Mesoscopic charge quantization

I.L. Aleiner\textsuperscript{1} and L.I. Glazman\textsuperscript{2}

\textsuperscript{1}NEC Research Institute, 4 Independence Way, Princeton, NJ 08540
\textsuperscript{2}Theoretical Physics Institute, University of Minnesota, Minneapolis MN 55455

We study the Coulomb blockade in a chaotic quantum dot connected to a lead by a single channel at nearly perfect transmission. We take into account quantum fluctuations of the dot charge and a finite level spacing for electron states within the dot. Mesoscopic fluctuations of thermodynamic and transport properties in the Coulomb blockade regime exist at any transmission coefficient. In contrast to the previous theories, we show that by virtue of these mesoscopic fluctuations, the Coulomb blockade is not destroyed even at perfect transmission. The oscillatory dependence of all the observable characteristics on the gate voltage is preserved, its period is still defined by the charge of a single electron. However, phases of those oscillations are random; because of the randomness, the Coulomb blockade shows up not in the averages but in the correlation functions of the fluctuating observables (e.g., capacitance or tunneling conductance). This phenomenon may be called “mesoscopic charge quantization”.

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I. INTRODUCTION

The effect of Coulomb blockade in chaotic quantum dots sets up an ideal stage for studying the interplay between the quantum chaos and interaction phenomena in a many-electron system. By tuning the connection between the leads and the quantum dot, one can study a rich variety of non-trivial effects. In the weak tunneling limit, discrete charging of the dot results in a sequence of sharp conductance peaks, which carry information about the chaotic motion of non-interacting electrons confined inside an almost closed dot. In the opposite limit of wide channels, charge quantization does not occur, and quantum chaos of free electrons in an open billiard may be studied. In a broad intermediate region, the charge quantization is gradually destroyed, and the chaotic electron motion is affected by fluctuations of charge of the cavity. The modern experimental technique allows one to continuously traverse between these regimes.

The effect of charging is conventionally described by Hamiltonian

\[ \hat{H}_C = \frac{e^2}{C} \left( \frac{\hat{Q}}{e} - N \right)^2, \quad E_C = \frac{e^2}{C}, \]

where \( C \) is the total capacitance of the dot, the dimensionless parameter \( N \) is related to the gate voltage \( V_g \) and gate capacitance \( C_g \) by \( N = V_g/eC_g \), and \( \hat{Q} \) is the dot charge. Usually, charging energy \( E_C \) is much larger than the one-electron mean level spacing of the dot, \( \Delta \). If the connection of the dot with the leads is weak and temperature \( T \) is small, \( T \ll E_C \), the charge is well quantized for almost all \( N \) except narrow vicinities of the charge degeneracy points (half-integer \( N \)). The behavior of the differential capacitance of the cavity, \( d\langle Q \rangle/dV_g \) and of the conductance through the cavity is quite different for the system tuned to the immediate vicinity of charge degeneracy points (Coulomb blockade peaks), or away from those points (Coulomb blockade valleys). The statistics of the peaks can be related to the properties of a single electron energy and wave function, so that the distribution functions for these quantities can be extracted from the well known Random Matrix theory (RMT). The transport in the valleys occurs by virtual transitions of an electron via excited states of the dot. The statistics of the conductance in this case was recently obtained in Ref. \[ \] and was confirmed experimentally in Ref. \[ \].

All the aforementioned results were obtained neglecting quantum fluctuations of the charge of the cavity. These fluctuations grow with the increase of the coupling between the dot and lead. Then, the difference between the peaks and valleys becomes less pronounced and eventually instead of the peak structure, one observes only a weak periodic modulation. Clearly, this modulation can be described neither by the properties of the single-electron wave function nor by the lowest order virtual transitions via the excited states.

The case of almost perfect transmission of a one-channel point contact connecting the quantum dot with the lead, see Fig. \[ \], was analyzed by Matveev and Flensberg in the framework of an effective one-dimensional Hamiltonian. Employing the bosonization technique, they showed that the Coulomb blockade disappears completely if the transmission coefficient of the point contact is exactly unity \( r = 0 \) and \( \Delta = 0 \). The explicit dependence of the differential capacitance of the system on \( N \), and on the small reflection coefficient \( 0 < |r|^2 \ll 1 \) was obtained by Matveev. It is important to emphasize that the Coulomb blockade in this situation is non-perturbative in charging energy effect and it can not be revealed in the standard Hartree-Fock or Random Phase approximations.

The properties of a quantum dot connected to reservoir by a channel was analyzed in a series of papers of Büttiker and collaborators. They have used...
the Random Phase approximation to calculate the frequency dependence of the linear response of the current $I$ through the channel to bias $V e^{i\omega t}$ applied to the reservoir: $I = G(\omega)V$. In this approach, the admittance $C = -\text{Im}(dG/d\omega)|_{\omega\rightarrow 0}$, coincides by construction with the thermodynamic capacitance of the non-interacting electrons. The quantum corrections and mesoscopic fluctuations of this quantity can be then analyzed by using the distribution of the Wigner delay times of the non-interacting system. This approach is perfectly valid for a large number of channels, but for a single channel, there is no any parameter justifying it. We will see below that the results obtained by well controlled procedure are significantly different, see Secs. V and VI.

For a finite reflection coefficient in the channel ($|r|^2 \neq 0$), we found a new contribution, in addition to the averaged differential capacitance calculated in Ref. [3]. This contribution is fluctuating and provides, in particular, the dependence of the differential capacitance on the magnetic field.

The paper is organized as follows. In Sec. I, we qualitatively discuss the mesoscopic fluctuations of the differential capacitance for spinless electrons. Section II is devoted to the formulation of the model and derivation of the effective action representation. We will also discuss the conditions of applicability of the model. Section III describes the bosonization procedure. Calculation of the ground state energy and differential capacitance is performed in Sec. IV. The tunneling conductance in a strongly asymmetric setup (one channel is reflectionless, and the other junction is of conductance $G_0 \ll e^2/\hbar$) is studied in Sec. V. Our findings are summarized in Conclusion.

II. QUALITATIVE DISCUSSION

Let us consider first a completely opened channel ($r = 0$). In the limit $\Delta \rightarrow 0$, the electron charge of the dot varies with the gate voltage as $\langle Q \rangle \sim N$, to assure the minimum of the electrostatic energy [4]. The interaction (2.1) depends only on the number of electrons crossing the dot-channel boundary. Therefore, the properties of the ground state can be characterized by the asymptotic behavior of the wave-functions far from the entrance to the dot. This behavior is described by the scattering phase, and at low energies can be understood from the following qualitative argument.

Entrance of an additional electron with energy $\epsilon$ (all the energies will be measured from the Fermi level) into the dot requires energy $E_C$. Therefore, the electron may spend in the dot time of the order of $\hbar/E_C$, and then the extra charge of the dot has to relax. There are two processes, that lead to the relaxation of the charge: i) elastic process where the same electron leaves the dot; and ii) inelastic process where some other electron is emitted from the dot. At low energies the probability of the inelastic process is small as $(\epsilon/E_C)^2$, by virtue of the smallness of the phase volume. (The last statement assumes the Fermi liquid behavior at low energies and, as we will see later, is valid only for spinless one-channel case.) Therefore, we may consider only elastic process. The same consideration is applicable also to an electron leaving the dot. Thus we conclude, that the low energy properties of the system can be mapped onto the dot effectively decoupled from the channel, and the phase of the scattering amplitude from the entrance of the dot is given by the Friedel sum rule

$$\delta = \pi \langle Q \rangle/e = \pi N.$$  

(2.1)

Equation (2.1) can be applied to electrons incident from inside the dot, as well as to electrons incident from

![FIG. 1. Schematic view of a quantum dot connected to a lead. Periodic orbit “A” encounters the entrance to the dot once, $n_A = 1$, and periodic orbit “B” does not encounter the entrance, $n_B = 0$.](image)
the channel. The outlined description resembles closely the Nozieres description of the unitary limit in the one-channel Kondo problem.

The outlined above qualitative picture based on the introduction of scattering phase \( \delta \) is somewhat intuitive; it will be verified by a calculation in Sec. IV. Here instead of rigorous proof, we demonstrate that this scheme reproduces the result

\[
E_g(N) \approx |r|E_C \cos 2\pi N
\]

obtained by Matveev for the ground state energy \( E_g(N) \) of spinless electrons in the limit of zero level spacing in the dot. Then, we apply the scheme to find the corrections to the ground state energy arising from a finite \( \Delta \). Those corrections will result in the mesoscopic fluctuations of the ground state energy.

We start with considering the limit \( \Delta = 0 \). First, we put also \( |r| = 0 \) and calculate the density of electrons in the channel \( \rho(x) \). Then we take into account the scattering potential \( V(x) \) that generates \( r \neq 0 \), in the first order of perturbation theory,

\[
E_g(N) = \int dx \rho(x)V(x).
\]

As we discussed, the Coulomb interaction leads to the perfect reflection of electron at low energies; wavefunctions have the form \( \psi_k(x) = \cos(k|x| - \delta) \), with phase shift \( \delta \) given by Eq. (2.1). It leads to the Friedel oscillation of the electron density \( \rho(x) = \sum_{v_F|k-k_F| \lesssim E_C} |\psi_k(x)|^2 \), where \( v_F \) and \( k_F \) are the Fermi velocity and Fermi wavevector respectively. We obtain

\[
\rho(x) = \begin{cases} 
\frac{E_C}{v_F} \cos(2k_F|x| - 2\delta), & |x| < v_F/E_C; \\
\frac{v_F}{2k_F} \sin(2k_F|x| - 2\delta)/|x|, & |x| > v_F/E_C.
\end{cases}
\]

Here we omitted the irrelevant constant part of the electron density. Substituting Eq. (2.2) into Eq. (2.3), assuming the magnitude of the potential around \( x = 0 \) smaller than \( v_F/E_C \), and using the standard expression \( |r| = |V(2k_F)|/v_F \), we obtain formula (2.2). Here \( V(k) \) is the Fourier transform of the potential \( V(x) \).

Having verified the suggested scheme for the case \( \Delta = 0 \), we proceed with evaluation of the ground state energy of a finite dot connected to a reservoir by a perfect channel. According to the above discussion, the channel is effectively decoupled from the dot due to the charging effect even though \( r = 0 \). Therefore, we have to relate the ground state energy of a closed dot to the scattering phase \( \delta \) of Eq. (2.1). For a chaotic dot, this problem is equivalent to finding a variation of the eigenenergies by introduction of impurity potential \( V(r) = (1/\pi\nu)\delta(r) \tan \delta \), where \( \nu \) is the averaged density of states per unit area. The relevant contribution to the ground state energy is given by

\[
E_g = \sum_{-E_C \leq \xi_i < 0} [\xi_i(\delta) + \mu],
\]

where \( \xi_i \) are the eigenenergies measured from the Fermi level \( \mu \). As soon as the scattering phase changes by \( \pi \), one more level enters under the Fermi level. Evolution of the energy levels with changing \( \delta \) is shown schematically in Fig. 2. The position of the level \( \xi_i(\delta) \) satisfies the gluing condition

\[
\xi_i(\delta + \pi) = \xi_{i+1}(\delta).
\]

From Eqs. (2.5) and Eq. (2.6) we see that the ground state energy depends almost periodically on \( \delta \):

\[
E_g(\delta) = E_g(\delta + \pi) + O(\Delta).
\]

As we will see below, the amplitude of the oscillation of the ground state energy with \( \delta \) is much larger than the mean level spacing and we will neglect the last term in Eq. (2.7).

In order to estimate the amplitude of the oscillations, we recall that the correlation function of the level velocities is given by

\[
\langle \partial_\delta \epsilon_i \rangle = \frac{\Delta}{\pi}, \quad \langle \partial_\delta \epsilon_i \partial_\delta \epsilon_j \rangle = \frac{\delta_{ij}}{\beta} \left( \frac{\lambda}{\pi} \right)^2,
\]

where \( \beta = 1,2 \) for the orthogonal and unitary ensembles respectively, and \( \langle \ldots \rangle \) stands for the ensemble averaging. Formula (2.8) can be easily understood from the first order perturbation theory. At \( \delta \ll 1 \), we have \( \epsilon_i(\delta) \approx \epsilon_i(0) + (\delta/\pi\nu)|\psi_i(0)|^2 \). For a chaotic system the exact wave functions \( \psi_i \) can be presented in the form

\[
|\psi_i(0)|^2 = \frac{1 + b_i}{A},
\]

where the area of the dot \( \mathcal{A} \) appears due to the normalization condition, and \( b_i \) characterizes the fluctuations of the chaotic wavefunctions. In accordance with the Porter-Thomas distribution,
The integrated density of states, \(N\), of the ground state is given by levels starting from the Gutzwiller trace formula only. It is instructive to evaluate the shift of the energy with changing the chemical potential with changing

\[\text{phase shift, } \delta\]

As we have already explained, energy (2.5) is a periodic function of \(\delta\) with period \(\pi\). On the other hand, for \(\delta \ll 1\), Eq. (2.9) is valid. Therefore the characteristic amplitude of the oscillations is of the order of \(\sqrt{E_C}\Delta\), and it is plausible to assume that the correlation function of energies at two different parameters \(N_1, N_2\) takes the form

\[\langle E_g(N_1) E_g(N_2) \rangle \approx \Delta E_C \cos 2\pi (N_1 - N_2), \quad (2.10)\]

where we use Eq. (2.1). It is important to notice that the variation of the energy of the ground state is much larger than the mean level spacing \(\Delta\). This observation enabled us not to consider in particular the variation of the chemical potential with changing \(\delta\), because it would generate a correction of the order of the level spacing only.

In order to explain the functional form of the correlation function (2.10) and make our argumentation more precise, it is instructive to evaluate the shift of the energy levels starting from the Gutzwiller trace formula\(^2\). The energy of the ground state is given by

\[E_g = -\int_{-\infty}^{0} d\epsilon N(\epsilon) K(\epsilon/E_C), \quad (2.11)\]

where \(\epsilon\) is measured from the Fermi level, and \(N(\epsilon)\) is the integrated density of states, \(N(\epsilon) = \sum_i \theta(\epsilon - \epsilon_i)\). Here \(K(x)\) is some function which decays rapidly at \(x > 1\), so only the states which can be described within the Fermi liquid theory contribute into the \(\delta\)-dependent part of the ground state energy [cf. Eq. (2.3)].

The integrated density of states can be expressed as a sum over the classical periodic orbit\(^2\).

\[N(\epsilon, \delta) = \text{Re} \sum_j R_j(\epsilon) \exp \left[\frac{i}{\hbar} S_j(\epsilon) + 2im_j (\delta_1 - \delta_2)\right]. \quad (2.12)\]

Here \(R_j\) is the weight associated with \(j\)-th orbit, \(S_j\) is the reduced action for this orbit. The last term in the exponent in Eq. (2.12) characterizes the reflection from the entrance of the cavity, and \(n_j\) is the number of such reflections for \(j\)-th orbit, see Fig. 1. We have omitted the mean value of \(N(\epsilon)\) which is independent of the phase shift (2.1).

Integrated density of states (2.12) is a strongly oscillating function of energy which vanishes upon ensemble (or energy) averaging. However, it contributes to the fluctuations of the density of states:

\[\langle N(\epsilon_1, \delta_1) N(\epsilon_2, \delta_2) \rangle = 2\text{Re} \sum_j |R_j|^2 \times \exp \left[\frac{i}{\hbar} (S_j(\epsilon_1) - S_j(\epsilon_2)) + 2im_j (\delta_1 - \delta_2)\right], \quad (2.13)\]

and we neglected the energy dependence of the pre-exponential factors \(R_j\) because it occurs on the energy scale of the order of the Fermi energy. In the double sum over the periodic orbits, arising in (2.13), one can retain only diagonal terms\(^3\) because different orbits have different actions; the non-diagonal terms oscillate strongly and vanish upon the averaging. Factor of 2 in Eq. (2.13) originates from the fact that, the electron trajectory, \(j\), and the trajectory time reversed to \(j\) have the same action.

In order to calculate correlation function of mesoscopic fluctuations of ground state energies (2.11), we use Eq. (2.13) and expand action as \(S_j(\epsilon_1) - S_j(\epsilon_2) = (\epsilon_1 - \epsilon_2) t_j\), with \(t_j\) being the period of \(j\)-th orbit. After integration over energies \(\epsilon_{1,2}\), we find

\[\langle E_g(\delta_1) E_g(\delta_2) \rangle = 2E_C^2 \sum_j |R_j|^2 \hat{K} \left(\frac{E_C t_j}{\hbar}\right) \cos [2n_j (\delta_1 - \delta_2)], \quad (2.14)\]

where \(\hat{K}(y) = \left|\int_{-\infty}^{0} dx K(x)e^{ixy}\right|^2\), is a function decaying at \(y > 1\). Coefficients \(|R_j|^2\) have a very simple physical meaning, and are related to the classical probability \(P(t)dt\) to find a periodic orbit with a period within the interval \([t; t + dt]\):

\[\sum_j |R_j|^2 \rightarrow \frac{1}{2\pi^2} \int_{0}^{\infty} \frac{dt}{t} P(t) \ldots \quad (2.15)\]
In the same fashion, we obtain from Eq. (2.14):
\[
\langle E_g(\delta_1)E_g(\delta_2) \rangle = \left( \frac{E_C^2}{\pi^2} \right) \sum_n \int_0^\infty \frac{dt}{t} \left( \frac{E_C t}{\hbar} \right) P_n(t) \cos[2n(\delta_1 - \delta_2)].
\] (2.16)

The classical probability \( P_n(t) \) differs from \( P(t) \) by satisfying an additional constraint: the corresponding periodic orbits are reflected from the dot entrance exactly \( n \) times. However, if the motion is chaotic, \( P_n(t) \) becomes universal,
\[
P_n(t) = \left( \frac{t \Delta / \hbar}{n!} \right)^n e^{-t \Delta / \hbar},
\] (2.17)
for the periods \( t \) much larger than the ergodic time \( \hbar/E_T \). Energy \( E_T \) associated with the time scale at which the classical dynamics becomes ergodic is the counterpart of the Thouless energy for the diffusive system. Typically \( E_T \approx E_C \), therefore we adopt approximation \( E_T \gg E_C \gg \Delta \). According to Eq. (2.14), the characteristic period of the semiclassical trajectory is of the order of \( \hbar/E_C \), which is much smaller than the Heisenberg time \( \hbar/\Delta \). Therefore, when calculating the oscillatory part of the correlation function of the ground state energies, it suffices to keep in Eq. (2.16) only the contribution of the trajectories reaching the entrance only once. As the result, we obtain the expression similar to our estimate (2.10).
\[
\langle E_g(N_1)E_g(N_2) \rangle = \alpha \Delta E_C \cos 2\pi(N_1 - N_2),
\] (2.18)
where we have used Eq. (2.4). The numerical coefficient \( \alpha = \frac{1}{\sqrt{2}} \int_0^\infty dx \hat{K}(x) \) depends on the particular form of function \( \hat{K} \), and cannot be found within the simple consideration.

Let us now discuss the correlation of the ground state energies as a function of magnetic field. The magnetic flux threading a periodic orbit adds a phase \( \phi = j = A_j H/\Phi_0 \) to the action in each term \( j \) in the Gutzwiller formula (2.12), where \( A_j \) is the directed area under the trajectory, \( H \) is the applied magnetic field, and \( \Phi_0 \) is the flux quantum. Correspondingly, formula (2.14) is modified to
\[
\langle E_g(\delta_1, H_1)E_g(\delta_2, H_2) \rangle = 2E_C^2 \sum_j |R_{j \bar{j}}|^2 \hat{K} \left( \frac{E_C t_{j \bar{j}}}{\hbar} \right) \times \cos \left( \frac{H_1 A_{j \bar{j}}}{\Phi_0} \right) \cos \left( \frac{H_2 A_{j \bar{j}}}{\Phi_0} \right) \cos[2n_j(\delta_1 - \delta_2)].
\] (2.19)
In analogy with Eq. (2.16), we transform Eq. (2.19) to
\[
\langle E_g(\delta_1, H_1)E_g(\delta_2, H_2) \rangle = \frac{E_C^2}{\pi^2} \sum_n \int dA \int_0^\infty \frac{dt}{t} P_n(t; A) \times \hat{K} \left( \frac{E_C t}{\hbar} \right) \cos \left( \frac{H_1 A}{\Phi_0} \right) \cos \left( \frac{H_2 A}{\Phi_0} \right) \cos[2n(\delta_1 - \delta_2)],
\] (2.20)
where \( P_n(t; A)dA \) differs from probability \( P_n(t) \) by one more constraint: the directed area swept by a trajectory lies within the interval \([A; A + dA]\).

In a chaotic system, the probability \( P_n(t; A) \) factorizes:
\[
P_n(t; A) = P_n(t; A). \] (2.21)
Here \( P_n(t) \) is defined by Eq. (2.17), and the distribution function of the areas is Gaussian:
\[
p(t; A) = \frac{1}{2(\sqrt{\pi})^2} \exp \left\{- \frac{A^2}{4(A^2(t))} \right\},
\] (2.22)
\[
\langle A^2(t) \rangle = \frac{E_T t}{\hbar} A^2.
\] (2.23)
The formula for \( \langle A^2(t) \rangle \) shows that the typical area under the electron trajectory differs from the area of the dot \( A \), and grows as \( \sqrt{t} \). This law is applicable at the time scale exceeding the ergodic time \( \hbar/E_T \), and reflects the time dependence of r.m.s. of the random winding number for the trajectory of an electron bouncing off the walls of the dot.

As we already discussed, the characteristic time an electron spends in the dot is \( \hbar/E_C \). The characteristic area accumulated during this time is \( \sqrt{E_T/E_C} \). A magnetic field produces an appreciable effect if a flux penetrating through this area is of the order of \( \Phi_0 \). Thus, the correlation magnetic field is controlled by the charging energy:
\[
H_c = \frac{\Phi_0}{A} \sqrt{\frac{E_C}{2\pi E_T}}.
\] (2.24)
Using Eqs. (2.20)-(2.23), we find the correlation function:
\[
\langle E_g(N_1, H_1)E_g(N_2, H_2) \rangle = E_C \Delta \sum_{\gamma=-+} \Lambda_E \left[ \frac{H_\gamma^2}{H^2} \right] \cos 2\pi n,
\] (2.25)
where we introduced the short hand notation \( H_\pm = H_1 \pm H_2 \) and \( n = N_1 - N_2 \). Calculation of the exact form of dimensionless function \( \Lambda_E(x) = \left\{ \int dA e^{-x \hat{K}}(y) \right\} \), and of the numerical coefficient in Eq. (2.23) requires more involved treatment which is a subject of the following sections.

Equations (2.18) and (2.25) constitute the main qualitative result of this section. We were able to demonstrate the oscillations of the ground state energy with the applied gate voltage. The phase of those oscillations is random, so that the oscillations can be revealed only in the correlation functions. Unfortunately, these simple qualitative arguments are not sufficient for finding the precise form of the correlation functions. Moreover, the assumption of the Fermi liquid behavior is valid only for the spinless electrons. It is known that the low-energy behavior of the \( s = 1/2 \) electrons is equivalent\$ to that of the two-channel Kondo problem in its strong coupling fixed point displaying a non-Fermi liquid behavior. Quantitative study of the system in this case will be presented later, see Sec. V.\]
III. THE MODEL

The main difficulty of the problem is in the non-perturbative nature of the Coulomb blockade effect. Derivation of an effective one-dimensional model is our first step in overcoming this difficulty. The interaction energy $\bar{H}$ depends only on the total number of electrons in the dot. The change of this number is associated with electrons propagating through the channel. Because the dynamics of the channel is one-dimensional, the charging effects of the system can be considered on the basis of a one-dimensional Hamiltonian $H$. However, original problem was at least two-dimensional, so backscattering of the electrons by the walls of the dot can not be accounted for by the one-dimensional Hamiltonian. Instead, of an effective Hamiltonian, we were able to find an effective action which depends only on the electron variables of the one-dimensional channel. If there were no interaction, such an approach would have no advantage; however, in the presence of interaction it becomes very powerful. The interaction will be exactly accounted for by means of bosonization, see Section IV.

Electrons are backscattered into the channel by the walls of the dot at random times, therefore the action we derive in Subsection II A has a non-local in time, random term. This term, however, can be treated perturbatively by virtue of the small parameter, $\Delta/E_c \ll 1$. With the help of the action, calculation of the correlation functions of energies and differential capacitances can be performed by the standard diagrammatic method. At energies less than $E_T$ the averages become universal. In this regime, it is also possible to formulate the model starting from the Random Matrix Hamiltonian, see Subsection II B.

The applicability of description of the interaction by Eq. (1.1), i.e., of the constant-interaction model, is discussed in Subsection II C. We will show that the corrections to this description are of the order of $1/g$, where $g \gg 1$ is the dimensionless conductance of the dot.

A. “Conventional” formulation.

We start with the Hamiltonian of the system,

$$\hat{H} = \hat{H}_F + \hat{H}_C,$$

where $\hat{H}_F$ is the Hamiltonian of non-interacting electrons,

$$\hat{H}_F = \int d\mathbf{r} \left[ \frac{1}{2m} \nabla \psi^\dagger \nabla \psi + (-\mu + U(\mathbf{r})) \psi^\dagger \psi \right].$$

The potential $U(\mathbf{r})$ describes the confinement of electrons to the dot and channel.

The interaction Hamiltonian $\hat{H}_C$ is given by Eq. (1.1), and the charge of the dot is

$$\frac{\hat{Q}}{e} = \int_{\text{dot}} d\mathbf{r} \psi^\dagger \psi,$$

where the integration is performed within the dot. Of course, the boundary separating the dot from the lead is not defined. However, this ambiguity can be absorbed into the definition of dimensionless gate voltage $\mathcal{N}$.

For the purpose of the evaluation it is more convenient, however, to change the definition of the charge. Noticing that the total number of particles in the system is an integer number which can be added to the parameter $\mathcal{N}$ without affecting any periodic in $\mathcal{N}$ observables, we write

$$\frac{\hat{Q}}{e} = -\int_{\text{channel}} d\mathbf{r} \psi^\dagger \psi = \frac{\hat{Q}}{e},$$

To calculate the ground state energy, we start with the thermodynamic potential,

$$\Omega = -\frac{1}{\beta} \ln \left( \text{Tr} e^{-\beta \hat{H}} \right),$$

where temperature $T = 1/\beta$.

We evaluate the trace in two steps, $\text{Tr} \ldots = \text{Tr}_1 \text{Tr}_2 \ldots$, where 1 and 2 indicate the fermionic operators belonging to the channel and dot respectively. Because all the interaction is attributed to the channel, see Eq. (3.4), the charging energy operator is not affected by the summation in the dot, and can be omitted in the intermediate formulas. The non-interacting Hamiltonian (3.2) can be presented as $\hat{H}_F = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$, where $\hat{H}_1$ and $\hat{H}_2$ are the noninteracting Hamiltonians of the channel and of the dot respectively, and $\hat{H}_{12}$ connects the dot with the channel. Thus, we write

$$\text{Tr}_2 e^{-\beta \hat{H}_F} = \text{Tr}_2 e^{-\beta (\hat{H}_1 + \hat{H}_2 + \hat{H}_{12})} =$$

$$e^{-\beta \hat{H}_1} \text{Tr}_2 \left[ e^{-\beta \hat{H}_2} T_r e^{-\int_0^{\beta} d\tau \hat{H}_1(\tau)} \right] =$$

$$e^{-\beta \Omega_2} e^{-\beta \hat{H}_1} T_r \left[ e^{\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 (\hat{H}_1(\tau_1) \hat{H}_1(\tau_2))} \right].$$

Here $\hat{H}_1(\tau) = e^{\tau (\hat{H}_1 + \hat{H}_2)} \hat{H}_2 e^{-\tau (\hat{H}_1 + \hat{H}_2)}$ is the interaction representation of the Hamiltonian connecting the dot and lead, $T_r$ stands for the chronological ordering, and $\Omega_2 = -T \ln \text{Tr}_2 e^{-\beta \hat{H}_2}$ is the thermodynamic potential of non-interacting electrons in the dot. Averaging $\langle \ldots \rangle_2$ over the electronic variables of the dot is defined by the relation: $\langle \ldots \rangle_2 = e^{\beta \Omega_2} \text{Tr}_2 \left( e^{-\beta \hat{H}_2} \ldots \right)$. Thermodynamic potential $\Omega_2$ does not depend on $\mathcal{N}$, and it will be omitted.

The operator Eq. (3.6) depends only on the electron variables of the channel. The evaluation of the last factor in Eq. (3.6) is performed in Appendix A. This yields

$$\frac{1}{2} \langle \hat{H}_{12}(\tau_1) \hat{H}_{12}(\tau_2) \rangle_2 = -\bar{\psi}(\tau_1;0) \bar{\psi}(\tau_2;0) \times$$

$$\frac{1}{4m^2} \int dy dy' \phi(y) \phi(y') \partial_{xx}^2 \mathcal{G} (\tau_1 - \tau_2; \mathbf{r}, \mathbf{r}'),$$
where \( \psi(\tau; x) = e^{\tau \hat{H}_1} \psi(x) e^{-\tau \hat{H}_1} \) are the one dimensional fermionic operators of the channel in the interaction representation, \( \psi(\tau) = \psi^\dagger(\tau) \), and \( \mathcal{G} \) is the exact Matsubara Green function of the closed dot subjected to the zero boundary condition. The wave function \( \phi(y) \) describes the transverse motion in the single-mode channel, and the coordinates \( x, x' \) in the derivative of the Green function \( \mathcal{G} \) are set to +0.

The detailed behavior of the Green function \( \mathcal{G} \) depends on the particular shape of the dot. It is convenient to separate the fluctuating part of the Green function, \( \mathcal{G} = \tilde{\mathcal{G}} + \mathcal{G} \), and to combine the proportional to \( \tilde{\mathcal{G}} \) sample-independent part of \( \langle \hat{H}_{12}(\tau_1) \hat{H}_{12}(\tau_2) \rangle_2 \) with the Hamiltonian \( \hat{H}_1 \) in Eq. (3.6).

We rewrite Eq. (3.7) in the form

\[
\langle \hat{H}_{12}(\tau_1) \hat{H}_{12}(\tau_2) \rangle_2 = \langle \hat{H}_{12}(\tau_1) \hat{H}_{12}(\tau_2) \rangle_2 - 2\psi(\tau_1; 0)\psi(\tau_2; 0)L(\tau_1 - \tau_2),
\]

where kernel \( L \) is given by

\[
L(\tau) = \frac{1}{4m^2} \int dy dy' \phi(y)\phi(y')\partial^2 \tilde{\mathcal{G}}(\tau; r, r').
\]

One can check by a direct calculation that

\[
e^{-\beta \hat{H}_1 T_\tau e^{-\hat{S}}} = \frac{1}{\sqrt{\text{Tr}_e^+ e^{-\beta \hat{H}_1 D}}} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle \hat{H}_{12}(\tau_1) \hat{H}_{12}(\tau_2) \rangle_2 \propto \text{Tr}_e^+ e^{-\beta \hat{H}_1 D},
\]

where \( \text{Tr}_e^+ \) stands for the trace of the one-dimensional fermionic operators on the positive half-axis, and

\[
\hat{H}_{1D} = \int_0^\infty \frac{dx}{2m} \left[ \frac{1}{2m} \nabla \psi^\dagger \nabla \psi - \mu \psi^\dagger \psi \right]
\]

is the one dimensional Hamiltonian defined on the whole real axis. The proportionality coefficient in Eq. (3.10) does not contribute to any observable quantity and we omit it. We substitute Eq. (3.8) into Eq. (5.6), use Eq. (3.10), restore the charging energy, see Eq. (3.4), and obtain

\[
\text{Tr}_e e^{-\beta \hat{H}} \propto \text{Tr} \left( e^{-\beta \hat{H}_0 T_\tau e^{-\hat{S}}} \right).
\]

One-dimensional effective Hamiltonian is given by

\[
\hat{H}_0 = \hat{H}_{1D} + E_C \left( N + \int_{-\infty}^0 : \psi^\dagger \psi : dx \right)^2,
\]

where \( : \ldots : \) stands for the normal ordering. The effective action \( \hat{S} \) in Eq. (3.13),

\[
\hat{S} = \int_0^\beta d\tau_1 d\tau_2 L(\tau_1 - \tau_2) \bar{\psi}(\tau_1; 0)\psi(\tau_2; 0),
\]

has kernel \( L \) defined by Eq. (3.9). If the electrons with spin are considered, the summation over spin indices is implied in the above formulas.

As we are interested in the dynamics of the system at energies much smaller than the Fermi energy, we can linearize the spectrum of one dimensional fermions. Writing \( \psi(x) = e^{-ik_F x}\psi_L(x) + e^{ik_F x}\psi_R(x) \), where \( \psi_L \) and \( \psi_R \) are the left- and right-moving fermions respectively, we obtain from Eqs. (3.11) and (3.13):

\[
\hat{H}_0 = i v_F \int_{-\infty}^\infty dx \left\{ \psi_L^\dagger \partial_x \psi_L - \psi_R^\dagger \partial_x \psi_R \right\} + \frac{E_C}{2} \left( \int_{-\infty}^0 dx : \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R : + N \right)^2,
\]

where \( v_F \) is the Fermi velocity in the channel. In Eq. (3.13) the fermionic fields are assumed to be smooth on the scale of the Fermi wavelength. The action has the following form in terms of the left- and right-movers:

\[
\hat{S} = \int_0^\beta d\tau_1 d\tau_2 L(\tau_1 - \tau_2) \times \left[ \bar{\psi}_L(\tau_1) + \bar{\psi}_R(\tau_1) \right] \left[ \psi_L(\tau_2) + \psi_R(\tau_2) \right].
\]

Finite reflection in the channel can be taken into account by adding to the Hamiltonian (3.13) one more term:

\[
c\hat{H}_{bs} = |r| v_F \left( \psi_L^\dagger(0) \psi_R(0) + \psi_R^\dagger(0) \psi_L(0) \right),
\]

where \(|r|^2 \ll 1\) is the reflection coefficient. The thermodynamic potential \( \Omega(N) \) can be found from

\[
\Omega(N) = -T \ln \text{Tr} \left( e^{-\beta (\hat{H}_0 + \hat{H}_{bs})} T_\tau e^{-\hat{S}} \right).
\]

The differential capacitance \( C_{\text{diff}}(N) \) of the system is then given by

\[
C_{\text{diff}}(N) = C \left( 1 - \frac{1}{E_C} \frac{\partial^2 \Omega}{\partial N^2} \right).
\]

Equations (3.13) and (3.17) were first suggested in Refs. [14][13].

So far we succeeded in reducing the original problem to the effective one-dimensional problem, where all the features of the chaotic motion of electrons in the dot are incorporated into the non-local in time action. The action fluctuates strongly from sample to sample, and we should study the statistics of these fluctuations.

\[1. \text{Statistics of } L(\tau). \]

It is convenient to use the Lehmann representation for the function \( L(\tau) \) of Eq. (3.3):

\[
L(\tau) = \int_{-\infty}^\infty \frac{dt}{2\pi} \left[ L^R(t) - L^A(t) \right] \frac{\pi T}{\sinh(\pi T(t + \imath \tau))},
\]

(3.20)
where the retarded and advanced kernels $L^{R,A}$ are given by Eq. (3.1), with $\tilde{G}$ replaced by the exact advanced and retarded Green functions $\tilde{G}^{R,A}$ respectively. It is well known that the averaged products of the type $\tilde{G}^{R} G^{R}$ and $\tilde{G}^{A} G^{A}$ vanish, and the products of the retarded and advanced Green functions can be expressed in terms of the classical propagators – diffusons $\mathcal{P}^D$ and Cooperons $\mathcal{P}^C$, 

\[
\begin{align*}
\tilde{G}_{H}^R(t_1; r_1^1, r_2^1) \tilde{G}_{H}^A(t_2; r_2^2, r_1^2) &= 2\pi \nu \delta(t_1 + t_2) \times \tag{3.21a} 
F(r_1^1, r_2^1) F(r_2^2, r_1^2) \mathcal{P}^D(t_1; R_1, R_2), \\
\tilde{G}_{H}^R(t_1; r_1^1, r_2^1) \tilde{G}_{H}^A(t_2; r_2^2, r_1^2) &= 2\pi \nu \delta(t_1 + t_2) \times \tag{3.21b} 
F(r_1^2, r_1^1) F(r_2^1, r_2^2) \mathcal{P}^C(t_1; R_1, R_2),
\end{align*}
\]

\[
\mathcal{F}(r_1, r_2) = \text{Im} \frac{G^A}{\pi} = J_0(k_F|r_1 - r_2|) - J_0\left(k_F(|r_1 - \hat{R} r_2|)\right),
\]

where $J_0(x)$ is the zeroth Bessel function, $r_1^\pm = R_i \pm r_i/2$, and $k_F$ is the Fermi wavevector. Point $\hat{R} r$ is the coordinate of the image charge created by the charge in the point $r$, so that the propagators (3.21) satisfy proper zero boundary conditions. Green functions here are taken at different values of magnetic fields $H_1, H_2$. In Eqs. (3.21), $\nu = m/2\pi$ is the density of states per unit area. Equations (3.21) are valid if the arguments of the Green functions are close to each other pairwise: $|r_1, r_2|$ must be much smaller than the elastic mean free path for a diffusive dot, and much smaller than the dot size for a ballistic dot. Boundary is assumed to be smooth on the scale of the Fermi wavelength.

Retarded classical propagators in the diffusive dot satisfy diffusion-like equation:

\[
\begin{align*}
\left\{ \frac{\partial}{\partial t} - D \left[ \nabla + \frac{ie}{c} \left( \mathbf{A}_1 - \mathbf{A}_2 \right) \right] \right\} \left\{ \mathcal{P}^D \right\} &= \delta(R_1 - R_2), \\
\left\{ \frac{\partial}{\partial t} - D \left[ \nabla + \frac{ie}{c} \left( \mathbf{A}_1 + \mathbf{A}_2 \right) \right] \right\} \left\{ \mathcal{P}^C \right\} &= \delta(R_1 - R_2),
\end{align*}
\]

(3.22)

where $D$ is the diffusion coefficient, and the vector potentials $\mathbf{A}_{1,2}$ are defined so that $\nabla \times \mathbf{A}_{1,2} = H_{1,2}$. For a ballistic dot, Eqs. (3.22) should be substituted by the corresponding Liouville (or, to be more precise, Perron-Frobenius) equation, and Eq. (3.21) should be somewhat changed. However, in the universal limit considered in this paper, there is no difference between the ballistic and diffusive dots.

The universal limit corresponds to a large time scale at which the semiclassical electron orbit covers all the available phase space. At such a time scale, the classical probabilities no longer depend on the coordinate and acquire the form:

\[
\mathcal{P}^D,C = \frac{1}{\mathcal{A}} \theta(t) e^{-t/\tau^D,C},
\]

(3.23)

where $\theta(x)$ is the step function, $\mathcal{A}$ is the area of the dot, and the decay times associated with the magnetic field are given by

\[
\frac{1}{\tau^D,C} = E_T \left( \frac{\Phi_1 + \Phi_2}{\Phi_0} \right)^2.
\]

Here $\Phi_0 = e/c$ is the flux quantum, $\Phi_{1,2} = A H_{1,2}$ are the fluxes through the dot corresponding to the fields $H_1$ and $H_2$, and the Thouless energy $E_T$ is of the order of $h v_F/\sqrt{\mathcal{A}}$ for a ballistic dot.

The correlation functions of the retarded and advanced parts of the kernel $L$ can be expressed, with the help of Eqs. (3.3) and (3.21), in terms of the diffuson and Cooperon. The kernel $L$, see Eq. (3.9), depends on the Green function at coinciding spatial arguments. Therefore, both pairings leading to the diffuson and Cooperon upon averaging, should be taken into account. In the universal regime, see Eq. (3.23), integrals in the transverse direction in Eq. (3.9) can be calculated using the normalization condition $\int \text{d}y \phi^2(y) = 1$. As the result, we find

\[
\langle L_{H_1}^R(t_1) L_{H_2}^A(t_2) \rangle = \frac{\nu^2 A^2}{2\pi} \delta(t_1 + t_2) \theta(t_1) \times \tag{3.25}
\left\{ e^{-t_1/\tau^D} + e^{-t_2/\tau^D} \right\},
\]

where $\Delta = 1/(\nu \mathcal{A}$) is the mean level spacing of the dot. Averages of the type $T R R$ vanish. Because we are interested in times smaller than the Heisenberg time $\hbar/\Delta$, the higher moments can be decoupled by using the Wick theorem and the pair correlation functions are defined by Eq. (3.25).

**B. Random matrix formulation.**

We start from dividing the entire system into two parts, the leads and the dot. In general, the Hamiltonian $\hat{H}$ of the system can be represented as

\[
\hat{H} = \hat{H}_L + \hat{H}_D + \hat{H}_{LD}.
\]

(3.26)

The Hamiltonian of the leads is of the form

\[
\hat{H}_L = v_F \sum_{k,j} k \psi_{k,j}^\dagger \psi_{k,j},
\]

(3.27)

where we linearized the electron spectrum in the leads, and measured all the energies from the Fermi level. Index $k$ is the longitudinal momentum in a mode propagating along the channel connected to the dot, and index $j = 1, \ldots, N$ labels these modes (summation over spin indices is implied whenever it is necessary). For the sake of simplicity, we assume the same Fermi velocity in all the modes, the general case can be reduced to Eq. (3.27) by the corresponding rescaling. Hamiltonian $\hat{H}_D = \hat{H}_n + \hat{H}_C$ of the dot consists of the non-interacting part.
\[ \hat{H}_n = \sum_{\alpha,\beta} \mathcal{H}_{\alpha,\beta} \psi^\dagger_{\alpha} \psi_{\beta}, \]  
(3.28)

and the interaction term \( \hat{H}_C \), which is described by the Hamiltonian (1.1) with the charge \( \hat{Q} \) given by

\[ \frac{\hat{Q}}{e} = \sum_{\alpha} \psi^\dagger_{\alpha} \psi_{\alpha}. \]  
(3.29)

(In this subsection, we reserve Greek and Latin letters for labeling the fermionic states in the dot and in the leads respectively.) For definiteness, we restrict the discussion to the case of orthogonal ensemble, generalization to other cases is straightforward. Elements \( \mathcal{H}_{\alpha,\beta} \) in Eq. (3.28) form a real random Hermitian matrix \( \mathcal{H} \) of size \( M \times M \), \((M \rightarrow \infty)\), belonging to the Gaussian ensemble

\[
P(\mathcal{H}) \propto \exp \left( -\frac{\pi^2}{4\Delta^2 M} \text{Tr} \mathcal{H}^2 \right),
\]

where \( \Delta \) is the mean level spacing near the center of the band. Finally, Hamiltonian \( \hat{H}_{LD} \) in Eq. (3.26) describes the coupling of the dot to the leads, and has the form

\[ \hat{H}_{LD} = \sum_{k,j,\alpha} \left( W_{\alpha,j} \psi^\dagger_{\alpha} \psi_{k,j} + \text{h.c.} \right), \]  
(3.31)

where the coupling constants \( W_{\alpha,j} \) is a real \( M \times N \) matrix \( W \).

Let us list the needed averaged quantities corresponding to the Gaussian ensemble Eq. (3.30). The averaged density of states is given by the semicircle law

\[ \rho(\epsilon) = \frac{\text{Tr}\delta(\epsilon - \mathcal{H})}{\Delta} = \text{Re} \frac{1}{\Delta} \sqrt{1 - \left( \frac{\pi\epsilon}{\Delta M} \right)^2}. \]  
(3.32)

We will need only properties of the system at energies \( \epsilon \) much smaller than the width of the band \( \Delta N/\pi \), and we will neglect the energy dependence of the averaged density of states. Average of the Green functions \( \bar{G}^{\mathcal{H}A}(\epsilon) = (\epsilon - \mathcal{H} \pm i0)^{-1} \) have the form

\[ \bar{G}^{\mathcal{H}A}_{\alpha,\beta} = \mp i\pi \delta_{\alpha,\beta} \frac{1}{M\Delta}. \]  
(3.33)

Random matrix counterparts of the diffuson and Cooperon propagators (3.21) can be written as

\[ \bar{G}^{\mathcal{R}A}_{\gamma,\delta}(\epsilon + \omega) \bar{G}^A_{\gamma,\delta}(\epsilon) = \frac{2\pi}{M\Delta} \left( \mathcal{P}^D(\omega) \delta_{\alpha,\delta} \delta_{\beta,\gamma} + \mathcal{P}^C(\omega) \delta_{\alpha,\gamma} \delta_{\beta,\delta} \right), \]  
(3.34)

where \( \bar{G} = \mathcal{G} - \bar{G} \). Averages of the type \( \bar{G}^{\mathcal{R}A} \bar{G}^{\mathcal{R}A} \) and \( \bar{G}^A \bar{G}^A \) vanish. Formulas (3.32) - (3.34) have the accuracy \( \sim 1/M \) and they neglect the oscillatory \( \omega \)-dependences on the scale of the order of \( \Delta \). This accuracy, however, is sufficient for us because, as we already discussed, the relevant results are contributed by the energy strip of the width \( E_C \gg \Delta \).

Our purpose now is to derive the effective action theory similar to Eqs. (3.15) - (3.18) starting from the random matrix model. Before doing so, let us review some useful properties of the system (3.26) in the absence of the interaction, \( E_C = 0 \). In this case, electron transmission at energy \( \epsilon \) is completely characterized by \( N \times N \) scattering matrix \( S(\epsilon) \):

\[ S(\epsilon) = 1 - 2\pi i\nu W^\dagger \left[ \epsilon - \mathcal{H} + i\pi\nu WW^\dagger \right]^{-1} W, \]  
(3.35)

where \( \nu = 1/(2\pi v_F) \) is the one-dimensional density of states in the leads. Coupling matrix \( W \) can be represented in the form

\[ W = \sqrt{\frac{\Delta M}{\pi^2\nu}} U O \tilde{W}, \]  
(3.36)

where \( U \) is an orthogonal \( M \times M \) matrix, \( \tilde{W} \) is a real \( N \times N \) matrix and \( O \) is \( M \times N \) matrix, \( O_{\alpha j} = \delta_{\alpha j}, \ 1 \leq \alpha \leq N \). Because the distribution function (3.26) is invariant under rotations \( \mathcal{H} \rightarrow U \mathcal{H} U^\dagger \), matrix \( U \) in Eq. (3.36) can be omitted. Substituting Eq. (3.36) into Eq. (3.35) and performing ensemble averaging with the help of Eq. (3.33), we obtain for the average scattering:

\[ \bar{S} = 1 - \tilde{W}^\dagger \tilde{W} \frac{1}{1 + \tilde{W}^\dagger \tilde{W}}. \]  
(3.37)

At \( \tilde{W}^\dagger \tilde{W} = 1 \) the average scattering matrix vanishes, see Eq. (3.37). It indicates that matrix \( S \) belongs to the circular ensemble (corresponding to the regime of “ideal contacts”). Deviation of \( \tilde{W}^\dagger \tilde{W} \) from unit matrix can be attributed to the scattering on the contacts between the leads and the dot. This scattering is described by unitary symmetric \( 2N \times 2N \) matrix

\[ S_\epsilon = \begin{pmatrix} r & t^\dagger \\ t & r^\dagger \end{pmatrix}, \]  
(3.38)

with \( N \times N \) matrices \( r, \ r', \ t \) being defined by

\[ r = 1 - \tilde{W}^\dagger \tilde{W} \frac{1}{1 + \tilde{W}^\dagger \tilde{W}}, \quad r' = -\tilde{W} \frac{1 - \tilde{W}^\dagger \tilde{W}}{1 + \tilde{W}^\dagger \tilde{W}}, \]  
\[ t = \tilde{W} \frac{2}{1 + \tilde{W}^\dagger \tilde{W}}. \]  
(3.39)

The explicit relations between the coupling matrices \( W \) and the scattering matrix in the contacts (3.38) were first obtained by Brouwer [23].

Now we turn to the derivation of the effective action. For technical reasons, [see discussion above Eq. (3.4)], we replace the charge operator (3.29) with

\[ \frac{\hat{Q}}{e} = -\sum_{k,j} \psi^\dagger_{k,j} \psi_{k,j}. \]  
(3.40)
After this replacement, the Hamiltonian of the system becomes quadratic in fermionic operators of the dot, so that this part of the system can be integrated out:

\[ \text{Tr}_D e^{-\beta \hat{H}} = \text{Tr}_D e^{-\beta (\hat{H}_L + \hat{H}_C + \hat{H}_0 + \hat{H}_{LD})} = \] (3.41)

\[ e^{-\beta (\hat{H}_L + \hat{H}_C)} e^{-\beta \Omega_{T}} e^{\frac{1}{\beta} \int_0^\beta d\tau_1 d\tau_2 (\hat{H}_{LD}(\tau_1) \hat{H}_{LD}(\tau_2))}. \]

Here \( \hat{H}_{LD}(\tau) \) is the interaction representation of the coupling operator \( \hat{H}_{LD} \) and averaging over Hamiltonian of the dot is defined as \( \langle \ldots \rangle_D = e^{\beta \Omega_{T}} \text{Tr}_D \left( e^{-\beta \hat{H}_n} \ldots \right) \).

Thermodynamic potential \( \Omega_T = T \ln \text{Tr}_D e^{-\beta \hat{H}_n} \) of the closed dot is independent of the gate voltage \( N \) and it will be omitted.

The average in the last factor in Eq. (3.41) is calculated with the help of Eq. (3.31) and of the definition of the Matsubara Green function for the closed dot:

\[ G_{\alpha \beta} = -\langle T \psi_\alpha (\tau) \bar{\psi}_\beta (0) \rangle_D = \sum_{\omega_n} e^{i \omega_n \tau} \left[ \frac{1}{[\omega_n - \xi_{\alpha \beta}]} \right], \] (3.42)

where \( \psi_\alpha (\tau) = e^{\hat{H}_n \tau} \psi_\alpha e^{-\hat{H}_n \tau} \), \( \bar{\psi}(\tau) = \psi^\dagger (\tau) \), and \( \omega_n = \pi T (2n + 1) \) is the fermionic Matsubara frequency. The result is

\[ \langle \hat{H}_{LD}(\tau_1) \hat{H}_{LD}(\tau_2) \rangle_D = \] (3.43)

\[ - \sum_{k_1,k_2,j_1,j_2} \langle \hat{b}_{k_1,j_1}(\tau_1) \rangle \langle \hat{W}^\dagger \hat{G}(\tau_1 - \tau_2) \hat{W} \rangle_{j_1,j_2} \hat{b}_{k_2,j_2}(\tau_2). \]

We separate the averaged part of the Green function \( \bar{G} = \bar{\mathcal{G}} + \bar{\mathcal{G}} \), use Eqs. (3.33) and (3.34), and obtain the Fourier transform of the kernel in Eq. (3.44):

\[ W \bar{G}(i \omega_n) \bar{W}^\dagger = -i \frac{\text{sgn} \omega_n}{\pi \nu} \bar{W}^\dagger \bar{W} + \frac{\Delta M}{\pi \nu} \bar{W}^\dagger \bar{G}(i \omega_n) \bar{W}. \] (3.44)

The first term in the RHS of Eq. (3.44) does not contain any information about the dot, and its frequency dependence is the same as of the Green function of free chiral fermions. It is, therefore, possible (and very convenient) to transform this part of the action to the Hamiltonian form by introducing fictitious fermionic fields \( b_{k,j} \), \( j = 1,...N \). Then, Eq. (3.44) acquires the form

\[ \text{Tr}_D e^{-\beta \hat{H}} \propto \text{Tr}_b \left( e^{-\beta \hat{H}_{eff}} T_r e^{-\beta \hat{S}} \right); \] (3.45)

the omitted \( N \)-independent proportionality coefficient is irrelevant. The effective Hamiltonian \( \hat{H}_{eff} \) is given by

\[ \hat{H}_{eff} = v_F \sum_{j,k} k \left[ \hat{b}_{k,j}^\dagger \hat{\psi}_j \psi_{k,j} + \hat{b}_{k,j}^\dagger \hat{b}_{k,j} \right] + \] \[ \frac{1}{\pi \nu} \sum_{k_1,k_2,j_1,j_2} \left[ \hat{b}_{k_1,j_1}^\dagger w_{j_1,j_2} \hat{b}_{k_2,j_2} \hat{\psi}_{k_2,j_2} \right] \left[ \hat{b}_{k_2,j_2}^\dagger \hat{b}_{k_1,j_1} \right] \] \[ E_C \left( \sum_{j,k} \hat{\psi}_{j,k}^\dagger \hat{\psi}_{j,k} + N \right)^2, \] (3.46)

where \( w_{j_1,j_2} \) are the elements of the Hermitian matrix \( w \) defined as

\[ w = \left( \hat{W}^\dagger \hat{W} \right)^{1/2}. \] (3.47)

Action \( S \) has the form

\[ S = 4 \sum_{k_1,k_2,j_1,j_2} \int_0^\beta d\tau_1 d\tau_2 \bar{\psi}_{k_1,j_1}(\tau_1) \hat{L}_{j_1,j_2}(\tau_1 - \tau_2) \bar{\psi}_{k_2,j_2}(\tau_2), \] (3.48)

where the kernel \( L \) is a \( N \times N \) matrix given by

\[ L(\tau) = \frac{\Delta M}{4 \pi^2 \nu} \bar{W}^\dagger \bar{G}(\tau) \bar{W}. \] (3.49)

Equation (3.45) can be easily checked by tracing out fermions \( b \) and using the relation

\[ \sum_{k_1,k_2} \langle T \bar{b}_{k_1,j_1}(\tau) \bar{b}_{k_2,j_2}(0) \rangle = i \pi \nu \delta_{j_1,j_2} \sum_{\omega_n} e^{i \omega_n \tau} \text{sgn} \omega_n. \]

Hamiltonian \( \hat{H}_{eff} \) can be rewritten in a more familiar form. Introducing the Fourier transform of the fermionic fields \( \psi_j(x) = \sum_k e^{-ikx} \psi_{j,k} \) and \( b_j(x) = \sum_k e^{-ikx} b_{j,k} \), we obtain from Eq. (3.46)

\[ \hat{H}_{eff} = iv_F \sum_{j} \int_{-\infty}^{\infty} dx \left( \psi_j^\dagger \partial_x \psi_j + b_j^\dagger \partial_x b_j \right) + \] \[ \frac{1}{\pi \nu} \sum_{j_1,j_2} \left[ \hat{b}_{j_1,0}^\dagger (0) \psi_{j_1,j_2}(0) + h.c. \right] \] \[ E_C \left( \sum_j \int_{-\infty}^{\infty} dx \psi_j^\dagger \psi_j + N \right)^2. \] (3.50)

We are interested in the case of almost open ideal contacts, \( || w - 1 || \ll 1 \). In this case, it is natural to change the variables and reveal the small parameter of the perturbation theory. Introducing left- and right-moving fermions,

\[ \psi_j(x) = \psi_{L,j}(x) \theta(-x) + \psi_{R,j}(-x) \theta(x), \] \[ \hat{b}_j(x) = i \left[ \psi_{R,j}(-x) \theta(-x) - \psi_{L,j}(x) \theta(x) \right] \]

(the ambiguity of this definition at the origin should be resolved as \( \psi(0) = [\psi(+0) + \psi(-0)]/2 \)), we obtain from Eq. (3.50).
\[ \hat{H}_{int} = \hat{H}_n + \hat{H}_{bs}; \]  
\[ \hat{H}_0 = i v_F \sum _j \int _{-\infty }^{\infty } dx \left( \psi \frac{1}{\partial x} \psi_R - \psi_R \frac{1}{\partial x} \psi_R \right) + \]  
\[ E_C \left( \sum _j \int _{-\infty }^{\infty } dx : \psi \frac{1}{\partial x} \psi_L + \psi_R \frac{1}{\partial x} \psi_R : + \beta \right)^2; \]  
\[ \hat{H}_{bs} = -i \sum _{ij} r_{ij} \psi \frac{1}{\partial x} \psi_L + h.c.; \]  
\[ \hat{\mathcal{S}} = \sum _{ij} \int _{ij} d\tau _1 d\tau _2 |L_{ij}| (\tau _1 - \tau _2) \times \]  
\[ \left[ \tilde{\psi}_L (\tau _1; 0) + \tilde{\psi}_R (\tau _1; 0) \right] \left[ \psi_L (\tau _2; 0) + \psi_R (\tau _2; 0) \right]. \]

Here we neglected the terms related to the discontinuities of the fermionic field at the origin which induces higher order terms in \( r \), and approximated the reflection matrix \( r \approx 1 - \omega \), as follows from Eqs. (3.43) and (3.33) for \( \| r \| \ll 1 \). Within the same approximation we can put \( W = 1 \) in Eq. (3.49). Formulas (3.51) are analogous to Eqs. (3.13) - (3.18) derived in a previous subsection.

Finally, we have to study the statistics of the kernel (3.49) which can be expressed in terms of its advanced and retarded counterparts by Lehmann formula (3.20). Performing time Fourier transform of Eq. (3.34), and using \( \nu = 1/(2\pi v_F) \), we obtain

\[ \langle L_{ij}^A (t_1) L_{rs}^A (t_2) \rangle = \frac{v_F^2}{2\pi} \Delta \delta (t_1 + t_2) \theta (t_1) \times \]  
\[ \left\{ \delta _{is} \delta _{jr} + \delta _{it} \delta _{jr} \right\}. \]

For the one-mode lead this result agrees with the zero magnetic field version of Eq. (3.22).

The statistics of the kernel at different magnetic fields can be obtained by adding purely imaginary Hermitian matrix to the original matrix \( \mathcal{H} \) in Eq. (3.29). This would lead to the result analogous to the exponential decay in Eq. (3.23). We will not describe details of such calculation here, and refer the reader to the extensive literature on parametric correlations [2, 49].

### C. Applicability of the model.

So far, we were using a very simple model of the interaction (1.1), which ascribes all the interaction effects to the variation of the number of particles in the dot. However, a natural question arises: what is the accuracy of this approximation? One may even think that the effects considered in this paper are completely washed out by remaining interaction terms which we neglected.

The purpose of this subsection is to show that the simple model (1.1) of interactions in the dot leaves out only small, \( \propto 1/g \), effects, where \( g = E_T / \Delta \) is the dimensionless conductance of the cavity (\( E_T \) is the Thouless energy). For a diffusive dot in the metallic regime, and for a ballistic non-integrable dot the conductance is large, \( g \gg 1 \). Mesoscopic charge quantization is adequately described by the model of interaction (1.1), as long as the \( g \gg 1 \), and the number of modes propagating to the lead is much smaller than \( g \). In other words, ergodic time of the dot \( h / E_T \) should be much larger than the escape time of electron from the dot.

The electrons in the dot are described by the Hamiltonian \( \hat{H} = \hat{H}_n + \hat{H}_{int} \), where the non-interacting part of the Hamiltonian, \( \hat{H}_n \), is given by Eq. (3.23). The validity of the random matrix theory for \( \hat{H}_n \), for the energy scale smaller than Thouless energy was proven in Ref. [2] for chaotic systems and in Ref. [30] for diffusive systems.

The general form of the interaction Hamiltonian is

\[ \hat{H}_{int} = \frac{1}{2} \sum _{\alpha} \phi _{\alpha}^2 \psi _{\alpha \sigma}^2 \psi _{\alpha \sigma}^2 \psi _{\beta \sigma} \psi _{\beta \sigma} \psi _{\beta \sigma} \psi _{\beta \sigma} \psi _{\beta \sigma}. \]

In this subsection, we will write explicitly the spin indices \( \sigma \) for the fermionic operators. The interaction Hamiltonian (1.1) corresponds to the approximation of the matrix \( V \) by

\[ V \approx E_C \delta _{\alpha \beta} \delta _{\alpha \gamma}. \]

Our goal now is to show that all the other matrix elements as well as the mesoscopic fluctuations of matrix elements (3.52) are small. Some of these calculations already appeared in the literature [2, 49], however, we will present brief derivation to make this paper self-contained.

The easiest way to study the statistics of the one-electron wave-function \( \phi _{\alpha} (r) \) is to relate them to the Green function and then use Eq. (3.21). By definition of the retarded and advanced Green functions we have

\[ G^A _{\alpha} (r_1, r_2) - G^R _{\alpha} (r_1, r_2) = 2\pi i \sum _{\alpha} \phi _{\alpha} (r_1) \phi _{\alpha} (r_2) \delta (\epsilon - \epsilon _{\alpha}), \]

where we assumed no magnetic field for simplicity. At given energy \( \epsilon \) only one function contributes into the sum in Eq. (3.55), so that the statistics of the Green functions is related to that of the wave-functions. Furthermore, it is known that there is no correlation between level statistics and wave function in the lowest order in \( 1/g \), see e.g. Ref. [30] so we can neglect the level correlations and average \( \delta \)-function in Eq. (3.55) independently. As the result, we can estimate

\[ \phi _{\alpha} (r_1) \phi _{\alpha} (r_2) \approx \frac{\Delta}{2\pi i} \left[ G^A _{\alpha} (r_1, r_2) - G^R _{\alpha} (r_1, r_2) \right]. \]

Now we can use Eq. (3.55) and (3.21) to study the average of different momenta of the matrix elements

\[ V_{\alpha \beta \gamma \delta} = \int d\mathbf{r}_1 d\mathbf{r}_2 V (r_1 - r_2) \phi _{\alpha} (r_1) \phi _{\beta} (r_1) \phi _{\gamma} (r_2) \phi _{\delta} (r_2). \]

Averaging this matrix element itself with the help of Eq. (3.56), we obtain
\[ V_{αβγδ} = V_{αβγδ}^{(0)} + V_{αβγδ}^{(1/2)} . \]  

(3.57)

First term in Eq. (3.57) comes from the product of the averaged Green functions

\[ \text{Im} \langle \mathcal{G}^A(r_1, r_2) \rangle = \pi ν \langle e^{i k(r_1 - r_2)} \rangle_{FS}, \]

and it is given by

\[ V_{αβγδ}^{(0)} = EC δ_{αδ} δ_{βγ} + F Δ (δ_{αγ} δ_{βδ} + δ_{αδ} δ_{βγ}), \]

(3.58)

\[ E_C = \frac{1}{A} \int dr_1 dr_2 V(r_1 - r_2), \]

\[ F = ν \langle \hat{V}(k) \rangle_{FS}. \]

Here \( A \) is the area of the dot, \( \langle \ldots \rangle_{FS} \) stands for the averaging over directions of the wave vector on the Fermi surface, \( ν \) is the averaged density of states per unit area and per one spin in the dot, and \( \hat{V}(k) \) is the Fourier transform of the two-particle interaction \( V(r) \).

Charging energy \( E_C \) in Eq. (3.58) is related to the zero mode of the interaction potential. This mode can not redistribute the electrons within the dot and that is why it is not screened (the redistribution of the electrons between the dot and the leads is taken into account by the model). As the result, \( E_C \) is much larger than the mean level spacing. On the other hand, coefficient \( F \) includes only non-zero modes which are perfectly screened, \( V(k) = V_0(k)/(1 + 2ν V_0(k)) \), where \( V_0(k) \) is the bare potential. (The use of the static screening here is possible because the screening is established during the characteristic time of the plasma propagation through the dot, which is much smaller than \( \hbar / E_T \).) Therefore, we estimate \( F \leq 1/2 \), so that the last two terms introduce a correction only of the order of level spacing, and may be neglected. (We will not consider here the case of the attractive interaction when the third term in Eq. (3.58) renormalizes to infinity due to the interaction in the Cooper channel.)

The second term in Eq. (3.57) originates from the product of the retarded and advanced Green functions, see Eqs. (3.21). In the absence of magnetic field, diffuson and Cooperon propagators coincide, and their spectral expansion for a diffusive system is

\[ \mathcal{P}_ω (r_1, r_2) = \frac{1}{(−iω + 0)A} + \sum_{γ_μ ≠ 0} \frac{f_μ(r_1) f_μ(r_2)}{-iω + γ_μ}, \]

(3.59)

where \( γ_μ \) and \( f_μ(r) \) are the corresponding eigenvalues and eigenfunctions. For a diffusive system, \( γ_μ = D Q_μ^2 \), where \( D \) is the diffusion constant, and wavevectors \( Q_μ \) depend on the shape of the system. For a rectangular diffusive dot of the size \( L_x × L_y \), one finds \( Q_μ^2 = π^2 (n_x^2 / L_x^2 + n_y^2 / L_y^2) \) with \( n_x, n_y \geq 0 \) being integer numbers. For chaotic systems, \( γ_μ \) are the eigenvalues of the Perron-Frobenius operator. Zero mode in Eq. (3.59) corresponds to the conservation of the number particles, and all the other modes describe the relaxation of any initial inhomogeneous distribution function by the virtue of classical chaotic dynamics, \( \text{Re} γ_μ > 0 \). If the system is integrable, or there are some additional symmetries of the system, other zero modes appear, however, we disregard such a possibility and consider only diffusive or classically chaotic systems.

Using Eqs. (3.21), (3.56) and (3.59), and taking into account that all the energies are smaller than the Thouless energy (or, in other words, the lowest non-zero eigenvalue of diffusion or Perron-Frobenius operator), \( |ε_a| < γ_1 \), we find

\[ V_{αβγδ}^{(1/2)} = \frac{Δ}{g} \left[ 2F_1 δ_{αδ} δ_{βγ} + (F_2 + F_1) (δ_{αγ} δ_{βδ} + δ_{αδ} δ_{βγ}) \right], \]

(3.60)

where the dimensionless conductance of the system is defined as

\[ g = \text{Re} \frac{γ_1}{Δ}, \]

(3.61)

and is assumed to be much larger than unity. Dimensionless coefficients in Eq. (3.60) are given by

\[ F_i = \frac{\text{Re}γ_1}{π} \int_{γ_μ ≠ 0} \frac{F_μ}{γ_μ}, \]

\[ F_μ^1 = ν \langle \hat{V}(k) \rangle_{FS}, \]

\[ F_μ^2 = ν \int dr_1 dr_2 V (r_1 - r_2) f_μ^*(r_1) f_μ(r_2). \]

For the screened interaction potential, coefficients \( F_{μ2}^1 = F_{μ2}^2 = 1/2 \) and, therefore, \( F_{1,2} \) are of the order of unity for chaotic systems and of the order of \( (1/4π^2) \ln(L/l) \) for the diffusive dot. Here \( L \) is the size of the dot and \( l \) is the transport elastic mean free path. Thus, we have shown that the corrections to the average matrix element (3.54) are parametrically small for the metallic regime.

Now, we wish to show that the fluctuations of the matrix elements are small. Indeed, with the help of Eq. (3.21), we obtain for a generic (i.e., with no pairwise equal indices) matrix element:

\[ \langle V_{αβγδ} \rangle^2 = c \left( \frac{Δ}{g} \right)^2, \]

(3.63)

The numerical coefficient \( c \) for the diffusive system is given by

\[ c = \left( \frac{γ_1}{π} \right)^2 \sum_{γ_μ ≠ 0} \left[ 2(F_1^μ)^2 + (F_2^μ)^2 \right] \left( \text{Re} \frac{1}{γ_μ} \right)^2, \]

[where coefficients \( F_{μ2}^1 \) are defined in Eq. (3.62)] and it is of the order of unity, so that the matrix elements are small at \( g ≫ 1 \). For chaotic systems, the expression for \( c \) is more cumbersome, but still have a similar structure. In the case of “diagonal” [in the sense of Eq. (3.60)] matrix
elements, the average in Eq. (8.63) should be replaced by the corresponding variance.

The main conclusion of this section is contained in Eqs. (8.60) and (8.63). These equations clearly show that the Coulomb blockade type interaction (1.1) is a parametrically justified description for the dynamics of the system at energies smaller than the Thouless energy.

Closing this subsection, let us mention numerical works that have been performed recently [22]. These papers addressed either dirty diffusive systems with small number of electrons \( g \simeq 1 \), or classically localized states \( \Phi \) and are not relevant for the metallic regime \( g \gg 1 \) we are dealing with. We believe that the large quantum dots studied in Refs. [22] belong to the metallic regime.

IV. BOSONIZATION PROCEDURE

Equations (3.15) – (3.18) reduce the initial system consisting of a dot and a single-mode channel to the effective one-dimensional model. To treat interaction in the model (the second term in Eq. (3.15)), we follow Ref. [3], and use the bosonization technique. In the bosonic variables, the entire Hamiltonian (3.13) becomes quadratic. The price for this convenience, is a strongly non-linear form that the backscattering terms acquire (in the language of left- and right movers, those are the terms \( \propto \psi^\dagger_L \psi_R \) in Eqs. (3.16) and (3.17)). Fortunately, the typical value of the kernel (3.9) is small \( (\Delta/E_C) \), and this enables us to use the perturbation theory, which will be presented in Secs. 5 and 6. In this section we present bosonization procedure in a form most suitable for our purposes.

A. Spinless electrons

One-dimensional fermionic fields can be presented in the form

\[
\psi_R(x) = \frac{\tilde{\eta}}{\sqrt{2\pi}\lambda} e^{i\tilde{\phi}_R(x)}; \quad \psi_L(x) = \frac{\tilde{\eta}}{\sqrt{2\pi}\lambda} e^{-i\tilde{\phi}_L(x)}, \tag{4.1}
\]

where \( \lambda \) is the high energy cut-off of the order of the Fermi wavelength, and \( \tilde{\eta} = \eta^\dagger \), \( \tilde{\eta}^2 = 1 \) is the Majorana fermion, its significance will be discussed later. One-dimensional bosonic fields \( \tilde{\phi}_{L,R}(x) \) satisfy the following commutation relations:

\[
\begin{align*}
[\tilde{\phi}_L(x), \tilde{\phi}_L(y)] &= -i\pi \text{ sgn}(x-y); \quad (4.2a) \\
[\tilde{\phi}_R(x), \tilde{\phi}_R(y)] &= i\pi \text{ sgn}(x-y); \quad (4.2b) \\
[\tilde{\phi}_R(x), \tilde{\phi}_L(y)] &= -i\pi. \quad (4.2c)
\end{align*}
\]

It is easy to check, using Eqs. (4.2), that the fermionic fields (1.1) obey the standard commutation relations. The expressions for the densities of left and right movers are

\[
\begin{align*}
\psi^\dagger_L(x)\psi_L(x) &= \frac{1}{2\pi} \frac{\partial_x \varphi_L(x)}{\partial \varphi_L(x)}; \\
\psi^\dagger_R(x)\psi_R(x) &= \frac{1}{2\pi} \frac{\partial_x \varphi_R(x)}{\partial \varphi_R(x)}. \tag{4.3}
\end{align*}
\]

With the help of Eq. (4.3), Hamiltonian (3.13) can be bosonized as

\[
\hat{H}_0 = \frac{v_F}{4\pi} \int_{-\infty}^{\infty} dx \left[ \left( \frac{\partial \tilde{\phi}_L}{\partial x} \right)^2 + \left( \frac{\partial \tilde{\phi}_R}{\partial x} \right)^2 \right] \tag{4.4}
\]

Backscattering Hamiltonian (3.17) takes the form:

\[
\hat{H}_{bs} = \frac{|r|v_F}{\pi \lambda} \cos [\tilde{\phi}_L(0) + \tilde{\phi}_R(0)]. \tag{4.5}
\]

Bosonized version of the effective action Eq. (3.16) is:

\[
\hat{S} = \frac{1}{2\pi \lambda} \int_0^\beta d\tau_1 d\tau_2 L (\tau_1 - \tau_2) \tilde{\eta}(\tau_1) \tilde{\eta}(\tau_2) \times \left[ e^{i\tilde{\phi}_L(\tau_1) + e^{-i\tilde{\phi}_R(\tau_1)}} \left[ e^{-i\tilde{\phi}_L(\tau_2) + e^{i\tilde{\phi}_R(\tau_2)}} \right] \right], \tag{4.6}
\]

where bosonic operators are taken at the origin \( x = 0 \). Majorana fermion \( \eta \) does not enter the effective Hamiltonian, and therefore it is not a dynamical field. Its role in the effective action is to take care of the difference in the definition of the operation of chronological ordering for the fermionic and bosonic operator. Equality

\[
\langle T_r \tilde{\eta}(\tau_1) \tilde{\eta}(\tau_2) \rangle = \text{sgn}(\tau_1 - \tau_2), \tag{4.7}
\]

and Wick’s theorem preserves the definition of chronological ordering for fermions in Eq. (3.18).

It is convenient to separate the part of the bosonic sector non affected by the Coulomb interaction and introduce new field \( \tilde{\phi}_+, \tilde{\phi}_-, \tilde{\phi} \) with the commutation relations

\[
\begin{align*}
[\tilde{\phi}_+(x), \tilde{\phi}_+(y)] &= -i\pi \text{ sgn}(x-y); \quad (4.8a) \\
[\tilde{\phi}_-(x), \tilde{\phi}_-(y)] &= i\pi \text{ sgn}(x-y); \quad (4.8b) \\
[\tilde{\phi}_+(x), \tilde{\phi}_-(y)] &= 0; \quad (4.8c) \\
\hat{\Phi}, \tilde{\phi}_+(x) &= i\pi. \quad (4.8d)
\end{align*}
\]

We express operators (4.2) in terms of new fields (4.8) as
\[ \dot{\phi}_L(x) = \frac{\dot{\phi}_+(x) + \dot{\phi}_-(x) + \dot{\Phi}}{\sqrt{2}} - \pi N; \]
\[ \dot{\phi}_R(x) = \frac{\dot{\phi}_-(x) - \dot{\phi}_-(x) - \dot{\Phi}}{\sqrt{2}} - \pi N; \]  
(4.9)

where \(c\)-number \(\pi N\) is incorporated into the definition of the field. It is easy to see that the commutation relations (4.4) are preserved. In the new variables, the Hamiltonian (4.14) is independent on the gate voltage (i.e., on \(N\)).

\[ \hat{H}_0 = \frac{v_F}{4\pi} \int_{-\infty}^{\infty} dx \left[ \left( \frac{\partial \hat{\phi}_+}{\partial x} \right)^2 + \left( \frac{\partial \hat{\phi}_-}{\partial x} \right)^2 \right] + \frac{E_C}{4\pi^2} \hat{\phi}_+^2(0). \]  
(4.10)

All the \(N\)-dependence is transferred now to the backscattering term in the Hamiltonian:
\[ \hat{H}_b = \frac{|r|v_F}{\pi\lambda} \cos \left[ \sqrt{2}\hat{\phi}_+(0) - 2\pi N \right], \]  
(4.11)

and to the action \(\hat{S}\) which is contributed by the return trajectories of electrons after multiple scattering within the dot:
\[ \hat{S} = \frac{1}{2\pi\lambda} \int_0^\beta d\tau_1 d\tau_2 L (\tau_1 - \tau_2) \hat{\eta}(\tau_1) \hat{\eta}(\tau_2) \times \]  
exp \left[ \frac{\hat{\phi}_-(\tau_1) - \hat{\phi}_-(\tau_2)}{\sqrt{2}} \right] \] 
exp \left[ \frac{\hat{\Phi}(\tau_1) - \hat{\Phi}(\tau_2)}{\sqrt{2}} \right] \times \]  
(4.12)

where averages are calculated with respect to Hamiltonian \(\hat{H}_0\) and all the bosonic fields are taken at \(x = 0\).

Standard calculation presented in Appendix B yields for \(T \ll E_C\) and \(\tau \gg \lambda/v_F \simeq 1/\epsilon_F\):

\[ D_\Phi (\tau) - D_\Phi (0) = \frac{1}{2} \int_0^\infty dx e^{-x} \ln \left[ \frac{\sin \left( i \frac{2\pi x}{E_C} + \tau \right) \pi T \sin \left( -i \frac{2\pi x}{E_C} + \tau \right) \pi T \right] \]  
(4.14a)

\[ D_- (\tau) - D_- (0) = \ln \left( \frac{\lambda}{v_F} \frac{\pi T}{\sin \pi T \tau} \right); \]  
(4.14b)

\[ D_+ (0) = \ln (\frac{2\pi v_F}{\lambda E_C \epsilon C}); \]  
(4.14c)

\[ D_+ (\tau) - D_+ (0) = [D_- (\tau) - D_- (0)] - [D_\Phi (\tau) - D_\Phi (0)]; \]  
(4.14d)

\[ D_{\Phi+} (\tau) = \frac{i}{2} \int_0^\infty dx e^{-x} \frac{2\pi^2 T^2}{E_C} \left[ \cot \left( \frac{2\pi x}{E_C} + \tau \right) \pi T + \cot \left( -\frac{2\pi x}{E_C} + \tau \right) \pi T \right], \]  
(4.14e)

where \(C \approx 0.577\) is the Euler constant.

To conclude this Subsection, let us prove the assumption of Sec. 1 about the Fermi liquid behavior of the system at low energies. In order to do this, we will calculate the fermionic Green functions, \(\langle \psi_L^\dagger (\tau) \psi_L (0) \rangle\) and \(\langle \psi_R^\dagger (\tau) \psi_R (0) \rangle\), using the definitions (4.3), (4.4), and the results (4.14). Averaging over the bosonic fields similar to the well-known calculation of the Debye-Waller factor, yields:

\[ \cos \left( \frac{\hat{\phi}_+(\tau_1)}{\sqrt{2}} + \frac{\pi}{4} - \pi N \right) \cos \left( \frac{\hat{\phi}_+(\tau_2)}{\sqrt{2}} + \frac{\pi}{4} - \pi N \right). \]
Substituting Eqs. (4.14) into Eq. (4.15), we obtain
\[ \langle \psi_R^i(x) \psi_L(0) \rangle = \frac{1}{2\pi v_F \sin \pi T} \eta \phi \int \frac{dx \exp(-\frac{E_C}{2\pi^2 T} \coth(x+i\pi T))}{\tau} \] (4.16a)
\[ \langle \psi_L^i(x) \psi_L(0) \rangle = \frac{1}{2\pi v_F} K(\tau) \] (4.16b)
\[ K(\tau) = \frac{\pi T}{\sin \pi T} \exp \left\{ -\int_0^\infty dx \frac{E_C}{2\pi^2 T} \coth(x+i\pi T) \right\} \]

The Green function (4.16a) is not affected by interactions at all. The reason is, that it is taken at coinciding arguments (xi = x2 = 0), e.g., outside the interaction region. Because \( \langle \psi_L^i(\tau) \psi_L(0) \rangle \) describes propagation of a chiral particle, the information about interaction is never carried back to the observation point \( x = 0 \). The Green function (4.16b) acquires the free-fermion form at \( \tau > E_C^{-1} \), which corresponds to the energies below the charging energy (note that \( T \ll E_C \)). In this energy range, \( \langle \psi_R^i(\tau) \psi_L(0) \rangle \) corresponds to a free fermion completely reflected from the dot. The phase factor \( \exp(i2\pi N) \) in Eq. (4.16b) represents the scattering phase \( \pi N \), which agrees with the Friedel sum rule (2.3). Thus, our intuitive picture of Sec. 1 is proven by explicit calculation of the fermionic propagators.

**B. Electrons with spin**

Similarly to the spinless case, we start here with the bosonization of electron operators:
\[ \psi_{R,\sigma}(x) = \frac{\eta_\sigma}{\sqrt{2\pi \lambda}} \exp \left( i\frac{\phi_R^\sigma(x) + \phi_L^\sigma(x)}{\sqrt{2}} \right) \]
\[ \psi_{L,\sigma}(x) = \frac{\eta_\sigma}{\sqrt{2\pi \lambda}} \exp \left( -i\frac{\phi_R^\sigma(x) + \phi_L^\sigma(x)}{\sqrt{2}} \right) \] (4.17)
where index \( \alpha = \pm 1 \) denotes the spin projections, and the Majorana fermions \( \eta_{\sigma,\pm} \) satisfy the anticommutation relations \( \{ \eta_{\sigma,1}, \eta_{-1} \} = 0 \). Boson fields \( \phi_{L,R}^\sigma \) corresponding to the charge and spin degrees of freedom respectively, satisfy the following commutation relations:
\[ [\phi_L^\sigma(x), \phi_L^\rho(y)] = -i\pi \text{sgn}(x-y)\delta_{ij} \] (4.18a)
\[ [\phi_R^\sigma(x), \phi_R^\rho(y)] = i\pi \text{sgn}(x-y)\delta_{ij} \] (4.18b)
\[ [\phi_R^\rho(x), \phi_L^\rho(y)] = -i\pi \delta_{ij}; \quad i,j = \rho, \sigma. \] (4.18c)

Like in the case of spinless fermions, it is convenient to introduce even and odd modes \( \phi_{\rho,s}^\sigma \) for the charge and spin sectors, and two \( x \)-independent fields, \( \Phi^\rho \) analogous to Eq. (4.9).

\[ \phi_L^\rho(x) = \frac{\phi_L^\rho(x) + \phi_L^\rho(-x)}{\sqrt{2}} \] (4.19)

The commutation relations for the new fields within the charge and spin sectors coincide with Eqs. (4.13); fields of different sectors commute with each other. In terms of the new fields, Hamiltonian (3.15) acquires the form independent on dimensionless gate voltage \( \tilde{N} \):
\[ \tilde{H}_0 = \frac{v_F}{\pi^2} \sum_{|i\rho,\sigma} \sum_{|N} \int_{-\infty}^\infty dx \left( \frac{\partial \tilde{\phi}}{\partial x} \right)^2 + \frac{E_C}{2\pi^2} \left[ \tilde{\phi}(0) \right]^2 . \] (4.20)

Backscattering Hamiltonian (4.17) takes the form
\[ \tilde{H}_{bs} = \frac{2|v_F|}{\pi \lambda} \cos (\tilde{\phi} + \pi \tilde{N}) \cos \tilde{\phi}(0), \] (4.21)
and the effective action (4.16c) can be rewritten as
\[ \tilde{S} = \frac{1}{\pi \lambda} \int_0^\beta d\tau_1 d\tau_2 L(\tau_1 - \tau_2) \sum_{\alpha = \pm 1} \tilde{\eta}_\alpha(\tau_1) \tilde{\eta}_\alpha(\tau_2) \times \]
\[ \cos \left[ \frac{\tilde{\phi}_1(\tau_1) + \tilde{\phi}_1(\tau_2)}{2} + \frac{\tilde{\phi}_2(\tau_1) + \tilde{\phi}_2(\tau_2)}{2} \right] - \pi \tilde{N} \]
\[ \sin \left[ \frac{\tilde{\phi}_1(\tau_1) - \tilde{\phi}_2(\tau_2)}{2} + \frac{\tilde{\phi}_2(\tau_1) - \tilde{\phi}_1(\tau_2)}{2} \right] \] (4.22)

Similarly to Eq. (4.13), we introduce the relevant bosonic correlation functions
\[ \tilde{D}^\rho_{\tau} = \left( T_{\tau} \tilde{\phi}_{-}(\tau)^i \tilde{\phi}_{-}(0)^i \right); \] (4.23a)
\[ \tilde{D}^\rho_{\tau} = \left( T_{\tau} \tilde{\phi}_{+}(\tau)^i \tilde{\phi}_{+}(0)^i \right); \] (4.23b)
\[ \tilde{D}_\tau^\rho = \left( T_{\tau} \tilde{\Phi}_{+}(\tau)^i \tilde{\Phi}_{0}(0)^i \right); \] (4.23c)
\[ \tilde{D}_\tau^\rho = \left( T_{\tau} \tilde{\Phi}_{0}(\tau)^i \tilde{\Phi}_{+}(0)^i \right), \] (4.23d)
where index \( i = \rho, \sigma \) labels charge and spin sectors respectively, bosonic fields are taken at \( x = 0 \), and averaging is performed over the Hamiltonian \( \tilde{H}_0 \) given by Eq. (4.20).

The calculation of these propagators can be performed immediately by noticing that the spin sector of the Hamiltonian (1.20) corresponds to the free bosons, and the charge sector differs from Eq. (4.10) only by replacement \( E_C \rightarrow 2E_C \). Thus, we obtain

\[ \tilde{D}^\rho_{\tau} = \frac{\tilde{D}^\rho_{\tau}}{2} \] (4.24a)
\[ \tilde{D}_\tau^\rho = \frac{\tilde{D}_\tau^\rho}{2} \] (4.24b)
\[ \tilde{D}^\rho_{\tau} = \frac{\tilde{D}^\rho_{\tau}}{2} \] (4.24c)
\[ \tilde{D}_\tau^\rho = \frac{\tilde{D}_\tau^\rho}{2} \] (4.24d)

Thus, we obtain

\[ \tilde{H}_0 = \frac{v_F}{\pi^2} \sum_{|i\rho,\sigma} \sum_{|N} \int_{-\infty}^\infty dx \left( \frac{\partial \tilde{\phi}}{\partial x} \right)^2 + \frac{E_C}{2\pi^2} \left[ \tilde{\phi}(0) \right]^2 . \] (4.20)
where $C \approx 0.577$ is the Euler constant.

As we will see below, see also Ref. 14, the main contribution to the observable quantities is associated with the time scale $\tau \sim 1/E_C$. At this time scale the effective theory can be further simplified. The mode $\hat{\phi}^n$ is “pinned” due to the charging energy, see Eq. (4.20). Therefore, the amplitude of quantum fluctuations of this mode is finite, see Eq. (1.24d), and the correlation function $D_n^\phi(\tau)$ decreases rapidly at $\tau \gtrsim 1/E_C$, as it follows from Eqs. (4.24a) and (1.24d). The decrease of correlations means that the average of a product, $\langle e^{i\hat{\phi}_n^\phi(\tau_1)} \ldots e^{i\hat{\phi}_n^\phi(\tau_n)} \rangle$, can be replaced by the product of averages, $\langle e^{i\hat{\phi}_n^\phi(\tau_1)} \rangle \ldots \langle e^{i\hat{\phi}_n^\phi(\tau_n)} \rangle$, if the intervals between the times $\tau_1, \ldots, \tau_n$ exceed $1/E_C$. In other words, the operator functions of $\hat{\phi}_n^\phi$ in Eq. (4.22) can be substituted with $c$-numbers, according the rule:

$$
e^{i\hat{\phi}_n^\phi} \rightarrow e^{-\frac{i}{2}D_n^\phi(0)}.$$  

(4.25)

On the other hand, it follows from Eqs. (4.24a) and (4.24d) that at $\tau > 1/E_C$

$$
\langle T_\tau \left( \hat{\phi}_n^\phi(\tau) + \hat{\Phi}_n^\phi(\tau) - \hat{\phi}_n^\phi(0) - \hat{\Phi}_n^\phi(0) \right)^2 \rangle = 2 \ln \left[ \frac{\pi \lambda}{v_F E_C C} \left( \frac{\pi T}{\sin \pi T \tau} \right)^2 \right],
$$

which means that such correlation function will be preserved if we introduce another free bosonic field $\hat{\phi}_p(x)$, with commutation relation $[\hat{\phi}_p(x); \hat{\phi}_p(y)] = -i \pi \text{sgn}(x - y)$, and substitute

$$
\hat{\phi}_n^\phi(x = 0) + \hat{\Phi}_n^\phi(x = 0) \rightarrow \sqrt{2} \hat{\phi}_p(x = 0).
$$

(4.26)

After substitutions (4.23) and (4.24d), Hamiltonian (4.20) becomes just a Hamiltonian of three free bosonic fields

$$
\hat{H}_0 = \int_{-\infty}^{\infty} dx \left[ \left( \frac{\partial \hat{\phi}_p}{\partial x} \right)^2 + \left( \frac{\partial \hat{\phi}_p^\phi}{\partial x} \right)^2 + \left( \frac{\partial \hat{\phi}_p^\phi}{\partial x} \right)^2 \right],
$$

(4.27)

backscattering term acquires the form

$$
\hat{H}_{bs} = \frac{2|\tau|}{\pi} \left( \frac{E_C e^C v_F}{\pi \lambda} \right)^{1/2} \sin \pi N \cos \phi_0^\phi(0),
$$

(4.28)

and the effective action is given by

$$
\dot{S} = \frac{1}{\pi \lambda} \int_0^\beta d\tau_1 d\tau_2 L (\tau_1 - \tau_2) \sum_{\alpha = \pm 1} \hat{\eta}_\alpha (\tau_1) \hat{\eta}_\alpha (\tau_2) \times
$$

$$
e^{\frac{i}{\alpha} \phi_{\tau_1}^\phi (\tau_1) + \phi_{\tau_2}^\phi (\tau_2)} e^{\frac{i}{\alpha} \phi_{\tau_1}^\phi (\tau_1) - \phi_{\tau_2}^\phi (\tau_2)} e^{-\frac{i}{\alpha} \phi_{\tau_1}^\phi (\tau_1) - \phi_{\tau_2}^\phi (\tau_2)} \times
$$

$$
\left\{ \cos \left[ \frac{\phi_{\tau_1}^\phi (\tau_1) + \phi_{\tau_2}^\phi (\tau_2)}{2} - \pi N \right] + \cos \left[ \frac{\pi}{4} + \alpha \phi_{\tau_1}^\phi (\tau_1) - \phi_{\tau_2}^\phi (\tau_2) \right] \right\}. 
$$

(4.29)

Correlation functions of the free bosonic fields are given by

$$
D_{\rho}^p (\tau) - D_{\rho}^p (0) = D_{\Phi}^\phi (\tau) - D_{\Phi}^\phi (0) = \ln \left( \frac{\tilde{\lambda}}{v_F |\sin \pi T \tau|} \right),
$$

(4.30)

where cut-off $\tilde{\lambda}$ is of the order of $v_F / E_C$ because the charging energy $E_C$ is the largest energy scale which can be considered with the help of Hamiltonian (1.27). It is easy to check also by an explicit calculation, that at time differences larger than $E_C^{-1}$ correlation functions of the electron operators evaluated with the help of the Hamiltonians (1.20) and (1.27) respectively coincide.

**V. THERMODYNAMICS OF THE “OPEN” DOT**

Coulomb blockade can be investigated experimentally by measuring the differential capacitance of a dot, see Eq. (3.19). In the regime of a developed blockade (weak tunneling between the dot and the electron reservoir), $C_{\text{diff}}(N)$ exhibits sharp peaks at half-integer values of $N$. In the opposite limit of no backscattering, the differential capacitance is an $N$–independent constant,
$C_{\text{diff}}(N) = C$. It was shown in Ref. 14 that weak reflection from a scatterer in the channel leads to the capacitance oscillations with a phase depending on the exact position of the scatterer. In this section we demonstrate that even at $r = 0$, the differential capacitance still depends on $N$ due to the electron backscatterings from inside the dot, which are described by the action $S$. The randomness of the backscattering events results in the randomness of the phase of the capacitance oscillations. We will relate the statistics of $C_{\text{diff}}(N)$ with the one of kernel $L(\tau)$.

The starting point for the calculation of the capacitance is Eq. (3.18) for the thermodynamic potential. In principle, Eq. (3.18) enables one to consider the backscattering off a barrier in the channel ($\hat{H}_{bs} \neq 0$), as well as off the dot ($\hat{S} \neq 0$). In the limit of weak backscattering, the perturbation theory in $\hat{H}_{bs}$ and $\hat{S}$ can be used to calculate $C_{\text{diff}}(N)$. The case of $\hat{S} = 0$ was considered by Matveev. He has shown that for spinless fermions, a non-vanishing result appears in the first-order perturbation theory, whereas for the spin-1/2 electrons this order gives zero result. Similarly, it is sufficient to account for the scatterings from inside the cavity in the first order of the perturbation theory for spinless fermions, but in the case of spin-1/2 electrons we have to expand the thermodynamic potential (3.18) up to the second order in $\hat{S}$, if $\hat{H}_{bs} = 0$.

Backscattering in the channel leads to a finite modulation of the average differential capacitance. The modulation amplitude can be estimated by expansion of Eq. (3.18) to the second order in $\hat{H}_{bs}$ in the spin-1/2 case. Electron scattering from inside the dot leads to the capacitance fluctuations superimposed on this modulation. The two contributions to the capacitance are not additive: the non-zero result appears only in the second-order perturbations to the Hamiltonian $\hat{H}_0$, when one expands Eq. (3.18). In the domain of a relatively strong backscattering in the channel, $|r|^2 \gtrsim \Delta/E_C$, the leading term in fluctuations is proportional to the product of $\hat{H}_{bs}$ and $\hat{S}$. We address the capacitance fluctuations at finite $|r|$ in the end of this section.

A. Spinless fermions

1. Reflectionless contact

The first-order expansion of Eq. (3.18) in $\hat{S}$ yields:

$$\delta\Omega = T\langle T_\tau \hat{S} \rangle.$$ (5.1)

We substitute Eq. (4.12) into Eq. (5.1), retain only $N$-dependent terms, and obtain with the help of Eqs. (4.14):

$$\delta\Omega = \frac{1}{2\pi v_F} \int_0^\beta d\tau L(\tau) \left[e^{i2\pi N}K(\tau) + \text{c.c.}\right],$$ (5.2)

where function $K(\tau)$ is defined in Eq. (4.16b). To perform the integration over $\tau$, we use the Lehmann representation (3.20) of the kernel $L$:

$$\delta\Omega = \frac{1}{2\pi v_F} \int_0^\infty dt \frac{\pi T}{\sinh[\pi T(t+i\tau)]} \left[e^{i2\pi N}K(\tau) + \text{c.c.}\right].$$ (5.3)

The integration over $\tau$ here can be now performed with the help of analytic properties of function $K(\tau)$. As it follows from Eq. (4.16b), the function $K(\tau)$ is analytic in the lower semiplane $\text{Im} \tau < 0$. To calculate the integral of the first term in the brackets, we deform the contour of integration over $\tau$ as shown in Fig. 3.

![Fig. 3. The integration contour used in the evaluation of $\delta\Omega$ in the spinless case, see Eq. (5.3). Branch cuts of $K(\tau)$ are shown by thick lines.](image)

Because of the periodicity of the integrand, the integrals over the parts of the contour running parallel to the imaginary axis, cancel out. As the result, only the pole contribution at $\tau = it$ remains at $t < 0$. At $t > 0$ the pole contribution disappears. Second term in Eq. (5.3) is integrated by using $K(\tau)^* = -K(-\tau)$ for the real $\tau$. With the help of the explicit expression (4.16b) for the function $K(\tau)$, we find:

$$\delta\Omega(N) = \frac{1}{2\pi iv_F} \int_0^\infty dt \frac{\pi T}{\sinh(\pi Tt)} \times$$ (5.4)

$$\exp \left[ -\int_0^\infty dx \exp \left[-2\pi^2 T\coth \left(\frac{2\pi x}{E_C} + t\right)\pi T \right] \right] \times L^R(t)e^{-i2\pi N} - L^A(-t)e^{i2\pi N}.$$ (5.5)

Finally, at low temperatures $T \ll E_C$, the $N$-dependent correction takes the form:

$$\delta\Omega(N) = \frac{1}{2\pi iv_F} \int_0^\infty dx \exp \left[-\int_0^\infty e^{-x} \frac{e^{-x}}{E_C x} \right] \times L^R(t)e^{i2\pi nN} - L^A(-t)e^{-2\pi i nN}.$$ (5.6)

Equation (5.3) relates the Coulomb blockade oscillation to the exact free-electron Green function in the dot. The variation of $\delta\Omega$ with the gate voltage is harmonic, however, its phase and amplitude are random quantities. To reveal this oscillatory dependence in the average quantities, one has to find the correlation function
At low temperatures, Eqs. (5.5) and (2.25) lead directly to the result (2.24) with the dimensionless function $\Lambda_E$ given by:

$$\Lambda_E(x) = \frac{1}{(2\pi)^4} \Lambda(x),$$

$$\Lambda(x) = \int_0^\infty \frac{dy}{y^2} \exp[-xy + 2e^y Ei(-y)],$$

where $Ei(x) = \int_x^\infty e^t/t$ is the exponential integral function. The correlation function of the differential capacitances Eq. (5.10) is given by:

$$\frac{\delta C_{\text{diff}}(1)\delta C_{\text{diff}}(2)}{C^2} = \frac{2\Delta}{E_C} \left[ \Lambda \left( \frac{H_k^2}{H_c^2} \right) + \Lambda \left( \frac{H_{k+1}^2}{H_c^2} \right) \right] \cos 2\pi n,$$

where we use the short hand notations $i = N_i H_i$, $n = N_1 - N_2$, and $H_k = H_1 \pm H_2$. Correlation magnetic field $H_i$ is controlled by the charging energy and it is given by Eq. (2.23).

The variance of the capacitance fluctuations at $H = 0$ is two times larger than in the unitary limit ($H \gg H_c$). The dimensionless crossover function $\Lambda(x)$ is plotted in Fig. 4.

![Graph showing $\Lambda(x)$](image)

**FIG. 4.** Function $\Lambda(x)$ determining correlation of differential capacitances at different values of magnetic fields, see Eqs. (5.7) and Eq. (5.10).

Correlation between the capacitances at different magnetic fields is suppressed if $H_-$ exceeds $H_c$. In our approximation, correlation “length” $n_c$ in the dimensionless gate voltage is infinite. To find $n_c$, one should take into account that varying of the gate voltage affects the chemical potential of electrons in the dot by the level spacing, each time when $N$ changes by one. If the chemical potential is shifted by $E_C$, a completely new set of levels determines the kernel $L$, thus suppressing the correlations. It results in the estimate: $n_c \approx E_C/\Delta$. Another effect which leads to the decrease of the correlation function at large $n$, is the variation of the dot shape with the gate voltage.

**2. Finite reflection in the contact**

At finite backscattering in the contact, $r \neq 0$, Hamiltonian (4.11) should be taken into account:

$$\delta \Omega(N) = \langle \hat{H}_{bs} \rangle.$$

Calculating this average with the help of Eq. (4.14c), and using Eq. (5.13), we arrive to the result for the oscillating contribution of the averaged capacitance,

$$\frac{\delta C(N)}{C} = 2e^C |r| \cos 2\pi N.$$

This result was first obtained in Ref. 14.

Average of the action Eq. (5.1) simply adds to this result. Thus, we conclude that finite reflection in this order of perturbation theory does not affect the mesoscopic fluctuations of the capacitance, see Eq. (5.7). We will see below that this is qualitatively different from the electrons with spin with finite reflection in the contact leads to the increase of the mesoscopic fluctuations of the differential capacitance.

**B. Electrons with spin**

**1. Reflectionless contact**

In the case of spin-1/2 fermions, the charging energy pins only one out of four modes. Fluctuations spin mode are not suppressed, see Eq. (4.27). These fluctuations average action (4.29) to zero. To obtain a finite result for the $N$-dependent part of the thermodynamic potential, we have to expand $\Omega$ up to the second order in $\hat{S}$:

$$\delta \Omega = -\frac{1}{2} T \langle \hat{S} \hat{S}^2 \rangle.$$

Upon the substitution of Eq. (1.29) into Eq. (5.1), we use the expressions (1.30) to perform the averaging over $H_0$. This cumbersome albeit straightforward calculation yields for the $N$-dependent part of the thermodynamic potential:

$$\delta \Omega = \frac{1}{(2\pi v_F)^2} \int_0^\beta d\tau_1 d\tau_2 d\tau_3 L(\tau_1) L(\tau_2) \sum_{\gamma = \pm 1} e^{-i2\pi \gamma N^2} \times \frac{\pi T}{\sin \pi T(\tau_1 + i\gamma 0) \sin \pi T(\tau_2 + i\gamma 0)}^{1/2} \times$$

$$\frac{\pi T}{\sin \pi T(\tau_1 + \tau_3 + i\gamma 0) \sin \pi T(\tau_2 - \tau_3 + i\gamma 0)}^{1/2}.$$
As in the Section [4], it is convenient to use the Lehmann representation \[ \langle \Omega \rangle \equiv \frac{1}{2\pi v_F} \int_{-\infty}^{\infty} dt_1 dt_2 \times \]
\[ (L_R(t_1) - L^A(t_1)) (L_R(t_2) - L^A(t_2)) \times \]
\[ \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \pi T \sinh \pi T (t_1 + i\tau_1) \sinh \pi T (t_2 + i\tau_2) \]
\[ \sum_{\gamma = \pm 1} e^{-i2\pi \gamma \Omega} \sinh \pi T \omega (\tau_1 + i\gamma 0) \sinh \pi T (\tau_2 - \tau_3 + i\gamma 0) \]
\[ \frac{1}{2\pi} \times \]
\[ \left( \sinh \pi T (t_1 + i\tau_1) \sinh \pi T (t_2 + i\tau_2) \right)^{1/2} \]
\[ \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \pi T \sinh \pi T (t_1 + i\tau_1) \sinh \pi T (t_2 + i\tau_2) \]
\[ \times \frac{1}{2\pi} \times \]
\[ \int_0^\pi d\phi \left[ \sinh \omega (x - i\phi) \sinh \omega (y + i\phi) \right]^{1/2} = 4e^{-|x+y|}K \left( e^{-|x+y|} \right) , \]
where \( K(k) \) is the complete elliptic integral of the first kind. We find for \( \delta\Omega \):
\[ \delta \Omega = \left( \frac{1}{2\pi v_F} \int_{-\infty}^{\infty} dt_1 dt_2 \pi T \sinh \omega (t_1 + i\tau_1) \sinh \omega (t_2 + i\tau_2) \right)^{1/2} \]
\[ \left[ L_R^A(t_1) L_R^A(t_2) e^{-i2\pi \Omega} + c.c. \right] e^{-\pi T (t_1 + t_2)} K \left( e^{-\pi T (t_1 + t_2)} \right) \]
Here, as in Eq. (5.7), we use the short hand notation \( i = \mathcal{H}_i, \mathcal{N}_i \) and \( n = \mathcal{N}_1 - \mathcal{N}_2 \). The diffusion and Cooperon decay times \( \tau_D \) and \( \tau_C \) are related to the magnetic field values \( \mathcal{H}_1, \mathcal{H}_2 \) by Eq. (3.24). Now we can use Eq. (3.19) to find the correlation function of the differential capacitances. In the leading logarithmic approximation we obtain
\[ \frac{\delta C_{V\Omega} (1) \delta C_{V\Omega} (2) - \delta C_{V\Omega} (2) \delta C_{V\Omega} (1)}{C^2} = \frac{8}{3\pi^2 E_C^2} \ln^4 \left( \frac{E_C}{T} \right) \cos 2\pi n \times \]
\[ \sum_{\gamma = \pm 1} \left[ 1 - \left( \frac{\ln \left( \frac{\mathcal{H}_1}{\mathcal{H}_2} \right)^2}{\ln \left( \frac{\mathcal{H}_1}{\mathcal{H}_2} \right)^2} \right)^{\gamma} \right] , \]
where the fields \( \mathcal{H}_\pm = \mathcal{H}_1 \pm \mathcal{H}_2 \) are assumed to be small compared to the correlation field \( \mathcal{H}_c \) given by Eq. (2.22). The new temperature dependent correlation field appearing in Eq. (5.13) is given by
\[ \mathcal{H}_c = \frac{\Phi_0}{A} \sqrt{\frac{T}{2\pi E_T}} \]
with \( \Phi_0, A, \) and \( E_T \) being the flux quantum, the geometrical area of the dot, and the Thouless energy respectively. Unlike the scale \( \mathcal{H}_c \), the characteristic field \( \mathcal{H}_c^\pm \) is independent on the charging energy. This smaller field scale appears due to the existence of the “free” excitation \( \varphi_+ \), which is not pinned by the effect of charging.

With the increase of the number of channels in the dot-lead junction, the number of free modes also increases. The role of charging (which still pins only one mode), and therefore of the field \( \mathcal{H}_c \), in the correlation functions should vanish gradually. The dependence of the correlation functions on the magnetic field at \( \mathcal{H} < \mathcal{H}_c \) becomes power law rather than logarithmic; however, this power law is still non-trivial, and approaches Fermi-liquid results only in the limit of an infinite number of channels.

If \( \mathcal{H}_\pm \gg \mathcal{H}_c \), the correlation field \( \mathcal{H}_c \) is given by Eq. (2.23). The correlation function of the fluctuation starts decreasing much faster, \( \propto 1/\mathcal{H}_c^4 \). Therefore, in order to get the representative statistics of the capacitance fluctuations, averaging should be performed in the interval of magnetic fields larger than the magnetic field determined by the charging energy (2.23). Finally, in the limit of a strong field, \( \mathcal{H}_1 = \mathcal{H}_2 \approx \mathcal{H}_c \) (unitary limit), the variance \( \delta C_{V\Omega}^2 \) of the differential capacitance becomes four times smaller than at \( \mathcal{H} = 0 \).

Our results for the correlation functions diverge logarithmically at \( T \rightarrow 0 \). At lower temperatures the pinning of the spin mode described by action \( \mathcal{S} \) should be taken into account. A variational estimate \[ \mathcal{E} \] shows that \( T \) at low temperatures should be replaced by \( \Delta \ln (E_C/\mathcal{D}) \) in the above results for \( \mathcal{H}_c^\pm \) and for the correlation function. We will elaborate on this point more in the end of the following subsection.
2. Finite reflection in the contact

The main effect of the backscattering in the channel is that the Coulomb blockade appears already in the averaged capacitance. Taking into account backscattering Hamiltonian (1.28) in the second order perturbation theory, we obtain from Eqs. (3.19):

$$\delta C(N) = \frac{8e^C}{\pi} |r|^2 \cos 2\pi N \ln \left( \frac{E_C}{T} \right).$$  \hspace{1cm} (5.17)

Due to the finite level spacing $\Delta$, this result acquires mesoscopic fluctuations. As we already mentioned in the introduction to this section, at $r \neq 0$ the leading term in fluctuations of the thermodynamic potential is first-order in both $H_{bs}$ and $\delta$,

$$\delta \Omega = -T \langle \tau \rangle \int_0^B d\tau \delta H_{bs}(\tau)\hat{S}. \hspace{1cm} (5.18)$$

To calculate the average over the unperturbed state, we use the bosonized representation of $\hat{H}_{bs}$ and $\hat{S}$ given by Eqs. (1.28) and (1.29) respectively, and then the expressions (3.30) for the correlation functions of the boson fields. Then, similar to the Section V.A, we switch to the Lehmann representation (1.24) for the kernel $L(\tau)$ to obtain:

$$\delta \Omega = \frac{2\sqrt{e^C} |r| \sqrt{E_C}}{\pi^2 \sqrt{\pi v_F}} \cos(\pi N) \int_{-\infty}^\infty dt \left( L^R(t) - L^A(t) \right) \times$$

$$\int_0^B d\tau d\tau_2 (\pi T)^2 \times$$

$$\frac{1}{\sin[\pi T(t + iT(\tau_1 - \tau_2))] \sin[\pi T(\tau_1) \sin(\pi T \tau_2)]^{1/2}} \times$$

$$\sum_{\gamma=\pm 1} [\sin \pi T(\tau_2 - \tau_1 + i\gamma 0)]^{1/2} e^{-i\pi(\gamma(N-1/4))}. \hspace{1cm} (5.19)$$

Here only the $N$-dependent part of the thermodynamic potential is taken into account. Integral over $\tau_1$ in Eq. (5.19) is determined by the contribution of the pole at $\tau_1 = \tau_2 - it$, which can be easily calculated:

$$\delta \Omega = \frac{2\sqrt{e^C} |r| \sqrt{E_C}}{\pi^2 \sqrt{\pi v_F}} \cos(\pi N) \int_{-\infty}^\infty dt \left( \pi T \right)^{1/2} \times$$

$$\left[ L^A(t) e^{i\pi(N-1/4)} + L^R(t) e^{-i\pi(N-1/4)} \right] \times$$

$$\int_0^B d\tau_2 \frac{\pi T}{\sin[\pi T^2 \tau_2 \sin(\pi T \tau_2 + it)]^{1/2}}. \hspace{1cm} (5.20)$$

Integration over $\tau_2$ now is completely similar to the one we performed over the variable $\tau_3$ in Eq. (5.12), and we find for $\delta \Omega$:

$$\delta \Omega = \frac{2\sqrt{e^C} |r| \sqrt{E_C}}{\pi^2 \sqrt{\pi v_F}} \cos(\pi N) \int_{-\infty}^\infty \left( \pi T \right)^{1/2} \ln(1/T^2 t^2) dt \times$$

$$\left[ L^A(t) e^{i\pi N} + L^R(t) e^{-i\pi N} \right]. \hspace{1cm} (5.21)$$

From Eq. (5.21), with the help of Eq. (3.25) and (3.19), we find the correlation function of mesoscopic fluctuation of the capacitances:

$$\frac{\delta C_{diff}(1) \delta C_{diff}(2)}{C^2} = \frac{32e^C |r|^2 \Delta}{3\pi^2} \ln^3 \left( \frac{E_C}{T} \right) \cos 2\pi n \times$$

$$\sum_{\gamma=\pm 1} \left[ \ln \left( \frac{1}{\ln \left( \frac{E_C}{T} \right)} \right) \right]^{3}, \hspace{1cm} (5.22)$$

where $i = H_i, n = N_1 - N_2$, and $H_{\pm} = H_1 \pm H_2$. Correlation field $H_{\pm}$ is defined in Eq. (6.16). The amplitude of fluctuations at a partial transmission of the channel is parametrically larger than at $r = 0$, cf Eq. (5.15). Furthermore, in the unitary limit the variance of the differential capacitance is suppressed only by a half of its zero-field value. This similarity with the case of spinless fermions is due to the backscattering in the channel, which leads to pinning of the spin mode.

Result (5.22) is valid at relatively high temperatures. As was shown by Matveev, the divergences should be cut at energy $\epsilon \sim |r|^2 E_C \cos^2 \pi N$ which corresponds to the pinning energy of the spin mode. The higher-order corrections in backscattering $\delta$ shows that at $T \lesssim |r|^2 E_C$ the logarithmic growth of fluctuations saturates. Simultaneously, the correlation functions start to depend not only on the difference $N_1 - N_2$, but also on each of these arguments separately. This weaker logarithmic dependence is beyond the scope of this paper. For an estimate of the differential capacitance variance at low temperature, one may replace $\ln(E_C/T)$ by $\ln(1/|r|^2)$ in Eq. (5.22).

Finally, we elaborate on the estimate of the characteristic energy scale which controls the low-temperature cutoff for the reflectionless contact. Comparing Eq. (5.17) with Eq. (5.21), or Eq. (5.21) with Eq. (5.15), we observe that the reflectionless case results can be obtained from formulas with finite reflection coefficient by putting $|r|^2 \rightarrow (\Delta/E_C) \ln(E_C/\Delta)$. It would correspond to the energy of pinning of the spin mode $\epsilon^* \rightarrow \Delta \ln(E_C/\Delta)$ which agrees with our variational estimate $E_C/\Delta$.

VI. TUNNELING CONDUCTANCE OF THE "OPEN" DOT

In the previous Section, we considered in detail the thermodynamics of the dot with one almost open channel, and studied mesoscopic effects related to the discreteness of the charge. However, the majority of experimental work deals not with thermodynamics, but rather with transport through a dot. Coulomb blockade shows up as an oscillatory gate voltage dependence of the conductance of the dot connected with two leads. The case of small transparency of the channels connecting the dot with leads is well studied. The con-
ductance in the valleys can be represented as the sum of two physically different contributions – elastic and inelastic cotunneling.\footnote{21} During the elastic cotunneling process, an electron enters the dot, spends there a time $\approx \hbar/E$, and then leaves the dot. Without interaction, this electron would be able to spend a time of the order of $\hbar/\Delta$. As the result, the conductance in the Coulomb blockade regime further, it is natural to expect that there are two physically different contributions – elastic and inelastic cotunneling.\footnote{22} In the limit $\Delta$ smaller than $E_C$, the strongly asymmetric limit corresponds to the fixed point of a system with an infinitesimally small initial asymmetry. Therefore we can expect that at $\Delta/E_C \ll 1$, this limit adequately describes dots with a finite initial degree of asymmetry.

\section{A. General formalism}

For calculation of such tunneling conductance, we have to modify derivation of Secs. III\textbf{A} and III\textbf{B} in order to take into account the tunneling between the dot and the second lead. In comparison with Hamiltonian (3.1), the total Hamiltonian of the system acquires two additional terms,

$$\hat{H} = \hat{H}_F + \hat{H}_C + \hat{H}_M + \hat{H}_T.$$  (6.1)

Here $\hat{H}_F$ describes the electron motion in the dot and in the left lead and is given by (3.2), interaction Hamiltonian $\hat{H}_C$ is given by Eq. (1.1), and $\hat{H}_M$ is the Hamiltonian of free electrons in the right lead

$$\hat{H}_M = \sum_p \xi_p \hat{a}^\dagger_p \hat{a}_p.$$  (6.2)

Tunneling Hamiltonian $\hat{H}_T$ describes the weak coupling between the right lead and the dot,

$$\hat{H}_T = v_t \hat{\psi}^\dagger (\mathbf{r}_t) \sum_p \hat{a}_p + h.c.,$$  (6.3)

where $\mathbf{r}_t$ is the coordinate of the tunneling contact, $\hat{\psi}^\dagger (\mathbf{r})$ describes the weak coupling between the right lead and the dot.

Because $G_R \ll e^2/(2\pi\hbar)$, we can consider the tunneling current $I$ as the function of applied voltage $V$ in the second order of perturbation theory in tunneling Hamiltonian (6.3). This gives us the standard result\footnote{23}

$$I(V) = i |J(\iota\Omega_n \rightarrow eV + i0) - J(\iota\Omega_n \rightarrow eV - i0)|,$$  (6.4)

where $\Omega_n = 2\pi T n$ is the bosonic Matsubara frequency, and Matsubara current $J$ is defined as

$$J(\iota\Omega_n) = ev_t^2 \nu \int_0^\beta d\tau e^{-\iota\Omega_n \tau} G_M(\tau) \Pi(\tau).$$  (6.5)

Here $\nu$ is the one-electron density of states per unit area and per one spin in the dot, $G_M$ is the Green function of the electrons in the leads,

$$G_M \equiv - \sum_{\mu_1, \mu_2} \langle T_\tau \hat{a}_{\mu_1}(\tau) \hat{a}^\dagger_{\mu_2}(0) \rangle = \nu_M \frac{\pi T}{\sin \pi T \tau}.$$  (6.6)
with $\nu_M$ being the one-electron density of states per one spin in the lead, and function $\Pi(\tau)$ is given by

$$\Pi(\tau) = \nu^{-1}(T, \bar{\psi}(\tau; r)) \psi(0; r)). \quad (6.7)$$

Averages in Eqs. (6.6) and (6.7) are performed with respect to the equilibrium distribution of the system without tunneling. We choose to introduce $\nu$ into Eq. (6.4) and into definition (6.4) to make function $\Pi(\tau)$ dimensionless.

In the absence of the interaction, $E_C = 0$, propagator $\Pi(\tau)$ is nothing but the Green function of non-interacting system; its ensemble average has the form analogous to Eq. (6.6),

$$\Pi(\tau) \bigg|_{E_C=0} = \frac{\pi T}{\sin \pi T \tau}. \quad (6.8)$$

Then, substitution of Eqs. (6.6) and (6.8) into Eq. (6.5) and analytic continuation (6.4) gives the tunneling current $I = s G_R V$ ($s$ is the spin degeneracy), where the tunneling conductance of the contact per one spin is

$$G_R = \frac{2\pi e^2}{h} v_{t}^2 \nu_M \nu. \quad (6.9)$$

With the help of Eq. (6.9) we can rewrite Eq. (1.5) in terms of the bare conductance of the point contact:

$$J(i\Omega_n) = \frac{G_R}{2 \pi e} \int_0^\beta dt \frac{\pi T e^{-i\Omega_n \tau}}{\sin \pi T \tau} \Pi(\tau). \quad (6.10)$$

As we will see below, function $\Pi(\tau)$ can be analytically continued from the real axis to the complex plane, so that the result is analytic in a strip $0 < \text{Re} \tau < \beta$, and has branch cuts along $\text{Re} \tau = 0, \beta$ lines. It allows one to deform the contour of integration as shown in Fig. 4, and to obtain

$$J(i\Omega_n) = \frac{G_R T}{2e} \int_{-\infty}^{\infty} dt e^{i\Omega_n t} \times \left( \frac{\Pi(it+0)}{\sinh \pi T (t-i0)} - \frac{\Pi(it-0)}{\sinh \pi T (t+i0)} \right). \quad (6.11)$$

Now the analytic continuation (6.4) can be performed, because the periodicity of the Matsubara Green functions was already taken into account. This gives

$$I(eV) = i \frac{G_R T}{2e} \int_{-\infty}^{\infty} dt e^{-i e V t} \times \left[ \frac{\Pi(it+0)}{\sinh \pi T (t-i0)} - \frac{\Pi(it-0)}{\sinh \pi T (t+i0)} \right]. \quad (6.12)$$

Next, we use the analyticity of $\Pi(\tau)$ in the strip $0 < \text{Re} \tau < \beta$, and shift the integration variable $t \rightarrow t - i \beta/2$ in the first term in brackets in Eq. (6.12), and $t \rightarrow t + i \beta/2$ in the second term. Bearing in mind that $\Pi(\tau) = -\Pi(\tau + \beta)$, we find

$$I = \left(T \sinh \frac{eV}{2T} \right) G_R \int_{-\infty}^{\infty} dt e^{-i e V t} \frac{\Pi(it + \frac{\beta}{2})}{\cosh \pi T t}. \quad (6.13)$$

Linear conductance $G$ is therefore given by

$$G = G_R \int_{-\infty}^{\infty} dt \frac{\Pi(it + \frac{\beta}{2})}{2 \cosh \pi T t}. \quad (6.14)$$

FIG. 6. The integration contour used in the evaluation of the conductance, see Eq. (6.11) for (a) $\Omega_n < 0$, and (b) for $\Omega_n > 0$. Branch cuts of the analytic continuation of $\Pi(\tau)$ are shown by thick lines.

Let us turn now to the actual calculation of the function $\Pi(\tau)$. It was shown in Ref. 27 that the interaction drastically affects the form of the function $\Pi(\tau)$, however, some contributions were not taken into account. Our purpose is to construct an effective action theory, similar to that of Sec. IV, for calculation of $\Pi(\tau)$. Once again, we wish to get rid of the fermionic degrees of freedom of the dot. Similar to Eq. (3.4), it is convenient to rewrite charge operator in terms of the variables of the channel. However, here we have to keep in mind the fact that the tunneling events described by operators $\hat{\psi}^\dagger(r_i)$ and $\hat{\psi}(r_i)$ change the charge in the system by $+e$ and $-e$. It can be taken into account by introducing three additional operators: Hermitian operator $\hat{n}$, and unitary operators $\hat{F}, \hat{F}^\dagger$ with the following commutation relations:

$$[\hat{n}, \hat{F}^\dagger] = \hat{F}.$$

We can definitely choose Hilbert subspace in a way such that operator $\hat{n}$ has integer eigenvalues. Finally, these operators commute with all the fermionic degrees of freedom. Then, we can change the definition of the charge operator [cf. Eq. (3.4)] to

$$\frac{\hat{Q}}{e} = -\int_{\text{channel}} d\mathbf{r} \psi^\dagger \psi + \hat{n}, \quad (6.16)$$

and rewrite Eq. (6.7) as

$$\Pi(\tau) = \nu^{-1}(T, \hat{F}(\tau) \bar{\psi}(r) \hat{F}(0) \psi(0; r_i)). \quad (6.17)$$

It is easy to see from Eqs. (6.16) and (6.15) that operators $\hat{F}, \hat{F}^\dagger$ in Eq. (6.17) change the charge by $+e$ and $-e$ respectively, in accordance with the initial definition of charge.

After this manipulation, the Hamiltonian of the system and correlation function (6.17) become quadratic in the fermionic operators of the dot, so that part of the system can be integrated out. We use the identity similar to Eq. (3.6):
where all $\psi_1^r, \psi_2$ are the fermionic operators of the dot, and the rest of the notation is the same as that in Eq. (3.6).

The calculation of the product $\langle \hat{H}_{12}(\tau_1)\hat{H}_{12}(\tau_2) \rangle_2$ was performed in Sec. IIIA, see Eq. (3.7), and all the steps leading to the derivation of the effective action (3.14) can be repeated here. Calculation of the remaining operator products can be performed along the lines of Appendix A. This yields

$$
\langle T_r\hat{\psi}(\tau_1; r_1)\hat{H}_{12}(\tau_2) \rangle_2 = -\psi(\tau_2, 0) R^*(\tau_2 - \tau_1), \tag{6.19a}
$$

$$
\langle T_r\hat{\psi}(\tau_1; r_1)\hat{H}_{12}(\tau_2) \rangle_2 = -\bar{\psi}(\tau_2, 0) R(\tau_2 - \tau_1), \tag{6.19b}
$$

where, similar to Eq. (3.7), $\psi(x, r) = e^{\tau H_1}\psi(x)e^{-\tau H_1}$ are the one dimensional fermionic operators of the channel in the interaction representation, $\bar{\psi}(\tau) = \bar{\psi}^\dagger(-\tau)$. Kernel $R$ describes the motion of an electron from the tunnel contact to the entrance of the single mode channel, and it is given by

$$
R(\tau) = \frac{1}{2m} \int dy \phi(y) \partial_x \hat{G}(r; r, r_t). \tag{6.20}
$$

Here, $\hat{G}$ is the exact Matsubara Green function of the closed dot subjected to the zero boundary condition. The wave function $\phi(y)$ describes the transverse motion in the single-mode channel, and the coordinates $x$ in the derivative of the Green function $\hat{G}$ is set to $+0$.

Kernel $R(\tau)$ is the random quantity with the zero averages. In the universal regime, products of retarded $R^R(t)$ and advanced $R^A(t)$ counterparts of $R(\tau)$ entering into the Lehmann representation (6.20) have the following non-vanishing averages:

$$
\frac{1}{\nu} \langle R^R_{H_1}(t_1)R^A_{H_2}(t_2) \rangle = \Delta v_F \delta(t_1 + t_2) \theta(t_1)e^{-\gamma_1/\nu}, \tag{6.21a}
$$

$$
\frac{1}{\nu} \langle R^R_{H_1}(t_1) [R^A_{H_2}(t_2)]^\dagger \rangle = \Delta v_F \delta(t_1 - t_2) \theta(t_1)e^{-\gamma_1/\nu}, \tag{6.21b}
$$

where the decay times $\gamma^{C, D}_{H_1, 2}$ associated with applied magnetic fields $H_{1, 2}$ are given by Eq. (3.24). All the higher momenta can be found by using the Wick theorem. Deriving Eq. (6.21a) we use Eqs. (3.21, 23), and the identity $\hat{G}^R(t; r_1, r_2) = [\hat{G}^A(-t; r_2, r_1)]^\dagger$.

To complete the derivation of the effective theory, we use Eqs. (6.18) and (6.19), introduce left and right moving fermions similarly to Sec. IIIA, and thus obtain the effective action representation for $\Pi(\tau)$ from Eq. (5.17):

$$
\Pi(\tau) = \Pi_{in}(\tau) + \Pi_{el}(\tau); \tag{6.22a}
$$

$$
\Pi_{in} = \frac{\hat{G}(-\tau; r, r_t)}{\nu(\tau e^{-\hat{S}})} \langle T_r\hat{\psi}(\tau) \hat{F}(\tau) \hat{F}(0) \rangle; \tag{6.22b}
$$

$$
\Pi_{el} = \frac{1}{\nu(\tau e^{-\hat{S}})} \int_0^\beta d\tau d\tau_r R(\tau_1 - \tau_2) R^*(\tau_2 - \tau_1) \times \tag{6.22c}
$$

$$
\langle T_r e^{-\hat{S}} \hat{F}(\tau) \hat{F}(0) [\hat{\psi}_L(\tau_1) + \hat{\psi}_R(\tau_1)] [\hat{\psi}_L(\tau_2) + \hat{\psi}_R(\tau_2)] \rangle.
$$

Here the averaging is performed with respect to the Hamiltonian

$$
\hat{H}_0 = iv_F \int_{-\infty}^\infty dx \left\{ \psi_L^\dagger \partial_x \psi_L - \psi_R^\dagger \partial_x \psi_R \right\} + \frac{E_C}{2} \left( \int_{-\infty}^0 dx : \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R : + N - \hat{n} \right)^2,
$$

and action $\hat{S}$ is given by Eq. (3.16). The difference between the Eq. (6.23) and Eq. (5.17), is caused by the different definitions of the charge operator in Eqs. (3.4) and (5.16).

Two contributions can be distinguished in the correlation function (6.22). Inelastic contribution (6.22b) was considered in Ref. 26 in the approximation corresponding to $\hat{S} = 0$, and with the Green function of the dot $\hat{G}$ replaced by its averaged value $\bar{\hat{G}}$. The obtained results vanish at low temperatures. The reason for the vanishing is that this term does not allow the introduced electron to leave the dot; charge of the dot at the moment of tunneling suddenly changes by $+e$ and all the other electrons have to redistribute themselves to accommodate this charge. The logarithmical divergence of the imaginary time action corresponding to such evolution (orthogonality catastrophe) completely suppresses this contribution at $T \to 0$. Conversely, the second contribution, $\Pi_{el}$ from Eq. (6.22c), contains the kernel $R(\tau)$ which promotes an electron from the tunneling contact to the channel. Because the very same tunneling electron is introduced to and then removed from the dot, there is no need in the redistribution of other electrons, so no orthogonality catastrophe occurs. As the result, the elastic...
contribution survives at $T \to 0$, analogously to the elastic cotunneling contribution for the weak coupling regime.

In what follows, we will be interested in the low temperature behavior of the system, so we will retain elastic contribution \( \text{(6.22)} \) only. Similarly to the Sec. IV results for electrons with spin and spinless electrons differ significantly, and we will consider those two cases separately.

### B. Spinless electrons

We follow the lines of Sec. IV A in the bosonization of the chiral fermionic fields. In order to account for the appearance of the operator $\hat{n}$ in the Hamiltonian [compare Eqs. (4.15) with Eq. (6.23)], we change slightly the transformation \( \text{(4.9)} \):

\[
\hat{\phi}_L(x) = \frac{\hat{\phi}_+(x) + \hat{\phi}_-(x) + \hat{\phi}}{\sqrt{2}} - \pi \mathcal{N} + \pi \hat{n};
\]
\[
\hat{\phi}_R(x) = \frac{\hat{\phi}_+(-x) - \hat{\phi}_-(-x) - \hat{\phi}}{\sqrt{2}} - \pi \mathcal{N} + \pi \hat{n},
\]

(6.24)

where operator $\hat{n}$ commutes with the bosonic fields $\hat{\phi}_\pm, \hat{\phi}$. In order to preserve the commutation relation $[\hat{F}^\dagger, \hat{\phi}_{L,R}] = 0$, we change the operator $\hat{F}^\dagger$ as

\[
\hat{F}^\dagger \mapsto \hat{F}^\dagger e^{-i\sqrt{2} \hat{\psi}_F}, \hat{F} \mapsto \hat{F} e^{i\sqrt{2} \hat{\psi}_F},
\]

(6.25)

The fact that $\hat{F}^\dagger$ commutes with bosonic fields $\hat{\phi}_{L,R}$ is obvious from Eqs. (6.14) and Eq. (4.8d). Substitution of Eqs. (4.1), (6.24) and (6.25) into Eqs. (6.22b) yields

\[
\Pi_{in}(\tau) = \frac{G(-\tau, \tau_1, \tau_2)}{\nu(T, e^{-}\hat{S}_F(\tau) F(\tau) F \hat{F}(0) e^{i\sqrt{2} \hat{\phi}(0) - \hat{\phi}(\tau)})} \times
\]

(6.26a)

for the inelastic part of the cotunneling, see also Ref. [27].

For the elastic contribution, we find

\[
\Pi_{el}(\tau) = \frac{2}{\pi \nu \lambda (T, e^{-}\hat{S}_F)} \int_0^\beta d\tau_1 d\tau_2 R(\tau_1 - \tau) R^*(\tau_2) \times
\]

(6.26b)

\[
\langle T_r e^{-\hat{S}_F(\tau) F(0) e^{i\sqrt{2} \hat{\phi}(0) - \hat{\phi}(\tau)}} \rangle \times
\]

\[
\hat{n}(\tau_1) \hat{n}(\tau_2)(-1)^{\hat{n}(\tau_1) + \hat{n}(\tau_2)} \exp \left[ i \frac{\hat{\phi}_-(\tau_1) - \hat{\phi}_-(\tau_2)}{\sqrt{2}} \right] \times
\]

\[
\prod_{i=1}^2 \cos \left[ \frac{\hat{\phi}_x(\tau_i)}{\sqrt{2}} + \frac{\pi}{4} - \pi \mathcal{N} \right].
\]

Averaging in Eqs. (6.26) is performed over the Hamiltonian given by Eq. (6.11), finite backscattering is described by \( \text{(1.11)} \), and action (4.12) is modified as

\[
\hat{S} = \frac{1}{2 \pi \lambda} \int_0^\beta d\tau_1 d\tau_2 L(\tau_1 - \tau_2) \hat{n}(\tau_1) \hat{n}(\tau_2)(-1)^{\hat{n}(\tau_1) + \hat{n}(\tau_2)} \times
\]

\[
\exp \left[ i \frac{\hat{\phi}_-(\tau_1) - \hat{\phi}_-(\tau_2)}{\sqrt{2}} \right] \times
\]

\[
\cos \left[ \frac{\hat{\phi}_+(\tau_1)}{\sqrt{2}} + \frac{\pi}{4} - \pi \mathcal{N} \right] \cos \left[ \frac{\hat{\phi}_+(\tau_2)}{\sqrt{2}} + \frac{\pi}{4} - \pi \mathcal{N} \right].
\]

We will consider only the elastic contribution \( \text{(6.26)} \) because it does not vanish at low temperatures.

#### 1. Reflectionless contact

In the lowest in $\Delta/E_C$ approximation we can neglect action $\hat{S}$ in Eq. (6.26) at all. Then averaging over bosonic fields can be performed with the help of Eqs. (1.14) and Eq. (1.14), average of the product Majorana fermions operator is given by Eq. (1.13), and the relevant correlation function of the operators $\hat{n}, \hat{F}, \hat{F}^\dagger$ is given by

\[
\langle T_r \hat{F}(\tau) F(\tau) e^{i\sqrt{2} \hat{\phi}(0) - \hat{\phi}(\tau)} \rangle = sgn(\tau - \tau_1)sgn(\tau - \tau_2).
\]

(6.27)

Equation (6.27) follows from Eq. (1.14) and from the fact that operators $\hat{n}, \hat{F}$ commute with Hamiltonian (4.11) and thus do not have their own dynamics. We obtain

\[
\Pi_{el}(\tau) = \frac{2\pi |K(\tau)|^2}{\nu v_F E_C e^{\frac{\pi T}{\nu v_F E_C}}} \int_0^\beta d\tau_1 d\tau_2 R(\tau_1 - \tau) R^*(\tau_2) \times
\]

\[
\left[ \frac{\pi T}{\nu \pi T (\tau_1 - \tau_2)} \frac{K(\tau_2) K(\tau_1 - \tau)}{K(\tau_1) K(\tau_2 - \tau)} + \frac{\pi T}{\nu \pi T (\tau_1 - \tau_2)} \frac{K(\tau_1) K(\tau_2 - \tau)}{K(\tau_1) K(\tau_2 - \tau)} + c.c. \right],
\]

(6.28)

where function $K(\tau)$ is defined by Eq. (4.16d), and $C$ is the Euler constant.

![FIG. 7. The integration contour used in the evaluation of the conductance in the spinless case, for the first (a) and the second term (b) in Eq. (6.29). Branch cuts of the integrand are shown by thick lines. Contribution of the semi-pole at $\tau_1 = \tau_2$ in the first term in brackets in Eq. (6.29) is canceled by the term complex conjugated to it.](image)
Before performing the analytic continuation, see Eq. (6.14), we have to transform integrals over imaginary times in Eq. (6.28) to the integrals over real time. In order to do so, we use Lehmann representation (3.20) for the kernel $R(\tau)$:

$$\Pi_{el} = \frac{2\pi |K(\tau)|^2}{\nu v_F E_C^2 e^{2C}} \int \frac{dt_1 dt_2}{2\pi} \left( R(t_1) - R^A(t_1) \right) \left( R(t_2) - R^A(t_2) \right)^* \times$$

$$\int_0^\beta d\tau_1 d\tau_2 \frac{\pi^2 t^2}{\sin[\pi T(t_1 + i\tau_1 - i\tau)] \sin[\pi T(t_2 + i\tau_2)]} \times$$

$$\left[ \frac{\pi T}{\sin[\pi T(\tau - \tau_2 + it_1)]} \right] \left( \frac{R(t_1)K(\tau_1 + \beta)K(\tau_2)K(\tau - \tau_2)}{K(it_1 + \tau)K(it_2 + \tau)} + e^{i2\pi N_K(\tau - \tau_2)K(\tau_1 + \beta)K(\tau - \tau_2) + c.c} \right).$$

Integration can be now performed in a manner similar to Sec. A. Using the fact that function $K(\tau)$ is analytical within the lower complex semiplane $\text{Im} \tau < 0$, we deform the contour of integration as shown in Fig. 7.

Because of the periodicity of the integrand, the integrals over the parts of the contour running parallel to the imaginary axis, cancel out. As the result, only the pole contribution at $\tau_1 = it_1 + \tau$ remains at $t_1 < 0$. At $t_1 > 0$ the pole contribution disappears. Analogously, the complex conjugated terms are contributed by pole $\tau_1 = it_1 + \tau$ at $t_1 > 0$. As the result, we obtain from Eq. (6.29)

$$\Pi_{el} = \frac{2\pi |K(\tau)|^2}{\nu v_F E_C^2 e^{2C}} \int \frac{dt_1 dt_2}{2\pi} \left( R(t_1) - R^A(t_1) \right) \left( R(t_2) - R^A(t_2) \right)^* \times$$

$$\int_0^\beta d\tau_2 \frac{\pi T}{\sin[\pi T(t_2 + i\tau_2)]} \times$$

$$\left[ \frac{\pi T}{\sin[\pi T(\tau - \tau_2 + it_1)]} \right] \left( \frac{R^A(t_1)K(\tau_1 + \beta)K(\tau_2)K(\tau - \tau_2)}{K(it_1 + \tau)K(it_2 + \tau)} + e^{i2\pi N_K(\tau - \tau_2)K(\tau_1 + \beta)K(\tau - \tau_2) + c.c} \right).$$

where we wrote explicit expressions for all the terms. Integration over $\tau_2$ can be now easily performed by deformation of the integration contours shown in Fig. 8 and we obtain

$$\Pi_{el} = \frac{2\pi |K(\tau)|^2}{\nu v_F E_C^2 e^{2C}} \int \frac{dt_1 dt_2}{2\pi} \left( R(t_1)R(t_2) + R^A(t_1)R^A(t_2) \right) \times$$

$$\int_0^\beta d\tau_2 \frac{\pi T}{\sin[\pi T(it_2 - it_1 + \tau)]} \times$$

$$\left[ \frac{\pi T}{\sin[\pi T(\tau - it_2 + it_1)]} \right] \left( \frac{R^A(t_1)K(it_1 + \tau)K(it_2 + \tau)}{K(it_1 + \tau + \beta)K(it_2 + \tau + \beta)} + e^{i2\pi N_K(it_1 + \tau)K(it_2 + \tau) + c.c} \right).$$

Finally, we substitute Eq. (6.31) into Eq. (6.14). For small temperatures $T \ll E_C$, we have from Eq. (4.16d) $K(\beta/2 + it) = \pi T/ \cosh \pi T t$. As the result, the first term in Eq. (6.31) produces a contribution $\propto T^2$ and can be neglected. The remainder can be recast into the formula

$$G = \frac{2\pi G_R}{\nu v_F E_C^2 e^{2C}} \left| \int_{-\infty}^0 dt K(it) \left[ R^A(t)e^{i2\pi N_K} + R(t) \right] \right|^2,$$

which gives non-averaged conductance of the dot. Here we used the fact that the characteristic scale of the integration over $t_1,t_2 \sim 1/E_C$ is much smaller than $\beta$. Equation (6.32) is reminiscent of the Landauer formula. However, the form-factor $K(it)$ entering into this formula indicates that a large number of states in the dot participate in the transport, unlike the case of non-interacting electrons.

![Fig. 8. The integration contours used in the evaluation of the integral over $\tau_2$ in Eq. (6.30) for (a) the first and third terms in brackets, and (b) for the second and fourth term. Only first and second term have poles at $\tau_2 = \tau + it$.](image-url)

One can see from Eq. (6.31) that function $\Pi_{el}(\tau)$ is indeed analytic in the strip $0 < \text{Re} \tau < \beta$, which justifies the steps leading to Eq. (6.14).
Now, we are prepared to study the statistics of the conductance. Using the explicit expression (4.16b) for function \( K \) and formula (6.21b), we find the average conductance
\[
\overline{G} = G_R \frac{2 \Delta}{\varepsilon_C} e^{-2 C \Lambda(0)}, \quad (6.33)
\]
where \( C \approx 0.577 \) is the Euler constant, and \( \Lambda(0) \approx 1.398 \) is given by Eq. (5.4). This expression is analogous to the elastic cotunneling for the case of weak coupling.

For the correlation function of the mesoscopic fluctuations of the conductance, we find with the help of Eqs. (6.24)
\[
\frac{\delta G(1) \delta G(2)}{G^2} = \left( \cos \frac{\pi n}{\Lambda(0)} \right)^2 \left[ \Lambda^2 \left( \frac{H_2^2}{H_c^2} \right) + \Lambda^2 \left( \frac{H_3^2}{H_c^2} \right) \right], \quad (6.34)
\]
where we use again the short hand notations \( i = N_i, H_i, \ n = N_1 - N_2, \) and \( H_\pm = H_1 \pm H_2. \) Correlation magnetic field \( H_c \) is defined in Eq. (2.23), and the dimensionless function \( \Lambda(x) \) is given by Eq. (5.4) and is plotted in Fig. 4. Once again, we see that even though the averaged conductance does not any longer oscillate with the gate voltage, the discreteness of charge manifests itself in the oscillatory behavior of the conductance correlation function. It is also noteworthy that the mesoscopic conductance fluctuations are of the order of the average, similarly to the weak coupling regime.

2. Finite reflection in the contact.

So far, we have shown that the conductance in the tunneling setup is non-vanishing at \( T \rightarrow 0 \), which is analogous to the elastic cotunneling in the weak tunneling regime. However, the oscillatory dependence of the conductance showed up not in the average conductance but rather in the correlation function of mesoscopic fluctuations. On the other hand, as we saw in Sec. V finite backscattering leads to the oscillatory dependence in the averaged quantities. The purpose of this subsection is to study how the finite reflection in the contact affects the elastic cotunneling, and to demonstrate that it indeed leads to the oscillatory dependence of the averaged conductance on the gate voltage.

To treat the finite reflection in the contact, we have to expand the denominator and numerator in Eq. (6.26b), up to the first order in the backscattering Hamiltonian (4.11). Performing averaging over the bosonic fields, we obtain with the help of Eqs. (4.14), (4.8), (6.27), and (4.11):

\[
\Pi_{\tau}(\tau) = -\left| r \right| \frac{|K(\tau)|^2}{2 \pi \nu \nu_F \varepsilon_C C} \int_0^\beta d\tau_1 d\tau_2 d\tau_3 R(\tau_1 - \tau) R^*(\tau_2 - \tau_3) \frac{\pi T}{\sin \pi T(\tau_1 - \tau_2)} \times (6.35)
\]
\[
\left\{ \frac{K(\tau_2)K(\tau_1 - \tau)}{K(\tau_1)K(\tau_2 - \tau)} \left( e^{-i2\pi N} K(\tau_3)K(\tau_3 - \tau) \sin \pi T(\tau_2 - \tau_3)K(\tau_2 - \tau_3) \right) + \frac{e^{i2\pi N} K(-\tau_3)K(\tau_3 - \tau)}{K(\tau_3)K(\tau - \tau_3)} \sin \pi T(\tau_2 - \tau_3)K(\tau_2 - \tau_3) \right) - 2 \cos 2 \pi N \right\} + c.c.
\]

Here we retained only the terms which do not vanish after ensemble averaging. Then, we can use the Lehmann representation for the kernel \( R(\tau) \) and perform the integration over \( \tau_1, \tau_2 \) in the manner of the previous subsection. It yields

\[
\Pi_{\tau}(\tau) = -\left| r \right| \frac{|K(\tau)|^2}{2 \pi \nu \nu_F \varepsilon_C C} \int dt_1 dt_2 \int_0^\beta d\tau_3 \frac{\pi T}{\sin \pi T(\tau - it_2 + it_1)} \times (6.36)
\]
\[
\left\{ \frac{e^{-i2\pi N} K(\tau_3)K(\tau - \tau_3)}{K(\tau_3)K(\tau - \tau_3)} \sin \pi T(\tau - it_2 - it_1)K(\tau + it_2 - \tau_3) \right\} + c.c.
\]

part of the ensemble averaged conductance
\[
G = \alpha_1 G_R \left( \frac{|r| \Delta}{\varepsilon_C} \right) \cos 2 \pi N. \quad (6.38)
\]

Here \( \alpha_1 \) is the numerical coefficient given by
\begin{align}
\alpha_1 &= \frac{4e^{-\mathcal{C}}}{\pi} \int_0^\infty \frac{dx}{x^2} e^{2\pi x E(x)} \sin(\pi x) \times \\
\sinh [e^{-y} E(y) + e^{x+y} E(-x-y) - e^{-y} E(-y)] &\approx 1.458,
\end{align}

with \( E(x) \) being the exponential-integral function.4

Equation (6.38) confirms our expectation that the finite backscattering leads to the oscillatory dependence of the averaged conductance on the gate voltage. Although the amplitude of oscillations (6.38) is small compared to the averaged value of the conductance (6.33), it still exceeds at low temperatures the contribution of the inelastic cotunneling to the conductance oscillations.

FIG. 9. The integration contour used in the evaluation of the integral over \( \tau_1 \) in Eq. (6.39). Integral along the part of the contour parallel to the real axis cancels out the \( 2 \cos 2\pi N \) term.

\begin{align}
\Pi_{\text{in}}(\tau) &= -\frac{2G(\rho,\sigma)}{\nu(T) e^{-S}} \langle T_+ e^{-S} \hat{F}(\tau) \hat{F}(0) e^{i\hat{\phi}(0)-\hat{\phi}(\tau)} \rangle, \\
\Pi_{\text{el}}(\tau) &= \frac{1}{\pi \nu \lambda(T) e^{-S}} \int_0^\beta d\tau_1 d\tau_2 R(\tau_1 - \tau) R^*(\tau_2) \sum_{\alpha=\pm 1} \langle T_+ e^{-S} \hat{F}(\tau) \hat{F}(0) \hat{\eta}_\alpha(\tau_1) \hat{\eta}_\alpha(\tau_2) e^{i\hat{\phi}(0)-\hat{\phi}(\tau)} \rangle \times \\
&\quad \left\{ \sin \left[ \frac{\hat{\phi}_+^\rho(\tau_1) - \hat{\phi}_+^\rho(\tau_2)}{2} + \frac{\hat{\phi}_+^\sigma(\tau_1) - \hat{\phi}_+^\sigma(\tau_2)}{2} + \frac{\pi}{2} \left( \hat{n}(\tau_1) - \hat{n}(\tau_2) \right) \right] - \\
&\quad \cos \left[ \frac{\hat{\phi}_+^\rho(\tau_1) + \hat{\phi}_+^\rho(\tau_2)}{2} + \frac{\hat{\phi}_+^\sigma(\tau_1) + \hat{\phi}_+^\sigma(\tau_2)}{2} + \frac{\pi}{2} \left( \hat{n}(\tau_1) + \hat{n}(\tau_2) \right) - \pi N \right] \right\} \}, 
\end{align}

C. Electrons with spin.

We use formulas of Sec. IV B to bosonize the chiral fermionic fields. In order to account for the appearance of the operator \( \hat{n} \) in the Hamiltonian [compare Eqs. (4.15) with Eq. (4.23)], we change the transformation (4.19):

\begin{align}
\hat{\phi}_L^\rho(x) &= \hat{\phi}_L^\rho(x) + \hat{\phi}^\rho - \delta_{\rho} \pi (N - \hat{n}), \\
\hat{\phi}_R^\rho(x) &= \hat{\phi}_R^\rho(x) - \hat{\phi}^\rho + \delta_{\rho} \pi (N - \hat{n}),
\end{align}

where \( \delta = \rho, \sigma \), and operator \( \hat{n} \) commutes with the bosonic fields \( \phi_{\pm,\rho}^\sigma \). To preserve the commutation relation \( [\hat{F}^\dagger, \phi_{L,R}^\sigma] = 0, \) we change the operators \( \hat{F}^\dagger, \hat{F} \) similarly to Eq. (6.24):

\begin{align}
\hat{F}^\dagger \rightarrow \hat{F}^\dagger e^{-i \hat{\phi}^\rho}, \quad \hat{F} \rightarrow \hat{F} e^{i \hat{\phi}^\rho}.
\end{align}

Substitution of Eqs. (6.17), (6.33) and (6.40) into Eqs. (6.22b) results in the formulas similar to Eq. (6.26):

\begin{align}
\hat{\phi}_L^\rho(x) &= \hat{\phi}_L^\rho(x) + \hat{\phi}^\rho - \delta_{\rho} \pi (N - \hat{n}), \\
\hat{\phi}_R^\rho(x) &= \hat{\phi}_R^\rho(x) - \hat{\phi}^\rho + \delta_{\rho} \pi (N - \hat{n}),
\end{align}

for the inelastic and the elastic cotunneling respectively. However, for the calculation it is more convenient to proceed directly to the low energy effective theory (4.28), because the main contribution to the conductance comes from the energy scale smaller than the charging energy \( E_C \). First, we integrate out the symmetric charge mode \( \hat{\phi}_k^\rho \). Then, we wish to use the substitution (4.26). The important difference brought into the problem by accounting for the second junction, is that Eqs. (6.41) contain the charged field \( \hat{\phi}^\rho \) itself, and not only the combination \( \hat{\phi}^\rho - \phi_0^\rho(x = 0) \), as we had before. Fortunately, the corresponding change can be simply accounted for by the introduction of one more chiral field \( \hat{\phi} \), so that we have

\begin{align}
\hat{\phi}_k^\rho(x = 0) + \hat{\phi}^\rho \rightarrow \sqrt{2} \hat{\phi}_k^\rho(x = 0), \\
\hat{\phi}^\rho - \phi_0^\rho(x = 0) \rightarrow \sqrt{2} \hat{\phi}(x = 0)
\end{align}

instead of Eq. (4.26). The field \( \hat{\phi} \) enters neither the effective action, nor the backscattering Hamiltonian, so it can be immediately integrated out, and we find the low energy representation

\begin{align}
\left( e^{i\hat{\phi}(\tau)} - \phi(0) \right) \rightarrow \left( e^{i\hat{\phi}(\tau)} - \phi(0) \right)/\sqrt{2},
\end{align}

\begin{align}
e^{i\phi(x) - \phi(0)} = e^{i\phi(x)}/\sqrt{2} e^{i\phi(0)}/\sqrt{2} = \\
\pi \frac{e^{-\mathcal{C}}}{\sqrt{\nu F}} \left( \sin \pi T \tau \right)^{1/2} e^{i\phi(\tau) - \phi(0)}/\sqrt{2}. 
\end{align}
The prefactor in the last formula can be found by requiring the averages calculated with the help of the effective theory and the initial theory to coincide. Using Eqs. (6.42) and (6.43), we obtain from Eq. (6.41)

\[
\Pi_{in}(\tau) = -\frac{2g(-\tau; r_1, r_1)}{\nu(T e^{-s})} \frac{\pi}{E_0 \lambda} \sqrt{\frac{v_F}{\lambda}} \left( \frac{\pi \tau}{\sin \pi \tau} \right)^{1/2} \langle T_{r} e^{-\hat{S}} \hat{\Phi}(\tau) \hat{\Phi}(0) e^{i[\hat{\varphi}(\tau) - \hat{\varphi}(0)]/\sqrt{\tau}} \rangle,
\]

(6.44a)

\[
\Pi_{el}(\tau) = \frac{1}{\nu \nu(T e^{-s})} \frac{\pi}{E_0 \lambda} \sqrt{\frac{v_F}{\lambda}} \left( \frac{\pi \tau}{\sin \pi \tau} \right)^{1/2} \int_{0}^{\beta} d\tau_1 d\tau_2 R(\tau_1 - \tau) R^*(\tau_2) \sum_{\alpha = \pm 1} \langle T_{r} e^{-\hat{S}} \hat{\Phi}(\tau) \hat{\Phi}(0) \rangle \times 
\]

(6.44b)

where the averaging is performed with respect to the Hamiltonian (4.27). The effective action (4.29) acquires the

\[
\hat{S} = \frac{1}{\pi \lambda} \int_{0}^{\beta} d\tau_1 d\tau_2 L(\tau_1 - \tau_2) \sum_{\alpha = \pm 1} \hat{\phi}_\alpha(\tau_1) \hat{\phi}_\alpha(\tau_2) \times e^{\frac{i \alpha}{\lambda} (\hat{\varphi}_+^\tau(\tau_1) - \hat{\varphi}_-^\tau(\tau_2))} e^{\frac{i \alpha}{\lambda} (\hat{\varphi}_+^\tau(\tau_1) - \hat{\varphi}_-^\tau(\tau_2))} \times 
\]

(6.45)

and the backscattering Hamiltonian \( \hat{H}_{bs} \) is given by

\[
\hat{H}_{bs} = \frac{2|\alpha|}{\pi} \left( \frac{E_0 v_F}{\pi \lambda} \right)^{1/2} (1) \cos \pi \tau \cos \hat{\varphi}_+^\tau(0),
\]

(6.46)

cf. Eq. (4.28). Cut-off \( \lambda \) in Eqs. (6.43) - (6.46) should be of the order of \( v_F / E_0 \), because the charging energy \( E_0 \) is the largest energy scale which can be considered with the help of Hamiltonian (4.27).

In the following subsections we will apply effective description (6.44) - (6.46) and (6.14) to find the tunneling conductance of the dot in the asymmetric setup.

1. Reflectionless contact.

Let us consider first the elastic contribution (6.44b) (we will see below that inelastic contribution should be taken into account in order to obtain the correct temperature dependence). In the lowest-order approximation we neglect the action \( \hat{S} \), and obtain with the help of Eqs. (4.30):

\[
\Pi_{el}^{(0)}(\tau) = \frac{\nu v_F E_0 v_F}{\pi \lambda} \frac{\pi T}{\sin \pi \tau} \int_{0}^{\beta} d\tau_1 d\tau_2 \times 
\]

(6.47)

\[
R(\tau_1 - \tau) R^*(\tau_2) \frac{\pi T}{\sin \pi \tau / (\tau_1 - \tau_2)} \times 
\]

\[
\sum_{\gamma = \pm 1} \left( \sin \pi \tau_1 \sin \pi \tau_2 / (\tau_2 - \tau + i \gamma 0) \right)^{1/2} \sin \pi \tau_1 \sin \pi \tau_2 / (\tau_1 - \tau + i \gamma 0)\right). 
\]

When deriving Eq. (6.47), we used the expression similar to Eq. (6.27),

\[
\langle T_{r} \hat{\Phi}(\tau) \hat{\Phi}(0) \rangle \cos \left[ \frac{\pi}{4} (\hat{n}(\tau_1) - \hat{n}(\tau_2)) \right] = \cos \left[ \frac{\pi}{4} (\hat{n}(\tau_1) - \hat{n}(\tau_2)) \right].
\]

(6.48)

All the further manipulations with Eq. (6.47) are absolutely analogous to the steps of Sec. VI A in the derivation of Eq. (6.32) from Eq. (6.28). Instead of Eq. (6.32), here we find

\[
G = \frac{G_R}{\nu v_F E_0 v_F} \int_{0}^{\infty} dt_1 dt_2 \frac{\pi T}{\sin \pi T T_1 \sin \pi T T_1} \times 
\]

(6.49)

\[
[R^A(-t_1) R^A(-t_2)]^* + R^R(t_1) [R^R(t_2)]^*,
\]

where the divergences (as we will see, logarithmical) should be cut off at times of the order of \( 1/E_0 \). Notice that there is no \( N \) dependence of the non-averaged conductance. The reason is similar to the absence of the oscillations of the capacitance in the first order in level spacing \( \Delta \) – the oscillations are washed out by the quantum fluctuations of the spin mode, which is not pinned. We will see later that the oscillatory term in the conductance is smaller than the leading non-oscillatory contribution to the conductance by a factor \( \sim (\Delta/T)^{1/2} \).

Let us now proceed with the statistics of the elastic cotunneling conductance (6.49). Ensemble averaging performed with the help of Eq. (6.21a) gives for the average conductance
where $C \approx 0.577$ is the Euler constant. At very low temperatures, $T$ should be substituted by $\Delta \ln(E_C/\Delta)$, see discussion in Sec. VI B.

Equation (6.50) deserves some discussion. Firstly, we notice the presence of the large logarithmic factor in comparison with Eq. (6.33). It can be understood using the arguments of the orthogonality catastrophe[29] similar to those applied in Ref. 27 for the inelastic cotunneling. Consider the elastic cotunneling process where an electron is introduced at time $t = 0$, and then taken away at time $t \geq E_C^{-1}$. Because the introduction of an electron costs extra energy $\approx E_C$, all the other electrons tend to redistribute themselves by moving one electron charge through the left point contact, Fig 5. One can describe such a tendency as sudden change of the boundary condition in each of the spin channels. Because all the spin channels are symmetric, each spin mode should transfer charge $e/2$. According to the Friedel sum rule, it corresponds to the additional scattering phase shift $\delta = \pm \pi/2$ in each spin mode. It is known[30] that such a sudden change causes creation of a large number of electron-hole excitations, so that the resulting state is orthogonal to the ground state. The probability for the system to retain its initial state during time $t$ is $P(t) \approx 1/(tE_C)^x$, where the index $x$ is related to the phase shift in all channels, $x = \sum(\delta/\pi)^2$. In our case we have four spin modes (two in the dot and two in the reservoir), therefore $x = 1$. In order to find the total conductance, we have to sum over all possible times that electron spends in the dot: $G \propto \int_{E_C^{-1}}^\beta dt/(E_Ct)$, which results in the logarithmic temperature dependence (6.50).

The similar argument for the spinless electrons gives the phase shift $\delta = \pi$ in each of the two channels. Thus, $x = 2$, all the relevant dynamics occurs during the times smaller than $1/E_C$, and the divergent factor disappears.

Secondly, even though the elastic contribution is dominant in the value of the conductance, in order to find the temperature dependence, one has to take into account the inelastic contribution (6.44a) which yields

$$G_{in} = G_R \frac{\pi^2Te^{-C}}{2E_C}. \quad (6.51)$$

We see that this term has stronger temperature dependence than (6.50) and therefore, the resulting conductance $G + G_{in}$ always grows as the temperature increases.

Mesoscopic fluctuations of the contribution (6.43) to the conductance can be obtained with the help of Eq. (6.21),

$$\frac{\delta G(H_1)\delta G(H_2)}{G^2} = \frac{1}{2} \sum_{\chi=\pm} \left( 1 - \frac{\ln \max \left( 1, \frac{H_2}{H_1} \right)^2}{\ln \left( \frac{E_C}{E_F} \right)} \right)^2. \quad (6.52)$$

Here the correlation field $H_\chi$ is defined by Eq. (6.16), and the field combinations $H_1 = H_1 \pm H_2$ are assumed to be much smaller than $H_c$ the charging correlation field $H_c$ is given by Eq. (2.24). The correlation function of the conductance fluctuations starts to decrease fast, as $1/H_1^2$, only at fields $H \gtrsim H_c$. Similarly to the case of capacitance fluctuations discussed in the previous Section, to obtain a representative statistics of the conductance, the magnetic field should be varied in a range wider than $H_c$.

Equation (6.52) shows that the amplitude of conductance fluctuations are of the order of its average value, like in the case of the spinless fermions. However, unlike Eq. (6.34), the correlation function (6.52) does not reveal any oscillations with the gate voltage $N$.

In order to reveal this oscillatory dependence on $N$, one has to expand $\Pi_{cl}(t)$ from Eq. (6.44) up to the first order in action (6.43). The procedure of averaging over all the relevant operators is absolutely similar to the derivation of Eq. (6.47), and we obtain

$$\Pi_{cl}^{(1)}(\tau) = -\frac{1}{\nu \nu_T^2 E_C e C} \frac{\pi T}{\sin \pi T \tau} \int_0^\beta d\tau_1 \ldots d\tau_4 R(\tau_1 - \tau) R^*(\tau_2) L(\tau_3 - \tau_4) \times$$

$$\sum_{\gamma = \pm 1} e^{i2\pi\gamma N} \left( \frac{\sin \pi T \tau_1 \sin \pi T(\tau_2 - \tau - i\gamma 0) \sin \pi T(\tau_4 - \tau + i\gamma 0) \sin \pi T\tau_3}{\sin \pi T\tau_2 \sin \pi T(\tau_1 - \tau + i\gamma 0) \sin \pi T\tau_4 \sin \pi T(\tau_3 - \tau + i\gamma 0)} \right)^{1/2} \times$$

$$\frac{\pi^2 T^2}{\sin \pi T(\tau_1 - \tau_2 + i\gamma 0) \sin \pi T(\tau_3 - \tau_4 + i\gamma 0) \sin \pi T(\tau_3 - \tau_2 + i\gamma 0) \sin \pi T(\tau_1 - \tau_4 + i\gamma 0)}^{1/2}. \quad (6.53)$$

Integration over imaginary times in Eq. (6.53) is rather straightforward, and technically very close to that in the Sec. VI B. Unlike Eq. (6.49), here the result for the non-averaged conductance is $N$-dependent. We obtain

$$G_{osc} = \frac{G_R}{\nu \nu_T^2 E_C e C} \frac{\alpha_2}{(2\pi^2 T)^{1/2}} \int_{E_C^{-1}}^\infty \left[ \prod_{i=1}^3 \left( \frac{\pi T}{\sin \pi T \tau_i} \right)^{1/2} \right]$$

for the oscillating contribution.
is given by Eq. (6.16), and the numerical coefficient \(\alpha_3\) is given by 
\[\alpha_3 = \frac{\alpha_2 e^{-2C}/(2\pi^4)}{0.207}.\]

The variance of the conductance fluctuations (6.56) in the unitary limit \((H \gg H_c)\) is suppressed by a factor of four compared to its zero-field value.

2. Finite reflection in the contact.

For the spinless electrons, finite reflection leads to the oscillations in the averaged conductance already in the first order of perturbation theory in \(r \ll 1\). On the contrary, for the electrons with spin, backscattering leads only to the enhancement of the oscillating part of the correlation function of the mesoscopic fluctuations. In order to demonstrate this, we expand Eq. (6.44b) up to the first order in the backscattering Hamiltonian (6.46). We obtain

\[
\Pi^{(b)}_{el}(\tau) = -\frac{1}{\nu_{FP} E_C e^C} \sqrt{\frac{E_C e^C}{\pi}} \frac{\pi T}{\sin \pi T} \int_{0}^{\beta} d\tau_1 \ldots d\tau_3 R(\tau_1 - \tau) R^*(\tau_2) \sum_{\gamma = \pm 1} e^{i2\pi \gamma \tau} \times
\]

\[
\left(\sin \pi T \tau \sin \pi T(\tau_2 - \tau - \gamma 0) / \sin \pi T \tau_2 \sin \pi T(\tau_1 - \tau + \gamma 0)\right)^{1/2} \left(\pi T\right)^{3/2} [\sin \pi T(\tau_1 - \tau_2 + i\gamma 0) \sin \pi T(\tau_3 - \tau_2 + i\gamma 0) \sin \pi T(\tau_1 - \tau_3 + i\gamma 0)]^{1/2}.
\]

Performing the contour deformation for the integration over \(\tau_{1,2}\), as we did before, and the analytic continuation (5.14), we find

\[
G = \frac{G_R}{\nu_{FP} E_C e^C} \frac{\alpha_2 |r| \sqrt{E_C e^C}}{(2\pi^2 T)^{1/2}} \times
\]

\[
\int_{E_1}^{E_2} dt_1 dt_2 \frac{\pi T}{\sin \pi T t_1 \sin \pi T t_2} [\sin \pi T t_1 \sin \pi T t_2]^{1/2} \times
\]

\[
\left\{ e^{-2\pi N} R(t_1) [R(t_2)]^* + c.c. \right\},
\]

where the numerical coefficient \(\alpha_2\) is given by Eq. (6.55). Average of Eq. (6.58) obviously vanishes, and for the mesoscopic fluctuations we obtain with the help of Eqs. (6.21)

\[
\frac{\delta G(1) \delta G(2)}{G_R^2} = \alpha_4 \left| r \right|^2 \frac{\Delta^2}{E_C T} \ln^2 \left(\frac{E_C}{T}\right) \left(\Lambda_+^2 + \Lambda_-^2\right) \cos 2\pi n,
\]

where we use the same short-hand notation as in Eq. (6.50). The numerical coefficient \(\alpha_4\) in Eq. (6.58) is given by 
\[\alpha_4 = \frac{\alpha_2 e^{-C}/\pi^4}{0.737}.\]

Calculation of the contribution of the backscattering into the average conductance requires accounting of magnetic field \(H_c^\xi\) is given by Eq. (6.16), and the numerical coefficient \(\alpha_3\) is given by

\[\alpha_3 = \frac{\alpha_2 e^{-2C}/(2\pi^4)}{0.207}.\]

The variance of the conductance fluctuations (6.56) in the unitary limit \((H \gg H_c)\) is suppressed by a factor of four compared to its zero-field value.

VII. CONCLUSION

In this paper, we considered mesoscopic effects in the Coulomb blockade (CB) regime. The emphasis was put on the case when the quantum dot is connected to a lead by a perfectly transparent single-mode channel. We have demonstrated that the earlier conclusion that the CB vanishes under this condition is only an approximation, which resulted from neglecting the electron trajectories...
 returning to the channel after traversing the dot. We have shown that the CB persists, and its period is still determined by a single electron charge. However, CB oscillations in all the observable quantities acquire a random phase and therefore it is revealed in the correlation functions of mesoscopic fluctuations. We constructed an analytic, well-controlled theory describing those fluctuations.

Our results are substantially different from the known results in noninteracting models of mesoscopic systems. For instance, the correlated ground state involves all the one-electron wave functions in the energy strip of the order of the charging energy $E_C$. The number of states in this strip is $n \sim E_C/\Delta \gg 1$ (here $\Delta$ is the level spacing). The correlation magnetic flux for the mesoscopic fluctuations $\Phi = \Phi_0 \sqrt{E_T/E_C}$ is controlled by the energy scale $E_C$. The large number of states in the relevant energy strip leads to the robustness of the oscillatory dependence over about $E_C/\Delta$ peaks.

We obtained the closed analytic expression for some experimentally relevant characteristics. Final results are summarized in Tables I and II for the thermodynamic quantities (differential capacitance of an almost open dot), and in Table III for the transport quantities (tunneling conductance of an almost open dot).

---

### Table I

| $s = 0$ | $\delta C_{d\!g}(N)/C$ | $\delta C_{d\!g}(N_1)\delta C_{d\!g}(N_2)/C^2$ |
|---------|-----------------|---------------------------------|
| $3.56 | r | \cos 2\pi N$ | $5.59/\beta \left( \frac{\Delta}{E_C} \right) \cos 2\pi (N_1 - N_2)$ |

$$\delta C_{d\!g}(N_1)\delta C_{d\!g}(N_2)/C^2 = \frac{0.54}{\beta} \left( \frac{\Delta}{E_C} \right) \ln^3 \left( \frac{E_C}{T} \right) \left[ \left( \frac{\Delta}{E_C} \right) \ln \left( \frac{E_C}{T} \right) + 7.12 |r|^2 \right] \cos 2\pi (N_1 - N_2)$$

### Table II

| $s = 0$ | $\overline{G(N)}/G_R$ | $\overline{\delta G(N_1)\delta G(N_2)}/G_R^2$ |
|---------|-----------------|---------------------------------|
| $\frac{\Delta}{E_C} \left[ \ln \left( \frac{E_C}{T} \right) + 7.75 \frac{T}{\Delta} + O(|r|^2) \right]$ | $0.78/\beta \left( \frac{\Delta}{E_C} \right)^2 \cos 2\pi (N_1 - N_2)$ |

$$\overline{\delta G(N_1)\delta G(N_2)}/G_R^2 = \left( \frac{\Delta^2}{E_C T} \right) \ln^2 \left( \frac{E_C}{T} \right) \left[ \frac{0.83}{\beta^2} \left( \frac{\Delta}{E_C} \right) \ln \left( \frac{E_C}{T} \right) + \frac{1.48 |r|^2}{\beta} \right] \cos 2\pi (N_1 - N_2)$$

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APPENDIX A: DERIVATION OF EQ. (3.7)

Let us discretize the space in the direction along the channel axis. The fermionic Hamiltonian \( \hat{H}_F \) acquires the form

\[
\hat{H}_F = \int dr_1 \sum_n \left( \frac{\psi^\dagger_n - \psi^\dagger_{n+1}}{2ma} (\psi_n - \psi_{n+1}) + a\psi^\dagger_n \hat{H}_\perp \psi_n \right),
\]

(A1)

where the transverse part of the motion is described by the operator

\[
\hat{H}_\perp = -\frac{\nabla^2}{2m} + U_n(r) - \mu,
\]

(A2)

and \( a \) is the discretization step. Fermionic operators in Eq. (A1) satisfy the anticommutation relation \( \{ \psi_n^{\dagger}(r) \psi_n'(r') \} = a^{-1} \delta_{nn'} \delta(r - r') \). The continuous limit of \( a \to 0 \), which will be taken in the end of the calculation corresponds to the usual Schrödinger equation. Let us separate the space into two regions; region “1” includes all the lattice sites with \( n \leq 0 \) and region “2” includes sites with \( n > 0 \). The terms entering into decomposition of the Hamiltonian, see also Eq. (B6), \( \hat{H}_F = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12} \), have the form

\[
\hat{H}_1 = \int dr_1 \sum_{n<0} \left[ \frac{(\psi^\dagger_n - \psi^\dagger_{n+1}) (\psi_n - \psi_{n+1})}{2ma} + a\psi^\dagger_n (\hat{H}_\perp + \frac{\delta_{nn}}{2ma}) \psi_n \right],
\]

(A3a)

\[
\hat{H}_2 = \int dr_1 \sum_{n \geq 1} \left[ \frac{(\psi^\dagger_n - \psi^\dagger_{n+1}) (\psi_n - \psi_{n+1})}{2ma} + a\psi^\dagger_n (\hat{H}_\perp + \frac{\delta_{nn}}{2ma}) \psi_n \right],
\]

(A3b)

\[
\hat{H}_{12} = -\int dr_1 \frac{\psi^\dagger_0 \psi_1 + \psi^\dagger_1 \psi_0}{2ma}.
\]

(A3c)

We obtain the average in Eq. (3.4) using Eq. (A3c):

\[
\frac{1}{2}\langle T_r \hat{H}_{12}(\tau_1) \hat{H}(\tau_2) \rangle_2 = \frac{1}{2m^2a^2} \int \frac{dr_1 dr'_1}{T_r \psi_0(r; \tau_1) \bar{\psi}_0(r'; \tau_2)} T_r \psi_1(r'; \tau_2) \bar{\psi}_1(r; \tau_1).
\]

(A4)

Using the definition of the Matsubara Green function

\[
-\langle \psi_n(\tau) \bar{\psi}_m(0) \rangle_2 = G_{nm}(\tau) = \frac{1}{[\partial_r - H_2]_{nm}}
\]

(A5)

and low energy representation for the fermionic operator (only one transverse mode \( \phi(r) \) in the channel)

\[
\psi_0(r_\perp) = \psi_0 \phi(r_\perp),
\]

we rewrite Eq. (A4) in the form

\[
\frac{1}{2} \langle T_r \hat{H}_{12}(\tau_1) \hat{H}(\tau_2) \rangle_2 = -T_r \psi_0(\tau_1) \bar{\psi}_0(\tau_2) \times \int \frac{dr_1 dr'_1}{4ma^2} \phi(r_\perp \phi(r'_\perp)} \hat{G}_{11} (\tau_1 - \tau_2; r_\perp, r'_\perp).
\]

(A6)

As it follows from Eqs. (A3) and (A7), Green function satisfies the equation

\[
\left\{ \partial_r - \hat{H}_\perp \right\} \delta_{nn'} - \frac{2\delta_{nn} - \delta_{nn+1} - \delta_{nn'-1}}{2ma} \right\} \hat{G}_{nn'} = \frac{a^{-1} \delta_{nn} \delta(r_1 - r_2)}{n > 1}; (A7a)

\[
\left\{ \partial_r - \hat{H}_\perp \right\} \delta_{nn'} - \frac{2\delta_{nn} - \delta_{nn+1} - \delta_{nn'-1}}{2ma} \right\} \hat{G}_{nn'} = \frac{a^{-1} \delta_{nn} \delta(r_1 - r_2)}{n = 1}; (A7b)

We see that the difference between Eq. (A7a) and Eq. (A7b) can be described by the boundary condition

\[
\hat{G}_{0n} = 0; \quad \hat{G}_{n0} = 0.
\]

(A8)

Now, we introduce coordinate \( x = an \), and take the continuous limit \( a \to 0 \). Substituting \( \hat{G}_{11} = a^2 \partial_{xx}^2 \hat{G}(x, x') \big|_{x, x' = 0} \) into Eq. (A6), we obtain Eq. (3.7).

APPENDIX B: DERIVATION OF EQUATIONS (4.14)

We introduce more general than Eq. (4.13) correlation functions

\[
D_-(\tau; x_1, x_2) = \langle T_r \phi(\tau; x_1) \phi_- (0; x_2) \rangle; \quad (B1a)

\[
D_+ (\tau; x_1, x_2) = \langle T_r \phi_+ (\tau; x_1) \phi_0 (0; x_2) \rangle; \quad (B1b)

\[
D_{\phi} (\tau) = \langle T_r \phi(\tau) \phi(0) \rangle; \quad (B1c)

\[
D_{\phi_+} (\tau; x) = \langle T_r \phi_+(\tau) \phi_+(0, x) \rangle. \quad (B1d)

\]

Correlation functions (4.13) are related to those from Eq. (B1) by

\[
D_\pm (\tau) = \int dx \lambda D_\pm (\tau; 0, x) \frac{x^2 + x^2}{\pi} \quad D_{\phi_+} (\tau) = D_{\phi} (\tau; 0), \quad (B2)
\]

where the high momenta cut-off is introduced consistently with Eqs. (4.1). Equations of motion for propagators (B1) follow from Eqs. (4.3) and (4.10) and they are given by
Performing imaginary time Fourier transform $D(\tau) = T \sum_{n} e^{-i\Omega_{n}\tau}D(\Omega_{n})$, (here $\Omega_{n} = 2\pi n T$ is the bosonic Matsubara frequency) we find the solution of Eq. (B3a)

$$\partial_{\tau} D_{-} (\tau; x, y) = -i \pi \delta (\tau) \text{sgn} (x - y); \quad (B3a)$$

$$\partial_{\tau} D_{+} (\tau; x, y) = i \text{sgn} x \frac{E_{C}}{2\pi} D_{+} (\tau; 0, y) \quad (B3b)$$

$$-i \pi \delta (\tau) \text{sgn} (x - y); \quad (B3c)$$

$$\partial_{\tau} D_{\Phi} (\tau) = i \frac{E_{C}}{2\pi} D_{\Phi+} (\tau; 0); \quad (B3c)$$

$$\partial_{\tau} D_{\Phi+} (\tau; x) = i \pi \delta (\tau) - i \frac{E_{C}}{2\pi} D_{+} (\tau; 0, x). \quad (B3d)$$

In Eqs. (B3), the cut-off function $f(x)$ is defined as

$$f(x) = \int_{0}^{\infty} 2dy \frac{e^{-x-y}}{x + y^2}. \quad (B6)$$

The propagator $D_{-}$ is found by putting $E_{C} = 0$ in Eq. (B3a),

$$D_{-}(\Omega_{n}; x, y) = \frac{\pi}{\Omega_{n}} \left\{ \text{SGN}(x - y) + 2\theta [\Omega_{n}(y - x)] e^{i\pi\delta (x - y)} \right\}$$

$$-\frac{E_{C}}{2\pi} D_{+}(\Omega_{n}; 0, y) \left\{ \text{SGN} x + 2\theta (-\Omega_{n}x) e^{i\pi\delta x} \right\}. \quad (B4)$$

Substituting $x = 0$ in the both sides of Eq. (B4), we find $D_{+}(\Omega_{n}; 0, y)$ and then Eq. (B2) yields:

$$D_{+}(\Omega_{n}) = \frac{\pi}{[\Omega_{n}] + \frac{E_{C}}{2\pi}} \eta \frac{[\Omega_{n}]\lambda}{v_{F}}. \quad (B5a)$$

Propagator $D_{-}$ is found by putting $E_{C} = 0$ in Eq. (B3a),

$$D_{-} (\Omega_{n}) = \frac{\pi}{[\Omega_{n}] \eta \frac{[\Omega_{n}]\lambda}{v_{F}}} \left[ \frac{n_{\Omega_{n}}} {\Omega_{n} + \frac{E_{C}}{2\pi}} \right] \right\}$$

and remaining propagators are found from the time Fourier transform of Eqs. (B3d) - (B3d) and Eq. (B5a):

$$D_{\Phi}(\Omega_{n}) = \frac{E_{C}}{2\pi[\Omega_{n}] + \frac{E_{C}}{2\pi}} \quad (B5c)$$

$$D_{\Phi+}(\Omega_{n}) = \frac{\pi \text{sgn} \Omega_{n}}{[\Omega_{n}] + \frac{E_{C}}{2\pi}} \quad (B5d)$$

In Eqs. (B3), the inverse Fourier transform of Eqs. (B3) gives Eqs. (14). Let us write here for completeness the result for the propagator $D_{+}(0)$ at arbitrary temperatures:

$$D_{+}(0) = \ln \left( \frac{2\pi v_{F}}{\lambda E_{C} e^{\lambda}} \right) + \int_{0}^{\infty} dx \left[ \cot x - \frac{1}{x} \right] e^{-x E_{C}/2\pi T}. \quad (B7)$$

At small temperatures $T \ll E_{C}$, Eq. (B7) reduces to Eq. (14). At large temperatures, $T \gg E_{C}$, we obtain

$$D_{+}(0) = \ln \left( \frac{v_{F}}{2\pi T \lambda} \right) + \frac{2\pi^{2}T}{E_{C}}. \quad \quad (B8)$$

All the results associated with the Coulomb blockade, see e.g. Eq. (15) contain exponential terms of the form $e^{-\Delta_{n}(0)}$. This gives the suppression of the charge quantization at high temperature by a factor of $e^{-2\pi^{2}T/E_{C}}$. This clearly indicates, that the effects considered in this paper and in Refs. [4,7] can not be obtained in any order of perturbation theory in charging energy $E_{C}$.
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