Computation of generalized inverses of tensors via $t$-product

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Abstract

Generalized inverses of tensors play increasingly important roles in computational mathematics and numerical analysis. It is appropriate to develop the theory of generalized inverses of tensors within the algebraic structure of a ring. In this paper, we study different generalized inverses of tensors over a commutative ring and a noncommutative ring. Several numerical examples are provided in support of the theoretical results. We also propose algorithms for computing the inner inverses, the Moore–Penrose inverse, and weighted Moore–Penrose inverse of tensors over a noncommutative ring. The prowess of some of the results is demonstrated by applying these ideas to solve an image deblurring problem.

Keywords

generalized inverses, image deblurring, Moore–Penrose inverse, ring, $t$-product, tensor

Mos Subject Classification

AMS Subject Classifications: 15A09; 1569

1 | Introduction

One of the basic operations in linear algebra is matrix multiplication; it expresses the product of two matrices to form a new matrix. A tensor is a higher-dimensional generalization of a matrix (i.e., a first-order tensor is a vector, a second-order tensor is a matrix), but there are more than one way to multiply two tensors.\textsuperscript{1,2} Even some basic matrix multiplication concepts cannot be generalized to tensor multiplication in a unique manner.\textsuperscript{3,4} This leads to the study of different types of tensor products,\textsuperscript{2,5} which have recently attracted a great deal of interest (see References 3,5-9). In this connection, the inverses and generalized inverses over different product of tensors\textsuperscript{10-13} have generated a tremendous amount of interest in mathematics, physics, computer science, and engineering.

In mathematics, a ring is an algebraic structure with two binary operations (addition and multiplication) over a set satisfying certain requirements. This structure facilitates the fundamental physical laws, such as those underlying special relativity and symmetry phenomena in molecular chemistry. In the literature, many different approaches have been proposed for the generalized inverse of a ring,\textsuperscript{5,14,15} where elements of the ring are scalars, vectors, or matrices. However, it is impractical to analyze quantities whose elements are addressed by more than two indices in a ring, (e.g., a vector has one index and a matrix has two indices). This inconvenience can be easily overcome, thanks to tensors.\textsuperscript{2,3} In connection with binary operations of the algebraic structure, we consider $+$ for addition, and use two different binary operations, that is, $\ast$ and $\otimes$, for multiplication. In this paper, we study the generalized inverses of tensors over a commutative ring with the binary operations $(+,\otimes)$ and a noncommutative ring with involution using $(+,\ast)$ as a binary operations.
Kilmer and Martin\textsuperscript{16} proposed a closed multiplication operation between tensors referred to as the \textit{t}-product. In fact, the multiplication of two tensors based on the \textit{t}-product takes advantage of the circulant-type structure,\textsuperscript{16,17} which allows one to compute efficiently using the Fast Fourier Transform and extend many concepts from linear algebra to tensors.

A summary of the main facets of this paper are given below by the following bullet points.

- Introduction of different generalized inverses of tensors (via \textit{t}-product) over a commutative ring and a noncommutative ring.
- Determination of necessary and sufficient conditions for reverse order law for the Moore–Penrose inverse and \{i,j,k\}-inverses of tensors over a noncommutative ring.
- Discussion of a few algorithms for computing inner inverses, the Moore–Penrose inverse and the weighted Moore–Penrose inverse of tensors over a noncommutative ring.
- Application of generalized inverses to color image deblurring.

Let $V$ and $W$ be vector spaces (both real or both complex). Let $A$ be a linear transformation from $V$ into $W$. Denote the null space of $A$ and the range of $A$ by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. Let $M$ be an algebraic complement of $\mathcal{N}(A)$ in $V$ and $S$ be an algebraic complement of $\mathcal{R}(A)$ in $W$, so $V = \mathcal{N}(A) \oplus M$ and $W = \mathcal{R}(A) \oplus S$. Let $P$ be the algebraic projector (linear idempotent transformation) of $V$ onto $M$, and let $Q$ be the algebraic projector of $W$ onto $S$. Then there exists a unique linear transformation $X$ on $W$ into $V$ that satisfies the following equations:

$$AXA = A, \quad XAX = X, \quad XA = P \text{ and } AX = Q.$$ 

The transformation $X$ is called the (algebraic) generalized inverse of $A$ relative to the projectors $P$ and $Q$, and is denoted by $A\overset{S}{P}_W$. In the case when $A$ is a bounded linear operator on a Hilbert space $V$ into Hilbert space $W$, the choice of orthogonal projector $P$ and $Q$ gives the Moore–Penrose inverse. “Inverses” that satisfy a subset of the above four equations are of interest in many applications. A linear operator $X$ is called an inner inverse if it satisfies $AXA = A$ and an outer inverse if it satisfies $XAX = X$. Other “inverses” are defined to satisfy one of the four equations coupled with additional equations, such as the Drazin inverse. Various generalized inverses serve different purposes. For detailed expositions of generalized inverses of matrices including computational aspects and applications, we refer to the following books: Ben-Israel and Greville,\textsuperscript{18} Rao and Mitra,\textsuperscript{19} Campbell and Meyer,\textsuperscript{20} Wei et al.,\textsuperscript{21} and Nashed.\textsuperscript{22-24} A comprehensive annotated bibliography of 1876 references is included in Nashed.\textsuperscript{25} For the theory of generalized inverses on (not necessarily finite dimensional) vector spaces and on Banach and Hilbert spaces, see Nashed.\textsuperscript{23} For various equivalent definitions of the Moore–Penrose of matrices and linear operators, see Nashed.\textsuperscript{26} 

The purpose of this paper is twofold. Firstly, we study generalized inverses of tensors over a commutative ring. This part is motivated by the work of Drazin\textsuperscript{27-29} and Zhu\textsuperscript{30,31} and Rakic\textsuperscript{14} in rings with involution. Here we relate the concept of a well-supported element in a ring with involution and study the group inverse of tensors along with the class of \((b, c)\)-inverses in a commutative ring. Secondly, we discuss the generalized inverse of tensors over noncommutative ring, which is an extension of a matrix over a ring (see References 6,32,33, also see the recent papers on tensors\textsuperscript{1,10,34,35}). Specifically, we focus on investigations of \{i,j,k\}-inverses, the Moore–Penrose inverse and the weighted Moore–Penrose inverse of tensors along with several characterizations of these inverses in a noncommutative ring with involution. Our aim is to focus on generalized inverses of tensors with a specific multiplication concept over a ring. We present algorithms for computing different generalized inverses of tensors in a noncommutative ring. This algorithms may open to the door to other type of tensor related problems. As an application, we use the tensor representation algorithm in image deblurring.

The outline of the paper is as follows. Section 2 introduces different generalized inverses of tensors over a commutative ring. In Section 3, we introduce different generalized inverses of tensors over a noncommutative ring with involution. Further, we discuss a few necessary and sufficient conditions for existence of such inverses in this section. We also develop algorithms for computing generalized inverses over a noncommutative ring. An application of the Moore–Penrose inverse on image reconstruction is discussed in Section 4. Section 5 devoted to a brief conclusion.

## 2 | Tensors over Commutative Ring

It is well-known that the tensors are a multidimensional array of numbers, sometimes it is called a multiway or a multimode array. For example, a matrix is a second order tensor or a two-way tensor. An element $a = (a_1, a_2, \cdots , a_p)^T$, 

where the entries $a_1, a_2, \ldots, a_p$ are elements from the field of real number $\mathbb{R}$, is called a first-order tensor with entries from $\mathbb{R}$. Following standard notation, the set of all first-order tensors with entries from the field of real number $\mathbb{R}$ is denoted by $\mathbb{R}^p$. This section is focused on $\mathbb{R}^p$. The circulant of an element $a = (a_1, a_2, \cdots, a_p)^T \in \mathbb{R}^p$ is defined as

$$\text{circ}(a) = \begin{bmatrix} a_1 & a_p & \cdots & a_2 \\ a_2 & a_1 & \cdots & a_3 \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p-1} & \cdots & a_1 \end{bmatrix}.$$ 

We recall the following result which will be used in Section 2.1.

**Lemma 1.** (Reference 34, theorem 3.2) $(\mathbb{R}^p, +, \circ)$ is a commutative ring with unity, where the addition “+”, multiplication “$\circ$” and the unity respectively defined by

$$a + b = (a_1, a_2, \cdots, a_p)^T + (b_1, b_2, \cdots, b_p)^T = (a_1 + b_1, a_2 + b_2, \cdots, a_p + b_p)^T,$$

$$a \circ b = \text{circ}(a)b = \begin{bmatrix} a_1 & a_p & \cdots & a_2 \\ a_2 & a_1 & \cdots & a_3 \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p-1} & \cdots & a_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix},$$

and $i = (1, 0, 0 \cdots, 0)^T$ is the unity of $\mathbb{R}^p$.

**Definition 1.** Let $(\mathbb{R}^p, +, \circ)$ be a commutative ring. An element $a(\neq 0) \in \mathbb{R}^p$ is said to be a zero-divisor if there exists an element $b(\neq 0) \in \mathbb{R}^p$, such that $a \circ b = 0$.

**Remark 1.** $(\mathbb{R}^p, +, \circ)$ is not an integral domain as shown in the next example.

**Example 1.** Let $a = (a, a, a)^T$ with $a \neq 0$. Consider $b = (1, -1, 0)^T \in \mathbb{R}^3$. Then

$$a \circ b = \text{circ}(a)b = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

Thus $a$ is zero-divisor in $\mathbb{R}^p$.

### 2.1 Generalized inverses

In this section we discuss different types of generalized inverses on $(\mathbb{R}^p, +, \circ)$. For convenience we use the known notation $\mathbb{R}^p$ for the ring $(\mathbb{R}^p, +, \circ)$.

**Definition 2.** Let $a \in \mathbb{R}^p$. The left and right ideals generated by $a$ are denoted by $\mathbb{R}^p \circ a$ and $a \circ \mathbb{R}^p$ which are, respectively, defined as

$$\mathbb{R}^p \circ a = \{x \circ a : x \in \mathbb{R}^p\} \quad \text{and} \quad a \circ \mathbb{R}^p = \{a \circ x : x \in \mathbb{R}^p\}.$$ 

We also denote $a \circ \mathbb{R}^p \circ b = \{a \circ x \circ b : x \in \mathbb{R}^p\}$. The left and right annihilator of an element are defined as follows.

**Definition 3.** The left and right annihilator of $a \in \mathbb{R}^p$, respectively, denoted by $\text{lann}(a)$ and $\text{rann}(a)$, are defined as

$$\text{lann}(a) = \{y \in \mathbb{R}^p : y \circ a = 0\} \quad \text{and} \quad \text{rann}(a) = \{y \in \mathbb{R}^p : a \circ y = 0\}.$$ 

**Definition 4.** Let $a, b, c \in \mathbb{R}^p$. An element $y \in \mathbb{R}^p$ is called a $(b, c)$-inverse of $a$ if it satisfies
(a) \( y \in b \odot R^p \odot y \cap y \odot R^p \odot c \),
(b) \( y \odot a \odot b = b \) and \( c \odot a \odot y = c \).

Example 2. Let \( a = (0,1)^t = c, b = (1,0)^t \in R^2 \). Consider an element \( y = (0,1)^t \in R^2 \). We can verify that

\[
\begin{align*}
a \odot b &= \text{circ}(a)b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } c \odot a = \text{circ}(c)b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
y \odot (a \odot b) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = b \text{ and } (c \odot a) \odot y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c.
\end{align*}
\]

Further, \( b \odot u \odot y = y \) and \( y \odot v \odot c = y \) for an \( u = (1,0)^t \) and a \( v = (0,1)^t \). Thus \( y \) is the \((b, c)\)-inverse of \( a \).

If \( y \) is the \((b, c)\) inverse of \( a \in R^p \), then we have

\[
b = y \odot a \odot b = b \odot s \odot y \odot a \odot b = b \odot x \odot b, \text{ where } s \in R^p \text{ and } x = s \odot y \odot a \in R^p.
\]

Similarly, \( c = c \odot z \odot c \) for some \( z \in R^p \). This leads to the following result.

**Proposition 1.** Let \( a, b, c \in R^p \). If the \((b, c)\) inverse of an element \( a \) exists, then \( b \) and \( c \) are both regular.

Next, we discuss the existence of the \((b, c)\)-inverse.

**Theorem 1.** Let \( a, b, c \in R^p \). Then the \((b, c)\)-inverse of \( a \) exists if and only if \( b, c \in R^p \odot c \odot a \odot b = c \odot a \odot b \odot R^p \).

**Proof.** Let \( x \) be the \((b, c)\)-inverse of \( a \). Then \( b = x \odot a \odot b, c \odot a \odot x = c \) and \( x = x \odot u \odot c = b \odot v \odot x \) for some \( u \) and \( v \in R^p \). Now \( b = x \odot u \odot c \odot a \odot b \in R^p \odot c \odot a \odot b \). Similarly, \( c = c \odot x \odot a \odot c \in R^p \odot c \odot a \odot b \). Conversely, let \( b \in R^p \odot c \odot a \odot b \) and \( c \in c \odot a \odot b \odot R^p \). Then \( b = u \odot c \odot a \odot b \) and \( c = c \odot a \odot b \odot v \) for some \( u, v \in R^p \). Further, \( u \odot c = b \odot v \). Let \( x = u \odot c = b \odot v \).

Thus \( x \) is a \((b, c)\)-inverse of \( a \).

We now establish the uniqueness of the \((b, c)\)-inverse.

**Theorem 2.** Let \( a, b, c \in R^p \). Then the \((b, c)\)-inverse of \( a \) is unique.

**Proof.** Suppose there exist two \((b, c)\)-inverses of \( a \), say \( x \) and \( y \). From Definition 4, we obtain \( x \odot a \odot b = b, c \odot a \odot y = c, x = x \odot u \odot c, y = b \odot v \odot y, x = b \odot v \odot x \) for some \( u \) and \( v \in R^p \).

\[
x = x \odot u \odot c = x \odot u \odot c \odot a \odot y = x \odot a \odot y = x \odot a \odot b \odot v \odot y = b \odot v \odot y = y.
\]

Note that \( x = b \odot v \odot x = x \odot a \odot b \odot v \odot x = x \odot a \odot x \).

**Theorem 3.** Let \( a, b, c \in R^p \). Then the following statements are equivalent:

(i) \((b, c)\)-inverse of \( a \) exists.
(ii) \( R^p = a \odot b \odot R^p \bigoplus \text{rann}(c) = R^p \odot c \odot a \odot \text{lann}(b) \).
(iii) \( R^p = a \odot b \odot R^p + \text{rann}(c) = R^p \odot c \odot a + \text{lann}(b) \).

**Proof.** It is enough to show (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) since (ii) \(\Rightarrow\) (iii) is trivial.

(i) \(\Rightarrow\) (ii) Let the \((b, c)\) inverse of \( a \) exists. Then
Let \( x \in \mathbb{R}^p \) be any arbitrary vector and consider \( z = x - a \odot b \odot v \odot x \). Then \( c \odot z = (c - c \odot a \odot b \odot v) \odot x = 0 \). Thus \( z \in \mathrm{rann}(c) \). In addition, \( x = a \odot b \odot v \odot x + a \odot b \odot \mathbb{R}^p + \mathrm{rann}(c) \). This implies

\[
\mathbb{R}^p = a \odot b \odot \mathbb{R}^p + \mathrm{rann}(c).
\]

Further, if \( y \in a \odot b \odot \mathbb{R}^p \cap \mathrm{rann}(c) \), then \( y = a \odot b \odot t \) and \( c \odot y = 0 \) for some \( t \in \mathbb{R}^p \). However, \( b \odot t = u \odot c \odot (a \odot b \odot t) = u \odot c \odot y = 0 \). Therefore, \( y = 0 \), which implies \( a \odot b \odot \mathbb{R}^p + \mathrm{rann}(c) = \{0\} \) and hence \( \mathbb{R}^p = a \odot b \odot \mathbb{R}^p + \mathrm{rann}(c) \). Similarly, we can show the other equality \( \mathbb{R}^p = \mathbb{R}^p \odot c \odot a \odot \mathrm{lann}(b) \).

(\( \text{iii} \)) \( \Rightarrow \) (\( \text{i} \)) Let \( \mathbb{R}^p = a \odot b \odot \mathbb{R}^p + \mathrm{rann}(c) \). Then \( c = c \odot i = c \odot (a \odot b \odot u + v) \), where \( u \in \mathbb{R}^p \) and \( v \in \mathrm{rann}(c) \). From \( v \in \mathrm{rann}(c) \), we get \( c \odot v = 0 \). This yields \( c = c \odot a \odot b \odot u \in c \odot a \odot b \odot \mathbb{R}^p \). Using \( \mathbb{R}^p \odot c \odot a + \mathrm{lann}(b) \), we can show \( b \in \mathbb{R}^p \odot c \odot a \odot b \). Thus by Theorem 1, the \((b, c)\) inverse of \( a \) exists.

We next define the annihilator inverses:

**Definition 5.** Let \( a, b, x \in \mathbb{R}^p \). An element \( x \) is called a left annihilator \( b \)-inverse of \( a \) if it satisfies

\[
x \odot a \odot x = x, \; \mathrm{lann}(x) = \mathrm{lann}(b).
\]

**Definition 6.** Let \( a, c, x \in \mathbb{R}^p \). An element \( x \) is called a right annihilator \( c \)-inverse of \( a \) if it satisfies

\[
x \odot a \odot x = x, \; \mathrm{rann}(x) = \mathrm{rann}(c).
\]

The uniqueness of the annihilator inverse is discussed in the following result.

**Theorem 4.** Let \( a, b, c \in \mathbb{R}^p \). If a left annihilator \( b \)-inverse (or right annihilator \( c \)-inverse) of \( a \) is exists then it is unique.

**Proof.** Assume to the contrary, let \( x \) and \( y \) be left annihilator \( b \)-inverse of \( a \). Then

\[
x \odot a \odot x = x, \; y \odot a \odot y = y, \; \mathrm{lann}(x) = \mathrm{lann}(b) = \mathrm{lann}(y).
\]

From \( i - x \odot a \in \mathrm{lann}(x) = \mathrm{lann}(y) \), we get \( y = x \odot a \odot y \). Similarly, we can show \( x = y \odot a \odot x \). This further implies \( x = x \odot a \odot y \) due to the fact that \( \mathbb{R}^p \) is commutative. Hence \( x = y \). Further, if \( x \) and \( y \) are two right annihilator \( c \)-inverse of \( a \). Then

\[
x \odot a \odot x = x, \; y \odot a \odot y = y, \; \mathrm{rann}(x) = \mathrm{rann}(c) = \mathrm{rann}(y).
\]

From \( i - a \odot x \in \mathrm{rann}(x) = \mathrm{rann}(y) \), we get \( y = y \odot a \odot x \). Similarly, we can show \( x = x \odot a \odot y = y \odot a \odot x \). Therefore, \( x = y \).

We now present an equivalent characterization of left annihilator \( b \)-inverse.

**Lemma 2.** Let \( a, b, x \in \mathbb{R}^p \). Then the following are equivalent:

(i) \( x \odot a \odot x = x \), \( \mathrm{lann}(x) = \mathrm{lann}(b) \).
(ii) \( x \odot a \odot b = b \), \( \mathrm{lann}(b) \subseteq \mathrm{lann}(x) \).

**Proof.** (i) \( \Rightarrow \) (ii) It is enough to show only \( x \odot a \odot b = b \). Let \( x \odot a \odot x = x \). Then \( i - x \odot a \in \mathrm{lann}(x) = \mathrm{lann}(b) \). This leads \( b = x \odot a \odot b \).

(ii) \( \Rightarrow \) (i) Let \( b = x \odot a \odot b \). Then \( i - x \odot a \in \mathrm{lann}(b) \subseteq \mathrm{lann}(x) \). Thus \( x \odot a \odot x = x \). Next we will claim that \( \mathrm{lann}(x) \subseteq \mathrm{lann}(b) \). Let \( v \in \mathrm{lann}(x) \). Then \( v \odot x = 0 \) and \( v \odot b = v \odot x \odot a \odot b = 0 \). Therefore, \( v \in \mathrm{lann}(b) \) and hence \( \mathrm{lann}(x) \subseteq \mathrm{lann}(b) \).

The next result for right annihilator \( c \)-inverse can be proved in a similar way.
Lemma 3. Let \(a, b, x \in \mathbb{R}^p\). Then the following are equivalent:

(i) \(x \odot a \odot x = x\), \(\text{rann}(x) = \text{lann}(c)\).
(ii) \(c \odot a \odot x = c\), \(\text{rann}(c) \subseteq \text{rann}(x)\).

Since \(\mathbb{R}^p\) is a associative ring, we define the annihilator \((b, c)\)-inverse as follows.

Definition 7. Let \(a, b, c, x \in \mathbb{R}^p\). An element \(x \in \mathbb{R}^p\) is called an annihilator \((b, c)\)-inverse of \(a\) if it satisfies

\[
x \odot a \odot x = x, \quad \text{lann}(x) = \text{lann}(b), \quad \text{rann}(x) = \text{rann}(c).
\]

We now give an example of the annihilator \((b, c)\)-inverse.

Example 3. Let \(a = \left(\frac{2}{3}, -\frac{1}{3}, 0\right)^T \in \mathbb{R}^3\) and \(b = c = (1, -1, 0)^T \in \mathbb{R}^3\). Consider \(x = (1, -1, 0)^T \in \mathbb{R}^3\). Then we can verify that

\[
x \odot a \odot x = \text{circ}(x)a = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}, \quad \text{and} \quad x \odot a \odot x = \text{circ}(x \odot a)x = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = x.
\]

Hence \(x\) is the annihilator \((b, c)\)-inverse of \(a\).

The uniqueness of annihilator \((b, c)\)-inverse easily follows from left or right annihilator inverses. Using Lemma 2 and Lemma 3, we state the next result concerning for annihilator \((b, c)\)-inverse.

Theorem 5. Let \(a, b, c, x \in \mathbb{R}^p\). Then the following statements are equivalent:

(i) \(x \odot a \odot x = x\), \(\text{lann}(x) = \text{lann}(b), \text{rann}(x) = \text{lann}(c)\).
(ii) \(x \odot a \odot b = b, \ c \odot a \odot x = c\), \(\text{lann}(b) \subseteq \text{lann}(x), \text{rann}(c) \subseteq \text{rann}(x)\).

The group inverses on \(\mathbb{R}^p\) is defined as follows.

Definition 8. Let \(a \in \mathbb{R}^p\). An element \(x \in \mathbb{R}^p\) is called a group inverse of \(a\) if it satisfies

\[
a \odot x \odot a = a, \quad x \odot a \odot x = x, \quad a \odot x = x \odot a.
\]

Note that the condition \(a \odot x = x \odot a\) is always satisfied, since \(\mathbb{R}^p\) is a commutative ring. The group inverse of an element \(a\) is unique if exists. We denote the group inverse of an element \(a \in \mathbb{R}^p\) by \(a^g\). An element \(a\) is called group invertible if \(a^g\) exists. We next give an example of the group inverse of \(a \in \mathbb{R}^p\).

Example 4. Let \(a = (1, 0, -1)^T \in \mathbb{R}^3\) and consider \(x = (1/3, -1/3, 0)^T \in \mathbb{R}^3\). Then we can verify that

\[
x \odot a \odot x = \text{circ}(x)a = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = x, \quad \text{and}
\]

\[
x \odot a \odot x = \text{circ}(x \odot a)x = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} = a.
\]

Hence \(x\) is the group inverse of \(a\).

The existence of group inverse is discussed in the theorem below:
Theorem 6. Let \( a \in \mathbb{R}^p \). Then \( a \) is group invertible if and only if \( a \in a^2 \circ \mathbb{R}^p \). Moreover, if \( a = x \circ a^2 = a^2 \circ y \), for some \( x, y \in \mathbb{R}^p \), then \( a^3 = x \circ a \circ y = x^2 \circ a = a \circ y^2 \).

Proof. Let \( a \) be group invertible. Then \( a = a \circ x \circ a = a^2 \circ x = x \circ a^2 \in \mathbb{R}^p \). Conversely, let \( a = x \circ a^2 = a^2 \circ y \), for some \( x, y \in \mathbb{R}^p \). Then \( x \circ a \circ y = a \circ a^2 \circ y = y \) and \( x \circ a \circ y = a \circ x \circ a^2 \circ y = x^2 \circ a \). Next we shall prove \( x \circ a \circ y \) is the group inverse of \( a \). Now \( a \circ (x \circ a \circ y) \circ a = x \circ a^2 \circ a \circ y = a^2 \circ y = a, x \circ a \circ y \circ (a) \circ x \circ a \circ y = x \circ a^3 \circ y \circ x \circ y = a^2 \circ y \circ x \circ y = a \circ x \circ y = x \circ a \circ y \), and the commutative property holds trivially. Thus \( a^3 = x \circ a \circ y \).

Remark 2. If \( \mathbb{R}^p \) is any commutative ring with unity, then the group inverse of an element \( a \in \mathbb{R}^p \) exists if and only if \( a \in a^2 \mathbb{R}^p \).

3 | TENSOR OVER A NONCOMMUTATIVE RING

3.1 | Notation and definitions

Let \( \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) be the set of order \( p \) and dimension \( n_1 \times n_2 \times \cdots \times n_p \) tensors over the real field \( \mathbb{R} \). Let \( A = (a_{i_1 \cdots i_p}) \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}, p > 1 \). For \( i = 1, \ldots, n_p \), denote by \( A_i \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{i-1} \times n_{i+1} \times \cdots \times n_p} \), the tensor whose order is \( (p - 1) \) and is created by holding the \( p \)-th index of \( A \) fixed at \( i \), which we called the frontal slices of the tensor \( A \). The generalization of matrix rows and columns are called fibers. Specifically, fixing all the indexes of a tensor \( A \) except one index. Now, define unfold(.) to take an \( n_1 \times n_2 \times \cdots \times n_p \) tensor\(^4\) and return an \( n_1 n_2 \times n_3 \times \cdots \times n_p \) block tensor in the following way:

\[
\text{unfold}(A) = \begin{bmatrix}
A_{p_1} \\
A_{p_2} \\
\vdots \\
A_{p_p}
\end{bmatrix}
\]

and fold(.) is the inverse operation,\(^5\) which takes an \( n_1 n_2 \times n_3 \times \cdots \times n_p \) block tensor and returns an \( n_1 \times n_2 \times \cdots \times n_p \) tensor. Then, fold(\text{unfold}(A)) = A. Now, one can easily see that,

\[
\text{circ(\text{unfold}(A))} = \begin{bmatrix}
A_{1} & A_{n_1} & \cdots & A_2 \\
A_{2} & A_{1} & \cdots & A_3 \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_p} & A_{n_{p-1}} & \cdots & A_1
\end{bmatrix}
\]

Definition 9. (Reference 36) Let \( A \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \). The transpose of \( A \) is denoted by \( A^T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) is defined by the tensor transposing each \( A_i \), for \( i = 1, 2, \ldots, n \) along with reversing the order of the \( A_i \) from 2 through \( n_p \).

We collect some useful definitions from\(^6\) as follows. A tensor \( D \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) is called a \( f \)-diagonal tensor if each frontal slice is diagonal. Similarly, a tensor is a \( f \)-upper triangular or \( f \)-lower triangular if each frontal slice is upper or lower triangular, respectively. Likewise, a tensor is called the identity tensor if each frontal slice is the identity matrix and all other frontal slices are zeros. The tensor whose entries are all zero is denoted by \( \emptyset \).

Now we construct a ring over the tensor space \( \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \).

Theorem 7. Let \( \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) be a \( p \)-th order tensor over a field \( \mathbb{R} \) with binary operations \((+, \ast)\) (for addition and multiplication), defined as

\[
A \ast B = \text{fold(\text{circ(\text{unfold}(A))} \ast \text{unfold}(B))}, \quad A + B = A_{\ast n_p} + B_{\ast n_p}, \quad \forall A, B \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}.
\]

Then \( \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) is a ring with unity.

Proof. It is easy to show that \( (\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}, +) \) is an group and it is straightforward. Using the definition of \( \ast \) and its associative properties, we can show that:
\[ A \ast (B + C) = \text{fold}(\text{circ}(\text{unfold}(A)) \ast \text{unfold}(B + C)) \]
\[ = \text{fold}(\text{circ}(\text{unfold}(A)) \ast [\text{unfold}(B) + \text{unfold}(C)]) \]
\[ = \text{fold}(\text{circ}(\text{unfold}(A)) \ast \text{unfold}(B)) + \text{fold}(\text{circ}(\text{unfold}(A)) \ast \text{unfold}(C)) \]
\[ = A \ast B + A \ast C. \]

Similarly, we can prove that, \((B + C) \ast A = B \ast A + C \ast A.\)

**Remark 3.** One can easily see that \((\mathbb{R}^{nxnxn} \times \mathbb{R}^{nxnxn}, \ast, +)\) is a noncommutative ring. For simplicity, we use the notation \((\mathbb{R}^{nxnxn} \times \mathbb{R}^{nxnxn}, \ast, +)\) to facilitate the presentation.

We next give an example of the noncommutative ring of tensors.

**Example 5.** Let \(A, B \in \mathbb{R}^{3 \times 3 \times 2}\) with frontal slices
\[
A_{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}, \quad B_{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}.
\]

Then we can see that
\[
\text{fold} \begin{pmatrix} 2 & 22 & 0 \\ 0 & 10 & 0 \\ 0 & 8 & 0 \\ 4 & 22 & 0 \\ 0 & 6 & 0 \\ 3 & 6 & 0 \end{pmatrix} = A \ast B \neq B \ast A = \text{fold} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 10 & 0 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \\ 0 & 6 & 0 \\ 1 & 4 & 3 \end{pmatrix}.
\]

### 3.2 Computation of generalized inverses

We now introduce the definition of \(i\)-inverses \((i = 1, 2, 3, 4)\) and the Moore–Penrose inverse of tensors \((t\text{-product})\) over \(\mathbb{R}^{nxnxn} \times \mathbb{R}^{nxnxn}\).

**Definition 10.** For any tensor \(A \in \mathbb{R}^{nxnxn} \times \mathbb{R}^{nxnxn}\), consider the following equations in \(Z \in \mathbb{R}^{nxnxn} \times \mathbb{R}^{nxnxn}:\)
\[
(1) \quad A \ast Z \ast A = A, \quad \text{(3) } (A \ast Z)^T = A \ast Z, \\
(2) \quad Z \ast A \ast Z = Z, \quad \text{(4) } (Z \ast A)^T = Z \ast A.
\]

Then \(Z\) is called

(a) an inner inverse of \(A\) if it satisfies (1) and is denoted by \(A^{(1)}\);

(b) an outer inverse of \(A\) if it satisfies (1) and (2), which is denoted by \(A^{(1,2)}\);

(c) a \(1, 3\) inverse of \(A\) if it satisfies (1) and (3), which is denoted by \(A^{(1,3)}\);

(d) a \(1, 4\) inverse of \(A\) if it satisfies (1) and (4), which is denoted by \(A^{(1,4)}\);

(e) the Moore–Penrose inverse of \(A\) if it satisfies all four conditions \((1) \sim (4)\), which is denoted by \(A^\dagger\).

It is worth mentioning that the discrete Fourier Transform plays a significantly role for the product of two tensors. For instance, if \(a = (a_1, a_2, \ldots, a_n)^T\) is an \(n \times 1\) vector, then \(F_n \circ (a) F_n^*\) is diagonal, where \(F_n\) is the \(n \times n\) discrete Fourier transform (DFT) matrix, and \(F_n^*\) is its conjugate transpose. To compute this diagonal, the fast Fourier transform (FFT) is used\(^{37}\) as follows.

\[ F_n \circ (a) F_n^* = \text{fft}(a) \]
Algorithm 1. Computation of transpose of a tensor $A$

1: procedure TRANPOSE($A$)
2:    Input $p, n, n_3, \ldots, n_p$ and the tensor $A \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p}$.
3:    for $i \leftarrow 3$ to $p$ do
4:        $A = \text{fft}(A, [\ ], i)$;
5:    end for
6:    $N = n_3n_4 \cdots n_p$
7:    for $i \leftarrow 1$ to $N$ do
8:        $Z(:,:,i) = \text{transpose}(A(:,:,i))$;
9:    end for
10:   for $i \leftarrow p$ to $3$ do
11:       $B = \text{ifft}(Z, [\ ], i)$;
12:   end for
13:   return $B$
14: end procedure

Further, the authors of References 16, 17, 34, 36 utilized it for tensors. Suppose $A \in \mathbb{R}^{n \times n \times n}$, then circ(unfold($A$)) is a block circulant matrix with each $A_i \in \mathbb{R}^{n \times n}$, for $1 \leq i \leq n_3$. Then

$$
(F_{n_3} \otimes I_n) \cdot \begin{pmatrix}
A_1 & A_{n_1} & \cdots & A_2 \\
A_2 & A_1 & \cdots & A_3 \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_3} & A_{n_3-1} & \cdots & A_1
\end{pmatrix} \cdot (F_{n_3}^* \otimes I_n) = \begin{pmatrix}
D_1 \\
D_2 \\
\vdots \\
D_{n_3}
\end{pmatrix},
$$

where $F_{n_3}$ is the $n_3 \times n_3$ DFT matrix, $F_{n_3}^*$ is the conjugate transpose of $F_{n_3}$, and `·` is the standard matrix multiplication. Here $D_i \in \mathbb{R}^{n \times n}$, for $1 \leq i \leq n_3$. Similarly for $A \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p}$, we can write

$$
(F_{n_p} \otimes F_{n_{p-1}} \otimes \cdots F_{n_3} \otimes I_n) \cdot \text{circ}(\text{unfold}(A)) \cdot (F_{n_p}^* \otimes F_{n_{p-1}}^* \otimes \cdots F_{n_3}^* \otimes I_n) = \begin{pmatrix}
\Sigma_1 \\
\Sigma_2 \\
\vdots \\
\Sigma_p
\end{pmatrix},
$$

(1)

is a block diagonal matrix with $\rho$ blocks each of size $n \times n$, where $\rho = n_3n_4 \cdots n_p$. Hence computation of tensors via the Fourier domain are obtained by systematically reorganizing the tensor into a matrix. Then, the benefits of the matrix computation results will be utilized for tensor computation. So once the matrix computation is performed on the Fourier domain we apply $(F_{n_p}^* \otimes F_{n_{p-1}}^* \otimes \cdots F_{n_3}^* \otimes I_n)$ to the left and $(F_{n_p} \otimes F_{n_{p-1}} \otimes \cdots F_{n_3} \otimes I_n)$ to the right of each of the block diagonal matrices in Equation (1). Folding up those results takes us back into the appropriate sized tensor results. In view of this representation, we next present the definition of the symmetric positive definite tensor which was introduced earlier in Reference 17. A tensor $A \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p}$ is called symmetric positive definite tensor if all the $\Sigma_i$ for $i = 1, 2, \cdots \rho$, are hermitian positive definite.

The FFT is utilized in Algorithms 1 and 2 for computing transpose of a tensor and inner inverses of tensors. Following the Definition 9, the “transpose” function is used in Algorithm 1 on line 8 to compute the transpose of matrices in the Fourier domain. Similarly, the functions (i.e., rank, rref) are used in Algorithm 2 on lines 8 and 9 to compute rank and reduced row-echelon form of matrices in the Fourier domain, respectively. Here, our purpose is not to compare our tensor-based approach to other methods, but rather to contribute to the class of algorithms used for this purpose.

Definition 11. The left and right ideals generated by $A \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p}$ are, respectively, defined by

$$
\mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p} * A = \{ Z * A : Z \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p} \} \quad \text{and} \quad A * \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p} = \{ A * Z : Z \in \mathbb{R}^{n \times n \times n_3 \times \cdots \times n_p} \}.
$$
Definition 12. The right annihilator of $A \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ denoted by $\text{rann}(A)$ and left annihilator of $A$ denoted by $\text{lann}(A)$ are defined respectively by

$$\text{rann}(A) = \{ \mathcal{X} \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p} : A \ast \mathcal{X} = \mathcal{O} \}$$

and

$$\text{lann}(A) = \{ \mathcal{X} \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p} : \mathcal{X} \ast A = \mathcal{O} \}.$$

Algorithm 2. Computation of inner inverse of a tensor $A$

```
1: procedure IINV(A)
2:   Input $p, n, n_3, \ldots, n_p$ and the tensor $A \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$.
3:   for $i \leftarrow 3$ to $p$ do
4:     $\mathcal{A} = \text{fit}(A, [, i])$;
5:   end for
6:   $N = n_3 n_4 \cdots n_p$
7:   for $i \leftarrow 1$ to $N$ do
8:     $r \leftarrow \text{rank}(A(:, :, i))$.
9:     $B = \text{ref}[A(:, :, i), I_p(:, :, i)]$.
10:    $\mathcal{G} \leftarrow \text{last r columns of } B(:, :, i)$.
11:    Find a permutation matrix $\mathcal{P}(:, :, i)$ such that $\mathcal{G}(:, :, i) \mathcal{A}(:, :, i) \mathcal{P}(:, :, i) = [I_p(:, :, i) \mathcal{K}(:, :, i)]$.
12:    Define an arbitrary $\mathcal{L}(:, :, i)$ such that $B(:, :, i) \mathcal{L}(:, :, i) = [I_p(:, :, i) \mathcal{O}(:, :, i) \mathcal{L}(:, :, i)]$.
13:    $\mathcal{Z}(:, :, i) = \mathcal{P}(:, :, i) B(:, :, i) \mathcal{G}(:, :, i)$.
14: end for
15: for $i \leftarrow p$ to $3$ do
16:   $\mathcal{X} = \text{iff}l(\mathcal{Z}, [, i])$;
17: end for
18: return $\mathcal{X}$ \hspace{1cm} $\triangleright \mathcal{X}$ is the inner inverse of $A$
19: end procedure
```

The annihilators of $A \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ satisfy the following properties.

**Proposition 2.** Let $A, B \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Then $A \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p} \subseteq B \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ if and only if $\text{lann}(B) \subseteq \text{lann}(A)$.

**Proof.** Let $U \in \text{lann}(B)$. Then $U \ast B = 0$ for some $U \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. From $A \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p} \subseteq B \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$, we get $A = B \ast \mathcal{V}$ for some $\mathcal{V} \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Now $U \ast A = U \ast B \ast \mathcal{V} = \mathcal{O}$. Thus $U \in \text{lann}(A)$ and hence $\text{lann}(B) \subseteq \text{lann}(A)$.

Conversely, from $(I - B \ast B^{(1)}) \ast B = B - B \ast B^{(1)} = \mathcal{O}$ and $\text{lann}(B) \subseteq \text{lann}(A)$, we have $(I - B \ast B^{(1)}) \ast A = \mathcal{O}$. Thus $A = B \ast B^{(1)} \ast A$. Now let $S \in A \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Then $S = A \ast U = B \ast B^{(1)} \ast A \ast U = B \ast \mathcal{T}$ for some $\mathcal{T} = B^{(1)} \ast A \ast U \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Hence $A \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p} \subseteq B \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$.

Using similar arguments, we can show the following result for the right annihilator.

**Proposition 3.** Let $C, D \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Then $\mathbb{R}^{n \times n_1 \times \cdots \times n_p} \ast C \subseteq \mathbb{R}^{n \times n_1 \times \cdots \times n_p} \ast D$ if and only if $\text{rann}(D) \subseteq \text{rann}(C)$.

The existence of solution of tensor equation through one-inverse is discussed in the following theorem, which can be easily proved.

**Theorem 8.** Let $\mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ be an associative ring with identity $I$. Let $A \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ and $\mathcal{X} \in \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$. Then the following statements are equivalent:

(i) $\mathcal{X} \ast B$ is a solution of the multilinear system $A \ast \mathcal{Y} = B$ whenever $B \in A \ast \mathbb{R}^{n \times n_1 \times \cdots \times n_p}$.

(ii) $A \ast \mathcal{X} \ast A = A$.

However, if $\mathbb{R}^{n \times n_1 \times \cdots \times n_p}$ is an associative ring without identity then the above theorem does not hold in general, as shown by the next example.
Example 6. Consider an associative ring of all $3 \times 3 \times 3$ real tensor. The first, second and third frontal slices are
\[
\begin{pmatrix}
a & a & a \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Consider $\mathcal{A}$ in the above set of frontal slice. But $\mathcal{A}^T$ is not in the above sets, that is, the first, second and third frontal slices of $\mathcal{A}^T$ are:
\[
\begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We now discuss the reverse order of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ inverses.

**Proposition 4.** Let $A, B \in \mathbb{R}^{n \times n \times n}$. Then the following are true.

(i) If $A^{(1)} * A * B = B * A^{(1)} * A$, then $(A * B)^{(1)} = B^{(1)} * A^{(1)}$

(ii) If $B * B^{(2)} * A^{(2)} = A^{(2)} * B * B^{(2)}$, then $(A * B)^{(2)} = B^{(2)} * A^{(2)}$.

**Proof.** (i) Let $A * A^{(1)} * B = B * A * A^{(1)}$. Then the result is follows from

\[
A * B * B^{(1)} * A^{(1)} * A = A * B * B^{(1)} * B * A^{(1)} * A = A * A^{(1)} * A * B = A * B.
\]

(ii) Let $B * B^{(2)} * A^{(2)} = A^{(2)} * B * B^{(2)}$. Then

\[
B^{(2)} * A^{(2)} * A * B * B^{(2)} * A^{(2)} = B^{(2)} * A^{(2)} * A * A^{(2)} * B * B^{(2)} = B^{(2)} * A^{(2)} * B * B^{(2)}
\]

The next result is for reflexive generalized inverse of a tensor $A \in \mathbb{R}^{n \times n \times n}$, which can be easily proved.

**Lemma 4.** Let $A, \chi \in \mathbb{R}^{n \times n \times n}$. If $Y, Z \in A[1]$ and $\chi = Y * A * Z$, then $\chi \in A[1, 2]$.

The next results are discussed for $\{1, 3\}$ and $\{1, 4\}$ inverses of a tensor $A \in \mathbb{R}^{n \times n \times n}$, which can be proved easily.

**Lemma 5.** Let $A \in \mathbb{R}^{n \times n \times n}$. Then the following are holds.

(i) If $A = \mathcal{X} * A^T * A$ for some $\mathcal{X} \in \mathbb{R}^{n \times n \times n}$, then $\mathcal{X}^T$ is a $\{1, 3\}$-inverse of $A$.

(ii) If $A = A * A^T * Y$ for some $Y \in \mathbb{R}$, then $Y^T$ is a $\{1, 4\}$-inverse of $A$.

**Theorem 9.** Let $A, B \in \mathbb{R}^{n \times n \times n}$. Then the following statements are equivalent:

(i) $B \in A[1, 3]$.

(ii) $A^T * A * B = A^T$.

(iii) $A * B = A * (A^T * A)^{(1)} * A^T$.

**Proof.** (i) $\Rightarrow$ (ii) Let $B \in A[1, 3]$. Then $A * B * A = A$ and $(A * B)^T = A * B$. Now

\[
A^T = (A * B * A)^T = A^T * (A * B)^T = A^T * A * B.
\]

(ii) $\Rightarrow$ (iii) Let $A^T * A * B = A^T$. Pre-multiplying by $A * (A^T * A)^{(1)}$, we obtain

\[
A * (A^T * A)^{(1)} * A^T = A * (A^T * A)^{(1)} * A^T * A * B = B^T * A^T * A * B = A * B.
\]

(iii) $\Rightarrow$ (i) First we will show that $\mathbb{R}^{n \times n \times n} * A^T \subseteq \mathbb{R}^{n \times n \times n} * A^T * A$. Let $Y \in \mathbb{R}^{n \times n \times n} * A^T$. Then $Y = A^T * Z$ for some $Z \in \mathbb{R}^{n \times n \times n} * A^T$. Now
\[ \mathcal{V} = A^T \star \mathcal{Z} = (A \star A^{(1,3)} \star A)^T \star \mathcal{Z} = A^T \star A \star \mathcal{T} \] where \( \mathcal{T} = A^{(1,3)} \star \mathcal{Z} \in \mathbb{R}^{n \times n \times n_{x,y}}. \)

Thus \( \mathbb{R}^{n \times n \times n_{x,y}} \ni A^T \subseteq \mathbb{R}^{n \times n \times n_{x,y}} \ni A^T \star A. \) From this condition we can show that \( A \star (A^T \star A)^{(1)} \star A^T \star A = A, \) so we have \( A \star B \star A = A. \) Further, \((A \star B)^T = A \star (A \star A^T)^{(1)} \star A^T = A \star B. \)

Similarly, for \((1,4)\)-inverse, we state the following result without proof.

**Theorem 10.** The following three conditions are equivalent:

(i) \( B \in A\{1,4\}. \)

(ii) \( B \star A \star A^T = A^T. \)

(iii) \( B \star A = A^T \star (A \star A^T)^{(1)} \star A. \)

A sufficient condition for the reverse order law of \((1,4)\)-inverses and \((1,3)\)-inverses of tensors is given in the next result.

**Theorem 11.** The following conditions are true for any \( A, B \in \mathbb{R}^{n \times n \times n_{x,y}}. \)

(i) If \( A^{(1,4)} \star A \star B \star B^T \) is symmetric, then \( (A \star B)^{(1,4)} = B^{(1,4)} \star A^{(1,4)}. \)

(ii) If \( A \star A^{(1,3)} \star B^T \star B \) is symmetric, then \( (A \star B)^{(1,3)} = B^{(1,3)} \star A^{(1,3)}. \)

**Proof.** (i) Let \( (A^{(1,4)} \star A \star B \star B^T)^T = A^{(1,4)} \star A \star B \star B^T. \) Then

\[
A^{(1,4)} \star A \star B \star B^T = (A^{(1,4)} \star A \star B \star B^T)^T = B \star B^T \star A \star A^{(1,4)} \star A.
\]

(2)

Using (2), we obtain

\[
A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star B = A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star B \star (B^{(1,4)} \star B^T) = A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star B \star (B^{(1,4)} \star B^T) = A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star (B^{(1,4)} \star B^T) = A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star B^T \star (B^{(1,4)} \star B^T) = A \star B \star B^{(1,4)} \star A^{(1,4)} \star A \star B.
\]

Hence \( B^{(1,4)} \star A^{(1,4)} \) is an \((1,4)\)-inverse of \( A \star B. \) Similarly, one can prove the part (ii).

The conditions of the above theorem are sufficient but not necessary for the reverse order law.

**Example 7.** Let \( A, B \in \mathbb{R}^{2 \times 2 \times 3} \) with frontal slices

\[
A_{(1)} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}, \quad A_{(3)} = \begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

We can verify that

\[
A^{(1,4)} = \text{fold} \begin{pmatrix}
3/80 & 0 \\
0 & -1/16
\end{pmatrix}, \quad B^{(1,4)} = \text{fold} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad A^{(1,4)} \star B^{(1,4)} = \text{fold} \begin{pmatrix}
3/80 & 0 \\
0 & -1/16
\end{pmatrix} = (A \star B)^{(1,4)}.
\]
We can verify that

\[ W \text{Lemma 7.} \]

Lemma 6.

used for proving our main result of the section.

TodiscussacharacterizationsoftheMoore–Penroseinverse,wefirstprovethefollowingauxiliaryresultswhichwillbe

\[ 3.3 \]

Results on the Moore–Penrose inverse

To discuss a characterization of the Moore–Penrose inverse, we first prove the following auxiliary results which will be

\[ \text{Lemma 6. Let } P, Q \in R^{nxnxn} \times \cdots \times n_f \text{ be idempotent tensors. If } R^{nxnxn} \times \cdots \times n_f \subseteq P \times Q \text{ and } Q \times R^{nxnxn} \times \cdots \times n_f, \text{ then } P = Q. \]

\[ \text{Proof. Let } R^{nxnxn} \times \cdots \times n_f \subseteq P \subseteq R^{nxnxn} \times \cdots \times n_f. \text{ Then } P = U^* * Q * U^* \text{ for some } U^* \in R^{nxnxn} \times \cdots \times n_f. \text{ Further, } P = U^* * Q = U^* * Q^2 = P * Q. \]

From the condition \( Q \times R^{nxnxn} \times \cdots \times n_f \subseteq P \times R^{nxnxn} \times \cdots \times n_f \), we have \( Q = P * V = P^2 * V = P * Q \text{ for some } V \in R^{nxnxn} \times \cdots \times n_f. \) Thus \( P = Q. \)

\[ \text{Lemma 7. Let } P, Q \in R^{nxnxn} \times \cdots \times n_f \text{ be symmetric and idempotent tensors. If } R^{nxnxn} \times \cdots \times n_f \times P = R^{nxnxn} \times \cdots \times n_f * Q \text{ or } Q \times R^{nxnxn} \times \cdots \times n_f = P \times R^{nxnxn} \times \cdots \times n_f, \text{ then } P = Q. \]

\[ \]

\[ ]
Proof. If $P_{m,n} \in \mathbb{R}^{m\times n_p}$ and $A \in \mathbb{R}^{n\times n_r}$, then $A \in \mathbb{R}^{m\times n_p}$ and $A = A^T$. Hence $A = A^T$. This yields

$$P = U \ast Q \ast U$$

Now $Q = Q^T = (Q \ast P)^T = P^T \ast Q^T = P \ast Q$.

**Theorem 12.** Let $A \in \mathbb{R}^{m\times n_p}$ and $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$. If $A \in \mathbb{R}^{m\times n_p}$, then there exist symmetric idempotents $P, Q \in \mathbb{R}^{m\times n_p}$ such that

(i) $P \ast P = P$ and $Q \ast Q = Q$

(ii) $P \ast Q = Q \ast P$

Proof. Let $A \in \mathbb{R}^{m\times n_p}$ and $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$. Then $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$. Hence $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$. If there exist symmetric idempotents $P, Q \in \mathbb{R}^{m\times n_p}$ such that $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$ and $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$, then $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$.

(i) $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$ and $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$ follow from Propositions 2 and 3.

(ii) $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$ and $\mathcal{A} \subset \mathbb{R}^{m\times n_p}$ follow from Lemmas 6 and 7.

The following result follows from the definition of the Moore–Penrose inverse.

**Theorem 13.** Let $A \in \mathbb{R}^{m\times n_p}$ be an associative ring with $I$. Let $A \in \mathbb{R}^{m\times n_p}$. Then the following statements are true.

(i) $A$ is symmetric and idempotent, then $A^T = A$.

(ii) $A \ast A^T \ast A, A \ast A^T \ast A, I - A \ast A^T$ and $I - A^T \ast A$ are all idempotent.

(iii) $A = A^T$ if and only if $A \ast A^T \ast A = A$.

A characterization of the Moore–Penrose inverse is given by the following:

**Theorem 14.** Let $A, \mathcal{A} \in \mathbb{R}^{m\times n_p}$. The following are equivalent:

(i) $A$ is the Moore–Penrose invertible and $\mathcal{A} = A^T$.

(ii) $A \ast \mathcal{A} = A$, $A \ast \mathcal{A} = A^T$ and $\mathbb{R}^{m\times n_p} \ast \mathcal{A} = \mathbb{R}^{m\times n_p} \ast A^T$.

(iii) $A \ast \mathcal{A} = A$, $\mathcal{A} \ast A = A$, $\mathcal{A} \ast A = A^T$ and $\mathbb{R}^{m\times n_p} \ast \mathcal{A} = \mathbb{R}^{m\times n_p} \ast A^T$.

(iv) $A \ast \mathcal{A} = A$, $\mathcal{A} \ast A = A$, $\mathcal{A} \ast A = A^T$ and $\mathbb{R}^{m\times n_p} \ast \mathcal{A} = \mathbb{R}^{m\times n_p} \ast A^T$.

(v) $A \ast \mathcal{A} = A$, $\mathcal{A} \ast \mathcal{A} \subset \mathcal{A}$ and $\mathcal{A} \ast \mathcal{A} \subset \mathcal{A}$.
Theorem 15. \( \mathcal{Y} \ast A \ast \mathcal{T} = \mathcal{T} \ast A \ast \mathcal{Y} \) and \( \mathcal{Y} = \mathcal{T} \ast \mathcal{Y}^{T} \ast \mathcal{Y} \).

Thus, \( \mathcal{Y} \ast \mathbb{R}^{n \times n \times \cdots \times n} = \mathcal{T} \ast \mathbb{R}^{n \times n \times \cdots \times n} \ast \mathcal{Y} \) and \( \mathbb{R}^{n \times n \times \cdots \times n} \ast \mathcal{Y} = \mathbb{R}^{n \times n \times \cdots \times n} \ast \mathcal{T} \).

(i) \( \Rightarrow \) (ii) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v). It follows from Propositions 2 and 3.

(v) \( \Rightarrow \) (i). From \( \mathcal{T} \ast \mathcal{Y}^{T} \ast \mathcal{T} = \mathcal{T} \), we obtain

\[
(I - \mathcal{Y}^{T} \ast \mathcal{T}) \in \text{rann}(\mathcal{T}) \subseteq \text{rann}(X) \quad \text{and} \quad (I - \mathcal{T} \ast \mathcal{Y}^{T}) \in \text{lann}(\mathcal{T}) \subseteq \text{lann}(\mathcal{Y}).
\]

Thus, \( \mathcal{Y} = \mathcal{Y} \ast \mathcal{Y}^{T} \ast \mathcal{T} \) and \( \mathcal{Y} = \mathcal{A} \ast \mathcal{Y} \ast \mathcal{Y} \ast \mathcal{Y} \ast \mathcal{A} \). This yields \( \mathcal{A} \ast \mathcal{Y} = \mathcal{A} \ast \mathcal{Y} \ast (\mathcal{A} \ast \mathcal{Y})^{T} \) and \( \mathcal{Y} \ast \mathcal{A} = (\mathcal{Y} \ast \mathcal{A})^{T} \ast \mathcal{Y} \ast \mathcal{A} \). Therefore, \( \mathcal{A} \ast \mathcal{Y} \) and \( \mathcal{Y} \ast \mathcal{A} \) are symmetric. Further, \( \mathcal{Y} \ast \mathcal{A} \ast \mathcal{Y} = \mathcal{Y} \ast (\mathcal{A} \ast \mathcal{Y})^{T} = \mathcal{Y} \ast \mathcal{Y}^{T} \ast \mathcal{T} = \mathcal{Y} \). Hence \( \mathcal{A}^{T} = \mathcal{Y} \).

This completes the proof.

Proposition 5. Let \( \mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n} \) and \( \mathcal{A} \ast \mathcal{A}^{T} = \mathcal{A}^{T} \ast \mathcal{A} \). Then

(i) there exists a \( \mathcal{X} \in \mathbb{R}^{n \times n \times \cdots \times n} \) such that \( \mathcal{A} \ast \mathcal{X} = \mathcal{T} \);
(ii) there exists a \( \mathcal{Y} \in \mathbb{R}^{n \times n \times \cdots \times n} \) such that \( \mathcal{T} \ast \mathcal{Y} = \mathcal{A} \).

Proof. Let \( \mathcal{A} \ast \mathcal{A}^{T} = \mathcal{A}^{T} \ast \mathcal{A} \). Then \( \mathcal{T} = (\mathcal{A} \ast \mathcal{A}^{T}) = (\mathcal{A} \ast \mathcal{A}^{T})^{T} = \mathcal{A} \ast \mathcal{A}^{T} \ast \mathcal{A} \), where \( \mathcal{A} = \mathcal{A}^{T} \ast \mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n} \). Similarly, the second part follows from \( \mathcal{A} = \mathcal{A} \ast \mathcal{A}^{T} \ast \mathcal{A} = \mathcal{A}^{T} \ast \mathcal{A} \ast \mathcal{A} = \mathcal{A} \ast \mathcal{A}^{T} \ast \mathcal{A} = \mathcal{A}^{T} \ast \mathcal{A} \), where \( \mathcal{Y} = (\mathcal{A}^{T})^{T} \ast \mathcal{A} \).

It is worth mentioning that Liang and Zheng explored in Reference 38 some identities for Moore–Penrose inverse of a tensor. Our next result discusses identities over a ring.

Theorem 15. Let \( \mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n} \). Then the following statements are true.

(i) \( (\mathcal{A} \ast \mathcal{A}^{T})^{T} = \mathcal{A}^{T} \ast (\mathcal{A}^{T})^{T} \) and \( (\mathcal{A} \ast \mathcal{A}^{T})^{T} = (\mathcal{A}^{T})^{T} \ast \mathcal{A} \).
(ii) \( \mathcal{A}^{T} = (\mathcal{A} \ast \mathcal{A})^{T} \ast \mathcal{A} = \mathcal{A}^{T} \ast (\mathcal{A} \ast \mathcal{A})^{T} \).

In the case of the Moore–Penrose inverse of tensors over a ring with involution, the reverse order law, that is, \( (\mathcal{A} \ast \mathcal{B})^{T} = \mathcal{B}^{T} \ast \mathcal{A}^{T} \), is not true in general. This can be seen from the Example 9 mentioned after our remark.

Remark 4. Theorem 15 (i) is not true if we replace \( \mathcal{A} \ast \mathcal{T} \) by any other tensor \( \mathcal{B} \), i.e., \( (\mathcal{A} \ast \mathcal{B})^{T} = \mathcal{B}^{T} \ast \mathcal{A}^{T} \), where \( \mathcal{A} \) and \( \mathcal{B} \in \mathbb{R}^{n \times n \times \cdots \times n} \).

Example 9. Let \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{2 \times 2 \times 2} \) with frontal slices

\[
\mathcal{A}(1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{A}(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{B}(2) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.
\]

Now

\[
\mathcal{A} \ast \mathcal{B} = \text{fold}(\text{circ}(\text{unfold}(\mathcal{A})) \ast \text{unfold}(\mathcal{B}))
\]

\[
= \text{fold} \left[ \begin{bmatrix} \mathcal{A}_{1} & \mathcal{A}_{2} \\ \mathcal{A}_{2} & \mathcal{A}_{1} \end{bmatrix} \ast \begin{bmatrix} \mathcal{B}_{1} \\ \mathcal{B}_{2} \end{bmatrix} \right] = \text{fold} \left[ \begin{bmatrix} \mathcal{A}_{1} \ast \mathcal{B}_{1} + \mathcal{A}_{2} \ast \mathcal{B}_{2} \\ \mathcal{A}_{2} \ast \mathcal{B}_{1} + \mathcal{A}_{1} \ast \mathcal{B}_{2} \end{bmatrix} \right] = \text{fold} \left[ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 3 \\ 1 & 2 \end{bmatrix} \right],
\]

\[
\mathcal{A}^{T} = \text{fold} \left[ \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & -1/8 \\ -1/2 & 1/4 \\ 1/4 & -1/8 \end{bmatrix} \right] \quad \text{and} \quad \mathcal{B}^{T} = \text{fold} \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \right].
\]
We can verify that

\[
\begin{pmatrix}
-2/5 & 1/2 \\
1/20 & -1/8 \\
-3/5 & 1/2 \\
9/20 & -1/8 \\
\end{pmatrix}
= (A \ast B)^\dagger \neq B^\dagger \ast A^\dagger = \text{fold}
\begin{pmatrix}
0 & 1/2 \\
1/4 & -1/8 \\
-1 & 1/2 \\
1/4 & -1/8 \\
\end{pmatrix}.
\]

Our next result deals with the commutative property of $A$ and $A^\dagger$.

**Theorem 16.** Let $A \in \mathbb{R}^{n \times n}$. If $A \ast A^T = A^T \ast A$, then $A \ast A^\dagger = A^\dagger \ast A$.

**Proof.** By using the definition of the Moore–Penrose inverse, we obtain $A \ast A^\dagger = (A^\dagger)^T \ast A^\dagger \ast A \ast A^T$. Using $A \ast A^T = A^T \ast A$, and Theorem 15, we get

\[
A \ast A^\dagger = (A^T)^\dagger \ast A \ast A^T = (A \ast A^T)^\dagger \ast A^T \ast A = (A^T \ast A)^\dagger \ast A^T \ast A = A^\dagger \ast A.
\]

The converse of the above result is not true in general as shown by the next example.

**Example 10.** Let $A \in \mathbb{R}^{2 \times 2}$, where

\[
A_{(1)} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix}, \quad A_{(3)} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}.
\]

Let $B$ be the transpose of $A$. Then

\[
B_{(1)} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{(3)} = \begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix}.
\]

We can verify that $A^\dagger \ast A = A \ast A^\dagger$,

\[
\text{fold}
\begin{pmatrix}
7 & 11 \\
11 & 43 \\
-1 & 3 \\
-3 & 23 \\
-1 & -3 \\
3 & 23 \\
\end{pmatrix}
= A^T \ast A \neq A \ast A^T = \text{fold}
\begin{pmatrix}
26 & 6 \\
6 & 24 \\
21 & 17 \\
15 & 1 \\
21 & 15 \\
17 & 1 \\
\end{pmatrix}.
\]

We next discuss how to compute the Moore–Penrose inverse through $f$-lower triangular tensors.

**Theorem 17.** Let $A \in \mathbb{R}^{n \times n}$. Suppose there exist a $f$-lower triangular tensor $L \in \mathbb{R}^{n \times n}$ such that $A \ast A^T = L \ast L^T$ and $L^T \ast L$ is invertible, then

\[
A^\dagger = A^T \ast L \ast (L^T \ast L)^{-2} \ast L^T.
\]

**Proof.** By Theorem 15, it is enough that show only $(L \ast L^T)^\dagger = L \ast (L^T \ast L)^{-2} \ast L^T$. Let $X = L \ast (L^T \ast L)^{-2} \ast L^T$ and $A = L \ast L^T$. Then

- $A \ast X \ast A = L \ast L^T \ast L \ast (L^T \ast L)^{-2} \ast L^T \ast L \ast L^T = L \ast L^T = A$,
- $X \ast A \ast X = L \ast (L^T \ast L)^{-2} \ast L^T \ast L \ast L^T \ast L \ast (L^T \ast L)^{-2} \ast L^T = L \ast (L^T \ast L)^{-2} \ast L^T = X$. 

Let \( \mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n} \). It is worth noting that the Matlab functions (i.e., zeros, sqrt) are used in Algorithm 3 on line 8 and line 13 to compute “zeros” and “square root” of matrices in the Fourier domain, respectively. Table 1 demonstrated the efficiency of the proposed Algorithm 3 in terms of time for computing the Moore–Penrose inverse by comparing the different order of random symmetric tensor with the algorithm 3 in Reference 35.

### 3.4 Weighted Moore–Penrose inverse

We introduce generalized weighted Moore–Penrose inverse an element over a ring \( \mathbb{R}^{n \times n \times \cdots \times n} \) as follows:

**Definition 13.** Let \( \mathcal{A}, \mathcal{M}, \mathcal{N} \in \mathbb{R}^{n \times n \times \cdots \times n} \), where \( \mathcal{M}, \mathcal{N} \) are invertible hermitian tensors. If a tensor \( \mathcal{Y} \in \mathcal{A}\{1, 2\} \) satisfies

\[
(3) (\mathcal{M} * \mathcal{A} * \mathcal{Y})^T = \mathcal{M} * \mathcal{A} * \mathcal{Y}; \quad (4) (\mathcal{N} * \mathcal{Y} * \mathcal{A})^T = \mathcal{N} * \mathcal{Y} * \mathcal{A},
\]

then \( \mathcal{Y} \) is called the generalized weighted Moore–Penrose inverse of \( \mathcal{A} \) and denoted by \( \mathcal{A}^{T}_{\mathcal{M}, \mathcal{N}} \).

The uniqueness of the generalized weighted Moore–Penrose inverse is proved in the next result.

**Proposition 6.** Let \( \mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n} \), and a pair of invertible hermitian tensors \( \mathcal{M} \in \mathbb{R}^{n \times n \times \cdots \times n} \) and \( \mathcal{N} \in \mathbb{R}^{n \times n \times \cdots \times n} \) be given. If the generalized weighted Moore–Penrose inverse exists then it is unique.

**Proof.** Suppose there exist \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) both satisfying the condition (1) – (4). Then

\[
\mathcal{X}_1 = \mathcal{X}_1 * \mathcal{A} * \mathcal{X}_1 = \mathcal{N}^{-1} * \mathcal{A}^T * \mathcal{X}_1^T * \mathcal{N} * \mathcal{X}_1 = \mathcal{N}^{-1} * \mathcal{A}^T * \mathcal{X}_2^T * \mathcal{A}^T * \mathcal{X}_1^T * \mathcal{N} * \mathcal{X}_1 \\
= \mathcal{N}^{-1} * \mathcal{A}^T * \mathcal{X}_2^T * \mathcal{N} * \mathcal{X}_1 \quad \text{and} \quad \mathcal{X}_1 = \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_1, \\
\mathcal{X}_2 = \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_2 = \mathcal{X}_2 * \mathcal{M}^{-1} * \mathcal{A}^T * \mathcal{M} = \mathcal{X}_2 * \mathcal{M}^{-1} * \mathcal{A}^T * \mathcal{X}_1^T * \mathcal{A}^T * \mathcal{M} \\
= \mathcal{X}_2 * \mathcal{M}^{-1} * \mathcal{A}^T * \mathcal{M} * \mathcal{A} * \mathcal{X}_1 = \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_1 = \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_1.
\]

Hence \( \mathcal{X}_1 = \mathcal{X}_2 * \mathcal{A} * \mathcal{X}_1 = \mathcal{X}_2. \)

The existence and computation of the generalized weighted Moore–Penrose inverse is discussed below.

| Order of \( \mathcal{A} \) | MT   | Algorithms                  |
|--------------------------|------|-----------------------------|
| 200 × 300 × 400          | 0.034663 | In Reference 35, algorithm 3 |
|                          | 0.022151 | Algorithm 3                 |
| 300 × 400 × 500          | 0.061981 | In Reference 35, algorithm 3 |
|                          | 0.037659 | Algorithm 3                 |
| 400 × 500 × 600          | 0.469042 | In Reference 35, algorithm 3 |
|                          | 0.135589 | Algorithm 3                 |
| 500 × 600 × 700          | 0.565222 | In Reference 35, algorithm 3 |
|                          | 0.149840 | Algorithm 3                 |
Algorithm 3. Computation of Moore–Penrose inverse of a tensor $A$

1: procedure MPI($A$)  
2: Input $p$, $n$, $n_3$, ..., $n_p$ and the tensor $A \in \mathbb{R}^{n \times n \times \cdots \times n_p}$.  
3: for $i \leftarrow 3$ to $p$ do  
4: $A = \text{fft}(A, [\ ], i)$;  
5: end for  
6: $C = n_3 n_4 \cdots n_p$.  
7: for $i \leftarrow 1$ to $C$ do  
8: $r = 0$; $L = \text{zeros(size}(A(:, :, i)))$;  
9: for $K \leftarrow 1$ to $n$ do  
10: $r = r + 1$;  
11: $L(k : n, r, i) = A(k : n, k, i) - L(k : n, 1 : (r - 1), i) \ast \text{transpose}(L(k, 1 : (r - 1), i))$;  
12: if then $L(k, r, i) > \epsilon$  
13: $L(k, r, i) = \text{sqrt}(L(k, r, i))$;  
14: if then $k < n$  
15: $L((k + 1) : n, r, i) = L((k + 1) : n, r, i) / L(k, r, i)$;  
16: end if  
17: else  
18: $r = r - 1$;  
19: end if  
20: end for  
21: $L(:, :, i) = L(:, 1 : r, i)$;  
22: end for  
23: Compute $W = A^T \ast L \ast (L^T \ast L)^{-2} \ast L^T$.  
24: for $i \leftarrow p$ to $3$ do  
25: $A' = \text{ifft}(W, [\ ], i)$;  
26: end for  
27: return $A'$  
28: end procedure

Theorem 18. Let $A \in \mathbb{R}^{n \times n \times \cdots \times n_p}$ and $M$, $N \in \mathbb{R}^{n \times n \times \cdots \times n_p}$ be inevitable hermitian tensors. Then the following statements are equivalent:

(i) $A_{M,N}^\dagger$ exists.

(ii) There exist unique idempotent tensors $P$, $Q \in \mathbb{R}^{n \times n \times \cdots \times n_p}$ such that

\[
M \ast P = (M \ast P)^T, \quad N \ast Q = (N \ast Q)^T, \quad P \ast \mathbb{R}^{n \times n \times \cdots \times n_p} = A \ast \mathbb{R}^{n \times n \times \cdots \times n_p}, \quad \text{and} \quad \mathbb{R}^{n \times n \times \cdots \times n_p} \ast Q = \mathbb{R}^{n \times n \times \cdots \times n_p} \ast A.
\]

If any one of the statements (i), (ii) holds, then $A_{M,N}^\dagger = Q \ast A^{(1)} \ast P$ and thus is invariant for any choice of $A^{(1)}$.

Proof. (i) $\Rightarrow$ (ii). Let $A' = A_{M,N}^\dagger$. If we define $P = A \ast A'$ and $Q = A' \ast A$, then $P \ast \mathbb{R}^{n \times n \times \cdots \times n_p} = A \ast \mathbb{R}^{n \times n \times \cdots \times n_p}$ and $\mathbb{R}^{n \times n \times \cdots \times n_p} \ast Q = \mathbb{R}^{n \times n \times \cdots \times n_p} \ast A$ can be shown easily. Further $P = A \ast A' = A \ast A' \ast A \ast A' = P^2$, $Q = A' \ast A = A' \ast A \ast A' = Q^2$. $M \ast P = M \ast A \ast A' = (M \ast A \ast A')^T = (M \ast P)^T$, and $N \ast Q = N \ast A \ast A' = (N \ast A \ast A')^T = (N \ast Q)^T$. To show the uniqueness of $P$ and $Q$, suppose there exists two idempotent pairs $(P, Q)$ and $(P_1, Q_1)$ which satisfies (b). Now from $\mathbb{R}^{n \times n \times \cdots \times n_p} \ast Q = \mathbb{R}^{n \times n \times \cdots \times n_p} \ast Q_1$, we have $Q = Q \ast Q_1$ and $Q_1 = Q_1 \ast Q$. From $Q = Q \ast Q_1$ and $Q_1 = Q_1 \ast Q$, we get

\[
N \ast Q = (N \ast Q)^* = (N \ast Q \ast N^{-1} \ast N \ast Q_1)^T = N \ast Q_1 \ast N^{-1} \ast N \ast Q = N \ast Q_1 \ast Q = N \ast Q_1.
\]

Premultiplying (4) by $N^{-1}$, we obtain $Q = Q_1$. Using the similar lines, we can prove the uniqueness of $P$. 
(ii) ⇒ (i). Let $P$ and $Q$ be the unique idempotent tensors such that $M * P = (M * P)^T$, $N * Q = (N * Q)^T$, $P * R^{n \times \ldots \times n} = A * R^{n \times \ldots \times n}$ and $R^{n \times \ldots \times n} * Q = R^{n \times \ldots \times n} * A$. Then $A = P * A = A * Q$, $P = A * U$, and $Q = V * A$ for some $U, V \in R^{n \times \ldots \times n}$. In addition, $P = A * A^{(1)} * A * U = A * A^{(1)} * P$ and $Q = Q * A^{(1)} * A$. Now, consider $\mathcal{Y} = Q * A^{(1)} * P$. Then $A_{M,N}^T = \mathcal{Y} = Q * A^{(1)} * P$ is follows from the following identities:

- $A * \mathcal{Y} * A = A * Q * A^{(1)} * P * A = A$,
- $\mathcal{Y} * A * \mathcal{Y} = Q * A^{(1)} * P * A * Q * A^{(1)} * P = \mathcal{Y}$;
- $M * A * \mathcal{Y} = M * A * Q * A^{(1)} * P = M * P = (M * A * \mathcal{Y})^T$,
- $N * \mathcal{Y} * A = N * Q * A^{(1)} * P * A = N * Q = (N * \mathcal{Y} * A)^T$.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two elements of $A(1)$. From $P * R^{n \times \ldots \times n} = A * R^{n \times \ldots \times n}$ and $R^{n \times \ldots \times n} * Q = R^{n \times \ldots \times n} * A$, we obtain

$$P = A * \mathcal{Y} \text{ and } Q = U * A \text{ for some } U, \mathcal{V} \in R^{n \times \ldots \times n}.$$  

Using (5), we have

$$A_{M,N}^T = Q * \mathcal{X}_1 * P = U * A * \mathcal{X}_1 * A = \mathcal{Y} = U * A * \mathcal{X}_2 * A * \mathcal{V} = Q * \mathcal{X}_2 * P.$$  

Thus $A_{M,N}^T$ is an invariant for any choice of $A(1)$.

Now we present an algorithm (see the Algorithm 4) for computing square root of a symmetric positive definite tensor, which will be used for computation of the weighted Moore–Penrose inverse by using the Moore–Penrose inverse. Here, the matrix computation Matlab functions (i.e., eig, sqrt) are utilized in Algorithm 4 to compute square root of a symmetric positive definite tensor $A \in R^{n \times \ldots \times n}$. In fact, the function “eig” uses to compute the eigenvalue of matrices in the Fourier domain.

An equivalent characterization for existence of the generalized weighted Moore–Penrose inverse is presented in the next result.

**Algorithm 4.** Computation of square root of a symmetric positive definite tensor $A$

1: procedure SQRT($A$)
2: Input $p, n, n_3, \ldots, n_p$ and the tensor $A \in R^{n \times \ldots \times n}$.
3: for $i \leftarrow 3$ to $p$ do
4: $A = \text{fft}(A, [\cdot], i)$;
5: end for
6: $C = n_3n_4 \ldots n_p$
7: for $i \leftarrow 1$ to $C$ do
8: $[\mathcal{V}(\cdot, \cdot, i), D(\cdot, \cdot, i)] = \text{eig}(A(\cdot, \cdot, i))$;
9: end for
10: for $i \leftarrow 1$ to $C$ do
11: $S(\cdot, \cdot, i) = \text{sqrt}(D(\cdot, \cdot, i))\text{inv}(D(\cdot, \cdot, i))$;
12: end for
13: for $i \leftarrow p$ to $3$ do
14: $\mathcal{X} = \text{ifft}(S, [\cdot], i)$;
15: end for
16: return $S$ \quad \triangleright $S$ is the equal to $A^{1/2}$
17: end procedure

**Theorem 19.** Let $M, N \in R^{n \times \ldots \times n}$ be an invertible hermitian tensors and $A \in R^{n \times \ldots \times n}$. If $M^{1/2}$ and $N^{-1/2}$ (the square root of $M$ and $N^{-1}$ respectively) are exists, then the generalized weighted Moore–Penrose inverse of $A$ exists.
Moreover

\[ A_{M,N}^\dagger = N^{-1/2} \ast (M^{1/2} \ast A \ast N^{-1/2})^\dagger \ast M^{1/2}. \]

Proof. Let \( \mathcal{Y} = (M^{1/2} \ast A \ast N^{-1/2})^\dagger \) and \( \mathcal{X} = N^{-1/2} \ast \mathcal{Y} \ast M^{1/2} \). From the conditions \( M^{1/2} \ast A \ast N^{-1/2} \ast \mathcal{Y} \ast M^{1/2} \ast A \ast N^{-1/2} = M^{1/2} \ast A \ast N^{-1/2} \ast \mathcal{Y} \ast M^{1/2} \) and \( \mathcal{Y} \ast M^{1/2} \ast A \ast N^{-1/2} \ast \mathcal{Y} = \mathcal{Y} \), we obtain \( A \ast \mathcal{X} \ast A = A \) and \( \mathcal{X} \ast A \ast \mathcal{X} = \mathcal{X} \). Further

\[ (M \ast A \ast \mathcal{X})^T = (M \ast A \ast N^{-1/2} \ast \mathcal{Y} \ast M^{1/2})^T = (M^{1/2} \ast M^{1/2} \ast A \ast N^{-1/2} \ast \mathcal{Y} \ast M^{1/2})^T = M \ast A \ast \mathcal{X}, \]

\[ (N \ast \mathcal{X} \ast A)^T = (N^{1/2} \ast \mathcal{Y} \ast M^{1/2} \ast A)^T = (N^{1/2} \ast \mathcal{Y} \ast M^{1/2} \ast A \ast N^{-1/2} \ast N^{1/2})^T = N^{1/2} \ast \mathcal{Y} \ast M^{1/2} \ast A = N \ast \mathcal{X} \ast A. \]

We employ algorithm 5 for computing the weighted Moore–Penrose inverse. In this algorithm we use the Matlab function “pinv” to compute the Moore-Penrose inverse of matrices in the Fourier domain.

**Algorithm 5.** Computation of Weighted Moore–Penrose inverse of a tensor \( A \)

1. **procedure** WMPI(A)
2. **Input** \( p, n, n_3, \ldots, n_p \) and the tensor \( A, M, N \in \mathbb{R}^{n \times n \times \cdots \times n_p} \),
3. **for** \( i \leftarrow 3 \) to \( p \) **do**
4. \( A = \text{fft}(A, [\ ], i); M = \text{fft}(M, [\ ], i); N = \text{fft}(N, [\ ], i); \)
5. **end for**
6. Compute \( S := M^{1/2}, T := N^{-1/2} \) by using Algorithm 4.
7. \( C = n_3 n_4 \cdots n_p. \)
8. **for** \( i \leftarrow 1 \) to \( C \) **do**
9. \( Z(:, :, i) = \text{pinv}(S \ast A \ast T)(:, :, i); \)
10. **end for**
11. **for** \( i \leftarrow 1 \) to \( C \) **do**
12. \( W(:, :, i) = (T \ast Z \ast S)(:, :, i); \)
13. **end for**
14. **for** \( i \leftarrow p \) to \( 3 \) **do**
15. \( \mathcal{X} \leftarrow \text{ifft}(W, [\ ], i); \)
16. **end for**
17. **return** \( \mathcal{X} \)
18. **end procedure**

Further, the Algorithm 5 is validated in the following example.

**Example 11.** Let \( A, B \in \mathbb{R}^{n \times n \times n \times \cdots \times n_p} \) with

\[ A_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}, \quad A_{(3)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \]

\[ M_{(1)} = \begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}, \quad M_{(2)} = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}, \quad M_{(3)} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}, \quad N_{(1)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad N_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
Using the Algorithm 4 we obtain,
\[ \mathcal{M}^{1/2} = \text{fold} \begin{pmatrix} 895/557 & 77/1098 \\ 77/1098 & 583/305 \\ 131/494 & 77/1098 \\ 179/346 & 1372/1349 \\ 131/494 & 179/346 \\ 77/1098 & 1372/1349 \end{pmatrix} \quad \text{and} \quad \mathcal{N}^{-1/2} = \text{fold} \begin{pmatrix} 1/6 & 7/24 \\ -1/3 & -7/12 \\ -1/6 & -1/24 \\ 1/3 & 5/12 \\ 1/6 & -3/8 \\ 0 & 5/12 \end{pmatrix}. \]

By applying Algorithm 5, we get
\[ \mathcal{A}_{\mathcal{M},\mathcal{N}}^{1} = \text{fold} \begin{pmatrix} 3/26 & -11/26 \\ -4/13 & 6/13 \\ 9/26 & -7/26 \\ 1/13 & 5/13 \\ 1/26 & -21/26 \\ 3/13 & 2/13 \end{pmatrix}. \]

4 | IMAGE DEBLURRING

Signal and image processing are still a major challenge and has stayed as a pre-occupation for the scientific community. The inclusion of ring theory to the spatial analysis of digital images and computer vision tasks has been carried out in.\(^{39}\) In this section, we apply the Moore–Penrose inverse in image reconstruction problem. The discrete model for the two-dimensional (2D) image blurring (see Reference 40) is represented as
\[ \mathcal{A}x = b, \quad (6) \]
where \( \mathcal{A} \) is the blurring matrix and has some special structure like a banded matrix, Toeplitz or block-Toeplitz matrix (see References 16,41). Here \( x \) is the true image and \( b \) is the blurred image. In practice, \( b \) is corrupted with noise and the blurred matrix \( \mathcal{A} \) is ill-conditioned. Such type of ill-posed problems are also observed in the discretization of Fredholm integral equations of the first kind, noisy image restoration, computer tomography, and inverse problems within electromagnetic flow. Ill-posed problems were extensively studied in the context of an inverse problem and image restorations. One can find more details on image restoration and deblurring in References 42-44.

The three-dimensional (3D) color image blurring problem, often occurs in medical or geographic imaging. It can be written as a tensor equation
\[ \mathcal{A} \ast \mathcal{X} = \mathcal{B}, \quad (7) \]
where \( \mathcal{A} \) is the known blurring tensor. Further, \( \mathcal{X} \) and \( \mathcal{B} \) are tensors representing the true image and the blurred image, often corrupt with noise, respectively. In image restoration, the main objective is to establish a blurred free image that requires the approximate solution of a multilinear system given by the Equation (7). To find the approximate solution of the ill-posed system namely system (7), several iterative methods such as preconditioned LSQR (see Reference 16), conjugate gradient (CG), \( t \)-singular value decomposition (\( t \)-SVD), Golub–Kahan iterative bidiagonalization (G-K-Bi-diag), for details see References 17,44.

It is well known that RGB image is a third-order tensor, and they often represent the intensities in the red, green, and blue scales. Consider a original error-free color image \( \mathcal{X} \) of size \( n \times n \times 3 \). Let \( X_{(1)}, X_{(2)}, \) and \( X_{(3)} \) be the slices of size \( n \times n \) that constitute the three channels of the image \( \mathcal{X} \) represent the color information. Similarly, consider \( B_{(1)}, B_{(2)}, \) and \( B_{(3)} \) are the slices of size \( n \times n \) that associated with error-free blurred color image \( \mathcal{B} \). Consider both cross-channel and
within-channel blurring take place in the blurring process of the original image.\textsuperscript{40} Now, define $\text{vec}$ to take an $n \times n$ matrix and return a $n^2 \times 1$ vector by stacking the columns of the matrix from left to right. We now describe the following blurring model as the equivalent form of the Equation (7),

$$
\begin{pmatrix}
A^c \otimes A^h \otimes A^v
\end{pmatrix}
\begin{pmatrix}
\text{vec}(X_{(1)}) \\
\text{vec}(X_{(2)}) \\
\text{vec}(X_{(3)})
\end{pmatrix}
= 
\begin{pmatrix}
\text{vec}(B_{(1)}) \\
\text{vec}(B_{(2)}) \\
\text{vec}(B_{(3)})
\end{pmatrix},
$$

(8)

where $\otimes$ denotes the Kronecker product of the matrices. $A^h \in \mathbb{R}^{n \times n}$ and $A^v \in \mathbb{R}^{n \times n}$ are horizontal and vertical within-channel blurring matrices, respectively. Further, $A^c$ is the cross-channel blurring matrix of size $3 \times 3$ as follows:

$$
A^c = 
\begin{pmatrix}
c_1 & c_3 & c_2 \\
c_2 & c_1 & c_3 \\
c_3 & c_2 & c_1
\end{pmatrix}
\quad \text{with} \quad \sum_{i=1}^{3} c_i = 1 \quad \text{and} \quad c_i \in \mathbb{R} \quad \text{for} \quad i = 1, 2, 3.
$$

Following the circulant structure of the cross-channel blurring matrix, we can write the following equivalent tensor-tensor model:

$$
A * X * C = B,
$$

(9)
where $A(\vdots, k) = c_k A^v$, for $k = 1, 2, 3$ with $C(\vdots, 1) = (A^h)^T$, and $C(\vdots, 2) = C(\vdots, 3) = 0$. We construct the blurring tensor $A$ using the following symmetric banded Toeplitz matrix:

$$A^v_{ij} = A^h_{ij} = \begin{cases} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(i-j)^2}{2\sigma^2}}, & |i-j| \leq k \\ 0, & \text{otherwise} \end{cases}$$

Here $\sigma$ controls the amount of smoothing, that is, the more ill posed the problem when $\sigma$ is larger. Further, we use $c_1 = 0.8$ and $c_2 = c_3 = 0.1$ in the cross-channel blurring matrix $A^c$. For numerical experiment, we consider two $256 \times 256 \times 3$ colour images, and present in Figure 1a,d. Using $\sigma = 4$ and $k = 6$, we generate the blurred image $B = A \ast A^c \ast C + \mathcal{N}$, where $\mathcal{N}$ is a noise tensor distributed normally with mean 0 and variance $10^{-3}$. The blurred noisy image is shown in the Figure 1b,d. Finally, we have reconstructed the true image using the Moore–Penrose inverse, that is, the Algorithm 3. The resulting reconstruction is given in the Figure 1c,e.

## 5 CONCLUSION

We have introduced the notion of generalized inverse of tensors over a ring. Our intention is to generalize some known results on generalized inverse of matrices to tensors over the algebraic structure of a ring. Since this tensor product is not a simple extension of the matrix product, we explore effective algorithms for computing inner inverses, the Moore–Penrose inverse and weighted Moore–Penrose inverse of tensors together with a few supporting algorithms for these inverses, including transpose of a tensor and square root of a symmetric positive definite tensor. Finally, the algorithm is used to restore the deblurring image via the Moore–Penrose inverse.

## ACKNOWLEDGEMENTS

The first and the third authors are grateful to the Mohapatra Family Foundation and the College of Graduate Studies of the University of Central Florida for their support for this research.

## CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

## DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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How to cite this article: Behera R, Sahoo JK, Mohapatra RN, Nashed MZ. Computation of generalized inverses of tensors via t-product. Numer Linear Algebra Appl. 2022;29:e2416. https://doi.org/10.1002/nla.2416