The Hilbert-Kunz Density Functions of Quadric Hypersurfaces

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Abstract. We show that the Hilbert-Kunz density function of a quadric hypersurface of Krull dimension \( n + 1 \) is a piecewise polynomial on a subset of \([0, n]\), whose complement in \([0, n]\) has measure zero. Our explicit description of the Hilbert-Kunz density function confirms a conjecture of Watanabe-Yoshida on the lower bound of the Hilbert-Kunz multiplicity of the quadric of dimension \( n + 1 \), provided the characteristic is at least \( n - 1 \). We also show that the Hilbert-Kunz multiplicity of a quadric of fixed dimension is an eventually strictly decreasing function of the characteristic confirming a conjecture of Yoshida.

The main input comes from the classification of Arithmetically Cohen-Macaulay bundles on the projective variety defined by the quadric via matrix factorizations.

1. Introduction

Let \( R \) be a Noetherian ring containing a field of characteristic \( p > 0 \) and \( I \) be an ideal of finite colength in \( R \). For such a pair P. Monsky (in [M]) had introduced a characteristic \( p \) invariant known as the Hilbert-Kunz (HK) multiplicity \( e_{HK}(R, I) \). This is a positive real number given by

\[
e_{HK}(R, I) = \lim_{n \to \infty} \frac{\ell(R/I^{[q^n]})}{q^{\dim R}} \geq 1.
\]

If \((R, m, k)\) is a formally unmixed Noetherian local ring then it was proved by Watanabe-Yoshida (Theorem 1.5 in [WY1]) that \( e_{HK}(R, m) = 1 \) if and only if \( R \) is regular. For the next best class of rings, namely the quadric hypersurfaces, they made the following conjecture in 2005:

Conjecture (1) (Conjecture 4.2 in [WY2]). Let \( p > 2 \) be prime and \( K = \overline{\mathbb{F}}_p \) and let \( R_{p,n+1} = K[x_0, \ldots, x_{n+1}]/(x_0^2 + \cdots + x_{n+1}^2) \) denote the quadric hypersurface of dimension \( n + 1 \). Then for any formally unmixed non regular local ring \((A, m_A, K)\) of dimension \( n + 1 \) we have

\[
e_{HK}(A, m_A) \geq e_{HK}(R_{p,n+1}, m) \geq 1 + m_{n+1},
\]

where \( m_{n+1} \) are the constants occurring as the coefficients of the following expression

\[
\sec(x) + \tan(x) = 1 + \sum_{n=0}^{\infty} m_{n+1} x^{n+1}, \quad \text{for} \quad |x| < \pi/2.
\]

In the same paper they showed that the conjecture holds for \( n \leq 3 \). The second inequality of the conjecture for \( n \leq 5 \) was proved by Yoshida in [Y]. Later the conjecture up to \( n \leq 5 \) was proved by Aberbach-Enescu in [AE2].

Enescu and Shimomoto in [ES] have proved the first inequality \( e_{HK}(A) \geq e_{HK}(R_{p,n+1}) \), where \( A \) belongs to the class of complete intersection local rings.

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On the other hand, around 2010, Gessel-Monsky in [GM], proved the following result:

$$\lim_{p \to \infty} e_{HK}(R_{p,n+1}, m) = 1 + m_{n+1}.$$  

Progresses on Conjecture (1) is discussed in Huneke’s survey article [Hu].

Various people ([AE1], [AE2], Celikbas-Dao-Huneke-Zhang in [CDHZ]) have given a lower bound of the form $e_{HK}(R, m) \geq C(d)$, where $C(d)$ is an explicit constant and $(R, m)$ is an arbitrary formally unmixed nonregular local ring of dimension $d$.

However the above result of Gessel-Monsky implies that the lower bounds such as $C(d)$ are weaker than the bound given in the above conjecture: Recall so far the best constant $C(d) = 1 + \frac{1}{d!}$ (see Section 8, Theorem 13 in [Hu]), whereas, as noted in [AE2],

$$1 + m_3 = 1 + \frac{1}{3!}, \quad 1 + m_4 = 1 + \frac{5}{4!}, \quad 1 + m_5 = 1 + \frac{16}{5!}, \quad 1 + m_6 = 1 + \frac{61}{6!} \text{ etc.}$$

Regarding the HK multiplicities of the quadric hypersurfaces, another conjecture was made by Yoshida ([Y]):

Conjecture (2). $e_{HK}(R_{p,n+1}, m)$ is a decreasing function in $p$ for a fixed $n$.

Note that it is not true in general that $e_{HK}(R_p)$ is a monotonic function of $p$. The first example which is given by Han-Monsky ([HM]) is the ring $R_p = k[x, y, z]/(x^4 + y^4 + z^4)$, where $k$ is a field of characteristic $p > 2$ and $e_{HK}(R_p, (x, y, z))$ is a function of the congruence class of $p$ (mod 8) and not monotonic.

Conjecture (1) and Conjecture (2) have been revisited in the recent paper [JNSWY].

Hilbert-Kunz density functions, which were introduced in [T1], generalize Hilbert-Kunz multiplicities in the graded setting. In this paper, our key new idea is to describe the Hilbert-Kunz density functions of quadric hypersurfaces.

As applications of our description in Section 4, we obtain upper and lower bounds of the values of the respective HK multiplicities of the quadric hypersurfaces.

In particular we prove the second inequality of the Conjecture (1) by showing

Theorem (A) (Theorem 4.3). Let $p \neq 2$ and let $p > n - 2$. Then, for $n \geq 3$,

$$1 + m_{n+1} + \frac{2n-4}{p} \geq e_{HK}(R_{p,n+1}, m) \geq 1 + m_{n+1}.$$  

In fact the second inequality is strict if $p \geq 3n - 4$.

In Section 8, we obtain a more refined description of the Hilbert-Kunz density function than in Section 4. As an application, we show that

Theorem (B) (Theorem 8.8 and Theorem 8.10). Given $n \geq 4$, there exist polynomials $p_n(t), q_n(t) \in \mathbb{Q}[t]$ of degrees $\leq (n + 1)^{n+3}$, such that for all $p > 2^{\lfloor n/2 \rfloor} (n - 2)$

$$e_{HK}(R_{p,n+1}) = 1 + m_{n+1} + \frac{p_n(t)}{q_n(t)}|_{t=1/p}.$$  

The same assertion holds if $n = 3$ and $p \geq 5$.

Given $n$, the polynomials $p_n(t)$ and $q_n(t)$ can be computed explicitly by Theorem 8.8 and Theorem 8.10. For a prime $p > 2$ and $n = 3, 4, 5$, the respective Hilbert-Kunz multiplicity is also computed in [WY2] and [Y].

As an application of our computation, we provide the following confirmation of Conjecture (2).
**Theorem (C) (Theorem 8.6).** Given any \( n \geq 3 \), there exists \( \epsilon > 0 \) such that \( e_{HK}(R_{p,n+1}) \) is a strictly decreasing function of \( p \) for \( p \geq 1/\epsilon \).

A value of \( \epsilon \) in Theorem (C) is explicitly determined in Section 8; see Theorem 8.6.

Now we outline our methods of obtaining the Hilbert-Kunz density function and provide descriptions of the Hilbert-Kunz density functions.

Recall that for a pair \((R,I)\), where \( I \) is a finite colength homogeneous ideal in a standard graded ring of dimension at least two, the HK density function \( f_{R,I} : \mathbb{R} \to [0, \infty) \) is a compactly supported continuous function given by

\[
f_{R,I}(x) = \lim_{s \to \infty} \frac{\ell(R/I^{[q]}[xq])}{q^{\dim R-1}}, \quad \text{where} \quad q = p^s.
\]

The HK density function recovers the HK multiplicity since

\[
e_{HK}(R,I) = \int_0^\infty f_{R,I}(x) \, dx.
\]

For further details we refer to Section 2.

The theory of Hilbert-Kunz density function was initiated in [T1] and subsequently developed in [TW2] (see Theorem 1.1 and the preceding definition in [T1] and [TW2]), and [T2] (see Proposition 2.1).

The HK density function of \((R,I)\) tracks the length of the graded pieces of \( R/I^{[p^n]} \) separately. This very feature of the HK density functions allows us to use the machinery of sheaf cohomology and other algebro-geometric gadgets on \( \text{Proj}(R) \) to describe the HK density functions; and thus in turn the HK multiplicity. These advantages of using HK density functions over the mere HK multiplicities are subsequently apparent.

Before going further we remark that for \( n = 1 \) and \( n = 2 \), the ring \( R_{p,n+1} \) is the homogeneous coordinate ring of \( \mathbb{P}^1_k \) and \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) respectively. In both the cases the invariants \( e_{HK}(R_{p,n+1}) \) and \( F_{R_{p,n+1},m} \) are independent of the characteristic (see Eto-Yoshida [EY] and [T1]).

Therefore in the rest of the paper we consider

\[
R_{p,n+1} = K[x_0,\ldots,x_{n+1}]/(x_0^2 + \cdots + x_{n+1}^2), \quad \text{where} \quad n \geq 3,
\]

where \( K \) is a perfect field of characteristic \( p > 0 \); \( R_{p,n+1} \) is equipped with the standard graded structure. Set \( Q_n = \text{Proj}(R_{p,n+1}) \).

One key input that facilitates calculation in the quadric hypersurface case is the complete classification of arithmetically Cohen-Macaulay bundles (abbreviated as ACM bundles) on \( Q_n \) as a direct sum of line bundles and twisted spinor bundles.

This classification follows from the classification of maximal Cohen-Macaulay modules on \( R_{p,n+1} \) due to Knorrer [K] and Buchweitz-Eisenbud-Herzog [BEH]; we recall the relevant details in Section 2. Since \( F^s_{\mathcal{O}(a)} \) is an ACM bundle on \( Q_n \), for every \( s^{th} \) iterated Frobenius map \( F^s : Q_n \to Q_n \) we have

\[
F^s_{\mathcal{O}(a)}(\mathcal{O}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{t^2(a)} \oplus \oplus_{t \in \mathbb{Z}} \mathcal{S}(t)^{t^2(a)},
\]

where \( \mathcal{O}(t) = \mathcal{O}_{Q_n}(t) \) and \( \mathcal{S}(t) \) is a twisted spinor bundle if \( n \) is odd, and a twisted direct sum of the two spinor bundles if \( n \) is even.

Later Achinger in [A] showed that the multiplicities of the bundles \( \mathcal{O}(t) \) and \( \mathcal{S}(t) \) occurring in (1.1) are related to the lengths of graded components of the ring \( R_{p,n+1}/m^{[q]} \), with \( q = p^s \), by the formula

\[
\ell(R_{p,n+1}/m^{[q]}) = \nu^q_0(a) + 2\lambda_0\mu^s_1(a),
\]
where \( \mathbf{m} = (x_0, \ldots, x_{n+1}) \) and \( \lambda_0 = 2^{[n/2]} \).

Here we crucially use this, and another result, namely Theorem 2 from [A] which determines the integers \( t_1 \) and \( t_2 \), in terms of \( q = p^s \) and \( a \), for which the bundles \( \mathcal{O}(t_1) \) and \( \mathcal{S}(t_2) \) occur with non-zero multiplicity in the decomposition of \( F^*_s(\mathcal{O}(a)) \) (or of \( F^*_s(\mathcal{S}(a)) \)). However [A] does not determine this multiplicity.

Using (1.2) and the theory of HK density functions we show that the functions \( \nu^{(p)}_i : \mathbb{R} \rightarrow \mathbb{R}, \mu^{(p)}_{i-1} : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
\nu^{(p)}_i(x) = \lim_{s \rightarrow \infty} \frac{\nu^a_{x_1}([xq])}{q^n}, \quad \mu^{(p)}_{i-1}(x) = \lim_{s \rightarrow \infty} \frac{2\lambda_0 \mu^a_{x_{i+1}}([xq])}{q^n}
\]

are well defined functions continuous functions and for any nonnegative integer \( i \)

\[
f_{R_{p,n+1}}(x + i) = \nu^{(p)}_i(x) + \mu^{(p)}_{i-1}(x), \quad \text{for } x \in [0,1).
\]

Next we analyze the functions \( \nu^{(p)}_i, \mu^{(p)}_{i-1} \).

We show that there exists a subset of \([0,1]\) which we call the difficult range in the interval \([0,1]\), provided \( p > n - 2 \), given as follows:

1. \( \left[ \frac{1}{2} - \frac{n-4}{2p}, \frac{1}{2} + \frac{n-4}{2p} \right] \), if \( n \geq 3 \) is an odd integer and
2. \( (0, \frac{n-4}{2p}) \cup \left[ 1 - \frac{n-2}{2p}, 1 \right) \), if \( n \geq 4 \) is an even integer.

We show (see Proposition 5.3) that the rank functions \( \mu^{(p)}_i, \mu^{(p)}_j \), when restricted to the complement of the difficult range in \([0,1]\), are piecewise polynomials. Moreover we see that these polynomials are independent of the characteristic \( p \). Such a description springs from the fact that the bundle \( F^*_s(\mathcal{O}([xq])) \) has at most one twist of spinor bundle(s) if \( x \) is outside the difficult range.

The dependence of the density function of \( R_{p,n+1} \) on the characteristic \( p > 0 \) is only via its description on the difficult range. We note that the length of the difficult range goes to zero as \( p \rightarrow \infty \). And in this range characteristic \( p \) shows up in the guise of functions \( \mu^{(p)}_i \), which is a bounded function (see Lemma 3.8). More precisely we have:

The first formulation of \( f_{R_{p,n+1}} \). Here we express \( f_{R_{p,n+1}} \) piecewise, as a sum of a polynomial in \( \mathbb{Q}[x] \) (and therefore characteristic free) and another nonnegative continuous function which is exactly the “pure characteristic \( p \)” contribution. We prove Proposition 3.3 and Corollary 1.2 in Section 4.

**Proposition 1.1.** Let \( p \neq 2 \) be a prime such that \( p > n - 2 \) and \( n_0 = \left[ n/2 \right] - 1 \). There exists an explicit set \( \{ Z_0(x), \ldots, Z_{n_0+1}(x), Y_{n_0+2}(x), \ldots, Y_{n-1}(x) \} \) of degree \( n \) polynomials in \( \mathbb{Q}[x] \) which are independent of the characteristic \( p \), and a set \( \{ \mu^{(p)}_{n_0-1}, \mu^{(p)}_{n_0}, \mu^{(p)}_{n_0+1} \} \) of bounded continuous functions from \([0,1] \rightarrow \mathbb{R}_{\geq 0} \) such that

1. if \( n \geq 4 \) is an even number then
Corollary 1.2. Moreover in both the cases the function \( f \) is given by

\[
f_{R_p,n+1}(x) = \begin{cases} 
Z_i(x), & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{n_0+1}(x), & \text{if } (n_0 + 1) \leq x < (n_0 + 2) - \frac{n-2}{2p} \\
Z_{n_0+1}(x) + \mu_{n_0-1}^{(p)}(x - n_0 - 1), & \text{if } 1 - \frac{n-2}{2p} \leq x - (n_0 + 1) < 1 \\
Y_{n_0+2}(x) + \mu_{n_0+1}^{(p)}(x - n_0 - 2), & \text{if } 0 \leq x - (n_0 + 2) < \frac{n-2}{2p} \\
Y_{n_0+2}(x), & \text{if } (n_0 + 2) + \frac{n-2}{2p} \leq x < (n_0 + 3) \\
Y_i(x), & \text{if } i \leq x < i + 1 \text{ and } n_0 + 3 \leq i < n
\end{cases}
\]

and \( f_{R_p,n+1}(x) = 0 \) elsewhere.

(2) If \( n \geq 3 \) is an odd number then

\[
f_{R_p,n+1}(x) = \begin{cases} 
Z_i(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{n_0+1}(x), & \text{if } (n_0 + 1) \leq x < (n_0 + \frac{3}{2}) - \frac{n-2}{2p} \\
Y_{n_0+1}(x) + \mu_{n_0}^{(p)}(x - n_0 - 1), & \text{if } \frac{1}{2} - \frac{n-2}{2p} \leq x - (n_0 + 1) < \frac{1}{2} + \frac{n-2}{2p} \\
Y_{n_0+1}(x), & \text{if } (n_0 + 1) + \frac{1}{2} + \frac{n-2}{2p} \leq x < (n_0 + 2) \\
Y_i(x), & \text{if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n
\end{cases}
\]

and \( f_{R_p,n+1}(x) = 0 \) elsewhere. Here we can also write

\[
f_{R_p,n+1}(x) = Z_{n_0+1}(x) + \mu_{n_0-1}^{(p)}(x - n_0 - 1), \quad \text{if } \frac{1}{2} - \frac{n-2}{2p} \leq x - (n_0 + 1) < \frac{1}{2}.
\]

Moreover in both the cases the function \( f_{R_p,n+1} \) is partially symmetric and the symmetry is given by

\[
f_{R_p,n+1}(x) = f_{R_p,n+1}(n - x) \quad \text{for } 0 \leq x \leq \frac{n-2}{2} \left( 1 - \frac{1}{p} \right).
\]

We can look at this result as follows

**Corollary 1.2.** If \( p > n - 2 \) and \( p \neq 2 \) then

\[
f_{R_p,n+1,m}(x) = f_{R_{\infty,n+1}}(x) \quad x \in [0, \frac{\frac{n+2}{2} - \frac{n-2}{2p}}{2}]
\]

\[
= f_{R_{\infty,n+1}}(x) + \mu_{n_0-1}^{(p)}(x - n_0 - 1) \quad x \in \left[ \frac{\frac{n+2}{2} - \frac{n-2}{2p}}{2}, \frac{\frac{n+2}{2}}{2} \right]
\]

\[
= f_{R_{\infty,n+1}}(x) + \mu_i^{(p)}(x - i - 1) \quad x \in \left[ \frac{\frac{n+2}{2}, \frac{\frac{n+2}{2} + \frac{n-2}{2p}}{2}}{2} \right]
\]

\[
= f_{R_{\infty,n+1}}(x) \quad x \in \left[ \frac{\frac{n+2}{2} + \frac{n-2}{2p}, \infty} \right]
\]
where \( i = n_0 + 1 \) if \( n \) is even and \( i = n_0 \) if \( n \) is odd. Moreover the function \( f_{R_{n+1}} : [0, \infty) \rightarrow [0, \infty) \) given by
\[
f_{R_{n+1}}^{\infty}(x) := \lim_{p \to \infty} f_{R_{p,n+1}, m}(x)
\]
is a well defined continuous function.

Theorem (A) follows from Proposition 1.1 and Corollary 1.2 as detailed in Theorem 4.3.

Further, this explicit description of support of the \( f_{R_{p,n+1}} \) along with the result of [TW1] gives the following result (in Corollary 4.4)

**Corollary** Let \( p > 2 \) be a prime number such that \( p > n - 2 \) then the \( F \)-threshold of the ring \( R_{p,n+1} \) is \( c^m(m) = n \).

The second formulation of \( f_{R_{p,n+1}} \). In Section 7, we further analyse the rank functions \( \nu^{(p)}_i, \mu^{(p)}_i \) on the difficult range.

- We find a countable collection of mutually disjoint semi open intervals, i.e., interval of the form \([a, b)\), which are contained in the difficult range. The complement of the union of these intervals in the difficult range has measure zero. Our indexing of the members in this countable collection is independent of the underlying characteristic. See Lemma 6.5 Lemma 6.7.
- In Proposition 7.2 and Proposition 7.4 we show that there is a finite set of matrices with entries in \( \mathbb{Q}[t], \mathbb{Q}[x] \) such that on a given semi open subinterval the rank functions \( \nu^{(p)}_i(x), \mu^{(p)}_i(x) \) are products of matrices from the above collection, evaluated at \( t = 1/p \). Given the semi open subinterval the choices of matrices in the product only depends on the chosen indexing of the interval.

The description of \( \mu^{(p)}_i \) in Proposition 7.2 and Proposition 7.4 gives us a ground where we can compare the integrals of \( \mu^{(p)}_i \) as \( p \) varies; the characteristic free indexing of the semi open intervals facilitates the comparison. Further, using detailed analysis, we choose one subinterval on which the integral of \( \mu^{(p)}_i \) is a strictly decreasing function of \( p \) and on all other subintervals it is a decreasing function of \( p \).

Moreover the integrals of \( \mu^{(p)}_i \) have nice features over such subintervals: For example, for given integer \( l \geq 1 \), \( \int \mu^{(p)}_i \) over the union of subintervals which are indexed by tuples of length \( l \), will correspond to \( A^l \) where \( A \) is the sum of the above mentioned finite set of matrices in \( \mathbb{Q}[t] \). In particular \( \int \mu^{(p)}_i \) over the difficult range involves a power series expression of the matrix. Now arguing that \( A \) has no eigenvalue \( \geq 1 \) leads to the polynomials as given in Theorem (B).

Using Theorem (B) and the result of [ES] one can answer the Conjecture (1) for the class of complete local rings in the following way:

Let \( n \geq 4 \) and let \( p > 2^{(n/2)(n-2)} \). Let \( (A, m_A, K) \) be a complete intersection but nonregular local ring of dimension \( n + 1 \). Then
\[
e_{HK}(A, m_A) \geq e_{HK}(R_{p,n+1}, m) = 1 + m_{n+1} + \frac{p_n(t)}{q_n(t)} \big|_{t=1/p} > 1 + m_{n+1},
\]
where for given \( n \), the polynomials \( p_n(t) \) and \( q_n(t) \) can be computed explicitly by Theorem 8.8 and Theorem 8.14.

In particular we have a sharp lower bound on the class of complete intersection rings.
In the end we explicitly write down the HK density function for $n = 3$ case in Theorem 9.2. Note that $n = 3$ is the first $n$ where the $\varepsilon_{HK}(R_{p,n+1})$ involves the characteristic $p$. Here the HK density function, which is a compactly supported continuous function, is non smooth at infinitely many points, however, intriguingly, it is a $C^2(\mathbb{R})$ function. This is different (see Remark 9.3) from the previous known cases such as two dimensional graded rings (see Example 3.3 in [T1]) or projective toric varieties (see [MT] Theorem 3.4 and the proof of Theorem 1.1, or Theorem 4.1.18 of [Mo]) where the set of non smooth points is a finite set.

**Question 1:** If $R$ is a standard graded ring over a perfect field and is of dimension $d \geq 2$ and $I \subset R$ is a graded ideal of finite colength, then when does the HK density function $f_{R,I}$ belong to $C^{d-2}(\mathbb{R})$?

An approach to address this question is mentioned in [Muk], where it is claimed that an affirmative answer to one of the questions in Question 5.6 shows that the HK density function is in $C^{d-2}(\mathbb{R})$.

Based on the examples of quadric hypersurfaces and also other above mentioned examples we pose a variant (suggested by the referee) of a question due to K.I. Watanabe (private communication).

**Question 2:** Let $(R, I)$ be a standard graded pair of dimension $d \geq 2$. Let $[0, \alpha]$ be the support of the HK density function $f_{R,I}$ of $(R, I)$. Does there exist a subset $S$ of $[0, \alpha]$ of Lebesgue measure zero such that

1. There is a countable collection of mutually disjoint semi open intervals $\{[a_n, b_n]\}_{n \in \mathbb{N}}$ such that $[0, \alpha] \setminus S = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$;

2. and for each $n$, there is a polynomial $P_n \in \mathbb{R}[x]$ such that on $[a_n, b_n]$ the density function is given by $P_n$?

Moreover, can $S$ be taken to a countable set?

Looking further, the methods used in this paper suggest possible computations for the HK density and related invariants in other situations, where we have information on ACM bundles using matrix factorizations.

We would like to thank the referee for a careful and thorough reading of the paper, and for detailed suggestions to improve the exposition.

**Notations 1.3.** In the rest of the paper

$$R_{p,n+1} = \frac{k[x_0, \ldots, x_{n+1}]}{(x_0^2 + \cdots + x_{n+1}^2)} \quad \text{and} \quad Q_n = \Proj R_{p,n+1},$$

where $n \geq 3$ and $k$ is a perfect field of characteristic $p > 2$.

2. Preliminaries

In this section we recall the relevant results which are known in the literature.

First we recall the following notion of HK density function for $(R, I)$, where $R$ is a standard graded ring and $I$ is a homogeneous ideal in $R$ of finite colength, which was introduced by the author ([T1]) for standard graded rings and later this notion was generalized by the author and Watanabe ([TW2]) for $\mathbb{N}$-graded rings.
Definition 2.1. Let $R$ be a Noetherian standard graded ring of dimension $d \geq 2$ over a perfect field of characteristic $p > 0$, and let $I \subset R$ be a homogeneous ideal such that $\ell(R/I) < \infty$. For $s \in \mathbb{N}$ and $q = p^s$, let $f_s : [0, \infty) \rightarrow [0, \infty)$ be defined as

$$f_s(R, I)(x) = \frac{\ell(R/I^q[x])}{q^{\dim R-1}} = \frac{\ell(R/I^q)}{q^{\dim R-1}}$$

where $q = p^s$.

Theorem (Theorem 1.1 of [T1]). The sequence $\{f_s(R, I)\}_{s \in \mathbb{N}}$ converges uniformly to a compactly supported continuous function $f_{R,I} : [0, \infty) \rightarrow [0, \infty)$, where

$$f_{R,I}(x) = \lim_{s \rightarrow \infty} f_s(R, I)(x)$$

Moreover $e_{HK}(R, I) = \int_0^\infty f_{R,I}(x)dx$.

We call $f_{R,I}$ to be the HK density function of $R$ with respect to the ideal $I$.

Definition 2.2. A vector bundle $E$ on a smooth $n$-dimensional hypersurface $X = \text{Proj} \ S/(f)$, where $S = k[x_0, \ldots, x_{n+1}]$ is called arithmetically Cohen-Macaulay (ACM) if $H^i(X, E(m)) = 0$, for $0 < i < n$ and for all $m$.

It is easy to check that a vector bundle $E$ on $X$ is ACM if and only if the corresponding graded $S/(f)$-module $\oplus_{j \in \mathbb{N}} H^0(X, E(j))$ is maximal Cohen-Macaulay (MCM).

Let $Q_n = \text{Proj} \ S/(f)$ be the quadric given by the hypersurface $f = x_0^2 + \cdots + x_{n+1}^2 = 0$ in $\mathbb{P}_{k}^{n+1} = \text{Proj} \ S$, where $n \geq 3$. Let $k$ be an algebraically closed field. Henceforth we assume $n > 2$.

By B-E-H classification ([BEH]) of indecomposable graded MCM modules over quadric we have: Other than free modules on $S/(f)$, there is (up to shift) only one indecomposable module $M$ (which corresponds to the single spinor bundle $\Sigma$ on $Q_n$) if $n$ is odd and there are only two of them $M_+$ and $M_-$ (which correspond to the two spinor bundles $\Sigma_+$ and $\Sigma_-$ on $Q_n$) if $n$ is even.

Moreover an MCM module over $S/(f)$ corresponds to a matrix factorization of the polynomial $f$ (such an equivalence is given by Eisenbud in [E], for more general hypersurfaces $(f)$), which is a pair $(\phi, \psi)$ of square matrices of polynomials, of the same size, such that $\phi \cdot \psi = f \cdot \text{id} = \psi \cdot \phi$ and the MCM module is the cokernel of $\phi$.

Now the matrix factorization $(\phi_n, \psi_n)$ for indecomposable bundles on $Q_n$ (see Langer [L], Section 2.2) gives an exact sequence of locally free sheaves on $\mathbb{P}_{k}^{n+1}$, where $i : Q_n \rightarrow \mathbb{P}_{k}^{n+1}$ is the natural inclusion.

(2.1) $0 \rightarrow \mathcal{O}_{\mathbb{P}_{k}^{n+1}}(-2)^{2\lfloor n/2 \rfloor + 1} \xrightarrow{\Phi_n} \mathcal{O}_{\mathbb{P}_{k}^{n+1}}(-1)^{2\lfloor n/2 \rfloor + 1} \rightarrow i_*\mathcal{S} \rightarrow 0,$

$\mathcal{S} = \Sigma$ and $\Phi_n = \phi_n = \psi_n$ for $n$ odd and $\mathcal{S} = \Sigma_+ \oplus \Sigma_-$ and $\Phi_n = \phi_n \oplus \psi_n$ for $n$ even. Moreover we have (see (2.4), (2.5) and (2.6) in [L]) the short exact sequences of vector bundles on $Q_n$: If $n$ is odd then

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_{Q_n}^{2\lfloor n/2 \rfloor + 1} \rightarrow \mathcal{S}(1) \rightarrow 0.$$ 

If $n$ is even then

$$0 \rightarrow \Sigma_- \rightarrow \mathcal{O}_{Q_n}^{2\lfloor n/2 \rfloor} \rightarrow \Sigma_+(1) \rightarrow 0$$

and

$$0 \rightarrow \Sigma_+ \rightarrow \mathcal{O}_{Q_n}^{2\lfloor n/2 \rfloor} \rightarrow \Sigma_-(1) \rightarrow 0.$$ 

We also have the natural exact sequence

(2.2) $0 \rightarrow \mathcal{O}_{\mathbb{P}_{k}^{n+1}}(-2) \rightarrow \mathcal{O}_{\mathbb{P}_{k}^{n+1}} \rightarrow \mathcal{O}_{Q_n} \rightarrow 0.$
We denote $\mathcal{O}_{Q_n}(m) = \mathcal{O}(m)$ and
\begin{equation}
R_{p,n+1} = \frac{k[x_0, \ldots, x_{n+1}]}{(x_0^2 + \cdots + x_{n+1}^2)} = \oplus_{m \geq 0} H^0(Q_n, \mathcal{O}(m)) \quad \text{and} \quad n \geq 3,
\end{equation}
where $k$ is a field of characteristic $p > 2$. In particular the $m^{\text{th}}$ graded component of $R_{p,n+1}$ is $H^0(Q_n, \mathcal{O}(m))$. We will be using the following set of equalities in our forthcoming computations.

\begin{equation}
\begin{cases}
h^0(Q_n, \mathcal{O}(m)) = h^0(\mathbb{P}^{n+1}_k, \mathcal{O}(m)) - h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{p,n+1}(m - 2)) \\
h^0(Q_n, S(m)) = 2\lambda_0 [h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{p,n+1}(m - 1)) - h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{p,n+1}(m - 2))],
\end{cases}
\end{equation}

where $2\lambda_0 = 2^{|n/2|} + 1$.

We note that $\omega_{Q_n} = \mathcal{O}(-n)$ and $S^\vee = S(1)$, where $\omega_{Q_n}$ denotes the canonical bundle of $Q_n$. Therefore by Serre duality

\begin{equation}
\begin{cases}
h^n(Q_n, \mathcal{O}(m)) = h^0(Q_n, \mathcal{O}(-m - n)) \\
h^n(Q_n, S(m)) = h^0(Q_n, S(1 - m - n)).
\end{cases}
\end{equation}

The rank of $Q_n$-bundle $S = \lambda_0 = 2^{|n/2|}$.

The statement and the proof of the following lemma is contained in [A].

**Lemma 2.3.** For given integer $a$ and $q = p^s$, if the nonnegative integers $\nu^s_t(a)$, $\mu^s_t(a)$ are the integers occurring in the decomposition

\[ F^s_*(\mathcal{O}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\nu^s_t(a)} \oplus \oplus_{t \in \mathbb{Z}} S(t)^{\mu^s_t(a)} \]

then

\[ \ell(R_{p,n+1}/m^a) = \nu^s_0(a) + 2\lambda_0\mu^s_0(a), \quad \text{where} \quad m = (x_0, \ldots, x_{n+1}). \]

**Proof.** Since $\mathcal{O}(a)$ and $S(a)$ are ACM bundles (follows from (2.1) and (2.2)), the projection formula implies that $F^s_*(\mathcal{O}(a))$ is an ACM bundle on $Q_n$. Thus for $q = p^s$ and $a \in \mathbb{Z}$, we indeed have a decomposition of the form

\begin{equation}
F^s_*(\mathcal{O}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\nu^s_t(a)} \oplus \oplus_{t \in \mathbb{Z}} S(t)^{\mu^s_t(a)}. 
\end{equation}

Restricting the Euler sequence in $\mathbb{P}^{n+1}_k$ to $Q_n$ we get the short exact sequence

\[ 0 \rightarrow \Omega_{p,n+1}^1(1) \mid Q_n \rightarrow \oplus^{n+2} \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow 0 \]

of sheaves of $\mathcal{O}$-modules, where the second map is given by $(a_0, \ldots, a_{n+1}) \rightarrow \sum a_i x_i$.

This gives the long exact sequence

\[ 0 \rightarrow H^0(Q_n, F^{ss}\Omega_{p,n+1}^1(1) \mid Q_n \otimes \mathcal{O}(a)) \rightarrow H^0(Q_n, \oplus^{n+2} F^{ss} \mathcal{O}(a) \mid Q_n \otimes \mathcal{O}(a+q)) \rightarrow H^1(Q_n, F^{ss}\Omega_{p,n+1}^1(1) \mid Q_n \otimes \mathcal{O}(a)) \rightarrow 0. \]

Therefore

\[ \ell\left( \frac{R_{p,n+1}}{m^a} \right)_{a+q} = \ell(\text{coker } \Psi_{a+q}) = h^1(Q_n, F^{ss}\Omega_{p,n+1}^1(1) \mid Q_n \otimes \mathcal{O}(a)) = h^1(Q_n, \Omega_{p,n+1}^1 \mid Q_n \otimes F^s\mathcal{O}(a+q)). \]
Now by Lemma 1.2 in [A] we have
\[ h^1(Q_n, \Omega^1_{p \pi + 1}(t)|Q_n) = \delta_{t,0} \quad \text{and} \quad h^1(Q_n, \mathcal{S} \otimes \Omega^1_{p \pi + 1}(t)|Q_n) = 2^{[n/2]+1}\delta_{t,1}. \]

Hence
\[ \ell(R_{p,n+1}/m[a])_a = \text{coker } \Psi_a = \nu^s_0(a) + 2\lambda_0\mu^s_1(a). \]

**Lemma 2.4.** Let \( i \geq 0 \) be an integer and \( x \in [i, i+1) \) then
\[ \nu^s_0([x]) = \nu^s_{-i}([x] - iq) \quad \text{and} \quad \mu^s_1([x]) = \mu^s_{-i+1}([x] - iq). \]

In particular
\[ f_{R_{p,n+1},m}(x) = \nu^s_i(x - i) + \mu^s_{i-1}(x - i) \quad \text{for} \quad x \in [i, i+1), \]
where \( \nu^s_i: \mathbb{R} \rightarrow \mathbb{R} \) and \( \mu^s_i: \mathbb{R} \rightarrow \mathbb{R} \) are the functions as defined in (1.3).

**Proof.** Note that for any integer \( m \geq 0 \), there is an integer \( i \geq 0 \) such that \( iq \leq m < (i + 1)q \). Hence by the projection formula
\[ F^*_s(\mathcal{O}(m)) \simeq F^*_s(\mathcal{O}(m - iq) \otimes F^*(\mathcal{O}(i))) \simeq F^*_s(\mathcal{O}(m - iq)) \otimes \mathcal{O}(i). \]

In particular
\[ \ell(R_{p,n+1}/m[a])_m = \nu^s_0(m) + 2\lambda_0\mu^s_1(m) = \nu^s_{-i}(m - iq) + 2\lambda_0\mu^s_{-i+1}(m - iq). \]

Therefore for \( x \in [i, i+1) \)
\[ f_{R_{p,n+1},m}(x) = \lim_{s \to \infty} \frac{\ell(R/m[a])_x}{q^n} = \lim_{s \to \infty} \frac{\nu^s_{-i}([x] - iq) + 2\lambda_0\mu^s_{-i+1}([x] - iq)}{q^n} = (\nu^s_i + \mu^s_{i-1})(x - i). \]

We also use the following result of Achinger (Theorem 2 in [A]) which determines, in terms of \( s, a \) and \( n \), when the numbers \( \nu^s_i(a) \) and \( \mu^s_i(a) \) are nonzero in the decomposition of \( F^*_s(\mathcal{O}(a)) \). Langer in [L] has given such formula for the occurrence of line bundles in the Frobenius direct image.

**Theorem** [A]. Let \( p \neq 2, s \geq 1 \) and \( n \geq 3 \) and
\[ F^*_s(\mathcal{O}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\nu^s_t(a)} \oplus \oplus_{t \in \mathbb{Z}} \mathcal{S}(t)^{\mu^s_t(a)}. \]
\[ F^*_s(\mathcal{S}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\tilde{\nu}^s_t(a)} \oplus \oplus_{t \in \mathbb{Z}} \mathcal{S}(t)^{\tilde{\mu}^s_t(a)}. \]

Then
1. \( F^*_s(\mathcal{O}(a)) \) contains \( \mathcal{O}(t) \) if and only if \( 0 \leq a - tq \leq n(q - 1) \).
2. \( F^*_s(\mathcal{O}(a)) \) contains \( \mathcal{S}(t) \) if and only if
\[ \left( \frac{(n-2)(p-1)}{2} \right) \frac{q}{p} \leq a - tq \leq \left( \frac{(n-2)(p-1)}{2} + n - 2 + p \right) \frac{q}{p} - n. \]
3. \( F^*_s(\mathcal{S}(a)) \) contains \( \mathcal{O}(t) \) if and only if \( 1 \leq a - tq \leq n(q - 1) \).
4. \( F^*_s(\mathcal{S}(a)) \) contains \( \mathcal{S}(t) \) if and only if
\[ \left( \frac{(n-2)(p-1)}{2} \right) \frac{q}{p} + 1 - \delta_{s,1} \leq a - tq \leq \left( \frac{(n-2)(p-1)}{2} + n - 2 + p \right) \frac{q}{p} - n + \delta_{s,1}. \]
3. Formula for the rank functions $\nu^{(p)}_i$ and $\mu^{(p)}_{i-1}$

Let the rank functions $\nu^{(p)}_i$, $\mu^{(p)}_{i-1}$ be as given in (1.3). As noted earlier to analyze the rank functions it is enough to study on the interval $[0, 1)$.

One of the goal of this section is to determine, for a given rank function a subinterval in $[0, 1)$ such that the function is polynomial, that is there is a polynomial $p(x) \in \mathbb{Q}[x]$ such that the rank function at $x$ is equal to $p(x)$ for $x$ in that subinterval. We give the description of such subintervals in Proposition 3.10.

Since this section is technical, the reader can choose to skip the proof of Lemmas 3.2, 3.5, 3.7 and 3.8 for the moment and move to the next section.

Notations 3.1.

1. For given integer $a$ and $q = p^s$, the nonnegative integers $\nu^{(s)}_i(a)$, $\mu^{(s)}_i(a)$ are the integers occurring in the decomposition

$$F^s_*(\mathcal{O}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\nu^{(s)}_i(a)} \oplus \oplus_{t \in \mathbb{Z}} S(t)^{\mu^{(s)}_i(a)}.$$ 

2. For the sake of abbreviation henceforth we will denote
   (a) $\mathcal{O}_{n}(m)) = \mathcal{O}(m)$,
   (b) $h^0(Q_n, \mathcal{O}(m)) = L_m$ and
   (c) $2\lambda_0 = 2(n/2)+1$.

3. Let $n_0$ and $\Delta$ be given as

$$n_0 = \bigl(\frac{(n-2)(p-1)}{2p}\bigr) \quad \text{and} \quad n_0 - \Delta = \frac{(n-2)(p-1)}{2p}.$$ 

Hence for $p > n - 2 \implies n_0 = \lfloor n/2 \rfloor - 1$

4. A bundle is a spinor bundle of type $t$ if it is isomorphic to $S(t)$, where $S = \Sigma$ if $n$ is odd and $S = \Sigma + \Sigma -$ if $n$ is even. We say two spinor bundles $S(t)$ and $S(t')$ are of the same type if $t = t'$.

First we make the observation that there can occur at most $\lfloor (n-2)/p \rfloor + 2$ twists of spinor bundles in the decomposition of $F^s_*(\mathcal{O}(a))$, hence for $p > n - 2$ the number reduces to three. The next result shows for $p > n - 2$ any $s$ and $a$, there can be at most two twists of Spinor bundles appearing in the decomposition of $F^s_*(\mathcal{O}(a))$.

**Lemma 3.2.** If $1 - n \leq a < q = p^s$ and $p > 2$ then

$$F^s_*(\mathcal{O}(a)) = \oplus_{t=0}^{n-1} \mathcal{O}(-t)^{\nu^{(s)}_i(a)} \oplus \oplus_{i=-1}^{\lfloor n/2 \rfloor} S(-n_0 - i)^{\mu^{(s)}_{i-1}(a)}.$$ 

(1) If $n \geq 4$ is an even number and $n - 2 < p$ then $n_0 = (n - 2)/2$ and

$$F^s_*(\mathcal{O}(a)) = \oplus_{t=0}^{n-1} \mathcal{O}(-t)^{\nu^{(s)}_i(a)} \oplus \oplus_{i=n_0}^{n+1} S(-i)^{\mu^{(s)}_i(a)}.$$ 

Moreover

(a) $\mu^{(s)}_{n_0+1}(a) \neq 0 \implies a/q \in \left[1 - \frac{n-2}{2p}, 1\right)$

(b) $\mu^{(s)}_{n_0}(a) \neq 0 \implies a/q \in \left[0, 1 \right)$

(c) $\mu^{(s)}_{n_0+1}(a) \neq 0 \implies a/q \in \left[0, \frac{n-2}{2p}\right)$.

(2) If $n \geq 3$ is an odd number and $n - 2 < p$ then $n_0 = (n - 1)/2$ and

$$F^s_*(\mathcal{O}(a)) = \oplus_{t=0}^{n-1} \mathcal{O}(-t)^{\nu^{(s)}_i(a)} \oplus S(-n_0 + 1)^{\mu^{(s)}_{n_0+1}(a)} \oplus S(-n_0)^{\mu^{(s)}_{n_0}(a)}.$$ 

Moreover

(a) $\mu^{(s)}_{n_0+1}(a) \neq 0 \implies a/q \in \left[\frac{1}{2} - \frac{n-2}{2p}, 1\right)$

(b) $\mu^{(s)}_{n_0}(a) \neq 0 \implies a/q \in \left[\frac{1}{2} + \frac{n-2}{2p}\right).
Proof. The formula for the decomposition for \( F_s^*(O(a)) \) for \( 1 - n < a < q \) follows from the assertion (1) of Theorem [A].

By the assertion (2) of [A], if \( S(t) \) occurs in the decomposition of \( F_s^*(O(a)) \) then

\[
(n_0 - \Delta)q \leq a - tq \leq (n_0 - \Delta)q + \frac{(n-2)q}{p} + q - n
\]

\[
\implies (n_0 - \Delta) \leq \frac{a}{q} - t \leq (n_0 - \Delta) + \frac{(n-2)}{p} + 1 - \frac{a}{q}.
\]

Hence

\[ 0 \leq \frac{a}{q} + \Delta - t - n_0 \leq \frac{a-2}{p} + 1 - \frac{a}{q}. \tag{3.3} \]

We note that \( 0 \leq a/q + \Delta < 2 \) and \( t \) and \( n_0 \) are integers. Therefore

\[ n_0 \leq a/q + \Delta - t < 2 - t \implies n_0 - 1 \leq -t \]

and

\[ -t - n_0 - 1 \leq a/q + \Delta - t - n_0 - 1 \leq \frac{n-2}{p} - \frac{a}{q} < \left[ \frac{n-2}{p} \right]. \]

This proves \(-n_0 - t \in \{-1, 0, \ldots, \left[ (n-2)/p \right] \}\).

In particular if \( n - 2 < p \) then \(-n_0 - t \in \{-1, 0, 1\}\).

Let \( n \) be even and \( n - 2 < p \) then \( n = 2m \) for some integer \( m \geq 2 \). Then

\[ n_0 = \left\lfloor \frac{(n-2)(p-1)}{2p} \right\rfloor = \left\lfloor \frac{(m-1)(p-1)}{p} \right\rfloor = \left\lfloor (m-1) - \frac{p-1}{p} \right\rfloor = m - 1 = \frac{n-2}{2}. \]

This implies \( \Delta = (n-2)/2p \).

Let \( n \) be odd and \( n - 2 < p \) then \( n = 2m + 1 \) for some integer \( m \geq 1 \). Now

\[ n_0 = \left\lfloor \frac{2m-1}(p-1) \right\rfloor = \left\lfloor \frac{2m(p-1)}{2p} - \frac{p-1}{2p} \right\rfloor = \left\lfloor m - \frac{2m+p-1}{2p} \right\rfloor = m = \frac{n-1}{2}, \]

where the second last equality follows as \( 0 < (2m + p - 1)/2p < 1 \). This implies

\[ \Delta = (n-2)/2p + 1/2. \]

Putting the three possible values of \( t \) in (3.3) we get the following three inequalities

1. If \(-t = n_0 - 1 \) then we have \( 0 \leq \frac{a}{q} + \Delta - 1 \leq 1 + \frac{n-2}{p} - \frac{a}{q} < 1 + \frac{n-2}{p}. \)

2. If \(-t = n_0 \) then we have \( 0 \leq \frac{a}{q} + \Delta \leq 1 + \frac{n-2}{p} - \frac{a}{q} < 1 + \frac{n-2}{p}. \)

3. If \(-t = n_0 + 1 \) then we have \( 0 \leq \frac{a}{q} + \Delta \leq \frac{n-2}{p} - \frac{a}{q} < \frac{n-2}{p}. \)

From these three equalities it is easy to obtain the assertions (1) (a), (1) (b), (1) (c), (2) (a) and (2) (b).

\[ \square \]

Notations 3.3. Fix an integer \( n \geq 3 \). Then for an integer \( a \geq 1 - n \)

\[ L_a = (2a + n) \frac{(a + n - 1) \cdots (a + 1)}{n!} = \frac{2a^n}{n!} + O(a^{n-1}). \]

We note that \( L_a = 0 \), for \( 1 - n \leq a \leq -1 \). Let \( q = p^s \), where \( s \geq 1 \). For \( 0 \leq i \leq n_0 + 1 \), we define iteratively the integers \( Z_{i-1}(a, q) \) as follows:

\[ Z_0(a, q) = Z_0(a) = L_a, \quad Z_{-1}(a, q) = Z_0(a + q) - L_1 Z_0(a) \]

and in general

\[ Z_{-i}(a, q) = Z_0(a + iq) - [L_1 Z_{-i+1}(a, q) + L_2 Z_{-i+2}(a, q) + \cdots + L_i Z_0(a)]. \]

Similarly, for \( n_0 + 1 \leq i \leq n - 1 \), we define iteratively another set of integers \( Y_{-i}(a, q) \) as follows:

\[ Y_{-n+1}(a, q) = L_{q-a-n} \]
Remark 3.4. By construction it follows that there exists unique rational numbers \( \{r_{ij}, s_{ik}\}_{j,k} \) such that for all \( a \in \mathbb{Z} \) and \( q = p^s \) we have \( Z_0(a) = L_a \) and for \( 0 \leq i \leq n_0 + 1 \)

\[
Z_{-i}(a, q) = r_{i0}L_a + r_{i1}L_{a+q} + \cdots + r_{i(i-1)}L_{a+(i-1)q} + L_{a+iq}
\]

and

\[
Y_{-i}(a, q) = s_{i0}L_{q-a-n} + s_{i1}L_{2q-a-n} + \cdots + s_{i(n-i-2)}L_{(n-i-1)q-a-n} + L_{(n-i)q-a-n}.
\]

Now let \( \{Z_i(x)\}_{0 \leq i \leq n_0+1} \) and \( \{Y_i(x)\}_{n_0+1 \leq i \leq n} \) denote the two sets of polynomials, where

\[
Z_i(x) = \frac{2}{n!} [r_{i0}(x-i)^n + r_{i1}(x-i+1)^n + \cdots + r_{i,i-1}(x-1)^n + (x)^n]
\]

and

\[
Y_i(x) = \frac{2}{n!} [s_{i0}(i+1-x)^n + s_{i1}(i+2-x)^n + \cdots + s_{i,n-i-2}(n-1-x)^n + (n-x)^n].
\]

Since the set \( \mathbb{Z}[1/p] \) is a dense subset of \( \mathbb{R} \) we have \( \lim_{q \to \infty} [xq]/q = x \) for \( x \geq 0 \). This implies

\[
Z_i(x+i) = \lim_{q \to \infty} \frac{Z_{-i}([xq], q)}{q^n} \quad \text{and} \quad Y_i(x+i) = \lim_{q \to \infty} \frac{Y_{-i}([xq], q)}{q^n}
\]

for \( x \in [0, 1) \).

Lemma 3.5. Let \( p \) be an odd prime such that \( p > n - 2 \). Then for given integer \( 1 - n \leq a < q = p^s \)

1. \( \nu^s_{-i}(a) + 2\lambda_0\mu^s_{-i+1}(a) = Z_{-i}(a, p^s) \), if \( 0 \leq i \leq n_0 \).

2. \( \nu^s_{-n_0-1}(a) + 2\lambda_0\mu^s_{-n_0}(a) = Z_{-n_0-1}(a, p^s) + 2\lambda_0\mu^s_{-n_0+1}(a) \).

3. \( \nu^s_{-i}(a) + 2\lambda_0\mu^s_{-i+1}(a) = Y_{-i}(a, p^s) \), if \( n_0 + 3 \leq i \leq n - 1 \).

4. \( \nu^s_{-n_0-2}(a) = Y_{-n_0-2}(a, p^s) \).

5. \( \nu^s_{-n_0-1}(a) = Y_{-n_0-1}(a, p^s) - 2\lambda_0\mu^s_{-n_0-1}(a) \).

6. \( \nu^s_{-i}(a) = 0 \), for \( i \geq n \) and \( \mu^s_{-j}(a) = 0 \) if \( j \not\in \{n_0 + 1, n_0, n_0 - 1\} \).

Proof. We fix \( 1 - n \leq a < q = p^s \).

Now, by Lemma 3.2

\[
F^s_*(\mathcal{O}(a)) = \mathcal{O}(-n + 1)^{\nu^s_{-n+1}(a)} \oplus \cdots \oplus \mathcal{O}(-1)^{\nu^s_{-1}(a)} + \mathcal{O}^{\nu^s_0(a)} \oplus \mathcal{S}(-n_0 + 1)^{\mu^s_{-n_0+1}(a)} \oplus \mathcal{S}(-n_0)^{\mu^s_{-n_0}(a)} \oplus \mathcal{S}(-n_0 - 1)^{\mu^s_{-n_0-1}(a)}.
\]

Tensoring the above equation by \( \mathcal{O}(i) \) and by the projection formula, we get

\[
F^s_*(\mathcal{O}(a + iq)) = \mathcal{O}(i - n + 1)^{\nu_{-n+i+1}(a)} \oplus \cdots \oplus \mathcal{O}(i - 1)^{\nu_{-1}(a)} + \mathcal{O}(i)^{\nu^s_0(a)} \oplus \mathcal{S}(i - n_0 + 1)^{\mu_{-n_0+i+1}(a)} \oplus \mathcal{S}(i - n_0)^{\mu_{-n_0}(a)} \oplus \mathcal{S}(i - n_0 - 1)^{\mu_{-n_0-1}(a)}.
\]
By (2.3), $h^0(Q, \mathcal{O}(m)) = 0$ for $m \leq -1$ and $h^0(Q, \mathcal{S}(m)) = 0$ for $m \leq 0$. Also $h^0(Q, \mathcal{O}(m)) = L_m$ for all $m \geq 1 - n$. Hence applying the functor $H^0(Q, -)$ to the above decomposition we get

\begin{align*}
\nu^s_0(a) &= L_a = Z_0(a, p^s) \quad \text{and} \\
\nu^s_{-1}(a) &= L_{a+q} - L_1 L_a = Z_{-1}(a, p^s).
\end{align*}

(3.4)

In general, for $i \leq n_0 - 1$,

$$L_{a+iq} = L_0 \nu^s_{-i}(a) + L_1 \nu^s_{-i+1}(a) + \cdots + L_i \nu^s_0(a)$$

which implies

$$\nu^s_{-i}(a) = \nu^s_{-i+1}(a) + 2\lambda_0 \mu^s_{-i+1}(a) = Z_{-i}(a, p^s).$$

Let $i = n_0$. Since $h^0(Q, \mathcal{S}(1)) = 2\lambda_0$ we have

$$L_{a+n_0q} = L_0 \nu^s_{-n_0}(a) + L_1 \nu^s_{-n_0+1}(a) + \cdots + L_{n_0} \nu^s_0(a) + 2\lambda_0 \mu^s_{-n_0+1}(a)$$

which implies

$$\nu^s_{-n_0}(a) + 2\lambda_0 \mu^s_{-n_0+1}(a) = Z_{-n_0}(a, p^s).$$

This proves assertion (1).

Let $i = n_0 + 1$. Since $h^0(Q, \mathcal{S}(2)) = 2\lambda_0(L_1 - L_0)$, we have

$$L_{a+(n_0+1)q} = L_0 \nu^s_{-n_0-1}(a) + L_1 \nu^s_{-n_0}(a) + \cdots + L_{n_0+1} \nu^s_0(a)$$

$$+ 2\lambda_0 \left[ (L_1 - L_0) \mu^s_{-n_0+1}(a) + \mu^s_{-n_0}(a) \right]$$

$$= (\nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a)) + \left[ L_1 (\nu^s_{-n_0}(a) + 2\lambda_0 \mu^s_{-n_0-1}(a)) \right]$$

$$+ L_2 \nu^s_{-n_0+1}(a) + \cdots + L_{n_0+1} \nu^s_0(a) - 2\lambda_0 \mu^s_{-n_0+1}(a)$$

which implies, by assertion (1) of the lemma,

$$\nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a) = Z_{-n_0-1}(a, p^s) + 2\lambda_0 \mu^s_{-n_0+1}.\quad (3.6)$$

This proves assertion (2).

Now if we tensor the decomposition of $F^s_*(\mathcal{O}(a))$ by $\mathcal{O}(-j)$, where $1 \leq j \leq n - n_0 - 1$, then we get

$$F^s_*(\mathcal{O}(a - jq)) = \mathcal{O}(-j - n + 1) \nu^s_{-n-1}(a) \oplus \cdots \oplus \mathcal{O}(-j - 1) \nu^s_{-1}(a) \oplus \mathcal{O}(-j) \nu^s_0(a)$$

$$\oplus \mathcal{S}(-j - n_0 + 1) \mu^s_{-n_0+1}(a) \oplus \mathcal{S}(-j - n_0) \mu^s_{-n_0}(a) \oplus \mathcal{S}(-j - n_0 - 1) \mu^s_{-n_0-1}(a).$$

Now applying the functor $H^0(Q_n, -)$ and then using the Serre duality

$$h^0(Q, \mathcal{O}(m)) = h^0(Q, \mathcal{S}(m)) = L_{-m-n}$$

and $h^n(Q, \mathcal{O}(m)) = h^0(Q, \mathcal{S}(1-m-n))$ (here $jq - a - n \geq 1 - n$) we get

$$L_{jq-a-n} = L_{j-1} \nu^s_{-n+1}(a) + L_{j-2} \nu^s_{-n+2}(a) + \cdots + L_0 \nu^s_{-n+j}(a)$$

$$+ \mu^s_{-n+1}(a) h^0(Q_n, \mathcal{S}(n_0+j-n)) + \mu^s_{-n_0}(a) h^0(Q_n, \mathcal{S}(n_0+j+1-n))$$

$$+ \mu^s_{-n_0-1}(a) h^0(Q_n, \mathcal{S}(n_0+j+2-n)).$$
Hence $1 \leq j \leq n - n_0 - 2$, we get
\[ L_{jq-a-n} = L_{j-1}\nu_{n+1}^s(a) + L_{j-2}\nu_{n+2}^s(a) + \cdots + L_0\nu_{n+j}^s(a). \]
Therefore
\[ \nu_{n+j}^s(a) = L_{jq-a-n} - [L_{j-1}\nu_{n+1}^s(a) + \cdots + L_{2}\nu_{n+j-2}^s(a) + L_1\nu_{n+j-1}^s(a) = Y_{n+j}(a,p^s). \]

(3.7) For $j = 1, \nu_{n+1}^s(a) = L_{q-a-n},$

In general
\[ \nu_i^s(a) = \nu_i^s(a) + 2\lambda_0\mu_{i+1}^s(a) = Y_i(a,p^s), \quad \text{for} \quad n_0 + 3 \leq i \leq n - 1 \]
\[ \nu_{n_0-2}^s(a) = Y_{n_0-2}(a,p^s). \]
This proves assertions (3) and (4).

For $j = n - n_0 - 1$ we get
\[ L_{(n-n_0-1)q-a-n} = L_{n-n_0-2}\nu_{n+1}^s(a) + \cdots + L_0\nu_{n_0-1}^s(a) + 2\lambda_0\mu_{n_0-1}^s(a). \]

(3.8) $\nu_{n_0-1}^s(a) = Y_{n_0-1}(a,p^s) - 2\lambda_0\mu_{n_0-1}^s(a).$
This proves assertion (5).

Remark 3.6. By Lemma 3.5 it follows that if there is an interval $I_1 \subset [0,1)$ with the property that $a/p^s \in I_1$ implies there is at the most one type of spinor bundle in the decomposition of $F^s_*(O(a))$, then all the functions $\nu_i^s(a)$ and $\mu_{i+1}^s(a)$ have polynomial expressions for $a/q \in I_1$. More precisely we have the following.

Lemma 3.7. Let $0 \leq a < q = p^s$, where $p \neq 2$ is a prime such that $p > n - 2$ and $n \geq 3$ is an integer. Let $n_0 = \lceil \frac{n}{2} \rceil - 1$. Then we have
\[ F^s_*(O(a)) = \bigoplus_{t=0}^{n_0-1} O(-t)\nu_t^s(a) \oplus \bigoplus_{t=n_0-1}^{n_0+1} S(-t)\mu_t^s(a). \]

Case (1). Let $n \geq 3$ be odd then

\[ 0 \leq a/q < 1 \implies \nu_i^s(a) = \begin{cases} Z_{-i}(a,p^s) & \text{for} \quad 0 \leq i \leq n_0 - 1 \smallskip \\
Y_{-i}(a,p^s) & \text{for} \quad n_0 + 1 \leq i < n \smallskip \\
\mu_i^s(a) = 0 & \text{for} \quad i \neq n_0 - 1,n_0, \end{cases} \]

\[ a/q \in \left[ 0, \frac{1}{2} - \frac{n-2}{2p} \right) \implies \begin{cases} \nu_{n_0}^s(a) = Z_{n_0}(a,p^s) \\
\mu_{n_0+1}^s(a) = 0 \\
\mu_{n_0}^s(a) = \frac{1}{2\lambda_0} [Z_{n_0-1}(a,p^s) - Y_{n_0-1}(a,p^s)], \end{cases} \]

\[ a/q \in \left( \frac{1}{2} + \frac{n-2}{2p}, 1 \right] \implies \begin{cases} \nu_{n_0}^s(a) = Z_{n_0}(a,p^s) - Y_{n_0-1}(a,p^s) + Z_{n_0-1}(a,p^s) \\
\mu_{n_0+1}^s(a) = \frac{1}{2\lambda_0} [Y_{n_0-1}(a,p^s) - Z_{n_0-1}(a,p^s)] \\
\mu_{n_0}^s(a) = 0. \end{cases} \]
Case (2). If $n \geq 4$ is even then

$$0 \leq a/q < 1 \implies \nu^s_{-i}(a) = \begin{cases} Z_{-i}(a, p^s) & \text{for } 0 \leq i \leq n_0 - 1 \\ Y_{-i}(a, p^s) & \text{for } n_0 + 2 \leq i < n \end{cases}$$

$$\mu^s_{-i}(a) = 0 \quad \text{for } i \neq n_0 - 1, n_0, n_0 + 1,$$

$$a/q \in \left[ \frac{n-2}{p}, \ 1 - \frac{n-2}{p} \right) \implies \begin{align*}
\nu^s_{n_0}(a) &= Z_{n_0}(a, p^s) \\
\nu^s_{n_0-1}(a) &= Y_{n_0-1}(a, p^s) \\
\mu^s_{n_0+1}(a) &= 0 \\
\mu^s_{n_0}(a) &= \frac{1}{2n_0} [Z_{n_0-1}(a, p^s) - Y_{n_0-1}(a, p^s)],
\end{align*}$$

$$a/q \in \left[ \frac{n-2}{p}, \ 1 \right) \implies \mu^s_{n_0-1}(a) = 0.$$

Proof. We prove the lemma for the case when $n$ is odd, as the proof is similar when $n$ is even.

Now by Lemma 3.2 (2) we have $\mu^s_j(a) = 0$ if $j \notin \{n_0 - 1, n_0\}$. Therefore by Lemma 3.5 (1) for $0 \leq i \leq n_0 - 1$

$$\nu^s_{-i}(a) = \nu^s_{-i}(a) + 2\lambda_0 \mu^s_{-i+1}(a) = Z_{-i}(a, p^s).$$

and for $n_0 + 2 \leq i < n$, by Lemma 3.5 (3) and (4),

$$\nu^s_{-i}(a) = Y_{-i}(a, p^s).$$

If $i = n_0 + 1$ then by Lemma 3.5 (5)

$$\nu^s_{n_0-1}(a) = Y_{n_0-1}(a, p^s) - 2\lambda_0 \mu^s_{n_0-1}(a) = Y_{n_0-1}(a, p^s).$$

Assertion (1). If $a/q \in \left[ 0, \frac{1}{2} - \frac{n-2}{2p} \right]$ then $\mu^s_{n_0+1}(a) = 0$, by Lemma 3.2 (2)(b).

This gives, by Lemma 3.5 (1) $\nu^s_{n_0}(a) = \nu^s_{n_0}(a) + \mu^s_{n_0+1}(a) = Z_{n_0}(a, p^s)$

and by Lemma 3.5 (2)

$$2\lambda_0 \mu^s_{n_0}(a) = Z_{n_0-1}(a, p^s) - \nu^s_{n_0-1}(a),$$

whereas by Lemma 3.5 (5), $\nu^s_{n_0-1}(a) = Y_{n_0-1}(a, p^s)$.

Assertion (2). If $a/q \in \left[ \frac{1}{2} + \frac{n-2}{2p}, \ 1 \right]$ then $\mu^s_{n_0}(a) = 0$, by Lemma 3.2 (2)(a). By Lemma 3.5 (2) and (5),

$$Y_{n_0-1}(a, p^s) = \nu^s_{n_0-1}(a) = Z_{n_0-1}(a, p^s) + 2\lambda_0 \mu^s_{n_0+1}(a)$$

which implies $\mu^s_{n_0+1}(a) = \frac{1}{2n_0} [Z^s_{n_0-1}(a) - Z^s_{n_0-1}(a)]$ and

$$\nu^s_{n_0}(a) = \begin{cases} Z_{n_0}(a, p^s) - 2\lambda_0 \mu^s_{n_0+1}(a) \\
\quad = Z_{n_0}(a, p^s) - Y_{n_0-1}(a, p^s) + Z_{n_0-1}(a, p^s).\end{cases}$$

Now using the fact that the HK density function $f_{R_{p,n+1}}$ is continuous and is the limit function of the sequence of functions $\{f_s(R, I)\}_{s \in \mathbb{N}}$, we deduce that the rank functions are well defined and continuous.

□
Lemma 3.8. The functions $\nu_i^{(p)} : \mathbb{R} \rightarrow \mathbb{R}$ and $\mu_i^{(p)} : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\nu_i^{(p)}(x) = \lim_{s \to \infty} \frac{\nu_i^s([xq])}{q^n} \quad \text{and} \quad \mu_i^{(p)}(x) = \lim_{s \to \infty} \frac{2\lambda_0 \mu_i^s([xq])}{q^n}$$

are well defined continuous functions.

Moreover, for each $i$ we have $0 \leq \nu_i^{(p)}(x) \leq 1$ and $0 \leq \mu_i^{(p)}(x) \leq 2$.

Proof. By (2.6), it is enough to prove the assertion for all $\nu_i^{(p)}$ and $\mu_i^{(p)}$ restricted to the interval $[0, 1]$.

By Theorem 1.1 of [T1] (see Definition 2.1) the sequence $\{f_s(R_{p,n+1}, m)\}_{s \in \mathbb{N}}$ converges uniformly to the HK density function $f_{R_{p,n+1}}$. Therefore

$$f_{R_{p,n+1}}(x) = \lim_{s \to \infty} f_s(R_{p,n+1}, m)(x) = \lim_{s \to \infty} \ell(R_{p,n+1}/m[q]/[xq])/q^n.$$

On the other hand if $x \in [0, 1)$ then by Lemma 2.3 and Lemma 2.4

(3.9) $$\ell(R_{p,n+1}/m[q])((x+i)/q)/q^n = \frac{1}{q^n} \left( \nu_i^s([xq]) + 2\lambda_0 \mu_i^s([xq]) \right).$$

By Lemma 3.7 $\mu_i^2([xq]) = 0$ if $i \notin \{n_0, n_0 + 1, n_0 + 2\}$. Hence

$$\nu_i^{(p)}(x) = \lim_{s \to \infty} \frac{\nu_i^s([xq])}{q^n} = f_{R_{p,n+1}}(x + i)$$

is a well defined continuous function. We can argue for other rank functions as follows. By Lemma 3.5 (4)

$$\nu_{n_0+2}^{(p)}(x) = \lim_{s \to \infty} \frac{\nu_{n_0+2}^s([xq])}{q^n} = \lim_{s \to \infty} \frac{Y_{n_0+2}([xq], q)}{q^n} = Y_{n_0+2}(x + n_0 + 2).$$

By (3.9)

$$\frac{\nu_{n_0+2}^s([xq])}{q^n} = \frac{\ell(R_{p,n+1}/m[q]/([x+n_0]q))}{q^n} - \frac{\nu_{n_0+2}^s([xq])}{q^n}.$$

Since both the terms on the right hand side has limit as $s \to \infty$ we get

(3.10) $$\mu_{n_0+1}^{(p)}(x) = f_{R_{p,n+1}}(x + n_0 + 2) - Y_{n_0+2}(x + n_0 + 2)$$

is a well defined and continuous function.

Using similar arguments and Lemma 3.5, we can show that the functions $\nu_{n_0+1}^{(p)}, \mu_{n_0}^{(p)}, \mu_{n_0+1}^{(p)}$ and $\nu_{n_0}^{(p)}$ are well defined and continuous:

$$\nu_{n_0+1}^{(p)}(x) = Y_{n_0+1}(x + n_0 + 1) - \mu_{n_0+1}^{(p)}(x), \quad \text{by Lemma 3.5 (5)},$$

$$\mu_{n_0}^{(p)}(x) = f_{R_{p,n+1}}(x + n_0 + 1) - \nu_{n_0+1}^{(p)}(x), \quad \text{by (3.9)}$$

(3.11) $$\mu_{n_0}^{(p)}(x) = \nu_{n_0+1}^{(p)}(x) + \mu_{n_0}^{(p)}(x) - Z_{n_0+1}(x + n_0 + 1), \quad \text{by Lemma 3.5 (2)}$$

$$= f_{R_{p,n+1}}(x + n_0 + 1) - Z_{n_0+1}(x + n_0 + 1).$$

and

$$\nu_{n_0}^{(p)}(x) = f_{R_{p,n+1}}(x + n_0) - \mu_{n_0+1}^{(p)}(x).$$

Hence all the ranks functions are well defined continuous functions.
To prove the second assertion of the lemma, we compute the ranks of the bundles on both sides of (3.1) and (3.2), and use the fact that the spinor bundle \( S \) has rank \( \lambda_0 \). This gives

\[
p^{sn} = \nu_0^s(a) + \cdots + \nu_s^{n+1}(a) + \lambda_0[\mu_{-n_0+1}^s(a) + \mu_{-n_0}^s(a) + \mu_{-n_0-1}^s(a)].
\]

In particular, we have \( 0 \leq \nu_i^s(a), \lambda_0\mu_{-i}^s(a) \leq q^n = p^{sn} \). Therefore

\[
\mu_i^{(p)}(x) = 2\lambda_0 \lim_{s \to \infty} \mu_{-i}^s([xq]) / q^n \leq 2 \quad \text{and} \quad \nu_i^{(p)}(x) = \lim_{s \to \infty} \nu_{-i}^s([xq]) / q^n \leq 1.
\]

**Definition 3.9.** We call \( \nu_i^{(p)} \) and \( \mu_i^{(p)} \) the rank functions of \( Q_n \) or of \( R_{p,n+1} \).

We say \( \nu_i^{(p)} \) is polynomial in the interval \( I_1 \subset [0,1) \) if there exists a polynomial \( F_i(X) \in \mathbb{Q}[x] \) such that \( \nu_i^{(p)}(x) = F_i(x) \), for all \( x \in I_1 \). We can define the similar notion for \( \mu_i^{(p)} \).

In the following Proposition, which is one of the key steps in the paper, we describe the range where these functions are in fact polynomial functions and hence are independent of \( p \) though the range itself might depend on \( p \).

**Proposition 3.10.** Let \( p > 2 \) such that \( p > n - 2 \). If \( n \geq 4 \) is even then for \( x \in [0,1) \) and for \( i \not\in \{n_0 - 1, n_0, n_0 + 1\} \) the function \( \mu_i^{(p)}(x) = 0 \), and

1. \( \nu_i^{(p)}(x) + \mu_i^{(p)}(x) = Z_i(x+i), \ \text{if} \ x \in [0,1) \ \text{and} \ 0 \leq i \leq n_0. \)
2. \( \nu_{n_0+1}^{(p)}(x) + \mu_{n_0}^{(p)}(x) = Z_{n_0+1}(x+n_0+1), \ \text{if} \ x \in [0,1 - n_2 + 1). \)
3. \( \nu_{n_0+2}^{(p)}(x) = Y_{n_0+2}(x+n_0+2), \ \text{if} \ x \in [0,1). \)
4. \( \nu_{n_0+1}^{(p)}(x) + \mu_{n_0+1}^{(p)}(x) = Y_{n_0+1}(x+n_0+1), \ \text{if} \ x \in [\frac{n_2 + 1}{2p}, 1). \)
5. \( \nu_i^{(p)}(x) + \mu_i^{(p)}(x) = Y_i(x+i), \ \text{if} \ x \in [0,1) \ \text{and} \ n_0 + 3 \leq i < n. \)

If \( n \geq 3 \) is odd then for \( x \in [0,1) \) and for \( i \not\in \{n_0 - 1, n_0\} \) the function \( \mu_i^{(p)}(x) = 0 \), and

1. \( \nu_i^{(p)}(x) + \mu_i^{(p)}(x) = Z_i(x+i), \ \text{if} \ x \in [0,1) \ \text{and} \ 0 \leq i \leq n_0. \)
2. \( \nu_{n_0+1}^{(p)}(x) + \mu_{n_0}^{(p)}(x) = Z_{n_0+1}(x+n_0+1), \ \text{if} \ x \in [0,1 - n_2 + 1). \)
3. \( \nu_{n_0+2}^{(p)}(x) = Y_{n_0+2}(x+n_0+2), \ \text{if} \ x \in [0,1). \)
4. \( \nu_{n_0+1}^{(p)}(x) = Y_{n_0+1}(x+n_0+1), \ \text{if} \ x \in [0,1). \)
5. \( \nu_{n_0+2}^{(p)}(x) = Y_{n_0+2}(x+n_0+2), \ \text{if} \ x \in [0,1). \)
6. \( \nu_i^{(p)}(x) + \mu_i^{(p)}(x) = Y_i(x+i), \ \text{if} \ x \in [0,1) \ \text{and} \ n_0 + 2 \leq i < n. \)

Moreover for both even and odd \( n \), \( \nu_i^{(p)}(x) = \mu_i^{(p)}(x) = 0 \), for all \( x \in [0,1) \) and \( i \geq n. \)

**Proof.** For both parities of \( n \) and for \( i = n_0 \) the assertion (1) follows from (3.5). Lemma 3.7 gives the vanishing of \( \mu_{-n_0+1}^s(a) \) for \( a \) in the certain range. This applied to (3.11) gives assertion (2). The rest of the assertions follow from Lemma 3.7 Lemma 3.5 along with Remark 3.4. \( \square \)
4. The HK density function $f_{R_{p,n+1}}$ and $f_{R_{n+1}^{\infty}}$

Now we are ready to give the first formulation of the HK density function $f_{R_{p,n+1}}$ provided $p > n - 2$.

We follow the Notations 3.1. By Lemma 3.2 for $q = p^s$ and an integer $0 \leq a < q$, there exist integers $\nu^i(a)$ and $\mu^i(a)$ given by the decomposition

$$F^s_a(\mathcal{O}(a)) = \bigoplus_{t=0}^{n-1} \mathcal{O}(-t)\nu^i(a) \bigoplus \bigoplus_{t=0}^{-n_0-1} \mathcal{S}(t)\mu^i(a).$$

Proof of Proposition 3.1. If $x \in [i, i + 1)$ then by Lemma 3.3

$$f_{R_{p,n+1}}(x) = \nu^i(x) + \mu^i(x - i).$$

Case (A). Suppose $n \geq 4$ is an even number.

Case (a). Let $i \in \{0, 1, \ldots, n_0\}$. Then by Proposition 3.10

$$f_{R_{p,n+1}}(x) = \nu^i(x) + \mu^i(x - i) = Z_i(x).$$

Case (b). Let $i = n_0 + 1$ and

Sub case (b) (1). If $x \in [n_0 + 1, n_0 + 2 - \frac{n-2}{2p})$, then by Proposition 3.10 (2)

$$f_{R_{p,n+1}}(x) = \nu^i(x - n_0 - 1) + \mu^i(x - n_0 - 1) = Z_{n_0+1}(x).$$

Sub case (b) (2). If $x \in [n_0 + 2 - \frac{n-2}{2p}, n_0 + 2)$, then, by (3.11),

$$f_{R_{p,n+1}}(x) = Z_{n_0+1}(x) + \mu^i(x - n_0 - 1).$$

Case (c). Let $i = n_0 + 2$ then by (3.10)

$$f_{R_{p,n+1}}(x) = Y_{n_0+2}(x) + \mu^i(x - n_0 - 2).$$

If in addition $x \in [n_0 + 2 + \frac{n-2}{2p}, n_0 + 3)$ then the last assertion of Lemma 3.7 implies

$$\mu^i(x - n_0 - 2) = 0.$$

Case (d). Let $i \in \{n_0 + 3, \ldots, n - 1\}$. Then by Proposition 3.10

$$f_{R_{p,n+1}}(x) = \nu^i(x - i) + \mu^i(x - i) = Y_i(x) \quad \text{for} \quad x \in [i, i + 1].$$

Case (e). Let $i \geq n$. Then by Lemma 3.5 (6),

$$f_{R_{p,n+1}}(x) = \nu^i(x - i) + \mu^i(x - i) = 0.$$

This completes the proof of the theorem when $n$ is an even number.

Assume $n \geq 3$ is odd. Let $x \in [i, i + 1)$ Then by Proposition 3.10 (1) and (6)

$$f_{R_{p,n+1}}(x) = Z_i(x), \quad \text{if} \quad i \leq n_0$$

$$= Y_i(x), \quad \text{if} \quad i \geq n_0 + 2.$$

If $i = n_0 + 1$ then

$$f_{R_{p,n+1}}(x) = \nu^i(x - n_0 - 1) + \mu^i(x - n_0 - 1), \quad \text{if} \quad x - n_0 - 1 \in [0, 1).$$
Now applying Proposition 3.10 (2), (4) and (3) we get
\[
\begin{align*}
\mathcal{f}_{n+1}(x) &= Z_{n+1}(x), & \text{if } x - n - 1 &\in (0, \frac{1}{2} - \frac{n-2}{2p}) \\
&= Y_{n+1}(x) + \mu_{n+1}(x) - n - 1, & \text{if } x - n - 1 &\in \left[\frac{1}{2} - \frac{n-2}{2p}, \frac{1}{2} + \frac{n-2}{2p}\right)
\end{align*}
\]
This completes the proof of the proposition.

**Theorem 4.1.** The function \( f_{n+1}^\infty : [0, \infty) \to [0, \infty) \) given by
\[
\begin{align*}
f_{n+1}^\infty(x) := \lim_{p \to \infty} f_{p+n+1}(x)
\end{align*}
\]
is partially symmetric continuous function, that is
\[
f_{n+1}^\infty(x) = f_{n+1}^\infty(n - x), \text{ for } 0 \leq x \leq (n-2)/2
\]
and is described as follows:

1. If \( n \geq 4 \) is even then
\[
f_{n+1}^\infty(x) = \begin{cases} 
Z_i(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 + 1 \\
Y_i(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n
\end{cases}
\]
and \( f_{n+1}^\infty(x) = 0 \) otherwise.

2. If \( n \geq 3 \) is an odd number then
\[
f_{n+1}^\infty(x) = \begin{cases} 
Z_i(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{n+1}(x) & \text{if } (n_0 + 1) \leq x < (n_0 + \frac{3}{2}) \\
Y_{n+1}(x) & \text{if } (n_0 + \frac{3}{2}) \leq x < (n_0 + 2) \\
Y_i(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n
\end{cases}
\]
and \( f_{n+1}^\infty(x) = 0 \) otherwise.

**Proof.** The description of the function \( f_{n+1}^\infty : [0, \infty) \to [0, \infty) \) follows from Proposition 4.1. To prove the symmetry, we consider the \( Z_i : [0, \infty) \to [0, \infty) \) and \( Y_i : [0, \infty) \to [0, \infty) \) and

**Claim.** \( Z_i(x) = Y_{n-i}(n - x) \), if \( j \leq x < j + 1 \), where \( j \leq n_0 - 1 \).

**Proof of the claim:** By induction on \( j \geq 0 \), first we prove the assertion that
\[
\lim_{q \to \infty} Z_{-j}(a,q)/q^n = \lim_{q \to \infty} Y_{-(n-j)}(q-a,q)/q^n \quad \text{for } 0 \leq a < q.
\]
If \( j = 0 \) then
\[
\lim_{q \to \infty} Z_0(a,q)/q^n = \lim_{q \to \infty} L_a/q^n = \lim_{q \to \infty} L_{a+n}/q^n = \lim_{q \to \infty} Y_0(q-a,q)/q^n.
\]
Assume that the assertion holds for all \( j \) where \( 0 \leq j < i \leq n_0 - 1 \). Now
\[
\lim_{q \to \infty} \frac{Z_{-i}(a, q)}{q^n} = \lim_{q \to \infty} \frac{L_a}{q^n} - \frac{[L_1Z_{-i+1}(a, q) + \cdots + L_iZ_0(a, q)]}{q^n}
\]
\[
= \lim_{q \to \infty} \frac{L_{a+n}}{q^n} - \frac{[L_1Y_{-(n-i)}(q - a, q) + \cdots + L_iY_{-(n-1)}(q - a, q)]}{q^n}
\]
\[
= \lim_{q \to \infty} \frac{Y_{-(n-i)}(q - a, q)}{q^n},
\]
where the second equality follows from the induction hypothesis.

Now to prove the claim, it is enough to prove for \( x = m/q \), where \( m \in \mathbb{Z}_{\geq 0} \). If \( j \leq x < j + 1 \) then \( m = a + jq \), where \( 0 \leq a < q \). Now
\[
Z_j(x) = \lim_{q \to \infty} \frac{Z_{-j}(m - jq, q)}{q^n} = \lim_{q \to \infty} \frac{Y_{-(n-1-j)}((j+1)q - m)}{q^n} = \lim_{q \to \infty} \frac{Y_{-(n-1-j)}((n-m)q - (nq - jq), q)}{q^n} = Y_{n-1-j}(n - x).
\]
This proves the claim.

If \( n \) is even then \( n_0 = (n/2) - 1 \). Let \( 0 \leq x < (n - 2)/2 = n_0 \) then \( i \leq x < (i+1) \) for some \( 0 \leq i < n_0 \). Now
\[
f_{R_{n+1}}(x) = Z_i(x) = Y_{n-1-i}(n - x) = f_{R_{n+1}}(n - x),
\]
where the second equality follows as \( n - (i+1) < n - x \leq n - i \).

If \( n \) is odd then \( n_0 = (n - 1)/2 \). Let \( 0 \leq x < (n - 2)/2 = n_0 - (1/2) \).

If \( i \leq x < (i+1) \), where \( i < n_0 \) then
\[
f_{R_{n+1}}(x) = Z_i(x) = Y_{n-1-i}(n - x) = f_{R_{n+1}}(n - x).
\]
If \( (n_0 - 1) \leq x < n_0 - 1/2 \) then again
\[
f_{R_{n+1}}(x) = Z_{n_0-1}(x) = Y_{n_0+1}(n - x) = f_{R_{n+1}}(n - x).
\]

\[\square\]

**Remark 4.2.** The same argument as above proves that \( f_{R_{p,n+1}} \) is partially symmetric and the symmetry is given by
\[
f_{R_{p,n+1}}(x) = f_{R_{p,n+1}}(n - x) \quad \text{for} \quad 0 \leq x \leq \frac{n-2}{2} \left(1 - \frac{1}{p}\right).
\]

**Proof of Corollary 4.2.** If \( n \) is even then \( n_0 = \frac{n-2}{2} \) and if \( n \) is odd then \( n_0 = \frac{n-1}{2} \). Therefore we can express the intervals as follows
\[
n \quad \text{is even} \quad \Rightarrow \quad \left[n_0 + 2 - \frac{n-2}{2p}, \; n_0 + 2 + \frac{n-2}{2p}\right] = \left[\frac{n+2}{2} - \frac{n-2}{2p}, \; \frac{n+2}{2} + \frac{n-2}{2p}\right]
\]
\[
n \quad \text{is odd} \quad \Rightarrow \quad \left[n_0 + \frac{3}{2} - \frac{n-2}{2p}, \; n_0 + \frac{3}{2} + \frac{n-2}{2p}\right] = \left[\frac{n+2}{2} - \frac{n-2}{2p}, \; \frac{n+2}{2} + \frac{n-2}{2p}\right].
\]
Now the corollary follows from Proposition 4.1 and Theorem 4.1. \[\square\]

**Theorem 4.3.** Let \( p \neq 2 \) and let \( p > n - 2 \). Then
\[
1 + m_{n+1} + \frac{2n-1}{p} \geq e_{HK}(R_{p,n+1}, \mathbf{m}) \geq 1 + m_{n+1}.
\]

In fact,

(1) if \( n \geq 3 \) odd and \( p \geq 3n - 4 \) or

(2) if \( n \geq 4 \) even and \( p \geq (3n - 4)/2 \)
then \(e_{HK}(R_{p, n+1}, m) > 1 + m_{n+1}\).

**Proof.** We prove the theorem for the case when \(n\) is odd, as the proof for the case when \(n\) is even is very similar. By Proposition \[1.1\] and Theorem \[4.1\]

\[0 \leq \int_0^\infty f_{R_{p, n+1}}(x)dx - \int_0^\infty f_{R_{n+1}}(x)dx = \int_0^\infty \frac{1}{2} + \frac{n-2}{2p} \mu_{n_0}(x)dx \leq \frac{2n-4}{p},\]

where the last inequality follows from Lemma \[8.8\]. On the other hand by Theorem \[1.1\] of \[T1\] we have

\[e_{HK}(R_{p, n+1}, m) = \int_0^\infty f_{R_{p, n+1}, m}(x)dx\]

and by the result of Gessel-Monsky [GM], \(\lim_{p \to \infty} e_{HK}(R_{p, n+1}, m) = 1 + m_{n+1}\). Hence by dominated convergence theorem

\[\int_0^\infty f_{R_{n+1}}(x)dx = \lim_{p \to \infty} \int_0^\infty f_{R_{p, n+1}, m}(x)dx = 1 + m_{n+1}.\]

The strict inequality assertion follows from Remark \[8.5\].

**Corollary 4.4.** Let \(p > 2\) be a prime number such that \(p > n - 2\) then the \(F\)-threshold of the ring \(R_{p, n+1}\) is \(c^m(m) = n\).

**Proof.** By Theorem E of [TW1], the \(F\)-threshold \(c^m(m) = \max \{x \mid f_{R_{p, n+1}}(x) \neq 0\}\). Now, by Proposition \[1.1\] \(f_{R_{p, n+1}}(x) = 0\), for \(x \geq n\) and for \(n - 1 \leq x < n\),

\[f_{R_{p, n+1}}(x) = Y_{n-1}(x) = \lim_{q \to \infty} \frac{Y_{n+1}(\lfloor xq \rfloor) - (n-1)q, q)}{q^n} = \frac{2(n-x)^n}{n!} \neq 0,\]

where the last equality follows as \(Y_{n+1}(a, p^n) = L_{q-a-n}\). This proves the corollary. \(\Box\)

5. Generating polynomials for \(f_{R_{p, n+1}}\)

In this and the next section we setup the ground to give second formulation of the HK density function \(f_{R_{p, n+1}}\).

Throughout the section we denote \(n_0 = \lceil \frac{n}{2} \rceil - 1\) and, unless stated otherwise, assume that the characteristic \(p \geq 3n - 4\) if \(n\) is odd and \(p \geq (3n - 4)/2\) if \(n\) is even.

By Proposition \[3.10\] we know that the every function \(\nu_i^{(p)} + \mu_i^{(p)} : [0, 1) \to [0, \infty)\) is a polynomial outside the difficult range which is

(1) \(\lceil \frac{n}{2} - \frac{n-2}{2p}, \frac{1}{2} + \frac{n-2}{2p} \rceil\), if \(n \geq 3\) is an odd integer and

(2) \(\left[0, \frac{n-2}{2p}\right) \cup \left[1 - \frac{n-2}{2p}, 1\right]\), if \(n \geq 4\) is an even integer.

In Proposition \[5.3\] we consider the same property for \(\nu_i^{(p)}\) and \(\mu_i^{(p)}\) instead of \(\nu_i^{(p)} + \mu_i^{(p)}\).

The formulation of \(f_{R_{p, n+1}}\) in Proposition \[1.1\] suggest that we only need to analyze the rank function \(\mu_{n_0}\) when \(n\) is odd, and the rank functions \(\mu_{n_0-1}\) and \(\mu_{n_0+1}\) when \(n\) is even. But the statements look less technical if we consider all the rank functions together in a tuple.
5.1. Polynomials for the rank functions $\nu_i^{(p)}$ and $\mu_j^{(p)}$.

**Notations 5.1.** For the sake of uniformity in the indexing, henceforth we would write the decomposition of $F_*(O(a))$ as follows: For $q = p^s$

$$F_*(O(a)) = \bigoplus_{i=0}^{n-1} O(-i)^{l_i(a,q)} \oplus S(-n_0 + 1)^{l_{n+1}(a,q)} \oplus S(-n_0 - 1)^{l_{n+2}(a,q)}.$$

We call the tuple $(l_0(a,q), \ldots, l_{n+2}(a,q))$ the rank tuple of $F_*(O(a))$ and now onwards we relabel the rank functions $\nu_0^{(p)}, \ldots, \nu_{n-1}^{(p)}, \mu_{n_0-1}^{(p)}, \mu_{n_0}^{(p)}, \mu_{n_0+1}^{(p)}$ as the functions $r_0^{(p)}, \ldots, r_{n-1}^{(p)}, r_n^{(p)}, r_{n+1}^{(p)}, r_{n+2}^{(p)}$ respectively.

Following set $\{l_i(x)\}_i \cup \{r_i(x)\}_i \subset \mathbb{Q}[x]$ of polynomials is going to be the first generating set of polynomials for the function $f_{R_{n,n+1}}$ when $n$ is odd. When $n$ is even the set will be $\{m_i(x)\}_i \subset \mathbb{Q}[x]$. The sets $\{F_i(t)\}_i, \{G_i(t)\}_i, \{\tilde{F}_i(t)\}_i \subset \mathbb{Q}[t]$ come from integrating these polynomials which will be used for the formulation of $e_{HK}(R_{p,n+1})$, but will list them here.

**Definition 5.2.** Let $Z_i(x)$ and $Y_i(x)$ be as in Remark 3.3. We recall that $n_0 = \frac{n-1}{2}$ if $n$ is odd, and $n_0 = \frac{n-2}{2}$ if $n$ is even.

(1) Let $n \geq 3$ be an odd integer. Then we define

$$l_i(x) = \begin{cases} Z_i(x + i) & \text{if } 0 \leq i \leq n_0 \\ Y_i(x + i) & \text{if } n_0 + 1 \leq i < n \\ 0 & \text{if } i = n \\ \frac{1}{2z_0} [Z_{n_0+1}(x + n_0 + 1) - Y_{n_0+1}(x + n_0 + 1)] & \text{if } i = n + 1 \end{cases}$$

and

$$r_i(x) = \begin{cases} Z_i(x + i) & \text{if } 0 \leq i < n_0 \\ Z_{n_0}(x + n_0) - [Y_{n_0+1}(x + n_0 + 1) - Z_{n_0+1}(x + n_0 + 1)] & \text{if } i = n_0 \\ Y_i(x + i) & \text{if } n_0 + 1 \leq i < n \\ \frac{1}{2z_0} [Y_{n_0+1}(x + n_0 + 1) - Z_{n_0+1}(x + n_0 + 1)] & \text{if } i = n \\ 0 & \text{if } i = n + 1. \end{cases}$$

Moreover we consider another two sets of polynomials

$$(5.1) \quad \{F_0(t), F_1(t), \ldots, F_{n+1}(t)\} \cup \{G_0(t), G_1(t), \ldots, G_{n+1}(t)\} \subset \mathbb{Q}[t]$$

given by

$$\int_0^{\frac{1}{2} + \frac{n-2}{2} t} l_i(x)dx = F_i(t), \quad \text{and} \quad \int_0^{\frac{1}{2} + \frac{n-2}{2} t} r_i(x)dx = G_i(t).$$

(2) Let $n \geq 4$ be an even integer, we define polynomials in $\mathbb{Q}[x]$ as follows.

$$m_i(x) = \begin{cases} Z_i(x + i) & \text{if } 0 \leq i \leq n_0 \\ Y_i(x + i) & \text{if } n_0 + 1 \leq i < n \\ 0 & \text{if } i = n \\ \frac{1}{2z_0} [Z_{n_0+1}(x + n_0 + 1) - Y_{n_0+1}(x + n_0 + 1)] & \text{if } i = n + 1 \\ 0 & \text{if } i = n + 2. \end{cases}$$
Also another set of polynomials
\begin{equation}
\{F_0(t), F_1(t), \ldots, \bar{F}_{n+2}(t)\} \subset \mathbb{Q}[t]
\end{equation}
given by
\[
\int_{n/2}^{n/2} \frac{n-2}{n-2} m_i(x)dx = \bar{F}_i(t).
\]

Note that when \(n\) is odd then \(r_{n+2}^{(p)} \equiv 0\) as \(\mu_{n+1}^{(p)}(a) = 0\) for \(0 \leq a < q\).

The proof of the following proposition is just the corollary of Lemma 3.7 and Lemma 3.8.

**Proposition 5.3.** Let \(p \neq 2\) such that \(p > n - 2\).

1. Let \(n \geq 3\) be an odd integer then each rank function \(r_i^{(p)} : [0, 1] \rightarrow [0, 2)\) is a polynomial function on the interval \( \left[ 0, \frac{1}{2} - \frac{n-2}{2p} \right] \cup \left[ \frac{1}{2} + \frac{n-2}{2p}, 1 \right) \) and is given by
   \[
   r_i^{(p)}(x) = \begin{cases} 
   l_i(x), & \text{for all } x \in \left[ 0, \frac{1}{2} - \frac{n-2}{2p} \right) \\
   r_i(x), & \text{for all } x \in \left[ \frac{1}{2} + \frac{n-2}{2p}, 1 \right).
   \end{cases}
   \]
   Moreover
   \[
   \int_{0}^{\frac{1}{2} - \frac{n-2}{2p}} r_i^{(p)}(x)dx = F_i(t) \quad \text{and} \quad \int_{\frac{1}{2} + \frac{n-2}{2p}}^{1} r_i^{(p)}(x)dx = G_i(t).
   \]
   In particular all \(F_i(1/p)\) and \(G_i(1/p)\) are nonnegative for \(p > n - 2\).

2. Let \(n \geq 4\) be even integer then each rank function \(r_i^{(p)} : [0, 1] \rightarrow [0, 2)\) is a polynomial function on the interval \( \left[ \frac{n-2}{2p}, 1 - \frac{n-2}{2p} \right) \) and is given by
   \[
   r_i^{(p)}(x) = m_i(x), \quad \text{for all } x \in \left[ \frac{n-2}{2p}, 1 - \frac{n-2}{2p} \right).
   \]
   Moreover
   \[
   \int_{\frac{n-2}{2p}}^{1 - \frac{n-2}{2p}} r_i^{(p)}(x)dx = \bar{F}_i(t) \geq 0.
   \]

5.2. **Polynomial matrices associated to** \(\nu_i^{(p)}(a), \mu_i^{(p)}(a), \bar{\nu}_i^{(p)}(a)\) and \(\bar{\mu}_i^{(p)}(a)\). Next we will consider a set of \(n - 2\) matrices with entries in \(\mathbb{Q}[t]\). This set along with the set given in Definition 5.2 will be the generating set of polynomials for the function \(f_{R_0, n+1}\).

We recall two sets of matrices as given in Notations 10.5 from the Appendix,

1. \(\{\bar{B}_0^{(p)}, \bar{B}_1^{(p)}, \ldots, \bar{B}_{n-3}^{(p)}\} \subset M_{n+2}(\mathbb{Z}_{\geq 0})\) when \(n \geq 3\) is odd and
2. \(\{C_0^{(p)}, C_1^{(p)}, \ldots, C_{n-3}^{(p)}\} \subset M_{n+2}(\mathbb{Z}_{\geq 0})\), when \(n \geq 4\) is even

We write
\[
\bar{B}_{n_i}^{(p)} = [b_{k_1, k_2}^{(n_i)}(p)]_{0 \leq k_1, k_2 \leq n+1}^{(n_i)} \quad \text{and} \quad \bar{C}_{n_i}^{(p)} = [c_{k_1, k_2}^{(n_i)}(p)]_{0 \leq k_1, k_2 \leq n+2}^{(n_i)}.
\]

**Definition 5.4.** Lemma 10.6 and Lemma 10.8 imply that each \(b_{k_1, k_2}^{(n_i)}(p)\) and \(c_{k_1, k_2}^{(n_i)}(p)\) has a polynomial expression, that is, there exist polynomials
\[
\{H_{k_1, k_2}^{(n_i)}(t) \in \mathbb{Q}[t] \mid 0 \leq n_i \leq n - 3 \quad \text{and} \quad 0 \leq k_1, k_2 \leq n + 1\}
\]
and
\[
\{\bar{H}_{k_1, k_2}^{(n_i)}(t) \in \mathbb{Q}[t] \mid 0 \leq n_i \leq n - 3 \quad \text{and} \quad 0 \leq k_1, k_2 \leq n + 2\}.
\]
such that

\[ b_{k_1,k_2}^{(n)}(p)/p^n = H_{k_1,k_2}^{(n)}(t) \big|_{t=1/p}, \text{ for all } p \geq 3n - 4 \]

and

\[ c_{k_1,k_2}^{(n)}(p)/p^n = \tilde{H}_{k_1,k_2}^{(n)}(t) \big|_{t=1/p}, \text{ for all } p \geq (3n - 4)/2. \]

We denote

1. if \( n \geq 3 \) is odd, then
   \[ \mathbb{H}^{(n)}(t) = \left[ H_{k_1,k_2}^{(n)}(t) \right]_{0 \leq k_1,k_2 \leq n+1} \]
   and

2. if \( n \geq 4 \) is even, then
   \[ \tilde{\mathbb{H}}^{(n)}(t) = \left[ \tilde{H}_{k_1,k_2}^{(n)}(t) \right]_{0 \leq k_1,k_2 \leq n+2}. \]

6. Decomposition of the difficult range

We showed in the previous section that the rank functions \( r_i^{(p)} : [0,1) \to [0,\infty) \) are piecewise polynomial functions in the complement of the difficult range, which is (1) the intervals \([0, \frac{1}{2} - \frac{n-2}{2p}] \) and \([\frac{1}{2} + \frac{n-2}{2p}, 1]\), when \( n \geq 3 \) is odd, and (2) the interval \([\frac{n-2}{2p}, 1 - \frac{n-2}{2p}]\) when \( n \geq 4 \) is even. Here with the help of matrices \( \{\mathbb{H}^{(n)}(t)\}_i \) and \( \{\tilde{\mathbb{H}}^{(n)}(t)\}_i \) as given in Definition 5.4 we will be able to describe these functions in the difficult range too.

To do this we almost cover the difficult range by countably infinitely many intervals which are indexed combinatorially such that each \( r_i^{(p)} \) and hence the HK density function, when restricted to one such interval, has neat expression in terms of the finite set of polynomials as given in Definition 5.2 and Definition 5.4.

Let \( M = \{0,1, \ldots, n-3\} \). For the sake of abbreviation we will denote a tuple \((n_1, \ldots, n_l) \in M^l\) by \( n \), provided there is no confusion. If \( n \geq 3 \) is odd then we construct a set of semi open intervals

\[ \{I_{\mathbb{H}}^{(n)}, J_{\mathbb{H}}^{(n)} \mid n = (n_1, \ldots, n_l) \in \bigcup_{l=1}^{L} M^l\} \]

such that

1. the indexing set is \( M^l \) which is independent of \( p \), although each interval \( I_{\mathbb{H}}^{(n)} \) and \( J_{\mathbb{H}}^{(n)} \) will depend on \( p \).
2. These semi open intervals \( I_{\mathbb{H}}^{(n)}, J_{\mathbb{H}}^{(n)} \) almost cover (i.e., the uncovered part is of measure 0) the difficult range \([\frac{1}{2} - \frac{(n-2)}{2p}, \frac{1}{2} + \frac{(n-2)}{2p}])\).
3. These semi open intervals are disjoint from each other.

Similarly, for even \( n \geq 4 \), we construct such a set of disjoint semi open intervals \( \{I_{\tilde{\mathbb{H}}}^{(n)}, J_{\tilde{\mathbb{H}}}^{(n)} \mid n \in \tilde{M}_0 \cup \tilde{M}_1 = \bigcup_{l \in \mathbb{N}} M^l\} \) which almost cover the difficult range, namely, the interval \([0, \frac{n-2}{2p}] \cup [1 - \frac{n-2}{2p}, 1]\). Now we start with a formal set up

**Definition 6.1.** Let \( n \geq 3 \) be an integer. Let \( A \subseteq \{0,1, \ldots, p-1\} \) be a set of \( n-2 \) elements indexed by the set \( M = \{0,1, \ldots, n-3\} \), that is every \( j_i \in A \) is indexed by unique \( n_i \in M \). Let \( B = \{b_0, b_0 + 1, \ldots, b_0 + t_0 - 1\} \subseteq \{0,1, \ldots, p-1\} \) be a fixed set of consecutive \( t_0 \) integers such that \( A \cap B = \phi \). Then, for \( l \geq 1 \), we define

\[ I_{(n_1, \ldots, n_l)}^{B(p)} = \left( \sum_{i=1}^{l} \frac{j_i}{p^i} + \frac{b_0}{p^{l+1}}, \frac{b_0 + t_0 - 1}{p^{l+1}} + \sum_{i=1}^{l} \frac{j_i}{p^i} \right). \]
Similarly if \( D = \{d_0, d_0 + 1, \ldots, d_0 + t_1 - 1\} \) is another such set of consecutive \( t_1 \) integers in \( \{0, \ldots, p - 1\} \) such that \( A \cap D = \phi \) then we define
\[
I_{(n_1, \ldots, n_l)}^{D(p)} = \left( \sum_{i=1}^{l} \frac{j_i}{p^i} + \frac{d_0}{p^{l+1}}, \frac{d_0 + t_1}{p^{l+1}} + \sum_{i=1}^{l} \frac{j_i}{p^i} \right).
\]

**Lemma 6.2.** Let \( n \geq 3 \) be an integer. If the sets \( A, D \) and \( B \), as given in Definition 6.3, are mutually disjoint sets then the set of semi open intervals
\[
\{I_{(n_1, \ldots, n_l)}^{B(p)} \mid (n_{i_1}, \ldots, n_{i_l}) \in \mathcal{M}^l, (n_{k_1}, \ldots, n_{k_1}) \in \mathcal{M}^l \} \quad l, l_1 \geq 1
\]
is a set of mutually disjoint intervals.

**Proof.** By construction, each of these semi open intervals is a subset of \([0, 1)\). Therefore, for any \( x \) in one of such intervals and for any \( q = p^s \), we have the \( p \)-adic expansion \([xq] = a_0 + a_1p + \cdots + a_{s-1}p^{s-1} \). On the other hand we can check that if \( a \) is a nonnegative integer with the \( p \)-adic expansion \( a = a_0 + a_1p + \cdots + a_{s-1}p^{s-1} \) and \( l_0 \) is an integer then
\[
\begin{align*}
(1) & \quad a/p^s < l_0/p \iff a_{s-1} < l_0 \\
(2) & \quad a/p^s \geq l_0/p \iff a_{s-1} \geq l_0.
\end{align*}
\]

Let us fix an element \( x \in \{I_{(n_{i_1}, \ldots, n_{i_l})}^{B(p)} \mid (n_{i_1}, \ldots, n_{i_l}) \in \mathcal{M}^l, (n_{k_1}, \ldots, n_{k_1}) \in \mathcal{M}^l \} \).

Then for \( q \geq p^{l+1} \), the \( p \)-adic expansion of \([xq]\) is given by
\[
[xq] = a_0 + a_1p + \cdots + a_{s-l-1}p^{s-l-1} + j_{i_0}p^{s-l} + \cdots + j_{i_l}p^{s-1},
\]
where \( a_{s-l-1} \in \mathcal{B} \) as \( b_0 \leq a_{s-l-1} < b_0 + t_0 \).

Now suppose \( x \in \{I_{(n_{k_1}, \ldots, n_{k_l})}^{B(p)} \mid (n_{k_1}, \ldots, n_{k_l}) \in \mathcal{M}^l, (n_{i_1}, \ldots, n_{i_l}) \in \mathcal{M}^l \} \).

Without loss of generality we assume \( l_1 \geq l \), then \( j_{i_t} = j_{k_t} \) for \( t \leq l \). If \( l_1 > l \) then \( j_{k_{l+1}} = a_{s-l-1} \in \mathcal{A} \), which contradicts the assumption that \( A \cap B = \phi \). Therefore \( l_1 = l \) and \((n_{k_1}, \ldots, n_{k_l}) = (n_{i_1}, \ldots, n_{i_l})\).

If \( x \in \{I_{(n_{k_1}, \ldots, n_{k_l})}^{D(p)} \mid (n_{k_1}, \ldots, n_{k_l}) \in \mathcal{M}^l, (n_{i_1}, \ldots, n_{i_l}) \in \mathcal{M}^l \} \), where without loss of generality we assume \( l_1 \geq l \) then \( j_{i_t} = j_{k_t} \) for \( t \leq l \). Now if \( l_1 = l \) then \( d_0 \leq a_{s-l-1} < d_0 + t_1 \), which implies \( a_{s-l-1} \in \mathcal{D} \), otherwise \( a_{s-l-1} \in \mathcal{A} \). Both outcome contradict the mutual disjointness property of the sets \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{D} \). Hence the lemma.

**Definition 6.3.** Let \( S_1 \subset S_2 \) be two Lebesgue measurable set in \( \mathbb{R} \). We say \( S_1 \) almost covers \( S_2 \) if the Lebesgue measure of the set \( S_2 \setminus S_1 \) is 0 in \( \mathbb{R} \).

We use Lemma 6.2 to construct a set of semi open intervals which almost cover the difficult range. We first construct such sets when \( n \geq 4 \) is an even integer.

### 6.1. Decomposition of the difficult range when \( n \) is even.

**Definition 6.4.** Let \( n \geq 4 \) be an even integer. Let \( p > n - 2 \) be a prime. Let
\[
\mathcal{A}_e = \{0, 1, \ldots, \frac{n}{2} - 2\} \cup \{p + 1 - \frac{n}{2}, \ldots, p - 1\}
\]
and let \( \mathcal{B} = \{\frac{n}{2} - 1, \ldots, p - \frac{n}{2}\} \) be the set of consecutive integers. Then \( \mathcal{A}_e \) and \( \mathcal{B} \) are disjoint.

We indexed the elements of the set \( \mathcal{A}_e \) by the set \( \mathcal{M} = \{0, 1, \ldots, n - 3\} \) as follows: An element \( j_i \in \mathcal{A}_e \) is indexed by \( n_i \in \mathcal{M} \) if
\[
j_i = \begin{cases} n_i & \text{if } n_i \leq \left(\frac{n}{2} - 2\right) \\ p - (n_i - \frac{n}{2} - 2) & \text{if } n_i > \left(\frac{n}{2} - 2\right). \end{cases}
\]
Henceforth in this subsection we write the set \( I^{B(p)}_{(n_1, \ldots, n_l)} \) as the set \( \tilde{I}^{(p)}_{(n_1, \ldots, n_l)} \). Following the Definition 6.1 for \( (n_1, \ldots, n_l) \in \mathcal{M}^l \), where \( l \geq 1 \), we define

\[
\tilde{I}^{(p)}_{(n_1, \ldots, n_l)} = \left( \sum_{i=1}^{l} \frac{n_i}{p} + \frac{n - 2}{2p^{l+1}} \left[ \sum_{i=1}^{l} \frac{n_i}{p} + \frac{n - 2}{2p^{l+1}} \right] \right)
\]

Let \( \tilde{\mathcal{M}}_0 = \{(n_1, \ldots, n_l) \in \mathcal{M}^l \mid n_1 \leq \frac{n}{2} \} \subseteq \mathbb{N} \) and \( \tilde{\mathcal{M}}_1 = \{(n_1, \ldots, n_l) \in \mathcal{M}^l \mid n_1 \geq \frac{n}{2} + 1 \} \subseteq \mathbb{N} \).

For the sake of abbreviation, provided there is no confusion, we will denote an element \( (n_1, \ldots, n_l) \in \mathcal{M}^l \) by \( n \).

It is easy to check that

\[
\bigcup_{n \in \tilde{\mathcal{M}}_0} \tilde{I}^{(p)}_{n} \subset [0, \frac{n-2}{2p}) \quad \text{and} \quad \bigcup_{n \in \tilde{\mathcal{M}}_1} \tilde{I}^{(p)}_{n} \subset [1 - \frac{n-2}{2p}, 1).
\]

**Lemma 6.5.**

1. The set \( \bigcup_{n \in \tilde{\mathcal{M}}_0} \tilde{I}^{(p)}_{n} \) almost covers the set \([0, \frac{n-2}{2p})\).
2. The set \( \bigcup_{n \in \tilde{\mathcal{M}}_1} \tilde{I}^{(p)}_{n} \) almost covers the set \([1 - \frac{n-2}{2p}, 1)\).

**Proof.** We prove the first assertion as the proof for the second assertion is very similar.

Let

\[
A_k = [0, \frac{n-2}{2p}) \setminus \bigcup_{\{(n_1, \ldots, n_l) \in \tilde{\mathcal{M}}_0 \mid l \leq k\}} \tilde{I}^{(p)}_{(n_1, \ldots, n_l)}.
\]

It is enough to prove that the Lebesgue measure of the set \( (A_k) \) is

\[
\mu(A_k) = (n - 2)^{k+1}/2p^{k+1}, \quad \text{for all} \quad k \geq 1.
\]

We prove the statement by induction on \( k \). For \( k = 1 \)

\[
A_1 = [0, \frac{n-2}{2p}) \setminus \bigcup_{0 \leq n_i < (n-2)/2} \left[ \frac{n_i}{p} + \frac{n - 2}{2p}, \frac{n_i + 1}{p} - \frac{n - 2}{2p} \right).
\]

Hence

\[
\mu(A_1) = \frac{n-2}{2p} - \frac{n-2}{2} \left[ \frac{1}{p} - \frac{n-2}{p^2} \right] = \frac{(n-2)^2}{2p^2}.
\]

We assume the statement for \( k - 1 \), that is \( \mu(A_{k-1}) = (n - 2)^k/2p^k \). Now

\[
A_k = A_{k-1} \setminus \bigcup_{(n_{i_1}, \ldots, n_{i_k}) \in \tilde{\mathcal{M}}_0} \tilde{I}^{(p)}_{(n_{i_1}, \ldots, n_{i_k})}.
\]

Since, by Lemma 6.2 these semi open intervals are mutually disjoint we have

\[
\mu(A_k) = \mu(A_{k-1}) - \sum_{(n_{i_1}, \ldots, n_{i_k}) \in \tilde{\mathcal{M}}_0} l(\tilde{I}^{(p)}_{(n_{i_1}, \ldots, n_{i_k})}).
\]

But

\[
\sum_{(n_{i_1}, \ldots, n_{i_k}) \in \tilde{\mathcal{M}}_0} l(\tilde{I}^{(p)}_{(n_{i_1}, \ldots, n_{i_k})}) = \frac{n-2}{2} (n - 2)^{k-1} \left( \frac{1}{p^k} - \frac{n-2}{p^{k+1}} \right) = \frac{(n-2)^k}{2p^k} - \frac{(n-2)^{k+1}}{2p^{k+1}},
\]

which implies \( \mu(A_k) = \frac{(n-2)^{k+1}}{2p^{k+1}} \). By induction this proves the result for all \( k \geq 1 \). \( \square \)
6.2. Decomposition of the difficult range when \( n \) is odd.

**Definition 6.6.** Let \( n \geq 3 \) be an odd integer and \( p > n - 2 \) a prime. Let
\[
m_0 = \frac{p}{2} - \frac{n - 2}{2} \quad \text{and} \quad A_o = \{m_0, m_0 + 1, \ldots, m_0 + n - 3\}
\]
and let \( B = \{0, 1, \ldots, m_0 - 1\} \) and \( D = \{m_0 + n - 2, \ldots, p - 1\} \) be two sets of consecutive integers. We indexed the set \( A_o \) by the set \( M = \{0, 1, \ldots, n - 3\} \) so that an element \( j_i \) of \( A_o \) is indexed by \( n_i \) if \( j_i = n_i + m_0 \). Here \( A_o, B \) and \( D \) are mutually disjoint sets.

Henceforth in this subsection we denote the interval \( I_{(n_1, \ldots, n_l)}^{B(p)} \) by \( I_{(n_1, \ldots, n_l)} \) and the interval \( I_{(n_1, \ldots, n_l)}^{D(p)} \) by \( J_{(n_1, \ldots, n_l)}^{(p)} \). Following Definition 6.1 for \((n_1, \ldots, n_l) \in M^l, \) where \( l \geq 1, \) we define two semi open intervals
\[
I_{(n_1, \ldots, n_l)}^{(p)} = \left[ \sum_{i=1}^{l} \frac{i}{p^i}, \sum_{i=1}^{l} \frac{i}{p^i} + \frac{m_0}{p^{l+1}} \right]
\]
and
\[
J_{(n_1, \ldots, n_l)}^{(p)} = \left[ \frac{m_0 + n - 2}{p^{l+1}} + \sum_{i=1}^{l} \frac{i}{p^i}, \sum_{i=1}^{l} \frac{i}{p^i} + \frac{p}{p^{l+1}} \right].
\]

For the sake of abbreviation, provided there is no confusion, we will denote an element \((n_1, \ldots, n_l) \in M^l \) by \( 0. \)

**Lemma 6.7.** Let \( n \geq 3 \) be an odd integer and \( p > n - 2 \) be an odd prime. Then the set
\[
\bigcup_{x \in M^l} I_{x}^{(p)} \cup \bigcup_{x \in M^l} J_{x}^{(p)} \quad \text{almost cover the interval} \quad \left( \frac{1}{2} - \frac{n - 2}{2p}, \frac{1}{2} + \frac{n - 2}{2p} \right).
\]

If \( n = 3 \) then this covering misses the point 2.5 as shown in Section 9.

**Proof.** It is obvious that
\[
\bigcup_{x \in M^l} I_{x}^{(p)} \cup J_{x}^{(p)} \subset \left( \frac{1}{2} - \frac{n - 2}{2p}, \frac{1}{2} + \frac{n - 2}{2p} \right).
\]

Let us denote
\[
A_k = \left( \frac{1}{2} - \frac{n - 2}{2p}, \frac{1}{2} + \frac{n - 2}{2p} \right) \setminus \bigcup_{\{n_1, \ldots, n_l \in M^l | l \leq k\}} I_{x}^{(p)} \cup J_{x}^{(p)}
\]
It is enough to prove that, for each \( k \geq 1, \) the Lebesgue measure of the set \( A_k = ((n - 2)/p)^{k+1}, \) which can be checked using the induction argument for \( k \) as done in Lemma 6.5.

7. POLYNOMIAL EXPRESSION FOR THE RANK FUNCTIONS \( r_{i}^{(p)} \) AND \( f_{R_{p,n+1}} \)

In the previous section, for \( n \) even and for \( n \) odd each, we have constructed an almost covering of the respective difficult range by semi open intervals such that the intervals are disjoint and combinatorially indexed.

By construction, the elements belonging to the same subinterval have a \( p \)-adic expression of certain ‘type’, we use this in the following key Proposition to show that the rank functions \( r_{i}^{(p)} \) when restricted to such intervals, are polynomials.

**Notations 7.1.** For \( n \geq 3 \) odd, let
\begin{enumerate}
    \item \( \{l_0(x), \ldots, l_{n+1}(x)\} \subset \mathbb{Q}[x] \) and \( \{r_0(x), \ldots, r_{n+1}(x)\} \subset \mathbb{Q}[x] \) be as in Definition 5.2.
    \item Let \( \left\{ H_{(n)}(t) \mid 0 \leq i \leq n - 3 \right\} \subset M_{n+2}(\mathbb{Q}[t]) \) be as in Definition 5.4.
    \item Let \( I_{(n_1, \ldots, n_l)}^{(p)} \) and \( J_{(n_1, \ldots, n_l)}^{(p)} \) be the semi open intervals as in Definition 6.6.
\end{enumerate}
(4) Let 
\[ \phi^{(p)} : \bigcup_{\bar{n} \in \mathcal{M}^{|l| \geq 1}} I^{(p)}_{\bar{n}} \cup J^{(p)}_{\bar{n}} \to [0, \frac{1}{2} - \frac{n-2}{2p}] \cup [\frac{1}{2} + \frac{n-2}{2p}, 1), \]

such that for each \( \bar{n} = (n_1, \ldots, n_l) \) the restriction maps
\[ \phi^{(p)} : I^{(p)}_{\bar{n}} \to [0, \frac{1}{2} - \frac{n-2}{2p}] \quad \text{and} \quad \phi^{(p)} : J^{(p)}_{\bar{n}} \to [\frac{1}{2} + \frac{n-2}{2p}, 1), \]

are the surjective maps given by \( x \to p^j (x - \sum_i \frac{j_i}{p^i}) \), where \( j_i = n_i + \frac{p}{2} - \frac{n-2}{2}. \)

**Proposition 7.2.** Let \( n \geq 3 \) is an odd integer, where \( p \geq 3n - 4 \). Then, for \( \bar{n} = (n_1, \ldots, n_l) \) the rank functions \( r^{(p)}_{\bar{n}} : I^{(p)}_{\bar{n}} \to \mathbb{R} \) are given by the formula

\[ (7.1) \quad \left[ r^{(p)}_{0}(x), \ldots, r^{(p)}_{n+1}(x) \right]_{1 \times n+2} = \left[ l_0(\phi^{(p)}(x)), \ldots, l_{n+1}(\phi^{(p)}(x)) \right]_{1 \times n+2} \cdot \mathbb{H}^{(n)}(t) \cdots \mathbb{H}^{(n)}(t) \bigg|_{t=1/p}. \]

Similarly the rank functions \( r^{(p)}_{1} : J^{(p)}_{(n_1, \ldots, n_l)} \to \mathbb{R} \) are given by the formula

\[ (7.2) \quad \left[ r^{(p)}_{0}(x), \ldots, r^{(p)}_{n+1}(x) \right]_{1 \times n+2} = \left[ r_0(\phi^{(p)}(x)), \ldots, r_{n+1}(\phi^{(p)}(x)) \right]_{1 \times n+2} \cdot \mathbb{H}^{(n)}(t) \cdots \mathbb{H}^{(n)}(t) \bigg|_{t=1/p}. \]

In particular we get a formula for \( \mu^{(p)}_{n_0}(x) = r^{(p)}_{n+1}(x) \).

**Proof.** We fix \( x \in J^{(p)}_{(n_1, \ldots, n_l)} \), then \( j_i = n_i + \frac{p}{2} - \frac{n-2}{2} \) and

\[ I^{(p)}_{(n_1, \ldots, n_l)} = \left[ \sum_{i=1}^{l} \frac{j_i}{p^i}, \sum_{i=1}^{l} \frac{j_i}{p^i} + \frac{p}{2^{l+1}} \right). \]

Therefore for any given \( q \geq p^{l+1} \) the \( p \)-adic expansion of \( [xq] \) is given by

\[ [xq] = a_0 + \cdots + a_{s-l-1}p^{s-l-1} + j_1 p^{s-l} + \cdots + j_l p^{s-1}, \]

where \( y := \phi^{(p)}(x) \in [0, \frac{1}{2} - \frac{n-2}{2p}) \) lies in the complement of the difficult range. Hence by Proposition 7.3

\[ r^{(p)}_{i}(y) = l_i(y), \quad \text{for all} \quad 0 \leq i \leq n+1. \]

Let us denote

\[ A_{s-l} = a_0 + \cdots + a_{s-l-1}p^{s-l-1} = \lfloor yp^{s-l} \rfloor. \]

Then \( \mathcal{O}([xq]) = \mathcal{O}(A_{s-l}) \otimes F^{s-l} \mathcal{O}(j_1 + \cdots + j_l p^{l-1}) \). Therefore, by projection formula

\[ F_{s}^{*}(\mathcal{O}([xq])) = F_{s}^{*} \left( F^{s-l}_{*}(\mathcal{O}(A_{s-l})) \otimes \mathcal{O}(j_1 + \cdots + j_l p^{l-1}) \right). \]

Let the rank tuple of \( F^{s-l}_{*}(\mathcal{O}(A_{s-l})) \) be denoted as

\[ (l_0, l_1, \ldots, l_{n+1}) = \left( l_{s-l}^{s-l}(A_{s-l}), l_{s-l}^{s-l}(A_{s-l}), \ldots, l_{n+1}^{s-l}(A_{s-l}) \right) \]

where

\[ F^{s-l}_{*}(\mathcal{O}(A_{s-l})) = \mathcal{O}^{l_0} \oplus \cdots \oplus \mathcal{O}((n+1)^{l_{n-1}} \oplus \mathcal{S}(-n_0 + 1)^{l_n} \oplus \mathcal{S}(-n_0)^{l_{n+1}}. \]

Then we can write
\[ F^s_{*}(\mathcal{O}(A_{j-1})) \otimes \mathcal{O}(j_1 + \cdots + j_1 p^{l-1}) \]
\[ = (\mathcal{O}(j_1)^0 \oplus \cdots \oplus \mathcal{O}(j_1 + n - 1)^{l_n-1} \oplus \mathcal{S}(j_1 - n_0 + 1)^{l_n} \oplus \mathcal{S}(j_1 - n_0)^{l_n+1}) \otimes F^s_{*} \mathcal{O}(j_1 + \cdots + j_1 p^{l-2}). \]

Again, by formula projection
\[ F^s_{*}(\mathcal{O}([xq])) = F^{l-1}_{*} \left( F_{*} \left( F^s_{*}(\mathcal{O}(A_{j-1})) \otimes \mathcal{O}(j_1 + \cdots + j_1 p^{l-1}) \right) \right) \]
\[ = F^{l-1}_{*} \left( (F_{*} \mathcal{O}(j_1)^0 \oplus \cdots \oplus F_{*} \mathcal{S}(-n_0 + j_1)^{l_n+1}) \otimes \mathcal{O}(j_1 + \cdots + j_1 p^{l-2}) \right). \]

Note that \( p \geq 3n - 4 \) implies that, for all \( 0 \leq t < n \) we have \( 0 \leq j_1 - t < p \). Hence by Lemma 3.2 (2) the decomposition of each \( F_{*} \mathcal{O}(j_1 - t) \) will only have bundles of type \( \mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(1 - n) \) and \( \mathcal{S}(-n_0 + 1) \) and \( \mathcal{S}(-n_0) \). Therefore we can write
\[ F_{*} \mathcal{O}(j_1)^0 \oplus \cdots \oplus F_{*} \mathcal{O}(j_1 - n + 1)^{l_n-1} \oplus F_{*} \mathcal{S}(-n_0 + 1 + j_1)^{l_n} \oplus F_{*} \mathcal{S}(-n_0 + j_1)^{l_n+1} \]
\[ = \mathcal{O}^0 \oplus \cdots \oplus \mathcal{O}((1 - n)^{l_n-1} \oplus \mathcal{S}(-n_0 + 1)^{l_n} \oplus \mathcal{S}(-n_0)^{l_n+1}, \]
where as a matrix
\[ [b_0, \ldots, b_{n+1}]_{1 \times n+2} = [l_0, \ldots, l_{n+1}]_{1 \times n+2} \cdot \mathbb{B}_{n_1}^{(p)}, \]
and where we recall
\[ \mathbb{B}_{n_1}^{(p)} = \left[ \begin{array}{cccc} \nu_0(j_1), \ldots, \nu_{-n+1}(j_1), \mu_{-n+1}(j_1), \mu_{-n+1}(j_1) \\ \vdots \\ \nu_0(j_1 - n + 1), \ldots, \nu_{-n+1}(j_1 - n + 1), \mu_{-n+1}(j_1 - n + 1), \mu_{-n+1}(j_1 - n + 1) \\ \nu_0(j_1 - n_0), \ldots, \nu_{-n+1}(j_1 - n_0), \mu_{-n+1}(j_1 - n_0), \mu_{-n+1}(j_1 - n_0) \end{array} \right]. \]

Iterating this we can write the rank tuple for \( F^s_{*}(\mathcal{O}([xq])) \) as
\[ (7.3) \quad \left[ \nu_0^s([xq]), \ldots, \nu_{-n+1}^s([xq]), \mu_{-n+1}^s([xq]), \mu_{-n+1}^s([xq]) \right]_{1 \times n+2} \]
\[ = [l_0, \ldots, l_{n+1}]_{1 \times n+2} \cdot \mathbb{B}_{n_1}^{(p)} \cdots \mathbb{B}_{n_1}^{(p)}. \]

We note that each matrix \( \mathbb{B}_{n_1}^{(p)} \) is independent of \( s \), where \( q = p^s \). Now
\[ \lim_{s \to \infty} \frac{l_i}{p^{s n}} = \lim_{s \to \infty} \frac{l^s_{i} - l^l_{s-l} \left( A_{s-l} \right)}{p^{s n} (p^{s-l})^{n}} = \lim_{s \to \infty} \frac{l^s_{i} - l^{s-l} \left( A_{s-l} \right)}{p^{s n} (p^{s-l})^{n}} = \frac{r_{i}^{(p)}(y)}{p^{n}} = \frac{r_{i}^{(p)}(y)}{p^{n}}, \]
where the second last equality follows from Lemma 3.8.

Hence multiplying both the sides of (7.3) by \( q^n = p^{s n} \) and taking the limit as \( s \to \infty \) we get
\[ \left[ \nu_{0}^{(p)}(x), \ldots, \nu_{-n+1}^{(p)}(x), \mu_{-n+1}^{(p)}(x), \mu_{-n+1}^{(p)}(x) \right]_{1 \times n+2} \]
\[ = [l_0(y), l_1(y), \ldots, l_{n+1}(y)]_{1 \times n+2} \cdot \left( \frac{1}{p^{n}} \cdot \mathbb{B}_{n_1}^{(p)} \right) \cdots \left( \frac{1}{p^{n}} \cdot \mathbb{B}_{n_1}^{(p)} \right). \]

This gives (7.4).

We can argue similarly for the function \( n_1^{(p)} : J_{n_1}^{(p)} \to \mathbb{R} \), where now \( y \) lies in the interval \( \left[ \frac{1}{2} + \frac{n-2}{2p}, 1 \right) \) and hence \( l_1 \) will be replaced by \( l_1 \).

The proof of the following proposition is along the same lines as for Proposition 7.2.

We recall the following.
Notations 7.3. For $n \geq 4$ an even integer,

1. let $\{m_0(x), \ldots, m_{n+2}(x) \in \mathbb{Q}[x]\}$ as in Definition 5.2
2. Let $\{\widetilde{\mathbb{H}}^{(n_i)}(t) \in M_{n+3}(\mathbb{Q}[t]) \mid 0 \leq n_i \leq n-3\}$ as in Definition 5.4
3. Let $I_{(n_1, \ldots, n_l)}$ be the interval as in Definition 5.4
4. Let $\psi^{(p)} : \bigcup_{\{n \in \mathbb{N}^l \mid l \geq 1\}} I_n^{(p)} \to \left[\frac{n-2}{2p}, 1 - \frac{n-2}{2p}\right)$,

such that for each $\underline{n} = (n_1, \ldots, n_l)$ the restriction map

$$\psi^{(p)} : I_{\underline{n}}^{(p)} \to \left[\frac{n-2}{2p}, 1 - \frac{n-2}{2p}\right), \text{ given by } y \to p^l(x - \sum_i \frac{y_i}{p})$$

is surjective.

Proposition 7.4. Let $n \geq 4$ be an even integer, where $p \geq (3n - 4)/2$. Then the functions $r_i^{(p)} : I_{(n_1, \ldots, n_l)} \to \mathbb{R}$ are given by the formula

$$\left[r_0^{(p)}(x), \ldots, r_{n+2}^{(p)}(x)\right]_{1 \times n+3} = \left[m_0(\psi^{(p)}(x)), \ldots, m_{n+2}(\psi^{(p)}(x))\right]_{1 \times n+3} \cdot \widetilde{\mathbb{H}}^{(n)}(t) \cdot \widetilde{\mathbb{H}}^{(n_1)}(t) \bigg|_{t=1/p}.$$  

In particular we get formulas for $\mu^{(p)}_{n_0+1}(x) = r_{n+2}^{(p)}(x)$ and for $\mu^{(p)}_{n_0+1}(x) = r_{n}^{(p)}(x)$.

Now we are ready to define the HK density function $f_{R_p,n+1} : [0, \infty) \to \mathbb{R}$ almost everywhere, that is outside a set of measure 0 in $\mathbb{R}$.

Notations 7.5. We recall that $\{Z_i(x)\}$ and $\{Y_i(x)\}$ are polynomials as in Notations 3.3 and $n_0 = \left\lceil\frac{n}{2}\right\rceil - 1$.

1. If $n$ is odd then, by Lemma 6.7 the set $\left[\frac{1}{2} - \frac{(n-2)}{2p}, \frac{1}{2} + \frac{(n-2)}{2p}\right)$ is almost covered by the set of disjoint intervals $\{I_n^{(p)} \mid n \in \mathcal{M}_l^l \}$, which are as in Definition 6.6.
2. If $n \geq 4$ is even then, by Lemma 6.5 the set $\left[1 - \frac{n-2}{2p}, 1\right)$ is almost covered by the set $\{I_n^{(p)} \mid n \in \mathcal{M}_0\}$, and the set $[1 - \frac{(n-2)}{2p}, 1)$ is almost covered by the set of intervals $\{I_n^{(p)} \mid n \in \mathcal{M}_1\}$, as in Definition 6.4.

Following theorem shows that outside a set of measure 0 the HK density function $f_{R_p,n+1}$ is a piecewise polynomial function though they will be infinitely many pieces.
Theorem 7.6. Let $n \geq 3$ be an odd integer and let $p \geq 3n - 4$. Then the HK density function is defined outside a set of measure 0 as follows:

$$f_{R_p,n+1}(x) = \begin{cases} 
Z_i(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{n_0+1}(x), & \text{if } (n_0 + 1) \leq x < (n_0 + \frac{3}{2}) - \frac{n-2}{2p} \\
Y_{n_0+1}(x) + \mu_{n_0}^{(p)}(x - n_0 - 1), & \text{if } \frac{1}{2} - \frac{n-2}{2p} \leq x - (n_0 + 1) < \frac{1}{2} + \frac{n-2}{2p} \\
Y_{n_0+1}(x), & \text{if } (n_0 + 1) + \frac{1}{2} + \frac{n-2}{2p} \leq x < (n_0 + 2) \\
Y_i(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n 
\end{cases}$$

where

(i) for $x - n_0 - 1 \in I_{(n_1,\ldots,n_{l-1})}^{(p)}$

(7.4) $\mu_{n_0}^{(p)}(x - n_0 - 1) = (n + 1)^{th}$ entry of the matrix

$$[l_0(\phi^{(p)}(x - n_0 - 1)), \ldots, l_{n+1}(\phi^{(p)}(x - n_0 - 1))]_{1 \times n+2} \cdot \mathbb{H}^{(n_1)}(t) \cdots \mathbb{H}^{(n_1)}(t) \big|_{t=1/p}.$$ 

(ii) For $x - n_0 - 1 \in J_{(n_1,\ldots,n_{l-1})}^{(p)}$

(7.5) $\mu_{n_0}^{(p)}(x - n_0 - 1) = (n + 1)^{th}$ entry of the matrix

$$[r_0(\phi^{(p)}(x - n_0 - 1)), \ldots, r_{n+1}(\phi^{(p)}(x - n_0 - 1))]_{1 \times n+2} \cdot \mathbb{H}^{(n_1)}(t) \cdots \mathbb{H}^{(n_1)}(t) \big|_{t=1/p}.$$

Theorem 7.7. If $n \geq 4$ is an even number and $p \geq (3n - 4)/2$ then the HK density function is defined outside measure 0 (in $\mathbb{R}$) as follows:

$$f_{R_p,n+1}(x) = \begin{cases} 
Z_i(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{n_0+1}(x), & \text{if } (n_0 + 1) \leq x < (n_0 + 2) - \frac{n-2}{2p} \\
Z_{n_0+1}(x) + \mu_{n_0-1}^{(p)}(x - n_0 - 1), & \text{if } 1 - \frac{n-2}{2p} \leq x - (n_0 + 1) < 1 \\
Y_{n_0+2}(x) + \mu_{n_0+1}^{(p)}(x - n_0 - 2), & \text{if } 0 \leq x - (n_0 + 2) < \frac{n-2}{2p} \\
Y_{n_0+2}(x), & \text{if } (n_0 + 2) + \frac{n-2}{2p} \leq x < (n_0 + 3) \\
Y_i(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 3 \leq i < n, 
\end{cases}$$

where
Lemma 8.1. 

(7.6) \( \mu_{n_0-1}(x - n_0 - 1) = n^{th} \) entry of the matrix

\[
\begin{pmatrix}
\psi_0(x - n_0 - 1), \ldots, \psi_{n+2}(x - n_0 - 1) \\
\end{pmatrix}
\cdot \left( \hat{H}_{(n)}(t) \cdots \hat{H}_{(n)}(t) \right)_{t=1/p}.
\]

(7.7) \( \mu_{n_0+1}(x - n_0 - 2) = (n + 2)^{th} \) entry of the matrix

\[
\begin{pmatrix}
\psi_0(x - n_0 - 2), \ldots, \psi_{n+2}(x - n_0 - 2) \\
\end{pmatrix}
\cdot \left( \hat{H}_{(n)}(t) \cdots \hat{H}_{(n)}(t) \right)_{t=1/p}.
\]

8. Applications

In the previous section we constructed an infinite set of semi open intervals such that they cover the domain of definition of the HK density function \( f_{R_{p,n+1}} \) except on a set of measure 0.

Since \( f_{R_{p,n+1}} \) is a continuous function, we can ignore the measure 0 part as far as the integration is concerned. Moreover \( f_{R_{p,n+1}} \) restricted to such semi open interval is a polynomial functions which are independent of \( p \). Though the length of each interval depends on \( p \), the set of indexing is independent of \( p \). These properties of the formulation of \( f_{R_{p,n+1}} \) allow us to compare the HK multiplicity \( e_{HK}(R_{p,n+1}) \) as \( p \) varies. As a consequence we settle an old conjecture of Yoshida (Conjecture (2) in the introduction) in the following form

**Theorem** (C) Let \( n \geq 3 \) be an integer then there exists \( \epsilon > 0 \) such that for \( p \geq 1/\epsilon \),

\[ e_{HK}(R_{p,n+1}) \]

is a strictly decreasing function of \( p \).

Moreover, following Notations 8.3, the \( \epsilon = \epsilon_1 \) or \( \epsilon = \epsilon_2 \) depending on the parity of \( n \).

Here we state and prove a basic result from analysis in the form which suits our purpose.

**Lemma 8.1.** If \( H(t) = t(t_0 + b_1 t + \cdots + b_m t^m) \in \mathbb{Q}[t] \) is a polynomial such that \( H(1/p) \geq 0 \) for all \( p \gg 0 \) then either \( H(t) \) is a zero polynomial or \( tH(t) \) is a strictly increasing function on the interval \([0, \epsilon_H])\), where we define

\[ \epsilon_H = \min \left\{ 1, \frac{b_0}{2|b_1| + 3|b_2| + \cdots + (m + 1)|b_m|} \right\}. \]

In particular \( tH(t) > 0 \) for \( t \in (0, \epsilon_H) \).

**Proof.** Suppose \( H(t) \) is not a zero polynomial. Further we can assume \( m > 0 \).

**Claim.** The number \( b_0 \) is strictly positive.

Proof of the claim: Suppose \( b_0 < 0 \).

Since \( H(t) \) is a nonzero polynomial it has only finitely many zeroes. Therefore there exists \( \epsilon_1 > 0 \) such that \( H(t) \) has no zero on the interval \((0, \epsilon_1)\). Now by choosing \( p_0 \geq 1/\epsilon_1 \) we can ensure that \( H(t) \) has no zero in the set \( \{1/p \mid p \geq p_0\} \). Since \(-b_0 > 0\), we can further choose \( \epsilon_1 \) such that for all \( t \in (0, \epsilon_1) \) we have \( t \left( |b_1| + \cdots + |b_m| t^{m-1} \right) < -b_0 \) which implies

\[ b_0 + b_1 t + \cdots + b_m t^m \leq b_0 + |b_1| t + \cdots + |b_m| t^m < 0. \]

But then it contradicts the hypothesis that \( H(1/p) \geq 0 \) for all \( p \gg 0 \). This proves the claim.
Let \( G(t) = t(b_0 + \cdots + b_m t^m) \), then
\[
\frac{d(G(t))}{dt} = b_0 + 2b_1 t + \cdots + (m+1)b_m t^m > 0, \quad \text{for } t \in [0, \epsilon_H),
\]
as
\[
-2b_1 t - 3b_2 t^2 - \cdots - (m+1)b_m t^m < t(2|b_1| + \cdots + (m+1)|b_m|) < b_0.
\]
Hence \( G(t) \) and therefore \( t(b_0 + \cdots + b_m t^m)(t') = tH(t) \) is a strictly increasing function on \([0, \epsilon_H)\). In particular \( tH(t) > 0 \) for \( t \in (0, \epsilon_H) \). □

In the rest of the section we use the notations as in Definitions 5.4 and 5.2 and Proposition 5.3.

The following lemma proves that the integral of \( r_i^{(p)} \), restricted to the each semi open interval, is determined by a polynomial in \( Q[t] \) evaluated at \( t = 1/p \).

**Lemma 8.2.** Let \( n \geq 3 \) be an odd integer and let \( p \geq 3n - 4 \). Then for \((n_1, \ldots, n_l) \in \mathcal{M}^l\), where \( l \geq 1 \)
\[
\left(8.1\right) \quad \left[ \int_{I_0(n_1, \ldots, n_l)} r_0^{(p)}(x)dx, \ldots, \int_{I_0(n_1, \ldots, n_l)} r_{n+1}^{(p)}(x) \right]_{1 \times n+2} = \left[ F_0(t), \ldots, F_{n+1}(t) \right]_{1 \times n+2} \cdot (t \cdot \mathbb{H}^{(n)}(t)) \cdots (t \cdot \mathbb{H}^{(n)}(t)) \bigg|_{t=1/p}.
\]
and
\[
\left(8.2\right) \quad \left[ \int_{J_0(n_1, \ldots, n_l)} r_0^{(p)}(x)dx, \ldots, \int_{J_0(n_1, \ldots, n_l)} r_{n+1}^{(p)}(x) \right]_{1 \times n+2} = \left[ G_0(t), \ldots, G_{n+1}(t) \right]_{1 \times n+2} \cdot (t \cdot \mathbb{H}^{(n)}(t)) \cdots (t \cdot \mathbb{H}^{(n)}(t)) \bigg|_{t=1/p}.
\]
Let \( n \geq 4 \) be an even integer and let \( p \geq (3n - 4)/2 \) then
\[
\left(8.3\right) \quad \left[ \int_{I_0(n_1, \ldots, n_l)} r_0^{(p)}(x)dx, \ldots, \int_{I_0(n_1, \ldots, n_l)} r_{n+2}^{(p)}(x) \right]_{1 \times n+3} = \left[ \bar{F}_0(t), \ldots, \bar{F}_{n+2}(t) \right]_{1 \times n+3} \cdot (t \cdot \bar{\mathbb{H}}^{(n)}(t)) \cdots (t \cdot \bar{\mathbb{H}}^{(n)}(t)) \bigg|_{t=1/p}.
\]

**Proof.** For \( I_0(n_1, \ldots, n_l) \) and \( J_0(n_1, \ldots, n_l) \) as in Definition 6.6
\[
\int_{I_0(n_1, \ldots, n_l)} 1_i \left( p'(x - \sum \frac{j_i}{p'}) \right) dx = \frac{1}{p^l} \int_0^1 \frac{n-2}{2p} 1_i(x)dx = t^l F_i(t) \bigg|_{t=1/p}
\]
and
\[
\int_{J_0(n_1, \ldots, n_l)} r_0 \left( p'(x - \sum \frac{j_i}{p'}) \right) dx = \frac{1}{p^l} \int_0^1 \frac{n-2}{2p} m_i(x)dx = t^l G_i(t) \bigg|_{t=1/p}.
\]
Therefore by (7.1) and (7.2) of Proposition 7.2 the assertions follow.
For even \( n \) the assertions follow by Proposition 7.3 □
Notations 8.3. For a polynomial \( H(t) \in \mathbb{Q}[t] \) we define \( \epsilon_H \) as given in Lemma 8.1. Following Definitions 5.2 and Definition 5.3 let
\[
S_{gp} = \{ H_{k_1,k_2}(t) \mid 0 \leq n_i \leq n - 3, \; 0 \leq k_1,k_2 \leq n + 1 \} \cup \{ F_i(t), G_i(t) \mid 0 \leq i \leq n + 1 \}
\]
and let \( \epsilon_1 = \min \{ \epsilon_H \mid H \in S_{gp} \} \).

Similarly let
\[
\tilde{S}_{gp} = \{ \tilde{H}_{k_1,k_2}(t) \mid 0 \leq n_i \leq n - 3, \; 0 \leq k_1,k_2 \leq n + 1 \} \cup \{ \tilde{F}_0(t), \ldots, \tilde{F}_{n+2}(t) \}
\]
and let \( \epsilon_2 = \min \{ \epsilon_H \mid H \in \tilde{S}_{gp} \} \).

**Lemma 8.4.** If \( n \geq 3 \) is odd number and \( p \geq 1/\epsilon_1 \). Then

(o1) for \( (n_1, \ldots, n_t) \in \cup_{i \geq 1} M^t \)
\[
\int_{I^p_{(n_1, \ldots, n_t)}} r^p_1(x)dx \quad \text{and} \quad \int_{J^p_{(n_1, \ldots, n_t)}} r^p_1(x)dx \quad \text{are decreasing functions of} \quad p.
\]

(o2) Moreover, for each \( n_1 \in \{0, \ldots, n - 3\} \)
\[
\int_{I^p_{(n_1)}} r^p_n(x)dx = \int_{I^p_{(n_1)}} \mu^{(p)}_{m_0-1}(x)dx \quad \text{is a strictly decreasing function of} \quad p.
\]

If \( n \geq 4 \) is an even integer and \( p \geq 1/\epsilon_2 \). Then

(e1) for \( (n_1, \ldots, n_t) \in \tilde{M}_0 \cup \tilde{M}_1 \)
\[
\int_{I^p_{(n_1, \ldots, n_t)}} r^p_1(x)dx \quad \text{is a decreasing function of} \quad p.
\]

(e2) Moreover, for each \( n_1 \in \{2n - 1, \ldots, n - 3\} \)
\[
\int_{I^p_{(n_1)}} \mu^{(p)}_{m_0-1}(x)dx \quad \text{is a strictly decreasing function of} \quad p.
\]

**Proof.** Assertion (o1) and Assertion (e1) follow from Lemma 8.1 and Lemma 8.2.

Assertion (o2). We recall that \( \mu^{(p)}_{m_0-1}(x) = r^{(p)}_{n+1}(x) \). Therefore, by Proposition 7.2 for \( p \geq 3n - 4 \), we have
\[
(8.4) \quad \int_{I^p_{(n_1)}} \mu^{(p)}_{m_0}(x)dx = \sum_{i=0}^{n+1} \left[ \int_{I^p_{(n_1)}} \mu^{(p)}_{m_0}(t) \right] F_i(t) \bigg|_{t=1/p} = \sum_{i=0}^{n+1} \left[ \int_{I^p_{(n_1)}} \mu^{(p)}_{m_0}(t)F_i(t) \right] \bigg|_{t=1/p},
\]
where \( n_1 \in \{0, \ldots, n - 3\} \).

By Lemma 8.1 it is sufficient to prove \( tH^{(n_1)}_{n-1,n+1}(t)F_{n-1}(t) \neq 0 \), which follows as we have
\[
H^{(n_1)}_{n-1,n+1}(1/p) = \frac{n_1(n_1)}{n_1-1,n+1}/p^n = \mu_{m_0}(j_1 - n + 1)/p^n > 0,
\]
where the last inequality holds as, by Lemma 10.3 the condition \( 0 \leq j_1 - n + 1 \leq m_0 - 2 \) gives \( \mu_{m_0}(j_1 - n + 1)/p^n > 0 \). Also
\[
F_{n-1}(\frac{1}{p}) = \int_0^{1/2} \frac{n-2}{2p} l_{n-1}(x)dx = \int_0^{1/2} \frac{n-2}{2p} \frac{2}{n!}(1-x)^n dx > 0
\]
as \( p > n - 2 \).
Assertion (e2). Now consider \( n_1 \in \{ \frac{p}{2} - 1, \ldots, n - 3 \} \) then following the notations of Proposition 5.5

\[
\tilde{F}_0(1/p) = \int_{n_1 - 2}^{n_1} Z_0(x)dx = \frac{2}{(n + 1)!} \left[ \left(1 - \frac{n}{2p} \right)^{n+1} - \left(\frac{n}{2p}\right)^{n+1} \right] > 0.
\]

Also, by Lemma 10.3 the condition \( j_1 = p + \frac{n}{2} - 2 - n_1 = \tilde{m}_0 + n - 3 - n_1 \geq \tilde{m}_0 \) gives

\[
\tilde{H}_{0,n}^{(n+1)}/p = c_{n_1}^{(n)} / p^n = \mu_{n_0+1}(j_1)/p^n > 0.
\]

Whereas, by Proposition 7.4 for \( p \geq (3n - 4)/2 \)

\[
(8.5) \int_{J_{(n_1)}} \mu_{n_0-1}^{(p)}(x)dx = \sum_{i=0}^{n+2} (t\tilde{H}_{i,n}^{(n_1)}(t)\tilde{F}_i(t)) |_{t=1/p}.
\]

Hence, by the same logic as above for \( n \) equal to odd case, this integral is a strictly increasing function of \( t \) for \( t \in [0, \varepsilon_2) \), where \( \varepsilon_2 \) is as in the above notations.

\[\Box\]

Remark 8.5. By Definition 5.4 the polynomials \( F_i(t)|_{1/p}, H_{k_1,n_2}^{(n)}(t)|_{1/p} \) are nonnegative for \( p \geq 3n - 4 \) and \( \tilde{F}_i(t)|_{1/p}, \tilde{H}_{k_1,n_2}^{(n)}(t)|_{1/p} \) are nonnegative for \( p \geq (3n - 4)/2 \). Therefore (8.4) and (8.5) imply that

1. If \( n \geq 3 \) odd and \( p \geq 3n - 4 \) then \( \mu_{n_0}^{(p)}(x)dx > 0 \).
2. If \( n \geq 4 \) even and \( p \geq (3n - 4)/2 \) then \( \mu_{n_0-1}^{(p)}(x)dx > 0 \).

Therefore there exists interval \( I_1 \subset [0,1) \) such that the rank function \( \mu_{n_0}^{(p)} \) are strictly positive on \( I_1 \) in case \( n \) is odd. Similar assertion holds for \( n \) even.

Theorem 8.6. If \( n \geq 3 \) then there exists \( \epsilon > 0 \) such that for \( p \geq 1/\epsilon \), the HK multiplicity \( e_{HK}(R_{p,n+1}) \) of the ring \( R_{p,n+1} \) is a strictly decreasing function of \( p \).

Moreover, following Notations 5.3 one can take \( \epsilon = \epsilon_1 \) or \( \epsilon = \epsilon_2 \) depending on the parity of \( n \).

Proof. Suppose \( n \geq 3 \) is an odd number. Then

\[
(8.6) e_{HK}(R_{p,n+1}) = \int_0^\infty f_{R_{p,n+1}}(x)dx = \int_0^\infty f_{\tilde{R}_{n+1}}(x)dx + \int_{n_1 - 2}^{1/2} \mu_{n_0}^{(p)}(x)dx,
\]

where the first equality follows from Theorem 1.1 of [T1] and the second from Proposition 1.1. Also by Proposition 1.1 the function \( \mu_{n_0}^{(p)} \) is a real valued continuous function on the interval \([0,1)\). Hence by Lemmas 6.2 and 6.7

\[
(8.7) \int_{n_1 - 2}^{1/2} \frac{1}{2} \frac{(n-2)}{2p} \mu_{n_0}^{(p)}(x)dx = \sum_{\nu \in \mathcal{L}_{1\geq 1} \mathcal{M}_1} \int_{J_\nu} \mu_{n_0}^{(p)}(x)dx + \sum_{\nu \in \mathcal{L}_{1\geq 1} \mathcal{M}_1} \int_{J_\nu} \mu_{n_0}^{(p)}(x)dx.
\]

Similarly, when \( n \geq 4 \) is even then

\[
(8.8) e_{HK}(R_{p,n+1}) = \int_0^\infty f_{\tilde{R}_{n+1}}(x)dx + \int_{1/2}^{n_1 - 2} \mu_{n_0}^{(p)}(x)dx + \int_{0}^{1} \frac{1}{2} \frac{(n-2)}{2p} \mu_{n_0+1}^{(p)}(x)dx,
\]

\[
= \int_0^\infty f_{\tilde{R}_{n+1}}(x)dx + \sum_{\nu \in \mathcal{M}_1} \int_{J_\nu} \mu_{n_0}^{(p)}(x)dx + \sum_{\nu \in \mathcal{M}_1} \int_{J_\nu} \mu_{n_0+1}^{(p)}(x)dx,
\]
where the last equality follows by Lemmas 6.2 and 6.5. Hence theorem follows by Lemma 8.4. □

In fact for \( p \gg 0 \), the HK multiplicity \( e_{HK}(R_{p,n+1}) \) can be expressed as a quotient of two rational polynomials. For this first we recall the Cofactor Formula for the Inverse of a Matrix.

Let \( R \) be a commutative ring with 1.

Let \( B \in M_{n+2}(R) \) and let \( B_c = [(−1)^{i+j} \det B_{ji}]_{i,j} \in M_{n+2}(R) \), where \( B_{ij} \in M_{n+1}(R) \) is obtained from \( B \) by deleting \( i \)th row and \( j \)th column. Then \( BB_c = B_cB = (\det B)I_{n+2} \).

In particular \( \det B \) is a unit in \( R \) if and only if \( B \) is unit in \( M_{n+2}(R) \) and in that case we have \( B^{-1} = \frac{1}{\det B} : B_c \in M_{n+2}(R) \).

**Notations 8.7.** Here \( F_i(t) \) and \( G_i(t) \) are polynomials in \( \mathbb{Q}[t] \) as in Proposition 5.3 and \( \mathbb{H}^{(n)}(t) \in M_{n+2}(\mathbb{Q}[t]) \) as in Definition 5.1. We define

\[
\mathbb{B}(t) = t \cdot \mathbb{H}^{(0)}(t) + t \cdot \mathbb{H}^{(1)}(t) + \cdots + t \cdot \mathbb{H}^{(n-3)}(t),
\]

Also for any matrix \( B \), we will denote \( \frac{1}{\det B} \cdot B_c \) by \( \frac{B_c}{\det B} \).

**Theorem 8.8.** Let \( n \geq 3 \) be an odd integer then let \( p \geq 5 \) and \( p > 2[n/2](n-2) \). Then

\[
e_{HK}(R_{p,n+1}) = (1 + m_{n+1}) + (n + 2)^{\text{th}} \text{ entry of the matrix}
\]

\[
\left[ F_0(t), \ldots, F_{n+1}(t) + G_{n+1}(t) \right]_{1 \times n+2} : \mathbb{B}(t) \cdot \left[ \frac{[\mathbb{I}_{n+2} - \mathbb{B}(t)]_c}{\det([\mathbb{I}_{n+2} - \mathbb{B}(t)])} \right]_{t=1/p},
\]

where \( [\mathbb{I}_{n+2} - \mathbb{B}(t)]_c \) is the cofactor matrix of \( \mathbb{I}_{n+2} - \mathbb{B}(t) \).

**Proof.** By (8.6) and (8.7)

\[
e_{HK}(R_{p,n+1}) = \int_0^\infty \int_{R_{n+1}}^\infty f_R(t) dx + \sum_{\mu \in U_{\geq 1} M^t} \int_{\mu_0}^{(p)} \mu_{\mu_0}(x) dx + \sum_{\mu \in U_{\geq 1} M^t} \int_{\mu_0}^{(p)} \mu_{\mu_0}(x) dx.
\]

If \( p \geq 3n - 4 \) then by Lemma 8.2 (8.1)

\[
\sum_{\mu \in M^t} \int_{\mu_0}^{(p)} \mu_{\mu_0}(x) dx = (n + 2)^{\text{th}} \text{ entry of}
\]

\[
\left[ F_0(t), \ldots, F_{n+1}(t) \right]_{1 \times n+2} : \left( t \cdot \mathbb{H}^{(0)}(t) + t \cdot \mathbb{H}^{(1)}(t) + \cdots + t \cdot \mathbb{H}^{(n-3)}(t) \right)
\]

Therefore

\[
\sum_{\mu \in U_{\geq 1} M^t} \int_{\mu_0}^{(p)} \mu_{\mu_0}(x) dx = (n + 2)^{\text{th}} \text{ entry of } \left[ F_0(t), \ldots, F_{n+1}(t) \right]_{1 \times n+2} : \sum_{l \geq 1} \mathbb{B}(t)^l |_{t=1/p}.
\]

Similarly, by Lemma 8.2 (8.2)

\[
\sum_{\mu \in U_{\geq 1} M^t} \int_{\mu_0}^{(p)} \mu_{\mu_0}(x) dx = (n + 2)^{\text{th}} \text{ entry of } \left[ F_0(t), \ldots, F_{n+1}(t) \right]_{1 \times n+2} : \sum_{l \geq 1} \mathbb{B}(t)^l |_{t=1/p}.
\]

Hence to prove the theorem it only remains to prove the following

**Claim.** \( \det([\mathbb{I}_{n+2} - \mathbb{B}(1/p)]) \neq 0 \) if \( p > 2[n/2](n-2) \) and \( p \geq 5 \).
Proof of the claim: We fix a prime \( p \) as in the claim, and consider the \( l^1 \) norm on \( \mathbb{Q}^{n+2} \), that is for \( v = (v_0, \ldots, v_{n+1}) \in \mathbb{Q}^{n+2} \) the \( l^1 \) norm of \( v \) is given by \( \|v\| = |v_0| + \cdots + |v_{n+1}| \).

Note that if the claim does not hold true then there exists a nonzero vector \( v \in \mathbb{Q}^{n+2} \) such that \( v \cdot (\mathbb{I}_{n+2} - \mathbb{B}(1/p)) = 0 \), which implies \( \|v\| = \|v \cdot \mathbb{B}(1/p)\| \).

Now we show that \( l^1 \) norm of the matrix \( \mathbb{B}(1/p) \) is \( < 1 \), that is \( \|v \cdot \mathbb{B}(1/p)\| < \|v\| \), for any nonzero vector \( v \). We recall

\[
\mathbb{B}(1/p) = 1/p \sum_{i=0}^{n-3} \mathbb{H}^{(n_i)}(1/p) = 1/p \sum_{i=0}^{n-3} \mathbb{B}^{(p)}_{n_i}/p^n.
\]

This gives

\[
\|v \cdot \mathbb{B}(1/p)\| \leq \frac{1}{p} \sum_{i=0}^{n-3} \|v \cdot \mathbb{B}^{(p)}_{n_i}/p^n\| \leq \frac{1}{p} \sum_{i=0}^{n-3} \sum_{j=0}^{n+1} |v_{j,n}^{(n_i)}(p) + \cdots + v_{j,n+1}^{(n_i)}(p)|/p^n,
\]

where \( (v_{j,0}^{(n_i)}(p), v_{j,1}^{(n_i)}(p), \ldots, v_{j,n+1}^{(n_i)}(p)) \) denotes the \( j \)th row of the matrix \( \mathbb{B}^{(p)}_{n_i} \) consisting of nonnegative integers.

On the other hand, for any integer \( m \), \( \text{rank}(F_*(O(m))) = p^n \) and \( \text{rank}(F_*(S(m))) = (2^{\lfloor n/2 \rfloor}) p^n \), which implies, following Notations \[10.5\] that

\[
v_{j,0}^{(n_i)}(p) + v_{j,1}^{(n_i)}(p) + \cdots + v_{j,n+1}^{(n_i)}(p) \leq 2^{\lfloor n/2 \rfloor} p^n \text{ for every } 0 \leq j \leq n + 1.
\]

Therefore

\[
\|v \cdot \mathbb{B}(1/p)\| \leq \frac{1}{p} \sum_{i=0}^{n-3} \sum_{j=0}^{n+1} |v_{j,n}^{(n_i)}| 2^{\lfloor n/2 \rfloor} \leq \frac{2^{\lfloor n/2 \rfloor} (n - 2)}{p} \|v\| < \|v\|.
\]

\( \square \)

Notations 8.9. Here \( \widetilde{F}_i(t) \) are as in Proposition 5.3 and where, for \( \mathbb{H}^{(n_i)}(t) \) as in Definition 5.4 we define

\[
\mathbb{C}(t) = t \cdot \mathbb{H}^{(0)}(t) + t \cdot \mathbb{H}^{(1)}(t) + \cdots + t \cdot \mathbb{H}^{(n-3)}(t) \\
\mathbb{C}_1(t) = t \cdot \mathbb{H}^{(0)}(t) + t \cdot \mathbb{H}^{(1)}(t) + \cdots + t \cdot \mathbb{H}^{(n-2)}(t).
\]

Theorem 8.10. Let \( n \geq 4 \) be an even integer then for \( p > 2^{\lfloor n/2 \rfloor} (n - 2) \)

\[
e_{HK}(R_{p,n+1}) = (1 + m_{n+1}) + (n + 1)\text{th entry of } [\widetilde{F}_0(t), \ldots, \widetilde{F}_{n+2}(t)]_{1 \times n+3} \cdot (\mathbb{C}(t) - \mathbb{C}_1(t)) \cdot \frac{[\mathbb{I}_{n+2} - \mathbb{C}(t)]_c}{\det(\mathbb{I}_{n+2} - \mathbb{C}(t))} \bigg|_{t=1/p}
\]

\[
+ (n + 3)\text{th entry of } [\widetilde{F}_0(t), \ldots, \widetilde{F}_{n+2}(t)]_{1 \times n+3} \cdot \mathbb{C}_1(t) \cdot \frac{[\mathbb{I}_{n+2} - \mathbb{C}(t)]_c}{\det(\mathbb{I}_{n+2} - \mathbb{C}(t))} \bigg|_{t=1/p}.
\]

Proof. By (8.8)

\[
e_{HK}(R_{p,n+1}) = \int_0^{\infty} f_{R_{n+1}}^\infty(x)dx + \sum_{\alpha \in \mathcal{M}_1} \int_{j^{(p)}_{\alpha}}^{\infty} \mu_{\alpha_{0-1}}^{(p)}(x)dx + \sum_{\beta \in \mathcal{M}_0} \int_{j^{(p)}_{\beta}}^{\infty} \mu_{\beta_{0+1}}^{(p)}(x)dx,
\]
Remark 8.11. In the above proof we have also proved the following.

This proves the theorem. □

Notations 9.1. Let $n = 3$ and let $p \geq 5$ be a prime. Here $n_0 = 1$. Following the notations as in Definition 6.6 we describe the almost cover of the difficult range $\left[\frac{1}{2} - \frac{1}{2p}, \frac{1}{2}\right) \cup \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2p}\right)$: Here $\mathcal{M} = \{0\}$ and $m_0 = \frac{p-1}{2}$. Let $F^{(p)}(l) =$
\( J^{(p)}_{(0\ldots l \text{ times})} \) and \( J^{(p)}(l) = J^{(p)}_{(0\ldots l \text{ times})} \). Then
\[
I^{(p)}(l) = \left( \sum_{i=1}^{l} \frac{m_{a}}{p}, \sum_{i=1}^{l+1} \frac{m_{a}}{p+1} \right) = \left[ \frac{1}{2} - \frac{1}{2p}, \frac{1}{2} - \frac{1}{2p+1} \right)
\]
and
\[
J^{(p)}(l) = \left[ \frac{m_{a}+1}{p+1} + \sum_{i=1}^{l} \frac{m_{a}}{p}, \sum_{i=1}^{l+1} \frac{m_{a}}{p+1} \right) = \left[ \frac{1}{2} + \frac{1}{2p+1}, \frac{1}{2} + \frac{1}{2p} \right).
\]

Hence it follows that \( \{ I^{(p)}(l), J^{(p)}(l) \}_{l \in \mathbb{N}} \) cover the difficult range everywhere except at the point \( \frac{1}{2} \). Therefore
\[
\left[ \frac{1}{2} - \frac{1}{2p}, \frac{1}{2} + \frac{1}{2p} \right) = \bigcup_{l \in \mathbb{N}} I^{(p)}(l) \cup \left\{ \frac{1}{2} \right\} \cup \bigcup_{l \in \mathbb{N}} J^{(p)}(l)
\]

For \( 0 \leq a < p \) we have
\[
F_{*}(O(a)) = O^{v_{\mu^1}(a)} \oplus O(-1)^{v_{\mu^1}(a)} \oplus O(-2)^{v_{\mu^1}(a)} \oplus S^{v_{\mu^1}(a)} \oplus S(-1)^{v_{\mu^1}(a)}
\]
and
\[
F_{*}(S(a)) = O^{v_{\mu^1}(a)} \oplus O(-1)^{v_{\mu^1}(a)} \oplus O(-2)^{v_{\mu^1}(a)} \oplus S^{v_{\mu^1}(a)} \oplus S(-1)^{v_{\mu^1}(a)}.
\]

Following Notations 3.3 and 10.1 we have
\[
L_{a} = \frac{1}{6}(2a^{3} + 9a^{2} + 13a + 6) \quad \text{and} \quad \bar{L}_{a} = \frac{2}{3}(a^{3} + 3a^{2} + 2a).
\]

By Lemma 10.3 we get
\[
\mu_{-1}(a) = \begin{cases} \frac{1}{4} \left[ L_{a+2p} - L_{1}L_{a+p} + (L_{1}^{2} - L_{2})L_{a} - L_{p-a-3} \right] & \neq 0, \text{ if } a \leq m_{0} - 2 \\
0 & \text{otherwise} \end{cases}
\]
\[
\mu_{0}(a) = \begin{cases} \frac{1}{4} \left[ L_{a+2p} - L_{1}L_{a+p} + (L_{1}^{2} - L_{2})L_{a} - L_{p-a-3} \right] & \neq 0, \text{ if } m_{0} \leq a \\
0 & \text{otherwise} \end{cases}
\]
\[
\bar{\mu}_{-1}(a) = \begin{cases} \frac{1}{4} \left[ \bar{L}_{a+2p} - L_{1}\bar{L}_{a+p} + (\bar{L}_{1}^{2} - L_{2})\bar{L}_{a} - \bar{L}_{p-a-3} \right] & \neq 0, \text{ if } a \leq m_{0} - 1 \\
0 & \text{otherwise} \end{cases}
\]
\[
\bar{\mu}_{0}(a) = \begin{cases} \frac{1}{4} \left[ \bar{L}_{a+2p} - L_{1}\bar{L}_{a+p} + (\bar{L}_{1}^{2} - L_{2})\bar{L}_{a} - \bar{L}_{p-a-3} \right] & \neq 0, \text{ if } m_{0} \leq a \\
0 & \text{otherwise} \end{cases}
\]

In particular
\[
(9.1) \quad \mu_{0} := \mu_{0}(m_{0}) = \frac{1}{4} \left[ L_{p-5} - L_{5p-1} + L_{1}L_{3p-1} - (L_{1}^{2} - L_{2})Y_{p-1} \right] \neq 0,
\]
\[
(9.2) \quad \mu_{-1} := \mu_{-1}(m_{0} - 2) = \frac{1}{4} \left[ L_{5p-5} - L_{1}L_{3p-5} + (L_{1}^{2} - L_{2})Y_{p-5} - L_{p-1} \right] \neq 0
\]
\[
(9.3) \quad \bar{\mu}_{0} := \bar{\mu}_{0}(m_{0}) = \frac{1}{4} \left[ \bar{L}_{p-5} - \bar{L}_{5p-1} + L_{1}\bar{L}_{3p-1} - (L_{1}^{2} - L_{2})\bar{L}_{p-1} \right] \neq 0.
\]
\[
(9.4) \quad \bar{\mu}_{-1} := \bar{\mu}_{-1}(m_{0} - 1) = \frac{1}{4} \left[ \bar{L}_{5p-3} - L_{1}\bar{L}_{3p-3} + (L_{1}^{2} - L_{2})\bar{L}_{p-3} - \bar{L}_{p-1} \right] \neq 0.
\]

Hence \( \mu_{0}, \mu_{-1}, \bar{\mu}_{0} \) and \( \bar{\mu}_{-1} \) are all positive integers.
Theorem 9.2. Let $k$ be a perfect field of characteristic $p \geq 5$ and let

$$R_{p,4} = \frac{k[x_0, x_1, x_2, x_3, x_4]}{(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2)}$$

and let $\mu, \mu_0, \mu_{-1}$ and $\bar{\mu}_{-1}$ be the positive integers given as above. Then

$$f_{R_{p,4},m}(x) = \begin{cases} 
\frac{x^3}{3} & \text{for } 0 \leq x < 1 \\
\frac{x^3}{3} - \frac{5}{3}(x-1)^3 & \text{for } 1 \leq x < 2 \\
\frac{1}{3}x^3 - \frac{5}{3}(x-1)^3 + \frac{14}{3}(x-2)^3 & \text{for } 2 \leq x < 2 + \frac{1}{2} - \frac{1}{2p} \\
\frac{(3-x)^3}{3} & \text{for } 2 + \frac{1}{2} + \frac{1}{2p} \leq x < 3 \\
0 & \text{for } x \geq 3, 
\end{cases}$$

and for $x \in \left[\frac{1}{2} - \frac{1}{2p}, \frac{1}{2} + \frac{1}{2p}\right)$

$$f_{R_{p,4},m}(x) = \frac{1}{3}x^3 - \frac{5}{3}(x-1)^3 + \frac{14}{3}(x-2)^3 + \frac{4}{3} \sum_{i=1}^{\infty} \left(x - 2 - \frac{1}{2} + \frac{1}{2p}\right)^3 (\mu_0(p_i)^{i-1})$$

if $x \in \left[\frac{1}{2} - \frac{1}{2p}, \frac{1}{2} + \frac{1}{2p}\right)$ and $j \geq 1$

and

$$f_{R_{p,4},m}(x) = \frac{(3-x)^3}{3} + \frac{1}{3} \sum_{i=1}^{\infty} \left(2 + \frac{1}{2} + \frac{1}{2p} - x\right)^3 (\mu_{-1}\bar{\mu}_{-1}^{i-1})$$

if $x \in \left[\frac{1}{2} + \frac{1}{2p+1}, \frac{1}{2} + \frac{1}{2p}\right)$ and $j \geq 1$.

Proof. By Proposition 1.1 for $n_0 = 1$ we get

$$f_{R_{p,4}}(x) = \begin{cases} 
Z_0(x) & \text{for } 0 \leq x < 1 \\
Z_1(x) & \text{for } 1 \leq x < 2 \\
Z_2(x) & \text{for } 2 \leq x < 2 + \frac{1}{2} - \frac{1}{2p} \\
Z_2(x) + \mu_0(p)(x-2) & \text{for } 2 + \frac{1}{2} - \frac{1}{2p} \leq x < 2 + \frac{1}{2} \\
Y_2(x) + \mu_1(p)(x-2) & \text{for } 2 + \frac{1}{2} \leq x < 2 + \frac{1}{2} + \frac{1}{2p} \\
Y_2(x) & \text{for } 2 + \frac{1}{2} + \frac{1}{2p} \leq x < 3 \\
Y_2(x) & \text{otherwise.} 
\end{cases}$$
Following Notations 3.3 and Remark 3.4 we get
\[
Z_0(x) = \frac{x^3}{3}, \quad Z_1(x) = \frac{x^3}{3} - \frac{5}{3}(x - 1)^3
\]
\[
Z_2(x) = \frac{x^3}{3} - \frac{5}{3}(x - 1)^3 + \frac{11}{3}(x - 2)^3 \quad \text{and} \quad Y_2(x) = \frac{(3 - x)^3}{3}.
\]

So it remains to compute \( \mu_0^{(p)}(x) \) for \( x \in I^{(p)}(j) \) and \( \mu_1^{(p)}(x) \) for \( x \in J^{(p)}(j) \), for \( j \geq 1 \).

Here instead of using the formula as stated in Proposition 7.2 we used the formula (7.3) given in the proof of the proposition.

Now let \( x \in I^{(p)}(j) \) or \( x \in J^{(p)}(j) \), then for \( q \geq p^{j+1} \), we have the \( p \)-adic expansion
\[
[xq] = a_0 + a_1 p + \cdots + a_{s-j-1} p^{s-j-1} + m_0 p^{s-j} + \cdots + m_0 p^{s-1}.
\]

Let \( A_{s-i} = a_0 + a_1 p + \cdots + a_{s-i-1} p^{s-i-1} \). Then \( A_{s-j} = a_0 + a_1 p + \cdots + a_{s-j-1} p^{s-j-1} \), where \( a_{s-j-1} < m_0 \) if \( x \in I^{(p)}(j) \) and \( a_{s-j-1} > m_0 \) if \( x \in J^{(p)}(j) \)

**Claim.** Let \( \mu_0, \mu_1, \mu_0 \) and \( \mu_1 \) be as given in the above notations. Then

1. \( x \in I^{(p)}(j) \) implies
   \[
   \mu_0^s([xq]) = \nu_0^{s-1}(A_{s-1})(\mu_0) + \nu_0^{s-2}(A_{s-2})(\mu_0 \mu_0) + \cdots + \nu_0^{s-j}(A_{s-j})(\mu_0 \mu_0 \cdots \mu_0).
   \]

2. \( x \in J^{(p)}(j) \) implies
   \[
   \mu_0^s([xq]) = \nu_0^{s-2}(A_{s-1})(\mu_1) + \nu_0^{s-2}(A_{s-2})(\mu_1 \mu_1) + \cdots + \nu_0^{s-j}(A_{s-j})(\mu_1 \mu_1 \cdots \mu_1).
   \]

**Proof of the claim:** By (7.3) in the proof of Proposition 7.2 we have

\[
B_0^{(p)} = \left[ \begin{array}{cccc}
\nu_0(m_0) & \nu_{-1}(m_0) & \nu_{-2}(m_0) & \mu_0 \\
\nu_0(m_0 - 1) & \nu_{-1}(m_0 - 1) & \nu_{-2}(m_0 - 1) & 0 \\
\nu_0(m_0 - 2) & \nu_{-1}(m_0 - 2) & \nu_{-2}(m_0 - 2) & \mu_{-1} \\
\tilde{\nu}_0(m_0) & \tilde{\nu}_{-1}(m_0) & \tilde{\nu}_{-2}(m_0) & \mu_0 \\
\tilde{\nu}_0(m_0 - 1) & \tilde{\nu}_{-1}(m_0 - 1) & \tilde{\nu}_{-2}(m_0 - 1) & \mu_{-1}
\end{array} \right].
\]

Moreover, by Lemma 3.7 if \( x \in I^{(p)}(j) \) then \( \mu_0^{s-j}(A_{s-j}) = 0 \) and if \( x \in J^{(p)}(j) \) then \( \mu_0^{s-j}(A_{s-j}) = 0 \).

Suppose \( x \in I^{(p)}(j) \). Then

\[
\nu_0^s(a), \nu_0^{s-1}(a), \nu_0^{s-2}(a), \mu_0(a), \mu_0^{s-1}(a)
\]

\[
= \nu_0^{s-j+1}(A_{s-j+1}), -, -\nu_0^{s-j}(A_{s-j})\mu_0, \mu_0^{s-j+1}(A_{s-j+1}) \cdot B_0^{(p)} \]

\[
= \nu_0^{s-j+2}(A_{s-j+2}), -, -\nu_0^{s-j+1}(A_{s-j+1})\mu_0 + \nu_0^{s-j}(A_{s-j})\mu_0 \mu_0, \mu_0^{s-j+2}(A_{s-j+2}) \cdot B_0^{(p)} \]

Iterating this \( j \) times we get the first identity given in the claim. The second identity follows similarly.
We recall that \( L_a = \frac{1}{6}(2a^3 + 9a^2 + 13a + 6) = a^3/3 + O(a^2) \). Hence
\[
\lim_{s \to \infty} \frac{\nu_{s-i}^s(A_{s-i})}{p^{3s}} = \frac{1}{3} \left[ x - \frac{1}{2} + \frac{1}{2p} \right]^3
\]
and
\[
\lim_{s \to \infty} \frac{\nu_{s-i}^s(A_{s-i})}{p^{3s}} = \lim_{s \to \infty} \frac{L_{p^s}^{-1}(a-n_0(p^{-i} + \cdots + p^{-1})-3)}{p^{3s}} = \frac{1}{3} \left[ \frac{1}{2} + \frac{1}{2p} - x \right]^3.
\]
Hence, for \( x \in I^{(p)}(j) \),
\[
\mu_{j}^{(p)}(x) = \frac{4}{3} \sum_{i=1}^{j} \left[ x - \frac{1}{2} + \frac{1}{2p} \right]^3 (\mu_{0,n_0-1})
\]
If \( x \in J^{(p)}(j) \) then
\[
\mu_{1}^{(p)}(x) = \frac{4}{3} \sum_{i=1}^{j} \left[ \frac{1}{2} + \frac{1}{2p} - x \right]^3 (\mu_{-1,n-1-1}).
\]
This proves the theorem.

\[\square\]

**Remark 9.3.** In the case of projective curves and projective toric varieties and therefore their arbitrary Segre products (see Proposition 2.17 in [T1]) the set of non smooth points of the HK density function is a finite set.

However, the above formulation in Theorem 9.2 gives that \( f_{R_{p,4}} \) is a \( C^2(\mathbb{R}) \) function everywhere but its third derivative does not exist at any point in the set \( \{ 2 + \frac{1}{2} - \frac{1}{2p}, 2 + \frac{1}{2} - \frac{1}{2p} : j \in \mathbb{N} \} \). In particular the set of non smooth points of \( f_{R_{p,4}} \) is infinite and has an accumulation point.

It would be an interesting direction to investigate the class of rings with \( C^{n-1}(\mathbb{R}) \) HK density function, where \( n+1 \) is the Krull dimension of the ring.

10. **Appendix**

Proposition 5.2 of [L] states that if \( s = 1 \) then \( F_*(\mathcal{O}(a)) \) and \( F_*(\mathcal{S}(a)) \) both have at most one type of spinor bundle, This also follows from the following result of [A] regarding the decomposition of Frobenius image of spinor bundles. Unlike the line bundle \( \mathcal{O}(a) \), where we set up the theory for \( F_s(\mathcal{O}(a)) \) for all \( s \geq 1 \), here we need the information only for the first Frobenius bundle \( F_*(\mathcal{S}(a)) \).

Consider the decomposition
\[
F_*(\mathcal{S}(a)) = \oplus_{t \in \mathbb{Z}} \mathcal{O}(t)\tilde{\nu}_{t}(a) \oplus \oplus_{t \in \mathbb{Z}} \mathcal{S}(t)\tilde{\nu}_{t}(a).
\]

By Theorem [A], for \( q = p \),

1. \( F_*(\mathcal{S}(a)) \) contains \( \mathcal{O}(t) \) if and only if \( 1 \leq a - tp \leq n(p-1) \).
2. \( F_*(\mathcal{S}(a)) \) contains \( \mathcal{S}(t) \) if and only if
\[
\left( \frac{(n-2)(p-1)}{2} \right) \leq a - tp \leq \left( \frac{(n-2)(p-1)}{2} + n - 2 + p \right) - n + 1.
\]

**Notations 10.1.** Fix an integer \( n \geq 3 \). Let \( n_0 = \lfloor n/2 \rfloor - 1 \). For an integer \( a \) such that \( 1 - n \leq a < p \)

We define
\[
\overline{L}_a = H^0(Q,S(a)) = 2\lambda_0 \left[ (a+n) - \binom{a+n-1}{n+1} \right] = 2\lambda_0 \frac{(a+n-1) \cdots a}{n!}.
\]
For $0 \leq i \leq n_0 + 1$, we define iteratively a set of integers $\tilde{Z}_0(a), \tilde{Z}_{-i}(a,p)$ as follows:

$$\tilde{Z}_0(a) = L_a, \quad \tilde{Z}_{-1}(a,p) = \tilde{Z}_0(a + p) - L_1 \tilde{Z}_0(a)$$

and in general

$$\tilde{Z}_{-i}(a,p) = \tilde{Z}_0(a + ip) - \left[ L_1 \tilde{Z}_{-i+1}(a,p) + L_2 \tilde{Z}_{-i+2}(a,p) + \cdots + L_i \tilde{Z}_0(a) \right].$$

Similarly, for $n_0 + 1 \leq i \leq n - 1$, we define iteratively another set of integers $\tilde{L}_{-i}(a,p)$ as follows:

$$\tilde{Y}_{-n+1}(a,p)) = \tilde{L}_{p-a-n+1} \quad \text{and for} \quad n_0 + 1 \leq i < n - 1$$

$$\tilde{Y}_{-i}(a,p) = \tilde{L}_{(n-i)p-a-n+1} - \left[ L_{n-i-1} \tilde{Y}_{-n+1}(a,p) + \cdots + L_1 \tilde{L}_{-i-1}(a,p) \right].$$

**Remark 10.2.** By construction, it is easy to check that there exists rational numbers $\{\tilde{r}_{ij}, \tilde{s}_{jk}\}_{j,k}$ such that for every pair $(a,p)$, where $0 \leq a < p$,

$$\tilde{Z}_0(a) = L_a$$

and

$$\tilde{Z}_{-i}(a,p) = \tilde{r}_{i0} L_a + \tilde{r}_{11} \tilde{L}_{a+p} + \cdots + \tilde{r}_{i(i-1)} \tilde{L}_{a+(i-1)p} + \tilde{L}_{a+ip}$$

and

$$\tilde{Y}_{-i}(a,p) = \sum_{j=1}^{n-i-1} \tilde{s}_{ij} \tilde{L}_{jp-a-n+1} + \tilde{L}_{(n-i)p-a-n+1}.$$ 

On the other hand, $L_m$ is a polynomial in $m$ of degree $n$ which implies, for given $i$

there exist polynomials $P_i(X,Y), Q_i(X,Y) \in \mathbb{Q}[X,Y]$, each of total degree $\leq n$, such that

$$Z_{-i}(a,p) = P_i(a,p) \quad \text{and} \quad Y_{-i}(a,p) = Q_i(a,p) \quad \text{for all} \quad 0 \leq a < p.$$ 

Similar assertions holds for each $\tilde{Z}_{-i}$ and $\tilde{Y}_{-i}$.

**Lemma 10.3.** Let $p > n - 2$, where $n \geq 3$ is an odd integer and $0 \leq a < p$ is an integer. Then

$$F_s(O(a)) = O(-n + 1)^{\nu_{-n+1}(a)} \oplus \cdots \oplus O^{n_0(a)} \oplus S(-n_0 + 1)^{m_{-n_0+1}(a)} \oplus S(-n_0)^{\mu_{-n_0}(a)}.$$ 

Let $m_0 = \frac{p}{2} - \frac{n-2}{2}$. Then

1. $\nu_{-i}(a) = \begin{cases} Z_{-i}(a,p), & \text{for } 0 \leq i \leq n_0 - 1 \\ Y_{-i}(a,p), & \text{for } n_0 + 1 \leq i < n \end{cases}$

2. $\mu_{-i}(a) = 0$, if $i \notin \{n_0 - 1, n_0\}$.

3. Moreover if $a \in [0, m_0 - 1]$ then
   a. $\nu_{-n_0}(a) = Z_{-n_0}(a,p)$
   b. $\mu_{-n_0+1}(a) = 0$
   c. $\mu_{-n_0}(a) = \frac{1}{2m_0} [Z_{-n_0-1}(a,p) - Y_{-n_0-1}(a,p)]$ and $\mu_{-n_0}(a) \neq 0 \iff a \leq m_0 - 2$.

4. Similarly, for $a \in [m_0, p]$ we have
   a. $\nu_{-n_0}(a) = Z_{-n_0}(a,p) - Y_{-n_0-1}(a,p) + Z_{-n_0-1}(a,p)$
   b. $\mu_{-n_0+1}(a) = \frac{1}{2m_0} [Y_{-n_0-1}(a,p) - Z_{-n_0-1}(a,p)] \neq 0.$
   c. $\mu_{-n_0}(a) = 0$.

Also we have

1. $\nu_{-i}(a) = \begin{cases} Z_{-i}(a,p), & \text{for } 0 \leq i \leq n_0 - 1 \\ Y_{-i}(a,p), & \text{for } n_0 + 1 \leq i < n. \end{cases}$
(2) Moreover if \( a \in [0, m_0 - 1] \) then
(a) \( \tilde{v}_{-n_0}(a) = \tilde{Z}_{-n_0}(a,p) \)
(b) \( \tilde{\mu}_{-n_0+1}(a) = 0 \) and
(c) \( \tilde{\mu}_{-n_0}(a) = \frac{1}{2n_0} \left[ \tilde{Z}_{-n_0}(a,p) - \tilde{Y}_{-n_0}(a,p) \right] \neq 0 \).

(3) For \( a \in [m_0, p) \)
(a) \( \tilde{v}_{-n_0}(a) = \tilde{Z}_{-n_0}(a,p) - \tilde{Y}_{-n_0}(a,p) + \tilde{Z}_{-n_0}(a,p) \),
(b) \( \tilde{\mu}_{-n_0+1}(a) = \frac{1}{2n_0} \left[ \tilde{Y}_{-n_0}(a,p) - \tilde{Z}_{-n_0}(a,p) \right] \neq 0 \)
(c) \( \tilde{\mu}_{-n_0}(a) = 0 \).

Proof. For \( \Delta \) as in Notations 3.1, \( \Delta = \frac{n-2}{2p} + \frac{1}{2} \). Now \( F_\ast \mathcal{O}(a) \) contains \( S(-n_0) \) if and only if
\[ 0 \leq a/p + \Delta \leq (p-2)/p \iff a \in [0, m_0 - 2] \].

The rest of the first set of assertions follows from Lemma 3.1.

Similarly \( F_\ast S(a) \) contains \( S(t) \) if and only if
\[ 0 \leq a/p + \Delta - t - n_0 \leq (p-1)/p \].

This implies \(-1 \leq -t - n_0 \leq 0 \) and hence if \( S(t) \) occurs in \( F_\ast S(a) \) then \( t = -n_0 \) or \( t = -n_0 \).

Moreover \( \tilde{\mu}_{-n_0}(a) \neq 0 \iff a \in [0, m_0 - 1] \) and \( \tilde{\mu}_{-n_0+1}(a) \neq 0 \iff a \in [m_0, p-1] \)

In Lemma 3.5 we have used repeatedly the equalities
\[ h^0(Q, F_\ast \mathcal{O}(m)) = h^0(Q, \mathcal{O}(m)) = L_m \]
\[ h^0(Q, F_\ast S(m)) = h^0(Q, S(m)) = h^0(Q, S(-m - n)). \]

Similarly equalities hold for spinor bundles:
\[ h^0(Q, F_\ast (S(m))) = h^0(Q, S(m)) = \tilde{L}_m \]
\[ h^0(Q, F_\ast (S(m))) = h^0(Q, S(m)) = h^0(Q, S(1 - m - n)) = \tilde{L}_{1-m-n}. \]

Now similar equalities as in Lemma 3.5 and Lemma 3.7 holds if we replace \( \nu_i(a) \) by \( \tilde{\nu}_i(a) \) and \( \mu_i(a) \) by \( \tilde{\mu}_i(a) \) in the statement. Hence the second set of assertions follow. \( \square \)

**Lemma 10.4.** Let \( p > n - 2 \), where \( n \geq 4 \) is an even integer and \( 1 - n \leq a < p \) is an integer. Then
\[ F_\ast (\mathcal{O}(a)) = \oplus_{i=0}^{n-1} \mathcal{O}(-t)^{\nu_{-i}(a)} \oplus S(-n_0+1)^{\mu_{-n_0+1}(a)} \oplus S(-n_0)^{\mu_{-n_0}(a)} \oplus S(-n_0-1)^{\mu_{-n_0-1}(a)}. \]

Let \( \tilde{m}_0 = p - \frac{n-2}{2} = p + 1 - \frac{n}{2} \). Then

(A1) \( \nu_{-i}(a) = \begin{cases} Z_{-i}(a,p), & \text{for } 0 \leq i \leq n_0 - 1 \\ Y_{-i}(a,p), & \text{for } n_0 + 1 \leq i < n \end{cases} \)

(A2) Moreover if \( a \in [1 - n, -\frac{n}{2}) \) then
(a) \( \nu_{-n_0}(a) = Z_{-n_0}(a,p) \)
(b) \( \mu_{-n_0+1}(a) = 0 \)
(c) \( \mu_{-n_0}(a) = 0 \) and
(d) \( \mu_{-n_0-1}(a) = \frac{1}{2n_0} [Y_{-n_0-1}(a,p) - Z_{-n_0-1}(a,p)] \)

(A3) if \( a \in [-\frac{n}{2}, \tilde{m}_0 - 1] \) then
(a) \( \nu_{-n_0}(a) = Z_{-n_0}(a,p) \)
(b) \( \mu_{-n_0+1}(a) = 0 \)
(c) \( \mu_{-n_0}(a) = \frac{1}{2n_0} [Z_{-n_0-1}(a,p) - Y_{-n_0-1}(a,p)] \)
(d) \( \mu_{-n_0-1}(a) = 0 \)

(A4) Similarly, for \( a \in [\tilde{m}_0, p) \) we have
(a) \( \nu_{-n_0}(a) = Z_{-n_0}(a,p) - Y_{-n_0-1}(a,p) + Z_{-n_0-1}(a,p) \),
(b) \( \mu_{-n_0+1}(a) = \frac{1}{\nu_{-n_0}} [Y_{-n_0-1}(a,p) - Z_{-n_0-1}(a,p)] \neq 0 \),
(c) \( \mu_{-n_0}(a) = 0 \),
(d) \( \mu_{-n_0-1}(a) = 0 \).

Also we have the decomposition

\[ F_\ast(O(a)) = \bigoplus_{t=0}^{n-1} O(-t)\tilde{v}_{-i}(a) \oplus S(-n_0+1)\tilde{\mu}_{-n_0+1}(a) \oplus S(-n_0)\tilde{\mu}_{-n_0}(a) \oplus S(-n_0-1)\tilde{\mu}_{-n_0-1}(a). \]

(B1) \( \tilde{v}_{-i}(a) = \begin{cases} \tilde{Z}_{-i}(a,p), & \text{for } 0 \leq i \leq n_0 - 1 \\ \tilde{Y}_{-i}(a,p), & \text{for } n_0 + 1 \leq i \leq n. \end{cases} \)

(B2) If \( a \in [-\frac{n}{2}, \tilde{m}_0 - 1] \) then

(a) \( \tilde{v}_{-n_0}(a) = \tilde{Z}_{-n_0}(a,p) \)
(b) \( \tilde{\mu}_{-n_0+1}(a) = 0 \) and
(c) \( \tilde{\mu}_{-n_0}(a) = \frac{1}{\nu_{-n_0}} \left[ \tilde{Z}_{-n_0-1}(a,p) - \tilde{Y}_{-n_0-1}(a,p) \right] \neq 0 \),
(d) \( \tilde{\mu}_{-n_0-1}(a) = 0 \).

(B3) For \( a \in [\tilde{m}_0, p) \)

(a) \( \tilde{v}_{-n_0}(a) = \tilde{Z}_{-n_0}(a,p) - \tilde{Y}_{-n_0-1}(a,p) + \tilde{Z}_{-n_0-1}(a,p) \),
(b) \( \tilde{\mu}_{-n_0+1}(a) = \frac{1}{\nu_{-n_0}} \left[ \tilde{Y}_{-n_0-1}(a,p) - \tilde{Z}_{-n_0-1}(a,p) \right] \neq 0 \),
(c) \( \tilde{\mu}_{-n_0}(a) = 0 \),
(d) \( \tilde{\mu}_{-n_0-1}(a) = 0 \).

Proof. Note that \( \Delta = \frac{n^2 - 2}{2p} \). If \( 1 - n \leq a < p \) then \(-1 < -\frac{n}{2p} \leq \frac{a}{p} + \Delta < 2 \). Now if \( F_\ast(O(a)) \) contains \( O(t) \) then

\[ 0 \leq a - tp \leq n(p - 1) \implies t \in \{0, -1, \ldots, -n + 1\}. \]

If \( F_\ast(O(a)) \) contains \( S(t) \) then

\[ (n_0 - \Delta)p \leq a - tp \leq (n_0 - \Delta)p + p - 2, \]

which implies \( 0 \leq \frac{a}{p} + \Delta - t - n_0 \leq 1 - \frac{2}{p} \). Hence \(-2 < -t - n_0 < 2 \), which gives \( t \in \{-n_0 + 1, -n_0, -n_0 - 1\} \).

If \( F_\ast(S(a)) \) contains \( O(t) \) then

\[ 1 \leq a - tp \leq n(p - 1) \implies t \in \{0, -1, \ldots, -n + 1\}. \]

Moreover

1. \( \tilde{\mu}_{-n_0-1}(a) \neq 0 \implies a \in \left[1 - n, -\frac{n}{2}\right], \)
2. \( \tilde{\mu}_{-n_0}(a) \neq 0 \implies a \in \left[-\frac{n}{2} + 1, \tilde{m}_0 - 1\right] \) and
3. \( \tilde{\mu}_{-n_0+1}(a) \neq 0 \implies a \in \left[\tilde{m}_0, p - 1\right]. \)

If \( F_\ast(S(a)) \) contains \( S(t) \) then

\[ (n_0 - \Delta)p \leq a - tp \leq (n_0 - \Delta)p + p - 1, \]

which implies \( 0 \leq \frac{a}{p} + \Delta - t - n_0 \leq 1 - \frac{1}{p} \) which gives \( t \in \{-n_0 + 1, -n_0, -n_0 - 1\} \).

Since the above decomposition of \( F_\ast(O(a)) \) and \( F_\ast(S(a)) \) hold for all \(-n + 1 \leq a < p \), all the equalities stated in Lemma 3.5 hold true for every \(-n + 1 \leq a < p \), where

\[ L_a = h^0(Q_a, O(a)) = (2a + n) \frac{(a + n - 1) \cdots (a + 1)}{n!}, \quad \text{for } n + 1 \leq a. \]

In particular, by Lemma 3.5 (2) and (5) we have

\[ Y_{-n_0-1}(a,p) - 2\lambda_0 \mu_{-n_0-1} + 2\lambda_0 \mu_{-n_0} = Z_{-n_0-1}(a,p) + 2\lambda_0 \mu_{-n_0+1}. \]
One can whack that if \( a \leq -\frac{3}{2} \) then \( \mu_{-n_0}(a) = 0 \) and for \( a \geq -\frac{3}{2} \) we have \( \mu_{-n_0-1}(a) = 0 \). This observation and the rest of the argument made for \( n \) odd case, will give all the assertion for \( F_*(\mathcal{O}(a)) \).

The set of assertion for \( F_*(\mathcal{S}(a)) \) is along the same lines so we leave the details to the reader. \( \square \)

**Notations 10.5.** Let \( n_i \in \{0, \ldots, n-3\} \). We define a set of matrices \( \mathbb{B}_{n_i}^{(p)} \) and \( \mathbb{C}_{n_i}^{(p)} \) as follows.

Let

\[
F_*(\mathcal{O}(m)) = (\oplus_{t=0}^{n-1} \mathcal{O}(-t)^{\nu_{-t}(m)}) \oplus \mathcal{S}(-n_0+1)^{\mu_{-n_0+1}(m)} \oplus \mathcal{S}(-n_0)^{\mu_{-n_0}(m)} \oplus \mathcal{S}(-n_0-1)^{\mu_{-n_0-1}(m)}
\]

\[
F_*(\mathcal{S}(m)) = (\oplus_{t=0}^{n-1} \mathcal{O}(-t)^{\nu_{-t}(m)}) \oplus \mathcal{S}(-n_0+1)^{\tilde{\nu}_{-n_0+1}(m)} \oplus \mathcal{S}(-n_0)^{\tilde{\nu}_{-n_0}(m)} \oplus \mathcal{S}(-n_0-1)^{\tilde{\nu}_{-n_0-1}(m)}.
\]

(1) \( n \geq 3 \) is an odd integer, we define \((n+2) \times (n+2)\) matrix

\[
\mathbb{B}_{n_i}^{(p)} = \begin{bmatrix}
\nu_0(j_i), \ldots, \nu_{-n+1}(j_i), \mu_{-n_0+1}(j_i), \mu_{-n_0}(j_i) \\
\vdots \\
\nu_0(j_i-n_0+1), \ldots, \nu_{-n+1}(j_i-n_0+1), \mu_{-n_0+1}(j_i-n_0+1), \mu_{-n_0}(j_i-n_0+1) \\
\tilde{\nu}_0(j_i-n_0), \ldots, \tilde{\nu}_{-n+1}(j_i-n_0), \tilde{\mu}_{-n_0+1}(j_i-n_0), \tilde{\mu}_{-n_0}(j_i-n_0)
\end{bmatrix}.
\]

Similarly

(2) If \( n \geq 4 \) is an even integer then we define \((n+3) \times (n+3)\) matrix

\[
\mathbb{C}_{n_i}^{(p)} = \begin{bmatrix}
\nu_0(j_i), \ldots, \nu_{-n_0+1}(j_i), \mu_{-n_0}(j_i), \mu_{-n_0-1}(j_i) \\
\vdots \\
\nu_0(j_i-n_0+1), \ldots, \nu_{-n_0+1}(j_i-n_0+1), \mu_{-n_0}(j_i-n_0+1), \mu_{-n_0-1}(j_i-n_0+1) \\
\tilde{\nu}_0(j_i-n_0), \ldots, \tilde{\nu}_{-n_0+1}(j_i-n_0), \tilde{\mu}_{-n_0}(j_i-n_0), \tilde{\mu}_{-n_0-1}(j_i-n_0)
\end{bmatrix}.
\]

We denote

\[
\mathbb{B}_{n_i}^{(p)} = \left[ b_{k_1,k_2}^{(n_i)}(p) \right]_{0 \leq k_1,k_2 \leq n+1} \quad \text{and} \quad \mathbb{C}_{n_i}^{(p)} = \left[ c_{k_1,k_2}^{(n_i)}(p) \right]_{0 \leq k_1,k_2 \leq n+3}.
\]

**Lemma 10.6.** For a given odd integer \( n \geq 3 \), There exists a set of polynomials

\[
\{H_{k_1,k_2}^{(n_i)}(t) \in \mathbb{Q}[t] \mid 0 \leq n_i \leq n-3 \quad \text{and} \quad 0 \leq k_1,k_2 \leq n+1 \}
\]

such that

\[
\frac{b_{k_1,k_2}^{(n_i)}(p)}{p^n} = H_{k_1,k_2}^{(n_i)}(\frac{1}{p}) \geq 0, \quad \text{for all} \quad p \geq 3n-4.
\]
Proof. Let $0 \leq k_1 < n$ then, by Lemma 10.3

$$b_{k_1,k_2}^{(n_1)}(p) = \begin{cases} Z_{-k_2}(j_i - k_1, p) & \text{if } 0 \leq k_2 \leq n_0 - 1 \\ Y_{-k_2}(j_i - k_1, p) & \text{if } n_0 + 1 \leq k_2 < n \end{cases}$$

$$b_{k_1,n_2}^{(n_2)}(p) = \begin{cases} Z_{-n_0}(j_i - k_1, p) & \text{if } n_i < k_1 \\ Z_{-n_0}(j_i - k_1, p) - Y_{-n_0-1}(j_i - k_1, p) + Z_{-n_0-1}(j_i - k_1, p) & \text{if } n_i \geq k_1 \end{cases}$$

$$b_{n_1,k_2}^{(n_1)}(p) = \begin{cases} 0 & \text{if } n_i < k_1 \\ \frac{1}{2^{n_0}} [Y_{-n_0-1}(j_1 - k_1, p) - Z_{-n_0-1}(j_1 - k_1, p)] & \text{if } n_i \geq k_1 \end{cases}$$

$$b_{k_1,n_2+1}^{(n_1)}(p) = \begin{cases} 0 & \text{if } n_i < k_1 \\ \frac{1}{2^{n_0}} [Z_{-n_0-1}(j_1 - k_1, p) - Y_{-n_0-1}(j_1 - k_1, p)] & \text{if } n_i \geq k_1 \end{cases}$$

By Remark 10.2, for given integer $i$, there exists a polynomial $P_i(X,Y) \in \mathbb{Q}[X,Y]$ of degree $\leq n$ such that $Z_{-i}(a,p) = P_i(a,p)$, for all $0 \leq a < p$ and for $p > n - 2$. For $Y_{-i}$, $\bar{Z}_{-i}$ and $\bar{Y}_{-i}$ we have similar polynomial expressions.

In particular, given $k_1$, $k_2$ and $n_i$, there exist polynomials of total degree $\leq n$

$$P_{k_1,k_2}^{(n_1)}(X,Y) = \sum_{m_1,m_2} \lambda_{m_1,m_2} X^{m_1} Y^{m_2},$$

which are independent of $p$ and for $0 \leq k_1 < n$

$$b_{k_1,k_2}^{(n_1)}(p) = \sum_{m_1,m_2} \lambda_{m_1,m_2}^{(k_1,k_2,n_1)} \left( \frac{p}{2} + n_i - k_1 + 1 - n/2 \right)^{m_1} (p^{m_2}),$$

$$b_{n_1,k_2}^{(n_2)}(p) = \sum_{m_1,m_2} \lambda_{m_1,m_2}^{(n_1,k_2,n_2)} \left( \frac{p}{2} + n_i - n_0 + 2 - n/2 \right)^{m_1} (p^{m_2}),$$
(10.3) \[ b_{n+1,k_2}(p) = \sum_{m_1,m_2} \lambda_{n+1,k_2,n_1}^{(m_1,m_2)} \left( \frac{p}{2} + n_i - n_0 + 1 - \frac{n}{2} \right)^{m_1} (p^{m_2}), \]

Now choosing

\[ H_{k_1,k_2}^{(n_1)}(t) = \sum_{m_1,m_2} \lambda_{k_1,k_2,n_1}^{(m_1,m_2)} \left( \frac{1}{2} + n_i t - \bar{k}_1 t + t - \frac{n t}{2} \right)^{m_1} t^{(n-m_1-m_2)}, \]

where \( \bar{k}_1 = n_0 - 1 \) if \( k_1 = n \) and \( \bar{k}_1 = n_0 \) if \( k_1 = n + 1 \) and \( \bar{k}_1 = k_1 \) for \( 0 \leq k_1 < n \), will prove the lemma.

**Notations 10.7.** Let \( n \geq 4 \) be an even integer and \( p \geq (3n - 4)/2 \). Let \( \tilde{S}_e = \{0, 1, \ldots, \frac{n-2}{2} \} \cup \{p - \frac{n-2}{2}, \ldots, p-1\} \), which is indexed by the set \( \mathcal{M} = \{0, 1, \ldots, n-3\} \) as in Definition 6.3. Here \( n_0 = \frac{n}{2} - 1 \). For given \( n_i \in \mathcal{M} \) we define a \((n+3) \times (n+3)\) matrix \( C_{n_i}^{(p)} = [c_{k_1,k_2}^{(n_i)}(p)]_{0 \leq k_1,k_2 \leq n+2} \) as follows:

\[
\begin{align*}
\nu_{-k_2}(j_i - k_1) & \quad \text{if } 0 \leq k_1, k_2 \leq n - 1 \\
\mu_{-n_0 - \delta_2}(j_i - k_1) & \quad \text{if } 0 \leq k_1 \leq n - 1, k_2 = n + 1 + \delta_2, -1 \leq \delta_2 \leq 1 \\
\nu_{-k_2}(j_i - n_0 - \delta_1) & \quad \text{if } 0 \leq k_2 \leq n - 1, k_1 = n + 1 + \delta_1, -1 \leq \delta_1 \leq 1 \\
\mu_{-n_0 - \delta_2}(j_i - n_0 - \delta_1) & \quad \text{if } k_i = n + 1 + \delta_i, -1 \leq \delta_1, \delta_2 \leq 1.
\end{align*}
\]

**Lemma 10.8.** There exists a set of polynomials

\[ \{ \tilde{H}_{k_1,k_2}^{(n_1)}(t) \in \mathbb{Q}[t] \mid 0 \leq n_i \leq n - 3 \text{ and } 0 \leq k_1, k_2 \leq n + 2 \} \]

such that

\[ \frac{c_{k_1,k_2}^{(n_i)}(p)}{p^n} = \tilde{H}_{k_1,k_2}^{(n_1)}(\frac{1}{p}) \geq 0, \quad \text{for all } p \geq (3n - 4)/2. \]

**Proof.** Case (1). \( n_i \in \{0, \ldots, \frac{n}{2} - 2\} \). Note that in this case \( j_i = n_i \).

1. Let \( 0 \leq k_1 \leq n - 1 \). Then \( 1 - n \leq j_i - k_1 \leq \frac{n}{2} - 2 \).

   \[ \text{If } k_1 \in \{n_i + \frac{n}{2}, \ldots, n_i + n - 1\}, \text{ then } 1 - n \leq j_i - k_1 \leq -\frac{n}{2}. \]

   Hence applying Lemma 10.4 (A2), we get polynomial expression for \( c^{(n_i)}_{k_1,k_2}(p) \), for all \( 0 \leq k_2 \leq n + 2 \).

   \[ \text{If } k_1 \in \{0, \ldots, n_i + \frac{n}{2} - 1\}, \text{ then } -\frac{n}{2} - 1 \leq j_i - k_1 \leq -\frac{n}{2} - 2. \]

   Hence applying Lemma 10.4 (A3), we get polynomial expression for \( c^{(n_i)}_{k_1,k_2}(p) \), for all \( 0 \leq k_2 \leq n + 3 \).

2. Let \( k_1 = n + 1 + \delta_1 \), where \( -1 \leq \delta_1 \leq 1 \). Then \( -\frac{n}{2} \leq n_i - n_0 - \delta_1 \leq 0 \). Hence applying Lemma 10.4 (B2) we get polynomial expression for \( c^{(n_i)}_{n+1+\delta_1,k_2}(p) \). 

Case (2). \( n_i \in \{\frac{n}{2} - 1, \ldots, n - 3\} \). Then \( j_i \in \{p - \frac{n-2}{2}, \ldots, p - 1\} \).

1. Let \( 0 \leq k_1 \leq n - 1 \). Then \( p - \frac{3n}{2} + 2 \leq j_i - k_1 = p - 1 \), which implies \( 0 \leq j_i - k_1 \leq p - 1 \) as \( p \geq (3n - 4)/2 \).

   \[ \text{If } k_1 \in \{n - 2 - n_i, \ldots, n - 1\}, \text{ then } 0 \leq j_i - k_1 \leq \bar{m}_0 - 1. \]

   Therefore applying Lemma 10.4 (A3), we get polynomial expression for \( c^{(n_i)}_{k_1,k_2}(p) \), for all \( 0 \leq k_2 \leq n + 2 \).

   \[ \text{If } k_1 \in \{0, \ldots, n - 3 - n_i\}, \text{ then } \bar{m}_0 \leq j_i - k_1 \leq p - 1 \text{ and hence applying } \]

   Lemma 10.4 (A4), we get polynomial expression for \( c^{(n_i)}_{k_1,k_2}(p) \), for all \( 0 \leq k_2 \leq n + 2 \).
Let \( k_1 = n + 1 + \delta_1 \), where \(-1 \leq \delta_1 \leq 1\).

Then \( p - n + 1 \leq j_i - n_0 - \delta_1 \leq \tilde{m}_0 \).

(a) If \( n_i = \frac{n}{2} - 1 \) and \( k_1 = n \) then \( j_i - n_0 - \delta_1 = \tilde{m}_0 \). In this case, by Lemma [10.1] (B3), we get polynomial expression for \( c^{(n_i)}_{n,k_2}(p) \).

(b) If either \( n_i \in \{\frac{n}{2}, \ldots, n - 3\} \) or \( k_1 \in \{n + 1, n + 2\} \) then \( p - n + 1 \leq j_i - n_0 - \delta_1 \leq \tilde{m}_0 - 1 \) and by Lemma [10.4] (B2), we get polynomial expression for \( c^{(n_i)}_{k_1,k_2}(p) \), for all \( 0 \leq k_2 < n + 2 \).

\[ \square \]

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