Use of optimal control in studying the dynamical behaviors of fractional financial awareness models

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Abstract
Around there, we new examination has been done on past investigations of perhaps the main numerical models that portray the worldwide monetary development and that is depicted as a non-straight fragmentary monetary model of mindfulness, where the investigations address the means following: One: The schematic of the model is proposed. Two: The sickness-free balance point and the soundness of the harmony point are talked about. Three: The strength of the model is satisfying by drawing the Lyapunov examples. Fourth: The presence of consistently stable arrangements is examined. Five: The Caputo is portrayed as the fragmentary subsidiary. Six: Fragmentary ideal control for NFFMA is examined, by explaining the partial ideal control through drawing when control. Seven: We are utilizing the calculation, summed up Adams–Bashforth–Moulton technique (GABMP) to tackle the is utilized to take the goal of an NFFMA. At last, we show that GABMP is profoundly indistinguishable. The mathematical strategy utilized in this composition to address this model has not been used by any creator before that. Additionally, this model with partial subordinates characterized in this manner has not been concentrated before that. The strategies used are not difficult to impact, regardless of whether logical or mathematical, and give great results.

Keywords Financial of awareness · Stability · Lyapunov exponents · Fractional optimal control · Hamiltonian

1 Foreword
It is completely perceived that the objective of the statement is to persuade purchasers to purchase items, depending on the overall need of these produces to show that they contrast as an unmistakable brand from different items to help purchasers to buy them (Sweilam et al. 2017). There are numerous approaches to turn the clients consider the items, and administrations offered to them. One of them is publicizing through messages. These messages are through physical media, like papers, magazines TVs, and radios. This mission can be through straightforward media, for example, sites and drawing out messages (Gepreel et al. 2019). It is vital to examine publicizing techniques to build deals to accomplish the most noteworthy benefit for the organization, see Fig. 1 (Mahdy and Higazy 2019). Consequently, it is a lot helpful to examine and make a fitting dynamic and to address time-sensitive selling and general assessment (Huang et al. 2012). There are additionally many methodology models to show up in the connection between promoting that distinguishes tangles from the
showcasing and financial administration perspective. Publicizing arrangements are examined over the long run by unique models depicted as differential conditions where sell parcel, arrangements, and all extreme conditions factors are continually evolving. With regard to time. The reason for publicizing is consistently extraordinary for instance, the motivation behind certain commercials is to analyze two, three, or more brands, and for another reason, for example, acquainting a novel item with the shop dependent on these objectives, promoting types are made. Usually, the activity of publicizing is perpetually late on schedule, and it is important to incorporate the memory of various models of an affirmation, so models that depend on the past cases in the current cases have not just their underlying past cases fitting to portray procedures for the announcement.

Hitherto, fractional calculus has gained extraordinary dissemination and importance because of its snappy execution as another model work in a combination of designing and logical spaces (Amer et al. 2017, 2018a, b; Diethelm et al. 2004b; El-Shahed et al. 2017; Freihat and Momani 2012; Gepreel et al. 2020; Jafari and Daftardar-Gejji 2006; Khader et al. 2014b; Liao 2005; Mahdy 2018; Mahdy et al. 2020b, c, f, 2020, 2021; Sweilam and Abou Hasan 2016, 2017, 2020; Sweilam et al. 2019c, 2017; Amer et al. 2018c;Mahdy et al. 2020e; Diethelm et al. 2004a), for example, viscoelasticity (Koeller 1984) and thermoelasticity (Povstenko 2015; Abbas 2012; Khader et al. 2014a; Khamis et al. 2021). The best strategy partial models are leaded as fragmentary differential conditions.

The great object of this composition is to propose a customized concentrate about GABMP for settling NFFMA (Sweilam et al. 2017; Diethelm et al. 2004a):

\[
\begin{align*}
\Delta^\epsilon w_1 &= -\alpha^\epsilon w_1 - \beta(N-w_1) + \mu^-w_1, \\
\Delta^\epsilon w_2 &= \alpha^\epsilon w_1 + \beta(N-w_1) - (\alpha^\epsilon + \alpha^\epsilon)w_2 + \delta^\epsilon w_3 - \mu^-w_2, \\
\Delta^\epsilon w_3 &= (\alpha^\epsilon + \alpha^\epsilon)w_2 - \delta^\epsilon w_3 - \mu^-w_3.
\end{align*}
\]  

with given initial case:

\[
w_1(t) = w_{10}, \quad w_2(t) = w_{20}, \quad w_3(t) = w_{30}
\]

\textbf{Definition 1} The $\Delta^\epsilon$ is Caputo fractional derivative is defined (Elsadany and Matouk 2015; El-Shahed et al. 2017; Mahdy et al. 2020a; Bulut et al. 2013; Sweilam et al. 2019b):

\[
\Delta^\epsilon W(h) = \begin{cases} 
\frac{1}{\Gamma(m-\varepsilon)} \int_0^t \frac{W^{(m)}(\eta)}{(h-\eta)^{\varepsilon-\eta+1}} d\eta, & 0 \leq m - 1 < \varepsilon < m, \quad 0 < \varepsilon \leq 1, \\
W^{(m)}(h), & \varepsilon = m \in N.
\end{cases}
\]

Fig. 1 The suggested graphical of the model

3402 A. M. S. Mahdy et al.
For additional exceptional about the basal definitions and benefits of fractional subsidiaries see (Elsadany and Matouk 2015; El-Shahed et al. 2017; Mahdy et al. 2020a; Bulut et al. 2013; Sweilam et al. 2019b).

The paper is organized into five areas. In segment 2, we study the balance focuses, strength, the presence of consistently stable arrangement nonlinear partial monetary models of mindfulness, explain the elements of the model between Lyapunov types, and Poincare maps. Ideal control for NFFMA is examined in area 3. In area 4, we show a guide to show the action of utilizing (GABMP) to address NFFMA. At long last, appropriate ends are attracted segment 5.

2 Equilibrium and Stability of nonlinear fragmentary monetary models of mindfulness

In this segment, we examine the harmony point and the security of nonlinear fragmentary monetary models of mindfulness (1.1).

2.1 Equilibrium points

We study the harmony points of the nonlinear fragmentary monetary models of mindfulness. The model has one harmony point, more insights concerning balance point and Strength of nonlinear fragmentary models see (Khader et al. 2015; El-Saka 2014; size 2014; El-Shahed et al. xxxx; the Hepatitis C with different types of Virus Genome 2016; Garsow et al. 2000).

Henceforth, we settle the accompanying conditions to decide the harmony point:

\[ \Delta'w_1 = -a'w_1 - \frac{k^c}{N}w_1(N - w_1) + \mu \delta w_1 - \mu \omega w_1 = 0, \]
\[ \Delta'w_2 = a'w_1 + \frac{k^c}{N}w_1(N - w_1) - (a' + \nu')w_2 + \delta w_3 - \mu w_2 = 0, \]
\[ \Delta'w_3 = (a' + \nu')w_2 - \delta w_3 - \mu w_3 = 0. \]

Equation (2.1) has to win the one equilibrium point.
2.2 Studying the stability

We calculate the Jacobian matrix $J$ for the model (1.1) as follows:

$$J = \begin{bmatrix} -u^e - \mu_d^e - k^e + 2k^e G - \lambda & 0 & 0 \\ u^e + k^e - 2k^e G & -a^e - v^e - \mu_d^e - \lambda & 0 \\ 0 & a^e + v^e & -\delta^e - \mu_d^e - \lambda \end{bmatrix}. $$

So, we get

$$|J - \lambda I| = \begin{bmatrix} -u^e - \mu_d^e - k^e + 2k^e G - \lambda & 0 & 0 \\ u^e + k^e - 2k^e G & -a^e - v^e - \mu_d^e - \lambda & 0 \\ 0 & a^e + v^e & -\delta^e - \mu_d^e - \lambda \end{bmatrix} = 0. $$

Then, the eigenvalues approaching by

$$\lambda_1 = -\mu_d^e, \quad \lambda_2 = -a^e - \delta^e - \mu_d^e - v^e, $$
$$\lambda_3 = -(u^e + \mu_d^e + k^e - 2k^e G).$$

The solution is stable.
2.3 Clarify Lyapunov exponents and Poincare map

Figures 2, 3, 4 explain Lyapunov types in various time spans. Figures 5, 6, 7, 8, 9, 10 explain the Poincare guide of the framework for three unique upsides of \( a, k, \) and delta. All of which included model security. All eigenvalues have negative which implies the steadiness of this fixed point and we ensure its security by plotting its Lyapunov models (LE1, LE2, LE3). The estimation used for choosing Lyapunov models had suggested in Mahdy et al. (2020a), see Figs. 2, 3, 4. From Fig. 2, 3, 4, we see that each Lyapunov models are negative after a little transient time that derives the structure has consistent and approaches its fixed point.

2.4 Existence of uniformly stable solution:

Let

\[
\begin{align*}
g_1 &= -u^w_1 - \frac{k^w_1}{N}w_1(N - w_1) + \mu_1^w N - \mu_d^w w_1, \\
g_2 &= u^w_1 + \frac{k^w_1}{N}w_1(N - w_1) - (a^v + v^w)w_2 + \delta w_3 - \mu_d^w w_2, \\
g_3 &= (a^v + v^w)w_2 - \delta w_3 - \mu_d^w w_3. \\
\end{align*}
\]

Let \( D = \{w_1, w_2, w_3 \in \mathbb{R} : |w_1, w_2, w_3| \leq a, \ t \in [0, T]\}. \)

This implies that every one of the three capacities satisfies the Lipschitz condition concerning the three cases, and afterward every one of the three capacities is ceaseless as for the three cases for extra of existence of uniformly see Mahdy et al. (2020a), El-Saka 2014, size (2014); Wolf et al. (1985); Abdel-Halim Hassan et al. (2009).

2.5 Optimal control for fractional financial models of awareness

Let us see the case model given in Eqs. (1.1), in \( \mathbb{R}^3 \), with the set of accepted control functions for more details in Charpentier et al. (2015), Sweilam et al. (2019a), Wolf et al. (1985):

\[
\Omega = \left\{(u(.), v(.)) \in \left(L^2(0, T_f)\right)^2 \mid 0 \leq u(.), v(.) \leq 1, \forall t \in [0, T_f]\right\},
\]

where \( T_f \) has the final time, \( u(.) \) and \( v(.) \) have controls functions.

The objective function is known as

\[
J(u(.), v(.)) = \int_0^{T_f} [Aw_1(t) + Bu^2(t) + Cv^2(t)]dt, \tag{3.1}
\]

where \( A, B, \) and \( C \) illustrate the rule constants.

The premier point in FOCPs is to get the optimal controls \( u(.) \) and \( v(.) \), which minimize the following objective function:

\[
J(u, v) = \int_0^{T_f} \eta[w_1, w_2, w_3, u, v, t]dt, \tag{3.2}
\]

subjected to the constraint

\[
\Delta^i w_1 = \xi_1, \\
\Delta^i w_2 = \xi_2, \quad \Delta^i w_3 = \xi_3, \quad \xi_i = \xi(w_1, w_2, w_3, u, v, t),
\]

\[i = 1, 2, 3. \tag{3.3}\]

The next starting conditions are fulfilled:

\[w_1(0) = w_{10}, \quad w_2(0) = w_{20}, \quad w_3(0) = w_{30}. \tag{3.4}\]

To realize the FOCP, let us think a revised objective (cost) function as directs:

\[
\mathcal{J} = \int_0^{T_f} \left[H(w_1, w_2, w_3, u, v, t) - \sum_{i=1}^{3} \lambda_i \xi_i(w_1, w_2, w_3, u, v, t)\right]dt, \tag{3.5}
\]

the Hamiltonian at the goal functional (3.5) and the control financial models of awareness (1.1) is given as follows:

\[
H(w_1, w_2, w_3, u, v, t) = \eta(w_1, w_2, w_3, u, v, t)
+ \sum_{i=1}^{3} \lambda_i \xi_i(w_1, w_2, w_3, u, v, t), \tag{3.6}\]

\[
H = Aw_1 + Bu^2 + Cv^2 + \lambda_1 \left[-u^w_1 - \frac{k^w_1}{N}w_1(N - w_1) + \mu_1^w N - \mu_d^w w_1\right]
+ \lambda_2 \left[u^w_1 + \frac{k^w_1}{N}w_1(N - w_1) - (a^v + v^w)w_2 + \delta w_3 - \mu_d^w w_2\right]
+ \lambda_3 [(a^v + v^w)w_2 - \delta w_3 - \mu_d^w w_3]. \tag{3.7}\]
From (3.5) and (3.7), we can deduce the necessary and sufficient conditions for FOPC as follows
\[
\Delta^i \lambda_j = \frac{\partial H}{\partial w_i}, \quad \Delta^i \lambda_2 = \frac{\partial H}{\partial w_2}, \quad \Delta^i \lambda_3 = \frac{\partial H}{\partial w_3},
\]
(3.8)
\[
\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0.
\]
(3.9)
\[
\Delta^i w_1 = \frac{\partial H}{\partial z_1}, \quad \Delta^i w_2 = \frac{\partial H}{\partial z_2}, \quad \Delta^i w_3 = \frac{\partial H}{\partial z_3},
\]
(3.10)
\[
\lambda_j, \quad (T_j) = 0.
\]
(3.11)
where \(\lambda_j, \quad j = 1, 2, 3\) have Lagrange multipliers. Equations (3.9) and (3.10) appear the necessary conditions in terms of a Hamiltonian for the FOPC.

We arrive at the following theorem:

**Theorem 1.** If \(u\) and \(v\) are optimal controls with the uniform state \(w^*_1, \quad w^*_2\) and \(w^*_3\) respectively there be adjoint variables \(\lambda^*_j, \quad i = 1, 2, 3\), fulfilled the next:

(i) Co-state equations (adjoint equations)

Laying the cases in the content hypothesis and applying conditions (3.8) (Charpentier et al. 2015; Sweilam et al. 2019a; Sweilam et al. 2019b Sweilam et al. 2019a), we obtain the accompanying three conditions, which can be composed as follows:-
\[
\Delta^i \lambda^*_1 = A + \lambda^*_1 \left(-u^* - k^* + \frac{2k^*}{N} w_1 - \mu^*_d\right)
+ \lambda^*_2 \left(u^* + k^* - \frac{2k^*}{N} w_1\right),
\]
(3.12)
\[
\Delta^i \lambda^*_2 = \lambda^*_2 \left(-a^* - v^* - \mu^*_d\right) + \lambda^*_3 \left(a^* + v^*\right),
\]
(3.13)
\[
\Delta^i \lambda^*_3 = \lambda^*_3 \left(\delta^* + \mu^*_d\right) + \lambda^*_4 \left(-\delta^* - \mu^*_d\right),
\]
(3.14)
(ii) Transversality conditions:
\[
\lambda^*_i (T_j) = 0, \quad i = 1, 2, 3.
\]
(3.15)
(iii) Optimality conditions
\[
H(w^*_1, \quad w^*_2, \quad w^*_3, \quad u^*, \quad v^*, \lambda^*_i) = \min_{0 \leq u, v \leq 1} \quad H(w^*_1, \quad w^*_2, \quad w^*_3, \quad u, \quad v, \lambda^*_i),
\]
(3.16)
As well, the control functions \(u^*, \quad v^*\) are offered by
\[
\frac{\partial H}{\partial u} = 0 \Rightarrow u^{* - 2} = \frac{2B}{zw^*_1(\lambda^*_1 - \lambda^*_2)} \Rightarrow u^* = \frac{2Bu^2}{zw^*_1(\lambda^*_1 - \lambda^*_2)},
\]
(3.17)
\[
\frac{\partial H}{\partial v} = 0 \Rightarrow v^{* - 2} = \frac{2C}{zw^*_2(\lambda^*_2 - \lambda^*_3)} \Rightarrow v^* = \frac{2Cv^2}{zw^*_2(\lambda^*_2 - \lambda^*_3)},
\]
(3.18)
\[
\lambda^*_i = \min \left\{ 1, \quad \max \left\{ 0, \frac{zw^*_1(\lambda^*_1 - \lambda^*_2)}{2B}\right\}\right\},
\]
(3.19)
\[
\lambda^*_i = \min \left\{ 1, \quad \max \left\{ 0, \frac{zw^*_2(\lambda^*_2 - \lambda^*_3)}{2C}\right\}\right\}.
\]
(3.20)

**Proof.** The co-state system Eqs. (3.12)–(3.14) are found from Eq. (3.10) where the Hamiltonian \(H^*\) is given by
\[
H^* = A(w^*_1 + B u^* - C v^* + \lambda^*_1 \Delta^i w^*_1 + \lambda^*_2 \Delta^i w^*_2
+ \lambda^*_3 \Delta^i w^*_3),
\]
(3.21)
Moreover, the case in Eq. (3.11) else fulfilled, and the optimal control in Eqs. (3.19)–(3.20) can be derived from Eq. (3.9).

Letting \(u^*\) and \(v^*\) in (1–1), the next case system will be found as:
\[
\Delta^i w^*_1 = -u^{* - 1} w^*_1 - \frac{k^*}{N} w^*_1 (N - w^*_1) + \mu^*_d N - \mu^*_d w^*_1,
\]
\[
\Delta^i w^*_2 = u^{* - 1} w^*_2 + \frac{k^*}{N} w^*_1 (N - w^*_1) - (a^* + v^*) w^*_2 + \delta^* w^*_3 - \mu^*_d w^*_2,
\]
\[
\Delta^i w^*_3 = (a^* + v^*) w^*_2 - \delta^* w^*_3 - \mu^*_d w^*_3.
\]
(3.22)

For extra properties of fractional optimal control see the references Charpentier et al. (2015), Wang et al. (2013), Sweilam and Al-Mekhlafi (2018), Sweilam et al. (2019a), Wolf et al. (1985).

### 3 Applications

Here GABMP is approaching in this here (Mahdy 2021; Gepreel et al. 2021). In this style, the GABMM is derived for obtaining the numerical solution of the FODEs. Put
\[
D^\varepsilon z(t) = g(t, z(t)), 0 \leq \varepsilon \leq T,
\]
(4.1)
\[
z^{(r)}(0) = z_0, \quad r = 0, 1, ..., [\varepsilon] - 1
\]
(4.2)
be a general case of FODEs. We gain the solution \(z(t)\) in think of implementation of fractional integral on (4.1).
\[
z(t) = \sum_{r=0}^{[\varepsilon]-1} z^{(r)}(t) + \int_{0}^{1} (t - \omega) g(\omega, z(\omega))d\omega.
\]
(4.3)

By setting \(k = \frac{\varepsilon}{m} \), \(t_n = nh, \quad n = 0, 1, ..., m\), Eq. (4.3) will be described as next for integer positive \(m\).
\[
z_n(t_{n+1}) = \sum_{r=0}^{[\varepsilon]-1} \frac{z^{(r)}(t_n)}{m} + \frac{k^*}{\Gamma(\varepsilon + 2)} g(t_{n+1}, z^{(\varepsilon)}(t_{n+1}))
+ \frac{k^*}{\Gamma(\varepsilon + 2)} \sum_{j=0}^{n} a_j z_j(t_j),
\]
(4.4)
The presence of consistently stable arrangements is addressing. The Caputo is portraying as the fractional subordinate. Fragmentary ideal control for NFFMA is examining, through explaining the partial ideal control through drawing when control. GABMP is utilizing to take the goal of an NFFMA. We are showing that GABMP is exceptionally indistinguishable. At last, a novel examination has been done on past investigations of quite possibly the most driving numerical models that name the worldwide financial development and that is depicted as an NFFMA, where the explored at the upper.

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Table 1 The boundary esteems and their definition

| Parameter | Definition |
|-----------|------------|
| N(t)      | Populace all out with time |
| x1(t)     | Some of a bunch of people who do not have the foggiest idea about the element of the produce |
| x2(t)     | Some of a bunch of people who think about the item, however, have not yet purchased it |
| x3(t)     | Some of the gathering of individuals who have purchased the item |
| u         | Knowledge, this changes the consumers from the intangible set x1(t) into the scope one x2(t) by informing them about the produce |
| v         | Try advertisement, this carries the consumers from the potential set x2(t) into the purchased one x3(t) by encouraging them to buy the produce |
| a         | Preliminary rate |
| k         | Associate proportion |
| δ         | Change proportion |
| µb         | Birth proportion |
| µd         | Passing rate |
| u x1(t)   | Total number of persons carry to the aware group x2(t) via declaration |
| (N(t) – x1(t)) | Connect and report a total of k(N(t) – x1(t)), out of which only a fraction of x1(t)/N(t) are latterly informed |

\[
a_{j,n+1} = \begin{cases} n^{+1} - (n + 1)^{+1} & \text{if } j = 0, \\
(n - j + 2)^{+1} - 2(n - j + 1)^{+1} + (n - j)^{+1} & \text{if } 0 < j \leq n, 1, \text{ if } j = n + 1.
\end{cases}
\]

In which the predicted value \( z'_k(t_{n+1}) \) may be derived as

\[
z'_k(t_{n+1}) = \sum_{j=0}^{[n]} z'_j \Gamma(\varepsilon) + \sum_{j=0}^{n} b_{j,n+1} g(t_j, z_j(t_j)), \quad (4.5)
\]

in which

\[
b_{j,n+1} = \frac{k^j[(n - j + 1)^{+1} - (n - j)^{+1}]}{\varepsilon}.
\]

The estimated error is.

\[
\max_{j=0,1,\ldots,m} |z(t_j) - z_k(t_j)| = o(k^p), \quad \text{in which } p = \min\{1 + \varepsilon, 2\}.
\]

In Figs. 5, 6, 7, 8, 9 and 10, show the approximate solutions of NFFMA by using GABMP at \( \varepsilon = 1, 0.85 \) show the illustrates the phase spaces.

It is no doubt that the activity of this way is greatly increased by the calculation of further terms \( w_1(t), w_2(t), \) and \( w_3(t) \) by using GABMP (Table 1).

4 Conclusion

In this paper, the graphical of the model is recommended. The sickness-free harmony point (DFE) and the steadiness of the balance point are explaining. The steadiness of the model is fulfilling by drawing the Lyapunov types and Poincare map. The presence of consistently
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