Reduction of XXZ model with generalized periodic boundary conditions

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We examine the XXZ model with generalized periodic boundary conditions and identify conditions for the truncation of the functional fusion relations of the transfer matrix fusion. After the truncation, the fusion relations become a closed system of functional equations. The energy spectrum can be obtained by solving these equations. We obtain the explicit form of the Hamiltonian eigenvalues for the special case where the anisotropy parameter \( q^4 = -1 \).

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1 Introduction

The Hamiltonian of the anisotropic Heisenberg-Ising model, or the XXZ model, has the form

\[
H_{XXZ} = \sum_{n=1}^{N} \left( \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{q + q^{-1}}{4} \sigma_n^z \sigma_{n+1}^z \right).
\]

(1)

We focus on the case where the module of the anisotropy parameter \( q \) is equal to 1, i.e., \( q = e^{i \eta} \). We consider the generalized periodic (twisted) boundary conditions

\[
\begin{align*}
\sigma_{N+1}^+ &= q^{k \beta} \sigma_1^+ \\
\sigma_{N+1}^- &= \sigma_1^-.
\end{align*}
\]

(2)

The spin projection \( S^z = \frac{1}{2} \sigma_1^z + \cdots + \frac{1}{2} \sigma_N^z \) on the \( z \) axis is conserved, i.e., \([S^z, H_{XXZ}] = 0\). As established in [2], the energies of chains with different twist parameters are interconnected such that the energy spectrum in the chain with the twist parameter \( \beta \) from the sector with \( S^z = \beta - 1 \) contains the energy spectrum of the chain with the twist parameter \( \beta - n \) from the sector \( S^z = \beta - 1 + n \), where \( n \) is an integer, i.e.,

\[
E_{S^z=\beta-1}^{(\beta-1+n)} = E_{S^z=\beta-1}^{(\beta-n)}.
\]

(3)

This properly follows from the quantum group symmetry \( U_q(sl(2)) \). We say that "one spectrum contains the other" because the number of vectors in the sector \( S^z = \beta - 1 + n \) is smaller than in \( S^z = \beta - 1 \).

The XXZ model is connected with the two dimensional classical statistical lattice six vertex model (the ice model) [1]. The transfer matrix \( \hat{t}_{1/2}(u) \) of the six vertex model commutes with the Hamiltonian, \([\hat{t}_{1/2}(u), H_{XXZ}] = 0\), for all values of the spectral parameter \( u \). This matrix also commutes with itself, \([\hat{t}_{1/2}(u), \hat{t}_{1/2}(v)] = 0\), and is therefore a generating function of the commuting integrals of motion that are in involution. In particular, the Hamiltonian itself can be expressed through the logarithmic derivative of \( \hat{t}_{1/2}(u) \) as

\[
H_{XXZ} = -\frac{N}{2} \cos(\eta) + \sin(\eta) \frac{d}{du} \log \hat{t}_{1/2}(u)|_{u=0}.
\]

(4)

Formula (4) allows obtain the energy spectrum of the XXZ chain if the transfer matrix eigenvalues are known.

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There exists an infinite family of commuting transfer matrices $t_j(u)$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. These matrices satisfy the infinite system of recursive functional fusion relations (3):

$$
\hat{t}_{1/2}(u - (j + 1/2)\eta) \hat{t}_j(u) = t_0(u - (j + 1)\eta) \hat{t}_{j-1/2}(u + \eta/2) + t_0(u - j\eta) \hat{t}_{j+1/2}(u - \eta/2),
$$

(5)

where $t_0(u) = \sin^N(u + \eta/2)$. In this article, we prove that in the case where the anisotropy parameter is the root of unity, $q^{p+1} = -1$, the transfer matrix in the representations with the spin $j = p/2$ has zero eigenvalues on the Bethe eigenvectors if the twist parameter $\beta$ and the sector $S^2 = s$ satisfy the conditions

$$
(-1)^{2\beta} = (-1)^N, \quad |s| < \min(\beta, p + 1 - \beta) \mod (p + 1)
$$

(6)

Once the eigenvalue of $t_{p/2}(u)$ vanishes, the infinite system of functional fusion relations (3) is truncated and becomes a functional equation with respect to $t_{1/2}(u)$. Solving this functional equation, it is possible to obtain the $XXZ$ energy spectrum through formula (4). We let the symbol $V_{p,\beta-s}$ denote the set of the Bethe eigenvectors on which $t_{p/2}(u)$ has zero eigenvalues. In this article, $V_{p,\beta-s}$ is fully determined in the terms of the generators of the quantum algebra $U_q(sl(2))$. We call the $XXZ$ model reduced onto the set $V_{p,\beta-s}$ the reduced $XXZ$ model. The energy spectrum of the reduced $XXZ$ model can be obtained by solving the functional equations for the transfer matrix of the siz-vertex model.

In Sec. 2, we prove the existence of the Bethe vectors on which the functional fusion relations are truncated if conditions (6) are satisfied. All such vectors are identified in Sec. 3, where we define $V_{p,\beta-s}$ in terms of the generators of the quantum algebra $U_q(sl(2))$. In Appendix A, we derive the formula for the transfer matrix $t_{p/2}(u)$. The transfer matrix eigenvalues and the energy spectrum of the reduced $XXZ$ model can be obtained by solving the functional equations for the transfer matrix of the siz-vertex model.

## 2 Zero eigenvalues

We prove the existence of zero eigenvalues of the transfer matrix $t_{p/2}(u)$ assuming that $q^{p+1} = -1$ and conditions (6) are satisfied. The transfer matrix of the siz-vertex model corresponding to the $XXZ$ model Hamiltonian with twisted boundary conditions has the form

$$
\hat{t}_{1/2}(u) = \text{tr} \left( q^{-\beta\sigma^z} R_N(u) \ldots R_1(u) \right),
$$

(7)

where the $R$-matrix is

$$
R_n(u) = \left( \begin{array}{cc} \sin(u + \frac{1}{2}(1 + \sigma_n^z)\eta) & \sin(\eta)\sigma_n^- \\ \sin(\eta)\sigma_n^+ & \sin(u + \frac{1}{2}(1 - \sigma_n^z)\eta) \end{array} \right).
$$

(8)

The well-known procedure of the algebraic Bethe anatz (quantum inverse scattering method) leads to the transfer matrix eigenvalues expressed through the roots $\{v_m\}$ of the Bethe equations,

$$
t_{1/2}(u) = q^{-\beta}t_0(u + \eta/2) \prod_{m=1}^{n} \frac{\sin(u - v_m - \eta)}{\sin(u - v_m)} + q^{\beta}t_0(u - \eta/2) \prod_{m=1}^{n} \frac{\sin(u - v_m + \eta)}{\sin(u - v_m)},
$$

(9)

where $n = \frac{N}{2} - s$ is the number of the flipped spins in the sector $S^2 = s$, and $t_0(u)$ is the transfer matrix in the representation with the spin $j = 0$. The Bethe equations are equivalent to the conditions that the transfer matrix eigenvalues determined by Eq. (6) have no poles in the complex plane of the variable $u$. In the variables

$$
Q(u) = \prod_{m=1}^{n} \sin(u - v_m).
$$

(10)

Eq. (5) is equivalent to the Baxter $T$-$Q$-equation

$$
t_{1/2}(u)Q(u) = q^{-\beta}t_0(u + \eta/2)Q(u - \eta) + q^{\beta}t_0(u - \eta/2)Q(u + \eta).
$$

(11)

As shown in (3), it is convenient to interpret this equation as a discrete realization of a second-order differential equations. In additions to the functions $Q(u)$, this equation should then have a second linear
independent solution $P(u)$ with the same eigenvalue of the transfer matrix $t(u)$. The transfer matrix eigenvalues can be expressed through the eigenvalues of the Baxter operator $Q(u)$ and the operator $P(u)$

$$t_j(u) = q^{2j-1}f(u-(j-1/2)\eta) \times$$

\begin{align*}
&\times [Q(u-(j+1/2)\eta)P(u+(j+1/2)\eta) - \\
&- Q(u+(j+1/2)\eta)P(u-(j+1/2)\eta)],
\end{align*}

where the function $f(u)$ is quasiperiodic,

$$f(u+\eta) = q^{-2\beta}f(u).$$

This function is introduced for convenience (the function $\tilde{P}(u) = f(u)P(u)$ could be introduced instead). Because the functions $t_j(u)$ satisfy the same functional fusion relation (13), we conclude that these functions are some eigenvalues of the corresponding transfer matrices $\tilde{t}_j(u)$.

We now consider the case $q^{p+1} = -1$, where $p$ is a natural number. For the transfer matrix in the representation with the spin $j = p/2$, we have

$$t_{p/2}(u + \pi/2) = (-1)^3 f(u)[P(u+\pi)Q(u) - P(u)Q(u+\pi)].$$

It then follows that if the functions $Q(u)$ and $P(u)$ satisfy the relation

$$\frac{P(u+\pi)}{P(u)} = \frac{Q(u+\pi)}{Q(u)},$$

then the eigenvalue of the transfer matrix $t_{p/2}(u)$ is zero.

By analogy with the procedure in [10], we can express the function $P(u)$ through the function $Q(u)$. We use the expansion

$$\frac{t_0(u)}{Q(u+\eta/2)Q(u-\eta/2)} = R(u) + q^\beta \frac{A(u+\eta/2)}{Q(u+\eta/2)} - q^{-\beta} \frac{A(u-\eta/2)}{Q(u-\eta/2)},$$

where $R(u)$ is a trigonometric polynomial of the order $N - 2n$, uniquely determined through the known trigonometric polynomials $t_0(u)$ and $Q(u)$, and the order of the polinomial $A(u)$ is smaller than $n$. Further, we use another expansion,

$$R(u) = q^\beta F(u+\eta/2) - q^{-\beta} F(u-\eta/2).$$

It is now easy to verify that the function $P(u)$ is determined by the expression

$$P(u) = \frac{1}{f(u)}(Q(u)F(u) + A(u)).$$

This expression implies the following constraint imposed on the values of the twist parameter $\beta$ in the boundary conditions of the $XXZ$ model:

$$(-1)^{2\beta} = (-1)^N. \quad (19)$$

This means that if the length of the spin chain of atoms is even, then the number $\beta$ must be integer, and if the length is odd, the number must be half-integer. No truncation of the functional relations occurs for other values of $\beta$. Representation (13) of the function $R(u)$ and $F(u)$ in terms of the functions $F(u)$ in which $F(u)$ behaves exactly as $R(u)$ under the argument shift $u \rightarrow u + \pi$ exists if the inequality

$$\prod_{m=-s}^{+s} \sin(\pi \frac{m + \beta}{p + 1}) \neq 0,$$

holds, because this product appears in the denominator if $F(u)$ is expanded in the Fourier series. The variable $m$ takes either integer or half-integer values from $-s$ to $+s$. Depending on whether the chain length is even or odd, the numbers $\beta$ and $s$ are either both integer or both half-integer, and their sum or difference is therefore always integer. Condition (20) is satisfied if

$$|s| < \min(\beta, p + 1 - \beta) \mod (p + 1). \quad (21)$$

Hence, analyzing the second linearly independent solution of the Baxter equation, we conclude that for the anisotropy parameter equal to the root of unity, $q^{p+1} = -1$, the transfer matrix with the spin $j = p/2$ in the sector $S^z = s$ has zero eigenvalues if conditions (13) and (21) are satisfied.
3 Algebraic structure

Above, we proved the existence of vectors such that functional relations (5) restricted to these vectors are truncated and have form of a closed system of equations for the eigenvalues of the transfer matrix $t_{1/2}(u)$. In this section, we find all vectors from the space of states of the model that satisfy this condition.

As mentioned above, the connection between the energies of the chains with different twist parameters (3) exists because of the presence of the quantum group symmetry $U_q(sl(2))$. The generators of this symmetry are the operators $S^\pm$, $X$ and $X^t$, where

$$X = \sum_{n=1}^{N} q^{\frac{1}{2}(\sigma_1^+ + \cdots + \sigma_{N-1}^+ + \sigma_n^- + \cdots + \sigma_N^-)}$$  \hspace{1cm} (22)

It is easy to verify that $X^{p+1} = 0$ for $q^{p+1} = -1$.

The operator $X$ acting on a vector from the sector $S^z = s$ transforms this vector to a vector from sector $S^z = s+1$: $[S^z, X] = X$. Acting with this operator on the Hamiltonian, we obtain

$$X \left( H_{XXZ} - H_{XXZ}^{(\beta-1)} \right) X = (\ldots)(1 - q^{2(S^z - \beta+1)}).$$  \hspace{1cm} (23)

In this formula and in Eq. (22) below, the dots stand for a finite number of operators whose explicit form is cumbersome and irrelevant to this consideration. Formula (23) proves relation (3). A similar relation holds for the transfer matrix.

The six vertex model transfer matrix related to the XXZ Hamiltonian is expressed through the matrix elements of the monodromy matrix $L(u) = R_N(u) \ldots R_1(u)$,

$$t_j(u) = \sum_{n=1}^{2j+1} q^{-2\beta(j+1-n)L_n(u)}.$$  \hspace{1cm} (24)

The $R$-matrix in the representation $\frac{1}{2} \otimes j$ on lattice site with the number $a$ has the form

$$R_a(u) = \begin{pmatrix} \sin(u + (\frac{1}{2} + \hat{H}_a)\eta) & \sin(\eta)\hat{F}_a \\ \sin(\eta)\hat{E}_a & \sin(u + (\frac{1}{2} - \hat{H}_a)\eta) \end{pmatrix}.$$  \hspace{1cm} (25)

The operators $\hat{E}$, $\hat{F}$, and $\hat{H}$ have the matrix elements

$$\pi_j(\hat{H})_m^n = (j + 1 - n) \delta_{m,n}, \hspace{0.5cm} m, n = 1, 2, \ldots, 2j + 1.$$  \hspace{1cm} (26)

where $\omega_n = \sqrt{[n]_q[2j+1-n]_q}$. In this formula, we use the standard notation $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. In particular, the operator $\hat{H}$ for $j = 1/2$ is $\frac{1}{4}\sigma^z$, and the operators $E$ and $F$ are respectively $\sigma^+$ and $\sigma^-$. (i.e., the generators of the quantum algebra $U_q(sl(2))$: $[\hat{H}, \hat{E}] = \hat{E}$, $[\hat{H}, \hat{F}] = -\hat{F}$, $[\hat{E}, \hat{F}] = [\hat{2}\hat{H}]_q$).

Using the Yang-Baxter equation

$$(R_{\frac{1}{2} \otimes j})^{ln}_{bm}(u-v)(L_{\frac{1}{2}})^{jh}_{lk}(u)(L_{1/2})^{nk}_{km}(v) = (L_{1/2})^{ln}_{bm}(u)(L_{1/2})^{jh}_{lk}(v)(R_{\frac{1}{2} \otimes j})^{bm}_{ln}(u-v),$$  \hspace{1cm} (27)

as in (8), we obtain the commutation relations between the matrix elements of the monodromy matrix and the operators $S^z$ and $X$,

$$q^{S^z L_k^n} = q^{n-k}L_k^n q^{S^z},$$  \hspace{1cm} (28)

$$XL_k^n(u) = q^{2(j+1-n-k)L_k^n(u)}X + \omega e^{+iu}q^{j+1-n}L_{k-1}^n(u)q^{-S^z} - \omega e^{-iu}q^{j+1-k}L_{k+1}^n(u)q^{+S^z}.$$  \hspace{1cm} (29)

To obtain commutation relations (28), we consider the monodromy matrix $(L_{1/2})_k^n(u)$ as $u \to -i\infty$. The matrix element $(L_{1/2})^{jh}_{lk}(u)$ is proportional to the operator $X$. We then pass to the limit $u \to -i\infty$ in Yang-Baxter equation (27). Commutation relations (28) allow expressing the transfer matrix itself through the operator $X$. The formula for the transfer matrix with the spin $j = p/2$ in the sector $S^z = s$ is (see Appendix A)

$$t_{p/2}(u) = \sum_{r,k=0}^{P} X^r \hat{M}_{rk} X^k,$$  \hspace{1cm} (29)
where

\[
\tilde{M}_{rk} = (-1)^r \omega^{-1} e^{-ipu} q^{-2s-(2\beta+1-r-k)p/2-2k} \times \sum_{n=0}^{m-p-r-k} q^{(2\beta-s+r-k)n} C^n_k C^{-n}_k \times \sum_{m=0}^{m-r-k-m} \lambda_{m-p-r-k}(L_{p/2})(u)^{1}_{p-r-k-m} X^{p-r-k-m},
\]

\(\omega = \prod_{k=1}^{2j} \omega_k\), the numbers \(C_N^N\) are the \(q\)-binomial coefficients

\[C_N^N = \frac{[N]_q!}{[n]_q! [N-n]_q!},\]

and the variables \(\lambda_{m-p-r-k}\) are determined by the formula

\[\lambda_{m,m'} = e^{imu} q^{p/2+1+s+(m'-m)(m'+s-3p/2-1)} C^{m'}_m \prod_{l=1}^{m} \omega_{m'-l}.\]

As can be seen from formula (30), the transfer matrix \(t_{p/2}(u)\) is expressed through of monomials of the form \(X^r \tilde{M}_{rk} X^k\).

We now investigate the numbers \(r\) and \(k\) for which the operator \(\tilde{M}_{rk}\) is nonzero in the case where \(q^{p+1} = -1\). The operator \(\tilde{M}_{rk}\) is proportional to the sum of the \(q\)-binomial coefficients

\[f_{rk} = \sum_{n=0}^{m} C^n_r C^{-n}_k q^{n(2l+r-k)},\]

where \(l = \beta - s\). Unfortunately, we failed to obtain an expression for the coefficients \(f_{rk}\) in a closed form. We therefore analyzed sum (31) numerically. We verified the hypothesis that for \(q^{p+1} = -1\), the coefficients \(f_{rk}\) are nonzero only in the closed domain \(r + k \leq p\) and \((r > p - l \text{ or } k > l - 1)\).

Hence, the transfer matrix \(t_{p/2}(u)\) can be expressed as

\[t_{p/2}(u) = X^{p+1-l}(\ldots) + (\ldots) X^l\]

and therefore vanishes on the cohomologies

\[V_{p,l} = \text{Ker } X^l / \text{Im } X^{p+1-l}.\]

This means that \(t_{p/2}(u)\) has zero eigenvalues on the eigenvectors \(v\) form the sector \(S^z = s\) that are annihilated by the operator \(X^l\),

\[X^l v = 0,\]

and cannot be expressed in the form \(X^{p+1-l} \chi\), where \(\chi\) is again vector,

\[v \neq X^{p+1-(\beta-s)} \chi.\]

We have thus identified all vectors on which the transfer matrix \(t_{p/2}(u)\) has zero eigenvalues and, as mentioned above, recursive system (33) takes the form of a closed system of equations on the eigenvalues of the transfer matrix \(t_{1/2}(u)\).

An example illustrating how the truncated functional relations can be used is given in Appendix B. Specifically, the eigenvalues of the transfer matrix and the Hamiltonian are obtained there for the case \(q^4 = -1\).
Formula (29) can be directly derived by using commutation relations (28), but this way is rather wearisome. The difficulties are connected with the action of the operator $X$ on $L_n^k(u)$ resulting in the appearance of two terms in the right-hand side, namely, $L_{n-1}^k(u)$ and $L_n^{k+1}(u)$. The calculation, however, can be simplified. We note that the commutation relations for $X$ and $L_n^k(u)$ are formally equivalent to the commutation relations for $X$ and the auxiliary operator $\psi^n(u)q^{-nS^z}\psi_m(u)q^{-mS^z}$, where the auxiliary objects $\psi$ commute with $X$ according to the rule

$$
X\psi^n_j - q^{j+1-2n}\psi^n_j X = -\omega_ne^{iu}\psi^n_{j+1}
$$

$$
X\psi^n_j - q^{j+1-2n}\psi^n_j X = \omega_ne^{iu}\psi^n_{j-1}
$$

$$
q^S\psi^n_j = q^{n-j-1}\psi^n_{j}q^{S^z}
$$

$$
q^S\psi^n_j = q^{j+1-n}\psi^n_{j}q^{S^z}
$$

With this definition of the auxiliary objects $\psi$, the correspondence rule

$$(L_j)_n^k(u) \sim \psi^n_j(u) q^{-nS^z}\psi_j^L(u) q^{-kS^z},$$

(37)

can be formulated. The symbol $\sim$ means that the commutation relations for the variables $X$ and $F(L_n^k)$, where $F$ is an arbitrary function, are exactly the same as those for $X$ and $F(\psi^n(u)q^{-nS^z}\psi_k(u)q^{-kS^z})$. The commutation relations for the latter variables, however, are easier to obtain because commutation relations (36) are simpler than relations (38). Hence, the symbol $\sim$ in the Eq. (37) should be interpreted as "transforms as". Hereinafter, we use the equality sign instead of the symbol $\sim$ for simplicity.

Relations (38) can be rewritten as

$$
\psi^n_j = -(\omega_{n-1}q)^{-1}e^{-iu}(X\psi^{n-1}_j - q^{j+1-2(n-1)}\psi^{n-1}_j X)
$$

$$
\psi^n_j = (\omega_{n+1})^{-1}e^{-iu}(X\psi^{n+1}_j - q^{j+1-2(n+1)}\psi^{n+1}_j X).
$$

These simple recursive relations allow expressing $\psi^n$ through $\psi^1$ and $\psi^n$ through $\psi_{2j+1}$. The solution of thes equations have the forms

$$
\psi^n_j = a_n \sum_{r=0}^{n-1}(-1)^r q^{r(j+1-n)}C^1_{n-1}X^{n-1-r}\psi^1_X^r
$$

$$
\psi^n_j = b_n \sum_{k=0}^{2j+1-n}(-1)^k q^{-k(j+1+n)}C^2_{n-1}X^{2j+1-n-k}\psi_{2j+1}X^k
$$

(39)

where the coefficients are

$$
a_n = \prod_{k=1}^{2j+1-n}(-e^{iu}\omega_{2j+1-n-k})^{-1}
$$

$$
b_n = \prod_{k=1}^{2j+1-n}(e^{-iu}\omega_{2j+1-n-k}).
$$

(40)

The transfer matrix is

$$
t_j(u) = \sum_{n=1}^{2j+1} q^{-2\beta(j+1-n)}(L_j)_j^n(u)
$$

$$
= \sum_{n=1}^{2j+1} q^{-2\beta(j+1-n)}\psi^n_j(u)q^{-nS^z}\psi^n_j(u)q^{-nS^z}.
$$

(41)

According to formulas (39), we obtain

$$
t_{p/2}(u) = \omega^{-1}e^{-i\mu q^{-2s-p(2\beta+1)/2}} \sum_{n=0}^{p} \sum_{r=0}^{p-n} (-1)^r q^{-pr/2-(p/2+2)k}
$$

$$
\times q^{(2(\beta-s)+r-k)}C^p_{n}C^p_{n}X^{r} (\psi^1X^{p-r-k}\psi_{p+1}) X^k.
$$

(42)

It remains to remove the auxiliary operators $\psi^1$ and $\psi_{p+1}$ that participate in the formula in the form $\psi^1X^{p-r-k}\psi_{p+1}$. To do this, we commute the operators $X^{p-r-k}$ and $\psi_{p+1}$. Using (36), it is easy to prove
by the induction that
\[ \psi_j^A X^M \psi_B = \sum_{m=0}^{M} e^{im\beta q} A^{(M-m)(j+1-m-2B)+A(j+1-B+m)} C_m \]
\[ \times \left( \prod_{l=1}^{m} \omega_{B-l} \right) L(u)^{A\beta-m} q^{(A+B-m)S} X^{M-m}. \] (43)

Using this formula and taking the relation \([S^x, X] = X\) into account, we obtain expression (29) for \(t_{p/2}(u)\), only \(X\) and \(L\) participate in the final formula; the auxiliary objects \(\psi\) are no longer needed.

5 Appendix B

We obtain the eigenvalues of the transfer matrix and the Hamiltonian for \(q^4 = -1\). We have
\[ t_{3/2}(u) = 0, \] (44)
\[ t_{1-\beta}(u) = -(-1)^{N+\beta-s} t_{\beta}(u + \pi/2). \]

To simplify the calculation, we introduce the function \(S_N(u) \equiv (-2)^{N/2} t_{1/2}(u - \beta/2)\). The functional relations become
\[ S_N(u + \frac{\pi}{8}) S_N(u - \frac{\pi}{8}) = \cos^N(2u) - (-1)^{N+1} \sin^N(2u). \] (45)

Equation (43) contains the numbers \(N\) and \(l = \beta - s\). We obtain four different equations, depending on whether these numbers are even or odd. Correspondingly, we obtain four different solutions:
\[ S_{4M}^{(n)}(u) = \prod_{m=1}^{M} (1 + (-1)^{N_m} \cos(\frac{2m - 1}{4M}) \cos(4u)), \] (46)
\[ S_{4M+2}^{(n)}(u) = \sin(2u) \sqrt{2} \prod_{m=1}^{M} (1 + (-1)^{N_m} \cos(\frac{2m}{4M + 2}) \cos(4u)), \]
\[ S_{4M-1}^{(n)}(u) = e^{-iu} (-i)^{M-1/2} \times \]
\[ \prod_{m=1}^{2M-1} \left( e^{(-1)^{N_m} \frac{\pi mn}{4M}} \sin(2u + \frac{\pi}{4}) - i e^{(-1)^{N_m} \frac{\pi mn}{4M}} \sin(2u - \frac{\pi}{4}) \right), \]
\[ S_{4M+1}^{(n)}(u) = e^{-iu} (-i)^M \times \]
\[ \prod_{m=1}^{2M} \left( e^{(-1)^{N_m} \frac{\pi mn}{4M}} \sin(2u + \frac{\pi}{4}) - i e^{(-1)^{N_m} \frac{\pi mn}{4M}} \sin(2u - \frac{\pi}{4}) \right). \]

In this formula, the set \(\{n_m\}\) is any set of numbers equal to either 0 or 1, for example, \(\{0, 1, 0, \ldots, 0, 1\}\). We interpret them as the fermion occupation numbers. The functions written above satisfy the equation
\[ S_{4M+\delta}(u + \frac{\pi}{8}) S_{4M+\delta}(u - \frac{\pi}{8}) = \cos^{4M+\delta}(2u) + i^{\delta} \sin^{4M+\delta}(2u). \] (47)

Hence, we obtained the solutions only for the case where \(\beta - s\) is an odd number. For the energy eigenvalues, we obtain
\[ E_{4M}^{(n)} = -\frac{4M}{2\sqrt{2}} \times \frac{4}{\sqrt{2}} \sum_{m=1}^{M} (-1)^{N_m} \cos(\frac{2m - 1}{4M}), \] (48)
\[ E_{4M+2}^{(n)} = -\frac{4M + 2}{2\sqrt{2}} \times \frac{2}{\sqrt{2}} \left( 1 - 2 \sum_{m=1}^{M} (-1)^{N_m} \cos(\frac{m}{4M + 1}) \right), \]
\[ E_{4M-1}^{(n)} = -\frac{4M - 1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} \left( i + 2i \sum_{m=1}^{2M-1} \exp(\frac{-2i(-1)^{N_m} \pi m}{4M - 1}) \right), \]
\[ E_{4M+1}^{(n)} = -\frac{4M + 1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} \left( i + 2i \sum_{m=1}^{2M} \exp(\frac{-2i(-1)^{N_m} \pi m}{4M + 1}) \right). \]

Although the imaginary unit explicitly participates in the last two formulas, the energy \(E\) is real in all the cases because the imaginary parts cancel each other in the sum. For spin chains whose length are even, \(N = 4M\) and \(N = 4M + 2\), we obtain \(2^M\) energy levels. For spin chains whose length are odd, \(N = 4M - 1\) and \(N = 4M + 1\), we respectively obtain \(2^{2M-1}\) and \(2^{2M}\) eigenvalues.
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