TOPOLOGICAL CENSORSHIP

KRISTIN SCHLEICH

and

DONALD M. WITT
Department of Physics, University of British Columbia
Vancouver, BC V6T 1Z1, Canada

ABSTRACT
Classically, all topologies are allowed as solutions to the Einstein equations. However, one does not observe any topological structures on medium range distance scales, that is scales that are smaller than the size of the observed universe but larger than the microscopic scales for which quantum gravity becomes important. Recently, Friedman, Schleich and Witt have proven that there is topological censorship on these medium range distance scales: the Einstein equations, locally positive energy, and local predictability of physics imply that any medium distance scale topological structures cannot be seen. More precisely, we show that the topology of physically reasonable isolated systems is shrouded from distant observers, or in other words there is a topological censorship principle.

1. Introduction
An interesting observed fact about our universe is that its spatial topology is trivial, that is continuously deformable to three dimensional Euclidean space, that is a region of $\mathbb{R}^3$. Thus is true on a remarkably large range of distance scales, ranging from fermis, the scale of interactions of high energy particles to megaparsecs, the scale of intergalactic distances. The theory of general relativity does not select a trivial topology; all 3-manifolds occur as the spatial topology of solutions to the Einstein equations. Thus general relativity allows spacetimes that contain arbitrarily large numbers of wormholes and other complicated topological structures. In fact, there even exist inflationary universes with an arbitrarily wide range of topology. Moreover, the dynamics of general relativity do not allow for the topology to change. Why don’t we see topological structures in our spacetime?

At first the answer to this question might seem obvious; we see no topological structures because they are not there. However, there is a more interesting possibility; such topological structures may indeed be present in our universe, but cannot be observed. This second possibility was more formally stated as the topological censorship conjecture by Friedman and Witt: The topology of any physically reasonable isolated system is shrouded. That is a spacetime may contain isolated topological structures such as wormholes; however, there is no method by which an experimenter can determine that a spacetime contains such non-Euclidean topology and report this result to a distant observer.

Recently Friedman, Schleich and Witt have proven this conjecture. Today,
instead of emphasizing the technical details used in proving the result, I would like to give an informal and intuitive sketch of how this proof works and mention some of its consequences for general relativity and cosmology.

2. Background

One of the fundamental tenets of classical physics is *physical predictability*, that is given knowledge of the initial condition of the system, the laws of physics allow one to determine the behavior of the system at all future (and past) times. For example the motion of a particle in a potential is physically predictable; given the position and velocity of the particle at one instant of time, the equations of motion when solved determine the trajectory of the particle for all time. Similarly a spacetime is physically predictable if information about its geometry and matter sources at one instant of time allows the determination of its spacetime geometry for all times via the Einstein equations. For those familiar with general relativity, physically predictable spacetimes are termed *globally hyperbolic* spacetimes. Such a definition is needed because not all solutions of the Einstein equations are physically predictable. For example, solutions that contain closed timelike curves or naked singularities are not physically predictable; however such solutions exhibit properties that patently violate the laws of physics. Thus it is the properties of physically predictable spacetimes that are of great interest as our universe is of this type.

Topology describes the properties of a space that are independent of the metric, that is the distances between points in the space. A familiar two dimensional example is given by the torus. Observe that there is one hole in this space; moreover, this hole remains under any continuous deformation of the surface, that is under any continuous twisting or stretching of the surface. A sphere has no hole; therefore, there is no continuous deformation of the torus that takes it into a sphere. This observation is formally stated in terms of topology: a sphere and a torus have different topology. This difference in topology can be seen in the properties of curves on the torus and the sphere: One can find closed curves on the torus that are noncontractible, that is they cannot be continuously shrunk to a point. Such curves are those that loop about the hole in the torus. However, all closed curves on the sphere are contractible. In fact, the properties of curves on the two spaces rigorously characterize their topology.

How would an experimenter go about determining the topology of spacetime? An important fact to remember is that experimenters, like all other massive objects, travel on timelike paths through their spacetime. Moreover, any measurement that an experimenter performs relies on information that also travels on a timelike or null path. Note especially that the path need not be everywhere timelike or everywhere null; for example, information could be carried from one point to another first by a massive particle travelling a timelike path that then releases a photon travelling a null path. Thus a determination of the topology of spacetime must be carried out
using information that travels a path that is either timelike or null at all points along its course. Such paths are referred to as causal curves.

1. A two dimensional spacetime with nontrivial topology. An observer can detect this topology through nontrivial timelike curves such as $c$ as shown on the left, or through nontrivial null curves, as shown on the right. The observer cannot detect the topology using a spatial curve, such as $s$ as information cannot travel such a curve.

For example, an experimenter living in the two dimensional spacetime illustrated in figure 1 can determine that the spatial topology is a circle by discovering that there are noncontractible causal curves in the spacetime. The experimenter can find such curves, for example, by laying out a rope while walking around the space counterclockwise as illustrated by curve $c$ in figure 1. The resulting loop of rope clearly will not be contractible. Note that the experimenter cannot use a noncontractible spacelike curve such as $s$ to determine the topology as it is physically impossible to lay out such a curve. It is especially important to note that the experimenter can probe the topology using distant objects that emit information such as massive particles or light that travels along causal curves. For example the experimenter may be

*An observant reader may worry that our universe is not two dimensional, but four dimensional; however, properties of curves in it still characterize its topology. Our universe is a physically predictable spacetime and thus its topology is completely carried by the topology of the three dimensional spatial hypersurface; physical predictability implies the topology at one time must determine it for all times. The topology of a three dimensional hypersurface can be characterized finding noncontractible curves as in the two dimensional case. Therefore, by laying out causal curves in the four dimensional spacetime, an experimenter can actively probe its topology.*

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able to use light travelling two distinct paths from a star to probe the topology as also illustrated in figure 1.

It is apparent that the causal structure of spacetime, that is which points of the spacetime can be connected to each other by causal curves, is important to the topological censorship theorem; after all the question at hand is whether or not causal curves can thread the topology and come out at any time, even infinitely far in the future. Therefore, a concrete way of talking about such curves is needed. This terminology and a very useful pictorial way of representing this causal structure, Penrose diagrams, is based off of the causal structure of Minkowski space.\footnote{Ref. 4, p. 118 provides a clear discussion of terminology used in causal structure and Penrose diagrams as well as a comprehensive discussion of many of the other concepts used in the theorem. References to the original literature also can be found in this text.}

2. The causal structure of Minkowski spacetime. On the left, Minkowski spacetime is represented in spherical coordinates as a two dimensional diagram by suppressing the angular coordinates. On the right is the Penrose diagram for Minkowski space. Points once off at infinite distance such as future and past timelike infinity and future and past null infinity now appear at finite distance.

An illustration of Minkowski spacetime with metric

\[
ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]  

is given in the left side of figure 2. Note that there is no concrete way in this diagram to represent where the timelike geodesics or the null geodesics end up at in the infinite future. For example, it is not clear from this diagram that null geodesics go to to
a different infinity than timelike geodesics. The reason is obvious; the coordinates \( r \) and \( t \) have infinite ranges; therefore there is no way to draw where infinity is. However, this is not a problem with studying causal structure; rather it is a problem with the choice of coordinates. But physics is coordinate invariant, so instead write the metric of Minkowski space in a set of coordinates such that these infinite points now occur at finite coordinate values. Defining \( t' \) and \( r' \) by

\[
2t = \tan\left(\frac{t' + r'}{2}\right) + \tan\left(\frac{t' - r'}{2}\right) \quad \quad 2r = \tan\left(\frac{t' + r'}{2}\right) - \tan\left(\frac{t' - r'}{2}\right)
\]

the Minkowski metric becomes

\[
ds^2 = \Omega^2(t', r')\left(-dt'^2 + dr'^2 + r'^2(d\theta + \sin^2\theta d\phi^2)\right)
\]

where \( \Omega(t', r') = \frac{1}{2}\sec\left(\frac{t'+r'}{2}\right)\sec\left(\frac{t'-r'}{2}\right) \). The new coordinates have finite ranges, \( r' \geq 0 \), \(-\pi < t' + r' < \pi\), and \(-\pi < t' - r' < \pi\); thus the full spacetime can now be represented in a finite diagram. Finally, one uses the fact that two spacetimes related by a conformal transformation have the same causal structure to further simplify the discussion; this fact means that one need not represent the factor of \( \Omega^2 \) in the above metric to concretely illustrate the causal structure. The resulting diagram, the Penrose diagram of Minkowski spacetime is given in the right side of figure 2. By changing coordinates, the infinite future and past are now clearly described. Timelike geodesics, for example the path travelled an observer who remains at a fixed radial coordinate position \( r \) begin at past timelike infinity, \( i^- \), and end at future timelike infinity, \( i^+ \). Similarly null geodesics, for example the path travelled by a photon travelling radially inward to the coordinate origin begin at past null infinity, \( \mathcal{I}^- \), and end at future null infinity, \( \mathcal{I}^+ \). It is now clear that the future infinities of that null geodesics and timelike geodesics are distinct. Observe that not all timelike curves end at \( i^+ \); curves corresponding to accelerated timelike observers can reach \( \mathcal{I}^+ \). Finally, radially directed photons travel along paths at 45 degree angles in this spacetime; thus any timelike or null curve leaving a point in this spacetime must have tangent lying between the inward directed radial null geodesic and outward directed radial null geodesic. Therefore, this Penrose diagram neatly encapsulates the information about the causal structure of Minkowski spacetime.

It is clear that technique used to concretely discuss the causal structure of Minkowski spacetime can be applied to illustrate the causal structure of other spacetimes. Of course, spacetimes with different metrics and topology will not have Penrose diagrams that are identical to figure 2. However, spacetimes that resemble or approach Minkowski space in regions of the spacetime will have similar causal structure in those regions. In particular, spacetimes containing isolated topological structures will have similar causal structure far away from the topology.

An isolated topological structure is, as implied by its name, one that can be isolated from the rest of the spacetime. More precisely, one can place a sphere around the topology at a given instant in time and "cut out" the topology by excising this sphere and everything inside it from the spatial hypersurface. When one does so, the remaining space has the topology of three dimensional Euclidean space minus a ball. Note that the sphere surrounding the topology might be very large and contain
many topological structures; for example, there might be a large number of wormholes inside the sphere. Additionally, the metric of the spacetime outside the evolution of the excised ball approaches Minkowski spacetime as one goes infinitely far away; that is one can find a set of coordinates such that as one goes to infinite spatial distance, the metric is Eq. (1) plus $1/r$ correction terms. A spacetime that satisfies these conditions is termed an *asymptotically flat spacetime*. The Schwarzschild solution is a canonical example of an asymptotically flat spacetime, though it is obvious that there are myriad examples of such spacetimes, including those with isolated topological structures.

It is important for understanding the theorem to observe that there may be more than one asymptotically flat region for a spacetime containing isolated topological structures; the topological structures potentially can connect several different copies of $\mathbb{R}^3$, each of which admits a metric approaching that of Minkowski space. Additionally note that as the spacetime is approaching Minkowski spacetime in each asymptotically flat region, intuitively its causal structure is also approaching that of Minkowski spacetime. Indeed this is the case; the Penrose diagram for an asymptotically flat spacetime will contain one or more regions which have the same causal structure as Minkowski spacetime.

### 3. The Topological Censorship Theorem

The Einstein equations can be coupled to a wide range of matter, but certain general properties characterize classical physical matter, that is sources that are not quantum in nature. These properties are called energy conditions. The energy condition used in the proof of the topological censorship theorem is the *null energy condition*. Physically, this condition states that an observer travelling along either a timelike or null curve measures the energy in their local frame to be positive at any point in the spacetime. This condition is satisfied by all classical sources of matter found in nature such as gas, dust, radiation, electromagnetic fields, as well as idealized sources such as classical scalar fields. It is implied by each of the other classical energy conditions: the weak energy condition, the strong energy condition and the dominant energy condition. Therefore, the null energy condition is a very reasonable and physical restriction.

Given the above, it is now possible to state the theorem:

**Theorem.** *If an asymptotically flat globally hyperbolic spacetime satisfies the null energy condition, then every causal curve from past null infinity to future null infinity is deformable to $\gamma_0$.***

The curve $\gamma_0$ is a representative causal curve with past endpoint at $\mathcal{I}^-$ and future endpoint at $\mathcal{I}^+$ that lies in the asymptotically flat region; i.e. it is a curve that does not pass through any of the topology of the spacetime.

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*A spatial analog of this feature is provided by figure 4, although it is included in this paper for other purposes. It contains two spatially asymptotically flat regions connected by a throat.*
3. A Penrose diagram illustrating the statement of the theorem. The theorem proves that any causal curve traversing the topology in the shaded region cannot reach future null infinity. Thus the curve $\gamma$ in this diagram either does not go through the topology or is spacelike at points along its course.

Figure 3 is a Penrose diagram of a spacetime used to illustrate the theorem; for convenience this spacetime is assumed to have only one asymptotic region. Note especially that the causal properties of the shaded region, that is the region containing topology are not faithfully illustrated; only those of the asymptotically flat region, that with the causal structure of the radially distant part of Minkowski spacetime, have been correctly diagrammed. The theorem states that any causal curve $\gamma$ that can reach an observer in the asymptotically flat region is deformable to the trivial curve $\gamma_0$. This means that $\gamma$ cannot traverse any topological structure because if it did, it could not be deformed into the trivial one as it would hook on the topology. It follows that all causal curves that enter a topological structure cannot come out again; therefore there is no way to actively probe the topology.

The proof of the theorem is based on a lemma that applies to simply connected spacetimes. A simply connected spacetime is one for which all closed curves are contractible. For example, a sphere and a plane are simply connected spacetimes, but a torus is not. Thus simply connected spacetimes have very special topology.

**Lemma.** Suppose one has an asymptotically flat simply connected spacetime that satisfies the null energy condition. Then no 2-surface $\tau$ that is outer trapped with respect to $\mathcal{J}$ can be seen from $\mathcal{J}^+$.

A surface is said to be *outer trapped* if radially outward directed null rays are converging. Note that radially outward is defined by the direction one goes to
reach the asymptotically flat region of the observer. In order to understand the motivation behind this definition, consider the behavior of light emitting surfaces in both Minkowski and curved spacetime. First take a sphere at one instant of time in Minkowski space and release radially outward directed photons from its surface. At some small interval of time later, say one second, consider the surface formed by these photons. As the spacetime geometry is flat, this photon sphere will have larger area than that of the original sphere. This result seems obvious; however, if one works in a curved spacetime the result can be very different and depends on the curvature of the spacetime in the region of the sphere. An example is illustrated in figure 4; the surface $\tau$ in this particular spacetime has a photon sphere that has larger area at later time, very similar to the situation in Minkowski space. However, the surface $\tau'$ has a photon sphere with smaller area at later time; the curvature of the spacetime in the neighborhood of $\tau'$ forces the light rays to converge even though the begin by heading in the radially outward direction. Thus this surface is outer trapped. Physically what is happening is the curvature of spacetime is so strong that even radially outward directed light rays are being forced inward.

![Illustration of an outer trapped surface](image)

4. Illustration of an outer trapped surface. The picture represents a three dimensional spatial slice of a spacetime; one dimension is suppressed. $\tau$ and $\tau'$ are spatial spheres in the hypersurface. $\tau$ is not outer trapped; $\tau'$ is outer trapped.

Given this physical picture of an outer trapped surface one intuitively gathers that no information from it or interior to it can escape out to asymptotic infinity to be detected by the observer; no signal travels faster than light and radially outward directed light is travelling outward at the maximal rate. Therefore if radially directed light is forced spatially inward by the curvature, all other signals, radial or nonradial will be as well. Thus you cannot see inside a trapped surface. This is precisely what the Lemma proves rigorously. I will not go into the proof here, but note that it uses standard techniques from the singularity theorems.5

This lemma is for a simply connected spacetime, but the theorem applies to all asymptotically flat spacetimes. The way that a connection is made is through the
concept of a \textit{universal cover}. A universal cover is a spacetime that is related to the original spacetime through properties of curves; curves that are noncontractible in the original space are unwrapped to become contractible curves in the universal cover. Again it is easiest to illustrate this concept through an example. An illustration of the cylinder spacetime and its universal cover are given in Figure 5. In the original spacetime, \( b \) is a trivial curve, \( c, s \) and \( d \) are all nontrivial curves. Its universal cover is a plane; one can wrap the plane around the cylinder to cover it an infinite number of times. Each point in the cylinder is covered by an infinite number of points in the plane, for example \( q \) corresponds to an infinite number of points in the plane as indicated. Now noncontractible curves in the cylinder spacetime are identified with contractible curves connecting different copies of the starting point \( q \) to \( p \) in the universal covering spacetime. For example the curve \( c \) is identified with a curve attaching one copy of \( q \) to \( p \), the curve \( d \) is identified with a curve attaching a different copy of \( q \) to \( p \). Clearly, by construction the universal cover is a simply connected spacetime.

5. An example of a universal covering space. On the right is the universal cover of the spacetime illustrated on the left. All noncontractible curves in the spacetime on the left are unwrapped to contractible curves in the universal cover.

Although we have illustrated the definition of a universal cover in a particular example, certain features of this example are generic. Of particular importance is the fact that noncontractible curves in the original spacetime correspond to contractible curves connecting copies of the original starting and ending points in the covering spacetime. This implies that the covering spacetime of an asymptotically flat spacetime with nontrivial topology always has multiple asymptotically flat regions even if the original spacetime had only one. Most importantly, any curve in the original spacetime that traverses the topology necessarily connects two distinct asymptotic regions in the covering spacetime.
6. An illustration of the proof of the theorem by contradiction. If the curve $\gamma$ of figure 3 traverses a topological structure, then it corresponds to a curve $\Gamma$ in the universal cover whose beginning is in a different copy of the asymptotic region than that containing its end and the trivial curve $\gamma_0$.

We now have all the tools to discuss the proof of the theorem. The proof is by contradiction.

Suppose the theorem is false. Then there is a causal curve $\gamma$ from $\mathcal{I}^-$ to $\mathcal{I}^+$ that passes through the topology and thus is not deformable to the trivial curve $\gamma_0$. (Recall that figure 3 provides an example of such a spacetime.) Now consider the universal covering space of the spacetime. (A schematic Penrose diagram of the universal covering space of figure 3 is given in figure 6.) As $\gamma$ passes through the topology, its corresponding curve $\Gamma$ in the covering space must begin in a different asymptotic region than that containing $\gamma_0$. In this asymptotic region, the spacetime is becoming asymptotically flat; therefore, the curve $\Gamma$ intersects arbitrarily large spatial spheres as it approaches the infinite past. Null curves that can reach $\mathcal{I}^+$ must be directed inward from these large spheres as indicated in the inset of figure 6. Therefore photon spheres released from these surfaces corresponding to these null curves are shrinking in area. Thus these large spheres near the origin of the curve $\Gamma$ are outer trapped with respect to an observer at $\mathcal{I}^+$. However, the assumption that $\Gamma$ reaches $\mathcal{I}^+$ means that an observer can see these spheres at $\mathcal{I}^+$. But this conclusion contradicts the Lemma! Therefore, our assumption that $\gamma$ is a causal curve reaching $\mathcal{I}^+$ must be wrong. Thus there is no causal curve that passes through the topology and reaches future null infinity. As any $\gamma$ not deformable to $\gamma_0$ must correspond to a curve that goes to another asymptotic region in the universal covering spacetime, we
conclude that all curves reaching future null infinity are deformable to $\gamma_0$ if they are causal. Q.E.D.

4. Discussion

The consequences of the topological censorship theorem can be seen by considering a physically predictable spacetime with non-Euclidean topology such as a handle attached to a plane. Note that this spacetime has one asymptotic region. Its universal cover will be a spacetime with multiple asymptotic regions. Now suppose that an experimenter wishes to probe the topology of this spacetime and communicate the results of the measurements to a distant observer near $J^+$; note that this distant observer could even be the experimenter herself if she sends signals or uses signals from distant objects in the spacetime. In order to detect the handle, the path of some signal must traverse the handle and exit to $J^+$; but this is forbidden by the theorem. Only causal paths that do not loop through the handle can communicate with $J^+$, and such causal curves do not detect the existence of non-Euclidean topology. Thus general relativity prevents one from actively probing the topology of spacetime.

So if the topology of the handle cannot be detected, what will the experimenter see? Note that curvature of spacetime such as that associated with a handle acts like a mass when viewed from a distant region. Moreover, one cannot probe the properties of this mass; the topology appears to be behind a horizon to the experimenter. Thus isolated topological structures appear to be black holes to outside observers, indistinguishable classically from black holes formed by the collapse of matter. Therefore, if our universe were full of isolated topological structures, they would appear to us as black holes.

The theorem was proven for asymptotically flat spacetimes; however, in cosmology one would like to apply it to spacetimes that do not have the precise asymptotic behavior of the metric described in the theorem. This is no difficulty for the case where the scale of the isolated topological structure is small, for example when it is the size of the solar system (or smaller!) or even of a galactic core. For these cases our universe is well approximated by an asymptotically flat spacetime. More generally, note that the key use of the asymptotic behavior of the metric is in showing that arbitrarily large spatial spheres are outer trapped in the region from which $\Gamma$ originates. Intuitively, one expects this behavior to occur in spacetimes with more general behavior in the asymptotic regions and that the theorem could be generalized to spacetimes that have metrics that allow arbitrarily large spatial spheres. Indeed this is the case. Therefore, the topological censorship theorem applies quite generally to cases of spacetime relevant to cosmology.

Finally, although the sketch of the topological censorship theorem uses the null energy condition, one can show that it actually can be rigorously proven for a weaker energy condition, the averaged null energy condition. Physically this energy condition states that the energy can be negative in small regions so long as it is
positive when averaged over the whole spacetime. This implies that the topological censorship theorem can be applied not only to spacetimes with classical matter but may also apply to spacetimes containing certain types of quantum matter.

5. References

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