The Gauss–Bonnet Theorem for Coherent Tangent Bundles over Surfaces with Boundary and Its Applications

Wojciech Domitrz · Michał Zwierzyński

Received: 25 February 2018 / Published online: 29 April 2019 © The Author(s) 2019

Abstract
In Saji et al. (J Math 62:259–280, 2008; Ann Math 169:491–529, 2009; J Geom Anal 222):383–409, 2012) the Gauss–Bonnet formulas for coherent tangent bundles over compact-oriented surfaces (without boundary) were proved. We establish the Gauss–Bonnet theorem for coherent tangent bundles over compact-oriented surfaces with boundary. We apply this theorem to investigate global properties of maps between surfaces with boundary. As a corollary of our results, we obtain a special version of Fukuda–Ishikawa’s theorem. We also study geometry of the affine-extended wave fronts for planar-closed non-singular hedgehogs (rosettes). In particular, we find a link between the total geodesic curvature on the boundary and the total singular curvature of the affine-extended wave front, which leads to a relation of integrals of functions of the width of a rosette.

Keywords Coherent tangent bundle · Wave front · Gauss–Bonnet formula

Mathematics Subject Classification Primary: 57R45 · Secondary: 53A05

1 Introduction
The local and global geometry of fronts and coherent tangent bundles, which are natural generalizations of fronts, has been recently very carefully studied in [19,29,30,35–38]. In particular in [35,36] the results of Kossowski [20,21] and Langevin et al. [24] were

The work of W. Domitrz and M. Zwierzyński was partially supported by NCN Grant No. DEC-2013/11/B/ST1/03080.

Michał Zwierzyński
zwierzynskim@mini.pw.edu.pl

Wojciech Domitrz
domitrz@mini.pw.edu.pl

1 Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland
generalized to the following Gauss–Bonnet-type formulas for the singular coherent tangent bundle $\mathcal{E}$ over a compact surface $M$ whose set of singular points $\Sigma$ admits at most peaks:

$$2\pi \chi(M) = \int_M K \, dA + 2 \int_{\Sigma} \kappa_s \, d\tau,$$

$$\frac{1}{2\pi} \int_M K \, d\hat{A} = \chi(M^+) - \chi(M^-) + \#P^+ - \#P^-.$$  \tag{1.1, 1.2}

In the above formulas $K$ is the Gaussian curvature, $\kappa_s$ is the singular curvature, $d\tau$ is the arc length measure on $\Sigma$, $d\hat{A}$ (respectively $dA$) is the signed (respectively unsigned) area form, $M^+$ (respectively $M^-$) is the set of regular points in $M$, where $d\hat{A} = dA$ (respectively $d\hat{A} = -dA$), $P^+$ (respectively $P^-$) is the set of positive (respectively negative) peaks (see [35] and Sect. 2 for details). Saji et al. also found several interesting applications of the above formulas (see especially [37]).

The classical Gauss–Bonnet theorem was formulated for compact-oriented surfaces with boundary. Therefore, it is natural to find the analogous Gauss–Bonnet formulas for coherent tangent bundles over compact-oriented surfaces with boundary (see Theorem 2.20). Coherent tangent bundles over compact oriented surfaces with boundary also appear in many problems. In this paper, we apply the Gauss–Bonnet formulas to study smooth maps between compact-oriented surfaces with boundary and affine-extended wave fronts of the planar non-singular hedgehogs (rosettes). As a result, we obtain a new proof of a special version of Fukuda–Ishikawa’s theorem [12] and we find a link between the total geodesic curvature on the boundary and the total singular curvature of the affine-extended wave front of a rosette. This leads to a relation between the integrals of the function of the width of the rosette, in particular of the width of an oval (see Theorem 5.24 and Conjecture 5.28).

In Sect. 2, we briefly sketch the theory of coherent tangent bundles and state the Gauss–Bonnet theorem for coherent tangent bundles over compact-oriented surfaces with boundary (Theorem 2.20), which is the main result of this paper. The proof of Theorem 2.20 is presented in Sect. 3. We apply this theorem to study the global properties of maps between compact-oriented surfaces with boundary in Sect. 4. The last section contains the results on the geometry of the affine-extended wave fronts of rosettes.

## 2 The Gauss–Bonnet Theorem

In this section, we formulate the Gauss–Bonnet-type theorem for coherent tangent bundles over compact-oriented surfaces with boundary. The proof of this theorem is presented in the next section. Coherent tangent bundles are intrinsic formulation of wave fronts. The theory of coherent tangent bundles were introduced and developed in [35–37]. We recall basic definitions and facts of this theory (for details see [35,37]).

**Definition 2.1** Let $M$ be a 2-dimensional compact-oriented surface (possibly with boundary). A **coherent tangent bundle** over $M$ is a 5-tuple $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \psi)$, where
The Gauss–Bonnet Theorem

$\mathcal{E}$ is an orientable vector bundle over $M$ of rank 2, $\langle \cdot, \cdot \rangle$ is a metric, $D$ is a metric connection on $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ and $\psi$ is a bundle homomorphism

$$\psi : TM \to \mathcal{E},$$

such that for any smooth vector fields $X, Y$ on $M$

$$DX \psi(Y) - DY \psi(X) = \psi([X, Y]). \quad (2.1)$$

The pull-back metric $d\hat{s}^2 := \psi^* \langle \cdot, \cdot \rangle$ is called the first fundamental form on $M$. Let $E_p$ denote the fiber of $E$ at a point $p \in M$. If $\psi_p := \psi_{|T_pM} : T_pM \to E_p$ is not a bijection at a point $p \in M$, then $p$ is called a singular point. Let $\Sigma$ denote the set of singular points on $M$. If a point $p \in M$ is not a singular point, then $p$ is called a regular point. Let us notice that the first fundamental form on $M$ is positive definite at regular points and it is not positive definite at singular points.

Let $\mu \in \text{Sec}(E^* \wedge E^*)$ be a smooth non-vanishing skew-symmetric bilinear section such that for any orthonormal frame $\{e_1, e_2\}$ on $E \mu(e_1, e_2) = \pm 1$. The existence of such $\mu$ is a consequence of the assumption that $E$ is orientable. A co-orientation of the coherent tangent bundle is a choice of $\mu$. An orthonormal frame $\{e_1, e_2\}$ such that $\mu(e_1, e_2) = 1$ (respectively $\mu(e_1, e_2) = -1$) is called positive (respectively negative) with respect to the co-orientation $\mu$.

From now on, we fix a co-orientation $\mu$ on the coherent tangent bundle.

**Definition 2.2** Let $(U; u, v)$ be a positively oriented local coordinate system on $M$. Then $d\hat{A} := \psi^* \mu = \lambda_\psi du \wedge dv$ (respectively $dA := |\lambda_\psi| du \wedge dv$) is called the signed area form (respectively the unsigned area form), where

$$\lambda_\psi := \mu(\psi_u, \psi_v), \psi_u := \psi\left(\frac{\partial}{\partial u}\right), \psi_v := \psi\left(\frac{\partial}{\partial v}\right).$$

The function $\lambda_\psi$ is called the signed area density function on $U$.

The set of singular points on $U$ is expressed as

$$\Sigma \cap U := \{p \in U : \lambda_\psi(p) = 0\}.$$

Let us notice that the signed and unsigned area forms, $d\hat{A}$ and $dA$, give globally defined 2-forms on $M$ and they are independent of the choice of positively oriented local coordinate system $(u, v)$. Let us define

$$M^+ := \left\{ p \in M \setminus \Sigma \mid d\hat{A}_p = dA_p \right\}, \quad M^- := \left\{ p \in M \setminus \Sigma \mid d\hat{A}_p = -dA_p \right\}.$$

We say that a singular point $p \in \Sigma$ is non-degenerate if $d\lambda_\psi$ does not vanish at $p$. Let $p$ be a non-degenerate singular point. There exists a neighborhood $U$ of $p$ such that the set $\Sigma \cap U$ is a regular curve, which is called the singular curve. The singular direction is the tangential direction of the singular curve. Since $p$ is non-degenerate,
the rank of $\psi_p$ is 1. The *null direction* is the direction of the kernel of $\psi_p$. Let $\eta(t)$ be the smooth (non-vanishing) vector field along the singular curve $\sigma(t)$ which gives the null direction.

Let $\wedge$ be the exterior product on $TM$.

**Definition 2.3** Let $p \in M$ be a non-degenerate singular point and let $\sigma(t)$ be a singular curve such that $\sigma(0) = p$. The point $p$ is called an $A_2$-point (or an intrinsic cuspidal edge) if the null direction at $p$ (i.e. $\eta(0)$) is transversal to the singular direction at $p$ (i.e. $\dot{\sigma}(0) := \frac{d\sigma}{dt}|_{t=0}$). The point $p$ is called an $A_3$-point (or an intrinsic swallowtail) if the point $p$ is not an $A_2$-point and

$$\frac{d}{dt}(\dot{\sigma}(t) \wedge \eta(t))|_{t=0} \neq 0.$$ 

**Definition 2.4** Let $p$ be a singular point $p \in M$ which is not an $A_2$-point. The point $p$ is called a peak if there exists a coordinate neighborhood $(U; u, v)$ of $p$ such that:

(i) if $q \in (\Sigma \cap U) \setminus \{p\}$ then $q$ is an $A_2$-point;
(ii) the rank of the linear map $\psi_p : T_p M \to \mathcal{E}_p$ at $p$ is equal to 1;
(iii) the set $\Sigma \cap U$ consists of finitely many $C^1$-regular curves emanating from $p$.

A peak is a non-degenerate if it is a non-degenerate singular point.

From now on, we assume that the set of singular points $\Sigma$ admits at most peaks, i.e. $\Sigma$ consists of $A_2$-points and peaks.

Furthermore, let us fix a Riemannian metric $g$ on $M$. Since the first fundamental form $ds^2$ degenerates on $\Sigma$, there exists a $(1, 1)$-tensor field $I$ on $M$ such that

$$ds^2(X, Y) = g(I X, Y),$$

for smooth vector fields $X, Y$ on $M$. We fix a singular point $p \in \Sigma$. Since $\Sigma$ admits at most peaks, the point $p$ is an $A_2$-point or a peak. Let $\lambda_1(p), \lambda_2(p)$ be the eigenvalues of $I_p := I|_{T_p M} : T_p M \to T_p M$. Since the kernel of $\psi_p$ is one-dimensional, the only one of $\lambda_1(p), \lambda_2(p)$ vanishes. Let us assume that $\lambda_1(p) = 0$. Then $\lambda_2(p) > 0$. Thus, there exists a neighborhood $V$ of $p$ such that for every point $q \in V$ the map $I_q$ has two distinct eigenvalues $\lambda_1(q), \lambda_2(q)$, such that $0 \leq \lambda_1(q) < \lambda_2(q)$. Furthermore, there exists a coordinate neighborhood $(U; u, v)$ of $p$ such that $U$ is a subset of $V$ and the $u$-curves (respectively $v$-curves) give the $\lambda_1$-eigendirections (respectively $\lambda_2$-eigendirections), because the eigenvectors of eigenvalues $\lambda_1(q), \lambda_2(q)$ depends smoothly on $q$. Such a local coordinate system $(U; u, v)$ is called a $g$-coordinate system at $p$.

**Definition 2.5** Let $\gamma(t) \ (0 \leq t < 1)$ be a $C^1$-regular curve on $M$ such that $\gamma(0) = p$. The $\mathcal{E}$-initial vector of $\gamma$ at $p$ is the following limit

$$\Psi_\gamma := \lim_{t \to 0^+} \frac{\psi(\dot{\gamma}(t))}{|\psi(\dot{\gamma}(t))|} \in \mathcal{E}_p \quad (2.2)$$

if it exists.
Remark 2.6 If \( p \) is a regular point of \( M \) then the \( \mathcal{E} \)-initial vector of \( \gamma \) at \( p \) is the unit tangent vector of \( \gamma \) at \( p \) with respect to the first fundamental form \( ds^2 \).

Proposition 2.7 (Proposition 2.6 in [35]). Let \( \gamma \) be a \( C^1 \)-regular curve emanating from an \( A_2 \)-point or a peak \( p \) such that \( \dot{\gamma}(0) \) is not a null vector or \( \gamma \) is a singular curve. Then, the \( \mathcal{E} \)-initial vector of \( \gamma \) at \( p \) exists.

Since, we study coherent tangent bundles over surfaces with boundary, we also consider a curve \( \gamma \) on the boundary which is tangent to the null direction at a singular point \( p \) on the boundary. We prove that in this case the \( \mathcal{E} \)-initial vector of \( \gamma \) at \( p \) exists if the singular direction is transversal to the boundary at \( p \).

Proposition 2.8 Let \((\mathcal{E}, \cdot , D, \psi)\) be a coherent tangent bundle over an compact oriented surface \( M \) with boundary. Let \( p \) be an \( A_2 \)-point in the boundary \( \partial M \). If the boundary \( \partial M \) is transversal to \( \Sigma \) at \( p \) and \( \gamma : (-\epsilon, \epsilon) \to \partial M \) is a \( C^2 \)-regular curve such that \( \gamma(0) = p \), \( \gamma((-\epsilon, \epsilon)) \cap \Sigma = \{p\} \) and \( \dot{\gamma}(0) \) is the null vector at \( p \), then the \( \mathcal{E} \)-initial vector \( \Psi_\gamma \) of \( \gamma \) at \( p \) exists, \( D_{\frac{\partial}{\partial t}} (\psi(\dot{\gamma}(t))) \big|_{t=0} \neq 0 \), and

\[
\Psi_\gamma = \frac{D_{\frac{\partial}{\partial t}} (\psi(\dot{\gamma}(t)))}{|D_{\frac{\partial}{\partial t}} (\psi(\dot{\gamma}(t)))|_{t=0}} \in \mathcal{E}_p. \tag{2.3}
\]

Proof Let \( \sigma : [0, \epsilon) \to \Sigma \) be a singular curve such that \( \sigma(0) = p \). Let \((U; u, v)\) be a \( g \)-coordinate system at \( p \) i.e. the null direction at \( \sigma(t) \) is spanned by \( \frac{\partial}{\partial u} \). Since \( \lambda_\psi(\sigma(t)) = 0 \), we get that

\[
\frac{d}{dt} (\lambda_\psi(\sigma(t))) \big|_{t=0} = d\lambda_\psi \big|_p \cdot \dot{\sigma}(0) = 0. \tag{2.4}
\]

Let us notice that

\[
\frac{d}{dt} (\lambda_\psi(\gamma(t))) \big|_{t=0} = d\lambda_\psi \big|_p \cdot \dot{\gamma}(0) \neq 0 \tag{2.5}
\]

since the vectors \( \dot{\sigma}(0) \) and \( \dot{\gamma}(0) \) span the space \( T_pM \) and \( d\lambda_\psi \big|_p \neq 0 \).

On the other hand, since \( \lambda_\psi(\gamma(t)) = \mu (\psi_u(\gamma(t)), \psi_v(\gamma(t))) \) and \( \psi_u(\gamma(0)) = 0 \), we get the following:

\[
\frac{d}{dt} (\lambda_\psi(\gamma(t))) \big|_{t=0} = \frac{d}{dt} \mu (\psi_u(\gamma(t)), \psi_v(\gamma(t))) \big|_{t=0} = \mu \left( D_{\frac{\partial}{\partial t}} (\psi_u(\gamma(t))) \big|_{t=0} , \psi_v(\gamma(0)) \right) \tag{2.6}
\]

By (2.5) and (2.6) we get that \( D_{\frac{\partial}{\partial t}} (\psi_u(\gamma(t))) \big|_{t=0} , \psi_v(\gamma(0)) \) are linearly independent.

The vector field \( \dot{\gamma} \) can be written in the following form \( \dot{\gamma}(t) = \dot{u}(t) \frac{\partial}{\partial u} + \dot{v}(t) \frac{\partial}{\partial v} \), where \( u(t) = t(a + h(t)), v(t) = t^2 g(t), a \neq 0 \) and \( h, g \) are some functions such that
\( h(0) = 0 \). Similarly, since \( \psi_u(\gamma(0)) = 0 \) and \( D_{\overline{\gamma}} (\psi_u(\gamma(t)))|_{t=0} \neq 0 \) we can write \( \psi_u(\gamma(t)) = t\xi(t), \) where \( \xi(t) \in \mathcal{E}_{\gamma(t)} \) and \( \xi(0) \neq 0 \).

Now, we will prove the formula (2.3).

\[
\lim_{t \to 0^+} \frac{\psi(\dot{\gamma}(t))}{\psi(\dot{\gamma}(t))} = \lim_{t \to 0^+} \frac{\dot{u}(t)\psi_u(\gamma(t)) + \dot{v}(t)\psi_v(\gamma(t))}{\dot{u}(t)\psi_u(\gamma(t)) + \dot{v}(t)\psi_v(\gamma(t))} = \lim_{t \to 0^+} \frac{t((a + h(t) + t\dot{h}(t))\xi(t) + (2g(t) + t\dot{g}(t))\psi_v(\gamma(t)))}{t((a + h(t) + t\dot{h}(t))\xi(t) + (2g(t) + t\dot{g}(t))\psi_v(\gamma(t)))} = \frac{a\xi(0) + 2g(0)\psi_v(\gamma(0))}{|a\xi(0) + 2g(0)\psi_v(\gamma(0))|},
\]

where the expression \( a\xi(0) + 2g(0)\psi_v(\gamma(0)) \) is non-zero since the vectors \( \xi(0) = D_{\overline{\gamma}} (\psi_u(\gamma(t)))|_{t=0} \) and \( \psi_v(\gamma(0)) \) are linearly independent.

Since \( D_{\overline{\gamma}} (\psi(\dot{\gamma}(t)))|_{t=0} = a\xi(0) + 2g(0)\psi_v(\gamma(0)) \), the equality (2.3) holds. \( \square \)

**Proposition 2.9** Under the assumptions of Proposition 2.8, if \( \overline{\gamma}(t) := \gamma(-t) \), then

\[
\Psi_{\gamma} = \Psi_{\overline{\gamma}}.
\]

**Proof** Since \( \overline{\gamma}(t) = \gamma(-t) \), we get that \( \overline{\gamma}(t) = -\dot{\gamma}(-t) \) and in particular \( \overline{\gamma}(0) = -\dot{\gamma}(0) \). Since

\[
D_{\overline{\gamma}} (\psi(\dot{\gamma}(t))) = D_{\overline{\gamma}} (-\psi(\dot{\gamma}(-t))) = -D_{\overline{\gamma}} (\psi(\dot{\gamma}(-t))) = D_{\overline{\gamma}} (\psi(\dot{\gamma}(-t))),
\]

the equality (2.7) holds. \( \square \)

**Definition 2.10** Let \( \gamma_1 \) and \( \gamma_2 \) be two \( C^1 \)-regular curves emanating from \( p \) such that \( \mathcal{E} \)-initial vectors of \( \gamma_1 \) and \( \gamma_2 \) at \( p \) exist. Then the angle

\[
\arccos(\langle \Psi_{\gamma_1}, \Psi_{\gamma_2} \rangle) \in [0, \pi]
\]

is called the *angle between the initial vectors* of \( \gamma_1 \) and \( \gamma_2 \) at \( p \).

We generalize the definition of singular sectors from [35] to the case of coherent tangent bundles over surfaces with boundary.

Let \( U \) be a (sufficiently small) neighborhood of a singular point \( p \). Let \( \sigma_1 \) and \( \sigma_2 \) be curves in \( U \) starting at \( p \) such that both are singular curves or one of them is a singular curve and the other one is in \( \partial M \). A domain \( \Omega \) is called a *singular sector* at \( p \) if it satisfies the following conditions

(i) the boundary of \( \Omega \cap U \) consists of \( \sigma_1, \sigma_2 \) and the boundary of \( U \).

(ii) \( \Omega \cap \Sigma = \emptyset \).

If the peak \( p \in M \setminus \partial M \) is an isolated singular point than the domain \( U \setminus \{p\} \) is a singular sector at \( p \), where \( U \) is a neighborhood of \( p \) such that \( U \cap \Sigma = \{p\} \). We
assume that singular direction is transversal to the boundary of $M$. Therefore, there are no isolated singular points on the boundary.

We define the interior angle of a singular sector. If $p$ is in $\partial M$, then the interior angle of a singular sector at $p$ is the angle of the initial vectors of $\sigma_1$ and $\sigma_2$ at $p$.

While the interior angle of a singular sector may take value greater than $\pi$ if $p \in M \setminus \partial M$, we can choose $\gamma_j$ for $j = 0, \ldots, n$ inside the singular sector in such a way that the angle between $\Psi_{\gamma_{j-1}}$ and $\Psi_{\gamma_j}$ is not greater than $\pi$.

Let $\Omega$ be a singular sector at the peak $p$. Then, there exists a positive integer $n$ and $C^1$-regular curves starting at $p$ $\gamma_0 = \sigma_0, \gamma_1, \ldots, \gamma_n = \sigma_1$ satisfying the assumptions of Proposition 2.7 and the following conditions:

(i) if $i \neq j$ then $\gamma_i \cap \gamma_j = \emptyset$ in $\Omega$,
(ii) for each $j = 1, \ldots, n$ there exists a sector domain $\omega_j \subset \Omega$ such that $\omega_j$ is bounded by $\gamma_{j-1}$ and $\gamma_j$ and $\omega_j \cap \gamma_i = \emptyset$ for $i \neq j - 1, j$,
(iii) if $n \geq 2$ the vectors $\dot{\gamma}_{j-1}(0), \dot{\gamma}_j(0)$ are linearly independent and form a positively oriented frame for $j = 1, \ldots, n$.

If the peak $p$ is an isolated singular point then there exist curves $\gamma_0, \gamma_1, \gamma_2$ satisfying the above assumptions and conditions (i)–(iii). We also put $\gamma_3 = \gamma_0$.

The interior angle of the singular sector $\Omega$ is

$$\sum_{j=1}^{n} \arccos \left( \langle \Psi_{\gamma_{j-1}}, \Psi_{\gamma_j} \rangle \right).$$

If $\Omega$ is a singular sector at a singular point $p$ then $\Omega$ is contained in $M^+$ or $M^-$. The singular sector $\Omega$ is called positive (respectively negative) if $\Omega \subset M^+$ (respectively $\Omega \subset M^-$).

**Definition 2.11** Let $p$ be a singular point. Then, $\alpha_+(p)$ (respectively $\alpha_-(p)$) is the sum of all interior angles of positive (respectively negative) singular sectors at $p$.

**Proposition 2.12** (Theorem A in [35]) Let $p \in M \setminus \partial M$ be a peak. The sum $\alpha_+(p)$ of all interior angles of positive singular sectors at $p$ and the sum $\alpha_-(p)$ of all interior angles of negative singular sectors at $p$ satisfy

$$\alpha_+(p) + \alpha_-(p) = 2\pi,$$

$$\alpha_+(p) - \alpha_-(p) \in \{-2\pi, 0, 2\pi\}.$$

**Theorem 2.13** Let $p \in \partial M$ be a singular point. We assume that the singular direction is transversal to the boundary $\partial M$ at $p$.

If the null direction is transversal to the boundary $\partial M$ at $p$, then

$$\alpha_+(p) + \alpha_-(p) = \pi,$$

$$\alpha_+(p) - \alpha_-(p) \in \{-\pi, \pi\}.$$
If the null direction is tangent to the boundary $\partial M$ at $p$, then
\[ \alpha_+(p) = \alpha_-(p). \]

**Proof** The first part of this theorem follows from Proposition 2.15 in [35]. By Proposition 2.9, we get the second part. □

**Definition 2.14** A peak $p$ in $M \setminus \partial M$ is called positive (null, negative, respectively) if $\alpha_+(p) - \alpha_-(p) > 0$ ($\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$, respectively).

**Definition 2.15** A singular point $p$ in $\partial M$ is called positive (null, negative, respectively) if $\alpha_+(p) - \alpha_-(p) > 0$ ($\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$, respectively).

**Remark 2.16** It is easy to see that a peak $p$ in $\partial M$ is not null if $\partial M$ is transversal to the singular direction at $p$ and an $A_2$-singular point $p$ in $\partial M$ is null if the null vector at $p$ is tangent to $\partial M$.

**Definition 2.17** Let $p$ be a peak in $\partial M$. We say that $p$ is in the positive boundary (respectively in the negative boundary) if there exists a neighborhood $U$ in $M$ of $p$ such that $(U \setminus \{p\}) \cap \partial M \subset M^+$ (respectively $(U \setminus \{p\}) \cap \partial M \subset M^-$).

Let $\sigma(t)$ ($t \in (a; b)$) be a $C^2$-regular curve on $M$. We assume that if $\sigma(t) \in \Sigma$ then $\dot{\sigma}(t)$ is transversal to the null direction at $\sigma(t)$. Then, the image $\psi(\dot{\sigma}(t))$ does not vanish. Thus, we take a parameter $\tau$ of $\sigma$ such that
\[ \left\langle \psi \left( \frac{d}{d\tau} \sigma(\tau) \right), \psi \left( \frac{d}{d\tau} \sigma(\tau) \right) \right\rangle \equiv 1. \]

**Definition 2.18** Let $n(\tau)$ be a section of $E$ along $\sigma(\tau)$ such that $\{\psi(\frac{d}{d\tau} \sigma(\tau)), n(\tau)\}$ is a positive orthonormal frame. Then
\[ \hat{\kappa}_g(\tau) := \left\{ D_{\frac{d}{d\tau}} \psi \left( \frac{d}{d\tau} \sigma(\tau) \right), n(\tau) \right\} = \mu \left( \psi \left( \frac{d}{d\tau} \sigma(\tau) \right), D_{\frac{d}{d\tau}} \psi \left( \frac{d}{d\tau} \sigma(\tau) \right) \right) \]
is called the $E$-geodesic curvature of $\sigma$, which gives the geodesic curvature of the curve $\sigma$ with respect to the orientation of $E$.

We assume that the curve $\sigma$ is a singular curve consisting of $A_2$-points. Take a null vector field $\eta(\tau)$ along $\sigma(\tau)$ such that $\{\frac{d}{d\tau} \sigma(\tau), \eta(\tau)\}$ is a positively oriented field along $\sigma(\tau)$ for each $\tau$. Then, the singular curvature function is defined by
\[ \kappa_s(\tau) := \text{sgn} (d\lambda_{\psi}(\eta(\tau))) \cdot \hat{\kappa}_g(\tau), \]
where $\text{sgn}(d\lambda_{\psi}(\eta(\tau)))$ denotes the sign of the function $d\lambda_{\psi}(\eta)$ at $\tau$. In a general parameterization of $\sigma = \sigma(t)$, the singular curvature function is defined as follows
\[ \kappa_s(t) = \text{sgn} (d\lambda_{\psi}(\eta(t))) \cdot \frac{\mu \left( \psi (\dot{\sigma}(t)), D_{\frac{d}{dt}} \psi (\dot{\sigma}(t)) \right)}{|\psi(\dot{\sigma}(t))|^3}, \]
where \( \dot{\xi} := \frac{d}{dt}, |\xi| := \sqrt{\langle \xi, \xi \rangle} \).

By Proposition 1.7 in [35] the singular curvature function does not depend on the orientation of \( M \), the orientation on \( E \), nor the parameter \( t \) of the singular curve \( \sigma(t) \).

By Proposition 2.11 in [35] the singular curvature measure \( \kappa_s d\tau \) is bounded on any singular curve, where \( d\tau \) is the arclength measure of this curve with respect to the first fundamental form \( ds^2 \). Now, we prove the following proposition concerning the geodesic curvature measure on the boundary of \( M \).

**Proposition 2.19** Let \( \gamma : [0, \varepsilon) \to \partial M \) be a \( C^2 \)-regular curve such that \( \Sigma \cap \gamma([0, \varepsilon)) = \{\gamma(0)\} \) is an \( A_2 \)-point and the vector \( \dot{\gamma}(0) \) is the null vector at \( \gamma(0) \). Then, the geodesic curvature measure \( \hat{\kappa}_g d\tau \) is continuous on \([0, \varepsilon)\), where \( d\tau \) is the arclength measure with respect to the first fundamental form \( ds^2 \).

**Proof** The point \( \gamma(0) \in \partial M \) is a null \( A_2 \)-point. By Proposition 2.8 we can write that \( \Psi(\dot{\gamma}(t)) = t \zeta(t) \) for \( t \in [0, \tilde{\varepsilon}) \) for sufficiently small \( \tilde{\varepsilon} \leq \varepsilon \), where \( \zeta(t) \in \mathcal{E}_{\gamma(t)} \) and \( \zeta(0) = D_{\dot{\gamma}} \Psi(\dot{\gamma}(t)) \big|_{t=0} \neq 0 \). The geodesic curvature in a general parameterization has the following form

\[
\hat{\kappa}_g(t) = \frac{\mu \left( \psi(\dot{\gamma}(t)), D_{\dot{\gamma}} \psi(\dot{\gamma}(t)) \right)}{|\psi(\dot{\gamma}(t))|^3}.
\]

Thus, the geodesic curvature measure

\[
\hat{\kappa}_g(t) d\tau = \hat{\kappa}_g(t) |\psi(\dot{\gamma}(t))| d\tau = \frac{\mu \left( \zeta(t), D_{\dot{\gamma}} \zeta(t) \right)}{|\zeta(t)|^2} d\tau
\]

is bounded and continuous on \([0, \tilde{\varepsilon})\). It implies that the geodesic curvature measure is continuous on \([0, \varepsilon)\) since \( \Sigma \cap \gamma([0, \varepsilon)) = \{\gamma(0)\} \). \( \Box \)

Let \( U \subset M \) be a domain and let \( \{e_1, e_2\} \) be a positive orthonormal frame field on \( E \) defined on \( U \). Since \( D \) is a metric connection, there exists a unique 1-form \( \omega \) on \( U \) such that

\[
D_X e_1 = -\omega(X)e_2, \quad D_X e_2 = \omega(X)e_1,
\]

for any smooth vector field \( X \) on \( U \). The form \( \omega \) is called the connection form with respect to the frame \( \{e_1, e_2\} \). It is easy to check that \( d\omega \) does not depend on the choice of a frame \( \{e_1, e_2\} \) and gives a globally defined 2-form on \( M \). Since \( D \) is a metric connection and it satisfies (2.1) we have

\[
d\omega = K d\hat{A} = \begin{cases} 
K dA & \text{on } M_+, \\
-K dA & \text{on } M_-.
\end{cases}
\]

where \( K \) is the Gaussian curvature of the first fundamental form \( ds^2 \) (see [35,36]).

The next theorem is a generalization of the Gauss–Bonnet theorem for coherent tangent bundles over smooth compact-oriented surfaces with boundary.
Theorem 2.20 (The Gauss–Bonnet type formulas) Let $\mathcal{E}$ be a coherent tangent bundle on a smooth compact-oriented surface $M$ with boundary whose set of singular points $\Sigma$ admits at most peaks. If the set of singular points $\Sigma$ is transversal to the boundary $\partial M$, then

$$2\pi \chi(M) = \int_M K d\hat{A} + 2 \int_{\Sigma} \kappa_s d\tau - \int_{\partial M \cap M^+} \hat{\kappa}_g d\tau - \int_{\partial M \cap M^-} \hat{\kappa}_g d\tau - \sum_{p \in \text{null}(\Sigma \cap \partial M)} (2\alpha_+(p) - \pi),$$

(2.8)

$$\int_M K d\hat{A} + \int_{\partial M} \hat{\kappa}_g d\tau = 2\pi \left( \chi(M^+) - \chi(M^-) \right) + 2\pi \left( \#P^+ - \#P^- \right) + \pi \left( \#(\Sigma \cap \partial M)^+ - \#(\Sigma \cap \partial M)^- \right) + \pi \left( \#P_{\partial M^+} - \#P_{\partial M^-} \right),$$

(2.9)

where $d\tau$ is the arc length measure, $P^+$ (respectively $P^-$) is the set of positive (respectively negative) peaks in $M \setminus \partial M$, $(\Sigma \cap \partial M)^+$ (respectively $(\Sigma \cap \partial M)^-$, $\text{null}(\Sigma \cap \partial M)$) is the set of positive (respectively negative, null) singular points in $\Sigma \cap \partial M$, $P_{\partial M^+}$ (respectively $P_{\partial M^-}$) is the set of peaks in the positive (respectively negative) boundary.

3 The Proof of Theorem 2.20

We use the method presented in the proof of Theorem B in [35]. First, we formulate the local Gauss–Bonnet theorem for admissible triangles.

Definition 3.1 A curve $\sigma(t)$ ($t \in [a, b]$) is admissible on the surface with boundary if it satisfies one of the following conditions:

1. $\sigma$ is a $C^2$-regular curve such that $\sigma([a, b])$ does not contain a peak, and the tangent vector $\dot{\sigma}(t)$ ($t \in [a, b]$) is transversal to the singular direction, the null direction if $\sigma(t) \in \Sigma$ and $\dot{\sigma}(t)$ is transversal to the boundary if $\sigma(t) \in \partial M$.
2. $\sigma$ is a $C^1$-regular curve such that the set $\sigma([a, b])$ is contained in the set of singular points $\Sigma$ and the set $\sigma((a, b))$ does not contain a peak.
3. $\sigma$ is $C^2$-regular curve such that the set $\sigma([a, b])$ is contained in the boundary $\partial M$, the set $\sigma((a, b))$ does not contain a singular point and the tangent vector $\dot{\sigma}(t)$ ($t \in [a, b]$) is transversal to the singular direction if $\sigma(t) \in \Sigma$.

Remark 3.2 A curve $\sigma(t)$ is admissible in the sense of Definition 2.12 in [35] if it satisfies conditions (1) or (2) in Definition 3.1. For the purpose of this paper we add (3) in Definition 3.1 and the transversality of the admissible curve to the boundary in (1).

Let $U$ be a domain in $M$. 

© Springer
Definition 3.3 (See Definition 3.1 in [35]) Let $\overline{T} \subset U$ be the closure of a simply connected domain $T$ which is bounded by three admissible arcs $\gamma_1, \gamma_2, \gamma_3$. Let $A$, $B$ and $C$ be the distinct three boundary points of $T$ which are intersections of these three arcs. Then $\overline{T}$ is called an admissible triangle on the surface with boundary if it satisfies the following conditions:

1. $\overline{T}$ admits at most one peak on $\{A, B, C\}$.
2. the three interior angles at $A$, $B$ and $C$ with respect to the metric $g$ are all less than $\pi$.
3. if $\gamma_j$ for $j = 1, 2, 3$ is not a singular curve, it is $C^2$-regular, namely it is a restriction of a certain open $C^2$-regular arc.

We write $\Delta ABC := \overline{T}$ and we denote by

$$BC := \gamma_1, \quad CA := \gamma_2, \quad AB := \gamma_3$$

the regular arcs whose boundary points are $\{B, C\}, \{C, A\}, \{A, B\}$, respectively.

We give the orientation of $\partial \Delta ABC$ compatible with respect to the orientation of $M$. We denote by $\angle A, \angle B, \angle C$ the interior angles (with respect to the first fundamental form $ds^2$) of the piecewise smooth boundary of $\Delta ABC$ at $A$, $B$ and $C$, respectively if $A$, $B$ and $C$ are regular points.

If $A \in M \setminus \partial M$ is a singular point and $(U; u, v)$ is a $g$-coordinate system at $A$, then we set (see Proposition 2.15 in [35])

$$\angle A := \begin{cases} \pi & \text{if the $u$-curve passing through $A$ separates $AB$ and $AC$,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma(t)$ be an admissible curve. We define a geometric curvature $\hat{k}_g(t)$ in the following way:

$$\tilde{k}_g(t) = \begin{cases} \hat{k}_g(t) & \text{if } \sigma(t) \in M^+, \\ -\hat{k}_g(t) & \text{if } \sigma(t) \in M^-, \\ \kappa_s(t) & \text{if } \sigma(t) \in \Sigma, \end{cases}$$

where $\hat{k}_g$ is the geodesic curvature with respect to the orientation of $M$ and $\kappa_s$ is the singular curvature.

Proposition 3.4 Let $\Delta ABC$ be an admissible triangle on the surface with boundary such that $A$ is an $A_2$-point, $AB \subset \partial M$ and $\Delta ABC \setminus AC$ lies in $M^+$ or in $M^-$. Suppose that the boundary $\partial M$ is transversal to $\Sigma$ at $A$ and let $T_A \partial M$ be a null direction at $A$. Then

$$\angle A + \angle B + \angle C - \pi = \int_{\partial \Delta ABC} \tilde{k}_g d\tau + \int_{\Delta ABC} K dA. \quad (3.1)$$

Proof Without loss of generality, let us assume that $\Delta ABC \setminus AC$ lies in $M^+$. If the arc $AC \subset \Sigma$ or the interior angle $\angle BAC$ with respect to the metric $g$ is greater than
Fig. 1 A decomposition of the triangle $ABC$ into admissible triangles

We decompose the triangle $\Delta ABC$ into admissible triangles $\Delta ABD$ and $\Delta ADC$ such that the interior angle $\angle BAD$ with respect to the metric $g$ is in the interval $(0, \frac{\pi}{2})$ and the arc $\overline{AD}$ is transversal to the arc $\overline{BC}$ at $D$, see Fig. 1. The formula (3.1) for $\Delta ADC$ follows from Theorem 3.3 in [35], so it is enough to prove the formula (3.1) for the triangle $\Delta ABD$.

We can take the arc $\overline{AD}$ and rotate it around $D$ with respect to the canonical metric $du^2 + dv^2$ on the $uv$-plane. Then, we obtain a smooth one-parameter family of $C^2$-regular arcs starting at $D$. Since the interior angle $\angle BAD$ is in $(0, \frac{\pi}{2})$ and $\overline{BD}$, $\overline{AD}$ are transversal at $D$, restricting the image of this family to the triangle $\Delta ABD$, we obtain a family of $C^2$-regular curves

$$\gamma_\varepsilon : [0, 1] \rightarrow \Delta ABD,$$

where $\varepsilon \in [0, 1]$ and:

(i) $\gamma_0$ parameterizes $\overline{AD}$ and $\gamma_0(0) = A$, $\gamma_0(1) = D$,
(ii) $\gamma_\varepsilon(1) = D$ for all $\varepsilon \in [0, 1]$,
(iii) the correspondence $\sigma : \varepsilon \mapsto \gamma_\varepsilon(0)$ gives a subarc of $\overline{AB}$. We set $A_\varepsilon = \gamma_\varepsilon(0)$, where $A_0 = A$.

Since $\Delta A_\varepsilon BD$ for $\varepsilon > 0$ is an admissible triangle, then by Theorem 3.3 in [35] we get that

$$\angle A_\varepsilon + \angle B + \angle A_\varepsilon BD - \pi = \int_{\partial \Delta A_\varepsilon BD} \tilde{k}_g \, d\tau + \int_{\Delta A_\varepsilon BD} K \, dA.$$

Since $\Delta ABD$ is admissible and $\tilde{k}_g$ is bounded on both $\overline{AB}$ and $\overline{AD}$, by taking the limit as $\varepsilon \to 0^+$, we have that

$$\lim_{\varepsilon \to 0^+} \angle A_\varepsilon + \angle B + \angle D - \pi = \int_{\partial \Delta ABD} \tilde{k}_g \, d\tau + \int_{\Delta ABD} K \, dA.$$
By Proposition 2.8 we have

$$
\lim_{\varepsilon \to 0^+} \angle A_{\varepsilon} = \lim_{\varepsilon \to 0^+} \arccos \frac{\left. \psi \left( \frac{d \gamma_{\varepsilon}(t)}{dt} \right) \right|_{t=0} \cdot \left. \psi \left( \frac{d \sigma(\varepsilon)}{d\varepsilon} \right) \right|_{t=0}}{\left. \psi \left( \frac{d \gamma_{\varepsilon}(t)}{dt} \right) \right|_{t=0} \cdot \left. \psi \left( \frac{d \sigma(\varepsilon)}{d\varepsilon} \right) \right|_{t=0}} = \arccos \langle \Psi_0, \Psi_\sigma \rangle.
$$

(3.2)

This completes the proof. \hfill \Box

\textbf{Remark 3.5} By Theorem 3.3 in [35] and Proposition 3.4 the equality (3.1) holds for any admissible triangle on a surface with boundary.

Let $\overline{X}$, $X^\circ$, $\partial X$, respectively, denote the closure of a subset $X$ of $M$, the interior of $X$ and the boundary of $X$, respectively.

Let us triangulate $M$ by admissible triangles such that each point in the set $P \cup (\Sigma \cap \partial M) =: P^*$ is a vertex, where $P$ is the set all peaks in $M^\circ$. Let $T$, $E$ and $V$, respectively, denote the set of all triangles, the set of all edges and the set all of vertices in the given triangulation, respectively.

\textbf{Lemma 3.6} The following relation holds:

$$
\# \left\{ v \in V \mid v \in (M^+)^\circ \right\} = \chi(M^+) + \frac{1}{2} \# \left\{ \Delta \in T \mid \Delta \subset \overline{M}^+ \right\} \\
+ \frac{1}{2} \# \left\{ e \in E \mid e \subset \partial M^+ \right\} \\
- \# \left\{ v \in V \mid v \in \partial M^+ \setminus P^* \right\} - \# P^*.
$$

(3.3)

\textbf{Proof} By the definition of Euler’s characteristic we get that

$$
\# \left\{ v \in V \mid v \in \overline{M}^+ \right\} = \chi(M^+) - \# \left\{ \Delta \in T \mid \Delta \subset \overline{M}^+ \right\} + \# \left\{ e \in E \mid e \subset \partial M^+ \right\}.
$$

(3.4)

Furthermore, it is easy to verify that

$$
\# \left\{ e \in E \mid e \subset \overline{M}^+ \right\} = \frac{3}{2} \# \left\{ \Delta \in F \mid \Delta \subset \overline{M}^+ \right\} + \frac{1}{2} \# \left\{ e \in E \mid e \subset \partial M^+ \right\}
$$

(3.4)

and

$$
\# \left\{ v \in V \mid v \in (M^+)^\circ \right\} = \# \left\{ v \in V \mid v \in \overline{M}^+ \right\} \\
- \# \left\{ v \in V \mid v \in \partial M^+ \setminus P^* \right\} - \# \left\{ p \in V \mid p \in P^* \right\}.
$$

(3.5)

Combining together (3.3), (3.4) and (3.5) we end the proof. \hfill \Box
Let us define the sum $\sum_{\Delta ABC \in T, \Delta \subseteq M^+} (\angle A + \angle B + \angle C - \pi)$ by $S_+$. Then,

$$S_+ = 2\pi \# \{ v \in V \mid v \in (M^+)_{\circ} \} + \pi \# \{ v \in V \mid v \in \partial M^+ \setminus P^* \}$$

$$+ \sum_{p \in P^*} \alpha_+(p) - \pi \# \{ \Delta \in T \mid \Delta \subseteq M^+ \}.$$ 

By Lemma 3.6 we get that

$$S_+ = 2\pi \chi(M^+) + \pi \# \{ e \in E \mid e \in \partial M^+ \} - \pi \# \{ v \in V \mid v \in \partial M^+ \setminus P^* \}$$

$$- 2\pi \# P^* + \sum_{p \in P^*} \alpha_+(p)$$

$$= 2\pi \chi(M^+) + \frac{\pi}{2} \sum_{v \in V, v \in \partial M^+} \deg_{\partial M^+}(v) - \pi \# \{ v \in V \mid v \in \partial M^+ \}$$

$$- \pi \# P^* + \sum_{p \in P^*} \alpha_+(p),$$

where $\deg_X(v) = \# \{ e \in E \mid e \subset X, v \in e \}$, where $X$ is a subset of $M$. Since $\partial M^+$ is an Eulerian graph, the number $\deg_{\partial M^+}(v)$ is even and let us write that $m_+(v) := \frac{1}{2} \deg_{\partial M^+}(v)$. Furthermore, if $v \in (V \cap \partial M^+) \setminus P^*$ then $\deg_{\partial M^+}(v) = 2$ and we get the relation

$$\frac{1}{2} \sum_{v \in V \cap \partial M^+} \deg_{\partial M^+}(v) - \# \{ v \in V \mid v \in \partial M^+ \} = \sum_{p \in P^*} (m_+(p) - 1).$$

Hence we get the following:

$$S_+ = 2\pi \chi(M^+) + \sum_{p \in P^*} (\alpha_+(p) + \pi m_+(p)) - 2\pi \# P^*. \quad (3.6)$$

Similarly we get that

$$S_- = 2\pi \chi(M^-) + \sum_{p \in P^*} (\alpha_-(p) + \pi m_-(p)) - 2\pi \# P^*. \quad (3.7)$$

where $S_- = \sum_{\Delta ABC \in T, \Delta \subseteq \overline{M}} (\angle A + \angle B + \angle C - \pi)$ and $m_-(v) := \frac{1}{2} \deg_{\partial M^-}(v)$.

It is easy to see that

$$m_+(p) = m_-(p) \text{ for } p \in P^* \setminus \partial M, \quad (3.8)$$

$$m_+(p) + m_-(p) = \deg_{\Sigma}(p) \text{ for } p \in P^* \setminus \partial M, \quad (3.9)$$

$$m_+(p) + m_-(p) = \deg_{\Sigma \cup \partial M}(p) - 1 \text{ for } p \in \Sigma \cap \partial M. \quad (3.10)$$
Furthermore if \( p \in \Sigma \cap \partial M \), then
\[
m_+(p) - m_-(p) = \begin{cases} 
1 & \text{if } p \text{ is a peak in the positive boundary}, \\
-1 & \text{if } p \text{ is a peak in the negative boundary}, \\
0 & \text{otherwise}. 
\end{cases}
\] (3.11)

**Lemma 3.7** The Euler characteristic of \( \Sigma \) is equal to
\[
\chi(\Sigma) = \#P^* - \frac{1}{2} \sum_{p \in P^*} (m_+(p) + m_-(p)) + \frac{1}{2}(\Sigma \cap \partial M).
\]

**Proof** We know that
\[
\chi(\Sigma) = \# \{ v \in V \mid v \in \Sigma \} - \# \{ e \in E \mid e \subset \Sigma \}
= \# \{ v \in V \mid v \in \Sigma \} - \frac{1}{2} \sum_{v \in V \cap \Sigma} \deg v.
\]
If \( p \in P \setminus \partial M \) then \( \deg p = \deg p_{\Sigma \cup \partial M} \) and if \( p \in \Sigma \cap \partial M \) then \( \deg p = \deg p_{\Sigma \cup \partial M} - 2 \). By (3.9) and (3.10) we get that
\[
\chi(\Sigma) = \#P^* - \frac{1}{2} \sum_{p \in P \setminus \partial M} (m_+(p) + m_-(p)) - \frac{1}{2} \sum_{p \in \Sigma \cap \partial M} (m_+(p) + m_-(p) - 1)
= \#P^* - \frac{1}{2} \sum_{p \in P^*} (m_+(p) + m_-(p)) + \frac{1}{2}(\Sigma \cap \partial M).
\]

\[\square\]

**Lemma 3.8** The following equality holds:
\[
S_+ + S_- = 2\pi \chi(M) + \sum_{p \in \text{null}(\Sigma \cap \partial M)} (2\alpha_+(p) - \pi).
\]

**Proof** Since \( \chi(M^+) + \chi(M^-) = \chi(M) + \chi(\Sigma) \), by (3.6), (3.7), Lemma 3.7 and Theorem 2.13 we get that:
\[
S_+ + S_- = 2\pi \chi(M) + 2\pi \chi(\Sigma) + \sum_{p \in P^*} (\alpha_+(p) + \alpha_-(p))
+ \pi \sum_{p \in P^*} (m_+(p) + m_-(p)) - 4\pi \#P^*
= 2\pi \chi(M) + \pi(\Sigma \cap \partial M) + \sum_{p \in P^*} (\alpha_+(p) + \alpha_-(p)) - 2\pi \#P^*
= 2\pi \chi(M) + \pi(\Sigma \cap \partial M) + \sum_{p \in (\Sigma \cap \partial M)^+ \cup (\Sigma \cap \partial M)^-} (\alpha_+(p) + \alpha_-(p))
\]
\[
\sum_{p \in P \setminus \partial M} \left( \alpha_+(p) + \alpha_-(p) \right) + \\
\sum_{p \in \text{null} \left( \Sigma \cap \partial M \right)} \left( \alpha_+(p) + \alpha_-(p) \right)
\]

\[
- 2\pi \#(\Sigma \cap \partial M) - 2\pi \#(P \setminus \partial M)
\]

\[
= 2\pi \chi(M) + \\
\sum_{p \in (\Sigma \cap \partial M) + \cup (\Sigma \cap \partial M)^-} \pi \quad + \quad \sum_{p \in P \setminus \partial M} 2\pi
\]

\[
+ \sum_{p \in \text{null} \left( \Sigma \cap \partial M \right)} 2\alpha_+(p) - \pi \#(\Sigma \cap \partial M) - 2\pi \#(P \setminus \partial M)
\]

\[
= 2\pi \chi(M) + \sum_{p \in \text{null} \left( \Sigma \cap \partial M \right)} (2\alpha_+(p) - \pi).
\]

\[\square\]

**Lemma 3.9** The following equality holds:

\[
S_+ - S_- = 2\pi \left( \chi(M^+) - \chi(M^-) \right) + 2\pi \left( \#P^+ - \#P^- \right) + \pi \left( \#(\Sigma \cap \partial M)^+ - \#(\Sigma \cap \partial M)^- \right) + \pi \left( \#P_{\partial M^+} - \#P_{\partial M^-} \right),
\]

where \(P^+\) (respectively \(P^-\)) is the set of positive (respectively negative) peaks in \(M \setminus \partial M\), \((\Sigma \cap \partial M)^+\) (respectively \((\Sigma \cap \partial M)^-\)) is the set of positive (respectively negative) singular points in \(\Sigma \cap \partial M\), \(P_{\partial M^+}\) (respectively \(P_{\partial M^-}\)) is the set of peaks in the positive (respectively negative) boundary.

**Proof** It is a consequence of (3.6), (3.7), Lemma 3.7 and Theorem 2.13 and the fact that \(\chi(M^+) - \chi(M^-) = \chi(M^+) - \chi(M^-)\).

Since the integration of the geometric curvature on curves which are not included in \(\Sigma \cup \partial M\) are canceled by opposite integrations and the singular curvature does not depend on the orientation of the singular curve, by Proposition 3.4 and Theorem 3.3 in [35] we get that

\[
S_{\pm} = \int_{M^\pm} K \, dA + \int_{\partial M^\pm} \kappa_\gamma \, d\tau = \int_{M^\pm} K \, dA + \int_{\Sigma} \kappa_\gamma \, d\tau \pm \int_{\partial M \cap M^\pm} \hat{\kappa}_\gamma \, d\tau.
\]

Hence

\[
S_+ + S_- = \int_{M} K \, dA + 2\int_{\Sigma} \kappa_\gamma \, d\tau + \int_{\partial M \cap M^+} \hat{\kappa}_\gamma \, d\tau - \int_{\partial M \cap M^-} \hat{\kappa}_\gamma \, d\tau, \quad (3.12)
\]

\[
S_+ - S_- = \int_{M} K \, d\hat{A} + \int_{\partial M} \hat{\kappa}_\gamma \, d\tau. \quad (3.13)
\]

By Lemma 3.8, Lemma 3.9, (3.12) and (3.13) we complete the proof of Theorem 2.20.
4 Applications of the Gauss–Bonnet Formulas to Maps

As a corollary of Theorem 2.20 we get a special version of Fukuda–Ishikawa’s theorem (Theorem 1.1 in [12], see also [22]), which is the generalization of Quine’s formula (Theorem 1 in [33]) for surfaces with boundary (see also Proposition 3.6 in [37]). We assume that the set of singular points of a map is transversal to the boundary of a surface.

Proposition 4.1 Let $M$ and $N$ both be compact oriented connected surfaces with boundary. Let $f : M \to N$ be a $C^\infty$-smooth map such that $f(\partial M) \subset \partial N$ and $f^{-1}(\partial N) = \partial M$ and whose set of singular points consists of folds and cusps. If the set of singular points of $f$ is transversal to $\partial M$ then the topological degree of $f$ satisfies

$$\operatorname{deg}(f) \chi(N) = \chi(M_f^+) - \chi(M_f^-) + S_f^+ - S_f^-,$$

where $M_f^+$ (respectively $M_f^-$) is the set of regular points at which $f$ preserves (respectively reverses) the orientation, $S_f^+$ (respectively $S_f^-$) is the number of positive cusps (respectively the number of negative cusps).

Proof Let $h$ be a Riemannian metric on $N$ and let $D$ be the Levi–Civita connection on $(N, h)$. Then, the tuple $(f^*TN, h, D, df)$ is a coherent tangent bundle on $M$ (see [37]). Since $f(\partial M) \subset \partial N$ and the set of singular points of $f$ is transversal to $\partial M$, there are no cusps in $\partial M$ and all folds in $\partial M$ are null singular points. Therefore, by Theorem 2.20 we get that:

$$\int_M K \, d\hat{A} + \int_{\partial M} \hat{\kappa}_g \, d\tau = 2\pi \left( \chi(M_f^+) - \chi(M_f^-) \right) + 2\pi \left( S_f^+ - S_f^- \right).$$

(4.2)

The following identity holds

$$\int_M K \, d\hat{A} = \int_M f^* \Omega_{12},$$

where $\Omega_{12}$ is a curvature 2-form.

Furthermore, it is well known that $\int_M f^* \Omega_{12} = \operatorname{deg}(f) \int_N \Omega_{12}$ (see for instance Remark 1 in [11] page 111). On the other hand, we have $\int_N \Omega_{12} = \int_N K_N dA$, where $K_N$ is the Gaussian curvature of $N$. By the Gauss–Bonnet theorem for $N$ we get $\int_N K_N dA = 2\pi \chi(N) - \int_{\partial N} \kappa_g d\tau$, where $\kappa_g$ is the geodesic curvature of $\partial N$ in $N$. Thus,

$$\int_M K \, d\hat{A} = \operatorname{deg}(f) \left( 2\pi \chi(N) - \int_{\partial N} \kappa_g d\tau \right).$$

(4.3)
Since $f(\partial M) \subset \partial N$ and $f^{-1}(\partial N) = \partial M$ and $(\cdot, \cdot)_p = h_{f(p)}(\cdot, \cdot)$ for $p$ in $M$, we obtain that

$$\int_{\partial M} \hat{k}_g d\tau = \deg(f|_{\partial M}) \int_{\partial N} \kappa_g d\tau.$$  \hspace{1cm} (4.4)

By Theorem 13.2.1 [11, p. 105] we get $\deg(f) = \deg(f|_{\partial M})$.

By (4.2)–(4.4) we obtain the formula (4.1). \hfill \Box

We can also get easily the generalization of Proposition 3.7 in [37] by the Gauss–Bonnet formulas.

**Proposition 4.2** Let $(N, h)$ be an oriented Riemannian 2-manifold, let $M$ be a compact oriented 2-manifold with boundary. Let $f : M \to N$ be a $C^\infty$-smooth map whose set of singular points consists of folds and cusps and is transversal to $\partial M$. Then the total singular curvature $\int_{\Sigma} \kappa_s d\tau$ with respect to the length element $d\tau$ (with respect to $h$) on the set of singular points $\Sigma$ is bounded, and satisfies the following identity

$$2\pi \chi(M) = \int_{M} (\tilde{K} \circ f) |f^*dA_h| + 2 \int_{\Sigma} \kappa_s d\tau$$

$$+ \int_{\partial M \cap M^+_f} \hat{k}_g d\tau - \int_{\partial M \cap M^-_f} \hat{k}_g d\tau - \sum_{p \in \text{null}(\Sigma \cap \partial M)} (2\alpha_+(p) - \pi),$$

where $M^+_f$ (respectively $M^-_f$) is the set of regular points at which $f$ preserves (respectively reverses) the orientation, $\tilde{K}$ is the Gaussian curvature function on $(N, h)$, $\hat{k}_g$ is a geodesic curvature, $|f^*dA_h|$ is the pull-back of the Riemannian measure of $(N, h)$ and

$$\alpha_+(p) = \arccos \left(h \left( \frac{D}{dt} \left( \frac{d}{dt} (f \circ \gamma)(t) \right) \right), \frac{d}{dt} (f \circ \sigma)(\tau) \right),$$

where $D$ is the Levi–Civita connection on $N$, $\gamma$ is a $C^2$-regular parameterization of the boundary $\partial M$ in the neighborhood of $p$ and $\sigma$ is a parameterization of $\Sigma$ in the neighborhood of $p$.

**5 Geometry of the Affine-Extended Wave Front**

In this section, we apply Theorem 2.20 to an affine-extended wave front of a planar non-singular hedgehog. Fronts are examples of coherent tangent bundles (see [35]).

Planar hedgehogs are curves which can be parameterized using their Gauss map. A hedgehog can be also viewed as the Minkowski difference of convex bodies (see [23, 25–28]). The non-singular hedgehogs are also known as the rosettes (see [2, 31, 44]).
The singularities and the geometry of affine $\lambda$-equidistants were very widely studied in many papers [1,7–9,16,18,34,41]. The envelope of affine diameters (the centre symmetry set) was studied in [5,13–15,17].

Let $C$ be a smooth parameterized curve on the affine plane $\mathbb{R}^2$, i.e. the image of the $C^\infty$-smooth map from an interval to $\mathbb{R}^2$. We say that a smooth curve is closed if it is the image of a $C^\infty$-smooth map from $S^1$ to $\mathbb{R}^2$. A smooth curve is regular if its velocity does not vanish. A closed regular curve is called an $m$-rosette if its signed curvature is positive and its rotation number is $m$. A convex curve is a 1-rosette.

**Definition 5.1** A pair of points $a, b \in C (a \neq b)$ is called a parallel pair if the tangent lines to $C$ at $a$ and $b$ are parallel.

**Definition 5.2** An affine $\lambda$-equidistant is the following set:

$$E_\lambda(C) = \left\{ \lambda a + (1 - \lambda)b \mid a, b \text{ is a parallel pair of } C \right\}.$$  

The set $E_{\frac{1}{2}}(C)$ will be called the Wigner caustic of $C$.

A chord passing through a parallel pair $a, b \in C$ is the following set:

$$\left\{ \lambda a + (1 - \lambda)b \mid \lambda \in [0, 1] \right\}.$$  

**Definition 5.3** The centre symmetry set of $C$, which we will denote as $CSS(C)$, is the envelope of all chords passing through parallel pairs of $C$.

If $C$ is a generic convex curve, then the Wigner caustic of $C$, $E_\lambda(C)$, for a generic $\lambda$, and $CSS(C)$ are smooth closed curves with at most cusp singularities [1,13,15,17], the number of cusps of the Wigner caustic and the centre symmetry set of $C$ are odd and not smaller than 3 [1,13], the number of cusps of $CSS(C)$ is not smaller than the number of cusps of $E_{\frac{1}{2}}(C)$ [5] and the number of cusps of $E_\lambda(C)$ is even for a generic $\lambda \neq \frac{1}{2}$ [10]. Moreover, cusp singularities of all $E_\lambda(C)$ are lying on smooth parts of $CSS(C)$ [15]. In addition, if $C$ is a convex curve, then the Wigner caustic is contained in a closure of the region bounded by the centre symmetry set ([3], see Fig. 2). The Wigner caustic also appears in one of the two constructions of bi-dimensional improper affine spheres. This construction can be generalized to higher even dimensions [4]. The oriented area of the Wigner caustic improves the classical planar isoperimetric inequality and gives the relation between the area and the perimeter of smooth convex bodies of constant width [42–44]. Recently, the properties of the middle hedgehog, which is a generalization of the Wigner caustic in the case of non-smooth convex bodies, were studied in [39,40].

**Definition 5.4** The extended affine space is the space $\mathbb{R}_\lambda^3 = \mathbb{R} \times \mathbb{R}^2$ with coordinate $\lambda \in \mathbb{R}$ (called the affine time) on the first factor and a projection on the second factor denoted by $\pi : \mathbb{R}_\lambda^3 \ni (\lambda, x) \mapsto x \in \mathbb{R}^2$. 

 Springer
Definition 5.5 Let $R_m$ be an $m$-rosette. The affine extended wave front of $R_m$ is the following set:

$$
\mathbb{E}(R_m) = \bigcup_{\lambda \in [0,1]} \{\lambda\} \times E_\lambda(R_m) \subset \mathbb{R}^3.
$$

$\mathbb{E}(R_m)$ is the union of all $E_\lambda(R_m)$ for $\lambda \in [0, 1]$, each embedded into its own slice of the extended affine space.

Note that, when $R_m$ is a circle on the plane, then $\mathbb{E}(R_m)$ is the double cone, which is a smooth manifold with the nonsingular projection $\pi$ everywhere, but at its singular point, which projects to the center of the circle (the center of symmetry).

We will study the geometry of $\mathbb{E}(R_m)$ through the support function of $R_m$ [2,44]. Take a point $O$ as the origin of our frame. Let $\theta$ be the oriented angle from the positive $x_1$-axis. Let $p(\theta)$ be the oriented perpendicular distance from $O$ to the tangent line at a point on $R_m$ and let this ray and $x_1$-axis form an angle $\theta$. The function $p$ is a single valued periodic function of $\theta$ with period $2m\pi$ and the parameterization of $R_m$ in terms of $\theta$ and $p(\theta)$ is as follows

$$
[0, 2m\pi) \ni \theta \mapsto \gamma(\theta) = \left( p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta \right) \in \mathbb{R}^2.
$$

(5.1)

Then, the radius of curvature $\rho$ of $R_m$ is in the following form

$$
\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta) > 0,
$$

(5.2)

or equivalently, the curvature $\kappa$ of $R_m$ is given by

$$
\kappa(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0.
$$

(5.3)

In Fig. 3 we illustrate (with different opacities) the surface $\mathbb{E}(R_1)$, where $R_1$ is an oval represented by the support function $p(\theta) = 11 - \frac{1}{2} \cos 2\theta + \sin 3\theta$. We also
present the following curves: \( \{0\} \times R_1, \{1\} \times R_1, \{\frac{1}{2}\} \times E_{\frac{1}{2}}(R_1), \{0\} \times E_{\frac{1}{2}}(R_1) \) and \( \{0\} \times \text{CSS}(R_1) \).

Let \( \Sigma \) be a set of singular points of \( \mathcal{E} \). It is well known that \( \pi(\Sigma(\mathcal{E}(R_1))) = \text{CSS}(R_1) \) and the map \( \Sigma(\mathcal{E}(R_1)) \ni p \mapsto \pi(p) \in \text{CSS}(R_1) \) is the double covering of \( \text{CSS}(R_1) \).

Let \( \mathcal{E}_k(R_m) \) for \( k = 1, \ldots, m \) be a branch of \( \mathcal{E}(R_m) \) which has the following parameterization

\[
f_k(\lambda, \theta) = (\lambda, \lambda \gamma(\theta) + (1 - \lambda) \gamma(\theta + k\pi)).
\]

(5.4)

We use the following notation:

\[
(f_k)_\lambda := \frac{\partial}{\partial \lambda} f_k(\lambda, \theta), \quad (f_k)_\theta := \frac{\partial}{\partial \theta} f_k(\lambda, \theta).
\]

(5.5)

In Figs. 4 and 5 we illustrate (with different opacities) the branches \( \mathcal{E}_1(R_2) \) and \( \mathcal{E}_2(R_2) \), respectively, where \( R_2 \) is a 2-rosette represented by the support function \( p(\theta) = 11 + \sin \frac{\theta}{2} - 7 \cos \frac{3\theta}{2} - \frac{1}{2} \sin 2\theta \).

Directly by Definition 5.5 we get the following proposition.

**Proposition 5.6** Every branch of \( \mathcal{E}(R_m) \) is a ruled surface.

It is well known that the Gaussian curvature of a ruled surface at a non-singular point is non-positive. By direct calculation we get the following proposition.
Proposition 5.7 Let $R_m$ be an $m$-rosette.

(i) A point $(\lambda, \theta)$ is a singular point of $E_k(R_m)$ if and only if

$$\frac{\kappa(\theta)}{\kappa(\theta + k\pi)} = (-1)^{k+1} \frac{\lambda}{1 - \lambda}. \quad (5.6)$$

(ii) A singular point $(\lambda_0, \theta_0)$ is a cuspidal edge if and only if

$$\left( \frac{\kappa(\theta + k\pi)}{\kappa(\theta)} \right)'_{(\lambda_0, \theta_0)} \neq 0. \quad (5.7)$$

(iii) A singular point $(\lambda_0, \theta_0)$ is a swallowtail if and only if

$$\left( \frac{\kappa(\theta + k\pi)}{\kappa(\theta)} \right)'_{(\lambda_0, \theta_0)} = 0 \text{ and } \left( \frac{\kappa(\theta + k\pi)}{\kappa(\theta)} \right)''_{(\lambda_0, \theta_0)} \neq 0. \quad (5.8)$$

Proof We use (5.4) as the parameterization of $E_k(R_m)$. Let us notice that $f_k$ is singular if and only if $(f_k)_\lambda \times (f_k)_\theta = 0$. This condition is equivalent to (5.6). By Fact 1.5 in [36] we get (5.7) and (5.8).

Remark 5.8 By Theorem 3.3 in [15] there exists an open and dense subset of the space of rosettes such that the affine extended wave front $E(R_m)$ has only $A_2$ and $A_3$ singularities (cuspidal edges and swallowtails) for any rosette $R_m$ in this subset. Thus, by Proposition 5.7 a rosette $R_m$ is called generic if there do not exist $\theta$ and $k \in \{1, \ldots, m\}$ such that

$$\left( \frac{\kappa(\theta + k\pi)}{\kappa(\theta)} \right)'_{(\lambda_0, \theta_0)} = 0 \text{ and } \left( \frac{\kappa(\theta + k\pi)}{\kappa(\theta)} \right)''_{(\lambda_0, \theta_0)} = 0. \quad (5.9)$$

By direct calculation we get the following proposition (see also Definition 2.2).

Proposition 5.9 If $R_m$ is generic then every singular point of $E(R_m)$ is non-degenerate.

Remark 5.10 In [6,10,44] we study in details the geometry of affine $\lambda$-equidistants of rosettes. We show among other things that there exist $m$ branches of $E_{\frac{1}{2}}(R_m)$ and
2m − 1 branches of $E_\lambda(R_m)$ for $\lambda \neq 0, \frac{1}{2}, 1$. Let $E_{\frac{1}{2}, k}(R_m)$ for $k = 1, 2, \ldots, m$ denote different branches of $E_{\frac{1}{2}}(R_m)$ and let $E_{\lambda, k}(R_m)$ for $k = 1, 2, \ldots, 2m − 1$ denote different branches of $E_{\lambda}(R_m)$ for $\lambda \neq 0, \frac{1}{2}, 1$. Then, the support function of $E_{\frac{1}{2}, k}(R_m)$ for $k = 1, \ldots, m$ is in the form (5.10), the support function of $E_{\lambda, k}(R_m)$ for $k = 1, 2, \ldots, m$ (respectively $k = m + 1, m + 2, \ldots, 2m − 1$) in the form (5.11) (respectively in the form (5.12)), where

\begin{align}
p_{\frac{1}{2}, k}(\theta) &= \frac{1}{2}(p(\theta) + (-1)^k p(\theta + k\pi)), \quad (5.10) \\
p_{\lambda, k}(\theta) &= \lambda p(\theta) + (-1)^k (1 - \lambda) p(\theta + k\pi), \quad (5.11) \\
p_{\lambda, k}(\theta) &= (1 - \lambda) p(\theta) + (-1)^k \lambda p(\theta + (k - m)\pi). \quad (5.12)
\end{align}

Let $\gamma_{\lambda, k}$ denote the parameterization of $E_{\lambda, k}$ in terms of the support function accordingly to (5.10), (5.11) and (5.12), respectively. Furthermore each branch of $E_\lambda(R_m)$, except $E_{\frac{1}{2}, m}(R_m)$, has the rotation number equal to $m$. The rotation number of $E_{\frac{1}{2}, m}(R_m)$ is equal to $\frac{m}{2}$. If $R_m$ is a generic $m$-rosette then for $\lambda \in (0, 1) - \{\frac{1}{2}\}$ only branches $E_{\lambda, k}(R_m)$ for $k = 1, 3, \ldots, 2\lceil\frac{1}{2}m\rceil - 1, m + 1, m + 3, \ldots, m + 2\lceil\frac{1}{2}m\rceil - 1$ can admit cusp singularities and branches $E_{\frac{1}{2}, k}(R_m)$ for $k = 1, 3, \ldots, 2\lceil\frac{1}{2}m\rceil - 1$ has cusp singularities. By [13] we known that if $a, b$ is parallel pair of $R_m$ and $R_m$ is parameterized at $a$ and $b$ in different directions and $\kappa(a), \kappa(b)$ denote the signed curvatures of $R_m$ at $a$ and $b$, respectively, then the point $\frac{2\kappa_1 + \kappa_2}{\kappa_1 + \kappa_2}$, which is lying on the line between $a$ and $b$, belongs to $CSS(R_m)$.

**Corollary 5.11** Let $R_m$ be a generic $m$-rosette. Then, $CSS(R_m)$ which is created from singular points of $E_\lambda(R_m)$ for $\lambda \in [0, 1]$ consists of exactly $2\lceil\frac{1}{2}m\rceil - 1$ branches.

**Proof** It is a consequence of Remark 5.10.

Let $CSS_k(R_m)$ for $k = 1, 3, \ldots, 2\lceil\frac{1}{2}m\rceil - 1$ denote a branch of $CSS(R_m)$. Then, the parameterization of $CSS_k(R_m)$ is in the following form

\begin{equation}
\gamma_{CSS_k(R_m)}(\theta) = \frac{\kappa(\theta)}{\kappa(\theta) + \kappa(\theta + k\pi)} \gamma(\theta) + \frac{\kappa(\theta + k\pi)}{\kappa(\theta) + \kappa(\theta + k\pi)} \gamma(\theta + k\pi), \quad (5.13)
\end{equation}

where if $k < m$ then $\theta \in [0, 2m\pi]$ and if $k = m$ then $\theta \in [0, m\pi]$.

**Lemma 5.12** Let $C$ be a closed smooth curve with at most cusp singularities and let the rotation number of $C$ be $m$. If $m$ is an integer, then the number of cusp singularities is even. If $m$ is the form $\frac{1}{2}d$, where $d$ is an odd integer, then the number of cusp singularities is odd.

**Proof** A continuous normal vector field to the germ of a curve with the cusp singularity is directed outside the cusp on the one of two connected regular components and is directed inside the cusp on the other component as it is shown in Fig. 6. If $m$ is an integer, then the number of cusps of $C$ is even, otherwise is odd. \(\square\)
Proposition 5.13 Let $R_m$ be a generic $m$-rosette. If $k = m$ and $m$ is an odd number, then the number of cusp singularities of $CSS_k(R_m)$ is odd and not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$, otherwise the number of cusp singularities of $CSS_k(R_m)$ is even and not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$, which is even and positive.

Proof The parity of the number of cusp singularities of $CSS_k(R_m)$ is a consequence of (5.13) and Lemma 5.12.

Let $m$ be even and $k \leq m$ or $m$ be odd and $k < m$. By Theorem 2.9 in [44] we know that $E_{\frac{1}{2},k}(R_m)$ has at least 2 cusp singularities. Because the cusp in $E_{\frac{1}{2}}$ appears when $\frac{\kappa(a)}{\kappa(b)} = 1$ and cusp in CSS appears when $\left(\frac{\kappa(a)}{\kappa(b)}\right)' = 0$ [5,13], where $a$, $b$ is a parallel pair and $'$ is used to denote the derivative with respect to the parameter along the corresponding segment of a curve. Therefore, by Roll’s theorem we get that the number of cusp singularities of $CSS_k(R_m)$ is not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$. The same arguments works when $m$ is odd and $k = m$. \(\square\)

Corollary 5.14 Let $R_m$ be an $m$-rosette. Then, the number of branches of $E(R_m)$ is equal to $m$ and a branch $E_k(R_m)$ is singular if and only if $k$ is odd.

In Figs. 4 and 5 we present two branches of $E(R_2)$: $E_1(R_2)$ and $E_2(R_2)$, respectively.

Proposition 5.15 Let $R_m$ be an $m$-rosette and let $p$ be a non-singular point of $E_k(R_m)$. Then, the Gaussian curvature of $E_k(R_m)$ at $p$ is equal to 0.

Proof The surface is parameterized by (5.4).

At a non-singular point $(\lambda, \theta)$ the Gaussian curvature $K$ of $E_k$ is equal to

$$K_k(\lambda, \theta) = \frac{\det ((f_k)_{\lambda \lambda}, (f_k)_{\lambda}, (f_k)_{\theta}) \cdot \det ((f_k)_{\theta \theta}, (f_k)_{\lambda}, (f_k)_{\theta}) - \det^2 ((f_k)_{\lambda \theta}, (f_k)_{\lambda}, (f_k)_{\theta}))}{\left(\left| (f_k)_{\lambda} \right|^2 \left| (f_k)_{\theta} \right|^2 - ((f_k)_{\lambda} \cdot (f_k)_{\theta})^2\right)^2}$$

(5.14)

Since $(f_k)_{\lambda \lambda} = 0$ and vectors $(f_k)_{\lambda}$ and $(f_k)_{\lambda \theta}$ are linearly dependent, the Gaussian curvature $K_k$ at a non-singular point of $E_k$ is equal to zero. \(\square\)
Definition 5.16 Let $R_m$ be an $m$-rosette. Let $k \in \{1, 2, \ldots, m\}$. Then, the $k$-width of $R_m$ for an oriented angle $\theta$ is the following

$$w_k(\theta) = p(\theta) - (-1)^k p(\theta + k\pi). \quad (5.15)$$

Remark 5.17 Let $R_m$ be a generic $m$-rosette and $k \leq m$ be an odd number. From now on we set

$$M := [0, 1] \times S^1,$$

$$M \ni (\lambda, \theta) \mapsto f_k(\lambda, \theta) \in \mathbb{E}_k(R_m) \subset \mathbb{R}^3,$$

$$M \ni (\lambda, \theta) \mapsto v_k(\lambda, \theta) := \left( \frac{(w_k(\theta), n(\theta))}{\sqrt{1 + w_k^2(\theta)}} \right) \in S^2.$$

The map $(f_k, v_k)$ is a front. Then, the coherent tangent bundle $\mathcal{E}^{f_k}$ over $M$ has the following fiber at $p \in M$

$$\mathcal{E}^{f_k}_p := \left\{ X \in T_{f_k(p)}\mathbb{R}^3 \mid \langle X, v_k(p) \rangle = 0 \right\}.$$

The set of singular points $\Sigma_k$ is parameterized by $(\lambda_k(\theta), \theta)$, where

$$\lambda_k(\theta) = \frac{\kappa(\theta)}{\kappa(\theta) + \kappa(\theta + k\pi)}.$$

Let us notice that

$$M^- = \left\{ (\lambda, \theta) \in M \mid \lambda < \lambda_k(\theta) \right\}, \quad M^+ = \left\{ (\lambda, \theta) \in M \mid \lambda > \lambda_k(\theta) \right\}.$$

Furthermore, if the function $\lambda_k(\theta)$ has a local minimum, then the point $(\lambda_k(\theta), \theta)$ is a negative peak and if $\lambda_k(\theta)$ has a local maximum, then this point is a positive peak. See Fig. 7.

Proposition 5.18 Let $R_m$ be a generic $m$-rosette. Let $k$ be an odd number and let $\lambda \in [0, 1]$. Then, the $\mathcal{E}^{f_k}$-geodesic curvature of a curve $\{\lambda\} \times S^1$ in $M$ at a non-singular point is equal to

$$\hat{k}_{k,g}(\theta) := \frac{w_k(\theta)}{|\lambda \rho(\theta) - (1 - \lambda) \rho(\theta + k\pi)| \sqrt{1 + w_k^2(\theta)}}. \quad (5.16)$$
Proof Let \( s_k(\lambda, \theta) := \lambda \rho(\theta) - (1-\lambda) \rho(\theta + k\pi) \). Then (5.16) follows from the formula

\[
\hat{k}_{k,s}(\theta) = \frac{\det(\gamma_k'_{\overline{\lambda}, \theta}, \gamma_k''_{\overline{\lambda}, \theta}, \nu_k(\lambda, \theta))}{|\gamma_k'_{\overline{\lambda}, \theta}(\theta)|^3}.
\]

\( \square \)

Proposition 5.19 Let \( R_m \) be a generic m-rosette. Let k be an odd number. Then the singular curvature on a cuspidal edge at a point \( (\kappa(\theta), \kappa(\theta + k\pi), \theta) \) is equal to

\[
\kappa_{k,s}(\theta) = \kappa_{CSS_k}(\theta) \cdot \frac{\sqrt{1 + w_k^2(\theta)}}{w_k(\theta)} \cdot \left( \frac{w_k^2(\theta) + w_k'^2(\theta)}{1 + w_k^2(\theta) + w_k'^2(\theta)} \right)^{\frac{3}{2}},
\]

(5.17)

where \( \kappa_{CSS_k}(\theta) \) is a curvature of \( CSS_k(R_m) \), which is given by the following formula:

\[
\kappa_{CSS_k}(\theta) = -\frac{(\kappa(\theta) + \kappa(\theta + k\pi))^3}{\kappa(\theta)\kappa(\theta + k\pi)|\kappa'(\theta + k\pi)\kappa(\theta) - \kappa'(\theta)\kappa(\theta + k\pi)|} \cdot \frac{w_k(\theta)}{(w_k^2(\theta) + w_k'^2(\theta))^{\frac{3}{2}}},
\]

(5.18)

Proof It is a direct consequence of the formula of the singular curvature and the formula of the curvature of the centre symmetry set (see Lemma 2.6 in [10]). \( \square \)

By Theorem 1.6 in [36] we know that the singular curvature does not depend on the orientation of the parameter \( \theta \), the orientation of \( M \), the choice of \( \nu \), nor the orientation of the singular curve. The sign of the singular curvature have a geometric interpretation, if the singular curvature is positive (respectively negative) then the cuspidal edge is positively (respectively negatively) curved. See Fig. 8.

We find a formula which gives us the relation between the total singular curvature on set of singular points and the total geodesic curvature on the boundary of \( M \). The integrals in (5.19)–(5.22) can be seen as integrals on \( f_k(\Sigma_k) \) and \( f_k([\lambda] \times S^1) = [\lambda] \times E_{k,\lambda}(R_m) \) since the arclength measure, the singular curvature and \( E^k_{\overline{\lambda}} \)-geodesic curvature are defined with respect to the first fundamental form \( ds^2 \) which is the pullback of metric \( \langle \cdot, \cdot \rangle \) on \( E_k(R_m) \subset \mathbb{R}^3 \).

\[\text{Fig. 8} \] Examples of positively (on the left) and negatively (on the right) curved cuspidal edges
Theorem 5.20 Let $k$ be an odd number. Let $R_m$ be a generic $m$-rosette. Then
\[ \int_{\Sigma_k} \kappa_{k,s} d\tau + \int_{[1] \times S^1} \hat{\kappa}_{k,g} d\tau = 0, \quad (5.19) \]
where $d\tau$ denote the arc length measure and the orientation of $[1] \times S^1$ is compatible with the orientation of $M$.

Proof By Remark 5.17 we get that $(f_k, v_k) : M \to \mathbb{R}^3 \times S^2$ is a front. The boundary of $M$ does not intersect the set of singular points $\Sigma$. By genericity of $R_m$ this front satisfies the assumptions of Theorem 2.20. Since $\lambda_k(\theta) + \lambda_k(\theta + k\pi) = 1$, we get that $M^+$ and $M^-$ are symmetric. Hence $\chi(M^+) = \chi(M^-)$ and $\#P^- = \#P^+$.

By Proposition 5.15 and Theorem 2.20 we get that
\[ \int_{[1] \times S^1} \hat{\kappa}_{k,g} d\tau = -\int_{[0] \times S^1} \hat{\kappa}_{k,g} d\tau \]
and then we get (5.19). \qed

Theorem 5.21 Let $k$ be an odd number, $R_m$ be a generic $m$-rosette and $\lambda \in [0, 1)$. If $E_{k, \lambda}(R_m)$ admits at most cusp singularities, then
\[ \int_{[\lambda] \times S^1} \hat{\kappa}_{k,g} d\tau = -\int_{[1] \times S^1} \hat{\kappa}_{k,g} d\tau, \quad (5.20) \]
\[ \int_{(\frac{1}{2}) \times S^1} \hat{\kappa}_{k,g} d\tau = \sum_{p \in C} \alpha_+(p) - \frac{1}{2}\pi #C - \frac{1}{2} \int_{[1] \times S^1} \hat{\kappa}_{k,g} d\tau, \quad (5.21) \]
\[ \int_{(\frac{1}{2}) \times S^1} \hat{\kappa}_{k,g} d\tau = -\sum_{p \in C} \alpha_+(p) + \frac{1}{2}\pi #C - \frac{1}{2} \int_{[1] \times S^1} \hat{\kappa}_{k,g} d\tau, \quad (5.22) \]
where the orientations of $S^1$ in the integrals on the left hand sides and the right-hand sides are opposite in the above formulas, $C = \Sigma_k \cap ((\frac{1}{2}) \times S^1)$, $d\tau$ is the arclength measure and
\[ \alpha_+(p) := \arccos \left( \sqrt{\frac{w_k^2(\theta) + w_k^2(\theta)}{1 + w_k^2(\theta) + w_k^2(\theta) \cos(\beta(\theta))}} \right). \quad (5.23) \]
where $p = (\frac{1}{2}, \theta)$ and $\beta(\theta)$ is the angle between the tangent vector to $R_m$ at $\gamma(\theta)$ and the vector $\gamma(\theta + k\pi) - \gamma(\theta)$.

Proof Let $M_\lambda := [\lambda, 1] \times S^1$. By Remark 5.17 we get that $(f_k, v_k)|_{M_\lambda} : M_\lambda \to \mathbb{R}^3 \times S^2$ is a front. It is easy to see that $\chi(M_\lambda^+) = 0$ and $\chi(M_\lambda^-) = \#P^- - \#P^+$ is the number of cusps of $E_{k, \lambda}(R_m)$ (that is $\#(\Sigma_k \cap ((\lambda) \times S^1))$). Since every point $p \in \Sigma_k \cap \partial M_\lambda$ is a null singular point, by Theorem 2.20 (see (2.9)) we get (5.20).
By the genericity of $R_m$ the front $(f_k, v_k)|_{M_{\frac{1}{2}}}^+$ satisfies the assumptions of Theorem 2.20. Since $\int_{\Sigma_k} \kappa_s d\tau = 2 \int_{\Sigma_k \cap M_{\frac{1}{2}}} \kappa_s d\tau$, we get (5.21) and (5.22).

The angle between initial vectors (see Definition 2.5) of the singular curve at $p$ and of the boundary curve at $p$ is $\alpha_+(p)$ (see Theorem 2.20). By Proposition 2.7 and Proposition 2.8 we get (5.23).

Furthermore, directly by (2.9) we get the following proposition.

**Proposition 5.22** Let $k$ be an odd number. Let $R_m$ be a generic $m$-rosette. Let $C^+$ (respectively $C^-$) be a simple regular curve in $M^+$ (respectively $M^-$) which is smoothly homotopic to $\{1\} \times S^1$ (respectively $\{0\} \times S^1$). If the orientations of $C^+$, $C^-$ are opposite then

$$\int_{C^+} \kappa_{k,g} d\tau + \int_{C^-} \kappa_{k,g} d\tau = 0,$$

where $d\tau$ denote the arc length measure.

By Theorem 5.20 we can get the relation between integrals of the curvature of the centre symmetry set, the curvature of the rosette and the width of the rosette.

**Corollary 5.23** Let $k$ be an odd number and let $R_m$ be a generic $m$-rosette. Then

$$\int_{R_m} \kappa(\theta(s)) \cdot \frac{w_k(\theta(s))}{\sqrt{1 + w_k^2(\theta(s))}} ds = \int_{CSS_k(R_m)(\theta(\ell))} \kappa_{CSS_k(R_m)}(\theta(\ell)) \cdot \frac{(\rho(\theta(\ell)) + \rho(\theta(\ell) + k\pi))\sqrt{1 + w_k^2(\theta(\ell))}}{(1 + w_k^2(\theta(\ell)) + w_k^2(\theta(\ell)))^\frac{3}{2}} d\ell,$$

(5.24)

where $s$ (respectively $\ell$) is the arc length parameter on $R_m$ (respectively on $CSS_m(R_m)$).

**Theorem 5.24** Let $k$ be an odd number and let $R_m$ be a generic $m$-rosette. Then

$$\int_0^{2m\pi} \frac{w_k(\theta)}{\sqrt{1 + w_k^2(\theta)}} d\theta = \int_0^{2m\pi} \left( w_k(\theta) + w_k''(\theta) \right) \cdot \sqrt{\frac{1 + w_k^2(\theta)}{1 + w_k^2(\theta) + w_k^2(\theta)}} d\theta.$$

(5.25)

**Proof** The proof is a straightforward use of (5.16), (5.17) and the fact that $\rho(\theta) + \rho(\theta + k\pi) = w_k(\theta) + w_k''(\theta)$. □

**Remark 5.25** Since $w_k(\theta) = \sinh(C_1\theta + C_2)$ for $C_1, C_2 \in \mathbb{R}$ is the general solution of

$$\frac{w_k(\theta)}{\sqrt{1 + w_k^2(\theta)}} = \left( w_k(\theta) + w_k''(\theta) \right) \cdot \sqrt{\frac{1 + w_k^2(\theta)}{1 + w_k^2(\theta) + w_k^2(\theta)}},$$

(5.26)
the only periodic solution of (5.26) is a constant function. Therefore, the relation (5.25)
is naively fulfilled only for rosettes of constant k-width.

**Remark 5.26** The condition that \( w \) is \( C^2 \)-smooth cannot be omitted. We can consider the function \( w(\theta) = 1 + |x - \pi|^3 \) and the interval \([0, 2\pi]\). One can check that relation (5.25) does not hold.

**Remark 5.27** By (5.15) the odd coefficients of the Fourier series of a width of an oval vanish. Thus, a function \( w(\theta) = 2 + \sin 3\theta \) is not a width of any oval but it satisfies the relation (5.25).

**Conjecture 5.28** Let \( w : \mathbb{R} \to \mathbb{R} \) be a \( 2\pi \)-periodic \( C^2 \)-smooth function. Then, \( w \) satisfies the relation

\[
\int_0^{2\pi} \frac{w(\theta)}{\sqrt{1 + w^2(\theta)}} \, d\theta = \int_0^{2\pi} \left( w(\theta) + w''(\theta) \right) \cdot \frac{\sqrt{1 + w^2(\theta)}}{1 + w^2(\theta) + w''^2(\theta)} \, d\theta. \tag{5.27}
\]

In [29,30] other invariants of cuspidal edges of fronts are introduced. Let \((f, \nu) : M \to \mathbb{R}^3 \times S^2\) be a front. Let \( \gamma \) be a singular curve near an \( A_2 \)-point (a cuspidal edge) and \( \eta \) be a null direction along \( \gamma \) such that the singular direction \( \gamma' \) and the null direction \( \eta \) form a positively oriented frame. We put \( \hat{\gamma} = f \circ \gamma, f_\eta = df(\eta), f_{\eta,\eta} = d(f_\eta)(\eta), f_{\eta,\eta,\eta} = d(f_{\eta,\eta})(\eta). \) Then, the limiting normal curvature along \( \gamma \) is defined in the following way

\[
\kappa_\nu(t) = \frac{[\hat{\gamma}''(t), \nu(\gamma(t))]}{|\hat{\gamma}'(t)|^2}. \tag{5.28}
\]

The **cuspidal curvature along** \( \gamma \) is defined as follows:

\[
\kappa_c(t) = \frac{|\hat{\gamma}(t)|^3 \det(\hat{\gamma}(t), f_{\eta,\eta}(\gamma(t)), f_{\eta,\eta,\eta}(\gamma(t)))}{|\hat{\gamma}(t) \times f_{\eta,\eta}(\gamma(t))|^5}. \tag{5.29}
\]

The **cusp-directional torsion** is defined by the formula

\[
\kappa_t(t) = \frac{\det(\hat{\gamma}', f_{\eta,\eta}(\gamma), (f_{\eta,\eta}(\gamma))^\prime)}{|\hat{\gamma}' \times f_{\eta,\eta}(\gamma)|^2}(t) - \frac{\det(\hat{\gamma}', f_{\eta,\eta}(\gamma), \hat{\gamma}'') \cdot [\hat{\gamma}', f_{\eta,\eta}(\gamma)]}{|\hat{\gamma}'|^2 |\hat{\gamma}' \times f_{\eta,\eta}(\gamma)|^2}(t). \tag{5.30}
\]

In [36], it was shown that a point \( p \) is a generic cuspidal edge if and only if \( \kappa_\nu(p) \) does not vanish. The curvature \( \kappa_c \) is exactly the cuspidal curvature of the cusp of the plane curve obtained as the intersection of the surface by the plane \( H \), where \( H \) is orthogonal to the tangential direction at a given cuspidal edge [30]. For the geometrical meaning of the cusp-directional torsion (5.30) see Proposition 5.2 in [29] and for global properties see Appendix A in [29]. By straightforward calculations we obtain the following lemma.
Lemma 5.29  Let $R_m$ be a generic $m$-rosette. Let $k$ be an odd number. Then the normal curvature $\kappa_{k,v}$, the cuspidal curvature $\kappa_{k,c}$ and the cusp-directional torsion $\kappa_{k,t}$ of the cuspidal edge of $E_k(R_m)$ at a point $\left(\frac{\kappa(\theta)}{\kappa(\theta) + k(\theta + k\pi)}, \theta\right)$ are given by the following formulas

$$\kappa_{k,v}(\theta) \equiv 0,$$

$$\kappa_{k,c}(\theta) = \frac{2\sqrt{\kappa(\theta)\kappa(\theta + k\pi)(\kappa(\theta) + \kappa(\theta + k\pi))}}{\sqrt{\left|\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right|'}} \cdot \left(1 + w_k^2(\theta) + w_k^2(\theta)\right)^\frac{3}{2},$$

$$\kappa_{k,t}(\theta) = -\frac{\left(\kappa(\theta) + \kappa(\theta + k\pi)\right)^2}{\kappa^2(\theta)} \cdot \left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)^\prime \cdot \frac{1}{1 + w_k^2(\theta)}.$$

(5.31) (5.32) (5.33)

Proposition 5.30  Let $R_m$ be a generic $m$-rosette. Let $k$ be an odd number. Then

(i) cuspidal edges of $E_k(R_m)$ are not generic,
(ii) the mean curvature of $E_k(R_m)$ is not bounded,
(iii) the total torsion of the image of singular curve $\hat{\gamma}_k(\theta)$ for $\theta \in [0, 2k\pi]$ is equal to $2n\pi$ for some integer $n$, i.e.

$$\int_{\gamma_k} \tau_k(s)ds = 2n\pi,$$

where $\gamma_k$ is the singular curve, $\tau_k$ is a torsion of $\hat{\gamma}_k$ and $s$ is the arc length parameter of $\hat{\gamma}_k$.

Proof  (i) It is a consequence of (5.31).
(ii) Since $\kappa_{k,c}(p) \neq 0$ for any cuspidal edge $p \in \Sigma$, then by Proposition 2.8 in [30] we get that the mean curvature of $CSS_k(R_m)$ is not bounded.
(iii) From Appendix A in [29] we know that in our case there is the following equality

$$\int_{\gamma_k} \kappa_{k,t}(s)ds = \int_{\gamma_k} \tau_k(s)ds - 2n\pi.$$

It is easy to see that $\int_{\gamma_k} \kappa_{k,t}(s)ds = 0$. Hence (5.34) holds.

Remark 5.31  For the geometrical meaning of the number $n$ in Corollary 5.30(iii) see Appendix A in [29]. In [32], authors show that the total torsion of a closed line of curvature on a surface (i.e. a closed curve on a surface whose tangents are always in the direction of a principal curvature) is $l\pi$, where $l$ is an integer. Furthermore, they show that if the total torsion of a closed curve is $l\pi$ for an integer $l$, then this curve can appear as a line of curvature on a surface and if $l$ is even, then it can appear as a line of curvature on a surface of genus 1.
Acknowledgements  The authors thank Kentaro Saji and Zbigniew Szafraniec for fruitful discussions and valuable comments. The authors also thank the reviewer for the very useful suggestions.

Open Access  This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Berry, M.V.: Semi-classical mechanics in phase space: a study of Wigner’s function. Philos. Trans. R. Soc. Lond. A 287, 237–271 (1977)
2. Cieślak, W., Mozgawa, W.: On rosettes and almost rosettes. Geom. Dedicata 24(2), 221–228 (1987)
3. Craizer, M.: Iteration of involutes of constant width curves in the Minkowski plane. Beiträge zur Algebra und Geometrie 55(2), 479–496 (2014)
4. Craizer, M., Domitrz, W., Rios, P.M.: Even dimensional improper affine spheres. J. Math. Anal. Appl. 421(2), 1803–1826 (2015)
5. Domitrz, W., Rios, P.M.: Singularities of equidistants and Global Centre Symmetry sets of Lagrangian submanifolds. Geom. Dedicata 169, 361–382 (2014)
6. Domitrz, W., Zwierzyński, M.: Singular points of the Wigner caustic and affine equidistants of planar curves. Bull. Braz. Math. Soc. New Series (2019). https://doi.org/10.1007/s00574-019-00141-4
7. Domitrz, W., Manoel, M., Rios, P.M.: The Wigner caustic on shell and singularities of odd functions. J. Geom. Phys. 71, 58–72 (2013)
8. Domitrz, W., Rios, P.M., Ruas, M.A.S.: Singularities of affine equidistants: projections and contacts. J. Singul. 10, 67–81 (2014)
9. Domitrz, W., Janeczko, S., Rios, P.M., Ruas, M.A.S.: Singularities of affine equidistants: extrinsic geometry of surfaces in 4-space. Bull. Braz. Math. Soc. (N.S.) 47(4), 1155–1179 (2016)
10. Domitrz, W., Zwierzyński, M.: The geometry of the Wigner caustic and affine equidistants of planar curves. arXiv:1605.05361v4
11. Dubrovin, B.A., Fomenko, A.T., Novikov, S.P.: Modern Geometry—Methods and Applications. Part II, The Geometry and Topology of Manifolds. Graduate Texts in Mathematics, vol. 104, Springer, New York
12. Fukuda, T., Ishikawa, G.: On the number of cusps of stable perturbations of a plane-to-plane singularity. Tokyo J. Math. 10(2), 375–384 (1987)
13. Giblin, P.J., Holtom, P.A.: The Centre Symmetry Set, Geometry and Topology of Caustics, vol. 50, pp. 91–105. Banach Center Publications, Warsaw (1999)
14. Giblin, P.J., Reeve, G.M.: Centre symmetry sets of families of plane curves. Demonstr. Math. 48, 167–192 (2015)
15. Giblin, P.J., Zakalyukin, V.M.: Singularities of centre symmetry sets. Proc. Lond. Math. Soc. (3) 90, 136–161 (2005)
16. Giblin, P.J., Warder, J.P., Zakalyukin, V.M.: Bifurcations of affine equidistants. Proc. Steklov Inst. Math. 267, 57–75 (2009)
17. Janeczko, S.: Bifurcations of the center of symmetry. Geom. Dedicata 60, 9–16 (1996)
18. Janeczko, S., Jelonek, Z., Ruas, M.A.S.: Symmetry defect of algebraic varieties. Asian J. Math. 18(3), 525–544 (2014)
19. Kokubu, M., Rossman, W., Saji, K., Umehara, M., Yamada, K.: Singularities flat fronts in hyperbolic 3-space. Pac. J. Math. 221, 265–299 (2005)
20. Kossowski, M.: The Boy–Gauss–Bonnet theorems for \( C^1 \)-singular surfaces with limiting tangent bundle. Ann. Glob. Anal. Geom. 21, 19–29 (2002)
21. Kossowski, M.: Realizing a singular first fundamental form as a nonimmersed surface in Euclidean 3-space. J. Geom. 81, 101–113 (2004)
22. Krzyżanowska, I., Szafraniec, Z.: On polynomial mappings from the plane to the plane. J. Math. Soc. Jpn. 66(3), 805–818 (2014)
23. Langevin, R., Levitt, G., Rosenberg, H.: Hérissons et multihérissons (enveloppes paramétrées par leur application de Gauss). (French) [Hedgehogs and multihedgehogs (envelopes parametrized by
their Gauss map] Singularities (Warsaw, 1985), pp. 245–253. Banach Center Publications, 20, PWN, Warsaw (1988)

24. Langevin, R., Levitt, G., Rosenberg, H.: Classes d’homotopie de surfaces avec rebroussements et queues d’aronde dans $\mathbb{R}^3$. Can. J. Math. 47, 544–572 (1995)
25. Martinez-Maure, Y.: Hedgehogs and Zonoids. Adv. Math. 158(1), 1–17 (2001)
26. Martinez-Maure, Y.: Théorie des hérissons et polytopes. Comptes Rendus de l’Académie des Sciences de Paris, Série I(336), 241–244 (2003)
27. Martinez-Maure, Y.: Geometric study of Minkowski differences of plane convex bodies. Can. J. Math. 58, 600–624 (2006)
28. Martinez-Maure, Y.: Hedgehog theory via Euler Calculus. Beitraege zur Algebra und Geometrie 56, 397–421 (2015)
29. Martins, L.F., Saji, K.: Geometric invariants of cuspidal edges. Can. J. Math. 68(2), 445–462 (2016)
30. Martins, L.F., Saji, K., Umehara, M., Yamada, K.: Behavior of Gaussian Curvature and Mean Curvature Near Non-degenerate Singular Points on Wave Fronts. Springer Proceedings in Mathematics & Statistics, vol. 154, pp. 247–281 (2016)
31. Miernowski, A., Mozgawa, W.: Isoptics of rosettes and rosettes of constant width. Note di Matematica 15(2), 203–213 (1995)
32. Qin, Y.A., Li, S.J.: Total torsion of closed lines of curvature. Bull. Aust. Math. Soc. 65(1), 73–78 (2002)
33. Quine, J.R.: A global theorem for singularities of maps between oriented 2-manifolds. Trans. Am. Math. Soc. 236, 307–314 (1978)
34. Reeve, G.M., Tari, F.: Minkowski Symmetry Sets of Plane Curves. Proc. Edinb. Math. Soc. (2) 60(2), 461–480 (2017)
35. Saji, K., Umehara, M., Yamada, K.: Behavior of corank-one singular points on wave fronts. Kyushu J. Math. 62, 259–280 (2008)
36. Saji, K., Umehara, M., Yamada, K.: The geometry of fronts. Ann. Math. 169, 491–529 (2009)
37. Saji, K., Umehara, M., Yamada, K.: Coherent tangent bundles and Gauss–Bonnet formulas for wave fronts. J. Geom. Anal. 22(2), 383–409 (2012)
38. Saji, K., Umehara, M., Yamada, K.: An index formula for a bundle homomorphism of the tangent bundle into a vector bundle of the same rank, and its applications. J. Math. Soc. Jpn. 69(1), 417–457 (2017)
39. Schneider, R.: Reflections of planar convex bodies. In: Convexity and Discrete Geometry Including Graph Theory. Springer Proceedings in Mathematics and Statistics, vol. 148, pp. 69–76. Springer, Cham (2016)
40. Schneider, R.: The middle hedgehog of a planar convex body. Beitrage zur Algebra und Geometrie 58, 235–245 (2017)
41. Zakalyukin, V.M.: Envelopes of families of wave fronts and control theory. Proc. Steklov Math. Inst. 209, 133–142 (1995)
42. Zwierzyński, M.: The improved isoperimetric inequality and the Wigner caustic of planar ovals. J. Math. Anal. Appl. 442(2), 726–739 (2016)
43. Zwierzyński, M.: The Constant Width Measure Set, the Spherical Measure Set and isoperimetric equalities for planar ovals. arXiv:1605.02930
44. Zwierzyński, M.: Isoperimetric equalities for rosettes. arXiv:1605.08304

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.