Invariant Differential Operators for Non-Compact Lie Groups: Euclidean Jordan Groups or Conformal Lie Groups

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Abstract. In the present paper we continue the project of systematic construction of invariant differential operators for non-compact semisimple Lie groups. Our starting point is the class of algebras, which we call 'conformal Lie algebras' (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this we introduce the new notion of parabolic relation between two non-compact semisimple Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}^\prime \) that have the same complexification and possess maximal parabolic subalgebras with the same complexification.

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allember, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory (for reviews, cf. e.g., [2],[3]. Thus, it is important for the applications in physics to study systematically such operators.

In a recent paper [4] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. A natural choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion [5] these are groups where holds:

\[ \text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}, \]

where \( \mathfrak{k} \) is the maximal compact subgroup of the non-compact group \( \mathfrak{g} \). Another formulation is to say that the Lie algebra \( \mathfrak{g} \) of \( \mathfrak{g} \) has a compact Cartan subalgebra.

Example: The groups \( SO(p,q) \) have discrete series, except when both \( p, q \) are odd numbers. ♦
This class is still rather big, thus, we decided to start with a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra \( K \) is of the form:

\[
K = \text{so}(2) \oplus K'.
\]

The Lie algebras from this class are:

\[
\begin{align*}
\text{so}(n,2), \quad \text{sp}(n,R), \quad \text{su}(m,n), \quad \text{so}^*(2n), \quad E_6(-14), \quad E_7(-25)
\end{align*}
\]

These groups/algebras have highest/lowest weight representations, and relatedly holomorphic discrete series representations \([6]\).

The most widely used of these algebras are the conformal algebras \( \text{so}(n,2) \) in \( n \)-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition \([7]\):

\[
\begin{align*}
\text{so}(n,2) &= \mathcal{P} \oplus \mathcal{N} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}' \oplus \bar{\mathcal{N}}, \\
\mathcal{M} &= \text{so}(n-1,1), \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \bar{\mathcal{N}} = n
\end{align*}
\]

that has direct physical meaning, namely, \( \text{so}(n-1,1) \) is the Lorentz algebra of \( n \)-dimensional Minkowski space-time, the subalgebra \( \mathcal{A} = \text{so}(1,1) \) represents the dilatations, the conjugated subalgebras \( \mathcal{N}, \bar{\mathcal{N}} \) are the algebras of translations, and special conformal transformations, both being isomorphic to \( n \)-dimensional Minkowski space-time.

The subalgebra \( \mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}' \cong \mathcal{M} \oplus \mathcal{A} \oplus \bar{\mathcal{N}} \) is a maximal parabolic subalgebra.

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

\[
\mathcal{K}^C = \text{so}(n,C) \oplus \text{so}(2,C) \cong \text{so}(n-1,1)^C \oplus \text{so}(1,1)^C = \mathcal{M}^C \oplus \mathcal{A}^C.
\]

In particular, the coincidence of the complexification of the semi-simple subalgebras:

\[
\mathcal{K}'^C = \mathcal{M}^C
\]

means that the sets of finite-dimensional (nonunitary) representations of \( \mathcal{M} \) are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of \( \mathcal{K}' \).

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of \( \text{so}(n,2) \). This subclass consists of:

\[
\begin{align*}
\text{so}(n,2), \quad \text{sp}(n,R), \quad \text{su}(n,n), \quad \text{so}^*(4n), \quad E_7(-25)
\end{align*}
\]

the corresponding analogs of Minkowski space-time \( V \) being:

\[
\begin{align*}
\mathbb{R}^{n-1,1}, \quad \text{Sym}(n,R), \quad \text{Herm}(n,C), \quad \text{Herm}(n,Q), \quad \text{Herm}(3,\mathbb{O})
\end{align*}
\]

where we use standard notation \( \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O} \) for the four division algebras (real, complex, quaternion, octonion).

In view of applications to physics, we proposed to call these algebras ’conformal Lie algebras’, (or groups) \([8]\).

They were used by other authors under different names. First, the corresponding groups are also called ’Hermitian symmetric spaces of tube type’ \([9]\). The same class was identified from different considerations in \([10]\) who called them ’conformal groups of simple Jordan algebras’. In
fact, the relation between Jordan algebras and division algebras was known long time ago. Our class was identified from still different considerations also in [11] where they were called ‘simple space-time symmetries generalizing conformal symmetry’.

We have started the study of the above class in the framework of the present approach in the cases: \(so(n,2), su(n,n), sp(n,\mathbb{R})\), \(E_{7(-25)}\), in [24], [12], [13], [8], resp., and we have considered also the algebra \(E_{6(-14)}\), [14].

Lately, in [15], which we review here, we discovered an efficient way to extend our considerations beyond this class introducing the notion of ‘parabolically related non-compact semisimple Lie algebras’.

- **Definition:** Let \(G, G'\) be two non-compact semisimple Lie algebras with the same complexification \(G^\mathbb{C} \cong G'^\mathbb{C}\). We call them *parabolically related* if they have parabolic subalgebras \(P = M \oplus A \oplus N\), \(P' = M' \oplus A' \oplus N'\), such that: \(M^\mathbb{C} \cong M'^\mathbb{C} \quad (\Rightarrow P^\mathbb{C} \cong P'^\mathbb{C})\). ♦

Certainly, there are many such parabolic relationships for any given algebra \(G\). Furthermore, two algebras \(G, G'\) may be parabolically related with different parabolic subalgebras.

For example, the exceptional Lie algebras \(E_{6(6)}\) and \(E_{6(2)}\) are parabolically related (and also to \(E_{6(-14)}\)) with maximal parabolics such that \(M^\mathbb{C} \cong sl(6,\mathbb{C})\). But these two algebras are related also by another pair of maximal parabolics such that \(M^\mathbb{C} \cong sl(3,\mathbb{C}) \oplus sl(3,\mathbb{C}) \oplus sl(2,\mathbb{C})\), see Section 7.

Another interesting example are the algebras \(so^*(2m)\) and \(so(p,q)\) which have a series of maximal parabolics with \(M\)-factors [4]:

\[
\begin{align*}
M_j &= su^*(2j) \oplus so^*(2m - 4j), & j \leq \left[\frac{m}{2}\right], \\
M'_j &= sl(2j, \mathbb{R}) \oplus so(p - 2j, q - 2j), & j \leq \left[\frac{q}{2}\right] \leq \left[\frac{p}{2}\right],
\end{align*}
\]

whose complexifications coincide for \(p + q = 2m\)

\[
(M_j)^\mathbb{C} = (M'_j)^\mathbb{C} = sl(2j, \mathbb{C}) \oplus so(2m - 4j, \mathbb{C}),
\]

\(j \leq \left[\frac{q}{2}\right] \leq \left[\frac{m}{2}\right] = \left[\frac{p + q}{2}\right]\).

As we know only for \(m = 2n\), i.e., for \(so^*(4n)\) is fulfilled relation (6), with \(M = M_n = su^*(2n)\) from (9), (recalling that \(K' \cong su(2n)\)). Obviously, \(so(p,q)\) is parabolically related to \(so^*(4n)\) with this \(M\)-factor only when \(p = q = 2n\), i.e., \(G' = so(2n,2n)\) with \(M'_{2n} = sl(2n,\mathbb{R})\) (which is outside the range of (10)).

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling (6) in the table below, where we have included also the algebra \(E_{6(-14)}\): we display only the semisimple part \(K'\) of \(K\); \(sl(n,\mathbb{C})_\mathbb{R}\) denotes \(sl(n,\mathbb{C})\) as a real Lie algebra, (thus, \((sl(n,\mathbb{C})_\mathbb{R})^\mathbb{C} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C}))\); \(e_6\) denotes the compact real form of \(E_6\); and we have imposed restrictions to avoid coincidences or degeneracies due to well known isomorphisms: \(so(1,2) \cong sp(1,\mathbb{R}) \cong su(1,1), so(2,2) \cong so(1,2) \oplus so(1,2), su(2,2) \cong so(4,2), sp(2,\mathbb{R}) \cong so(3,2), so^*(4) \cong so(3) \oplus so(2,1), so^*(8) \cong so(6,2)\).
**Table** of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling (6) and the corresponding parabolically related algebras $\mathcal{G}'$

| $\mathcal{G}$ | $\mathcal{K}'$ | $\mathcal{M}$ | $\mathcal{G}'$ | $\mathcal{M}'$ |
|----------------|----------------|---------------|---------------|---------------|
| $so(n, 2)$, $n \geq 3$ | $so(n)$ | $so(n - 1, 1)$, $n$ | $so(p, q)$, $p + q = n + 2$ | $so(p - 1, q - 1)$ |
| $su(2k, 2k)$, $k \geq 2$ | $su(2k) \oplus su(2k)$ | $sl(2k, \mathbb{C})_{\mathbb{R}}$ | $su^*(4k)$ | $su^*(2k) \oplus su^*(2k)$ |
| $sp(2r, \mathbb{R})$, rank $= 2r \geq 4$ | $su(2r)$ | $sl(2r, \mathbb{R})$ | $sp(r, r)$ | $su^*(2r)$ |
| $so^*(4n)$, $n \geq 3$ | $su(2n)$ | $su^*(2n)$ | $so(2n, 2n)$ | $sl(2n, \mathbb{R})$ |
| $E_{7(-25)}$ | $e_6$ | $E_{6(-26)}$ | $E_{7(7)}$ | $E_{6(6)}$ |
| below not CLA! |
| $E_{6(-14)}$ | $so(10)$ | $su(5, 1)$ | $E_{6(6)}$ | $sl(6, \mathbb{R})$ |
| | | | $21$ | $E_{6(2)}$ |
| | | | | $su(3, 3)$ |
2. Preliminaries

Let $G$ be a semisimple non-compact Lie group, and $K$ a maximal compact subgroup of $G$. Then we have an Iwasawa decomposition $G = KA_0N_0$, where $A_0$ is Abelian simply connected vector subgroup of $G$, $N_0$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A_0$. Further, let $M_0$ be the centralizer of $A_0$ in $K$. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of $G$. A parabolic subgroup $P = M'A'N'$ is any subgroup of $G$ which contains a minimal parabolic subgroup.

Further, let $G, K, P, M, A, N$ denote the Lie algebras of $G, K, P, M, A, N$, resp.

For our purposes we need to restrict to maximal parabolic subgroups $P = MAN$, i.e. rank $A = 1$, resp. to maximal parabolic subalgebras $P = M \oplus A \oplus N$ with dim $A = 1$.

Let $\nu$ be a (non-unitary) character of $A$, $\nu \in A^*$, parameterized by a real number $d$, called the conformal weight or energy.

Further, let $\mu$ fix a discrete series representation $D^\mu$ of $M$ on the Hilbert space $V_\mu$, or a finite-dimensional (non-unitary) representation of $M$.

We call the induced representation $\chi = \text{Ind}_K^G(\mu \otimes \nu \otimes 1)$ an elementary representation of $G$ [16]. (These are called generalized principal series representations (or limits thereof) in, e.g., [6].) Their spaces of functions are:

$$C_\chi = \{ F \in C^\infty(G, V_\mu) \mid F(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) F(g) \}$$

where $a = \exp(H) \in A^*$, $H \in \mathfrak{a}^*$, $m \in M^*$, $n \in N^*$. The representation action is the left regular action:

$$(T^\chi(g)F)(g') = F(g^{-1}g'), \quad g, g' \in G.$$  

• An important ingredient in our considerations are the highest/lowest weight representations of $G^C$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $G^C$, where $\Lambda \in (\mathfrak{h}^C)^*$, $\mathfrak{h}^C$ is a Cartan subalgebra of $G^C$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from $\chi$ [17].

Actually, since our ERs may be induced from finite-dimensional representations of $M$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules $V^\Lambda$ such that the role of the highest/lowest weight vector $v_0$ is taken by the (finite-dimensional) space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight $d$. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• Another important ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets [18],[19]. A multiplet is depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines (arrows) between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In the case of a conformal Lie algebra $G$ (cf. (7)) the number of ERs in a multiplet is equal to [15]:

$$|W(G^C, \mathfrak{h}^C)| / |W(M^C, \mathfrak{h}^C_m)|$$

where $\mathfrak{h}^C, \mathfrak{h}^C_m$ are Cartan subalgebras of $G^C, M^C$, resp.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair $(\beta, m)$, where $\beta$ is a (non-compact) positive root of $G^C$, $m \in \mathbb{N}$, such that the BGG Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee = 2\beta/(\beta, \beta)$$

(14)
\( \rho \) is half the sum of the positive roots of \( G^C \). When the above holds then the Verma module with shifted weight \( V^{\lambda - m\beta} \) (or \( V^{\Lambda - m\beta} \) for GVM and \( \beta \) non-compact) is embedded in the Verma module \( V^{\lambda} \) (or \( V^{\Lambda} \)). This embedding is realized by a singular vector \( v_s \) determined by a polynomial \( P_{m,\beta}(G^-) \) in the universal enveloping algebra \((U(G^-))v_0, \ G^- \) is the subalgebra of \( G^C \) generated by the negative root generators \[20\]. More explicitly, \( v^s_{m,\beta} = P_{m,\beta} v_0 \) (or \( v^s_{m,\beta} = P_{m,\beta} V_{\mu} v_0 \) for GVMs) \[17\]. Then there exists \[17\] an intertwining differential operator

\[ D_{m,\beta} : \mathcal{C}_{\chi(\lambda)} \rightarrow \mathcal{C}_{\chi(\lambda - m\beta)} \]  

given explicitly by:

\[ D_{m,\beta} = P_{m,\beta}(\hat{G}^-) \]  

where \( \hat{G}^- \) denotes the right action on the functions \( \mathcal{F} \).

In most of these situations the invariant operator \( D_{m,\beta} \) has a non-trivial invariant kernel in which a subrepresentation of \( G \) is realized. Thus, studying the equations with trivial RHS:

\[ D_{m,\beta} f = 0, \quad f \in \mathcal{C}_{\chi(\lambda)}, \]  

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for \( m = m_{\beta} = 1 \), these equations are called conservation laws, and the elements \( f \in \ker D_{m,\beta} \) are called conserved currents.

The above construction works also for the subsingular vectors \( v^s_{ssv} \) of Verma modules. Such a vector is also expressed by a polynomial \( P_{ssv}(\hat{G}^-) \) in the universal enveloping algebra: \( v^s_{ssv} = P_{ssv}(\hat{G}^-) v_0 \). Thus, there exists a conditionally invariant differential operator given explicitly by: \( D_{ssv} = P_{ssv}(\hat{G}^-) \), and a conditionally invariant differential equation, for many more details, see \[21\]. (Note that these operators (equations) are not of first order.)

Below in our exposition we shall use the so-called Dynkin labels:

\[ m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, \ldots, n, \]  

where \( \Lambda = \Lambda(\chi), \ \rho \) is half the sum of the positive roots of \( G^C \).

We shall use also the so-called Harish-Chandra parameters \[22\]:

\[ m_{\beta} \equiv (\Lambda + \rho, \beta), \]  

where \( \beta \) is any positive root of \( G^C \). These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition.)
3. Conformal algebras \( so(n,2) \) and parabolically related

Let \( G = so(n,2), \ n > 2 \). We label the signature of the ERs of \( G \) as follows:

\[
\chi = \{ n_1, \ldots, n_k; c \}, \quad n_j \in \mathbb{Z}/2, \quad c = d - \frac{n}{2}, \quad \hat{h} \equiv \left[ \frac{n}{2} \right],
\]

where the last entry of \( \chi \) labels the characters of \( A \), and the first \( \hat{h} \) entries are labels of the finite-dimensional nonunitary irreps of \( M \sim so(n-1,1) \).

The reason to use the parameter \( c \) instead of \( d \) is that the parametrization of the ERs in the multiplets is given in a simple intuitive way:

\[
\begin{align*}
\chi_{1}^{\pm} &= \{en_1, \ldots, n_{k}; \pm n_{h+1}\}, \quad n_{h} < n_{h+1}, \\
\chi_{2}^{\pm} &= \{en_1, \ldots, n_{k-1}, n_{h+1}; \pm n_{h}\} \\
\chi_{3}^{\pm} &= \{en_1, \ldots, n_{k-2}, n_{h}, n_{h+1}; \pm n_{h-1}\} \\
&\vdots \\
\chi_{\hat{h}}^{\pm} &= \{en_1, n_3, \ldots, n_{h-1}, n_{h}, n_{h+1}; \pm n_2\} \\
\chi_{\hat{h}+1}^{\pm} &= \{en_2, \ldots, n_{h-1}, n_{h}, n_{h+1}; \pm n_1\}
\end{align*}
\]

\( \epsilon = \pm, \quad n \text{ even} \)

\( \epsilon = 1, \quad n \text{ odd} \)

(\( \epsilon = \pm \) is correlated with \( \chi^{\pm} \)).

Further, we denote by \( \hat{C}^{\pm}_{i} \) the representation space with signature \( \chi_{i}^{\pm} \).

The number of ERs in the multiplets according to (13) is equal to:

\[
|W(G^C, \mathcal{H}^C)| / |W(M^C, \mathcal{H}^C_m)| = 2(1 + \hat{h})
\]

All the above is valid for \( so(p, q) \) for \( p + q = n + 2 \), (with \( M \sim so(p-1, q-1) \)), cf. [23] (for \( q = 1 \), [15].

The multiplets look a little differently for \( n \) even and \( n \) odd.

We give two diagrams for \( n \) even. The first shows the multiplets as they appeared for \( so(n + 1,1) \) [23] and \( so(n,2) \) [24]. The next picture is for the general case \( so(p, q) \) with \( p + q = n + 2 \) [15].

Then we give the two diagrams for \( n \) odd.
$SO(n,2)$ and $SO(n+1,1)$ for $n \geq 4$ even, $\tilde{h} = \frac{n}{2}$

(arrows are differential operators, dashed arrows are integral operators)
\( SO(p, q) \) for \( p + q = n + 2 = 2\ell = 2(h + 1) \geq 6 \) with maximal parabolic subgroup \( P = MAN \), where \( M = SO(p - 1, q - 1) \) (arrows are differential operators, dashed arrows are integral operators) \( \varepsilon_1 \pm \varepsilon_k \) are the non-compact roots, \( m_j \) are Harish-Chandra parameters.
$SO(n, 2)$ and $SO(n + 1, 1)$ for $n \geq 3$ odd, $\tilde{h} = \frac{1}{2}(n - 1)$
(arrows are differential operators, dashed arrows are integral operators)
$SO(p, q)$ for $p + q = n + 2 = 2\ell + 1 = 2\hat{h} + 3 \geq 5$ with maximal parabolic subgroup $P = MAN$, where $M = SO(p - 1, q - 1)$ (arrows are differential operators, dashed arrows are integral operators) $\varepsilon_1 \pm \varepsilon_k$, $\varepsilon_1$ are the non-compact roots, $m_j$ are Harish-Chandra parameters
The ERs in the multiplets are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein [25]. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. These operators intertwine the pairs $\mathcal{C}_i^\pm$

$$G_i^\pm : \mathcal{C}_i^\pm \rightarrow \mathcal{C}_i^\pm , \quad i = 1, \ldots , 1 + \hat{h} \quad (23)$$

The intertwining differential operators correspond to non-compact positive roots of the root system of $so(n + 2, \mathbb{C})$, cf. [17]. [In the current context, compact roots of $so(n + 2, \mathbb{C})$ are those that are roots also of the subalgebra $so(n, \mathbb{C})$, the rest of the roots are non-compact.] The degrees of these intertwining differential operators are given just by the differences of the entries:

$$\deg d_i = \deg d_i' = n_{k+2-i} - n_{k+1-i} , \quad i = 1, \ldots , \hat{h}, \quad \forall n ,$$

$$\deg d_{\hat{h}+1} = n_2 + n_1 , \quad n \text{ even} \quad (24)$$

Matters are arranged so that in every multiplet only the ER with signature $\chi_k^-$ contains a finite-dimensional nonunitary subrepresentation in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n + 2)$ with signature $\{n_1, \ldots , n_{\hat{h}}, n_{\hat{h}+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator $G_1^+$, and is the image of the operator $G_1^-$. Although the diagrams are valid for arbitrary $so(p, q)$ ($p+q \geq 5$) the content is very different. We comment only on the ER with signature $\chi_1^+$. In all cases it contains an UIR of $so(p, q)$ realized on an invariant subspace $\mathcal{D}$ of the ER $\chi_1^+$. That subspace is annihilated by the operator $G_1^-$, and is the image of the operator $G_1^+$. (Other ERs contain more UIRs.) If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which are realized a holomorphic discrete series representation and its conjugate anti-holomorphic discrete series representation, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Note that the $\deg d_i , \deg d_i'$, are Harish-Chandra parameters corresponding to the non-compact positive roots of $so(n + 2, \mathbb{C})$. From these, only $\deg d_1$ corresponds to a simple root, i.e., is a Dynkin label.

Above we restricted to $n > 2$. The case $n = 2$ is reduced to $n = 1$ since $so(2, 2) \cong so(1, 2) \oplus so(1, 2)$. The case $so(1, 2)$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_k^\pm$ in all pictures that we presented. And they have the properties that we described. That case was the first given already in 1946-7 independently by Gel’fand et al [26] and Bargmann [27].

4. The Lie algebra $su(n, n)$ and parabolically related

Let $\mathcal{G} = su(n, n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n, \mathbb{C})_\mathbb{R}$.

The signature of the ERs of $\mathcal{G}$ is [12]:

$$\chi = \{ n_1, \ldots , n_{n-1} , n_{n+1} \ldots , n_{2n-1} ; c \} , \quad n_j \in \mathbb{N} , \quad c = d - n \quad (25)$$
The Knapp-Stein restricted Weyl reflection is given by:

\[ G_{KS} : C_\chi \longrightarrow C_{\chi'}, \]

\[ \chi' = \left\{ (n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^*; -c \right\}, \]

\[ (n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^* = (n_{n+1}, \ldots, n_{2n-1}, n_1, \ldots, n_{n-1}) \]

Further, we use the root system of the complex algebra \( sl(2n, \mathbb{C}) \). The positive roots \( \alpha_{ij} \) in terms of the simple roots \( \alpha_i \) are:

\[ \alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i < j \leq 2n - 1, \]

\[ \alpha_{ii} \equiv \alpha_i, \quad 1 \leq i \leq 2n - 1 \]

from which the non-compact are:

\[ \alpha_{ij}, \quad 1 \leq i \leq n, \quad n \leq j \leq 2n - 1 \]

The correspondence between the signatures \( \chi \) and the highest weight \( \Lambda \) is through the Dynkin labels:

\[ n_i = m_i \equiv \left( \Lambda + \rho, \alpha_i^\vee \right) = \left( \Lambda + \rho, \alpha_i \right), \quad i = 1, \ldots, 2n - 1, \]

\[ c = -\frac{1}{2} (m_\alpha + m_n) = -\frac{1}{2} (m_1 + \cdots + m_{n-1} + 2m_n + m_{n+1} + \cdots + m_{2n-1}), \]

\[ \Lambda = \Lambda(\chi), \quad \tilde{\alpha} = \alpha_1 + \cdots + \alpha_{2n-1} \text{ is the highest root.} \]

The number of ERs in the corresponding multiplets by (13) is equal to:

\[ |W(G^C, \mathcal{H}_m^C)| / |W(M^C, \mathcal{H}_m^C)| = \binom{2n}{n} \]

In our diagrams we need also the Harish-Chandra parameters for the non-compact roots using the following notation:

\[ m_{ij} \equiv m_{\alpha_{ij}} = m_i + \cdots + m_j, \quad i < j \]

We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol \( i_{j...k} \) encoding the root \( \beta_{j...k} \) and the number \( m_{\beta_{j...k}} \) which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data \( \beta, m_\beta \) is involved in the embedding \( V^\Lambda \longrightarrow V^{\Lambda-m_\beta,\beta} \) turns out to involve only the \( m_i \) corresponding to simple roots, i.e., for each \( \beta, m_\beta \) there exists \( i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, r\} \), \( (r = \text{rank} \ G) \), such that \( m_\beta = m_i \). Hence the data \( \beta_{j...k}, m_{\beta_{j...k}} \) is represented by \( i_{j...k} \) on the arrows.

Below we give the diagrams for \( su(n,n) \) for \( n = 2, 3, 4 \). For \( n = 2k \) these are diagrams also for the parabolically related \( su^*(4k) \) and \( sl(4k, \mathbb{R}) \). We present these introducing a new diagram look which can handle the more complicated cases that follow further. In this new look only the invariant differential operators are presented explicitly. The integral Knapp-Stein operators, more precisely the restricted Weyl reflection action is understood by a symmetry of the picture, either w.r.t. a central point, or w.r.t. middle line.

We give first the diagram for \( n = 2 \) where the Knapp-Stein symmetry is w.r.t. to the bullet in the middle of the figure. The diagram describes also \( su^*(4) \) and \( sl(4, \mathbb{R}) \). And since we have the isomorphisms \( su^*(4) \cong so(5, 1) \), \( su(2, 2) \cong so(4, 2) \), \( sl(4, \mathbb{R}) \cong so(3, 3) \), we repeat in the
second picture the diagram as it was given originally in the cases $so(p, q)$ for $p + q = 6$: for $q = 1$ in [28], and for $q = 2$ in [29].

Main multiplets for $su(2, 2) \cong so(4, 2)$, $su^*(4) \cong so(5, 1)$, and $sl(4, \mathbb{R}) \cong so(3, 3)$
Sextet of partially equivalent ERs and intertwining operators
given first for $so(5, 1)$, then for $so(4, 2)$ (valid also for $so(3, 3)$)
(arrows are differential operators, dashed arrows are integral operators)
Main multiplets for $su(3, 3)$
Main multiplets for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$
5. The Lie algebras $sp(n, \mathbb{R})$ and $sp(\frac{n}{2}, \frac{n}{2})$ ($n$–even)

Let $n \geq 2$. Let $G = sp(n, \mathbb{R})$, the rank $n$ split real form of $sp(n, \mathbb{C}) = G^C$. The maximal compact subgroup is $K \cong u(1) \oplus su(n)$, while $M = sl(n, \mathbb{R})$.

The signature of the ERs of $G$ is \([13]\):

$$\chi = \{ n_1, \ldots , n_{n-1}; c \}, \quad n_j \in \mathbb{N} .$$  \hspace{1cm} (32)

The Knapp-Stein restricted Weyl reflection acts as follows:

$$G_{KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \quad \chi' = \{(n_1, \ldots , n_{n-1})^*; -c\}, \quad (n_1, \ldots , n_{n-1})^* = (n_{n-1}, \ldots , n_1)$$  \hspace{1cm} (33)

In terms of an orthonormal basis $\epsilon_i$, $i = 1, \ldots , n$, the positive roots are:

$$\Delta^+ = \{ \epsilon_i \pm \epsilon_j, \quad 1 \leq i < j \leq n; \quad 2\epsilon_i, \quad 1 \leq i \leq n \}$$  \hspace{1cm} (34)

the simple roots are:

$$\pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad 1 \leq i \leq n-1, \quad \alpha_n = 2\epsilon_n \}$$  \hspace{1cm} (35)

the non-compact roots:

$$\beta_{ij} = \epsilon_i + \epsilon_j, \quad 1 \leq i \leq j \leq n ,$$  \hspace{1cm} (36)

the Harish-Chandra parameters: $m_\beta \equiv (\Lambda + \rho, \beta)$ for the noncompact roots are:

$$m_{\beta_{ij}} = \left( \sum_{s=i}^{n} + \sum_{s=j}^{n} \right) m_s , \quad i < j ,$$  \hspace{1cm} (37)

$$m_{\beta_{ii}} = \sum_{s=i}^{n} m_s$$

The correspondence between the signatures $\chi$ and the highest weight $\Lambda$ is:

$$n_i = m_i , \quad c = -\frac{1}{2}(m_\bar{\alpha} + m_n) = -\frac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n)$$  \hspace{1cm} (38)

where $\bar{\alpha} = \beta_{11}$ is the highest root.

The number of ERs in the corresponding multiplets by (13) is:

$$|W(G^C, \mathcal{H}^C)| / |W(M^C, \mathcal{H}_{M}^C)| = 2^n$$  \hspace{1cm} (39)

Below we give pictorially the multiplets for $sp(n, \mathbb{R})$ for $n = 2, 3, 4, 5, 6$. For $n = 2r$ these are also multiplets for $sp(r, r)$, $r = 1, 2, 3$.

Recalling the isomorphisms $sp(1, 1) \cong so(4, 1)$, $sp(2) \cong so(3, 2)$, we repeat the $n = 2$ diagram as it was given originally in the cases $so(p, q)$ with $p + q = 5$ : for $q = 1$ in [16] and for $q = 2$ in [30].
Main multiplets for $Sp(2, \mathbb{R})$ and $Sp(1, 1)$
Quartet of partially equivalent ERs and intertwining operators given first for $so(4,1) \cong sp(1,1)$, then for $so(3,2) \cong sp(2,\mathbb{R})$ (arrows are differential operators, dashed arrows are integral operators)
Main multiplets for $Sp(3, \mathbb{R})$
Main multiplets for $Sp(4, \mathbb{R})$ and $Sp(2, 2)$
Main multiplets for $Sp(5, I\mathbb{R})$
Main multiplets for $Sp(6,\mathbb{R})$ and $Sp(3,3)$
6. The Lie algebras $E_7(-25)$ and $E_7(7)$

Let $\mathcal{G} = E_7(-25)$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$. We work with maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{N}$ with $\mathcal{M} \cong E_6(-26)$.

The signatures of the ERs of $\mathcal{G}$ are \cite{8}:

$$\chi = \{ n_1, \ldots, n_6; c \}, \quad n_j \in \mathbb{N}, \quad (40)$$

expressed through the Dynkin labels:

$$n_i = m_i, \quad c = -\frac{1}{2}(m_\alpha + m_\gamma) = -\frac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7)$$

The same holds for the parabolically related exceptional Lie algebra $E_7(7)$ (with $\mathcal{M}$-factor $E_6(6)$).

The noncompact roots of the complex algebra $E_7$ are:

$$\alpha_7, \alpha_{17}, \ldots, \alpha_{67}, \alpha_{1,37}, \alpha_{2,47}, \alpha_{17,4}, \alpha_{27,4}, \alpha_{17,34}, \alpha_{17,35}, \alpha_{17,36}, \alpha_{17,45}, \alpha_{17,46}, \alpha_{27,45}, \alpha_{27,46}, \alpha_{17,25,4}, \alpha_{17,26,4}, \alpha_{17,35,4}, \alpha_{17,36,4}, \alpha_{17,26,45}, \alpha_{17,26,46}, \alpha_{17,26,35,4}, \alpha_{17,26,45,4}, \alpha_{17,16,35,4} = \tilde{\alpha}, \quad (42)$$

using compact notation:

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad i < j, \quad (43)$$

$$\alpha_{ij,k} = \alpha_{k,ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k, \quad i < j, \quad \text{etc.}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $E_7$, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs in the corresponding multiplets by (13) is equal to:

$$|W(\mathcal{G}^C, \mathcal{H}^C)| / |W(\mathcal{M}^C, \mathcal{H}_m^C)| = 56 \quad (44)$$

The Knapp-Stein operators $G_\chi^\pm$ act pictorially as reflection w.r.t. the bullet intertwining each $T_\chi^-$ member with the corresponding $T_\chi^+$ member.
Main Type for $E_{7(-25)}$ and $E_{7(7)}$
The Lie algebras $E_6(-14)$, $E_6(6)$ and $E_6(2)$

Let $G = E_6(-14)$. The maximal compact subalgebra is $K \cong so(10) \oplus so(2)$, while $M \cong su(5, 1)$.

The signature of the ERs of $G$ is $[14]$: $\chi = \{ n_1, n_3, n_4, n_5, n_6; c \}$, $c = d - \frac{11}{2}$, (45)

expressed through the Dynkin labels as:

$n_i = m_i$, $-c = \frac{1}{2} m_\bar{\alpha} = \frac{1}{2} (m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6)$ (46)

The same holds for the parabolically related exceptional Lie algebras $E_6(6)$ and $E_6(2)$ with $M$–factors $sl(6, R)$ and $su(3, 3)$, resp.

Further, we need the noncompact roots of the complex algebra $E_6$:

$\alpha_2, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{24}, \alpha_{25}, \alpha_{26}$
$\alpha_{2,4}, \alpha_{2,45}, \alpha_{2,46}, \alpha_{25,4}, \alpha_{15,4}, \alpha_{26,4}$
$\alpha_{16,4}, \alpha_{15,34}, \alpha_{26,45}, \alpha_{16,34}, \alpha_{16,45}$
$\alpha_{16,35}, \alpha_{16,35,4}, \alpha_{16,25,4} = \bar{\alpha}$

(47)

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $G$, i.e., they will be labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$. It turns out that each such multiplet contains 70 ERs/GVMs - see the figure below.

The Knapp-Stein operators $G^\pm_\chi$ act pictorially as reflection w.r.t. the dotted line separating the $T^-_\chi$ members from the $T^+\chi$ members.

Note that there are five cases when the embeddings correspond to the highest root $\bar{\alpha}$: $V^\Lambda^- \rightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_\bar{\alpha} \bar{\alpha}$. In these five cases the weights are denoted as: $\Lambda_\pm^\nu$, $\Lambda_\pm^\nu$, $\Lambda_\pm^\mu$, $\Lambda_\pm^\nu$, then: $m_\bar{\alpha} = m_1, m_3, m_4, m_5, m_6$, resp. Thus, their action coincides with the action of the Knapp-Stein operators $G^+\chi$ which in the above five cases degenerate to differential operators as we discussed for $so(3, 2)$.

Note that the figure has the standard $E_6$ symmetry, namely, conjugation exchanging indices $1 \leftrightarrow 6, 3 \leftrightarrow 5$. 




Main Type for $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$
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