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| 氏名 | Author |
|-------|--------|
| 田原, 伸彦 |

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An augmentation of the phase space of the system of type $A_4^{(1)}$

($A_4^{(1)}$型微分方程式系の相空間の拡張について)
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An augmentation of the phase space of the system of type \(A_{4}^{(1)}\)

Nobuhiko TAHARA

Abstract. We investigate the differential system with affine Weyl group symmetry of type \(A_{4}^{(1)}\) and construct a space which parametrizes all meromorphic solutions of it. To demonstrate our method based on singularity analysis and affine Weyl group symmetry, we first study the system of type \(A_{2}^{(1)}\), which is equivalent to the fourth Painlevé equation, and obtain the space which augments the original phase space of the system by adding spaces of codimension 1. In the case of type \(A_{4}^{(1)}\), codimension 2 spaces should be added to the phase space of the system of type \(A_{4}^{(1)}\) in addition to codimension 1 spaces.

1. Introduction

The differential system of type \(A_{l}^{(1)}\) (\(l = 2, 3, \ldots\)), proposed by Noumi and Yamada [3], is a system of autonomous ordinary differential equations for \((l + 1)\) unknown functions \(f_0, \ldots, f_l\) with complex parameters \(\alpha_0, \ldots, \alpha_l\) satisfying \(\alpha_0 + \cdots + \alpha_l = 1\). The system has the symmetry of the affine Weyl group of type \(A_{l}^{(1)}\), where \(\alpha_0, \ldots, \alpha_l\) are considered as simple roots of the affine root system of type \(A_{l}^{(1)}\). It should be noted that the system of type \(A_{l}^{(1)}\) is equivalent to the fourth Painlevé equation when \(l = 2\), and to the fifth when \(l = 3\).

The purpose of this paper is to construct a parameter space of all meromorphic solutions (including holomorphic solutions) of the system of type \(A_{4}^{(1)}\). We call this space an augmented phase space, and the process to obtain an augmented phase space an augmentation of the original phase space, which is the parameter space of all holomorphic solutions. Such spaces have been constructed in the case of Painlevé equations as “spaces of initial conditions” [4]. They are constructed by successive blowing-up procedures at accessible singular points. But in our case, namely for the system of type \(A_{4}^{(1)}\), the calculations in blowing-up procedures are very complicated and we take another approach based on singularity analysis, that is, the construction of the formal meromorphic solutions of the system in the form of Laurent expansions at an arbitrary point containing several arbitrary constants. We also analyse the noteworthy connection between the formal meromorphic solutions and the affine Weyl group symmetry the system has.
In the case of \( l = 2n \) \((n = 1, 2, \ldots)\), the differential system of type \( A_l^{(1)} \) is defined by

\[
(A_l^{(1)}): \quad f'_i = f_i \left( \sum_{1 \leq r < n} f_{i+r-1} + \sum_{1 \leq r < n} f_{i+r+1} \right) + \alpha_i,
\]

where \( i = 0, \ldots, 2n \) and ’ stands for the derivation with respect to the independent variable \( t \). The indices of \( f \) and \( \alpha \) should be read as elements of \( \mathbb{Z}/(l + 1)\mathbb{Z} \). The system \( (A_l^{(1)}) \) gives a structure of a differential field to the field of rational functions \( \mathbb{C}(\alpha; f) \) of \( \alpha = (\alpha_0, \ldots, \alpha_l) \) and \( f = (f_0, \ldots, f_l) \) and admits an action of the extended affine Weyl group \( \overline{W}(A_l^{(1)}) \) of type \( A_l^{(1)} \) as Bäcklund transformation group. Here Bäcklund transformation means an automorphism of the differential field \( \mathbb{C}(\alpha; f) \) which commutes with the derivation. The group \( \overline{W}(A_l^{(1)}) \) is generated by the automorphisms \( s_0, \ldots, s_l \) and \( \pi \) with fundamental relations

\[
s_i^2 = 1, \quad s_isjs_i = s_jsjs_i \quad (j \neq i \pm 1), \quad s_is_j = s_js_i \quad (j \neq i \pm 1), \quad \pi^{l+1} = 1, \quad \pi s_i = s_i\pi
\]

for \( i, j = 0, 1, \ldots, l \). The actions of \( s_i \) and \( \pi \) on \( \alpha_j \) and \( f_j \) are given by

\[
s_i(\alpha_j) = \alpha_j - \alpha_ia_i, \quad s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad \pi(f_j) = f_{j+1}
\]

for \( i, j = 0, 1, \ldots, l \), where \( A = (a_{ij})_{0 \leq i, j \leq l} \) is the generalized Cartan matrix of type \( A_l^{(1)} \):

\[
a_{jj} = 2, \quad a_{ij} = -1 \quad (j = i \pm 1), \quad a_{ij} = 0 \quad (j \neq i, i \pm 1),
\]

and \( U = (u_{ij})_{0 \leq i, j \leq l} \) is the orientation matrix:

\[
u_{ij} = \pm 1 \quad (j = i \pm 1), \quad u_{ij} = 0 \quad (j \neq i \pm 1).
\]

In this paper we investigate the system \( (A_l^{(1)}) \) in the case of \( l = 2 \) and \( l = 4 \), namely, the system \( (A_2^{(1)}) \) and the system \( (A_4^{(1)}) \), respectively. Although the main object of the paper is the system \( (A_4^{(1)}) \), we first study the system \( (A_2^{(1)}) \) in Section 2, to make clear what our method is. Section 2 is divided into three subsections. In the first subsection, we obtain all formal meromorphic solutions of the system \( (A_2^{(1)}) \), observing where arbitrary constants appear. These solutions are classified into three families of formal meromorphic solutions, each of which corresponds to Bäcklund transformation \( s_i \) for some \( i = 0, 1, 2 \). In the second subsection, we choose an appropriate coordinate system for each family of formal meromorphic solutions in order to extract arbitrary constants, or more precisely, in order that the new coordinates express the formal meromorphic solutions as formal holomorphic solutions and so that the arbitrary constants appear in their constant terms. By the use of such appropriate coordinate system, the convergence of formal meromorphic solutions is shown. In fact, for each family of meromorphic solutions, we obtain two different coordinate systems. In the third subsection, we construct a fiber space \( \mathbb{E} \) over
the space of parameters $\alpha$, so that each fiber of the space $E$ parametrizes all holomorphic solutions and meromorphic solutions. Holomorphic solutions correspond to the points of a 3-dimensional affine subspace of the fiber and three families of meromorphic solutions correspond to three 2-dimensional affine subspaces. We also study mappings from $E$ to itself associated with Bäcklund transformations. It should be noted that our coordinate systems of $E$ are convenient for the studying these mappings.

The latter sections from Section 3 to Section 5 are devoted to the study of the system $(A_4^{(1)})$. Sections 3, 4 and 5 are $A_4^{(1)}$ versions of Subsections 2.1, 2.2 and 2.3, respectively. In Section 3, we study formal meromorphic solutions of the system $(A_4^{(1)})$. There are fifteen families of formal meromorphic solutions, which are divided into three classes corresponding to $s_i$, $s_js_i$, and $s_k s_j s_i$, or rather the type $(i)$, $(ij)$ and $(ijk)$ for some $i, j, k = 0, \ldots, 4$. In Section 4, we choose a suitable coordinate system for each family of formal meromorphic solutions. The case of type $(i)$ is almost the same as for the system $(A_2^{(1)})$. In the case of type $(ij)$, we first apply the process corresponding to the type $(i)$ and then proceed to the next and final process. And in the case of type $(ijk)$, we apply the process of the type $(ij)$, and then proceed to the final process. In Section 5, we construct a fiber space $E$ over the space of parameters $\alpha$ of the system $(A_4^{(1)})$ in the same way as in Subsection 2.3, and study the mappings from $E$ to itself. In this case, each fiber $E(\alpha)$ consists of a 5-dimensional space, five 4-dimensional spaces and ten 3-dimensional spaces, any two of which do not intersect. The 5-dimensional space is a parameter space of the holomorphic solutions and each 4-dimensional or 3-dimensional space is a parameter space of a 4-parameter or a 3-parameter family of meromorphic solutions, respectively.

The results in Section 5 are easily translated to the so-called defining manifold (a fiber space over the space of independent variable $t$, whose fibers are the spaces of initial conditions) of the Hamiltonian system associated with the system $(A_4^{(1)})$. In the last section, Section 6, we give a list of local coordinate systems of the defining manifold and the form of the Hamiltonian functions on the charts.

2. The system of type $A_2^{(1)}$

The system of differential equations with affine Weyl group symmetry of type $A_2^{(1)}$ is explicitly written as

$$
\begin{align*}
    f_0' &= f_0(f_1 - f_2) + \alpha_0, \\
    f_1' &= f_1(f_2 - f_0) + \alpha_1, \\
    f_2' &= f_2(f_0 - f_1) + \alpha_2,
\end{align*}
$$

(2.1)

where $' = d/dt$. 


2.1. Formal meromorphic solutions

For an arbitrarily fixed $t_0 \in \mathbb{C}$, let us consider a formal meromorphic solution of the system (2.1) of the form

$$f_i = \sum_{n=-r}^{\infty} c_i^n T^n, \quad T := t - t_0 \quad (i = 0, 1, 2) \tag{2.2}$$

where $r$ is a positive integer and $c_{-r} = (c_{-r}^0, c_{-r}^1, c_{-r}^2) \neq (0, 0, 0)$.

Substituting this into (2.1) and comparing the coefficients of $T^{n-1}$ on both sides, we have

$$nc_i^n = \sum_{k=-r}^{n-r-1} c_i^k G_{n-k-1}^i + \delta_{0,n-1} \alpha_i \quad (i = 0, 1, 2) \tag{2.3}$$

for $n \geq -2r + 1$ where $G_n^i = c_n^{i+1} - c_n^{i+2}$ and $\delta_{., .}$ is the Kronecker delta. Note that $c_n = (c_n^0, c_n^1, c_n^2) = (0, 0, 0)$ for any $n \leq -r - 1$, by convention.

We first see $r = 1$ by deriving a contradiction. For this purpose, we assume $r > 1$ and look into the equations (2.3) for $n = -2r + 1, -2r + 2, \ldots, -r$:

$$0 = c_{-r}^i G_{-r-r}^i,$$

$$0 = c_{-r}^i G_{-r-r+1}^i + c_{-r+1}^i G_{-r-r}^i,$$

$$0 = c_{-r}^i G_{-r-r+2}^i + c_{-r+1}^i G_{-r-r+1}^i + c_{-r+2}^i G_{-r-r}^i,$$

$$\vdots$$

$$0 = c_{-r}^i G_{-r-2}^i + c_{-r+1}^i G_{-r-3}^i + \cdots + c_{-r}^i G_{-r-r+1}^i + c_{-r+2}^i G_{-r-r}^i,$$

$$-rc_{-r}^i = c_{-r}^i G_{-r-1}^i + c_{-r+1}^i G_{-r-2}^i + \cdots + c_{-r+2}^i G_{-r-r+1}^i + c_{-r+1}^i G_{-r-r}^i.$$  

The first equation, for $i = 0, 1, 2$, is solved as

$$c_{-r} = (a, a, a), \ (a, 0, 0), \ (0, a, 0), \ (0, 0, a),$$

where $a$ is an arbitrary non-zero constant. Note that $G_{-r}^i + c_{-r}^i \neq 0$ ($i = 0, 1, 2$) in every case of the values of $c_{-r}$. Then these equations enables us to find out either $c_{-r}^1 = 0$ or $G_{-r-1}^i = -r$, for each $i = 0, 1, 2$, as follows. If $c_{-r}^i = 0$ for some $i$, then $G_{-r}^i \neq 0$ by $G_{-r}^i + c_{-r}^i \neq 0$ and the above equations yield $c_{-r+1}^i = \cdots = c_{-r+2}^i = 0$. On the other hand, if $c_{-r}^i \neq 0$ for some $i$, we obtain $G_{-r}^i = 0$ from the first equation, and hence, by the other equations, $G_{-r+1}^i = G_{-r+2}^i = \cdots = G_{-r+2}^i = 0$ and $G_{-r-1}^i = -r$. Therefore $c_{-r}^i = 0$ for some $i$ implies $c_{-r}^i = 0$ and $c_{-r}^i \neq 0$ for some $i$ implies $G_{-r-1}^i = -r$.

Let us derive a contradiction in each case. In the case of $c_{-r} = (0, 0, a)$, for $a \neq 0$, we obtain $c_{-1}^0 = c_{-1}^1 = 0$, $G_{-2}^i = -r$ by the above and, since $G_{-2}^2 = c_{-1}^0 - c_{-1}^1$ by definition, $r = 0$, which contradicts the assumption $r > 1$. We have the contradiction in the case of $c_{-r} = (0, a, 0)$ or $c_{-r} = (a, 0, 0)$ similarly. In the case of $c_{-r} = (a, a, a)$, for $a \neq 0$, we have
\( G^0_{-1} = G^1_{-1} = G^2_{-1} = -r \), which contradicts \( G^0_{-1} + G^1_{-1} + G^2_{-1} = (c^1_{-1} - c^2_{-1}) + (c^2_{-1} - c^0_{-1}) + (c^0_{-1} - c^1_{-1}) = 0 \). Thus we have shown that \( r = 1 \).

Now we shall determine the coefficients \( c_n = (c^0_n, c^1_n, c^2_n) \) in (2.2) for \( n \geq -1 \) by the equations (2.3). For \( n = -1 \), we have

\[
(-1)c^i_{-1} = c^i_{-1}(c^i+1_{-1} - c^{i+2}_{-1}) \quad (i = 0, 1, 2)
\]

and it follows that

\[
c_{-1} = (-1, 0, 1), \quad (1, -1, 0), \quad (0, 1, -1).
\]

For \( n \geq 0 \) the equations (2.3) can be written as a system of linear equations

\[
(n - G^i_{-1})c^i_n - c^i_{-1}c^{i+1}_n + c^i_{-1}c^{i+2}_n = \sum_{k=0}^{n-1} c^i_k G^i_{n-k-1} + \delta_{0,n-1} \alpha_i \quad (i = 0, 1, 2)
\]

with respect to \( c_n = (c^0_n, c^1_n, c^2_n) \), and hence the coefficients \( c_i \) of the expansion (2.2) are successively determined as polynomials of \( \{c^i_k, \alpha_i; i = 0, 1, 2, k = 0, \ldots, n-1\} \) unless \( \det P_n = 0 \) where

\[
P_n := \begin{bmatrix}
    n - G^0_{-1} & -c^0_{-1} & c^0_{-1} \\
    c^1_{-1} & n - G^1_{-1} & -c^1_{-1} \\
    -c^2_{-1} & c^2_{-1} & n - G^2_{-1}
\end{bmatrix}.
\]

In the case that \( \det P_n = 0 \) for some \( n \), if the linear system has a solution, then it contains arbitrary constants, the number of which is equal to \( \dim \ker P_n \).

Let us observe the expansion (2.2) precisely in the case of \( c_{-1} = (-1, 0, 1) \). The expansions in the other cases are easily obtained by the use of cyclic rotations. We first note that

\[
P_n = \begin{bmatrix}
    n+1 & 1 & -1 \\
    0 & n-2 & 0 \\
    -1 & 1 & n+1
\end{bmatrix} \quad ; \quad \det P_n = (n+2)n(n-2).
\]

We can see that \( \dim \ker P_0 = 1 \) and the linear system for \( n = 0 \) is solved as \( c^0_1 = 0 \) and \( c^0_2 = c^2_0 \) which can be an arbitrary constant. The coefficients \( c^i_1 \) for \( i = 0, 1, 2 \) are uniquely determined depending on the value of \( c^0_0 = c^2_0 \). For \( n = 2 \), we can verify that \( \dim \ker P_2 = 1 \) and the solution \( c_2 = (c^0_2, c^1_2, c^2_2) \) of the linear system is determined so that \( c^1_2 \) is an arbitrary constant while the other \( c^0_2 \) and \( c^2_2 \) are uniquely depending on the value of \( c^2_2 \). The coefficients \( c_n \) for \( n \geq 3 \) are uniquely determined by \( c^0_0, c^1_2 \) and \( \alpha_i \). Thus we have obtained the expansion of the formal meromorphic solution as

\[
f_0 = -\frac{1}{T} + c^0_0 + \frac{(2\alpha_0 + 3\alpha_1 + \alpha_2) - (c^0_0)^2}{3} T + O(T^2),
\]

\[
f_1 = -\alpha_1 T + c^1_2 T^2 + O(T^3), \tag{2.4}
\]

\[
f_2 = \frac{1}{T} + c^0_0 + \frac{(\alpha_0 + 3\alpha_1 + 2\alpha_2) + (c^0_0)^2}{3} T + O(T^2),
\]
where \( c_0 \) and \( c_1 \) are arbitrary constants which are free of the system. This formal solution depends on two arbitrary constants and then it is called a 2-parameter family of formal solutions. Note that we do not consider the position of the pole, \( t_0 \), as an arbitrary constant, since the system (2.1) is autonomous.

Let us set

\[
\text{Res } f = (\text{Res}_{t=t_0} f_0, \text{Res}_{t=t_0} f_1, \text{Res}_{t=t_0} f_2)
\]

for a set of formal meromorphic functions \( f = (f_0, f_1, f_2) \) at \( t = t_0 \). Then the above solution (2.4) can be indicated by \( \text{Res } f = (-1, 0, 1) \). The other families of formal solutions are also indicated by \( \text{Res } f = (1, -1, 0) \) or by \( \text{Res } f = (0, 1, -1) \).

In the rest of this subsection, we mention the relations between the three 2-parameter families of formal meromorphic solutions and the Bäcklund transformations \( s_i \) for \( i = 0, 1, 2 \). For example, let \( f \) be the solution with \( \text{Res } f = (-1, 0, 1) \) and let \( g_i = s_i(f_i) \), \( \beta_i = s_i(\alpha_i) \) for \( i = 0, 1, 2 \) be its transform, namely, from (1.1), (1.2) and (1.3),

\[
g_0 = f_0 - \frac{\alpha_1}{f_1}, \quad g_1 = f_1, \quad g_2 = f_2 + \frac{\alpha_1}{f_1}; \quad \beta_0 = \alpha_0 + \alpha_1, \quad \beta_1 = \alpha_1, \quad \beta_2 = \alpha_2 + \alpha_1.
\]

Then \( g = (g_0, g_1, g_2) \) satisfies the system

\[
g'_i = g_i(g_{i+1} - g_{i+2}) + \beta_i, \quad (i = 0, 1, 2)
\]

and it has the series expansion given by

\[
g_0 = \left(c_0^0 + \frac{c_1^1}{\beta_1}\right) + O(T), \quad g_1 = 0 - \beta_1 T + c_2^1 T^2 + O(T^3), \quad g_2 = \left(c_0^0 - \frac{c_1^1}{\beta_1}\right) + O(T)
\]

without terms of negative powers of \( T \). This fact means that, in case of \( \alpha_1(= \beta_1) \neq 0 \), the formal solution with \( \text{Res } f = (-1, 0, 1) \) is related via Bäcklund transformation \( s_1 \) to the formal holomorphic solution of the system (2.1) with parameters \( s_1(\alpha) = \beta = (\beta_0, \beta_1, \beta_2) \) under the initial condition

\[
g_0(t_0) = c_0^0 + \frac{c_1^1}{\beta_1}, \quad g_1(t_0) = 0, \quad g_2(t_0) = c_0^0 - \frac{c_1^1}{\beta_1}.
\]

In this sense, the solution with \( \text{Res } f = (-1, 0, 1) \) corresponds to the Bäcklund transformation \( s_1 \) and we say that the formal solution is of type (1). Then the solutions with \( \text{Res } f = (1, -1, 0), (0, 1, -1) \) are of type (2) and (0), respectively. Generalizing the terminology, we say that the 3-parameter family of holomorphic solutions is of type (0) (see Table 1).
Table 1: Classification of the families of solutions of the system \((A_2^{(1)})\)

| type | Res \(f\) | corresponding BT | # of arbitrary constants |
|------|-----------|------------------|-------------------------|
| (∅)  | (0, 0, 0) | id               | 3                       |
| (1)  | (−1, 0, 1) | \(s_1\)          |                         |
| (2)  | (1, −1, 0) | \(s_2\)          | 2                       |
| (0)  | (0, 1, −1) | \(s_0\)          |                         |

2.2. Coordinates for formal meromorphic solutions

In this subsection, we prove the convergence of the formal meromorphic solutions \((2.4)\) by choosing suitable coordinate systems so that the arbitrary constants in the formal solutions are interpreted as initial conditions of the holomorphic differential systems in the new coordinate systems.

Let \(f = (f_0, f_1, f_2)\) be the solution \((2.4)\) with \(\text{Res } f = (−1, 0, 1)\). We first notice that \(f_0 + f_1 + f_2 = 2c_0^1 + T\) and that \(f_0\) has a simple pole while \(f_1\) has a simple zero at \(t = t_0\), or at \(T = 0\). Then \(1/f_0\) has a simple zero, and moreover, we see that \(f_0f_1\) is of the form

\[ f_0f_1 = \alpha_1 - (c_2^1 + \alpha_1 c_0^0)T + O(T^2). \]

Therefore in order that the arbitrary constant \(c_2^1\) appears in a constant term, we multiply \(\alpha_1 - f_0f_1\) by \(f_0\). Then we have

\[ f_0(\alpha_1 - f_0f_1) = (c_2^1 + \alpha_1 c_0^0) + O(T), \]

where \(c_2^1\) is contained in the initial value at \(t = t_0\). Based on such an observation, we introduce the transformation

\[ u_0 = 1/f_0, \quad u_1 = f_0(\alpha_1 - f_0f_1), \quad u_0 + u_1 + u_2 = f_0 + f_1 + f_2. \quad (2.5) \]

We can verify that the transformation \((2.5)\) changes the system \((2.1)\) into the holomorphic system

\[
\begin{align*}
    u'_0 &= 2u_0^3u_1 - (\alpha_0 + 2\alpha_1 - 1)u_0^2 + (u_1 + u_2)u_0 - 1, \\
    u'_1 &= -3u_0^2u_1^2 + (2\alpha_0 + 4\alpha_1 - 1)u_0u_1 - (u_1 + u_2)u_1 - (\alpha_0 + \alpha_1)\alpha_1, \\
    u'_0 + u'_1 + u'_2 &= \alpha_0 + \alpha_1 + \alpha_2. 
\end{align*}
\]

(2.6)

Now take a solution of the new system \((2.6)\) holomorphic at \(t = t_0\) with \(u_0(t_0) = 0, u_1(t_0) = h_1, u_2(t_0) = h_2\) by Cauchy’s existence and uniqueness theorem. Then, since

\[ u'_0(t_0) = -1, \quad u''_0(t_0) = -(h_1 + h_2), \]
we have the Taylor expansion of the holomorphic solution as
\[ u_0 = -T - \frac{1}{2}(h_1 + h_2)T^2 + O(T^3), \]
\[ u_1 = h_1 + O(T), \]
\[ u_2 = h_2 + O(T). \]

Transform this solution \( u(t) = (u_0(t), u_1(t), u_2(t)) \) by (2.5), then we have a meromorphic solution of (2.1) expanded as
\[ f_0 = 1/u_0 = -\frac{1}{T} + \frac{1}{2}(h_1 + h_2) + O(T), \]
\[ f_1 = u_0(\alpha_1 - u_0u_1) = -\alpha_1 T - (h_1 + \frac{1}{2}\alpha_1(h_1 + h_2))T^2 + O(T^3), \]
\[ f_2 = u_0 + u_1 + u_2 - 1/u_0 - u_0(\alpha_1 - u_0u_1) = \frac{1}{T} + \frac{1}{2}(h_1 + h_2) + O(T), \]

which coincides formally with the formal solutions (2.4) under the one-to-one correspondence
\[ c_0^0 = \frac{1}{2}(h_1 + h_2), \quad c_1^1 = h_1 + \frac{1}{2}\alpha_1(h_1 + h_2). \]

Thus we have proved the convergence of the formal meromorphic solution (2.4) with \( \text{Res} f = (-1, 0, 1) \), or of type (1).

From the above result, it follows that the domain of definition of the system (2.1) can be extended from the affine space \( \{(f_0, f_1, f_2) \in \mathbb{C}^3\} \) to the space obtained by identification of \( \{(f_0, f_1, f_2) \in \mathbb{C}^3\} \) and \( \{(u_0, u_1, u_2) \in \mathbb{C}^3\} \) via (2.5). The identified space is considered to be a disjoint union of the original phase space \( \{(f_0, f_1, f_2) \in \mathbb{C}^3\} \) and the two-dimensional affine space \( \{(u_0, u_1, u_2) \in \mathbb{C}^3; u_0 = 0\} \). The added two-dimensional affine space is considered to be a parameter space of the 2-parameter family of solutions with \( \text{Res} f = (-1, 0, 1) \). We call such an extension of the domain of definition an augmentation of the phase space.

We note that the same argument can be done by another transformation
\[ v_2 = 1/f_2, \quad v_1 = f_2(-\alpha_1 - f_0f_1), \quad v_0 + v_1 + v_2 = f_0 + f_1 + f_2. \]  

The affine space \( \{(v_0, v_1, v_2) \in \mathbb{C}^3; v_2 = 0\} \) is also the parameter space of the same 2-parameter family of solutions with \( \text{Res} f = (-1, 0, 1) \) and isomorphic to \( \{u_0 = 0\} \).

In the end of this subsection, we observe how \( s_i \) \( (i = 0, 1, 2) \) act on the variables \( u_0, u_1, u_2 \) and \( v_0, v_1, v_2 \). The actions of \( s_0 \) on \( u_0, u_1, u_2 \) are calculated as
\[ s_0(u_0) = s_0(1/f_0) = 1/s_0(f_0) = 1/f_0 = u_0, \]
\[ s_0(u_1) = s_0(f_0(\alpha_1 - f_0f_1)) \]
\[ = s_0(f_0)(s_0(\alpha_1) - s_0(f_0)s_0(f_1)) \]
\[ = f_0(\alpha_0 + \alpha_1 - f_0(f_1 + \frac{2f_0}{f_0}) \]
\[ = f_0(\alpha_1 - f_0f_1) = u_1, \]
\[ s_0(u_2) = (f_0 + f_1 + f_2) - (u_0 + u_1) = u_2. \]
Similarly, we obtain the action of \( s_1 \) as
\[
\begin{align*}
  s_1(u_0) &= u_0 - \frac{\alpha_1}{u_1}, \\
  s_1(u_1) &= u_1, \\
  s_1(u_2) &= u_2 + \frac{\alpha_1}{u_1}.
\end{align*}
\]
However we see that the forms of \( s_2(u_i) \) for \( i = 0, 1, 2 \) are not so simple as above. On the other hand, by (2.7), we see
\[
\begin{align*}
  s_1(v_0) &= v_0 - \frac{\alpha_1}{v_1}, \\
  s_1(v_1) &= v_1, \\
  s_1(v_2) &= v_2 + \frac{\alpha_1}{v_1},
\end{align*}
\]
but the forms of \( s_0(v_i) \) for \( i = 0, 1, 2 \) are complicated. Hence the coordinate system \( u \) is convenient to observe the action of \( s_0 \) while \( v \) is convenient to see the action of \( s_2 \). This is the reason why we take both systems \( u \) and \( v \). In the next subsection, \( u \) and \( v \) will be distinguished by the lables with \(-\) and \(+\), respectively.

We can obtain the similar results in the cases of \( \text{Res} f' = (1, -1, 0) \) and of \( \text{Res} f' = (0, 1, -1) \), obviously.

### 2.3. Augmentation of the phase space and Bäcklund transformations

We now define a fiber space \( \mathbb{E} \) over the parameter space
\[
V = \{ \alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{C}^3; \alpha_0 + \alpha_1 + \alpha_2 = 1 \},
\]
each fiber \( \mathbb{E}(\alpha) \) of which is the augmented phase space parametrizing all meromorphic solutions of the system (2.1).

Let \( I = \{ \varnothing, 0_i, 1_i, 2_i, 0, 1, 2 \} \) be a label set and \( W_\ast \) for each \( \ast \in I \) be seven copies of \( V \times \mathbb{C}^3 \) with coordinates \((\alpha, x_\ast) = (\alpha_0, \alpha_1, \alpha_2; x^0_\ast, x^1_\ast, x^2_\ast) \in W_\ast \). Then we define the space \( \mathbb{E} \) by gluing \( W_\ast \) via the following identification equations
\[
\begin{align*}
  x^+_{i+1} &= 1/x^+_i, & x^i_i &= x^i_i(-\alpha_i - x^i_i x^i_\varnothing), & x^i_i - x^i_i + x^i_i &= x_{i+1}^i + x^i_\varnothing + x^i_\varnothing, \\
  x^i_{i-1} &= 1/x^i_{i-1}, & x^i_i &= x^i_i(+\alpha_i - x^i_i x^i_\varnothing), & x^i_i + x^i_i + x^i_i &= x_{i-1}^i + x^i_\varnothing + x^i_\varnothing
\end{align*}
\]
for \((\alpha, x_\varnothing) \in W_\varnothing \) and \((\alpha, x_{i-}) \in W_{i-} \), and
\[
\begin{align*}
  x^i_{i+1} &= 1/x^i_{i+1}, & x^i_i &= x^i_i(-\alpha_i - x^i_i x^i_\varnothing), & x^i_i - x^i_i + x^i_i &= x_{i+1}^i + x^i_\varnothing + x^i_\varnothing, \\
  x^i_{i-1} &= 1/x^i_{i-1}, & x^i_i &= x^i_i(+\alpha_i - x^i_i x^i_\varnothing), & x^i_i + x^i_i + x^i_i &= x_{i-1}^i + x^i_\varnothing + x^i_\varnothing
\end{align*}
\]
for \((\alpha, x_\varnothing) \in W_\varnothing \) and \((\alpha, x_{i+}) \in W_{i+} \), namely,
\[
\mathbb{E} = \left( \bigsqcup_{\ast \in I} W_\ast \right) / \sim,
\]
where \( \sim \) is the equivalence relation generated by the above equations. Here \( x_\varnothing \) shall be considered as the original coordinate system \( f \) in (2.1). Then we see that \( x_{i-} = u \) and \( x_{i+} = v \), where \( u \) and \( v \) are the coordinate systems defined by (2.5) and (2.7), respectively.
We denote by \( \pi_V \) the natural projection \( \pi_V : \mathbb{E} \to V \) and let \( \mathbb{E}(\alpha) := \pi_V^{-1}(\alpha) \) be the fiber of \( \mathbb{E} \) over \( \alpha \in V \).

For \( (\alpha, x_*) \in W_s (* \in I) \), denote its equivalence class by \([\langle \alpha, x_* \rangle]\) and let

\[
U_* := \{ \langle \alpha, x_* \rangle ; (\alpha, x_*) \in W_s \}.
\]

We also denote the coordinate mappings from \( U_* \) to \( W_s \) by \( \varphi_* \), namely,

\[
\varphi_* : p \in U_* \mapsto (\alpha_0(p), \alpha_1(p), \alpha_2(p); x^0_*(p), x^1_*(p), x^2_*(p)) \in W_s = V \times \mathbb{C}^3
\]

for each \(* \in I\). Note that \( \alpha_i \) and \( x^i_* \) are here considered to be coordinate functions. Let \( \mathbb{E}_\emptyset \) and \( \mathbb{E}_i (i = 0, 1, 2) \) be the subsets of \( \mathbb{E} \) defined by

\[
\mathbb{E}_\emptyset := U_\emptyset, \quad \mathbb{E}_i := U_i \setminus U_\emptyset = (\varphi_i)^{-1} (\{ x^i_\emptyset = 0 \}) = U_i \setminus U_\emptyset = (\varphi_i)^{-1} (\{ x^i_\emptyset = 0 \}).
\]

Then \( \mathbb{E}_\emptyset \cong V \times \mathbb{C}^3, \mathbb{E}_i \cong V \times \mathbb{C}^2 \) and the space \( \mathbb{E} \) is decomposed as

\[
\mathbb{E} = \mathbb{E}_\emptyset \sqcup \mathbb{E}_0 \sqcup \mathbb{E}_1 \sqcup \mathbb{E}_2.
\]

Each \( \mathbb{E}_i \) \((i = 0, 1, 2)\) is the parameter space of the corresponding 2-parameter family of meromorphic solutions of type \((i)\) and \( \mathbb{E}_\emptyset \) is that of the 3-parameter family of holomorphic solutions of type \((\emptyset)\) of the system \((2.1)\).

The system of differential equations \((2.1)\) defines a vector field

\[
X_p = \sum_{i=0}^{2} \left( x^i_\emptyset (p) (x^{i+1}_\emptyset (p) - x^{i+2}_\emptyset (p)) + \alpha_i (p) \right) \frac{\partial}{\partial x^i_\emptyset} \quad , \quad p \in \mathbb{E}(\alpha) \cap U_\emptyset
\]

for each \( \alpha \in V \). As we have already shown, the vector field can be holomorphically extended to \( \mathbb{E}(\alpha) \).

We next observe how Bäcklund transformations act on the space \( \mathbb{E} \). For \( w \in \overline{W}(A_2^{(1)}) \), we define \( \sigma_w \) from \( \mathbb{E} \) to itself by

\[
(\alpha_j \circ \sigma_w)(p) = (w(\alpha_j))(p), \quad (x^i_* \circ \sigma_w)(p) = (w(x^i_*))(p) \quad (p \in \mathbb{E}, j = 0, 1, 2)
\]

for any \(* \in I\) as far as the right-hand sides are defined for \( p \). Here \( w(x^i_*)(j = 0, 1, 2) \) are rational functions of \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \) and \( x_* = (x^0_*, x^1_*, x^2_*) \) determined by \( w : \mathbb{C}(\alpha, f) \to \mathbb{C}(\alpha, f) \) and the isomorphism from \( \mathbb{C}(\alpha, f) \) to \( \mathbb{C}(\alpha, x_*) \).

We can verify that \( \sigma_w \) is birational and \( \sigma_w' \circ \sigma_w = \sigma_w' \circ \sigma_w \) for \( w, w' \in \overline{W}(A_2^{(1)}) \). Note that \( \overline{W}(A_2^{(1)}) \) is generated by \( s_0, s_1, s_2 \) and \( \pi \), and that \( \sigma_{\pi} \) is obviously extended to a biholomorphic mapping from \( \mathbb{E}(\alpha) \) to \( \mathbb{E}(\pi(\alpha)) \), which maps \( \mathbb{E}_s \) \((s = \emptyset, 0, 1, 2)\) as

\[
\sigma_{\pi}(\mathbb{E}_\emptyset) = \mathbb{E}_\emptyset, \quad \sigma_{\pi}(\mathbb{E}_0) = \mathbb{E}_1, \quad \sigma_{\pi}(\mathbb{E}_1) = \mathbb{E}_2, \quad \sigma_{\pi}(\mathbb{E}_2) = \mathbb{E}_0.
\]
Therefore we now consider $\sigma_i := \sigma_{s_i}$ for $i = 0, 1, 2$ in more detail. Let $i = 0, 1, 2$ be fixed. We can verify that $\sigma_i$ can be extended to a biholomorphic mapping from $E(\alpha)$ to $E(s_i(\alpha))$ for any $\alpha \in V$ and its images of $E_\ast(\ast = \varnothing, 0, 1, 2)$ are given by

$$
\sigma_i(E_\varnothing \setminus D_i) = E_\varnothing \setminus D_i, \quad \sigma_i(E_s \cap D_i) = E_s', \quad \sigma_i(E_{i\pm 1}) = E_{i\pm 1} 
$$

where

$$
E_\varnothing' := E_\varnothing \setminus \pi_\varnothing^{-1}(\{x_\varnothing = 0\}), \quad D_i := (\varnothing_\varnothing)^{-1}(\{x_\varnothing = 0\}).
$$

These assertions are verified by direct calculation using

$$
s_i(x_j^\ast) = \begin{cases} 
    x_j^\ast + \frac{\alpha_i}{u_{ij}} & (* = \varnothing, i_+, i_-), \\
    x_j^\ast & (* = (i - 1)_+, (i + 1)_-),
\end{cases}
$$

where $u_{ij}$ are given by (1.3).

We remark that such relations, say $\sigma_1(E'_1) = E_\varnothing \cap D_1$, explain the correspondence of the meromorphic solutions of type (1) and the holomorphic solutions of type ($\varnothing$). $\sigma_1$ maps the integral curve through a point of $E_1'$ to an integral curve through a point of $E_\varnothing \cap D_1$.

### 3. Formal meromorphic solutions of the system $(A_4^{(1)})$

In this section, we obtain all formal meromorphic solutions of the system of differential equations with affine Weyl group symmetry of type $A_4^{(1)}$:

$$
f_i' = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i \quad (i = 0, \ldots, 4). \quad (3.1)
$$

We show that (i) the order of the pole of every formal solution is one, (ii) there are fifteen families of such formal solutions, and (iii) five of them contain four arbitrary constants and the other ten contain three arbitrary constants.

We proceed in the same manner as Subsection 2.1 of the case of type $A_2^{(1)}$. Let $f = (f_0, \ldots, f_4)$ be a formal solution of the form

$$
f_i = \sum_{n=-r}^{\infty} c_n^i T^n, \quad T := t - t_0 \quad (i = 0, \ldots, 4) \quad (3.2)
$$

where $r$ is a positive integer. Direct substitution of these series into (3.1) gives

$$
nc_n^i = \sum_{k=-r}^{n+r-1} c_k^i G_n^{i-k_{n-k-1} + \delta_{0,n-1} \alpha_i} \quad (i = 0, \ldots, 4) \quad (3.3)
$$

where

$$
G_n^i = \sum_{k=1}^{4} (-1)^{k-1} c_n^{i+k} = c_n^{i+1} - c_n^{i+2} + c_n^{i+3} - c_n^{i+4}.
$$
Proposition 3.1. If \( f_i \) has a pole at \( t = t_0 \) then the order of the pole is one, i.e., \( r = 1 \).

Proof. Suppose that \( r > 1 \). Consider a series of equations (3.3) for \( n = -2r + 1, -2r + 2, \ldots, -r \). The first equation \( 0 = c_{-r}^i G_{-r}^i \) \((i = 0, \ldots, 4)\) has the following solutions up to cyclic rotation

\[
c_{-r} = (c_{-r}^0, \ldots, c_{-r}^4) = (a, a, a, 0, 0), (a, a, a, 0, 0), (a, a, a, a, a)
\]

where \( a \neq 0 \). It is verified that \( G_{-r}^i + c_{-r}^i \neq 0 \) for each \( i = 0, \ldots, 4 \) and each solution \( c_{-r} \).

Therefore, as in Subsection 2.2, the remaining equations yield that \( c_{-r}^i = 0 \) implies \( c_{-r}^i = 0 \) and \( c_{-r}^i \neq 0 \) implies \( G_{-r}^i = -r \) for each \( i = 0, \ldots, 4 \). Then, in each case of \( c_{-r} \), we derive the contradiction that \( r = 0 \) as follows:

(i) In the case of \( c_{-r} = (a, 0, 0, 0, 0) \), we deduce that \( c_{-r} = \cdots = c_{-r}^4 = 0 \) and \( G_{-r}^0 = -r \).

Then it follows that \( r = 0 \) from \( G_{-1}^0 = c_{-1}^1 - c_{-1}^2 + c_{-1}^3 - c_{-1}^4 \).

(ii) In the case of \( c_{-r} = (a, a, a, 0, 0) \), we have \( G_{-1}^0 = G_{-1}^1 = G_{-1}^2 = -r, c_{-1}^3 = c_{-1}^4 = 0 \) and \( G_{-1}^0 + G_{-1}^1 + G_{-1}^2 = c_{-1}^3 + c_{-1}^4 \), and hence \( r = 0 \).

(iii) In the case of \( c_{-r} = (a, a, 0, -a, 0) \), we have \( G_{-1}^0 = G_{-1}^1 = -r, c_{-1}^2 = c_{-1}^3 = 0 \) and \( G_{-1}^0 + G_{-1}^1 - G_{-1}^3 = c_{-1}^2 - c_{-1}^3 \), and hence \( r = 0 \).

(iv) In the case of \( c_{-r} = (a, a, a, a, a) \), we have \( G_{-1}^0 = \cdots = G_{-1}^4 = -r \) and \( G_{-1}^0 + \cdots + G_{-1}^4 = \sum_{i=0}^4 \sum_{k=1}^1 (-1)^{i-1} c_{n} = 0 \), and hence \( r = 0 \).

Let us now determine the coefficients \( c_n \) \((n \geqslant -1)\) of the expansion (3.2) from (3.3). For \( n = -1 \), the equations (3.3) are written by

\[
(-1)c_{-1}^0 = (c_{-1}^0 + c_{-1}^1 - n - c_{-1}^2 + n - c_{-1}^3 + c_{-1}^4) \quad (i = 0, \ldots, 4)
\]

which has fifteen solutions \( c_{-1} = (c_{-1}^0, \ldots, c_{-1}^4) \), each of which equals to one of

\[
(-1, 0, 1, 0, 0), \ (-1, 0, 0, 0, 1), \ (-1, -3, 0, 3, 1),
\]

by suitable cyclic rotations.

For \( n \geqslant 0 \), the equations (3.3) with respect to \( c_n \) are written by the following linear system

\[
(n - G_{-1}^i)c_{-1}^i - (c_{-1}^i - c_{-1}^2 + c_{-1}^3 - c_{-1}^4) = \sum_{k=0}^{n-1} c_{-k}^i G_{n-k}^i + \delta_{n-1} a_i. \quad (i = 0, \ldots, 4)
\]

Let

\[
P_n := \begin{bmatrix}
    n - G_{-1}^0 & -c_{-1}^0 & c_{-1}^0 & -c_{-1}^0 & c_{-1}^0 \\
    c_{-1}^1 & n - G_{-1}^1 & -c_{-1}^1 & c_{-1}^1 & -c_{-1}^1 \\
    -c_{-1}^2 & c_{-1}^2 & n - G_{-1}^2 & -c_{-1}^2 & c_{-1}^2 \\
    c_{-1}^3 & -c_{-1}^3 & c_{-1}^3 & n - G_{-1}^3 & -c_{-1}^3 \\
    -c_{-1}^4 & c_{-1}^4 & -c_{-1}^4 & c_{-1}^4 & n - G_{-1}^4
\end{bmatrix}.
\]
Then \(c_n\) is uniquely determined by \(c_1, \ldots, c_{n-1}\) unless \(\det P_n = 0\). Since \(P_n\) only depends on \(n\) and \(c_{-1}\), it suffices to consider the above three typical cases of the values of \(c_{-1}\).

(i) The case of \(c_{-1} = (-1, 0, 1, 0, 0)\): We have

\[
\det P_n = (n + 2)n^3(n - 2)
\]

and we have the relations \(c_1^0 = 0, c_0^4 - c_0^0 + c_0^2 - c_0^3 = 0\) among \(c_0^0, \ldots, c_0^4\). In this case the system has a formal meromorphic solution of the form

\[
f_0 = -\frac{1}{T} + c_0^0 + \frac{1}{3}[2\alpha_0 + 3\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - (c_0^0)^2 + 2(c_0^3)^2 - 2c^2] T + O(T^2),
\]

\[
f_1 = -\alpha_1 T + c_1^1 T^2 + O(T^3),
\]

\[
f_2 = \frac{1}{T} + c_0^3 + \frac{1}{3}[(\alpha_0 + 3\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - (c_0^0)^2 - 2(c_0^3)^2 + 2c^2] T + O(T^2),
\]

\[
f_3 = (c_0^0 + c) + [\alpha_3 - (c_0^3)^2 + c^2] T + O(T^2),
\]

\[
f_4 = (c_0^0 + c) + [\alpha_4 + (c_0^3)^2 - c^2] T + O(T^2),
\]

where \(c := c_0^4 - c_0^0 = c_0^3 - c_0^2\). This formal solution depends on four arbitrary constants \(c_0^0, c_0^2, c_0^3, c_1^1\) and therefore defines a 4-parameter family of formal meromorphic solutions.

(ii) The case of \(c_{-1} = (-1, 0, 0, 1)\): We have

\[
\det P_n = (n + 2)^2n(n - 2)^2
\]

and obtain a formal solution containing three arbitrary constants \(c_0^0, c_1^1, c_2^3\) written by

\[
f_0 = -\frac{1}{T} + c_0^0 + \frac{1}{3}[2\alpha_0 + 3\alpha_1 + \alpha_2 + 3\alpha_3 + \alpha_4 - (c_0^0)^2] T + O(T^2),
\]

\[
f_1 = -\alpha_1 T + c_1^1 T^2 + O(T^3),
\]

\[
f_2 = \frac{1}{3}\alpha_2 T + O(T^3),
\]

\[
f_3 = -\alpha_3 T + c_2^3 T^2 + O(T^3),
\]

\[
f_4 = \frac{1}{T} + c_0^0 + \frac{1}{3}[(\alpha_0 + 3\alpha_1 + \alpha_2 + 3\alpha_3 + 2\alpha_4 + (c_0^0)^2] T + O(T^2).
\]

This is a 3-parameter family of formal meromorphic solutions.

(iii) The case of \(c_{-1} = (-1, -3, 0, 3, 1)\): We have

\[
\det P_n = (n + 4)(n + 2)n(n - 2)(n - 4)
\]
and get a formal solution including three arbitrary constants \( c_0, c_2, c_4 \) written by

\[
\begin{align*}
f_0 &= -\frac{1}{T} + c_0^0 + \frac{1}{3} \left[ 2\alpha_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + \alpha_4 - (c_0^0)^2 \right] T - c_2^2 T^2 + O(T^3), \\
f_1 &= -\frac{3}{T} + \frac{1}{5} \left[ c_2 - \frac{1}{2} \left( (\alpha_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + \alpha_4) c_0^0 \right) \right] T^2 + O(T^3), \\
f_2 &= -\frac{1}{3} \alpha_2 T + \frac{1}{45} \alpha_2 \left[ -\alpha_0 - 3\alpha_1 + 3\alpha_3 + \alpha_4 + 2(c_0^0)^2 \right] T^3 + c_4^2 T^4 + O(T^5), \\
f_3 &= \frac{3}{T} - \frac{1}{5} \left[ c_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 - \alpha_4 - 2(c_0^0)^2 \right] T + c_2^2 T^2 + O(T^3), \\
f_4 &= \frac{1}{T} + c_0^0 + \frac{1}{3} \left[ (\alpha_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 2\alpha_4) + (c_0^0)^2 \right] T \\
&\quad - \left[ c_2 - \frac{1}{2} (\alpha_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + \alpha_4) c_0^0 \right] T^2 + O(T^3).
\end{align*}
\]

(3.6)

This is another 3-parameter family of solutions.

As the system of type \( A^{(1)}_2 \), we denote the coefficients \( c_{-1} = (c_{-1}^0, \ldots, c_{-1}^4) \) by \( \text{Res} f = (\text{Res } f_0, \ldots, \text{Res } f_4) \). The above results are stated as:

**Proposition 3.2.** The system (3.1) has fifteen families of formal solutions \( f = (f_0, \ldots, f_4) \) with simple pole and the types of the formal solutions are determined by \( \text{Res } f \).

The fifteen families of formal meromorphic solutions are divided into five 4-parameter families and ten 3-parameter families. We further classify the families from the viewpoint of the actions of \( \text{B"acklund transformations} \).

Let \( f \) be the formal solution with \( \text{Res } f = (-1, -3, 0, 3, 1) \), i.e., of case (iii). Substituting it into \( g_i = s_2(f_i) \) \((i = 0, \ldots, 4)\), namely into

\[
g_0 = f_0, \quad g_1 = f_1 - \frac{\alpha_2}{f_2}, \quad g_2 = f_2, \quad g_3 = f_3 + \frac{\alpha_2}{f_2}, \quad g_4 = f_4,
\]

we obtain the series expansion

\[
\begin{align*}
g_0 &= -\frac{1}{T} + O(1), \quad g_1 = -(\alpha_1 + \alpha_2) T + O(T^2), \\
g_2 &= -\frac{1}{3} \alpha_2 T + O(T^3), \quad g_3 = -(\alpha_3 + \alpha_2) T + O(T^2), \quad g_4 = \frac{1}{T} + O(1),
\end{align*}
\]

which is the formal solution with \( \text{Res } g = (-1, 0, 0, 0, 1) \) of the system (3.1) with parameter \( s_2(\alpha) = (s_2(\alpha_0), \ldots, s_2(\alpha_4)) \). Note that this does not hold in the case of \( \alpha_2 = 0 \), since \( g = f \) in that case. Let us denote this fact as

\[
\text{Res } f = (-1, -3, 0, 3, 1) \Rightarrow \text{Res } s_2(f) = (-1, 0, 0, 0, 1) \quad \text{if } \alpha_2 \neq 0.
\]
Table 2: Classification of the families of solutions of the system \( (A_4^{(1)}) \)

| type | \( \text{Res} f \) | \( \text{corresponding BT} \) | \# of arbitrary constants |
|------|-----------------|-----------------|-----------------|
| \( \emptyset \) | \( (0,0,0,0,0) \) | \( \text{id} \) | 5 |
| (1) | \( (-1,0,1,0,0) \) | \( s_1 \) | |
| (2) | \( (0,-1,0,1,0) \) | \( s_2 \) | 4 |
| (3) | \( (0,0,-1,0,1) \) | \( s_3 \) | |
| (4) | \( (1,0,0,-1,0) \) | \( s_4 \) | |
| (0) | \( (0,1,0,0,-1) \) | \( s_0 \) | |
| (13) | \( (-1,0,0,0,1) \) | \( s_3 s_1 \) | |
| (24) | \( (1,-1,0,0,0) \) | \( s_4 s_2 \) | |
| (30) | \( (0,1,-1,0,0) \) | \( s_0 s_3 \) | 3 |
| (41) | \( (0,0,1,-1,0) \) | \( s_1 s_4 \) | |
| (02) | \( (0,0,0,1,-1) \) | \( s_2 s_0 \) | |
| (132) | \( (-1,-3,0,3,1) \) | \( s_2 s_3 s_1 \) | |
| (243) | \( (1,-1,-3,0,3) \) | \( s_3 s_4 s_2 \) | |
| (304) | \( (3,1,-1,-3,0) \) | \( s_4 s_0 s_3 \) | 3 |
| (410) | \( (0,3,1,-1,-3) \) | \( s_0 s_1 s_4 \) | |
| (021) | \( (-3,0,3,1,-1) \) | \( s_1 s_2 s_0 \) | |

Similarly, we obtain

\[
\text{Res} f = (-1,0,0,0,1) \Rightarrow \text{Res} s_3(f) = (-1,0,1,0,0) \quad \text{if} \ \alpha_3 \neq 0,
\]
\[
\text{Res} f = (-1,0,0,0,1) \Rightarrow \text{Res} s_1(f) = (0,0,-1,0,1) \quad \text{if} \ \alpha_1 \neq 0,
\]
\[
\text{Res} f = (-1,0,1,0,0) \Rightarrow \text{Res} s_1(f) = (0,0,0,0,0) \quad \text{if} \ \alpha_1 \neq 0.
\]

Here \( \text{Res} s_1(f) = (0,0,0,0,0) \) means that \( s_1(f) \) is a formal holomorphic solution. Hence, generically, each formal meromorphic solution can be transformed into a holomorphic solution of the system \( (3.1) \) with different parameter by an appropriate Bäcklund transformation. Therefore it is convenient to distinguish each family of formal meromorphic solutions assigning to it a Bäcklund transformation expressed as a product of \( s_i \) \( (i = 0,\ldots,4) \). For instance, we assign Bäcklund transformations \( s_1, s_3 s_1 \) and \( s_2 s_3 s_1 \) to the families of formal meromorphic solutions with \( \text{Res} f = (-1,0,1,0,0), (-1,0,0,0,1) \) and \( (-1,-3,0,3,1) \), respectively. We also say simply that they are of type \( (1), (13) \) and \( (132) \), respectively. Such classification of all the families of formal meromorphic solutions of \( (3.1) \) is given in Table 2, where the family of holomorphic solutions is denoted as type \( \emptyset \). The position \( t_0 \) of pole is not counted into the arbitrary constants as before.
4. Coordinates for formal meromorphic solutions of the system \((A_4^{(1)})\)

Now we shall prove the convergence of the formal meromorphic solutions obtained in the preceding section. This section is devoted to the proof of the following proposition.

**Proposition 4.1.** Any formal meromorphic solution of the system (3.1) converges.

The idea of the proof is almost the same as the case of the system of type \(A_4^{(1)}\) described in Subsection 2.2: we will introduce appropriate change of variables so that the system (3.1) is transformed to a holomorphic system and the arbitrary constants contained in the formal meromorphic solution of (3.1) are considered as an initial condition of the holomorphic solution of the converted system.

While the system (3.1) has the fifteen families of formal meromorphic solutions, we discuss the three typical families of type (1), (13) and (132) in sequence. The convergence of the solutions of the remaining types can be shown by the rotations of indices.

4.1. The family of type (1)

Let \(f = (f_0, \ldots, f_4)\) be the formal meromorphic solution of type (1). Then the series expansion of \(f\) is the form of (3.4), and \(f_0\) has simple pole at \(t = t_0\) while \(f_1\) has a simple zero. We note that \(1/f_0\) has a simple zero and \(f_0f_1\) has no pole:

\[
f_0f_1 = a_1 - (c_2^1 + a_1c_0^0)T + O(T^2).
\]

Since \(a_1 - f_0f_1\) has a simple zero, multiplying \(f_0\), we have the expansion

\[
f_0(a_1 - f_0f_1) = (c_2^1 + a_1c_0^0) + O(T),
\]

which has no terms of negative power of \(T\) but has the constant term containing the arbitrary constant \(c_2^1\). We also note that \(f_0 + f_2\) has no pole, even though \(f_2\) has a simple pole. Therefore we introduce the following transformation:

\[
\begin{align*}
u_0 &= 1/f_0, \\
u_1 &= f_0(a_1 - f_0f_1), \\
u_0 + u_2 &= f_0 + f_2, \\
u_3 &= f_3, \\
u_0 + \cdots + u_4 &= f_0 + \cdots + f_4.
\end{align*}
\]
We can verify that this biholomorphic mapping from \( \mathbb{C}^5 \setminus \{ f_0 = 0 \} \) to \( \mathbb{C}^5 \setminus \{ u_0 = 0 \} \) transforms the system (3.1) to the system

\[ u_0' = 2u_0^3u_1 - (\alpha_0 + 2\alpha_1 - 1)u_0^2 + (u_1 + u_2 - u_3 + u_4)u_0 - 1, \]
\[ u_1' = -3u_0^2u_1^2 + (2\alpha_0 + 4\alpha_1 - 1)u_0u_1 - (u_1 + u_2 - u_3 + u_4)u_1 - (\alpha_0 + \alpha_1)\alpha_1, \]
\[ u_2' = -2u_0^3u_1 + (\alpha_0 + 2\alpha_1 - 1)u_0^2 - (u_1 + u_2 - u_3 + u_4)u_0 \]
\[ + (u_3 - u_4 - u_1)(u_0 + u_2) - 2u_0u_1 + \alpha_0 + 2\alpha_1 + \alpha_2 + 1, \]
\[ u_3' = (u_4 - u_0 + u_1 - u_2)u_3 + \alpha_3, \]
\[ u_4' = 3u_0^2u_1^2 - (2\alpha_0 + 4\alpha_1 - 1)u_0u_1 + (u_1 + u_2 - u_3 + u_4)u_1 + (\alpha_0 + \alpha_1)\alpha_1 \]
\[ + (u_4 + u_1)(u_0 + u_2 - u_3) + 2u_0u_1 + \alpha_4 - \alpha_1. \]

Here notice that \( u_0' + \cdots + u_4' = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \)

Since the right-hand sides of the above equations are polynomials in \( u = (u_0, \ldots, u_4) \), there exists a unique holomorphic solution \( u = u(t) \) with the initial condition \( u(t_0) = (u_0(t_0), \ldots, u_4(t_0)) = (0, h_1, h_2, h_3, h_4) \), where \( h_1, h_2, h_3, h_4 \) are arbitrary complex numbers. Let \( F \) be a polynomial of \( u \) such that \( Fu_0 - 1 \) is the right-hand side of the first equation in (4.2), namely, \( u_0' = Fu_0 - 1 \). Then it follows that \( F|_{u_0=0} = u_1 + u_2 - u_3 + u_4 \), and hence, setting \( k = F(t_0) = F(u(t_0)) = h_1 + h_2 - h_3 + h_4 \), we have

\[ u_0'(t_0) = F(t_0)u_0(t_0) - 1 = -1, \quad u_0''(t_0) = F'(t_0)u_0(t_0) + F(t_0)u_0'(t_0) = k. \]

Therefore the Taylor expansion of \( u_0 = u_0(t) \) is

\[ u_0 = u_0(t) = 0 + u_0'(t_0)T + \frac{1}{2}u_0''(t_0)T^2 + O(T^3) \]
\[ = -T \left( 1 + \frac{1}{2}kT + O(T^2) \right). \]

We also have the Taylor expansion of \( u_1, \ldots, u_4 \) as

\[ u_1 = h_1 + O(T), \quad u_2 = h_2 + O(T), \quad u_3 = h_3 + O(T), \quad u_4 = h_4 + O(T), \]

and, from (4.1), we see that the transform \( f = (f_0, \ldots, f_4) \) of \( u = (u_0, \ldots, u_4) \) is of the form

\[ f_0 = \frac{1}{u_0} = -\frac{1}{T} + \frac{1}{2}k + O(T), \]
\[ f_1 = u_0(\alpha_1 - u_0u_1) \]
\[ = -T \left( 1 + \frac{1}{2}kT + O(T^2) \right) \cdot (\alpha_1 + h_1T + O(T^2)) \]
\[ = -\alpha_1T - (h_1 + \frac{1}{2}\alpha_1k)T^2 + O(T^3), \]
\[ f_2 = u_0 + u_2 - 1/u_0 = \frac{1}{T} + (h_2 - \frac{1}{2}k) + O(T), \]
\[ f_3 = u_3 = h_3 + O(T), \]
\[ f_4 = (u_1 + u_4) - f_1 = h_1 + h_4 + O(T). \]
This expansion of $f$ coincides formally with the formal meromorphic solution (3.4) of (3.1) under the change of the constants

$$c_0^0 = \frac{1}{2}k, \quad c_2^1 = -h_1 - \frac{1}{2}\alpha_1 k, \quad c_0^2 = h_2 - \frac{1}{2}k, \quad c_3^3 = h_3, \quad c_0^4 = h_1 + h_4 = k - h_2 + h_3.$$  

Hence the formal meromorphic solution (3.4) must converge.

### 4.2. The family of type (13)

First, transform the series (3.5) of type (13) formally by (4.1), then we have

$$u_0 = -T - c_0^0 T^2 + O(T^3),$$

$$u_1 = -(c_1^1 + \alpha_1 c_0^0) + O(T),$$

$$u_0 + u_2 = -\frac{1}{T} + c_0^0 + O(T),$$

$$u_3 = -\alpha_3 T + c_2^3 T^2 + O(T^3),$$

$$u_1 + u_4 = \frac{1}{T} + c_0^0 + O(T).$$

The variable $u_1$ already contains the arbitrary constant $c_2^1$ in its constant term, so we need to pull out the $c_2^3$ appearing in $u_3$. Noting that $u_0 + u_2$ has a simple pole while $u_3$ has a simple zero, we construct Taylor expansions successively as follows:

$$(u_0 + u_2)u_3 = \alpha_3 - (c_2^3 + \alpha_3 c_0^0) T + O(T^2),$$

$$(u_0 + u_2)(\alpha_3 - (u_0 + u_2)u_3) = -(c_2^3 + \alpha_3 c_0^0) + O(T).$$

Hence we introduce the following transformation:

$$v_0 = u_0,$$

$$v_1 = u_1,$$

$$v_0 + v_2 = 1/(u_0 + u_2),$$

$$v_3 = (u_0 + u_2)(\alpha_3 - (u_0 + u_2)u_3),$$

$$v_0 + \cdots + v_4 = u_0 + \cdots + u_4.$$  

(4.3)

Then we can verify that the transformation from $f = (f_0, \ldots, f_4)$ to $v = (v_0, \ldots, v_4)$ is expressed by

$$v_0 = 1/f_0,$$

$$v_1 = f_0 f_1 (\alpha_1 - f_0 f_1),$$

$$v_0 + v_2 = 1/(f_0 + f_2),$$

$$v_3 = (f_0 + f_2)(\alpha_3 - (f_0 + f_2)f_3),$$

$$v_0 + \cdots + v_4 = f_0 + \cdots + f_4.$$  

(4.4)
and the variable \( v \) satisfies the system
\[
\begin{align*}
v'_0 &= 2v_0^3v_1 - (\alpha_0 + 2\alpha_1)v_0^2 + 2v_0(v_0 + v_2)((v_0 + v_2)v_3 - \alpha_3) \\
&\quad + v_0(v_0 + v_2) + v_0(v_1 + v_3 + v_4) - 1, \\
v'_1 &= -3v_0^2v_1 + 2(\alpha_0 + 2\alpha_1)v_0v_1 - 2v_1(v_0 + v_2)((v_0 + v_2)v_3 - \alpha_3) \\
&\quad - v_1(v_0 + v_2) - v_1(v_1 + v_3 + v_4) - (\alpha_0 + \alpha_1)\alpha_1, \\
v'_0 + v'_2 &= 2(v_0 + v_2)^2(2v_0v_1 - \alpha_0 - 2\alpha_1 + 2(v_0 + v_2)v_3 - \alpha_2 - 2\alpha_3) \\
&\quad + (v_0 + v_2)^2 + (v_1 + v_3 + v_4)(v_0 + v_2) - 1, \\
v'_3 &= -3(v_0 + v_2)^2v_3^2 - 4v_0v_1(v_0 + v_2)v_3 + 2(\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3)(v_0 + v_2)v_3 \\
&\quad - (v_0 + v_2)v_3 - (v_1 + v_3 + v_4)v_3 + 2\alpha_3v_0v_1 - (\alpha_0 + 2\alpha_1 + 2\alpha_2 + 3\alpha_3)\alpha_3, \\
v'_0 + \cdots + v'_4 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,
\end{align*}
\]
from which it follows that
\[
\begin{align*}
v'_2 &= 2(v_0 + v_2)((v_0 + v_2)v_3 - \alpha_3)v_2 \\
&\quad + (2v_0 + v_2)(2(v_0v_1 - \alpha_1) - (\alpha_0 + \alpha_2))v_2 + (v_0 + v_2)v_2 \\
&\quad + (v_1 + v_3 + v_4)v_2 - \alpha_0v_0^2.
\end{align*}
\]
Now let \( v = v(t) \) be the unique holomorphic solution with
\[
v(t_0) = (v_0(t_0), \ldots, v_4(t_0)) = (0, h_1, 0, h_3, h_4).
\]
Then we have
\[
v'_0(t_0) = -1, \quad v'_0(t_0) = -k; \quad v'_2(t_0) = v'_2(t_0) = 0, \quad v'_2(t_0) = -2\alpha_2, \quad v''_2 = -8\alpha_2k,
\]
where \( k := h_1 + h_3 + h_4 \). This gives the Taylor expansion of \( v \) as follows:
\[
\begin{align*}
v_0 &= -T (1 + \frac{1}{2}kT + O(T^2)), \quad v_1 = h_1 + O(T), \\
v_2 &= -\frac{1}{3}\alpha_2T^2(1 + kT) + O(T^5), \quad v_3 = h_3 + O(T), \quad v_4(t) = h_4 + O(T).
\end{align*}
\]
Noting that
\[
v_0 + v_2 = -T (1 + \frac{1}{2}kT + O(T^2)),
\]
we have
\[
\begin{align*}
f_0 &= 1/v_0 = -\frac{1}{T} + \frac{1}{2}k + O(T), \\
f_1 &= v_0(\alpha_1 - v_0v_1) = -\alpha_1T - (h_1 + \frac{1}{2}\alpha_1k)T^2 + O(T^3), \\
f_2 &= \frac{1}{v_0 + v_2} - \frac{1}{v_0} = -\frac{v_2}{v_0(v_0 + v_2)} = \frac{1}{3}\alpha_2T + 0 \cdot T^2 + O(T^3), \\
f_3 &= (v_0 + v_2)(\alpha_3 - (v_0 + v_2)v_3) = -\alpha_3T - (h_3 + \frac{1}{2}\alpha_3k)T^2 + O(T^3), \\
f_4 &= (v_0 + \cdots + v_4) - (f_1 + \cdots + f_4) = \frac{1}{T} + \frac{1}{2}k + O(T).
\end{align*}
\]
The formal meromorphic solution (3.5) formally coincides with this meromorphic solutions if

$$c^0 = \frac{1}{2}k, \quad c^1 = -h_1 - \frac{1}{2}a_1k, \quad c^2 = -h_3 - \frac{1}{2}a_3k,$$

and hence the formal solution (3.5) must be convergent.

4.3. The family of type (132)

We need transform the formal meromorphic solutions (3.6) by the change of variables (4.4) and observe the arbitrary constants $c^0, c^3$ and $c^2$, which will appear in the expansion of $v = (v_0, \ldots, v_3)$.

First of all, we observe that

$$v_0 = 1/f_0 = -T - c^0 T^2 + \frac{1}{3} \left[ 2\alpha_0 + 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + \alpha_4 + 2(c^0)^2 \right] T^3 + O(T^4),$$

and $v_0$ has a simple zero. Secondly, from

$$v_3 = (f_0 + f_2)(\alpha_3 - (f_0 + f_2)f_3)
= -\frac{3}{75}(1 - 2c^0 T - \frac{1}{3} \left[ 7\alpha_0 + 11\alpha_1 + 15\alpha_2 + 9\alpha_3 + 3\alpha_4 - 9(c^0)^2 \right] T^2
- \left[ \frac{1}{15}(22\alpha_0 + 36\alpha_1 + 50\alpha_2 + 29\alpha_3 + 8\alpha_4)c^0 - \frac{14}{5}(c^0)^3 + 7c^3 \right] T^3 \right) + O(T),$$

it follows that $1/v_3$ has a zero of order three. We will select $v_0$ and $1/v_3$ as coordinates with initial value 0. Since

$$v_1 + v_3 = f_0(\alpha_1 - f_0f_1) + (f_0 + f_2)(\alpha_3 - (f_0 + f_2)f_3)
= f_0(\alpha_1 + \alpha_3 - f_0(f_1 + f_3) - 2f_2f_3) + f_2(\alpha_3 - f_2f_3)
= \left[ \frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)c^0 - 2c^2 \right] + O(T),$$

the value of the arbitrary constant $c^3$ can be recovered from the initial value of $v_1 + v_3$. 
For the remaining arbitrary constant $c_4^2$, we observe that

\[
v_2 = \frac{1}{f_0 + f_2} - \frac{1}{f_0} = \frac{-f_2}{f_0(f_0 + f_2)}
\]

\[
= \frac{1}{3} \alpha_2 T^3 + \frac{2}{3} \alpha_2 c_0^0 T^4 + \frac{1}{15} \alpha_2 [7 \alpha_0 + 11 \alpha_1 + 15 \alpha_2 + 9 \alpha_3 + 3 \alpha_4 + 11(c_0^0)^2] T^5
\]

\[
+ \left[ \frac{1}{45} \alpha_2 ((62 \alpha_0 + 96 \alpha_1 + 135 \alpha_2 + 84 \alpha_3 + 28 \alpha_4) c_0^0 + 26(c_0^0)^3 - 30c_2^2 - c_4^2 \right] T^6
\]

\[
+ O(T^7),
\]

and then write it as

\[
v_2 + c_4^2 T^6 = \frac{1}{3} \alpha_2 T^3(1 + 2c_0^0 T + \frac{1}{3} [7 \alpha_0 + 11 \alpha_1 + 15 \alpha_2 + 9 \alpha_3 + 3 \alpha_4 + 11(c_0^0)^2] T^2
\]

\[
+ \frac{1}{15} ((62 \alpha_0 + 96 \alpha_1 + 135 \alpha_2 + 84 \alpha_3 + 28 \alpha_4) c_0^0 + 26(c_0^0)^3 - 30c_2^2 \right] T^3)
\]

\[
+ O(T^7).
\]

Then, multiplying $v_3$, we obtain

\[
(v_2 + c_4^2 T^6)v_3 = -\alpha_2 \left(1 + 0 \cdot T + 0 \cdot T^2 + \frac{1}{3} [c_2^3 + (\alpha_2 + \alpha_3) c_0^0] T^3 \right) + O(T^4),
\]

\[
= -\alpha_2 - \frac{1}{3} \alpha_2 [c_2^3 + (\alpha_2 + \alpha_3) c_0^0] T^3 + O(T^4).
\]

Noting that $c_4^2 T^6 \cdot v_3 = -3c_4^2 T^3 + O(T^4)$, we have

\[
v_2 v_3 = -\alpha_2 - \left[ \frac{1}{3} \alpha_2 (c_2^3 + (\alpha_2 + \alpha_3) c_0^0) - 3c_4^2 \right] T^3 + O(T^4)
\]

and hence

\[
-\alpha_2 - v_2 v_3 = \left[ \frac{1}{3} \alpha_2 (c_2^3 + (\alpha_2 + \alpha_3) c_0^0) - 3c_4^2 \right] T^3 + O(T^4).
\]

Thus we are now prepared to define the following transformation:

\[
\begin{align*}
w_0 &= v_0, \\
w_1 + w_3 &= v_1 + v_3, \\
w_2 &= v_3(-\alpha_2 - v_2 v_3), \\
w_3 &= 1/v_3, \\
w_0 + \cdots + w_4 &= v_0 + \cdots + v_4.
\end{align*}
\]
We can verify that the system for \( w = (w_0, \ldots, w_4) \) is of the following holomorphic form:

\[
\begin{align*}
\dot{w}_0' &= 2w_0^3(w_1 + w_3) - (a_0 + 2\alpha_1)w_0^2 - 4w_0^2w_2w_3 - 2(2\alpha_2 + \alpha_3)w_0^2 \\
&\quad + 2w_0w_3(w_2w_3 + \alpha_2)(w_2w_3 + \alpha_2 + \alpha_3) \\
&\quad + w_0(w_0 + w_2 + w_4) + w_0(w_1 + w_3) - 1, \\
\dot{w}_1' + \dot{w}_3' &= -2(w_1 + w_3)w_3(w_2w_3 + \alpha_2)(w_2w_3 + \alpha_2 + \alpha_3) \\
&\quad - 3w_0^2(w_1 + w_3)^2 + 8w_0(w_1 + w_3)w_2w_3 - w_2^2w_3^2 \\
&\quad + 2(a_0 + 2\alpha_1 + 4\alpha_2 + 2\alpha_3)w_0(w_1 + w_3) - 2(a_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3)w_2w_3 \\
&\quad - 2w_0w_2 - (w_0 + w_2 + w_4)(w_1 + w_3) - (w_1 + w_3)^2 \\
&\quad - (a_0 + 2\alpha_2 + \alpha_3)(a_0 + 2\alpha_2 + \alpha_3) + \alpha_2(a_2 + \alpha_3), \\
\dot{w}_2' &= -4w_0^3w_1^2 + [6w_0(w_1 + w_3) - 3(a_0 + 2\alpha_1) - 6(2\alpha_2 + \alpha_3)]w_2w_3^2 \\
&\quad + 2w_0w_2w_3^2 + 4(2\alpha_2 + \alpha_3)w_0(w_1 + w_3)w_2w_3 \\
&\quad - 2[(a_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3)(2\alpha_2 + \alpha_3) + \alpha_2(a_2 + \alpha_3)]w_2w_3 \\
&\quad - 2w_0(w_1 + w_3)[2w_0w_2 - \alpha_2(a_2 + \alpha_3)] \\
&\quad - (w_0 + w_2 + w_4)w_2 - (w_1 + w_3)w_2 \\
&\quad + (a_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3)[2w_0w_2 - \alpha_2(a_2 + \alpha_3)], \\
\dot{w}_3' &= 3w_0^2w_3^2 - 4w_0(w_1 + w_3)w_2w_3^2 + 2(a_0 + 2\alpha_1 + 4\alpha_2 + 2\alpha_3)w_2w_3^2 \\
&\quad - 2w_0w_2w_3^2 - (4\alpha_2 + 2\alpha_3)w_0(w_1 + w_3)w_3^2 \\
&\quad + [(a_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3)(2\alpha_2 + \alpha_3) + \alpha_2(a_2 + \alpha_3)]w_3^2 \\
&\quad + 4w_0^2(w_1 + w_3)w_3 - (2\alpha_0 + 4\alpha_1 + 4\alpha_2 + 2\alpha_3)w_0w_3 \\
&\quad + (w_0 + w_2 + w_4)w_3 + (w_1 + w_3)w_3 - w_0^2, \\
\dot{w}_4' &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.
\end{align*}
\]

Let \( w = w(t) \) be a unique holomorphic solution of the above system with \( w(t_0) = (w_0(t_0), \ldots, w_4(t_0)) = (0, h_1, h_2, 0, h_4) \).

Then, by a tedious calculation, we can obtain the expansion of \( w \) as

\[
\begin{align*}
w_0 &= -T(1 + \frac{1}{2}kT + a_2 T^2 + a_3 T^3 + O(T^4)), \quad w_1 = h_1 + O(T), \quad w_2 = h_2 + O(T), \\
w_3 &= -\frac{1}{2}T^3(1 + kT + b_2 T^2 + b_3 T^3 + O(T^4)), \quad w_4 = h_4 + O(T),
\end{align*}
\]

where

\[
\begin{align*}
k &= h_1 + h_2 + h_4, \\
a_2 &= \frac{1}{6}k^2 + \frac{1}{3}(2a_0 + 3a_1 + 5a_2 + 3a_3 + a_4), \\
a_3 &= \frac{1}{24}k^3 + \frac{1}{2}h_1 + \frac{1}{24}(13a_0 + 21a_1 + 37a_2 + 21a_3 + 5a_4)k, \\
b_2 &= \frac{11}{36}k^2 + \frac{1}{6}(7a_0 + 11a_1 + 15a_2 + 9a_3 + 3a_4), \\
b_3 &= \frac{13}{60}k^3 + \frac{7}{6}h_1 + \frac{1}{120}(213a_0 + 349a_1 + 485a_2 + 281a_3 + 77a_4)k.
\end{align*}
\]
From (4.1), (4.3) and (4.5), it follows that

\[ f_0 = 1/w_0 = -\frac{1}{T} + \frac{1}{2}k + O(T), \]

\[ f_1 = w_0(a_1 - w_0w_1) - w_0^2w_3 + w_0^2/w_3 
   = -\frac{3}{2} + 0 - \left[ \frac{1}{3}(-\alpha_0 + 2\alpha_1 + 5\alpha_2 + 3\alpha_3 + \alpha_4) + \frac{1}{10}k^2 \right] T
   - \left[ \frac{1}{2}h_1 + \frac{1}{2}k(\alpha_0 + 5\alpha_1 + 9\alpha_2 + 5\alpha_3 + \alpha_4) \right] T^2 + O(T^3), \]

\[ f_2 = \frac{1}{w_0 - w_3(w_3w_2 + \alpha_2)} - \frac{1}{w_0} = \frac{w_3(w_3w_2 + \alpha_2)}{w_0(w_0 - w_3(w_3w_2 + \alpha_2))} 
   = -\frac{1}{3}a_2T + 0 \cdot T^2 - \frac{1}{3}a_2\left[ \frac{1}{12}(\alpha_0 + 3\alpha_1 + 0\alpha_2 - 3\alpha_3 - 4\alpha_4) - \frac{1}{50}k^2 \right] T^3 
   + \frac{1}{2}h_2 - \frac{1}{2}a_2h_1 + \frac{1}{8}a_2(\alpha_0 + \alpha_1 + 5\alpha_2 + 5\alpha_3 + \alpha_4)k \right] T^4 + O(T^5), \]

\[ f_3 = -w_0^2/w_3 + w_0(2a_2 + a_3 + 2w_2w_3) - w_3(a_2 + w_2w_3)(a_2 + \alpha_3 + w_2w_3) 
   = \frac{3}{T} + 0 - \frac{1}{3}\left[ \alpha_0 + 3\alpha_1 + 5\alpha_2 + 2\alpha_3 - \alpha_4 - \frac{1}{2}k^2 \right] T 
   + \left[ \frac{1}{8}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)k - \frac{1}{2}h_1 \right] T^2 + O(T^3), \]

\[ f_4 = \frac{1}{T} + \frac{1}{2}k + O(T). \]

Therefore, by the same argument as in the preceding subsections, the formal meromorphic solutions (3.6) converge. The relations between the arbitrary constants in (3.6) and \( h_1, h_2, h_4 \) are given by

\[ c_0^0 = \frac{1}{2}k, \quad c_2^3 = -\frac{1}{2}h_1 + \frac{1}{8}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)k, \]

\[ c_4^2 = \frac{1}{2}h_2 - \frac{1}{18}a_2h_1 + \frac{1}{72}a_2(\alpha_0 + \alpha_1 + 5\alpha_2 + 5\alpha_3 + \alpha_4)k. \]

5. **Augmentation of the phase space of the system** \((A_4^{(1)})\)

As for the system \((A_2^{(1)})\), we define a fiber space \( E \) over the parameter space

\[ V = \{ \alpha = (\alpha_0, \ldots, \alpha_4) \in \mathbb{C}^5; \alpha_0 + \cdots + \alpha_4 = 1 \}, \]

of which each fiber \( E(\alpha) \) for \( \alpha \in V \) is the augmented phase space of the system (3.1). Then we observe how Bäcklund transformations act on the space \( E \).

First, we define the space \( E \) and give its properties. Let

\[ I = \{ \emptyset, 0, 1, \ldots, 4, 02, 13, \ldots, 41, 021, 132, \ldots, 410, 0, 1, \ldots, 4, 02, 13, \ldots, 41, 021, 132, \ldots, 410 \}, \]

be a label set and let \( W_* \) \((* \in I)\) be thirty-one copies of \( V \times \mathbb{C}^5 \) with coordinate system \((\alpha, x_\lambda) = (\alpha_0, \ldots, \alpha_4; x_0^\alpha, \ldots, x_4^\alpha) \in W_* \). In order to express identifying relations in simple
form, we use some auxiliary mappings. Let \( \Psi_{i*}, \Psi_{i,}, \overline{\Psi}_{i*}, \overline{\Psi}_{i,} \) (\( i = 0, \ldots, 4 \)) be birational mappings given by \( \Psi_{i*}(\alpha, \xi) = (\alpha, \eta) \) and \( \overline{\Psi}_{i*}(\alpha, \overline{\xi}) = (\alpha, \overline{\eta}) \) where

\[
\eta^{\pm 1} = 1/\xi^{\pm 1}, \quad \eta/\eta^{\pm 1} = \mp \alpha_i - \xi^{\pm 1}, \\
\eta^{\pm 1} + \eta^{\mp 1} = \xi^{\pm 1} + \xi^{\mp 1}, \quad \eta^{\mp 2} = \xi^{\pm 2}, \quad \sum_j \eta^j = \sum_j \xi^j; \\
\overline{\eta}^{\pm 3} = \overline{\xi}^{\pm 3}, \quad \overline{\eta}^{\pm 2} = \overline{\xi}^{\pm 2}, \quad \overline{\eta}^{\pm 2} + \overline{\eta}^{\pm 1} = 1/\overline{\xi}^{\pm 2} + \overline{\xi}^{\pm 1}, \\
\overline{\eta} (\overline{\eta}^{\pm 2} + \overline{\eta}^{\pm 1}) = \mp \alpha_i - \overline{\xi} (\overline{\xi}^{\pm 2} + \overline{\xi}^{\pm 1}), \quad \sum_j \overline{\eta}^j = \sum_j \overline{\xi}^j .
\]

Then we define the space \( \mathbb{E} \) by gluing \( W_* \) via the following identifying equations among coordinates \((\alpha, x_*), (\alpha, x_*)\)

\[
\Psi_{i*}(\alpha, x_*) = (\alpha, x_{i*}), \quad \overline{\Psi}_{i*}(\alpha, x_{i*+1}) = (\alpha, x_{i-1,i+1}), \quad \Psi_{i,}(\alpha, x_{i-1,i+1}) = (\alpha, x_{i-1,i+1}), \\
\Psi_{i,}(\alpha, x_*) = (\alpha, x_{i*}), \quad \overline{\Psi}_{i,}(\alpha, x_{i*+1}) = (\alpha, x_{i-1,i+1}), \quad \Psi_{i,}(\alpha, x_{i-1,i+1}) = (\alpha, x_{i-1,i+1}).
\]

That is,

\[ \mathbb{E} = \left( \bigsqcup_{* \in I} W_* \right) / \sim, \]

where \( \sim \) is the equivalence relation generated by the above equations. Here, we notice that the coordinate systems in (4.1), (4.3) and (4.5) are written as

\[ f = x_*, \quad u = x_{i*}, \quad v = x_{13*}, \quad w = x_{132*}, \]

in our new notation.

In the same way as in Subsection 2.3, we define the subset \( U_* \) of \( \mathbb{E} \) for each \( * \in I \) and coordinate mappings

\[ \varphi_\#: p \in U_* \leftrightarrow (\alpha_0(p), \ldots, \alpha_4(p); x_0(p), \ldots, x_4(p)) \in W_* = V \times \mathbb{C}^5 \quad (* \in I). \]

Then the space \( \mathbb{E} \) is described by the atlas \( \{(\varphi_*, U_*)\} \). We also have the natural projection \( \pi_\#: \mathbb{E} \rightarrow V \). The fiber \( \mathbb{E}(\alpha) := \pi_\#^{-1}(\alpha) \) is a five dimensional complex manifold for each \( \alpha \in V \).

Now we see that the space \( \mathbb{E} \) is decomposed into the disjoint union of subsets, each of which corresponds to a family of meromorphic solutions. Let

\[
\begin{align*}
\mathbb{E}_\# &= U_\#
\\
\mathbb{E}_1 &= U_{1*} \setminus U_\# = (\varphi_\#)^{-1}([x^0_{1*} = 0]) \\
&= U_{1*} \setminus U_\# = (\varphi_\#)^{-1}([x^2_{1*} = 0]), \\
\mathbb{E}_{13} &= U_{13*} \setminus U_\# = (\varphi_{13*})^{-1}([x^0_{13*} = x^2_{13*} = 0]) \\
&= U_{13*} \setminus U_\# = (\varphi_{13*})^{-1}([x^2_{13*} = x^2_{13*} = 0]), \\
\mathbb{E}_{132} &= U_{132*} \setminus U_\# = (\varphi_{132*})^{-1}([x^0_{132*} = x^3_{132*} = 0]) \\
&= U_{132*} \setminus U_\# = (\varphi_{132*})^{-1}([x^1_{132*} = x^1_{132*} = 0]).
\end{align*}
\]
Then
\[ \mathbb{E}_\sigma \cong V \times \mathbb{C}^5, \mathbb{E}_4 \cong V \times \mathbb{C}^4, \mathbb{E}_{13} \cong V \times \mathbb{C}^3, \mathbb{E}_{132} \cong V \times \mathbb{C}^3. \]

Similarly, we define the following subspaces by cyclic rotation
\[ \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4, \mathbb{E}_0; \mathbb{E}_{24}, \mathbb{E}_{30}, \mathbb{E}_{41}, \mathbb{E}_{02}; \mathbb{E}_{243}, \mathbb{E}_{304}, \mathbb{E}_{410}, \mathbb{E}_{021}. \]

Then the space \( \mathbb{E} \) is decomposed by these subsets as
\[
\mathbb{E} = \mathbb{E}_\sigma \sqcup \mathbb{E}_0 \sqcup \mathbb{E}_4 \sqcup \cdots \sqcup \mathbb{E}_4
\]
\[
\sqcup \mathbb{E}_{02} \sqcup \mathbb{E}_{13} \sqcup \cdots \sqcup \mathbb{E}_{41}
\]
\[
\sqcup \mathbb{E}_{021} \sqcup \mathbb{E}_{132} \sqcup \cdots \sqcup \mathbb{E}_{410}.
\]

By arguments analogous to those in Section 4, we have:

**Theorem 5.1.** The vector field
\[
X_p = \sum_{i=0}^{4} \left( x^i(p)\left( x^{i+1}_\sigma(p) - x^{i+2}_\sigma(p) + x^{i+3}_\sigma(p) - x^{i+4}_\sigma(p) \right) + \alpha_i(p) \right) \left( \frac{\partial}{\partial x^i_\sigma} \right)_p
\]
defined on \( \mathbb{E}(\alpha) \cap U_\sigma \) extends to the entire fiber \( \mathbb{E}(\alpha) \) as a holomorphic vector field.

Secondly, we investigate how Bäcklund transformations act on the space \( \mathbb{E} \). For any \( w \in \tilde{\mathbb{W}}(A_4^{(1)}) \), we define a birational mapping \( \sigma_w \) from \( \mathbb{E} \) to itself by
\[
(\alpha_j \circ \sigma_w)(p) = (w(\alpha_j))(p), \quad (x^i_j \circ \sigma_w)(p) = (w(x^i_j))(p), \quad p \in \mathbb{E}, j = 0, \ldots, 4
\]
for any \( * \in I \). Since \( \sigma_{w'} \circ \sigma_w = \sigma_{w'w} \) for \( w, w' \in \tilde{\mathbb{W}}(A_4^{(1)}) \), and since \( \tilde{\mathbb{W}}(A_4^{(1)}) \) is generated by \( s_0, \ldots, s_4 \) and \( \pi \), we need only to study \( \sigma_i := \sigma_{s_i} \) (\( i = 0, \ldots, 4 \)) and \( \sigma_{\pi} \).

We first note that:

**Theorem 5.2.** For any \( w \in \tilde{\mathbb{W}}(A_4^{(1)}) \), \( \sigma_w \) is a biholomorphic mapping from \( \mathbb{E}(\alpha) \) to \( \mathbb{E}(w(\alpha)) \) for any \( \alpha \in V \), and the vector field \( X \) is invariant by \( \sigma_w \).

**Proof.** It is sufficient to show the proposition for \( w = s_0, \ldots, s_4, \pi \) and these special cases can be shown by explicit calculation. By use of the coordinates \( u, v \) and \( w \) defined in the preceding section, we can verify that

\[
s_i(u_j) = \begin{cases} 
    u_j + \frac{\alpha_i}{u_i}u_{ij}, & i = 1, 3, \quad (u_j = x^i_j) \\
    u_j, & i = 0.
\end{cases}
\]

\[
s_i(v_j) = \begin{cases} 
    v_j + \frac{\alpha_i}{v_i}u_{ij}, & i = 1, 2, 3, \quad (v_j = x^i_{13}) \\
    v_j, & i = 0.
\end{cases}
\]

\[
s_i(w_j) = \begin{cases} 
    w_j + \frac{\alpha_i}{w_i}u_{ij}, & i = 2, \quad (w_j = x^i_{132}) \\
    w_j, & i = 0, 3.
\end{cases}
\]

The remaining coordinates can be computed similarly. \( \square \)
Remark 5.3. For any \( \alpha \in V \) and \( w \in \tilde{W}(A_4^{(1)}) \), the augmented phase spaces \( \mathbb{E}(\alpha) \) and \( \mathbb{E}(w(\alpha)) \) are isomorphic.

Now we study how the mapping \( \sigma_i \) \((i = 0, \ldots, 4)\) acts on each component of the decomposition of \( \mathbb{E} \). We observe here only the case of \( i = 2 \), since the other cases are obtained by cyclic rotations. Let

\[
D_2 = \{ x_{\varnothing}^2 = 0 \} = \{ x_{\varnothing}^2 = 0 \} \cup \{ x_{0}^2 = 0 \} \cup \{ x_{4}^2 = 0 \} \cup \{ x_{13}^2 = 0 \}
\]

and

\[
\mathbb{E}' = \mathbb{E} \setminus \pi^{-1}_V(\{ \alpha_2 = 0 \}).
\]

Here \((\varphi_\alpha)^{-1}(\{ x_{\varnothing}^2 = 0 \})\) is simply denoted by \( \{ x_{\varnothing}^2 = 0 \} \) and the closure is that in the space \( \mathbb{E} \). Then the points of \( \mathbb{E} \setminus \pi^{-1}_V(\{ \alpha_2 = 0 \}) \) are mapped as follows:

\[
\begin{align*}
\sigma_2(\mathbb{E}_\varnothing \setminus D_2) &= \mathbb{E}_\varnothing \setminus D_2, & \sigma_2(\mathbb{E}_\varnothing \cap D_2) &= \mathbb{E}_2, & \sigma_2(\mathbb{E}_2') &= \mathbb{E}_\varnothing \cap D_2, \\
\sigma_2(\mathbb{E}_0 \setminus D_2) &= \mathbb{E}_0 \setminus D_2, & \sigma_2(\mathbb{E}_0' \cap D_2) &= \mathbb{E}_0', & \sigma_2(\mathbb{E}_0') &= \mathbb{E}_0' \cap D_2, \\
\sigma_2(\mathbb{E}_4 \setminus D_2) &= \mathbb{E}_4 \setminus D_2, & \sigma_2(\mathbb{E}_4' \cap D_2) &= \mathbb{E}_4', & \sigma_2(\mathbb{E}_4') &= \mathbb{E}_4' \cap D_2, \\
\sigma_2(\mathbb{E}_13 \cap D_2) &= \mathbb{E}_13, & \sigma_2(\mathbb{E}_13') &= \mathbb{E}_13' \cap D_2, \\
\sigma_2(\mathbb{E}_41) &= \mathbb{E}_41, & \sigma_2(\mathbb{E}_41') &= \mathbb{E}_41', & \sigma_2(\mathbb{E}_410) &= \mathbb{E}_410, & \sigma_2(\mathbb{E}_410') &= \mathbb{E}_410', \\
\sigma_2(\mathbb{E}_3) &= \mathbb{E}_3, & \sigma_2(\mathbb{E}_30) &= \mathbb{E}_30, & \sigma_2(\mathbb{E}_30') &= \mathbb{E}_30', & \sigma_2(\mathbb{E}_243) &= \mathbb{E}_243.
\end{align*}
\]

These relationships explain the correspondence between the types of meromorphic solutions and the Bäcklund transformations.

To end this section, we remark that the space \( \mathbb{E} \) contains the augmented phase space for the system \((A_2^{(1)})\) as a submanifold with an appropriate restriction. In fact, setting \( \alpha_3 = \alpha_4 = 0 \), the system (3.1) can be restricted to \( f_3 = f_4 = 0 \). Therefore \((\varphi_\varnothing)^{-1}(\{ \alpha_3 = \alpha_4 = x_{\varnothing}^3 = x_{\varnothing}^4 = 0 \}) \subset \mathbb{E} \) is isomorphic to the fiber space for the system \((A_2^{(1)})\).

6. The \( A_4^{(1)} \) Hamiltonian system

The differential system (3.1) is equivalent to the Hamiltonian system, as is shown in [3]. This is shown by introducing a new coordinate system \((p_1, q_1, p_2, q_2; t)\) in the original phase space \( \{ f \in \mathbb{C}^5 \} \) fixing \( f_0 + \cdots + f_4 = t \). Then the system (3.1) is written as a Hamiltonian system

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2)
\]

with a polynomial Hamiltonian function \( H \) of \( p_1, q_1, p_2, q_2 \) and \( t \). We can also choose a coordinate system of each \( U_\varnothing \), for \( * \in I \) so that the transformation from the coordinate system of \( U_\varnothing \) to that of \( U_* \) is isomorphic. This means that the Hamiltonian system in \( U_\varnothing \) extends to the whole space \( \mathbb{E} \).
We list below the Hamiltonian functions $H_*$ for $\ast = \emptyset, 1_+, 1_-, 13_+, 13_-, 132_+, 132_-$, where the canonical coordinates in $U_\ast$ are denoted by the same notation $(p_1, q_1, p_2, q_2)$ for simplicity. These canonical coordinates are written by the original coordinates $f$ of the system (3.1) as well as by our new coordinates $x_\ast$. Note that we fix $x_0^0 + \cdots + x_4^4 = t$ in every case.

**The Hamiltonian $H_\emptyset$ in $U_\emptyset$**

$$H_\emptyset = (t - q_1 - p_1)q_1p_1 + (t - q_2 - p_2)q_2p_2 - 2q_1p_1q_2 - \alpha_1q_1 + \alpha_2p_1 - (\alpha_1 + \alpha_2)q_2 + \alpha_4p_2,$$

where

$$p_1 = f_1, q_1 = f_2, p_2 = f_1 + f_3, q_2 = f_4;$$

$$(p_1, q_1, p_2, q_2) = (x_1^0, x_2^0, x_2^1 + x_3^3, x_4^4).$$

**The Hamiltonian $H_{1_+}$ in $U_{1_+}$**

$$H_{1_+} = (t - q_1 - p_1)q_1p_1 - t(q_2p_2 + \alpha_1) - q_2(q_2p_2 + \alpha_2) + (\alpha_0 + 2\alpha_1 + \alpha_2)p_1 - 4q_1p_1 + p_2,$$

where

$$p_1 = f_4, q_1 = f_0 + f_2, p_2 = f_2(-\alpha_1 - f_1f_2), q_2 = 1/f_2;$$

$$(p_1, q_1, p_2, q_2) = (x_1^{1_+}, x_1^{0_+}, x_0^{1_+}, x_1^{1_+}).$$

**The Hamiltonian $H_{1_-}$ in $U_{1_-}$**

$$H_{1_-} = (t - q_2 - p_2)q_2p_2 - t(q_1p_1 - \alpha_1) - p_1(\alpha_1 - q_1p_1)(\alpha_0 + \alpha_1 - q_1p_1) + 2q_1p_1q_2 + q_1 - (\alpha_0 + 2\alpha_1 + \alpha_2)q_2 + \alpha_3p_2,$$

where

$$p_1 = 1/f_0, q_1 = f_0(\alpha_1 - f_0f_1), p_2 = f_0 + f_2, q_2 = f_3;$$

$$(p_1, q_1, p_2, q_2) = (x_1^{1_-}, x_1^{0_-}, x_0^{1_-}, x_1^{1_-}).$$

**The Hamiltonian $H_{13_+}$ in $U_{13_+}$**

$$H_{13_+} = -q_1(q_1p_1 + \alpha_1)(q_1p_1 + \alpha_1 + \alpha_2) - q_2(q_2p_2 + \alpha_3)(q_2p_2 + \alpha_3 + \alpha_4) - q_1(q_1p_1 + \alpha_1)(2q_2p_2 + 2\alpha_3 + \alpha_4) - t(q_1p_1 + \alpha_1 + q_2p_2 + \alpha_3) + p_1 + p_2,$$

where

$$p_1 = (f_2 + f_4)(-\alpha_1 - (f_2 + f_4)f_1), q_1 = 1/(f_2 + f_4), p_2 = f_4(-\alpha_3 - f_3f_4), q_2 = 1/f_4;$$

$$(p_1, q_1, p_2, q_2) = (x_1^{13_+}, x_2^{13_+}, x_3^{13_+}, x_4^{13_+}).$$
The Hamiltonian $H_{13_1}$ in $U_{13_1}$.

$$H_{13_1} = -p_1(\alpha_1 - q_1p_1)(\alpha_0 + \alpha_1 - q_1p_1) - p_2(\alpha_3 - q_2p_2)(\alpha_2 + \alpha_3 - q_2p_2)$$

$$- p_2(\alpha_3 - q_2p_2)(\alpha_0 + 2\alpha_1 - 2q_1p_1) + t(\alpha_1 - q_1p_1 + \alpha_3 - q_2p_2) + q_1 + q_2,$$

where

$$p_1 = 1/f_0, \quad q_1 = f_0(\alpha_1 - f_0f_1), \quad p_2 = 1/(f_0 + f_2), \quad q_2 = (f_0 + f_2)(\alpha_3 - (f_0 + f_2)f_3);$$

$$(p_1, q_1, p_2, q_2) = (x^0_{13_1}, x^1_{13_2}, x^0_{13_3} + x^2_{13_3} + x^3_{13_3}).$$

The Hamiltonian $H_{13_2}$ in $U_{13_2}$.

$$H_{13_2} = q_2(q_2p_2 + \alpha_2)(q_2p_2 + \alpha_2 + \alpha_3)(\alpha_0 + 2\alpha_1 - 2q_1p_1 + q_2p_2 + 2\alpha_2 + \alpha_3)$$

$$- p_1(\alpha_1 - q_1p_1 + q_2p_2 + 2\alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 - q_1p_1 + q_2p_2 + 2\alpha_2 + \alpha_3)$$

$$- p_1(q_2p_2(\alpha_0 + 2\alpha_1 - 2q_1p_1) - \alpha_2(\alpha_2 + \alpha_3) + p_1p_2)$$

$$+ t(\alpha_1 - q_1p_1 + q_2p_2 + \alpha_2 + \alpha_3) + q_1,$$

where

$$p_1 = 1/f_0, \quad q_1 - 1/q_2 = f_0(\alpha_1 - f_0f_1),$$

$$q_2(-\alpha_2 - q_2p_2) + p_1 = 1/(f_0 + f_2), \quad 1/q_2 = (f_0 + f_2)(\alpha_3 - (f_0 + f_2)f_3);$$

$$(p_1, q_1, p_2, q_2) = (x^0_{13_2}, x^1_{13_2}, x^3_{13_2} + x^3_{13_3}).$$

The Hamiltonian $H_{13_2}$ in $U_{13_2}$.

$$H_{13_2} = -p_1(\alpha_2 - q_1p_1)(\alpha_1 + \alpha_2 - q_1p_1)(\alpha_1 + 2\alpha_2 - q_1p_1 + 2q_2p_2 + 2\alpha_3 + \alpha_4)$$

$$- q_2(\alpha_1 + 2\alpha_2 - q_1p_1 + q_2p_2 + \alpha_3)(\alpha_1 + 2\alpha_2 - q_1p_1 + q_2p_2 + \alpha_3 + \alpha_4)$$

$$+ q_2(q_1p_1(2q_2p_2 + 2\alpha_3 + \alpha_4) + (\alpha_1 + \alpha_2)\alpha_2 - q_1q_2)$$

$$- t(\alpha_1 + \alpha_2 - q_1p_1 + q_2p_2 + \alpha_3) + p_2,$$

where

$$1/p_1 = (f_2 + f_4)(-\alpha_1 - (f_2 + f_4)f_1), \quad p_1(\alpha_2 - p_1q_1) + q_2 = 1/(f_2 + f_4),$$

$$p_2 - 1/p_1 = f_4(-\alpha_3 - f_3f_4), \quad q_2 = 1/f_4;$$

$$(p_1, q_1, p_2, q_2) = (x^1_{13_2}, x^2_{13_2}, x^3_{13_2} + x^3_{13_2} + x^4_{13_2}).$$
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Nobuhiko Tahara
Graduate School of Science and Technology
Kobe University
Rokko, Kobe
657-8501 Japan
(E-mail: tahara@math.sci.kobe-u.ac.jp)