Resolvent algebra in Fock-Bargmann representation

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Abstract

The resolvent algebra $\mathcal{R}(X, \sigma)$ associated to a symplectic space $(X, \sigma)$ was introduced by D. Buchholz and H. Grundling as a convenient model of the canonical commutation relation (CCR) in quantum mechanics. We first study a representation of $\mathcal{R}(\mathbb{C}^n, \sigma)$ with the standard symplectic form $\sigma$ inside the full Toeplitz algebra over the Fock-Bargmann space. We prove that $\mathcal{R}(\mathbb{C}^n, \sigma)$ itself is a Toeplitz algebra. In the sense of R. Werner’s correspondence theory we determine its corresponding shift-invariant and closed space of symbols. Finally, we discuss a representation of the resolvent algebra $\mathcal{R}(\mathcal{H}, \tilde{\sigma})$ for an infinite dimensional symplectic separable Hilbert space $(\mathcal{H}, \tilde{\sigma})$. More precisely, we find a representation of $\mathcal{R}(\mathcal{H}, \tilde{\sigma})$ inside the full Toeplitz algebra over the Fock-Bargmann space in infinitely many variables.

keywords: canonical commutation relation, Toeplitz $C^*$ algebra, correspondence theorem, Gaussian measure on Hilbert space

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1 Introduction

The classical CCR algebra (see [5]) provides a standard $C^*$ algebraic model for the canonical commutation relation (CCR) in quantum mechanics. In typical applications, however, these algebras have significant disadvantages. In particular, CCR algebras in general do not contain bounded functions of the Hamiltonian (physical observables). Only in rare and physically less relevant cases they are stable under dynamics. More precisely, it was noticed in [11] that time evolution of the Hamiltonian $H := -\Delta + V$ on $L^2(\mathbb{R}^n)$ in the standard Schrödinger representation gives rise to *-automorphisms of the CCR algebra over the standard symplectic space $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ only in the trivial case of an identically vanishing potential function $V$. In order to resolve such problems, D. Buchholz and H. Grundling have proposed a new approach to CCR by introducing the resolvent algebra $\mathcal{R}(X, \sigma)$ associated to a symplectic space $(X, \sigma)$ in [7]. This algebra abstractly is defined through certain algebraic relations that
mimic properties of the resolvents of the (i.e. unbounded) canonical operators (see (2.1)).

In the present paper, we first consider complex $n$-space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ equipped with the standard symplectic structure. As is well-known, in this setting the CCR algebra is identified with the $C^\ast$ algebra generated by Toeplitz operators with symbols in the space $TP$ of trigonometric polynomials. It is an interesting observation that this Toeplitz $C^\ast$ algebra coincides with the linear closure of Toeplitz operators with symbols in $TP$ (see [8] for a precise statement and further extensions of these results). In a similar manner, our goal is to study the resolvent algebra in the Fock-Bargmann representation and to describe it as a concrete algebra generated by Toeplitz operators with shift-invariant symbol space. Being a shift-invariant $C^\ast$ algebra, it has turned out that a convenient mathematical framework for the analysis of the resolvent algebra is R. Werner’s quantum harmonic analysis [20] and its extension by R. Fulisch in [13] to Toeplitz operator theory on the Fock-Bargmann space. A particularly useful tool is the correspondence theorem (Theorem 4.5) in [20, 13] which ensures the existence and uniqueness of a closed and shift-invariant space of bounded uniformly continuous functions on $\mathbb{C}^n$, which are the symbols of Toeplitz operators linearly generating the resolvent algebra.

In dimension $n = 1$ we show that the space corresponding to the resolvent algebra $\mathcal{R}(\mathbb{C}, \sigma)$ is the uniform closure of the classical resolvent functions. However, in higher dimensions $n > 1$ we only can prove a weaker result (see Proposition 4.26). Nevertheless, we show that $\mathcal{R}(\mathbb{C}^n, \sigma)$ coincides with a $C^\ast$ algebra generated by a concrete set of Toeplitz operators with bounded symbols. It remains an open problem whether or not the closure of classical resolvent functions forms the corresponding space to the resolvent algebra in every complex dimension $n \in \mathbb{N}$ and in the sense of Theorem 4.5.

In the second part of the paper, we replace $\mathbb{C}^n$ by a separable infinite dimensional complex symplectic Hilbert spaces $(\mathcal{H}, \tilde{\sigma})$. In [15] the authors have proposed two definitions of Toeplitz operators on the infinitely many variable Fock-Bargmann space. Here, we follow the measure theoretical approach and consider $C^\ast$ algebras generated by Toeplitz operators over $\mathcal{H}$. Due to the non-nuclearity of $\mathcal{H}$ (as well as of the space of entire functions over $\mathcal{H}$ with the compact-open topology) as topological vector spaces and caused by features of the measure theory on infinite dimensional spaces, a variety of new effects can be observed in the theory of Toeplitz operators. In particular, in this setting a correspondence theorem so far is unknown, even though it may exist in a suitable formulation. Our main result of the last section (Theorem 5.4) states that there is a representation of the resolvent algebra corresponding to a symplectic Hilbert space $\mathcal{H}_{1/2}$ with Hilbert-Schmidt embedding $\mathcal{H}_{1/2} \hookrightarrow \mathcal{H}$ inside the full Toeplitz algebra over $\mathcal{H}$.

The structure of the paper is as follows: In Section 2 we recall the definition of the CCR and resolvent algebra [7]. We restate a well-known representation of the CCR algebra associated to $\mathbb{C}^n$ with the standard symplectic structure as a $C^\ast$ algebra generated by Toeplitz operators.

Section 3 provides some basic facts on Fock-Bargmann spaces $F^2_1$ over $\mathbb{C}^n$ and Toeplitz operators acting on $F^2_1$. We define the Berezin transform and then express Weyl operators in form of Toeplitz operators with bounded symbols.
Most of the material is standard and further details on the role of Toeplitz operators in quantum mechanics can be found in [3, 8].

Section 4 describes the Fock-Bargmann representation of the resolvent algebra. We present the correspondence theorem [20] in this setting (see [13]). In passing, we mention some applications to the analysis of Toeplitz operators on Fock-Bargmann spaces. We discuss the unique shift-invariant closed symbol space corresponding to the resolvent algebra. An essential ingredient to the analysis is an integral form of the Berezin transform on products of resolvents (Proposition 4.12). Moreover, in our proofs we essentially use a specific compactification of the affine Grassmannian which was introduced in [19].

Finally, in Section 5, we discuss the resolvent algebra associated to an infinite dimensional symplectic separable Hilbert space in the framework of Toeplitz operators on the Fock-Bargmann space of Gaussian square integrable entire functions in infinitely many variables.

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2 CCR and Resolvent Algebra

Let \((X, \sigma)\) be a symplectic vector space. Consider a Hilbert space \(H\) and a real linear map \(\phi\) from \((X, \sigma)\) into the space of (unbounded) self-adjoint operators on \(H\). Let us assume that all operators \(\phi(f)\) are essentially self-adjoint on a common domain \(D\) which forms a core for each \(\phi(f)\). Recall that the canonical commutation relation (CCR) have the form:

\[
[\phi(f), \phi(g)] = i\sigma(f, g), \quad f, g \in X.
\]

We call \(\phi(f)\) and \(\phi(g)\) canonical operators. Especially in an algebraic setup the analysis of unbounded operators involves some difficulties (compare e.g. Chapters VIII.5 and VIII.6). Therefore, one may look for \(C^*\) algebraic models which suitably encode (CCR). One standard idea consists in replacing \(\phi(f)\) above by suitable bounded functions of \(\phi(f)\). A classical approach due to H. Weyl amounts in considering the \(C^*\) algebra \(\text{CCR}(X, \sigma)\) generated by the relations

\[
W(f)W(g) = e^{-i\sigma(f, g)}W(f + g), \quad f, g \in X
\]

\[
W(f)^* = W(-f).
\]

Such \(C^*\) algebras are usually called CCR algebras, see [5].

With \(z, w \in \mathbb{C}^n\) put \(\langle w, z \rangle := w_1\overline{z}_1 + \ldots + w_n\overline{z}_n\) and consider the standard symplectic space \((\mathbb{C}^n, \sigma)\) with symplectic form

\[
\sigma(w, z) := \text{Im}(w, z).
\]
One obtains a representation of CCR\((C^n, \sigma)\) as operators acting on the Fock-Bargmann space \(F^2_1\) (see [8] and Section 3 for precise definitions). In fact, with our previous notation we put:

\[ \phi(z) := T_{2\sigma(\cdot, z)} , \]

where \(T_f\) is a Toeplitz operator with complex valued symbol \(f\) on \(C^n\). The corresponding CCR algebra is generated by the unitary Weyl operators below which satisfy

\[ W_z = \exp \left( i T_{2\sigma(\cdot, z)} \right) , \]

i.e. \(iT_{2\sigma(\cdot, z)}\) is the generator of the unitary one-parameter group \((W_{tz})_{t \in \mathbb{R}}\). Hence,

\[ \text{CCR}(C^n, \sigma) \cong \mathbb{C}^* (W_z : z \in \mathbb{C}^n) . \] (2.2)

The algebra CCR\((C^n, \sigma)\) is a classical object and in a more general setup it is discussed in [5]. In particular, the Fock-Bargmann representation of CCR\((C^n, \sigma)\) maps into the full Toeplitz algebra \(T\), i.e. the \(C^*\) algebra generated by Toeplitz operators with arbitrary bounded measurable symbols.

**Theorem 2.1** (L. A. Coburn [8]). It holds

\[ \text{CCR}(C^n, \sigma) \cong \{ T_\phi : \phi \in TP \} , \]

where TP is the space of trigonometric polynomials:

\[ TP := \text{span} \{ w \mapsto \exp \left( i \sigma(w, z) \right) : z \in \mathbb{C}^n \} \subset L^\infty(\mathbb{C}^n) . \]

Here we write \(\mathcal{M}\) for the operator norm closure of a given set \(\mathcal{M} \subset L(F^2_1)\).

As was mentioned in the introduction, there are certain drawbacks of using CCR algebras as quantum mechanical models. A new algebraic framework was suggested in [7] which partly overcomes the above mentioned obstructions. Therein, the authors consider the resolvent algebra, which is the \(C^*\) algebra generated by the resolvents of the (in general unbounded) operators \(\phi(f)\).

**Definition 2.2** (D. Buchholz, H. Grundling [7]). Given a symplectic space \((X, \sigma)\) we define \(\mathcal{R}_0(X, \sigma)\) as the universal unital \(+\)-algebra generated by the set \(\{ R(\lambda, f) : \lambda \in \mathbb{R} \setminus \{0\}, f \in X \} \) together with the relations:

1. \(R(\lambda, 0) = -\frac{i}{\lambda} 1\),
2. \(R(\lambda, f)^* = R(-\lambda, f)\),
3. \(\nu R(\nu \lambda, \nu f) = R(\lambda, f)\),
4. \(R(\lambda, f) - R(\mu, f) = i(\mu - \lambda) R(\lambda, f)R(\mu, f)\),
5. \([R(\lambda, f), R(\mu, g)] = i\sigma(f, g) R(\lambda, f)R(\mu, g)^2 R(\lambda, f)\),
6. \(R(\lambda, f)R(\mu, g) = R(\lambda + \mu, f + g) [R(\lambda, f) + R(\mu, g) + i\sigma(f, g) R(\lambda, f)^2 R(\mu, g)]\),

for \(\lambda, \mu, \nu \in \mathbb{R} \setminus \{0\}\) and \(f, g \in X\).

The resolvent algebra \(\mathcal{R}(X, \sigma)\) is defined as the closure of \(\mathcal{R}_0(X, \sigma)\) with respect to a certain semi-norm obtained through the GNS construction (cf. [2]).
Remark 2.3. 1. Note that Equations (1) and (2) encode \( \phi(0) = 1 \) and the self-adjointness of \( \phi(f) \), respectively. The third and sixth equation reflect the R-linearity of \( \phi \), whereas Equation (4) is the resolvent identity. Finally, the fifth equation is the substitute for the (CCR).

2. When one considers a representation of \( \mathcal{R}_0(X, \sigma) \) as a concrete \( \ast \)-algebra generated by resolvents of self-adjoint operators on a Hilbert space, then \( \mathcal{R}(X, \sigma) \) coincides with the operator norm closure of \( \mathcal{R}_0(X, \sigma) \).

In the present paper we study the Fock-Bargmann representation of the resolvent algebra over the standard symplectic space \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) as well as on an infinite dimensional Hilbert space with a symplectic structure. In the finite dimensional setting of \( \mathbb{C}^n \) it is well-known (cf. [3, 8] and Section 3) that the generators are resolvents of self-adjoint Toeplitz operators \( T_{2\sigma(z),z} \), \( z \in \mathbb{C}^n \) defined below.

Therefore, the resolvent algebra \( \mathcal{R}(\mathbb{C}^n, \sigma) \) is the \( \mathbb{C}^* \) algebra generated by the resolvents of Toeplitz operators:

\[
\mathcal{R}(\mathbb{C}^n, \sigma) \cong L^2(\mathbb{C}^n, \mu_t) \cap \text{Hol}(\mathbb{C}^n),
\]

where \( \mu_t \) denotes the Lebesgue measure on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). The Fock-Bargmann space \( F_t^2 = F_t^2(\mathbb{C}^n) \) is defined as

\[
F_t^2 := L^2(\mathbb{C}^n, \mu_t) \cap \text{Hol}(\mathbb{C}^n),
\]

where \( \text{Hol}(\mathbb{C}^n) \) denotes the space of entire functions on \( \mathbb{C}^n \). The reader experienced with the analysis on such spaces may wonder why we introduce the parameters \( t_1, \ldots, t_n \) in the above definition, as the standard approach corresponds to the choice \( t_1 = t_2 = \cdots = t_n = t > 0 \). We will use the notation \( F_t^2 = F(t, \ldots, t) \) when we are in this standard situation. Conceptually, the more general setup does not cause any problems (apart from longer notations), but will be convenient when we pass to infinite dimensional symplectic spaces.

Throughout the paper we denote the standard inner product of \( L^2(\mathbb{C}^n, \mu_t) \) and of \( F_t^2 \) by

\[
\langle f, g \rangle := \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\mu_t(z)
\]

and we write \( \|f\| := \langle f, f \rangle^{1/2} \) for the induced norm. As is well-known, \( F_t^2 \) is a reproducing kernel Hilbert space with kernel function

\[
K_{t}(z, w) := e^{\frac{\left| z_1 \right|^2}{t_1} + \cdots + \frac{\left| z_n \right|^2}{t_n}},
\]

In fact, this will be the starting point of our analysis.

3 Fock-Bargmann space and Toeplitz operators

Let \( n \in \mathbb{N} \) and consider positive real numbers \( t_j > 0 \) where \( j = 1, \ldots, n \). We write \( t := (t_1, \ldots, t_n) \) and on \( \mathbb{C}^n \) we define the probability measure \( \mu_t \) by

\[
d\mu_t(z) = \frac{1}{\pi^{n}t_1 \cdots t_n} e^{-\left(\frac{|z_1|^2}{t_1} + \cdots + \frac{|z_n|^2}{t_n}\right)} dV(z), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.
\]

Here, \( V \) denotes the Lebesgue measure on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). The Fock-Bargmann space \( F^2_t = F^2_t(\mathbb{C}^n) \) is defined as

\[
F^2_t := L^2(\mathbb{C}^n, \mu_t) \cap \text{Hol}(\mathbb{C}^n),
\]
where \( w = (w_1, \ldots, w_n) \in \mathbb{C}^n \). In order to simplify notations, we write
\[
(z, w)^t := \frac{z_1 \cdot w_1}{t_1} + \cdots + \frac{z_n \cdot w_n}{t_n}
\]
and
\[
\|z\|^2_t := \langle z, z \rangle_t
\]
(3.2)
such that \( K^t_w(z) = e^{\langle z, w \rangle_t} \). We express the normalized reproducing kernels \( k^t_w \in F^t_2 \) as
\[
k^t_w(z) := \frac{K^t_w(z)}{\|K^t_w\|} = e^{\langle z, w \rangle_t - \frac{1}{2} \|w\|^2_t}.
\]

Let \( A \in \mathcal{L}(F^t_2) \) be a bounded linear operator on \( F^t_2 \). We define the Berezin transform \( \tilde{A} \) of \( A \) by
\[
\tilde{A}(z) := \langle Ak^t_z, k^t_z \rangle, \quad z \in \mathbb{C}^n.
\]
Note that \( \tilde{A}(z) \) is a bounded real-analytic function. Moreover, the map \( A \rightarrow \tilde{A} \) is known to be injective, [12].

As \( F^t_2 \) is a closed subspace of \( L^2(\mathbb{C}^n, \mu_t) \), there is an orthogonal projection \( P^t : L^2(\mathbb{C}^n, \mu_t) \rightarrow F^t_2 \) acting on \( f \in L^2(\mathbb{C}^n, \mu_t) \) as
\[
P^t(f)(w) = \langle f, K^t_w \rangle, \quad w \in \mathbb{C}^n.
\]

For a measurable function \( \varphi : \mathbb{C}^n \rightarrow \mathbb{C} \), we let \( T^t_\varphi \) denote the Toeplitz operator with symbol \( \varphi \), which is defined by
\[
T^t_\varphi(g)(z) := P^t(\varphi g)
\]
on the natural domain \( D(T^t_\varphi) = \{ g \in F^t_2 : \varphi g \in L^2(\mathbb{C}^n, \mu_t) \} \).

If \( \varphi \in L^\infty(\mathbb{C}^n) \), then \( T^t_\varphi \) is a bounded operator. For a given set \( S \subset L^\infty(\mathbb{C}^n) \), we will denote by \( T^t(S) \) the \( C^* \) algebra generated by all Toeplitz operators with symbols in \( S \) and set
\[
T^t := T^t(L^\infty(\mathbb{C}^n))
\]
for the full Toeplitz algebra. We also define \( T^t_{\mathrm{lin}}(S) \) to be the closed linear span of Toeplitz operators with symbols in \( S \).

Let us introduce Weyl operators on the Fock-Bargmann space \( F^t_2 \): if \( z \in \mathbb{C}^n \) then we define \( W^t_z \in \mathcal{L}(F^t_2) \) by
\[
W^t_z(g)(w) = k^t_z(w)g(w - z).
\]
(3.3)

Weyl operators are well-known to be unitary. Moreover, they fulfill the relation
\[
(W^t_z)^* = (W^t_z)^{-1} = W^t_{-z} \quad \text{and} \quad W^t_z W^t_w = e^{-i\sigma_t(z, w)}W^t_{z+w}.
\]
Here, the symplectic form \( \sigma_t \) on \( \mathbb{C}^n \) with parameter \( t \) is defined by
\[
\sigma_t(w, z) := \frac{\Im(w_1 \cdot \ov{z_1})}{t_1} + \cdots + \frac{\Im(w_n \cdot \ov{z_n})}{t_n} = \Im\langle w, z \rangle_t.
\]

As a matter of fact, Weyl operators are Toeplitz operators themselves. If we define the family \( (g^t_z)_{z \in \mathbb{C}^n} \) of bounded functions on \( \mathbb{C}^n \) by
\[
g^t_z(w) := e^{\frac{1}{2} \|w\|^2 + 2i\sigma_t(w, z)},
\]
then it holds
\[
W^t_z = T^t_{g^t_z},
\]
(3.4)

In fact, (3.4) follows by showing that the Berezin transform of both sides coincide (see [3, 8] and the references therein).
4 Resolvent algebra in Fock-Bargmann representation

In this section, we study the resolvent algebra \( R(\mathbb{C}^n, \sigma_t) \) in its Fock-Bargmann representation. Of course, it is not hard to reduce the analysis of \( R(\mathbb{C}^n, \sigma_t) \) to that of \( R(\mathbb{C}^n, \sigma) \). Again, we emphasize that the extra flexibility coming from the parameter set \( t = (t_1, \ldots, t_n) \) will be useful when passing to the infinite dimensional limit \( n \to \infty \). Therefore, we take the (mostly notational) burden upon us to carry \( t \) all the way through this section. For readability, we will denote the Fock-Bargmann representation of the resolvent algebra also by \( R(\mathbb{C}^n, \sigma_t) \).

Since the Toeplitz operators \( T^t_{2\sigma_t(z)} \sigma_t(z) \) fulfill the canonical commutation relation (equivalently, the Weyl operators \( W^t_z \) fulfill the relation of the CCR algebra), the resolvent algebra is the \( C^* \) algebra generated by resolvents of Toeplitz operators. The following integral representations allow us to study \( R(\mathbb{C}^n, \sigma_t) \) more in detail:

**Lemma 4.1.** For \( z \in \mathbb{C}^n \setminus \{0\} \) the map \( \mathbb{R} \ni s \mapsto W^t_{sz} \) defines a strongly continuous unitary one-parameter group with generator \( iT^t_{2\sigma_t(z)} \). In particular, for \( \lambda > 0 \) the following integral representations of the resolvents hold in the strong sense:

\[
(T^t_{2\sigma_t(z)} + i\lambda)^{-1} = (T^t_{2\sigma_t(z)+i\lambda})^{-1} = -i \int_0^\infty e^{-\lambda s} W^t_{sz} ds
\]

(4.1)

and

\[
(T^t_{2\sigma_t(z)} - i\lambda)^{-1} = (T^t_{2\sigma_t(z)-i\lambda})^{-1} = i \int_0^\infty e^{-\lambda s} W^t_{sz} ds.
\]

(4.2)

**Proof.** It is easy to check that \( R \ni s \mapsto W^t_{sz} \) is indeed a strongly continuous unitary one-parameter group. For determining its generator we compute

\[
\lim_{s \to 0} \frac{W^t_{sz} f - f}{s} = \lim_{s \to 0} \frac{T^t_{g_{sz}} f - f}{s} = \lim_{s \to 0} T^t_{(g_{sz})^{-1}/s} f
\]

for all \( f \in F^2_t \) such that the limit exists. According to Stone’s Theorem there is a unique self-adjoint operator \( A \) with domain \( D(A) \) such that \( iA \) generates the group, i.e. for \( f \in D(A) \) it holds

\[
\lim_{s \to 0} T^t_{(g_{sz})^{-1}/s} f = iAf
\]

(4.3)

in the norm sense. Since norm convergence in the reproducing kernel Hilbert space \( F^2_t \) implies pointwise convergence, it suffices to determine the pointwise limit of the right hand side of (4.3). Let \( u \in \mathbb{C}^n \) and note that:

\[
\frac{T^t_{g_{sz}} - 1}{s} f(u) = \int_{\mathbb{C}^n} \frac{1}{s} \left( e^{\frac{1}{2} ||z||^2 + 2i\sigma_t(w,z)} - 1 \right) e^{(u,w)_t} f(w) d\mu_t(w).
\]

The integrand converges as \( s \to 0 \):

\[
\frac{1}{s} \left( e^{\frac{1}{2} ||z||^2 + 2i\sigma_t(w,z)} - 1 \right) \xrightarrow{s \to 0} 2i\sigma_t(w,z).
\]
An easy application of the dominated convergence theorem yields

\[ T_{(s^2 - 1)/s} f(u) \to iT_{2\sigma_t(z)} f(u) \]

for those \( f \in F_t^2 \) such that \( w \mapsto \sigma_t(w, z) f(w) \in L^2(C^n, \mu_t) \). Hence,

\[ T_{(s^2 - 1)/s} f \to iT_{2\sigma_t(z)} f = iAf \quad \text{for} \quad f \in D(A). \]

Therefore, \( iT_{2\sigma_t(z)} \) is the generator of the one-parameter group \( s \mapsto W_t z \). Since all operators \( W_t z \) are unitary, the group \( (W_t z) \) has growth bound \( \omega_0 = 0 \). For \( \lambda > 0 \) it follows (cf. [10, Theorem I.1.10])

\[ \left( \lambda - iT_{2\sigma_t(z)} \right)^{-1} = \int_0^\infty e^{-\lambda s} W_t z ds \]

strongly, i.e.

\[ \left( T_{2\sigma_t(z)} + i\lambda \right)^{-1} = -i \int_0^\infty e^{-\lambda s} W_t z ds. \]

The integral representation (4.2) follows from:

\[ \left( T_{2\sigma_t(z)} - i\lambda \right)^{-1} = \left( (T_{2\sigma_t(z)} + i\lambda)^{-1} \right)^* = i \int_0^\infty e^{-\lambda s} W_{-2\sigma_t(z)} ds, \]

which completes the proof. \( \square \)

**Remark 4.2.** An inspection of the above arguments shows that the limit in Equation (4.3) exists for all \( f = K^t_w, w \in C^n \). Moreover, one easily verifies that the space \( \mathcal{X} := \text{span}\{K^t_w : w \in C^n\} \) is invariant under the action of \( W_t z \). Then, [17] Theorem VIII.10 implies that \( T_{2\sigma_t(z)} \) is essentially self-adjoint on \( \mathcal{X} \).

Define for \( z \in C^n, \lambda \in C \setminus i\mathbb{R} \):

\[ R_t(\lambda, z) := (T_{2\sigma_t(z)} - i\lambda)^{-1}. \]

For simplicity, we will occasionally suppress \( t \) in the notation and shortly write \( R(\lambda, z) = R_t(\lambda, z) \). Before we continue with our investigation, we need to recall the following well-known expansion of the resolvent.

**Lemma 4.3.** Let \( \lambda_0, \lambda \in C \setminus i\mathbb{R} \) such that \( |\lambda_0 - \lambda| < |\lambda_0| \). Then, it holds

\[ R(\lambda, z) = \sum_{k=0}^\infty (\lambda - \lambda_0)^k i^k R(\lambda_0, z)^{k+1}, \quad (4.4) \]

where the series converges in operator norm. In particular:

\[ R(\lambda_0, z)^k = \frac{i^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} |\lambda = \lambda_0| R(\lambda, z). \quad (4.5) \]

The previous lemma has the following important consequence:

**Corollary 4.4.** \( \mathcal{R}(C^n, \sigma_t) = C^*(R_t(\lambda, z) : \lambda \in C \setminus i\mathbb{R}, z \in C^n) \).
We further note that Equations (4.1) and (4.2) extend to $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ with the same proof as in the case $\lambda \in \mathbb{R} \setminus \{0\}$:

$$R(\lambda, z) = i \int_0^\infty e^{-\lambda s} W^t_{sz} ds, \quad \text{Re}(\lambda) > 0$$

and

$$R(\lambda, z) = -i \int_0^\infty e^{\lambda s} W^t_{sz} ds, \quad \text{Re}(\lambda) < 0.$$

We mention two ways of analyzing the connection between the algebra $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ in its Fock-Bargmann space representation and the theory of Toeplitz operators and, more precisely, its realization as a Toeplitz algebra. The computational-heavy method studies the resolvents through their Laplace transform representation, using that the Weyl operators themselves are Toeplitz operators. Alternatively, one can use “soft analysis” arguments from the theory of Toeplitz operators, most prominently those arising in the theory of quantum harmonic analysis [20, 13]. Both approaches have their advantages and so we will try to shed some light on either of them.

4.1 Correspondence Theory in the Fock-Bargmann space

We recall some aspects of quantum harmonic analysis in the setting of the Fock-Bargmann space and explain a particular part of it: the correspondence theory. We refer to [20, 13] for further details. Roughly speaking, correspondence theory relates certain subspaces of $\text{BUC}(\mathbb{C}^n)$, the $\mathcal{C}^*$ algebra of bounded, uniformly continuous functions on $\mathbb{C}^n$, with corresponding subspaces of $L^2(F_t^2)$. We start with some notation. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function and $z \in \mathbb{C}^n$, then put:

$$\alpha_z(f)(w) := f(w - z).$$

Given an operator $A \in L^2(F_t^2)$ we define its shift by $z \in \mathbb{C}^n$ through

$$\alpha_z(A) = W^t_zAW^t_z = W^t_zA(W^t_z)^*,$$

where $W^t_z$ is a Weyl operator. Clearly, $\text{BUC}(\mathbb{C}^n)$ is the subalgebra of $L^\infty(\mathbb{C}^n)$ consisting of functions $f$ for which $z \mapsto \alpha_z(f)$ is continuous with respect to the $L^\infty$-norm. The analogous space on the operator side is

$$\mathcal{C}_t^1 = \{ A \in L^2(F_t^2) : z \mapsto \alpha_z(A) \text{ is } \| \cdot \|_{\text{op}-\text{continuous}} \}.$$

The Fock-Bargmann space formulation of the correspondence theorem due to R. Werner in [20] specifically tailored for Toeplitz operators can be found in [13] within the standard situation $t_1 = t_2 = \cdots = t_n = t > 0$. The case of positive weight parameters $t_1, \ldots, t_n$ in the definition of the Gaussian measure $\mu_t$ follows by identical proofs and obvious modifications:

**Theorem 4.5** (Correspondence Theorem, [20, 13]). Let $\mathcal{D}_t \subset \mathcal{C}_t^1$ be a closed, $\alpha$-invariant subspace (meaning $\alpha_z(A) \in \mathcal{D}_t$ for every $z \in \mathbb{C}^n$, $A \in \mathcal{D}_1$). Then, there is a unique $\alpha$-invariant closed subspace $\mathcal{D}_0$ of $\text{BUC}(\mathbb{C}^n)$ such that

$$\mathcal{D}_1 = T_{\text{lin}}(\mathcal{D}_0).$$
Further, for an operator $A \in C^1_t$ the following statements are equivalent:

$$A \in D_1 \iff \tilde{A} \in D_0.$$ 

Finally, $D_0$ can be computed as follows: $D_0 = \{ A : A \in D_1 \}$. In the following we call $D_0$ and $D_1$ corresponding spaces.

Here, we used the notation

$$T_{lin}^t(D_0) = \{ T_f : f \in D_0 \}.$$ 

The previous result is a useful tool in the study of Toeplitz operators and Toeplitz algebras when it is combined with the following:

**Theorem 4.6** ([13]). The following $C^*$ algebras coincide: $C^1_t = T^t$.

We mention that the previous two results extend to the case of the $p$-Fock space $F^p_t$ where $p \neq 2$, cf. [13, 14]. They have immediate applications such as simple proofs of a compactness characterization for operators acting on the Fock-Bargmann space. We only present the case $p = 2$.

Let $K(\mathcal{H})$ denote the ideal of compact operators on a Hilbert space $\mathcal{H}$.

**Theorem 4.7** ([1]). Let $A \in \mathcal{L}(F^2_t)$. Then, the following are equivalent:

$$A \in K(F^2_t) \iff A \in \mathcal{T}^t \text{ and } \tilde{A} \in C_0(C^n).$$

Here, $C_0(C^n)$ denotes the space of continuous complex valued functions vanishing at infinity.

The short proofs of the last theorems based on correspondence theory can be found in [13]. We now want to demonstrate how Theorem 4.6 can be applied in order to gain a better understanding of the resolvent algebra. First, we note:

**Lemma 4.8.** Let $z, w \in C^n$ and $\lambda \in C \setminus iR$. Then:

$$\alpha_w(R(\lambda, z)) = R(\lambda + 2i\sigma_t(z, w), z).$$

**Proof.** Applying the CCR of Weyl operators, we get for $\text{Re}(\lambda) > 0$:

$$\alpha_w(R(\lambda, z)) = W^t_{w^*} \int_0^\infty e^{-\lambda s} W^t_{-sz} ds W^t_{-sw} = i \int_0^\infty e^{-\lambda s} W^t_{w^*} W^t_{sz} W^t_{-w} ds = i \int_0^\infty e^{-\lambda t} e^{-2i\sigma_t(z, w)} W^t_{sz} ds = R(\lambda + 2i\sigma_t(z, w), z).$$

The case $\text{Re}(\lambda) < 0$ follows analogously. □

**Proposition 4.9.** $\mathcal{R}(C^n, \sigma_t)$ is a closed, $\alpha$-invariant subspace of $C^1_t$. 

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Proof. The $\alpha$-invariance immediately follows from the previous lemma. Further, the resolvent algebra is closed by definition. We only need to prove that it is contained in $C^1_t$. It suffices to show that $R_t(\lambda, z) \in C^1_t$ for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ and $z \in \mathbb{C}^n$. This is an easy consequence of Lemma 4.3 and Lemma 4.8. In fact, for $|w|$ sufficiently small such that simultaneously

$$2|\sigma_t(z, w)| < |\lambda| \quad \text{and} \quad |2\sigma_t(z, w)||R(\lambda, z)| < 1$$

it follows:

$$||R(\lambda, z) - \alpha_w(R(\lambda, z))|| = ||R(\lambda, z) - R(\lambda + 2i\sigma_t(z, w), z)||$$

$$\leq \sum_{k=1}^{\infty} |2\sigma_t(z, w)|^k ||R(\lambda, z)||^{k+1}$$

$$= \frac{2|\sigma_t(z, w)||R(\lambda, z)||^2}{1 - 2|\sigma_t(z, w)||R(\lambda, z)||}.$$}

This last expression tends to 0 as $|w| \rightarrow 0$. □

Proposition 4.9 and the correspondence theorem (Theorem 4.5) imply that there is a closed, $\alpha$-invariant subspace $D^t_0$ of $BUC(C^n)$ (possibly depending on the parameter tuple $t$) such that

$$R(C^n, \sigma_t) = T^t_{lin}(D^t_0).$$

Since $R(C^n, \sigma_t)$ itself is a $C^*$ algebra, we get:

**Theorem 4.10.** $R(C^n, \sigma_t)$ is a Toeplitz algebra. More precisely,

$$R(C^n, \sigma_t) = T_f^t(D^t_0) = C^*(\{f \in D^t_0\})$$

for a suitable $\alpha$-invariant subspace $D^t_0 \subset BUC(C^n)$.

Our next task is to determine the space $D^t_0$ explicitly.

### 4.2 Computing $D^t_0$

Correspondence theory has the nice flavour that in several examples the corresponding spaces (in the sense of Theorem 4.5) are what one might naively expect; for example, $C_0(C^n)$ corresponds to the compact operators $K(F^2_t)$ and the almost periodic functions correspond to the CCR algebra. In this respect, one might hope that the space $D^t_0$ corresponding to $R(C^n, \sigma_t)$ is:

$$R(C^n, \sigma_t) = C^* \left( \{(\lambda - 2i\sigma_t(\cdot, z))^{-1} : \lambda \in \mathbb{C} \setminus i\mathbb{R}, z \in \mathbb{C}^n \} \right).$$

Note that $R_{cl}$ does not depend on the choice of the parameter set $t = (t_1, \ldots, t_n)$ and we can replace $\sigma_t$ by $\sigma = \sigma_1$ in its definition (the “contribution” of the parameters $t_k$ can be absorbed into $z$).

Since $R_{cl}$ is indeed a subalgebra of $BUC(C^n)$, which is further $\alpha$-invariant as well as invariant under the parity operation $f \mapsto f(-\cdot)$, Theorem 3.13 in [13] implies that:

$$T^t_{lin}(R_{cl}) = T^t(R_{cl}).$$

Therefore, $D^t_0 = R_{cl}$ if and only if $R(C^n, \sigma_t) = T^t(R_{cl})$. Next, we collect some useful facts:
Proposition 4.11. The following inclusions hold:

1. $C_0(\mathbb{C}^n) \subset R_{cl}$.

2. $K(F^2_t) \subset R(\mathbb{C}^n, \sigma_t)$.

Proof. The first statement is an easy application of the Stone-Weierstrass theorem; the second statement is [7, Theorem 5.4].

In what follows some computations can no longer be avoided. The Berezin transform of a product of resolvents can be computed explicitly, which turns out to be quite useful. We will not need the formula in full generality, but still provide the complete expression here.

Proposition 4.12. Let $m \in \mathbb{N}$ and $z_1, \ldots, z_m \in \mathbb{C}^n$. Given a multi-index $k = (k_1, \ldots, k_m) \in \mathbb{N}^m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C} \setminus i\mathbb{R}$ we have:

$$\left(R(\lambda_1, z_1)^{k_1} \cdots R(\lambda_m, z_m)^{k_m}\right)^\sim (w) = C \prod_{j=1}^m \lambda_j^{k_j}$$

where

$$C = C_{k, \lambda, m} := (-1)^{|k| - m} \prod_{j=1}^m \text{sign}(\text{Re}(\lambda_j))^{k_j} \in \{-1, 1\}.$$

Here, we have used the standard multi-index notation together with:

$$1 := (1, \ldots, 1) \in \mathbb{N}^m,$$

$$\Lambda := (\text{sign}(\text{Re}(\lambda_1))\lambda_1, \ldots, \text{sign}(\text{Re}(\lambda_m))\lambda_m),$$

$$s \cdot z = s_1 z_1 + \cdots + s_m z_m \in \mathbb{C}^n.$$

Proof. Using the relation $R(-\lambda, z) = -R(\lambda, -z)$, we can reduce the proof to the case where $\text{Re}(\lambda_j) > 0$ for $j = 1, \ldots, m$. From Lemma 4.3 we obtain that

$$\left(R(\lambda_1, z_1)^{k_1} \cdots R(\lambda_m, z_m)^{k_m}\right)^\sim (w) = \frac{i^{k-1} \prod_{j=1}^m \text{sign}(\text{Re}(\lambda_j))^{k_j}}{(k-1)!} \left. \frac{\partial^{k-1}}{\partial \mu_k^{k-1}} \right|_{\mu = \Lambda} R(\mu_1, z_1)^{k_1} \cdots R(\mu_m, z_m)^{k_m}.$$

where we use the short notation:

$$\frac{\partial^{k-1}}{\partial \mu_k^{k-1}} \left|_{\mu = \Lambda} := \prod_{\mu_{\ell} = \lambda_\ell}^{k_{\ell}-1} \right.$$

According to Lemma 4.5, we have:

$$\left(R(\mu_1, z_1) \cdots R(\mu_m, z_m)\right)^\sim (w) = \alpha_w \left(R(\mu_1, z_1) \cdots R(\mu_m, z_m)\right)^\sim (0)$$

$$= \left(\alpha_w R(\mu_1, z_1) \cdots \alpha_w R(\mu_m, z_m)\right)^\sim (0)$$

$$= \left(R(\mu_1 - 2i\sigma(t) z_1, z_1) \cdots R(\mu_m - 2i\sigma(t) z_m, z_m)\right)^\sim (0).$$
By using analyticity of the resolvent maps \( \mu_j \mapsto R(\mu_j, z_j) \in \mathcal{L}(F^2) \) it follows that the difference quotients converge in operator norm. Hence, differentiation can be interchanged with the inner product, which yields:

\[
\frac{(k - 1)!}{i^{|k| - m}} \left( R(\lambda_1, z_1) R(\lambda_2, z_2) \ldots R(\lambda_m, z_m) \right) \langle w \rangle
= \frac{\partial^{k-1}}{\partial \mu^{k-1}} \biggr|_{\mu = \lambda} \left( R(\mu_1, z_1) \ldots R(\mu_m, z_m) \right) \langle w \rangle
= \frac{\partial^{k-1}}{\partial \mu^{k-1}} \biggr|_{\mu = \lambda} \left( R(\mu_1 - 2i\sigma(t, z_1), z_1) \ldots R(\mu_m - 2i\sigma(t, z_m), z_m) \right) \langle (0) \rangle.
\]

Now, we insert the integral expression of the resolvent in (4.6):

\[
(*) = \int e^{-\sum_j |\mu_j + 2i\sigma(t, z_j)|^2} \int_{\Omega_m} e^{i\sum_j \mu_j \sigma_j} \langle W_{s, z}^t 1, 1 \rangle \, ds.
\]

The inner product in the integrand can be calculated explicitly:

\[
\langle W_{s, z}^t 1, 1 \rangle = e^{-\frac{1}{2}||s||^2_t}.
\]

Finally, the assertion follows by inserting this expression into the last integral and performing the \( \mu \)-derivatives.

Similarly, one computes the Berezin transform of the classical resolvent functions:

**Lemma 4.13.** Let \( m \in \mathbb{N}, z_1, \ldots, z_m \in \mathbb{C}^n \) and \( k := (k_1, \ldots, k_m) \in \mathbb{N}^m \). For any set of complex numbers \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \setminus i\mathbb{R} \) consider the function

\[
g := (\lambda_1 - 2i\sigma(t, z_1))^{-k_1} \ldots (\lambda_m - 2i\sigma(t, z_m))^{-k_m}.
\]

With the notation in Proposition 4.12 the Berezin transform of \( g \) is given by:

\[
\tilde{g}^{(t)}(w) := (T^t_x)^{-1}(w) = \frac{1}{(k - 1)!} \int_{(0, \infty)^m} s^{k-1} e^{-\Lambda s + 2i\sigma(t, s, z)} \langle w, s \rangle ds.
\]

**Proof.** The lemma follows by a direct calculation. First, note that

\[
\tilde{g}^{(t)}(w) = \langle \alpha_w(g) 1, 1 \rangle = \int_{\mathbb{C}^n} g(v + w) d\mu_t(v). \tag{4.8}
\]

Without loss of generality may assume that Re(\( \lambda_j \)) > 0 for \( j = 1, \ldots, m \) such that \( g \) has an integral representation (Laplace transform):

\[
g(w) = \frac{1}{(k - 1)!} \int_{(0, \infty)^m} s^{k-1} e^{-\Lambda s + 2i\sigma(t, s, z)} ds.
\]

Inserting the last expression into (4.8) and interchanging the order of integrations shows:

\[
\tilde{g}^{(t)}(w) = \frac{1}{(k - 1)!} \int_{(0, \infty)^m} s^{k-1} e^{-\Lambda s + 2i\sigma(t, s, z)} \int_{\mathbb{C}^n} e^{2i\sigma(t, s, z)} d\mu_t(v) \, ds.
\]
The inner integration can be evaluated explicitly:

\[
\int_{\mathbb{C}^n} e^{2i\sigma_t(v,z)} \, d\mu_t(v) = \langle K_s^t, K_{-s}^t \rangle = e^{-\|s\|^2_t},
\]

which implies the statement of the lemma.

We now prove a first inclusion of algebras:

**Lemma 4.14.** It holds \( R(\lambda, z) \in T^t(\mathcal{R}_{cl}) \) for every \( \lambda \in \mathbb{C} \setminus i\mathbb{R} \) and \( z \in \mathbb{C}^n \). In particular,

\[
\mathcal{R}(\mathbb{C}^n, \sigma_t) \subset T^t(\mathcal{R}_{cl}).
\]

**Proof.** Since we already know that \( R(\mathbb{C}^n, \sigma_t) \subset C^1_t \), it suffices to show that \( \tilde{R}(\lambda, z) \in \mathcal{R}_{cl} \) by the correspondence theorem. Using \( R(\lambda, z) = -R(-\lambda, -z) \) we may assume without loss of generality that \( \text{Re}(\lambda) > 0 \). By transformation of the integral and according to Lemma 4.13:

\[
\tilde{R}(\lambda, z)(w) = -i \int_0^\infty e^{-\lambda s - 2i\sigma_t(w,z) - \frac{s^2}{2}\|z\|^2_t} \, ds
\]

\[
= i\sqrt{2} \int_0^\infty e^{-\lambda\sqrt{2}s - 2i\sigma_t(\sqrt{2}w,z) - \frac{s^2}{2}\|z\|^2_t} \, ds
\]

\[
= i\sqrt{2} \tilde{g}^{(t)}(\sqrt{2}w),
\]

where \( g \) is a classical resolvent, namely:

\[
g(w) = (\sqrt{2}\lambda - 2i\sigma_t(w,-z))^{-1}.
\]

Note that the Berezin transform \( \tilde{\sim}^{(t)} \) behaves under dilations as follows:

\[
f \circ (\rho w) = \rho^2 \cdot \tilde{\sim}^{(t/\rho^2)}(w), \quad \rho > 0,
\]

where \( f \in L^\infty(\mathbb{C}^n) \), \( f_\rho(w) := f(\rho w) \) and \( t/\rho^2 = (t_1/\rho^2, \ldots, t_n/\rho^2) \). Hence, we obtain:

\[
\tilde{R}(\lambda, z)(w) = i2\sqrt{2} \cdot \tilde{g}^{(t/2)}(\sqrt{2}w).
\]

Since \( \tilde{g}\sqrt{\pi} \) again is a resolvent function and \( \mathcal{R}_{cl} \) is invariant under the Berezin transform \( \tilde{\sim}^{(t/2)} \) (which is simply the convolution by an appropriate Gaussian function), the inclusion \( R(\lambda, z) \in \mathcal{R}_{cl} \) follows.

Recall that an isotropic subspace \( V \subset \mathbb{C}^n \) is a (real) subspace such that \( \sigma_t(z,w) = 0 \) for all \( z,w \in V \). Every isotropic subspace \( V \) is of real dimension \( \leq n \), and if \( \dim_{\mathbb{R}}(V) = n \), then \( V \) is called Lagrangian. To every Lagrangian subspace \( V \subset \mathbb{C}^n \) there exists a complementary Lagrangian subspace \( V' \subset \mathbb{C}^n \), i.e. \( \mathbb{C}^n = V \oplus V' \). Indeed, one can choose \( V' := \{iz : z \in V\} \) and we will make this choice in the following for convenience.

If we now fix a Lagrangian subspace \( V \), then Proposition 4.12 shows that the unital \( C^* \) algebra

\[
\mathcal{R}_V := C^*(R(\lambda, z) : \lambda \in \mathbb{C} \setminus i\mathbb{R}, \, z \in V)
\]

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is commutative. Note that $\mathcal{R}_V$ is also $\alpha$-invariant according to Lemma 4.8.

It is our next aim to show that $\mathcal{R}_V$, in the sense of Theorem 4.5, corresponds to the space

$$\mathcal{R}_{cl,V} := C^*\left((\lambda - 2i\sigma t(\cdot, z))^{-1} : \ z \in V, \ \lambda \in \mathbb{C} \setminus i\mathbb{R}\right),$$

i.e.

$$\mathcal{R}_V = T^*_{lin}(\mathcal{R}_{cl,V}).$$

Before we approach this goal, note the following facts:

First, by Theorem 3.13 of [14], we have $T^*_{lin}(\mathcal{R}_{cl,V}) = T^*(\mathcal{R}_{cl,V})$, i.e. it suffices to prove that $\mathcal{R}_V = T_t(\mathcal{R}_{cl,V})$. Secondly, since $\mathcal{R}_{cl,V}$ is also invariant under dilations $f \mapsto f(\lambda \cdot)$ where $\lambda > 0$, we obtain $\mathcal{R}_V \subseteq T_t(\mathcal{R}_{cl,V})$ as in the proof of Lemma 4.14. Therefore, we only need to prove that $T_t g \in \mathcal{R}_V$ where $g$ is a product of classical resolvent functions $(\lambda - 2i\sigma t(\cdot, z))^{-1}$ with $z \in V$. Since the operator algebra $\mathcal{R}_V$ is commutative, we have more techniques at hand for obtaining this goal. In particular, Gelfand theory turns out to be useful here. The first conceptual goal is therefore describing the Gelfand spectrum of $\mathcal{R}_V$.

Using the explicit formulas for the Berezin transform, one observes that point evaluations of the Berezin transforms are multiplicative linear functionals on $\mathcal{R}_V$. Since $\tilde{A}(v + v') = \tilde{A}(v')$ for every $v \in V$ and $v' \in V'$ and $A \in \mathcal{R}_V$, there is no loss of generality in considering only point evaluations of the Berezin transform at $V'$. Since the Berezin transform is injective, we can expect the Gelfand spectrum of $\mathcal{R}_V$ to be a suitable compactification of $V'$. We will describe this compactification now and start by recalling a compactification of a real inner product space first constructed in [19]. Therein the author described the maximal ideal space of $\mathcal{R}_{cl}$ as such a compactification of $\mathbb{R}^{2n}$.

Let $X$ be a finite-dimensional real inner product space. We denote by $P_Y$ the orthogonal projection onto a given subspace $Y \subset X$. By $\text{Graff}(X)$ we denote the affine Grassmannian of $X$, i.e. the set of all affine subspaces of $X$. As a set, this can be written as

$$\text{Graff}(X) = \{x + Y : \text{Y a linear subspace of } X \text{ and } x \perp Y\}.$$  

The precise topology with which $\text{Graff}(X)$ is endowed can be found in [19] and will not be described here. We will denote by $\gamma X$ the set $\text{Graff}(X)$ endowed with this particular topology. We collect some facts about $\gamma X$ in the following lemma:

**Lemma 4.15** ([19]). Let $X$ be a finite-dimensional real inner product space.

1. $\gamma X$ is a compact Hausdorff space.

2. Together with the embedding $X \ni x \mapsto x + \{0\} \in \gamma X$, $\gamma X$ is a compactification of $X$.

3. A net $(x_i + Y_i)_{i \in I} \subset \gamma X$ converges to $x + Y \in \gamma X$ if and only if the following hold:

---

1To be more precise, therein it was only described for $\mathbb{R}^n$. It is straightforward to generalize the procedure to any finite dimensional real inner product space.

2In [19], the compactification is denoted by $\Omega$. We chose to name it differently, as the symbol $\Omega$ is somewhat ambiguous in a symplectic context.
\[ P_{Y \perp} x_i, \ i \leq k, \]
\[ \text{eventually } Y_i \subseteq Y \]
\[ \text{There is no affine subspace } x' + Y' \subseteq x + Y \text{ such that there exists a}
\[ \text{subnet of } (x_i + Y_i) \text{ with } P_{(Y')^\perp} x_i \to x' \text{ and } Y_i \subseteq Y' \text{ eventually (along}
\[ \text{the subnet).} \]

One of the main theorems of [19] is the following:

**Theorem 4.16 (19).** The Gelfand spectrum \( \mathcal{M}(\mathcal{R}_{cl}) \) of \( \mathcal{R}_{cl} \) can be identified with the above compactification of \( \mathbb{R}^{2n} \), i.e. \( \mathcal{M}(\mathcal{R}_{cl}) \cong \gamma \mathbb{R}^{2n} \).

Let us briefly describe the identification of \( \mathcal{M}(\mathcal{R}_{cl}) \) with \( \gamma \mathbb{R}^{2n} \) in more detail: Given a function \( f \in \mathcal{R}_{cl} \) and an affine line \( x + \text{span}\{ y \} \subset \mathbb{R}^{2n}, y \in \mathbb{R}^{2n} \), it is not hard to verify that \( \lim_{n \to \infty} f(x + ny) \) exists. For an affine subspace \( x + Y \subset \mathbb{R}^{2n} \) and any \( y \in Y \) with \( \|y\| = 1 \), the value of this limit is almost everywhere, with respect to the surface measure on \( \{ y \in Y : \|y\| = 1 \} \), the same. If we denote this value by \( \varphi_{x+Y}(f) \), then this defines a multiplicative linear functional and every element of \( \mathcal{M}(\mathcal{R}_{cl}) \) can be obtained in this way.

Indeed, the same can be done for \( \mathcal{R}_{cl,V} \), and the following holds true:

**Proposition 4.17.** The Gelfand spectrum \( \mathcal{M}(\mathcal{R}_{cl,V}) \) of \( \mathcal{R}_{cl,V} \) can be identified with the above compactification of \( \mathbb{R}^{2n} \), i.e. \( \mathcal{M}(\mathcal{R}_{cl,V}) \cong \gamma V' \).

Here, we consider \( V' \) as a real inner product space with the \( t \)-weighted inner product induced from \( \mathbb{R}^{2n} \cong \mathbb{C}^n \), i.e. \( (z, w) = \sum_{j=1}^{n} \frac{\text{Re}(z_j \bar{w}_j)}{t_j} \). We will not need the result in its full strength and we leave it as an exercise to adapt the arguments from [19]. It is sufficient and elementary to verify that via the embedding \( V' \ni v \mapsto v + \{ 0 \} \in \gamma V' \), every classical resolvent \( (\lambda - 2i\sigma t, \cdot, z)^{-1} \) with \( z \in V \) extends to a function in \( C(\gamma V') \).

As we will see next, the Gelfand spectrum of \( \mathcal{R}_V \) is indeed the same as the one of \( \mathcal{R}_{cl,V} \):

**Proposition 4.18.** The Gelfand spectrum \( \mathcal{M}(\mathcal{R}_V) \) of \( \mathcal{R}_V \) can be identified with \( \mathcal{M}(\mathcal{R}_V) \cong \mathcal{M}(\mathcal{R}_{cl,V}) \cong \gamma V' \), where all multiplicative linear functionals on \( \mathcal{R}_V \) have the form:

\[ \psi_{x+Y}(A) = \varphi_{x+Y}(\tilde{A}|_{V'}). \]

Here, we denote by \( \varphi_{x+Y} \) the multiplicative functional of \( \mathcal{R}_{cl,V} \) introduced above.

Before proving Proposition 4.18 let us derive our intended result from this:

**Theorem 4.19.** It is \( \mathcal{R}_V = T_{lin}(\mathcal{R}_{cl,V}) \).

**Proof.** The inclusion \( \mathcal{R}_V \subset T_{lin}(\mathcal{R}_{cl,V}) \) was already stated above. Let \( g \in \mathcal{R}_{cl,V} \). Since \( \mathcal{R}_{cl,V} \) is translation invariant and closed, we conclude that \( \tilde{g}(t) = T_t g \in \mathcal{R}_{cl,V} = C(\mathcal{M}(\mathcal{R}_V)) \). Therefore, \( \tilde{g}(t) \) is the Gelfand transform of an operator in \( \mathcal{R}_V \). This operator must be \( T_{g} \), i.e. \( T_{g} \in \mathcal{R}_V \).

We will present a sequence of lemmas that lead to a proof of Proposition 4.18.

**Lemma 4.20.** For any resolvent \( R(\lambda, z) \in \mathcal{R}_V \), where \( z \in V \), and \( x, y \in V' \) it holds:

\[ \tilde{R}(\lambda, z)(x + y) = \tilde{R}(\lambda, z)(x + P_{\text{span}\{z\}} y). \]
Proof. Let $z \in V$ and first observe that $\sigma_t(z, y) = 0$ iff $y \perp iz$ (note that $iz \in V'$ by our choice $V' = \{iz : z \in V\}$). Hence, we have

$$R(\lambda, z)(x + y) = R(\lambda, z)(x + (I - P_{\text{span}(iz)})y + P_{\text{span}(iz)}y)$$

$$= R(\lambda, z)(x + P_{\text{span}(iz)}y),$$

where the last equality follows from the formula for the Berezin transform of a resolvent in Proposition 4.12.

Lemma 4.21. For any resolvent $R(\lambda, z) \in \mathcal{R}_V$ where $z \in V$ and any affine line $x + \text{span}\{y\} \in \gamma V'$ the limit $\lim_{\alpha \to \infty} R(\lambda, z)(x + \alpha y)$ exists and is given by

$$\lim_{\alpha \to \infty} R(\lambda, z)(x + \alpha y) = \begin{cases} R(\lambda, z)(x), & \sigma_t(z, y) = 0, \\ 0, & \sigma_t(z, y) \neq 0. \end{cases}$$ (4.9)

Proof. If $\sigma_t(z, y) = 0$, then $y \perp iz$ such that $P_{\text{span}(iz)}y = 0$ and the equality follows from the previous lemma. Assume that $\sigma_t(z, y) \neq 0$ and let $\text{Re}(\lambda) > 0$ (the case $\text{Re}(\lambda) > 0$ follows similarly). According to Proposition 4.12 the Berezin transform of the resolvent at $x + \alpha y$ has the value:

$$R(\lambda, z)(x + \alpha y) = \int_0^\infty e^{-\lambda s - 2i\alpha \sigma_t(y, z) - 2i\sigma_t(z, z) - \frac{s^2}{2}t^{2}} ds.$$

We use integration by parts in order to show that the right hand side converges to 0 as $\alpha \to \infty$. With the short notation $\sigma := \sigma_t(y, z) \neq 0$ we have:

$$R(\lambda, z)(x + \alpha y)$$

$$= -\frac{1}{2i\alpha} \int_0^\infty \frac{d}{ds} e^{-2i\alpha \sigma} e^{-\lambda s - 2i\alpha \sigma_t(x, z) - \frac{s^2}{2}t^{2}} ds$$

$$= -\frac{1}{2i\alpha} \left( e^{-2i\alpha \sigma - \lambda s - 2i\sigma_t(x, z) - \frac{s^2}{2}t^{2}} \right)_{s=0}$$

$$- \int_0^\infty e^{-2i\alpha \sigma} (-\lambda - 2i\sigma_t(x, z) - s\|z\|_t^2) e^{-\lambda s - 2i\sigma_t(x, z) - \frac{s^2}{2}t^{2}} ds.$$
to \(\gamma V'\) is defined by the right hand side of (1.10). For a given affine spaces \(x + Y\) with \(\dim_{\mathbb{R}}(Y) \geq 2\), we distinguish two cases: If \(Y \subseteq \{w \in V' : \sigma_{t}(w, z) = 0\}\), then the value of \(R(\lambda, z)(x + Y)\) coincides with \(R(\lambda, z)(x)\). Otherwise, if \(Y \not\subseteq \{w \in V' : \sigma_{t}(w, z) = 0\}\), then we set it to be zero.

Note that these choices are in accordance with the definition of the multiplicative linear functional \(\varphi_{x+y}\). We explain what we mean by this only in the last example: If \(Y \not\subseteq \{w \in V' : \sigma_{t}(w, z) = 0\}\), then \(\{w \in Y : \sigma_{t}(w, z) = 0\}\) is a subspace of \(Y\) of dimension strictly smaller than the dimension of \(Y\). Hence, the unit sphere of \(\{w \in Y : \sigma_{t}(w, z) = 0\}\) is a zero set with respect to the surface measure of the unit sphere of \(Y\). Hence, \(\varphi_{x+y}(R(\lambda, z))|_{Y'} = 0\).

It remains to prove that the above extension of \(R(\lambda, z))|_{Y'}\) from \(V'\) to the compactification \(\gamma V'\) is in fact continuous. According to Bourbaki’s Extension Theorem, [1, Theorem 1, p. 82], it suffices to show that

\[
\widetilde{R(\lambda, z)}(x_{i}) \to \varphi(\lambda, z)(R(\lambda, z))
\]  

(4.10)

for any net \((x_{i})_{i \in I} \subset V'\) such that \(x_{i} \to 0\). We distinguish several cases:

1. If \(x_{i} \to 0\), then (4.10) follows from the continuity of the Berezin transform \(R(\lambda, z)\) on \(V'\).

2. If \(x_{i} \to 0\) with \(Y \subseteq \{w \in V' : \sigma_{t}(w, z) = 0\}\), then:

\[
\widetilde{R(\lambda, z)}(x_{i}) = \widetilde{R(\lambda, z)}(P_{Y}x_{i}) = \widetilde{R(\lambda, z)}(P_{Y}x_{i}) \to \widetilde{R(\lambda, z)}(x) = \varphi_{x+y}(R(\lambda, z)).
\]

3. If \(x_{i} \to 0\) with \(Y \not\subseteq \{w \in V' : \sigma_{t}(w, z) = 0\}\), then we conclude that

\[
\widetilde{R(\lambda, z)}(x_{i}) \to 0 = \varphi_{x+y}(R(\lambda, z)).
\]

In fact, assuming the opposite there exists a subnet, also denoted by \((x_{i})\), such that \(\widetilde{R(\lambda, z)}(x_{i})\) is bounded away from zero. Take \(w_{0} \in Y\) such that \(\sigma_{t}(w_{0}, z) \neq 0\) and put \(Z \coloneqq \text{span}(w_{0}) \subset Y\). By \(Y' \subset Y\) denote the orthogonal complement of \(Z\) in \(Y\). We obtain a decomposition of \(V'\):

\[
V' = Y^{\perp} \oplus Y = Y^{\perp} \oplus Y' \oplus Z.
\]  

(4.11)

(a) If \(P_{Z}x_{i}\) was unbounded, we could pass to a subnet such that \(\|P_{Z}x_{i}\| \to \infty\). For this subnet, one could show by using the method of stationary phase (as in the proof of Lemma 4.21) that \(\widetilde{R(\lambda, z)}(x_{i}) \to 0\). However, this is impossible as we assumed that \(\widetilde{R(\lambda, z)}(x_{i})\) is bounded away from zero.

(b) If \(P_{Z}x_{i}\) is bounded, we can pass to a subnet such that \(P_{Z}x_{i}\) converges (say, to some \(x_{0} \in Z\)). From the orthogonal decomposition (4.11) we see that:

\[
P_{Y'}x_{i} = P_{Y}x_{i} + P_{Z}x_{i} \to x + x_{0} \in Y^{\perp} \oplus Z.
\]

From \(x_{0} \in Z \subset Y\) it follows: \(x + x_{0} + Y' \subset x + Y\). However this contradicts the assumption \(x_{i} \to 0\). \(\square\)
Since the Berezin transform is multiplicative on $V'$ and $A_k \to A$ in $\mathcal{R}_V$ implies $A_k \to \tilde{A}$ uniformly, we conclude that $\tilde{A}$ for every $A \in \mathcal{R}_V$ extends to a continuous function on $\gamma V'$. Now we can give the proof of Proposition 4.18.

Proof of Proposition 4.18. We claim that the map

$$\Phi : \mathcal{R}_V \to C(\gamma V') : \Phi(A) = [(x + Y) \mapsto \varphi_{x+Y}(\tilde{A}_{|_V})]$$

is a bijective and unital homomorphism of $C^*$ algebras. By what has been said before, $\Phi$ defines a continuous and injective unital $*$-homomorphism. We are left with verifying surjectivity. Since the range of $\Phi$ is an $*$-algebra, we only need to show that it separates points of $\gamma V'$ in order to apply the Stone-Weierstrass Theorem. Showing that the points are separated by $\Phi(\mathcal{R}_V)$ is easy, as we can always choose the Berezin transform of a resolvent for separating two given sets. Careful inspection of the proof of the previous lemma indeed gives a guideline on how to choose the resolvent. We briefly explain this here. In what follows, we always assume that $\text{Re}(\lambda) > 0$.

1. $0 + V'$ is separated from any other affine subspace in the following way: If $Y \subseteq V'$ is a proper subspace, then let $0 \neq v \perp Y$ and consider $R(\lambda, i\alpha v)$ for $\alpha \in \mathbb{R}$. Then, $\alpha$ can be chosen such that

$$\Phi(R(\lambda, i\alpha v))(v + Y) = \psi_{v+Y}(R(\lambda, i\alpha v)) = R(\lambda, i\alpha v)(v) \neq 0.$$

However, $\psi_{0+V'}(R(\lambda, i\alpha v)) = 0$ as long as $\alpha \neq 0$. Hence the affine spaces $v + Y$ and $0 + V'$ are separated. Separating $0 + V'$ from $0 + Y$ can be done by considering $R(\lambda, iv)$.

2. Affine spaces $x_1 + Y \neq x_2 + Y$ are separated as follows: We necessarily have $Y \neq V'$ and $x_1 \neq x_2$. If $x_1 = \rho x_2$ for some $\rho \in \mathbb{R}$, then consider $R(\lambda, i\alpha x_2)$ for suitable $\alpha \in \mathbb{R}$. Otherwise, we can find $z_0 \perp Y$ such that $P_{\text{span}(z_0)}x_1 \neq P_{\text{span}(z_0)}x_2$. Let $z = iz_0 \in V$ such that $Y \subseteq \{w \in V' : \sigma_1(w, z) = 0\}$. Hence for $\alpha \in \mathbb{R} \setminus \{0\}$:

$$\psi_{x_1+Y}(R(\lambda, \alpha z)) = R(\lambda, \alpha z)(P_{\text{span}(z_0)}x_1),$$

$$\psi_{x_2+Y}(R(\lambda, \alpha z)) = R(\lambda, \alpha z)(P_{\text{span}(z_0)}x_2).$$

Now, one has to choose $\alpha$ accordingly such that these values are different (see the formula in Proposition 4.12).

3. If $Y_1 \neq Y_2$ are proper subspaces of $V'$, then we consider two cases:

(a) Recall that we always have $x \perp Y$ for an affine subspace $x + Y$. If $x_1 = \rho x_2$ with $\rho \in \mathbb{R}$ are real multiples of each other, then we obtain with $\alpha \in \mathbb{R}$:

$$\psi_{x_1+Y_1}(R(\lambda, i\alpha x_2)) = R(\lambda, i\alpha x_2)(\rho x_2) = \int_0^\infty e^{-\lambda s} e^{-2i\alpha ||x_2||^2 s - \frac{\rho^2}{2} ||x_2||^2} ds,$$

$$\psi_{x_2+Y_2}(R(\lambda, i\alpha x_2)) = R(\lambda, i\alpha x_2)(x_2) = \int_0^\infty e^{-\lambda s} e^{-2i\alpha ||x_2||^2 s - \frac{\rho^2}{2} ||x_2||^2} ds.$$

Now, arrange $\alpha$ such that these are not equal.
(b) If \( x_1 \) and \( x_2 \) are not real multiples of each other, then it must be either \( P_{\text{span}(x_1)}x_2 \neq x_1 \) or \( P_{\text{span}(x_2)}x_1 \neq x_2 \). Assume that \( P_{\text{span}(x_1)}x_2 \neq x_1 \). For \( \alpha \in \mathbb{R} \):

\[
\psi_{x_1+y_1}(R(\lambda, i\alpha x_1)) = R(\lambda, i\alpha x_1)(x_1)
= \int_0^\infty e^{-\lambda s}e^{-2i\alpha\|x_1\|^2 - \frac{2\lambda}{r^2}\|x_1\|^2} ds.
\]

The value at \( x_2 + y_2 \) now depends on whether \( x_1 \) is orthogonal to \( y_2 \) or not: If \( x_1 \perp y_2 \), then

\[
\psi_{x_2+y_2}(R(\lambda, i\alpha x_1)) = R(\lambda, i\alpha x_1)(x_2)
= \int_0^\infty e^{-\lambda s}e^{-2i\alpha\|x_2\|^2(\text{span}(x_2, x_1)) - \frac{2\lambda}{r^2}\|x_1\|^2} ds.
\]

Since \( \sigma_t(P_{\text{span}(x_1)}x_2, x_1) \neq \|x_1\|^2 \), we can arrange \( \alpha \) such that the values of \( \psi_{x_j+y_j}(R(\lambda, i\alpha x_1)) \) for \( j = 1, 2 \) are different. If, on the other hand, \( x_1 \not\perp y_2 \), then

\[
\psi_{x_2+y_2}(R(\lambda, i\alpha x_1)) = 0.
\]

Hence, we have to arrange \( \alpha \) such that \( \psi_{x_1+y_1}(R(\lambda, i\alpha x_1)) \neq 0 \) (e.g. by letting \( \alpha = 0 \)).

Before continuing, we state a consequence of the previous considerations:

**Corollary 4.23.** Let \( m \in \mathbb{N} \) and \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \setminus i\mathbb{R} \). Consider the function

\[
g = (\lambda_1 - 2i\sigma_t(, z_1))^{-k_1} \cdots (\lambda_m - 2i\sigma_t(, z_m))^{-k_m},
\]

where \( z_1, \ldots, z_m \in V \) and \( V \subset \mathbb{C}^n \) is Lagrangian. Then, \( T_g^* \in \mathcal{R}^{-1}(\mathbb{C}^n, \sigma_t) \).

**Proof.** By definition we have \( g \in \mathcal{R}_{cl,V} \). Now Theorem 4.19 implies that \( T_g^* \in \mathcal{T}_{\mathbb{C}^n}(\mathcal{R}_{cl,V}) = \mathcal{R}_V \subset \mathcal{R}(\mathbb{C}^n, \sigma_t) \).

Now we consider resolvent functions for which the vectors \( z_j \) are either taken from \( V \) or from \( V' \).

**Corollary 4.24.** Let \( V \) and \( V' \) be two Lagrangian subspaces of \( \mathbb{C}^n \) such that \( \mathbb{C}^n = V \oplus V' \). Let \( m, \ell \geq n \) and \( z_1, \ldots, z_m \in V \) and \( w_1, \ldots, w_\ell \in V' \) such that \( V = \text{span}_\mathbb{R}\{z_1, \ldots, z_m\} \) and \( V' = \text{span}_\mathbb{R}\{w_1, \ldots, w_\ell\} \). Set

\[
g = \prod_{j=1}^{m} (\lambda_j - 2i\sigma_t(, z_j))^{-k_j} \prod_{j=1}^{\ell} (\lambda'_j - 2i\sigma_t(, w_j))^{-k'_j},
\]

where \( k_j, k'_j \in \mathbb{N} \) and \( \lambda_j, \lambda'_j \in \mathbb{C} \setminus i\mathbb{R} \). Then, \( T_g^* \in \mathcal{R}(\mathbb{C}^n, \sigma_t) \).

**Proof.** For \( v = v_1 + v_2 \in V \oplus V' = \mathbb{C}^n \) we have

\[
g(v) = \prod_{j=1}^{m} (\lambda_j - 2i\sigma_t(v_2, z_j))^{-k_j} \prod_{j=1}^{\ell} (\lambda'_j - 2i\sigma_t(v_1, w_j))^{-k'_j}.
\]

As \( v_1 \to \infty \), the first factor tends to 0. As \( v_2 \to \infty \), the second factor tends to 0. In conclusion, \( g \in \mathcal{C}_0(\mathbb{C}^n) \). Hence, \( T_g^* \in \mathcal{K}(\mathbb{C}^n) \). Since \( \mathcal{K}(\mathbb{C}^n) \subset \mathcal{R}(\mathbb{C}^n, \sigma_t) \) according to Proposition 4.11, the statement follows.
Theorem 4.25. Let $n = 1$, then $D_0^t = R_{cl}$.

Proof. If $n = 1$, then any classical resolvent function $g$ either fulfills the assumption of Corollary 4.23 or the assumptions of Corollary 4.24. In either case, we obtain $T^t_g \in R(C^n, \sigma_t)$. □

For $n > 1$, the situation is more complicated. We will show only a weaker result for this case.

Since every $z \in C^n$ is contained in some Lagrangian subspace $V$, we conclude that each resolvent $R(\lambda, z)$ is contained in some $R_V$. Thus, we see that

$$R_0 := \sum_{V \text{ Lagrangian subspace}} R_V$$

contains every resolvent $R(\lambda, z)$. Further, since $R_V = T^t_{\text{lin}}(R_{cl,V})$, we obtain

$$\overline{R_0} = T^t_{\text{lin}}(\overline{R_{cl,0}}),$$

where $R_{cl,0}$ is given by

$$R_{cl,0} := \sum_{V \text{ Lagrangian subspace}} R_{cl,V}.$$

As a weak substitute for Theorem 4.25 in higher dimensions, we get:

Proposition 4.26. $R(C^n, \sigma_t) = T^t(\overline{R_{cl,0}})$.

It might of course be true that $R_{cl,0} = R_{cl}$. In this case, the previous result would imply $D_0^t = R_{cl}$. Nevertheless, so far this remains an open question.

5 Infinite dimensional symplectic space

As is well-known there are significant differences between the resolvent algebras of finite and infinite dimensional symplectic spaces: in finite dimensions every two regular irreducible representations are unitarily equivalent. However, in the case of infinite dimensional symplectic spaces, this statement is false. Nevertheless, we can build particular representations of the resolvent algebra by using ideas from the previous sections.

Let $t = (t_k)_{k=1}^{\infty} \in \ell^1(\mathbb{N})$ be a summable sequence of strictly positive real numbers. Then, we set $\mathcal{H} := \ell^2(\mathbb{N})$ with the usual norm $\| \cdot \|_{\ell^2}$ and define:

$$\mathcal{H}_{1/2}^t := \left\{ z \in \ell^2(\mathbb{N}) : \sum_{n=1}^{\infty} \frac{|z_n|^2}{t_n} < \infty \right\},$$

$$\mathcal{H}_1^t := \left\{ z \in \ell^2(\mathbb{N}) : \sum_{n=1}^{\infty} \frac{|z_n|^2}{t_n} < \infty \right\}.$$

We may think of $\mathcal{H}_1^t$ and $\mathcal{H}_{1/2}^t$ as the range of $B$ and $\sqrt{B}$, respectively, where $B$ is the trace class operator obtained by linearly extending the map $e_n \mapsto t_n e_n$ with $\{e_n\}$ being the standard orthonormal basis of $\ell^2(\mathbb{N})$. We consider $\mathcal{H}_{1/2}^t$ as a symplectic Hilbert space equipped with the symplectic form:

$$\sigma_t(z, w) := \sum_{n=1}^{\infty} \frac{\text{Im}(z_n \cdot \overline{w_n})}{t_n}, \quad z = (z_n), w = (w_n) \in \mathcal{H}_{1/2}^t.$$

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Note that $\sigma_t(z,w)$ is well-defined even for $z \in \mathcal{H}^{1/2}_t$ and $w \in \mathcal{H}$. We can write $\sigma_t = \text{Im} \langle \cdot , \cdot \rangle_{1/2}$, where $\langle \cdot , \cdot \rangle_{1/2}$ is the canonical inner product of $\mathcal{H}^{1/2}_t$:

$$\langle z, w \rangle_{1/2} := \sum_{n=1}^{\infty} \frac{z_n \cdot w_n}{t_n}.$$ 

Moreover, $\| \cdot \|_{1/2}$ denotes the corresponding norm: $\| z \|^2_{1/2} = \langle z, z \rangle_{1/2}$.

We recall some basic facts about Gaussian measures on infinite dimensional Hilbert spaces, cf. [18, 9, 2]. The infinite product measure $\mu_t := \prod_{k=1}^{\infty} \mu_{t_k}$ of the Gaussian measures

$$d\mu_{t_k}(z) := \frac{1}{\sqrt{\pi t_k}} e^{-\frac{|z|^2}{2t_k}} dV(z), \quad z \in \mathbb{C}$$

gives a well-defined probability measure on $\mathbb{C}^{\infty} = \prod_{k=1}^{\infty} \mathbb{C}$. It is concentrated on $\mathcal{H} = \ell^2(\mathbb{N}) \subset \mathbb{C}^{\infty}$. By $\mu_t$, we also denote its restriction to the measurable space $(\mathcal{H}, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $\mathcal{H}$. Note that $\mathcal{B}$ agrees with the $\sigma$-algebra generated by cylindrical Borel sets. The measure $\mu_t$ is called the (centered) Gaussian measure on $\mathcal{H}$ with covariance operator $B$: This is due to the fact that $B$ is naturally related to $\mu_t$ by

$$\int_{\mathcal{H}} e^{i \text{Re}(\langle x, y \rangle)} d\mu_t(x) = e^{-\frac{1}{2} \langle B y, y \rangle}, \quad (y \in \mathcal{H}). \quad (5.1)$$

The space $\mathcal{H}^{1/2}_t$ introduced above is also known as the Cameron-Martin space of $\mu_t$. It is a measurable set of measure zero: $\mu_t(\mathcal{H}^{1/2}_t) = 0$. Hence, so is $\mathcal{H}$. We will now introduce the Fock-Bargmann space of holomorphic functions in infinitely many variables and Berezin-Toeplitz quantization in this setting. Two different approaches have been proposed in [15, 16]: the first being of measure theoretic nature and the second based on an inductive limit construction. Both approaches yield different theories and we deal with the measure theoretic approach here.

We will write $Z_0$ for the set of all sequences $(\alpha_n)_{n=1}^{\infty}$ with values in $\mathbb{N}_0$ such that all but finitely many entries are zero. In the following we use the standard multi-index notation: let $z = (z_1, z_2, \ldots) \in \ell^2(\mathbb{N})$ and $\alpha \in Z_0$, then we put:

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots, \quad \alpha! = \alpha_1! \alpha_2! \ldots, \quad t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots.$$ 

The monomials $\mathcal{M}$ below form an orthonormal system of complex analytic functions on $\mathcal{H}$ inside $L^2(\mathcal{H}, \mu_t)$:

$$\mathcal{M} := \left\{ e_\alpha(z) := \frac{1}{\sqrt{t^\alpha \alpha!}} z^\alpha : \alpha \in Z_0 \right\}.$$ 

We write $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ for the standard inner product and norm in $L^2(\mathcal{H}, \mu_t)$, respectively. Elements in the linear span of $\mathcal{M}$ are referred to as analytic polynomials. In analogy to the finite dimensional setting of $\mathbb{C}^n$ we define the Bargmann-Fock space $F^2_t(\mathcal{H})$ to be the $L^2$-closure of analytic polynomials:

$$F^2_t(\mathcal{H}) := \text{span} \mathcal{M}.$$
Let \( z \in \mathcal{H}_1 \) and \( p \in \text{span} \mathcal{M} \) be an analytic polynomial. Extending (3.3) we define the Weyl operator \( W^*_z \) on \( F^2(\mathcal{H}) \) by:

\[
W^*_z p(w) := k^1_z(w)p(w-z), \quad w \in \mathcal{H}.
\]

Here, \( k^1_z \) denotes the normalized reproducing kernel defined by:

\[
k^1_z(w) = e^{i(w,z)1/2}\|z\|_{1/2}^{-1}.
\]

Let \( N \in \mathbb{N} \) and assume that the analytic polynomial \( p \) only depends on the variables \( z_1, \ldots, z_N \). We write \( \mu_t = \mu_{\nu} \otimes \mu_{t_{N+1}} \), where \( t' := (t_1, \ldots, t_N) \) and \( t_{N+1} = (t_{N+1}, t_{N+2}, \ldots) \). Define the projections \( \pi_N(z) := (z_1, \ldots, z_N) \) for a sequence \( z = (z_j) \in \mathcal{H} \). Since \( p = p \circ \pi_N \) we obtain:

\[
\int_{\mathcal{H}} |W^*_zp(w)|^2 d\mu_t(w) = \int_{\mathcal{C}^\infty} \left| k^1_{\pi_N(z)} \circ \pi_N(w) \circ \pi_N(w-z) \right|^2 |k^1_{t'N+1}(w)|^2 d\mu_t(w)
\]

\[
= \int_{\mathcal{C}^N} |W^*_{\pi_N(z)} p(w)|^2 d\mu_{t'}(w) \times \int_{\mathcal{C}^N} \left| k^1_{(I-\pi_N)(z)}(w) \right|^2 d\mu_{t_{N+1}}(w)
\]

\[
= \int_{\mathcal{H}} |p(w)|^2 d\mu_t(w).
\]

Therefore, \( W^*_z p \in L^2(\mathcal{H}, \mu_t) \) and \( W^*_z \) acts isometrically (say, on analytic polynomials). In finite dimensions, it would now be clear that \( W^*_z p \in F^2(\mathcal{H}) \), since \( W^*_z p \) defines a holomorphic function. In infinite dimensions, this statement is not entirely trivial, as \( F^2 \) is defined as the closure of the analytic polynomials. Nevertheless, \( W^*_z p \in F^2(\mathcal{H}) \) remains true, cf. [15]. Hence, \( W^*_z \) extends to an isometric operator on \( F^2(\mathcal{H}) \) for every \( z \in \mathcal{H}_1 \).

Further, note that for \( (z_k) \subset \mathcal{H}_1 \), \( z_k \to z \) in \( \mathcal{H}_1 \) we clearly have

\[
k^1_{z_k}(w)p(w-z_k) \to k^1_z(w)p(w-z)
\]

pointwise almost everywhere on \( \ell^2(\mathbb{N}) \). Hence, Scheffé’s Lemma shows that the assignment \( z \mapsto W^*_z p \) is continuous from \( \mathcal{H}_1 \) to \( F^2(\mathcal{H}) \) for every analytic polynomial \( p \). Since these are dense in \( F^2(\mathcal{H}) \), the Weyl operators \( W^*_z \) extend to isometric operators on \( F^2(\mathcal{H}) \) such that \( z \mapsto W^*_z \) is continuous on \( \mathcal{H}_1 \) with respect to the strong operator topology.

An even stronger statement holds, namely the map \( z \mapsto W^*_z \) is even strongly continuous with respect to the coarser topology of \( \mathcal{H}_{1/2} \), cf. [15]. Therefore, it extends to a map from \( \mathcal{H}_{1/2} \) to the unitary operators on \( F^2(\mathcal{H}) \).

By applying the Weyl operators with \( z, w \in \mathcal{H}_1 \) to analytic polynomials, it is not hard to see that

\[
W^*_z W^*_w = e^{-i\langle z, w \rangle} W^*_{z+w} \quad \text{and} \quad (W^*_z)^* = (W^*_z)^{-1} = W^*_{-z}.
\]

These relations extend to \( z, w \in \mathcal{H}_{1/2} \), as well:
Lemma 5.1 ([15][16]). The Weyl operators $W^t_z \in \mathcal{L}(F^2(t)(\mathcal{H}))$ depend continuously on $z \in (\mathcal{H}_{1/2}; \|\cdot\|_{1/2})$ with respect to the strong operator topology and

$$W^t_z W^t_w = e^{-i\sigma(z, w)} W^t_{z+w}$$

for every $z, w \in \mathcal{H}_{1/2}$.

Fix a bounded operator $A \in \mathcal{L}(F^2(t)(\mathcal{H}))$ and $z \in \mathcal{H}_1$. We define the Berezin transform of $A$ at $z$ by

$$\tilde{A}(z) = \langle Ak^t_z, k^t_z \rangle = \langle W^t_z A W^t_z 1, 1 \rangle.$$ 

Lemma 5.2. The map $A \mapsto \tilde{A}$ is injective on $\mathcal{L}(F^2(t)(\mathcal{H}))$.

Proof. If $\tilde{A} = 0$, then $\tilde{A}(z) = 0$ for every $z = (z_1, \ldots, z_N, 0, 0, \ldots)$. We denote by $F^2(t)$ the closed linear span in $F^2(t)(\mathcal{H})$ of the set of all analytic polynomials in the finitely many variables $z_1, \ldots, z_N$. Moreover, we write $P_N \in \mathcal{L}(F^2(t)(\mathcal{H}))$ for the orthogonal projection onto $F^2(t)$. Then, $k^t_z \in F^2(t)$ for $z = (z_1, \ldots, z_N, 0, 0, \ldots)$ and hence:

$$\tilde{A}(z) = \langle Ak^t_z, k^t_z \rangle = \langle P_N Ak^t_z, k^t_z \rangle.$$

Note that $F^2(t)$ can be naturally identified with the $N$-variable Fock-Bargmann space $F^2(N) = F^2(\mathbb{C}^N)$ with weight parameter $t = (t_1, \ldots, t_N)$. Moreover,

$$P_N A|_{F^2(t)} \in \mathcal{L}(F^2(t)) \cong \mathcal{L}(F^2(N)) \quad \text{and} \quad 0 = \tilde{A}(z) = (P_N A|_{F^2(t)}) \sim (z_1, \ldots, z_N).$$

On the right hand side of the last formula we have use the Berezin transform on $F^2(t)$. Since the Berezin transform is injective on $\mathcal{L}(F^2(t))$ (see [14]), we conclude that $P_N A|_{F^2(t)} = 0$. Hence, $P_N A p = 0$ for any analytic polynomial $p$ and $N \in \mathbb{N}$ sufficiently large. Finally, we have

$$P_N A p \rightarrow A p, \quad N \rightarrow \infty,$$

in $F^2(t)(\mathcal{H})$, i.e. $A p = 0$ for analytic polynomials $p$ showing that $A = 0$. \hfill \Box

We denote by $P^t$ the orthogonal projection from $L^2(\mathcal{H}, \mu_t)$ onto $F^2(t)(\mathcal{H})$. Given $\varphi \in L^\infty(\mathcal{H}, \mu_t)$ we define the Toeplitz operator $T^t_\varphi \in \mathcal{L}(F^2(t)(\mathcal{H}))$ by:

$$T^t_\varphi(g) = P^t(\varphi g).$$

By applying the injectivity of the Berezin transform in Lemma 5.2 it is not hard to verify that the Weyl operators $W^t_z$ for $z \in \mathcal{H}_1$ have a representation as Toeplitz operators. More precisely:

$$W^t_z = T^t_{g_z} \quad \text{where} \quad g_z(w) = e^{2i\sigma(t, w)} e^{\frac{i}{2} z^2 / t}.$$  \hfill (5.2)

Due to the importance of this fact, we derive an explicit formula for the Berezin
We have to explain in what sense the map \( W_z^t(w) = e^{-\|w\|^2/2} \langle W_z^t K_w^t, K_w^t \rangle \)

We therefore obtain:

\[
W_z^t(w) = e^{-\|w\|^2/2} \langle W_z^t K_w^t, K_w^t \rangle = e^{-\|w\|^2/2 - \frac{1}{2} \|z\|^2} e^{-\langle z, w \rangle / 2} \langle W_z^t K_w^t, K_w^t \rangle = e^{-\|w\|^2/2 - \frac{1}{2} \|z\|^2} e^{-\langle z, w \rangle / 2} K_w^t + z \cdot \langle W_z^t K_w^t, K_w^t \rangle.
\]

\[
\tilde{g}_z(t)(w) = e^{\frac{1}{2} \|z\|^2 - \|w\|^2} e^{2\langle z, w \rangle} K_w^t = e^{\frac{1}{2} \|z\|^2 - \|w\|^2} e^{2\langle z, w \rangle} K_w^t - \langle z, w \rangle + z \cdot \langle K_w^t - z, K_w^t - z \rangle = e^{\frac{1}{2} \|z\|^2 - \|w\|^2} e^{2\langle z, w \rangle} W_z^t(w).
\]

It is an important but non-trivial fact that \( \tilde{g}_z \) extends to \( z \in \mathcal{H}_{1/2} \). First, we have to explain in what sense the map \( w \mapsto \langle w, z \rangle_{1/2} \) defines a measurable function on \( \mathcal{H} \) for \( z \in \mathcal{H}_{1/2} \). Clearly, the expression \( \langle z, \cdot \rangle \) is pointwise well-defined on \( \mathcal{H}_{1/2} \). However, since \( \mathcal{H}_{1/2} \) is a set of measure zero, this is of no big help. We need a preliminary result:

**Lemma 5.3.** \( \| \langle \cdot, z \rangle \|_2^2 = 2\|z\|^2_{1/2} \) for \( z \in \mathcal{H}_1 \).

**Proof.** We introduce a complex parameter \( \lambda \in \mathbb{C} \). Then, it is not hard to verify the following equalities through standard results on differentiation of parameter integrals, where we use that \( \langle \cdot, z \rangle \in L^2(\mathcal{H}, \mu_t) \) (this is easily established):

\[
\frac{\partial^2}{\partial \lambda \partial \lambda} \int_{\mathcal{H}} e^{i2 \Re \langle w, z \rangle_{1/2}} \, d\mu_t(w) = \frac{\partial^2}{\partial \lambda \partial \lambda} \int_{\mathcal{H}} e^{\varphi(w, \lambda z)_{1/2}} \, d\mu_t(w) = -\int_{\mathcal{H}} \langle z, w \rangle_{1/2}^2 e^{\varphi(w, \lambda z)_{1/2}} \, d\mu_t(w).
\]

On the other hand, using Equality (5.1):

\[
\frac{\partial^2}{\partial \lambda \partial \lambda} \int_{\mathcal{H}} e^{i2 \Re \langle w, z \rangle_{1/2}} \, d\mu_t(w) = \frac{\partial^2}{\partial \lambda \partial \lambda} \int_{\mathcal{H}} e^{i Re \langle w, 2\lambda B^{-1} z \rangle_{1/2}} \, d\mu_t(w) = \frac{\partial^2}{\partial \lambda \partial \lambda} e^{-\frac{i}{2} \langle 2\lambda B^{-1} z, 2\lambda B^{-1} z \rangle_{1/2}} = \frac{\partial^2}{\partial \lambda \partial \lambda} e^{-2\lambda \varphi(z, z)_{1/2}} = \frac{\partial^2}{\partial \lambda \partial \lambda} e^{-2\lambda \varphi(z, z)_{1/2}} = \left(-2\|z\|^2_{1/2} + 4\|\lambda\|^2 \|z\|^2_{1/2}\right) e^{-2\lambda \varphi(z, z)_{1/2}}.
\]

We therefore obtain:

\[
\int_{\mathcal{H}} |\langle z, w \rangle_{1/2}|^2 e^{i2 \Re \langle w, z \rangle_{1/2}} \, d\mu_t(w) = (2 - 4\|\lambda\|^2 \|z\|^2_{1/2} \|z\|^2_{1/2} e^{-2\lambda \varphi(z, z)_{1/2}}).
\]

Choosing \( \lambda = 0 \) yields the desired equality. \( \square \)
Hence, if \((z_k) \subset \mathcal{H}_1\) is a Cauchy sequence with respect to \(\| \cdot \|_{1/2}\), then \(\langle \cdot, z_k \rangle_{1/2}\) is Cauchy in \(L^2(\mathcal{H}, \mu_k)\). In conclusion, the family of functions \(\langle \cdot, z \rangle_{1/2} \in L^2(\mathcal{H}, \mu_k)\) continuously extends to \(z \in \mathcal{H}_{1/2}\). In particular, for each \(z \in \mathcal{H}_{1/2}\), \(\langle w, z \rangle_{1/2}\) is a well-defined expression for almost every \(w \in \mathcal{H}\) and, as a function of \(w\), measurable. In particular,

\[
g_z(w) = e^{2i\sigma(z,w) + \frac{1}{2}\|z\|_{1/2}^2} = e^{\langle w, z \rangle_{1/2}} e^{-(z,w)_{1/2} e^{-\frac{1}{2}\|z\|_{1/2}^2}}
\]

is an almost everywhere well-defined function with

\[
\|g_z\|_{\infty} \leq e^{\frac{1}{2}\|z\|_{1/2}^2}, \quad z \in \mathcal{H}_{1/2}.
\]

We also note that, since \(\langle \cdot, z \rangle_{1/2} \in F^2_t(\mathcal{H})\) for every \(z \in \mathcal{H}_1\), the same is true for \(z \in \mathcal{H}_{1/2}\). Iterating the procedure from the proof of Lemma 5.3, one easily obtains:

\[
\|\langle \cdot, z \rangle_{1/2}^k\|^2 = 2^k \|z\|_{1/2}^{2k} \quad \text{for} \quad k \in \mathbb{N}.
\]

Next, we calculate:

\[
\sum_{k=0}^{\infty} \frac{\|\langle \cdot, z \rangle_{1/2}^k\|}{k!} = \sum_{k=0}^{\infty} \frac{2^k \|z\|_{1/2}^k}{k!} = e^{\sqrt{2}\|z\|_{1/2}}.
\]

We obtain:

\[
\sum_{k=0}^{\infty} \frac{\langle \cdot, z \rangle_{1/2}^k}{k!} = e^{\langle \cdot, z \rangle_{1/2}} = K_z^t \in F^2_t(\mathcal{H})
\]

for every \(z \in \mathcal{H}_{1/2}\). By a standard density argument, we have:

\[
(K_z, K_w) = e^{\langle w, z \rangle_{1/2}} \quad \text{for} \quad z, w \in \mathcal{H}_{1/2}.
\]

Now, one can compute the Berezin transform of \(g_z\) as above. For every \(z \in \mathcal{H}_{1/2}\) and \(w \in \mathcal{H}_1\) one obtains:

\[
\overline{T}_{g_z}(w) = e^{\frac{1}{2}\|z\|_{1/2}^2 - \|w\|_{1/2}^2} e^{(w-z, w+z)_{1/2}}.
\]

Further, since \(W_z^t\) continuously (in strong operator topology) depends on the parameter \(z \in \mathcal{H}_{1/2}\), we conclude that the Berezin transform \(\overline{W}_z^t\) continuously (in the topology of pointwise convergence) depends on \(z \in \mathcal{H}_{1/2}\), which gives

\[
\overline{W}_z^t(w) = e^{\frac{1}{2}\|z\|_{1/2}^2 - \|w\|_{1/2}^2} e^{(w-z, w+z)_{1/2}}
\]

for every \(z \in \mathcal{H}_{1/2}\) and \(w \in \mathcal{H}_1\). Comparing Berezin transforms, we have shown that \(T_{g_z} = W_z^t\) extends to \(z \in \mathcal{H}_{1/2}\).

In an abuse of notation, we will write \(\mathcal{R}(\mathcal{H}_{1/2}, \sigma_t)\) for the representation of the resolvent algebra on \(F^2_t(\mathcal{H})\) with respect to the symplectic space \((\mathcal{H}_{1/2}, \sigma_t)\). Since the Weyl operators \(W_z^t\), \(z \in \mathcal{H}_{1/2}\) satisfy the CCR in Lemma 5.3, we start again by expressing the resolvent \(R(\lambda, z)\) in form of a Laplace transform of the one-parameter group \((W_{iz})_{t \in \mathbb{R}}\). For \(\text{Re}(\lambda) > 0\) we have:

\[
R(\lambda, z) = i \int_0^\infty e^{-\lambda s} W_{-sz}^t \, ds,
\]

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which exists as an integral in strong operator topology. There is an analogous formula in the case \( \text{Re}(\lambda) < 0 \). Consider the space of (classical) symbols:

\[
\mathcal{R}_{cl}^t = C^* \left\{ (\lambda - 2i\sigma_t, z)^{-1} : \lambda \in \mathbb{C} \setminus i\mathbb{R}, z \in \mathcal{H}_{1/2} \right\}.
\]

Moreover, we write \( T^t(\mathcal{R}_{cl}^t) \) for the \( C^* \) algebra generated by Toeplitz operators over \( \mathcal{R}_{cl}^t(\mathcal{H}) \) with symbols in \( \mathcal{R}_{cl}^t \).

**Theorem 5.4.** The following inclusion holds true: \( \mathcal{R}(\mathcal{H}_{1/2}, \sigma_t) \subset T^t(\mathcal{R}_{cl}^t) \).

**Proof.** Without loss of generality we assume \( \text{Re}(\lambda) > 0 \). We will verify that the representation of the resolvent in form of a Laplace transform of Weyl operators defines an element in the Toeplitz algebra. The following integrals are to be understood as improper Riemann integrals in strong operator topology:

\[
\int_0^\infty e^{-\lambda s} W^t_{-sz} \, ds = \int_0^\infty e^{-\lambda s} T^t_{-sz} \, ds = \int_0^\infty e^{-\lambda s} e^{\frac{s^2}{2}} \|z\|^2 T^t_{\exp(-2isz\sigma_t)} \, ds = \int_0^\infty e^{-\lambda s} \sum_{k=0}^\infty \frac{s^{2k}}{2^k k!} T^t_{\exp(-2isz\sigma_t)} \, ds.
\]

Since the Weyl operators are unitary, and hence satisfy \( \|W^t_z\| = 1 \), it follows that:

\[
\|T^t_{\exp(-2isz\sigma_t)}\| = e^{-\frac{s^2}{2}} \|z\|^2.
\]

Therefore, the dominated convergence theorem gives as \( m \to \infty \):

\[
\left\| iR(\lambda, z) - \sum_{k=0}^m \frac{s^{2k}}{2^k k!} \int_0^\infty e^{-\lambda s} T^t_{\exp(-2isz\sigma_t)} \, ds \right\| \leq \int_0^\infty e^{-\lambda s} \left| \frac{s^{2k}}{2^k k!} \right| ds \to 0, \quad m \to \infty.
\]

We have therefore seen that

\[
iR(\lambda, z) = \sum_{k=0}^\infty \frac{s^{2k}}{2^k k!} \int_0^\infty T^t_{s^{2k} \exp(-\lambda s + 2isz\sigma_t)} \, ds,
\]

where the series converges in operator norm. Fix for the moment \( N > 0 \). Since the Riemann integral \( \int_0^N T^t_{s^{2k} \exp(-\lambda s + 2isz\sigma_t)} \, ds \) exists as a limit of Riemann sums in strong operator topology (this easily follows from the fact that the mapping \( s \mapsto W^t_{sz} \) is strongly continuous), we obtain for the Berezin transform:

\[
\left( \int_0^N T^t_{s^{2k} \exp(-\lambda s + 2isz\sigma_t)} \, ds \right)^\sim(u) = \left\langle T^t_{u^N} \sum_{k=0}^N T^t_{u^{2k} \exp(-\lambda s + 2isz\sigma_t)} ds^k \right\rangle_{u^N, u^t}.
\]

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Therefore, injectivity of the Berezin transform (Lemma 5.2) shows:

$$\int_0^N T_s^{\mathcal{A}} \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds = T_s^{\mathcal{A}} \int_0^\infty \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds.$$ 

Since

$$\int_0^N s^{2k} \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds \overset{N \to \infty}{\rightarrow} \int_0^\infty s^{2k} \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds,$$

uniformly, this gives

$$\int_0^\infty T_s^{\mathcal{A}} \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds = T_s^{\mathcal{A}} \int_0^\infty s^{2k} \exp(-\lambda s + 2i\sigma \mathbf{t}(\cdot, z)) ds.$$

For the symbol, standard facts on the Laplace transform yield

$$\int_0^\infty s^{2k} e^{-\lambda s + 2i\sigma \mathbf{t}(w, z)} ds = (2k)! (\lambda - 2i\sigma \mathbf{t}(w, z))^{-(2k+1)}.$$

This is now, as a function of $w \in \mathcal{H}$, bounded and measurable, which finishes the proof.

The careful reader may have noticed that the proofs of Proposition 4.12 and Lemma 4.13 generalize to the infinite dimensional phase space, i.e. analogous formulas for the Berezin transforms of products of resolvents and the classical resolvent functions, respectively, are valid. Nevertheless, since a correspondence theorem in the infinite dimensional setup (similar to Theorem 4.5) is not available at the moment, this observation is of no further use for proving a refinement of Theorem 5.4. The biggest issue is the lack of a Haar measure on the infinite dimensional symplectic space. In the finite dimensional framework of $\mathbb{C}^n$, this measure coincides with the Lebesgue measure and is at the heart of quantum harmonic analysis. An important ingredient to the theory is a correspondence between the spaces $L^1(\mathbb{C}^n)$ and $\mathcal{T}^1(\mathbb{F}_2)$ (the latter being the space of trace class operators). It is not clear what an appropriate interpretation of “$L^1(\mathbb{H}_{1/2})$” should be in order to develop a quantum harmonic analysis and correspondence theory for infinite dimensional phase spaces. This will be a key issue for our future work.

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