VIVE LA DIFFÉRÉNCE III

SAHARON SHELAH

ABSTRACT. We show that, consistently, there is an ultrafilter \( \mathcal{F} \) on \( \omega \) such that if \( N_0^\ell = (P_0^\ell \cup Q_0^\ell, P_0^\ell, Q_0^\ell, R_0^\ell \) (for \( \ell = 1, 2, n < \omega \)), \( P_0^\ell \cup Q_0^\ell \subseteq \omega \), and \( \prod_{n<\omega} N_0^1/\mathcal{F} \equiv \prod_{n<\omega} N_0^2/\mathcal{F} \) are models of the canonical theory \( \text{ind} \) of the strong independence property, then every isomorphism from \( \prod_{n<\omega} N_0^1/\mathcal{F} \) onto \( \prod_{n<\omega} N_0^2/\mathcal{F} \) is a product isomorphism.

0. Introduction

In a previous paper [Sh 326] we gave two constructions of models of set theory in which the following isomorphism principle fails in various strong respects:

1. \textbf{(Iso 1):} If \( M, N \) are countable elementarily equivalent structures and \( \mathcal{F} \) is a non-principal ultrafilter on \( \omega \), then the ultrapowers \( M^*, N^* \) of \( M, N \) with respect to \( \mathcal{F} \) are isomorphic.

As is well known, this principle is a consequence of the Continuum Hypothesis. Recall that Keisler celebrated theorem (from [Ke67]) says that two models, \( M, N \) of cardinality at most \( \lambda^+ \) are elementarily equivalent iff for some ultrafilter \( \mathcal{F} \) of \( \lambda \), the ultrapowers \( M^\lambda/\mathcal{F}, N^\lambda/\mathcal{F} \) are isomorphic, this has given an algebraic characterization of elementary equivalence.

In [Sh 405] our aim originally was to give a related example in connection with the well-known isomorphism theorem of Ax and Kochen. In its general formulation, that result states that a fairly broad class of Henselian fields of characteristic zero satisfying a completeness (or saturation) condition are classified up to isomorphism by the structure of their residue fields and their value groups. The case that interest us in the second paper in this series [Sh 405], was:

2. \textbf{(Iso 2):} If \( \mathcal{F} \) is a non-principal ultrafilter on \( \omega \), then the ultraproducts \( \prod_p \mathbb{Z}_p/\mathcal{F} \) and \( \prod_p \mathbb{F}_p[[t]]/\mathcal{F} \) are isomorphic.

And more generally:

\textbf{Theorem 0.1} (See [Sh 405]). It is consistent with the axioms of set theory that there is a non-principal ultrafilter \( \mathcal{F} \) on \( \omega \) such that for any two
sequences of discrete rank 1 valuation rings \((R^i_n)_{n=1,2,...}\) having countable residue fields, any isomorphism \(F : \prod R^1_n/F \to \prod R^2_n/F\) is an ultraproduct of isomorphisms \(F_n : R^1_n \to R^2_n\) (for a set of \(n\)'s contained in \(\mathcal{F}\)). In particular, \(\mathcal{F}\)–majority of the pairs \(R^1_n, R^2_n\) are isomorphic.

In the case of the rings \(\mathbb{F}_p[[t]]\) and \(\mathbb{Z}_p\), we see that (Iso 2) fails. For this our main work was to show the following statement which actually from model theoretic point of view is more basic and interesting.

**Theorem 0.2** (See [Sh 405]). It is consistent with the axioms of set theory that there is a nonprincipal ultrafilter \(\mathcal{F}\) on \(\omega\) such that for any two sequences of countable trees \((T^i_n)_{n=1,2,...}\) for \(i = 1, 2\), with each tree \(T^i_n\) countable with \(\omega\) levels, and with each node having at least two immediate successors, if \(\mathcal{T}^i = \prod T^i_n/\mathcal{F}\), then for any isomorphism \(F : \mathcal{T}^1 \to \mathcal{T}^2\) there is an element \(a \in \mathcal{T}^1\) such that the restriction of \(F\) to the cone above \(a\) is the restriction of an ultraproduct of maps \(F_n : T^1_n \to T^2_n\).

From a model theoretic point of view this still is not the right level of generality for a problem of this type. There are two natural ways to pose the problem:

**Problem 1.** Characterize the pairs of countable models \(M, N\) such that in some forcing extension, \(\prod M/\mathcal{F} \neq \prod N/\mathcal{F}\) for some non-principal ultrafilter \(\mathcal{F}\).

**Problem 2.** Characterize the pairs of countable models \(M, N\) with non-isomorphic ultrapowers mod \(\omega\) in some forcing extension. (I.e., such that there is no forcing extension in which for some non-principal ultrafilter \(\mathcal{F}\) on \(\omega\) we have \(M^\omega/\mathcal{F} \simeq N^\omega/\mathcal{F}\).)

[There are two variants of the second problem: the ultrapowers may be formed either using one ultrafilter twice (called \(2(A)\)), or may consider using any two ultrafilters (called \(2(B)\), but see below.]

**Problem 3.** Let us write \(M \leq N\) whenever in every forcing extension, if \(\mathcal{F}\) is an ultrafilter on \(\omega\) such that \(N^\omega/\mathcal{F}\) is saturated, then \(M^\omega/\mathcal{F}\) is also saturated. Characterize this relation.

This is somewhat like the Keisler order (see Keisler [Ke67], or [Sh:13], or [Sh:15], Chapter VI), but does not depend on the fact that the ultrafilter is regular, so some of the results there apply to Problem 3 this in turn implies results on Problem 2(A). We can replace \(\aleph_0\) here by any cardinal \(\kappa\) satisfying \(\kappa^\kappa = \kappa\).

Now, by [Sh 13], there is an ultrafilter \(\mathcal{D}\) on \(2^{\aleph_0}\) such that for countable models \(M, N\)

\[M \equiv N \implies M^{2^{\aleph_0}}/\mathcal{D} \simeq N^{2^{\aleph_0}}/\mathcal{D}.\]
Also, if \(2^{\aleph_0} = \aleph_1\), \(\mathcal{F}\) is a non-principal ultrafilter on \(\omega\) and \(M_1 \equiv M_2\) are countable, then \(M_1^\ell/\mathcal{F} \simeq M_2^\ell/\mathcal{F}\) (as they are saturated); similarly if \(M_n^\ell\) are countable models (for \(\ell = 1, 2, n < \omega\)), \(M_\ell = \prod_{n<\omega} M_n^\ell/\mathcal{F}_\ell\), and \(\mathcal{F}_\ell\) are non-principal ultrafilters on \(\omega\), then \(M_1 \equiv M_2\) \(\Rightarrow\) \(M_1 \cong M_2\). On the other hand, if \(2^{\aleph_0} > \aleph_1\), then by [Sh:c, Ch VI] for every regular cardinal \(\theta\), \(\aleph_1 \leq \theta < 2^{\aleph_0}\) we have an ultrafilter \(\mathcal{F}_\theta\) on \(\omega\) such that the down cofinality of \((\omega, <)\)\(^\omega/\mathcal{F}_\theta\) above \(\omega\) is \(\theta\) (so \(\theta_1 \neq \theta_2 \Rightarrow (\omega, <)\)\(^\omega/\mathcal{F}_{\theta_1} \not\simeq (\omega, <)\)\(^\omega/\mathcal{F}_{\theta_2}\)).

The present paper is dedicated to adding some further light. Working on [Sh 405] we had hoped to continue it sometime. However, we actually began only when Jarden asked:

\[(*)\] Suppose that \(F_n^\ell\) are finite fields (for \(n < \omega\), \(\ell = 1, 2\)). Can we have (a universe and) an ultrafilter \(\mathcal{F}\) on \(\omega\) such that \(\prod_{n<\omega} F_n^1/\mathcal{F}\) and \(\prod_{n<\omega} F_n^2/\mathcal{F}\) are elementarily equivalent but not isomorphic.

That was not an arbitrary question: he knew that many such pairs of ultraproducts are elementarily equivalent, because the first order theory of a field \(F\) which is isomorphic to an ultraproduct of finite fields is determine by its characteristic and its subfield of algebraic elements. Hence we can find an equivalence relation \(E_k\) on the family of finite fields for \(k < \omega\), each with finitely many equivalence classes such that if \(F_n^1, F_n^2\) are finite fields for \(n < \omega\) and \(\mathcal{F}\) is a non principal ultrafilter on \(\omega\) and for each \(k\) the set \(\{n < \omega : (F_n^1)E_k(F_n^2)\}\) belongs to \(\mathcal{F}\) then the respective ultraproducts are isomorphic.

Jarden asked me, I inquire whether it has the strong independence property and told him what it is, he says yes. Years later finishing the work on the paper he deny any knowledge on this, and this is my recollection. Cherlin, to whom he refer me, give me the reference to the strong independence property for finite field: Duret [Du81, pp. 136–157].

Here we continue [Sh 326, §3], [Sh 405, §1]. To give an affirmative answer to (*) we show that after adding \(\aleph_3\) Cohen reals to a suitable ground model, one gets a universe with an ultrafilter \(\mathcal{F}\) on \(\omega\) and a sequence of models \(\langle M_n : n < \omega\rangle\) on \(\omega\) such that

\[(**)\] if \(N_n^\ell = (P_n^\ell \cup Q_n^\ell, P_n^\ell, Q_n^\ell, R_n^\ell)\) (for \(\ell = 1, 2, n < \omega\)), \(P_n^\ell \cup Q_n^\ell \subseteq \omega\), and \(\prod_{n<\omega} N_n^1/\mathcal{F} \equiv \prod_{n<\omega} N_n^2/\mathcal{F}\) are models of the canonical theory \(T^{ind}\) of the strong independence property (see Definition 1.3),

then every isomorphism from \(\prod_{n<\omega} N_n^1/\mathcal{F}\) onto \(\prod_{n<\omega} N_n^2/\mathcal{F}\) is (first order) definable in \(\prod_{n<\omega} M_n/\mathcal{F}\) for some expansions \(M_n\) of \(N_n^1, N_n^2\) simultaneously, or what is equivalent but hopefully more transparent if \(F\) is an isomorphism from \(N^1 = \prod_{n<\omega} N_n^1/\mathcal{F}\) onto \(N^2 = \prod_{n<\omega} N_n^2/\mathcal{F}\) then we can find unary functions \(F_n\) from \(N_n^1\) onto \(N_n^2\) for every \(n < \omega\) such that
the set of $n$ for which $F_n$ is an isomorphism from $N_1^n$ onto $N_2^n$ belongs to the ultrafilter and $\prod_{n<\omega} (N_1^n, N_2^n, F_n)/F$ is $(N_1, N_2, F)$.

Out forcing is adding $\aleph_3$ Cohen reals, but we need that our model of set theory, i.e. the universe, satisfies some conditions over which we force. There are two ways to get a “suitable” ground model. The first way involves taking any ground model which satisfies a portion of the GCH, and extending it by an appropriate preliminary forcing, which generically adds the name for an ultrafilter which will appear after addition of the Cohen reals. The alternative approach, which we consider more model-theoretic, is to start with an $L$-like ground model and use instances of diamond (or related weaker principles) to prove that a sufficiently generic name already exists in the ground model. We will fully present the first approach - the second one should be then an easy modification of the arguments presented in [Sh 405, §1].

Our presentation is slightly more general than needed for (**) By allowing more what we call “bigness” properties to be involved in the definition of $\mathbb{App}$, we leave room for getting analogs of (**) for more classes of models (getting the conclusion for all of them at once, or possibly only for some) - as long as the respective bigness notions are like in 1.4. This, we hope, would be helpful in connection with Problems 1, 2 above. For the problem on fields only the case associated with the strong independence property is needed; general bigness notions appear for possible general treatment.

Let us comment on our general point of view. In this paper we try to advance in Problems 1+2(A) and for this, it seemed, we can take the maximal $\Gamma$, i.e., allow all $\aleph_0$-bigness notions. However, concerning Problem 3 (investigating the partial order $\leq$ or models), for showing $M \nless N$, the construction causes $N^\omega/F$ to be almost always non $\aleph_3$-saturated. We need stronger tools for them.

The two previous papers benefited from Gregory Cherlin, the present one benefited from Andrzej Rosłanowski, thank you!

We continue this investigations in [Sh:F503].

Notation 0.3. Our notation is standard and compatible with that of classical textbooks (like Chang and Keisler [CK] and Jech [J]). In forcing we keep the older convention that a stronger condition is the larger one.

1. We will use two forcing notions denoted by $C_{\aleph_3}$ and $\mathbb{App}$ (see Definitions 2.2 and 2.4, respectively). Conditions in these forcing notions will be called $p, q, r$ (with possible sub/super-scripts).
2. All names for objects in forcing extensions will be denoted with a tilde below (e.g., $\tilde{a}$).
3. The letter $\tau$ (with possible sub/super-scripts) stands for a vocabulary of a first order language; me may also write $\tau(M), \tau(T)$ for a model $M$ or theory $T$ with the obvious meaning. We will use letter $p$ (with sub/super-scripts) to denote types.
4. The universe of a model $M$ will be denoted $|M|$, but we will often abuse this notation and write, e.g., $a \in M$. The cardinality of a set $A$ will be denoted $|A|$, and, for a model $M$, $|M|$ will stand for the cardinality of its universe.

1. **Bigness notions**

In this section we will quote relevant definitions and results from [Sh:1, Chapters X, XI] (= [Sh 384], [Sh 482]), but we somewhat restrict ourselves here. The reader interested in the field case only may jump directly to Definition 1.7.

**Definition 1.1 (See [Sh:1, Chapter XI, §1]).** Let $T$ be a complete first order theory (in a vocabulary $\tau$), and $\mathcal{K} = \mathcal{K}_T$ be a class of models of $T$ partially ordered by $\prec$. Also let $t$ be a first order theory with a countable vocabulary $\tau(t)$ (including equality, treating function symbols as predicates).

1. We say that $\mathcal{K}'$ is an $A$–place in $\mathcal{K}$ if
   (a) $\mathcal{K}' \subseteq \mathcal{K}$,
   (b) if $M \in \mathcal{K}'$, then $A \subseteq M$,
   (c) if $M \prec N$ are from $\mathcal{K}$ and $A \subseteq M$, then $M \in \mathcal{K}' \iff N \in \mathcal{K}'$,
   (d) if $M \in \mathcal{K}'$ and $A \subseteq N \in \mathcal{K}$ and $M, N$ are isomorphic over $A$, then $M \in \mathcal{K}' \iff N \in \mathcal{K}'$.

2. For $A \subseteq M \in \mathcal{K}$ we let $\mathcal{K}' = \mathcal{K}_{A,M}$ be the class
   $$\{N : A \subseteq N \text{ and } \bar{a} \in A \Rightarrow \text{tp}(\bar{a}, \emptyset, M) = \text{tp}(\bar{a}, \emptyset, N)\}.$$ We call it the $(A, M)$–place.

3. A local bigness notion $\Gamma$ for $\mathcal{K}$ (without parameters, in one variable $x$) is a function with domain $\mathcal{K}$ which for every model $M \in \mathcal{K}$ gives
   $$\Gamma_M^-(M) \subseteq \{\varphi(x, \bar{a}) : \varphi \in \mathcal{L}(\tau) \& \bar{a} \subseteq M\},$$
   $$\Gamma_M^+(M) = \{\varphi(x, \bar{a}) : \varphi \in \mathcal{L}(\tau) \& \bar{a} \subseteq M\} \setminus \Gamma_M^-$$
   such that
   (a) $\Gamma_M^-$ is preserved by automorphisms of $M$,
   (b) $\Gamma_M^-$ is a proper ideal, i.e., $\Gamma_M^+ \neq \emptyset$ and
   (α) if $M \models (\forall x)(\varphi(x, \bar{a}) \Rightarrow \psi(x, \bar{b}))$ and $\psi(x, \bar{b}) \in \Gamma_M^-$, then
   $\varphi(x, \bar{a}) \in \Gamma_M^-.$
   (β) if $\varphi_1(x, \bar{a}_1), \varphi_2(x, \bar{a}_2) \in \Gamma_M^-$, then $\varphi_1(x, \bar{a}_1) \lor \varphi(x, \bar{a}_2) \in \Gamma_M^-.$

   Elements of $\Gamma_M^-$ are called $\Gamma$–small in $M$, members of $\Gamma_M^+$ are $\Gamma$–big.

   A local bigness notion $\Gamma$ for $\mathcal{K}$ with parameters\footnote{Alternatively use the monster model.} from $A$ is defined similarly but $\text{Dom}(\Gamma)$ is an $A$–place $\mathcal{K}'$ in $\mathcal{K}$.

4. We say that a local bigness notion $\Gamma$ is invariant if for $M \prec N$ from $\mathcal{K}$ we have $\Gamma_M \subseteq \Gamma_N$ and $\Gamma_M^+ \subseteq \Gamma_N^+.$

5. A $\Gamma$–big type $p(x)$ in $M$ is a set of formulas $\psi(x, \bar{a})$ all of whose finite conjunctions are $\Gamma$–big in $M$.
6. A pre $t$–bigness notion scheme $\Gamma$ is a sentence $\psi_T$ (in possibly infinitary logic) in the vocabulary $\tau(t) \cup \{P^*\}$, where $P^*$ is a unary predicate.

7. An interpretation with parameters of $t$ in a model $M \in K$ is $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$, where $\varphi_R \in \mathcal{L}(\tau)$ and $\bar{a}$ is a sequence of appropriate length of elements of $M$. So $R$ is interpreted as

$$\{ \bar{b} : M \models \varphi_R(\bar{b}, \bar{a}_R), \ lg(\bar{b}) = \lg(\bar{y}_R) (= \text{the arity of } R) \}.$$ 

The interpreted model is called $M[\bar{\varphi}]$ and we demand that it is a model of $t$.

8. For a pre $t$–bigness notion scheme $\Gamma = \psi_T$ and an interpretation $\bar{\varphi}$ of $t$ in $M \in K$ with parameters from $A \subseteq M$, we define the $\bar{\varphi}$–derived local bigness notion $\Gamma[\bar{\varphi}]$ with parameters from $A \subseteq M$ (in the $K_{A,M}$–place) as follows:

Given $M' \in K_{(A,M)}$. A formula $\vartheta(x, \bar{b})$ in $\mathcal{L}(\tau)$ (with parameters from $M$, of course) is $\Gamma[\bar{\varphi}]$–big in $M$ if for any quite saturated $N^*$, $M \prec N^*$, letting

$$P^* = \{ a \in N^*[\bar{\varphi}] : N^* \models \vartheta(a, \bar{b}) \}$$

we have $(N^*[\bar{\varphi}], P^*) \models \psi_T$.

We write $\Gamma = \Gamma(\varphi, \vartheta, \psi_T)$.

9. We omit the “pre” if every $\Gamma[\bar{\varphi}]$ is an invariant local bigness notion (for our fixed $K$). So it is enough in (8) above if we define $\Gamma_M$ when $M \prec M'$.

**Proposition 1.2.** If $\Gamma$ is a local bigness notion for $K$ with parameters in $A$, $M \in K_{A,M'}$ and $\varphi(x)$ is a $\Gamma$–big type in $M$, then it can be extended to $\Gamma$–big notion $\varphi$ in $M$ which is a complete type over $M$.

**Proposition 1.3.** For $T, K = K_T$ and $t$ as in [1,7],

($\exists$) if $N \prec M$ are from $K$, and $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$ is an interpretation of $t$ in $N$, then $\bar{\varphi}$ is an interpretation of $t$ in $M$ (i.e., $M[\bar{\varphi}] \models t$).

The following definition illuminates the most important aspect of Definition [3] (which is central for our present paper), and also it is needed for more general results.

**Definition 1.4.** Let $t$ be a first order theory in a vocabulary $\tau(t)$. Suppose that $\Gamma$ is a $t$–bigness notion scheme, $P \in \tau(t)$ is a unary predicate, and $\vartheta(y, x)$ is a $\tau(t)$–formula. We say that $\Gamma$ is $(N_2, N_1)$–$(P, \vartheta)$–separative with a witness $X$ whenever the following condition ($\bigodot_{\Gamma}^{P, \vartheta}$) holds.

($\bigodot_{\Gamma}^{P, \vartheta}$) Assume that $M$ is an $N_2$–compact $\tau$–model, $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$ is an interpretation of $t$ in $M$. Then $X \subseteq |M|$ is of cardinality at most $N_1$, includes all parameters of $\bar{\varphi}$ and
if \( N \prec M, X \subseteq |N|, ||N|| \leq \aleph_1 \), and \( p(x) \) is a \( \Gamma[\bar{\phi}] \)-big type over \( N \), 
\( ||p(x)|| \leq \aleph_1 \), and \( a_1, a_2 \) are distinct members of \( |M| \setminus |N| \) with 
\[ M \models \phi_p[a_1] \land \phi_p[a_2] \]
then the type \( p(x) \cup \{ \vartheta(a_1, x) \equiv \neg \vartheta(a_2, x) \} \) is \( \Gamma[\bar{\phi}] \)-big.
We may omit the “(\( \aleph_2, \aleph_1 \))”.

**Definition 1.5** (See [Sh:3], Def. 3.4, 3.5, Chapter XI). 1. \( t_{\text{ind}} = t_0^{\text{ind}} \) is the first order theory in vocabulary \( \tau(t_{\text{ind}}) = \{ P, Q, R \} \), where \( P, Q \)
are unary predicates and \( R \) is a binary predicate, including formulas
\[ (\forall x)(\forall y)(x R y \Rightarrow P(x) \land Q(y)), \quad \text{and} \]
\[ (\forall x)(P(x) \lor Q(x)) \]
and saying that for each \( n < \omega \) and any distinct elements \( a_1, \ldots, a_{2n} \in P \), there is \( c \in Q \) such that
\[ a_i R^c M \text{ if and only if } i \leq n. \]
\( t_1^{\text{ind}} \) is \( t_{\text{ind}} \) plus
\[ (\forall x)(\forall y)(\exists z)(Q(x) \land Q(y) \land x \neq y \Rightarrow P(z) \land (z R x \equiv \neg z R y)). \]
2. We define a pre \( t_{\text{ind}} \)-bigness notion scheme \( \Gamma_{\text{ind}} \) as follows. The sentence \( \psi_{\Gamma_{\text{ind}}} \) says that \( P^* \subseteq Q \) and \( (P, Q, R, P^*) \) satisfies:
for every \( n < \omega \), there is a finite set \( A \subseteq P \) such that
for every distinct \( a_1, \ldots, a_{2n} \in P \setminus A \) there is \( c \in P^* \)
satisfying
\[ a_\ell R^c M \text{ for } \ell \leq n, \quad \text{and} \quad \neg a_\ell R^c M \text{ for } n < \ell \leq 2n. \]
(If \( \psi_{\Gamma_{\text{ind}}} \) is not first order.)
3. We say that a first order theory \( T \) has the strong independence property
if some formula \( \vartheta(x, y) \) define a two place relation which is a model of \( t_{\text{ind}}^{\text{ind}} \) with \( P, Q \) choosen as \( x + x \).
Plainly,

**Proposition 1.6.** 1. For a model \( M \) of \( t_{\text{ind}}^{\text{ind}} \), an automorphism \( \pi \) of \( M \)
is determined by \( \pi \upharpoonright P^M \) (i.e., if \( \pi_1, \pi_2 \in \text{Aut}(M) \) are such that \( \pi_1 \upharpoonright P^M = \pi_2 \upharpoonright P^M \), then \( \pi_1 = \pi_2 \)).
2. Moreover, if \( \bar{\phi} \) is an interpretation of \( t_{\text{ind}}^{\text{ind}} \) in \( M^* \), \( M = M^*[\bar{\phi}], \pi \in \text{Aut}(M) \) and \( \pi \upharpoonright P^M \) is definable with parameters in \( M^* \), then so is \( \pi \).

**Proposition 1.7** (See [Sh:3], Chapter XI, §3). \( \Gamma_{\text{ind}} \) is a \( t_{\text{ind}} \)-bigness notion scheme. It is \( (\aleph_2, \aleph_1) \)-(\( P, \vartheta \))-separative with witness \( \emptyset \), where \( P \) is given and
\( \vartheta(y, x) = y R x \).

**Definition 1.8.** A mapping \( F : N^1 \rightarrow N^2 \) is a \( \Delta \)-embedding from \( N^1 \) to \( N^2 \) whenever \( \Delta \) is a set of formulas in \( L_{\omega, \omega}(\tau(N^1) \cap \tau(N^2)) \) such that 
if \( \varphi \in \Delta \) and \( N^1 \models \varphi[a_1, \ldots, a_n] \),
then $N^2 \models \varphi[F(a_1), \ldots, F(a_n)]$.

[If $\Delta$ is closed under negation, then we have “if and only if”.

2. THE FORCING NOTION App

As explained in the introduction, we work in a Cohen generic extension of a suitable ground model. In this section we present how that ground model can be obtained: we start with $V \models \text{GCH}$ and we force with the forcing notion $\text{App}$ defined in [2.4] below, the App comes for approximations, as themembers are approximations to a name for an ultrafilter as we desire.

Definition 2.1. 1. The Cohen forcing adding $\aleph_3$ Cohen reals is denoted by $\mathcal{C}_{\aleph_3}$. Thus a condition $p$ in $\mathcal{C}_{\aleph_3}$ is a finite partial function from $\aleph_3 \times \omega$ to $\omega$, and the order of $\mathcal{C}_{\aleph_3}$ is the natural one. The canonical $\mathcal{C}_{\aleph_3}$–name for the $\beta$th Cohen real will be called $\check{x}_\beta$.

2. Let $A \subseteq \aleph_3$. For a condition $p \in \mathcal{C}_{\aleph_3}$, its restriction to $A \times \omega$ is called $p \upharpoonright A$, and we let $\mathcal{C}_{\aleph_3} \upharpoonright A = \mathcal{C}_A = \{p \upharpoonright A : p \in \mathcal{C}_{\aleph_3}\}$. Also, we let $\omega^*_A = (\omega^\omega)^{\mathcal{C}_{\aleph_3} \upharpoonright A}$.

3. For a sequence $\langle A_n : n < \omega \rangle$ of non-empty sets (and $A \subseteq \aleph_3$), we define

$$\prod^A_{n<\omega} A_n = \{f \in V^{\mathcal{C}_{\aleph_3} \upharpoonright A} : f \text{ is a function with domain } \omega,$$

and such that $f(n) \in A_n$ for all $n\},$$

and similarly for models.

4. For $A \subseteq \aleph_3$ and $m < \omega$, let $I^m_A$ be the set of all $\omega$–sequences of canonical $\mathcal{C}_A$–names for subsets of $\omega^m$. Let $Q_s$ (for $s \in I^m_A$, $m < \omega$) be an $m$–ary predicate, $Q_{s_0} \neq Q_{s_1}$ whenever $s_0 \neq s_1$, and let

$$\tau_A = \{Q_s : s \in I^m_A \& m < \omega\}$$

(so $||\tau_A|| = \aleph_1 \cdot ||A||$). Let $M^A_s$ be a $\mathcal{C}_A$–name for the $\tau_A$–model on $\omega$ such that if $s = \langle s_n : n < \omega \rangle \in I^m_A$, then $\models_{\mathcal{C}_A} (Q_s) M^A_s = s_n$.

Definition 2.2. 1. A function $G$ is called an $(\aleph_3, \aleph_2)$–bigness guide if the domain $\operatorname{Dom}(G)$ of $G$ is

$$\{(A, F) : A \subseteq \aleph_3, ||A|| \leq \aleph_1, \text{ and } F \text{ is a } \mathcal{C}_A \text{–name of a non principal ultrafilter on } \omega\},$$

and

(a) $G(A, F)$ is a set of triples $(t, \Gamma, \varphi)$, where $t$ is a first order theory (or just a $\mathcal{C}_A$–name of a first order theory), $\Gamma$ is a $\mathcal{C}_A$–name of $t$–bigness notion scheme, and $\varphi$ is (a $\mathcal{C}_A$–name for) an interpretation of $t$ in $\prod^A_{n<\omega} M^A_s / F$, and $||G(A, F)|| \leq \aleph_2$, and

(b) if $(A^\ell, F^\ell) \in \operatorname{Dom}(G)$ for $\ell = 1, 2$, $A^1 \subseteq A^2$ and $||\mathcal{C}_A^1 F^1 \subseteq F^2$, then $G(A^1, F^1) \subseteq G(A^2, F^2)$.

2. An $(\aleph_3, \aleph_2)$–bigness guide $G$ is ind–full if
(γ) for every \((A, F) \in \text{Dom}(G)\) and a \(C_A\)-name \(\tilde{\varphi}\) for an interpretation of \(t_{\text{ind}}\) in \(\prod_{n<\omega} M^n_A/F\) we have \((t_{\text{ind}}, \Gamma_{\text{ind}}, \tilde{\varphi}) \in G(A, F)\).

3. We say that \(G\) is full whenever the following condition holds.

(Ⅲ) Assume \((A, F) \in \text{Dom}(G)\) and \(t\) is a \(C_A\)-name of a first order theory in the vocabulary \(\tau(t) \in \mathcal{H}(\mathbb{N})\), \(\psi\) is a \(C_A\)-name for a pre \(t\)-bigness notion scheme, \(\tilde{\psi} \in L_{\mathbb{N}_1, \mathbb{N}_1}(\tau(t) \cup \{P^s\})\). Let \(\tilde{\varphi}\) be a \(C_A\)-name for an interpretation of \(t\) in \(\prod_{n<\omega} M^n_A/F\). Suppose also that \(\Delta\)

is a set of \(L_{\omega, \omega}(\tau(t))\)-formulas such that \((\tilde{\varphi}, t, \tilde{\psi})\) defines a bigness notion \(\Gamma = \Gamma_{(\tilde{\varphi}, t, \tilde{\psi})}\) Then \((t, \Gamma, \tilde{\varphi}) \in G(A, F)\).

[The main case for us is \(t = t_{\text{ind}}, \Gamma = \Gamma_{\text{ind}}\).]

The clause 2.2(2) is added for our particular application. It can be replaced by use of different bigness notions.

**Proposition 2.3.**
1. There is a full \((\mathbb{N}_3, \mathbb{N}_2)\)-bigness guide \(G\).
2. If a bigness guide \(G\) is full, then it is ind-full.

**Proof.** Trivial. \(\square\)

**Definition 2.4.** Let \(G\) be an \((\mathbb{N}_3, \mathbb{N}_1)\)-bigness guide. We define the forcing notion \(\text{App} = \text{App}_G\). (When \(G\) is fixed, as typically in the present paper, we do not mention it.)

1. A condition \(q\) in \(\text{App}\) is a triple \(q = (A, F, \tilde{\Gamma}) = (A^q, F^q, \tilde{\Gamma}^q)\) such that:

   (a) \(A\) is a subset of \(\mathbb{N}_3\) of cardinality \(\leq \mathbb{N}_1\);

   (b) \(F\) is a canonical \(C_A\)-name of a non-principal ultrafilter on \(\omega\), such that for \(\beta \in A\),

   \[\mathcal{F} \upharpoonright (A \cap \beta) \overset{\text{def}}{=} \mathcal{F} \cap \{a : a \text{ is a } C_{A \cap \beta}\text{-name of a subset of } \omega\}\]

   is a \(C_{A \cap \beta}\)-name (of an ultrafilter on \(\omega\));

   (c) \(\tilde{\Gamma} = \Gamma_{\beta} : \beta \in A \& \text{cf}(\beta) = \mathbb{N}_2\), where each \(\Gamma_{\beta}\) is a local bigness notion \(\Gamma[\varphi]\) for some \((t, \Gamma, \varphi) \in G(A \cap \beta, F \upharpoonright (A \cap \beta))\);

   (d) If \(\text{cf}(\beta) = \mathbb{N}_2, \beta \in A\), then it is forced (i.e., \(\Vdash_{\mathbb{N}_3}\)) that:

   - the type realized by \(x_{\beta}\) over the model \(\prod_{n<\omega} M^n_{A \cap \beta}/(\mathcal{F} \upharpoonright (A \cap \beta))\)

   (so it is a type in the vocabulary \(\tau_{A \cap \beta}\)) is \(\Gamma_{\beta}\)-big complete, and moreover this type is a \(C_{A \cap \beta}\)-name. We call it “the type induced by \(x_{\beta}\) according to \(q\)”.

2. The order \(\leq_{\text{App}} = \leq_{\text{App}} = \leq_{\text{App}}\) is the natural one: \(q_1 \leq q_2\) if and only if \(A^{q_1} \subseteq A^{q_2}, \models_{C_{A^{q_2}}} F^{q_1} \subseteq F^{q_2}\), and \(\tilde{\Gamma}^{q_1} \upharpoonright A^{q_1} = \tilde{\Gamma}^{q_2}\).

3. We say that \(q_2 \in \text{App}\) is an end extension of \(q_1 \in \text{App}\), and we write \(q_1 \leq_{\text{end}} q_2\), if \(q_1 \leq q_2\) and \(\sup(A^{q_1}) \leq \min(A^{q_2} \setminus A^{q_1})\).\(^{2}\)

\(^{2}\)We can fix a \(C_{\mathbb{N}_3}\)-name of countable first order theory; really \(\mathcal{F}\) serves simultaneously for all.
4. For a condition \( q \in \text{App} \) and an ordinal \( \beta \in \aleph_3 \) we define \( q \upharpoonright \beta = (A^q \cap \beta, F^q \upharpoonright (A^q \cap \beta), \Gamma^q \upharpoonright (A^q \cap \beta)) \).

5. For \( \beta < \aleph_3 \) we let \( \text{App} \upharpoonright \beta = \{ q \in \text{App} : A^q \subseteq \beta \} \) with inherited order.

If \( G \subseteq \text{App} \) is generic over \( V \), then we let \( G \upharpoonright \beta = G \cap (\text{App} \upharpoonright \beta) \).

One easily checks that

**Proposition 2.5.**

1. If \( q \in \text{App} \), \( \beta < \aleph_3 \), then \( q \upharpoonright \beta \in \text{App} \) and \( q \upharpoonright \beta \leq \text{end} \ q \).

2. Both \( \leq_{\text{App}} \) and \( \leq_{\text{end}} \) are partial orders on \( \text{App} \).

**Lemma 2.6.** If \( \langle q_\zeta : \zeta < \xi \rangle \) is an increasing sequence of members of \( \text{App} \), \( \xi \leq \aleph_1 \), and \( q_\zeta \leq_{\text{end}} q_{\zeta'} \) for \( \zeta < \zeta' \), then there is \( q \in \text{App} \) such that \( A^q = \bigcup_{\zeta < \xi} A^{q_\zeta} \) and \( q \leq_{\text{end}} q \) for all \( \zeta < \xi \).

**Proof.** We may assume that \( \xi > 0 \) is a limit ordinal. If \( \text{cf}(\xi) > \aleph_0 \), then we let \( A^q = \bigcup_{\zeta < \xi} A^{q_\zeta}, F^q = \bigcup_{\zeta < \xi} F^{q_\zeta} \) and \( \Gamma^q = \bigcup_{\zeta < \xi} \Gamma^{q_\zeta} \). If \( \text{cf}(\xi) = \aleph_0 \), then additionally we have to extend \( \bigcup_{\zeta < \xi} F^{q_\zeta} \) to a \( C_{A^q} \)–name of an ultrafilter on \( \omega \), which is no problem. \( \square \)

**Lemma 2.7.** Suppose that \( q \in \text{App} \), \( A^q \subseteq \gamma \in \aleph_3 \), and \( p \) is a \( C_{A^q} \)–name of a type over the model \( \prod_{n<\omega} A_n / F^n \) (so in the vocabulary \( \tau_{A^q} \), finitely satisfiable in \( \prod_{n<\omega} A_n / F^n ) \). Then:

1. If \( \text{cf}(\gamma) < \aleph_2 \), then there is a condition \( r \in \text{App} \) stronger than \( q \) such that \( A^r = A^q \cup \{ \gamma \} \), and

\[ \models_{C_{A^r}} " x_\gamma/F^r \text{ realizes } p \text{ in } \prod_{n<\omega} A_n / F^n " \].

2. If \( \text{cf}(\gamma) = \aleph_2 \), \( (t, \Gamma, \bar{\varphi}) \in G(A^q, F^q) \) and the type \( p \) is (forced to be) \( \Gamma[\bar{\varphi}] \)–big, then there is a condition \( r \in \text{App} \) as in (1) and such that \( \Gamma^r = \Gamma[\bar{\varphi}] \).

**Proof.** 1) Extend \( F^q \) to \( F^r \) so that \( x_\gamma/F^r \) realizes the required type.

2) Note that every \( \Gamma[\bar{\varphi}] \)–big type can be extended to a complete \( \Gamma[\bar{\varphi}] \)–big one by [1.2]. \( \square \)

**Lemma 2.8.** 1. Suppose \( q_0, q_1, q_2 \in \text{App} \), \( q_0 = q_2 \upharpoonright \beta \), \( q_0 \leq q_1 \), \( A^{q_1} \subseteq \beta \). Suppose further that \( A^{q_2} \setminus A^{q_0} = \{ \beta \} \) and \( \text{cf}(\beta) = \aleph_2 \). Assume that \( p_1 \) is a \( C_{A^{q_1}} \)–name for a complete \( \Gamma^{q_2}_\beta \)–big type over \( (\prod_{n<\omega} A_n / F^n) \) such that \( p_1 \) contains the type \( p_0 \) induced by \( x_\beta \) according to \( q_2 \). Then there is \( q_3 \geq q_1, q_2 \) with \( A^{q_3} = A^{q_1} \cup \{ \beta \} \), such that \( x_\beta \) induces \( p_1 \) on \( (\prod_{n<\omega} A_n / F^n) \) (according to \( q_3 \)).
2. Assume \( q_0, q_1, q_2 \in \text{App} \), \( q_0 = q_2 \upharpoonright \beta \), \( q_0 \leq q_1 \) and \( A^{q_1} \subseteq \beta \). If \( A^{q_2} \setminus A^{q_0} = \{ \beta \} \) and \( cf(\beta) < \aleph_2 \), then there is \( q_3 \in \text{App} \), \( q_3 \geq q_1, q_2 \) such that \( A^{q_3} = A^{q_1} \cup A^{q_2} \).

3. Assume that \( \delta_1, \delta_2 < \aleph_2 \), and \( \langle \beta_j : j < \delta_2 \rangle \) is a non-decreasing sequence of ordinals below \( \aleph_3 \). Let \( \langle p_i : i < \delta_1 \rangle \) be an increasing sequence from \( \text{App} \). Suppose that \( q_j \in \text{App} \upharpoonright \beta_j \) (for \( j < \delta_2 \)) are such that:

\[
p_i \upharpoonright \beta_j \leq q_j \text{ for } i < \delta_1, \ j < \delta_2, \quad q_j \leq \text{end } q_{j'} \text{ for } j < j' < \delta_2.
\]

Then there is an \( r \in \text{App} \) with \( p_i \leq r \) and \( q_j \leq \text{end } r \) for all \( i < \delta_1 \) and \( j < \delta_2 \).

4. If \( \bar{p} = \langle p_i : i < \delta_1 \rangle \) an increasing sequence in \( \text{App} \), \( \delta_1 < \aleph_2 \), then \( \bar{p} \) has an upper bound in \( \text{App} \).

**Proof.** 1) Let \( A_i = A^{q_i} \) and let \( \mathcal{F}_i = \mathcal{F}^{q_i} \) for \( i < 3 \), and \( A_3 = A_1 \cup A_2 = A_1 \cup \{ \beta \} \). The only possibly not clear part is to show that, in \( V^{C_{A_3}} \), there is an ultrafilter extending \( \mathcal{F}_1 \cup \mathcal{F}_2 \) which contains \( \mathcal{F}' \), the family of all the sets

\[
\{ n < \omega : M^n_{A_3} \models \varphi[x_\beta(n), \bar{a}(n)] \}
\]

for \( \varphi(x, \bar{y}) \in p_1 \), \( \ell g(\bar{y}) = m \), and a \( C_{A_1} \)-name \( \bar{a} \) of an \( m \)-tuple from \( \omega^*_A \) (and in our notation above \( \bar{a}(n) \) is a \( C_{A_1} \)-name for an \( m \)-tuple of elements of \( \omega \)). As \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}' \) are (forced, i.e., \( \models_{C_{A_3}} \)) to be closed under intersections (of two, and hence of finitely many), clearly if this fails, then there are \( m < \omega \), a condition \( p \in C_{A_3} \), a \( C_{A_1} \)-name \( \bar{a} \) of a member of \( \mathcal{F}_1 \), a \( C_{A_2} \)-name \( \bar{b} \) of a member of \( \mathcal{F}_2 \), a \( (\text{name for } a) \) \( \tau_{A_1} \)-formula \( \varphi \) and a \( C_{A_1} \)-name for an \( m \)-tuple \( \bar{a} \) from \( \omega^*_A \) such that

\[
p \upharpoonright A_1 \models_{C_{A_1}} " \varphi(x, \bar{a}) \in p_1 " \quad \text{and} \quad p \upharpoonright C_{A_3} \models " \bar{a} \cap \bar{b} \cap c = \emptyset " ,
\]

where

\[
c = \{ n : M^n_{A_3} \models \varphi[x_\beta(n), \bar{a}(n)] \} .
\]

We may easily eliminate parameters, so we may assume that we have \( \varphi[x_\beta(n)] \) only (remember the definition of \( \tau_{A_1} \)). Let \( p_i = p \upharpoonright A_i \) for \( i = 0, 1, 2 \), and let \( H^0 \subseteq C_{A_0} \) be generic over \( V \) such that \( p_0 \in H^0 \). For \( n < \omega \) let \( A_n \) be a \( C_{A_0} \)-name such that

\[
A_n[H^0] = \{ y \in M^n_{A_2} : \text{ there is } p_2' \in C_{A_2} \text{ such that } p_2 \leq p_2', \ p_2' \upharpoonright A_0 \in H^0 \text{ and } p_2' \models " x_\beta(n) = y \text{ and } n \in \bar{b} " \}
\]

(recall \( y \in M^n_{A_2} \) means \( y \in \omega \)). Let \( A^* = \prod_{n<\omega} A_n[H^0] / \mathcal{F}_0 \). So \( A^*[H^0] \) is (the interpretation of) an unary predicate from \( \tau_{A_0} \); in fact \( Q_{(A^*_n : n < \omega)} \) is such a predicate. Thus, in \( V[H_0^0] \), either \( A^*(x) \in p_0 \) or \( \neg A^*(x) \in p_0 \). The latter is impossible by the choice of \( p_0 \), so necessarily \( A^*(x) \in p_0 \). As also \( \models_{C_{A_1}} \)
“\(\varphi(y) \in p_1\)”, clearly if \(H^1 \subseteq C_{A_1}\) is generic over \(V\) and \(H^0 \cup \{p_1\} \subseteq H^1\), then in \(V[H^1]\) we have
\[\{n \in \omega : M^n_{A_1} \models (\exists y)(A^*(y) \& \varphi(y))\} \in \mathcal{F}_1[H^1]\]
(remember \(p_1\) is a type over \(\prod_{n < \omega} M^n_{A_2}/\mathcal{F}_1\) extending \(p_0\)). Consequently, we may find a condition \(p'_1 \in H_1 \subseteq C_{A_1}\) stronger than \(p_1\), an integer \(n < \omega\), and an element \(y \in M^n_{A_1}\) (so \(y \in \omega\)) such that
\[p'_1 \upharpoonright A_0 \in H^0, \quad \text{and} \quad p'_1 \Vdash \gamma_1 \in (A^*(y) \& \varphi(y))\text{ and }n \in a\].
As \(\gamma_1\) is a \(C_{A_0}\)-name, we really have \(y \in A^n_1[H^0]\), and hence (by its definition) for some \(p_2 \in C_{A_2}\) we have
\[p_2 \leq p'_2, \quad p'_2 \upharpoonright A_0 \in H^0, \quad \text{and} \quad p'_2 \Vdash \gamma_1 = x_\beta(n)\text{ and }n \in b\].
Now for our \(n\) we can force \(n \in a \cap b \cap c\) by amalgamating the corresponding conditions \(p'_1, p'_2\), getting a contradiction. As said above this finishes the proof of the existence of \(q_3\).

2) The proof is essentially contained in the previous one (use the very trivial bigness notion: \(\varphi(x, \bar{a})\) is big in \(M\) if and only if \(M \models (\exists x)\varphi(x, \bar{a})\), so we may use a \(p_1\)). See also the end of the proof of (3).

3) We will prove by induction on \(\gamma \in \mathcal{N}_3\) that if all \(\beta_j \leq \gamma\) and all \(p_i\) belong to \(\mathbb{A}_{\text{pp}} \upharpoonright \gamma\), then the assertion in (3) holds for some \(r \in \mathbb{A}_{\text{pp}} \upharpoonright \gamma\).

We may assume that \(\delta_1 > 0\) (otherwise apply 2.8) and \(\delta_2 > 0\) (otherwise let \(\delta_2 = 1\), \(\beta_0 = 0\), \(q'_0 \in \mathbb{A}_{\text{pp}} \upharpoonright \beta_0\) be above \(p_i \upharpoonright 0\) for \(i < \delta_1\); so it just means \(F_{\gamma_0}^0\) is the ultrafilter \(F_{\gamma_i}^{\mathbb{n}}\) for \(i < \delta_1\); now if \(\gamma = 0\), then \(r = q_0\) is as required and otherwise we have reduced the case \(\delta_2 = 0\) to the case \(\delta_2 = 1\).

We may assume that \(\beta_j = \sup\{\alpha + 1 : \alpha \in A_j\}\) (for \(j < \delta_2\)), and also that the sequence \(\langle \beta_j : j < \delta_2 \rangle\) is strictly increasing. Let \(\beta = \sup \beta_j\) and let
\[q = (\bigcup_{j < \delta_2} A_j, \bigcup_{j < \delta_2} F_j, \bigcup_{j < \delta_2} F_j).
\]

We first assume \(\text{cf}(\gamma) \neq \aleph_0\).

If \(\gamma = \beta\), then \(q \in \mathbb{A}_{\text{pp}}\) and we may take \(r = q\). So let us assume \(\beta < \gamma\). If \(\delta_2\) is a successor ordinal, or a limit ordinal of uncountable cofinality, then we let \(q^* = q\) (clearly \(q^* \in \mathbb{A}_{\text{pp}} \upharpoonright \beta\)). If \(\text{cf}(\delta_2) = \aleph_0\), then we may first apply the inductive hypothesis to \(\langle p_i \upharpoonright \beta : i < \delta_1 \rangle\) (and \(\langle \beta_j, q_j : j < \delta_2 \rangle\)) to get a condition \(q^* \in \mathbb{A}_{\text{pp}} \upharpoonright \beta\) which is stronger than all \(p_i \upharpoonright \beta\) and which end-extends all \(q_j\). So in all these cases, we have a condition \(q^* \in \mathbb{A}_{\text{pp}} \upharpoonright \beta\) (end extending all \(q_j\) for \(j < \delta_2\)) stronger than all \(p_i \upharpoonright \beta\) for \(i < \delta_1\) (and we are looking for its end-extension which is a bound to all \(p_i \upharpoonright \beta\)).

**The case \(\gamma = \gamma_0 + 1\), a successor**
In this case our inductive hypotheses applies to the \(p_i \upharpoonright \gamma_0, q^*\), and \(\gamma_0\), yielding \(r_0\) in \(\mathbb{A}_{\text{pp}} \upharpoonright \gamma_0\) with \(p_i \upharpoonright \gamma_0 \leq r_0\) and \(q^* \leq_{\text{end}} r_0\). What remains to be done is an amalgamation of \(r_0\) with all of the \(p_i\), where \(A^{p_i} \subseteq A^{r_0} \cup \{\gamma_0\},\)
and where one may as well suppose that \( \gamma_0 \) is in \( A_{p_1} \) for all \( i \). This is a slight variation on (1) or (2). For instance, suppose \( \text{cf}(\gamma_0) = \aleph_2 \). We let

- \( A_2 = \bigcup_{i < \delta_1} A_{p_i}, A_0 = A_2 \setminus \{ \gamma_0 \}, A_1 = A_{r_0}, A_3 = A_2 \cup A_1. \)
- \( F_1 = F^{r_0}, F_2 = \bigcup_{i < \delta_1} F^{p_i} \) (the latter might be only a \( C_{A_2} \)-name of a filter).
- For \( i < \delta_1 \) let \( p_i \) be the \( C_{A_{p_i} \cap r_0} \)-name for the \( (\Gamma^\gamma_{r_0} \text{-big}) \) type induced by \( x_{\gamma_0} \) over the model \( \prod_{n < \omega} A_{p_i}, A_{r_0}, A_{r_0}/F^{\text{r}_0} \gamma_0 \). Then let \( p_0 = \bigcup_{i < \delta_1} p_i \), and note that it is a \( C_{A_0} \)-name for a \( \Gamma^\gamma_{\gamma_0} \)-big type over the model \( \prod_{n < \omega} A_{r_0}, A_{r_0}/F_0 \).
- Let \( p_1 \) be (a \( C_{A_1} \)-name for) a complete \( \Gamma^\gamma_{\gamma_0} \)-big type over \( \prod_{n < \omega} A_{r_1}, A_{r_0}/F_0 \) extending \( p_0 \). (Exists by [1.2].)

Now, in \( V^{C_{A_1}} \), we want to extend \( F_1 \cup F_2 \) to an ultrafilter \( F' \) containing the sets of the form \( \{ n < \omega : M^n_{A_3} \models \varphi'(x_\gamma_0(n)) \} \) for all \( \varphi'(x) \in p_1 \). If this fails, then as

\[
\Vdash_{C_{A_1}} \text{" } \langle F^{p_i} : i < \delta_1 \rangle \text{ is increasing } \quad \text{"}
\]

we find a condition \( p \in C_{A_3} \), a \( C_{A_1} \)-name \( a \) of a member of \( F_1 \), and \( i < \delta_1 \), and a \( C_{A_2} \)-name \( b \) for a member of \( F_i \), and \( \varphi \) such that

\[
p \upharpoonright A_1 \Vdash \varphi(x) \in p_i \subseteq p_1 \quad \text{and} \quad p \Vdash_{C_{A_3}} a \cap b \cap \{ n : M^n_{A_3} \models \varphi(x_\beta(n)) \} = \emptyset.
\]

Next we continue exactly as in the proof of (1).

**The case \( \gamma \) is a limit ordinal of cofinality \( \aleph_2 \)**

Since \( \delta_1 < \aleph_2 \) there is some \( \gamma_0 < \gamma \) such that all \( p_i \) lie in \( \text{App} \upharpoonright \gamma_0 \) and \( \beta < \gamma_0 \), and the induction hypothesis then yields the claim.

**The case \( \gamma \) is a limit ordinal of cofinality \( \aleph_1 \)**

Choose a strictly increasing and continuous sequence \( \langle \gamma_j : j < \aleph_1 \rangle \) with supremum \( \gamma \), starting with \( \gamma_0 = \beta \). By induction on \( j \) choose \( r_j \in \text{App} \upharpoonright \gamma_j \) (for \( j < \aleph_1 \)) such that:

- \( r_0 = q^*; \)
- \( r_j \leq_{\text{end}} r_{j'} \) for \( j < j' < \aleph_1 \);
- \( p_i \upharpoonright \gamma_j \leq r_j \) for \( i < \delta_1 \) and \( j < \aleph_1 \).

[Thus, at a successor stage \( j + 1 \), the inductive hypothesis is applied to \( p_i \upharpoonright \gamma_j+1, \gamma_j, \gamma_{j+1} \), and \( \gamma_{j+1} \). At a limit stage \( j \), we apply the inductive hypothesis to \( p_i \upharpoonright \gamma_j \) for \( i < \delta_1, r_{j'} \) for \( j' < j \), \( \gamma_{j'} \) for \( j' < j \), and \( \gamma_j \).] Finally, we let \( r = ( \bigcup_{j < \aleph_1} A_{r_j}, \bigcup_{j < \aleph_1} F^{r_j}, \bigcup_{j < \aleph_1} \Gamma^{r_j} \) Clearly \( r \in \text{App} \) as required.

Now we are going to consider the remaining case:

**The case \( \gamma \) is a limit ordinal of cofinality \( \aleph_0 \)**

If \( \beta < \gamma \) (where \( \beta \) is as defined at the beginning of the proof), then we first pick a strictly increasing sequence \( \langle \gamma_j : j < \aleph_0 \rangle \) of ordinals such that
have an antichain

The only perhaps unclear part is the chain condition. Suppose we
Proof.

to vary over ordinals of cofinality \(\aleph_\gamma\).

Thus we have reduced this sub-case to the only one remaining: \(\beta = \gamma\).

Now if for some \(j < \delta_2\) we have \(\beta_j = \gamma\), then \(r = q_j\) is as required, so without
loss of generality \((\forall j < \delta_2)(\beta_j < \gamma)\). Then necessarily cf\(\delta_2 = \aleph_0\) and we may
equally well assume that \(\delta_2 = \aleph_0\).

We take \(q\) as defined earlier (so it is the “union” of all \(q_j\)), but it does not
have to be a condition in \(\mathbb{A}_{pp}\): the filter \(\bigcup_{j<\aleph_0} \mathcal{F}_{q_j}\) does not have to be
an ultrafilter, and we need to extend it to one that contains also \(\bigcup_{j<\delta_1} \mathcal{F}_{p_j}\).

Note that \(A^+ = \bigcup_{i<\delta_1} A^p_i \subseteq \bigcup_{j<\aleph_0} A^q_j\), but there might be \(C_{A_j}\)-names
for elements of \(\bigcup_{i<\delta_1} \mathcal{F}_{p_i}\) that are not \(C_{A_{q_j}}\)-names for any \(j < \aleph_0\), so it could
happen that one name like that is forced to be disjoint from some element of
\(\mathcal{F}_{q_i}\). So assume toward contradiction, that there are a condition \(p \in C_{A^+}\),
ordinals \(i < \delta_1\) and \(j < \aleph_0\), a \(C_{A_{p_i}}\)-name \(a\), and a \(C_{A_{q_j}}\)-name \(b\) such that

\[ p \models C_{A^+} \quad \text{“} a \in \mathcal{F}_{p_i} \& b \in \mathcal{F}_{q_j} \& a \cap b = \emptyset \quad \text{“}. \]

Increasing \(j\) if necessary, we may also assume that \(p \in C_{A_{q_j}}\) so \(\text{Dom}(p) \subseteq \beta_j \times \omega\).

\(H^0 \subseteq C_{A_{p_i}} \cap \beta_j\) be generic over \(V\) such that \(p \upharpoonright A^p_i \in H^0\), and
let

\[ c = \{ n \in \omega : \text{there is a condition } p' \in C_{A_{p_i}} \text{ stronger than } p \upharpoonright A^p_i \text{ and } \]

\(\text{such that } p' \upharpoonright (A^p_i \cap \beta_j) \in H^0 \text{ and } p'' \models C_{A_{p_i}} \text{ “} \)n \in a \text{ “} \}.

Clearly, \(c \in V[H^0]\) is a set from \((\mathcal{F}_{p_i} \upharpoonright (A^p_i \cap \beta_j))[H^0]\). Since \(p_i \upharpoonright \beta_j \leq q_j\),
we find a condition \(p'' \in C_{A_{q_j}}\) and \(n \in c\) such that

\[ p \leq p'' \quad \& \quad p'' \upharpoonright (A^p_i \cap \beta_j) \in H^0 \quad \& \quad p'' \models C_{A_{q_j}} \text{ “} n \in b \text{ “}. \]

For this \(n\) we take \(p' \in C_{A_{p_i}}\) witnessing that \(n \in c\) and next we let \(p^* = p' \cup p''\). Clearly \(p^* \models c\), a contradiction.

4) Follows, i.e., it is the case \(\delta_2 = 0\) of part (3).

\[ \square \]

**Lemma 2.9.** Assume \(V \models \text{GCH}\). The forcing notion \(\mathbb{A}_{pp}\) satisfies the
\(\aleph_3\)-chain condition, it is \(\aleph_2\)-complete, \(\|\mathbb{A}_{pp}\| = \aleph_3\) and \(\|\mathbb{A}_{pp} \upharpoonright \gamma\| \leq \aleph_2\) for
every \(\gamma \in \aleph_3\). Consequently, the forcing with \(\mathbb{A}_{pp}\) does not collapse cardinals
nor changes cofinalities, and \(\|\mathbb{A}_{pp}\| \text{ GCH}\).

**Proof.** The only perhaps unclear part is the chain condition. Suppose we
have an antichain \(\{q_\alpha : \alpha \in \aleph_3 \& \text{cf}(\alpha) = \aleph_2\} \subseteq \mathbb{A}_{pp}\) (the index \(\alpha\) is taken
to vary over ordinals of cofinality \(\aleph_2\) just for convenience). An important
point is that \(G\) can “offer” at most \(\aleph_2\) candidates for the bigness notion at
\(\delta < \aleph_3\), \(\text{cf}(\delta) = \aleph_2\), hence for each \(\gamma \in \aleph_3\) the restricted forcing \(\mathbb{A}_{pp} \upharpoonright \gamma\)
has cardinality \(\leq \aleph_2\). Applying Fodor’s lemma twice, we find a stationary
set $S \subseteq \{ \alpha \in \aleph_3 : \text{cf}(\alpha) = \aleph_2 \}$ and a condition $q^* \in \text{App}$ such that $(\forall \alpha \in S)(q_\alpha \upharpoonright \alpha = q^*)$. Pick $\alpha_1, \alpha_2 \in S$ such that $\sup(A^{\alpha_1}) < \alpha_2$; it follows from Lemma 2.8 that the conditions $q_{\alpha_1}, q_{\alpha_2}$ are compatible, a contradiction.

**Proposition 2.10.** 1. For each $p \in \text{App}$ and $\alpha \in \aleph_3$, there is a condition $q \in \text{App}$ stronger than $p$ and such that $\alpha \in A^q$.

2. $F = \bigcup \{ F^r : r \in G_{\text{App}} \}$ is a $\aleph_3$-name for an ultrafilter on $\omega$. Also, for each $r \in G_{\text{App}}$ we have: $F \cap \mathcal{P}(\omega)(V^*)^{A^p} = F^r$.

**Proof.** Should be clear (for (1) use 2.7 + 2.8(1); then (2) follows).

**Definition 2.11.** 1. Suppose $G_{\text{App}} \subseteq \text{App}$ is generic over $V$, $V^* = V[G_{\text{App}}]$. We let $F^\delta$ be the $\aleph_\delta$-name for the restriction $F \upharpoonright \delta$ of the ultrafilter $F$ to the sets from the universe $(V^*)^{\aleph_\delta}$.

2. We define an $\text{App}$-name $\Gamma^\delta$ of a $\aleph_\delta$-name as $\Gamma^p_{\delta}$ for every $p \in G_{\text{App}}$ such that $\delta \in A^p$. (So it is an $\text{App} \ast \aleph_\delta$-name.)

**Lemma 2.12.** 1. Suppose that $G_{\text{App}} \subseteq \text{App}$ is generic over $V$, $V^* = V[G_{\text{App}}]$, and $\delta < \aleph_3$, $\text{cf}(\delta) = \aleph_2$, and $H^\delta \subseteq \aleph_\delta$ is generic over $V^*$. Then, in $V[G_{\text{App}} \cap (\text{App} \upharpoonright \delta)][H^\delta]$, we have:

$$\prod_{n < \omega} M^n_{\aleph_\delta} / F^\delta[H^\delta]$$

is $\aleph_2$-compact.

2. Also if $H \subseteq \aleph_\delta$ is generic over $V^*$, $H \supseteq H^\delta$, then in $V^*[H]$:

(a) $\prod_{n < \omega} M^n_{\aleph_\delta} / F[H]$ is $\aleph_2$-compact,

(b) $\hat{\pi}_\delta[H]/F[H] \in \prod_{n < \omega} M^n_{\aleph_\delta} / F[H]$ realizes a $\Gamma^\delta[G][H^\delta]$–big type over $\prod_{n < \omega} M^n_{\aleph_\delta} / F^\delta[H^\delta]$.

**Proof.** By 2.7(1). We can use some $\hat{\pi}_\beta$ with $\beta$ of cofinality less than $\aleph_2$ to realize each type.

3. **Definability**

**Hypothesis 3.1.** In this section we assume that $G$ is an $(\aleph_3, \aleph_2)$–bigness guide, $\text{App} = \text{App}_G$, $G \subseteq \text{App}$ is a generic filter over $V$, and $V^* = V[G]$.

For an ordinal $\delta < \aleph_3$, we let $G_\delta = G \cap (\text{App} \upharpoonright \delta)$. Also, $H, H^\delta$ are the canonical $\aleph_\delta$– and $\aleph_\delta$–names of the generic subsets of $\aleph_\delta$ and $\aleph_\delta$, respectively. We work mostly in $V^*$.

[Note that, by Lemma 2.9, $V^* \models \text{GCH}$]

**Definition 3.2.** 1. We say that $m$ is an $(\aleph_3, \aleph_2)$–isomorphism candidate (or just an isomorphism candidate, in $V$ or in $V^*$, see below; letting

3Note: $M^\delta$ is $M^\delta_A$ for $A = \delta$
\( \mathbf{m}^\ominus = \langle t, \tilde{\varphi}, \psi, \Delta, \langle N_n : n < \omega, \ell \in \{1, 2\} \rangle \rangle \), note that as \( \text{App} \) is \( \aleph_2 \)-complete, this forcing does not add new \( \mathbf{m}^\ominus \), i.e., \( \mathbf{V} \) and \( \mathbf{V}^* \) have the same set of \( \mathbf{m}^\ominus \), though we have an \( \text{App} \)-name \( \mathbf{m} \) of such object if:

(i) \( \mathbf{m} \) consists of \( A^* = A^*[\mathbf{m}] \subseteq [\aleph_3]^{<\aleph_2} \), \( P^* = P^*[\mathbf{m}] \), \( N_n^\ell = N_n^\ell[\mathbf{m}] \) (for \( n < \omega, \ell \in \{1, 2\} \), \( F = F[\mathbf{m}] \), \( \Gamma = \Gamma[\mathbf{m}] \) and \( (t, \tilde{\varphi}, \psi, \Delta) = (t[\mathbf{m}], \tilde{\varphi}[\mathbf{m}], \psi[\mathbf{m}], \Delta[\mathbf{m}] ) \);

(ii) \( t, \tilde{\varphi}, \psi, \Delta \) are \( C_{A^*} \)-names as in \( \mathbf{2.2} \) and \( \Gamma = \Gamma(\varphi, \Delta, \psi) \) is a bigens notion as there, \( \tau(t) \) countable for simplicity;

(iii) \( N_n^\ell \), for \( n < \omega \) and \( \ell \in \{1, 2\} \), are \( C_{A^*} \)-names for countable models of a (countable) theory \( t_n^\ell \), and the universes \( [N_n^\ell] \) are subsets of \( \omega \).

Also it is forced (i.e., \( \vDash_{\aleph_3} \)) that \( t = \text{Th}(\prod_{n<\omega} N_n^3/F) = \text{Th}(\prod_{n<\omega} N_n^3/F) \), so the \( \prod_{n<\omega} \mathcal{N}_n^\mathcal{F} \) is \( \prod_{n<\omega} \mathcal{N}_n^\mathcal{F} \).

(iv) We have predicates \( Q^R_\ell \in \tau_{A^*} \) (for \( R \in \tau(t) \)) such that \( \tilde{\varphi}^\ell = \langle Q^R_\ell : R \in \tau(t) \rangle \) is the interpretation of \( t \) in \( \prod_{n<\omega} M_n^A/F \) giving \( \prod_{n<\omega} N_n^3/F \). (Remember \( \mathbf{2.1} \), \( \mathbf{1.3} \); so by the choice of \( \tau_{A^*} \) actually \( \tilde{\varphi}^\ell = \tilde{\varphi}^\ell \).

(v) \( F \) is a \( \aleph_3 \)-name (more accurately an \( \text{App} \)-name of such name, but we sometimes write \( F \) instead of \( F[G] \)) as when \( G \) is constant) and \( p^* \in \mathcal{C}_{\aleph_3} \) is a condition such that:

\[ \vDash_{\aleph_3} p^* \vDash_{\aleph_3} \text{"} F \text{ is a map from } \prod_{n<\omega} N_n^1 \text{ into } \prod_{n<\omega} N_n^2 \text{ which represents a } \Delta \text{-embedding modulo } F \text{"}. \]

2. \( \mathbf{m} \) is \( (P, \vartheta) \)-separative if \( P, \vartheta \) are \( C_{A^*} \)-names and there is a witness \( X \subseteq \omega_{A^*} \) for \( \Gamma^m \) in the intended model, i.e., this is forced, \( \vDash_{\aleph_3} \).

[If \( \mathbf{m} \) is clear from the context we may omit it.]

**Observation 3.3.** Assume, in \( \mathbf{V} \), that \( \mathbf{m} \) is an \( (\aleph_3, \aleph_2) \)-isomorphism candidate, \( \Gamma = \Gamma[\mathbf{m}] = \Gamma(t, \varphi, \psi) \). Then there is a stationary set of ordinals \( \delta < \aleph_3 \) such that:

(a)\( \delta \ A^* \subseteq \delta \), \( \text{cf}(\delta) = \aleph_2 \), and \( p^* \in \mathcal{C}_{\aleph_3} \upharpoonright \delta \), and for some \( q \in G \) we have that \( \Gamma^\delta_\ell = \Gamma[\varphi] \) (for \( t, \Delta, \psi \) from \( \mathbf{2.2} \)),

(b)\( \delta \) for every \( \mathcal{C}_{\aleph_3} \upharpoonright \delta \)-name \( x \) for an element of \( \prod_{n<\omega} N_n^1 \), \( F(x) \) is a \( \mathcal{C}_{\aleph_3} \upharpoonright \delta \)-name,

[recall \( \text{App} \) satisfies the \( \aleph_3 \)-ccc]

(c)\( \delta \) similarly for \( F^{-1} \) and for \( " y \in \text{Rang}(F) \)",

(d)\( \vDash_{\aleph_3} \{ n < \omega : x_\delta(n) \in N_n^1 \} \in F \) (so \( x_\delta/F \in \prod_{n<\omega} N_n^1/F \)).

For such \( \delta \), we let \( y^*_\delta = y_\delta^*_F = y^*_\delta \mathbf{m} \) be \( F(x_\delta) \in \prod_{n<\omega} N_n^2 \).
The Main Isomorphism Theorem 3.4. Assume that \( m \) is an \((\aleph_3,\aleph_2)\)-isomorphism candidate as in \( \$3 \), and \( \delta < \aleph_3 \) is as there. Then there are \( q_\delta \), \( y \) such that

(a) \( q_\delta \in \text{App} \), moreover \( q_\delta \in G \),
(b) \( q_\delta \Vdash \text{App} \quad “F(x_\delta) = y^*” \), where \( y^* \) is a \( C_{Aq_\delta} \)-name of a member of \( \omega^\omega \),
(c) \( A^* \subseteq A^{q_\delta} \), \( A_\delta \defeq A^{q_\delta} \cap \delta \),
(d) in \( \mathsf{V}[G][H^\delta] \) we have:
   (i) \( \mathcal{F}_\delta = \mathcal{F}_\delta[G][H^\delta] \) is a non-principal ultrafilter on \( \omega \).
   (ii) The model \( M_\delta = \prod_{n<\omega} M_n^\delta/\mathcal{F}_\delta \) with the vocabulary \( \tau_\delta \) is \( \aleph_2 \)-compact.
   (iii) The vocabulary \( \tau_{A_\delta} \subseteq \tau_\delta \) is of cardinality \( \leq \aleph_1 \).
   (iv) \( M_{A_\delta} = \prod_{n<\omega} A_\delta[M_n^\delta/\mathcal{F}_\delta[H^\delta]] \prec M_\delta \mid \tau_{A_\delta} \).
   (v) \( F_\delta = (F \upharpoonright \delta)[H^\delta] = ((F \upharpoonright \delta)(A_{\text{App}} \upharpoonright \beta)))[H^\delta] \) is a \( \Delta \)-embedding
      from the model \( \prod_{n<\omega} N_n^1/\mathcal{F}_\delta \) into \( \prod_{n<\omega} N_n^2/\mathcal{F}_\delta \),
   (vi) Let \( p_\delta = p_\delta(x) \) be the \( (C_\delta \text{-name of the}) \ 1 \text{-type in the vocabulary} \)
      \( \tau_{A_\delta} \) such that

"\( p_\delta(x) \) is the type realized by \( x_\delta \) over \( M_{A_\delta} \) in \( \prod_{n<\omega} A_\delta[M_n^\delta/\mathcal{F}_\delta[H^\delta]] \)."

[Clearly it is a \( C_\delta \)-name, or an \( \text{App} \ast C_\delta \)-name; see clause (d) of
   Definition \( \$2 \)(1).]

Then \( p_\delta \) is \( \Gamma \)-big.

(vii) Let \( N_\delta^0 = \prod_{n<\omega} N_n^0/\mathcal{F}_\delta \) (they are in \( \mathsf{V}^*[H^\delta] \), even in \( \mathsf{V}[G \upharpoonright \delta][H^\delta] \)).

We define \( R_{\delta,m} \subseteq (N_1^1)^m \times (N_2^2)^m \) for \( m < \omega \) so that

(\( \oplus \)) \( R_{\delta,m} \) includes the graph of \( F_\delta \), i.e., if \( \bar{a} \) is an \( m \)-tuple from \( N_1^1 \), then \( (\bar{a},F_\delta(\bar{a})) \in R_{\delta,m} \),
(\( \oplus \)) \( \delta \) the truth value of \( (\bar{a},\bar{b}) \in R_{\delta,m} \) depends only on \( L_{\omega,\omega}(\tau_{A_\delta}) \)-type
      realized by \( (\bar{a},\bar{b}) \) over \( M_{A_\delta} \) in \( M_\delta \),
(\( \oplus \)) \( \delta \) in fact \( R_{\delta,m} \) is minimal such that (\( \oplus \)) \( \delta \) hold.

(viii) The relations \( R_{\delta,m} \) mentioned above satisfy:

(\( \odot \)) \( \delta \) if \( \bar{a}_1,\bar{a}_2 \) are finite sequences of the same length \( m \) of members
      of \( N_1^1 \), and \( p_\delta \cup \{ \vartheta^{N_1}(x,\bar{a}_1),\neg \vartheta^{N_1}(x,\bar{a}_2) \} \) is a \( \Gamma \)-big type over \( M_\delta \), and \( \vartheta,\neg \vartheta \in \Delta[m] \), where \( \vartheta^{N_1} \) is \( \vartheta \) as interpreted in
      the interpretation \( \bar{a}^1 \),
      then \( (\bar{a}_1,F_\delta(\bar{a}_2)) \notin R_{\delta,m} \).
(\( \odot \)) \( \delta \) Above, we may replace \( \vartheta,\neg \vartheta \) by any pair \( \vartheta_0,\vartheta_1 \) of contradictory
      formulas from \( \Delta[m] \).
(ix) Note that also

\[ (*)_{\gamma^*, \delta} \quad p^* \Vdash_{C_{\aleph_3}} \text{"the } \Delta \text{-type which } y^* \text{ realizes over } N_\delta^2 = (\prod_{n<\omega} N_n^2/F)^{(\forall^*_{\aleph_3})} \]

in the model \( N^2 = (\prod_{n<\omega} N_n^2/F)^{(\forall^*_{\aleph_3})} \) includes the image under \( F \) of the \( \Delta \text{-type which} \)

\( \bar{x}_\delta/F \text{ realizes over } N_\delta^1 = (\prod_{n<\omega} N_n^1/F)^{(\forall^*_{\aleph_3})} \)

in the model \( N^1 = (\prod_{n<\omega} N_n^1/F)^{(\forall^*_{\aleph_3})} \)

(x) If \( \Delta \) is closed under negation we can naturally note that there are equivalence relations \( E_\ell \) on \( N_\delta^1 \) satisfying \((\circ)_2\) induced by \( (*)_{\gamma^*, \delta} \).

Also notice that the clauses \((b)_\delta, (c)_\delta\) of \([3.3]\) above say that \( F^\delta[G] \) is really a \( C_\delta \)-name for a function from \( (\prod_{n<\omega} N_n^1/F)^{\forall^*_{\aleph_3}} \) into \( (\prod_{n<\omega} N_n^2/F)^{\forall^*_{\aleph_3}} \) preserving \( \Delta \)-formulas; in the main case it is "onto".

**The proof of the Main Isomorphism Theorem.** The proof of \([3.3]\) is broken into several steps and lemmas. Note that we use the countability of \( \aleph_3 \).

Take a condition \( q_\delta \in G \) such that

(A)\(^{q_\delta}\) \( A^* \subseteq A^{q_\delta} \), \( \bar{x}_\delta, y^* \) are \( C_{A^{q_\delta}} \)-names (so \( \delta \in A^{q_\delta} \)), and \( p^* \in C_{A^{q_\delta}} \), and

(B)\(^{q_\delta}\) the condition \( q_\delta \) forces (in \( A^{q_\delta} \)) that clauses \((b)_\delta, (c)_\delta\) and \((d)_\delta\) from \([3.3]\) hold true (so in particular \( q_\delta \) forces that \( \bar{x}_\delta/F \in (\prod_{n<\omega} N_n^1/F), y^* \in \prod_{n<\omega} N_n^2 \) and \((*)_{\gamma^*, \delta} \) holds), and

(C)\(^{q_\delta}\) if \( \bar{x} \) is a \( C_{A^{q_\delta}} \)-name for a member of \( (\prod_{n<\omega} N_n^1) (\prod_{n<\omega} N_n^2) \), respectively,

then \( F(\bar{x}) (F^{-1}(\bar{x}), \text{respectively}) \) is also a \( C_{A^{q_\delta}} \)-name.

[In fact, also \( X \subseteq \prod_{n<\omega} A^* M_{A^*} \) by \([3.3]\).]

Before we continue with the proof of \([3.4]\) let us note the following.

**Lemma 3.5.** Let \( \delta < \aleph_3, q_\delta \in A^{\delta} \) and \( y^*, p^* \) be as above. Suppose that

\( q_\delta \vdash \delta = q \leq q' \in G \cap (A^{\delta} \upharpoonright \delta) \).

Let \( \bar{y}^* \) be a \( C_{A^*} \)-name of a \( \tau(t) \)-formula. Assume further that \( x', x'' \) and \( y', y'' \) are \( C_{A^{\bar{y}^*}} \)-names, and \( p^* \leq p \in C_{\aleph_3} \), and the condition \( p \upharpoonright A^{\bar{y}^*} \) forces (in \( C_{A^{\bar{y}^*}} \)) that

(\alpha) \( x', x'' \in \prod_{n<\omega} N_n^1 \), and \( y', y'' \in \prod_{n<\omega} N_n^2 \), and

(\beta) the types of \( (x', y') \) and \( (x'', y'') \) over \( \prod_{n<\omega} A^* M_{A^*} \) in the model

\( \prod_{n<\omega} A^* M_{A^*} \) (i.e., the vocabulary and the \( \omega \) structures are from \( V[G][H \cap C_{A^{\bar{y}^*}}] \), the ultraproduct is taken in \( V[G][H \cap C_{A^{\bar{y}^*}}] \) are equal.
Then the following conditions are equivalent.

(A) There is \( r^0 \in \mathcal{A} \) \( A \) such that \( q_\delta, q' \leq r^0, r^0 \upharpoonright \delta \in G \cap (\mathcal{A} \upharpoonright \delta), \) and

\[
p \Vdash_{\mathcal{A}^r} \phi \quad \text{if and only if} \quad \prod_{n < \omega} N^1_n / F^{r^0} \models \bar{\varphi}_n[x'/F^{r^0}, x_\delta / F^{r^0}] \quad \text{and} \quad \prod_{n < \omega} N^2_n / F^{r^0} \models \bar{\varphi}_n[y'/F^{r^0}, y_\delta / F^{r^0}].
\]

(B) There is \( r^1 \in \mathcal{A} \) such that \( q_\delta, q' \leq r^1, r^1 \upharpoonright \delta \in G \cap (\mathcal{A} \upharpoonright \delta) \) and

\[
p \Vdash_{\mathcal{A}^r} \phi \quad \text{if and only if} \quad \prod_{n < \omega} N^1_n / F^{r^1} \models \bar{\varphi}_n[x''/F^{r^1}, x_\delta / F^{r^1}] \quad \text{and} \quad \prod_{n < \omega} N^2_n / F^{r^1} \models \bar{\varphi}_n[y''/F^{r^1}, y_\delta / F^{r^1}].
\]

Proof. By symmetry it suffices to show that (A) implies (B). So suppose that \( r^0 \) is as in (A). We may also assume that \( A^r = A^q \cup A^{q'} \) (just replace \( q' \) by the stronger condition \( r_\delta \upharpoonright \delta \) if needed). We want to define a respective condition \( r^1 \) so that \( A^r = A^r \) \( \text{def} \ A \), and for this we need to extend \( F^q \cup F^{q'} \) to an ultrafilter containing the set

\[
\{ n \in \omega : N^1_n \models \bar{\varphi}_n[x''(n), x_\delta(n)] \} \cap \{ n \in \omega : N^2_n \models \bar{\varphi}_n[y''(n), y_\delta(n)] \}.
\]

Suppose toward contradiction that this is impossible and thus we have a condition \( p' \in \mathcal{C}_A \) stronger than \( p \), a \( \mathcal{C}_{A^q} \)–name \( a \) of a member of \( F^q \), and a \( \mathcal{C}_{A^{q'}} \)–name \( b \) of a member of \( F^{q'} \) such that \( p' \) forces

\[
\text{“} a \cap b \cap \{ n \in \omega : N^1_n \models \bar{\varphi}_n[x''(n), x_\delta(n)] \} \& N^2_n \models \bar{\varphi}_n[y''(n), y_\delta(n)] \} = \emptyset \text{”}.
\]

Let \( A_1 = A^q, A_2 = A^{q'}, A_0 = A_1 \cap A_2 = A^q, \) and \( p_i = p' \upharpoonright A_i \) (for \( i = 0, 1, 2 \)). Let \( H^0 \subseteq \mathcal{C}_{A_0} \) be generic over \( V \) such that \( p_0 \in H^0 \), and for \( n \in \omega \) let \( A_0^r \) be the \( \mathcal{C}_{A_0} \)–name such that

\[
A_0^r[H^0] = \{(u, v) : \text{there is a condition} \ p'_1 \in \mathcal{C}_{A_1} \text{ such that} \ p_1 \leq p'_1, p'_1 \upharpoonright A_0 \in H^0 \text{ and} \ p'_1 \text{ forces} \ n \in a \& N^1_n \models \bar{\varphi}_n[x''(n), x_\delta(n)] \& N^2_n \models -\bar{\varphi}_n[y''(n), y_\delta(n)] \}.
\]

Let \( Q_{\bar{\varphi}} \) be the predicate in \( \tau_{A_0} \) corresponding to the sequence \( \bar{A}^r = \langle A_0^r : n < \omega \rangle \), see [2,1](4). Note that

\[
(\otimes') \quad p_2 \Vdash_{\mathcal{C}_{A_2}} \prod_{n < \omega} M^n / F^{q'} \models Q_{\bar{\varphi}}(x', y').
\]

[Why? Assume not and let \( H' \subseteq \mathcal{C}_A \) be a generic over \( V \) such that \( H^0 \subseteq H', \ p' \in H', \) and we have (in \( V^{H'} \))

\[
\mathfrak{c} \overset{\text{def}}{=} \{ n : \neg A_0^r(x''(n), y''(n)) \} \in F^{q'} \subseteq F^{r^0}, \text{ and } \ a \in F^q \subseteq F^{r^0}.
\]

By our assumption

\[
\mathfrak{d} \overset{\text{def}}{=} \{ n : N^1_n \models \bar{\varphi}_n[x''(n), x_\delta(n)] \& N^2_n \models -\bar{\varphi}_n[y''(n), y_\delta(n)] \} \in F^{r^0}.
\]
so \( a \cap c \cap d \neq \emptyset \). Consequently, we may find \( u, v \) and a condition \( p^{\diamond} \in H' \) stronger than \( p' \) such that

\[
p^{\downarrow} \models_{C_A} x'(n) = u \land y'(n) = v \land \neg A^*_n(u, v) \land n \in a \land 
N_n \models \bar{y}^*[u, x_\delta(n)] \land N_n^2 \models \neg \bar{y}^*[v, y_\delta(n)].
\]

Then in particular \( (u, v) \notin A^*_n[H^0] \), but also \( p^{\downarrow} \upharpoonright A_1 \) witnesses \( (u, v) \in A^*_n[H^0] \), a contradiction.

Therefore, by assumption \((\beta)\),

\[
(\otimes)^{\prime \prime} \quad p_2 \models_{C_{A_2}} \prod_{n < \omega} A'^n_{A^\delta}/F' \models A^*(x'', y'').
\]

Thus we may choose \( n, u, v \) and a condition \( p'_2 \in C_{A^\delta} \) such that \( p_2 \leq p'_2 \), \( p'_2 \upharpoonright A_0 \in H^0 \), and

\[
p'_2 \models_{C_{A_2}} \quad n \in b \land x''(n) = u \land y''(n) = v \land A^*_n(u, v).
\]

Since \( (u, v) \in A^*_n[H^0] \), we may pick a condition \( p'_1 \in \mathbb{C}_{A_1} \) stronger than \( p_1 \) and such that \( p'_1 \upharpoonright A_0 \in H^0 \) and

\[
p'_1 \models_{C_{A_1}} \quad n \in a \land N_n^1 \models \bar{y}^*[u, x_\delta(n)] \land N_n^2 \models \neg \bar{y}^*[v, y_\delta(n)].
\]

Then \( p'' \overset{\text{def}}{=} p' \cup p'_1 \cup p'_2 \in \mathbb{C}_A \) is a condition stronger than \( p' \) and

\[
p'' \models \quad n \in a \cap b \land N_n^1 \models \bar{y}^*[x''(n), x_\delta(n)] \land N_n^2 \models \neg \bar{y}^*[y''(n), y_\delta(n)],
\]

a contradiction. \(\square\)

Let us go back to the proof of 3.3. We define some \( C_\delta \)-names; recall \( H^\delta \subseteq \mathbb{C}_{N_3} \upharpoonright \delta \) is generic over \( \mathbf{V}^*, \mathcal{F}[H^\delta] = \bigcup \{ \mathcal{F}'[H^\delta] : r' \in G_\delta \} \), and

\[
M^\delta_n = \prod_{n < \omega} M^0_n/F^\delta, \quad \text{and} \quad N^\delta_n = \prod_{n < \omega} N^1_n/F^\delta \quad \text{for } (\ell = 1, 2).
\]

Let

\[
Z^0_\delta[H^\delta] = \left\{ \left( x/F^\delta, y/F^\delta \right) \in N^1_\delta \times N^2_\delta : \right. \text{there are a } \tau(t) \text{-formula } \varrho \in \Delta \text{ and conditions } p \in \mathbb{C}_{N_3} \text{ and } r^0 \in \text{App such that } \left. p^* \leq p, p \upharpoonright \delta \in H^\delta, x, y \text{ are } \mathbb{C}_{A'^\delta \cap \delta} \text{-names, and } \right.
\]

\[
q^\delta \leq r^0, \quad r^0 \upharpoonright \delta \in G \cap (\text{App} \upharpoonright \delta), \quad \text{and } \left. p \models_{C_{A'^\delta}} \prod_{n < \omega} A'^n_n/F^0 \models \varrho[x/F^0, x_\delta/F^0] \text{ and } \prod_{n < \omega} A'^n_n/F^0 \models \neg \varrho[y/F^0, y_\delta/F^0] \right\}.
\]

\[
Z^0_\delta[H^\delta] = (N^1_\delta \times N^2_\delta) \setminus Z^1_\delta.
\]

Now, it follows from 3.5 (and 2.8) that

\[(\square)_\delta \text{ in } \mathbf{V}[G \cap (\text{App} \upharpoonright \delta)][H^\delta], \text{ if the types realized by } (x/F^\delta, y/F^\delta) \text{ and } (x''/F^\delta, y''/F^\delta) \text{ over the model } \prod_{n < \omega} M^\delta_n/F^\delta \delta \text{ in the model } \prod_{n < \omega} M^\delta_n/F^\delta \delta\text{ are equal, then}
\]

\[
(x'/F^\delta, y'/F^\delta) \in Z^0_\delta \quad \text{if and only if} \quad (x''/F^\delta, y''/F^\delta) \in Z^0_\delta.
\]
Now, most clauses of §3.4 should be clear; we say more on (d)(vii,viii), for simplicity for \( m = 1 \).

We let \( R_{\delta,1} = Z^n_{\delta} \), so clause (d)(vii)(\( \oplus \)) holds.

Since \( F \) is (an \( \mathbb{A} \text{pp} \ast \mathbb{C}_{\mathbb{N}_3} \)-name for) a \( \Delta \)-embedding from \( \prod_{n<\omega} N_n^1 / F \) onto \( \prod_{n<\omega} N_n^2 / F \), if \( \mathcal{H} / F^\delta \in N_\delta^1 \), then \( \vDash_{\mathcal{C}_\delta} " (\mathcal{H} / F^\delta, F(\mathcal{H}) / F^\delta) \in Z^n_{\delta} " \). Hence clause (d)(viii)(\( \oplus \)) holds.

Thus the proof of §3.4 is completed. \( \blacksquare \)

**Conclusion 3.6.** In \( V[G][H^n_{\aleph_3}] \), for each \( m \), there is a stationary set \( S \subseteq \{ \delta < \aleph_3 : \text{cf}(\delta) = \aleph_2 \} \) and conditions \( \varphi, \delta \in \mathbb{A} \text{pp} \) such that for each \( \delta \in S \):

- clauses (a)\( \delta \)-\( \delta \) of §3.3 are satisfied,
- \( \varphi \notin G \), \( \varphi \not\vDash \delta = \varphi, \varphi, \varphi \) as in §3.3,
- the conclusion of §3.4 holds,
- for every \( \delta_1, \delta_2 \in S \) there is a one-to-one order preserving function \( h : A^{\varphi_1} \rightarrow A^{\varphi_2} \) (so it is the identity on \( A^{\varphi} \)) which maps \( \delta_1, \mathcal{H}^{\delta_1}, F(\mathcal{H}^{\delta_1}) = y^{\delta_1} \) onto \( \delta_2, \mathcal{H}^{\delta_2}, F(\mathcal{H}^{\delta_2}) = y^{\delta_2} \),
- in particular \( p_\delta = p^\ast \) for \( \delta \in S \), so the last clause in §3.4 holds for \( M_{\aleph_3} \).

**Proof.** Straightforward. \( \square \)

4. BACK TO MODEL THEORY

In this section we present just enough to solve the problem on finite fields.

**Definition 4.1.** Let \( M \) be a model. Assume \( N_1 = M^{\varphi^1} \), \( N_2 = M^{\varphi^2} \) are models of \( t_0 \) interpreted in \( M \) by the sequences \( \varphi^1, \varphi^2 \) of formulas with parameters from \( M \), and they have the same vocabulary \( \tau^* = \tau(N_1) = \tau(N_2) \). Furthermore, let \( \Gamma \) be an invariant bigness notion in \( M \) (over some set \( A_0 \) of \( < \kappa \) parameters, more exactly in \( K_{(M,A_0)} \), and \( \Delta \subseteq L_{\omega,\omega}(\tau(N_1)) \) and \( \kappa > \aleph_0 \) (for simplicity).

1. We say that \( (N_1, N_2) \) is \( (\kappa, \Gamma, \Delta) \)-complicated in \( M \) if:
   - for every \( \Delta \)-embedding \( F \) of \( N_1 \) into \( N_2 \), and for every \( \Gamma \)-big type \( p_0(x) \) inside \( M \) of cardinality \( < \kappa \), there is a \( \Gamma \)-big type \( p_1(x) \) inside \( M \) of cardinality \( < \kappa \) which includes \( p_0(x) \) and such that, letting \( \tau(p) \subseteq \tau(M) \) consist of those mentioned in \( p(x) \) (so \( |\tau(p)| < \kappa \)) and \( A \subseteq M \) be the set of parameters of \( p_0 \) (so \( A_0 \subseteq A \) ), we have \( (\ast)p(x) \) if

\[
R_m = \{ (\bar{a}, \bar{b}) : \bar{a} \in m(N_1), \bar{b} \in m(N_2) \text{ and for some } \bar{c} \in m(N_1) \text{ we have} \\
\text{tp}_{L_{\omega,\omega}(\tau(p))}(\bar{a} \sim \bar{b}, A, M) = \text{tp}_{L_{\omega,\omega}(\tau(p))}(\bar{c} \sim F(\bar{c}), A, M) \}.
\]

then we have the parallel of §3.4(\( \ast \)), so

\( (\oplus) \) if \( \bar{a}_1, \bar{a}_2 \) are finite finite sequences of the same length \( m \) of members of \( N_\delta^1 \), and \( p \cup \{ \vartheta N_1(x, \bar{a}_1), \neg \vartheta N_1(x, \bar{a}_2) \} \) is a \( \Gamma \)-big type over \( M_\delta \), then \( (\bar{a}_1, F_\delta(\bar{a}_2)) \notin R_m \).
(⊕)2 Above, we may replace \( \vartheta, \neg \vartheta \) by any pair \( \vartheta_0, \vartheta_1 \) of contradictory formulas from \( \Delta \).

2. In part (1):
   (i) We do not mention \( \Delta \) if it is the set of quantifier free formulas (of \( L_{\omega,\omega}(\tau(N_1)) \)).
   (ii) We replace \( \Gamma \) by \( (t, \psi) \) if we mean "for all bigness notions of the form \( \Gamma = \Gamma(t, \varphi, \psi) \), where \( \varphi \) is an interpretation of \( t \) in \( M \) with \( < \kappa \) parameters and \( |t| < \kappa, \psi \in L_{\kappa,\omega} \)" (i.e., \( \psi \in L_{\mu+1,\omega} \) for some \( \mu < \kappa \)).
   (iii) We omit \( \Gamma \) if we mean "for all \( \Gamma \)'s as in (ii)".
   (iv) We say \( M \) is \( \kappa \)-complicated (or: \( (\kappa, \Gamma, \Delta) \)-complicated) and omit \( N_1, N_2 \) if this holds for all \( N_1, N_2 \) as in our assumptions, but with \( |\tau(N_1)| < \kappa \).

Remark 4.2. We can add about the equivalence relation, implicit in 3.4, see [Sh:F503].

Theorem 4.3.
1. Let \( G \) be a full \( (\aleph_3, \aleph_2) \)-bigness guide (see 2.2; recall there is one by 2.3). Assume that \( G \subseteq \text{App}_G \) is generic over \( V \) and \( H \subseteq C_{\aleph_3} \) is generic over \( V[G] \) and \( F = F_{\aleph_3}[G][H] \), and let \( \langle M_n = M^n_{\aleph_3} : n < \omega \rangle \) be a sequence of models as above, that is each with a countable universe being the set of natural numbers for simplicity, all with the same vocabulary such that for every \( k \) and a sequence \( \langle R_n : n < \omega \rangle \) with \( R_n \) being a \( k \)-place relation on \( M_n \) there is a \( k \)-place predicate in the common vocabulary satisfying \( R^M_n = R_n \) for each \( n \). Then in \( V[G][H] \) the model \( M = \prod_{n<\omega} M^n_{\aleph_3}/F \) is \( \aleph_2 \)-complicated and \( \aleph_2 \)-compact.
2. We can change the demands on \( G \) accordingly to the version of \( \aleph_2 \)-complicated.
3. If \( N_1, N^2 \) are models of \( t^{\text{ind}}_1 \) interpreted in \( M \), then any isomorphism \( \pi \) from \( N^1 \) onto \( N^2 \) is definable in \( M \).
4. If \( N^\ell = \prod_{n<\omega} N^n_{\ell}/F, \) each \( N^n_{\ell} \) is countable, and \( N^\ell \) is a model of \( t^{\text{ind}}_1 \) (for \( \ell = 1,2 \)), and \( N^1, N^2 \) are isomorphic, then there are \( A \in F \) and isomorphisms \( \pi_n \) from \( N^n_1 \) onto \( N^n_2 \) (for \( n \in A \)) such that \( \pi = \prod_{n<\omega} \pi_n/F \).
5. Above we replace : \( N^\ell \) is a model of \( t^{\text{ind}}_1 \) by "for some formula \( \phi(x,y) \) in the vocabulary of \( N^1 \) which is equal to that of \( N^2 \), has the strong independence property (in their common theory; \[\Box\])

\[\text{\footnotesize of course of the strong independence property holds when we restrict ourselves to say a predicate } P \text{ we get less, but see } [\text{Sh:F503}]\]
6. If $N_n^\ell$ are finite fields (for $\ell = 1, 2$ and $n < \omega$), and $\prod_{n<\omega} N_n^1/F$ is isomorphic to $\prod_{n<\omega} N_n^2/F$, then the set $\{n < \omega : N_n^1 \simeq N_n^2\}$ belongs to $F$.

Proof. (1) By 3.6.
(2) The same proof.
(3) By 4.4 below and 1.6(2).
(4) Without loss of generality, the universe of $N_n^\ell$ is $\alpha_n^\ell \leq \omega$. Now, for $\ell = 1, 2$, we can find $P_\ell \in \tau_M$ such that $(P_\ell)^{M_{\aleph_3}} = |N_n^\ell|$ and for $Q \in \tau(N_n^\ell)$ there is $Q[\ell] \in \tau_M$ with $(Q[\ell])^{M_{\aleph_3}} = Q^{N_n^\ell}$. Therefore, $N_n^\ell = \prod_{n<\omega} N_n^\ell/F$ can be viewed as an interpretation in $M$ by $\bar{\varphi}^\ell$. Now apply part (3) for $\Gamma = \Gamma(\text{ind}, \bar{\varphi}^1, \psi\text{ind})$. (5) This follows by part (4), as the vocabulary is finite, being a nn isomorphism is expressible by a first order sentence. (6) This is a particular case of part (5). By part (4) it suffice for infinite ultraproducts $N_n^\ell$ of finite fields to find a formula $\vartheta(x, y)$ in the vocabulary of fields which has the strong independence property. First we deal with the case that the fields are of characteristic $p > 2$. Consider the formula $\vartheta(x, y)$ saying that $x + y$ has a square root in the field.

We relay a theorem of Duret, [Du80, p. 982, Lemma 10], choosing $p = 2$, a its hypothesis holds as the field contains all $p$-th root of the unit (that is 1 and 1). The conclusion says that for $n$ and any pairwise distinct elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of the field there is an element $c$ such that $a_m + c$ has a square root and $b_m + c$ does not have a square root for $m = 1, \ldots, n$. So the formula $\vartheta(x, y) = (\exists z)(z^2 = x + y)$ is as required.

Of course, if the characteristic of the field is 2, then we naturally use the same theorem but choosing $p = 3$, so of course maybe the field fail to have all $p$-th root of the unit, however, as Duret does, we consider an algebraic extension of $N_n^\ell$ of order 3 by adding a root of $x^3 - 1$ hence all of them getting a new field $N_n^\ell$. Now the set of elements of $a_1^\ell N$ can be represented as the set of triples of elements of $N_n^\ell$, and the operations of $N_n^\ell$ are definable in $N^\ell$. So our problem is almost notational. E.g., we can note that recalling $N^\ell = \prod_{n<\omega} N_n^\ell/F$ then $N^\ell = \prod N_{n, n}^\ell/F$ where $N_{n, n}^\ell$ is equal to $N_n^\ell$ if $N_n^\ell$ has three 3-th roots of the unit and an algebraic extension of $N_n^\ell$ of order three which has this property otherwise. Again the first order theory of $N_n^\ell$ has the strong and for $N_n^1, N_n^2$ we get the desired conclusion but any isomorphism from $N_n^1$ onto $N_n^2$ can be extended to an isomorphism from $N_n^1$ onto $N_n^2$ and we can easily finish.

\[\square\]

**Proposition 4.4.** Assume that $M$ is a $\kappa$–complicated $\kappa$–compact model. Let $N_1, N_2$ be interpretations of $\text{ind}_{\kappa}$ in $M$. Then for any isomorphism $\pi$
from $N_1$ onto $N_2$, the function $\pi \upharpoonright P^{N_1}$ is definable in $M$ by a first order formula (with parameters).

Proof. Let $N_1 = M^\varphi$ (so $\varphi$ has parameters in $M$) and let $F$ be an isomorphism from $N_1$ onto $N_2$.

Let $\Gamma$ be the bigness notion $\Gamma_{(\text{ind}, \varphi, \psi)}$ (so $\psi \in L_{\omega_1, \omega}$). Let $A \subseteq M$, $|A| < \kappa$ and $\tau^* \subseteq \tau_M$, $|\tau^*| < \kappa$ be given by the definition of being $\kappa$–complicated (applied to $F$). Without loss of generality, $A$ includes the parameters of $\varphi^1, \varphi^2$ and is closed under $F$ and $F^{-1}$, and for every $n$ includes the finite set mentioned in (1).

Let $R_1$ be as in (1). Clearly, recalling Definition (2), there are no distinct $a_1, a_2 \in P^{N_1} \setminus A$ and $b \in N_2$ such that $(a_1, b), (a_2, b) \in R_1$. Hence

\[
\{ (a, b) : (a, b) \in R_1 \text{ and } a \in P^{N_1} \}\n\]

is the graph of a partial function from $P^{N_2}$ into $P^{N_1}$ which includes the graph of $F^{-1} \upharpoonright P^{N_1}$. Therefore, $R_1 \upharpoonright (P^{N_1} \times P^{N_2})$ is the graph of $F \upharpoonright P^{N_1}$. But $R_1 \upharpoonright P^{N_1}$ is definable in $(M \upharpoonright \tau^*, c)_{c \in A}$ by a formula from $L_{\infty, \kappa}$, so also $F \upharpoonright P^{N_1}$ is, and thus if $N_1, N_2$ are models of $\text{ind}$ also $F$ is (by (3)). Applying [Sh 72, 1.9] (or [Sh:e, Ch XI]) we conclude that it is definable by a first order formula with parameters from $M$, as required. \hfill \Box

Similarly we can show the following.

Proposition 4.5. Assume that $\Gamma$ is a $(\aleph_2, \aleph_1)$–$(P, \vartheta)$–separable bigness notion. Suppose that $N_1, N_2$ are interpretations of $t$ in $M$, and $M$ is $\kappa$–compact $\kappa$–complicated (or just $\kappa$–complicated for $\Gamma$), $\kappa > \aleph_0$.

1. If $F$ is an isomorphism from $N_1$ onto $N_2$, then

   ($*1$) $F \upharpoonright P^{N_1}$ is definable in $(M \upharpoonright \tau^*, c)_{c \in A}$ by a formula from $L_{\infty, \kappa}$, recalling $\tau \subseteq \tau_M$, $|\tau| < \kappa$, $A \subseteq M$, $|A| < \kappa$.

2. If $F$ is an embedding of $N_1$ into $N_2$, then

   ($*2$) there is a partial function $f$ from $P^{N_2}$ into $P^{N_1}$ which extends $F^{-1}$ and is definable in $(M \upharpoonright \tau^*, c)_{c \in A}$ by a formula from $L_{\infty, \kappa}$, where $\tau^*, A$ are as above.

Remark 4.6. 1. The proposition /ref4.4 should be the beginning of an analysis of first order theories $T$. For more in this direction see [Sh 702], [Sh:F503].

2. As stated in the introduction, we may avoid the preliminary forcing with $\text{App}$ and construct the name $F$ in the ground model $V$, provided $V$ is somewhat $L$–like. Assuming $\diamondsuit = \aleph_2$ is enough, but we may also use the weaker principle from [HLSh 162] and [Sh 405, Appendix].

3. We may vary the cardinals, e.g., we may replace $\aleph_2, \aleph_3$ by $\kappa, \lambda$, respectively, provided $\lambda = 2^\kappa$, $\kappa = \kappa^{< \kappa}$ (so an approximation has size $< \kappa$).

So let us assume that

$\theta = \theta^{< \theta} < \kappa = \kappa^{< \kappa} < \lambda = \kappa^+$. 


(a) For $A \subseteq \lambda$ let $C(A) = \mathcal{C}_A = \{p : p$ is a partial function from $\text{Dom}(p) \in [A]^{<\theta}$ to $\theta^2 \} \text{ ordered by}$

\[ p_1 \leq_{C(A)} p_2 \text{ iff } \text{Dom}(p_1) \subseteq \text{Dom}(p_2) \& (\forall \alpha \in \text{Dom}(p_1))(p_1(\alpha) \leq p_2(\alpha)). \]

(b) We define $\text{App}_G$ as the set of $q = (A^q, \mathcal{F}^q)$ where $A^q \in [\lambda]^{<\kappa}$ and $\mathcal{F}^q$ is a $\mathcal{C}_{A^q}$–name of a regular ultrafilter on $\theta$ such that for each $\alpha < \lambda$, $\mathcal{F}^q \cap \mathcal{P}(\theta)^{V(G(C(A)))}$ is a $\mathcal{C}_{A^q\cap \alpha}$–name.

(c) For $\alpha \in A \in [\lambda]^{<\kappa}$, $\mathcal{F}_\alpha$ is the $\mathcal{C}_A$–name $\bigcup \{p(\alpha) : p \in G(C(A)) \text{ of a member of } \theta \}$.

(d) We define $N^\varepsilon_A$ for $\varepsilon < \theta$, $A \in [\lambda]^{<\kappa}$ as the following $\mathcal{C}_A$–name:

it is a model with universe $\theta$,

\[ \tau^\varepsilon_A = \{P^\varepsilon_B : \mathcal{R}_\varepsilon = (R^\varepsilon : \varepsilon < \theta, \text{ for some } m \text{ each } R^\varepsilon \text{ is an } m\text{-place relation } \}, \]

\[ (P^\varepsilon_B)^N_A = R^\varepsilon. \]

[So we may think of $\tau^\varepsilon_A$ to be an old object whose members are indexed as $P^\varepsilon_B$, where each $R^\varepsilon$ is a $\mathcal{C}_A$–name. Or we can consider $\tau^\varepsilon_A$ to be a name and interpret it in $V[G(C(A))]$.]

References

[CK] Chen C. Chang and Jerome H. Keisler. Model Theory, volume 73 of Studies in Logic and the Foundation of Math. North Holland Publishing Co., Amsterdam, 1973.

[Du80] Jean-Louis Duret. Les corps faiblement algébriquement clos non séparablement clos ont la propriété d’indépendance. In Model theory of algebra and arithmetic (Proc. Conf., Karpacz, 1979), volume 834 of Lecture Notes in Math., pages 136–162. Springer, Berlin – New York, 1980.

[HLSh 162] Bradd Hart, Claude Laflamme, and Saharon Shelah. Models with second order properties, V: A General principle. Annals of Pure and Applied Logic, 64:169–194, 1993. math.LO/9311211

[J] Thomas Jech. Set theory. Academic Press, New York, 1978.

[Ke67] Jerome H. Keisler. Ultraproducts which are not saturated. Journal of Symbolic Logic, 32:23–46, 1967.

[Sh 482] Saharon Shelah. In Non structure theory, Ch XI, accepted. Oxford University Press.

[Sh 384] Saharon Shelah. Compact logics in ZFC : Complete embeddings of atomless Boolean rings. In Non structure theory, Ch X, accepted. Oxford University Press.

[Sh:e] Saharon Shelah. Non-structure theory, accepted. Oxford University Press.

[Sh 702] Saharon Shelah. On what I do not understand (and have something to say), model theory. Mathematica Japonica, to appear. math.LO/9910156

[Sh 13] Saharon Shelah. Every two elementarily equivalent models have isomorphic ultrapowers. Israel Journal of Mathematics, 10:224–233, 1971.

[Sh:a] Saharon Shelah. Classification theory and the number of nonisomorphic models, volume 92 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam-New York, xvi+544 pp, $62.25, 1978.

5References of the form math.XX/··· refer to the xxx.lanl.gov archive
Saharon Shelah. Models with second-order properties. I. Boolean algebras with no definable automorphisms. *Annals of Mathematical Logic*, **14**:57–72, 1978.

Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.

Saharon Shelah. Vive la différence I: Nonisomorphism of ultrapowers of countable models. In *Set Theory of the Continuum*, volume 26 of *Mathematical Sciences Research Institute Publications*, pages 357–405. Springer Verlag, 1992. 

Saharon Shelah. Vive la différence II. The Ax-Kochen isomorphism theorem. *Israel Journal of Mathematics*, **85**:351–390, 1994.

Shelah, Saharon. t-rigid models of T.

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

E-mail address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah