Gluing of categories and Krull–Schmidt partners

D. O. Orlov

Let $k$ be a field and let $\mathcal{A}$ be a small $k$-linear differential graded (DG) category. We define a (right) DG $\mathcal{A}$-module as a DG-functor $M: \mathcal{A}^{\text{op}} \to \text{Mod}-k$, where $\text{Mod}-k$ is the DG-category of complexes of $k$-vector spaces. Let $\text{Mod}-\mathcal{A}$ be the DG-category of right DG $\mathcal{A}$-modules, and let $\text{AC-}\mathcal{A}$ be the full DG subcategory consisting of all acyclic DG modules. The homotopy category $H^0(\text{Mod}-\mathcal{A})$ is triangulated, and $H^0(\text{AC-}\mathcal{A})$ forms a full triangulated subcategory. The derived category $D(\mathcal{A})$ is defined as the Verdier quotient $D(\mathcal{A}) := H^0(\text{Mod-}\mathcal{A})/H^0(\text{AC-}\mathcal{A})$.

Each object $Y \in \mathcal{A}$ defines the representable DG-module $h^Y_\mathcal{A}(-) := \text{Hom}_\mathcal{A}(-, Y)$. The DG-module $P$ is said to be free if it is isomorphic to a direct sum of DG modules of the form $h^Y_\mathcal{A}[n]$, and it is said to be semi-free if it has a filtration $0 = \Phi_0 \subset \Phi_1 \subset \cdots = P$ with free quotients $\Phi_{i+1}/\Phi_i$. The full DG subcategory of semi-free DG-modules is denoted by $\text{SF-}\mathcal{A}$. The canonical DG-functor $\text{IF-}\mathcal{A} \hookrightarrow \text{Mod-}\mathcal{A}$ induces an equivalence of triangulated categories $H^0(\text{IF-}\mathcal{A}) \simeq D(\mathcal{A})$ [1], [2]. We denote by $\text{IF}_{1g-}\mathcal{A} \subset \text{IF-}\mathcal{A}$ the full DG-subcategory of finitely generated semi-free DG-modules, that is, $\Phi_n = P$ for some $n$, and $\Phi_{i+1}/\Phi_i$ is a finite direct sum of the modules $h^Y_\mathcal{A}[n]$. A DG-category of perfect modules $\text{Perf-}\mathcal{A}$ is the full DG subcategory of $\text{IF-}\mathcal{A}$ consisting of all DG modules that are homotopy equivalent to direct summands of objects in $\text{IF}_{1g-}\mathcal{A}$. Denote by $\text{Perf-}\mathcal{A}$ the homotopy category $H^0(\text{Perf-}\mathcal{A})$. It is triangulated, and it is equivalent to the subcategory of compact objects $D(\mathcal{A})^c \subset D(\mathcal{A})$.

**Definition 1.** Let $\mathcal{A}$ and $\mathcal{B}$ be small DG categories, and let $S$ be a $\mathcal{B}-\mathcal{A}$-bimodule. Define an upper triangular DG category $C = \mathcal{A} \downarrow S \mathcal{B}$ as follows:

1) $\text{Ob}(C) = \text{Ob}(\mathcal{A}) \sqcup \text{Ob}(\mathcal{B})$,

2) $\text{Hom}_C(X, Y) = \begin{cases} \text{Hom}_\mathcal{A}(X, Y), & X, Y \in \mathcal{A}, \\ \text{Hom}_\mathcal{B}(X, Y), & X, Y \in \mathcal{B}, \\ S(Y, X), & X \in \mathcal{A}, Y \in \mathcal{B}, \\ 0, & X \in \mathcal{B}, Y \in \mathcal{A}, \end{cases}$

with the composition law coming from $\mathcal{A}$ and $\mathcal{B}$, and the bimodule structure on $S$ [3], [4].

The gluing $\text{Perf-}\mathcal{A} \mathcal{C} \text{Perf-}\mathcal{B}$ (respectively, $D(\mathcal{A}) \mathcal{C} D(\mathcal{B})$) is defined as $\text{Perf-}\mathcal{C}$ (respectively, $D(\mathcal{C})$), where $\mathcal{C} = \mathcal{A} \downarrow S \mathcal{B}$.

The natural inclusions $a: \mathcal{A} \hookrightarrow \mathcal{A} \downarrow S \mathcal{B}$ and $b: \mathcal{B} \hookrightarrow \mathcal{A} \downarrow S \mathcal{B}$ define the fully faithful DG functors $a^*$ and $b^*$ from $\text{Perf-}\mathcal{A}$ and $\text{Perf-}\mathcal{B}$ to $\text{Perf-}\mathcal{A} \mathcal{C} \text{Perf-}\mathcal{B}$, which induce fully faithful functors $a^*$ and $b^*$ from $D(\mathcal{A})$ and $D(\mathcal{B})$ (respectively, $\text{Perf-}\mathcal{A}$ and $\text{Perf-}\mathcal{B}$) to $D(\mathcal{C})$ (respectively, $\text{Perf-}\mathcal{C}$) and give semi-orthogonal decompositions of triangulated categories $D(\mathcal{C}) = \langle D(\mathcal{A}), D(\mathcal{B}) \rangle$ and $\text{Perf-}\mathcal{C} = \langle \text{Perf-}\mathcal{A}, \text{Perf-}\mathcal{B} \rangle$. The natural restriction functors $a_*$ and $b_*$ from $\text{IF-}(\mathcal{A} \downarrow S \mathcal{B})$ to $\text{IF-}\mathcal{A}$ and $\text{IF-}\mathcal{B}$ induce derived functors $a_*$ and $b_*$ from $D(\mathcal{A} \downarrow S \mathcal{B})$ to $D(\mathcal{A})$ and $D(\mathcal{B})$ that send perfect modules to perfect modules. Let $T$ be a $\mathcal{B}-\mathcal{A}$-bimodule. For each DG $\mathcal{B}$-module $N$ we can construct a DG $\mathcal{A}$-module $N \otimes_\mathcal{B} T$. The DG-functor $(-) \otimes_\mathcal{B} T: \text{Mod-}\mathcal{B} \to \text{Mod-}\mathcal{A}$ does not respect quasi-isomorphisms in general, but if $T$ is a semi-free $\mathcal{B}-\mathcal{A}$-bimodule, then we...
obtain a DG-functor \((-\otimes_{\mathcal{B}} T) : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})\) that on the homotopy level induces the derived functor \((-\otimes_{\mathcal{B}} T) : D(\mathcal{B}) \to D(\mathcal{A})\).

Any \(\mathcal{B}-\mathcal{A}\)-bimodule \(P\) gives a \(\mathcal{B}-\mathcal{C}\)-bimodule \(\mathcal{P}\) such that \(\mathcal{P}(B, C) = P(B, C)\) if \(C \in \mathcal{A}\) and \(= 0\) otherwise. Any morphism \(\phi : S \to T\) of a \(\mathcal{B}-\mathcal{A}\)-bimodule induces a \(\mathcal{B}-\mathcal{C}\)-bimodule \(\mathcal{T}\) by the rule that \(\mathcal{T}(B, C) = \mathcal{T}(B, C)\) if \(C \in \mathcal{A}\) and \(\mathcal{T}(B, C) = \text{Hom}_{\mathcal{A}}(C, B)\) if \(C \in \mathcal{B}\), with the natural bimodule structure from the composition \(\text{Hom}_{\mathcal{A}}(C, B) \otimes S(C, A) \to S(B, A) \overset{\phi}{\to} T(B, A)\) for \(C \in \mathcal{B}\). There is an isomorphism \(\mathcal{S}(B, -) \cong \text{Hom}_{\mathcal{C}}(-, B) = h_{\mathcal{B}}^B\), and the functor \((-\otimes_{\mathcal{B}} S) : D(\mathcal{B}) \to D(\mathcal{C})\) is isomorphic to the fully faithful functor \(b^* : D(\mathcal{B}) \to D(\mathcal{C})\).

**Theorem 1.** Let the DG categories \(\mathcal{A}\) and \(\mathcal{B}\), the DG \(\mathcal{B}-\mathcal{A}\)-bimodules \(S\) and \(T\), and the morphism \(\phi : S \to T\) be as above, and let \(R = \text{Cone}(\phi)\) be the cone of \(\phi\). Suppose that for any DG \(\mathcal{B}\)-modules \(M\) and \(N\) the following condition holds:

\[
\text{Hom}_{D(\mathcal{B})}(M \otimes_{\mathcal{B}} R, N \otimes_{\mathcal{B}} T) = 0.
\]

Then the derived functor \((-\otimes_{\mathcal{B}} T) : D(\mathcal{B}) \to D(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})\) is fully faithful, and it induces a semi-orthogonal decomposition \(D(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) = \langle D(\mathcal{E}), D(\mathcal{B}) \rangle\) for some small DG category \(\mathcal{E}\).

**Proof.** The morphism \(\phi\) induces a morphism \(\phi : \mathcal{S} \to \mathcal{T}\) whose cone is quasi-isomorphic to the \(\mathcal{B}-\mathcal{C}\)-bimodule \(\mathcal{R}\). The functor \((-\otimes_{\mathcal{B}} R)\) is isomorphic to the composition of \(a^*\) and \((-\otimes_{\mathcal{B}} R)\), while the functor \((-\otimes_{\mathcal{B}} T)\) is isomorphic to \(a_*(-\otimes_{\mathcal{B}} \mathcal{T})\). Thus, we obtain the following isomorphisms:

\[
\text{Hom}_{D(\mathcal{E})}(M \otimes_{\mathcal{B}} R, N \otimes_{\mathcal{B}} T) = 0,
\]

\[
\text{Hom}_{D(\mathcal{E})}(M \otimes_{\mathcal{B}} \mathcal{S}, N \otimes_{\mathcal{B}} R) = 0.
\]

Therefore, for any DG \(\mathcal{B}\)-modules \(M\) and \(N\) we have isomorphisms

\[
\text{Hom}_{D(\mathcal{E})}(M \otimes_{\mathcal{B}} \mathcal{T}, N \otimes_{\mathcal{B}} \mathcal{T}) = \text{Hom}_{D(\mathcal{E})}(M \otimes_{\mathcal{B}} \mathcal{S}, N \otimes_{\mathcal{B}} \mathcal{T}) = \text{Hom}_{D(\mathcal{E})}(M \otimes_{\mathcal{B}} \mathcal{S}, N \otimes_{\mathcal{B}} \mathcal{T}).
\]

The functor \((-\otimes_{\mathcal{B}} \mathcal{T})\) is fully faithful, and hence so is the functor \((-\otimes_{\mathcal{B}} \mathcal{T})\).

**Theorem 2.** Let the DG categories \(\mathcal{A}\) and \(\mathcal{B}\), the DG \(\mathcal{B}-\mathcal{A}\)-bimodules \(S\), \(T\), and \(R\), and the morphism \(\phi\) be as above, and assume the condition (1). If the functors \((-\otimes_{\mathcal{B}} \mathcal{T})\) and \(R \text{Hom}_{\mathcal{C}}(\mathcal{T}, -)\) send perfect modules to perfect modules, then there is a semi-orthogonal decomposition of the form \(\text{Perf}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) = \langle \text{Perf}(\mathcal{E}), \text{Perf}(\mathcal{B}) \rangle\) for some small DG category \(\mathcal{E}\).

The category \(\text{Perf}(\mathcal{E})\) (respectively, \(\text{Perf}(\mathcal{E})\)) in Theorem 2 will be called a Krull–Schmidt partner (KS partner) for \(\text{Perf}(\mathcal{A})\) (respectively, \(\text{Perf}(\mathcal{A})\)). Since the composition of \((-\otimes_{\mathcal{B}} \mathcal{T})\) and \(R \text{Hom}_{\mathcal{C}}(\mathcal{S}, -)\) is isomorphic to the identity functor, the K-theories for \(\text{Perf}(\mathcal{E})\) and \(\text{Perf}(\mathcal{A})\) are the same. Moreover, their K-motives are isomorphic.

Let \(X\) be a smooth projective scheme, and let \(P_s \in \text{Perf}(X), s = 1, 2,\) be two perfect complexes such that their supports supp\(P_s\) and supp\(P_2\) do not meet. Consider the gluing \(\mathcal{D} = \text{Perf}(X) \otimes_\mathcal{S} \text{Perf}(k)\) with \(\mathcal{S} = P_1 \oplus P_2\) and take \(T = P_2\). By Theorem 2 the functor \((-\otimes_{\mathcal{B}} \mathcal{T})\) from \(\text{Perf}(k)\) to \(H^0(\mathcal{D})\) is fully faithful, and we obtain a semi-orthogonal decomposition \(H^0(\mathcal{D}) = \langle \text{Perf}(\mathcal{E}), \text{Perf}(\mathcal{B}) \rangle\). Thus, we get a KS-partner \(\text{Perf}(\mathcal{E})\) for \(\text{Perf}(X)\).
For example, let $X = C$ be a smooth projective curve of genus $g = g(C)$, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be torsion coherent sheaves of lengths $l_s = \text{length } \mathcal{P}_s$, $s = 1, 2$, such that $\text{supp} \mathcal{P}_1 \cap \text{supp} \mathcal{P}_2 = \emptyset$. It can easily be checked that no KS partner $\mathcal{Perf} - \mathcal{E}$ is equivalent to $\mathcal{Perf} - C$. Indeed, the integral bilinear form $\chi(E, F) = \sum_m (-1)^m \dim \text{Hom}(E, F[m])$ on $K_0(C)$ factors through $\mathbb{Z}^2 = H^{ev}(C, \mathbb{Z})$. In this case the forms $\chi$ on $K_0(C)$ and $K_0(\mathcal{E})$ are equal to $\chi_C = \begin{pmatrix} 1 - g & 1 \\ -1 & 0 \end{pmatrix}$ and $\chi_\mathcal{E} = \chi_t := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}$, $t = 1 - g - l_1 l_2$. The integral bilinear forms $\chi_t$ are not equivalent for different $t$. Hence the categories $\mathcal{Perf} - C$ and $\mathcal{Perf} - \mathcal{E}$ that are KS partners of $\mathcal{Perf} - C$ are not equivalent for different $t = 1 - g - l_1 l_2$. The case of $\mathbb{P}^1$ and two points $\mathbb{P}_s = p_s$, $s = 1, 2$, is discussed in [5], §3.1.

Bibliography

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Dmitri O. Orlov
Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: orlov@mi.ras.ru

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