AMPLITUDE, PHASE, AND COMPLEX ANALYTICITY

D. Cabrera,*, P. Fernández de Córdoba**, and J.M. Isidro***
Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, Valencia 46022, Spain
* dacabur@upvnet.upv.es, ** pfernandez@mat.upv.es
*** joissan@mat.upv.es

Abstract Expressing the Schrödinger Lagrangian \( \mathcal{L} \) in terms of the quantum wavefunction \( \psi = \exp(S + iI) \) yields the conserved Noether current \( \mathbf{J} = \exp(2S) \nabla I \). When \( \psi \) is a stationary state, the divergence of \( \mathbf{J} \) vanishes. One can exchange \( S \) with \( I \) to obtain a new Lagrangian \( \tilde{\mathcal{L}} \) and a new Noether current \( \tilde{\mathbf{J}} = \exp(2I) \nabla S \), conserved under the equations of motion of \( \tilde{\mathcal{L}} \). However this new current \( \tilde{\mathbf{J}} \) is generally not conserved under the equations of motion of the original Lagrangian \( \mathcal{L} \). We analyse the role played by \( \tilde{\mathbf{J}} \) in the case when classical configuration space is a complex manifold, and relate its nonvanishing divergence to the inexistence of complex-analytic wavefunctions in the quantum theory described by \( \mathcal{L} \).

1 Introduction

Madelung, Brillouin, Kramers and Wentzel with their WKB approximation, and later Bohm, all pioneered the factorisation of the complex wavefunction \( \psi \) into amplitude and phase (see the book [4] for a full account). Invoking the correspondence principle, this factorisation expresses the phase of \( \psi \) as the (complex) exponential of the classical action \( I \). For the amplitude of \( \psi \) one invokes Boltzmann’s principle and Born’s rule in order to write it as the (real) exponential of the entropy \( S \) [3]. Altogether one writes the wavefunction as

\[
\psi = \exp \left( \frac{S}{2k_B} + i\frac{I}{\hbar} \right),
\]

where \( k_B \) is Boltzmann’s constant\(^1\). Obviously the factorisation (1) breaks down at the zeroes of \( \psi \), where one formally sets \( S = -\infty \). It is convenient to introduce the dimensionless entropy \( S \) and the dimensionless mechanical action \( I \),

\[
S := \frac{S}{2k_B}, \quad I := \frac{I}{\hbar},
\]

in order to write more neatly

\[
\psi = \exp \left( S + iI \right).
\]

\(^1\)That Boltzmann’s constant \( k_B \) qualifies as a quantum of entropy does not seem to have been widely recognised in the literature; see however ref. [5].
Then the quantum mechanics of $\psi$ can be very conveniently pictured as the fluid mechanics of a quantum probability fluid, the velocity field $v$ being given by

$$v = \frac{\hbar}{m} \nabla I.$$  

(4)

Using the decomposition (3), in ref. [3] we have analysed the properties of nonstationary quantum states. In the present letter we will analyse stationary states instead. We will establish that when classical configuration space is $\mathbb{R}^{2k}$, quantum effects cause a certain current $J$ (to be defined in Eq. (16) below) to develop a nonvanishing value of $\nabla \cdot J$. In the stationary regime, a nonvanishing value of $\nabla \cdot J$ indicates the breakdown of a conservation law. Specifically, we will establish conditions under which $\nabla \cdot J \neq 0$ will imply the impossibility of having complex–analytic wavefunctions on $\mathbb{R}^{2k}$, although the latter qualifies as a complex–analytic manifold. Analytic wavefunctions are common in the theory of coherent states [8], but they are defined on phase space instead.

We will see that the quantum effect responsible for the lack of complex analyticity in the quantum theory is the appearance of a natural length scale $\lambda_B = \frac{\hbar}{(mv)}$ associated with a quantum particle, namely the de Broglie wavelength. One the contrary, no natural length scale exists in classical mechanics. This does not imply that classical mechanics can always be endowed with a complex–analytic structure. A necessary condition for complex analyticity is that the dimension of configuration space be even, so odd–dimensional spaces are ruled out already from the start.

For the rest of this letter we will consider a quantum particle with $\mathbb{R}^{2k}$ as its classical configuration space. Then the stationary wavefunction $\psi$ will be a complex–valued function $\psi : D \subset \mathbb{R}^{2k} \to \mathbb{C}$ depending on $2k$ real coordinates $x_j$. Let $u = u(x_j)$ and $v = v(x_j)$ be real–valued functions on the domain $D$. We recall that the Cauchy–Riemann equations for the complex function $u + iv$ are equivalent to the orthogonality condition $\nabla u \cdot \nabla v = 0$, where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_{2k}})$.

### 2 Analyticity and the equations of motion

We will describe the quantum motion of a particle of mass $m$ under the stationary, external potential $V = V(x_j)$ by means of a fluid flow in a domain $D$ within configuration space $\mathbb{R}^{2k}$. It is convenient to start with the Schrödinger Lagrangian,

$$L = i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - V \psi^* \psi,$$  

(5)

and perform a canonical analysis in terms of the variables $S$ and $I$, as in Eq. (3). Substituting $\psi = \exp(S + iI)$ we find

$$L = \exp(2S) \left\{ i\hbar (\partial_t S + i\partial_t I) - \frac{\hbar^2}{2m} \left[ (\nabla S)^2 + (\nabla I)^2 \right] - V \right\}.$$  

(6)

This Lagrangian is manifestly invariant under the transformations $I \to I + \alpha$, where $\alpha \in \mathbb{R}$. These transformations induce rigid phase rotations $\psi \to \exp(i\alpha)\psi$ of the
wavefunction. The corresponding conserved Noether current is the probability density current,

\[ J = \frac{\hbar}{m} \left( 2S \right) \nabla I = \rho \mathbf{v}, \tag{7} \]

where the dimensionless density function is \( \rho = \exp(2S) \), and the velocity field \( \mathbf{v} \) is given in Eq. (4).

From the Lagrangian (6) one derives the equations of motion for \( S \) and \( I \). The former is also known as the quantum Hamilton–Jacobi equation,

\[ \hbar \frac{\partial I}{\partial t} + \frac{\hbar^2}{2m} (\nabla I)^2 + V + U = 0, \tag{8} \]

where \( U \) denotes the quantum potential,

\[ U := -\frac{\hbar^2}{2m} \left[ (\nabla S)^2 + \nabla^2 S \right]. \tag{9} \]

The equation of motion for \( I \) is the continuity equation for the quantum probability fluid,

\[ \frac{\partial S}{\partial t} + \frac{\hbar}{m} \nabla S \cdot \nabla I + \frac{\hbar}{2m} \nabla^2 I = 0. \tag{10} \]

Stationarity means \( \partial S/\partial t = 0 \) and \( \partial I/\partial t = -E \), thus Eqs. (8) and (10) respectively become

\[ \frac{\hbar^2}{2m} (\nabla I)^2 + V + U = E \tag{11} \]

and

\[ \nabla S \cdot \nabla I = -\frac{1}{2} \nabla^2 I. \tag{12} \]

If we now compute the divergence of the probability density current (7) we find that, by Eq. (12), it vanishes identically as had to be the case:

\[ \nabla \cdot J = \frac{\hbar}{m} \exp(2S) \left( 2\nabla S \cdot \nabla I + \nabla^2 I \right) = 0. \tag{13} \]

The conservation law (13) implies that the functions \( S \) and \( I \) can be arranged into an analytic function

\[ g := S + iI \tag{14} \]

if and only if \( I \) is harmonic. Let us summarise:

**Property 1** In the quantum mechanics of a particle in \( D \subset \mathbb{R}^{2k} \), the following three statements are equivalent:

i) the action \( I \) is a harmonic function on \( D \);

ii) the complex–valued function \( g = S + iI \) is analytic on \( D \);

iii) the stationary wavefunction \( \psi = \exp(S + iI) \) is analytic on \( D \).

The case when the quantum amplitude is spatially constant deserves special attention:

**Property 2** When \( \nabla S = 0 \) on \( D \), the following three statements hold:

i) the action \( I \) is a harmonic function on \( D \);

ii) the complex–valued function \( g \) is analytic on \( D \);

iii) the stationary wavefunction \( \psi = \exp(S + iI) \) is analytic on \( D \).
3 The analytic current

The previous properties follow from an analysis of the conservation law $\nabla \cdot \mathbf{J} = 0$, which holds exactly both classically and quantum mechanically.

Let us exchange the variables $S$ and $I$ in the Lagrangian $\mathcal{L}$. Denoting the result by $\tilde{\mathcal{L}}$, we have

$$\tilde{\mathcal{L}} = \exp(2I) \left\{ i\hbar(\partial_t I + i\partial_t S) - \frac{\hbar^2}{2m} \left[ (\nabla S)^2 + (\nabla I)^2 \right] - V \right\}. \quad (15)$$

The above Lagrangian is manifestly invariant under the transformations $S \rightarrow S + \beta$, where $\beta \in \mathbb{R}$. These transformations induce rigid phase rotations $\tilde{\psi} = \exp(i\beta)\psi$ of the wavefunction. The corresponding conserved Noether current is

$$\tilde{\mathbf{J}} = \frac{\hbar}{m} \exp(2I) \nabla S. \quad (16)$$

Indeed,

$$\nabla \cdot \tilde{\mathbf{J}} = \frac{\hbar}{m} \exp(2I) \left( 2\nabla I \cdot \nabla S + \nabla^2 S \right), \quad (17)$$

and the bracketed term vanishes by virtue of Eq. (12), after exchanging $S$ and $I$ in the latter.

A remark is in order. The statement the current $\tilde{\mathbf{J}}$ is conserved means conserved under the equations of motion of the Lagrangian $\tilde{\mathcal{L}}$; it does not imply conservation under the motions corresponding to $\mathcal{L}$. More precisely, the equations of motion of $\mathcal{L}$ may, but need not, preserve the property $\nabla \cdot \mathbf{J} = 0$. By the same token, the current $\mathbf{J}$ is conserved under the Lagrangian $\mathcal{L}$, but not necessarily under $\tilde{\mathcal{L}}$.

The current $\tilde{\mathbf{J}}$ generates scale transformations on the fields of the Lagrangian $\mathcal{L}$, where the stationary wavefunction is $\psi = \exp(S + iI)$. By Eq. (17) we have:

**Property 3** The current $\tilde{\mathbf{J}}$ is conserved by the motions corresponding to the Lagrangian $\mathcal{L}$ if and only if

$$\nabla I \cdot \nabla S = -\frac{1}{2} \nabla^2 S. \quad (18)$$

**Property 4** Whenever $S$ is a harmonic function on $D$, the following three statements are equivalent:

i) the divergence $\nabla \cdot \tilde{\mathbf{J}}$ vanishes identically on $D$;

ii) the function $g = S + iI$ is analytic on $D$;

iii) the stationary wavefunction $\psi = \exp(S + iI)$ is analytic on $D$.

A particular case of the above property occurs when $S$ is spatially constant. When $\nabla S = 0$ on $D$, then $\nabla^2 S = 0$, and the quantum potential (9) vanishes identically. Moreover the quantum Hamilton–Jacobi equation (8) reduces to its classical counterpart, while the continuity equation for the quantum probability fluid, Eq. (10), reduces to $\nabla^2 I = 0$. The two necessary conditions for analyticity of $g = S + iI$, namely $\nabla^2 S = 0$ and $\nabla^2 I = 0$, are satisfied.

**Property 5** Whenever $S$ is constant on $D$, it holds that $\nabla \cdot \tilde{\mathbf{J}} = 0$, and the stationary wavefunction $\psi$ is analytic on $D$. 
4 Discussion

We need to identify the quantum effects responsible for the stationary wavefunction \( \psi \) generally not being a complex-analytic map \( \psi : \mathbb{C}^k \to \mathbb{C} \), where \( \mathbb{C}^k \) is classical configuration space \( \mathbb{R}^{2k} \).

The classical mechanics of a point particle possesses no natural length scale. Indeed, the classical density function of a point particle is a Dirac delta function, which naturally carries the dimensions of an inverse volume. On the contrary, the quantum mechanics of a particle of mass \( m \) carries a natural length scale associated, namely the de Broglie wavelength \( \lambda_B = \hbar/(mv) \). Quantum density distributions are usually not sharply localised in space. Instead of being a Dirac delta, the real part of the function \( g = S + iI \) in Eq. (14) is spread out, and a necessary requirement of analyticity (that the real and imaginary parts of \( g \) be harmonic functions) need not be satisfied. As a consequence, the quantum wavefunction may, but need not always be, analytic.

These conclusions can be neatly reexpressed through the introduction of a new current \( \tilde{J} \), defined in Eq. (16), and the corresponding divergence \( \nabla \cdot \tilde{J} \). The new current \( \tilde{J} \) is obtained from the standard probability density current \( J \) by the exchange of \( S \) and \( I \). We have established necessary, sufficient, and necessary and sufficient conditions that relate the (non)vanishing divergence \( \nabla \cdot \tilde{J} \) to the (non)analyticity of the function \( g = S + iI \) and of the stationary wavefunction \( \psi = \exp(g) \).

We have refrained from calling the nonvanishing divergence \( \nabla \cdot \tilde{J} \) an anomaly. Our situation does not exactly match the textbook definition of an anomaly [10], in the sense that we are not always dealing with a classical symmetry that breaks down at the quantum level. Classical mechanics need not always be complex-analytic (e.g., when configuration space is odd-dimensional), nor need \( \nabla \cdot \tilde{J} \neq 0 \) always hold in the quantum theory. Still, let us temporarily accept calling a nonvanishing value of \( \nabla \cdot \tilde{J} \) an anomaly. Then the adjectives holomorphic and analytic come to mind. Now calling our nonvanishing divergence \( \nabla \cdot \tilde{J} \) the analytic anomaly might cause confusion with the well-established holomorphic anomaly of string theory. Indeed, topological string theory [6] and the holomorphic anomaly have been used in ref. [2] to analyse the WKB expansion of quantum mechanics. Altogether, calling \( \nabla \cdot \tilde{J} \) an analytic divergence seems more appropriate.

The relation exhibited in ref. [2] between quantum mechanics and topological theories [7, 9] raises an interesting question [1]: could it be that quantum mechanics arises as some kind of topological sector of some underlying theory?

Acknowledgements Research supported by grant no. ENE2015-71333-R (Spain).

References

[1] D. Cabrera, P. Fernández de Córdoba, J.M. Isidro and J. Vázquez Molina, Entropy, Topological Theories and Emergent Quantum Mechanics, [arXiv:1611.07357] [quant-ph].
[2] S. Codesido and M. Mariño, *Holomorphic Anomaly and Quantum Mechanics*, arXiv:1612.07687 [hep-th].

[3] P. Fernández de Córdoba, J.M. Isidro and J. Vázquez Molina, *Schrödinger vs. Navier–Stokes*, Entropy 18 (2016) 34, arXiv:1409.7035 [math-ph].

[4] P. Holland, *The Quantum Theory of Motion*, Cambridge University Press, Cambridge (1993).

[5] R. Landauer, *Irreversibility and Heat Generation in the Computing Process*, IBM Journal of Research and Development 5 (1961) 183.

[6] M. Mariño, *Chern–Simons Theory, Matrix Models, and Topological Strings*, International Series of Monographs on Physics 131, Oxford University Press, Oxford (2005).

[7] C. Nash, *Differential Topology and Quantum Field Theory*, Academic Press, London (1994).

[8] A. Perelomov, *Generalized Coherent States and their Applications*, Texts and Monographs in Physics, Springer, Berlin (1986).

[9] A. Schwarz, *Quantum Field Theory and Topology*, Grundlehren der Mathematischen Wissenschaften 307, Springer, Berlin (2010).

[10] S. Weinberg, *The Quantum Theory of Fields*, vol. II, Cambridge University Press, Cambridge (1996).