Products of characters and derived length II

by

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1 Introduction

Let $G$ be a finite group. Let $\chi$ and $\psi$ be irreducible complex characters, i.e. irreducible characters over the complex numbers. Since a product of characters is a character, $\chi\psi$ is a character. Then the decomposition of the character $\chi\psi$ into its distinct irreducible constituents $\alpha_1, \alpha_2, \ldots, \alpha_n$ has the form

$$\chi\psi = \sum_{i=1}^{n} a_i\alpha_i$$

where $n > 0$ and $a_i > 0$ is the multiplicity of $\alpha_i$ for each $i = 1, \ldots, n$. Let $\eta(\chi\psi) = n$ be the number of distinct irreducible constituents of the character $\chi\psi$. Denote by $\text{Ker}(\chi\psi)$ the kernel of the character $\chi\psi$. Also denote by $\text{dl}(\text{Ker}(\alpha)/\text{Ker}(\chi\psi))$ the derived length of the group $\text{Ker}(\alpha)/\text{Ker}(\chi\psi)$.

Given $\chi \in \text{Irr}(G)$, let $\chi$ be the complex conjugate of $\chi$, i.e. $\chi(g) = \overline{\chi(g)}$ for all $g \in G$. In Theorem A of [1], it is proved that there exist universal constants $C_0$ and $D_0$ such that for any finite solvable group $G$ and any $\chi \in \text{Irr}(G)$ we have that $\text{dl}(G/\text{Ker}(\chi)) \leq C_0\eta(\chi\chi) + D_0$. The following generalizes that result and it is the main result of this paper.

**Theorem A.** There exist universal constants $C$ and $D$ such that for any finite solvable group $G$, any irreducible characters $\chi$ and $\psi$ of $G$, and any irreducible constituent $\alpha$ of $\chi\psi$, we have

$$\text{dl}(\text{Ker}(\alpha)/\text{Ker}(\chi\psi)) \leq C\eta(\chi\psi) + D.$$

In Theorem 2.8 we prove that we may take $C = 2$ and $D = -1$ if the group $G$ in Theorem A is, in addition, supersolvable.

**Theorem B.** Let $G$ be a finite group and $\chi, \psi \in \text{Irr}(G)$ be characters. If $(\chi(1), \psi(1)) = 1$, then $\text{dl}(\text{Ker}(\alpha)/\text{Ker}(\chi\psi)) \leq 1$ for any irreducible constituent $\alpha$ of the product $\chi\psi$.  

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Thus if $(\chi(1),\psi(1)) = 1$, then the irreducible constituents of the product $\chi\psi$ are “almost” faithful characters of the group $G/\ker(\chi\psi)$.

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2 Proof of Theorem A

We will be using the notation of [2]. In addition, we write $M \subseteq N$ if $M$ is a subgroup of $N$. Also $M \subset N$ denotes that $M$ is a proper subgroup of $N$.

Definition 2.1. Assume that $G$ acts on a finite vector space $V$. We define $m(G,V)$ as the number of orbits of nonzero vectors under the action of $G$.

The key tool for the proof of Theorem A is the following.

Lemma 2.2 (Keller). There exist universal constants $A$ and $B$ such that for any finite solvable group $G$ acting faithfully on a finite vector space $V$ we have that

$$dl(G) \leq Am(G,V) + B.$$ 

Proof. See [3]. \qed

We emphasize that the result of Keller is much stronger than Lemma 2.2. He proved that the derived length of $G$ is bounded by a logarithmic function of the number of different sizes of $G$-orbits on $V$.

Definition 2.3. We define the function $f$ by

$$f(n) = An + B,$$
for any integer \( n \geq 1 \), where \( A \) and \( B \) are as in Lemma 2.2.

**Definition 2.4.** Let \( G \) be a finite group and \( M \) and \( N \) be normal subgroups of \( G \) such that \( M \subset N \). Let \( \theta \in \text{Irr}(N) \). We say that \((N, M, \theta)\) is an extreme triple of \( G \) if for any normal subgroup \( K \) of \( G \) such that \( M \subset K \subseteq N \), we have that \( \theta_K \) is irreducible but \( \theta_M \) is reducible.

Observe that given \( N \trianglelefteq G \) and a nonlinear character \( \theta \in \text{Irr}(N) \) of \( N \), we can always find \( M \triangleleft G \) such that \((N, M, \theta)\) is an extreme triple.

**Lemma 2.5.** Let \((N, M, \theta)\) be an extreme triple of \( G \) and suppose \( M \subset L \subseteq N \), where \( L \triangleleft G \) and \( L/M \) is abelian. Then \( C_{N/M}(L/M) \) is abelian.

**Proof.** By definition of extreme triple, we have that \( \theta_L \in \text{Irr}(L) \) and \( \theta_M \) is reducible. Since \( L/M \) is abelian, by Theorem 6.22 of [2] we have that there exists a subgroup \( U \) containing \( M \) such that \( U \) has prime index in \( L \) and \( \theta_U \) is reducible. Since \( \theta_U \) is reducible and \( |L : U| = p \), we have that \( \theta_U \) is a sum of \( p \) distinct irreducible characters of \( U \). Let \( \varphi \in \text{Irr}(U) \) be an irreducible constituent of \( \theta_U \).

Let \( C = C_N(L/M) \). Note that \( L \subseteq C \) and thus \( U \triangleleft C \) and \( \theta_C \in \text{Irr}(C) \). It follows that \(|C : C_\varphi| = p\) and \( C = LC_\varphi \). Since \( L \) centralizes \( C/M \), we see that \( C_\varphi \trianglelefteq C \) and \( C/C_\varphi \) is abelian. Then \( MC' \subseteq C_\varphi \) and thus \( \theta_{MC'} \) is reducible. Since \( MC' \subseteq G \), it follows that \( MC' = M \). Thus \( C' \subseteq M \) as wanted.

We introduce more notation. If \( N/M \) is a normal section of \( G \) and \( \Delta \) is a character of \( G \), we write \( S_{\Delta}(N/M) \) to denote the (possibly empty) set of those irreducible constituents \( \alpha \) of \( \Delta \) such that \( M \subseteq \text{Ker}(\alpha) \) and \( N \not\subseteq \text{Ker}(\alpha) \).
Lemma 2.6. Let \((N, M, \theta)\) be an extreme triple of \(G\), where \(N/M\) is solvable. Assume that \(\theta\) is (not necessarily irreducible) constituent of \(\Delta_N\), for some character \(\Delta\) of \(G\). Then \(S_\Delta(N/M)\) is nonempty, and the derived length of \(N/M\) is at most \(1 + f(|S_\Delta(N/M)|)\), where \(f\) is the function as in Definition 2.3.

Proof. Since \(N/M\) is solvable and non-trivial, we can choose \(L \trianglelefteq G\) with \(M \subset L \subset N\) such that \(L/M\) is a chief factor of \(G\). Also, as in the previous lemma, we know that there exists a subgroup \(U\) containing \(M\) and of prime index \(p\) in \(L\) such that \(\theta_U\) is a sum of \(p\) distinct irreducible constituents. Thus \(\theta\) vanishes on \(L \setminus U\). Since \(\theta_L\) is \(N\)-invariant, \(\theta\) vanishes on \(L \setminus D\), where \(D\) is the intersection of the \(N\)-conjugates of \(U\). Since \(M \subset D \subset L\) and \(D \lhd N\), we can choose a chief factor \(L/K\) of \(N\) such that \(D \subset K\), and thus \(\theta\) vanishes on \(L \setminus K\).

Write \(\varphi = \theta_L \in \text{Irr}(L)\) and let \(\lambda\) be a nonprincipal linear character of \(L/K\). Then \(\varphi \lambda = \varphi\), and this \(\lambda\) is a constituent of \(\varphi\), which in turn is a constituent of \(\Delta_L\). It follows that \(\lambda\) is a constituent of \(\alpha_L\), where \(\alpha\) is an irreducible constituent of \(\Delta\). In particular, we see that \(\alpha \in S_\Delta(N/M)\), which is therefore nonempty.

Set \(H = N_G(K)\). Observe that \(H \supseteq N\). All members of the \(H\)-orbit of \(\lambda\) are linear constituents of \(\alpha_L\) having kernels containing \(K\). Conversely, we show that if \(\nu\) is any linear constituent of \(\alpha_L\) such that \(K \subseteq \text{Ker}(\nu)\), then \(\nu\) lies in the \(H\)-orbit of \(\lambda\). Certainly \(\nu = \lambda^g\) for some element \(g \in G\), and thus \(\nu\) is nonprincipal. Also, since \(K \subseteq \text{Ker}(\lambda)\), we see that \(K^g \subseteq \text{Ker}(\nu)\). We assumed that \(K \subseteq \text{Ker}(\nu)\), and thus \(KK^g \subseteq \text{Ker}(\nu) \subseteq L\). Observe that \(KK^g \lhd N\) since \(K^g \subseteq N^g = N\) and \(K\) is normal in \(N\). Since \(KK^g \lhd N\) and \(L/K\) is a chief factor of \(N\), it follows that \(K = K^g\) and thus \(g \in H\). We see
now that for each $H$-orbit of nonprincipal linear characters of $L/K$, there is a member of $S_\Delta(N/M)$ that lies over all members of this orbit and over no other linear character of $L/K$. It follows that the number of $H$-orbits on $L/K$, which is the same as the number of $H$-orbits of linear characters of $L/K$, is at most $|S_\Delta(N/M)|$. By Lemma 2.2, therefore, the derived length of $H/C_H(L/K)$ is at most $f(|S_\Delta(N/M)|)$. Set $m = f(|S_\Delta(N/M)|)$. Then the derived length of $N/C_N(L/K)$ is at most $m$ and thus $N^{(m)} \subseteq C_N(L/K)$.

Lemma 2.7. Let $G$ be a finite group and $\chi, \psi \in \text{Irr}(G)$. Let $\alpha \in \text{Irr}(G)$ be an irreducible constituent of the product $\chi \psi$ and $K = \text{Ker}(\alpha)$. Then there exists some $\theta \in \text{Irr}(K)$ such that $[\chi_K, \theta] \neq 0$ and $[\psi_K, \theta] \neq 0$.

Proof. Since $K = \text{Ker}(\alpha)$, we have that $\alpha_K = \alpha(1)1_K$. Thus

$$\alpha(1)[(\chi \psi)_K, 1_K] = [(\chi \psi)_K, (\alpha)_K] > 0.$$ 

Therefore there exist $\theta, \sigma \in \text{Irr}(K)$ such that $[\chi_K, \theta] \neq 0$, $[\psi_K, \sigma] \neq 0$ and $[\theta \sigma, 1_K] \neq 0$. Since $\theta$ and $\sigma$ are irreducible characters and $[\theta \sigma, 1_K] = [\theta, \sigma] > 0$, we conclude that $\theta = \sigma$. Thus $[\psi_K, \theta] \neq 0$. \qed

Proof of Theorem A. Write $\Delta = \chi \psi$ and $n = \eta(\chi \psi)$. Note that we can assume that $\Delta$ is faithful. Let $K = \text{Ker}(\alpha)$ so that our task is to show that the derived length $dl(K)$ is bounded by a linear function on $n$. By Lemma 2.7 we can choose a character $\nu \in \text{Irr}(K)$ that lies under both $\chi$ and $\psi$. Also we see that if $N \triangleleft G$ and $N \subseteq \text{Ker}(\nu)$, then $N \subseteq \text{Ker}(\Delta)$ and thus $N = 1$.  

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Write $K_0 = K$. If $\nu$ is nonlinear, we can choose a subgroup $L$ such that $(K, L, \nu)$ is an extreme triple of $G$. Write $K_1 = L$ and let $\nu_1$ be an irreducible constituent of $\nu_L$. If $\nu_1$ is nonlinear, we can repeat the process and choose an extreme triple of $G$ of the form $(K_1, K_2, \nu_1)$. Continuing like this, we obtain a series $K = K_0 \supset K_1 \supset \cdots \supset K_r$ of normal subgroups of $G$ and characters $\nu_i \in \text{Irr}(K_i)$ such that $(K_i, K_{i+1}, \nu_i)$ is an extreme triple for $0 \leq i < r$ and where $\nu_r$ is linear. Also, we know that for each subscript $i$, the character $\nu_i$ is a constituent of $\nu_{K_i}$, and thus $\nu_i$ lies under both $\chi$ and $\bar{\psi}$. It follows that $\nu_i \nu_i$ is a constituent of $\Delta_{K_i}$. Therefore by Lemma 2.6, the sets $S_{\Delta}(K_i/K_{i+1})$ are nonempty for $0 \leq i < r$. Since they are certainly disjoint and $\alpha$ lies in none of the sets $S_{\Delta}(K_i/K_{i+1})$, we have that $r < n$. Also, writing $s_i = |S_{\Delta}(K_i/K_{i+1})|$, we see that $\sum s_i < n$.

By Lemma 2.6 the derived length of $K_i/K_{i+1}$ is at most $1 + f(s_i)$. Also, because $\nu_r$ is linear, we see that $(K_r)' \subseteq \text{Ker}(\nu_r)$. Since $(K_r)' \lhd G$, it follows that $(K_r)' \subseteq \text{Ker}(\Delta)$, and hence $(K_r)' = 1$ and $K_r$ is abelian. We see now that the derived length $\text{dl}(K)$ is at most $1 + \sum_{i=0}^{r-1} (1 + f(s_i))$. Since $f(x) = Ax + B$, we have that $\text{dl}(K) \leq A\sum s_i + rB + r + 1$. But $\sum s_i < n$ and $r < n$, and so $\text{dl}(K) < (A + B + 1)n + 1$. The proof is now complete. \qeda

**Theorem 2.8.** Let $G$ be a finite supersolvable group and $\chi \psi \in \text{Irr}(G)$ be characters of $G$. Let $\alpha \in \text{Irr}(G)$ be any constituent of the product $\chi \psi$. Then

$$\text{dl}(\text{Ker}(\alpha)/\text{Ker}(\chi \psi)) \leq 2\eta(\chi \psi) - 1.$$ 

**Proof.** Set $K = \text{Ker}(\alpha)$ and $n = \eta(\chi \psi)$. We choose a character $\nu \in \text{Irr}(K)$ that lies under both $\chi$ and $\bar{\psi}$. As in the proof of Theorem A, we obtain a series $K = K_0 \supset K_1 \supset \cdots \supset K_r$ of normal subgroups of $G$ and characters $\nu_i \in \text{Irr}(K_i)$ such that $(K_i, K_{i+1}, \nu_i)$ is an extreme triple for $0 \leq i < r$.
and where $\nu_r$ is linear. As before, we have that $r < n$ and $K_r/Ker(\chi \psi)$ is abelian.

For $0 \leq i < r$, let $L_i \triangleleft G$ such that $K_i \subseteq L_i \subseteq K_{i+1}$ and $L_i/K_i$ is a chief factor of $G$. Since $G$ is supersolvable, we have that $L_i/K_i$ is cyclic of prime order and therefore $G/C_G(L_i/K_i)$ is abelian. Thus $dl(K_i/K_{i+1}) \leq 2$ for $0 \leq i < r$. Therefore

$$dl(K/Ker(\chi \psi)) \leq \sum_{i=1}^{r} dl(K_{i-1}/K_i) + dl(K_r/Ker(\chi \psi)) \leq 2r + 1.$$ 

Since $r < n$, we conclude that $dl(K/Ker(\chi \psi)) \leq 2(n-1) + 1 = 2n - 1$ and the proof is complete. \qed

3 Proof of Theorem B

**Proof of Theorem B.** Set $K = Ker(\alpha)$. Let $\theta \in Irr(K)$ be as in Lemma 2.7. Since $[\chi_K, \theta] \neq 0$ and $[\psi_K, \theta] \neq 0$, and $K$ is normal in $G$, we have that $\theta(1)$ divides both $\chi(1)$ and $\psi(1)$. Since $(\chi(1), \psi(1)) = 1$, it follows that $\theta(1) = 1$.

Observe that $[K, K] \subseteq Ker(\theta) \subseteq K$ since $\theta(1) = 1$. Also observe that $[K, K]$ is normal in $G$ since $K$ is normal in $G$. Since $\theta$ is an irreducible constituent of $\chi$, $[K, K] \subseteq Ker(\theta)$ and $[K, K]$ is normal in $G$, we have that $\chi_{[K, K]} = \chi(1)1_{[K, K]}$. Similarly we can check that $\psi_{[K, K]} = \psi(1)1_{[K, K]}$. Thus $[K, K] \subseteq Ker(\chi \psi) \subseteq K$. Therefore $K/Ker(\chi \psi)$ is abelian and thus $dl(Ker(\alpha)/Ker(\chi \psi)) \leq 1$. \qed

**Example 3.1.** Let $G$ be a solvable group and $\chi, \psi \in Irr(G)$. Assume that

$$dl(Ker(\alpha)/Ker(\chi \psi)) \leq 1$$

for all irreducible constituents $\alpha$ of the product $\chi \psi$. Then we do not necessarily have that $(\chi(1), \psi(1)) = 1$. 

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Proof. Let $p$ be an odd prime. Let $E$ be an extra-special group of order $p^3$. Choose a character $\chi \in \text{Irr}(E)$ such that $\chi(1) = p$. Let $\psi = \overline{\chi}$. Observe that $\text{Ker}(\chi \psi) = \mathbb{Z}(E)$. Also observe that $(\chi(1), \psi(1)) = \chi(1) = p$. Since $E/\mathbb{Z}(E)$ is abelian, for all the irreducible constituents $\alpha$ of the product $\chi \psi$, we have that $\text{dl}(\text{Ker}(\alpha)/\text{Ker}(\chi \psi)) = 1$. \hfill $\square$

4 Further results

Assume that $G$ is a finite group and $\chi, \psi$ are irreducible characters of $G$ such that the product $\chi \psi$ has a linear constituent $\alpha$. Observe that $\alpha \overline{\chi} \in \text{Irr}(G)$ since $\alpha(1) = 1$ and $\chi \in \text{Irr}(G)$. Since $[\psi, \alpha \overline{\chi}] = [\chi \psi, \alpha] > 0$, we have that $\psi = \alpha \overline{\chi}$. In particular $\chi(1) = \psi(1)$. If, in addition, the group $G$ is solvable, more information is available about $\chi(1)$. Since $\alpha(1) = 1$, by Exercise 4.12 of [2] we have that $[\chi \psi, \alpha] = 1$. If $\chi \psi(1) \neq 1$, by Proposition 4.3 there exists a character $\beta \in \text{Irr}(G)$ such that $\beta \neq \alpha$ and $[\beta, \chi \psi] = 1$.

The main results of this section are Propositions 4.2 and 4.3. Those are corollaries of Theorem B and C of [1] and Lemma 4.1.

Lemma 4.1. Let $G$ be a finite group and $\chi \psi \in \text{Irr}(G)$. Set

$$
\chi \psi = \sum_{i=1}^{n} a_i \alpha_i
$$

where $n > 0$ and $a_i > 0$ is the multiplicity of $\alpha_i \in \text{Irr}(G)$ in $\chi \psi$, for each $i = 1, \ldots, n$.

If $\alpha_1(1) = 1$, then the irreducible constituents of the character $\chi \overline{\chi}$ are $1_G$, $\overline{\alpha_1} \alpha_2$, $\overline{\alpha_1} \alpha_2$, $\ldots$, $\overline{\alpha_1} \alpha_n$, and

$$
\chi \overline{\chi} = a_1 1_G + \sum_{i=2}^{n} a_i (\overline{\alpha_1} \alpha_i)
$$

where $n > 0$ and $a_i > 0$ is the multiplicity of $\alpha_1 \alpha_i$ in $\chi \overline{\chi}$. In particular, $\eta(\psi \chi) = \eta(\chi \overline{\chi})$.

**Proof.** Observe that $\alpha_1 \overline{\chi} \in \text{Irr}(G)$ since $\alpha_1(1) = 1$ and $\chi \in \text{Irr}(G)$. Since $[\psi, \alpha_1 \overline{\chi}] = [\chi \psi, \alpha_1] > 0$, we have that $\psi = \alpha_1 \overline{\chi}$.

Since $\psi = \alpha_1 \overline{\chi}$ and $\alpha_1$ is a linear character, we have that $(\alpha_1)^{-1} = \overline{\alpha_1}$ and

$$\chi \overline{\chi} = \sum_{i=1}^{n} a_i (\overline{\alpha_1} \alpha_i).$$

Observe that $\overline{\alpha_1} \alpha_i \in \text{Irr}(G)$ since $\overline{\alpha_1}$ is a linear character and $\alpha_i \in \text{Irr}(G)$.

Also observe that $\overline{\alpha_1} \alpha_i \neq \overline{\alpha_1} \alpha_j$ if $\alpha_i \neq \alpha_j$. Thus the distinct irreducible constituents of $\chi \overline{\chi}$ are $1_G = \overline{\alpha_1} \alpha_1$, $\overline{\alpha_1} \alpha_2$, $\overline{\alpha_1} \alpha_3$, ..., $\overline{\alpha_1} \alpha_n$, and $a_i$ is the multiplicity of $\overline{\alpha_1} \alpha_i$ in $\chi \overline{\chi}$. □

**Proposition 4.2.** Let $G$ be a finite solvable group. Let $\chi, \psi \in \text{Irr}(G)$ with $\chi(1) > 1$. If the product $\chi \psi$ has a linear constituent, then $\chi(1) = \psi(1)$ and $\chi(1)$ has at most $\eta(\psi \chi) - 1$ distinct prime divisors.

If, in addition, $G$ is supersolvable, then $\chi(1)$ is a product of at most $\eta(\psi \chi) - 2$ primes.

**Proof.** By Theorem C of [1], we have that $\chi(1)$ has at most $\eta(\chi \overline{\chi}) - 1$ distinct prime divisors. Also by Theorem C of [1] if, in addition, $G$ is supersolvable, then $\chi(1)$ is a product of at most $\eta(\chi \overline{\chi}) - 2$ primes. Thus the result follows from Lemma 4.1. □

**Proposition 4.3.** Let $G$ be a finite solvable group and $\chi, \psi \in \text{Irr}(G)$ with $\chi(1) > 1$. Assume that the product $\chi \psi$ has a linear constituent $\alpha_1$ and the decomposition of the character $\chi \psi$ into its distinct irreducible constituents
\(\alpha_1, \alpha_2, \ldots, \alpha_n\) has the form

\[\chi \psi = \sum_{i=1}^{n} a_i \alpha_i\]

where \(n > 0\) and \(a_i > 0\) is the multiplicity of \(\alpha_i\) for each \(i = 1, \ldots, n\).

Then \(a_1 = [\alpha_1, \chi \psi] = 1\) and \(1 \in \{a_i \mid i = 2, \ldots, n\}\).

Proof. Let \(\{\theta_i \in \text{Irr}(G)^\# \mid i = 2, \ldots, \eta(\chi \psi)\}\) be the set of non-principal irreducible constituents of \(\chi \psi\). If \(\text{Ker}(\theta_j)\) is maximal under inclusion among the \(\text{Ker}(\theta_i)\) for \(i = 2, \ldots, n\), then \([\chi \psi, \theta_j] = 1\) by Theorem C of [1]. Thus \(1 \in \{[\chi \psi, \theta_i] \mid i = 2, \ldots, n\}\). The result then follows from Lemma 4.1.

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