On the long time behavior of stochastic Schrödinger evolutions

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We discuss the time evolution of the wave function which is solution of a stochastic Schrödinger equation describing the dynamics of a free quantum particle subject to spontaneous localizations in space. We prove global existence and uniqueness of solutions. Observing that there exist three time regimes, namely the collapse regime, after which the wave function is localized in space; the classical regime, during which the collapsed wave function moves along a classical path and the diffusive regime, in which diffusion overlaps significantly the deterministic motion we study the long time behavior of the wave function. We assert that the general solution converges a.s. to a diffusing Gaussian wave function having a finite spread both in position as well as in momentum. This paper corrects and completes earlier works on this.

I. INTRODUCTION

Stochastic differential equations (SDEs) in infinite dimensional spaces are a subject of growing interest within the mathematical physics and physics communities working in quantum mechanics; they are currently used in models of spontaneous wave function collapse [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], in the theory of continuous quantum measurement [15, 16, 17, 18, 19, 20, 21, 22, 23], and in the theory of open quantum systems [24, 25, 26]. In the first case, the Schrödinger equation is modified by adding appropriate non-linear and stochastic terms which induce the (random) collapse of the wave function in space; in this way, one achieves the goal of a unified description of microscopic quantum phenomena and macroscopic classical ones, avoiding the occurrence of macroscopic quantum superpositions. In the second case, using the projection postulate, stochastic terms in the Schrödinger equation are used to describe the effect of a continuous measurement. In
the third case, slightly generalising the notion of continuous measurement to generic interactions with environments, SDEs are used as phenomenological equations describing the interaction of a quantum system with an environment, the stochastic terms encoding the effect of the environment on the system. Looking directly at the stochastic differential equation for the wave function, rather than the deterministic equation of the Lindblad type for the statistical operator has some advantages with respect to the standard master equation approach, e.g. for faster numerical simulations [27].

Among the different SDEs which have been considered so far, the following equation, defined in the Hilbert space $\mathcal{H} \equiv L^2(\mathbb{R})$, is of particular interest [16, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]:

$$d\psi_t = \left[ -\frac{i}{\hbar} \frac{p^2}{2m} dt + \sqrt{\lambda} (q - \langle q \rangle_t) dW_t - \frac{\lambda}{2} (q - \langle q \rangle_t)^2 dt \right] \psi_t, \quad \psi_0 = \psi. \quad (1)$$

The first term on the right-hand-side represents the usual quantum Hamiltonian of a free particle in one dimension, $p$ being the momentum operator. The second and third terms of the equation, as we shall see, induce the localization of the wave function in space; $q$ is the position operator and $\langle q \rangle_t$ denotes the quantum expectation $\langle \psi_t | q | \psi_t \rangle$ of $q$ with respect to $\psi_t$. The parameter $\lambda$ is a fixed positive constant which sets the strength of the collapse mechanism, while $W_t$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t, t \geq 0\}$.

Eq. (1) plays a special role among the SDEs in Hilbert spaces because it is the simplest exactly solvable equation describing the time evolution of a non-trivial physical system. Within the theory of continuous quantum measurement, it describes a measurement-like process designed to measure the position of a free quantum particle; within decoherence theory it represents one of the possible unravellings of the master equation first derived by Joos and Zeh [38]. Within collapse models (like GRW-models), it may describe the evolution of a free quantum particle (or the center of mass of an isolated system) subject to spontaneous localizations in space [1], [2] in the following sense. Realistic models of spontaneous wave function collapse are based on a more complicated stochastic differential equation: The difference between Eq. (1) and the equations of the standard localization models such as GRW [1] and CSL [2] is most easily described on the level of the Lindblad equations for the respective statistical operators $\rho_t := \mathbb{E}_\rho[|\psi_t\rangle\langle\psi_t|]$, induced by the stochastic dynamics of the wave function. By virtue of Eq. (1) (see e.g. [9]):

$$\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [p^2, \rho_t] - \frac{\lambda}{2} [q, [q, \rho_t]], \quad (2)$$

with the “Lindblad term” in position representation

$$\frac{\lambda_{\text{GRW}} \alpha}{4} (x - y)^2 \rho_t(x, y). \quad (3)$$
For the GRW dynamics as described in [1] the corresponding Lindblad term of the GRW master equation in the position representation reads:

$$-\lambda_{GRW} \left[ 1 - e^{-\alpha(x-y)^2/4} \right] \rho_t(x,y).$$

(4)

When the distances involved are smaller than the length $1/\sqrt{\alpha} \simeq 10^{-5}$ cm characterizing the model we have that

$$-\lambda_{GRW} \left[ 1 - e^{-\alpha(x-y)^2/4} \right] \simeq \frac{\lambda_{GRW} \alpha}{4} (x-y)^2 \quad \text{for: } |x-y| \ll 1/\sqrt{\alpha}. \tag{5}$$

Accordingly, the stochastic dynamics of Eq. (1) approximates—at least on the statistical level—the GRW dynamics for all atomic and subatomic distances. Since this is a regime of growing interest [39, 40, 41, 42] it is reasonable to study now first the simpler equation Eq. (1).

Eq. (1) is non-linear. Non-linearity is a fundamental ingredient because only in this way it is possible to reproduce the collapse of the wave function. It is well known how to “linearize” the equation, i.e. how to express its solutions as a function of the solutions of a suitable linear SDE [29, 43]. We briefly review this procedure.

Let us consider the following linear SDE:

$$d\phi_t = \left[ -i \frac{p^2}{2m} dt + \sqrt{\lambda} q d\xi_t - \frac{\lambda}{2} q^2 dt \right] \phi_t, \quad \phi_0 = \phi, \tag{6}$$

defined in the same Hilbert space $\mathcal{H} \equiv L^2(\mathbb{R})$; the stochastic process $\xi_t$ is a standard Wiener process with respect to the probability space $(\Omega, \mathcal{F}, Q)$ and filtration $\{\mathcal{F}_t, t \geq 0\}$, where $Q$ is a new probability measure whose relation with $P$ will soon be established. This equation does not conserve the norm of the state vector, as the evolution is not unitary; we therefore introduce the normalized state vectors:

$$\psi_t = \begin{cases} \phi_t/\|\phi_t\| & \text{if: } \|\phi_t\| \neq 0, \\ 0 & \text{otherwise}; \end{cases} \tag{7}$$

A standard application of Itô calculus shows that, if $\phi_t$ solves Eq. (6), then $\psi_t$ defined in (7) solves the following non-linear SDE:

$$d\psi_t = \left[ -i \frac{p^2}{2m} dt + \sqrt{\lambda}(q - \langle q \rangle_t)(d\xi_t - 2\sqrt{\lambda}\langle q \rangle_t dt) - \frac{\lambda}{2}(q - \langle q \rangle_t)^2 dt \right] \psi_t, \tag{8}$$

for the same initial condition $\psi = \phi$.

Eq. (8) is a well defined collapse equation, however it is not suitable for physical applications, as the collapse does not occur with the correct quantum probabilities. This can be seen by analyzing
the time evolution of particular solutions, such as Gaussian wave functions; it can also be easily understood by noting that there is no fundamental difference between Eq. (8) and Eq. (6), since any solution of Eq. (8) can be obtained from a solution of Eq. (6) simply by normalizing the wave function. In turn, Eq. (6) does not contain any information as to why the wave function should collapse according to the Born probability rule, i.e. the Wiener process $\xi_t$ is not forced to pick most likely those values necessary to reproduce quantum probabilities, during the collapse process.

The way to include such a feature into the dynamical evolution of the wave function is to replace the measure $Q$ with a new measure (which will turn out to be the measure $P$ previously introduced) so that the process $\xi_t$, according to the new measure, is forced to take with higher probability the values which account for quantum probabilities. This is precisely the key idea behind the original GRW model of spontaneous wave function collapse [1]: the wave function is more likely to collapse where it is more appreciably different from zero. The mathematical structure of the GRW model suggests that the square modulus $\|\phi_t\|^2$ should be used as density for the change of measure. We now formalize these steps.

In [20], Holevo has proven that $\|\phi_t\|^2$ is a martingale satisfying the equation:

$$\|\phi_t\|^2 = \|\phi_0\|^2 + 2\sqrt{\lambda} \int_0^t \langle q \rangle_s \|\phi_s\|^2 d\xi_s; \quad (9)$$

when $\|\phi_0\|^2 = 1$, and from now on we will always assume that this is the case, $\|\phi_t\|^2$ can be used as a Radon-Nikodym derivative to generate a new probability measure $P$ from $Q$, according to the usual formula:

$$P[E] := EQ[1_E\|\phi_t\|^2] \quad \forall \ E \in \mathcal{F}_t \quad \forall \ t < +\infty, \quad (10)$$

where $1_E$ is the indicator function relative to the measurable subset $E$. The martingale property, together with the property $EQ[\|\phi_t\|^2] = 1$, guarantee consistency among different times, so that (10) defines a unique probability measure $P$. In the following, for simplicity we will write $dP/dQ \equiv \|\phi_t\|^2$.

One can then show that Eq. (8), with the stochastic dynamics defined on the probability space $(\Omega, \mathcal{F}, P)$ in place of $(\Omega, \mathcal{F}, Q)$, correctly describes the desired physical situations.

A drawback of the change of measure is that the equation is defined in terms of the stochastic process $\xi_t$, which is not anymore a Wiener process with respect to the measure $P$, as it was with respect to the measure $Q$. This can be a source of many difficulties, e.g. when analyzing the properties of the solutions of the equation. The disadvantage can be removed by resorting to Girsanov’s theorem, which connects Wiener processes defined on the same measurable space, but
with respect to different probability measures. According to this theorem, the process
\[ W_t := \xi_t - 2\sqrt{\lambda} \int_0^t \langle q \rangle_s ds, \tag{11} \]
is a Wiener process with respect to \((\Omega, \mathcal{F}, \mathbb{P})\) and filtration \(\{\mathcal{F}_t, t \geq 0\}\), and thus is the natural
process for describing the stochastic dynamics with respect to the measure \(\mathbb{P}\). It is immediate to
see that, once written in terms of \(W_t\), Eq. (8) reduces to Eq. (1), thus the link between Eq. (6)
and (1) is established. The above discussion should also have given a first idea of why SDEs like
Eq. (1) are those which are used in Quantum Mechanics to described the collapse of the wave
function; we will come back on this point later in the paper.

The first important problem to address concerns the status of the solutions of Eq. (6). In \cite{29},
Holevo has proven the existence and uniqueness of topological weak solutions of a rather general
class of SDEs with unbounded operators, to which Eq. (6) belongs. (See the end of the section
for the notation.) The problem of the existence and uniqueness of topological strong solutions
of Eq. (6) has been addressed in \cite{28}; there however, the proof relies on the expansion of wave
functions in terms of Gaussian states, which in general is problematic and requires special care, as
shown in \cite{44}. An explicit representation of the strong strong solution of Eq. (6) has been given
in \cite{35}; the representation is written in terms of path integrals and is not particularly suitable for
analyzing the time evolution of the general solution. A much more convenient representation, given
in terms of the Green’s function of Eq. (6), has been first derived in \cite{30,33}; the Green’s function
reads:
\[ G_t(x,y) = K_t \exp \left[ -\frac{\alpha_t}{2} (x^2 + y^2) + \beta_t xy + \overline{a}_t x + \overline{b}_t y + \overline{c}_t \right]; \tag{12} \]
the coefficients \(K_t, \alpha_t\) and \(\beta_t\) are deterministic and equal to
\[ K_t = \sqrt{\frac{\lambda}{v \pi \sinh vt}}; \tag{13} \]
\[ \alpha_t = \frac{2\lambda}{v} \coth vt; \tag{14} \]
\[ \beta_t = 2 \frac{\lambda}{v} \sinh^{-1} vt; \tag{15} \]
while the remaining coefficients are functions of the Wiener process \(\xi_t\):
\[ \overline{a}_t = \sqrt{\lambda} \sinh^{-1} vt \int_0^t \sinh vs \, d\xi_s; \tag{16} \]
\[ \overline{b}_t = 2ih \frac{\lambda}{m} \int_0^t \frac{\overline{a}_s}{\sinh vs} \, ds; \tag{17} \]
\[ \overline{c}_t = \frac{ih}{m} \int_0^t \overline{a}_s^2 \, ds. \tag{18} \]
In the above expressions, we have introduced the following two constants:

\[ \nu \equiv \frac{1 + i}{2} \omega, \quad \omega \equiv 2 \sqrt{\frac{\hbar \lambda}{m}}. \]  

(19)

As we shall see, the parameter \( \omega \), which has the dimensions of a frequency, will set the time scales for the collapse of the wave function. The representation in terms of the Green’s function (12), as we said, is particularly suitable for analyzing the time evolution of the general solution of Eq. (6), and thus of Eq. (1), even though we will see that, when studying the long time behavior, another representation is more convenient.

Our first result concerns the meaning of the solution of Eq. (6) in terms of

\[ \phi_t(x) := \int dy G_t(x, y) \phi(y) \]  

(20)

for given initial condition \( \phi \).

**Theorem 1 (Solution):** let \( \phi_t \) be defined as in (20); then the following three statements hold true with \( \mathbb{Q} \)-probability 1:

1. \( \phi \in L^2(\mathbb{R}) \Rightarrow \phi_t \in L^2(\mathbb{R}) \),

(21)

2. \( \phi \in L^2_B(\mathbb{R}) \Rightarrow \phi_t \) is a topological strong solution of Eq. (6),

(22)

3. \( \phi \in L^2(\mathbb{R}) \Rightarrow \lim_{t \to 0} \| \phi_t - \phi \| = 0 \),

(23)

where \( L^2_B(\mathbb{R}) \) is the subspace of all \textit{bounded} functions of \( L^2(\mathbb{R}) \).

Having the explicit solution of the Eq. (6), and thus of Eq. (1), the next relevant problem is to unfold its physical content. Previous analysis of similar equations \([2, 8, 10, 14, 36]\) have shown that one can identify three regimes, which are more or less well separated depending on the value of the parameters \( \lambda \) and \( m \).

1. **Collapse regime:** A wave function having an initial large spread, localizes in space, the localization occurring in agreement with the Born probability rule.

2. **Classical regime:** The localized wave function moves in space like a classical free particle, since the fluctuations due to the Wiener process can be safely ignored.

3. **Diffusive regime:** Eventually, the random fluctuations become dominant and the wave function starts to diffuse appreciably.
It is not an easy task to spell out rigorously these regimes and their properties. We shall however be a bit more specific on this in the following section. We shall afterwards focus on the simplest regime, namely the diffusive one, which in fact has been intensively looked at in the previous years [7, 17, 24, 32, 33, 36] and we shall prove a remarkable property of the solutions of Eq. (1):

Any solution converges almost surely to a Gaussian state wave function having a fixed spread.

**Theorem 2 (Large time behavior):** let $\psi_t$ be a solution of Eq. (1); then under conditions which we will specify, the following property holds true with $P$-probability 1:

$$\lim_{t \to \infty} \|\psi_t - \psi_t^\infty\| = 0,$$

where $\psi_t^\infty$, defined in (116), is a Gaussian wave function with a fixed spread both in position and momentum.

Theorem 1 and Theorem 2 have been extensively discussed before in the literature [7, 17, 24, 28, 32, 33, 34, 36], proving that the community has devoted much attention to the problem. However, these proofs are not complete or flawed. Concerning Theorem 1, in particular Statement 3 was not proven [28, 32, 33, 34]. While Statements 1 and 2 are rather straightforward conclusions from the Gaussian kernel of the propagator, the third Statement is much more subtle and does not follow from purely analytical arguments. Concerning Theorem 2, none of the previous proofs is decisive. In [33, 34], the major flaw was that it was overlooked that the eigenfunction expansion of the relevant dissipative operator (not self-adjoint) does not give rise to an orthonormal basis. In [17], the long time behavior was analyzed by expanding the general solution in terms of coherent states, while in [24, 36] it was analyzed by scrutinizing the time evolution of the spread in position of the solution; in [44] it has been shown that both approaches are not conclusive. Finally, [7] proposed Theorem 2 as a conjecture, but shows stability of $\psi_t^\infty$ only against small perturbations. Building on previous work of Holevo, Mora and Rebolledo recently enhanced in [45] and [46] the general theory of stochastic Schrödinger equations. In particular they developed criteria for the existence of regular invariant measures for a large class of stochastic Schrödinger equations as an important step towards an understanding of the large time behavior. Until now however the only complete and detailed results on the large time behavior seem to be Theorems 1 and 2.

We conclude this introductory section by summarizing the content of the paper. In Sec. II we will present a qualitative analysis of the time evolution of the general solution of Eq. (1); we will discuss the three regimes previously introduced, giving also numerical estimates, and we will set the main problems which we aim at solving. In Sec. III we will analyze the structure of the Green’s
function (12) and prove theorem 1. In Sec. IV we will introduce another representation of the general solution of Eq. (1), which is more suitable for analyzing its long time behavior. Sec. V will be devoted to the proof of theorem 2. Finally, Sec. VI will contain some concluding remarks and an outlook.

**Notation.** We will work in the complex and separable Hilbert space $L^2(\mathbb{R})$, with the norm and the scalar product given, respectively, by $\| \cdot \|$ and $\langle \cdot | \cdot \rangle$. We will also consider the subspace $L^2_B(\mathbb{R})$ of all bounded functions of $L^2(\mathbb{R})$. Given an operator $O$, we denote with $\mathcal{D}(O)$ its domain and with $\mathcal{R}(O)$ its range.

Since in some expression the real and imaginary parts of some coefficients appear, we introduce for ease of readability the symbols $z^R$ or $z^I$ will denote the real part of the complex number $z$, while $z^I$ or $z^I$ will denote its imaginary part.

Given the linear SDE (6), a *topological strong solution* is an $L^2$-values process such that for any $t > 0$,

$$
\phi_t = \phi - \frac{i}{\hbar} \int_0^t \frac{p^2}{2m} \phi_s ds + \sqrt{\lambda} \int_0^t q \phi_s dW_s - \frac{\lambda}{2} \int_0^t q^2 \phi_s ds
$$

holds with $\mathbb{Q}$-probability 1. A *topological weak solution* instead is an $L^2$-values process such that for any $t > 0$ and for any $\chi \in \mathcal{D}(p^2) \cap \mathcal{D}(q^2)$,

$$
\langle \chi | \phi_t \rangle = \langle \chi | \phi \rangle - \frac{i}{\hbar} \int_0^t \frac{1}{2m} \langle p^2 \chi | \phi_s \rangle ds + \sqrt{\lambda} \int_0^t \langle q \chi | \phi_s \rangle dW_s - \frac{\lambda}{2} \int_0^t \langle q^2 \chi | \phi_s \rangle ds
$$

holds with $\mathbb{Q}$-probability 1. Topological strong and week solutions for the nonlinear SDE (1) are defined in a similar way.

There is also a distinction between strong and weak solutions in a stochastic sense [47], depending on whether the probability space, the filtration and the Wiener process are given *a priori* (strong solution) or whether they can be constructed in such a way to solve the required SDE (weak solution). Throughout the paper we will deal only with strong solutions in the stochastic sense.

**II. TIME EVOLUTION OF THE GENERAL SOLUTION**

We begin our discussion with a qualitative analysis of the time evolution of the general solution of Eq. (1); we will spot out the regimes we introduced in the previous section, corresponding to three different behaviors of the wave function. These regimes of course depend on the value of the mass $m$ of the particle and also on the value of the coupling constant $\lambda$ which sets the strength of
the collapse mechanism. As discussed e.g. in [36], it is physically appropriate to take \( \lambda \) proportional to the mass \( m \) according to the formula:

\[
\lambda := \lambda_0 \frac{m}{m_0},
\]

(27)

where \( \lambda_0 \) is now assumed to be a *universal* coupling constant, while \( m_0 \) is taken equal to the mass of a nucleon \((\simeq 1.67 \times 10^{-27} \text{ kg})\). To be definite, in the following we take \( \lambda_0 \simeq 1.00 \times 10^{-2} \text{ m}^{-2} \text{ sec}^{-1} \), so that the localization mechanism has the same strength as that of the GRW model [1]. Though, as we discussed in the introduction, Eq. (1) is used also in the context of the theory of continuous measurement as well as in the theory of decoherence, for brevity and clarity in the following we will only make reference to its application within models of spontaneous wave function collapse.

1. **The collapse regime.** The first important effect of the dynamics embodied in Eq. (1) is that a wave function, which initially is well spread out in space, becomes rapidly localized. This is most easily seen through the Green’s function representation of the solution. The Green’s function \( G_t(x, y) \) in (12) can be rewritten as follows

\[
G_t(x, y) = K_t \exp \left[ -\frac{\tilde{\alpha}_t}{2} x^2 + \tilde{a}_t x + \tilde{c}_t \right] \exp \left[ -\frac{\alpha_t}{2} (y - Y_t^x)^2 \right]
\]

(28)

where we have introduced the new parameters:

\[
\begin{align*}
\tilde{\alpha}_t &= \alpha_t - \frac{\beta_t^2}{\alpha_t} = \frac{2\lambda}{v} \tanh vt, \\
\tilde{a}_t &= \tilde{a}_t + \frac{\beta_t b_t}{\alpha_t}, \\
\tilde{c}_t &= \tilde{c}_t + \frac{\beta_t^2}{2\alpha_t}, \\
Y_t^x &= \frac{\beta_t x + b_t}{\alpha_t}.
\end{align*}
\]

(29-32)

The \( y \)-part of \( G_t(x, y) \) is a Gaussian function whose spread in position (equal to \( 1/\sqrt{\alpha_t^R} \)) rapidly decreases in time, and afterwards remains very small. In particular, we have:

\[
\alpha_t^R = \frac{2\lambda \sinh \omega t - \sin \omega t}{\omega \cosh \omega t - \cos \omega t} = \begin{cases} 
\frac{2}{3} \lambda t \simeq (3.99 \times 10^{24} \text{ m}^{-2} \text{ Kg}^{-1} \text{ sec}^{-1}) t & t \ll \omega^{-1}, \\
\frac{2\lambda}{\omega} \simeq (2.39 \times 10^{29} \text{ m}^{-2} \text{ Kg}^{-1}) t & t \to +\infty,
\end{cases}
\]

(33)

with \( \omega \simeq 5.01 \times 10^{-5} \text{ sec}^{-1} \) independent of the mass of the particle.

Let us introduce a length \( \ell \), and let say that a wave function is *localized* when its spread is smaller than \( \ell \). For sake of definiteness, we take \( \ell \simeq 1.00 \times 10^{-7} \text{ m} \), corresponding to the width of
the collapsing Gaussian of the GRW model. By means of this length, we can define the \textit{collapse time} \( t_1 \) as the time when the spread of the \( y \)-part of the Green’s function \( G_t(x,y) \) becomes smaller than \( \ell \). By using the small time approximation of \( \alpha_t^R \) given in (33), we can set:

\[
t_1 := \frac{3}{2\ell^2 \lambda} \simeq \frac{2.51 \times 10^{-11}}{m} \text{ Kg sec},
\]

(34)

As we see, and as we expect, this time decreases for increasing masses, i.e. for increasing values of \( \lambda \), and is very small for macroscopic particles.

Let us assume that the initial state \( \phi(x) \) is not already localized, and in particular that it does not change appreciably on the scale set by \( \ell \); this is a physically reasonable assumption when \( \phi \) represents the state of the center of mass of a macroscopic object. In this case, from the time \( t_1 \) on, the \( y \)-part of the Green’s function \( G_t(x,y) \) acts like a Dirac-delta on \( \phi(Y_t^x) \), and the solution at time \( t \) of the linear equation can be written as follows:

\[
\phi_t(x) \simeq \sqrt{\frac{2\pi}{\alpha_t}} K_t \exp \left[ -\frac{\tilde{\alpha}_t}{2} x^2 + \tilde{a}_t x + \tilde{c}_t \right] \phi(Y_t^x);
\]

(35)

This is a Gaussian state whose spread is controlled by \( \tilde{\alpha}_t \), which evolves in time in a way similar to \( \alpha_t \); in particular:

\[
\tilde{\alpha}_t^R = \frac{2\lambda \sinh \omega t + \sin \omega t}{\omega \cosh \omega t + \cos \omega t} = \begin{cases} 
2\lambda t & \simeq (1.20 \times 10^{25} \text{ m}^{-2} \text{ Kg}^{-1} \text{ sec}^{-1})m t \ll \omega^{-1}, \\
\frac{2\lambda}{\omega} & \simeq (2.39 \times 10^{29} \text{ m}^{-2} \text{ Kg}^{-1})m t \to +\infty.
\end{cases}
\]

(36)

As we see, the spread \( 1/\sqrt{\tilde{\alpha}_t^R} \) is well below \( \ell \), for any \( t \geq t_1 \). We can the conclude that, for times greater than the collapse time, any state initially well spread out in space is mapped into a very well localized wave function.

An important issue is \textit{where} the wave function collapses to, given that the initial state is spread out in space. We now show that the position of the wave function after the collapse is distributed in very good agreement with the Born probability rule.

A reasonable measure of where the wave function is, after it has collapsed, is given by the quantum average of the position operator \( \langle q \rangle_t \). Accordingly, the probability for the collapsed wave function to lie within a Borel measurable set \( A \) of \( \mathbb{R} \) can be simply defined to be \( \mathbb{P}^{\text{coll}}_t[A] := \mathbb{P}[\omega : \langle q \rangle_t \in A] \). Though this probability is mathematically well defined for any Borel measurable subset \( A \), it is physically meaningful only when \( A \) represents an interval \( \Delta \) much larger than the spread of the wave function itself, or a sum of such intervals. In such a case, as discussed in [48], one can show that:

\[
\mathbb{P}^{\text{coll}}_t[A] \simeq \mathbb{E}_\tilde{\beta}[\mathbb{P}_\Delta \psi_t]^2 \equiv \int_\Delta p_t(x) dx, \quad (37)
\]
where $P_{\Delta}(x)$ is the characteristic function of the interval $\Delta$ of the real axis and $p_t = \mathbb{E}[|\psi_t(x)|^2]$. The idea behind the approximate equality (37) is that when $\psi_t$ lies within $\Delta$, then $P_{\Delta}\psi_t \simeq \psi_t$, so that $\|P_{\Delta}\psi_t\|^2$ is almost equal to 1, while when it lies outside $\Delta$, it is practically 0. The critical situations, which require special care, are those when the wave function lies at the edges of $\Delta$.

In [36] it has been proven that:

$$p_t(x) = \sqrt{\frac{\mu_t}{\pi}} \int dy e^{-\mu_t|\psi_t^\text{Sch}(x+y)|^2}, \quad \mu_t = \frac{3m^2\lambda_0}{2\hbar^2} \simeq (2.27 \times 10^{43} \text{m}^{-2} \text{Kg}^{-1} \text{sec}^3) \frac{m}{t^3}, \quad (38)$$

where $p_t^\text{Sch}(x) = |\psi_t^\text{Sch}(x)|^2$ and $\psi_t^\text{Sch}(x)$ is the solution of the standard free-particle Schrödinger equation, for the given initial condition $\phi(x)$. For the times we are considering ($t = t_1$), the Gaussian term in (38) is much more peaked than any typical quantum probability distribution $p_t^\text{Sch}(x)$, and consequently acts like a Dirac-delta on it; accordingly, $p_t(x) \simeq p_t^\text{Sch}(x)$. Finally, for macroscopic systems and for the times we are considering, the wave function solution of the free-particle Schrödinger equation does not change appreciably, implying that $p_t^\text{Sch}(x) \simeq p_0^\text{Sch}(x) = |\phi(x)|^2$, which means precisely that the collapse probability is distributed in agreement with the Born probability rule.

2. The classical regime. After time $t_1$, we are left with a wave function which, when $m$ is the mass of a macroscopic particle, is very well localized in space, almost point-like. This is the way in which collapse model reproduce the particle-like behavior of classical systems, within the framework of a wave-like dynamics. The relevant question now is to unfold the time evolution of the position and momentum of the wave function, to see whether it matches Newton’s laws.

When the wave function is well localized in space ($t > t_1$), one can reasonably assume that it can be approximated with the Gaussian state to which—as we shall see—it asymptotically converges to. We will analyze the time evolution of such a Gaussian state in the following, and we will see that its mean position $\bar{x}_t$ and momentum $\hbar \bar{k}_t$ evolve in time as follows (see Eqs. (142) and (143)):

$$\bar{x}_t = \bar{x}_{t_1} + \frac{\hbar}{m} \bar{F}_{t_1}(t - t_1) + \sqrt{\lambda} \frac{\hbar}{m} \int_{t_1}^t W_s ds + \sqrt{\lambda} \frac{\hbar}{m} (W_t - W_{t_1}), \quad (39)$$

$$\bar{k}_t = \bar{k}_{t_1} + \sqrt{\lambda} (W_t - W_{t_1}). \quad (40)$$

We can easily recognize in the deterministic parts of the above equations the free-particle equations of motions of classical mechanics describing a particle moving along a straight line with constant velocity; the remaining terms are the fluctuations around the classical motion, driven by the Brownian motion $W_t$. The important feature of the above equations is that these fluctuations, for macroscopic masses, are very small, for very long times. As a matter of fact, if we estimate the
Brownian motion fluctuations by setting $W_t \sim \sqrt{t}$, we have for the stochastic terms in Eq. (39):

$$\sqrt{\lambda} \frac{\hbar}{m} \int_{t_1}^t W_s ds \simeq \frac{2}{3} \sqrt{\lambda} \frac{\hbar}{m} t^{3/2} \simeq (1.63 \times 10^{-22} \text{m Kg}^{1/2} \text{sec}^{-3/2}) \frac{t^{3/2}}{\sqrt{m}},$$  

(41)

$$\sqrt{\frac{\hbar}{m}}(W_t - W_{t_1}) \simeq \sqrt{\frac{\hbar}{m}} t \simeq (1.02 \times 10^{-17} \text{m Kg}^{1/2} \text{sec}^{-1/2}) \sqrt{t}. \quad (42)$$

We see that the random fluctuations decrease with the square root of the mass $m$ of the particle, which means that the bigger the system, the more deterministic its motion. This is how collapse models recover classical determinism at the macroscopic level, from a fundamentally stochastic theory.

We can introduce a time $t_2$, defined as the time after which the fluctuations become larger than $L$; we can set e.g. $L \simeq 1.00 \times 10^{-3} \text{m}$. Since the fluctuations in (41) grow faster as those in (42), we can set:

$$t_2 \simeq \left(\frac{3}{2} \frac{L}{\sqrt{\lambda}} \frac{m}{\hbar}\right)^{2/3} \simeq (3.55 \times 10^{12} \text{sec m}^{-1/3}) \sqrt{m} \simeq (1.13 \times 10^5 \text{y m}^{-1/3}) \sqrt{m}. \quad (43)$$

The time $t_2$ defines the time interval $[t_1, t_2]$ during which the classical regime holds. As we see, for macroscopic systems this is a very long time, much longer than the time during which a macro-object can be kept isolated from the rest of the universe, so that its dynamics is described by Eq. (1).

To summarize, during the classical regime, which for macroscopic systems lasts very long, the wave function behaves, for all practical purposes, like a point moving deterministically in space according to Newton’s laws. In other words, the wave function reproduces the motion of a classical particle.

3. THE DIFFUSIVE REGIME. After time $t_2$, two new effects become dominant: First, the wave function converges towards a Gaussian state, as we shall prove. Second, the motion becomes more and more erratic: the dynamics begins to depart from the classical one, showing its intrinsic stochastic nature.

A thorough mathematical analysis of these time regimes and their main properties is still lacking. In this paper, as we have anticipated, we focus now only on the long time behavior of the solutions of Eq. (1), leaving the study of the remaining properties as open problems for future research.

III. SOLUTION OF THE EQUATION

In the first part of this section we derive the Green’s function (12) in a way which will make clear the connection between Eq. (6) and the equation of the so called non-self-adjoint (NSA)
harmonic oscillator \[51, 52, 53\]. This connection is important for two reasons; from a physical point of view, it will bring a deep insight on how the collapse of the wave function actually works. From a mathematical point of view, it will allow to prove rigorously both the theorem 1 and 2 presented in the introductory section.

A way to connect Eq. (6) with that of the NSA harmonic oscillator is to apply suitable transformations to the wave function in such a way to transform the SDE in a Schrödinger-like equation. We will do this in two steps. We present this section in detail for convenience although the approach goes back to Kolokoltsov \[33\].

1. **Reduction of Eq. (6) to a linear differential equation with random coefficients.**

The idea is to remove the stochastic differential term \(\sqrt{\lambda q} d\xi_t\) from Eq. (6): borrowing the language of quantum mechanics, we shift to a sort of interaction picture by defining a suitable operator which maps the solution of Eq. (6) to the solution of a new equation which does not have that stochastic term. To this end, let us consider the operator \(Q_a : \mathcal{D}(Q_a) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})\) defined as follows:

\[
Q_a \phi(x) = e^{ax} \phi(x), \quad a \in \mathbb{C};
\]

where \(\mathcal{D}(Q_a)\) is defined as the set of all \(\phi(x) \in L^2(\mathbb{R})\) such that \(e^{ax} \phi(x) \in L^2(\mathbb{R})\). It should be noted that, in general, the operator \(Q_a\) is unbounded and its domain \(\mathcal{D}(Q_a)\) is dense in \(L^2(\mathbb{R})\) but not coincide with it. We will settle all technical issues in the second part of the section. We now define the vector:

\[
\phi^{(1)}_t = Q_{-\sqrt{\lambda} \xi_t} \phi_t;
\]

an easy application of Itô calculus shows that \(\phi^{(1)}_t\) satisfies the differential equation:

\[
d\phi^{(1)}_t = \left[ -\frac{i}{\hbar} Q_{-\sqrt{\lambda} \xi_t} \frac{p^2}{2m} Q_{-\sqrt{\lambda} \xi_t}^{-1} - \lambda q^2 \right] \phi^{(1)}_t dt, \quad \phi^{(1)}_0 = \phi.
\]

The stochastic differential \(\sqrt{\lambda q} d\xi_t\) has disappeared; in turn, the free Hamiltonian \(p^2/2m\) has been replaced by the operator \(Q_{-\sqrt{\lambda} \xi_t} (p^2/2m) Q_{-\sqrt{\lambda} \xi_t}^{-1}\) which, due to the specific commutation relations between \(q\) and \(p\), takes the simple form:

\[
Q_{-\sqrt{\lambda} \xi_t} p^2 Q_{-\sqrt{\lambda} \xi_t}^{-1} = p^2 - 2i\hbar \sqrt{\lambda} \xi_t p - \lambda \hbar^2 \xi_t^2;
\]

Eq. (46) can then be re-written as follows:

\[
-i\hbar \frac{d}{dt} \phi^{(1)}_t = \left[ \frac{p^2}{2m} - i\hbar \lambda q^2 \sqrt{\lambda} \xi_t p - \frac{\lambda \hbar^2}{2m} \xi_t^2 \right] \phi^{(1)}_t.
\]
This is a standard differential equation with random coefficients; note that the operator on the right hand side is not self-adjoint, due to the presence of the second and third term. The last term of Eq. (48) is a multiple of the identity operator and can be removed by defining:

\[ \phi^{(2)}_t = \exp \left[ -\frac{i\hbar \lambda}{2m} \int_0^t \xi^2 \, ds \right] \phi^{(1)}_t; \]  

(49)

we then obtain:

\[ i\hbar \frac{d}{dt} \phi^{(2)}_t = \left[ \frac{p^2}{2m} - i\hbar \lambda q^2 - \frac{i\hbar}{m} \sqrt{\lambda} \xi_t p \right] \phi^{(2)}_t. \]  

(50)

The third term on the right-hand-side contains a time dependent coefficient, and the next step aims at removing it.

2. Reduction of Eq. (50) to a differential equation with constant coefficients. The idea we now follow is to perform a transformation similar to a boost. We introduce the operator \( \mathcal{P}_a : \mathcal{D}(\mathcal{P}_a) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) defined as:

\[ \mathcal{P}_{ia/\hbar} \phi(x) = \phi(x + a), \quad a \in \mathbb{C}, \]  

(51)

where \( \mathcal{D}(\mathcal{P}_a) \) is the set of all \( \phi(x) \in L^2(\mathbb{R}) \) which can be analytically continued to the line \( x + a \) in the complex space \( \mathbb{C} \), and such that \( \phi(x + a) \in L^2(\mathbb{R}) \). Similarly to \( \mathcal{Q}_a \), also \( \mathcal{P}_a \) is in general an unbounded operator and its domain \( \mathcal{D}(\mathcal{P}_a) \), though being dense, does not coincide with \( L^2(\mathbb{R}) \); we will come back to this point later in this section. We define the operator:

\[ \mathcal{V}_t = \exp (-ia_t/\hbar) \mathcal{P}_{ib_t/\hbar} \mathcal{Q}_{-ic_t/\hbar}, \]  

(52)

where the coefficients \( a_t, b_t \) and \( c_t \), yet to be determined, will turn out to be complex random functions of time. One can easily verify that:

\[ \mathcal{V}_t q \mathcal{V}_t^{-1} = q + b_t, \]  

(53)

\[ \mathcal{V}_t p \mathcal{V}_t^{-1} = p + c_t, \]  

(54)

and similarly for higher powers of \( q \) and \( p \). Let us define the vector:

\[ \varphi_t = \mathcal{V}_t \phi^{(2)}_t, \]  

(55)

which solves the equation:

\[ i\hbar \frac{d}{dt} \varphi_t = \left[ \frac{p^2}{2m} - i\hbar \lambda q^2 - \left( \dot{b}_t - \frac{1}{m} c_t + \frac{i\hbar}{m} \sqrt{\lambda} \xi_t \right) p + \right. \]
\[ + \left. \left( \dot{c}_t - 2i \hbar \lambda b_t \right) q + \left( \dot{a}_t + \dot{c}_t b_t + \frac{1}{2m} c_t^2 - \frac{i\hbar}{m} \sqrt{\lambda} \xi_t c_t - i\hbar \lambda b_t^2 \right) \right] \varphi_t. \]  

(56)
The time-dependent part of the equation can be removed by requiring that \( a_t, b_t \) and \( c_t \) satisfy the first-order differential equations:

\[
\begin{align*}
\dot{m}b_t - c_t &= -i\hbar\sqrt{\lambda}\xi_t \quad b_0 = 0, \\
\dot{c}_t - 2i\hbar\lambda b_t &= 0 \quad c_0 = 0
\end{align*}
\]  

and

\[
\dot{a}_t + i\hbar\lambda b_t^2 + \frac{1}{2m} c_t^2 - i\hbar \sqrt{\lambda}\xi_t c_t = 0, \quad a_0 = 0.
\]  

The first two equations form a non-homogeneous linear system of first order differential equations, which has a unique \( Q \)-a.s. continuous random solution; the third equation instead determines the global factor \( a_t \), which is also random. With such a choice for the three parameters, Eq. (56) becomes:

\[
i\hbar \frac{d}{dt} \varphi_t = \left[ \frac{p^2}{2m} - i\hbar\lambda q^2 \right] \varphi_t, \quad \varphi_0 = \phi,
\]  

which is the equation of the so-called non-self-adjoint (NSA) harmonic oscillator, whose solution and most important properties are well known. Before continuing, we note that in the case of a more general Hamiltonian \( H = p^2/2m + V(q) \) appearing in Eq. (1) in place of just the free evolution \( p^2/2m \), the potential \( V(q) \) would have been transformed, when going from Eq. (50) to Eq. (59), according to the rule: \( \mathcal{V}_t V(q) \mathcal{V}_t^{-1} = V(q + b) \); in this case, we would not be able to remove completely the time-dependent terms from the equation and we would not be able to reduce the original equation to one, whose solution is known. However, besides the free particle case, all equations containing terms at most quadratic in \( q \) and \( p \) (among them, the important case of the harmonic oscillator) can be solved in a similar way.

The solution of Eq. (59) admit a representation in terms of the Green’s function:

\[
G_t^{\text{NSA}}(x, y) = \sqrt{\frac{\lambda}{v\pi \sinh vt}} \exp \left[ -\frac{\lambda}{v} (x^2 + y^2) \coth vt + 2\frac{\lambda}{v} xy \sinh^{-1} vt \right],
\]  

with \( v \) and \( \omega \) defined as in (19). In this way we have established the link between the solutions of the SDE (6) and those of the equation for the NSA harmonic oscillator (59), which we summarize in the following lemma, whose proof is straightforward.

**Lemma 11.1:** Let \( T_t^{\text{NSA}} \) be the evolution operator represented by the Green’s function \( G_t^{\text{NSA}}(x, y) \) and \( T_t \) the one represented by \( G_t(x, y) \); then:

\[
T_t = \exp \left( i\frac{\partial_t}{\hbar} \right) Q \sqrt{\mathcal{X}_{t+}}(i\partial_t \hbar) \mathcal{P}_{-i\partial_t \hbar} T_t^{\text{NSA}},
\]  

with \( \mathcal{P} \) being the projection onto the positive frequency subspace.
where the two random functions $b_t$ and $c_t$ solve the linear system (57), and $\vartheta_t$, which includes all global, i.e. independent of $x$, phase factors, solves the equation:

$$
\dot{\vartheta}_t = -i\hbar \lambda b_t^2 - \frac{1}{2m} c_t^2 + \frac{i\hbar}{m} \sqrt{\lambda}\xi_t c_t + \frac{\lambda\hbar^2}{2m}\xi_t^2, \quad \theta_0 = 0. \quad (62)
$$

We now proceed to prove in which sense $\varphi_t := T_t \varphi$ is the topological strong solution of Eq. (6) for the given initial condition $\varphi$. We first need to set some properties of the Green’s function $G_{t}^{NSA}(x, y)$ which will be necessary for the subsequent theorem.

**Lemma III.2:** The absolute value of $G_{t}^{NSA}(x, y)$ is equal to:

$$
|G_{t}^{NSA}(x, y)| = \sqrt{\frac{2\lambda}{\pi\omega\sqrt{\cosh \omega t - \cos \omega t}}} \exp \left[-\frac{\lambda}{\omega}(x^2 + y^2)p_t + \frac{\lambda}{\omega}xyq_t\right], \quad (63)
$$

where we have introduced the following quantities:

$$
p_t = \frac{\sinh \omega t - \sin \omega t}{\cosh \omega t - \cos \omega t}, \quad (64)
$$

$$
q_t = \frac{\sinh \omega t/2 \cos \omega t/2 - \cosh \omega t/2 \sin \omega t/2}{\cosh \omega t - \cos \omega t}; \quad (65)
$$

note that the function $p_t$ is positive for any $t > 0$. The integral of $|G_{t}^{NSA}(x, y)|^2$ with respect to $y$ is equal to:

$$
\int dy |G_{t}^{NSA}(x, y)|^2 = \sqrt{\frac{2\lambda}{\pi\omega(\sinh \omega t - \sin \omega t)}} \exp \left[-\frac{2\lambda}{\omega} p_t^2 - \frac{4q_t^2}{p_t} x^2\right]. \quad (66)
$$

A simple calculation shows that $p_t^2 - 4q_t^2 > 0$ for any $t > 0$; this means that $G_{t}^{NSA}(x, \cdot)$, taken as a function of $y$, belongs to $L^2(\mathbb{R})$ for any $x \in \mathbb{R}$ and $t > 0$; moreover:

$$
\int dx \|G_{t}^{NSA}(x, \cdot)\|^2 < +\infty \quad \text{for any } t > 0. \quad (67)
$$

Finally, the following expression holds true:

$$
\int dy |e^{bx}G_{t}^{NSA}(x + a, y)|^2 = \sqrt{\frac{2\lambda}{\pi\omega(\sinh \omega t - \sin \omega t)}} \exp \left[-2\frac{\lambda}{\omega} \left[\frac{p_t^2 - 4q_t^2}{p_t} x^2 \right.\right.

+ 2 \left(p_t a_R + p_t a_I - 4q_t (q_t a_R + q_t a_I) \right) x

+ \left. \left. p_t (a_R^2 - a_I^2) + 2p_t a_R a_I - 4 \left(\frac{q_t a_R + q_t a_I}{p_t}\right)^2\right]\right], \quad (68)
$$

with

$$
\bar{p}_t = \frac{\sinh \omega t + \sin \omega t}{\cosh \omega t - \cos \omega t}, \quad (69)
$$

$$
\bar{q}_t = \frac{\sinh \omega t/2 \cos \omega t/2 + \cosh \omega t/2 \sin \omega t/2}{\cosh \omega t - \cos \omega t}. \quad (70)
$$
The above formulas imply that, for any \(a,b \in \mathbb{C}\), for any \(x \in \mathbb{R}\) and for any \(t > 0\), the function \(e^{bx}G_{t}^{\text{NSA}}(x + a, \cdot)\) belongs to \(L^2(\mathbb{R})\) and:

\[
\int dx \| e^{bx}G_{t}^{\text{NSA}}(x + a, \cdot) \|^2 < +\infty.
\] (71)

We are now in a position to state and prove the main theorem of this section.

**Theorem 11.1:** Let \(P_a\) and \(Q_a\) be defined, respectively, as in (51) and (44); let \(b_t\) and \(c_t\) solve the linear system (57) and \(\theta_t\) be the solution of Eq. (62). Finally, let \(\phi_t = T_t\phi\), with \(\phi \in L^2(\mathbb{R})\) and \(T_t\) defined as in (61). Then the following three statements hold true with probability 1:

1. \(T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) defines a bounded operator for every \(t > 0\) \hspace{1cm} (72)
2. \(\phi \in L^2_B(\mathbb{R}) \implies \phi_t\) is a topological strong solution of Eq. (6) \hspace{1cm} (73)
3. \(\phi \in L^2(\mathbb{R}) \implies \lim_{t \to 0} \| \phi_t - \phi \| = 0\). \hspace{1cm} (74)

**Proof of statement 1.** Let \(\phi\) belong to \(L^2(\mathbb{R})\); since also \(G_{t}^{\text{NSA}}(x, \cdot)\) belongs to \(L^2(\mathbb{R})\) for any \(x \in \mathbb{R}\) and \(t > 0\), Hölder’s inequality implies that \(G_{t}^{\text{NSA}}(x, \cdot)\phi\) belongs to \(L^1(\mathbb{R})\); accordingly, the operator \(T_t^{\text{NSA}}\) is well defined for any \(t > 0\), and maps any \(L^2(\mathbb{R})\)-function into a measurable function. By using Schwartz inequality together with relation (67), we have:

\[
\int dx \left| \int dy G_{t}^{\text{NSA}}(x, y)\phi(y) \right|^2 \leq \| \phi \|^2 \int dx \| G_{t}^{\text{NSA}}(x, \cdot) \|^2 < +\infty;
\] (75)

thus \(T_t^{\text{NSA}}\phi\) belongs to \(L^2(\mathbb{R})\) for any \(\phi\) in \(L^2(\mathbb{R})\) and for any \(t > 0\).

In a similar way, since also \(G_{t}^{\text{NSA}}(x + a, \cdot)\) belongs to \(L^2(\mathbb{R})\) for any \(a \in \mathbb{C}\) and because of (71), one proves that \(P_a T_t^{\text{NSA}}\phi\) belongs to \(L^2(\mathbb{R})\) for any \(\phi \in L^2(\mathbb{R})\), for any complex \(a\) and for any \(t > 0\), i.e. that \(\mathcal{D}(P_a)\) contains \(\mathcal{R}(T_t^{\text{NSA}})\). Using once more the same inequalities and (71), one shows also that \(Q_b P_a T_t^{\text{NSA}}\phi\) belongs to \(L^2(\mathbb{R})\) for any \(\phi\) in \(L^2(\mathbb{R})\), for any \(a,b \in \mathbb{C}\) and \(t > 0\).

**Remark:** Actually a stronger statement is true, as can be readily seen from the Gaussian form of the Green’s function \(G_t\) of the operator \(T_t\): For positive \(t\) it maps \(L^2(\mathbb{R})\) to Schwartz space \(\mathcal{S}(\mathbb{R})\). We shall need this information in the proof of statement 3.

**Proof of statement 2.** Let us consider the vector \(\varphi_t := T_t^{\text{NSA}}\phi\), with \(\phi \in L^2_B(\mathbb{R})\). By construction, \(\varphi_t\) solves Eq. (59), once one proves that the integration

\[
\int dy G_{t}^{\text{NSA}}(x, y)\phi(y)
\] (76)

can be exchanged with the first and second partial derivatives with respect to \(x\) and with the first partial derivative with respect to \(t\). We note that the function \(G_{t}^{\text{NSA}}(x, y)\phi(y)\) satisfies the
following two properties: i) The function $y \mapsto G_{t}^{\text{NSA}}(x,y)\phi(y)$ is measurable and integrable on $\mathbb{R}$ for any $t > 0$ and for any $x \in \mathbb{R}$; ii) The first and second partial derivatives with respect to $x$ and the first partial derivatives with respect to $t$ are exists for any $t > 0$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and can be bounded uniformly with respect to $t$ and $x$. Accordingly, one can apply e.g. theorem 12.13 pag. 199 of [49] to conclude that the operations of integration and differentiation can be exchanged.

Having proved that $\varphi_{t}$ solve Eq. (59), a direct application of Itô calculus proves that $\phi_{t}$, defined as in (61), is a topological strong solution of Eq. (6).

**Proof of statement 3.** Let $\phi = \phi_{0} \in C_{c}^{\infty}(\mathbb{R})$ be given. Since $\phi_{t}$ solves Eq. (6) in a strong sense, it also solves the SDE in a weak sense; hence, using e.g. Eq. (1.1) of [29], one has:

$$\lim_{t \to 0} \langle \varphi | \phi_{t} \rangle = \langle \varphi | \phi_{0} \rangle \quad \forall \varphi \in C_{c}^{\infty}(\mathbb{R}).$$

(77)

We extend (77) to the general case of $\varphi \in L^{2}(\mathbb{R})$. Being dense in $L^{2}(\mathbb{R})$, there exist a sequence $\{ \varphi_{n} \in C_{c}^{\infty}(\mathbb{R}), n \in \mathbb{N} \}$ which approximates any $\varphi \in L^{2}(\mathbb{R})$. By triangle and Schwarz inequality we get

$$|\langle \varphi | \phi_{t} \rangle - \langle \varphi | \phi_{0} \rangle| \leq |\langle \varphi_{n} | \phi_{t} \rangle - \langle \varphi_{n} | \phi_{0} \rangle| + \| \varphi - \varphi_{n} \| \| \phi_{t} \| + \| \varphi - \varphi_{n} \| \| \phi_{0} \|.$$  

(78)

The first term on the right-hand-side can be made arbitrarily small because of (77); the second and third term can also be made arbitrarily small by choosing $n$ sufficiently large, while $\| \phi_{t} \|$ can be bounded as it converges to $\| \phi_{0} \|$ for $t \to 0$, due to Eq. (9). This proves that:

$$\lim_{t \to 0} \langle \varphi | \phi_{t} \rangle = \langle \varphi | \phi_{0} \rangle \quad \forall \varphi \in L^{2}(\mathbb{R}).$$

(79)

Statement 3 for test functions $\phi \in C_{c}^{\infty}(\mathbb{R})$ now follows directly from Eq. (9), Eq. (79) and observing $\| \phi_{t} \| - \| \phi_{0} \|^{2} = \| \phi_{t} \|^{2} + \| \phi_{0} \|^{2} - 2\langle \phi_{0} | \phi_{t} \rangle$. It remains to extend the strong continuity of $T_{t}$ from the subspace $C_{c}^{\infty}(\mathbb{R})$ to $L^{2}(\mathbb{R})$. For this observe that for $\phi \in C_{c}^{\infty}(\mathbb{R})$ ($\| \phi_{t} \|_{t \geq 0}$ defines a stochastic process with continuous paths and by Holevo’s result (cf. Eq. (9)) it is a martingale. For given $f \in L^{2}(\mathbb{R})$ choose a sequence $(\varphi^{n})_{n \in \mathbb{N}} \subset C_{c}^{\infty}(\mathbb{R})$, which converges to $f$ in $L^{2}(\mathbb{R})$. Doob’s inequality for submartingales implies that for all $n, m \in \mathbb{N}$, $T > 0$ and $\lambda > 0$

$$\mathbb{Q}\left(\sup_{0 \leq t \leq T} \| \varphi_{t}^{n} \|^{2} - \| \varphi_{t}^{m} \|^{2} > \lambda\right) \leq \frac{1}{\lambda}\mathbb{E}_{\mathbb{Q}}\left[\| \varphi_{T}^{n} \|^{2} - \| \varphi_{T}^{m} \|^{2}\right].$$

(80)

We now show that

$$\lim_{n,m \to \infty} \mathbb{E}_{\mathbb{Q}}\left[\| \varphi_{T}^{n} \|^{2} - \| \varphi_{T}^{m} \|^{2}\right] = 0.$$  

(81)
The elementary inequality

$$
\|\varphi^n_t\|_2^2 - \|\varphi^m_t\|_2^2 \leq (\|\varphi^n_t\|_2 + \|\varphi^m_t\|_2)\|\varphi^n_t - \varphi^m_t\|_2
$$

implies that

$$
\mathbb{E}_Q[\|\varphi^n_t\|_2^2 - \|\varphi^m_t\|_2^2] \leq \mathbb{E}_Q[(\|\varphi^n_t\|_2 + \|\varphi^m_t\|_2)\|\varphi^n_t - \varphi^m_t\|_2] \\
\leq (\mathbb{E}_Q[\|\varphi^n_t\|_2 + \|\varphi^m_t\|_2])^{\frac{1}{2}} (\mathbb{E}_Q[\|\varphi^n_t - \varphi^m_t\|_2^2])^{\frac{1}{2}} \\
\leq \sqrt{2}(\mathbb{E}_Q[\|\varphi^n_t\|_2^2] + \mathbb{E}_Q[\|\varphi^m_t\|_2^2])^{\frac{1}{2}} \|\varphi^n - \varphi^m\|_2^2
$$

The right hand side converges to 0 as $n, m \to \infty$. Therefore the sequence of stochastic processes $(\|\varphi^n_t\|_2^2)_{t \geq 0}$ is a Cauchy sequence in the complete metric space $(\mathbb{D}, d)$ of adapted processes with right continuous paths having left limits, where the metric $d$ is defined as (see page 56 – 57 in [50] for background concerning this topology)

$$
d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{E}_Q \left[ \min \left( 1, \sup_{0 \leq s \leq n} |(X - Y)_s| \right) \right] (X,Y \in \mathbb{D}).
$$

Therefore $(\|\varphi^n_t\|_2^2)_{t \geq 0}$ converges locally uniformly in probability to a stochastic process. This stochastic process again has to be continuous almost surely, since a subsequence of $(\|\varphi^n_t\|_2^2)_{t \geq 0}$ converges locally uniformly with probability one. Since $\lim_{n \to \infty} \|\varphi^n_t\|^2 = \|f_t\|^2$ almost surely we know that $[0, \infty) \ni t \mapsto \|f_t\|^2$ is continuous, in particular $\lim_{t \to 0} \|f_t\| = \|f\|$ almost surely and defines by the lemma of Fatou a positive continuous supermartingale. Therefore it has a unique decomposition $\|f_t\|^2 = M_t - A_t$, where $(M_t)_{t \geq 0}$ is a continuous martingale and $(A_t)_{t \geq 0}$ is increasing process. In fact as we shall show now, the increasing process is identically 0, i.e. $\|f_t\|^2_{t \geq 0}$ is a positive martingale for every $f \in \mathcal{L}^2(\mathbb{R})$. For that we observed in the Remark above that for positive $\varepsilon$ the function $f_\varepsilon$ almost surely belongs to the Schwartz space and in particular to the domain of the generator. By Holevo's result cited above $(\|T_{t-\varepsilon} f\|)_{t \geq \varepsilon}$ is a continuous martingale. Therefore $A_t = 0$ for $t > 0$ and hence it equals 0 almost surely. In order to ensure strong convergence $\lim_{t \to \infty} \|f_t - f\| = 0$ we need only show that weak convergence holds, i.e. $\lim_{t \to \infty} \langle \phi | f_t \rangle = \langle \phi | f \rangle$. Observing

$$
|\langle \psi | f_t \rangle - \langle \phi | f \rangle| \leq |\langle \phi | f_t \rangle - \langle \phi | \varphi^n_t \rangle| + |\langle \phi | \varphi^n_t \rangle - \langle \phi | \varphi^n \rangle| + |\langle \phi | \varphi^n \rangle - \langle \phi | f \rangle|
$$

it suffices to show that for some $T > 0$ $\lim_{n \to \infty} \sup_{t \leq T} |\langle \phi | f_t \rangle - \langle \phi | \varphi^n_t \rangle| = 0$. But $\sup_{t \leq T} |\langle \phi | f_t \rangle - \langle \phi | \varphi^n_t \rangle| \leq \|\phi\| \sup_{t \leq T} \|f_t - \varphi^n_t\|$. Therefore we need only establish that $\lim_{n \to \infty} \sup_{t \leq T} \|f_t - \varphi^n_t\| = 0$. 

This is done by a similar argument as above, namely we show that for every $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{Q}\left( \sup_{t \leq T} \| f_t - \varphi^n_t \|^2 > \varepsilon \right) = 0,$$

because then there exists a subsequence which is almost surely convergent to 0. But as we showed above ($\| g_t \|^2$)$_{t \geq 0}$ is a martingale for every $g \in L^2(\mathbb{R})$. Hence ($\| f_t - \varphi^n_t \|^2$)$_{t \geq 0}$ is a martingale and we can again apply Doob’s inequality as before.

**Remark 1.** The Gaussian form of the Green’s function (12) is a consequence of the fact that

Eq. (6) contains terms which are at most quadratic in $q$ and $p$. This also implies preserves shape

of initially Gaussian wave functions; in fact, as shown e.g. in [28, 32, 33, 36], a state

$$\phi_t(x) = \exp\left[ -\sigma_t(x - x^m_t)^2 + ik^m_t x + \varsigma_t \right], \quad (82)$$

is solution of Eq. (6) provided that the two real parameters $x^m_t$, $k^m_t$ and the two complex parameters

$\sigma_t$, $\varsigma_t$ satisfy the following stochastic differential equations:

\[
\begin{align*}
    d\sigma_t &= \left[ -i\frac{2\hbar}{m} (\sigma_t)^2 \right] dt, \quad (83) \\
    dx^m_t &= \frac{\hbar}{m} k^m_t dt + \frac{\sqrt{\lambda}}{2\sigma^R_t} \left[ d\xi_t - 2\sqrt{\lambda} x^m_t dt \right], \quad (84) \\
    dk^m_t &= -\sqrt{\lambda} \frac{\sigma^R_t}{\sigma_t} \left[ d\xi_t - 2\sqrt{\lambda} x^m_t \right], \quad (85) \\
    d\sigma^R_t &= \left( \frac{\lambda}{m} (x^m_t)^2 + \frac{\hbar}{m} \sigma^R_t + \frac{\lambda}{4\sigma^R_t} \right) dt + \sqrt{\lambda} x^m_t \left[ d\xi_t - 2\sqrt{\lambda} x^m_t dt \right], \quad (86) \\
    d\varsigma_t &= \left( -\frac{\hbar}{2m} (k^m_t)^2 - \frac{\hbar}{m} \sigma^R_t + \frac{\lambda \sigma^R_t}{4(\sigma^R_t)^2} \right) dt + \sqrt{\lambda} \frac{\sigma^R_t}{\sigma_t} x^m_t \left[ d\xi_t - 2\sqrt{\lambda} x^m_t dt \right]. \quad (87)
\end{align*}
\]

In particular, the solution of Eq. (83) is $\sigma_t = (\lambda/\nu) \coth(\nu t + \kappa)$, where $\kappa$ sets the initial condition. These results will be useful in the subsequent analysis.

**IV. REPRESENTATION OF THE SOLUTION IN TERMS OF EIGENSTATES OF THE NSA HARMONIC OSCILLATOR**

We now turn to the problem of analyzing the long time behavior of the solution of the (norm-preserving) non-linear Eq. (1). The representation of the solution $\phi_t$ of Eq. (6) in terms of the Green’s function (12) is not suitable for controlling the long time behavior; it turns out to be more convenient to express $\phi_t$ in terms of the *eigenstates* of the NSA harmonic oscillator, resorting to the connection which we previously established between Eq. (6) and (59). In this way, as we shall see, the collapse process will be manifest: the coefficients of the superposition will decrease exponentially
in time, the damping being the faster, the higher the associated eigenstate. Accordingly—when normalization is also taken into account—in the large time limit only the ground state survives, which has a Gaussian shape.

We first recall a few basic features of the Hamiltonian of the NSA harmonic oscillator,

\[ H \equiv \frac{p^2}{2m} - i\hbar\lambda q^2 \]  

which has been studied in particular by Davies in a series of papers [51, 52] and reviewed in his recent book [53]. The eigenvalues of \( H \) are complex and equal to:

\[ \lambda_n \equiv \frac{1 - i}{2} \hbar \omega_n, \quad \omega_n \equiv \left( n + \frac{1}{2} \right) \omega, \]  

and the corresponding eigenvectors are:

\[ \phi^{(n)}(x) \equiv \sqrt{z} e^{-z^2 x^2 / 2} \overline{H}_n(zx), \quad z^2 \equiv (1 - i) \sqrt{\frac{\lambda m}{\hbar}} \]  

where \( \overline{H}_n(x) \) is the normalized Hermite polynomial of degree \( n \). Since the argument of \( \overline{H}_n \) in (90) is complex, these eigenstates are not orthogonal; it can be shown that they are linearly independent and form a complete set, however they do not form a basis. As such, they can not directly used to expand an initial state into a superposition of the eigenstates of \( H \). This problem can be circumvented in the following way, also discussed by Davies.

It is easy to see that the sequences \( \{ \phi^{(n)} \} \) and \( \{ \phi^{(n)*} \} \) form a bi-orthonormal system; one then defines the (non-orthogonal) projection operators:

\[ P_n \phi \equiv \langle \phi | \phi^{(n)*} \rangle \phi^{(n)} = \alpha_n \phi^{(n)}, \]  

which satisfy the relations:

\[ P_n P_m = \delta_{n,m} P_n, \quad \| P_n \| = \| \phi^{(n)} \|^2 \quad \text{and} \quad \lim_{n \to +\infty} \frac{\ln \| P_n \|}{n} = 2c, \]  

where \( c \) is an appropriate constant [53]. As we see, although the states \( \phi^{(n)} \) are normalized, in the sense that

\[ \int_{-\infty}^{+\infty} \phi^{(n)}(x) \phi^{(m)}(x) \, dx = \delta_{n,m}, \]  

the norm of the projection operators \( P_n \) grows exponentially as \( n \to +\infty \). Finally, the following equality holds true [53]:

\[ T_{t\text{NSA}} = \sum_{n=0}^{\infty} e^{-(1+i)\omega_n t/2} P_n \quad \text{for } t > 4c/\omega. \]  

\[ T_{t\text{NSA}} = \sum_{n=0}^{\infty} e^{-(1+i)\omega_n t/2} P_n \quad \text{for } t > 4c/\omega. \]
A remarkable property of the above representation of the solution of Eq. (59) in terms of the eigenstates of the operator (88) is that it holds not for any \( t \geq 0 \), as one would naively expect, but only for \( t > 4c/\omega \). The reason is that the norm of the projection operators \( P_n \) grows exponentially with \( n \), so one has to wait for \( t \) to be large enough in order for the term \( e^{-n\omega t/2} \) to suppress the exponential growth of the projectors. From a physical point of view, recalling the discussion of sec II, since the constant \( c \) is of order 1 [53] and \( \omega \approx 5.01 \times 10^{-5} \) sec\(^{-1}\), we see that the representation (94) holds true only in part of the classical regime and in the diffusive regime, which is the one we are interested in studying now, but not in the physically more crucial collapse regime.

We now apply the above results to our problem; we will first proceed in an informal way, and at the end we will prove the relevant theorems. Let \( \phi \in L^2(\mathbb{R}) \); then, according to (61) and (94):

\[
\phi_t(x) = T_t \phi = e^{\sqrt{\lambda} \xi_t + ict/\hbar} x + i\theta_t/\hbar \sum_{n=0}^{+\infty} \alpha_n e^{-(1+i)n\omega t/2} \phi^{(n)}(x - b_t)
\]

(95)

\[
e^{-z^2(x-x_t^2)/2+ix_k^2x+\gamma t} \sum_{n=0}^{+\infty} \alpha_n e^{-(1+i)n\omega t/2} \Pi_n[z(x - b_t)],
\]

(96)

where \( \alpha_n = \langle \phi | \phi^{(n)} \rangle \) (see Eq. (91)), while the two real parameters \( \xi_t, \kappa_t \) and the complex parameter \( \gamma_t \) are defined as follows:

\[
\xi_t = b_t^R + b_t^I - (2/m\omega)e_t^R + (\omega/2\sqrt{\lambda})\xi_t,
\]

(97)

\[
\kappa_t = (m\omega/\hbar)b_t^R + (1/\hbar)(e_t^R - e_t^I) + \sqrt{\lambda}\xi_t,
\]

(98)

\[
\gamma_t = -(1 - i)(m\omega/4\hbar)(b_t^R - \kappa_t^2) + (i/\hbar)\theta_t.
\]

(99)

By resorting to Eqs. (57) and (62), and after a rather long calculation, we obtain the following set of SDEs for these parameters:

\[
d\xi_t = \frac{\hbar}{m} \kappa_t dt + \sqrt{\frac{\hbar}{m}} [d\xi_t - 2\sqrt{\lambda} \xi_t dt],
\]

(100)

\[
d\kappa_t = \sqrt{\lambda} [d\xi_t - 2\sqrt{\lambda} \xi_t dt],
\]

(101)

\[
d\gamma_t^R = \left[ \frac{\lambda \xi_t^2 + \omega}{4} \right] dt + \sqrt{\lambda} \xi_t [d\xi_t - 2\sqrt{\lambda} \xi_t dt],
\]

(102)

\[
d\gamma_t^I = -\left[ \frac{\hbar}{2m} \kappa_t^2 + \frac{\omega}{4} \right] dt - \sqrt{\lambda} \xi_t [d\xi_t - 2\sqrt{\lambda} \xi_t dt];
\]

(103)

the initial conditions are: \( \xi_0 = \kappa_0 = \gamma_0 = 0 \). Note that these equations are equivalent to (84)–(87), with \( \sigma_t = \sigma_\infty = \lambda/\nu = z^2/2, \xi_t = x_t^R, \kappa_t = k_t^R \) and \( \gamma_t = \varsigma_t + (1 + i)\omega/4 \); as a matter of fact, the above equations describe the time evolution (according to Eq. (6)) of the ground state of the NSA.
harmonic oscillator, which is:

$$\phi^\infty_t(x) = \exp \left[ -\frac{z^2}{2} (x - \bar{x}_t)^2 + i k t x + \gamma - \frac{1 + i}{4} \omega t \right], \quad \phi^0_0(x) = \phi^{(0)}(x). \quad (104)$$

As we shall prove in the next section, this is the state to which—apart from normalization—any initial state converges to, in the long time limit, hence the name $\phi^\infty_t$.

As we see, due to the stochastic part of the dynamics, the argument the Gaussian weighting factor and that of the Hermite polynomials of Eq. (96) are different functions of time, while for analyzing the long time behavior of the wave function, it is more convenient that both arguments display the same time dependence. We thus modify the argument of the Hermite polynomials, to make it equal to that of the weighting factor. To this end, let us define $\zeta_t = x_t - b_t$; we can then write:

$$\overline{H}_n[z(x - b_t)] = \frac{1}{\sqrt{\pi 2^n n!}} H_n[z(x - \bar{x}_t) + z\zeta_t]$$

$$= \frac{1}{\sqrt{\pi 2^n n!}} \sum_{m=0}^{n} \binom{n}{m} (2z\zeta_t)^{n-m} H_m[z(x - \bar{x}_t)]$$

$$= \sum_{m=0}^{n} \frac{\sqrt{m!}}{\sqrt{m!(n-m)!}} (\sqrt{2z\zeta_t})^{n-m} \overline{H}_m[z(x - \bar{x}_t)], \quad (105)$$

where $H_m$ is the standard (not normalized) Hermite polynomial of degree $m$; in going from the first to the second line, we have used property (A2). Resorting to the above relation, we can rewrite Eq. (96) as follows:

$$\phi_t(x) = e^{i k t x + \gamma - (1+i)\omega t / 4} \sum_{m=0}^{+\infty} \alpha_t^{(m)} e^{-(1+i)m\omega t / 2} \phi^{(m)}(x - \bar{x}_t); \quad (106)$$

the functions $\phi^{(m)}$ are the eigenstates defined in (90), while the time dependent coefficients $\alpha_t^{(m)}$ are defined as follows:

$$\alpha_t^{(m)} = \sum_{k=0}^{+\infty} \alpha_{k+m} \frac{\sqrt{(k + m)!}}{\sqrt{m!} k!} (\sqrt{2z\zeta_t})^k, \quad (107)$$

where we have introduced the new quantity $\zeta_t \equiv e^{-(1+i)\omega t / 2} \zeta_t$.

Eqs. (106) and (107) represent the two main formulas, which we will use in the next section to analyze the large time behavior. Before doing this, we need to set these formulas on a rigorous ground; we will do these with the following two lemmata.

**Lemma IV.1:** Let $\phi \in L^2(\mathbb{R})$ and $\alpha_n = \langle \phi | \phi^{(n)*} \rangle$, with $\phi^{(n)}$ defined as in (90). Then the series (107) defining $\alpha_t^{(m)}$ is a.s. convergent for any $m$ and any $t > 0$. Moreover, one has the following bound
on the coefficients:

$$|\alpha^{(m)}_t| \leq N_t e^{(c+1/2)m}, \quad N_t \equiv A \sum_{k=0}^{+\infty} e^{k(c+1)|\sqrt{2\pi\zeta_t}|^k} \quad \text{a.s.,}$$  \hspace{1cm} (108)

where $A$ is a constant independent of the Brownian motion $\xi_t$.

**Proof:** Because of (92), there exists a constant $C_1$ such that:

$$|\alpha_n| \leq \|\phi\|\|\phi^{(n)}\| = \|\phi^{(n)}\| \leq C_1 e^{nc}. \quad \text{(109)}$$

Secondly, using Stirling formula, there exists a constant $C_2$ such that:

$$C_2^{-1}\sqrt{2\pi nn^ne^{-n}} < n! < C_2\sqrt{2\pi nn^ne^{-n}}, \quad \text{(110)}$$

for $n > 1$; we can then write the following estimate:

$$\sqrt{\frac{(k+m)!}{m!k!}} \leq \frac{C_2^2}{\sqrt{2\pi}} \frac{\sqrt{k+m}}{m^k} \frac{(k+m)(k+m)/2e^{-(k+m)/2}}{m^{m/2}e^{-m/2}k^{k/2}e^{-k}} \leq \frac{C_2^2}{\sqrt{2\pi}} e^{-k(lnk-2)/2+m/2}, \quad \text{(111)}$$

in the second line, we have used the inequality $(k+m)ln(k+m) \leq klnk + mlnm + k + m$. Using Eqs. (109) and (111), we have the following bound:

$$\left| \alpha_{k+m}\frac{\sqrt{(k+m)!}}{\sqrt{m!k!}} (\sqrt{2\pi\zeta_t})^k \right| \leq \frac{C_1 C_2^2}{\sqrt{2\pi}} e^{k(c+1)|\sqrt{2\pi\zeta_t}|^k} \frac{e^{k(c+1)}}{\sqrt{k^k}}, \quad k, m \geq 1. \quad \text{(112)}$$

The cases $k = 0$ and $m = 0$ can be treated separately, giving the same bound, with the only possible difference of an overall constant factor. This proves convergence of the series defined in (107) and the bound (108).

**Theorem IV.1:** Let the conditions of Lemma IV.1 be satisfied; let moreover $\zeta_t \equiv e^{-(1+i)\omega t/2}\xi_t$, where $\zeta_t = \bar{x}_t - b_t$ with $\bar{x}_t$ and $b_t$ solutions of Eq. (100) and (57), respectively. Then the series defined in (106) is a.s. norm convergent for $t > \bar{T} \equiv (4c+1)/\omega$. In addition, the following equality holds true:

$$T_t \phi = e^{\bar{x}_k \xi + \gamma_{1+1}t} \sum_{m=0}^{+\infty} \alpha^{(m)}_t e^{-(1+i)\omega t/2}\phi^{(m)}(x - \bar{x}_t), \quad t > \bar{T}, \quad \text{(113)}$$

where $T_t$ is the evolution operator associated to the Green’s function (12).

**Proof:** According to (92) and (108), one has:

$$\left\| \alpha^{(m)}_t e^{-(1+i)\omega t/2}\phi^{(m)}[z(x - \bar{x}_t)] \right\| \leq C_1 N_t e^{(2c+1/2-\omega t/2)m}, \quad \text{(114)}$$

from which the conclusion follows. Comparing the two expressions of Eq. (61) and Eq. (106) when the initial state $\phi$ is an eigenstate $\phi^{(n)}$, we see that they coincide on the dense subspace of all finite linear combinations of $\phi^{(n)}$, and hence on the whole of $L^2(\mathbb{R})$. 
V. THE LONG TIME BEHAVIOR

We are now in a position to study the long time behavior of the solution of Eq. (1). Looking at expressions (106) for the solution $\psi_t$ and (107) for the coefficients $\alpha_t^{(m)}$, it should be clear what the long time behavior of the normalized solution $\psi_t = \phi_t / \|\phi_t\|$ is: whatever the initial condition, at any time $t > 0$ the wave function $\phi_t$ picks up a component on the ground state $\phi^{(0)}(x - \bar{x}_t)$, since $\alpha^{(0)}_t \neq 0$ as long as at least one of the coefficients $\alpha_k$ is not null, which is always the case. Eq. (106) on the other hand shows that each term of the superposition has an exponential damping factor, which is the bigger, the higher the eigenvalue. Accordingly, after normalization, only the eigenstate with the weakest damping factor survives, which is the ground state. Hence we expect that the general solution of Eq. (1) converges a.s., in the large time limit, to the ground state $\phi^{(0)}(x - \bar{x}_t)$, which is a Gaussian state. That this is true is proven in the following theorem.

**Theorem V.1:** Let $\phi_t$ be a strong solution of Eq. (6) that admits, for $t > \bar{t}$ a representation as in (113). Let $\psi_t \equiv \phi_t / \|\phi_t\|$ (when $\|\phi_t\| \neq 0$), which can be written as follows:

$$\psi_t = \psi_t^\infty + \sum_{m=1}^{+\infty} r_t \frac{\alpha_t^{(m)}}{r_t} e^{-(1+i)m\omega t/2} \phi_m(x - \bar{x}_t),$$

(115)

with:

$$\psi_t^\infty := \frac{\tau_t^{(0)}}{r_t} e^{i(x^t + \gamma - \omega t/4)} \phi_0(x - \bar{x}_t),$$

(116)

$$r_t := \left\| \sum_{m=0}^{+\infty} \alpha_t^{(m)} e^{-(1+i)m\omega t/2} \phi_m(x - \bar{x}_t) \right\|.$$  

(117)

Then, with $\mathbb{P}$-probability 1:

$$\lim_{t \to \infty} \|\psi_t - \psi_t^\infty\| = 0.$$  

(118)

Note that, apart from global factors, $\psi_t^\infty$ is the ground state of the NSA harmonic oscillator, randomly displaced both in position space as well as in momentum space.

**Proof.** According to Eq. (115), all we need to prove is that, with $\mathbb{P}$-probability 1:

$$\lim_{t \to \infty} \left\| \sum_{m=1}^{+\infty} \frac{\alpha_t^{(m)}}{r_t} e^{-(1+i)m\omega t/2} \phi_m(x - \bar{x}_t) \right\| = 0.$$  

(119)

Resorting to (114), one can write the following bound:

$$\left\| \sum_{m=1}^{+\infty} \frac{\alpha_t^{(m)}}{r_t} e^{-(1+i)m\omega t/2} \phi_m(x - \bar{x}_t) \right\| \leq C_1 \frac{N_t}{r_t} e^{-\omega(t-\bar{t})}.$$  

(120)
thus all we need to set is the long time behavior of \( r_t \) and \( N_t \). Lemmas \( \textbf{V.1} \) and \( \textbf{V.2} \) (see Eqs. \( \textbf{121} \) and \( \textbf{126} \)) state that, with \( \mathbb{P} \)-probability 1, \( r_t \) converges asymptotically to a finite and non-null random variable, while \( N_t \) converges to a finite random variable. From these properties, the conclusion of the theorem follows immediately.

In the remaining of the section, we prove the required lemmas.

**Lemma \( \textbf{V.1} \):** Let \( r_t \) be defined as in \( \textbf{117} \). Then, with \( \mathbb{P} \)-probability 1,

\[
\lim_{t \to \infty} r_t = r_\infty \text{ finite and not null.} \tag{121}
\]

**Proof.** According to Eq. \( \textbf{106} \) and \( \textbf{117} \), the following equality holds:

\[
\|\phi_t\| = e^{\gamma R_t - \omega t^4/r_t} \tag{122}
\]

resorting to the stochastic differentials \( \textbf{9} \) and \( \textbf{102} \) for \( \|\phi_t\|^2 \) and \( \gamma_t \) respectively, one can write down the following stochastic differential equation for \( r_t^2 \):

\[
dr_t^2 = \left[2\sqrt{\lambda}(\langle q \rangle_t - \overline{x}_t) d\xi_t + 4\lambda(\overline{\rho}_t^2 - \langle q \rangle_t \overline{x}_t) dt\right] r_t^2, \quad r_0^2 = 1. \tag{123}
\]

By using relation \( \textbf{11} \), the above equation can be re-written in terms of the Wiener process \( W_t \) as follows:

\[
dr_t^2 = \left[2\sqrt{\lambda}(\langle q \rangle_t - \overline{x}_t) dW_t + 4\lambda(\langle q \rangle_t - \overline{x}_t)^2 dt\right] r_t^2, \quad r_0^2 = 1, \tag{124}
\]

whose solution is:

\[
r_t^2 = \exp\left[2\sqrt{\lambda} \int_0^t (\langle q \rangle_s - \overline{x}_s) dW_s + 2\lambda \int_0^t (\langle q \rangle_s - \overline{x}_s)^2 ds\right]. \tag{125}
\]

The crucial point is to establish the behavior of the difference \( \langle q \rangle_t - \overline{x}_t \) between the mean position of the general solution \( \psi_t \) and the mean position of the “asymptotic” state \( \psi_t^\infty \). Since \( \psi_t \) converges to \( \psi_t^\infty \), we expect \( \langle q \rangle_t - \overline{x}_t \) to vanishes asymptotically. That this is actually true with \( \mathbb{P} \)-probability 1 is proven in Lemma \( \textbf{V.3} \) (see Eq. \( \textbf{129} \)), where indeed it is shown that the convergence is exponentially fast. This fact, together with \( \textbf{125} \), concludes the proof of the lemma.

**Lemma \( \textbf{V.2} \):** Let \( N_t \) be defined as in \( \textbf{108} \). Then, with \( \mathbb{P} \)-probability 1,

\[
\lim_{t \to \infty} N_t = N_\infty \text{ finite.} \tag{126}
\]
Proof. Looking back at Eq. (108), we see that in order to prove this lemma it is sufficient to show that \( \zeta_t \) tends to a finite limit as \( t \to \infty \), with \( \mathbb{P} \)-probability 1. According to our previous definition, \( \zeta_t \) is equal to:

\[
\zeta_t = e^{-(1+i)\omega t/2}(\pi_t - b_t);
\]

(127)

Eqs. (57) and (97), together with the change of measure (11), lead to the following stochastic differential equation for \( \zeta_t \) in terms of the Wiener process \( W_t \):

\[
d\zeta_t = \frac{\omega}{2\sqrt{\lambda}}e^{-(1+i)\omega t/2}\left[dW_t + 2\sqrt{\lambda}(\langle q \rangle_t - \pi_t)dt\right], \quad \zeta_0 = 0.
\]

(128)

Once again, the large time behavior of \( \langle q \rangle_t - \pi_t \) (see Eq. (129)) yields the conclusion of the lemma.

Lemma V.3: Let \( \langle q \rangle_t \equiv \langle \psi_t|q|\psi_t \rangle \) and \( \pi_t \) defined in (97). Then, with \( \mathbb{P} \)-probability 1:

\[
h_t \equiv \langle q \rangle_t - \pi_t = O(e^{-\omega t/2}).
\]

(129)

Proof. Let us consider the Gaussian solution of Eq. (5):

\[
\phi_t^G(x) \equiv G_t(x,0) = K_t \exp\left[-\frac{\alpha_t}{2}x^2 + \alpha_t x + \bar{c}_t\right]
\]

(130)

\[
= K_t \exp\left[-\frac{\alpha_t}{2}(x - \pi_t^G)^2 + i\alpha_t^G x + \bar{c}_t\right]
\]

(131)

where \( G_t(x,y) \) is the Green’s function defined in (12) and

\[
\pi_t^G = \frac{\pi_t^R}{\alpha_t^G}, \quad \bar{c}_t = \bar{c}_t - \frac{\alpha_t^G}{\alpha_t^R} \pi_t^G, \quad \bar{c}_t = \bar{c}_t + \frac{\alpha_t}{2}(\pi_t^G)^2.
\]

(132)

Note that \( \pi_t^G \) is the mean position of the Gaussian state \( \phi_t^G \), while \( \hbar \pi_t^G \) is its average momentum. Obviously we can write:

\[
h_t = (\langle q \rangle_t - \pi_t^G) + (\pi_t^G - \pi_t);
\]

(133)

Lemma [31] proves that \( \langle q \rangle_t - \pi_t^G \) has the required asymptotic behavior (see Eq. (131)), so all we need to show is that also \( \pi_t^G - \pi_t \) behaves as required. Lemma [31] was first proven in [33]; for completeness, we reproduce it in Appendix [41] adapting it to our notation. The proof of the lemma is instructive because it makes clear why it is convenient to analyze \( \langle q \rangle_t - \pi_t^G \) separately from \( \pi_t^G - \pi_t \).

By letting the ground state of the NSA harmonic oscillator evolve according to the Green’s function \( G_t(x,y) \), one can express \( \pi_t \) in terms of the functions (13)–(18); a straightforward calculation leads to the following result:

\[
\pi_t - \pi_t^G = \frac{\omega}{2\lambda}\left[(p_t^{-1} - 1)\pi_t^R - \left(\frac{\beta_t}{\alpha_t + \alpha_\infty}\right)^R\right],
\]

(134)
where $\alpha_\infty \equiv \lim_{t \to \infty} \alpha_t = 2\lambda/\nu$. By inspecting expressions (64) and (15), we recognize that $p_t^{-1} - 1 = O(e^{-\omega t})$ and $|\beta_t| = O(e^{-\omega t/2})$, thus in order to prove the lemma all we have to do is to control the long time behavior of $a_t$, which in turn sets the asymptotic behavior of $b_t$ through (17). Inverting Eq. (132) we get:

$$a_t = \alpha_t \bar{x}_t^G + i \bar{k}_t^G,$$

(135)

thus we can control $\bar{x}_t$ by controlling $\bar{x}_t^G$ and $\bar{k}_t^G$. These two quantities, being the average position and (modulo $\hbar$) average momentum of the Gaussian solution (131), satisfy the stochastic differential equations (84) and (85), with $\alpha_t/2$ in place of $\sigma_t$. By using the change of measure (11), we can re-express these equations in terms of the Wiener process $W_t$ as follows:

$$d\bar{x}_t^G = \left[ \frac{\hbar}{m} \bar{k}_t^G + \frac{2\lambda}{\alpha_t^R} f_t \right] dt + \frac{\sqrt{\lambda}}{\alpha_t^R} dW_t,$$

(136)

$$d\bar{k}_t^G = -2\sqrt{\lambda} \frac{\alpha_t}{\alpha_t^R} f_t dt - \sqrt{\lambda} \frac{\alpha_t}{\alpha_t^R} dW_t,$$

(137)

with $f_t \equiv \langle q \rangle_t - \bar{x}_t^G$. By integrating the second equation, by using the strong law of large numbers applied to $W_t$, Eq. (131) for $f_t$ and the fact that $\alpha_t$ has an asymptotic finite limit, one can show that, with $\mathbb{P}$-probability 1, the process $\bar{k}_t^G$ grows slower than $t^2$, for $t \to \infty$. By integrating now the first equation, and by using the same properties as before, one can show that $\bar{x}_t^G$ grows slower than $t^3$, for $t \to \infty$ and again with $\mathbb{P}$-probability 1. According to Eq. (135) and (17), we then have, with $\mathbb{P}$-probability 1:

$$\bar{x}_t = o(t^3) \text{ as } t \to \infty, \quad \lim_{t \to \infty} \bar{b}_t = \bar{b}_\infty \text{ finite.}$$

(138)

This proves that $\bar{x}_t - \bar{x}_t^G$ has the required asymptotic behavior, hence the conclusion of the lemma.

In this way we have proven that any initial state is $\mathbb{P}$-a.s. norm convergent to the Gaussian state (116), which can be written as follows:

$$\psi_\infty^\omega = \sqrt{\frac{\pi}{z_R}} \exp \left[ -\frac{z^2}{2} (x - \bar{x}_t)^2 + i \bar{k}_t x + i \left( \frac{1}{4} t - \omega \frac{t}{4} \right) \right],$$

(139)

which has a fixed finite spread both in position and in momentum, given by (36):

$$\Delta_q = \langle \psi_\infty^\omega | (q - \bar{x}_t)^2 | \psi_\infty^\omega \rangle^{1/2} = \sqrt{\frac{\hbar}{m\omega}},$$

(140)

$$\Delta_p = \langle \psi_\infty^\omega | (p - i\bar{k}_t)^2 | \psi_\infty^\omega \rangle^{1/2} = \sqrt{\frac{\hbar m\omega}{2}}.$$  

(141)

This corresponds almost to the minimum allowed by Heisenberg’s uncertainty relations, as $\Delta_q \Delta_p = \hbar/\sqrt{2}$. Note also that, the more massive the particle, the smaller the spread in position of the
asymptotic Gaussian state: this is a well known effect of the localizing property of Eq. (1). Finally, Eqs. (100) and (101), together with the change of measure (11), tell how the average position $x_t$ and momentum $\hbar k_t$ evolve in time, as a function of the Wiener process $W_t$:

$$d\bar{x}_t = \frac{\hbar}{m} k_t dt + \omega h_t dt + \frac{\omega}{2\sqrt{\lambda}} dW_t,$$

(142)

$$d\bar{k}_t = 2\lambda h_t dt + \sqrt{\lambda} dW_t,$$

(143)

which imply that there exist two random variables $X$ and $K$ such that:

$$\bar{x}_t = X + \frac{\hbar}{m} K t + \sqrt{\lambda} \frac{\hbar}{m} \int_0^t W_s ds + \sqrt{\frac{\hbar}{m}} W_t + O(e^{-\omega t/2}),$$

(144)

$$\bar{k}_t = K + \sqrt{\lambda} W_t + O(e^{-\omega t/2}).$$

(145)

These parameters fully describe the time evolution of the Gaussian state (139).

VI. CONCLUSIONS AND OUTLOOK

In section II we have spotted three interesting time regimes during which the wave function, depending on the values of the parameters $\lambda$ and $m$, evolves in a different way. In the central sections of this paper we have analyzed the long time behavior, which pertains to the third regime, the diffusive one. There are many other properties of the solutions of Eq. (1) which deserve to be analyzed, and in this conclusive section we would like to point out a number of interesting open problems.

I: Collapse regime. Let $\ell$ be the length which discriminates between a localized and a non-localized wave function, i.e. such that, defining with $\Delta^\psi q$ the spread in position of a wave function $\psi$, we say that $\psi$ is localized in space whenever $\Delta^\psi q \leq \ell$. In our case, we must take $\ell > \sqrt{\hbar/m\omega}$, where $\sqrt{\hbar/m\omega}$ is the asymptotic spread (see Eq. (140)).

**Problem I.1:** collapse time. Let $\psi_t$ be the solution of Eq. (1), for a given initial condition $\psi \in L^2(\mathbb{R})$ such that $\Delta^\psi q > \ell$. Let us define the collapse time $T^\psi_{\text{COL}}$ as the first time at which the wave function is localized in space:

$$T^\psi_{\text{COL}} := \min \{ t : \Delta^\psi q \leq \ell \}.$$

(146)

**Question I.1.1:** How is $T^\psi_{\text{COL}}$ distributed, as a random variable? In particular, is it finite with $\mathbb{P}$-probability 1, as we expect it to be? What are its mean $\mathbb{E}_\mathbb{P}[T^\psi_{\text{COL}}]$ and variance $\mathbb{V}_\mathbb{P}[T^\psi_{\text{COL}}]$?

**Question I.1.2:** How does $T^\psi_{\text{COL}}$ depend on the initial spread $\Delta^\psi q$, as well as on the parameters $\lambda$ and $m$?
Question I.1.3: What is the probability that, for \( t > T_{\text{COL}}^{\psi} \), the wave function de-localizes in space, i.e. acquires a spread greater than \( \ell \), namely \( \Delta^{\psi} q \geq \ell + \epsilon \), where \( \epsilon \) is an arbitrary positive quantity? This kind of analysis is important because it gives a measure of how stable the localization process is.

Problem I.2: Collapse probability. Let \( \overline{\psi} := \psi_t \), for \( t = \mathbb{E}_\theta[T_{\text{COL}}^{\psi}] \). Let \( \overline{x} := (\overline{\psi}|q|\overline{\psi}) \) be the position of the wave function at the average time at which it is localized in space.

Question I.2.1: How is \( \overline{x} \) distributed as a random variable?

Question I.2.2: Let \( p(x) \) be the probability density of \( \overline{x} \); let \( |\psi(x)|^2 \) be the collapses probability density given by the Born probability rule. When does it happen that

\[
d(x) := |p(x) - |\psi(x)||^2 \leq \delta,
\]

where \( \delta \) is an appropriately small number? How does this depend on the values of the parameters \( \lambda \) and \( m \)?

II: Classical regime. in the classical regime, the wave function is expected to move, on the average, like a classical free particle.

Problem II.1: Classical motion. Let \( \overline{q}_t \) and \( \overline{p}_t \) be the (quantum) average position and momentum of \( \psi_t \). Let \( t > T_{\text{COL}}^{\psi} \).

Question II.1.1: How are \( \overline{q}_t \) and \( \overline{p}_t \) distributed, as random variables? In particular, what are their mean \( \mathbb{E}_\theta[\overline{q}_t] \), \( \mathbb{E}_\theta[\overline{p}_t] \) and variances \( \mathbb{V}_\theta[\overline{q}_t] \), \( \mathbb{V}_\theta[\overline{p}_t] \)?

Question II.1.2: How do they depend on the values of \( \lambda \) and \( m \)?

Question II.1.3: Let \( T_{\text{DIF}}^{\psi} \) be the time at which the motion departs from the classical one

\[
T_{\text{DIF}}^{\psi} := \min\{t : \mathbb{V}_\theta[\overline{q}_t] \geq \Lambda_q \lor \mathbb{V}_\theta[\overline{p}_t] \geq \Lambda_p\},
\]

where \( \Lambda_q \) and \( \Lambda_p \) are suitable parameters measuring the fluctuations of the position and momentum, respectively, of the wave function. How does \( T_{\text{DIF}}^{\psi} \) depend on the parameters of the model?

III: Diffusive regime. This regime begins after \( T_{\text{DIF}}^{\psi} \), and it has been analyzed in this paper: as we have seen, the wave keeps diffusing in the Hilbert space, eventually taking a Gaussian shape, as described in Sec. V.

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APPENDIX A: PROPERTIES OF HERMITE POLYNOMIALS

We list here the main properties of Hermite polynomials, which are used in the paper. The primary definition of the Hermite polynomials is

\[ H_n(z) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2z)^{n-2m}}{m!(n-2m)!}, \quad (A1) \]

where \( z \) is any complex number. These polynomials satisfy the following addition rule

\[ H_n(z_1 + z_2) = \sum_{m=0}^{n} \binom{n}{m} (2z_2)^{n-m} H_m(z_1). \quad (A2) \]

When the argument is real \((z = x \in \mathbb{R})\), they form an orthogonal set with respect to the weight \(\exp[-x^2]\); the normalized Hermite polynomials are:

\[ \overline{H}_n(x) = \frac{1}{N_n} H_n(x), \quad N_n = \sqrt{\pi 2^n n!}. \quad (A3) \]

APPENDIX B: LEMMA

**Lemma B.1**: Let \( \phi \in L^2(\mathbb{R}) \), \( \| \phi \| = 1 \) and let \( \phi_t = T_t \phi \). Then, with \( \mathbb{P} \)-probability 1:

\[ f_t \equiv \langle q \rangle_t - \pi_t^2 = O(e^{-\omega t/2}), \quad (B1) \]

where \( \langle q \rangle_t = \langle \psi_t | q | \psi_t \rangle \), and \( \pi_t^2 \) has been defined in \( (132) \).

**Proof.** Using the expression \( (12) \) for \( G_t(x, y) \) together with Schwartz inequality, we can derive the following bound on \( \phi_t \):

\[ |\phi_t(x)|^2 \leq |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ -2\lambda \frac{t}{\omega} - 4\beta_t^2 x^2 + 2 \left( \frac{\alpha_t^R}{p_t} \frac{\beta_t}{p_t} + 8 \frac{\beta_t^2}{\alpha_t^R} \frac{q_t}{p_t} \right) x + 2\pi_t^2 + \frac{\omega (\pi_t^2)^2}{2\lambda} \right], \quad (B2) \]

which holds for any \( t > 0 \). The above inequality implies that it is sufficient to consider \( \phi \in L^2(\mathbb{R}) \) such that:

\[ |\phi(x)| \leq C e^{-Ax^2}, \quad (B3) \]

where \( C \) and \( A \) are random variables. A direct calculation leads to the following expression for the quantum average \( \langle \phi_t | q | \phi_t \rangle \):

\[ \langle \phi_t | q | \phi_t \rangle = |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ 2\pi_t^2 - \frac{\beta_t^2}{2\alpha_t^R} \right] \int dy_1 dy_2 \phi(y_1) \phi(y_2)^* \left[ \frac{\beta_t y_1 + \beta_t^* y_2}{2\alpha_t^R} + \frac{\beta_t^* y_1}{\alpha_t^R} \right] \]

\[ \cdot \exp \left[ \frac{1}{2} \left( \alpha_t - \frac{\beta_t^2}{2\alpha_t^R} \right) y_1^2 - \frac{1}{2} \left( \alpha_t^R - \frac{\beta_t^2}{2\alpha_t^R} \right) y_2^2 \right] \]

\[ \cdot \exp \left[ \left( \frac{\beta_t}{\alpha_t^R} \right) y_1 + \left( \frac{\beta_t^*}{\alpha_t^R} \right) y_2 + \frac{\beta_t^2}{2\alpha_t^R} y_1 y_2 \right]. \quad (B4) \]
As we shall soon see, all exponential terms in the above expression can be controlled. The crucial factors are the two within brackets: the first term decays exponentially in time, since \( \beta_t = O(e^{-\omega t/2}) \), while \( \alpha_t \) has a finite asymptotic limit; the term \( \bar{\alpha}_t^R/\alpha_t^R \), instead, does not decay in time (see the discussion in connection with the proof of lemma \([X]3\)). Since \( \|\phi_t\|^2 \) is equal to the expression \( (B4) \) without the terms in square brackets, and because of \( (B3) \), we have that

\[
|f_t||\phi_t|^2 = \langle \phi_t|q|\phi_t \rangle - \frac{\bar{\alpha}_t^R}{\alpha_t^R}||\phi_t||^2 = 
\]

\[
= |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ \frac{\beta_t^R}{\alpha_t^R} \right] \int dy_1 dy_2 \phi(y_1) \phi(y_2) \left[ \frac{\beta_t y_1 + \beta_t^2 y_2}{2\alpha_t^R} \right] 
\]

\[
\cdot \exp \left[ -\frac{1}{2} \left( \frac{\alpha_t - \beta_t^2}{2\alpha_t^R} \right) y_1^2 - \frac{1}{2} \left( \alpha_t^* - \frac{\beta_t^2}{2\alpha_t^R} \right) y_2^2 \right] 
\]

\[
\cdot \exp \left[ \left( \frac{\beta_t}{\alpha_t^R} \right) y_1 + \left( \frac{\beta_t^2}{\alpha_t^R} \right) y_2 + \left( \frac{\beta_t^2}{2\alpha_t^R} \right) \right]. \tag{B6}
\]

According to the discussion above, we expect the quantity \( |f_t||\phi_t|^2 \) to decay exponentially in time, as we shall now prove; this is the reason why, in proving lemma \([X]3\), it was convenient to split the difference \( h_t \) as done in Eq. \( (B3) \).

Using the inequality \( y_1 y_2 \leq \frac{1}{2} (y_1^2 + y_2^2) \) we can write:

\[
|f_t||\phi_t|^2 = \langle \phi_t|q|\phi_t \rangle - \frac{\bar{\alpha}_t^R}{\alpha_t^R}||\phi_t||^2 
\]

\[
\leq |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ \frac{\beta_t^R}{\alpha_t^R} \right] \int dy_1 dy_2 |\phi(y_1)||\phi(y_2)||y_1 + y_2||g(y_1)g(y_2), \tag{B8}
\]

with:

\[
g(y) \equiv \exp \left[ -\frac{1}{2} \left( \alpha_t^R - \frac{\beta_t^2}{\alpha_t^R} \right) y^2 + \left( \frac{\beta_t^2}{\alpha_t^R} \right) y \right]. \tag{B9}
\]

Next, by using the inequality \( g(y_1) + g(y_2) \leq g(y_1^2 + g(y_2^2)/2 \) and the symmetry between \( y_1 \) and \( y_2 \), we have:

\[
|f_t||\phi_t|^2 \leq |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ \frac{\beta_t^R}{\alpha_t^R} \right] \int dy_1 dy_2 |\phi(y_1)||\phi(y_2)||y_1 + y_2||g(y_1)^2. \tag{B10}
\]

Now, a direct computation shows that

\[
\|G_t(\cdot, y)\|^2 \equiv \int dx |G_t(x, y)|^2 = |K_t|^2 \sqrt{\frac{\pi}{\alpha_t^R}} \exp \left[ \frac{\beta_t^R}{\alpha_t^R} \right] g(y)^2; \tag{B11}
\]

the key point is that, since \( G_t(x, y) \) solves Eq. \( (B) \), then \( \|G_t(\cdot, y)\|^2 \) is a positive martingale with respect to the measure \( Q \), for any value of \( y \); we call \( \text{Mar}_Q(t, y) \) this martingale. We can then write:

\[
|f_t||\phi_t|^2 \leq \int dy_1 dy_2 |\phi(y_1)||\phi(y_2)||y_1 + y_2||\text{Mar}_Q(t, y) 
\]

\[
\leq |K_t| \int dy e^{-\omega y^2} (A_1|y| + A_2) \text{Mar}_Q(t, y), \tag{B12}
\]
where $A_1$ and $A_2$ are suitable constants. In going from the first to the second line, we have used (B3). The quantity

$$\frac{1}{2\epsilon t^2} \int dy e^{-Ay^2} (A_1|y| + A_2) \text{Mar}_Q(t, y)$$

is another positive martingale with respect to $Q$, which we call $\text{Mar}'_Q(t)$. We arrive in this way at the inequality:

$$|f_t| \leq |\beta_t| \frac{\text{Mar}'_Q(t)}{\|\phi_t\|^2}.$$  \hfill (B14)

Since $\text{Mar}'_Q(t)$ is a positive martingale with respect to $Q$, then $\text{Mar}_P(t) = \text{Mar}'_Q(t)/\|\phi_t\|^2$ is a positive martingale with respect to $P$ which, by Doob’s convergence theorem, has a $P$-a.s. finite limit for $t \to +\infty$. The conclusion of the lemma then follows from Eq. (15), according to which $\beta_t = O(e^{-\omega t/2})$.

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