Field Theory of \( N \) Entangled Polymers

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We formulate a field theory capable of describing the entanglement in a canonical ensemble of \( N \) polymers in terms of Feynman diagrams.

I. INTRODUCTION

The discovery of the intimate relationship between the statistical mechanics of long polymer molecules and certain field theories describing critical systems has been crucial in achieving the present understanding of the physics of unentangled polymer chains \[1\].

Under certain experimental conditions, polymers form knotted configurations which remain stable in time. In this case, the topological relationships among the molecules become relevant. The principal theoretical tools for investigating these were set up by S. Edwards \[2\] and Brereton and Shaw \[3\], and formulated by one of the authors (HK) with the help of the topological field theory of Chern and Simons \[4\]. While the tools are available, a satisfactory theoretical treatment of an ensemble of topologically linked polymers is still missing, in spite of much work \[5–10]\.

Definite progress has recently been achieved by the present authors \[11,12\] by mapping the statistical mechanics of two fluctuating polymers to a topological Ginzburg-Landau model, and applying Feynman diagram techniques to calculate the average of the square winding number. In the present work we extend the field theoretic formulation of \[11,12\] to a system of \( N \) polymers.

Excluded volume forces, although physically important, have so far been neglected, and will be ignored also in this work.

We start in Section II from the path integral description of polymers in which these are viewed as Brownian trajectories. The topological constraints are imposed using the simplest link invariant; the Gauss link integral for pairs of trajectories. The topological interactions are complicated, and several nontrivial steps are necessary to arrive at a useful field theoretic description of the system. For this purpose we introduce in Section III a set of auxiliary abelian Chern-Simons fields. These allow us to make the theory local and to convert the polymer path integral to a Markovian form. Actually, there are several possible abelian Chern-Simons field theories which could accomplish this task, differing from each other by the number of fields. However, we prove that these are equivalent after exploiting the field equations and the freedom of performing linear transformations of the fields.

The arbitrariness in choosing the auxiliary topological field theory is used in Section VI to overcome a technical problem which was absent in the two polymer case: the coupling constants of the interactions between Chern-Simons fields and polymer trajectories are related to the parameters for the topological constraints by non-linear algebraic equations. The latter are too complicated to be solved analytically apart from particular cases, in which the parametrization simplifies considerably. A particular choice of the Chern-Simons fields removes this problem.

With these methods we convert the polymer path integral to a Markovian form and it becomes possible to complete the mapping of the polymer problem into a field theoretical model of topological entanglement. This is achieved in Section VII by exploiting the method of replica \[13\].

II. PATH INTEGRAL APPROACH TO TOPOLOGICAL POLYMERS

Let \( P_1, \ldots, P_n \) be a set of topologically linked polymers of lengths \( L_1, \ldots, L_N \) respectively. In order to keep our treatment as general as possible we consider both, open and closed chains of polymers. Strictly speaking, the entanglement of open chains is not really topological. Over a long time scale, it is always possible to disentangle open polymers, and the topology of the system is not conserved. However, if we restrict ourselves to intermediate time scales short compared to that of complete molecular rearrangements, open chains will behave almost like closed chains. In this approximate sense we shall be able to apply our topological methods also to open chains. Moreover, the study of the difference between the behaviors of open and closed chains will make it possible to gain some insight in the nature of topological interactions \[3\]. Thus, we shall start our considerations with

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open polymers $P_i$ with $i = 1, \ldots, N$, whose end points lie at $x^i, y^i$. For coinciding end points $x^i = y^i$, they become closed polymers running through $x^i$.

The relevant quantity to describe the statistical mechanics of the polymers is the configurational probability $G_{(m)}(\vec{x}, \vec{y}; \vec{L})$, which measures the probability to find the polymers $P_i$ with end points at $x^i, y^i$ in a given topological configuration $\{m\}$. To compactify the notations, we have collected the set of all end points in $N$-dimensional multi-vectors $\vec{x} = (x^1, \ldots, x^N)$, $\vec{y} = (y^1, \ldots, y^N)$, with an analogous vector notation for the lengths of the polymers: $\vec{L} = (L_1, \ldots, L_N)$. To distinguish different topological configurations, we shall use the Gauss link invariant:

$$\chi(P_i, P_j) = \frac{1}{\sqrt{J(N)}} \int_0^{L_i} ds_i \int_0^{L_j} ds_j \delta(s_i) \left[ \frac{\hat{x}^i(s_j) - \hat{x}^j(s_i)}{\hat{y}^i(s_i) - \hat{y}^j(s_i)} \right]^3,$$

defined for each pair $P_i, P_j$ of polymers with $i \neq j$.

The integral (3) is well-defined also for open trajectories, but it becomes a real topological invariant only for a pair of closed polymers with $x^i \rightarrow y^i$, $x^i \rightarrow y^i$. In the latter case it counts how many times $P_i$ winds up around $P_j$.

Following the approach of Edwards [22], the configurational probability can be expressed as a path integral over all possible configurations $x^i(s_i)$ with $0 \leq s_i \leq L_i$ and periodic boundary conditions $x^i(0) = y^i$ and $x^i(L_i) = x^i$:

$$G_{(m)}(\vec{x}, \vec{y}; \vec{L}) = \int_{x^1}^{x^1} D\vec{x}'(s_1) \cdots \int_{x^N}^{x^N} D\vec{x}'(s_N) e^{-\int_0^L \nu}$$

$$\times \prod_{i=1}^{N-1} \prod_{j=i+1}^N \delta(\chi(P_i, P_j) - m_{ij}),$$

where $A_0$ is the euclidean action of a random-chain

$$A_0 = \frac{3}{2a} \sum_{i=1}^N \int_0^{L_i} \hat{x}'^2 ds_i,$$

and

$$A_v = \frac{1}{2a^2} \sum_{i,j=1}^N \int_0^{L_i} ds_i \int_0^{L_j} ds_j v_{ij}^0 \delta^{(3)}(x^i(s_i) - x^j(s_j))$$

(4)

the steric repulsion between the chain elements. The parameter $a$ denotes the length of the chain elements and $v_{ij}^0$ represents an $N \times N$ matrix of coupling constants with the dimension of a volume.

The $\delta$-functions in the integrand of Eq. (3) enforce the topological constraints that the pairs of chains $P_i, P_j$ wind around each other a number of times $m_{ij}$. Since we are describing open polymers up to this point, these numbers are continuous. Only for closed polymers will they become integer numbers. Then the Dirac $\delta$-functions in Eq. (3) have to be replaced by Kronecker $\delta$'s.

In the following, it will be convenient to introduce an auxiliary probability $G_{(\lambda)}(\vec{x}, \vec{y}; \vec{L})$, from which the original $G_{(m)}(\vec{x}, \vec{y}; \vec{L})$ is obtained by a Fourier transformation with respect to the topological numbers:

$$G_{(m)}(\vec{x}, \vec{y}; \vec{L}) = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \prod_{j=2}^{N} \frac{d\lambda_{ij}}{2\pi} e^{-\lambda_{ij} m_{ij}} G_{(\lambda)}(\vec{x}, \vec{y}; \vec{L}).$$

(5)

The auxiliary probability has the advantage that its path integral representation

$$G_{(\lambda)}(\vec{x}, \vec{y}; \vec{L}) = \int_{\vec{y}} \mathcal{D}\vec{x}'(s_1) \cdots \int_{\vec{y}} \mathcal{D}\vec{x}'(s_N) e^{\lambda_{ij} m_{ij}},$$

(6)

accounts for the topological constraints among the polymers by a source like term:

$$A_{\text{top}} = \int \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \sum_{i=1}^{N} \chi(P_i, P_j) - 1 \right) \lambda_{ij}$$

(7)

Note that if in a formulation for closed polymers, where $m_{ij}$ are integer numbers and the Dirac $\delta$-functions in Eq. (3) are Kronecker symbols, the Fourier variable $\lambda_{ij}$ would be cyclic with a range $\lambda_{ij} \in (0, 2\pi)$. Then we would use angular variables $\varphi_{ij}$ rather than $\lambda_{ij}$, and with Eq. (3) as

$$G_{(m)}(\vec{x}; \vec{L}) = \int_0^{2\pi} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \frac{d\varphi_{ij}}{2\pi} e^{-i\varphi_{ij} m_{ij}} G_{(\lambda)}(\vec{x}; \vec{L}).$$

(8)

where

$$G_{(m)}(\vec{x}; \vec{L}) = \lim_{\vec{x} \rightarrow \vec{y}} G_{(m)}(\vec{x}, \vec{y}; \vec{L}).$$

(9)

Returning to (3), we see that it has the form of a path integral over the trajectories of a system of $N$ particles performing a random walk. Thus we may exploit the duality between particles and fields, which is valid in statistical mechanics as well as quantum mechanics, and express the path integral in terms of fields. Although the general techniques are well-known [23], the specific task here is complicated by the presence of the topological term (3), which is non-Markoffian and rather complicated. We shall solve this problem by introducing auxiliary fields, which allow us to rewrite the right-hand side of (3) in a more tractable form.

### III. AUXILIARY FIELDS AND THE DECOUPLING OF TRAJECTORIES

Before coming to this we first reformulate the excluded volume interaction $A_v$ in Eq. (3) is a standard way, since
its non-Markoffian character. The trajectories are coupled to each other by the two-body potential in Eq. (3). The interactions can be disentangled by introducing $N$ real scalar fields $\phi_1 \ldots \phi_N$ with the euclidean action

$$A_\phi = \frac{g^2}{2} \sum_{i,j=1}^{N} \int d^3x \phi_i(x)[(\nabla^2)^{-1}]_{ij} \phi_j(x).$$  \hspace{1cm} (10)$$

It is obviously possible to rewrite the exponential of $-A_\phi$ in Eq. (2) with the help of the following identity [4]:

$$e^{-A_\phi} = \int D\phi_1 \cdots D\phi_N e^{A_\phi} \prod_{i=1}^{N} \exp \left[ -i \int_0^{L_i} ds_i \phi_i \left( x^i(s_i) \right) \right].$$  \hspace{1cm} (11)$$

On the right-hand side, each trajectory $x^i(s_i)$ interacts only with a single field $\phi_i(x)$, so that the polymers move in individual random fields. Thus, the contribution of the excluded volume forces is converted to a Markoffian form.

### IV. TOPOLOGICAL INTERACTIONS

Let us now turn to the topological interactions, where we search for auxiliary fields to simplify $A_{\text{top}}$ in the path integral (2). Such auxiliary fields are provided by Chern-Simons theories [4]. Let $A_\alpha^a$ with $a = 1, \ldots, N'$ be a set of $N'$ Chern-Simons fields with euclidean spatial indices $\mu = 1, 2, 3$. They will allow us to write the identity

$$e^{-A_{\text{top}}} = \frac{\int D\Lambda e^{-iA_{\text{CS}} + i\sum_{i=1}^{N} \int d^3x h_{i\alpha} J^i A^\alpha}}{\int D\Lambda e^{-iA_{\text{CS}}}}$$  \hspace{1cm} (12)$$

where $A_{\text{CS}}$ is a the Chern-Simons action

$$A_{\text{CS}} = \sum_{\alpha, \beta=1}^{N'} \int d^3x A^\alpha \cdot (\nabla \times A^\beta) g_{\alpha\beta}$$  \hspace{1cm} (13)$$

and $J^i$ are currents

$$J^i(x) = \int_0^{L_i} ds_i x^i(s_i) \delta^{(3)}(x - x^i(s_i)).$$  \hspace{1cm} (14)$$

The measure of functional integration $D\Lambda$ is short for the product $\prod_{\alpha=1}^{N'} D\Lambda^\alpha$. We assume $g_{\alpha\beta}$ to be a suitable $N' \times N'$-symmetric matrix to be specified later, which possesses an inverse $(g^{-1})^{\alpha\beta}$, while $h_{i\alpha}$ is a $N \times N'$ matrix.

The right-hand side of Eq. (12) has the described Markoffian form. To calculate it explicitly, we quantize the Chern-Simons fields $A_\alpha^a$ in the Lorentz gauge, where they are completely transverse, and have the correlation functions

$$G^{\alpha\beta}_{\mu\nu}(x,y) = \langle A_\alpha^a(x) A_\beta^b(y) \rangle$$

$$= \frac{(g^{-1})^{\alpha\beta}}{4\pi} \varepsilon_{\mu\nu\rho}(x - y)^\rho |x - y|^3.$$  \hspace{1cm} (15)$$

After some calculations we find

$$\int D\Lambda e^{-iA_{\text{CS}} + i\sum_{i=1}^{N} \int d^3x h_{i\alpha} J^i A^\alpha}$$

$$= \exp \left[ -i \frac{1}{4} \sum_{i,j=1}^{N} h_{i\alpha}(g^{-1})^{\alpha\beta} h_{j\beta} \chi(P_i, P_j) \right].$$  \hspace{1cm} (16)$$

The right-hand side is equal to that of Eq. (2) with the topological action (3) if we satisfy the equation

$$h_{i\alpha}(g^{-1})^{\alpha\beta} h_{j\beta} = 2\lambda_{ij}.$$  \hspace{1cm} (17)$$

Since the matrix $\lambda_{ij}$ has vanishing diagonal elements, this condition automatically eliminates the appearance of non-topological terms of the form $\chi(P_i, P_i)$ with $i = 1, \ldots, N$.

At this point, it is convenient to introduce the linear combinations of the Chern-Simons fields

$$C^i = \sum_{\alpha=1}^{N'} h_{i\alpha} A^\alpha;$$  \hspace{1cm} (18)$$

such that Eq. (12) becomes

$$e^{-A_{\text{top}}} = \frac{\int D\Lambda e^{-iA_{\text{CS}} + i\sum_{i=1}^{N} \int d^3x J^i C^i}}{\int D\Lambda e^{-iA_{\text{CS}}}}$$  \hspace{1cm} (19)$$

The absence of the non-topological contributions $\chi(P_i, P_i)$ in Eq. (16) is ensured by requiring purely off-diagonal correlation with

$$\langle C^i_\mu(x) C^j_\nu(y) \rangle = 0, \hspace{1cm} i = 1, \ldots, N.$$  \hspace{1cm} (20)$$

The linking of a polymer $P_i$ with a different polymer $P_j$ is described by the off-diagonal correlation functions. the related propagator should be different from zero:

$$\langle C^i_\mu(x) C^j_\nu(y) \rangle \neq 0 \hspace{1cm} i \neq j.$$  \hspace{1cm} (21)$$

This condition allows us to conclude that in the most general situation in which all elements $\lambda_{ij}$ are non-vanishing, the number $N'$ of Chern-Simons fields should be at least equal to $N$.

Indeed let us assume the contrary: $N' < N$. In this case we can always rearrange the indices such that the first $N'$ fields $C^\sigma_\mu(x)$ with $\sigma = 1, \ldots, N'$ are independent. The remaining fields

$$\tilde{C}^\tau_\mu(x) \equiv C^{N'+\tau}_\mu(x), \hspace{1cm} \tau = 1, \ldots, N - N',$$  \hspace{1cm} (22)$$

should then be linear combinations of the first $N'$ fields.

$$\tilde{C}^\tau_\mu(x) = \sum_{\sigma=1}^{N'} s^\tau_\sigma C^\sigma_\mu(x).$$  \hspace{1cm} (23)$$

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From the property (22), we find the correlation functions
\[
\langle \tilde{C}_\mu^\sigma(x)\tilde{C}_\nu^{\sigma'}(y) \rangle = \sum_{\sigma,\sigma' = 1}^{N'} s_\sigma^* s_{\sigma'} \langle C_\mu^\sigma(x)C_\nu^{\sigma'}(y) \rangle = 0 \tag{24}
\]
By hypothesis, however, we have
\[
\langle C_\mu^\sigma(x)C_\nu^{\sigma'}(y) \rangle \propto \lambda_{\sigma\sigma'} \left( \frac{x-y}{\|x-y\|^3} \right) \neq 0.
\]
As a consequence, the most general solution of Eq. (24) is:
\[
s_\sigma^* \neq 0, \quad s_\sigma^* = 0 \quad \text{if} \quad \sigma \neq \bar{\sigma}, \tag{26}
\]
where \( \bar{\sigma} \) is a fixed integer with \( 1 \leq \bar{\sigma} \leq N' \). Therefore, the fields \( C^\sigma(x) \) and \( \bar{C}^\sigma(x) \) coincide apart from an irrelevant factor \( s_\sigma^* \). In this way, if \( N' < N \), we obtain for \( N' < N \) the following contradiction:
\[
0 = \langle C_\mu(x)C_\nu^{\sigma'}(y) \rangle = s_\sigma^* \langle C_\mu(x)C_\nu^{\sigma'}(y) \rangle
\quad = \langle \tilde{C}_\mu^\sigma(x)\tilde{C}_\nu^{\sigma'}(y) \rangle = \langle C_\mu^{N'+1}(x)C_\nu^{\sigma'}(y) \rangle
\quad \propto \lambda_{N'+\bar{\sigma}} \left( \frac{x-y}{\|x-y\|^3} \right) \neq 0 \tag{27}
\]
In the opposite case of \( N' > N \), it is always possible to reduce the number of Chern-Simons fields to \( N \). To show this we consider the action
\[
A^i_{CS} = A_{CS} - \sum_{i=1}^{N'} \int d^3x \, J^i \cdot C^i \tag{28}
\]
appearing in Eq. (29). In the above action we perform a change of variables in which a number \( N \) of fields \( A^\alpha \) is expressed as a linear combination of the fields \( C^\alpha \)’s and of the remaining \( A^\lambda \)’s. Without loss of generality, we may suppose (18) to be invertible, with the solutions
\[
A^i = \sum_{\beta = 1}^{N'} \left( h^{-1} \right)^{ij} \left( C^j - \sum_{\alpha = N+1}^{N'} h_{\gamma \alpha} A^\alpha \right). \tag{29}
\]
Substituting this into (28), we find
\[
A^i_{CS} = \int d^3x \left\{ C^{ij} J_j^i + \varepsilon^{i\nu\rho} \sum_{i,j=1}^{N} M_{ij} C_\mu^i \partial_\nu C_\rho^j \right. \right.
\quad - 2 \sum_{\alpha = N+1}^{N'} N_{\alpha i} A^\alpha \partial_\nu C^i_\nu + \sum_{\alpha,\beta = N+1}^{N'} O_{\alpha \beta} A^\alpha \partial_\nu A^\beta \right. \right. \tag{30}
\]
where the constant coefficients \( M_{ij}, N_{\alpha i} \) and \( O_{\alpha \beta} \) are functions of the matrix elements \( g^{\alpha\beta} \) and \( h_{\alpha \gamma} \). The mixed terms in \( A^i_{CS} \), which are proportional to \( N_{\alpha i} \), are eliminated by introducing the new field variables
\[
A^{i\alpha} = A^\alpha - \sum_{\beta = N+1}^{N'} \sum_{i=1}^{N} \left( O^{-1} \right)^{\alpha\beta} N_{\beta i} C^i_\nu. \tag{31}
\]
It is easy to see that the redundant fields \( A^{i\alpha} \) with \( \alpha = N+1, \ldots, N' \) can now be integrated out in Eq. (13), so that we arrive at a Chern-Simons field theory with \( N \) fields \( C^1, \ldots, C^N \).

This result could be expected from the fact that only the fields \( C^\alpha \) have sources in the action (28), while the remaining fields are free, and may be eliminated via the equations of motion. As a consequence, the freedom in choosing the number of Chern-Simons fields is only apparent, because any Chern-Simons field theory with \( N' > N \) substituting Eq. (19) is equivalent to a Chern-Simons field theory with \( N \) fields only. In principle, there is still some arbitrariness in the choice of the matrix elements \( g^{\alpha\beta} \) and \( h_{\alpha \gamma} \) once \( N' \) has been fixed, since their values are only constrained by (17). However, this arbitrariness reflects merely the possibility of performing linear transformations of the \( A^i \)’s, and it is thus irrelevant.

To conclude this section, we compute the denominator in the right hand side of (14). For this purpose we note that since \( g_{\alpha \beta} \) is a \( N' \times N' \) symmetric matrix, it can always be expressed as follows:
\[
g_{\alpha \beta} = \sum_{\beta = 1}^{N'} \eta_{\alpha \beta} \eta_{\beta \alpha} \tag{32}
\]
where \( \eta_{\alpha \beta} \) is again a \( N' \times N' \) symmetric matrix. Performing in Eq. (13) the substitution:
\[
a_\alpha = \sum_{\beta = 1}^{N'} \eta_{\alpha \beta} A^\beta \tag{33}
\]
the Chern-Simons action becomes
\[
A_{CS} = \sum_{\alpha = 1}^{N'} \int d^3x \varepsilon^{ij\rho} a^\alpha \partial_\rho a^\alpha \tag{34}
\]
i.e., the dependence on \( g_{\alpha \beta} \) disappears. Thus:
\[
\int DA e^{-iA_{CS}(A^\alpha)} = |\det(g_{\alpha \beta})|^{1/2} c \tag{35}
\]
where \( c = \int DA e^{-iA_{CS}(A^\alpha)} \) is an irrelevant constant factor.

V. FIELD THEORY

In the previous section we have seen that the path integral over the polymer trajectories can be converted to a Markovian form via auxiliary fields. In rewriting the topological interactions there is some freedom in choosing the auxiliary fields by varying their number. Also the
parameters \( g_{\alpha \beta} \) and \( h_{ij} \) are not completely fixed by the system of equations \([33]\).

On the other hand, it has been shown that all abelian Chern-Simons field theories for which the relevant identity \([19]\) is satisfied are equivalent after exploiting the equations of motion and performing linear transformations of the vector fields.

The simplest way to ensure \([33]\) is to choose

\[
N' = N \quad g_{ij} = \frac{\kappa \lambda_{ij}}{4\pi} \quad h_{ij} = \frac{\kappa^{1/2}}{4\pi} \lambda_{ij}
\]

for \( i, j = 1, \ldots, N \). This choice, however, makes the Fourier transformation \([3]\) from the auxiliary probability \( G(\lambda) (\vec{x}, \vec{y}; \vec{L}) \) to the original configurational probability \( G(m)(\vec{x}, \vec{y}; \vec{L}) \) too difficult for an analytic treatment. As a matter of fact, starting from Eq. \((36)\), we find with the help of \([35]\):

\[
e^{-A_{\text{top}}} = \left[ (\det \lambda_{ij})^{1/2} c \right]^{-1} \int D \mathbf{A} \times \exp \left\{ - \int d^3 x \sum_{i=1}^{N} \sum_{j \neq i}^{N} \lambda_{ij} \left[ \frac{\kappa}{4\pi} \epsilon_{\mu \nu \rho} A_i^\mu \partial_\nu A_j^\rho + \frac{\kappa^{1/2}}{4\pi} \int_0^{L_i} dx^i (s_i) A^j (x^j (s_i)) \right] \right\}.
\]

The denominator in the right hand side contains the term \( (\det \lambda_{ij})^{1/2} \), which complicates the integration over the parameters \( \lambda_{ij} \) in \([3]\).

Moreover, in the ansatz \([36]\) the requirement that the matrix \( g_{ij} \) should be invertible cannot be guaranteed for all possible matrices \( \lambda_{ij} \).

The situation does not improve if we choose

\[
N' = N \quad g_{ij} = \frac{\kappa}{4\pi} \delta_{ij} \quad h_{ij} = \frac{\kappa^{1/2}}{4\pi} \eta_{ij}.
\]

Here the elements \( \eta_{ij} \) have a complicated dependence on the variables \( \lambda_{ij} \), via the algebraic equations \([13]\).

To solve these difficulties, we exploit the freedom of enlarging the number of topological vector fields. The simplest Chern-Simons field theory for our purpose contains \( N' = 2(N - 1) \) fields \( A^1, \ldots, A^{N-1} \) and \( B^1, \ldots, B^{N-1} \). The action \( A_{\text{CS}} \) is given by:

\[
A_{\text{CS}} = \kappa \sum_{i=1}^{N-1} e^{\mu \nu \rho} \int d^3 x A_i^\mu \partial_\nu B_i^\rho.
\]

Equation \([19]\) becomes now

\[
e^{-A_{\text{top}}} = c^{-1} \int DA DB e^{-i A_{\text{CS}}} \exp \left\{ \frac{N}{\kappa} \int d^3 x \mathbf{J}^i \cdot \mathbf{F}^i \right\},
\]

where the currents \( J^i_j (x) \) have been already defined in Eq. \((14)\) and the fields \( C^i \) of Eq. \((18)\) have the following explicit expressions:

\[
C^1 = B^1, \quad C^N = \kappa \sum_{i=1}^{N-1} \lambda_i A^i
\]

\[
C^i = \kappa \sum_{j=1}^{i-1} \lambda_{ij} A^j + B^i, \quad i = 2, \ldots, N - 1.
\]

The factor \( c^{-1} \) in Eq. \((41)\) is an irrelevant constant independent of \( \lambda_{ij} \).

Using Eq. \((11)\) and \((40)\) in the expression of the auxiliary probability \([6]\) we obtain the path integral representation

\[
G(\lambda) (\vec{x}, \vec{y}; \vec{L}) = \left\{ \prod_{i=1}^{N} G (x^i, y^i; L_i | \phi_i, C^i) \right\},
\]

where

\[
G (x^i, y^i; L_i | \phi_i, C^i) = \int_{x^i}^{y^i} D x^i (s_i) \times e^{- \int_0^{L_i} ds_i L_i (x^i (s_i)) + \int_0^{L_i} ds_i x^i (s_i) C^i (x^i (s_i))},
\]

and

\[
L_i (x^i (s_i)) = \frac{3}{2\mu} x^i (s_i)^2 + i \phi_i (x^i (s_i)).
\]

In Eq. \((42)\), the expression in the expectation symbol \( \langle \cdot \rangle \) must be averaged with respect to the fields \( \phi_i \) with \( i = 1, \ldots, N \) and the Chern-Simons fields \( A^i, B^j \) with \( j = 1, \ldots, N - 1 \).

The path integral in Eq. \((43)\) describes a Markovian random walk of a particle immersed in an electromagnetic field \( C^i, i \phi_i \). In analogy with the evolution kernel of a particle in quantum mechanics, \( G (x^i, y^i; L_i | \phi_i, C^i) \) satisfies a Schrödinger-like equation

\[
\frac{\partial}{\partial L_i} - \frac{\mu}{6} D_i^2 + i \phi_i \right] \int D x^i, y^i; L_i | \phi_i, C^i = \delta (L_i) \delta (x^i - y^i)
\]

where

\[
D_i = \nabla + i C^i.
\]

It is now convenient to consider its Laplace-transformed in the length parameter \( L_i \):

\[
G(\lambda) (\vec{x}, \vec{y}; \vec{\mu}) = \int_0^{+\infty} dL_1 \cdots dL_N e^{- \sum_{i=1}^N \mu_i L_i} G(\lambda) (\vec{x}, \vec{y}; \vec{L}).
\]

The parameters \( \mu_i \) in Boltzmann-like factors control the growth of the polymers. Applying the Laplace transformations to both sides of Eq. \((42)\), we find
where $G(x_i^i, y_i^i; \mu_i | \phi_i, C^i)$ is the Laplace transformed correlation function \[ (43) \], obeying the stationary equation
\[ \left[ \mu_i - \frac{a}{6} D_i^2 + i \phi_i \right] G(x_i^i, y_i^i; \mu_i | \phi_i, C^i) = \delta(x_i^i - y_i^i). \] \[ (49) \]
Equation \[ (49) \] can be solved in terms of fluctuating polymer fields $\psi_i^*, \psi_i$ as
\[ G(x_i^i, y_i^i; \mu_i | \phi_i, C^i) = \frac{1}{Z_i} \int D\psi_i D\psi_i^* \psi_i(x^i) \psi_i^*(y^i) e^{-A_{pol}[\psi_i^*, \psi_i]} \] \[ (50) \]
where $A_{pol}$ is the polymer action
\[ A_{pol}[\psi_i^*, \psi_i] = \int d^3x \left[ \frac{a}{6} D_i^2 |\psi_i|^2 + (\mu_i + i \phi_i) |\psi_i|^2 \right] \] \[ (51) \]
and $Z_i$ is the associated partition function
\[ Z_i = \int D\psi_i D\psi_i^* e^{-A_{pol}[\psi_i^*, \psi_i]} \] \[ (52) \]
The integrations over the auxiliary fields $\phi_i$ is complicated by the presence of the factor $Z_i^{-1}$ in Eq. \[ (50) \], which makes them non-gaussian. This problem can be solved exploiting the method or replicas.

**VI. REPLICA FORMULATION**

For each pair of fields $\psi_i, \psi_i^*$ we introduce a set of $n_i$ replica field $\psi_i^{a_i}, \psi_i^{*a_i}$ with $a_i = 1, \ldots, n_i$. It is convenient to group the replica fields in $n_i$-tuplets
\[ \Psi_i = (\psi_1^1, \ldots, \psi_i^{n_i}), \quad \Psi_i^* = (\psi_1^{*1}, \ldots, \psi_i^{*n_i}) \].
With these fields the correlation function \[ (50) \] may be rewritten as follows:
\[ G(x_i^i, y_i^i; \mu_i | \phi_i, C^i) = \lim_{n_i \to 0} \int D\Psi_i D\Psi_i^* e^{-A_{rep}[\Psi_i, \Psi_i^*]} \] \[ (53) \]
where we have set
\[ \int D\Psi_i D\Psi_i^* = \int \prod_{a_i=1}^{n_i} D\psi_i^{a_i} D\psi_i^{*a_i} \] \[ (54) \]
and defined the replica field action
\[ A_{rep}[\Psi_i, \Psi_i^*] = \sum_{a_i=1}^{n_i} \left[ \frac{a}{6} |D_i \psi_i^{a_i}|^2 + (\mu_i + i \phi_i) |\psi_i^{*a_i}|^2 \right] \] \[ (55) \]
The index $\bar{a}_i$ in \[ (53) \] is a fixed replica index chosen arbitrarily in the range $1 \leq \bar{a}_i \leq n_i$.

The limit of zero replica number in \[ (53) \] is performed by an analytic extrapolation. The path integral on the right hand side is calculated for integer values of $n_i$ and the result is then extrapolated analytically to the point $n_i = 0$.

Combining everything, the auxiliary probability \[ (48) \] has the functional integral representation
\[ G(x_i^i, y_i^i; \mu) = \lim_{n_i \to 0} \int D\Psi_i D\Psi_i^* e^{-A_{pol} - A_{rep}[\Psi_i, \Psi_i^*]} \] \[ (56) \]
Note that in this expression the integration over the auxiliary fields $\phi_i$ has become Gaussian. After a suitable rescaling of the fields $\psi_i^{a_i}, \psi_i^{*a_i}$, we give the final result in the following form:
\[ G(x_i^i, y_i^i; \mu) = \lim_{n_i \to 0} \int D\Psi_i D\Psi_i^* \psi_i^{*a_i}(x^i) \psi_i^{a_i}(y^i) e^{-A_{tot}} \] \[ (57) \]
where
\[ A_{tot} = i A_{CS} + \sum_{i=1}^{N} \int d^3x \left[ \psi_i^* \left( -D_i^2 + m_i^2 \right) \psi_i \right. \] \[ + \sum_{i,j=1}^{N} \left. \frac{2 M^2 v_0}{a^2} |\Psi_i|^2 |\Psi_j|^2 \right] \] \[ (58) \]
and
\[ m_i^2 = 2 M \mu_i \] \[ (59) \]
With respect to Eq. \[ (53) \], the fields $\psi_i^{a_i}, \psi_i^{*a_i}$ have been rescaled by a factor $\sqrt{2/M}$, so that they acquire the canonical dimension $[\psi_i^{a_i}] = [\psi_i^{*a_i}] = 1/2$ of usual scalar fields.

Following \[ [4] \], we have introduced a mass parameter $M$ and a parameter playing a similar role as the Planck constant in quantum mechanical path integrals $\hbar = \mu_i/3$. The value of $M$ has been fixed with respect to the step length $a$ by requiring the condition $\hbar = 1$.

In this way, the action \[ (53) \] becomes that of a standard field theory with unit Planck constant. Moreover, $A_{tot}$ is a quadratic form with respect to the parameters $\lambda, \mu$, so that the inverse Fourier transformations leading to the original configurational probability can be performed after a diagrammatic evaluation of the correlation functions \[ (53) \].

The actual calculations are left to a future publication.

\[ \text{ } \]
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