi s|^2 + \gamma_d |\Pi d|^2 + \gamma_v [s^* (\Pi^2_y - \Pi^2_x) d + c.c.]$$  \hspace{1cm} (1)

where $\Pi = -i \nabla - e^* A$ (with $c = \hbar = 1$). Using equations following from this free energy or using more fundamental equations [8], one obtains a characteristic four-lobe structure for the $s$-wave inside a single vortex [7,11]. The distribution of the magnetic field was also obtained recently [9,7]. The $s$-wave vanishes outside the core, while the $d$-wave distribution becomes rotationally invariant, indistinguishable from the usual Abrikosov solution. Since the fourfold vortex core structure is seemingly in conflict with the symmetry of the triangular lattice, the asymmetry of vortices therefore can, in principle, distort the usual triangular vortex lattice. If one were to look for differences in the behavior of vortices between this case and that of the conventional $s$-wave superconductors, one would like to be closer to $H_{c2}$, so that the influence of the core will be more significant. Another phenomenon in which the core plays a major role is the dissipation in the course of the flux flow.

In this paper we study the above two phenomena using time independent and time dependent effective GL equations within a simplified one component framework. The simplicity of the formulation allows us to clarify, essentially without loss of generality and the use of numerical methods, various delicate questions about single vortex and the vortex lattice. The degrees of freedom we include in the analysis contain: (1) an arbitrary rotation angle $\varphi$
between the crystal lattice and the vortex lattice and (2) all the possible lattices, not only the rectangular ones considered before \[6,7,11\]. This is the first time that the lattice is demonstrated to be centered rectangular (CR) with the most general lattice included in the analysis. Moreover the treatment can be extended to moving flux lattices, which, as is well known \[12,13\] are more demanding, as far as calculational complexity is concerned.

Within the two-field formulation, Soininen et al \[7\] observed that, in a predominantly d-wave superconductor, the s-wave component is generally very small: it is "induced" by the variations of the larger d component. In the bulk, only the d field acquires a nonzero value, while, near the core, the rotationally noninvariant gradient term \(s^*(\Pi_y^2 - \Pi_x^2) d^*\) "communicates" the deviations from the condensate value of d to that of s. Even near the core, where d is small, it is still larger then s by a factor of 20. Since the s field is "induced" by d, it varies on the scale of the d-wave coherence length \(\xi_d = \sqrt{\gamma_d/\alpha_d}\). Then \(\gamma_s\Pi^2 s \sim (\gamma_s/\xi_d^2) s\) is small compared to \(\alpha_s s\) if \((\gamma_s/\gamma_d)(\alpha_d/\alpha_s) \ll 1\) (typically \(\gamma_s/\gamma_d \sim 1\)). This is an excellent approximation in both near and far regions from the core \[7,10,9\]. Therefore, the field s is, to the first order in \(1/\alpha_s\),

\[
s = -\gamma^2_v/\xi^2_d (\Pi_y^2 - \Pi_x^2) d.
\]

Substituting this back to GL equations, we obtain, to first order in \(1/\alpha_s\), the effective free energy

\[
f_{eff} = \frac{1}{2m_d} |\Pi d|^2 - \alpha_d |d|^2 + \beta |d|^4 - \eta d^* \left( \Pi_y^2 - \Pi_x^2 \right)^2 d.
\]

Here we have replaced \(\gamma_d\) by a more conventional notation \(1/2m_d\) and defined \(\eta \equiv \gamma^2_v/\alpha_s\). The last term should be treated as a perturbation.

Note that the two-field equations are highly nontrivial even at the linearized level, so that authors of Ref. \[4\] resort to the variational estimate in order to find a solution. The linearized one-field equation can be solved perturbatively in \(\eta\). The advantage of this equation, especially as far as relation to experiments is concerned, is that the number of coefficients is much smaller: instead of 10 parameters in the two-field free energy, there is just 3. One can further motivate the use of the effective free energy Eq.(2) even with no connection to the two-field formalism. Generally superconductivity is a phenomenon of "spontaneous gauge symmetry breaking" phenomenon irrespective of the mechanism of pairing or channels in
which it occurs. The symmetry should be represented by a single order parameter: the superconducting phase. While in pure s-wave or d-wave superconductors the phase is simply identified with the phase of the gap function, in more complicated microscopic theories with few channels opened, the superconducting phase is just the common phase of various gap functions. Therefore, quantities other than the phase which enter various phenomenological GL type equations, although useful, are not directly related to the spontaneous gauge symmetry breaking.

The general form of the effective free energy can be obtained just by the dimensional analysis and symmetry. It is well known that in a $D_4h$ symmetric field theory, at the level of the dimension three (relevant) terms, full rotational symmetry is restored. Therefore, one has to consider "irrelevant" dimension five terms in order to break the rotational symmetry down to $D_{4h}$. $d^*(\Pi_y^2 - \Pi_x^2) d$ is the only such term which does not respect rotational symmetry and therefore is important for studying anisotropy effects. Consequently only one coefficient $\eta$ is needed to describe the anisotropy effects. The origin of $\eta$ might, in principle, come from sources other than the d-s mixing. The one field formulation also avoids the problem of the artificial second phase transition at $T_s$ assuming $\alpha_s = \alpha'(T_s - T)$ in the two-field formulation.

**Single vortex solution near $H_{c1}$**. In strongly type II materials, we can safely ignore the magnetic field. The GL equation can be solved perturbatively as follows: Write $d = d_0 + \lambda d_1$, where $d_0 = f_0(r)e^{i\phi}$ is the solution of the standard unperturbed GL equation and $\lambda \equiv 4\eta m_d^2 \alpha_d$ is a dimensionless small parameter. The angular dependence of $d_1$ is easily found to contain only three harmonics $e^{-3i\phi}$, $e^{+i\phi}$ and $e^{5i\phi}$

\[
d_1(r, \phi) = f_{-3}(r)e^{-3i\phi} + f_1(r)e^{i\phi} + f_5(r)e^{5i\phi}.
\]

This is consistent with the fourfold symmetry which is built into the theory. The analytic expression for $f_0$ does not exist, but there are a number of known approximations. Using one of them, $f_0(r) = r/\sqrt{r^2 + \xi_0^2}$, the set of linear equations for $f_1, f_2, f_3$ are then solved numerically (the $f_3$ equation decouples from the other two) [15]. The d-wave configuration is nearly indistinguishable from that of the two-field formalism as expected, while the s-wave
configuration calculated with \( s = -\frac{\gamma_v}{\alpha_s} (\Pi_y^2 - \Pi_z^2) d \) has the four-lobe structure with four zeroes on the axes (same as in [7] but different from [10]).

**Vortex lattice.** In the opposite limit near \( H_{c2} \) one first neglects the nonlinear terms in the GL equation and finds the lowest energy solutions \( \Psi_{kn}(x,y) \) of the linearized equation. The lattice solution is constructed from the linear superposition of \( \Psi_{kn}(x,y) \) in such a way that it is invariant under the corresponding symmetry group of the lattice: \( d(x,y) = \sum_n C_n \Psi_{kn}(x,y) \). A general lattice in 2D can be specified by three parameters \( a, b \) and \( \alpha \), where \( a \) and \( b \) are the two lattice constants, while \( \alpha \) is the angle between the two primitive lattice vectors as defined in Ref. [14]. Flux quantization provides a constraint: \( H_a b \sin \alpha = \Phi_0 \). In the d-wave superconductors the rotational symmetry is broken, therefore the relative orientation of the vortex lattice to the underlying lattice becomes important. We denote \( \varphi \) to be the angle between \( \vec{a} \) and one of the axes of the underlying lattice. The most general lattice is built into the solution via appropriate relations among \( C_n \) [14]. The lattice structure is then found by minimizing the free energy or equivalently, the Abrikosov’s parameter \( \beta_A \equiv \langle |d|^4 \rangle / \langle |d|^2 \rangle^2 \). Here \( \langle ... \rangle \) is the average over space.

We solve the linearized GL equation perturbatively in the anisotropy parameter \( \eta \) in the Landau gauge, \( A = (0, H x) \) (see details in [13]):

\[
\psi(x,y) = \left( \frac{1}{\pi l_H^2} \right)^{1/4} \exp \left( i y k_n \right) \left[ 1 + \eta m_d e^* H \frac{e^{4i\varphi}}{16} H_4 \left( \frac{x}{l_H} - k_n^2 H \right) \right] \exp \left[ -\frac{1}{2} \left( \frac{x}{l_H} - k_n^2 H \right)^2 \right], \tag{4}
\]

where \( H_4 \) is the fourth Hermit polynomial and \( l_H \equiv \sqrt{1/e^* H} \). The upper critical field is:

\[
H_{c2}(T) = \frac{2m_d \alpha'}{e^*} \left[ (T_c - T) + 8\eta m_d^2 \alpha' (T_c - T)^2 \right]. \tag{6}
\]

Note that \( H_{c2}(T) \) bends upwards around \( T_c \), in agreement with the two-field results [7]. This effect has been reported in some experiments. However one should be cautioned against taking this too seriously. As we have discussed, there is another rotationally invariant term of the same dimensionality \( \tau d^* (\Pi^2)^2 d \) which gives contribution similar to Eq.(6):
\[ \Delta H_{c2}(T) = -\tau \frac{2ma'}{e} (T_c - T)^2. \] This means that for large enough positive \( \tau \) the sign of the \( H_{c2}(T) \) curvature changes.

To present \( \beta_A \) it is convenient to use variables: \( \rho + i\sigma \equiv \zeta \equiv \frac{b}{a} \exp(i\alpha) \) [14]. After some rather nontrivial calculations we obtain, \( \beta_A = \beta^0_A + \eta' \beta^1_A \) with

\[
\beta^0_A = \sqrt{\sigma} \left\{ \left| \sum_{n=-\infty}^{\infty} w_n(\zeta) \right|^2 + \left| \sum_{n=-\infty}^{\infty} w_{n+1/2}(\zeta) \right|^2 \right\},
\]

\[
\beta^1_A = \frac{1}{4}\sqrt{\sigma} \text{Re} \left\{ e^{4i\varphi} \left[ \sum_{n'} w_{n'}^*(\zeta) \left[ \sum_n w_n(\zeta) G_n(\sigma) \right] + \left( n \rightarrow n + \frac{1}{2}, n' \rightarrow n' + \frac{1}{2} \right) \right] \right\}
\]

where \( \eta' \equiv \eta m_a e^* H \) is the dimensionless parameter appropriate for this situation, \( w_n(\zeta) = \exp(2\pi i\zeta n^2) \) and \( G_n(\sigma) \equiv (64\pi^2\sigma^2 n^4 - 48\pi\sigma n^2 + 3) \). Having calculated the Abrikosov parameter \( \beta_A \), one finds the vortex structure by minimizing it with respect to \( \varphi, \rho, \) and \( \sigma \). The minimization with respect to the angle \( \varphi \) is easily done analytically, while further minimization of \( \beta^0_A(\rho, \sigma) \) is done numerically. The angle \( \alpha \) as function of \( \eta' \) is given on Fig.1. The corresponding \( \varphi \) is zero. For \( \alpha = 74^\circ \) one gets \( \eta' \simeq 0.02 \). There is a phase transition from CR to square lattice at \( \eta'_c = 0.0235 \). Note that this calculation, unlike that for the single vortex, is valid for arbitrary \( \kappa \). Using standard methods, one can take into account variations of the magnetic field and calculate corrections to the magnetization curve using the \( \beta_A \) calculated here [14].

**Moving lattice.** The time dependent GL equation describing the time evolution of the order parameter the one-field formulation is

\[
\gamma \left( \frac{\partial}{\partial t} + ie^* \Phi \right) d = -\left( \frac{1}{2m_d} \Pi^2 - \alpha_d \right) d + \eta(\Pi_y^2 - \Pi_x^2) d - 2\beta|d|^2 d,
\]

where \( \Phi \) is the electric potential. It involves just one additional parameter \( \gamma \) compared to the \( 2 \times 2 \) matrix for the two-field formalism. Although, in principle, this parameter, describing various dissipation effects, can have a complex part [12], we will consider only real values. To generalize the above procedure for finding the structure for a moving vortex lattice near \( H_{c2} \), one considers the motion caused by an electric field \( \mathbf{E} \) making an angle \( \Theta \) with the
crystal [1,0,0] axis. The vortex lattice velocity is: \( \mathbf{E} = -\mathbf{v} \times \mathbf{B} \). For a general direction of the electric field the fourfold symmetry of the system is completely (explicitly) broken.

Even for the simple s-wave the problem of finding the moving lattice solution is nontrivial. However in that case there exists the “Galilean boost” trick to solve the linearized problem, which is not applicable to the d-wave. Our approach is to use perturbation theory in \( \eta' \) to find a complete set of solutions of the linearized equations and then impose the periodicity conditions to construct the vortex lattice. It is more convenient to perform the first step in the gauge aligned in the direction of the electric field, in order to make both the scalar and vector potentials independent of \( y \) and \( t \). For the second step however, it is preferable to use a gauge aligned in the direction of the vortex lattice. However in general the vortex lattice will not be periodic along this special direction. To construct this general periodic solution, one has to solve a periodicity constraint equation for the coefficients \( C_k \), where \( k \) is now a continuous index. We then combined the two steps using gauge transformation. The results are as follows. The upper critical field of course depends on velocity

\[
H_{c2}(T) = h_0 + h_1(T_c - T) + h_2(T_c - T)^2
\]

with \( h_0 = (m_d^2/e^*)[-1 + (9 + \cos 4\Theta) \eta m_d^3 \gamma^2 v^2] \gamma^2 v^2 \), \( h_1 = 2\alpha'(m_d/e^*) (1 - 12\eta m_d^3 \gamma^2 v^2) \), \( h_2 = 16\alpha'^2 \eta m_d^3 / e^* \). Note that the curvature \( h_2 \) hasn’t changed compared to the static case, but we have two new effects. First of all, the electric field (or, equivalently, electric current) shifts \( H_{c2} \) by a negative constant (proportional to \( E^2 \)) to a lower value. It simply depends on the angle . This is expected. Secondly, although the curvature \( h_2 \) doesn’t change compared with the static case, the slope \( h_1 \) acquires a negative contribution proportional to \( E^2 \). In the s-wave case the phase boundary was first found and discussed in Ref. [13]. There are a couple of peculiarities associated with it like the existence of a metastable normal state and the unstable superconductive state. The same applies to the present case. As far as we know, these peculiarities haven’t been directly observed in low \( T_c \) materials. It would be interesting to reconsider this question for the high \( T_c \) materials.
After a lengthy and nontrivial calculation [15], the vortex lattice solution is amazingly simple:

\[
\psi(x, y) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{\sqrt{L}} \left( \frac{H}{\pi} \right)^{1/4} \exp \left( ik_n y \right) \exp \left[ -\frac{1}{2l_H^2} \left( x - k_n l_H^2 \right)^2 \right] \times \\
\left[ 1 + \eta' \sum_{m=1}^{4} c_m e^{im(\Theta + \varphi)} \sqrt{2m!} \right] 
\]

with

\[
c_1 = -\sqrt{2}ig \left[ (1 + e^{-4i\Theta}) g^2 - 2 \right], \quad c_2 = -\frac{\sqrt{2}}{2} \left( 1 + 3e^{-4i\Theta} \right) g, \quad c_3 = \frac{4\sqrt{3}}{3}ige^{-4i\Theta}, \quad c_4 = \frac{\sqrt{6}}{2}e^{-4i\Theta} \quad \text{and} \quad g \equiv \gamma v. 
\]

The standard Abrikosov's procedure to develop an approximation for small order parameter around \( H_{c2} \) can be applied also in the flux flow case (see [13]). Using this expression the correction term in the expansion of the Abrikosov parameter \( \beta_A \) changes and now the function \( G_n(\sigma) \) in Eq.(7) depends in a very simple way on electric field:

\[
G_n(\sigma) = e^{4i\varphi} \left( 64\pi^2\sigma^2n^4 - 48\pi\sigma n^2 + 3 \right) - 8e^{2i\varphi} g^2 \cos 2\Theta(8\pi\sigma n^2 - 1). 
\]

One immediately observes a surprising fact - the dependence on the angle \( \Theta \) and velocity \( g \) is only via the combination \( g^2 \cos 2\Theta \). For example, the resulting lattice for \( \Theta = \pi/4 \) and arbitrary \( g \) will be the same as without electric field at all! Also apparent complete breaking of the rotational symmetry by general direction of the electric field is not felt by \( \beta_A \). Indeed, the lattice for some arbitrary \( \Theta \) and \( g^2 \) is the same as for \( \Theta = 0 \) and \( g^2' = g^2 \cos 2\Theta \). However, the fourfold symmetry has been reduced. For fixed \( \eta' \) and \( g^2 \cos 2\Theta \), the minimization was performed numerically and we again obtain only \( CR \) lattices. The angle \( \alpha \) turns out only weakly depend on the combination \( g^2 \cos 2\Theta \).

This is the first time that the lattice is demonstrated to be \( CR \) type using the most general lattice in the analysis. It turns out that general oblique lattices have higher energy than the \( CR \) ones despite the fact that rotational symmetry is completely broken by both the electric field and by the underlying atomic lattice. Intuitively in the symmetric case this fact is understandable: \( CR \) lattices are more symmetric, however for rotationally nonsymmetric or moving lattices this is no longer so.

\( I - V \) curves for the flux flow. Now we consider the dissipation in vortex cores due to flux flow. As it is well known, the fourfold symmetry forces the conductivity tensor \( \sigma_{ij} \) defined
by \( J_i = \sigma_{ij} E_j \), to be rotationally symmetric. Namely, \( \sigma_{ij} = \sigma_{ij}^H + \sigma_{ij}^H \varepsilon_{ij} \). Here \( \sigma \) is the usual (Ohmic) conductivity, \( \sigma^H \) is the Hall conductivity and \( \varepsilon_{ij} \) is the antisymmetric tensor. The additional term in the free energy corrects the values of \( \sigma \) and \( \sigma^H \), but the correction is of the order \( \eta \) and therefore small. So, to see anisotropy, we definitely would like to go beyond linear response. This has been done for simple s-wave TDGL [3] near \( H_{c2} \). We will neglect pinning and consider motion of a very large bundle. While there is a normal component of the conductivity, here we will concentrate on the contribution of the supercurrent only. For the discussion of the relative contribution of the two see [3].

We therefore calculate the current as a function of \( E \) beyond linear response. Note that in this case the condensate has to be properly normalized: \( \langle d^* d \rangle = \alpha_d / (2\beta_\lambda \Lambda) \). Using the normalized \( d \), we found the anisotropic direct and Hall currents

\[
\Delta J_{dir} = \eta \frac{2m_d \gamma^3 E^3}{\beta^3 e^* H^4} (1 + \cos 4\Theta), \tag{11}
\]

\[
\Delta J_{Hall} = -\eta \frac{2m_d \gamma^3 E^3}{\beta^3 e^* H^4} \sin 4\Theta. \tag{12}
\]

Note that both direct and Hall currents depend only on the fourth harmonics of the angle between \( E \) and the crystal lattice orientation. Only the cubic power of \( E \) contributes, all the higher orders terms are cancelled.

To summarize, we studied the static and dynamical anisotropy effects in d-wave superconductors using the effective one component model in which these effects are parametrized by a single parameter \( \eta \). For static lattice we clarified several issues and found the critical value of \( \eta' c = .0235 \) at which thermodynamic transition to the square lattice occurs. (The sample of [3] is found to be in CR phase while that of [2] is in square phase). The results of our study of the moving lattices for the direct current, Hall current magnetization and phase boundary are very simple and are given in Eqs.11, 12, 7 and 9 respectively. It would be interesting to determine \( \eta' \) of the same sample from a few of the above effects.

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Figure Caption:

Fig. 1 The angle $\alpha$ of the centered rectangular lattice as a function of the parameter $\eta'$ describing the asymmetry. There is a phase transition from centered rectangular to square lattice at $\eta'_c = 0.0235$. 
Fig. 1