The retarded van der Waals potential - revisited

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Abstract

The retarded van-der-Waals potential, as first obtained by Casimir and Polder, is usually computed on the basis of nonrelativistic QED. The hamiltonian describes two infinitely heavy nuclei, charge $e$, separated by a distance $R$ and two spinless electrons, charge $-e$, nonrelativistically coupled to the quantized radiation field. Casimir and Polder use the dipole approximation and small coupling to the Maxwell field. We employ here the full hamiltonian and determine the asymptotic strength of the leading $-R^{-7}$ potential, which is valid for all $e$. Our computation is based on a path integral representation and expands in $1/R$, rather than in $e$.

1 Introduction

Neutral atoms and molecules interact through the long range, attractive van der Waals potential which has a decay as $-R^{-6}$ for large separation $R$. The quantum origin of this force was first recognized by London [1]. Lieb and Thirring [2] supply a non-perturbative proof valid for very general charge configurations. If one goes beyond the static Coulomb interaction in quantizing the Maxwell field, then the action is no longer instantaneous but travels with the speed of light between atoms. In a now very famous paper [3] Casimir and Polder establish that thereby the effective interaction potential decays somewhat faster, namely as $-R^{-7}$, which is known as the retarded van der Waals potential. For two hydrogen atoms, the cross over between $R^{-6}$ and $R^{-7}$ sets in at roughly 100 Bohr radii. There are both direct and indirect measurements [4] which confirm the theoretical prediction.

The starting point of Casimir and Polder is nonrelativistic QED for two atoms separated by a distance $R$. Within dipole approximation they expand to fourth order in the coupling to the Maxwell field and obtain a prefactor of $-R^{-7}$ which is proportional to the square of the electric dipole moment of a single atom. Later on alternative routes and simplified derivations were proposed. For an extensive discussion we refer to the book by Milonni [5], see also the monograph
by Margenau and Kestner \cite{6} and the lecture notes of Martin and Bünzli \cite{7}. Feinberg and Sucher \cite{8, 9} reconsider the issue by employing a dispersion-theoretic approach. Their prefactor turns out to be quadratic in the electric and magnetic dipole moment of a single atom. Somewhat later Boyer \cite{10} derived the same prefactor using quantum zero-point energy and semiclassical expressions for the level shifts due to the presence of the atoms. In our note we stick to nonrelativistic QED, no dipole approximation and no assumption on small coupling, and expand in $1/R$. We use the path integral formulation, in which the subtraction of the ground state energy at $R = \infty$ is particularly transparent. As in previous studies the strength of the retarded van der Waals potential is quadratic in the electric and magnetic dipole moments, but with modified coefficients as compared to \cite{9, 10}.

In mathematical physics there has been a revived interest in nonrelativistic QED \cite{11}. It is conceivable that some parts of the argument can be elevated to a rigorous proof. In our paper we mostly ignore this line of research, but will provide a more detailed discussion in the conclusions.

2 Hamiltonian and van der Waals potential

We consider a single hydrogen atom with an infinitely heavy nucleus located at the origin. The nucleus has charge $e$, $e > 0$, the electron has charge $-e$. We will use units in which $\hbar = 1$, $c = 1$, and the bare mass of the electron $m = 1$. In Section 3 we will restore the proper physical units. Let $x, p$ be position and momentum of the electron. Then the nonrelativistic QED hamiltonian for this system reads

$$H = \frac{1}{2} \left( p - e A(x) \right)^2 - e^2 V(x) + H_f.$$  \hspace{1cm} (2.1)

For $H$ to make sense the electron is assumed to have a prescribed charge distribution $\varphi$ with the following properties: $\varphi$ is normalized, $\int dx \varphi(x) = 1$, rotation invariant, $\varphi(x) = \varphi_{\text{rad}}(|x|)$, of rapid decrease, and its Fourier transform, $\hat{\varphi}$, is real. Then $V$ is the smeared Coulomb potential

$$V(x) = \int \! dk \, |\hat{\varphi}(k)|^2 |k|^{-2} e^{-ik \cdot x}.$$  \hspace{1cm} (2.2)

$A(x)$ is the quantized vector potential and $H_f$ is the field energy. These are defined through a two-component Bose field $a(k, \lambda), k \in \mathbb{R}^3, \lambda = 1, 2$, with commutation relation

$$[a(k, \lambda), a(k', \lambda')^*] = \delta_{\lambda \lambda'} \delta(k - k').$$  \hspace{1cm} (2.3)

Explicitly

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \! dk \, \omega(k) a(k, \lambda)^* a(k, \lambda)$$  \hspace{1cm} (2.4)
with dispersion relation

$$\omega(k) = |k|$$  \hspace{1cm} (2.5)

and

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \frac{\hat{\phi}(k)}{\sqrt{2\omega(k)}} \varepsilon(k, \lambda) \left( e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^* \right)$$  \hspace{1cm} (2.6)

with the standard dreibein $\varepsilon(k, 1), \varepsilon(k, 2), \hat{k} = k/|k|$. Thus the Hilbert space for $H$ is

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F},$$  \hspace{1cm} (2.7)

where $\mathfrak{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. From the quantization of the classical system of charges coupled to the Maxwell field it follows that for the smearing of $A(x)$ and of $V$ the same charge distribution has to be used. We refer to [11] for details. As proved by Griesemer, Lieb, and Loss [12, 13, 14], $H$ has a unique ground state, denoted here by $\psi$, with ground state energy $E_H$, $H\psi = E\psi$.

The asymptotic strength $\kappa$ of the van der Waals potential depends on the properties of a single hydrogen atom only through its electric and magnetic dipole moment, $\alpha_E$ and $\alpha_M$. They are defined through the energy, $W$, of our system for weak external uniform electric and magnetic fields according to

$$W = -\frac{1}{2} \alpha_E E_{ex}^2 - \frac{1}{2} \alpha_M B_{ex}^2.$$  \hspace{1cm} (2.8)

To say, $H$ is perturbed by $eE_{ex} \cdot x$ and the vector potential is perturbed by $\frac{1}{2} B_{ex} \wedge x$. Then by second order perturbation theory it follows that

$$\alpha_E = 2 \left( \frac{1}{4} \langle \psi, x \cdot (H - E)^{-1} x \psi \rangle \right) e^2,$$

$$\alpha_M = -\frac{1}{4} \left( \frac{1}{2} \langle \psi, x^2 \psi \rangle \right) e^2.$$  \hspace{1cm} (2.9)

As a convention, $\langle \cdot, \cdot \rangle$ denotes always the inner product on the respective Hilbert space.

To investigate the van der Waals potential we consider two hydrogen atoms, one located at 0 and the other at $r = (0, 0, R), R \geq 0$. It will be convenient to define the position of the second electron relative to $r$. Then $x_1, x_2 + r$ are positions and $p_1, p_2$ the momenta of the two electrons. The two-electron hamiltonian reads

$$H_R = \frac{1}{2} \left( p_1 - eA(x_1) \right)^2 - e^2 V(x_1) + \frac{1}{2} \left( p_2 - eA(x_2 + r) \right)^2 - e^2 V(x_2)$$

$$+ H_I + e^2 V_R(x_1, x_2)$$  \hspace{1cm} (2.10)

with the interaction potential

$$V_R(x_1, x_2) = -V(x_1 - r) - V(x_2 + r) + V(r) + V(r + x_2 - x_1)$$

$$= \int_{\mathbb{R}^3} dk |\hat{\phi}(k)|^2 e^{ik \cdot r} |k|^{-2} (1 - e^{-ik \cdot x_1})(1 - e^{ik \cdot x_2}).$$  \hspace{1cm} (2.11)
$H_R$ acts on the Hilbert space $L^2(\mathbb{R}^3_{x_1}) \otimes L^2(\mathbb{R}^3_{x_2}) \otimes \mathbb{F}$. $H_R$ has a unique ground state with energy $E(r)$.

In the Born-Oppenheimer approximation $E(r)$ is the effective potential between the two neutral hydrogen atoms in their ground state. Thus the issue at hand is to investigate $E(r)$ for large $R$. By rotation invariance $E(r)$ depends only on $|r| = R$ and we also write $E(r) = E(R)$. For $R \to \infty$ the two atoms become independent and one can show that $E(R)$ converges to $2E$. The Casimir-Polder result is that, for small $e$,

$$\lim_{R \to \infty} R^7(E(R) - 2E) = -\frac{23}{4\pi} \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{2}\alpha_{E,at}\right)^2.$$  \hspace{1cm} (2.12)

The factor $(1/2\pi)^2$ results from our use of the Lorentz-Heaviside units. We remark that Casimir and Polder omit in their definition of $\alpha_E$ the factor 2, which accounts for the extra $1/2$. $\alpha_{E,at}$ is the dipole moment of a decoupled hydrogen atom. It is defined through

$$\alpha_{E,at} = 2\left(\frac{1}{3}(\psi_{at}, x \cdot (H_{at} - E_{at})^{-1} x \psi_{at})\right)e^2$$  \hspace{1cm} (2.13)

with

$$H_{at} = \frac{1}{2}p^2 - e^2 V(x)$$  \hspace{1cm} (2.14)

and $\psi_{at}$ the ground state of the hydrogen atom, $H_{at}\psi_{at} = E_{at}\psi_{at}$.

In our set-up, the natural dimensionless coupling constant is the Sommerfeld fine-structure constant

$$\alpha = \frac{e^2}{4\pi \hbar c}.$$  \hspace{1cm} (2.15)

The energy unit is set by the ionization energy of the hydrogen atom, which is $\alpha^2mc^2$ and the length unit is the Bohr radius $r_B = \hbar/\alpha mc$. Anticipating a decay as $R^{-7}$, the dimensionless coupling strength, $\kappa$, is defined through

$$E(R) - 2E = -\kappa(\alpha, \lambda \Lambda)\alpha mc^2(R/r_B)^{-7}$$  \hspace{1cm} (2.16)

valid for large $R$. $\kappa$ depends on $\alpha$ and on the ultraviolet cutoff $\Lambda$ in units of the inverse Compton wave length $\lambda_c = \hbar/mc$. For this interpretation the form factor $\hat{\varphi}$ is chosen as $\hat{\varphi}(k) = \hat{\varphi}_1(\Lambda^{-1}k)$, where $\hat{\varphi}_1$ rapidly interpolates between $\hat{\varphi}_1 = (2\pi)^{-3/2}$ for $|k| \leq 1 - \delta$ and $\hat{\varphi}_1 = 0$ for $|k| \geq 1$.

The goal of our note is to obtain an exact expression for the strength $\kappa$. Readers not so much interested in the details of the computation may skip ahead to Section 5 where the result is discussed.

### 3 Path integration

As noted by Feynman [15], in the functional integral representation of $e^{-tH_R}, t \geq 0$, the interaction with the radiation field is linear in $A$. Therefore one can carry
out the Gaussian integration over the fluctuating photon field. This form will be particularly convenient for the Born-Oppenheimer energy \( E(R) - 2E \). After such a detour we will return to operators. Our notation is formal, but rigorous versions are available [16, 17].

We denote by \( q_j(t) \in \mathbb{R}^{3} \) the path of electron \( j \). In case of a single electron we omit the index \( j \). For the ground state energy of the two-electron system one obtains

\[
E(r) = -\lim_{T \to \infty} \frac{1}{2T} \log \int \left[ \Pi dq_1(\cdot) \right] \int \left[ \Pi dq_2(\cdot) \right] \exp \left[ -\int_{-T}^{T} dt \left( \sum_{j=1,2} \left( \frac{1}{2} \dot{q}_j(t)^2 - e^2 V(q_j(t)) \right) + e^2 V_R(q_1(t), q_2(t)) \right) \right] - e^2 \int_{-T}^{T} ds \int_{-T}^{T} dt \dot{q}_1(s) \cdot W_0(q_1(s) - q_2(t), s - t) \dot{q}_2(t) \right] \].

Here \( \int [\Pi dq_j(\cdot)] \) is the “sum over all paths” and \( W_R \) is the photon propagator,

\[
W_R(q, t) = \int_{\mathbb{R}^3} dk |\hat{\psi}(k)|^2 \frac{1}{2\omega(k)} (\mathbb{1} - |\hat{k}\rangle \langle \hat{k}|) e^{-\omega(k)|t|} e^{ik \cdot r} e^{-ik \cdot q}
\]
as a \( 3 \times 3 \) matrix. Here \( \mathbb{1} \) is the unit matrix and \( |\hat{k}\rangle \langle \hat{k}| \) the orthogonal projection onto \( \hat{k}, \hat{k} = k/|k| \).

Correspondingly for a single electron

\[
E = -\lim_{T \to \infty} \frac{1}{2T} \log \int \left[ \Pi dq(\cdot) \right] \exp \left[ -\int_{-T}^{T} dt \left( \frac{1}{2} \dot{q}(t)^2 - e^2 V(q(t)) \right) \right] - \frac{1}{2} e^2 \int_{-T}^{T} ds \int_{-T}^{T} dt \dot{q}(s) \cdot W_0(q(s) - q(t), s - t) \dot{q}(t) \right].
\]

Therefore

\[
E(R) - 2E = -\lim_{T \to \infty} \frac{1}{2T} \log \mathbb{E} \times \mathbb{E} \left[ \exp \left[ -e^2 \int_{-T}^{T} dt V_R(q_1(t), q_2(t)) \right] - e^2 \int_{-T}^{T} ds \int_{-T}^{T} dt \dot{q}_1(s) \cdot W_R(q_1(s) - q_2(t), s - t) \dot{q}_2(t) \right] \].
\]

Here \( q_1(t) \) and \( q_2(t) \) are two independent copies of the ground state process \( q(t) \) with a path measure as written in (3.3). \( q(t) \) is stationary and the distribution of \( q(t) \) at a fixed time is the electronic density computed from the ground state \( \psi \) of \( H \). The average with respect to the ground state process is denoted by \( \mathbb{E}[\cdot] \) and the average over the two independent copies by \( \mathbb{E} \times \mathbb{E}[\cdot] \).
We note that in the expression (3.4) the only $ R $ dependence sits in $ V_R $ and $ W_R $, which in a certain sense are small. Thus it is natural to use the cumulant expansion. Denoting the exponent by $ X_T(R) $, one arrives at

$$ -\frac{1}{2T} \log \mathbb{E} \times \mathbb{E}[e^{-e^{2}X_T(R)}] = -\frac{1}{2T} \left( e^{2}C_{1}(R, T) + \frac{1}{2} e^{4}C_{2}(R, T) + \cdots \right), $$

$$ C_{1}(R, T) = -\mathbb{E} \times \mathbb{E}[X_T(R)], $$

$$ C_{2}(R, T) = \mathbb{E} \times \mathbb{E}[X_T(R)^{2}] - \mathbb{E} \times \mathbb{E}[X_T(R)]^{2}. \quad (3.5) $$

As we will see below $ \mathbb{E} \times \mathbb{E}[X_T(R)^{2}] / 2T = \mathcal{O}(R^{-6}) $, not using the cancellation between $ V_R $ and $ W_R $ terms. Since we are heading for a decay as $ R^{-7} $ for large $ R $, it should suffice to stop the expansion at the second cumulant $ C_{2}(R, T) $. This will be our main assumption. $ \mathbb{E} \times \mathbb{E}[X_T(R)] / 2T $ is exponentially small. The large $ R $ behavior of the second cumulant will be investigated in detail in the following section.

We note that the expectation under the $ k $-integral always factorizes with respect to $ q_{1}, q_{2} $. Using their independence we will need to compute only a few expectations for the ground state process of a single electron. They are listed now for later convenience and proved in Appendix $ \Lambda $. We set $ \varepsilon = \varepsilon(k, \lambda), \varepsilon_{j} = \varepsilon(k_{j}, \lambda_{j}) $ for either $ \lambda_{j} = 1 $ or $ \lambda_{j} = 2 $ and $ \theta(t) $ the step function, $ \theta(t) = -1 $ for $ t \leq 0, \theta(t) = 1 $ for $ t > 0 $.

(i) $ \mathbb{E}[e^{ik_{1}q(t)}] = \langle \psi, e^{ik_{1}x}\psi \rangle $,

(ii) $ \mathbb{E}[e^{ik_{1}q(t)}(\varepsilon(k, \lambda) \cdot \dot{q}(t))] = -\langle \psi, e^{ik_{1}x}(H - E)(\varepsilon(k, \lambda) \cdot x)\psi \rangle = 0 $,

(iii) $ \mathbb{E}[e^{-ik_{1}q(s)}e^{ik_{2}q(t)}] = \langle \psi, e^{-ik_{1}x}e^{-|s-t|(H-E)}e^{ik_{2}x}\psi \rangle $,

(iv) $ \mathbb{E}[(\varepsilon_{1} \cdot \dot{q}(s))e^{-ik_{1}q(s)}(1 - e^{-ik_{2}q(t)})] $

$$ = \theta(t - s)\langle \psi, (\varepsilon_{1} \cdot x)(H - E)e^{-ik_{1}x}e^{-|s-t|(H-E)}(1 - e^{-ik_{2}x})\psi \rangle, $$

(v) $ \mathbb{E}[(\varepsilon_{1} \cdot \dot{q}(s))e^{-ik_{1}q(s)}e^{-ik_{2}q(t)}(\varepsilon_{2} \cdot \dot{q}(t))] $

$$ = -\langle \psi, (\varepsilon_{1} \cdot x)(H - E)e^{-ik_{1}x}e^{-|s-t|(H-E)}e^{-ik_{2}x}(H - E)(\varepsilon_{2} \cdot x)\psi \rangle $$

$$ + \delta(t - s)(\varepsilon_{1} \cdot \varepsilon_{2})\langle \psi, e^{-ik_{1}x}e^{-ik_{2}x}\psi \rangle. $$

In (v) the second term arises because locally $ q(t) $ is like a standard Brownian motion for which $ dq(s) \otimes dq(t) = \Pi d\delta(s - t)dsdt $.

The first cumulant can be dealt with immediately. Using (ii) in the above list one obtains

$$ \mathbb{E} \times \mathbb{E}[\dot{q}_{1}(s) \cdot W_{R}(q_{1}(s) - q_{2}(t), s - t)\dot{q}_{2}(t)] = 0. \quad (3.6) $$
For the potential term it holds
\[
\mathbb{E} \times \mathbb{E}[V_R(q_1(t), q_2(t))]
= \int_{\mathbb{R}^3} dk \, |\hat{\varphi}(k)|^2 \omega(k)^{-2} e^{ik \cdot r} \langle \psi, (1 - e^{-ik \cdot x}) \psi \rangle \langle \psi, (1 - e^{ik \cdot x}) \psi \rangle
= \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \left( (\varrho - \delta) * \varphi \right)(x) (4\pi |x + r - y|)^{-1} \left( (\varrho - \delta) * \varphi \right)(y).
\] (3.7)

Here \( \varrho(x) \) is the electron density for \( \psi \), which is known to have an exponential decay \[13\]. \( \delta \) is the Dirac delta and * denotes convolution. Since \( \int_{\mathbb{R}^3} dx \, (\varrho(x) - \delta(x)) = 0 \) and since \( \varrho \) decays rapidly, by Newton’s theorem it holds that there are suitable constants \( c_1, c_2 \) such that
\[
|\mathbb{E} \times \mathbb{E}[V_R(q_1(t), q_2(t))]| \leq c_1 e^{-c_2 R}.
\] (3.8)

4 The second cumulant

The second cumulant consists of three terms. They are all proportional to \( 2T \) by the stationarity of \( q_1(t), q_2(t) \). We could take the limit \( T \to \infty \) first, but the symmetric version is more convenient. All expectations are written in terms of \( H \) and its ground state \( \psi \). For notational simplicity we replace \( H - E \) by \( H \), hence \( H \psi = 0 \). For inverses as \( \langle \phi_1, H^{-n} \phi_2 \rangle \) we make sure that either \( \langle \phi_1, \psi \rangle = 0 \) or \( \langle \phi_2, \psi \rangle = 0 \). Note that the ground state is nondegenerate \[16\]. But \( \langle \phi_1, H^{-n} \phi_2 \rangle \) could still be infinite. If \( H \) is replaced by \( H_{at} \), then \( H_{at} \) has a spectral gap and therefore an inverse on the orthogonal complement of \( \psi_{at} \).

We set
\[
C_2(R, T) = I_{VV} + 2I_{VW} + I_{WW}
\] (4.1)

and compute each term separately. The second step is a partial time-integration through which one can understand how the \( R^{-6} \) decay from the interaction potential is canceled. In a final step we collect terms according to their number of time-integrations and discuss their \( R \)-dependence.
4.1 Expectations

a) $I_{VV}$. Setting $\hat{\varphi}(k_j) = \hat{\varphi}_j, \omega(k_j) = \omega_j$, we have

$$I_{VV} = \int dt_1 dt_2 \int dk_1 dk_2 |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 (|k_1|^2 |k_2|^2)^{-1} e^{i(k_1 + k_2) \cdot r} \times \left\{ |\mathbb{E}[(1 - e^{-ik_1 \cdot \varphi_1(t_1)})(1 - e^{-ik_2 \cdot \varphi_1(t_2)}])|^2 \right. $$

$$- \left. |\mathbb{E}[(1 - e^{-ik_1 \cdot \varphi_1(t_1)}]|\mathbb{E}[(1 - e^{-ik_2 \cdot \varphi_1(t_2)}])|^2 \right\} = \int dt_1 dt_2 \int dk_1 dk_2 |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 (|k_1|^2 |k_2|^2)^{-1} e^{i(k_1 + k_2) \cdot r} \times \left\{ |\langle \psi, (1 - e^{-ik_1 \cdot x})e^{-|t_1 - t_2|H} (1 - e^{-ik_2 \cdot x}) \psi \rangle|^2 \right. $$

$$- \left. |\langle \psi, (1 - e^{-ik_1 \cdot x})\psi \rangle \langle \psi, (1 - e^{-ik_2 \cdot x}) \psi \rangle|^2 \right\}. \quad (4.2)$$

Note that the integrand in (4.2) decays to zero, since

$$e^{-|t_1 - t_2|H} \rightarrow |\psi \rangle \langle \psi| \quad (4.3)$$

as $|t_1 - t_2| \rightarrow \infty$.

b) $I_{VV}$. Setting $\varepsilon_j = \varepsilon(k_j, \lambda_j)$ and noting

$$\sum_{\lambda=1,2} |\varepsilon(k, \lambda)\rangle \langle \varepsilon(k, \lambda)| = \mathbb{1} - |\hat{k} \rangle \langle \hat{k}|, \quad (4.4)$$

one has

$$I_{VV} = \int dt_1 dt_2 dt_3 \sum_{\lambda_1=1,2} \int dk_1 dk_2 |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 (2\omega_1 |k_2|^2)^{-1} e^{i(k_1 + k_2) \cdot r} e^{-\omega_1 |t_1 - t_2|} \times $$

$$\times \mathbb{E}[(\varepsilon_1 \cdot \hat{q}_1(t_1))e^{-ik_1 \cdot \varphi_1(t_1)}(1 - e^{-ik_2 \cdot \varphi_1(t_2)})] \times \mathbb{E}[(\varepsilon_1 \cdot \hat{q}_2(t_2))e^{ik_1 \cdot \varphi_2(t_2)}(1 - e^{ik_2 \cdot \varphi_2(t_3)})]$$

$$= \int dt_1 dt_2 dt_3 \sum_{\lambda_1=1,2} \int dk_1 dk_2 |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 (2\omega_1 |k_2|^2)^{-1} e^{i(k_1 + k_2) \cdot r} e^{-\omega_1 |t_1 - t_2|} \times $$

$$\times \theta(t_1 - t_3) \langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x}e^{-|t_1 - t_3|H}(1 - e^{-ik_2 \cdot x}) \psi \rangle \times \theta(t_2 - t_3) \langle \psi, (\varepsilon_1 \cdot x)He^{ik_1 \cdot x}e^{-|t_2 - t_3|H}(1 - e^{ik_2 \cdot x}) \psi \rangle. \quad (4.5)$$

As proved in Appendix B, it holds

$$\langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x} \psi \rangle = i \langle \psi, (\varepsilon_1 \cdot (p - eA(x)))e^{-ik_1 \cdot x} \psi \rangle = 0. \quad (4.6)$$

Thus no truncation of the expectation is needed.
c) $I_{WW}$. One has
\[
I_{WW} = \int dt_1 dt_2 dt_3 dt_4 \sum_{\lambda_1, \lambda_2} \int dk_1 dk_2 |\hat{\phi}_{1,2}|^2 (2\omega_1 2\omega_2)^{-1} e^{i(k_1+k_2)\cdot r} \\
\times e^{-\omega_1|t_1-t_2|} e^{-\omega_2|t_3-t_4|} \mathbb{E} \left[ (\varepsilon_1 \cdot \hat{\phi}_1(t_1)) e^{-ik_1 \cdot q_1(t_1)} (\varepsilon_2 \cdot \hat{\phi}_2(t_3)) e^{-ik_2 \cdot q_1(t_3)} \right] \\
\times \mathbb{E} \left[ (\varepsilon_1 \cdot \hat{\phi}_2(t_2)) e^{ik_2 \cdot q_2(t_2)} (\varepsilon_2 \cdot \hat{\phi}_1(t_4)) e^{ik_1 \cdot q_2(t_4)} \right] \\
= \int dt_1 dt_2 dt_3 dt_4 \sum_{\lambda_1, \lambda_2} \int dk_1 dk_2 |\hat{\phi}_{1,2}|^2 (2\omega_1 2\omega_2)^{-1} e^{i(k_1+k_2)\cdot r} \\
\times e^{-\omega_1|t_1-t_2|} e^{-\omega_2|t_3-t_4|} \\
\times \left( - \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} e^{-\varepsilon_1} H e^{-ik_2 \cdot x} H (\varepsilon_2 \cdot x) \psi \rangle \\
+ \delta(t_1-t_3) (\varepsilon_1 \cdot \varepsilon_2) \langle \psi, e^{-i(k_1+k_2) \cdot x} \psi \rangle \\
+ \delta(t_2-t_4) (\varepsilon_1 \cdot \varepsilon_2) \langle \psi, e^{i(k_1+k_2) \cdot x} \psi \rangle \right) \tag{4.7}
\]

### 4.2 Partial time integration

The next step is a partial time integration for $I_{VW}$ and $I_{WW}$. For the integrand of $I_{VW}$ we use the identity
\[
\theta(s-t) \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} e^{-\varepsilon_1} H e^{-ik_2 \cdot x} H (\varepsilon_2 \cdot x) \psi \rangle \\
= -\frac{\partial}{\partial s} \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} H^{-1} e^{-\varepsilon_1} H e^{-ik_2 \cdot x} \psi \rangle \\n\tag{4.8}
\]

and for the integrand of $I_{WW}$ the identity
\[
- \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} e^{-\varepsilon_1} H e^{-ik_2 \cdot x} H (\varepsilon_2 \cdot x) \psi \rangle \\
= \frac{\partial}{\partial s} \frac{\partial}{\partial t} \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} H^{-2} e^{-\varepsilon_1} H e^{-ik_2 \cdot x} H (\varepsilon_2 \cdot x) \psi \rangle \\
- 2\delta(s-t) \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} H^{-1} e^{-ik_2 \cdot x} H (\varepsilon_2 \cdot x) \psi \rangle. \tag{4.9}
\]

We insert these expectations in $I_{VW}, I_{WW}$, integrate partially in time, and use
\[
\frac{\partial}{\partial s} \frac{\partial}{\partial t} e^{-\omega|s-t|} = -\omega^2 e^{-\omega|s-t|} + 2\omega \delta(s-t). \tag{4.10}
\]

The boundary terms vanish. For $I_{VW}$ one obtains
\[
I_{VW} = \int dt_1 dt_2 dt_3 \sum_{\lambda_1} \int dk_1 dk_2 |\hat{\phi}_{1,2}|^2 (2\omega_1 |k_2|^2)^{-1} e^{i(k_1+k_2)\cdot r} \\
\times \left( - \omega_1^2 e^{-\omega_1|t_1-t_2|} + 2\omega_1 \delta(t_1-t_2) \right) \\
\times \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} H^{-1} e^{-\varepsilon_1} H (1 - e^{-ik_2 \cdot x}) \psi \rangle \\
\times \langle \psi, (\varepsilon_1 \cdot x) H e^{ik_1 \cdot x} H^{-1} e^{-\varepsilon_1} H (1 - e^{ik_2 \cdot x}) \psi \rangle. \tag{4.11}
\]
Finally $I_{WW}$ is given by

$$I_{WW} = \int dt_1 dt_2 dt_3 dt_4 \sum_{\lambda_1, \lambda_2} \int dk_1 dk_2 [\hat{\varphi}_1]^2 |\hat{\varphi}_2|^2 (2\omega_1 2\omega_2)^{-1} e^{i(k_1+k_2)\cdot r} \times \left\{ \left( -\omega_1^2 e^{-\omega_1|t_1-t_2|} + 2\omega_1 \delta(t_1 - t_2) \right) \left( -\omega_2^2 e^{-\omega_2|t_3-t_4|} + 2\omega_2 \delta(t_3 - t_4) \right) \right. $$

$$\times \langle \psi, (\varepsilon_1 \cdot x) He^{-ik_1\cdot x} H^{-2} e^{-|t_1-t_3|H} e^{-ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle \times \langle \psi, (\varepsilon_1 \cdot x) He^{ik_1\cdot x} H^{-2} e^{-|t_2-t_4|H} e^{ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle$$

$$+ \theta(t_1 - t_2) \theta(t_3 - t_4) \omega_2 e^{-\omega_1|t_1-t_2|} e^{-\omega_2|t_3-t_4|}$$

$$\times \left( \langle \psi, (\varepsilon_1 \cdot x) He^{-ik_1\cdot x} H^{-2} e^{-|t_1-t_3|H} e^{-ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle \times \delta(t_2 - t_4) B_+(k_1, \lambda_1; k_2, \lambda_2) + \langle \psi, (\varepsilon_1 \cdot x) He^{ik_1\cdot x} H^{-2} e^{-|t_2-t_4|H} e^{ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle \times \delta(t_1 - t_3) B_-(k_1, \lambda_1; k_2, \lambda_2) \right)$$

$$+ e^{-\omega_1|t_1-t_2|} e^{-\omega_2|t_3-t_4|} \delta(t_1 - t_3) \delta(t_2 - t_4)$$

$$\times B_-(k_1, \lambda_1; k_2, \lambda_2) B_+(k_1, \lambda_1; k_2, \lambda_2) \right\}, \quad (4.12)$$

where

$$B_-(k_1, \lambda_1; k_2, \lambda_2) = \langle \varepsilon_1 \cdot \varepsilon_2 \rangle \langle \psi, e^{-i(k_1+k_2)\cdot x} \psi \rangle - 2 \langle \psi, (\varepsilon_1 \cdot x) He^{-ik_1\cdot x} H^{-1} e^{-ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle, \quad (4.13)$$

$$B_+(k_1, \lambda_1; k_2, \lambda_2) = \langle \varepsilon_1 \cdot \varepsilon_2 \rangle \langle \psi, e^{i(k_1+k_2)\cdot x} \psi \rangle - 2 \langle \psi, (\varepsilon_1 \cdot x) He^{ik_1\cdot x} H^{-1} e^{ik_2\cdot x} H(\varepsilon_2 \cdot x) \psi \rangle \quad (4.14)$$

We use (4.11) and collect the terms of $C_2(R,T)$ according to the number of their time-integrations, divide by $1/2T$, and take the limit as $T \to \infty$. To prepare for the limit $R \to \infty$, we rescale the momentum integration as $k_j \sim k_j / R, j = 1, 2,$ and introduce the unit vector $\hat{n} = r / R = (0,0,1)$. Note that $\varepsilon(k/R, \lambda) = \varepsilon(k, \lambda)$. The two-time, three-time, and four-time integrations are treated separately.
4.3 Two-time integrations

The sum of all terms involving two-time integrations is denoted by $I_2(R)$. One has

$$I_2(R) = \int dk_1 dk_2 |\hat{\phi}(k_1/R)|^2 |\hat{\phi}(k_2/R)|^2 e^{i(k_1 + k_2) \cdot \hat{n}}$$

$$\times \left[ \int dt \left\{ R^{-2} |k_1|^2 |k_2|^2 - 1 \left( \left| \langle \psi, (1 - e^{-ik_1 \cdot x/R})e^{-|t|H}(1 - e^{-ik_2 \cdot x/R})\psi \rangle \right|^2 - \left| \langle \psi, (1 - e^{-ik_1 \cdot x/R})\psi \rangle \langle \psi, (1 - e^{-ik_2 \cdot x/R})\psi \rangle \right| \right) \right. + 2 \sum_{\lambda_1} R^{-4} |k_2|^{-2} \langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x/R}H^{-1}e^{-|t|H}(1 - e^{-ik_2 \cdot x/R})\psi \rangle$$

$$\times \langle \psi, (\varepsilon_1 \cdot x)He^{ik_1 \cdot x/R}H^{-1}e^{-|t|H}(1 - e^{ik_2 \cdot x/R})\psi \rangle + \sum_{\lambda_1, \lambda_2} R^{-6} \langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x/R}H^{-2}e^{-|t|H}e^{ik_2 \cdot x/R}H(e_2 \cdot x)\psi \rangle \left\} \right.$$

$$\times \left. \left( \sum_{\lambda_1, \lambda_2} R^{-3} (2\omega_1 \omega_2 (\omega_1 + \omega_2))^{-1} \right. \right.$$

$$\times \left. B_+(k_1/R, \lambda_1; k_2/R, \lambda_2)B_-(k_1/R, \lambda_1; k_2/R, \lambda_2) \right]$$

$$= I_{2,1}(R) + I_{2,2}(R). \quad (4.15)$$

We consider the sum, $I_{2,1}(R)$, of the first three terms and expand in $1/R$, which yields expectations of the form $\langle \psi, (a \cdot x)e^{-|t|H}(b \cdot x)\psi \rangle$ with $a, b \in \mathbb{R}^3$. By rotation invariance

$$\langle \psi, (a \cdot x)e^{-|t|H}(b \cdot x)\psi \rangle = (a \cdot b) \frac{1}{3} \langle \psi, x \cdot e^{-|t|H}x\psi \rangle. \quad (4.16)$$

Using

$$\sum_{\lambda_1, \lambda_2} (\varepsilon_1 \cdot \varepsilon_2)^2 = 1 + (\hat{k}_1 \cdot \hat{k}_2)^2, \quad \sum_{\lambda_1}(\varepsilon_1 \cdot \hat{k}_2)^2 = 1 - (\hat{k}_1 \cdot \hat{k}_2)^2, \quad (4.17)$$

one arrives at the lowest order

$$R^{-6} \left( \frac{1}{3} \langle \psi, x \cdot e^{-|t|H}x\psi \rangle \right)^2 \left( (\hat{k}_1 \cdot \hat{k}_2)^2 + 2 - 2(\hat{k}_1 \cdot \hat{k}_2)^2 + 1 + (\hat{k}_1 \cdot \hat{k}_2)^2 \right). \quad (4.18)$$

The nonsmooth contributions, containing $(\hat{k}_1 \cdot \hat{k}_2)^2$, are canceled exactly, while the smooth part, at this order, is given by

$$I_{2,1}(R) = \int dt \int dk_1 dk_2 |\hat{\phi}(k_1/R)|^2 |\hat{\phi}(k_2/R)|^2 e^{i(k_1 + k_2) \cdot \hat{n}} R^{-6} \left( \frac{1}{3} \langle \psi, x \cdot e^{-|t|H}x\psi \rangle \right)^2, \quad (4.19)$$
which inherits the rapid decay in $R$ from $\varphi$. At the next order one picks up the quadratic contributions $k_1^2, k_1 \cdot k_2, k_2^2$ with coefficients integrable in $t$. By power counting one arrives at $I_{2,1}(R) \approx R^{-8}$ as $R \to \infty$.

For the second summand, $I_{2,2}(R)$, we use that to leading order in $1/R$,

$$B_{\pm}(k_1/R, \lambda_1; k_2/R, \lambda_2) = R^{-2}|k_1||k_2|(\varepsilon_1 \cdot \hat{k}_2)(\varepsilon_2 \cdot \hat{k}_1)\left(\frac{1}{3}\langle \psi, x^2 \psi \rangle\right) + \mathcal{O}(R^{-4}),$$

(4.20)

as in shown in Appendix [B]. Using that $|\hat{\phi}(0)|^2 = (2\pi)^{-3}$ one arrives at

$$I_{2,2}(R) = R^{-7}\left(\frac{1}{2}e\psi, x^2 \psi\right)^2(2\pi)^{-6}\frac{1}{2}\int dk_1dk_2(\omega_1\omega_2(\omega_1 + \omega_2))^{-1}|k_1|^2|k_2|^2 \times e^{i(k_1+k_2)\cdot \hat{n}}(1 - (\hat{k}_1 \cdot \hat{k}_2)^2)^2 + \mathcal{O}(R^{-9})$$

$$= \frac{128}{\pi}\left(\frac{1}{2\pi}\right)^2e^{-4\alpha_M^2}R^{-7} + \mathcal{O}(R^{-9}).$$

(4.21)

The evaluation of the numerical coefficient is discussed in Appendix [C]. Altogether

$$\lim_{R \to \infty} R^7 I_2(R) = \frac{128}{\pi}\left(\frac{1}{2\pi}\right)^2e^{-4\alpha_M^2}.$$  

(4.22)

### 4.4 Three-time integrations

The sum of all terms involving three-time integrations is denoted by $I_3(R)$. One obtains

$$I_3(R) = \int dt_1dt_2 \int dk_1dk_2 |\hat{\phi}(k_1/R)|^2|\hat{\phi}(k_2/R)|^2e^{i(k_1+k_2)\cdot \hat{n}}$$

$$\times \left\{ - R^{-5}\omega_1|k_2|^{-2}e^{-\omega_1|t_1-t_2|/R} \right.$$

$$\times \left( \sum_{\lambda_1} \langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x/R}H^{-1}e^{-|t_1|H}(1 - e^{-ik_2 \cdot x/R})\psi \rangle \right)$$

$$\times \langle \psi, (\varepsilon_1 \cdot x)He^{ik_1 \cdot x/R}H^{-1}e^{-|t_2|H}(1 - e^{ik_2 \cdot x/R})\psi \rangle \right\}$$

$$- R^{-7}\left(\frac{1}{2}\omega_1e^{-\omega_1|t_1-t_2|/R} + \frac{1}{2}\omega_2e^{-\omega_2|t_1-t_2|/R} \right)$$

$$\times \left( \sum_{\lambda_1, \lambda_2} \langle \psi, (\varepsilon_1 \cdot x)He^{-ik_1 \cdot x/R}H^{-2}e^{-|t_1|H}e^{-ik_2 \cdot x/R}H(\varepsilon_2 \cdot x)\psi \rangle \right)$$

$$\times \langle \psi, (\varepsilon_1 \cdot x)He^{ik_1 \cdot x/R}H^{-2}e^{-|t_2|H}e^{ik_2 \cdot x/R}H(\varepsilon_2 \cdot x)\psi \rangle$$

(4.23)
\[ + R^{-6} \frac{1}{4} \theta(t_1)e^{-\omega_1 |t_1|/R} \theta(t_2)e^{-\omega_2 |t_2|/R} \times \left( \sum_{\lambda_1, \lambda_2} \left( \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x/R} H^{-2}e^{-|t_1 - t_2|/R} H e^{-ik_2 \cdot x/R} H(\varepsilon_2 \cdot x) \psi \rangle \times B_+(k_1, \lambda_1; k_2, \lambda_2) + \langle \psi, (\varepsilon_1 \cdot x) H e^{ik_1 \cdot x/R} H^{-2}e^{-|t_1 - t_2|/R} H e^{ik_2 \cdot x/R} H(\varepsilon_2 \cdot x) \psi \rangle \times B_-(k_1, \lambda_1; k_2, \lambda_2) \right) \right) \]

\[ = I_{3,1}(R) + I_{3,2}(R). \]  

We expand the integrand in 1/R which yields

\[
\int dt_1 dt_2 \int dk_1 dk_2 |\hat{\phi}(k_1/R)|^2 |\hat{\phi}(k_2/R)|^2 e^{i(k_1 + k_2) \cdot \hat{n}} \times \left\{ - R^{-7} \omega_1 e^{-\omega_1 |t_1 - t_2|/R} \left( 1 - (\hat{k}_1 \cdot \hat{k}_2)^2 \right) \right.
\]

\[ \times \left\{ \frac{1}{3} \langle \psi, x \cdot e^{-|t_1|H} x \psi \rangle \frac{1}{2} \langle \psi, x \cdot e^{-|t_2|H} x \psi \rangle - R^{-7} \left( \frac{1}{2} \omega_1 e^{-\omega_1 |t_1 - t_2|/R} + \frac{1}{2} \omega_2 e^{-\omega_2 |t_1 - t_2|/R} \right) \left( 1 + (\hat{k}_1 \cdot \hat{k}_2)^2 \right) \right.
\]

\[ \times \left\{ \frac{1}{3} \langle \psi, x \cdot e^{-|t_1|H} x \psi \rangle \frac{1}{2} \langle \psi, x \cdot e^{-|t_2|H} x \psi \rangle + R^{-6} \theta(t_1) e^{-\omega_1 |t_1|/R} \theta(t_2) e^{-\omega_2 |t_2|/R} \times \left( \sum_{\lambda_1, \lambda_2} \frac{1}{3} \langle \psi, x \cdot e^{-|t_1 - t_2|H} x \psi \rangle \times \langle \varepsilon_1 \cdot \varepsilon_2 \rangle (B_+(k_1/R, \lambda_1; k_2/R, \lambda_2) + B_-(k_1/R, \lambda_1; k_2/R, \lambda_2)) \right) \right\} \]

\[ = - R^{-7} \int dt_1 dt_2 \int dk_1 dk_2 |\hat{\phi}(k_1/R)|^2 |\hat{\phi}(k_2/R)|^2 e^{i(k_1 + k_2) \cdot \hat{n}} \omega_1 e^{-\omega_1 |t_1 - t_2|/R} \times \left\{ \frac{1}{3} \langle \psi, x \cdot e^{-|t_1|H} x \psi \rangle \frac{1}{2} \langle \psi, x \cdot e^{-|t_2|H} x \psi \rangle + R^{-6} \int dt_1 dt_2 \int dk_1 dk_2 |\hat{\phi}(k_1/R)|^2 |\hat{\phi}(k_2/R)|^2 \left( \sum_{\lambda_1, \lambda_2} e^{i(k_1 + k_2) \cdot \hat{n}} \frac{1}{4} \theta(t_1) e^{-\omega_1 |t_1|/R} \theta(t_2) e^{-\omega_2 |t_2|/R} \frac{1}{3} \langle \psi, x \cdot e^{-|t_1 - t_2|H} x \psi \rangle \times \langle \varepsilon_1 \cdot \varepsilon_2 \rangle (B_+(k_1/R, \lambda_1; k_2/R, \lambda_2) + B_-(k_1/R, \lambda_1; k_2/R, \lambda_2)) \right) \right\}. \]  

For the first summand the nonsmooth contributions are canceled exactly, as in Section 4.3, while the smooth contributions decay rapidly since \( \varphi \) does so. Its next order picks up an extra factor \( R^{-2} \). Thus \( I_{3,1}(R) \approx R^{-9} \) for large \( R \). For the
There is only a single term with four-time integrations, namely

\[ I_{3,2}(R) = R^{-6} \int dk_1dk_2|\hat{\varphi}(k_1,R)|^2|\hat{\varphi}(k_2,R)|^2 \left( e^{ik_1+k_2} \int R(\omega_1 + \omega_2)^{-1} \frac{1}{3} \langle \psi, x \cdot (H + R^{-1}\omega_1)^{-1}(H + R^{-1}\omega_2)^{-1}x\rangle \right) \times \sum_{\lambda_1,\lambda_2} (\varepsilon_1 \cdot \varepsilon_2) (B_+(k_1/R, \lambda_1; k_2/R, \lambda_2) + B_-(k_1/R, \lambda_1; k_2/R, \lambda_2)) \right). \]

Since \( B_{\pm} = R^{-2}|k_1||k_2|(\varepsilon_1 \cdot \hat{k}_2)(\varepsilon_2 \cdot \hat{k}_1)(\frac{1}{3}\langle \psi, x^2\psi \rangle) + \mathcal{O}(R^{-4}) \), as shown in Appendix B, we conclude that

\[ I_{3,2}(R) = - R^{-7} \left( \frac{1}{3}\langle \psi, x^2\psi \rangle \right) \left( \frac{1}{3}\langle \psi, x \cdot H^{-1}x\rangle \right) \times 2(2\pi)^{-6} \int dk_1dk_2 e^{i(k_1+k_2)\cdot \hat{n}}(\omega_1\omega_2(\omega_1 + \omega_2))^{-1} \times |k_1|^2|k_2|^2(\hat{k}_1 \cdot \hat{k}_2)(1 - (\hat{k}_1 \cdot \hat{k}_2)^2) + \mathcal{O}(R^{-9}) \]

\[ = \frac{52}{\pi} \left( \frac{1}{2\pi} \right)^2 e^{-4\alpha_E\alpha_M} R^{-7} + \mathcal{O}(R^{-9}). \]  

The evaluation of the numerical coefficient is discussed in Appendix C. Altogether one has

\[ \lim_{R \to \infty} R^7 I_3(R) = \frac{52}{\pi} \left( \frac{1}{2\pi} \right)^2 e^{-4\alpha_E\alpha_M}. \]  

### 4.5 Four-time integrations

There is only a single term with four-time integrations, namely

\[ I_4(R) = \int dt_1dt_2dt_3 \sum_{\lambda_1,\lambda_2} \int dk_1dk_2|\hat{\varphi}_1|^2|\hat{\varphi}_2|^2 e^{i(k_1+k_2)\cdot \hat{n}} \times \frac{1}{4} \omega_1\omega_2 e^{-\omega_1|t_1-t_2+t_3|} e^{-\omega_2|t_3|} \times \langle \psi, (\varepsilon_1 \cdot x) H e^{-ik_1 \cdot x} H^{-2} e^{-|t_1|H} e^{-ik_2 \cdot x} H \rangle \langle \varepsilon_2, x \rangle \psi \times \langle \psi, (\varepsilon_1 \cdot x) H e^{ik_1 \cdot x} H^{-2} e^{-|t_2|H} e^{ik_2 \cdot x} H \rangle \langle \varepsilon_2, x \rangle \psi \right). \]

We rescale \( k_j \sim k_j/R \) as before and in addition \( t_3 \sim t_3 R \). Then the exponentially decaying terms are

\[ e^{-\omega_1|t_3+(t_1-t_2)/R|} e^{-\omega_2|t_3|} \sim e^{-\omega_1|t_3|} e^{-\omega_2|t_3|} \]  

\( 4.49 \)
for large $R$. Expanding in $1/R$ yields

\[
I_4(R) = R^{-7} \left( \frac{1}{2} \langle \psi, x \cdot H^{-1} x \psi \rangle \right)^2 \int_0^\infty dt \int dk_1 dk_2 |\hat{\varphi}(k_1/R)|^2 |\hat{\varphi}(k_2/R)|^2 \\
\times e^{i(k_1+k_2 \cdot \hat{n})/\omega_1 \omega_2} e^{-t(\omega_1+\omega_2)} (1 + (\hat{k}_1 \cdot \hat{k}_2)^2) + O(R^{-9})
\]

\[
= R^{-7} (2\pi)^{-6} \left( \frac{1}{3} \langle \psi, x \cdot H^{-1} x \psi \rangle \right)^2 \int dk_1 dk_2 \\
\times e^{i(k_1+k_2 \cdot \hat{n})/\omega_1 \omega_2} (1 + (\hat{k}_1 \cdot \hat{k}_2)^2) + O(R^{-9})
\]

\[
= R^{-7} \left( \frac{1}{2\pi} \right)^2 e^{-4} \frac{23}{8\pi} \alpha_E^2 + O(R^{-9}).
\]

(4.30)

The prefactor of $R^{-7}$ agrees with the one computed already by Casimir and Polder.

4.6 Sum of all terms

We add the limits listed in (4.22), (4.27), and (4.30), which yields

\[
\kappa(e) = \lim_{R \to \infty} R^7 \left( \lim_{T \to \infty} \frac{1}{2T} e^{1/2} C_2(R, T) \right) = \left( \frac{1}{2\pi} \right)^2 \left( \frac{23}{16\pi} \alpha_E^2 + \frac{26}{\pi} \alpha_E \alpha_M + \frac{64}{\pi} \alpha_M^2 \right).
\]

(4.31)

4.7 Comparison with previous results

Magnetic contributions to the $-R^{-7}$ decay were first considered by Feinberg and Sucher [8, 9] and by Boyer [10]. They find that $\alpha_{E,at}^2$ and $\alpha_{M,at}^2$ have the same coefficient, namely $23/16\pi$, while the one of $\alpha_{E,at}^2\alpha_{M,at}^2$ is $7/8\pi$. This raises the issue on the origin for the discrepancy.

The $\alpha_{E,at}^2$ term can be most easily obtained through the dipole approximated hamiltonian

\[
H_{\text{dip}} = \frac{1}{2} (p_1 - eA(0))^2 - e^2 V(x_1) + \frac{1}{2} (p_2 - eA(r))^2 - e^2 V(x_2) + H_f \\
+ e^2 V_R(x_1, x_2).
\]

(4.32)

One shifts $p_1$ by $eA(0)$ and $p_2$ by $eA(r)$ through the unitary $U = \exp[-i(x_1 \cdot A(0) + x_2 \cdot A(r))]$. Then

\[
U H_{\text{dip}} U^{-1} = \frac{1}{2} p_1^2 - e^2 V(x_1) + \frac{1}{2} p_2^2 - e^2 V(x_2) + H_f \\
- ex_1 \cdot E_\perp(0) - ex_2 \cdot E_\perp(r) + e^2 V_R(x_1, x_2) \\
+ e^2 \int dk |\hat{\varphi}(k)|^2 \left( \frac{1}{3} x_1^2 + \frac{1}{3} x_2^2 + e^{ik \cdot r} (x_1 \cdot x_2 - |k|^2 (x_1 \cdot k)(x_2 \cdot k)) \right),
\]

(4.33)
where $E_\perp(r)$ is the quantized transverse electric field. Note that the long range part of $V_R$ is cancelled. From the 4-th order perturbation in $-e x_1 \cdot E_\perp(0) - e x_2 \cdot E_\perp(r)$ one obtains the Casimir-Polder result.

To include magnetic effects one proceeds to the next order of the multipole expansion and defines

$$H_{\text{mul}} = \frac{1}{2} \left( p_1 - e A(0) - e x_1 \cdot \nabla_r A(0) \right)^2 - e^2 V(x_1)$$

$$+ \frac{1}{2} \left( p_2 - e A(r) - e x_2 \cdot \nabla_r A(r) \right)^2 - e^2 V(x_2)$$

$$+ H_f + e^2 V_R(x_1, x_2),$$

(4.34)

which, as before, is unitarily transformed to

$$U H_{\text{mul}} U^{-1} = \frac{1}{2} \left( p_1 - e x_1 \cdot \nabla_r A(0) \right)^2 - e^2 V(x_1)$$

$$+ \frac{1}{2} \left( p_2 - e x_2 \cdot \nabla_r A(r) \right)^2 - e^2 V(x_2)$$

$$+ H_f - e x_1 \cdot E_\perp(0) - e x_2 \cdot E_\perp(r) + e^2 V_R^{\text{cor}}(x_1, x_2),$$

(4.35)

where $V_R^{\text{cor}}$ is the interaction potential from (4.33). Note that $\nabla_\perp \cdot (x \cdot \nabla_r A(r)) = B(r) = \nabla_\perp \cdot A(r)$. Thus the hamiltonian (4.35) clearly displays the fluctuating electric and magnetic fields. Expanding in the terms proportional to $\nabla_r A(0)$, $\nabla_r A(r), E_\perp(0), E_\perp(r)$ to 4-th order leads to an energy with a large $R$ asymptotics in agreement with (4.31). In spirit the authors of [8, 9, 10] make a further gauge transformation through the unitary $\exp[-i(x_1 \cdot \nabla_r A(0)) + x_2 \cdot \nabla_r (x_2 \cdot A(r))]$. Thereby $x_1 \cdot \nabla_r A(0)$ is transformed to $\frac{1}{2} x_1 \cdot B(0)$ and correspondingly for $x_2 \cdot \nabla_r A(0)$. In addition there are terms coming from the shifting of $E_\perp$. In the 4-th order perturbation only the former terms are taken into account. More precisely the term $\langle \psi_{\text{at}} \otimes \Omega, (x_1 \cdot B(0))^2 P H^{-1} P (x_2 \cdot B(r))^2 \psi_{\text{at}} \otimes \Omega \rangle$, which yields indeed the integrand (C.1) and thus 23/16$r$ for the prefactor. Here $P = \mathbb{1} - |\psi\rangle \langle \psi|$, and $\Omega$ is the Fock vacuum. For the cross-term they use $\langle \psi_{\text{at}} \otimes \Omega, x_1 \cdot E_\perp(0) P H^{-1} P (x_2 \cdot B(r))^2 \psi_{\text{at}} \otimes \Omega \rangle$, which yields the integral (C.2) with the term $(1 - (\hat{k}_1 \cdot \hat{k}_2)^2)$ omitted and thus $7/8\pi$ for the prefactor of $\alpha_{E,\text{at}} \alpha_{M,\text{at}}$.

## 5 The dimensionless strength

We restore the physical units in $H$ of (2.1). Then

$$H = \frac{1}{2m} \left( p - \frac{e}{\hbar} A(x) \right)^2 - e^2 V(x) + H_f$$

(5.1)

with $p = -i \hbar \nabla_x$,

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \hbar c |k| a(k, \lambda)^* a(k, \lambda),$$

(5.2)
and

\[ A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \sqrt{\frac{\alpha}{2|k|}} \hat{\varphi}(k) \varepsilon(k, \lambda) \left( e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^* \right). \]  

(5.3)

\( H \) is transformed to atomic units through the canonical transformation \( U \) defined as

\[ U^* a(k, \lambda) U = (\alpha^{-2} \lambda c)^{3/2} a(\alpha^{-2} \lambda c k, \lambda), \]

\[ U^* x U = \alpha^{-1} r_B x, \quad U^* p U = \alpha r_B^{-1} p. \]  

(5.4)

Then

\[ U^* H U = \alpha^2 m c^2 \left( \frac{1}{2} (-i \nabla_x - \sqrt{4\pi \alpha^3 / 2} \tilde{A}(\alpha x))^2 - \tilde{V}(x) + \tilde{H}_f \right) \]

\[ = \alpha^2 m c^2 \tilde{H}, \]  

(5.5)

where

\[ \tilde{H}_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk |k| a(k, \lambda)^* a(k, \lambda) \]  

(5.6)

and

\[ \tilde{V}(x) = 4\pi \int_{\mathbb{R}^3} dk \hat{\varphi}(\alpha^2 \lambda^{-1} c k)^2 |k|^{-2} e^{-ik \cdot x}, \]  

(5.7)

\[ \tilde{A}(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \frac{\hat{\varphi}(\alpha^2 \lambda^{-1} c k)}{\sqrt{2|k|}} \varepsilon(k, \lambda) \left( e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^* \right). \]  

(5.8)

Energy is now in units of \( \alpha^2 m c^2 \) and distances are in units of the Bohr radius \( r_B \).

From (5.6), (5.8) the photon propagator in atomic units is obtained as

\[ \tilde{W}_R(q,t) = 4\pi \int_{\mathbb{R}^3} dk |k| a(k, \lambda)^* a(k, \lambda) \]

(5.9)

Comparing with (3.2) this amounts to replacing \( e^2 \) by \( 4\pi \) and \( t \) by \( t/\alpha \). One merely has to follow through this change in the computation of Section 4. The final result is

\[ E(R) - 2E = -\kappa(\alpha, \lambda c \Lambda) \alpha m c^2 (R/r_B)^{-7} \]  

(5.10)

for large \( R \) with the strength \( \kappa \) obtained as

\[ \kappa(\alpha, \lambda c \Lambda) = \frac{4}{\pi} \left( \frac{23}{16} \tilde{\alpha}_E^2 + 26 \tilde{\alpha}_E \tilde{\alpha}_M + 64 \tilde{\alpha}_M^2 \right). \]  

(5.11)

Here, the dimensionless electric and magnetic dipole moments are

\[ \tilde{\alpha}_E = 2 \left( \frac{1}{3} \langle \psi, x \cdot (\tilde{H} - E)^{-1} \psi \rangle \right) \]  

(5.12)
\[ \bar{\alpha}_M = -\alpha^2 \frac{1}{4} \left( \frac{1}{3} \langle \psi, x^2 \psi \rangle \right) \] (5.13)

with \( \psi \) the ground state of \( \tilde{H}, \tilde{H}\psi = E\psi \). The ultraviolet cut-off is implemented by replacing \( \hat{\phi}(k) \) by the form factor \( \hat{\phi}_1(k/\Lambda) \): \( \hat{\phi}_1 \) decreases rapidly at \( |k| = 1 \) from \( (2\pi)^{-3/2} \) to 0.

Following Bethe [19] a physically reasonable choice is \( \lambda_c \Lambda = 1 \). One would like to remove the ultraviolet cut-off through \( \Lambda \to \infty \). But this limit is not well understood. In any case, the bare mass \( m \) would have to substituted by the renormalized mass. For fixed \( \Lambda \), \( \lambda_c \Lambda = 1 \), in the limit \( \alpha \to 0 \), \( \tilde{H} \) decouples from the radiation field and \( \kappa(\alpha, \lambda_c \Lambda) \) tends to the strength obtained by Casimir and Polder. Then \( \bar{\alpha}_E \simeq \frac{9}{4} \) and \( \bar{\alpha}_M \simeq -\alpha^2 \) with quality for the strict Coulomb potential. For a systematic expansion in \( \alpha \) one would need the correction of order \( \alpha \) to \( \bar{\alpha}_E \) and to the bare mass \( m \).

### 6 Conclusions and Outlook

The reader may wonder how much is still missing to a complete proof. For the full model, a central difficulty is that the ground state process \( q(t) \) is not so well under control. Since \( H \) has no spectral gap, the two-point correlation has a slow decay, presumably as \( E[q(t) \cdot q(0)] \simeq |t|^{-4} \) for large \( t \). This means that higher order cumulants are difficult to control. In fact, we cannot even prove that \( \bar{\alpha}_E < \infty \).

From the statistical mechanics point of view an interesting case would be to replace the ground state process \( q(t) \) by \( q_{at}(t) \), namely the one governed by \( H_{at} \). This process is exponentially mixing which should help in controlling the error term. On the other hand, the double stochastic integral in the action causes extra difficulties. Unfortunately there is no obvious hamiltonian corresponding to this approximation.

A further variant would be the dipole approximation of equation (4.32). In the path integral (3.41) this corresponds to replacing \( W_R(x, t) \) by \( W(0, t) \). The effective action is quadratic in \( \dot{q}_j(t) \) and the partial time integration can be easily implemented. It results in a diagonal term which cancels the slowly decaying part of \( V_R \) and the remainder action is given by \( \int dsdtq(s) \cdot \dot{W}_R(0, s - t)q(t) \). Since \( \dot{W}_R(0, t) \simeq t^{-4} \) for large \( t \), the difficulties mentioned above remain. To have an exactly solvable Gaussian model, on top one would have to use the quadratic approximation for \( V \) and \( V_R \).

For ground state properties a powerful method is the Feshbach projection together with a successive integration over high \( k \)-modes of the radiation field [20, 21]. It would be interesting to understand whether this technique could be used for a rigorous control on the van der Waals potential.

**Acknowledgement:** We are grateful to P. Milonni for very useful hints on the literature.
A  Proof of (i) to (v)

The proof of (i) to (v) is based on the identity

$$E\left[\prod_{j=1}^{m} f_j(q(t_j))\right] = \langle \psi, f_1(x)e^{-|t_1-t_2|(H-E)} \cdots f_{m-1}(x)e^{-|t_{m-1}-t_m|(H-E)} f_m(x)\psi \rangle$$  \hspace{1cm} (A.1)

for the time order $t_1 < t_2 < \cdots < t_m$. $\dot{q}(t)dt$ is the Ito stochastic integral as defined through the forward discretization. For this purpose we introduce the lattice spacing $\delta$ and let $[t]_\delta$ be the integer part of $t$ modulo $\delta$. Then $\dot{q}(t) = \lim_{\delta \to 0} \delta^{-1}(q([t]_\delta) + \delta) - q([t]_\delta))$. We only establish (ii) and (v). The other items are proved by the same procedure.

ad (ii): Using stationarity,

$$E[e^{ik\cdot \dot{q}(\tau)\cdot \dot{q}(\tau)}] = \lim_{\delta \to 0} \frac{1}{\delta} E[e^{ik\cdot q(0)\cdot (q(\delta) - q(0))}]$$

$$= \lim_{\delta \to 0} \frac{1}{\delta} \langle \psi, e^{ik\cdot x(e^{-\delta(H-E)} - 1)}(\varepsilon \cdot x)\psi \rangle$$

$$= -\langle \psi, e^{ik\cdot x(H-E)}(\varepsilon \cdot x)\psi \rangle, \hspace{1cm} (A.2)$$

which vanishes as proved in Proposition B.4.

ad (v): For $t \neq s$ we proceed as in ad (ii), which yields the first summand of (v). For the “diagonal part” one has, for small $\delta$,

$$E\left[(\varepsilon_1 \cdot (q(\delta) - q(0)))e^{-ik_1\cdot q(0)}e^{-ik_2\cdot q(0)}(\varepsilon_2 \cdot (q(\delta) - q(0)))\right]$$

$$= \langle \psi, e^{-ik_1\cdot x}e^{-ik_2\cdot x}(e^{-\delta(H-E)}(\varepsilon_1 \cdot x)(\varepsilon_2 \cdot x) - (\varepsilon_1 \cdot x)e^{-\delta(H-E)}(\varepsilon_2 \cdot x) - (\varepsilon_1 \cdot x)e^{-\delta(H-E)}(\varepsilon_2 \cdot x)\psi \rangle$$

$$= -\delta \langle \psi, e^{-ik_1\cdot x}e^{-ik_2\cdot x}[(\delta^{-1}(1 - e^{-\delta(H-E)})], (\varepsilon_1 \cdot x), (\varepsilon_2 \cdot x)]\psi \rangle$$

$$\simeq \delta(\varepsilon_1 \cdot \varepsilon_2)\langle \psi, e^{-ik_1\cdot x}e^{-ik_2\cdot x}\psi \rangle. \hspace{1cm} (A.3)$$

B  Ground state expectations

In this appendix, we mainly work in the Schrödinger representation $L^2(\mathbb{R}^3_x) \otimes \mathcal{F} = L^2(\mathbb{R}^3_x) \otimes L^2(Q)$, see [11][16]. In this representation it holds

(a) $A(x)$ is a real-valued multiplication operator.

(b) Let $\vartheta = \Gamma(e^{i\varpi/2})$. Then $\vartheta e^{-tH} \vartheta^{-1}$ is positivity improving. Hence $\tilde{\psi} = \vartheta \psi$ is strictly positive.

Here for a unitary operator $U$, $\Gamma(U)$ is defined by

$$\Gamma(U)a(f_1)^* \cdots a(f_n)^* \Omega = a(Uf_1)^* \cdots a(Uf_n)^* \Omega$$
for all \( n \in \mathbb{N} \) and \( f_1, \ldots, f_n \in \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \), where \( \Omega \) is the Fock vacuum. A linear operator \( A \) “improves the positivity” if, for all \( 0 \leq f \in (L^2(\mathbb{R}^3_+) \otimes L^2(Q)) \backslash \{0\} \), \( e^{-tA}f > 0 \) for all \( t > 0 \). Let \( J \) be the natural conjugation in \( L^2(\mathbb{R}^3_+) \otimes L^2(Q) \), namely, \( J\Psi = \overline{\Psi} \) for all \( \Psi \in L^2(\mathbb{R}^3_+) \otimes L^2(Q) \). A linear operator \( A \) is called to be \( J \)-real if \( A \) commutes with \( J \), i.e., \( JA = AJ \). Then, from (b), it follows that

\[
(\text{c}) \quad \tilde{H} = \vartheta H \vartheta^{-1} \text{ is a } J \text{-real operator, i.e., } J \tilde{H} = \tilde{H} J.
\]

**Lemma B.1** *(Vanishing ground state expectation I).* Let \( F(x) \) a measurable function on \( \mathbb{R}^3 \) such that \( \psi \in \text{dom}(F) \). Then one has

\[
\langle \psi, F(x)A(x)\psi \rangle = 0. \tag{B.1}
\]

**Remark B.2** From the proof one infers the stronger property

\[
\langle \psi, F(x)A(x)_{i_1}e^{-s_1 H} \cdots e^{-s_{2n} H} A(x)_{i_{2n+1}} \psi \rangle = 0
\]

for all \( n \in \{0\} \cup \mathbb{N}, s_1, \ldots, s_{2n} > 0 \) and \( F \in L^\infty(\mathbb{R}^3) \). (Note that we have to use (c).)

**Proof.** Note that

\[
J\vartheta = \Gamma(e^{i\pi})\vartheta J. \tag{B.2}
\]

Since \( A(x) \) is a real-valued multiplication operator, one has \( A(x)J = JA(x) \). Hence using (B.2), one sees that \( E(x) = \vartheta A(x)\vartheta^{-1} \) is purely \( J \)-imaginary, that is, \( JE(x) = -E(x)J \). Remark that

\[
\langle \psi, A(x)\psi \rangle = \langle \tilde{\psi}, E(x)\tilde{\psi} \rangle.
\]

Since \( \tilde{\psi} \) is real-valued by (b), the right hand side is purely imaginary. On the other hand, by the self-adjointness of \( A(x) \), the left hand side is real. Thus the only possibility is \( \langle \psi, A(x)\psi \rangle = 0 \). Similarly one can show that

\[
\langle \psi, \Re F(x)A(x)\psi \rangle = 0 = \langle \psi, \Im F(x)A(x)\psi \rangle
\]

which implies the assertion. □

**Lemma B.3** *(Vanishing ground state expectation II).* One has the following:

i) For all \( a \in \mathbb{R}^3 \), \( \langle \psi, (a \cdot x)^{2n+1}\psi \rangle = 0 \).

ii) If \( m + n \) is odd, then \( \langle \psi, (a \cdot x)H(b \cdot x)^m H^{-1}(c \cdot x)^n H(d \cdot x)\psi \rangle = 0 \) for all \( a, b, c, d \in \mathbb{R}^3 \).
Proof. For the proof, we return to the Fock representation $L^2(\mathbb{R}^3_+) \otimes \mathfrak{g}$. Let $J_2$ be the involution defined by
\[ J_2 \Psi = \sum_{n \geq 0} \Psi_n(-x; k_1, \ldots, k_n, \lambda) \] (B.3)
for all $\Psi = \sum_{n \geq 0} \Psi_n(x; k_1, \ldots, k_n, \lambda) \in L^2(\mathbb{R}^3_+) \otimes \mathfrak{g}$. Then as proved in [22], we can check that
\[ J_2 \Psi = -x J_2, \] (B.4)
\[ J_2 H = H J_2, \] (B.5)
which implies $\langle \psi, (a \cdot x)^{2n+1} \rangle = 0$. Similarly if $m + n$ is odd, then
\[ \langle \psi, (a \cdot x)H(b \cdot x)^m H^{-1}(c \cdot x)^n H(d \cdot x) \rangle \]
\[ = -\langle J_2 \psi, (d \cdot x)H(c \cdot x)^m H^{-1}(b \cdot x)^n H(a \cdot x) \rangle \]
\[ = -\langle \psi, (d \cdot x)H(c \cdot x)^n H^{-1}(b \cdot x)^m H(a \cdot x) \rangle. \] (B.7)
On the other hand, in the Schrödinger representation, we can see that
\[ \langle \tilde{\psi}, (d \cdot x)\tilde{H}(c \cdot x)^n \tilde{H}^{-1}(b \cdot x)^m \tilde{H}(a \cdot x) \tilde{\psi} \rangle \]
\[ = \langle \tilde{\psi}, (a \cdot x)\tilde{H}(b \cdot x)^m \tilde{H}^{-1}(c \cdot x)^n \tilde{H}(d \cdot x) \tilde{\psi} \rangle \] (B.8)
because every operator appearing in the expectation is $J$-real. Therefore
\[ \langle \psi, (d \cdot x)H(c \cdot x)^n H^{-1}(b \cdot x)^m H(a \cdot x) \rangle \]
\[ = \langle \psi, (a \cdot x)H(b \cdot x)^m H^{-1}(c \cdot x)^n H(d \cdot x) \rangle. \] (B.9)
Combining this with (B.7), we conclude ii). □

**Proposition B.4** One has the following

i) $\langle \psi, e^{ik \cdot x} H(\varepsilon(k, \lambda) \cdot x) \psi \rangle = 0$,

ii) $B_{\pm}(k_1/R, \lambda_1; k_2/R, \lambda_2) = R^{-2} |k_1||k_2| (\varepsilon_1 \cdot \hat{k}_2)(\varepsilon_2 \cdot \hat{k}_1) (\frac{1}{3} \langle \psi, x^2 \psi \rangle) + O(R^{-4})$ as $R \to \infty$. 

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Proof. \( \textbf{i) By (B.13) below and by } \langle \psi, e^{-ik \cdot x} A(x) \psi \rangle = 0 \text{ according to Lemma } \textbf{B.1} \text{ one has} \)
\[
\langle \psi, e^{ik \cdot x} H(\varepsilon(k, \lambda) \cdot x) \psi \rangle = -i \langle \psi, e^{ik \cdot x}(\varepsilon(k, \lambda) \cdot (p - eA(x))) \psi \rangle \\
= -i \langle \psi, e^{ik \cdot x}(\varepsilon(k, \lambda) \cdot p) \bar{\psi} \rangle \\
= -\frac{i}{2} \int dx d\mu(A) e^{ik \cdot x} \varepsilon(k, \lambda) \cdot (-i \nabla_x \bar{\psi}^2)(x, A) \\
= -\frac{i}{2} \int dx d\mu(A) e^{ik \cdot x} \varepsilon(k, \lambda) \cdot k \bar{\psi}^2(x, A) \\
= 0,
\]
since \( k \cdot \varepsilon(k, \lambda) = 0. \)

\( \textbf{ad ii) By Lemma } \textbf{B.3} \text{ the order } R^{-1} \text{ vanishes and it suffices to check the order } R^0 \text{ and } R^{-2}. \)

\( \textbf{ii-a) order } R^0. \text{ One has} \)
\[
\lim_{R \to \infty} B_-(k_1/R, \lambda_1; k_2/R, \lambda_2) = (\varepsilon_1 \cdot \varepsilon_2) - 2 \langle \psi, (\varepsilon_1 \cdot x) H(\varepsilon_2 \cdot x) \psi \rangle. \quad (B.10)
\]
Furthermore, using the identities
\[
\begin{align*}
[a \cdot \tilde{p}, b \cdot x] &= -ia \cdot b, \\
[H, a \cdot x] &= -ia \cdot \tilde{p}, \\
H(a \cdot x) \psi &= -ia \cdot \tilde{p} \psi
\end{align*}
\]
for all \( a, b \in \mathbb{C}^3 \), where \( \tilde{p} = p - eA(x) \), one obtains
\[
2 \langle \psi, (\varepsilon_1 \cdot x) H(\varepsilon_2 \cdot x) \psi \rangle = i \langle \psi, ((\varepsilon_1 \cdot \tilde{p})(\varepsilon_2 \cdot x) - (\varepsilon_1 \cdot x)(\varepsilon_2 \cdot \tilde{p})) \psi \rangle = (\varepsilon_1 \cdot \varepsilon_2). \quad (B.14)
\]
Remark that we have used \( \textbf{B.1} \) to conclude that \( \langle \psi, (\varepsilon_1 \cdot A(x))(\varepsilon_2 \cdot x) \psi \rangle = 0 \)
\( = \langle \psi, (\varepsilon_1 \cdot x)(\varepsilon_2 \cdot A(x)) \psi \rangle. \)

\( \textbf{ii-b) order } R^{-2}. \text{ We expand as} \)
\[
R^2 B_-(k_1/R, \lambda_1; k_2/R, \lambda_2) = c_1 k_1^2 + c_2 k_2^2 + d(k_1, \lambda_1; k_2, \lambda_2) + O(R^{-2}). \quad (B.15)
\]
For \( c_1 k_1^2 \), the expansion coefficient is
\[
(\varepsilon_1 \cdot \varepsilon_2) \langle \psi, (k_1 \cdot x)^2 \psi \rangle - 2 \langle \psi, (\varepsilon_1 \cdot x) H(k_1 \cdot x)^2(\varepsilon_2 \cdot x) \psi \rangle. \quad (B.16)
\]
Using \( \textbf{B.1} \) and that \( \bar{\psi} \) and \( \bar{H} \) can be chosen as real, for the second term one obtains
\[
\langle \psi, (\varepsilon_1 \cdot x) H(k_1 \cdot x)^2(\varepsilon_2 \cdot x) \psi \rangle + \langle \psi, (\varepsilon_2 \cdot x)(k_1 \cdot x)^2 H(\varepsilon_1 \cdot x) \psi \rangle \\
= i \langle \psi, (\varepsilon_1 \cdot \tilde{p})(k_1 \cdot x)^2(\varepsilon_2 \cdot x) \psi \rangle - i \langle \psi, (\varepsilon_2 \cdot x)(k_1 \cdot x)^2(\varepsilon_1 \cdot \tilde{p}) \psi \rangle \\
= i \langle \psi, (k_1 \cdot x)^2[(\varepsilon_1 \cdot \tilde{p}), (\varepsilon_2 \cdot x)] \psi \rangle \\
= (\varepsilon_1 \cdot \varepsilon_2) \langle \psi, (k_1 \cdot x)^2 \psi \rangle. \quad (B.17)
\]
Thus \( c_1 = 0 = c_2 \).

The expansion coefficient of the mixed term \( d(k_1, \lambda_1; k_2, \lambda_2) \) is given by

\[
-(\varepsilon_1 \cdot \varepsilon_2)\langle \psi, (k_1 \cdot x)(k_2 \cdot x)\psi \rangle + 2\langle \psi, (\varepsilon_1 \cdot x)H(k_1 \cdot x)H^{-1}(k_2 \cdot x)H(\varepsilon_2 \cdot x)\psi \rangle.
\]  

(B.18)

Using (B.1) and that \( \tilde{H} \) and \( \tilde{\psi} \) can be chosen to be real, we have

\[
\begin{align*}
\langle \psi, (\varepsilon_1 \cdot x)H(k_1 \cdot x)H^{-1}(k_2 \cdot x)H(\varepsilon_2 \cdot x)\psi \rangle + (1 \leftrightarrow 2) \\
= \frac{1}{2} \left\{ -i\langle \psi, (\varepsilon_1 \cdot x)(k_1 \cdot x)(k_2 \cdot x)(\varepsilon_2 \cdot \tilde{p})\psi \rangle \\
- \langle \psi, (\varepsilon_1 \cdot x)(k_1 \cdot \tilde{p})H^{-1}(k_2 \cdot x)(\varepsilon_2 \cdot \tilde{p})\psi \rangle \\
+ i\langle \psi, (\varepsilon_1 \cdot \tilde{p})(k_1 \cdot x)(k_2 \cdot x)(\varepsilon_2 \cdot \tilde{p})\psi \rangle \\
+ \langle \psi, (\varepsilon_1 \cdot \tilde{p})(k_1 \cdot x)H^{-1}(k_2 \cdot \tilde{p})(\varepsilon_2 \cdot x)\psi \rangle + (1 \leftrightarrow 2) \right\} \\
= \frac{1}{2} \left\{ i\langle \psi, (k_1 \cdot x)((\varepsilon_1 \cdot \tilde{p})(\varepsilon_2 \cdot x) - (\varepsilon_1 \cdot x)(\varepsilon_2 \cdot \tilde{p}))(k_2 \cdot x)\psi \rangle \\
- \langle \psi, (\varepsilon_1 \cdot x)(k_1 \cdot \tilde{p})H^{-1}(k_2 \cdot x)(\varepsilon_2 \cdot \tilde{p})\psi \rangle \\
+ \langle \psi, (\varepsilon_1 \cdot \tilde{p})(k_1 \cdot x)H^{-1}(\varepsilon_2 \cdot x)(k_2 \cdot \tilde{p})\psi \rangle + (1 \leftrightarrow 2) \right\} \\
= (\varepsilon_1 \cdot \varepsilon_2)\langle \psi, (k_1 \cdot x)(k_2 \cdot x)\psi \rangle + (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_1)\left(\frac{1}{3}\langle \psi, x^2\psi \rangle\right) \\
= \frac{1}{3}\langle \psi, x^2\psi \rangle ((\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2) + (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_1)).
\end{align*}
\]

(C.19)

This proves the assertion. \( \square \)

C Numerical coefficients

In this appendix, we will explain how to compute the following integrals appearing in the main text:

\[
S_1 = \int dk_1 dk_2 \frac{|k_1||k_2|}{|k_1| + |k_2|} e^{i(k_1 + k_2) \cdot \hat{n}} (1 + (\hat{k}_1 \cdot \hat{k}_2)^2),
\]

(C.1)

\[
S_2 = \int dk_1 dk_2 \frac{|k_1||k_2|}{|k_1| + |k_2|} e^{i(k_1 + k_2) \cdot \hat{n}} (\hat{k}_1 \cdot \hat{k}_2)(1 - (\hat{k}_1 \cdot \hat{k}_2)^2),
\]

(C.2)

\[
S_3 = \int dk_1 dk_2 \frac{|k_1||k_2|}{|k_1| + |k_2|} e^{i(k_1 + k_2) \cdot \hat{n}} (1 - (\hat{k}_1 \cdot \hat{k}_2)^2).\]

(C.3)

These integrals are of the form

\[
S_j = \int dk_1 dk_2 \frac{|k_1||k_2|}{|k_1| + |k_2|} e^{i(k_1 + k_2) \cdot \hat{n}} F_j(\hat{k}_1 \cdot \hat{k}_2),
\]

(C.4)

which we rewrite as

\[
S_j = \int_0^\infty dt \int dk_1 dk_2 |k_1||k_2| e^{-t(|k_1| + |k_2|)} e^{i(k_1 + k_2) \cdot \hat{n}} F_j(\hat{k}_1 \cdot \hat{k}_2).
\]

(C.5)
Let us switch to polar coordinates \((r, \varphi, \vartheta)\) by
\[
\hat{k} = (Y \cos \varphi, Y \sin \varphi, X),
\]
\[
X = \cos \vartheta, \quad Y = \sin \vartheta.
\] (C.6)

Clearly \(X^2 + Y^2 = 1\). Then we have
\[
\hat{k}_1 \cdot \hat{k}_2 = \cos(\varphi_1 - \varphi_2)Y_1Y_2 + X_1X_2,
\] (C.7)

and hence
\[
S_j = \int_0^\infty \int_0^\infty dr_1r_1^3e^{-tr_1} \int_{-1}^1 dX_1 e^{ir_1X_1} \times \int_0^\infty dr_2r_2^3e^{-tr_2} \int_{-1}^1 dX_2 e^{ir_2X_2} \mathfrak{S}_j(X_1, X_2),
\] (C.8)

where
\[
\mathfrak{S}_j(X_1, X_2) = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 F_j(\cos(\varphi_1 - \varphi_2)Y_1Y_2 + X_1X_2)
\]
\[
= 2\pi \int_0^{2\pi} d\varphi F_j(\cos \varphi Y_1Y_2 + X_1X_2).
\] (C.9)

Set
\[
\langle A \rangle = 2\pi \int_0^\infty dr \, r^3 e^{-tr} \int_{-1}^1 dX \, e^{irX} A(X).
\] (C.10)

After performing \(\varphi\)-integration in (C.9), \(S_j\) can be expressed as
\[
S_1 = \int_0^\infty dt \left\{ \frac{3}{2} \langle 1 \rangle - \langle 1 \rangle \langle X^2 \rangle + \frac{3}{2} \langle X^2 \rangle \langle X^2 \rangle \right\},
\] (C.11)

\[
S_2 = \int_0^\infty dt \left\{ -\frac{1}{2} \langle X \rangle \langle X \rangle + 3 \langle X \rangle \langle X^3 \rangle - \frac{5}{2} \langle X^3 \rangle \langle X^3 \rangle \right\},
\] (C.12)

\[
S_3 = \int_0^\infty dt \left\{ \frac{3}{8} \langle 1 \rangle + \frac{1}{2} \langle 1 \rangle \langle X^2 \rangle + \frac{3}{2} \langle X^2 \rangle \langle X^2 \rangle + \frac{3}{4} \langle 1 \rangle \langle X^4 \rangle - \frac{15}{2} \langle X^2 \rangle \langle X^2 \rangle + \frac{27}{8} \langle X^4 \rangle \langle X^4 \rangle \right\},
\] (C.13)

Using Mathematica, one obtains
\[
\langle 1 \rangle = -\frac{4 + 12t}{(1 + t^2)^3},
\] (C.14)

\[
\langle X \rangle = \frac{16t}{(1 + t^2)^3},
\] (C.15)

\[
\langle X^2 \rangle = \frac{4(-3 + t^2)}{(1 + t^2)^3},
\] (C.16)

\[
\langle X^3 \rangle = 4i \left\{ \frac{t(9 + 8t^2 + 3t^4)}{(1 + t^2)^3} - 3 \arccot(t) \right\},
\] (C.17)

\[
\langle X^4 \rangle = \frac{4(3 + 27t^2 + 32t^4 + 12t^6)}{(1 + t^2)^3} - 48t \arccot(t).
\] (C.18)
Inserting these formulas to (C.11)-(C.13) and using Mathematica again, one arrives at

\[ S_1 = 92\pi^3, \quad S_2 = 208\pi^3, \quad S_3 = 256\pi^3. \]  

(C.19)

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