Analytic test configurations and geodesic rays

Julius Ross and David Witt Nyström

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Abstract
Starting with the data of a curve of singularity types, we use the Legendre transform to construct weak geodesic rays in the space of locally bounded metrics on an ample line bundle $L$ over a compact manifold. Using this we associate weak geodesics to suitable filtrations of the algebra of sections of $L$. In particular this works for the natural filtration coming from an algebraic test configuration, and we show how in this case we recover the weak geodesic ray of Phong-Sturm.

1 Introduction

Let $\mathcal{H}(L)$ be the space of smooth strictly positive hermitian metrics on an ample line bundle $L$ over a compact manifold $X$. Then, by the work of Mabuchi, Semmes and Donaldson (see [25], [34], [19]), $\mathcal{H}(L)$ has the structure of an infinite dimensional symmetric space with a canonical Riemannian metric. Thus a natural way to study this space is through its geodesics, an approach that has been taken up by a number of authors (e.g. Chen-Tian, Donaldson, Phong-Sturm, Mabuchi and Semmes among others).

In this paper we give a general method for constructing weak geodesics in the space of locally bounded positive metrics on $L$. The initial data consists of a fixed smooth positive metric $\phi$ and a curve of singular positive metrics $\psi_\lambda$ on $L$ for $\lambda \in \mathbb{R}$ that is concave in $\lambda$. We are really only interested in the singularity type of $\psi_\lambda$, so we consider the equivalence class of $\psi_\lambda$ under the relation $\psi_\lambda \sim \psi'_\lambda$ if $\psi_\lambda - \psi'_\lambda$ is bounded globally on $X$. We define the maximal envelope of this data to be

$$\phi_\lambda := \sup\{\psi : \psi \leq \phi \text{ and } \psi \sim \psi_\lambda\}^*$$

where the supremum is over positive metrics $\psi$ with the same singularity type as $\psi_\lambda$, and the star denotes the operation of taking the upper-semicontinuous regularization.

Theorem 1.1. Suppose $\psi_\lambda$ is a test curve (as defined in (5.1)) and $\phi \in \mathcal{H}(L)$, and consider the Legendre transform of its maximal envelope $\phi_\lambda$ given by

$$\hat{\phi}_t := \sup_{\lambda} \{\phi_\lambda + \lambda t\}^* \text{ for } t \in [0, \infty).$$

Then $\hat{\phi}_t$ is a weak geodesic ray in the space of locally bounded positive metrics on $L$ that emanates from $\phi$.
1 INTRODUCTION

We recall what is meant by a weak geodesic. Let $A := \{e^a < |z| < e^b\}$ be an annulus and let $\pi$ be the projection $X \times A \to X$. Given a curve $\phi_t : \mathcal{A} \to \mathcal{X}$ of positive metrics, consider the metric $\Phi(x, w) := \phi_{\log|w|}(x)$ on $\pi^*(L)$. Then a simple calculation reveals that if the $\phi_t$ are smooth then the geodesic equation for $\phi_t$ is equivalent to the degenerate homogeneous Monge-Ampère equation

$$\Omega^{n+1} = 0 \quad \text{on} \quad X \times A \quad \text{for all} \quad \lambda,$$

where $\Omega = \pi^*\omega_0 + dd^c\Phi$ and $\omega_0$ is the curvature of the initial metric. A curve of locally bounded positive metrics is said to be a weak geodesic if it solves (1) in sense of currents.

The first step in our approach to Theorem 1.1 is showing that the Monge-Ampère measure of the maximal envelope $\phi_\lambda$ has the crucial property

$$MA(\phi_\lambda) = 1_{\{\phi_\lambda = \phi\}} MA(\phi_\lambda) \quad \text{for all} \quad \lambda,$$

where $1_S$ denotes the characteristic function of a set $S$. We say that a positive metric $\phi_\lambda$ bounded by $\phi$ and having property (2) is maximal with respect to $\phi$ (see Definition 4.5), and a test curve $\phi_\lambda$ where $\phi_\lambda$ is maximal with respect to $\phi$ for all $\lambda$ is referred to as a maximal test curve. We show that the Aubin-Mabuchi energy of the Legendre transform of a maximal test curve is linear in $t$, which is well known to be equivalent to (1) once it is established the curve is a subgeodesic.

A now standard conjecture, originally due to Yau, states that for a smooth projective manifold it should be possible to detect the existence of a constant scalar curvature Kähler metric algebraically. Through ideas developed by many authors (e.g. Chen, Donaldson, Mabuchi, Tian) a general picture has emerged in which such metrics appear as critical points of certain energy functionals that are convex along smooth geodesics. The input from algebraic geometry arises through Donaldson’s notion of a test configuration which, roughly speaking, is a one-parameter algebraic degeneration of our original projective manifold.

In a series of papers Phong-Sturm show how one can naturally associate a weak geodesic ray to a test configuration [26, 27, 29]. (See also [1] by Arezzo-Tian, [12, 13] by Chen, [15] by Chen-Tang and [14] by Chen-Sun for other constructions of geodesic rays related to test configurations.) We show how the geodesic constructed above can be viewed as a generalization of the geodesic of Phong-Sturm.

Generalizing slightly, suppose that $\mathcal{F}_{k,\lambda}$, for $k \in \mathbb{N}, \lambda \in \mathbb{R}$ is a multiplicative filtration of the graded algebra $\oplus_k H^0(X, kL)$. Using our underlying smooth positive metric $\phi$ we have an $L^2$-inner product on each $H^0(X, kL)$, and thus can consider the associated Bergman metric

$$\phi_{k,\lambda} = \frac{1}{k} \ln \sum |s_\alpha|^2$$

where $\{s_\alpha\}$ is an orthonormal basis for $\mathcal{F}_{k,\lambda k} \subset H^0(X, kL)$. 
Theorem 1.2. Suppose that $F_{k,\lambda}$ is left continuous and decreasing in $\lambda$ and bounded (see (7.2)). Then there is a well-defined limit

$$
\phi^F = (\lim_{k \to \infty} \phi_{k,\lambda})^*.
$$

Furthermore this limit is maximal except possibly for one critical value of $\lambda$, and its Legendre transform is a weak geodesic ray.

In particular this applies to a natural filtration associated to a test configuration, and thus we have associated a weak geodesic to any such test configuration. We prove that, in this case, we recover the construction of Phong-Sturm, thus reproving the main result of [27]. Hence one interpretation of Theorem 1.1 is that in the problem of finding weak geodesics, the algebraic data of a test configuration can be replaced with a curve of singularity types which we thus refer to as an analytic test configuration.

It should be stressed that in the problem of finding constant scalar curvature metrics it is important to have control of the regularity of geodesics under consideration. By using approximations to known regularity results of solutions of Monge-Ampère equations, Phong-Sturm prove that their weak geodesic is in fact $C^{1,\alpha}$ for $0 < \alpha < 1$ (see [29]). It is interesting to ask whether such regularity holds more generally, which is a topic we hope to address in a future work. The Legendre transform approach also has applications to the Cauchy problem for the homogeneous Monge-Ampère equation, see [33].

Organization: We start in Section 2 with some motivation from convex analysis, and Section 3 contains preliminary material on the space of singular metrics, the Monge-Ampère measure and the Aubin-Mabuchi functional. The real work starts in Section 4 where we consider the maximal envelopes associated to a given singularity type. Along the way we prove a generalization of a theorem of Bedford-Taylor which says that such envelopes are maximal (Theorem 4.9). This is then extended to the case of a test curve of singularities, and in Section 6 we discuss the Legendre transform and prove Theorem 1.1.

Following these analytic results, we move on to the algebraic picture. In Section 7 we associate a test curve to a suitable filtration of the coordinate ring of $(X, L)$, and prove Theorem 1.2. We then recall how such filtrations arise from test configurations, and in Section 9 show how this agrees with the construction of Phong and Sturm.

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2 Convex motivation

This section contains some motivation from convex analysis in the study of the homogeneous Monge-Ampère equation. Much of this material is standard; our main
references are the two papers [31] and [32] by Rubinstein-Zelditch. Although this is logically independent of the rest of the paper, the techniques used are very similar. We shall presently see how solutions to this equation can be found using the Legendre transform in two different, but ultimately equivalent, ways.

Let \( \text{Conv}(\mathbb{R}^n) \) denote the space of convex functions on \( \mathbb{R}^n \). We take the convention that the function identically equal to \( -\infty \) is in \( \text{Conv}(\mathbb{R}^n) \).

**Definition 2.1.** Let \( \phi \) be a \( C^2 \) convex function on an open subset of \( \mathbb{R}^n \). The (real) Monge-Ampère measure of \( \phi \), denoted by \( MA(\phi) \), is the Borel measure defined as

\[
MA(\phi) := d \frac{\partial \phi}{\partial x_1} \wedge \ldots \wedge d \frac{\partial \phi}{\partial x_{n+1}}.
\]

Furthermore, \( MA \) has an unique extension to a continuous operator on the cone of (finite-valued) convex functions (see [32] for references). If \( \phi \) is \( C^2 \) then

\[
MA(\phi) = \det(\nabla^2 \phi) dx = (\nabla \phi)^* dx,
\]

i.e. the Monge-Ampère measure is the pullback of the Lebesgue measure under the gradient map.

If \( \phi \in \text{Conv}(\mathbb{R}^n) \), let \( \Delta_\phi \) denote the set of subgradients of \( \phi \), i.e. the set of points \( y \) in \( \mathbb{R}^n \) such that the convex function \( \phi - x \cdot y \) is bounded from below. So, if \( \phi \) is differentiable, then \( \Delta_\phi \) is simply the image of \( \nabla \phi \). One can easily check that \( \Delta_\phi \) is convex, that if \( r > 0 \) then \( \Delta_{r\phi} = \Delta_\phi \) and \( \Delta_{\phi + \psi} \subseteq \Delta_\phi + \Delta_\psi \).

When \( \phi \) is \( C^2 \) it follows from equation (3) that the total mass of the Monge-Ampère measure \( MA(\phi) \) equals the Lebesgue volume of the set of gradients \( \Delta_\phi \). An important fact [32] is that this is true for all convex function \( R^n \) with linear growth, i.e.

\[
\int_{\mathbb{R}^n} MA(\phi) = \text{vol}(\Delta_\phi).
\]

We say two convex functions \( \phi \) and \( \psi \) are equivalent if \( |\phi - \psi| \) is bounded, and denote this by \( \phi \sim \psi \). Since for two equivalent convex functions \( \phi \) and \( \psi \) with linear growth we clearly have that

\[
\Delta_\phi = \Delta_\psi,
\]

it follows from (4) that

\[
\int_{\mathbb{R}^n} MA(\phi) = \int_{\mathbb{R}^n} MA(\psi) \quad \text{whenever } \phi \sim \psi.
\]

**Definition 2.2.** Let \( \phi \in \text{Conv}(\mathbb{R}^n) \) and let \( \dot{\phi} \) be a bounded continuous function on \( \mathbb{R}^n \). A curve \( \phi_t \) in \( \text{Conv}(\mathbb{R}^n) \), \( t \in [a, b] \), is said to solve the Cauchy problem for the homogeneous real Monge-Ampère equation, abbreviated as HRMA, with initial data \( (\phi, \dot{\phi}) \), if the function \( \Phi(x, t) := \phi_t(x) \) is convex on \( \mathbb{R}^n \times [a, b] \), and satisfies the equation

\[
MA(\Phi) = 0 \quad \text{on the strip } \mathbb{R}^n \times (a, b),
\]
2 CONVEX MOTIVATION

Let \( \phi_0 \) and \( \phi_1 \) be two equivalent convex functions with linear growth, and \( \phi_t \) be the affine curve between them. The energy of \( \phi_1 \) relative to \( \phi_0 \), denoted by \( E(\phi_1, \phi_0) \) is defined as

\[
E(\phi_1, \phi_0) := \int_0^1 \left( \int_{\mathbb{R}^n} (\phi_1 - \phi_0) MA(\phi_t) \right) dt.
\]

We observe that by the linear growth assumption it follows that the relative energy \( E(\phi_1, \phi_0) \) is finite. This energy has a cocycle property, namely if \( \phi_0, \phi_1 \) and \( \phi_2 \) are equivalent with finite energy then

\[
E(\phi_2, \phi_0) = E(\phi_2, \phi_1) + E(\phi_1, \phi_0),
\]

which is easily seen to be equivalent to the fact that

\[
\frac{\partial}{\partial t} E(\phi_t, \phi) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t).
\]

The energy along a smooth curve of convex functions with linear growth \( \phi_t \) is related to the Monge-Ampère measure of \( \Phi(x, t) := \phi_t(x) \) by the identity

\[
\int_{\mathbb{R}^n \times [a,b]} MA(\Phi) = \frac{\partial}{\partial t}_{|t=b} E(\phi_t, \phi_a) - \frac{\partial}{\partial t}_{|t=a} E(\phi_t, \phi_a).
\] (5)

Thus a smooth curve \( \phi_t \) of equivalent convex functions of linear growth solves the HRMA equation if and only if \( \Phi \) is convex and the energy \( E(\phi_t, \phi_a) \) is linear in \( t \).

As is noted in [31] the Cauchy problem is not always solvable. Nevertheless there is a standard way to produce solutions \( \phi_t \) to the homogeneous Monge-Ampère equation with given starting point \( \phi_0 = \phi \) and \( t \in [0, \infty) \) using the Legendre transform. We give a brief account of this.

For simplicity assume from now on that \( \phi \) is differentiable and strictly convex. Recall that the Legendre transform of \( \phi \), denoted by \( \phi^* \), is the function on \( \Delta \phi \) defined as

\[
\phi^*(y) := \sup_x \{ x \cdot y - \phi(x) \}
\]

(which we can also think of as being defined on the whole of \( \mathbb{R}^n \), by being \( +\infty \) outside of \( \Delta \phi \)). Since \( \phi^* \) is defined as the supremum of the linear functions \( x \cdot y - \phi(x) \), it is convex. In fact, one can show that \( \phi \) being differentiable and strictly convex implies that \( \phi^* \) is also differentiable and strictly convex.

For a given \( y \in \Delta \phi \), the function \( x \cdot y - \phi(x) \) is strictly concave, and is maximized at the point where the gradient is zero. Thus we get that

\[
\phi^*(y) = x \cdot y - \phi(x) \quad \text{where} \quad \nabla \phi(x) = y,
\]

and hence

\[
\nabla \phi^*(y) = x \quad \text{where} \quad \nabla \phi(x) = y.
\]
2 CONVEX MOTIVATION

The Legendre transform is an involution. For using the above formula for $\phi^{**}$ we deduce that $\nabla \phi^{**}(x) = y$, for $x$ such that $\nabla \phi^*(y) = x$ which holds when $\nabla \phi(x) = y$, i.e.

$$\nabla \phi^{**}(x) = \nabla \phi(x).$$

If $\nabla \phi(x) = y$, then $\phi^*(y) = x \cdot y - \phi(x)$, therefore

$$\phi^{**}(x) = x \cdot y - \phi^*(y) = x \cdot y - (x \cdot y - \phi(x)) = \phi(x),$$

and hence $\phi^{**} = \phi$.

Lemma 2.3. If $\phi_t$ is a curve of convex functions, then for any point $y \in \Delta_{\phi_t}$,

$$\frac{\partial}{\partial t} \phi_t^*(y) = - \frac{\partial}{\partial t} \phi_t(x),$$

where $x$ is the point such that $\nabla \phi(x) = y$.

Proof. Let $x_t$ be the solution to the equation $\nabla \phi_t(x_t) = y$. By the implicit function theorem $x_t$ varies smoothly with $t$. By equation (6) we know

$$\frac{\partial}{\partial t} \phi_t^*(y) = \frac{\partial}{\partial t} (x_t \cdot y - \phi_t(x)) = \frac{\partial}{\partial t} (x_t \cdot y - \phi(x)) - \frac{\partial}{\partial t} \phi_t(x).$$

Since $x_t \cdot y - \phi(x)$ is maximized at $x = x_0$, the derivative of that part vanishes at $t = 0$, so we get the lemma for $t = 0$, and similarly for all $t$.

This leads us to the following formula relating the energy with the Legendre transform

Lemma 2.4. We have that

$$E(\phi_t, \phi) = \int_{\Delta_{\phi}} (\phi^* - \phi_t^*)dy. \quad (7)$$

Proof. We noted above that the derivative with respect to $t$ of the left-hand side of (7) is equal to

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t).$$

On the other hand, differentiating the right-hand side yields

$$\frac{\partial}{\partial t} \int_{\Delta_{\phi}} (\phi^* - \phi_t^*)dy = \int_{\Delta_{\phi}} \frac{\partial}{\partial t} \phi_t^* dy = \int_{\Delta_{\phi}} \frac{\partial}{\partial t} \phi_t (\nabla \phi_t^{-1}(y)) dy =$$

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t (\nabla \phi_t)^* dy = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t),$$

where we used Lemma 2.3 and the fact that $(\nabla \phi_t)^* dy = MA(\phi_t)$. Since both sides of the equation (7) is zero when $\phi_t = \phi$ and the derivatives coincide, we get that they must be equal for all $t$. 

Now fix a smooth bounded strictly concave function $u$ on $\Delta \phi$ and let
\[
\tilde{\phi}_t := (\phi^* - tu)^*.
\]
By the involution property of the Legendre transform $(\tilde{\phi}_t)^* = \phi^* - tu$.

**Proposition 2.5.** The curve $\tilde{\phi}_t$, $t \in [0, \infty)$ solves the HRMA equation.

To see this note that from (7) it follows that
\[
E(\tilde{\phi}_t, \phi) = \int_{\Delta \phi} (\phi^* - \tilde{\phi}_t^*)dy = \int_{\Delta \phi} (\phi^* - \phi^* + tu)dy = t \int_{\Delta \phi} udy,
\]
which is linear in $t$. The convexity of $\tilde{\Phi}(x,t) = \tilde{\phi}_t(x)$ can of course be shown directly, but it also follows from another characterization of $\tilde{\phi}_t$ that also involves a Legendre transform, but in the $t$-coordinate instead of in the $x$-coordinates which we now discuss.

Let $A_\lambda$ be the subset of $\Delta \phi$ where $u$ is greater than or equal to $\lambda$ and let $\phi_\lambda$ be defined as
\[
\phi_\lambda := \sup\{\psi \leq \phi : \psi \in \text{Conv}(\mathbb{R}^n), \Delta \psi \subseteq A_\lambda\}.
\]

**Lemma 2.6.** The curve of functions $\phi_\lambda$ is concave in $\lambda$ and
\[
\{\phi_\lambda = \phi\} = \{x : \nabla \phi(x) \in A_\lambda\}.
\]

**Proof.** Let $\psi_i \leq \phi$ be such that $\Delta \psi_i \subseteq A_\lambda$ with $i = 1, 2$. Let $0 < t < 1$. From our discussion above it follows that $t \psi_1 + (1 - t) \psi_2 \leq \phi$ and
\[
\Delta_t \psi_1 + (1 - t) \psi_2 \subseteq t \Delta \psi_1 + (1 - t) \Delta \psi_2 \subseteq t A_{\lambda_1} + (1 - t) A_{\lambda_2} \subseteq A_{t \lambda_1 + (1 - t) \lambda_2},
\]
where the last inclusion follows from the fact that $u$ was assumed to be concave. For the second statement, it is easy to see that in fact $\phi_\lambda$ is equal to the supremum of affine functions $x \cdot y + C$ bounded by $\phi$ and $y$ lying in $A_\lambda$. \qed

**Definition 2.7.** For $t \geq 0$ let $\hat{\phi}_t$ be defined as
\[
\hat{\phi}_t := \sup_{\lambda} \{\phi_\lambda + t \lambda\}.
\]

Since for each $\lambda$ the function $(x, t) \mapsto \phi_\lambda(x) + t \lambda$ is convex in all its variables, and the supremum of convex functions is convex, we get $\hat{\Phi}(x, t) := \hat{\phi}_t(x)$ is convex.

**Proposition 2.8.** We have that $\hat{\phi}_t = \hat{\phi}_t$. In particular this proves that $\hat{\Phi}$ is convex, thereby proving $\hat{\phi}_t$ solves the HRMA equation (Proposition 2.5).

**Proof.** We claim
\[
\frac{\partial}{\partial t} \hat{\phi}_t(x) = u(\nabla \hat{\phi}_t(x)).
\]
To see this first consider the right-derivative at \( t = 0 \). As we noted above, the gradient of a Legendre transform is the point where the maximum is attained, thus in this case

\[
\frac{\partial}{\partial t_{=0+}} \hat{\phi}_t(x) = \sup \{ \lambda : \phi_\lambda(x) = \phi(x) \}.
\]

By the second statement in Lemma 2.6 it follows that this supremum is equal to \( u(\nabla \phi(x)) \), and we are done for \( t = 0 \). On the other hand it is easy to see that

\[
\hat{\phi}_{t_1 + t_2} = \hat{\psi}_2,
\]

with \( \psi := \hat{\phi}_t \). Using this we get that the equation (8) holds for all \( t \). Thus by Lemma 2.3 the Legendre transform of \( \hat{\phi}_t \) is equal to \( \phi - tu \), so by the involution property of the Legendre transform \( \hat{\phi}_t \) coincides with \( \hat{\phi} \).

Now the above discussion can be applied as follows. Let \( \psi_\lambda \) be a concave curve in \( \text{Conv}(\mathbb{R}^n) \), with \( |\psi_\lambda - \phi| \) bounded for \( \lambda < -C \) and \( \psi_\lambda \equiv -\infty \) for \( \lambda > C \) for some constant \( C \). We call such a curve a test curve. Define \( \phi_\lambda \) as

\[
\phi_\lambda := \sup\{ \psi : \psi \leq \phi, \psi \leq \psi_\lambda + o(1), \psi \in \text{Conv}(\mathbb{R}^n) \}.
\]

Let also \( u \) be the function on \( \Delta_\phi \) defined by

\[
u(y) := \sup\{ \lambda : y \in \Delta_\psi \}.
\]

Since \( \psi_\lambda \) was assumed to be concave it follows that \( u \) is concave, and in fact we get that

\[
\phi_\lambda = \sup\{ \psi \leq \phi : \psi \in \text{Conv}(\mathbb{R}^n), \Delta_\psi \subseteq \{ u \geq \lambda \} \}.
\]

From Proposition 2.8, \( \hat{\phi}_t \) solves the homogeneous real Monge-Ampère equation. Thus in order to get solutions to the HRMA, instead of starting with a concave function \( u \) on \( \Delta_\phi \) we can just as well start with a test curve \( \psi_\lambda \). In the subsequent sections we will show how this construction carries over in the context of positive metrics on line bundles.

3 Preliminary Material

We collect here some preliminary material on the space of positive metrics, the (non pluripolar) Monge-Ampère measure and the Aubin-Mabuchi energy functional. Most of this material is standard, and we give proofs only for those results for which we did not find a convenient reference.

3.1 The space of positive singular metrics

Let \( X \) be a compact Kähler manifold of complex dimension \( n \), and let \( L \) be an ample line bundle on \( X \). A continuous (or smooth) hermitian metric \( h = e^{-\phi} \) on \( L \) is a
continuous (or smooth) choice of scalar product on the complex line \( L_p \) at each point \( p \) on the manifold. If \( f \) is a local holomorphic frame for \( L \) on \( U_f \), then one writes

\[ |f|_h^2 = h_f = e^{-\phi_f}, \]

where \( \phi_f \) is a continuous (or smooth) function on \( U_f \). We will use the convention to let \( \phi \) denote the metric \( h = e^{-\phi} \), thus if \( \phi \) is a metric on \( L \), \( k\phi \) is a metric on \( kL := L^\otimes k \).

The curvature of a smooth metric is given by \( dd^c \phi \) which is the \((1,1)\)-form locally defined as \( dd^c \phi_f \), where \( f \) is any local holomorphic frame. Here \( d^c \) is short-hand for the differential operator

\[ \frac{i}{2\pi} (\partial - \bar{\partial}), \]

so \( dd^c = i/\pi \partial \bar{\partial} \). A classic fact is that the curvature form \( dd^c \phi \) of a smooth metric \( \phi \) is a representative for the first Chern class of \( L \), denoted by \( c_1(L) \). The metric \( \phi \) is said to be strictly positive if the curvature \( dd^c \phi \) is strictly positive as a \((1,1)\)-form, i.e. if for any local holomorphic frame \( f \), the function \( \phi_f \) is strictly plurisubharmonic. We let \( \mathcal{H}(L) \) denote the space of smooth strictly positive (i.e. locally strictly plurisubharmonic) metrics on \( L \), which is non-empty since we assumed that \( L \) was ample.

A positive singular metric \( \psi \) is a metric that can be written as \( \psi := \phi + u \), where \( \phi \) is a smooth metric and \( u \) is a \( dd^c \phi \)-psh function, i.e. \( u \) is upper semicontinuous and \( dd^c \psi := dd^c \phi + dd^c u \) is a positive \((1,1)\)-current. For convenience we also allow \( u \equiv -\infty \). We let \( PSH(L) \) denote the space of positive singular metrics on \( L \).

As an important example, if \( \{s_i\} \) is a finite collection of holomorphic sections of \( kL \), we get a positive metric \( \psi := \frac{1}{k} \ln(\sum |s_i|^2) \) which is defined by letting for any local frame \( f \),

\[ e^{-\psi_f} := \frac{|f|^2}{(\sum |s_i|^2)^{1/k}}. \]

We note that \( PSH(L) \) is a convex set, since any convex combination of positive metrics yields a positive metric. Another important fact is that the upper semicontinuous regularization of the supremum \( \sup\{\psi_i : i \in I\} \), denoted by \( (\sup\{\psi_i : i \in I\})^* \), where \( \psi_i \in PSH(L) \) for all \( i \in I \), lie in \( PSH(L) \), as long as all \( \psi_i \) are bounded from above by some fixed positive metric. If \( \psi \) lie in \( PSH(L) \), then all its translates \( \psi + c \) lies in \( PSH(L) \), where \( c \) is any real number. For any \( \psi \in PSH(L) \), \( dd^c \psi \) is a closed positive \((1,1)\)-current, and from the \( dd^c \) lemma it follows that any closed positive current cohomologous with \( dd^c \psi \) can be written as \( dd^c \phi \) for some \( \phi \) in \( PSH(L) \). By the maximum principle this \( \phi \) is uniquely determined up to translation.

If there exists a constant \( C \) such that \( \psi \leq \phi + C \), we say that \( \psi \) is more singular than \( \phi \), and we will write this as

\[ \psi \geq \phi. \]

If both \( \psi \geq \phi \) and \( \phi \geq \psi \) we say that \( \psi \) and \( \phi \) are equivalent, which we write as \( \psi \sim \phi \).

Following [10] an equivalence class \( [\psi] \) is called a singularity type, and we introduce the notation \( \text{Sing}(L) \) for the set of singularity types. If \( \psi \) is equivalent to an element in \( \mathcal{H}(L) \) we say that \( \psi \) is locally bounded.

The singularity locus of a positive metric \( \psi \) is the set where \( \psi \) is minus infinity, i.e. the set where \( \psi_f = -\infty \) when \( f \) is a local frame. The unbounded locus of \( \psi \) is the
set where \( \psi \) is not locally bounded. Recall that a set is said to be complete pluripolar if it is locally the singularity locus of a plurisubharmonic function. In [10] BEGZ (Boucksom-Eyssidieux-Guedj-Zeriahi) give the following definition.

**Definition 3.1.** A positive metric \( \psi \) is said to have small unbounded locus if its unbounded locus is contained in a closed complete pluripolar subset of \( X \).

We note that metrics of the form \( \frac{1}{k} \ln(\sum |s_i|^2) \) have small unbounded locus, since they are locally bounded away from the algebraic set \( \cup_i \{ s_i = 0 \} \) which is a closed pluripolar set.

### 3.2 Regularization of positive singular metrics

If \( f \) is a plurisubharmonic function on an open subset \( U \) of \( \mathbb{C}^n \), using convolution we can write \( f \) as the limit of a decreasing sequence of smooth plurisubharmonic functions on any relatively compact subset of \( U \).

If \( \psi \) is a positive singular metric, we can use a partition of unity with respect to some open cover \( U_f_i \) to patch together the smooth decreasing approximations of \( \psi_f_i \). Thus any positive singular metric can be written as the pointwise limit of a decreasing sequence of smooth metrics, but of course because of the patching these smooth approximations will in general not be positive.

A fundamental result due to Demailly [17] is that any positive singular metric can be approximated by metrics of the form \( k^{-1} \ln(\sum |s_i|^2) \), where \( s_i \) are sections of \( kL \). Let \( \mathcal{I}(\psi) \) denote the multiplier ideal sheaf of germs of holomorphic functions locally integrable against \( e^{-\psi} dV \), where \( f \) is a local frame for \( L \) and \( dV \) is an arbitrary volume form. We get a scalar product \( (.,.)_{k\psi} \) on the space \( H^0(kL \otimes \mathcal{I}(k\psi)) \) by letting

\[
||s||^2_{k\psi} := \int_X |s|^2 e^{-k\psi} dV.
\]

Let \( \{ s_i \} \) be an orthonormal basis for \( H^0(kL \otimes \mathcal{I}(k\psi)) \) and set

\[
\psi_k := \frac{1}{k} \ln(\sum |s_i|^2).
\]

**Theorem 3.2.** The sequence of metrics \( \psi_k \) converge pointwise to \( \psi \) as \( k \) tends to infinity, and there exists a constant \( C \) such that for large \( k \),

\[
\psi \leq \psi_k + \frac{C}{k}.
\]

As a reference see [18], but the results of Demailly are in fact much stronger than that stated here, and hold in greater generality [17]. When \( \psi \) is assumed to be smooth and strictly positive, by a celebrated result by Bouche-Catlin-Tian-Zelditch [8] [11] [36] [40] on Bergman kernel asymptotics we have that \( \psi_k \) in fact converges to \( \psi \) in any \( C^m \) norm.

Using a variation of this construction Guedj-Zeriahi prove in [22] that any positive singular metric on an ample line bundle is the pointwise limit of a decreasing sequence of smooth positive metrics.
3 PRELIMINARY MATERIAL

3.3 Monge-Ampère measures

Let $\psi_i$, $1 \leq i \leq n$, be an n-tuple of positive metrics, so for each $i$, $dd^c \psi_i$ is a positive $(1,1)$-current. If all $\psi_i$ are smooth one can consider the wedge product

$$dd^c \psi_1 \wedge ... \wedge dd^c \psi_n,$$

which is a positive measure on $X$. The fundamental work of Bedford-Taylor shows that one can still take the wedge product of positive currents $dd^c \psi_i$ to get a positive measure as long as the metrics $\psi_i$ are all locally bounded. The Monge-Ampère measure of a locally bounded positive metric $\psi$, is then defined as the positive measure

$$MA(\psi) := (dd^c \psi)^n.$$

This measure does not put any mass on pluripolar sets (i.e., sets that are locally contained in the unbounded locus of a local plurisubharmonic function). We recall the following important continuity property, proved in [2].

**Theorem 3.3 (Bedford-Taylor).** If $\psi_{i,k}$, $1 \leq i \leq n + 2$, $k \in \mathbb{N}$, are sequences of locally bounded positive metrics such that each $\psi_{i,k}$ decreases to a locally bounded positive metric $\psi_i$, then the signed measures $(\psi_{1,k} - \psi_{2,k})dd^c \psi_{3,k} \wedge ... \wedge dd^c \psi_{n+2,k}$ converge weakly to $(\psi_1 - \psi_2)dd^c \psi_3 \wedge ... \wedge dd^c \psi_{n+2}$. If each sequence of locally bounded positive metrics $\psi_{i,k}$ instead increase pointwise a.e. to a positive metric $\psi_i$, then again the measures $(\psi_{1,k} - \psi_{2,k})dd^c \psi_{3,k} \wedge ... \wedge dd^c \psi_{n+2,k}$ converge weakly to $(\psi_1 - \psi_2)dd^c \psi_3 \wedge ... \wedge dd^c \psi_{n+2}$.

Since the curvature form $dd^c \phi$ of any smooth metric $\phi$ is a representative of $c_1(L)$, we see that if $\phi_i$ is any n-tuple of smooth metrics then

$$\int_X dd^c \phi_1 \wedge ... \wedge dd^c \phi_n = \int_X c_1(L)^n$$

which is just a topological invariant of $L$. Since any positive metric can be approximated from above in the manner of Theorem 3.3 by positive metrics that are smooth, we see that (10) still holds if the $\phi_i$ are merely assumed to be locally bounded instead of smooth.

Recall that a plurisubharmonic function is, by definition, upper semicontinuous, so if $\psi$ is a positive metric then for each local frame $f$ the function $\psi_f$ is upper semicontinuous. The plurifine topology is defined as the coarsest topology in which all local plurisubharmonic functions are continuous; a basis for this topology is given by sets of the form $A \cap \{ u > 0 \}$, where $A$ is open in the standard topology and $u$ is a local plurisubharmonic function. This topology has the quasi-Lindelöf property [3 Thm 2.7], meaning that an arbitrary union of plurifine open sets differs from a countable subunion by at most a pluripolar set. Any basis set $A \cap \{ u > 0 \}$ is Borel, so it follows from the quasi-Lindelöf property that the plurifine open (and closed) sets lie in the completion of the Borel $\sigma$-algebra with respect to any Monge-Ampère measure [3 Prop 3.1].
Definition 3.4. A function $f$ is said to be quasi-continuous on a set $\Omega$ if for every $\epsilon > 0$ there exists an open set $U$ with capacity less than $\epsilon$ so that $f$ is continuous on $\Omega \setminus U$.

We refer to [2] for the definition of capacity. By [3] Thm 4.9 plurisubharmonic functions are quasi-continuous.

If $f_k$ is a sequence of non-negative continuous functions increasing to the characteristic function of an open set $A$ then the characteristic function of a basis set $A \cap \{u > 0\}$ is the increasing limit of the non-negative quasi-continuous functions

$$k f_k(\max\{u, 0\} - \max\{u - 1/k, 0\}).$$

From this fact and the quasi-Lindelöf property it follows that the characteristic function of any plurifine open set differs from an increasing limit of non-negative quasi-continuous functions at most on a pluripolar set.

A fundamental property of the Bedford-Taylor product is that it is local in the plurifine topology, so if $\psi_i = \psi'_i$ for all $i$ on some plurifine open set $O$ then

$$1_O dd^c \psi_1 \wedge ... \wedge dd^c \psi_n = 1_O dd^c \psi'_1 \wedge ... \wedge dd^c \psi'_n,$$

where $1_O$ denotes the characteristic function of $O$. We also have that the convergence in Theorem 3.3 is local in this topology [3] Thm 3.2], i.e. we get convergence when testing against bounded quasi-continuous functions.

Lemma 3.5. Let $\psi_k$ be a sequence of locally bounded positive metrics that decreases pointwise (or increases a.e.) to a locally bounded positive metric $\psi$, and let $O$ be a plurifine open set. Then we have that

$$1_O MA(\psi) \leq \liminf_{k \to \infty} 1_O MA(\psi_k),$$

where the $\liminf$ is to be understood in the weak sense, i.e. when testing against non-negative continuous functions.

Proof. Let $u_i$ be a sequence of quasi-continuous functions increasing to $1_O$ except on a pluripolar set. Let $f$ be a non-negative continuous function. Since $u_i MA(\psi_k)$ converges weakly to $u_i MA(\psi)$, and $MA(\psi_k)$ does not put any mass on a pluripolar set, we get that

$$\int_X f u_i MA(\psi) = \lim_{k \to \infty} \int_X f u_i MA(\psi_k) \leq \liminf_{k \to \infty} \int_O f MA(\psi_k). \quad (11)$$

Now $u_i$ increases to the characteristic function of $O$ except possibly on a pluripolar set, so letting $i$ tend to infinity in (11) yields that

$$\int_O f MA(\psi) \leq \liminf_{k \to \infty} \int_O f MA(\psi_k).$$
For singular $\phi_i$, there is a (non pluripolar) product constructed by Boucksom-Eyssidieux-Guedj-Zeriahi [10], building on a local construction due to Bedford-Taylor [3]. Fix a locally bounded metric $\phi$, and consider the auxiliary metrics
\[
\psi_{i,k} := \max\{\psi_i, \phi - k\}
\]
for $k \in \mathbb{N}$, and the sets $O_k := \bigcap_i \{\psi_i > \phi - k\}$. The non-pluripolar product of the currents $dd^c \psi_i$, here denoted by
\[
dd^c \psi_1 \wedge ... \wedge dd^c \psi_n
\]
is defined as the limit
\[
1_{O_k} \lim_{k \to \infty} dd^c \psi_{1,k} \wedge ... \wedge dd^c \psi_{n,k}.
\]
Since we are assuming that $X$ is compact this limit is well defined [10, Prop. 1.6]. The (non-pluripolar) Monge-Ampère measure of a positive metric is $\psi$ is defined as
\[
MA(\psi) := \left(\dd c \psi\right)^n.
\]
Essentially by construction, the non-pluripolar product is local in the plurifine topology [10, Prop. 1.4], and is multilinear [10, Prop 4.4].

Clearly from the definition and (10), for any $n$-tuple of positive metrics $\psi_i$,
\[
\int_X \dd^c \psi_1 \wedge ... \wedge \dd^c \psi_n \leq \int_X c_1(L)^n,
\]
however the inequality may well be strict.

Combining Lemma 3.5 with the fact that the Monge-Ampère measure is local in the plurifine topology yields the following continuity result.

**Lemma 3.6.** Let $\psi_k$ be a sequence of positive metrics decreasing to a positive metric $\psi$, and let $\phi$ be some locally bounded positive metric. If $O$ is a plurifine open set contained in $\{\psi > \phi - C\}$ for some constant $C$ then
\[
1_O MA(\psi) \leq \liminf_{k \to \infty} 1_O MA(\psi_k),
\]
where again the limit is to be understood in the weak sense. If $\psi_k$ instead is increasing a.e. to $\psi$, and $O$ is a plurifine open set contained in $\{\psi_j > \phi - C\}$ for some natural number $j$ and some constant $C$ then once again
\[
1_O MA(\psi) \leq \liminf_{k \to \infty} 1_O MA(\psi_k).
\]

**Proof.** First assume that $\psi_k$ is decreasing to $\psi$. Let $\psi'_k := \max\{\psi_k, \phi - C\}$ and $\psi' := \max\{\psi, \phi - C\}$. From Lemma 3.5 it follows that
\[
1_O MA(\psi') \leq \liminf_{k \to \infty} 1_O MA(\psi'_k),
\]
and since by assumption $\psi' = \psi$ and $\psi'_k = \psi_k$ on $O$ the lemma follows from the locality of the non-pluripolar product. The case where $\psi_k$ is increasing a.e. follows by the same reasoning.

In [10, Thm 1.16] BEGZ prove the following monotonicity property of the non-pluripolar product when restricted to metrics with small unbounded locus.

**Theorem 3.7.** Let $\psi_i, \psi'_i$ be two $n$-tuples of positive metrics with small unbounded locus, and suppose that for all $i$, $\psi_i$ is more singular than $\psi'_i$. Then
\[
\int_X \dd^c \psi_1 \wedge ... \wedge \dd^c \psi_n \leq \int_X \dd^c \psi'_1 \wedge ... \wedge \dd^c \psi'_n.
\]
BEGZ also prove a comparison principle for metrics with small unbounded locus \[10\text{ Cor 2.3}\] and a domination principle \[10\text{ Cor 2.5}\]. When combined with the comparison principle, the proof of the domination principle in \[10\] in fact yields a slightly stronger version:

**Theorem 3.8.** Let \(\phi\) be a positive metric with small unbounded locus and suppose that there exists a positive metric \(\rho\), more singular than \(\phi\), with small unbounded locus and such that \(MA(\rho)\) dominates a volume form. If \(\psi\) is a positive metric more singular than \(\phi\) and such that \(\psi \leq \phi\) a.e. with respect to \(MA(\phi)\), then it follows that \(\psi \leq \phi\) on the whole of \(X\).

### 3.4 The Aubin-Mabuchi Energy

The **Aubin-Mabuchi energy** bifunctional maps any pair of equivalent positive metrics \(\psi_1\) and \(\psi_2\) to the number

\[
E(\psi_1, \psi_2) := \frac{1}{n+1} \sum_{i=0}^{n} \int_X (\psi_1 - \psi_2)(dd^c \psi_1)^i \wedge (dd^c \psi_2)^{n-i}.
\]

Observe

\[
E(\psi + t, \psi) = t \int_X MA(\psi).
\]

The Aubin-Mabuchi energy restricted to the class of locally bounded metrics has a cocycle property (see, for example, \[6\text{ Cor 4.2}\]), namely if \(\phi_0, \phi_1\) and \(\phi_2\) are locally bounded equivalent metrics then

\[
E(\phi_0, \phi_2) = E(\phi_0, \phi_1) + E(\phi_1, \phi_2).
\]

In fact the proof in \[6\] of the cocycle property extends to the case where the equivalent metrics are only assumed to have small unbounded locus, since the integration-by-parts formula of \[10\] used in the proof holds in that case.

This leads to an important monotonicity property. If \(\psi_0, \psi_1\) and \(\psi_2\) are equivalent with small unbounded locus, and \(\psi_0 \geq \psi_1\), then

\[
E(\psi_0, \psi_2) \geq E(\psi_1, \psi_2)
\]

since \(E(\psi_0, \psi_2) = E(\psi_0, \psi_1) + E(\psi_1, \psi_2)\), and \(E(\psi_0, \psi_1) \geq 0\) as it is the integral of the positive function \(\psi_0 - \psi_1\) against a positive measure.

We also record the following lemma, which comes from the locality of the non-pluripolar product in the plurifine topology.

**Lemma 3.9.** Let \(\psi_1 \sim \psi_2\) be such that \(\psi_1 \geq \psi_2\). Let \(\psi_1'\) and \(\psi_2'\) be two other metrics such that \(\psi_1' \sim \psi_2'\) and assume that \(\{\psi_1' = \psi_2'\} = \{\psi_1 = \psi_2\}\) and that \(\psi_1' = \psi_1\) and \(\psi_2' = \psi_2\) on the set where \(\psi_1 > \psi_2\). Then

\[
E(\psi_1', \psi_2') = E(\psi_1, \psi_2).
\]
Following Phong-Sturm in [26] we can relate weak geodesics to the energy functional. Let\( A := \{e^a \leq |z| \leq e^b\} \) be an annulus and let \( \pi \) denote the standard projection from \( X \times A \) to \( X \). A curve of metrics \( \phi_t \) of \( L, a \leq t \leq b \), can be identified with the rotation invariant metric on \( \pi^* L \) whose restriction to \( X \times \{w\} \) equals \( \phi_{\ln |w|} \).

**Definition 3.10.** A curve of positive metrics \( \phi_t, a \leq t \leq b \), is said to be a weak subgeodesic if there exists a locally bounded positive metric \( \Phi \) on \( \pi^* L \) that is rotation invariant and whose restriction to \( X \times \{w\} \) equals \( \phi_{\ln |w|} \). A curve \( \phi_t \) is said to be a weak geodesic if it is a weak subgeodesic and furthermore \( \Phi \) solves the HCMA equation, i.e.

\[
MA(\Phi) = 0
\]
on \( X \times A^\circ \).

As in the convex setting (5) there is a formula [7, 6.3] relating the Aubin-Mabuchi energy of a locally bounded subgeodesic \( \phi_t \) with the Monge-Ampère measure of \( \Phi \), namely

\[
\frac{dd_c}{c} E(\phi_t, \phi_a) = \pi^*(MA(\Phi)),
\]

where \( \pi_*(MA(\Phi)) \) denotes the push-forward of the measure \( MA(\Phi) \) with respect to the projection \( \pi \). From this we immediately get the following lemma.

**Lemma 3.11.** A curve \( \phi_t \) of locally bounded positive metrics is a weak geodesic if and only if it is a subgeodesic and the Aubin-Mabuchi energy \( E(\phi_t, \phi_a) \) is linear in \( t \).

## 4 Envelopes and maximal metrics

In studying, for example, the Dirichlet problem for the HCMA equation, it is often possible to give a solution as an envelope in some space of plurisubharmonic functions (or positive metrics). Such envelopes will be crucial in our setting as well.

**Definition 4.1.** If \( \phi \) is a continuous metric, not necessarily positive, let \( P\phi \) denote the envelope

\[
P\phi := \sup\{\psi \leq \phi, \psi \in PSH(L)\}.
\]

Since \( \phi \) is assumed to be continuous it follows that \( (P\phi)^* \leq \phi \), thus \( P\phi = (P\phi)^* \), so \( P\phi \in PSH(L) \).

The next theorem is essentially just a reformulation of a local result of Bedford-Taylor [2, Corollary 9.2] in our global setting. It follows as a special case of [6, Prop 1.10] (letting \( K = X \)).

**Theorem 4.2.** If \( \phi \) is a continuous metric then \( P\phi = \phi \) a.e. with respect to \( MA(P\phi) \).

Recall that if \( A \) is a closed set and \( \mu \) is a Borel measure we say that \( \mu \) is said to be concentrated on \( A \) if \( 1_A \mu = \mu \), or equivalently \( \mu(A^c) = 0 \). Thus another way of formulating Theorem 4.2 is to say that \( MA(P\phi) \) is concentrated on \( \{P\phi = \phi\} \). We now extend this result to more general envelopes that arise from the additional data of singularity type.
Definition 4.3. Given a positive metric $\psi \in PSH(L)$ let $P_\psi$ denote the projection operator on $PSH(L)$ defined by

$$P_\psi \phi := \sup \{ \psi' \leq \min \{ \phi, \psi \}, \psi' \in PSH(L) \}.$$ 

We also let $P[\psi]$ be defined by

$$P[\psi] \phi := \lim_{C \to \infty} P_{\psi+C} \phi = \sup \{ \psi' \leq \phi, \psi' \sim \psi, \psi' \in PSH(L) \}.$$ 

Clearly $P_\psi \phi$ is monotone with respect to both $\psi$ and $\phi$. Since $\min \{ \phi, \psi \}$ is upper semicontinuous, it follows that the upper semicontinuous regularization of $P_\psi \phi$ is still less than $\min \{ \phi, \psi \}$, and thus $P_\psi \phi \in PSH(L)$. By this it follows that $P_\psi(P_\psi \phi) = P_\psi \phi$, i.e. that $P_\psi$ is indeed a projection operator on $PSH(L)$. One also notes that the upper semicontinuous regularization of $P[\psi]$ lies in $PSH(L)$ and is bounded by $\phi$.

Definition 4.4. The maximal envelope of $\phi$ with respect to the singularity type $[\psi]$ is defined to be

$$\phi[\psi] := (P[\psi] \phi)^*.$$ 

Definition 4.5. If $\psi \in PSH(L)$, then $\psi$ is said to be maximal with respect to a metric $\phi$ if $\psi \leq \phi$ and furthermore $\psi = \phi$ a.e. with respect to $MA(\psi)$. Similarly, if $A$ is a measurable set, we say that $\psi$ is maximal with respect to $\phi$ on $A$ if $\psi \leq \phi$ and $\psi = \phi$ a.e. on $A$ with respect to $MA(\psi)$.

The terminology is justified by a proof below that the maximal envelope of a continuous metric $\phi$ is maximal with respect to $\phi$. Note that we do not know whether the maximal envelope $\phi[\psi]$ is equivalent to $\psi$. Therefore we cannot at this point use the method in the proof of Theorem 4.2 in [6], so instead we will use an approximation argument. The reason for the use of the word maximal is motivated by the following property:

Proposition 4.6. Let $\psi$ be maximal with respect to a metric $\phi$. Suppose also that there exists a positive metric $\rho$, more singular than $\psi$, with small unbounded locus and such that $MA(\rho)$ dominates a volume form. Then for any $\psi' \sim \psi$ with $\psi' \leq \phi$ we have $\psi' \leq \psi$.

Proof. Since $\psi' \leq \phi$ the maximality assumption yields that $\psi' \leq \psi$ a.e. with respect to $MA(\psi)$, so the proposition thus follows from the domination principle (Theorem 3.8).\]

The next two lemmas are the main steps in showing that maximal envelopes are maximal.

Lemma 4.7. Let $\psi_k$ be a sequence of positive metrics increasing a.e. to a positive metric $\psi$, and assume that all $\psi_k$ are maximal with respect to a fixed continuous metric $\phi$ on some plurifine open set $O$. Then $\psi$ is maximal with respect to $\phi$ on $O$.
Proof. Since $\phi$ was assumed to be continuous we have that $\psi \leq \phi$. For all $k$,

$$\{\psi_k = \phi\} \subseteq \{\psi = \phi\}$$

and thus by the the maximality of $\psi_k$, we know $1_O MA(\psi_k)$ is concentrated on $\{\psi = \phi\}$. Since $\psi \leq \phi$ we have that $\{\psi = \phi\} = \{\psi \geq \phi\}$, and since $\phi$ is continuous this is a closed set. Let $C$ be a constant. The set $O \cap \{\psi_1 > \phi - C\}$ is plurifinely open, so by Lemma 3.6 it follows that

$$1_O 1_{\{\psi_1 > \phi - C\}} MA(\psi) \leq \liminf_{k \to \infty} 1_O 1_{\{\psi_1 > \phi - C\}} MA(\psi_k). \tag{14}$$

It is easy to see that if $\mu_k$ is a sequence of measures all concentrated on a closed set $A$, and

$$\mu \leq \liminf_{k \to \infty} \mu_k$$

in the weak sense, then $\mu$ is also concentrated on $A$. It thus follows from (14) that $1_O 1_{\{\psi_1 > \phi - C\}} MA(\psi)$ is concentrated on $\{\psi = \phi\}$. Since $MA(\psi)$ puts no mass on the pluripolar set $\{\psi_1 = -\infty\}$ the lemma follows by letting $C$ tend to infinity. \qed

**Lemma 4.8.** Let $\psi \in PSH(L)$ and let $\phi$ be a continuous metric. Then the envelope $P_{\psi,\phi}$ is maximal with respect to $\phi$ on the plurifine open set $\{\psi > \phi\}$.

**Proof.** By definition $P_{\psi,\phi} \leq \phi$. Now let $\phi_k$ be a sequence of continuous metrics decreasing pointwise to $\min\{\phi, \psi\}$, so that $\phi_k \leq \phi$ for all $k$ and $\phi_k = \phi$ on the set $\{\psi > \phi\}$. For example let $\phi_k := \min\{\phi, \psi_k\}$ where $\psi_k$ is a sequence of smooth metrics decreasing pointwise to $\psi$. From Theorem 4.2 it follows that $MA(P\phi_k)$ is concentrated on $P\phi_k = \phi_k$, and since $\phi_k = \phi$ when $\psi > \phi$ we get that $1_{\{\psi > \phi\}} MA(P\phi_k)$ is concentrated on $P\phi_k = \phi$. Now $P\phi_k$ is decreasing in $k$ and $\lim_{k \to \infty} P\phi_k \leq \min\{\phi, \psi\}$. At the same time, for any $k \in \mathbb{N}$ we clearly have that $P_{\psi,\phi} \leq P\phi_k$, which taken together means that

$$\lim_{k \to \infty} P\phi_k = P_{\psi,\phi}.$$

Since $P\phi_k \leq \phi$ this implies that $\{P\phi_k = \phi\}$ is decreasing in $k$ and

$$\{P_{\psi,\phi} = \phi\} = \bigcap_{k \in \mathbb{N}} \{P\phi_k = \phi\}. \tag{15}$$

Let $O$ denote the plurifine open set $\{\psi > \phi\} \cap \{P_{\psi,\phi} > \phi - C\}$. By Lemma 3.6 we have that

$$1_O MA(P_{\psi,\phi}) \leq \liminf_{k \to \infty} 1_O MA(P\phi_k),$$

and thus we conclude that $1_O MA(P_{\psi,\phi})$ is concentrated on $\{P\phi_k = \phi\}$ for any $k$, so by (15) we get that $1_O MA(P_{\psi,\phi})$ is concentrated on $\{P_{\psi,\phi} = \phi\}$. Since $MA(P_{\psi,\phi})$ puts no mass on the pluripolar set $\{P_{\psi,\phi} = -\infty\}$, letting $C$ tend to infinity yields the lemma. \qed

**Theorem 4.9.** Let $\psi \in PSH(L)$ and let $\phi$ be a continuous metric. Then $\phi[\psi]$ is maximal with respect to $\phi$, i.e. $\phi[\psi] = \phi$ a.e. with respect to $MA(\phi[\psi])$. 

Proof. $P_{[\psi]} \phi = \phi_{[\psi]}$ a.e., and since $P_{\psi+C} \phi$ increases to $P_{[\psi]} \phi$, it thus increases to $\phi_{[\psi]}$ a.e.. By Lemma 4.3 we get that $P_{\psi+C} \phi$ is maximal with respect to $\phi$ on the plurifine open set $\{ \psi > \phi - C \}$ and also on any set $\{ \psi > \phi - C' \}$ whenever $C' \leq C$. From Lemma 4.7 it thus follows that $\phi_{[\psi]}$ is maximal with respect to $\phi$ on the set $\{ \psi > \phi - C \}$ for any $C$. Since $MA(\phi_{[\psi]})$ puts no mass on $\{ \psi = -\infty \}$ the theorem follows.

Example 4.10. Consider the case that $s$ is a section of $rL$ that vanishes along a divisor $D$, and set $\psi = \frac{1}{r} \ln |s|^2$. Then the maximal envelope $\phi_{[\psi]}$ is considered by Berman [5, Sec. 4], and equals

$$\sup \{ \psi' \leq \phi : \psi' \in PSH(L), \nu_D(\psi') \geq 1 \}^*$$

where $\nu_D$ denotes the Lelong number along $D$. This metric governs the Bergman kernel asymptotics of sections of $kL$ for $k \gg 0$ that vanish along the divisor $D$. The more general case when $\psi$ has analytic singularities is also considered in [5].

The maximal property gives the following bounds on the energy functional which will be crucial for our construction of weak geodesics (Theorem 6.7).

Proposition 4.11. Suppose that $\psi$ is maximal with respect to a positive metric $\phi$ with small unbounded locus, and let $t > 0$. Then we have that

$$t \int_X MA(\psi) \leq E(\max \{ \psi + t, \phi \}, \phi) \leq t \int_X MA(\phi). \quad (16)$$

Proof. Since by assumption $\psi \leq \phi$ we have that $\max \{ \psi + t, \phi \} \leq \phi + t$, so from the monotonicity of the Aubin-Mabuchi energy it follows that

$$E(\max \{ \psi + t, \phi \}, \phi) \leq E(\phi + t, \phi) = t \int_X MA(\phi)$$

which gives the upper bound. For the lower bound, first choose an $\epsilon$ with $0 < \epsilon < t$. Again by monotonicity,

$$E(\max \{ \psi + t, \phi \}, \phi) \geq E(\max \{ \psi + t, \phi \}, \max \{ \psi + \epsilon, \phi \}). \quad (17)$$

Now clearly

$$E(\max \{ \psi + t, \phi \}, \max \{ \psi + \epsilon, \phi \}) \geq (t - \epsilon) \int_{\{ \psi + \epsilon > \phi \}} MA(\psi). \quad (18)$$

By the assumption that $\psi$ is maximal with respect to $\phi$

$$\int_{\{ \psi = \phi \}} MA(\psi) = \int_X MA(\psi)$$

and since $\{ \psi = \phi \} \subseteq \{ \psi + \epsilon > \phi \}$, the combination of (17) and (18) yields

$$E(\max \{ \psi + t, \phi \}, \phi) \geq (t - \epsilon) \int_X MA(\psi).$$

Since $\epsilon > 0$ was chosen arbitrarily the lower bound in (16) follows. \qed
5 Test curves and analytic test configurations

Definition 5.1. A map $\lambda \mapsto \psi_\lambda$ from $\mathbb{R}$ to $PSH(L)$ is called a test curve if there is a constant $C$ such that

1. $\psi_\lambda$ is equal to some locally bounded positive metric $\psi_{-\infty}$ for $\lambda < -C$,
2. $\psi_\lambda \equiv -\infty$ for $\lambda > C$,
3. $\psi_\lambda$ has small unbounded locus whenever $\psi_\lambda \not\equiv -\infty$, and
4. $\psi_\lambda$ is concave in $\lambda$.

Observe also that since $\psi_\lambda$ is concave and constant for $\lambda$ sufficiently negative it is decreasing in $\lambda$.

Note that the set of test curves forms a convex set, by letting

$$ \left( \sum r_i \gamma_i \right)(\lambda) := \sum r_i \gamma_i(\lambda). $$

It is also clear that any translate $\gamma_\alpha(\lambda) := \gamma(\lambda - \alpha)$ of a test curve $\gamma$ is a new test curve.

We introduce the notation $\lambda_c$ for the critical value of a test curve defined as

$$ \lambda_c := \inf \{ \lambda : \psi_\lambda \equiv -\infty \}. $$

We record for later use two continuity properties of test curves.

Lemma 5.2.

1. A test curve $\psi_\lambda$ is left-continuous in $\lambda$ for $\lambda < \lambda_c$.
2. Suppose that $\lambda < \lambda_c$ and $\lambda_k$ is a decreasing sequence that tends to $\lambda$. Then

$$ \lim_{k \to \infty} \psi_{\lambda_k} = \psi_\lambda $$

(19)

(so a test curve is right continuous modulo taking an upper semicontinuous regularization).

Proof. For (1), let $\lambda_k$ increase to some $\lambda < \lambda_c$, and we need to show that

$$ \lim_{k \to \infty} \psi_{\lambda_k} = \psi_\lambda. $$

By our hypothesis there exists a $\lambda'$ such that $\lambda < \lambda' < \lambda_c$, and thus $\psi_{\lambda'} \not\equiv -\infty$. Since $\psi_\lambda(x)$ is concave in $\lambda$ it is continuous for all $x$ such that $\psi_{\lambda'}(x) \not\equiv -\infty$. Thus $\psi_{\lambda_k}$ converges to $\psi_\lambda$ at least away from a pluripolar set, i.e. a.e. with respect to a volume form. On the other hand we have that $\psi_{\lambda_k}$ is decreasing in $k$, so the limit is a positive metric. Now if two positive metrics coincide a.e. with respect to a volume form it follows that they are equal everywhere, because this is true locally for plurisubharmonic function.

The proof of (2) is essentially the same. This time $\lambda_k$ is a decreasing sequence, so as $\lambda < \lambda_c$ we may as well assume that each $\lambda_k < \lambda'$ and so in particular $\psi_{\lambda_k} \not\equiv -\infty$. 

Then the $\psi_{\lambda_k}$ form an increasing sequence so the left hand side of (19) is a positive metric. But for the same reason as above, the limit $\lim_{k \to \infty} \psi_{\lambda_k}$ equals $\psi_{\lambda}$ away from a pluripolar set, and thus the left and right hand side of (19) agree a.e. with respect to a volume form, and thus are equal everywhere.

Definition 5.3. A map $\gamma$ from $\mathbb{R}$ to $\text{Sing}(L)$ is called an analytic test configuration if it is the composition of a test curve with the natural projection of $\text{PSH}(L)$ to $\text{Sing}(L)$.

As with the set of test curves, the set of analytic test configurations is convex. We now extend the definition of the maximal envelope (Definition 4.4) to test curves.

Definition 5.4. Let $\psi_{\lambda}$ be a test curve and $\phi$ an element in $\mathcal{H}(L)$. The maximal envelope of $\phi$ with respect to $\psi_{\lambda}$ is the map

$$\lambda \mapsto \phi_{[\psi_{\lambda}]} := (P_{[\psi_{\lambda}]} \phi)^{\ast}.$$ 

It is easy to see that $\phi_{\lambda}$ only depends on $\phi$ and the analytic test configuration $[\psi_{\lambda}]$, since if $\psi_{\lambda}' \sim \psi_{\lambda}$ we trivially have $\hat{\phi}_{[\psi_{\lambda}]} = \hat{\phi}_{[\psi_{\lambda}']}$. Observe also that since $\psi_{-\infty}$ is locally bounded, we have $\phi_{\lambda} = \phi$ for $\lambda < -C$.

Definition 5.5. We say that a test curve $\psi_{\lambda}$ is maximal if for all $\lambda$ the metric $\psi_{\lambda}$ is maximal with respect to $\psi_{-\infty}$.

Since $\psi_{\lambda}$ is decreasing in $\lambda$,

$$\{ \psi_{\lambda'} = \psi_{\lambda} \} \supseteq \{ \psi_{\lambda'} = \psi_{-\infty} \} \quad \text{if} \quad \lambda \leq \lambda'.$$

It follows that if $\psi_{\lambda}$ is a maximal test curve, $\psi_{\lambda'}$ is maximal with respect to $\psi_{\lambda}$ whenever $\lambda \leq \lambda'$.

Proposition 5.6. The maximal envelope $\phi_{\lambda}$ is a maximal test curve.

Proof. We first show it is a test curve. Pick a real number $C$. Let $\lambda$ and $\lambda'$ be two real numbers, and let $0 \leq t \leq 1$. By the concavity of $\psi_{\lambda}$,

$$tP_{\psi_{\lambda} + C \phi} + (1 - t)P_{\psi_{\lambda'} + C \phi} \leq t\psi_{\lambda} + (1 - t)\psi_{\lambda'} + C \leq \psi_{\lambda + (1 - t)\lambda'} + C.$$ 

Thus from the definition of the projection operator,

$$tP_{\psi_{\lambda} + C \phi} + (1 - t)P_{\psi_{\lambda'} + C \phi} \leq P_{\psi_{\lambda + (1 - t)\lambda'} + C \phi},$$

which means that $P_{\psi_{\lambda} + C \phi}$ is concave in $\lambda$ for all $C$. Since $P_{\psi_{\lambda} + C \phi}$ increases to $P_{[\psi_{\lambda}]} \phi$, and an increasing sequence of concave functions is concave, we get that $P_{[\psi_{\lambda}]} \phi$ is concave, and because of the monotonicity of the upper semicontinuous regularization it follows that $P_{[\psi_{\lambda}]} \phi^{\ast} = \phi_{\lambda}$ also is concave. The other properties of a test curve are immediate.

Clearly $\phi_{-\infty} = \phi$, so that $\phi_{\lambda}$ is maximal follows from Theorem 4.9. 

\[\square\]
The Legendre transform and geodesic rays

If \( f \) is a convex function in the real variable \( \lambda \), the set of subderivatives of \( f \), denoted by \( \Delta f \), is the set of \( t \in \mathbb{R} \) such that \( f(\lambda) - t\lambda \) is bounded from below. If \( f \) happens to be differentiable, then the set subderivatives coincides with the image of the derivative of \( f \). By convexity of \( f \), the set of subderivatives is convex, i.e. an interval. Recall that the Legendre transform of \( f \), here denoted by \( \hat{f} \), is the function on \( \Delta f \), defined as

\[
\hat{f}(t) := \sup_{\lambda} \{ t\lambda - f(\lambda) \}.
\]

Since \( \hat{f} \) is defined as the supremum of the linear functions \( t\lambda - f(\lambda) \), it follows that \( \hat{f} \) is convex.

If \( f \) is concave instead of convex, then of course \( -f \) is convex, and one can define the Legendre transform of \( f \), also denoted by \( \hat{f} \), as the Legendre transform of \( -f \), i.e.

\[
\hat{f}(t) := \sup_{\lambda} \{ f(\lambda) + t\lambda \},
\]

which is thus convex.

Definition 6.1. The Legendre transform of a test curve \( \psi_\lambda \), denoted by \( \hat{\psi}_t \), is given by

\[
\hat{\psi}_t := (\sup_{\lambda \in \mathbb{R}} \{ \psi_\lambda + t\lambda \})^*,
\]

where \( t \in [0, \infty) \).

Recall that the * means that we are taking the upper semicontinuous regularization of the supremum.

Lemma 6.2. Let \( \psi_\lambda \) be any test curve (not necessarily maximal). Then the Legendre transform \( \hat{\psi}_t \) is locally bounded for all \( t \), and the map \( t \mapsto \hat{\psi}_t \) is a subgeodesic ray emanating from \( \psi_{-\infty} \).

Proof. By assumption, for some \( \lambda \), \( \psi_\lambda \) is locally bounded, and trivially \( \hat{\psi}_t \geq \psi_\lambda + t\lambda \), thus \( \hat{\psi}_t \) is locally bounded. It is clear that for a fixed \( \lambda \), the curve \( \psi_\lambda + t\lambda \) is a subgeodesic. Clearly \( \sup_{\lambda \in \mathbb{R}} \{ \psi_\lambda + t\lambda \} \) is convex and Lipschitz in \( t \), and the same is easily seen to hold for \( \psi_\lambda \). Thus \( \hat{\psi}_t \) is upper semicontinuous in the \( X \) directions and Lipschitz in \( t \), which implies that it is upper semicontinuous on the whole product space. Therefore \( \hat{\psi}_t \) coincides with the usc regularization on the product space of \( \sup_{\lambda \in \mathbb{R}} \{ \psi_\lambda + t\lambda \} \). Taking the upper semicontinuous regularization of the supremum of subgeodesics yields a subgeodesic, as long as it is bounded from above. We observed above that \( \psi_\lambda \leq \psi_{-\infty} \). Now for some constant \( C, \psi_C \equiv -\infty \). It follows that \( \hat{\psi}_t \leq \psi_{-\infty} + tC \), so it is bounded from above and thus it is a subgeodesic.

Finally by definition \( \hat{\psi}_0 = (\sup_{\lambda} (\psi_\lambda))^* \), which clearly is equal to \( \psi_{-\infty} \) since \( \psi_\lambda \leq \psi_{-\infty} \) (\( \psi_\lambda \) being decreasing in \( \lambda \)) and \( \psi_{-\infty}^* = \psi_{-\infty} \).

One can also consider the inverse Legendre transform, going from subgeodesic rays to concave curves of positive metrics.
**Definition 6.3.** The Legendre transform of a subgeodesic ray $\phi_t$, $t \in [0, \infty)$, denoted by $\hat{\phi}_\lambda$, $\lambda \in \mathbb{R}$, is defined as

$$\hat{\phi}_\lambda := \inf_{t \in [0, \infty)} \{ \phi_t - t\lambda \}.$$  

It follows from Kiselman’s minimum principle (see [23]) that for any $\lambda \in \mathbb{R}$, $\hat{\phi}_\lambda$ is a positive metric (we would like to thank Bo Berndtsson for this observation). Furthermore it is clear that $\hat{\phi}_\lambda$ is concave and decreasing in $\lambda$. From the involution property of the (real) Legendre transform it follows that the Legendre transform of $\hat{\phi}_\lambda$ is $\phi_t$, thus any subgeodesic ray is the Legendre transform of a concave curve of positive metrics.

The goal of this section is to prove that if $\psi_\lambda$ is a maximal test curve then the Legendre transform $\hat{\psi}_t$ of $\psi_\lambda$ is a weak geodesic ray emanating from $\psi_{-\infty}$. By Lemma 6.2 we know $\hat{\psi}_t$ is a subgeodesic ray emanating from $\psi_{-\infty}$. What remains then is to show that if $\psi_\lambda$ is maximal then the Aubin-Mabuchi energy $E(\hat{\psi}_t, \hat{\psi}_0)$ is linear in $t$, which we now do with an approximation argument.

For $N \in \mathbb{N}$ consider the approximation $\hat{\psi}_t^N$ to $\hat{\psi}_t$, given by

$$\hat{\psi}_t^N := \sup_{k \in \mathbb{Z}} \{ \psi_{k2^{-N}} + tk2^{-N} \}.$$  

Since $\psi_\lambda$ is concave it is continuous in $\lambda$ at all points such that $\psi_\lambda(x) > -\infty$. From the continuity it follows that $\hat{\psi}_t^N$ will increase pointwise to $\hat{\psi}_t$ a.e. as $N$ tends to infinity. Also let $\hat{\psi}_t^{N,M}$ denote the curve

$$\hat{\psi}_t^{N,M} := \sup_{k \in \mathbb{Z}, k \leq M} \{ \psi_{k2^{-N}} + tk2^{-N} \}.$$  

Once again, $\hat{\psi}_t^N$ and $\hat{\psi}_t^{N,M}$ are all locally bounded.

**Lemma 6.4.** Let $M < M'$ be two integers. Then

$$\hat{\psi}_t^{N,M'} = \psi_{M2^{-N}} + tM'2^{-N}$$  

implies that

$$\hat{\psi}_t^{N,M} = \psi_{M2^{-N}} + tM2^{-N}.$$  

**Proof.** Certainly $f(\lambda) := \psi_\lambda(x) + t\lambda$ is concave in $\lambda$. If

$$\hat{\psi}_t^{N,M} > \psi_{M2^{-N}} + tM2^{-N}$$  

at $x$, then $f$ would be strictly decreasing at $\lambda = M2^{-N}$, so by concavity we would get that $f(M'2^{-N}) < f(M2^{-N}) < \hat{\psi}_t^{N,M}(x)$, which would be a contradiction. \hfill $\Box$

**Lemma 6.5.** If $\psi_\lambda$ is a maximal test curve then

$$t2^{-N} \int_X MA(\psi_{(M+1)2^{-N}}) \leq E(\hat{\psi}_t^{N,M+1}, \hat{\psi}_t^{N,M}) \leq t2^{-N} \int_X MA(\psi_{M2^{-N}}).$$
Proof. By Lemma 6.4 it follows that \( \hat{\psi}_t^{N,M} = \psi_{M2^{-N}} + tM2^{-N} \) on the support of \( \hat{\psi}_t^{N,M+1} - \hat{\psi}_t^{N,M} \) and thus Lemma 3.9 yields
\[
\mathcal{E}(\hat{\psi}_t^{N,M+1}, \hat{\psi}_t^{N,M}) = \mathcal{E}(\max\{\psi_{M2^{-N}}, \psi_{(M+1)2^{-N}} + t2^{-N}\}, \psi_{M2^{-N}}).
\]
(20)
Since we assumed that \( \psi_{\lambda} \) was maximal we get that \( \psi_{(M+1)2^{-N}} \) is maximal with respect to \( \psi_{M2^{-N}} \), and thus the lemma follows immediately from Lemma 4.11.

Let \( \psi_{\lambda} \) be a maximal test curve, and let \( F(\lambda) \) denote the function
\[
F(\lambda) := \int_X MA(\psi_{\lambda}).
\]
Whenever \( \lambda < \lambda' \), \( \psi_{\lambda'} \leq \psi_{\lambda} \) and therefore it follows from Theorem 3.7 that \( F(\lambda) \) is decreasing in \( \lambda \), hence \( F(\lambda) \) is Riemann integrable.

Proposition 6.6. If \( \psi_{\lambda} \) is a maximal test curve then
\[
\mathcal{E}(\hat{\psi}_t, \hat{\psi}_0) = -t \int_{\lambda=-\infty}^{\infty} \lambda dF(\lambda).
\]
(21)

Proof. Suppose first \( m \in \mathbb{Z} \) is such that \( \psi_m = \psi_{-\infty} \). For a given \( N \in \mathbb{N} \) set \( M = m2^N \). Then
\[
\hat{\psi}_t^{N,M} = \psi_{-\infty} + tm = \hat{\psi}_0 + tm.
\]
By repeatedly using the cocycle property of the Aubin-Mabuchi energy in combination with Lemma 6.5 we get that
\[
t \sum_{k>M} 2^{-N} F((k+1)2^{-N}) \leq \mathcal{E}(\hat{\psi}_t^{N}, \hat{\psi}_t^{N,M}) \leq t \sum_{k>M} 2^{-N} F(k2^{-N}).
\]
(22)
We noted above that \( \hat{\psi}_t^{N} \) increases pointwise to \( \hat{\psi}_t \) a.e. as \( N \) tends to infinity. By the continuity of the Aubin-Mabuchi energy under a.e. pointwise increasing sequences \( (3.6) \),
\[
\mathcal{E}(\hat{\psi}_t, \hat{\psi}_0 + tm) = t \int_{\lambda=m}^{\infty} \lambda F(\lambda) d\lambda,
\]
since both the left- and the right-hand side of \( (22) \) converges to this. Again using the cocycle property we get that
\[
\mathcal{E}(\hat{\psi}_t, \hat{\psi}_0 + tm) = \mathcal{E}(\hat{\psi}_t, \hat{\psi}_0 + tm) + \mathcal{E}(\hat{\psi}_0 + tm, \hat{\psi}_0) = t \int_{\lambda=m}^{\infty} \lambda F(\lambda) d\lambda + tm \int_X MA(\psi_{-\infty}) = t \int_{\lambda=m}^{\infty} \lambda F(\lambda) d\lambda + tm F(m).
\]
(23)
Since by our assumption the measure \( dF \) is zero on \((-\infty, m)\), integration by parts yields
\[
-t \int_{\lambda=-\infty}^{\infty} \lambda dF(\lambda) = -\lambda F(\lambda) \bigg|_{\lambda=m}^{\infty} + \int_{\lambda=m}^{\infty} F(\lambda) d\lambda = tm F(m) + \int_{\lambda=m}^{\infty} F(\lambda) d\lambda.
\]
(24)
The proposition follows from combining equation \( (23) \) and equation \( (24) \).
Theorem 6.7. The Legendre transform \( \hat{\psi}_t \) of a maximal test curve \( \psi_\lambda \) is a weak geodesic ray emanating from \( \psi_{-\infty} \).

Proof. That \( \hat{\psi}_t \) is a subgeodesic emanating from \( \psi_{-\infty} \) was proved in Lemma 6.2. According to Proposition 6.6 the energy \( E(\hat{\psi}_t, \hat{\psi}_0) \) is linear in \( t \), and therefore by Lemma 3.11 we get that \( \hat{\psi}_t \) is a geodesic ray.

These weak geodesics are continuous in \( \phi \) in the following sense:

Proposition 6.8. Let \( \psi_\lambda \) be a test curve and \( \phi, \phi' \in H(L) \). Suppose \( \phi_\lambda \) is the maximal curve of \( \phi \) (with respect to \( \psi_\lambda \)) and similarly for \( \phi'_\lambda \). If \( ||\phi - \phi'||_\infty < C \) then

\[
||\hat{\phi}_t - \hat{\phi'}_t||_\infty < C \text{ for all } t.
\]

Proof. We claim that \( ||\phi_\lambda - \phi'_\lambda||_\infty < C \) for all \( \lambda \). But this is clear since \( \phi \leq \phi' \) implies that \( \phi_\lambda \leq \phi'_\lambda \) for all \( \lambda \). It is also clear that \( (\phi + C)_\lambda = \phi_\lambda + C \) when \( C \) is a constant. Now we noted above that \( \phi \leq \phi' \) implies that \( \phi_\lambda \leq \phi'_\lambda \) for all \( \lambda \), and so it follows that \( \hat{\phi}_t \leq \hat{\phi'}_t \) for all \( t \). We also noted that \( (\phi + C)_\lambda = \phi_\lambda + C \) when \( C \) is a constant, so consequently \( \hat{\phi} + C_t = \hat{\phi}_t + C \) which proves the lemma.

Let \( [\psi_\lambda] \) be an analytic test configuration, and let \( \phi_\lambda \) be an associated maximal test curve. Then \( [\phi_\lambda] \) defines a new analytic test configuration. This could possibly differ from \( [\psi_\lambda] \), but the following proposition tells us that the associated geodesic rays are the same.

Proposition 6.9. Let \( \phi' \in H(L) \). Then the Legendre transform of \( \phi'_{[\phi_\lambda]} \) coincides with the Legendre transform of \( \phi'_{[\phi_\lambda]} \).

Proof. Since \( \phi'_{[\phi_\lambda]} \sim \phi_\lambda \) we get that \( \phi'_{[\phi_\lambda]} = \phi'_{[\phi_\lambda]} \), thus without loss of generality we can assume that \( \phi' = \phi \). Recall that the critical value \( \lambda_c \) was defined as

\[
\lambda_c := \inf \{ \lambda : \phi_\lambda \equiv -\infty \}.
\]

If \( \lambda < \lambda_c \) there exists a \( \lambda' \) such that \( \lambda < \lambda' < \lambda_c \), and thus by the assumption \( \phi_{\lambda'} \) has small unbounded locus. Let \( C \) be a constant less than \( \lambda \) such that \( \phi_{\lambda'} = \phi \). By concavity it follows that

\[
\phi_\lambda \geq r \phi + (1 - r)\phi_{\lambda'},
\]

where \( 0 < r < 1 \), is chosen such that

\[
\lambda = rC + (1 - r)\lambda'.
\]

If we let

\[
\rho := r \phi + (1 - r)\phi_{\lambda'},
\]

by the multilinearity of the Monge-Ampère operator it follows that \( MA(\rho) \) dominates the volume form \( r^n MA(\phi) \). Furthermore \( \rho \) has small unbounded locus and is more singular than \( \phi_\lambda \). Thus by Proposition 4.6 we get that

\[
P_{\phi_\lambda + C} \phi \leq \phi_\lambda.
for any constant $C$ and therefore

$$\phi_{[\phi,\lambda]} = \phi_{\lambda}, \quad (25)$$

whenever $\lambda < \lambda_c$. If $\lambda > \lambda_c$ then clearly equation \((25)\) holds as well since both sides are identically equal to minus infinity. It follows that for any $\epsilon > 0$,

$$\phi_{\lambda} \leq \phi_{[\phi,\lambda]} \leq \phi_{\lambda - \epsilon},$$

which implies that

$$\hat{(\phi_{\lambda})}_t \leq \hat{(\phi_{[\phi,\lambda]})}_t \leq \hat{(\phi_{\lambda - \epsilon})}_t = \hat{(\phi_{\lambda})}_t + \epsilon t.$$ 

Since $\epsilon > 0$ was arbitrary the proposition follows. \qed

7 Filtrations of the ring of sections

First we recall what is meant by a filtration of a graded algebra.

**Definition 7.1.** A filtration $F$ of a graded algebra $\bigoplus_k V_k$ is a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,

$$F : (t, k) \mapsto F_t V_k,$$

such that for any $k$, $F_t V_k$ is a family of subspaces of $V_k$ that is decreasing and left-continuous in $t$.

In [9] Boucksom-Chen consider certain filtrations which behaves well with respect to the multiplicative structure of the algebra. They give the following definition.

**Definition 7.2.** Let $F$ be a filtration of a graded algebra $\bigoplus_k V_k$. We shall say that

(i) $F$ is multiplicative if

$$(F_t V_k)(F_s V_m) \subseteq F_{t+s} V_{k+m}$$

for all $k, m \in \mathbb{N}$ and $s, t \in \mathbb{R}$.

(ii) $F$ is (linearly) bounded if there exists a constant $C$ such that $F_{-kC} V_k = V_k$ and $F_{kC} V_k = \{0\}$ for all $k$.

The goal in this section is to associate an analytic test configuration $\phi^F_{\lambda}$ to any bounded multiplicative filtration of the section ring $R(L) = \bigoplus_k H^0(kL)$.

Let $\phi \in \mathcal{H}(L)$, and let $dV$ be some smooth volume form on $X$ with unit mass. This gives the $L^2$-scalar product on $H^0(kL)$ by letting

$$(s, t)_k := \int_X s(z)\overline{t(z)}e^{-k\phi(z)}dV(z).$$

For any $\lambda \in \mathbb{R}$ let $\{s_{i,\lambda}\}$ be an orthonormal basis for $F_{k\lambda} H^0(kL)$ and define

$$\phi_{k,\lambda} := \frac{1}{k} \ln(\sum |s_{i,\lambda}|^2),$$

which is a positive metric on $L$. 

Lemma 7.3. For any \( \lambda \), the sequence of metrics \( \phi_{k,\lambda} \) converges to a limit as \( k \) tends to infinity, and the usc regularization of the limit

\[
\phi_\lambda^* := \left( \lim_{k \to \infty} \phi_{k,\lambda} \right)^*
\]

is a positive metric.

Proof. Since

\[
K_\lambda(z, w) := \sum_i s_i,\lambda(z) \overline{s_i,\lambda(w)}
\]

is a reproducing kernel of \( \mathcal{F}_{k\lambda} H^0(kL) \) with respect to \( (\cdot, \cdot)_{k\phi} \), as for the full Bergman kernel we have the following useful characterization

\[
\sum |s_i,\lambda(z)|^2 = \sup \{ |s|^2 : s \in \mathcal{F}_{k\lambda} H^0(kL), \|s\|_{k\phi}^2 \leq 1 \}. \tag{26}
\]

Let \( \|s\|_\infty^2 := \sup_{z \in X} \{ |s(z)|^2 e^{-k\phi} \} \) and define

\[
F_{k,\lambda}(z) := \sup \{ |s(z)|^2 : s \in \mathcal{F}_{k\lambda} H^0(kL), \|s\|_\infty^2 \leq 1 \}.
\]

We trivially have the upper bound

\[
F_{k,\lambda}(z) \leq e^{-k\phi(z)}.
\]

It follows that

\[
\left( \frac{1}{k} \ln F_{k,\lambda} \right)^* = \left( \sup \left\{ \frac{1}{k} \ln |s|^2 : s \in \mathcal{F}_{k\lambda} H^0(kL), \|s\|_{k\phi}^2 \leq 1 \right\} \right)^*
\]

is a positive metric. Let \( \lambda \) be fixed, pick a point \( z \in X \), and let for all \( k, s_k \in \mathcal{F}_{k\lambda} H^0(kL) \) be such that \( \|s_k\|_\infty = 1 \) and

\[
F_{k,\lambda}(z) = |s_k(z)|^2.
\]

Since the product \( s_k s_m \) lies in \( \mathcal{F}_{(k+m)\lambda} H^0((k + m)L) \) by the multiplicativity of \( \mathcal{F} \), and \( \|s_k s_m\|_\infty \leq \|s_k\|_\infty \|s_m\|_\infty \), we get that

\[
F_{k+m,\lambda}(z) \geq F_{k,\lambda}(z) F_{m,\lambda}(z), \tag{27}
\]

i.e. the map \( k \mapsto F_{k,\lambda}(z) \) is supermultiplicative. The existence of a limit

\[
\lim_{k \to \infty} \frac{1}{k} \ln F_{k,\lambda}(z)
\]

thus follows from Fekete’s lemma (see e.g. [38]). Since we assumed that \( dV \) had unit mass, we get that for any section \( s \)

\[
\|s\|_{k\phi}^2 \leq \|s\|_{\infty}^2,
\]

and thus by equation \(26\)

\[
\sum |s_{i,\lambda}(z)|^2 \geq F_{k,\lambda}(z).
\]
On the other hand, by the Bernstein-Markov property of any volume form $dV$ we have that for any $\epsilon > 0$ there exists a constant $C_\epsilon$ so that
\[ ||s||^2_\infty \leq C_\epsilon e^{ck} ||s||^2_{k_\phi}, \]
and thus
\[ \sum |s_{i,\lambda}(z)|^2 \leq C_\epsilon e^{ck} F_{k,\lambda}(z), \] (28)
(see [38]). It follows that the difference $\phi_{k,\lambda}(z) - \frac{1}{k} \ln F_{k,\lambda}(z)$ tends to zero as $k$ tends to infinity, thus the convergence of $\phi_{k,\lambda}$ follows.

By the supermultiplicativity we get that for any $k \in \mathbb{N}$
\[ \frac{1}{k} \ln F_{k,\lambda} \leq \lim_{l \to \infty} \frac{1}{l} \ln F_{l,\lambda} = \lim_{l \to \infty} \phi_{l,\lambda}, \]
and thus
\[ \left( \frac{1}{k} \ln F_{k,\lambda} \right)^* \leq \left( \lim_{l \to \infty} \phi_{l,\lambda} \right)^* =: \phi_F^\lambda. \] (29)
On the other hand, clearly
\[ \lim_{l \to \infty} \phi_{l,\lambda} \leq \sup_k \{ \left( \frac{1}{k} \ln F_{k,\lambda} \right)^* \}, \]
and it follows that
\[ \phi_F^\lambda = \left( \sup_k \{ \left( \frac{1}{k} \ln F_{k,\lambda} \right)^* \} \right)^* \]
so $\phi_F^\lambda$ is indeed a positive metric.

**Remark 7.4.** Since all volume forms $dV$ on $X$ are equivalent, the limit $\phi_\lambda$ does not depend on the choice of volume form $dV$.

**Lemma 7.5.** We have that
\[ \phi_{k,\lambda} \leq \phi_F^\lambda + \epsilon(k), \]
where $\epsilon(k)$ is a constant independent of $\lambda$ that tends to zero as $k$ tends to infinity.

**Proof.** By combining the inequalities (28) and (29) from the proof of the the previous lemma we get that for any $\epsilon > 0$ there exists a constant $C_\epsilon$ independent of $\lambda$ such that
\[ \phi_{k,\lambda} \leq \phi_F^\lambda + \epsilon + \frac{1}{k} \ln C_\epsilon. \]
This yields the lemma.

**Proposition 7.6.** The map $\lambda \mapsto \phi_F^\lambda$ is a test curve.

**Proof.** Let $\lambda$ be such that $F_{k,\lambda} \mathcal{H}^0(kL) = \mathcal{H}^0(kL)$ for all $k$. Then $\phi_{k,\lambda}$ is the usual Bergman metric, and by the result on Bergman kernel asymptotics due to Bouche-Catlin-Tian-Zelditch (see Section 3) we get that $\phi_{k,\lambda}$ converges to $\phi$. Trivially we see that if $F_{k,\lambda} \mathcal{H}^0(kL) = \{0\}$ for all $k$ then $\phi_F^\lambda \equiv -\infty$. By the boundedness of the filtration we thus have $\phi_F^\lambda \equiv \phi$ for $\lambda < -C$ and $\phi_F^\lambda \equiv -\infty$ for $\lambda > C$. 


7  FILTRATIONS OF THE RING OF SECTIONS

By the multiplicativity of the filtration we get that \( \phi_\lambda \equiv -\infty \) iff for all \( k \),
\[
\mathcal{F}_{k\lambda} H^0(kL) = \{0\}.
\]

Pick a \( \lambda \) such that \( \phi^F_\lambda \not\equiv -\infty \), then for some \( k \), \( \mathcal{F}_{k\lambda} H^0(kL) \) is non-trivial. From Lemma 7.5 it follows that \( \phi^F_\lambda \) has small unbounded locus since \( \phi_{k,\lambda} \) has small unbounded locus.

It remains to prove concavity. Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) and let \( t \) be a rational point in the unit interval. Let \( m \) be a natural number such that \( mt \) is an integer. Given a point \( z \in X \), let \( s_1 \in \mathcal{F}_{k\lambda_1} H^0(kL) \) and \( s_2 \in \mathcal{F}_{k\lambda_2} H^0(kL) \) be two sections with \( ||s_1|| \infty = ||s_2|| \infty = 1 \) such that
\[
F_{k,\lambda_1} = |s_1(z)|^2
\]
and
\[
F_{k,\lambda_2} = |s_2(z)|^2.
\]

By the multiplicativity of the filtration we have that
\[
s_1^{mt} s_2^{m(1-t)} \in \mathcal{F}_{mk(t\lambda_1+(1-t)\lambda_2)} H^0(mkL),
\]
and trivially \( ||s_1^{mt} s_2^{m(1-t)}|| \infty \leq 1 \). It follows that
\[
F_{mk,t\lambda_1+(1-t)\lambda_2}(z) \geq F_{k,\lambda_1}(z)^{mt} F_{k,\lambda_2}(z)^{m(1-t)}.
\]

Taking the logarithm on both sides, dividing by \( mk \), and taking the limit yields that
\[
\phi^F_{t\lambda_1+(1-t)\lambda_2} \geq t\phi^F_{\lambda_1} + (1-t)\phi^F_{\lambda_2}
\]
except possibly on the pluripolar set where the limits are not equal to their upper semi-continuous regularization. But it is easily seen that if a positive metric is larger than or equal to another except on a pluripolar set then it is in fact larger than or equal on the whole space. Thus we get that (30) holds on the whole of \( X \). Recall that \( t \) was assumed to be rational. If \( \lambda_1 \leq \lambda_2 \), the left-hand side of (30) is decreasing in \( t \) since clearly \( \phi^F_\lambda \) is decreasing in \( \lambda \). The right-hand side of (30) is continuous in \( t \), so it follows that the equation (30) holds for all \( t \in (0, 1) \), i.e. \( \phi^F_\lambda \) is concave in \( \lambda \).

Lemma 7.7. For any two \( \phi, \psi \in \mathcal{H}(L) \) and any \( \lambda \in \mathbb{R} \) we have \( \phi^F_\lambda \sim \psi^F_\lambda \).

Proof. Assume that \( \phi \leq \psi \), then it is immediate that for all \( k \) and \( \lambda \) we have that \( \phi_{k,\lambda} \leq \psi_{k,\lambda} \), and we thus get that \( \phi^F_\lambda \leq \psi^F_\lambda \). Also it is clear that \( (\phi+C)_{k,\lambda} = \phi_{k,\lambda}+C \). When combining these two facts we get the lemma.

Definition 7.8. We call the map \( \lambda \mapsto [\phi^F_\lambda] \) the analytic test configuration associated to the filtration \( \mathcal{F} \).

So by the previous lemma this analytic test configuration depends only on \( \mathcal{F} \) and not on the choice of \( \phi \in \mathcal{H}(L) \). Our next goal is to show the curve \( \phi^F_\lambda \) is maximal for \( \lambda < \lambda_c \), for which we will need a Skoda-type division theorem.
Theorem 7.9. Let $L$ be an ample line bundle. Assume that $L$ has a smooth positive metric $\phi$ with the property that $dd^c \phi \geq dd^c \phi_{K_X}$ for some smooth metric $\phi_{K_X}$ on the canonical bundle $K_X$. Let $\{s_i\}$ be a finite collection of holomorphic sections of $L$ and $m > n + 2$ where $n = \dim X$.

Suppose $s$ is a section of $mL$ such that
\[
\int_X \frac{|s|^2}{(\sum |s_i|^2)^m} dV < \infty.
\]
Then there exists sections $h_\alpha \in H^0((n + 1)L)$ such that
\[
s = \sum_\alpha h_\alpha s^\alpha,
\]
where $\alpha$ is a multiindex $\alpha = (\alpha_i)$ with $\sum_i \alpha_i = m - n - 1$, and $s^\alpha$ are the monomials $s^\alpha := \prod_is_i^{\alpha_i}$.

Proof. Let $k$ be an integer such that $n + 2 \leq k \leq m$. Then given a section $t \in H^0(kL)$ with
\[
\int_X \frac{|t|^2}{(\sum |s_i|^2)^k} dV < \infty
\]
an application of the Skoda division theorem [37, Thm. 2.1] yields sections $\{t_i\}$ of $(k - 1)L$ such that $t = \sum t_is_i$ and
\[
\int_X \frac{|t_i|^2}{(\sum |s_i|^2)^{k-1}} dV < \infty.
\]
(To apply the cited theorem replace $F, E, \psi, \eta$ with $kL - K_X, L, k\phi - \phi_{K_X}, \phi$ respectively and replace $\alpha q$ with $k - 1 \geq n + 1$.)

Now we first apply the above with $k = m$ to the section $s$, and then apply again with $k = m - 1$ to each of the sections $t_i$. Repeating this process with $k = m, m - 1, \ldots, n + 2$ we see that $s$ can be written as a linear sum of monomials in the $s_i$ as required. \qed

Proposition 7.10. For $\lambda$ less than the critical value $\lambda_c$ we have that
\[
\phi_\lambda^K = \lim_{k \to \infty} \phi_{[\phi_{k,\lambda}]}.
\]

Proof. Let $\phi_k := \phi_{k,-\infty}$, i.e. the Bergman metric $1/k \ln(\sum |s_i|^2)$, where $\{s_i\}$ is an orthonormal basis for the whole space $H^0(kL)$ with respect to $(\cdot, \cdot)_{k\phi}$. By the Bernstein-Markov property of any volume form $dV$ (see e.g. [38]), or simply the maximum principle, we get that
\[
\phi_k \leq \phi + \epsilon_k,
\]
where $\epsilon_k$ tends to zero as $k$ tends to infinity. Since $\phi_{k,\lambda}$ is decreasing in $\lambda$, the inequality [31] still holds when $\phi_k$ is replaced by $\phi_{k,\lambda}$, i.e. $\phi_{k,\lambda} - \epsilon_k \leq \phi$. Therefore $\phi_{k,\lambda} - \epsilon_k$ belongs to the class of metrics the supremum of which yields $P_{[\phi_{k,\lambda}]} \phi$, and thus clearly
\[
\phi_{k,\lambda} \leq P_{[\phi_{k,\lambda}]} \phi + \epsilon_k.
\]
Letting \( k \) tend to infinity yields
\[
\phi_{\lambda}^F \leq \left( \lim_{k \to \infty} P_{\phi_{k,\lambda}^F} \phi \right)^*.
\]

For the other inequality it is enough to show that for any constant \( C \),
\[
P_{\phi_{k,\lambda} + C} \phi \leq \phi_{\lambda}^F. \tag{32}
\]

By the assumption that \( \lambda < \lambda_c \) we have that \( \phi_{\lambda}^F \neq -\infty \). Let \( \psi \) be a positive metric dominated by both \( \phi_{k,\lambda} + C \) and \( \phi \), where \( k \) is large enough so that \( kL \) fulfills the requirements of Theorem 7.9. We denote by \( J(\psi) \) the multiplier ideal sheaf of germs of holomorphic functions locally integrable against \( e^{-k\psi} \). Let \( \{s_i\} \) be an orthonormal basis of \( H^0(kL \otimes J(\psi)) \), and denote by \( \psi_k \) the Bergman metric
\[
\psi_k := \frac{1}{k} \ln(\sum |s_i|^2).
\]

By Theorem 3.2 we have that
\[
\psi \leq \psi_k + \delta_k
\]
where \( \delta_k \) tends to zero as \( k \) tends to infinity, and \( \psi_k \) converges pointwise to \( \psi \). If \( s \) lies in \( H^0(kL \otimes J(k\psi)) \), specifically we must have that
\[
\int_X |s|^2 \sum |s_{i,\lambda}|^2 dV < \infty,
\]
since we assumed that \( \psi \) was dominated by \( \phi_{k,\lambda} + C = 1/k \ln(\sum |s_{i,\lambda}|^2) + C \). Similarly if \( s \) lies in \( H^0(kmL \otimes J(km\psi)) \) we have
\[
\int_X \left( \sum |s_{i,\lambda}|^2 \right)^m dV < \infty.
\]

From Theorem 7.9 applied to the sections \( \{s_{i,\lambda}\} \) it thus follows that
\[
s = \sum h_{\alpha} s^\alpha,
\]
where \( h_{\alpha} \in H^0(k(n+1)L) \), and the \( s^\alpha \) are monomials in the \( \{s_{i,\lambda}\} \) of degree \( m-n-1 \). Because of the multiplicativity of the filtration each \( s^\alpha \) lies in \( F_{k(m-n-1)\lambda} H^0(k(m-n-1)L) \), and by the boundedness of the filtration we also have that each \( h_{\alpha} \) lies in \( F_{-k(n+1)C} H^0(k(n+1)L) \) for some fixed constant \( C \). We thus get that \( H^0(kmL \otimes J(km\psi)) \) is contained in
\[
(F_{-k(n+1)C} H^0(k(n+1)L)) \cap (F_{k(m-n-1)\lambda} H^0(k(m-n-1)L)) \subseteq F_{k(m-n-1)\lambda-k(n+1)C} H^0(kmL). \tag{33}
\]

Since we assumed that \( \psi \leq \phi \) we have that \( \psi_{km} \) is less than or equal to the Bergman metric using an orthonormal basis for \( H^0(kmL \otimes J(km\psi)) \) with respect to \( \phi \). Because of (33) this Bergman metric is certainly less than or equal to \( \phi_{km,\lambda'} \), where
\[
\lambda' := \frac{1}{km} (k(m-n-1)\lambda - k(n+1)C).
\]
Hence
\[ \psi_{km} \leq \phi_{km,\lambda'} \]
On the other hand, by Lemma 7.5 we have that
\[ \phi_{km,\lambda'} \leq \phi_{\lambda'}^F + \epsilon (km) \]
where \( \epsilon (km) \) is a constant independent of \( \lambda' \) that tends to zero as \( km \) tends to infinity.
Since \( \lambda' \) tends to \( \lambda \) as \( m \) tends to infinity we get that \( \psi \leq \lim_{\lambda' \to \lambda} \phi_{\lambda'}^F \), and thus by Lemma 5.2 \( \psi \leq \phi_{\lambda'}^F \). Taking the supremum over all such \( \psi \) completes the proof.

**Corollary 7.11.** The test curve \( \phi_{\lambda'}^F \) is maximal for \( \lambda < \lambda_c \) and its Legendre transform is a geodesic ray.

**Proof.** Theorem 4.9 tells us that \( \phi_{[\phi_{k,\lambda}]} \) is maximal with respect to \( \phi = \phi_{-\infty} \). By Lemma 4.7 it follows that this is true for the limit \( \phi_{\lambda'}^F = \lim_{k \to \infty} \phi_{[\phi_{k,\lambda}]} \) as well. Let \( \phi_{\lambda} \) be the test curve defined by \( \phi_{\lambda} := \phi_{\lambda'}^F \) for \( \lambda < \lambda_c \) and \( \phi_{\lambda} \equiv -\infty \) for \( \lambda \geq \lambda_c \). Then we get that \( \phi_{\lambda} \) is a maximal test curve, thus its Legendre transform is a geodesic ray.

On the other hand, for every \( \epsilon > 0 \) we have that
\[ \phi_{\lambda} \leq \phi_{\lambda'}^F \leq \phi_{\lambda - \epsilon} \]
and therefore
\[ \hat{\phi}_t \leq (\phi_{\lambda'}^F)_t \leq \hat{\phi}_t + \epsilon t. \]
Since \( \epsilon \) was arbitrary we get that the Legendre transform of \( \phi_{\lambda'}^F \) coincides with that of \( \phi_{\lambda} \), and thus it is a geodesic ray.

**Remark 7.12.** Given an analytic test configuration \([\psi_{\lambda}]\) there is a naturally associated filtration \( \mathcal{F} \) of the section ring, defined as
\[ \mathcal{F}_{k,\lambda} H^0(kL) := H^0(kL \otimes \mathcal{O}(k\psi_{\lambda})). \]
This filtration is bounded, but in general not multiplicative.

### 8 Filtrations associated to algebraic test configurations

We recall briefly Donaldson’s definition of a test configurations \([20, 21]\). In order to not confuse them with the our analytic test configurations, we will in this article refer to them as algebraic test configuration.

**Definition 8.1.** An algebraic test configuration \( \mathcal{T} \) for an ample line bundle \( L \) over \( X \) consists of:

(i) a scheme \( \mathcal{X} \) with a \( \mathbb{C}^\times \)-action \( \rho \),

(ii) a \( \mathbb{C}^\times \)-equivariant line bundle \( \mathcal{L} \) over \( \mathcal{X} \),

(iii) and a flat \( \mathbb{C}^\times \)-equivariant projection \( \pi : \mathcal{X} \rightarrow \mathbb{C} \) where \( \mathbb{C}^\times \) acts on \( \mathbb{C} \) by multiplication, such that \( \mathcal{L} \) is relatively ample, and such that if we denote by \( X_1 := \pi^{-1}(1) \), then \( \mathcal{L}|_{X_1} \rightarrow X_1 \) is isomorphic to \( rL \rightarrow X \) for some \( r > 0 \).
By rescaling we can without loss of generality assume that \( r = 1 \) in the definition. An algebraic test configuration is called a product test configuration if there is a \( C^\times \)-action \( \rho' \) on \( L \to X \) such that \( \mathcal{L} = L \times \mathbb{C} \) with \( \rho \) acting on \( L \) by \( \rho' \) and on \( \mathbb{C} \) by multiplication. An algebraic test configuration is called trivial if it is a product test configuration with the action \( \rho' \) being the trivial \( C^\times \)-action.

Since the zero-fiber \( X_0 := \pi^{-1}(0) \) is invariant under the action \( \rho \), we get an induced action on the space \( H^0(kL_0) \), also denoted by \( \rho \), where we have denoted the restriction of \( \mathcal{L} \) to \( X_0 \) by \( L_0 \). Specifically, we let \( \rho(\tau) \) act on a section \( s \in H^0(kL_0) \) by
\[
(\rho(\tau)(s))(x) := \rho(\tau)(s(\rho^{-1}(\tau)(x))).
\]
(34)

By standard theory any vector space \( V \) with a \( C^\times \)-action can be split into weight spaces \( V_{\lambda_i} \) on which \( \rho(\tau) \) acts as multiplication by \( \tau^{\lambda_i} \), (see e.g. [20]). The numbers \( \lambda_i \) with non-trivial weight spaces are called the weights of the action. Thus we may write \( H^0(kL_0) \) as
\[
H^0(kL_0) = \oplus \lambda \, V_{\lambda}
\]
with respect to the induced action \( \rho \).

In [27] Lem. 4] Phong-Sturm give the following linear bound on the absolute value of the weights.

**Lemma 8.2.** Given a test configuration there is a constant \( C \) such that
\[
|\lambda_i| < Ck
\]
whenever \( \dim V_{\lambda_i} > 0 \).

In [39] the second author showed how to get an associated filtration \( \mathcal{F} \) of the section ring \( \oplus_k H^0(kL) \) given a test configuration \( \overline{T} \) of \( L \) which we now recall.

First note that the \( C^\times \)-action \( \rho \) on \( \mathcal{L} \) via the equation (34) gives rise to an induced action on \( H^0(\mathcal{X}, k\mathcal{L}) \) as well as \( H^0(\mathcal{X} \setminus X_0, k\mathcal{L}) \), since \( \mathcal{X} \setminus X_0 \) is invariant. Let \( s \in H^0(k\mathcal{L}) \) be a holomorphic section. Then using the \( C^\times \)-action \( \rho \) we get a canonical extension \( \tilde{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L}) \) which is invariant under the action \( \rho \), simply by letting
\[
\tilde{s}(\rho(\tau)x) := \rho(\tau)s(x)
\]
(35)
for any \( \tau \in C^\times \) and \( x \in X \).

We identify the coordinate \( z \) with the projection function \( \pi(x) \), and we also consider it as a section of the trivial bundle over \( \mathcal{X} \). Exactly as for \( H^0(\mathcal{X}, k\mathcal{L}) \), \( \rho \) gives rise to an induced action on sections of the trivial bundle, using the same formula (34). We get that
\[
(\rho(\tau)z)(x) = \rho(\tau)(z(\rho^{-1}(\tau)x)) = \rho(\tau)(\tau^{-1}z(x)) = \tau^{-1}z(x),
\]
(36)
where we used that \( \rho \) acts on the trivial bundle by multiplication on the \( z \)-coordinate. Thus
\[
\rho(\tau)z = \tau^{-1}z,
\]
which shows that the section \( z \) has weight \(-1\).

By this it follows that for any section \( s \in H^0(kL) \) and any integer \( \lambda \), we get a section \( z^{-\lambda}\tilde{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L}) \), which has weight \( \lambda \).
Lemma 8.3. For any section \( s \in H^0(kL) \) and any integer \( \lambda \) the section \( z^{-\lambda} \bar{s} \) extends to a meromorphic section of \( k\mathcal{L} \) over the whole of \( \mathcal{X} \), which we also will denote by \( z^{-\lambda} \bar{s} \).

Proof. It is equivalent to saying that for any section \( s \) there exists an integer \( \lambda \) such that \( z^\lambda \bar{s} \) extends to a holomorphic section \( S \in H^0(X, k\mathcal{L}) \). By flatness, which was assumed in the definition of a test configuration, the direct image bundle \( \pi_* \mathcal{L} \) is in fact a vector bundle over \( \mathbb{C} \). Thus it is trivial, since any vector bundle over \( \mathbb{C} \) is trivial. Therefore there exists a global section \( s' \in H^0(X, k\mathcal{L}) \) such that \( s = s'|_X \). On the other hand, as for \( H^0(kL_0) \), \( H^0(X, k\mathcal{L}) \) may be decomposed as a direct sum of invariant subspaces \( W_\lambda \) such that \( \rho(\tau) \) restricted to \( W_\lambda \) acts as multiplication by \( \tau^\lambda \). Let us write

\[
S' = \sum S'_\lambda,
\]

where \( S'_\lambda \in W_\lambda \). Restricting the equation (37) to \( X \) gives a decomposition of \( s \),

\[
s = \sum s_\lambda,
\]

where \( s_\lambda := S'_\lambda|_X \). From (35) and the fact that \( S'_\lambda \) lies in \( W_\lambda \) we get that for \( x \in X \) and \( \tau \in \mathbb{C}^\times \) we have that

\[
\bar{s}_\lambda(\rho(\tau)(x)) = \rho(\tau) s_\lambda(x) = (\rho(\tau) S'_\lambda(\rho(\tau)(x))) = \tau^\lambda S'_\lambda(\rho(\tau)(x)),
\]

and therefore \( \bar{s}_\lambda = \tau^\lambda S'_\lambda \). Since trivially

\[
\bar{s} = \sum \bar{s}_\lambda,
\]

it follows that \( t^\lambda \bar{s} \) extends holomorphically as long as \( \lambda \geq \max -\lambda' \). \( \square \)

Definition 8.4. Given a test configuration \( \mathcal{T} \) we define a vector space-valued map \( \mathcal{F} \) from \( \mathbb{Z} \times \mathbb{N} \) by letting

\[
(\lambda, k) \mapsto \{ s \in H^0(kL) : z^{-\lambda} \bar{s} \in H^0(X, k\mathcal{L}) \} =: \mathcal{F}_\lambda H^0(kL).
\]

It is immediate that \( \mathcal{F}_\lambda \) is decreasing since \( H^0(X, k\mathcal{L}) \) is a \( \mathbb{C}[z] \)-module. We can extend \( \mathcal{F} \) to a filtration by letting

\[
\mathcal{F}_\lambda H^0(kL) := \mathcal{F}_{[\lambda]} H^0(kL)
\]

for non-integers \( \lambda \), thus making \( \mathcal{F} \) left-continuous. Since

\[
z^{-(\lambda + \lambda')} s_\lambda s'_\lambda = (z^{-\lambda} \bar{s})(z^{-\lambda'} \bar{s}') \in H^0(X, k\mathcal{L}) H^0(X, m\mathcal{L}) \subseteq H^0(X, (k + m)\mathcal{L})
\]

whenever \( s \in \mathcal{F}_\lambda H^0(kL) \) and \( s' \in \mathcal{F}_{\lambda'} H^0(kL) \), we see that

\[
(\mathcal{F}_{\lambda} H^0(kL))(\mathcal{F}_{\lambda'} H^0(mL)) \subseteq \mathcal{F}_{\lambda + \lambda'} H^0((k + m)L),
\]

i.e. \( \mathcal{F} \) is multiplicative.

Recall that we had the decomposition of \( H^0(kL_0) \) into weight spaces \( V_\lambda \).
Lemma 8.5. For each \( \lambda \), we have that
\[
\dim F_\lambda H^0(kL) = \sum_{\lambda' \geq \lambda} \dim V_{\lambda'}.
\]

Proof. We have the following isomorphism:
\[
(\pi_*kL)|_{\{0\}} \cong H^0(\mathcal{X}, kL)/zH^0(\mathcal{X}, kL),
\]
the right-to-left arrow being given by the restriction map, see e.g. [30]. Also, for \( k \gg 0 \),
\[
(\pi_*kL)|_{\{0\}} = H^0(kL_0),
\]
therefore we get that for large \( k \)
\[
H^0(kL_0) \cong H^0(\mathcal{X}, kL)/zH^0(\mathcal{X}, kL),
\]
(38)

We also had a decomposition of \( H^0(\mathcal{X}, kL) \) into the sum of its invariant weight spaces \( W_\lambda \). By Lemma 8.3 it is clear that a section \( S \in H^0(\mathcal{X}, kL) \) lies in \( W_\lambda \) if and only if it can be written as \( z^{-\lambda} \bar{s} \) for some \( s \in H^0(kL) \), in fact we have that \( s = S|_{\mathcal{X}} \). Thus we get that
\[
W_\lambda \cong F_\lambda H^0(kL),
\]
and by the isomorphism (38) then
\[
V_\lambda \cong F_\lambda H^0(kL)/F_{\lambda+1} H^0(kL).
\]
Thus we get
\[
\dim F_\lambda H^0(kL) = \sum_{\lambda' \geq \lambda} \dim V_{\lambda'}.
\]
(39)

Using Lemma 8.5 together with Lemma 8.2 shows that the filtration \( \mathcal{F} \) is bounded.

9 The geodesic rays of Phong and Sturm

In [27] Phong-Sturm show how to construct a weak geodesic ray, starting with a \( \phi \in \mathcal{H}(L) \) and an algebraic test configuration \( \mathcal{T} \) (see also [35] for how this works in the toric setting). In the previous section we showed how to associate an analytic test configuration \( [\phi_\lambda^F] \) to an algebraic test configuration, and thus get a weak geodesic using the Legendre transform of its maximal envelope. Recall by Proposition 6.9 this geodesic is the same as the Legendre transform of the original test curve \( \phi_\lambda^F \). The goal in this section is to prove that this ray coincides with the one constructed by Phong-Sturm.

To describe what we aim to show, recall that if \( V \) is a vector space with a scalar product, and \( \mathcal{F} \) is a filtration of \( V \), there is a unique decomposition of \( V \) into a direct sum of mutually orthogonal subspaces \( V_{\lambda_i} \) such that
\[
\mathcal{F}_\lambda V = \bigoplus_{\lambda_i \geq \lambda} V_{\lambda_i}.
\]
Furthermore we allow for $\lambda_i$ to be equal to $\lambda_j$ even when $i \neq j$, so we can assume that all the subspaces $V_{\lambda_i}$ are one dimensional. This additional decomposition is of course not unique, but it will not matter in what follows.

Let $\phi \in H(L)$ and $H^0(kL) = \oplus V_{\lambda_i}$ be the decomposition of $H^0(kL)$ with respect to the scalar product $(\cdot, \cdot)_{k\phi}$ coming from the volume form $(dd^c\phi)^n$. Consider next the filtration coming from an algebraic test configuration (note that then the collection of $\lambda_i$ will depend also on $k$ but we omit that from our notation) and define the normalized weights to be

$$\bar{\lambda}_i := \frac{\lambda_i}{k},$$

which form a bounded family by Lemma 8.2.

Now if $s_i$ is a vector of unit length in $V_{\lambda_i}$, then $\{s_i\}$ will be an orthonormal basis for $H^0(kL)$. Since the filtration $F$ encodes the $C^*$-action on $H^0(kL)$ it is easy to see that the basis $\{s_i\}$ is the same one as in \cite[Lem 7]{27}. In terms of the notation in the previous sections we have

$$\phi_{k,\lambda} = \sum_{\lambda_i \geq k\lambda} |s_i|^2 \quad \text{and} \quad \phi^F_{\lambda} = (\lim_{k \to \infty} \phi_{k,\lambda})^*.$$

**Definition 9.1.** Let

$$\Phi_k(t) := \frac{1}{k} \ln \left( \sum_{i} e^{t\lambda_i} |s_i|^2 \right)$$

The *Phong-Sturm* ray is the limit

$$\Phi(t) := \lim_{k \to \infty} (\sup_{t\geq k} \Phi_k(t))^*.$$

(40)

Our goal is the following:

**Theorem 9.2.** Let $\phi^F$ be the analytic test configuration associated to the filtration $F$ from a test configuration. Then

$$\Phi(t) = (\phi^F)_t.$$

In particular, the results from the previous section thus yield another proof of \cite[Thm 1]{27} which says that $\Phi(t)$ is a weak geodesic ray emanating from $\phi$.

**Lemma 9.3.**

$$\Phi(t) = \lim_{k \to \infty} (\sup_{t\geq k} \Phi_k(t))^* = \lim_{k \to \infty} (\sup_{t\geq k} \max_{i} \{\phi_{t,\bar{\lambda}_i} + t\bar{\lambda}_i\})^*. \quad (41)$$

**Proof.** Our proof will be based on the elementary fact that if $\{a_{t,i} : i \in I_t\}$ is a set of real numbers then

$$\max_{i \in I_t} a_{t,i} \leq \frac{1}{I} \ln \left( \sum_{i \in I_t} e^{a_{t,i}} \right) \leq \max_{i \in I_t} a_{t,i} + \frac{1}{I} \ln |I_t|.$$

(42)
Now pick \( x \in X \) and \( t > 0 \). Let
\[
a_{l,i} := \frac{1}{l} \ln |s_i(x)|^2 + t \bar{\lambda}_i
\]
and \( I_l \) be the indexing set for the \( \lambda_i \). Then \( |I_l| = O(l^n) \) and
\[
\Phi_l(t) = \frac{1}{l} \ln \left( \sum_i e^{a_{l,i}} \right).
\]

Thus by (42)
\[
\max_i \{a_{l,i}\} \leq \Phi_l(t) \leq \max_i \{a_{l,i}\} + \frac{|I_l|}{l}. \tag{43}
\]

Now set
\[
b_{l,i} := \phi_{l,\bar{\lambda}_i} + t \bar{\lambda}_i = \frac{1}{l} \ln \sum_{\lambda_j \geq \lambda_i} |s_j(x)|^2 + t \bar{\lambda}_i.
\]

For fixed \( i \), pick any \( j_0 \) such that
\[
\max_{\lambda_j \geq \lambda_i} |s_j(x)|^2 = |s_{j_0}|^2 \quad \text{and} \quad \lambda_{j_0} \geq \lambda_i.
\]

Then
\[
b_{l,i} \leq \frac{1}{l} \ln(|I_l||s_{j_0}|^2 + t \bar{\lambda}_i) \leq \frac{1}{l} \ln |s_{j_0}|^2 + t \bar{\lambda}_{j_0} + \frac{\ln |I_l|}{l} = a_{j_0,i} + \frac{\ln |I_l|}{l}.
\]

Clearly \( a_{l,i} \leq b_{l,i} \) for all \( i \), so we in fact have
\[
\max_i \{a_{l,i}\} \leq \max_i \{b_{l,i}\} \leq \max_i \{a_{l,i}\} + \frac{\ln |I_l|}{l},
\]

which combined with (43) yields
\[
\max_i \{b_{l,i}\} - \frac{\ln |I_l|}{l} \leq \Phi_l(t) \leq \max_i \{b_{l,i}\} + \frac{\ln |I_l|}{l}.
\]

Now taking the supremum over all \( l \geq k \) followed by the upper semicontinuous regularization and then the limit as \( k \) tends to infinity gives the result since \( k^{-1} \ln |I_k| \) tends to zero.

**Lemma 9.4.** Suppose that the test configuration \( \mathcal{F} \) is non-trivial. Then there exists a \( \delta > 0 \) such that every point in the interval \( (\lambda_c - \delta, \lambda_c) \) is a limit of some sequence of normalized weights \( \bar{\lambda}_i \) as \( k \) tends to infinity.

**Proof.** If \( \bar{\lambda}_i \) is a normalized weight, clearly \( \phi_{k,\bar{\lambda}_i} \neq -\infty \). Thus e.g. by Lemma 7.5 we get an upper bound on the normalized weights
\[
\bar{\lambda}_i \leq \lambda_c.
\]

On the other hand, since for any \( \delta > 0 \) we have that \( \phi_{\lambda_c-\delta} \neq \phi_{\lambda_c+\delta} \) there must exist normalized weights arbitrarily close to \( \lambda_c \). It follows that \( \lambda_c \) equals the supremum of the normalized weights.
We rely on a construction of the second author [39] called the concave transform of a test configuration. This is a concave, hence continuous, function \( g \) on an open convex subset \( A \subset \mathbb{R}^n \) (the interior of the Okounkov body of \( L \)) with three important properties: first any value of \( g \) is the limit of some sequence of normalized weights \( \lambda_i \) with \( k \) tending to infinity, second, the supremum of \( g \) is equal to the supremum of the normalized weights, i.e \( \lambda_c \), and third, for any \( p \),

\[
\frac{1}{k^n} \sum_i \bar{\lambda}_i^p = \int_A g^p dx + O(1/k),
\]

where \( dx \) denotes the Lebesgue measure.

Now set \( \Lambda_k := \frac{1}{|I_k|} \sum_i \bar{\lambda}_i \)

Following Donaldson [21] we can write

\[
\frac{1}{k^n} \sum_i (\bar{\lambda}_i - \Lambda_k)^2 = N^2 + O(1/k),
\]

where \( N \geq 0 \) with equality if and only if the test configuration is trivial. We also have that

\[
\frac{I_k}{k^n} = vol(A) + O(1/k)
\]

(see [39]) so by \( 44 \) we get

\[
\frac{1}{k^n} \sum_i (\bar{\lambda}_i - \Lambda_k)^2 = \int_A (g - \bar{g})^2 dx + O(1/k),
\]

where \( \bar{g} := \frac{1}{vol(A)} \int_A g dx \).

Suppose now that the test configuration is non-trivial, so the above implies \( g \) is not constant. The supremum of \( g \) is equal to \( \lambda_c \), and from the continuity of \( g \) it follows that the image of \( g \) contains some interval \( (\lambda_c - \delta, \lambda_c) \) with \( \delta > 0 \). Since any value of \( g \) is the limit of some sequence of normalized weights, the lemma follows.

**Proof of Theorem 9.2** From Lemma 7.5 we know that there is a constant \( \epsilon(l) \) such that

\[
\phi_{t, \bar{\lambda}_i} + t\bar{\lambda}_i \leq \phi^F_{\lambda_i} + t\bar{\lambda}_i + \epsilon(l),
\]

where \( \epsilon(l) \) is independent of \( \lambda_i \) and tends to zero as \( l \) tends to infinity. Thus we certainly have

\[
\max_i \{\phi_{t, \bar{\lambda}_i} + t\bar{\lambda}_i\} \leq \sup_{\lambda} \{\phi^F_{\lambda} + t\lambda\} + \epsilon(l),
\]

and so

\[
(\sup_{i \geq k} \max_i \{\phi_{t, \bar{\lambda}_i} + t\bar{\lambda}_i\})^* \leq (\phi^F)^*_t + \sup_{l \geq k} \epsilon(l).
\]

Hence taking the limit as \( k \) tends to infinity and using Lemma 9.3 gives

\[
\Phi(t) \leq (\phi^F)^*_t.
\]
For the opposite inequality, suppose first that $\phi_{\lambda}^F = \phi$ for $\lambda < \lambda_c$ and $\phi_{\lambda}^F = -\infty$ for $\lambda > \lambda_c$. Then, as is easy to check

$$\Phi(t) = \phi + t\lambda_c.$$ 

If the test configuration is non-trivial then Lemma 9.4 provides a $\delta > 0$ such that every point in ($\lambda_c - \delta, \lambda_c$) is a limit of some normalized weights as $k$ tends to infinity. So picking $0 < \delta' < \delta$ there is a sequence $i(k)$ such that $\lambda_{i(k)} \in (\lambda_c - \delta', \lambda_c - \delta'/2)$ for $k \geq k_0$, which implies

$$\sup_{i} \max_{l \geq k_0} \{ \phi_{i,k}^F + t\lambda_{i(k)} \}^* \geq \phi_{k,k_i}^F + t\lambda_{i(k)} \geq \phi_{k,\lambda_{i}} - \delta'/2 + t((\lambda_c - \delta')).$$

Thus taking the limit as $k$ tends to infinity gives

$$\Phi(t) \geq \phi_{\lambda_{i}} - \delta'/2 + t(\lambda_c - \delta') = \phi + t(\lambda_c - \delta'),$$

and since $\delta'$ was arbitrary we thus have

$$\Phi(t) \geq \phi + t\lambda_c = (\phi^F)^t,$$

which completes the proof in this case.

If the test configuration is trivial then there is a constant $\eta$ with $\lambda_{i} = k \eta$ for all $i$, so

$$\Phi_{t}(t) = \frac{1}{t} \log(\sum \{ \phi_{i,k}^F + t\lambda_{i} \}) = \phi_{t,-\infty} + \eta.$$ 

Now by standard properties of Bergman kernel asymptotics, $(\sup_{i} \max_{l \geq k_0} \phi_{i,k}^F, \eta^*)$ converges to $\phi$ as $k$ tends to infinity, which gives the result in the trivial case.

Thus we now may assume that there is a $\lambda' < \lambda_c$ such that $\phi \neq \phi_{\lambda'}^F$. Set

$$\eta = \sup \{ \lambda : \phi_{\lambda}^F = \phi \}$$

so, by assumption, $\eta$ is strictly less than $\lambda_c$.

Consider first $\lambda$ such that $\phi_{\lambda}^F \neq \phi$ and $\phi_{\lambda}^F \neq -\infty$. Then for any $\delta > 0$ there are, for arbitrarily large $k$, normalized weights $\lambda_{i(k)}$ in $(\lambda - \delta, \lambda)$, since otherwise we would have $\phi_{\lambda}^F - \delta = \phi_{\lambda}^F$, which is impossible by concavity of $\phi^F$. Thus we have for $k \geq k_0$,

$$\sup_{i} \max_{l \geq k_0} \{ \phi_{i,k}^F + t\lambda \} \geq \phi_{k,\lambda_{i(k)}}(x) + t\lambda_{i(k)} \geq \phi_{k,\lambda}(x) + t(\lambda - \delta).$$

So arguing just as above

$$\Phi(t) \geq \phi_{\lambda}^F + t\lambda \quad \text{if} \quad \phi_{\lambda}^F \neq \phi, \text{ and } \phi_{\lambda}^F \neq -\infty,$$

and in particular

$$\Phi(t) \geq (\sup_{\lambda > \eta} \{ \phi_{\lambda}^F + t\lambda \})^*.$$

Now if $\lambda$ is such that $\phi_{\lambda}^F = \phi$ then using the right continuity part of Lemma 5.2

$$\phi_{\lambda}^F + t\lambda = \phi + t\lambda = (\sup_{\lambda > \eta} \{ \phi_{\lambda}^F \})^* + t\lambda \leq (\sup_{\lambda' > \eta} \{ \phi_{\lambda'}^F + t\lambda' \})^* \leq \Phi(t),$$

which along with (45) completes the proof. \hfill \Box
Remark 9.5. Phong-Sturm prove in [27] that the geodesic ray one gets from a non-trivial test configuration is never of the form $\phi + t\eta$ with $\eta$ constant. Thus from the above we see that the test configuration is trivial iff the associated analytic test configuration is trivial, i.e. if there exists a number $\lambda_c$ such that $\phi_\lambda = \phi$ when $\lambda < \lambda_c$ and $\phi_\lambda \equiv -\infty$ when $\lambda > \lambda_c$.

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JULIUS ROSS, UNIVERSITY OF CAMBRIDGE, UK.
J.ROSS@DPMMS.CAM.AC.UK

DAVID WITT NYSTROM, UNIVERSITY OF GOTHENBURG, SWEDEN.
WITTNYST@CHALMERS.SE