TRIVIALITY OF VECTOR BUNDLES ON SUFFICIENTLY TWISTED
IND-GRASSMANNIANS

IVAN PENKOV AND ALEXANDER S. TIKHOMIROV

Abstract. Twisted ind-Grassmannians are ind-varieties $G$ obtained as direct limits of Grassmannians $G(r_m, V^{r_m})$, for $m \in \mathbb{Z}_{>0}$, under embeddings $\varphi_m : G(r_m, V^{r_m}) \rightarrow G(r_{m+1}, V^{r_{m+1}})$ of degree greater than one. It has been conjectured in [PT] and [DP] that any vector bundle of finite rank on a twisted ind-Grassmannian is trivial. We prove this conjecture under the assumption that the ind-Grassmannian $G$ is sufficiently twisted, i.e. that $\lim_{m \to \infty} \deg \varphi_1 \cdots \deg \varphi_m = 0$.

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1. Introduction

Ind-Grassmannians are ind-varieties defined by chains of embeddings

$$G(r_1, V^{n_1}) \hookrightarrow G(r_2, V^{n_2}) \hookrightarrow \cdots \hookrightarrow G(r_m, V^{n_m}) \hookrightarrow \cdots,$$

where $G(r, V)$ denotes the Grassmanian of $r$-planes in a finite dimensional vector space $V$. Any of the embeddings $\varphi_m$ has a well defined degree $\deg \varphi_m$, and the ind-Grassmannian defined by (1) is twisted if $\deg \varphi_m > m$ for infinitely many indices. In the special case when $r_m = 1$ and $\deg \varphi_m = 1$ for all $m$, the study of finite rank vector bundles on ind-Grassmannians goes back to W.Barth, A.Van de Ven and A.N.Tyurin, [BV], [T]. In this case the ind-Grassmannian is just the infinite projective space $\mathbb{P}^\infty$, and the remarkable Barth-Van de Ven-Tyurin Theorem claims that any vector bundle of finite rank on $\mathbb{P}^\infty$ is isomorphic to a direct sum of line bundles.

Historically, this is the first manifestation of a general phenomenon that seems to take place for ind-varieties defined via sequences of embeddings similar to (1) with $G(r_m, V_m)$ replaced by arbitrary compact homogeneous spaces: in all cases known, the restriction of any finite rank vector bundle on the ind-variety to a large enough finite dimensional homogeneous subspace is equivariant. Around the same time this phenomenon occurred also in the important work of E. Sato who gave an independent proof of the Barth-Van de Ven-Tyurin Theorem, [S1]. Shortly after that Sato established a more general result which applies in particular to the ind-Grassmannian $G(r, V)$ of $r$-planes in a countable dimensional vector space $V$, [S2].

More recently the subject has been revisited in the papers [DP], [CT] and [PT]. In particular, in [PT] a general conjecture about finite rank vector bundles on ind-Grassmannians has been stated. In fact, as we show in [PT], if the ind-Grassmannian is not twisted (which is easily seen to be equivalent to assuming that $\deg \varphi_m = 1$ for all $m$), this conjecture is a relatively straightforward corollary of Sato’s result. This leaves open the case of a twisted ind-Grassmannian, in which case the conjecture claims simply that finite rank vector bundle on such an ind-Grassmannian is trivial. So far this latter conjecture is established in the following three cases: for a rank two bundle on any twisted ind-Grassmannian [PT], for any finite rank bundle on any twisted projective ind-space (a twisted projective ind-space can be defined via the sequence (1) where $r_m = 1$ and $\deg \varphi_m > 1$ for all $m$) [DP] and for an arbitrary finite rank bundle on some special twisted ind-Grassmannians (here $\varphi_m$ are twisted extensions as defined in [DP]).

In the present paper we consider the case of arbitrary finite rank vector bundle on arbitrary twisted ind-Grassmannians satisfying the condition $\dim r_m = \text{const}$ for all $m$. In fact, we work with a more general class of twisted ind-Grassmannians which we call sufficiently twisted. They
are defined via the condition
\begin{equation}
\lim_{m \to \infty} \frac{r_m}{\deg \varphi_1 \ldots \deg \varphi_{m-1}} = 0.
\end{equation}

Our main idea is that the relatively simple proof of the conjecture in the case of a twisted ind-projective space, [DP], admits an interesting generalization. More precisely, the original method is based on the study of certain morphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ into larger and larger projective spaces. Under the assumption that a twisted infinite projective space admits a non-trivial vector bundle, one pulls it back to $\mathbb{P}^1 \times \mathbb{P}^1$, and for a sufficiently large projective space the pull-back is forced to have numerical invariants which yield a contradiction. The main technical achievement of the present paper is the introduction of an appropriate class of maps of ind-projective space, $\mathbb{P}^1$, admits an interesting generalization. More precisely, the original proof of the conjecture in the case of a twisted infinite projective space was based on the study of certain morphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ into larger and larger projective spaces.

Our notation is mostly standard. The ground field is $\mathbb{C}$. All vector bundles considered are assumed to have finite rank. We do not make a distinction between locally free sheaves of finite rank and vector bundles. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety $X$, $\mathcal{F}^n$ denotes the direct sum of $n$ copies of $\mathcal{F}$, $H^i(\mathcal{F})$ denotes the $i$th cohomology group of $\mathcal{F}$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ stands for the dual bundle, i.e., $\mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $Z \subset X$ is a subvariety, $I_{Z,X}$ denotes the sheaf of ideals corresponding to $Z$.

By $G(r, V)$ we denote the Grassmannian of $r$-dimensional subspaces of a vector space $V$; unless the contrary is stated explicitly, we assume that $\dim V < \infty$, $r \neq 1, r \neq \dim V - 1$.

By a rational curve we always mean a curve isomorphic to $\mathbb{P}^1$, i.e., for convenience we assume a rational curve to be smooth. If $C$ is a rational curve, $\mathcal{O}_C(i)$ stands for a line bundle on $C$ with first Chern class equal to $i \in \mathbb{Z}$. A line in $G(r, V)$ is a rational curve of degree 1 and is determined by a flag of $V_1 \subset V_2$ of subspaces in $V$ with $\dim V_1 = r - 1$, $\dim V_2 = r + 1$.

If $C \subset X$ is a rational curve in an algebraic variety $X$ and $E$ is a vector bundle on $X$, then by a classical theorem of Grothendieck, $E|_C$ is isomorphic to $\bigoplus_i \mathcal{O}_C(d_i)$ for some $d_1 \geq d_2 \geq \cdots \geq d_{rkE}$. We call the ordered $rkE$-tuple $(d_1, \ldots, d_{rkE})$ the splitting type of $E|_C$ and denote it by $d_E(C)$.

We call a curve $C = \bigcup_{i=1}^r C_i$, where $C_i$ are rational curves, a chain of rational curves, if, for each $i < r$, the intersection $C_i \cap C_{i+1}$ is a transversal intersection at a single point and there are no other intersections of the curves $C_i$. If $C$ is a chain of rational curves, $\mathcal{O}_C(n_1, \ldots, n_r)$ denotes a line bundle on $C$ such that $\mathcal{O}_C(n_1, \ldots, n_r)|_{C_i} \simeq \mathcal{O}_{C_i}(n_i)$.

Finally, under a partition of $n \in \mathbb{Z}_{>0}$ (respectively, a strict partition of $n$) we understand a $k$-tuple $(n_1, \ldots, n_k) \in \mathbb{Z}_{>0}^k$ (respectively, $(n_1, \ldots, n_k) \in \mathbb{Z}_{>0}^k$) with $\sum_{i=1}^k n_i = n$. 

3. An estimate for $D_E(C)$

For a vector bundle $E_C$ on a rational curve $C$ with splitting type $d(E_C) = (d_1(E_C), \ldots, d_{rkE}(E_C))$, set $D_E(E_C) := d_1(E_C) - d_{rkE}(E_C)$. Our objective in this section is to prove the following theorem.
Theorem 3.1. Let \( \pi : X \to B \) be a flat family whose fibers are chains of rational curves and whose generic fiber is a rational curve. Assume that, for a point \( 0 \in B, C_0 := \pi^{-1}(0) \) is a chain of rational curves \( C_1 \cup \ldots \cup C_r, \ r \geq 1 \). Let \( E \) be a vector bundle on \( X \). Then there exists a neighbourhood \( U \) of the point \( 0 \) in \( B \) such that for any \( t \in U \) for which \( C_t = \pi^{-1}(t) \) is a rational curve, one has

\[
D(E|_{C_t}) \leq \sum_{i=1}^{r} D(E|_{C_i}).
\]

Proof. The proof is based on two auxiliary results, namely Corollary 3.4 and Lemma 3.5, which we prove later on in this section. First, Corollary 3.4 implies that for any integers \( n_1, \ldots, n_r \) there exists a neighbourhood \( U' \subset B \) of the point \( 0 \) and a line bundle \( \mathcal{L} \) on \( \pi^{-1}(U') \) such that \( \mathcal{L}|_{C_0} \simeq \mathcal{O}_{C_0}(n_1, \ldots, n_r) \). Hence \( E|_{C_0} \otimes \mathcal{L} \simeq E|_{C_0}(n_1, \ldots, n_r) \) and \( E|_{C_i} \otimes \mathcal{L} \simeq E|_{C_i}(n_1 + \ldots + n_r) \) for any \( t \in U' \) for which \( C_t = \pi^{-1}(t) \) is a rational curve. By semicontinuity,

\[
h^0((E|_{C_0})(n_1, \ldots, n_r)) \geq h^0((E|_{C_i})(n_1 + \ldots + n_r)).
\]

Therefore, for \( n_i = -d_i(E|_{C_i}) - \delta_i \), where \( \delta_i \) are as in Lemma 3.5 the inequality (4) and Lemma 3.5 imply

\[
h^0(E|_{C_i}(-\sum_{i=1}^{r} d_i(E|_{C_i}) - \sum_{i=1}^{r} \delta_i)) = 0.
\]

In particular, (5) holds for the following \( r \) choices of \( \delta_1, \ldots, \delta_r: \ \delta_0 = 1, \ldots, \delta_j = 0 \) for \( j \neq i_0, i_0 \), running from 1 to \( r \). Therefore, for \( t \) in the intersection \( U \) of the corresponding \( r \) neighbourhoods \( U' \) we have

\[
d_1(E|_{C_i}) \leq \sum_{i=1}^{r} d_1(E|_{C_i}).
\]

Since \( d_{rk}(E) = -d_1(E') \), inequality (3) applied to \( E' \) instead of \( E \) yields

\[
d_{rk}(E|_{C_i}) \geq \sum_{i=1}^{r} d_{rk}(E|_{C_i})
\]

for \( t \in U \). The desired inequality (3) follows from (6) and (7).

We now proceed to the auxiliary statements used above. Given a strict partition \((n_1, \ldots, n_r)\) of \( n \in \mathbb{Z}_{>0} \), we define a polarized chain (of rational curves) as a pair \((C, \mathcal{O}_C(n_1, \ldots, n_r))\), where \( C = C_1 \cup \ldots \cup C_r \) is a chain of rational curves.

Lemma 3.2. For any strict partition \((n_1, \ldots, n_r)\) of \( n \in \mathbb{Z}_{>0} \) and any polarized chain \((C, \mathcal{O}_C(n_1, \ldots, n_r))\) there exists a linearly normal embedding \( i : C \hookrightarrow \mathbb{P}^n \) such that \( \mathcal{O}_C(n_1, \ldots, n_r) \simeq i^*\mathcal{O}_{\mathbb{P}^n}(1) \).

Proof. We use induction on \( r \). For \( r = 1 \) the desired embedding \( i : C \hookrightarrow \mathbb{P}^n \) is clearly given by the complete linear series \( |\mathcal{O}_C(n_1)| \). Assume now that the claim is true for \( r - 1 \). If we decompose \( C \) as \( C = C' \cup C_r \), where \( C' := C_1 \cup \ldots \cup C_{r-1} \), and set \( n' := n - n_r \), then by the induction assumption there exists a linearly normal embedding \( i' : C' \hookrightarrow \mathbb{P}^{n'} \) such that \( \mathcal{O}_C(n_1, \ldots, n_{r-1}) = i'^*\mathcal{O}_{\mathbb{P}^{n'}}(1) \). Next, consider the linearly normal embedding \( i_r : C_r \hookrightarrow \mathbb{P}^{n_r} \) by the complete linear series \( |\mathcal{O}_{C_r}(n_r)| \) and embed the spaces \( \mathbb{P}^{n'} = \text{Span}(i'(C')) \) and \( \mathbb{P}^{n_r} = \text{Span}(i_r(C_r)) \) into the projective space \( \mathbb{P}^n \) in such a way that their intersection \( \mathbb{P}^{n'} \cap \mathbb{P}^{n_r} \) in \( \mathbb{P}^n \) is a point. We may assume, after possible projective linear transformations of \( \mathbb{P}^{n'} \) and \( \mathbb{P}^{n_r} \), that this point equals \( i'(C') \cap i_r(C_r) \). Thus we obtain an embedding \( i : C \hookrightarrow \mathbb{P}^n \) such that \( i|_{C'} = i' \), \( i|_{C_r} = i_r \) and, by the construction, \( \mathcal{O}_C(n_1, \ldots, n_r) = i^*\mathcal{O}_{\mathbb{P}^n}(1) \). \( \square \)

\(^1\)Recall that \( i \) is linearly normal if \( \text{Span}(i(C)) = \mathbb{P}^n \).
Next, we recall that a k-pointed chain (of rational curves) is a datum \((C, B_1, ..., B_k)\) consisting of: (i) a chain \(C = C_1 \cup ... \cup C_r\) of rational curves, (ii) a set of \(k\) distinct points \(B_1, ..., B_k \in C\) which for \(r > 1\) are also distinct from the points \(A_i = C_i \cap C_{i+1}, \ i = 1, ..., r - 1\). For any k-pointed chain \((C, B_1, ..., B_k)\) we denote its isomorphism class by \([C, B_1, ..., B_k]\). The set \(M_{0,k}\) of isomorphism classes of k-pointed chains is the well known moduli space of k-pointed chains (of rational curves).

Denote by \(\text{Hilb}^{n+1}\mathbb{P}^n\) the Hilbert scheme of subschemes of \(\mathbb{P}^n\) with Hilbert polynomial \(n + 1\). Fix \(n + 2\) points \(B_1, ..., B_{n+2} \in \mathbb{P}^n\) in general position, i.e. such that no \(n + 1\) points lie in a hyperplane of \(\mathbb{P}^n\). Consider the set \(V(B_1, ..., B_{n+2}) := \{C \in \text{Hilb}^{n+1} \mathbb{P}^n|(C, B_1, ..., B_k)\) is a k-pointed chain in \(\mathbb{P}^n\}\) and the morphism

\[
\theta : V(B_1, ..., B_{n+2}) \to M_{0,n+2}, \ C \mapsto [(C, B_1, ..., B_{n+2})].
\]

In addition, put \(\Gamma(B_1, ..., B_{n+2}) := \{(x, C) \in \mathbb{P}^n \times V(B_1, ..., B_{n+2}) | x \in C\}\).

Now we invoke results of Kapranov [K] concerning k-pointed chains. Together with Lemma 3.2 these results yield the following proposition.

**Proposition 3.3.**

1) The morphism \(\theta\) is an isomorphism, hence it induces an embedding \(i_\Gamma : \Gamma(B_1, ..., B_{n+2}) \hookrightarrow \mathbb{P}^n \times \overline{M}_{0,n+2}\).

2) For any strict partition \((n_1, ..., n_r)\) of \(n \in \mathbb{Z}_{\geq 0}\) and any polarized chain \((C_0, O_{C_0}(n_1, ..., n_r))\), there exist points \(B_1, ..., B_{n+2} \in C_0\) such that the point \(c_0 = [(C_0, B_1, ..., B_{n+2})] \in M_{0,n+2}\) satisfies the condition \(O_{C_0}(n_1, ..., n_r) \simeq i_*^r(O_{\mathbb{P}^n}(1) \boxtimes O_{\overline{M}_{0,n+2}}(\alpha_0))\).

3) The family of curves \(\pi : \Gamma(B_1, ..., B_{n+2}) \cong \mathbb{P}^n \times \overline{M}_{0,n+2} \to \overline{M}_{0,n+2}\) is a semiuniversal deformation of the curve \(C_0\), i.e., for any flat family \(\pi : X \to B\) of chains of rational curves such that \(C_0 = \pi^{-1}(0)\) for some point \(0 \in B\), there exists a neighbourhood \(U \ni 0 \in B\) and a morphism \(f : U \to \overline{M}_{0,n+2}\) with \(f(0) = c_0\) and \(\pi^{-1}(U) = \Gamma(B_1, ..., B_{n+2}) \times_{\overline{M}_{0,n+2}} U\).

4) The line bundle \(L := \Phi i_*^r(O_{\mathbb{P}^n}(1) \boxtimes O_{\overline{M}_{0,n+2}})\), where \(\Phi : \pi^{-1}(U) \to \Gamma(B_1, ..., B_{n+2})\) is the induced morphism, satisfies the property \(L|_{C_0} \simeq O_{C_0}(n_1, ..., n_r)\).

**Corollary 3.4.** Let \(\pi : X \to B\) be a flat family of chains of rational curves. Let \(0 \in B\) be a fixed point. For any line bundle \(L_0\) on the fiber \(C_0 = \pi^{-1}(0)\) there exists a neighbourhood \(U \ni 0 \in B\) and a line bundle \(L\) on \(\pi^{-1}(U)\) such that \(L|_{C_0} \simeq L_0\).

**Proof.** If \(L_0\) is ample, its restrictions to each irreducible component of the fiber \(C_0\) define a strict partition \((n_1, ..., n_r)\), and our statement is an immediate consequence of Proposition 3.3.2. Since any line bundle \(L_0\) on \(C_0\) can be represented as \(L_0' \otimes L_0''\) for some ample bundles \(L_0', L_0''\), the Corollary follows.

**Lemma 3.5.** Let \(C = C_1 \cup ... \cup C_r\), be a chain of rational curves and let \(E\) be a vector bundle on \(C\). Then, for any \(\delta_1, ..., \delta_r \in \mathbb{Z}_{\geq 0}\) with \(\sum \delta_i > 0\), one has

\[
h^0(E(-\delta_1(E|_{C_1}) - \delta_1, ..., -\delta_1(E|_{C_r}) - \delta_r)) = 0.
\]

**Proof.** We use induction on \(r\). For \(r = 1\) the statement is clear from the definition of \(\delta_1(E)\). For the step of induction we just consider the case \(r = 2\), since for arbitrary \(r\) the argument goes through without changes. Let \(r = 2\) and \(\delta_1 > 0\). Then clearly \(h^0(E(-\delta_1(E|_{C_1}) - \delta_1)) = 0\). Hence the natural exact triple

\[
0 \to (E|_{C_2})(-\delta_1(E|_{C_2}) - \delta_2 - 1) \to E(-\delta_1(E|_{C_1}) - \delta_1, -\delta_1(E|_{C_2}) - \delta_2) \to (E|_{C_1})(-\delta_1(E|_{C_1}) - \delta_1) \to 0
\]

implies the equality \(h^0(E(-\delta_1(E|_{C_1}) - \delta_1, -\delta_1(E|_{C_2}) - \delta_2)) = 0\). \(\square\)
Let $E$ be a rank $k$ vector bundle on $G = G(r, V)$. Denote
\[
D(E) := \max \{D(E|_l) \mid l \text{ is a line in } G\}.
\]
Our aim in this section is to prove that, for any point $y_0 \in G$, the inequality $D(E|_C) \leq 2rD(E)$ holds on a dense open subset of a suitably defined subscheme of the Hilbert scheme $H_{2r}(y_0)$ of rational curves $C$ of degree $2r$ on $G$ passing through the point $y_0$.

We start with the following construction. Under the assumption that $\dim V \geq 3$, let $V', V'', V'''$ be three $r$-dimensional subspaces of $V$ such that
\[
V' \cap V'' = V' \cap V''' = V'' \cap V''' = \{0\}, \quad V''' \subset V' + V''.
\]
In addition, fix $r$ linearly independent one-dimensional subspaces $V'_i$, $i = 1, ..., r$, in $V'$. This datum defines linearly independent one-dimensional subspaces $V''_i := V'' \cap (V'' + V'_i)$ in $V''$, as well as linearly independent one-dimensional subspaces $V'''_i := V''' \cap (V' + V''_i)$. We obtain $r$ projective lines $\mathbb{P}_i^1 := P(V_i' \oplus V_i'')$, with points $V_i', V_i'', V_i'''$, $i = 1, ..., r$, on them. On each of the lines $\mathbb{P}_i^1$ there is an affine coordinate $t_i$ uniquely determined by the condition
\[
V'_i = \{t_i = 0\}, \quad V''_i = \{t_i = \infty\}, \quad V'''_i = \{t_i = 1\}, \quad i = 1, ..., r.
\]
Let $V_{tr} \in \mathbb{P}_i^1$ be the point with affine coordinate $t_i$. By construction, the points $V_{t_1}, ..., V_{t_r}$, considered as one-dimensional subspaces of $V$, are linearly independent in $V$ and their span $V_{t} := V_{t_1} \oplus ... \oplus V_{t_r}$ is an $r$-dimensional subspace of $V$. Thus we have an embedding
\[
\varphi_1 : \mathbb{P}_i^1 \to G, \quad t \mapsto V_t^i
\]
such that $\varphi_1(0) = V'$, $\varphi_1(\infty) = V''$, $\varphi_1(1) = V'''$, $\varphi_1^*\mathcal{O}_G(1) = \mathcal{O}_1(r)$. Note that the degree $r$ curve $C^i_t$ is an embedding depends only on the choice of the triple of one-dimensional spaces $V', V'', V'''$. We call the curve $C^i_t$ the Segre curve associated to $V', V'', V'''$. Moreover, the subspaces $V', V'', V'''$ define an embedding
\[
s = s(V', V'', V''') : \mathbb{P}_i^1 \times \mathbb{P}_r^1 \to \mathbb{P}_r^{2r-1} \xrightarrow{\ell_{1,1}} \mathbb{P}_r^{2r-1} \to P(V),
\]
where $s_{1,1}$ is the Segre embedding and $j$ is an embedding with $\im j = P(V' \oplus V'')$. By construction, $\varphi_1(t) = s(V', V'', V''')(t) \times \mathbb{P}_r^{r-1}$. We call $s(V', V'', V''')$ the extended Segre embedding associated to $V', V'', V'''$.

More generally, for any $(t_2, ..., t_r) \in (k^*)^{r-1}$ the triple of spaces $V', V'', V^{(t_2, ..., t_r)}$, where
\[
V^{(t_2, ..., t_r)} := V_{t_1} \oplus V_{t_2} \oplus ... \oplus V_{t_r},
\]
satisfies condition (8) with $V'''$ replaced by $V^{(t_2, ..., t_r)}$ and hence yields a Segre curve
\[
C^i_{(t_2, ..., t_r)} := \im s(V', V'', V^{(t_2, ..., t_r)})
\]
in $G$. In particular, $C^i_{(1, ..., 1)}$ coincides with the Segre curve $C^i_1 = \im \varphi_1$. In addition,
\[
y_0 := \{V'\} \in C^i_{(t_2, ..., t_r)}.
\]
Let $l_1$ be the line in $G$ determined by the flag $(V_1'' \oplus V_1' \subset V_1'' \oplus V_1' \oplus \ldots \oplus V_1')$ and let $l_i$, for $i = 2, ..., r$, be the line in $G$ determined by the flag $(V_i'' \oplus \ldots \oplus V_i'' \oplus V_i' \oplus \ldots \oplus V_i') \subset V_i'' \oplus \ldots \oplus V_i'' \oplus V_i' \oplus \ldots \oplus V_i')$. These lines constitute a chain of rational curves
\[
C^i_0 = l_1 \cup \ldots \cup l_r
\]
in $G$. Moreover, setting $(t_2, t_3, ..., t_r) = (t, t^2, ..., t^r-1)$, one easily proves the following lemma.
Lemma 4.1. Let $G = G(r, V)$, with $\dim V \geq 3r$. Consider the surface $S^r := \{(x, t) \in G \times \mathbb{k}^r \mid x \in C^r_t := C^r_{(t, \ldots, t^{r-1})}\}$ with projection $\pi^r : S^r \to \mathbb{k}^r$, $(x, t) \mapsto t$. Then the following statements hold.

(i) The fibers $C^r_t = (\pi^r)^{-1}(t)$ are rational curves passing through the point $y_0 \in G$.

(ii) Let $\mathbb{k}^r \to \mathbb{k}$ be the standard inclusion and let $S$ be the closure of $S^r$ in $G \times \mathbb{k}$. The extended projection $\pi : S \to \mathbb{k}$ is a flat morphism and the natural morphism $\nu : S \to G$, $(x, t) \mapsto x$ is birational and gives an isomorphism $\nu : \pi^{-1}(0) \simeq C^r_y$.

We note next that the condition $\dim V \geq 3r$ in Lemma [4.1] can be removed. In fact, let $\dim V < 3r$. By our assumption (see Section [2]), $r + 2 \leq \dim V$. Fix a $3r$-dimensional vector space $V$ and let $V^\prime, V^\prime\prime, V^\prime\prime\prime$ be three $r$-dimensional subspaces of $V$ satisfying the conditions (8). By performing the above constructions for this datum we obtain a surface $\tilde{S}$ with projections $\mathbb{A}^1 \xrightarrow{\pi} \tilde{S} \xrightarrow{\nu} \tilde{G} := G(r, \tilde{V})$ as in Lemma [4.1]. For $y \in \tilde{S}$, let $V_y$ be the $r$-dimensional subspace of $\tilde{V}$ corresponding to the point $\tilde{\nu}(y)$ in $\tilde{G}$. Since $\dim \tilde{S} = 2$, it follows from the inequality $r + 2 \leq \dim V$ that there exists a subspace $L$ of dimension $3r - \dim V$ in $\tilde{V}$ such that

\[ L \cap (\bigcup_{y \in \tilde{S}} V_y) = \{0\}. \]

Fix an isomorphism $h : \tilde{V}/L \simeq V$ and consider the rational morphism $\tilde{f} : \tilde{G} \dasharrow G$, $V^r \mapsto h((V^r + L) \mod L)$. Then [11] implies that the morphism $f : \tilde{S} \to G \times \mathbb{A}^1, (x, t) \mapsto (\tilde{f}(x), t)$ is an embedding, i.e. that there exists an isomorphism $g : S := f(\tilde{S}) \simeq \tilde{SS}$ such that $f \circ g = \text{id}_S$. Hence the surface $S$ with its projections $\pi := \tilde{\pi} \circ g : S \to \mathbb{A}^1$ and $\nu := f \circ \tilde{\nu} \circ g : S \to G$ satisfies the assertion of Lemma [4.1].

Theorem 3.1. Lemma [4.1] and this latest argument directly imply the following corollary.

Corollary 4.2. Let $E$ be a rank $k$ vector bundle on the Grassmannian $G$. There exists an open subset $U(E)$ of $\mathbb{k}^r$ such that, in the notation of Lemma [4.1], the inequality

\[ D(E_{C^r_t}) \leq rD(E) \]

holds for any $t \in U(E)$.

Fix $t \in U(E)$. According to (9), the extended Segre embedding $s_t := s(V^\prime, V^\prime\prime, V^\prime\prime\prime) : \mathbb{P}^1 \times \mathbb{P}^{r-1} \hookrightarrow P(V)$ induces an embedding $\psi_t : \mathbb{P}^1 \to G$, $u \mapsto s_t(\{u\} \times \mathbb{P}^{r-1})$, such that

\[ \text{im} \psi_t = \pi^{-1}(t) = C^r_t \]

is the Segre curve from Lemma [4.1]. We will now construct another Segre curve $C^r_i$ in $G$ such that $C^r_t \cup C^r_i$ is a chain of rational curves (see below).

For this, assume again temporarily that $\dim V \geq 3r$. Set $W^r := V^r$, $W_i := V_i$, $i = 1, ..., r$, and choose two $r$-dimensional subspaces $W^r, W^\prime$ in $V$ satisfying the conditions similar to (8)

\[ W^r \cap W^\prime = W^r \cap W^\prime \cap W^\prime \cap W^\prime = \{0\}, \quad W^\prime \subset W^r \oplus W^\prime, \]

and the condition

\[ W^r \cap (V^r \oplus V^\prime) = \{0\}. \]

We repeat the above construction for the datum $(W^r, W^\prime, W^\prime \cap W^\prime)$ instead of $(V^r, V^\prime, V^\prime \cap V^\prime)$. First, there are uniquely defined linearly independent one-dimensional subspaces $W^r_t := W^r \cap (W^\prime \oplus W^\prime)$ in $W^r$, as well as linearly independent one-dimensional subspaces $W^\prime_i := W^\prime \cap (W^r \oplus W^\prime)$ in $W^\prime$. Furthermore, for $i = 1, ..., r$ there is a uniquely defined affine coordinate $t_i$ on the projective line $P(W^r_t \oplus W^\prime_i)$ such that $W^r_t = \{t = 0\}$, $W^\prime_i = \{t = \infty\}$, $W^\prime_i = \{t = 1\}$. Denote by $W^r_t$ the point on $P(W^r_t \oplus W^\prime_i)$ with coordinate $t_i$. Finally, set $W^{(t_2, \ldots, t_r)} := W^r \oplus W^r_{t_2} \oplus \cdots \oplus W^r_{t_r}, (t_2, ..., t_r) \in (\mathbb{k}^*)^{r-1}$. 


Moreover, for any \( t \in \mathbb{K}^* \) consider the extended Segre embedding \( s'_t := s(W', W''', W(t, t', \ldots, t^{r-1})) : \mathbb{P}^t \times \mathbb{P}^{r-1} \hookrightarrow P(V) \), where \( \mathbb{P}^t \) is a copy of \( \mathbb{P}^1 \). This yields a map \( \psi'_t : \mathbb{P}^1 \to G, u \mapsto s'_t(\{u\} \times \mathbb{P}^{r-1}) \), such that

\[
C''_t = \text{im} \psi'_t
\]

is a degree-\( r \) Segre curve and, as in Corollary 12, there exists an open subset \( U'(E) \) of \( \mathbb{K}^* \) such that

\[
\text{D}(E|_{C''_t}) \leq r \text{D}(E), \quad t \in U'(E).
\]

Moreover, for \( t \in U_0(E) := U(E) \cap U'(E) \) we have \( C'_t \cap C''_t = \{ V'' \} \), i.e.

\[
C'_t \cup C''_t, \quad t \in U_0(E),
\]

is a chain of rational curves in \( G \) passing through the point \( y_0 = \{ V' \} \).

Let \( y \in U \) and \( y \neq 0 \). Assume first that \( \dim V \geq 3r \) and consider a pencil of conics \( C' \) in \( V \) passing through the point \( y \). Set \( \tilde{Y} := Y^{\text{triv}} \) and a marked point \( z_0 \in \tilde{Y} \) such that \( \tilde{s}_t(\{ z_0 \} \times \mathbb{P}^{r-1}) = P(V') \). Note that, as a consequence of (14), there exists an embedding of \( g : Y \hookrightarrow \mathbb{P}^2 \) such that \( \tilde{s}_t \) fits into the composition of maps

\[
\tilde{s}_t : Y \times \mathbb{P}^{r-1} \overset{\text{triv}}{\hookrightarrow} \mathbb{P}^2 \times \mathbb{P}^{r-1} \overset{s_{1,1}}{\hookrightarrow} \mathbb{P}^{3r-1} \overset{\psi}{\hookrightarrow} P(V),
\]

where \( s_{1,1} \) is the Segre embedding by the linear series \( |O_{\mathbb{P}^2}(1) \otimes O_{\mathbb{P}^{r-1}}(1)| \) and \( \psi \) is an embedding.

Next, consider the pencil of conics \( \{ Y_\tau \}_{\tau \in \mathbb{P}^1} \) in \( \mathbb{P}^2 \) satisfying the conditions: (i) \( Y_0 = Y \), (ii) all conics of the pencil pass through the point \( z_0 \) and (iii) the generic conic in the pencil is smooth (i.e. a rational curve). Set \( U' = \{ \tau \in \mathbb{P}^1 \setminus \{ 0 \} \mid Y_\tau \text{ is smooth} \} \) (this is a dense open subset of \( \mathbb{P}^1 \)). In view of (13) there exists a dense open subset \( U^* \) such that, for any \( \tau \in U^* \) the composition \( \tilde{s}_t : \mathbb{P}^1 \times \mathbb{P}^{r-1} \overset{\text{triv}}{\hookrightarrow} Y_\tau \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^{r-1} \overset{s_{1,1}}{\hookrightarrow} \mathbb{P}^{3r-1} \overset{\psi}{\hookrightarrow} P(V) \), coincides with the embedding \( f_{t,\tau} : \mathbb{P}^1 \times \mathbb{P}^{r-1} \hookrightarrow P(V) \) by a subseries of the linear series \( |O_{\mathbb{P}^1}(2) \otimes O_{\mathbb{P}^{r-1}}(1)| \). This implies that the induced map

\[
\varphi_{t,\tau} : \mathbb{P}^1 \to G, u \mapsto f_{t,\tau}(\{u\} \times \mathbb{P}^{r-1})
\]

satisfies the property

\[
\varphi_{t,\tau}^* Q \simeq (O_{\mathbb{P}^1}(2))^r, \quad \tau \in U^*.
\]
Put $\mathcal{U} := \mathcal{U}^* \cup \{0\}$ and consider the total space $\Pi_t := \{(x, \tau) \in \mathbb{P}^2 \times \mathcal{U} | x \in Y_\tau\}$ of the above pencil of conics, together with the projections $\mathbb{P}^2 \xrightarrow{\rho} \Pi_t \xrightarrow{\rho} \mathcal{U}$. We obtain a morphism $\varphi_t : \Pi_t \to G, (x, \tau) \mapsto s_{1,1}(\{x\} \times \mathbb{P}^{r-1})$.

By construction, $\rho^{-1}(\tau) = Y_\tau$ and for $\tau \in \mathcal{U}^*$ the map $\varphi_{\tau, Y_\tau}$ coincides with $\varphi_{t, \tau}$ from (19). Moreover, by (12) and (15) we have an isomorphism

\[\varphi_{\tau, Y_\tau} : Y_0 \cong C_{t_0}^{2r} := C_t^r \cup C_{t'}^r.\]

This means that $\varphi_{\tau, Y_\tau}$ is an embedding near $\tau = 0$, i.e. the set $\mathcal{U}_0 = \{ \tau \in \mathcal{U} | Y_\tau$ is smooth and $\varphi_{\tau, Y_\tau}$ is an embedding $\}$ is dense in $\mathcal{U}$. We thus obtain isomorphisms

\[\varphi_t : \mathbb{P}^1 \xrightarrow{\sim} C_{t, \tau}^r \xrightarrow{\text{im}} \mathcal{U}_0, \quad \tau \in \mathcal{U}_0.\]

The isomorphisms (21) and (22) show that $\{C_{t, \tau}^r\}_{\tau \in \mathcal{U}_0 \cup \{0\}}$ is a flat family of curves in $G$ whose fiber at 0 is a chain of rational curves of the form $C_{t_0}^{2r} = C_t^r \cup C_{t'}^r$ and whose other fibers are rational curves $C_{t, \tau}^r$. Hence, applying Theorem 3.1 to $\varphi_t$ we obtain that $\mathcal{U}(E) := \{ \tau \in \mathcal{U}_0 \cup \{0\} | \text{im}(D(E_{C_{t, \tau}^r})) \leq \text{im}(D(E_{C_{t_0}^{2r}}) + D(E_{C_{t'}^r})) \}$ is a dense open subset of $\mathcal{U}_0 \cup \{0\}$. Combining this with (11) and (14), and using (21), we obtain the assertion of the Proposition for any curve $C := C_{t, \tau}^r$, $(t, \tau) \in \mathcal{U}_0(E) \times \mathcal{U}_0(E)$.

Finally, it remains to remove the assumption $\dim V \geq 3r$. Let $\dim V < 3r$. Take a space $\tilde{V}$ of dimension $3r$ and choose its subspace $L$ of dimension $3r - \dim V$ satisfying the condition

$$P(L) \cap s_{1,1}(\mathbb{P}^2 \times \mathbb{P}^{r-1}) = \emptyset,$$

where $s_{1,1}$ is the Segre embedding defined in (13) and where the intersection is taken in the space $\mathbb{P}^{3r-1}$ which is identified with $P(\tilde{V})$ in view of the condition (14). (Note that $L$ always exists as $\dim V \geq r + 2$.) The rest of the argument goes through as in the remark preceding the Proposition.

We are now ready to discuss Hilbert schemes. Recall that any rational curve of given degree $k$ in $G$ can be considered as a point in the Hilbert scheme $\text{Hilb}^{k+1} G$. Set

\[H_k := \{ C \in \text{Hilb}^{k+1} G | C \text{ is a rational curve of degree } k \text{ in } G \}, \]

\[R_k := \{ \varphi : \mathbb{P}^1 \to G | \varphi \text{ is an embedding} \}.\]

It is well known (see, e.g., [51, Theorem 2.1]) that $H_k$ is a smooth irreducible open subset of $\text{Hilb}^{k+1} G$ and that the natural morphism

\[g_k : R_k \to H_k, \varphi \mapsto \text{im} \varphi\]

is a principal $\text{PGL}(2)$-bundle. Next, consider the vector space $\text{Hom}(V^\vee, k^r \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2)))$ and its dense open subset

\[W := \{ e \in \text{Hom}(V^\vee, k^r \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) | \text{ the composition } \]

\[\tilde{e} : V^\vee \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{ev} k^r \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{ev} k^r \otimes \mathcal{O}_{\mathbb{P}^1}(2) \text{ is an epimorphism} \} ,\]

where $ev$ is the evaluation map. Let $\gamma : V^\vee \otimes \mathcal{O}_G \to Q$ be the natural epimorphism. By the universality of the Grassmannian $G$ any element $e \in W$ defines a pair

\[\chi_e : \mathbb{P}^1 \xrightarrow{\varphi_e} G, \quad \chi_e : \varphi_e^* Q \xrightarrow{\sim} k^r \otimes \mathcal{O}_{\mathbb{P}^1}(2) \]

such that

\[\chi_e \circ \varphi_e^* \gamma = \tilde{e},\]

where $\tilde{e}$ is defined in (25). Conversely, the element $e$ is recovered by the pair $(\varphi_e, \chi_e)$ since clearly $e$ is obtained from $\tilde{e}$ by passing to sections:

\[e = H^0(\tilde{e}).\]
Now put $k = 2r$ in (24) and consider the set

$$H^*_2 := \{ C \in H_{2r} \mid D_0(C) = 0, \text{ i.e. } Q_C \simeq k^r \otimes O_{P^1}(2) \}.$$

By semicontinuity, $H^*_2$ is an open subset of $H_{2r}$. Moreover, $H^*_2$ is nonempty (and hence dense in $H_{2r}$) since it contains all curves $C_{t,r}^{2r}$ from Proposition 4.3.

**Theorem 4.4.** Fix a point $y_0 \in G$ and put

$$H_{2r}^*(y_0) := \{ C \in H^*_{2r} \mid y_0 \in C \}.$$  

For any vector bundle $E$ on $G$, $B(E, y_0) := \{ C \in H^{*}_{2r}(y_0) \mid D(E_C) \leq 2rD(E) \}$ is a dense open subset of the irreducible variety $H^*_2(y_0)$.

**Proof.** Since $H^*_2$ is smooth and irreducible, and the group $PGL(V^\vee)$ acts transitively on $G$, it follows that $H^*_2(y_0)$ is an irreducible (and smooth) subvariety of $H^*_2$, which contains the curve $C = C_{t,r}^{2r}$ from Proposition 4.3. Moreover, since the condition $D(E_C) \leq 2rD(E)$ is open on $C \in H^*_2(y_0)$ by semicontinuity, Proposition 4.3 immediately implies the Theorem.

Now take an arbitrary curve $C \in H^*_2$, and pick an embedding $\varphi_C : P^1 \hookrightarrow G$ such that $\text{im}\varphi_C = C$. In addition, pick an isomorphism $\chi_C : C_{r,0} \otimes q \simeq k^r \otimes O_{P^1}(2)$. These data define an element $e = H^0(\chi_C \circ \varphi_C^-) \in W$ (cf. (27) and (28)) such that $\varphi_0 = \varphi_C$. Moreover, $e$ belongs to the subset

$$W^* := \{ e \in W \mid \varphi_0 : P^1 \to G \text{ is an embedding} \}$$

of $W$. This nonempty subset is clearly open in $W$, hence it is dense in $W$. Moreover, setting $R^*_2 := g_{2r}^{-1}(H^*_2)$ (see (24)), we obtain a principal $GL(r)$-bundle $\theta_{2r} : W^* \to R^*_2$, $e \mapsto \varphi_0$. Since (24) is a principal $PGL(2)$-bundle, the composition

$$w_{2r} := g_{2r} \circ \theta_{2r} : W^* \to H^*_2$$

is a smooth surjective morphism.

Fix a point $y_0 \in G$ and consider the set

$$W^*(E, y_0) := w_{2r}^{-1}(B(E, y_0)).$$

By (20), (29) and (29) any point $e \in W^*(E, y_0)$ defines an embedding $\varphi_e : P^1 \hookrightarrow G$ with $y_0 \in \text{im}\varphi_e$. Let $z := \varphi_e^{-1}(y_0)$. We have an exact triple $0 \to k^r \otimes O_{P^1}(1) \to k^r \otimes O_{P^1}(2) \to k^r \otimes k(z) \to 0$. By applying $\text{Hom}(V^\vee \otimes O_{P^1}, -)$ and passing to sections we obtain an exact triple

$$0 \to \text{Hom}(V^\vee, k^r \otimes H^0(O_{P^1}(1))) \to \text{Hom}(V^\vee, k^r \otimes H^0(O_{P^1}(2))) \xrightarrow{res_z} \text{Hom}(V^\vee, k^r \otimes k(z)) \to 0.$$

By construction, the set

$$W(E, y_0) := res_z^{-1}(res_z(e)) \simeq \text{Hom}(V^\vee, k^r \otimes H^0(O_{P^1}(1)))$$

depends only on $E$ and $y_0$ and contains $W^*(E, y_0)$ as a dense open subset. We thus obtain the following corollary.

**Corollary 4.5.** $W^*(E, y_0) = w_{2r}^{-1}(B(E, y_0))$ is a dense open subset of $W(E, y_0) \simeq \text{Hom}(V^\vee, k^r \otimes H^0(O_{P^1}(1)))$. 


5. Construction of a special vector bundle on $\mathbb{P}^1 \times \mathbb{P}^1$

A key ingredient in the proof of our main result, Theorem 7.1, is a specific morphism of $\mathbb{P}^1 \times \mathbb{P}^1$ into $G = G(r, V)$. It defines a vector bundle of rank $r$ on $\mathbb{P}^1 \times \mathbb{P}^1$ as the pullback of the antitautological bundle $Q$ on $G$. We construct this morphism in two steps. We first construct a suitable vector bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and then prove that this bundle arises from an appropriate morphism of $\mathbb{P}^1 \times \mathbb{P}^1$ to $G$.

We now proceed to the construction of a special vector bundle on $X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $pr_i : X \to \mathbb{P}^1$, $i = 1, 2$, be the natural projections, and let 0 and $\infty$ be two fixed points on $\mathbb{P}^1$. Set $P := pr_1^{-1}(0)$, $P' := pr_1^{-1}(\infty)$.

Fix a partition $(a_1, a_2, \ldots, a_r)$ of $d \in \mathbb{Z}_{\geq 0}$ such that $a_1 \geq a_2 \geq \ldots \geq a_r$, and consider a set $Z'$ of $d - a_1$ distinct points on $\mathbb{P}^1$ of the form

$$Z' = \bigcup_{i=2}^{r} Z_i', \quad Z_i' = \bigcup_{j=1}^{a_i} x_{ij}$$

($Z_i' \neq \emptyset$ for $a_i = 0$). The set $Z'$ determines the set

$$Z := P' \cap pr_2^{-1}(Z') = \bigcup_{i=2}^{r} Z_i, \quad Z_i := P' \cap pr_2^{-1}(Z_i') = \bigcup_{j=1}^{a_i} (\infty, x_{ij}).$$

In what follows we think of $Z_i$ as reduced 0-dimensional subschemes of $X$.

Setting

$$E_1 := \mathcal{O}_X(0, a_1)$$

we will recursively construct sheaves $E_k$ for $2 \leq k \leq r$ via the exact triples

$$0 \to E_{k-1} \to E_k \to I_{Z_k, X}(0, a_k) \to 0, \quad 2 \leq k \leq r.$$

**Proposition 5.1.** Assume that for $2 \leq k \leq r$ the sheaf $E_{k-1}$ is locally free of rank $k - 1$ and satisfies the conditions

$$h^2(E_{k-1}(0, -a_i)) = 0, \quad k \leq i \leq r,$$

$$h^2(E_{k-1}(0, -1)) = 0,$$

$$h^1(E_{k-1}) = h^1(E_{k-1}(0, -1)) = h^1(E_{k-1}(1, 0)) = 0,$$

$$h^1(E_{k-1}(1, -1)) = 0,$$

$$h^0(E_{k-1}(0, -1)) = a_1, \quad h^0(E_{k-1}(1, -1)) = 2a_1 + a_2 + \ldots + a_{k-1}, \quad h^0(E_{k-1}) = a_1 + k - 1,$$

$$h^0(E_{k-1}(1, 0)) = 2a_1 + a_2 + \ldots + a_{k-1} + 2(k - 1), \quad h^0(E_{k-1}(2, 0)) = 3a_1 + 2a_2 + \ldots + 2a_{k-1} + 3(k - 1),$$

$$E_{k-1|pr_2^{-1}(x)} \simeq (\mathcal{O}_{\mathbb{P}^1})^{k-1}, \quad x \notin Z_2' \sqcup \ldots \sqcup Z_{k-1}',$$ 

$$k \geq 3,$$

$$E_{k-1|pr_2^{-1}(x)} \simeq (\mathcal{O}_{\mathbb{P}^1})^{k-3} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \quad x \in Z_2' \sqcup \ldots \sqcup Z_{k-1}',$$ 

$$k \geq 3,$$

Then

(i) there is an epimorphism

$$\Ext^1(I_{Z_k, X}(0, a_k), E_{k-1}) \xrightarrow{\beta} H^0(\Ext^1(I_{Z_k, X}(0, a_k), E_{k-1})).$$
and moreover

\[(45) \quad H^0(\mathcal{E}xt^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1})) \simeq H^0(\mathcal{O}_{Z_k}^{k-1}) \simeq \bigoplus_{x \in Z_k} k(x)^{k-1};\]

(ii) there exists an element \( \xi \in \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) \) such that the sheaf \( \mathcal{E}_k \) defined by the corresponding exact triple \((43)\) is locally free of rank \( k \) and satisfies the conditions \((34)-(43)\) with \( k \) substituted for \( k-1 \); we label the so modified conditions as \((43)\).

Proof. (i) The existence of the epimorphism \((44)\) follows from the standard exact sequence of local and global Ext’s

\[0 \to H^1(\text{Hom}(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1})) \to \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) \to H^0(\mathcal{E}xt^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1})) \to \]

and from \((40)\) in view of the canonical isomorphism \(\text{Hom}(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) = \mathcal{E}_{k-1}(0, -a_k)\). The isomorphisms in \((43)\) are standard.

(ii) Pick an element \( \xi \in \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) \) and consider the extension \((33)\) defined by \( \xi \). Put \( S_x := pr_1^{-1}(x) \). If \( x \notin Z'_k \), then in view of \((40)\) the restriction of \((33)\) onto \( S_x \) is \( 0 \to (\mathcal{O}_{P_1})^{k-1} \to \mathcal{E}_{k|S_x} \to \mathcal{O}_{P_1} \to 0 \), i.e. \( \mathcal{E}_{k|S_x} \simeq (\mathcal{O}_{P_1})^k \). This implies \((44)\).

If \( x \in Z'_k \), then in view of \((41)\) the restriction of \((33)\) onto \( S_x \) is \( 0 \to (\mathcal{O}_{P_1})^{k-3} \oplus \mathcal{O}_{P_1}(1) \oplus \mathcal{O}_{P_1}(-1) \to \mathcal{E}_{k|S_x} \to \mathcal{O}_{P_1} \to 0 \), i.e. \( \mathcal{E}_{k|S_x} \simeq (\mathcal{O}_{P_1})^{k-2} \oplus \mathcal{O}_{P_1}(1) \oplus \mathcal{O}_{P_1}(-1) \), and we obtain \((44)\) for \( x \in Z'_k \).

For \( x \in Z'_k \), one has \( \mathcal{I}_{Z_k,X}(0,a_k)|_{S_x} \simeq \mathcal{L}(\bar{x}) \oplus \mathcal{O}_{P_1}(-1) \), where \( \bar{x} := (\infty, x) \). Therefore \((40)\) yields an exact sequence

\[(46) \quad 0 \to (\mathcal{O}_{P_1})^{k-1} \to \mathcal{E}_{k|S_x} \to \mathcal{L}(\bar{x}) \oplus \mathcal{O}_{P_1}(-1) \to 0.\]

Here the extension \((46)\) is given by an element

\[(47) \quad \xi^1 \in \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k)|_{S_x}, \mathcal{E}_{k-1|S_x}) \simeq \text{Ext}^1(\mathcal{L}(\bar{x}) \oplus \mathcal{O}_{P_1}(-1), (\mathcal{O}_{P_1})^{k-1}) \simeq \]

\[H^0(\mathcal{E}xt^1(\mathcal{L}(\bar{x}) \oplus \mathcal{O}_{P_1}(-1), (\mathcal{O}_{P_1})^{k-1})) \simeq (\mathcal{L}(\bar{x})^{k-1}).\]

Note that a sufficient condition for \((41)\) is that \( \xi^1 \neq 0 \) for \( x \in Z'_k \). Note in addition that the restriction of \((33)\) onto \( S_x \) defines a natural restriction map \( \psi_x : \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) \to \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k)|_{S_x}, \mathcal{E}_{k-1|S_x}) \) such that

\[\xi^1 = \psi_x(\xi).\]

The map \( \psi_x \) together with \((44)\) and \((47)\) fits in the diagram

\[(48) \quad \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k), \mathcal{E}_{k-1}) \xrightarrow{\psi_x} \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k)|_{S_x}, \mathcal{E}_{k-1|S_x}) \]

\[\beta \downarrow \quad \simeq \downarrow \quad \text{Res}_{S_x} \]

\[H^0(\mathcal{O}_{Z_k}^{k-1}) \xrightarrow{\text{res}_{S_x}} (\mathcal{L}(\bar{x})^{k-1}),\]

where \( \text{res}_{S_x} \) is the restriction epimorphism defined by the inclusion \( \bar{x} \hookrightarrow Z_k \).

Next, \((33)\) and \((42)\) give

\[0 \to \mathcal{O}_{P_1}(a_1) \oplus \ldots \oplus \mathcal{O}_{P_1}(a_k-1) \to \mathcal{E}_{k|P} \to \mathcal{O}_{P_1}(a_k) \to 0\]

Since \( a_1 \geq a_2 \geq \ldots \geq a_k \), this extension splits and yields \((42)\). Furthermore, by \((33)\) and \((43)\) we have

\[(49) \quad 0 \to \mathcal{O}_{P_1}(a_1 + \ldots + a_{k-1}) \oplus (\mathcal{O}_{P_1})^{k-2} \to \mathcal{E}_{k|P'} \to \mathcal{O}_{Z_k} \oplus \mathcal{O}_{P_1} \to 0.\]

The extension \((49)\) is given by an element

\[(50) \quad \xi' \in \text{Ext}^1(\mathcal{I}_{Z_k,X}(0,a_k)|_{P'}, \mathcal{E}_{k-1|P'}) \simeq \text{Ext}^1(\mathcal{O}_{Z_k} \oplus \mathcal{O}_{P_1}, \mathcal{O}_{P_1}(a_1 + \ldots + a_{k-1}) \oplus (\mathcal{O}_{P_1})^{k-2}) \simeq \]

\[\text{Ext}^1(\mathcal{O}_{Z_k} \oplus \mathcal{O}_{P_1}, \mathcal{O}_{P_1}(a_1 + \ldots + a_{k-1}) \oplus (\mathcal{O}_{P_1})^{k-2}).\]
For any $a_1 + \ldots + a_{k-1} \geq 0$, there is a distinguished injection
\[ g_k = \bigoplus_{x \in Z_k} g_k(x) : \bigoplus_{x \in Z_k} k(x) \simeq \text{Ext}^1(O_{Z_k} \oplus O_{p^1}, O_{p^1}(a_1 + \ldots + a_{k-1})) \hookrightarrow \text{Ext}^1(I_{Z_k}, X(0, a_k)|_{P'}, \mathcal{E}_{k-1}|_{P'}) \]
\[ \simeq \bigoplus_{x \in Z_k} k(x) \, k^{-1}. \]
Furthermore, twisting (33) by $h'$ (60)
\[ h \, h (57) \]
It follows from these conditions that (41) holds and the extension (39) implies (33).
To prove the remaining equalities (34), (39) for the vector bundle $\mathcal{E}_k$ defined by $\xi$ as the extension (33), we consider the standard Koszul resolution
\[ 0 \to O_X(-1, 0) \to O_X \oplus O_X(-1, a_k) \to I_{Z_k}, X(0, a_k) \to 0. \]
Twisting (52) by $O_X(a, b)$ for appropriate $a, b$ and keeping in mind that $a_k \geq 0$, we obtain
\[ h^2(I_{Z_k}, X(0, a_k - a_i)) = 0, \quad k + 1 \leq i \leq r, \]
\[ h^2(I_{Z_k}, X(0, a_k - 1)) = 0, \]
\[ h^1(I_{Z_k}, X(0, a_k)) = h^1(I_{Z_k}, X(1, a_k)) = 0, \]
\[ h^1(I_{Z_k}, X(1, a_k - 1)) = 0, \]
\[ h^0(I_{Z_k}, X(0, a_k - 1)) = 0, \quad h^0(I_{Z_k}, X(1, a_k - 1)) = a_k, \quad h^0(I_{Z_k}, X(0, a_k)) = 1, \]
\[ h^0(I_{Z_k}, X(1, a_k)) = a_k + 2, \quad h^0(I_{Z_k}, X(1, a_k)) = 2a_k + 3. \]
Furthermore, twisting (33) by $O_X(a, b)$ we quickly see that: (34) and (53) imply (34); (35) and (54) imply (35); (36) and (55) imply (36); (37) and (56) imply (37); (38) and (57) imply (38); (39) and (58) imply (39). The Proposition is proved.

As a corollary of Proposition 5.1 one obtains the following theorem.

**Theorem 5.2.** For any $r \in \mathbb{Z}_{>0}$, $r \geq 2$, there exists a rank $r$ vector bundle $\mathcal{F}$ on the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the following properties:

\[ \mathcal{F}_{|_{P^1_{r-1}(x)}} = (O_{\mathbb{P}^1}(2))^r, \quad x \notin Z', \]
\[ \mathcal{F}_{|_{P^1_{r-1}(x)}} = (O_{\mathbb{P}^1}(2))^{r-2} \oplus O_{\mathbb{P}^1}(3) \oplus O_{\mathbb{P}^1}(1), \quad x \in Z', \]
\[ \mathcal{F}_{|_{P^1}} = O_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus O_{\mathbb{P}^1}(a_r), \]
\[ \mathcal{F}_{|_{P^1}} = O_{\mathbb{P}^1}(d) \oplus (O_{\mathbb{P}^1})^{r-1}, \]
\[ h^1(\mathcal{F}(-1, -1)) = h^1(\mathcal{F}(-2, 0)) = h^1(\mathcal{F}(-2, -1)) = h^1(\mathcal{F}(-1, 0)) = 0, \]
We define \( l \) as a line \((64)\). The restriction of the canonical epimorphism \( V \) to \( G \) has the form \( g : V \otimes \mathcal{O}_G \rightarrow Q \). Hence the epimorphism \( \psi^* g \) has the form \( \psi^* g : V \otimes \mathcal{O}_G \rightarrow \mathcal{O}_G (d) \oplus (\mathcal{O}_G)^{r-1} \).

Passing to sections we obtain an element

\[
g_\psi \in \text{Hom}(V, H^0(\mathcal{O}_G (d) \oplus (\mathcal{O}_G)^{r-1})).
\]

Note that, similarly to (26), \( \psi \) is determined by the element \( g_\psi \).

Next, we put \( k = d \) in (23) and we fix a curve \( C_0 \in H_d \) together with an isomorphism \( \theta : \mathbb{P}^1 \cong C_0 \). The composition \( \varphi_{C_0} : \mathbb{P}^1 \cong C_0 \rightarrow G \) defines an epimorphism \( \varphi_{C_0}^* \gamma : V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \varphi_{C_0} Q \). Moreover,

\[
\varphi_{C_0} Q \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)
\]

for some partition \((a_1, \ldots, a_r)\) of \( d \). (The nonnegativity of the integers \( a_i \) follows from the surjectivity of \( \varphi_{C_0}^* \gamma \).) Pick an isomorphism \( \chi_{C_0} : \varphi_{C_0} Q \rightarrow \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r) \). The datum \((\varphi_{C_0}, \chi_{C_0})\) defines an element

\[
e(C_0) = H^0(\chi_{C_0} \circ \varphi_{C_0}^* \gamma) \in W(C_0) := \text{Hom}(V, H^0(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)))
\]

(cf. (27) and (28)).

The set \( W(C_0)^\text{epi} := \{ e \in W(C_0) \mid \text{the composition } V \otimes \mathcal{O}_{\mathbb{P}^1} \overset{e \otimes \text{id}}{\rightarrow} H^0(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)) \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r) \text{ is an epimorphism} \} \) is a dense open subset of \( W(C_0) \) containing \( e(C_0) \), and (by the universality property of \( G \)) any element \( e \in W(C_0)^\text{epi} \) determines a morphism

\[
\varphi_e : \mathbb{P}^1 \rightarrow G
\]

(cf. (26)). For any vector bundle \( E \) on \( G \) we put

\[
B_P(E, C_0) := \{ C \in H_d \mid D(E|_C) \leq D(E|_{C_0}) \}.
\]

By construction \( C_0 \in B_P(E, C_0) \), and moreover by semicontinuity, \( B_P(E, C_0) \) is an open subset of \( H_d \). Since \( e(C_0) \in W(C_0)^\text{epi} \) it follows that

\[
W_P(E, C_0) := \{ e \in W(C_0)^\text{epi} \mid \text{im} \varphi_e \in B_P(E, C_0) \}
\]

is a dense open subset in \( W(C_0)^\text{epi} \), respectively, in \( W(C_0) \), and we obtain a natural surjection \( W_P(E, C_0) \rightarrow B_P(E, C_0), e \mapsto \text{im} \varphi_e \).

We put also

\[
\pi := (pr_2)^{-1} : \mathbb{P}^1 \rightarrow P, \quad \pi' := (pr_2')^{-1} : \mathbb{P}^1 \rightarrow P', \quad \rho := (pr_1|_S)^{-1} : \mathbb{P}^1 \rightarrow S.
\]
Theorem 6.1. Let $y_0 \in l_0$ and $w \in \psi^{-1}(y_0)$ be fixed points. Then, for any vector bundle $E$ on $G$ there exists a morphism $f : X \to G$ such that:

(i) $f^*Q \simeq F$, where $F$ is defined in Theorem 5.2;
(ii) $f \circ \pi' = \psi$;
(iii) $f \circ \pi : \mathbb{P}^1 \hookrightarrow G$ is an embedding such that $(f \circ \pi)^*Q \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(a_r)$, respectively, $f \circ \rho : \mathbb{P}^1 \hookrightarrow G$ is an embedding such that $(f \circ \rho)^*Q \simeq (\mathcal{O}_{\mathbb{P}^1}(2))^r$;
(iv) $D(E|_{f(P)}) \leq D(E|_{C_2})$ and $D(E|_{f(S)}) \leq 2rD(E)$.

Proof. Recall (Theorem 5.2) that

\[ F|_P \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(a_r), \quad F|_{P'} \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus (\mathcal{O}_{\mathbb{P}^1})^{r-1}, \quad F|_S \simeq (\mathcal{O}_{\mathbb{P}^1}(2))^r. \]
Furthermore, let us introduce the following notation:

\[ z_0 := P' \cap S, \quad W_0 := \text{Hom}(V^\vee, H^0(\mathcal{F})), \quad W_{-1} := \text{Hom}(V^\vee, H^0(\mathcal{F}(-1, 0))), \]

(70) \[ W_P := \text{Hom}(V^\vee, H^0(\mathcal{F}|_P)), \quad W_{-1,P} := \text{Hom}(V^\vee, H^0(\mathcal{F}(-1, 0)|_P)) \cong W_P, \]

\[ W_{0,PS} := \text{Hom}(V^\vee, H^0(\mathcal{F}|_{PS})), \quad W_{-1,PS} := \text{Hom}(V^\vee, H^0(\mathcal{F}(-1, 0)|_{PS})). \]

(71) \[ W_{-1,S} := \text{Hom}(V^\vee, H^0(\mathcal{F}(-1, 0)|_S)) \cong \text{Hom}(V^\vee, H^0((\mathcal{O}_{\mathbb{P}^1}(1))^r)), \]

(72) \[ W_{P'} := \text{Hom}(V^\vee, H^0(\mathcal{F}|_{P'})) \cong \text{Hom}(V^\vee, H^0((\mathcal{O}_{\mathbb{P}^1}(d) \oplus (\mathcal{O}_{\mathbb{P}^1})^{-r})). \]

(The right-hand isomorphisms in (71) and (72) follow from (69)).

By using (69), we see that the functor \( \text{Hom}_{\mathcal{O}_X}(V^\vee \otimes \mathcal{O}_X, -) \) applied to the exact sequence \( 0 \to \mathcal{F}(-1, 0) \to \mathcal{F} \to \mathcal{F}|_{P'} \to 0 \) yields and exact sequence \( 0 \to \text{Hom}_{\mathcal{O}_X}(V^\vee \otimes \mathcal{O}_X, \mathcal{F}(-1, 0)) \to \text{Hom}_{\mathcal{O}_{P'}}(V^\vee \otimes \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d) \oplus (\mathcal{O}_{\mathbb{P}^1})^{-r}) \to 0 \). Passing to cohomology and using (63), we obtain the exact sequence

\[ 0 \to W_{-1} \xrightarrow{i_W} W_0 \xrightarrow{\text{res}_{P'}} W_{P'} \to 0. \]

Next, for any \( s \in W_{-1} \) we consider the composition morphism \( e_s : V^\vee \otimes \mathcal{O}_X \xrightarrow{s \otimes \text{id}} H^0(\mathcal{F}(-1, 0)) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} \mathcal{F}(-1, 0), \) and, for any \( z \in X \setminus P' \), we consider the composition

\[ e_s(z) : V^\vee \otimes \mathcal{O}_X \xrightarrow{e_s} \mathcal{F}(-1, 0) \xrightarrow{\text{res}_z} \mathcal{F}(-1, 0) \otimes k(z) \cong k(z)^r. \]

Passing to sections in the exact sequence \( 0 \to \mathcal{F}(-2, -1) \to \mathcal{F}(-2, 0) \oplus \mathcal{F}(-1, -1) \to \mathcal{F}(-1, 0) \xrightarrow{\text{res}_z} \mathcal{F}(-1, 0) \otimes k(z) \to 0 \) and using (63) and (64), we obtain an epimorphism \( H^0(\mathcal{F}(-1, 0)) \xrightarrow{\text{res}_z} H^0(\mathcal{F}(-1, 0) \otimes k(z)) \cong k^r \), and hence an induced epimorphism

\[ r(z) : W_{-1} \to \text{Hom}(V^\vee, H^0(\mathcal{F}(-1, 0) \otimes k(z))) \cong \text{Hom}(V^\vee, k(z)^r) := W_z. \]

Put \( Y(z) := \{ s \in W_{-1} \mid e_s(z) \) is not surjective \} and \( Y_0(z) := \{ u \in W_z \mid u : V^\vee \to k(z)^r \) is not surjective \}. By definition \( Y(z) = r(z)^{-1}(Y_0(z)) \), and one easily checks that \( \text{codim}_{W_z} Y_0(z) = \text{dim } V - r + 1 \). Therefore the surjectivity of \( r(z) \) yields

\[ \text{codim}_{W_z} Y(z) = \text{codim}_{W_z} Y_0(z) = \text{dim } V - r + 1. \]

If \( Y := \bigcup_{z \in X \setminus P'} Y(z) \), (75) implies

\[ \text{codim}_{W_z} Y \geq \text{dim } V - r - 1 > 0. \]

(Note that \( \text{dim } V - r - 1 > 0 \) according to our assumption from Section 2.)

For each \( z \in X \setminus P' \) the exact sequence (73) and the map (74) fit in the diagram

\[ \begin{array}{ccc}
0 & \xrightarrow{i_W} & W_0 \\
\downarrow r(z) & & \downarrow r(z) \\
W_z & \xrightarrow{\text{Hom}(V^\vee, H^0(\mathcal{F} \otimes k(z)))} & \end{array} \]

the right vertical map \( r(z) \) being the natural restriction map. This diagram together with (72) and the inequality (76) shows that, for the element \( g_{p_1} \in \text{Hom}(V^\vee, H^0(\mathcal{O}_{\mathbb{P}^1}(d) \oplus (\mathcal{O}_{\mathbb{P}^1})^{-r})) = W_{P'} \)

given in (66) and for a generic element

\[ \varepsilon \in \text{res}_{P'}^{-1}(g_{p_1}) \cong W_{-1}, \]

the composition \( \varepsilon(z) : V^\vee \otimes \mathcal{O}_X \xrightarrow{\varepsilon \otimes \text{id}} H^0(\mathcal{F}) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} \mathcal{F} \xrightarrow{\text{res}_z} \mathcal{F} \otimes k(z) \) is an epimorphism for any \( z \in X \setminus P' \). Moreover, since we can consider \( g_{p_1} \) as an epimorphism : \( V^\vee \otimes \mathcal{O}_X \to \mathcal{F}|_{P'}, \varepsilon(z) \) is also an epimorphism for any \( z \in P' \). This means that \( \varepsilon(z) \) is an epimorphism for any \( z \in X \),
i.e. that the morphism $\varepsilon : V^\vee \otimes \mathcal{O}_X \to \mathcal{F}$ is an epimorphism. By the universality property of $G$ this means that there exists a morphism $f = f_\varepsilon : X \to G$ such that $\varepsilon = f_\varepsilon^* \gamma$, $\mathcal{F} \simeq f_\varepsilon^* \mathcal{Q}$, where $\gamma : V^\vee \otimes \mathcal{O}_G \to \mathcal{Q}$ is the canonical epimorphism. This together with (69) yields (iii). In addition, since $\psi$ is determined by $g_\psi$, the equality $g_\psi = \text{res}_P(\varepsilon)$ directly implies (ii).

Next, we apply the functor $\text{Hom}_{\mathcal{O}_X}(V^\vee \otimes \mathcal{O}_X, \mathcal{F})$ to the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{F}(-2, -1) \\
\downarrow \\
\mathcal{F}(-1, 0) \\
\downarrow \\
\mathcal{F}(-1, 0)_{|\text{PS}} \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{F}(-2, -1) \\
\downarrow \\
\mathcal{F}(-1, 0) \\
\downarrow \\
\mathcal{F}_{|\text{PS}} \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{F}(-1, 0)_{|\text{PS}} \\
\downarrow \\
\mathcal{F}_{|\text{PS}} \\
\downarrow \\
\mathcal{F} \otimes k(z_0) \\
\downarrow \\
0
\end{array}
\]

Using (63), (64), (70)-(72), we obtain the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
W_{-1} \\
\downarrow_{\text{res}_P} \\
W_0 \\
\downarrow \\
W_{-1,PS} \\
\downarrow_{\text{res}_q} \\
W_{0,PS} \\
\downarrow \\
W_{0} \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
W_{-1} \\
\downarrow_{\text{res}_P} \\
W_0 \\
\downarrow \\
W_{-1,PS} \\
\downarrow_{\text{res}_q} \\
W_{0,PS} \\
\downarrow \\
W_{0} \\
\downarrow \\
0
\end{array}
\]

Moreover, setting $\varepsilon_{z_0} := r(z_0)(g_\psi)$, we have

\[
\text{res}_{z_0}^{-1}(\varepsilon_{z_0}) \simeq W_{-1,PS}.
\]

Similarly to (77), using Theorem 5.2 and (70)-(71), we obtain the surjective restriction maps

\[
\text{res}_P W_P \xleftarrow{\text{res}_P} W_{-1,PS} \xrightarrow{\text{res}_q} W_{-1,PS} \simeq \text{Hom}(V^\vee, H^0((\mathcal{O}_{\mathbb{P}}(1)))').
\]

Now (77) and (78) together with Corollary 4.5 show that

\[
U := \text{res}_{PS}^{-1}((\text{res}_P)^{-1}(W_P(E, C_0)) \cap (\text{res}_S)^{-1}(W^*(E, y_0)))
\]

is a dense open subset of $W_{-1}$. Hence, for a generic element $\varepsilon \in U$ the corresponding morphism $f = f_\varepsilon : X \to G$ satisfies the conditions (a) $e_\pi := \text{res}_P(\text{res}_PS(\varepsilon)) \in W_P(E, C_0)$ and (b) $e_\rho := \text{res}_S(\text{res}_PS(\varepsilon)) \in W^*(E, y_0)$. Here, by construction, we have $\varphi_{e_\pi} = f_\pi$ and, respectively, $\varphi_{e_\rho} = f_\rho$. Now (a) together with (67) and (68) means that $f(P) = \text{im} f \circ \pi = \text{im} \varphi_{e_\pi} \in B_P(E, C_0)$, i.e. that $D(E_{f(P)}) \leq D(E_{C_0})$; respectively, (b) together with (61) and Corollary 4.4 means that $D(E_{f(S)}) \leq 2rD(E)$. This yields (iv). The claim (i) is clear from the construction. □

7. MAIN RESULT

We now proceed to the main construction. Consider a twisted ind-Grassmannian $G$ defined by (1), together with a vector bundle on $G$, i.e. with vector bundles $E_m$ on $G_m := G(r_m, V^{n_m})$ of fixed rank such that $\varphi^*_m E_{m+1} = E_m$ for $m \geq 1$. We assume that $G_m$ is not isomorphic to a projective space for any $m$. Set

\[
\Phi_m := \varphi_{m-1} \circ \cdots \circ \varphi_1 : G_1 \hookrightarrow G_m.
\]
The basic assumption (2) that \( G \) is sufficiently twisted can be rewritten as

\[
\lim_{m \to \infty} \frac{r_m}{\deg \Phi_m} = 0.
\]

Note that (79) is always satisfied if \( r_m \) doesn’t depend on \( m \).

For a given integer \( m > 1 \), fix a line \( l_0 \) in \( G_m \) such that

\[
D(E_{m|l_0}) = \max_{l \in G_m} D(E_{m|l}) =: D_m.
\]

Let also \( l_1 \) be any line in \( G_1 \) such that

\[
D(E_{1|l_1}) = \max_{l \in G_1} D(E_{1|l}) =: D_1.
\]

(Clearly, such lines \( l_0 \) and \( l_1 \) exist by semicontinuity).

Put \( C_0 := \Phi_m(l_1) \), \( d := \deg C_0 = \deg \Phi_m \). Fix a point \( y_0 \in l_0 \) and a curve \( C \subset B(E, y_0) \) (see Theorem [4,3]) and fix a degree \( d \) morphism \( \psi : \mathbb{P}^1 \to l_0 \hookrightarrow G_m \). Consider the surface \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) together with the distinguished fibers \( P, P' \) of the projection \( pr_1 : X \to \mathbb{P}^1 \) and \( S \) of the projection \( pr_1 : X \to \mathbb{P}^1 \). Applying Theorem 6.1 to this datum, we obtain a morphism \( f : X \to G_m \) such that, for \( E_X := f^* E_m \), we have:

(i) the morphism \( \mathbb{P}^1 \to \mathbb{P}^1 \) \( pr_1 \) \( f \) \( G_m \) coincides with \( \psi \), hence by \( (80) \) and \( (81) \) the vector bundle \( E_X \) satisfies the equality

\[
D(E_{X|P'}) = \deg(f|P')D(E_{m|l_0}) = (\deg \Phi_m)D_m;
\]

(ii) \( f|P \) and \( f|S \) are embeddings such that

\[
D(E_{X|P}) = D(E_{m|f(P)}) \leq D(E_{m|C_0}) = D(E_{m|\Phi_m(l_1)}) = D(E_{1|l_1}) = D_1,
\]

\[
D(E_{X|S}) = D(E_{m|f(S)}) \leq 2r_m D_m.
\]

Now applying the inequality (3.11) from [DP] to \( (82) \)-\( (84) \) we obtain

\[
(\deg \Phi_m)D_m = dD_m \leq 4\text{rk}E_X(D_1 + 2)(2r_m D_m + 1) - 2r_k E_X.
\]

But this inequality clearly contradicts to (79) for large enough \( m \) if \( D_m \neq 0 \). Hence \( D_m = 0 \). Therefore, by [PT] Proposition 4.1 \( E_m \) is trivial. We thus have proved our main result.

**Theorem 7.1.** There are no nontrivial vector bundles of finite rank on a sufficiently twisted ind-Grassmannian \( G \).

We conclude this paper by a class of natural examples of twisted ind-Grassmannians \( (1) \) with \( r_m = r = \text{const} \). Recall that any embedding of \( \mathbb{P}^n \) to \( \mathbb{P}^{n''} \) is given by a subsystem of a complete linear system, i.e. by a composition of a Veronese embedding of \( \mathbb{P}^n \) into \( \mathbb{P}^{n''} \), \( n'' \geq n' \), and subsequent projection of \( \mathbb{P}^{n''} \) to \( \mathbb{P}^{n'} \). In fact, this procedure extends to Grassmannians of \( r \)-dimensional subspaces, \( r \) being fixed. More precisely, for each \( m \geq 1 \) fix an integer \( k_m > 1 \) and construct the Grassmannians \( G_m = G(r, V_m) \) and their successive embeddings \( \varphi_m : G_m \to G_{m+1} \) inductively by the following procedure. Consider the flag variety \( \Gamma_m = FL(1, r; V_m) \) together with the natural embedding \( \Gamma_m \hookrightarrow P(V_m) \times G_m \) and the sheaf \( \mathcal{O}_{\Gamma_m}(1, k_m) := (\mathcal{O}_{P(V_m)}(1) \boxtimes \mathcal{O}_{G_m}(k_m))|_{\Gamma_m} \). Set \( W_{m+1} := H^0(\mathcal{O}_{\Gamma_m}(1, k_m)) \). The embedding \( \theta_m : \Gamma_m \hookrightarrow P(W_{m+1}) \) by the complete linear series \( |\mathcal{O}_{\Gamma_m}(1, k_m)| \) is (by construction) induced by a homogeneous embedding (in the sense of [DP] Sect. 4) \( \psi_m : G_m \hookrightarrow G(r, W_{m+1}) \). By composing \( \psi_m \) with a possible rational projection of the form \( \pi : G(r, W_{m+1}) \twoheadrightarrow G(r, W_{m+1}) \), where \( V_{m+1} \) is an appropriate quotient of \( W_{m+1} \), we obtain an embedding \( \varphi_m : G_m \to G_{m+1} \).

Note that Theorem 7.1 was proved in [DP] for ind-Grassmannians \( (1) \) defined via certain homogeneous morphisms \( \varphi_m \) called twisted extensions, see [DP] Sect. 4.2. It is not difficult to check that the above constructed embeddings \( \varphi_m \) are not twisted extensions, hence Theorem 7.1 is new for the corresponding ind-Grassmannians \( G \).
References

[BV] W. Barth and A. Van de Ven, On the geometry in codimension 2 in Grassmann manifolds, Lecture Notes in Math. 412 (1974), 1-35.

[CT] I. Coanda I., G. Trautmann The splitting criterion of Kempf and the Babylonian tower theorem, arXiv:math.AG/0411636 v.1, 29 Nov 2004.

[DiP] I. Dimitrov, I. Penkov, Ind-varieties of generalized flags as homogeneous spaces for classical ind-groups, IMRN 2004, No. 55, 2935-2953.

[DP] J. Donin, I. Penkov, Finite rank vector bundles on inductive limits of grassmannians, IMRN 2003, No. 34, 1871-1887.

[K] M. Kapranov, Veronese curves and Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$, J. Algebraic Geometry 2 (1993), 239-262.

[PT] I. Penkov, A. S. Tikhomirov, Rank 2 vector bundles on ind-Grassmannians, Manin Festschrift, Birkhäuser 2007, to appear.

[S1] E. Sato, On the decomposability of infinitely extendable vector bundles on projective spaces and Grassmann varieties, J. Math. Kyoto Univ. 17 (1977), 127-150.

[S2] E. Sato, On infinitely extendable vector bundles on $G/P$, J. Math. Kyoto Univ. 19 (1979), 171-189.

[St] S. A. Strømme, On parametrized rational curves in Grassmann varieties, Lecture Notes in Math 1266 (1987), 251-272.

[T] A. N. Tyurin, Vector bundles of finite rank over infinite varieties, Math. USSR Izvestija, 10 (1976), 1187-1204.

Jacobs University Bremen, School of Engineering and Science, Campus Ring 1, 28759 Bremen, Germany
E-mail address: ivanpenkov@jacobs-university.de

Department of Mathematics, State Pedagogical University, Republikanskaya Str. 108
150 000 Yaroslavl, Russia
E-mail address: astikhomirov@mail.ru