A New $k$-th Derivative Estimate for Exponential Sums via Vinogradov’s Mean Value

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In celebration of the 125th
anniversary of the birth of
Ivan Matveevich Vinogradov

1 Introduction

The familiar van der Corput $k$-th derivative estimate for exponential sums (Titchmarsh [10, Theorems 5.9, 5.11, & 5.13], for example), may be stated as follows. Let $k \geq 2$ be an integer, and suppose that $f(x) : [0,N] \to \mathbb{R}$ has continuous derivatives of order up to $k$ on $(0,N)$. Suppose further that $0 < \lambda_k \leq f^{(k)}(x) \leq A\lambda_k$ on $(0,N)$. Then

$$\sum_{n \leq N} e(f(n)) \ll A^{2 - k} N\lambda_k^{1/(2^k - 2)} + N^{1 - 2^{-k}} \lambda_k^{-1/(2^k - 2)},$$

with an implied constant independent of $k$. One usually chooses $k$ so that the first term dominates, and one often has $A^{2 - k} \ll 1$, so that the bound is merely $O(N\lambda_k^{1/(2^k - 2)})$. Clearly one can only get a non-trivial bound when $\lambda_k < 1$. A typical application is the series of estimates

$$\zeta(\sigma + it) \ll t^{1/(2^k - 2)} \log t, \quad \left(\sigma = 1 - \frac{k}{2^k - 2}, \ t \geq 2\right)$$

for $k = 2, 3, \ldots$. Again the implied constant is independent of $k$.

One can improve on the standard $k$-th derivative bound somewhat. Thus Robert and Sargos [7] show roughly that if $k = 4$ then

$$\sum_{n \leq N} e(f(n)) \ll \varepsilon N^\varepsilon (N\lambda_4^{1/14} + \lambda_4^{-7/13}),$$

for any $\varepsilon > 0$. In the corresponding version of (1) one would have a term $N\lambda_4^{1/14}$ in place of $N\lambda_4^{1/13}$. Similarly for $k = 8$ and 9, Sargos [9, Theorems 3 & 4] gives bounds

$$\sum_{n \leq N} e(f(n)) \ll \varepsilon N^\varepsilon (N\lambda_8^{1/204} + \lambda_8^{-95/204}),$$

and

$$\sum_{n \leq N} e(f(n)) \ll \varepsilon N^\varepsilon (N\lambda_9^{7/2640} + \lambda_9^{-1001/2640}),$$

and so on.
respectively. Here the exponents $1/204$ and $7/2640$ should be compared with
the values $1/254$ and $1/510$ produced by (1).

There are also results in which the second term in the van der Corput bound
is improved. Thus when $k = 3$ the bound (1) reduces to an estimate $O(N^{\lambda_3^{1/6}})$
for $\lambda_3 \geq N^{-3/2}$. However it was shown independently by Gritsenko [3] and
Sargos [8] that the weaker hypothesis $\lambda_3 \geq N^{-2}$ suffices.

There are quite different approaches to exponential sums, using estimates
for the Vinogradov mean value integral

$$J_{s,l}(P) = \int_{0}^{1} \ldots \int_{0}^{1} \left| \sum_{n \leq P} e(\alpha_1 n + \ldots + \alpha_l n^l) \right|^{2s} d\alpha,$$

(2)

see Vinogradov [11], [12], and Korobov [4], amongst others. The first of these
methods is described by Titchmarsh [10, Chapter 6] for example. The Vinogradov-
Korobov machinery has been used by Ford [2, Theorem 2] to show that

$$\sum_{N<n \leq 2N} n^{-it} \ll N^{1-1/134k^2}$$

(3)

for $N^k \geq t \geq 2$. (Ford’s result is somewhat more precise, and more general.)

One may think of this as corresponding very roughly to a bound of the form (1)
with first term $N^{\lambda_3^{1/6}}$.

A slightly refined version of the original method of Vinogradov [11] coupled
with new estimates for the Vinogradov mean value integral, leads to distinctly
stronger bounds. For example, Wooley [13, Theorem 1.2] gives

$$J_{s,l}(P) \ll \varepsilon, l P^{2s-l(l+1)/2+\varepsilon} \quad (s \geq l(l-1)), $$

and Robert [6, Theorem 10] used this to show that if $k \geq 4$ then

$$\sum_{n \leq N} e(f(n)) \ll A, k, \varepsilon \ N^{1+\varepsilon(1/2(k-1)(k-2) + N^{-1/2(k-1)(k-2)})}$$

for $N \geq \lambda_k^{-1-1/2(k-3)}$. This is a remarkable improvement on the classical $k$-th
derivative estimate. The exponent of $\lambda_k$ is better than $1/(2k - 2)$ for all $k \geq 4$, and
decreases quadratically rather than exponentially.

The purpose of this paper is to further refine the original method of Vino-
gradov [11] and to input the very recent optimal bounds for the Vinogradov
mean value integral, due to Wooley [13] (for $l = 3$), and to Bourgain, Demeter
and Guth [1] (for $l \geq 4$). These theorems show that

$$J_{s,l}(P) \ll \varepsilon, l P^{2s-l(l+1)/2+s} \quad (s \geq \frac{1}{2}l(l+1), \ l \geq 1), $$

(4)

the cases $l = 1$ and $l = 2$ being elementary. The range for $s$ is optimal, and it is
this feature that represents the dramatic culmination of many previous works
over the past 80 years. Unfortunately neither result gives an explicit dependence
on $l$ and $s$, nor gives an explicit form for the factor $P^\varepsilon$. Results prior to the
advent of Wooley’s efficient congruencing method had required $s$ to be larger,
but had given an explicit dependence on \(l\). Thus for example, Ford [2, Theorem 3] implies in particular that
\[
J_{s,l}(P) \ll l^{3/4} p^{s-(l+1)/2+t^2/1000} \quad (s \geq \frac{5}{2}l^2, \ l \geq 129),
\]
in which one has an additional term \(l^2/1000\) in the exponent, and more restrictive conditions on \(l\) and \(s\). An important application of bounds for Weyl sums is to the zero-free region for \(\zeta(s)\), as described by Ford. However for this it is crucial to have a suitable dependence on the parameter \(l\), so that the new result of Bourgain, Demeter and Guth is not applicable.

Our first result gives a new \(k\)-th derivative estimate

**Theorem 1** Let \(k \geq 3\) be an integer, and suppose that \(f(x) : [0,N] \to \mathbb{R}\) has continuous derivatives of order up to \(k\) on \((0,N)\). Suppose further that

\[
0 < \lambda_k \leq f^{(k)}(x) \leq A\lambda_k, \ x \in (0,N).
\]

Then
\[
\sum_{n \leq N} e(f(n)) \ll_{A,k} N^{1+\varepsilon} (\lambda_k^{1/k(k-1)} + N^{-1/k(k-1)} + N^{-2/k(k-1)} \lambda_k^{-2/k^2(k-1)}).
\]

If one thinks of \(N\lambda_k^{1/k(k-1)}\) as being the leading term here, then one needs to compare the exponent \(1/k(k-1)\) with the corresponding exponent \(1/(2^k - 2)\) in (1). These agree for \(k = 3\), but for larger values of \(k\) the new exponent tends to zero far more slowly than the old one. It may perhaps be something of a surprise that an analysis via Vinogradov’s mean value integral reproduces the same term \(N\lambda_k^{1/6}\) as in the classical third-derivative estimate.

We should emphasize that the strength of Theorem 1 comes almost entirely from the new bound (4). One could have injected (4) into the method of Robert [6], to produce an estimate with the same terms \(\lambda_k^{1/k(k-1)} + N^{-1/k(k-1)}\) as in Theorem 1, but valid only for \(N \geq \lambda_k^{-(k-1)/(2k-3)}\). Our result, incorporating a slightly better way of using the Vinogradov mean value, gives the terms \(\lambda_k^{1/k(k-1)} + N^{-1/k(k-1)}\) in the substantially longer range \(N \geq \lambda_k^{-2/k}\). However for our application to Theorems 2–5 below, Robert’s range would have been very nearly sufficient.

The secondary terms in the bound given by Theorem 1 are somewhat awkward. The classical estimate (1) leads easily to an exponent pair,
\[
\left( \frac{1}{2^k - 2}, \frac{2^k - k - 1}{2^k - 2} \right)
\]
in which the term \(N^{1-2^{2-k}} \lambda_k^{-1/(2^k - 2)}\) has no effect. However the situation with Theorem 1 is more complicated. None the less we are able to produce a series of new exponent pairs.

Before stating the result we remind the reader of the necessary background. Let \(s\) and \(c\) be positive constants, and let \(\mathcal{F}(s,c)\) be the set of quadruples \((N, I, f, y)\) where \(y \geq N^s\) are positive real numbers, \(I\) is a subinterval of \((N, 2N]\), and \(f\) is an infinitely differentiable function on \(I\), with
\[
\left| f^{(n+1)}(x) - \frac{d^n}{dx^n} (y x^{-s}) \right| \leq c \left| \frac{d^n}{dx^n} (y x^{-s}) \right|
\]

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for $x \in I$, for all $n \geq 0$. We then say that $(p, q)$ is an exponent pair, if $p$ and $q$ lie in the range $0 \leq p \leq \frac{1}{2} \leq q \leq 1$, and for each $s$ there is a corresponding $c = c(p, q, s) > 0$ such that

$$
\sum_{n \in I} e(f(n)) \ll_{p, q, s} (yN^{-s})^p N^q,
$$

uniformly for all quadruples $(N, I, f, y) \in F(s, c)$.

We then have the following.

**Theorem 2** For any integer $k \geq 3$ and any real $\varepsilon > 0$ there is an exponent pair given by

$$
p = \frac{2}{(k - 1)^2(k + 2)}, \tag{5}
$$

and

$$
q = \frac{k^3 + k^2 - 5k + 2}{k(k - 1)(k + 2)} + \varepsilon = 1 - \frac{3k - 2}{k(k - 1)(k + 2)} + \varepsilon. \tag{6}
$$

In fact we are able to handle a much weaker condition on $f$. Let $A = (a_k)_{k=3}^{\infty}$ and $B = (b_k)_{k=3}^{\infty}$ be sequences of positive real numbers, and let $G(A, B)$ be the set of quadruples $(N, I, g, T)$ where $T \geq N$ are positive real numbers, $I$ is a subinterval of $(N, 2N]$, and $g$ is an infinitely differentiable function on $I$, with

$$
a_k TN^{-k} \leq \left| g^{(k)}(x) \right| \leq b_k TN^{-k}
$$

for $x \in I$, and for all $k \geq 3$. We then have the following.

**Theorem 3** For any integer $k \geq 3$ and any real $\varepsilon > 0$, let $p$ and $q$ be given by (5) and (6). Then

$$
\sum_{n \in I} e(g(n)) \ll_{k, \varepsilon, A, B} (TN^{-1})^p N^q,
$$

uniformly for $(N, I, g, T) \in G(A, B)$.

If $(N, I, g, y) \in F(s, \frac{1}{4})$, then $(N, I, g, yN^{1-s}) \in G(A, B)$ with

$$
a_k = \frac{3 \times 2^{1-2-k}}{4s(s + 1) \ldots (s + k - 2)} \quad \text{and} \quad b_k = \frac{5}{4s(s + 1) \ldots (s + k - 2)}.
$$

The sequences $A$ and $B$ depend only on $s$, and we immediately see that Theorem 2 follows from Theorem 3.

We next present a slightly weaker version of Theorem 3, which is somewhat more immediately intelligible. It will be convenient to write $T = N^\tau$.

**Theorem 4** Let sequences $A$ and $B$, and a real number $\varepsilon > 0$ be given, then

$$
\sum_{n \in I} e(g(n)) \ll_{\varepsilon, A, B} N^{1-49/(80\tau^2) + \varepsilon},
$$

uniformly for quadruples $(N, I, g, T) \in G(A, B)$ with $N \leq T^{1/2}$.
The constant $49/80$ arises from the use of an exponent pair 
$\left(\frac{1}{20}, \frac{33}{40}\right) = A^2 B A^2 B(0, 1)$
when $\tau = \frac{7}{2}$. One could improve the constant slightly by employing a better exponent pair. As will be clear from the proof, the constant $\frac{49}{80}$ may be replaced by $1 - \delta$ for any small $\delta > 0$, if we restrict to sufficiently large values $\tau \geq \tau(\delta)$.

As an example of Theorem 4, if $t \geq 2$ we find that

$$\sum_{n \in I} n^{-it} \ll \varepsilon \sqrt{\tau}^{-49/80 + \varepsilon},$$

for $\tau = (\log t)/\log N \geq 2$. This should be compared with (3). Using (7) we produce the following result.

**Theorem 5** Let $\kappa = \frac{8}{63} \sqrt{15} = 0.4918\ldots$. Then for any fixed $\varepsilon > 0$ we have

$$\zeta(\sigma + it) \ll \varepsilon t^{\kappa(1-\sigma)3/2 + \varepsilon}$$

uniformly for $t \geq 1$ and $\frac{1}{2} \leq \sigma \leq 1$. Moreover we have

$$\zeta(\sigma + it) \ll \varepsilon t^{\frac{1}{2}(1-\sigma)3/2 + \varepsilon}$$

uniformly for $t \geq 1$ and $0 \leq \sigma \leq 1$.

One sees from the proof that $\kappa$ may be reduced to $2/\sqrt{27} + \delta = 0.3849\ldots$ for any small $\delta > 0$, if we restrict $\sigma$ to a suitably small range $\sigma(\delta) \leq \sigma \leq 1$. The corresponding result in the work of Ford [2, Theorem 1] states that

$$|\zeta(\sigma + it)| \leq 76.2 t^{4.45(1-\sigma)3/2} (\log t)^{2/3}$$

for $t \geq 3$ and $\frac{1}{2} \leq \sigma \leq 1$. Thus we have reduced the constant 4.45 to 0.4918\ldots. Unfortunately our result yields no useful information when $\sigma$ tends to 1, which is a critical situation in many applications. Moreover we do not have the explicit order constant that Ford finds.

As Ford explains, there are a number of interesting corollaries, for which we merely have to replace the constant $B = 4.45$ by $B = 0.492$ in the arguments given in [2, Pages 566 and 567]. We can feed our bound into the zero-density theorem of Montgomery [5, Theorem 12.3] (with $1 - \alpha = 4.93 (1-\sigma)$ as used by Ford [2, Page 566]) to give the following.

**Corollary 1** We have

$$N(\sigma, T) \ll \varepsilon T^{6.42(1-\sigma)3/2 + \varepsilon}$$

for $0 \leq \sigma \leq 1$.

For moments of the Riemann Zeta-function we have:

**Corollary 2** For any positive integer $k$ one has

$$\int_0^T |\zeta(\sigma + it)|^{2k} dt \sim T \sum_{n=1}^{\infty} d_k(n)^2 n^{-2\sigma},$$

as $t \to \infty$, for any fixed $\sigma \geq 1 - 0.534 k^{-2/3}$.
For the generalized divisor problem we have:

**Corollary 3** For any positive integer $k$ the error term $\Delta_k(x)$ in the generalized divisor problem satisfies

$$\Delta_k(x) \ll_k x^{1-0.849k^{-2/3}}.$$ 

In Section 2 we will reduce the proof of Theorem 1 to a two-variable counting problem involving fractional parts of the derivatives $f^{(j)}(n)$. Section 3 shows how this counting problem is tackled, and finally Section 4 completes the proof of our theorems.

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## 2 Initial Steps

Our goal in the first stage of the proof is to estimate the sum

$$\Sigma = \sum_{n \leq N} e(f(n))$$

in terms of $J_{s,l}(P)$, together with, a counting function involving the fractional parts of numbers of the form $f^{(j)}(n)/j!$.

**Lemma 1** Let $k \geq 2$ be an integer, and suppose that $f(x) : [0,N] \to \mathbb{R}$ has continuous derivatives of order up to $k$ on $(0,N)$. Suppose further that $0 < \lambda_k \leq f^{(k)}(x) \leq A\lambda_k, \ x \in (0,N)$, and that $A\lambda_k \leq 1/4$. Then

$$\Sigma \ll H + k^2 N^{1-1/s} N^{1/2s} \left( H^{-2s+k(k-1)/2} J_{s,k-1}(H) \right)^{1/2s},$$

where $H = \lfloor (A\lambda_k)^{-1/k} \rfloor$ and

$$N = \# \left\{ m,n \leq N : \left| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right| \leq 2H^{-j} \text{ for } 1 \leq j \leq k-1 \right\}.$$

If $J_{s,k-1}(H) \ll_{c,k} H^{2s-k(k-1)/2+\epsilon}$ as in (4) the estimate in the lemma reduces to

$$\Sigma \ll_{c,k} H + N^{1-1/s+\epsilon} N^{1/2s}. \quad (10)$$

Here we would want to choose $s$ to be as small as possible, and since we are taking $l = k - 1$ this means that we will have $s = k(k-1)/2$.

The lemma is clearly trivial if $H \geq N$, and we may therefore suppose for the proof that $H \leq N$. For any positive integer $H \leq N$ we will have

$$H\Sigma = \sum_{h \leq H} \sum_{h < n \leq N-h} e(f(n+h)) = \sum_{h \leq H} \sum_{1 \leq n \leq N-H} e(f(n+h)) + O(H^2),$$

so that

$$\Sigma = H^{-1} \sum_{n \leq N-H} \sum_{h \leq H} e(f(n+h)) + O(H). \quad (11)$$
We proceed to approximate \( f(n+h) \) by the polynomial

\[
f_n(h) := f(n) + f'(n)h + \ldots + \frac{f^{(k-1)}(n)}{(k-1)!} h^{k-1}.
\]

To do this we set \( g_n(x) = f(n + x) - f_n(x) \) and use summation by parts to obtain the bound

\[
\sum_{h \leq H} e(f(n+h)) \ll |S_n(H)| + \int_0^H |S_n(x)g_n'(x)|dx,
\]

where we have written

\[
S_n(x) = \sum_{h \leq x} e(f_n(h))
\]

for convenience.

If \( 0 \leq x \leq H \) we may use Taylor’s Theorem with Lagrange’s form of the remainder to show that

\[
f'(n + x) = f'(n) + \frac{f^{(k)}(\xi)}{(k-1)!} x^{k-1}
\]

for some \( \xi \in (n, n+x) \subseteq (0, N) \). It follows that

\[
g_n'(x) \ll A\lambda_k H^{k-1}
\]

on \([0, H]\). With the choice \( H = [(A\lambda_k)^{-1/k}] \) we find that

\[
\sum_{h \leq H} e(f(n+h)) \ll |S_n(H)| + H^{-1} \int_0^H |S_n(x)|dx.
\]

The bound (11) now yields

\[
\Sigma \ll H + H^{-1} \sum_{n \leq N-H} |S_n(H)| + H^{-2} \int_0^H \left\{ \sum_{n \leq N-H} |S_n(x)| \right\} dx.
\]

It then follows that there is a positive integer \( H_0 \leq H \) such that

\[
\Sigma \ll H + H^{-1} \sum_{n \leq N-H} |S_n(H_0)|.
\]

Now suppose that \( \alpha \in [0, 1]^{k-1} \) and

\[
\|f^{(j)}(n)/j! - \alpha_j\| \leq H^{-j} \text{ for } 1 \leq j \leq k - 1
\]

where

\[
\|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n|
\]

as usual. We proceed to replace \( f_n(h) \) by

\[
f(h; \alpha) = \alpha_1 h + \ldots + \alpha_{k-1} h^{k-1}
\]
as follows. Firstly we remove the constant term \( f(n) \) from \( f_n(h) \). This has no effect on \( |S_n(H_0)| \). Next, we replace each coefficient \( f^{(j)}(n)/j! \) by \( c_j \), say, with \( f^{(j)}(n)/j! - c_j \in \mathbb{Z} \), so that \(|c_j - \alpha_j| \leq H^{-j}\), and denote the resulting polynomial by \( f_n'(h) \). If we write
\[
S_n'(H_0) = \sum_{h \leq H_0} e(f_n'(h))
\]
then clearly \( |S_n(H_0)| = |S_n'(H_0)| \). Moreover
\[
\frac{d}{dx} (f(x; \alpha) - f_n'(x)) \ll k^2 \max_{j \leq k-1} |c_j - \alpha_j| H^{j-1} \ll k^2 H^{-1}.
\]
It therefore follows on summing by parts that
\[
S_n'(H_0) \ll |S(H_0; \alpha)| + k^2 H^{-1} \int_0^{H_0} |S(x; \alpha)| \, dx,
\]
where we have set
\[
S(x; \alpha) = \sum_{h \leq x} e(f(h; \alpha)).
\]
We may therefore conclude that
\[
S_n'(H_0) \ll 2^{-k} H^{k(k-1)/2} \left\{ \int_\alpha |S(H_0; \alpha)| \, d\alpha + k^2 H^{-1} \int_0^{H_0} \int_\alpha |S(x; \alpha)| \, d\alpha \, dx \right\},
\]
where the integral over \( \alpha \) is for vectors in \([0,1]^{k-1}\) satisfying (13). For each \( \alpha \in [0,1]^{k-1} \) we now define
\[
\nu(\alpha) = \# \{ n \leq N - H : \| f^{(j)}(n)/j! - \alpha_j \| \leq H^{-j} \text{ for } 1 \leq j \leq k-1 \}.
\]
We then find that
\[
\sum_{n \leq N-H} |S_n(H_0)| \ll 2^{-k} H^{k(k-1)/2} \left\{ I(H_0) + k^2 H^{-1} \int_0^{H_0} I(x) \, dx \right\},
\]
with
\[
I(x) = \int_0^1 \ldots \int_0^1 |S(x; \alpha)| \nu(\alpha) \, d\alpha.
\]
We easily see that
\[
\int_0^1 \ldots \int_0^1 \nu(\alpha) \, d\alpha = 2^{k-1} H^{-k(k-1)/2}(N-H),
\]
and that
\[
\int_0^1 \ldots \int_0^1 \nu(\alpha)^2 \, d\alpha \leq 2^{k-1} H^{-k(k-1)/2} N,
\]
where \( N \) is defined in Lemma 1. Moreover
\[
\int_0^1 \ldots \int_0^1 |S(x; \alpha)|^2 \, d\alpha = J_{s,k-1}(x)
\]
in the notation of (2). Since \( J_{s,k-1}(P) \) is non-decreasing in \( P \) this last integral may be bounded by \( J_{s,k-1}(H) \).
Hence, by Hölder’s inequality, for any positive integer $s$ we have
\[ I(x) \ll 2^k H^{-(k-1)/2} N^{1-1/s} N^{1/2s} \left\{ H^{k(k-1)/2} J_{s,k-1}(H) \right\}^{1/2s}. \]
Thus (14) yields
\[ \sum_{n \leq N-H} |S_n(H_0)| \ll k^2 N^{1-1/s} N^{1/2s} \left\{ H^{k(k-1)/2} J_{s,k-1}(H) \right\}^{1/2s} \]
and (12) gives us
\[ \Sigma \ll H + k^2 N^{1-1/s} N^{1/2s} \left\{ H^{-2s+k(k-1)/2} J_{s,k-1}(H) \right\}^{1/2s} \]
as required.

3 The counting function $N$

Naturally our next task is to bound $N$. The original approach taken by Vinogradov, as described in Titchmarsh [10, Chapter 6], merely used an $L^\infty$ bound for $\nu(\alpha)$. One discards all the information on $f^{(j)}(n)/j!$ for $j \leq k-2$ and uses only the case $j = k-1$. One then employs a standard procedure given by the following trivial variant of [10, Lemma 6.11], for example.

Lemma 2 Let $N$ be a positive integer, and suppose that $g(x) : [0, N] \to \mathbb{R}$ has a continuous derivative on $(0, N)$. Suppose further that
\[ 0 < \mu \leq g'(x) \leq A_0 \mu, \quad x \in (0, N). \]
Then
\[ \# \{ n \leq N : \| g(n) \| \leq \theta \} \ll (1 + A_0 \mu N)(1 + \mu^{-1}\theta). \]
We fix $m$ and take
\[ g(x) = \frac{f^{(k-1)}(x) - f^{(k-1)}(m)}{(k-1)!} \]
and $\mu = \lambda_k/(k-1)!$, $A_0 = A$. This leads to a bound
\[ N \ll (k-1)! N(1 + AN\lambda_k)(1 + H^{-1-k}\lambda_k^{-1}). \]
Under the assumption $A\lambda_k \leq \frac{1}{4}$ in Lemma 1 we have
\[ H^{-1-k}\lambda_k^{-1} \asymp (A\lambda_k)^{1-1/k}\lambda_k^{-1} = A(A\lambda_k)^{-1/k} \geq A \geq 1, \]
whence our bound produces
\[ N \ll A^2 (k-1)! N\lambda_k^{-1/k}(1 + N\lambda_k). \quad (15) \]
If one inserts this into (10) with $s = k(k-1)/2$ one gets an estimate
\[ \Sigma \ll_{\varepsilon,k} \left( A\lambda_k \right)^{-1/k} + N^{1-1/s+\varepsilon} \left\{ A^2 N\lambda_k^{-1/k}(1 + N\lambda_k) \right\}^{1/2s} \ll_{\varepsilon,k} AN\left\{ \lambda_k^{-1/k} + N^{1-1/k(k-1)}\lambda_k^{-1/k^2(k-1)} + N\lambda_k^{1/k^2} \right\}. \]
In fact the first term can be dropped, giving
\[ \sum \ll_{\varepsilon,k} A N^{\varepsilon} \{ N \lambda_k^{1/k^2} + N^{1-1/k(k-1)} \lambda_k^{-1/k^2(k-1)} \}. \] (16)

To see this we note that we have
\[ \sum \ll N^{1-1/k(k-1)} \lambda_k^{-1/k^2(k-1)} \]
trivially unless
\[ N^{1-1/k(k-1)} \lambda_k^{-1/k^2(k-1)} \leq N. \]
In this latter case however one sees that
\[ \lambda_k^{-1/k} \leq N^{1-1/k(k-1)} \lambda_k^{-1/k^2(k-1)}. \]

We may therefore regard (16) as being the result that Vinogradov’s method achieves, given the results of Wooley [13] and Bourgain, Demeter and Guth [1]. It is already a remarkable improvement on (1), replacing the critical exponent \( 1/(2k-2) \) by \( 1/k \). Thus, in appropriate circumstances, we get an improvement as soon as \( k \geq 5 \). Our goal in this section is to make the following small further sharpening in the estimation of \( N \).

**Lemma 3** When \( k \geq 3 \) we have
\[ N \ll ((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N. \]

Apart from the term \( \lambda_k^{-2/k} \), which is insignificant in applications, this represents an improvement of (15) by a factor \( \ll_A \lambda_k^{1/k} \).

On the one hand our proof will use the fact that \( N \) is a counting function of two variables \( m \) and \( n \). On the other we shall use information about both \( f^{(k-1)} \) and \( f^{(k-2)} \). The reader may find it slightly surprising in the light of this that our bound depends on \( \lambda_k \) only, and not on estimates for other derivatives \( f^{(j)} \). The introduction of \( N \), and our procedure for estimating it, are the only really new aspects to this paper.

We begin our analysis by assuming that \( k \geq 3 \) and noting that \( N \) is at most
\[ N_1 = \# \left\{ m,n \leq N : \left| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right| \leq 2H^{-j} \text{ for } j = k-2,k-1 \right\}. \]

We proceed to show that it suffices to consider pairs \( m,n \) of integers that are relatively close. It will be convenient to write \( B = 4H^{2-k} \) and \( C = 4H^{1-k} \) and to set
\[ g_1(x) = \frac{f^{(k-2)}(x)}{(k-2)!}, \quad g_2(x) = \frac{f^{(k-1)}(x)}{(k-1)!}. \]

We also define the doubly-periodic function
\[ \phi(x,y) = \max \{ 1 - B^{-1}\|x\|, 0 \} \max \{ 1 - C^{-1}\|y\|, 0 \}, \]
so that
\[ N_1 \ll \sum_{m,n \leq N} \phi(g_1(m) - g_1(n), g_2(m) - g_2(n)). \]
The function \( \phi(x, y) \) has an absolutely convergent Fourier series

\[
\phi(x, y) = \sum_{r, s \in \mathbb{Z}} c_{r,s} e(rx + sy)
\]

with non-negative coefficients

\[
c_{r,s} = BC \left( \frac{\sin(\pi rB) \sin(\pi sC)}{\pi^2 rsBC} \right)^2.
\]

Thus

\[
N_1 \ll \sum_{r, s \in \mathbb{Z}} c_{r,s} \left| e(r(g_1(m) - g_1(n)) + s(g_2(m) - g_2(n))) \right|^2
\]

\[
= \sum_{r, s \in \mathbb{Z}} c_{r,s} \left| \sum_{m, n \leq N} e(rg_1(m) + sg_2(n)) \right|^2.
\]

Let \( K \) be a positive integer parameter, to be chosen later. We proceed to partition the range \((0, N]\) into \( K \) intervals \( I_i = (a_i, b_i] \) for \( i \leq K \), having integer endpoints, and length \( b_i - a_i \leq 1 + N/K \). An application of Cauchy's inequality then yields

\[
N_1 \ll K \sum_{i \leq K} \sum_{r, s \in \mathbb{Z}} c_{r,s} \left| \sum_{m, n \in I_i} e(rg_1(n) + sg_2(n)) \right|^2
\]

\[
= K \sum_{i \leq K} \sum_{r, s \in \mathbb{Z}} c_{r,s} \left| \sum_{m, n \in I_i} e(rg_1(m) + sg_2(n)) \right|^2
\]

\[
= K \sum_{i \leq K} \sum_{m, n \in I_i} \phi(g_1(m) - g_1(n), g_2(m) - g_2(n))
\]

\[
\leq K \sum_{|m-n| \leq N} \phi(g_1(m) - g_1(n), g_2(m) - g_2(n)).
\]

We may therefore conclude that \( N_1 \ll KN_2 \), where \( N_2 \) counts pairs of integers \( m, n \leq N \) with \( |m - n| \leq 1 + N/K \) for which

\[
\left| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right| \leq 4H^{-j} \text{ for } j = k - 2, k - 1.
\]

If \( |m - n| \leq 1 + N/K \) we will have

\[
\left| \frac{f^{(k-1)}(m)}{(k-1)!} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| \leq \frac{|m - n|}{(k-1)!} \sup |f^{(k)}| \leq A\lambda_k(1 + N/K),
\]

by the mean-value theorem. We will choose

\[
K = 1 + [4A\lambda_k N],
\]

so that

\[
\left| \frac{f^{(k-1)}(m)}{(k-1)!} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| \leq \frac{1}{2}.
\]
in view of our assumption that $A\lambda_k \leq \frac{1}{4}$. Thus if

$$\left| \frac{f^{(k-1)}(m)}{(k-1)!} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| \leq 4H^{1-k}$$

we must have

$$\left| \frac{f^{(k-1)}(m)}{(k-1)!} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| \leq 4H^{1-k}.$$  

However the mean-value theorem also tells us that

$$\left| \frac{f^{(k-1)}(m)}{(k-1)!} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| \geq |m-n| \inf |f^{(k)}| \geq \lambda_k \frac{|m-n|}{(k-1)!}.$$  

We therefore conclude that

$$|m-n| \leq \frac{4(k-1)!}{\lambda_k H^{k-1}}$$

for any pair $m, n$ counted by $N_2$.

There are $N$ pairs $m = n$ counted by $N_2$. We consider the remaining pairs with $m > n$, the alternative case producing the same estimates by symmetry. Then $m = n + d$ with $1 \leq d \leq D$, where

$$D = \min \left( N, \left[ \frac{4(k-1)!}{\lambda_k H^{k-1}} \right] \right).$$

For each available value of $d$ we estimate the number of corresponding integers $n$ via Lemma 2, taking

$$g(x) = \frac{f^{(k-2)}(x+d) - f^{(k-2)}(x)}{(k-2)!}.$$  

Then

$$g'(x) = \frac{f^{(k-1)}(x+d) - f^{(k-1)}(x)}{(k-2)!},$$

so that

$$d \frac{\lambda_k}{(k-2)!} \leq d \inf |f^{(k)}| \frac{1}{(k-2)!} \leq g'(x) \leq d \sup |f^{(k)}| \frac{1}{(k-2)!} \leq d \frac{A\lambda_k}{(k-2)!},$$

by the mean-value theorem. We therefore apply the lemma with $\mu = \lambda_k d/(k-2)!$ and $A_0 = A$. This shows that each $d \geq 1$ contributes

$$\ll (k-2)!(1 + AN\lambda_k d)(1 + H^{2-k}\lambda_k^{-1}d^{-1})$$

$$\ll (k-2)!(1 + AN\lambda_k D)(D + H^{2-k}\lambda_k^{-1})d^{-1}$$

$$\ll \left( (k-1)! \right)^3 A(1 + NH^{1-k})H^{2-k}\lambda_k^{-1}d^{-1}$$

$$\ll \left( (k-1)! \right)^3 (1 + N\lambda_k^{-1/k})\lambda_k^{-2/k}d^{-1}.$$  

Summing for $d \leq D$ we therefore find that

$$N_2 \ll N + \left( (k-1)! A \right)^3 (1 + N\lambda_k^{-1/k})\lambda_k^{-2/k} \log D.$$  

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Since $k \geq 3$, $\lambda_k \leq 1$ and $D \leq N$ this simplifies to give

$$N_2 \ll ((k - 1)!A)^3 (N + \lambda_k^{-2/k}) \log N,$$

whence

$$N \leq N_1 \ll KN_2 \ll (1 + A\lambda_k N)((k - 1)!A)^3 (N + \lambda_k^{-2/k}) \log N \ll ((k - 1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k} + N\lambda_k^{1-2/k}) \log N.$$

Since $k \geq 3$ and $\lambda_k \leq 1$ we have $N\lambda_k^{1-2/k} \leq N$, and Lemma 3 follows.

4 Proof of the Theorems

If we insert Lemma 3 into Lemma 1, and use the bound (4) with the choices $l = k - 1, s = k(k - 1)/2$, we see that

$$\Sigma \ll_{A,k,\varepsilon} N^\varepsilon (\lambda_k^{-1/k} + N^{1-1/k(k-1)} + N\lambda_k^{1/k(k-1)} + N^{1-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}).$$

The term $\lambda_k^{-1/k}$ may be omitted, since the resulting bound

$$\Sigma \ll_{A,k,\varepsilon} N^\varepsilon (N^{1-1/k(k-1)} + N\lambda_k^{1/k(k-1)} + N^{1-2/k(k-1)}\lambda_k^{-2/k^2(k-1)})$$

holds trivially when $N \leq N^{1-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}$, while

$$\lambda_k^{-1/k} \leq N^{1-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}$$

when $N \geq N^{1-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}$. This suffices for Theorem 1.

We turn next to Theorem 3. Suppose that $(\mathcal{N}, I, g, T) \in \mathcal{G}(A, B)$, and let $I$ have end points $N_0$ and $N_0 + N_1$, so that $N_1 \leq N$. We apply Theorem 1 to the function $f(x) = g(N_0 + x)$, taking $\lambda_k = a_kTN^{-k}$ and $A = b_k/a_k$. (Since $f^{(k)}$ is differentiable it is continuous, and hence it cannot change sign if $|f^{(k)}(x)| \geq a_kTN^{-k} > 0$. Taking complex conjugates of our sum if necessary we may therefore assume that $f^{(k)}(x)$ is positive on $I$.) It follows that if $k \geq 3$ then

$$\sum_{n \in I} e(g(n)) \ll_{\varepsilon, A, B, C} N^{1+\varepsilon} (\lambda_k^{-1/k} + N^{-1/k(k-1)} + N^{-2/k(k-1)}\lambda_k^{-2/k^2(k-1)})$$

$$\ll_{\varepsilon, A, B, C} N^{1+\varepsilon} (N^{-1/k(k-1)}T^{1/k(k-1)} + N^{-1/k(k-1)} + T^{-2/k^2(k-1)}).$$

We use the above bound for

$$\frac{(k - 1)^2 + 1}{k} \leq \tau < \frac{k^2 + 1}{k + 1}$$

where we define $\tau$ by $T = N^\tau$. For this range of $\tau$ we find that

$$\max \left( \frac{\tau - k}{k(k - 1)}, \frac{-1}{k(k - 1)}, \frac{-2\tau}{k^2(k - 1)} \right)$$

$$= \begin{cases} 
-1/k(k - 1), & (k - 1)^2 + 1/k \leq \tau \leq k - 1, \\
(t - k)/k(k - 1), & k - 1 \leq \tau < (k^2 + 1)/(k + 1), 
\end{cases}$$

$$\leq A_k\tau + B_k,$$
where the coefficients $A_k$ and $B_k$ are chosen so that

$$A_k \frac{(k-1)^2 + 1}{k} + B_k = \frac{-1}{k(k-1)}$$

and

$$A_k \frac{k^2 + 1}{k+1} + B_k = \frac{(k^2 + 1)/(k+1) - k}{k(k-1)} = \frac{-1}{k(k+1)}.$$ 

One then calculates that

$$A_k = \frac{2}{(k-1)^2(k+2)} \quad \text{and} \quad B_k = -\frac{3k^2 - 3k + 2}{k(k-1)^2(k+2)}.$$ 

If we now define $\phi(\tau) : [2, \infty) \to \mathbb{R}$ by taking $\phi(\tau) = A_k \tau + B_k$ on

$$\left[\frac{(k-1)^2 + 1}{k}, \frac{k^2 + 1}{k+1}\right]$$

for each integer $k \geq 3$, we conclude that

$$\sum_{n \in I} e(g(n)) \ll_{\epsilon, \tau_0, A, B} N^{1+\phi(\tau) + \epsilon},$$

uniformly for $2 \leq \tau \leq \tau_0$. The function $\phi$ is continuous, and since the coefficients $A_k$ are monotonic decreasing $\phi$ is also convex. It follows that $\phi(\tau) \leq A_k \tau + B_k$ for any $\tau \in [2, \infty)$ and any $k \geq 3$. Thus

$$\sum_{n \in I} e(g(n)) \ll_{\epsilon, \tau_0, A, B} N^{1+B_k+\epsilon} T^{A_k} = (TN^{-1})^p N^q,$$

with $p, q$ given by (5) and (6). As before, this is uniform in any finite range $2 \leq \tau \leq \tau_0$. However if we set $\tau_0 = 1 + (1-q)/p$ then $\tau_0$ will depend on $\epsilon$ and $k$ alone. Moreover, if $\tau \geq \tau_0$ then we trivially have

$$\sum_{n \in I} e(g(n)) \ll N \leq (TN^{-1})^p N^q.$$ 

Finally, if $\tau \leq 2$ we use the well known exponent pair $(\frac{1}{6}, \frac{2}{3})$ to show that

$$\sum_{n \in I} e(g(n)) \ll T^{1/6} N^{1/2}.$$ 

When $k \geq 3$ one easily verifies that $q \geq p + 1/2$ and $p + q \geq 5/6$ for the values (5) and (6), whence $T^{1/6} N^{1/2} \leq T^p N^{q-p}$ for $N \geq T^{1/2}$. It then follows that

$$\sum_{n \in I} e(g(n)) \ll T^{1/6} N^{1/2} \leq T^p N^{q-p}$$

for the remaining range $1 \leq \tau \leq 2$. This completes the proof of Theorem 3.

We move now to the proof of Theorem 4. Let $\tau_0 = \sqrt{49/80 \epsilon^2}$. Then if $\tau \geq \tau_0$ we will trivially have

$$\sum_{n \in I} e(g(n)) \ll N \leq N^{1-49/80 \tau^2 + \epsilon}.$$
When $\tau \leq \tau_0$ we begin by handling the range $\frac{13}{3} \leq \tau \leq \tau_0$, for which we claim that $\phi(\tau) \leq -\frac{49}{80}\tau^2$. This will clearly suffice, in view of the estimate (18). Since $\phi(\tau)$ is piecewise linear, while the function $-\frac{49}{80}\tau^2$ is convex, it suffices to verify that $\phi(\tau) \leq -\frac{49}{80}\tau^2$ at each of the points $\tau = (k^2 + 1)/(k + 1)$, for $k \geq 5$. This condition is equivalent to

$$\frac{(k^2 + 1)^2}{k(k + 1)^3} \geq \frac{49}{80}.$$ 

However the fraction on the right is increasing for $k \geq 5$, and takes the value $\frac{169}{270} > \frac{49}{80}$ at $k = 5$.

When $\frac{7}{2} \leq \tau \leq \frac{13}{3}$ we will use the bounds

$$\sum_{n \in I} e(g(n)) \ll \varepsilon N^{1-1/20+\varepsilon}, \quad (\frac{7}{2} \leq \tau \leq 4)$$

and

$$\sum_{n \in I} e(g(n)) \ll \varepsilon N^{1-(5-\tau)/20+\varepsilon}, \quad (4 \leq \tau \leq \frac{13}{3})$$

which come from the case $k = 5$ of (17). Note that the first of these is valid in the longer range $\frac{17}{5} \leq \tau \leq 4$, but we shall only use it when $\frac{7}{2} \leq \tau \leq 4$. We therefore need to verify that $\frac{2\tau - 9}{40} \leq -\frac{49}{80}\tau^2$ for $\frac{7}{2} \leq \tau \leq 4$ and that $-(5 - \tau)/20 \leq -\frac{49}{80}\tau^2$ for $4 \leq \tau \leq \frac{13}{3}$. This is routine, but we observe that we have equality at $\tau = \frac{7}{2}$.

We next consider the case in which $\frac{59}{22} \leq \tau \leq \frac{7}{2}$, for which we use the bound

$$\sum_{n \in I} e(g(n)) \ll (T/N)^{1/20} N^{33/40} = N^{1+(2\tau - 9)/40}$$

corresponding to the exponent pair $(\frac{1}{20}, \frac{33}{40})$. (This pair is $A^2BA^2B(0, 1)$ in the usual notation, see Titchmarsh [10, §5.20], for example.) Again, it is routine to check that

$$\frac{2\tau - 9}{40} \leq -\frac{49}{80}\tau^2, \quad (\frac{59}{22} \leq \tau \leq \frac{7}{2}).$$

Finally we examine the range $2 \leq \tau \leq \frac{59}{22}$, and here we use the bound

$$\sum_{n \in I} e(g(n)) \ll (T/N)^{1/9} N^{13/18} = N^{1+(2\tau - 7)/18}$$

corresponding to the exponent pair $(\frac{1}{9}, \frac{13}{18})$. (This pair is $ABA^2B(0, 1)$ in the usual notation, see Titchmarsh [10, §5.20], for example.) Another routine check shows that

$$\frac{2\tau - 7}{18} \leq -\frac{49}{80}\tau^2, \quad (2 \leq \tau \leq \frac{59}{22}),$$

thereby completing the proof of Theorem 4.

We turn now to Theorem 5. If $\tau \geq 2$ we may use (7) along with a partial summation to obtain

$$\sum_{n \in I} n^{-\sigma-it} \ll \varepsilon N^{1-49/80\tau^2-\sigma+\varepsilon} \leq (1-\sigma) \tau^{-1} \frac{49}{80}\tau^{-3+\varepsilon/2}.$$
for any $\sigma \in [\frac{1}{2}, 1]$, and for any interval $J \subseteq (N, 2N]$ the exponent of $t$ is maximal at

$$
\tau = \sqrt{\frac{147}{80(1 - \sigma)}},
$$

whence

$$
\sum_{n \in J} n^{-\sigma - it} \ll \varepsilon t^{\kappa(1-\alpha)^{3/2} + \varepsilon/4}.
$$

Using a dyadic subdivision of $(0, N]$ we therefore have

$$
\sum_{n \leq N} n^{-\sigma - it} \ll \varepsilon t^{\kappa(1-\alpha)^{3/2} + 3\varepsilon/4}
$$

for any $N \leq t^{1/2}$. A further summation by parts then shows that

$$
\sum_{n \leq M} n^{-1+\sigma - it} \ll \varepsilon M^{2\sigma - 1} t^{\kappa(1-\alpha)^{3/2} + \varepsilon} \ll \varepsilon t^{\sigma - \frac{1}{2} + \kappa(1-\alpha)^{3/2} + \varepsilon}
$$

for any $M \leq t^{1/2}$. The required bound (8) then follows from the approximate functional equation for $\zeta(s)$.

The bound (9) follows from (8) when $\frac{1}{2} \leq \sigma \leq 1$, since $\kappa < \frac{1}{2}$. For the remaining range we use the functional equation, which shows that

$$
\zeta(\sigma + it) \ll t^{\frac{1}{2} - \sigma}|\zeta(1-\sigma + it)| \ll \varepsilon t^{\frac{1}{2} - \sigma + \frac{1}{2} \sigma^{3/2}} + \varepsilon.
$$

However one can readily verify that

$$
\frac{1}{2} - \sigma + \frac{\sigma^{3/2}}{2} \leq \frac{(1 - \sigma)^{3/2}}{2}
$$

for $0 \leq \sigma \leq \frac{1}{2}$, which completes the proof of Theorem 5.

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