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Loïc Teyssier

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MEROMORPHIC INFINITESIMAL AFFINE ACTIONS OF THE PLANE*

LOÎC TEYSSIER

Abstract. We study complex Lie algebras spanned by pairs \((Z, Y)\) of germs of a meromorphic vector field of the complex plane satisfying \([Z, Y] = \delta Y\) for some \(\delta \in \mathbb{C}\). This topic relates to Liouville–integrability of the differential equation induced by the foliation underlying \(Z\). We give a direct geometric proof of a result by M. Berthier and F. Touzet characterizing germs of a foliation admitting a first–integral in a Liouvillian extension of the standard differential field. In so doing we study transverse and tangential rigidity properties when \(Z\) is holomorphic and its linear part is not nilpotent. A second part of the article is devoted to computing the Galois–Malgrange groupoid of meromorphic \(Z\).

1. Introduction

Most reduced singularities of holomorphic foliations \(\mathcal{F}\) in the complex plane are locally analytically conjugate to a simple polynomial model, either its linear part or a Dulac–Poincaré normal form [Dul09], all of whom admit Liouvillian first–integrals. The other cases consist in quasi–resonant saddles (small divisors problem) and resonant singularities (encompassing resonant saddles and saddle–nodes). M. Berthier and F. Touzet have studied in [BT99] these difficult cases, characterizing those admitting a Liouvillian first–integral \(H\). By carefully studying the transverse dynamics they worked out that \(\mathcal{F}\) needs to be of a certain type. For quasi–resonant singularities \(H\) is of moderate growth along the separatrices of the foliation (Nilsson class); in that case \(\mathcal{F}\) is analytically linearizable (see [Cer91] as well). For resonant singularities at least one separatrix of \(\mathcal{F}\) is a line of essential singularities for \(H\); here \(\mathcal{F}\) is conjugate to a rather simple model and the Martinet–Ramis modulus [MR83, MR82] is an affine map (in a suitable ramified chart). In any cases the local form of \(\mathcal{F}\) is induced by a first–order Bernoulli differential equation. This article mainly provides a geometrical proof of that fact, simplifying the arguments through the use of a unified framework which is more explicit and does not need the study of functional moduli or transverse dynamics. This formalism focuses on meromorphic vector fields \(Z\) tangent to \(\mathcal{F}\) and allows to provide normal forms for holomorphic ones. We are also interested in computing the Galois–Malgrange groupoid [Cas06, Mal01] associated to \(Z\).

Our main object of study are Lie algebras \(\mathcal{L}(Z, Y)\) over \(\mathbb{C}\) generated by two germs at \((0,0)\) of a non–zero, meromorphic vector field. We will be interested only in finite–dimensional algebras, more precisely those admitting a ratio \(\delta \in \mathbb{C}\) such that

\[ [Z, Y] = \delta Y. \]

We speak of Abelian Lie algebras when \(\delta = 0\) and affine Lie algebras when \(\delta \neq 0\). In the latter case the ratio may be chosen equal to 1 by considering \(\frac{1}{\delta}Z\) instead. The most

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interesting cases arise when the locus of tangency between $Z$ and $Y$ is of codimension
at least 1, in which case we shall write $Z \pitchfork Y$ and speak of non–degenerate
affine Lie algebras. These objects correspond to (local infinitesimal) affine actions of the plane $C^2$.

Save for explicit mention to the contrary we shall always assume that $Z \pitchfork Y$. The holomorphic foliation $\mathcal{F}_Z$, whose leaves are the integral curves of $Z$, admits a Godbillon–Vey sequence [GV71] of length 1 or 2. It is given by the dual basis $(\tau_Z, \tau_Y)$ of $(Z, Y)$: the pair of meromorphic 1–forms indeed satisfies
\[
\begin{align*}
d\tau_Y &= \delta \tau_Y \wedge \tau_Z, \\
d\tau_Z &= 0.
\end{align*}
\]

The 1–form $\tau_Y$ defines the same foliation as $Z := A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$. Because of these properties, the solutions to the ordinary differential equation
\[
y' = \frac{B(x, y)}{A(x, y)}
\]
are given implicitly by level curves of the first–integral $H$ lying in a Liouvillian extension $\mathbb{L}$ over the differential field $\mathbb{K}_0 := \mathbb{C}((x, y))$ of germs at $(0, 0)$ of a meromorphic function [Cas06, BT99], equipped with the standard derivations $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. Such an extension is constructed by taking a finite number of extensions $\mathbb{K}_{n+1} := \mathbb{K}_n \langle f_n \rangle$ of the following three types : $f_n$ is algebraic over $\mathbb{K}_n$, a derivative of $f_n$ lies in $\mathbb{K}_n$, a logarithmic derivative of $f_n$ lies in $\mathbb{K}_n$. The function defined by
\[
H := \int \exp(-\delta \int \tau_Z) \tau_Y
\]
is a multiform first–integral of $Z$ belonging to a Liouvillian extension of $\mathbb{K}_0$ (one can check that $\exp(-\delta \int \tau_Z) \tau_Y$ is a multiform closed 1–form and that the Lie derivative $Z \cdot H$ of $H$ along $Z$ vanishes).

A few years ago B. Malgrange gave a candidate Galois theory for non–linear differential equations [Mal01], by building an object (a $D$–groupoid) which encodes information regarding whether or not a differential equation, for instance of the form 1.1, can be solved by quadrature. He proved that in the case of a linear system this groupoid coincides with the linear group of Picard–Vessiot associated to the system. G. Casale then studied intensively the Galois–Malgrange groupoid for codimension 1 foliations, and related indeed the existence of Liouvillian first–integrals, the existence of short Godbillon–Vey sequences and the smallness of the groupoid [Cas06]. We contribute here to this work by computing explicitly the Galois–Malgrange groupoid of the vector field $Z$. This result complements a previous work of E. Paul about Galois–Malgrange groupoids of quasi–homogeneous foliations [Pau09].

### 1.1. Globalizing holomorphic vector fields $Z$.

We begin with giving general results on non–degenerate Lie algebras of ratio $\delta$ in Section 2 when $Z$ is holomorphic. If $Z$ is regular at $(0, 0)$ then $\psi^* Z = \frac{\partial}{\partial y}$ for some analytic change of local coordinates, as follows from the theorem of rectification. On the contrary when $Z(0, 0) = 0$ we say that $Z$ is non–nilpotent if its linear part\footnote{The linear part of a vector field $(ax + by + \cdots) \frac{\partial}{\partial x} + (cx + dy + \cdots) \frac{\partial}{\partial y}$ is identified with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.} at $(0, 0)$ is neither. In
that case we label the eigenvalues \( \{ \lambda_1, \lambda_2 \} \) in such a way that \( \lambda_2 \neq 0 \). A direct computation (Lemma 2.8) ensures that \( Z \) is non-nilpotent when \( \delta \neq 0 \).

**Globalization Theorem.** Let \( \mathcal{L}(Z,Y) \) be a non-degenerate Lie algebra of ratio \( \delta \) such that \( Z \) is non-nilpotent. Then \( Z \) is glocal, in the sense that it admits a polynomial form in some local analytic chart around \((0,0)\).

If some non-nilpotent vector field \( Z \) admits a non-isolated singularity at \((0,0)\) it defines a regular foliation \( \mathcal{F}_Z \) and there exists a local system of coordinates in which \( Z = \lambda_2 y \frac{\partial}{\partial y} \) with \( \lambda_2 \neq 0 \) (Corollary 2.10). We focus next our argument to the case of an isolated singularity. For most \( \lambda := \lambda_1/\lambda_2 \) it is well known [Dul09] that either \( Z \) is conjugate to \( Z_0 \) (linearizable) or to a Dulac–Poincaré polynomial form. We will therefore explore only the following non-trivial cases, allowing to cover the remaining cases of non-nilpotent singularities and prove the globalization theorem:

- quasi-resonant saddles (\( \lambda \in \mathbb{R}_{\leq 0} \setminus \mathbb{Q} \)), studied in Section 3, where we recover the fact that \( Z \) is actually linearizable,
- resonant singularities (\( \lambda \in \mathbb{Q}_{\leq 0} \), non-linearizable), studied in Section 4, where we establish that \( Z \) is given in an appropriate local chart by a Bernoulli vector field with polynomial coefficients.

**Remark.**

1. We give in the course of the article an explicit family of polynomial normal forms for resonant vector fields meeting the hypothesis of the theorem, generalizing that of [Tey04] obtained for foliations.
2. I do not know whether the globalization theorem still holds when \( Z \) is meromorphic. As there exists non-glocal foliations [GT10] this question is non-trivial.

1.1.1. **Normalization procedure.**

The techniques of this article can be applied to other non-nilpotent singularities in order to obtain normal forms \((Z,Y)\) unique up to the action of a finite-dimensional space. This question is non-trivial in the presence of a vector field \( Z \) with a non-constant meromorphic first-integral. It is possible to reduce \( Y \) to a rational vector field using isotropies of \( Z \). These computations are reasonably easy and therefore not included in the present paper. We nonetheless present the general procedure of normalization of the pair \((Z,Y)\) when \( Z \) is holomorphic and admits an isolated singularity at \((0,0)\) with a non-nilpotent linear part. We will apply this scheme for resonant and quasi-resonant singularities.

**Preparation** We can associate to \( Z \) a non-degenerate Lie algebra \( \mathcal{L}(X_0, Y_0) \) of ratio \( \delta_0 \) (the latter is generically zero and always a holomorphic first-integral of \( Y_0 \)) such that in a suitable analytic chart

\[
Z = U(X_0 + RY_0)
\]

for holomorphic \( Y_0 \), \( U \) and \( R \) where \( U(0,0) = 1 \) and \( R \) vanishes sufficiently. It will turn out that \( Y_0 = y \frac{\partial}{\partial y} \) or \( Y_0 = \frac{\partial}{\partial y} \). Define

\[
X := X_0 + RY_0.
\]

**Formal normalization** We show that \( Z \) is formally conjugate by \( \hat{\psi} \) to a polynomial vector field

\[
Z_0 := QX_0
\]
where \( Q(0,0) = 1 \). Using first a conjugacy formed with the flow along \( Y_0 \) we conjugate \( X_0 \) to \( X \) (we speak of transverse normalization), then using a conjugacy given by the flow of a suitable \( QX \) we conjugate the latter to \( Z \) (we speak of tangential normalization).

**Rigidity** A formal computation ensures that the a priori formal vector field \( \hat{Y} := \hat{\psi}^*Y \) is actually meromorphic. We deduce that \((Z,Y)\) is analytically conjugate to some couple \((\hat{Z},\hat{Y})\) where

\[
\hat{Z} := Z_0 + K\hat{Y}
\]

with \( K \) in the kernel of \( \hat{Y} \). Those are Berthier–Touzet foliations.

**Reduction** We next use isotropies of \( K\hat{Y} \) to reduce \( K \) to a polynomial. Then using isotropies of \( \hat{Z} \), we conjugate \( \hat{Y} \) to a global vector field \( \tilde{Y} \).

1.1.2. Auxiliary rigidity results.

The first result of rigidity addresses the rigidity of tangential transforms and is of general interest since its scope reaches beyond the case of transversely affine / Abelian vector fields. The following theorem is a consequence of Corollary 3.8 and Proposition 4.4.

**Theorem 1.1.** Assume that \( Z \) is a holomorphic vector field with an isolated, non-nilpotent singularity at \((0,0)\). There exists a closed meromorphic 1–form \( \tau \) such that \( \tau(Z) = 1 \) if, and only if, \( Z \) is analytically conjugate to a vector field \( QX \) introduced in the previous paragraph.

The 1-form \( \tau \) is called a time–form for \( Z \) and we will prove that \( QX \) always admits a closed time–form: the 1–form \( \tau_0 \) given by the dual basis of \((Z_0,Y_0)\).

**Corollary 1.2.** Assume \( \mathcal{L}(Z,Y) \) is a Lie algebra of ratio \( \delta \) with \( Z \) non-nilpotent. If \( \delta \neq 0 \) or \( \mathcal{L}(Z,Y) \) is non-degenerate then \( Z \) is analytically conjugate to \( QX \).

**Proof.** If the Lie algebra \( \mathcal{L}(Z,Y) \) of ratio \( \delta \neq 0 \) is degenerate then \( Y = FZ \) for some meromorphic \( F \neq 0 \) and a direct computation shows that \( F \) satisfies

\[
Z \cdot F = \delta F.
\]

The meromorphic 1–form \( \frac{dF}{\delta} \) is a closed time–form for \( Z \) so that, according to the previous theorem, it is analytically conjugate to \( QX \). In general it is not possible to give simpler models. In the case of a non–degenerate affine Lie algebra the dual basis \((\tau_2,\tau_Y)\) associated to \((Z,Y)\) provides a closed time–form \( \tau_Z \) for \( Z \), which is therefore analytically conjugate to \( QX \). \(\square\)

We then use transverse changes of coordinates to send \( X_0 \) onto \( X \). The second result addresses the rigidity of transverse transforms within non–degenerate algebras. When the web \( \Omega := (Z_0,Y_0,\hat{Y}) \) is non–degenerate (i.e. the vector fields are generically pairwise transverse to each–others) the convergence will follow, more precisely:

**Theorem 1.3.** Let \( \mathcal{L}(Z,Y) \) be a non–degenerate Lie algebra of ratio \( \delta \) and consider the associated 3–web \( \Omega := (Z_0,Y_0,\hat{Y}) \). If \( Z \) is non-nilpotent then:

1. either \( \Omega \) is non–degenerate and there exists an analytic conjugacy between \( Z \) and \( Z_0 \).
2. either \( \hat{Y} \) is not transverse to \( Y_0 \) and, in general, any formal conjugacy between \( Z \) and \( Z_0 \) diverges.
Case (1) corresponds to the case where the foliation \( F_Z \) admits a Godbillon–Vey sequence of length 1, i.e. there exists a closed meromorphic 1–form \( \tau \) such that \( \tau(Z) = 0 \) or, equivalently, a transverse commuting vector field. This situation somehow relates to results of \([\text{Cer91}]\) or \([\text{Sto05}]\). The case (2) corresponds to the case where \( F_Z \) admits a Godbillon–Vey sequence of length 2, and no sequence of lesser length in general. This theorem is a consequence of the study carried out in Sections 3 and 4.

The key to this theorem is the following lemma:

**Lemma 1.4.** Let \( X \) and \( Y \) be meromorphic vector fields on a domain \( U \subset \mathbb{C}^2 \) such that \( X \lhd Y \), and \( N \) be a formal power series based at a point \( p \in U \). Assume that \( X \cdot N \) and \( Y \cdot N \) are meromorphic on \( U \). Then \( N \) is a convergent power series.

**Proof.** Let \( (\tau_X, \tau_Y) \) be the dual basis of \( (X, Y) \). Then \( dN = (X \cdot N)\tau_X + (Y \cdot N)\tau_Y \) is meromorphic on \( U \). Hence \( N \) is a convergent power series at \( p \). \( \square \)

Surprisingly enough this elementary lemma really is the point of the result of transverse rigidity we present here.

### 1.2. Galois–Malgrange groupoid of meromorphic vector fields \( Z \).

The last part of this paper, Section 6, introduces briefly the definition of the Galois–Malgrange groupoid for meromorphic vector fields given in \([\text{Mal01}, \text{Mal02}]\) or \([\text{Cas06}]\). It is particularly a subgroupoid of the sheaf \( \text{Aut}(Z) \) of germs of a biholomorphic symmetry of \( Z \) at points near \((0,0)\) (perhaps outside an analytic hypersurface). We identify the groupoid by considering all transverse structures of \( Z \):

\[
\text{TS}(Z) := \{ (\delta, Y) : Y \lhd Z, [Z,Y] = \delta Y \}.
\]

The set of all possible such \( \delta \) is a \( \mathbb{Z} \)-lattice which classifies most of the dynamical type of \( Z \) (Section 5).

**Integrability Theorem.** Assume \( Z \) is a germ of a meromorphic vector field which admits a non–degenerate affine Lie algebra. For \( (\delta, Y) \in \text{TS}(Z) \) call \( \text{Aut}(\mathcal{L}(Z,Y)) \) the sheaf of germs of biholomorphic symmetries of the Lie Algebra \( \mathcal{L}(Z,Y) \). The Galois–Malgrange groupoid \( \text{Gal}(Z) \) of \( Z \) is then given by

\[
\text{Gal}(Z) = \text{Aut}(Z) \cap \bigcap_{(\delta, Y) \in \text{TS}(Z)} \text{Aut}(\mathcal{L}(Z,Y)).
\]

**Remark.** We also give explicit equations of \( \text{Gal}(Z) \) in canonical forms, expressed as linear differential equations in the Lie derivatives \( Z \cdot \) and \( Y \cdot \) acting on the sheaf of germs of a holomorphic function.

### 1.3. Notations.

Let us introduce some notations and conventions.

- \( \mathbb{N} := \{1, 2, \ldots \} = \mathbb{Z}_{>0} \).
- The spaces \( \mathbb{C}[x_1, \ldots, x_m] \) and \( \mathbb{C}\{x_1, \ldots, x_m\} \) stand respectively for the spaces of germs at \((0, \cdots, 0) \in \mathbb{C}^m \) of a holomorphic function and of a meromorphic function.
- We write \( W \cdot F \) for the action of a vector field \( W \) as a derivation on the \( \mathbb{C} \)-algebra \( \mathbb{C}\{[x,y]\} \) of formal power series \( F \) at \((0,0)\). Its action is extended component–wise to vector functions.
We write $\Phi^t_W$ as the 1–parameter (pseudo–)group associated to a vector field $W$ (its flow). In the sequel the notation $\Phi^F_W$, where $F$ is a function (or a formal power series), will stand for the (formal) map $(x,y) \mapsto \Phi^F_W(x,y)$.

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2. Preliminary results

In this section $Z$ is a germ of a holomorphic vector field near $(0,0)$. If $Z(0,0) = 0$ we define $\{\lambda_1, \lambda_2\}$ the spectrum of the linear part of $Z$ at $(0,0)$. When $Z$ is not nilpotent we label the eigenvalues in such a way that $\lambda_2 \neq 0$ and set $\lambda := \lambda_1 / \lambda_2$. The vector field $Z$ admits a reduced singularity at $(0,0)$ when $\lambda \notin \mathbb{Q}_{>0}$.

2.1. Separatrices and first–integrals.

Let us recall briefly basic facts about separatrices.

Definition 2.1. A separatrix of $Z$ at $(0,0)$ is a curve $\{f = 0\} \ni (0,0)$ where $f$ is a germ of a (non–constant) holomorphic function at $(0,0)$ such that there exists a germ of a holomorphic function $K$ (called the cofactor) satisfying

$$Z \cdot f = Kf.$$
By abuse of language we may also say that \( f \) is a separatrix of \( Z \).

- Equivalently this means the curve \( \{ f = 0 \} \) is tangent to \( Z \), and therefore the adherence of a finite union of leaves of \( \mathcal{F}_Z \).
- A classical result of C. Camacho and P. Sad asserts that every vector field of \( (\mathbb{C}^2, 0) \) admits at least one separatrix [CS82].
- The level curves of any non–constant first–integral \( h \) of \( Z \) are separatrices, since \( Z \cdot h = 0 \).
- In the case of a non–nilpotent singularity, if \( \lambda \) is not a rational number or \( \mathcal{F}_Z \) is not formally linearizable then \( Z \) admits no non–constant (formal or convergent) meromorphic first–integral. Hence if \( Z \cdot a = 0 \) for a meromorphic then \( a \in \mathbb{C} \).

**Lemma 2.2.** Let \( Z \) be a germ of a singular vector field at \((0,0)\). Then any non–zero germ of a meromorphic function \( F \) at \((0,0)\) such that \( Z \cdot F = KF \) with \( K \) holomorphic is of the form

\[
F = \beta \prod_j f_j^{\alpha_j}, \quad \alpha_j \in \mathbb{Z}
\]

where \( \beta \neq 0 \) is some germ of a holomorphic function at \((0,0)\) which is not a separatrix of \( Z \) and the \( f_j \)'s form a finite family of distinct irreducible branches of a separatrix of \( Z \).

**Proof.** Write \( F = \frac{p}{q} \) with \( p, q \) coprime germs at \((0,0)\) of a holomorphic function. If \( p \) or \( q \) does not vanish at \((0,0)\) the result is trivial. In the other case from

\[
qZ \cdot p = (Kq + Z \cdot q)p
\]

we deduce that \( Z \cdot p \) vanishes along the curve \( \{ p = 0 \} \) and \( p \) is a separatrix. The conclusion follows. \( \square \)

**Lemma 2.3.** Let \( \mathcal{L}(Z,Y) \) be a Lie algebra of ratio \( \delta \) with \( Y \) not holomorphic. The polar locus \( \{ p = 0 \} \) of \( Y \) is a separatrix of \( Z \).

**Proof.** The proof is the same as the previous lemma’s, this time using the equality

\[
(2.1) \quad \rho [Z,W] = (\delta \rho + Z \cdot \rho) W
\]

where \( W = \rho Y \) for coprime and holomorphic \( \rho, W \). \( \square \)

### 2.2. Non–degenerate algebras.

We assume here that the locus of tangency \( \{ \det(Z,Y) = 0 \} \) is of codimension at least 1, that is \( Z \cap Y \). Let us first show that the foliation \( \mathcal{F}_Z \) admits a Godbillon–Vey sequence of length 1 or 2:

**Lemma 2.4.** Denote by \( (\tau_Z, \tau_Y) \) the dual basis of \((Z,Y)\). Then \( d\tau_Z = 0 \) and \( d\tau_Y = \delta \tau_Z \wedge \tau_Y \).

**Proof.** We use the formula

\[
d\tau(Z,Y) = Z \cdot \tau(Y) - Y \cdot \tau(Z) - \tau([Z,Y])
\]

to obtain:

\[
d\tau_Z(Z,Y) = Z \cdot 0 - Y \cdot 1 - \delta \tau_Z(Y) = 0.
\]

Using the formula once more we conclude:

\[
d\tau_Y(Z,Y) = -\delta = \delta \tau_Y \wedge \tau_Z(Z,Y).
\]
The polar locus of $\tau_Z$ and $\tau_Y$ is contained in the union of the polar locus of $Y$ and the locus of tangency $\{\det(Z, Y) = 0\}$ because of Cramer’s formula. Taking Lemma 2.3 into account we obtain the

**Lemma 2.5.** The locus of tangency $\{\det(Z, Y) = 0\}$ is a union of separatrices of $Z$. Consequently the polar locus of $(\tau_Z, \tau_Y)$ also is.

**Proof.** Write $Z = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$ and $Y = C \frac{\partial}{\partial x} + D \frac{\partial}{\partial y}$ so that $\Delta := AD - BC = \det(Z, Y)$. Because $[Z, Y] = \delta Y$ we obtain immediately that

$$Z \cdot \Delta = \left(\delta + \frac{\partial}{\partial x} A + \frac{\partial}{\partial y} B\right) \Delta.$$

\[\square\]

### 2.3. Changes of coordinates.

We will mostly use two kinds of changes of coordinates to normalize a non-degenerate Lie algebra $L(Z, Y)$ of ratio $\delta$, formed with two formal power series $T, N \in \mathbb{C}[[x, y]]$ with $T(0, 0) = N(0, 0) = 0$.

- The first type of conjugacy will be called **tangential** since it is constructed with the flow $\Phi^t_Z$ of $Z$:
  $$T(x, y) := \Phi^t_Z(x, y).$$
- The second type, obtained with the flow of the vector field $Y_0$, is called **transversal**:
  $$N(x, y) := \Phi^N_Y(x, y).$$

Since the flows $\Phi^t_Z$ and $\Phi^t_Y$ are convergent power series, these changes of coordinates are convergent if $T$ and $N$ are. Let us explain how they act by conjugacy on $(Z, Y)$.

**Proposition 2.6.** Assume that $[Z, Y] = DY$ with $Y \cdot D = 0$ and $Y$ holomorphic, not necessarily transverse.

1. Let $W$ be a holomorphic vector field near $(0, 0)$ and $F \in \mathbb{C}[[x, y]]$ such that $F(0, 0) = 0$.
   Then $\Phi^t_W$ is a formal change of coordinates if, and only if, $(W \cdot F)(0, 0) \neq -1$. It fixes $(0, 0)$ and when $W$ is singular it is tangent to the identity.
2. $N^*Z = Z - (Z \cdot N + DN) N^*Y$ where $N^*Y = \frac{1}{1 + Z \cdot N} Y$.
3. $T^*Z = \frac{1}{1 + T \cdot Z} Z$. If moreover $D \in \mathbb{C}$ and if $Z \pitchfork Y$ then $T^*Y = e^{DT} (Y - (Y \cdot T) T^*Z)$.

**Remark 2.7.** If $W$ is singular at $(0, 0)$ then $(W \cdot F)(0, 0) = 0$.

**Proof.**

1. Define the convergent vector power series at $(0, 0, 0)$
   $$g(x, y, t) := \Phi^t_W(x, y)$$
   which can be written
   $$g(x, y, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} W^n (x, y)$$
where $W^{n+1} := W \cdot W^n$ and $W^0 := Id$. The object $g(x, y, F(x, y))$ is then a formal power series since $F(0, 0) = 0$. When $W$ is singular a simple computation ensures that $\Phi_W^0$ is tangent to identity and thus is a formal change of coordinates. The fact that the Jacobian of $\Phi_W^0$ at $(0, 0)$ is $1 + (W \cdot F)(0, 0)$ is proved in the next point.

(2) Consider the vector formal power series

$$
\pi : (x, y) \mapsto (x, y, N(x, y)),
$$

as well as the convergent power series

$$
g(x, y, t) := \Phi_t^0(x, y).
$$

Writing $N = g \circ \pi$ we obtain:

$$
DN = DgD\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ \pi.
$$

The determinant of this matrix is

$$
det(DN) = \det \left( \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial t} & \frac{\partial g}{\partial t} \end{pmatrix} \circ \pi \right) + \det \left( \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial t} & \frac{\partial g}{\partial t} \end{pmatrix} \circ \pi \right) \frac{\partial N}{\partial x}
$$

$$
+ \det \left( \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial t} & \frac{\partial g}{\partial t} \end{pmatrix} \circ \pi \right) \frac{\partial N}{\partial y}
$$

which we evaluate at $(0, 0)$:

$$
det(DN(0, 0)) = 1 + (N \cdot \pi)(0, 0),
$$

since $\frac{\partial g}{\partial t}(x, y, 0) = Y(x, y)$ (component-wise) and $g(\cdot, 0) = Id$.

Define $Z := Z + Ry$; the conjugacy equation $N^*Z = Z$ writes, component-wise,

$$
Z \circ g \circ \pi = Z \cdot (g \circ \pi) + Ry \cdot (g \circ \pi)
$$

$$
= DN(Z + Ry)
$$

(2.2)

$$
= \left( Z \cdot g + (Z \cdot N) \frac{\partial g}{\partial t} + R Y \cdot (Y \cdot N) \frac{\partial g}{\partial t} \right) \circ \pi
$$

Since $g(\cdot, t)Y = Y$ we have $\frac{\partial g}{\partial t} = Y \circ g = Y \cdot g$. On the other hand,

$$
Z \cdot g = \sum_{n \geq 0} \frac{t^n}{n!} Z \cdot Y^n Id.
$$

For $Y \cdot D = 0$ and $[Z, Y] = DY$ the equality $Z \cdot Y^n = Y^n Z \cdot + nDY^n$ holds, which further yields:

$$
Z \cdot g = Z \circ g + tDY \circ g.
$$

Equation (2.2) writes now

$$
Z \circ g \circ \pi = Z \circ g \circ \pi + (Z \cdot N + DN + R(1 + Y \cdot N)) Y \circ g \circ \pi
$$
and is satisfied if, and only if,
\[ R = -\frac{Z \cdot N + DN}{1 + Y \cdot N}. \]

(3) The first statement is actually given by (2) when \( Y := Z, D := 0 \) and \( N := T \). Write \( T^* Y = U (Y + RZ) \) and \( \delta := D \). With the corresponding notations, we need to solve
\[ Y \circ g \circ \pi = U (Y \cdot g + (Y \cdot T + R(1 + Z \cdot T))Z \circ g) \circ \pi. \]

Taking into account that
\[ Y \cdot Z \cdot n = \sum_{p \leq n} C_p^n (-\delta)^{n-p} Z \cdot p \cdot Y, \]
we obtain
\[ Y \cdot g = \sum_{0 \leq n} \sum_{p \leq n} \frac{t^n}{n!} C_p^n (-\delta)^{n-p} Z \cdot p \cdot Y \cdot Id \]
\[ = \sum_{0 \leq p} \left( \sum_{p \leq n} \frac{(-\delta t)^{n-p}}{(n-p)!} \right) t^p Z \cdot p \cdot Y \cdot Id \]
\[ = e^{-\delta t} Y \circ g \]
which yields the result since \( Z \not\equiv Y \).

\[ \square \]

2.4. Affine Lie algebras.

We first show that \( Z \) cannot be too degenerate. Then we deal with a non-isolated singularity as a showcase situation for the subtler setting of (quasi-)resonant singularities.

If \( Z \) is singular let \( Z_0 \) denote the linear part of \( Z \) at \( (0,0) \). Up to a linear change of variables we can put \( Z_0 \) under Jordan normal form
\[ Z_0(x,y) = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 (y + \varepsilon x) \frac{\partial}{\partial y}, \varepsilon \in \{0,1\}, \]
with \( \varepsilon = 0 \) if \( \lambda_1 \neq \lambda_2 \). In all the paragraph we consider some meromorphic vector field \( Y = \frac{1}{p} W \) where \( W \not\equiv 0 \) is a holomorphic vector field and \( p \) a holomorphic function, co-prime with \( W \). If \( Y \) is actually holomorphic we make the convention \( p := 1 \). We suppose that \( L(Z,Y) \) is affine, which may be degenerate. We recall that formula (2.1) and Lemma 2.3 imply the existence of a holomorphic \( K \) with
\[ Z \cdot \rho = K \rho \]
\[ [Z,W] = (\delta + K) W. \]

Lemma 2.8. The linear part \( Z_0 \) is non-nilpotent.

Proof. Assume \( Z_0 = 0 \). Looking at the least homogeneous degree on both sides of \( Z \cdot \rho = K \rho \) implies that \( K(0,0) = 0 \). On the other hand, noting \( W_0 \neq 0 \) the least homogeneous part of \( W \):
\[ 0 = Z_0 \cdot W_0 - W_0 \cdot Z_0 = \delta W_0. \]
Therefore \( \delta = 0 \).
Assume now $Z_0 \neq 0$ but $\lambda_1 = \lambda_2 = 0$ with $\epsilon = 1$. We mention the following immediate lemma:

**Lemma 2.9.** Let $\alpha \in \mathbb{C}$ and $f \in \mathbb{C}[x,y]$ be a homogeneous polynomial such that $\psi \frac{\partial f}{\partial x} = \alpha f$. Then $\alpha f = 0$.

If $K(0,0) \neq 0$ the relation $Z \cdot p = K \cdot p$ reads $Z_0 \cdot p_0 = \psi \frac{\partial p_0}{\partial x} = K(0,0)p_0$ for the least homogeneous degree. Since $p_0 \neq 0$ this contradicts the above lemma, so that $K(0,0) = 0$. Now the relation

$$Z_0 \cdot W_0 - W_0 \cdot Z_0 = \delta W_0$$

holds and writing $W_0 := C \frac{\partial}{\partial x} + D \frac{\partial}{\partial y}$ we derive $\psi \frac{\partial D}{\partial x} C = \delta C$ and $\psi \frac{\partial C}{\partial x} D = \delta D + C$. Using again the lemma we deduce $C = D = 0$, a contradiction. \hfill $\square$

**Corollary 2.10.** Let $Z$ be a holomorphic singular vector field with a non–isolated singularity at $(0,0)$. If there exists an affine Lie algebra $\mathcal{L}(Z,Y)$ of ratio $\delta$, maybe degenerate, then there exists a local analytic change of coordinates $\psi$ such that $\psi^* Z = \lambda_2 \psi \left( 1 + \mu x^k \right) \frac{\partial}{\partial y}$ with $\lambda_2 \neq 0$, $k \in \mathbb{N}$ and $\mu \in \mathbb{C}$. If moreover $Y \cap Z$ then $\mu = 0$ and we have, for a unique $m \in \mathbb{Z}$ and meromorphic germs $a, b \in \mathbb{C}[(x)], b \neq 0$:

$$\psi^* Y = y^m \left( a(x) \psi^* Z + b(x) \frac{\partial}{\partial x} \right)$$

$$\delta = m \lambda_2.$$

**Proof.** The previous lemma states that $Z_0 \neq 0$ so if $|s = 0|$ is the singular locus of $Z$ then the homogeneous valuation of $s$ at $(0,0)$ is 1 and $Z = sX$ with $X(0,0) \neq 0$. Up to change the local coordinates we may assume $X = \frac{\partial}{\partial y}$ and $s(x,y) = \lambda_2 y + \cdots$ with $\lambda_2 \neq 0$. According to the implicit function theorem we can locally write

$$\{ s(x,y) = 0 \} = \{ y = u(x) \}$$

for some holomorphic $u$, so that $(x,y) \mapsto (x,y - u(x))$ provides a conjugacy between $Z$ and $\tilde{Z}(x,y) := \lambda_2 \tilde{s}(x,y) \frac{\partial}{\partial y}$ with $\tilde{s}(0,0) = 1$. Now, according to Proposition 2.6, conjugating $\hat{Z}(x,y) := \lambda_2 \tilde{s}(x,0) \frac{\partial}{\partial y}$ to $Z$ through a tangential change of coordinates $T := \Phi_T^Z$ is equivalent to solving

$$\lambda_2 y \frac{\partial T}{\partial y}(x,y) = \frac{1}{\tilde{s}(x,y)} - \frac{1}{\tilde{s}(x,0)}.$$

A holomorphic solution $T$ always exists.

Finally we conjugate some $\lambda_2 \left( 1 + \mu x^k \right) \frac{\partial}{\partial y}$ to $\lambda_2 \tilde{s}(x,0) \frac{\partial}{\partial y}$ by using a transverse change of coordinates $\mathcal{N}(x,y) := \Phi_{\mathcal{N}(x)}^Z(x,y)$. This change of coordinates is an isotropy of $\lambda_2 y \frac{\partial}{\partial y}$ so that we only need to solve

$$\tilde{s}(x,0) = 1 + \mu x^k \exp(p \mathcal{N}(x)).$$

Either $\tilde{s}(x,0)$ is constant and we set $\mu := 0$, or $\tilde{s}(x,0) = 1 + \mu \left( x^k + \cdots \right)$ with $\mu \neq 0$. In any case we obtain a holomorphic solution $\mathcal{N}$. 
Assume that the coordinates are changed accordingly: \( Z(x, y) = \lambda_2 y \left( 1 + \mu x^k \right) \frac{\partial}{\partial y} \) and write \( Y = AZ + B \frac{\partial}{\partial x} \) for meromorphic germs \( A, B \) with \( B \neq 0 \). The relation \([Z,Y] = \delta Y\) becomes

\[
\begin{cases}
Z \cdot A &= \delta A + \frac{k \mu x^{k-1}}{1+\mu x^k} B \\
Z \cdot B &= \delta B
\end{cases}
\]

If \( \delta = 0 \) the second equation yields \( B \in \mathbb{C}\{x\} \) and taking \( y := 0 \) in the first one yields \( \mu = 0 \), then \( A \in \mathbb{C}\{x\} \). On the contrary when \( \delta \neq 0 \) we have \( B(x,y) = b(x) y^{\frac{\delta}{2(1+\mu x^k)}} \) so that the meromorphy of \( B \) imposes \( \mu = 0 \) and \( m := \frac{\delta}{\lambda_2} \in \mathbb{Z} \). Plugging this relation into the first equation finally gives \( A(x,y) = a(x) y^{1+m} \).

### 3. Quasi–resonant vector fields

We consider here

\[
Z(x, y) = (\lambda_1 x + \ldots) \frac{\partial}{\partial x} + (\lambda_2 y + \ldots) \frac{\partial}{\partial y}
\]

where «\ldots» stands for terms of homogeneous degree greater than 1 and \( \lambda := \lambda_1/\lambda_2 \in \mathbb{R}_{<0} \setminus \mathbb{Q} \). We denote the linear part of \( Z \) by \( Z_0 \). In this section we aim to prove the following result.

**Theorem 3.1.** Assume that \( \lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q} \). Any non degenerate Lie algebra \( \mathcal{L}(Z,Y) \) of ratio \( \delta \) is analytically conjugate to some \( \mathcal{L}(Z_0, \tilde{Y}) \) where \( Z_0 := \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} \) is the linear part of \( Z \) and \( \tilde{Y} \) is as follows. There exist a unique \((n,m) \in \mathbb{Z}^2 \) and \((d,c) \in \mathbb{C}\) such that

\[
\tilde{Y}(x,y) = x^n y^m \left( dx \frac{\partial}{\partial x} + cy \frac{\partial}{\partial y} \right).
\]

The ratio of \( \mathcal{L}(Z,Y) \) is given by

\[
\delta = n \lambda_1 + m \lambda_2.
\]

#### 3.1. Preparation.

It is well known that the foliation induced by \( Z \) admits two and only two reduced separatrices, say \( S_x \) and \( S_y \) (see [CS82]). They are tangent to the eigenvectors of \( Z_0 \), respectively \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Hence for a sufficiently small open polydisc \( V \) centered at \((0,0)\) there exist two holomorphic functions \( f_x \in \mathcal{O}(V \cap \{x = 0\}) \) and \( f_y \in \mathcal{O}(V \cap \{y = 0\}) \) such that \( f_x(0) = f_y(0) = 0 \) and :

\[
S_x \cap V = \left\{ x = f_x(y) \right\}, \quad S_y \cap V = \left\{ y = f_y(x) \right\}.
\]

The analytic change of coordinates

\[
\psi : V \to \mathbb{C}^2, \quad (x,y) \mapsto (x-f_x(y), y-f_y(x))
\]
brings $Z$ into the vector field
\[ \psi^* Z = \lambda_1 x (1 + A) \frac{\partial}{\partial x} + \lambda_2 y (1 + B) \frac{\partial}{\partial y} \]
with $A(0, 0) = B(0, 0) = 0$. As of now we consider vector fields $Z$ written in the form:
\[ Z = UX \]
\[ X = Z_0 + Ry \frac{\partial}{\partial y} \]
where
\[ Z_0 := \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} \]
\[ U := (1 + A) \]
\[ R := \lambda_2 \left( \frac{1 + B}{1 + A} - 1 \right). \]
Notice that $U(0, 0) = 1$ and $R(0, 0) = 0$.

**Remark 3.2.** The only separatrices of $Z$ are now the branches of $\{xy = 0\}$.

### 3.2. Formal normalization.

We are going to show that $Z$ is formally conjugate to $Z_0$. Although this fact is classical we include a short proof here in order to introduce an ingredient that will play an important role in the following. According to Proposition 2.6 we want to find formal power series $T, N \in \mathbb{C}[[x, y]]$ such that $T(0, 0) = N(0, 0) = 0$ and
\[ X \cdot T = \frac{1}{U} - 1 \]
\[ X \cdot N = -R \]
since then $\psi_T := \Phi_T X$ and $\psi_N := \Phi_N y \frac{\partial}{\partial y}$ satisfies
\[ \psi_N^* Z_0 = X \]
\[ \psi_T^* X = UX \]
(notice that $\left[ X_0, y \frac{\partial}{\partial y} \right] = 0$ so that $D = 0$ in the above-mentioned proposition).

**Lemma 3.3.** Let $G \in \mathbb{C}[[x, y]]$ be given. There exists $F \in \mathbb{C}[[x, y]]$ such that $X \cdot F = G$ if, and only if, $G(0, 0) = 0$. The power series $F - F(0, 0)$ is unique.

**Proof.** Write $G(x, y) = \sum_{a,b} g_{a,b} x^a y^b$ and $F(x, y) = \sum_{a,b} f_{a,b} x^a y^b$. Then
\[ (a \lambda_1 + b \lambda_2) f_{a,b} + o(a, b) = g_{a,b}, \]
where $o(a, b)$ stands for terms involving $f_{c,d}$ for $c + d < a + b$ only. Since $a \lambda_1 + b \lambda_2 \neq 0$ if $(a, b) \neq (0, 0)$ the only constraint is $g_{0,0} = 0$; moreover $F$ is unique up to the choice of $f_{0,0}$. $\square$

**Corollary 3.4.** The vector field $Z$ is formally conjugate to its linear part $Z_0$. 
3.3. Lie algebras of $Z_0$.

Starting with $L(Z,Y)$ we have now to consider $\hat{\psi}^*L(Z,Y) = L(Z_0,\hat{Y})$ with $\hat{Y} := \hat{\psi}^*Y$ a formal meromorphic vector field. An important part of the rigidity property we will study below needs to know that in fact $\hat{Y}$ is a meromorphic vector field. As a matter of fact we will show that it is essentially unique.

**Proposition 3.5.** Let $\hat{Y} \neq 0$ be a formal meromorphic vector field satisfying $\left[Z_0,\hat{Y}\right] = \delta\hat{Y}$ with $\delta \in \mathbb{C}$. There exists two unique pairs $(n,m) \in \mathbb{Z}^2$ and $(c,d) \in \mathbb{C}^2$ such that

$$\delta = n\lambda_1 + m\lambda_2$$

and

$$\hat{Y}(x,y) = x^n y^m \left(dx \frac{\partial}{\partial x} + cy \frac{\partial}{\partial y}\right).$$

**Proof.** Write $\hat{Y} = D \frac{\partial}{\partial x} + C \frac{\partial}{\partial y}$ for some $C, D \in \mathbb{C}(\mathbb{R}^2)$. Then:

$$Z_0 \cdot D - \lambda_1 D = \delta D$$

$$Z_0 \cdot C - \lambda_2 C = \delta C.$$

According to Lemma 2.2 we can write

$$D(x,y) = x^N y^M \hat{\gamma}(x,y)$$

$$C(x,y) = x^{N'} y^{M'} \hat{\gamma}(x,y)$$

with either $\gamma = 0$ or $\gamma(0,0) \neq 0$ (and the same for $\hat{\gamma}$). Assume that $\gamma \neq 0$ (so that $\log \gamma \in \mathbb{C}[[x,y]]$); we immediately obtain that

$$(\lambda_1 N + \lambda_2 M)D + x^N y^M Z_0 \cdot \gamma = (\delta + \lambda_1) D,$$

which we divide by $D$:

$$Z_0 \cdot \log \gamma = (\delta + \lambda_1 (1-N) - \lambda_2 M).$$

We now apply Lemma 3.3: on the one hand

$$\lambda_1 (N-1) + \lambda_2 M = \delta$$

which determines $(N,M)$ uniquely in terms of $(\lambda_1, \lambda_2, \delta)$, while on the other hand

$$\log \gamma \in \mathbb{C}.$$

The same argument applies for $C$ yielding $\lambda_1 N' + \lambda_2 (M' - 1) = \delta$, so we conclude that:

$$D(x,y) = dx^{N-1} y^M$$

$$C(x,y) = cx^{N'} y^{M'-1}$$

with $c, d \in \mathbb{C}$. Set now $n := N - 1 = N'$ and $m := M = M' - 1$. □

3.4. Rigidification.

We intend to show that the power series $N$ and $T$ constructed in Section 3.2 are actually convergent power series provided the existence of a non–degenerate Lie algebra $L(Z,Y)$ of ratio $\delta$. Notice that under some arithmetic condition on $\lambda$ (see [Per92]), called “Bruno condition” and satisfied for $\lambda$ belonging to a set of full measure in $\mathbb{R}$, it is not necessary to assume the existence of such a $\hat{Y}$ as every $Z$ is analytically conjugate to $Z_0$. We are not concerned here with such considerations.
3.4.1. Tangential rigidity.

According to Lemma 2.4 there exists a closed meromorphic 1–form $\tau_Z$ such that $\tau_Z(Z) = 1$ (a closed time–form). On the other hand the closed 1–form $\tau_X := \frac{d}{dt}X$ is a time–form for $X$. Hence

$$\tau := \tau_Z - \tau_X$$

is a closed meromorphic 1–form. Because the poles of $\tau_Z$ and $\tau_X$ are included in $\{xy = 0\}$ (Remark 3.2 and Lemma 2.5) we necessarily have that

$$\tau = d\left(\frac{A(x,y)}{x^N y^M} + \alpha \log x + \beta \log y\right)$$

with $(\alpha, \beta, N, M) \in \mathbb{C}^2 \times \mathbb{Z}^2_{\geq 0}$ and $A$ holomorphic near $(0,0)$. Observe that

$$X \cdot F = \tau(X) = \tau_Z(X) - 1$$

$$= \frac{1}{U} - 1$$

so we have built a meromorphic, multivalued solution to the equation of tangential normalization.

**Lemma 3.6.** Let $A$ be the linear space over $\mathbb{C}$ of all formal expressions

$$\frac{A(x,y)}{x^N y^M} + \alpha \log x + \beta \log y$$

with $A \in \mathbb{C}[[x,y]]$ and $(\alpha, \beta, N, M) \in \mathbb{C}^2 \times \mathbb{Z}^2_{\geq 0}$. Assume that $F \in A$ is such that $X \cdot F = G \in \mathbb{C}[[x,y]]$. Then there exists $\hat{F} \in \mathbb{C}[[x,y]]$ and a unique $\alpha \in \mathbb{C}$ such that

$$F(x,y) = \hat{F}(x,y) + G(0,0) \log y + \alpha \left(\log x - \lambda \log y\right).$$

The power series $\hat{F} - \hat{F}(0,0)$ is unique. On the other hand every $G \in \mathbb{C}[[x,y]]$ writes $X \cdot F$ for some $F \in A$ with $\alpha = 0$.

Before giving the proof of this lemma we conclude our discussion about tangential conjugacy. Since $\tau = dF$ and $F, T \in A$ with $X \cdot F = X \cdot T \in \mathbb{C}[[x,y]]$ we obtain

$$T = \hat{F} - \hat{F}(0,0).$$

Hence $T$ is a convergent power series since $F$ is a convergent object. What we have in fact proved is the following:

**Proposition 3.7.** Assume that $Z$ admits a closed time–form whose polar locus is included in $\{xy = 0\}$. Then $T$ is a convergent power series.

**Corollary 3.8.** There exists a closed time–form for $Z$ if, and only if, $Z$ is analytically conjugate to $X$ by a tangential change of coordinates.

This result proves Theorem 1.1 for quasi–resonant vector fields (setting $Q := 1$).

**Proof.** Because $d\tau_X = 0$ the existence of an analytical conjugacy gives rise to the closed time–form $\tau := \psi^* \tau_X$ for $Z$. Assume conversely that there exists some $\tau$ with $\tau(Z) = 1$.
and $d\tau = 0$. To apply the previous proposition we only need to show that the polar locus of $\tau$ is empty or a separatrix of $Z$. Since $Z = UX = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$ we can write

$$\tau = \frac{1}{U} \tau_X + \frac{f}{\rho} (Ay - Bdx)$$

with $f, \rho$ coprime and holomorphic. The case $\rho(0,0) \neq 0$ is trivial so we may as well assume $\rho(0) \neq \emptyset$. Taking into account that $\tau$ and $\tau_X$ are closed we derive the relations

$$0 = d\tau = \left( Z \cdot \frac{f}{\rho} + \frac{f}{\rho} \text{div} Z \right) dx \wedge dy - \frac{dU}{U^2} \wedge \tau_X$$

$$0 = \rho Z \cdot f - f Z \cdot \rho + \rho \text{div} Z - \frac{\rho^2}{\lambda_1 x U^2} \frac{\partial U}{\partial y}.$$ 

From this we deduce that $Z \cdot \rho$ vanishes along the curve $\rho(0) \neq \emptyset$, meaning that $\rho$ is a separatrix of $Z$. $\square$

We now give the proof of Lemma 3.6.

**Proof.** Write

$$F(x,y) = \frac{A(x,y)}{x^Ny^M} + a(\log x - \lambda \log y) + \beta \log y$$

where $(a, \beta) \in \mathbb{C}^2$ and

$$A(x,y) = \sum_{a,b \geq 0} f_{a,b} x^a y^b.$$ 

The equality $X \cdot F = G$ can be rewritten in the following way:

$$(3.1) \quad \quad \quad \quad \quad \quad \quad \quad X \cdot A - A (N \lambda (1 + R) + M) = x^N y^M (G - \beta - \alpha R).$$

The coefficient $h_{a,b}$ of $x^a y^b$ in the previous equality is given by

$$(3.2) \quad \quad \quad \quad \quad \quad \quad \quad (a-N) \lambda_1 + (b-M) \lambda_2 - N \lambda_2 \sum_{c=0}^{a} \sum_{d=0}^{b} f_{c,d} r_{a-c,b-d} = h_{a,b}$$

where $R(x,y) = \sum_{a+b > 0} r_{a,b} x^a y^b$. Obviously $h_{a,b} = 0$ if $a < N$ or $b < M$. Reasoning by induction on $a < N$ we show that $f_{a,b} = 0$ for all $b$. Indeed assume that $f_{c,d} = 0$ for all $c \leq a$ and $d < b$ (or $d \in \mathbb{Z} \geq 0$ when $c < a$). Because $r_{0,0} = 0$ and $\lambda \in \mathbb{Q}$ the expression (3.2) implies that $f_{a,b} = 0$. The same argument proves also that $f_{a,b} = 0$ if $b < M$ for all $a$. As a consequence there exists some $\tilde{A} \in \mathbb{C}[[x,y]]$ such that

$$A = x^N y^M \tilde{A}.$$ 

Equation (3.1) divided by $x^N y^M$ and evaluated at $(0,0)$ yields $\beta = G(0,0)$. The remaining of the claim is now clear. $\square$
3.4.2. Transversal rigidity.

Corollary 3.8 together with Lemma 2.4 allow us to restrict our study to Lie algebras $L(Z,Y)$ of ratio $\delta$ with

$$Z = X = Z_0 + RY_0$$

where $Y_0 = y \frac{\partial}{\partial y}$. We know that the formal power series $N$ satisfies

$$X \cdot N = -R.$$

In order to apply Lemma 1.4 we only need to know that $Y_0 \cdot N$ is a convergent power series. When this is the case we deduce that $N$ is convergent because $X \pitchfork Y_0$.

**Lemma 3.9.** Assume that $\hat{Y} \pitchfork Y_0$. Then $Y_0 \cdot N$ is a convergent power series.

**Proof.** Write $Y_0 = aZ_0 + b\hat{Y}$ for some meromorphic functions $a \neq 0$ and $b$. The discussion in Section 2.1 yields $a \in \mathbb{C}_{\neq 0}$. Since $N = \Phi_{Y_0}^N$ and $N^*(Z_0, \hat{Y}) = (Z, Y)$ we deduce from Proposition 2.6 that

$$N^* Y_0 = \frac{1}{1 + Y_0 \cdot N} Y_0 = aZ + b \circ N Y.$$

As a consequence $a(1 + Y_0 \cdot N)$ is convergent because $Z \pitchfork Y$. Since $a \neq 0$ it follows that $Y_0 \cdot N$ is convergent.

**Corollary 3.10.** The pair $(X, Y)$ is analytically conjugate to $(Z_0, \hat{Y})$.

**Proof.** If $\hat{Y} \pitchfork Y_0$ then the previous lemma proves the claim. Conversely assume that

$$\hat{Y} = cx^{n-1} y^m \frac{\partial}{\partial y}, \ c \in \mathbb{C}.$$

Let $Y_1 := x \frac{\partial}{\partial x}$ which is transverse to $\hat{Y}$ and commutes to $Z_0$. By repeating the construction at the beginning of the section, we can write

$$Z = \hat{U} \hat{X}, \quad \hat{X} = Z_0 + \lambda_1 R_1$$

with $R$ holomorphic and $R(0,0) = 0$. The conclusion follows.

4. Resonant singularities

The spirit of this section is the same as the previous one's. Here we assume that

$$\lambda := \frac{\lambda_1}{\lambda_2} = -p/q \in \mathbb{Q}_{\leq 0} \text{ with } p \wedge q = 1 \text{ if } p \neq 0 \text{ and } q := 1 \text{ if } p = 0.$$ For $(k, \mu) \in \mathbb{N} \times \mathbb{C}$ set

$$u(x, y) := x^k y^\mu, \quad W_0(x, y) := \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}, \quad X_0(x, y) := u^k x \frac{\partial}{\partial x} + \left(1 + \mu u^k\right) W_0(x, y).$$
Theorem 4.1. Assume that \( \mathcal{L}(Z,Y) \) is a non–degenerate Lie algebra of ratio \( \delta \) with \( \lambda \in \mathbb{Q}_{\leq 0} \) and such that \( Z \) is not formally linearizable. Then there exists a unique \((n,m) \in \mathbb{Z}^2 \) such that \((Z,Y)\) is analytically conjugate to \((\tilde{Z},\tilde{Y})\) given by:
\[
\begin{align*}
\tilde{Y} &= x^n y^m (Q \circ u)^n (c(Q \circ u)X_0 + dW_0) \\
\tilde{Z} &= (Q \circ u)X_0 + (P \circ u)x^n y^m W_0 \\
\gamma' &= \frac{\delta \mu + \lambda_2 n}{qk\lambda_2} \\
\delta &= n\lambda_1 + m\lambda_2
\end{align*}
\]
where \( P \) is a polynomial of degree at most \( k - 1 \) and \( Q \) at most \( k \), with \( Q(0,0) = 1 \), and \((c,d) \in \mathbb{C}^2 \). The polynomial \( P \) vanishes if \( c \neq 0 \) or \( \delta = 0 \), but is otherwise unspecified. Moreover
\[
\begin{itemize}
\item if \( \delta \neq 0 \) then \( Q(u) = 1 - (\lambda_2 \frac{\mu}{\delta} + \mu) u^k \) and \((n,m) \in (q,p) \mathbb{Z} \),
\item if \( \delta = 0 \) then \( Q \) is unspecified and \( n = m = 0 \).
\end{itemize}
These expressions are unique up to diagonal changes of coordinates \((x,y) \mapsto (\alpha x, \beta y)\) with \( u(\alpha, \beta)^k = 1 \).

Note that \( \tilde{Z} \) is polynomial and \( \tilde{Y} \) is a global object with either 0 or \( k \) points of ramification. One could adapt the upcoming proof to show that the result remains true with \( \tilde{Z} = (Q \circ u)(X_0 + (P \circ u)u^a x^n y^m W_0) \) (for a different \( P \), of course); the later expression might be more useful for practical purposes.

4.1. Preparation and general facts.

We need the following well known ingredients:
\[
\begin{itemize}
\item According to a result by H. Dulac [Dul109] there exists local analytic coordinates in which \( Z \) writes
\[
\begin{align*}
Z &= UX \\
X &= X_0 + RY_0 \\
Y_0 &:= y^\varepsilon W_0
\end{align*}
\]
where \( U(0,0) = 1 \), \( R \) is divisible by \( u^{k+1} \) and \( \varepsilon \in \{-1,1\} \). We define \( \varepsilon := 1 \) if \( Z \) admits two transverse separatrices while if \( Z \) has only one separatrix then necessarily \( \lambda = 0 \) and we set \( \varepsilon := -1 \). Notice that \([X_0,Y_0] = \delta_0 Y_0 \) with
\[
\delta_0 := \varepsilon \lambda_2 (1 + \mu u^k) \in \ker Y_0.
\]
\item Any meromorphic function \( h \) such that \( Z \cdot h = 0 \) is constant. Any meromorphic function \( b \neq 0 \) such that \( Z \cdot b = Kb \) for some holomorphic \( K \) is of the form
\[
b = x^N y^M \beta(x,y), (N,M) \in \mathbb{Z}^2 \text{ and } \beta(0,0) \neq 0
\]
because \( \{x = 0\} \) and \( \{y = 0\} \) are the only candidate separatrices of \( Z \).
\end{itemize}
\]

Lemma 4.2. Let \( G \in \mathbb{C}[[x,y]] \) and \( \varepsilon \in \{-1,0,1\} \) be given. There exists \( F \in \mathbb{C}[[x,y]] \) such that
\[
X \cdot F + \varepsilon \lambda_2 (1 + \mu u^k) F = G
\]
if, and only if, \( G \) does not contains terms \( y^{-n} u^n \) for \( 0 \leq n \leq k \). If \( \varepsilon \leq 0 \) the term \( c := \frac{\partial F}{\partial y^{\varepsilon}}(0,0) \) is free and \( F - c \) is unique.
Proposition 4.3. Formal normal forms.

Here we study the Lie algebras \( \mathfrak{g}\). If \( (a, b) \neq 0 \) the term \( f_{a,b} \) is defined uniquely. On the contrary if \( (a, b) = 0 \) we deduce that \( f_{a,b} = 0 \) for the monomial \( x^a y^b \).

Proof. Write \( f(x, y) := \sum f_{a,b} x^a y^b \) and \( G(x, y) := \sum g_{a,b} x^a y^b \). The equation writes, for the monomial \( x^a y^b \),

\[
(\lambda_1 a + \lambda_2 (b + \varepsilon)) f_{a,b} + (a - kq) f_{a-kq,b-kp} + o(a,b) = g_{a,b}
\]

where \( o(a,b) \) stands for terms involving \( f_{c,d} \) only for \( c + d < a + b - kq - kp \). We proceed by induction on the homogeneous degree \( a+b \): while \( \lambda_1 a + \lambda_2 (b + \varepsilon) \neq 0 \) the term \( f_{a,b} \) is defined uniquely. On the contrary if \( (a, b + \varepsilon) = l(q,p) \) with \( l \in \mathbb{Z} \):

1. if \( 0 \leq l < k \) then \( f_{a-kq,b-kp} = 0 \) so necessarily \( g_{a,b} = 0 \).
2. if \( l = k \) then \( g_{kq,kp} = 0 \) and \( f_{0,-\varepsilon} \) is free if \( \varepsilon \leq 0 \) or is zero if \( \varepsilon = 1 \).
3. if \( l > k \) the term \( f_{l-k,p,l-k|q} \) is well determined.

\[\square\]

Hence there exists a polynomial \( Q \in \mathbb{C}_k[u] \) (of degree at most \( k \)) satisfying \( Q(0) = 1 \) such that \( Z \) is formally tangentially conjugate to \( (Q \circ u) X \) by \( T := \Phi_{QX}^T \), according to Proposition 2.6. Namely \( Q \) is the natural projection of \( U \) on \( \mathbb{C}_k[u] \) and

\[ X \cdot T = \frac{1}{Q \circ u} - \frac{1}{U}. \]

If \( Z \) admits two separatrices, because \( [X_0, Y_0] = \delta_0 Y_0 \) and \( Y_0 \cdot u = 0 \) we deduce that \( (Q \circ u) X_0 \) is formally transversely conjugate to \( (Q \circ u) X \) by

\[
N(x,y) := \Phi_{Y_0}^{N(x,y)}(x,y) \quad (4.1)
\]

where

\[
X \cdot N + \delta_0 N = -R. \quad (4.2)
\]

The triple \((Q,k,\mu)\) is the formal invariant of conjugacy [Bru89] for \( Z \), up to the equivalence

\[(Q,k,\mu) \sim (\hat{Q},\hat{k},\hat{\mu}) \iff (k,\mu) = (\hat{k},\hat{\mu}) \text{ and } Q(u) = \hat{Q}(au) \text{ with } a^k = 1.\]

In the following we fix \((k,\mu)\), which is the formal invariant for the foliation underlying \( Z \). We define

\[ Z_0 := (Q \circ u) X_0. \]

4.2. Formal normal forms.

Here we study the Lie algebras \( \mathcal{L}(Z_0, \hat{Y}) \).

Proposition 4.3. Assume that \( \hat{Y} \neq 0 \) is a formal meromorphic vector field such that \([Z_0, \hat{Y}] = \delta \hat{Y} \) and \( \hat{Y} \pitchfork Z_0 \).

1. If \( \delta \neq 0 \) there exists a unique \((n,m) \in \mathbb{Z}^2 \setminus (q,p) \mathbb{Z} \) such that

\[
\delta = n \lambda_1 + m \lambda_2 \quad Q(u) = 1 - \left( \frac{n}{\delta} + \mu \right) u^k \quad \hat{Y} = x^m y^n (Q \circ u)^\gamma (c Z_0 + d W_0) \]

with \((c,d) \in \mathbb{C} \times \mathbb{C}^* \) and \( \gamma = -(\delta \mu + n)/kq \).
(2) If \( \delta = 0 \) there exists \((c,d)\) \(\in\mathbb{C} \times \mathbb{C}^\ast\) such that

\[
\dot{Y} = cZ_0 + dW_0.
\]

Proof. Observe that \(W_0\) is holomorphic and commutes with \(Z_0\). Write \(\dot{Y} = aZ_0 + bW_0\) with \(a\) and \(b\) in \(\mathbb{C}(x,y)\). Then

\[
Z_0 \cdot a = \delta a, \\
Z_0 \cdot b = \delta b.
\]

Since \(b \neq 0\) we can write \(b(x,y) = x^n y^m \beta(x,y)\) for \((n,m) \in \mathbb{Z}^2\) and \(\beta(0,0) \neq 0\). We find:

\[
(Q \circ u)(n \lambda_1 + m \lambda_2)(1 + \mu u^k) + nu^k \beta \quad = \quad \delta \beta.
\]

Define \(B := \log \beta \in \mathbb{C}[[x,y]]\). Lemma 4.2 implies that

\[
Z_0 \cdot B = \delta - (Q \circ u)(n \lambda_1 + m \lambda_2)(1 + \mu u^k) + nu^k
\]

cannot contain terms of the form \(u^l\) for \(0 \leq l \leq k\). We immediately get that \(\delta = n \lambda_1 + m \lambda_2\).

Write \(Q(u) = 1 + \sum_{j=1}^k v_j u^j\) and assume first that \(\delta \neq 0\). We immediately derive that \(v_j = 0\) for \(0 < j < k\), while

\[
n = -\delta (v_k + \mu).
\]

Thus

\[
Q \circ u = 1 - \left(\frac{n}{\delta} + \mu\right) u^k,
\]

which determines \(n\) completely (and thus \(m\) since \(\delta = n \lambda_1 + m \lambda_2\)). On the one hand for all \(\gamma \in \mathbb{C}\)

\[
Z_0 \cdot \gamma \log Q \circ u = \gamma qu^{k+1} Q' \circ u = \gamma q k \lambda_2 v_k u^{2k}
\]

whereas on the other hand

\[
Z_0 \cdot B = -\delta v_k^2 u^{2k}.
\]

The uniqueness condition in Lemma 4.2 allows us to find \(C \in \mathbb{C}\) such that

\[
B = \gamma \log Q \circ u + C
\]

with

\[
\gamma := \frac{\delta \mu + n}{qk}.
\]

To conclude

\[
b = dx^m y^m (Q \circ u) x^2 \quad a = cx^m y^m (Q \circ u) x.
\]

Finally when \(\delta := 0\) the above computations show that \(n = 0\), then \(m = 0\) and \(\beta\) must be constant. \(\square\)
4.3. Tangential rigidity.

Here we do not assume anything on the number of separatrices of $Z$ or the existence of a non-degenerate Lie algebra $L(Z, Y)$. We end the proof of Theorem 1.1:

**Proposition 4.4.** There exists a closed time–form $\tau$ for $Z$ if, and only if, $Z$ is analytically conjugate to $Q(u)X$ by a tangential change of coordinates.

We proceed along the same steps as taken in Section 3.4.1.

1. If the polar locus of $\tau_0 := \tau - \frac{1}{Q \circ u} \tau_X$ is a separatrix of $Z$, where $\tau_X := \frac{du}{u}$, we have
   \[ \tau_0 = d \left( \frac{A(x, y)}{x^N y^M} + \alpha \log x + \beta \log y \right) = dF \]
   with $A$ holomorphic and $(N, M, \alpha, \beta) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{C}^2$. Besides
   \[ X \cdot F = \frac{1}{Q \circ u} - \frac{1}{U} = X \cdot T. \]

2. We need to show that $A = x^N y^M \tilde{A}$ and $\alpha = \beta = 0$. Once this is done the uniqueness condition in Lemma 4.2 implies that $T = \tilde{A} - \tilde{A}(0, 0)$ is a convergent power series.

3. Any closed time–form $\tau$ of $Z$ is of the form
   \[ \tau = \frac{1}{U} \tau_X + \frac{f}{\rho} \omega \]
   for some holomorphic 1–form $\omega$ with $\omega(Z) = 0$. The polar locus $\{ \rho = 0 \}$ is again a separatrix of $Z$.

We state now in detail the second step but we skip the proof since it is very similar to the one of Lemma 3.6.

**Lemma 4.5.** Let $A$ be the linear space over $\mathbb{C}$ of all formal expressions
   \[ \frac{A(x, y)}{x^N y^M} + \alpha \log x + \beta \log y \]
   with $A \in \mathbb{C}[ [x, y] ]$ and $(\alpha, \beta, N, M) \in \mathbb{C}^2 \times \mathbb{Z}_{\geq 0}^2$. Assume that $F \in A$ is such that $X \cdot F = G \in \mathbb{C}[ [x, y] ]$. Then there exists $\hat{F} \in \mathbb{C}[ [x, y] ]$ and a unique $\alpha \in \mathbb{C}$ such that
   \[ F(x, y) = \hat{F}(x, y) - G(0, 0) \frac{1}{qk^k} \alpha \left( (1 - \mu p) \log y - \mu q \log x + \frac{1}{k^k} \right). \]

The power series $\hat{F} - \hat{F}(0, 0)$ is unique. On the other hand every $G \in \mathbb{C}[ [x, y] ]$ writes $X \cdot F$ for some $F \in A$ with $\alpha = 0$.

4.4. Transversal rigidity.

Let us write

\[ \hat{Y} = L(cZ_0 + dy^{-1}Y_0) \]
\[ L := x^{n}y^{m}(Q \circ u) \]


with \( c \in \mathbb{C}, \, d \in \mathbb{C}^* \) as in Proposition 4.3. Since \( Y = N' \hat{Y} \) is meromorphic we deduce that
\[
cL \circ N \quad \text{and} \quad (y^{-\varepsilon}L) \circ N' (1 + Y_0 \cdot N)^{-1}
\]
are meromorphic. Taking (4.1) into account we derive that
\[
L \circ N = L(1 - \varepsilon \lambda_2 y^\varepsilon N)^{-\varepsilon(m+n)}
\]
\[
(y^{-\varepsilon}L) \circ N = y^\varepsilon L(1 - \varepsilon \lambda_2 y^\varepsilon N)^{1-\varepsilon(m+n)}
\]
which means, if \( \varepsilon \neq 0 \), that we are done: \( N \) is convergent since \( \varepsilon(m+n) = \varepsilon \delta / \lambda_2 \neq 0 \). On the other hand, if \( \varepsilon = 0 \), then
\[
(1 + Y_0 \cdot N)(1 - \varepsilon \lambda_2 y^\varepsilon N)^{\varepsilon \delta / \lambda_2 - 1} = A
\]
with \( A \) meromorphic. Indeed, either \( \varepsilon = 1 \) and the claim is clear, or \( \varepsilon = -1 \) and \( \varepsilon \delta / \lambda_2 = -m \) so that \( (1 + \lambda_2 y^{-1}N)^{-\delta / \lambda_2 - 1} = y^{m+1} (y + \lambda_2 N)^{-m-1} \).

**Lemma 4.6.** There exists a convergent power series \( \hat{N} \) satisfying equation (4.4) and such that \( \hat{N}(0,0) = Y_0 \cdot \hat{N}(0,0) = 0 \).

**Proof.** Noticing that \( Y_0 \cdot y^{-\varepsilon} = -\varepsilon \lambda_2 \) we deduce that
\[
Y_0 \cdot (y^{-\varepsilon} - \varepsilon \lambda_2 N)^{\varepsilon \delta / \lambda_2} = -\delta y^{\varepsilon - \varepsilon \lambda_2} (1 + Y_0 \cdot N)
\]
\[
= -\delta y^{-\delta / \lambda_2 + \delta} A.
\]
Assume first that \( \varepsilon = 1 \) and write \( (1 - \lambda_2 y N(x,y))^{\delta / \lambda_2} = \sum_{a+b=0} N_{a,b} x^a y^b \). The previous equation writes
\[
\sum_{a+b=0} (\lambda_1 a + \lambda_2 b - \delta) N_{a,b} x^a y^b = -\delta A(x,y)
\]
and we know that a formal solution exists. As a consequence \( A(x,y) = \sum_{a,b \geq 0} A_{a,b} x^a y^b \) is holomorphic and, since \( \delta = \lambda_1 n + \lambda_2 m \),
\[
\frac{\lambda_2}{q} (-p(a-n) + q(b-m)) N_{a,b} = A_{a,b}.
\]
When \( (a,b) \not\in A := (q,p)\mathbb{Z} + (n,m) \) the estimate
\[
\lambda_2 \left| N_{a,b} \right| \leq q \left| A_{a,b} \right|
\]
proves the convergence of
\[
N_0(x,y) := \sum_{(a,b) \in \mathbb{Z}^2 \setminus A} N_{a,b} x^a y^b.
\]
We finally set
\[
\hat{N}(x,y) := \frac{1}{\lambda_2 y} \left( 1 - N_0^{\lambda_2 / \delta} \right)
\]
which is of course a formal power series.

Assume now that \( \varepsilon = -1 \) so that \( Y_0 = \lambda_2 \frac{\partial}{\partial y} \). Let \( \{B = 0\} \) be the (formal) vanishing locus of \( y + \lambda_2 N \). The power series \( B \) is not zero since \( X_0 \) admits \( y = 0 \) for separatrix. Because \( N(0,0) = (Y_0 \cdot N)(0,0) = 0 \) we deduce that \( B(x,y) = y - \delta(x) \) for some \( \delta \in \mathbb{C}[[x]] \) so that there exists \( \hat{N} \in \mathbb{C}[[x,y]] \) such that
\[
y + \lambda_2 N = (y - \delta) \exp \hat{N}.
\]
We obtain that
\[ y^{-m-1} A = Y_0 \cdot (y + \lambda_2 N)^{-m} = (y - \hat{s})^{-m-1} \exp \left( -m \hat{N} \right) \left( 1 + (y - \hat{s}) \frac{\partial}{\partial y} \hat{N} \right) \]
so that the hypothetical poles of \( y^{-m-1} A \) are of the form \( \{ B = 0 \} \) and of order \( m + 1 \neq 1 \). Hence \( y^{-m-1} A \) admits a meromorphic primitive \( E \) with respect to \( \frac{\partial}{\partial y} \) such that \( E(x, 0) = 0 \) and
\[ E = (y - \hat{s})^{-m} F \]
with \( F \in \mathbb{C}[x, y] \), \( F(0, 0) = 1 \). Hence \( (y - \hat{s}) F^{-1/m} \) is holomorphic and the following holomorphic function solves our problem:
\[ \tilde{N}(x, y) := \frac{E^{-1/m} - y}{\lambda_2} \].

**Corollary 4.7.** The couple \((Z, Y)\) is analytically conjugate to some \((Z_0 + K \hat{Y}, \hat{Y})\) where \( \hat{Y} \cdot K = 0 \). If moreover \( Y \equiv Y_0 \) then \( K = 0 \).

**Proof.** On the one hand we have already discussed the case \( Y \equiv Y_0 \). On the other hand we just proved that \( \tilde{N} := \Phi_{Y_0}^{\tilde{N}} \) is an analytic conjugation between \( \hat{Y} \) and \( Y \). Set \( \tilde{Z} := \tilde{N}, Z = Z_0 + K \hat{Y} \) with \( K \) meromorphic. Since \( \left[ \tilde{Z}, \hat{Y} \right] = \delta \hat{Y} \) it turns out that \( \hat{Y} \cdot K = 0 \). \( \square \)

### 4.5. Final reduction: proof of Theorem 4.1.

We only give a sketch of proof: computations for saddle–node s have already been done in [Tey04], and they generalize straightforwardly to resonant saddles. Assume that \( X = X_0 + (K \circ u) x^n y^m W_0 \) and that \( \tilde{Y} = dx^n y^m (Q \circ u) \hat{Y} W_0 \) (we already know that \( K = 0 \) if \( \hat{Y} \equiv W_0 \)). By playing with linear transformations \( (x, y) \mapsto (x, \alpha y) \), one can choose \( d = 1 \). We seek a function \( F \in \mathbb{C}[x, y] \) such that
\[ N^*_F := \Phi_{Y_0}^{F \circ u} \]
satisfies
\[ N^*_F X = X_0 + (P \circ u) x^n y^m W_0. \]
Because \( W_0 \circ (Q \circ u) = 0 \) we can write \( N^*_F = \Phi_{Y_0}^{\tilde{F} \circ u} \) where \( Y_0 = x^n y^m W_0 \), so that \( N^*_F ((K \circ u) Y_0) = (K \circ u) Y_0 \). We end up with the cohomological equation
\[ X_0 \cdot (FQ \circ u) + \delta_0 FQ \circ u = (K - P) \circ u \]
where \( [X_0, Y_0] = \delta_0 Y_0 \), which can be rewritten
\[ X_0 \cdot (x^n y^m FQ \circ u) = -\delta_0 x^n y^m (K - P) \circ u. \]
In the above–mentioned reference the reader will find a way of (explicitly) constructing a polynomial \( P \), satisfying the sought properties, such that the equation admits an analytic solution \( F \).
Remark 4.8. In fact we prove in [Tey04] that \( \dot{Z} \) is analytically conjugate to \( Z_0 \) if, and only if, \( P = 0 \). Hence \( \mathcal{L}(Z_0, \dot{Y}) \) and \( \mathcal{L}(Z_0 + (P \circ u) \dot{Y}, \dot{Y}) \) are formally conjugate affine Lie algebras which are not analytically conjugate. This remark provides a family of examples complementing those of [Cer91].

In the result of rigidity we presented here the main obstruction regarding the convergence of the formal objects we built is the collinearity of the vector fields \( \dot{Y} \) and \( Y_0 \). In other terms denote by \( \Omega : = (Z_0, Y_0, \dot{Y}) \) the (singular) 3–web corresponding to the formal normal form of \( \mathcal{L}(Z, Y) \): if \( \Omega \) is non–degenerate then \( \mathcal{L}(Z, Y) \) is analytically conjugate to \( \mathcal{L}(Z_0, \dot{Y}) \) whereas in the opposite case such a result is not true in general. The geometrical meaning of that fact is that we have constructed the formal conjugacies using the flows along \( Z \) and \( Y \), while the convergence of the power series involved related to those vector fields being transverse to each–others. When this is no longer the case we cannot infer the convergence of the whole power series, but only of a sub–series which «does not contain» first–integrals of \( Y_0 \) since the only information we have is of the type «\( Y_0 \cdot N \) is convergent».

5. Classification by the delta–lattice

Definition 5.1. Let \( Z \) be a germ of a meromorphic vector field at \((0,0)\). We define the transverse structures of \( Z \) as the set

\[
\text{TS}(Z) := \{ (\delta, Y) : Y \cap Z, [Z, Y] = \delta \}
\]

and call \( \Delta(Z) \) its delta–lattice:

\[
\Delta(Z) := \{ \delta : (\exists Y) (\delta, Y) \in \text{TS}(Z) \}.
\]

In this section we focus once more on holomorphic germs of a vector field. As before if \( Z(0,0) = 0 \) we define \( \{\lambda_1, \lambda_2\} \) the spectrum of the linear part of \( Z \) at \((0,0)\). When \( Z \) is not nilpotent we label the eigenvalues in such a way that \( \lambda_2 \neq 0 \) and set \( \lambda := \lambda_1/\lambda_2 \). The delta–lattice of a holomorphic vector fields characterize its dynamical class.

Proposition 5.2. Assume that \( \Delta(Z) \neq \emptyset \). Then \( \Delta(Z) \) is an affine \( \mathbb{Z} \)–lattice corresponding to exactly one of the following cases.

1. \( \Delta(Z) = \mathbb{C} \), if, and only if, \( Z \) is regular at \((0,0)\).
2. \( \Delta(Z) = \lambda_1 \mathbb{Z} \oplus \lambda_2 \mathbb{Z} \) of rank 2 if, and only if, \( Z \) is not nilpotent and \( \lambda \in \mathbb{R} \).
3. \( \Delta(Z) = \lambda_1 \mathbb{Z} \oplus \lambda_2 \mathbb{Z} \) dense in a real line if, and only if, \( Z \) is not nilpotent, analytically linearizable and \( \lambda \in \mathbb{R}(\lambda) \).
4. \( \Delta(Z) = \delta \mathbb{Z} \) of rank 1 \((\delta \neq 0)\) if, and only if, \( Z \) is not nilpotent, \( \lambda \in \mathbb{Q} \) and \( Z \) is analytically conjugate to its formal normal form.
5. \( \Delta(Z) = \{ \delta \} \), if, and only if, \( Z \) is a resonant saddle or a saddle–node not analytically conjugate to its formal normal form, or has nilpotent linear part. In the latter case \( \delta = 0 \).

Remark 5.3. In (4) the formal normal form considered is either the linear part of \( Z \) (non–resonant singularity) or its Dulac–Poincaré normal form. The latter case encompasses resonant saddles \((\lambda \in \mathbb{Q}_{>0})\) and saddle–nodes \((\lambda = 0)\) studied in Theorem 4.1 as well as resonant nodes \((\lambda \in \mathbb{N} \cup 1/\mathbb{N}) \) and the normal form is given e.g. when \( \lambda \in \mathbb{N} \) by \( \lambda_1 (x + y^{-1}) \frac{\partial^2}{\partial x} + \lambda_2 y \frac{\partial^2}{\partial y} \).
Proof. Assume there exists \((\delta, Y) \in TS(Z)\) and write \(\tilde{Y} = aZ + bY\) with \(a, b \in \mathbb{C}\{x, y\}\) and \(b \neq 0\). Then \(Z \cdot a = \delta a\) and \(Z \cdot b = (\delta - \delta)b\). For \(n \in \mathbb{Z}\) the meromorphic germ \(b^n\) solves \(Z \cdot b^n = n(\delta - \delta)b^n\) so that \(\left(\delta + n(\delta - \delta), b^nY\right) \in TS(Z)\) and \(\Delta(Z)\) is an affine \(\mathbb{Z}\)-lattice.

The remaining of the proof is only a restatement of both classical results and results established in the previous sections. We indicate shortly below why the condition on \(Z\) is sufficient, the necessity of said condition then follows from the exhaustion of all dynamical types of germs of a vector field by the above discrimination. In case (1) if \(Z(0,0) \neq 0\) the rectification theorem states that, in some appropriate local chart, \(Z = \frac{d}{dy}\). Hence being given any \(\delta \in \mathbb{C}\) the vector field \(Y := e^{\delta y} \frac{d}{dx}\) induces a non-degenerate Lie algebra \(L(Z,Y)\) of ratio \(\delta\). Case (2) is a straightforward consequence of Poincaré’s linearization theorem and of direct computations analogous to those we carried out for the last part of the proof of Corollary 2.10. In fact all linearizable cases are dealt with in the same way, which accounts for (3) via the use of Theorem 3.1, as well as (4) for non-resonant singularities. Cases (4) and (5) for resonant singularities are taken care of by Theorem 4.1. Notice that if \(Z\) is nilpotent then Lemma 2.8 ensures that \(\Delta(Z) = \{0\}\) (here the holomorphy of \(Z\) is an essential part of the argument). □

6. Computation of Galois–Malgrange’s groupoid

We assume in this section that \(Z\) is a germ of a meromorphic vector field at \((0,0)\).

6.1. Definitions.

We do not introduce in full details what the Galois–Malgrange groupoid for a meromorphic vector field \(Z\) (or foliation \(\mathcal{F}_Z\)) is. We refer to [Mal01] and [Cas06] for precise definitions and general properties. Roughly speaking a \(\mathcal{D}\)-groupoid is a groupoid (pseudo–group) of germs of a diffeomorphism \(\Gamma : (\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, q)\) solutions to some differentially stable (sheaf of) ideal \(I\) of partial differential equations, meromorphic in \(\Gamma\) and polynomial in its derivatives, and where \(p, q\) are taken in a poly-disc \(\Delta\). The important point here is that \(I\) may only be defined outside an analytic hypersurface, which in our case will be located along the polar locus of \(Z\) and \(Y\), as well as along their tangency locus. To such a \(\mathcal{D}\)-groupoid is associated a \(\mathcal{D}\)-algebroid, which is a (sheaf of) Lie algebra whose elements are solutions to the linearized equations near the identity element. Not every \(\mathcal{D}\)-algebroid is integrable (i.e. that of a \(\mathcal{D}\)-groupoid).

Definition 6.1. Let \(Z\) be a germ of a holomorphic vector field in \(\mathbb{C}^2\) with underlying foliation \(\mathcal{F}_Z\).

1. The **Galois–Malgrange groupoid** of \(\mathcal{F}_Z\) is the smallest \(\mathcal{D}\)-groupoid \(\mathbb{G}a1(\mathcal{F}_Z)\) whose \(\mathcal{D}\)-algebroid \(\mathbb{G}a1(\mathcal{F}_Z)\) contains the sheaf \(O_Z\) of vector fields tangent to \(\mathcal{F}_Z\).

2. The **Galois–Malgrange groupoid** of the meromorphic vector field \(Z\) is the smallest \(\mathcal{D}\)-groupoid \(\mathbb{G}a1(Z)\) whose \(\mathcal{D}\)-algebroid \(\mathbb{G}a1(Z)\) contains \(Z\).

Remark 6.2.

1. Obviously \(\mathbb{G}a1(Z) < \mathbb{G}a1(\mathcal{F}_Z)\) as \(\mathbb{G}a1(Z) < \mathbb{G}a1(\mathcal{F}_Z)\).

2. The sheaf of germs of an isotropy of \(\mathcal{F}_Z\) (resp. \(Z\)) form a \(\mathcal{D}\)-groupoid \(\text{Aut} (\mathcal{F}_Z)\) (resp. \(\text{Aut} (Z)\)) containing \(\mathbb{G}a1(F_Z)\) (resp. \(\mathbb{G}a1(Z)\)).
Knowing whether $\text{Gal}(\mathcal{F}_Z)$ is a proper subgroupoid of $\text{Aut}(\mathcal{F}_Z)$ pertains to the problem of Liouvillian integrability of the underlying differential equation, as we explain now. In the case of a foliation or vector field equations of $\mathcal{I}$ defining elements $\Gamma \in \text{Gal}(\mathcal{F}_Z)$ can be restricted to the transverse direction, yielding an ideal of equations $\mathcal{I}_\delta$ of a transverse $D$–groupoid (see [Cas06, Lemme 2.4]). This transverse $D$–groupoid encodes all the information regarding the question of Liouvillian integrability of the foliation. The information is concentrated in the transverse rank $\ell_Z$ of $\mathcal{F}_Z$, which is the rank of $\mathcal{I}_\delta$. This rank does not depend on the transverse disc considered nor on the local analytic charts in which $Z$ is expressed.

**Theorem of Casale–Malgrange.** Let $\ell_Z$ be the transverse rank of $\mathcal{F}_Z$. The following properties are equivalent.

1. $\ell_Z < \infty$.
2. $\ell_Z \leq 3$.
3. $\text{Gal}(\mathcal{F}_Z)$ is a proper subgroupoid of $\text{Aut}(\mathcal{F}_Z)$.
4. $\mathcal{F}_Z$ admits a Godbillon–Vey sequence of finite length $\ell_Z$ and no sequence of lesser length.

### 6.2. Integrability theorem.

**Definition 6.3.** Let $Z$ be a meromorphic vector field on a small polydisc $\Delta$ around $(0,0)$. We denote by $\text{Sol}_\delta(Z)$ the coherent sheaf of germs of a holomorphic function $f$ at points of $\Delta \backslash \{(0,0)\}$, called $\delta$–solutions, such that $Z \cdot f = \delta f$.

In particular $\text{Sol}_0(Z)$ is the sheaf of holomorphic first–integrals of $Z$.

**Remark 6.4.** A straightforward computation shows the inclusion, for every $\alpha, \beta \in \mathbb{C}$,

$$\text{Sol}_\alpha(Z) \text{Sol}_\beta(Z) \subseteq \text{Sol}_{\alpha+\beta}(Z).$$

Besides if $f \in \text{Sol}_\alpha(Z)$ then $Y \cdot f \in \text{Sol}_{\alpha+\delta}(Z)$ for any vector field $Y$ such that $[Z,Y] = \delta Y$.

We are now able to state our theorem:

**Theorem 6.5.** Assume that the transverse structures $\text{TS}(Z)$ is not empty and the delta–lattice is not reduced to $\{0\}$. For $(\delta,Y) \in \text{TS}(Z)$ denote by $\text{Aut}(Z,\mathcal{L}(Z,Y))$ the pseudo–group $\text{Aut}(Z) \cap \text{Aut}(\mathcal{L}(Z,Y))$ of invariance of the pair $(Z,\mathcal{L}(Z,Y))$. Then the following properties hold.

1. $\text{Gal}(Z) = \bigcap_{(\delta,Y)\in \text{TS}(Z)} \text{Aut}(Z,\mathcal{L}(Z,Y))$.

2. There exists a sheaf of Lie algebras of dimension at most 2, $\text{Trans}(Z)$, such that

$$\text{Gal}(Z) = \mathbb{C}Z \oplus \text{Trans}(Z).$$

It is actually an integrable $D$–algebra whose corresponding $D$–groupoid $\text{Trans}(Z)$ is isomorphic to the transverse $D$–groupoid of $\mathcal{F}_Z$. The dimension of $\text{Trans}(Z)$ is therefore $\ell_Z$.

In order to characterize elements $\Gamma \in \text{Aut}(Z)$ actually belonging to $\text{Gal}(Z)$ we need to find a minimal set of proper equations satisfied by $\Gamma$. These equations are obtained by studying the action of $\text{Aut}(Z)$ on the other generators $Y$ of non–degenerate Lie algebras $\mathcal{L}(Z,Y)$ of ratio $\delta \in \mathbb{C}$.
Lemma 6.6. \( \Gamma \in \text{Aut}(Z, \mathcal{L}(Z, Y)) \) if, and only if,
\[
\begin{cases}
\Gamma^* Z = Z \\
\Gamma^* Y = d_T Z + c_T Y 
\end{cases}
\]
for some \((c_T, d_T) \in \mathbb{C}_{\neq 0} \times \mathbb{C}\) with \(\delta d_T = 0\).

Proof. We have \([Z, \Gamma^* Y] = \delta \Gamma^* Y\) which means \(\Gamma^* Y = d_T Z + c_T Y\) for unique \(d_T \in \text{So1}_0(Z)\) and \(c_T \in \text{So1}_0(Z)\). The condition \(\mathcal{L}(Z, \Gamma^* Y) = \mathcal{L}(Z, Y)\) then implies \(c_T, d_T \in \mathbb{C}\) and finally \(d_T = 0\) if \(\delta \neq 0\). \(\square\)

6.2.1. Action of \(\text{Aut}(Z)\) on \(\mathcal{L}(Z, Y)\).

Lemma 6.7. Let \(\mathcal{L}(Z, Y)\) be a non–degenerate Lie algebra of ratio \(\delta\), meromorphic on a small polydisc \(\Delta\) centered at \((0, 0)\). Take \(\Gamma \in \text{Aut}(Z)\) defined on a connected neighborhood \(U' \subset \Delta\) of a point \(p_T \in \Delta \setminus \{(0,0)\}\) ranging in \(U \supset U'\), such that:
- \(Z\rvert_U\) and \(Y\rvert_U\) are holomorphic and transverse,
- \(Z\rvert_U\) is rectifiable.

Then the following properties hold.

1. There exists \(T_T \in \text{So1}_0(Z)\) and \(N_T \in \text{So1}_{-\delta}(Z)\) such that
   \[\Gamma = \Phi_{Y}^{N_T} \circ \Phi_{Z}^{T_T}.\]

2. Moreover
   \[\Gamma^* Y = \frac{\exp(\delta T_T)}{1 + Y \cdot N_T} (Y - (Y \cdot T_T) Z)\]
   with \(Y \cdot N_T \in \text{So1}_0(Z)\) and \(Y \cdot T_T \in \text{So1}_0(Z)\).

Proof.

1. Let \(\psi : U \to U'\) be a rectifying chart of \(Z\) and denote \((t, z)\) a system of coordinates on \(U',\) that is \(\psi^* Z = \tilde{Z} := \frac{\partial}{\partial t}\) and \(z\) is the «transverse» coordinates. We ask that \(\psi\) be holomorphic, one to one and send \(p_T = (0,0)\); set \(\tilde{q} := \psi(q_T)\). Since \(\tilde{\Gamma} := \psi^* \Gamma\) belongs to \(\text{Aut}(\frac{\partial}{\partial t})\) we have
   \[\tilde{\Gamma}(t, z) = (t + \alpha(z), \beta(z))\]
   for some \(\alpha, \beta \in O(\psi(U'))\) with \((\alpha(0), \beta(0)) = \tilde{q}\).

Write \(\tilde{Y} := \psi^* Y\), so that \([\tilde{Z}, \tilde{Y}] = \delta \tilde{Y}\). For \(T, N \in O(\psi(U'))\) apply Proposition 2.6 with \(N := \Phi_{Y}^{N_T}\) and \(T := \Phi_{Z}^{T_T}\). We seek \(\eta \in \mathcal{C}[z]\) such that \(N(t, z) := \eta(z) \exp(-\delta t) \in \text{So1}_{-\delta}(\frac{\partial}{\partial t})\) solves
   \[\pi_z \circ N(t, z) = \beta(z),\]
where \(\pi_z\) is the natural projection \((t, z) \mapsto z\). This function is given by the implicit function theorem applied to the map
   \[g(t, z, r) := \pi_z \circ \Phi_{Y}^{N_T}(t, z),\]
   since \(\frac{\partial g}{\partial t}(t, z, r) = \pi_z \circ \tilde{Y} \circ \Phi_{Z}^{T_T}(t, z)\) does not vanish for \(\tilde{Z}\) and \(\tilde{Y}\) are transverse on \(U\).

Now \((N^{-1} \circ \tilde{\Gamma}) \in \text{Aut}(\frac{\partial}{\partial t})\) is the identity in the transverse variable and therefore
is of the form \((t,z) \mapsto \Phi^\delta_z(t,z) = (t + \delta(z), z)\) with \(\delta \in \mathcal{O}(\psi(U'))\). Back in the original coordinates we deduce the result with \(N_\Gamma := N \circ \psi^{-1}\) and \(T_\Gamma := T \circ \psi^{-1}\).

(2) Write \(\Gamma = \mathcal{N} \circ T\) with \(\mathcal{N} := \Phi^N_Z\) and \(T := \Phi^T_Z\). From Proposition 2.6 we deduce the expressions

\[
\begin{align*}
N^* Y &= \frac{1}{1 + Y \cdot N_\Gamma} Y \\
T^* Y &= \exp(\delta T_\Gamma)(Y - (Y \cdot T_\Gamma) Z).
\end{align*}
\]

But \(Y \cdot N_\Gamma \in \text{Sol}_0(Z)\) so that \((Y \cdot N_\Gamma) \circ T = Y \cdot N_\Gamma\), which yields the result. 

\[\square\]

**Corollary 6.8.** Outside the tangency and polar loci of \(Z\) and \(Y\), the sheaf of \(C\)-linear systems

\[
\begin{align*}
Z \cdot T_\Gamma &= 0 \\
\delta Y \cdot T_\Gamma &= 0 \\
Y \cdot Y \cdot T_\Gamma &= 0 \\
Z \cdot N_\Gamma + \delta N_\Gamma &= 0 \\
Y \cdot Y \cdot N_\Gamma &= 0
\end{align*}
\]

defines the \(D\)-groupoid of invariance \(\text{Aut}(Z, L(Z, Y))\). Let us define the respective transverse and tangential \(D\)-groupoids

\[
\begin{align*}
\text{Trans}(Z, Y) &:= \text{Aut}(Z, L(Z, Y)) \cap \{\Gamma : T_\Gamma = 0\} \\
\text{Tang}(Z, Y) &:= \text{Aut}(Z, L(Z, Y)) \cap \{\Gamma : N_\Gamma = 0\}
\end{align*}
\]

so that

\[
\text{Aut}(Z, L(Z, Y)) = \text{Trans}(Z, Y) \circ \text{Tang}(Z, Y).
\]

Then the corresponding \(D\)-algebroid of each factor is a sheaf of Lie algebras of dimension at most 2.

**Proof.** The first and third equations come from Lemma 6.7 (1). According to (2) of said lemma the condition \(\Gamma^* Y = d_T Z + c_T Y\), with \((c_T, d_T) \in \mathbb{C}_{\neq 0} \times \mathbb{C}\) as in Lemma 6.6, is equivalent to

\[
\begin{align*}
\frac{\exp(\delta T_\Gamma)}{1 + T_\Gamma \cdot N_\Gamma} &= c_T \\
-c_T Y \cdot T_\Gamma &= d_T.
\end{align*}
\]

Observe that because \(T_\Gamma \in \text{Sol}_0(Z)\) we have \(Y \cdot T_\Gamma \in \text{Sol}_0(Z)\) and the second equation yields \(\delta Y \cdot T_\Gamma = 0\), as well as \(Y \cdot Y \cdot T_\Gamma = 0\). Now applying \(Y \cdot \) to the first equation we obtain \(Y \cdot Y \cdot N_\Gamma = 0\). The converse direction is clear.

By construction \(T_\Gamma\) and \(N_\Gamma\) are obtained from \(\Gamma\) through the implicit function theorem, and therefore depend meromorphically on \(\Gamma\) and on none of its derivatives. Both independent sub–systems of \((\star)\) characterizes a \(D\)-groupoid, which has rank at most 2 as can be checked using \(Z \pitchfork Y\). \[\square\]
6.2.2. Proof of Theorem 6.5.

For \((\delta, Y) \in \text{TS}(Z)\), that is \(Y \in Z\) and \([Z, Y] = \delta Y\), define

\[ g_{(\delta, Y)} := \{ TZ + NY : (T, N) \text{ solution of } \bullet \} . \]

**Lemma 6.9.** The (sheaf of) Lie algebra \( g_{(\delta, Y)} \) is the \( D \)-algebra of \( G_{(\delta, Y)} := \text{Aut}(Z, \mathcal{L}(Z, Y)) \). In particular \( X \in g_{(\delta, Y)} \) if, and only if,

\[
\begin{align*}
[Z, X] & = 0 \\
[Y, X] & = d_X Z + c_X Y \text{ for some } c_X, d_X \in \mathbb{C}
\end{align*}
\]

with \( \delta d_X = 0 \).

**Proof.** \( g_{(\delta, Y)} \) is clearly a (sheaf of) \( \mathbb{C} \)-linear space. We compute

\[
[TZ + NY, \bar{T}Z + \bar{N}Y] = \begin{bmatrix} TZ, \bar{T}Z \end{bmatrix} + \begin{bmatrix} TZ, \bar{N}Y \end{bmatrix} - \begin{bmatrix} \bar{T}Z, NY \end{bmatrix} + \begin{bmatrix} NY, \bar{N}Y \end{bmatrix}
\]

\[
= (\bar{N}Y \cdot T - NY \cdot \bar{T})Z + (NY \cdot \bar{N} - \bar{N}Y \cdot N)Y.
\]

Now

\[
Y \cdot (NY \cdot \bar{N} - \bar{N}Y \cdot N) = 0
\]

\[
\delta Y \cdot (\bar{N}Y \cdot T - NY \cdot \bar{T}) = 0
\]

so that \([TZ + NY, \bar{T}Z + \bar{N}Y] \in g_{(\delta, Y)}\) and \( g_{(\delta, Y)} \) is a Lie algebra. The fact it is the \( D \)-algebra of \( G_{(\delta, Y)} \) is an immediate consequence of the system \((\bullet)\) already being linear. Take now \( X = TZ + NY \in g_{(\delta, Y)}\); a trivial computation ensures

\[
\begin{align*}
[Z, X] & = (Z \cdot N) Y + N [Z, Y] = 0 \\
[Y, X] & = (Y \cdot T) Z + (Y \cdot N - \delta T) Y.
\end{align*}
\]

Because \( Y \cdot N - \delta T \in \text{Sol}_0(Z) \cap \text{Sol}_0(Y) \) we deduce that the function is a constant \( c_X \). Likewise \( Z \cdot Y \cdot T = \delta Y \cdot T = 0 = Y \cdot Y \cdot T \) implies \( d_X := Y \cdot T \) is constant. This constant vanishes if \( \delta \neq 0 \). The converse is clear. \( \square \)

Define

\[
\hat{G} := \bigcap_{(\delta, Y) \in \text{TS}(Z)} G_{(\delta, Y)}
\]

and the corresponding \( D \)-algebroid \( \hat{\mathfrak{g}} \) which, because of Corollary 6.8, is a sheaf of Lie algebras of dimension at most 3. Indeed the hypothesis that the delta–lattice \( \Delta(Z) \) differs from \([0]\) implies that

\[
\hat{\mathfrak{g}} \cap \mathcal{O}Z = \mathbb{C}Z.
\]

The inclusion \( \text{Gal}(Z) \subset \hat{G} \) follows from the definition. The reverse inclusion is obtained by considering a \( D \)-groupoid \( G \) whose \( D \)-algebroid \( \mathfrak{g} \) contains \( Z \) and by showing that \( \hat{\mathfrak{g}} \subset \mathfrak{g} \). Let \( E \neq 0 \) be a local linear partial differential equation of order \( k \) partially defining \( \mathfrak{g} \), that is

\[
E(T, N) = \sum_{|\alpha| \leq k} \lambda_\alpha \partial^{\alpha}T + \mu_\alpha \partial^{\alpha}N,
\]
where \( k \in \mathbb{Z}_{\geq 0} \), \( \alpha \in \mathbb{Z}_{\geq 0}^2 \), \( \lambda_\alpha \) and \( \mu_\alpha \) belong to the sheaf of meromorphic functions near \((0,0)\), such that
\[
TZ + NY \in \mathfrak{g} \implies E(T,N) = 0.
\]
We assume that this equation is given near a point \( p \) outside the polar locus of \( Z \) and \( Y \), as well as outside their locus of tangency.

Because \( \mathbb{C}Z < \mathfrak{g} \) we have
\[
E(0,N) = 0
\]
whenever \( NY \in \mathfrak{g} \). We can find local analytic coordinates \((t,z)\) in which \( Z = \frac{d}{dt} \) and \( Y = e^{bt} \frac{d}{dz} \). Because \([Z, NY] = 0\) we have
\[
N(t,z) = e^{-bt} \eta(z)
\]
with \( \eta \) meromorphic near \( 0 \), and
\[
\partial^{(n,m)} N(t,z) = (-\delta)^n e^{-\delta t} \eta^{(m)}(z).
\]
The relation \( E(0,N) = 0 \) corresponds therefore to some linear differential equation \( \hat{E}(\eta) = 0 \) of a \( D \)-algebroid on a transverse disk. According to Casale–Malgrange's theorem it contains the transverse \( D \)-algebroid of the foliation. More precisely the construction of G. Casale carried out in [Cas06, Théorème 3.2] allows to recover a meromorphic vector field \( \hat{Y} \) near \((0,0)\) from \( E \) such that \([Z, \hat{Y}] = \delta \hat{Y}\) for some \( \delta \in \mathbb{C} \). If we assume \( NY \in \mathfrak{g} \) then in particular \( N \in \mathfrak{g}^{(\delta, \hat{Y})}\) and it also solves the equation \( \hat{E} \) in the local coordinates \((t,z)\).

We finally proved

\[
\text{References}
\]
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Laboratoire I.R.M.A., 7 rue R. Descartes, Université de Strasbourg, 67084 Strasbourg cedex, France

E-mail address: teyssier@math.unistra.fr

URL: http://math.u-strasbg.fr/~teyssier/