ON THE COMPACT REAL FORMS OF THE LIE ALGEBRAS OF TYPE $E_6$ AND $F_4$

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Abstract. We give a construction of the compact real form of the Lie algebra of type $E_6$, using the finite irreducible subgroup of shape $3^4:3\cdot SL_3(3)$, which is isomorphic to a maximal subgroup of the orthogonal group $O_7(3)$. In particular we show that the algebra is uniquely determined by this subgroup. Conversely, we prove from first principles that the algebra satisfies the Jacobi identity, and thus give an elementary proof of existence of a Lie algebra of type $E_6$. The compact real form of $F_4$ is exhibited as a subalgebra.

1. Introduction

The standard construction of the complex simple Lie algebras using their root systems (see for example Carter’s book [4]) yields a so-called Chevalley basis, with respect to which the structure constants of the algebra are all integers. This basis can therefore be used to define Lie algebras over any field. In particular, these algebras over the real numbers are known as the split real forms of the simple Lie algebras. The corresponding Lie groups are not compact.

On the other hand, Lie groups which arise in practical applications often are compact, and it is desirable to have good constructions of these groups and the corresponding algebras, both of which are known as the compact real forms. It is well-known that every complex simple Lie algebra has a unique compact real form, and a suitable basis may be obtained from a Chevalley basis by replacing each pair \{\(e_r, e_{-r}\}\) of root vectors by the pair \(\{e_r + e_{-r}, \sqrt{-1}(e_r - e_{-r})\}\), and replacing each basis vector \(h_r\) of the Cartan subalgebra by \(\sqrt{-1}h_r\) (see for example p. 149 of [7]).

The simplest example of all is the algebra of type $A_1$. The split real form is usually taken with respect to a basis \(\{e, f, h\}\) and Lie products

\[
[h, e] = 2e, \quad [h, f] = -2f.
\]

Defining

\[
i = \sqrt{-1}h/2, \quad j = (e - f)/2, \quad k = \sqrt{-1}(e + f)/2,
\]

we easily compute \([i, j] = k, [j, k] = i, \text{ and } [k, i] = j\), so that we obtain the familiar ‘cross product’ on Euclidean 3-space. More generally, in the case of the orthogonal groups, it makes sense to take an orthonormal basis \(\{v_1, \ldots, v_n\}\) for the Euclidean space, on which the (compact) orthogonal group acts naturally. Then
the Lie algebra is essentially the exterior square of this module, so has a basis \( \{ v_i \wedge v_j = -v_j \wedge v_i \} \), which is more or less equivalent to the modified Chevalley basis described above. The Lie product is easily described with respect to this basis by

\[
[v_i \wedge j, v_j \wedge k] = v_i \wedge v_k
\]

for distinct \( i, j, k \), all other products of basis vectors being 0.

In the case of the five exceptional simple Lie algebras, \( G_2, F_4, E_6, E_7 \) and \( E_8 \), however, a more complicated change of basis may reveal some more interesting structure. There has been quite a lot of work on the compact real forms by the Russian school (see for example the book by Kostrikin and Tiep \[8\]). But even in this work, really nice constructions are hard to find. In \[9\] I gave a construction of the compact real form of \( G_2 \) using only the action of the group \( 2^3 \cdot L_3(2) \). In particular I showed that this group determines the algebra. It acts by permuting 7 mutually orthogonal Cartan subalgebras, and the Lie multiplication is given by a single easy formula and its images under the group.

Turning now to \( E_6 \), it is well-known that the complex Lie group \( E_6(\mathbb{C}) \) has a finite subgroup of shape \( 3^{3+3} \cdot SL_3(3) \). Moreover, this subgroup is isomorphic to the stabilizer of a maximal isotropic subspace (of dimension 3) in the finite simple orthogonal group \( \Omega_7(3) \). Since it acts irreducibly on the 78-dimensional Lie algebra, it preserves a unique (up to scalars) Hermitian form. Moreover, the representation is real, and if we write it as such then the form becomes a quadratic form, which is (positive or negative) definite. Therefore, this form is (again up to scalars) the Killing form, and it follows that the Killing form is negative definite, and the given 78-dimensional real Lie algebra is the compact real form of \( E_6 \).

In this paper, I construct this algebra from scratch using nothing more than the structure of this finite group. In particular, the algebra is uniquely determined (up to an overall scalar factor) by the group. It may be hoped that this provides a useful way to calculate within the compact real form of \( E_6 \). Moreover, the embedding of \( F_4 \) in \( E_6 \) is reflected in the embedding of \( 3^{3} \cdot SL_3(3) \) in \( 3^{3+3} \cdot SL_3(3) \), and therefore we obtain also a simple description of the compact real form of \( F_4 \). This is particularly revealing, as it is expressed in terms of a 13-dimensional space of quaternions, although of course the Lie product is not (bi-)linear over quaternions.

There are a few related constructions in the literature, most notably that of Burichenko \[1\] (see also Burichenko and Tiep \[3\]). Our work overlaps with theirs, but goes a bit further: our formulae are a little more concrete and explicit; we prove the existence of a Lie algebra of type \( E_6 \) independently of the Chevalley construction; we generalise to arbitrary fields of characteristic not 3; and we express the subalgebra of type \( F_4 \) in terms of the Hurwitz ring of integral quaternions.

There is also a very nice construction of the triple cover \( 3 \cdot E_6(\mathbb{C}) \) in its 27-dimensional representation by Griess \[6\], using a Moufang loop of order \( 3^4 \). This loop has an automorphism group \( 3^3 \cdot SL_3(3) \), and is analogous to the Moufang loop of \( 2^4 \) octonions \( \{ \pm 1, \pm i_0, \ldots, \pm i_6 \} \) which has automorphism group \( 2^3 \cdot SL_3(2) \). Burichenko \[2\] has a similar construction of the 27-dimensional representation of \( 3 \cdot E_6(\mathbb{C}) \), and a related 27-dimensional representation of \( 3 \cdot \Omega_7(3) \), which are also briefly described in \[8\ Section 14.1\].
2. The group \(3^{3+3}:\text{SL}_3(3)\)

The easiest way to define the required group \(3^{3+3}:\text{SL}_3(3)\), as an abstract group, is to say that it is isomorphic to the stabilizer in the simple orthogonal group \(\Omega_7(3)\) of a maximal isotropic subspace (of dimension 3) in the natural module. However, I shall not be using this description here (although it was used as input to some computer calculations which led to the definitions below). Instead I shall proceed directly to describing the action of this group on 78-dimensional real Euclidean space. The disadvantage of this approach, however, is that it is not easy to see that our group has exactly the above structure, at least until a very late stage in the argument.

Recall that \(L_3(3)\) (which can be thought of as any of \(\text{SL}_3(3)\), \(\text{PSL}_3(3)\) or \(\text{PGL}_3(3)\), according to preference) is a group of automorphisms of the projective plane of order 3. This plane consists of 13 points and 13 lines, with each line consisting of four points. The points may be labelled by the elements of the field \(\mathbb{F}_{13}\) of order 13, in such a way that the lines are

\[
\{t, t + 1, t + 3, t + 9\}
\]

for each \(t \in \mathbb{F}_{13}\). As a permutation group on these 13 points, \(L_3(3)\) may be generated by the three permutations

\[
\begin{align*}
a &= t \mapsto t + 1 \\
b &= t \mapsto 3t \\
c &= (3, 9)(4, X)(5, 6)(7, E)
\end{align*}
\]

where we write \(X = 10, E = 11, T = 12\) to avoid confusion later.

We take 13 Euclidean spaces of dimension 6, labelled \(V_0, \ldots, V_{12}\) with subscripts in \(\mathbb{F}_{13}\) as before. Let \(V\) be the orthogonal direct sum of the \(V_t\). Each 6-space is written as a 3-dimensional complex space, with

\[
\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2, \quad \theta = \sqrt{-3} = \omega - \overline{\omega},
\]

and Euclidean norm equal to the usual Hermitian norm. Then the 72 roots of \(E_6\) may be taken as the images, under coordinate permutations and multiples of each coordinate by powers of \(\omega\), of

\[
\pm(\theta, 0, 0) \quad (18 \text{ of these}), \quad \pm(1, 1, 1) \quad (54 \text{ of these}).
\]

For any vector \(v \in \mathbb{C}^3\), we write \(v_t\) for the corresponding vector in \(V_t\).

We are now ready to describe the actions of some elements on the 78-space \(V\). First, the element \(a\) of \(L_3(3)\) lifts to an element of order 13 (also called \(a\)) which maps each \(v_t\) to \(v_{t+1}\), so that for example \((\theta, 0, 0)_0 \mapsto (\theta, 0, 0)_1\). Second, the element \(b\) maps \(v_t\) to \(v_{3t}\) and then multiplies by the diagonal matrix \(\text{diag}(\omega, \overline{\omega}, \overline{\omega})\), so that for example \((1, 1, 1)_2 \mapsto (\omega, \overline{\omega}, \overline{\omega})_6\). The action of the element \(c\) is harder to describe: let \(M_1, M_2, M_3\) and \(M_4\) be the matrices

\[
\frac{\theta}{3} \begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}, \quad \frac{\theta}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}, \quad \frac{\theta}{3} \begin{pmatrix} \omega & \omega & \omega \\ \omega & \omega & 1 \\ \omega & 1 & \omega \end{pmatrix}, \quad \frac{\theta}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega & \omega & \omega \\ \omega & \omega & \omega \end{pmatrix}
\]
respectively. Since the \(M_i\) are unitary we have \(M_i^{-1} = M_i^\top\), so
\[
M_i^{-1} = M_i^\top.
\]
Then \(c\) is defined by
\[
\begin{align*}
(x, y, z)_0 & \mapsto -(\overline{x}, \overline{y}, y) \\
(x, y, z)_1 & \mapsto -(x, z, y) \\
(x, y, z)_3 & \mapsto -(x\omega, \overline{z}\omega, \overline{y}\omega) \\
(x, y, z)_4 & \mapsto (\overline{x}, \overline{y}, \overline{z})_X \\
(x, y, z)_T & \mapsto (\overline{x}, \overline{y}, \overline{z})_T \\
(x, y, z)_2 & \mapsto (\overline{x}, \overline{y}, \overline{z})_2 M_1 \\
(x, y, z)_8 & \mapsto (\overline{x}, \overline{y}, \overline{z})_8 M_2 \\
(x, y, z)_5 & \mapsto (\overline{x}, \overline{y}, \overline{z})_6 M_3 \\
(x, y, z)_7 & \mapsto (\overline{x}, \overline{y}, \overline{z})_E M_4
\end{align*}
\]
For clarity we add also:
\[
\begin{align*}
(x, y, z)_9 & \mapsto -(x\omega, \overline{z}\omega, y\omega) \\
(x, y, z)_X & \mapsto (\overline{x}, \overline{y}, \overline{z})_4 \\
(x, y, z)_6 & \mapsto (\overline{x}, \overline{y}, \overline{z})_3 M_3^\top \\
(x, y, z)_E & \mapsto (\overline{x}, \overline{y}, \overline{z})_7 M_4^\top 
\end{align*}
\]
It is clear that the permutation action of the group given by these generators on the 13 subspaces \(V_0, \ldots, V_T\), is exactly the standard permutation action of \(\SL_3(3)\). In fact, the kernel of this action is trivial, so that \(a, b\) and \(c\) generate a group isomorphic to \(\SL_3(3)\), but we shall not need this fact, and we shall not prove it here. (It is in any case straightforward to show that all the generators preserve the set of \(13 \times 72 = 936\) roots of the 13 copies of \(E_6\), after which the order of the group can be easily obtained computationally.)

Next we need to specify some elements generating the normal subgroup \(3^{3+3}\). First, the normal subgroup of order \(3^3\) is generated by conjugates of an element \(d\) acting as powers of \(\omega\) on each 6-space, as follows:
\[
(1, \omega, 1, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1).
\]
That is, \(v_0 \mapsto v_0, v_1 \mapsto \omega v_1, \) and so on.

**Lemma 1.** The group \(\langle D, a \rangle\) is of shape \(3^3:13\), in which the normal subgroup of shape \(3^3\) is generated by \(d, d^a, d^{a^2}\).

**Proof.** First note that under pointwise multiplication the product of
\[
(1, \omega, 1, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1)
\]
with
\[
(\omega, 1, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1, 1)
\]
is
\[
(\omega, \omega, \omega, \overline{\omega}, \overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1, 1, \omega, 1).
\]
Thus
\[
d d^{a^{-1}} = d^{a^{-3}},
\]
which implies that the minimum polynomial of \(a\) in its action on the conjugates of \(d\) is \(x^3 + x^2 - 1\). In particular, the conjugates of \(d\) by powers of \(a\) generate an elementary abelian group of order \(3^3\). \(\square\)
Modulo this group $D$, the next $3^3$-factor acts monomially on each 6-space, generated by conjugates of an element $e$ which acts as follows:

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $\omega$ | $\omega$ | $\omega$ | $\omega$ |

Lemma 2. The group $H = \langle a, b, d, e \rangle$ has the shape $3^{3+3}:3:13:3$, in which the normal subgroup $E \cong 3^{3+3}$ is generated by $e, e^a, e^{a^2}$.

Proof. A similar calculation applied to $e$ yields

$$e^a.e = e^{a^3}.d^{-1},$$

so that modulo the group $D = \langle d, d^a, d^{a^2} \rangle \cong 3^3$, the conjugates of $e$ generate another $3^3$. This time the minimum polynomial of the action of $a$ is $x^3 - x - 1$. Therefore the group generated by $a, d$ and $e$ has the shape $3^{3+3}:13$. It remains to check that $b$ normalizes this group. In fact it is easy to see that $a^b = a^3$, and not much harder to check that $e^b = e$, so this completes the proof. $\square$

Let $G$ be the group (in fact of shape $3^{3+3}L_3(3)$) generated by $a, b, c, d, e$, and let $L$ denote the group $\langle a, b, c \rangle$, which is in fact isomorphic to $SL_3(3)$ (although we have not proved this here). The element $c$ fixes the points $0, 1, 2, 8, T$, and therefore the subgroup

$$F = \langle E, c, c^{a^{-1}}, c^{a^{-2}}, c^a \rangle$$

fixes the point $0$. In the action of $F$ on $V_0$, we see that $d, d^a, e$ lie in the kernel, and $e^a, e^{a^2}$ generate an extraspecial group of order $3^3$, which is the image of $E$ so is obviously normal in $F$ modulo the kernel of the action. A little calculation shows that $F$ acts on $V_0$ as $3^{1+2}:2S_4$, which is sometimes known as $\Gamma U_3(2)$, and is isomorphic to a maximal subgroup of the Weyl group of type $E_6$, which is itself isomorphic to $\Sigma U_4(2)$. It is easy to see that $F$ acts irreducibly on $V_0$, and therefore $G$ acts irreducibly on $V$.

Similarly, the five lines fixed by $c$ are

$$\{0, 1, 3, 9\}, \{1, 2, 4, X\}, \{5, 6, 8, 1\}, \{E, T, 1, 7\}, \{T, 0, 2, 8\},$$

so that the line $\{0, 1, 3, 9\}$ is fixed by

$$c, c^{a^{-1}}, c^{a^{-5}}, c^{a^2}, c^a.$$  

These elements act on the line as the permutations $(3, 9), (3, 9), (0, 1), (0, 9)$, and $1$ respectively, so induce the full $S_4$ of permutations. Indeed, they generate the full line stabilizer $3^2:2S_4$ inside $L_3(3)$. The elements of $D$ which act non-trivially on this line act as

$$(1, \omega, \omega, \omega), (\omega, 1, \omega, \overline{\omega}), (\omega, \omega, \overline{\omega}, 1), (\omega, \overline{\omega}, 1, \omega)$$

or their inverses. These elements will be used frequently in the sequel.
3. The Lie product

In this section we show that there is (up to real scalar multiplication) a unique bilinear product invariant under the action of the group $G = \langle a, b, c, d, e \rangle$ on the 78-space $V$, and moreover that this product satisfies the Jacobi identity. First we show that there is at most one such product, and then we show that any product so defined is anti-symmetric, before using this to prove that there is indeed a non-zero such product invariant under $G$, and that this product satisfies the Jacobi identity.

**Lemma 3.** Up to a real scalar multiplication, there is a unique $G$-invariant bilinear product on $V$.

**Proof.** Since $G$ acts 2-transitively on the 13 spaces $V_i$, it is enough to determine the product on $V_0 \times V_0$ and on $V_0 \times V_1$. Now the action of the conjugates of $d$ shows immediately that the product is zero on $V_0 \times V_0$, and that the product of any vector in $V_0$ with any vector in $V_1$ lies in $V_3 + V_9$.

Since the group $3^{3+3}$ acts irreducibly on $V_0$, we only need to consider products of $(1, 0, 0)_0$ with $V_1$. Moreover, the stabilizer in $3^{3+3}$ of $(1, 0, 0)_0$ permutes the nine spanning vectors

$$(\omega^i, 0, 0)_1, (0, \omega^i, 0)_1, (0, 0, \omega^i)_1$$

of $V_1$ transitively, so it is sufficient to determine the product of $(1, 0, 0)_0$ with $(1, 0, 0)_1$.

The element $e^\theta$ fixes $(1, 0, 0)_0$ and $(1, 0, 0)_1$, and maps

$$\begin{align*}
(x, y, z)_3 &\mapsto (y\omega, z, x\omega)_3, \\
(x, y, z)_9 &\mapsto (z\omega, x\overline{\omega}, y)_9.
\end{align*}$$

Therefore the product of $(1, 0, 0)_0$ with $(1, 0, 0)_1$ lies in $\mathbb{C}(\overline{\omega}, 1, 1)_3 + \mathbb{C}(\omega, 1, 1)_9$.

Next, $e^\theta$ fixes $(1, 0, 0)_0$ and negates $(1, 0, 0)_1$, and acts on $\mathbb{C}(\overline{\omega}, 1, 1)_3$ by fixing $\theta(\overline{\omega}, 1, 1)_3$ and negating $(-1, \overline{\omega}, \overline{\omega})_9$. Similarly, it fixes $\theta(1, \overline{\omega}, \overline{\omega})_9$ and negates $(-1, \overline{\omega}, \overline{\omega})_9$. Therefore the given product lies in $\mathbb{R}(\overline{\omega}, 1, 1)_3 + \mathbb{R}(1, \overline{\omega}, \overline{\omega})_9$.

Finally, $c$ itself acts by negating both $(1, 0, 0)_0$ and $(1, 0, 0)_1$, and interchanging $(\overline{\omega}, 1, 1)_3$ with $-1(\overline{\omega}, \overline{\omega})_9$. Therefore the given product is a real multiple of

$$(\overline{\omega}, 1, 1)_3 - (1, \overline{\omega}, \overline{\omega})_9.$$ 

Hence there is up to scalars at most one bilinear product invariant under $G$, as claimed.

\[\square\]

Our strategy for showing that such a (non-zero) product actually exists divides into four steps, which are dealt with in Lemmas 4, 5, 6 and 7 respectively:

1. show that any such product is anti-symmetric;
2. find two particular values of the product whose images under $H$ are sufficient to define the whole product;
3. prove that this product is well-defined, that is, it is invariant under $H$;
4. prove that this product is invariant under $c$.

**Lemma 4.** Any $G$-invariant bilinear product on $V$ is anti-symmetric.

**Proof.** From the proof of Lemma 3 we may assume that the product, written $[u, v]$, satisfies

$$[(1, 0, 0)_0, (1, 0, 0)_1] = (\overline{\omega}, 1, 1)_3 - (1, \overline{\omega}, \overline{\omega})_9.$$
Applying $c_5a^5$ to this equation gives
\[
\frac{\theta}{3}(\omega, \omega, \omega)_{1}, \frac{\theta}{3}(\omega, \omega, \omega)_{0} = (\omega, 1, 1)_{3}M_{2} + (1, \omega, \omega)_{9} \\
= (\omega, \omega, \omega)_{3} + (1, \omega, \omega)_{9}
\]
We now calculate the product of these two vectors the other way round. Applying $e$ to the defining equation gives
\[
[(1, 0, 0)_{0}, (1, 0, 0)_{1}] = [(1, 0, 0)_{0}, (0, 1, 0)_{1}] = [(1, 0, 0)_{0}, (0, 0, 1)_{1}].
\]
Similarly, applying other conjugates of $e$ and $d$ we obtain the following multiplication table:

|        | $(1, 0, 0)_{0}$ | $(0, 1, 0)_{0}$ | $(0, 0, 1)_{0}$ |
|--------|----------------|----------------|----------------|
| $(1, 0, 0)_{0}$ | $(\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}$ | $(\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}$ | $(\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}$ |
| $(0, 1, 0)_{0}$ | $(1, \omega, 1)_{3} - (\omega, \omega, \omega)_{9}$ | $(\omega, \omega, \omega)_{3} - (1, 1, \omega)_{9}$ | $(\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}$ |
| $(0, 0, 1)_{0}$ | $(1, \omega, \omega)_{3} - (\omega, \omega, \omega)_{9}$ | $(\omega, \omega, \omega)_{3} - (1, 1, \omega)_{9}$ | $(\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}$ |

In fact, applying conjugates of $d$ is quite easy: if we multiply $v_{0}$ by $\omega$ and fix $v_{1}$ then we must multiply $v_{3}$ by $\omega$ and $v_{9}$ by $\overline{\omega}$. On the other hand, if we fix $v_{0}$ and multiply $v_{1}$ by $\omega$, then we must multiply both $v_{3}$ and $v_{9}$ by $\omega$. This leads quickly to the equations
\[
[(\omega, 0, 0)_{0}, (\omega, \omega, \omega)_{1}] = 3(\omega, \omega, \omega)_{3} - 3(1, \omega, \omega)_{9} \\
[(0, \omega, 0)_{0}, (\omega, \omega, \omega)_{1}] = 0 \\
[(0, 0, \omega)_{0}, (\omega, \omega, \omega)_{1}] = 0
\]
from which we obtain
\[
\frac{\theta}{3}(\omega, \omega, \omega)_{0}, \frac{\theta}{3}(\omega, \omega, \omega)_{1} = -3(\omega, \omega, \omega)_{3} - (1, \omega, \omega)_{9} \\
= -\frac{\theta}{3}(\omega, \omega, \omega)_{0}, \frac{\theta}{3}(\omega, \omega, \omega)_{1}
\]
Since, by Lemma $\S$, this single non-zero value of the product defines the whole multiplication, it follows that the whole multiplication is anti-symmetric. $\square$

**Lemma 5.** Any $G$-invariant product on $V$ is determined by anti-symmetry and the images under $H$ of just two products, which may be taken (up to an overall scalar multiplication) to be
\[
[(1, 0, 0)_{0}, (1, 0, 0)_{1}] = (\omega, 1, 1)_{3} - (1, \omega, \omega)_{9}, \\
[(1, 0, 0)_{1}, (1, 0, 0)_{0}] = -(1, \omega, \omega)_{0} + (\omega, \omega, \omega)_{3}.
\]

**Proof.** Since the group 13:3 generated by $a$ and $b$ has just two (regular) orbits on unordered pairs of the 13 points, represented by $\{0, 1\}$ and $\{1, 9\}$, it suffices to determine the product on $V_{1} \times V_{9}$. A similar argument to that given in the second paragraph of the proof of Lemma $\S$ shows that it is sufficient to determine $[(1, 0, 0)_{1}, (1, 0, 0)_{0}]$.

Applying $e$ to the equation
\[
[(1, 0, 0)_{0}, (1, 0, 0)_{0}] = (\omega, 1, 1)_{E} - (1, \omega, \omega)_{4}
\]
gives
\[
-\frac{\theta}{3}(1, 1, 1)_{8}, (\omega, 0, 0)_{3} = (\omega, 1, 1)_{7}M_{4} - (1, \omega, \omega)_{X} \\
= (\theta(\omega, 0, 0)_{7} - (1, \omega, \omega)_{X}
\]
To assist with computations, we provide a fuller version of the multiplication table in Table 1. This must be used in combination with the action of $D$, which shows how to compute the products of vectors with coordinates $\omega$ or $\overline{\omega}$.

*Lemma 7.* The product defined in Lemma 2 is invariant under $G$.

*Proof.* It suffices now to prove that this product is invariant under $e$. Since $e$ normalizes the group $3^{3+3}:3$ generated by $b$ together with conjugates of $d$ and $e$, it suffices to check the product on one pair of basis vectors in each orbit under the latter group. There are 26 such orbits, represented by

\[
[(1,0,0)_t, (1,0,0)_{t+1}],
\]

\[
[(1,0,0)_t, (1,0,0)_{t+2}],
\]

for each $t \in \mathbb{F}_{13}$.

The easiest cases are those where $c$ acts monomially, that is on the coordinates $t = 0, 1, 3, 4, 9, X, T$. There are seven such cases, namely $T0, 01, 34$ and $9X$ of the form $t, t + 1$, and $XT$, $T1$ and $13$ of the form $t, t + 2$. In the 01 case we have

\[
[(1,0,0)_0, (1,0,0)_1] = (\overline{\omega}, 1, 1)_3 - (1, \overline{\omega}, \overline{\omega})_9
\]
and under $c$ the left-hand-side is fixed (since both factors are negated), while on the right-hand-side the two terms are swapped. Thus this instance of the product is preserved by $c$, as required. In the $T0$ case we have

$$[(1, 0, 0)_{T}, (1, 0, 0)_{0}] = (\omega, 1, 1)_{2} - (1, \overline{\omega}, \overline{\omega})_{8},$$

and this time the left-hand-side is negated by $c$, while the right-hand-side maps to

$$(\omega, 1, 1)_{2}M_{1} - (1, \omega, \omega)_{8}M_{2} = -(\omega, 1, 1)_{2} + (1, \overline{\omega}, \overline{\omega})_{8}$$

as required, since $2 + \overline{\omega} = \overline{\omega}$ and $1 + 2\omega = \theta$. Now consider the case $9X$: we have

$$[(1, 0, 0)_{9}, (1, 0, 0)_{X}] = (\omega, 1, 1)_{T} - (1, \overline{\omega}, \overline{\omega})_{5}$$

in which the left-hand-side maps under $c$ to $-[(\overline{\omega}, 0, 0)_{3}, (1, 0, 0)_{4}]$ and the right-hand side maps to

$$(\omega, 1, 1)_{T} - (1, \omega, \omega)_{6}M_{3} = (\omega, 1, 1)_{T} - (\omega, \overline{\omega}, \overline{\omega})_{6}$$

which checks out with Table 1 after applying a suitable element of $D$.

| $\omega, 0, 0)$ | $(1, 0, 0)_{1}$ | $(0, 1, 0)_{1}$ | $(0, 0, 1)_{1}$ |
|----------------|----------------|----------------|----------------|
| $(\omega, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ |
| $(\omega, 1, 1)_{3}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ |
| $(\omega, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ | $\omega, 1, 1)_{3} - (1, \overline{\omega}, \overline{\omega})_{9}$ |

Table 1. The Lie bracket on $E_{6}$
In the case 13 we have

\[(1, 0, 0)_1, (1, 0, 0)_{3}] = (\omega, 1, 1)_0 - (\overline{\omega}, \overline{\omega}, \overline{\omega})_9\]

and the left-hand-side is mapped by \(c\) to \([(1, 0, 0)_1, (\omega, 0, 0)_9]\), while the right-hand-side is mapped to \(- (\overline{\omega}, 1, 1)_0 + (\omega, 1, 1)_3\), which again checks out with Table 1.

Similarly the right-hand side of

\[(1, 0, 0)_X, (1, 0, 0)_T]\]

maps to

\[-(1, \omega, \omega)_3 - (\omega, \omega, \omega)_4 M_3 = -(1, \omega, \omega)_3 + (\overline{\omega}, \omega, \omega)_6\]

which is the value of \([(1, 0, 0)_4, (1, 0, 0)_T]\) as required. In the case \(T1\) we have

\[(1, 0, 0)_T, (1, 0, 0)_1]\]

and the second term on the right-hand-side is mapped by \(c\) to

\[-(\omega, \omega, \omega)_E M_4 = - (\omega, 1, 1)_E\]

Since \(c\) has order 2 it also maps the first term on the right-hand-side to the negative of the second, and so \(c\) preserves this instance of the product also.

The other 19 of the 26 calculations are slightly more awkward since \(c\) no longer acts monomially on the left-hand-side, and are left as exercises for the reader. The calculations can be reduced from 19 cases to 14 by using the fact that \(c\) has order 2. These are the cases

\[12/45/56/78/XE/02/57/79/E0\]

and one from each of the pairs

\[23/35, 24/46, 67/68, 89/9E, 8X/ET\]

Indeed, one of these calculations was essentially done in Lemma 5 above, where the equivalence of the following was shown:

\[[(1, 0, 0)_8, (1, 0, 0)_9] = (\omega, 1, 1)_E - (1, \overline{\omega}, \overline{\omega})_4\]

\[[(1, 0, 0)_8, (1, 0, 0)_x] = -(1, \omega, \omega)_7 + (\overline{\omega}, \omega, \omega)_X\]

Applying \(b\) to the second equation we obtain

\[[(\omega, 0, 0)_E, (\omega, 0, 0)_9] = -(\omega, \overline{\omega}, \overline{\omega})_8 + (1, \overline{\omega}, \overline{\omega})_4\]

and thence

\[[(1, 0, 0)_9, (1, 0, 0)_E] = (\omega, \overline{\omega}, \overline{\omega})_8 - (\overline{\omega}, \omega, \omega)_4\]

\(\square\)

To summarise the results of this section so far, we have now proved the following.

**Theorem 1.** Up to scalar multiplication, there is a unique bilinear product on the 78-space \(V\) which is invariant under \(G\). This product is defined by

\[[(1, 0, 0)_0, (1, 0, 0)_1] = (\overline{\omega}, 1, 1)_3 - (1, \overline{\omega}, \overline{\omega})_9\]

and is anti-symmetric.
The only remaining serious calculation is to verify that our product satisfies the Jacobi identity,
\[ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \]
By linearity and anti-symmetry, and the symmetry of the formula under cyclically permuting \( x, y, z \), it suffices to check this for unordered triples \( \{x, y, z\} \) of distinct basis vectors.

**Proposition 1.** The product defined in Lemma 6 satisfies the Jacobi identity.

**Proof.** We need to check that
\[ [[x_r, y_s], z_t] + [[y_s, z_t], x_r] + [[z_t, x_r], y_s] = 0 \]
for suitable choices of (linearly independent) vectors \( x_r, y_s, z_t \). If \( r, s, t \) are not collinear, then we may assume
\[
\begin{align*}
x_r &= (1, 0, 0)_0, \\
y_s &= (1, 0, 0)_1, \\
z_t &= (1, 0, 0)_2.
\end{align*}
\]
If \( r, s, t \) are collinear and distinct, then we may assume
\[
\begin{align*}
x_r &= (1, 0, 0)_0, \\
y_s &= (1, 0, 0)_1, \text{ and} \\
z_t &= (1, 0, 0)_3 \text{ or } (\omega, 0, 0)_3.
\end{align*}
\]
If \( r, s, t \) are collinear and two of them are equal, we may assume that
\[
\begin{align*}
x_r &= (1, 0, 0)_0, \\
y_s &= (1, 0, 0)_1, \\
z_t &= (\omega, 0, 0)_0 \text{ or } (0, 1, 0)_0.
\end{align*}
\]
Thus we have five cases to check. The four cases when \( r, s, t \) are collinear are relatively easy, and left as exercises. The hard case is when they are not collinear, and we calculate as follows.
\[
\begin{align*}
&\ [(1, 0, 0)_0], (1, 0, 0)_1], (1, 0, 0)_2] \\
+&[[(1, 0, 0)_1], (1, 0, 0)_2], (1, 0, 0)_0] \\
+&[[(1, 0, 0)_2], (1, 0, 0)_0], (1, 0, 0)_1] = \\
&[\theta, (1, 0, 0)_3], (1, 0, 0)_2] - [(1, \omega, \omega)_9], (1, 0, 0)_2] \\
&+ [(\omega, 1, 1)_4], (1, 0, 0)_0] - [(\omega, \omega, \omega)_7], (1, 0, 0)_0] \\
&- [(\omega, 1, 1)_7], (1, 0, 0)_0] + [(\omega, \omega, \omega)_8], (1, 0, 0)_1]
\end{align*}
\]
Next we calculate the individual terms on the right-hand side as follows (details of the calculations are omitted).
\[
\begin{align*}
&[\theta, (1, 0, 0)_3], (1, 0, 0)_2] = -\theta(\omega, \omega, \omega)_5 + \theta(\omega, \omega, \omega)_5 \\
&[\theta, (1, 1)_4], (1, 0, 0)_0] = 3(\omega, 0, 0)_5 + \theta(1, 1)_5 \\
&[(\omega, \omega, \omega)_7], (1, 0, 0)_0] = -\theta(\omega, \omega, \omega)_6 + 3(\omega, 0, 0)_5 \\
&[(1, 0, 0)_2], (1, \omega, \omega)_9] = \theta(\omega, 1, 1)_6 + 3(\omega, 0, 0)_7 \\
&[(\omega, 1, 1)_7], (1, 0, 0)_1] = \theta(1, \omega, \omega)_7 + \theta(\omega, \omega, \omega)_7 \\
&[(\omega, \omega, \omega)_8], (1, 0, 0)_1] = \theta(1, \omega, \omega)_5 + \theta(1, \omega, \omega)_5
\end{align*}
\]
Finally we combine these results and find that all the terms cancel out, giving 0 as required. \( \square \)
4. Identification of the Algebra with $E_6$

It is easy to see that the Lie product is not identically zero on any subspace properly containing $V_0$, and therefore $V_0$ is a Cartan subalgebra. The stabilizer of $V_0$ in $G$ is a group of shape $3^{3+3}:3^2:2S_4$ which acts on $V_0$ as $3^{3+2}:2S_4$. Since this group has no faithful complex representation of degree less than 6, it acts absolutely irreducibly on $V_0$. Therefore the Lie algebra is simple, so by the classification theorem it is $E_6$. Since $G$ acts irreducibly, the only invariant quadratic forms are (positive or negative) definite. In particular the Killing form is negative definite, so the algebra is the compact real form of $E_6$.

An alternative proof may be obtained from first principles by extending the field to $\mathbb{C}$ and explicitly diagonalizing the action of $V_0$ by multiplication on $L$, and thereby obtaining the root spaces. These may then be explicitly identified with the 72 roots of the $E_6$ root system, and the products of the root vectors explicitly calculated. One would then see that the complexification of the algebra is the same as the usual complex Lie algebra of type $E_6$.

As a first step, we compute the eigenspaces of the action of $V_0$. We find that one of them is spanned by

$$v = (1, 1, 1)_1 + (\omega, \overline{\omega}, \overline{\omega})_3 + (\omega, 1, 1)_9.$$  

To verify this we compute

\[
[ (1, 0, 0)_0, v ] = 3(\overline{\omega}, 1, 1)_3 - 3(1, \overline{\omega}, \overline{\omega})_9 + 3(\overline{\omega}, \omega, \omega)_9 - 3(\overline{\omega}, \overline{\omega}, \overline{\omega})_1
\]

\[
+ 3(\omega, \omega, \omega)_1 - 3(1, \omega, \omega)_3
\]

\[
= 3\theta v
\]

\[
[(\omega, 0, 0)_0, v] = 3(1, \omega, \omega)_3 - 3(\overline{\omega}, \omega, \omega)_9 + 3(1, \overline{\omega}, \overline{\omega})_9 - 3(\omega, \omega, \omega)_1
\]

\[
+ 3(\overline{\omega}, \overline{\omega}, \overline{\omega})_1 - 3(\overline{\omega}, 1, 1)_3
\]

\[
= -3\theta v
\]

and

\[
[(0, 1, 0)_0, v] = [(0, \omega, 0)_0, v] = [(0, 0, 1)_0, v] = [(0, 0, \omega)_0, v] = 0.
\]

Moreover, we see that this complex eigenspace corresponds to the pair of roots $\pm(1 - \omega, 0, 0)_0 = (\overline{\omega}, 0, 0)_0$. Applying suitable elements of $D$ we obtain the correspondence

$$\pm(\theta, 0, 0) \leftrightarrow \langle 1, 1, 1 \rangle_1 + (\overline{\omega}, 1, 1)_3 + (1, \overline{\omega}, \overline{\omega})_9,$$

$$(\omega, \omega, \omega)_1 + (1, \omega, \omega)_3 + (\omega, 1, 1)_9.$$  

Notice that ‘scalar multiplication’ is always interpreted as applying a suitable element of $D$, so is not always the same as scalar multiplication by $\omega$. The other orbits of $E$ give rise to the following correspondences:

$$\pm(1, 1, 1)_0 \leftrightarrow \langle \theta, 0, 0 \rangle_T + (\overline{\omega}, 1, 1)_2 - (1, \overline{\omega}, \overline{\omega})_8,$$

$$\langle \omega \theta, 0, 0 \rangle_T + (1, \omega, \omega)_2 - (\overline{\omega}, \omega, \omega)_8 \rangle.$$  

$$\pm(1, \omega, \omega)_0 \leftrightarrow \langle \theta, 0, 0 \rangle_X - (\omega, \omega, \omega)_E + (\omega, 1, 1)_6,$$

$$\langle \omega \theta, 0, 0 \rangle_X - (1, 1, 1)_E + (\overline{\omega}, \omega, \omega)_6 \rangle.$$  

$$\pm(1, \overline{\omega}, \overline{\omega})_0 \leftrightarrow \langle \theta, 0, 0 \rangle_4 + (1, 1, 1)_5 - (\overline{\omega}, 1, 1)_7,$$

$$\langle \omega \theta, 0, 0 \rangle_4 + (\omega, \omega, \omega)_5 - (\omega, \overline{\omega}, \overline{\omega})_7 \rangle.$$  

Now we can use elements of the stabiliser of $V_0$ in $3^{3+3}:SL_3(3)$ to obtain all the other root spaces, labelled with the corresponding roots. The pointwise stabiliser
of $V_0$ is an elementary abelian group of order $3^5$, generated by
\[ d, d^a, e, e^a, \frac{e^a - 2}{e^a + 2} b. \]

It follows (or one can check directly) that $v$ is an eigenvector for this group. The elements $e^a, e^{a^2}$ then map the given eigenspace to the nine eigenspaces which lie inside $V_1 + V_3 + V_9$. The eigenspaces lying in the other ‘lines’ containing 0 can be computed by applying other conjugates of $c$ which fix the point 0.

Recall that our 39-dimensional complex notation denotes a 78-dimensional real vector space. In order to find a Chevalley basis, of course, one needs to extend the scalars to $\mathbb{C}$ (without confusing the real vector $\omega$ with the complex scalar $e^{2\pi i/3}$). Then each of our ‘eigenspaces’ becomes a 2-dimensional space, in which one can distinguish two root vectors, corresponding to a root and its negative.

5. The subalgebra of type $F_4$

The subspace $W_t$ of $V_t$ consisting of the vectors $(x, y, y)_t$ has (real) dimension 4. The direct sum $W$ of the $W_t$ is a space of dimension 52, which is easily seen to be invariant under the action of $a, b, c, d$. These elements in fact generate a symmetry group of shape $3^3: L_3(3)$.

This group induces on each $W_t$ a group of shape $(3 \times 2A_4):2$, which acts irreducibly. Using the symmetry group it is not hard to show that $W$ is closed under the Lie product. Hence $W$ is a simple Lie algebra of rank 4 and dimension 52 and can only be the compact real form of $F_4$.

The short roots of the $F_4$ root system may be taken as the 24 vectors of the form
\[
\pm \omega^n(\theta, 0, 0), \\
\pm \omega^n(1, 1, 1), \\
\pm \omega^n(1, \omega, \omega), \\
\pm \omega^n(1, \overline{\omega}, \overline{\omega}).
\]

We may label these vectors by unit quaternions by defining
\[
1 = -(\theta, 0, 0), \\
i = (1, 1, 1), \\
j = (1, \omega, \omega), \\
k = (1, \overline{\omega}, \overline{\omega}).
\]

and identifying (left-)multiplication by the complex number $\omega$ with left-multiplication by the quaternion $\omega = (-1 + i + j + k)/2$. Let $q_t$ denote the quaternion $q$ in the space $W_t$.

With this notation, the compact real form of the Lie algebra of type $F_4$ becomes a 13-dimensional object over quaternions. It is of course not linear, but the quaternions do provide a compact notation both for the multiplication and for the action of certain automorphisms. For example, $d$ becomes left-quaternion multiplication by
\[
(1, \omega, 1, \omega, \omega, \overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1)
\]
on the 13 spaces $W_t$. Similarly, the element $b$ becomes $q_t \mapsto q_{3t} \omega$, that is, the combination of right-quaternion-multiplication by $\omega$ with the coordinate permutation
\[
(1, 3, 9)(2, 6, 5)(4, T, X)(8, E, 7).
\]
Similarly, the matrices $M_1, M_2, M_3, M_4$ defined earlier induce right-multiplication by the quaternions $j\omega, i, k\omega, j$, respectively. Complex conjugation induces the negative of the automorphism $\ast$ which negates $i$ and swaps $j$ with $-k$. Then $c$ maps

\[
\begin{align*}
q_0 &\mapsto q_0^* \\
q_1 &\mapsto -q_1 \\
q_2 &\mapsto -(q_j \omega)_2 \\
q_3 &\mapsto -q_3^* \\
q_4 &\mapsto -q_4^* \\
q_5 &\mapsto -q_5^* \\
q_6 &\mapsto -(q_i \omega)_5 \\
q_7 &\mapsto -q_7^* \\
q_8 &\mapsto -(q_i \omega)_6 \\
q_9 &\mapsto -q_9^* \\
q_T &\mapsto -(q_j \beta)_1 \\
q_X &\mapsto -q_X^*
\end{align*}
\]

For convenience, we note also

\[
\begin{align*}
q_9 &\mapsto -(q_j \omega)_3 \\
q_X &\mapsto -(q_j \omega)_3 \\
q_6 &\mapsto -(q_i \omega)_5 \\
q_E &\mapsto -(q_j \beta)_1 \\
q_T &\mapsto -(q_j \beta)_1 \\
q_X &\mapsto -(q_j \beta)_1 \\
q_E &\mapsto -(q_j \beta)_1 \\
q_T &\mapsto -(q_j \beta)_1
\end{align*}
\]

The Lie product suitably scaled (in fact, this is the previous product divided by $-3$) is written out in more detail in Table 2. All products of roots in the $W_t$ can be obtained from this table by applying elements of the group $D \cong 3 \times 3$. Indeed, this table may be useful for calculating the product in $E_6$ as well, by applying elements of $E \cong 3^{3+3}$ to the entries.

**Remark 1.** One might expect that a result similar to Theorem 1 should hold also for $F_4$. That is, one might conjecture that there is a unique algebra invariant under the appropriate 52-dimensional representation of $3^3 : L_3(3)$. However, this is not the case.

In $3^3 : SL_3(3)$, the pointwise stabilizer of $W_0$ is an elementary abelian group of order $3^4$, generated by

\[d, a, e^a, e^{a^{-1}}c^a, (e^{a^2}c^a e^{-a^2}c)^2b.\]

We may re-compute the action of $e^{a^{-1}}c^a$ in $F_4$, as follows:

\[
\begin{align*}
q_0 &\mapsto q_0, \\
q_1 &\mapsto -(q_k \omega)_1, \\
q_2 &\mapsto (q_j \omega)_T, \\
q_3 &\mapsto -(q_j \omega)_3, \\
q_4 &\mapsto (q_i \omega)_7, \\
q_5 &\mapsto (q_i \omega)_4, \\
q_6 &\mapsto (q_k \omega)_X, \\
q_7 &\mapsto -(q_j \beta)_5, \\
q_8 &\mapsto -(q_i \omega)_2, \\
q_9 &\mapsto -(q_j \beta)_9, \\
q_X &\mapsto (q_j \beta)_E, \\
q_E &\mapsto -(q_k \omega)_6, \\
q_T &\mapsto (q_j \beta)_8.
\end{align*}
\]
## Table 2. The multiplication table of the Lie algebra of type $F_4$

|   | $\omega_1$ | $\omega_1$ | $\omega_1$ | $\omega_1$ |
|---|-------------|-------------|-------------|-------------|
| $\omega_0$ | $j_3 + k_9$ | $\theta j_3 + \theta k_9$ | $-j_3 - k_9$ | $j_3 + k_9$ |
| $\omega_0$ | $-i_3 + j_9$ | $-j_3 + k_9$ | $1_3 - j_9$ | $k_3 - i_9$ |
| $\omega_0$ | $k_3 + 1_9$ | $-j_3 + k_9$ | $i_3 + i_9$ | $-j_3 + j_9$ |
| $\omega_0$ | $-i_3 - i_9$ | $-j_3 + k_9$ | $-j_3 - j_9$ | $-i_3 + 1_9$ |

|   | $\omega_3$ | $\omega_3$ | $\omega_3$ | $\omega_3$ |
|---|-------------|-------------|-------------|-------------|
| $1_0$ | $k_9 + i_1$ | $k_9 + i_1$ | $\theta k_9 + \theta i_1$ | $-k_9 - i_1$ |
| $i_0$ | $-1_9 - j_1$ | $-j_9 + 1_1$ | $-k_9 + i_1$ | $-i_9 - k_1$ |
| $j_0$ | $-j_9 + k_1$ | $i_9 - j_1$ | $-j_9 + i_1$ | $1_9 - i_1$ |
| $k_0$ | $i_9 + 1_1$ | $-1_9 + k_1$ | $-k_9 + i_1$ | $j_9 + j_1$ |

|   | $\omega_3$ | $\omega_3$ | $\omega_3$ | $\omega_3$ |
|---|-------------|-------------|-------------|-------------|
| $1_0$ | $i_1 + j_3$ | $-i_1 - j_3$ | $i_1 + j_3$ | $\theta i_1 + \theta j_3$ |
| $\omega_0$ | $j_1 + 1_3$ | $k_1 + k_3$ | $-1_1 + i_3$ | $-i_1 + j_3$ |
| $\omega_0$ | $-i_1 - k_3$ | $-j_1 - i_3$ | $-k_1 + 1_3$ | $-i_1 + j_3$ |
| $\omega_0$ | $-j_1 + i_3$ | $1_1 - 1_3$ | $j_1 - k_3$ | $-i_1 + j_3$ |

|   | $\omega_3$ | $\omega_3$ | $\omega_3$ | $\omega_3$ |
|---|-------------|-------------|-------------|-------------|
| $1_1$ | $-k_0 - i_9$ | $-i_0 + j_9$ | $1_0 - k_9$ | $j_0 - i_9$ |
| $i_1$ | $-i_0 - k_9$ | $-1_0 - k_9$ | $-\theta i_0 + \theta k_9$ | $1_0 + k_9$ |
| $j_1$ | $i_0 - 1_9$ | $j_0 + i_9$ | $-1_0 + k_9$ | $-k_0 + j_9$ |
| $k_1$ | $-j_0 + j_9$ | $-k_0 + 1_9$ | $1_0 - k_3$ | $i_0 + i_9$ |

|   | $\omega_3$ | $\omega_3$ | $\omega_3$ | $\omega_3$ |
|---|-------------|-------------|-------------|-------------|
| $1_3$ | $-i_0 - j_1$ | $k_0 - 1_1$ | $-j_0 + k_1$ | $1_0 - i_1$ |
| $\omega_3$ | $-k_0 + k_1$ | $j_0 + j_1$ | $-i_0 + 1_1$ | $1_0 - i_1$ |
| $\omega_3$ | $-1_0 - i_1$ | $1_0 + i_1$ | $-1_0 - i_1$ | $-\theta i_0 + \theta i_1$ |
| $\omega_3$ | $j_0 - 1_1$ | $-i_0 + k_1$ | $k_0 + j_1$ | $-1_0 + i_1$ |

|   | $\omega_3$ | $\omega_3$ | $\omega_3$ | $\omega_3$ |
|---|-------------|-------------|-------------|-------------|
| $\omega_1$ | $-j_0 - k_3$ | $1_0 - j_3$ | $i_0 - 1_3$ | $-k_0 + i_3$ |
| $\omega_1$ | $k_0 - 1_3$ | $-1_0 + j_3$ | $-j_0 + i_3$ | $i_0 + k_3$ |
| $\omega_1$ | $-i_0 + i_3$ | $1_0 - j_3$ | $k_0 + k_3$ | $-j_0 + 1_3$ |
| $\omega_1$ | $-1_0 - j_3$ | $-\theta i_0 + \theta j_3$ | $1_0 + j_3$ | $-1_0 - j_3$ |

The final element $(e^{a_{-3}} e^{a_{-5}} e^{a_{-2}} c^2 b)$ acts as follows:

- $q_0 \mapsto q_0$
- $q_1 \mapsto (q_0 j_0)$
- $q_2 \mapsto (q_1 j_1)$
- $q_3 \mapsto -(q_1 i_1)$
- $q_4 \mapsto (q_0 k_0)$
- $q_5 \mapsto -(q_1 k_0)$
- $q_6 \mapsto (q_0 k_0)$
- $q_7 \mapsto (q_1 k_0)$
If we replace \( \mathbb{R} \) by \( \mathbb{F}_p \), or indeed by any field \( \mathbb{F} \) of characteristic \( p \) in particular, Theorem 1 and Proposition 1 hold in this more general setting. Throughout, \( \mathbb{C} \) is replaced by a 2-dimensional space over \( \mathbb{F} \), with basis \( \{1, \omega\} \). (One must be careful to distinguish between \( \omega \) and an element of order 3 in \( \mathbb{F} \), if there is one.)

In characteristic 3, however, the whole strategy fails for many reasons: \( G \) has no faithful irreducible representations in characteristic 3, a 2-dimensional space does not support a fixed-point-free linear map \( \omega \) of order 3, and the definition of \( \epsilon \) requires dividing by 3, to give just a few examples.

### 6. Reducing Modulo \( p \)

We can find the common ‘eigenvectors’ of this 3\(^4\) (where, again, scalar multiplication is defined by an element of \( D \)). These are the images under \( D \) and \( a \) of

\[
\begin{align*}
\pm 1_0 & \mapsto (i_1 + \overline{\omega}j_3 + k_9, \omega i_1 + j_3 + \omega k_9), \\
\pm i_0 & \mapsto (1_T - \overline{\omega}j_2 + k_8, \omega T - j_2 + \overline{\omega}k_8), \\
\pm j_0 & \mapsto (1_X + \omega i_E - \omega k_8, \omega X + i_E - \overline{\omega}k_6), \\
\pm k_0 & \mapsto (1_E - i_5 + \overline{\omega}j_7, \omega_4 - \omega i_5 + \omega j_7).
\end{align*}
\]

The other eigenvectors can be found by applying elements of the stabilizer of \( W_0 \).

More specifically, we can use the part of the Weyl group of \( F_4 \) that lies inside \( 3^3:\text{SL}_3(3) \). This is a group \( (3 \times 2: A_4):2 \), which is the centralizer of an involution in \( 3^3:\text{SL}_3(3) \). If we take the involution \( c^{a^{-1}} \), then it is centralized by \( d^{a^{-1}} \) in the normal \( 3^3 \), and by \( c^{a^{-2}} \) and \( (bc^{a^{-1}}b)^2 \), generating \( 2S_4 \), inside \( \text{SL}_3(3) \). For convenience we exhibit the element \( (bc^{a^{-1}}b)^2 \) explicitly:

\[
\begin{align*}
q_0 & \mapsto q\omega_0, \\
q_1 & \mapsto q_1, \\
q_3 & \mapsto q\omega_3, \\
q_9 & \mapsto -q\overline{\omega_9}, \\
q_2 & \mapsto qj_2, \\
q_X & \mapsto q\overline{\omega}X, \\
q_4 & \mapsto -q\omega_2, \\
q_5 & \mapsto q^*j_8, \\
q_8 & \mapsto q^*\omega_6, \\
q_6 & \mapsto q\overline{\omega_5}, \\
q_7 & \mapsto q^*k_T, \\
q_T & \mapsto q^*i_E, \\
q_E & \mapsto -qk_7.
\end{align*}
\]

If \( \epsilon \) is a scalar of order 3, we may pick a root vector

\[
e_1 = i_1 + \overline{\omega}j_3 + k_9 - \epsilon(\omega i_1 + j_3 + \omega k_9),
\]

where the subscript denotes the corresponding root in \( W_0 \). Then we can apply elements of the above group \( (3 \times 2: A_4):2 \) to get the remaining (long) root vectors. We have

\[
\begin{align*}
e_1 & = -\omega T + j_2 - \overline{\omega}k_8 - \epsilon(-\overline{\omega}T + \omega j_2 - \omega k_8), \\
e_j & = -\overline{\omega}X - \overline{\omega}i_E + k_6 - \epsilon(-1X - \omega i_E + \omega k_6), \\
e_k & = -1_4 + i_5 - \overline{\omega}j_7 - \epsilon(-\omega_4 + \omega i_5 + \omega j_7),
\end{align*}
\]

and the corresponding negative roots are obtained by swapping the coefficients 1 and \( -\epsilon \) of the two halves of the vector. Left-multiples by \( \omega \) and \( \overline{\omega} \) are easily obtained by applying the element \( d^{a^{-1}} \).
In characteristic 2, the proof of Lemma 3 fails, but the construction of the Lie algebra goes through. Restricting to $F_4$ one finds that the group $3^3:\text{SL}_3(3)$ is no longer irreducible in characteristic 2, but has two constituents, each of degree 26. This is reflected in the fact that the Lie algebra of type $F_4$ is no longer simple, but contains an ideal of dimension 26. This ideal can be seen using the (2-sided) ideal $\langle 1 + i \rangle$ in $\mathbb{Z}[i,\omega]$. The latter ideal is spanned additively by the long roots of the $F_4$ root system, and modulo $2\mathbb{Z}[i,\omega]$ contains just three non-zero cosets, containing respectively $i + j, j + k$ and $k + i$. Writing
\[
a_t = i_t + j_t, \\
b_t = j_t + k_t, \\
c_t = k_t + i_t,
\]
we have the following multiplication table for the ideal:
\[
\begin{array}{ccc}
a_0 & b_1 & c_1 \\
\hline
a_0 & a_3 + a_9 & a_3 + a_9 & b_3 + b_9 \\
b_0 & a_3 + b_9 & b_3 + c_9 & c_3 + a_9 \\
c_0 & b_3 + a_9 & c_3 + b_9 & a_3 + c_9
\end{array}
\]
Apart from the notation, this is the same as the multiplication constructed in [10] for the exceptional Jordan algebra in characteristic 2. (This is the only characteristic in which the exceptional Jordan algebra is also a Lie algebra.)

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