Algebraic cones of LCK manifolds with potential

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Abstract
A complex manifold $X$, $\dim X > 2$, is called “an LCK manifold with potential”, if it can be realized as a complex submanifold of a Hopf manifold. Let $\tilde{X}$ be its $\mathbb{Z}$-covering, considered as a complex submanifold in $\mathbb{C}^n \setminus 0$. We prove that $\tilde{X}$ is algebraic. We call the manifolds obtained this way the algebraic cones, and show that the affine algebraic structure on $\tilde{X}$ is independent from the choice of $X$. We give several intrinsic definitions of an algebraic cone, and prove that these definitions are equivalent.

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1 Introduction

1.1 LCK manifolds with potential

An LCK manifold with potential ([OV2]) can be defined in terms of a certain Hermitian metric (Definition 3.3). For our purpose, another definition is more convenient. In complex dimension \(>2\), an LCK manifold with potential can be defined as a compact complex manifold which admits a holomorphic embedding to a linear Hopf manifold (Definition 2.10).\(^3\)

Let \(\tilde{M}\) be an LCK manifold with potential, embedded to a Hopf manifold \(\mathbb{C}^n \setminus \mathbb{D}\). Then the \(\mathbb{Z}\)-cover \(\tilde{M}\) of \(\tilde{M}\) is a complex submanifold in \(\mathbb{C}^n \setminus \mathbb{D}\).

The closure \(\tilde{M}_c\) of \(\tilde{M} \subset \mathbb{C}^n \setminus \mathbb{D}\) in \(\mathbb{C}^n\) is a Stein variety with an isolated singularity, called the (weak) Stein completion of \(\tilde{M}\) (Definition 2.3). The nature of this singularity was not explored so far; in the present paper, we relate its geometry to the classical subject of projective normality, due to Zariski, Muhly and Hodge-Pedoe ([Mu, Za, HP]).

The purpose of this paper is to study the algebraic geometry of the variety \(\tilde{M}_c\). We use the LCK geometry mostly as a motivation; almost all the results are based on arguments of complex-analytic and algebraic

\(^3\)The equivalence of these two definitions is proven in dimension \(>2\) ([OV4]); for complex surfaces it holds up to Kato conjecture, which is one of the greatest unresolved conjectures in the geometry of complex surfaces. Note that the Kato conjecture is widely believed to be true.
geometric nature which do not invoke the differential geometry of an LCK manifold.

1.2 Algebraic cones

Locally conformally Kähler manifolds with potential are in a sense very “algebraic”. The difference between complex projective and Kähler geometry is understood through the Kodaira embedding theorem: a Kähler manifold with rational Kähler class admits a complex embedding into a projective space. In this setup, one can say that the geometry of complex projective manifolds is part of the Kähler geometry, and this part is certainly very algebraic.

The analogue of Kodaira embedding theorem is already known: an LCK manifold with potential is precisely a complex manifold which admits a holomorphic embedding to a Hopf manifold. This notion is already very close to algebraic geometry. For example, there exists a $p$-adic version of the theory of Hopf manifolds and their complex subvarieties, in the framework of the rigid analytic spaces ([Sc]). The $p$-adic Hopf manifolds are known and well-studied ([Mus, Vo]).

The passage from LCK geometry to complex algebraic geometry is based on the notion of the “algebraic cone”, introduced in [OV3]. The closed algebraic cone of an LCK manifold with potential is its (weak) Stein completion $\tilde{M}_c$ (Definition 2.3), equipped with a structure of an affine variety, and the open algebraic cone is $\tilde{M}$, that is, $\tilde{M}_c$ without its origin (apex) point. We give an intrinsic definition of the closed and open algebraic cones in Subsection 2.2.

1.3 Projective normality

Let $X$ be a projective variety, and $R := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}(i))$ the corresponding graded ring. Since $R$ is finitely generated, its spectrum is an affine variety, called the affine cone of $X$.

Recall that a complex variety $X$ is called normal if any bimeromorphic finite map $X' \to X$ is an isomorphism. We discuss the normality in some detail in Section 8.

This is related to the notion of “projective normality” much studied in classical algebraic geometry. A projective variety $X \subset \mathbb{C}P^n$ is called projectively normal if the ring $R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}(i))$ of the regular

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\textsuperscript{4}We actually use an equivalent definition (Subsection 8): a variety is normal if all locally bounded meromorphic functions are holomorphic.
functions in its affine cone is integrally closed; this is equivalent to the affine cone being normal. The projective normality is a very subtle notion, because it greatly depends on the choice of projective embedding ([Ha, Exercise I.3.18]). Before the advent of the Hodge theory, projective normality was used to prove things such as the Riemann-Roch formula. O. Zariski and his student H. T. Muhly, who were the first to study the projective normality in a systematic manner, called it “arithmetic normality” ([Mu, Za]); however, the “projective normality” stuck because it was used in the textbook by Hodge and Pedoe, [HP].

1.4 New results

In this subsection, we summarize the new results of this paper.

From the Remmert-Stein theorem ([De, Chapter 2, §8.2]) it follows that the closure $\tilde{M}_c$ of $\tilde{M}$ in $\mathbb{C}^n$ is a Stein variety with a singular point in the origin (Remark 4.2). We call this closure a weak Stein completion of $\tilde{M}$. Note that a weak Stein completion $\tilde{A}$ a priori depends on the embedding $\tilde{M} \to H$; however, for an appropriate embedding, the completion $\tilde{M}_c$ is normal, and this normal variety, called the Stein completion, is uniquely determined by the complex geometry of $\tilde{M}$ (Remark 2.2).

We prove that the Stein completion $\tilde{M}_c$ homeomorphically and holomorphically projects to all weak Stein completions (Proposition 4.3).

We consider $\tilde{M}_c$ as a Stein variety with a unique isolated singular point. In [OV2, OV3] we proved that this Stein variety is in fact an affine cone over a projective variety (Theorem 3.5). In this paper we describe this cone from the algebraic-geometric point of view.

In Theorem 2.9, we prove that the (affine) algebraic structure of $\tilde{M}$ is uniquely determined by its complex analytic structure, as long as the $\mathbb{Z}$-action is algebraic. Ipso facto this algebraic structure is independent from the choice of the LCK manifold with potential $M = \tilde{M}/\mathbb{Z}$ used to construct $\tilde{M}$.

We define a closed algebraic cone as an affine cone over a projective orbifold, which is smooth outside of the origin. The variety $\tilde{M}_c$ is an example of a closed algebraic cone. In Theorem 2.11 we prove that this definition is equivalent to the definition of the closed algebraic cone given in [OV3] in terms of LCK manifolds with potential.

In this paper we study the set of closed algebraic cones which correspond to the same open algebraic cone.

We use the Stein completion to prove that any open algebraic cone $\tilde{M}$ can be associated with a unique closed algebraic cone $\tilde{M}_c$, which is normal.
(Proposition 4.3). We also prove that the other closed algebraic cones associated to \( \hat{M} \) are homeomorphic to \( \hat{M}_c \), because they are all obtained by adding a point in the origin. We prove that the natural homeomorphism map \( n : \hat{M}_c \to \hat{M}'_c \) to another closed algebraic cone associated to \( \hat{M} \) is holomorphic. Moreover, \( n \) is the normalization map.

2 Closed and open algebraic cones

In this section we give several definitions of closed and open algebraic cones. In the main body of this paper we prove that all these definitions are equivalent.

2.1 Stein completions

Let \( K \subset \mathbb{C}^n \) be a compact subset. Hartogs theorem implies that any holomorphic function on \( \mathbb{C}^n \setminus K \) can be extended to \( \mathbb{C}^n \), if \( n \geq 2 \). Due to Rossi, the same result is true for any Stein variety:

**Theorem 2.1:** Let \( X \) be a normal Stein variety, \( \dim_{\mathbb{C}} X > 1 \), and \( K \subset X \) a compact subset. Then every holomorphic function on \( X \setminus K \) can be extended to \( X \).

**Proof:** [Ro, Theorem 6.6].

**Remark 2.2:** Let \( A \) be a commutative Fréchet algebra over \( \mathbb{C} \) and \( \text{Spec}(A) \) the continuous spectrum of \( A \), defined as the set of all continuous \( \mathbb{C} \)-linear homomorphisms \( A \to \mathbb{C} \). By [Fo1, Fo2], \( \text{Spec}(H^0(\mathcal{O}_X)) = X \) for any Stein variety \( X \), where \( H^0(\mathcal{O}_X) \) is the algebra of holomorphic functions equipped with the topology of uniform convergence on compacts. From [Fo2] it also follows that a Stein manifold is determined uniquely from \( H^0(\mathcal{O}_X) \) considered as a ring with \( C^0 \)-topology.

**Definition 2.3:** Let \( X \) be a normal Stein variety, and \( K \subset X \) a compact subset. By **Theorem 2.1**, the ring of functions on \( X \) is identified with \( H^0(\mathcal{O}_{X\setminus K}) \); by **Remark 2.2**, this ring with \( C^0 \) topology uniquely defines \( X \). Following [AS], we call \( X \) the Stein completion of \( X \setminus K \). A weak Stein completion of \( X \setminus K \) is any Stein variety \( X' \) containing a compact \( K' \) such that \( X'\setminus K' \cong X\setminus K \).

**Remark 2.4:** Clearly, if \( X \setminus K \) is normal, the normalization of a weak Stein completion of \( X' \setminus K \) is the Stein completion of \( X \setminus K \).
Example 2.5: Let $M \subset H$ be a submanifold in a Hopf manifold $H = \mathbb{C}^N \setminus \mathbb{Z}$, and $\tilde{M} \subset \mathbb{C}^N \setminus \mathbb{Z}$ its $\mathbb{Z}$-cover. By the Remmert-Stein theorem (Remark 4.2), the closure of $\tilde{M}$ is Stein; however, it is not necessarily normal (Example 8.6). This is why we need the notion of a “weak Stein completion”.

Example 2.6: Let $P$ be a projective orbifold, and $L$ an ample line bundle on $P$. Consider the total space $\text{Tot}^\circ(L)$ of all non-zero vectors in $L$. By Proposition 3.6 and Proposition 4.3 below, its (weak) Stein completion is obtained by adding a single point, called the apex, or the origin. The weak Stein completion of $\text{Tot}^\circ(L)$ is not unique, as Example 8.6 shows; however, all weak Stein completions of $\text{Tot}^\circ(L)$ are homeomorphic (Proposition 4.3).

2.2 Algebraic structures on algebraic cones

The notion of an algebraic cone has two flavours: there are “closed algebraic cones”, which can be obtained as weak Stein completions of “open algebraic cones”. The latter can be defined in algebraic terms as follows.

Definition 2.7: Let $P$ be a projective orbifold, and $L$ an ample line bundle on $P$. Assume that the total space $\text{Tot}^\circ(L)$ of all non-zero vectors in $L$ is smooth. An open algebraic cone is $\text{Tot}^\circ(L)$. The corresponding closed algebraic cone is its weak Stein completion $Z$, such that the natural $\mathbb{C}^*$-action can be extended to $Z$ holomorphically. By Proposition 4.3, the closed algebraic cone is obtained by adding one point, called “the apex”, or “the origin”.

Further on, we shall rely on the notion of “holomorphic contraction”. To simplify the terminology, we tacitly assume that the “contractions” are invertible; however, most of the arguments remain valid without this assumption.

Definition 2.8: A contraction of a topological space $X$ to a point $x \in X$ is a homeomorphism $\varphi : X \to X$ such that for any compact subset $K \subset X$ and any open set $U \ni x$, there exists $N > 0$ such that for all $n > N$, the map $\varphi^n$ maps $K$ to $U$.

An algebraic structure on a complex analytic variety $Z$ is a subsheaf of the sheaf of holomorphic functions which can be realized as a sheaf of
regular functions for some biholomorphism between $Z$ and a quasi-projective variety.

One of the main results of this paper is the following theorem, proven in Section 6 below. Note that any closed algebraic cone $X$ admits an algebraic contraction $\gamma: X \rightarrow X$ taking $v$ to $\lambda v$, for $\lambda$ an invertible complex number, $|\lambda| < 1$.

**Theorem 2.9:** Let $X$ be a closed algebraic cone, and $\mathcal{A}_1$, $\mathcal{A}_2$ two algebraic structures on $X$. Assume that the algebraic varieties $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$ are affine and admit (invertible) algebraic contractions $\gamma_1$ and $\gamma_2$, that is, holomorphic contractions compatible with the respective algebraic structures. Then $\mathcal{A}_1 = \mathcal{A}_2$.

**Proof:** By Theorem 2.11 below, $X$ can be obtained in two different ways as a weak Stein completion of $Z$-coverings of LCK manifolds with potential associated with the holomorphic contractions $\gamma_1$ and $\gamma_2$. Then $\mathcal{A}_1 = \mathcal{A}_2$ by Theorem 6.5. ■

### 2.3 Algebraic cones and subvarieties in Hopf manifolds

The algebraic cones can be defined in terms of projective orbifolds as above; there are two more definitions equivalent to this one. We could define open algebraic cones as $Z$-coverings of submanifolds of a linear Hopf manifold (which is done in the present section) or define the closed algebraic cones as Stein varieties with an isolated singularity admitting a holomorphic contraction, as in Subsection 2.4 below.

**Definition 2.10:** A **linear Hopf manifold** is a complex manifold $H := \mathbb{C}^n \setminus \{A\}$, where $A \in \text{GL}(n, \mathbb{C})$ is an invertible linear contraction. A **diagonal Hopf manifold** is a linear Hopf manifold such that $A$ is diagonalizable, and a **classical Hopf manifold** is a linear Hopf manifold such that $A$ is a scalar matrix.

**Theorem 2.11:** Let $M \subset H$ be a submanifold in a Hopf manifold, and $\tilde{M} \subset \mathbb{C}^n \setminus 0$ its $Z$-covering. Then $\tilde{M}$ is an open algebraic cone. Moreover, any open algebraic cone can be obtained this way.

**Proof:** Let $\tilde{M}$ be an open algebraic cone equipped with the standard $\mathbb{C}^*$-action, and $\gamma \in \text{Aut}(\tilde{M})$ the automorphism associated with $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$. ■
1. By Theorem 7.11, $\tilde{M}$ admits an embedding to a linear Hopf manifold. To obtain the converse assertion, we use the locally conformally Kähler metrics with potential, defined below (Definition 3.3). As shown in [OV3], any submanifold of a linear Hopf manifold is LCK with (proper) potential. Theorem 7.9 implies that a Kähler $\mathbb{Z}$-cover of an LCK manifold with proper potential is an open algebraic cone. ■

2.4 Algebraic cones and holomorphic contractions

Another equivalent definition of algebraic cones uses holomorphic contractions. Note that this definition does not refer to algebraic structures, however, an algebraic structure compatible with a contraction is unique by Theorem 2.9.

Theorem 2.12: Let $X$ be a Stein variety with a unique isolated singularity, admitting a holomorphic contraction. Then $X$ is biholomorphic to a closed algebraic cone. Conversely, any closed algebraic cone can be obtained this way.

Proof: By the same argument as used in the proof of Proposition 4.3, a closed algebraic cone $X$ can be obtained as the closure of the preimage $\pi^{-1}(P)$, where $P \subset \mathbb{C}P^n$ is a projective orbifold and $\pi : \mathbb{C}^{n+1}\setminus 0 \to \mathbb{C}P^n$ the standard projection. Then $X$ admits a contraction $v \mapsto \lambda v$, where $|\lambda| < 1$.

Conversely, suppose that $X$ is a Stein variety with a unique isolated singularity $x$ admitting a holomorphic contraction $\gamma$ to $x$. Then $X \setminus \{x\} \gamma$ is a complex manifold admitting a holomorphic embedding to a Hopf manifold (Theorem 7.11), hence it is an LCK manifold with proper potential (Theorem 3.5). By Theorem 7.9, $X \setminus \{x\}$ is an open algebraic cone. ■

3 Locally conformally Kähler manifolds

The main motivation for this paper comes from the theory of LCK (locally conformally Kähler) manifolds, though we could state and prove everything in it without using this important notion.

Definition 3.1: A complex manifold $(M, I)$ is called locally conformally Kähler (LCK, for short) if it admits a covering $(\tilde{M}, I)$ equipped with a Kähler metric $\tilde{\omega}$ such that the deck group of the cover acts on $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. An LCK metric on an LCK manifold is an
Hermitian metric on $(M, I)$ such that its pullback to $\tilde{M}$ is conformal with $\tilde{\omega}$.

**Remark 3.2:** Equivalently, a Hermitian manifold $(M, I, g)$ is LCK if there exists a closed 1-form $\theta$ (called the Lee form) such that the fundamental form $\omega(x, y) := g(Ix, y)$ satisfies $d\omega = \theta \wedge \omega$.

The special class of LCK manifolds we are concerned with is the following:

**Definition 3.3:** An LCK manifold has a **proper LCK potential** if it admits a Kähler $\mathbb{Z}$-covering on which the Kähler metric has a global and positive potential function $\psi$ such that the deck group multiplies $\psi$ by a constant.\(^5\) In this case, $M$ is called an **LCK manifold with proper potential**.

In the sequel, we will often tacitly omit the word “proper”.

**Example 3.4:** The linear Hopf manifolds are LCK with potential, \([OV6]\).

For a classical Hopf manifold $H := (\mathbb{C}^n \setminus 0)/\langle A \rangle$, $A = \lambda \text{Id}$, $|\lambda| > 1$, the flat Kähler metric $\tilde{g}_0 = \sum dz_i \otimes d\overline{z}_i$ on $\mathbb{C}^n$ is multiplied by $\lambda^2$ by the deck group $\mathbb{Z}$. Also, $\tilde{g}_0$ has the global automorphic potential $\psi := \sum |z_i|^2$.

The main properties of LCK manifolds with (proper) potential are the following.

**Theorem 3.5:** ([OV2, OV3]) Let $M$ be a compact LCK manifold with proper potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-cover. If $\dim_{\mathbb{C}} M \geq 3$, then the metric completion $\tilde{M}_c$ is identified with the Stein completion of $\tilde{M}$, the complement $\tilde{M}_c \setminus \tilde{M}$ is just one point. Moreover, $\tilde{M}_c$ is an affine algebraic variety obtained as an affine cone over a projective orbifold.

This is used to prove the following result (for an alternative, more algebraic, proof, see **Proposition 4.3**).

**Proposition 3.6:** Let $P$ be a projective orbifold, $\dim_{\mathbb{C}} P > 1$, and $L$ an ample line bundle on $P$. Assume that the total space $\text{Tot}^0(L)$ of all non-zero

\(^5\)Such a function is called **automorphic**.
vectors in $L$ is smooth. Then a weak Stein completion of $\text{Tot}^o(L)$ is obtained by adding a point (called “the origin” or “the apex” elsewhere).

**Proof. Step 1:** Let $P$ be a principal $\mathbb{C}^*$-bundle over $X$, with $\mathbb{C}^*$-action denoted by $\rho_P(t) : P \rightarrow P$. Consider the dual $\mathbb{C}^*$-bundle $P^*$, which has the same underlying complex manifold but the group $\mathbb{C}^*$ acts on $P^*$ via $\rho_{P^*}(t) = \rho_P(t^{-1})$. There is a natural duality between $P$ and $P^*$ mapping $P \times_X P^*$ to a trivial $\mathbb{C}^*$-bundle $\Theta^*(M)$, taking a pair $(p, p)$ of sections of $P$ and $P^*$, identified with $P$, to the unit section of $\Theta^*(X)$.

Let $\text{Tot}^o(L)$ be the set of non-zero vectors in an ample line bundle $L$ over a projective orbifold. Clearly, its dual bundle can be identified with $\text{Tot}^o(L^*)$, hence the spaces $\text{Tot}^o(L)$ and $\text{Tot}^o(L^*)$ are biholomorphic.

**Step 2:** Replacing $L$ by $L^*$, we can assume that $L$ is anti-ample and it is equipped with a metric with negative curvature. Then the function $v \rightarrow |v|^2$ is strictly plurisubharmonic on $\text{Tot}^o(L)$, by [Ca, (2.6)] or [Be, (15.19)], and defines an LCK potential on the compact manifold $M := \frac{\text{Tot}^o(L)}{\langle \gamma \rangle}$, where $\gamma(x) = \lambda x$, $|\lambda| > 1$. Applying Theorem 3.5 to $\text{Tot}^o(L)$, considered as a $\mathbb{Z}$-covering of $M$, we obtain that the Stein completion of $\text{Tot}^o(L)$ is obtained by adding a point in the origin. By Remark 2.4, $\tilde{M}_c$ is the normalization of any weak Stein completion $\tilde{Z}$ of $\tilde{M}$. However, the normalization map $\nu : \tilde{M}_c \rightarrow \tilde{Z}$ is continuous, and identity outside of the origin, because $\tilde{Z}$ is smooth outside of the origin. Therefore, $\nu$ is a diffeomorphism.

We give a more algebraic proof of Proposition 3.6, independent from the LCK geometry, in Section 4. Note that the argument in Section 4 is stronger, because it works even when $\dim_{\mathbb{C}} P = 1$.

Theorem 3.5 was used to prove the following Kodaira type embedding theorem for LCK manifolds with potential. Recall that a linear Hopf manifold is a manifold $H := \frac{\mathbb{C}^{m\times n}}{ \langle A \rangle}$, where $A \in \text{GL}(\mathbb{C}^m)$ is a linear contraction. In [OV3] it was shown that all linear Hopf manifolds admit an LCK metric with potential.

**Theorem 3.7:** ([OV2, OV4]) Let $(M, I, \omega)$ be a compact LCK manifold with potential, $\dim_{\mathbb{C}} M \geq 3$. Then $(M, I)$ admits a holomorphic embedding to a linear Hopf manifold. ■
Remark 3.8: The converse assertion is also clearly true: a submanifold of a linear Hopf manifold is LCK with potential ([OV3]). Therefore, in dimension $\geq 3$, an LCK manifold with (proper) potential can be defined as a manifold which admits a holomorphic embedding into a linear Hopf manifold. In dimension 2, this is also true, if we assume the Kato conjecture (also known as “the global spherical shell conjecture”); see [OV4, Section 5] for details.

4 The total space of an ample bundle, Remmert-Stein theorem and weak Stein completions

In this section, we give an alternative proof of Proposition 3.6, independent from the LCK geometry. We start by recalling the Remmert-Stein theorem.

Theorem 4.1: (Remmert-Stein theorem)
Let $B$ be a complex analytic variety, and $C \subset B$ and $A \subset B \setminus C$ complex analytic subvarieties. Assume that all irreducible components $A_i$ of $A$ satisfy $\dim A_i > \dim C$. Then the closure of $A$ is complex analytic in $B$.

Proof: [De, §II.8.2].

Remark 4.2: We will use Theorem 4.1 in one special case, which is also the case used in the proof of Chow theorem ([De, Theorem II.8.10]). Suppose that $A \subset \mathbb{C}^n \setminus 0$ is a complex analytic subset without 0-dimensional connected components; then its closure $\overline{A}$ in $\mathbb{C}^n$ is complex analytic. A posteriori, $\overline{A}$ is isomorphic to a weak Stein completion of $A$, Definition 2.3.

At this point we can give an alternative proof of Proposition 3.6.

Proposition 4.3: Let $P$ be a projective orbifold, and $L$ an ample line bundle on $P$. Assume that the total space $\text{Tot}^0(L)$ of all non-zero vectors in $L$ is smooth. Then a weak Stein completion of $\text{Tot}^0(L)$ is obtained by adding a point (called “the origin” or “the apex” elsewhere). Moreover, the natural normalization map from the Stein completion to a weak Stein completion is a homeomorphism.

Proof: The space $\text{Tot}^0(L)$ can be interpreted algebraically as follows. Consider the standard $\mathbb{C}^*$-action on $\text{Tot}^0(L)$ understood as a principal $\mathbb{C}^*$-bundle. A finite-dimensional $\mathbb{C}^*$-representation is a direct sum of 1-dimen-
dimensional irreducible representations, with \( \mathbb{C}^* \) acting by \( \rho(t) = t^w \); the number \( w \) is called the **weight** of the representation. Therefore, the category of \( \mathbb{C}^* \)-representations is equivalent to the category of graded vector spaces. A ring with \( \mathbb{C}^* \)-action is the same as a graded ring. In particular, the ring of fiberwise polynomial holomorphic functions on \( \text{Tot}^\circ(L) \) is graded, with the functions of weight \( w \) being polynomials of degree \( w \) on each \( \mathbb{C}^* \)-orbit. We identify the space \( (\mathcal{O}_{\text{Tot}^\circ(L)})_w = (\mathcal{O}_{\text{Tot}^\circ(L^*)})_w \) of functions of degree \( w \) with the space of sections of \( L^\otimes w \). The variety \( P \subset \mathbb{P}(H^0(L^N)^*) \) is identified with the graded spectrum of the graded ring \( \bigoplus_w H^0(X, L^\otimes w N) \), whenever \( L^N \) is very ample. Then, the manifold \( \text{Tot}^\circ(L) \) is the preimage of \( P \) under the natural projection \( H^0(L^N)^* \setminus 0 \to \mathbb{P}(H^0(L^N)^*) \). The Stein completion of \( \text{Tot}^\circ(L) \) is its closure in \( H^0(L^N)^* \), obtained by adding the zero (called “the origin” or “the apex” elsewhere). It is complex analytic by Remmert-Stein theorem (Remark 4.2), and normal by Proposition 8.4 below.

By Remark 2.4, the Stein completion of \( \text{Tot}^\circ(L) \) is the normalization of any weak Stein completion \( Z \) of \( \tilde{M} \). Since the normalization map \( \nu : \text{Tot}^\circ(L) \to Z \) is continuous, and identity outside of the origin, \( \nu \) it is a homeomorphism.

### 5 Montel theorem and Riesz-Schauder theorem

In this section, we recall several basic notions of complex analysis and functional analysis.

#### 5.1 Montel theorem

**Definition 5.1:** Let \( M \) be a complex manifold, and \( \mathcal{F} \subset H^0(\mathcal{O}_M) \) a family of holomorphic functions. We call \( \mathcal{F} \) a **normal family** if for each compact \( K \subset M \) there exists \( C_K > 0 \) such that for each \( f \in \mathcal{F} \), \( \sup_K |f| \leq C_K \).

**Definition 5.2:** The \( C^0 \)-**topology** on the space of functions on \( M \) is the topology of uniform convergence on compacts.

**Theorem 5.3:** (Montel)

Let \( \mathcal{F} \subset H^0(\mathcal{O}_M) \) be a normal family of functions, and \( \overline{\mathcal{F}} \subset H^0(\mathcal{O}_M) \) its closure in the \( C^0 \)-topology. Then \( \overline{\mathcal{F}} \) is compact in \( C^0 \)-topology.

**Proof:** [Wu, Lemma 1.4]. □
5.2 The Banach space of holomorphic functions

We briefly recall some notions of functional analysis, which are standard.

Remark 5.4: Let $C^0(X)$ be the space of continuous functions on a compact space, with the norm $\|f\|_{\sup} := \sup_{x \in X}|f(x)|$. It is not hard to see that this space is Banach. Similarly, if $X$ is a compact smooth manifold equipped with a connection $\nabla$, the space $C^k(M)$ of $k$ times differentiable functions with the norm $\|f\|_{C^k} := \sum_{i=0}^k \|\nabla^i(f)\|_{\sup}$ is also Banach.

Theorem 5.5: Let $M$ be a complex manifold, and let $H^0_b(O_M)$ the space of all bounded holomorphic functions, equipped with the sup-norm $|f|_{\sup} := \sup_M |f|$. Then $H^0_b(O_M)$ is a Banach space.

Proof: Let $\{f_i\} \in H^0_b(O_M)$ be a Cauchy sequence in the $\sup$-norm. Then $\{f_i\}$ converges to a continuous function $f$ in the $\sup$-topology.

Since $\{f_i\}$ is a normal family (see Definition 5.1), it has a subsequence which converges in $C^0$-topology to $\tilde{f} \in H^0(O_M)$, by Montel’s Theorem 5.3. However, the $C^0$-topology is weaker than the $\sup$-topology, hence $\tilde{f} = f$. Therefore, $f$ is holomorphic. ■

5.2.1 Compact operators

Recall that a subset $X$ of a topological space $Y$ is called precompact, or relatively compact in $Y$, if its closure is compact.

Definition 5.6: A subset $K \subset V$ of a topological vector space is called bounded if for any open set $U \ni 0$, there exists a number $\lambda_U \in \mathbb{R}^>0$ such that $\lambda_U K \subset U$.

Definition 5.7: Let $V, W$ be topological vector spaces. A continuous operator $\varphi : V \to W$ is called compact if the image of any bounded set is precompact.

5.2.2 Holomorphic contractions

Theorem 5.8: Let $X$ be a complex variety, and $\gamma : X \to X$ a holomorphic contraction to $x \in X$ such that $\gamma(X)$ is precompact. Consider the Banach space $V = H^0_b(O_X)$ of bounded holomorphic functions with the sup-norm, and let $V_x \subset V$ be the space of all $v \in V$ vanishing in $x$. Then $\gamma^* : V \to V$...
is compact, and the eigenvalues of its restriction to \( V_x \) are strictly less than 1 in absolute value.\(^6\)

**Proof. Step 1:** For any \( f \in H^0(\mathcal{X}) \) we have

\[
|\gamma^* f|_{\sup} = \sup_{x \in \gamma(X)} |f(x)|.
\]

This implies that \( \gamma^*(f) \) is bounded. Therefore, for any sequence \( \{ f_i \} \subset H^0(\mathcal{X}) \) converging in the \( C^0 \)-topology, the sequence \( \{ \gamma^* f_i \} \) converges in the \( \sup \)-topology. The set \( B_C := \{ v \in V \mid |v|_{\sup} \leq C \} \) is precompact in the \( C^0 \)-topology, because it is a normal family. Then \( \gamma^* B_C \) is precompact in the \( \sup \)-topology.\(^7\) This proves that the operator \( \gamma^* : V \to V \) is compact.

It remains to show that its operator norm is \(<1\) on \( V_x \).

**Step 2:** Since \( \sup_X |\gamma^* f| = \sup_{\gamma(X)} |f| \leq \sup_X |f| \), one has \( \|\gamma^*\| \leq 1 \). If this inequality is not strict, for some sequence \( \{ f_i \} \) of holomorphic functions \( f_i \in V_x \) with \( \sup_X |f_i| \leq 1 \) (that is, \( f_i \in B_1 \)) one has \( \lim_i \sup_{x \in \gamma(X)} |f_i(x)| = 1 \). Since \( B_1 \) is a normal family, \( f_i \) has a subsequence converging in \( C^0 \)-topology to \( f \). Then \( \{ \gamma(f_i) \} \) converges to \( \gamma(f) \) in \( \sup \)-topology, giving

\[
\lim_i \sup_{x \in \gamma(X)} |f_i(x)| = \sup_{x \in \gamma(X)} |f(x)| = 1.
\]

Since \( f(x) = 0 \), \( f \) is non-constant. By the maximum modulus principle, a non-constant holomorphic function has no local maxima; this means that \( |f(x)| > 1 \) somewhere on \( X \). Then \( f \) cannot be the \( C^0 \)-limit of \( \{ f_i \} \subset B_1 \). ■

### 5.2.3 The Riesz-Schauder theorem

The following result is a Banach analogue of the usual spectral theorem for compact operators on Hilbert spaces.

**Theorem 5.9:** (Riesz-Schauder, [Fr, Section 5.2])

Let \( F : V \to V \) be a compact operator on a Banach space. Then for each non-zero \( \mu \in \mathbb{C} \), there exists a sufficiently big number \( N \in \mathbb{N} \) such that for each \( n > N \) one has \( V = \ker(F - \mu \text{Id})^n \oplus \text{im}(F - \mu \text{Id})^n \), where

---

\(^6\)Since \( \gamma^* \) maps constants to constants identically, one cannot expect that \( \|\gamma^*\| < 1 \) on \( V \). However, if we add a condition which excludes constants, such as \( v(x) = 0 \), we immediately obtain \( \|\gamma^*\| < 1 \).

\(^7\)The \( C^0 \)-convergence for holomorphic functions is strictly weaker than the \( \sup \)-convergence; for example, the sequence of bounded holomorphic functions on an open disk, \( f_i := z^i \) converges in \( C^0 \)-topology, but does not converge in the \( \sup \)-topology.
\text{im}(F - \mu \text{Id})^n$ is the closure of the image. Moreover, $\ker(F - \mu \text{Id})^n$ is finite-dimensional and independent on $n$. 

**Remark 5.10:** Define the root space of an operator $F \in \text{End}(V)$, associated with an eigenvalue $\mu$, as $\bigcup_{n \in \mathbb{Z}} \ker(F - \mu \text{Id})^n$. In the finite-dimensional case, the root spaces coincide with the Jordan cells of the corresponding matrix. Then Theorem 5.9 can be reformulated by saying that any compact operator $F \in \text{End}(V)$ admits a Jordan cell decomposition, with $V$ identified with a completed direct sum of the root spaces, which are all finite-dimensional; moreover, the eigenvalues of $F$ converge to zero.

We need the following corollary of the Riesz-Schauder theorem, which is obtained using the same arguments as in the finite-dimensional case.

**Theorem 5.11:** Let $F : V \to V$ be a compact operator on a Banach space. We say that $v$ is a root vector for $F$ if $v$ lies in a root space of $F$, for some eigenvalue $\mu \in \mathbb{C}$. Then the space generated by the root vectors is dense in $V$.

**Definition 5.12:** Let $F \in \text{End}(V)$ be an endomorphism of a vector space. A vector $v \in V$ is called $F$-finite if the space generated by $v, F(v), F(F(v)), \ldots$ is finite-dimensional.

**Remark 5.13:** Clearly, a vector is $F$-finite if and only if it is obtained as a combination of root vectors. Then Theorem 5.11 implies the following.

**Corollary 5.14:** Let $F : V \to V$ be a compact operator on a Banach space, and $V_0 \subset V$ the space of all $F$-finite vectors. Then $V_0$ is dense in $V$.

We also need the following lemma, giving “a relative case” of Riesz-Schauder.

**Lemma 5.15:** Consider the commutative square of continuous operators of Banach spaces

$$
\begin{array}{ccc}
W_1 & \xrightarrow{K_1} & W_1 \\
R \downarrow & & \downarrow R \\
W_2 & \xrightarrow{K_2} & W_2
\end{array}
$$
with $R$ a surjective map, and both $K_i$, $i = 1, 2$, are compact. Restricting $R$ to the space $W^K_i$ of $K_i$-finite vectors, we obtain a map $R_f : W^K_1 \rightarrow W^K_2$. Then $R_f$ is surjective.

Proof: Let $U_2 \subset W_2$ be a finite-dimensional space of $K_2$-finite vectors, and $U_1 \subset W_1$ its preimage. Since $R$ is surjective, the natural map $U_1 \rightarrow U_2$ is surjective. Since $U^K_1$ is dense in $U_1$, and $U_2$ is finite-dimensional, the restriction of $R$ to $U^K_1$ is also surjective. ■

6 Algebraic structures on weak Stein completions

In this section, we put an algebraic structure on a weak Stein completion of a Kähler $\mathbb{Z}$-covering of a submanifold in a Hopf manifold. We also prove that this algebraic structure is unique. Later on, this algebraic structure is used to show that this weak Stein completion is an algebraic cone.

6.1 Ideals of the embedding to a Hopf manifold

Let $M \subset H$ be a subvariety of a linear Hopf manifold $H = \mathbb{C}^n \setminus \{0\}$. In this subsection we prove that the preimage of $M$ in $\mathbb{C}^n \setminus 0$ can be defined by a set of polynomial equations.

We start with the following lemma.

Lemma 6.1: Let $\gamma$ be an invertible linear contraction of $\mathbb{C}^n$. A holomorphic function on $\mathbb{C}^n$ is $\gamma$-finite if and only if it is polynomial.

Proof: Clearly, a polynomial function is $\gamma$-finite. The operator $\gamma^*$ acts on homogeneous polynomials of degree $d$ with eigenvalues $\alpha_{i_1, i_2, \ldots, i_d}$, where $\alpha_{i_j}$ are the eigenvalues of $\gamma$ on $\mathbb{C}^n$. Since $\gamma$ is a contraction, all $\alpha_{i_j}$ satisfy $|\alpha_{i_j}| < 1$. Therefore, any sequence $\{\alpha_{i_1, i_2, \ldots, i_d}\}$ converges to 0 as $d$ goes to infinity. We obtain that every given number can be realized as an eigenvalue of $\gamma^*$ on homogeneous polynomials of degree $d$ for finitely many choices of $d$ only. Therefore, any root vector of $\gamma^*$ is a finite sum of homogeneous polynomials. This implies that the Taylor decomposition of a $\gamma$-finite function $f$ can have only finitely many components, otherwise the eigenspace decomposition of $f$ with respect to the action of $\gamma^*$ is infinite. ■
This lemma can be used to prove the following theorem (see also Theorem 6.3 below).

**Theorem 6.2:** ([OV3, Theorem 2.8], [OV5, Theorem 5.5])

Let \( M \hookrightarrow H \) be a complex subvariety of a linear Hopf manifold, and \( \tilde{M}_c \rightarrow \mathbb{C}^N \) the corresponding map of weak Stein completions, with \( \tilde{M}_c \) obtained as the closure of \( M \subset \mathbb{C}^N \) by adding the zero (Remark 4.2). Then \( \tilde{M}_c \) is an algebraic subvariety, that is, a set of common zeroes of a system of polynomial equations.

### 6.2 Algebraic structures on Stein completions: the existence

Let \( X \subset M \subset \mathbb{C}P^n \) be projective subvarieties of \( \mathbb{C}P^n \). Then \( Z = M \setminus X \) is called a quasi-projective variety. The Zariski topology on \( Z \) is the topology such that the closed subsets of \( Z \) are closed algebraic subvarieties. The regular functions on a Zariski open subset \( W \subset X \) are restrictions of the rational functions on \( \mathbb{C}P^n \) which have no poles on \( W \). This defines the sheaf of regular functions on \( Z \) as a subsheaf of the sheaf of holomorphic functions in the Zariski topology on \( Z \).

Recall that an algebraic structure on a complex analytic variety \( Z \) is a subsheaf of the sheaf of holomorphic functions which can be realized as a sheaf of regular functions for some biholomorphism between \( Z \) and a quasi-projective variety.

Note that the algebraic structure is not unique, indeed, even the complex manifold \( (\mathbb{C}^*)^2 \) has more than one algebraic structure ([Si]). In [Je], Z. Jelonek has constructed an uncountable set of pairwise non-isomorphic algebraic structures on certain Stein manifolds.

It turns out that a Kähler \( \mathbb{Z} \)-cover of an LCK manifold with potential is equipped with a distinguished (affine) algebraic structure, which is uniquely determined by the complex structure (Theorem 6.3, Theorem 6.5).

The main result of this section is the following theorem.

**Theorem 6.3:** Let \( M \) be an LCK manifold with proper potential, and \( \tilde{M}_c \) the Stein variety obtained as a weak Stein completion of its Kähler \( \mathbb{Z} \)-cover \( \tilde{M} \). Denote by \( \gamma \in \text{Aut}(\tilde{M}_c) \) the generator of the \( \mathbb{Z} \)-action on \( \tilde{M} \) extended to \( \tilde{M}_c \). This map is an invertible holomorphic contraction centered in the apex \( c \) of \( \tilde{M}_c \). Then \( \tilde{M}_c \) admits an algebraic structure such that the regular functions are precisely the \( \gamma \)-finite functions.
Proof. Step 1: Lemma 6.1 implies Theorem 6.3 when \( \gamma \) is a linear contraction and \( \tilde{M}_c = \mathbb{C}^n \).

Step 2: By definition, \( M \) admits a holomorphic embedding to a linear Hopf manifold. Consider the corresponding embedding \( \tilde{M}_c \to \mathbb{C}^n \). By Theorem 6.2, \( \tilde{M}_c \) is an algebraic subvariety of \( \mathbb{C}^n \) given by a finite set of polynomial equations.

To finish the proof of Theorem 6.3, it remains to show that a holomorphic function on \( \tilde{M}_c \) is polynomial if and only if it is \( \gamma \)-finite. It is clear that any holomorphic function on \( \tilde{M}_c \) obtained from a polynomial function on \( \mathbb{C}^n \) is \( \gamma \)-finite. It remains to show the converse. Consider a \( \gamma \)-finite function on \( \tilde{M}_c \). By Lemma 6.1, to prove that it is polynomial, it would suffice to prove that it can be extended to a \( \gamma \)-finite function on \( \mathbb{C}^n \).

Consider the exact sequence of sheaves of holomorphic functions on \( \mathbb{C}^n \)

\[
0 \to I_{\tilde{M}_c} \to \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}_{\tilde{M}_c} \to 0,
\]

where \( I_{\tilde{M}_c} \) is the ideal sheaf of \( \tilde{M}_c \). The corresponding long exact sequence gives

\[
0 \to H^0(\mathbb{C}^n, I_{\tilde{M}_c}) \to H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \to H^0(\mathbb{C}^n, \mathcal{O}_{\tilde{M}_c}) \to H^1(\mathbb{C}^n, I_{\tilde{M}_c}).
\]

Since \( \mathbb{C}^n \) is Stein, \( H^1(\mathbb{C}^n, I_{\tilde{M}_c}) = 0 \), hence the function \( f \) is the restriction of a holomorphic function \( \tilde{f} \in H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \). Finally, by Lemma 5.15, \( \tilde{f} \) can be chosen \( \gamma \)-finite.

### 6.3 Algebraic structures on Stein completions: the uniqueness

In this subsection, we prove that the algebraic structure constructed on \( \tilde{M}_c \) in Theorem 6.3 is uniquely determined by the complex geometry of the Stein variety \( \tilde{M}_c \).

We start with the following lemma, showing that a function is \( \gamma \)-finite (that is, polynomial) if and only if it admits a certain growth condition. This is similar to a well-known result of complex analysis which states that an entire holomorphic function on \( \mathbb{C}^n \) is polynomial if and only if it has polynomial growth.

**Lemma 6.4:** Let \( M \) be an LCK manifold with proper potential, and \( \tilde{M}_c \) the Stein variety, obtained as a weak Stein completion of its Kähler \( \mathbb{Z} \)-cover \( \tilde{M} \). Denote by \( \gamma \in \text{Aut}(\tilde{M}_c) \) the holomorphic contraction generating the \( \mathbb{Z} \)-action
on $\tilde{M}_c$. Choose a compact set $K \subset \tilde{M}_c$ containing an open neighbourhood of the apex. Denote by $B \subset H^0(\mathcal{O}_{\tilde{M}_c})$ the following ring of functions on $\tilde{M}_c$:

$$B := \{ f \in H^0(\mathcal{O}_{\tilde{M}_c}) \mid \exists C > 0 \text{ such that } \forall i \sup_K |(\gamma^*)^{-i}|f| < C^i \}. \quad (6.1)$$

Then $B$ coincides with the space of $\gamma$-finite functions.\footnote{We call the function satisfying (6.1) the function of polynomial growth. This terminology is justified because for $\gamma$ a linear contraction of $\mathbb{C}^n$, (6.1) is equivalent to having polynomial growth.}

**Proof.** Step 1: Let $f$ be a $\gamma$-finite function, and $W$ the space generated by $\{ f, (\gamma^*)f, (\gamma^*)^2f, \ldots \}$. Let $\| \cdot \|_K$ be the norm on $W$ defined by $\| f \|_K := \sup_K |f|$, and let $C := \sup_{\| f \|=1} \| (\gamma^*)^{-1}f \|$ be the operator norm of the operator $(\gamma^*)^{-1} \in \text{End}(W)$ in this norm. Then $\sup_K |(\gamma^*)^{-i}f| \leq C^i \sup_K |f|$, and $f$ has polynomial growth, in the sense of (6.1).

Step 2: Suppose that $f$ has polynomial growth, in the sense of (6.1), and let $W$ be the space of all functions generated by $\{ f, (\gamma^*)f, (\gamma^*)^2f, \ldots \}$. Then all elements of $W$ have the same growth as $f$, with the same bound $C$, hence the closure $\overline{W}$ of $W$ in the norm $\| f \| := \sup_K |f|$ consists of functions with polynomial growth.

The condition (6.1) holds for all $f \in W$ if and only if $(\gamma^*)^{-1}$ has finite norm on $W$. Therefore, $\gamma^*|_{\overline{W}}$ is invertible. To finish the proof of Lemma 6.4, we need to prove that the norm of $(\gamma^*)^{-1}$ is infinite on $\overline{W}$ if $W$ is infinite-dimensional.

The operator $\gamma^*$ on $\overline{W}$ is compact; by the Riesz-Schauder theorem, it has the Jordan cell decomposition with eigenvalues converging to 0, unless $\overline{W}$ is finite-dimensional. The norm of a linear operator $A$ with eigenvalues $\alpha_i$ satisfies $\| A \| \geq \sup |\alpha_i|$. Therefore, a compact operator cannot be invertible on an infinitely-dimensional Banach space: the inverse operator would have infinite norm. \hfill \[ \square \]

**Theorem 6.5:** Let $\tilde{M}_c$ be a weak Stein completion of a Kähler $\mathbb{Z}$-covering of an LCK manifold with potential. Suppose that $\tilde{M}_c$ can be obtained from two different LCK manifolds, and let $\gamma_1, \gamma_2 \in \text{Aut}(\tilde{M}_c)$ be the corresponding holomorphic contractions. Consider the affine algebraic structures $\mathcal{A}_1$ and $\mathcal{A}_2$ associated with the LCK structures on $\tilde{M}_{(\gamma_1)}$ and $\tilde{M}_{(\gamma_2)}$ as in Theorem 6.3. Then $\mathcal{A}_1 = \mathcal{A}_2$.\footnote{We call the function satisfying (6.1) the function of polynomial growth. This terminology is justified because for $\gamma$ a linear contraction of $\mathbb{C}^n$, (6.1) is equivalent to having polynomial growth.}
Proof. Step 1: Let $\gamma_1, \gamma_2 \in \text{Aut}(\tilde{M}_c)$ be holomorphic contractions centered in the apex $c$, associated with two LCK manifolds with proper potential such that their Kähler $\mathbb{Z}$-covers have the same weak Stein completion $\tilde{M}_c$. We prove Theorem 6.5 using Lemma 6.4. To show that the algebraic structures induced by $\gamma_1$ and $\gamma_2$ are the same, it would suffice to show that the spaces of $\gamma_1$- and $\gamma_2$-finite functions on $\tilde{M}_c$ coincide. By Lemma 6.4, this would follow if the growth estimates (6.1) for $\gamma_1$ and $\gamma_2$ are equivalent.

We have reduced Theorem 6.5 to the following statement. Let $K \subset \tilde{M}_c$ be a compact subset which contains an open neighbourhood of the apex $c \in \tilde{M}_c$. Then a function $f \in H^0(\mathcal{O}_\tilde{M}_c)$ which satisfies $\sup_K (\gamma_1)^{-n} |f| < C_1^n$, also satisfies $\sup_K (\gamma_2)^{-n} |f| < C_2^n$ for some other constant $C_2 > 0$ and all $n > 0$.

Step 2: Suppose that there exists an integer $d > 0$ such that $\gamma_1^d(K) \subset \gamma_2(K)$. Then the polynomial growth estimate (6.1) for $\gamma_2$ follows from the growth estimate for $\gamma_1^d$, which is equivalent to the growth estimate for $\gamma_1$. Therefore, Theorem 6.5 would follow if we prove that the integer $d$ always exists.

Recall that a continuous map $f : M \rightarrow M$ fixing $x \in M$ is called a contraction centered in $x \in M$ if for each compact subset $K \subset M$, and each open neighbourhood of $x$, a sufficiently big iteration of $f$ gives $f^N(K) \subset U$. The maps $\gamma_1, \gamma_2$ are contractions centered in the apex $x$, hence for some $N > 0$, the set $\gamma_1^N(K)$ lies in the interior of $\gamma_2(K)$ which is a neighbourhood of $x$. Theorem 6.5 is proven.

7 Algebraic cones and LCK manifolds

In Section 2, we gave three equivalent definitions of an algebraic cone: an open algebraic cone is the total space of an ample $\mathbb{C}^*$-bundle, or a Kähler $\mathbb{Z}$-covering of a submanifold in a Hopf manifold, or a Stein variety admitting a holomorphic contraction, with the apex removed. In this section, we prove this equivalence. We start by recalling some basic facts about the weighted projective spaces.

7.1 Weighted projective spaces

For an introduction and background material about the weighted projective spaces, see [BG, §4.5].
Recall that any representation $V$ of $\mathbb{C}^*$ is a direct sum of 1-dimensional representations isomorphic to $\rho_w$, with $\mathbb{C}^*$ acting by $\rho_w(t)(z) = t^w z$. Such a representation is called representation of weight $w$, and the decomposition of $V$ into subrepresentation of constant weight – the weight decomposition.

The following trivial claim is left as an exercise to the reader.

**Claim 7.1:** Let $\rho$ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$. Assume that $\rho$ contains a contraction. Then all weights of $\rho$ are positive or negative.

**Claim 7.2:** Let $\rho$ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, \ldots, w_n \in \mathbb{Z}_{>0}$. Then its orbit space $\mathbb{C}P^{n-1}(w_1, \ldots, w_n)$ is equipped with a structure of a projective orbifold, and $\mathbb{C}^n\setminus0$ can be identified with the total space of an ample $\mathbb{C}^*$-bundle over $\mathbb{C}P^{n-1}(w_1, \ldots, w_n)$.

**Proof:** [BG, Proposition 4.5.3].

**Definition 7.3:** The orbifold $\mathbb{C}P^{n-1}(w_1, \ldots, w_n)$ is called the weighted projective space.

**Claim 7.4:** Let $\rho$ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, \ldots, w_n \in \mathbb{Z}_{>0}$, and $Z \subset \mathbb{C}^n\setminus0$ be a $\rho$-invariant submanifold. Then the orbit space $Z/\mathbb{C}^*$ is a projective orbifold in the corresponding weighted projective space $\mathbb{C}P^{n-1}(w_1, \ldots, w_n)$.

**Proof:** [BG, Proposition 4.6.2].

**Corollary 7.5:** Let $\rho$ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, \ldots, w_n \in \mathbb{Z}_{>0}$, and $Z \subset \mathbb{C}^n\setminus0$ a $\rho$-invariant submanifold. Then $Z$ is an open algebraic cone.

### 7.2 Jordan-Chevalley decomposition

We recall a few definitions and results from the algebraic group theory, generalizing the Jordan normal form to arbitrary algebraic group.

**Definition 7.6:** An element of an algebraic group $G$ is called semisimple if its image is semisimple for some exact algebraic representation of $G$, and is called unipotent if its image is unipotent (that is, exponential of a nilpotent) for some exact algebraic representation of $G$.

**Remark 7.7:** For any algebraic representation of an algebraic group $G$, the
image of any semisimple element is a semisimple operator, and the image of any unipotent element is a unipotent operator ([Hu, §15.3]).

**Theorem 7.8:** (Jordan-Chevalley decomposition, [Hu, §15.3])

Let $G$ be an algebraic group, and $A \in G$. Then there exists a unique decomposition $A = SU$ of $A$ in a product of commuting elements $S$ and $U$, where $U$ is unipotent and $S$ semisimple.

7.3 Algebraic cone obtained from an LCK manifold with potential

**Theorem 7.9:** Let $M$ be an LCK manifold with proper potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-cover. Then $\tilde{M}$ is an open algebraic cone.

**Proof:** Let $\gamma$ be the generator of the $\mathbb{Z}$-action on $\tilde{M}$. Using Theorem 3.7, we can embed $M$ to a linear Hopf manifold. This gives a holomorphic map $\tilde{M} \to \mathbb{C}^n \setminus \{0\}$ taking $\gamma$ to a linear contraction $A \in \text{End}(\mathbb{C}^n)$. Let $\tilde{M}_c$ be the closure of $\tilde{M}$ in $\mathbb{C}^n$. By Theorem 6.2, $\tilde{M}_c$ is equipped with a natural algebraic structure. To prove Theorem 7.9 it remains only to show that this variety admits a holomorphic $\mathbb{C}^*$-action, free outside of the origin $c$, and acting with eigenvalues $|\alpha_i| < 1$ on the Zariski tangent space $T_c \tilde{M}_c$.

Let $G_A$ be the Zariski closure of $\langle A \rangle$ in $\text{GL}(\mathbb{C}^n)$. This is a commutative algebraic group, acting on the variety $\tilde{M}_c \subset \mathbb{C}^n$. To prove that $\tilde{M}_c$ is an algebraic cone, we find a $\mathbb{C}^*$-action contracting $\tilde{M}_c$ to the origin and apply Corollary 7.5.

Consider the map taking any $A_1 \in G_A$ to its unipotent component $U_1$. Since $G_A$ is commutative, this map is a group homomorphism. Therefore, its kernel $G_s$ (that is, the set of all semisimple elements in $G_A$) is an algebraic subgroup of $G_A$. A connected commutative algebraic subgroup of $\text{GL}(\mathbb{C}^N)$ consisting of semisimple elements is always isomorphic to $(\mathbb{C}^*)^k$ ([BT, Proposition 1.5]). The one-parametric subgroups $\mathbb{C}^* \subset (\mathbb{C}^*)^k$ are dense in $(\mathbb{C}^*)^k$ because one-parametric complex subgroups $\mathbb{C}^* \subset (\mathbb{C}^*)^k$ can be obtained as complexification of subgroups $S^1 \subset U(1)^k \subset (\mathbb{C}^*)^k$, and those are dense in $\mathbb{C}^*$.
U(1)\(^k\). Therefore, the contraction \(S \in \mathcal{G}_s = (\mathbb{C}^*)^k\) can be approximated by an element of \(\mathbb{C}^*\) acting on \(\tilde{M}_c\).

We have obtained a \(\mathbb{C}^*\)-action \(\rho\) on \(\tilde{M}_c\) containing a contraction. By 

Claim 7.1, all weights of this \(\mathbb{C}^*\)-action are positive (or negative, in which case we replace \(\rho\) by its opposite). Then, Corollary 7.5 implies that \(\tilde{M}\) is an open algebraic cone.

### 7.4 Submanifolds of Hopf manifolds obtained from algebraic cones

**Lemma 7.10:** Let \(\gamma\) act on a complex variety \(\tilde{M}_c\) by contractions, contracting \(\tilde{M}_c\) to a point \(c\). Then the corresponding \(\mathbb{Z}\)-action on \(\tilde{M} := \tilde{M}_c \setminus \{c\}\) is properly discontinuous, hence \(\tilde{M}/\mathbb{Z}\) is Hausdorff; it is a manifold when \(\tilde{M}\) is a manifold.

**Proof:** By definition, the \(\mathbb{Z}\)-action is properly discontinuous if every point has a neighbourhood \(U\) such that the set \(\{g \in \mathbb{Z} \mid g(U) \cap U \neq \emptyset\}\) is finite. Let \(x \in \tilde{M}\) and \(K\) be the compact closure of an open neighbourhood of \(U \subset \tilde{M}\) containing \(x\). Since \(\tilde{M}_c\) is Hausdorff, there exists a neighbourhood \(W \ni c\) such that its closure does not intersect \(K\). By definition of contractions, there exists \(N > 0\) such that \(\gamma^n(K) \subset W\) for all \(n \geq N\). This implies that \(\gamma^n(K) \cap K = \emptyset\). This also implies that \(K \cap \gamma^{-n}(K) = \emptyset\). We have shown that \(\gamma^n(K) \cap K = \emptyset\) for all \(n \notin [-N, N]\).

The following theorem, applied to a weak Stein completion of an open algebraic cone, allows us to obtain a holomorphic embedding of its \(\mathbb{Z}\)-quotient to a Hopf manifold.

**Theorem 7.11:** Let \(\tilde{M}_c\) be a Stein variety equipped with a holomorphic contraction \(\gamma\), contracting it to the point \(c\). Assume that the complement \(\tilde{M} := \tilde{M}_c \setminus c\) is smooth. By Lemma 7.10, \(\tilde{M}/\langle \gamma \rangle\) is a complex manifold. Then there exists a holomorphic embedding \(j : \tilde{M}/\langle \gamma \rangle \hookrightarrow H\) to a Hopf manifold.

**Proof:** Let \(R\) be the ring of \(\gamma\)-finite holomorphic functions on \(\tilde{M}_c\), \(I\) the maximal ideal of \(c\), and and \(V \subset I\) a finite-dimensional \(\gamma\)-invariant space generating \(I\). By Theorem 5.8, the action of \(\gamma^*\) is compact on \(I\) and has all eigenvalues < 1. By Riesz-Schauder theorem, \(R\) is dense in \(\mathcal{O}_{\tilde{M}_c}\) (Corollary 5.14). Therefore, the functions in \(V\) separate the points in \(\tilde{M}\), for \(V\) sufficiently big.
By Montel’s theorem, \( R \) is dense in \( C^1 \)-topology whenever it is dense in \( C^0 \)-topology; therefore, the differentials of the functions \( f \in R \) generate \( T^*_x \tilde{M} \) for all \( x \in \tilde{M} \). This implies that the tautological map \( \tilde{M} \rightarrow V^* \) is a holomorphic embedding. This map is by construction \( \gamma \)-equivariant, hence it induces a holomorphic embedding \( \tilde{M}/(\gamma) \rightarrow H = V^*/(\gamma) \).

8 Closed algebraic cones and normal varieties

Definition 8.1: Recall that a complex variety \( X \) is called normal if any locally bounded meromorphic function on an open subset \( U \subset X \) is holomorphic ([De, Definition II.7.4]). In algebraic geometry, a variety is normal if all its local rings are integrally closed; these two notions are equivalent for a complex variety obtained from an algebraic one ([De, Theorem II.7.3], [Ku, Satz 4, p. 122]).

Remark 8.2: Each complex variety \( A \) admits the normalization, that is, a normal variety \( B \) equipped with a finite holomorphic map \( n : B \rightarrow A \). The normalization is unique up to an isomorphism. In the sequel, we will usually consider the normalization maps \( n : B \rightarrow A \) which are homeomorphisms. The concept might seem alien, but in fact it is very natural. For example, consider the plane curve \( C \) defined by the equation \( x^2 = y^3 \). This curve is clearly singular in zero. It can be parametrized by a variable \( t \) with \( t^2 = y \) and \( t^3 = x \); this parametrization defines a map \( n : \mathbb{C} \rightarrow C \) which is finite, hence \( n \) is the normalization of \( C \). However, \( n \) is clearly a homeomorphism.

Claim 8.3: Let \( G \) be a finite group holomorphically acting on a normal complex variety \( Y \), and \( X = Y/G \) the quotient variety. Then \( X \) is normal.

Proof: Let \( f \) be a locally bounded meromorphic function on \( X \). Its pullback \( \tau^* f \) to \( Y \) is clearly meromorphic and locally bounded, hence holomorphic. Since \( \tau^* f \) is \( G \)-invariant, it is holomorphic on \( Y \).

We are interested in the normality of a closed algebraic cone.

Let now \( X \) be a projective variety and \( L \) an ample line bundle on \( X \). The homogeneous coordinate ring of \( X \) is the ring \( \bigoplus_i H^0(X, L^\otimes i) \). This ring is finitely generated whenever \( L \) is ample ([La, Theorem 2.3.15]). Indeed, for

\[^9\text{We are grateful to Francesco Polizzi for this reference; please see the excellent Mathoverflow thread https://mathoverflow.net/questions/303406/algebraic-vs-analytic-normality/ for more details about the complex analytic and complex algebraic normality.}\]
an ample line bundle $L$ and $i$ sufficiently big, the cohomology $H^{>0}(X, L^{⊗i})$ vanishes, hence the number $\dim H^0(X, L^{⊗i})$ is equal to the holomorphic Euler characteristic $\chi(L^{⊗i})$. The latter is polynomial in $i$ by Riemann-Roch-Hirzebruch theorem, which implies that finitely many generators are enough to generate this ring.

A projective variety $(X, L)$ for which the ring $\bigoplus_i H^0(X, L^{⊗i})$ is integrally closed is called \textbf{projectively normal}. By definition, the affine variety associated with this ring is \textbf{the affine cone of} $X$, which is a closed algebraic cone in our parlance.

The closed algebraic cone, obtained from an open algebraic cone by taking the Stein completion, is normal by definition of Stein completion. However, the projective normality is a very tricky condition, depending on the choice of the bundle $L$.

This was the subject of Exercise I.3.18 from Hartshorne’s “Algebraic Geometry” ([Ha]), where Hartshorne considers the same smooth complex curve with two different projective embeddings; the affine cone of the first is normal, and the affine cone of the second is not.

In Exercise II.5.14, [Ha], Hartshorne gives a criterion for projective normality:

\textbf{Proposition 8.4}: A projective variety $X \subset \mathbb{C}P^n$ is projectively normal if and only if $X$ is normal, and the restriction map

$$H^0(\mathbb{C}P^n, \mathcal{O}(i)) \longrightarrow H^0 \left( X, \mathcal{O}(i) \right|_X \right)$$

is surjective for all $i \geq 0$. ■

\textbf{Remark 8.5}: This proposition can be applied to algebraic cones, because the quotient singularities are normal (Claim 8.3), and the orbifolds have only quotient singularities.

\textbf{Example 8.6}: Let $X \subset \mathbb{C}P^n$ be a projective manifold, which is not contained in a projective subspace $\mathbb{C}P^k \subset \mathbb{C}P^n$ of smaller dimension. Consider a linear projection $p : \mathbb{C}P^n \longrightarrow \mathbb{C}P^{n-1}$ centered in a point $z \notin X$. Then the restriction $p \big|_X$ is holomorphic. Assume that $p : X \longrightarrow \mathbb{C}P^{n-1}$ is also injective (this can be always achieved if $2\dim X < n - 1$ for an appropriately general choice of $z$; indeed, $p$ is not injective if and only if the center $z$ does not belong to the secant variety\footnote{A secant variety of $X \subset \mathbb{C}P^n$ is the Zariski closure of the union of all lines $\mathbb{C}P^m \subset \mathbb{C}P^n$ intersecting $X$ in at least two points.} of $X$, which has dimension $\leq 2\dim X + 1$).
Consider the natural map of affine cones $u : C(X) \to C(p(X))$. Outside of the origin $c$, this map is biholomorphic, hence it is birational and finite. By definition, a variety $Z$ is normal if any bimeromorphic finite map $Z_1 \to Z$ is an isomorphism. Were $C(p(X))$ normal, this would imply that $u$ is an isomorphism. However, the Zariski tangent space $T_cC(X)$ is $n + 1$-dimensional, because $C(X) \subset C(\mathbb{CP}^n)$ generates the vector space $C(\mathbb{CP}^n) = \mathbb{C}^{n+1}$ (otherwise $X$ would have been contained in a smaller dimensional projective subspace). On the other hand, $\dim T_cC(p(X)) \leq n$, hence $u : C(X) \to C(p(X))$ is not an isomorphism.$^{11}$

This example makes sense, if one considers the following statement.

**Claim 8.7:** Let $\tilde{M}$ be an open algebraic cone, and $\mathcal{G}$ the set of all closed algebraic cones $\tilde{M}_c$ obtained by adding the origin to $\tilde{M}$. Then there exists only one closed algebraic cone $Z \in \mathcal{G}$ which is normal, and for any other $Z' \in \mathcal{G}$, its normalization is $Z$. Moreover, the normalization map $Z \to Z'$ is a homeomorphism.

**Proof:** Let $Z$ be the Stein completion of $\tilde{M}$; by definition, it is normal. Since all holomorphic functions on $\tilde{M}$ can be extended to $Z$, this variety is equipped with a finite, bijective, bimeromorphic map to any closed algebraic cone $Z'$ associated with $\tilde{M}$. Indeed, the holomorphic functions on $Z'$ are holomorphic on $\tilde{M}$, hence they can be extended to its Stein completion. This implies that $Z$ is the normalization of all other $Z' \in \mathcal{G}$. ■

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$^{11}$We are grateful to Yu. Prokhorov for this example.
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