$K$-theory formulas for orthogonal and symplectic orbit closures

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Abstract

The complex orthogonal and symplectic groups both act on the complete flag variety with finitely many orbits. We study two families of polynomials introduced by Wyser and Yong representing the $K$-theory classes of the closures of these orbits. Our polynomials are analogous to the Grothendieck polynomials representing $K$-classes of Schubert varieties, and we show that like Grothendieck polynomials, they are uniquely characterized among all polynomials representing the relevant classes by a certain stability property. We show that the same polynomials represent the equivariant $K$-classes of symmetric and skew-symmetric analogues of Knutson and Miller’s matrix Schubert varieties. We derive explicit expressions for these polynomials in special cases, including a Pfaffian formula relying on a more general degeneracy locus formula of Anderson. Finally, we show that taking an appropriate limit of our representatives recovers the $K$-theoretic Schur $Q$-functions of Ikeda and Naruse.

1 Introduction

Our results in this paper concern two families of polynomials representing $K$-theory classes of orbit closures in the complete flag variety, which we call orthogonal and symplectic Grothendieck polynomials. For motivation, we start by reviewing the classical story of Grothendieck polynomials, which represent the $K$-theory classes of type A Schubert varieties.

Let $n$ be a positive integer and write $\mathbf{GL}_n = \mathbf{GL}_n(\mathbb{C})$ for the general linear group of invertible $n \times n$ complex matrices. Define $B \subseteq \mathbf{GL}_n$ to be the Borel subgroup of invertible lower triangular matrices.

Suppose $X$ is a smooth complex algebraic variety. Let $K(X)$ denote the Grothendieck group of coherent sheaves on $X$ equipped with a ring structure induced from the tensor product. This the usual $K$-theory ring of $X$.

We write $CK(X)$ for the connective $K$-theory ring of $X$ introduced by Cai [8]. This is a certain graded algebra over the coefficient ring $\mathbb{Z}[\beta]$, which can be interpreted as the connective $K$-theory ring of a point. For any closed equidimensional subscheme $Y \subseteq X$, there is an associated $K$-theory class $[Y]_K \in$
$K(X)$, namely the class of the structure sheaf of $Y$, and an associated connective $K$-theory class $[Y]_{CK} \in CK(X)$.

We define the complete flag variety $Fl_n := B \backslash GL_n$ to be the set of right cosets of $B$ in $GL_n$. The ordinary $K$-theory ring of $Fl_n$ can be realized as

$$K(Fl_n) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I_{\Lambda_n}$$

(1.1)

and the connective $K$-theory ring as

$$CK(Fl_n) \cong \mathbb{Z}[\beta][x_1, x_2, \ldots, x_n]/I_{\Lambda_n}[\beta].$$

(1.2)

where $\beta, x_1, x_2, \ldots$ are commuting indeterminates and $I_{\Lambda_n} \subseteq \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is the ideal generated by symmetric polynomials without constant term in the variables $x_1, x_2, \ldots, x_n$; see [2, 3].

Let $S_n$ denote the symmetric group of permutations of $[n] := \{1, 2, \ldots, n\}$ and identify $w \in S_n$ with the permutation matrix in $GL_n$ with 1 in position $(i, w(i))$. It follows by elementary linear algebra that the opposite Borel subgroup $B^+$ of upper triangular matrices in $GL_n$ acts on $Fl_n$ on the right with $n! = |S_n|$ distinct orbits. The orbit closures $X_w := BwB^+$ for $w \in S_n$ are the Schubert varieties in $Fl_n$ and one is interested in describing the classes $[X_w] \in CK(Fl_n)$.

For $v \in S_n$ and $w \in S_m$, let $v \times w \in S_{n+m}$ be the permutation with $i \mapsto v(i)$ for $i \in [n]$ and $n + i \mapsto n + w(i)$ for $i \in [m]$. We also write $w^m$ for the $m$-fold product $w \times w \times \cdots \times w$, so that $1^m$ is the identity in $S_m$. Many different polynomials correspond to each class $[X_w] \in CK(Fl_n)$ under the isomorphism (1.2), but if one also requires a certain compatibility condition with respect to the maps $w \mapsto w \times 1^m$, then there is a unique family of such polynomials:

**Theorem 1.1.** There are unique polynomials $\mathfrak{G}_w \in \mathbb{Z}[\beta][x_1, x_2, \ldots]$ for $n \in \mathbb{P}$ and $w \in S_n$ such that $\mathfrak{G}_w + I_{\Lambda_n}[\beta] = [X_w] \in CK(Fl_n)$ and $\mathfrak{G}_w = \mathfrak{G}_{w \times 1^m}$.

This statement combines several known results reviewed in Section 2.2. The polynomials $\mathfrak{G}_w$ are the (generalized) Grothendieck polynomials introduced in [13]. The Schubert polynomials (see [33] Chapter 2) are the special case of these functions with $\beta = 0$. Setting $\beta = -1$ and replacing each variable $x_i$ by $1 - x_i$, alternatively, recovers Lascoux and Schützenberger’s original definition of Grothendieck polynomials in [30, 31].

It is a remarkable observation of Fomin and Kirillov [13] that the sequence of polynomials $\mathfrak{G}_{1^n \times \underline{w}}$ converges as $m \to \infty$ to a symmetric function:

**Theorem 1.2** ([13] Theorem 2.3). There are unique symmetric functions $G_w$ for each $n \in \mathbb{P}$ and $w \in S_n$ such that $G_w(x_1, \ldots, x_n) = \mathfrak{G}_{1^n \times \underline{w}}(x_1, \ldots, x_n)$ for all $N \geq n$.

Following established practice, we refer to the symmetric functions $G_w$ as stable Grothendieck polynomials. These power series have a number of other interesting properties and are studied in [6, 7, 13].
The preceding results have interesting counterparts for the orbit closures of the orthogonal and symplectic groups acting on $\mathbb{F}l_n$. These actions are particularly natural to consider: they both have finitely many orbits, and correspond to two of the three families of type A symmetric varieties [42]. (The third family comes from the action of $GL_p \times GL_{n-p}$ on $\mathbb{F}l_n$; $K$-theory representatives for the relevant orbit closures are studied in [45].)

Fix nondegenerate symmetric and skew-symmetric bilinear forms on $\mathbb{C}^n$. We define the orthogonal group $O_n$ and the symplectic group $Sp_n$ as the subgroups of $GL_n$ preserving these forms. Note that $n$ must be even in the skew-symmetric case. As explained in [42, §10], the $O_n$-orbits on $\mathbb{F}l_n$ are in bijection with the set of involutions $I_n := \{w \in S_n : w = w^{-1}\}$ while the $Sp_n$-orbits are in bijection with the set of fixed-point-free involutions

$$I_n^{FPF} := \{z \in I_n : z(i) \neq i \text{ for all } i \in [n]\}.$$ 

We write $\{X^O_z : z \in I_n\}$ and $\{X^{Sp}_z : z \in I_n^{FPF}\}$ for the respective families of $K_n$-orbit closures, where $K$ is one of the symbols $O$ or $Sp$; see [23] for explicit descriptions of these varieties.

We can now state symplectic and orthogonal analogues of Theorem 1.1:

**Theorem 1.3** (Wyser and Yong [46]). There are unique polynomials

$$G^{Sp}_z(\beta) \in \mathbb{Z}[\beta][x_1, x_2, \ldots] \quad \text{for } n \in 2\mathbb{P} \text{ and } z \in I_n^{FPF}$$

such that $G^{Sp}_z + I\Lambda_n[\beta] = [X^{Sp}_z] \in CK(\mathbb{F}l_n)$ and $G^{Sp}_z = G^O_z \times 2^{21}$.

The derivation of this statement from the results in [46], which is not entirely trivial, is explained in Section 3.3. The following theorem is new:

**Theorem 1.4.** There are unique polynomials

$$G^O_z(\beta) \in \mathbb{Z}[\beta][x_1, x_2, \ldots] \quad \text{for } n \in \mathbb{P} \text{ and } z \in I_n$$

such that $G^O_z + I\Lambda_n[\beta] = [X^O_z] \in CK(\mathbb{F}l_n)$ and $G^O_z = G^O_z \times 1$.

We refer to $G^{Sp}_z$ and $G^O_z$ as symplectic and orthogonal Grothendieck polynomials. Setting $\beta = 0$ transforms these functions to the (fixed-point-free) involution Schubert polynomials $G^{FPF}_z$ and $G_z$ studied in [17, 19, 21, 46]. The latter represent the cohomology classes of the orbit closures $X^{Sp}_z$ and $X^O_z$.

Wyser and Yong [46] give a recursive method for computing $G^{Sp}_z$ involving divided difference operators; see Theorem 3.10. By contrast, no simple algebraic formulas for computing $G^O_z$ are known for general $z \in I_n$. This notably differs from the situation for the involution Schubert polynomials $G_z$, which can again be characterized using divided differences [46].

We prove Theorems 1.3 and 1.4 in a uniform way in Section 3.1 by adapting an idea of Knutson and Miller [23]. The $B^+$-orbits on $B \setminus GL_n = \mathbb{F}l_n$ are naturally
in bijection with the $B \times B^+$-orbits on $GL_n$. The closures $M_w$ of the latter orbits in the space $Mat_n$ of $n \times n$ matrices are known as matrix Schubert varieties.

Let $T$ denote the torus of invertible diagonal matrices in $GL_n$. Knutson and Miller prove that the class $[M_w]_T$ in the $T$-equivariant $K$-theory ring $K_T(Mat_n) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is the polynomial obtained from $\Phi_w$ by setting $\beta = -1$. Using this fact, one can show that $\Phi_w + I\Lambda_n[\beta] = [X_w] \in CK(Fl_n)$.

The $K_n$-orbits on $Fl_n$ are in bijection with the $B$-orbits on $GL_n/K_n$. Embedding $GL_n/K_n$ as an open dense subset of the space of symmetric matrices $Mat^O_n$, or skew-symmetric matrices $Mat^{Sp}_n$, as appropriate, and taking the closures of these $B$-orbits gives a family of (skew-)symmetric matrix Schubert varieties $MX^O_n$; see Definition 2.15.

The following is a consequence of our results in Section 2.4.

**Theorem 1.5.** Fix $K \in \{O, Sp\}$ and let $z \in I_n$. Assume $n$ is even and $z \in I^{FPP}_n$ if $K = Sp$. The $T$-equivariant class $[MX^O_n]_T \in K_T(Mat^K_n) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is then the polynomial obtained from $\Phi^K_z$ by setting $\beta = -1$.

We now turn to analogues of Theorem 1.2. The next result follows from Theorem 1.2 and Corollary 1.6.

**Theorem 1.6.** There are unique symmetric functions $GP^Sp_z$ for each $n \in 2\mathbb{P}$ and $z \in I^{FPP}_n$ such that $GP^Sp_z(x_1, \ldots, x_n) = \Phi^Sp_{(21)}(x_1, \ldots, x_n)$ for all $N \geq n$.

In the orthogonal case, we have only succeeded in proving a partial analogue of Theorem 1.2. A permutation is vexillary if it avoids the pattern 2143. The following is a corollary of Theorem 1.11.

**Theorem 1.7.** There are unique symmetric functions $GQ^O_z$ for each $n \in \mathbb{P}$ and vexillary $z \in I_n$ such that $GQ^O_z(x_1, \ldots, x_n) = \Phi^O_{(1)}(x_1, \ldots, x_n)$ for all $N \geq n$.

Our proof of Theorem 1.7 relies on an explicit Pfaffian formula for $\Phi^O_z$ when $z$ is vexillary, which we derive by realizing $X^O_z$ as a type C Grassmannian degeneracy locus and applying a formula of Anderson [2] for the $K$-theory classes of such loci.

The main result of [35] shows that $GP^Sp_z$ is an $\mathbb{N}[\beta]$-linear combination of the $K$-theoretic Schur $P$-functions $GP_\lambda$ of Ikeda and Naruse [26]. When $z$ is vexillary, we prove that $GQ^O_z$ is likewise a $K$-theoretic Schur $Q$-function $GQ_\lambda$, also introduced in [26]. The degeneracy locus formulas in [2] are very complicated and we find it amazing that the expressions we derive for $GQ^O_z$ in the vexillary case coincide exactly with symmetric functions already considered in the literature.

We expect that Theorem 1.7 holds for all involutions $z \in I_n$, and that the resulting power series $GQ^O_z$ are $\mathbb{N}[\beta]$-linear combinations of $GQ_\lambda$’s. Section 5 discusses several other related open problems.

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2 Preliminaries on connective $K$-theory

This section provides an expository overview of connective $K$-theory and then describes a general method of constructing polynomial $K$-theory representatives for orbit closures in the complete flag variety.

Throughout, the symbols $\beta$, $a_1$, $a_2$, $\ldots$, $x_1$, $x_2$, $\ldots$ denote commuting indeterminates. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{P} = \{1, 2, 3, \ldots\}$ for the sets of nonnegative and positive integers, and define \([n] := \{i \in \mathbb{P} : i \leq n\}\) for $n \in \mathbb{N}$. Given $n \in \mathbb{P}$, let $S_n$ denote the usual symmetric group of bijections $[n] \to [n]$. The length of a permutation $w$ is $\ell(w) := |\{(i, j) : i < j \text{ and } w(i) > w(j)\}|$.

2.1 Connective $K$-theory

Let $X$ be a smooth complex variety. Recall that the ordinary $K$-theory ring of $X$ is the Grothendieck group $K(X)$ of coherent sheaves on $X$, equipped with a ring structure induced by the tensor product. The structure sheaf of any closed subscheme $Z \subseteq X$ has a class in $K(X)$ which we denote by $[Z]_K$.

Let $K(X, c)$ be the Grothendieck group of coherent sheaves whose support has codimension at least $c \in \mathbb{Z}$, so that $K(X, c) = K(X)$ whenever $c \leq 0$ and $K(X, c) = 0$ whenever $c > \dim(X)$. The tensor product again induces a product on $K(X, c)$. The next definition originates in [8] but our notation follows [2, 24].

**Definition 2.1** (See [2, Appendix A] or [24 §2.1]). The connective $K$-theory ring of $X$ is the graded $\mathbb{Z}[\beta]$-algebra

$$CK(X) := \bigoplus_{c \in \mathbb{Z}} CK^c(X)$$

in which $CK^c(X)$ is the image of the natural map $K(X, c) \to K(X, c-1)$, so that $CK^c(X) = K(X)$ whenever $c \leq 0$. The maps $K(X, c) \to K(X, c-1)$ induce maps $CK^c(X) \to CK^{c-1}(X)$, and the $\mathbb{Z}[\beta]$-algebra structure on $CK(X)$ is defined by letting $CK^c(X) \to CK^{c-1}(X)$ be multiplication by $-\beta$.

**Example 2.2.** A coherent sheaf on $X = pt$ is a map $pt \to \{V\}$ for some finite-dimensional complex vector space $V$. The Grothendieck group $K(pt) = \mathbb{Z}$ is generated by the sheaf $pt \to \{\mathbb{C}\}$. All sheaves on $pt$ have codimension zero so $K(pt, c) = \mathbb{Z}$ for $c \leq 0$ and $K(pt, c) = 0$ for $c > 0$. Identifying $CK^c(pt) \cong \mathbb{Z}$ for $c \leq 0$ with the free abelian group $\mathbb{Z}$-span\{−$\beta$−\} lets us write $CK(pt) = \mathbb{Z}[\beta]$.

Suppose $Z \subseteq X$ is a closed subscheme; its structure sheaf $\mathcal{O}_Z$ has support $Z$, so there is a corresponding class in $K(X, \text{codim}(Z))$, whose image under the natural map $K(X, \text{codim}(Z)) \to K(X)$ is $[Z]_K$. The connective $K$-class of $Z$ is the image of the former class under the natural map $K(X, \text{codim}(Z)) \to CK^\text{codim}(Z)(X)$, which we denote by $[Z]_{CK}$. We drop the subscripts from $[Z]_K$ or $[Z]_{CK}$ when these are clear from context. These classes are related as follows:
**Proposition 2.3.** There is a \( \mathbb{Z}[\beta] \)-algebra morphism

\[
\psi : K(X)[\beta, \beta^{-1}] \rightarrow CK(X)[\beta^{-1}]
\]

with \( \psi([Z]_K) = (-\beta)^{\operatorname{codim}(Z)} [Z]_{CK} \) for any closed subscheme \( Z \subseteq X \).

**Proof.** The map \( \psi \) is induced from the identity map \( K(X) \rightarrow CK^0(X) \). By definition, the image of \([Z]_{CK} \in CK^{\operatorname{codim}(Z)}(X)\) in \( CK^0(X) \) is \((-\beta)^{\operatorname{codim}(Z)}[Z]_{CK}\), and this element of \( CK^0(X) = K(X) \) is just \([Z]_K\).

\[\square\]

### 2.2 Grothendieck polynomials for Schubert varieties

Fix a positive integer \( n \). As in the introduction, write \( GL_n \) for the complex general linear group and \( B \) for the Borel subgroup of lower triangular matrices in \( GL_n \). We are primarily interested in the preceding definitions applied to the complete flag variety \( Fl_n := B \backslash GL_n \). For this choice of \( X \), one can realize \( K(X) \) and \( CK(X) \) as quotients of a polynomial ring.

For each \( i \in [n] \), there is a natural line bundle \( L_i \) on \( Fl_n \), whose fiber over an orbit \( Bg \in Fl_n \) is the quotient \( F_i/F_{i-1} \), where \( F_i \) is the subspace of \( \mathbb{C}^n \) spanned by the first \( i \) rows of \( g \in GL_n \). Let \( I\Lambda_n \) denote the ideal in \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \) generated by the symmetric polynomials without constant term.

**Theorem 2.4 ([22, Theorem 2.6]).** There are isomorphisms

\[
K(Fl_n) \xrightarrow{\sim} \mathbb{Z}[x_1, \ldots, x_n]/I\Lambda_n \quad \text{and} \quad CK(Fl_n) \xrightarrow{\sim} \mathbb{Z}[\beta][x_1, \ldots, x_n]/I\Lambda_n[\beta]
\]

mapping the first Chern class \( c_1(L_i^\vee) \) of the line bundle dual to \( L_i \) to \( x_i \).

From now on, we identify the rings \( K(Fl_n) = \mathbb{Z}[x_1, x_2, \ldots, x_n]/I\Lambda_n \) and \( CK(Fl_n) = \mathbb{Z}[\beta][x_1, x_2, \ldots, x_n]/I\Lambda_n[\beta] \) via the preceding theorem. For a closed subscheme \( Z \subseteq Fl_n \), it is then natural to ask for a polynomial whose image in these quotient rings gives \([Z]_K \) or \([Z]_{CK}\).

This question is well-understood for the Schubert varieties \( X_w \). Recall that these varieties are the closures of the double cosets \( BwB^+ \subseteq Fl_n \), where \( B^+ \subseteq GL_n \) is the subgroup of upper triangular matrices and \( w \) ranges over the symmetric group \( S_n \), viewed as the subgroup of permutation matrices in \( GL_n \).

Let \( s_i = (i, i+1) \in S_n \) for each \( i \in [n-1] \). Given \( f \in \mathbb{Z}[\beta][x_1, \ldots, x_n] \), let \( s_i f \) be the polynomial formed from \( f \) by interchanging \( x_i \) and \( x_{i+1} \), and define

\[
\partial_i f := \frac{f-s_if}{x_i-x_{i+1}} \quad \text{and} \quad \partial_i^{(3)} f := \partial_i((1+\beta x_{i+1})f) = -\beta f + (1+\beta x_{i+1})\partial_i f. \quad (2.1)
\]

We refer to \( \partial_i \) and \( \partial_i^{(3)} \) as divided difference operators. Write \( w_1w_2\cdots w_n \) for the permutation in \( S_n \) with the formula \( i \mapsto w_i \).

**Theorem-Definition 2.5 (See [13]).** The Grothendieck polynomials \( \{G_w\}_{w\in S_n} \) are the unique family in \( \mathbb{Z}[\beta][x_1, \ldots, x_n] \) with \( G_{\epsilon_n} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1} \) and \( \partial_i^{(3)} G_w = G_{ws_i} \) for all \( w \in S_n \) and \( i \in [n-1] \) such that \( w(i) > w(i+1) \).
It follows from the last property that $\partial_i^{(\beta)} \mathcal{G}_w = -\beta \mathcal{G}_w$ if $w(i) < w(i+1)$. It is also not hard to check that $\mathcal{G}_w = \mathcal{G}_{w \times 1}$ for all $w \in S_n$, where $w \times 1$ denotes the permutation in $S_{n+1}$ with $i \mapsto w(i)$ for $i \in [n]$ and $n + 1 \mapsto n + 1$. Less obviously, one always has $\mathcal{G}_w \in \mathbb{N}[\beta][x_1, x_2, \ldots, x_n]$ [13, Theorem 2.3].

**Example 2.6.** The Grothendieck polynomials for $w \in S_3$ are

- $\mathcal{G}_{123} = 1$,
- $\mathcal{G}_{132} = x_1 + x_2 + \beta x_1 x_2$,
- $\mathcal{G}_{312} = x_1^2$,
- $\mathcal{G}_{213} = x_1$,
- $\mathcal{G}_{231} = x_1 x_2$,
- $\mathcal{G}_{321} = x_1^2 x_2$.

Work of Hudson, extending earlier results of Fulton and Lascoux, shows that the polynomials $\mathcal{G}_w$ represent the Schubert classes $[X_w]$ in connective $K$-theory. Specifically, the following is the special case of [23, Theorem 1.2] obtained by taking $V$ to be a trivial vector bundle of rank $n$ over $X = pt$:

**Theorem 2.7** ([23, Theorem 1.2]). For each $w \in S_n$, it holds that

$$\mathcal{G}_w + I\Lambda_n[\beta] = [X_w] \in CK(Fl_n).$$

We typically suppress the parameter $\beta$ in our notation, but for the moment write $\mathcal{G}_w^{(\beta)} = \mathcal{G}_w$ for $w \in S_n$. The Schubert polynomial $\mathcal{G}_w$ of a permutation $w \in S_n$ (see [23, Chapter 2]) is then $\mathcal{G}_w^{(0)}$. It follows that $\{\mathcal{G}_w\}_{w \in S_n}$ are linearly independent by [33, Proposition 2.5.3].

Some references use the term “Grothendieck polynomial” to refer to the polynomials $\mathcal{G}_w^{(-1)}$. One loses no generality in setting $\beta = -1$ since one can show by downward induction on permutation length that

$$(-\beta)^{t(w)} \mathcal{G}_w^{(-1)} = \mathcal{G}_w^{(-1)}(-\beta x_1, -\beta x_2, \ldots, -\beta x_n). \quad (2.2)$$

This lets us translate any formulas in $\mathcal{G}_w^{(-1)}$ to formulas in $\mathcal{G}_w = \mathcal{G}_w^{(\beta)}$. The specialization $\beta = -1$ is natural since it corresponds to ordinary $K$-theory:

**Theorem 2.8** ([13, Theorem 3]). For each $w \in S_n$, it holds that

$$\mathcal{G}_w^{(-1)} + I\Lambda_n = [X_w] \in K(Fl_n).$$

We can now describe the map in Proposition 2.3 for $X = Fl_n$.

**Corollary 2.9.** If $X = Fl_n$, then the map $\psi : K(Fl_n)[\beta, \beta^{-1}] \to CK(Fl_n)[\beta^{-1}]$ in Proposition 2.3 is the ring homomorphism sending $x_i \mapsto -\beta x_i$ for $i \in [n]$.

**Proof.** Since $\mathrm{codim}(X_w) = \ell(w)$, it follows from Theorems 2.7 and 2.8 that $\psi(\mathcal{G}_w^{(-1)} + I\Lambda_n) = (-\beta)^{t(w)} \mathcal{G}_w^{(\beta)} + I\Lambda_n[\beta]$. By (2.2), this agrees with the ring homomorphism $\mathbb{Z}[x_1, \ldots, x_n]/I\Lambda_n \to \mathbb{Z}[\beta][x_1, \ldots, x_n]/I\Lambda_n[\beta]$ sending $x_i \mapsto -\beta x_i$ for $i \in [n]$. As $\{\mathcal{G}_w^{(-1)} + I\Lambda_n : w \in S_n\}$ is a basis for $K(Fl_n)$ by [33, Proposition 2.5.3 and Corollary 2.5.6], we conclude that $\psi$ is equal to the latter map. \(\square\)
2.3 Matrix Schubert varieties

For the remainder of this section, K denotes one of the symbols O or Sp. Fix \( n \in \mathbb{P} \) and write \( I_n \) and \( I_n^{\text{PF}} \) for the respective sets of involutions and fixed-point-free involutions in the finite symmetric group \( S_n \). If \( n \) is odd then \( I_n^{\text{PF}} = \emptyset \).

If \( V_1, V_2 \) are complex vector spaces and \( \alpha : V_1 \times V_2 \to \mathbb{C} \) is a bilinear form, then we let \( \text{rank}(\alpha) \) denote the rank of the map \( V_2 \to V_1^* \) given by \( v \mapsto \alpha(\cdot, v) \). Let \( \alpha_n^O \) be a fixed symmetric nondegenerate bilinear form on \( \mathbb{C}^n \). When \( n \) is even, let \( \alpha_n^S \) be a fixed skew-symmetric nondegenerate bilinear form on \( \mathbb{C}^n \).

Define \( O_n \) to be the subgroup of \( \text{GL}_n \) preserving \( \alpha_n^O \) and \( \text{Sp}_n \) the subgroup preserving \( \alpha_n^S \). Write \( A_{ij}[j] \) for the upper-left \( i \times j \) corner of a matrix \( A \).

Given \( E = Bg \in \mathcal{F}_n \) and \( i \in [n] \), define \( E_i \subseteq \mathbb{C}^n \) to be the subspace spanned by the first \( i \) rows of \( g \in \text{GL}_n \); these spaces do not depend on the choice of \( g \).

**Definition 2.10.** Given \( K \in \{O, Sp\} \) and \( z \in I_n \), let

\[
X_z^K := \{ E \in \mathcal{F}_n : \text{rank}(\alpha_n^K|_{E_i \times E_j}) < \text{rank}(z_{i,j}) \text{ for } i, j \in [n] \},
\]

where we identify \( z \) with its permutation matrix.

Each \( X_z^K \) is a closed subvariety of \( \mathcal{F}_n \). The correspondence \( z \mapsto X_z^K \) is a bijection from \( I_n \) to the set of closures of the \( O_n \)-orbits on \( \mathcal{F}_n \); when \( n \) is even, \( z \mapsto X_z^S \) is likewise a bijection from \( I_n^{\text{PF}} \) to the set of closures of the \( \text{Sp}_n \)-orbits on \( \mathcal{F}_n \) [44]. Although we are primarily interested in \( X_z^S \) in the case when \( z \) is fixed-point-free, we have defined \( X_z^K \) for any involution \( z \in I_n \) and this flexibility will occasionally be convenient.

Many of the rank conditions in Definition 2.10 are redundant. The essential rank conditions can be read off from the following diagrams.

**Definition 2.11.** Let \( z \in I_n \). The **orthogonal Rothe diagram** of \( z \) is

\[
D^O(z) := \{(i, z(j)) : (i, j) \in [n] \times [n] \text{ and } z(i) > z(j) \leq i < j\}.
\]

The **symplectic Rothe diagram** of \( z \) is

\[
D^S(z) := \{(i, z(j)) : (i, j) \in [n] \times [n] \text{ and } z(i) > z(j) < i < j\}.
\]

The sets \( D^O(z) \) and \( D^S(z) \) contain the positions below the main diagonal in the **Rothe diagram** \( D(z) := \{(i, z(j)) : (i, j) \in [n] \times [n], z(i) > z(j), \text{ and } i < j\} \). For \( K \in \{O, Sp\} \) and \( z \in I_n \), then

\[
X_z^K = \{ E \in \mathcal{F}_n : \text{rank}(\alpha_n^K|_{E_i \times E_j}) \leq \text{rank}(z_{i,j}) \text{ for } (i, j) \in \text{Ess}(D^O(z)) \}.
\]

Moreover, if \( n \) is even and \( z \in I_n^{\text{PF}} \) then

\[
X_z^S = \{ E \in \mathcal{F}_n : \text{rank}(\alpha_n^S|_{E_i \times E_j}) \leq \text{rank}(z_{i,j}) \text{ for } (i, j) \in \text{Ess}(D^S(z)) \}.
\]
Example 2.14. Let \( z = (1, 3) \in I_3 \). Then
\[
D^O(z) = \{(1, 1), (2, 1)\} = \begin{bmatrix}
\circ & \times \\
\circ & \circ & \times \\
\times & \times & \times 
\end{bmatrix}
\]
where elements of \( D^O(z) \) are drawn with \( \circ \), points \((i, z(i))\) with \( \times \), and the diagram is shown in matrix coordinates with \((1, \left[0\right])\) at the upper left. We have \( \operatorname{Ess}(D^O(z)) = \{(2, 1)\} \) and \( \operatorname{rank}(z) = 0 \), so
\[
X^O_z = \{ E \in \mathcal{F}_3 : \operatorname{rank}(z(E_1, E_2)) \leq 0 \} = \{ E \in \mathcal{F}_3 : E_2 \subseteq E_1^\perp \}.
\]

Let \( \operatorname{Mat}_n^O \) (respectively, \( \operatorname{Mat}_n^{Sp} \)) be the set of complex \( n \times n \) matrices that are symmetric (respectively, skew-symmetric). The space \( \operatorname{Mat}_n^K \) contains a family of varieties closely related to \( X^K_z \).

Definition 2.15. Given \( K \in \{O, Sp\} \) and \( z \in I_n \), let
\[
MX^K_z := \left\{ A \in \operatorname{Mat}_n^K : \operatorname{rank}(A_{i,j}) \leq \operatorname{rank}(z_{i,j}) \text{ for } i, j \in [n] \right\}.
\]

We call the closed subvariety \( MX^O_z \) (respectively, \( MX^{Sp}_z \)) a symmetric matrix Schubert variety (respectively, skew-symmetric matrix Schubert variety). If one allows arbitrary \( z \in S_n \) and arbitrary matrices in Definition 2.15, then one recovers Knutson and Miller’s notion of a matrix Schubert variety from [28].

The variety \( MX^K_z \) is also an orbit closure, but now for the Borel subgroup \( B \subseteq \operatorname{GL}_n \), which acts on \( A \in \operatorname{Mat}_n^K \) by \( b : A \mapsto bAb^T \). The maps \( z \mapsto MX^O_z \) and \( z \mapsto MX^{Sp}_z \) are bijections from \( I_n \) and \( I_n^{FPF} \) to the closures of the \( B \)-orbits in \( \operatorname{Mat}_n^O \) and \( \operatorname{Mat}_n^{Sp} \), respectively; see [8, 9]. There is an analogue of Proposition 2.16.

Proposition 2.16. Let \( K \in \{O, Sp\} \) and \( z \in I_n \). Then
\[
MX^K_z = \left\{ A \in \operatorname{Mat}_n^K : \operatorname{rank}(A_{i,j}) \leq \operatorname{rank}(z_{i,j}) \text{ for } (i, j) \in \operatorname{Ess}(D^O(z)) \right\}.
\]
Moreover, if \( n \) is even and \( z \in I_n^{FPF} \) then
\[
MX^{Sp}_z = \left\{ A \in \operatorname{Mat}_n^{Sp} : \operatorname{rank}(A_{i,j}) \leq \operatorname{rank}(z_{i,j}) \text{ for } (i, j) \in \operatorname{Ess}(D^{Sp}(z)) \right\}.
\]

Proof. One can almost repeat the proof of [17] Proposition 3.16 verbatim; the argument in [17] goes through after replacing “\( y \)” by “\( z \)” and redefining “\( C_{ij} \)” to be the set of symmetric (when \( K = O \)) or skew-symmetric (when \( K = Sp \)) \( n \times n \) matrices \( A \) with \( \operatorname{rank}(A_{i,j}) \leq \operatorname{rank}(z_{i,j}) \). \qed

2.4 Grothendieck polynomials for orbit closures

The orthogonal and symplectic matrix Schubert varieties have canonical polynomial representatives in equivariant \( K \)-theory. Here, we use these polynomials to
give a uniform definition of the Grothendieck polynomials $\mathfrak{G}_G^O$ and $\mathfrak{G}_G^Sp$ described in the introduction.

Suppose $G$ is a linear algebraic group acting on a smooth complex variety $X$. The $G$-equivariant $K$-theory ring $K_G(X)$ is the Grothendieck group of $G$-equivariant vector bundles on $X$ with tensor product as multiplication, or equivalently the Grothendieck group of $G$-equivariant coherent sheaves on $X$. If $Z \subseteq X$ is a $G$-invariant subscheme, then we write $[Z] \in K_G(X)$ for the class of its structure sheaf.

A $G$-equivariant vector bundle over a point is just a representation of $G$, so $K_G(pt)$ is the Grothendieck ring $R(G)$ of finite-dimensional complex rational representations of $G$. If $T \cong (\mathbb{C}^*)^n$ is a torus then $K_T(pt)$ can be identified with the ring $\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}]$; the one-dimensional representation on which $(t_1, t_2, \ldots, t_n) \in T$ acts as multiplication by $t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ has class $a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}$, and every $T$-representation is a direct sum of such one-dimensional representations.

We summarize a few other properties we will need from [11, §5.2]:

- A $G$-equivariant map $f : X \to Y$ between smooth complex varieties defines a pullback $f^* : K_G(Y) \to K_G(X)$, and this assignment is functorial. The pullback of $X \to pt$ makes the ring $K_G(X)$ into an algebra over $K_G(pt) \cong R(G)$. The pullbacks $f^*$ are $R(G)$-algebra homomorphisms.

- If $f : X \to Y$ is a flat morphism (e.g., the projection of a fiber bundle or inclusion of an open subset), then $f^*([Z]) = [f^{-1}(Z)]$ for any $G$-invariant subscheme $Z$. Here, $f^{-1}(Z)$ is the scheme-theoretic inverse image, but if $Z$ and the fibers of $f$ are reduced, then the flatness of $f$ implies $f^{-1}(Z)$ reduced [13 Proposition 11.3.13]. This will always be the case for us, so we can take $f^{-1}(Z)$ to be the set-theoretic inverse image.

- If $V$ is a finite-dimensional linear representation of $G$, then there are isomorphisms $K_G(V) \cong K_G(pt) \cong R(G)$ [11 Corollary 5.4.21].

- Given a group homomorphism $\phi : H \to G$, there is a ring homomorphism $K_G(X) \to K_H(X)$ sending $[Z]$ to $[Z]$, since one can view a $G$-equivariant vector bundle as $H$-equivariant via $\phi$. In particular, taking $H$ to be the trivial subgroup of $G$, there is such a map $K_G(X) \to K(X)$.

- If $G$ acts freely on $X$, then the pullback of the quotient $X \to X/G$ defines an isomorphism $K(X/G) \cong K_G(X)$.

For each symbol $K \in \{O, Sp\}$, fix an $n \times n$ matrix $\Omega^K_n$ with $\alpha^K_n(v, w) = v^T \Omega^K_n w$ for all $v, w \in \mathbb{C}^n$. Define $T$ to be the torus of invertible diagonal matrices in $\text{GL}_n$. Each $t \in T$ acts on $\text{GL}_n$ by left multiplication and on $A \in \text{Mat}_n^K$ by $t : A \mapsto t A t$. Let $\sigma^K_n : \text{GL}_n \to \text{Mat}_n^K$ be the $T$-equivariant map with $\sigma^K_n(g) := g \Omega^K_n g^T$ and write $(\sigma^K_n)^* : K_T(\text{Mat}_n^K) \to K_T(\text{GL}_n)$ for its pullback.
If \( S \subseteq \text{Fl}_n = B^1 \text{GL}_n \), then we let \( B \cdot S := \{ g \in \text{GL}_n \mid Bg \in S \} \). The pullback of the quotient \( \text{GL}_n \to T \backslash \text{GL}_n \) is an isomorphism \( K(T \backslash \text{GL}_n) \xrightarrow{\sim} K_T(\text{GL}_n) \) since \( T \) acts freely on \( \text{GL}_n \). The forgetful map \( K_B(\text{GL}_n) \to K_T(\text{GL}_n) \) is also an isomorphism \[ \text{II} \, 5.2.18 \]. Composing these maps gives an isomorphism

\[
K(\text{Fl}_n) = K(B \backslash \text{GL}_n) \xrightarrow{\sim} K_T(\text{GL}_n)
\]

sending \( [Z] \) to \( [B \cdot Z] \). Let \( \phi : K_T(\text{GL}_n) \xrightarrow{\sim} K(\text{Fl}_n) \) be the inverse of this map.

**Theorem 2.17.** Choose a symbol \( K \in \{ O, \text{Sp} \} \) and assume \( n \) is even if \( K = \text{Sp} \). The composition \( K_T(\text{Mat}_n^{K}) \xrightarrow{(\sigma_n^K)^*} K_T(\text{GL}_n) \xrightarrow{\phi} K(\text{Fl}_n) \) maps \( [MX_z^K] \mapsto [X_z^K] \) for each \( z \in I_n \).

**Proof.** We just need to show that \( (\sigma_n^K)^* \) maps \( [MX_z^K] \mapsto [B \cdot X_z^K] \) for any \( z \in I_n \). Write \( K_n \) for the group \( O_n \) or \( \text{Sp}_n \) corresponding to the symbol \( K \). The map \( \sigma_n^K \) factors as the quotient map \( \text{GL}_n \to \text{GL}_n/K_n \) followed by the map \( \text{GL}_n/K_n \to \text{Mat}_n^K \) sending \( gK_n \to g\text{GL}_n g^T \), which is an isomorphism from \( \text{GL}_n/K_n \) onto the open subset of invertible matrices in \( \text{Mat}_n^K \). This implies that \( \sigma_n^K \) is a flat morphism, because it is the composition of two flat morphisms: the projection of a fiber bundle and the inclusion of an open subset.

It now suffices to show that \( (\sigma_n^K)^{-1}(MX_z^K) = B \cdot X_z^K \). Indeed, we have \( g \in B \cdot X_z^K \) if and only if the rows \( g_1, g_2, \ldots, g_n \) of \( g \) are such that the matrix \( A = [a_{ij}(g_p, g_q)]_{p,q \in [n]} \) has rank \( (A_{ij}) \leq \text{rank}(z_{ij}) \) for any \( i, j \in [n] \). But \( A = \sigma_n^K(g) \), so this condition is equivalent to \( \sigma_n^K(g) \in MX_z^K \). \( \square \)

One way to realize the isomorphism \( K(\text{Fl}_n) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I\Lambda_n \) in Theorem 2.4 is as follows. Let \( \text{Mat}_n \) denote the algebra of complex \( n \times n \) matrices. Since \( \text{Mat}_n \) is a finite-dimensional representation of \( T \) under the action \( t : A \mapsto tAt \), the equivariant \( K \)-theory ring \( K_T(\text{Mat}_n) \) is the representation ring \( R(T) \cong \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}] \), and the following diagram commutes:

\[
\begin{array}{ccc}
K_T(\text{Mat}_n) & \xrightarrow{\sim} & \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}] \\
\downarrow \iota^* & & \downarrow \\
K(\text{Fl}_n) \cong K_T(\text{GL}_n) & \xrightarrow{\sim} & \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}] \\
& \text{(in } d(a_1, a_2, \ldots, a_n) - \binom{n}{d} : d \in [n]) & 
\end{array}
\]

Here, the vertical map on the left is the pullback of the inclusion \( t : \text{GL}_n \hookrightarrow \text{Mat}_n \). Sending \( a_i \mapsto 1 - x_i \) gives an isomorphism from the ring in the lower right to \( \mathbb{Z}[x_1, x_2, \ldots, x_n]/I\Lambda_n \). This change of variables reflects a general relationship between \( K(X) \) and the Chow ring of \( X \).

Now suppose \( Y \subseteq \text{Mat}_n \) and \( Z \subseteq \text{Fl}_n \) are closed subschemes such that \( \iota^*[Y] := [\iota^{-1}(Y)] = [Z] \in K(\text{Fl}_n) \). The class \( [Y] \in K_T(\text{Mat}_n) \) may be canonically identified with a Laurent polynomial in \( \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}] \) via the diagram above, and it can be shown that this element is actually a polynomial in \( a_1, \ldots, a_n \) \[ \text{II} \, 6.6 \]. After applying the change of variables \( a_i \mapsto 1 - x_i \), this
polynomial becomes a representative for \([Z]\) in the quotient \(\mathbb{Z}[x_1, \ldots, x_n]/I\Lambda_n \cong K(\mathcal{F}_n)\).

If \(V\) is any finite-dimensional linear representation of \(T\) and \(\sigma : \text{Mat}_n \to V\) is a \(T\)-equivariant map, then composing the pullback \(\sigma^* : K_T(V) \to K_T(\text{Mat}_n)\) with the isomorphism \(K_T(\text{Mat}_n) \cong R(T)\) coincides with the canonical isomorphism \(K_T(V) \cong R(T)\) described by [11 Corollary 5.4.21]. Therefore, taking \(\sigma\) to be the map \(\text{Mat}_n \to \text{Mat}_K^n\) with \(g \mapsto g\Omega^K_n g^T\), we can repeat everything in the previous paragraph for closed subschemes \(Y \subseteq \text{Mat}_K^n\). In particular, to obtain “canonical” polynomial representatives for the varieties \(Z = X^K_z\), we can apply the preceding construction with \(Y = MX^K_z\).

**Definition 2.18.** For each \(K \in \{O, Sp\}\) and \(z \in I_n\), let

\[ \Theta^K_z \in \mathbb{Z}[\beta, \beta^{-1}][x_1, x_2, \ldots, x_n] \]

be the polynomial obtained from \([MX^K_z] \in K_T(\text{Mat}_n^K) \cong \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]\) by substituting \(a_i \mapsto 1 + \beta x_i\) for \(i \in [n]\) and then dividing by \((-\beta)^{\text{codim}(MX^K_z)}\).

It is helpful to note that if \(y \in I_n\) and \(z \in I^{FPF}_n\) then \(\text{codim}(MX^K_y) = |D^O(y)|\) and \(\text{codim}(MX^K_{Sp}) = |D^{Sp}(z)|\) [39 Lemma 5.4]. A method for computing \(\text{codim}(MX^K_{Sp})\) for \(z \in I_n \setminus I^{FPF}_n\) is implicit in the proof of [39 Theorem 6.11], though this is slightly nontrivial. We refer to \(\Theta^K_O\) and \(\Theta^K_{Sp}\) as orthogonal and symplectic Grothendieck polynomials.

The polynomials \(\Theta^K_z\) can actually be defined without inverting \(\beta\):

**Theorem 2.19.** For each \(K \in \{O, Sp\}\) and \(z \in I_n\), it holds that

\[ \Theta^K_z \in \mathbb{Z}[\beta][x_1, x_2, \ldots, x_n] \quad \text{and} \quad \Theta^K_z + I\Lambda_n[\beta] = [X^K_z] \in CK(\mathcal{F}_n). \]

**Proof.** By the preceding discussion and Theorem 2.17 applying the change of variables \(a_i \mapsto 1 - x_i\) to \([MX^K_z] \in K_T(\text{Mat}_n^K) \cong \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]\) gives a polynomial in \(\mathbb{Z}[x_1, \ldots, x_n]\) whose image in \(K(\mathcal{F}_n) = \mathbb{Z}[x_1, \ldots, x_n]/I\Lambda_n\) is \([X^K_z]\). Since one obtains \((-\beta)^{\text{codim}(MX^K_z)}\) \(\Theta^K_z\) from this polynomial by substituting \(x_i \mapsto -\beta x_i\) by Corollary 2.9, Proposition 2.3 implies that we have \(\Theta^K_z + I\Lambda_n[\beta, \beta^{-1}] = [X^K_z] \in CK(\mathcal{F}_n)/[\beta^{-1}].\)

To finish the proof, it is enough to show that after substituting \(a_i \mapsto 1 - x_i\), the polynomial \([MX^K_{Sp}]\) has no terms of degree less than \(\text{codim}(MX^K_{Sp})\) in the \(x_i\) variables. However, as will be explained in more detail in Section 3.2, this polynomial can be computed in terms of multigraded Hilbert series, and from this perspective the needed degree property is exactly [37 Claim 8.54].

**Example 2.20.** The symplectic Grothendieck polynomials for \(z \in I^{FPF}_4\) are

\[ \Theta^O_{243} = 1, \]
\[ \Theta^O_{3412} = x_1 + x_2 + \beta x_1 x_2, \]
\[ \Theta^O_{4321} = x_1^2 + x_1 x_2 + x_1 x_3 + x_2 x_3 + 2\beta x_1 x_2 x_3 + \beta x_1^2 x_2 + \beta x_1 x_2^2 + \beta x_2^2 x_3 + \beta^2 x_1^2 x_2 x_3. \]
The smallest example of $\mathfrak{S}_z^{Sp}$ where $z$ is not $Sp$-dominant (see Theorem 3.8) is

$$
\mathfrak{S}_{215634} = x_1 + x_2 + x_3 + x_4 + \beta x_1 x_2 + \beta x_1 x_3 + \beta x_1 x_4 + \beta x_2 x_3 + \beta x_2 x_4
$$

+ $\beta x_3 x_4 + \beta^2 x_1 x_2 x_3 + \beta^2 x_1 x_2 x_4 + \beta^2 x_1 x_3 x_4 + \beta^2 x_2 x_3 x_4$

We have computed these examples using Theorem 3.10.

**Example 2.21.** The orthogonal Grothendieck polynomials for $z \in I_3$ are

- $\mathfrak{S}_1^{O} = 1$,
- $\mathfrak{S}_2^{O} = 2x_1 + \beta x_1^2$,
- $\mathfrak{S}_3^{O} = 2x_1 + 2x_2 + \beta x_1^2 + 4\beta x_1 x_2 + \beta x_2^2 + 2\beta^2 x_1^2 x_2 + 2\beta x_1 x_2^2 + \beta^3 x_1^2 x_2^2$,
- $\mathfrak{S}_4^{O} = 2x_1^2 + 2x_1 x_2 + \beta x_1^3 + 3\beta x_1^3 x_2 + \beta^2 x_1^3 x_2^2$.

We have computed these examples using Theorem 3.6 and Macaulay2.

### 3 More on Grothendieck polynomials

Continue to let $n$ be a fixed positive integer. Our goal in this section is to outline the notable properties of the orthogonal and symplectic Grothendieck polynomials $\mathfrak{S}_z^{O}$ and $\mathfrak{S}_z^{Sp}$. The results here will also explain more direct methods of computing these polynomials.

#### 3.1 Stability

To start, we prove that the polynomials $\mathfrak{S}_z^K$ for $K \in \{O, Sp\}$ are stable under the natural inclusions $I_n \hookrightarrow I_{n+1}$ and $F_{n,FPF}^n \hookrightarrow F_{n+1,FPF}^n$ (applied to the indices $z$). In the $K = Sp$ case, this corresponds to [57, Theorem 4].

Define a map $p: Mat_n^{K} \to Mat_n^{K}$ by $p(A) = A[n][n]$. To distinguish between the tori in $GL_n$ and $GL_{n+1}$, write $T_n = T$ for the subgroup of invertible diagonal matrices in $GL_n$. Letting the last factor of $T_{n+1}$ act on $Mat_n^{K}$ trivially, the map $p$ is then $T_{n+1}$-equivariant, and the projection $T_{n+1} \to T_n$ induces a ring homomorphism $K_{T_n}(Mat_n^K) \to K_{T_{n+1}}(Mat_{n+1}^K)$ with $[Z] \mapsto [Z]$.

**Lemma 3.1.** Choose a symbol $K \in \{O, Sp\}$. The composition

$$
K_{T_n}(Mat_n^K) \to K_{T_{n+1}}(Mat_{n+1}^K) \xrightarrow{p^*} K_{T_{n+1}}(Mat_{n+1}^K)
$$

maps $[MX_z^K] \mapsto [MX_{z+1}^{K}]$ for each $z \in I_n$.

**Proof.** Since $Ess(D^K(z)) = Ess(D^K(z \times 1))$, it follows in view of Proposition 2.16 that $p^{-1}(MX_z^K) = MX_{z+1}^{K}$, which suffices as $p^*[MX_z^K] = [p^{-1}(MX_z^K)]$.

If $w \in S_n$ then we write $w \times 21$ for the permutation in $S_{n+2}$ that maps $i \mapsto w(i)$ for $i \in [n]$, $n+1 \mapsto n+2$, and $n+2 \mapsto n+1$. 

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Theorem 3.2. For each $K \in \{O, Sp\}$ and $z \in I_n$ it holds that $\Phi^K_{z \times 1} = \Phi^K_z$. Moreover, if $n$ is even and $z \in I^n_{FPF}$ then $\Phi^{Sp}_{z \times 21} = \Phi^K_z$.

Proof. The composition (3.1) can be identified with

$$R(T_n) \rightarrow R(T_{n+1}) \xrightarrow{id} R(T_{n+1}),$$

where the first arrow is the linear map that sends each representation $\pi : T_n \rightarrow GL(V)$ to $\pi \circ p : T_{n+1} \rightarrow GL(V)$; the second arrow must be the identity map since this is the unique $R(T_{n+1})$-algebra morphism $R(T_{n+1}) \rightarrow R(T_{n+1})$. After identifying $R(T_n)$ with $\mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$, (3.2) becomes the inclusion

$$\mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}] \rightarrow \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}] \xrightarrow{id} \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}],$$

so the first claim in theorem follows from Lemma 3.1.

For the second claim, assume $n$ is even and $z \in I^n_{FPF}$. If $u = z \times 21$ and $v = z \times 1^2$, then we have $\operatorname{rank}(u^{[n+1][n+1]}) + 1 = \operatorname{rank}(v^{[n+1][n+1]}) = n + 1$ while $\operatorname{rank}(u^{[i][j]}) = \operatorname{rank}(v^{[i][j]})$ for all $i, j \in \{n+2\} \times \{n+2\}$. Since $\operatorname{rank}(A^{[n+1][n+1]})$ is necessarily even if $A$ is skew-symmetric, it follows by Definition 2.15 that $M_{X_{z \times 21}}^{Sp} = M_{X_{z \times 1^2}}^{Sp}$, so $\Phi^{Sp}_{z \times 21} = \Phi^{Sp}_{z \times 1^2} = \Phi^{Sp}_z$. \(\square\)

As an application, we can now prove Theorems 1.3 and 1.4. We require one lemma. Recall that $IA_n$ is the ideal in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ generated by the elements that are symmetric in $x_1, x_2, \ldots, x_n$ and have zero constant term.

Lemma 3.3. Suppose $n_1, n_2, n_3, \ldots$ is a sequence of positive integers with $\lim_{i \to \infty} n_i = \infty$. Then $\bigcap_{i=1}^{\infty} IA_{n_i} = 0$.

The following argument is similar to the proof of [39, Lemma 2.11]. Write $\{\Phi_w\}_{w \in S_n}$ for the usual family of Schubert polynomials (see [33, Chapter 2]).

Proof. If $f \in \bigcap_{i=1}^{\infty} IA_{n_i}$ then $f \in \mathbb{Z}$-span$\{\Phi_w : w \in S_N\}$ for some $N = n_i$. [33, Proposition 2.5.4]. As $\{\Phi_w + IA_N : w \in S_N\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \ldots, x_N]/IA_N$ [33, Proposition 2.5.3 and Corollary 2.5.6], this can only happen if $f = 0$. \(\square\)

Proof of Theorems 1.3 and 1.4. The existence assertions in these results are Theorems 2.19 and 3.2. The uniqueness of $\Phi^K_2$ and $\Phi^{Sp}_2$ follows from Lemma 3.3 which implies that $\bigcap_{i=1}^{\infty} IA_{n_i+1}[\beta] = \bigcap_{i=1}^{\infty} IA_{n_i+2i}[\beta] = 0$ for any $n \in \mathbb{P}$. \(\square\)

3.2 Dominant formulas

Continue to let $T = T_n$ be the torus of invertible diagonal matrices in $GL_n$. When $V$ is a rational representation of $T$ and $Z \subseteq V$ is a $T$-invariant subscheme, there is a useful algebraic method for computing the polynomial $[Z] \in K_T(V)$, which we will use to derive an explicit product formula for certain instances of the polynomials $\Phi^K_i$.

Let $X(T) = \text{Hom}(T, \mathbb{C}^\times)$ be the character group of $T$. For $\lambda \in X(T)$, let

$$V_{\lambda} = \{ v \in V : tv = \lambda(t)v \text{ for } t \in T \} \subseteq V$$
be the $\lambda$-weight space of $V$. Choosing coordinates on $T$ uniquely identifies integers $m_1, \ldots, m_n$ with $\lambda(t) = t_1^{m_1} \cdots t_n^{m_n}$ for all $t = (t_1, \ldots, t_n) \in T$. Accordingly, we identify $\lambda$ with $(m_1, \ldots, m_n)$, and write $a^\lambda$ for the monomial $a_1^{m_1} \cdots a_n^{m_n}$.

**Definition 3.4.** Suppose $V$ is a rational representation of $T$ such that each weight space $V_\lambda$ is finite-dimensional. The Hilbert series of $V$ is then

$$\mathcal{H}(V, a) := \sum_{\lambda \in X(T)} \dim(V_\lambda) a^\lambda.$$ 

When the variables are clear from context, we write $\mathcal{H}(V)$ in place of $\mathcal{H}(V, a)$.

**Example 3.5.** Let $t \in T_n$ act on $\mathbb{C}[z_1, \ldots, z_n]$ as the algebra morphism sending $z_i$ to $t_i z_i$. The nonzero weight spaces in $\mathbb{C}[z]$ are $\mathbb{C}[z]_{(i)}$ for $i \geq 0$, each of which is one-dimensional, so the Hilbert series is defined and equal to

$$\mathcal{H}(\mathbb{C}[z]) = \sum_{i=0}^{\infty} a_i^i = 1/(1 - a).$$

If $V$ and $W$ are representations of $T_m$ and $T_n$, then the Hilbert series of $V \otimes_{\mathbb{C}} W$ as a $T_m \times T_n$-module is $\mathcal{H}(V, a_1, \ldots, a_m) \mathcal{H}(W, a_{m+1}, \ldots, a_{m+n})$. In particular,

$$\mathcal{H}(\mathbb{C}[z_1, \ldots, z_n]) = \prod_{i=1}^{n} 1/(1 - a_i).$$

Let $\mathbb{C}^\lambda$ be the one-dimensional representation of $T$ on which $t \in T$ acts as multiplication by $\lambda(t)$. The weights of $V$ are the elements of the unique multiset $\{\lambda_1, \ldots, \lambda_d\}$ such that $V \cong \bigoplus_i \mathbb{C}^{\lambda_i}$ as a $T$-module. Let $I(Z)$ be the ideal of $Z$ in the coordinate ring $\mathbb{C}[V] := \text{Sym}(V^*)$. The decomposition of $V$ into one-dimensional weight spaces determines (up to scalars) an isomorphism $\mathbb{C}[V] \cong \mathbb{C}[z_1, \ldots, z_d]$ with $z_i \in V_{\lambda_i}$. The $T$-action on $V$ defines a $T$-action on $\mathbb{C}[V]$; and since $Z$ is $T$-invariant, so is the ideal $I(Z)$.

**Theorem 3.6** ([11], §6.6). Suppose the cone generated by the weights of $V$ does not contain $0$. Let $Z \subseteq V$ be a $T$-invariant subscheme. Then the quotient $\mathcal{H}(\mathbb{C}[V]/I(Z))/\mathcal{H}(\mathbb{C}[V])$ is a well-defined polynomial, which corresponds to the class $[Z] \in K_T(V)$ under the isomorphism $K_T(V) \cong R(T) \cong \mathbb{Z}[a_1, \ldots, a_n]$.

The denominator $\mathcal{H}(\mathbb{C}[V])$ is easily computed: as in Example 3.5, it is the product $\prod_{i=1}^{n} 1/(1 - a_i)$ where $\lambda_1, \ldots, \lambda_d$ are the weights of $V$.

**Example 3.7.** Take $V = \text{Mat}^O_n$ with $T$-action $t : A \mapsto t A t$ as above. If $e_{ij}$ is the matrix with 1 in entry $(i, j)$ and 0 in all other entries, then

$$\text{Mat}^O_n = \bigoplus_{1 \leq i \leq j \leq n} \text{C-span}\{e_{ij} + e_{ji}\}$$

decomposes $\text{Mat}^O_n$ into one-dimensional weight spaces. Therefore the monomials $a^\mu$ as $\mu$ varies over all weights are $a_i a_j$ for $1 \leq i \leq j \leq n$. 

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Let $Z$ be the variety of matrices $A \in \text{Mat}^O_n$ with $A_{11} = A_{21} = A_{12} = 0$. Let $z_{ij} : \text{Mat}^O_n \to \mathbb{C}$ be the map $A \mapsto A_{ij}$, so that $\mathbb{C}[\text{Mat}^O_n] = \mathbb{C}[z_{ij} : 1 \leq i \leq j \leq n]$. Then $I(Z) = (z_{11}, z_{12})$, so

$$\mathbb{C}[V]/I(Z) \cong \mathbb{C}[z_{ij} : 1 \leq i \leq j \leq n, (i,j) \neq (1,1), (1,2)]$$

and hence

$$[Z] = \frac{\mathcal{H}(\mathbb{C}[V]/I(Z))}{\mathcal{H}(\mathbb{C}[V])} = \prod_{1 \leq i \leq j \leq n} \frac{1/(1-a_i a_j)}{1/(1-a_i a_j)} = (1-a_1^2)(1-a_1 a_2).$$

Note that $Z$ is the symmetric matrix Schubert variety $X^O_z$ for $z = 321 \in I_n$. The preceding calculation shows that $\mathcal{O}_{321} = (2x_1 + \beta x_1^2)(x_1 + x_2 + \beta x_1 x_2)$.

For any polynomials $x$ and $y$, let

$$x \oplus y := x + y + \beta xy \quad \text{and} \quad x \otimes y := \frac{x-y}{1+\beta y} \quad (3.3)$$

We say that an involution $z \in I_n$ is $O$-dominant if it holds that

$$D^O(z) = \{(i+j-1, j) \in \mathbb{P} \times [k] : 1 \leq i \leq \mu_j\}$$

for a strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0)$. Similarly, we define $z \in I_n$ to be $\text{Sp}$-dominant if $z$ is $O$-dominant or it holds that

$$z \in I_n^{\text{FPP}} \quad \text{and} \quad D^\text{Sp}(z) = \{(i+j, j) \in \mathbb{P} \times [k] : 1 \leq i \leq \mu_j\}$$

for a strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0)$. One can show that an involution is $O$-dominant if and only if it is dominant in the classical sense of being a 132-avoiding permutation [17, Proposition 3.25].

**Theorem 3.8.** Let $K \in \{O, \text{Sp}\}$ and suppose $z \in I_n$ is $K$-dominant. Then

$$\mathcal{O}^K_z = \prod_{(i,j) \in D^K(z)} x_i \otimes x_j.$$

**Proof.** Assume that $z \in I_n^\text{FPP}$ if $K = \text{Sp}$. It then follows from Proposition 2.10 that $A \in \text{Mat}^K_n$ and $\mathbb{F}$, the fact that $(i,z(i)) \notin D^K(z)$ for all $i \in [n]$ that $MX^K_z$ is just the set of matrices $A \in \text{Mat}^K_n$ with $A_{ij} = 0$ for all $(i,j) \in D^K(z)$. Thus, $I(MX^K_z) = \langle z_{ij} : (i,j) \in D^K(z) \rangle$ so codim$(MX^K_z) = |D^K(z)|$. Exactly as in Example 3.7, this implies that $[MX^K_z] = \prod_{(i,j) \in D^K(z)} (1-a_i a_j) \in K_T(\text{Mat}^K_n)$, which becomes $\prod_{(i,j) \in D^K(z)} x_i \otimes x_j$ on making the transformations in Definition 2.18.

If $K = \text{Sp}$ and $z \in I_n$ is $O$-dominant, then the same argument shows that $MX^\text{Sp}_z$ is the set of matrices $A \in \text{Mat}^\text{Sp}_n$ with $A_{ij} = 0$ for all $(i,j) \in D^O(z)$. In this case, since all elements of $\text{Mat}^\text{Sp}_n$ are skew-symmetric, the conditions $A_{ij} = 0$ determined by the diagonal positions $(i,i) \in D^O(z) \setminus D^\text{Sp}(z)$ are redundant, so we have $I(MX^\text{Sp}_z) = \langle z_{ij} : (i,j) \in D^\text{Sp}(z) \rangle$ and we can proceed as before. $\square$
As a special case, we recover two formulas of Wyser and Yong.

**Corollary 3.9** (Wyser and Yong [46]). For any positive integer \( n \) it holds that

\[
\mathcal{G}_z^{D_{n-321}} = \prod_{1 \leq i \leq j \leq n-i} x_i \oplus x_j \quad \text{and} \quad \mathcal{G}_z^{Sp_{n-321}} = \prod_{1 \leq i \leq j \leq n-i} x_i \oplus x_j.
\]

**Proof.** This follows by calculating \( D^0(n \cdots 321) \) and \( D^{Sp}(n \cdots 321) \).

\[\Box\]

### 3.3 Symplectic Grothendieck polynomials

Throughout this section, we assume that \( n \in 2\mathbb{P} \) is even. Here, we investigate some properties of the polynomials \( \mathcal{G}_z^{Sp} \) that are particular to the symplectic case.

Results of Wyser and Yong [46] show that the family \( \{\mathcal{G}_z^{Sp}\}_{z \in F_n^{FF}} \) can be completely characterized in terms of divided difference operators:

**Theorem 3.10** (Wyser and Yong [46]). The symplectic Grothendieck polynomials \( \{\mathcal{G}_z^{Sp}\}_{z \in F_n^{FF}} \) are the unique family in \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \) with

\[
\mathcal{G}_z^{Sp_{n-321}} = \prod_{1 \leq i \leq j \leq n-i} (x_i + x_j + \beta x_i x_j)
\]

and \( \partial_i^{(\beta)} \mathcal{G}_z^{Sp} = \mathcal{G}_{s_i s_j}^{Sp} \) for all \( i \in [n-1] \) such that \( i + 1 \neq z(i) > z(i+1) \neq i \).

The derivation of Theorem 3.10 from [46] requires some explanation. The **fixed-point-free involution length** of \( z \in F_n^{FF} \) is

\[
\ell_{FF}(z) := \left| \{(i, j) \in [n] \times [n] : z(i) > z(j) < i < j \} \right|.
\]

One has \( \ell_{FF}(z \times 21) = \ell_{FF}(z) = |D^{Sp}(z)| \), and the only element \( z \in F_n^{FF} \) with \( \ell_{FF}(z) = 0 \) is \( z = s_1 s_3 s_5 \cdots s_{n-1} \). It also holds that

\[
\ell_{FF}(s_i s_j) = \begin{cases} 
\ell_{FF}(z) + 1 & \text{if } z(i) < z(i+1) \\
\ell_{FF}(z) & \text{if } i + 1 = z(i) > z(i+1) = i \\
\ell_{FF}(z) - 1 & \text{if } i + 1 \neq z(i) > z(i+1) \neq i.
\end{cases}
\]

It follows by induction that \( \ell_{FF}(z) = \min \{ \ell(w) : w \in S_n \text{ and } w^{-1} \Theta w = z \} \).

**Proof of Theorem 3.10** Let \( a_i = 1 - x_i \) and \( D_i = \partial_i^{(-1)} \) for \( i \in \mathbb{P} \). Wyser and Yong [46] Theorem 4] prove that there exists a unique family of polynomials \( \{\mathcal{G}_z^{Sp}\}_{z \in F_n^{FF}} \subseteq \mathbb{Z}[x_1, x_2, \ldots, x_n] \) with \( \mathcal{G}_z^{Sp_{n-321}} = \prod_{1 \leq i \leq j \leq n-i} (1 - a_i a_j) \) and \( D_i \mathcal{G}_z^{Sp} = \mathcal{G}_{s_i z}^{Sp} \), for all \( i \in [n-1] \) with \( i + 1 \neq z(i) > z(i+1) \neq i \). (In [46], the variable \( a_i \) is written as \( x_i \).) It is straightforward to check that the elements

\[
\mathcal{G}_z^{Sp} := \beta^{-\ell_{FF}(z)} \mathcal{G}_z^{Sp}(\beta x_1, \beta x_2, \ldots, \beta x_n)
\]

belong to \( \mathbb{Z}[\beta][x_1, x_2, \ldots, x_n] \) and make up the unique family with the properties described in Theorem 3.10.
It remains to show that these polynomials are the same as the ones in Definition 2.13. In view of Theorem 1.3 and Proposition 2.3, it suffices to verify that \( \{ \Upsilon_i \}_{i \in \mathbb{N}} \) represent the classes of the structure sheaves of \( \{ X^i \}_{i \in \mathbb{N}} \) in ordinary \( K \)-theory and that \( \Upsilon_2 = \Upsilon_{2 \times 2} \). This is [46] Theorems 3 and 4. \( \square \)

We can describe the action of any \( \partial_i^{(\beta)} \) on \( \mathfrak{S}_{z}^{\mathbb{P}} \).

**Proposition 3.11.** Let \( z \in I_{\mathbb{N}}^{\mathbb{P}} \) and \( i \in [n-1] \). Then

\[
\partial_i^{(\beta)} \mathfrak{S}_{z}^{\mathbb{P}} = \begin{cases} 
\mathfrak{S}_{z}^{\mathbb{P}} & \text{if } i+1 \neq z(i) > z(i+1) \neq i \\
-\beta \mathfrak{S}_{z}^{\mathbb{P}} & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( y = s_i z s_i \). We have \( \partial_i^{(\beta)} \mathfrak{S}_{z}^{\mathbb{P}} = \mathfrak{S}_{y}^{\mathbb{P}} \) if \( i+1 \neq z(i) > z(i+1) \neq i \) by Theorem 3.10. If \( z(i) < z(i+1) \) then \( \mathfrak{S}_{z}^{\mathbb{P}} = \partial_i^{(\beta)} \mathfrak{S}_{y}^{\mathbb{P}} \) so \( \partial_i^{(\beta)} \mathfrak{S}_{z}^{\mathbb{P}} = -\beta \mathfrak{S}_{z}^{\mathbb{P}} \) since \( \partial_i^{(\beta)} \mathfrak{S}_{y}^{\mathbb{P}} = -\beta \mathfrak{S}_{z}^{\mathbb{P}} \).

Now suppose \( i+1 = z(i) > z(i+1) = i \). To show that \( \partial_i^{(\beta)} \mathfrak{S}_{z}^{\mathbb{P}} = -\beta \mathfrak{S}_{z}^{\mathbb{P}} \), it suffices by (3.6) to check that \( s_i \Upsilon_{z}^{\mathbb{P}} = \Upsilon_{z}^{\mathbb{P}} \). To show this, we resort to a geometric argument.

The action of \( S_n \) on \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \) descends to an action on \( K(\mathcal{F}_{n}) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I\Lambda \). Since \( \Upsilon_{z}^{\mathbb{P}} = [X^i_{\mathbb{P}}] \in K(\mathcal{F}_{n}) \) [46] Theorem 3, it follows by Lemma 3.3 that we can just show that \( s_i [X^i_{\mathbb{P}}] = [X^i_{\mathbb{P}}] \in K(\mathcal{F}_{n}) \).

Let \( q : T \setminus \mathcal{G}_{n} \to B \setminus \mathcal{G}_{n} = : \mathcal{F}_{n} \) be the quotient map. The left action of \( S_n \) on \( \mathcal{G}_{n} \) which permutes rows descends to \( T \setminus \mathcal{G}_{n} \) and induces an \( S_n \)-action on \( K(T \setminus \mathcal{G}_{n}) \). As noted in [2.3], the pullback \( q^* : K(\mathcal{F}_{n}) \to K(T \setminus \mathcal{G}_{n}) \) is an isomorphism; pulling back the \( S_n \)-action on \( K(T \setminus \mathcal{G}_{n}) \) gives the action of \( S_n \) on \( K(\mathcal{F}_{n}) \) described in the previous paragraph (see [46] §6).

It is enough to show that \( q^*[X^i_{\mathbb{P}}] = [q^{-1}(X^i_{\mathbb{P}})] \) is \( s_i \)-invariant. We prove this by showing that the variety \( q^{-1}(X^i_{\mathbb{P}}) \) itself is \( s_i \)-invariant. Recall that \( \mathcal{G}_{n} \) is defined as the subgroup of \( \mathcal{G}_{n} \) preserving the fixed skew-symmetric nondegenerate bilinear form \( \alpha^{\mathbb{P}}_{x} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \). Proposition 2.13 implies that if \( g \in \mathcal{G}_{n} \) has rows \( g_1, g_2, \ldots, g_n \), then \( T g \in q^{-1}(X^i_{\mathbb{P}}) \) if and only if the matrix \( A = [\alpha_{x}^{\mathbb{P}}(g_p, g_q)] \) has \( \operatorname{rank}(A_{z_{i}[z_{j}]}(i,j)) \leq \operatorname{rank}(\alpha_{x}^{\mathbb{P}}(z_{i}[z_{j}])) \) for any \( (i, j) \in \mathbb{D}^{n}(z) \). These rank conditions are invariant under permuting rows \( i \) and \( i + 1 \) of \( g \) long as row \( i \) of \( \mathbb{G}(\mathbb{P}) \) is empty. The latter holds since if \( (i, j) \in \mathbb{D}^{n}(z) \), then we have \( j \leq z(i) = i + 1 \) and \( j < i < z(j) \), and therefore also \( j < z(i + 1) = i \) and \( j < i + 1 < z(j) \), so \( (i + 1, j) \in \mathbb{D}^{n}(z) \). Thus \( q^*[X^i_{\mathbb{P}}] \) is \( s_i \)-invariant, and we conclude that \( \partial_i^{(\beta)} \mathfrak{S}_{z}^{\mathbb{P}} = -\beta \mathfrak{S}_{z}^{\mathbb{P}} \). \( \square \)

Any element of \( \mathbb{Z}[\beta][[x_1, x_2, \ldots]] \) whose homogeneous terms are polynomials (treating \( \beta \) as a scalar of degree zero) can be uniquely expressed as a possibly infinite \( \mathbb{Z}[\beta] \)-linear combination of ordinary Grothendieck polynomials.

Our next main result shows that the symplectic Grothendieck polynomials have a stronger property: each \( \mathfrak{S}_{z}^{\mathbb{P}} \) is actually a finite linear combination of \( \mathfrak{S}_{z}^{\mathbb{P}} \)'s with coefficients in \( \{ 1, \beta, \beta^2, \ldots \} \). In principle, this could also be deduced from general results of Brion [5] Theorem 4. One advantage to our approach is
that it will let us identify the summands appearing in the expansion of $\mathfrak{S}_w^{\text{Sp}}$ in terms of $\mathfrak{S}_w$, somewhat explicitly.

Let $U_n$ denote the free $\mathbb{Z}$-module with a basis given by the symbols $U_w$ for $w \in S_n$. Set $U_i := U_{s_i}$ for $i \in \mathbb{P}$. The abelian group $U_n$ has a unique ring structure with multiplication satisfying
\[
U_w U_i := \begin{cases} U_{w s_i} & \text{if } w(i) < w(i+1) \\ U_w & \text{if } w(i) > w(i+1) \end{cases} \quad \text{for } w \in S_n, \ i \in [n - 1].
\]

This is the usual Iwahori-Hecke algebra of $S_n$ with $q = 0$; see [24, Chapter 7].

Let $N^n$ be the free $\mathbb{Z}$-module with basis $\{N_z : z \in I_n^{\text{PPF}}\}$. Results of Rains and Vazirani (namely, [31, Theorems 4.6 and 7.1] with $q = 0$) imply that $N^n$ has a unique structure as a right $U_n$-module with
\[
N^n U_i := \begin{cases} N_{s_i z} & \text{if } z(i) < z(i+1) \\ N_z & \text{if } i + 1 \neq z(i) > z(i+1) \neq i \\ 0 & \text{if } i + 1 = z(i) > z(i+1) = i \end{cases} \quad \text{for } z \in I_n^{\text{PPF}}, \ i \in [n - 1].
\]

It is shown in [33] that (the undegenerated form of) $N^n$ has a “quasi-parabolic Kazhdan-Lusztig basis”; it would be interesting to relate this basis to the polynomials $\mathfrak{S}_z^{\text{Sp}} := \mathfrak{S}_w^{\text{Sp}} |_{\beta = 0}$ and $\mathfrak{S}_z^{\text{Sp}}$, in analogy with results in [3].

For each $z \in I_n^{\text{PPF}}$, define $B_{\text{PPF}}(z) = \{w \in S_n : N_0 U_w = z\}$. This set is nonempty with $\ell_{\text{PPF}}(z) \leq \ell(w)$ for all $w \in B_{\text{PPF}}(z)$. Define $A_{\text{PPF}}(z) = \{w \in B_{\text{PPF}}(z) : \ell_{\text{PPF}}(z) = \ell(w)\}$. We refer to the elements of $A_{\text{PPF}}(z)$ and $B_{\text{PPF}}(z)$ as atoms and Hecke atoms for $z$, respectively. The set $A_{\text{PPF}}(z)$ consists of the permutations $w \in S_n$ of minimal length with $z = w^{-1} s_1 s_3 s_5 \cdots s_{n-1} \cdot w$.

**Theorem 3.12.** If $z \in I_n^{\text{PPF}}$ then $\mathfrak{S}_z^{\text{Sp}} = \sum_{w \in B_{\text{PPF}}(z)} \beta_{\ell(w) - \ell_{\text{PPF}}(z)} \mathfrak{S}_w$.

This result makes it clear that the family $\{\mathfrak{S}_z^{\text{Sp}} \}_{z \in I_n^{\text{PPF}}}$ is linearly independent.

**Proof.** Define $\Sigma_z := \sum_{w \in B_{\text{PPF}}(z)} \beta_{\ell(w) - \ell_{\text{PPF}}(z)} \mathfrak{S}_w$ for $z \in I_n^{\text{PPF}}$. We claim that
\[
\partial_{\ell}^{(\beta)} \Sigma_z = \begin{cases} \Sigma_{s_i z} & \text{if } i + 1 \neq z(i) > z(i+1) \neq i \\ -\beta \Sigma_z & \text{otherwise} \end{cases} \quad (3.7)
\]

for all $z \in I_n^{\text{PPF}}$ and $i \in [n - 1]$. To show this, fix $z \in I_n^{\text{PPF}}$ and $i \in [n - 1]$ and let $y = s_i z s_i \in I_n^{\text{PPF}}$. There are three cases to consider.

First assume $i + 1 \neq z(i) > z(i+1) \neq i$. If $w \in B_{\text{PPF}}(z)$ and $w(i) > w(i+1)$, then $ws_i \in B_{\text{PPF}}(x)$ for some $x \in I_n^{\text{PPF}}$ and $N_z = N_0 U_w = N_0 U_{ws_i} U_i = N_z U_i$, so $x \in \{y, z\}$ and $ws_i \in B_{\text{PPF}}(y) \cup B_{\text{PPF}}(z)$. Alternatively, if $v \in B_{\text{PPF}}(y)$ then $N_0 U_i U_i = N_y U_i = N_z$, so $U_i U_i \neq U_v$ and $w(i) < v(i+1)$ and $v(i+1)$. The disjoint union
\[
\{v s_i : v \in B_{\text{PPF}}(y)\} \cup \{u s_i : u \in B_{\text{PPF}}(z), \ u(i) < u(i+1)\}.
\]
Now, since $\partial_i^{(\beta)} G_w = -\beta G_w$ if $w(i) < w(i+1)$, we have

$$\partial_i^{(\beta)} \Sigma_z = \sum_{w \in B_{FPF}(\beta)} \beta(w) - \ell_{FPF}(z) \partial_i^{(\beta)} \Sigma_z - \sum_{w \in B_{FPF}(\beta)} \beta(w) - \ell_{FPF}(z) + 1 \partial_i^{(\beta)} \Sigma_z.$$

Since $\ell_{FPF}(z) = \ell_{FPF}(y) + 1$, it follows that the first sum on the right is

$$\sum_{v \in B_{FPF}(y)} \beta(v) - \ell_{FPF}(y) \partial_i^{(\beta)} \Sigma_z + \sum_{w \in B_{FPF}(\beta)} \beta(w) - \ell_{FPF}(z) + 1 \partial_i^{(\beta)} \Sigma_z.$$

Substituting this into the previous equation gives $\partial_i^{(\beta)} \Sigma_z = \Sigma_y$.

If $z(i) < z(i+1)$ then $i+1 \neq y(i) > y(i+1) \neq i$, so the previous paragraph implies that $\Sigma_z = \partial_i^{(\beta)} \Sigma_y$ and $\partial_i^{(\beta)} \Sigma_z = -\beta \Sigma_z$ as $\partial_i^{(\beta)} \partial_i^{(\beta)} = -\beta \partial_i^{(\beta)}$. Finally assume that $i+1 = z(i) > z(i+1) = i+1$. If $w \in B_{FPF}(\beta)$ has $w(i) > w(i+1)$, then $w \in B_{FPF}(\beta)$ for some $x \in I_n^{FPF}$ and $\beta(z(i)) = \beta(z(i+1)) = 0$, which implies the contradiction $0 = N_z U_i = N_z U_i^2 = N_z U_i = N_z$. Thus every $w \in B_{FPF}(\beta)$ has $w(i) < w(i+1)$, so $\partial_i^{(\beta)} \Sigma_z = -\beta \Sigma_z$. Thus (3.7) holds.

We argue by contradiction that $\mathcal{S}_z^\beta = \Sigma_z$ for all $z \in I_n^{FPF}$. Let $\Delta_z := \mathcal{S}_z^\beta - \Sigma_z$ and suppose $z \in I_n^{FPF}$ is of minimal length $\ell_{FPF}(z)$ such that $\Delta_z \neq 0$. We cannot have $z = s_1 s_2 s_3 \cdots s_n$ since then $\mathcal{S}_z^\beta = \Sigma_z = 1$. The set of indices $I := \{i \in [n-1] : i+1 \neq z(i) > z(i+1) \neq i\}$ is therefore nonempty. By Proposition 3.11, 3.7, and induction, we have $\partial_i^{(\beta)} \Delta_z = 0$ for all $i \in I$ and $\partial_i^{(\beta)} \Delta_z = -\beta \Delta_z$ for all $i \notin I$. This means that for each $i \in [n-1]$, either $\Delta_z$ or $(1 + \beta x_{i+1}^l) \Delta_z$ is symmetric in $x_i$ and $x_{i+1}$.

The homogeneous term of $\Delta_z$ of lowest degree (with $\deg(x_i) := 1$ and $\deg(\beta) := 0$) must therefore be symmetric in $x_1, x_2, \ldots, x_n$. Since $\Delta_z = \Delta_z \times 21$, it follows that $\Delta_z$ must actually be symmetric in all the $x_i$-variables for $i \in \mathbb{P}$. Since $\Delta_z$ is a polynomial, this can only occur if $\Delta_z$ has a nonzero constant term. But it is easy to show by induction that both $\mathcal{S}_z^\beta$ and $\Sigma_z$ have no homogeneous terms of degree less than $\ell_{FPF}(z) \geq 1$, so we reach a contradiction. Hence no such $z$ can exist, so $\mathcal{S}_z^\beta = \Sigma_z$ for all $z \in I_n^{FPF}$. 

Given $w \in S_z$, write $\mathcal{H}(w)$ for the set of finite integer sequences $i_1 i_2 \cdots i_t$ with $U_w = U(i_1) U(i_2) \cdots U(i_t)$. Define $\mathcal{H}_{sp}(z) = \bigcup_{w \in B_{sp}(z)} \mathcal{H}(w)$ for $z \in I_n^{FPF}$. We refer to the elements of $\mathcal{H}(w)$ (respectively, $\mathcal{H}_{sp}(z)$) as (symplectic) Hecke words. Evidently, one has $i_1 i_2 \cdots i_t \in \mathcal{H}_{sp}(z)$ if and only if $N_z = N_{\delta} U(i_1) U(i_2) \cdots U(i_t)$.

**Corollary 3.13.** Given a subset $S = \{(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)\} \subseteq \mathbb{P} \times \mathbb{P}$ with $a_1 \leq a_2 \leq \cdots \leq a_l$ and $b_k > b_k + 1$ whenever $a_k = a_{k+1}$, define

$$i_k := b_k - (a_k - 1) \quad \text{and} \quad \delta(S) := i_1 i_2 \cdots i_l \quad \text{and} \quad x^S := x_{a_1} x_{a_2} \cdots x_{a_l}.$$

If $z \in I_n^{FPF}$ then $\mathcal{S}_z^\beta = \sum_{S \subseteq [n] \times [n]} \beta(S) - \ell_{FPF}(z) x^S$. 

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Proof. The formula \( \mathcal{G}_w = \sum_{S \subseteq [n] \times [n], \delta(S) \in \mathcal{H}(w)} \beta^{|S|} - \ell(w) x^S \) for \( w \in S_n \) is [13, Theorem 2.3] (cf. [27, Corollary 5.4]), so this follows from Theorem 3.12. \( \square \)

Corollary 3.14. If \( z \in I_n^{\text{FFP}} \) then \( \mathcal{G}_z^{\text{FPF}} \in \mathbb{N}[\beta][x_1, x_2, \ldots, x_n] \).

Proof. This holds as \( \mathcal{G}_w \in \mathbb{N}[\beta][x_1, \ldots, x_n] \) for all \( w \in S_n \) [13, Theorem 2.3]. \( \square \)

We can describe \( \mathcal{B}_{\text{FFP}}(z) \) more concretely. Fix an involution \( z \in I_n^{\text{FFP}} \) and suppose \( a_1 < a_2 < \ldots \) are the integers \( a \in [n] \) such that \( a < z(a) \), arranged in increasing order. Let \( b_i = z(a_i) \) for each \( i \) and define

\[
\alpha_{\text{FFP}}(z) = (a_1 b_1 a_2 b_2 a_3 b_3 \cdots)^{-1} \in S_n.
\]

Write \( w_i = w(i) \) for \( w \in S_n \) and \( i \in [n-1] \). Let \( \approx_{\text{FFP}} \) be the strongest equivalence relation on \( S_n \) with \( v^{-1} \approx_{\text{FFP}} w^{-1} \) whenever there is an even index \( i \in 2\mathbb{N} \) and integers \( a < b < c < d \) such that \( v_i v_i+2 v_i+3 v_i+4 \) and \( w_i w_i+2 w_i+3 w_i+4 \) both belong to \{adbc, bcad, bdac\} and \( v_j = w_j \) for all \( j \notin \{i+1, i+2, i+3, i+4\} \).

Proposition 3.15 ([35, Theorem 2.5]). If \( z \in I_n^{\text{FFP}} \) then

\[
\mathcal{B}_{\text{FFP}}(z) = \{ w \in S_n : \alpha_{\text{FFP}}(z) \approx_{\text{FFP}} w \}.
\]

There is a complementary result for \( \mathcal{A}_{\text{FFP}}(z) \). Let \( \prec_{\text{FFP}} \) be the transitive closure of the relation on \( S_n \) with \( v^{-1} \prec_{\text{FFP}} w^{-1} \) whenever there is an even index \( i \in 2\mathbb{N} \) and integers \( a < b < c < d \) such that \( v_i v_i+2 v_i+3 v_i+4 = adbc \) and \( w_i w_i+2 w_i+3 w_i+4 = bcad \) and \( v_j = w_j \) for all \( j \notin \{i+1, i+2, i+3, i+4\} \).

Proposition 3.16 ([18, Theorem 6.10]). If \( z \in I_n^{\text{FFP}} \) then

\[
\mathcal{A}_{\text{FFP}}(z) = \{ w \in S_n : \alpha_{\text{FFP}}(z) \prec_{\text{FFP}} w \}.
\]

Example 3.17. The elements of \( \mathcal{B}_{\text{FFP}}(z) \) for \( z = 4321 \in I_4^{\text{FFP}} \) are

\[
\alpha_{\text{FFP}}(4321) = 1342 = (1423)^{-1} \prec_{\text{FFP}} 3124 = (2314)^{-1} \approx_{\text{FFP}} 3142 = (2413)^{-1}.
\]

We have \( l_{\text{FFP}}(4321) = \ell(1342) = \ell(3124) = \ell(3142) - 1 = 2 \) and

\[
\mathcal{G}_{1342} = x_1 x_2 + x_1 x_3 + x_2 x_3 + 2\beta x_1 x_2 x_3,
\]

\[
\mathcal{G}_{3124} = x_1^2,
\]

\[
\mathcal{G}_{3142} = x_1^2 x_3 + x_1^2 x_2 + \beta x_1^2 x_2 x_3.
\]

Comparing with Example 2.20 shows that \( \mathcal{G}_{1342}^{\text{SP}} = \mathcal{G}_{1342} + \mathcal{G}_{3124} + \beta \mathcal{G}_{3142} \).

3.4 Degeneracy locus formulas

In contrast to the symplectic case, the polynomials \( \mathcal{G}_Z^O \) do not have an inductive description in terms of divided difference operators, and it is an open problem to find a general formula for \( \mathcal{G}_Z^O \) that improves on Theorem 3.6. We will give
some partial results to this problem Section 5.5. As preparation, we review some more general formulas from [2, 24] in this section.

Let $X$ be a smooth complex variety. Each vector bundle $V$ over $X$ has Chern classes $c_d(V) \in CK^d(X)$ for $d \in \mathbb{N}$ and a Chern polynomial

$$c(V, t) := \sum_{d \geq 0} c_d(V)t^d \in CK(X)[t]$$

with the following properties (see [2, Appendix A]):

(a) It holds that $c_0(V) = 1$ and $c_d(V) = 0$ for $d > \text{rank}(V)$.

(b) We have $c(V, t) = c(U, t)c(W, t)$ if $0 \to U \to V \to W \to 0$ is a short exact sequence of vector bundles over $X$.

(c) If $\text{rank}(V) = 1$, then $c_1(V^*) = \frac{-c_1(V)}{1 + \beta c_1(V)}$.

(d) If $f : Y \to X$ is a morphism, then $c(f^*V, t) = f^*c(V, t) \in CK(Y)[t]$.

Since $CK^d(X) = 0$ for $d > \dim(X)$, property (a) implies that $c(V, t)$ is invertible in $CK(X)[t]$, and any vector bundle $V$ over $X = pt$ must have $c(V, t) = 1$. It follows from property (d), with the morphism $Y \to X$ replaced by $X \to pt$, that if $V$ is a trivial vector bundle over $X$ then $c(V, t) = 1$.

Although the difference "$V - W$" for two vector bundles $V$ and $W$ over $X$ is not defined, we set

$$c(V - W, t) := c(V, t)/c(W, t) \in CK(X)[[t]].$$

We regard "-" defined in this way as a formal inverse of "⊕," which makes sense as property (b) implies that $c((V \oplus U) - (W \oplus U), t) = c(V - W, t)$ for any vector bundle $U$ over $X$.

Let $c_0, c_1, c_2, \ldots$ be indeterminates. The raising operator $T$ associated to such a sequence is the linear operator on the space of arbitrary linear combinations of the $c_i$’s that sends $c_i \mapsto c_{i+1}$ for each $i$, and $\sum_{i \in \mathbb{N}} a_i c_i \mapsto \sum_{i \in \mathbb{N}} a_i c_{i+1}$ for arbitrary coefficients $a_i$. We adopt the following conventions to make it easier to work with complicated expressions involving these operators:

- If $f(x)$ is a function with a Laurent expansion $\sum_{m \in \mathbb{Z}} a_m x^m$ at $x = 0$, then we take $f(T)$ to mean $\sum_{m \in \mathbb{Z}} a_m T^m$. For instance,

  $$(1 - \beta T)^{-1} c_i := \sum_{m \geq 0} \beta^m T^m c_i = \sum_{m \geq 0} \beta^m c_{m+i}.$$ 

- We write $T^{-1}$ for the operator sending $\sum_{i \in \mathbb{N}} a_i c_i \mapsto \sum_{i \in \mathbb{N}} a_{i+1} c_i$, so that $T^{-1}(c_i) = c_{i-1}$ for $i > 0$ and $T^{-1}(c_0) = 0$. The composition $T^{-1}T$ is the identity operator while $TT^{-1}$ sends $c_i \mapsto c_i$ for $i > 0$ and $c_0 \mapsto 0$. 

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Given a finite collection of sequences of indeterminates \( c_0^{(i)}, c_1^{(i)}, c_2^{(i)}, \ldots \) for \( i \in [n] \), we write \( T^{(i)} \) for the raising operators that act on monomials by
\[
T^{(i)} c_{d_1} \cdots c_{d_i} \cdots c_{d_n} = c_{d_1} \cdots c_{d_i+1} \cdots c_{d_n},
\]
\[
T^{(i)} c_{d_1} \cdots c_{d_{i-1}} c_{d_i+1} \cdots c_{d_n} = 0.
\]

In other words, \( T^{(i)} \) acts as zero on each monomial that does not involve any of \( c_0^{(i)}, c_1^{(i)}, c_2^{(i)}, \ldots \) On sums of monomials, \( T^{(i)} \) acts linearly in the usual way. For example, \( (T^{(1)})^{-1}(c_2^{(1)} c_1^{(2)} + c_0^{(1)} c_3^{(2)} + c_1^{(2)} c_3^{(3)}) = c_1^{(1)} c_2^{(1)} \).

To declutter our notation, we sometimes write \( 1/T^{(i)} \) in place of \( (T^{(i)})^{-1} \).

Later we will apply the raising operators \( T \) to expressions involving \( c_i \) which already have some assigned meaning: in such expressions, we treat the \( c_i \) as indeterminates, apply the raising operators, and then replace the symbols \( c_i \) with their assigned values.

Continue to let \( X \) denote a smooth complex variety. Fix \( n \in \mathbb{P} \) and let \( \pi : V \to X \) be a vector bundle of even rank \( 2n \) over \( X \). For \( x \in X \), write \( V_x = \pi^{-1}(x) \) for the fiber of \( V \) over \( x \). Assume \( V \) is equipped with a nondegenerate skew-symmetric bilinear form, meaning that we have fixed a section of the bundle \( \Lambda^2 V^* \) which is nondegenerate on each fiber of \( V \).

A subbundle \( F \subseteq V \) is isotropic with respect to this form if \( F \subseteq F^\perp \), where \( F^\perp \) is the vector bundle whose fiber over \( x \in X \) is the orthogonal complement of \( V_x \) under the associated form. Assume \( F^n \subseteq \cdots \subseteq F^1 \subseteq V \) is an isotropic flag of subbundles, where \( \text{rank}(F^i) = n - i + 1 \), and let \( G \subseteq V \) be a maximal isotropic subbundle \( G \subseteq V \), necessarily of rank \( n \).

**Definition 3.18.** For a strict partition \( \lambda \) with \( \lambda_1 \leq n \), with \( V, G, F^\bullet \) as above, define the associated Lagrangian Grassmannian degeneracy locus to be
\[
\Omega^L_X(V, G, F^\bullet) := \{ x \in X : \dim(G_x \cap F^\lambda_x) \geq i \text{ for } i \in [\ell(\lambda)] \}.
\]

Among the components of \( F^\bullet \), only the bundles \( F^\lambda \) play a role in the definition of \( \Omega^L_X(V, G, F^\bullet) \). Moreover, as we will discuss in Remark 3.21, many of the rank conditions \( \dim(G_x \cap F^\lambda_x) = i \) in (3.8) turn out to be superfluous.

Anderson [2] and Hudson, Ikeda, Matsumura, and Naruse [24] give explicit formulas for the classes \( \Omega^L_X(V, G, F^\bullet) \in CK(X) \) in terms of Chern classes.

**Notation.** In the next theorem, we define certain power series \( c^{(i)} \in CK(X)[[t]] \) in the variable \( t \). Let \( c_d^{(i)} \in CK(X) \) be such that \( c^{(i)} = \sum_{d \geq 0} c_d^{(i)} t^d \), and write \( T^{(i)} \) for the raising operator acting on \( c_d^{(i)} \). For \( i < j \), we also define
\[
R^{(i,j)} := \frac{1 - T^{(i)}/T^{(j)}}{1 + T^{(i)}/T^{(j)} - \beta T^{(i)}}.
\]

This operator should be expanded in \( T^{(i)} \) as
\[
R^{(i,j)} = \sum_{k \geq 0} \left( \beta T^{(i)} \right)^k + \sum_{k \geq 0} \sum_{l > 0} (-1)^l \binom{k+l-1}{k} \binom{k+l}{k} \left( \beta T^{(i)} \right)^k \left( T^{(i)} / T^{(j)} \right)^l.
\]
Finally, denote the Pfaffian of a skew-symmetric matrix \( A = (A_{ij})_{i,j \in [n]} \) by

\[
\text{pf}(A) := \sum_{z \in I_n^{PP}} (-1)^{\ell_D(z)} \prod_{z(i) < i \in [r]} A_{z(i), i}.
\]

One has \( \det(A) = \text{pf}(A)^2 \). If \( n \) is odd then \( \text{pf}(A) = 0 \) since \( I_n^{PP} \) is empty.

**Theorem 3.19** ([1, Theorem 2]). Suppose \( \lambda \) is a strict partition and \( \mathcal{V}, \mathcal{G}, \) and \( \mathcal{F}^* \) are given such that \( \Omega^G_{\lambda}(\mathcal{V}, \mathcal{G}, \mathcal{F}^*) \) is a Lagrangian Grassmannian degeneracy locus in a smooth complex variety \( X \) with \( \text{codim}(\Omega^G_{\lambda}(\mathcal{V}, \mathcal{G}, \mathcal{F}^*)) = |\lambda| \). Let \( r \) be the smallest even integer with \( \ell(\lambda) \leq r \). Let \( S \) be a subset of \([r]\) containing

\[
\mathcal{C}(\lambda) := \{ i \in [\ell(\lambda) - 1] : \lambda_i > \lambda_{i+1} + 1 \} \cup \{ \ell(\lambda) \}.
\]

For \( i \in [\ell(\lambda)] \), define \( c(i) = c(\mathcal{V} - \mathcal{G} - \mathcal{F}^{\lambda_i}, t) \) where \( s \in S \) is minimal with \( i \leq s \), and let \( c^{(r)} = 1 \) and \( \lambda_r = 0 \) if \( r = \ell(\lambda) + 1 \). Then \( [\Omega^G_{\lambda}(\mathcal{V}, \mathcal{G}, \mathcal{F}^*)] \in \text{CK}(X) \) is the Pfaffian of the \( r \times r \) skew-symmetric matrix whose \((i, j)\) entry for \( i < j \) is

\[
R^{(i,j)} \left( 1 - \beta T^{(j)} \right)^{r - i - \lambda_i} \left( 1 - \beta T^{(j)} \right)^{r - j - \lambda_j} c^{(i)}_{\lambda_i} c^{(j)}_{\lambda_j}.
\]

The exponents \( r - i - \lambda_i \) and \( r - j - \lambda_j \) in (3.10) may be negative. Since our statement is slightly different from the one in [1], we sketch a proof below.

**Remark 3.20.** In the preceding theorem and the proof which follows, we are citing the arXiv version [1] of Anderson’s paper rather than the published version [2]. At the time of writing, there is an error in the statement of [2] Theorem 2 which has been corrected in [1] Theorem 2.

**Proof.** When \( \ell(\lambda) \) is even, this is the special case of [1] Theorem 2 with \( s = |S| \) and \( S = \{ k_1 < k_2 < \cdots < k_s \} \), with \( p_i = 1 \) and \( q_i = \lambda_{k_i} \) for \( i \in [s] \), and with \( E_{p_i} = \mathcal{G} and F_{q_i} = \mathcal{F}^{\lambda_i} \) for \( i \in [s] \). When \( \ell(\lambda) \) is odd, our matrix is different from the matrix in [1] Theorem 2, but we claim that it has the same Pfaffian. To see this, for any \( l \in \mathbb{P} \) define \( P^{(l)} := \prod_{1 \leq i < j \leq l} R^{(i,j)} \prod_{i=1}^{l} (1 - \beta T^{(i)})^{l - i - \lambda_i} c^{(i)}_{\lambda_i} \)

where we set \( c^{(i)} = 1 \) and \( \lambda_i = 0 \) if \( i > \ell(\lambda) \). If \( l \geq \ell(\lambda) \), then

\[
\left( \frac{1}{T^{(l+1)}} \right)^{(l+1)} c^{(l+1)}_{\lambda_{l+1}} = \left( \frac{1}{T^{(l+1)}} \right)^{(l+1)} c^{(l+1)}_0 = 0
\]

and therefore

\[
P^{(l+1)} = \prod_{1 \leq i < j \leq l} R^{(i,j)} \prod_{i=1}^{l} R^{(i,l+1)} \prod_{i=1}^{l} (1 - \beta T^{(i)})^{l+1 - i - \lambda_i} c^{(i)}_{\lambda_i} = \prod_{1 \leq i < j \leq l} R^{(i,j)} \prod_{i=1}^{l} (1 - \beta T^{(i)})^{-1} \prod_{i=1}^{l} (1 - \beta T^{(i)})^{l+1 - i - \lambda_i} c^{(i)}_{\lambda_i} = P^{(l)}.
\]

It is shown in the proof of [1] Theorem 2 that \( P^{(\ell(\lambda))} = [\Omega^G_{\lambda}(\mathcal{V}, \mathcal{F}, \mathcal{G}^*)] \), and that if \( l \) is even then \( P^{(l)} \) is the Pfaffian of the \( l \times l \) skew-symmetric matrix with entries (3.10). In particular, if \( \ell(\lambda) \) is odd then \( P^{(\ell(\lambda)+1)} \) is the Pfaffian in the statement of the theorem, and \( P^{(\ell(\lambda)+1)} = P^{(\ell(\lambda))} = [\Omega^G_{\lambda}(\mathcal{V}, \mathcal{F}, \mathcal{G}^*)] \).
Remark 3.21. The statement of Theorem 3.19 would be simpler if we fixed $S = [r]$. It is useful to allow some flexibility in our choice of $S$, however, since not all of the rank conditions in (3.8) are necessary: restricting $i$ to be an element of $S \subseteq [r]$ defines the same locus $\Omega^G_\lambda(\mathcal{V}, \mathcal{G}, \mathcal{F}^*)$ whenever $\mathcal{C}(\lambda) \subseteq S$. Theorem 3.19 gives a formula for $[\Omega^G_\lambda(\mathcal{V}, \mathcal{G}, \mathcal{F}^*)]$ that does not involve the bundles $\mathcal{F}^\lambda_i$ for $i \notin S$ that are irrelevant to the definition of $\Omega^G_\lambda(\mathcal{V}, \mathcal{G}, \mathcal{F}^*)$.

3.5 Orthogonal Grothendieck polynomials

Here, we use Theorem 3.19 to derive a formula for the polynomials $\mathcal{O}^\alpha_G$ indexed by involutions $z \in I_n$ that are vexillary in the sense of being 2143-avoiding. These permutations have a useful alternate characterization; recall the definitions of Ess(D) and $D^\alpha(z)$ from Section 2.3.

Lemma 3.22 ([29 Lemma 4.18]). An involution $z \in I_n$ is vexillary if and only if the essential set $\text{Ess}(D^\alpha(z))$ is a chain under the partial order $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ with $(a, b) \preceq (i, j)$ if and only if $i \leq a$ and $b \leq j$.

Write $\mathbb{C}^n^*$ for the dual space of $\mathbb{C}$-linear maps $\mathbb{C}^n \rightarrow \mathbb{C}$. We represent elements of the direct sum $\mathbb{C}^n \oplus \mathbb{C}^n^*$ as pairs $(v, \omega)$ where $v \in \mathbb{C}^n$ and $\omega \in \mathbb{C}^n^*$. Define $\langle \cdot, \cdot \rangle^-$ to be the skew-symmetric bilinear form on $\mathbb{C}^n \oplus \mathbb{C}^n^*$ with

$$\langle (v_1, \omega_1), (v_2, \omega_2) \rangle^- := \omega_1(v_2) - \omega_2(v_1).$$

Let $L^G_{2n}$ be the Lagrangian Grassmannian with respect to this form, so that

$$L^G_{2n} = \{ U \in \text{Gr}(n, \mathbb{C}^n \oplus \mathbb{C}^n^*) : \langle \cdot, \cdot \rangle^-|_{U \times U} \equiv 0 \}.$$  

The graph of a bilinear form $\alpha : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ is

$$\Gamma(\alpha) := \{(v, \alpha(v, \cdot)) : v \in \mathbb{C}^n \} \in \text{Gr}(n, \mathbb{C}^n \oplus \mathbb{C}^n^*).$$

(3.12)

Such a form $\alpha$ is symmetric if and only if $\Gamma(\alpha) \in L^G_{2n}$. Recall from Section 2.3 that $\alpha^O_G$ is a fixed symmetric nondegenerate bilinear form on $\mathbb{C}^n$, and that we define $O_n$ to be the subgroup of $G_n$ preserving $\alpha^O_G$.

As explained in [46 §2.2], geometric obstructions prevent us from being able to completely characterize the family $\{\mathcal{O}^O_G\}_{z \in I_n}$ using divided difference operators as we did for $\{\mathcal{O}^{5p}_z\}_{z \in I_n^{1p}}$ in Theorem 3.10. We mention in passing one special situation where the operators $\partial^{(\beta)}_i$ do act on $\mathcal{O}^O_G$ as one would expect.

Proposition 3.23. Let $z \in I_n$ and $i \in [n-1]$ be such that $z(i) > z(i+1)$. Assume $z \neq s_i z s_i$ are both vexillary. Then $\partial^{(\beta)}_i \mathcal{O}^O_G = \mathcal{O}^O_{z s_i}.$

Proof. The operators $\partial^{(\beta)}_1, \partial^{(\beta)}_2, \ldots, \partial^{(\beta)}_{n-1}$ preserve $\text{IA}_n[\beta]$ so descend to operators on $CK(\mathfrak{Fl}_n) = \mathbb{Z}[\beta][x_1, \ldots, x_n]/\text{IA}_n[\beta]$. It is enough to prove $\partial^{(\beta)}_i [X^O_z] = [X^O_{z s_i}] \in CK(\mathfrak{Fl}_n)$, since then Theorems 2.19 and 3.2 imply that $\partial^{(\beta)}_i \mathcal{O}^O_z - \mathcal{O}^O_{z s_i} \in \text{IA}_N[\beta]$ for all $N \geq n$, which is only possible if $\partial^{(\beta)}_i \mathcal{O}^O_z = \mathcal{O}^O_{z s_i}$ by Lemma 3.3.
As explained in [46, §2.2], to prove that \( \partial^O_i[X^O_i] = \{X^O_{s_izs_j}\} \) it suffices to show that \( X^O_{s_izs_j} \) have rational singularities. Following our earlier convention, given an orbit \( E = Bg \in Fl_n \) where \( g \in GL_n \), let \( E_i \) be the subspace of \( C^n \) spanned by the first \( i \) rows of \( g \). For a subspace \( V \subseteq C^n \), write \( V^\perp \) for the subspace of linear maps in \( C^{n*} \) that vanish on \( V \). Define \( \tilde{X}^O_i \) to be the closure in \( LG_{2n} \times Fl_n \) of the set of pairs \( (U, E) \in LG_{2n} \times Fl_n \) satisfying
\[
\dim(U \cap (E_j \oplus E_i^\perp)) = j - \text{rank}(z_{i[j]}^j) \quad \text{for all } (i, j) \in \text{Ess}(D^O(z)).
\]
Since \( z \) is vexillary, the elements of \( \text{Ess}(D^O(z)) \) form a chain \( (i_1, j_1), \ldots, (i_s, j_s) \) in the order \( \leq \) from Lemma 3.22. If \( E \in Fl_n \) then
\[
E_{j_1} \oplus E_{i_1}^\perp \subseteq \cdots \subseteq E_{j_s} \oplus E_{i_s}^\perp
\]
is an isotropic flag in \( C^n \oplus C^{n*} \). This makes it clear than the fiber over \( E \in Fl_n \) of the obvious projection \( \tilde{X}^O_i \rightarrow Fl_n \) is isomorphic to a Schubert variety in \( LG_{2n} \). Schubert varieties have rational singularities [29 §8.2.2], so the same is true of \( \tilde{X}^O_i \) by [12 Théorème 2].

Let \( \iota : Fl_n \hookrightarrow LG_{2n} \times Fl_n \) be the inclusion \( E \mapsto (\Gamma(\alpha^O), E) \). We claim that \( X^O_i \) is the scheme-theoretic fiber \( \iota^{-1}(\tilde{X}^O_i) \). It follows from Lemma 3.24 that \( \iota^{-1}(\tilde{X}^O_i) \) and \( X^O_i \) agree as sets, so it suffices to show that \( \iota^{-1}(\tilde{X}^O_i) \) is reduced. Let \( \pi : \tilde{X}^O_i \rightarrow LG_{2n} \) be projection onto the first factor. Then \( \iota \) is an isomorphism \( \iota^{-1}(\tilde{X}^O_i) \rightarrow \pi^{-1}(\Gamma(\alpha^O)) \), and the fiber \( \pi^{-1}(\Gamma(\alpha^O)) \) is reduced because \( \pi \) is a fiber bundle over an open subset of \( LG_{2n} \) containing \( \Gamma(\alpha^O) \) [39, Lemma 5.3]. This establishes the claim, so \( X^O_i \) has rational singularities by [12 Théorème 3] and the fact that \( \tilde{X}^O_i \) has rational singularities. The same argument applied to \( s_izs_j \) shows that \( X^O_{s_izs_j} \) also has rational singularities, so we have \( \partial^O_i[X^O_i] = \{X^O_{s_izs_j}\} \in CK(Fl_n) \) by the discussion in [46 §2.2].

The orthogonal code of \( z \in I_n \) is the sequence \( c^O(z) = (c_1, c_2, \ldots, c_n) \) where \( c_i \) is the number of positions in the \( i \)th row of \( D^O(z) \). The orthogonal shape \( \lambda^O(z) \) of \( z \in I_n \) is the transpose of the partition sorting \( c^O(z) \). These objects are denoted \( \hat{c}(z) \) and \( \mu(z) \) in [46 §4.3]. If \( z = n \cdots 321 \in I_n \), for example, then we have \( \lambda^O(z) = (n-1, n-3, n-5, \ldots) \).

**Lemma 3.24** (See [39 §5.2]). Suppose \( z \in I_n \) is vexillary so that
\[
\text{Ess}(D^O(z)) = \{(i_1, j_1) \prec (i_1, j_1) \prec \cdots \prec (i_s, j_s)\}
\]
where \( \prec \) is the order in Lemma 3.22. Let \( \lambda, X, V, G, \) and \( F^* \) be given as follows:

(i) Define \( \lambda = \lambda^O(z) \) and \( X = Fl_n \).

(ii) Define \( V \) to be the trivial bundle \( C^n \oplus C^{n*} \) over \( Fl_n \) equipped with the skew-symmetric form \( \langle \cdot, \cdot \rangle^\perp \).

(iii) Define \( G \) to be the trivial bundle \( \Gamma(\alpha^O) \) over \( Fl_n \).

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Lemma 3.25. Assume positions \((p, q)\) is at most \(c\) distinct. By Lemma 3.22, we may assume that \(p\) this would imply that \((i, z)\) is vexillary. Then, in the notation of Definition 3.18, we have \(X^O_i = \Omega^O_{\lambda}(\mathcal{V}, \mathcal{G}, F^*)\).

**Proof.** It suffices to show that the rank conditions defining \(X^O_i\) (given in terms of \(z\)) are equivalent to the rank conditions defining \(\Omega^O_{\lambda}(\mathcal{V}, \mathcal{G}, F^*)\) (given in terms of \(\lambda\)), and this follows from [39] Lemmas 5.5 and 5.6. \(\square\)

Let \(\bar{x} := 0 \oplus x = \frac{x}{1 + x_m t}\) where \(\oplus\) is as in (3.3). For each \(z \in I_n\), let

\[S(z) := \{ q - \text{rank}(z_{[p][q]}^{(i)}) : (p, q) \in \text{Ess}(D^O(z)) \}\].

**Lemma 3.25.** Assume \(z \in I_n\) is vexillary. Then \(|S(z)| = |\text{Ess}(D^O(z))|\).

**Proof.** The Rothe diagram \(D(z)\) is formed by removing from \([n] \times [n]\) all positions \((i + j, z(i))\) and \((i, z(i) + j)\) for \(i \in [n]\) and \(j \geq 0\), and \(D^O(z)\) is the subset of positions \((p, q)\) \(D(z)\) with \(p \geq q\). Suppose \((p_1, q_1), (p_2, q_2) \in \text{Ess}(D^O(z))\) are distinct. By Lemma 3.22 we may assume that \(p_1 \geq p_2\) and \(q_1 \leq q_2\).

It suffices to show that rank \((z_{[p_2][q_2]}^{(i)} - z_{[p_1][q_1]}^{(i)}) < q_2 - q_1\). If \(q_1 = q_2\) then clearly rank \((z_{[p_2][q_2]}^{(i)} - z_{[p_1][q_1]}^{(i)}) \leq 0\) and we cannot have equality since this would imply that \((i, q_1) = (i, q_2) \in D^O(z)\) for all \(p_2 \leq i \leq p_1\), contradicting \((p_2, q_2) \in \text{Ess}(D^O(z))\). If \(q_1 < q_2\) then rank \((z_{[p_2][q_2]}^{(i)} - z_{[p_1][q_1]}^{(i)})\) is bounded above by the number of pairs \((i, z(i))\) with \(1 \leq i \leq p_2\) and \(q_1 < z(i) \leq q_2\), which is at most \(q_2 - q_1 - 1\) since no such pair has \(z(i) = q_2\) as \((p_2, q_2) \in D^O(z)\). \(\square\)

**Theorem 3.26.** Suppose \(z \in I_n\) is a vexillary involution with shape \(\lambda = \lambda^O(z)\). Let \(r\) be the smallest even integer with \(\ell(\lambda) \leq r\). For each \(i \in [\ell(\lambda)]\), let

\[c^{(i)} := \sum_{d \geq 0} c^{(i)} d^d = \prod_{m=1}^p (1 + x_m t) \prod_{m=1}^q (1 + \bar{x}_m t)^{-1} \] (3.13)

where \((p, q) \in \text{Ess}(D^O(z))\) is such that \(q - \text{rank}(z_{[p][q]}^{(i)}) = \min\{s \in S(z) : i \leq s\}\). If \(r = \ell(\lambda) + 1\) then also let \(c^{(r)} = 1\). The polynomial \(\Theta_i^O\) is then the Pfaffian of the \(r \times r\) skew-symmetric matrix whose \((i, j)\) entry for \(i < j\) is

\[R^{(i,j)} \left(1 - \beta T^{(i)}\right)^{r-i-\lambda_i} \left(1 - \beta T^{(j)}\right)^{r-j-\lambda_j} c^{(i)}_{\lambda_i} c^{(j)}_{\lambda_j} \] (3.14)

where \(T^{(i)}\) is the raising operator acting on \(c^{(i)}\) and \(R^{(i,j)}\) is defined by (3.3).

**Proof.** Since \(\text{codim}(X^O_i) = |\lambda^O(z)|\) (see [42] Theorem 4.6), Theorem 3.19 and Lemma 3.24 imply that \([X^O_i] \in CK(\text{Fl}_n)\) is the Pfaffian of the \(r \times r\) skew-symmetric matrix \(\mathfrak{R}\) with entries (3.10) where \(\lambda = \lambda^O(z), S = S(z)\), and \(c^{(i)} := c(\mathcal{V} - \mathcal{G} - (\mathcal{E}_q \oplus \mathcal{E}_q^+), t)\), where \(p\) and \(q\) are such that \((p, q) \in \text{Ess}(D^O(z))\) and \(q - \text{rank}(z_{[p][q]}^{(i)}) = \min\{s \in S(z) : i \leq s\}\). Using the triviality of \(\mathcal{V}\) and \(\mathcal{G}\),
the canonical isomorphism $E^*_l \cong (C^*/E_l)^*$, and the basic properties of Chern classes presented at the start of this section, we deduce that

$$c^{(i)} = \frac{1}{c(G, t)c(E_q, t)c(E_p, t)} = \frac{c(E_p, t)}{c(E_q, t)} = \frac{\prod_{m=1}^n (1 + x_m t)}{\prod_{m=1}^n (1 + x_m t)}.$$  

Thus $c^{(i)}$ is as in (3.13), so $\mathfrak{I}$ is the skew-symmetric matrix with entries (3.14), and we have $\text{pf}(\mathfrak{I}) = [X^O_2] \in CK(F_{1n}).$

Let $\mathfrak{I}'$ denote the ideal in $\mathbb{Z}[[x_1, \ldots, x_n]]$ generated by the symmetric formal power series, so that $\mathfrak{I}' = \mathfrak{I}' \cap \mathbb{Z}[x_1, \ldots, x_n]$. The entries of $\mathfrak{I}$ and therefore also $\text{pf}(\mathfrak{I})$, belong to the ring of formal power series $\mathbb{Z}[\beta][[x_1, \ldots, x_n]]$, and the assertion $\text{pf}(\mathfrak{I}) = [X^O_2] \in CK(F_{1n})$ means that $\text{pf}(\mathfrak{I}) \in \mathfrak{O}^O_2 + \mathfrak{I}'[[\beta]]$. We claim that in fact $\text{pf}(\mathfrak{I}) = \mathfrak{O}^O_2$ as polynomials.

Theorem 3.2 and Lemma 3.3 imply that $\mathfrak{O}^O_2$ is the unique polynomial with $\mathfrak{O}^O_2 + \mathfrak{I}'[[\beta]] = [X^O_2] \in CK(F_{1n})$ for all $N \geq n$. In fact, it follows that $\mathfrak{O}^O_2$ is unique among formal power series in $\mathbb{Z}[\beta][[x_1, x_2, \ldots]]$ that are polynomials in each fixed degree such that $\mathfrak{O}^O_2 + \mathfrak{I}'[[\beta]]$ coincides with the image of $[X^O_2]$ under the inclusion $\mathbb{Z}[\beta][x_1, \ldots, x_n]/\mathfrak{I}'[[\beta]] \to \mathbb{Z}[\beta][[x_1, \ldots, x_n]]/\mathfrak{I}'[[\beta]]$ for all $N \geq n$. But $\mathfrak{O}^O_2$ also has this property, since $\mathfrak{I}$ does not change if we replace $z$ by $z \times 1$. We must therefore have $\text{pf}(\mathfrak{I}) = \mathfrak{O}^O_2$.

**Example 3.27.** Let $z = 21 \in I_2$ so $\lambda = (1), r = 2$, and $\text{Ess}(D^O(z)) = \{(1, 1)\}$. Then $c^{(1)} = \frac{1}{1 + x_1 x_2}$ and $c^{(2)} = 1$ and $\mathfrak{O}^O_{21} = \text{pf}\left[\begin{smallmatrix} 0 & f \\ -f & 0 \end{smallmatrix}\right] = f := R^{(1, 2)}c^{(1)}c^{(2)}$. Since $1/T^{(2)}$ annihilates $c_1^{(1)}c_0^{(2)}$ and since $c_0^{(2)} = 1$, we have

$$f = \frac{1}{1 - \beta T^{(2)}}c^{(1)} = \sum_{m \geq 0} \beta^m c^{(1)}_{m+1} = \beta^{-1} \left(1 + x_1^2\beta + x_2^2\beta - 1\right) = 2x_1 + \beta x_2^2.$$  

This gives $\mathfrak{O}^O_{21} = 2x_1 + \beta x_2^2$, which agrees with Example 2.21.

Example 3.27 required a little algebra to simplify the infinite sums resulting from Theorem 3.26 to polynomials. We now describe a change of variable which handles these simplifications in general.

We have been working with certain expressions $c^{(i)}_m$ that we often view as formal indeterminates. Let $D_1, D_2, D_3 \ldots$ be another sequence of commuting indeterminates, and if $f$ is a linear combination of monomials $c^{(1)}_{m_1} \cdots c^{(1)}_{m_l}$, then define $\Phi(f)$ to be the formal sum obtained by replacing each $c^{(1)}_{m_i}$ by $D^{m_i}_i/m_i!$. Then $\Phi((1/T^{(i)})f) = \frac{\partial}{\partial x_i} \Phi(f)$ and $\Phi(Tf) = \int \Phi(f) dD_i$ where we write

$$\int g(D_1, \ldots, D_1) dD_i := \int_0^1 g(D_1, \ldots, u_i, \ldots, D_i) du.$$  

(3.15)

For example, we have

$$\Phi\left(\left(1 - \beta T^{(1)}\right)^{-1} c^{(1)}_1\right) = \Phi\left(\sum_{m \geq 0} \beta^m c^{(1)}_{m+1}\right) = \sum_{m \geq 0} \frac{\beta^m D^{m+1}_1}{(m+1)!} = \frac{e^{\beta D_1} - 1}{\beta}.$$  

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For integers \( r \in \mathbb{P} \) and \( a \in \mathbb{Z} \), define
\[
F_{r,a}(D) := \frac{1}{(r-1)!} \left( \frac{\partial}{\partial D} \right)^{r-a-1} \left( D^{r-1} e^{\beta D} \right)
\]
where \( \left( \frac{\partial}{\partial D} \right)^m := \int^{-m} \) if \( m < 0 \). We also set \( F_{0,a}(D) = D^a/a! \).

**Proposition 3.28.** For any integers \( r, s \in \mathbb{P} \) and \( a, b \in \mathbb{Z} \), the expression
\[
\Phi \left( R^{(1,2)} \left( 1 - \beta T^{(1)} \right)^{-r} \left( 1 - \beta T^{(2)} \right)^{-s} c_a^{(1)} c_b^{(2)} \right)
\]
is equal to
\[
e^{\beta D_1} \int_0^{D_1} e^{-\beta u} \left( F_{r,a-1}(u) F_{s,b}(u + D_2 - D_1) - F_{r,a}(u) F_{s,b-1}(u + D_2 - D_1) \right) du.
\]

**Proof.** Using the fact that \( (1 - \beta x)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{k} \beta^k x^k \) it is routine to verify \( \Phi((1 - \beta T^{(1)})^{-r} e_a) = F_{r,a}(D) \). Set \( M := (1 - \beta T^{(1)})^{-r} (1 - \beta T^{(2)})^{-s} c_a^{(1)} c_b^{(2)} \) so that \( \Phi(M) = F_{r,a}(D_1) F_{s,b}(D_2) \), and define
\[
\Theta(D_1, D_2) = \frac{\partial \Phi(M)}{\partial D_1} - \frac{\partial \Phi(M)}{\partial D_2} = F_{r,a-1}(D_1) F_{s,b}(D_2) - F_{r,a}(D_1) F_{s,b-1}(D_2).
\]
Now let \( G(D_1, D_2) \) be the expression in (3.16). We have
\[
\left( \frac{\partial}{\partial D_1} + \frac{\partial}{\partial D_2} - \beta \right) G(D_1, D_2) = \Phi \left( (1/T^{(1)} + 1/T^{(2)} - \beta) R^{(1,2)} M \right)
\]
\[
= \Phi \left( (1/T^{(1)} - 1/T^{(2)}) M \right)
\]
\[
= \Theta(D_1, D_2).
\]
The rational function in \( T^{(1)} \) and \( T^{(2)} \) appearing inside \( \Phi \) in (3.16) only involves nonnegative powers of \( T^{(1)} \) when expanded as a Laurent series in \( T^{(1)} \), so we have \( G(0, D_2) = 0 \). Thus, if we define \( \hat{G}(u) := G(u, u + D_2 - D_1) \), then \( \hat{G}(0) = 0 \). By the multivariate chain rule and our expression for \( \left( \frac{\partial}{\partial D_1} + \frac{\partial}{\partial D_2} - \beta \right) G(D_1, D_2) \) derived above, we deduce that
\[
\frac{\partial}{\partial u} \hat{G} - \beta \hat{G} = \frac{\partial}{\partial D_1}(u, u + D_2 - D_1) + \frac{\partial}{\partial D_2}(u, u + D_2 - D_1) - \beta \hat{G}
\]
\[
= \Theta(u, u + D_2 - D_1).
\]
Therefore \( \hat{G}(u) \) is the unique solution to the initial value problem \( \frac{\partial}{\partial u} \hat{G} - \beta \hat{G} = \Theta(u, u + D_2 - D_1) \) and \( \hat{G}(0) = 0 \), which one checks to be
\[
e^{\beta u} \int e^{-\beta u} \Theta(u, u + D_2 - D_1) du
\]
with integration as in (3.15). As \( G(D_1, D_2) = \hat{G}(D_1) \), the result follows.

The next proposition gives an algorithm for computing the inverse map \( \Phi^{-1} \).
Proposition 3.29. Suppose \( G \) is a formal infinite linear combination of monomials in the \( D_i \) with coefficients in \( \mathbb{Q}[\beta] \). The following properties then hold:

(a) Interpreting \( D_i \) as \( \frac{\partial}{\partial t_i}\big|_{t_i=0} \), we have \( \Phi^{-1}(G) = G \left( c^{(1)}(t_1) \cdots c^{(i)}(t_i) \right) \).

(b) Interpreting \( D_i \) as \( \frac{\partial}{\partial t_i}\big|_{t_i=0} \), we have \( \Phi^{-1}(D_i e^{\beta D_i}) = \frac{\partial^m}{\partial t^m} c^{(i)} \big|_{t=\beta} \).

Proof. For part (a), observe that \( \Phi^{-1}(D_i e^{\beta D_i}) = c_m(i) = \frac{\partial^m}{\partial t^m} c^{(i)} \big|_{t=0} \). Part (b) holds since we have \( D_i e^{\beta D_i} = \sum d \geq 0 \frac{\partial^d}{\partial t^d} \frac{\partial^{m+d}}{\partial t^{m+d}} c^{(i)} \big|_{t=0} = \frac{\partial^m}{\partial t^m} c^{(i)} \big|_{t=\beta} \). \( \square \)

Example 3.30. Let us compute \( \Theta^O \) for \( z = 4321 = (1,4)(2,3) \). We have

\[
D^O(z) = \begin{pmatrix}
\circ & \times & \circ & \times \\
\circ & \circ & \times & \cdot \\
\times & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot 
\end{pmatrix}
\]

so \( \text{Ess}(D^O(z)) = \{(2,2)\} \) and \( \lambda^O(z) = (2,1) \). Theorem 3.26 implies that

\[
\Theta^O_{4321} = R^{(1,2)} \left( 1 - \beta T^{(1)} \right)^{-1} \left( 1 - \beta T^{(2)} \right)^{-1} c_1^{(1)} c_2^{(2)}
\]

where \( c^{(1)} = c^{(2)} = \frac{1}{1-x_1} \frac{1}{1-x_2} \). Following Proposition 3.28 we compute

\[
\Theta(D_1, D_2) = F_{1,1}(D_1) F_{1,2}(D_2) - F_{1,2}(D_1) F_{1,1}(D_2)
= \frac{1}{\beta} D_1 e^{\beta D_2} - \frac{1}{\beta^2} e^{\beta D_1} + \frac{1}{\beta^2}
\]

and then

\[
\Phi(\Theta^O_{4321}) = e^{\beta D_1} \int_0^{D_1} e^{-\beta u} \Theta(u, u + D_2 - D_1) du
= \frac{1}{\beta^2} D_1^2 e^{\beta D_2} - \frac{1}{\beta^3} D_1 e^{\beta D_1} + \frac{1}{\beta^3} e^{\beta D_1} - \frac{1}{\beta^3}.
\]

Finally, using Proposition 3.29 we compute that

\[
\Theta^O_{4321} = \Phi^{-1} \left( \frac{1}{\beta^2} D_1^2 e^{\beta D_2} - \frac{1}{\beta^3} D_1 e^{\beta D_1} + \frac{1}{\beta^3} e^{\beta D_1} - \frac{1}{\beta^3} \right)
= \frac{1}{\beta^2} \frac{\partial^2}{\partial t^2} c^{(1)} \big|_{t=0} - \frac{1}{\beta^3} \frac{\partial}{\partial t} c^{(1)} \big|_{t=\beta} + \frac{1}{\beta^3} \cdot c^{(1)} \big|_{t=\beta} - \frac{1}{\beta^3}
= 4x_1^2 x_2^2 + 4x_1^2 x_2 + 2\beta x_1 x_3^2 + 8\beta^2 x_1 x_2^2 + 2\beta x_1 x_2 + 3\beta x_1 x_2^2 + 3\beta^3 x_1^3 x_2^3
= (x_1 + x_1)(x_1 + x_2)(x_2 + x_2)
\]

which agrees with Theorem 3.38.
Example 3.31. Let $z = 4571263 = (1, 4)(2, 5)(3, 7) \in I_7$, so that

$$D^O(z) = \begin{pmatrix}
\circ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \cdot & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \circ & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \circ & \circ & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}$$

and $\text{Ess}(D^O(z)) = \{(6, 3), (3, 3)\}$ and $\lambda^O(z) = (4, 2, 1)$. In the notation of Theorem 3.26, one has $r = 4$, $S = \{1, 3\}$, $c(4) = 1$,

$$c^{(1)} = \frac{(1+x_1 t)(1+x_2 t)(1+x_3 t)}{(1+x_1 t)(1+x_2 t)(1+x_3 t)}, \quad \text{and} \quad c^{(2)} = c^{(3)} = \frac{(1+x_1 t)(1+x_2 t)(1+x_3 t)}{(1+x_1 t)(1+x_2 t)(1+x_3 t)}.$$

Theorem 3.26 tells us that $\mathbf{\Theta}^O = \text{pf}(\mathfrak{M})$ where $\mathfrak{M}$ is the $4 \times 4$ skew-symmetric matrix with entries

$$\mathfrak{M}_{12} := R^{(1, 2)} \left(1 - \beta T^{(1)}\right)^{-1} \left(1 - \beta T^{(2)}\right)^0 c^{(1)}_4 c^{(2)}_2,$$

$$\mathfrak{M}_{13} := R^{(1, 2)} \left(1 - \beta T^{(1)}\right)^{-1} \left(1 - \beta T^{(3)}\right)^0 c^{(1)}_4 c^{(1)}_1,$$

$$\mathfrak{M}_{14} := R^{(1, 2)} \left(1 - \beta T^{(1)}\right)^{-1} \left(1 - \beta T^{(4)}\right)^0 c^{(1)}_4 c^{(1)}_0,$$

$$\mathfrak{M}_{23} := R^{(1, 2)} \left(1 - \beta T^{(2)}\right)^0 \left(1 - \beta T^{(3)}\right)^0 c^{(2)}_2 c^{(3)}_1,$$

$$\mathfrak{M}_{24} := R^{(1, 2)} \left(1 - \beta T^{(2)}\right)^0 \left(1 - \beta T^{(4)}\right)^0 c^{(2)}_2 c^{(4)}_0,$$

$$\mathfrak{M}_{34} := R^{(1, 2)} \left(1 - \beta T^{(3)}\right)^0 \left(1 - \beta T^{(4)}\right)^0 c^{(3)}_1 c^{(4)}_0.$$

Calculating as in Example 3.30 we find that $\mathbf{\Theta}^O_{4571263} \in \mathbb{N}[\beta][x_1, x_2, \ldots, x_6]$ is a polynomial with 865 terms which begins as

$$\beta^{11} x_1^5 x_2^5 x_3 x_4 x_5 x_6 + \beta^{10} x_1^5 x_2^5 x_3^2 x_4 x_5 + \beta^{10} x_1^5 x_2^5 x_3 x_4 x_6 + \beta^{10} x_1^5 x_2^5 x_3^2 x_5 x_6 + 5 \beta^{10} x_1^5 x_2^5 x_3 x_4 x_5 x_6 + 5 \beta^{10} x_1^5 x_2^5 x_3^2 x_4 x_5 x_6$$

$$+ \beta^{10} x_1^5 x_2^5 x_3^3 x_4 x_5 x_6 + \beta^{10} x_1^4 x_2^4 x_3^3 x_4 x_5 x_6 + \beta^{10} x_1^4 x_2^4 x_3 x_4 x_5 x_6 + \beta^{10} x_1^4 x_2^4 x_3^2 x_4 x_5 x_6 + \beta^{10} x_1^4 x_2^4 x_3^3 x_4 x_5 x_6$$

$$+ \beta^{10} x_1^4 x_2^4 x_3^4 x_4 x_5 x_6 + \beta^{10} x_1^4 x_2^4 x_3^5 x_4 x_5 x_6 + \beta^{10} x_1^5 x_2^5 x_3^4 x_4 x_5 x_6 + \beta^{10} x_1^5 x_2^5 x_3^5 x_4 x_5 x_6$$

$$+ ( \ldots \text{terms of lower degree in } \beta \ldots ).$$

This polynomial has only 35 distinct nonzero coefficients, given by

$$\left\{ 1, 2, 4, 5, 7, 8, 9, 10, 12, 16, 24, 28, 30, 32, 34, 41, 43, 64, 65, 72, 80, 109, 110, 116, 121, 128, 142, 159, 173, 177, 180, 246, 261, 292, 344 \right\},$$

The entries of $\mathfrak{M}$ are not all polynomials, although $\text{pf}(\mathfrak{M})$ is a polynomial.
4 Stable Grothendieck polynomials

The limit of a sequence of polynomials or formal power series is defined to converge if the sequence of coefficients of any fixed monomial is eventually constant. Let \( n \in \mathbb{P} \) and \( w \in S_n \). Given \( m \in \mathbb{N} \), define \( 1^m \times w \in S_{m+n} \) to be the permutation that maps \( i \mapsto i \) for \( i \leq m \) and \( i + m \mapsto w(i) + m \) for \( i \in \mathbb{P} \). The stable Grothendieck polynomial of \( w \) is then

\[
G_w := \lim_{n \to \infty} \mathcal{S}_{1^n \times w} \in \mathbb{Z}[\beta][[x_1, x_2, \ldots]].
\] (4.1)

Remarkably, this limit always converges and the resulting power series is a symmetric function in the \( x_i \) variables with many notable properties \([6, \S 2]\).

In this section, we study the natural analogues of (4.1) for orthogonal and symplectic Grothendieck polynomials.

4.1 \( K \)-theoretic symmetric functions

We start by reviewing some properties of \( G_w \) and related symmetric functions. If \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \) is an integer partition, then a set-valued tableau of shape \( \lambda \) is a map \( T : (i, j) \mapsto T_{ij} \) from the Young diagram

\[
D_{\lambda} := \{(i, j) \in \mathbb{P} \times \mathbb{P} : j \leq \lambda_i \}
\]

to the set of finite, nonempty subsets of \( \mathbb{P} \). For such a map \( T \), define

\[
x^T := \prod_{(i, j) \in D_{\lambda}} \prod_{k \in T_{ij}} x_k \quad \text{and} \quad |T| := \sum_{(i, j) \in D_{\lambda}} |T_{ij}|.
\]

A set-valued tableau \( T \) is semistandard if one has \( \max(T_{ij}) \leq \min(T_{i,j+1}) \) and \( \max(T_{ij}) < \min(T_{i+1,j}) \) for all relevant \( (i, j) \in D_{\lambda} \). Let SetSSYT(\( \lambda \)) denote the set of semistandard set-valued tableaux of shape \( \lambda \).

**Definition 4.1.** The stable Grothendieck polynomial of a partition \( \lambda \) is

\[
G_{\lambda} := \sum_{T \in \text{SetSSYT}(\lambda)} \beta^{|T| - |\lambda|} x^T \in \mathbb{Z}[\beta][[x_1, x_2, \ldots]].
\]

This definition sometimes appears in the literature with the parameter \( \beta \) set to \( \pm 1 \). This specialization is immaterial to most results since if we write \( G^{(\beta)}_{\lambda} = G_{\lambda} \) then \( (-\beta)^{\lambda \lambda} G^{(\beta)}_{\lambda} = G^{(\beta^{-1})}_{\lambda}(-\beta x_1, -\beta x_2, \ldots) \). Setting \( \beta = 0 \) transforms \( G_{\lambda} \) to the usual Schur function \( s_{\lambda} \).

The symmetric functions \( G_{\lambda} \) are related to \( G_w \) for \( w \in S_n \) by the following theorems. Given a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \) with \( k + \lambda_1 \leq n \), define \( w_{\lambda} \in S_n \) to be the unique permutation with \( w_{\lambda}(i) = i + \lambda_{k+1-i} \) for \( i \in [k] \) and \( w_{\lambda}(i) < w_{\lambda}(i+1) \) for all \( k < i \leq n \). Write \( \mathcal{P} \) for the set of all partitions.

**Theorem 4.2** ([6 Theorem 3.1]). If \( \lambda \) is any partition then \( G_{w_{\lambda}} = G_{\lambda} \).

**Theorem 4.3** ([7 Theorem 1]). If \( w \in S_n \) then \( G_w \in \mathbb{N}[\beta]-\text{span}\{G_{\lambda} : \lambda \in \mathcal{P}\} \).
Buch [6] also derives a Littlewood-Richardson rule for the stable Grothendieck polynomials $G_\lambda$, which shows that the product $G_\lambda G_\mu$ is always a finite $\mathbb{N}[\beta]$-linear combination of $G_\nu$’s.

There are shifted analogues of $G_\lambda$ that will be related in a similar way to our orthogonal and symplectic analogues of [11]. Define the marked alphabet to be totally ordered set $M := \{1' < 1 < 2' < 2 < \ldots \}$, and write $|i'| := |i| = i$ for $i \in \mathbb{P}$. If $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0)$ is a strict partition, then a shifted set-valued tableau of shape $\lambda$ is a map $T: (i, j) \mapsto T_{ij}$ from the shifted diagram $SD_\lambda := \{(i, i+j-1) \in \mathbb{P} \times \mathbb{P}: 1 \leq j \leq \lambda_i\}$ to the set of finite, nonempty subsets of $M$. Given such a map, define

$$\begin{align*}
x^T := \prod_{(i,j) \in SD_\lambda} \prod_{k \in T_{ij}} x_{|k|} \quad \text{and} \quad |T| := \sum_{(i,j) \in SD_\lambda} |T_{ij}|.
\end{align*}$$

A shifted set-valued tableau $T$ is semistandard if for all relevant $(i, j) \in SD_\lambda$:

1. $\max(T_{ij}) \leq \min(T_{i,j+1})$ and $T_{ij} \cap T_{i,j+1} \subseteq \{1, 2, 3, \ldots\}$.
2. $\max(T_{ij}) \leq \min(T_{i+1,j})$ and $T_{ij} \cap T_{i+1,j} \subseteq \{1', 2', 3', \ldots\}$.

In such tableaux, an unprimed number can appear at most once in a column, while a primed number can appear at most one in a row. Let SetSSMT($\lambda$) denote the set of semistandard shifted set-valued tableaux of shape $\lambda$.

**Definition 4.4.** The $K$-theoretic Schur $P$-function and $K$-theoretic Schur $Q$-function of a strict partition $\lambda$ are the formal power series

$$\begin{align*}
GP_\lambda := \sum_{T \in \text{SetSSMT}(\lambda)} \beta^{|T|-|\lambda|} x^T \quad \text{and} \quad GQ_\lambda := \sum_{T \in \text{SetSSMT}(\lambda)} \beta^{|T|-|\lambda|} x^T.
\end{align*}$$

The summation defining $GP_\lambda$ is over shifted set-valued tableaux with no primed numbers in any position on the main diagonal.

These definitions are due to Ikeda and Naruse [26], who also show that $GP_\lambda$ and $GQ_\lambda$ are symmetric in the $x_i$ variables [25, Theorem 9.1]. Setting $\beta = 0$ transforms $GP_\lambda$ and $GQ_\lambda$ to the Schur $P$- and $Q$-functions $P_\lambda$ and $Q_\lambda$.

**Example 4.5.** We have $GP_{(1)} = G_{(1)} = s_{(1)} + \beta s_{(1,1)} + \beta^2 s_{(1,1,1)} + \cdots$ while

$$\begin{align*}
GQ_{(1)} := \sum_{m \in \mathbb{P}} \sum_{1 \leq i_1 < i_2 < \cdots < i_m} \beta^{n-1} (x_{i_1} + x_{i_1})(x_{i_2} + x_{i_2}) \cdots (x_{i_m} + x_{i_m})
\end{align*}$$

where $x \oplus y := x + y + \beta xy$ as in (3.3).

Clifford, Thomas, and Yong prove a Littlewood-Richardson rule for the $GP_\lambda$’s in [10], which shows that each product $GP_\lambda GP_\mu$ is a finite $\mathbb{N}[\beta]$-linear combination of $GP_\nu$’s with positive coefficients; see the discussion in [16, §1]. A general Littlewood-Richardson rule for the $K$-theoretic Schur $Q$-functions $GQ_\lambda$ is not yet known. Each product $GQ_\lambda GQ_\mu$ is a linear combination of $GQ_\nu$’s [26, Proposition 3.5], but it is an open problem to determine if these combinations are always finite [26, Conjecture 3.2].
4.2 Orthogonal and symplectic variants

Assume \( n \) is even and let \( z \in I_n^{\text{PFF}} \) be a fixed-point-free involution in \( S_n \). Given \( m \in \mathbb{N} \), let \((21)^m \times z = 21 \times 21 \times \cdots \times 21 \times z \in I_{n+2m}^{\text{PFF}} \) denote the involution that maps \( i \mapsto i - (−1)^i \) for \( i \leq 2m \) and \( i + 2m \mapsto z(i) + 2m \) for \( i \in \mathbb{P} \). We define the 

\textit{symplectic stable Grothendieck polynomials} of \( z \) to be the limit

\[
GP^S_z := \lim_{m \to \infty} \varnothing^{Sp}_{(21)^m \times z}.
\]

(4.2)

These limits are always defined and have the following formula:

**Corollary 4.6.** If \( z \in I_n^{\text{PFF}} \) then \( GP^S_z = \sum_{w \in B_{\text{PFF}}(z)} \beta^{\ell(w)−\ell_{\text{PFF}}(z)}G_w \).

**Proof.** Proposition [3.15] implies that \( B_{\text{PFF}}((21)^m \times z) = \{1^{2m} \times w : w \in B_{\text{PFF}}(z)\} \) for all \( z \in I_n^{\text{PFF}} \) and \( m \in \mathbb{P} \), so this follows from Theorem 3.12.

Let \( \mathcal{P}_{\text{strict}} \) denote the set of all strict partitions. The symmetric functions \( GP^S_z \) were studied in [35], which proves the following analogue of Theorem 4.3:

**Theorem 4.7** ([35, Theorem 1.9]). If \( z \in I_n^{\text{PFF}} \) then

\[
GP^S_z \in \mathbb{N}[\beta]-\text{span} \{GP_\lambda : \lambda \in \mathcal{P}_{\text{strict}}\}.
\]

There is also a symplectic analogue of Theorem 4.2, which shows that every \( K \)-theoretic Schur \( P \)-function occurs as \( GP^S_z \) for some \( n \in 2\mathbb{P} \) and \( z \in I_n^{\text{PFF}} \); see [36]. We mention one corollary of [35, Theorem 1.9 and Corollary 3.27]:

**Corollary 4.8** (See [35]). If \( n \in 2\mathbb{P} \) then \( GP^S_{n-321} = GP_{(n-2,n-4,n-6,\ldots,2)} \).

For the rest of this section let \( n \in \mathbb{P} \) be arbitrary and suppose \( z \in I_n \). We wish to define the 

\textit{orthogonal stable Grothendieck polynomial} of \( z \) by

\[
GQ^O_z := \lim_{m \to \infty} \varnothing^O_{1^m \times z}.
\]

(4.3)

Unlike (4.2), it is not clear that this limit exists for an arbitrary involution, though we expect that this is always the case.

By Theorem 5.26 we at least know that \( GQ^O_z \) is a well-defined power series when \( z \in I_\infty \) is vexillary, since then \( 1^m \times z \) is also vexillary with

\[
D^O(1^m \times z) = \{(i + m, j + m) : (i, j) \in D^O(z)\}
\]

for all \( m \in \mathbb{N} \), so the corresponding sequence of Pfaffian formulas for \( \varnothing^O_{1^m \times z} \) obviously converges. Since the matrix entries (3.14) are symmetric when \( p, q \to \infty \), the power series \( GQ^O \) is also symmetric when \( z \) is vexillary. Our last main result will show that in this case \( GQ^O_z \) is actually a single \( K \)-theoretic Schur \( Q \)-function.

For this, we require the following theorem of Nakagawa and Naruse [38, Theorem 5.2.4]. Write \( pf[a_{ij}]_{1 \leq i < j \leq m} \) for the Pfaffian of the \( m \times m \) skew symmetric matrix \( A \) whose entries satisfy \( A_{ij} = −A_{ji} = a_{ij} \) for \( i < j \). Define

\[
GQ(u, v) := \frac{1}{1 + \beta u} \prod_{j=1}^\infty \frac{1 + \beta x_j}{1 - x_j u} \cdot (1 + (u + \beta)x_j) \in \mathbb{Z}[\beta][[u, v, x_1, x_2, \ldots]]
\]
and
\[ \Delta(u, v) := \frac{1 - uv}{(1 + \beta u)(1 + \beta u + uv)} \in \mathbb{Z}[\beta][[u, v]]. \]

Finally, for any \(a, b \in \mathbb{N}\) let
\[ GQ_{(a, b)} := [u^{-a}v^{-b}]GQ(u^{-1}, u)GQ(v^{-1}, v)\Delta(u, v^{-1}). \]

**Theorem 4.9** ([RS Theorem 5.7]). Let \(\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0)\) be a strict partition with \(r \in 2\mathbb{P}\) parts, the last of which may be zero. Then
\[ GQ_\lambda = \text{pf} \left[ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \beta^{k+l} \binom{i + 1 - r}{k} \binom{j - r}{l} GQ_{(\lambda_i + k, \lambda_j + l)} \right]_{1 \leq i < j \leq r}. \]

In the next two results, let \(c^{(i)}(u) = \prod_{j=1}^{\infty} \frac{1 + x_j u}{1 + x_j v}\) for \(x \in \mathbb{P}\) where \(x := \frac{-x}{1 + y}\), and define \(R^{(i,j)}\) for \(i, j \in \mathbb{P}\) as in (3.9).

**Lemma 4.10.** If \(a, b \in \mathbb{N}\) then
\[ GQ_{(a, b)} = R^{(1,2)} \left( 1 - \beta T^{(1)} \right)^{-a+1} \left( 1 - \beta T^{(2)} \right)^{-b} c_a^{(1)} c_b^{(2)}. \]

**Proof.** Abbreviate by setting \(T := T^{(1)}\) and \(c(u) = \sum_{j=0}^{\infty} c_j u^j := c^{(i)}(u)\), and note that we then have \(GQ(u, v) = \frac{1}{1 + \beta v} \cdot c(u + \beta).\) We compute
\[
\begin{align*}
    c(u + \beta) &= \sum_{j \geq 0} c_j \sum_{m=0}^{j} \binom{j}{m} \beta^{m-j} u^m \\
    &= \sum_{m \geq 0} u^m \sum_{j \geq m} \binom{j}{m} c_j \beta^{m-j} \\
    &= \sum_{m \geq 0} u^m \sum_{j \geq m} \binom{j}{m} \beta^{-m} T^{j-m} c_m = \sum_{m \geq 0} (1 - \beta T)^{-m-1} c_m u^m. 
\end{align*}
\]

From this, it follows that if \(i \in \mathbb{Z}\) then
\[
    u^i c(u^{-1} + \beta) = (1 - \beta T)^{-i} \sum_{m \geq -i} (1 - \beta T)^{-m-1} c_{m+i} u^{-m}. 
\]

Since \(c_{m+i} = T^i c_m\) if \(m \geq \max\{0, -i\}\), we deduce that
\[
    [u^m] \left( u^i c(u^{-1} + \beta) \right) = [u^m] \left( T^i (1 - \beta T)^{-i} c(u^{-1} + \beta) \right) 
\]
for all \(i \in \mathbb{Z}\) and \(m \leq 0\).

One can check that substituting \(u \mapsto T^{(1)}_{1-\beta T^{(1)}}\) and \(v \mapsto T^{(2)}_{1-\beta T^{(2)}}\) transforms
\[
    \Delta(u, v^{-1}) \cdot \frac{1}{1 + \beta u} \cdot \frac{1}{1 + \beta v} \quad \mapsto \quad R^{(1,2)} \left( 1 - \beta T^{(1)} \right)^2 \left( 1 - \beta T^{(2)} \right). 
\]
Fix \(a, b \in \mathbb{N}\). Since \(GQ_{(a,b)}\) is the coefficient \(u^{-a}v^{-b}\) in
\[
\Delta(u, v^{-1}) \cdot \frac{1}{1 + \beta u} \cdot \frac{1}{1 + \beta v} \cdot c^{(1)}(u^{-1} + \beta)c^{(2)}(v^{-1} + \beta),
\]
it follows from (4.5) that \(GQ_{(a,b)}\) is also the coefficient of \(u^{-a}v^{-b}\) in
\[
R^{(1,2)} \left(1 - \beta T^{(1)}\right)^2 \left(1 - \beta T^{(2)}\right) c^{(1)}(u^{-1} + \beta)c^{(2)}(v^{-1} + \beta).
\]
The result is now clear after using (4.4) to rewrite this last expression as
\[
\sum_{m,n \geq 0} R^{(1,2)} \left(1 - \beta T^{(1)}\right)^{-m+1} \left(1 - \beta T^{(2)}\right)^{-n} u^{-m}v^{-n}.
\]

We may now state our final theorem.

**Theorem 4.11.** If \(z \in I_n\) is vexillary then \(GQ_{z}^O = GQ_{\lambda^O(z)}^O\).

**Proof.** Fix a vexillary involution \(z \in I_n\). Let \(\lambda = \lambda^O(z)\) and define \(r\) to be the smallest even integer with \(r \geq \ell(\lambda)\). As noted at the beginning of this section, Theorem 3.26 implies that \(GQ_{\lambda}^O\) is the Pfaffian of the \(r \times r\) skew-symmetric matrix whose \((i, j)\) entry for \(i < j\) is
\[
R^{(i,j)} \left(1 - \beta T^{(i)}\right)^{-i} \left(1 - \beta T^{(j)}\right)^{-j} c^{(i)}_{\lambda, c^{(j)}_{\lambda}}.
\]
Thus, it suffices to show that \(GQ_{\lambda}^O\) is given by the same Pfaffian. It follows from Theorem 4.9 and Lemma 4.10 that \(GQ_{\lambda}^O\) is the Pfaffian of the \(r \times r\) skew-symmetric matrix whose \((i, j)\) entry for \(i < j\) is
\[
R^{(i,j)} \sum_{k,l \geq 0} \beta^{k+l} \binom{i+1-r}{k} \binom{j-r}{l} \left(1 - \beta T^{(i)}\right)^{-i} \left(1 - \beta T^{(j)}\right)^{-j} c^{(i)}_{\lambda, c^{(j)}_{\lambda}}.
\]
But we have
\[
\sum_{k \geq 0} \beta^k \binom{i+1-r}{k} \left(1 - \beta T^{(i)}\right)^{-i} c^{(i)}_{\lambda, c^{(i)}_{\lambda}} = \left( \sum_{k \geq 0} \beta T^{(i)} \right)^k \binom{i+1-r}{k} \left(1 - \beta T^{(i)}\right)^{i+1-r-k} \left(1 - \beta T^{(i)}\right)^{-i} c^{(i)}_{\lambda, c^{(i)}_{\lambda}} = \left(1 - \beta T^{(i)} + \beta T^{(i)}\right) \binom{i+1-r}{k} \left(1 - \beta T^{(i)}\right)^{-i} c^{(i)}_{\lambda, c^{(i)}_{\lambda}} = \left(1 - \beta T^{(i)}\right)^{-i} c^{(i)}_{\lambda, c^{(i)}_{\lambda}}
\]
and similarly
\[
\sum_{l \geq 0} \beta^l \binom{j-r}{l} \left(1 - \beta T^{(j)}\right)^{-j} c^{(j)}_{\lambda, c^{(j)}_{\lambda}} = \left(1 - \beta T^{(j)}\right)^{-j} c^{(j)}_{\lambda, c^{(j)}_{\lambda}}.
\]
Thus (4.6) is equal to \(R^{(i,j)} \left(1 - \beta T^{(i)}\right)^{-i} \left(1 - \beta T^{(j)}\right)^{-j} c^{(i)}_{\lambda, c^{(j)}_{\lambda}}\) which suffices to prove the theorem. \(\square\)
Corollary 4.12. If \( n \in \mathbb{P} \) then \( GQ_{n \cdots 321} = GQ_{(n-1, n, n-3, n-5, \ldots)} \).

Proof. It suffices to observe that \( \lambda^0(n \cdots 321) = (n-1, n, n-3, n-5, \ldots). \)

Following [20], we say that an involution \( z \in I_n \) is \( I \)-Grassmannian if there are integers \( r \in \mathbb{N} \) and \( 1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n \) such that
\[
z = (\phi_1, n+1)(\phi_2, n+2) \cdots (\phi_r, n+r).
\]

The case \( n = r = 0 \) corresponds to \( z = 1 \). Computing \( \lambda^0(z) \) gives the following:

Corollary 4.13. If \( z \in I_n \) is \( I \)-Grassmannian of the form (4.7), then
\[
GQ^0_z = GQ_{(n+1-\phi_1, n+1-\phi_2, \ldots, n+1-\phi_r)}.
\]

Thus, every \( K \)-theoretic Schur \( Q \)-function occurs as \( GQ^0_z \) for some \( z \), since for any strict partition \( \lambda \) there is an \( I \)-Grassmannian involution of shape \( \lambda \).

5 Open problems

We conclude with a list of related open problems.

Each \( \Theta^\mathcal{Sp} \) is a finite linear combination of \( \Theta_w \)'s, whose summands are described by Proposition 3.15. It remains to find analogous results for \( \Theta^O_z \):

Problem 5.1. Describe the set of summands expanding \( \Theta^O_z \) as a \( \mathbb{Z}[\beta] \)-linear combination of \( \Theta_w \)'s. Are the coefficients in this expansion all nonnegative?

Lenart [32] proves a “transition formula” which expands \( (1 + \beta x_j) \Theta_w \) as a finite, \( \mathbb{N}[\beta] \)-linear combination of \( \Theta_v \)'s. The sequel to this paper [36] describes an analogous formula involving the symplectic Grothendieck polynomials \( \Theta^\mathcal{Sp} \).

Problem 5.2. Is there a transition formula in the sense of [32, 36] for \( GQ^O_z \)?

It remains to show that \( GQ^O_z \) is well-defined with \( z \in I_n \) not vexillary.

Problem 5.3. Show that \( GQ^O_z := \lim_{m \to \infty} \Theta^O_{1^m \cdot z} \) converges for all \( z \in I_n \).

Recall that \( \mathcal{P} \) and \( \mathcal{P}_{\text{strict}} \) denote the sets of arbitrary and strict partitions.

Problem 5.4. Does it always hold that \( GQ^O_z \in \bigoplus_{\lambda \in \mathcal{P}_{\text{strict}}} \mathbb{N}[\beta]GQ_{\lambda} \)?

It is known that if \( \lambda, \mu \in \mathcal{P}_{\text{strict}} \) then \( GQ_{\lambda} GQ_{\mu} \in \sum_{\nu \in \mathcal{P}_{\text{strict}}} \mathbb{Z}[\beta]GQ_{\nu} \), where the sum could involve infinitely many \( GQ_{\nu} \)'s. The following problem, asserting that the sum is always finite, is [26 Conjecture 3.2].

Problem 5.5. Show that if \( \lambda, \mu \in \mathcal{P}_{\text{strict}} \) then \( GQ_{\lambda} GQ_{\mu} \in \bigoplus_{\nu \in \mathcal{P}_{\text{strict}}} \mathbb{N}[\beta]GQ_{\nu} \).

If \( \lambda \in \mathcal{P}_{\text{strict}} \) then \( GQ_{\lambda} \in \sum_{\mu \in \mathcal{P}} \mathbb{Z}[\beta]G_{\mu} \) since this is true with \( GQ_{\lambda} \) replaced by any power series in \( \mathbb{Z}[\beta][[x_1, x_2, \ldots]] \) that is symmetric in the \( x_i \) variables. This expansion could be an infinite sum, but we expect that it is also finite:

Problem 5.6. Show that if \( \lambda \) is a strict partition then \( GQ_{\lambda} \in \bigoplus_{\mu \in \mathcal{P}} \mathbb{N}[\beta]G_{\mu} \).
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