The Nonlinear Multiplet Revisited

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Abstract

Using a reformulation of the nonlinear multiplet as a gauge multiplet, we discuss its dynamics. We show that the nonlinear “duality” that appears to relate the model to a conventional $\sigma$-model introduces a new sector into the theory.

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1 Introduction

Conformal field theories provide string backgrounds. Different supermultiplets in general give rise to different superconformal field theories. The nonlinear multiplet was first introduced as a compensating multiplet for $N = 2$ $D = 4$ supergravity [1]. It was later used to construct hyperkähler metrics [2]. In this letter we study it as a $D = 2$ dynamical system in its own right. We find a number of novel features: It can be formulated as a theory with a sector linear in a gauge field analogous to a $\Phi F$ topological theory [3, 4]. It is “dual” to an ordinary $\sigma$-model via a nonlinear “duality” that introduces new solutions to the classical field equations (and presumably new states in the quantum theory).

We begin this letter with a quick review of the nonlinear multiplet and $N = 4$ superspace. We then reformulate the nonlinear multiplet as a kind of $U(1)$ gauge multiplet, and give a construction of a nontrivial string theory background (a hyperkähler manifold). Next we define gauge-covariant components and give the component action. Finally, we discuss the nonlinear “duality”.

2 Review of the nonlinear multiplet

We begin with a quick review of higher $N$ superspaces in two dimensions. Dimensional analysis of the superspace measure and the superfield component content implies that to construct superspace actions one needs to find invariant subspaces and corresponding restricted measures. Such subspaces are analogous to $N = 1$, $D = 4$ chiral and antichiral superspaces. To this end, in a series of papers [5, 6, 7, 8], we have constructed and used a projective superspace. In the present $N = 4$ context, it is introduced as follows [2, 8] (see also [3, 4]): The complex $SU(2)$ doublet spinor derivatives $D_\alpha^\pm$, $\bar{D}_\beta^\pm$ that describe $N = 4$ supersymmetry obey the commutation relations

$$\{D_\alpha^\pm, \bar{D}_\beta^\pm\} = i\delta^b_\alpha \delta^b_\beta \partial^\pm$$  \hspace{1cm} (1)

(all others vanish). When we work with $N = 2$ superfields, we identify $D^\pm \equiv D_1^\pm$ as the $N = 2$ spinor covariant derivative and $Q^\pm \equiv D_2^\pm$ as the
generator of the nonmanifest supersymmetries. We use a complex variable $\zeta$ to define a set of anticommuting spinor derivatives:

$$\nabla_{\pm} = D_{\pm} + \zeta Q_{\pm}, \quad \bar{\nabla}_{\pm} = D_{\pm} - \zeta^{-1} \bar{Q}_{\pm}. \quad (2)$$

A real structure $R$ acts on $\zeta$ by hermitian conjugation composed with the antipodal map, i.e.;

$$R \zeta = -\bar{\zeta}^{-1}. \quad (3)$$

Since $\nabla_{\pm} = R \bar{\nabla}_{\pm}$, the map $R$ preserves the subspaces annihilated by the derivatives (2). To describe the nonlinear multiplet $[1, 2]$, we consider a superfield $\eta(\zeta)$, with particular $\zeta$ dependence and a reality condition:

$$\eta = \frac{a + b\zeta}{1 + c\zeta}, \quad R\bar{\eta} = -\frac{1}{\eta}. \quad (4)$$

The reality condition implies that the $N = 4$ superfields $a, b, c$ obey

$$\bar{a} = -\frac{c}{b}, \quad \bar{b} = \frac{1}{b} \quad (5)$$

which we solve by writing

$$\eta = \frac{\bar{\Phi} + e^{i\gamma} \zeta}{1 - \Phi e^{i\gamma} \zeta}. \quad (6)$$

Finally, we require that $\eta$ is annihilated by the derivatives in (2); these $N = 4$ constraints lead to the $N = 2$ component relations

$$D_{\pm} \bar{\Phi} = 0, \quad Q_{\pm} \Phi = 0,$$

$$Q_{\pm} \bar{\Phi} = -D_{\pm}[(1 + \Phi \bar{\Phi})e^{i\gamma}], \quad Q_{\pm}[(1 + \Phi \bar{\Phi})e^{-i\gamma}] = D_{\pm} \Phi, \quad (7)$$

which imply

$$D^{2}[(1 + \Phi \bar{\Phi})e^{i\gamma}] = 0, \quad (8)$$

where $D^{2} \equiv D_{+}D_{-}$.

To construct $N = 4$ actions for $N = 4$ superfields we use a second set of linearly independent covariant spinor derivatives:

$$\Delta_{\pm} = D_{\pm} - \zeta Q_{\pm}, \quad \bar{\Delta}_{\pm} = \bar{D}_{\pm} + \zeta^{-1} \bar{Q}_{\pm}. \quad (9)$$

\footnote{In \(\mathbb{R} \mathbb{F}\), \(\nabla_{\pm}\) are rescaled by a factor: \(\nabla_{\pm} \rightarrow -\zeta \nabla_{\pm}\).}
An action may then be written as
\[ S = \frac{1}{16} \int d^2 x \int_C d\zeta \Delta^2 \bar{\Delta}^2 L(\eta(\zeta); \zeta) \tag{10} \]
where \( C \) is an appropriate contour. Using
\[ \Delta_\pm = 2D_\pm - \nabla_\pm, \quad \bar{\Delta}_\pm = 2\bar{D}_\pm - \bar{\nabla}_\pm, \tag{11} \]
and (of course) \( \nabla_\pm \eta = \bar{\nabla}_\pm \eta = 0 \), the \( N = 2 \) superspace form of the action (10) is:
\[ S = \int d^2 x D^2 \bar{D}^2 \int_C d\zeta L(\eta(\zeta); \zeta) \tag{12} \]
Up to a sign \( (\eta \rightarrow -\eta) \) and the relabelling \( (X \rightarrow Y, \chi \rightarrow \Phi) \), this is the description of the nonlinear multiplet given in [2].

We may also consider an \( N = 2 \) nonlinear multiplet that obeys only the constraint (8) and
\[ D_\pm \Phi = 0, \tag{13} \]
and has an action
\[ S = \int d^2 x D^2 \bar{D}^2 L(\Phi, \bar{\Phi}, Y) \tag{14} \]
for arbitrary \( L \). Clearly, the \( N = 4 \) symmetric action (12) is a special case of this, and our subsequent discussion applies to the general case (14).

We do not know how to solve the constraints (8) and (13) in superspace (Actually, not even in \( N = 1 \) superspace!). Consequently, it is not clear what the superfield equations are, and in [2] we did not look directly at the dynamics of the nonlinear multiplet. To do so, we would have needed to go to components, and the component expansion of the action (14) subject to the constraints (8), (13) is very tedious. In the next section, we reformulate the nonlinear multiplet in a way that makes the computation of the component action tractable.

In [2], we studied the action in another way: We found a “duality” transformation to a formulation in terms of ordinary \( N = 2 \) chiral superfields; in terms of these, the dynamics can be understood straightforwardly. The “duality” transformation resembles the well known target space duality of string theory, but is nonlinear. It is performed as follows: we replace the action by a first-order action
\[ S_1 = \int d^2 x D^2 \bar{D}^2 \left( L(\Phi, \bar{\Phi}, \Psi) + (1 + \Phi \bar{\Phi})(\chi e^{-i\Psi} + \bar{\chi} e^{i\Psi}) \right), \tag{15} \]
where \( Y \rightarrow \Psi \) is now an unconstrained superfield, and \( \chi \) is a chiral superfield: 
\[
\bar{D}_\pm \chi = 0.\]
Integrating out \( \chi \), we recover the constraint (\( \mathbb{R} \)); integrating out 
\( \Psi \) gives \( \Psi(\Phi, \bar{\Phi}, \chi, \bar{\chi}) \), and substituting back gives a standard \( N = 2 \) action for the chiral superfields \( \Phi, \chi \).

However, as noted in \( \mathbb{2} \), this is a peculiar "duality": the trivial action \( S = 0 \) gives rise to a nontrivial free action for \( \Phi, \chi \). We resolve this paradox in the section 4.

### 3 The nonlinear multiplet as a gauge multiplet.

As discussed in the introduction, the formalism can be simplified by rewriting the nonlinear multiplet as a gauge multiplet. We do this by rescaling

\[
\Phi \rightarrow \frac{\Phi_1}{\Phi_2}, \quad e^{iY} \rightarrow e^{iY} \frac{\Phi_2}{\Phi_2}. \quad (16)
\]
Then the superfield \( \eta \) becomes

\[
\eta = \frac{\bar{\Phi}_1 + e^{iY} \Phi_2 \zeta}{\Phi_2 - \Phi_1 e^{iY} \bar{\zeta}}, \quad (17)
\]
and the constraints and transformations (\( \mathbb{7} \)) become (suppressing the ± indices on the spinor operators \( D, Q \)):

\[
\varepsilon^{ab} \bar{\Phi}_a D \Phi_b = 0, \quad \varepsilon_{ab} \Phi^a Q \Phi^b = 0, 
\]

\[
\varepsilon^{ab} \bar{\Phi}_a Q \Phi_b = -(\Phi^a e^{iY}) \bar{D} \Phi_a, \quad \Phi^a \bar{Q}(e^{-iY} \bar{\Phi}_a) = -\varepsilon_{ab} \Phi^b D \Phi^b. \quad (18)
\]
The superfield \( \eta \) is left unchanged by a gauge transformation:

\[
\Phi^a \rightarrow e^A \Phi^a, \quad Y \rightarrow Y + i(\Lambda - \bar{\Lambda}). \quad (19)
\]
Because of this invariance, the transformations (\( \mathbb{18} \)) do not determine the transformations of the fields \( \Phi^a \); We can consistently choose:

\[
Q \Phi^a = 0, \quad D \Phi_a = 0, \quad Q(|\Phi|^2 e^{-iY}) = -\varepsilon_{ab} \Phi^a D \Phi^b \quad (20)
\]
and

\[
Q \bar{\Phi}_a = -\varepsilon^{ab} \frac{\bar{\Phi}_b}{\bar{\Phi}_c \Phi_c} D(|\Phi|^2 e^{iY}). \quad (21)
\]
(Note that (21) breaks the manifest SU(2) invariance of (18) to an SO(2) subgroup.) The transformations and constraints (20), (21) are invariant under the gauge transformations (19) as long as \( \Lambda \) is chiral: \( \bar{D}_\pm \Lambda = 0 \). In this case, \( Y \) transforms as an ordinary \( N = 2 \) \( U(1) \) gauge supermultiplet. This allows us to find component actions in a Wess-Zumino gauge, greatly simplifying our calculations.

We work in a chiral representation of the gauge group, and define

\[
\hat{\Phi}^a \equiv \Phi_a e^{-iY} \Rightarrow \eta = \frac{\hat{\Phi}^1 + \Phi^2 \zeta}{\Phi^2 - \Phi^1 \zeta} .
\]  

(22)

We also define gauge covariant derivatives \( \nabla \)

\[
\nabla = e^{-iqY} De^{iqY}, \quad \bar{\nabla} = \bar{D},
\]  

(23)

where \( q \) is the charge of the field that \( \nabla \) acts on (\( \Phi \) and \( \hat{\Phi} \) both have \( q = 1 \)). These derivatives obey the usual algebra

\[
\{ \nabla_+ , \nabla_- \} = 0 , \quad \{ \bar{\nabla}_+ , \bar{\nabla}_- \} = 0 ,
\]

\[
\{ \nabla_+ , \bar{\nabla}_+ \} = \nabla_+ , \quad \{ \nabla_- , \bar{\nabla}_- \} = \nabla_- ,
\]

\[
\{ \nabla_+ , \nabla_- \} = -\bar{W}q , \quad \{ \bar{\nabla}_+ , \bar{\nabla}_- \} = Wq ,
\]

\[
[\nabla_+ , \nabla_-] = (\nabla_- W)q , \quad [\bar{\nabla}_+ , \nabla_-] = - (\bar{\nabla}_- W)q ,
\]

\[
[\nabla_- , \nabla_+] = -(\nabla_+ W)q , \quad [\bar{\nabla}_- , \nabla_+] = (\bar{\nabla}_+ W)q ,
\]

\[
[\nabla_+ , \nabla_-] = f q , \quad f \equiv \nabla_+ \nabla_- W - \nabla_- \nabla_+ W .
\]  

(24)

Here \( W = i\bar{D}_+ D_- Y \) is the superfield strength of \( Y \), and is a twisted chiral superfield with charge \( q = 0 \): \( \bar{D}_+ W = D_- W = 0 \).

The \( N = 2 \) action (14) becomes:

\[
S = \int d^2x D^2 \bar{D}_2 L(\Phi^a , \hat{\Phi}^a) ,
\]  

(25)

where \( L \) is restricted to be gauge invariant:

\[
\Phi^a L_a + \hat{\Phi}^a \bar{L}_a = 0 ,
\]  

(26)

\(^5\)Actually, gauge invariance of the action implies invariance of \( L \) only up to superspace total derivatives; this has no effect on our analysis.
and where \( L_a \equiv \frac{\partial L}{\partial \Phi^a} \) and \( \bar{L}_{\dot{a}} \equiv \frac{\partial L}{\partial \hat{\Phi}^{\dot{a}}} \). We use (26) and its derivatives frequently below. The condition that the action (25) has \( N = 4 \) supersymmetry is Laplace’s equation

\[
L_{a\dot{a}} = 0.
\]

(27)

The constraint (8) (and its complex conjugate) become simply

\[
\nabla^2(\Phi^a \hat{\Phi}^a) = \bar{\nabla}^2(\Phi^a \hat{\Phi}^a) = 0.
\]

(28)

These constraints are actually gauge covariant; thus when we go to first order form and impose them via chiral Lagrange multipliers \( \chi \) and \( \hat{\chi} \equiv \bar{\chi} e^{2\Psi} \) (recall that we are replacing \( Y \) with the unconstrained gauge superfield \( \Psi \)), the action (29) is gauge invariant if \( \chi, \hat{\chi} \) both have charge \( q = -2 \):

\[
S_1 = \int d^2x D^2 \bar{D}^2 \left[ L(\Phi^a, \hat{\Phi}^a) + (\chi + \hat{\chi})\Phi^a \hat{\Phi}^a \right].
\]

(29)

The gauge invariance of the action allows us to choose a gauge that is very convenient for performing the “duality” transformation: \( \chi = \frac{1}{2} \). In this gauge, we may for example consider \( L = i ln \left( \frac{\Phi}{\hat{\Phi}} \right) \); this gives

\[
S_1 = \int d^2x D^2 \bar{D}^2 \left[ \Psi + |\Phi|^2 cos(\Psi) \right].
\]

(30)

(Equivalently, we may choose as the action a Fayet-Iliopoulos term for \( \Psi \).) Eliminating \( \Psi \) by the equation \( sin(\Psi) = |\Phi|^{-2} \), we get the Kähler potential for the Eguchi-Hansen gravitational instanton with the “wrong” sign of the mass-parameter. This gives a nontrivial example of a hyperkähler metric constructed using the nonlinear multiplet and “duality” transformation.

4 Components

In this section, we descend from superspace to spacetime, and compute the component action. We work with the first order system (29), and derive the component form of the constraints by integrating out the component Lagrange multiplier fields. After giving the full action in a compact geometric formulation, we focus on the bosonic sector and resolve the paradox that our nonlinear “duality” introduced.
As usual, we define component fields as $\theta$ independent projections of the superfields and their spinor derivatives. The Wess-Zumino gauge components of the gauge superfield are:

$$\nabla_+| = \partial_+ + V_+ , \quad \nabla_-| = \partial_- + V_- , \quad W| = w , \quad \bar{W}| = \bar{w} ,$$

$$\nabla_+ W| = \lambda_+ , \quad \nabla_- W| = \lambda_- , \quad \nabla_+ \bar{W}| = \bar{\lambda}_+ , \quad \nabla_- \bar{W}| = \bar{\lambda}_- ,$$

$$(\nabla_+ \nabla_- \bar{W} - \nabla_+ \nabla_- W)| = f \equiv \partial_+ V_- - \partial_- V_+ ,$$

$$(\nabla_+ \nabla_- \bar{W} + \nabla_+ \nabla_- W)| = i\mathbb{D} , \quad (31)$$

where $V$ is the component gauge field, and $w, \lambda, \mathbb{D},$ etc., are various gauge invariant superpartners of $V$.

The components of the chiral superfields $\Phi^a, \chi$ (recall that $\chi$ is the Lagrange multiplier field) are:

$$\Phi^a| = A^a , \quad \nabla_\pm \Phi^a| = \psi_\pm^a , \quad i\nabla_+ \nabla_- \Phi^a| = F^a ,$$

$$\chi| = C , \quad \nabla_\pm \chi| = \chi_\pm , \quad i\nabla_+ \nabla_- \chi| = G . \quad (32)$$

We denote the set $(\Phi^a, \chi)$ collectively by $\Phi^i$ (similarly, we denote $A^i = (A^a, C),$ etc.), and the total super-Lagrangian $\left( L(\Phi^a, \tilde{\Phi}^a) + (\chi + \tilde{\chi})|\Phi|^2 \right)$ by the Kähler potential $K(\Phi^i, \tilde{\Phi}^i)$.

The component Lagrangian, after integrating by parts and eliminating the auxiliary fields $F^i$ to bring out the geometric features, is:

$$\mathcal{L} = G_{ij} \left( \frac{1}{2}(\nabla_+ A^i \nabla_- \bar{A}^j + \nabla_- A^i \nabla_+ \bar{A}^j) + w\bar{w}k^i\bar{k}^j + w\psi_+^i \bar{k}^j \tilde{\psi}_-^j - \bar{w}\bar{\psi}_+^i k^j \psi_-^j \right. \right.$$

$$\left. - \psi_+^i D_+ \bar{\psi}_-^j - \psi_-^i D_+ \bar{\psi}_+^j + \bar{\lambda}_+^i \bar{k}^j \psi_+^j + \psi_-^i \bar{k}^j \lambda_- - \lambda_+^i k^j \bar{\psi}_-^j - \bar{\psi}_+^i k^j \lambda_- \right)$$

$$\left. + R_{ijkl} \psi_+^i \psi_-^k \bar{\psi}_+^j \bar{\psi}_-^l + i \frac{1}{2}(\bar{k}^j K_i - k^i \bar{K}_j)\mathbb{D} , \right. \quad (33)$$

where the metric $G_{ij}$, the Levi-Civita connection $\Gamma^i_{jk}$, and the curvature $R_{ijkl}$ of the Kähler manifold with potential $K(A^i, \bar{A}^i)$ are

$$G_{ij} = K_{ij} = \begin{pmatrix} L_{ab} + \delta_{ab}(C + \bar{C}) & A^a \\ A^b & 0 \end{pmatrix} ,$$

$$\mathbb{D} = \begin{pmatrix} \mathbb{D}^a & 0 \\ 0 & \mathbb{D}^b \end{pmatrix}.$$
\[ \Gamma_{jk} = (G_{il})^{-1} K_{lk} , \]
\[ R_{i j k l} = K_{i k j l} - (G_{m n})^{-1} K_{m l j k} , \]

(for Kähler manifolds, these and their complex conjugates are the only non-vanishing components). Further, \( k^i \) is the killing vector \( k^i = (A^a, -2C) \), \( k^i_j = \partial_i k^j + \Gamma^i_{jk} k^j \), and the world-sheet covariant derivatives

\[ \nabla A^i = \partial A^i + V k^i , \]
\[ D \bar{\psi}^i = \nabla \bar{\psi}^i + \Gamma^i_{jk} \nabla \bar{A}^j \bar{\psi}^k \]
\[ = \partial \bar{\psi}^i + V \bar{k}^i_j \bar{\psi}^j + \Gamma^i_{jk} \partial \bar{A}^j \bar{\psi}^k \]

are both gauge and diffeomorphism covariant. In deriving the Lagrangian \( \mathcal{L} \) (33), we have used the following identities:

\[ \nabla A^i \nabla^2 \hat{\Phi}^i = \left( \nabla_+ \nabla_- + (\nabla_+ \bar{W})q + \bar{W} q \nabla_+ \right) \hat{\Phi}^i , \]
\[ \nabla_- \nabla^2 \hat{\Phi}^i = \left( -\nabla_+ \nabla_- + (\nabla_- W)q + W q \nabla_- \right) \hat{\Phi}^i , \]
\[ \nabla^2 \nabla^2 \hat{\Phi}^i = \left( -\frac{1}{2} \Box + \frac{1}{2}(\nabla_+ \nabla_- W + \nabla_+ W q \nabla_- ) q \right. \]
\[ -W \bar{W} q^2 - \nabla_+ \bar{W} q \nabla_- + \nabla_+ W q \nabla_- ) \hat{\Phi}^i , \]

where \( q \) is the charge of \( \hat{\Phi}^i \). To get the pure nonlinear multiplet action, we separate out the dependence on the Lagrange-multiplier multiplet \( \chi \), and integrate it out. This gives the constraints:

\[ A^a \bar{F}^a = 0 , \quad \psi^a \bar{F}^a + i A^a (\nabla_+ \bar{\psi}^a + \bar{\lambda}_+ \bar{A}^a + \bar{w} \bar{\psi}^a) = 0 , \]
\[ \psi^a \bar{F}^a + i A^a (-\nabla_- \bar{\psi}^a + \lambda_- \bar{A}^a + w \bar{\psi}^a) = 0 , \]
\[ F^a \bar{F}^a - \psi^a (-\nabla_- \bar{\psi}^a + \lambda_- \bar{A}^a + w \bar{\psi}^a) + \psi^a (\nabla_+ \bar{\psi}^a + \bar{\lambda}_+ \bar{A}^a + \bar{w} \bar{\psi}^a) \]
\[ -A^a (-\frac{1}{2} \Box \bar{A}^a + (\frac{3}{2} \Box - w \bar{w}) \bar{A}^a - \bar{\lambda}_+ \bar{\psi}^a + \lambda_+ \bar{\psi}^a) = 0 , \]

and their complex conjugates.
The basic features of the model can be seen in the bosonic sector of the Lagrangian (33); after integrating by parts and collecting terms, we find

\[ L_{\text{bose}} = L_{ab} \left( A^a A^b \bar{w} - \frac{i}{2} \left( \nabla_+ A^a \nabla_- \bar{A}^b + \nabla_- A^a \nabla_+ \bar{A}^b \right) \right) 
+ \frac{i}{4} (\bar{A}^a L_\bar{a} - A^a L_a) \mathcal{D} - (C + \bar{C}) \left( \bar{w}A^2 + \frac{i}{4} (A^a \Box A^a + \bar{A}^a \Box A^a) \right) 
- \frac{i}{2} (\bar{C} - C) (\mathcal{D} |A|^2 - \frac{i}{2} \bar{A}^a \Box A^a), \]  

(38)

where \( \Box \equiv \{ \nabla_+, \nabla_- \} \). This Lagrangian is gauge invariant; one may pick various gauges, e.g., \(|A^1| = 1 \) or \(|C| = 1 \). Integrating out the gauge multiplet fields \( w, V, \mathcal{D} \) gives the “dual” theory: the \( w \) field equation sets \( \bar{w} = 0 \), the \( V \) field equation gives \( V(A, C) \), and the \( \mathcal{D} \) field equation gives \( C - \bar{C} \) in terms of \( A \). After substituting back into the Lagrangian \( L_{\text{bose}} \), in the gauge \( |C| = 1 \) this is just an ordinary \( \sigma \)-model with target space coordinates \( A^a, \bar{A}^a \); the actual form of the resulting \( L_{\text{dual}} \) is most easily found by going back to the superspace action (29) in the gauge \( \chi = \frac{1}{2} \) and integrating out \( \Psi \) to find the Kähler potential.

Integrating out the Lagrange multiplier field \( C \) gives

\[ \mathcal{D} = -\frac{i \bar{A}^a \Box A^a}{2 |A|^2}, \quad \bar{w} = -\frac{A^a \Box \bar{A}^a + \bar{A}^a \Box A^a}{4 |A|^2}. \]  

(39)

Substituting back into \( L_{\text{bose}} \), we find

\[ L_{\text{nonlin}} = L_{ab} \left( -A^a A^b \frac{\bar{A}^a \Box A^a + \bar{A}^a \Box A^a}{4 |A|^2} + \frac{i}{2} (\nabla_+ A^a \nabla_- \bar{A}^b + \nabla_- A^a \nabla_+ \bar{A}^b) \right) 
- \frac{i}{8} (\bar{A}^a L_\bar{a} - A^a L_a) \frac{\bar{A}^a \Box A^a}{|A|^2}. \]  

(40)

This still depends on the gauge field \( V \); however, the dependence is now only linear, and hence \( V \) cannot be integrated out. Thus we appear to have found a “duality” between a conventional \( \sigma \)-model and a model with a sector that is first order in derivatives. The most extreme form of this paradox occurs when the superspace lagrangian \( L \) (25) vanishes: then \( L_{\text{nonlin}} = 0 \), but its “dual” is \( L_{\text{dual}} = -\frac{1}{2} A^a \Box \bar{A}^a \). To understand this peculiar result, we must examine how we integrate out \( w \) more carefully. Varying \( w \) in \( L_{\text{bose}} \) (38) gives

\[ \bar{w} \left[ L_{ab} A^a \bar{A}^b - (C + \bar{C}) |A|^2 \right] = 0. \]  

(41)

This has two solutions: The one we naively took — which one normally takes when eliminating auxiliary fields — namely, \( \bar{w} = 0 \), and \( C + \bar{C} = \frac{L_{ab} A^a \bar{A}^b}{|A|^2} \). If
we take the first solution $w = 0$, then we get the ordinary $\sigma$-model; however, the second solution $C + \bar{C} = \frac{L_{\text{nonlin}}}{|A|^2}$ gives precisely the peculiar model with Lagrangian $L_{\text{nonlin}}$. Thus the two models are actually two different sectors of the first order theory described by $L_{\text{bose}}$ (38). In sum: the first order Lagrangian has two sectors. One, which is reached by integrating out the Lagrange multiplier multiplet, is the nonlinear multiplet model. When one instead attempts to integrate out the gauge multiplet from the first order action, one finds that depending on the order of integration and a choice of solutions to a nonlinear algebraic equation, one gets either the original nonlinear multiplet model (sector one), or a new sector corresponding to an ordinary $\sigma$-model.

5  Conclusions

We have reformulated the nonlinear multiplet as a gauge multiplet. This allowed us to compute the component Lagrangian in a Wess-Zumino gauge. We found a system with a subsystem linear in the gauge field (40). We considered a nonlinear “duality” transformation that eliminates this unusual subsystem and gives an ordinary $\sigma$-model. We found that the two theories are not dual in the sense of representing different formulations of the same theory, but correspond to different sectors of the first order model (38).

Our analysis has been entirely classical; it would be interesting to study the nonlinear multiplet at the quantum level. In particular, one would like to see if the nonlinear “dual” of a superconformal field theory is superconformal, and if it is, what, if any, relation the two theories have. A second unsolved problem is classical and more geometric: which $\sigma$-models admit a nonlinear “duality” to a nonlinear multiplet formulation.

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