Vortex solutions of the evolutionary Ginzburg-Landau type equations

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Abstract

We consider two types of the time-dependent Ginzburg-Landau equation in 2D bounded domains: the heat-flow equation and the Schrödinger equation. We study the asymptotic behaviour of the vortex solutions of these equations when the vortex core size is much smaller than the inter-vortex distance.

Using the method of the asymptotic expansion near the vortices, we obtain the systems of ordinary differential equations (ODEs) governing the evolution of the vortices. The expressions for these equations in the circle and in the annular domain are presented. We study the motion of the vortices in these two domains. It is shown that there exist the stationary points for both types of the equations and these points are determined for some particular cases.

The heat-flow equation describes the particles that move like electric charges. If the vortices have different signs (i.e. Poincaré indices) then they attract each other and collide. If they have the same signs then they repulse. The particles always move away from the stationary points. The motion is very slow and non-periodic.

The Schrödinger equation describes the motion of particles that behave like hydrodynamics vortices. The vortices with the same signs move in the same direction. If the signs are different then the vortices move in the opposite directions. In particular, if the initial positions are near the stationary points, then the particles move along the elliptic trajectories. The motion is not stable with respect to the initial data. It is always periodic or quasiperiodic. Examples of such trajectories are presented.

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1 Introduction

We start with the free energy in Ginzburg-Landau theory:

\[ F(u) = \int_{\Omega} \left( \frac{\hbar^2}{2m} |\nabla u|^2 + \alpha |u|^2 + \frac{1}{2} \beta |u|^4 \right). \]  

(1.1)

Here in (1.1) \( u(x,t) \) \((x = (x, y)) \) is a complex-valued wave function, \( \Omega \) is the domain occupied by (super)fluid, \( m \) is an atomic mass, \( 2\pi\hbar \) is a Planck’s constant, \( \alpha \) and \( \beta \) are the temperature-dependent parameters such that 

\[ \alpha(T) = \alpha_0 \left( \frac{T}{T_c} - 1 \right), \quad \beta(T) > 0, \quad \alpha_0 > 0 \]

with \( T_c \) being a temperature of phase transition, i.e. \( \alpha < 0 \) for \( T < T_c \).

In the absence of gradients \( F(u) \) represents the condensation energy with the density 

\[ f(u) = \alpha |u|^2 + \frac{1}{2} \beta |u|^4. \]

We assume that there are \( N \) vortices in the domain, located at the points \( \xi_1(t), \ldots, \xi_N(t) \), \( \xi_j = (\zeta_j, \eta_j) \) with degrees \( n_1, \ldots, n_N \), respectively. The vortices are stable if and only if \( n_j = \pm 1 \). If \( \varepsilon \) is a size of vortex core, than inter-vortex distance is supposed to be of the order \( O(1) \).
Let us pass to dimensionless variables $u = u_0 u', \ x = \varepsilon x'$, where $u_0^2 = -\frac{\alpha}{\beta}$ corresponds to the minimum of the condensation energy and $\varepsilon = \sqrt{\frac{\hbar^2}{2m|\alpha|}}$ is a coherence length (size of vortex core). Then the free energy takes the form

$$F(u') = \int_{\Omega} |\nabla' u'|^2 + \frac{1}{2} (1 - |u'|^2)^2$$  \hspace{1cm} (1.2)

Note that in these variables the vortex core size is $O(1)$ and inter-vortex distance is $O(1/\varepsilon)$.

Using standard techniques of the calculus of variation, the minimization of $F$ with respect to variation in $u'$ yields the stationary G-L equation:

$$\Delta u' + u'(1 - |u'|^2) = 0.$$

In this paper we study two types of non-stationary Ginzburg-Landau equations: the nonlinear Schrödinger equation

$$\Delta u' + u'(1 - |u'|^2) = -iu';$$  \hspace{1cm} (1.4)

and the nonlinear heat flow equation:

$$\Delta u' + u'(1 - |u'|^2) = u'_t.$$

Under the usual diffusive scaling $x'' = x' \varepsilon, \ t'' = \varepsilon^2 t$, Eq. (1.4) takes the form

$$\Delta u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon(1 - |u^\varepsilon|^2) = -iu^\varepsilon$$

and for (1.5) we obtain

$$\Delta u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon(1 - |u^\varepsilon|^2) = u^\varepsilon_t$$  \hspace{1cm} (1.7)

We have again the vortex core size $O(\varepsilon)$ and inter-vortex distance $O(1)$.

In this paper we consider these equations in the bounded domain with the Neumann boundary condition on $\partial \Omega$.

We look for two approximate solutions of (1.6) and (1.7) (see [1]): outer solution

$$v^\varepsilon(x, t) = u_0(x, t) + \varepsilon u_1(x, t) + \varepsilon^2 u_2(x, t) + \ldots$$  \hspace{1cm} (1.8)

in the region outside the vortex cores, but for small $r = |x - \xi_j(t)|$, and inner solution for the core of the $j$th vortex:

$$U^\varepsilon(x, t) = U_0 \left( \frac{x - \xi_j(t)}{\varepsilon} \right) + \varepsilon U_1 \left( \frac{x - \xi_j(t)}{\varepsilon} \right) + \varepsilon^2 U_2 \left( \frac{x - \xi_j(t)}{\varepsilon} \right) + \ldots$$  \hspace{1cm} (1.9)

Substituting the expansions (1.8) and (1.9) in the equations (1.6) and (1.7), we obtain the series of equations for the functions $u_0(x, t), \ u_1(x, t), \ u_2(x, t), \ldots,$ and $U_0(x, t), \ U_1(x, t), \ldots$.

We construct the approximate form for the solution (1.8) near the $j$th vortex. For the inner solution (1.9) we pass to the stretched variable $X = \frac{x - \xi_j(t)}{\varepsilon}$ and find the solution for large $R = |X|$. Asymptotic matching of these solutions yields the equations governing the motion of
the particles. Taking into account that \( r = \varepsilon R \), we obtain that the vortices described by the Schrödinger equation are governed by the system

\[
\dot{\xi}_j(t) = 2K^{(j)},
\]

(1.10)

where the vector \( K^{(j)} \) depends on the form of the domain \( \Omega \) and can be expressed, for example, through the derivative of the complex potential \( \tilde{W}(z) \) of the fluid with vortices out of \( j \)th vortex:

\[
K^{(j)} = (\text{Re} \tilde{W}'(z_j), -\text{Im} \tilde{W}'(z_j)).
\]

(1.11)

In particular, for annular domain we have (see formula (2.10) below):

\[
\tilde{W}'(z_j) = \frac{1}{z_j} \sum_{k=1, k\neq j}^N n_k \left[ \zeta \left( i \ln \frac{z_j}{z_k} \right) - \zeta \left( i \ln \frac{z_j z_k}{Z_k^2} \right) \right] - \frac{n_j}{z_j} \zeta \left( 2i \ln \frac{z_j}{R} \right) + \frac{2i \eta}{\omega_1 z_j} \sum_{k=1}^N n_k \ln \frac{r_k}{R}.
\]

We obtain by the same matching procedure for the heat-flow equation that for the standard time scaling \( t'' = \varepsilon^2 t \) the vortices are stationary at the leading order. To obtain a non-trivial dynamics we have to rescale the time by the factor \((\log 1/\varepsilon)\). In this long-time scaling the dynamics of the vortices is described by the system

\[
\dot{\xi}^\perp_j(t) = 2n_j K^{(j)},
\]

(1.12)

where \( \dot{\xi}^\perp_j = (-\dot{\eta}_j, \dot{\zeta}_j) \) and \( K^{(j)} \) is determined by (1.11).

We construct the numerical solutions of the systems (1.10) and (1.12) in the circle and in the ring for given initial positions of vortices and given indexes. We show that the Schrödinger equation (1.10) describes the motion of the hydrodynamical vortices and the heat-flow equation (1.12) governs the motion of some charged-similar particles. The trajectories of the particles for different initial data are shown in the Figs.1-9.

2 Schrödinger equation

Let us consider the Schrödinger equation (1.4). Substituting (1.8) into (1.6), we obtain the following the series of equations:

\[
\varepsilon^{-2} : \quad u_0(1 - |u_0|^2) = 0;
\]

(2.1)

\[
\varepsilon^{-1} : \quad u_1 \overline{u_0} + u_0 \overline{u_1} = 0;
\]

(2.2)

\[
\varepsilon^{0} : \quad -iu_{0t} = \Delta u_0 - |u_1|^2 u_0.
\]

(2.3)

Here \( \overline{\pi} \) means the complex conjugation of \( u \). The first of these equations yields

\[
u_0(x, t) = e^{i\Phi_0(x, t)}.
\]

(2.4)

Now from (2.3) and (2.1) we get

\[
\Phi_{0t} = i\Delta \Phi_0 - (\nabla \Phi_0)^2 - |u_1|^2.
\]
Separating real and imaginary part, we obtain
\[ \Delta \Phi_0 = 0. \]

Asymptotic boundary conditions at the cores are
\[ \Phi_0(x, t) \to n_j \tan^{-1} \left( \frac{y - \eta_j}{x - \zeta_j} \right) = n_j \theta \]
as \( x \to \xi_j \), where \((r, \theta)\) are the polar coordinates related to the \(j\)th vortex: \( r = |x - \xi_j| \), \( \theta = \tan^{-1} \left( \frac{y - \eta_j}{x - \zeta_j} \right) \).

Hence, a boundary-value problem for \( \Phi_0(x, t) \) takes the form:
\[
\begin{cases}
\Delta \Phi_0(x, t) = 0, & \text{in } \Omega \setminus \{ \cup_{j=1}^N \xi_j \}; \\
\Phi_0(x, t) \to n_j \theta, & x \to \xi_j, \quad j = 1, 2, \ldots, N \\
\frac{\partial \Phi_0}{\partial n} = 0 & x \in \partial \Omega.
\end{cases}
\]

(2.5)

The last boundary condition follows from the condition for \( u^\varepsilon \) on \( \partial \Omega \). It is well-known (see e.g. \([2]\)) that the phase of condensate wave function \( \Phi(x, t) \) is related to the hydrodynamic velocity potential \( \phi(x, t) \)
\[ \phi(x, t) = \frac{\hbar}{m} \Phi(x, t). \]

Then we obtain \( \mathbf{v} = \nabla \phi = \frac{\hbar}{m} \nabla \Phi \), and the boundary condition is the standard hydrodynamics condition of impermeability, \( \mathbf{v} \cdot \mathbf{n} = 0 \). Thus we can consider problem (2.5) as a usual hydrodynamic problem describing the potential of the ideal liquid with the vortices.

The solution of this problem can be found for any form of the domain \( \Omega \) by the method of “reflexed vortices” or by some other method. In particular, for the annular domain \( R_1 \leq r \leq R_2 \) the velocity potential \( \Phi_0(x, t) \) of an ideal fluid with \( N \) vortices of vortex strengths \( n_k \) can be written in the complex form (see \([3]\))
\[
\Phi_0(z) = \text{Re} W(z) = \text{Re} \left\{ -i \sum_{k=1}^{N} n_k \left[ \ln \sigma \left( i \ln \frac{z}{z_k} \right) - \ln \sigma \left( i \ln \frac{zz_k}{Z_k^2} \right) - \frac{2\eta}{\omega_1} \ln \frac{r_k}{R_2} \ln z \right] \right\} + \text{const.}
\]

(2.6)

Here \( z \) is a complex variable: \( z = x + iy = r \exp(i\tilde{\theta}) \), \( z_k = \zeta_k + i\eta_k = r_k \exp(i\tilde{\theta}_k) \) are the points of vortex locations, \( r = |x| \), the polar angle \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \) is measured from the center of the ring, \( W(z) \) is a complex potential, \( Z_k = R_2 \exp(i\tilde{\theta}_k) \), \( \zeta(z) \) and \( \sigma(z) \) are the Weierstrass zeta- and sigma-function with the half-periods \( \omega_1 = \pi \), \( \omega_2 = i \ln(R_2/R_1) \); \( \eta = \zeta(\omega_1) \) (see e.g. \([4]\)).

To obtain the approximate expression for the wave function \( u_0 \) near the \( j \)th vortex, let us expand \( \Phi_0 \) in a Taylor series around \( j \)th vortex. We have to take into account that \( \Phi_0 \) has singularities
we obtain the relation

$$\Phi_0(x, t) = \Phi_0(\xi_j, t) + \nabla \Phi_0(\xi_j, t) \cdot (x - \xi_j) + \ldots, \quad (2.7)$$

we obtain the relation

$$u_0(x, t) = e^{i\Phi_0} \approx e^{in_j\theta_i\theta_0} [1 + iK^{(j)} \cdot (x - \xi_j) + O(r^2)], \quad (2.8)$$

where

$$\theta_0 = \Phi_0(\xi_j, t)$$

and

$$K^{(j)} = \nabla \Phi_0(\xi_j, t) = (Re\hat{W}'(z_j), -Im\hat{W}'(z_j)). \quad (2.9)$$

Here $\hat{W}'(z_j)$ is a derivative of the complex potential in the point of the vortex location.

In particular, for annular domain we have (see [4])

$$\hat{W}'(z_j) = \frac{1}{z_j} \sum_{k=1}^{N} n_k \left[ \zeta \left( i \ln \frac{z_j}{z_k} \right) - \zeta \left( i \ln \frac{z_j z_k}{Z_k} \right) \right] - \frac{n_j}{z_j} \zeta \left( 2i \ln \frac{r_j}{R_2} \right) + \frac{2i\eta}{\omega_1 z_j} \sum_{k=1}^{N} n_k \ln \frac{r_k}{R_2}. \quad (2.10)$$

One can show that the expression $(2.10)$ gives the velocity potential of the ideal liquid with the vortices it the circle when $R_1 \to 0$ (see Section 5)

$$\hat{W}'(z_j) = \sum_{k=1}^{N} \frac{n_k}{i(z_j - z_k)} - \sum_{k=1}^{N} \frac{n_k}{i(z_j - z_k)} \cdot \frac{r_k}{R_2}, \quad (2.11)$$

where $z_k' = \frac{R_1^2}{z_k}$ are the coordinates of the vortices symmetric to $z_k$ with respect to the circumference $r = R_2$.

Thus, the exterior solution $u_0(x, t)$ is defined by $(2.8)$, $(2.9)$ and $(2.10)$ (or $(2.11)$).

Let us consider now the core structure of the vortices. We denote by $X = \frac{x - \xi_j(t)}{\varepsilon}$ the stretched variable and by $(R, \theta)$ are the polar coordinates of $X$. Then the inner solution $(1.4)$ takes a form

$$U^c(x, t) = U_0(X) + \varepsilon U_1(X) + \varepsilon^2 U_2(X) + \ldots$$

Substituting this expansion in the equation $(1.7)$ we obtain the series of equations for $U_0(X), U_1(X), U_2(X), \ldots$

$$\Delta U_0 + U_0(1 - |U_0|^2) = 0, \quad (2.12)$$

$$\Delta U_1 + (1 - 2|U_0|^2)U_1 - U_0^2 \nabla U_0 = i\xi_j(t) \cdot \nabla U_0, \quad (2.13)$$

As in the consideration above (see. $(2.8)$) we look for the solution of $(2.13)$ in the form:

$$U_0(R, \theta, t) = f_0(R)e^{in_j\theta_i\theta_0}.$$

Then $f_0(R)$ satisfies the equation

$$f_0'' + \frac{1}{R} f_0' - \frac{n_j^2}{R^2} f_0 + f_0(1 - f_0^2) = 0.$$
with boundary conditions

\[
\begin{align*}
  f_0(0) &= 0, \\
  f_0(\infty) &= 1.
\end{align*}
\]

As in \cite{1} it can be shown that

\[
f_0(R) \approx 1 - \frac{n_j^2}{2R^2} + O\left(\frac{1}{R^4}\right) \tag{2.14}
\]

for \( R \gg 1 \).

As for \( U_1(\mathbf{x}) = U_1(R, \theta) \), we look for it in the form

\[
U_1(R, \theta) = f_1(R, \theta)e^{in_j\theta+i\phi_0} = [(A_r(R) + iA_i(R)) \cos \theta + (B_r(R) + iB_i(R)) \sin \theta]e^{in_j\theta+i\phi_0}. \tag{2.15}
\]

Substituting \( U_0 \) and \( U_1 \) in (2.13), gathering the terms with \( \sin \theta \) and \( \cos \theta \) and separating real and imaginary parts, we obtain the system of equations for \( A_r, A_i, B_r \) and \( B_i \):

\[
\begin{align*}
  -n_j \frac{\dot{f}_0}{R} &= A_r'' + \frac{1}{R} A_r' + \left(1 - 3f_0^2 - \frac{2}{R^2}\right) A_r - n_j \frac{2B_i}{R^2}, \\
  n_j \frac{\dot{f}_0}{R} &= B_r'' + \frac{1}{R} B_r' + \left(1 - 3f_0^2 - \frac{2}{R^2}\right) B_r + n_j \frac{2A_i}{R^2}, \\
  f_0' \dot{\phi}_j &= A_i'' + \frac{1}{R} A_i' + \left(1 - f_0^2 - \frac{2}{R^2}\right) A_i + n_j \frac{2B_r}{R^2}, \\
  f_0' \dot{\phi}_j &= B_i'' + \frac{1}{R} B_i' + \left(1 - f_0^2 - \frac{2}{R^2}\right) B_i - n_j \frac{2A_r}{R^2}.
\end{align*}
\]

We look for the solutions \( A_r, A_i, B_r, \) and \( B_i \) in the form

\[
\begin{align*}
  A_r &= -n_j \dot{\phi}_j W(R), & A_i &= \dot{\phi}_j Z(R), \\
  B_r &= n_j \dot{\phi}_j W(R), & B_i &= \dot{\phi}_j Z(R). \tag{2.16}
\end{align*}
\]

Substituting these expressions into equations for \( A_r, B_i \), we derive two equations for \( W(R) \) and \( Z(R) \):

\[
\begin{align*}
  \frac{f_0}{R} &= W'' + \frac{1}{R} W' + W \left(1 - 3f_0^2 - \frac{2}{R^2}\right) + \frac{2}{R^2} Z, \\
  f_0' &= Z'' + \frac{1}{R} Z' + W \left(1 - f_0^2 - \frac{2}{R^2}\right) + \frac{2}{R^2} W. \tag{2.17}
\end{align*}
\]

Taking into account (2.14), we conclude that

\[
f_0' = \frac{1}{R^3} + O\left(\frac{1}{R^5}\right)
\]

for large \( R \).
Expanding $W(R)$ and $Z(R)$ in the series with respect to the degrees of $R$, we can find the solution $U_1(R, \theta, t)$ for large $R$:

$$U_1(R, \theta, t) = e^{i n_j \theta + i \theta_0} \left[ \frac{n_j C_1}{R} \left( \frac{\xi_j}{R} \right) + i R \left( \frac{\xi_j}{R} \right) + \frac{i A_1}{R} \left( \frac{\xi_j}{R} \right) + O\left( \frac{1}{R^3} \right) \left( \frac{\xi_j}{R} \right) \right],$$

where $\xi_j = (\xi_j, \eta_j)$, $\xi_j^\perp = (-\eta_j, \xi_j)$, $C_1, A_1$ are constants.

Then the solution $U(X) = U_0(X) + \varepsilon U_1(X)$ can be represented as

$$U(X) = e^{i n_j \theta + i \theta_0} \left[ 1 + \frac{i \varepsilon}{2} (\xi_j \cdot X) + O\left( \frac{1}{R^2} \right) \left( 1 + \varepsilon \frac{\xi_j^\perp \cdot X}{R} \right) \right],$$

for large $R$.

The function $U(X)$ is an approximate solution in the region near the $j$th vortex, where $R = |(x - \xi_j)|/\varepsilon$ is large.

To guarantee consistency of the solutions $(2.8)$ and $(2.19)$ it is necessary to suppose that the leading terms of the expansion be equal for $x - \xi_j(t) = \varepsilon X$, i.e.

$$K^{(j)} \cdot \varepsilon X = \frac{1}{2} \xi_j \cdot \varepsilon X.$$

This gives the equations

$$\dot{\xi}_j = 2K^{(j)}, \quad j = 1, 2, \ldots, N.$$

This is the desired system describing the motion of the vortices in the domain $\Omega$.

3 Heat flow equation

Now we consider the heat-flow equation $(1.7)$. To construct the solution near the $j$th vortex in outer region we substitute $(1.8)$ in $(1.7)$ and obtain the series of equations for $u_0, u_1, \ldots$:

$$u_0(1 - |u_0|^2) = 0;$$

$$u_1 u_0 + u_0 u_1 = 0;$$

$$u_{0t} = \Delta u_0 - |u_1|^2 u_0.$$

For the function $u_0$ we have

$$u_0(x, t) = e^{i\Phi_0(x, t)},$$

as it was in the case of the the Schrödinger equation, but now $\Phi_0$ satisfies the equation:

$$\Phi_{0t} = \Delta \Phi_0.$$ (3.2)

Passing to the moving coordinates $\tilde{x} = x - \xi_j(t)$ we obtain the following equation for $\Phi_0$ that describes the local behaviour of the solution near the $j$th vortex:

$$\begin{cases}
\Phi_{0t} - \dot{\xi}_j(t) \cdot \nabla \Phi_0 = \Delta \Phi_0(\tilde{x}, t), \\
\Phi_0(\tilde{x}, t) \to n_j \tan^{-1} \left( \frac{\tilde{y}}{\tilde{x}} \right) = n_j \theta, \quad \tilde{x} \to 0;
\end{cases}$$

(3.3)
We denote by \((r, \theta)\) the polar coordinates corresponding to \(\tilde{x}\). The function \(\Phi_0\) near the vortex is described by the power series of \(r\) and \(\log r\) for small \(r\). It can be shown that
\[
\Phi_0(\tilde{x}, t) = n_j \theta + \frac{1}{2} n_j (\log r) \tilde{\xi}_j \cdot \tilde{x} + S_j \cdot \tilde{x} + O(r^2 \log r),
\]
where \(S_j\) is an unknown vector. For the function \(u_0\) we have
\[
u_0(\tilde{x}, t) = e^{n_j \theta} \left[ 1 + \frac{i}{2} n_j (\log r) \tilde{\xi}_j \cdot \tilde{x} + i S_j \cdot \tilde{x} + O(r^2 \log r) \right].
\]
This gives a representation of the outer solution near the \(j\)th vortex.

The inner solution for the heat flow equation in the stretched variables leads to the system of equations for \(U_0, U_1, \ldots\), similar to (2.12)-(2.13):
\[
\Delta U_0 + U_0(1 - |U_0|^2) = 0,
\]
\[
-\dot{\xi}_j(t) \cdot \nabla U_0 = \Delta U_1 + (1 - 2|U_0|^2)U_1 - U_0^2 \overline{U_1}.
\]

The first equation is the same as (2.12) and hence we have again (2.14). We look for the function \(U_1(x)\) in the form (2.15). Then Eq.(3.7) in terms of \(f_0, f_1\) becomes
\[
-\dot{\xi}_j \cdot (f_0' \nabla R + in_j f_0 \nabla \theta) = \Delta f_1 + 2in_j (\nabla f_1 \cdot \nabla \theta) - \frac{n_j^2}{R^2} f_1 + f_1(1 - 2f_0^2) - f_0^2 \overline{f_1}.
\]

The following system for \(A_r(R), A_i(R), B_r(R), B_i(R)\) is valid:
\[
-\dot{\xi}_j f_0' = A_r'' + \frac{1}{R} A_r' + \left( 1 - 3f_0^2 - \frac{2}{R^2} \right) A_r - n_j \frac{2B_i}{R^2},
\]
\[
-\dot{\xi}_j f_0' = B_r'' + \frac{1}{R} B_r' + \left( 1 - 3f_0^2 - \frac{2}{R^2} \right) B_r - n_j \frac{2A_i}{R^2},
\]
\[
-\dot{\xi}_j f_0' = A_i'' + \frac{1}{R} A_i' + \left( 1 - f_0^2 - \frac{2}{R^2} \right) A_i + n_j \frac{2B_r}{R^2},
\]
\[
-\dot{\xi}_j f_0' = B_i'' + \frac{1}{R} B_i' + \left( 1 - f_0^2 - \frac{2}{R^2} \right) B_i - n_j \frac{2A_r}{R^2}.
\]

We seek the solution of this system in the form that is similar to (2.16):
\[
A_r = \dot{\xi}_j Z, \quad A_i = \dot{\xi}_j n_j W, \quad B_r = \dot{\xi}_j Z, \quad B_i = -\dot{\xi}_j n_j W,
\]
where \(Z\) and \(W\) satisfy
\[
-f_0' = Z'' + \frac{1}{R} Z' + \left( 1 - 3f_0^2 - \frac{2}{R^2} \right) Z + \frac{2W}{R^2},
\]
\[
-\frac{f_0}{R} = W'' + \frac{1}{R} W' + \left( 1 - f_0^2 - \frac{2}{R^2} \right) W + \frac{2Z}{R^2}.
\]
The solutions $W$ and $Z$ can be found in the power series of $R$ and $\log R$. We are interested in the behaviour of the solutions for $R \gg 1$:

$$W = -\frac{1}{2} R \log R + C_0 R + O(\log R),$$

$$Z = -\frac{1}{2R} \log R + O\left(\frac{1}{R}\right).$$

Here $C_0$ is a constant. So, we have for $R \gg 1$

$$U_1(X) = e^{in j} \left[ \frac{1}{2} i n_j (\log R) (\dot{\xi}_j^\perp \cdot X) + in_j C_0 (\dot{\xi}_j^\perp \cdot X) + O\left(\frac{\log R}{R}\right) \right].$$  \hspace{0.5cm} (3.9)

Putting together the solutions $U_0$ and $U_1$ we obtain as $R \to \infty$

$$U_0(X) + \varepsilon U_1(X) = e^{in j} \left[ f_0(R) + \frac{1}{2} i \varepsilon n_j (\log R) (\dot{\xi}_j^\perp \cdot X) + i \varepsilon n_j C_0 (\dot{\xi}_j^\perp \cdot X) + O\left(\frac{\varepsilon \log R}{R}\right) \right].$$  \hspace{0.5cm} (3.10)

Matching (3.10) with the outer solution (3.5) and taking into account that $\bar{x} = \varepsilon X$, $r = \varepsilon R$, we can see that

$$\frac{1}{2} (\log \varepsilon) \dot{\xi}_j^\perp + n_j S_j = C_0 \dot{\xi}_j^\perp$$

or

$$\dot{\xi}_j^\perp = \frac{S_j}{\frac{1}{2} \log \varepsilon + C_0} = O(1/\log(1/\varepsilon)).$$  \hspace{0.5cm} (3.11)

This equation shows that in this time scale the vortices are stationary at the leading order. The time scale of the vortices is slower than the time scale of the phase field.

### 4 Heat flow equation: long time scale

To obtain a non-trivial dynamics of the vortices, we have to rescale the time variable by a factor $O(\log(1/\varepsilon))$, i.e. instead of diffusive scaling $x'' = x' \varepsilon$, $t'' = \varepsilon^2 t$ we consider the scaling $x'' = x' \varepsilon$, $t'' = [\varepsilon^2 / \log(1/\varepsilon)] t$. Then we obtain the equation (see [3])

$$\Delta u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon (1 - |u^\varepsilon|^2) = \frac{1}{\log(1/\varepsilon)} u^\varepsilon_t. \hspace{0.5cm} (4.1)$$

We consider the approximate solutions of this equation in the form (1.8) for outer region and (1.9) for inner region. Thus we obtain as in the previous case the equations (2.4) for $u_0$. For the function $\Phi_0$ we have

$$\frac{1}{\log(1/\varepsilon)} \Phi_0_t = \Delta \Phi_0. \hspace{0.5cm} (4.2)$$

To investigate the behaviour of the outer solution near the $j$th vortex we expand $\Phi_0$ in powers of $\delta = (\log(1/\varepsilon))^{-1}$:

$$\Phi_0 = \Phi_{00}(x, t) + \delta \Phi_{01}(x, t) + \delta^2 \Phi_{02}(x, t) + \ldots$$

Then we get the series of equations for $\Phi_{0i}$:

$$\Delta \Phi_{00} = 0, \hspace{0.5cm} (4.3)$$

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\[ \Phi_{00,t} = \Delta \Phi_{01}, \ldots \quad (4.4) \]

Eq. (4.3) should be completed by the boundary conditions at the vortices:

\[ \Phi_{00}(x, t) \rightarrow n_j \tan^{-1} \left( \frac{y - \eta_j}{x - \zeta_j} \right) \]

and the Neumann condition on the boundary \( \partial \Omega \), i.e.

\[ \frac{\partial \Phi_{00}}{\partial n} = 0. \]

So, we have problem (2.5) for \( \Phi_{00}(x, t) \) and hence \( \Phi_{00}(x, t) \) is given by (2.6). Thus, near the \( j \)th vortex

\[ \Phi_{00}(x, t) = n_j \theta + \theta_0 + K_j \cdot (x - \xi_j(t)), \]

where \( K_j \) is defined by (2.10).

For the function \( \Phi_{01} \) we have

\[ \Phi_{01}(x, t) = \frac{1}{2} n_j (\log r) (\hat{\xi}_j \cdot \hat{x}) + P_{1j} \cdot \hat{x} + O(r^2 \log r) \]

for \( r = |\hat{x}| \ll 1 \), \( P_{1j} \) is a constant vector depending on the history of the vortices. Extracting the singularity of \( \Phi_{01}(x, t) \) in the \( j \)th vortex (see (2.7)), we obtain

\[ u_0(\hat{x}, t) = e^{in_j \theta} \left( 1 + iK(j) \cdot \hat{x} + \frac{in_j}{2 \log(1/\varepsilon)} (\log r)(\hat{\xi}_j \cdot \hat{x}) + \frac{in_j}{\log(1/\varepsilon)} (\log r)(P_{1j} \cdot \hat{x}) \right) + \]

\[ + O(\delta^2 + \delta r^2 \log r + r^2) \quad (4.5) \]

as \( r \) is sufficiently small.

It follows from (4.4) that for the inner solution

\[ - \frac{\hat{\xi}_j(t)}{\log(1/\varepsilon)} \cdot \nabla U_0 = \Delta U_1 + (1 - 2|U_0|^2)U_1 - U_0^2. \quad (4.6) \]

By the same way as it was done for (3.7) we obtain

\[ U_0(X) + \varepsilon U_1(X) = e^{in_j \theta} \left[ f_0(R) + \frac{in_j}{2 \log(1/\varepsilon)} (\log R)(\hat{\xi}_j \cdot X) + \frac{in_j C_0}{\log(1/\varepsilon)} (\hat{\xi}_j \cdot X) + O \left( \frac{\varepsilon \log R}{R \log(1/\varepsilon)} \right) \right] \quad (4.7) \]

as \( R \to \infty \).

Matching (1.5) and (4.7) gives

\[ K(j) + \frac{n_j}{2 \log(1/\varepsilon)} (\log \varepsilon) \hat{\xi}_j + \frac{n_j}{\log(1/\varepsilon)} P_{1j} = \frac{n_j C_0}{\log(1/\varepsilon)} \hat{\xi}_j. \quad (4.8) \]

Therefore we obtain the desired equation

\[ \hat{\xi}_j = 2n_j K(j) + O(1/\log(1/\varepsilon)). \]

In this time scale the dynamics of vortices is not trivial and it is described by the system of equations

\[ \hat{\xi}_j = 2n_j K(j), \quad j = 1, 2, \ldots, N. \quad (4.9) \]

This equation is similar to (2.20) for the Schrödinger equation, but it includes the turned coordinates \( \hat{\xi}_j \) and the factor \( n_j \). Notice that the time scale in (4.9) is slower by the factor \( O(1/\log(1/\varepsilon)) \).
5 Motion of the vortices in the circle

Let us show that from (2.6) we obtain the velocity of the liquid with the vortices in the circle:

\[
\tilde{W}'(z) = \sum_{k=1}^{N} \frac{n_k}{i(z - z_k)} - \sum_{k=1}^{N} \frac{n_k}{i(z - z_k')},
\]

(5.1)
as \(R_1 \to 0\). Here \(z_k' = \frac{R_2^2}{z_k}\) are the coordinates of the vortices symmetric to \(z_k\) with respect to the circumference \(r = R_2\).

It follows from (2.6) that

\[
W'(z) = \frac{1}{z} \sum_{k=1}^{N} n_k \left[ \zeta \left( i \ln \frac{z}{z_k} \right) - \zeta \left( i \ln \frac{zz_k}{Z_k^2} \right) \right] - \frac{n_j}{z} \zeta \left( 2i \ln \frac{r}{R_2} \right) + \frac{2i\eta}{\omega_1 z} \sum_{k=1}^{N} n_k \ln \frac{r_k}{R_2},
\]

(5.2)
Notice that as \(R_1 \to 0\) the denominator \(q\) tends to zero. Therefore, the \(\zeta\)-function

\[
\zeta(z) = \frac{\eta z}{\omega_1} + \frac{\pi}{2\omega_1} \left[ \cot \left( \frac{i2}{z} \ln \frac{z}{z_k} \right) - \frac{n_j}{z} \zeta \left( 2i \ln \frac{r}{R_2} \right) \right] + \frac{2i\eta}{\omega_1 z} \sum_{k=1}^{N} n_k \ln \frac{r_k}{R_2}.
\]

is reduced in the sum of the linear part and cotangents. Hence

\[
W'(z) = \sum_{k=1}^{N} \frac{n_k}{2z} \left[ \cot \left( \frac{i2}{z} \ln \frac{z}{z_k} \right) - \cot \left( \frac{i2}{z} \ln \frac{zz_k}{Z_k^2} \right) \right].
\]

Calculating all the cotangents, we obtain the expression

\[
W'(z) = \frac{i}{2} \sum_{k=1}^{N} n_k \left[ \frac{z + z_k}{z(z - z_k)} - \frac{zz_k + Z_k^2}{z(zz_k - Z_k^2)} \right].
\]

Denoting by \(z_k' = Z_k^2/z_k\) the coordinates of the vortex symmetric with the vortex \(z_k\) with respect to the circumference \(r = R_2\) \((Z_k = R_2 \exp(\theta_k))\) and expanding the terms into the simplest fractions we derive the desired formula (5.1). The systems (2.20) and (4.9) describing this motion can be represented as

\[
\begin{aligned}
\dot{\zeta}_j &= 2K_{j1}, \\
\dot{\eta}_j &= -2K_{j2},
\end{aligned}
\]

(5.4)
for the Schrödinger equation and as

\[
\begin{align*}
\dot{\zeta}_j &= -2n_j K_{j2}, \\
\dot{\eta}_j &= -2n_j K_{j1},
\end{align*}
\]

(5.5)

for the heat-flow equation.

Here \( K_j = (K_{j1}, K_{j2}) = (\Re W'(z_j), \Im W'(z_j)) \) and \( W'(z_j) \) is defined by (5.3).

### 5.1 Schrödinger equation

Notice that the particles described by the Schrödinger equation behave as hydrodynamic vortices. So, we can expect that they move along the closed trajectories and never meet the walls.

If at the initial moment we have the symmetric locations of the vortices with alternating signs, we obtain the symmetric closed trajectories (see Fig.1 for \( N = 2 \) and \( N = 4 \)). The closer are vortices to the wall at \( t = 0 \) the wider are trajectories.

**Fig.1a. Motion “vortex-antivortex”.**

\( N = 2. \)

**Fig.1b. Motion ”vortices-antivortices”.**

\( N = 4. \)

If one of the vortices is close to the center and the other is near the wall we have more complicated motion: the vortex near the wall moves almost along the circumference and the inner vortex passes a “star” trajectory (Fig.2a). But if all the vortices have the same signs and are on the same circumference at \( t = 0 \), they move independently along this circumference (Fig.2b).

The motion is much more complicated for odd number of the vortices with the different signes (Fig.3a, 3b): vortices move in the chaotic way even if the initial positions are on the same circumference (comp. with Fig.2b).

### 5.2 Heat flow equation

The figures below represent the motion of the particles governed by the heat flow equation. In fact, we have only two types of motions: attraction, if the vortices have different signs and in
the initial time are close enough to each other, and repulsion, in all other cases.

So, if we have the even number of particles with alternative signs, each particle is attracted to the nearest neighbour and annihilated (Fig.4a, 4b).

For the odd number of particles the picture is symmetric if the initial data is symmetric. In Fig.5a we have the circle of particles with alternative signs. In this case the nearest particles with different signs are attracted and annihilated.

Finally, any number of particles with the same signs is attracted to the boundary and annihilated with the imaginary vortices (Fig. 5b).

Thus the motion of the particles in the framework of the heat flow equation is always attractive or repulsive. There are no periodic and quasiperiodic trajectories.
6 Motion of the particles in the ring

6.1 Stationary points

We investigate the motion of the vortices in the ring. The motion is governed by (5.4) and (5.5) but $W'(z_j)$ is defined by (2.10).

Consider the expression for $W'(z_j)$.

A. Let us suppose that there are only two vortices: $N = 2$. Representing $z_j = r_j e^{i \theta_j}$, we obtain

\[
W'(z_j) = \frac{1}{z_j} \left[ n_k \zeta \left( i \ln \frac{r_j}{r_k} - (\theta_j - \theta_k) \right) - n_k \zeta \left( i \ln \frac{r_k r_j}{R_j^2} - (\theta_j - \theta_k) \right) \right] -
\]
\[-n_j \zeta \left( 2i \ln \frac{r_j}{R_2} + \frac{2i\eta}{\omega_1} \left( n_1 \ln \frac{r_1}{R_2} + n_2 \ln \frac{r_2}{R_2} \right) \right) = \frac{1}{z_j} F_j. \quad (6.1)\]

Here \( j, k = 1, 2, k \neq j \) (if \( j = 1 \), then \( k = 2 \) and vice versa, if \( j = 2 \), then \( k = 1 \)).

Notice that \( \omega_1 = \pi \) and \( \omega_2 = i \ln(R_2/R_1) \) are the half-periods of the \( \zeta \)-functions.

If we put \( r_1 = r_2 = r_0 \) (vortices are on the same circumference) and take into account that \( \zeta(-z) = -\zeta(z) \), we obtain

\[ W'(z_j) = \frac{1}{z_j} \left[ n_k \zeta (\theta_k - \theta_j) - n_k \zeta \left( 2i \ln \frac{r_0}{R_2} - (\theta_j - \theta_k) \right) - n_j \zeta \left( 2i \ln \frac{r_0}{R_2} + \frac{2i\eta}{\omega_1} (n_1 + n_2) \ln \frac{r_0}{R_2} \right) \right]. \]

Using the addition formula for the \( \zeta \)-function \((6.3)\) on p.635 we obtain

\[ \zeta \left( 2i \ln \frac{r_0}{R_2} - (\theta_j - \theta_k) \right) = \zeta \left( 2i \ln \frac{r_0}{R_2} \right) + \zeta (\theta_k - \theta_j) + \frac{1}{2} \frac{\mathcal{P}' \left( 2i \ln \frac{r_0}{R_2} \right) - \mathcal{P}' (\theta_k - \theta_j)}{\mathcal{P} \left( 2i \ln \frac{r_0}{R_2} \right) - \mathcal{P} (\theta_k - \theta_j)}, \]

where \( \mathcal{P}(z) \) is Weierstrass’ \( \mathcal{P} \)-function \((6.4)\). Thus

\[ W'(z_j) = \frac{1}{z_j} \left[ -(n_k + n_j) \left[ \zeta \left( 2i \ln \frac{r_0}{R_2} \right) - \frac{2i\eta}{\omega_1} \ln \frac{r_0}{R_2} \right] - \frac{n_k}{2} \frac{\mathcal{P}' \left( 2i \ln \frac{r_0}{R_2} \right) - \mathcal{P}' (\theta_k - \theta_j)}{\mathcal{P} \left( 2i \ln \frac{r_0}{R_2} \right) - \mathcal{P} (\theta_k - \theta_j)} \right]. \]

Notice that the function \( \mathcal{P}'(\pm \omega_i) = 0 \) vanishes at the points of the half-periods \( \pm \omega_i, \ i = 1, 2 \).

Thus, if we put

\[ 2i \ln \frac{r_0}{R_2} = -\omega_2 = -i \ln \frac{R_2}{R_1} \quad \text{and} \quad \theta_k - \theta_j = \omega_1 = \pi \]

the last term will be equal zero. Hence, for

\[ r_1 = r_2 = \sqrt{R_1 R_2} , \quad \theta_j - \theta_k = \pi \]

we have

\[ W'(z_j) = \frac{n_1 + n_2}{z_j} \left[ \eta' - \eta \omega_2 \right] / \omega_1 \frac{\pi i}{2\omega_1} = \begin{cases} 0, & n_1 = -n_2, \\ i, & n_1 = n_2 = 1. \end{cases} \quad (6.2) \]

Here we have used the Legendre’s relation \( \eta \omega_2 - \eta' \omega_1 = \pi i / 2; \ \eta' = \zeta(\omega_2) \) \((6.3)\).

Expression \((6.2)\) means that there exist the stationary points \( r_1 = r_2 = \sqrt{R_1 R_2} , \ \theta_1 - \theta_2 = \pi \)

for both equations if \( n_1 = -n_2 \). Without loss of generality we can suppose that \( n_1 = -n_2 = 1 \) and that the stationary points are

\[ (x_1^0, y_1^0) = (\sqrt{R_1 R_2}, 0), \quad (x_2^0, y_2^0) = (-\sqrt{R_1 R_2}, 0). \quad (6.3) \]

If we take the initial positions of the vortices at these points, the vortices will not move at all.

**B.** Let us obtain the similar points for any *even* number of vortices \( 2N \). It was shown \((6.3)\) that if we have \( N \) vortices with the equal intensities \( n_j = n \) located uniformly on the same circumference \( r = r_0 \), i.e. \( z_j = r \exp(i\theta_j), \ \theta_j = 2\pi j / N, \ j = 1, 2, \ldots, N \), the expression for \( W'(z) \) take the form

\[ W'(z) = \frac{n_j}{z} \left[ \zeta \left( i \ln \frac{z}{z_j} \right) - \zeta \left( i \ln \frac{z z_j}{R_2^2} \right) - \frac{2i\eta}{\omega_1} \ln \frac{r}{R_2} \right], \quad (6.4) \]
where \( z_j = r_0 \exp(i\theta_j) \) is the position of any vortex and the half-periods of the \( \zeta \)-function are \( \omega_1 = \pi/N \) and \( \omega_2 = i \ln(R_2/R_1) \). This case corresponds to the vortex chaine in the ring.

If we have \( N \) pairs “vortex-antivortex” located uniformly on the circumference, one can interprete this as two vortex chaines. The complex-conjugated velocity has the form

\[
W'(z) = \sum_{k=1}^{2} \frac{n_k}{z} \left[ \zeta \left( i \ln \frac{z}{z_k} \right) - \zeta \left( i \ln \frac{z z_k}{R_2^2} \right) - \frac{2i\eta}{\omega_1} \ln \frac{r}{R_2} \right],
\]

(6.5)

where \( z_1 \) (respectively \( z_2 \)) is the coordinate of a vortex (respectively antivortex) in the chaines and \( n_1 = 1, n_2 = -1 \). For the sake of simlicity we can take \( z_1 = r_0 \) and \( z_2 = r_0 \exp(\pi i/N) \).

To obtain the velocity of the \( j \)th vortex, we should substract the term corresponding to this vortex and then put \( z = z_j \). We have

\[
W'(z_j) = \frac{n_k}{z_j} \left[ \zeta \left( i \ln \frac{z_j}{z_k} \right) - \zeta \left( i \ln \frac{z_j z_k}{R_2^2} \right) - \frac{2i\eta}{\omega_1} \ln \frac{r_j}{R_2} \right],
\]

(6.6)

where if \( j = 1 \), then \( k = 2 \) and if \( j = 2 \), then \( k = 1 \).

Notice that the real semi-period of the \( \zeta \)-functions is equal to \( \omega_1 = \pi/N \) while the imaginary semi-period does not change, i.e. \( \omega_2 = i \ln(R_2/R_1) \). As in the part A we obtain that

\[
W'(z_j) = \frac{n_1 + n_2}{z_j} \left[ \eta' - \frac{\eta \omega_2}{\omega_1} \right] = -\frac{n_1 + n_2}{z_j} \cdot \frac{\pi i}{2\omega_1} = \begin{cases} 0, & n_1 = -n_2, \\ -\frac{iN}{z_j}, & n_1 = n_2 = 1 \end{cases}
\]

(6.7)

for \( r_1 = r_2 = \sqrt{R_1 R_2} \), \( \theta_j - \theta_k = \pi/N \). This means that the stationary points for two chains of alternating-sign particles are

\[
\begin{align*}
z_k &= r_0 e^{i\pi k/N}, \\
n_k &= (-1)^{k+1}, \\
r_0 &= \sqrt{R_1 R_2}, \\
k &= 1, 2, \ldots, 2N,
\end{align*}
\]

(6.8)

It occurs that there exist the stationary points in the case when all the vortices have the same signs \( n_1 = n_2 = \ldots = n_N = 1 \) and are on the same circumference \( r = r_0 \). To find these points we have to solve the equation \( W'(z_j) = 0 \), where \( W'(z_j) \) is the velocity of \( j \) vortex obtained from (6.4) by subtractions of the term corresponding to the vortex,

\[
W'(z_j) = \frac{n}{z_j} \left[ -\zeta \left( 2i \ln \frac{z_j}{R_2} \right) + \frac{2i\eta}{\omega_1} \ln \frac{r_0}{R_2} \right].
\]

(6.9)

We have

\[
\coth \left( N \ln \frac{r_0}{R_2} \right) = 4 \sum_{t=1}^{\infty} \frac{q^{2t}}{1 - q^{2t}} \sinh \left( 2tN \ln \frac{r_0}{R_2} \right),
\]

where \( q = \exp \left( -\frac{\pi |\omega_2|}{\omega_1} \right) \) is the denominator of the \( \zeta \)-functions.

This equation has a root for any \( N \) and any \( R_1, R_2 \). For example, if we take \( R_1 = 0.5, R_2 = 1.5 \) and \( N = 2 \) (these parameters were used in the numerical simulations below) then \( r_0 = 0.651159 \). This value coincides with the numerically calculated stationary point of the systems (5.4), (5.5).
6.2 Motion "vortex-antivortex": linear problems

Let us investigate the motion of the pair “vortex-antivortex” in the ring, i.e. we have \( n_1 = -n_2 = 1 \) and the stationary points are defined by (6.3).

If the initial data are close to the stationary points one can describe the motion of two vortices in an analytic way. Taking into account that the stationary points correspond to the solution of the equation \( W'(z_j) = 0 \) we obtain

\[
W'(z_j) \approx \frac{1}{z_j} \left[ \frac{\partial F_j}{\partial x_1}(x_1 - x_1^0) + \frac{\partial F_j}{\partial x_2}(x_2 - x_2^0) + \frac{\partial F_j}{\partial y_1}(y_1 - y_1^0) + \frac{\partial F_j}{\partial y_2}(y_2 - y_2^0) \right],
\]

where \( F_j \) is defined in (6.1), \( (x_1^0, y_1^0), (x_2^0, y_2^0) \) are the stationary points, \( Z_k = R_2 \exp(i\theta_k) \), and

\[
\frac{\partial F_j}{\partial x_j} = \frac{ix_j}{R_1R_2}\left(n_k(e_2 - e_1) + 2n_j e_3 + \frac{2\eta \pi}{\pi}\right); \quad \frac{\partial F_k}{\partial x_k} = \frac{x_k n_k}{R_1R_2}\left(e_1 + e_2 + \frac{2\eta}{\pi}\right), \quad k \neq j;
\]

\[
\frac{\partial F_k}{\partial y_k} = \frac{x_k n_k}{R_1R_2}(e_2 - e_1); \quad \frac{\partial F_j}{\partial y_j} = -\frac{x_j n_k}{R_1R_2}(e_2 - e_1).
\]

Here we have taken into account that \( z_1 = x_1^0 = r_0 = \sqrt{R_1R_2}, z_1 = x_2^0 = -r_0 = -\sqrt{R_1R_2}, \)

\( \zeta'(z) = -\mathcal{P}(z) [ ] \) and

\[
\mathcal{P}\left(i \ln \frac{r_j}{r_k} - (\theta_j - \theta_k)\right) = \mathcal{P}(\omega_1 + \omega_2) = e_2;
\]

\[
\mathcal{P}\left(i \ln \frac{r_j}{r_k} - (\theta_j - \theta_k)\right) = \mathcal{P}(\omega_1) = e_1; \quad \mathcal{P}\left(2i \ln \frac{r_j}{R_2}\right) = \mathcal{P}(\omega_2) = e_3
\]

at the stationary points \( (x_1^0, y_1^0), (x_2^0, y_2^0) \). Here \( e_1, e_2, e_3 \) are the roots of the characteristic equation (6.4) and \( e_1 + e_2 + e_3 = 0 \). Denoting

\[
a_1 = \frac{2}{R_1R_2}(e_1 - e_2),
\]

\[
a_2 = \frac{2}{R_1R_2}\left(-e_3 + \frac{2\eta}{\pi}\right),
\]

\[
a_3 = \frac{2}{R_1R_2}\left(e_2 - e_1 - 2e_3 - \frac{2\eta}{\pi}\right),
\]

we can represent \( W'(z_j) \) in the form

\[
W'(z_1) = \frac{1}{2} \left[-a_1(y_1 - y_1^0) - a_1(y_2 - y_2^0) - a_3(x_1 - x_1^0) + a_2(x_2 - x_2^0)\right];
\]

\[
W'(z_2) = \frac{1}{2} \left[a_1(y_1 - y_1^0) + a_1(y_2 - y_2^0) - a_2(x_1 - x_1^0) + a_3(x_2 - x_2^0)\right];
\]

Denote \( x_1 - x_1^0 = \tilde{x}_1, x_2 - x_2^0 = \tilde{x}_2 \),... the deviations from the stationary points. Then we obtain the following systems
a) for the heat-flow equation

\[
\begin{aligned}
\frac{d\tilde{x}_1}{dt} &= a_3\tilde{x}_1 - a_2\tilde{x}_2, \\
\frac{d\tilde{x}_2}{dt} &= -a_2\tilde{x}_1 + a_3\tilde{x}_2, \\
\frac{d\tilde{y}_1}{dt} &= a_1\tilde{y}_1 + a_1\tilde{y}_2, \\
\frac{d\tilde{y}_2}{dt} &= a_1\tilde{y}_1 + a_1\tilde{y}_2,
\end{aligned}
\]

(6.10)

b) for the Schrödinger equation

\[
\begin{aligned}
\frac{d\tilde{x}_1}{dt} &= -a_1\tilde{y}_1 - a_1\tilde{y}_2, \\
\frac{d\tilde{x}_2}{dt} &= a_1\tilde{y}_1 + a_1\tilde{y}_2, \\
\frac{d\tilde{y}_1}{dt} &= a_3\tilde{x}_1 - a_2\tilde{x}_2, \\
\frac{d\tilde{y}_2}{dt} &= a_2\tilde{x}_1 - a_3\tilde{x}_2.
\end{aligned}
\]

(6.11)

Notice that \( e_3 < e_2 \leq 0 < e_1 \) (3), hence, \( a_1 > 0, a_2 > 0 \) and the sign of \( a_3 \) depends on the parameters of the ring. We obtain the systems of ODEs with the constant coefficients.

Let the initial positions of the particles be at the points \((\delta_i, \varepsilon_i) \ (i = 1, 2)\), where \( \delta_i, \varepsilon_i \) are small parameters. Then we can obtain an analytic form for the law of small motions near the stationary points.

6.3 Heat-flow equation

The system (6.10) splits in two independent systems for \( x_1, x_2 \) and \( y_1, y_2 \) and we have

\[
\begin{aligned}
x_1(t) &= C_1e^{(a_3-a_2)t} + C_2e^{(a_3+a_2)t}, \\
x_2(t) &= C_1e^{(a_3-a_2)t} - C_2e^{(a_3+a_2)t}
\end{aligned}
\]

with

\[
\begin{aligned}
C_1 &= \frac{\delta_1 + \delta_2}{2}, \\
C_2 &= \frac{\delta_1 - \delta_2}{2}.
\end{aligned}
\]

Since

\[
a_3 + a_2 = \frac{4}{R_1R_2}(e_2 - e_3) > 0,
\]

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\[ a_3 - a_2 = \frac{2}{R_1 R_2} \left( 2e_2 - \frac{4\eta}{\pi} \right) < 0, \]

we have \( e^{(a_3+a_2)t} \to \infty \) and \( e^{(a_3-a_2)t} \to 0 \) as \( t \to \infty \). If the initial positions of the particles are symmetric with respect to the stationary points, \( \delta_1 = -\delta_2 = \delta \), we obtain \( x_1(t) = -x_2(t) = \delta e^{(a_3+a_2)t} \to \infty \), i.e. particles move away from the stationary points. For antisymmetric initial positions \( \delta_1 = \delta_2 = \delta \), and \( x_1(t) = x_2(t) = \delta e^{(a_3-a_2)t} \to 0 \). This means that the particles described by the linearized equation are attracted to the stationary points.

From the system for \( y_i \) we have:

\[
\begin{align*}
  y_1(t) &= \varepsilon_2 e^{2a_1 t}, \\
  y_2(t) &= \varepsilon_2 e^{2a_1 t} - (\varepsilon_2 - \varepsilon_1).
\end{align*}
\]

Both the solutions are divergent. If at the initial moment the particles are at the \( x \)-axis (\( \varepsilon_1 = \varepsilon_2 = 0 \)) they move along this axis.

More detail numerical investigation of the motion of the pairs “vortex-antivortex” (not only for \( N = 2 \) but for any even \( N \)) shows that all the particles move from the stationary points to the nearest boundary or to the nearest neighbours and annihilate. For example, initial positions are far enough (the vortices are distributed uniformly on the same circumference), then they “don’t fill” each other, move to the boundary and annihilate with their reflections (see Fig. 6a).

If they are close enough, they attract and annihilate with the nearest neighbour (see Fig. 6b). Such a behaviour is similar to the motion of the particles in the circle and corresponds to the usual physical picture: the charges with the different signs attract.

![Alternating signs, N=6](image1)  ![Alternating signs, non-symmetric initial data](image2)

**Fig. 6a.** Even number of the alternative-sign vortices

**Fig. 6b.** Particles are close enough.

If we have all the particles with the equal signs they always move to the \textit{outer} boundary and annihilate. The picture of motion is the same as for circle (see Fig. 5b).
6.4 Schrödinger equation

In the case of the Schrödinger equation the system does not split and we have more complicated motion. We see that $\tilde{x}_2 = \tilde{x}_1 + C$ and the second part of the system takes a form

$$\frac{\partial \tilde{y}_1}{\partial t} = a_3 \tilde{x}_1 - a_2 (C - \tilde{x}_1),$$

$$\frac{\partial \tilde{y}_2}{\partial t} = a_2 \tilde{x}_1 - a_3 (C - \tilde{x}_1).$$

Eliminating $\tilde{y}_1, \tilde{y}_2$ from the equation for $\tilde{x}_1$ we obtain

$$\frac{d^2 \tilde{x}_1}{dt^2} = -2a_1 (a_2 + a_3) \tilde{x}_1 - Ca_1 (a_3 + a_2).$$

In the standard way we find the solution of the system satisfying the initial data $\tilde{x}_1(0) = \delta_1$, $\tilde{x}_2(0) = \delta_2$, $\tilde{y}_1(0) = \varepsilon_1$, and $\tilde{y}_2(0) = \varepsilon_2$:

$$\begin{cases}
\tilde{x}_1(t) = \frac{\delta_1 - \delta_2}{2} \cos kt - \frac{\delta_1 + \delta_2}{2} \sin kt + \frac{\delta_1 - \delta_2}{2} t, \\
\tilde{x}_2(t) = -\frac{\delta_1 - \delta_2}{2} \cos kt - \frac{\delta_1 + \delta_2}{2} \sin kt + \frac{\delta_1 - \delta_2}{2} t.
\end{cases}$$

Here $k$ equals $k = \sqrt{2a_1 (a_2 + a_3)}$ and $d_2$ is a constant that will be determined below.

The system for $y_i$ with initialy conditions $y_1(0) = \varepsilon_1$ and $y_2(0) = \varepsilon_2$ yields

$$\begin{cases}
\tilde{y}_1(t) = \frac{(\delta_1 - \delta_2)(a_2 + a_3)}{2k} \sin kt + \frac{\varepsilon_1 + \varepsilon_2}{2} \cos kt + \frac{(\delta_1 + \delta_2)(a_3 - a_2)}{2} t + \frac{\varepsilon_1 - \varepsilon_2}{2}, \\
\tilde{y}_2(t) = \frac{(\delta_1 - \delta_2)(a_2 + a_3)}{2k} \sin kt + \frac{\varepsilon_1 + \varepsilon_2}{2} \cos kt + \frac{(\delta_1 + \delta_2)(a_2 - a_3)}{2} t - \frac{\varepsilon_1 - \varepsilon_2}{2}.
\end{cases} \tag{6.12}$$

We obtain the constant $d_2$ from the equation for $\tilde{x}_2$: $\dot{x}_2(0) = a_1 (\varepsilon_1 + \varepsilon_2)$. Namely,

$$d_2 = -\frac{a_1 (\varepsilon_1 + \varepsilon_2)}{k}.$$ 

We have periodic solutions for all the variables if $\delta_1 = -\delta_2 = \delta$ and $\varepsilon_1 = \varepsilon_2 = 0$, i.e. if the initial positions of the vortices are symmetric with respect to y-axis. Thus

$$\begin{cases}
\tilde{x}_1 = \delta \cos kt, \quad & \tilde{y}_1 = \frac{\delta(a_2 + a_3)}{k} \sin kt, \\
\tilde{x}_2 = -\delta \cos kt, \quad & \tilde{y}_2 = \frac{\delta(a_2 + a_3)}{k} \sin kt.
\end{cases}$$

The vortices describe elliptic trajectories.

In general any number of the pairs “vortex-antivortex” with symmetric initial positions moves along the closed trajectories (see Fig.7a for $N = 2$ and Fig.7b for $N = 6$). If the initial positions are close to the stationary points, the trajectories are ”narrower” and ”shorter”.

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In general any number of the pairs “vortex-antivortex” with symmetric initial positions moves along the closed trajectories (see Fig.7a for $N = 2$ and Fig.7b for $N = 6$). If the initial positions are close to the stationary points, the trajectories are ”narrower” and ”shorter”.
Non-symmetric initial positions lead to a very complicated motion due to the terms which are linear in $t$ (see (6.12)). The velocities of the vortices are different and the picture of the motion for the linear problem strongly depends on the initial data.

Thus, if we have the particles with the different signs with arbitrary initial positions we obtain chaotic non-periodic motion (see Fig.8a for $N = 5$). Even the initial positions are symmetric but the number of particles is odd, the motion is chaotic (Fig.8b).

Motion "vortex-vortex". The stationary points in this case is not so convenient to calculate the coefficients of the linear system. But it is obvious that if the initial positions are symmetric the vortices move by the same way. It means that the particles move independently in the same direction on the same circumference as it was for the circle (see. Fig.2b). If the initial positions are not symmetric, the vortices move globally in the same directions but along the complicated trajectories with the loops (see, for example, Fig. 9 for $N = 3$).
So, for Schrödinger equation we have very different pictures of motion depend on the initial positions of the vortices.

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