Custodial symmetry violation in the Georgi-Machacek model

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We study the effects of custodial symmetry violation in the Georgi-Machacek (GM) model. The GM model adds isospin-triplet scalars to the Standard Model in a way that preserves custodial symmetry at tree level; however, this custodial symmetry has long been known to be violated at the one-loop level by hypercharge interactions. We consider the custodial-symmetric GM model to arise at some high scale as a result of an unspecified ultraviolet completion, and quantify the custodial symmetry violation induced as the model is run down to the weak scale. The measured value of the electroweak $\rho$ parameter (along with unitarity considerations) lets us constrain the scale of the ultraviolet completion to lie below tens to hundreds of TeV. Subject to this constraint, we quantify the size of other custodial-symmetry-violating effects at the weak scale, including custodial symmetry violation in the couplings of the 125 GeV Higgs boson to $W$ and $Z$ boson pairs and mixings and mass splittings among the additional Higgs bosons in the theory. We find that these effects are small enough that they are unlikely to be probed by the Large Hadron Collider, but may be detectable at a future $e^+e^-$ collider.
I. INTRODUCTION

With the discovery of a Standard Model (SM)-like Higgs boson at the CERN Large Hadron Collider (LHC) in 2012 [1], we have the first direct access to the dynamics of electroweak symmetry breaking. The simplest implementation of this dynamics is through a single complex scalar field transforming as a doublet under the weak SU(2)$_L$ gauge symmetry; this is consistent with experimental data to date [2].

While at least one SU(2)$_L$ doublet is required to generate the masses of the SM fermions in a gauge-invariant way, the masses of the $W$ and $Z$ bosons can in principle also receive contributions from scalars in larger representations of SU(2)$_L$. Such an extension to the Higgs sector is severely constrained by measurements of the $\rho$ parameter [3], defined as the ratio of the strengths of the neutral and charged weak currents in the low-energy limit and measured to very high precision via the global electroweak fit [4]. Indeed, unless the vacuum expectation values (vevs) of the larger representations are negligibly small, the only viable models are those that preserve $\rho = 1$ at tree level:

i. models with extra SU(2)$_L$ doublet(s) and/or singlet(s);

ii. a model with an extra SU(2)$_L$ septet with appropriately-chosen hypercharge [5]; and

iii. the Georgi-Machacek (GM) model [7, 8] and its generalizations to larger SU(2)$_L$ representations [9, 13].

In this paper, we consider the GM model. In addition to the usual SU(2)$_L$-doublet scalar fields, arranged in such a way that the scalar potential is invariant under a global SU(2)$_L$×SU(2)$_R$ symmetry; upon electroweak symmetry breaking, this global symmetry breaks down to its diagonal subgroup [known as the custodial SU(2)] and $\rho = 1$ is thereby preserved. The GM model gives rise to a rich and exotic phenomenology, including singly- and doubly-charged scalars that couple to vector boson pairs at tree level and the possibility that the SM-like Higgs boson’s couplings to $WW$ and $ZZ$ could be larger than in the SM. It has been used as a benchmark by the LHC experiments for interpreting searches for singly-charged Higgs bosons decaying into vector boson pairs [14, 15].

However, it has been known since the early ‘90s that the custodial symmetry in the GM model holds only at tree level [10]: the global SU(2)$_R$ symmetry is explicitly violated by the gauging of hypercharge, which leads to an uncontrolled violation of the custodial symmetry at one loop. The most obvious manifestation of this is that the standard calculation of the Peskin-Takeuchi $T$ parameter [17] yields an infinite result; this infinity is to be cancelled by a counterterm that is absent in the SU(2)$_L$×SU(2)$_R$-invariant potential of the GM model but appears in the full gauge-invariant but custodial-symmetry-violating theory [10].

A further manifestation, most relevant for our purposes, is that it is not possible to compute a consistent set of renormalization group equations (RGEs) for the Lagrangian parameters of the custodial-symmetric GM model unless one sets the hypercharge gauge coupling to zero [18]. Instead, we use the RGEs for the full gauge-invariant but custodial-symmetry-violating potential. These imply that it is possible to choose the Lagrangian parameters to preserve the custodial symmetry, but only at one energy scale. As one moves away from that special scale, custodial symmetry violation builds up due to the renormalization group running. Reference [18] studied this effect by assuming that the theory is custodial-symmetric at the weak scale and quantifying the amount of custodial symmetry violation that develops as one runs to higher scales.

In this paper, we consider a different approach. We imagine that the custodial-symmetric GM model arises at some high scale, for example as a theory of composite scalars with an accidental global SU(2)$_L$×SU(2)$_R$ symmetry in the scalar sector. (Such models have been constructed in the context of little Higgs theories in Refs. [19, 20].) Below the compositeness scale, custodial symmetry violation accumulates through the running of the Lagrangian parameters down to the weak scale. Weak-scale measurements of the $\rho$ parameter can then be used to constrain how high the custodial-symmetric scale can be. Subject to this constraint, we can also quantify the physical effects of custodial symmetry violation in Higgs-sector observables, such as the ratio of the SM-like Higgs boson couplings to $WW$ and $ZZ$ and custodial-violating mixings and mass splittings among the additional scalars in the GM model. Working within a particular benchmark scenario for concreteness, we will show that the custodial-symmetric scale can be as high as tens to hundreds of TeV, and that the effects of custodial symmetry violation at the weak scale are typically too small to be detected at the LHC. The custodial-violation-induced mass splittings may however be detectable at a future $e^+e^-$ collider. The fermiophobic scalars of the GM model acquire small fermion couplings due to custodial-violation-induced mixing, but the resulting branching ratios remain below the percent level in the benchmark that we study. We leave to future work a careful study of the fermionic decays of the would-be fermiophobic scalars for masses below about 160 GeV, where fermionic decays could compete against the loop-induced diphoton decays that otherwise put strong experimental constraints on such light scalars.

This paper is organized as follows. In Sec. II we review the GM model with exact custodial symmetry in order to set our notation. In Sec. III we write down the most general gauge invariant scalar potential for the custodial-violating theory with the same field content. In Sec. IV we compute the masses and mixing angles of the physical scalars in...
the custodial-violating theory and derive formulas for the most interesting custodial-violating couplings. In Sec. V we collect the one-loop RGEs for the custodial-violating theory. In Sec. VI we describe our calculational procedure and give our numerical results. In Sec. VII we conclude. In Appendix A we give a translation between our notation and that of Ref. [18], and in Appendix B we give some details of our calculation of the RGEs.

II. GEORGI-MACHACEK MODEL WITH EXACT CUSTODIAL SYMMETRY

The scalar sector of the GM model [17,8] consists of the usual complex doublet \((\phi^+, \phi^0)\) with hypercharge \(U(Y) = 1\), a real triplet \((\xi^+, \xi^0, \xi^-)\) with \(Y = 0\), and a complex triplet \((\chi^+, \chi^0, \chi^0)\) with \(Y = 2\). The doublet is responsible for the fermion masses as in the SM. In order to make the global SU(2)L a real triplet, \(\xi\), upon electroweak symmetry breaking, the global SU(2)L Z boson masses constrain the conventions of Ref. [21], by which is the custodial SU(2) symmetry. In Appendix B we give our numerical results. In Sec. VII we conclude. In Appendix A we give a translation between our notation and that of Ref. [18], and in Appendix B we give some details of our calculation of the RGEs.

The physical fields can be organized by their transformation properties under the custodial SU(2) symmetry into a doublet in the form of a bidoublet \(\Phi\) and combine the triplets to form a bitriplet \(X\) for the fermion masses as in the SM. In order to make the global SU(2)L a real triplet, \(\xi\), upon electroweak symmetry breaking, the global SU(2)L Z boson masses constrain the conventions of Ref. [21], by which is the custodial SU(2) symmetry.

The most general gauge-invariant scalar potential involving these fields that conserves custodial SU(2) is given, in the conventions of Ref. [21], by

\[
V(\Phi, X) = \frac{\mu_1^2}{2} \text{Tr}(\Phi^\dagger \Phi) + \frac{\mu_2^2}{2} \text{Tr}(X^\dagger X) + \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 + \lambda_2 \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(X^\dagger X) + \lambda_3 \text{Tr}(X^\dagger XX^\dagger X) + \lambda_4 [\text{Tr}(X^\dagger X)]^2 - \lambda_5 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) \text{Tr}(X^\dagger t^a X t^b) - M_1 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) (UXU^\dagger)_{ab} - M_2 \text{Tr}(X^\dagger t^a X t^b) (UXU^\dagger)_{ab}.
\]

Here the SU(2) generators for the doublet representation are \(\tau^a = \sigma^a/2\) with \(\sigma^a\) being the Pauli matrices, the generators for the triplet representation are

\[
t^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

and the matrix \(U\), which rotates \(X\) into the Cartesian basis, is given by

\[
U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 \end{pmatrix}.
\]

The minimization conditions for the scalar potential read

\[
0 = \frac{\partial V}{\partial v_\phi} = v_\phi \left[ \mu_2^2 + 4\lambda_1 v_\phi^2 + 3(2\lambda_2 - \lambda_3) v_\chi^2 - \frac{3}{2} M_1 v_\chi \right],
\]
\[
0 = \frac{\partial V}{\partial v_\chi} = 3\mu_3^2 v_\chi + 3(2\lambda_2 - \lambda_3) v_\phi^2 v_\chi + 12(\lambda_3 + 3\lambda_4) v_\chi^3 - \frac{3}{4} M_1 v_\phi^2 - 18 M_2 v_\chi^2.
\]

The physical fields can be organized by their transformation properties under the custodial SU(2) symmetry into a

\footnote{We use \(Q = T^3 + Y/2\).}

\footnote{A translation table to other parameterizations in the literature has been given in the appendix of Ref. [21].}
fiveplet, a triplet, and two singlets. The fiveplet and triplet states are given by

\[ H_5^{++} = \chi^{++}, \quad H_5^{+} = \frac{(\chi^+ - \xi^+)}{\sqrt{2}}, \quad H_5^0 = \sqrt{\frac{2}{3}} \xi^{0,r} - \sqrt{\frac{1}{3}} \chi^{0,r}, \]

\[ H_3^+ = -s_H \phi^+ + c_H \frac{(\chi^+ + \xi^+)}{\sqrt{2}}, \quad H_3^0 = -s_H \phi^{0,i} + c_H \chi^{0,i}, \tag{7} \]

where the vevs are parameterized by

\[ c_H \equiv \cos \theta_H = \frac{v_\phi}{v}, \quad s_H \equiv \sin \theta_H = \frac{2\sqrt{2} v_\chi}{v}, \tag{8} \]

and we have decomposed the neutral fields into real and imaginary parts according to

\[ \phi^0 \to \frac{v_\phi}{\sqrt{2}} + \frac{\phi^{0,r} + i\phi^{0,i}}{\sqrt{2}}, \quad \chi^0 \to v_\chi + \frac{\chi^{0,r} + i\chi^{0,i}}{\sqrt{2}}, \quad \xi^0 \to v_\chi + \xi^{0,r}. \tag{9} \]

The masses within each custodial multiplet are degenerate at tree level and can be written (after eliminating \( \mu_2^2 \) and \( \mu_3^2 \) in favor of the vevs) as

\[ m_2^2 = \frac{M_1}{4v_\chi} v_\phi^2 + 12M_2 v_\chi + \frac{3}{2} \lambda_5 v_\phi^2 + 8\lambda_3 v_\chi^2, \]

\[ m_3^2 = \frac{M_1}{4v_\chi} (v_\phi^2 + 8v_\chi^2) + \frac{\lambda_5}{2} (v_\phi^2 + 8v_\chi^2) = \left( \frac{M_1}{4v_\chi} + \frac{\lambda_5}{2} \right) v^2. \tag{11} \]

The two custodial SU(2)–singlet mass eigenstates are given by

\[ h = \cos \alpha \phi^{0,r} - \sin \alpha H_1^{0r}, \quad H = \sin \alpha \phi^{0,r} + \cos \alpha H_1^{0r}, \tag{12} \]

where

\[ H_1^{0r} = \sqrt{\frac{1}{3}} \xi^{0,r} + \sqrt{\frac{2}{3}} \chi^{0,r}. \tag{13} \]

Their mixing angle and masses are given by

\[ \sin 2\alpha = \frac{2M_1^2}{m_H^2 - m_h^2}, \quad \cos 2\alpha = \frac{M_{22}^2 - M_{11}^2}{m_H^2 - m_h^2}, \]

\[ m_{h,H}^2 = \frac{1}{2} \left[ M_{11}^2 + M_{22}^2 + \sqrt{(M_{11}^2 - M_{22}^2)^2 + 4(M_{12}^2)^2} \right], \tag{14} \]

where we choose \( m_h < m_H \), and

\[ M_{11}^2 = 8\lambda_1 v_\phi^2, \]

\[ M_{12}^2 = \frac{\sqrt{3}}{2} v_\phi \left[ -M_1 + 4(2\lambda_2 - \lambda_3) v_\chi \right], \]

\[ M_{22}^2 = \frac{M_1 v_\phi^2}{4v_\chi} - 6M_2 v_\chi + 8(\lambda_3 + 3\lambda_4) v_\chi^2. \tag{15} \]

We will later apply constraints on the parameters of the custodial-symmetric scalar potential from perturbative unitarity of two-to-two scalar scattering amplitudes and bounded-from-belowness of the scalar potential. Perturbative

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3 Note that the ratio \( M_1/v_\chi \) is finite in the limit \( v_\chi \to 0 \),

\[ \frac{M_1}{v_\chi} = \frac{4}{v_\phi^2} \left[ \mu_2^2 + (2\lambda_2 - \lambda_3)v_\phi^2 + 4(\lambda_3 + 3\lambda_4)v_\chi^2 - 6M_2 v_\chi \right], \tag{10} \]

which follows from the minimization condition \( \partial V/\partial v_\chi = 0 \).
unitarity requires that the \( \lambda_i \) obey the following relations \[21, 22\]:

\[
\sqrt{(6\lambda_1 - 7\lambda_3 - 11\lambda_4)^2 + 36\lambda_2^2} + |6\lambda_1 + 7\lambda_3 + 11\lambda_4| < 4\pi,
\]

\[
\sqrt{(2\lambda_1 + \lambda_3 - 2\lambda_4)^2 + \lambda_5^2} + |2\lambda_1 - \lambda_3 + 2\lambda_4| < 4\pi,
\]

\[
|2\lambda_3 + \lambda_4| < \pi,
\]

\[
|\lambda_2 - \lambda_5| < 2\pi.
\]

(16)

Requiring that the scalar potential is bounded from below imposes the following constraints \[21\]:

\[
\lambda_1 > 0,
\]

\[
\lambda_4 > \begin{cases} 
-\frac{1}{3}\lambda_3 & \text{for } \lambda_3 \geq 0, \\
-\lambda_3 & \text{for } \lambda_3 < 0,
\end{cases}
\]

\[
\lambda_2 > \begin{cases} 
\frac{1}{2}\lambda_5 - 2\sqrt{\frac{1}{3}\lambda_3 + \lambda_4} & \text{for } \lambda_5 \geq 0 \text{ and } \lambda_3 \geq 0, \\
\omega_+(\zeta)\lambda_5 - 2\sqrt{\lambda_1(\zeta\lambda_3 + \lambda_4)} & \text{for } \lambda_5 \geq 0 \text{ and } \lambda_3 < 0, \\
\omega_-(\zeta)\lambda_5 - 2\sqrt{\lambda_1(\zeta\lambda_3 + \lambda_4)} & \text{for } \lambda_5 < 0,
\end{cases}
\]

(17)

where

\[
\omega_\pm(\zeta) = \frac{1}{6}(1 - B) \pm \frac{\sqrt{2}}{3} \left[ (1 - B) \left( \frac{1}{2} + B \right) \right]^{1/2},
\]

\[
B \equiv \sqrt{\frac{3}{2} \left( \zeta - \frac{1}{3} \right)} \in [0, 1],
\]

(18)

and Eq. (17) must be satisfied for all values of \( \zeta \in \left[ \frac{1}{3}, 1 \right] \).

### III. CUSTODIAL VIOLATION AND THE MOST GENERAL GAUGE-IN Variant SCALAR POTENTIAL

In order to allow for custodial symmetry violation, we rewrite the scalar potential in Eq. (3) in the most general \( SU(2)_L \times U(1)_Y \) gauge invariant form, following Ref. [16]. We define the scalar fields in \( SU(2)_L \) vector notation as

\[
\phi = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right), \quad \chi = \left( \begin{array}{c} \chi^{++} \\ \chi^+ \\ \chi^0 \end{array} \right), \quad \xi = \left( \begin{array}{c} \xi^+ \\ \xi^0 \\ \xi^{++} \end{array} \right),
\]

(19)

with vevs given by [compare Eq. (9)],

\[
\phi^0 \rightarrow \tilde{v}_\phi \frac{\phi^0}{\sqrt{2}}, \quad \chi^0 \rightarrow \tilde{v}_\chi \frac{\chi^0 + i\chi^0_i}{\sqrt{2}}, \quad \xi^0 \rightarrow \tilde{v}_\xi + \xi^{0,r}.
\]

(20)

We use tildes to denote the vevs, parameters, and mass eigenstates of the custodial-violating theory. The vevs of these three fields will be determined by \( G_F \) according to [compare Eq. (2)]

\[
\tilde{v}_\phi^2 \equiv \frac{1}{\sqrt{2}G_F} = \frac{v^2}{v^2},
\]

(21)

and will be constrained by the \( \rho \) parameter,

\[
\rho = \frac{\tilde{v}_\phi^2 + 4\tilde{v}_\chi^2 + 4\tilde{v}_\xi^2}{\tilde{v}_\phi^2 + 8\tilde{v}_\chi^2} = \frac{v^2}{v^2 + 4(\tilde{v}_\chi^2 - \tilde{v}_\xi^2)}.
\]

(22)
For convenience, we define the conjugate multiplets,
\[ \tilde{\phi} \equiv C_2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^* = \begin{pmatrix} \phi^0* \\ -\phi^{**} \end{pmatrix}, \]
\[ \tilde{\chi} \equiv C_3 \chi^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \chi^* = \begin{pmatrix} \chi^0* \\ -\chi^{**} \\ \chi^{++*} \end{pmatrix}. \]  
(23)

We also define the following matrix forms of the triplet fields,
\[ \Delta_2 \equiv \sqrt{2} \tau^a U_a \chi = \begin{pmatrix} \chi^+ / \sqrt{2} \\ \chi^0 \\ -\chi^{++} / \sqrt{2} \end{pmatrix}, \]
\[ \Delta_0 \equiv \sqrt{2} \tau^a U_a \xi = \begin{pmatrix} \xi^0 / \sqrt{2} \\ -\xi^{**} \\ -\xi^0 / \sqrt{2} \end{pmatrix}, \]
\[ \tilde{\Delta}_0 \equiv -t^a U_a \xi = \begin{pmatrix} -\xi^0 \\ \xi^{**} \\ 0 \\ \xi^+ \\ 0 \\ \xi^0 \end{pmatrix}. \]  
(24)

The most general gauge invariant scalar potential can then be written as
\[ V(\phi, \chi, \xi) = \tilde{\mu}^2 \phi^\dagger \phi + \tilde{\mu}_0^2 \chi^\dagger \chi + \tilde{\mu}_3^2 \xi^\dagger \xi \
+ \tilde{\lambda}_1 (\phi^\dagger \phi)^2 + \tilde{\lambda}_2 (\chi^\dagger \chi)^2 + \tilde{\lambda}_3 (\phi^\dagger \tau^a \phi)(\chi^\dagger t^a \chi) + \left[ \tilde{\lambda}_4 (\phi^\dagger \tau^a \phi)(\chi^\dagger t^a \xi) + \text{h.c.} \right] \
+ \tilde{\lambda}_5 (\phi^\dagger \phi)(\chi^\dagger \chi) + \tilde{\lambda}_6 (\phi^\dagger \phi)(\xi^\dagger \xi) + \tilde{\lambda}_7 (\chi^\dagger \chi)^2 + \tilde{\lambda}_8 (\xi^\dagger \xi)^2 + \tilde{\lambda}_9 (\chi^\dagger \chi)(\xi^\dagger \xi) \
- \frac{1}{2} \left[ \tilde{M}_1^\dagger \Delta_2 \phi + \text{h.c.} \right] + \frac{\tilde{M}_1}{\sqrt{2}} \phi^\dagger \Delta_0 \phi - 6 \tilde{M}_2 \xi^\dagger \tilde{\Delta}_0 \chi. \]  
(25)

Note that \( \tilde{\lambda}_4 \) and \( \tilde{\lambda}_1^\prime \) are complex in general, while the rest of the parameters are real. We have adopted the same notation as in Eq. (3.2) of Ref. [16] for the coefficients of the quartic terms, and we have added the trilinear terms that were eliminated in Ref. [16] by the imposition of a \( Z_2 \) symmetry. This scalar potential has also been written down (for real \( \tilde{\lambda}_4 \) and \( \tilde{\lambda}_1^\prime \)) in Ref. [18]; we give a translation table to their notation in Appendix A.

We note that the last term in Eq. (25) can also be written as
\[ -6 \tilde{M}_2 \chi^\dagger \tilde{\Delta}_0 \chi = -6 \tilde{M}_2 \epsilon_{ijk} \tilde{\chi}_i \tilde{\xi}_j \chi_k, \]  
(26)

where \( \epsilon_{ijk} \) is the totally antisymmetric tensor with \( \epsilon_{123} = +1. \)

In the custodially-symmetric limit, the Lagrangian parameters in the gauge-invariant scalar potential in Eq. (25)
reduce to those in the custodially-symmetric potential in Eq. (3) according to
\[
\begin{align*}
\tilde{\mu}_2^2 &= \mu_2^2 \\
\tilde{\mu}_3^2 &= \mu_3^2 \\
\tilde{\mu}_3^2 &= \mu_3^2 \\
\tilde{\lambda}_1 &= 4\lambda_1 \\
\tilde{\lambda}_3 &= 2\lambda_3 \\
\tilde{\lambda}_3 &= -2\lambda_5 \\
\tilde{\lambda}_4 &= -\sqrt{2}\lambda_5 \\
\tilde{\lambda}_5 &= 4\lambda_2 \\
\tilde{\lambda}_6 &= 2\lambda_2 \\
\tilde{\lambda}_7 &= 2\lambda_3 + 4\lambda_4 \\
\tilde{\lambda}_8 &= \lambda_3 + \lambda_4 \\
\tilde{\lambda}_9 &= 4\lambda_3 \\
\tilde{\lambda}_{10} &= 4\lambda_4 \\
\tilde{M}_1' &= M_1 \\
\tilde{M}_1 &= M_1 \\
\tilde{M}_2 &= M_2,
\end{align*}
\]
(27)
in agreement with Ref. [10].

Replacing the fields with their vevs and assuming CP conservation, the most general scalar potential becomes
\[
V(\phi, \chi, \xi) = \frac{\tilde{\mu}_2^2}{2}\tilde{\phi}^2 + \tilde{\mu}_3^2\tilde{\phi}_\chi^2 + \frac{\tilde{\mu}_3^2}{2}\tilde{\phi}_\xi^2 \\
+ \frac{\tilde{\lambda}_1}{4}\tilde{\phi}_\chi^4 + \frac{\tilde{\lambda}_3}{4}\tilde{\phi}_\chi^2\tilde{\phi}_\xi^2 + \frac{\tilde{\lambda}_4}{\sqrt{2}}\tilde{\phi}_\chi\tilde{\phi}_\xi \\
+ \frac{\tilde{\lambda}_5}{2}\tilde{\phi}_\chi^2\tilde{\phi}_\xi^2 + \frac{\tilde{\lambda}_6}{2}\tilde{\phi}_\chi\tilde{\phi}_\xi + \tilde{\lambda}_7\tilde{\phi}_\chi^4 + \tilde{\lambda}_8\tilde{\phi}_\xi^4 + \tilde{\lambda}_{10}\tilde{\phi}_\chi^2\tilde{\phi}_\xi^2 \\
- \tilde{M}_1'\tilde{\phi}_\chi - \frac{\tilde{M}_1}{4}\tilde{\phi}_\phi - 6\tilde{M}_2\tilde{\phi}_\xi.
\]
(28)

Minimizing this potential yields three equations:
\[
0 = \frac{\partial V}{\partial \tilde{\phi}} = \tilde{\phi}_\phi \left[ \tilde{\mu}_2^2 + \tilde{\lambda}_1\tilde{\phi}_\chi^2 + \frac{\tilde{\lambda}_3}{2}\tilde{\phi}_\chi^2 + \frac{\tilde{\lambda}_4}{\sqrt{2}}\tilde{\phi}_\chi\tilde{\phi}_\xi + \tilde{\lambda}_5\tilde{\phi}_\chi\tilde{\phi}_\xi + \tilde{\lambda}_6\tilde{\phi}_\chi^2 + \tilde{\lambda}_7\tilde{\phi}_\chi^4 + \tilde{\lambda}_8\tilde{\phi}_\xi^4 + \tilde{\lambda}_{10}\tilde{\phi}_\chi^2\tilde{\phi}_\xi^2 - \frac{\tilde{M}_1'}{2}\tilde{\phi}_\phi - \frac{\tilde{M}_1}{4}\tilde{\phi}_\phi - 6\tilde{M}_2\tilde{\phi}_\xi \right],
\]
(29)
\[
0 = \frac{\partial V}{\partial \tilde{\phi}_\chi} = 2\tilde{\mu}_3^2\tilde{\phi}_\chi + \frac{\tilde{\lambda}_3}{2}\tilde{\phi}_\phi^2\tilde{\phi}_\chi + \frac{\tilde{\lambda}_4}{\sqrt{2}}\tilde{\phi}_\phi\tilde{\phi}_\chi + \frac{\tilde{\lambda}_5}{2}\tilde{\phi}_\phi\tilde{\phi}_\chi + 4\tilde{\lambda}_7\tilde{\phi}_\chi^3 + 2\tilde{\lambda}_{10}\tilde{\phi}_\chi^2\tilde{\phi}_\xi - \frac{\tilde{M}_1'}{2}\tilde{\phi}_\phi - 12\tilde{M}_2\tilde{\phi}_\xi, 
\]
(30)
\[
0 = \frac{\partial V}{\partial \tilde{\phi}_\xi} = \tilde{\mu}_3^2\tilde{\phi}_\xi + \frac{\tilde{\lambda}_4}{\sqrt{2}}\tilde{\phi}_\phi\tilde{\phi}_\xi + \frac{\tilde{\lambda}_6}{2}\tilde{\phi}_\phi\tilde{\phi}_\xi + 4\tilde{\lambda}_8\tilde{\phi}_\xi^3 + 2\tilde{\lambda}_{10}\tilde{\phi}_\chi^2\tilde{\phi}_\xi - \frac{\tilde{M}_1}{4}\tilde{\phi}_\phi - 6\tilde{M}_2\tilde{\phi}_\chi. 
\]
(31)

When the SU(2)_L × SU(2)_R symmetry is imposed, these conditions reduce to those in Eq. (6).

**IV. PHYSICAL MASSES AND MIXING IN THE CUSTODIAL SYMMETRY VIOLATING THEORY**

Isolating all terms quadratic in scalar fields from the potential and using Eqs. (29–31) to eliminate \(\tilde{\mu}_2^2\), \(\tilde{\mu}_3^2\) and \(\tilde{\mu}_3^2\) in favour of the vevs yields the following mass matrices for the physical scalars.

There is only one doubly-charged scalar, \(\tilde{H}_5^{++} = \chi^{++} = H_5^{++}\), and its mass is given by
\[
m_{\tilde{H}_5^{++}}^2 = 4\lambda_2\tilde{\phi}_\chi^2 - \frac{\tilde{\lambda}_3\tilde{\phi}_\phi^2}{2} - \frac{\tilde{\lambda}_4\tilde{\phi}_\phi^2}{2\sqrt{2}\tilde{\phi}_\chi} + \frac{\tilde{M}_1'}{4}\tilde{\phi}_\phi^2 + 12\tilde{M}_2\tilde{\phi}_\xi.
\]
(32)
There are two CP-odd neutral scalars (one of which becomes the neutral Goldstone boson), whose mass-squared matrix in the basis $(\chi^{0,i}, \phi^{0,i})$ is given by

$$\mathcal{M}_i^2 = \begin{pmatrix} \mathcal{M}_{i,11}^2 & \mathcal{M}_{i,12}^2 \\ \mathcal{M}_{i,12}^2 & \mathcal{M}_{i,22}^2 \end{pmatrix},$$  \hspace{1cm} (33)

where

$$\mathcal{M}_{i,11}^2 = -\frac{\hat{\lambda}_4 \tilde{v}_\phi^2 \hat{v}_\chi}{2\sqrt{2} \tilde{v}_\phi} + \frac{\tilde{M}_1^2 \hat{v}_\phi^2}{4 \tilde{v}_\phi},$$

$$\mathcal{M}_{i,12}^2 = -2\sqrt{2} \lambda_4 \tilde{v}_\chi \hat{v}_\chi + 2 \tilde{M}_1' \hat{v}_\chi,$$

$$\mathcal{M}_{i,22}^2 = \hat{\lambda}_4 \tilde{v}_\phi \hat{v}_\chi - \frac{\tilde{M}_1'}{\sqrt{2}} \hat{v}_\phi.$$

(34)

Note that the mass-squared matrix for the neutral imaginary states can be written as

$$\mathcal{M}_i^2 = \begin{pmatrix} \tilde{M}_1' & \tilde{v}_\phi \\ \tilde{v}_\phi & \tilde{M}_1 \end{pmatrix} \begin{pmatrix} \tilde{M}_1' & \tilde{v}_\phi \\ \tilde{v}_\phi & \tilde{M}_1 \end{pmatrix}^T,$$  \hspace{1cm} (35)

This matrix is easily diagonalized, yielding exact mass eigenstates

$$\tilde{G}_0 = \frac{\tilde{v}_\phi \phi^{0,i} + \sqrt{8} \tilde{v}_\phi \chi^{0,i}}{\sqrt{\tilde{v}_\phi^2 + 8 \tilde{v}_\chi^2}},$$

$$\tilde{H}_3 = \frac{-\sqrt{8} \tilde{v}_\phi \phi^{0,i} + \tilde{v}_\phi \chi^{0,i}}{\sqrt{\tilde{v}_\phi^2 + 8 \tilde{v}_\chi^2}},$$  \hspace{1cm} (36)

where $\tilde{G}_0$ is the (massless) neutral Goldstone boson and the mass of $\tilde{H}_3$ is given by

$$m^2_{\tilde{H}_3} = \left[ \frac{\tilde{M}_1'}{\tilde{v}_\phi} - \frac{\hat{\lambda}_4 \tilde{v}_\chi}{2\sqrt{2} \tilde{v}_\phi} \right] \left( \tilde{v}_\phi^2 + 8 \tilde{v}_\chi^2 \right).$$  \hspace{1cm} (37)

There are three singly-charged scalars (one of which becomes the charged Goldstone boson), whose mass-squared matrix in the basis $(\chi^+, \xi^+, \phi^+)$ is given by

$$\mathcal{M}_+^2 = \begin{pmatrix} \mathcal{M}_{+,11}^2 & \mathcal{M}_{+,12}^2 & \mathcal{M}_{+,13}^2 \\ \mathcal{M}_{+,12}^2 & \mathcal{M}_{+,22}^2 & \mathcal{M}_{+,23}^2 \\ \mathcal{M}_{+,13}^2 & \mathcal{M}_{+,23}^2 & \mathcal{M}_{+,33}^2 \end{pmatrix},$$  \hspace{1cm} (38)

where

$$\mathcal{M}_{+,11}^2 = -\frac{\hat{\lambda}_4 \tilde{v}_\phi^2 \hat{v}_\chi}{2\sqrt{2} \tilde{v}_\phi} + \tilde{\lambda}_9 \tilde{v}_\phi^2 + \frac{\tilde{M}_1'}{\tilde{v}_\phi} \tilde{v}_\phi^2 + 6 \tilde{M}_2 \tilde{v}_\xi,$$

$$\mathcal{M}_{+,22}^2 = -\frac{\tilde{\lambda}_4 \tilde{v}_\phi^2 \hat{v}_\chi}{\sqrt{2} \tilde{v}_\phi} + \tilde{\lambda}_9 \tilde{v}_\phi^2 + \frac{\tilde{M}_1'}{\tilde{v}_\phi} \tilde{v}_\phi^2 + 6 \tilde{M}_2 \tilde{v}_\xi,$$

$$\mathcal{M}_{+,33}^2 = -\frac{\hat{\lambda}_3 \tilde{v}_\phi^2 \hat{v}_\chi}{\sqrt{2} \tilde{v}_\phi} - \tilde{\lambda}_9 \tilde{v}_\phi^2 + \tilde{M}_1 \tilde{v}_\chi + \tilde{M}_1' \tilde{v}_\chi,$$

$$\mathcal{M}_{+,12}^2 = \frac{\hat{\lambda}_4 \tilde{v}_\phi^2 \hat{v}_\chi}{2\sqrt{2}} - \tilde{\lambda}_9 \tilde{v}_\phi \tilde{v}_\xi - 6 \tilde{M}_2 \tilde{v}_\chi,$$

$$\mathcal{M}_{+,13}^2 = \frac{\tilde{\lambda}_3 \tilde{v}_\phi \hat{v}_\chi}{2} - \frac{\tilde{M}_1'}{2} \tilde{v}_\phi,$$

$$\mathcal{M}_{+,23}^2 = \frac{\hat{\lambda}_4 \tilde{v}_\phi \hat{v}_\chi}{\sqrt{2}} - \frac{\tilde{M}_1'}{2} \tilde{v}_\phi.$$

(39)

We first transform this mass-squared matrix into the basis of custodial-symmetric states $(H_0^+, H_3^+, G^+)$ using

$$\mathcal{M}_+^2 = R_+ \mathcal{M}_+^2 R_+^T,$$  \hspace{1cm} (40)
The compositions of the mass eigenstates are given to first order using
\[
\begin{pmatrix}
H^+_5 \\
H^+_3 \\
G^+
\end{pmatrix} = R_+ \begin{pmatrix}
\chi^+ \\
\xi^+ \\
\phi^+
\end{pmatrix},
\]
with
\[
R_+ = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \sqrt{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{2}
\end{pmatrix}.
\]

Because the custodial-symmetry-violating effects will be small, for practical purposes we can diagonalize the mass-squared matrix \(\mathcal{M}^2\) using first-order perturbation theory. To first order in the custodial violation, the masses of the singly-charged physical mass eigenstates \(\tilde{H}_5^+\) and \(\tilde{H}_3^+\) are just given by the diagonal elements of the mass-squared matrix,
\[
m^2_{\tilde{H}_5^+} = \mathcal{M}^2_{+,11}, \quad m^2_{\tilde{H}_3^+} = \mathcal{M}^2_{+,22}.
\]
The compositions of the mass eigenstates are given to first order using
\[
\tilde{H}_n = H_n + \sum_{m \neq n} \frac{\mathcal{M}^2_{nm}}{\mathcal{M}^2_{nn} - \mathcal{M}^2_{mm}} H_m,
\]
where \(\mathcal{M}^2\) is the mass-squared matrix in the appropriate basis. Applying this to the singly-charged states and using the fact that \(\mathcal{M}^2_{+,33} = 0\), we get,
\[
\tilde{H}_5^+ = H_5^+ + \frac{\mathcal{M}^2_{+,12}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} H_3^+ + \frac{\mathcal{M}^2_{+,13}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} G^+
\]
\[
= \frac{\chi^+ - \xi^+}{\sqrt{2}} + \left[ c_H \frac{\mathcal{M}^2_{+,13}}{\mathcal{M}^2_{+,11}} - s_H \frac{\mathcal{M}^2_{+,12}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} \right] \phi^+ + \left[ s_H \frac{\mathcal{M}^2_{+,13}}{\mathcal{M}^2_{+,11}} + c_H \frac{\mathcal{M}^2_{+,12}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} \right] \frac{\chi^+ + \xi^+}{\sqrt{2}}.
\]
\[
\tilde{H}_3^+ = H_3^+ + \frac{\mathcal{M}^2_{+,12}}{\mathcal{M}^2_{+,22} - \mathcal{M}^2_{+,11}} H_5^+ + \frac{\mathcal{M}^2_{+,23}}{\mathcal{M}^2_{+,22} - \mathcal{M}^2_{+,11}} G^+,
\]
\[
\tilde{G}^+ = G^+ + \frac{\mathcal{M}^2_{+,13}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} H_5^+ + \frac{\mathcal{M}^2_{+,23}}{\mathcal{M}^2_{+,22} - \mathcal{M}^2_{+,11}} H_3^+.
\]
We highlight the composition of \(\tilde{H}_5^+\) in particular because the custodial symmetry violation results in an admixture of \(\phi^+\) into this state. This allows \(\tilde{H}_5^+\) to couple to fermions, which does not occur in the custodial-symmetric GM model. Indeed, we can write the Feynman rule for the \(\tilde{H}_5^+ \bar{u} d\) vertex as
\[
\tilde{H}_5^+ \bar{u} d : \quad i \frac{\sqrt{2}}{v} V_{ud} \kappa_f \tilde{H}_5^+ (m_u P_L - m_d P_R),
\]
where the coupling to fermions induced by the custodial symmetry violation is, to first order,
\[
\kappa_f^+ = \frac{\mathcal{M}^2_{+,13}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}} - \tan \theta_H \frac{\mathcal{M}^2_{+,12}}{\mathcal{M}^2_{+,11} - \mathcal{M}^2_{+,22}}.
\]
For comparison, in the custodial-symmetric GM model we can write the analogous coupling of \(H_3^+\) to fermion pairs as \(\kappa_f^{H_3^+} = -\tan \theta_H\).

Finally, there are three CP-even neutral scalars, whose mass-squared matrix in the basis \((\chi^{0,r}, \xi^{0,r}, \phi^{0,r})\) is given
The mixing angle that achieves this diagonalization is given by

$$\sin 2\tilde{\alpha} = \frac{2M^2_{r,23}}{m^2_{h} - m^2_{H}}, \quad \cos 2\tilde{\alpha} = \frac{M^2_{r,22} - M^2_{r,33}}{m^2_{h} - m^2_{H}},$$

where the states are given in terms of $\tilde{\alpha}$ by

$$h_{\tilde{\alpha}} = c_{\tilde{\alpha}} \phi^{0,r} - s_{\tilde{\alpha}} H_1^{0'}, \quad H_{\tilde{\alpha}} = s_{\tilde{\alpha}} \phi^{0,r} + c_{\tilde{\alpha}} H_1^{0'},$$

and we have defined $c_{\tilde{\alpha}} = \cos \tilde{\alpha}, \ s_{\tilde{\alpha}} = \sin \tilde{\alpha}$. (Note that these are not yet the mass eigenstates: there is still a small
mixing with $H_5^0$ to be dealt with below.) We introduce a second orthogonal rotation matrix $R_\tilde{\alpha}$, defined according to

$$
\begin{pmatrix}
H_5^0 \\
H_\tilde{\alpha} \\
h_\tilde{\alpha}
\end{pmatrix}
= R_\tilde{\alpha}
\begin{pmatrix}
H_5^0 \\
H_\tilde{\alpha} \\
\phi_0^{0,r}
\end{pmatrix},
$$

(59)

with

$$
R_\tilde{\alpha}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & c_\tilde{\alpha} & s_\tilde{\alpha} \\
0 & -s_\tilde{\alpha} & c_\tilde{\alpha}
\end{pmatrix}.
$$

(60)

The mass-squared matrix in the basis $(H_5^0, H_\tilde{\alpha}, h_\tilde{\alpha})$ is then given by

$$
\mathcal{M}^{\mu\nu}_{r} = R_\tilde{\alpha} \mathcal{M}^{\mu\nu}_{r} R_\tilde{\alpha}^{T} = \begin{pmatrix}
M_{r,11}^{\mu\nu} & M_{r,12}^{\mu\nu} & M_{r,13}^{\mu\nu} \\
M_{r,12}^{\mu\nu} & M_{r,22}^{\mu\nu} & 0 \\
M_{r,13}^{\mu\nu} & 0 & M_{r,33}^{\mu\nu}
\end{pmatrix}.
$$

(61)

Note that $\mathcal{M}^{\mu\nu}_{r,11} = \mathcal{M}^{\mu\nu}_{r,11}$. The masses of $\tilde{h}$ and $\tilde{H}$ can then be written (to first order in the custodial symmetry violation) in terms of the diagonal elements of this matrix as

$$
m_{\tilde{h}}^2 = M_{r,33}^{\mu\nu}, \quad m_{\tilde{H}}^2 = M_{r,22}^{\mu\nu}.
$$

(62)

We now use Eq. (44) to write the compositions of the CP-even neutral mass eigenstates to first order in the custodial violation as

$$
\tilde{H}_5^0 = H_5^0 + \frac{M_{r,12}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,22}^{\mu\nu}} H_\tilde{\alpha} + \frac{M_{r,13}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,33}^{\mu\nu}} h_\tilde{\alpha},
$$

$$
= \left[ \sqrt{\frac{2}{3}} \xi^{0,r} - \sqrt{\frac{1}{3}} \phi^{0,r} \right] + \left[ s_\tilde{\alpha} \frac{M_{r,12}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,22}^{\mu\nu}} + c_\tilde{\alpha} \frac{M_{r,13}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,33}^{\mu\nu}} \right] \phi^{0,r},
$$

(63)

$$
\tilde{H} = H_\tilde{\alpha} + \frac{M_{r,12}^{\mu\nu}}{M_{r,22}^{\mu\nu} - M_{r,11}^{\mu\nu}} H_5^0,
$$

(64)

$$
\tilde{h} = h_\tilde{\alpha} + \frac{M_{r,13}^{\mu\nu}}{M_{r,33}^{\mu\nu} - M_{r,11}^{\mu\nu}} H_5^0.
$$

(65)

We highlight the composition of $\tilde{H}_5^0$ in particular because the custodial symmetry violation results in an admixture of $\phi^{0,r}$ into this state. This allows $\tilde{H}_5^0$ to couple to fermions, which does not occur in the custodial-symmetric GM model. The coupling of $\tilde{H}_5^0$ to $\bar{f} f$, normalized to the corresponding coupling of the SM Higgs boson, is then given to first order in the custodial symmetry violation by

$$
\kappa_{\tilde{H}_5^0}^{\tilde{H}} = \frac{1}{c_H} \left[ s_\tilde{\alpha} \frac{M_{r,12}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,22}^{\mu\nu}} + c_\tilde{\alpha} \frac{M_{r,13}^{\mu\nu}}{M_{r,11}^{\mu\nu} - M_{r,33}^{\mu\nu}} \right].
$$

(66)

Finally, the mixing of a small amount of custodial-fiveplet $H_5^0$ into the physical Higgs boson $\hat{h}$, together with $\tilde{\nu}_\chi \neq \tilde{\nu}_\xi$, leads to a violation of custodial symmetry in the couplings of $\hat{h}$ to $WW$ and $ZZ$. This is parameterized in terms of the physical observable

$$
\lambda_{WWZ} \equiv \frac{\kappa_{\hat{h}}^W}{\kappa_{\hat{h}}^Z},
$$

(67)

where $\kappa_{\hat{h}}^W$ and $\kappa_{\hat{h}}^Z$ are the couplings of $\hat{h}$ to $WW$ and $ZZ$, respectively, normalized to the corresponding couplings of
the SM Higgs boson. We can write this in terms of the vevs and the mixing with $H_0^0$ as follows:

$$\tilde{\lambda}_{WZ} = \frac{\tilde{\kappa}_W^h + \tilde{\kappa}_W^Z}{\tilde{\kappa}_Z^h + \tilde{\kappa}_Z^W},$$

(68)

where the couplings of $h_0$ to $W$ and $Z$ boson pairs, including the effects of $\tilde{v}_\chi \neq \tilde{v}_\xi$, are given by

$$\tilde{\kappa}_W^h = c_\lambda \frac{\tilde{v}_\phi}{v} - s_\lambda \frac{4}{\sqrt{3}} \frac{\tilde{v}_\chi + \tilde{v}_\xi}{v}, \quad \tilde{\kappa}_Z^h = c_\lambda \frac{\tilde{v}_\phi}{v} - s_\lambda \frac{8}{\sqrt{3}} \frac{\tilde{v}_\chi}{v},$$

(69)

the couplings of $H_0^0$ to $W$ and $Z$ boson pairs are given by

$$\tilde{\kappa}_W^h = \sqrt{\frac{2}{3}} \frac{4\tilde{v}_\chi - 2\tilde{v}_\xi}{v} \simeq \frac{1}{\sqrt{3}} \log s_H, \quad \tilde{\kappa}_Z^h = -\sqrt{\frac{2}{3}} \frac{4\tilde{v}_\chi}{v} \simeq \frac{2}{\sqrt{3}} s_H,$$

(70)

and the mixing of $H_0^0$ into $h$ from Eq. (65) is

$$\epsilon = \frac{M_{r,13}^{1/2}}{M_{r,13}^{1/2} - M_{r,11}^{1/2}}.$$

(71)

V. RENORMALIZATION GROUP EQUATIONS FOR LAGRANGIAN PARAMETERS

In order to run the parameters down from a custodial-symmetric high scale to the weak scale, we need the RGEs. We determine these using the formalism presented in Ref. [23], some details of which are given in Appendix B. The resulting equations are then (with $t \equiv \log \mu$, where $\mu$ is the energy scale),

$$16\pi^2 \frac{d (\tilde{\mu}_2)}{dt} = \frac{3}{2} \tilde{M}_1^2 + 3|\tilde{M}_1|^2 + \tilde{\mu}_2^2 \left(6y_b^2 + 6y_t^2 + 2g_2^2 - \frac{9}{10} g_1^2 - \frac{9}{2} g_2^2 + 12\tilde{\lambda}_1\right) + 6\tilde{\mu}_3^2 \tilde{\lambda}_6 + 6\tilde{\mu}_3 \tilde{\lambda}_5,$$

(72)

$$16\pi^2 \frac{d (\tilde{\mu}_3^2)}{dt} = |\tilde{M}_1|^2 + 144\tilde{M}_2^2 + \tilde{\mu}_3^2 \left(8\tilde{\lambda}_2 + 16\tilde{\lambda}_7 - \frac{18}{5} g_1^2 - 12g_2^2\right) + 4\tilde{\mu}_2^2 \tilde{\lambda}_5 + 2\tilde{\mu}_3 \left(\tilde{\lambda}_9 + 3\tilde{\lambda}_{10}\right),$$

(73)

$$16\pi^2 \frac{d (\tilde{\mu}_3)}{dt} = \tilde{M}_1^2 + 144\tilde{M}_2^2 + 4\tilde{\mu}_3^2 \left(10\tilde{\lambda}_8 - 3g_2^2\right) + 8\tilde{\mu}_3^2 \tilde{\lambda}_6 + 4\tilde{\mu}_3 \left(\tilde{\lambda}_9 + 3\tilde{\lambda}_{10}\right),$$

(74)

$$16\pi^2 \frac{d (\tilde{\lambda}_1)}{dt} = -6y_b^2 - 6y_t^2 - 2g_2^2 + \tilde{\lambda}_1 \left(12y_b^2 + 12y_t^2 + 4g_2^2 - \frac{9}{5} g_1^2 - g_2^2 + 24\tilde{\lambda}_1\right)$$

$$+ \frac{27}{200} g_1^4 + \frac{9}{8} g_2^4 + \frac{9}{20} g_1^2 g_2^2 + \frac{1}{2} \tilde{\lambda}_3^2 + 2|\tilde{\lambda}_4|^2 + 3\tilde{\lambda}_5^2 + 6\tilde{\lambda}_6^2,$$

(75)

$$16\pi^2 \frac{d (\tilde{\lambda}_2)}{dt} = 3g_2^4 - \frac{36}{5} g_1^2 g_2^2 + 12\tilde{\lambda}_2 \left(\tilde{\lambda}_2 + 2\tilde{\lambda}_7 - \frac{3}{5} g_1^2 - 2g_2^2\right) - \frac{1}{2} \tilde{\lambda}_3^2 + \tilde{\lambda}_6^2,$$

(76)

$$16\pi^2 \frac{d (\tilde{\lambda}_3)}{dt} = \tilde{\lambda}_3 \left(6y_b^2 + 6y_t^2 + 2g_2^2 + 4\tilde{\lambda}_1 - 8\tilde{\lambda}_2 + 8\tilde{\lambda}_5 + 4\tilde{\lambda}_7 - \frac{9}{2} g_1^2 - \frac{33}{2} g_2^2\right) + \frac{36}{5} g_2^4 g_1^2 + 4|\tilde{\lambda}_4|^2,$$

(77)

$$16\pi^2 \frac{d (\tilde{\lambda}_4)}{dt} = \tilde{\lambda}_4 \left(6y_b^2 + 6y_t^2 + 2g_2^2 - \frac{27}{10} g_1^2 - \frac{33}{2} g_2^2 + 4\tilde{\lambda}_1 + 2\tilde{\lambda}_3 + 4\tilde{\lambda}_5 + 8\tilde{\lambda}_6 - 2\tilde{\lambda}_9 + 4\tilde{\lambda}_{10}\right),$$

(78)
16\pi^2 \frac{d\tilde{\lambda}_5}{dt} = \tilde{\lambda}_5 \left( 6y_b^2 + 6y_t^2 + 2y_t^2 + 4\lambda_5 + 12\lambda_1 + 8\lambda_2 + 16\lambda_7 - \frac{9}{10}g_1^2 - \frac{33}{2}g_2^2 \right) + \frac{27}{25}g_1^4 + 6g_2^4 + 2\lambda_2^2 + 4|\lambda_4|^2 + 4\lambda_5\lambda_9 + 12\lambda_6\lambda_{10},

(79)

16\pi^2 \frac{d\tilde{\lambda}_6}{dt} = \tilde{\lambda}_6 \left( 6y_b^2 + 6y_t^2 + 2y_t^2 + 8\lambda_6 + 12\lambda_1 + 40\lambda_8 - \frac{9}{10}g_1^2 - \frac{33}{2}g_2^2 \right) + 3g_2^4 + 4|\lambda_4|^2 + 2\lambda_5\lambda_9 + 6\lambda_5\lambda_{10},

(80)

16\pi^2 \frac{d\tilde{\lambda}_7}{dt} = \frac{54}{25}g_1^4 + 9g_2^4 + \frac{36}{5}g_2^2 g_1^2 + \left( -\frac{36}{5}g_1^2 - 24g_2^2 + 16\lambda_2 + 28\lambda_7 \right) + 16\tilde{\lambda}_7^2 + \frac{1}{2}\lambda_3^2 + 2\lambda_5^2 + \lambda_5^2 + 2\lambda_{10} \left( 3\lambda_{10} + 2\lambda_9 \right),

(81)

16\pi^2 \frac{d\tilde{\lambda}_8}{dt} = 3g_2^4 + 8\lambda_8 \left( -3g_2^2 + 11\lambda_8 \right) + 2\lambda_6^2 + \lambda_9 \left( \tilde{\lambda}_9 + 2\tilde{\lambda}_{10} \right) + 3\lambda_{10},

(82)

16\pi^2 \frac{d\tilde{\lambda}_9}{dt} = 6g_2^4 + 2\tilde{\lambda}_9 \left( -12g_2^2 - \frac{9}{5}g_1^2 + 5\lambda_9 + 4\lambda_2 + 2\lambda_7 + 8\lambda_8 + 8\lambda_{10} \right) - 2|\tilde{\lambda}_4|^2,

(83)

16\pi^2 \frac{d\tilde{\lambda}_{10}}{dt} = 6g_2^4 + 2\tilde{\lambda}_{10} \left( -\frac{9}{5}g_1^2 - 12g_2^2 + 4\lambda_2 + 8\lambda_7 + 20\lambda_8 + 4\lambda_{10} \right) + 2|\tilde{\lambda}_4|^2 + 2\lambda_5^2 + 4\lambda_5\lambda_6 + 4\lambda_9 \left( \tilde{\lambda}_7 + 2\lambda_8 \right),

(84)

16\pi^2 \frac{dM_1'}{dt} = \tilde{M}_1' \left( 6y_b^2 + 6y_t^2 + 2y_t^2 - \frac{27}{10}g_1^2 - \frac{21}{2}g_2^2 + 4\lambda_1 + 4\lambda_3 + 4\lambda_5 \right) + 4\sqrt{2}\lambda_4 \left( \tilde{M}_1 + 6\tilde{M}_2 \right),

(85)

16\pi^2 \frac{dM_1}{dt} = \tilde{M}_1 \left( 6y_b^2 + 6y_t^2 + 2y_t^2 - \frac{9}{5}g_1^2 - \frac{21}{2}g_2^2 + 4\lambda_1 + 8\lambda_6 \right) + 24\tilde{M}_2\lambda_3 + 8\sqrt{2}\text{Re} \left[ \tilde{M}_1'\tilde{\lambda}_4 \right],

(86)

16\pi^2 \frac{dM_2}{dt} = \tilde{M}_2 \left( -\frac{18}{5}g_1^2 - 18g_2^2 - 8\lambda_2 + 4\lambda_7 - 4\lambda_9 + 8\lambda_{10} \right) + \frac{1}{6} \tilde{M}_1\lambda_3 + \frac{1}{3}\sqrt{2}\text{Re} \left[ \tilde{M}_1'\tilde{\lambda}_4 \right],

(87)

where \( g_1 \) and \( g_2 \) are gauge couplings (see below) and \( y_b, y_t, \) and \( y_r \) are Yukawa couplings, normalized according to \( y_f = \sqrt{2}m_f/\bar{v}_f. \) These RGEs agree with those of Ref. [18] (for real \( \tilde{\lambda}_4 \) and \( M_1' \)) after translating the notation for the Lagrangian parameters as in Appendix A. A few potential symmetries are apparent in these RGEs. Setting \( M_1' = M_1 = M_2 = 0, \) the potential becomes invariant under \( (\chi, \xi) \to (\chi, -\xi) \) and therefore these three parameters are not regenerated by the running. Setting instead \( \tilde{\lambda}_4 = M_1' = 0, \) the potential becomes invariant under \( \chi \to -\chi \) and therefore these two parameters are not regenerated by the running. Setting \( \tilde{\lambda}_4 = M_1 = M_2 = 0, \) the potential becomes invariant under \( \xi \to -\xi \) and therefore these three parameters are not regenerated by the running. Finally, if all the Lagrangian parameters are taken to be real at some scale, as will be the case when the most general potential is matched onto the intrinsically CP-conserving custodial-symmetric Georgi-Machacek model, they remain real at all scales.

Throughout we use the GUT normalization \( g' = \sqrt{\frac{3}{5}}g_1, g = g_2, \) and \( g_s = g_3. \) The renormalization group equations for the electroweak gauge couplings, including all the particle content of the GM model in the spectrum, are [24],

\[
16\pi^2 \frac{dg_1}{dt} = \frac{47}{10}g_1^3 \quad \text{or equivalently} \quad 16\pi^2 \frac{dg'}{dt} = \frac{47}{6}g'^3,
\]

(88)

\[
16\pi^2 \frac{dg_2}{dt} = -\frac{13}{6}g_2^3,
\]

(89)
and that for the strong gauge coupling is the same as in the SM (including the top quark contribution),

\[ 16\pi^2 \frac{dg_3}{dt} = -7g_3^3. \] (90)

The RGEs for the Yukawa couplings are identical to those of the SM [25],

\[ 16\pi^2 \frac{dy_t}{dt} = \left( -\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \frac{3}{2}y_b^2 + \frac{9}{2}y_t^2 + y_\tau^2 \right) y_t, \] (91)

\[ 16\pi^2 \frac{dy_b}{dt} = \left( -\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \frac{9}{2}y_b^2 + \frac{3}{2}y_t^2 + y_\tau^2 \right) y_b, \] (92)

\[ 16\pi^2 \frac{dy_\tau}{dt} = \left( -\frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 + 3y_b^2 + 3y_t^2 + \frac{5}{2}y_\tau^2 \right) y_\tau. \] (93)

In our numerical work we will ignore \( y_b \) and \( y_\tau \).

As a consistency check, we can turn off the custodial-violating parts of the RGEs by setting \( g_1 = 0 \) and substituting the relations given in Eq. (27). We then find a self-consistent set of RGEs for the custodial-preserving Lagrangian parameters:

\[ 16\pi^2 \frac{d(\mu_2^2)}{dt} = \frac{9}{2}M_1^2 + \mu_2^2 \left( 6y_b^2 + 6y_t^2 + 2y_\tau^2 - \frac{9}{2}g_2^2 + 48\lambda_1 \right) + 36\mu_3^2\lambda_2, \] (94)

\[ 16\pi^2 \frac{d(\mu_3^2)}{dt} = M_1^2 + 144M_2^2 + 16\mu_2^2\lambda_2 + \mu_3^2 \left( -12g_2^2 + 56\lambda_3 + 88\lambda_4 \right), \] (95)

\[ 16\pi^2 \frac{d\lambda_1}{dt} = -\frac{3}{2}g_b^4 - \frac{3}{2}g_t^4 - \frac{1}{2}y_\tau^4 + \lambda_1 \left( 12y_b^2 + 12y_t^2 + 4y_\tau^2 - 9g_2^2 + 96\lambda_1 \right) + \frac{9}{32}g_2^4 + 18\lambda_2^2 + \frac{3}{2}\lambda_3^2, \] (96)

\[ 16\pi^2 \frac{d\lambda_2}{dt} = \lambda_2 \left( 6y_b^2 + 6y_t^2 + 2y_\tau^2 - \frac{33}{2}g_2^2 + 48\lambda_1 + 16\lambda_2 + 56\lambda_3 + 88\lambda_4 \right) + \frac{3}{2}g_2^4 + 4\lambda_3^2, \] (97)

\[ 16\pi^2 \frac{d\lambda_3}{dt} = \frac{3}{2}g_2^4 + \lambda_3 \left( -24g_2^2 + 80\lambda_3 + 96\lambda_4 \right) - \lambda_3^2, \] (98)

\[ 16\pi^2 \frac{d\lambda_4}{dt} = \frac{3}{2}g_2^4 + \lambda_4 \left( -24g_2^2 + 136\lambda_4 + 112\lambda_3 \right) + 8\lambda_2^2 + 24\lambda_3^2 + \lambda_5^2, \] (99)

\[ 16\pi^2 \frac{d\lambda_5}{dt} = \lambda_5 \left( 6y_b^2 + 6y_t^2 + 2y_\tau^2 - \frac{33}{2}g_2^2 + 16\lambda_1 + 32\lambda_2 - 8\lambda_3 + 16\lambda_4 - 4\lambda_5 \right), \] (100)

\[ 16\pi^2 \frac{dM_1}{dt} = M_1 \left( 6y_b^2 + 6y_t^2 + 2y_\tau^2 - \frac{21}{2}g_2^2 + 16\lambda_1 + 16\lambda_2 - 16\lambda_5 \right) - 48M_2\lambda_5, \] (101)

\[ 16\pi^2 \frac{dM_2}{dt} = -M_1\lambda_5 + M_2 \left( -18g_2^2 - 24\lambda_3 + 48\lambda_4 \right). \] (102)
and boundedness-from-below as derived for the custodial-symmetric theory [21] as given at the end of Sec. II.

whether the potential has become unbounded from below; this turns out not to happen in our scans of the H5plane
run into a Landau pole). This allows us to determine the maximum scale allowed by perturbativity. We also check
the quartic scalar couplings has grown large enough to violate perturbative unitarity (indicating that we have almost
The result of this is a custodial-symmetric scalar potential at the scale Λ. At this stage we can check whether any of
scalars potential, causing violation of the custodial-symmetry relations of Eq. (27) among the parameters of the most

g RGEs in Eqs. (72–91) with

set to zero. We also run the gauge couplings (including the actual value of g1) and the top Yukawa
coupling from m3 to Λ using Eqs. (88–91). For the running we use fourth-order Runge-Kutta with a small step size.
The result of this is a custodial-symmetric scalar potential at the scale Λ. At this stage we can check whether any of
the quartic scalar couplings has grown large enough to violate perturbative unitarity (indicating that we have almost
run into a Landau pole). This allows us to determine the maximum scale allowed by perturbativity. We also check
whether the potential has become unbounded from below; this turns out not to happen in our scans of the H5plane
benchmark. Because the potential is still custodial-symmetric, we can use the requirements for perturbative unitarity
and boundedness-from-below as derived for the custodial-symmetric theory [21] as given at the end of Sec. II.

From the custodial-symmetric scalar potential at scale Λ, we then run back down to the scale m5 using the full
RGEs in Eqs. (72–91) with g1 ̸= 0. The nonzero hypercharge coupling induces custodial symmetry violation in the
scalar potential, causing violation of the custodial-symmetry relations of Eq. (27) among the parameters of the most
general gauge invariant scalar potential. Having determined the custodial violating parameters we can now solve the
minimization conditions in Eqs. (29), (30), and (31) for the custodial-violating vevs ˜vφ, ˜vχ, and ˜vξ. First we solve
Eq. (29) for ˜vϕ in terms of the other vevs and plug this in to Eqs. (30) and (31), which we then solve numerically using
a two-dimensional Newton’s method. For the initial guess we take ˜vχ = ˜vξ = ˜vχ, where vχ is the custodial-symmetric
triplet vev in our original weak-scale input point.

However, this procedure suffers from a complication. The definition of the original weak scale input point in the

| Fixed Parameters | Variable Parameters | Dependent Parameters |
|------------------|---------------------|----------------------|
| GF = 1.1663787 × 10^{-5} GeV^{-2} | m5 ∈ [200, 3000] GeV | λ2 = 0.4 m5/(1000 GeV) |
| mh = 125 GeV | sH ∈ (0, 1) | M1 = √2 mH (m5^2 + v^2)/v |
| λ3 = −0.1 | | M2 = M1/6 |
| λ4 = 0.2 | | |

TABLE I: Parameter definitions for the H5plane benchmark scenario in the custodial-symmetric GM model.

VI. NUMERICAL RESULTS

A. Calculational procedure

In this paper we imagine that the custodially-symmetric GM model emerges at some scale Λ as an effective theory of
some unspecified ultraviolet (UV) completion. For example, the scalars in the GM model could be composites and the
custodial symmetry an accidental global symmetry resulting from the particle content of the UV theory. The running
of the scalar potential parameters down to the weak scale induces custodial symmetry violation. We can then use the
experimental constraint on the ρ parameter at the weak scale to set an upper bound on the scale Λ. Subject to this
constraint, we can also predict the size of other custodial symmetry violating effects such as mass splittings among
the members of the custodial fiveplet and triplet scalars, mixing between scalars in different custodial-symmetry
representations (which, for example, can induce fermionic decays of the otherwise fermiophobic H5 states), and the
value of the ratio λWZ ≡ κW/κZ of the 125 GeV Higgs boson (predicted as λWZ = 1 in custodial-symmetric theories).

For concreteness, we work within the context of the so-called H5plane benchmark, which is a two-dimensional slice
through the custodial-symmetric GM model parameter space as defined in Table I at the weak scale. This benchmark
was introduced in Ref. [26] for interpretation of LHC searches for H5 ± and H5 ± ±, and its phenomenology was studied
in some detail in Ref. [27]. The H5plane benchmark takes m5 and sH as its two free parameters: this will allow us to
plot our results as contours in the m5–sH plane. The benchmark is defined for m5 values of 200 GeV and higher.
We leave to future work a detailed study of the custodial violating effects at lower m5 values.

We perform the calculations as follows. We start by specifying an input point in the custodial-symmetric GM model
at the weak scale, using the H5plane benchmark. Because it is not possible to separate the scale of the GM model
states from the SM weak scale so long as the triplets contribute to electroweak symmetry breaking, for the purposes
of renormalization group running we will define the “weak scale” to be m5 as defined in the custodial-symmetric
low-scale input parameter set. We define the electroweak gauge couplings at the weak scale in terms of the inputs GF,
MW, and MZ, and we take αs(MZ) = 0.118 to define the strong coupling at the weak scale (we ignore the running of
the strong coupling between MZ and m5; this is a small effect because the strong coupling only enters in the running
of the top Yukawa coupling). We extract the value of the top Yukawa coupling using the relation yt = √2 mH/vϕ evaluated in terms of the custodial-symmetric input parameters at the weak scale. For simplicity, we set yt = yr = 0; their effects would be very small.

We then run the parameters of the custodial-symmetric scalar potential up to a scale Λ using the RGEs in Eqs. (72–
87) but with g1 set to zero. We also run the gauge couplings (including the actual value of g1) and the top Yukawa
coupling from m5 to Λ using Eqs. (88–91). For the running we use fourth-order Runge-Kutta with a small step size.

From the custodial-symmetric scalar potential at scale Λ, we then run back down to the scale m5 using the full
RGEs in Eqs. (72–91) with g1 ̸= 0. The nonzero hypercharge coupling induces custodial symmetry violation in the
scalar potential, causing violation of the custodial-symmetry relations of Eq. (27) among the parameters of the most
general gauge invariant scalar potential. Having determined the custodial violating parameters we can now solve the
minimization conditions in Eqs. (29), (30), and (31) for the custodial-violating vevs ˜vφ, ˜vχ, and ˜vξ. First we solve
Eq. (29) for ˜vϕ in terms of the other vevs and plug this in to Eqs. (30) and (31), which we then solve numerically using
a two-dimensional Newton’s method. For the initial guess we take ˜vχ = ˜vξ = ˜vχ, where vχ is the custodial-symmetric
triplet vev in our original weak-scale input point.

However, this procedure suffers from a complication. The definition of the original weak scale input point in the

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H5plane benchmark uses the measured $m_h$ and $G_F$ as input parameters. These are used to fix $\lambda_1$ and $\mu_3^2$ in the weak-scale custodial-symmetric theory. After running the parameters up to the scale $\Lambda$ using the custodial-symmetric RGEs (with $g'$ set to zero) and then running them back down to the weak scale with the full custodial violating RGEs, the new weak-scale calculations of $m_\tilde{G}$ and $G_F = 1/\sqrt{2(\tilde{v}_R^2 + 4\tilde{v}_L^2 + 4\tilde{v}_F^2)}$ yield numbers that do not match the original input values. To address this, we need to adjust the custodially-symmetric weak-scale input values for $\lambda_1$ and $\mu_3^2$ (while keeping all the other weak-scale inputs fixed) until we obtain the correct experimental values of $m_h$ and $G_F$ after implementing the custodial symmetry violation. We do this by defining two functions, $f_1 = m_h^{\text{calc}}(\lambda_1, \mu_3^2) - m_h^{\text{expt}}$ and $f_2 = G_F^{\text{calc}}(\lambda_1, \mu_3^2) - G_F^{\text{expt}}$, where $\lambda_1$ and $\mu_3^2$ are the inputs at the weak scale, $m_h^{\text{calc}}$ and $G_F^{\text{calc}}$ are calculated using the procedure described above, and $m_h^{\text{expt}}$ and $G_F^{\text{expt}}$ are the desired (experimental) values. The solution is the point at which $f_1 = f_2 = 0$, which we find iteratively using a two-dimensional Newton’s method. This involves running the full RGE machinery up and down multiple times and is the slowest part of our numerical work.

Having solved for the appropriate input values of $\lambda_1$ and $\mu_3^2$, we now have a self-consistent set of scalar potential input parameters at the weak scale ($\mu = m_5$), corresponding to a custodial-symmetric theory at the high scale ($\mu = \Lambda$), which we then run back down to obtain the custodial-violating theory at the weak scale (again $m_5$) with the correct predictions for $m_h$ and $G_F$. We then calculate our desired observables including the $\rho$ parameter, the mass splittings among the states of the would-be custodial multiplets, and the effects of the mixing among the would-be custodial eigenstates.

In the rest of this section we present our results as contour plots in the H5plane benchmark in the $m_5-s_H$ plane. We emphasize that $m_5$ and $s_H$ here are defined as part of the weak-scale custodial-symmetric input parameter point, and do not directly correspond to the physical masses, couplings, or vevs of the corresponding parameter point in the weak-scale custodial-violating theory.

### B. Constraints on the cutoff scale from perturbativity and the $\rho$ parameter

We begin by determining the maximum scale allowed for the custodial-symmetric ultraviolet completion by running the custodial-symmetric model up until we hit a Landau pole. This is shown in the left panel of Fig. 1 in the H5plane benchmark. The shaded region at large $s_H$ in these plots is excluded by theoretical constraints on the custodial-symmetric model. We define the Landau pole as the scale at which any of the custodial-symmetric quartic couplings $\lambda_i$ becomes larger than $10^6$; the true divergence happens extremely close to this scale. In the right panel of Fig. 1 we also show the scale at which the quartic couplings in the custodial-symmetric theory violate any of the conditions for perturbative unitarity of two-to-two scattering amplitudes given in Eq. (15). We can see that the scale at which perturbative unitarity is violated is roughly an order of magnitude below the scale of the Landau pole. Within the H5plane benchmark, if the theory is to remain perturbative the ultraviolet completion has to appear at 290 TeV or below, and the maximum scale of the Landau pole in this benchmark is around 2600 TeV. For $m_5 \gtrsim 400$ GeV, the upper bound on $s_H$ from theory constraints in the H5plane benchmark is due to the perturbative unitarity constraint; therefore along this boundary the scale of perturbative unitarity violation is essentially the same as $m_5$, and the Landau pole occurs around 10 TeV.

We also note that in the H5plane benchmark, the value of $\lambda_2$ at the weak scale grows linearly with $m_5$ (see Table I). This is responsible for the decrease in the scale of perturbative unitarity violation and the subsequent Landau pole with increasing $m_5$ at small $s_H$ values, and is a quirk of the H5plane benchmark.

In what follows, we take the scale of perturbative unitarity violation to be an upper bound on the scale of the custodial-symmetric theory, and we do not run above this scale.

The maximum allowed scale of the custodial-symmetric ultraviolet completion can also be constrained by the stringent experimental limits on the $\rho$ parameter, as defined in Eq. (22). For this calculation (and all that follow), we bring to bear the full computational machinery described in the previous section, including adjusting the input values of $\lambda_1$ and $\mu_3^2$ to obtain the correct measured values of $G_F$ and $m_h$ in the custodial-violating theory at the weak scale. We take the current value of $\rho$ from the 2016 Particle Data Group electroweak fit [4],

$$\rho = 1.00037 \pm 0.00023,$$

and require that the value of $\rho$ in the weak-scale custodial-violating theory be within $2\sigma$ of this value; i.e., between $\rho_{\text{lower}} = 0.99991$ and $\rho_{\text{upper}} = 1.00083$. Because the deviation in the $\rho$ parameter in the custodial-violating weak-scale theory grows as the scale of the custodial-symmetric ultraviolet completion increases, this constraint puts a stronger upper bound on the scale of the ultraviolet completion in part of the H5plane benchmark parameter space, as shown in the left panel of Fig. 2 where we also plot the upper bound from requiring perturbative unitarity. The $\rho$ parameter constraint is stronger than that from perturbative unitarity for moderate $s_H$ values and $m_5$ below about 850 GeV.
FIG. 1: Constraints on the custodial-symmetric cutoff scale due to perturbativity of the model in the H5-plane benchmark. Left: the scale of the Landau pole, defined as the scale at which any of the $\lambda_i$ in the custodial-symmetric theory becomes larger than $10^3$. This scale varies between 2.5 TeV and 2594.2 TeV over the benchmark considered. Right: the highest scale at which the perturbative unitarity constraints of Eq. (16) in the custodial-symmetric theory remain satisfied. This scale varies between 346.8 GeV and 291.1 TeV over the benchmark considered.

FIG. 2: Values of and constraints due to the $\rho$ parameter in the H5-plane benchmark. Left: the highest scale at which the perturbative unitarity constraints of Eq. (16) in the custodial-symmetric theory remain satisfied as in the right panel of Fig. 1 (solid lines), showing also the highest allowed custodial-symmetric scale after requiring that the $\rho$ parameter remain within $\pm 2\sigma$ of its experimental value [Eq. (103)] in the custodial-violating weak-scale theory (dashed lines). The range of scales allowed after imposing the $\rho$ parameter constraint remains the same as in Fig. 1. Right: the value of $\rho$ in the weak-scale custodial-violating theory when the custodial-symmetric scale is taken as large as possible subject to perturbative unitarity at the high scale and the experimental limits on $\rho$. The values of $\rho$ range between the $\pm 2\sigma$ limits of 0.99991 and 1.00083.

In the right panel of Fig. 2 we plot contours of $\rho$ at the weak scale in the custodial-violating theory after running down from the maximum scale allowed by the stronger of the perturbative unitarity and $\rho$ parameter constraints. $\rho > 1$ in almost all of the H5-plane benchmark, except for a tiny sliver of parameter space at low $m_5 < 250$ GeV and $s_H$ below 0.4.

C. Custodial violation in couplings

Custodial symmetry violation can modify the phenomenology of the GM model by changing the decay patterns of the physical Higgs bosons. The most experimentally-interesting manifestations of this are in the ratio of the couplings...
of the SM-like Higgs boson mass eigenstate \( \tilde{h} \) to \( W \) boson and \( Z \) boson pairs, \( \lambda_{WZ}^h \equiv \kappa_W^h/\kappa_Z^h \) [Eq. (67)], and in the couplings of the otherwise-fermiophobic mass eigenstates \( \tilde{H}_5^\pm \) and \( \tilde{H}_5^0 \) to fermion pairs induced by custodial-violating mixing among the custodial-symmetry eigenstates [Eqs. (49) and (66)]. In what follows we maximize the custodial-violating effects by taking the scale of the custodial-symmetric theory as high as possible, subject to the constraints from perturbative unitarity and the \( \rho \) parameter.

In Fig. 3 we plot the deviation of \( \lambda_{WZ}^h \) from its SM value of 1 in the H5plane benchmark. The effect is tiny, reaching at most half a percent in a small region of the H5plane benchmark with \( m_5 \lesssim 250 \text{ GeV} \) and moderate values of \( s_H \); for larger \( m_5 \), the deviation is below two per mille. This deviation is well below the sensitivity of the current experimental measurement at the LHC, \( \lambda_{WZ}^h = 0.88^{+0.10}_{-0.09} \) [2]. It is also below the expected sensitivity obtained by combining the projections for the measurement precision of the SM Higgs couplings \( \kappa_W \) and \( \kappa_Z \) at the High-Luminosity LHC (a few percent) and the proposed International Linear Collider (ILC) (roughly half a percent) as summarized in Ref. [28]. The proposed Future Circular Collider (FCC-ee) could begin to reach the required precision, with projected sensitivity for \( \kappa_W \) of 1.5 to 2 per mille [29].

In Figs. 4 and 5 we plot the custodial-violation-induced couplings and branching ratios of \( \tilde{H}_5^0 \) and \( \tilde{H}_5^\pm \) to fermions, respectively. The \( \tilde{H}_5^0 \) coupling to fermions \( \kappa_{f}^{\tilde{H}_5^0} \) reaches a magnitude of at most 0.04 in the H5plane benchmark, leading to fermion-induced (e.g., gluon fusion) production cross sections at most \( (0.04)^2 = 1.6 \times 10^{-3} \) times that of a SM Higgs of the same mass. Potentially more interesting is the effect of this coupling on the \( \tilde{H}_5^0 \) decays: as shown in the right panel of Fig. 4, the branching ratio of \( \tilde{H}_5^0 \) to fermions can reach almost half a percent in the H5plane benchmark. For \( \tilde{H}_5^0 \) masses above 350 GeV, these fermionic decays are overwhelmingly into \( t\bar{t} \) pairs.

Similarly, the \( \tilde{H}_5^\pm \) coupling to fermions \( \kappa_{f}^{\tilde{H}_5^\pm} \) reaches a magnitude of at most 0.052 in the H5plane benchmark. Again, production processes involving \( \tilde{H}_5^\pm \) coupling to fermions, such as associated production with a top quark, will have cross sections that are far too small to be interesting at the LHC. The branching ratio of \( \tilde{H}_5^\pm \to t\bar{t} \) can reach 1.2%, as shown in the right panel of Fig. 5.

The custodial-violation-induced decays of \( \tilde{H}_5^0 \) and \( \tilde{H}_5^\pm \) to fermion pairs do not dramatically alter the phenomenology within the H5plane benchmark, which is defined only for \( m_5 \geq 200 \text{ GeV} \). Potentially more interesting is the effect

\[ 4 \] Because these methods are based on measurements of Higgs production cross sections and decay branching ratios, they probe only the magnitude of \( \lambda_{WZ}^h \), not the sign; a method involving the dependence of the \( h \to 4\ell \) decay distributions on the \( hWW \) coupling at one loop provides sensitivity to the sign of \( \lambda_{WZ}^h \), but can achieve a precision only of order 20-50% at the High-Luminosity LHC [30].
FIG. 4: The coupling of $\tilde{H}_0^5$ to fermions and the resulting fermionic branching ratio in the H5plane benchmark, taking the scale of the custodial-symmetric theory to be as large as possible subject to perturbative unitarity and the $\rho$ parameter constraint. Left: contours of $\kappa_{f}^{\tilde{H}_0^5}$ [defined above Eq. (66)]. The allowed values range between $-4.0 \times 10^{-2}$ and $2.0 \times 10^{-3}$. Right: contours of the branching ratio of $\tilde{H}_0^5$ to fermions. We compute only the partial width to the heaviest kinematically accessible pair of fermions; i.e., to $tt$ for $m_{\tilde{H}_0^5} > 2m_t$ and $bb$ otherwise. The branching ratio of $\tilde{H}_0^5$ to fermions ranges from $3.5 \times 10^{-11}$ to $4.8 \times 10^{-3}$.

FIG. 5: The coupling of $\tilde{H}_+^5$ to fermions and the resulting fermionic branching ratio in the H5plane benchmark, taking the scale of the custodial-symmetric theory to be as large as possible subject to perturbative unitarity and the $\rho$ parameter constraint. Left: contours of $\kappa_{f}^{\tilde{H}_+^5}$ [defined in Eq. (48)]. The allowed values range between $1.0 \times 10^{-4}$ and $5.2 \times 10^{-2}$. Right: contours of the branching ratio of $\tilde{H}_+^5$ to fermions, including only the decay to $t\bar{b}$. This branching ratio ranges from $2.0 \times 10^{-8}$ to $1.2 \times 10^{-2}$.

of fermionic decays of these particles for masses below the WW or WZ thresholds, when the dominant diboson decays of these scalars go off shell. In the custodial-symmetric GM model, $H_0^5$ decays to $\gamma\gamma$ and $H_+^5$ decays to $W^+\gamma$ become interesting for these low masses [31, 32]; competition from custodial-violation-induced fermionic decays could dramatically change the phenomenology in this mass region. We leave a detailed study to future work.

D. Custodial-violating mass splittings

Custodial symmetry violation also induces splittings between the masses of the otherwise-degenerate custodial fiveplet and triplet states. These splittings follow a universal pattern everywhere within the H5plane benchmark; we
fig. 6: The mass splittings within the custodial triplet in the H5plane benchmark, taking the scale of the custodial-symmetric theory to be as large as possible subject to perturbative unitarity and the p parameter constraint. Left: m_{R_3^+} - m_{R_5^0}. This quantity is negative because H_{3}^+ is lighter than H_5^0. The mass splitting ranges between zero and 5.3 GeV. Right: m_{R_0^3} - m_3, where m_3 is the weak-scale custodial-symmetric input value of the custodial triplet mass. m_{R_3^0} and m_{R_5^+} are both larger than m_3 over the entire benchmark. In our numerical scan, the difference between m_{R_3^0} and m_3 ranges between 4 MeV and 9.1 GeV.

leave it to future work to determine whether this pattern holds in general scans of the entire GM model parameter space. We again maximize the custodial-violating effects in what follows by taking the scale of the custodial-symmetric theory as high as possible, subject to the constraints from perturbative unitarity and the p parameter.

Among the custodial-triplet mass eigenstates, H_3^0 is heavier than H_3^+, and both of these masses are shifted up relative to the weak-scale custodial-symmetric input value of m_3. The splittings are small, as shown in Fig. 6: the mass difference between H_3^0 and H_3^+ reaches at most 5.3 GeV (left panel of Fig. 6). The shift of the H_3^0 mass upward from the input value of m_3 is shown in the right panel of Fig. 6 and is at most 9.1 GeV. The shift of H_3^+ from the input m_3 value is smaller, reaching at most 3.9 GeV.

Among the custodial-fiveplet mass eigenstates, H_5^{++} is the heaviest, followed by H_5^0 and then H_5^- . Again the mass splittings are small, as shown in Fig. 6. The top left panel of Fig. 6 shows the mass difference between H_5^{++} and H_5^0, which is at most 7.2 GeV. The mass of H_5^- falls between these two, but closer to the lighter H_5^0 state: the mass difference between H_5^- and H_5^0 reaches at most 1.8 GeV, as shown in the top right panel of Fig. 7. The mass of H_5^0 remains within 2.3 GeV of the weak-scale custodial-symmetric input value of m_{5^0}, but can be heavier or lighter: this is plotted in the bottom left panel of Fig. 7. The mass of H_5^{++} is always larger than m_5^0, with the difference reaching a maximum of 9.0 GeV, as shown in the bottom right panel of Fig. 7. The smallness of these shifts of the physical H_5 masses relative to the weak-scale custodial-symmetric input value of m_{5^0} justifies our use of this input value on the x axis of the plots.

Finally, in Fig. 8 we plot the shift of the mass of the physical mass eigenstate H relative to the weak-scale custodial-symmetric input value of m_H. The H mass is shifted upwards over almost all of the H5plane benchmark, and the shift is by at most 5.6 GeV. We conclude that, within the H5plane benchmark and even allowing for custodial symmetry violation, the custodial-symmetric predictions for the masses of the scalars in the model are reliable to within better than 10 GeV.

Experimentally checking the mass degeneracy of the scalars within the custodial triplet and the custodial fiveplet has been proposed as a way to test the custodial symmetry in the GM model [33, 34]. At the LHC, mass reconstruction of the H3 states relies on their decays to dijets, H_3^+ \rightarrow c\bar{s}, H_5^0 \rightarrow b\bar{b} [33]. Considering that the dijet invariant mass resolution at the LHC is not sufficient to kinematically separate the hadronic decays of the W and the Z with their 11 GeV mass difference, it will not be possible to resolve a custodial-symmetry-violation-induced mass splitting between H_3^+ and H_5^0 of at most 5.3 GeV within the H5plane benchmark. Mass reconstruction of the H5 states at the LHC relies on their decays to vector boson pairs VV. Reference [33] studied the fully-leptonic final states, in which the masses of H_5^{++}, H_5^+, and H_5^0 could be determined from the endpoint of the transverse mass distribution of the VV final state. The resolution is worse than for a dijet resonance. The ATLAS experiment has performed a search for H_5^0 \rightarrow W^+Z \rightarrow jj\ell^+\ell^- [14], in which reconstruction of a mass peak for H_5^{++} becomes possible; however, the mass resolution is still limited by the dijet invariant mass resolution of the LHC, which is too poor to resolve the...
Prospects are somewhat better at the ILC, as studied in Ref. [34]. \( \tilde{H}_5^0 \) and \( \tilde{H}_5^\pm \) can be singly produced in \( e^+ e^- \) collisions via vector boson fusion, or in association with a Z or \( W^\pm \) boson, respectively. In the clean lepton collider environment, the \( H_5 \) decays to dibosons can be reconstructed using the fully hadronic final states. With the ILC target dijet energy resolution of \( \sigma_E = 0.3 \times \sqrt{E_{jj}} \) GeV [35], the dijet resolution will be \( \sigma_E \simeq 3 \) GeV for \( E_{jj} \simeq 100 \) GeV, famously allowing for \( W \) and \( Z \) bosons to be distinguished in the all-hadronic channel. Unfortunately, even this excellent mass resolution is too poor to resolve the custodial-symmetry-violation-induced mass splitting between \( \tilde{H}_5^\pm \) and \( \tilde{H}_5^0 \), which reaches at most 1.8 GeV in the H5plane benchmark. One could hope to do better by using the leptonic decays of \( H_5^\pm \rightarrow ZZ \rightarrow 4\ell \) and \( H_5^\pm \rightarrow W^\pm Z \rightarrow \ell^\pm E_T^{miss} \ell^\mp \); these suffer from smaller branching fractions, but may offer good enough mass resolution to detect the mass splitting effect of the custodial symmetry violation.

### E. Direct search constraints

The most stringent direct search constraint on the custodial-symmetric H5plane benchmark comes from a CMS search for \( H_5^{\pm \pm} \) produced in vector boson fusion and decaying to \( W^\pm \rightarrow \ell^\pm E_T^{miss} \ell^\mp \) [30]. This constraint excludes...
FIG. 8: $m_{\tilde{H}} - m_H$, where $m_H$ is the mass of the heavier custodial singlet $H$ in the weak-scale custodial-symmetric theory. We work in the H5plane benchmark and take the scale of the custodial-symmetric theory to be as large as possible subject to perturbative unitarity and the $\rho$ parameter constraint. This mass difference ranges between $-5.9 \times 10^{-2}$ GeV and 5.6 GeV.

FIG. 9: The fractional change in $\tilde{v}_\chi$ relative to the weak-scale custodial-symmetric input $v_\chi$, defined as $\frac{\tilde{v}_\chi}{v_\chi} - 1$. We work in the H5plane benchmark and take the scale of the custodial-symmetric theory to be as large as possible subject to perturbative unitarity and the $\rho$ parameter constraint. The fractional change is always negative and its absolute value reaches a maximum of 1.0%.

$s_H$ above 0.25 for $m_5 = 200$ GeV, rising to $s_H = 0.55$ at $m_5 = 800$ GeV. We can apply this straightforwardly to the model with custodial symmetry violation by noting the following. First, as shown in the bottom right panel of Fig. 7, the physical mass of $H_5^{++}$ is at most 5 GeV higher than $m_5$ in the region of interest. Second, we show in Fig. 9 the shift in $\tilde{v}_\chi$, which controls the $\tilde{H}_5^{\pm}W^\mp W^\pm$ coupling and hence the vector boson fusion production cross section, relative to the value of $v_\chi$ in the weak-scale custodial-symmetric theory. This shift is negative and amounts to less than a percent, so that the cross section is suppressed by no more than 2% due to the custodial symmetry violation. Finally, the custodial-symmetry-violation-induced mass splitting between $H_5^{++}$ and $H_5^+$ is less than 5 GeV in the region of interest, too small for the cascade decay $\tilde{H}_5^{\pm} \rightarrow W^\pm H_5^0$ to compete significantly with the dominant $\tilde{H}_5^{\pm} \rightarrow W^\pm W^\pm$ signal channel. Thus we conclude that this direct search constraint on the custodial symmetry violating parameter space studied in this paper will be almost identical to that in the custodial-symmetric H5plane benchmark.

Very recent LHC searches for $H_5^0 \rightarrow Zh$ and $H \rightarrow hh$ may further constrain the custodial-symmetric H5plane benchmark, and are worth examining more closely in future work.
VII. CONCLUSIONS

In this paper we studied the effects of custodial symmetry violation in the Georgi-Machacek model. We assumed that the exactly custodial-symmetric GM model emerges at some high scale \( \Lambda \) as an effective low energy theory of an unspecified ultraviolet completion, and then ran the model down to the weak scale, giving rise to custodial symmetry violation from hypercharge interactions at one loop. The amount and pattern of custodial symmetry violation at the weak scale is uniquely determined by the parameters of the high-scale custodial-symmetric theory and the value of the scale \( \Lambda \).

Starting from the the most general gauge invariant scalar potential, we derived the minimization conditions for the vevs and expressions for the physical scalar mass eigenstates. These allow us to calculate the custodial symmetry violating couplings of the physical \( \tilde{H}_0^5 \) and \( \tilde{H}_5^\pm \) states to fermions, as well as the parameter \( \lambda_{WZ} \equiv \kappa_{\tilde{H}_0^5} / \kappa_{\tilde{H}_0^5} \) for the 125 GeV Higgs boson. We rederived the RGEs for the parameters of the most general scalar potential including CP violation, and confirm the results of Ref. [18] in the CP-conserving limit. Our numerical implementation of these results was complicated by the need to adjust the custodial-symmetric inputs to obtain the correct values of the physical 125 GeV Higgs mass and \( G_F \) in the custodial-violating theory.

Working for concreteness in the H5plane benchmark scenario, we determined the maximum allowed scale of the custodial-symmetric theory imposing perturbative unitarity of two-to-two scalar scattering amplitudes and the experimental constraint on the \( \rho \) parameter. This allowed us to quantify the maximum possible deviation of \( \lambda_{WZ} \) from its SM value, as well as the branching ratios of the otherwise-fermiophobic \( \tilde{H}_0^5 \) and \( \tilde{H}_5^\pm \) scalars into fermions and the mass splittings within the custodial triplet and fiveplet. We found that the scale of the custodial-symmetric theory could be as high as tens to hundreds of TeV, with an upper bound of 290 TeV in the H5plane benchmark. Subject to this upper bound, we showed that \( \lambda_{WZ} \) can deviate from its SM value by at most two per mille, and that the mass splittings within the custodial triplet and the custodial fiveplet are in the range of 1–8 GeV. Both of these effects are too small to be probed at the LHC, but may be detectable at a future \( e^+ e^- \) collider. We also showed that the fermionic branching ratios of \( \tilde{H}_5^\pm \) and \( \tilde{H}_5^3 \) can reach of order 1\% in the H5plane benchmark, which is defined for \( m_5 \geq 200 \) GeV. These fermionic decays may be more interesting for \( \tilde{H}_5 \) masses below the \( WW \) and \( WZ \) thresholds, where they can compete directly with the loop-induced \( \gamma \gamma \) and \( W\gamma \) decay modes; we leave a detailed study of this low mass region to future work.

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Appendix A: Translation from the notation of Ref. [18]

The most general gauge invariant scalar potential for the custodial symmetry-violating extension of the Georgi-Machacek model was also written down in Ref. [18]. The notation of that paper can be translated to our notation as follows:
\[
\sigma_1 = -\tilde{\lambda}_3 + \tilde{\lambda}_5,
\]
\[
\sigma_2 = \tilde{\lambda}_3,
\]
\[
\sigma_3 = \tilde{\lambda}_6,
\]
\[
\sigma_4 = \tilde{\lambda}_4,
\]
\[
\lambda = \tilde{\lambda}_1,
\]
\[
\rho_1 = 2\tilde{\lambda}_2 + \tilde{\lambda}_7,
\]
\[
\rho_2 = -2\tilde{\lambda}_2,
\]
\[
\rho_3 = 2\tilde{\lambda}_8,
\]
\[
\rho_4 = \tilde{\lambda}_{10},
\]
\[
\rho_5 = \tilde{\lambda}_9,
\]
\[
\mu_1 = \frac{M_1}{\sqrt{2}},
\]
\[
\mu_2 = \frac{M_1'}{2},
\]
\[
\mu_3 = -6\sqrt{2}M_2,
\]
\[
m_\phi^2 = \tilde{\mu}_2^2,
\]
\[
m_\chi^2 = \tilde{\mu}_2^3,
\]
\[
m_\xi^2 = \frac{\tilde{\mu}_3^2}{2}.
\]

(A1)

Appendix B: Calculating the renormalization group equations

We calculate the one-loop renormalization group equations (RGEs) in this paper using the formalism of Cheng, Eichten, and Lee [23]. They considered a Lagrangian for nonabelian gauge fields \( A^a_{\mu} \), real scalar fields \( \phi_i \), and fermionic fields \( \psi_\alpha \) of the form

\[
\mathcal{L} = -\frac{1}{4} F^{a\mu\nu}_{\mu\nu} + \frac{1}{2} (D_{\mu} \phi)_i (D^\mu \phi)_i + i \bar{\psi} \gamma^\mu D_{\mu} \psi - \bar{\psi} m_0 \psi - \bar{\psi} h_i \psi \phi_i - V(\phi),
\]

(B1)

where the gauge field strength tensor and covariant derivatives are

\[
F^{a\mu\nu}_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} - g C^{abc} A^b_{\mu} A^c_{\nu},
\]

(B2)

\[
(D_{\mu} \phi)_i = \partial_{\mu} \phi_i + i g \theta^a_{ij} \phi_j A^a_{\mu},
\]

(B3)

\[
(D_{\mu} \psi)_\alpha = \partial_{\mu} \psi_\alpha + i g t^a_{\alpha\beta} \psi_\beta A^a_{\mu}.
\]

(B4)

Here \( g \) is the gauge coupling, \( \theta^a_{ij} \) and \( t^a_{\alpha\beta} \) are the generators of the gauge group acting on the scalar and fermion representations, respectively, and \( C^{abc} \) are the structure constants of the gauge group. The fermion masses \( m_0 \) and Yukawa couplings \( h_i \) are matrices in the space of fermions. The (quartic) scalar potential is given by

\[
V(\phi) = \sum_{ijkl} \frac{1}{4!} f_{ijkl} \phi_i \phi_j \phi_k \phi_l.
\]

(B5)

The quartic scalar couplings \( f_{ijkl} \) are defined to be symmetric under interchange of any pair of indices; after collecting terms in the scalar potential, they can be extracted using

\[
f_{ijkl} = 4! \times \text{coefficient of } \phi_i \phi_j \phi_k \phi_l \text{ in } V \over \text{number of permutations of } (ijkl).
\]

(B6)
The trilinear couplings and quadratic mass-squared coefficients in Eq. (25) can be integrated into this formalism by inserting one or two factors of a nondynamical scalar field $\phi_0$ that has no gauge or fermion couplings, e.g., $\mu^2 \phi_0 \phi_1 \rightarrow \mu^2 \phi_0 \phi_0 \phi_1$. The trilinear and quadratic coefficients can then be treated in the same way as the quartic coupling coefficients $f_{ijkl}$, setting one or two of $ijkl$ equal to 0.

The RGEs for the quartic scalar couplings are given by Eq. (2.8) of Ref. [23].

$$16\pi^2 \frac{df_{ijkl}}{dt} = \beta_{ijkl},$$

with $t = \log \mu$ where $\mu$ is the energy scale and

$$\beta_{ijkl} = f_{ijkl} + f_{ikmn}f_{mnjl} + f_{ikmn}f_{mnlj} + f_{ilmn}f_{mijn} - 12g^2 S_2(S)f_{ijkl} + 3g^4 A_{ijkl} + 8 \text{Tr} [h_i h_m] f_{mjkl} - 12H_{ijkl}.$$  \hspace{1cm} (B8)

Repeated indices are to be summed over. In this expression the first three terms come from one-loop diagrams with two quartic scalar vertices, the fourth term comes from diagrams in which an external leg is decorated with a gauge boson loop, the fifth term is a four-scalar coupling induced by a closed loop of gauge bosons, the sixth term comes from diagrams in which an external leg is decorated with a fermion loop, and the last term is a four-scalar coupling induced by a closed box of fermions (see Fig. 3 in Ref. [23]). The new symbols in Eq. (B8) are defined as [23]:

$$S_2(S)\delta_{ij} = [\theta^a \theta^a]_{ij},$$

$$A_{ijkl} = \{\theta^a, \theta^b\}_{ij} \{\theta^a, \theta^b\}_{kl} + \{\theta^a, \theta^b\}_{ik} \{\theta^a, \theta^b\}_{jl} + \{\theta^a, \theta^b\}_{il} \{\theta^a, \theta^b\}_{jk},$$

with repeated gauge indices summed over, and

$$H_{ijkl} = \frac{1}{3!} \text{Tr} [h_i h_j h_k h_l] + h_i h_k [h_j, h_l] + h_i h_l [h_j, h_k] .$$

The formalism in Ref. [23] assumes a single gauge group and a single representation containing all the scalars. This can be straightforwardly generalized to our theory in which the scalars transform under SU(2)$_L$ and U(1)$_Y$ as a doublet and two triplets as follows. We first write out all the scalar fields in terms of their real components, using

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$$

as a complex scalar.

The covariant derivative for the scalars can then be written as

$$(\mathcal{D}_\mu \phi)_i = \partial_\mu \phi_i + ig \theta^a_{ij} \phi_j W^a_\mu + ig \frac{Y_i}{2} \phi_1 B_\mu,$$  \hspace{1cm} (B12)

where $g$ and $g'$ are now the SU(2)$_L$ and U(1)$_Y$ gauge couplings and $\theta^a_{ij}$ and $Y_i/2$ are the SU(2)$_L$ and U(1)$_Y$ generators written as big matrices in the space of the 13 real scalars $\phi_i$ in our model (plus one nondynamical scalar field $\phi_0$).

Equation (B8) must then be modified slightly to take into account the two gauge groups:

$$\beta_{ijkl} = f_{ijkl} + f_{ikmn}f_{mnjl} + f_{ikmn}f_{mnlj} + f_{ilmn}f_{mijn} - 12g^2 S'_2(S)f_{ijkl} - 12g^2 S_2(S)f_{ijkl} + 3A_{ijkl} + 8 \text{Tr} [h_i h_m] f_{mjkl} - 12H_{ijkl} .$$

The new gauge terms are given as follows. The $S'_2(S)$ term comes from diagrams in which a U(1)$_Y$ gauge boson loop decorates one of the external scalar legs. Using Eq. (B9) with $\theta^a_{ij} = (Y_i/2)\delta_{ij}$, this term is given for each $ijkl$ by

$$-12g^2 S'_2(S) = -3g^2 \sum_{\text{legs}} \left[ \frac{Y_i}{2} \frac{Y_j}{2} \right]_{ij} = -3g^2 \left[ \frac{(Y_i/2)^2}{2} + \left( \frac{Y_j}{2} \right)^2 + \left( \frac{Y_k}{2} \right)^2 + \left( \frac{Y_l}{2} \right)^2 \right].$$

The $S_2(S)$ term comes from diagrams in which an SU(2)$_L$ gauge boson loop decorates one of the external scalar legs. It will have different values depending on the SU(2)$_L$ representation of the scalar on each leg. Using the SU(2)$_L$ generators for doublets and triplets, we obtain from Eq. (B9) for each leg

$$S_2(S)_{\text{leg}} \delta_{ij} = [\theta^a \theta^a]_{ij} = \begin{cases} 3/4 \delta_{ij} & \text{doublet} \\ 2\delta_{ij} & \text{triplet} \\ [(n^2 - 1)/4]\delta_{ij} & \text{triplet} \end{cases}$$

(B15)
Summing over the four legs then gives, for each $ijkl$,

$$-12g^2 S_2(S) = -3g^2 \left[ S_2(S)_i + S_2(S)_j + S_2(S)_k + S_2(S)_l \right] = -\frac{3}{4}g^2 \left( n_i^2 + n_j^2 + n_k^2 + n_l^2 - 4 \right), \quad (B16)$$

where $n_i = 2T_i + 1$ is the dimensionality of the $SU(2)_L$ representation of the $i$th leg.

The $3\bar{A}_{ijkl}$ term in Eq. (B13) yields terms in the RGEs of order $g^4$, $g'^4$, and $g^2g'^2$. The couplings that give rise to these terms are the quartic scalar-scalar-vector-vector vertices, which can be found by examining the anti-commutation relations among the generators of the relevant gauge groups. We derive the form of $\bar{A}_{ijkl}$ as follows. First, starting from Eq. (B10) we absorb the gauge coupling into the generators and define

$$\tilde{\theta}^1 = g t^1, \quad \tilde{\theta}^2 = g t^2, \quad \tilde{\theta}^3 = g t^3, \quad \tilde{\theta}^4 = g Y \frac{g'}{2} I_{n \times n}, \quad (B17)$$

where $t^a$ are the appropriate $SU(2)_L$ generators acting on the relevant subspaces of the scalars and $I_{n \times n}$ is the unit matrix on the subspace of scalars with a common hypercharge. Then,

$$\bar{A}_{ijkl} \equiv \{\tilde{\theta}^a, \tilde{\theta}^b\}_{ij} \{\tilde{\theta}^c, \tilde{\theta}^d\}_{kl} + \{\tilde{\theta}^a, \tilde{\theta}^b\}_{ik} \{\tilde{\theta}^c, \tilde{\theta}^d\}_{jl} + \{\tilde{\theta}^a, \tilde{\theta}^b\}_{il} \{\tilde{\theta}^c, \tilde{\theta}^d\}_{jk}, \quad (B18)$$

To actually calculate this, we write

$$\bar{A}_{ijkl} = \sum_{a,b=1}^{4} \alpha_{ij}^{ab} \alpha_{kl}^{ab} + \alpha_{ik}^{ab} \alpha_{jl}^{ab} + \alpha_{il}^{ab} \alpha_{kj}^{ab}, \quad (B19)$$

where for a real scalar multiplet the gauge-covariant terms yield,

$$\Phi^T (\tilde{\theta}^a \tilde{\theta}^b + \tilde{\theta}^b \tilde{\theta}^a) \Phi = \sum_{i,j} \phi_i \phi_j \alpha_{ij}^{ab}, \quad (B20)$$

and for a complex scalar multiplet they give,

$$2 \Phi^T (\tilde{\theta}^a \tilde{\theta}^b + \tilde{\theta}^b \tilde{\theta}^a) \Phi = \sum_{i,j} \phi_i \phi_j \alpha_{ij}^{ab}. \quad (B21)$$

Note that $\alpha_{ij}^{ab}$ is symmetric under interchange of $i$ and $j$; care must be taken with factors of two in extracting the $\alpha_{ij}^{ab}$ from terms involving two different real scalar fields.

Finally for the fermion contributions, it is most straightforward to separate the contributions into a sum of terms each involving only leptons, down-type quarks, or up-type quarks. In our model only the $SU(2)_L$ doublet couples to fermions, as in the SM, and we can write the Yukawa matrices in the fermion mass basis as

$$Y_i^u = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \quad Y_i^d = \begin{pmatrix} y_d & 0 & 0 \\ 0 & y_o & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad Y_i^e = \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad (B22)$$

for $i$ being one of the four real fields in the scalar doublet, and

$$Y_i^f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (B23)$$

for $f \in \{u, d, e\}$ and $i$ being any other scalar field.

The contribution from diagrams in which an external leg is decorated with a fermion loop is then given for each $ijkl$ by

$$8 \text{Tr} [h_i h_m] f_{mijkl} = (Y_i + Y_j + Y_k + Y_l) f_{ijkl}, \quad (B24)$$
The multiple redundant solutions for each couplings with $N$ where

$$\lambda_i$$ in Eq. (25), one can write the $f_{ijkl}$ as linear combinations of the $\lambda_i$ and solve the set of linear equations. The multiple redundant solutions for each $\lambda_i$ can be used as a check of the algebraic implementation.

\[ \Upsilon_m = \text{Tr} \left[ \sum_{f \in \{u,d,e\}} N_f^i Y_m^f Y^f_m \right], \]  

(B25)

with $N^f$ being the number of colors of fermion type $f$.

This yields the RGEs for the coefficients $f_{ijkl}$ defined in Eq. (B5). To obtain the RGEs for the individual quartic couplings $\lambda_i$ in Eq. (25), one can write the $f_{ijkl}$ as linear combinations of the $\lambda_i$ and solve the set of linear equations.

\[ -12 H_{ijkl} = -4 \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \frac{1}{\# \text{ of permutations of } (ijkl)} \text{Tr} \left[ \sum_{f \in \{u,d,e\}} \sum_{\text{permutations of } (ijkl)} N_f^i Y^f_j Y^f_k Y^f_l \right]. \]  

(B26)

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