The Riemann-Lanczos Problem as an Exterior Differential System with Examples in 4 and 5 Dimensions

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Abstract

The key problem of the theory of exterior differential systems (EDS) is to decide whether or not a system is in involution. The special case of EDSs generated by one-forms (Pfaffian systems) can be adequately illustrated by a 2-dimensional example.

In 4 dimensions two such problems arise in a natural way, namely, the Riemann-Lanczos and the Weyl-Lanczos problems. It is known from the work of Bampi and Caviglia that the Weyl-Lanczos problem is always in involution in both 4 and 5 dimensions but that the Riemann-Lanczos problem fails to be in involution even for 4 dimensions. However, singular solutions of it can be found.

We give examples of singular solutions for the Gödel, Kasner and Debever-Hubaut spacetimes. It is even possible that the singular solution can inherit the spacetime symmetries as in the Debever-Hubaut case. We comment on the Riemann-Lanczos problem in 5 dimensions which is neither in involution nor does it admit a 5-dimensional involution of Vessiot vector fields in the generic case.

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I. Introduction

The problem of generating the spacetime Weyl conformal curvature tensor $C_{abcd}$ from a tensor potential is called the Weyl-Lanczos problem and the analogous problem for the Riemann curvature tensor the Riemann-Lanczos problem.

The Lanczos tensor potential admits the following index symmetries

\[ L_{[ab]c} = L_{abc} , \]  

(1)

where $a, b, c, s = 0, 1, 2, 3$ and

\[ L_{[abc]} = 0 . \]  

(2)

Apart from these, we may impose two gauge conditions: the differential gauge condition

\[ L_{ab};s = 0 , \]  

(3)

where "\text{';}" indicates covariant differentiation, and the algebraic gauge or trace free condition

\[ L_{a}^{s}s = 0 . \]  

(4)

Lanczos discovered the Weyl-Lanczos equations [30], where he introduced a Lagrangian based on the double dual of the Riemann tensor $R_{abcd}$. The Lanczos tensor $L_{abc}$ arose as a Lagrange multiplier for this Lagrangian. Lanczos found an expression for the Weyl tensor in terms of certain Lagrange multipliers $L_{abc}$, namely,

\[ C_{abcd} = L_{ab}[c;d] - L_{ab;c} + L_{cda;b} - L_{cda;b} - g_{bc}L_{(ad)} + g_{ad}L_{(bc)} - g_{bd}L_{(ac)} - g_{ac}L_{(bd)} - g_{bd}L_{(ac)} - g_{ac}L_{(bd)} . \]  

(5)

Many solutions to (7) are known and solutions for vacuum spacetimes can be found in [12]. We can also attempt to express the Riemann curvature tensor in terms of a comparable tensor potential $\hat{L}_{abc}$ which leads to the Riemann-Lanczos problem for spacetimes and which has been discussed at length by Bampi and Caviglia [3, 4]. Udeschini Brinis [8] had wanted to describe the spacetime Riemann tensor in terms of a Lanczos tensor $\hat{L}_{abc}$ and proposed the Riemann-Lanczos relations

\[ R_{abcd} = \hat{L}_{ab}[c;d] - \hat{L}_{ab;c} + \hat{L}_{cda;b} - \hat{L}_{cda;b} - g_{bc}\hat{L}_{(ad)} + g_{ad}\hat{L}_{(bc)} - g_{bd}\hat{L}_{(ac)} + g_{ac}\hat{L}_{(bd)} . \]  

(8)

But the difficulties with the relations (8) were pointed out in two papers by Bampi and Caviglia [3, 4] where they proved existence theorems for solutions
of (5) and (8). Whereas the Weyl-Lanczos problem is always in involution, the Riemann-Lanczos problem is not and only singular solutions of it can occur if the problem is not modified. Bampi and Caviglia showed that:

i) The Weyl-Lanczos problem (5) or (7) has non-singular solutions for \( n = 4, 5 \), where \( n \) is the dimension of the space-time manifold \( M \);

ii) For \( n = 4 \) the Riemann-Lanczos problem (8) has no non-singular solutions but it does have “singular” solutions which means that the Cartan characters do not adopt their maximal values;

iii) The differential gauge condition (3) has no effect on the existence or non-existence of solutions of either (5) or (8).

Bampi and Caviglia [4] also suggested a prolongation of the Riemann-Lanczos equations to a second-order system which makes them a system in involution.

In the Riemann-Lanczos problem, we meet equation (2) and possibly (3) but not (4), which leaves us with 20 independent components for the \( \hat{L}_{abc} \) in 4 dimensions. We always assume that the cyclic conditions (2) hold but not the trace-free conditions (4). If equations (4) were to hold, we would have

\[
R = 4\hat{L}_{nk}^{\,k;n} = -4\hat{L}_{n;k}^{\,n;k} = 0,
\]

which would lead to inconsistencies. Because we are only going to talk about the Riemann-Lanczos problem in this paper, we shall change notation from \( \hat{L}_{abc} \) to \( L_{abc} \) from now on. We shall also write the Riemann-Lanczos equations in solved form as

\[
f^{(R)}_{abcd} := R_{abcd} - L_{abc;d} + L_{abd;c} - L_{cdab} + L_{cdba}.
\]

The paper by Massa and Pagani [32] concerning a modification of the Riemann-Lanczos problem used a totally different approach to the above work of Bampi and Caviglia. Accordingly, we do not consider [32] here. As reference [15] is based on [32] it is also not applicable here.

II. Exterior Differential Systems and Cartan Characters

It is necessary to introduce some notions from the theory of exterior differential systems (EDS) in order to understand the structure of the Riemann-Lanczos problem. We denote a formal \( N \)-dimensional manifold by \( \mathcal{M} \) of which the spacetime manifold \( M \) is an \( n \)-dimensional submanifold. Then, for a collection of differential forms we define an exterior differential system (EDS).

**Definition 1** An exterior differential system \( EDS \Sigma \) on \( \mathcal{M} \) consists of

\[
\begin{align*}
1 &- \text{forms } \alpha_{i_1}^{(1)} & (1 \leq i_1 \leq k_1), \\
2 &- \text{forms } \alpha_{i_2}^{(2)} & (1 \leq i_2 \leq k_2), \\
\vdots & & \\
p &- \text{forms } \alpha_{i_p}^{(p)} & (1 \leq i_p \leq k_p),
\end{align*}
\]

and sometimes 0-forms on \( \mathcal{M} \).

Any EDS \( \Sigma \) is a subset of the set of all differential forms on \( \mathcal{M} \). There is a further set of differential forms on \( \mathcal{M} \) formed by a subset of \( \Sigma \) called its
associated ideal \( I(\Sigma) \). The associated ideal \( I(\Sigma) \) is a \textit{differential ideal} \( I \), that is, \( dI \subset I \). This means that such an ideal is closed under exterior differentiation. If the exterior differential system (EDS) \( \Sigma \) itself is closed under exterior differentiation, we say that \( \Sigma \) is a \textit{closed exterior differential system}. Next, we wish to look at Pfaffian systems and Vessiot vector fields as they are of great importance to our applications later on.

### A. Pfaffian Systems and Vessiot Vector Fields

A \textbf{Pfaffian system} \( \mathcal{P} \) is a special EDS containing only one-forms \( \theta^\alpha \) and 0-forms. We denote the \textit{rank} of \( \mathcal{P} \) by \( s \) which is given by the number \( s \) of independent one-forms in \( \mathcal{P} \). Consider now the collection of all vector fields on \( \mathcal{M} \), \( \mathcal{X}(\mathcal{M}) \). There is a subset of \( \mathcal{X}(\mathcal{M}) \) given by all those vector fields which annihilate the Pfaffians in \( \mathcal{P} \)

\[
\mathcal{D} := \{ X \in \mathcal{X}(\mathcal{M}) | \theta^\alpha(X) = 0, \quad \alpha = 1, \ldots, s \}
\]

so that the number of independent vector fields in \( \mathcal{D} \) is \( N - s \). We call \( \mathcal{D} \) the \textbf{dual system} to \( \mathcal{P} \). For any Pfaffian system \( \mathcal{P} \), we can also look at its \textbf{derived system} \( \mathcal{P}' \) for which we can write

\[
\mathcal{P}' := \{ \theta^\alpha \in \mathcal{P} | d\theta^\alpha(X, Y) = 0, \forall X, Y \in \mathcal{D}, \quad \alpha = 1, \ldots, s \},
\]

where \( \mathcal{P}' \subset \mathcal{P} \) always holds.

Pfaffian systems are classified according to the ease with which they can be integrated. The most familiar Pfaffian Systems are also the simplest, namely, the \textbf{complete Pfaffian systems} to which the celebrated \textit{Frobenius theorem} applies. We say that a Pfaffian system \( \mathcal{P} \) is \textit{complete} if \( \mathcal{P} = \mathcal{P}' \) holds. A necessary and sufficient condition for a complete Pfaffian system is now given by the Frobenius theorem.

**Theorem 1 Frobenius Theorem**

Let the \( s \) independent 1-forms \( \theta^1, \ldots, \theta^s \) generate a closed ideal \( I(\mathcal{P}) \), then, we can find coordinates in a neighbourhood of any point \( x^1, \ldots, x^N \) such that all forms are generated by the \( s \) coordinate differentials \( dx^{N-s+1}, \ldots, dx^N \).

A good account of Pfaffian systems can be found in [21, 2] or in a more theoretical approach in [20]. In general, Pfaffian systems are not complete but each Pfaffian system can be enlarged by adding further 1-forms until the enlarged system is complete or the enlarged system becomes inconsistent. The minimal enlarged Pfaffian system which is complete is known as the \textbf{associated system} \( \mathcal{A}(\mathcal{P}) \) of \( \mathcal{P} \) and its dimension is called the \textbf{class} \( c \) of \( \mathcal{P} \). Sometimes, \( \mathcal{A}(\mathcal{P}) \) is also called the \textbf{Cartan system} \( \mathcal{C}(\mathcal{P}) \) [28].

A Pfaffian system \( \mathcal{P} \) can dually be characterised using \textbf{Vessiot vector field systems} as discussed in [41]. This approach was developed in Vessiot [44] and can be found in [18]. First, we take a vector field system \( \mathcal{V} \), which we here choose to be the dual system \( \mathcal{D} \) of a Pfaffian system \( \mathcal{P} \) so that \( \mathcal{V} = \mathcal{D} \) is given by \( N - s \) vector fields \( Y^i \) and we define in turn its \textbf{first and second derived systems} as

\[
\begin{align*}
\mathcal{D}' & := \mathcal{D} + [\mathcal{D}, \mathcal{D}] \\
\mathcal{D}'' & := \mathcal{D}' + [\mathcal{D}', \mathcal{D}'] \\
\vdots & 
\end{align*}
\]

(10)
and so on [1]. For the derived system, the inclusion \( D \subset \mathcal{D}' \) is always valid and \((\mathcal{P}')^\perp = \mathcal{D}'\) for a Pfaffian system, where \( D = \mathcal{P}^\perp\). We can also formulate the notion of completeness in terms of vector field systems and versions of the Frobenius theorem in this language are numerous. We therefore define the space \( \bar{D} \subset D \) as

\[
\bar{D} := \{ Y \in D | d\theta^\alpha(X,Y) = 0 \forall X \in D, \alpha = 1, \ldots, s \}.
\]

Here, we state the dual version of the Frobenius theorem as

**Theorem 2 Complete Pfaffian Systems**

Given any two vector fields \( X, Y \in D \), then \( P \) is a complete Pfaffian system means that the the commutator vector field \([X,Y]\) is also a vector field in \( D \).

A Vessiot vector field system \( D \) is complete means then that \([D,D] \subseteq D\) for all \( Y \in D \) which is equivalent to \( \bar{D} = D \).

When a vector field system \( D \) or Pfaffian system \( P \) fails to be complete, a slightly weaker condition may hold on some subsystem of \( D \). Such a possibility is that a vector field system is in involution, which is defined as follows:

**Definition 2 Involutory Subsystem of a Vector Field System**

An involutory subsystem \( T \) of a vector field system \( D \) is a subsystem \( T \) of \( D \) such that \([T,T] \subseteq D\), which means that \( T \) is closed relative to \( D \) but not relative to \( T \) itself.

This means that if we are given two vector fields \( X, Y \in T \), it is then valid for all \( \theta^\alpha \in P \)

\[
d\theta^\alpha(X,Y) = \theta^\alpha(X)Y - \theta^\alpha(Y)X - \frac{1}{2}\theta^\alpha([X,Y])
\]

\[
= -\frac{1}{2}\theta^\alpha([X,Y])
\]

\[
\neq 0.
\]

This is because we have \( \theta^\alpha(X) = 0 \) and \( \theta^\alpha(Y) = 0 \) holding identically but not \( \frac{1}{2}\theta^\alpha([X,Y]) = 0 \) since \([X,Y]\) need not lie inside \( T \). However, if we allow for the vector fields in \( D - T \), then \( \theta^\alpha([X,Y]) \) can vanish i.e.

\[
d\theta^\alpha(X,Y) = 0 \pmod{J(P)}.
\]

Note that when we take \( T = D \), we get a complete vector field system but when \( T \) is strictly contained in \( D \) it can fail to be complete [1].

Further, the dual space of \( C(\mathcal{P}) \) which is usually denoted by \( C(D) = C(\mathcal{P})^\perp \) is the space of all Cauchy characteristic vector fields, where such a vector field is defined as

**Definition 3 Cauchy Characteristic Vector Field**

\( Y \) is a Cauchy characteristic vector field of a vector field ideal \( \mathcal{I}(P) \) means that

\[
Y | \theta^\alpha = 0
\]

\[
Y | d\theta^\alpha = \lambda_\gamma \theta^\gamma = 0 \pmod{\mathcal{P}}
\]

for all one-forms \( \theta^\alpha \) in \( P \), where \( \lambda_\gamma \) are scalar multipliers for each \( \theta^\alpha \).
Both $C(D)$ and $C(P)$ are complete Pfaffian systems and it is $C(D) = \tilde{D}$. For their dimensions it has to hold $\text{dim}(C(D)) + \text{dim}(C(P)) = N$. Cauchy characteristic vector fields can also be characterised as $[6]$.

**Theorem 3** Given a Pfaffian system $\mathcal{P}$ and its associated ideal $\mathcal{I}(\mathcal{P})$ and let $\mathcal{D}$ be the dual space of $\mathcal{P}$. $Y \in \mathcal{D}$ is Cauchy means that $[X, Y] \in \mathcal{D}$ $\forall X \in \mathcal{D}$.

For EDS and Pfaffian systems in particular, it can be of interest to determine its symmetries and especially their infinitesimal generators. In the same way as Killing vector fields are symmetries of the metric tensor and hence of Einstein’s equations, any EDS or system of PDEs can adopt symmetries. When an EDS adopts symmetries it means that there exist vector fields $Y$ whose Lie derivatives applied to each form in the EDS annihilates them or those new forms are contained in the EDS itself. These infinitesimal symmetry generators $Y$ are called isovectors and defined as

**Definition 4** Isovector

$Y$ is an isovector of an EDS $\Sigma$ means that $\mathcal{L}_Y \Sigma \subset \Sigma$ and dually $Y$ is an isovector of a vector field system $\mathcal{D}$ means that $\mathcal{L}_Y \mathcal{D} \subset \mathcal{D}$.

Cauchy characteristics form a special sub-algebra of all isovectors of a system $[6]$. Isovectors $Y$ do not necessarily have to be such that $Y \in \mathcal{D}$. Often there exist isovectors $Y$ such that $Y \notin \mathcal{D}$. But for those $Y$ with $Y \in D$, it is known that $[6]$.

**Theorem 4** Cauchy Characteristics

Given a vector field ideal $\mathcal{I}(\mathcal{P})$. Then, $Y \in \mathcal{D}$ is an isovector means that $Y$ is a Cauchy characteristic vector field.

Note that if $\mathcal{P}$ is complete, then all its vector fields $Y \in \mathcal{D}(= \tilde{D})$ are isovectors ($= \text{Cauchy characteristics}$). The formula of H. Cartan

$$\mathcal{L}_Y \alpha = Y \lrcorner d\alpha + d(Y \lrcorner \alpha) \quad (13)$$

is used when we determine the isovectors of p-forms $\alpha$. See $[7, 14, 40]$ for discussions of symmetries and isovectors. An algebraic computing package assisting with symmetry determination is the REDUCE package DIMSYM by Sherring $[26]$.

We illustrate the above theory with the example of a coordinate transformation to Euclidean coordinates in two dimensions. For the line element we therefore have

$$ds^2 = y^2 dx^2 + dy^2 = du^2 + dU^2,$$

where $u, U$ are the Euclidean coordinates. For this to hold we must have

$$y^2 = p^2 + P^2,$$

$$0 = pq + PQ,$$

$$1 = q^2 + Q^2,$$

where we introduced the Monge notation

$$p := \frac{\partial u}{\partial x}, \quad q := \frac{\partial u}{\partial y}, \quad r := \frac{\partial^2 u}{\partial x^2}, \quad s := \frac{\partial^2 u}{\partial x \partial y}, \quad t := \frac{\partial^2 u}{\partial y^2};$$

$$P := \frac{\partial U}{\partial x}, \quad Q := \frac{\partial U}{\partial y}, \quad R := \frac{\partial^2 U}{\partial x^2}, \quad S := \frac{\partial^2 U}{\partial x \partial y}, \quad T := \frac{\partial^2 U}{\partial y^2}. $$
These local coordinates form a jet bundle $\mathcal{J}^2(\mathbb{R}^2, \mathbb{R}^2)$ with $N = 14$ formal dimensions, where we have $n = 2$ independent and $m = 2$ dependent variables. After differentiating equations (14) with respect to $x$ and $y$ and rearranging them, we obtain a Pfaffian system $\mathcal{P}$ with $s = 12$ independent 1-forms which can locally be expressed as

$$
\theta^1 = du - pdx - qdy \\
\theta^2 = dU - Pdx - Qdy \\
\theta^3 = dp - rdx - sdy \\
\theta^4 = dP - Rdx - Sdy \\
\theta^5 = dq - sdx - tdy \\
\theta^6 = dQ - Sdx - Tdy \\
\theta^7 = pdr + rdp + PdR + RdP \\
\theta^8 = qdr + rdq + QdR + RdQ + dy \\
\theta^9 = pds + sdp + PdS + SdP - dy \\
\theta^{10} = qds + sdq + QdS + SdQ \\
\theta^{11} = pdt + tdp + PdT + TdP \\
\theta^{12} = qdt + tdq + QdT + TdQ .
$$

As the rank of $\mathcal{D}$ is $N - s = 14 - 12 = 2$, the system (13) can dually be characterised by two Vessiot vector fields $V^1, V^2$ generating $\mathcal{D}$, where

$$
V^i = V_x^i \frac{\partial}{\partial x} + V_y^i \frac{\partial}{\partial y} + (pv^i_y + qv^i_x) \frac{\partial}{\partial u} + (pv^i_x + qv^i_y) \frac{\partial}{\partial U} + (rv^i_x + sv^i_y) \frac{\partial}{\partial P} + (sv^i_x + tv^i_y) \frac{\partial}{\partial q} \\
+ (sv^i_x + tv^i_y) \frac{\partial}{\partial P} + (sv^i_y + tv^i_y) \frac{\partial}{\partial q} + (sv^i_x + tv^i_y) \frac{\partial}{\partial Q} + V_r^i \frac{\partial}{\partial R} + V_s^i \frac{\partial}{\partial S} + V_t^i \frac{\partial}{\partial T} , i = 1, 2
$$

(16)

and the coefficients $V_r^i, V_R^i, V_s^i, V_S^i, V_t^i, V_T^i$ are determined through the condition that the six 1-forms $\theta^7$ to $\theta^{12}$ are annihilated by $V^1, V^2$. Therefore, we obtain

$$
V_r^i = \frac{1}{\alpha} \left[ \left( \frac{P}{Q} (rs + RS) - r^2 - R^2 \right) V_x^i + \left( \frac{P}{Q} (rt + RT + 1) - rs - RS \right) V_y^i \right], \\
V_R^i = \frac{1}{Q} \left[ \left( rs + RS + \frac{q}{\alpha} \frac{P}{Q} (rs + RS - r^2 - R^2) \right) V_x^i + \left( rt + RT + 1 \right) \frac{v^y}{\alpha} \frac{P}{Q} (rt + RT + 1) - rs - RS \right] V_y^i, \\
V_s^i = \frac{1}{\alpha} \left[ \left( \frac{P}{Q} (s^2 + S^2) - rs - RS \right) V_x^i + \left( \frac{P}{Q} (st + ST) - s^2 - S^2 + 1 \right) V_y^i \right], \\
V_S^i = \frac{1}{Q} \left[ \left( s^2 + S^2 + \frac{q}{\alpha} \frac{P}{Q} (s^2 + S^2) - rs - RS \right) V_x^i + \left( st + ST \right) \frac{v^y}{\alpha} \frac{P}{Q} (st + ST) - s^2 - S^2 + 1 \right] V_y^i, \\
V_t^i = \frac{1}{\alpha} \left[ \left( \frac{P}{Q} (st + ST) - rt - RT \right) V_x^i + \left( \frac{P}{Q} (t^2 + T^2) - st - ST \right) V_y^i \right], \\
V_T^i = \frac{1}{Q} \left[ \left( st + ST + \frac{q}{\alpha} \frac{P}{Q} (st + ST) - rt - RT \right) V_x^i + \left( (t^2 + T^2) - st - ST \right) \frac{v^y}{\alpha} \frac{P}{Q} (t^2 + T^2) - st - ST \right] V_y^i ,
$$

(17)
The Plücker coordinate meaning that all multiples of a $p$-vector are in the same equivalence class. This means that the system \((15)\) is complete and its derived system consists of the two Vessiot vector fields $V^1, V^2$ spanning $D$ so that $D = D'$. It also means that $C(P) = D$ so that $c = 2$ and therefore, both $V^1$ and $V^2$ are Cauchy characteristics. At the same time, they form an involution of maximal dimension $g = 2$. Note that the $V^1, V^2$ are also isovectors but that other isovectors which are not in $D$ such as $Y^i = C^i \frac{\partial}{\partial u} + C^i \frac{\partial}{\partial v}$ $n = 1, 2$ do occur.

**B. Integral Elements, Cartan Characters and Genus**

All definitions in this section are based on \cite{[8]} which relies on Cartan’s original work whereas the work in \cite{[27]} varies slightly from \cite{[10]} in some definitions. When an EDS $\Sigma$ or a Pfaffian system $\mathcal{P}$ is given, we try to find manifolds on which all the differential forms of $\Sigma$ or $\mathcal{P}$ are annihilated. Firstly, we shall-construct tangent spaces to such manifolds, where the dimension of each of them can be deduced from a sequence of non-negative integers called the Cartan characters.

Assume, we are given a vector space $\mathbb{R}^N$ of which we want to find all possible distinct $p$-dimensional subspaces. The set of such subspaces forms a $p(N-p)$—dimensional manifold called the Grassmann manifold $G(N, p)$. We can characterise $G(N, p)$ as a quotient space of orthogonal groups as follows:

$$G(N, p) \simeq O(N, p)/O(N) \times O(p).$$

Alternatively, one can write each $p$-dimensional vector space as a $p$-vector $v = \lambda e^1 \wedge \cdots \wedge e^p$, where $\{e^1, \ldots, e^p\}$ is its basis. Using this definition we obtain

$$G(N, p) \simeq \{(v^1, \ldots, v^p)\}/\sim,$$

where $(v^1 \wedge \cdots \wedge v^p) \sim (w^1 \wedge \cdots \wedge w^p) \Leftrightarrow (w^1, \ldots, w^p) = \lambda (v^1, \ldots, v^p)$ meaning that all multiples of a $p$-vector are in the same equivalence class. The Plücker coordinate of a $p$-vector $v = \frac{1}{N!}a_1 \cdots a_p e^{a_1} \wedge \cdots \wedge e^{a_p}$ is given by the equivalence class $[(a_1 \cdots a_p)]$ of which each equivalence class represents a distinct $p$-plane.

Because we are not only looking at a single $G(N, p)$ but at a whole set for a given manifold $\mathcal{M}$, we define the Grassmann bundle as

$$G_p(\mathcal{M}) := \bigcup_{x \in \mathcal{M}} G(N, p)|_x,$$

where $G(N, p)|_x$ is the Grassmannian manifold at a point $x \in \mathcal{M}$. We shall use these definitions later on to help to describe integral manifolds but first we introduce integral elements as

**Definition 5** Integral Element

A $p$-dimensional subspace $(E^p)_x$ of the tangent space $T_x(\mathcal{M})$ at a point $x$ on a $N$-dimensional manifold $\mathcal{M}$ is called an integral element of an EDS $\Sigma$ if $\alpha_j(E^p)_x = 0$ at $x$ for any form $\alpha_j$ in $\Sigma_p$, $1 \leq i \leq p$, $1 \leq j \leq k_p$, which means that all differential forms of $\Sigma$ are annihilated on $(E^p)_x$ at $x \in \mathcal{M}$.

Each $p$-dimensional integral element $(E^p)_x$ is then a particular element of $G(N, p)|_x$. The set of all $p$-dimensional integral elements $(E^p)_x$ for all $x \in \mathcal{M}$ is then denoted by $V_p(\Sigma)$, where $V_p(\Sigma) \subset G_p(\mathcal{M})$. 

8
Once we have found such a \( p \)-dimensional integral element \((E^p)_x \subset T_x \mathcal{M}\) spanned by \(V^1, \ldots, V^p\), we are looking for vectors \(V_x^{p+1} \in T_x \mathcal{M}\) in such a way that the space generated by \((E^p)_x\) and \(V_x^{p+1}\) form a \((p+1)\)-dimensional integral element. The conditions on such a tangent vector \(V_x^{p+1}\) at the point \(x \in \mathcal{M}\) are:

\[
\begin{align*}
\alpha^1_{i_1}(V_x^{p+1}) &= 0, & 1 \leq i_1 \leq k_1, \\
\alpha^2_{i_2}(V_x^{j_2}, V_x^{p+1}) &= 0, & 1 \leq i_2 \leq k_2, 1 \leq j_1 \leq p, \\
&\vdots& \\
\alpha^{p+1}_{i_{p+1}}(V^1, V^2, \ldots, V^p, V_x^{p+1}) &= 0, & 1 \leq i_{p+1} \leq k_{p+1}.
\end{align*}
\tag{18}
\]

The vectors \(V_x^{p+1}\) that satisfy the above polar system \(H((E^p)_x)\) of linear equations generate the polar space \(H((E^p)_x)^{\perp}\) of \((E^p)_x\). Depending on the ranks of the polar systems generated, we can divide integral elements into 3 classes which we call regular, ordinary and singular. We look at the subsystem \(\alpha^1_1(V_x^1) = \cdots = \alpha^1_{k_1}(V_x^1) = 0\) of \([\mathbb{R}]\) where \(s_0(x)\) is its rank. From this, we get the integer

\[
s_0 := \max_{x \in \mathcal{M}}(r(H((E^0)_x))) = s_0(x)
\]

called the zeroth Cartan character. We define a regular point \(x \in \mathcal{M}\) to be a point where \(s_0(x) = s_0\). Then, a 1-dimensional integral element \((E^1)_x\) is an ordinary integral element if \(x\) is a regular point. Let the polar system of \((E^1)_x\) in \(T_x \mathcal{M}\) have rank \(r(H((E^1)_x)) = s_1(x) + s_0\). From this, we define

\[
s_1 := \max_{x \in \mathcal{M}, V^2 \in T_x \mathcal{M}}(r(H((E^1)_x))) - s_0,
\]

which is called the first Cartan character. If, for \(x \in \mathcal{M}\), \(s_1(x) = s_1\) holds, then the integral element \((E^1)_x\) is called regular. A 2-dimensional integral element \((E^2)_x\) then is called ordinary if it contains at least one regular 1-dimensional integral element. We define inductively

**Definition 6** Cartan Characters

The \(p\)th Cartan character is inductively defined as:

\[
s_p = \max_{x \in \mathcal{M}, V^{p+1} \in T_x \mathcal{M}}(r(H((E^p)_x))) - \sum_{i=0}^{p-1} s_i.
\]

Using the notion of Cartan characters we can now precisely define

**Definition 7** Regular, Ordinary and Singular Integral Elements

A \(p\)-dimensional integral element \((E^p)_x\) is called ordinary if it contains at least one \((p-1)\)-dimensional regular integral element. A \(p\)-dimensional integral element is called regular if its polar system \(H((E^p)_x)\) has maximal rank \(r(H((E^p)_x)) = s_p(x) = s_p\). A \(p\)-dimensional integral element is singular if it is neither regular nor ordinary.

Note that a sequence of integral elements \((E^0)_x \subset (E^1)_x \subset \cdots \subset (E^p)_x\) at a point \(x \in \mathcal{M}\) is called a regular chain of integral elements if all its \((E^k)_x, \ 1 \leq k \leq p - 1, \) are regular and \((E^p)_x\) is at least ordinary. The maximal dimension an ordinary integral element can adopt is called the genus \(g\) of \(\Sigma\) or \(\mathcal{P}\).

Integral elements are the tangent planes to manifolds which are solution manifolds for a given EDS \(\Sigma\) and Pfaffian systems \(\mathcal{P}\) and defined as
\textbf{Definition 8 Integral Manifolds}

An integral manifold of an EDS $\Sigma$ on $\mathcal{M}$ is a $p$-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ such that each $k$-dimensional vector subspace $(E^k)_x \subset T_x \mathcal{N}$ for $1 \leq k \leq p$ annihilates all the $k$-forms in the EDS $\Sigma$.

Next, we wish to illustrate this theory using example (15) again. There, a 1-dimensional integral element $(E^1)_x$ can for instance be created by means of setting $V^1_x := 1$, $V^1_y := 0$ and computing the other components according to (10) and (17). Its polar space is then spanned by $H((E^1)_x) = \{V^2\}$ for any $V^2$ satisfying (16) and (17) other than $V^1$ itself. Then, $(E^2)_x$ is given by $(E^2)_x = \{V^1, V^2\}$ which is the maximal dimension an integral element can adopt for this example, where we can for instance choose $V^2_x := 0$ and $V^2_y := 1$. The polar matrix of the polar space $H(E^0)_x$ is given by

\[

dx \ dy \\
-pq \\
-pQ \\
-rS \\
-rT \\
-1S \\
-1T \\
-0 \\
-0 \\
-0
\]

The rank $r(H(E^0)_x)$ of the polar matrix is equal to the rank $s$ of our Pfaffian system (13) and therefore we obtain $s_0 = s = 12$. The two Vessiot vector fields $V^2$ which form a 2-dimensional involution for (13) also constitute a regular chain of integral elements where $(E^1)_x$ is spanned by either of the $V^i$ and $(E^2)_x$ by both of them.

Next, we wish to determine the integral manifold for (15) on which (14) holds as well. From the 6 equations obtained from differentiating (14) with respect to $x$ and $y$, we see that $t = \frac{\partial^2 u}{\partial y^2} = 0$ and $T = \frac{\partial^2 u}{\partial y^2} = 0$ so that the solution must be linear in $y$

\[
\begin{align*}
  u &= yf(x) + g(x) \\
  U &= yF(x) + G(x)
\end{align*}
\]

(19)

with $f(x), g(x), F(x), G(x)$ arbitrary functions. From this we find by very elementary calculations that the general solution is given by

\[
\begin{align*}
u &= y \cos(x + \epsilon) + a, \quad U = y \sin(x + \epsilon) + b, \\
p &= -y \sin(x + \epsilon), \quad P = y \cos(x + \epsilon), \\
q &= \cos(x + \epsilon), \quad Q = \sin(x + \epsilon), \\
r &= -y \cos(x + \epsilon), \quad R = y \cos(x + \epsilon), \\
s &= -\sin(x + \epsilon), \quad S = \cos(x + \epsilon), \\
t &= 0, \quad T = 0,
\end{align*}
\]
where \{a, b, e\} are arbitrary constants corresponding to the translational and rotational degrees of freedom. This also gives a local parameterisation of the 2-dimensional integral manifold which corresponds to this solution.

C. Independence Condition, Reduced Characters and Involution

Until now, all variables of a given EDS \(\Sigma\) were treated equally. But if we wish to look for specific integral manifolds transversal to some given submanifold, we must specify this fact for our EDS. This is achieved by introducing an independence condition \(\Omega = \omega^1 \wedge \cdots \wedge \omega^n \neq 0\), which is sometimes also called the volume element, where the \(\omega^i\) are 1-forms characterising such a submanifold. If we select \(x^1, \cdots, x^n\) as local coordinates, where we shall use brackets to indicate powers of \(x^i\) such as in \((x^i)^n\), then \(\Omega\) is given by \(\Omega = dx^1 \wedge \cdots \wedge dx^n\).

All integral elements on which \(\Omega\) does not vanish are called admissible integral elements according to [45] and using this concept we can define

**Definition 9 Involutive Systems**

An EDS with independence condition \((\Sigma, \Omega)\) is in involution with respect to \(\Omega\) at a point \(x \in M\) means that there exists an admissible ordinary integral element of dimension \(n\) of \((\Sigma, \Omega)\) at \(x\).

For practical reasons new integers called the reduced Cartan characters are introduced so as to decide whether a given EDS \(\Sigma\) is in involution or not with respect to an independence condition \(\Omega\). In order to determine them, we need to introduce the reduced polar systems \(H^{red}((E^p)_x)\) which are defined as the polar systems \(H^{red}((E^p)_x)\), where all terms involving any of the \(\omega^1, \cdots, \omega^n\) are suppressed. We define the reduced rank \(s'_0(x) := r(H^{red}((E^0)_x))\) and \(s'_0 := \max_{x \in M} s'_0(x)\) so that we can define the reduced characters inductively.

**Definition 10 Reduced Cartan Characters**

The reduced Cartan characters \(s'_p\) are inductively defined as

\[
s'_p := \max_{\{x \in M, V^{p+1}_x \in T_x M\}} r(H^{red}((E^p)_x)) - \sum_{i=0}^{p-1} s'_i.
\]

The coincidence of the set of characters with that of the set of reduced characters is a necessary condition for a system to be in involution. In practice, we can use Cartan’s test for involution based on a comparison of polar elements with the reduced polar elements. But first, we must enlarge our account of Pfaffian systems with the use of more general concepts from the literature. Given a Pfaffian system \(P\) consisting of \(s\) 1-forms \(\theta^\alpha\) with independence condition \(\Omega = \omega^1 \wedge \cdots \wedge \omega^n \neq 0\), we denote by \(\pi^\lambda\) all the extra forms such that \((\theta^\alpha, \omega^1, \cdots, \omega^n, \pi^\lambda, \ldots)\) form a coframe on our formal \(N\)-dimensional manifold \(M\). Once we have chosen such forms \(\pi^\lambda\), we can write each \(d\theta^\alpha\) as

\[
d\theta^\alpha = A^\alpha_{\lambda i} \pi^\lambda \wedge \omega^i + \frac{1}{2} B^\alpha_{ij} \omega^i \wedge \omega^j + \frac{1}{2} C^\alpha_{\lambda \kappa} \pi^\lambda \wedge \pi^\kappa \mod (I(P)) .
\]

In equations (20), the \(A^\alpha_{\lambda i}\) form the tableau matrix and the \(B^\alpha_{ij}\) are called the torsion terms\(^1\). The Pfaffian system is called quasi-linear if all the
\( C_{\lambda \kappa}^\alpha = 0 \). The following changes of coframe preserve linearity

\[
\begin{align*}
\tilde{\theta}^\alpha &= T^\beta_\alpha \theta^\beta \\
\tilde{\omega}^j &= T^j_\alpha \omega^j + T^j_\alpha \theta^\alpha \\
\tilde{\pi}^\lambda &= T^\lambda_\kappa \pi^\kappa + T^\lambda_\alpha \omega^j + T^\lambda_\alpha \theta^\alpha. 
\end{align*}
\]

We say that the torsion can be absorbed if a suitable transformation \( \Phi \)

\[\Phi: \pi^\lambda \rightarrow \pi^\lambda + p_\lambda^\kappa \omega^\kappa \]

with

\[\tilde{B}^\alpha_{ij} = B^\alpha_{ij} + A^\alpha_{\lambda j} p_\lambda^i - A^\alpha_{\lambda i} p_\lambda^j \]

(22) can be found such that \( \tilde{B}^\alpha_{ij} = 0 \). When \( B^\alpha_{ij} \neq 0 \) in every coframe \((\theta^\alpha, \omega^\kappa, \pi^\lambda)\), the system possesses integrability conditions which will prevent our given Pfaffian system \( \mathcal{P} \) from being in involution. If a transformation \( \Phi \) above exists such that all above \( \tilde{B}^\alpha_{ij} = 0 \), then there are no integrability conditions to stop the system being involutive.

We can rewrite (20) in the dual language of Vessiot vector fields. If we choose a frame dual to the coframe \((\omega^i, \theta^\alpha, \pi^\lambda)\), which we denote by \((E^i, F^\alpha, G^\lambda)\) such that

\[
\begin{align*}
\omega^i (E^j) &= \delta^i_j, \quad (i, j = 1, \cdots, n) \\
\theta^\alpha (F^\beta) &= \delta^\alpha_\beta, \quad (\alpha, \beta = 1, \cdots, s) \\
\pi^\lambda (G^\kappa) &= \delta^\lambda_\kappa, \quad (\lambda, \kappa = 1, \cdots, N - n - s) 
\end{align*}
\]

(23) two Vessiot vector fields \( X, Y \in \mathcal{D} \) can then be recast as

\[
\begin{align*}
X &= X^i_\omega E^i + X^\alpha_\theta F^\alpha + X^\lambda_\pi G^\lambda, \\
Y &= Y^j_\omega E^j + Y^\beta_\theta F^\beta + Y^\kappa_\pi G^\kappa.
\end{align*}
\]

(24)

When we calculate \( d\theta^\alpha (X, Y) \), we obtain that

\[
\begin{align*}
d\theta^\alpha (X, Y) &= A^\alpha_{\lambda j} (X^i_\omega Y^j_\omega - X^i_\omega Y^\lambda_\lambda) + B^\alpha_{ij} X^i_\omega Y^j_\omega + C^\alpha_{\lambda \kappa} X^\lambda_\omega Y^\kappa_\pi \\
&= 0 \mod (\mathcal{I}(\mathcal{P})),
\end{align*}
\]

(25)

where (24) is a system of equations for \( X, Y \in \mathcal{D} \) based on the dual approach to (21).

From the tableau matrix \( A^\alpha_{\lambda j} \), we can determine the reduced Cartan characters directly \([15], [1] \), where for the ranks \( r(A^\alpha_{\lambda 1}) := s'_1 \) and so on. There are many versions of Cartan’s original test for involution \([10] \) some of which one can find in \([1] \) or in \([13] \), where a version for Pfaffian systems is discussed. We state it as:

**Theorem 5** Cartan’s Test for Involution

A Pfaffian system \((\mathcal{P}, \Omega)\) is in involution means that \( s_0 = s'_0, s_1 = s'_1, \cdots, s_p = s'_p \) and a coframe transformation can be found such that all its torsion terms \( \tilde{B}^\alpha_{ij} \) vanish identically.

The involutive genus \( g \) determines the maximal dimension a regular integral manifold can adopt and is given by \( g = N - \sum_{i=0}^{n} s_i \). Note that the notion of an involutory subsystem of a vector field systems is not the same as the one used in discussing differential forms. In order to obtain the equivalent of this, we must add an independence condition and define
what it means for a vector field system to be in involution with respect to $x^1, \ldots, x^n$. The precise definition can be found in [11]. We can also use Groebner bases in developing criteria for involutivity. This is carried out in many places for instance in a paper by Mansfield [31].

In our example (15), we choose $w^1 := dx$ and $w^2 := dy$ so that $\Omega = dx \wedge dy$ whereas the $u$ and $U$ constitute the dependent variables. Then, the reduced Cartan characters happen to coincide with their full counterparts so that $s = s_0 = s_0' = 12$ while all higher reduced characters vanish. Because the 12 $\theta^\alpha$ in (15) together with $\omega^1 = dx$ and $\omega^2 = dy$ form a complete coframe of dimension $N = 14$ already, we do not need to add any $\pi^\lambda$ and the tableau matrices vanish identically with all components $A^\alpha_{\lambda i} = 0$. This proves that $s'_i = 0, \forall i > 0$. The two torsion terms $B^4_{12}$ and $B^6_{12}$ vanish identically. Next, we must compute the torsion terms which do not vanish identically and they are

$$B^4_{12} = B^4_{12} = B^6_{12} = B^6_{12} = rt + RT + 1 - s^2 - S^2.$$  

But for our system transforming polar into Cartesian coordinates we have $t = T = 0$ and $s^2 + S^2 = 1$ so that all the torsion terms vanish identically and the system is in involution, and, the involutive genus $g$ is given by $g = 14 - 12 = 2$.

**D. Existence Theorems and Prolongation**

An important question is whether and when integral manifolds exist and whether, given some initial data, they are unique. The Cartan-Kähler theorems [9, 45] give some answers to these questions for real-analytic systems. They reduce to the Cauchy-Kovalevskaya theorem when the number of equations equals the number of unknowns which is given by $m$. A version for first-order systems is given in [23] which is sufficient because every system of PDEs can be transformed into a first-order system of PDEs.

The first Cartan-Kähler theorem specifies under which conditions a unique integral manifold of dimension $p$ can be constructed from a $(p-1)$-dimensional one. For the second Cartan-Kähler existence theorem, we need our EDS $\Sigma$ to be given in normal form. Then, the second theorem simply states that under certain conditions a whole chain of regular integral manifolds of increasing dimensions exists (see Appendix A).

If there is no general solution for a given EDS $(\Sigma, \Omega)$, then there could be identities or even integrability conditions so that we must prolong the system by adding in these conditions and by enlarging the space of our formal jet coordinates to higher order. Good instructive examples of prolongation methods were given in [24, 27] and an example of an infinite dimensional prolongation of vector fields can be found in [19]. Next, we shall discuss Cartan’s classical approach for an EDS briefly. But if we are dealing with a system of PDEs, then prolongation means just adding the partial derivatives of order $q + 1$ as new jet-coordinates and supplementing our original system of PDEs with all partial derivatives of the equations in the system [35], [38].

Cartan’s classical procedure [10], [1] or [12] starts with an EDS with independence condition $(\Sigma, \Omega)$ given as

$$\Sigma_0 = (F_1, \ldots, F_{k_0}), k_0 \text{ 0-forms}$$
$$\Sigma_1 = (\alpha^{(1)}_1, \ldots, \alpha^{(1)}_{k_1}), k_1 \text{ 1-forms}$$
$$\Sigma_2 = (\alpha^{(2)}_1, \ldots, \alpha^{(2)}_{k_2}), k_2 \text{ 2-forms}$$
and so on. We now complete $(\alpha_{i_1}^{(1)}, \omega^i)$ to a coframe $(\alpha_{i_1}^{(1)}, \omega^i, \pi^\lambda)$ on $\mathcal{M}$. We then choose an admissible integral element $(E^n)_x$ from the set of all admissible integral elements over $\mathcal{M}$, which we denote by $\mathcal{V}(\Sigma, \Omega) \subset \mathcal{V}_n(\Sigma)$, on which
\[ \alpha_{i_1}^{(1)} = t_{i_1,i} \omega^i = 0 \quad \text{and} \quad \pi^\lambda = t_{i_1}^\lambda \omega^i \]
has to hold for some $(t_{i_1,i}; t_{i_1}^\lambda)$ which are our new coordinates on the fiber $G_p(\mathcal{M})_x \cong G(N, p)$, where $T_x(\mathcal{M}) \cong \mathbb{R}^p$. Based on this we can define the first prolongation of $(\Sigma, \Omega)$ as

**Definition 11 First Prolongation**

The first prolongation $\Sigma^{(1)}$ of $(\Sigma, \Omega)$ is formed by
i) the original EDS $(\Sigma, \Omega)$ restricted to $\mathcal{V}(\Sigma, \Omega)$ and
ii) the closed system on $\mathcal{V}(\Sigma, \Omega)$ given by
\[ \tilde{\pi}^\lambda = \pi^\lambda - t_{i_1}^\lambda \omega^i, \]
and
\[ d\tilde{\pi}^\lambda. \]

We essentially added an extra number of contact conditions and their exterior derivatives to the above EDS $(\Sigma, \Omega)$. Higher-order prolongations are defined accordingly and for a detailed account on prolongations see [9]. The key result for prolongations in the real analytic case [29] is given by the Cartan-Kuranishi theorem

**Theorem 6 Cartan-Kuranishi Theorem**

An EDS $(\Sigma, \Omega)$ which is not in involution and not inconsistent becomes either involutive or inconsistent after a finite number of prolongations.

An upper limit for the number of prolongations for involutivity or inconsistency is given by a number $\hat{q}$ which can be determined inductively (c.f. [36], [38] or [29]). In the next sections, we shall apply some of the above theory to the Riemann-Lanczos problems in 4 and 5 dimensions.

Concluding Remarks: We should point out that there is a quantity similar to the Cartan characters also introduced by Cartan [10] called the degré d’arbitraire. Seiler derived a relation between this degree of arbitrariness and the Cartan characters for nonlinear, overdetermined higher-order systems of PDEs [39]. Einstein introduced the strength of a system of equations in his last unified field theory of electromagnetism and gravity in order to measure how strongly such a system restricts the solution. A discussion between him and Cartan on this subject is presented in [11]. Sué finally compared Cartan’s degree of arbitrariness and Einstein’s strength in [42, 43] for linear systems of equations.

**III. The Riemann-Lanczos Problem as an EDS and Singular Solutions**

In the Bampi-Caviglia papers [3, 4] the Riemann-Lanczos problem was written as an EDS and found not to be in involution in 4 dimensions as mentioned earlier. In their second paper [4] they introduced a prolongation to create an
involutive system. Now, we are going to form the Pfaffian system resulting from the Riemann-Lanczos equations (3). The Pfaffian system we are going to introduce slightly differs from that in [3] because we incorporate the cyclic conditions (2) as well. The system consists of 20 exterior derivatives of the Riemann-Lanczos equations together with the 20 contact conditions and their exterior derivatives and we can omit the differential gauge conditions which do not change our results qualitatively. Using the exterior differentials of equations (9) in solved form together with the contact conditions, we obtain the system

\[ df^{(R)}_{abcd} = (R_{abcd,e} + \alpha_{abced})dx^e - dP_{abcd} + dP_{abde} - dP_{cdab} + dP_{cdba} + \gamma_{ad}^{n}(dL_{nbc} + dL_{ncb}) - \gamma_{ac}^{n}(dL_{nbde} + dL_{ndbe}) + \gamma_{bc}^{n}(dL_{nade} + dL_{ndae}) - \gamma_{bd}^{n}(dL_{nace} + dL_{ncae}). \]

The (\(x^i, L_{abc}, P_{abcd}\)) are local jet coordinates on \(J^1(\mathbb{R}^4, \mathbb{R}^{20})\) with \(P_{abcd} = \frac{\partial L_{abc}}{\partial x^d}\) when projected onto our spacetime manifold, and the \(K_{abc}\) are the 20 contact conditions. A Vessiot vector field \(V \in \mathcal{D}\) is locally given as

\[ V = V^e \frac{\partial}{\partial x^e} + V^e P_{abce} \frac{\partial}{\partial L_{abc}} + V_{abcd} \frac{\partial}{\partial P_{abcd}}, \]

where \(\alpha\) is given by

\[ \alpha_{abced} = \Gamma_{ad}^{n}(L_{nbc} + L_{ncb}) - \Gamma_{ac}^{n}(L_{nbde} + L_{ndbe}) + \Gamma_{bc}^{n}(L_{nade} + L_{ndae}) - \Gamma_{bd}^{n}(L_{nace} + L_{ncae}). \]

The reduced Cartan characters can be computed using the REDUCE computer code in [22] based on the EDS package by Hartley [24] and slight modifications of it. For a number of spacetimes such as conformally flat, Kasner, Gödel, some Debever types and plane wave spacetimes we verified that the reduced Cartan characters \((s^i_0, s^i_1, s^i_2, s^i_3, s^i_4)\) were \((40, 20, 19, 15, 6)\) when the differential gauge condition (3) was not included and \((40, 20, 19, 15, 0)\) when it was included. In both cases, the systems were clearly not in involutive because some of the reduced characters differ from the Cartan characters - a fact reflected by results given by the computer code.

Even though the Riemann-Lanczos problem in 4 dimensions is not in involutive, singular solutions do exist. Singular solutions occur when some
of the integral elements, which are tangent to an integral manifold, are *singular* leading to singular integral manifolds, which means that the Cartan characters of the corresponding integral elements are not maximal.

In the remaining sections of this paper, we shall give a few singular solutions for the Riemann-Lanczos problem in 4 dimensions for a number of spacetimes and briefly discuss the Riemann-Lanczos problem in 5 dimensions.

### A. Debever-type Spacetimes: Example

We expect singular solutions to occur for most of the Debever-type spacetimes for which a line element is given by

\[ ds^2 = dt^2 - f^2(t, y, z)dx^2 - dy^2 - dz^2 \]  

with \( f(t, y, z) \) being one of the functions given in [12] or in [34]. Specifically, for a Debever spacetime with \( f = y^2 \) of Petrov type D we are interested in singular solutions such that

\[ \mathcal{L}_\xi L_{abc} = 0 \]  

is possible using \( \mathcal{L}_\xi R_{abcd} = 0 \) and the fact that \( \nabla \mathcal{L}_\xi = \mathcal{L}_\xi \nabla \) for \( \xi \) any Killing vector field (=KV). For convenience we shall denote the local coordinates by \( x^1 := t, x^2 := x, x^3 := y, x^4 := z \) for this example. Using CLASSI [1] we can determine the dimension of the isometry group which is 4 in this case. There are 3 KVs corresponding to the ignorable coordinates \( t, x \) and \( z \) when the above local coordinate frame is used and the components of the fourth KV are \( \xi^4 = (z, 0, 0, t) \). Imposing \( \mathcal{L}_\xi L_{abc} = 0 \) on the \( L_{abc} \) leads to

\[
\begin{align*}
L_{ttx} &= 2L_{txz}, \\
L_{tzy} &= 2L_{tyz}, \\
L_{txt} &= L_{xzz}, \\
L_{tzt} &= L_{yzz}.
\end{align*}
\]  

The only non-vanishing independent Riemann-Lanczos equations then are

\[
\begin{align*}
R_{txty} + 2\Gamma_{xy}^x L_{txt} - P_{txty} &= 0, \\
\frac{1}{2}R_{xyxy} + \Gamma_{xy}^x L_{xyx} - P_{xyxy} &= 0, \\
R_{xzyz} + P_{xzzy} - 2\Gamma_{xy}^x L_{xzz} &= 0.
\end{align*}
\]  

One solution which also does also satisfy the 6 components of the differential gauge condition (3) is given by

\[
\begin{align*}
L_{txt} &= C_1 y^4, & L_{xyx} &= C_2 y^2 + y^3, \\
L_{tzt} &= C_3, & L_{txz} &= C_4, \\
L_{tyz} &= C_5 y^{-2}, & L_{xyy} &= C_6.
\end{align*}
\]  

where \( C_1, \ldots, C_6 \) are constants. If we choose \( C_3 = C_4 = C_6 = 0 \) then only 3 independent components will remain. On a submanifold with \( f_{abcd}^{(R)} = 0 \) of (27) for the above line element this solution corresponds to a singular
integral manifold for which the (reduced) characters are $s_0 = s'_0 = 3$ while all higher characters vanish. Such an integral manifold is parameterised by the above expressions for the $L_{abc}$ and their corresponding $P_{abcd}$. The Vessiot vector fields which span the tangent spaces of this singular solution manifold can locally be given by

$$
V^1 = \frac{\partial}{\partial t},
$$

$$
V^2 = \frac{\partial}{\partial x},
$$

$$
V^3 = \frac{\partial}{\partial y} + 4C_1y^3 \frac{\partial}{\partial L_{txt}} - 2C_2y^{-3} \frac{\partial}{\partial L_{tyz}} - 4C_2y^{-3} \frac{\partial}{\partial L_{tzy}} + (2C_3y + 3y^2) \frac{\partial}{\partial P_{txy}} + 4C_1y^3 \frac{\partial}{\partial P_{txy}} + 12C_2y^{-4} \frac{\partial}{\partial P_{txy}} + 12C_2y^{-4} \frac{\partial}{\partial P_{txy}} + 12(C_3 + 3y) \frac{\partial}{\partial P_{txy}} + 12y^2 \frac{\partial}{\partial P_{txy}},
$$

$$
V^4 = \frac{\partial}{\partial z},
$$

(36)

where we substituted

$$
V^3_{txt} = 12y^2,
V^3_{tyz} = 6C_2y^{-4},
V^3_{xyy} = 2(C_3 + 3y),
$$

(37)

and where $V^3_{txt} = V^3_{tzz}$ and $V^3_{tyz} = 2V^3_{tzy}$, because of (34). In the next section, we are going to discuss examples of spacetimes where $\mathcal{L}_\xi L_{abc} = 0$ cannot be implemented but where other singular solutions can occur.

**B. Singular Solution for Gödel Spacetime**

Gödel spacetime is a perfect fluid spacetime admitting closed time-like curves with line element

$$
ds^2 = a^2(dt^2 - dx^2 - dz^2 + \frac{1}{2}e^{2x}dy^2 + 2e^x dt dy).
$$

(38)

Again, we are going to replace the local coordinates in such a way that $x^1 := t, x^2 := x, x^3 := y, x^4 := z$. It is well known [23] that this spacetime admits a $G_5$ as an isometry group of which 3 Killing vectors $\xi^1, \xi^2, \xi^3$ are based on the ignorable coordinates $t, y, z$. The other two are given by

$$
\xi^4 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad \xi^5 = -2e^{-x} \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} + (e^{-2x} - \frac{1}{2}y^2) \frac{\partial}{\partial y}.
$$

But here, the Riemann-Lanczos problem does not admit singular solutions which can inherit all the spacetime symmetries. This is the case because not all elements of the isometry group of the spacetime are symmetries of the Riemann-Lanczos problem.

However, it is possible to find singular solutions with $\mathcal{L}_\xi L_{abc} = 0$ imposed for the 3 Killing vectors based on ignorable coordinates. An Ansatz where
some components are proportional to exponentials $e^x, e^{2x}$ while all other components vanish \cite{22} leads to the solution

\[
L_{txy} = -\frac{a^2}{8}e^x, \quad L_{tyx} = \frac{a^2}{8}e^x,
\]
\[
L_{txt} = -\frac{a^2}{8}, \quad L_{xyy} = 3\frac{a^2}{16}e^{2x}.
\]

This solution does not satisfy all the differential gauge conditions \cite{22}. This singular solution manifold is again parameterised by the above components $L_{abc}$ and their derivatives $P_{abcd}$. Its tangent spaces are spanned by 4 Vessiot vector fields which can locally be given as

\begin{align*}
V^1 &= \frac{\partial}{\partial t}, \\
V^2 &= \frac{\partial}{\partial x} + \frac{a^2}{8}e^x \frac{\partial}{\partial L_{txy}} + \frac{a^2}{8}e^x \frac{\partial}{\partial L_{tyx}} + \frac{3a^2}{8}e^{2x} \frac{\partial}{\partial L_{xyy}} \\
&+ \frac{1}{3}e^x \left(\frac{5a}{16} - \frac{1}{2} - e^x \frac{3a}{16}\right) \frac{\partial}{\partial P_{txyt}} - \frac{1}{3}e^x \left(\frac{5a}{16} - \frac{1}{2} - e^x \frac{3a}{16}\right) \frac{\partial}{\partial P_{tyxt}} \\
&+ \frac{3}{4a}e^{2x} \frac{\partial}{\partial P_{xyyx}}, \\
V^3 &= \frac{\partial}{\partial y}, \\
V^4 &= \frac{\partial}{\partial z},
\end{align*}

(39)

where it is $V^2_{tyxx} = V^2_{txxy}$. This solution is singular and all Cartan characters vanish even $s_0 = s'_0 = 0$ because no constants remain in the above solution.

C. Kasner Spacetime

Kasner spacetime admits a $G_3$ based on the 3 ignorable coordinates $x, y, z$, where we again replaced $x^1 := t, x^2 := x, x^3 := y, x^4 := z$. If we impose $\xi \ell_{abc} = 0$, the 6 non-vanishing components of the Riemann-Lanczos equations result in

\begin{align*}
1 \quad 0 &= R_{txtx} + 2P_{txxt} - 2\frac{P_1}{t}L_{txt} \\
2 \quad 0 &= R_{tyty} + 2P_{tyyt} - 2\frac{P_2}{t}L_{tyy} \\
3 \quad 0 &= R_{tztz} + 2P_{tzzt} - 2\frac{P_3}{t}L_{tzz} \\
4 \quad 0 &= R_{xyxy} - 2t^2p_1t^{-1}p_2L_{txx} - 2t^2p_1^{-1}p_1L_{tyy} \\
5 \quad 0 &= R_{xxzz} - 2t^2p_1^{-1}p_3L_{txt} - 2t^2p_1^{-1}p_1L_{tzz} \\
6 \quad 0 &= R_{xzxz} - 2t^2p_1^{-1}p_3L_{txy} - 2t^2p_1^{-1}p_2L_{tzz}.
\end{align*}

(40)

We can now easily see that solving the last 3 equations of (40) algebraically leaves us with a solution for $L_{txx}, L_{tyy}, L_{tzz}$. But, inserting this solution into the first 3 equations of (40) and solving for the $P_{abcd}$ leads to

\begin{align*}
L_{txt} &= -\frac{1}{4}p_1t^{2p_1-1}, \quad P_{txxt} = (\frac{1}{2} - \frac{3}{4}p_1)p_1t^{2p_1-2}, \\
L_{tyy} &= -\frac{1}{4}p_2t^{2p_2-1}, \quad P_{tyyt} = (\frac{1}{2} - \frac{3}{4}p_2)p_2t^{2p_2-2}, \\
L_{tzz} &= -\frac{1}{4}p_3t^{2p_3-1}, \quad P_{tzzt} = (\frac{1}{2} - \frac{3}{4}p_3)p_3t^{2p_3-2}.
\end{align*}
We see that when we project this down onto the spacetime manifold, we obtain that \( P_{ttxx} \neq L_{ttxx,t} \), \( P_{tyty} \neq L_{tyty,t} \) and \( P_{tzzt} \neq L_{tzzz,t} \) therefore not being a solution to \( (\text{I}) \). Therefore, for Kasner spacetime no singular solution inheriting \( L_{abc} = 0 \) exists. But there are other singular solutions for which some of the components \( L_{abc} \) are linear in either \( x, y \) or \( z \) with no Lie symmetries along Killing directions. We decide to make the Ansatz

\[
L_{txx} = C_1 t^{n_1} x, \quad L_{txx} = C_4 t^{n_4}, \quad L_{ttxx} = C_1 t^{n_1}, \quad L_{tyy} = C_5 t^{n_5}, \\
L_{tzz} = C_3 t^{n_3} z, \quad L_{tzz} = C_6 t^{n_6}, \quad L_{xyz} = C_7 t^{n_7} y, \quad L_{xyz} = C_10 t^{n_{10}} x, \\
L_{txx} = C_8 t^{n_8} z, \quad L_{txx} = C_11 t^{n_{11}} x, \quad L_{xyz} = C_9 t^{n_9} z, \quad L_{xyz} = C_{12} t^{n_{12}} y,
\]

where \( C_1, \ldots, C_{12} \) and \( n_1, \ldots, n_{12} \) are arbitrary constants. Inserted into the Riemann-Lanczos equations, we obtain

\[
\begin{align*}
1 \quad 0 &= R_{ttxx} - 2P_{ttxx} + 2P_{ttxx} - \frac{2p_1}{t}L_{txx} \\
2 \quad 0 &= R_{tyty} - 2P_{tyty} + 2P_{tyty} - \frac{2p_2}{t}L_{tyy} \\
3 \quad 0 &= R_{tzzz} - 2P_{tzzz} + 2P_{tzzz} - \frac{2p_3}{t}L_{tzz} \\
4 \quad 0 &= R_{xyxy} - 2P_{xyxy} + 2P_{xyxy} - 2t^{2p_1-1}p_2 L_{txx} - 2t^{2p_1-1}p_2 L_{tyy} \\
5 \quad 0 &= R_{zzzz} - 2P_{zzzz} + 2P_{zzzz} - 2t^{2p_1-1}p_2 L_{txx} - 2t^{2p_1-1}p_2 L_{tyy} \\
6 \quad 0 &= R_{yzyz} - 2P_{yzyz} + 2P_{yzyz} - 2t^{2p_1-1}p_2 L_{txx} - 2t^{2p_1-1}p_2 L_{tyy} \\
7 \quad 0 &= R_{txxy} + L_{tyty} t^{2p_1-1}p_2 + L_{txxy} t^{2p_1-1} (p_1 + 2p_2) + P_{xyxt} \\
8 \quad 0 &= R_{tyxy} - L_{txxy} t^{2p_1-1}p_2 - L_{txxy} t^{2p_1-1} (p_1 + 2p_2) + P_{xyxt} \\
9 \quad 0 &= R_{txzz} + L_{ttxx} t^{2p_1-1}p_2 - L_{ttxx} t^{2p_1-1} (p_1 + 2p_2) + P_{xzxt} \\
10 \quad 0 &= R_{tyzy} + L_{txzy} t^{2p_1-1}p_2 - L_{txzy} t^{2p_1-1} (p_1 + 2p_2) + P_{yzyt} \\
11 \quad 0 &= R_{tzzy} + L_{tzyz} t^{2p_1-1}p_2 - L_{tzyz} t^{2p_1-1} (p_1 + 2p_2) + P_{yzyt},
\end{align*}
\]

where we labelled the equations as explained in Appendix B. Using the above Ansatz with \( n_1 = 2p_1 - 2, n_2 = 2p_2 - 2, n_3 = 2p_3 - 2, n_4 = n_1 + 1, n_5 = n_2 + 1, n_6 = n_3 + 1 \) and \( n_7 = n_{10} = -2p_3, n_8 = n_{11} = -2p_2, n_9 = n_{12} = -2p_1 \), we obtain a singular solution for Kasner spacetime with the following constants for the above Ansatz

\[
\begin{align*}
C_4 &= \frac{C_1}{(p_1 - 1)} - \frac{p_1}{2}, \quad C_5 = \frac{C_2}{(p_2 - 1)} - \frac{p_2}{2}, \\
C_6 &= \frac{C_3}{(p_3 - 1)} - \frac{p_3}{2}, \quad C_7 = \frac{p_1 C_2}{2 - p_1}, \\
C_8 &= \frac{p_2 C_3}{2 - p_1}, \quad C_9 = \frac{p_3 C_4}{2 - p_2}, \\
C_{10} &= \frac{-p_3 C_4}{2 - p_2}, \quad C_{11} = \frac{-p_2 C_3}{2 - p_3}, \\
C_{12} &= \frac{-p_3 C_4}{2 - p_3}.
\end{align*}
\]

A computer code, which determines the rather longish expressions for \( C_1, C_2, C_3 \) in terms of \( p_1, p_2, p_3 \), can be found in Appendix B or in \([22]\). For this solution all characters and their reduced counterparts vanish identically so that \( s_0 = s_0' = 0 \) because all constants are completely determined by \( p_1, p_2, p_3 \).
IV. Comment on the Riemann-Lanczos Problem in 5 Dimensions

On a 5-dimensional spacetime, we can again state the Riemann-Lanczos problem. The structure of the Riemann-Lanczos equations remain the same as (3) but, we obtain a larger number, namely 50 independent equations due to the 50 independent components of the Riemann tensor in 5 dimensions. Again, we omit the here 10 components of the differential gauge condition $L_{ab\,\,s} = 0$ in what follows.

We also obtain 50 Lanczos components $L_{abc}$ but imposing the cyclic conditions (3) reduces the number to 40 independent components. Looking at the dimension of the jet bundle $J^1(\mathbb{R}^5, \mathbb{R}^{20})$, we find 5 spacetime dimensions, 40 independent Lanczos components $L_{abc}$ and 200 independent $P_{abcd}$ totalling $N = 245$ formal dimensions for the manifold $\mathcal{M}$. We also find the total number of independent 1-forms is given by $s \leq 90$, where we have up to 50 independent $d\mathcal{f}^{(R)}_{abcd}$ and 40 contact conditions $K_{abc}$. The number of independent components of a Vessiot vector field $V$ is given by $p = N - s$ which here amounts to $p \geq 155$. Such a Vessiot vector field can again be written as

$$V = V^e \frac{\partial}{\partial x^e} + V^e P_{abc} \frac{\partial}{\partial L_{abc}} + V_{abcd} \frac{\partial}{\partial P_{abcd}}$$

(42)

with $V_{abcd} = V_{(abcd)}$, where $\{abcd\}$ indicates Riemann type symmetries over the indices $abcd$. The EDS is again given by (27), the only difference being the number of equations involved and the range of the indices.

We wish to find the maximal dimension of a possible involution of Vessiot vector fields. The first Vessiot vector field $V^1$ has at least 155 free (=parametric) components since $\text{dim}(\mathcal{D}) = p \geq 155$. But all additional Vessiot vector fields $V^2, \ldots, V^5$ need to satisfy a further 40 conditions of the form

$$dK_{abc}(V^i, V^j) = 0 ,$$

(43)

where $K_{abc} = dL_{abc} - P_{abc} dx^e$ are the contact conditions in local coordinates. This leaves the next three Vessiot vector fields with $V^2$ having at least 115, $V^3$ at least 75 and $V^4$ at least 35 parametric components. The last condition for $V^5$ can cause problems however, as there may not be enough free components to form a $V^5$ in some cases. If we wish to find a suitable $V^5$, we must have $s_0 + s_1 + s_2 + s_3 + s_4 \leq 240$. We can illustrate this as follows:

- $s_0 \leq 90 \implies N - s_0 \geq 155 ,$
- $s_0 + s_1 \leq 130 \implies N - s_0 - s_1 \geq 115 ,$
- $s_0 + s_1 + s_2 \leq 170 \implies N - s_0 - s_1 - s_2 \geq 75 ,$
- $s_0 + s_1 + s_2 + s_3 \leq 210 \implies N - s_0 + s_1 + s_2 + s_3 \geq 35 ,$
- $s_0 + s_1 + s_2 + s_3 + s_4 \leq 250 \implies N - s_0 - s_1 - s_2 - s_3 - s_4 \geq -5 ,$

where the second column gives the number of free components for $V^1, V^2$ and so on. Note that the fact that the sum of all free components $155 + 115 + 75 + 35 = 380 \geq 245$ is permissible because 380 is the minimal total number of free but not necessarily independent components for all $V^1, \ldots, V^5$.

If we abandoned the cyclic conditions for a moment, the numbers would be: $N = 250, s \leq 100, p \geq 150$. Then $V^1$ would have at least 150 components left, $V^2$ at least 100 components, $V^3$ at least 50 components and $V^4$ would have no parametric components left in the worst case when all the $s_i$ are
maximal. There would not be any parametric components left to form $V^5$. Therefore, in the generic case no 5-dimensional involution of Vessiot vector fields exists. This means that in the generic case we cannot even form a singular 5-dimensional chain of integral elements but only a 4-dimensional at most. Therefore, neither the first nor the second of the Cartan-Kähler theorems hold in this case.

Apart from that, we expect 20 identities of the same kind as the identities in [3] to occur. Using the computer code in Appendix C given in [22] adapted to the 5-dimensional case, we could find the following results for the reduced Cartan characters for a number of spacetimes such as Minkowski space, the 5-dimensional conformally flat spacetime and the Debever spacetimes. Their reduced characters $(s'_1, s'_2, s'_3, s'_4, s'_5)$ when not including the differential gauge condition (3) were $(40, 39, 35, 26, 10)$ and the second set including the differential gauge conditions (3) was $(40, 39, 35, 26, 0)$. The computer code also confirmed that neither of the systems were in involution. We conclude that in order to be a viable problem in 5 dimensions, the Riemann-Lanczos problem needs to be modified significantly.

**Conclusion**

We gave a review of the theory of exterior differential systems and Pfaffian systems in particular. We illustrated this theory with the 2-dimensional example of a coordinate transformation expressed as a complete Pfaffian system.

Singular solutions for the Riemann-Lanczos problem in 4 dimensions do exist and we found some for Kasner, Gödel and for a Debever type spacetime.

For the Riemann-Lanczos problem in 5 dimensions, 20 internal identities of the same type as those in 4 dimensions can occur as a computer code suggests and a 5-dimensional involution of Vessiot vector fields cannot be achieved in a generic case.

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**Appendix A: The Cartan-Kähler Existence Theorems**

The Cartan-Kähler existence theorems [15, 27, 10] are stated here. The first Cartan-Kähler theorem specifies under which conditions a unique integral manifold of dimension $(p+1)$ can be found from a $p$-dimensional one:

**Theorem 7 First Cartan-Kähler Existence Theorem**

Let $\mathcal{M}$ be a formal $M$-dimensional manifold and $\mathcal{N}$ be a $p$-dimensional integral manifold with a regular integral element $T_x(\mathcal{N})$, where $(E^p)_x = T_x(\mathcal{N})$ at a point $x$ on $M$. Further, there exists a submanifold $\mathcal{F}$, where $\mathcal{N} \subset \mathcal{F} \subset \mathcal{M}$ such that \( \dim(\mathcal{F}) = M - t_{p+1}, \) \( \dim(T_x(\mathcal{F}) \cap H((E^p)_x)) = p + 1, \) where
$t_{p+1} = \dim(H((E^p)_x)) - p - 1$. Then, there exists a unique integral manifold $\mathcal{X}$ around $x$ such that $\dim(\mathcal{X}) = p + 1$ and $\mathcal{X} \supset \mathcal{F} \supset \mathcal{N}$.

This means that for a given $p + 1$-dimensional manifold $\mathcal{F}$, we are looking for the submanifold $\mathcal{X} \subset \mathcal{F}$ such that $\dim(\mathcal{X}) = p + 1$ and the theorem tells us that $\mathcal{X}$ exists and is unique. Here, the integer $t_{p+1}$ is defined as $t_{p+1} = \dim((E^p)_x) - p - 1$, $0 \leq p < n$ and $s_p = t_p - t_{p+1} - 1$. For $t_0$ we have $t_0 = \sum_{i=0}^{n} s_i + n = \dim(\mathcal{I}(\Sigma_0))$. For the second Cartan-Kähler existence theorem, we need our EDS $\Sigma$ to be given in normal form which in this context means that there exists a local coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^r)$, where $r = N - n$, such that:

i) $\mathcal{I}(\Sigma_0)$ is defined by $y^{t_0-n+1} = \ldots = y^{N-n} = 0$,

ii) $H((E^p)_x) = \{ \partial_{x^1}, \ldots, \partial_{x^n}, \partial_{y^{(t_0+n+s_{p-1}+1)}}, \ldots, \partial_{y^{(t_0-n)}} \}$, $0 \leq p < n$,

iii) $(E^p)_x = (\partial_{x^1}, \ldots, \partial_{x^n})$, $1 \leq p \leq n$,

iv) The integral point $(E^0)_x$ is given by $(E^0)_x = (0, \ldots, 0)$, so that the point $x$ coincides with $(0, \ldots, 0)(N$-times) on $\mathcal{M}$.

Then, we can state the second Cartan-Kähler existence theorem:

**Theorem 8** Second Cartan-Kähler Existence Theorem

Given a regular chain of integral elements $(E^0)_x \subset \cdots \subset (E^n)_x$ of an EDS $\Sigma$ given in normal form. Consider a set of initial data $f_1, \ldots, f_{s_0}, (s_0 \text{ arbitrary constants}), f_{s_0+1}(x^1), \ldots, f_{s_0+s_1}(x^1), (s_1 \text{ arbitrary functions of 1 variable each}), \ldots, f_{s_0+\ldots+s_{n-1}+1}(x^1, \ldots, x^n), f_{t_0-n}(x^1, \ldots, x^n), (s_n \text{ arbitrary functions of } n \text{ variables } x^1, \ldots, x^n)$.

For sufficiently small values of all the above $f_i$ and their first-order derivatives there exists a unique integral manifold defined by $y^i(x^1, \ldots, x^n)$, $0 \leq i \leq t_0 - n \leq j \leq N - n$, such that $y^i(x^1, \ldots, x^n, 0, \ldots, 0) = f_i(x^1, \ldots, x^n)$ for $s_0 + \ldots + s_{p-1} < i \leq s_0 + \ldots + s_p$, $0 \leq p \leq n$.

The highest non-vanishing Cartan character say $s_k$, $k \leq n$ is an invariant of an EDS and its value gives the number of arbitrary functions of $k$ variables involved in the general solution. The other characters have the same meaning only if we used normal local coordinates in the above sense.

**Appendix B:** Labelling of the $f_{abcd}^{(R)}$ and Constants for Kasner Spacetime

The labelling of the equations for the Riemann-Lanczos problem in 4 dimensions as used in section III.C was carried out in the following way:

1 $\leftrightarrow f_{1212}^{(R)}$, 2 $\leftrightarrow f_{1313}^{(R)}$, 3 $\leftrightarrow f_{1414}^{(R)}$, 4 $\leftrightarrow f_{2323}^{(R)}$, 5 $\leftrightarrow f_{2424}^{(R)}$, 6 $\leftrightarrow f_{2434}^{(R)}$, 7 $\leftrightarrow f_{1213}^{(R)}$, 8 $\leftrightarrow f_{1223}^{(R)}$, 9 $\leftrightarrow f_{1323}^{(R)}$, 10 $\leftrightarrow f_{1324}^{(R)}$, 11 $\leftrightarrow f_{1214}^{(R)}$, 12 $\leftrightarrow f_{1314}^{(R)}$, 13 $\leftrightarrow f_{1224}^{(R)}$, 14 $\leftrightarrow f_{2324}^{(R)}$, 15 $\leftrightarrow f_{1424}^{(R)}$, 16 $\leftrightarrow f_{1234}^{(R)}$, 17 $\leftrightarrow f_{1334}^{(R)}$, 18 $\leftrightarrow f_{2334}^{(R)}$, 19 $\leftrightarrow f_{1434}^{(R)}$, 20 $\leftrightarrow f_{2434}^{(R)}$. 
Next, we would like to give the computer code used to determine the constants for the singular solution for Kasner spacetime. The expressions for the constants $C_4,\ldots,C_{12}$ given in section III.C above can also be found in [22]. They will be used as input for the REDUCE code which determines $C_1, C_2, C_3$ completely in terms of $p_1, p_2, p_3$. This code is given by:

\[
\text{\%Constants for a singular solution for the Riemann-Lanczos problem for Kasner spacetime:} \\
\text{\%p1+p2+p3=1; p1**2+p2**2+p3**2=1;} \\
\text{\%First we solve equations (1) to (3) and (7) to (12) and obtain:} \\
C4:=C1/(p1-1)-p1/2; \\
C5:=C2/(p2-1)-p2/2; \\
C6:=C3/(p3-1)-p3/2; \\
C7:=(C2*p1/(2*p3+2*p2+p1)); \\
C8:=(C3*p1/(2*p3+2*p2+p1)); \\
C9:=(C3*p2/(2*p3+2*p1+p2)); \\
C10:=(-C1*p2/(2*p1+2*p3+p2)); \\
C11:=(-C1*p3/(p3+2*p2+2*p1)); \\
C12:=(-C2*p3/(p3+2*p2+2*p1)); \\
\text{\%We write (4), (5) and (6) for the remaining C1, C2 and C3 and solve:} \\
s4:=2*C7-2*C10+2*p2*C4+2*p1*C5+p1*p2; \\
s5:=2*C8-2*C11+2*p3*C4+2*p1*C6+p1*p3; \\
s6:=2*C9-2*C12+2*p3*C5+2*p2*C6+p2*p3; \\
solve(s4=0,C1); \\
C1:=rhs first ws; \\
solve(s5=0,C2); \\
C2:=rhs first ws; \\
solve(s6=0,C3); \\
C3:=rhs first ws; \\
\text{off nat, nero, echo;} \\
\text{out "H:/constants";} \\
C1:=C1; C2:=C2; C3:=C3; \\
shut "H:/constants"; \\
on nat, nero, echo; \\
end; \\
\]

The computer code above produces an output file called “constants” with the constants $C_1, C_2$ and $C_3$ expressed in terms of the $p_i$.

1 The term torsion in the context of EDS has nothing to do with the torsion occurring in the theory of affine connections.

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