We consider modifications of General Relativity obtained by adding the logarithm of some curvature invariants to the Einstein-Hilbert action. These non-linear actions can explain the late-time acceleration of the universe giving an expansion history that differs from that of a pure cosmological constant. We show that they also modify the Newtonian potential below a fixed acceleration scale given by the late-time Hubble constant times the speed of light. This is exactly what is required in MOND, a phenomenological modification of the Newtonian potential that is capable of explaining galactic rotation curves without the need to introduce dark matter. We show that this kind of modification also predicts short distance deviations of Newton’s law at the sub-mm scale and an anomalous shift in the precession of the Moon’s orbit around the Earth, both effects of a size that is less than an order of magnitude below current bounds.

1 Modified gravity: motivation

The validity of General Relativity (GR) has been extensively tested not only in the Solar System but also in other astrophysical systems that involve stronger gravitational fields. And while there is a widespread belief among theoretical physicists that the Einstein equation will have further corrections that will be computable once a consistent quantization of gravity is achieved, these corrections are expected to modify the behaviour of the classical solutions only at very short distances, unaccessible to present and possibly future experiments. According to this picture there would be a big “gravitational desert” for curvatures much smaller than the Planck mass where GR would provide an accurate description of gravity.

On the other hand, cosmological and astrophysical observations are widely inconsistent with GR if we consider as the source in the energy-momentum tensor the visible matter in many systems. Since we have no obvious theoretical motivation for expecting an infrared modification of gravity, the first hypothesis one is led to is that there are extra sources in the energy-momentum tensor...
tensor that do not interact with photons. In this case we need two such components: dark energy and dark matter. The simplest way to incorporate these components into the theory is to assume the existence of a small vacuum energy (or cosmological constant) of magnitude $\Lambda^4_{\text{vac}} \sim (10^{-3} \text{eV})^4$ and an extra weakly interacting massive particle that makes up most of the matter density of the Universe. This particle is assumed to be cold (i.e. non-relativistic) and can be naturally produced with the right abundance in the early universe if it is associated with the electro-weak scale. This is the working hypothesis for the standard $\Lambda$CDM model. This model is capable of explaining the features observed in the temperature fluctuations of the CMB and is in general good agreement with other cosmological probes (although one could mention some tension between determinations of the power spectrum amplitude coming from CMB and Lyman-\(\alpha\) forest).

But the discrepancies between theory and experiment begin to grow as we go to shorter length scales and we compare the simulations of structure formation in CDM models with observations. The most obvious problems come from the “cuspy core” in the central parts of halos or the abundance of substructures produced in the simulations, none of which seem compatible with observations. This has prompted the consideration of other flavors of dark matter like warm or self-interacting dark matter in order to reduce the halo sub-structure and its density in the innermost part. The problem with these alternatives is that they tend to delay structure formation in the early universe, and it is not clear to what extent such possibilities are compatible with the observed constraints coming from evolved structure seen at high redshift. But despite of these problems, the most serious problem that the dark matter paradigm faces is probably to explain the strong correlation between the luminous matter content and the dark matter density that is inferred by the study of the rotation curves of galaxies. For spiral galaxies these rotation curves imply that the visible matter content completely determines the dynamics and predicts the detailed rotation curve even when the visible matter makes only a very small fraction of the dynamical mass. These correlations are not restricted to spiral galaxies and are exemplified in several well-known empirical relations like the Tully-Fisher or the Faber-Jackson laws. The degree of precision of these correlations, compatible with zero intrinsic scatter, is hardly expected from a stochastic structure formation scenario of the kind envisaged in dark matter models. Remarkably, a simple modification of the Newtonian potential for small accelerations can account surprisingly well for this phenomenology (although the simplest fitting formula does seem to find problems both for larger cluster scales and small satellites of galaxies). This is the so-called MOND (for MOdified Newtonian Dynamics), proposed by Milgrom in 1983 which simply states that below a fixed critical acceleration the force of gravity decays with an $1/r$ law instead of the Newtonian $1/r^2$. An interesting observation is that the critical acceleration implied by the data, $a_0 \sim 1.2 \times 10^{-10} \text{m/s}^2$, is of the order of the Hubble constant times the speed of light, which is determined by the dark energy density. This coincidence strongly suggests, within a modified theory of gravity, a link between both phenomenons. It is clear thus what kind of properties we should look for in a modification of gravity if it is to replace dark matter.

A relativistic theory with a MOND-like Newtonian limit that agrees with Solar System tests has been proposed. It is built by adding extra fields to the action with particular couplings and supplementing them with constraints by introducing also Lagrange multipliers. This theory has been shown to be consistent with other cosmological observations with the help of massive neutrinos, but the relation $a_0 \sim cH_0$ remains unexplained. Here we will present a modification of the Einstein-Hilbert action for the space-time metric such that we get a modification of the Newtonian potential below a fixed acceleration scale and where the relation $a_0 \sim cH_0$ is naturally explained. Moreover we will see that these kind of theories make predictions for deviations (with respect to GR) in measurable quantities at the Solar System and laboratory levels. In fact we will see that some of these effects should be on the edge of detection, and this opens the door to the possibility of getting an experimental validation in the laboratory of modified theories of
gravity intended to address cosmological and astrophysical phenomena.

2 Modifying gravity below a fixed acceleration scale

The actions we will be interested on are of the type

\[ S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left\{ R - \mu^2 \text{Log} [f(R, Q - 4P)] \right\}, \tag{1} \]

where

\[ P \equiv R_{\mu\nu} R^{\mu\nu} \quad \text{and} \quad Q \equiv R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \tag{2} \]

and \( f \) is a function for which we will only assume that \( f \to 0 \) for \( R_{\mu\nu\lambda\rho} \to 0 \), and we can approximate \( f \approx Q/Q_0 \) whenever \( Q \gg R^2, P \). Minkowski spacetime will not be a solution of the theory but there will typically exist de Sitter solutions with curvature \( R \sim \mu^2 \). We see then that if we want to explain the late time acceleration of the Universe we have to take \( \mu \sim H_0 \), with \( H_0 \) being the value of Hubble’s constant today. But even if these theories could explain the acceleration of the Universe, they raise several serious questions that should be addressed before one could consider them as a candidate to explain such acceleration. For instance, since the equations of motion for the spacetime metric now contain up to fourth order derivatives, one can worry about the unwanted appearance of ghosts that would render the vacuum unstable. Also one can expect that the extra propagating degrees of freedom introduced by these higher derivatives would modify the Newtonian limit, and one should check that it is modified in a manner compatible with observation. For studying these issues it is convenient to discuss the linearisation, or particle content, of these modified theories of gravity.

2.1 Particle content of modified gravity

In general, if we consider an action that is an arbitrary function of the invariants \( R, P \) and \( Q \) we can expect eight propagating degrees of freedom in vacuum\(^a\). These are grouped as: two in a massless spin two particle, one in a scalar excitation and five in a massive spin two ghost\(^b\).

It is easy to obtain the properties of these degrees of freedom by realizing that at the bilinear level, the expansion over a maximally symmetric spacetime of any action defined through a Lagrangian such as \( L = F(R, P, Q) \) is the same\(^b\) as the expansion of

\[ S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left\{ -\Lambda + \delta R + \frac{1}{6m_0^2} R^2 - \frac{1}{2m_2^2} C_{\mu\nu\lambda\sigma} C_{\mu\nu\lambda\sigma} \right\}, \tag{3} \]

where \( C_{\mu\nu\lambda\sigma} \) is the Weyl tensor and the parameters appearing in these action can be obtained as functions of \( F(R, P, Q) \) and its derivatives with respect to \( R, P \) and \( Q \) evaluated in the background. \( m_0 \) is the mass of the scalar and \( m_2 \) is the mass of the ghost. In particular \( m_2^2 = -\langle \partial_\nu F + 4\partial_\nu F \rangle_0 \) so for functions of the type \( F(R, P, Q) = F(R, Q - 4P) \), the ghost is absent. This is the reason why we took this particular combination of \( P \) and \( Q \) in our original action\(^a\). For this action we find that in vacuum the mass of the scalar is given by \( m_0 \sim H_0^2/\mu \sim \mu \), so the extra scalar is almost massless. This would appear to rule out the theory, since we know that at the Solar System level gravity is mediated only by a massless spin two graviton. But a closer inspection of the expansion reveals that this conclusion is not correct.

The reason is that for corrections to the Einstein-Hilbert action that become important at small curvatures (but are negligible at large curvatures), the expansion of the action in powers of the fluctuations breaks down at a very small energy scale\(^c\). This means that for actions of the type

\(^a\)We assume that the overall sign is such that the massless spin two graviton and the scalar are not ghosts.

\(^b\)Notice that this equivalence only applies to the expansion of the action up to the bilinear level.
the spherically symmetric solution found in the linearised approximation can not be trusted at distances smaller than \( r_V \equiv (G_N M/\mu^3)^{1/4} \). For a star like the Sun this distance is of the order of 10 kpc, many orders of magnitude larger than the Solar System.

2.2 Newtonian limit of modified gravity

If the linearised approximation is not valid, and the full non-linear equations are difficult to solve, how can we proceed? We can get some insight on the expected behaviour of the solutions by applying the following argument. If we have an extra degree of freedom with mass \( m_s \), we can expect that it will only affect the solution whenever \( r < m_s^{-1} \). For longer distances the mass effectively decouples it. We can then estimate the mass of the extra scalar particle in a generic background by applying the expression that we found when linearising in maximally symmetric spacetimes. On a Schwarzschild background we find that the mass depends on the distance as \( m_0^2 \sim Q/\mu^2 \sim (G_N M)^2/\left(r^6 \mu^2 \right) \). The relation \( r < m_0^{-1} \) now turns into

\[
 r > \left( \frac{G_N M}{\mu} \right)^{1/2} \equiv r_c. \tag{4}
\]

So due to the dependence of the scalar mass on the distance we see that we can expect a long distance modification of gravity. Moreover this long distance corresponds to a fixed Newtonian acceleration scale \( a_0 \sim \mu \), precisely of the order of the late-time Hubble constant times the speed of light (that we are setting to 1), as required in MOND. Notice that for the Sun \( r_c \sim 10^3 AU \), where 1AU is the Sun-Earth distance.

One can check this result more rigorously applying an approximation procedure that is complementary to the linearised approximation for this type of theories. This alternative expansion is valid whenever the extra term that one adds to the Einstein tensor produces only a small correction of the GR background. What one can do then is to take as the 0-th order solution the solution of GR. Then compute the correction term to the GR equations evaluated in this background and solve for the backreaction in the Einstein tensor. One can iterate this process and provided that the backreaction produces only a small perturbation in the original background, one can expect that the procedure will produce a good approximation to an exact solution if we iterate the process a sufficient number of times. Doing this it was shown that the corrections to the Schwarzschild geometry are small at small distances (\( r \ll r_c \)) and take the form

\[
ds^2 \simeq - \left[ 1 - \frac{2G_N M}{r} \left( 1 + \frac{4}{3} \left( \frac{r}{r_c} \right)^4 + O \left( \left( \frac{r}{r_c} \right)^8 \right) \right) \right] dt^2 + \left[ 1 - \frac{2G_N M}{r} \left( 1 - 2 \left( \frac{r}{r_c} \right)^4 + O \left( \left( \frac{r}{r_c} \right)^8 \right) \right) \right]^{-1} dr^2 + r^2 d\Omega_2^2. \tag{5}\]

From this expression it is clear that the modifications of the gravitational field of the Sun at the Solar System level are very small. But in the Solar System there are very stringent tests of GR that we will have to face, the most precise coming probably form the Lunar Laser Ranging experiment. Using this the Moon-Earth distance is known with a precision of a centimeter. Any anomalous precession is bound to be less than \( 2.4 \times 10^{-11} \) radians per revolution. Considering the correction to the gravitational potential of the Earth given by the expression above one can estimate the expected anomalous precession in radians per revolution as

\[
\frac{d}{dr} \left( \frac{r^2 d}{dr} \left( \frac{\delta V}{rV_N} \right) \right) \simeq 16\pi \left( \frac{T_{(\text{Moon-Earth})}}{T_{c(\text{Earth})}} \right)^4 \sim 10^{-12} \tag{6}\]
which is just a factor of five below the current bound. So this theory passes the tests coming from precision astrometrical measurements in the Solar System, but what about the tests of gravity at smaller scales? The $1/r$ form of the potential for gravity has been tested down to scales of the order of 0.2 millimeters\(\overset{\frown}{17}\). What kind of short distance deviations, if any, can we expect in our case? In order to get some intuition about the expected scale where anomalies could appear for this kind of experiments we can again estimate the mass of the scalar field in the dominant background of a massive object of mass $M$, at a distance $r_0$ from its centre, as $m_0 \sim G_NM/10^3 \mu$. If we take $M$ as the mass of the Earth and $r_0$ as its radius we see that we do not expect modifications in the Newtonian potential on the laboratory at distances bigger than $m_0^{-1} |_{\text{Earth}} \sim 0.1 mm$. Since this is just compatible with current bounds it is interesting to explore this issue a bit further, and compute the first correction to the Newtonian potential for a probe mass $m$ situated at a distance $r_0$ of a big massive object of mass $M$. We will consider a coordinate system $(t, x, y, z)$ where the masses are separated in the $z$ direction so the 0-th order GR solution reads (in the weak field limit, for small $r \equiv |x| = \sqrt{x^2 + y^2 + z^2} \ll r_0$)

$$ds^2 = - (1 + 2\Phi^{(0)})dt^2 + (1 - 2\Psi^{(0)})dx^2$$  \hspace{1cm} (7)

where

$$\Phi^{(0)} \approx \Psi^{(0)} \approx - \frac{G_N m}{r} - \frac{G_NM}{r_0} (1 - \frac{z}{r_0} + \frac{3}{2} \frac{z^2}{r_0^2} - \frac{r^2}{2r_0^2}).$$  \hspace{1cm} (8)

We can now find the first order perturbation of the metric by solving

$$G^{(1)}_{\mu\nu} = - \mu^2 H^{(0)}_{\mu\nu},$$  \hspace{1cm} (9)

where $\mu^2 H_{\mu\nu}$ is the extra term introduced in the equations for the metric by the logarithmic part of the action. Taking $g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}$, the first order perturbation of the 00 and trace components of the Einstein tensor are given by

$$G^{(1)}_{00} \approx 2\nabla^2 \Psi^{(1)}, \quad G^{(1)}_{\mu\mu} \approx 2\nabla^2 \Phi^{(1)} - 4\nabla^2 \Psi^{(1)},$$  \hspace{1cm} (10)

while for $\mu^2 H_{\mu\nu}$ evaluated on the 0-th order background we find

$$\mu^2 H^{(0)}_{00} \approx - \frac{3}{4} \frac{G_N m}{m^2_s r^3} (3 - 30\frac{z^2}{r^2} + 35\frac{z^4}{r^4}), \quad \mu^2 H^{(0)}_{\mu\mu} \approx 0$$  \hspace{1cm} (11)

where we have defined $m_s \equiv \frac{G_NM}{r_0 \mu}$. Taking appropriate care in order to avoid the introduction of spurious sources, we can solve the previous equations yielding

$$\Phi^{(1)} = 2\Psi^{(1)} = - \frac{3}{8} \frac{G_N m}{m^2_s r^3} \left(1 - 6\frac{z^2}{r^2} + 5\frac{z^4}{r^4}\right).$$  \hspace{1cm} (12)

$\Phi^{(1)}$ is the first correction to the Newtonian potential in an expansion in powers of $1/(m_s r)$. Notice that when the correction to the Newtonian potential becomes of order one (for $r \sim m_s^{-1}$), the expansion that we have used breaks down. This was anticipated by our discussion above, since the method that we have used is only valid whenever the modification of the GR solution is small. But this computation is enough to show that the corrections are suppressed at distances bigger that $m_s^{-1}$ and they are anisotropic. This anisotropy is an expected property for this theory since, from an effective field theory point of view, the properties of the excitations that we have depend crucially on the background. So the corrections that we can expect to get from them will also reflect the symmetries of the background geometry. In Fig. 1 we present a density plot of this correction. The gravitational force is the gradient of this field, and it points from the darker towards the lighter regions. It is apparent that the attractive/repulsive nature of the correction depends on the direction of the measurement.
3 Conclusions

We have proposed a class of modifications of GR that have the potential of addressing the issues of dark matter and dark energy as manifestations of the same phenomenon. These actions naturally modify the Newtonian potential below a fixed acceleration scale given by the late-time Hubble constant (times the speed of light), as required in MOND. We have seen that they are stable and ghost free and pass all tests of gravity at the Solar System and laboratory levels. Interestingly, these models predict gravitational anomalies at a level that is accessible in current and planned experiments. Their gravitational phenomenology is markedly characteristic, and to our knowledge these are the only models that predict observable corrections to Newton’s potential reflecting the geometry of the underlying background geometry. Were such anisotropic corrections of Newton’s potential to be found, they would provide in our view a smoking gun for the existence of a MOND-like modification of gravity along these lines that would be responsible for the structure observed in rotation curves of galaxies and the acceleration of the Universe.

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References

1. C. M. Will, arXiv:gr-qc/0510072.
2. D. N. Spergel et al., arXiv:astro-ph/0603449.
3. U. Seljak, A. Slosar and P. McDonald, arXiv:astro-ph/0604335.
4. J. P. Ostriker and P. J. Steinhardt, Science 300 (2003) 1909 arXiv:astro-ph/0306402.
5. Z. Haiman, R. Barkana and J. P. Ostriker, arXiv:astro-ph/0103050 J. R. Primack, Nucl. Phys. Proc. Suppl. 124 (2003) 3 arXiv:astro-ph/0205391.
6. M. Persic et al. Mon. Not. Roy. Astron. Soc. 281 (1996) 27 arXiv:astro-ph/9506004.
7. R. H. Sanders and S. S. McGaugh, Ann. Rev. Astron. Astrophys. 40 (2002) 263.
8. H. Zhao, arXiv:astro-ph/0508635.
9. M. Milgrom, Astrophys. J. 270 (1983) 365.
10. J. D. Bekenstein, Phys. Rev. D 70 (2004) 083509 arXiv:astro-ph/0403694.
11. C. Skordis et al., Phys. Rev. Lett. 96 (2006) 011301 arXiv:astro-ph/0505519.
12. I. Navarro and K. Van Acoleyen, arXiv:gr-qc/0512109.
13. A. Hindawi et al., Phys. Rev. D 53 (1996) 5597 arXiv:hep-th/9509147.
14. I. Navarro and K. Van Acoleyen, JCAP 0603 (2006) 008 arXiv:gr-qc/0511045.
15. I. Navarro and K. Van Acoleyen, Phys. Lett. B 622 (2005) 1 arXiv:gr-qc/0506096.
16. G. Dvali et al., Phys. Rev. D 68 (2003) 024012 arXiv:hep-ph/0212069.
17. C. D. Hoyle et al., Phys. Rev. D 70 (2004) 042004 arXiv:hep-ph/0405262.