THE ATOMS OF THE FREE ADDITIVE CONVOLUTION OF
TWO OPERATOR-VALUED DISTRIBUTIONS

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Abstract. We find the atoms of the free additive convolution of two operator-valued distributions. This result allows one, via the linearization trick, to determine the atoms of the distribution of a selfadjoint polynomial in two free selfadjoint random variables.

1. Introduction

Consider a tracial von Neumann algebra \( A \) containing a von Neumann subalgebra \( B \), and let \( E: A \to B \) be the trace-preserving conditional expectation. Assume that \( X = X^*, Y = Y^* \in A \) are free with amalgamation over \( B \), and that \( p := \ker(X + Y) \neq \{0\} \). In this paper, we study what consequences this assumption has on the spectral distribution of \( X \) and/or \( Y \). This type of question has been first answered in the case of scalar-valued distributions (i.e. when \( B = \mathbb{C} \cdot 1 \)) by Bercovici and Voiculescu in [9]: the existence of \( p \) is equivalent to the existence of projections \( q \) and \( r \) and of a real number \( \gamma \) such that \( q = \ker(X - \gamma) \), \( r = \ker(Y + \gamma) \), and \( \tau(p) + 1 = \tau(q) + \tau(r) \) (see [9, Theorem 7.4]). The proof uses the analytic subordination functions of Voiculescu and Biane [20, 10].

In this paper, we provide a characterization in terms of Voiculescu’s operator-valued subordination functions [22, 23] of elements \( X, Y \) for which the above hypothesis is satisfied. We are able to provide a complete answer if either (i) \( E[p] > 0 \) in \( B \) (that is, \( E[p] \) is positive and invertible in \( B \)), or (ii) \( B \) is a finite dimensional algebra, with no restriction on \( E[p] \). As a corollary to this second case, we give a complete answer to the question under what circumstances a selfadjoint noncommutative rational expression \( P(X, Y) \in A \) evaluated in two selfadjoint bounded variables \( X \) and \( Y \) which are free over \( \mathbb{C} \) can have a nontrivial kernel.

In recent years there were numerous results on the lack of atoms in the distributions of sums of operator valued random variables and of polynomials in free random variables [18, 12, 15, 3], as well as the occurrence of “trivial” (in the above sense) invariant projections [18, 10]. As of now, with the exception of [4], we are not aware of results that indicate the existence and properties of invariant projections for \( X, Y \).

2. Analytic tools

Consider a tracial von Neumann algebra \((A, \tau)\) containing a von Neumann subalgebra \( B \). We shall assume throughout the paper that \( A \) acts on the Hilbert space \( \mathcal{H} := L^2(A, \tau) \), which is the completion of \( A \) with respect to the inner product \( \langle \xi, \eta \rangle = \tau(\eta^* \xi) \). It is known (see, for instance, [19]) that there exists a unique trace-preserving conditional expectation \( E: A \to B \), which appears as the restriction to \( A \) of the orthogonal projection from \( L^2(A, \tau) \) onto \( L^2(B, \tau) \). If \( c \in A \), we
write $c \geq 0$ if $c = c^*$ and the spectrum $\sigma(c) \subseteq [0, +\infty)$, and we write $c > 0$ if $c \geq 0$ and $\sigma(c) \subseteq (0, +\infty)$. For any $c \in \mathcal{A}$, we have $c = \Re c + i\Im c$, where $\Re c = \frac{c + c^*}{2}$ and $\Im c = \frac{c - c^*}{2i}$. We define

$$H^+(\mathcal{A}) = \{ c \in \mathcal{A}; \Im c > 0 \},$$

$H^-(\mathcal{A}) = -H^+(\mathcal{A})$ and similar for $B$, or any other von Neumann algebra. As there will be no risk of confusion, we will use the same notations to define the noncommutative extensions of these sets (see [14]).

2.1. Analytic transforms. Assume that $X = X^*, Y = Y^* \in \mathcal{A}$ are free over $B$ with respect to the conditional expectation $E$ (see [21]). Define the analytic map on the noncommutative upper half-plane

$$G_X : H^+(B) \to H^-(B), \quad G_X(b) = E[(b - X)^{-1}].$$

As shown in [24], $G_X$ is a free noncommutative map in the sense of [14], which fully encodes the distribution of $X$ with respect to $E$. The map $w \mapsto G_X(w^{-1})$ extends to the noncommutative ball of radius $\|X\|^{-1}$, centered at zero: indeed, $G_X(w^{-1}) = \sum_{n=0}^{\infty} E[w(Xw)^n]$ converges in norm for $\|w\| < 1/\|X\|$. We shall call $G_X$ the noncommutative Cauchy transform of the (distribution of) $X$.

Since we use it often, it will be convenient to denote the reciprocal of $G_X$ by $F_X$:

$$F_X(b) = G_X(b)^{-1}, \quad b \in H^+(B) \text{ or } \|b^{-1}\| < \|X\|^{-1}.$$ 

It has been shown in [7, Remark 2.5] that $\exists F_X(b) \geq 3b, b \in H^+(B)$.

As in [22], let $B(X)$ denote the von Neumann algebra generated by $B$ and $X$. Denote by $E_X : \mathcal{A} \to B(X)$ the unique trace-preserving conditional expectation from $\mathcal{A}$ to $B(X)$. It is shown in [22] that there exists a free noncommutative analytic map $\omega_1 : H^+(B) \to H^+(B)$ such that

$$E_X[(b - X - Y)^{-1}] = (\omega_1(b) - X)^{-1}, \quad b \in H^+(B) \text{ or } \|b^{-1}\| < \|X + Y\|^{-1}.$$ 

A similar statement holds for a map $\omega_2$, if we interchange $X$ and $Y$. By applying $E$ to (1) and using Voiculescu’s $R$-transform [21, 24], it is shown in [8] that

$$F_{X+Y}(b) = F_X(\omega_1(b)) = F_Y(\omega_2(b)) = \omega_1(b) + \omega_2(b) - b, \quad b \in H^+(B).$$

(See [9] for the scalar version of this relation.) The above relation extends to elements $b$ such that $\|b^{-1}\| < \|X + Y\|^{-1}$. It is also shown in [22] that

$$\exists \omega_j(b) \geq 3b, \quad \omega_j(b^*) = \omega_j(b)^*, \quad b \in H^+(B), j = 1, 2.$$ 

2.2. Kernels from Borel functional calculus. Assume $T = T^* \in \mathcal{A}$. Denote by $\lim_{z \to a}$ the limit as $z$ approaches $a \in \mathbb{R}$ from the complex upper half-plane non-tangentially to $\mathbb{R}$. If $f : \sigma(T) \to \mathbb{C}$ is a bounded Borel function, we denote by $f(T)$ the operator in the von Neumann algebra generated by $T$ given by the Borel functional calculus (see, for instance, [2]).

Lemma 2.1. We have

$$\lim_{z \to a} (z - a)(z - T)^{-1} = \chi_K(T),$$

in the strong operator (so) topology, where $\chi_K$ denotes the characteristic function of the Borel set $K \subseteq \mathbb{R}$.
Proof. This is a consequence of the strong operator (so) continuity of the Borel functional calculus. The essential part of the proof can be found for instance in [9]. We sketch it here for convenience. For any vector \( \xi \in \mathcal{H} \) of \( L^2 \)-norm equal to one, we write
\[
\| (z-a) (z-T)^{-1} \|_2^2 = \langle (z-a) (z-T)^{-1} \xi, (z-a) (z-T)^{-1} \xi \rangle \\
= \langle (x-a)^2 + y^2 \rangle \langle (x-T)^2 + y^2 \rangle^{-1} \xi, \xi \rangle \\
= \int_{\mathbb{R}} \frac{(x-a)^2 + y^2}{(x-t)^2 + y^2} \, d\mu_{\xi,T}(t),
\]
where \( z = x + iy \) is the decomposition in real and imaginary parts of \( z \) and \( \mu_{\xi,T} \) is the distribution of the selfadjoint random variable \( T \) with respect to the expectation (state) \( \cdot \mapsto \langle \xi, \xi \rangle \). The dominated convergence theorem guarantees that
\[
\lim_{z \rightharpoonup a} \int_{\mathbb{R}} \frac{(x-a)^2 + y^2}{(x-t)^2 + y^2} \, d\mu_{\xi,T}(t) = \mu_{\xi,T}(\{a\}),
\]
allowing us to conclude. \( \square \)

Remark 2.2. As \( E,E_X \) are weak operator (wo) and strong operator (so) continuous, the above lemma implies
\[
\lim_{z \rightharpoonup a} (z-a) E [(z-T)^{-1}] = E[p], \quad \lim_{z \rightharpoonup a} (z-a) E_X [(z-T)^{-1}] = E_X[p],
\]
in the so topology. Similarly, we have
\[
\lim_{z \rightharpoonup a} \Re (z-a) (z-T)^{-1} = p, \quad \lim_{z \rightharpoonup a} \Im (z-a) (z-T)^{-1} = 0.
\]
In particular,
\[
\lim_{y \downarrow 0} y (a-T) ((a-T)^2 + y^2)^{-1} = 0, \quad \lim_{y \downarrow 0} y^2 ((a-T)^2 + y^2)^{-1} = p.
\]

2.3. The noncommutative Julia-Carathéodory derivative. One other important analytic tool available to us is the noncommutative version of the Julia-Carathéodory Theorem (see [6] Theorem 2.2). We reproduce here the statements from [6] [5] that are relevant to our proofs below.

Theorem 2.3. Let \( \mathcal{M}, \mathcal{N} \) be two von Neumann algebras and let \( f : H^+(\mathcal{M}) \to H^+(\mathcal{N}) \) be a free noncommutative map. If there exists \( v \in \mathcal{M}, v > 0 \), such that
\[
\liminf_{y \to 0} \varphi \left( \frac{\Im f(\alpha + iyw)}{y} \right) < \infty
\]
for all wo continuous states \( \varphi \) on \( \mathcal{N} \), then
\[
(1) \text{ so-} \lim_{y \to 0} \frac{\Im f(\alpha \otimes 1_n + iyw)}{y} \text{ exists and is strictly positive for any } n \in \mathbb{N}, w \in M_n(\mathcal{N}), w > 0; \\
(2) \lim_{y \to 0} f(\alpha \otimes 1_n + iyw) = f(\alpha) \otimes 1_n \text{ exists in the norm topology of } M_n(\mathcal{N}) \text{ and is selfadjoint}; \\
(3) \lim_{y \to 0} \Im \frac{f(\alpha \otimes 1_n + iyw) - f(\alpha) \otimes 1_n}{y} = 0.
\]

We need one more (very simple) fact about the functions that behave like reciprocals of noncommutative Cauchy transforms.
Lemma 2.4. Assume that $f: \mathbb{C}^+ \to H^+(B)$ is a free noncommutative function in the sense of [14]. For any $a \in \mathbb{R}$, the so limit
\[
\lim_{y \to 0} \left[ \frac{\Im f(a + iy)}{y} \right]^{-1}
\]
even exists and is finite.

Proof. The proof is based on the representation of free noncommutative maps of noncommutative half-planes provided by [17, 25]: there exists a completely positive map $\rho: \mathcal{C}(X) \to B$, an element $a = a^*$ and $\beta \geq 0$ in $B$ such that
\[
f(z) = a + z\beta + \rho \left[ (X - z)^{-1} \right], \quad z \in H^+(\mathbb{C}).
\]
Then $\Im f(z) = \Im z\beta + \rho \left[ (X - z)^{-1}\Im(X - z)^{-1} \right] = \Im z\beta + \rho \left[ \frac{\Im z}{X - z + (\Im X)^{-1}} \right]$. Here $X$ is a selfadjoint operator. Thus,
\[
y(\Im f(a + iy))^{-1} = (\beta + \rho \left[ (X - a - iy)^{-1}(X - a + iy)^{-1} \right])^{-1}.
\]
Trivially the map $y \mapsto (X - a - iy)^{-1}(X - a + iy)^{-1}$ is decreasing. This concludes the proof. \[\square\]

Unsurprisingly, we shall apply Theorem 2.3 to the reciprocal of the noncommutative Cauchy transform.

Lemma 2.5. Let $B$ be an arbitrary von Neumann subalgebra of an arbitrary von Neumann algebra $A$ such that there exists an so-continuous unit-preserving, faithful conditional expectation $E: A \to B$. If $T = T^* \in A$ and $\gamma = \gamma^* \in B$ are such that $\ker(T - \gamma) = p \neq 0$ and $E[p] > 0$, then
\[
\lim_{y \to 0} \frac{\Im Fr(\gamma + iy)}{y} = E[p]^{-1}, \quad \lim_{y \to 0} \frac{\Re Fr(\gamma + iy)}{y} = 0.
\]
If in addition $B$ is finite dimensional and $E[p]$ is not invertible, let $q = \ker(E[p])^\perp$.
Then $E_q: qAq \to qBq$, $E_q[x] = E[qx]$, $x \in qAq$, is an so-continuous unit preserving, faithful conditional expectation, $qp = pq = p$, $E_q[p] > 0$ in $qBq$, and the map $F_q E_q: H^+(qBq) \to H^+(qBq)$ satisfies
\[
\lim_{y \to 0} \frac{\Im Fr(T q + i y q)}{y} = E_q[p]^{-1}, \quad \lim_{y \to 0} \frac{\Re Fr(T q + i y q)}{y} = 0.
\]
Proof. The proof is straightforward. As seen in Remark 2.2 so-lim$_{y \to 0} y \Im G_T(\gamma + iy) = -E[p]$, so-lim$_{y \to 0} y \Re G_T(\gamma + iy) = 0$, so that
\[
\lim_{y \to 0} \frac{\Im Fr(\gamma + iy)}{y} = \lim_{y \to 0} \frac{\Re Fr(\gamma + iy) - y \Re G_T(\gamma + iy)(y \Im G_T(\gamma + iy))^{-1} y \Re G_T(\gamma + iy)}{y E[p]^{-1},
\]
since $y \Re G_T(\gamma + iy)(y \Im G_T(\gamma + iy))^{-1} y \Re G_T(\gamma + iy)$ tends to $0 \cdot E[p]^{-1} \cdot 0 = 0$. According to Theorem 2.3(2), the norm limit $\lim_{y \to 0} F_T(\gamma + iy) = Fr(\gamma)$ exists and is selfadjoint. Since $\lim_{y \to 0} F_T(\gamma + iy) = \lim_{y \to 0} iy(y \Im G_T(\gamma + iy))^{-1} = 0 \cdot E[p]^{-1} = 0$, it follows that $F_T(\gamma) = 0$. By Theorem 2.3(3), we have
\[
0 = \lim_{y \to 0} \frac{\Re F_T(\gamma + iy) - F_T(\gamma)}{y} = \lim_{y \to 0} \frac{\Re F_T(\gamma + iy)}{y},
\]
Lemma 2.5, we have \[ \lim_{y \to 0} (1 - q)pp(1 - q) = (1 - q)E[p](1 - q) = 0, \]
so, by \( E \)'s faithfulness, \( (1 - q)pp(1 - q) = 0 \), i.e. \( (1 - q)p = 0 = p(1 - q) \), which is equivalent to \( pq = qp = p \). Thus, \( (q\gamma q - qTq)p = q(\gamma - T)qp = (\gamma - T)qp = (\gamma - T)pq = q0q = 0 \) and \( (q\gamma q - qTq)p^* = q(\gamma - T)qp^* = p^*q(\gamma - T)q = qp^*(\gamma - T)q = q(\gamma - T)q \). So \( p = \ker(q\gamma q - qTq) \) in \( q\mathcal{H} \). Since \( E[p] = qE[p]q \) is invertible in \( qBq \), it follows that the above apply to \( GqTq \) and \( FqTq \) viewed as maps on \( H^+(qBq) \). \( \square \)

3. The case of arbitrary scalar von Neumann algebra \( B \) and invertible \( E[p] \)

We remind the reader the context of the problem and our hypotheses: \( (A, \tau) \) is a tracial von Neumann algebra (with normal faithful tracial state \( \tau \)), \( B \subset A \) is a von Neumann subalgebra of \( A \), \( E : A \to B \) is the unique trace-preserving conditional expectation from \( A \) to \( B \), and \( X = X^*, Y = Y^* \in A \) are two bounded selfadjoint random variables which are free with respect to \( E \) over \( B \). Also, \( B(X) \) (respectively \( B(Y) \)) is the von Neumann algebra generated by \( B \) and \( X \) (respectively \( B \) and \( Y \)) in \( A \), and \( E_X : A \to B(X) \) (respectively \( E_Y : A \to B(Y) \)) is the unique trace-preserving conditional expectation from \( A \) onto \( B(X) \) (respectively \( B(Y) \)). Finally, \( A \) acts faithfully on the Hilbert space \( \mathcal{H} = L^2(A, \tau) \), which is the completion of \( A \) with respect to the inner product \( \langle \xi, \eta \rangle = \tau(\eta^* \xi) \).

We assume that \( a = a^* \in B \) and \( \ker(X + Y - a) = p \neq 0 \). As seen in Lemma 2.4 we have so-lim \( z^{-1}(z - a)E[(z - X - Y)^{-1}] = E[p] \), and, by Remark 2.2

\[
\lim_{z \to a} (z - a)E[(z - X - Y)^{-1}] = E[p], \quad \lim_{z \to a} (z - a)E_X[(z - X - Y)^{-1}] = E_X[p].
\]

A similar statement holds if we interchange \( X \) and \( Y \). According to [2], [3], and Remark 2.5], we have \( \exists F_{X+Y}(b) \geq \exists \omega_j(b), \) \( j = 1, 2 \). Moreover, according to Lemma 2.4 we have \( \lim_{y \to 0} F_{X+Y}(a+iy) = 0 \) in norm, so that, by [2] and Theorem 2.3

1. \( \lim_{y \to 0} \omega_j(a + iy) = \omega_j(a), \) \( j = 1, 2 \) exist in the norm topology and are selfadjoint;
2. so-lim \( y \to 0 \frac{\omega_j(a+iy)}{y} = \omega_j \) exist and are strictly positive;
3. \( \omega_1(a) + \omega_2(a) = a; \)
4. so-lim \( y \to 0 \frac{\rho_\omega(a+iy) - \omega_j(a)}{y} = 0, \) \( j = 1, 2 \).

We can now state and prove the main result of this section.

**Theorem 3.1.** Let \( X, Y \) be two selfadjoint random variables in the tracial \( W^* \)-probability space \( (A, \tau) \). Assume that \( B \) is a von Neumann subalgebra of \( A \), and \( X, Y \) are free with amalgamation over \( B \) with respect to the trace-preserving conditional expectation \( E \). If \( \ker(X + Y - a) = p \) and \( E[p] > 0 \), then

\[
\ker\left( \omega_1^{-\frac{1}{2}}(X - \omega_1(a))\omega_1^{-\frac{1}{2}} \right) = E_X[\omega_1^{-\frac{1}{2}}p\omega_1^{-\frac{1}{2}}]
\]

and

\[
\ker\left( \omega_2^{-\frac{1}{2}}(Y - \omega_2(a))\omega_2^{-\frac{1}{2}} \right) = E_Y[\omega_2^{-\frac{1}{2}}p\omega_2^{-\frac{1}{2}}].
\]
Moreover, \[ 1 + \tau(p) = \tau(E_X[\varpi_1^{\frac{1}{2}} p \varpi_1^{\frac{1}{2}}]) + \tau(E_Y[\varpi_2^{\frac{1}{2}} p \varpi_2^{\frac{1}{2}}]). \]

**Proof.** Let us write Voiculescu’s subordination relation (1) according to our needs:

\[ E_X[p] = \lim_{y \to 0} iy E_X[(iy + a - X - Y)^{-1}] = \lim_{y \to 0} iy (\omega_1(iy + a) - X)^{-1}. \]

Consider now the following difference:

\[ i y (i y \varpi_1 - (X - \omega_1(a)))^{-1} - i y (\omega_1(a + iy) - X)^{-1} = \]

\[ i y (i y \varpi_1 - (X - \omega_1(a)))^{-1} (\omega_1(a + iy) - X - iy \varpi_1 + X - \omega_1(a)) \]

\[ \times (\omega_1(a + iy) - X)^{-1} = i y (i y \varpi_1 - (X - \omega_1(a)))^{-1} \]

\[ \times \left( \frac{\Re \omega_1(a + iy) - \omega_1(a)}{iy} + \frac{3 \omega_1(a + iy)}{iy} - \varpi_1 \right) iy (\omega_1(a + iy) - X)^{-1} \]

(4)

As seen above, so-\( \lim_{y \to 0} iy (\omega_1(a + iy) - X)^{-1} = E_X[p] \) boundedly. Since

\[ \lim_{y \to 0} iy (i y \varpi_1 - (X - \omega_1(a)))^{-1} = \varpi_1^{-\frac{1}{2}} \lim_{y \to 0} iy \left( i y - \varpi_1^{-\frac{1}{2}} (X - \omega_1(a)) \varpi_1^{-\frac{1}{2}} \right)^{-1} \varpi_1^{-\frac{1}{2}} \]

\[ = \varpi_1^{-\frac{1}{2}} \ker \left( \varpi_1^{-\frac{1}{2}} (X - \omega_1(a)) \varpi_1^{-\frac{1}{2}} \right) \varpi_1^{-\frac{1}{2}}, \]

again boundedly in the so topology, the convergence of the middle factor in (4) to zero in the so topology, guaranteed by items 2 and 4 above, allows us to conclude that the difference in (4) converges to zero, and thus ker \( \left( \varpi_1^{-\frac{1}{2}} (X - \omega_1(a)) \varpi_1^{-\frac{1}{2}} \right) \) = \( \varpi_1^{-\frac{1}{2}} E_X[p] \varpi_1^{-\frac{1}{2}} = E_X[\varpi_1^{\frac{1}{2}} p \varpi_1^{\frac{1}{2}}] \), as stated in our theorem (recall that \( \varpi_1^{\frac{1}{2}} \in B \subseteq B(X) \)). Similarly, ker \( \left( \varpi_2^{-\frac{1}{2}} (Y - \omega_2(a)) \varpi_2^{-\frac{1}{2}} \right) = \varpi_2^{-\frac{1}{2}} E_Y[p] \varpi_2^{-\frac{1}{2}} = E_Y[\varpi_2^{\frac{1}{2}} p \varpi_2^{\frac{1}{2}}] \).

Multiplying by \( G_{X+Y}(iy + a) \) in (2) and taking into consideration item 3 we obtain, as in (3),

\[ 1 + iy G_{X+Y}(iy + a) = \omega_1(a + iy) - \omega_1(a) G_X(\omega_1(a + iy)) + \omega_2(a + iy) - \omega_2(a) G_Y(\omega_2(a + iy)). \]

The left hand side converges to 1 + \( E[p] \) in the so topology. In the right hand side, \( (\omega_1(a+iy) - \omega_1(a)) G_X(\omega_1(a+iy)) = \frac{\Re \omega_1(a+iy) - \omega_1(a) + i 3 \omega_1(a + iy)}{iy} iy G_{X+Y}(a+iy) \),

which provides us again (via items 4 and 2 above) with \( \varpi_1 E[p] \). Thus,

\[ 1 + E[p] = (\varpi_1 + \varpi_2) E[p] = E[p](\varpi_1 + \varpi_2). \]

Taking trace, we obtain the relation

\[ 1 + \tau(p) = \tau(E[\varpi_1^{\frac{1}{2}} p \varpi_1^{\frac{1}{2}}]) + \tau(E[\varpi_2^{\frac{1}{2}} p \varpi_2^{\frac{1}{2}}]) = \tau(E_X[\varpi_1^{\frac{1}{2}} p \varpi_1^{\frac{1}{2}}]) + \tau(E_Y[\varpi_2^{\frac{1}{2}} p \varpi_2^{\frac{1}{2}}]). \]

\[ \square \]

**Remark 3.2.** In the proof of the first part of the above theorem, we have only used the fact that \( E_X[(iy + a - X - Y)^{-1}] = (\omega_1(a + iy) - X)^{-1} \), with \( \omega_1 \) being a self-map of the upper half-plane of \( B \). If there were non-tracial probability spaces in which this property holds, the formulae for the kernels of \( \varpi_1^{-\frac{1}{2}} (X - \omega_1(a)) \varpi_1^{-\frac{1}{2}} \)
and \( \varpi_2^{-\frac{1}{2}} (Y - \omega_2(a)) \varpi_2^{-\frac{1}{2}} \) would remain true. At this moment, we are not aware of any examples of non-tracial probability spaces in which this property would hold.

However, some non-trivial conclusions can be drawn even if Voiculescu’s relation

\[ E_X ((a + iy - X - Y)^{-1}) = (\omega_1(a + iy) - X)^{-1} \]

does not hold (or even if \( E_X \) does not exist). It is shown in [8] that (2) holds whenever \( X, Y \) are free over \( B \) with respect to \( E \) (in particular, the weaker version of Voiculescu’s relation, namely

\[ E [(iy + a - X - Y)^{-1}] = E [(\omega_1(iy + a) - X)^{-1}] \]

holds). Thus, items [1][4] above still hold for the two subordination functions. Under the assumptions that \( E \) is so continuous and faithful, this allows us to immediately establish relation (5). In addition, we have

\[
E[p] = \lim_{y \to 0} iy E[(iy + a - X - Y)^{-1}]
\]

\[
= \lim_{y \to 0} iy E[(\omega_1(a + iy) - X)^{-1}]
\]

\[
= \lim_{y \to 0} \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}}
\]

\[
\times i E \left[ \left( i - \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} (X - \Re \omega_1(a + iy)) \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} \right)^{-1} \right]
\]

\[
\times \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}}
\]

\[
= \varpi_1^{-\frac{1}{2}}
\]

\[
\times \lim_{y \to 0} iy E \left[ \left( iy - \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} (X - \Re \omega_1(a + iy)) \left[ \frac{3\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} \right)^{-1} \right]
\]

(6)

Generally, if a family of operators \( A_y, y > 0 \), converges in the so topology to the bounded operator \( A \) and there exists a sequence \( y_n \) converging to zero such that

\[ iy_n(iy_n - A_{y_n})^{-1} \]

converges in the weak operator topology to a non-zero operator \( r \) as \( n \to \infty \), then it follows quite easily that \( ker A \neq 0 \), and in fact \( Ar = rA = 0 \) (see, for instance, [11] Lemma 3.2). Applying this to \( A_y = \left[ \frac{3\omega_1(iy)}{y} \right]^{-\frac{1}{2}} (X - \Re \omega_1(iy)) \left[ \frac{3\omega_1(iy)}{y} \right]^{-\frac{1}{2}} \) and using the faithfulness of the conditional expectation \( E \), we conclude that \( ker \left( \varpi_1^{-\frac{1}{2}} (X - \omega_1(0)) \varpi_1^{-\frac{1}{2}} \right) \neq 0 \).

4. The case of finite dimensional scalar von Neumann algebra \( B \) and possibly non-invertible \( E[p] \)

We consider the context from Section [3], with the additional assumption that \( B \) is a finite dimensional von Neumann algebra, and thus isomorphic to an algebra of matrices. As in Lemma [2][3], we denote by \( q = ker(E[p])^\perp \), and recall that \( q \geq p \) and that \( ker(qXq + qYq) = p \) in \( qAq \). In addition, we easily see that the expectation \( E_q \) defined in Lemma [2][3] is the unique trace-preserving conditional expectation with respect to the normal faithful tracial state \( \tau_q(\cdot) = \frac{\tau(q\cdot q)}{\tau(qq)} \) on \( qAq \). If \( X \) and \( Y \) are free with amalgamation over \( B \) with respect to \( E \), then it follows trivially that \( qXq \) and \( qYq \) are free with amalgamation over \( qBq \) with respect to \( E_q \). Since we have
Corollary 4.1. Let $X, Y$ be two self-adjoint random variables in the tracial $W^*$-probability space $(A, \tau)$. Assume that $B$ is a finite dimensional von Neumann sub-algebra of $A$, $a = a^* \in B$, $X, Y$ are free with amalgamation over $B$ with respect to the trace-preserving conditional expectation $E$, and $\ker(X + Y - a) = p \neq 0$. Let $q = \ker(E[p])^\perp$.

1. The limits $\varpi_j^{-1} := \lim_{y \to 0} \left[ \frac{\Im \omega_j(a + iy)}{y} \right]^{-1}$ and

$$\varpi_j := \lim_{y \to 0} \left[ \frac{\Im \omega_j(a + iy)}{y} \right]^{-\frac{1}{2}} \Re \omega_j(a + iy) \left[ \frac{\Im \omega_j(a + iy)}{y} \right]^{-\frac{1}{2}}, \quad j = 1, 2,$$

exist, and $\varpi_j^{-1} \geq E[p]$;

2. $\ker\left( \varpi_1^{-\frac{1}{2}} X \varpi_1^{-\frac{1}{2}} - \varpi_1 \right) \ominus \ker(\varpi_1^{-1}) \neq \{0\}$;

$$\ker\left( \varpi_2^{-\frac{1}{2}} Y \varpi_2^{-\frac{1}{2}} - \varpi_2 \right) \ominus \ker(\varpi_2^{-1}) \neq \{0\}.$$

3. Let $\omega_{1,q}$ and $\omega_{2,q}$ be Voiculescu’s analytic subordination functions corresponding to $qXq$ and $qYq$, respectively, as self-maps of $H^+(qBq)$. Then the results of Theorem 3.1 hold, with $\ker(qXq + qYq) = p \in qAq$ such that

$$\ker\left( \varpi_{1,q}^{-\frac{1}{2}} (qXq - \omega_{1,q}(qaq)) \varpi_{1,q}^{-\frac{1}{2}} \right) = E_{qXq}[\varpi_{1,q}^{-\frac{1}{2}} p \varpi_{1,q}^{-\frac{1}{2}}],$$

$$\ker\left( \varpi_{2,q}^{-\frac{1}{2}} (qYq - \omega_{2,q}(qaq)) \varpi_{2,q}^{-\frac{1}{2}} \right) = E_{qYq}[\varpi_{2,q}^{-\frac{1}{2}} p \varpi_{2,q}^{-\frac{1}{2}}],$$

and

$$1 + \tau_q(p) = \tau_q(E_{qXq}[\varpi_{1,q}^{-\frac{1}{2}} p \varpi_{1,q}^{-\frac{1}{2}}]) + \tau(E_{qYq}[\varpi_{2,q}^{-\frac{1}{2}} p \varpi_{2,q}^{-\frac{1}{2}}]).$$

Proof. Part (3) is an immediate consequence of Lemma 2.5 and Theorem 3.1. The existence of the limit $\varpi_j^{-1} := \lim_{y \to 0} \left[ \frac{\Im \omega_j(a + iy)}{y} \right]^{-1}$ follows from Lemma 2.4. The limit in (7) has been proved to exist in the weak operator topology in [4]. Since $B$ is finite dimensional, the weak operator topology is equivalent to the norm topology. Using Remark 2.2 and the fact that

$$\Im \left( (\omega_1(z) - X)^{-1} \right) = - \left( \Im \omega_1(z) + (X - \Re \omega_1(z))(\Im \omega_1(z))^{-1}(X - \Re \omega_1(z)) \right)^{-1},$$
we obtain

\[ E[p] = \lim_{y \to 0} yE \left[ \frac{y}{(X + Y - a)^2 + y^2} \right] = -\lim_{y \to 0} y\Im \left[ (iy + a - X - Y)^{-1} \right] \]
\[ = -\lim_{y \to 0} yE \left[ \Im(\omega_1(a + iy) - X)^{-1} \right] \]
\[ = \lim_{y \to 0} \frac{\Im\omega_1(a + iy)}{y} \times E \left[ \left( 1 + \left( \Im\omega_1(a + iy) \right)^{-\frac{1}{2}} (X - \Re\omega_1(a + iy)) \left( \Im\omega_1(a + iy) \right)^{-\frac{1}{2}} \right)^{-1} \right] \]
\[ \times \frac{\Im\omega_1(a + iy)}{y} \]
\[ \leq \lim_{y \to 0} \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-1} = \varpi_1^{-1}. \]

Finally, we show that \( \ker(\varpi_1^{-\frac{1}{2}}X\varpi_1^{-\frac{1}{2}} - \varpi_1) \cap \ker(\varpi_1^{-1}) \neq \{0\} \). It has been shown in [4] that the set \( \left\{ \left\| \Re\omega_1(a + iy) \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} \right\| : y \in (0, 1) \right\} \) is bounded. This implies that \( \ker(\varpi_1) \geq \ker(\varpi_1^{-1}) \). In particular, we immediately obtain that \( \ker(\varpi_1^{-\frac{1}{2}}X\varpi_1^{-\frac{1}{2}} - \varpi_1) \geq \ker(\varpi_1^{-1}) \). We show that this inequality must be strict.

We have

\[ E_X[p] = \lim_{y \to 0} \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-1/2} \]
\[ \times iy \left( iy - \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-1/2} (X - \Re\omega_1(a + iy)) \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-1/2} \right)^{-1} \]
\[ \times \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-1/2} \]
\[ = \varpi_1^{-\frac{1}{2}} \lim_{y \to 0} iy \left( iy - \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} (X - \Re\omega_1(a + iy)) \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} \right)^{-1} \]
\[ \times \varpi_1^{-\frac{1}{2}}. \]

Since \( \left\| (i - \Im\omega_1(a + iy))^{-1/2} (X - \Re\omega_1(a + iy)) \left[ \Im\omega_1(a + iy) \right]^{-1/2} \right\| < 1 \) for all \( y > 0 \), there exists a sequence \( y_n \) converging to zero so that

\[ r := \lim_{n \to \infty} iy \left( iy - \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} (X - \Re\omega_1(a + iy)) \left[ \frac{\Im\omega_1(a + iy)}{y} \right]^{-\frac{1}{2}} \right)^{-1} \]

exists. Since \( E_X[p] \neq 0 \), by [4] we necessarily have \( r \neq 0 \). As shown in [4] Lemma 3.2, \( r \left( \varpi_1^{-\frac{1}{2}}X\varpi_1^{-\frac{1}{2}} - \varpi_1 \right) = \left( \varpi_1^{-\frac{1}{2}}X\varpi_1^{-\frac{1}{2}} - \varpi_1 \right) r = 0 \). This implies that \( \ker(\varpi_1^{-\frac{1}{2}}X\varpi_1^{-\frac{1}{2}} - \varpi_1) \geq \text{ran}(r) \). However, since \( 0 \leq E_X[p] = \varpi_1^{-\frac{1}{2}} r \varpi_1^{-\frac{1}{2}} \), we conclude that the range of \( r \) must be strictly bigger than \( \ker(\varpi_1^{-1}) \), as claimed. The same holds for \( Y \). This concludes our proof. \( \square \)
5. Application to linearizations of polynomials

It has been a longstanding question, first answered in [18], whether the distribution of a nontrivial selfadjoint polynomial in two bounded selfadjoint random variables whose distributions have no atom can or cannot have itself an atom. Since then, there were several results regarding the lack of atomic part for distributions of selfadjoint polynomials in free random variables. Here we show, via the results of the previous sections, under what conditions the distribution of a polynomial \(P(X, Y)\) has a nontrivial atomic part. This will be the consequence of the linearization process (see [18]) and the following abstract result. (In the following, we shall make all the assumptions on our probability space and random variables outlined at the beginning of Section 3)

**Proposition 5.1.** Let \(L = \begin{bmatrix} 0 & u^* \\ u & Q \end{bmatrix} \in M_{n+1}(A)\) be such that \(Q = Q^*\) is invertible in \(M_n(A)\). Assume that \(\ker(u^*Q^{-1}u) = p \neq 0\) in \(A\). Then \(\pi := \ker(L)\) is Murray-von Neumann equivalent to the projection \(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(A)\).

**Proof.** By Schur’s formula, we have

\[
\begin{bmatrix} iy & u^* \\ u & Q \end{bmatrix}^{-1} = \begin{bmatrix} (iy - u^*Q^{-1}u)^{-1} & -(iy - u^*Q^{-1}u)^{-1}uQ^{-1} \\ -Q^{-1}u(iy - u^*Q^{-1}u)^{-1}Q^{-1} + Q^{-1}u(iy - u^*Q^{-1}u)^{-1}uQ^{-1} \end{bmatrix},
\]

and

\[
\begin{bmatrix} iy & u^* \\ u & Q + iy \end{bmatrix}^{-1} = \begin{bmatrix} (iy - u^*(Q + iy)^{-1}u)^{-1} & -(iy - u^*(Q + iy)^{-1}u)^{-1}u*(Q + iy)^{-1} \\ -(Q + iy)^{-1}u(iy - u^*(Q + iy)^{-1}u)^{-1}(Q + iy)^{-1}u(iy - u^*(Q + iy)^{-1}u)^{-1}u*(Q + iy)^{-1} \end{bmatrix}.
\]

As shown in Lemma 2.1, we have \(p = \lim_{y \to 0} iy(iy - u^*Q^{-1}u)^{-1}\) and \(\pi = \lim_{y \to 0} iy(iy + L)^{-1}\). Let \(p'\) be the (1,1) entry of \(\pi\). Then \(p' = \lim_{y \to 0} iy(iy - u^*(Q + iy)^{-1}u)^{-1}\). Thus,

\[
\pi = \begin{bmatrix} p' & -p'uQ^{-1} \\ -Q^{-1}up' & Q^{-1}up'u*Q^{-1} \end{bmatrix}.
\]

The fact that \(\pi\) is a projection provides the relation \(p' = (p')^2 + p'uQ^{-2}up'\). We observe that

\[
p - p' = \lim_{y \to 0} iy(iy - u^*Q^{-1}u)^{-1} - iy(iy - u^*(Q + iy)^{-1}u)^{-1} = \lim_{y \to 0} iy(iy - u^*Q^{-1}u)^{-1}u*Q^{-1}i(y(Q + iy)^{-1}u(iy - u^*(Q + iy)^{-1}u)^{-1}
\]

\[
= pu*Q^{-2}up' = p'uQ^{-2}up.
\]

Thus, \(p = p'(1 + u*Q^{-2}up) = 1 + pu*Q^{-2}u)p'\) and \(p' = p(1 - u*Q^{-2}up') = (1 - p'u*Q^{-2}up)\). These equalities together with the above \(p' = (p')^2 + p'uQ^{-2}up'\) guarantee that \(pp' = p'p = p'\) and moreover, \(p'\) is invertible in \(pA_{p}\). Thus, in particular, \(p = p' + (p')^2 u*Q^{-2}u(p')^2\). We have

\[
\pi = \begin{bmatrix} p' & -p'uQ^{-1} \\ -Q^{-1}up' & Q^{-1}up'u*Q^{-1} \end{bmatrix} = \begin{bmatrix} (p')^2 & 0 \\ -Q^{-1}u(p')^2 & 0 \end{bmatrix} \begin{bmatrix} (p')^2 & 0 \\ 0 & (p')^2u*Q^{-1} \end{bmatrix},
\]

and
and
\[
\begin{bmatrix}
  p & 0 \\
  0 & 0
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{2}p^* Q^{-1} u(p')^\frac{1}{2} & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  (p')^\frac{1}{2} & -(p')^\frac{1}{2} u(p')^{-1} \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  (p')^\frac{1}{2} & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix}.
\]

This concludes the proof of our proposition. \(\square\)

As it was shown in [1], if \(P(X_1, X_2)\) is a rational selfadjoint expression (in particular a selfadjoint polynomial) in two noncommuting variables with scalars from the finite dimensional algebra \(B\), then there exist \(n \in \mathbb{N}\), \(\gamma_0 \in M_{n+1}(B)\), \(\gamma_1, \gamma_2 \in M_{n+1}(\mathbb{C})\), all selfadjoint, so that
\[
(z - P(X, Y))^{-1} = \left[\left((z\varepsilon_{1,1} + \gamma_0) \otimes 1_A + \gamma_1 \otimes X + \gamma_2 \otimes Y\right)^{-1}\right]_{1,1}.
\]

If \(X, Y\) are free over \(B\) with respect to \(E\), then \(\gamma_1 \otimes X\) and \(\gamma_2 \otimes Y\) are free over \(M_{n+1}(B)\) with respect to \(1_{\mathbb{N}} \otimes E\). Moreover, one may choose \(\gamma_0, \gamma_1, \gamma_2\) such that
\[
\gamma_0 \otimes 1_A + \gamma_1 \otimes X + \gamma_2 \otimes Y = \begin{bmatrix}
  0 & u^* \\
  u & Q
\end{bmatrix} \in M_{n+1}(A),
\]
with \(Q = Q^*\) invertible in \(M_n(A)\). In particular, Corollary 4.4(3) applies to, say, \(\gamma_0 \otimes 1_A + \gamma_1 \otimes X\) and \(\gamma_2 \otimes Y\) whenever
\[
\begin{bmatrix}
  0 & u^* \\
  u & Q
\end{bmatrix}
\]
has a nontrivial kernel. However, Proposition 5.1 guarantees that if \(P(X, Y)\) has a kernel, then so does
\[
\begin{bmatrix}
  0 & u^* \\
  u & Q
\end{bmatrix},
\]
and the two kernels are Murray-von Neumann equivalent when viewed in \(M_{n+1}(A)\).

(Applying Proposition 5.1 does not allow us generally to work with unbounded operators. In particular, our method requires that each inverse taken in the expression \(P\) is an inverse in \(A\), and not the algebra of operators affiliated to it.)

Thus, Corollary 4.4(3) together with Proposition 5.1 allow us in principle to determine precisely the formulae relating the kernel of \(P(X, Y)\) and affine deformations of the kernels of \(\gamma_0 \otimes 1_A + \gamma_1 \otimes X\) and \(\gamma_2 \otimes Y\). While the process is very unwieldy, one cannot reasonably expect to find a simpler one: indeed, the algebra of scalars is the smallest object in terms of which one can expect to describe joint distributions of free random variables.

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