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Tunneling time and superposition principle

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Abstract We show that scattering a quantum particle on a one-dimensional potential barrier as well as scattering the electromagnetic wave on a quasi-one-dimensional layered structure (both represent scattering problems with one ‘source’ and two ‘sinks’) violate the superposition principle; the role of nonlinear elements is played here by the potential barrier and the layered structure, splitting the incident (probability and electromagnetic) wave into two parts (transmitted and reflected). This explains why all attempts to solve the tunneling time problem within the framework of the standard (linear) models of these processes, both in quantum mechanics and in classical electrodynamics, have been unsuccessful. We revise the traditional formulation of the superposition principle, present a new (nonlinear) wave model, by the example of the quantum-mechanical scattering process, and show that concepts of the tunneling time developed on its basis are free from the Hartman paradox.

Keywords tunneling time · Hartman paradox · superposition and causality principles

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1 Introduction

Scattering a quantum particle on a one-dimensional potential barrier, where the particle has two mutually exclusive possibilities – either to pass (tunnel) through the barrier or to be reflected from it – is one of the simplest scattering problems in quantum mechanics (QM). But the simplicity of this “two-channel” scattering process is deceptive, since the study of its temporal aspects leads to the tunneling time problem (TTP), with its key question “How long does it take to tunnel through the barrier?”, which remains unresolved up to date. Experts on this problem (see, for example, the reviews [1,2,3,4,5,6,7]) analyze a huge number of candidates for the role of the tunneling time and show that none of them is really suitable for this purpose because, as was said in [1], “All [the known tunneling-time concepts] have been found to suffer one logical flaw or another, flaws sufficiently serious that must be rejected”.

Now we can also add that all the hypotheses expressed in [1,2,3,4,5,6,7] regarding the underlying cause that makes this problem intractable also turned out to be far from the truth (this is indirectly explained by the fact that all the recent tunneling-time approaches (see, for example, [8,9,10,11,12]), like the previous ones, ”suffer one logical flaw or another” (see, e.g., [13])). And, perhaps, only those experts who linked the difficulties of solving the TPP with the fundamental problems of the QM itself were the closest thing to the truth. For example, as was stressed in [5], ”A very important aspect, not technical but fundamental, is that the existing solutions [of the TTP], or even the identification of the difficulties, are closely linked to particular interpretations of quantum mechanics…” [Thus, no

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2 Tunneling and superposition principle

We begin our analysis with the stationary Schrödinger equation that describes the process of scattering a particle on a one-dimensional potential barrier \( V(x) \), nonzero in the spatial interval \([a, b]\). For the particle with the energy \( E = \hbar^2 k^2 / 2m \), where \( m \) is the particle’s mass and \( \hbar k \) is its momentum, the general solution \( \psi(x, k) \) outside the interval \([a, b]\) can be written in the form

\[
\psi(x, k) = \begin{cases} 
    A e^{i k x} + B e^{-i k x} : & x \leq a \\
    A e^{i k x} + B e^{-i k x} : & x \geq b
\end{cases}
\]  

(1)

\( k = \sqrt{2mE}/\hbar \). The main requirement imposed on a searched-for wave function \( \psi(x, k) \) is that this function and its first \( x \)-derivative must be continuous everywhere on the \( OX \)-axis. Thus, this function must obey four real linear conditions of continuity (two real conditions for the complex-valued \( \psi(x, k) \) and two real conditions for its first \( x \)-derivative) at the points \( x = a \) and \( x = b \), where \( V(x) \) is discontinuous.

According to the transfer matrix approach, the wave amplitudes in Eqs. (1) are linked by the transfer matrix \( \textbf{Y} \):

\[
\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \textbf{Y} \begin{pmatrix} A_r \\ B_r \end{pmatrix} ; \quad \textbf{Y} = \begin{pmatrix} q & p \\ p^* & q^* \end{pmatrix} ; \quad |q|^2 - |p|^2 = 1 ;
\]  

(2)

the matrix elements \( q \) and \( p \) are uniquely determined by the potential function \( V(x) \). Besides, two (independent) amplitudes are determined here by the boundary conditions at the regions \( x < a \) and \( x > b \). For example, for a particle impinging on the barrier from the left, the standard boundary conditions are as follows: \( A_l = 1, B_r = 0 \). In this case, from Eq. (2) it follows that

\[
\psi(x, k) = \begin{cases} 
    e^{i k x} + \frac{1}{q} e^{-i k x} : & x \leq a \\
    \frac{1}{q} e^{i k x} : & x \geq b
\end{cases}
\]  

(3)

Thus, the in- and out-asymptotes of the corresponding time-dependent solution \( \psi(x, t) \) are

\[
\psi_{\text{inc}}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{-i E(k) t / \hbar} dk ; \quad \psi_{\text{out}}(x, t) = \psi_{tr}(x, t) + \psi_{ref}(x, t); \quad \psi_{tr}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i k x - i E(k) t \hbar / \hbar} dk,
\]  

(4)

\[
\psi_{ref}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A(k)}{q(k)} e^{-i k x - i E(k) t \hbar / \hbar} dk ; \quad \psi_{tr}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A(k) p^*(k)}{q(k)} e^{-i k x + i E(k) t \hbar / \hbar} dk ;
\]
where $\mathcal{A}(k)$ is a complex-valued function determined by the initial condition (it is assumed that for ‘physical initial states’ the function $\mathcal{A}(k)$ belongs to the Schwartz space). That is, in the limit $t \to -\infty$ the time-dependent solution $\psi(x,t)$ approaches the left in-asymptote $\psi_{inc}(x,t)$ that represents a single incident wave packet, while in the limit $t \to +\infty$ the wave function $\psi(x,t)$ approaches the out-asymptote $\psi_{out}(x,t)$ that represents the superposition of the right out-asymptote $\psi_{tr}(x,t)$ (a transmitted wave packet) and the left out-asymptote $\psi_{ref}(x,t)$ (a reflected wave packet).

It is generally accepted that this scattering process a priori respects the superposition principle, and this standard (linear) quantum-mechanical model is internally consistent. But is it?

As is seen, they are such that $\psi$ the incident waves to the transmitted wave $1$ and $-1$ for 'physical initial states' the function where $A$ incoming and outgoing waves, in each pair of wave functions, differ from each other.

$p$ transmitted wave packet) and the left out-asymptote $\psi$ the out-asymptote $\psi$ superposition principle, then the fact that the out-asymptote 'sink' (because the Schrödinger’s dynamics is reversible in time), this problem can also be considered and this standard (linear) quantum-mechanical model is internally consistent. But is it?

But the standard model, with its linear continuity conditions, does not allow for a semitransparent potential barrier the existence of stationary solutions which would have one incoming wave and one outgoing wave. That is, it does not imply the existence of incident waves for each subprocess. In the final analysis, it does not imply the individual description of the transmission (tunneling) and reflection subprocesses at all stages of scattering.

At first glance, we may fill this gap and resolve this problem, remaining within the standard model, with making use of two stationary solutions $\psi_1(x,k)$ and $\psi_2(x,k)$: the first has a single outgoing wave being the transmitted wave $\psi_{tr}(x,k)$, and the second has a single outgoing wave being the reflected wave $\psi_{ref}(x,k)$:

$$
\psi_1(x,k) = \begin{cases} 
\psi_1^1 e^{ikx} & : x \leq a \\
\psi_1^2 e^{-ikx} & : x \geq b 
\end{cases} \quad \psi_2(x,k) = \begin{cases} 
(\frac{p}{q})^2 e^{ikx} + \frac{e^{ikx}}{\sqrt{q}} & : x \leq a \\
(\frac{e^{-ikx}}{\sqrt{q}}) e^{ikx} & : x \geq b 
\end{cases}
$$

As is seen, they are such that $\psi_1(x,k) + \psi_2(x,k) = \psi(x,k)$. In this case the superposition of the incident waves $\psi_1(x,k)$ and $\psi_2(x,k)$ in the spatial domain $x \geq b$ is destructive, resulting in their complete disappearance. While the superposition of their incident waves in the region $x \leq a$ is constructive, giving the incident wave of the initial solution $\psi(x,k)$ (see [3]). But all this does not at all mean that the incident wave $\psi_{inc}(x,k)$ is causally connected to the transmitted wave $\frac{p}{q} e^{ikx}$, and the incident wave $\psi_{inc}(x,k)$ is causally connected to the reflected wave $\frac{e^{-ikx}}{\sqrt{q}}$. This is so, because the probability current densities corresponding to the incoming and outgoing waves, in each pair of wave functions, differ from each other.

This fact as well as the fact that the superposition of the wave functions $\psi_1(x,k)$ and $\psi_2(x,k)$ (each of them is associated with two ‘sources’ and one ‘sink’) leads to their cardinal reconstruction (their superposition – the wave function $\psi(x,k)$ – is associated with one (left) ‘source’ and two (left and right) ‘sinks’) mean that this scattering process violates the superposition principle and its standard (linear) model is internally inconsistent. Thus, the standard formulation of the superposition principle as well as the standard (linear) model of this quantum mechanical scattering process must be revised.

Of course, revising is not needed in the particular case, when the potential barrier is either fully transparent or fully opaque (a one-channel scattering). We must imply that, as before, a coherent superposition of two or more wave functions that describe possible states of a pure quantum ensemble moving in the physical context formed by the left ‘source’ and (right or left) ‘sink’ gives a new state of this pure ensemble; in this case, the interference between states forming this superposition – a new possible state determined by this physical context – makes them indistinguishable; characteristic times and other physical quantities can be defined only for this new state of the pure ensemble.

However, in the general case (when both left and right "sinks" are involved), a new formulation of the superposition principle should imply that the coherent superposition of pure states associated with different (left and right) "sinks" is a state of a mixture of two pure quantum ensembles determined by different physical contexts; two pure states forming this superposition remain distinguishable, despite
the interference between them; characteristic times and other physical quantities can be now defined only for these two pure states, but not for their mixture. We have to stress that this result is in a full agreement with our recent study [13], where we have presented a superselection rule which restricts the validity of the superposition principle in the rigged Hilbert space of states of a particle scattering on a one-dimensional potential barrier. All this requires a new (nonlinear) model of this two-channel scattering process which would describe the individual dynamics of the transmitted and reflected wave packets at all stages of scattering.

So, the main results of this section, extended onto the corresponding scattering process in CED, can be summarized as follows:

- "...our [non] understanding of the emergence of the classical world of events from the quantum world of possibilities" is not the root cause that makes the solution of the TTP impossible; rather, our false understanding of the role of the principle of superposition in the problem of scattering a classical electromagnetic wave on a quasi-one-dimensional layered structure makes it impossible both the solution of the TTP and "...our understanding of the emergence of the classical world of events from the quantum world of possibilities";
- scattering a quantum particle on a one-dimensional potential barrier and scattering the electromagnetic wave on a quasi-one-dimensional layered structure are nonlinear phenomena;
- in both cases the role of nonlinear elements is played by a potential barrier and layered structure that split the incident wave (wave packet) into the transmitted and reflected waves (wave packets);
- the standard models of these two scattering processes, both in QM and in CED, are linear and, thus, they do not give an adequate description of these processes; the adequate (nonlinear) model of each of these two two-channel scattering processes must answer the question of how to uniquely represent the incident wave associated with the whole process in the form of a superposition of the incident wave, which would be causally related only to the transmitted wave, and the incident wave, which would be causally related only to the reflected wave;
- for each subprocess (transmission and reflection), a causal relationship between the incident wave and the corresponding outgoing wave can be realized only on the basis of nonlinear continuity conditions that have yet to be formulated.

Note that the relevant nonlinear tunneling models, both in QM and CED, have already been developed, based on intuitive considerations, and presented in our articles [15][16] and [17], respectively. These models give the 'subprocess' wave functions (SWFs) which allow one to describe the transmission and reflection subprocesses at all stages of scattering. And on their basis the transmission (tunneling) and reflection times have been defined. In this paper, a new (nonlinear) quantum-mechanical model is presented on a more rigorous basis with the addition of new important details.

3 Standard model of scattering a particle on a system of two identical rectangular potential barriers

In connection with the subsequent analysis of the generalized Hartman paradox, a standard quantum mechanical description of tunneling will be presented by the example of scattering a quantum particle on a one-dimensional system of two identical rectangular potential barriers (when the gap between barriers is zero, this model describes the tunneling of a particle through a single rectangular potential barrier). In doing so, we will use our version [13] of the transfer matrix approach.

Let a particle with a given momentum $\hbar k$ ($k > 0$) impinge from the left on the system of two identical rectangular potential barriers located at the intervals $[a_1, b_1]$ and $[a_2, b_2]$; $0 < a_1 < b_1 < a_2 < b_2$. The height of both barriers is $V_0$, $b_1 - a_1 = b_2 - a_2 = d$ is their width; $L = a_2 - b_1$ is the distance between the barriers; $b_2 - a_1 = D$ is the width of this two-barrier system.

The wave function $\Psi_{tot}(x, k)$ that describes the state of the quantum ensemble of such particles can be written as follows:

$$
\Psi_{tot}(x, k) = \begin{cases} 
\begin{array}{ll}
A_{tot}^{(1)} \sinh[\kappa(x - a_1)] + B_{tot}^{(1)} \cosh[\kappa(x - a_1)] & : x \in (-\infty, a_1] \\
A_{tot}^{(2)} \sinh[\kappa(x - x_c)] + B_{tot}^{(2)} \cosh[\kappa(x - x_c)] & : x \in [a_1, b_1] \\
A_{tot}^{(3)} \sinh[\kappa(x - b_2)] + B_{tot}^{(3)} \cosh[\kappa(x - b_2)] & : x \in [a_2, b_2] \\
A_{out} e^{i k (x - D)} & : x \in [b_2, \infty)
\end{array}
\end{cases}
$$

(6)
here \( \kappa = \sqrt{2m(V_0 - E)/\hbar}; E = \hbar^2 k^2/2m; x_c = (b_2 + a_1)/2 \). We have to stress that the formalism presented here is valid not only for \( E < V_0 \) (when the Hartman paradox appears in the opaque barrier limit \( d \to \infty \)) but also for \( E \geq V_0 \) (in this case, \( \kappa \) is a purely imaginary quantity).

According to [18], for any semitransparent potential barrier located in the interval \([a, b]\), the elements \( q \) and \( p \) of the transfer matrix \( Y \) (see [2]) can be presented in the form

\[
q = \frac{1}{\sqrt{T_{(a,b)}}} \exp \{ i [k(b - a) - J(a,b)] \}, \quad p = i \sqrt{\frac{R_{(a,b)}}{T_{(a,b)}}} \exp \{ i [F(a,b) - k(b + a)] \},
\]

where (see [18]) the transmission coefficient \( T_{(a,b)} \) and phases \( J(a,b) \) and \( F(a,b) \) are determined either by explicit analytical expressions (e.g., for the rectangular barrier and the \( \delta \)-potential) or by the recurrence relations (for many-barrier structures); \( R_{(a,b)} = 1 - T_{(a,b)} \); for any symmetric system of barriers, when \( V(x - x_c) = V(x_c - x) \), the phase \( F(a,b) \) can take only two values, either 0 or \( \pi \).

For the transfer matrices \( Y_{\text{two}}, Y_1 \) and \( Y_2 \) that describe this two-barrier system as well as its left and right barriers, respectively, we have

\[
\begin{align*}
B_{\text{out}} e^{2ika_1} & = Y_{\text{two}} \begin{pmatrix} 1 \\ A_{\text{out}} e^{-ikD} \\ 0 \end{pmatrix}; & Y_{\text{two}} Y_1 & = Y_2, \quad Y_n = \begin{pmatrix} q_n & p_n \\ p_n & q_n \end{pmatrix}
\end{align*}
\]

where \( q_n = q \cdot \exp[ik(b_n - a_n)], p_n = ip \cdot \exp[-ik(b_n + a_n)] \) \((n = 1, 2)\);

\[
q = \frac{e^{-ij}}{\sqrt{T}}, \quad p = \sqrt{\frac{R}{T}} e^{iF}; \quad q_{\text{two}} = \frac{1}{\sqrt{T_{\text{two}}}} e^{i[k(b_2 - a_1) - J_{\text{two}}]}, \quad p_{\text{two}} = i \sqrt{\frac{R_{\text{two}}}{T_{\text{two}}}} e^{i[F_{\text{two}} - k(b_2 + a_1)]}
\]

For rectangular barriers the 'one-barrier' parameters \( T, J \) and \( F \) are (see also [18])

\[
T = \left[ 1 + \theta^2 \sinh^2(\eta d) \right]^{-1}, \quad J = \arctan(\theta \tanh(\eta d)) + J^{(0)}, \quad \theta^{(0)} = \frac{1}{2} \left( \frac{k}{\kappa} \pm \frac{\kappa}{k} \right);
\]

\( J^{(0)} = 0 \), if \( \cosh(\eta d) > 0 \); otherwise, \( J^{(0)} = \pi \) (this can occur for \( E \geq V_0 \)); \( F = 0 \), if \( \theta^{(1)} \sinh(\eta d) > 0 \); otherwise, \( F = \pi \). From the latter it follows that the parameter \( p \) is real. It can be rewritten in the form \( p = \eta \sqrt{R/T} \); where \( \eta = +1, \) if \( \theta^{(1)} \sinh(\eta d) > 0 \); otherwise, \( \eta = -1 \).

The 'two-barrier' parameters \( T_{\text{two}}, J_{\text{two}} \) and \( F_{\text{two}} \) are determined by Eq. [8] (see also the recurrence relations for the scattering parameters in [18]):

\[
T_{\text{two}}^{-1} = 1 + 4 \frac{R}{T^2} \cos^2 \chi, \quad J_{\text{two}} = J + \arctan \left( \frac{1 - R}{1 + R} \tan \chi \right) + F_{\text{two}}^{(0)}, \quad F_{\text{two}} = F + F_{\text{two}}^{(0)};
\]

\[
\text{here } \chi = J + kL; \quad F_{\text{two}}^{(0)} = 0, \text{ if } \cos \chi \geq 0; \text{ otherwise, } F_{\text{two}} = \pi \text{ (the piecewise constant function } F_{\text{two}}(k) \text{ is discontinuous at the resonance points where } T_{\text{two}} = 1)\).
\]

Now the coefficients in Exp. [6] can be written in terms of these one-barrier and two-barrier parameters scattering. For this purpose it is suitable to rewrite the wave function \( \Psi_{\text{tot}}(x, k) \) in the interval \([b_1, a_2]\) in the form \( \Psi_{\text{tot}}(x, k) = A_{\text{tot}}^{\text{gap}} \exp(ikx) + B_{\text{tot}}^{\text{gap}} \exp(-ikx) \) where

\[
\begin{align*}
\left( \begin{array}{c}
\tilde{A}_{\text{tot}}^{\text{gap}} \\
\tilde{B}_{\text{tot}}^{\text{gap}}
\end{array} \right) & = Y_2 \left( \begin{array}{c}
A_{\text{out}} e^{-ikD} \\
0
\end{array} \right) = Y_1^{-1} \left( \begin{array}{c}
B_{\text{out}} e^{2ika_1}
\end{array} \right).
\end{align*}
\]

Since \( A_{\text{tot}}^{\text{gap}} = i \left( \tilde{A}_{\text{tot}}^{\text{gap}} e^{ikx} - \tilde{B}_{\text{tot}}^{\text{gap}} e^{-ikx} \right) \) and \( B_{\text{tot}}^{\text{gap}} = \tilde{A}_{\text{tot}}^{\text{gap}} e^{ikx} + \tilde{B}_{\text{tot}}^{\text{gap}} e^{-ikx} \), from the first equality in [11] it follows that \( A_{\text{tot}}^{\text{gap}} \) and \( B_{\text{tot}}^{\text{gap}} \) in [6] are determined by the expressions

\[
\begin{align*}
A_{\text{tot}}^{\text{gap}} & = -A_{\text{out}} P e^{ikx_1}, \quad B_{\text{tot}}^{\text{gap}} = A_{\text{out}} Q e^{ikx_1}; \\
\end{align*}
\]

\[
\text{here } Q = q^{*} \exp(ikL/2) + ip \exp(-ikL/2), \quad P = iq^{*} \exp(ikL/2) + p \exp(-ikL/2). \text{ Besides, by 'sewing' the solutions } \text{at the points } x = a_1 \text{ and } x = b_2, \text{ we obtain}
\]

\[
\begin{align*}
A_{(1)}^{(1)} & = i(1 - B_{\text{out}}) \frac{k}{\kappa} e^{ikx_1}, \quad B_{(1)}^{(1)} = (1 + B_{\text{out}}) e^{ikx_1}; \quad A_{(2)}^{(2)} = iA_{\text{out}} \frac{k}{\kappa} e^{ikx_1}, \quad B_{(2)}^{(2)} = A_{\text{out}} e^{ikx_1}.
\end{align*}
\]
The amplitudes $A_{\text{out}}$ and $B_{\text{out}}$ can be expressed either through the one-barrier parameters, with help of the second equality in (11), or through the two-barrier ones, with help of the relationship

$$\begin{pmatrix} B_{\text{out}} e^{2ikx_1} \\ 1 \end{pmatrix} = Y_{\text{two}} \begin{pmatrix} A_{\text{out}} e^{-ikD} \\ 0 \end{pmatrix}.$$ 

As a result, we have two equivalent forms for each amplitude,

$$A_{\text{out}} = \frac{1}{2} \left( \frac{Q}{Q^*} - \frac{P^*}{P} \right) = \sqrt{R_{\text{two}}} e^{iJ_{\text{two}}},$$

$$B_{\text{out}} = -\frac{1}{2} \left( \frac{Q}{Q^*} + \frac{P^*}{P} \right) = -i\sqrt{R_{\text{two}}} e^{i(J_{\text{two}} - F_{\text{two}})} \quad (13)$$

(both the forms will be useful for developing a new model of this process.

4 A new, nonlinear model of scattering a particle on a system of two identical rectangular potential barriers

4.1 Stationary wave functions for transmission and reflection

According to [15], for any symmetric two-barrier system the total wave function $\Psi_{\text{tot}}(x, k)$ to describe the whole scattering process can be uniquely presented, for any values of $x$ and $k$, as a superposition of two SWFs $\psi_{tr}(x, k)$ and $\psi_{ref}(x, k)$ to describe the transmission and reflection subprocesses, respectively. Both possess the following properties:

(a) $\psi_{tr}(x, k) + \psi_{ref}(x, k) = \Psi_{\text{tot}}(x, k)$;

(b) each SWF has only one outgoing wave and only one incoming wave; in this case the transmitted wave in (6) serves as the outgoing wave in $\psi_{tr}(x, k)$, while the reflected one represents the outgoing wave in $\psi_{ref}(x, k)$;

(c) the incoming wave and the corresponding outgoing wave of each SWF, extended into the barrier region, join together at some point $x_{\text{join}}(k)$, where the SWF and the corresponding probability density are continuous.

From (c) it follows that the continuity conditions used to sew incoming (falling) and outgoing waves for each subprocess represent three (real) continuity conditions: two (real) conditions for the complex-valued SWF itself and one (real) continuity condition for the corresponding probability current density. Thus, these three (real) continuity conditions are nonlinear as it should be, according to a new formulation of the superposition principle! In this case the first $x$-derivatives of both SWFs are discontinuous at the point $x_c$.

A simple analysis shows that for any symmetric two-barrier system $x_{\text{join}}(k)$ coincides, for any value of $k$, with the midpoint $x_c$ of the barrier region: $x_c = (b_2 + a_1)/2$. In this case, $\psi_{ref}(x, k) \equiv 0$ and $\psi_{tr}(x, k) \equiv \Psi_{\text{tot}}(x, k)$ for $x \geq x_c$. This means that particles reflected by the symmetric two-barrier system do not enter into the region $x > x_c$, and the SWF $\psi_{ref}(x, k)$ is a currentless wave function.

Calculations yield that in the region $x < x_c$ the wave function $\psi_{ref}(x, k)$ can be written as follows,

$$\psi_{ref}(x, k) = \begin{cases} A_{\text{ref}} \exp^{ikx} + b_{\text{out}} \exp^{i(k(2b_1-x))} : & x \in (-\infty, a_1) \\ a_{\text{ref}}^{(1)} \sinh[k(x - b_1)] + b_{\text{ref}}^{(1)} \cosh[k(x - b_1)] : & x \in [a_1, b_1] \\ a_{\text{ref}}^{(2)} \sin[k(x - x_c)] : & x \in [b_1, x_c] \end{cases} \quad (14)$$

Again, as in Section [3] in order to find the amplitudes in these expressions it is suitable to rewrite the function $\psi_{ref}(x, k)$ in the interval $[b_1, x_c]$ in the form $\psi_{ref}(x, k) = A_{\text{ref}} \exp(lkx) + B_{\text{ref}} \exp(-lkx)$. The amplitudes in this expression are linked as follows

$$\begin{pmatrix} A_{\text{ref}}^{(a)} \\ B_{\text{ref}}^{(a)} \end{pmatrix} = Y^{-1}_a \begin{pmatrix} A_{\text{ref}}^{(b)} \\ B_{\text{out}} e^{2ika_1} \end{pmatrix} \quad (15).$$
Then, making use of the relationships
\[ \alpha_{\text{ref}}^{\text{gap}} = i \left( A_{\text{ref}}^{\text{gap}} e^{ikx} - B_{\text{ref}}^{\text{gap}} e^{-ikx} \right), \quad A_{\text{ref}}^{\text{gap}} e^{ikx} + B_{\text{ref}}^{\text{gap}} e^{-ikx} = 0 \]  
we find the unknown amplitudes in Exps. (14).

From the second equality in (16) it follows that \( A_{\text{ref}}^{\text{in}} = -b_{\text{out}} Q^*/Q \). Then, taking into account Exps. (15) and (17), we obtain
\[ A_{\text{ref}}^{\text{in}} = b_{\text{out}} (b_{\text{out}}^* - a_{\text{out}}) = \sqrt{R_{\text{two}}} (\sqrt{R_{\text{two}} + i\eta_{\text{two}} \sqrt{T_{\text{two}}}}) \equiv \sqrt{R_{\text{two}}} \exp(i\lambda) \]
where \( \eta_{\text{two}} = +1, \) if \( T_{\text{two}} = 0; \) otherwise, \( \eta_{\text{two}} = -1. \) This means that the phases of the incident waves in \( \Psi_{\text{tot}}(x, k) \) and \( \psi_{\text{ref}}(x, k) \) differ from each other by the amount \( \lambda = \eta_{\text{two}} \cdot \arctan(\sqrt{T_{\text{two}}}/R_{\text{two}}(k)). \)

Then, taking into account, in (16), Exps. (15) and (17), we obtain
\[ \alpha_{\text{ref}}^{\text{gap}} = -2Pb_{\text{out}} a_{\text{out}} e^{ikx}. \]
And lastly, making use of the continuity conditions at the point \( x = b_1, \) we obtain
\[ \begin{align*}
\alpha_{\text{ref}}^{(1)} & = \frac{k}{\kappa} \alpha_{\text{ref}}^{\text{gap}} \cos \left( \frac{kL}{2} \right), \\
b_{\text{ref}}^{(1)} & = -\alpha_{\text{ref}}^{\text{gap}} \sin \left( \frac{kL}{2} \right).
\end{align*} \]

Since \( \psi_{\text{ref}}(x, k) \) is now known, we have \( \psi_{\text{tr}}(x, k) = \Psi_{\text{tot}}(x, k) - \psi_{\text{ref}}(x, k). \) In particular,
\[ A_{\text{in}}^{\text{tot}} = 1 - A_{\text{in}}^{\text{ref}} = \sqrt{T_{\text{two}}} \left( \sqrt{T_{\text{two}} - i\eta_{\text{two}} \sqrt{R_{\text{two}}} \right) \equiv \sqrt{T_{\text{two}}} \exp \left[ i \left( \lambda - \eta_{\text{two}} \frac{\pi}{2} \right) \right]. \]
As is seen, not only \( A_{\text{in}}^{\text{tot}}(k) + A_{\text{in}}^{\text{ref}}(k) = 1, \) but also \( |A_{\text{in}}^{\text{tot}}(k)|^2 + |A_{\text{in}}^{\text{ref}}(k)|^2 = 1. \) Besides,
\[ |\psi_{\text{tr}}(x - x_c, k)| = |\psi_{\text{tr}}(x - x_c, k)|. \]

4.2 Time-dependent wave functions for transmission and reflection

Let us now proceed to the time-dependent process described by the wave packet
\[ \Psi_{\text{tot}}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\Psi_{\text{tot}}(x, k)e^{-iE(k)t/\hbar} dk. \]
At this point (see also (4)) we assume \( A(k) \) to be the Gaussian function \( A(k) = (2\gamma/k)^{1/4} \exp \left[ -i\gamma^2(k - \bar{k})^4 \right]. \)
In this case
\[ \bar{x}_{\text{tot}}(0) = 0, \quad \bar{p}_{\text{tot}}(0) = \hbar \bar{k}, \quad \bar{x}_{\text{tot}}^2(0) = l_0^2. \]
hereinafter, for any observable \( F \) and time-dependent localized state \( \Psi_{\text{tot}}^A(t) \)
\[ \bar{F}_{\text{tot}}^A(t) = \frac{\langle \Psi_{\text{tot}}^A(t)|\hat{F}|\Psi_{\text{tot}}^A(t) \rangle}{\langle \Psi_{\text{tot}}^A(t)|\Psi_{\text{tot}}^A(t) \rangle}. \]
(if \( \bar{F}_{\text{tot}}^A(t) \) is constant its argument will be omitted). We assume that the parameters \( l_0 \) and \( \bar{k} \) obey the conditions for the scattering process: the rate of scattering the transmitted and reflected wave packets is assumed to exceed the rate of widening each packet; so that the transmitted and reflected wave packets non-overlap each other. We also assume that the origin of coordinates, from which the "center of mass" (CM) \( \bar{x}_{\text{tot}} \) of the wave packet \( \Psi_{\text{tot}}(x, t) \) starts, lies sufficiently far from the left boundary of the two-barrier system: \( a_1 \gg l_0. \)

Besides, let the expression
\[ \psi_{\text{tr}, \text{ref}}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\psi_{\text{tr}, \text{ref}}(x, k)e^{-iE(k)t/\hbar} dk \]

give the wave functions \( \psi_{tr}(x,t) \) and \( \psi_{ref}(x,t) \) to describe, respectively, the time-dependent transmission and reflection subprocesses. It is evident (see the requirement (a) in Section 4.1) that the sum of these two functions yields, at any value of \( t \), the total wave function \( \Psi_{tot}(x,t) \),
\[
\Psi_{tot}(x,t) = \psi_{tr}(x,t) + \psi_{ref}(x,t).
\]  
(23)

So, at the first stage, the process is described by the incident packet
\[
\Psi_{tot}(x,t) \simeq \Psi_{tot}^{inc}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \exp[i(kx - E(k)t/\hbar)]dk,
\]
and its transmission and reflection subprocesses are described by the wave packets
\[
\psi_{tr,ref} \simeq \psi_{tr,ref}^{inc} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{tr,ref}^{inc}(k) A(k) \exp[i(kx - E(k)t/\hbar)]dk.
\]
Considering Exps. \( (17) \) and \( (18) \) for the amplitudes of the incident waves in \( \psi_{tr}(x,k) \) and \( \psi_{ref}(x,k) \), it is easy to show that
\[
\hat{x}^{inc}_{tr}(0) = -\chi^{inc}_{tr},
\]
\[
\hat{x}^{inc}_{ref}(0) = -\chi^{inc}_{ref},
\]
(24)
where the prime denotes the derivative on \( k \).
That is, in the general case the CMs of the wave packets \( \Psi_{tot}(x,t) \), \( \psi_{tr}(x,t) \) and \( \psi_{ref}(x,t) \) start at \( t = 0 \) from the different spatial points.

Similarly, for the final stage of the process
\[
\psi_{tr} \simeq \psi_{tr}^{out} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) a_{out}(k) e^{i[k(x_D - E(k)t/\hbar)]} dk,
\]
\[
\psi_{ref} \simeq \psi_{ref}^{out} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) b_{out}(k) e^{i[k(2\lambda_1 - x) - E(k)t/\hbar]} dk.
\]
Thus, since \( |A_{tr}^{inc}(k)|^2 = |a_{out}(k)|^2 = T_{two}(k) \) and \( |A_{ref}^{inc}(k)|^2 = |b_{out}(k)|^2 = R_{two}(k) \) (see \( (13), (17) \) and \( (18) \)), for the initial and final stages of scattering we have
\[
\langle \psi_{tr}^{inc} | \psi_{tr}^{inc} \rangle = \langle \psi_{tr}^{out} | \psi_{tr}^{out} \rangle = \int_{-\infty}^{\infty} T_{two}(k) |A(k)|^2 dk \equiv T_{as},
\]
\[
\langle \psi_{ref}^{inc} | \psi_{ref}^{inc} \rangle = \langle \psi_{ref}^{out} | \psi_{ref}^{out} \rangle = \int_{-\infty}^{\infty} R_{two}(k) |A(k)|^2 dk \equiv R_{as}.
\]
In its turn, since \( T_{two}(k) + R_{two}(k) = 1 \) and \( \langle \Psi_{tot} | \Psi_{tot} \rangle = \int_{-\infty}^{\infty} |A(k)|^2 dk = 1 \), from the above it follows that the sum of the constant norms \( T_{as} \) and \( R_{as} \) is equal to unity:
\[
T_{as} + R_{as} = 1.
\]  
(25)

From the fact that the transmission and reflection subprocesses obey, at both these stages of scattering, the probabilistic "either-or" rule \( (25) \) it follows that they behave as alternative subprocesses at both the stages, despite interference between them at the first stage. We can add to the equality \( (25) \) that
\[
\langle \psi_{tr}^{inc} | \psi_{ref}^{inc} \rangle = \int_{-\infty}^{\infty} |A(k)|^2 \left[ A_{tr}^{inc}(k) \right]^{*} A_{ref}^{inc}(k) dk = i \int_{-\infty}^{\infty} |A(k)|^2 [\eta_{two}(k) \sqrt{T_{two}(k)R_{two}(k)}] dk
\]
(the piece-wise constant function \( \eta_{two}(k) \) is defined in \( (17) \)). Thus, \( \langle \psi_{tr}^{inc} | \psi_{ref}^{inc} \rangle + \langle \psi_{ref}^{inc} | \psi_{ref}^{inc} \rangle = 0 \).

At the very stage of scattering, when the wave packet \( \psi_{tr}(x,t) \) crosses the point \( x_c \), the norm \( \mathbf{T} = \langle \psi_{tr} | \psi_{tr} \rangle \) can change. The point is that the nonlinear continuity conditions (a)-(c) in Section 4.1 guarantee the balance between the input \( I_{tr}(x_c - 0, k) \) and output \( I_{tr}(x_c + 0, k) \) probability flows only for the stationary waves \( \psi_{tr}(x,k) \), of which the wave packet \( \psi_{tr}(x,t) \) is built. However, for the packet
itself the interference between the main 'harmonic' \( \psi_{tr}(x,k) \) and 'subharmonics' \( \psi_{tr}(x,k) \), whose first derivatives are discontinuous at the point \( x_c \), leads to the imbalance between the input and output flows at the point \( x_c \): \( dT/dt = I_{tr}(x_c + 0, t) - I_{tr}(x_c - 0, t) \neq 0 \). Since the role of subharmonics is essential only at the leading and trailing fronts of the wave-packet, this effect takes place only when these fronts cross the midpoint \( x_c \). Otherwise, \( T + R \approx 1 \) even at the very stage of scattering, when the CM of the wave packet \( \psi_{tr}(x,t) \) moves inside the barrier region.

Note that the total alteration of the norm \( T \), accumulated in the course of the scattering process, is zero. As regards \( R \), this norm remains constant even at the very stage of scattering: \( R \equiv R_\infty \). This follows from the fact that \( I_{ref}(x_c + 0, t) = I_{ref}(x_c - 0, t) = 0 \) since \( \psi_{ref}(x_c, t) = 0 \) for any value of \( t \).

5 Local and asymptotic group ("phase") times for transmission and reflection

Now, when the dynamics of the sub-processes of transmission and reflection at all stages of scattering became known, we can proceed to study temporal aspects of each subprocess. We begin with the presentation of the local (exact) and asymptotic (extrapolated) group times for transmission and reflection. For example, the local transmission group time \( \tau^\text{loc}_{tr} \) that characterizes the dynamics of the CM of the wave packet \( \psi_{tr}(x,t) \) inside the region \([a_1, b_2]\) is defined as follows (see [15]):

\[
\tau^\text{loc}_{tr} = t^\text{exit}_{tr} - t^{\text{entry}}_{tr}, \quad \text{where } t^{\text{entry}}_{tr} \text{ and } t^\text{exit}_{tr} \text{ are such instants of time that }
\]

\[
\bar{x}_{tr}(t^{\text{entry}}_{tr}) = a_1, \quad \bar{x}_{tr}(t^{\text{exit}}_{tr}) = b_2.
\]

Similarly, for reflection \( \tau^\text{loc}_{ref} = t^\text{exit}_{ref} - t^{\text{entry}}_{ref} \), where \( t^{\text{entry}}_{ref} \) and \( t^\text{exit}_{ref} \) are two different roots, if any, of the same equation (\( t^{\text{entry}}_{ref} < t^\text{exit}_{ref} \)):

\[
\bar{x}_{ref}(t^{\text{entry}}_{ref}) = a_1, \quad \bar{x}_{ref}(t^\text{exit}_{ref}) = a_1.
\]

If this equation has no more than one root, \( t^\text{loc}_{ref} = 0 \).

Note that the local group times \( \tau^\text{loc}_{tr} \) and \( \tau^\text{loc}_{ref} \) do not give a complete description of the temporal aspects of each subprocess, because the two-barrier system affects the subensembles of transmitted and reflected particles not only when the CMs of the wave packets \( \psi_{tr}(x,t) \) and \( \psi_{ref}(x,t) \) move in the region \([a_1, b_2]\). Of importance is also to define the asymptotic group times to describe these subprocesses in the asymptotically large spatial region \([0, b_2 + \Delta X]\) where \( \Delta X \gg l_1 \).

In doing so, we have to take into account that each wave packet does not interact with the system when its CM is at the boundaries of this spatial interval. That is, the asymptotic transmission time can be defined in terms of the transmitted \( \psi_{tr}^\text{inc} \) and to-be-transmitted \( \psi_{tr}^\text{out} \) wave packets. Similarly, the asymptotic reflection time can be introduced in terms of the wave packets \( \psi_{ref}^\text{out} \) and \( \psi_{ref}^\text{inc} \).

We begin with the transmission subprocess. For the CM’s position \( \bar{x}_{tr}(t) \) at the initial stage of scattering we have (see also [24])

\[
\bar{x}_{tr}(t) \simeq \bar{x}_{tr}^\text{inc}(t) = \frac{\hbar k_{tr}}{m} t - \lambda(k)_{tr},
\]

\begin{equation}
(26)
\end{equation}

here \( \bar{k}_{tr} = \bar{k}_{tr}^\text{out} = \bar{k}_{tr}^\text{inc} \). At the final stage

\[
\bar{x}_{tr}(t) \simeq \bar{x}_{tr}^\text{out}(t) = \frac{\hbar k_{tr}}{m} t - \frac{J_{tr}^\text{two}(k)_{tr}}{\hbar k_{tr}} + D.
\]

Thus, the time \( \tau_{tr}^y(0, b_2 + \Delta X) \) spent by the CM of \( \psi_{tr}(x,t) \) in \([0, b_2 + \Delta X] \) is

\[
\tau_{tr}^y(0, b_2 + \Delta X) = \frac{m}{\hbar k_{tr}} \left[ \frac{J_{tr}^\text{two}(k)_{tr}}{\hbar k_{tr}} - \lambda(k)_{tr} + a_1 + \Delta X \right],
\]

where \( t^y_{arr} \) and \( t^y_{dep} \) obey the equations

\[
\bar{x}_{tr}^\text{inc}(t^y_{arr}) = 0; \quad \bar{x}_{tr}^\text{out}(t^y_{dep}) = b_2 + \Delta X.
\]
The quantity $\tau_{as}^{\alpha} = \tau_{tr}^{\alpha}(a_1, b_2)$, which is associated with the region $[a_1, b_2]$, will be referred to as the asymptotic (extrapolated) transmission group time:

$$\tau_{tr}^{as} = \frac{m}{\hbar k_{tr}} \left[ J_{t_2o}(k)_{tr}^{\text{out}} - \lambda(k)_{tr}^{\text{inc}} \right].$$  \hspace{1cm} (27)

Similarly, for reflection we have

$$x_{ref}^{as} = \frac{m}{\hbar k_{ref}} \left[ J_{t_2o}(k)_{ref}^{\text{out}} - \lambda(k)_{ref}^{\text{inc}} \right];$$  \hspace{1cm} (28)

$$k_{ref}^{inc} = -k_{ref}^{out} = k_{ref}.$$

Let us now consider narrow (in $k$-space) wave packets (in this case we will omit the upper line in the notation $k$):

$$a_1, DX \gg l_0 \gg D.$$  \hspace{1cm} (29)

Now Exps. (27) and (28) give the same value:

$$\tau_{tr}^{as}(k) = \tau_{ref}^{as}(k) \equiv \tau_{as}(k) = \frac{m}{\hbar k} \left[ J_{t_2o}(k) - \lambda(k) \right];$$  \hspace{1cm} (30)

$$t_{dep}^{tr}(k) = t_{dep}^{ref}(k) \equiv t_{dep}(k) = m\lambda(k)/\hbar k; \quad x_{inc}^{tr}(0) = x_{inc}^{ref}(0) \equiv x_{start} = -\lambda(k).$$

Here (see Exps. (10) and (17))

$$J_{t_2o}^{tr} = J' + \frac{T_{t_2o}}{T^2} \left[ T(1 + R) (J' + L) + T' \sin[2(J + kL)] \right],$$

$$\lambda' = 2\eta \frac{T_{t_2o}}{\sqrt{R} T^2} \left[ T'(1 + R) \cos(J + kL) + 2RT(J' + L) \sin(J + kL) \right].$$

These expressions are valid for any symmetric two-barrier system. For rectangular barriers we can obtain explicit expressions for one-barrier functions. Namely, from Exps. (9) it follows that

$$J' = \frac{T}{K} \left[ \theta_{(+)}^2 \sinh(2\kappa d) + \theta_{(-)} \kappa d \right], \quad T' = 2\theta_{(+)}^2 \frac{T^2}{K} \left[ 2\theta_{(-)} \sinh^2(\kappa d) + \kappa d \kappa \sinh(2\kappa d) \right].$$

Note that the corresponding expressions for the Wigner phase time $\tau_{ph}$ [20] has been obtained in [19]. In our notations it can be written as $\tau_{ph} = mJ_{t_2o}(k)/\hbar k$. This concept is based on the assumption that transmitted particles start, on average, from the point $x_{tot}(0)$ [21] which equals to zero in our target setting. However, our approach says that these particles start, on average, from the point $x_{start}$ which does not coincide with $x_{tot}(0)$ in the general case. That is, our approach does not confirm the validity of the Wigner-time concept in the general case. With taking into account of (30), the relationship between the asymptotic transmission group time $\tau_{as}(k)$ introduced in our approach and the Wigner phase time $\tau_{ph}$ introduced in [19] on the basis of the standard model of the process can be written as follows,

$$\tau_{as}(k) = \tau_{ph}(k) - t_{dep}(k).$$

For $L = 0$, when the two-barrier system is reduced to a single rectangular barrier of width $D$, we have (see [19])

$$\tau_{as}(k) = \frac{4m}{\hbar k \kappa} \left[ k^2 + \kappa^2 \sinh^2(\kappa D/2) \right] \frac{[\kappa^2 \sinh(\kappa D) - k^2 \kappa D]}{4k^2 \kappa^2 + \kappa^2 \sinh^2(\kappa D)};$$

$$x_{start}(k) = -\frac{\kappa^2}{\kappa} \frac{(k^2 - k^2 \kappa D) + k^2 \kappa D \cosh(\kappa D)}{4k^2 \kappa^2 + \kappa^2 \sinh^2(\kappa D)}. \hspace{1cm} (31)$$

where $\kappa_0 = \sqrt{2mV_0}/\hbar$ (note, focusing on the Hartman effect we assumed that $V_0 > 0$; however, the formalism presented is valid also for $V_0 < 0$ when both $\kappa_0$ and $\kappa$ are purely imaginary quantities).
As is seen from (31), $x_{\text{start}} \to 0$ in the opaque-barrier limit, i.e., when $\kappa D \to \infty$ (providing that the requirements (29) are fulfilled). Thus, in this limit, the above-mentioned assumption that underlies the concept of the Wigner time is well justified – $\tau_{\text{ph}}(k) \approx \tau_{\text{as}}(k) \approx \frac{2\pi}{h\kappa(k)}$. The fact that the Wigner time $\tau_{\text{ph}}(k)$ does not depend, in this limit, on the width of the rectangular barrier is known as the Hartman effect.

Note that in the limit of opaque barriers the equality $\tau_{\text{ph}}(k) \approx \tau_{\text{as}}(k) \approx \frac{2\pi}{h\kappa(k)}$ holds also for the two-barrier system when $L \neq 0$; that is, again, in this limit $x_{\text{start}} \to 0$. The main peculiarity of the two-barrier system is that now $\tau_{\text{ph}}(k)$ does not depend not only on the width of two identical barriers, but also on the distance $L$ between them – the generalized Hartman effect \[19\]. So that the existence of both Hartman effects predicted on the basis of the standard model of the process is also confirmed by our concept of the asymptotic group times derived for the transmission subprocess.

However, unlike the standard model ours does not associate these effects with superluminal velocities of a particle in the region $[a_1, b_2]$ (see Section 7). In this region, the velocity of a tunneling particle with a definite energy is associated, according to our approach, with the dwell time and local group time, rather than with the asymptotic group time.

6 Dwell times for transmission and reflection

Our next step is to consider the stationary scattering problem and introduce the dwell times for both subprocesses. For the two-barrier system the dwell times $\tau_{\text{dwell}}^{\text{tr}}$ and $\tau_{\text{dwell}}^{\text{ref}}$ for transmission and reflection, respectively, are defined as follows

\[
\tau_{\text{dwell}}^{\text{tr}} = \frac{m}{\hbar kT_{\text{two}}} \int_{a_1}^{b_2} |\psi_{\text{tr}}(x, k)|^2 \, dx \equiv \tau_{\text{tr}}^{(1)} + \tau_{\text{tr}}^{(2)} + \tau_{\text{dwell}}^{\text{tr}},
\]

\[
\tau_{\text{dwell}}^{\text{ref}} = \frac{m}{\hbar kR_{\text{two}}} \int_{a_1}^{x_c} |\psi_{\text{ref}}(x, k)|^2 \, dx \equiv \tau_{\text{ref}}^{(1)} + \tau_{\text{ref}}^{(2)},
\]

here $\tau_{\text{tr}}^{(1)}$ and $\tau_{\text{ref}}^{(1)}$ describe the left rectangular barrier located in the interval $[a_1, b_1]$; $\tau_{\text{tr}}^{\text{gap}}$ and $\tau_{\text{ref}}^{\text{gap}}$ characterize the free space $[b_1, a_2]$; $\tau_{\text{tr}}^{(2)}$ relates to the right rectangular barrier located in the interval $[a_2, b_2]$.

Calculations yield (see Section 3)

\[
\tau_{\text{tr}}^{(1)} = \tau_{\text{tr}}^{(2)} = \frac{m}{4\hbar^2 k^2 \kappa^4} \left[ 2k \delta(k^2 - k^2) + \kappa_0^2 \sinh(2k \delta) \right],
\]

\[
\tau_{\text{tr}}^{\text{gap}} = \frac{m}{\hbar k T} \left[ kL(1 + R) + 4\eta R \sin \left( \frac{kL}{2} \right) \sin \left( \frac{J + kL}{2} \right) \right],
\]

\[
\tau_{\text{ref}}^{(1)} = \frac{m T_{\text{two}}|P|^2}{2\hbar^2 k^2 \kappa^4} \left\{ 2k \delta^2 (k^2 - k^2 \cos(kL)) + 4k \kappa \sin(kL) \sinh^2(k \delta) \right\} + \left[ \kappa_0^2 - (k^2 - k^2 \cos(kL)) \sinh(2k \delta) \right],
\]

\[
\tau_{\text{ref}}^{\text{gap}}(k) = \frac{m T_{\text{two}}|P|^2}{\hbar k^2} \left[ kL - \sin(kL) \right];
\]

here $|P|^2 = |1 + R - 2\eta \sqrt{R} \sin(J + kL)|/T$ (see Exp. 12).

Note that $\tau_{\text{tr}}^{\text{dwell}}(k) \neq \tau_{\text{ref}}^{\text{dwell}}(k)$ while $\tau_{\text{tr}}^{\text{as}}(k) = \tau_{\text{ref}}^{\text{as}}(k)$. Another feature is that $\tau_{\text{tr}}^{(2)} = \tau_{\text{tr}}^{(1)} \equiv \tau_{\text{bar}}$ (see \[19\]). If $\tau_{\text{tr}}^{\text{left}}$ and $\tau_{\text{tr}}^{\text{right}}$ denote the transmission dwell times for the intervals $[a_1, x_c]$ and $[x_c, b_2]$, respectively, then

\[
\tau_{\text{tr}}^{\text{left}} = \tau_{\text{tr}}^{\text{right}} = \tau_{\text{bar}} + \tau_{\text{tr}}^{\text{gap}}/2 = \tau_{\text{tr}}^{\text{dwell}}/2.
\]

That is, the (stationary) transmission time obeys the natural physical requirement: for any barrier possessing the mirror symmetry, this characteristic time must be the same for both symmetrical parts of the barrier.
For comparison we also present for this two-barrier system the additive characteristic time $\tau_{dwell}$ defined in the spirit of Buttiker's dwell time [21]:

$$
\tau_{dwell} = \frac{m}{\hbar k} \int_{a_1}^{b_2} |\Psi_{tot}(x, k)|^2 dx \equiv \tau_{tot}^{(1)} + \tau_{gap}^{(2)} + \tau_{tot}^{(2)};
$$

(35)

here the contributions $\tau_{tot}^{(1)}$, $\tau_{tot}^{(2)}$ and $\tau_{gap}^{(2)}$ describe, respectively, the left and right barriers as well as the gap between them. Calculations yield

$$
\tau_{tot}^{(1)} = \frac{mT_{two}}{4\hbar k^2 T} \left[ 2kd \left( (k^2 - \kappa^2)(1 + R) + 2\sqrt{R}\kappa^2 \sin(J + kL) \right) + \kappa^2 (1 + R) + 2\sqrt{R}(k^2 - \kappa^2) \sin(J + kL) \right] \sinh(2kd) + 8\kappa \sqrt{R} \cos(J + kL) \sinh^2(\kappa d) \right] \right),
$$

(36)

$$
\tau_{tot}^{(2)} = \tau_{two}^{(2)} T_{two}.
$$

As is seen, unlike $\tau_{dwell}^{(2)}$ the Buttiker dwell time $\tau_{dwell}$ does not possess the property (34). Besides, from Exps. (33) and (35) it follows that $\tau_{dwell}$ describes neither transmitted nor reflected particles.

Let us consider the limit of opaque barriers when $kd \to \infty$ and $\cos^2(J(\infty) + kL) \gg T^2/4R$; here $J(\infty) = \arctan(\theta_{\infty})$. In this case, we will imply that $d \to \infty$ but other parameters are fixed. Omitting the exponentially small terms in the expressions for each scattering time, for all three contributions in Exp. (33) to $\tau_{dwell}$ as well as for $\tau_{ref}$ we obtain

$$
\tau_{tr}^{(1)} \approx \frac{m\eta_0^2}{4\hbar k^2 \kappa^2} 2kd; \quad \tau_{gap}^{(2)} \approx \frac{m\theta_{(+)}^2}{2h\kappa^2} \left[ kL + 2 \sin \left( \frac{kL}{2} \right) \sin \left( J(\infty) + \frac{kL}{2} \right) \right] e^{2kd};
$$

$$
\tau_{ref} \approx \tau_{(1)}^{(1)} \approx \frac{m}{2\hbar k^2 \theta_{(+)}^2} \left[ \kappa^2 - (k^2 - \kappa^2) \cos(kL) + k\kappa \kappa \sin(kL) \right].
$$

As regards the Buttiker's dwell time $\tau_{dwell}$, in this limit $\tau_{dwell}$ is determined by $\tau_{tot}^{(1)}$ which saturates in this limit.

That is, the dwell time $\tau_{dwell}$ for reflection and $\tau_{as}$ (as well as $\tau_{dwell}$ and $\tau_{ph}$ that appear in the standard approach) saturate in the opaque-barrier limit, while $\tau_{dwell}$ grows exponentially in this case! Thus, two scattering times, $\tau_{as}$ and $\tau_{dwell}$, that describe in the nonlinear model of the process the transmission subprocess, demonstrate a qualitatively different behaviour in the opaque-barrier limit. Unlike the asymptotic group time $\tau_{as}$, the dwell time $\tau_{dwell}$ does not lead to the Hartman effects. In particular, for the two-barrier system it depends on $L$ in the opaque-barrier limit (see the expression for $\tau_{ref}$).

7 Numerical results for characteristic times

So, we have introduced three characteristic times for the transmission subprocess: in the stationary case, this is the dwell time $\tau_{dwell}^{(2)}$ that characterizes its dynamics in the interval $[a_1, b_2]$ occupied by the two-barrier system; for wave packets, they are the local group time $\tau_{loc}^{(2)}$ which, too, characterizes its dynamics in the interval $[a_1, b_2]$, as well as the asymptotic group time $\tau_{as}^{(1)}$ that characterizes its dynamics in the asymptotically large interval $[0, b_2 + \Delta X]$. Our next step is to compare these characteristic times for the rectangular barrier of width $D = 2d$ (that is, one has to take $L = 0$ in the corresponding expressions for the two-barrier system) with the Buttiker dwell time $\tau_{dwell}$ and Wigner phase time $\tau_{ph}$.

As is known, in the standard approach $\tau_{ph}$ diverges and $\tau_{dwell}$ diminishes in the low energy region, but both these quantities coincide with each other, in the high energy region (see, e.g., fig. 3 in [21]). Figs. [13] show that in our model the same behavior is manifested by the asymptotic group time $\tau_{as}$ (in all these figures, the quantity $\tau_{free}/\tau_0$, where $\tau_{free} = mD/\hbar k$ and $\tau_0 = mD/\hbar R_0$, is presented as a 'reference' one). Unlike the standard time scales $\tau_{ph}$ and $\tau_{dwell}$, as well as $\tau_{as}$, the transmission dwell
Fig. 1 $\frac{\tau_{dwell}}{\tau_0}$ (bold full line), $\tau_{dwell}/\tau_0$ (full line), $\tau_{ph}/\tau_0$ (dots), $\tau_{dep}/\tau_0$ (dash-dot) and $\tau_{free}/\tau_0$ (broken line) as functions of $k$ for a system with $L = 0$ and $2\kappa_0 d = 3\pi$ (see also fig. 3 in [21]).

time $\tau_{dwell}$ never leads to 'anomalously' short tunneling times. As is seen from fig. 1, all the analyzed time scales (of course, excluding $\tau_{dep}$) approach the free-passage time $\tau_{free}$ in the high energy region. However, in the low energy region $\tau_{dwell} \gg \tanh 2\left(\kappa_0 D^2\right) \cdot \tau_{ph} \approx \tau_{as} \approx 2\kappa_0 D \tanh 2\left(\kappa_0 D^2\right) \cdot \tau_{free} \gg \tau_{dwell}$.

Note that, on the $k$-axis the quantity $\tau_{dep}$ changes its sign at the points located between resonant points, where $R_{two} = 0$. At resonance points, the function $|\tau_{dep}(k)|$, like the dwell time $\tau_{dwell}$ and the Wigner phase time $\tau_{ph}$, takes extreme values (see figs. [1][3]). At these points, the Buttiker dwell time $\tau_{dwell}(k)$ coincides with $\tau_{dwell}^k(k)$ and takes maximal values in the vicinity of these points. Note that the behavior of $\tau_{as}(k)$ is more complicated: like $\tau_{dwell}(k)$ it takes maximal values at the even resonance points (in this case $\tau_{as}(k) \approx 2\tau_{dwell}(k)$, while $\tau_{as}(k)$ has no maxima at the odd resonance points, including the first (lowest) resonance point (see fig. [3]).

The CMs of the wave packet $\psi_{tr}(x, t)$, peaked on the $k$-scale at the resonance points with the even numbers, starts earlier ($\tau_{dep}(k) < 0$) than the CM of the total wave packet $\Psi_{tot}(x, t)$. While at the resonance points with odd numbers we find the opposite situation. Moreover, at such energy points, the local maxima of the function $\tau_{ph}(k)$ transform into the local minima of the function $\tau_{as}(k) = \tau_{ph}(k) - t_{tr}^{dep}(k)$.

When $L \neq 0$ and $E > V_0$, all scattering times show the tendency to increase in the limit $L \to \infty$ (see fig. [2]). However, in the tunneling regime ($E < V_0$), only the dwell time $\tau_{dwell}^k(k)$ monotonously increases (see fig. [3]). Other characteristic times do not in fact depend on $L$ in the opaque-barrier limit (the generalized Hartman effect). Moreover, for $\tau_{as}(k)$ this takes place also at the odd resonance points.)
Fig. 2  $\tau_{dwell}/\tau_0$ (bold full line), $\tau_{dwell}/\tau_0$ (full line), $\tau_{ph}/\tau_0$ (dots), $\tau_{as}/\tau_0$ (dash-dot) and $\tau_{free}/\tau_0$ (broken line) as functions of $L$ for $2\kappa_0d=3\pi$ and $k=1.5\kappa_0$.

So, for the two-barrier system with opaque rectangular barriers, the dwell time $\tau_{dwell}$ is much larger than the asymptotic group time $\tau_{as}$ which does not depend on $L$ in this case. However, this fact does not at all mean that our approach leads to mutually contradictory tunneling time concepts, with one of them violating special relativity. We have to remember that, unlike $\tau_{dwell}$, the asymptotic characteristic time $\tau_{as}$ is not a tunneling time. In order to demonstrate the difference between these two characteristic times let us consider in the time-dependent case the function $\bar{x}_{tr}(t)$ to describe scattering the Gaussian wave packet (20) on the rectangular potential barrier (i.e., now $L=0$): $l_0=10nm$, $E=(\bar{h}\bar{k})^2/2m=0.05eV$, $a_1=200nm$, $b_2=215nm$, $V_0=0.2eV$.

For this case calculations yield $\tau_{loc}\approx0.155ps$, $\tau_{as}\approx0.01ps$, $\tau_{free}\approx0.025ps$ (see fig. 4). This figure shows explicitly a qualitative difference between the local $\tau_{loc}$ and asymptotic $\tau_{as}$ transmission group times. While the former gives the time spent by the CM of this packet in the region $[a_1,b_2]$, the latter describes the influence of the barrier on this CM in the asymptotically large interval $[0,b_2+\Delta X]$. Thus, the quantity $\tau_{as}-\tau_{free}$ is a time delay which is acquired by this CM in the course of the whole scattering process; $\tau_{free}=mD/\bar{h}\bar{k}$. It describes the relative motion of the CMs of the transmitted wave packet and the CM of a reference packet that moves freely at all stages of scattering and departs from the point $x=0$ at the time $\tau_{dep}$. When the barrier is opaque, $\tau_{dep}$ coincides approximately with the departure time of the total wave packet $\psi_{tot}(x,t)$.

Thus, the influence of an opaque rectangular barrier on the transmitted wave packet has a complicated character. The local group time $\tau_{loc}$, together with the dwell time $\tau_{dwell}$, says that the opaque barrier retards the motion of the CM in the barrier region, while the asymptotic group time $\tau_{as}$ tells us that, in the course of the whole scattering process, the total influence of the opaque barrier on the transmitted wave packet has an accelerating character: at the final stage of scattering, this packet moves ahead the schedule of the reference packet: $\tau_{as}-\tau_{free}\approx-0.015ps$. 
Fig. 3 $\frac{\tau_{\text{dwell}}}{\tau_0}$ (bold full line), $\frac{\tau_{\text{dwell}}}{\tau_0}$ (full line), $\frac{\tau_{\text{ph}}}{\tau_0}$ (dots), $\frac{\tau_{\text{as}}}{\tau_0}$ (dash-dot) and $\frac{\tau_{\text{free}}}{\tau_0}$ (broken line) as functions of $L$ for $2\kappa_0 d = 3\pi$ and $k = 0.97\kappa_0$.

Note, for any finite value of $l_0$, the velocity of the CM of the wave packet $\psi_{tr}(x,t)$ is constant only at the initial and final stages of scattering. However, when the value of $l_0$ is large enough (a quasi-monochromatic wave packet) this takes place also at the very stage of scattering, when the CM of this packet moves inside the region $[a_1, b_2]$ while its leading and trailing fronts are far beyond this region. At this stage, only the main harmonic $k$ determines the input and output probability flows at the point $x_c$. As a result, these flows balance each other, and hence the norm $T$ is constant at this stage. In this case the local group time $\tau_{tr}^{\text{as}}$ approaches the dwell time $\tau_{tr}^{\text{dwell}}$; in the opaque-barrier limit, both predict the effect of retardation of tunneling particles in the region $[a_1, b_2]$.

Another situation arises in this limit when either the leading or trailing front of the wave packet $\psi_{tr}(x,t)$ crosses the point $x_c$. In the first case, due to the interference effect discussed in Section 4.2 this point serves as a 'source' of particles, what leads to the acceleration of its CM located at this stage to the left of the barrier. While in the second case this spatial point effectively acts as an 'absorber' of to-be-transmitted particles; what leads again to the acceleration of this CM which is located now to the right of the barrier (see fig 4). In the case presented on this figure, the velocity of the CM, prior to its entering into the barrier region and after its exit from this region, is larger three times compared with its velocity at the first and final stages of scattering. In the last analysis, this effect leads to the saturation of the asymptotic transmission group time, in the opaque-barrier limit.

Thus, in our approach the usual Hartman effect does not at all mean that a particle tunnels through the opaque potential barrier with a superluminal velocity. Rather, it means that the subensemble of tunneling particles is accelerated by the opaque barrier when the average distance between particles and the nearest boundary of the barrier equals to $l_0/2$. And what is important is that this acceleration does not lead to superluminal velocities of a particle.
One has to bear in mind that the accuracy of determining the coordinates of the leading and trailing fronts of the wave packet $\psi_{tr}(x, t)$ is proportional to its width. Thus, the wider is the packet, the larger is the size $L_{accel}$ of spatial intervals where the velocity of its CM grows. This means, in particular, that $L_{accel}$ grows in the opaque-barrier limit $d \to \infty$. This is so because the propagation of a quasi-monochromatic wave packet in the asymptotically large interval $[0, b_2 + \Delta X]$, in the opaque-barrier limit $d \to \infty$, implies the validity of the inequalities $a_1, \Delta X \gg l_0 \gg D$ (29). Thus, in this limit, the wave-packet’s width $l_0$ (and hence $L_{accel}$) grows together with $D = 2d + L$. That is, the opaque-barrier limit leads to the growth of the spatial and temporal scales of the curve $\bar{x}_{tr}(t)$, rather than to the unbounded growth of the average velocity of particles passing through the opaque barrier.

8 Conclusion

We showed that the superposition principle is violated in the quantum mechanical process of scattering a particle on a one-dimensional potential barrier as well as in the process of scattering the electromagnetic wave on a quasi-one-dimensional layered structure. The barrier and the structure, dividing the incident wave into two parts (transmitted and reflected), play the role of nonlinear elements in the corresponding scattering problems. Thus, both in QM and in CED, these scattering phenomena are nonlinear, in reality. As a consequence, the standard (linear) models of these two scattering processes are not adequate to them, which makes the study of the temporal aspects of each of these two processes an intractable problem. This explains why the TTP remains a controversial issue right up to our days.

By the example of scattering a quantum particle on a one-dimensional potential barrier we present a new, nonlinear model of this process. In particular, we find the (stationary) wave functions for the transmission and reflection subprocesses which allow one to study them at all stages of scattering. On their basis we define the dwell times and (“phase”) group times for each subprocess. As was shown, only the dwell time and the local group time, defined for the transmission subprocess, can be treated as the transmission (tunneling) time, that is, the time spent by a transmitted (tunnelled) particle in the barrier region. These times correlate with each other and do not lead to the Hartman paradox. As it follows from our model, a direct measurement of the tunneling time is impossible.

From our point of view the Hartman paradox stands alongside with the Schrödinger cat paradox and the mystery of the double-slit experiment. They have a common root cause. According to our approach, the Schrödinger cat paradox is not a measurement problem. It must be resolved at the
in quantum mechanics, the decay model of radioactive nuclei should be nonlinear, like the model of scattering of a quantum particle on a one-dimensional potential barrier.

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