Cross-Sectional Dynamics Under Network Structure:
Theory & Macroeconomic Applications

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2024 ASSA Annual Meeting, San Antonio

January 5, 2024
Motivation

• Common in economics: cross-section of units/agents, linked by network ties

• Theory and empirics: network amplifies unit-level shocks, implies comovement of cross-sectional variables

• How does network-induced comovement play out over time?

• Literature: Two restrictive cases:
  
  • innovations transmit contemporaneously
e.g. Acemoglu et al. 2012, Elliott et al. 2014

    → static model, links of all order play out simultaneously

  • innovations transmit one link per period
e.g. Long & Plosser 1983, Golub & Jackson 2010

    → tenable in theory, less so in empirics
Contribution

- **Econometric framework that can speak to dynamics implied by networks**
  - VAR parameterized s.t. innovations transmit cross-sectionally via bilateral links
  - Can accommodate general patterns on how innovations travel through network over time

- **Applicable in two distinct lines of empirical work with cross-sectional time series**
  - estimate dynamic network (peer) effects, with network given or estimated (+ shrink to observed links)
  - dimensionality-reduction technique for modeling (c.s.) time series
  → **Two applications**
Related Literature: Model

Networks in econometrics

- **Spatial Autoregressive Models:**
  - **identify network effects in static framework**
    Manski 1993, Lee 2007, Bramouillé et al. 2009, de Paula et al. 2020, ...
    → I look at dynamic, contagion-like network effects
  - **some work on lagged/dynamic network effects**
    Knight et al. 2016, Zhu et al. 2017, Yang & Lee 2019, ...
    → I relate TS properties to network and timing of network effects, generalize mapping, & show how to conduct inference on both

- **Networks in time series (TS) econometrics:**
  - **represent TS model output as network**
    Diebold & Yilmaz 2009, 2014, Barigozzi & Brownlees 2018, ...
    → I use network to obtain a TS model
  - **restrict TS models using networks**
    Pesaran et al. 2004, Chudik & Pesaran 2011, Caporin et al. 2022, ...
    → I focus on simpler/clearer case & assume transmission via links → analytical results
Bilateral Connections in Networks

\[ A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix} \]

shows direct links

\[ A^2 = \begin{bmatrix} 0 & .64 & 0 \\ 0 & .48 & .56 \\ .56 & 0 & .48 \end{bmatrix} \]

shows 2nd order connections

...
Lagged Innovation Transmission via Bilateral Links

**VAR(1):**

\[ y_t = Ay_{t-1} + u_t , \]

\[ \rightarrow y_{it} = \sum_{j=1}^{n} a_{ij}y_{j,t-1} + u_{it} \]

- Interpret \( A \) as network: innovations travel one link per period

\[ \rightarrow \text{Granger Causality at horizon } h = 1, 2, \ldots \text{ given by } h\text{th order network connections:} \]

\[ \frac{\partial y_{i,t+h}}{\partial y_{j,t}} = (A^h)_{ij}. \]

- Used in theory:
  - Long & Plosser (1983): sectoral output under one period delay in I-O conversion
  - Golub & Jackson (2010): study of societal opinion formation through friendship ties

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Lagged Innovation Transmission via Bilateral Links

**NVAR**\((p, 1)\): (particular version of NAR\(p\) in Zhu et al. 2017)

\[
\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau , \quad \alpha = (\alpha_1, ..., \alpha_p)' \in \mathbb{R}^p .
\]

- Assuming \(\alpha_l \neq 0 \forall l\), \(\tilde{y}_j\) Granger-causes \(\tilde{y}_i\) at horizon \(h\) iff there exists a connection from \(i\) to \(j\) of at least one order \(k \in \{k, k+1, ..., h\}\), where \(k = ceil(h/p)\).

\[
\frac{\partial \tilde{y}_{i,\tau+h}}{\partial \tilde{u}_{j,\tau}} = c^h_k(\alpha) [A^k]_{ij} + \ldots + c^h_h(\alpha) [A^h]_{ij} .
\]

- i.e. \(\tilde{y}_\tau\) driven by lagged network effects, with transmission spread out over \(p\) periods
- \(\alpha\) shows time profile of transmission
Lagged Innovation Transmission via Bilateral Links

\[ \tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau, \quad \alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p. \]

If \( \tilde{y}_\tau \) observed every \( q > 1 \) periods, then \( \{y_t\}_{t=1}^T = \{\tilde{y}_{tq}\}_{t=1}^T \)

- for GC at horizon \( h \), links of order \( k \in \{k, k+1, \ldots, hq\} \) matter, \( k = \text{ceil}(hq/p) \)
- network-induced correlation in observed innovations \( u_t \)
- holds for \( q \in \mathbb{Q}_{++} \), and also for flow variables under \( q \in \mathbb{N} \)

\( \rightarrow \) “\( \text{NVAR}(p, q) \)” stationarity relation to contemp. transmission VARMA approx.
Inference: $\alpha | A$, in $\text{NVAR}(p, 1)$

\[ y_t = \alpha_1 Ay_{t-1} + \ldots + \alpha_p Ay_{t-p} + u_t = X_t(A)\alpha + u_t \]

- LS estimator for $\alpha$:

\[ \hat{\alpha} | A = \left[ \sum_{t=1}^{T} X_t'\Sigma^{-1} X_t \right]^{-1} \left[ \sum_{t=1}^{T} X_t'\Sigma^{-1} y_t \right] \]

\[ X_t = [Ay_{t-1}, \ldots, Ay_{t-p}] \]

- OLS ($\Sigma = I$): consistent and asympt. Normal for $n, T \& (n, T) \to \infty$ conditions
Inference: $\alpha \mid A$, $\text{NVAR}(p, q), q > 1$

$$\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau = X_\tau(A)\alpha + \tilde{u}_\tau, \quad \tau = 1 : T_\tau,$$

$$y_{\tau/q} = \tilde{y}_\tau \quad \text{if } \tau/q \in \mathbb{N},$$

- Data augmentation. But: point ID not guaranteed; e.g. for $q = 2, p = 1$, can identify $\alpha_1$ up to sign: $y_t = \alpha_1^2 A^2 y_{t-1} + \eta_t$
- Akin to AR($p$) observed every $q > 1$ periods (Palm & Nijman 1984)
- Shrink towards lower-dimensional function; e.g. $\alpha_l \sim N (\mu_l, \lambda^{-1}_\alpha), \mu_l = (1, l, l^2) \beta_\alpha$
- Gives full-sample posterior $\alpha_l \mid \tilde{Y}_{1:T_\tau} \sim N (\tilde{\alpha}, \tilde{V}_\alpha)$ with

$$\tilde{V}_\alpha = \left[ \sum_{\tau=1}^{T_\tau} \tilde{X}_\tau' \tilde{\Sigma}^{-1} \tilde{X}_\tau + \lambda_\alpha I_p \right]^{-1}, \quad \tilde{\alpha} = \tilde{V}_\alpha \left[ \sum_{\tau=1}^{T_\tau} \tilde{X}_\tau' \tilde{\Sigma}^{-1} \tilde{y}_\tau + \lambda_\alpha I_p \mu \right].$$

- Uniform hyperpriors for $\beta_\alpha$ and $\lambda_\alpha$: shrink towards MLE/OLS $\hat{\beta}_\alpha$, optimizing predictive ability (Giannone, Lenza & Primiceri 2015)
Inference: \((\alpha, A)\)

\[
\tilde{y}_\tau = \alpha_1 A\tilde{y}_{\tau-1} + \ldots + \alpha_p A\tilde{y}_{\tau-p} + \tilde{u}_\tau = A z_\tau(\alpha) + \tilde{u}_\tau, \quad \tau = 1 : T_\tau,
\]

\[
y_{\tau/q} = \tilde{y}_\tau \quad \text{if } \tau/q \in \mathbb{N},
\]

**NVAR**\((p, 1):\)

- Ridge-prior \(a_{ij} \sim N(b_{ij}, \lambda_a^{-1})\) gives posterior \(A|\alpha, \Sigma) \sim MN (\bar{A}', \Sigma, \bar{U}_A)\) with

\[
\bar{U}_A = \left[Z'Z + \lambda_a \Sigma \right]^{-1}, \quad \bar{A} = \bar{U}_A \left[Z'Y + \lambda_a B'\Sigma \right].
\]

- Can shrink to actual links: set \(b_{ij} = w_{ij}^b \beta_b\)

- Iterate on posteriors (or modes) of \(A|\alpha\) and \(\alpha|A\), normalizing \(||\alpha||_1 = 1\) (e.g.)

- \((\hat{\alpha}, \hat{A})_{OLS}\) consistent and asymp. Normal for \(T \to \infty\)

**NVAR**\((p, q):\) add data augmentation step (Carter-Kohn Gibbs sampler / EM algo)
Application 1: Motivation

Macro literature on production networks:

- assuming contemporaneous input-output-conversion, shows:
  Horvath 2000, Acemoglu et al. 2012, 2016, Bouakez et al. 2014, ...
  
  - supply chain network amplifies sectoral shocks
  - strength of effect on aggregates depends on sector’s position in network

- exception: one period-lagged I-O-conversion $\rightarrow$ NVAR(1,1)
  Long & Plosser (1983), Foerster et al. (2011), Carvalho & Reischer (2021)
  
  - generates endogenous BCs (persistence in aggregates)
  - model-persistence matches empirics,
    calibrated model gives improved forecasts of agg. IP relative to statistical models

This application: setup  theory  data

- How does amplification materialize over time?
- Does network-position shape timing of effect?
- Estimate roles of exogenous shock persistence vs. lagged IO conversion Foerster et al. (2011)
Introduction

Network-VAR

Inference

App 1: $\alpha | A$

App 2: $(\alpha, A)$

Conclusion

Results: Impulse Responses & Their Composition

more results

Relevance of Link-Orders Across Horizons

Input-Output Links to Utilities Sector

Figure: Transmission of Price Shocks via Supply-Chain Links (1)

Recall:

$$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c_{h}^{k}(\alpha) \left( A^{k} \right)_{ij} + \ldots + c_{h}^{h}(\alpha) \left( A^{h} \right)_{ij}, \quad k = ceil(h/p).$$

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IRF of Chemical Products to Utilities

IRF of Truck Transportation to Utilities

Figure: Transmission of Price Shocks via Supply-Chain Links (2)

Recall:

$$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c_k^h(\alpha) \left( A_k^h \right)_{ij} + \ldots + c_h^h(\alpha) \left( A_h^h \right)_{ij}, \quad k = \text{ceil}(h/p).$$

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Application 2: Motivation

How to model industrial production dynamics across 44 countries?

• Even for this moderate size of cross-section, unrestricted VAR not feasible

• $\text{NVAR}(p, q)$: well-performing, simple-to-estimate and interpretable alternative

→ Estimate $(\alpha, A)$, $A$ sparse!

• Assumption: a few bilateral links drive dynamics of whole cross-section
Relation to Alternative Dimensionality-Reduction Techniques

- Combines insights from factor models / RR regression (Velu et al. 1986, Stock & Watson 2002, ...) and sparse / shrinkage methods (Tibshirani 1996, ...)

Recall \( \text{NVAR}(p, 1) \):
\[
y_t = A [y_{t-1}, ..., y_{t-p}] \alpha + u_t.
\]

- Equivalence betw. factor model & \( \text{NVAR}(p, 1) \), with \# factors = \text{rank}(A):
  - \( y_t \sim \text{NVAR}(p, 1) \) \( \Rightarrow \) \( y_t \sim \text{FM} \)
  - \( y_t \sim \text{FM} + f_t \sim \text{NVAR}(p, 1) \) \( \Rightarrow \) \( y_t \sim \text{NVAR}(p, 1) \), for \( n \) large

- Expect: Network-VAR preferred when dynamics driven by many micro links rather than few influential units (see Boivin & Ng, 2006)

- Rationalize sparse factors as locally important units in network

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Results: Forecasting

Figure: Out-Of-Sample Forecasting Performance: NVAR(4, 1) vs. Factor Model

Notes: Plot depicts percentage difference between out-of-sample Mean Squared Errors generated by NVAR(4, 1) to those generated by Principal Components Factor Model.
Conclusion

- I propose econometric framework for cross-sectional time series exploiting network structure
- I apply it to estimate how supply shocks propagate through US supply chain network and affect dynamics of sectoral prices
- I apply it to forecast cross-country IP dynamics, assuming & estimating network
Bilateral Connections in Networks

• Network: $n \times n$ adjacency matrix $A$ with elements $a_{ij}$

• Directed and weighted: $a_{ij} \in [0, 1]$ shows strength of (direct) link from $i$ to $j$

• Walk: product of direct links $a_{ij}$ that lead from $i$ to $j$ over some intermediary units
e.g. $a_{i,k_1}a_{k_1,k_2}a_{k_2,j}$: walk from $i$ to $j$ of length 3

• $(A^K)_{ij}$: sum of all walks from $i$ to $j$ of length $K$ (“$K$th order connection from $i$ to $j$”)
e.g.

$$A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & .64 & 0 \\ .48 & .56 & 0 \\ .56 & 0 & .48 \end{bmatrix}, \quad A^3 = \begin{bmatrix} .448 & 0 & .384 \\ .336 & .448 & .288 \\ 0 & .384 & .448 \end{bmatrix}.$$
Lagged Innovation Transmission via Bilateral Links

Figure: Example Generalized Impulse Responses For NVAR(1, 1)

Notes: Panel \((i, j)\) shows \((A^h)_{ij}\) in blue, \(\alpha^h\) in red and \(GC^h_{ij}\), their product, in purple.
Time Aggregation of Lagged Transmission Patterns

- Let \( \tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \alpha_2 A \tilde{y}_{\tau-2} + \alpha_3 A \tilde{y}_{\tau-3} + \tilde{u}_\tau \), and \( \{y_t\}_{t=1}^{T} = \{\tilde{y}_t\}_{t=1}^{T} \).

- We get

\[
\tilde{y}_\tau = \left[ \alpha_2 A + \alpha_1^2 A^2 \right] \tilde{y}_{\tau-2} + \left[ (\alpha_1 \alpha_2 + 2 \alpha_1 \alpha_3) A^2 \right] \tilde{y}_{\tau-4}
+ \tilde{u}_\tau + \alpha_1 A \tilde{u}_{\tau-1} + (\alpha_3 A + \alpha_1 \alpha_2 A^2) \tilde{u}_{\tau-3} + \text{terms in } \tilde{y}_{\tau-6}, \tilde{y}_{\tau-7}.
\]

\[\rightarrow y_t \approx \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Theta_0 u_t + \Theta_1 u_{t-1},\]

with \( \Phi_1 = \alpha_2 A + \alpha_1^2 A^2 \), \( \Phi_2 = (\alpha_1 \alpha_2 + 2 \alpha_1 \alpha_3) A^2 \)
\( u_t = [\tilde{u}'_\tau, \tilde{u}'_{\tau-1}]', \quad u_{t-1} = [\tilde{u}'_{\tau-2}, \tilde{u}'_{\tau-3}]', \)
\( \Theta_0 = [I_n, \alpha_1 A], \quad \Theta_1 = [0_n, \alpha_3 A + \alpha_1 \alpha_2 A^2]. \)
• Under contemporaneous network interactions,

\[ x = Ax + \varepsilon = (A + A^2 + A^3 + ...) \varepsilon. \]

→ Acemoglu et al. (2012): network \( A \) amplifies granular shocks \( \varepsilon_j \), implies cross-sectional comovement in \( \{x_i\}_{i=1}^{n} \)

• Result: for \( \text{NVAR}(p, 1) \), \( y_t = \alpha_1 Ay_{t-1} + ... + \alpha_p Ay_{t-p} + u_t \), we have that

\[ \lim_{h \to \infty} \sum_{j=0}^{h} \frac{\partial y_{t+h}}{\partial u_{t+j}} = \frac{\partial x}{\partial \varepsilon} = (I - A)^{-1}, \quad \text{(for } \sum_{l=1}^{p} \alpha_l = 1 \text{)} \]

→ Taking stance on timing of network effects, \( y_t \) can speak to (transition) dynamics
Stationarity of NVAR\((p, 1)\)

Let \(\tilde{y}_\tau\) follow an NVAR\((p, 1)\)

\[
\tilde{y}_\tau = \alpha_1 A \tilde{y}_{\tau-1} + \ldots + \alpha_p A \tilde{y}_{\tau-p} + \tilde{u}_\tau,
\]

where \(\tilde{u}_\tau \sim WN\), and assume \(\alpha_l \neq 0\) for at least one \(l\). Let \(a = \sum_{l=1}^{p} |\alpha_l|\).

1a \(\tilde{y}_\tau\) is WS if for all Eigenvalues \(\lambda_i\) of \(A\) it holds that \(|\lambda_i| < 1/a\).

1b If in addition \(\alpha_1, \ldots, \alpha_p \geq 0\), this condition is both necessary and sufficient.

2 \(\tilde{y}_\tau\) is WS iff the univariate AR\((p)\)

\[
\tilde{x}_\tau = \lambda_i \alpha_1 \tilde{x}_{\tau-1} + \ldots + \lambda_i \alpha_p \tilde{x}_{\tau-p} + \tilde{v}_\tau
\]

is WS for all Eigenvalues \(\lambda_i\) of \(A\).
Asymptotics: $\hat{\alpha}_{OLS}|A$ in $NVAR(p, 1)$

\[
T \to \infty
\]

- Model correct: \( y_t = X_t \alpha + u_t \)
- \( \mathbb{E}_{t-1}[u_t] = 0 \), \( \mathbb{E}_{t-1}[u_t u'_t] = \Sigma \)
- \( y_t \) ergodic and strictly stationary

\[
n \to \infty
\]

- Model correct: \( y_{it} = x'_{it} \alpha + u_{it} \)
- \( \mathbb{E}_{t-1}[u_t] = 0 \), \( \mathbb{E}_{t-1}[u_{it} u_{is}] = \sigma^2 \) if \( t = s \) and zero otherwise
- \( A_n \) converges to some limit s.t.
  - \( \frac{1}{n} \sum_{i=1}^{n} (A_{n,i} y_{t-l})' (A_{n,i} y_{t-k}) \to \mathbb{E} [(A_i y_{t-l})' (A_i y_{t-k})] \)
  - \( \frac{1}{n} \sum_{i=1}^{n} (A_{n,i} y_{t-l})' u_{it} \to \mathbb{E} [(A_i y_{t-l})' u_{it}] \)
  - \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (A_{n,i} y_{t-l})' u_{it} \Rightarrow \mathcal{N} \left( \mathbb{E} [(A_i y_{t-l})' u_{it}] , \mathbb{V} [(A_i y_{t-l})' u_{it}] \right) \)
• Generalized version of LP: firms use inputs produced in last \( p \) periods → at some model-frequency, sectoral prices \( \sim \) NVAR(\( p, 1 \)):
\[
\tilde{y}_\tau \approx \alpha_1 A\tilde{y}_{\tau-1} + \ldots + \alpha_p A\tilde{y}_{\tau-p} + \tilde{u}_\tau ,
\]
with \( \alpha_l \geq 0 \) ∀ \( l \) and \( \sum_{l=1}^{p} \alpha_l = 1 \) and \( A = I-O\)-matrix.

• Observation freq. potentially ≠ network interaction freq.: \( \{y_t\}_{t=1}^{T} = \{\tilde{y}_{qt}\}_{t=1}^{T} \)
→ I consider \( q \in \{1/3, 1/2, 1, 2, 4\} \), i.e. quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions

• 51 sectors, Jan 2005 - Aug 2022, annual I-O-matrix from 2010

• For now, let \( \tilde{u}_{i\tau} \overset{iid}{\sim} N(0, \sigma_i^2) \), get \( (\hat{\alpha}, \hat{\sigma})_{MLE} \) for different \( (p, q) \) & select model via IC

• Work in progress: \( \tilde{u}_{i\tau} = \lambda_i f_\tau + \varepsilon_{i\tau} \), \( f_\tau, \varepsilon_{i\tau} \sim AR(2) \)
→ Determine roles of exogenous shock persistence vs. lagged I-O-conversion
Theory

Assume \( n \) sectors, rep. firm produces variety \( i \) by using labor and inputs \( j = 1 : n \):

\[
y_{i\tau} = z_{i\tau} l_{i\tau}^{b_i} \prod_{j=1}^{n} x_{ij\tau}^{a_{ij}} , \quad b_i > 0 , \quad a_{ij} \geq 0 , \quad b_i + \sum_{j=1}^{n} a_{ij} = 1 .
\]

• If \( x_{ij\tau} \) is variety \( j \) bought at \( \tau \): \( p_{\tau} = A p_{\tau} + \varepsilon_{\tau} , \varepsilon_{\tau} = -\log(z_{\tau}) \) (e.g. Acemoglu et al., 2012)

• If \( x_{ij\tau} \) is variety \( j \) bought at \( \tau - 1 \): \( p_{\tau} = A p_{\tau-1} + \varepsilon_{\tau} \) (Long & Plosser 1983, Carvalho & Reischer 2021)

→ If \( x_{ij\tau} \) is CES-aggregate of variety \( j \) bought at \( \{\tau - p, ..., \tau - 1\} \):

\[
p_{\tau} \approx \alpha_1 A p_{\tau-1} + ... + \alpha_p A p_{\tau-p} + \varepsilon_{\tau} , \text{ for some } \alpha_l \geq 0 , \ l = 1 : p , \text{ and } \sum_{l=1}^{p} \alpha_l = 1
\]
Input-Output Matrix from Bureau of Economic Analysis (BEA)

- 64 mostly 3- and 4-digit sectors (due to PPI availability)
- I take data for 2010
- Following Acemoglu et al. (2016), links defined as $a_{ij} \equiv \frac{sales_{j \rightarrow i}}{sales_i}$ (valid for general $p$ as $\beta \to 1$)

Monthly sector-level PPI data from Bureau of Labor Statistics (BLS)

- 51 BEA-sectors, January 2005 - August 2022
- I take logs and subtract sector-specific linear time trend and seasonality (since the assumed process is stationary)
Data: Input-Output Network

- Density: 16.88 %
- Average shortest path: 2.41, longest shortest path: 7

(a) Weighted In-Degrees

(\text{wd}_{\text{in}} \text{ of } i = \sum_j a_{ij})

(b) Weighted Out-Degrees

(\text{wd}_{\text{out}} \text{ of } j = \sum_i a_{ij})

Notes: Left panel plots weighted in-degrees (column-wise sums of $A$), shows sectors’ differing reliance on intermediate inputs. Right panel plots weighted out-degrees (row-wise sums of $A$), shows sectors’ differing importance as suppliers to other sectors.
Data: Input-Output Network

Figure: Network Distance And The Correlation of Sectoral Inflation

Notes: Figure plots average correlation of sectoral prices for different distances between them. Lightest blue line refers to contemporaneous correlations. Darker lines show average correlation of sector $i$ with lagged values of sector $j$ as function of distance from $i$ to $j$. Lags from 1 to 12 months. Series are de-trended and de-seasonalized log PPIs.

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Data: PPI

Figure: Aggregate & Sectoral PPIs

Notes: Left panel shows raw PPI series for few selected sectors. Right panel compares aggregate PPI (FRED Database) and output-weighted average of PPIs of studied sectors.
## Estimation Results: Model Selection

### Table: Model Selection: Log MDD

| \( q \) | 1/3 | 1/2 | 1  | 2  | 4  | 5  | 6  |
|-------|-----|-----|----|----|----|----|----|
| 1/3   | 19079 | 19384 | 20153 | 20056 | 19879 | 18899 | 19044 |
| 1/2   | 19044 | 18768 | 19675 | 19248 | 20142 | 18662 | 18690 |
| 1     | 20153 | 20056 | 19675 | 19879 | 18899 | 20218 |
| 2     | 17546 | 19570 | 20142 | 18662 | 19636 |
| 4     | 18517 | 19808 | 19754 | 19655 | 18904 | 19301 |

*Notes:* Table shows log Marginal Data Density (MDD) across model specifications. Values for \( q \) (from top to bottom) refer to quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions, while \( p = mq \) implies last \( m \) months matter for dynamics.
### Table: Estimation Results: $\alpha$

|     | MLE    | Mean   | Low   | High   |
|-----|--------|--------|-------|--------|
| $\alpha_1$ | 0.1550 | 0.1557 | 0.1370 | 0.1745 |
| $\alpha_2$ | 0.3460 | 0.3382 | 0.3168 | 0.3605 |
| $\alpha_3$ | 0.2816 | 0.2865 | 0.2644 | 0.3129 |
| $\alpha_4$ | 0.0915 | 0.0991 | 0.0785 | 0.1174 |
| $\alpha_5$ | 0.1045 | 0.0975 | 0.0837 | 0.1135 |

*Notes:* First column shows Maximum Likelihood or Maximum A-Posteriori (MAP) Estimator, second refers to posterior mean, and Low and High report the bounds of the 95% Bayesian HPD credible sets.
Application 2: Motivation

NVAR\((p, q)\): sparse, flexible and interpretable dimensionality-reduction

\[
\tilde{y}_{\tau} = \sum_{l=1}^{p} \alpha_l A \tilde{y}_{\tau-l} + \tilde{u}_{\tau}, \quad \{y_t\}_{t=1}^{T} = \{\tilde{y}_t\}_{Q_t=1}^{T}.
\]

• Sparsity:
  - \(y_{it\tau} = x_{it\tau}' \alpha + u_{it\tau}\) with \(X_{\tau} = A[\tilde{y}_{\tau-1}, \ldots, \tilde{y}_{\tau-p}]_{(n \times p)}\)
  - \(\rightarrow\) reduce \(n^2\) parameters in VAR to \(n^2 + p - 1\) parameters in NVAR
  - \(A\) can be sparse: higher-order network effects through \(A^2, A^3, \ldots\)

• Flexibility:
  - estimated network + general time dimension of network effects
  - like functional approximation using \(A\) as basis (recall: \(y_t \overset{\text{approx.}}{\sim}\) restricted VARMA)

• Interpretability:
  - dynamics driven by innovation transmission along bilateral links
  - estimate network & whole set of spillover and spillback effects
Relation to Factor Model

NVAR \rightarrow FM

- \( y_t = A[\alpha_1 y_{t-1} + \alpha_2 y_{t-2}] + u_t \) with \( A \) of rank \( r \in 1 : n \)
- Write \( A = B_{n \times r} C_{r \times n} \)
- \( y_t = \Lambda f_t + u_t \), with \( \Lambda = B \) and \( f_{kt} = \alpha_1 C_k y_{t-1} + \alpha_2 C_k y_{t-2} \) for \( k = 1 : r \)
- (not unique: \( A = BC = BQQ^{-1}C = \tilde{B}\tilde{C} \) for any \( r \times r \) full-rank matrix \( Q \))
Relation to Factor Model

**FM → NVAR**

- \( y_t = \Lambda f_t + \xi_t \), \( f_t = \Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t \), with \( f_t \in \mathbb{R}^r \)

- Take \( r \) distinct vectors of weights \( w^k = (w^k_1, \ldots, w^k_n) \), \( k = 1 : r \), and consider \( \sum_{i=1}^n w^k_i y_{it} = \sum_{i=1}^n w^k_i \Lambda_i f_t + \sum_{i=1}^n w^k_i \xi_{it} \)

- If \( n \) large enough, \( \bar{\xi}_t^k \equiv \sum_{i=1}^n w^k_i \xi_{it} \sim O_p(n^{-1/2}) \) is negligible \( \rightarrow Wy_t = W\Lambda f_t \)

\[
y_t = \Lambda (\Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t) + \xi_t
= \Lambda \Phi_1 (W\Lambda)^{-1} W y_{t-1} + \Lambda \Phi_2 (W\Lambda)^{-1} W y_{t-2} + u_t ,
\]

- If \( \Phi_l = \phi_l \Phi \) for \( l = 1, 2 \) (i.e. \( f_t \sim NVAR(2,1) \)), then

\[
y_t = \Lambda \Phi (W\Lambda)^{-1} W [\phi_1 y_{t-1} + \phi_2 y_{t-2}] + u_t
\]

- Let \( A = \Lambda \Phi (W\Lambda)^{-1} W \), \( \alpha_l = \phi_l \)
Data & Forecasting Setup

Data:
- Use IMF & OECD data on monthly IP series
- Compute growth rate relative to same month previous year, subtract mean
- January 2001 - January 2020, 44 countries

Forecasting Exercise:
- Use sample end dates from December 2017 to December 2019
- Consider forecasts of up to 24 months ahead (COVID-19 excluded)
- For $p = 1 : 6$, compare
  - $\text{NVAR}(p, 1) + \text{Lasso-shrinking of } a_{ij} \text{ to zero, select } \lambda \text{ based on BIC (Zou, Hastie & Tibshirani 2007)}$
  - $\text{PC-FM: select } \# \text{ of factors based on Bai & Ng (2002), fit } \text{VAR}(p) \text{ for factors}$
Estimation

\[ y_t = \sum_{l=1}^{p} \alpha_l A y_{t-l} + u_t , \quad \alpha \equiv (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p , \quad a_{ij} \in [0, 1] , \]

- To identify \((\alpha, A)\), normalize \(||\alpha||_1 = 1\) and change domain of \(a_{ij}\) to \(\mathbb{R}_+\)
- Consider OLS with Lasso penalty \((\lambda)\) on \(a_{ij}\)
- Get \((\hat{\alpha}, \hat{A})\) by iterating on

\[
\hat{\alpha}|A = \left[ \sum_{t=1}^{T} X'_t X_t \right]^{-1} \left[ \sum_{t=1}^{T} X'_t y_t \right] ,
\]

\[
\hat{a}_{ij}|(\alpha, A, -j) = \max \{0, \tilde{a}_{ij}\} , \quad \tilde{a}_{ij} = \frac{\sum_{t=1}^{T} (y_{it} - A_{i,-j} z_{-j,t}) z_{jt} - \lambda}{\sum_{t=1}^{T} z_{jt}^2} .
\]
Results: Estimated Network

Figure: Weighted Outdegrees In The Estimated Network

Notes: Plot shows weighted outdegrees in estimated network as relevant for cross-country monthly IP dynamics.

Marko Mlikota, Cross-Sectional Dynamics Under Network Structure
Results: Impulse Responses & Their Composition

Relevance of Link-Orders Across Horizons

| Link-Order | Horizon [Months Since Shock] |
|------------|-----------------------------|
| 1          | 0 3 6 9 12                  |
| 2          |                             |
| 3          |                             |
| 4          |                             |
| 5          |                             |
| 6          |                             |
| 7          |                             |

Links to United States

| Link-Order | Link-Strength |
|------------|--------------|
| 1          | Germany      |
| 2          | Finland      |

Figure: Network-Induced Transmission of Industrial Production Innovations (1)

Notes: Left panel shows importance of different connection-orders for transmission as function of time elapsed since shock took place. Right panel shows connections of different order from Germany and Finland to United States.
Results: Impulse Responses & Their Composition

IRF of Germany to United States

IRF of Finland to United States

Figure: Network-Induced Transmission of Industrial Production Innovations (2)

Notes: The two panels show the Impulse-Response Functions (IRFs) of German and Finnish IP growth, respectively, to a one standard deviation increase in US IP growth.