Limits of Recursive Triangle and Polygon Tunnels

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Abstract.
In this paper we present unsolved problems that involve infinite tunnels of recursive triangles or recursive polygons, either in a decreasing or in an increasing way. The “nedians or order i in a triangle” are generalized to “nedians of ratio r” and “nedians of angle $\alpha$” or “nedians at angle $\beta$”, and afterwards one considers their corresponding “nedian triangles” and “nedian polygons”. This tunneling idea came from physics. Further research would be to construct similar tunnel of 3-D solids (and generally tunnels of n-D solids).

A) Open Question 1 (Decreasing Tunnel).

1. Let $\triangle ABC$ be a triangle and let $\triangle A_1B_1C_1$ be its orthic triangle (i.e. the triangle formed by the feet of its altitudes) and $H_1$ its orthocenter (the point on intersection of its altitudes).
Then, let’s consider the triangle $\triangle A_2B_2C_2$, which is the orthic triangle of triangle $\triangle A_1B_1C_1$, and $H_2$ its orthocenter.
And the recursive tunneling process continues in the same way.
Therefore, let’s consider the triangle $\triangle A_nB_nC_n$, which is the orthic triangle of triangle $\triangle A_{n-1}B_{n-1}C_{n-1}$, and $H_n$ its orthocenter.
a) What is the locus of the orthocenter points $H_1, H_2, ..., H_n, ...$? {Locus means the set of all points satisfying some condition.}
b) Is this limit:
$$\lim_{n \to \infty} \triangle A_nB_nC_n$$
convergent to a point? If so, what is this point?
c) Calculate the sequences
$$a_n = \frac{\text{area}(\triangle A_nB_nC_n)}{\text{area}(\triangle A_{n-1}B_{n-1}C_{n-1})} \quad \text{and} \quad \beta_n = \frac{\text{perimeter}(\triangle A_nB_nC_n)}{\text{perimeter}(\triangle A_{n-1}B_{n-1}C_{n-1})}.$$
d) We generalize the problem from triangles to polygons. Let $AB...M$ be a polygon with $m \geq 4$ sides. From $A$ we draw a perpendicular on the next polygon’s side $BC$, and note its intersection with this side by $A_1$. And so on. We get another polygon $A_1B_1...M_1$.
We continue the recursive construction of this tunnel of polygons and we get the polygon sequence $A_nB_n...M_n$.
d1) Calculate the limit:
\[ \lim_{n \to \infty} \Delta A_n B_n \ldots M_n \]

d2) And the ratios of areas and perimeters as in question c).

e) A version of this polygonal extension d) would be to draw a perpendicular from \( A \) not necessarily on the next polygon’s side, but on another side (say, for example, on the third polygon’s side) – and keep a similar procedure for the next perpendiculars from all polygon vertices \( B, C, \) etc.

In order to tackle the problem in a easier way, one can start by firstly studying particular initial triangles \( \Delta ABC \), such as the equilateral and then the isosceles.

**B) Open Question 2 (Decreasing Tunnel).**

2. Same problem as in Open Question 1, but replacing “orthic triangle” by “medial triangle”, and respectively “orthocenter” by “center of mass (geometric centroid)”, and “altitude” by “median”. Therefore:

Let \( \Delta ABC \) be a triangle and let \( \Delta A_1 B_1 C_1 \) be its **medial triangle** (i.e. the triangle formed by the feet of its medians on the opposite sides of the triangle \( \Delta ABC \)) and \( H_1 \) its center of mass (or geometric centroid) (the point on intersection of its medians).

Then, let’s consider the triangle \( \Delta A_2 B_2 C_2 \), which is the medial triangle of triangle \( \Delta A_1 B_1 C_1 \), and \( H_2 \) its center of mass.

And the recursive tunneling process continues in the same way.

Therefore, let’s consider the triangle \( \Delta A_n B_n C_n \), which is the medial triangle of triangle \( \Delta A_{n-1} B_{n-1} C_{n-1} \), and \( H_n \) its center of massy.

a) What is the locus of the center of mass points \( H_1, H_2, \ldots, H_n, \ldots \) ?

{This has an easy answer; all \( H_i \) will coincide with \( H_1 \) (FS, IP).}

b) Is this limit:

\[ \lim_{n \to \infty} \Delta A_n B_n C_n \]

convergent to a point? If so, what is this point?

{Same response; the limit is equal to \( H_1 \) (FS, IP).}

c) Calculate the sequences

\[ \alpha_n = \frac{\text{area}(\Delta A_n B_n C_n)}{\text{area}(\Delta A_{n-1} B_{n-1} C_{n-1})} \quad \text{and} \quad \beta_n = \frac{\text{perimeter}(\Delta A_n B_n C_n)}{\text{perimeter}(\Delta A_{n-1} B_{n-1} C_{n-1})}. \]

d) We generalize the problem from triangles to polygons. Let \( AB \ldots M \) be a polygon with \( m \geq 4 \) sides. From \( A \) we draw a line that connects \( A \) with the midpoint of \( BC \), and note its intersection with this side by \( A_1 \). And so on. We get another polygon \( A_1 B_1 \ldots M_1 \).

We continue the recursive construction of this tunnel of polygons and we get the polygon sequence \( A_n B_n \ldots M_n \).
d1) Calculate the limit:

\[ \lim_{n \to \infty} \Delta A_n B_n ... M_n. \]

d2) And the ratios of areas and perimeters of two consecutive polygons as in question c).
e) A version of this polygonal extension d) would be to draw a line that connects A not necessarily on the midpoint of the next polygon’s side, but with the midpoint of another side (say, for example, of the third polygon’s side) – and keep a similar procedure for the next lines from all polygon vertices B, C, etc.

C) Open Questions 3-7 (Decreasing Tunnels).

3. Same problem as in Open Question 1, but considering a tunnel of **incentral triangles** and their incentral points, and their interior angles’ bisectors.
   Incentral triangle is the triangle whose vertices are the intersections of the interior angle bisectors of the reference triangle $\Delta ABC$ with the respective opposite sides of $\Delta ABC$.

4. Same problem as in Open Question 1, but considering a tunnel of **contact triangles** (intouch triangles) and their incircle center points, and their interior angles’ bisectors.
   A contact triangle is a triangle formed by the tangent points of the triangle sides to its incircle.

5. Same problem as in Open Question 1, but considering a tunnel of **pedal triangles** and a fixed point $P$ in the plane of triangle $\Delta ABC$.
   A pedal triangle of $P$ is formed by the feet of the perpendiculars from $P$ to the sides of the triangle $\Delta ABC$.

6. Same problem as in Open Question 1, but considering a tunnel of **symmedial triangles**.
   “The symmedial triangle $\Delta K_A K_B K_C$ (a term coined by E.W. Weisstein [4]), is the triangle whose vertices are the intersection points of the symmedians with the reference triangle $\Delta A B C$.”

7. Same problem as in Open Question 1, but considering a tunnel of **cyclocevian triangles**.
   A cyclocevian triangle of triangle $\Delta ABC$ with respect to the planar point $P$ is the Cevian triangle of the cyclocevian conjugate of $P$.

D) Open Questions 8-12 (Increasing Tunnels).

8. Similar problem as in Open Question 1, but considering a tunnel of **anticevian triangles** of the triangle $\Delta ABC$ with respect to the same planar point $P$. For question c) and d1) only.
   The anticevian triangle of the given triangle $\Delta ABC$ with respect to the given point $P$ is the triangle of which $\Delta ABC$ is the Cevian triangle with respect to $P$.

9. Similarly, but considering a tunnel of **tangential triangles**.
   The tangential triangle to the given triangle $\Delta ABC$ is a triangle formed by the tangents to the circumcircle of $\Delta ABC$ at its vertices.

10. Similarly, but considering a tunnel of **antipedal triangles**.
The antipedal triangle of the given triangle $\Delta ABC$ with respect to the given point $P$ is the triangle of which $\Delta ABC$ is the pedal triangle with respect to $P$.

11. Similarly, but considering a tunnel of **excentral triangles**.
   The excentral triangle (or tritangent triangle) of the triangle $\Delta ABC$ is the triangle with vertices corresponding to the excenters of $\Delta ABC$.

12. Similarly, but considering a tunnel of **anticomplementary triangles**.
   The anticomplementary (or antimedian) triangle of the triangle $\Delta ABC$ is the triangle formed by the parallels drawn through the vertices of the triangle $\Delta ABC$ to the opposite sides.

**E) Open Questions Involving Nedians 13-14.**

a) One calls **nedians of order** $i$ [see 4] of the triangle $\Delta ABC$ the lines that pass through each of the vertices of the triangle $\Delta ABC$ and divide the opposite side of the triangle into the ratio $i/n$, for $1 \leq i \leq n-1$.
   Let’s generalize this to **nedians of ratio** $r$, which means lines that pass through each of the vertices of the given triangle $\Delta ABC$ and divide the opposite side of the triangle into the ratio $r$.
   We introduce the notion of **nedian triangles**, first the interior nedian triangle of order $i$ (or more general interior nedian triangle of ratio $r$), which is the triangle formed by the three points of intersections of the three nedians of order $i$ (or respectively of the three nedians of ratio $r$), taken two by two;
   and that of exterior nedian triangle of order $i$ (or more general exterior nedian triangle of ratio $r$), which is the triangle $\Delta A'B'C'$ such that $A' \in BC$, $B' \in CA$, and $C' \in AB$ - where $AA'$, $BB'$, and $CC'$ are nedians of order $i$ (respectively of ratio $r$) in the triangle $\Delta ABC$.

b) Another notion to introduce: **nedians of angle** $\alpha$ (or $\alpha$-nedians), which are nedians that each of them forms the same angle $\alpha$ with its respective side of the triangle, i.e.
   \[ \angle (AA', AB) = \angle (BB', BC) = \angle (CC', CA) = \alpha. \]
   And associated with this we have interior $\alpha$-nedian triangle and exterior $\alpha$-nedian triangle.

c) And one more derivative to introduce now: **nedians at angle** $\beta$ to the opposite side (or nedian-$\beta$), which are of course nedians that form with the opposite side of the triangle $\Delta ABC$ the same angle $\beta$.
   \{As a particular case we have the altitudes, which are nedians at an angle of 90° or 90°-nedians.\}
   And associated with this we have interior nedian-$\beta$ triangle and exterior nedian-$\beta$ triangle.

d) All these notions about nedians in a triangle can be extended to **nedians in a polygon**, and to the formation of corresponding nedian **polygons**.

Then:

13. Let $\Delta ABC$ be a triangle and let $\Delta A_1B_1C_1$ be its **interior nedian triangle of ratio** $r$.
   Then, let’s consider the triangle $\Delta A_2B_2C_2$, which is the interior nedian triangle of order $i$ of triangle $\Delta A_1B_1C_1$.
   And the recursive tunneling process continues in the same way.
Therefore, let’s consider the triangle $\Delta A_nB_nC_n$, which is the interior nedian triangle of ordered $\triangle A_{n-1}B_{n-1}C_{n-1}$.

Same questions b)-e) as in Open Question 1.

14. Similar questions for **exterior nedian triangle of ratio $r$**.

15-16. Similar questions for **interior $\alpha$-nedian triangle** and **exterior $\alpha$-nedian triangle**.

16-17. Similar questions for **interior nedian-$\beta$ triangle** and **exterior nedian-$\beta$ triangle**.

18-23. Similar questions as the above 13-17 for the corresponding **nedian polygons**.

**F) More Open Questions.**

The reader can exercise his or her research on other types of decreasing or increasing tunnels of special triangles (if their construction may work), such as the: extangential triangle, cotangential triangle, antisupplementary triangle, automedial triangle, altimedial triangle, circumpedal triangle, antiparallel triangle, Napoléon triangles, Vecten triangles, Sharygin triangles, Brocard triangles, Smarandache-Pătraşcu triangles (or orthohomological triangles), Carnot triangle, Fuhrmann triangle, Morley triangle, Tiţeica triangle, Lucas triangle, Lionnet triangle, Schroeter triangle, Grebe triangle, etc.

{We don’t present their definitions since the reader can find them in books of *Geometry of Triangle* or in mathematical encyclopedias, see for examples [1] and [6].}

**G) Construction.**

Further research would be to construct similar tunnels of 3-D solids (and, more general, **tunnels of n-D solids** in $\mathbb{R}^n$).

**References:**

1. Cătălin Barbu, *Teoreme fundamentale din geometria triunghiului*, Ed. Unique, Băcău, 2008.
2. Mihai Dicu, The Smarandache-Pătraşcu Theorem of Orthohomological Triangles (to appear), http://fs.gallup.unm.edu//Smarandache-PatrascuOrthogonalToreomTriangles.pdf
3. Ion Pătraşcu, *E-mails to the Author*, February-March 2010.
4. Florentin Smarandache, *Généralisations et généralités*, Ed. Nouvelle, Fès, Morocco, 1984, http://fs.gallup.unm.edu//Generalisations.pdf.
5. Viorel Gh. Vodă, *Surprize în matematica elementară*, Ed. Albatros, Bucharest, 1981.
6. Eric W. Weisstein, *MathWorld -- A Wolfram Web Resource*, http://mathworld.wolfram.com/.

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¹We call two triangles, which are simultaneously orthological and homological, *orthohomological triangles* (or Smarandache-Pătraşcu triangles [2]); for example: if the triangle $\Delta ABC$ is given and $P$ is a point inside it such that its pedal triangle $\Delta A_1B_1C_1$ is homological with $\Delta ABC$, then we say that the triangles $\Delta ABC$ and $\Delta A_1B_1C_1$ are orthohomological.