GLOBAL DYNAMICS OF COMPETITION MODELS WITH NONSYMmetric NONLOCAL DISPERSALS WHEN ONE DIFFUSION RATE IS SMALL

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ABSTRACT. In this paper, we study the global dynamics of a general 2 × 2 competition models with nonsymmetric nonlocal dispersal operators. Our results indicate that local stability implies global stability provided that one of the diffusion rates is sufficiently small. This paper extends the work in [3], where Lotka-Volterra competition models with symmetric nonlocal operators are considered, to more general competition models with nonsymmetric operators.

1. Introduction. In this paper, we mainly consider 2×2 reaction diffusion systems with nonlocal diffusion of the following type:

\[
\begin{align*}
&u_t = dK[u] + uf(x, u, v) \quad \text{in } \Omega \times [0, \infty), \\
v_t = D\mathcal{P}[v] + vg(x, u, v) \quad \text{in } \Omega \times [0, \infty), \\
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{in } \Omega,
\end{align*}
\]

(1)

where \(d, D > 0\), \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n \geq 1\), \(K\) and \(\mathcal{P}\) represent nonlocal operators, which will be defined later.

When nonlocal operators are replaced by differential operators, two-component systems like (1) allow for a large range of possible phenomena in chemistry, biology, ecology, physics and so on and have been extensively studied. One of the most famous phenomena is the idea, which was first proposed by Alan Turing, that a stable state in the local system can become unstable in the presence of diffusion. This remarkable idea is called “diffusion-driven instability”, which is one of the most classical theories in the studies of pattern formations. Another important phenomena comes from ecology, where random diffusion is introduced to model dispersal strategies [26] and there are tremendous studies in this direction, see the

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Indeed, dispersal strategies of organisms have been a central topic in ecology. However, in ecology, in many situations (e.g. [4, 7, 8, 25]), dispersal is better described as a long range process rather than as a local one, and integral operators appear as a natural choice. A commonly used form that integrates such long range dispersal in ecology is the following nonlocal diffusion operator [1, 11, 12, 14, 19, 20, 21, 22, 24]:

\[
\mathcal{L}u := \int_{\Omega} k(x, y)u(y)dy - a(x)u(x),
\]

where the dispersal kernel \(k(x, y) \geq 0\) describes the probability to jump from one location to another.

To be more specific, this paper is motivated by the studies of Lotka-Volterra type weak competition models with spatial inhomogeneity, that is

\[
f(x, u, v) = m(x) - u - cv, \quad g(x, u, v) = m(x) - bu - v
\]

with \(0 < bc < 1\) in (1). For this case, \(u(t, x), v(t, x)\) are the population densities of two competing species, \(d, D > 0\) are their dispersal rates, which measure the total number of dispersal individuals per unit time, respectively, while \(m(x)\) is nonconstant and represents spatial distribution of resources. This type of models reflects the interactions among dispersal strategies, spatial heterogeneity of resources and interspecific competition abilities on the persistence of species and has received extensive studies from both mathematicians and ecologists for the last three decades. For models with random diffusion, see [5, 9, 13, 17, 18] and the references therein, while for models with nonlocal dispersals, see [2, 3, 10, 16] and the references therein.

Inspired by the nature of this type of models, in [18], an insightful conjecture was proposed and partially verified:

**Conjecture A.** *The locally stable steady state is globally asymptotically stable.*

For symmetric PDE case, this conjecture has been completely resolved in [9]. Moreover, if random diffusion is replaced by symmetric nonlocal operators or symmetric mixed dispersal strategies, this conjecture is also verified in [3]. In the proofs of these results, the symmetry property of operators and the particular form of reaction terms are crucial. This naturally leads us to investigate the system (1) with nonsymmetric operators and more general reaction terms.

To better present the main results, let us designate the definitions of nonlocal operators in (1) and assumptions imposed on \(f, g\). Denote

\[
\mathcal{X} = C(\bar{\Omega}), \quad \mathcal{X}^+ = \{u \in \mathcal{X} \mid u \geq 0\}, \quad \mathcal{X}^{++} = \mathcal{X}^+ \setminus \{0\}.
\]

For \(\phi \in \mathcal{X}\), define

\[
(D) \quad \mathcal{K}[\phi] = \int_{\Omega} k(x, y)\phi(y)dy - \phi(x), \quad \mathcal{P}[\phi] = \int_{\Omega} p(x, y)\phi(y)dy - \phi(x),
\]

\[
(N) \quad \mathcal{K}[\phi] = \int_{\Omega} k(x, y)\phi(y)dy - \int_{\Omega} k(y, x)d\phi(x), \quad \mathcal{P}[\phi] = \int_{\Omega} p(x, y)\phi(y)dy - \int_{\Omega} p(y, x)d\phi(x),
\]

where the kernels \(k(x, y), p(x, y)\) describe the rate at which organisms move from point \(y\) to point \(x\). Here the operators defined in (D) and (N) correspond to nonlocal operator with lethal boundary condition and no flux boundary condition respectively. See [11] for the derivation of different types of nonlocal operators.

Throughout this paper, unless designated otherwise, we assume that
(A0) $k(x, y), p(x, y) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ are nonnegative. $k(x, x) > 0, p(x, x) > 0$ in $\mathbb{R}^n$. Moreover, $\int_{\mathbb{R}^n} k(x, y) dy = \int_{\mathbb{R}^n} k(y, x) dy = \int_{\mathbb{R}^n} p(x, y) dy = \int_{\mathbb{R}^n} p(y, x) dy = 1.$

(A1) $f(x, u, v), g(x, u, v), f_u(x, u, v), f_v(x, u, v), g_u(x, u, v), g_v(x, u, v) \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+).$

(A2) $f_u < 0, g_u < 0$ in $\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+.$

(A3) $f_v < 0, g_u < 0$ in $\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+.$

(A4) There exists $M > 0$ such that $f(x, u, v) < 0$ for $u \geq M, g(x, u, v) < 0$ for $v \geq M.$

(A5) $f_v g_u < f_u g_v$ in $\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+.$

In (A0), the assumption $k(x, x), p(x, x) > 0$ corresponds to the strict ellipticity condition for differential operators, which guarantees strong maximum principle. Notice that the reaction terms $f(x, u, v) = m(x) - u - cv, g(x, u, v) = m(x) - bu - v$ satisfy (A2)-(A4) provided that $m \in L^\infty(\Omega)$ and the assumption (A5) corresponds to $0 < bc < 1.$ Moreover, different from PDE case, for models with nonlocal operators, the optimal regularity of solutions is at most the same as the regularity of the reaction terms. Hence, (A1) is imposed to guarantee that the solutions could be continuous in space variable.

For clarity, let $(U(x), V(x))$ denote a nonnegative steady state of (1), then there are at most three possibilities:

- $(U, V) = (0, 0)$ is called a trivial steady state.
- $(U, V) = (\theta_d, 0)$ or $(U, V) = (0, \eta_D)$ is called a semi-trivial steady state, where $\theta_d, \eta_D$ are the positive solutions to single-species models
  
  $dK[U] + Uf(x, U, 0) = 0,$

  
  $Dp[V] + Vg(x, 0, V) = 0$

  respectively.
- $U > 0, V > 0,$ and we call $(U, V)$ a coexistence/positive steady state.

The main result in this paper gives a classification of the global dynamics to the competition system (1) under the assumptions (A0)-(A5) provided that one diffusion rate is small.

**Theorem 1.1.** Assume that (A0)-(A5) hold. Also assume that (1) admits two semi-trivial steady states $(\theta_d, 0)$ and $(0, \eta_D).$ Then for the global dynamics of the system (1) with nonlocal operators defined in (D) or (N), we have the following statements provided that $d$ is sufficiently small:

- (i) If $\mu_0 > 0, \nu^D_0 > 0$ and in addition $k(x, y) > 0$ for $x, y \in \bar{\Omega},$ then the system (1) admits a unique positive steady state in $X \times X,$ which is globally asymptotically stable relative to $X_{++} \times X_{++};$
- (ii) If $\mu_0 > 0$ and $\nu^D_0 < 0,$ then $(0, \eta_D)$ is globally asymptotically stable relative to $X_{++} \times X_{++};$
- (iii) If $\mu_0 < 0,$ then $(\theta_d, 0)$ is globally asymptotically stable relative to $X_{++} \times X_{++}.$

Here $\mu_0$ and $\nu^D_0$ are defined in (12).

Based on the definitions of $\mu_0$ and $\nu^D_0$ defined in (12) and Lemma 2.3, indeed Theorem 1.1 verifies Conjecture A for more general reaction-diffusion systems provided that one diffusion rate is small.

To prove Theorem 1.1, the min-max and max-min characterizations of nonlocal eigenvalue problems obtained in [15] play an important role.
In particular, the proof of Theorem 1.1(i) is much more complicated. It is motivated by the techniques originally developed in the proof of [2, Theorem 1.1], where a simplified version of nonlocal operators
\[ \mathcal{K}[\phi] = \mathcal{P}[\phi] = \frac{1}{|\Omega|} \int_{\Omega} \phi(x)dx - \phi(x) \]
is considered.

We need first investigate the properties of the limiting system to the system (1) as \( d \to 0^+ \). To be more specific, notice that due to (A2) and implicit function theorem, there exists \( u = F(x, v) \in C(\bar{\Omega} \times \mathbb{R}_+) \) with \( F_v(x, v) \in C(\bar{\Omega} \times \mathbb{R}_+) \) such that
\[ f(x, F(x, v), v) = 0 \text{ in } \bar{\Omega} \times \mathbb{R}_+, \]
and
\[ F_v(x, v) = -\frac{f_v(x, u, v)}{f_u(x, u, v)} \text{ in } \bar{\Omega} \times \mathbb{R}_+. \] (4)
Also denote \( F_+(x, v) = \max\{0, F(x, v)\} \) in \( \bar{\Omega} \times \mathbb{R}_+ \). Then formally we derive the limiting system of the stationary problem of the system (1) as \( d \to 0^+ \) as follows
\[ \begin{cases} u = F_+(x, v), \\ D\mathcal{P}[v] + vg(x, u, v) = 0, \end{cases} \] (5)
and characterize the existence and uniqueness of solutions \((u, v)\) with \( v \) being positive to this system.

**Theorem 1.2.** Assume that (A0)-(A5) hold. The system (5) admits a unique solution \((U_0, V_0)\) with \( V_0 \) being positive if and only if \( \mu_0 > 0 \). Moreover, \( u \equiv 0 \) if and only if \( \nu_0 \geq 0 \). Here \( \mu_0 \) and \( \nu_0 \) are defined in (12).

As demonstrated in Theorem 2.5, to establish Theorem 1.1(i), the key point is to verify the uniqueness of positive steady states, if exists, to the system (1) when \( d \) is sufficiently small. According to Theorem 1.2, it is naturally expected that this property holds for \( d \) small. However, it is highly nontrivial to realize it mathematically since it is possible that multiple positive steady states to the system (1) converge to the same solution of the limiting system (5) as \( d \to 0^+ \). To exclude this possibility, we need more precise characterization of the asymptotic behavior of the positive steady states of the system (1) as \( d \to 0^+ \).

**Theorem 1.3.** Suppose that (A0)-(A5) hold and \((u_d, v_d)\) is a positive solution of the stationary problem of the system (1)
\[ \begin{cases} d\mathcal{K}[u] + uf(x, u, v) = 0 \text{ in } \Omega, \\ D\mathcal{P}[v] + vg(x, u, v) = 0 \text{ in } \Omega, \end{cases} \] (6)
If \( \mu_0 > 0 \), there exist \( C_1, C_2 > 0 \), independent of \( d \), such that
\[ (U_0 - C_1d)_+ < u_d < U_0 + C_1\sqrt{d}, \quad (V_0 - C_2\sqrt{d})_+ < v_d < V_0 + C_2d \quad \text{ in } \bar{\Omega}, \]
where \((U_0, V_0)\) is the unique solution to (5) with \( V_0 \) being positive.

Obviously Theorem 1.3 indicates that indeed the system (5) is the limiting system of the system (6) as \( d \to 0^+ \). On the basis of Theorem 1.3, we make use of the proof by contradiction and some thorough estimates to derive the uniqueness of positive steady states, if exists, to the system (1) for \( d \) small. During this process, it seems that the extra condition that \( k(x, y) > 0 \) for \( x, y \in \bar{\Omega} \) in Theorem 1.1(i) is imposed.
for technical reasons. However, it remains unknown that whether the complexity of nonsymmetric operators could result in multiple positive steady states under the assumption (A5). We will return to this topic in future work.

In this paper, we only demonstrate the proof of Theorem 1.1 for the nonlocal operators defined in (D), since the proof for nonlocal operators defined in (N) is almost the same.

This paper is organized as follows. In Section 2, some useful properties related to single equations, nonlocal eigenvalue problems and monotone systems are prepared. Then the limiting system and the asymptotic behavior of the positive steady states of the system (1) as $d \to 0^+$ is investigated in Section 3. At the end, Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminary. Throughout this paper, when we discuss the spectrum of linear operators, we always think of them as operators from $X$ to $X$.

2.1. Properties of single equations. It is known that under assumptions (A0), (A1), (A2) and (A4), the single equation (2) admits a unique positive solution in $X$, denoted by $\theta_d$, if and only if

$$\sup \{ \text{Re } \lambda \, | \, \lambda \in \sigma(\text{d}K + f(x, 0, 0)) \} > 0,$$

while the single equation (3) admits a unique positive solution in $X$, denoted by $\eta_D$, if and only if

$$\sup \{ \text{Re } \lambda \, | \, \lambda \in \sigma(\text{DP} + g(x, 0, 0)) \} > 0.$$

See [3, Theorem 2.1] for details.

For further analysis, we need estimate the asymptotic behavior of $\theta_d$ as $d \to 0^+$.

**Proposition 1.** There exists $C_3 > 0$, independent of $d$, such that

$$(F_+(x, 0) - C_3d)_+ < \theta_d < F_+(x, 0) + C_3\sqrt{d} \text{ in } \Omega$$

for $0 < d < 1$.

**Proof.** First of all, by (A4) and strong maximum principle, it is easy to show that $0 < \theta_d < M$. Then due to (A2), there exists $c_0$ such that

$$f_u(x, u, 0) \leq -c_0 \text{ for } x \in \bar{\Omega}, u \in [0, M].$$

On the one side, set $\hat{u} = F_+(x, 0) + c_1\sqrt{d}$, $c_1 > 0$. We compute as follows

$$dK[\hat{u}] + \hat{u}f(x, \hat{u}, 0)$$

$$= d \int_{\Omega} k(x, y)(F_+(y, 0) + c_1\sqrt{d})dy - d(F_+(x, 0) + c_1\sqrt{d})$$

$$+ (F_+(x, 0) + c_1\sqrt{d})f(x, F_+(x, 0) + c_1\sqrt{d}, 0)$$

$$\leq d \int_{\Omega} k(x, y)(F_+(y, 0) + c_1\sqrt{d})dy$$

$$+ (F_+(x, 0) + c_1\sqrt{d}) \left( f(x, F_+(x, 0) + c_1\sqrt{d}, 0) - f(x, F_+(x, 0), 0) \right)$$

$$\leq d \int_{\Omega} k(x, y)(F_+(y, 0) + c_1\sqrt{d})dy - c_0c_1\sqrt{d}(F_+(x, 0) + c_1\sqrt{d})$$

$$< 0,$$

provided that $c_1$ is large enough.
On the other side, set \( u = (F_+(x, 0) - c_2d)_+ \). Then compute as follows
\[
dK[\phi] + uf(x, u, 0) \\
\geq -d(F_+(x, 0) - c_2d)_+ + (F_+(x, 0) - c_2d)_+ f(x, (F_+(x, 0) - c_2d)_+, 0) \\
= -d(F_+(x, 0) - c_2d)_+ \\
+ (F_+(x, 0) - c_2d)_+ (f(x, F_+(x, 0) - c_2d, 0) - f(x, F_+(x, 0), 0)) \\
\geq 0,
\]
provided that \( c_2 \) is large enough.

At the end, choose \( C_3 = \max\{c_1, c_2\} \) and the desired conclusion follows.

2.2. Properties of nonlocal eigenvalue problems. First of all, the linearized operator of (1) at \((\theta_d, 0)\) is
\[
\mathcal{L}(\theta_d,0) \left( \phi \over \psi \right) = \left( \begin{array}{c}
dK[\phi] + [f(x, \theta_d, 0) + \theta_df_u(x, \theta_d, 0)]\phi + \theta_df_v(x, \theta_d, 0)\psi \\
DP[\psi] + g(x, \theta_d, 0)\psi \end{array} \right).
\]
(7)
Also, the linearized operator of (1) at \((0, \eta_D)\) is
\[
\mathcal{L}(0,\eta_D) \left( \phi \over \psi \right) = \left( \begin{array}{c}
dK[\phi] + f(x, 0, \eta_D)\phi \\
DP[\psi] + [g(x, 0, \eta_D) + \eta_Dg_u(x, 0, \eta_D)]\psi + \eta_Dg_v(x, 0, \eta_D)\phi \end{array} \right).
\]
(8)
Denote
\[
\mu(\theta_d,0) = \sup \left\{ \Re \lambda \mid \lambda \in \sigma(DP + g(x, \theta_d, 0)) \right\},
\]
(9)
\[
\nu(0,\eta_D) = \sup \left\{ \Re \lambda \mid \lambda \in \sigma(dK + f(x, 0, \eta_D)) \right\}.
\]
(10)

It is known that the signs of \( \mu(\theta_d,0) \) and \( \nu(0,\eta_D) \) determine the local stability/instability of \((\theta_d, 0)\) and \((0, \eta_D)\) respectively. This is explicitly stated as follows and the proof is omitted since it is standard.

**Lemma 2.1.** Assume that the assumptions \((A0), (A1)\) hold. Then
(i) \((\theta_d, 0)\) is locally unstable if \( \mu(\theta_d,0) > 0 \); \((\theta_d, 0)\) is locally stable if \( \mu(\theta_d,0) < 0 \).
(ii) \((0, \eta_D)\) is locally unstable if \( \nu(0,\eta_D) > 0 \); \((0, \eta_D)\) is locally stable if \( \nu(0,\eta_D) < 0 \).

**Remark 1.** \((\theta_d, 0)\) is called neutrally stable if \( \mu(\theta_d,0) = 0 \), while \((0, \eta_D)\) is called neutrally stable if \( \nu(0,\eta_D) = 0 \).

The characterization of \( \mu(\theta_d,0) \) and \( \nu(0,\eta_D) \) plays an important role in this paper. We present the related results from [15] as follows for the convenience of readers.

**Theorem 2.2** ([15]). Assume that \( \gamma \in \mathcal{X} \) and let
\[
\lambda_1(d) = \sup \left\{ \Re \lambda \mid \lambda \in \sigma(dK + \gamma(x)) \right\}.
\]
Then we have
\[
\lambda_1(d) = \sup_{\phi \in \mathcal{X}, \phi > 0 \text{ on } \Omega} \inf_{x \in \Omega} \frac{dK[\phi] + \gamma\phi}{\phi} = \inf_{\phi \in \mathcal{X}, \phi > 0 \text{ on } \Omega} \sup_{x \in \Omega} \frac{dK[\phi] + \gamma\phi}{\phi}.
\]
(10)

In particular,
\[
\lim_{d \to 0^+} \lambda_1(d) = \max_{x \in \Omega} \gamma(x).
\]
(11)
Proof. Indeed, (10) is proved in [15] and we only explain how to derive (11). Note that
\[
\lambda_1(d) = \inf_{\phi \in \mathbb{X}, \phi > 0 \text{ on } \Omega} \sup_{x \in \Omega} \frac{dK[\phi] + \gamma(x)\phi(x)}{\phi(x)}.
\]
Then it is easy to see that
\[
\lim_{d \to 0^+} \lambda_1(d) \leq \lim_{d \to 0^+} \sup_{x \in \Omega} (dK[1] + \gamma(x)) = \max_{x \in \Omega} \gamma(x).
\]
Moreover,
\[
\lim_{d \to 0^+} \lambda_1(d) \geq \lim_{d \to 0^+} \sup_{x \in \Omega} (-d + \gamma(x)) = \max_{x \in \Omega} \gamma(x).
\]
It is proved.

Now we prepare two simple properties, which will be repeatedly used in future.

Lemma 2.3. Both \(\lim_{d \to 0^+} \mu(\theta_d, 0)\) and \(\lim_{d \to 0^+} \nu(0, \eta_D)\) exist and denote
\[
\mu_0 = \lim_{d \to 0^+} \mu(\theta_d, 0), \quad \nu_D^0 = \lim_{d \to 0^+} \nu(0, \eta_D).
\]
Moreover,
\[
\mu_0 = \sup \{ Re \lambda | \lambda \in \sigma(D\mathcal{P} + g(x, F_+(x, 0), 0)) \}, \quad \nu_D^0 = \max_{x \in \Omega} f(x, 0, \eta_D). \tag{12}
\]

This lemma follows directly from Proposition 1 and Theorem 2.2 and we omit the details of its proof.

Lemma 2.4. Assume that \(\gamma \in \mathbb{X}\) and let \(\mathcal{L} = d\mathcal{K} + \gamma(x)\). If there exists an eigenpair \((\lambda, \phi)\) of \(\mathcal{L}\) with \(\phi > 0\) in \(\bar{\Omega}\), then \(\lambda = \sup \{ Re \lambda | \lambda \in \sigma(\mathcal{L}) \}\).

Notice that indeed Lemma 2.4 is different from [15, Theorem 2.1(iv)]. Thus we prove it for the convenience of readers.

Proof of Lemma 2.4. Denote \(\lambda_1 = \sup \{ Re \lambda | \lambda \in \sigma(\mathcal{L}) \}\). Obviously, \(\lambda \leq \lambda_1\). Suppose that \(\lambda < \lambda_1\). Then by Theorem 2.2, there exists \(\phi_1 \in \mathbb{X}\) with \(\phi_1 > 0\) in \(\Omega\) such that
\[
\inf_{x \in \bar{\Omega}} \frac{\mathcal{L}[\phi_1]}{\phi_1} > \lambda \quad \text{in } \bar{\Omega},
\]
i.e., \(\mathcal{L}[\phi_1] - \lambda \phi_1 > 0\) in \(\bar{\Omega}\). By Touching Lemma ([15, Lemma 2.1]) with \(\mathcal{L}_1\) replaced by \(-\mathcal{L} + \lambda, u = \phi, v = -\phi_1\), we have \(\phi\) is a constant multiple of \(\phi_1\) and \(\mathcal{L}[\phi_1] - \lambda \phi_1 = 0\) in \(\bar{\Omega}\), which is a contradiction. \(\square\)

2.3. Properties of monotone systems. The following result explains how to characterize the global dynamics of the competition model (1) with two semi-trivial steady states.

Theorem 2.5. Assume that the assumptions (A0)-(A4) hold and the system (1) admits two semi-trivial steady states, denoted by \((\theta_d, 0)\) and \((0, \eta_D)\). We have the following three possibilities:

(i) If both \(\mu(\theta_d, 0) > 0\) and \(\nu(0, \eta_D) > 0\), the system (1) at least has one positive steady state in \(L^\infty(\Omega) \times L^\infty(\Omega)\). If in addition, assume that the system (1) has a unique positive steady state in \(\mathbb{X} \times \mathbb{X}\), then it is globally asymptotically stable relative to \(\mathbb{X}_{++} \times \mathbb{X}_{++}\).

(ii) If \(\mu(\theta_d, 0) > 0\) and no positive steady states of the system (1) exist, then the semi-trivial steady state \((0, \eta_D)\) is globally asymptotically stable relative to \(\mathbb{X}_{++} \times \mathbb{X}_{++}\).
subsection is to study the existence and uniqueness of solution \((u,v)\) limit system and asymptotic behavior as \(3\).

\[ \begin{align*}
3.1. & \quad \text{Existence and uniqueness of limiting system.} \\
& \quad \text{The arguments are almost the same as that of [2, Theorem 2.1], where a}
\end{align*} \]

\[ \text{simplified nonlocal operator is considered.} \]

\[ \text{Hugh of} \quad \text{Lemma 2.6. Assume that (A0) holds and} f_v g_u \leq f_u g_v \quad \text{in} \quad \Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \]

\[ \text{Then any positive steady state of the system (1) in} \quad L^\infty(\Omega) \times L^\infty(\Omega) \quad \text{belongs to} \quad \mathbb{X} \times \mathbb{X}. \]

\[ \text{This lemma can be verified easily by applying implicit function theorem and we}
\]

\[ \text{omit the details.} \]

3. Limiting system and asymptotic behavior as \(d \to 0^+\). In this section, we

\[ \text{will investigate properties of the limiting system (5) and characterize the}
\]

\[ \text{asymptotic behavior of the solutions of (6) as} \quad d \to 0^+. \]

3.1. Existence and uniqueness of limiting system. The main purpose in this

\[ \text{subsection is to study the existence and uniqueness of solution} \quad (u,v) \quad \text{with} \quad v \quad \text{being}
\]

\[ \text{positive to the system (5) and establish Theorem 1.2.} \]

\[ \text{Notice that the system (5) can be rewritten as}
\]

\[ \begin{align*}
& \quad \text{DP}\{v\} + vg(x, F_v(x,v), v) = 0. \quad (13) \\
\end{align*} \]

\[ \text{First of all, we need establish a property of} \quad g(x, F_v(x,v), v), \quad \text{which will be used}
\]

\[ \text{repeatedly throughout this paper.} \]

\[ \text{Lemma 3.1. Assume that (A2), (A3) and (A5) hold. Then} \quad g(x, F_v(x,v), v) \quad \text{is}
\]

\[ \text{strictly decreasing in} \quad v \geq 0. \quad \text{Moreover, for any fixed} \quad M_1 > 0, \quad \text{there exists} \quad \sigma_1 = \sigma_1(M_1) > 0 \quad \text{such that for} \quad 0 \leq v < v_1 \leq M_1 \]

\[ g(x, F_v(x,v), v) - g(x, F_v(x,v), v) \leq -\sigma_1(v_1 - v). \]

\[ \text{Proof. Notice that due to (A2), (A3) and (4),} \quad F_v(x,v) \quad \text{is strictly decreasing in}
\]

\[ v \geq 0. \quad \text{Thus we only need discuss the following three situations.} \]

\[ \begin{align*}
& \quad \text{In} \quad \{x \in \Omega \mid F_v(x,v_1) \geq 0, F(x,v) > 0\}, \\
& \quad g(x, F_v(x,v_1), v_1) - g(x, F_v(x,v), v) \\
& = \quad g(x, F_v(x,v_1), v_1) - g(x, F_v(x,v), v) \\
& \leq (v_1 - v) \left[ -g_u(x, F(x,\alpha), \alpha) \frac{f_u(x, F(x,\alpha), \alpha)}{f_u(x, F(x,\alpha), \alpha)} + g_v(x, F(x,\alpha), \alpha) \right] \\
& \leq -c_1(v_1 - v),
\end{align*} \]

\[ \text{for some} \quad c_1 > 0 \quad \text{due to (A5), where} \quad 0 \leq v \leq \alpha \leq v_1 \leq M_1. \]

\[ \begin{align*}
& \quad \text{In} \quad \{x \in \Omega \mid F_v(x,v_1) < 0, F(x,v) \geq 0\}, \\
& \quad g(x, F_v(x,v_1), v_1) - g(x, F_v(x,v), v) \\
& = \quad g(x, F_v(x,v_1), v_1) - g(x, F_v(x,v), v) \\
& \leq (v_1 - v) \left[ -g_u(x, F(x,\alpha), \alpha) \frac{f_u(x, F(x,\alpha), \alpha)}{f_u(x, F(x,\alpha), \alpha)} + g_v(x, F(x,\alpha), \alpha) \right] \\
& \leq -c_2(v_1 - v),
\end{align*} \]
for some $c_2 > 0$ due to (A5), where $0 \leq v \leq \alpha \leq v_1 \leq M_1$.

- In $\{x \in \Omega \mid F(x, v_1) < 0, F(x, v) < 0\}$,
  
  \[
g(x, F_+(x, v_1), v_1) - g(x, F_+(x, v), v) = g(x, 0, v_1) - g(x, 0, v) = (v_1 - v)g_v(x, 0, \zeta) \leq -c_3(v_1 - v)
  \]

  for some $c_3 > 0$ due to (A5), where $0 \leq v \leq \zeta \leq v_1 \leq M_1$.

Set $\sigma_1 = \min\{c_1, c_2, c_3\}$ and the proof is complete.

Next, we prepare the comparison principle for (13).

**Lemma 3.2.** Suppose that (A0)-(A3) and (A5) hold, $v^*, v \in \mathbb{R}$ are nonnegative, $v^* \neq 0$, $v \neq 0$ and $v^*, v$ satisfy the following inequalities respectively

\[
D\mathcal{P}[v] + vg(x, F_+(x, v), v) \geq 0 \text{ in } \Omega,
\]

\[
D\mathcal{P}[v^*] + v^*g(x, F_+(x, v^*), v^*) \leq 0 \text{ in } \Omega.
\]

Then either $v^* > v$ or $v^* = v$ in $\Omega$.

**Proof.** First, we claim that $v^* > 0$ in $\Omega$. Let $\Omega_0 = \{x \in \bar{\Omega} \mid v^* = 0\}$. Suppose that the claim is not true, then $\Omega_0 \neq \emptyset$. Obviously, $\Omega_0$ is closed in $\bar{\Omega}$. Choose any $x_1 \in \Omega_0$, then at $x = x_1$,

\[
0 \geq D\mathcal{P}[v^*](x_1) + v^*(x_1)g(x_1, F_+(x_1, v^*(x_1)), v^*(x_1))
\]

\[
= D\int_{\Omega} p(x_1, y)v^*(y)dy.
\]

Due to (A0), this implies that $v^* = 0$ in a small neighborhood of $x_1$ in $\bar{\Omega}$. Thus $\Omega_0$ is open in $\bar{\Omega}$. Hence $\Omega_0 = \bar{\Omega}$, which is a contradiction to $v^* \neq 0$. The claim is proved.

Now since $v^* > 0$ in $\bar{\Omega}$, we have $\ell v^* > v$ in $\bar{\Omega}$ for $\ell$ large. Define

\[
\ell_* = \inf \{\ell \mid \ell v^* > v \text{ in } \bar{\Omega}\}.
\]

It is clear that $\ell_* > 0$ since $v \neq 0$.

We will further prove that $\ell_* \leq 1$. Suppose that $\ell_* > 1$. Let $z = \ell_* v^* - v$. It is clear that $z \geq 0$ and there exists $x_2 \in \bar{\Omega}$ such that $z(x_2) = 0$. Thus by Lemma 3.1, direct computation yields that

\[
D\mathcal{P}[z] = \ell_* D\mathcal{P}[v^*] - D\mathcal{P}[v]
\]

\[
\leq -\ell_* v^* g(x, F_+(x, v^*), v^*) + vg(x, F_+(x, v), v)
\]

\[
< -\ell_* v^* g(x, F_+(x, \ell_* v^*), \ell_* v^*) + vg(x, F_+(x, \ell_* v^*), v)
\]

\[
= -g(x, F_+(x, \ell_* v^*), \ell_* v^*)z - g(x, F_+(x, \ell_* v^*), \ell_* v^*)z - g(x, F_+(x, \ell_* v^*), v)z.
\]

Hence at $x = x_2$, we have

\[
D\int_{\Omega} p(x_2, y)z(y)dy < 0,
\]

which is impossible. Therefore, $0 < \ell_* \leq 1$.

At the end, if $\ell_* < 1$, then $v^* > \ell_* v^* \geq v$ in $\bar{\Omega}$. If $\ell_* = 1$, then similar to the arguments in the proof of the claim, by using (14), it follows that $z = v^* - v \equiv 0$ is the only possibility. The proof is complete.

Now, the existence and uniqueness of positive solutions to (13) will be characterized as follows.
Proposition 2. Assume that (A0)–(A5) hold. The problem (13) admits a unique positive solution $V_0(x)$ in $\mathbb{X}$ if and only if $\mu_0 > 0$, where $\mu_0$ is defined in (12).

Proof. On the one hand, assume that $V_0(x)$ is a positive solution to (13). Then

$$
\mu_0 = \sup \left\{ \Re \lambda \mid \lambda \in \sigma(\mathcal{D}\mathcal{P} + g(x, F_+(x, 0), 0)) \right\} = \sup \inf_{\phi \in \mathbb{X}, \phi > 0 \text{ on } \bar{\Omega}} \frac{D\mathcal{P}[\phi] + g(x, F_+(x, 0), 0)\phi(x)}{\phi(x)} \geq \inf_{x \in \Omega} \frac{D\mathcal{P}[V_0] + g(x, F_+(x, 0), 0)V_0(x)}{V_0(x)} = \inf_{x \in \Omega} [-g(x, F_+(x, V_0), V_0) + g(x, F_+(x, 0), 0)] > 0,
$$

since $g(x, F_+(x, v), v)$ is strictly decreasing in $v > 0$ due to (A2), (A3), (A5) and (4).

On the other hand, assume that $\mu_0 > 0$. We first point out that $g(x, F_+(x, \alpha), v)$ is strictly increasing in $\alpha > 0$ due to (A2), (A3) and (4). This simple property will be used repeatedly for the rest of the proof.

By [3, Theorem 2.1], $\mu_0 > 0$ implies that

$$
D\mathcal{P}[v] + vg(x, F_+(x, 0), v) = 0 \quad (15)
$$

admits a unique positive solution, denoted by $V_1 \in \mathbb{X}$. For $k \geq 2$, let $V_k$ denotes the unique positive solution to

$$
D\mathcal{P}[v] + vg(x, F_+(x, V_{k-1}), v) = 0. \quad (16)
$$

Note that due to (A2), (A3) and (4)

$$
\sup \left\{ \Re \lambda \mid \lambda \in \sigma(\mathcal{D}\mathcal{P} + g(x, F_+(x, V_{k-1}), 0)) \right\} \geq \mu_0 > 0.
$$

Thus the existence and uniqueness of $V_k$ is guaranteed by [3, Theorem 2.1].

Moreover, due to (A2), (A3) and (4) again, it is routine to show that

$$
0 < V_1 < \ldots < V_k < \eta_D, \quad k \geq 1.
$$

Hence $V = \lim_{k \to \infty} V_k$ is a positive solution of (13) in $L^\infty(\Omega)$. Moreover, due to Lemma 3.1, it is standard to verify that $V \in \mathbb{X}$ by applying implicit function theorem.

At the end, suppose that (13) has another positive solution $V^* \in \mathbb{X}$ and $V^* \neq V$. First, based on the equations (13) and (15), one sees that $V_1 \leq V^*$ due to (A2), (A3) and (4). Then by the equations (13) and (16), (A2), (A3) and (4) yield that $V_k \leq V^*$, $k = 2, 3, \ldots$, which implies that $V \leq V^*$. Thanks to Lemma 3.2, we have $V < V^*$ in $\Omega$.

Define

$$
\ell_1 = \inf \left\{ \ell \mid \ell V > V^* \text{ in } \bar{\Omega} \right\}.
$$

Obviously, $\ell_1 > 1$ since $V < V^*$ in $\Omega$. Then thanks to Lemma 3.1,

$$
D\mathcal{P}[\ell_1 V] + \ell_1 Vg(x, F_+(x, \ell_1 V), \ell_1 V) < D\mathcal{P}[\ell_1 V] + \ell_1 Vg(x, F_+(x, V), V) = 0. \quad (17)
$$

Again, by Lemma 3.2, we have either $\ell_1 V \equiv V^*$ or $\ell_1 V > V^*$ in $\bar{\Omega}$. The former case contradicts to (17), while the latter case is a contradiction to the definition of $\ell_1$. Therefore, the positive solution of (13) is unique. The proof is complete. \qed

Finally, we are ready to demonstrate the main result in this subsection.
Proof of Theorem 1.2. Notice that if \( V_0 \) is a positive solution of (13), then \((u, v) = (F_+(x, V_0), V_0)\) is a solution of the limiting system (5). Thus it is clear that the first part is equivalent to Proposition 2.

It remains to verify that \( u \equiv 0 \) if and only if \( \nu_D^0 \leq 0 \). This is obvious since \( u \equiv 0 \) if and only if \( v = \eta_D \).

3.2. Asymptotic behavior of the solutions of (6) as \( d \to 0^+ \). This subsection is devoted to the proof of Theorem 1.3, which indicates that (5) is the limiting system of (6) as \( d \to 0^+ \) by characterizing the asymptotic behavior of the solutions of (6) as \( d \to 0^+ \).

Proof of Theorem 1.3. For clarity, we divide the proof into several steps. Note that \( 0 < u_d, v_d < M \) due to (A4). Then due to (A2), there exists \( c_0 > 0 \) such that

\[
\hat{f}_u(x, u, v) \leq -c_0 \quad \text{for } x \in \hat{\Omega}, u, v \in [0, M + 1].
\]

**Step 1.** Regard \( v_d \) as a given function first and thus \( u_d \) is unique due to (A2). Set \( \hat{u} = F_+(x, v_d) + c_1 \sqrt{d} \) with \( c_1 > 0 \). Then by direct computation, we have

\[
dK[\hat{u}] + \hat{u} f(x, \hat{u}, v_d) \\
\leq d \int_{\Omega} k(x, y) \left( F_+(y, v_d(y)) + c_1 \sqrt{d} \right) dy \\
+ \left( F_+(x, v_d) + c_1 \sqrt{d} \right) f(x, F_+(x, v_d) + c_1 \sqrt{d}, v_d) \\
\leq d \int_{\Omega} k(x, y) \left( F_+(y, v_d(y)) + c_1 \sqrt{d} \right) dy \\
+ \left( F_+(x, v_d) + c_1 \sqrt{d} \right) \left[ f(x, F_+(x, v_d) + c_1 \sqrt{d}, v_d) - f(x, F_+(x, v_d), v_d) \right] \\
\leq d \int_{\Omega} k(x, y) \left( F_+(y, v_d(y)) + c_1 \sqrt{d} \right) dy - c_0 c_1 \sqrt{d} \left( F_+(x, v_d) + c_1 \sqrt{d} \right) \\
< 0,
\]

if \( c_1 \) is large enough and fix it.

Set \( \bar{u} = (F_+(x, v_d) - c_2 d)_+ \) with \( c_2 > 0 \). Then it follows that

\[
dK[\bar{u}] + \bar{u} f(x, \bar{u}, v_d) \\
\geq -d(F_+(x, v_d) - c_2 d)_+ + (F_+(x, v_d) - c_2 d)_+ f(x, (F_+(x, v_d) - c_2 d)_+, v_d) \\
= -d(F_+(x, v_d) - c_2 d)_+ \\
+(F_+(x, v_d) - c_2 d)_+ \left[ f(x, F_+(x, v_d) - c_2 d, v_d) - f(x, F_+(x, v_d), v_d) \right] \\
\geq (F_+(x, v_d) - c_2 d)_+ (-d + c_0 c_2 d) \\
\geq 0
\]

if \( c_2 \) is large enough and fix it.

Now we have derived that

\[
(F_+(x, v_d) - c_2 d)_+ < u_d < F_+(x, v_d) + c_1 \sqrt{d} \quad \text{in } \hat{\Omega}
\]

by upper/lower solution method.

**Step 2.** Let us give a preliminary estimate of \( v_d \). Since \( \mu_0 > 0 \), then similar to the proof of Proposition 2, for \( d \) sufficiently small, the following problem

\[
D\mathcal{P}[v] + v g(x, (F_+(x, v) - c_2 d)_+, v) = 0
\]
admits a unique positive solution, denoted by \( \hat{v} \). Similarly, for \( d \) sufficiently small, the problem
\[
D\mathcal{P}[v] + v g(x, F_+(x, v) + c_1 \sqrt{d}, v) = 0
\]  
(20)
admits a unique positive solution, denoted by \( \underline{v} \). According to (18), it is standard to check that \( v_d \) is a lower solution of (19). Together with the fact that \( \hat{v} \) is the unique solution of (19), one sees that \( v_d < \hat{v} \) in \( \Omega \). Similarly, we have \( \underline{v} < v_d \) in \( \Omega \). Hence
\[
\underline{v} < v_d < \hat{v} \quad \text{in} \quad \Omega.
\]  
(21)

**Step 3.** Now we could improve the estimate of \( v_d \) and relate it to \( V_0 \). On the one side, consider \( (1 + c_3 d)V_0 \) with \( c_3 > 0 \) and compute as follows
\[
D\mathcal{P}[(1 + c_3 d)V_0] + (1 + c_3 d)V_0 g(x, (F_+(x, (1 + c_3 d)V_0) - c_2 d) + (1 + c_3 d)V_0)
\]
\[
= (1 + c_3 d)V_0 [-g(x, F_+(x, V_0), V_0) + g(x, F_+(x, (1 + c_3 d)V_0) - c_2 d) + (1 + c_3 d)V_0)]
\]
\[
\leq (1 + c_3 d)V_0 [-g(x, F_+(x, V_0), V_0) + g(x, F_+(x, (1 + c_3 d)V_0) - c_2 d, (1 + c_3 d)V_0)]
\]
\[
= (1 + c_3 d)V_0 [-g(x, F_+(x, V_0), V_0) + g(x, F_+(x, (1 + c_3 d)V_0), (1 + c_3 d)V_0)]
\]
\[
- c_2 d (1 + c_3 d)V_0 g_u(x, \xi, (1 + c_3 d)V_0)
\]
\[
\leq -d (1 + c_3 d)V_0 [\sigma_1 c_2 V_0 + c_2 g_u(x, \xi, (1 + c_3 d)V_0)]
\]
\[
< 0,
\]  
(22)
if \( c_3 \) is large enough and fix it, where
\[
F_+(x, (1 + c_3 d)V_0) - c_2 d \leq \xi = \xi(x) \leq F_+(x, (1 + c_3 d)V_0),
\]
the second inequality is due to Lemma 3.1 and \( \sigma_1 > 0 \) is some constant.

This indicates that \( (1 + c_3 d)V_0 \) is a upper solution of (19). Since \( \hat{v} \) is the unique solution of (19), we have
\[
\hat{v} < (1 + c_3 d)V_0 \quad \text{in} \quad \Omega.
\]  
(23)

On the other side, consider \( (1 - c_4 \sqrt{d})V_0 \) with \( c_4 > 0 \) and compute as follows
\[
D\mathcal{P}[(1 - c_4 \sqrt{d})V_0] + (1 - c_4 \sqrt{d})V_0 g(x, F_+(x, (1 - c_4 \sqrt{d})V_0) + c_1 \sqrt{d},
\]
\[
(1 - c_4 \sqrt{d})V_0)
\]
\[
= (1 - c_4 \sqrt{d})V_0 [-g(x, F_+(x, V_0), V_0)
\]
\[
+ g(x, F_+(x, (1 - c_4 \sqrt{d})V_0) + c_1 \sqrt{d}, (1 - c_4 \sqrt{d})V_0)]
\]
\[
= (1 - c_4 \sqrt{d})V_0 [-g(x, F_+(x, V_0), V_0)
\]
\[
+ g(x, F_+(x, (1 - c_4 \sqrt{d})V_0), (1 - c_4 \sqrt{d})V_0)]
\]
\[
+ c_1 \sqrt{d}(1 - c_4 \sqrt{d})V_0 g_u(x, \xi, (1 - c_4 \sqrt{d})V_0)
\]
\[
\geq \sqrt{d}(1 - c_4 \sqrt{d})V_0 [\sigma_2 c_4 V_0 + c_1 g_u(x, \xi, (1 - c_4 \sqrt{d})V_0)]
\]
\[
> 0,
\]
if \( c_4 \) is large enough and fix it, where
\[
(1 - c_4 \sqrt{d})V_0 \leq \xi = \xi(x) \leq (1 - c_4 \sqrt{d})V_0 + c_1 \sqrt{d},
\]
the inequality is due to Lemma 3.1 and \( \sigma_2 > 0 \) is some constant.

Similarly, this implies that \( (1 - c_4 \sqrt{d})V_0 \) is a lower solution of (20). Thus the uniqueness of the positive solution to (20) yields that
\[
(1 - c_4 \sqrt{d})V_0 < \underline{v} \quad \text{in} \quad \Omega.
\]  
(24)

At the end, it follows from (21), (23) and (24) that
\[
(1 - c_4 \sqrt{d})V_0 < \underline{v} < v_d < \hat{v} < (1 + c_3 d)V_0 \quad \text{in} \quad \Omega.
\]  
(25)
This, together with (18), gives the desired conclusion.

4. **Proof of Theorem 1.1.** For clarity, we prove Theorem 1.1(ii) and (iii) first since it is easier.

**Proof of Theorem 1.1(ii).** The assumptions in Theorem 1.1(ii), Lemma 2.3 and Theorem 2.5 together indicate that to prove Theorem 1.1(ii), it suffices to show that the system (1) admits no positive steady states if $d$ is sufficiently small.

Suppose that there exists a sequence $\{d_i\}_{i\geq 1}$ with $\lim_{i \to \infty} d_i = 0^+$ such that for $d = d_i$, the problem as follows admits a positive solution $(u_i, v_i) \in X \times X$:

$$
\begin{aligned}
&d_i K[u] + uf(x, u, v) = 0 \quad \text{in } \Omega, \\
&DP[v] + vg(x, u, v) = 0 \quad \text{in } \Omega.
\end{aligned}
$$

(26)

Thanks to Theorem 1.3, we have

$$
\lim_{i \to \infty} u_i = U_0 = F_+(x, V_0), \quad \lim_{i \to \infty} v_i = V_0 \quad \text{in } X.
$$

Note that by Theorem 1.2, the positivity of $V_0$ is guaranteed by $\mu_0 > 0$, while $\nu_D^0 < 0$ implies that $U_0 \equiv 0$. Thus, in fact, $V_0 = \eta_D$. Then due to the first equation in (26) and Lemma 2.4, one sees that

$$
0 = \inf_{\phi \in X, \phi > 0 \text{ on } \bar{\Omega}} \sup_{x \in \Omega} \left\{ d_i K[\phi] + f(x, u_i, v_i) \phi \right\} \leq \sup_{x \in \Omega} \left\{ d_i K[1] + f(x, u_i, v_i) \right\}.
$$

Therefore,

$$
0 \leq \lim_{i \to \infty} \sup_{x \in \Omega} \left\{ d_i K[1] + f(x, u_i, v_i) \right\} = \max_{x \in \Omega} f(x, 0, \eta_D) = \nu_D^0 < 0.
$$

This is a contradiction. \hfill \Box

**Proof of Theorem 1.1(iii).** First of all, we claim that if $\mu_0 < 0$, then $\nu_D^0 > 0$.

Recall that

$$
0 > \mu_0 = \sup \left\{ \Re \lambda \left| \lambda \in \sigma(DP + g(x, F_+(x, 0), 0)) \right. \right\}
= \sup_{\phi \in X, \phi > 0 \text{ on } \bar{\Omega}} \inf_{x \in \Omega} \frac{DP[\phi] + g(x, F_+(x, 0), 0) \phi}{\phi}
\geq \inf_{x \in \Omega} \frac{DP[\eta_D] + g(x, F_+(x, 0), 0) \eta_D}{\eta_D}
= \inf_{x \in \Omega} \left[ -g(x, 0, \eta_D) + g(x, F_+(x, 0), 0) \right]
> \inf_{x \in \Omega} \left[ -g(x, 0, \eta_D) + g(x, F_+(x, \eta_D), \eta_D) \right],
$$

where the last inequality is due to Lemma 3.1. This indicates that there exists $x^* \in \bar{\Omega}$ such that

$$
F_+(x^*, \eta_D(x^*)) = F(x^*, \eta_D(x^*)) > 0.
$$

Hence according to (A2) and the definition of $F$, it follows that

$$
\nu_D^0 = \max_{x \in \Omega} f(x, 0, \eta_D) \geq f(x^*, 0, \eta_D(x^*)) > 0.
$$

The claim is proved.

Now by Lemma 2.3 and Theorem 2.5, to prove Theorem 1.1(iii), it suffices to show that the system (1) admits no positive steady states if $d$ is sufficiently small.
Suppose that there exists a sequence \( \{d_i\} \geq 1 \) with \( \lim_{i \to \infty} d_i = 0^+ \) such that for \( d = d_i \), the problem (26) admits a positive solution \((u_i, v_i) \in X \times X\). Then

\[
0 > \mu_0 = \sup \{ \Re \lambda \mid \lambda \in \sigma(DP + g(x, F_+(x, 0), 0)) \} > \sup_{\phi \in X, \varphi > 0} \inf_{\Omega \in \Omega} \frac{DP[\phi] + g(x, F_+(x, 0), 0)\phi}{\phi} \geq \inf_{v_i} \frac{DP[v_i] + g(x, F_+(x, 0), 0)v_i}{v_i} = \inf_{\xi \in \xi} \left\{ \begin{array}{l}
-\inf_{\xi \in \xi} \left[ -g(x, u_i, v_i) + g(x, F_+(x, 0), 0) \right] + g(x, F_+(x, v_i), 0) \right\},
\end{array} \right.
\]

where the last inequality is due to Lemma 3.1.

Moreover, notice that regardless of whether \( \mu_0 > 0 \) or not, the estimate in (18) in the proof of Theorem 1.3 always holds. Hence it follows from (27) that

\[
0 > \mu_0 \geq \inf_{x \in \xi} \left[ -g(x, u_i, v_i) + g(x, F_+(x, v_i), v_i) \right] \geq \inf_{\xi \in \xi} \left[ -g(x, F_+(x, v_i) - c_2d_i), v_i \right] + g(x, F_+(x, v_i), v_i),
\]

which yields a contradiction by letting \( d_i \to 0^+ \).

The rest of this section is devoted to the proof of Theorem 1.1(i).

Proof of Theorem 1.1(i). According to Lemma 2.3, Lemma 2.6 and Theorem 2.5, it is clear that we only need verify the uniqueness of positive steady states in \( X \times X \) when \( d \) is sufficiently small.

Suppose that there exists a sequence \( \{d_i\} \geq 1 \) with \( \lim_{i \to \infty} d_i = 0^+ \) such that the system (1) with \( d = d_i \) admits two different positive steady states \((u_i, v_i)\) and \((u_i^*, v_i^*)\) in \( X \times X \). Then similar to the discussion at the beginning of the proof of [3, Theorem 1.1(i)], we assume that w.l.o.g., \( u_i < u_i^*, v_i > v_i^* \).

Denote

\[
\phi_i = \frac{u_i - u_i^*}{\|u_i - u_i^*\|_{L^2(\Omega)} + \|v_i - v_i^*\|_{L^2(\Omega)}} < 0, \quad \psi_i = \frac{v_i - v_i^*}{\|u_i - u_i^*\|_{L^2(\Omega)} + \|v_i - v_i^*\|_{L^2(\Omega)}} > 0.
\]

It is routine to check that

\[
\begin{cases}
D_iK[\phi_i] + f(x, u_i, v_i)\phi_i = -u_i^* \left[ f(x, u_i, v_i) - f(x, u_i^*, v_i) \phi_i + \frac{f(x, u_i^*, v_i) - f(x, u_i^*, v_i^*)}{v_i - v_i^*} \psi_i \right] \\
DP[\psi_i] + g(x, u_i, v_i)\psi_i = -v_i^* \left[ g(x, u_i, v_i) - g(x, u_i^*, v_i) \phi_i + \frac{g(x, u_i^*, v_i) - g(x, u_i^*, v_i^*)}{v_i - v_i^*} \psi_i \right]
\end{cases}
\]

in \( \Omega \),

\[
\begin{cases}
-\inf_{\xi \in \xi} \left[ -\inf_{\xi \in \xi} \left[ -g(x, u_i, v_i) + g(x, F_+(x, 0), 0) \right] + g(x, F_+(x, v_i), 0) \right],
\end{cases}
\]

where

\[
u_i \leq \xi_i, \alpha_i \leq u_i^*, \quad v_i^* \leq \zeta_i, \beta_i \leq v_i.
\]

For simplicity, denote

\[
k \ast \phi = \int_{\Omega} k(x, y)\phi(y)dy.
\]
Direct calculation gives that
\[ 0 > \phi_i = \frac{-f_v(x, u_i^*, \zeta) u_i^* \psi_i - d_i k * \phi_i - u_i}{f(x, u_i, v_i) + f_u(x, \xi, v_i) u_i^* - d_i}, \]
\[ = \frac{-f_v(x, u_i^*, \zeta) u_i^* \psi_i - d_i u_i k * \phi_i}{-d_i k * u_i + f_u(x, \xi, v_i) u_i^*}, \]
\[ \geq \frac{-f_v(x, u_i^*, \zeta) \psi_i}{f_u(x, \xi, v_i)}, \quad \text{for } (29) \]

On the other hand, due to the second equation in (26) and Lemma 2.4, one has
\[ 0 = \sup \{ \Re \lambda | \lambda \in \sigma(DP + g(x, u_i, v_i)) \}
\]
\[ = \sup \inf_{\phi \in \mathbb{X}, \phi > 0} \frac{DP[\phi] + g(x, u_i, v_i)\phi}{\phi} \]
\[ \geq \inf_{x \in \Omega} \frac{DP[\psi_i] + g(x, u_i, v_i)\psi_i}{\psi_i} \]
\[ = \inf_{x \in \Omega} -v_i^* \left[ g_u(x, \alpha_i, v_i) \frac{\phi_i}{\psi_i} + g_v(x, u_i^*, \beta_i) \right]. \]

This, together with (29), yields that
\[ 0 \geq \inf_{x \in \Omega} \left[ -g_u(x, \alpha_i, v_i) \phi_i - g_v(x, u_i^*, \beta_i) \psi_i \right] \]
\[ \geq \inf_{x \in \Omega} \left( g_u(x, \alpha_i, v_i) \frac{f_v(x, u_i^*, \zeta)}{f_u(x, \xi, v_i)} - g_v(x, u_i^*, \beta_i) \right) \psi_i \]
\[ -g_u(x, \alpha_i, v_i) \left( \frac{k * u_i}{u_i} - f_u(x, \xi, v_i) \frac{u_i^*}{d_i} \right)^{-1} k * \phi_i. \quad (30) \]

We claim that
\[ \lim_{i \to \infty} \left( \frac{k * u_i}{u_i} - f_u(x, \xi, v_i) \frac{u_i^*}{d_i} \right)^{-1} = 0 \text{ in } \mathbb{X}. \quad (31) \]

Since \( \mu_0 > 0 \), by Theorem 1.3, one has
\[ \lim_{i \to \infty} u_i = U_0 = F_+(x, V_0) \text{ in } \mathbb{X} \]
and \( U_0 \not\equiv 0 \) because of Theorem 1.2. Then the extra assumption that \( k(x, y) > 0 \) for \( x, y \in \Omega \) indicates that \( k * U_0 > 0 \) in \( \Omega \). Hence it follows that
\[ \lim_{i \to \infty} \left( \frac{k * u_i}{u_i} - f_u(x, \xi, v_i) \frac{u_i^*}{d_i} \right)^{-1} \leq \lim_{i \to \infty} \left( \frac{k * u_i}{u_i} - f_u(x, \xi, v_i) \frac{u_i}{d_i} \right)^{-1} \]
\[ \leq \lim_{i \to \infty} \left( -f_u(x, \xi, v_i) \frac{k * u_i}{d_i} \right)^{-1/2} = 0. \]

The claim is proved.

Next, by Theorem 1.3 and the assumption (A5), it is easy to check that
\[ \lim_{i \to \infty} \left( g_u(x, \alpha_i, v_i) \frac{f_v(x, u_i^*, \zeta)}{f_u(x, \xi, v_i)} - g_v(x, u_i^*, \beta_i) \right) \]
\[ = g_u(x, U_0, V_0) \frac{f_v(x, U_0, V_0)}{f_u(x, U_0, V_0)} - g_v(x, U_0, V_0) \geq c_0 > 0 \text{ in } \mathbb{X}, \quad (32) \]
where \( c_0 \) is some constant.
Then (30), (31) and (32) together imply that for $i$ large enough, there exists $x_i \in \bar{\Omega}$ such that
\[
\lim_{i \to \infty} \psi_i(x_i) = 0. \tag{33}
\]
This, combined with (29) and (31), gives that
\[
\lim_{i \to \infty} \phi_i(x_i) = 0. \tag{34}
\]
Back to the equation satisfied by $\psi_i$ in (28), it follows from (33) and (34) that
\[
\lim_{i \to \infty} D(p * \psi_i)(x_i) = 0. \tag{35}
\]
We claim that there exists a sequence of $\{\psi_i\}_{i \geq 1}$, still denoted by $\{\psi_i\}_{i \geq 1}$, such that
\[
\lim_{i \to \infty} p * \psi_i = 0 \text{ in } \mathbb{X}. \tag{36}
\]
Obviously, by passing to a subsequence if necessary, there exist $x_0 \in \bar{\Omega}$ and $\Psi_0 \in \mathbb{X}$ such that
\[
\lim_{i \to \infty} x_i = x_0 \text{ and } \lim_{i \to \infty} p * \psi_i = \Psi_0 \text{ in } \mathbb{X}.
\]
Thus $\Psi_0(x_0) = 0$. Denote $\Omega_0 = \{x \in \bar{\Omega} \mid \Psi_0(x) = 0\}$. To prove the claim, it suffices to show that $\Omega_0$ is both open and closed in $\bar{\Omega}$. According to (A0), there exist $c > 0$ and $\delta > 0$ such that
\[
p(x, y) \geq c > 0 \text{ if } x, y \in \bar{\Omega}, |x - y| \leq \delta.
\]
Hence
\[
0 = \Psi_0(x_0) = \lim_{i \to \infty} (p * \psi_i)(x_0) \geq c \lim_{i \to \infty} \int_{\{x \in \bar{\Omega}, |x - x_0| \leq \delta\}} \psi_i(x) dx \geq 0,
\]
which yields that
\[
\lim_{i \to \infty} \psi_i = 0 \text{ a.e. in } \{x \in \bar{\Omega}, |x - x_0| \leq \delta\}.
\]
Moreover, based on the equations satisfied by $v_i$ and $\psi_i$ respectively, it is routine to check that
\[
p * \psi_i = \left( \frac{p * v_i}{v_i} - \frac{g_u(x, u_i^*, \beta_i)}{D} \right) \psi_i - \frac{g_u(x, \alpha_i, v_i)}{D} v_i^* \phi_i \leq \left( \frac{p * v_i}{v_i} - \frac{g_u(x, u_i^*, \beta_i)}{D} \right) \psi_i,
\]
which yields that
\[
\lim_{i \to \infty} p * \psi_i = 0 \text{ a.e. in } \{x \in \bar{\Omega}, |x - x_0| \leq \delta\}.
\]
Hence $\Psi_0 = 0$ in $\{x \in \bar{\Omega}, |x - x_0| < \delta\}$ and thus $\Omega_0$ is open. On the other side, the closedness of $\Omega_0$ is obvious. (36) is verified.

At the end, based on the equations satisfied by $v_i$ and $\psi_i$ again, we derive that
\[
Dv_i p * \psi_i = D\psi_i p * v_i - v_i v_i^* g_u(x, \alpha_i, v_i) \phi_i - v_i v_i^* g_u(x, u_i^*, \beta_i) \psi_i \geq \left[ Dp * v_i + v_i v_i^* \left( g_u(x, \alpha_i, v_i) f_u(x, u_i^*, \zeta_i) - g_u(x, u_i^*, \beta_i) \right) \right] \psi_i - v_i v_i^* g_u(x, \alpha_i, v_i) \left( k * u_i - f_u(x, \zeta_i, v_i) \frac{u_i^*}{d_i} \right)^{-1} k * \phi_i.
\]
Then thanks to (A5), Theorem 1.3, (31) and (36), one immediately sees that
\[
\lim_{i \to \infty} \psi_i = 0 \text{ in } L^\infty(\Omega),
\]
which, combined with (29) and (31), implies that
\[
\lim_{i \to \infty} \phi_i = 0 \text{ in } L^\infty(\Omega).
\]
This is a contradiction to the definitions of \(\phi_i\) and \(\psi_i\).

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