LIMIT THEOREMS ON LARGE DEVIATIONS FOR SEMIMARTINGALES

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Abstract. We consider a sequence \( X^n = (X^n_t)_{t \geq 0}, n \geq 1 \) of semimartingales. Each \( X^n \) is a weak solution to an Itô equation with respect to a Wiener process and a Poissonian martingale measure and is in general non-Markovian process. For this sequence, we prove the large deviation principle in the Skorokhod space \( D = D_{[0,\infty)} \). We use a new approach based on exponential tightness. This allows us to establish the large deviation principle under weaker assumptions than before.

Main notations

\begin{align*}
(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P), & \quad \text{a stochastic basis;} \\
T_L(\mathbb{F}), & \quad \text{the set of stopping times (relative to a filtration } \mathbb{F} \text{ not exceeding } L; \\
D = D_{[0,\infty)}(D_{0,T}), & \quad \text{the Skorokhod space of all right continuous, having left hand limits real valued functions } X = (X_t)_{t \geq 0} (X = (X_t)_{0 \leq t \leq T}); \\
C = C_{[0,\infty)}(C_{0,T}], & \quad \text{the space of all right continuous functions from } D_{[0,\infty)}(D_{0,T}); \\
(E, \mathcal{E}) & \quad \text{a Blackwell space;} \\
\mathcal{B}(R), \mathcal{B}(R_+), \mathcal{B}(R_0), & \quad \text{the Borel } \sigma\text{-fields on } R, R_+, R_0 := R \setminus \{0\}; \\
\mathcal{D} = (\mathcal{D})_{t \geq 0}, & \quad \text{the family of } \sigma\text{-algebras } \mathcal{D}_t = \sigma(X_s, s \leq t), X \in D; \\
\rho(\ldots), & \quad \text{the Lindvall-Skorokhod metric on } D; \\
\overset{P}{\rightarrow}, & \quad \text{convergence in probability;} \\
x \wedge y = \min(x,y), & \quad x \vee y = \max(x,y); \\
X_t^* = \sup_{s \leq t} |X_s|, & \quad X_{t-}^* = \sup_{s < t} |X_s|, \quad \triangle X_t = X_t - X_{t-}, \quad X \in D;
\end{align*}

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1. Setting, basic concept, method

1. Let \( X^n = (X^n_t)_{t \geq 0}, n \geq 1 \) be a sequence of stochastic processes with path in \( D \).
   For each \( n \), \( X^n \) is a semimartingale on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}^n = (\mathcal{F}^n_t)_{t \geq 0}, P)\) and is a weak solution to the Ito equation \([1]\)
   \[
   X^n_t = x + \int_0^t a(s, X^n_s)ds + \frac{1}{\sqrt{n}} \int_0^t b(s, X^n_s)dW^n_s \\
   + \frac{1}{n} \int_0^t \int_E f(s, X^n_s, u)[p^n - q^n](ds, du), \tag{1.1}
   \]

where \( x \in R \), \((W^n_t)_{t \geq 0}\) is a Winer process, \( p^n(dt, du) \) is an integer valued random measure on \((R_+ \times E, \mathcal{B}(R_+) \otimes \mathcal{E})\), \( q^n(dt, du) \) is the compensator of \( p^n(dt, du) \) with
\[
q^n(dt, du) = ndtq(du), \tag{1.2}
\]

\( q(du) \) is a measure on \((E, \mathcal{E})\) (for all the definitions and facts from the martingale theory we refer the reader to \([2]\) and \([3]\).

We assume that functionals \( a(t, X), b(t, X) \) and \( f(t, X, u) \) \((t \in R_+, X \in D, u \in E)\) are \( \mathcal{P}(\mathbb{D})\)- and \( \mathcal{P}(\mathbb{D})\)-measurable and satisfy the following conditions:

**I.** (linear growth). For all \( X \in D \)
\[
|a(t, X)| \leq l_t(1 + X^n_t), \quad |b(t, X)| \leq l_t(1 + X^n_t), \quad |f(t, X, u)| \leq h(u)l_t(1 + X^n_t)
\]

for almost all \( t \in R_+ \) with respect to the Lebesgue measure, where \( l_t \) depends only on \( t \) and is a nondecreasing function of \( t \), \( h(u) \) is a \( \mathcal{E} \)-measurable function which satisfies the following condition of the Cramer type: the function
\[
K(\lambda) = \int_E (e^{\lambda h(u)} - 1 - \lambda h(u))q(du) \tag{1.3}
\]
is finite for all \( \lambda \in R; \)

**II.** (C-continuity in \( X \)). For all \( u \in E, X \in C \) and almost all \( t \in R_+ \) with respect to the Lebesgue measure, the following holds: if a sequence \( X^{(k)}(t) \in D, k \geq 1 \) is such that
\[
\lim_{k \to \infty} \sup_{s \leq t} |X^{(k)}_s - X_s| = 0,
\]
then (as \( k \to \infty \))
\[
a(t, X^{(k)}) \to a(t, X), \quad b(t, X^{(k)}) \to b(t, X), \quad f(t, X^{(k)}, u) \to f(t, X, u).
\]

If we further assume that the differential equation
\[
\dot{Y}_t = a(t, Y), \quad Y_0 = x \tag{1.4}
\]
has a unique solution, then (see Lemma 5.1) we the following ergodic property: for all \( T > 0 \) as \( n \to \infty \)
\[
\sup_{t \leq T} |X^n_t - Y_t| \overset{P}{\to} 0. \tag{1.5}
\]

This means that for any open set \( A \subset D \) which contains \((Y_t)_{t \geq 0}\)
\[
\lim_{n \to \infty} n^{-1} \log P(X^n \in A) = 0.
\]

In this paper we study large deviations of \( X^n, n \geq 1 \), i.e. the asymptotics (as \( n \to \infty \)) of \( n^{-1} \log P(X^n \in A) \) for sets \( A \in D \) not necessarily containing \( Y \).

The problem of large deviations (in \( D_{[0, T]} \)) for Markov processes of the form \([1]\) was considered by Wentzell and Freidlin \([4]\) and Wentzell \([5]\). They assumed that the functions \( a = a(t, x), b = b(t, x) \) and \( f = f(t, x, u) \) are bounded and continuous in \((t, x), x \in R \). Here we prove the large deviation principle in \( D \).
(\(D_{[0,T]}\) is a consequence) for the case when \(a, b\) and \(f\) may depend on the whole past. We also allow \(a, b\) and \(f\) to grow linearly in \(X\) and to be only measurable in \(t\) (Theorems 2.1 and 2.2). Also we give explicit conditions on \(a, b\) and \(f\) which ensure the assumptions of our main theorems while the conditions imposed in [4] and [5] are not easy to interpret in terms of the coefficients. We wish to present main ideas in a most refined form, so we consider the one-dimensional case. The multidimensional case does not seem to be principally different. Another distinctive feature of the paper is that we use a completely new approach based on a counterpart of the Prokhorov theorem.

For processes without jumps (of the diffusion type), along with similar lines as ours the result of [4] and [5] have been generalized by a number of authors. Stroock [6], Azencott [7], Baldi, P. and Chaleat-Maurel [8], Narita [9] and Friedman [10] studied homogeneous diffusions with unbounded coefficients. The case of non-homogeneous diffusion with the coefficients depending on the past in a general manner (but bounded) was considered by Cutland [11] who invoked the infinitesimal technique. All these results relate to \(C_{[0,T]}\). Micami [12] obtained the result from [5] and [4] under weaker assumptions.

The case of processes with independent increments was considered in [13-15].

The paper is organized as follows: in the rest of Section 1 we outline our approach to obtaining the large deviation principle. In Section 2 main results (Theorems 2.1 and 2.2) are stated. Section 3 contains the proof of the fundamental property of \(C\)-exponential tightness for \(X^n\). In Sections 4 and 5 the ground is laid for proving upper and lower bounds which are obtained in Sections 6 and 7. In Section 8 we prove Theorems 2.1 and 2.2. Section 9 gives explicit conditions on the functions \(a, b\) and \(f\) which are sufficient for the assumptions Theorems 2.1 and 2.2 to hold.

2. Recall main definitions. We assume that \(D\) is supplied with the Skorokhod-Lindvall metric and is thus a complete separable metric space. Following Varadhan [16], [17] we say that a sequence \(X^n, n \geq 1\) of processes with paths in \(D\) obeys the large deviation principle (in \(D\)) if

0) there exists a function \(I = I(\phi), \phi \in D\), with values in \([0, \infty]\) such that for any \(\alpha \geq 0\) the set \(\Phi(\alpha) := \{\phi \in D : I(\phi) \leq \alpha\}\) is compact in \(D\);

1) for any open set \(G \subset D\)

\[
\lim_{n} n^{-1} \log P(X^n \in G) \geq -\inf_{\phi \in G} I(\phi);
\]

1) for any closed set \(F \subset D\)

\[
\lim_{n} n^{-1} \log P(X^n \in F) \leq -\inf_{\phi \in F} I(\phi).
\]

A function \(I = I(\phi)\) meeting in (0) is called a rate function [17] (or good rate function, [18]).

Equivalent to (1) and (2) under (0) are the following conditions introduced by Wentzell and Freidlin [11]:

1') for all \(\delta > 0\) and \(\phi \in D\)

\[
\lim_{n} n^{-1} \log P(\rho(X^n, \phi) < \delta) \geq -I(\phi);
\]

2') for all \(\delta > 0\) and \(\alpha \geq 0\)

\[
\lim_{n} n^{-1} \log P(\rho(X^n, \Phi(\alpha) \geq \delta) \leq -\alpha.
\]

The large deviation principle in \(D_{[0,T]}\) is defined similarly.

Our approach for proving a large deviation principle relies on the concept of exponential tightness.
**Definition 1.1.** (15). A sequence $X^n = (X^n_t)_{t \geq 0}, n \geq 1$ of processes with paths in $D$ is said to be exponential tight if for any $C$ there exists a compact $K_C$ in $D$ such that

$$
\lim_{n} n^{-1} \log P(X^n \in D \setminus K_C) \leq -C.
$$

Other names for exponential tightness are “large deviation tightness” [15] and “strong tightness” [19].

The fundamental importance of this concept was made clear from Pukhalskii [19] who proved for large deviations of an analog of the Prokhorov theorem on the equivalence of weak relative compactness and tightness for a family of probability measures (Theorem 1.1 below. We first recall the following definition from [19].

**Definition 1.2.** A sequence $X^n = (X^n_t)_{t \geq 0}, n \geq 1$ of processes with paths in $D$ is said to obey the partial large deviation principle if any subsequence $\{n'\}$ of $\{n\}$ contains a further subsequence $\{n''\}$ such that $X^n_{n''}$ obeys the large deviation principle (with some rate function).

**Theorem 1.1.** (19). A sequence $X^n = (X^n_t)_{t \geq 0}, n \geq 1$ of processes with paths in $D$ is exponentially tight if and only if it obeys the partial large deviation principle.

**Remark 1.** The theorem holds for any arbitrary Polish space [19].

By Theorem 1.1 the following definition makes sense.

**Definition 1.3.** An exponentially tight sequence $X^n = (X^n_t)_{t \geq 0}, n \geq 1$ is said to be $C$-exponentially tight if for any subsequence $X^n_{n'}$ which obeys the large deviation principle with the rate function $I'(\phi)$, we have $I'(\phi) = \infty$ for all $\phi \in D \setminus C$.

Next we introduce the notion of the local large deviation principle.

**Definition 1.4.** A sequence $X^n, n \geq 1$ obeys the local large deviation principle (in $D$) if for any $\phi = (\phi_t)_{t \geq 0} \in D$ we have

$$
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta)
= \lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta)(= -J(\phi))
$$

(1.6)

(it can be $-\infty = -\infty$).

It is known [4] that the large deviation principle implies the local large deviation principle. The following theorem from [19] establishes that under exponential tightness the converse is also true.

**Theorem 1.2.** If a sequence $X^n = (X^n_t)_{t \geq 0}, n \geq 1$ of processes with paths in $D$ is exponentially tight and obeys the local large deviation principle (in $D$), then the function $J = J(\phi)$ from (1.6) is a rate function, and the sequence $X^n, n \geq 1$ obeys the large deviation principle with the rate function $I = J$.

**Proof of Theorem 1.2.** Assume that an exponential tight sequence $X^n, n \geq 1$ obeys the large deviation principle (1.6). By Theorem 1.1 we can choose a subsequence $\{n\}$ such that the subsequence $X^n_{n}$ obeys the large deviation principle with a rate function $I$. By [4] the sequence $X^n_{n}$ obeys the local large deviation principle with a rate function $I$. On the other hand, since

$$
\lim_{n} \leq \lim_{n} \leq \lim_{n} \leq \lim_{n},
$$
This means that \( \tilde{I} = J \) and hence \( J \) is a rate function.

Let \( G \subset D \) be an open set. Choose a subsequence \( \{n'\} \) of \( \{n\} \) such that
\[
\lim_{n} n^{-1} \log P( (X^n \in G) = \lim_{n'} (n')^{-1} \log P( (X^{n'} \in G) \tag{1.8}
\]
and from \( \{n'\} \) by Theorem 1.1 choose a subsequence \( \{n''\} \) such that \( X^{n''} \) obeys the large deviation principle with \( I'' \). Then by (1)
\[
\lim(n'')^{-1} \log P( (X^{n''} \in G) \geq - \inf_{\phi \in G} \ I''(\phi). \tag{1.9}
\]
Now we take \( \{n''\} \) as the subsequence \( \{\tilde{n}\} \) above. Then \( I'' = J \) and by (1.8) and (1.9)
\[
\lim_{n} n^{-1} \log P( (X^n \in G) \geq - \inf_{\phi \in G} \ J(\phi). \tag{1.10}
\]
We have proved (1), (2) is proved in the same way.

To establish (1.6) may be difficult since the Skorokhod metric is not easy to deal with. However in our particular case the following modification of the scheme suggested by Theorem 1.2 applies:

\(a)\) check that the sequence \( X^n, n \geq 1 \) is \( C\)-exponentially tight,

\(b)\) for all \( T > 0 \) and all \( \phi = (\phi_t)_{t \geq 0} \in C \) calculate
\[
J_T(\phi) = - \lim_{\delta \to 0} \lim_{n} n^{-1} \log P( (\sup_{t \leq T} |X^n_t - \phi_t| \leq \delta) \tag{1.11}
\]
\[
= - \lim_{\delta \to 0} \lim_{n} n^{-1} \log P( (\sup_{t \leq T} |X^n_t - \phi_t| \leq \delta), \tag{1.12}
\]
\(c)\) then
\[
I(\phi) = \begin{cases} 
\sup_T J_T(\phi) & \phi \in C, \\
\infty & \phi \in D \setminus C.
\end{cases}
\]
This is a consequence of the following theorem.

**Theorem 1.3.** Let a sequence \( X^n, n \geq 1 \) be \( C\)-exponentially tight, and let (1.10) hold for all \( \phi \in C \) and \( T > 0 \). Then \( X^n \) obeys the large deviation principle with the rate function given by \( \gamma \).

**Proof.** By Theorem 1.2 and the definition of \( C\)-exponential tightness it suffices to prove that (1.10) implies (1.6) for \( \phi \in C \). First we recall the definition of the Skorokhod-Lindvall metric in \( D \) (see, e.g. [2]).

Let for \( k = 1, 2, \ldots \),
\[
g_k(t) = I(t \leq k) + (k + 1 - t)I(k < t \leq k + 1), \quad t \geq 0.
\]

For \( X = (X_t)_{t \geq 0} \in D \) and \( \phi = (\phi_t)_{t \geq 0} \in D \) define \( X^k = (X^k_t)_{0 \leq t \leq 1} \in D_{[0,1]} \) and \( \phi^k = (\phi^k_t)_{0 \leq t \leq 1} \in D_{[0,1]} \), \( k = 1, 2, \ldots, \) by
\[
X^k_t = X_{a(t)g_k(a(t))}, \quad 0 \leq t \leq 1, \quad X^k_0 = 0,
\]
\[
\phi^k_t = \phi_{a(t)g_k(a(t))}, \quad 0 \leq t \leq 1, \quad \phi^k_0 = 0,
\]
where
\[
a(t) = - \log(1 - t), \quad 0 \leq t < 1. \tag{1.11}
\]
Let \( d_0 \) be the complete metric in \( D_{[0,1]} \) introduced by Prokhorov: if \( Y = (Y_t)_{0 \leq t \leq 1} \in D \) and \( Z = (Z_t)_{0 \leq t \leq 1} \in D \) then
\[
d_0(Y, Z) = \inf_{\mu \in \mathcal{M}} \left\{ \sup_{t \leq 1} |Y_t - Z_{\mu(t)}| + \sup_{0 \leq s < t \leq 1} \left| \log \frac{\mu(t) - \mu(s)}{t - s} \right| \right\},
\] (1.12)
where \( \mathcal{M} \) is a set of strictly increasing continuous functions \( \mu = (\mu(t))_{0 \leq t \leq 1} \) with \( \mu(0) = 0, \mu(1) = 1 \).

The Skorokhod-Lindvall metric is given by
\[
\rho(X, \phi) = \sum_{k=1}^{\infty} 2^{-k} \frac{\rho_k(X, \phi)}{1 + \rho_k(X, \phi)},
\] (1.13)
where
\[
\rho_k(X, \phi) = d_0(X^k, \phi^k), \quad k = 1, 2, \ldots
\] (1.14)
Since obviously
\[
d_0((Y, Z) \leq sup_{t \leq 1} |Y_t - Z_t|,
\]
we have from (1.14) and the definitions of \( X^k \) and \( \phi^k \) that for all \( k = 1, 2, \ldots \)
\[
\rho_k(X, \phi) \leq sup_{t \leq k} |X_t - \phi_t|,
\]
and by (1.13)
\[
\rho(X, \phi) \leq sup_{t \leq k+1} |X_t - \phi_t| + \frac{1}{2^k}, \quad k = 1, 2, \ldots
\] (1.15)
Next we show that for \( \delta \leq 1/4 \)
\[
\{ \rho_k(X, \phi) \leq \delta \} \subseteq \left\{ sup_{t \leq k} |X_t - \phi_t| \leq 2\delta + W_k(\phi, 4\delta_k) \right\},
\] (1.16)
where
\[
W_k(\phi, \sigma) = \sup_{\frac{u-v}{\sigma} \leq \sigma} |\phi_{ug_k(u)} - \phi_{v\delta_k(v)}|,
\] (1.17)
\( \delta_k = \delta / (1 + \alpha^{-1}(k+1)) \),
\( (\alpha^{-1} \) is the inverse of \( \alpha \).

First note that if
\[
\sup_{0 \leq s < t \leq 1} \left| \log \frac{\mu(t) - \mu(s)}{t - s} \right| \leq \frac{1}{4},
\]
then ((20, ch. 3, §14))
\[
\frac{1}{2} sup_{t \leq 1} |\mu(t) - t| \leq \sup_{0 \leq s < t \leq 1} \left| \log \frac{\mu(t) - \mu(s)}{t - s} \right|.
\]
Therefore if \( \rho_k(X, \phi) \leq 1 \), then by (1.12) and (1.14)
\[
\rho_k(X, \phi) \geq \inf_{\mu \in \mathcal{M}} \left\{ sup_{t \leq 1} |X^k_t - \phi^k_{\mu(t)}| + \frac{1}{2} sup_{t \leq 1} |\mu(t) - t| \right\},
\] (1.19)
So for any \( \delta \leq 1/4 \) we can find \( \mu_\delta \in \mathcal{M} \) such that
\[
\rho_k(X, \phi) \geq \left\{ sup_{t \leq 1} |X^k_t - \phi^k_{\mu_\delta(t)}| + \frac{1}{2} sup_{t \leq 1} |\mu_\delta(t) - t| \right\} - \delta.
\] (1.20)
Thus
\[
\{ \rho_k(X, \phi) \leq \delta \} \subseteq \{ \frac{1}{2} sup_{t \leq 1} |\mu_\delta(t) - t| \leq 2\delta \}.
\] (1.21)
From (1.11) it is obvious that for \( s, t \leq \alpha^{-1}(k+1) \)
\[
|\alpha(t) - \alpha(s)||t - s|/(1 - \alpha^{-1}(k+1))
\]
which easily implies in view of the definition of \( X^k \) and \( \phi^k \), and by (1.17), (1.18) and (1.21) that if \( \rho_k(X, \phi) \leq \delta \) then
\[
\sup_{t \leq 1} |\phi^k_t - \phi_{\mu_k(t)}| \leq W_k(\phi, 4\delta k).
\]
The latter by (1.20) and by the definition of \( X^k \) and \( \phi^k \) implies that \( \rho_k(X, \phi) \leq \delta(\leq \frac{1}{4}) \) then
\[
\rho_k(X, \phi) \geq \sup_{t \leq 1} |X^k_t - \phi^k_t| - W_k(\phi, 4\delta k) - \delta \geq \sup_{t \leq k} |X_t - \phi_t| - W_k(\phi, 4\delta k) - \delta
\]
which gives (1.16).

(1.15) and (1.16) together with (1.10) lead to (1.6) almost immediately. Indeed from (1.15) and (1.10) we have for all \( \delta > 0 \) and \( k = 1, 2, \ldots \)
\[
\lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta + 1/2^k) \geq \lim_{n} n^{-1} \log \left( \sup_{t \leq k+1} |X^n_t - \phi_t| \leq \delta \right) \geq -J_{k+1}(\phi).
\]

Since by \( \gamma \) \( J_{k+1}(\phi) \leq J(\phi) \) we obtain
\[
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta) \geq -J(\phi).
\]

Conversely (1.13) and (1.16) yield for \( \delta \leq 1/4 \) and \( k = 1, 2, \ldots \)
\[
\left\{ \rho(X, \phi) \leq \frac{\delta}{1 + \delta/2^k} \right\} \subseteq \{ \rho_k(X_k, \phi) \leq \delta \} \subseteq \left\{ \sup_{t \leq k} |X_t - \phi_t| \leq 2\delta + W_k(\phi, 4\delta k) \right\}.
\]

Since the function \( \phi, g_k(t) \) is continuous, so in view of (1.17) and (1.18)
\[
W_k(\phi, 4\delta k) \to 0 \text{ as } \delta \to 0,
\]
and then by (1.23) and (1.10),
\[
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta) \leq \lim_{\delta \to 0} \lim_{n} n^{-1} \log P\left( \sup_{t \leq k} |X^n_t - \phi_t| \leq \delta \right) \leq -J_k(\phi).
\]

Since by \( \gamma \) \( J(\phi) = \sup_k J_k(\phi) \), (1.22) and (1.24) prove (1.6) for \( \phi \in C \).

Remark 1. Note that the fact in the theorem has been noticed by Dawson and Gärtner ([24, Th. 5.1-5.3]) for the particular problem they studied.

Remark 2. Along with the Skorokhod topology one can also consider the local uniform topology on \( D \) (see e.g. [3]) and study the large deviation principle for this topology, should \( X^n \) remain measurable with respect to the corresponding \( \sigma \)-field, which the case of the solution (1.1). As the local uniform topology is stronger than the Skorokhod topology, the large deviation principle for the local uniform topology implies that for the Skorokhod topology. In fact, the result [4] and [5] are for the uniform topology on \( D_{[0, T]} \). However, as far as the solution of (1.1) are concerned the corresponding rate function (for the Skorokhod topology) is infinity at discontinuous elements of \( D \) (Theorems. 2.1 and 2.2 below), and in such a case the large deviation principles for both topologies are easily seen to be equivalent (the situation is the same as for weak convergence, see [20, ch. 3, §18]). In particular in Theorem 1.3 we have the large deviation principle for the local uniform topology as well (provided the measurability is preserved).
2. Cumulant. Legendre-Fenchel transform. Main results

1. For \( \lambda \in R, t \in R_+, X \in D \) define the cumulant

\[
G(\lambda; t, X) = \lambda a(t, X) + \frac{\lambda^2}{2} b^2(t, X) + \int_E (e^{\lambda f(t,u)} - 1 - \lambda f(t,u))q(du). \tag{2.1}
\]

By condition I, \( G(\lambda; t, X) \) is finite and smooth in \( \lambda \) for all \( \lambda \in R \) and \( X \in D \), and almost all \( t \in R_+ \) with respect to the Lebesgue measure, with the first two derivatives expressed as

\[
g(\lambda; t, X) := G_\lambda'(\lambda; t, X) = a(t, X) + \lambda b^2(t, X) + \int_E f(t, X, u)(e^{\lambda f(t,u)} - 1 - \lambda f(t,u))q(du).
\]

\[
G''_{\lambda\lambda}(\lambda; t, X) = b^2(t, X) + \int_E f^2(t, X, u)e^{\lambda f(t,u)}q(du).
\] \tag{2.2}

In particular, the cumulant is convex in \( \lambda \).

**Remark.** Throughout the paper we assume that \( R_+ \) is supplied by the Lebesgue measure and so in the sequel we omit specific reference on it.

Define the Legendre-Fenchel transform of \( G(\lambda; t, X) \) (cf., e.g. [21])

\[
H(y; t, X) = \sup_{\lambda \in R} \{\lambda y - G(\lambda; t, X)\}. \tag{2.3}
\]

Since the cumulant is continuous in \( \lambda \) “sup” in (2.3) may be taken only over rational \( \lambda \in R \) for all \( y \in R, t \in R_+, X \in D \). Therefore, \( H(y; t, X) \) is \( B(R) \otimes \mathcal{P}(\mathbb{D}) \)-measurable. Besides \( H(0; t, X) = 0 \).

2. For our main result we need two additional conditions.

For \( \phi = (\phi_t)_{t \geq 0} \in D \) and \( T > 0 \) define

\[
I_T(\phi) = \begin{cases} 
\int_0^T H(\dot{\phi}_t; t, \phi)dt, & \text{if } \phi = x(\equiv X_0^t), \\
\infty, & \text{otherwise}.
\end{cases}
\]

The following condition enables us to calculate “sup” in (2.3).

**III** (solvability condition). For all \( T > 0 \) and for all \( \phi \in D \) with \( I_T(\phi) < \infty \) there exist \( \delta_{T, \phi} > 0 \) and a \( B(R) \otimes \mathcal{P}(\mathbb{D}) \)-measurable function \( \Lambda_{T, \phi}(y; t, X) \), \( y \in R, t \in R_+, X \in D \), such that for all \( y \in R \) and \( X \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta_{T, \phi} \) we have

\[
y = g(\Lambda_{T, \phi}(y; t, X), t, X) \tag{2.4}
\]

for almost all \( t \leq T \). In additional, \( \Lambda_{T, \phi}(y; t, X) \) has the following properties.

i) (local boundedness). For every \( N > 0 \) there exists \( r \) (depending on \( N, \phi, \delta_{T, \phi} \) and \( T > 0 \)), such that for all \( y \in R \) with \( |y| \leq N \) and \( X \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta_{T, \phi} \) we have

\[
|\lambda_{T, \phi}(y; t, X)| \leq r
\]

for almost all \( t \leq T \).

ii) (C-continuity). For all \( y \in R \), \( \Lambda_{T, \phi}(y; t, X) \) is \( C_{[0,T]} \)-continuous in \( X \) at \( X = \phi \in C \) for almost all \( t \leq T \), i.e. if \( X^{(k)} \in D, k \geq 1 \), then the implication holds

\[
\lim_{k \to \infty} \sup_{t \leq T} |X^{(k)}_t - \phi_t| = 0 \Rightarrow \lim_{k \to \infty} \Lambda_{T, \phi}(y; t, X^{(k)}) = \Lambda_{T, \phi}(y; t, \phi).
\]

**Remark.** (2.2) implies that under condition **III** “sup” in (2.3) is attained at \( \lambda = \Lambda_{T, \phi}(y; t, X) \): for all \( y \in R \) and \( X \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta_{T, \phi} \), we have for almost all \( t \leq T \)

\[
H(y; t, X) = \Lambda_{T, \phi}(y; t, X)y - G(\Lambda_{T, \phi}(y; t, X); t, X). \tag{2.5}
\]
IV. For all absolutely continuous functions $\phi = (\phi_t)_{t \geq 0}$ with $\phi_0 = x$ and all $T > 0$ the implication holds
$$\int_0^T H(\dot{\phi}_t; t, \phi_0) dt < \infty \Rightarrow \int_0^T \sup_N H(\dot{\phi}_t^N; t, \phi_0^N) dt < \infty,$$
where
$$\phi_t^N = x + \int_0^t \phi_s I(|\dot{\phi}_s| \leq N) ds.$$

**Remark.** Conditions III and IV are basically conditions of nondegeneracy (see Theorem 9.1 below).

4. Now we state main results.

**Theorem 2.1.** Let conditions I-IV hold. Then the sequence $X^n, n \geq 1$, of semimartingales defined by (1.1) obeys the large deviation principle in $D$ with the rate function
$$I(\phi) = \begin{cases} \int_0^\infty H(\dot{\phi}_t; t, \phi_0) dt, & \phi \text{ is absolutely continuous, } \phi_0 = x(\equiv X^n_0), \\ \infty, & \text{otherwise.} \end{cases}$$

5. Theorem 2.1 does not include the Poisson process. So we consider separately the following case covers processes of the Poisson type: for all $T \in \mathbb{R}^+$ all $X \in D$ and $u \in E$
$$b(t, X) = 0 \text{ a.e., \quad (2.6)}$$
$$f(t, X, u) > \epsilon_T \text{ a.e. on } [0, T], \quad (2.7)$$
$$q(E) < \infty. \quad (2.8)$$

**Theorem 2.2.** Let (2.6)-(2.8) hold and the equation
$$d\psi/dt = a(t, \psi) - \int_E f(t, \psi, u)q(du), \quad \psi_0 = x \quad (2.9)$$
has a unique solution. If conditions I, II and IV are satisfied and condition III may fail only for the function $\phi$ which is a solution of (2.9), then the assertion of Theorem 2.1 remains true.

**Remark 1.** Since for $X \in D$, $X^*_t = X^*_t$ almost everywhere in condition I $X^*_t$ may be replaced by $X^*_t$.

**Remark 2.** Assumption (2.7) in Theorem 2.2 may be replaced by the assumption $f(t, X, u) \geq q(t, u) > 0$ with
$$\int_0^T \int_E g^{-1}(t, u) q(du) < \infty.$$ Analogously one can consider the case when $f$ is negative.

**Remark 3.** Since $I(\phi) = \infty$ for $\phi \in D \setminus C$ it is easy to deduce applying the continuous mapping theorem (see [18, Lemma 2.1.4], [19, Th. 2.2]) that for all $T > 0$ the sequence $((X^n_t)_{0 \leq t \leq T})_n \geq 1$ obeys the large deviation principle in $D_{[0, T]}$ with the rate function $I_T$ which is defined above (since $I_T$ depends on the values of $\phi$ up to $T$, we can as well consider it as a function on $D_{[0, T]}$).

**Remark 4.** All the results are retained for the (local) uniform topology in $D(D_{[0, T]})$ (see the remark at the end of Section 1).
3. Exponential tightness

1. In this section we prove that under condition \( \mathbf{I} \) the sequence \( X^n, n \geq 1 \), defined by (1.1), is \( C \)-exponentially tight. For this, we use the next theorem on \( C \)-exponential tightness of a sequence of adapted processes.

**Theorem 3.1.** Let \( X^n = (X^n_t)_{t \geq 0}, n \geq 1 \) be a sequence of processes with paths in \( D \). Each \( X^n \) is defined on a stochastic basis \( (\Omega, F, \mathbb{P}, P) \). Assume that the following is satisfied: for all \( L > 0 \) and \( \eta > 0 \)

\[
i \lim_{c \to \infty} \lim_{n \to \infty} n^{-1} \log P(X^n_L \geq c) = -\infty, 
\]

\[
ii \lim_{\delta \to 0} \sup_{\tau \in T_{\alpha}(\mathbb{F}^n)} n^{-1} \log P(\sup_{t \leq \delta} |X^n_{\tau+t} - X^n_{\tau}| \geq \eta) = -\infty.
\]

Then the sequence \( X^n, n \geq 1 \) is \( C \)-exponentially tight.

**Proof.** By Theorem 4.4 [19], i) and ii) imply that the sequence \( X^n, n \geq 1 \) is exponentially tight. By Theorem 4.6 [19] it is proved that ii) also implies \( C \)-exponential tightness. We reproduce the proof here in more details.

Let \( \phi \in D \setminus C \), We can find \( k \in \{2, 3, \ldots\} \) such that there exists \( s_0 \in (0, k-1] \) with \( \Delta \phi_{s_0} \neq 0 \). Let us show that for \( \forall \phi \in D \) we have if \( \delta \leq (1 - \alpha^{-1}(k) - \alpha^{-1}(k-1))/4 \) (we use the notation of the proof of Theorem 1.3)

\[
\{ \rho_k(Y, \phi) \leq \delta \} \subseteq \left\{ \sup_{|s_0-t| \leq 4\delta(k)} |Y_{s_0} - Y_t| \geq |\Delta \phi_{s_0}| - 2\delta \right\}, \tag{3.1}
\]

where \( \delta(k) = \delta/(1 - \alpha^{-1}(k)) \).

As in the proof of Theorem 1.3, we can choose \( \mu_\delta = (\mu_\delta(t))_{0 \leq t \leq 1} \in \mathfrak{M} \). so that

\[
\rho_k(Y, \phi) \geq \sup_{t \leq 1} |Y_{\mu_\delta(t)}^k - \phi_t| + \frac{1}{2} \sup_{t \leq 1} |\mu_\delta(t) - t| - \delta.
\]

Then if \( \rho_k(Y, \phi) \leq \delta \) we have

\[
\frac{1}{2} \sup_{t \leq 1} |\mu_\delta(t) - t| \leq 2\delta
\]

and hence, taking \( t_0 = \alpha^{-1}(s_0) \)

\[
|\Delta Y_{\mu_\delta(t_0)}^k| \leq 2 \sup_{|t-s| \leq 4\delta \atop 0 \leq s, t \leq 1} |Y_{t_0}^k - Y_t^k|. \tag{3.2}
\]

As by the triangular inequality

\[
|\Delta \phi_{t_0}^k| \leq |\Delta Y_{\mu_\delta(t_0)}^k - \phi_{t_0}^k| + |\Delta Y_{\mu_\delta(t_0)}^k| \leq 2 \sup_{t \leq 1} |\Delta Y_{\mu_\delta(t_0)}^k - \phi_t^k| + |\Delta Y_{\mu_\delta(t_0)}^k|,
\]

so we deduce from (3.2) that if \( \rho_k(Y, \phi) \leq \delta \), then

\[
\sup_{|t_0-t| \leq 4\delta \atop 0 \leq t \leq 1} |Y_{t_0}^k - Y_t^k| \geq |\Delta \phi_{t_0}^k|/2 - \sup_{t \leq 1} |Y_{\mu_\delta(t)}^k - \phi_t^k|,
\]

and therefore

\[
\sup_{|t_0-t| \leq 4\delta \atop 0 \leq t \leq 1} |Y_{t_0}^k - Y_t^k| \geq |\Delta \phi_{t_0}^k|/2 - (\rho_k(Y, \phi) + \delta).
\]

Now \( \alpha(t_0) \leq k - 1 \) and \( 4\delta \leq \alpha^{-1}(k) - \alpha^{-1}(k-1) \). So by the definition of \( Y^k \) (see the proof of Theorem 3.1)

\[
\sup_{|t_0-t| \leq 4\delta \atop 0 \leq t \leq 1} |Y_{t_0}^k - Y_t^k| = \sup_{|t_0-t| \leq 4\delta \atop 0 \leq t \leq 1} |Y_{\alpha(t_0)}^k - Y_{\alpha(t)}^k|.
\]
Noticing that
\[ |\alpha(t) - \alpha(t_0)| \leq 4\delta/(1 - \alpha^{-1}(k)) \]
for \(|t - t_0| \leq 4\delta\), we obtain (3.1).

Obviously since \(s_0 < k\)
\[ P\left( \sup_{|s_0 - t| \leq 4\delta(k)} |X^n_{s_0} - X^n_t| \geq \eta \right) \leq 2 \sup_{\tau \leq T(k)} P\left( \sup_{s \leq 4\delta(k)} |X^n_{\tau + s} - X^n_s| \geq \eta \right). \]

Therefore, using (3.1) and ii) we conclude that
\[ \lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho_k(X^n, \phi) \leq \delta) = -\infty. \]

Since \(\rho(X^n, \phi) \geq 2^{-k} \rho_k(X^n, \phi)/(1 + \rho_k(X^n, \phi))\) it then follows that
\[ \lim_{\delta \to 0} \lim_{n} n^{-1} \log P(\rho(X^n, \phi) \leq \delta) = -\infty, \phi \in D \setminus C. \]

Now assume that a subsequence \(X^n\) obeys the large deviation principle with a rate function \(I\). Then as it obeys the local large deviation principle, by the above we have \(I(\phi) = \infty, \phi \in D \setminus C\).

2. Now we state the main result on \(C\)-exponential tightness of the sequence of solutions of (1.1).

**Theorem 3.2.** Under condition I (linear growth) the sequence \(X^n, n \geq 1\) of solutions of (1.1) is \(C\)-exponentially tight in \(D\).

3. The proof is based on a number of lemmas.

**Lemma 3.1.** Let \(Y = (T_t)_{t \geq 0}\) be a semimartingale. Denote by \((B, C, \nu)\) the triplet of predictable characteristic of \(Y\) ([2]), and assume that \(\nu\) satisfies the following analogue of the Cramer condition
\[ \int_{0}^{t} \int_{R_0} [e^{\lambda x} - 1 - \lambda x] \nu(ds, dx) < \infty, \ P-a.s., \ t > 0, \ \lambda \in R. \tag{3.3} \]

Assume that for \(T > 0\) there exists a convex function \(H = H(\lambda), \lambda \in R\) with \(H(0) = 0\) and such that for all \(\lambda \in R\) and \(t \leq T\)
\[ \lambda \overline{B}_t + \lambda^2 C_t / 2 + \int_{0}^{t} \int_{R_0} [e^{\lambda x} - 1 - \lambda x] \nu(ds, dx) \leq H(\lambda \xi), \ P-a.s., \]
where \(\overline{B}_t = B_t + \int_{0}^{t} \int_{R_0} x I(|x| > 1) \nu(ds, dx)\) and \(\xi\) is a nonnegative random variable defined on the same probability space as \(Y\).

Then for all \(c > 0\) and \(\eta > 0\)
\[ P(Y^n_T \geq \eta) \leq P(\xi > c) + \exp \left\{ - \sup_{\lambda \in R} \lambda \eta - TH(\lambda \xi) \right\}. \]

The proof is analogous to the proof of Theorem 4.13.2 in [2].

**Lemma 3.2.** Assume that \(X^n = (X^n_t)_{t \geq 0}\) for each \(n\) is a special semimartingale on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}^n, P)\) with the decomposition
\[ X^n_t = x + A^n_t + M^n_t, \ x \in R. \tag{3.4} \]

Assume that the predictable process of locally bounded variation \(A^n = (A^n_t)_{t \geq 0}\) and the local martingale \(M^n = (M^n_t)_{t \geq 0}\) have the following properties:

1) for all \(T > 0\) there exists \(a_0 = a_0(T)\) such that for all \(t \leq T\)
\[ \text{Var}(A^n)_t \leq a_0 \int_{0}^{t} (1 + X^n_s^2) ds; \tag{3.5} \]
2) $(M^n)^{2n} (M^n)^{2n}_{t \geq 0}$ is a special semimartingale with the decomposition

$$(M^n)^{2n} = V^n + L^n,$$  \hspace{1cm} (3.6)

where $L^n = (L^n_t)_{t \geq 0}$ is a local martingale, and $V^n = (V^n_t)_{t \geq 0}$ is an increasing predictable process such that for all $T > 0$ there exists $a_1 = a_1(T)$ and $a_2 = a_2(T)$ for which

$$V^n_t \leq (a_1 + a_2 n) \int_0^t \{1 + (M^n_s)^{2n}\}ds$$  \hspace{1cm} (3.7)

if $t \leq T$. Then for all $T > 0$

$$\lim_{n \to \infty} n^{-1} \log E(X^n_T)^{2n} < \alpha,$$

where $\alpha$ depends only on $T$, $a_0$, $a_1$, $a_2$.

Proof. If $s \leq t \leq T$, then by (3.4) and (3.5)

$$X^n_s \leq |x| + a_0 \int_0^s \{1 + X^n_u\}du + M^n_s.$$

This and the Gronwall inequality imply that

$$X^n_t \leq (|x| + a_0 t + M^n_t) \exp(a_0 t)$$  \hspace{1cm} (3.8)

and hence, by Jensen inequality

$$(X^n_t)^{2n} \leq 3^{2n-1} \{(|x|^{2n} + (a_0 t)^{2n} + (M^n_t)^{2n} \exp(2a_0 t), t \leq T.$$

It then follows that for fixed $T$ we can find a constant $C$ which depends on $x$, $a_0$ and $T$, and such that

$$(X^n_T)^{2n} \leq 1/2(C)^n \{1 + (M^n_T)^{2n}\}.$$

Hereafter

$$E(X^n_T)^{2n} \leq 1/2(C)^n \{1 + E(M^n_T)^{2n}\} \leq 1/2(C)^n \{1 + \sqrt{E(M^n_T)^{2n}}\}.$$  \hspace{1cm} (3.9)

Now we estimate $E(M^n_T)^{2n}$. Assume that $t \leq T$. By the version of the Doob inequality in [2] (Th. 19.2), decomposition (3.6) and inequality (3.7) we have

$$E(M^n_T)^{2n} \leq (2n/(2n-1))^{2n}V^n_T \leq 4(a_1 + a_2 n) \int_0^t \{1 + C_s^{2n}\}ds.$$

Therefore, if $E(M^n_T)^{2n} < \infty$, then the Gronwall inequality yields

$$E(M^n_T)^{2n} \leq (a_1 + a_2 n)T \exp\{4(a_1 + a_2 n)T\}$$

(in general case, (3.6) implies the local integrability of $(M^n_T)^{2n}$ and the above inequality follows by localization).

The latter and (3.9) imply that

$$\lim_{n \to \infty} 1/n \log E(X^n_T)^{2n} \leq \log C + 4a_2 T (= \alpha).$$

4. We prove Theorem 3.2 by verifying conditions i) and ii) of Theorem 3.1.

To check i) we use Stroock's idea ([6]), Lemma 4.12, ch. 4). By the Chebyshev inequality

$$P(X^n_L > c) \leq c^{-2n} \log(1/n).$$

Hence i) holds if

$$\lim_{n \to \infty} 1/n \log E(X^n_L)^{2n} < \infty.$$  \hspace{1cm} (3.10)

To prove (3.10), we apply Lemma 3.2. By (1.1) we have decomposition (1.4) in which

$$A^n_t = \int_0^t a(s, X^n)\,ds,$$  \hspace{1cm} (3.11)
\[ M^n_t = 1/\sqrt{n} \int_0^t b(s, X^n) dW^n_s + 1/n \int_0^t \int_E f(s, X^n, u)[p^n - q^n](ds, du). \]  

(3.12)

Conditions (3.5) of Lemma 3.2 in view of condition I (of linear growth).

Now, we verify (3.6) and (3.7). By the Itô formula (cf., e.g. [2], [3]),

\[ (M^n_t)^{2n} = 2n \int_0^t \int_0^t (M^n_{s-})^{2n-1} dM^n_s + (2n - 1) \int_0^t (M^n_s)^{2n-2}b^2(s, X^n)ds + U^n_t, \]

(3.13)

where

\[ U^n_t = \sum_{s \leq t} \{(M^n_s)^{2n} - (M^n_{s-})^{2n} - 2n(M^n_{s-})^{2n-1} \triangle M^n_s\} \]

(3.14)

is an increasing process. If the process \( U^n_t = (U^n_t)_{t \geq 0} \) is locally integrable, then we have (3.6) with

\[ V^n_t = (2n - 1) \int_0^t (M^n_s)^{2n-2}b^2(s, X^n)ds + \tilde{U}^n_t \]

(3.15)

and

\[ L^n_t = 2n \int_0^t (M^n_{s-})^{2n-1} dM^n_s + U^n_t - \tilde{U}^n_t, \]

where \( \tilde{U}^n_t = (\tilde{U}^n_t)_{t \geq 0} \) is the compensator of \( U^n_t \). Let us show that \( U^n_t \) is locally integrable. By (1.1), \( \triangle X^n_s = \triangle M^n_s \) where in view of (1.1) and (1.2)

\[ \triangle X^n_s = 1/n \int f(s, X^n, u)p^n(\{s\}, du). \]

Thus if we denote

\[ H^n(s, u) = \{M^n_{s-} + 1/nf(s, X^n, u)\}^{2n} - (M^n_{s-})^{2n} - 2(M^n_{s-})^{2n-1}f(s, X^n, u), \]

then

\[ U^n_t = \int_0^t H^n(s, u)p^n(ds, du). \]

Since \( H^n(s, u) \geq 0 \) and \( H^n(s, u) \) is \( \bar{P} \)-measurable, and \( q^n(ds, du) \) is the compensator of \( p^n(ds, du) \), the process \( U^n_t \) is locally integrable and the compensator

\[ \tilde{U}^n_t = \int_0^t \int_E H^n(s, u)q^n(ds, du), \]

(3.16)

provided the integral in (3.16) is finite \( P \)-a.s. for all \( t > 0 \). Indeed, for any stopping time \( \tau \)

\[ EU^n_\tau = E\tilde{U}^n_\tau \]

(3.17)

(cf., e.g. Th. 3.2.1 in [2]). Furthermore, \( \tilde{U}^n_t \) being an increasing finite-valued predictable process, is locally bounded (Lemma 1.6.1 in [2]). It follows then that (3.17) implies both the local integrability of \( U^n_t \) and the fact that \( \tilde{U}^n_t \) is the compensator of \( U^n_t \).

Now we prove that that integral in (3.16) is finite \( P \)-a.s. To this end denote

\[ a = 1 \vee M^n_{s-}, \quad b = |1/nf(s, X^n, u)|. \]

(3.18)

Applying the mean value theorem to \( H^n(s, u) \) we have

\[ H^n(s, u) \leq n(2n - 1)(a + b)^{2n-2}b^2 = n(2n - 1)a^{2n-2}(1 + b/a)^{2n-2}b^2 \]

\[ \leq n(2n - 1)a^{2n-2} \exp\{(2n - 2)b/a\}\mathbb{I}. \]

(3.19)

Since \( X^n \) admits a representation (3.4) with \( A^n \) from (3.11) and \( M^n \) from (3.12), and \( A^n \) by condition I satisfies (3.5), it follows from the proof of Lemma 3.2 that
for $X_t^{n*}$ an inequality of the type (3.8) holds. Hence, there exists $c > 1$ such that for $s \leq T$

$$X_s^{n*} \leq c(1 \vee M_s^{n*}).$$

(3.20)

Therefore (see (3.18)), $a \geq 1 \vee (c^{-1}X_s^{n*}) (\equiv a')$. So by (3.19), we have for $n \geq 2$

$$H^n(s, u) \leq n(2n - 1)(2n - 2)^{-2}a^{2n}\exp\{(2n - 2)b/a'[2(2n - 2)b/a']^2. $$

(3.21)

Now, we estimate $(2n - 2)b/a'$. By condition I

$$(2n - 2)b/a' = (2n - 2)n^{-1}|f(s, X^n, u)|/[1 \vee (c^{-1}X_s^{n*})]
\leq (2n - 2)n^{-1}l_sh(u)(1 + X_s^{n*})/[1 \vee (c^{-1}X_s^{n*})].$$

Since (recall that $c > 1$)

$$1 \vee (c^{-1}X_s^{n*}) \geq 1/2(1 + c^{-1}X_s^{n*}) \geq (2c)^{-1}(1 + X_s^{n*})$$

we have

$$(2n - 2)b/a' \leq 4cl_sh(u) \leq 4cl_T(h(u) (= r_T h(u)).$$

So (3.2) implies that

$$H^n(s, u) \leq 2a^{2n}(r_T h(u))^2 \exp(r_T h(u)), \ n \geq 2.$$}

The latter allows us to obtain an estimate for the integral in (3.16). We have (see (3.18)) and the definition of $a$)

$$\widetilde{U}_t^n \leq 2 \int_0^t \int_E (1 \vee M_s^{n*})^{2n}(r_T h(u))^2 \exp(r_T h(u))q^n(ds, du).$$

Choose $\beta > 0$ such that the inequality $x2e^x \leq e^{\beta x} - 1 - \beta x$ holds for all $x > 0$. Then by (1.3)

$$\gamma = \int_E (r_T h(u))^2 \exp(r_T h(u))q(du) < \infty.$$}

Then, since $q^n((ds, du)$ equals $nds\{du\}$ (see (1.2)), so

$$\widetilde{U}_t^n \leq 2\gamma nt(1 + M_t^{n*})^{2n} < \infty, \ P - a.s.$$ (3.22)

So we have established that $U^n$ is locally integrable, and as a sequence, (3.6) holds with $V^n$ from (3.15).

Now we prove (3.7). To this end note that in view of (3.22)), the process $V^n$ from (3.15) can be estimated as

$$V_t^n \leq (2n - 1)\int_0^t (M_s^{n*})^{2n-2}b^2(s, X^n)ds + 2\gamma n \int_0^t (1 + M_s^{n*})^{2n}ds.$$ (3.23)

By condition I, $b^2(s, X^n) \leq b_T(1 + X_s^{n*})^2$, so using (3.20), we deduce that $V_t^n$ satisfies (3.7) for some $a_1$ and $s_2$ which depend only on $T$.

Thus, all the conditions of Lemma 3.2 are met, and hence, (3.10) holds. This ends the proof of condition i) of Theorem 3.1.

Check ii). Denote

$$A_{n, \delta, \tau} = \{\sup_{t \leq \delta}|X^n_{\tau+t} - X^n_{\tau}| > \eta\}$$

and

$$A_{n, \delta, \tau}^+ = \{\sup_{t \leq \delta}(X^n_{\tau+t} - X^n_{\tau}) > \eta\}$$
$$A_{n, \delta, \tau}^- = \{\sup_{t \leq \delta}(X^n_{\tau} - X^n_{\tau+t}) > \eta\}.$$ 

Since

$$P(A_{n, \delta, \tau}) = P(A_{n, \delta, \tau}^+ \cup A_{n, \delta, \tau}^-) \leq 2\{P(A_{n, \delta, \tau}^+) \lor P(A_{n, \delta, \tau}^-)\},$$
it suffices to show that
\[
\lim_{\delta \to 0} \lim_{n} 1/n \log \sup_{\tau \in T_L(F_{\delta})} P(A_{n,\delta,\tau}^\pm) = -\infty. \tag{3.24}
\]

We are going to establish (3.24) for $A_{n,\delta,\tau}^+$. We shall apply Lemma 3.1. Denote $Y_t^n = X_{n+t}^n - X_t^n$. The process $Y_t^n = (Y_t^n)_{t \geq 0}$ is a semimartingale with respect to the filtration $(\mathcal{F}^n_{t+})_{t \geq 0}$, and by (1.1) and (1.2), its triplet of predictable characteristics $(B^n, C^n, \nu^n)$ is as follows
\[
B_t^n = \int_{\tau}^{\tau+t} a(s, X^n)ds - \int_{\tau}^{\tau+t} f(s, X^n, u)I(|f(s, X^n, u)| > n)q(du)du,
\]
\[
\left(\tilde{B}_t^n\right) = \int_{\tau}^{\tau+t} a(s, X^n)ds,
\]
\[
C_t^n = n^{-1} \int_{\tau}^{\tau+t} b^2(s, X^n)ds,
\]
\[
\nu^n((0, t], \Gamma) = n \int_{\tau}^{\tau+t} \int_{R} I(f(s, X^n, u)/n \in \Gamma)q(du)ds, \quad \Gamma \in \mathcal{B}(R_0).
\]

Then
\[
\lambda \tilde{B}_t^n + \lambda^2/2C_t^n + \int_{0}^{t} \int_{E} (e^{\lambda x} - 1 - \lambda x)\nu^n(ds, dx)
\]
\[
= \lambda \int_{\tau}^{\tau+t} a(s, X^n)ds + \lambda^2/2n \int_{\tau}^{\tau+t} b^2(s, X^n)ds
\]
\[
+ n \int_{\tau}^{\tau+t} \int_{E} (e^{\lambda f(s, X^n, u)} - 1 - \lambda f(s, X^n, u))q(du)ds.
\]

Condition I implies that $Y^n$ satisfies the assumptions of Lemma 3.1 for $T = \delta \leq 1$ with the function
\[
H(\lambda) = |\lambda| + \lambda^2/2n + n \int_{E} (e^{|\lambda| h(u)/n} - 1 - |\lambda| h(u)/n)q(du)
\]
and the random variable $\xi = l_{L+1}(1 + X_{L+1}^n)$.

Choose arbitrary $c > 0$. We have
\[
\sup_{\lambda \in R} [\lambda \eta - \delta H^n(\lambda c)] = \sup_{\lambda > 0} [\lambda \eta - \delta H^n(\lambda c)] = n\mathcal{L}(\delta),
\]
where
\[
\mathcal{L}(\delta) = \sup_{\mu > 0} \left[ \mu (\eta - \delta c) - \delta \mu^2 c^2/2 - \delta \int_{E} (e^{\mu h(u)} - 1 - \mu h(u))q(du) \right].
\]

It is easy to see that
\[
\lim_{\delta \to 0} \mathcal{L}(\delta) = \infty. \tag{3.25}
\]

Hence by lemma 3.1 we have for $\delta < 1$
\[
P\left\{ \sup_{t \leq \delta} (X_{t+}^n - X_t^n) > \eta \right\} \leq P\{l_{L+1}(1 + X_{L+1}^N) > c\} + e^{-n\mathcal{L}(\delta)}
\]
\[
\leq 2[P\{X^* > c/l_{L+1} - 1\} \lor e^{-n\mathcal{L}(\delta)}].
\]

This in view of (3.25) yields
\[
\lim_{\delta \to 0} \lim_{n} 1/n \log P\left\{ \sup_{t \leq \delta} (X_{t+}^n - X_t^n) > \eta \right\}
\]
\[
\leq \lim_{n} 1/n \log P\{X_{L+1}^n > c/l_{L+1} - 1\}.
\]
And consequently, (3.24) holds for \( A_{n,\delta,\tau}^+ \), as we proved earlier

\[
\lim_{c \to \infty} \lim_{n \to \infty} 1/n \log P \{ X_{L+1}^n > c/l_{L+1} - 1 \} = -\infty.
\]

(3.24) for \( A_{n,\delta,\tau}^- \) is established similarly.

4. Multiplicative decomposition. Change of measure

1. Let \( X^n = (X^n_t)_{t \geq 0} \) be a semimartingale defined in (1.1). In what follows, one of the main actors is the process

\[
U^n_t(\lambda) = \exp \left( n \int_0^t \lambda(s)xX^n_s \right),
\]

where \( \lambda = (\lambda(t))_{t \geq 0} \), \( \lambda(t) \leq r \) for almost all \( t \), is a predictable bounded function.

Applying the Itô formula, it is easy to see that the process \( Z^n(\lambda) = (Z^n_t(\lambda))_{t \geq 0} \) with

\[
Z^n_t(\lambda) = \exp \left( n \left[ \int_0^t \lambda(s)dX^n_s - \int_0^t G(\lambda(s); s, X^n)ds \right] \right),
\]

where \( G(\lambda; t, X) \) is the cumulant (see (2.3)), is a positive local martingale. So we have the following multiplicative decomposition

\[
U^n_t(\lambda) = Z^n_t(\lambda) \exp \left( n \int_0^t G(\lambda(s); s, X^n)ds \right)
\]

(cf., [2], ch. 2).

Let \( \phi = (\phi_t)_{t \geq 0} \in D, \phi_0 = x \) \((\equiv X^n_0)\) and \( T > 0 \). For \( \gamma > 0 \) define a stopping time by

\[
\tau = \inf \left( t \leq T : \sup_{s \leq t} |X^n_s - \phi_s| > \gamma \right), \quad (\inf \{ \emptyset \} = T),
\]

and introduce the stopped process \( Z^{n,\tau}(\lambda) = (Z^{n,\tau}_t(\lambda))_{t \geq 0} \) with \( Z^{n,\tau}_t(\lambda) = Z^n_{t\wedge \tau}(\lambda) \).

**Lemma 4.1.** Under condition I, \( Z^{n,\tau}(\lambda) \) is a square integrable martingale with

\[
EZ^{n,\tau}_\infty(\lambda) = 1.
\]

**Proof.** Show that

\[
E(Z^{n,\tau}_\infty(\lambda))^2 < \infty.
\]

Using (3.3) (with \( 2\lambda \) and (3.3) we have

\[
E(Z^{n,\tau}_\infty(\lambda))^2 = E(U^n(2\lambda) \exp \left( -2n \int_0^\tau G(\lambda(s); s, X^n)ds \right))
\]

\[
= E \left\{ Z^n(2\lambda) \exp \left[ n \int_0^\tau G(2\lambda(s); s, X^n)ds - 2n \int_0^\tau G(\lambda(s); s, X^n)ds \right] \right\}.
\]

The process \( Z^n(2\lambda) = (Z^n_t(2\lambda))_{t \geq 0} \) is a positive local martingale and hence, it is a positive supermartingale ([2], Problem 1.4.4). So \( EZ^n(2\lambda) \leq 1 \), and (4.5) holds, provided \( |G(\alpha; s, X)| \) is uniformly bounded in \( s \) for almost all \( s \leq \tau \) for any \( \alpha > 0 \). The latter is a consequence of condition I since

\[
|G(\alpha; s, X^n)| \leq \alpha l_T(1 + \gamma + \phi_T^n)^2 + K(\alpha l_T(1 + \gamma + \phi_T^n)), \quad s < \tau.
\]

(4.5) implies that \( Z^{n,\tau}(\lambda) \) is a uniformly integrable martingale and hence

\[
EZ^{n,\tau}_\infty(\lambda) = EZ^{n,\tau}_0(\lambda) = 1.
\]
2. Assume that conditions I and III hold, \( \phi = (\phi_t)_{t \geq 0} \) is absolutely continuous function with \( \phi_0 = x (\equiv X_0^n) \) and its derivative \( \dot{\phi}_t \) is bounded almost everywhere: \( |\dot{\phi}_t| \leq N \).

For \( T > 0 \) define the stopping time \( \tau \) by (4.4) with \( \gamma = \delta_{T,\phi} \) from condition III.

We set
\[
\Lambda(t) = \Lambda_{T,\phi}(\dot{\phi}_t; t, X^n) I(t \leq \tau),
\]
(4.6)
where \( \Lambda_{T,\phi}(y; t, X) \) is defined in condition III. By III(i) \( \Lambda(t) \equiv (\Lambda(t), t \geq 0) \) from (4.6) ia predictable and bounded (almost everywhere in \( t \)). Let \( Z_{n,\tau}^{n,\gamma} \) be defined by (4.2) Then by III(i) and Lemma 4.1 \( E Z_{n,\tau}^{n,\gamma} = 1 \) (in what follows we fix \( \Lambda \) from (4.6) and omit dependence on it).

On the measurable space \((\Omega, \mathcal{F})\) define a new probability measure \( Q^{n,\phi} \) by
\[
dQ^{n,\phi} = Z_{n,\tau}^{n,\gamma} dP. \tag{4.7}
\]
The definition of \( Z_{n,\tau}^{n,\gamma} \) implies that \( Z_{n,\tau}^{n,\gamma} > 0 \), \( P\)-a.s. So the measures \( P \) and \( Q^{n,\phi} \) are equivalent and
\[
dP = (Z_{n,\tau}^{n,\gamma})^{-1} dQ^{n,\phi}. \tag{4.8}
\]
Since an absolutely continuous change measure preserves the semimartingale property[2], \( X^n \) is a semimartingale on the basis \((\Omega, \mathcal{F}, \mathbb{F}^n, Q^{n,\phi})\). The structure of \( X^n \) is given in the next theorem.

**Theorem 4.1.** Let conditions I and III(i) hold. Then \( X^n \) on \((\Omega, \mathcal{F}, \mathbb{F}^n, Q^{n,\phi})\) has the decomposition
\[
X^n_t = x + \int_0^t I(s \leq \tau) a(s, X^n) ds + M^{n,\phi}_t, \tag{4.9}
\]
where \( M^{n,\phi} = (M^{n,\phi}_t)_{t \geq 0} \) is a local square integrable martingale with the predictable quadratic variation process
\[
\langle M^{n,\phi} \rangle_t = n^{-1} \int_0^t b^2(s, X^n) ds + n^{-1} \int_0^t \int_E f^2(s, X^n, u) \times \exp[\Lambda_{T,\phi}(s, X^n)] I(s \leq \tau) f(s, X^n, u) q(du) ds. \tag{4.10}
\]

**Proof.** By Lemma 4.1, \( Z_{n,\tau}^{n,\gamma} \) is a square integrable martingale. From the definition of \( Z_t^n \), we have by the Itô formula
\[
Z_{n,\tau}^{n,\gamma} = 1 + \int_0^t I(s \leq \tau) Z_{s-}^{n,\gamma} dL_s^n, \tag{4.11}
\]
where \( L^n = (L^n_t)_{t \geq 0} \) is a local square integrable martingale with
\[
L^n_t = \sqrt{n} \int_0^{t \wedge \tau} \Lambda_{T,\phi}(\dot{\phi}_s, s, X^n) b(s, X^n) dW_s^n + \int_0^{t \wedge \tau} \int_E \{ \exp[\Lambda_{T,\phi}(\dot{\phi}_s, s, X^n)] f(s, X^n, u) - 1 \} [p^n - q^n] 9ds, du. \tag{4.12}
\]

By (1.1), \( X^n_t \) has the following decomposition (with respect to \((\mathbb{F}^n, P)\):
\[
X^n_t = x + \int_0^t a(s, X^n) ds + M^n_s, \tag{4.13}
\]
where the local square integrable martingale \( M^n = (M^n_t)_{t \geq 0} \) is of the form:
\[
M^n_t = \frac{1}{\sqrt{n}} \int_0^t b(s, X^n) dW_s^n + \frac{1}{n} \int_0^t \int_E f(s, X^n, u) [p^n - q^n] 9ds, du. \tag{4.14}
\]
It follows from (4.11), (4.12) and (4.14) that the predictable quadratic covariance process \((Z^{n,\tau}, M^n)\) is given by
\[
(Z^{n,\tau}, M^n)_t = \int_0^{t \wedge \tau} Z_{s-}^{n,\tau} u \left[ \Lambda_{T,\phi}(\dot{\phi}_s, X^n) b^2(s, X^n) + \int_E f(s, X^n u) \right. \\
\times \left( \exp[\Lambda_{T,\phi}(\dot{\phi}_s, X^n) f(s, X^n u)] - 1 \right) q(du) \, ds.
\]
This implies by the definition of \(\Lambda_{T,\phi}(\dot{\phi}_s, X^n)\) (see (2.2) and (2.4)) that
\[
\langle Z^{n,\tau}, M^n \rangle_t = \int_0^{t \wedge \tau} Z_s^{n,\tau}(\dot{\phi}_s - a(s, X^n)) \, ds. \tag{4.15}
\]
By Theorem 4.5.2 [2], the process \((M^{n,\phi}_t)_{t \geq 0}\) which is defined by
\[
M^{n,\phi}_t = M^n_t - \int_0^t (Z_{s-}^{n,\tau})^{-1} d\langle Z^{n,\tau}, M^n \rangle_s,
\]
is a local martingale with respect to \((\mathcal{F}^n, Q^{n,\phi})\). Together with (4.15) this implies that
\[
M^{n,\phi}_t = M^n_t - (\phi_{t \wedge \tau} - x) + \int_0^{t \wedge \tau} a(s, X^n) \, ds. \tag{4.16}
\]
Decomposition (4.9) follows from (4.16) and (4.13).

We prove now (4.10). Let us first calculate the quadratic variation process
\[
[M^{n,\phi}, M^{n,\phi}].
\]
Let \(M^{n,\phi,c}\) and \(M^{n,c}\) be the continuous martingale components on \(M^{n,\phi}\) and \(M^n\). The processes \((M^{n,\phi})\) and \((M^n)\) are \(Q^{n,\phi}\)-indistinguishable ([2], Theorem 4.5.2), and hence by (4.4) we have
\[
\langle M^{n,\phi,c} \rangle_t = \frac{1}{n} \int_0^t b^2(s, X^n) \, ds. \tag{4.17}
\]
Next, by (4.16) and (4.14)
\[
(\Delta M^{n,\phi}_t)^2 = n^{-2} \int_E f^2(s, X^n u) p^n(\{s\}, du). \tag{4.18}
\]
Therefore the quadratic variation process \([M^{n,\phi}, M^{n,\phi}]\) is \([M^{n,\phi}, M^{n,\phi}]_{t \geq 0}\) is
\[
[M^{n,\phi}, M^{n,\phi}]_t = n^{-1} \int_0^t b^2(s, X^n) \, ds + n^{-2} \int_0^t \int_E f^2(s, X^n u) p^n(du, ds). \tag{4.19}
\]
Now we prove that the increasing process \([M^{n,\phi}, M^{n,\phi}]\) is locally integrable with the compensator given by (4.10). Let \(q^{n,\phi}(ds, du)\) be the compensator of \(p^n(ds, du)\) with respect to \((\mathcal{F}^n, Q^{n,\phi})\). By Theorem 4.5.1 [2],
\[
q^{n,\phi}(ds, du) = Y(s, u) p^n(ds, du), \tag{4.20}
\]
where
\[
Y(s, u) = M^P_{\rho^n} \left( 1 + \frac{\Delta Z^{n,\phi}}{Z^{n,\phi}} \right) \mathcal{P}(\mathcal{F}^n)(s, u),
\]
and \(M^P_{\rho^n}(\mathcal{P}(\mathcal{F}^n))\) is the conditional expectation of Doleans-Dade measure
\[
M^P_{\rho^n} = M^P_{\rho^n}(d\omega, dt, du) = P(\omega) p^n(dt, du)
\]
with respect to \(\mathcal{P}(\mathcal{F}^n)\). The definition of \(Z^{n,\tau}\) yields
\[
1 + \frac{\Delta Z^{n,\tau}}{Z^{n,\tau}} = \exp \left[ I(s \leq \tau) \Lambda_{T,\phi}(\dot{\phi}_s, X^n) \int_E f(s, X^n u) p^n(\{s\}, du) \right].
\]
It is easy to deduce that \(M^P_{\rho^n}\)-a.s.
\[
Y(s, u) = \exp[I(s \leq \tau) \Lambda_{T,\phi}(\dot{\phi}_s, s, X^n) f(s, X^n u)]. \tag{4.21}
\]
(4.20) and (4.21) imply that
\[ q^{n,\phi}(ds, du) = n \exp[I(s \leq \tau)\Lambda_{T,\phi}(\phi_{s}^{\tau}, s, X^{n})]q(du)ds. \]  
(4.22)
Consider the following increasing process (cf. (4.22))
\[ \alpha_{t} = n^{-2} \int_{0}^{t} \int_{E} f^{2}(s, X^{n}, u)q^{n,\phi}(ds, du) \]
\[ = n^{-2} \int_{0}^{t} \int_{E} f^{2}(s, X^{n}, u) \exp[I(s \leq \tau)\Lambda_{T,\phi}(\phi_{s}^{\tau}, s, X^{n})]q(du)ds. \]
In view of I and III(i), \( \alpha_{t} < \infty \) \( \mathbb{P} \)-a.s. for all \( t > 0 \). Therefore, by the equivalence of \( Q^{n,\phi} \) and \( P, \alpha_{t} < \infty Q^{n,\phi} \)-a.s. The process \( (\alpha_{t})_{t \geq 0} \), being increasing and continuous, is locally integrable (with \( \sigma_{m} = \inf\{t : \alpha_{t} \geq m\}, m \geq 1 \) as a localization sequence). Hence, \( (\alpha_{t})_{t \geq 0} \) is the compensator of the process
\[ \left(n^{-2} \int_{0}^{t} \int_{E} f(s, X^{n}, u)p^{n}(ds, du)\right)_{t \geq 0} \]
(with respect to \( (\mathbb{F}^{n}, Q^{n,\phi}) \)). Then by (4.19) \( [M^{n,\phi}, M^{n,\phi}] \) is locally integrable and its compensator is given by (4.10). This also means that \( M^{n,\phi} \) is a local square integrable martingale (see the Burkholder-Gandi inequality for \( p = 2, [2], \) ch. 1), and so its predictable quadratic variation process is the compensator of its quadratic variation process (Theorem 1.8.1 [2]).

5. Ergodic properties

Let \( T > 0 \) and let \( \phi = (\phi_{t})_{t \geq 0} \) be an absolutely continuous function with \( \phi_{0} = x \) \( (\equiv X_{0}^{n}) \) and bounded everywhere derivative \( \dot{\phi}_{t} \) on \( [0, T] \). We also assume that \( I_{T}(\phi) < \infty \) and that condition III(i) holds. Define the measure \( Q^{n,\phi} \) by (4.7), where \( \overline{X}(t) \) is from (4.6) and \( \tau \) from (4.4) with \( \gamma = \delta_{T,\phi}, X^{n} \) is a semimartingale both under \( P \) and \( Q^{n,\phi} \) with the respective decomposition (4.13) and (4.9).

In this section we prove that
\[ \lim_{n} P\left(\sup_{t \leq T} |X_{t}^{n} - Y_{t}| > \varepsilon\right) = 0, \forall \varepsilon > 0 \]  
(5.1)
and
\[ \lim_{n} Q^{n,\phi}\left(\sup_{t \leq T} |X_{t}^{n} - \phi_{t}| > \delta\right) = 0, \forall \delta > 0, \]  
(5.2)
where \( (Y_{t})_{0 \leq t \leq T} \) is a (unique) solution of the equation
\[ \dot{Y}_{t} = a(t, Y), \ Y_{0} = x. \]  
(5.3)

Lemma 5.1. Assume that conditions I and II hold, and (5.3) has a unique solution. Then (5.1) holds for all \( T > 0 \).

Proof. We denote \( Q^{X^{n}} \) the distribution of \( X^{n} \) under \( P \), in other words \( Q^{X^{n}} \) is the measure on \( (\mathcal{D}, \mathcal{D}) \) defined as
\[ Q^{X^{n}}(\Gamma) = P(X^{n} \in \Gamma), \ \Gamma \in \mathcal{D}. \]
We have shown in Section 3 that the sequence \( (Q^{X^{n}})_{n \geq 1} \) is exponentially tight, so since exponential tightness obviously implies tightness the sequence \( (Q^{X^{n}}) \) is tight.

Now we prove that
\[ M_{T}^{n} \overset{P}{\to} 0 \ (n \to \infty). \]  
(5.4)
By the Lenglart-Rebolledo inequality (see, e.g. [2], ch. 1), (5.4) will follow from
\[ (M^{n})_{T} \overset{P}{\to} 0 \ (n \to \infty) \]  
(5.5)
(Problem 1.9.2 [2]). (4.14) implies by condition I that
\begin{align*}
\langle M^n \rangle_t &= n^{-1} \int_0^t b^n(s, X^n)ds + n^{-1} \int_0^t f^n(s, X^n, u)q(du)ds \\
& \leq 2n^{-1}t^2 \int_0^t (1 + (X^n)^2)ds + 2n^{-1}t^2 \int_E h^2(u)q(du) \int_0^t (1 + (X^n)^2)ds.
\end{align*}
Then there exists \( k \) such that
\[ \langle M^n \rangle_T \leq kn^{-1}(1 + (X^n_T)^2). \]
Hence
\[ \langle M^n \rangle_T I(X^n_T \leq C) \xrightarrow{P} 0 \quad (n \to \infty), \quad \forall C > 0, \]
and (5.5) follows since by the tightness of \( Q^n \)
\[ \lim_n P(X^n_T > C) \to 0 \quad (C \to \infty). \]

Denote for \( X \in D \)
\[ M_t(X) = X_t - X_0 - \int_0^t a(s, X)ds. \]
Then by (5.4)
\[ Q^n \left( \sup_{t \leq T} |M_t(X)| > \varepsilon \right) = P(M^n_T > \varepsilon) \to 0 \quad (n \to \infty). \quad (5.6) \]
The sequence \( Q^n \) is tight, so it is relatively compact. Let a subsequence \( Q^{n'} \) converge weakly to a probability measure \( Q' \) on \( (D, D) \). Then by (5.6)
\[ \lim Q^{n'} \left( \sup_{t \leq T} |M_t(X)| > \varepsilon \right) = 0. \]
Since \( \sup_{t \leq T} |X^n_T| \xrightarrow{P} 0 \quad (n \to \infty) \), \( Q^n \) is \( C \)-tight [3]. Therefore, by condition II, \( \sup_{t \leq T} |M_t(X)| = 0 \) \( Q' \)-a.s. for every \( T > 0 \), i.e. \( X = (X_t)_{t \geq 0} \) is a solution of (5.3) \( Q' \)-a.s. The solution being unique, \( Q' \) is the Dirac measure concentrated on \( Y \) for any sequence \( (n') \).

**Theorem 5.1.** Let conditions I and III(i) hold. Then for all \( T > 0 \) we have (5.2).

**Proof.** Show that for \( \delta \leq \delta_{T, \phi} \)
\[ Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| > \delta \right) \leq 2Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| \geq \delta \right). \quad (5.7) \]
Since
\[ \{ \tau < T \} \subseteq \left\{ \sup_{t \leq \tau} |X^n_t - \phi_t| \geq \delta_{T, \phi} \right\}, \]
we have
\[ Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| > \delta \right) \leq Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| \geq \delta, \tau = T \right) + Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| \geq \delta \right) \]
and (5.7) follows. So it suffices for us to show that for \( \delta \leq \delta_{T, \phi} \)
\[ \lim_{n} Q^{n, \phi} \left( \sup_{t \leq T} |X^n_t - \phi_t| \geq \delta \right) = 0. \quad (5.8) \]
We have from (4.9) that
\[ \sup_{t \leq \tau} |X^n_t - \phi_t| = M^n_t^{\phi}, \]
and, so all we need to prove is

\[ Q^n,\phi(M^n,\phi^* \geq \delta) = 0. \]  \hfill (5.9)

By the Lenglart-Rebolledo inequality,

\[ Q^n,\phi(M^n,\phi^* \geq \delta) \leq \beta / \delta^2 + Q^n,\phi((M^n,\phi)_\tau \geq \beta). \]  \hfill (5.10)

By Theorem 4.1 and in view of conditions I and III(i),

\[ \langle M^n,\phi \rangle_t \leq 2n^{-1}l^2_T (1 + (X^n_{t+})^2) + 2n^{-1}l^2_T \int_E h^2(u) \]
\[ \times \exp\{rh(u)(1 + X^n_{t+})q(du)(1 + (X^n_{t-})^2). \]  \hfill (5.11)

Since \( X^n_{t+} \leq \phi^*_t + \delta T, \phi \), by (5.1) there exists \( k \) such that \( \langle M^n,\phi \rangle_T \leq k/n \), which implies in view of (5.10) that

\[ \lim_n Q^n,\phi(M^n,\phi^* \geq \delta) \leq \beta / \delta^2 \rightarrow 0 \ (\beta \rightarrow 0), \]

which proves (5.9).

6. Upper bound

1. We define \( I_T(\phi) \) as in Section 2. In this section we prove the following

Theorem 6.1. Let condition I and II hold. Then for all \( \phi \in D \) with \( \phi_0 = x \equiv X^n_0 \)

\[ \lim_{\delta \rightarrow 0} \lim_n n^{-1} \log P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \leq -I_T(\phi). \]

For the proof we will need an auxiliary result.

2. Consider a step function \( (\lambda(t))_{t \geq 0} \) of the form \( 0 \leq t_0 < t_1 < \ldots < t_k \)

\[ \lambda(t) = \sum_{i=1}^k \lambda_i [t_{i-1}, t_i](t). \]  \hfill (6.1)

For \( \phi \in D \) set by definition

\[ \int_0^t \lambda(s) d\phi_s = \sum_{i=1}^k \lambda_i [\phi_{t \wedge t_i} - \phi_{t \wedge t_i-1}], \] \hfill (6.2)

and denote

\[ I_T(\phi) = \sup \left[ \int_0^T \lambda(t) d\phi_t - \int_0^T G(\lambda(t); t, \phi) dt \right], \] \hfill (6.3)

with “sup” taken over all \( \lambda(t) \) with representation (6.1).

Lemma 6.1. If \( \phi = (\phi_t)_{t \geq 0} \in D \) and \( \phi_0 = x \equiv X^n_0 \), then for all \( T > 0 \)

\[ I_T(\phi) = \bar{I}_T(\phi). \]

Proof. On first step is that

\[ \bar{I}_T(\phi) = \infty. \]  \hfill (6.4)

if \( \phi \) is not absolutely continuous on \([0, T]\).

By the definition of absolutely continuity [22], for such \( \phi \) we can choose \( \varepsilon > 0 \), so that for any \( \gamma > 0 \) there exists non overlapping intervals \((t''_i, t'_i), i \geq 1\) on \([0, T]\) satisfying

\[ \sum_i (t''_i - t'_i) \leq \gamma, \quad \sum_i |\phi_{t''_i} - \phi_{t'_i}| > \varepsilon. \] \hfill (6.5)

We set an integer \( N \)

\[ \gamma(N) = |NL_T(1 + \phi_N^*) + (NL_T)^2 (1 + (\phi_N^*)^2) + K(NL_T(1 + \phi_N^*))|^{-1}, \]
where $K(\lambda)$ and $l_T$ are defined in condition I, and choose $(t'_i, t''_i)$ for each $N$ so that (6.5) holds with $\gamma = \gamma(N)$. Let

$$\lambda^0(t) = N \sum_{i=1}^{k} \text{sign} (\phi_{i'} - \phi_{i''}) I_{[t'_i, t''_i]}(t).$$  \hspace{1cm} (6.6)

By condition I, $|G(\lambda^0(t); t, \phi)| \leq \gamma^{-1}(N)$ for all $t \leq T$, and hence

$$\int_{0}^{T} |G(\lambda^0(t); t, \phi)|dt = \int_{0}^{T} I(\lambda^0(t); t, \phi)|dt \leq \gamma^{-1}(N) \sum_{i=1}^{k} (t''_i - t'_i) \leq 1. \hspace{1cm} (6.7)$$

On the other hand,

$$\int_{0}^{T} \lambda^0(t) d\phi_t = N \sum_{i} I_{[t'_i, t''_i]}(\phi_{i'} - \phi_{i''}) > N \varepsilon.$$

It then follows by the definition of $\tilde{I}_T(\phi)$ and (6.2), that

$$\tilde{I}_T(\phi) \geq \int_{0}^{T} \lambda^0(t) d\phi_t - \int_{0}^{T} G(\lambda^0(t); t, \phi) dt \geq \int_{0}^{T} \lambda^0(t) d\phi_t - \int_{0}^{T} |G(\lambda^0(t); t, \phi)| dt \geq N \varepsilon - 1 \to \infty, \hspace{1cm} N \to \infty$$

which yields (6.4).

Now let $\phi$ be absolutely continuous on $[0, T]$. We show that

$$\tilde{I}_T(\phi) = \int_{0}^{T} H(\phi_t; t, \phi) dt. \hspace{1cm} (6.8)$$

First, we have by the definition of $H(\phi_t; t, \phi)$ (see (2.3))

$$\tilde{I}_T(\phi) \leq \int_{0}^{T} H(\phi_t; t, \phi) dt.$$

So, it is left to prove

$$\tilde{I}_T(\phi) \geq \int_{0}^{T} H(\phi_t; t, \phi) dt. \hspace{1cm} (6.9)$$

It is rather obvious that (6.9) holds, provided we can choose for each $c > 0$ a finite function $\lambda^c(t)$ satisfying

$$\lambda^c(t) \phi_t - G(\lambda^c(t); t, \phi) \geq [(c \wedge H(\phi_t; t, \phi)) - 1/c] \vee 0 \hspace{1cm} (6.10)$$

for almost all $t \leq T$ taking $\lambda^c(t) = \lambda^c(t) I(|\lambda^c(t)| \leq c))$. Indeed there exists a sequence $\lambda_n(t)$, $n \geq 1$ of step-functions of the form (6.1) for which

$$|\lambda_n(t)| \leq \sup_{t \leq T} |\tilde{\lambda}^c(t)|, \hspace{1cm} t \leq T, \hspace{1cm} (6.11)$$

$$\lim_n \int_{0}^{T} |\lambda_n(t) - \tilde{\lambda}^c(t)| dt = 0 \hspace{1cm} (6.12)$$

(Lemma 4.4, ch. 4 in [23]). By condition I and (6.11)

$$[|\lambda_n(t)\phi_t - G(\lambda_n(t); t, \phi)| - |\lambda_n(t)\phi_t - G(\lambda_n(t); t, \phi)|] \leq |\lambda_n(t) - \tilde{\lambda}^c(t)|(|\phi_t| + r), \hspace{1cm} (6.13)$$
where \( r \) depends on \( \phi \), on the parameters in condition \( \text{I} \) and on \( c \). So, provided (6.10) holds, we have using the definition \( \bar{I}_T(\phi) \)

\[
\bar{I}_T(\phi) \geq \int_0^T \lambda_n(t)d\phi_t - \int_0^T G(\lambda_n(t); t, \phi)dt
\]

\[
\geq \int_0^T \tilde{\lambda}^c(t)d\phi_t - \int_0^T G(\tilde{\lambda}^c(t); t, \phi)dt
\]

\[
- \int_0^T |\lambda_n(t) - \tilde{\lambda}^c(t)||\dot{\phi}_t| + r)dt
\]

\[
\geq \int_0^T I\{|\lambda^c(t)| \leq c\}\{[(c \wedge H(\dot{\phi}_t; t, \phi)) - 1/c] \lor 0\}dt
\]

\[
- \int_0^T |\lambda_n(t) - \tilde{\lambda}^c(t)||\dot{\phi}_t| + r)dt.
\]

(6.14)

We next show that

\[
\lim\limits_n \int_0^T |\lambda_n(t) - \tilde{\lambda}^c(t)||\dot{\phi}_t| + r)dt = 0.
\]

(6.16)

(6.15) holds by the inequality (see (6.11))

\[
\int_0^T |\lambda_n(t) - \tilde{\lambda}^c(t)||\dot{\phi}_t| + r)dt \leq (N + r) \int_0^T |\lambda_n(t) - \tilde{\lambda}^c(t)|dt
\]

\[+2 \sup_{t \leq T} |\tilde{\lambda}^c(t)| \int_0^T I\{|\dot{\phi}_t| > N\}dt,
\]

since in view of (6.12) and by the absolute continuity of \( \phi \) “\( \lim\limits_n \lim\limits_n \)” of the latter expression is 0.

Thus, (6.14) and (6.15) imply that under (6.10) for any \( c > 0 \)

\[
\bar{I}_T(\phi) \geq \int_0^T I\{|\tilde{\lambda}^c| \leq c\}\{[(c \wedge H(\dot{\phi}_t; t, \phi)) - 1/c] \lor 0\}dt.
\]

Since \( H(\dot{\phi}_t; t, \phi) \) is nonnegative, we have

\[
\lim_{c \to \infty} \int_0^T I\{|\tilde{\lambda}^c| \leq c\}\{[(c \wedge H(\dot{\phi}_t; t, \phi)) - 1/c] \lor 0\}dt = \int_0^T H(\dot{\phi}_t; t, \phi)dt
\]

which proves (6.9).

It is left to find \( \tilde{\lambda}^c(t) \) meeting (6.10). We denote

\[
G_0(\lambda; t, \phi) = G(\lambda; t, \phi) - \lambda a(t, \phi)
\]

and

\[
U'(t, \lambda) = \lambda|\dot{\phi}_t - a(t, \phi)| - G_0(\lambda; t, \phi),
\]

\[
U''(t, \lambda) = -\lambda|\dot{\phi}_t - a(t, \phi)| - G_0(\lambda; t, \phi).
\]

Introduce the functions

\[
\lambda'(t) = \inf\{\lambda \geq 0 : U'(t, \lambda) \geq [(c \wedge H(\dot{\phi}_t; t, \phi)) - 1/c] \lor 0\}
\]

\[
\lambda''(t) = \inf\{\lambda \geq 0 : U''(t, \lambda) \geq [(c \wedge H(\dot{\phi}_t; t, \phi)) - 1/c] \lor 0\},
\]

(\( \inf(\emptyset) = \infty \)). Then

\[
\tilde{\lambda}^c(t) = \lambda'(t)I(\dot{\phi}_t > a(t, \phi))\lambda''(t)I(\dot{\phi}_t < a(t, \phi)).
\]
Proof of Theorem 6.1. Let \( \lambda(t) \) be of the form (6.1). Define a positive local martingale \( Z^n = (Z^n_t)_{t \geq 1} \) by (4.3). By Lemma 4.1, \( EZ^n_1 = 1 \) and hence

\[
1 \geq E \left( I \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) Z^n_T \right).
\]

The expression under the expectation is less than

\[
I \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \exp \left\{ n \left[ \int_0^T \lambda(t) \phi_t dt - \int_0^T G(\lambda(t); t, \phi) dt \right] - n \left[ \int_0^T \lambda(t) d(X^n_t - \phi_t) \right] - \int_0^T G(\lambda(t); t, \phi) dt - \int_0^T G(\lambda(t); t, X) dt \right\}.
\]

Since \( \lambda(t) \) is piecewise constant, we have on \( \{ \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \} \)

\[
\left| \int_0^T \lambda(t) d(X^n_t - \phi_t) \right| \leq 2 \sup_{t \leq T} |\lambda(t)| \delta.
\]

By conditions I and II, \( \int_0^T G(\lambda(t); t, X) dt \) is continuous in \( X \in D \) at each \( \phi \in C \), i.e. if \( \sup_{t \leq T} |X^{(k)}_t - \phi_t| \to 0 \) \( (k \to \infty) \) for \( X^{(k)} \in D \), \( k \geq 1 \), then

\[
\int_0^T G(\lambda(t); t, X^{(k)}) dt \to \int_0^T G(\lambda(t); t, \phi) dt \quad (k \to \infty).
\]

It follows that for each for each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon, \phi) \) such that for \( \delta < \delta(\varepsilon, \phi) \) and \( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \)

\[
\left| \int_0^T G(\lambda(t); t, X^{(k)}) dt - \int_0^T G(\lambda(t); t, \phi) dt \right| \leq \varepsilon.
\]

Hence for \( \delta < \delta(\varepsilon, \phi) \)

\[
I \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) Z^n_T \geq I \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \times \exp \left\{ n \left[ \int_0^T \lambda(t) \phi_t dt - \int_0^T G(\lambda(t); t, \phi) dt - 2 \sup_{t \leq T} |\lambda(t)| \delta - \varepsilon \right] \right\}.
\]

This and (6.16) imply since \( \lambda(t) \) and \( \varepsilon \) are arbitrary, that

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \log P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \leq - \sup \left[ \int_0^T \lambda(t) \phi_t dt - \int_0^T G(\lambda(t); t, \phi) dt \right],
\]

with “sup” taken over all step-functions \( \lambda(t) \) of the form (6.1).

The theorem follows by Lemma 6.1.

7. Lower bound

Lemma 7.1. Let \( \phi = (\phi_t)_{t \geq 0} \) be absolutely continuous on \([0, T]\) with \( \phi_0 = x \) \((\equiv X^n_0)\) and bounded almost everywhere derivative: \( |\dot{\phi}| \leq N \). If conditions I, II and III hold then

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \log P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \geq -I_T(\phi).
\]
Define $Q^{n,\gamma}$ by (4.7) with $\lambda(t)$ from (4.6) and $\tau$ from (4.4) with $\gamma = \delta_{T,\gamma}$. Then by (4.8)

$$P\left(\sup_{t \leq T}|X^n_t - \phi_t| \leq \delta\right) = \int \Omega \left(I\left(\sup_{t \leq T}|X^n_t - \phi_t| \leq \delta\right)\right)(\tilde{Z}_{\infty}^{n,\gamma})^{-1}dQ^{n,\gamma},$$  \hspace{1cm} (7.2)

where

$$Z_{\infty}^{n,\gamma} = \exp\left\{n\left[\int_0^T \Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)\lambda dt - \int_0^T G(\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n); t, t^n)dt\right]\right\}. \hspace{1cm} (7.3)$$

We estimate the right hand side of (7.2). Note that by condition III

$$\int_0^T \{\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)\hat{\phi}_t - G(\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n); t, X^n)\}dt = \int_0^T H(\hat{\phi}_t; t)dt = I_T(\phi). \hspace{1cm} (7.4)$$

Let $M^{n,\gamma} = (M^{n,\gamma}_t)_{t \geq 0}$ be local square integrable martingale (under $Q^{n,\gamma}$) from Theorem 4.1.

By condition I and Theorem 5.1, and since $|\hat{\phi}_t|$ is bounded, $\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)$ is uniformly bounded for almost all $t \leq \tau$ and so the integral

$$\int_0^T \Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)dM^{n,\gamma}_t$$

is well defined. Then (7.3) and (7.4) yield

$$(Z_{\infty}^{n,\gamma})^{-1} \geq \exp\left\{-n\left[I_T(\phi) + \int_0^T \Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)dM^{n,\gamma}_t\right.ight.$$

$$\left.+ \int_0^T |\hat{\phi}_t||\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n) - \Lambda_{T,\phi}(\hat{\phi}_t; t, \phi)|dt + \int_0^T |G(\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n); t, X^n) - G(\Lambda_{T,\phi}(\hat{\phi}_t; t, \phi); t, \phi)|dt\right]\}. \hspace{1cm} (7.5)$$

Conditions I - III and boundedness of $\hat{\phi}_t$ imply that for every $\varepsilon > 0$ there exists $\delta(\varepsilon, T, \phi) \leq \delta_{T,\phi}$ (which depends on $\varepsilon, T$ and $\phi$) such that, provided

$$\sup_{t \leq T}|X^n_t - \phi_t| \leq \delta(\varepsilon, T, \phi),$$

we have

$$\int_0^T |\hat{\phi}_t||\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n) - \Lambda_{T,\phi}(\hat{\phi}_t; t, \phi)|dt + \int_0^T |G(\Lambda_{T,\phi}(\hat{\phi}_t; t, X^n); t, X^n) - G(\Lambda_{T,\phi}(\hat{\phi}_t; t, \phi); t, \phi)|dt \leq \varepsilon. \hspace{1cm} (7.6)$$

(7.2) and (7.3), and (7.5) and (7.6) yield for $\delta \leq \delta_{\varepsilon, T, \phi}$:

$$P\left(\sup_{t \leq T}|X^n_t - \phi_t| \leq \delta\right) \geq \exp\left(-n[I_T(\phi) + \varepsilon]\right)$$

$$\times \exp\left(-n\int_0^T \Lambda_{T,\phi}(\hat{\phi}_t; t, X^n)dM^{n,\gamma}_t\right)\right)dQ^{n,\gamma}. \hspace{1cm} (7.7)$$
We have for the integral of the right hand side of (7.7) for $\beta > 0$
\[
\int_{\Omega} \left[ I \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \exp \left( -n \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right) \right] dQ^{n,\phi}
\]
\[
\geq \exp \left\{ -n\beta \right\} Q^{n,\phi} \left\{ \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right\} \leq \beta, \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right\}. \tag{7.8}
\]
(7.7) and (7.8) imply that for $\delta \leq \delta(\varepsilon, T, \phi)$
\[
n^{-1} \log P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \geq -(I_T(\phi) + \varepsilon + \beta)
\]
\[
+n^{-1} \log Q^{n,\phi} \left\{ \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right\} \leq \beta, \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right\}. \tag{7.9}
\]
By (7.9) and Theorem 5.1, it suffices for us to prove that for any $\beta > 0$
\[
Q^{n,\phi} \left\{ \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right\} > 0 \right\} = 0. \tag{7.10}
\]
Since by condition III $|\Lambda_{T,\phi}(\phi_t; t, X^n)| \leq r$ for almost all $s \leq \tau$, the Lenglart-Rebolledo inequality gives
\[
Q^{n,\phi} \left\{ \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right\} \leq \alpha / \beta^2 + Q^{n,\phi} \left\{ \int_0^T \Lambda_{T,\phi}(\phi_t; t, X^n) dM_t^{n,\phi} \right\} \geq \beta
\]
\[
\leq \alpha / \beta^2 + Q^{n,\phi} \left\{ (M^{n,\phi}) \geq \alpha / r^2 \right\}. \tag{7.11}
\]
While proving Theorem 5.1 we saw that $\langle M^{n,\phi} \rangle \leq k/n$, where $k$ does not depend on $n$. Taking on the right side of (7.11) “lim $n \to 0$ lim $n$” we obtain (7.10).

**Theorem 7.1.** Let $\phi = (\phi_t)_{t \geq 0} \in D$ and $\phi_0 = x \equiv X^n_0).$ Under I - IV we have
\[
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \geq -I_T(\phi). \tag{7.12}
\]

**Proof.** Obviously we can assume (7.1), in particular $\phi$ is absolutely continuous on $[0, T]$. Define $\phi^n = (\phi^n_t)_{t \geq 0}$ as in condition IV:
\[
\phi^n_t = x + \int_0^t I(|\phi_s| \leq N) \phi_s ds. \tag{7.13}
\]
Applying Lemma (7.1) to $\phi^n$ we have for all $T > 0$
\[
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P \left( \sup_{t \leq T} |X^n_t - \phi^n_t| \leq \delta \right) \geq -I_T(\phi^n). \tag{7.14}
\]
Since
\[
\lim_{n} \sup_{t \leq T} |\phi_t - \phi^n_t| = 0,
\]
\[
P \left( \sup_{t \leq T} |X^n_t - \phi_t| + \sup_{t \leq T} |\phi_t - \phi^n_t| \leq \delta \right) \geq P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right)
\]
we have from (7.14) that
\[
\lim_{\delta \to 0} \lim_{n} n^{-1} \log P \left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \geq -\lim_{n} I_T(\phi^n),
\]
which implies that the assertion will follow if
\[
\lim_{n} I_T(\phi^n) \leq I_T(\phi). \tag{7.15}
\]
Hence, it suffices that \( t \) exists for \( I \). It is not difficult to see using conditions \( I \) which is a consequence of condition \( III \) and the Lebesgue dominated convergence theorem.

**Theorem 7.2.** Let the conditions of Theorem 2.2 hold and \( \varphi \) be the solution of (2.9). Then for all \( T > 0 \)

\[
\lim_{n \to 0} \frac{1}{n} \log P \left( \sup_{t \leq T} |X_t^n - \phi_t| \leq \delta \right) \geq -I_T(\phi) = q(E)T.
\]

**Proof.** We first show that

\[
I_T(\phi) = q(E)T.
\]  

(7.17)

Since \( \phi \) solves (2.9), we have (see (2.1))

\[
\lambda \dot{\phi}_t - G(\lambda; t, \phi) = \int_E (1 - e^{\lambda f(t, \phi, u)}) q(du)
\]

and so by (2.3) \( H(\dot{\phi}_t; t, \phi) = q(E) \), which gives (7.17).

We now prove the required. For \( \epsilon > 0 \) consider the equation

\[
\dot{\phi}_t^\epsilon = a(t, \phi^\epsilon) + \int_E (e^{-f(t, \phi^\epsilon, u)} - 1) f(t, \phi^\epsilon, u) q(du).
\]

It is not difficult to see using conditions \( I \) and \( II \) that the solution \( \phi_t^\epsilon = (\phi_t^\epsilon)_{t \geq 0} \) exists for \( t \in [0, \infty) \) and it suffices the conditions of Lemma 7.1. Then

\[
\lim_{\delta \to 0} \lim_{n \to \infty} n^{-1} \log P \left( \sup_{t \leq T} |X_t^n - \phi_t^\epsilon| \leq \delta \right) \geq -I_T(\phi^\epsilon),
\]

(7.18)
where
\[
I_T(\phi^c) = \int_0^T \int_E [1 - (c + 1)e^{-c f(t, \phi, u)}]q(du)dt.
\]

Obviously
\[
I_T(\phi^c) \to q(E)T \quad (c \to \infty).
\]

The inequality
\[
P\left( \sup_{t \leq T} |X^n_t - \phi_t| + \sup_{t \leq T} |\phi_t - \phi^c_t| \leq \delta \right) \geq P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right),
\]

and (7.18) and (7.19) imply that it suffices to prove that
\[
\sup_{t \leq T} |\phi_t - \phi^c_t| \to 0 \quad (c \to \infty), \quad \forall \ T > 0.
\]

It is easy to see applying the Arzela-Ascoli theorem that the family \((\varphi^n_t)_{0 \leq t \leq T}\), \(c > 0\) is relatively compact in \(C_{[0,T]}\). Since by conditions \(I\) and \(II\) any subsequential limit of \(\phi^c\) as \(c \to \infty\) solves (2.9), and the solution of (2.9) is unique, (7.29) is proved.

8. Proof of main result

Proof of Theorem 2.1. In Theorem 3.2 we proved that the sequence \(X^n, n \geq 1\) is \(C\)-exponentially tight. Besides, by Theorem 6.1 for any \(\phi \in D\) with \(\phi_0 = x \equiv X^n_0\)
\[
\lim_{\delta \to 0} \lim_{n \to \infty} n^{-1} \log P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \leq -I_T(\phi),
\]

where \(I_T(\phi)\) is defined in defined in Section 2. On the other hand, by Theorem 7.1 we have for any \(\phi \in D\) with \(\phi_0 = x \equiv X^n_0\)
\[
\lim_{\delta \to 0} \lim_{n \to \infty} n^{-1} \log P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) \geq -I_T(\phi),
\]

Therefore for any \(\phi \in C\) with \(\phi \in D\) with \(\phi_0 = x \equiv X^n_0\)
\[
I_T(\phi) = -\lim_{\delta \to 0} \lim_{n \to \infty} n^{-1} \log P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right) = -\lim_{\delta \to 0} \lim_{n \to \infty} n^{-1} \log P\left( \sup_{t \leq T} |X^n_t - \phi_t| \leq \delta \right),
\]

i.e. (1.11) holds with \(J_T(\phi) = I_T(\phi)\) for \(\phi \in C\) with \(\phi_0 = x \equiv X^n_0\).

Obviously we have (8.1) with \(\phi_0 \neq x \equiv X^n_0\) as well, since in that case each term in (8.1) is equal to \(\infty\).

So according to the scheme in Section 1 property \(\beta\) holds and the rate function for \(X^n, n \geq 1\) is given in \(\gamma\).

The proof of Theorem 2.2 is analogous with the use of Theorem 7.2 in addition to Theorem 7.1.

9. Explicit conditions on coefficients

In this section we give simple sufficient conditions on \(a(t, X), b(t, X), f(t, X, u)\) which imply conditions \(III\) and \(IV\).

We begin with some definitions. Say that \(f^+(t, X, u) (= f \vee 0)\) is nondegenerate with respect to \(q(du)\) uniformly on \([0, T]\) in a neighborhood of \(\phi = (\phi_t)_{t \geq 0} \in C\), if there exists \(\delta > 0\) and \(\gamma > 0\) such that for all \(X = (X_t)_{t \geq 0} \in D\) with \(\sup_{t \leq T} |X_t - \phi_t| \leq \delta\)
\[
\sup_{t \leq T} \int_E f^+(t, X, u)I(f^+(t, X, u) > \gamma)q(du) > 0.
\]
Say that \( f^+(t, X, u) = f \vee 0 \) is degenerate with respect to \( q(du) \) uniformly on \([0, T]\) in a neighborhood of \( \phi = (\phi_t)_{t \geq 0} \in C \), if there exists \( \delta > 0 \) such that for all \( X = (X_t)_{t \geq 0} \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta \)

\[
\sup_{t \leq T} \int_E f^+(t, X, u)q(du) = 0.
\]

Similarly the nondegeneracy and degeneracy of \( f^-(t, X, u) = -f \wedge 0 \) are defined.

First we prove two lemmas.

**Lemma 9.1.** If for any \( T > 0 \) and any \( \phi = (\phi_t)_{t \geq 0} \) with \( I_T(\phi) < \infty \) there exists \( \delta > 0 \) such that for all \( X = (X_t)_{t \geq 0} \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta \)

\[
c_+ = \inf_{t \leq T} (b^2(t, X) + \int_E (f^+(t, X, u))^2q(du)) > 0.
\]

\[
c_- = \inf_{t \leq T} (b^2(t, X) + \int_E (f^-(t, X, u))^2q(du)) > 0,
\]

then condition III holds with \( \delta_{T, \phi} \delta \).

**Proof.** We have to prove that the equation

\[
y = g(\lambda; t, X)
\]

(9.1) where (see Section 2)

\[
g(\lambda; t, X) = a(t, X) + \lambda b^2(t, X) + \int_E f(t, X, u)(e^{\lambda f(t, X, u)} - 1)q(du),
\]

(9.2) has a solution for all \( y \in \mathbb{R} \), for all \( t \leq T \) and \( X \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta_{T, \phi} \) for some \( \delta_{T, \phi} > 0 \) and this solution satisfies III.

We begin with proving the existence. Assume that \( y \geq a(t, X) \) where \( X \) is as in the conditions of the lemma. As for \( \lambda > 0 \)

\[
\lambda b^2(t, X) + \int_E f(t, X, u)(e^{\lambda f(t, X, u)} - 1)q(du) \geq \lambda b^2(t, X) + \int_E (f^+(t, X, u))^2q(du),
\]

so \( g(\lambda; t, X) > y \) for all \( t \leq T \) where

\[
\lambda = \frac{y - a(t, X)}{b^2(t, X) + \int_E (f^+(t, X, u))^2q(du)}.
\]

Since \( g(0; t, X) \leq y \) and \( g(\lambda; t, X) \) is continuous in \( \lambda \) a solution \( \lambda_{T, \phi}(y; t, X) \) of (9.1) exists and satisfies the inequality

\[
0 \leq \lambda_{T, \phi}(y; t, X) \leq \frac{y - a(t, X)}{b^2(t, X) + \int_E (f^+(t, X, u))^2q(du)}.
\]
This solution is unique since under assumptions of the lemma
\[ g_\lambda'(\lambda; t, X) = b^2(t, X) + \int_E f^2(t, X, u)e^{\lambda f(t, X, u)q(du)} > 0. \] (9.3)

The case when \( y < a(t, X) \) is considered similarly.
Thus the solution of (9.1) (with respect \( \lambda \)) is unique and for all \( t \leq T \)
\[ \Lambda_{T, \phi}(y; t, X) \leq \frac{|y - a(t, X)|}{C_+ \wedge e} \] (9.4)
for all \( X = (X_t)_{t \geq 0} \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta \). The latter and condition I (for \( a(t, X) \)) imply \( \Lambda_{T, \phi}(y; t, X) \) meets part (i) of condition III (with \( \delta = \delta_{T, \phi} \)).

Set \( \Lambda_{T, \phi}(y; t, X) = \Lambda_{T, \phi}(y; t, X)I(t \leq T)I(\sup_{s \leq t} |X_s - \phi_s| \leq \delta) \). \( \Lambda_{T, \phi}(y; t, X) \) is \( B(R) \otimes P(D) \)-measurable. Indeed for \( \lambda > 0 \)
\[ \{(y, t, X) : \Lambda_{T, \phi}(y; t, X) > \lambda \} = \left\{(y, t, X) : g(\lambda; t, X) < y, \ t \leq T, \ \sup_{s \leq t} |X_s - \phi_s| \leq \delta \right\} \in B(R) \otimes P(D). \]
The case \( \lambda < 0 \) does not differ.

Finally we prove that \( \Lambda_{T, \phi}(y; t, X) \) is \( C_{[0, T]} \) in \( X = \phi \) for \( y \in R \) and all \( t \leq T \). Let \( \sup_{t \leq T} |X_t^{(k)} - \phi_t| \to \infty \) where \( X^{(k)} = (X_t^{(k)})_{t \geq 0} \in D \). Then we can assume that \( \sup_{t \leq T} |X_t^{(k)} - \phi_t| \leq \delta \) for all \( k \) and hence by (9.4) the sequence \( \Lambda^{(k)} = \Lambda_{T, \phi}(y; t, X^{(k)}) \) is bounded in \( k \) for all \( t \in [0, T] \). Fix \( t \) and let \( \Lambda^{(k)} \) be a converging subsequence. Then denoting
\[ \Lambda^0 = \lim_{k' \to \infty} \Lambda^{(k')} \]
we have by conditions I and II, and by (9.2)
\[ y = \lim_{k' \to \infty} g(\Lambda^{(k')}; t, X^{(k')}) = g(\Lambda^0; t, \phi). \]
Since the solution of (9.1) for \( X = \phi \) is unique, we obtain that \( \Lambda^0 = \Lambda_{T, \phi}(y; t, \phi) \).

**Lemma 9.2.** Let condition III hold. Assume that for any absolutely continuous function \( \phi = (\phi_t)_{t \geq 0} \) with \( I_T(\phi) < \infty \) there exists a nonnegative function \( g(T, \phi)(y) \) which satisfies the linear growth condition
\[ g(T, \phi)(y) \leq C(1 + |y|), \ C > 0, \]
and is such that for all \( t \leq T \) and all \( X = (X_t)_{t \geq 0} \in D \) with \( \sup_{t \leq T} |X_t - \phi_t| \leq \delta_{T, \phi} \)
we have for some \( C_1 > 0 \) and \( C_2 > 0 \)
\[ C_1|\Lambda_{T, \phi}(y; t, X)| \geq g_T(y) \leq C_2(|\Lambda_{T, \phi}(y; t, \phi)| + 1). \]
Then condition IV holds.

**Proof.** First prove that
\[ \int_0^T |\Lambda_{T, \phi}(\phi_t; t, \phi)\dot{\phi}_t|dt < \infty. \] (9.5)
It suffices to verify (9.5) separately for \( \Lambda^+_{T, \phi} = \Lambda_{T, \phi} \vee 0 \) and for \( \Lambda^-_{T, \phi} = -[\Lambda_{T, \phi} \wedge 0] \).
By the definition of \( I_T(\phi) \)
\[ \int_0^T \left( \Lambda^+_{T, \phi}(\phi_t; t, \phi)\dot{\phi}_t - G(\Lambda^+_{T, \phi}(\phi_t; t, \phi); t, \phi) \right)dt \]
\[ \leq \int_0^T \sup_\lambda (\lambda \dot{\phi}_t - G(\lambda; t, \phi))dt = I_T(\phi) < \infty. \] (9.6)
We saw in the proof of Lemma 9.1, (see (9.1) and (9.2)) that if \( \Lambda_{T,\phi}(\hat{\phi}_t; t, \phi) \geq 0 \), then for all \( \lambda \) with \( 9 \leq \lambda \leq \Lambda^+_T(\hat{\phi}_t; t, \phi) \) for all \( t \leq T \)
\[
\hat{\phi}_t \geq g(\lambda; t, \phi),
\]
and hence for any \( m > 1 \)
\[
\Lambda^+_T(\hat{\phi}_t; t, \phi) - G(\Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi) = \int_0^{\Lambda^+_T(\hat{\phi}_t; t, \phi)} (\hat{\phi}_t - g(\lambda; t, \phi)) d\lambda \geq \int_0^{\Lambda^+_T(\hat{\phi}_t; t, \phi)} (\hat{\phi}_t - g(\lambda; t, \phi)) d\lambda \]
g(\lambda; t, \phi) is increasing in \( \lambda \) since (see (9.3)) \( g'_\lambda(\lambda; t, \phi) \geq 0 \) and so by (9.7)
\[
\Lambda^+_T(\hat{\phi}_t; t, \phi) - G(\Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi) \geq \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi) \left( \hat{\phi}_t - g\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi \right) \right). \tag{9.8}
\]
Since by the definition of \( \Lambda_{T,\phi} \), \( \hat{\phi}_t = g(\Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi) \), if \( \Lambda_{T,\phi} > 0 \), (9.2) implies that if \( \Lambda^+_T(\hat{\phi}_t; t, \phi) > 0 \) then
\[
\hat{\phi}_t - g\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi \right) = \left( 1 - \frac{1}{m} \right) \Lambda^+_T(\hat{\phi}_t; t, \phi)b^2(t, \phi)
\]
\[
+ \int_E f(t, \phi, u) \left( \exp(\Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) q(du)
\]
\[
- \int_E f(t, \phi, u) \left( \exp\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) \right) q(du). \tag{9.9}
\]
Using the inequality \( e^x - 1 \geq m(e^{x/m} - 1), \ x \geq 0 \), we have
\[
\int_E f^+(t, \phi, u) \left( \exp(\Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) q(du)
\]
\[
- \int_E f^+(t, \phi, u) \left( \exp\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) \right) q(du)
\]
\[
\geq \ (m - 1) \int_E f^+(t, \phi, u) \left( \exp\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) \right) q(du). \tag{9.10}
\]
and since \( e^{x/m} \geq e^x, \ x \leq 0 \), so
\[
\int_E f^-(t, \phi, u) \left( \exp(\Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) q(du)
\]
\[
- \int_E f^-(t, \phi, u) \left( \exp\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) \right) q(du) \leq 0. \tag{9.11}
\]
Now we substitute (9.10) and (9.11) in turn into (9.8) to obtain
\[
\Lambda^+_T(\hat{\phi}_t; t, \phi) - G(\Lambda^+_T(\hat{\phi}_t; t, \phi); t, \phi) \geq \frac{m - 1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi) \left[ \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)b^2(t, \phi)
\]
\[
+ \int_E f^+(t, \phi, u) \left( \exp\left( \frac{1}{m} \Lambda^+_T(\hat{\phi}_t; t, \phi)f((t, \phi, u)) - 1 \right) \right) q(du). \tag{9.12}
\]
(9.12) and (9.6) imply that
\[
\int_0^T b^2(t, \phi)(\Lambda^+_T(\hat{\phi}_t; t, \phi))^2 dt < \infty \tag{9.13}
\]
\[
\int_0^T \left[ \int_E f^+(t, \phi, u) \left( \exp \left( \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) f((t, \phi, u)) \right) - 1 \right) q(du) \right] \times \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) dt < \infty.
\]  
(9.14)

(9.6), (9.8), and (9.13) and (9.14) lead in view of (9.2) to lead to

\[
\int_0^T \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) \left[ \dot{\phi}_t - a(t, \phi) + \int_E f^+(t, \phi, u) \times \left( \exp \left( - \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) f^+((t, \phi, u)) \right) - 1 \right) q(du) \right] dt < \infty.
\]  
(9.15)

By condition I for \( f \) and the inequality \( 1 - e^{-x} \leq x \), we have

\[
\int_E f^+(t, \phi, u) \left[ 1 - \exp \left( - \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) f^+((t, \phi, u)) \right) \right] q(du) \leq \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) \int_E (f^+((t, \phi, u))^2 q(du) \leq \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi)(1 + \phi_t^2)^2 \int_E h^2(u) q(du).
\]  
(9.16)

So far \( m \) is arbitrary, we take it such that

\[
m \geq 2C_1C(1 + \phi_T^2)^2 \int_E h^2(u) q(du),
\]

with \( C_1 \) and \( C \) from the assumptions of the lemma. Then (9.16) and assumptions of the Lemma yield

\[
\int_E f^+(t, \phi, u) \left[ 1 - \exp \left( - \frac{1}{m} \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) f^+((t, \phi, u)) \right) \right] q(du) \leq 1 + \frac{1}{2} |\dot{\phi}_t|.
\]

This together with (9.15) implies (note that the integrand in (9.15) is nonnegative)

\[
\int_0^T \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) I(\dot{\phi}_t > 0) \left( \dot{\phi}_t - a(t, \phi) - \frac{1 + \dot{\phi}_t}{2} \right) dt < \infty.
\]  
(9.17)

As by condition I, \( |a(t, \phi)| \leq l_T(1 + \phi_T^2) \), \( t \leq T \), so

\[
\left( \dot{\phi}_t - a(t, \phi) - \frac{1 + \dot{\phi}_t}{2} \right) I(\dot{\phi}_t \geq 4l_T(1 + \phi_T^2) + 2) \geq \frac{\dot{\phi}_t}{4} I(\dot{\phi}_t \geq 4l_T(1 + \phi_T^2) + 2)
\]

and hence by (9.17)

\[
\int_0^T \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi)|\dot{\phi}_t| I(\dot{\phi}_t \geq 4l_T(1 + \phi_T^2) + 2) dt < \infty.
\]  
(9.18)

Obviously \( \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi) = 0 \) if \( \phi_t \leq -l_T(1 + \phi_T^2) \), also by the conditions of the lemma \( |\Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi)| \leq C_1 C(1 + |\dot{\phi}_t|) \), and so by (9.18)

\[
\int_0^T \Lambda_{T,\phi}^+(\dot{\phi}_t; t, \phi)|\dot{\phi}_t| dt < \infty.
\]

Argument for \( \Lambda_{T,\phi}^-(\dot{\phi}_t; t, \phi) \) is similar. (9.5) is proved.

We end the proof. Let \( \phi^N = (\phi^N_t)_{t \geq 0} \) be defined in condition IV. Since \( \phi \in D \) is absolutely continuous so \( \int_0^T |\dot{\phi}_t| dt < \infty \) and we may regard that

\[
\sup_{t \leq T} |\phi^N_t - \phi_t| \leq \int_0^T |\dot{\phi}_t| I(|\dot{\phi}_t| > N) dt \leq \delta_{\phi}.
\]  
(9.19)
By Remark to condition \textbf{III} (see (2.1))
\[
H(\dot{\phi}_t; t, \phi^N) = A_{T,\phi}(\dot{\phi}_t; t, \phi^N)\dot{\phi}_t - G(A_{T,\phi}(\dot{\phi}_t; t, \phi^N); t, \phi^N) \\
\leq |A_{T,\phi}(\dot{\phi}_t; t, \phi^N)\dot{\phi}_t| + |A_{T,\phi}(\dot{\phi}_t; t, \phi^N)a(t, \phi^N)|. \tag{9.20}
\]
In view of conditions \textbf{I} and \textbf{II}, and (9.19) we may well assume that for large \(N\)
\[
\sup_{t \leq T} a(t, \phi^N) \leq l_T(2 + \phi^N),
\]
and then (9.20) yields for almost all \(t \leq T\) that
\[
H(\dot{\phi}_t; t, \phi^N) \leq C'|A_{T,\phi}(\dot{\phi}_t; t, \phi^N)|(1 + |\dot{\phi}_t|)
\]
for some \(C' > 0\). As the assumptions of the lemma and (9.19) imply that
\[
|A_{T,\phi}(\dot{\phi}_t; t, \phi^N)| \leq C_1^{-1}g_{T,\phi}(\dot{\phi}_t),
\]
we thus obtain
\[
H(\dot{\phi}_t; t, \phi^N) \leq C C_1^{-1}g_{T,\phi}(\dot{\phi}_t)(1 + |\dot{\phi}_t|). \tag{9.21}
\]
By the assumption of the lemma and (9.5)
\[
\int_0^T g_{T,\phi}(\dot{\phi}_t)(1 + |\dot{\phi}_t|)dt < \infty,
\]
and so (9.21) gives condition \textbf{IV}.

\textit{Proof of Theorem 9.1.} Condition \textbf{III} holds by Lemma 9.1. We now prove that conditions of Lemma 9.2 hold.

Assume first that \(f^+(t, X, u)\) and \(f^-(t, X, u)\) are nondegenerate with respect to \(q(du)\) uniformly on \([0, T]\) in a neighborhood of \(\phi\).

If \(y \geq l_T(2 + \phi^N)\), then by conditions \textbf{I} and \textbf{II} there exists \(\delta \leq \delta_{T,\phi}\) such that
\[
\sup_{t \leq T} |a(t, X)| \leq y \quad \text{for all } X = (X_t)_{t \geq 0} \in D \text{ with } \sup_{t \leq T} |X_t - \phi_t| \leq \delta.
\]
The definition of \(A_{T,\phi}\) for these \(X\), \(A_{T,\phi}(y; t, X) \geq 0\) and hence for any \(\varepsilon > 0\)
\[
y \geq -l_T(2 + \phi^N) + [\exp(A_{T,\phi}(y; t, X)\varepsilon) - 1] \int_E f^+(t, X, u)I(f^+(t, X, u) \geq \varepsilon)q(du),
\]
which implies that there exists \(r > 0\) such that
\[
A_{T,\phi}(y; t, X) \leq r \ln(1 + y) \tag{9.22}
\]
for all \(t \leq T\), all \(X = (X_t)_{t \geq 0} \in D\) with \(\sup_{t \leq T} |X_t - \phi_t| \leq \delta\) and \(y \geq l_T(2 + \phi^N)\).

The nondegeneracy of \(f^-(t, X, u)\) provides us with an estimate similar to (9.22) for \(y \leq -l_T(2 + \phi^N)\).

Hence we can find \(r_1\) such that for all \(y \in R\), all \(X = (X_t)_{t \geq 0}\) with
\[
\sup_{t \leq T} |X_t - \phi_t| \leq \delta
\]
and all \(t \leq T\) we have
\[
|A_{T,\phi}(y; t, X)| \leq r_1 \ln(1 + |y|). \tag{9.23}
\]

On the other hand, the definition of \(A_{T,\phi}\) and condition \textbf{I} imply that for all \(t \leq T\)
\[
|y| \leq l_T(1 + \phi^N) + |A_{T,\phi}(y; t, X)|l_T^2(1 + \phi^N)^2 \\
+ (|A_{T,\phi}(y; t, X)| + 1)l_T^2(1 + \phi^N)^2 \int_E h^2(u)q(du) \\
+ K(||A_{T,\phi}(y; t, X)|| + 1)l_T(1 + \phi^N) \tag{9.24}
\]
\((K(\lambda)\text{ is defined in condition } \textbf{I}). By assumptions of the theorem there exist \(A_1\) and \(A_2\) such that \(K(\lambda) \leq A_1 \exp(\lambda A_2)\) for all \(\lambda \geq 0\), and it then follows in view of (9.24) that for some \(r_1 > 0\) which depends on \(T\) and \(\phi\) we have
\[
|A_{T,\phi}(y; t, X)| \geq r_1 \ln(1 + |y|) \tag{9.25}
\]
for all $t \leq T$.

(9.23) and (9.25) show that the conditions of Lemma 9.2 are met by $g_{T,\phi}(y) = \ln(1 + |y|)$.

Now let $f^+(t, X, u)$ and $f^-(t, X, u)$ be degenerate with respect to $q(du)$ uniformly on $[0, T]$ in a neighborhood of $\phi$. Then by (9.1) and (9.2) and the definition of $\Lambda_{T,\phi}$ we have for all $t \leq T$ and all $X = (X_t)_{t \geq 0} \in D$ with $\sup_{t \leq T} |X_t - \phi(t)| \leq \delta$

$$y \geq -l_T(2 + \phi^*_T) + \Lambda_{T,\phi}(y; t, X)b^2(t, X)$$

if $y \geq l_T(\phi^*_T)$, and

$$y \leq l_T(2 + \phi^*_T) - \Lambda_{T,\phi}(y; t, X)b^2(t, X)$$

if $y \leq -l_T(\phi^*_T)$.

So if $b(t, X)$ is uniformly nondegenerate, than

$$|\Lambda_{T,\phi}(y; t, X)| \leq L(1 + |y|)$$

for all the above $t$ and $X$ and all $y \in \mathbb{R}$ and some $L > 0$. As, obviously

$$|y| \leq l_T(1 + \phi^*_T) + |\Lambda_{T,\phi}(y; t, X)|b^2(t, X),$$

so conditions of Lemma 9.3 hold with $g_{T,\phi}(y) = 1 + |y|$.

Finally, it is easily follows from the above that if $f^+(t, X, u)$ is degenerate with respect to $q(du)$ uniformly on $[0, T]$ in a neighborhood of $\phi$, $f^-(t, X, u)$ and $b(t, X)$ are both nondegenerate then we can take

$$g_{T,\phi}(y) = \begin{cases} 
1 + y, & y \geq 0, \\
1 + \ln(1 - y), & y \leq 0,
\end{cases}$$

and if degenerate is $f^-(t, X, u)$ and $f^+(t, X, u)$ and $b(t, X)$ are nondegenerate then conditions of Lemma 9.2 are met by

$$g_{T,\phi}(y) = \begin{cases} 
1 + \ln(1 + y), & y \geq 0, \\
1 - y, & y \leq 0.
\end{cases}$$

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