HIDA THEORY FOR SPECIAL ORDERS

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Abstract. This note is devoted to the study of families of quaternionic modular forms arising from orders defined by Pizer. In this situation, the Hecke-eigenspaces are 2-dimensional contrary to the classical case of Eichler orders. The main result is a Control Theorem in the spirit of Hida, interpolating these 2-dimensional Hecke-eigenspaces. We restrict our attention to a definite rational quaternion algebra ramified at a single odd prime \( \ell \).

2010 Mathematics Subject Classification: 11F11, 11R52.
Key words: Quaternion algebras, Hijikata–Pizer–Shemanske orders, Hida families, control theorems.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

Let \( \ell \) be an odd prime, \( N \geq 1 \) an integer prime to \( \ell \) and \( k \geq 2 \) an integer. Let \( B \) be the unique, up to isomorphism, quaternion algebra over \( \mathbb{Q} \) ramified exactly at \( \ell \) and \( \infty \). Take \( R \) to be an Eichler order of level \( N \) in \( B \) and consider the space of \( \mathbb{C} \)-valued cuspidal quaternionic newforms with level \( R \), denoted by \( \mathcal{S}_{new}^k(R, \mathbb{C}) \); we recall its precise definition in Section 2.6 but, roughly speaking, it represents the space of non-Eisenstein quaternionic modular forms which do not satisfy a lower level structure. The Jacquet–Langlands correspondence ensures an injective transfer between automorphic representations of the algebraic group associated with \( B \times \) and \( GL_2/\mathbb{Q} \), but at the level of automorphic forms it takes an explicit realization, often referred to as the Eichler–Jacquet–Langlands correspondence. More precisely, one obtains the Hecke-equivariant isomorphism

\[
\mathcal{S}_{new}^k(R, \mathbb{C}) \cong S_{new}^k(\Gamma_0(N\ell), \mathbb{C}).
\]

On the other hand, in order to study modular forms with higher level structure at \( \ell \), one needs more general orders, namely the special orders defined by Pizer and Hijikata–Pizer–Shemanske. As the precise definition of special orders does not contribute necessarily to the understanding of this introduction, we prefer to avoid technicalities and postpone it to Definition 2.1.1. In the case where \( R \) is such a special order in \( B \), corresponding to level structure \( N\ell^{2r} \) with \( r \geq 1 \), the relation between automorphic forms changes considerably. Before stating the precise relation, in order to be consistent with the notation present in [15], let us introduce the following (unfortunately unconventional but rather useful) piece of notation. Throughout the whole document, for any module \( M \), we write \( 2M \) for the direct sum \( M \oplus M \). The Jacquet–Langlands correspondence, for \( r \geq 2 \), takes the explicit form of the following Hecke-isomorphism

\[
2S_{new}^k(\Gamma_1(N\ell^{2r}), \mathbb{C}) \cong \mathcal{S}_{new}^k(R, \mathbb{C}) \oplus \bigoplus_\chi 2S_{new}^k(\Gamma_1(N\ell^r), \chi^2, \mathbb{C})^{\otimes \chi},
\]
coefficients of $S_2(\Gamma_1(N\ell^{2r}))$, $\mathbb{C}$ must be taken into account and the sum of the twisted spaces $S^{new}_{k}(\Gamma_1(N\ell^{r})), \chi^2, \mathbb{C})^{\otimes r}$ (see beginning of Section 3.3) runs over certain primitive characters modulo $\ell^r$. Equation (1.1) has a slightly more complicated expression for $r = 1$, but the above situation is already explanatory of the general phenomenon; an exhaustive statement can be found in Theorem 3.3.2. For any choice of an isomorphism in Eq. (1.1) we see that any classical newform of level $N\ell^{2r}$, which is twist-minimal at $\ell$ (as in Definition 3.3.4), lifts to two linearly independent quaternionic newforms with the same Hecke-eigenvalues away from the level. The above-mentioned situation has been extensively studied in a series of works by H. Hijikata, A. Pizer, and T. Shemanske, most notably [26] and [15]. In the light of this multiplicity, it is natural to ask whether these quaternionic modular forms still live in $p$-adic families, for $p$ an odd prime different from $\ell$ and prime to $N$. In this note we provide a positive answer to this question.

As introduced above, let $\ell$, $p$ and $N$ be, in the order, two distinct odd primes and a positive integer prime to both $\ell$ and $p$. For the sake of simplicity, we restrict here to the case of trivial character at $\ell$ and exponent $2r \geq 4$; we refer the reader to Theorem 4.2.4 for the general statement. As in Definition 2.4 of [9], we consider $\mathcal{R}$ to be the universal ordinary $p$-adic Hecke algebra of tame level $N\ell^{2r}$. Considering the Iwasawa algebra $\mathbb{Z}_p[\mathbb{Z}_p^\times]$, $\mathcal{R}$ represents the $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-algebra of Hecke-operators acting on Hida families of tame level $N\ell^{2r}$. For any continuous group homomorphism $\kappa : \mathcal{R} \rightarrow \overline{\mathbb{Q}}_p$, we say that $\kappa$ is an arithmetic homomorphism if its restriction to $1 + \mathbb{Z}_p \subseteq \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ defines a character of the form $z \mapsto z^{k-2}\epsilon(z)$, for $k \in \mathbb{Z}_{\geq 2}$ and $\epsilon$ a $p$-adic character of conductor $p^n$, $n \geq 0$. We recall that these homomorphisms are identified with points on the so-called weight space. As usual, we associate to each arithmetic homomorphism $\kappa$ the couple $(k, \epsilon)$. For any $\kappa$ we also denote its kernel by $\mathcal{P}_\kappa$ and the corresponding localization of $\mathcal{R}$ by $\mathcal{R}_{\mathcal{P}_\kappa}$. Let $f$ be a classical modular newform in $S_k(\Gamma_0(N\ell^{2r})), \mathbb{C})$ and assume that $f$ is twist-minimal at $\ell$. Moreover, if $f$ is $p$-ordinary, we can consider the unique Hida family $f_{\infty}$ associated with $f$ by the works of Hida and Wiles. For each arithmetic homomorphism $\kappa$ we denote by $f_{\kappa}$ the specialization of $f_{\infty}$ at $\kappa$. We set $F$ (resp. $F_{\kappa}$) to be the field extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of $f$ (resp. $f_{\kappa}$) and take $\mathcal{O}$ (resp. $\mathcal{O}_\kappa$) to be its ring of integers. Fix now $R$ to be a maximal order in the quaternion algebra $B$ which contains the family of nested orders $\{R_n\}$.

\[(1.2) \quad \cdots \subset R^{n+1} \subset R^n \subset \cdots R^0 \subset R, \quad R^n \text{ is a special order of level } Np^n\ell^{2r}.\]

At all places $q \neq \ell, \infty$, we fix isomorphisms $\iota_q : B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ such that $R^n \otimes \mathbb{Z} \mathbb{Q}_q$ is identified with the upper triangular matrices modulo $Np^n$. For each $n$ we consider the compact open subgroup $U_n \subset \tilde{R}^n := R^n \otimes \tilde{\mathbb{Z}}$ defined as

\[(1.3) \quad U_n = \left\{ g = (g_q) \in \tilde{R}^{n\times} \mid \iota_q(g_q) \equiv (\delta_q 1) \pmod{Np^n\mathbb{Q}_q}, \text{ for } q \mid Np^n \right\}.\]

Let $V_{k-2}(\mathcal{O}_\kappa)$ be the dual of $L_{k-2}(\mathcal{O}_\kappa)$, the space of homogeneous polynomials in $\mathcal{O}_\kappa[X, Y]$ of degree $k - 2$, endowed with the action $|_{u_p}$ of $GL_2(\mathbb{Z}_p)$, induced by the left multiplication \((a/b): (X, Y)^t = (aX + bY, cX + dY)^t\). Denoting $\tilde{B} = B \otimes \mathbb{A}_{\kappa, f}$ for $\mathbb{A}_{\kappa, f}$ the finite adèles of $\mathbb{Q}$, we consider, as in Definition 2.4.3, the space of quaternionic $p$-adic modular forms of weight $k \geq 2$, character $\epsilon$ (of conductor $p^n$) and level $U_n$,

\[(1.4) \quad S_k(U_n, \epsilon, \mathcal{O}_\kappa) := \left\{ \varphi : \tilde{B}^\times \rightarrow V_{k-2}(\mathcal{O}_\kappa) \mid \varphi(bbuz) = \epsilon(z)z^{k-2}\varphi(b)|_{u_p}, \right\}\]

for $b \in B^\times, \tilde{b} \in \tilde{B}^\times, u \in U_n, z \in \mathbb{A}_{\kappa, f}^\times$.

More generally, let $X$ be the subset of primitive vectors in $\mathbb{Z}_p^2$, namely the subset of vectors with at least one component which is not divisible by $p$, and consider the space $\mathcal{M}(X, \mathcal{O})$ of $\mathcal{O}$-valued measures on $X$. We construct, following Definition 4.1.1, the space of measure-valued quaternionic modular forms.
\( S_2(U_0, \mathcal{M}(X, \mathcal{O})) := \left\{ \varphi : \hat{B}^\times \to \mathcal{M}(X, \mathcal{O}) \mid \varphi(b_0uz) = \varphi(b)|_{u_p}, \right\} \)

for \( b \in B^\times \), \( \hat{b} \in \hat{B}^\times \), \( u \in U_0 \), \( z \in \mathbb{A}_Q^2 \).

The action of \( GL_2(\mathbb{Z}_p) \) induced by the left multiplication on the variables. By integration, we induce, for any arithmetic homomorphism \( \kappa = (k, \varepsilon) \), a specialization map

\[
\nu_\kappa : S_2(U_0, \mathcal{M}(X, \mathcal{O})) \to S_k(U_n, \varepsilon, \mathcal{O})
\]

such that

\[
\nu_\kappa(\varphi)(\hat{b})(P) := \int_{\mathbb{Z}_p^\times} \varepsilon^{-1}(y)P(x, y)d(\varphi(\hat{b}))(x, y)
\]

for \( \varphi \) and \( \hat{b} \) as above, and any \( P \in L_{k-2}(\mathcal{O}_f) \); all details can be found in Section 4.1. Considering the ordinary component of \( S_2(U_0, \varepsilon, \mathcal{M}(X, \mathcal{O})) \), which we denote by \( \mathcal{W} \), the specialization maps descend to maps between the ordinary components

\[
\nu_\kappa^{ord} : \mathcal{W} \to S_k(U_n, \varepsilon, \mathcal{O})^{ord}
\]

for \( S_k(U_n, \varepsilon, \mathcal{O})^{ord} \) the subspace of \( p \)-ordinary quaternionic forms in \( S_k(U_n, \varepsilon, \mathcal{O}) \). As the algebra \( \mathcal{R} \) acts on the space of Hida families, we can consider the \( f_\infty \)-isotypic component

\[
(\mathcal{W} \otimes \mathcal{O}_{[1+p\mathbb{Z}_p]} \mathcal{R})[f_\infty],
\]

that is, the component of \( \mathcal{W} \otimes \mathcal{O}_{[1+p\mathbb{Z}_p]} \mathcal{R} \) where the Hecke-operators act with the same \( \mathcal{R} \)-eigenvalues of \( f_\infty \). Up to a mild condition on the level \( N^{2r} \), as explained in Remark 4.1.5, one can assume the \( \mathcal{R} \)-module \( \mathcal{W} \otimes \mathcal{O}_{[1+p\mathbb{Z}_p]} \mathcal{R} \) to be free, as we do here. We can hence state our main result under the above simplifying restrictions for the character and the power of \( \ell \); the general statements are the content of Theorems 4.2.4 and 4.2.5.

**Theorem A** (Control theorem for special orders). With the above notation, suppose that \( f \) is twist-minimal at \( \ell \). For any arithmetic homomorphism \( \kappa : \mathcal{R} \to \mathcal{O}_p \) the map \( \nu_\kappa \) (of Proposition 4.2.1, induced by the specialization map) induces an isomorphism of \( 2 \)-dimensional \( F_{\kappa} \)-vector spaces

\[
(\mathcal{W} \otimes \mathcal{O}_{[1+p\mathbb{Z}_p]} \mathcal{R})[f_\infty] \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\ell}/\mathcal{P}_{\kappa}\mathcal{R}_{\mathcal{P}_\ell} \cong (S_k(U_n, \varepsilon, \mathcal{O})^{ord})[f_\kappa].
\]

The two main ingredients needed for the proof of Theorem A are the above isomorphism (1.1) and the seminal paper [13]. The results proved in [13] for definite quaternion algebras over totally real fields different from \( \mathbb{Q} \) remain true in the case of definite quaternion algebras over \( \mathbb{Q} \), as already noticed in Section 3 of [21] and Section 4 of [17], and in the case of special orders, as remarked in Remarks 4.1.5 and 4.2.3. The strategy of the proof is then a generalization of the work [21], from which we take inspiration.

Theorem A extends the foundational results of Hida theory to the case of quaternionic modular forms with special level structure, allowing to consider quaternionic \( p \)-adic families with tame level \( \ell^{2r} \) over the quaternion algebra \( B \), which, we remark, is ramified at \( \ell \). We highlight again that the situation discussed here differs markedly from the classical case of Eichler orders, where all the local non-archimedean automorphic representations at \( \ell \) are 1-dimensional. In particular, the novelty of this result lies in the rank of the Hecke-eigenspaces being and no more 1 as in the classical Eichler case.

An additional motivation for such a Control Theorem originates from the desire to study points outside the interpolation region of the triple product \( p \)-adic \( L \)-function, as treated in [6]. More precisely, the present study together with [6] are motivated by a conjecture of Bertolini–Seveso–Venerucci and by the wish to provide computational support to it.

The present work leaves open some questions which we briefly discuss in Section 4.3. The main one is whether it is possible to distinguish the 2-dimensional spaces of quaternionic modular forms attached
to special orders, first in the case of a classical eigenspace, and then for families.

Acknowledgments: This note presents the content of a section of the author’s doctoral dissertation [7]; he expresses his gratitude to his supervisor Massimo Bertolini. Many thanks go also to Matteo Longo for several helpful discussions, among the others on [21] and [13]. The author is grateful to the anonymous referee for the valuable comments, suggestions, and questions which led to a significant improvement of the exposition. The author also wishes to thank Matteo Tamiozzo, for numerous mathematical conversations, and Jonas Franzel, for reading an early draft of this work and finding out various typos. The author is grateful to the Universität Duisburg–Essen, where the main part of this work has been carried out, as well as to the Università degli Studi di Padova (Research Grant funded by PRIN 2017 “Geometric, algebraic and analytic methods in arithmetic”) for their financial support.

2. Quaternionic orders and modular forms

We begin by recalling the definitions of the various quaternionic orders as well as the definition of quaternionic modular forms, both $p$-adic and classical, for a definite quaternion algebra over $\mathbb{Q}$. Special orders are a generalization of the classical Eichler orders, which are needed for studying both higher ramification and the presence of a character at primes where the quaternion algebra is ramified. We refer the reader to [15] and [16] for all the details that we are not recalling here.

We fix, once and for all, a choice of field embeddings $\mathbb{Q} \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{C}$. For any prime $q$ we denote $q$-adic valuation by $\nu_q$, normalized so that $\nu_q(q) = 1$.

2.1. Special orders. Let $B$ the unique, up to isomorphism, quaternion algebra over $\mathbb{Q}$ with discriminant $D$. Fix an isomorphism $\iota_q : B_q := B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for each $q \mid D$. We denote the reduced norm of $b \in B$ by $n(b) \in \mathbb{Q}$. Let $q$ be an odd rational prime, and fix $u \in \mathbb{Z}$ to be a quadratic non-residue modulo $q$. The local field $\mathbb{Q}_q$ has a unique quadratic unramified extension $\mathbb{Q}_q(\sqrt{u})$ and two quadratic ramified ones, $\mathbb{Q}_q(\sqrt{-u})$ and $\mathbb{Q}_q(\sqrt{u})$. For $L_q$ one of these quadratic extensions, we denote by $\mathcal{O}_{L_q}$ its ring of integers. Set

$$
M_2^0(\mathbb{Z}_q):= \{ \gamma \in M_2(\mathbb{Z}_q) \mid \gamma \equiv (\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}) \pmod{\mathbb{Z}_q} \},
$$

and, for $r \geq 1$,

$$
M(L_q; r) := \mathcal{O}_{L_q} + \{ x \in B_q \mid n(x) \in q^r \mathbb{Z}_q \}^{r-1} = \mathcal{O}_{L_q} + \{ x \in B_q \mid n(x) \in q^r \mathbb{Z}_q \}.
$$

We notice that, for $r = 1$, $M(L_q, 1)$ is the unique maximal ideal of $B_q$.

Definition 2.1.1 ([16], Def. 6.1). An order $R$ in $B$ is said to be a special order of level $M = N \cdot \prod_{\ell \mid D} \ell^{m_\ell}$ if

(i) $R_q := R \otimes \mathbb{Z}_q$ is conjugate to $M_2^0(\mathbb{Z}_q)$ by an element of $B_q^*$ (via $\iota_q$), for each $q \mid D$;

(ii) there exists a quadratic extension $L_{\ell}$ of $\mathbb{Q}_{\ell}$ such that $R_{\ell}$ is conjugate to $M(L_{\ell}, m_{\ell})$, for each $\ell | D$.

In the following, we choose for each $q \mid D$ the isomorphism $\iota_q$ such that $\iota_q(R_q) = M_2^0(\mathbb{Z}_q)$. If in the above definition we take the level $M$ to be such that $D||M$, we obtain the usual definition of an Eichler order of level $N$ (see [25], page 344).

From now on, we assume $D$ to be odd and we fix a particular choice of special orders and quadratic field extensions. We follow the thorough summary given in Section 2.2 of [19], which is based on a careful analysis of [15] and takes into account more general orders. Let $f$ be any newform in $S_2(\Gamma_0(N \prod_{\ell \mid D} \ell^{e_\ell}), \mathbb{C})$. In order to be able to lift $f$ to a quaternionic modular form, we fix the choice of the special order $R$ such that the quadratic extensions $L_{\ell}$ of $\mathbb{Q}_{\ell}$ and the exponents $m_{\ell}$ are as follows. For any (odd) prime $\ell | D$, if

...
3.3 \[ m(2.3) \]

The choice of character in Assumption \ref{assumption:2.2.1} follows.

Assume that \( \text{Assumption 2.2.1.} \)

\( \text{(1) } \epsilon_\ell \text{ is odd, we take } L_\ell \text{ to be the unramified extension of } \mathbb{Q}_\ell \text{ and } m_\ell = \epsilon_\ell; \)

\( \text{(2) } \epsilon_\ell \text{ is even, we take } L_\ell \text{ to be one of the two ramified extension of } \mathbb{Q}_\ell \text{ and } m_\ell = \epsilon_\ell. \)

The case of even discriminant presents some further difficulties and, as our main case of interest is the case of \( \ell \) odd, we omit it and refer the reader to \cite{15} and \cite{19}.

We recall a part of the notation already used in the Introduction.

\textbf{Notation 2.1.2.} Let \( \hat{\mathbb{Z}} \) be the profinite completion of \( \mathbb{Z} \). We set \( \hat{B} = B \otimes_{\mathbb{Q}} \mathcal{A}_{Q,f} \) and \( \hat{R} = R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \), where \( \mathcal{A}_{Q,f} = \mathbb{Q} \hat{\mathbb{Z}} \) are the finite ad`eles of \( \mathbb{Q} \).

Special orders satisfy properties similar to the Eichler ones.

\textbf{Lemma 2.1.3.} \( \hat{R}^\times \) is a compact open subgroup of \( \hat{B}^\times \). In fact, this is true for each one of its local components.

\textbf{Proof.} This lemma is a classical result whenever \( R \) is an Eichler order (see e.g. Sections 5.1 and 5.2 of \cite{23}). We consider the case of a special order. Lemma 5.1.1 of \cite{23} tells us that \( R^\times \) is compact in \( B^\times \), independently of the order. By definition of special order, it is enough to consider \( M(L_\ell, r) \). Since the reduced norm is continuous, \( \{ x \in B_\ell \mid n(x) \in \ell^{r-1}\mathbb{Z}_\ell \} \) is open and thus, as the sum is a continuous homomorphism, we deduce the claim. \qed

\textbf{Proposition 2.1.4} [\cite{24}, Proposition 2.13]. All special orders have finite class number. Moreover, it depends only on the level and not on the specific choice of the special order.

\textbf{Lemma 2.1.5} [\cite{16}, Lemma 7.4]. Let \( R \) be a special order of level \( N \). Then there exists a set of ideal class representatives \( \{ I_1, \ldots, I_h \} \) for the left \( R \)-ideal classes, such that \( I_i \otimes \mathbb{Z}_q = R_q \) for all \( q \) dividing the level.

\subsection*{2.2. Characters.}

Let \( R \) be a special order of level \( M = N \cdot \prod \ell \mid D \ell^{m_\ell} \) and let \( \chi \) be a Dirichlet character with conductor \( C \).

\textbf{Assumption 2.2.1.} Assume that \( \nu_q(N) \geq \nu_q(C) \) for all \( q \mid N \) and, for each \( \ell \mid D \), that

\( 2 \nu_\ell(C) - 1 \) if \( L_\ell \) is unramified over \( \mathbb{Q}_\ell \),

\( 2 \nu_\ell(C) \) if \( L_\ell \) is ramified over \( \mathbb{Q}_\ell \).

The choice of character in Assumption 2.2.1 is motivated by the explicit form of the Jacquet–Langlands correspondence recalled in the Introduction (see Section 3.3 for details).

We want to extend \( \chi \) to a character \( \tilde{\chi} \) of \( \hat{R}^\times \) and for this purpose we must deal with several sub-cases. First of all, we decompose \( \chi = \prod \chi_q \) by the Chinese Reminder Theorem and we define each \( \chi_q \) as follows.

\( (1) \) If \( q \mid M \) and \( q \nmid C \), we set \( \chi_q(\alpha) = 1 \) for each \( \alpha \in \mathcal{O}_q \).

\( (2) \) If \( q \mid N \) and \( q \mid C \), we set \( \chi_q(\alpha) = \chi_q(d) \) for \( \alpha = (a/b, c \, d) \in \mathcal{O}_q = M_2(\mathbb{Z}_q) \).

\( (3) \) If \( q \) is odd, \( q \mid M/N \) and \( q \mid C \) we deal with three further sub-cases:

\( (i) \) If \( q \mid C \) and \( \chi_q \) is odd, we can always find a lift to \( \mathcal{O}_q/mL_q \supseteq \mathbb{Z}_q/q\mathbb{Z}_q \) and thus to \( \mathcal{O}_q \), which we call \( \chi_{L_q} \); here \( mL_q \) is the maximal ideal of \( \mathcal{O}_q \). Because of Assumption 2.2.1, \( R_q = M(L_q, m_q) \) is contained in \( M(L_q, 2\nu_q(C) - 1) \) (if \( L_q \) is unramified) or in \( M(L_q, 2\nu_q(C)) \), hence we set \( \chi_q(\alpha + \beta) = \chi_{L_q}(\alpha) \) for each \( \alpha + \beta \in R_q = M(L_q, m_q) \) (and \( \alpha \in \mathcal{O}_{L_q} \)).

\( (ii) \) If \( q^e || C \), with \( e \geq 1 \), and \( \chi_q \) is even, we can always find a character \( \psi \) such that \( \psi^2 = \chi_q \) and with conductor \( \text{cond}(\psi) = \text{cond}(\chi_q) \). As remarked in Section 7.2 of \cite{16}, the choice of this character is not important, but the fact that a particular choice is fixed is. We set \( \tilde{\chi}_q(\alpha) = \psi(n(\alpha)) \).
(iii) If \( q^e \parallel C \), with \( e > 1 \), and \( \chi_q \) is odd, we write \( \chi_q = \varepsilon \cdot \phi \) for a fixed choice of characters \( \varepsilon \) odd and with \( \text{cond}(\varepsilon) = q \), and \( \phi \) even. Thus, proceeding analogously to the previous sub-cases, we set \( \tilde{\chi}_q = \tilde{\varepsilon} \cdot \tilde{\phi} \).

(4) If \( q = 2 \), \( 2 \mid N/M \) and \( 2 \mid C \), one proceeds in a similar way as in case 3.a.

Patching together the local lifts, we define
\[
\tilde{\chi}(b) := \prod_{q \nmid N} \tilde{\chi}_q(b_q),
\]
for any \( b \in \tilde{R}^\times \). In particular, if \( I \) is a lattice in \( B \) such that \( I_q = I \otimes \mathbb{Z}_q = R_q \) for each \( q \nmid N \), and \( b \in I \), we have
\[
\tilde{\chi}(b) = \prod_{q \nmid N} \tilde{\chi}_q(b).
\]
We refer to [16], Section 7.2 for all the details.

2.3. Quaternionic modular forms of weight 2. Take \( B \) as in the above Section 2.1, with \( R \subset B \) a special order of level \( M \). Recall that we fixed isomorphisms \( \iota_q : B_q := B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q) \) for each \( q \mid D \), such that \( \iota_q : R_q := R \otimes \mathbb{Z}_q \cong M_2^0(\mathbb{Z}_q) \). Set \( B(\mathbb{A}_\mathbb{Q})^\times = (B \otimes \mathbb{A}_\mathbb{Q})^\times \) and
\[
\hat{R}(\mathbb{A}_\mathbb{Q})^\times = \{ r \in B(\mathbb{A}_\mathbb{Q})^\times \mid \langle r_q \rangle_{q < \infty} \in \tilde{R}^\times \}.
\]
We extend any character \( \tilde{\chi} \) as in the above Section 2.2 to \( \hat{R}(\mathbb{A}_\mathbb{Q})^\times \) imposing \( \tilde{\chi}(b) = \tilde{\chi}(b_f) \), where \( b_f \in \tilde{R}^\times \) is the finite part of \( b \in \hat{R}(\mathbb{A}_\mathbb{Q})^\times \).

**Definition 2.3.1.** We define the space of weight-2 quaternionic modular forms with level structure \( \hat{R}(\mathbb{A}_\mathbb{Q})^\times \), character \( \chi \) satisfying Assumption 2.2.1 and \( \mathbb{C} \)-coefficients, as the \( \mathbb{C} \)-vector space \( S_2(R, \tilde{\chi}) \) of all continuous functions \( \varphi : \hat{R}(\mathbb{A}_\mathbb{Q})^\times \to \mathbb{C} \) satisfying
\[
\varphi(\tilde{b}br) = \tilde{\chi}^{-1}(r)\varphi(b),
\]
for all \( b \in B^\times \), \( \tilde{b} \in \hat{R}(\mathbb{A}_\mathbb{Q})^\times \) and \( r \in \hat{R}(\mathbb{A}_\mathbb{Q})^\times \).

As in Chapter 5 of [15], we can decompose \( \hat{R}(\mathbb{A}_\mathbb{Q})^\times \) as a finite union of distinct double cosets
\[
\hat{R}(\mathbb{A}_\mathbb{Q})^\times = \bigoplus_{i=1}^h B^\times x_i R(\mathbb{A}_\mathbb{Q})^\times
\]
where \( h = h(R) \) is the class number of \( R \). Since \( B \) is definite, the analogous decomposition holds for \( \tilde{B}^\times \), namely \( \tilde{B}^\times = \bigoplus_{i=1}^h B^\times \tilde{x}_i \tilde{R}^\times \), with \( \tilde{x}_i = (x_{i,q})_{q < \infty} \). By the above Lemma 2.1.5, the representatives \( x_i = (x_{i,q})_q \in \hat{R}(\mathbb{A}_\mathbb{Q})^\times \) can be taken to lie in \( R(\mathbb{A}_\mathbb{Q})^\times \), in particular \( x_{i,q} \in \tilde{R}_q^\times \) for each prime \( q \mid M \). If we fix the representatives in this way, we have an explicit description of quaternionic modular forms. By the definition of a quaternionic modular forms and the double coset decomposition, a quaternionic modular form \( \varphi \) is uniquely determined by its values on the representatives. More precisely, for \( i = 1, \ldots, h \), let \( \tilde{\gamma}_{x_i} := B^\times \cap x_i^{-1} R^\times x_i \) and define
\[
\mathbb{C}_{\tilde{\chi},i} := \left\{ c \in \mathbb{C} \mid \tilde{\chi}(\gamma) \cdot c = c, \text{ for each } \gamma \in \tilde{\gamma}_{x_i} \right\}.
\]
As thoroughly explained in loc.cit., the above observations yield the identification
\[
S_2(R, \tilde{\chi}) \cong \bigoplus_{i=1}^h \mathbb{C}_{\tilde{\chi},i},
\]
given by \( \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_h)) \). We are allowed to consider different coefficients, in fact the above identification still holds when we replace \( \mathbb{C} \) by \( \mathbb{Q}(\tilde{\chi}) \), the field extension of \( \mathbb{Q} \) generated by the values of the character \( \tilde{\chi} \). By extension of scalars we recover \( S_2(R, \tilde{\chi}) = S_2(R, \tilde{\chi}; \mathbb{Q}(\tilde{\chi})) \otimes \mathbb{C} \) and we can consider
p-adic coefficients \( S_2(R, \tilde{\chi}; \mathbb{Q}_p) = S_2(R, \tilde{\chi}; \mathbb{Q}(\chi)) \otimes \mathbb{Q}_p \), for \( p \) a prime which does not divide the reduced discriminant of \( B \).

**Remark 2.3.2.** All the above constructions and definitions are, up to isomorphism, independent of the specific choice of the special order. Moreover, fixing compatible choices of the lifting characters \( \tilde{\chi} \), all the constructions are compatible with respect to the inclusion of special orders.

We end this section with the following fact: often the groups \( \tilde{\Gamma}_{x_i} \) have cardinality 2, i.e. \( \tilde{\Gamma}_{x_i} = \{ \pm 1 \} \).

**Proposition 2.3.3** ([26], Proposition 5.12). Let \( R \) be a special order of level \( M \ell^2 \) in the quaternion algebra over \( \mathbb{Q} \) ramified exactly at \( \ell \) and \( \infty \). Then

\[
\# R^\times = \begin{cases} 
2 & \text{if } \ell > 3, \\
\text{either 2 or 6} & \text{if } \ell = 3.
\end{cases}
\]

Moreover, if \( \ell = 3 \) and \( 2 | M \) or \( M \) is divisible by a prime \( q \equiv 2 \pmod{3} \), then \( \# R^\times = 2 \).

### 2.4. p-adic quaternionic modular forms for special orders.

Let \( p \) and \( \ell \) be two distinct rational odd primes. From now on, we denote by \( B \) the (unique up to isomorphism) definite quaternion algebra over \( \mathbb{Q} \) with discriminant \( \ell \) and we fix \( R \) to be a maximal order in \( B \). For \( N \) a fixed positive integer, prime to both \( p \) and \( \ell \), consider a family of nested special orders \( \{ R^n \}_{n \geq 0} \) satisfying

\[
\cdots \subset R^{n+1} \subset R^n \subset \cdots \subset R^0 \subset R, \quad R^n \text{ is a special order of level } N p^n \ell^{2r},
\]

where \( r \geq 1 \). Up to conjugation, we can suppose that the orders \( R_n \) are all canonical orders of level \( N p^n \ell^{2r} \), that is, as in Definition 2.1.1. For any prime \( q \) different from \( \ell \), we assume that the fixed isomorphism \( \iota_q : B_q \cong M_2(\mathbb{Q}_q) \) satisfies \( \iota_q R_n^\times \cong \text{GL}_2(\mathbb{Z}_q) \) and

\[
\iota_q(R_n^\times)^\times \cong \Gamma_0(N p^n \mathbb{Z}_q) := M_2^0(N p^n \mathbb{Z}_q) \cap \text{GL}_2(\mathbb{Z}_q).
\]

**Definition 2.4.1.** We define (cf. Lemma 2.1.3) open compact subgroups \( U_n \subset \tilde{B}^\times \),

\[
U_n := U_1(R^n) := \left\{ g = (g_q) \in \tilde{R}^n^\times \mid \iota_q(g_q) \equiv (a_1^+ 1) \pmod{N p^n \mathbb{Z}_q}, \text{ for } q \mid N p^n \right\}.
\]

By construction, \( U_{n+1} \subset U_n \subset \ldots \subset U_0 \). Given any special order \( R' \), we denote by \( U_1(R') \) the corresponding open compact subgroup defined analogously to \( U_n \).

For any commutative ring \( A \), we consider the left action of \( M_2(A) \) on the polynomial ring \( A[X,Y] \), defined as

\[
\gamma \cdot P(X,Y) := P((X,Y)\gamma),
\]

for \( P \in A[X,Y] \) and \( \gamma \in M_2(A) \). We denote by \( L_m(A) \) the submodule of \( A[X,Y] \) consisting of homogeneous polynomials of degree \( m \); by definition, \( L_m(A) \) is stable under the action of \( M_2(\mathbb{Z}) \). Its dual module \( V_m(A) \) is endowed with the right action

\[
\mu |_\gamma(P(X,Y)) := \mu(\gamma \cdot P((X,Y))),
\]

for any \( \mu \in V_m(A) \) and \( P \in L_m(A) \).

We take now \( O \) to be a finite flat extension of \( \mathbb{Z}_p \), which we assume to contain all the \( \phi(N p^n \ell^{2r}) \)-th roots of unity, where \( \phi \) is Euler’s totient function. Given an \( O \)-algebra \( A \), any \( A \)-valued Dirichlet character \( \psi \) modulo \( N p^n \ell^{2r} \), can be lifted to \( \psi_A : \mathbb{Q}^\times \backslash \mathbb{A}^\times_Q \to A^\times \), its ad\'elization, that is, the unique finite order Hecke character

\[
\psi_A : \mathbb{Q}^\times \mathbb{A}_Q^\times / \mathbb{R}_+ (1 + N p^n \ell^{2r} \mathbb{Z})^\times \to A^\times.
\]
such that \( \psi_k(\varpi_q) = e^{-1}(q) \), for every \( q \mid Np^n\ell^2r \) and \( \varpi_q = (\varpi_{q,l} \mid \varpi_{q,l} = 1 \text{ if } l \neq q, \varpi_{q,q} = q) \in \mathbb{A}_{Q,f}^\times \).

We fix such a Dirichlet character \( \psi \) modulo \( Np^n\ell^2r \) with small conductor at \( \ell \). More precisely, as in [15], we enforce the following assumption.

**Assumption 2.4.2.** The \( \ell \)-component of \( \psi \), \( \psi_{\ell^2r} \), is either the trivial character modulo \( \ell \) or an odd character of conductor exactly \( \ell \).

**Definition 2.4.3.** Let \( k \geq 2 \) be an integer and let \( U_n \) be as in Definition 2.4.1. For an \( \mathcal{O} \)-algebra \( A \) and an \( A \)-valued Dirichlet character \( \psi = \psi_{Np^n}\psi_{\ell^2r} \) (for \( \psi_{Np^n} \) modulo \( Np^n \) and \( \psi_{\ell^2r} \) modulo \( \ell^2r \)) we define the space of \( p \)-adic quaternionic modular forms of weight \( k \), level \( Np^n\ell^2r \) and character \( \psi \) as

\[
S_k(U_n, \psi, A) := \left\{ \varphi : \hat{B}^\times \to V_{k-2}(A) \mid \varphi(buz) = \psi_{Np^n,A}(z)\psi_{\ell^2r,A}(z)z_p^{k-2}\overline{\psi_{\ell^2r}}(u_{\ell})\varphi(\hat{b})|_{u_p}, \right. \\
\left. \text{for } b \in B^\times, \hat{b} \in \hat{B}^\times, u \in U_n, z \in \mathbb{A}_{Q,f}^\times \right\}.
\]

This space can be identified with the space of functions \( \varphi : B^\times \setminus \hat{B}^\times \to V_{k-2}(A) \) satisfying

\[
\varphi(\hat{b}) = \psi_{Np^n,A}(z)\psi_{\ell^2r,A}(z)z_p^{k-2}\varphi(\hat{b})
\]

for \( \hat{b} \in B^\times \setminus \hat{B}^\times \) and \( z \in \mathbb{A}_{Q,f}^\times \), and such that \( (u \cdot \varphi)(\hat{b}) := \varphi|_{u_p^{-1}}(bu) = \overline{\varphi_{\ell^2r}}(u_{\ell})\varphi(\hat{b}) \), for any \( u \in U_n \) and any \( \hat{b} \in \hat{B}^\times \).

**Remark 2.4.4.** The term \( p \)-adic quaternionic modular forms refers to the fact that we are considering \( p \)-adic coefficients, and the action at the place \( p \) instead of the action at infinity (see the next Section 2.5).

2.5. Quaternionic modular forms of higher weight. The definition of classical quaternionic modular forms for higher weight is similar to the one for weight 2. We fix an identification \( \iota_{\infty} : B_{\infty} \to M_2(\mathbb{C}) \) in order to compare \( p \)-adic and classical quaternionic modular forms.

**Definition 2.5.1.** We define the space of weight-\( k \), \( k \geq 2 \), quaternionic modular forms with level structure \( R^n(\mathbb{A}_Q)^\times \), character \( \chi \) satisfying Assumption 2.2.1 and \( \mathbb{C} \)-coefficients, as the \( \mathbb{C} \)-vector space \( S_k(R^n, \chi) \) of all continuous functions \( \varphi_\infty : B(\mathbb{A}_Q)^\times \to V_{k-2}(\mathbb{C}) \) satisfying

\[
\varphi_\infty(b\overline{b}_\infty r) = \chi^{-1}(r)|n(b|_{\hat{\mathbb{A}}})(k-2)/2b_{\infty}^{-1} \cdot \varphi_\infty(\hat{\mathbb{b}})
\]

for all \( b \in B^\times \), \( b_{\infty} \in B_{\infty}^\times \), \( \mathbb{b} \in B_{\mathbb{A}_Q}^\times \) and \( r \in R^n(\mathbb{A}_Q)^\times \).

As explained in Chapter 2 of [13], we can identify classical and \( p \)-adic modular forms. We identify \( \mathbb{C}_p \) with \( \mathbb{C} \) compatibly with the fixed inclusion \( \mathbb{C}_p \hookrightarrow \mathbb{C} \) and associate (see also [17], Eq. (4.4)) to \( \varphi \in S_k(U_n, \psi, \mathbb{C}_p) \) the form \( \Phi_\infty(\varphi) \in S_k(R^n, \widetilde{\psi}) \), defined as

\[
\Phi_\infty(\varphi)(\mathbb{b}_\infty) = |n(\overline{\mathbb{b}}_\infty)|_{\hat{\mathbb{A}}}^{(k-2)/2}b_{\infty}^{-1} \cdot (\mathbb{b}_p \cdot \varphi(\hat{\mathbb{b}})),
\]

for \( b_{\infty} \in B_{\infty}^\times \) and \( \hat{\mathbb{b}} \in B_{\mathbb{A}_Q}^\times \).

2.6. Quaternionic Eisenstein series and newforms. We recall the notions of quaternionic Eisenstein series and quaternionic newforms as presented in [15], Chapters 5 and 7. For this section, we take \( A \) to be a \( \mathbb{Q}_p \)-module. We begin with the Eisenstein series part of \( S_k(U_n, \psi, A) \), that is

\[
S^E_k(U_n, \psi, A) := \left\{ f \in S_k(U_n, \psi, A) \mid \exists g : \mathbb{A}_{Q,f}^\times \to A^{k-1} \text{ s.t. } f(\hat{b}) = g(n(\hat{b})) \right\},
\]

where \( n : \hat{B}^\times \to \mathbb{A}_{Q,f}^\times \) is the extension of the quaternionic norm to \( \hat{B} \). In other words, \( S^E_k(U_n, \psi, A) \) is the space of quaternionic modular forms factoring through the reduced norm map. As proved by
Propositions 5.2, 5.3 and the discussion after Proposition 5.4 in loc.cit., this space is often trivial, in fact

\[
S^E_k(U_n, \psi, A) = \begin{cases} 
[0] & \text{if } k > 2 \text{ or } \psi_{Np^n} \text{ is non trivial,} \\
A(\mathbb{Z}^n: n((R^+)^{*})) & \text{if } k = 2 \text{ and } \psi_{Np^n} \text{ is trivial.}
\end{cases}
\]

In particular, \(S^E_k(U_n, \psi, A)\) has at most rank 2. Defining the Petersson inner product as in [27] or [10], one can consider the orthogonal complement of \(S^E_k(U_n, \psi, A)\) in \(S_k(U_n, \psi, A)\), namely

\[
\mathcal{H}_k(U_n, \psi, A) := \begin{cases} 
S_k(U_n, \psi, A)/S^E_k(U_n, \psi, A) & \text{if } k = 2 \text{ and } \psi_{Np^n} \text{ is trivial,} \\
S_k(U_n, \psi, A) & \text{otherwise.}
\end{cases}
\]

**Definition 2.6.1.** We call \(\mathcal{H}_k(U_n, \psi, A)\) the space of \(A\)-valued cuspidal quaternionic modular forms of level \(U_n\) and character \(\psi\).

Inside of this space, we find the so-called space of old forms, \(\mathcal{H}^\text{old}(U_n, \psi, A)\), defined to be the subspace of \(\mathcal{H}_k(U_n, \psi, A)\) spanned by all \(\mathcal{H}_k(U_1(R'), \psi, A)\) for each special order \(R' \subset R_n\) for which \(S_k(U_1(R'), \psi, A)\) makes sense. One should pay attention to fix a suitable ramified extension of \(\mathbb{Q}_\ell\), but we point the reader to Remarks 7.13 and 7.14 of [15] for further details. Finally, we define the space of quaternionic newforms \(\mathcal{H}_k^\text{new}(U_n, \psi, A)\) as the orthogonal complement of \(\mathcal{H}_k^\text{old}(U_n, \psi, A)\), with respect to the Petersson inner product, inside \(\mathcal{H}_k(U_n, \psi, A)\).

### 3. Hecke algebras and lifts to quaternionic modular forms

One of Hida’s main results is the extension of the classical duality between the Hecke algebra and the space of classical modular forms to \(p\)-adic families. The analogous result can be recovered in the quaternionic setting, when one considers Eichler orders or special orders with odd exponent at the primes of ramification, but in the case of special orders with even exponent, this is no more true (see Remark 3.3.3). Even though one cannot speak about duality anymore, it is indeed possible to recover the correct dimension result for proving a rank-2 Hida theory.

#### 3.1. Hecke operators

For any prime \(q\), recall the element introduced in Section 2.4, \(\varpi_q \in \mathbb{A}_Q^\infty\) such that \(\varpi_{q,q} = q\) and 1 otherwise. Let \(A\) be again an \(O\)-algebra and take \(\varphi \in S_k(U_n, A)\). On this quaternionic space we have (for any \(\tilde{b} \in \tilde{B}^\infty\)) the Hecke operators \(T_q\),

\[
T_q \varphi(\tilde{b}) = \begin{cases} 
\varphi \left( \tilde{b} \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi_q \end{array} \right) \right) + \sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi \left( \tilde{b} \left( \begin{array}{cc} \varpi_q a & 0 \\ 0 & 1 \end{array} \right) \right) & \text{for each } q \nmid Np^n\ell^{2r}, \\
\varphi \left( \tilde{b} \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi_q \end{array} \right) \right) + \sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi \left( \tilde{b} \left( \begin{array}{cc} \varpi_q a & 0 \\ 0 & 1 \end{array} \right) \right) & \text{for } q = p \text{ and } n = 0,
\end{cases}
\]

and the Hecke operators \(U_q\),

\[
U_q \varphi(\tilde{b}) = \begin{cases} 
\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi \left( \tilde{b} \left( \begin{array}{cc} \varpi_q a & 0 \\ 0 & 1 \end{array} \right) \right) & \text{for } q \mid N, \\
\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \varphi \left( \tilde{b} \left( \begin{array}{cc} \varpi_q a & 0 \\ 0 & 1 \end{array} \right) \right) & \text{for } q = p \text{ and } n > 0.
\end{cases}
\]

We also consider the quaternionic operator at \(\ell\), \(\tilde{U}_\ell\) which is defined as

\[
\tilde{U}_\ell \varphi(\tilde{b}) = \varphi \left( \tilde{b} \tilde{x}_\ell \right),
\]

for \(\tilde{x}_\ell\) such that \(\tilde{x}_\ell \ell\) is a units in the maximal order at \(\ell\) of norm \(n(\tilde{x}_\ell \ell) = \ell\), and \(\tilde{x}_\ell \ell = 1\) elsewhere. For each \(d \in \Delta_{Np^n\ell^{2r}} := (\mathbb{Z}/Np^n\ell^{2r}\mathbb{Z})^\times\), we also recall the diamond operator \(\langle d \rangle\) with its usual definition
on classical modular forms and straightforwardly extended to the $p$-adic quaternionic case. On the space of classical modular forms $S_k(\Gamma_1(Np^n\ell^2r), \psi, A)$ we have the usual operators with a similar expression to the quaternionic ones except at $\ell$, where the definition of $U_{\ell}$ is analogous to the above $U_q$ operators.

3.2. Hecke algebras. For each $n \geq 1$, let $H_n^1(A)$ be the Hecke algebra generated over $A$ by all Hecke and diamond operators away from the level, which acts on $S_k(\Gamma_1(Np^n\ell^2r), A)$. We denote by $H_n(A)$ the direct summand of $H_n^1(A)$ acting on $S_k(\Gamma_1(Np^n\ell^2r), \psi, A)$ and by $h_n(A)$ the Hecke algebra acting on the space of newforms $S_k^{new}(\Gamma_1(Np^n\ell^2r), \psi, A)$. For each $m > n$ we have the projection maps $H_n^1(A) \twoheadrightarrow H_m^1(A)$ and the same holds true for the subalgebras $H_n(A)$ and $h_m(A)$. We construct the projective limits with respect to these maps,

$$H_\infty^1(A) = \lim_{\leftarrow} H_n^1(A), \quad H_\infty(A) = \lim_{\leftarrow} H_n(A), \quad h_\infty(A) = \lim_{\leftarrow} h_n(A),$$

together with the projection maps $H_\infty^1(A) \twoheadrightarrow H_\infty(A) \twoheadrightarrow h_\infty(A)$. For any $n \geq 1$, we define $H_n^{1,ord}(A)$ to be the product of all the localizations of $H_n^1(A)$ on which $U_p$ is invertible, and denote by $e_n$ the corresponding projector $e_n : H_n^1(A) \twoheadrightarrow H_n^{1,ord}(A)$. Similarly we define $H_n^{ord}(A)$ and $h_n^{ord}(A)$, together with the corresponding ordinary projectors, which we denote by the same symbol $e_n$.

Passing to the limit we obtain $H_\infty^{1,ord}(A)$, $H_\infty^{ord}(A)$, and $h_\infty^{ord}(A)$, each of them equipped with the corresponding ordinary projector $e_\infty = \lim_{\leftarrow} e_n$.

**Remark 3.2.1.** It is well known (see e.g. [18], Theorem 3) that Assumption 2.4.2 forces the Hecke operator $U_{\ell}$ to be trivial on the space of classical modular forms of level $\ell^2r$. We can then identify each $h_n(A)$ with the Hecke $A$-algebra

$$h_n^{(O)}(A) \subseteq \text{End}(S_k(\Gamma_1(Np^n\ell^2r), \psi, A))$$
generated by all diamond and Hecke operators, except $U_{\ell}$.

On the quaternionic side we proceed similarly. For each $n \geq 1$, let $H_n^B(A)$ be the Hecke algebra acting on $S_k(U_n, \psi, A)$, generated over $A$ by the Hecke and diamond operators. We denote by $h_n^B(A)$ the component acting on the space of newforms $S_k^{new}(U_n, \psi, A)$. For each $n \geq 1$ we have the projection maps $H_n^B(A) \twoheadrightarrow h_n^B(A)$ and we construct the projective limits with respect to these maps,

$$H_\infty^B(A) = \lim_{\leftarrow} H_n^B(A), \quad h_\infty^B(A) = \lim_{\leftarrow} h_n^B(A),$$

together with the projection map $H_\infty^B(A) \twoheadrightarrow h_\infty^B(A)$. In the end, we define as above the ordinary Hecke algebras $H_n^{B,ord}(A)$ and $h_n^{B,ord}(A)$, and obtain $H_\infty^{B,ord}(A)$ and $h_\infty^{B,ord}(A)$ as inverse limit of the $H_n^{B,ord}(A)$ and $h_n^{B,ord}(A)$ respectively.

The Jacquet–Langlands correspondence provides a compatible morphism between the classical and the quaternionic side, that is

$$J_L_{\infty} : h_\infty(A) \longrightarrow H_\infty^B(A),$$

which preserves the Hecke and diamond operators away from the discriminant of the quaternion algebra.

Let $\mathcal{A} = \mathcal{O}[\mathbb{Z}_p^\infty]$ be the finite flat extension of the Iwasawa algebra $\mathcal{O}[\mathbb{Z}_p^\infty]$ obtained from $\mathcal{O}$. We remark that by construction, the algebra $H_\infty^{B,ord}$ is naturally a $\mathcal{A}$-algebra; moreover, one can prove that it is finitely generated over $\mathcal{A}$. We define the two universal $\mathcal{A}$-adic Hecke algebras

$$H_{univ}^B := \mathcal{A}[T_q, U_{\ell}, (d), \text{ for } q \nmid Np^n, l \mid Np^2r, d \in \Delta_{Np^2r}]$$

$$H_{univ}^{B,ord} := \mathcal{A}[U_{\ell}, T_q, U_{\ell}, (d), \text{ for } q \nmid Np^n, l \mid Np, d \in \Delta_{Np^2r}]$$

and, as in [21], we obtain the compatible morphisms

$$H_{univ}^B \longrightarrow H_\infty^{B,ord}(A) \quad \text{and} \quad H_{univ}^{B,ord} \longrightarrow H_\infty^{B,ord}(A).$$
3.3. Quaternionic lifts of modular forms and the failure of the duality. We analyze more carefully [15], recalling the results which we need. Let \((\frac{-\ell}{\ell})\) be the Kronecker character at \(\ell\) and \(F = \text{Frac}(\mathcal{O})\), a field extension of \(\mathbb{Q}_p\). For any space of modular forms \(S_k(M, \varepsilon, F)\) and each Dirichlet character modulo \(M\), we denote by \(S_k(M, \varepsilon, F) \otimes \chi\) the space of all the modular forms which are twists by \(\chi\) of modular forms in \(S_k(M, \varepsilon, F)\).

**Theorem 3.3.1** ([15], Theorem 7.10). Let \(R'\) be a special order of level \(M\ell^{2r+1}\) (so \(L_\ell\) is the unramified quadratic extension of \(Q_\ell\)). Let \(\varepsilon\) be a character modulo \(N\) such that \(\varepsilon\) is either the trivial character modulo \(\ell\) or an odd character modulo \(\ell\). Suppose moreover that \(\varepsilon\) is even and that \(r \geq \nu_{\ell}(\text{cond}(\varepsilon))\).

Then there exist a Hecke-equivariant isomorphism

\[
\mathcal{S}^{\text{new}}(U_1(R'), \varepsilon, C) \cong S^{\text{new}}_k(\Gamma_1(M\ell^{2r+1}), \varepsilon, C).
\]

The above theorem proves that, as in the case of Eichler orders, there is a one-to-one correspondence for special orders with odd exponent at \(\ell\). The situation for even exponent is more complicated.

**Theorem 3.3.2** ([15], Theorems 7.16 & 7.17). Let \(\ell\) be an odd prime, and let \(r \geq 1\) and \(k \geq 2\) be integers. Let \(\psi\) be a character modulo \(Np^r\ell^2\) such that \(\psi(-1) = (-1)^k\), it satisfies Assumptions 2.2.1 and 2.4.2, and such that \(\text{cond}_{\ell}(\psi) \leq 2r - 1\). Then the following decomposition of \(h_n^{(Np^r\ell^2)}(C)\)-modules holds true.

(a) If \(r = 1\) and \(\psi_{12}\) is the trivial character:

\[
2S^{\text{new}}_k(\Gamma_1(Np^r\ell^2), \psi, C) \cong \mathcal{S}^{\text{new}}_k(U_n, \psi, C) \oplus S^{\text{new}}_k(\Gamma_1(Np^r\ell), \psi, C) \otimes (\frac{-\ell}{\ell}) \oplus 2S^{\text{new}}_k(\Gamma_1(Np^r\ell), \chi^2\psi, C) \otimes \chi
\]

where the sum \(\bigoplus_{\chi/\sim}\) runs over all the \(\frac{1}{2}(\ell - 3)\) classes of primitive characters modulo \(\ell\) excepting \((\frac{-\ell}{\ell})\), modulo the equivalence \(\chi \sim \chi\).

(b) If \(r = 1\) and \(\psi_{12}\) is a odd character modulo \(\ell\):

\[
2S^{\text{new}}_k(\Gamma_1(Np^r\ell^2), \psi, C) \cong \mathcal{S}^{\text{new}}_k(U_n, \tilde{\psi}, C) \oplus \bigoplus_{\chi/\sim} 2S^{\text{new}}_k(\Gamma_1(Np^r\ell), \chi^2\psi, C) \otimes \chi
\]

where \(\tilde{\psi}\) is a lift of \(\psi\) as in Section 2.2 and the sum \(\bigoplus_{\chi/\sim}\) runs over all the \(\frac{1}{2}(\ell - 3)\) classes of primitive characters modulo \(\ell\) excepting \(\psi_{12}\), modulo the equivalence \(\chi \sim \chi\).

(c) If \(r \geq 2\) and \(\psi_{12}\) is either trivial or odd of conductor \(\ell\):

\[
2S^{\text{new}}_k(\Gamma_1(Np^r\ell^2), \psi, C) \cong \mathcal{S}^{\text{new}}_k(U_n, \tilde{\psi}, C) \oplus \bigoplus_{\chi} 2S^{\text{new}}_k(\Gamma_1(Np^r\ell'), \chi^2\psi, C) \otimes \chi
\]

where \(\tilde{\psi}\) is a lift of \(\psi\) as in Section 2.2 and the sum \(\bigoplus_{\chi}\) runs over all the \(\ell' = 2\ell^{r-1} + \ell^{r-2}\) classes of primitive characters modulo \(\ell'\), modulo the equivalence \(\chi \sim \chi^{\ell'\ell^2}\).

**Remark 3.3.3.**

(a) In the above theorem the decomposition is given as \(h_n^{(Np^r\ell^2)}(C)\)-modules, but strong multiplicity one for classical modular newforms guarantees the decomposition to hold (at least) as \(h_n^{(\ell)}(C)\)-modules. As already noticed in Remark 3.2.1, the Hecke algebra \(h_n^{(\ell)}(C)\) coincides with \(h_n(C)\) since the Hecke operator \(U_n\) is the 0-operator on this space.

(b) The theorem implies that the duality between the Hecke algebra and the space of modular forms does not necessarily hold true for special orders with level \(\ell^{2r}\). This situation represents the main difference between this setting and the case of classical modular forms (and special orders with an
Let \( \pi \) be a representation of \( \pi \), and \( \chi \) the same Hecke eigenvalues for each twist-minimal modular eigenform in Corollary 3.3.5.

We recall the definition of a twist-minimal modular form.

**Definition 3.3.4.** A modular form is twist-minimal if \( \cond(\pi) \geq \cond(\pi \otimes \chi) \) for all \( q \)-adic characters \( \chi \), where \( \pi = \otimes_q \pi_q \) is the automorphic representation attached to \( f \) and for any automorphic representation \( \pi = \otimes_q \pi_q \) we denote \( \cond(\pi_q) \) the conductor of \( \pi_q \).

**Corollary 3.3.5.** Each twist-minimal modular eigenform in \( S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, \mathbb{C}) \) lifts to (up to linear combinations) exactly two linearly independent quaternionic modular eigenforms in \( \mathcal{S}_k^{\text{new}}(U_n, \widetilde{\psi}, \mathbb{C}) \) with the same Hecke eigenvalues for \( h_n^k(\mathbb{C}) \).

Regardless of Remark 3.3.3.(b), one can still obtain an isomorphism between the space of quaternionic modular forms and the square of a suitable Hecke algebra, as in the following proposition.

**Proposition 3.3.6.** Under the hypotheses of Theorem 3.3.2, there exists a \( \mathbb{C} \)-vector subspace \( T_k(n, r, \psi) \) of \( S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, \mathbb{C}) \), which is a \( h_n^k(\mathbb{C}) \)-submodule satisfying

\[
2T_k(n, r, \psi) \cong \begin{cases} 
\mathcal{S}_k^{\text{new}}(U_n, \psi, \mathbb{C}) \oplus S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, \mathbb{C}) \otimes (\overline{r}) & \text{if } r = 1 \text{ and } \psi_r \text{ is trivial}, \\
\mathcal{S}_k^{\text{new}}(U_n, \psi, \mathbb{C}) & \text{otherwise}.
\end{cases}
\]

Moreover, for \( h_n^k(\mathbb{C}) \) the Hecke-subalgebra of \( h_n(\mathbb{C}) \) acting on \( T_k(n, r, \psi) \), we have an isomorphism of \( \mathcal{S}_k^{\text{new}}(U_n, \psi, \mathbb{C}) \)-modules,

\[
(h_n^k(\mathbb{C}))^2 \cong \begin{cases} 
\mathcal{S}_k^{\text{new}}(U_n, \psi, \mathbb{C}) \oplus S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, \mathbb{C}) \otimes (\overline{r}) & \text{if } r = 1 \text{ and } \psi_r \text{ is trivial}, \\
\mathcal{S}_k^{\text{new}}(U_n, \psi, \mathbb{C}) & \text{otherwise}.
\end{cases}
\]

**Proof.** The first statement follows directly from Theorem 3.3.2 as noticed in Chapter 8 of [15]. The second part follows from \( \text{Hom}(2T_k(n, r, \psi), \mathbb{C}) \cong \text{Hom}(T_k(n, r, \psi), \mathbb{C})^2 \cong 2T_k(n, r, \psi) \), where the first isomorphism is due to the properties of \( \text{Hom}(\cdot, \mathbb{C}) \) and the second is the Hecke-duality for classical modular forms restricted to \( T_k(n, r, \psi) \) (since the decomposition is Hecke-equivariant away from \( \ell \)).

As in Section 3.2, taken an \( \mathcal{O} \)-algebra, we define \( h_n^k(\mathcal{O}) = \lim_{\ell} h_n^k(\mathcal{O}) \) and \( h_n^k(\mathcal{O})^\text{ord}(A) = \lim_{\ell} h_n^k(A)_\text{ord}(A) \).

We obtain injective homomorphisms \( h_n^k(A) \hookrightarrow h_n^k(A) \) and \( h_n^k(A) \hookrightarrow h_n^k(A)_\text{ord}(A) \).

**Definition 3.3.7.** For any module \( M \) with an action of a suitable Hecke algebra, and any classical eigenform \( g \), we denote by \( M[g] \) the \( g \)-isotypic component of \( M \), i.e. the biggest submodule of \( M \) on which the Hecke algebra acts with the same eigenvalues of \( g \).

**Proposition 3.3.8.** Let \( g \) be a newform in \( S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, F) \) with \( k \geq 2 \) and \( \psi(-1) = (-1)^k \). Write \( \widetilde{\psi} = \psi_{Np^\ell} \psi_{\overline{r}_2} \) for \( \psi_{Np^\ell} \) and \( \psi_{\overline{r}_2} \) the component of \( \psi \), respectively, modulo \( Np^\ell \) and \( \ell^2r_2 \).

(a) If \( r = 1 \) and \( \psi_{\overline{r}_2} \) is the trivial character modulo \( \ell \),

\[
\dim \mathcal{S}_k^{\text{new}}(U_n, \widetilde{\psi}, F)[g] = \begin{cases} 
2 & \text{if } g \text{ is twist-minimal at } \ell, \\
1 & \text{if } g \in S_k^{\text{new}}(\Gamma_1(Np^\ell), \psi, F) \otimes (\overline{r}), \\
0 & \text{otherwise}.
\end{cases}
\]
(b) If either $\psi_{12r}$ is a non-trivial character modulo $\ell$ or $r \geq 2$,

\begin{equation}
(3.18) \quad \dim_F \big( \mathcal{H}^\text{new}_k(U_n, \tilde{\psi}, F)[g] \big) = \dim_F \big( \mathcal{H}_k(U_n, \tilde{\psi}, F)[g] \big) = \begin{cases} 2 & \text{if } g \text{ is twist-minimal at } \ell, \\
0 & \text{otherwise}. \end{cases}
\end{equation}

**Proof.** This is a straightforward consequence of Theorem 3.3.2 combined with the fact that strong multiplicity one applies to $g$. \hfill $\blacksquare$

### 3.4. Choice of a modular form.

Let $f \in S_2(\Gamma_1(Np^n\ell^{2r}), \psi, \mathbb{C})$ be a fixed $p$-ordinary newform, for $n \geq 1$ and with $\psi$ a Dirichlet character modulo $Np^n\ell^{2r}$ satisfying Assumptions 2.2.1 and 2.4.2. In this way, the automorphic representation associated with $f$ admits a Jacquet–Langlands lift. Moreover, we assume that the $p$-adic Galois representation associated with $f$ is residually absolutely irreducible and $p$-distinguished. Let $F = \mathbb{Q}_p(f)$ be the finite extension of $\mathbb{Q}_p$ defined by $f$ and take $\mathcal{O}$ to be its ring of integers; note that $\mathcal{O}$ is a finite flat extension of $\mathbb{Z}_p$. We denote by $f_\infty$ the unique Hida family passing through $f$. By duality with the ordinary Hecke algebra, we know that $f_\infty$ defines a character, which we denote with the same symbol $f_\infty$,

\begin{equation}
(3.19) \quad f_\infty : H^{\text{ord}}_\infty(F) \rightarrow \mathcal{R},
\end{equation}

where $\mathcal{R}$ is the universal ordinary $p$-adic Hecke algebra of tame level $N\ell^{2r}$ as in Definition 2.4 of [9]. The Jacquet–Langlands correspondence ensures that such character factors through the morphism to $H^{\text{ord}}_\infty(F)$; we keep denoting the corresponding map by $f_\infty : H^{\text{ord}}_\infty(F) \rightarrow \mathcal{R}$.

### 4. The control theorem

In this last section, we prove a control theorem for special orders of even conductor at $\ell$. We introduce a space that is suitable for the $p$-adic interpolation and we define some specialization maps. We consider again $\mathcal{O}$ to be the ring of integers of a fixed finite extension of $\mathbb{Q}_p$ and we take an $\mathcal{O}$-algebra which we denote again by $\mathcal{A}$.

#### 4.1. Specialization maps.

Let $X = (\mathbb{Z}_p \times \mathbb{Z}_p)^{prim}$ be the set of primitive row vectors, that is, the vectors in $\mathbb{Z}_p^2$ which have at least one component not divisible by $p$. Denote by $\mathcal{C}(X, \mathcal{A})$ the space of $\mathcal{A}$-valued continuous functions on $X$ and by $\mathcal{M}(X, \mathcal{A})$ the space of $\mathcal{A}$-valued measures on $X$. We have a left $\mathbb{M}_2(\mathbb{Z}_p)$-action on $\mathcal{C}(X, \mathcal{A})$

\begin{equation}
(4.1) \quad \gamma \cdot f(x, y) = f((x, y)\gamma),
\end{equation}

for $f \in \mathcal{C}(X, \mathcal{A})$ and $\gamma \in \mathbb{M}_2(\mathbb{Z}_p)$, and the induced right action on $\mathcal{M}(X, \mathcal{A})$ as

\begin{equation}
(4.2) \quad \mu|_\gamma(f(x, y)) = \mu(\gamma \cdot f(x, y)),
\end{equation}

for $\mu \in \mathcal{M}(X, \mathcal{A})$. Considering the action by $U_n$, with $n \geq 1$, we can notice that the subspace $p\mathbb{Z}_p \times \mathbb{Z}_p^\times \subset X$ satisfies

\begin{equation}
(4.3) \quad (p\mathbb{Z}_p \times \mathbb{Z}_p^\times) \cdot (U_n)_p = (p\mathbb{Z}_p \times \mathbb{Z}_p^\times) \left( \frac{\mathbb{Z}_p^\times}{p^n\mathbb{Z}_p 1 + p^n\mathbb{Z}_p} \right) = p\mathbb{Z}_p \times \mathbb{Z}_p^\times.
\end{equation}

**Definition 4.1.1.** Let $\psi$ be a Dirichlet character modulo $N\ell^{2r}$ satisfying Assumptions 2.2.1 and 2.4.2. We define the measure-valued quaternionic modular forms with character $\psi$ as the space

\begin{equation}
(4.4) \quad S_2(U_0, \psi, \mathcal{M}(X, \mathcal{A})) := \left\{ \varphi : \tilde{B}^\times \rightarrow \mathcal{M}(X, \mathcal{A}) \mid \varphi(\overline{b}bu) = \psi_{N,A}^{-1}(z)\psi_{\ell^{2r},A}^{-1}(z)(\overline{\psi_{\ell^{2r}}(u)})\varphi(b)|_{u_p}, \right. \\
\left. \quad \text{for } b \in B^\times, \overline{b} \in \tilde{B}^\times, u \in U_0, z \in \mathbb{A}_{\ell, f}^\times \right\}.
\end{equation}

This space can be identified with the space of functions $\varphi : B^\times \setminus \tilde{B}^\times \rightarrow \mathcal{M}(X, \mathcal{A})$ satisfying

\begin{equation}
(4.5) \quad \varphi(z\overline{b}) = \psi_{N,A}^{-1}(z)\psi_{\ell^{2r},A}^{-1}(z)\varphi(\overline{b}),
\end{equation}
for $\tilde{b} \in B^\times \setminus \hat{B}^\times$ and $z \in \mathbb{A}_{Q,f}^\times$, and such that $\varphi|_{u_p}(bu) = \widetilde{\psi}_{\varphi}(u)b\big)$ for any $u \in U_0$ and any $\tilde{b} \in B^\times \setminus \hat{B}^\times$.

Take $k \geq 2$ and let $\varepsilon : \mathbb{Z}_p^\times \to A^\times$ be any character which factors through $\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$. We extend $\varepsilon$ multiplicatively to $\mathbb{Z}_p$ imposing $\varepsilon(p) = 0$. We define the specialization map

$$\nu_{k,\varepsilon} : S_2(U_0, \psi, \mathcal{M}(X, A)) \to S_k(U_n, \psi, \mathcal{M}(A)(\varepsilon))$$

such that

$$\nu_{k,\varepsilon}(\varphi)(\tilde{b})(P) := \int_{\mathbb{A}^\times \times \mathbb{Z}_p^\times} \varepsilon(y)(u^{-1}_p) \cdot P(x, y)d(\varphi(\tilde{b}))(x, y),$$

where $n = \max\{1, m\}$, $\varphi \in S_2(U_0, \psi, \mathcal{M}(X, A))$, $\tilde{b} \in B^\times \setminus \hat{B}^\times$ and $P \in L_{k-2}(A)$.

**Proposition 4.1.2.** The specialization maps $\nu_{k,\varepsilon}$ are well-defined and Hecke-equivariant for $H^0_{univ}$, where the equivariance at $p$ is meant as $\nu_{k,\varepsilon}(T_p\varphi) = U_p\nu_{k,\varepsilon}(\varphi)$.

**Proof.** Let $\varphi \in S_2(U_0, \psi, \mathcal{M}(X, A))$. Then, for any $\tilde{b} \in B^\times \setminus \hat{B}^\times$, $z \in \mathbb{A}_{Q,f}^\times$, $u \in U_n$ and $P \in L_{k-2}(A)$, we have

$$\nu_{k,\varepsilon}(\varphi)(\tilde{b}uz)_{u_p^{-1}}(P) = \int_{\mathbb{A}^\times \times \mathbb{Z}_p^\times} \varepsilon(y)(u^{-1}_p) \cdot P(x, y)d(\varphi(\tilde{b}uz))(x, y)$$

$$= \psi_{N,\varepsilon}^{-1}(z)\psi_{\varepsilon,\varepsilon}^{-1}(z)\psi_{\varphi}(u) \cdot \int_{\mathbb{A}^\times \times \mathbb{Z}_p^\times} (u_p z_p) \cdot (\varepsilon(y)P((x, y)u^{-1}_p))d(\varphi(\tilde{b}))(x, y)$$

and, since $(x, y)u^{-1}_p = (\ast, y + p\ast)$, $\varepsilon$ is extended to $\mathbb{Z}_p$ and $P((x, y)z_p) = z_p^{k-2}P(x, y)$, we obtain

$$\psi_{N,\varepsilon}^{-1}(z)\psi_{\varepsilon,\varepsilon}^{-1}(z)\psi_{\varphi}(u) \cdot \int_{\mathbb{A}^\times \times \mathbb{Z}_p^\times} \varepsilon(y)(z_p) \cdot P((x, y)u^{-1}_p z_p) d(\varphi(\tilde{b}))(x, y)$$

$$= \psi_{N,\varepsilon}^{-1}(z)\varepsilon_{\varepsilon}(z)\psi_{\varepsilon,\varepsilon}^{-1}(z)\psi_{\varphi}(u) z_p^{k-2} \int_{\mathbb{A}^\times \times \mathbb{Z}_p^\times} \varepsilon(y) d(\varphi(\tilde{b}))(x, y)$$

The equivariance with respect to the $T_q$ operators is obvious, as well as that for the operators $U_q$ with $q \neq p$ (also for $\tilde{U}_\varepsilon$). To prove the equivariance at $p$ it is enough to note that we have

$$\nu_{k,\varepsilon}(\varphi\big|_{\phi}(1, p) \big) \big(\tilde{b} \big(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\big) \big) \big)(P) = \int_X \chi_p \zeta_p \cdot \zeta_p^{\times}(x, y) \varepsilon(y) P(x, y) d \left( \varphi\big|_{\phi}(1, p) \big) \big(\tilde{b} \big(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\big) \big) \big)(x, y) = 0$$

for $\chi_p \zeta_p \cdot \zeta_p^{\times}(x, y)$ the characteristic function of $p\zeta_p \cdot \zeta_p^{\times}$.

We must now investigate the properties of the space $S_2(U_0, \psi, \mathcal{M}(X, A))$. We begin by noticing that the action on $\mathcal{H}(X, A)$ is exactly the one induced by the right action, defined by right multiplication, of $M_2(\mathbb{Z}_p)$ on $X$. We proceed similarly to Proposition 7.5 of [20] or Chapter 6 of [9] and, denoting by $X_n$ the set of primitive vectors in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, we recover $X = \lim X_n$, with respect to the canonical projection maps. We obtain then $\mathcal{M}(X, A) = \lim \mathcal{M}(X_n, A)$ (see e.g. Section 7 of [22]). Since $X_n$ is a finite set, $\mathcal{M}(X_n, A)$ is identified with the space $\text{Hom}(X_n, A)$ of step functions. The action of $U_0$ on $X_n$ is transitive and the stabilizer of $(0, 1)$ is

$$\text{Stab}_{U_0}(\{(0, 1)\}) = \{ \gamma \in U_0 \mid (0, 1)\gamma = (0, 1) \} = \{(e, d) = (0, 1)\} = U_n.$$
ring of finite adèles away from \( p \). By Shapiro’s Lemma we obtain
\[
S_2(U_0, 1, \mathcal{M}(X_n, A)) \cong \left( \text{Hom}_\mathcal{O}(\mathcal{O}[\hat{B}^{p, \infty}], \mathcal{M}(X_n, A)) \right) \simeq S_2(U_0, 1, A),
\]
as well as the analogous isomorphism when we consider a character \( \psi \). Equation (4.12) implies that
\[
S_2(U_0, \psi, \mathcal{M}(X, A)) = \lim_{\leftarrow} S_2(U_n, \psi, A),
\]
where the identification is \( H^\mathcal{B}_{\text{nc}} \)-equivariant. We hence deduce that \( S_2(U_0, \psi, \mathcal{M}(X, A)) \) is a compact \( \mathcal{O} \)-module, since \( \mathcal{O} \) is \( p \)-adically complete and each \( S_2(U_n, \psi, A) \) is a finitely generated free \( \mathcal{O} \)-module. This allows us to define its ordinary part \( S_2(U_0, \psi, \mathcal{M}(X, A))^{\ord} \) as usual (see Section 2.4 of [21] and the references therein) as its direct summand on which the Hecke operator \( T_p \) acts invertibly. We shorten the notation and denote by \( \mathcal{W} \) the space \( S_2(U_0, \psi, \mathcal{M}(X, \mathcal{O}))^{\ord} \). In particular, the Hecke-equivariance in the inverse limit construction of \( S_2(U_0, \psi, \mathcal{M}(X, \mathcal{O}))^{\ord} \), implies that \( S_2(U_0, \psi, \mathcal{M}(X, \mathcal{O}))^{\ord} = \lim_{\leftarrow} S_2(U_n, \psi, A) \), where \( T_p \) is replaced by \( U_p \) on each component of the inverse limit. Proposition 4.1.2 shows that the specialization maps descend to Hecke-equivariant specialization maps between the ordinary components,
\[
\nu_{k, \varepsilon}^{\ord} : \mathcal{W} \longrightarrow S_k(U_n, \psi, \mathcal{O}(\varepsilon))^{\ord},
\]
with the same definition of \( \nu_{k, \varepsilon} \) and where \( \mathcal{O}(\varepsilon) \) is the finite extension of \( \mathcal{O} \) generated by the values of \( \varepsilon \).

**Notation 4.1.3.** We need to introduce some more notation.

(a) For any \( m \geq 1 \) and any character \( \chi : \mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}_p}^\times \), let \( \Psi_{m, \chi} : \chi \longrightarrow \overline{\mathbb{Q}_p}^\times \) such that
\[
\Psi_{m, \chi}(x, y) = \begin{cases} 
\chi(y) & \text{if } x \in p^m \mathbb{Z}_p, \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( \Psi_{m, \chi} \) is homogeneous of degree \( \chi \) for each \( m \).

(b) Let \( \Lambda = \mathcal{O}[1 + p \mathbb{Z}_p] \) be the extension of the classical Iwasawa algebra \( \mathbb{Z}_p[1 + p \mathbb{Z}_p] \), obtained by \( \mathcal{O} \). We set \( \mathcal{W}_\Lambda := \mathcal{W} \otimes_{\Lambda} \mathcal{O} \) for any \( \Lambda \)-algebra \( \mathcal{O} \).

(c) We say that a homomorphism \( \kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p} \) is an arithmetic homomorphism if its restriction to \( \mathbb{Z}_p^\times \) is of the form \( \kappa|_{\mathbb{Z}_p^\times}(x) = x^k \varepsilon(x) \) for \( k \geq 2 \) and \( \varepsilon : \mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}_p}^\times \) a character which factors through \( \mathbb{Z}_p^\times/(1 + p^n \mathbb{Z}_p) \), with \( n \) minimal. In this situation, we say that \( \kappa \) has weight \( k \) and character \( \varepsilon \) of conductor \( p^n \).

**Lemma 4.1.4.** Let \( \kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p} \) be an arithmetic homomorphism of weight \( k \) and character \( \varepsilon \) of conductor \( p^n \). Let \( F_\kappa \) be the field extension of \( F \) containing the values of \( \kappa \). The map \( \nu_{k, \varepsilon}^{\ord} \) induces the injective Hecke-equivariant homomorphism
\[
\nu_{k, \varepsilon}^{\ord} : \mathcal{W} \mathcal{R}/\mathcal{P}_\kappa \mathcal{W} \mathcal{R} \hookrightarrow S_k(U_n, \psi, \mathcal{F}_\kappa)^{\ord},
\]
where \( \mathcal{P}_\kappa \) is the kernel of \( \kappa \) in \( \mathcal{R} \).

**Proof.** We begin noting that \( \mathcal{P}_\kappa S_2(U_0, \psi, \mathcal{M}(X, \mathcal{O})) = S_2(U_0, \psi, \mathcal{P}_\kappa \mathcal{M}(X, \mathcal{O})) \), as it can be seen by applying twice Lemma 1.2 of [1] to \( S_2(U_0, \psi, \mathcal{M}(X, \mathcal{O})) = H^0(U_0^p, H^1(F[\hat{B}^{p, \infty}], \mathcal{M}(X, \mathcal{O}))) \). We prove now that \( \mathcal{P}_\kappa \mathcal{W} = \ker(\rho_{k, \varepsilon}^{\ord}) \). Let \( \varphi \in \mathcal{P}_\kappa \mathcal{W} \); therefore \( \varphi(\tilde{b}) \) lies in \( \mathcal{P}_\kappa \mathcal{M}(X, \mathcal{O}) \) for any \( \tilde{b} \in B^\times \setminus \hat{B}^\times \). Lemma 6.3 of [9] shows that \( \varphi(\tilde{b}) \in \mathcal{P}_\kappa \mathcal{M}(X, \mathcal{O}) \) if and only if \( \varphi(\tilde{b})(f) = 0 \) for each homogeneous function of degree \( \kappa \). For each \( P \in L_{k-2}(F_\kappa) \), \( \varepsilon(y) P(x, y) \) is homogeneous of degree \( \kappa \) and hence \( \rho_{k, \varepsilon}^{\ord}(\varphi(\tilde{b}))(P) = 0 \) for each \( \tilde{b} \) and \( P \). Take now \( \varphi \in \ker(\rho_{k, \varepsilon}^{\ord}) \) and let \( m \geq 1 \). Since \( T_p \) is invertible, let \( \mu \in \mathcal{W} \) be such that \( T_p^m \mu = \varphi \). Let \( \gamma_a := \binom{a}{0} \) for \( a = 0, \ldots, p - 1 \) and \( \gamma_\infty := \binom{1}{0} \). The definition of the Hecke operator \( T_p \) in Section 3.1 does come from the coset decomposition \( U_0 \gamma_\infty U_0 = \bigsqcup_{a=0,\ldots,p-1} \gamma_a U_0 \). Then \( T_p^m \) corresponds to a decomposition of the form \( \bigsqcup_{a=1}^{m} \gamma_{m,i} U_0 \), where each \( \gamma_{m,i} \) is a product of \( m \) matrices \( \gamma_a \).
for $\alpha = 0, \ldots, p - 1, \infty$. We compute
\begin{equation}
(4.17) \quad \int_{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \Psi_{m,k}(x, y)d(\varphi(\tilde{b}))(x, y) = \sum_i \int_{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \Psi_{m,k_i}((x, y)\gamma_{m,i})d(\mu(\tilde{b} \cdot \gamma_{m,i}))(x, y).
\end{equation}
For each $m$, $\Psi_{m,k}(x, y)\gamma_\infty = 0$ thus, $\Psi_{m,k}(x, y)\gamma_{m,i} = 0$ whenever $\gamma_m$ contains a copy of $\gamma_\infty$. Therefore, only the matrices $\gamma_{m,i} = \Pi_{j=0, m-1} \gamma_{a_j} = \left( \left\lfloor \sum_i \alpha_i^j \right\rfloor \right)$ contribute to the integral and we recognize that
\begin{equation}
(4.18) \quad \sum_i \int_{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \gamma_{m,i} \cdot (\varepsilon(y)y^{k-2})d(\mu(\tilde{b} \cdot \gamma_{m,i}))(x, y)
= \int_{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \varepsilon(y)y^{k-2}d(\mu(\tilde{b}))(x, y) = U_p^m \nu_{k,\varepsilon}(\mu(\tilde{b}))(y^{k-2}).
\end{equation}
By construction, $0 = \nu_{k,\varepsilon}^{ord}(\varphi(\tilde{b})) = \nu_{k,\varepsilon}^{ord}(T_p U(\psi)) = U_p^m \nu_{k,\varepsilon}^{ord}(\mu(\tilde{b}))$ and since $U_p$ is invertible on the space $S_k(U_n, \psi, F)_{ord}$, $\nu_{k,\varepsilon}^{ord}(\mu(\tilde{b})) = 0$ and hence $\nu_{k,\varepsilon}^{ord}(\varphi(\tilde{b}))(y^{k-2}) = 0$. Lemma 6.3 of [9] implies that $\varphi(\tilde{b}) \in \mathcal{P}_\kappa \mathcal{W}$. \hfill \blacksquare

**Remark 4.1.5.** As in the case of Eichler orders, the space of quaternionic modular forms $\mathcal{Z}_k(U_n, \tilde{\psi}, \mathcal{O})$ is often finitely generated over $\mathbb{Z}_p$ and free, as it follows from the discussion in Section 2.3. In particular, this holds true under the conditions discussed in Proposition 2.3.3. Therefore, Lemma 4.1.4 implies that $\mathcal{W}_R/\mathcal{P}_\kappa \mathcal{W}_R$ is $\mathbb{Z}_p$-finitely generated and free. The discussion in Section 2.3 shows also that $S_2(U_0, \psi, \mathcal{M}(X, A))$ is often $\hat{\Lambda}$-free and finitely generated, once again, for example under the conditions in Proposition 2.3.3. One can argue as in the proofs of Theorem 10.1, Corollary 10.3, and Corollary 10.4 of [13], since the results proved there for quaternionic modular forms over definite quaternion algebras hold in more generality for special orders which are split at the interpolation prime $p$ (see also Remark 4.2.3).

4.2. The proof of the control theorem. As in Section 3.4, we fix a $p$-ordinary newform $f$ in $S_2^{new}(\Gamma_1(Np^r)), \psi, \mathcal{C})$, for $n \geq 1$ and with $\psi$ a Dirichlet character modulo $Np^n\ell^r$ satisfying Assumptions 2.1.2 and 2.1.4. We also assume that its associated $p$-adic Galois representations is residually absolutely irreducible and $p$-distinguished. Let $f_\infty : H^B_{ord}(F) \longrightarrow \mathcal{R}$ the homomorphism associated in Section 3.4 with the Hida family passing through $f$. Recall that $F = \mathbb{Q}_p(f)$ and that $\mathcal{O}$ is its ring of integers. For any $\mathcal{P}_\kappa$ as in the above Lemma 4.1.4, we denote by $f_\kappa^B$ the composition
\begin{equation}
(4.19) \quad f_\kappa^B : H^B_{univ} \longrightarrow H^B_{ord}(F) \xrightarrow{f_\infty} \mathcal{R} \longrightarrow \mathcal{R}_{\mathcal{P}_\kappa},
\end{equation}
where the first map is the compatible morphism of Section 3.2 and the last map is the one to the localization of $\mathcal{R}$ at the prime $\mathcal{P}_\kappa$. We recall that $\mathcal{R}$ is a $\hat{\Lambda}$-algebra; we identify any $\hat{\Lambda}$-algebra as a $\Lambda$-algebra via the inclusion $\Lambda \hookrightarrow \hat{\Lambda}$ and write
\begin{equation}
(4.20) \quad \mathcal{W}_\kappa := (\mathcal{W} \otimes_{\hat{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}) [f_\kappa^B]
\end{equation}
for the isotypic component of the $\mathcal{R}_{\mathcal{P}_\kappa}$-module $\mathcal{W} \otimes_{\hat{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}$, where the Hecke operators act as determined by $f_\kappa^B$.

**Proposition 4.2.1.** With the notation of Lemma 4.1.4, there is an induced injective homomorphism
\begin{equation}
(4.21) \quad \nu_\kappa : \mathcal{W}_\kappa/\mathcal{P}_\kappa \mathcal{W}_\kappa \longrightarrow (S_k(U_n, \psi, F)_{ord}) [f_\kappa],
\end{equation}
for $f_\kappa$ the weight-$\kappa$ specialization of $f_\infty$.

**Proof.** The proof is the same as of Proposition 3.5 in [21], since it does not depend on the choice of the quaternionic order. \hfill \blacksquare
As one can note from Theorem 3.3.2, the case of level \(\ell^2\) and trivial character has to be handled with more care. The theory of Hida families for classical modular forms is well known and we can restrict our attention to the Hecke-submodules \(S_{k}^{\text{new}}(\Gamma_1(Np^n\ell), \psi, F)^{\text{ord}}\) with \(\psi\) a Dirichlet character modulo \(Np^n\), with \(n \geq 1\). We do not provide details here, but we refer to Chapter 7 of [14] and Section 2 of [21]. We construct the space of \(\Lambda\)-adic modular newforms, level \(Np^n\ell\) and character \(\psi\), as \(W_{k} := \lim S_{k}^{\text{new}}(\Gamma_1(Np^n\ell), \psi, F)^{\text{ord}}\). Moreover, we can twist its Hecke action by the character \(\tau\) obtaining the corresponding space \(W_{k}(\tau) := \lim S_{k}^{\text{new}}(\Gamma_1(Np^n\ell), \psi, F)^{\text{ord}}\). As in Section 3.2 we have an action of the universal Hecke algebra \(H_{\text{univ}}\) on \(W_{k}(\tau)\). In particular, taking \(f\) in \(S_{k}^{\text{new}}(\Gamma_1(Np^n\ell), \psi, F)^{\text{ord}}\), the module \(\left(W_{k}(\tau) \otimes_{\Lambda} \mathcal{R}_{\mathcal{P}_k}\right)[f]\) is a free rank-1 \(\mathcal{R}_{\mathcal{P}_k}\)-module (see Proposition 2.17 and the proof of Theorem 2.18 in [21]).

**Lemma 4.2.2.** Assume \(W\) to be \(\Lambda\)-free and finitely generated (see Remark 4.1.5). Suppose that \(f \in T_{2}(n, r, \psi)\) and set, for any arithmetic homomorphism \(\kappa = (k, \varepsilon)\),

\[
W_{\kappa} := \begin{cases} \overline{W}_{\kappa} \oplus \left(W_{k}(\tau) \otimes_{\Lambda} \mathcal{R}_{\mathcal{P}_k}\right)[f_{\kappa}] & \text{if } r = 1 \text{ and } \psi_{\kappa} \text{ is the trivial character}, \\ \overline{W}_{\kappa} & \text{otherwise}, \end{cases}
\]

where we let \(H_{\text{univ}}\) act on \(W_{\kappa}\) via the homomorphism \(H_{\text{univ}} \to H_{\text{univ}}^{\Lambda}\) induced by the Jacquet–Langlands correspondence. Then \(W_{\kappa}\) is a free rank-2 \(\mathcal{R}_{\mathcal{P}_k}\)-module.

**Proof.** We start dealing with the case \(W_{\kappa} = \overline{W}_{\kappa}\). We consider the \(p\)-divisible abelian group (cf. Remark 4.1.5 and Section 2.3) \(V := \lim S_{2}^{\text{ord}}(U_n, \wtilde{\psi}, F/\mathcal{O})\), where the inductive limit is taken with respect to the restriction maps induced by the inclusions \(U_{n+1} \subset U_n\). The Hecke and diamond operators (at least away from \(\ell\)) act on \(V\) since, as in the case of Eichler orders, the restriction maps in [13] (see Eqs. (2.9a), (2.9b) and (3.5)) are compatible with the Hecke action. Taking the Pontryagin dual of \(V\) we obtain the Hecke-equivariant isomorphism \(\wtilde{V} \approx W\) (cf. Eqs. 4.12 and 4.13), which shows it is a free \(\Lambda\)-module of finite rank. We denote by \(F_{\kappa}\) the field extension of \(F\) generated by the values of \(\varepsilon\) and by \(O_{\kappa}\) its ring of integers, which we can assume to be finite flat over \(\mathbb{Z}_p\). Up to a scalar and up to taking the tensor product by \(O_{\kappa}\), we can suppose \(F_{\kappa}\) to have coefficients in \(O\). We observe that \(\overline{W}_{\kappa} = W[f_{\kappa}] \otimes_{\Lambda} \mathcal{R}_{\mathcal{P}_k}\), as the action of the Hecke algebra is on the first component and the tensor product is just an extension of scalars. We can hence apply Theorem 9.4 of [13] (cf. Remark 4.2.3) to \(W[f_{\kappa}]\) and obtain the isomorphism of \(h_{k}^{\text{ord}}(O)\)-modules,

\[
W[f_{\kappa}] \cong \wtilde{V}[f_{\kappa}] \cong S_{k}^{\text{new}}(U_n, \wtilde{\psi}, F/\mathcal{O})[f_{\kappa}].
\]

We remark that the last Hecke-equivariant isomorphism in the above Eq. (4.23) (as well as in Eq. (4.24)), comes from the restriction to \(T_{k}(n, r, \psi, \varepsilon)\) of the Pontryagin duality established in Lemma 7.1 of [11]; under the hypotheses of Lemma 2.3.3 one has the isomorphism \(S_{k}(U_n, \wtilde{\psi}, F/\mathcal{O}) \cong S_{k}(U_n, \wtilde{\psi}, O) \otimes F/\mathcal{O}\), as in the proof of Theorem 10.1 in [13], and then Proposition 3.3.6 recovers the needed Hecke-isomorphism for quaternionic modular forms. Similarly to the above discussion for \(W^{t}\), we can follow Section 2 of [21] and construct the interpolation module \(W^{t} = \lim S_{2}^{\text{ord}}(\Gamma_1(Np^n\ell^2), \psi, F)^{\text{ord}}\), relative to the ordinary subspaces \(S_{k}(\Gamma_1(Np^n\ell^2), \psi, F)^{\text{ord}}\). We notice that under the hypothesis of Proposition 2.3.3, the space \(W^{t}\) is free of finite rank. In particular, we can reproduce the above chain of isomorphisms and obtain \(h_{k}^{\text{ord}}(O)\)-isomorphisms

\[
W^{t}[f_{\kappa}] \cong S_{k}(\Gamma_1(Np^n\ell^2), \psi, F)^{\text{ord}}[f_{\kappa}] \cong T_{k}(n, r, \psi, \varepsilon)^{\text{ord}}[f_{\kappa}].
\]

Applying Propositions 3.3.6 and 3.3.8, we deduce the isomorphism of \(h_{L}^{\text{ord}}(O)\)-modules,

\[
W[f_{\kappa}] \cong 2W^{t}[f_{\kappa}].
\]
Tensoring over $\Lambda$ with $\mathcal{R}_{P_u}$, we obtain the isomorphism of $h^{T,\ord}_\infty(\mathcal{O}) \otimes_\Lambda \mathcal{R}_{P_u}$-modules,
\begin{equation}
\mathbb{W}_\kappa \cong 2 \left( \mathbb{W}^{\mathfrak{r}} \otimes_\Lambda \mathcal{R}_{P_u} \right) [f_\kappa].
\end{equation}
As in the proof of Theorem 2.18 of [21], Proposition 2.17 of loc. cit. guarantees that $\left( \mathbb{W}^{\mathfrak{r}} \otimes_\Lambda \mathcal{R}_{P_u} \right) [f_\kappa]$ is a free $\mathcal{R}_{P_u}$-module of rank 1, therefore $\mathbb{W}_\kappa$ is a free $\mathcal{R}_{P_u}$-module of rank 2.

The case of $r = 1$ and trivial character at $\ell$ is carried out similarly, once we define the $p$-divisible abelian group
\begin{equation}
\mathcal{V} := \operatorname{lim}_{\leftarrow} \left( S^\text{new}_2(U_n, \widetilde{\psi}, F/\mathcal{O})^{\ord} \oplus S^\text{new}_2(\Gamma_1(\mathfrak{N}p^n\ell), \psi, F/\mathcal{O})(\overline{\tau})^{\ord} \right),
\end{equation}
whose Pontryagin dual is $\mathbb{W} \oplus \mathbb{W}^{\ell}(\overline{\tau})$.

**Remark 4.2.3.** (a) The congruence subgroup we consider, away from $\ell$, is the one denoted by $V(Np^n)$ in [13] and one passes from this choice to the one used there by changing all the actions via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(b) We point out that Theorem 9.4 of [13] is stated under more strict hypotheses but, in the case of definite quaternion algebras, such hypotheses can be relaxed; this has been already noticed in [21] and [17] in order to work with Eichler orders for algebras over $\mathbb{Q}$, but Theorem 9.4 of [13] holds true also for special orders. This is due to the degree of generality in which the results of Chapter 8 of [13] are proved (as well as Lemma 2.1.3), together with the necessity of a controlled behavior only at the interpolation prime $p$. Let us remark that we did not take into account the case of indefinite algebras, but that it seems to require a generalization of the spectral sequences approach contained in Chapter 9 of [13].

We can finally state the sought for Hida control theorem in the case of special orders of level $\ell^{2\mathfrak{r}}$.

**Theorem 4.2.4** (Control theorem for special orders). With the above notation, suppose $f$ to be twist-minimal at $\ell$. For any arithmetic homomorphism $\kappa : \mathcal{R} \rightarrow \overline{\mathbb{Q}}_p$, the map $\nu_\kappa$ of Proposition 4.2.1 induces an isomorphism of 2-dimensional $F_\kappa$-vector spaces
\begin{equation}
\mathbb{W}_\kappa/\mathcal{P}_\kappa \mathbb{W}_\kappa \xrightarrow{\cong} \left( S^\text{new}_k(\Gamma_1(\mathfrak{N}p^n\ell), \psi, F_\kappa) \otimes (\overline{\tau}) \right) [f_\kappa].
\end{equation}
If $r = 1$, $f$ has trivial character at $\ell$ and lies in $S^\text{new}_k(\Gamma_1(\mathfrak{N}p^n\ell), \psi, F_\kappa) \otimes (\overline{\tau})$ (in particular, it is not twist-minimal at $\ell$), then the above isomorphism still holds, but the $F_\kappa$-vector spaces are 1-dimensional.

**Proof.** Suppose $f$ to be twist-minimal. Because of Propositions 4.2.1 and 3.3.8 we know that
\begin{equation}
\dim_{F_\kappa} \left( \mathbb{W}_\kappa/\mathcal{P}_\kappa \mathbb{W}_\kappa \right) \leq 2
\end{equation}
and thus it is enough to prove the opposite inequality. Lemma 4.2.2 shows that $\mathbb{W}_\kappa$ is a free $\mathcal{R}_\kappa$-module of rank 2. The case of $r = 1$, trivial character at $\ell$ and $f \not\in S^\text{new}_k(\Gamma_1(\mathfrak{N}p^n\ell), \psi, F_\kappa) \otimes (\overline{\tau})$ follows similarly. The remaining case accounts to the fact that the Jacquet–Langlands correspondence preserves twists. \hfill \blacksquare

We can consider the finitely generated $\mathcal{R}$-module
\begin{equation}
\mathbb{W}_\infty := \begin{cases} 
\left( \mathbb{W} \oplus \mathbb{W}^{\ell}(\overline{\tau}) \otimes_\Lambda \mathcal{R} \right) [f_\infty] & \text{if } r = 1 \text{ and } \psi_\ell \text{ is the trivial character}, \\
\left( \mathbb{W} \otimes_\Lambda \mathcal{R} \right) [f_\infty] & \text{otherwise}.
\end{cases}
\end{equation}
Proceeding similarly as in the proof of the above theorem we notice that $\mathbb{W}_\infty \otimes \operatorname{Frac}(\Lambda)$ is a 2-dimensional $\mathcal{K}$-vector space, where $\mathcal{K}$ is the finite field extension of $\operatorname{Frac}(\Lambda)$ called the primitive component associated with the Hida family $f_\infty$ (see Section 3 in [12] in particular, Theorem 3.5 and also Theorem 2.6a of [9]).

As noticed in Section 2.2 of [21], we point out that $\mathcal{R}$ is the integral closure of $\Lambda$ in $\mathcal{K}$. We can then formulate Theorem 4.2.4 highlighting this global $\mathcal{R}$-module.
Theorem 4.2.5. With the above notation, suppose $f$ to be twist-minimal at $\ell$. For any arithmetic homomorphism $\kappa : R \rightarrow \mathbb{Q}_p$, the map $\nu_\kappa$ of Proposition 4.2.1 induces an isomorphism of 2-dimensional $F_\kappa$-vector spaces

$$W_{\infty} \otimes_R \mathcal{R}_{P_\kappa}/P_\kappa \mathcal{R}_{P_\kappa} \xrightarrow{\cong} (S_k(U_n, \psi_\kappa, F_\kappa)^{ord}) [f_\kappa].$$

If $r = 1$, $f$ has trivial character at $\ell$ and lies in $S_k^{new}(\Gamma_1(Np^d\ell), \psi_\kappa)$ (in particular, it is not twist-minimal at $\ell$), then the isomorphism of 1-dimensional $F_\kappa$-vector spaces holds:

$$((W \otimes_\Lambda \mathcal{R}) [f_{\infty}] ) \otimes_R \mathcal{R}_{P_\kappa}/P_\kappa \mathcal{R}_{P_\kappa} \xrightarrow{\cong} (S_k(U_n, \psi_\kappa, F_\kappa)^{ord}) [f_\kappa].$$

Corollary 4.2.6. Let $f_{\infty}$ be a primitive Hida family of tame level $N\ell^{2r}$, $r \geq 1$, tame character $\psi$ with its $\ell$-component, $\psi_\ell$, as in Assumption 2.4.2. Suppose moreover $f_{\infty}$ to be twist-minimal at $\ell$. Then there exist two $\mathcal{R}$-linearly independent elements $\phi_{1{\infty}}^1$ and $\phi_{1{\infty}}^2$ in $(W \otimes_\Lambda R) [f_{\infty}]$, which form a basis for $((W \otimes_\Lambda \mathcal{R}) [f_{\infty}] ) \otimes K$. Moreover, for any arithmetic homomorphism $\kappa$, $\nu_\kappa(\phi_{1{\infty}}^1)$ and $\nu_\kappa(\phi_{1{\infty}}^2)$ form a $F_\kappa$-basis for $(S_k(U_n, \psi_\kappa, F_\kappa)^{ord}) [f_\kappa]$.

Definition 4.2.7. We denote by $W_{f_{\infty}}$ the $\mathcal{R}$-linear span of $\phi_{1{\infty}}^1$ and $\phi_{1{\infty}}^2$ and call it the subspace of special quaternionic Hida families associated with $f_{\infty}$.

4.3. A small remark on related works and open questions. The mathematical literature about this situation of higher ramification at the primes at which the quaternion algebra ramifies is quite meager. Excluding the (singular and collective) works of Pizer, Hijikata and Shemanske, there are few other works considering special orders and they all share working with indefinite algebras. We already referred to [19], but we wish to point the reader’s attention also to the two works [5] and [8]. In particular, in [26], Pizer defines certain local operators acting on the quaternionic modular forms. The present note leaves unanswered whether the two linearly independent quaternionic modular forms, and then the two Hida families, can be distinguished via some of these local operators. We wish to address carefully this question in the near future.

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