Remarks on some new Models of Interacting Quantum Fields with Indefinite Metric

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Abstract

We study quantum field models in indefinite metric. We introduce the modified Wightman axioms of Morchio and Strocchi as a general framework of indefinite metric quantum field theory (QFT) and present concrete interacting relativistic models obtained by analytical continuation from some stochastic processes with Euclidean invariance. As a first step towards scattering theory in indefinite metric QFT, we give a proof of the spectral condition on the translation group for the relativistic models.

1 Introduction

By an argument due to Strocchi (for a review see e.g.\cite{14}) it became clear that in quantized gauge theories which fulfill a so-called Gauss law (as e.g. quantum electrodynamics) the postulate of locality of the fields is in contradiction with the postulate of positivity of the inner product of the Hilbert space associated to the quantum system. Technical reasons (cf. \cite{14} p. 87) seem to indicate that it is better to drop the latter condition and keep the former since the problems related to the physical interpretation of a quantum system with indefinite inner product ("metric") can be circumvented with the help of the formalism of Gupta-Bleuler gauge.

This motivated the study of Quantum Field Theories (QFT) in indefinite metric. An axiomatic framework for indefinite metric QFT is given by the modified Wightman axioms of Morchio and Strocchi \cite{9}. Here the requirement of positivity of the inner product of the Hilbert space is replaced by
the so-called Hilbert space structure condition, which makes sure that the fields can still be realized as operators on a Hilbert space. However, the Wightman functions of the theory are "vacuum expectation values" w.r.t. an inner product on the Hilbert space which differs from the Hilbert space scalar product by a "metric operator".

In recent works [1, 2, 6] we studied interacting relativistic local quantum fields in the framework of indefinite metric QFT in any space-time dimension. We proved that these models fulfill the modified Wightman axioms. As far as we know, these are the first models with these properties which are constructed rigorously in a mathematical sense.

A natural question is that for a scattering theory for these models. This question has two aspects: $S$-matrix theory and Haag-Ruelle theory. While for related vector models [2, 5] we proved non-triviality of the associated scattering-$(S)$-matrix [3], a similar result is difficult to obtain for the scalar models considered here, due to the conceptual difficulties of scattering theory for quantum fields in which the one-particle states have a "smeared mass".

On the other hand, this "smeared mass" in the case of the scalar models smoothes the singularities of the Wightman functions, which turn out to be locally integrable and thus the Wightman functions are measures (in contrast to this situation the Wightman functions in the vector case are first order distributional derivatives of measures). This makes it possible to construct unitary representations of the translation group (but not of the entire Poincaré group!) for these models. These theories thus have an energy-momentum operator, and we may ask for the spectrum of this operator. We show, that it is located in the closed forward light cone.

Since the spectrum of the generator of the translation group and the locality of the fields are the main ingredients of Haag-Ruelle theory in the positive metric case, this might be a starting point for a Haag-Ruelle theory in the indefinite metric case.

The article is organized as follows: In Section 2 we briefly recall the axiomatic framework for indefinite metric QFT. Section 3 sketches how to obtain Wightman functions including some (non classical) interaction via analytic continuation from a corresponding Euclidean random field with some non Gaussian component. In the last Section we give an outline how to verify the axioms of indefinite metric QFT for these Wightman functions and sketch the proof of the existence of unitary representations for the translation group. We also show, that the energy momentum operator has spectrum in the closed forward light cone.
Here we only treat the case of scalar chargeless fields (the generalization to higher spin and charged fields is straightforward). Let \( s + 1 \) be the dimension of space-time. By \( S_n \) we denote the space of complex Schwartz functions on \( \mathbb{R}^{(s+1)n} \) with the associated Schwartz topology and let \( S'_n \) be the topological dual spaces of tempered distributions.

The following modified Wightman axioms for QFT in indefinite metric are due to Morchio and Strocchi:

**Axioms.**

**A1)** Temperedness: Let \( \{W_n\}_{n \in \mathbb{N}_0} \) be a sequence of Wightman functions with \( W_0 = 1 \) and \( W_n \in S'_n \) for \( n \in \mathbb{N} \).

**A2)** Poincaré invariance: For any transformation \( \Lambda \) in the proper, orthochronous Lorentz group \( L^\uparrow \) (\( \mathbb{R} \times \mathbb{R}^s \)) and any vector \( a \in \mathbb{R}^{s+1} \) and \( n \in \mathbb{N} \)

\[
W_n(x_1, \ldots, x_n) = W_n(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a)) \quad (1)
\]

holds.

**A3)** Spectral condition: By the translation invariance (c.f. A2)) we can define a tempered distribution \( w_n \in S'_{n-1} \) as \( w_n(y_1, \ldots, y_{n-1}) = W_n(x_1, \ldots, x_n) \) with \( y_j = x_j - x_{j+1} \). For any \( n \in \mathbb{N} \) the Fourier transform of \( w_n \) has support in \( (\bar{V}_0^-)^{\times(n-1)} \), where \( \bar{V}_0^- \) (resp. \( \bar{V}_0^+ \)) is the closed backward (resp. forward) light cone in \( \mathbb{R} \times \mathbb{R}^s \) (different conventions in some of the physical textbooks sometimes leads to the interchange of forward and backward cones).

**A4)** Locality: For \( n \geq 2 \) and \( x_j, x_{j+1} \) space-like separated, we have the Bosonic commutation relations

\[
W_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = W_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n). \quad (2)
\]

**A5)** Hermiticity: For \( n \in \mathbb{N} \) we have

\[
W_n(x_1, \ldots, x_n) = \overline{W_n(x_n, \ldots, x_1)}. \quad (3)
\]

Up to this point the above axioms are the same as in standard QFT. In order to get a indefinite metric QFT we have to drop the axiom of ”positivity” of the Wightman functions and replace it by the following:
A6) Hilbert space structure condition: For each \( n \in \mathbb{N} \) there exists a Hilbert seminorm \( p_n \) on \( S_n \) such that for any \( f \in S_j \) and \( g \in S_l \) the following inequality holds:

\[
\left| \int_{\mathbb{R}^{(s+1)(j+l)}} W_{j+l}(x_1, \ldots , x_{j+l}) f(x_1, \ldots , x_j) \times g(x_{j+1}, \ldots , x_{j+l}) \, dx_1 \ldots dx_{j+l} \right| \leq p_j(f) \, p_l(g). \tag{4}
\]

Theorem 2.1 (Morchio and Strocchi [9]) Let \( \{W_n\}_{n \in \mathbb{N}_0} \) be a sequence of Wightman functions which fulfill A1)-A6). Then there exist

(i) a Hilbert space \( \mathcal{H} \) with scalar product \( (,.) \), a distinguished vacuum vector \( \Omega \in \mathcal{H} \) and an indefinite inner product \( \langle ., . \rangle \) which differs from \( (,.) \) only by a self-adjoint metric operator \( T \) with \( T^2 = 1 \), i.e. \( \langle ., . \rangle = (,T.) \);

(ii) a \( T \)-symmetric and local quantum field \( \phi \), which is a distribution valued field operator \( \phi(x) \) acting on a dense core \( \mathcal{D} \subset \mathcal{H} \) with \( \phi(x)^* = T\phi(x)T \) and

\[
[\phi(x), \phi(y)]_\lambda = 0 \tag{5}
\]

for \( x \) and \( y \) space-like separated. Furthermore \( \phi \) is connected with the Wightman functions of the theory by

\[
W_n(x_1, \ldots , x_n) = \langle \Omega, \phi(x_1) \cdots \phi(x_n) \Omega \rangle. \tag{6}
\]

(iii) a \( T \)-unitary representation \( \mathcal{U} \) of the connected orthochronous Poincaré group on \( \mathcal{H} \), i.e. a representation with \( T \mathcal{U}^* \mathcal{T} = \mathcal{U}^{-1} \), s.t. \( \Omega \) is invariant under \( \mathcal{U} \) and \( \phi(x) \) transforms covariantly

\[
\mathcal{U}(a, \Lambda) \phi(x) \mathcal{U}(a, \Lambda)^{-1} = \phi(\Lambda^{-1} (x-a)). \tag{7}
\]

Furthermore, \( \mathcal{U} \) fulfills the following spectral condition:

\[
\int_{\mathbb{R}^{s+1}} \langle \Phi, \mathcal{U}(a, 1) \Psi \rangle \, e^{iqa} \, da = 0 \quad \text{for all } \Phi, \Psi \in \mathcal{D} \tag{8}
\]

if \( q \not\in \overline{V_0} \).

A quadrupel \( (\mathcal{H}, (,.) \Omega, T, \phi, \mathcal{U}) \) is called a QFT in indefinite metric.
3 From Euclidean Random Fields to Relativistic Wightman Functions

Since the work of Nelson [10] it has become a paradigm in QFT to construct relativistic QFT’s by analytic continuation (“reversing the Wick rotation”) from Euclidean random fields. In this picture, Gaussian random fields correspond to non-interacting relativistic quantum fields. Furthermore, random fields corresponding to positive metric QFT fulfill some additional properties, as e.g. the Markoff property. Dropping this requirement, we construct non-Gaussian random fields which turn out to be the Euclidean analogue of some indefinite metric QFT.

In this section we want to discuss heuristically the Euclidean random fields introduced in [1, 6] (related vector models are studied in [5]). Let $\eta$ be a generalized white noise[7], i.e. a $S'_1$-valued random variable which is independently and identically distributed in different nonintersecting regions of the space-time $\mathbb{R}^{s+1}$ and has an infinitely divisible (not necessarily Gaussian) probability law. Let $L$ be a continuously invertible and symmetric (pseudo-) partial differential operator on $S'$. We can then solve the (pseudo-) stochastic partial differential equation

$$L\varphi(x) = \eta(x). \quad (9)$$

Since $\eta(x)$ is by construction Euclidean invariant (i.e. $\eta(x) = \eta(\Lambda^{-1}(x - a))$ in law for all $a \in \mathbb{R}^{s+1}$ and $\Lambda$ orthogonal), a necessary and sufficient condition for $\varphi(x)$ being Euclidean invariant is the Euclidean invariance of $L$. In the following we always assume this. Let $G(x)$ denote the Green function of $L$ and let $E$ be the expectation value w.r.t. the probability space on which $\eta$ lives. By an explicit calculation one can then figure out the Schwinger (moment) functions $S_n \in S'_n$ of $\varphi(x)$

$$S_n(x_1, \ldots, x_n) := \mathbb{E} [\varphi(x_1) \cdots \varphi(x_n)]$$

$$= \sum_{I \in \mathcal{P}(n)} \prod_{\{j_1, \ldots, j_l\} \in I} c_l \int_{\mathbb{R}^{s+1}} G(x_{j_1} - x) \cdots G(x_{j_l} - x) \, dx \ . \quad (10)$$

Here the sum is over all partitions $I$ of $\{1, \ldots, n\}$ into disjoint subsets and $c_l$ are constants which depend on the law of $\eta(x)$. In particular, the law of $\eta(x)$ is Gaussian if and only if $c_l = 0$ for all $l > 2$. The distributions $S'^T_l$ are called
the truncated Schwinger functions of $\varphi(x)$. Thus, the higher order $S_l^T$, $l > 2$, represent the interaction forces present in this model.

**Remark.** In order to understand this interaction better, let us have a look at the lattice analogue of $\varphi(x)$, i.e. we replace $\mathbb{R}^{s+1}$ by a bonded region $\Gamma$ in $\mathbb{Z}^{s+1}$. For all $x \in \Gamma$, $\eta(x)$ are independently identically distributed random variables with infinitely divisible law. Let $L_\Gamma$ be the lattice analogue of $L$. Then the space of all possible configurations of the lattice field $\varphi(x) = L_\Gamma^{-1}\eta(x)$ is $\mathbb{R}^{\Gamma}$. Let $\rho$ denote the density of $\eta(x)$ w.r.t. the Lebesgue measure and let $c$ be the second moment of $\eta(x)$. For the expectation value of a bounded measurable function $C : \mathbb{R}^{\Gamma} \to \mathbb{R}$ one then gets the following path integral representation [6]:

$$E[C \circ \varphi] = Z^{-1} \int_{\mathbb{R}^{\Gamma}} C(\varphi)e^{-\int_{\mathbb{R}^{s+1}} c(\varphi(x)L_\Gamma^2\varphi(x)) - V(L_\Gamma \varphi(x))}d_{\Gamma}x \mathcal{D}_\Gamma \varphi.$$  (11)

Here $d_{\Gamma}x = \sum_{x' \in \Gamma} \delta(x - x')dx$ is the lattice Lebesgue measure and $V(t) := \log \rho(t) + ct^2$ is the "potential function".

Obviously, for $L = (-\triangle + m^2)^{\frac{1}{2}}$ the quadratic term in the above action is the usual kinetic energy term of the free field and thus $V$ is responsible for the interaction. For $\eta(x)$ to be Gaussian we clearly get $V \equiv 0$ and there is no interaction. But if $\eta(x)$ contains a non Gaussian component we have $V \not\equiv 0$. Nevertheless, the potential term $V(L_\Gamma \varphi(x))$ is different from the usual potential term e.g. in $P(\varphi)_{2}$-theory which is of the form $\tilde{V}(\varphi(x))$. Since $L$ in general is a nonlocal operator, the interaction in our model exhibits a kind of "non locality" which has to be made responsible for the lack of the Markoff property of the field $\varphi(x)$ [1]. On the other hand, this "non locality" does not destroy the relativistic requirement of "locality", since, roughly speaking, in the Euclidean region there are only space-like distances and thus the non-local smearing in the fields due to $L$ only smears space-like separated amplitudes of the field. On the other hand, this "smearing" might be responsible for the analytic well behaviour of these models, since "smearing" implies some regularization of the distributional fields $\varphi(x)$. However, a precise mathematical formulation for these "explanations" has not been obtained yet.

Let us explain now how to do the analytical continuation from purely imaginary Euclidean time back to purely real relativistic time. This can be done by representing the truncated Schwinger functions $S_n^T(x_1, \ldots, x_n)$
as Laplace transform of the Fourier transform $\hat{W}^T_n(x_1, \ldots, x_n)$ of the truncated Wightman functions $W^T_n$ of the model, where all $W^T_n$, $n \in \mathbb{N}$, fulfill the spectral condition:

$$S^T_n(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{\sum_{l=1}^n ik_l^0(x_l^0)} dW_n^T(k_1, \ldots, k_n) \, dk_1 \cdots dk_n,$$

for $x_1^0 < x_2^0 < \cdots < x_n^0$. To get $W^T_n$ one then just has to replace on the right hand side $ix_l^0$ by $x_l^0$.

Let us take the particular case where $L = (-\Delta + m^2)^{\alpha}$ for $0 < \alpha \leq 1/2$. By a lengthy but explicit calculation we proved in [1] that such a representation indeed does exist. The formulae for the $\hat{W}^T_n$, $n > 2$, are:

$$\hat{W}^T_n(k_1, \ldots, k_n) := c'_n \left\{ \sum_{j=1}^n \prod_{l=1}^{j-1} \mu_\alpha^+(k_l) \mu_\alpha(k_j) \prod_{l=j+1}^n \mu_\alpha^-(k_l) \right\} \delta(\sum_{l=1}^n k_l).$$

with $c'_n = c_n 2^{n-1}(2\pi)^{s+1}$,

$$\mu_\alpha^+(k) := (2\pi)^{\frac{(s+1)n}{2}} \sin \pi \alpha 1_{\{k^2 > m^2, k^0 > 0\}}(k) \frac{1}{(k^2 - m^2)^{\alpha}}$$

$$\mu_\alpha^-(k) := (2\pi)^{\frac{(s+1)n}{2}} \sin \pi \alpha 1_{\{k^2 > m^2, k^0 < 0\}}(k) \frac{1}{(k^2 - m^2)^{\alpha}}$$

$$\mu_\alpha(k) := (2\pi)^{\frac{(s+1)n}{2}} (\cos \pi \alpha 1_{\{k^2 > m^2\}}(k) + 1_{\{k^2 < m^2\}}(k)) \frac{1}{|k^2 - m^2|^\alpha}$$

for $n \geq 3$ as well as $n = 2$, $\alpha < 1/2$ and in the case $n = 2$, $\alpha = 1/2$

$$\hat{W}_2^T = 2\pi c_2 1_{\{k_0^2 < 0\}}(k_1) \delta(k^2 - m^2) \delta(k_1 + k_2).$$

Here we have used the notation $k^2 = k_0^2 - |\vec{k}|^2$.

Let the Wightman functions $W_n$ be defined by the truncated Wightman functions $W^T_n$ in the way of formula (10). Then the Wightman functions are given as the analytic continuation of the Schwinger functions of the model. The spectral property of the $W_n$ can be checked by hand using the Eqn. (13)-(15) and the Schwinger functions by definition are real, symmetric and Euclidean invariant. The following theorem therefore follows from the Osterwalder – Schrader reconstruction theorem [11]:

**Theorem 3.1 (Main Theorem of [1])** The sequence of Wightman functions $\{W_n\}_{n \in \mathbb{N}_0}$ fulfills the axioms A1)-A5).
4 Hilbert Space Structures with Unitary Translation Group

In order to prove that the Wightman functions \( \{ W_n \}_{n \in \mathbb{N}_0} \) belong to an indefinite metric QFT, it remains to verify A6). This can be done as follows. First, for \( n \in \mathbb{N} \), we choose a special system of Schwartz norms

\[
\| f \|_{K,N} := \sup_{x_1, \ldots, x_n \in \mathbb{R}^{s+1}} \left[ \prod_{l=1}^{n} (1 + |x_l|^2)^{N/2} \left( \frac{\partial}{\partial x_l} \right)^{\beta_l} \right] |f(x_1, \ldots, x_n)|.
\]

(16)

on \( S_n \). Based on a kind of "quantitative nuclear theorem" related to this system of norms, it is possible to prove the following sufficient Hilbert space structure condition for the truncated Wightman functions:

**Theorem 4.1** ([2]) Let \( K, N \in \mathbb{N}_0 \) be fixed. If for all \( n \in \mathbb{N} \) \( \hat{W}_n^T \) are continuous with respect to \( \| \cdot \|_{K,N} \), then the sequence of Wightman functions \( \{ W_n \}_{n \in \mathbb{N}_0} \) fulfills A6). \( \blacksquare \)

For a generalization of this theorem see [8].

By explicit calculation using the formulae (13)-(15) we proved in [2] the following inequalities: For \( n \in \mathbb{N} \) there exist constants \( a_n > 0 \) s.t.

\[
\left| \int_{\mathbb{R}^{s+1}} \hat{W}_n(k_1, \ldots, k_n) f(k_1, \ldots, k_n) \, dk_1 \cdots dk_n \right| \leq a_n \| f \|_{0,2s+2}
\]

(17)

holds for all \( f \in S_n \). Now from the Theorems 2.1, 3.1 and 4.1 we immediately get

**Theorem 4.2** (Main Theorem of [2]) The Wightman functions \( \{ W_n \}_{n \in \mathbb{N}_0} \) constructed in Section 3 fulfill all modified Wightman axioms A1)-A6). Thus, there exists a indefinite metric QFT with Wightman functions \( \{ W_n \}_{n \in \mathbb{N}_0} \). \( \blacksquare \)

For a given sequence of Wightman functions there can be different Hilbert space structures, which are, however, determined by the Hilbert seminorms on the right hand side of Eq. (4). In particular, if these seminorms can be chosen to be translation invariant, the metric operator \( T \) is also translation invariant, i.e. \([U(a, 1), T]_\cdot = 0\) for all \( a \in \mathbb{R}^{s+1} \). This implies that at least the translations are represented by unitary operators. In this case it makes sense
to speak of the generator $P$ of the translation group, which is a self adjoint vector operator on $H$ with spectrum $\text{spec}(P)$. The following theorem is a new result on the existence of such Hilbert space structures. For a detailed proof we refer to [4].

**Theorem 4.3** For the given sequence of Wightman functions $\{W_n\}_{n \in \mathbb{N}_0}$ the Hilbert seminorms in the Hilbert space structure condition Eq. (4) can be chosen translation invariant. Thus, there is a Hilbert space structure s.t. the representation of the translation group $\mathcal{U}(a, 1)$ is unitary. If $P$ denotes the generator of $\mathcal{U}(a, 1)$, then $\text{spec}(P) \subseteq \overline{V}_0^+$.

**Proof.** First let us assume that Eq. (4) is fulfilled with $p_j, j \in \mathbb{N}$, translation invariant. Then the positive semidefinite inner product on $D$ constructed in [9] is translation invariant and thus the obtained representation of the translations $\mathcal{U}(a, 1)$ on the Hilbert space related to this inner product and the metric operator $T'$ commute. However, in general $T'$ is not continuously invertible and thus does not fulfill $T'^2 = 1$. But from the given metric operator one can pass to a new metric operator with $T^2 = 1$ using the procedure described in [9] (see also [2]). But the steps of this procedure consist of changing the inner product two times by a function of the metric operator. Since $\mathcal{U}$ commutes with such functions (since it commutes with $T'$), we get that $\mathcal{U}$ commutes with the resulting metric operator $T$. Thus we get a unitary representation of the translation group.

Let $P$ be the generator of this representation. In order to prove $\text{spec}(P) \subseteq \overline{V}_0^+$ we first write Eq. (8) in the equivalent form

$$\int_{\mathbb{R}^{s+1}} (\Phi, \mathcal{U}(a, 1)\Psi) \hat{f}(a) da = 0 \quad (18)$$

for all $\Phi \in \mathcal{T}D, \Psi \in D$ and $f \in S_1$ with supp$f \cap \overline{V}_0^+ = \emptyset$. Since $TD$ and $D$ are dense in $\mathcal{H}$ we have sequences $\Phi_n \in TD$ and $\Psi_n \in D$ with $\Phi_n \to \Phi$ and $\Psi_n \to \Psi$ in $\mathcal{H}$. Thus

$$\left| \int_{\mathbb{R}^{s+1}} (\Phi, \mathcal{U}(a, 1)\Psi) \hat{f}(a) da \right|$$

$$\leq \sup_{a \in \mathbb{R}^{s+1}} |(\Phi, \mathcal{U}(a, 1)\Psi) - (\Phi_n, \mathcal{U}(a, 1)\Psi_n)| \int_{\mathbb{R}^{s+1}} |\hat{f}(a)| da$$

$$\leq \left( \sup_{n \in \mathbb{N}} \|\Phi_n\| \|\Psi - \Psi_n\| + \|\Psi\| \|\Phi - \Phi_n\| \|\hat{f}\|_{L^1(\mathbb{R}^{s+1}, dx)} \right) \to 0 \quad (19)$$
as \( n \to \infty \). Thus Eq. (18) holds for all \( \Phi, \Psi \in \mathcal{H} \). Let \( E \) denote the spectral measure of \( P \). Then the left hand side of Eq. (18) is equal to

\[
(2\pi)^{-(s+1)/2} \int_{\mathbb{R}^{s+1}} f(\lambda) d(\Phi, E(\lambda) \Psi),
\]

which shows that \( \text{spec}(P) = \text{supp}(E) \subseteq \tilde{V}_0^+ \).

It remains to show the existence of translation invariant Hilbert semi norms \( p_j, j \in \mathbb{N} \), s.t. Eq. (4) holds. Note that from the formulae (10) and (13)-(15) it follows that \( \hat{W}_n \) is a positive measure for \( n \in \mathbb{N} \). From Parseval theorem, the Cauchy Schwarz inequality and Fubini theorem we therefore get that the left hand side of Eq. (4) is smaller or equal than

\[
\left( \int_{\mathbb{R}^{(s+1)j}} |\hat{f}(k_1, \ldots, k_j)|^2 \left[ \prod_{r=1}^j (1 + |k_r|^2)^{s+1} \int_{\mathbb{R}^{(s+1)l}} \hat{W}_{j+l}(k_1, \ldots, k_{j+l}) \right] \right) \times \left( \int_{\mathbb{R}^{(s+1)l}} |\hat{g}(k_{j+1}, \ldots, k_{j+l})|^2 \left[ \prod_{r=j+1}^{j+l} (1 + |k_r|^2)^{s+1} \right] \times \int_{\mathbb{R}^{(s+1)j}} \hat{W}_{j+l}(k_1, \ldots, k_{j+l}) \right) \times \left( \prod_{r=1}^j (1 + |k_r|^2)^{-(s+1)} \right) \] \( dk_1 \cdots dk_j \right) \] \( \int_{\mathbb{R}^{(s+1)j}} |\hat{f}|^2 [\ldots] dk_1 \cdots dk_j \leq C_{j,l} \int_{\mathbb{R}^{(s+1)j}} |\hat{f}|^2 dM_j < \infty \)

we have finished the proof, since we may define

\[
p_j(f) = \left( \max_{1 \leq r, q \leq j} C_{r,q} + 1 \right) \left( \int_{\mathbb{R}^{(s+1)j}} |\hat{f}|^2 dM_j \right)^{1/2},
\]

and from (21)-(23) we easily get Eq. (4). \( p_j \) obviously is translation invariant, since the translation group acts on \( \hat{f} \) by multiplication with \( \exp \{ i \sum_{r=1}^j k_r \cdot a \} \) which does not change \( |\hat{f}|^2 \) and hence does not change \( p_j(f) \).
The proof of the existence of the measures $M_j$ is technical and not very instructive. Here we only give formulae for suitable measures $M_j$ for the case $\alpha < 1/2$ which is a little easier to treat than the case $\alpha = 1/2$. Using estimates similar to those in Subsection 4.1 of [2] one can prove that the measures

$$dM_j(k_1, \ldots, k_j) = \sum_{I \in P(j)} \prod_{j=1}^{k_j} \prod_{r=1}^{l} (1 + |k_{jr}|^2)^{s+1}$$

$$\times \left[ \delta_{c,1} \frac{1}{|k_{j1}^2 - m^2|^{2\alpha}} + \delta_{c,2} \frac{1}{|k_{j1}^2 - m^2|^{2\alpha}} \delta(k_{j1} + k_{j2}) + (1 - \delta_{c,2}) \prod_{r=1}^{l} \frac{1}{|k_{jr}^2 - m^2|^{\alpha}} \right. $$

$$\left. + (1 - \delta_{c,1} - \delta_{c,2}) |(\sum_{r=1}^{l} k_{jr})^2 - m^2|^{-\alpha} \prod_{r=1}^{l} \frac{1}{|k_{jr}^2 - m^2|^{\alpha}} \right] dk_1 \ldots dk_j \quad (24)$$

fulfill the requirements of Eq. (23). (Note that by $2\alpha < 1$ $M_j$ has a locally integrable density w.r.t. the Lebesgue measure which implies that the second estimate in (22) holds.)

Since the spectrum of $P$ together with the locality of the fields in the positive metric case are the main properties on which Haag-Ruelle theory (i.e. the proof of existence of asymptotic ”scattering” states) is based [12], we consider Theorem 4.3 as a first step towards an analogous theory in the indefinite metric case. However, there are other properties of indefinite metric QFT which differ significantly from the case of positive metric. E.g. the uniqueness of the vacuum cannot hold in case $T$ commutes with $U$, since $T\Omega \not\in \mathbb{C}\Omega$ is translation invariant, too. Therefore, the operation of truncation has to be carried out w.r.t. the projection onto the 0-eigenspace of $P$ instead of the projection onto $\mathbb{C}\Omega$. But then the truncated objects are operator valued functions acting on the 0-eigenspace rather than distributions. Possibly, these operators on a space of at least two dimensions do not commute, which could destroy ”locality” of the truncated objects. But that property plays a crucial rôle in Haag Ruelle theory [12]. We will investigate these problems in [4].

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