On the symmetry of the vacuum in theories with spontaneous symmetry breaking

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Abstract

We review the usual account of the phenomena of spontaneous symmetry breaking (SSB), pointing out the common misunderstandings surrounding the issue, in particular within the context of quantum field theory. In fact, the common explanations one finds in this context, indicate that under certain conditions corresponding to the situation called SSB, the vacuum of the theory does not share the symmetries of the Lagrangian. We explain in detail why this statement is incorrect in general, and in what limited set of circumstances such situation could arise. We concentrate on the case of global symmetries, for which we found no satisfactory exposition in the existing literature, and briefly comment on the case of gauge symmetries where, although insufficiently publicized, accurate and complete descriptions exist. We briefly discuss the implications for the phenomenological manifestations usually attributed to the phenomena of spontaneous symmetry breaking, analyzing which might be affected by our analysis and which are not. In particular we describe the mass generation mechanism in a fully symmetric scheme (i.e., with a totally symmetric vacuum), and briefly discuss the implications of this analysis to the problem of formation of topological defects in the early universe.

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I. INTRODUCTION

The understanding of theories with “spontaneous symmetry breaking” (SSB) is currently taken as belonging to what one might call “the established part” of theoretical physics, not less due to the fact that it comprises nowadays a fundamental aspect of the standard model of particle physics, and plays an important role in the understanding of well known phenomena like super-fluidity, superconductivity, and of the behavior of multitude of systems displaying phase transitions of various kinds. In fact a Nobel price was just awarded to work on the subject. It is thus surprising that something is left to be said on this subject at this point in time. In fact, it well might be that the points we will be making in this manuscript are well understood by some of the experts on the field, nevertheless we find it remarkable that no fully satisfactory exposition of the subject can be found in the literature, and that most people working in related areas do not seem to have a complete understanding of the issue at hand.

Before we take on this discussion it is convenient to reassure the reader that all the successful phenomenology usually attributed to SSB will be recovered in the picture we will present here. The well established physical conclusions will be unaltered despite the fact that we will argue that the standard picture is not only confusing but also misleading in various aspects, and that all the standard results can be understood within the context of the analysis we will be presenting throughout this manuscript.

In almost all presentations of the subject one is informed that one is dealing with a situation in which the symmetries of the theory are not shared by the ground state of the system, the standard example being of course a system with one degree of freedom, standard kinetic term and double well potential: classically, the lowest energy states correspond to the system at rest at one of the two minima of the potential. The problem arises when we want to treat the system quantum mechanically, as it is well known, the ground state does not correspond to a localized wave function at either of the two minima but rather to a wave function representing a symmetric superposition of localized wave packets. In this (finitely many degrees of freedom) example, the quantum ground state does indeed share the symmetries of the theory. When extrapolating this result to the field theory case, the questions one is forced to confront are: does the analogy with the one degree of freedom work?, if not, why not? and if it does, what happens then to all the phenomenology one learns to associate with the existence of the large set of vacua—all of them asymmetric, among which, a single one is thought to represent the state of our universe (or region thereof)—such as Goldstone bosons, the emergence of masses for particles in the standard model, and the prediction of topological defects resulting from the phase transitions in the early universe?

We will see, in this paper, that most of the standard predictions usually associated with SSB remain unchanged when these issues are dealt with, taking into account the quantum nature of the vacua, and its resulting symmetries. That is, we will argue that in many situations usually associated with a “spontaneous breaking of the symmetry”, the true vacuum is in fact symmetric, and that this modifies in no way most of the standard results usually associated with SSB. One of the few exceptions seems to be the prediction of topological defects, as inevitable result of the breaking of symmetries in the early universe.

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1 In fact the issue of which one is not deem to be relevant as they are all equivalent in the sense of being connected amongst themselves by symmetry transformations.
In fact as it is well known, no such objects have ever been observed in association with the fundamental theories of particle physics, but only in connection with the phase transitions in systems described with statistical mechanics. We will see that, in a certain sense, the last two are fundamentally different and will explain how those differences might account for the emergence of topological defects in one case and not in the other.

The paper is organized as follows. In the reminder of this section we consider the issue of symmetries of the ground state in quantum mechanical systems. In section II we consider the issue in the field theoretical case of global symmetries. In section III we give a review of the appropriate treatment of the problem in the case of gauge symmetries. In section IV we consider the problem of spontaneous symmetry breaking in statistical mechanics. In section V we examine the relevance of the issues treated here to the topic of of topological defects, and we end with a brief discussion in section VI.

A. Symmetry breaking in isolated quantum mechanical systems

Let us discuss the question of symmetry breaking in the context of standard quantum mechanics. We start by considering the case of a system with two degrees of freedom possessing a $U(1)$ symmetry: for instance a two dimensional anharmonic oscillator with Lagrangian

$$L = \frac{1}{2} m (\dot{X}_1^2 + \dot{X}_2^2) - \lambda (X_1^2 + X_2^2 - v^2)^2$$

In order to find its ground state it is useful to change to polar variables $r, \theta$ such that $X_1 = r \sin(\theta)$ and $X_2 = r \cos(\theta)$, where as usual, we must recall the limitations of this change of coordinates reflected in the fact that we must keep the identification of $\theta$ and $\theta + 2\pi$. The minimum of the potential corresponds to the circle $X_1^2 + X_2^2 = v$. Therefore, classically there is a circle-worth of ground states related by the $U(1)$ symmetry transformation. Classically, the possible ground states of the system break in this sense the $U(1)$ symmetry of the Hamiltonian. When this happens we will say that a classical spontaneous symmetry breaking (CSSB) is at play.

Let us now go to the quantum theory. The Lagrangian for the system is now written as

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \lambda (r^2 - v^2)^2,$$

and the Hamiltonian is

$$H = \frac{1}{2m} P_r^2 + \frac{1}{2mr^2} P_\theta^2 + \lambda (r^2 - v^2)^2.$$

Upon quantization the ground state of the system is:

$$\Psi_0(r, \theta) = \frac{1}{\sqrt{2\pi}} \Phi_0(r),$$

where $\Phi_0(r)$ is the ground state wave function for the one degree of freedom system with Hamiltonian $H^{(1)} = \frac{1}{2m} P_r^2 + \lambda (r^2 - v^2)^2$. In other words, the ground state wave function $\Psi_0(r, \theta)$ is evenly distributed among the values of $\theta$ and is thus invariant under rotations as is the Hamiltonian. It is easy to see that any localization of the variable $\theta$, would lead to an increase in the expectation value of the energy of the system.
If $\lambda$ is large enough so that $\Phi_0(r)$, is sharply peaked around $r = v$, then we can write some eigenstates of the Hamiltonian, which are close to the vacuum, approximately as $\Psi(r, \theta) = \frac{1}{2\pi} \Phi_0(r) e^{ik\theta}$ were $k = n/2\pi$, with $n$ integer. If $E_0$ is the energy of the ground state, then these states have approximate energies given by $E_0 + \frac{k^2}{2mv^2}$. This degree of freedom has only a kinetic term and thus would correspond to what we normally would call a “Goldstone Boson”, in a field theory analoge. In fact we can write the approximate effective Hamiltonian for this angular degree of freedom as

$$H_{eff}(\theta, P_\theta) = \langle \Phi_0 | H | \Phi_0 \rangle = \langle \Phi_0 | H^{(1)} | \Phi_0 \rangle + \langle \Phi_0 | \frac{1}{2mrv^2} P_\theta^2 | \Phi_0 \rangle \approx E_0 + \frac{1}{2mrv^2} P_\theta^2.$$  (5)

Of course the fact that the degree of freedom $\theta$ is free is already apparent from the exact Hamiltonian (3). The language of effective Hamiltonian will be more useful in the following example.

Now let’s consider a system of three degrees of freedom to illustrate the mass generating mechanism associated with “spontaneous symmetry breaking”. The Lagrangian is

$$L' = \frac{1}{2} m (\dot{X}_1^2 + \dot{X}_2^2) + (\mu/2) \dot{X}_3^2 - \lambda (X_1^2 + X_2^2 - v^2)^2 - \alpha (X_1^2 + X_2^2) X_3^2$$  (6)

Note that there is no term that is simply quadratic in $X_3$ so this degree of freedom can not be though of as associated with an harmonic oscillator. The Hamiltonian (after the same change of variables as before) can be written as:

$$H' = \frac{1}{2m} P_r^2 + \frac{1}{2mrv^2} P_\theta^2 + \frac{1}{2\mu} P_3^2 + \frac{1}{2\mu v^2} P_3 - \lambda (r^2 - v^2)^2 + \alpha r^2 X_3^2.$$  (7)

Now the wave function of the ground state of the system can be written as

$$\Psi_0'(r, \theta, X_3) = \frac{1}{2\pi} \Phi_0'(r, X_3),$$  (8)

where $\Phi_0'(r, X_3)$ is the ground state wave function for the two degree of freedom system with Hamiltonian $H^{(2)} = \frac{1}{2m} P_r^2 + \lambda (r^2 - v^2)^2 + \frac{1}{2\mu} P_3^2 + \alpha r^2 X_3^2$. Again the wave function is independent of $\theta$ and thus invariant under rotations in the $X_1 - X_2$ plane. We might use the fact that $H'(r, \theta, X_3, P_r, P_\theta, P_3) = H(r, \theta, P_r, P_\theta) + \frac{1}{2\mu} P_3^2 + \alpha r^2 X_3^2$ to write an approximate effective Hamiltonian for the degree of freedom $X_3$ in the ground state (of the other variables) as $H_{eff}'(X_3, P_3) = \langle \Psi_0 | H' | \Psi_0 \rangle$ where $\Psi_0$ is the ground state wave function of eq. (4), thus

$$H_{eff}'(X_3, P_3) = \langle \Psi_0 | H | \Psi_0 \rangle + \langle \Psi_0 | (\frac{1}{2\mu} P_3^2 + \alpha r^2 X_3^2) | \Psi_0 \rangle \approx E_0 + \frac{1}{2\mu} P_3^2 + \alpha v^2 X_3^2.$$  (9)

which now exhibits the standard harmonic oscillator form, with spring constant $K = 2\alpha v^2$. We have therefore given a non vanishing spring constant (the analog of mass in field theoretical Higgs mechanism) to $X_3$ without breaking the symmetry. This simple quantum mechanical example contains the main idea whose extension to field theory will be considered in the remainder of this manuscript. In fact we will argue in the following that in most of the cases considered paradigmatic of the phenomena of SSB, the physical vacuum, is indeed symmetric.

We just illustrated how continuous symmetries are unbroken by the ground state in standard quantum mechanics, for a large class of systems exhibiting CSSB. At this point
we want to consider the question: If symmetry is not broken by the ground state in QM, is there a characteristic feature of the ground state in systems with CSSB? The answer should be apparent:

**Observation 1:** In systems with finitely many degrees of freedom exhibiting CSSB the quantum ground state is symmetric, and therefore, highly non- semiclassical.

Roughly speaking, the ground state cannot be picked around some classical configuration as it must give equal amplitude to an infinite set of classical configurations associate to the classical ground states of the system. In particular in the example of this section is clear that the ground state wave function (4) is very far from satisfying any notion of semi-classicality. In particular the semiclassical property

$$\langle \psi | X^2_1 + X^2_2 | \psi \rangle = \langle \psi | X_1 | \psi \rangle \langle \psi | X_1 | \psi \rangle + \langle \psi | X_2 | \psi \rangle \langle \psi | X_2 | \psi \rangle + O(\hbar)$$

is highly violated for the ground state (4), as we have $$\langle \psi | X^2_1 + X^2_2 | \psi \rangle \approx v^2$$ while $$\langle \psi | X_1 | \psi \rangle = \langle \psi | X_2 | \psi \rangle = 0.$$ This is a key feature of the quantum ground state of systems with CSSB: it implies that we cannot investigate the properties of the vacuum state of such systems using semiclassical analysis. Nevertheless, there are text books where the analog of the previous condition in field theory is used to argue for the non vanishing of the vacuum expectation value of the analog of $$X_2$$ (see eq. 12); a signal of SSB [6] eq. 12.9. The point is that the ground state of a quantum system is a very quantum mechanical state, the assignment to it, of semiclassical properties is very delicate issue in general, and is mostly wrong in the context of systems with CSSB in particular.

Before going into the field theoretical context let us illustrate another important point with a different toy model. Consider a system with Hamiltonian $$H$$ defined on a two dimensional Hilbert space $$\mathcal{H}$$ with a discrete symmetry represented by a unitary operator $$P : \mathcal{H} \rightarrow \mathcal{H}$$ with $$PHP^\dagger = H$$. This toy model corresponds to a minimalistic version of the double well potential problem mentioned in the introduction. We choose a basis of two orthonormal states $$|R\rangle$$ and $$|L\rangle$$ Let us further assume that $$H$$ is symmetric under the replacement $$|R\rangle \leftrightarrow |L\rangle$$. By symmetry, one would then expect the ground state to be some linear combination of these two states with equal probabilities: $$|0\rangle = 2^{-1/2}(|R\rangle + e^{i\alpha}|L\rangle)$$. The issue is what would choose the value of the phase $$\alpha$$? The answer lies in the Hamiltonian. Suppose $$\langle R | H | L \rangle = c \neq 0$$. Let us write $$c = |c|e^{i\beta}$$ Consider then the value of $$\langle 0 | H | 0 \rangle = (1/2)[(R|H|R) + (L|H|L) + 2Re(e^{i\alpha}⟨R|H|L⟩)]$$ the assumption that H is symmetric under the replacement $$|R\rangle \leftrightarrow |L\rangle$$ implies that $$⟨R|H|R⟩ = ⟨L|H|L⟩ = h$$ where $$h$$ is a real number. Then we have $$\langle 0 | H | 0 \rangle = h + |c| \cos(\alpha + \beta)$$. So the requirement that the ground state should have the minimum expectation value for the energy leads to a unique choice for $$\alpha$$ namely $$\alpha = \pi - \beta$$.

What happens if $$c = 0$$? In that case the vacuum becomes degenerate, as the value of $$\alpha$$ is undetermined. In addition the non symmetric states $$|R\rangle$$ or $$|L\rangle$$ are also possible ground states. This situation corresponds then to the case where, if the system was prepared initially in the state $$|R\rangle$$ it would have zero probability of “tunneling” to the state $$|L\rangle$$. However, if the value of $$c$$ is determined by some physical mechanism, for which $$c = 0$$ represents some idealized situation (for example, we can think of the limiting situation where the central bump of the double well potential becomes of infinite hight adiabatically in an example with time dependent Hamiltonian), then the degeneracy of the vacuum at $$c = 0$$ can be understood as a singular feature of that limit. In other words, no matter how small $$c$$ is
the vacuum is uniquely determined and induces a unique choice of vacuum in the limit
\(c \to 0\), which is, in fact, symmetric. If \(c = 0\) does not correspond to such idealized situation
(for instance if there is a barrier which is actually of infinite height), then the degeneracy is
genuinely there. Nevertheless, among the set of ground states we still have the symmetric
states (the only exception seems to corresponds situations where the Hilbert space itself
contains no symmetric states at all, as discussed in section V.B ). In the cases where there
are degenerate ground states, one could think of evoking super-selection rules\(^3\) implying
that one should only work within one sector of the theory (e.g. a putative value for \(\alpha\) in the
example above).

**Observation 2:** There can be regions in parameter space where the ground state becomes
degenerate, yet symmetric ground states can always be found among the set of degenerate
ground states.

A similar phenomenon will take place in field theoretical systems. In fact we will show how
the ground state of QFT’s with CSSB is indeed symmetric. However, the energy of certain
non symmetric states approaches that of the symmetric ground state in the limit where
the spatial extension of the system becomes infinite. In that limit, the vacuum becomes
degenerate and non symmetric ground states exist; however, the symmetric ground state is
always there for all values of the spatial extension of the system.

Next we consider the analogous issues in the field theoretical context.

II. CONTINUOUS SYMMETRIES IN FIELD THEORIES

The simplest example of spontaneous symmetry breaking in field theories is perhaps the
linear sigma model. This is a theory of an n-tuplet of scalar fields \(\Phi_i, \ i = 1, \ldots n\), with a
global \(O(n)\) symmetry, with Lagrangian density;

\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \partial_{\mu} \Phi_i \partial^{\mu} \Phi_i + V \left( \sum_{j=1}^{n} \Phi_j \Phi_j \right),
\]

(11)

The symmetry of the theory is \(\Phi_i \to R_{ij} \Phi_j\) where \(R_{ij}\) is an orthogonal \(n \times n\) matrix. When
the potential has the typical Mexican hat form \(V = \frac{1}{4} (\Phi_i \Phi_i - v^2)^2\), one might expect to have a
spontaneously broken symmetry situation, i.e., to have a ground state which does not share
the symmetries of the theory. Indeed it is usual to see the ground state \(|0\rangle\) characterized by

\[
\langle 0 | \Phi_i | 0 \rangle = 0, \quad i = 1, \ldots n-1, \quad \langle 0 | \Phi_n | 0 \rangle = v
\]

(12)

which is clearly not invariant under the \(O(n)\) rotations. Is such symmetry breaking state
the true ground state of the system? In what follows we show that this is not the case.

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\(^2\) We must be careful not to confuse the limit \(c \to 0\) with the case \(c = 0\) as there is in general no warrantee
of “continuity”.

\(^3\) We must be careful with the usage of the words “super-selection rules” as their original meaning was to be
attached to situations where the superposition principle was to restrict the application of the superposition
principle in order to avoid arriving to contradictions. The usage was later extended beyond that original
notion.
For the sake of simplicity, and definiteness, let us consider the case of a theory of two scalar fields so \( n = 2 \), and the standard Mexican hat potential \( V = \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 - v^2)^2 \) (the general case can be dealt with in a similar fashion). In this case it is convenient to introduce a new parametrization of the fields by writing \( \phi_1 = (\rho + v)\cos(\theta) \), \( \phi_2 = (\rho + v)\sin(\theta) \), leading to a Lagrangian:

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} (\rho + v)^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda}{4} (\rho^2 + 2v\rho)^2
\]

The conjugate momenta are \( \pi_\rho = \dot{\rho} \) and \( \pi_\theta = (\rho + v)^2 \dot{\theta} \). The Hamiltonian density is then:

\[
\mathcal{H} = \frac{1}{2} \pi_\rho^2 + \frac{1}{2(\rho + v)^2} \pi_\theta^2 + \frac{1}{2} (\partial_\rho \rho)^2 + \frac{1}{2}(\rho + v)^2(\partial_\theta \theta)^2 + \frac{\lambda}{4} (\rho^2 + 2v\rho)^2
\]

This can be separated into a free part and an interacting part, with the free part given by

\[
\mathcal{H}_f = \frac{1}{2} \pi_\rho^2 + \frac{1}{2v^2} \pi_\theta^2 + \frac{1}{2} (\partial_\rho \rho)^2 + \frac{1}{2}(\rho + v)^2(\partial_\theta \theta)^2 + \frac{1}{2} m^2 \rho^2.
\]

where \( m^2 = 2\lambda v^2 \). The free theory can now be quantized in the standard fashion taking care of the special treatment for the zero mode corresponding to the space independent value of \( \theta \). To make things more transparent let us consider space to be a finite box of side \( L \) and impose periodic boundary conditions\(^4\). Before proceeding it is convenient to make a canonical transformation to the new variables

\[
\phi = v\theta \quad \text{and} \quad \pi_\phi = (1/v)\pi_\theta.
\]

The free Hamiltonian is therefore:

\[
H_f = \int d^3x \left[ \frac{1}{2} \pi_\rho^2 + \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (\partial_\rho \rho)^2 + \frac{1}{2} (\partial_\phi \phi)^2 + \frac{1}{2} m^2 \rho^2 \right]
\]

Here we must recall that this description is appropriate under the assumption that the interactions can be treated as a perturbation and as long as the fields do not leave the corresponding ranges implicit in their definitions i.e. \( \rho \in (-v, \infty) \), and with the understanding that there is an implicit identification of \( \phi \) and \( \phi + 2n\pi v \).

### A. The vacuum state in a (spatially) compact universe

As mentioned in the introduction, the spontaneous symmetry breaking phenomenon is presented as a feature of field systems of infinite extension. In order to study such statement in detail it will be convenient to define the field theory for a finite universe and study the limit in which the infrared cut-off is removed. Consequently, in this subsection we will analyze with care the free part of the linear sigma model introduced above, focussing on the situation where the field theory is defined on a finite three dimensional square box of size \( L \) with periodic boundary conditions.

\(^4\) We will investigate also what happens in the limit \( L \to \infty \) but note that the finite size of the box has no bearing on the infinite number of d.o.f. which is usually called upon as part of the explanation for the differences between field theory and the simple case of standard quantum mechanics.
We start by writing a mode decomposition of the field and momentum conjugate:

\[ \rho(\vec{x}, t) = \sum_{\vec{k}} \frac{(a_{\vec{k}}(t)e^{-i\vec{k}\vec{x}} + a_{\vec{k}}^\dagger(t)e^{i\vec{k}\vec{x}})}{\sqrt{2\omega_k L^3}}, \quad \pi_\rho(\vec{x}, t) = -i \sum_{\vec{k}} \sqrt{\frac{\omega_k}{2L^3}}(a_{\vec{k}}(t)e^{-i\vec{k}\vec{x}} - a_{\vec{k}}^\dagger(t)e^{i\vec{k}\vec{x}}) \]

where \( \omega_k = \sqrt{\vec{k} \cdot \vec{k} + m^2} \) and \([a_{\vec{k}}, a_{\vec{k}}^\dagger] = \delta_{\vec{k},\vec{k}'}\). Here we have ignored the bounds of the range of \( \rho \), because the mass associated with that field indicates that large departures from its zero value would require very large energies\(^5\), and thus could be safely ignored when interested in features of the ground state and nearby states. The sum is over the \( \vec{k} \) with components \( k_i \) of the form \( k_i = 2n_i\pi/L \) for \( n_i \in \mathbb{Z} \). Similarly we write the Fourier expansion for the field \( \phi \) except, that in this case there are two crucial differences: First and foremost, special care must be taken of the zero mode which needs to be treated separately, but second we must recall that the range of the field variables is limited to \((0, 2\pi v)\), and given that this field is massless, one can not rely on arguments as simple as those employed when considering \( \rho \). This is a fact that will have to be kept in mind when attempting a mode decomposition.

We start by writing a mode decomposition of the field and momentum conjugate:

\[ \phi = \frac{\phi_0(t)}{L^{3/2}} + \frac{1}{L^{3/2}} \sum_{\vec{k} \neq 0} \phi_{\vec{k}}(t)e^{i\vec{k}\vec{x}}, \quad \pi_\phi = \frac{1}{L^{3/2}}\pi_0(t) + \frac{1}{L^{3/2}} \sum_{\vec{k} \neq 0} (\pi_{\phi})_{\vec{k}}(t)e^{-i\vec{k}\vec{x}} \]

The Hamiltonian is then:

\[ H_f = \int \mathcal{H}_f \ d^3x = \frac{1}{2} \pi_0^2 + \sum_{\vec{k}} \frac{1}{2}((\pi_\phi)_{\vec{k}})^2 + \frac{k^2}{2} (\phi_{\vec{k}})^2 + \sum_{\vec{k}} \frac{1}{2}((\pi_\rho)_{\vec{k}})^2 + \frac{k^2 + m^2}{2} (\rho_{\vec{k}})^2 \]

We see that this is the Hamiltonian of one free particle together with an infinite collection of harmonic oscillators. This view is deceptive, because of the constraints associated with the finite range of the variable \( \phi(x) \) which, in terms of the mode decomposition leads among other things to \( \phi_{\vec{k}} \in [0, 2\pi vL^{3/2}] \). We call this issue (especially the nontrivial constraints involving collectively the multiple modes) the “range constraint” (RC). It should be clear that the precise treatment of the constraints would be a rather complicated analysis involving all the modes, except the zero mode\(^6\). This is an important point, the zero mode has a bounded range but is otherwise completely decoupled, in the constraints, from the other modes.

Let us assume for the moment that we can ignore the (non-perturbative) complications associated with the RC (for instance by considering only small amplitude excitations for all modes). This assumption will be justified below. **The first thing we must note is that we find the mass-less scalar excitations or Goldstone bosons usually associated with SSB even though the vacuum is, as we will next show, symmetric.**

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\(^5\) Note that this holds for small fluctuations of either sign.

\(^6\) The exact nature of these constraints is that if two field configurations \( \phi_1(x) \) and \( \phi_2(x) \) differ by \( 2\pi n(x)vL^{3/2} \), where \( n(x) \) is an integer-valued function, then the wave functional should satisfy \( \Psi[\phi_1(x)] = \Psi[\phi_2(x)] \). This condition implies complicated constraints for the modes in general. However, taking \( n(x) = \text{constant} \) one gets that the zero mode should be in the range \([0, 2\pi vL^{3/2}] \). The constraints on the other modes are independent of the value of the zero mode if its amplitude is constant (as in the case of the vacuum state considered below).
Furthermore, we see that in that case we could write for all the modes, except for \( \phi_0 \), the standard harmonic oscillator creation and annihilation operators, namely

\[
\phi = \frac{1}{L^{3/2}} \phi_0(t) + \sum_{\vec{k} \neq 0} \left( b_k(t) e^{-i\vec{k} \cdot \vec{x}} + b_k^\dagger(t) e^{i\vec{k} \cdot \vec{x}} \right)
\]

where \( \omega_k' = |\vec{k}| \), and the Hamiltonian becomes

\[
H_f = \int \mathcal{H}_f \, d^3x = \frac{1}{2} \pi_0^2 + \sum_{\vec{k}} \omega_k (a_k^\dagger a_k + \frac{1}{2}) + \omega_k' (b_k^\dagger b_k + \frac{1}{2}).
\]  \hspace{1cm} (19)

It is easy to see that the state with minimal energy will be characterized by \( a_k |0\rangle = 0, b_k |0\rangle = 0, \) and \( \pi_0 |0\rangle = 0. \) We can now write the corresponding vacuum state wave functional, i.e., the minimum energy eigenstate of the Hamiltonian, as

\[
\Psi_0 = \prod_k \psi^\phi_{\vec{k}}[\phi_k] \psi^\rho_{\vec{k}}[\rho_k],
\]  \hspace{1cm} (20)

where

\[
\psi^\rho_{\vec{k}}[\rho_k] = N_k \exp\left[-\frac{\sqrt{\vec{k}^2 + m^2 \rho_k^2}}{2}\right],
\]  \hspace{1cm} (21)

and

\[
\psi^\phi_{\vec{k}}[\phi_k] = N'_k \exp\left[-\frac{\sqrt{\vec{k}^2 \phi_k^2}}{2}\right], \quad \text{for all } \vec{k} \neq 0 \text{ while } \psi^\phi_{0}[\phi_0] = \frac{1}{(2\pi v L^{3/2})^{1/2}}.
\]  \hspace{1cm} (22)

where \( N_k \) and \( N'_k \) are normalization factors. The last equation shows that the zero mode of the field has a constant wave functional. Despite of the composite nature of the operators \( \phi_1 = (\rho + v) \cos(\phi/v) \), and \( \phi_2 = (\rho + v) \sin(\phi/v) \), one can easily prove that due to the contribution of the zero mode, their expectation values in the above vacuum state, vanishes for any finite value of the (in principle necessary) UV cut-off \( (L) \), namely, that,

\[
\langle 0 | \phi_i | 0 \rangle = 0 \quad i = 1, 2.
\]  \hspace{1cm} (23)

This equation shows that the symmetry is not broken by the vacuum in contrast with the naive expectation (12). Moreover, one can explicitly show that

\[
\exp(i\epsilon \int \pi_\phi(x) dx) |0\rangle = |0\rangle,
\]  \hspace{1cm} (24)

where the operator acting on the left is the \( U(1) \) rotation operator with angle \( \epsilon \) around the bottom of the Mexican hat potential.

The symmetric ground state (20) has zero charge \( \pi_0 \Psi_0 = 0. \) Note that the symmetry implies the conservation of \( \pi_0 \), namely \( [\pi_0, H] = 0 \) where \( H \) is the full Hamiltonian. Therefore, no process in a universe, preserving the symmetry of the Lagrangian (11), would be able to change or be sensitive to the energy stored in the zero mode\(^7\). The eigenvalues \( \pi_0 \) would

\(^7\) We should note however that although, we do not expect it to affect the symmetry, gravitation is expected to be sensitive to the energy of the quantum field.
label, in this sense, disjoint super-selection sectors (see footnote 3). Nevertheless, even if we take another eigenstates of $\pi_0$ the symmetry would not be broken in any sense as the value of the field $\phi$ is completely undetermined due to the uncertainty principle—alternatively, the probability density $\rho(\phi_0)$ is constant and all $\phi_0$ are equally likely. Conversely, there is no process in a symmetry preserving universe that can prepare a state in a symmetry violating configuration starting from a symmetric state.

Note that in order to satisfy equation (12) one would have to modify the zero mode wave function from a constant to some localized packet. For instance, one could take

$$\psi_0^\phi = (\pi \sigma^2)^{-1/4} \exp[-\phi_0^2/\sigma^2],$$

for some chosen value of the dispersion $\sigma$. This would clearly increase the expectation value of the energy at least by the amount $\Delta E = \langle \pi_0^2/2 \rangle \approx \sigma^{-2}/2$. A state satisfying (12) is an excited state. Essentially the reason for the energy difference between the two states lies in the extra contribution from the momentum terms to the localization of the wave packet along the bottom of the potential. It is thus clear that all things being equal the state without that energy contribution will have lower energy. That is, the true vacuum of the theory is one for which the zero mode is described by $\pi_0 = 0$, i.e. a wave functional which is uniformly spread over the bottom of the potential well, and thus fully symmetric, i.e. the state (20) above.

Leaving aside for the moment the question of the preparation of symmetry violating state, one could in fact consider a construction of perturbation theory based on a state which is ‘non symmetric’, i.e., where the zero mode is initially sharply peaked about a certain angle, and thus is not the true vacuum. That would seem to correspond to what is done in the usual treatments of this problem; however, evolution will destroy the initial sharp asymmetry of the zero mode wave function. This can be explicitly checked using elementary quantum mechanics of a free particle on a circle of radius $v$. Indeed if we start with a state for $\phi_0$ localized somewhere around the circle with spread $\sigma_{\phi_0}$, then, after a time of the order of $t_0 = \hbar^{-1}(2\pi v L^{3/2} \sigma_{\phi_0} - \sigma_{\phi_0}^2)$ the original localization is completely lost. Of course, such a state does break the symmetry. However, it does so in weaker sense as no special value of $\phi_0$ is sharply selected at all times by the corresponding probability distribution. Therefore, if we consider the zero mode as a quantum degree of freedom there is no way to make sense of a stationary state satisfying (12).

Moreover, if we include the gravitational interaction in our considerations, recall that without breaking the symmetry of (11), gravity is sensitive to the total value of energy of the system, rather than to energy differences. In other words, unless the zero mode is in the symmetric state (20), there would be excess energy stored in the field which would behave as a non trivial source of gravity, and which is therefore, in principle, detectable.

Finally, when considering a non-symmetric state, we must keep in mind that any interaction, no matter how small, that violates the symmetry (and which is not taken into account in (11)) would tend to drive the system in the direction of relaxing to the true vacuum (20). The extent of this relaxation would of course depend on the full details of the initial state of the entire multi-system, and of the interactions between its parts.

Incidentally, we should note that the issues associated with the ranges and constraints in $\phi_k$ cannot be ignored. In fact the vacuum state above, has for the variable $\phi_k$ a width $\sigma_k = 1/\sqrt{|k|}$, and that would have the support of the wave function exceeding the range of variable $\phi_k$ for $|\vec{k}| < [(2\pi)^2 v^2 L^3]^{-1}$. This would not seem to be a serious problem for nonzero

\[\text{footnote 3} \]

\[\text{footnote 8} \]

This point is not affected by the fact we can not describe the state fully due to the nontrivial constraints (RC) present among the modes $\phi_k$. 

10
modes, in any relevant situation. Recall that $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$ so that the problematic condition would correspond to $n_1^2 + n_2^2 + n_3^2 < \left[ \frac{(2\pi)^3 L^2 v^3}{v^2} \right]^{-1}$ which for large enough $L$ would not apply to any of the non-zero modes. Therefore, the perturbative assumption made here (i.e. to ignore the RC) is consistent as long as we take a sufficiently large box.

1. The infinite size limit

Let us now consider the infinite size limit i.e. $L \to \infty$. The first observation is that the localization of the wave function in the variable $\phi$ is not the relevant thing to consider as the value of the field is characterized by the variable $\theta$ whose range is $[0, 2\pi]$. Localizing the corresponding global variable in the narrow band $\delta \theta$ corresponds to localizing the variable $\phi$ in the range $\sigma = \delta \theta v L^{3/2}$ and that would increase the energy of the state by an amount $E = \frac{1}{2 v^2 L^3 (\delta \theta)^2}$. This quantity goes to zero as $L \to \infty$, as long as $(\delta \theta)$ does not go to 0 as $1/L^{3/2}$ or faster. This indicates that in the limit, we are lead to a situation in which the ground state is highly degenerate. That is, as candidates for lowest energy state, we have, on the one hand, the symmetric state, and on the other, a large collection of asymmetric states where the zero mode is localized to an arbitrarily finite degree on any arbitrary point in the range $\theta \in [0, 2\pi]$, and in fact many others.

The second observation is that the vacuum associated with the Fock construction about any of these states would correspond to a wave packet concentrated, within the allowed range for the variable $\phi_k$, only for those values of $k$ such that $|\vec{k}| > \left[ \frac{(2\pi)^2 v^2 L^3}{v^2} \right]^{-1}$. However, in the limit $L \to \infty$ they would include all the values of $\vec{k} \neq 0$.

However nothing of what we have said so far indicates which of the vacua should one use as the “true vacua”. The symmetric state is at least as good a state, as the others, and it is in fact, the one that corresponds to the true vacua for the cases where the value of $L$ is finite. The degeneracy appears only as the result of the infinite limit on the size of the box. However, in the infinite size limit, among the degenerate set of lowest energy states one might choose a state which breaks the symmetry. The situation is just the analog of the one considered in the QM example that motivated the Observation 2 in Section I A.

Finally say one wants to focus on the issue of which one of those states is most appropriate to use when attempting to represent the ground state of a particular system: in so doing we need to face several issues. First and foremost at the practical level, it is clear that when dealing with a localized experiment, the size of our universe should not matter and one might be tempted to conclude that for all practical purposes it could be taken to be infinite. However we must be careful about how this limit is to be taken:

1. should we consider the expression for the Hamiltonian with finite $L$, then take the limit, and then find its eigenstates and in particular its vacuum?, or

2. should we consider the expression for the Hamiltonian with finite $L$, then find its eigenvalues and eigenvectors, and in particular the vacuum, and then consider the limit when $L$ goes to infinity?

This is a very relevant question as it is clear, from both, the example above and that discussed in the context of Observation 2, that the two procedures do not always commute. The answer lies in the fact that what we have, in general, in such situation is a system with finite extent and for which we want to use the simplifications that arise by considering one
of the length parameters of the problem to be much larger than all the others, and thus whenever the two alternatives are different we must recognize that the realistic situation corresponds to the second one. From the above discussion, it seems clear that the possibility of a vacuum that does not share the symmetries of the theory, is, in this, the most common field theoretical context in which SSB is considered, associated not only with a system with infinite degrees of freedom, but a system with infinite spatial extent. Moreover we have seen that the analysis of the idealizations that lie behind the standard approach indicates that the consideration of SSB in the case of infinitely extended systems (i.e. the selection of an asymmetric vacuum state within option 1) is not justified when its conclusions differ from those one arrives to in taking option 2).

2. The zero mode, causality and clustering

We have seen that the question of spontaneous symmetry breaking is intimately related with the behavior of the wave functional of the zero mode. However, as shown by the above argument some of the physical contributions of the zero mode (e.g. the contribution to the energy density) become negligible in the large volume limit. Does that mean that we can simply neglect that mode in the quantization and consider it fully classical? If so, then the question of symmetry breaking will become a classical question concerning the zero mode position around the bottom of the mexican hat potential. In such context condition (12) would be a clear-cut semiclassical statement. In this subsection we show that (in the spatially compact case) the zero mode cannot be neglected in the quantization process. Doing so would be in direct conflict with local causality.

In order to see this explicitly let us compute the commutator

\[
[\phi(x), \phi(y)] = \sum_{k \neq 0} \frac{1}{2\omega_k L^3} \exp(-ik \cdot (x-y)) - \sum_{k \neq 0} \frac{1}{2\omega_k L^3} \exp(i k \cdot (x-y)) + 2i \frac{(y^0 - x^0)}{L^3},
\]

where the last term comes from the contribution of the zero mode, namely

\[
[\phi_0(x^0), \phi_0(y^0)] = [\phi_0(x^0), \exp(i(y^0 - x^0)\pi_0^2)\phi_0(x^0) \exp(-i(y^0 - x^0)\pi_0^2)] = 2i(y^0 - x^0).
\]

In the non compact case the sums become integrals, and the functional form of the commutator is in that limit \([\phi(x), \phi(y)] = D(x-y) - D(y-x)\). Invoking Lorentz invariance of the function \(D(x)\), one can easily see that the commutator vanishes if \(x-y\) is space-like. In the compact case the commutator is still zero when \(x-y\) is space-like (as expected from causality). However, the Lorentz invariance of the sums in the expressions above can not be warranted because Lorentz invariance, in this situation, is broken by the boundaries, and in fact an explicit calculation indicates that \(D(x)\) is no longer invariant and that the causality requirement is satisfied only thanks to the precise cancellation of the problematic term resulting from the sums and the contribution to the commutator coming from the zero mode\(^9\). The point is that the zero mode (which is the one that carries the information about

\(^9\) The simplest way to explicitly verify this is to work in 1 + 1 dimensions. Taking \(L = 1\) space-time events are labeled by \(x = (\varphi, t) \in S^1 \times \mathbb{R}\) and \(k = n \in \mathbb{Z}\). We have the following identities for the first series in (25):
the symmetry of the vacuum) is a genuine quantum degree of freedom for any value of the volume and must be treated accordingly in considering the infinite volume limit.

However, if the zero mode of the chosen vacuum state is in a quantum state with non-vanishing spread in \( \phi_0 \) the clustering property is violated. This can be seen from the computation of the equal time correlation function \( \langle \phi(x,t) \phi(y,t) \rangle \), namely, using equation (19) we have

\[
\langle \phi(x,t) \phi(y,t) \rangle = \frac{1}{L^3} \phi_0^2 + \frac{1}{L^3} \sum_{\vec{k} \neq 0} b^\dagger_k b_k \exp(-i\vec{k} \cdot \vec{x}) \exp(i\vec{k} \cdot \vec{y}) = \frac{1}{L^3} \langle \phi_0^2 \rangle + \delta(\vec{x}, \vec{y})
\]

which means that the correlation

\[
\Delta(x, y) = \langle \phi(x, t) \phi(y, t) \rangle - \langle \phi(x, t) \rangle \langle \phi(y, t) \rangle = \langle \phi_0^2 \rangle / L^3 + \delta(x, y) - \langle \phi_0 \rangle^2 / L^3 = \sigma_{\phi_0}^2 / L^3 + \delta(x, y)
\]

(26)

which does not depend of \( x \) and \( y \) as long as the points are different. Here \( \sigma_{\phi_0} \) denotes the width of the wave function of the zero mode. We note that if the zero mode is in the minimum energy state, i.e., the wave function \( \Psi(\phi_0) = 1/(2\pi v L^{3/2}) \) (recall that from its definition (17) one has \( 0 \leq \phi_0 \leq 2\pi v L^{3/2} \)) then \( \sigma_{\phi_0}^2 = (1/3)\pi^2 v^2 L^3 \). Hence, in the true vacuum there are long distance correlations given by

\[
\Delta(x, y) = \frac{\pi^2 v^2}{3} + \delta(x, y).
\]

(27)

Thus the clustering property is violated independently of the size of the universe. Notice that if we were to consider the lack of clustering as a problem we must be aware that it will not be solved by considering the translationally invariant state with sharp orientation in \( \phi_0 \) (the quantum counterpart of the state usually taken as the emblematic state of SSB) as the ground state, because in that case the vacuum will violate the clustering property when considering the n-point functions of \( \pi(x) \) rather than \( \psi(x) \): In fact it is easy to see that for a state which has a width \( \sigma_{\pi_0}^2 \) in the variable \( \pi_0 \), one has:

\[
\Delta_\pi(x, y) = \langle \pi(x, t) \pi(y, t) \rangle - \langle \pi(x, t) \rangle \langle \pi(y, t) \rangle = \langle \pi_0^2 \rangle / L^3 + \delta(x, y) - \langle \pi_0 \rangle^2 / L^3 = \sigma_{\pi_0}^2 / L^3 + \delta(x, y)
\]

and, the Heisenberg uncertainty relation between \( \psi_0 \) and \( \pi_0 \) (note that these two quantities have standard commutation relations) warrantees that a sharp value of the former requires a large value of the uncertainty of the latter. Therefore, we must conclude that the clustering violation exhibited here—and in particular the \( L \)-independent long distance correlation (27) produced by the symmetric ground state (20)—is a clear-cut manifestation of the quantum nature of the zero mode that survives the infinite size limit.

\[
\frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{2k} \exp(-ik \cdot x) - \frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{2k} \exp(ik \cdot x) = \\
- \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(n(\varphi + t))}{n} - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(n(-\varphi + t))}{n} = -\frac{1}{4\pi} \log \frac{1-e^{-i(\varphi + t)}}{1-e^{i(\varphi + t)}} + \frac{1}{4\pi} \log \frac{1-e^{-i(-\varphi + t)}}{1-e^{i(-\varphi + t)}} = \\
\frac{it}{2\pi} + \frac{1}{2} \left( \text{Int}\left(\frac{t+2\varphi}{2\pi}\right) + \text{Int}\left(\frac{t-2\varphi+4\pi}{2\pi}\right) \right),
\]

where \( \text{Int}(\cdot) \) denotes the integer part function. With all this we see that the zero mode contribution cancels the term linear in \( t = x^0 - y^0 \) and \( [\phi(x), \phi(y)] \) vanishes at space-like separated points.
We have illustrated with a simple model how no spontaneous symmetry breaking is possible in QFT’s defined on spatially compact space-times. The only way to have the ground state breaking the symmetry of the system would be to have a zero mode that behaves classically. In the compact case the latter possibility is inconsistent as it would lead to conflict with local causality. However, the situation changes drastically in the non-compact case where the technical necessity to impose certain boundary conditions in the mathematical treatment of problem, ‘freeze’ the zero mode at some position around the Mexican hat potential. That is, in this treatment the boundary conditions requiring (quantum) fields to decay at spatial infinity, simply drop the zero mode from the set of (quantum) degrees of freedom from the onset. Consequently, the ground state of such model breaks the symmetry in a way that is mathematically in strict correspondence with the CSSB scenario: what breaks the symmetry is the localization of the zero mode which is a classical degree of freedom. The viewpoint taken in such treatment is essentially perturbative in nature. In other words, one quantizes fluctuations around some classical solution of the field equations which violates the symmetry from the beginning: the homogeneous field configuration defined by a point on the bottom of the Mexican hat potential in our example.

In view of our discussion of the previous sections it is clear that the symmetry breaking being considered here is introduced by our mathematical limitations concerning the treatment of quantum fields in a non-compact space. Indeed, one could decide to model the physics that is tested in the lab by assuming that the universe is spatially compact (yet very large). It is clear that such model will introduce all sorts of complications of practical nature; it will for instance make the definition of scattering theory rather involved. However, it should be clear that these formal difficulties are irrelevant from the physical viewpoint, and that for sufficiently large spacial dimensions all the local phenomenology of, say, particle physics, could be recovered. Therefore, ignoring some practical difficulties, the alternative treatment in terms of a compact space, indicates that the symmetry is retained by the ground state. Thus, our view is that in the standard treatments one is not really justified in stating that the symmetry is broken, because the symmetric states have been removed from consideration by our boundary conditions (the latter being a choice of the physicist and not of nature).

In the references [6] and [22] the authors formally argue that the state
\[ |\epsilon\rangle \equiv \exp(i\epsilon \int \pi_0(x)dx)|0\rangle \]
is in fact orthogonal to \(|0\rangle\): a manifest illustration of the breaking of the symmetry by the ground state. This is done by using a “regulated” expression for the integral in the argument of the above exponential, namely:
\[ \pi_0^{(L)} \equiv \int e^{-x^2/L^2} \pi_0(x)dx. \]
We would like to note that the formal character of such derivation is very misleading.
subsection), but at the same time, demands us to restrict ourselves to the algebra of local operators, by arguing that in practice all of our laboratories and apparatuses are of finite extent. One is, *ab initio*, setting up the problem in a way where the zero (or homogeneous) modes of the system, would end up being treated classically. We have seen that this is not justified by physical considerations, and not needed for the construction of the quantum theory.

Finally we note, that in fact, by showing that Nambu-Goldstone bosons do emerge under the conditions usually thought to be associated with SSB even if the vacuum is symmetric, the analysis we have presented opens a path for reconsidering the implications of the so called Mermin-Wagner-Coleman theorem [15]. What the theorem actually shows that, in 1+1 dimensions, the vacuum state must be invariant under any continuous symmetry of the action. This is often interpreted as implying the absence of Nambu-Goldstone bosons. As we have shown, it is generically the case that the vacuum is invariant under the continuous symmetries, and that this is by no means in conflict with the existence of Nambu-Goldstone bosons associated with the symmetry. What is special about two dimensions is that there is no way of making sense of a mass-less scalar field in two dimensional Minkowsky space-time[16]. The problem is closely related to infrared divergences that seem impossible to regularize in any sensible way. These divergences imply vacuum must be invariant in [15]. The present work indicates that the option of Nambu Goldstone bosons (in a theory that is not free) when the vacuum is invariant, has not been ruled out by that theorem. On the other hand, there is closely related situation (which could be consider as an IR regularization of the former although with a breaking of Lorentz invariance) that clearly allows Nambu-Golstone excitations: the spacially compat model studied in Section II A. In such case, the vacuum is perfectly symmetric and mass-less Nambu-Goldstone bosons are present.

C. The case of approximate symmetries.

It is clear that if the symmetry one is considering is only an approximate symmetry, a big part of the analysis we have been making throughout this manuscript would not longer apply. On the other hand, physicists in various fields do often consider symmetries that are only approximate symmetries of the system they want to study. The reason for that is the idea that one can consider, as a starting point of the analysis, the case where the symmetry is exact, which is in fact often easier to study than the realistic situation, and where the latter is regarded as involving the “small corrections” associated with the say, Lagrangian terms, that break the symmetry and are to be treated by a suitable perturbative analysis. Perhaps the most famous of these cases corresponds to the so called “chiral symmetry” in the light quark sector of the standard model.

In considering these situations we should be careful to avoid confusing the sequence in which the analysis is carried out with any sort of implicit time sequence describing when things do occur in nature.

In order to discuss these situations it is convenient to have a specific example in mind, and to that end we consider the model of Section II were we add to the Lagrangian a small perturbation that explicitly breaks the $O(n)$ symmetry of that example. We take for instance a linear term in one of the fields, such as $a^3 \Phi_1$ where $a$ has dimension of mass and a value that is small as compared with all other nonzero mass scales of the system. It is clear that this modification leads to potential that resembles a mexican hat but with a slight tilt. What used to be the flat bottom of the potential, has now acquired a small
inclination. The standard way to approach this problem is to start off by disregarding the small term that explicitly breaks the symmetry, and to study the symmetric case which has the characteristics that are thought to lead to the phenomena of SSB. Among these we have, for instance, the emergence of mass-less Nambu-Goldstone bosons. The perturbation analysis then indicates that the symmetry breaking term would give the Nambu-Goldstone bosons a small mass characterized by the parameter $a$. As a result of the full analysis one ends up with a picture where the vacuum is not symmetric and where there is a set of very light particles called pseudo-Nambu-Goldstone Bosons, which, in the case corresponding to the chiral symmetry mentioned above, are identified with the light mesons (specifically the three pions, corresponding to the $SU(2)$ chiral symmetry of the pair of almost massless quarks $u$ and $d$, or to the $SU(3)$ chiral symmetry leading to the octet of light mesons involving also the quark $s$), the analysis of which was recently awarded the physics Nobel Prize.

The fact is, however, that as we have seen, one can carry out exactly the same type of perturbative analysis but starting out instead with the recognition that after the identification of the approximate lagrangian (obtained by setting the parameter $a$ to zero), we have an exact symmetry and that the ground state is thus symmetric$^{10}$, as discussed in Section II. This situation leads nevertheless to the conclusion about the existence of the corresponding Nambu-Goldstone bosons. Once having treated this approximated description in a rigorous way, we can next use it as the starting point of the perturbation characterized by a nonzero value of the parameter $a$. This perturbative analysis would then lead to two very important conclusions: The first one is the acquisition of a small mass by the initially (in the sense of the analysis not of timing of events in nature) mass-less bosons, of a small mass characterized by the parameter $a$ and the second, is a deformation of the vacuum, which will now be characterized among other things by the selection of the $\Psi_1$ field direction as the one characterizing the vacuum expectation value of the fields, in accordance with the equation 12. The point is that we end again with the same final picture: non-symmetric vacuum, and light pseudo Nambu Goldstone Bosons, without ever coming into conflict with the overall analysis of this manuscript. Moreover by doing so, we gain not only conceptually, but also in the understanding that the picture is independent of whether the universe is finite or not.

At this point we would like to discuss a simple issue that might lead to confusion in distinguishing the physics associated to the limit $a \to 0$ with that corresponding to the symmetric case $a = 0$. For simplicity let us focus on the QM model of Section I A where we add to the Lagrangean (1) a linear term $aX_1$ (this amounts to the analog explicit breaking of the $U(1)$ symmetry referred to above that slightly tilts the mexican hat potential along the $X_1$ direction). The effective Hamiltonian for the angular degree of freedom in equation (5) acquires an extra term $\delta = a \nu \sin(\theta)$ which can be used to evaluate in first order perturbation theory the new ground state wave function for the variable $\theta$, which is given by $\Psi_\theta \approx$

$^{10}$ In particular we have shown that in the case where the box is finite ($L$ is finite), then there is no degeneration in the ground state, and that it is exactly symmetric. It would be highly undesirable to take the position (of relying on the viable alternative of choosing by hand one of the asymmetric vacua, that we saw, do exist when the universe is infinite) that the size of the Universe has any bearing on whether the analysis of the breaking of the chiral symmetry is correct or not. Indeed, relying on such posture one would be lead to conclude that as “we know that the symmetry is spontaneously broken” the universe must be finite.
In order to separate the mathematical issue from the physics, consider the set real valued nonnegative $\lambda$. Thus for large values of $\lambda$ and for $a$ small, we can write the ground state of the full asymmetric theory, approximately as $\Psi_0'(r, \theta) = \Phi_0(r) \frac{1}{\sqrt{2\pi}} (1 - 2a m v^3 \sin(\theta)) + O(a^2)$, which can be contrasted with the ground state of the symmetric theory given by equation (4). Thus we have (for $a > 0$) that

$$\langle 0 | X_1 | 0 \rangle = -v + O(a), \quad \langle 0 | X_2 | 0 \rangle = 0 \quad \forall \quad 1 >> a > 0.$$  

Therefore, even when the ground state is symmetric at $a = 0$, the limit $\lim_{a \to 0} \langle 0 | X_1 | 0 \rangle = -v$, and $\lim_{a \to 0} \langle 0 | X_2 | 0 \rangle = 0$ yields non vanishing order parameters. Note that the wave functional of the ground state converges pointwise to the symmetric state (4) when $a \to 0$. The above situation is a prototypical example of non commutativity of integrals with limiting procedures. It is clear that the above argument can be directly translated to the case of linear sigma model of Section II and to any field theoretical situation where symmetries are only approximate. Therefore, the fact that our analysis implies the exact symmetry of the ground state in the symmetric world ($a = 0$) is by no means in contradiction with the existence of non vanishing order parameters in a world with approximate symmetries (in the limit $a \to 0$).

An important example where the above discussion applies is the famous case of the chiral symmetry in QCD. Ignoring the details of the electroweak Lagrangian, as well as the heavy quarks, we can write an effective Lagrangian for the up-down sector as follows:

$$\mathcal{L} = -\frac{1}{4} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] + \sum_{i = u, d} \left[ \psi_R^{(i)} D_\mu \gamma^\mu \psi_R^{(i)} + \psi_L^{(i)} D_\mu \gamma^\mu \psi_L^{(i)} \right] + \sum_{i = u, d} m_i \left[ \bar{\psi}_R^{(i)} \psi_L^{(i)} + \bar{\psi}_L^{(i)} \psi_R^{(i)} \right] \quad (29)$$

where $F_{\mu\nu}$ is the Lie Algebra valued field strength tensor of the gauge fields, $\psi_R^{(i)}, \psi_L^{(i)}$ are the right and left handed components of the quark field with flavor $i$, $D_\mu$ are the color gauge covariant derivatives, $\gamma^\mu$ the Dirac matrices, and $m_i$ the quark masses. This Lagrangian has besides the Poincaré symmetries, the $SU(3)$ color gauge symmetry (whose indices we are not explicitly shown for simplicity), and the two $U(1)$ flavor symmetries. If we set the two quark masses to the same value the Lagrangian has an additional $SU(2)$ global flavor symmetry and if we set these masses to zero it has an $SU(2)_L \times SU(2)_R$ where the first term corresponds to the global flavor symmetry for the left handed components, and the second for the right handed components. In this later situation we should have two $SU(2)$

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In order to separate the mathematical issue from the physics, consider the set real valued nonnegative functions on the line, viewed as probability distributions of the variable $x$ and the corresponding average $\langle x \rangle$. For concreteness take a Gaussian distributions of width $1/a$ and centered at, say $x = 5$. It is clear that the mean value $\langle x \rangle = 5$ independently of the value of $a$, so it follows that $\lim_{a \to 0} \langle x \rangle = 5$. However, the distribution corresponding to $a = 0$ is the constant distribution for which the mean value of $x$ is not defined. In order to avoid this issue consider now distributions on a circle given by (periodic functions) $f(\theta)$ for $\theta \in [0, 2\pi]$. Now the expectation value will be finite for any bounded $f(\theta)$. The analog of the previous example are for instance Gaussian distributions with center in $\theta = \pi$ and width $1/a$. As before $\langle \theta \rangle = \pi$ and $\lim_{a \to 0} \langle \theta \rangle = \pi$. In the limit $a \to 0$, however the distribution is constant and thus gives to all values of $\theta$ the same likelihood, and it would make no sense to interpret this as indicating that somehow the symmetry under rotations in the circle is broken by the constant distribution. The point of course is that in such case the uncertainty in the value of $\theta$ is equal to the full range of its possible values.
Lie algebra valued conserved currents, which is customary to write the combinations $J_\mu^a(x)$ and $J_\mu^a(x)$. The first stands for the current (called the vector current) is associated with the simultaneous identical $SU(2)$ rotation of both the left and right handed components, while the second (the axial vector current or axial current in short) corresponds to the rotation of the left handed components with the inverse of the rotation that is applied to the right handed components.

The basic idea for the treatment of most problems in this area is to consider the realistic case where $m_i \neq 0$ as a small perturbation from the mass-less case which as we saw has an extensive set of symmetries. This leads one to write at the phenomenological level the well known expression characterizing of the partial conservation of axial current (PCAC):

$$\langle 0|\partial_\mu J_\mu^a(x)|k, b\rangle = -m_\pi^2 f_\pi \delta^{ab} \psi_k(x)$$

where $J_\mu^a(x)$ is the axial current where $|k, b\rangle$ is a one pseudo-Goldston boson (pion) state with momentum $k, b = 1, 2, 3$ is an $SU(2)$ Lie algebra index, $m_\pi$ is the Pion mass, $f_\pi$ is the pion decay constant, and $\psi_k(x)$ is the normalized wave function for the scalar particle of momentum $k$.

It is often argued that when considering the behavior of the equation above in the limit when the quark masses go to zero, there left hand side must vanish, and that there are two options for the way the right hand side should vanish. Either $m_\pi \to 0$ or $f_\pi \to 0$. The standard posture is to say that is that $m_\pi \to 0$ corresponding to the mass-less Goldstone bosons characteristic of SSB, while $f_\pi$ remains finite in this limit, something that is associated with the breaking of the symmetry by the vacuum of the theory. Indeed, one can characterize the pion decay constant as an order parameter of the form

$$f_\pi^2 \delta^{ab} = -\frac{i}{3} \int d^4x \langle 0|T(J_\mu^a(x) \cdot J_\mu^b(0))|0\rangle.$$  \hspace{1cm} (31)

The reason one calls the previous expression an order parameter is that it vanishes if the vacuum is invariant under an axial transformation. More explicitly, under an axial transformation $J_\mu^a$ is ‘rotated’ as a vector in one direction while $J_\mu^b$ is ‘rotated’ the same amount but in the opposite direction. Note that there is no non-trivial bi-linear invariant under transformation that can be constructed in terms of the left and right currents. Therefore, the operator whose expectation value we are computing in the previous expression has no singlet component under axial transformations and its expectation value in symmetric states should vanish.

Consequently, the fact that the pion decay rate $f_\pi$ measured in the lab is large (in comparison to the light quark masses) could be (misleadingly) taken as an observational evidence of the existence of a spontaneous symmetry breaking by the QCD vacuum. However, as we point out above even when the ground state (e.g. in a spatially compact universe) might converge to a symmetric state in the limit $m_i \to 0$ the order parameters of the kind described in the equation (31) might remain large as the limit is taken.

In other words, the limit

$$\lim_{m_i \to 0} \left( -\frac{i}{3} \int d^4x \langle 0|T(J_\mu^a(x) \cdot J_\mu^b(0))|0\rangle \right)$$

which is identified with the non zero quantity $f_\pi^2 \delta^{ab}$ should not be confused with the evaluation of the LHS of (31) in the theory with $m_i = 0$ which does
vanish. It is worthwhile pointing out that the uncertainties in the vacuum state of the operator \( \int d^4x T(J_a^L(x) \cdot J_b^R(0)) \) grow as the symmetry breaking parameters go to zero (see footnote 11). As the light quark masses are very small, this may have some phenomenological implications whose study is outside the scope of this paper.

From the point of view discussed in this manuscript, the case of chiral symmetry can be concisely described by saying that the explicit breaking of the symmetry associated with the light quarks masses leaves an imprint in the vacuum of the theory that does not disappear smoothly in the limit when these masses are artificially made to tend to zero. The result is that the correct results can be obtained either by starting off with an approximation where the light quark masses are taken as zero and maintaining (without justification at this level) the non-symmetric nature of the vacuum, and then using perturbation theory to treat the nonzero value of those masses, or alternatively, and more transparently, following the approach discussed above.

We now turn for completeness to a brief review of the situation of spontaneous symmetry breaking in gauge theories, where, in contrast to what happened in the case considered here, we find ourselves in complete agreement with the analysis found in [14].

### III. GAUGE SYMMETRIES AND THE HIGGS MECHANISM

The situation of SSB of gauge theories could be described in a single paragraph: as it is well known gauge degrees of freedom are not physical. They are not accessible to any physical interaction and are not governed by any equation of motion. They are simply redundant fields that enter our mathematical description of certain physical interactions. Consequently, a gauge symmetry cannot be broken not only by the vacuum, but by any state in the theory. Nevertheless the main phenomenological features one associates with a SSB in gauge theories (i.e. mass generation for gauge bosons and for fermions) remain valid despite this fact. This statement resumes the conceptual content of this section and the treatment of [13].

In those cases where the physical degrees of freedom can be explicitly identified, the illustration of the above statements is obvious when expressing the dynamics in terms of gauge invariant quantities. In this section we will study this issue in the Abelian Higgs model where one encounters the Higgs phenomena of mass generation for gauge fields. The simplest example is provided by a theory with a $U(1)$ gauge symmetry, gauge field $A_\mu$ and a charged scalar field $\phi$ with Lagrangian density;

\[
\mathcal{L}_{AH} = D_\mu \phi D^\mu \phi^* + V(\phi^* \phi) + F_\mu^\nu F^{\mu\nu}
\]

where $D_\mu \phi = \partial_\mu \phi + ig A_\mu \phi$ and $F_\mu^\nu = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge transformations of the theory correspond to $\phi \rightarrow \phi e^{ig \alpha(x)}, A_\mu \rightarrow A_\mu - \partial_\mu \alpha$. It is clear that the Lagrangian is gauge invariant.

We can think of the states as represented by wave functionals $\Psi[\phi, A_\mu]$. When the potential has the typical Mexican hat form $V = \lambda(\phi^* \phi - v^2)^2$ is when we expect to have a spontaneously broken symmetry situation, i.e. to have a ground state which does not share the symmetries of the theory. However in this case, this is not only an incorrect choice of the ground state but is also a choice that is not allowed by the constraints of the theory. This is particularly transparent in the Hamiltonian formulation of the above theory. Upon
the 3+1 decomposition the action becomes

\[
S = \int dt \int d^3 x \, \tilde{E} \tilde{A} + \pi \dot{\phi} - H - A_0 (\tilde{\nabla} \cdot \tilde{E} + i g (\phi \pi - \phi^* \pi^*))
\]  

(34)

where \(\pi\) and the electric field \(\tilde{E}\) are the momenta conjugate to \(\phi\) and \(\tilde{A}\) and the Hamiltonian density \(H\) takes the form

\[
H = \frac{1}{2} \left( \tilde{E}^2 + B^2 + \pi \pi^* + (\tilde{\nabla} \phi + i g \tilde{A} \phi) \cdot (\tilde{\nabla} \phi^* - i g \tilde{A} \phi^*) \right) + V(\phi \phi^*),
\]

with \(\tilde{B} = \nabla \times \tilde{A}\). The zero component of the vector potential \(A_0\) is a Lagrange multiplier associated with the Gauss constraint:

\[
\tilde{\nabla} \cdot \tilde{E} + i g (\phi \pi - \phi^* \pi^*) = 0
\]  

(35)

It is easy to check that a classical solution minimizing the total energy of the system is given by \(\tilde{A} = \nabla \alpha, \tilde{E} = 0, \pi = 0\) and \(\phi = \exp(i g \alpha) \phi_0\) for \(\phi_0 = \text{constant}\) corresponding to the minimum of the Mexican hat potential \(V(\phi \phi^*)\), and \(\alpha(\vec{x}, t)\) and \(A_0(\vec{x}, t)\) arbitrary space-time fields. The fields \(A_0(\vec{x}, t)\) and \(\alpha(\vec{x}, t)\) are pure gauge degrees of freedom: they are not observable and their dynamics is undetermined. They have no physical reality, since they cannot be measured by any physical system constructed out of \(U(1)\) gauge invariant interactions. This is so unless the gauge symmetry is truly broken, which means that the full Lagrangian of nature breaks explicitly \(U(1)\) gauge symmetry\(^{12}\). Therefore, the distinction between different choices of \(A_0(\vec{x}, t)\) and \(\alpha(\vec{x}, t)\) only exist in the notebook of the physicist but have otherwise no objective meaning. Mathematically, for any two different choices of fields \(A_0(\vec{x}, t)\) and \(\alpha(\vec{x}, t)\) we do not have different solutions to the theory above but one and the same solution written in different ‘field coordinates’ or gauges.

This is why the solution written above does not break the gauge symmetry in any way. Only if we think of \(\alpha\) as a physical quantity then we can be mislead to the conclusion (unfortunately found in many basic textbooks on the subject) that a certain point on the bottom of the Mexican hat potential is selected by the classical solution. Customarily, one represents the classical solution written above in a particular gauge (e.g. \(\alpha(t, \vec{x}) = 0\)); however, the “breaking of the symmetry” is just in the choice of gauge and has nothing to do with any dynamical consideration (such as the form of the potential). Notice that the situation is different from the global symmetries case already at the classical level. Even classically, there is no physical meaning to saying that there is a classical solution sitting ‘somewhere’ around the \(S^1\) defining the minimum of the potential. Physically, each point around the \(S^1\) (gauge orbit) is to be identified as one and the same state!

The situation in the quantum theory is exactly the same as in the classical one. This is particularly transparent in the Dirac canonical quantization of the gauge theory. Let us illustrate this in the case of our electromagnetic example. What follows is rather formal as in conventional treatments one starts by fixing the gauge in some way which eliminates the question by construction from the starting point\(^{13}\).

\(^{12}\) This is not the case in the Higgs mechanism used in the construction of the standard model.
\(^{13}\) A manifestly gauge invariant treatment is in principle possible in terms of some non perturbative treatment such as that of lattice gauge theory.
In the Dirac program one defines the so-called auxiliary Hilbert space $\mathcal{H}$ of wave functionals of the chosen configuration variable (e.g. $\Psi[A, \phi, \phi^*] \in \mathcal{H}$). The canonical variables also become self adjoint operators: in the polarization chosen here $A$ and $\phi$ act by multiplication, while the electric field $E = -i\hbar\delta/\delta A$ and $\pi = -i\hbar\delta/\delta \phi$. The (first class) constraints of the theory are promoted to self adjoint operators which become the generators of infinitesimal gauge transformations in the quantum theory. Finally, physical states are required to be annihilated by the (first class) constraints. So in our case one requires

$$\left[ \int_{\mathbb{R}^3} i\hbar \vec{\nabla} \alpha \cdot \frac{\delta}{\delta A} - g\hbar \alpha (\phi \frac{\delta}{\delta \phi} - \phi^* \frac{\delta}{\delta \phi^*}) \right] \Psi[A, \phi, \phi^*] = 0, \quad \forall \alpha(x), \quad (36)$$

where $\alpha(x)$ ranges over a complete set are suitable test fields.

The previous expression can be written in a more familiar form. It just requires that

$$\Psi[A, \phi, \phi^*] = \Psi[\vec{A} - \vec{\nabla} \alpha, (1 + i\alpha)\phi, (1 - i\alpha)\phi^*], \quad \forall \alpha \quad (37)$$

which is just the condition that the state be gauge invariant. Therefore, physical states cannot break the gauge symmetry of the theory independently of whatever the form of the interaction potential appearing in the Lagrangian. This conclusion is general as long as the Lagrangian that defines the theory contains gauge symmetries. In particular, the state minimizing the energy of the system must satisfy the above condition which means that it cannot correspond to the field being “localized” anywhere on the bottom of the Mexican hat potential. A more appropriate image is that of a wave function whose amplitude is exactly homogeneously distributed around the hat.

We can proceed further by changing the coordinates in field space (the space of functions $\mathcal{F}$ on which the wave functional is defined). In the discussion above $\mathcal{F}$ was taken as the space of triplets $(\vec{A}, \phi, \phi^*)$ where $\vec{A}$ is a smooth vector field on $\mathbb{R}^3$ and $\phi$ is a smooth complex scalar field on $\mathbb{R}^3$. We will re-parametrize $\mathcal{F}$ so that now each point is represented by $(\vec{A}, f, \theta)$ where $f = |\phi|$ and $\theta = \arg(\phi)$. We must of course be mindful of the multiple parametrization associated with the change $\theta \rightarrow \theta + 2\pi$. Now, we can parametrize the physical configuration space in terms of the gauge invariant fields

$$\vec{C}(x) = \vec{A}(x) - \vec{\nabla} \theta(x) \quad \text{and} \quad f(x) \quad (38)$$

In accordance with the discussion above the functional representing any state of the system must satisfy:

$$\Psi[\vec{C}, f, \theta] = \Psi[\vec{C}, f, \theta + \alpha] = \Phi[\vec{C}, f], \quad \forall \alpha \quad (39)$$

i.e. physical states are independent of $\theta$ which is a pure gauge degree of freedom. In fact in terms of the new variables the Gauss constraint simply becomes $\pi_\theta = 0$, where $\pi_\theta$ is the momentum conjugate to $\theta$. The Hamiltonian density acting on such states takes the form:

$$\mathcal{H} \Psi[\vec{C}, f, \theta] =$$

$$= \frac{1}{2} \left( -\hbar^2 \frac{\delta^2}{\delta \vec{C}^2} + (\vec{\nabla} \times \vec{C})^2 - \hbar^2 \frac{\delta^2}{\delta f^2} - \frac{\hbar^2}{f} \frac{\delta^2}{\delta \phi^2} + (\vec{\nabla} f)^2 + f^2 \vec{C} \cdot \vec{C} + V(f^2) \right) \Psi[\vec{C}, f, \theta]$$

$$= \frac{1}{2} \left( -\hbar^2 \frac{\delta^2}{\delta \vec{C}^2} + (\vec{\nabla} \times \vec{C})^2 - \hbar^2 \frac{\delta^2}{\delta f^2} + (\vec{\nabla} f)^2 + f^2 \vec{C} \cdot \vec{C} + V(f^2) \right) \Phi[\vec{C}, f],$$

14 This is known as a Stuckelberg transformation.
where in the last line we have used the fact that the states satisfy the Gauss constraint (eq. (39)). Note that the term \( f^2 \vec{C} \cdot \vec{C} \) will be responsible for the appearance of an effective mass for the vector field \( \vec{C} \) in perturbation theory around the minimal energy state for a Mexican hat potential \( V(f^2) \). More precisely, we can write

\[
V(f^2) = V(v^2) + (1/2) V''(v^2) (f-v)^2 + O[(f-v)^3],
\]

where \( v \) is the value of the field \( f \) at the minimum of the potential. Then one can treat the terms \( O[(f-v)^3] \) as (higher order) self-interactions in the perturbation theory. If we do so the ground state at lowest order perturbations around the minimum of \( V(f) \) becomes

\[
\Psi[\vec{C}, f, \theta]_0 = \mathcal{N} \exp[-(4\hbar^2) \int C_a(x) [(\triangle + v^2) \delta_{ab} - \partial_a \partial_b]^{-1} C_b(x) - (f-v)[\triangle - V''(v)]^{-1} (f-v) dx],
\]

Eq. (40)

The Gaussian profile with spread \( \sigma = v/(2\hbar) \) for the dependence of the wave function for the vector field \( \vec{C} \) implies the massive character of the latter with an acquired mass \( m = v/\hbar \). Notice that for the state (40) \( \langle \phi(x) \rangle = \int [\mathcal{D}A_i][\mathcal{D}f][\mathcal{D}\theta] f^\alpha \Psi[\vec{A}, f, \theta]_0^\alpha \rangle = 0 \) as must be the case for any gauge invariant, and thus physical state. However it is quite clear that the expectation value \( \langle \phi(x)\phi(x) \rangle = \int [\mathcal{D}A_i][\mathcal{D}f][\mathcal{D}\theta] f^\alpha \Psi[\vec{A}, f, \theta]_0^\alpha \rangle \approx v^2 \) which gives rise to the mass of \( \vec{C} \).

**Fermion masses**

In the standard model fermions are also thought to acquire mass as a result of the SSB and the asymmetric vacuum expectation value of the Higgs field. The key difference with the case of the bosons is that these masses turn out to be proportional to \( v \) rather than to \( v^2 \). There is a very simple way to describe the mass generation [13] in a fully gauge invariant way. Assume that the fermion fields are \( \psi_L, \psi_R \) and that they are characterized with the Higgs-electromagnetic Lagrangian (33) through:

\[
\mathcal{L}_f = i \bar{\psi}_L D_\mu \gamma^\mu \psi_L + i \bar{\psi}_R \partial_\mu \gamma^\mu \psi_R + \lambda \phi \bar{\psi}_L \psi_R
\]

Eq. (41)

so that under a gauge transformation \( \psi_R \) is invariant and \( \psi_L \rightarrow \exp i\alpha \psi_L \). One can define new (manifestly) gauge invariant fermions \( \Psi_R = \psi_R \) and \( \Psi_L = \phi^* \psi_L/(\sqrt{\phi^2}) \) (note that the normalization in the previous definition is needed in order to have the correct fermion dimension). In terms of manifestly gauge invariant fields the Yukawa coupling becomes

\[
\mathcal{L}_{yuk} = \lambda (\sqrt{\phi^2}) \Psi_L \Psi_R = \lambda f \Psi_L \Psi_R.
\]

Eq. (42)

Due to the fact that \( \langle 0 | (\sqrt{\phi^2}) | 0 \rangle = v \) the previous term generates a fermion mass for a manifestly symmetric vacuum.

We should point out that even if a potential does not have the typical “Mexican hat” shape, but rather, say, has a simple parabolic profile, the vacuum expectation value of \( \phi^2 \) can not be zero (as that would require an infinitely sharp value for \( \phi \) which would imply an infinite uncertainty of the conjugate momenta of that filed) and thus the possibility of obtaining the standard SSB phenomenology exists, in principle, even with such simple potentials\(^{15} \).

\(^{15} \) The detailed estimates for the corresponding values that emerge in any straight forward analysis, do not seem to work in the Standard Model of Particle Physics.
IV. THE STANDARD LORE

In the usual accounts of the paradigm of SSB in field theories one encounters an emphasis on the difference between ordinary quantum mechanical systems of finite number of degrees of freedom ($QM$) and quantum mechanical systems with infinitely many degrees of freedom ($QM_\infty$) such as those treated in the thermodynamical limit of statistical mechanics and field theory.

The arguments one finds in the literature on the subject, regarding the central difference between the two situations, rely on the existence, in the latter case of unitary in-equivalent representations of the algebra of observables [1]. This is a well known feature of quantum field theory, most conspicuously found in the context of quantum field theory in curved space-times [2]. The situation arises when the comparison between the states involved indicates that in going from one to the other necessitates the application—in the mean\textsuperscript{16}—of an infinite number of creation/annihilation operators. One is thus, in general, lead to take the so called “algebraic approach” in which all the states in all representation are collectively considered in a unified fashion, and in practice one relies on physical considerations and Fell’s theorem\textsuperscript{17} to justify working with any one of the Hilbert space representations that are associated with a particular problem. Nevertheless it is clear from these same analyses that there could exist in principle physical processes mapping states from one irreducible representation to another. This will be the case, when the process involves the creation or annihilation of an “infinite number of particles”, as it occurs, for instance, in the contexts of cosmological evolution, or systems in interaction with certain “external potentials”.

The fact that one needs to deal with physical processes involving transition from one to the other of the in-equivalent constructions, or folia, is what lead to the development of the so called Algebraic Approach, which allows for the simultaneous treatment of all states in all folia (see for instance [2]). We should be aware that even in describing one single simple physical situation from two different points of view, requires one to pass from one QFT construction to an in-equivalent one, as exemplified by the analysis of the Unruh effect [3]. Thus, it seems clear that, in general, physics might not force us to describe the state of the system, as belonging at all times, to one of these representations. In fact, if that was the case, we could not consider the phenomena of phase transitions associated with a spontaneously breaking of the symmetry, as transitions occurring in physical time, which is precisely what one often wants to do. These considerations indicate that the central question we need to confront is: What are the physical, rather than mathematical, issues behind the “phenomena of spontaneous symmetry breaking”. Stating that we face “in-equivalent representations of the algebra of observables” does nothing to indicate the physical reason for the breaking or not of a symmetry present in the laws governing the relevant phenomena by the lowest energy states of the system in question. We must understand when does this happen, if at all, and in those cases, the physical reasons behind it.

We note that in those same accounts of the issues such in-equivalence of representations are often invoked as the fundamental difference between $QM$ and $QM_\infty$ which makes pos-

\textsuperscript{16}By “in the mean” here refer to the difference in the expectation value of the total particle number operator between the two states.

\textsuperscript{17}Fell’s theorem states, loosely speaking, that for any representation of the canonical commutation relations, and for any set of possible outcomes of measurements, one can find a state that could be associated with these results to any desired, but finite, accuracy. See for instance [2] page 81.
sible the phenomena of SSB in the latter but not in the former [13]. Those analyses start by tying this rather mathematical issue (the in-equivalence of representations) to various, apparently sacrosanct physical characteristics, namely: i) the “clustering property” and the ii) “irreducibility of the representation” ([13] page 119).

Let us review these points here briefly to uncover what assumptions are really behind the standard picture. The starting point is to consider the algebra of observables generated (by algebraic plus Cauchy completion) by the local observables $A_{\text{loc}}$. Such algebra satisfies, by construction, the property of asymptotic locality\(^{19}\), justified by the realization that in a laboratory, only those states that are associated with local operations can ever be created or measured (pages 34, 142 and 156 of [13]). Then, one uses the result (page 35 of [13]) that warrentes that, given an algebra satisfying the asymptotical locality conditions, and assuming that one has a cyclic\(^{20}\) translational invariant vacuum, and if moreover the representation is irreducible, then there can be no other translational invariant state. In the case of relativistic fields, the cluster property is strongly tied to the uniqueness of the ground state, by the Arakki-Hepp-Ruelle Theorem, (see for instance page 65 of [13]).

The issue we need to face, is then: the physical relevance of the following properties:

1. Asymptotic locality:

As we indicated, the assumption of validity of the property of asymptotic locality is generally well justified in the usual applications, by the realization that, in a laboratory, only those states that are associated with local operations can ever be created or measured. The problem is, of course, that when we want to concern ourselves with issues that involve processes that are relevant for the evolution of our universe, such limitations to local operations have no clear justification. The issue of whether in our universe the vacuum is invariant or not (assuming that the true theory is invariant under a certain continuous global symmetry with the conditions normally associated with SSB), must then be answered on different grounds.

In particular, the notion of asymptotic locality looses its mathematical meaning in the framework of a spatially compact universe (as the example treated in Section II). We have explicitly shown in that example that the ground state is indeed symmetric, that the symmetry is unitarily implemented, and have argued that the standard local physics can be described in such framework. Of course, loosing asymptotic locality might take us out of the hypothesis of some of the strong theorems of local quantum field theory. One of the standard results that is no longer valid is clustering decomposition.

2. The clustering property:

This property corresponds to the the condition where the two point field correlation for the vacuum state decays to zero at long distances. The requirement of this property is usually justified in physical terms that are often misleading. The demand that

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\(^{18}\) This algebra is constructed by integrals of the local field operators over compact space-time regions.

\(^{19}\) This property, sometimes called “Asymptotic Abelianess”, states that given two local observables $A\hat{k}\hat{B}$. if we define by $A_{x}$ the displaced observable $e^{i\hat{p}\cdot x}Ae^{-i\hat{p}\cdot x}$ then in the limit $x \to \infty$ the observables commute i.e. $\lim_{x \to \infty}[A_{x}, B] = 0$.

\(^{20}\) A state is said to be cyclic, if when acted upon by all the elements of the algebra in question, it leads to a dense set on the Hilbert space.
local experiments should “not depend” of what happens at space-like separated regions, taken in unqualified terms, is a severe and unphysical restriction in quantum mechanical context: we know that non-local correlations do occur in situations such as an EPR experiment, and that they have indeed been observed experimentally [4]. Moreover, contrary to what is often loosely stated, and in close analogy with what occurs in the EPR correlations, the violation of the clustering property (in the sense of the existence of correlations, whether or not they are associated with the vacuum state) does not need to lead to conflict with relativistic causality. The statement should then be replaced by a more limited requirement that information should not propagate superluminally\(^2\) and these two different requirements should not be confused. What is clear from the analysis of such situations is that such correlations cannot be used to send a-causal signals by any observer to another \(^2\).

From the perspective of the Wightman axioms, the clustering property is a theorem rather than a requirement. All the same, the restrictive conditions that lead to this result are hidden in the axioms themselves. Of crucial importance is the requirement of the uniqueness of the Poincaré invariant state directly tied to the irreducibility of the representation (namely, property (2) above).

We have seen that the spatially compact model of Section II violates the previous two properties as a consequence of the quantum nature of the zero mode and that this leads to no problems at all. It should thus be clear that the violation of such properties in the \(L \to \infty\) limit can not lead to any contradiction either, and that the consideration of the symmetric vacuum in this case is not prevented by any physical argument. A particularly clear exhibition of a characteristic of such state is the infinite range field correlations in the symmetric vacuum shown in equation (27).

3. Irreducibility of the representation:

This requirement is linked to the assumption (one of Wightman axioms [5]) that the state space of a relativistic quantum field theory is a separable Hilbert space. Despite of the fact that separability of the Hilbert space is a powerful mathematical tool leading to a variety of important theorems applicable in a vast variety of physical situations, it is clear that this does not include all physical situations\(^2\). Contrary to the previous two properties, the present one is not violated by the spatially compact model analyzed in Section II. However, this property is often violated in physically relevant situations (e.g. thermodynamics).

In order to illustrate the issues mentioned above in a concrete and simple context, let us consider the one dimensional infinite spin chain model (infinite Ising model). It is the theory of an infinite number of spins placed along the real line and interacting with the

\(^{2}\) Consideration of this issue will necessarily bring into consideration the question of measurement in quantum field theory, an issue which has difficulties of its own[19].

\(^{2}\) In fact there seems to be an unjustified time-bias in our description of physics regarding the possibility of correlations. We naturally expect them to arise after an interaction but \textit{a priori} dismiss the possibility of them exiting before the interaction[20].

\(^{2}\) Perhaps the best known example of a physical theory where one needs to deal with a non-separable Hilbert space, is the case of Loop Quantum Gravity [18].
The in-equivalence one refers to, in this context, is the fact that the operators realizing the mapping from one sector of the full Hilbert space to the other, have vanishing matrix element when restricted to states in each sector. The different sector are on the other hand “unitarily equivalent” in the sense of the existence, among the sectors, of a bijective mapping which preserves the inner product.
somewhat exotic in the standard QFT applications is that they do not satisfy the famous clustering property; essential among other things for the definition of scattering theory. As indicated before this property has been overvalued, and assertions that without it we could not do physics cannot be sustained (recall EPR discussion above).

At first sight, and in the mindset of Fell’s theorem, there seems to be no way to distinguish through finitely many local measurements whether we are in a pure or mixed phase. In the above example it is clear that no matter what local quantity one considers, it is always possible to obtain the same expectation values, probabilities, etc. by taking a mixed phase \( \Psi = \alpha \Psi^+(\vec{n}) + \beta \Psi^+(\vec{m}) \) or a suitable state in a pure phase (one just needs to tune the state in the pure phase to get the right answers). Does this mean that breaking or not breaking the symmetry becomes a meaningless statement? We will argue that the answer to this question is, in general, in the negative. We must start by recognizing that in our practice of physics, we do often consider things that are not strictly related to what we measure or will measure. In the situation at hand, we find, for instance, that certain infinitely long range correlations (clustering violation) are present in the mixed phase and we might want to consider such state as the state appropriate for describing the extrapolation beyond the actual measurements (which, in practice, can only concern finite distances), of a situation we might conceive find by local experiments. More concretely, in the field of theoretical cosmology we often consider situations based on the cosmological principle, and it would be inappropriate then to shackle oneself to employ a quantum field theory description in which such principle can not be incorporated. More generally, if we have a situation in which the internal symmetries of the field theory can only be realized for certain type of states, one can not argue that the symmetry must be broken, simply because one has a priori limited (by construction) the range of states under consideration based on such “measurability arguments”.

In the simple setting of the present example it is clear that even when many of the usual questions can be addressed using the irreducible representations \( \mathcal{H}_{\vec{n}, \pm} \), the full physics in the one dimensional spin chain is contained in \( \mathcal{H}_\infty \). In fact some of the justifications often

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25 For instance we might consider the behavior of certain quantity, as we measure it at different scales, and if the result of all previous measurements indicate that such quantity does not change, we might consider extrapolating, and as a working hypothesis consider that they never change. Strictly speaking the corresponding measurements would be possible, only through nonlocal experiments, and it seems clear that the “limit when the distances between the observations regions tend to infinity” can not, in practice, be carried out by humans. Nevertheless, we can envision experiments carried out at very large distances whereby one could “reasonably infer” the behavior of the correlations as the distances between the points involved increase without bounds, and these inferences can be taken to give meaning to the notion of “correlations between points infinitely separated”. This way of reasonably accessing in practice, these sort of limits, are reminiscent of the issue of “practical measurements” of probabilities, which, in the frequency interpretation, are defined in terms of infinite number of trials. It is clear that we never make experiments involving infinite trials but, nevertheless, we do consider, as measurable, certain probabilities which are defined based on such idealizations. It is clear then, that we do often set different criteria of “measurability” in different fields of physics, and we must, then, be very careful when considering the corresponding intersections

26 I.e. there is no reason to a priori exclude from consideration as a possible description of the world, any of the states in \( \mathcal{H}_\infty \)
invoked in the literature to try to justify a consideration of one sub-sector of the theory, in these sort of situation can be turned around. For instance, the very same argument showing that, once in a pure phase $\mathcal{H}_{\vec{n},\pm}$, there exist no local process that can take us away from it, serves to show that if we take the system to start off in a mixed phase $\mathcal{H}_{\vec{n},\pm} \oplus \mathcal{H}_{\vec{m},\pm}$ there is no local process, which will allow the system to evolve into a pure phase (elements of say $\mathcal{H}_{\vec{n},+}$).

Let us return to the question of symmetries in the context of our simple infinite spin chain model. Let us focus for simplicity on the discrete symmetry $z \rightarrow -z$. The “mixed state” $|\text{mixed}\rangle = \Psi^+(\hat{z}) + \Psi^-(\hat{z})$ is clearly symmetric under that discrete symmetry. However, there exist other translational invariant states respecting that symmetry. In particular, let us consider the state in which each individual spin in the chain is in the state $|\hat{x},+\rangle = \frac{1}{\sqrt{2}}(|\hat{z},+\rangle + |\hat{z},-\rangle)$. This can be taken as a “vacuum state” $|\text{pure}\rangle = \Psi^+(\hat{x})$ and is clear that it is symmetric under $z \rightarrow -z$. It corresponds to a “pure phase”. Its is worthwhile warning the reader against confusing the two states $|\text{mixed}\rangle$ and $|\text{pure}\rangle$, or thinking that a translational invariant symmetric state in a theory like the one considered in the present example, must necessarily, be a state of the form $|\text{mixed}\rangle$.

Having said all this we must also point out that in a theory like the one under consideration, all the discussion above, does not have any bearing on deciding which is the true vacuum: a state that breaks the symmetry, such as $\Psi^+(\hat{z})$, one that does not but respects the clustering property, such as $|\text{pure}\rangle$, or one like $|\text{mixed}\rangle$ which is symmetric but breaks the clustering property. It is clear that either of the first two can be used as a reference state to lie at the basis of the perturbative construction of a Hilbert space. Either can be taken as the unique translationally invariant in the physically relevant representation. Regarding the third, we believe that there are no convincing physical arguments indicating that it should not be considered as a possibility. Of course, the one which is appropriate to a specific situation will depend in general on the details of that situation.

V. FURTHER CONSIDERATIONS: ENERGETICS AND THE THERMODYNAMICAL LIMIT

Having analyzed in detail the quantum field theoretical constructions most commonly associated with SSB, we believe that is appropriate at this point to take a more global view of the relevant issues and discuss some aspects that often lead to misinterpretations and to inappropriate “reasonings by analogy”. The following discussion is motivated, in large part, by our own state of confusion before embarking on this analysis, and it is our hope that it will serve our colleagues who have been equally puzzled by the “standard lore” on the subject.

The usual discussions on the present subject start by stating that, in the case of a quantum mechanical system with a finite number of degrees of freedom, the vacuum state is unique\textsuperscript{27} (i.e. non-degenerate) and possesses the symmetries of the Hamiltonian. That situation is often then contrasted with what occurs when the system has an infinite number of d.o.f.. The way this contrast is often presented, starts by considering a quantum mechanical particle in a double well potential. It is then argued that if the system is prepared to be localized

\textsuperscript{27} As we have seen that requires in particular that the Hilbert space does contain symmetric states. We have encountered already simple cases where this does not occur.
in one of the wells it would tend to tunnel into the other well with a nonzero probability thus indicating that it was not in the ground state, and that such state would indeed be symmetric. This basic picture is not believed to cover systems with an infinite number of degrees of freedom like those contemplated in the thermodynamical limit in statistical mechanics or in field theories due to the infinite energy barrier that separates the different minima of the energy functional. This difference is often invoked to argue that in a system with an infinite number of degrees of freedom the ground state need not share the symmetries of the theory and thus that the symmetry could be spontaneously broken.

We find this argument misleading as it mixes up two issues that must be considered separately. One is the issue of how long would a system that is prepared in a non-symmetric state stay in that state, and a very different one concerning which is the lowest energy state. It might well be that the system once prepared into a non-symmetric state would never evolve towards the full symmetric state, or even that it might stay in the initial state forever, but that does not mean that the initial state was the true ground state.

Let us start by considering a system made of infinite number of particles each of which is in a double well potential. The naive “energetics” argument can be used to argue that if the system is initially prepared in a state with all the particles wave functions picked on the left well will be quasi-stationary (i.e. would take an infinite amount of time to tunnel to a state in which all the particles are in the right well), something that is obviously wrong. However, one might in fact compute the energy of the barrier and argue that an infinite time of tunneling should be expected in association with the fact that this barrier is infinite. However, in this case, we can treat each of the noninteracting degrees of freedom separately and evaluate the wave function of each of them at any finite time after the preparation time. On the other hand we note that each particle is initially in a state described by a wave function that has support only on the left, and as the different degrees of freedom are not interacting, the subsequent evolution of each particles’ wave function, cannot differ from the corresponding evolution extracted from the evolution of the whole system and thus we conclude that at a finite time after the preparation of the full system, the state will not be the initial one, and thus, that it could not possibly have been the ground state. We see in this trivial example that the standard arguments based on heuristic energy considerations cannot be taken at face value.

Of course the above example lacks something that is common to the systems with infinite number of degrees of freedom that are usually considered in this regard: An interaction that tends to drive all the degrees of freedom of the system towards the same individual state for that degree of freedom d.o.f. This is the case for instance for a ferromagnetic sample of infinite extent, where the interaction amongst the neighboring magnetic moments tends to align them. This feature then implies that there can be no transitions associated to an individual d.o.f.28, but the allowed transitions involve a coordinated and simultaneous change in all the system. Thus it is the interaction amongst all the individual subsystems that stabilizes the asymmetric state of the global system. This however does not indicate that the system was in its ground state.

There can exist, in principle, stable states other than the true ground state, and all that is needed for such stability is the absence, for some reason, of perturbations connecting these states with others of lower energy. The existence of an infinite energy barrier in systems

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28 We are assuming that there are no sources of energy capable of providing the energy represented by a single d.o.f. becoming misaligned with the rest.
known to have a finite energy is the best known example of such situation. In the case
of systems that are infinite in extent however, the issue of finiteness of the systems energy
is something that cannot be treated lightly, and more careful analysis is required to reach
reliable conclusions one way or the other. In fact the thermodynamic limit involves taking
the number of particles $N \to \infty$ (often stated as $V \to \infty$ limit while keeping the value of
the intensive quantities (like pressure $P$ or temperature $T$) fixed awhile those of the extensive
quantities scale with $N$ so $E/N$ or $E/V$ (where $E$ is the system’s energy and $V$ its volume)
is kept constant. In those situations the total energy of the infinite system is infinite and
the arguments involving infinite energy barriers cannot be taken as reliable.

A. Thermodynamical limit and a system in equilibrium with a thermal reservoir

One of the paradigmatic examples often used to illustrate the phenomena of SSB occurs
in the thermodynamical treatment of various systems. The lessons extracted are then trans-
lated essentially unchanged to the field theoretical situations. We would like to clarify the
similarities and differences between the two situations in order to uncover some misleading
analogies often found in the literature. In order to do this it is convenient to review the
setup that lies behind the statistical mechanical treatment as it involves often forgotten
assumptions that are not always valid in the quantum field theoretical context.

The starting point of such treatment is to consider the system of interest as one within
an ensemble of identical systems (as far as their dynamics is concerned) and then study
the behavior of the ensemble averages of the quantities of interest. Under certain physical
conditions (which are often, explicitly stated, in defining the statistical mechanical systems),
the ensemble averages are expected to coincide with the time averages of the corresponding
quantities in the particular system (i.e. element of the ensemble). The point however is that
important new assumptions go into the theoretical description the quantum mechanical
statistical mechanical system (among which the large, or even infinite number of degrees
of freedom is only one). To start with, we must keep in mind the fundamental role of the
interaction of the system with external d.o.f.. At the practical level, this is characterized by
the impossibility of complete isolation of the system, and in particular by the need interact
with it to carry out observations, in order to draw any conclusions. At the theoretical level,
the many external d.o.f. with which the system of interest interacts are represented in the
statistical treatments by a thermal bath. Notice that even in the case where we set $T=0$,
the external environment has a fundamental influence, being the physical source of the de-
coherence that is brought into the analysis in the form of the hypothesis of \textit{a priori} random
phases which is needed to justify the construction the micro-canonical, canonical, and grand
canonical ensembles in the statistical mechanics of quantum systems (see [11] page 109,
Ch5.2 A). Therefore, one key feature of thermodynamical systems is their intrinsically open
nature.

As mentioned above, one of the basic assumptions of statistical mechanics is the ergodic
hypothesis: that is the notion that the average over the members of the ensemble—thought
to be spread over the accessible micro-states—coincides with the time average for an en-
semble member over the curse of its dynamical evolution. In addition to the assumption
of equilibrium, the ergodic hypothesis relies, generically, on the existence of a large enough
set of uncontrollable external disturbances that affect the system in minute but essentially
random fashion. These disturbances prevent the system from remaining in a relatively small
set of states during the course of its evolution and are though to imply ergodicity (see [30]).
The intrinsically open nature of thermodynamical systems is a crucial difference with SSB situation in quantum field theory studied in Section II. This justifies the description of system “in equilibrium” in interaction with a thermal reservoir at fixed temperature $T$ by the Gibbs density matrix $\rho(T)$ (instead of pure states) given by,

$$
\rho(T) = Z^{-1}e^{-\frac{H}{kT}} = Z^{-1}\sum_i e^{-\frac{E_i}{kT}} |\psi_i\rangle\langle \psi_i|,
$$

(47)

where $\{|\psi_i\rangle\}$ is the set of the energy eigen-basis with eigenvalues $E_i$. The previous equation will be important for discussion of SSB in the ferromagnetic system. The latter is often used as a paradigmatic example of SSB in general; however, as we argue in the following section the analogy is misleading.

B. The ferromagnetic phenomena.

One situation that is often used as an illustrative example of the phenomenon of spontaneous symmetry breaking is the spontaneous magnetization of ferromagnetic materials. In this section we review this example and show in what sense symmetry is broken by the ferromagnet below critical temperature. The model often considered is given by a system with large number $N$ of spin half atoms arranged in a lattice where the translational degrees of freedom are taken as frozen. Other possible internal degrees of freedom of the individual atoms are ignored. Each atom is assumed to have a ferromagnetic interaction with its nearest neighbors. If one labels the spin of the $a$-th atom by $\vec{s}_a$, the Hamiltonian of the system in interaction with an external magnetic field $\vec{B}$ is:

$$
H = -J \sum_{a, b=1}^N \vec{s}_a \cdot \vec{s}_b - \mu \vec{B} \cdot \sum_{a=1}^N \vec{s}_a.
$$

(48)

where $J$ and $\mu$ are coupling constants. It is customary to simplify the analysis of the system and concentrate instead on the Ising model. The system is analyzed in terms of the canonical ensemble at fixed temperature $T$. In this situation one considers the Helmholtz free energy $A(T, B)$, and one obtains the magnetization as $M(B, T) = -\left.\left(\frac{\partial A}{\partial B}\right)\right|_T$. The spontaneous magnetization corresponds to the situation in which $M(0, T) \neq 0$ which occurs in general for $T < T_c$ for a certain critical temperature $T_c$. This type of analysis is in general in excellent accord with the experimentally observed features of ferromagnetic materials, such as details of the phase transitions and the appearance of magnetization domains, etc.

There is a precise analogy between statistical (thermodynamical) systems and certain quantum field theory situations even when the latter describe quantum mechanically closed systems. This analogy comes from the familiar Euclidean formulation of QFT where expectation values (in the appropriate vacuum state) of observables can be computed by the formula

$$
<\phi(x_1)\cdots\phi(x_n)> = \int D\phi(x_1)\cdots\phi(x_n) \exp(-S_{eucl}[\phi]),
$$

(46)

where for simplicity we have chosen as an example a scalar field theory. The assumptions implying the validity of the previous equation (justifying the validity of Wick rotations) are unrelated to those implying the validity of (47).
Let us consider the issue of symmetry breaking. What symmetry is broken when spontaneous magnetization occurs? The first important aspect that should be emphasized (and which is a main difference with the examples considered in Section II) is that the Hilbert space of each of the individual d.o.f. does not possess symmetric states. More precisely, suppose we start with the single spin state \( |\vec{n}, +\rangle \)—in the context of the spin chain example of Section IV and in the same notation—one could try to construct a rotational invariant single particle state by superimposing spin states in all possible directions. This could be concretely achieved by acting on \( |\vec{n}, +\rangle \) with a Wigner rotation matrix \( D^{1/2}(g) \) for \( g \in SU(2) \) and then group averaging by integrating the resulting state on all possible rotations with the invariant measure (the \( SU(2) \) Haar measure in this case). The invariant state would look like

\[
|\text{invariant}\rangle \equiv \int_{SU(2)} dg \, D^{1/2}(g) |\vec{n}, +\rangle.
\] (49)

However, it is easy to see that the previous average vanishes identically, namely \( |\text{invariant}\rangle = 0 \). Moreover, one can easily show that the absence of a non-trivial rotationally symmetric state also holds for a system with finite number of spins. Thus there cannot be a rotationally symmetric state in the full Hilbert space (with infinitely many spins) simply because there is no symmetric state in the Hilbert space of the individual d.o.f. or any finite number of d.o.f..

This establishes an important difference between a ferromagnet (or the spin chain of Section IV), which exhibits phenomena such as the spontaneous magnetization, and the scalar field theories of Section II with their characteristic "Nambu-Goldstone boson" phenomena, or the gauge theories of Section III where one finds the Higgs phenomenon.

Thus the question remains: if there are no quantum mechanical symmetric states in what sense symmetry is broken by the ferromagnet below critical temperature? The key point is the statistical mechanical nature of the treatment of the system. Indeed the ferromagnet at temperature \( T \) represents an open quantum mechanical system in equilibrium with an external environment, and is therefore described by a density matrix of the form (47) instead of a pure state. Now, despite of the fact that no quantum (pure) state of the ferromagnet can be rotationally invariant, the density matrix can indeed be symmetric. For example, at \( T = 0 \) the sphere-worth (degenerate) ground states of the ferromagnet dominate (47) and the resulting density matrix is rotationally invariant. For instance the density matrix \( \rho^{(1)}_{\text{inv}} \) for a single spin

\[
\rho^{\text{inv}} \equiv \int_{SU(2)} dg \, D^{1/2}(g) |\vec{n}, +\rangle \langle D^{1/2}(g) |\vec{n}, +\rangle = \frac{1}{2} \mathbb{1}.
\] (50)

For \( N \) spins one gets \( \rho^{(N)}_{\text{inv}} = P_{N/2}/(N + 1) \) where \( P_{N/2} \) is the projection operator into the irreducible representation space of spin \( N/2 \) contained in the tensor product space of \( N \) spin half particles, as expected. Thus the density matrix describing the low temperature magnet is spherically symmetric yet the possible states in which the magnet can be found when an element of the ensemble is singled out by observation breaks the rotational symmetry by selecting one of the sphere-worth of degenerate vacua\(^{30} \). This is the sense in which rotational symmetry is broken by the ferromagnet below critical temperature.

\(^{30} \) Here we must be careful and avoid confusing two situations that are often mixed up. The density matrix can be used to represent the state of an ensemble of identical systems, where each individual element of the ensemble is in a pure state. One can use the statistical matrix to represent a system that has no
Therefore, two important facts lead to a notion SSB in the case of the ferromagnet: On the one hand the statistical mechanical treatment of the system; which is valid under the conditions that justify the assumptions of ergodicity, and \textit{a priori} uncorrelated phases. On the other hand, the absence of symmetric pure states in the quantum theory which lies at the heart of the phenomenon. These features are not shared by many other systems, often compared with the example of the ferromagnet, as for instance those that are usually presented in discussions of the emergence of a Goldstone mode and in the associated discussions of the Higgs phenomena.

C. Two Kinds of Phase Transitions

We should be aware that there are in fact two notions of phase transitions that despite having several features in common are conceptually quite different and it is thus important to keep those differences in mind when considering analogies. The thermodynamic phase transitions which have been studied extensively and the purely quantum phase transitions that have received much less attention (see however [24, 25]). The latter are often characterized as “changes in the character the ground state of the system and are described via unitary reversible dynamics”. Thus they are thought to be quite different from the ordinary thermodynamical phase transitions that are in general associated with an irreversible process. To think of a concrete example we might consider the quantum Ising model corresponding to a set of collection of spins subject to an interaction among nearest neighbors corresponding and also to an external magnetic field with a Hamiltonian of the following form:

\[ H = -gB \sum_i \sigma_i^x - J \sum_{<i,j>} \sigma_i^z \sigma_j^z \]

(51)

where \( g \) and \( J \) are a coupling constants \( B \) is external magnetic field along a fixed direction \( x \), the \( \sigma \)s are the standard Pauli spin matrices and the second sum is over near neighbor pairs. We note that in this model the spins are thought to interact only through their \( z \) components while the external field acts in a perpendicular direction. Thus while the internal interaction tends to align the \( z \) components of the various spins, the external magnetic field tends to align the \( x \) components of all spins in the direction of the field.

We first concentrate in the case where the system is not in contact with any other external degree of freedom, and thus can be treated through the exact Schroedinger equation for an isolated system as there is no channel through which it can loose quantum phase coherence. We can consider the changes in the nature of the ground state of the system and see that by changing the magnitude of the field we can bring about a change in the character of the ground state. When \( B \) is sufficiently large the ground state corresponds to a situation in which all spins are align in either the \(+x\) or \(-x\) direction (depending on the sign of \( B \)), while in the case where \( B \) vanishes, there is in fact a two dimensional space of ground states with a basis given by the two degenerate states corresponding to all spins pointing in the \(+z\) direction, and all spins pointing in the \(-z\) direction. This degeneracy is ignored in the

individual state of its own, as in the case where the system of interest is part of a larger system which, as a whole, is in a state involving correlations between the subsystems in such a way that the subsystem of interest can only be described through its reduced density matrix. Finally we could have an ensemble of subsystems as in the previous case.

33
discussion of [24], probably because the focus there is on different issues. As we now consider
the changes in the nature of the manifold of ground states as $B$ is continuously changed we
encounter a clear example of a quantum phase transition. What we must note here, however,
is that for all values of $B$, the manifold of ground states includes states that are invariant
under reversal of the $z$ axis, exactly as we found in the discussion of section I C.

Let us now consider the same system when in contact with a thermal reservoir at a finite
temperature $T$. We can now consider the standard statistical mechanical treatment of such
situation. We know that if we consider changing this temperature (for a fixed value of $B$)
we encounter the usual thermodynamic phase transitions.

The first difference we encounter among these two types of transitions, is that when
studying a system in contact with a thermal reservoir, the most stable state is not the state
of minimal energy but a state with minimal free energy. These free energy minimizing states
can be thought to be found among the possible states of the system by a suitable optimization
process in which besides the obvious changes which lower the energy and increase the entropy,
one might include changes that involve increasing the systems energy if they are accompanied
by a sufficiently large increase in the entropy. The fact that the lowest energy state of
symmetric theories corresponds to the most symmetric state (or at least includes such a
state in the set degenerate vacua) might, in these situations, be superseeded—as far as
the determination of most stable state is concerned—by a large amount of entropy that is
associated with the possible complicated correlations of the systems degrees of freedom with
the larger set of less symmetric states for the d.o.f. of the environment.

One might be tempted to think that in the $T = 0$ limit the statistical mechanical treatment
would always coincide with the quantum mechanical of quantum field theoretical
treatments, and that thermodynamic phase transitions go smoothly into quantum phase
transitions. That this is not necessarily the case has been observed before, for instance in
the discussion is section 1.2 of [24]. We should point however that the consideration of the
issues of symmetry forces us to look with additional light on this matter. The point is that
in the case where the vacuum state is degenerate, the statistical mechanical analysis, by its
reliance on representative ensembles, assumes that all the vacuum are equally weighted in
the description of the system, while a quantum mechanical treatment would normally not
involve such assumption and could consider one individual system as described by any of the
various degenerate vacua. In fact the statistical treatment in such situation does not allow
within its formalism (i.e. in the partition function description) the consideration of issues
such as which basis should be used to characterize the various manifold of vacua sates of the
individual system (i.e turning to the example of section I C should the natural description
be in terms of non-symmetric states, symmetric pure states or symmetric mixed states?).
By default, one describes the whole ensemble with a density matrix which would be sym-
metric even in the cases where there are no symmetric states as discussed in the previous
subsection. There is however no way to even consider the issue of what is the state of one
of the individual elements of the ensemble.

One of the points where it is clear that the statistical mechanical treatment can hide
some of the issues we have been confronting in this paper lies in the construction of the
micro-canonical statistical ensembles for quantum mechanical systems. There one brings in
an ad-hoc assumption about the relative phases of the states of the various systems in the
ensemble called the hypothesis of random a-priori phases (See section 5.2 A of [11]). This
hypothesis is akin to imposing de-coherence in the particular and, in principle, arbitrarily
preselected basis, when one turns to apply the formalism to describe the state of an individual
Thus the set of differences between the situations described within quantum field theory for an isolated system and those described with the quantum theory of an open system are sometimes rather stark, and we must keep them in mind if we want to be able to trace to the appropriate source of some behavior we are trying to understand. It might well be that in most circumstances, the statistical mechanical treatment of open systems is the appropriate one, and in fact most of the examples of observed phase transitions are described within that paradigm. Among the few experimentally observed exceptions of an example of a truly quantum phase transitions exists (see [24]).

The above considerations and in particular the caveats in the analogies and the several distinctions between the two sets of phase transitions and their treatment, show that it would be clearly erroneous to consider that the well known examples of thermodynamic phase transitions represent, in any sense, some sort of incontrovertible evidence about the correctness of all of the standard lore about the subject of SSB in quantum field theories.

VI. THE COSMOLOGICAL SETTING

This should be, in some sense, the paradigmatic object of our discussion, as the universe is perhaps the only truly isolated quantum system\(^{31}\). However, as we have already mentioned, this setting is in fact rather the inappropriate one to consider the question of what is the true vacuum of the theory, as the fact that in this situation we do not have at our disposal a well defined notion of energy (because the general situation is not stationary), the question of which is the state that minimizes the energy, becomes, strictly speaking, meaningless. Moreover, if any definition of global energy of the system would become available it seems clear the states that are compatible with large scale homogeneity and isotropy (i.e. the cosmological principle) would all have infinite energy unless we were dealing with a finite universe.

One possibility that one has to have in mind in discussing these issues in the cosmological context is that the universe might be closed, and thus, all the issues associated with the infinite extent of the region where two states differ, which we saw are fundamental for some of the conclusions, and the arguments based on boundary conditions which lie at the basis of the analysis of Section II C, would loose their relevance. In fact it seems natural to think that the issue of the universes large scale topology should have no bearing on the conclusions.

Nonetheless we face in this context, and in particular in the inflationary scenarios, one of the earliest examples of what one would like to call the spontaneous breaking of a symmetry. The point is that inflation is supposed to take the universe (or at least the relevant part of it) to a state that is totally homogeneous and isotropic, in both its classical background (the inflating FRW with the slow rolling inflaton field) and its quantum fluctuations (of course we view this as an approximated description of a full quantum state of the system, which is however thought to share the symmetries of its simplified description). The usual account for the origin of the initial spectrum of fluctuations sees them as emerging from

\(^{31}\) Even the most isolated laboratory system can not be decoupled from the gravitational d.o.f. and the extent of such coupling has been argued to be enough to force one to consider the system as open \([23]\).
the quantum uncertainties of an homogeneous and isotropic vacuum state. We have argued elsewhere [28] that the standard accounts of this process fall short of being fully satisfactory and that additional elements must be called upon, beyond those usually considered in this context. The present analysis should serve to remove the standard account of SSB as part of the answer to the conundrums raised in [28].

When dealing with cosmology, on the other hand, one might need to face the question of whether our universe should be considered infinite or finite, and one might be tempted to wonder whether these type of issues would offer a possible avenue to empirically address such questions. Unfortunately the fact is that realistic considerations would require not only the addressing of a full set of unsettled cosmological issues, but the need to be in possession of a well defined notion of energy, which is generically not available in the situation at hand due to the lack of time translation invariance of the space-time and the impossibility to assign an energy to the gravitational degrees of freedom. Therefore it is clear that nothing of that sort can be envisaged at this time.

Another related issue is what can we take to be the initial conditions of our universe in regard to the symmetries in question (those we would have wanted to associate with the zero mode), i.e. were the initial conditions symmetric or not?. This takes us to the realm of quantum cosmology, where one can hope to find some relatively convincing information about the initial state in one of various proposals [26], or alternatively to the realm of inflationary cosmology [27] in which one expects the universe to eventually reach a stage where the conditions become independent of such initial data, at least in a large enough region containing all of our causal past. It seems clear that in such situations, even if one can simply take the universe to be spatially infinite, the question of which is the appropriate “vacuum” has changed dramatically as compared to the other situations treated in this manuscript, as one can no longer rely on simple energy considerations, and then the question of whether the state of the system is or not symmetric, must be addressed on different grounds. In this sense, there does not seem to be any argument that would justify the conclusion that the “vacuum” must break the symmetry of the theory. We will briefly touch on some related issues in the context of our analysis of topological defects.

A. Topological defects

One of the issues that is closely tied with the standard accounts of SSB is that of the generation of topological defects. In the case where the vacuum is degenerated, the idea that the theory (or more correctly, the system under the appropriate conditions) “selects” locally one vacuum, among the various possible ones, naturally leads to the possibility that the vacua selected in various spatial regions are not only distinct but that they are, collectively, “knotted” in such a way that the resulting configuration becomes stable due to a large energy barrier that separates it from the, un-knotted global minima of the energy. These “knotted” configurations are called topological defects and are supposed to be the inescapable consequence of the dynamics of the early universe. Similar considerations can be applied to some condensed matter systems which lead to analogous topological defects in the context of some thermodynamical phase transitions (see for instance [29]).

The treatment of these issues is carried out in the classical language that, we have seen, can be very misleading in dealing with the situation in quantum field theory. In particular we have seen in section II, that the true vacuum, for the restriction of the system to any local region (which has finite spatial extent), is unique and is symmetric, whenever the theory’s
Lagrangian is symmetric. System is therefore locally allowed to relax to the vacuum, the different regions will be brought to the same state, and no knotting can be expected. This raises the question: Do we, in such cases expect the formation of topological defects? We will focus here on the case of continuous global symmetries and leave the case of Gauge Theories for future analysis. Let us consider, for concreteness the case of a theory with an $O(4)$ symmetry in a 3+1 static homogeneous and isotropic space-time background with closed spatial slices (Einstein’s static universe). In the cosmological setting we would be interested in considering an expanding universe, but for the sake of our discussion we consider here the static example in order to have a well defined energy associated with every field configuration. Let $t$ be the Killing “time” parameter and consider the foliation of this space-time by the 3 spheres of constant $t$: $\Sigma_t$.

Now let us consider the usual account of the generation of topological defects. To do so we imagine the field configuration to be created a time $t_0$. For the sake of our discussion, we imagine that the $\Sigma_{t_0}$ is divided in $N$ patches, the configuration is created independently in each patch and then some interpolating smoothing is carried out on a region of width $D$ in the intersection around of the patches. The first thing to note is that, as in the quantum setting the vacuum state is unique, in contrast with what happens in the classical account, if in each patch the state reproduces the vacuum locally, then it will reproduce the vacuum globally. Thus in this case one should not expect the generation of any topological defect. But let us assume for a moment that for some reason, the local states reproduce, not the vacuum but one of those states in which the field is sharply peaked around one of the classical vacuum, a state similar to that of eq. 12. Let us refer to such state as “centered on the classical vacuum pointing in the direction $\phi^{(0)}_1$” and denote them by $|\phi^{(0)}_1\rangle$. Let us imagine, in fact, that the global state is indeed close to a classical configuration with nontrivial topological winding number. The issue is the topological stability of the quantum configuration. The point is that, while classically the field configuration space is a mapping $\phi: S^3 \rightarrow R^4$ which has a degenerate set of minima, with topology $S^3$, and thus the configurations that are restricted to be locally at the minimum correspond to maps from $S^3$ to $S^3$ which we know are labeled by the winding number, at the quantum level what we have is a state, characterized by a wave function: a mapping from $\Psi: F(S^3, R^4) \rightarrow C$ where $F(S^3)$ is the set of functions $\phi$ from $S^3$ to $R^4$. The topological characteristics of the two sets of mappings need not be the same.

Let us consider for instance the temporal evolution of the state $|\phi^{(0)}_1\rangle$. The state is homogeneous and thus should have no component in the mode $k$. Such state corresponds to a wave packet of eigen-states of $\pi_0$, and since this operator commutes with the Hamiltonian, each of the components will evolve just in its phase, as in the case of a free particle. It then follows that the wave packet will spread in field space (as a free particle spreads in position space). As this happens the wave packet will become less and less concentrated about $\phi^{(0)}_1$. When we recall that the components corresponding to the various $\pi_0$ eigenvalues

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32 The remaining issue is whether there is some mechanism that would create the correlations between the local states, and which are needed if these local states are to come together and form a global vacuum [10].

33 At this point one could become concerned with the issue of relative phases among the local states, but this is an issue that is different from the one we are dealing with here. In fact the phases have group structure $U(1)$ independently of the group structure of the symmetry which is supposed to undergo SSB.

34 This particular type of topological defect is called a texture [31].
remain present throughout the evolution we conclude that no matter how large this spreading becomes, the evolution will not transform the state in the true vacuum, a conclusion that is evident from energetic considerations. For that to happen one would need some interaction that leads to dissipative phenomena as far as this mode is concerned. In a more realistic cosmological context the expansion of the universe could be one source of effective dissipation, although by itself it could not disrupt the charge conservation associated with $\pi_0$.

Having gained this insight we now turn to the situation that is relevant to the issue of topological defects. Let us consider first the case in which we have only two contiguous regions such that in each one of them the state locally resembles the state $\phi_1^{(0)}$ and $\phi_2^{(0)}$ respectively. How would this state evolve? First of all it is clear that far away from the boundary the state should evolve in the same way as in the preceding discussion, i.e. the wave packet would spread and the sharp value of $\phi_1^{(0)}$ and $\phi_2^{(0)}$ would become less and less sharp. Now we note that in this case we are not really dealing with the zero modes as the spatial dependency of the state indicates the strong excitement of the modes with wavelengths $\lambda$ of the order of the size of the two regions.

These modes are not the ones associated with charge conservation and so there is nothing to prevent the energy to flow into other modes. This de-excitation of the modes can be expected to continue until the energy is equally spread between all modes not protected by charge conservation. This suggests that the evolution of the state will consist of a simultaneous spreading of the wave packet, and de-excitation of the $k \sim \lambda^{-1}$ modes leading to an erasing of any “memory” of the values of $\phi_1^{(0)}$ and $\phi_2^{(0)}$. Thus except for the relative small excitations in all modes, the global state will evolve into something closely resembling the true vacuum. If we add to this any source of dissipation, as that associated for instance with the cosmological expansion, we can expect to approximate the true vacuum to be approaches to an increasing degree as the system evolves.

This view is of course in sharp contrast with that inspired by the classical picture. Particularly striking is the contrast in expectation of what is the fate of a state that approximates the classical configurations that correspond to topological defects. The previous discussion suggests that such configuration would evolve towards the vacuum and that the classical notion of winding number, is made irrelevant because at the quantum level such notion is not well defined and the number we can associate to a state in some approximate semiclassical description, is simply not conserved.

How can we view the unwinding of a state that approximates a classical configuration which corresponds to a topological defect? An important aspect that must be kept in mind to help understanding how that can occur lies in the distinction between $\langle V(\phi) \rangle$ and $V(\langle \phi \rangle)$. If we are dealing with a classical state corresponding to a static configuration containing a topological defect, the unwinding would require the field to pass trough configurations that have at some points the value $\phi = 0$ and such evolution can be made impossible by the large energy density associated with $V(0)$. In the quantum language, on the other hand, the state of the field is described by a wave functional, and as such, the relevant quantity in such consideration is the behavior of $\langle V(\phi) \rangle$ as the state of the system evolves. As it is clear from the example provided by the true vacuum state in sub-section II A, this quantity can be very different from $V(0)$, even when the expectation value for the filed is zero. Thus the state of the system can evolve in such a way that the expectation value of the local field $\langle \phi(x) \rangle$ can go trough zero in some points—and even to zero globally, if there are some sources of dissipation— unwinding in the process the “topological defect” without the impediment
posed by the large energy barrier associated with $V(0)$.

The study of the actual evolution of a state which approximates a classical configuration corresponding to topological defect is a very interesting problem, which we hope to approach with suitable numerical techniques in future works. Given the expected complexity of such analysis its study lies clearly outside the scope of the present article. Nevertheless it is easy to present a toy evolution that exhibits the possibility of a quantum unwinding of a configuration that is classically "knotted", and thus impossible to unwind classically. Let $\Phi[\phi(x)]_{\text{def}}$ be a wave function for the filed $\phi(x)$ which corresponds to a classical topological defect, and let $\Phi[\phi(x)]_0$. Now consider the wave function $\Phi[\phi(x)](t) = \alpha(t)\Phi[\phi(x)]_{\text{def}} + \beta(t)\Phi[\phi(x)]_0$ where $\alpha(t)$ is a function such that $\alpha(0) = 1$ and $\alpha(\infty) = 0$ and $\beta(t)$ is a function such that $\beta(0) = 0$ and $\beta(\infty) = 1$ and are normalized to preserve the norm of the state throughout the evolution. This mock evolution, while completely unrealistic, illustrates the important differences between the quantum and classical pictures that stands behind the distinct fates of the topological defects as can be expected from the two kinds of analysis. Needless is to say that the physically relevant description, is the quantum version because we currently view the classical description of any system as nothing more than a useful approximation valid only under certain conditions and as long as its results do not conflict with the corresponding quantum analysis.

B. Topological defects in Statistical Mechanics

Topological defects are known to arise as a result of phase transitions in many systems. There is a widespread notion that this is a common feature of quantum phase transitions and statistical mechanical phase transitions. However we must recall that in the case of the former, the issue one must address in considering the equilibrium state of the system is not just related to the minimization of the energy, but with the tendency of entropy to increase, and thus one must consider minimizing the Helmholtz free energy, rather than the energy of a pure quantum state. Under such circumstances the precise correlations that are exhibited by the true vacuum of the isolated quantum mechanical system, are statistically unlikely and as such, they would be associated with a very low value of the entropy. One can think of this as due to the essential random nature of the interaction of the systems d.o.f. with the "environment".

The point is that the states that minimize that kind of quantity are in general not pure quantum mechanical states but rather mixed states, which are characterized, among other things by a large and complicated set of correlations between the system of interest and the collection of systems which play the role of thermal reservoir or "environment". Under those conditions one can envisage a situation where individual regions that arrive at local equilibrium and that are thus described by a thermodynamical state –and the point is that in this case there are in fact many different such states– that has locally minimized the free energy, end up in a global configuration such that each local state would differ from those corresponding to the other regions in such a way that the global state would correspond to a topological "knot". This is in fact what lies behind the appearance of textures, domain walls, strings and monopoles, in suitable systems when they undergo thermodynamical phase transitions of different kinds. The point however is that this situation is quite different from the one we have been interested on throughout this manuscript and which concerns the case of quantum mechanical isolated systems.
VII. DISCUSSION

In the present manuscript we have considered the type of situations where SSB is usually thought to arise. We have seen that the claim that such situations are characterized by a vacuum state that does no share the symmetries of the theory, are often incorrect.

In particular in the field theoretical settings, where the quantum system has an infinite number of d.o.f., and is spacially compact with no boundaries (box with periodic boundary conditions) the true vacuum of the theory is unique and symmetric. This remains true in the case where the spatial extent of the system is infinite while lowest energy states become highly degenerate including both symmetric and asymmetric states. We have seen this by considering in detail an example showing that the quantization of the mode associated with the phase or orientation of the vector in internal space corresponding to the bottom of the potential is like that of the so called zero mode of a mass-less scalar field. In contrast with the other modes, that are quantized like harmonic oscillators, this mode must be quantized like a free particle. In the case of interest considered in section III the field corresponds to a particle on a circle (i.e. a bounded region) and so its ground state corresponds to a constant (i.e. a wave function that is independent of the value of the field) where the field has no defined value.

In the case of quantum field theory on a spacially non compact space, the symmetry breaking nature of the customary notions of vacuum is a feature of the mathematical model of QFT and not a physical phenomenon. The question of whether the vacuum state is symmetric is simply ruled out as the zero mode of the field is not one of the dynamical degrees of freedom in local quantum field theories. The symmetry issue has infrared subtelties in QFT. If one wants to properly talk about the symmetry of the vacuum in QFT one needs to bring in a infrared regulating structure that would allow one to consider the zero mode as a fully dynamical degree of freedom. The simplest way to do this is to define QFT in a box with periodic boundary conditions.

We have shown that the standard phenomena usually associated with SSB like the emergence of Nambu-Goldstone bosons, the Higgs mechanism and Fermion mass generation, and the analysis of approximate symmetries such as the Chiral Symmetry of the QCD with light quarks, can all be recovered in a treatment where the vacuum state is symmetric. That is, that all the successful phenomenology usually attributed to SSB is fully recovered if we define QFT in an infrared regulating box (with periodic boundary conditions) for a sufficiently large box, and thus all physical conclusions remain unaltered and can be understood within the context of the analysis presented throughout this manuscript in which the main feature is the complete symmetry of the vacuum in symmetric theories.

Possible observational consequences of our analysis are not ruled out. One example is the prediction of long range correlations (clustering violation) independent of the infrared regulator Equation (27), or the suggestion (Section II C) that the uncertainties in the vacuum of certain operators in QCD might be large due to the approximate axial symmetry of the ground state.

We have argued, on the other hand, that the standard account for generation of topological defects in connection with the SSB in the field theoretical context is rather suspect, and that the classical topological stability of the resulting objects, might not be extensible to a quantum treatment.

We have further considered briefly a non-isolated quantum mechanical system and its possible states, together with the set of hypothesis that underly the statistical mechanics
considerations, which are in general not applicable to the case of an isolated quantum system. We have indicated that in those circumstances something akin to a spontaneous symmetry breaking might occur when considering the most stable states, which should not be confused with the true ground state of the system. One might well argue that the real world examples are better represented by the former and that the latter is an unjustified idealization. This might indeed be right for the vast majority of cases of interest, but even then, it is worthwhile understanding where does the justification for the treatment lie, i.e. in the fact that the system is not isolated? or in the characteristics of ground state of the isolated system’s Hamiltonian? Moreover, keeping these issues in mind is fundamental to uncover situations where the usual treatment might not be justified, and where other considerations might have to be brought to bear in understanding the behavior of the system in question. Such is the case where one is considering issues like the spontaneous symmetry breaking of quantum field theories which one would hope characterize the complete set of existing fields, and where no obvious external d.o.f. can be called upon to play the role of thermal environment, or aspects of quantum cosmology where the system under consideration is the whole universe and thus one is dealing, by definition, with an isolated quantum mechanical system. It might be that one can, in some of those situations, find a different way to justify the standard treatments, but it is important not to deceive oneself into believing that the standard justifications apply automatically to those cases as well.

We end by emphasizing that the view we have presented here, does not only clarify the issues but liberates us from the very uncomfortable position of having to rely on the infinitude of our universe as part of the justification of the standard accounts for the possibility for the phenomena of SSB that one finds in the more careful accounts of these issues [14].

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