A PROPERTY OF THE BROWN-YORK MASS IN SCHWARZSCHILD MANIFOLDS

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Abstract. We will extend partially our previous results about the limit of the Brown-York mass of a family of convex revolution surfaces in the Schwarzschild manifold such that these surfaces may have unbounded ratios of their radii.

1. Introduction

In this paper, we will continue to study the limiting behavior of the Brown-York mass of a family of convex revolution surfaces in the Schwarzschild manifold and extend our previous results in [8]. Throughout this paper, we will denote $(\mathbb{R}^3, \delta_{ij})$ as the 3-dimensional Euclidean space, $x^1, x^2, x^3$ as the standard coordinates of $\mathbb{R}^3$, $r$ and $\partial$ as the Euclidean distance and the standard derivative operator on $\mathbb{R}^3$ respectively. For the sake of convenience, let us first recall some definitions. First of all, we will adopt the following definition of asymptotically flat manifolds.

Definition 1.1. A complete three dimensional manifold $(M, \lambda)$ is said to be asymptotically flat (AF) of order $\tau$ (with one end) if there is a compact subset $K$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates in $\mathbb{R}^3$, the metric $\lambda$ satisfies:

\[
\lambda_{ij} = \delta_{ij} + \sigma_{ij}
\]

with

\[
|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2|\partial \partial \sigma_{ij}| + r^3|\partial \partial \partial \sigma_{ij}| = O(r^{-\tau}),
\]

for some constant $1 \geq \tau > \frac{1}{2}$.

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A coordinate system of \( M \) near infinity so that the metric tensor in this system satisfy the above decay conditions is said to be admissible. In such a coordinate system, we can define the ADM mass as follows.

**Definition 1.2.** The Arnowitt-Deser-Misner (ADM) mass (see [1]) of an asymptotically flat manifold \((M, \lambda)\) is defined as:

\[
m_{\text{ADM}}(M, \lambda) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (\lambda_{ij,i} - \lambda_{ii,j}) \nu^j d\Sigma^0_r,
\]

where \( S_r \) is the Euclidean sphere, \( d\Sigma^0_r \) is the volume element induced by the Euclidean metric, \( \nu \) is the outward unit normal of \( S_r \) in \( \mathbb{R}^3 \) and the derivative is the ordinary partial derivative.

We always assume that the scalar curvature is in \( L^1(M) \) so that the limit exists in the definition. In [2], Bartnik showed that the ADM mass is a geometric invariant.

On the other hand, there have been many studies on the relation between the ADM mass of an AF manifold and the so called quasi-local mass. Let us recall the definition of the Brown-York quasi-local mass. Suppose \((\Omega, \mu)\) is a compact three dimensional manifold with smooth boundary \( \partial \Omega \), if moreover \( \partial \Omega \) has positive Gauss curvature, then the Brown-York mass of \( \partial \Omega \) is defined as (see [5, 6]):

**Definition 1.3.**

\[
m_{\text{BY}}(\partial \Omega) = \frac{1}{8\pi} \int_{\partial \Omega} (H_0 - H) d\sigma
\]

where \( H \) is the mean curvature of \( \partial \Omega \) with respect to the outward unit normal and the metric \( \mu \), \( d\sigma \) is the volume element induced on \( \partial \Omega \) by \( \mu \) and \( H_0 \) is the mean curvature of \( \partial \Omega \) when embedded in \( \mathbb{R}^3 \).

The existence of an isometric embedding in \( \mathbb{R}^3 \) for \( \partial \Omega \) was proved by Nirenberg [17], the uniqueness of the embedding was given by [13, 19, 18], so the Brown-York mass is well-defined.

It can be proved that the Brown-York mass and the Hawking quasi-local mass [11] of the coordinate spheres tends to the ADM mass in some AF manifolds, see [6, 12, 4, 3, 22, 9], even of nearly round surfaces [21], and of a family of convex revolution surfaces in an asymptotically Schwarzschild manifold [8] for the Brown-York mass. The ratios of the radii of these surfaces are all bounded. In this paper, we will generalize some results in [8] partially in that we allow the ratios of the radii of the family of surfaces to be unbounded. We will consider a kind of AF manifolds, called Schwarzschild manifolds, defined as follows:
Definition 1.4. \((N, g)\) is called a Schwarzschild manifold if \(N = \mathbb{R}^3 \setminus K\), \(K\) is a compact set containing the origin, and in the standard coordinates of \(\mathbb{R}^3\),

\[
g_{ij} = \phi^4 \delta_{ij}, \phi = 1 + \frac{2m}{r}, m > 0.
\]

Clearly, \((N, g)\) is an AF manifold and the scalar curvature of \((N, g)\) is zero \[15\] (page 283). Moreover, the ADM mass is equal to \(m\).

Let \(w(\varphi), z(\varphi)\) be smooth functions on \([0, l]\) such that the surface of revolution generated by \(w\) and \(z\):

\[
(w(\varphi) \cos \theta, w(\varphi) \sin \theta, z(\varphi))
\]

is a smooth convex surface diffeomorphic to \(S^2\) and

\[
\begin{aligned}
C_1^2 &\leq w^2 + z^2 \leq C_2^2, \text{ for } C_1, C_2 > 0 \\
w^2 + z^2 &= 1 \\
w &\geq 0 \text{ on } [0, l] \text{ and } z(0) > z(l).
\end{aligned}
\]

Let \(f(a)\) be a function such that \(f(a) \geq 1\) for all \(a \geq 1\). We define the family of surfaces \(S_a\) by the parametrization

\[
(aw(\varphi) \cos \theta, aw(\varphi) \sin \theta, ah_a(\varphi))
\]

where \(h_a(\varphi) = f(a)z(\varphi)\). Note that \(S_a\) forms an exhaustion of \(N\) as \(a \to \infty\).

We will prove the following:

**Theorem 1.1.** Suppose \((S_a, g|_{S_a})\) has positive Gaussian curvature, then the Brown York mass of \(S_a\) tends to the ADM mass of \((N, g)\). That is

\[
\lim_{a \to \infty} m_{BY}(S_a) = m.
\]

One example of surfaces satisfying the conditions in Theorem 1.1 is the family of ellipsoids:

\[
S_a = \left\{(x_1)^2 + (x_2)^2 + \frac{2m(x_3)^2}{a} = a^2\right\}
\]

which has unbounded ratios of their radii as \(a \to \infty\).

From Theorem 1.1 we have

**Corollary 1.1.** Suppose \(\frac{L^2}{a} = o(1)\) for sufficiently large \(a\), then

\[
\lim_{a \to \infty} m_{BY}(S_a) = m.
\]

Clearly the above example shows that the condition \(\frac{L^2}{a} = o(1)\) is not necessary.

This paper is organized as follows. In Section 2 we will prove Theorem 1.1. Corollary 1.1 will be proven in Section 3.
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2. Proof of Theorem 1.1

Let us first prove some lemmas. We can assume \( w \) is anti-symmetric about 0 and \( l \), \( z \) is symmetric about 0 and \( l \). This implies

\[
(2.1) \quad w(0) = w(l) = z'(0) = z'(l) = 0.
\]

The Gaussian curvature \( K \) of \([1.5]\) with respect to \( \delta \) is (\[7\] p.162)

\[
K = \frac{z''(w'z'' - w''z')}{w} \quad \text{for} \quad \varphi \in (0, l).
\]

So by \([1.6]\), \( z' < 0 \) on \((0, l)\).

We will sometimes regard \( \phi \) as function of \( \varphi \) (by restricting on \( S_a \)) by abuse of notation. We define

\[
(2.2) \quad D = \sqrt{w'^2 + h'^2} = \sqrt{w'^2 + f^2z'^2}.
\]

Similar to Lemma 2.1 in \([8]\), we have

**Lemma 2.1.** The functions \( \frac{w}{z} \) and \( \frac{w''}{z} \) can be extended continuously to \([0, l]\).

**Proof.** The Gaussian curvature of the surface given by \([1.5]\) at \((0, 0, z(0))\) is \( z''(0)^2 > 0 \), so \( \lim_{\varphi \to 0} \frac{w}{z} = \frac{w'(0)}{z'(0)} < \infty \). The case for \( \varphi = l \) is the same.

The plane curve \((w(\varphi), z(\varphi))\) has curvature \( k = w''z' - z''w' = (w', z') \cdot (-z'', w'') \). Since \((w'', z'') \perp (w', z')\) and \((w'', z'') \perp (-z'', w'')\), we have \( w'' = kz' \). From this we see that \( \frac{w''}{z} \) can be extended to \( k(0) \) at \( \varphi = 0 \). The case for \( \varphi = l \) is the same. \(\square\)

**Lemma 2.2.** The following functions can be extended continuously to \([0, l]\) such that

\[
\frac{w}{h'} = O \left( \frac{1}{f} \right), \quad \frac{w''}{h'} = O \left( \frac{1}{f} \right),
\]

\[
D' = O \left( \frac{f^2}{D} \right), \quad \frac{D'}{h'} = O \left( \frac{f}{D} \right)
\]

\[
(2.3) \quad \frac{\phi'}{h'} = O \left( \frac{fa^2}{r^3} \right), \quad \frac{\phi'}{h'} = O \left( \frac{fa^2}{r^3} \right), \quad \phi'' = O \left( \frac{f^2a^2}{r^3} \right).
\]
Proof. The first line of (2.3) follows from Lemma 2.1. As $D' = \frac{w'w''+f^2z''}{h'}$, $|D'| \leq \frac{Cf^2}{D}$. Also, using the fact that $\frac{w}{h} = O(f^{-1}) = O(1)$, $\frac{D'}{D} = \frac{1}{D}(w''w' + f z'') = O\left(\frac{f}{D}\right)$.

For (2.4), we have $\phi' = -\frac{ma^2}{2r^3}(ww' + f^2zz')$. So
\[
|\phi'| \leq \frac{ma^2}{2r^3}(w^2 + f^2z^2)\frac{1}{2}(w^2 + f^2z^2)^{\frac{1}{2}} = O\left(\frac{fa}{r^2}\right).
\]

Using (2.3) again,
\[
|\phi'| = \frac{ma^2}{2r^3}\left(w'w + fz\right) = O\left(\frac{fa^2}{r^3}\right).
\]

Finally,
\[
|\phi''| = \frac{3m}{2}r^{-5}a^4(ww' + hh')^2 - \frac{m}{2}r^{-3}a^2((w')^2 + (h')^2 + w w'' + h h'')
\leq \frac{3m}{2}r^{-5}a^4(w^2 + h^2)((w')^2 + (h')^2)
+ \frac{m}{2}r^{-3}a^2(((w')^2 + (h')^2) + (w^2 + h^2)^{\frac{1}{2}}((w'')^2 + (h''^2)^{\frac{1}{2}})
= \frac{3m}{2}r^{-5}a^2((w')^2 + (h')^2) + O(r^{-3}a^2 f^2) = O(r^{-3}a^2 f^2).
\]

Hence Lemma 2.2 holds. □

We want to compute the mean curvatures. By Lemma 2.4 in [8], we have

Lemma 2.3. For a smooth revolution surface in $(\mathbb{R}^3, \delta)$ parametrized by
\[
(2.5) \quad (au(\varphi) \cos \theta, au(\varphi) \sin \theta, av(\varphi)), \quad 0 < \varphi < 1, 0 < \theta < 2\pi,
\]
its mean curvature $\overline{H}$ with respect to $\delta$ is
\[
(2.6) \quad \overline{H} = \frac{u''}{a T v'} - \frac{T' u'}{a T^2 v'} - \frac{v'}{a T u} \quad \text{where} \quad T = \sqrt{u'^2 + v'^2}.
\]

Similar to Lemma 2.5 in [8], we can get

Lemma 2.4. The mean curvature $H$ of $S_a$ with respect to $g$ is
\[
H = \frac{w''}{a \phi^2 Dh'} - \frac{D' w'}{a \phi^2 D^2 h'} - \frac{h'}{a \phi^2 D w} + 4 \phi^{-3} n(\phi)
\]
where $n$ is the outward unit normal vector of $S_a$ with respect to $\delta$. 

Proof. It is similar to the proof of Lemma 2.5 in [8]. For completeness, we sketch it here. By Lemma 2.4, the mean curvature of $S_a$ with respect to $\delta$ is

$$H = \frac{w''}{aDh'} - \frac{D'w'}{aD^2h'} - \frac{h'}{aDw}.$$  

The mean curvature $H$ of $S_a$ with respect to $g$ is ([20], page 72):

$$H = \phi^{-2} \left( \bar{H} + 4\phi^{-1} n(\phi) \right)$$

where $n$ is the outward unit normal vector of $S_a$ with respect to $\delta$. Submitting (2.7) to (2.8), we can get Lemma 2.4. □

Lemma 2.5. Suppose $(S_a, g|_{S_a})$ has positive Gaussian curvature such that it can be uniquely isometrically embedded into $\mathbb{R}^3$ (for sufficiently large $a$), then the embedding is given by

$$u = \phi^2w \quad \text{and} \quad v' = \phi^2h' \left(1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2h'^2}\right)^{\frac{1}{2}}.$$  

Proof. The proof is similar to that of Lemma 2.6 in [8]. In $(\varphi, \theta)$ coordinates, the metric on $S_a$ induced by $g$ can be written as:

$$ds^2 = a^2\phi^4(w'^2 + h'^2)d\varphi^2 + a^2\phi^4w^2d\theta^2.$$  

We can regard $(S_a, ds^2)$ as the sphere with the metric $ds^2$. We want to find two functions $u, v$ such that the surface written as the form (2.9) is an embedded surface $S^e_a$ of $S_a$ into $(\mathbb{R}^3, \delta)$. First of all, the induced metric by the Euclidean metric on the surface which is of the form (2.9) can be written as:

$$ds^2_e = a^2(u'^2 + v'^2) d\varphi^2 + a^2u^2d\theta^2.$$  

Comparing this with (2.11), one can choose

$$u = \phi^2w \quad \text{and} \quad u'^2 + v'^2 = \phi^4D^2.$$  

Consider

$$\phi^4(w'^2 + h'^2) - u'^2 = \phi^2(\phi^2(w'^2 + h'^2) - (2\phi'w + \phi w')^2)$$

$$= \phi^2(\phi^2h'^2 - 4\phi\phi'ww' - 4\phi'^2w^2)$$

$$= \phi^4h'^2 \left(1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2h'^2}\right).$$  

(2.13)
A property of the Brown-York mass

By Lemma 2.2, the functions $\frac{\phi'w'w'}{\phi h^2}, \frac{\phi'^2w^2}{\phi^2h^2}$ can be extended continuously on $[0, l]$ with $\frac{\phi'w'w'}{\phi h^2} = O(a^{-1}), \frac{\phi'^2w^2}{\phi^2h^2} = O(a^{-2})$. So $1 - \frac{4\phi'w'w'}{\phi h^2} - \frac{4\phi'^2w^2}{\phi^2h^2} > 0$ for sufficiently large $a$. For these $a$, we can take

$$v' = \phi^2h' \left( 1 - \frac{4\phi'w'w'}{\phi h^2} - \frac{4\phi'^2w^2}{\phi^2h^2} \right)^{\frac{1}{2}},$$

so that $u'^2 + v'^2 = \phi^4(w'^2 + h'^2)$. Note that $v'$ is an odd function for $\phi \in [-l, l]$. By choosing an initial value, one can get an even function $v$. By the above argument, one has

$$v' = \phi^2h' \left( 1 - \frac{2\phi'w'w'}{h'^2} + O(a^{-2}) \right).$$

From (2.12) and (2.13), near $\phi = 0$, $u, v$ can be extended naturally to $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Since $u$ is an odd function in $\phi$, $v$ is an even function in $\phi$, and $u'^2 + v'^2 = T^2 > 0$, the generating curve in $\{x^2 = 0\}$ is symmetric with respect to $x^3$-axis, and is smooth at $\phi = 0$. Similarly, it is also smooth at $\phi = l$. Hence the revolution surface determined by the choice of $u, v$ as above can be extended smoothly to a closed revolution surface, which is the embedded surface of $S_a$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $F = \phi^2D$, then by Lemma 2.3 and Lemma 2.4.

$$H_0 - H = \left( \frac{u''}{aFv'} - \frac{w''}{a\phi^2Dh'} \right) - \left( \frac{F'u'}{aF^2v'} - \frac{D'w'}{a\phi^2D^2h'} \right) - \left( \frac{v'}{aFv} - \frac{h'}{a\phi^2Dw} \right) - 4\phi^{-3}n(\phi).$$

By Lemma 2.2, we have

$$\frac{u''}{aFv'} - \frac{w''}{a\phi^2Dh'} = \frac{4\phi'w'}{aDh'} + \frac{2\phi'w'w''}{aDh'^3} + O \left( \frac{fa^3}{Dr^6} \right),$$

$$\frac{F'u'}{aF^2v'} + \frac{D'w'}{a\phi^2D^2h'} = - \frac{2\phi'D'w'}{aD^2h'} - \frac{2\phi'w'}{aDh'} - \frac{2\phi'w'w'^3D^2}{aD^2h'^3} + O \left( \frac{fa^3}{Dr^5} \right) + O \left( \frac{fa^3}{Dr^6} \right),$$

(2.15)
and

\begin{equation}
(2.17) \quad -\frac{v'}{aFu} + \frac{h'}{a\phi^2 Dw} = \frac{2\phi'w'}{aDh'} + O\left(\frac{fa^3}{Dr^6}\right).
\end{equation}

Summing (2.15), (2.16) and (2.17), comparing with (2.14), we have

\begin{equation}
(2.18) \quad H_0 - H
= \left(\frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh'^2}\right) - 4\phi^{-3}n(\phi) + O\left(\frac{f^2a^2w}{D^3r^5}\right) + O\left(\frac{fa^3}{Dr^6}\right).
\end{equation}

Note that by Lemma 2.2,

\begin{equation}
H_0 - H = O\left(\frac{fa}{Dr^3}\right) + O\left(\frac{f^2a^2w}{D^3r^5}\right).
\end{equation}

We claim that

\begin{equation}
(2.19) \quad \lim_{a\to\infty} \int_{S_a} (H_0 - H)d\sigma = \lim_{a\to\infty} \int_{S_a} (H_0 - H)d\sigma_0.
\end{equation}

Noting that $d\sigma - d\sigma_0 = O(r^{-1})d\sigma_0$, it suffices to show that

\begin{equation}
(2.20) \quad \lim_{a\to\infty} \int_{S_a} O\left(\frac{fa}{Dr^3}\right) O(r^{-1})d\sigma_0 = O(a^{-1})
\end{equation}

and

\begin{equation}
(2.21) \quad \int_{S_a} O\left(\frac{f^2a^2w}{D^3r^5}\right) O(r^{-1})d\sigma_0 = O(a^{-1}),
\end{equation}

which in turn is implied by the stronger result

\begin{equation}
(2.22) \quad \int_{S_a} O\left(\frac{f^2a^2w}{D^3r^5}\right) d\sigma_0 = O(a^{-1}).
\end{equation}
To prove (2.20). Since $d\sigma = a^2 Dw d\phi d\theta$, let $f^2 = 1 + \alpha^2$, consider

$$0 \leq \int_{S_a} f^2 a^2 w d\sigma = \frac{2\pi}{a} \int_0^l \frac{fw}{(w^2 + f^2 z^2)} d\phi \leq \frac{2\pi}{a} \int_0^l \frac{(1 + \alpha)w}{(w^2 + z^2 + \alpha^2 z^2)} d\phi \leq \frac{2\pi}{a} \left( l + \int_0^l \frac{\alpha w}{(C_1^2 + \alpha^2 z^2)} d\phi \right)$$

$$= \frac{2\pi}{a} \left( l + \frac{1}{C_1^4} \int_{s(0)}^{s(l)} \frac{\alpha w/s'}{(1 + \alpha^2 z^2)} ds \right) \leq \frac{2\pi}{a} \left( l + \frac{C}{C_1^4} \int_{s(0)}^{s(l)} \frac{\alpha}{(1 + \alpha^2 z^2)} ds \right) \leq \frac{2\pi}{a} \left( l + \frac{C}{C_1^4} \int_{y_1}^{y_2} \frac{1}{(1 + y^2)^{5/2}} dy \right) \leq \frac{2\pi}{a} \left( l + \frac{C\pi}{C_1^3} \right)
$$

where we have used the fact that $\frac{w}{z'}$ can be extended to a continuous function on $[0, l]$ which is bounded by $C$. For (2.22), using similar computations, we have

$$0 \leq \int_{S_a} f^2 a^2 w D^3 r^3 d\sigma \leq \frac{C_3}{a} + \frac{C_3}{a} \int_0^l \frac{\alpha^2 w^2}{(1 + \alpha^2 z^2)^2 (1 + \alpha^2 z'^2)^{5/2}} d\phi \leq \frac{C_3}{a} + \frac{C_3}{a} \int_0^l \frac{\alpha^2 w^2}{(2\alpha z')(1 + \alpha^2 z^2)^{5/2}} d\phi \leq \frac{C_3}{a} + \frac{C_3 C'}{2a} \int_0^l \frac{\alpha w}{(1 + \alpha^2 z^2)^{5/2}} d\phi \leq \frac{C_4}{a}.$$

Hence (2.19) is true:

$$\lim_{a \to \infty} \int_{S_a} (H_0 - H) d\sigma = \lim_{a \to \infty} \int_{S_a} (H_0 - H) d\sigma_0.$$
Next, by (2.18)

\[ H_0 - H = \left( \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh^2} \right) - 4\phi^{-3}n(\phi) + O \left( \frac{f^2a^2w}{D^3r^5} \right) + O \left( \frac{fa^3}{Dr^6} \right). \]

Consider

\[
\int_{s_a} \left( \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh^2} \right) d\sigma_0 \\
= 2\pi a \int_0^l \left( \frac{4\phi'ww'}{h'} + \frac{2\phi''w^2}{h'} - \frac{2\phi'w^2h''}{h'^2} \right) d\phi \\
= 2\pi a \int_0^l \frac{d}{d\phi} \left( \frac{2\phi'w^2}{h'} \right) d\phi \\
= 0.
\]

So by (2.20) and (2.21), we have

\[
\frac{1}{8\pi} \int_{s_a} (H_0 - H) d\sigma_0 = -\frac{1}{2\pi} \int_{s_a} \phi^{-3}n(\phi) d\sigma_0 + O (a^{-1}) \\
= \frac{1}{4\pi} \int_{s_a} n(\phi^{-2}) d\sigma_0 + O (a^{-1}).
\]

For each \( a \), choose \( \partial B_a \) which is a Euclidean coordinate sphere enclosing \( S_a \) and let \( \Omega_a \) be the region between \( \partial B_a \) and \( S_a \). The ADM mass of \( N \) is defined as

\[
m_{ADM} = \lim_{a \to \infty} \frac{1}{16\pi} \int_{\partial B_a} (g_{ij,i} - g_{ii,j})n^j d\sigma_0 = -\lim_{a \to \infty} \frac{1}{2\pi} \int_{\partial B_a} \phi^3n(\phi) d\sigma_0
\]

where \( n \) is the unit outward normal of \( \partial B_a \) with respect to \( \delta \). As \( n(\phi) = O (r^{-2}) \),

\[ \phi^3n(\phi) = n(\phi) + O(r^{-3}). \]

Clearly

\[ \lim_{a \to \infty} \int_{\partial B_a} O(r^{-3}) d\sigma_0 = 0, \]

so

\[(2.25) m_{ADM} = -\lim_{a \to \infty} \frac{1}{2\pi} \int_{\partial B_a} n(\phi) d\sigma_0. \]
By divergence theorem and the fact that $\Delta \phi = 0$,
\[
\frac{1}{4\pi} \int_{S_a} n(\phi^{-2})d\sigma_0 = \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2})d\sigma_0 - \frac{1}{4\pi} \int_{\Omega_a} \Delta(\phi^{-2})dV_0
\]
\[
= \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2})d\sigma_0 - \frac{1}{4\pi} \int_{\Omega_a} 6|\nabla \phi|^2
\]
\[
= \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2})d\sigma_0 + \int_{\Omega_a} O(r^{-4})dV_0
\]
\[
= -\frac{1}{2\pi} \int_{\partial B_a} n(\phi)d\sigma_0 + \int_{\Omega_a} O(r^{-3})d\sigma_0 + O(a^{-1})
\]
\[
= -\frac{1}{2\pi} \int_{\partial B_a} n(\phi)d\sigma_0 + O(a^{-1}).
\]
So by (2.24) and (2.25), we have
\[
\lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H)d\sigma_0 = m_{ADM}.
\]
Since
\[
\lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H)d\sigma = \lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H)d\sigma_0,
\]
we can conclude that
\[
\lim_{a \to \infty} m_{BY}(S_a) = m_{ADM}(N, g).
\]
This completes the proof of our result. \qed

3. Proof of Corollary 1.1

First of all, we have the following:

**Lemma 3.1.** The Gaussian curvature $K(\delta)$ of $S_a$ with metric induced by $\delta$ is positive.

**Proof.** Let $d\bar{s}^2$ be the metric on $S_a$ induced by $\delta$. The Gaussian curvature of the revolution surface given by (1.5) is $K_0 = -\frac{w''}{w} > 0$. On the other hand,
\[
d\bar{s}^2 = a^2((w^2 + f^2z'^2)d\varphi^2 + w^2d\theta^2) = \overline{E}d\varphi^2 + \overline{G}d\theta^2.
\]
The Gaussian curvature of \( d\mathbb{S}^2 \) is then given by

\[
K(\delta) = -\frac{1}{2\sqrt{EG}} \left( \left( \frac{E\theta}{\sqrt{EG}} \right)_{\theta} + \left( \frac{G\varphi}{\sqrt{EG}} \right)_{\varphi} \right)
\]

\[
= -\frac{1}{a^2 w\sqrt{w'^2 + h'^2}} \left( \frac{w'}{\sqrt{w'^2 + f^2 z'^2}} \right)'
\]

\[
= -\frac{1}{a^2 w\sqrt{w'^2 + h'^2}} \left( \frac{w''}{\sqrt{w'^2 + f^2 z'^2}} - \frac{w'(w'w'' + f^2 z'z'')}{(w'^2 + f^2 z'^2)^{\frac{3}{2}}} \right)
\]

\[
= -\frac{-w''f^2}{a^2 w(w'^2 + f^2 z'^2)^2}
\]

\[
= \frac{K_0 f^2}{a^2 D^4} > 0,
\]

where we have used \( w'w'' + z'z'' = 0 \).

**Proof of Corollary 1.1.** By Theorem 1.1, we just need to show that \( g|_{S_a} \) has positive Gaussian curvature as \( a \gg 1 \). By abuse of notations, we denote \( \delta|_{S_a} \) simply by \( \delta \) and \( g|_{S_a} \) by \( g \). Noting that \( g = \phi^4 \delta \). Similar to (2.4) in [10] or (2.14) in [10], we have

\[
K(g) = \phi^{-4}(K(\delta) - 4\Delta_S(\log \phi))
\]

where \( \Delta_S \) is the Laplacian on \( (S_a, \delta|_{S_a}) \). We have the following formula:

\[
\Delta_S \psi = \Delta_{\mathbb{R}^3} \psi - \nabla^2_{\mathbb{R}^3} \psi(n, n) - \overline{H} n(\psi)
\]

where \( \overline{H} \) and \( n \) are the mean curvature and the unit outward normal vector of \( S_a \) with respect to \( \delta \) respectively. Letting \( \psi = \log \phi \), we have

\[
K(g) = \phi^{-4}(K(\delta) - 4\Delta_{\mathbb{R}^3} \psi + 4\nabla^2_{\mathbb{R}^3} \psi(n, n) + 4\overline{H} n(\psi)).
\]

Since \( \Delta_{\mathbb{R}^3} \phi = 0 \), we have

\[
\Delta_{\mathbb{R}^3} \psi = \frac{\Delta_{\mathbb{R}^3} \phi}{\phi} - \frac{|
abla \phi|^2}{\phi^2} = -\frac{|
abla \phi|^2}{\phi^2} \leq 0.
\]

By direct calculations, in Euclidean coordinates, for \( \psi = \psi(r) \), we have

\[
(\nabla^2_{\mathbb{R}^3} \psi)_{ij} = \psi'' \frac{x^i x^j}{r^2} + \psi' \frac{\delta_{ij}}{r} - \psi' \frac{x^i x^j}{r^3}.
\]
Therefore
\[ \nabla^3 R_{\mathbb{R}^3}(n, n) = \frac{\psi''}{r^2} (X, n)^2 + \frac{\psi'}{r} - \frac{\psi'}{r^3} (X, n)^2 \]
where \( X \) is the position vector. For \( \psi = \log \phi = \log(1 + \frac{m}{2r}) \), it is easy to see that the negative part of \( \nabla^2 R_{\mathbb{R}^3}(n, n) \) is of the order \( O(\frac{1}{r^3}) \):
\[ \langle \nabla^2 R_{\mathbb{R}^3}(n, n) \rangle_\mathbb{R} = O \left( \frac{1}{r^3} \right) . \]

By Lemma 2.3, \( \bar{H} = O \left( \frac{f}{ar^2} \right) \), thus
\[ \bar{H} n(\psi) = O \left( \frac{f}{ar^2 D} \right) . \]

By (1.6), we have
\[ \frac{1}{r^3} \leq \frac{f}{C_1a r^2 D} . \]

We conclude that the negative part of \(-4\Delta S(\log \phi)\) is of order \( O(\frac{f}{ar^2D}) \). Lemma 3.1 shows that \( K(\delta) \) is of order \( O(\frac{f^2}{ar^2 D^3}) \). As \( f \lesssim D \) and \( a \lesssim r \), it is easy to see that \( K(\delta) \) dominates the negative part of \(-4\Delta S(\log \phi)\) if \( \frac{f^2}{a} = o(1) \). Hence the Gaussian curvature of \( S_a \) is positive for \( a \) large enough.

\[ \square \]

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