Continuous frames in Krein spaces

Elmar Wagner¹ · Diego Carrillo² · Kevin Esmeral³

Received: 24 April 2021 / Accepted: 4 December 2021 / Published online: 19 January 2022
© Tusi Mathematical Research Group (TMRG) 2022

Abstract
The purpose of this paper is to propose a definition of continuous frames of rank $n$ for Krein spaces and to study their basic properties. Similarly to the Hilbert space case, continuous frames are characterized by the analysis, the pre-frame and the frame operator, where the latter gives rise to a frame decomposition theorem. The paper includes a discussion of similar, dual and Parseval frames and of reproducing kernels. In addition, the importance of the fundamental symmetry in the formula for the frame operator in a Krein space is clarified. As prime examples, it is shown how to transfer continuous frames for Hilbert spaces to Krein spaces arising from a possibly non-regular Gram operator.

Keywords Continuous frames · Krein spaces · Reproducing kernels · Non-regular Gram operator

Mathematics Subject Classification 42C15 · 46B15 · 46C20

Communicated by Michael Frank.

Elmar Wagner
elmar@ifm.umich.mx

Diego Carrillo
diego.carrillo@cecar.edu.co

Kevin Esmeral
kevin.esmeral@ucaldas.edu.co

1 Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, C.P. 58040 Morelia, Michoacán, Mexico

2 Departamento de Ciencias Básicas, Corporación Universitaria del Caribe, Sincelejo, Sucre, Colombia

3 Department of Mathematics, Universidad de Caldas, P.O. 170004 Manizales, Colombia
1 Introduction

A generalization of the concept of *frame in a Hilbert space* to a family indexed by a locally compact space with a Radon measure was introduced by Kaiser [32] and, independently, by Ali, Antoine and Gazeau [2]. Such frames are known as *continuous frames* in distinction from usual frames that are given by a countable spanning set. Continuous frames appear under different names in the literature. For example, Gabardo and Han [28] call them frames associated to measure spaces, and Askari-Hemmat, Dehghan, and Radjabalipour [5] call them generalized frames. Moreover, most of the (vector) coherent states in mathematical physics are actually described by the continuous frames [1, 40]. For more details, see e.g. [2, 5, 11, 14, 15, 17, 26, 28, 45].

The possibility of being over-complete makes frames more flexible than orthonormal bases and, for this reason, a powerful tool in signal processing, electrical engineering and several branches of mathematics [12, 13, 16, 18–20, 22, 27, 29, 41], as well as in physics [1–4, 19, 22, 39]. On the other hand, Krein spaces arise naturally in mathematics, for instance in relation to signed measure spaces, and in mathematical physics, for instance in quantum field theory. It is therefore natural to extend frame theory for Hilbert spaces to Krein spaces [24, 30, 34, 39].

Discrete frames in a Krein space were introduced simultaneously by Giribet, Maestripieri, Martínez Pería and Massey [30], and Esmeral, Ferrer and Wagner [24], albeit with slightly different definitions. The definition in [30] manifests directly the independence of a \( J \)-norm, whereas in [24], frames in a Krein space are essentially the same objects as frames for the associated Hilbert space although the property of being a frame also does not depend on the \( J \)-norm. Other approaches can be found in [33, 34, 36, 39].

The present paper introduces continuous frames of rank \( n \) for Krein spaces by weak integrals (Definition 3.4) following the Hilbert space approach developed in [2]. As in [24], continuous frames in Krein spaces are essentially the same objects as continuous frames in the associated Hilbert space (Theorem 3.5), and are characterized by the analysis, the pre-frame and the frame operator (see Sect. 3.2). The most important result in frame theory is the decomposition theorem [12, 13, 18–20, 22, 29, 35, 41], so it is clearly desirable to extend this result to the continuous case. This will be achieved in Theorem 3.6. In Sect. 3.3, we discuss the importance of the fundamental symmetry for the invertibility of the frame operator. A treatment of reproducing kernels will be presented in Sect. 3.4. Unfortunately, we cannot speak about reproducing kernel Krein spaces unless the image of the analysis operator is orthocomplemented. Sections 3.5 and 3.6 are devoted to the study of similar, dual and Parseval frames, where we also comment briefly on the notion of coherent state. In the final section, it is shown how to transfer continuous frames in a Hilbert space to an associated Krein space arising from a possibly non-regular and unbounded Gram operator \( W \). This will be realized by extending the square root \( \sqrt{\langle W \rangle} \) to a \( J \)-unitary operator. The motivation for using \( \sqrt{|W|} \) stems from the fact that a frame in the Hilbert space can never be a frame for the associated Krein space if the Gram operator is unbounded or if 0 belongs to its spectrum.
Our main motivation for studying continuous frames in Krein spaces comes from physics, in particular from quantum mechanics with non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians, where $\mathcal{P}$ and $\mathcal{T}$ stand for the parity and the time reversal operator, respectively, see e.g. the review articles [9, 37] and the references therein. A convenient way to treat these non-Hermitian Hamiltonians is to study them in a Krein space, where the Gram operator for the definition of the indefinite inner product is either given by the parity operator or by an additional charge conjugation. Then a superselection rule identifies the positive definite part in the fundamental decomposition with the physical states and forbids a mixing with vectors from the negative definite part. Our goal is the construction of coherent states for a unitary group representations compatible with the fundamental symmetry. In [44], Sierra-Acosta and the first author constructed SU(1, 1)-coherent states starting from unitary representations of SU(1, 1) in finite-dimensional Krein spaces. Here, the novel feature is that all finite-dimensional irreducible representations of the complex simple Lie algebra sl(2, $\mathbb{C}$) can be lifted to unitary representations of SU(1, 1) in a Krein space with an essentially unique fundamental symmetry, whereas SU(1, 1), being the non-compact, does not admit finite-dimensional unitary representations in a Hilbert space. Lie algebraic su(1, 1)-coherent states occur for instance in 2D Dirac materials like graphene [21]. However, on the coherent states produced from a Hilbert space representation, the ladder operators have complex eigenvalues. Since the Minkowski metric is indefinite, above ideas may be further explored for unitary Krein space representations of the Lorentz group, see e.g. [8]. Furthermore, coherent states in Krein spaces appear naturally in the fermionic theory of the general boundary formulation [38], again as a consequence of the use of the indefinite Minkowski metric in quantum field theory. Let us mention that Krein spaces show up in the Gupta-Bleuler formalism of quantum electrodynamic, where the “states” of time-like polarizations with negative inner product are regarded unphysical. A Krein space construction of covariant coherent states for massless particles (photons) in $1 + 1$ dimensions can be found in [1, Section 10.1.3]. Lastly, Krein spaces also appear in solid state physics. In [43], Schulz–Baltes and Villegas-Blas describe transfer operators for half-space topological insulators that are $J$-unitary in a Krein space, where $J$ can be thought of as arising from a symplectic structure.

An important aspect of these theories is that the fundamental symmetry is determined by physical considerations and therefore fixed. By the same token, we fix a fundamental symmetry in our definition of continuous frames in Krein spaces. Although this is mathematically a drawback, we nevertheless believe that our paper lays useful foundations for physical applications.

2 Preliminaries

The purpose of this section is to fix notations and to recall the basic elements of frame theory. For more details on Krein spaces, we refer the reader to [7, 10]. A comprehensive introduction to frame theory can be found in [14].
2.1 Krein spaces

Throughout this paper, \((\mathcal{K}, [, , ]_J)\) denotes a Krein space with fundamental decomposition \(\mathcal{K}_+ [+] \mathcal{K}_-\) and fundamental symmetry \(J\) given by
\[
J(k^+ + k^-) = k^+ - k^-, \quad k^+ + k^- \in \mathcal{K}_+[+] \mathcal{K}_-,
\]
such that the \(J\)-inner product
\[
[h^+ + h^-, k^+ + k^-]_J := [h^+ + h^-, J(k^+ + k^-)] = [h^+, k^+] - [h^-, k^-], \quad h^\pm, k^\pm \in \mathcal{K}_\pm,
\]
turns \((\mathcal{K}, [, , ]_J)\) into a Hilbert space. The positive definite inner product \([, , ]_J\) defines a topology on \(\mathcal{K}\) by the \(J\)-norm
\[
\|k\|_J := \sqrt{[k, k]_J} = \sqrt{[k, Jk]}, \quad k \in \mathcal{K},
\]
and \(\mathcal{K}_+[+] \mathcal{K}_-\) becomes the orthogonal sum of Hilbert spaces. Note that \(J^2 = 1\) by (2.1).

The Hilbert space \((\mathcal{K}, [, , ]_J)\) is used to study linear operators acting on the Krein space \((\mathcal{K}, [, , ])\). Topological concepts such as continuity, closedness of operators, spectral theory and so on refer to the topology induced by the \(J\)-norm given in (2.3). Therefore, we may apply the same definitions as in operator theory of Hilbert spaces. The unique adjoint \(T^* : (\mathcal{K}_2, [, , ]_2) \to (\mathcal{K}_1, [, , ]_1)\) of a bounded linear operator \(T : (\mathcal{K}_1, [, , ]_1) \to (\mathcal{K}_2, [, , ]_2)\) is always taken with respect to the specified inner products, i.e.,
\[
[T^* h, k]_1 = [h, T k]_2 \quad \text{for all} \quad k \in \mathcal{K}_1, \quad h \in \mathcal{K}_2.
\]
Furthermore, the linear operator \(T^{[*]} = J_1 T^* J_2\) is called the \(\text{\(J\)-adjoint}\) of \(T\), where \(J_i, i = 1, 2\), denotes a (fixed) fundamental symmetry on \(\mathcal{K}_i\). An operator \(T \in \mathcal{B}(\mathcal{K})\) is said to be self-adjoint if \(T = T^*\), and \(J\)-self-adjoint if \(T = T^{[*]}\). A similar distinction will be made between unitary and \(J\)-unitary operators. By a \(J\)-orthogonal projection we mean a \(J\)-self-adjoint idempotent.

To illustrate this in some examples, it follows from Eqs. (2.1) and (2.2) that \(J : (\mathcal{K}, [, , ]) \to (\mathcal{K}, [, , ])\) and \(J : (\mathcal{K}, [, , ]_J) \to (\mathcal{K}, [, , ]_J)\) are self-adjoint, i.e., \(J^* = J = J^{[*]}\). Moreover, the identity operator
\[
1_J : (\mathcal{K}, [, , ]) \to (\mathcal{K}, [, , ]_J), \quad 1_J k = k,
\]
has the adjoint \(J : (\mathcal{K}, [, , ]_J) \to (\mathcal{K}, [, , ])\) since \([1_J k, h]_J = [k, h]_J = [k, J h]\) for all \(h, k \in \mathcal{K}\).

A subspace \(V \subset \mathcal{K}\) is said to be uniformly \(J\)-positive (resp. uniformly \(J\)-negative) if there exists an \(\varepsilon > 0\) such that \([v, v] \geq \varepsilon \|v\|^2\) (resp. \([-v, v] \geq \varepsilon \|v\|^2\)) for all \(v \in V\). A self-adjoint operator \(A = A^*\) on \((\mathcal{K}, [, , ]_J)\) is called uniformly \(J\)-positive, if \([k, A k] \geq \varepsilon [k, k]_J\) for a suitable constant \(\varepsilon > 0\) and all \(k \in \mathcal{K}\). Equivalently, since \([k, A k] = [k, J A k]_J\), we have \(J A \geq \varepsilon\) on the Hilbert space \((\mathcal{K}, [, , ]_J)\). As a consequence, \(A\) has a bounded inverse.

The fundamental projections

\[\text{Birkhäuser}\]
P_+ := \frac{1}{2}(1 + J), \quad P_- := \frac{1}{2}(1 - J), \quad 1 : \mathcal{K} \to \mathcal{K} \tag{2.5}

act on \mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- by P_+(k^+ + k^-) = k^+, P_-(k^+ + k^-) = k^- and 1(k) = k. Equation (2.5) implies immediately that P_\pm and J commute. Moreover, P_+ and P_- are orthogonal projections, i.e. \( P_\pm^2 = P_\pm = P_\pm^* \), regardless of whether we consider \([\cdot , \cdot ]\) or \([\cdot, \cdot ]_J\) on \( \mathcal{K} \).

**Example 1** Let \( \nu \) be a signed measure on \( (\mathcal{M}, \mathcal{B}) \) and \( \{\mathcal{M}_+, \mathcal{M}_-\} \) be the Hahn decomposition of \( \mathcal{M} \) with respect to \( \nu \). By the Jordan decomposition theorem, there exist positive measures \( \nu^+ \) and \( \nu^- \) on \( (\mathcal{M}, \mathcal{B}) \) such that \( \nu = \nu^+ - \nu^- \), \( |\nu| = \nu^+ + \nu^- \) and \( \nu^+(\mathcal{M}_-) = \nu^-(\mathcal{M}_+) = 0 \).

Next, let \( \mathfrak{L}_2(\mathcal{M}, |\nu|) \) denote the Hilbert space of complex valued functions on \( \mathcal{M} \) that are square integrable with respect to the positive measure \( |\nu| \). Then, for each \( f \in \mathfrak{L}_2(\mathcal{M}, |\nu|) \), we may write \( f = \chi_{\mathcal{M}_+}f + \chi_{\mathcal{M}_-}f \), where \( \chi_{\mathcal{M}} \) stands for the indicator function of \( \mathcal{M} \in \mathcal{B} \). Thus,

\[ \mathfrak{L}_2(\mathcal{M}, |\nu|) = \mathfrak{L}_2(\mathcal{M}_+, \nu^+) [+] \mathfrak{L}_2(\mathcal{M}_-, \nu^-). \tag{2.6} \]

On \( \mathfrak{L}_2(\mathcal{M}, |\nu|) \), consider the indefinite inner product

\[ [\cdot , \cdot ]_{\mathfrak{L}_2} : \mathfrak{L}_2(\mathcal{M}, |\nu|) \times \mathfrak{L}_2(\mathcal{M}, |\nu|) \to \mathbb{C}, \quad [f, g]_{\mathfrak{L}_2} := \int_{\mathcal{M}} f(x)g(x)d\nu(x). \tag{2.7} \]

As easily seen, \( (\mathfrak{L}_2(\mathcal{M}_+, \nu^+), [\cdot, \cdot]_{\mathfrak{L}_2}) \) and \( (\mathfrak{L}_2(\mathcal{M}_-, \nu^-), -[\cdot, \cdot]_{\mathfrak{L}_2}) \) are Hilbert spaces. Therefore, \( (\mathfrak{L}_2(\mathcal{M}, |\nu|), [\cdot, \cdot]_{\mathfrak{L}_2}) \) is a Krein space, which we abbreviate to \( \mathfrak{L}_2(\mathcal{M}, \nu) \).

The projections \( P_\pm : \mathfrak{L}_2(\mathcal{M}, \nu) \to \mathfrak{L}_2(\mathcal{M}_\pm, \nu_\pm) \) are given by \( P_\pm f = \chi_{\mathcal{M}_\pm}f \) and hence the fundamental symmetry associated to the decomposition (2.6), denoted by \( J_{\mathfrak{L}_2} \), can be expressed in terms of the multiplication operator with the Radon–Nikodym derivative \( J_{\mathfrak{L}_2} f := \chi_{\mathcal{M}_+} - \chi_{\mathcal{M}_-} \) of \( \nu \) with respect to \( |\nu| \), i.e.,

\[ J_{\mathfrak{L}_2} f = j_{\mathfrak{L}_2} f = \chi_{\mathcal{M}_+}f - \chi_{\mathcal{M}_-}f, \quad f \in \mathfrak{L}_2(\mathcal{M}, \nu). \tag{2.8} \]

Observe that, for all \( f, g \in \mathfrak{L}_2(\mathcal{M}, \nu) \),

\[ [f, g]_{J_{\mathfrak{L}_2}} = \int_{\mathcal{M}} \overline{f(x)}J_{\mathfrak{L}_2}g(x)d\nu(x) = \int_{\mathcal{M}} \overline{f(x)}g(x)|\nu|(x) = : \langle f, g \rangle_{\mathfrak{L}_2(\mathcal{M}, |\nu|)}. \tag{2.9} \]

Let \( V \) be a closed subspace of a Krein space \( (\mathcal{K}, [\cdot, \cdot]) \). The subspace

\[ V^\perp = \{ x \in \mathcal{K} : [x, y] = 0 \text{ for all } y \in V \}\tag{2.10} \]

is called the **orthogonal complement of \( V \) with respect to \([\cdot, \cdot] \)**. A closed subspace \( V \) of a Krein space \( (\mathcal{K}, [\cdot, \cdot]) \) such that \( V \cap V^\perp = \{ 0 \} \) and \( V + V^\perp = \mathcal{K} \) is said to be **ortho-complemented**.

**Proposition 2.1** [7, Theorem 7.16] Let \( (\mathcal{K}, [\cdot, \cdot]) \) be a Krein space and let \( V \) be a closed subspace of \( \mathcal{K} \). The following statements are equivalent:
(i) $V$ is ortho-complemented.
(ii) $(V, [\cdot, \cdot]_V)$ is a Krein space, where $[\cdot, \cdot]_V := [\cdot, \cdot]_{V\times V}$.
(iii) Any vector in $K$ admits at least one $J$-orthogonal projection onto $V$.

3 Continuous frames in Krein spaces

3.1 Weak Integral in Krein spaces

Definition 3.1 Let $(K_1, [\cdot, \cdot]_1)$, $(K_2, [\cdot, \cdot]_2)$ be Krein spaces and $(\mathcal{M}, \mathcal{B})$ be a measurable space. A function $F : \mathcal{M} \to \mathcal{B}(K_1, K_2)$ is said to be weakly measurable in $\mathcal{B}(K_1, K_2)$ if the map $\mathcal{M} \ni x \mapsto [k, F(x)h]_2 \in \mathbb{C}$ is measurable for all $h \in K_1$ and $k \in K_2$.

Remark 1 For a Krein space $(K, [\cdot, \cdot])$, the Riesz representation theorem establishes an isomorphism $(K, [\cdot, \cdot]) \ni h \mapsto \varphi_h \in K' := \mathcal{B}(K, \mathbb{C})$, where the linear functional $\varphi_h$ is defined by $\varphi_h(\cdot) = [h, \cdot]_1 = [h, J(\cdot)]$. Thus, given a map $T : \mathcal{M} \to K$, we may consider $U : \mathcal{M} \to \mathcal{B}(K, \mathbb{C})$, $U(x) := \varphi_{JT(x)}$, so that $\langle \lambda, U(x)k \rangle_{\mathbb{C}} = T(x), k \rangle_{\mathbb{C}}$ for all $k \in K$ and $\lambda \in \mathbb{C}$. As a consequence, if $U$ is weakly measurable in $K'$, then the complex function $\mathcal{M} \ni x \mapsto [T(x), k]$ is measurable for each $k \in K$. In this case, we say that $T : \mathcal{M} \to K$ is weakly measurable in $K$.

Definition 3.2 Let $(K_1, [\cdot, \cdot]_1)$, $(K_2, [\cdot, \cdot]_2)$ be Krein spaces, $(\mathcal{M}, \mathcal{B},\nu)$ a measure space and $F : \mathcal{M} \to \mathcal{B}(K_1, K_2)$ a weakly measurable function. Then we say that $F$ is weakly integrable in $\mathcal{B}(K_1, K_2)$ if there exists an $A \in \mathcal{B}(K_1, K_2)$ such that for all $h \in K_2$ and $k \in K_1$,

$$\int_{\mathcal{M}} [h, F(x)k]_2 \, d\nu(x) = [h, Ak]_2. \quad (3.1)$$

The bounded operator $A$ is called weak integral of $F$ and will be written as

$$\int_{\mathcal{M}} F(x) \, d\nu(x) := A.$$

Remark 2 Let $(K, [\cdot, \cdot])$, $(\mathcal{M}, \mathcal{B},\nu)$ and $T$ be as in Remark 1. Then, given any $h \in K$ such that $\int_{\mathcal{M}} \langle \lambda, U(x)k \rangle_{\mathbb{C}} \, d\nu(x) = \langle \lambda, \varphi_{h}(k) \rangle_{\mathbb{C}}$ for all $k \in K$ and $\lambda \in \mathbb{C}$, it follows that

$$\tilde{\lambda} \int_{\mathcal{M}} [T(x), k] \, d\nu(x) = \int_{\mathcal{M}} \langle \lambda, U(x)k \rangle_{\mathbb{C}} \, d\nu(x) = \tilde{\lambda} [h, k].$$

Motivated by this observation, we define the weak integral of $T : \mathcal{M} \to K$ in terms of the weak integral of $U : \mathcal{M} \to \mathcal{B}(K, \mathbb{C})$, i.e.,
\[
\int_{\mathcal{M}} T(x) \, dv(x) := h \quad \text{if and only if} \quad \int_{\mathcal{M}} [T(x), k] \, dv(x) = [h, k] \quad \text{for all } k \in \mathcal{K}.
\]

The next proposition is an immediate consequence of Definition 3.2. Note that the integrability of the functions \( \mathcal{M} \ni x \mapsto [h, F(x)k]_2 \in \mathbb{C}, \; k \in \mathcal{K}_1, \; h \in \mathcal{K}_2, \) implies the integrability of \( \mathcal{M} \ni x \mapsto [h, F(x)Tk_0]_2 \in \mathbb{C} \) for all \( T \in \mathcal{B} (\mathcal{K}_0, \mathcal{K}_1) \) and \( k_0 \in \mathcal{K}_0, \) and of \( \mathcal{M} \ni x \mapsto [h_0, SF(x)k]_3 = [S^*h_0, F(x)k]_2 \in \mathbb{C} \) for all \( S \in \mathcal{B} (\mathcal{K}_2, \mathcal{K}_3) \) and \( h_0 \in \mathcal{K}_3, \) where \((\mathcal{K}_i, [\cdot, \cdot]), i = 0, \ldots, 3,\) denote Krein spaces.

**Proposition 3.3** Let \((\mathcal{M}, \mathfrak{B}, \nu)\) be a measure space, \((\mathcal{K}_i, [\cdot, \cdot]), i = 0, \ldots, 3,\) Krein spaces, \( F: \mathcal{M} \to \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)\) a weakly measurable map, and \( A \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)\) a linear operator such that \( \int_{\mathcal{M}} F(x) \, dv(x) = A.\) Then

(i) \[\int_{\mathcal{M}} F(x)k \, dv(x) = Ak \quad \text{for all } k \in \mathcal{K}_1.\]

(ii) For all linear operators \( T \in \mathcal{B}(\mathcal{K}_0, \mathcal{K}_1) \) and \( S \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_3),\)

\[\int_{\mathcal{M}} F(x)T \, dv(x) = AT \quad \text{and} \quad \int_{\mathcal{M}} SF(x) \, dv(x) = SA.\]

(iii) \[\int_{\mathcal{M}} F(x)^* \, dv(x) = A^*.\]

### 3.2 Continuous frames of rank \( n \)

From now on, we will work with a signed measure space \((\mathcal{M}, \mathfrak{B}, \nu)\) as in Example 1. Our paper is based on the following fundamental definition.

**Definition 3.4** A family of linearly independent vectors \( \{\eta^i_1, \ldots, \eta^i_n\}_{i=1}^{\infty} \) in a Krein space \((\mathcal{K}, [\cdot, \cdot])\) with fundamental symmetry \( J\) is said to be a continuous frame of rank \( n \in \mathbb{N} \) with respect to \((\mathcal{M}, \mathfrak{B}, \nu)\) if the functions \( \eta^i: \mathcal{M} \to \mathcal{K}, \; \eta^i(x) := \eta^i_1, \) are weakly measurable for all \( i = 1, \ldots, n,\) and if there exist positive constants \( 0 < a \leq b,\) called frame bounds, such that

\[
a ||k||^2_j \leq \sum_{i=1}^{n} \int_{\mathcal{M}} ||[\eta^i, k]||^2 \, dv(x) \leq b ||k||^2_j \quad \text{for all } k \in \mathcal{K}. \tag{3.2}
\]

In case \( a = b = 1,\) the frame is said to be \( J\)-tight, and it is called a \( J\)-Parseval frame, if \( a = b = 1.\)

**Remark 3** Clearly, the \( J\)-norm of a Krein space depends on the fundamental decomposition. However, two \( J\)-norms corresponding to different fundamental decompositions of a Krein space are equivalent, see e.g. [7, §7, Theorem 7.19]. Therefore,
although the frame bounds may change, the *property of being* a continuous frame in a Krein space is independent from the fundamental decomposition.

On the other hand, as the frame bounds may change, the properties of being *tight* and a *Parseval frame* depend substantially on the fundamental decomposition. Consider for instance the Krein space \((\mathbb{R}^2, [\cdot, \cdot])\) with canonical basis \(\{e_1, e_2\} \subset \mathbb{R}^2\), inner product \([e_j, e_k] := (-1)^{j-1} \delta_{jk}\) (Kronecker delta) and fundamental decomposition \(\mathbb{R}^2 = \mathbb{R}e_1[+] \mathbb{R}e_2\). Then \(\{e_j\}_{j \in \{1,2\}}\) is a \(J_0\)-Parseval frame of rank 1 with respect to the measure space \((\{1,2\}, \mathcal{B}^{\{1,2\}}, \mu_2)\), where \(\mu_2\) denotes the counting measure and \(J_0\) is given by \(J_0 e_j = (-1)^{j-1} e_j\). Now we define a family of fundamental decompositions \(\mathbb{R}^2 = \mathbb{R}b_1(t)[+] \mathbb{R}b_2(t)\), where \(b_1(t) := e_1 + te_2\) and \(b_2(t) := te_1 + e_2\) for \(t \in [0, 1)\). Let \(J_t\) denote the corresponding fundamental symmetry. Then the Gram operator for \([\cdot, \cdot]_{J_t}\) with respect to the euclidean inner product \(\langle \cdot, \cdot \rangle = [\cdot, \cdot]_{J_0}\) (see (4.1) in Sect. 4) has eigenvalues \(\frac{1-t}{1+t}\) and \(\frac{1+t}{1-t}\). Therefore the optimal frame bounds change to \(a_t = \frac{1-t}{1+t}\) and \(b_t = \frac{1+t}{1-t}\). This shows that we can’t even give a lower and upper bound for the new frame bounds.

Next we state the analogue of Theorem 3.3 in [24] for continuous frames of rank \(n\). This result shows that continuous frames for a Krein space are essentially the same objects as continuous frames for the associated Hilbert space. The proof differs from that in [24] only in notation and will be omitted. More information on the proof can be found in the remarks below Proposition 3.11.

**Theorem 3.5** Let \(n \in \mathbb{N}\) and let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space with fundamental symmetry \(J\). Given a family of vectors \(\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}} \subset \mathcal{K}\), the following statements are equivalent:

(i) \(\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for the Krein space \((\mathcal{K}_x, [\cdot, \cdot])\) with frame bounds \(a \leq b\).

(ii) \(\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for the Hilbert space \((\mathcal{K}_x, [\cdot, \cdot]_{J_x})\) with frame bounds \(a \leq b\).

(iii) \(\{J_1^x, \ldots, J_n^x\}_{x \in \mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for the Krein space \((\mathcal{K}_x, [\cdot, \cdot])\) with frame bounds \(a \leq b\).

(iv) \(\{J_1^x, \ldots, J_n^x\}_{x \in \mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for the Hilbert space \((\mathcal{K}_x, [\cdot, \cdot]_{J_x})\) with frame bounds \(a \leq b\).

**Example 2** (Wavelets) Let \((L_2(\mathbb{R}), \langle \cdot, \cdot \rangle)\) be the Hilbert space consisting of all measurable functions that are square integrable with respect to the Lebesgue measure of \(\mathbb{R}\), where \(\langle \cdot, \cdot \rangle\) denotes the usual inner product given by \(\langle f, g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t)} \, dt\).

On \(L_2(\mathbb{R})\), we consider the following sesquilinear form and fundamental symmetry:

\[
[f, g] := \int_{\mathbb{R}} f(t) \overline{g(-t)} \, dt, \quad (Jg)(t) := g(-t), \quad f, g \in L_2(\mathbb{R}).
\]
Note that \( \langle f, g \rangle = [f, g] \). Hence \( (L_2(\mathbb{R}), [\cdot, \cdot]) \) defines a Krein space with fundamental symmetry \( J \) such that \( (L_2(\mathbb{R}), \langle \cdot, \cdot \rangle) \) is the associated Hilbert space. Let \( \psi \in L_2(\mathbb{R}) \) be an admissible wavelet, i.e., a function in \( L_2(\mathbb{R}) \) satisfying \( C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(s)|^2}{|s|} ds < \infty \), where \( \hat{\psi} \) denotes the Fourier transform of \( \psi \). For \( (a, b) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \), set

\[
\psi_{a,b}(t) := \frac{1}{|b|^{1/2}} \psi \left( \frac{t-a}{b} \right), \quad t \in \mathbb{R}.
\]

It is well known that \( \{\psi_{a,b}\}_{(a,b) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}} \) defines a tight continuous frame of rank 1 with frame bound \( C_\psi \) for the Hilbert space \( (L_2(\mathbb{R}), \langle \cdot, \cdot \rangle) \) and with respect to the measure space \( (\mathbb{R} \times \mathbb{R} \setminus \{0\}, \mathcal{B}, \frac{da db}{b^2}) \), see [14, Corollary 11.1.2]. From Proposition 3.5, we conclude that \( \{\psi_{a,b}\}_{(a,b) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}} \) yields a \( J \)-tight continuous frame for the Krein space \( (L_2(\mathbb{R}), \langle \cdot, \cdot \rangle) \) with the same frame bound \( C_\psi \) and with respect to the same measure space.

Our next aim is to define the analogue of the frame operator for continuous frames. Let \( \{\eta^1_i, \ldots, \eta^n_i\}_{i \in \mathcal{M}} \) be a continuous frame of rank \( n \in \mathbb{N} \) for the Krein space \( (K, [\cdot, \cdot]) \) with frame bounds \( 0 < a \leq b \) and with respect to the measure space \( (\mathcal{M}, \mathfrak{B}, \nu) \). Combining Definition 3.1, Remark 1 and Definition 3.4 shows that the function of rank 1 operators \( \mathcal{M} \ni x \mapsto |\eta^1_i| [\eta^1_i] \in \mathcal{B}(\mathcal{K}) \) is weakly measurable, where \( |\eta^1_i| [\eta^1_i] (k) := |\eta^1_i|, k |\eta^1_i| \) for \( k \in \mathcal{K} \) and \( i = 1, \ldots, n \). Consider the the sesquilinear form \( \Psi : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{C} \) given by

\[
\Psi(k_1, k_2) = \sum_{i=1}^{n} \int_{\mathcal{M}} [k_1, \eta^1_i] [\eta^1_i, k_2] d\nu(x), \quad k_1, k_2 \in \mathcal{K}.
\]

Applying the Cauchy–Schwarz inequality, we obtain \( |\Psi(k_1, k_2)| \leq b \|k_1\|_J \|k_2\|_J \) for all \( k_1, k_2 \in \mathcal{K} \), so \( \|\Psi\| \leq b \). Thus, the Riesz representation theorem ensures that there exists a unique bounded linear operator \( S : \mathcal{K} \rightarrow \mathcal{K} \) such that \( \|S\| = \|\Psi\| \) and \( \Psi(k_1, k_2) = [k_1, Sk_2] \). Since

\[
[k_1, Sk_2] = \sum_{i=1}^{n} \int_{\mathcal{M}} [k_1, \eta^1_i] [\eta^1_i, k_2] d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} [k_1, j_{\eta^1_i}(x) \eta^1_i] [\eta^1_i, k_2] d\nu(x),
\]

where \( j_{\eta^1_i} := \chi_{\mathcal{M}^+} - \chi_{\mathcal{M}^-} \) denotes the Radon–Nikodym derivative of \( \nu \) with respect to \( \nu \), it follows from Definition 3.2 that

\[
S = \sum_{i=1}^{n} \int_{\mathcal{M}} |\eta^1_i| [\eta^1_i] d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} j_{\eta^1_i}(x) |\eta^1_i| [\eta^1_i] d\nu(x), \quad (3.3)
\]

is the weak integral of the weakly measurable functions \( \mathcal{M} \ni x \mapsto |\eta^1_i| [\eta^1_i] \in \mathcal{B}(\mathcal{K}) \) with respect to \( \nu \), and of \( \mathcal{M} \ni x \mapsto j_{\eta^1_i}(x) |\eta^1_i| [\eta^1_i] \in \mathcal{B}(\mathcal{K}) \) with respect to \( \nu \).

We call the weak integral \( S \) in (3.3) the frame operator and \( Sk \) the frame transform of \( k \in \mathcal{K} \). From (3.2) and (3.3), it follows that

\[
a [k, k]_J \leq [k, Sk] \leq b [k, k]_J \quad \text{for all } k \in \mathcal{K}.
\]

\( \copyright \) Birkhäuser
As a consequence, $S$ is uniformly $J$-positive and therefore invertible. Moreover, the inverse operator $S^{-1}$, regarded as an operator on the Hilbert space $(K, [\cdot, \cdot]_J)$, satisfies

$$0 < b^{-1} \mathbf{1} \leq S^{-1}J \leq a^{-1} \mathbf{1} \quad \text{and} \quad 0 < b^{-1} \mathbf{1} \leq JS^{-1} \leq a^{-1} \mathbf{1}, \quad (3.5)$$

where the second relation follows from the first by applying $J^2 = \mathbf{1}$.

To describe the frame operator in terms of an analogue of the so-called analysis operator, let $\bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu)$ be the $n$-fold direct orthogonal sum of the Krein space $\mathcal{Q}_2(M, \nu)$ from Example 1. For elements $f \in \bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu)$, we write $f = (f_i)_{i=1}^n$ and $f(x) = (f_i(x))_{i=1}^n$, where $f_i \in \mathcal{Q}_2(M, \nu)$. Consider the inner product $[\cdot, \cdot]_\Theta$ in $\bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu)$ given by

$$[(g_i)_{i=1}^n, (f_i)_{i=1}^n]_\Theta := \sum_{i=1}^n \int_M g_i(x)f_j(x)\nu(x).$$

Then $\left(\bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu), [\cdot, \cdot]_\Theta \right)$ yields a Krein space with fundamental decomposition

$$\bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu) = \bigoplus_{i=1}^n \mathcal{Q}_2(M_+, \nu_+) \bigoplus \bigoplus_{i=1}^n \mathcal{Q}_2(M_-, \nu_-)$$

and fundamental symmetry $J_{\mathcal{Q}_2}^\Theta (f_i)_{i=1}^n := J_{\mathcal{Q}_2}(f_i)_{i=1}^n$, where $J_{\mathcal{Q}_2}$ denotes the fundamental symmetry of the Krein space $(\mathcal{Q}_2(M, \nu), [\cdot, \cdot])$ as described in (2.8). The associated Hilbert space $\left(\bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu), [\cdot, \cdot]_\Theta^\oplus \right)$ has the inner product

$$[(g_i)_{i=1}^n, (f_i)_{i=1}^n]_{\Theta^\oplus} = [(g_i)_{i=1}^n, J_{\mathcal{Q}_2}^\Theta (f_i)_{i=1}^n)]_{\Theta} = \sum_{j=1}^n \langle g_j, f_j \rangle_{\mathcal{Q}_2(M, \nu)}. \quad (3.6)$$

Define

$$T : K \to \bigoplus_{i=1}^n \mathcal{Q}_2(M, \nu), \quad (Tk)(x) := ([\eta^i_x, k])_{i=1}^n, \quad k \in K. \quad (3.7)$$

It follows immediately from (3.2) that $T$ is well defined and bounded. In analogy to the Hilbert space case [1, 2, 6], we refer to $T$ as the analysis operator, to $Tk$ as the analysis transform of $k \in K$, and to $\hat{k}(x) := [\eta^i_x, k]$ as the $i$-th analysis transform of $k$. Straightforward calculations show that its adjoint operator, the so-called pre-frame operator, is given by the weak integral

$$T^*(f_i)_{i=1}^n = \sum_{i=1}^n \int_M f_i(x)\eta^i_x d\nu(x), \quad f_i \in \mathcal{Q}_2(M, \nu). \quad (3.8)$$

With the bounded operators $T^*_{i} : \mathcal{Q}_2(M, \nu) \to K$, $T^*_{i}f_i := \int_M f_i(x)\eta^i_x d\nu(x)$, (3.8) becomes $T^*(f_i)_{i=1}^n = \sum_{i=1}^n T^*_{i}f_i$. Furthermore, $T^*J_{\mathcal{Q}_2}^\Theta Tk = \sum_{i=1}^n \int_M j_{\mathcal{Q}_2}^i(x)[\eta^i_x, k]\eta^i_x d\nu(x)$ for all $k \in K$, so that

$$S = T^*J_{\mathcal{Q}_2}^\Theta T \quad (3.9)$$
by (3.3).

The most important result in frame theory is the frame decomposition theorem. For continuous frames, it states that each vector in the Krein (Hilbert) space can be represented by a weak integral obtained from the continuous frame and the inverse of the frame operator. An equivalent way of stating the frame decomposition theorem is that this weak integral yields the identity operator.

**Theorem 3.6** (Frame decomposition) Let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space with fundamental symmetry \(J\) and let \(\{\eta^1_x, \ldots, \eta^n_x\}\) be a continuous frame of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\). Then

\[
1_{\mathcal{K}} = \sum_{i=1}^{n} \int_{\mathcal{M}} |\eta^i_x| [S^{-1} \eta^i_x] \, d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} |S^{-1} \eta^i_x| [\eta^i_x] \, d\nu(x). \tag{3.10}
\]

**Proof** It has been observed in (3.5) that \(S\) has a bounded inverse. Clearly, by (3.9), we have \(S^* = S\) and thus \(S^{-1} = S^{-1}\). From (3.3) and Proposition 3.3(ii), it follows that, for all \(k \in \mathcal{K}\),

\[
k = S^{-1} Sk = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta^i_x, k] S^{-1} \eta^i_x \, d\nu(x),
\]

\[
k = SS^{-1} k = \sum_{i=1}^{n} \int_{\mathcal{M}} [S^{-1} \eta^i_x, k] \eta^i_x \, d\nu(x).
\]

Now the definition of the weak integral gives the result. \(\square\)

By Theorem 3.5, each continuous frame \(\{\eta^1_x, \ldots, \eta^n_x\}\) of rank \(n\) for \((\mathcal{K}, [\cdot, \cdot])\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) gives rise to three other frames with slightly different frame operators. In the following, we will relate these frame operators to \(S\) presented in (3.3). First, consider the frame \(\{J\eta^1_x, \ldots, J\eta^n_x\}\) for \((\mathcal{K}, [\cdot, \cdot])\). Denoting the corresponding frame operator by \(S_0\), we get from (3.3)

\[
S_0 k = \sum_{i=1}^{n} \int_{\mathcal{M}} [J\eta^i_x, k] J\eta^i_x \, d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta^i_x, Jk] J\eta^i_x \, d\nu(x), \quad k \in \mathcal{K}. \tag{3.11}
\]

Comparing (3.11) with (3.3) and applying Proposition 3.3(ii) shows that the two frame operators are related by \(S_0 = JSJ\). Next, let \(S_1\) be the frame operator of the frame \(\{\eta^1_x, \ldots, \eta^n_x\}\) for the Hilbert space \((\mathcal{K}, [\cdot, \cdot])\). Then, by (3.3), we have

\[
S_1 k = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta^i_x, k] \eta^i_x \, d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta^i_x, Jk] \eta^i_x \, d\nu(x), \quad k \in \mathcal{K}, \tag{3.12}
\]

and thus \(S_1 = SJ\). Finally, with \(S_2\) denoting the frame operator of the continuous frame \(\{J\eta^1_x, \ldots, J\eta^n_x\}\) for the Hilbert space \((\mathcal{K}, [\cdot, \cdot])\), we get from (3.3)
\[ S_2k = \sum_{i=1}^{n} \int_{\mathcal{M}} [J]\eta_i^j, k, ]J\eta_i^j d\nu(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta_i^j, k, ]J\eta_i^j d\nu(x), \quad k \in \mathcal{K}. \tag{3.13} \]

so that \( S_2 = JS \).

The next two propositions show that continuous frames can be characterized by properties of the analysis and pre-frame operator. By Theorem 3.5, the proofs are literally the same as in the Hilbert space case, see e.g. [32].

**Proposition 3.7** Let \((\mathcal{M}, \mathcal{B}, \nu)\) be a measure space and let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space. Given a collection of weakly measurable functions \(\mathcal{M} \ni x \mapsto \eta_i^j \in \mathcal{K}, \quad j = 1, \ldots, n\) the family \(\{\eta_i^1, \ldots, \eta_i^n\}_{\mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) if and only if the analysis operator \(T\) defined in (3.7) is injective.

**Proposition 3.8** Let \((\mathcal{M}, \mathcal{B}, \nu)\) be a measure space and let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space. Given a collection of weakly measurable functions \(\mathcal{M} \ni x \mapsto \eta_i^j \in \mathcal{K}, \quad j = 1, \ldots, n\) the family \(\{\eta_i^1, \ldots, \eta_i^n\}_{\mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) if and only if the pre-frame operator \(T^*\) given in (3.8) is surjective.

In practice, it is not necessary to verify the frame condition (3.2) for all \(k \in \mathcal{K}\), it suffices to show (3.2) on a dense subspace. This will be proven in the next proposition.

**Proposition 3.9** Let \((\mathcal{M}, \mathcal{B}, \nu)\) be a measure space and let \(\{\eta_i^1, \ldots, \eta_i^n\}_{\mathcal{M}}\) be a family of vectors in a Krein space \((\mathcal{K}, [\cdot, \cdot])\) such that the maps \(\mathcal{M} \ni x \mapsto \eta_i^j \in \mathcal{K}, \quad i = 1, \ldots, n\) are weakly measurable. If the frame condition (3.2) is satisfied for all elements in a dense subset \(\mathcal{G}\) of \(\mathcal{K}\), then \(\{\eta_i^1, \ldots, \eta_i^n\}_{\mathcal{M}}\) is a continuous frame of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\).

**Proof** Assume that (3.2) is satisfied for all \(k \in \mathcal{G}\) and for fixed positive constants \(a \leq b\). Using (3.6) and (3.7), we can write (3.2) as

\[ a\|k\|^2_{J} \leq \|Tk\|^2_{\mathcal{B}} \leq b\|k\|^2_{J} \quad \text{for all } k \in \mathcal{G}. \tag{3.14} \]

It follows that \(T\) is bounded and thus uniformly continuous. By uniform continuity, (3.14) holds for all \(k \in \mathcal{K}\), which is equivalent to (3.2).

We close this section by showing that any continuous frame of rank \(n\) can be reduced to a continuous frame of rank 1 with respect to a suitable measure space.

**Proposition 3.10** Any continuous frame \(\{\eta_i^1, \ldots, \eta_i^n\}_{\mathcal{M}}\) of rank \(n \in \mathbb{N}\) for a Krein space \((\mathcal{K}, [\cdot, \cdot])\) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) may be viewed as a continuous frame \(\{\sigma(y)\}_{\mathcal{M}}\) of rank 1 for \(\mathcal{K}\) with respect to some measure space \((\mathcal{M}, \mathcal{B}, \nu)\).
Proof Assume that $\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}}$ is a continuous frame of rank $n \in \mathbb{N}$ for the Krein space $(\mathcal{K}, [\cdot, \cdot])$, with respect to $(\mathcal{M}, \mathcal{B}, \nu)$. Set $\mathcal{M}_n := \{1, \ldots, n\} \times \mathcal{M}$ and consider the counting measure $\mu_n$ on the power set $2^{\{1, \ldots, n\}}$. Let $\nu_n := \mu_n \otimes \nu$ denote the product measure on the product $\Sigma$-algebra $\mathcal{B}_n := 2^{\{1, \ldots, n\}} \otimes \mathcal{B}$. Define $\sigma : \mathcal{M}_n \rightarrow \mathcal{K}$ by $\sigma((i, x)) := \eta_i^x$. Since the functions $\mathcal{M} \ni x \mapsto \eta_i^x \in \mathcal{K}$, $i = 1, \ldots, n$, are weakly measurable, it follows that $\sigma$ is weakly measurable. Note that $|\nu_n| = \mu_n \otimes |\nu|$. Now, by Tonelli’s theorem, we have

$$\int_{\mathcal{M}_n} ||\sigma(i, x), k||^2 d|\nu_n|((i, x)) = \sum_{i=1}^n \int_{\mathcal{M}} ||\eta_i^x, k||^2 d|\nu|(x)$$

for all $k \in \mathcal{K}$. From this, we conclude that $\{\sigma(y)\}_{y \in \mathcal{M}_n}$ is a continuous frame of rank 1 for $\mathcal{K}$ with respect to $(\mathcal{M}_n, \mathcal{B}_n, \nu_n)$ admitting the same frame bounds as $\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}}$. \hfill \square

3.3 Role of the fundamental symmetry in the formula of the frame operator

Given a continuous frame $\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}}$ of rank $n$ for the Krein space $(\mathcal{K}, [\cdot, \cdot])$, let $T$ and $T^*$ denote the analysis and pre-frame operator defined in (3.7) and (3.8), respectively. In this section, we will discuss the importance of the fundamental symmetry $J_{\mathcal{B}}^\otimes$ in the formula (3.9) of the frame operator: $S = T^*J_{\mathcal{B}}^\otimes T$.

For comparison, consider the bounded linear operator $\mathcal{S} : \mathcal{K} \rightarrow \mathcal{K}$ given by $\mathcal{S} := T^*T$. As the frame operator $S$, it is obviously self-adjoint in the Krein space $(\mathcal{K}, [\cdot, \cdot])$. Furthermore, by (3.7) and (3.8),

$$[h, \mathcal{S}k] = \sum_{i=1}^n \int_{\mathcal{M}} [\eta_i^x, k][h, \eta_i^x] d\nu(x)$$

(3.15)

for all $h, k \in \mathcal{K}$. Hence $\mathcal{S}$ coincides with the weak integral

$$\mathcal{S} = \sum_{i=1}^n \int_{\mathcal{M}} [\eta_i^x][\eta_i^x] d\nu(x).$$

The operators $\mathcal{S}$ and $S$ have different properties, for instance, $S$ is always invertible, but $\mathcal{S}$ may not be and therefore cannot be used in the frame decomposition theorem. We illustrate this in the following example.

Example 3 Let $(\mathcal{M}_0, \mathcal{B}_0, \mu_0)$ be a finite measure space and $\{\eta_1^x, \ldots, \eta_n^x\}_{x \in \mathcal{M}_0}$ a continuous frame of rank $n \in \mathbb{N}$ for a Krein space $(\mathcal{K}, [\cdot, \cdot])$ with respect to $(\mathcal{M}_0, \mathcal{B}_0, \mu_0)$ and with frame bounds $0 < a \leq b$. Consider a two point set $\{p_-, p_+\}$ and the signed measure $\nu$ on the power set $2^{[p_-, p_+]}$ determined by $\nu(\{p_\pm\}) = -1$ and $\nu(\{p_\pm\}) = 1$. Let $\mathcal{B} := \mathcal{B}_0 \otimes 2^{[p_-, p_+]}$ denote the product $\Sigma$-algebra and $\nu := \mu_0 \otimes \nu$ the product measure on $\mathcal{B}$. For $(x, p_\pm) \in \mathcal{M} := \mathcal{M}_0 \times \{p_-, p_+\}$, set $\eta_{(x, p_\pm)}^i := \eta_i^x$. Then the map
$\mathcal{M} \ni (x, p) \mapsto \eta^i_{(x,p)} \in \mathcal{K}$ is weakly measurable for all $i = 1, \ldots, n$ and, by Tonelli’s Theorem,

$$\sum_{i=1}^{n} \int_{\mathcal{M}} |[\eta^i_{(x,p)}, k]|^2 \, d|\nu|(x, p) = \sum_{i=1}^{n} \int_{\mathcal{M}} \left( \int_{(p,x)} |[\eta^i_{(x,p)}, k]|^2 \, d|\nu|(p) \right) \, d\mu_0(x)$$

Hence $\{\eta^1_{(x,p)}, \ldots, \eta^n_{(x,p)}\}_{x \in \mathcal{M}}$ is a continuous frame of rank $n \in \mathbb{N}$ for the Krein space $\mathcal{K}$ with frame bounds $0 < 2a \leq 2b$. Now, by Fubini’s theorem, Eq. (3.15) yields

$$[h, \mathcal{S}k] = \sum_{i=1}^{n} \int_{\mathcal{M}} [\eta^i_{(x,p)}, k][h, \eta^i_{(x,p)}] \, d\nu((x, p))$$

for all $h, k \in \mathcal{K}$. As $(\mathcal{K}, [\cdot, \cdot])$ is non-degenerate, it follows that $\mathcal{S} = T^* T = 0$. In particular, $\mathcal{S}$ is not invertible.

### 3.4 Reproducing kernels

Let $\{\eta^1, \ldots, \eta^n\}_{x \in \mathcal{M}}$ be a continuous frame of rank $n$ for the Krein space $(\mathcal{K}, [\cdot, \cdot])$ with frame bounds $a \leq b$. Then, by Theorem 3.5, we may view $\{\eta^1, \ldots, \eta^n\}_{x \in \mathcal{M}}$ as a continuous frame in the Hilbert space $(\mathcal{K}, [\cdot, \cdot])$ with the same frame bounds. Considering the analysis operator $T$ defined in Eq. (3.7) as a Hilbert space operator $T : (\mathcal{K}, [\cdot, \cdot]) \rightarrow \left( \bigoplus_{i=1}^{n} \mathcal{H}_2(\mathcal{M}, \nu), [\cdot, \cdot]^{\oplus}_{\mathcal{H}_2} \right)$, it follows from (3.14) that $T$ is bounded, injective, and $\text{Ran} T$ is closed. In particular, $T(\mathcal{K}) := \text{Ran} T$ with the inner product $[\cdot, \cdot]^{\oplus}_{\mathcal{H}_2}$ is a Hilbert space by itself. As shown in [1] and [3], $T(\mathcal{K}, [\cdot, \cdot]^{\oplus}_{\mathcal{H}_2})$ is actually a reproducing kernel Hilbert space. To see this, note that

$$([\eta^i_x, k])_{i=1}^{n} = \left( \sum_{j=1}^{n} \int_{\mathcal{M}} [\eta^i_x, S^{-1}\eta^j_x] [\eta^j_x, k] \, d\nu(y) \right)_{i=1}^{n}$$

for all $k \in \mathcal{K}$, $x \in \mathcal{M}$, by (3.10). Setting $K_{ij}(x, y) := [\eta^i_x, S^{-1}\eta^j_y]$ and $\mathbf{K}(x, y) := (K_{ij}(x, y))_{i,j=1}^{n} \in \text{Mat}_{n \times n}(\mathbb{C})$, we have for all $\Phi = (\phi_i)_{i=1}^{n} \in T(\mathcal{K})$

$$\Phi(x) = \int_{\mathcal{M}} \mathbf{K}(x, y) \circ \Phi(y) \, d\nu(y) = \left( \sum_{j=1}^{n} \int_{\mathcal{M}} K_{ij}(x, y) \phi_j(y) \, d\nu(y) \right)_{i=1}^{n}, \quad (3.16)$$

where $\circ$ denotes matrix multiplication. Moreover,
\[ K_{ij}(x, y) = [\eta_j^i, S^{-1} \eta_i^j] = K_{ji}(y, x), \]  

(3.17)

it follows that

\[ \| \Phi(x) \|_{C_0}^2 \leq \left( \sum_{i=1}^n \sum_{j=1}^n \left[ \int_{\mathcal{M}} [\eta_j^i, S^{-1} \eta_i^j] [\eta_j^i, S^{-1} \eta_i^j] \, d|\nu|(y) \right] \| \Phi \|_{j_{\mathcal{K}_2}}^2 \right)^n = \left( \sum_{i=1}^n [\eta_j^i, S^{-1} \eta_i^j] \right) \| \Phi \|_{j_{\mathcal{K}_2}}^2. \]

As a consequence, the evaluation map \( T(\mathcal{K}) \ni \Phi \mapsto \phi_i(x) \in \mathbb{C} \) is continuous for all \( x \in \mathcal{M} \) and \( i = 1, \ldots, n \). Moreover, Eqs. (3.7), (3.16) and (3.17) show that this evaluation map can be given by the following inner product:

\[ \phi_i(x) = [T(S^{-1} \eta_i^j), \Phi]_{j_{\mathcal{K}_2}} = \left( K_{ij}(x, \cdot) \right)_{i=1}^n. \]  

(3.18)

Since we are working in the Krein space setting, it would be desirable to have a Krein space analogue reproducing kernel Hilbert space. Clearly, (3.18) can be written

\[ \phi_i(x) = [J_{\mathcal{K}_2}T(S^{-1} \eta_i^j), \Phi] = \left( (j_{\mathcal{K}_2} \cdot) K_{ij}(x, \cdot) \right)_{i=1}^n, \]  

(3.19)

and then (3.16) becomes

\[ \Phi(x) = \int_{\mathcal{M}} J_{\mathcal{K}_2}T(x, y) \Phi(y) \, dv(y) \]

\[ = \left( \sum_{j=1}^n \int_{\mathcal{M}} j_{\mathcal{K}_2}(y) K_{ij}(x, y) \phi_j(y) \, dv(y) \right)_{i=1}^n. \]  

(3.20)

However, although \( T(\mathcal{K}) \) is always a closed subspace of \( \bigoplus_{i=1}^n \mathcal{K}_2(\mathcal{M}, \nu), [\cdot, \cdot]_{\mathcal{K}_2} \), the restriction of \([\cdot, \cdot]_{\mathcal{K}_2} \) to \( T(\mathcal{K}) \) does not necessarily yield a Krein space. For instance, in Example 3, we have \([T_h, T_k] = [h, S\nu] = 0 \) for all \( h, k \in \mathcal{K} \) so that \( T(\mathcal{K}) \cap T(\mathcal{K})^\perp = T(\mathcal{K}) \). Therefore, by Proposition 2.1, we may view \( (T(\mathcal{K}), [\cdot, \cdot]_{\mathcal{K}_2}) \) as a reproducing kernel Krein space only if any element in \( \bigoplus_{i=1}^n \mathcal{K}_2(\mathcal{M}, \nu) \) admits at least one \( J_{\mathcal{K}_2} \)-orthogonal projection onto \( T(\mathcal{K}) \), or equivalently, if \( T(\mathcal{K}) \) is ortho-complemented. In this case, (3.19) and (3.20) hold with the Krein space kernel function

\[ J_{\mathcal{K}_2} : \mathcal{M} \times \mathcal{M} \to \text{Mat}_{n \times n}(\mathbb{C}), \]

\[ J_{\mathcal{K}_2}(x, y) := \left( j_{\mathcal{K}_2}(y) K_{ij}(x, y) \right)_{i=1}^n = \left( j_{\mathcal{K}_2}(y) [\eta_j^i, S^{-1} \eta_i^j] \right)_{i=1}^n. \]

Moreover, if \( J_{\mathcal{K}_2} : T(\mathcal{K}) \to T(\mathcal{K}) \), then

\[ K^i_x := J_{\mathcal{K}_2}T(S^{-1} \eta_i^j) = \left( j_{\mathcal{K}_2}(\cdot) K_{ij}(\cdot, \cdot) \right)_{i=1}^n \in T(\mathcal{K}) \]

and
\[
\sum_{i=1}^{n} \int_{M} |[K_{x}^{i}, \Phi]|^{2} \, d\nu(x) = \sum_{i=1}^{n} \int_{M} \left( \sum_{j=1}^{n} K_{ij}(x,y) \phi_{j}(y) \right)^{2} \, d\nu(y) = \|\phi\|_{f_{g_{2}}}^{2},
\]

so that \( \{K_{1}^{x}, \ldots, K_{x}^{n}\}_{x \in M} \) defines a \( J^{\oplus}_{g_{2}} \)-Parseval frame for \( (T(K), \cdot, \cdot)^{\oplus} \).

### 3.5 Similar continuous frames

It is not difficult to see that the proof of the equivalences in Theorem 3.5 hinges on the fact that \( J \) happens to be a unitary operator with respect to \([\cdot,\cdot]_{I}\) and \([\cdot,\cdot]_{J}\). Theorem 3.5 can be generalized by an application of the inverse mapping theorem as follows.

**Proposition 3.11** Let \((K_{1}, [\cdot,\cdot]_{1})\) and \((K_{2}, [\cdot,\cdot]_{2})\) be Krein spaces with chosen fundamental symmetries. Assume that \( \{\eta_{1}^{i}, \ldots, \eta_{n}^{i}\}_{x \in M} \) is a continuous frame of rank \( n \in \mathbb{N} \) for \( K_{1} \) with respect to \((M, \mathcal{B}, \nu)\) and with frame bounds \( a \leq b \). Then, for any bijective linear operator \( A \in \mathcal{B}(K_{1}, K_{2}) \), the family \( \{A\eta_{1}^{i}, \ldots, A\eta_{n}^{i}\}_{x \in M} \) is a continuous frame of rank \( n \) for \( K_{2} \) with frame bounds \( a\|A^{-1}\|^{-2} \leq b\|A\|^{2} \).

As \( \|[A\eta_{i}^{j}, k]\| = \|\eta_{i}^{j}, A^{*}k\|, \|A^{*}k\|_{J_{2}} \leq \|A\| \|k\|_{J_{1}}, \|k\|_{J_{2}} \leq \|A^{-1}\| \|A^{*}k\|_{J_{1}} \) for all \( k \in K_{2} \), the proof of the proposition is straightforward. Proposition 3.11 motivates the following definition.

**Definition 3.12** Let \((K_{1}, [\cdot,\cdot]_{1})\) and \((K_{2}, [\cdot,\cdot]_{2})\) be Krein spaces. Two continuous frames \( \{\eta_{1}^{i}, \ldots, \eta_{n}^{i}\}_{x \in M} \) and \( \{\theta_{1}^{i}, \ldots, \theta_{n}^{i}\}_{x \in M} \) of rank \( n \in \mathbb{N} \) with respect to \((M, \mathcal{B}, \nu)\) for the Krein spaces \( K_{1} \) and \( K_{2} \), respectively, are said to be **similar** if there exists a bijective operator \( A \in \mathcal{B}(K_{1}, K_{2}) \) such that \( A\eta_{i}^{j} = \theta_{i}^{j} \) for all \( x \in M \) and all \( i = 1, \ldots, n \). The continuous frames are called **unitarily equivalent** if \( A \) is a \( J \)-unitary operator.

Equations (3.4) and (3.5) ensure that \( S \) belongs to \( \mathcal{B}(K) \) and has a bounded inverse. Therefore, the family \( \{S^{-1}\eta_{i}^{1}, \ldots, S^{-1}\eta_{i}^{n}\}_{x \in M} \) in the Frame Decomposition Theorem 3.6 yields a continuous frame of rank \( n \) that is similar to \( \{\eta_{1}^{1}, \ldots, \eta_{n}^{n}\}_{x \in M} \).

The next proposition provides a criterion for a pair of frames to be similar. It is an analogue of Deguang and Larson’s result for Hilbert spaces [23].

**Proposition 3.13** Let \((K_{1}, [\cdot,\cdot]_{1})\) and \((K_{2}, [\cdot,\cdot]_{2})\) be Krein spaces. Two continuous frames \( \{\eta_{1}^{i}, \ldots, \eta_{n}^{i}\}_{x \in M} \) and \( \{\theta_{1}^{j}, \ldots, \theta_{n}^{j}\}_{x \in M} \) of rank \( n \in \mathbb{N} \) with respect to \((M, \mathcal{B}, \nu)\) for the Krein spaces \( K_{1} \) and \( K_{2} \), respectively, are similar if and only if their analysis operators \( T_{1} : K_{1} \rightarrow \bigoplus_{i=1}^{n} Q_{2}(M, \nu) \) and \( T_{2} : K_{2} \rightarrow \bigoplus_{i=1}^{n} Q_{2}(M, \nu) \) have the same range.
Proof The proof of the “if” part is the same as in [25, Theorem 3.3] for $\mathcal{K}_G = \mathcal{K}_1$ and $\mathcal{H}_W = \mathcal{K}_2$. It follows from $[\mathcal{A}\eta_i^h, h]_2 = [\eta_i^h, \mathcal{A}^* h]_1$ for all $h \in \mathcal{K}_2$, $i = 1, \ldots, n$, and the bijectivity of $\mathcal{A} \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$.

To prove the opposite direction, assume that $V := \text{Rang} T_1 = \text{Rang} T_2$, where $V \subset \bigoplus_{i=1}^n \mathcal{L}_2(\mathcal{M}, \nu)$. Similarly to (3.14), we obtain from (3.2), (3.6) and (3.7) that
\[
a_j \|k_j\|^2_{J_j} \leq \|T_j k_j\|^2_{\mathcal{L}_2^*} \leq b_j \|k_j\|^2_{J_j} \quad \text{for all } k_j \in \mathcal{K}_j,
\]
(3.21)
where $J_j$ denotes a chosen fundamental symmetry on $\mathcal{K}_j$ and $0 < a_j \leq b_j$ are frame bounds for the frame in $\mathcal{K}_j$, $j = 1, 2$. From (3.21), we conclude that $T_1$ and $T_2$ are bounded and injective, and their range $V$ is closed in $\bigoplus_{i=1}^n \mathcal{L}_2(\mathcal{M}, \nu)$. It follows that $T_1 \in \mathcal{B}(\mathcal{K}_1, V)$ and $T_2 \in \mathcal{B}(\mathcal{K}_2, V)$ possess bounded inverses and the same holds for $T_1^* \in \mathcal{B}(V, \mathcal{K}_1)$ and $T_2^* \in \mathcal{B}(V, \mathcal{K}_2)$. Given $k_1 \in \mathcal{K}_1$, set $k_2 := T_2^{-1} T_1 k_1$. Then obviously $T_2 k_2 = T_1 k_1$, which is equivalent to $[\theta^j x, k_2]_2 = [\eta_i^j x, k_1]_1 \in \mathcal{L}_2(\mathcal{M}, |\nu|)$ for all $i = 1, \ldots, n$. Recall from Sect. 3.4 that $V = \text{Rang} T_1 = \text{Rang} T_2 \subset \bigoplus_{i=1}^n \mathcal{L}_2(\mathcal{M}, |\nu|)$ is a reproducing kernel Hilbert space, so $[\theta^j x, k_2]_2 = [\eta_i^j x, k_1]_1$ for all $x \in \mathcal{M}$ by (3.18). With $k_2 = T_2^{-1} T_1 k_1$, we obtain
\[
[\eta_i^j x, k_1]_1 = [\theta^j x, T_2^{-1} T_1 k_1]_2 = [T_1^* T_2^{-1} \theta^j x, k_1]_1 \quad \text{for all } k_1 \in \mathcal{K}_1.
\]
Since $(\mathcal{K}_1, [\cdot, \cdot]_1)$ is non-degenerate, it follows that $\mathcal{A} \theta^j x = \eta_i^j x$ for all $x \in \mathcal{M}$ and $i = 1, \ldots, n$, where $\mathcal{A} := T_1^* T_2^{-1} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ is a bounded bijective operator. This shows that the two continuous frames are similar. \qed

3.6 Dual and Parseval continuous frames

This section is devoted to the description of dual and Parseval continuous frames and their relations to the fundamental symmetry $J$, the frame operator $S$ and coherent states. A similar analysis for discrete $J$-frames can be found in [31]. We start with the definition of dual continuous frames.

Definition 3.14 Let $\{\eta_i^1, \ldots, \eta_i^n\}_{x \in \mathcal{M}}$ be a continuous frame of rank $n \in \mathbb{N}$ for the Krein space $(\mathcal{K}, [\cdot, \cdot])$ with respect to $(\mathcal{M}, \mathcal{B}, \nu)$. A continuous frame $\{\theta^1 x, \ldots, \theta^n x\}_{x \in \mathcal{M}}$ of rank $n$ for $(\mathcal{K}, [\cdot, \cdot])$ with respect to $(\mathcal{M}, \mathcal{B}, \nu)$ is called a dual frame (or sometimes a $v$-dual frame) of $\{\eta_i^1, \ldots, \eta_i^n\}_{x \in \mathcal{M}}$ if
\[
1_{\mathcal{K}} = \sum_{i=1}^n \int_{\mathcal{M}} |\theta^j x| |\eta_i^j| \, d\nu(x).
\]
Note that we used the measure $\nu$ instead of $|\nu|$ to emphasize the Krein space setting. Obviously, this can be compensated by multiplying $\theta^i$ by the value of the Radon-Nikodym derivative $J_{\nu}$ at the point $x \in \mathcal{M}$. The following proposition is analogous to [24, Proposition 3.8] in the discrete case. However, the
Radon–Nikodym derivative $j_{g_2}$ will appear in the formulas since the definition of the frame operator forced us to include the fundamental symmetry $J_{g_2}^0$, see Sect. 3.3.

**Proposition 3.15** Let $(K, [\cdot, \cdot])$ be a Krein space with fundamental symmetry $J$ and let $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$ be a continuous frame of rank $n \in \mathbb{N}$ for $(K, [\cdot, \cdot])$ with respect to $(M, \mathcal{B}, \nu)$. With $S$ denoting the corresponding frame operator given in (3.3) and $j_{g_2}$ denoting the Radon–Nikodym derivative of $\nu$ with respect to $|\nu|$, there are the following dual frames:

(i) The continuous frame $\{j_{g_2}(x)S^{-1}\eta_1^n, \ldots, j_{g_2}(x)S^{-1}\eta^n_n\}_{x \in M}$ is a dual frame of $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$ for the Krein space $(K, [\cdot, \cdot])$.

(ii) The continuous frame $\{j_{g_2}(x)JS^{-1}\eta_1^n, \ldots, j_{g_2}(x)JS^{-1}\eta^n_n\}_{x \in M}$ is a dual frame of $\{J\eta_1^n, \ldots, J\eta^n_n\}_{x \in M}$ for the Krein space $(K, [\cdot, \cdot])$.

(iii) The continuous frame $\{j_{g_2}(x)JS^{-1}\eta_1^n, \ldots, j_{g_2}(x)JS^{-1}\eta^n_n\}_{x \in M}$ is a dual frame of $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$ for the Krein space $(K, [\cdot, \cdot])$.

(iv) The continuous frame $\{j_{g_2}(x)S^{-1}\eta_1^n, \ldots, j_{g_2}(x)S^{-1}\eta^n_n\}_{x \in M}$ is a dual frame of $\{J\eta_1^n, \ldots, J\eta^n_n\}_{x \in M}$ for the Hilbert space $(K, [\cdot, \cdot])$.

Furthermore, if $a \leq b$ are frame constants for $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$, then $b^{-1} \leq a^{-1}$ are frame constants for all these dual frames.

**Proof** Let $a \leq b$ be frame bounds for the frame $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$. Since

$$
\sum_{i=1}^{n} \int_{\mathcal{M}} |[j_{g_2}(x)S^{-1}\eta_i^n, k]|^2 d|\nu|(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} [S^{-1} k, \eta_i^n] [\eta_i^n, S^{-1} k] d|\nu|(x) = [S^{-1} k, SS^{-1} k] = [k, JS^{-1} k]_J
$$

by (3.3) and $b^{-1}||k||^2_J \leq [k, JS^{-1} k]_J \leq a^{-1}||k||^2_J$ by the second relation in (3.5), it follows that $\{j_{g_2}(x)S^{-1}\eta_1^n, \ldots, j_{g_2}(x)S^{-1}\eta^n_n\}_{x \in M}$ defines a continuous frame for $(K, [\cdot, \cdot])$ with frame bounds $b^{-1} \leq a^{-1}$. Furthermore, from Equation (3.3) and Proposition 3.3(ii), we get

$$
1_K = S^{-1}S = \sum_{i=1}^{n} \int_{\mathcal{M}} [j_{g_2}(x)S^{-1}\eta_i^n][\eta_i^n] d\nu(x),
$$

so that it is indeed a dual frame of $\{\eta_1^n, \ldots, \eta^n_n\}_{x \in M}$. Replacing $S$ by $S_0 = JSJ$, $S_1 = SJ$ and $S_2 = JS$ from Sect. 3.2 shows (ii), (iii) and (iv), respectively. \(\square\)

The basic step in the proof of the last proposition was to multiply the frame operator by its inverse from the left and to apply Proposition 3.3(ii). Likewise, if we multiply the frame operator from both sides with the inverse of a self-adjoint square root, then we will get a “self-dual” frame. However, this can only be done if a self-adjoint square root of the frame operator exists, for instance, by (3.5), for the positive Hilbert space operators $S_1 = SJ$ and $S_2 = JS$. According to Theorem 3.5, the
resulting frames for the associated Hilbert space will also yield frames for the Krein space. This observation is the starting point of the next proposition.

**Proposition 3.16** Let \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) be a Krein space with fundamental symmetry \(J\) and \(\{\eta^1_x, \ldots, \eta^n_x\}_{x \in \mathcal{M}}\) a continuous frame of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\) with respect to \((\mathcal{M}, \mathfrak{B}, \nu)\). Then \(\{(SJ)^{-1/2}\eta^1_x, \ldots, (SJ)^{-1/2}\eta^n_x\}_{x \in \mathcal{M}}\) and \(\{(JS)^{-1/2}J\eta^1_x, \ldots, (JS)^{-1/2}J\eta^n_x\}_{x \in \mathcal{M}}\) define \(J\)-Parseval continuous frames of rank \(n \in \mathbb{N}\) for \(\mathcal{K}\), where \(S\) denotes the frame operator given in (3.3) and \(K\) is viewed either as the Krein space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) or as the Hilbert space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\).

**Proof** Let \(0 < a \leq b\) be frame bounds for \(\{\eta^1_x, \ldots, \eta^n_x\}_{x \in \mathcal{M}}\) and thus, by Theorem 3.5, also for \(\{J\eta^1_x, \ldots, J\eta^n_x\}_{x \in \mathcal{M}}\). As is well-known from \(C^\ast\)-algebra theory, (3.5) implies

\[
0 < b^{-1/2}1 \leq (JS)^{-1/2} \leq a^{-1/2}1 \quad \text{and} \quad 0 < b^{-1/2}1 \leq (SJ)^{-1/2} \leq a^{-1/2}1.
\]

As a consequence, \((JS)^{\pm1/2}\) and \((SJ)^{\pm1/2}\) are bounded operators. Hence, by Proposition 3.11, \(\{(SJ)^{-1/2}\eta^1_x, \ldots, (SJ)^{-1/2}\eta^n_x\}_{x \in \mathcal{M}}\) and \(\{(JS)^{-1/2}J\eta^1_x, \ldots, (JS)^{-1/2}J\eta^n_x\}_{x \in \mathcal{M}}\) define continuous frames for \(\mathcal{K}\). Next, we compute for \(k \in \mathcal{K}\) that

\[
\sum_{i=1}^n \int_{\mathcal{M}} \|(SJ)^{-1/2}\eta^i_x, k\|^2 d\nu(x) = \sum_{i=1}^n \int_{\mathcal{M}} [J(SJ)^{-1/2}Jk, \eta^i_x] [\eta^i_x, J(SJ)^{-1/2}Jk] d\nu(x) = [J(SJ)^{-1/2}Jk, SJ(SJ)^{-1/2}Jk] = [Jk, SJ(SJ)^{-1/2}Jk] = [Jk, Jk]_J = \|k\|_J^2,
\]

where we used \((SJ)^{-1/2}\)^\ast = \(J((SJ)^{-1/2})[^s]J = J(SJ)^{-1/2}J\) in the first equality and (3.3) in the second. This shows that \(\{(SJ)^{-1/2}\eta^1_x, \ldots, (SJ)^{-1/2}\eta^n_x\}_{x \in \mathcal{M}}\) yields a \(J\)-Parseval continuous frame for \(\mathcal{K}\), where, by Theorem 3.5, we may view \(\mathcal{K}\) either as the Krein space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) or as the Hilbert space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\).

Likewise, since \(S_2 = JS\) functions as the frame operator for \(\{J\eta^1_x, \ldots, J\eta^n_x\}_{x \in \mathcal{M}}\) with respect to the Hilbert space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\), we obtain from (3.13) that

\[
\sum_{i=1}^n \int_{\mathcal{M}} \|(JS)^{-1/2}J\eta^i_x, k\|^2 d\nu(x) = \sum_{i=1}^n \int_{\mathcal{M}} [(JS)^{-1/2}Jk, J\eta^i_x] [J\eta^i_x, (JS)^{-1/2}Jk] d\nu(x) = [(JS)^{-1/2}Jk, JS(JS)^{-1/2}Jk] = [Jk, (JS)^{-1/2}Jk]_J = [Jk, Jk]_J = \|k\|_J^2,
\]

which shows that \(\{(JS)^{-1/2}J\eta^1_x, \ldots, (JS)^{-1/2}J\eta^n_x\}_{x \in \mathcal{M}}\) defines a \(J\)-Parseval continuous frames for \(\mathcal{K}\). \(\square\)
There are different notations of coherent states in the physics literature. For instance, Perelomov [40, Section 2.1] relates coherent states with unitary group representations, and Ali, Antoine and Gazeau [1, Definition 5.4.5] use the reproducing kernel property to define generalized coherent states. What both definitions have in common is that, in favorable situations (e.g. compact groups), a family of coherent states yields a resolution of the identity, commonly written as \( 1 = \int_x |x\rangle \langle x| \, dv(x) \).

**Example 4** (Coherent states in Krein spaces) Let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space with fundamental symmetry \(J\) and let \(\{\eta_x\}_{x \in \mathcal{M}}\) be a continuous frame of rank 1 with respect to the positive measure space \((\mathcal{M}, \mathfrak{B}, \nu)\). Define \(J \circ [J]^{-1/2} \eta_x\). Then, by Proposition 3.16, \(\{\sigma(x)\}_{x \in \mathcal{M}}\) yields a continuous \(J\)-Parseval frame for \(\mathcal{K}\). Note that, for any \(J\)-Parseval frame with frame operator \(S\), (3.5) implies \(S^{-1}J = 1\), hence \(S = J\). Thus (3.3) gives

\[
J = \int_{\mathcal{M}} |\sigma(x)| \, dv(x).
\]

From this perspective, we may view the family \(\{\sigma(x)\}_{x \in \mathcal{M}}\) as a system of coherent states for the Krein space \((\mathcal{K}, [\cdot, \cdot])\).

In the light of Definition 3.12, Proposition 3.15 states that any continuous frame is similar to a dual frame, and Proposition 3.16 shows that any continuous frame is similar to a \(J\)-Parseval frame. Moreover, combining Proposition 3.10 and Example 4, we can say that any continuous frame with respect to a positive measure space determines a set of coherent states.

The conclusion in Example 4 that the frame operator \(S\) of any \(J\)-Parseval frame satisfies \(S = J\) remains valid for continuous frames of rank \(n\). Then (3.3) and Proposition 3.3(ii) yield

\[
1_{\mathcal{K}} = \sum_{i=1}^n \int_{\mathcal{M}} [j_{\mathcal{K}_i}(x)Jn_i^j][n_i^j] \, dv(x).
\]

From this, we conclude that, for any \(J\)-Parseval continuous frame \(\{n_1^i, \ldots, n_n^i\}_{x \in \mathcal{M}}\), a dual frame is given by \(\{j_{\mathcal{K}_i}(x)n_1^j, \ldots, j_{\mathcal{K}_i}(x)n_n^j\}_{x \in \mathcal{M}}\).

### 4 Continuous frames in Krein spaces arising from a non-regular \(W\)-metric

Each Krein space \((\mathcal{K}, [\cdot, \cdot])\) with fundamental symmetry \(J\) can be given by “starting” with the Hilbert \((\mathcal{K}, [\cdot, \cdot])\) and setting \([h, k] := [k, Jh]_J\) for all \(k, h \in \mathcal{K}\). More generally, if \(W\) is a bounded self-adjoint operator on a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) with polar decomposition \(W = J|W|\) and such that \(0 \notin \text{spec}(W)\), then

\[
[f, g] := \langle f, Wg \rangle, \quad f, g \in \mathcal{H},
\]

(4.1)
defines a non-degenerate inner product on \( \mathcal{H} \). Since, by assumption, \( 0 \notin \text{spec}(|W|) \), there exists an \( \epsilon > 0 \) such that \( \epsilon I \leq |W| \leq ||W|| I \). It follows that the Hilbert space norms \( \| \cdot \|_J = \sqrt{\langle \cdot, |W| \cdot \rangle} \) and \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \) are equivalent. Therefore, \( (\mathcal{H}, [\cdot, \cdot]) \) yields a Krein space with fundamental symmetry \( J \) and fundamental decomposition \( \mathcal{H} = P_+ \mathcal{H} \oplus P_- \mathcal{H} \), where \( P_{\pm} = \frac{1}{2} (I \pm J) \), see for instance [7]. Clearly, Krein or Hilbert spaces with the topology given by equivalent norms admit the same collection of (continuous) frames. Hence, by Proposition 3.11 and Theorem 3.5, any continuous frame for \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) of rank \( n \in \mathbb{N} \) yields one for \( (\mathcal{H}, [\cdot, \cdot]) \) and vice versa.

As in [24], the aim of our last section is to show how to transfer continuous frames for a Hilbert space \( \mathcal{H} \) to a Krein space with a \( W \)-metric like the one defined in (4.1), but with a possibly unbounded operator \( W \) and allowing \( 0 \in \text{spec}(W) \). To begin, let \( \mathbf{W} \) denote a self-adjoint operator with domain \( \text{dom}(\mathbf{W}) \subset \mathcal{H} \), polar decomposition \( \mathbf{W} = J|\mathbf{W}| \), and integral representation \( \mathbf{W} = \int \lambda \, d\mathbf{E}(\lambda) \), where \( \mathbf{E} \) stands for the corresponding projection-valued measure on the Borel \( \Sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \). We assume that \( \ker(\mathbf{W}) = \mathbf{E}(\{0\}) = \{0\} \), which implies that \( J \) is a unitary self-adjoint operator. Analogous to (4.1), we define

\[
[f, g] := \langle f, \mathbf{W} g \rangle, \quad f, g \in \text{dom}(\mathbf{W}).
\]  

(4.2)

Then \( \text{dom}(\mathbf{W}) \) becomes a decomposable non-degenerate inner product space with fundamental decomposition \( \text{dom}(\mathbf{W}) = D_+ \oplus D_- \) and fundamental symmetry \( J \), where

\[
D_+ := \mathbf{E}(0, \infty) \text{dom}(\mathbf{W}), \quad D_- := \mathbf{E}(-\infty, 0) \text{dom}(\mathbf{W}),
\]

\[
J = \mathbf{E}(0, \infty) - \mathbf{E}(-\infty, 0).
\]  

(4.3)

Here, \( \ker(\mathbf{W}) = \{0\} \) is necessary since otherwise \( \text{dom}(\mathbf{W}) \) would be degenerate. From \( J^2 = I \), the polar decomposition \( \mathbf{W} = J|\mathbf{W}| \) and (4.2), it follows that

\[
[f, g]_J = \langle f, |\mathbf{W}| g \rangle, \quad f, g \in \text{dom}(\mathbf{W}).
\]  

(4.4)

Taking the closure under the norm defined by \( [\cdot, \cdot]_J \) and extending \( J \) to the closure (without changing the notation), we obtain a Krein space \( (\mathcal{H}_W, [\cdot, \cdot]) \) with fundamental symmetry \( J \) and fundamental decomposition \( \mathcal{H}_W = \mathcal{H}_+ \oplus \mathcal{H}_- \) such that \( D_+ \) and \( D_- \) are dense in \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), respectively. The linear operator \( \mathbf{W} \) in (4.1) and (4.2) is called Gram operator. In case \( \mathbf{W} \in \mathcal{B}(\mathcal{H}) \) and \( 0 \notin \text{spec}(\mathbf{W}) \), the Gram operator \( \mathbf{W} \) and the Krein space \( (\mathcal{H}_W, [\cdot, \cdot]) \) are said to be regular.

The next proposition summarizes some properties of \( \mathcal{H}_W \). Its proof can be found in [24].

**Proposition 4.1** Let \( \mathbf{W} : \text{dom}(\mathbf{W}) \longrightarrow \mathcal{H} \) be a self-adjoint operator on the Hilbert space \( \mathcal{H} \) such that \( \ker(\mathbf{W}) = \{0\} \). Then

(i) \( \text{dom}(\sqrt{|\mathbf{W}|}) \) is complete in the norm \( \| \cdot \|_J \) if and only if \( 0 \notin \text{spec}(\mathbf{W}) \). In this case, \( \mathcal{H}_W \) can be identified with \( \text{dom}(\sqrt{|\mathbf{W}|}) \subset \mathcal{H} \).

\[\copyright\text{ Birkhäuser}\]
(ii) $\sqrt{|W|} : \text{dom}(\sqrt{|W|}) \to \mathcal{H}$ defines an isometric operator that admits an
extension to a $J$-unitary operator $U := \sqrt{|W|} : \mathcal{H}_W \to \mathcal{H}$.

Proposition 4.1(i) shows that we may set $\mathcal{H}_W := \text{dom}(\sqrt{|W|})$ if $0 \not\in \text{spec}(W)$, and if $0 \in \text{spec}(W)$, then the completion process will always require to add abstract
elements to dom$(\sqrt{|W|})$.

**Example 5** Let $(\mathcal{M}, \mathfrak{B}, \mu)$ be measure space with a positive and $\Sigma$-finite measure $\mu$. Given a measurable real function $\varphi : \mathcal{M} \to \mathbb{R}$, set

$$\text{dom}(W_{\varphi}) := \left\{ g \in \mathfrak{L}_2(\mathcal{M}, \mu) : \int_{\mathcal{M}} |g(p)|^2 |\varphi(p)|^2 d\mu(p) < \infty \right\},$$

$$(W_{\varphi}f)(p) := \varphi(p)f(p), \quad f \in \text{dom}(W_{\varphi}),$$

and

$$[f, g] := \langle f, W_{\varphi}g \rangle_{\mathfrak{L}_2(\mathcal{M}, \mu)} = \int_{\mathcal{M}} \overline{f(p)}g(p)\varphi(p)d\mu(p), \quad f, g \in \text{dom}(W_{\varphi}). \quad (4.5)$$

If $0 < \text{ess inf}|\varphi| \leq \text{ess sup}|\varphi| < \infty$, then (4.5) defines an (indefinite) inner product on $\mathcal{H}_{W_{\varphi}} = \text{dom}(W_{\varphi}) = \mathfrak{L}_2(\mathcal{M}, \mu)$ such that $(\mathfrak{L}_2(\mathcal{M}, \mu), [\cdot, \cdot])$ becomes a Krein space with fundamental symmetry $J_{\mathfrak{L}_2}$ given by multiplication by the sign function $\text{sign}(\varphi)$ of $\varphi$, i.e., $(J_{\mathfrak{L}_2}f)(p) := \text{sign}(\varphi)(p)f(p)$. Note that this Krein space coincides with that in Example 1 for $\nu := \varphi\mu$ and $|\nu| = |\varphi|\mu$.

Now assume that $\text{ess inf}|\varphi| > 0$ and $\text{ess sup}|\varphi| = \infty$. In this case,

$$\mathcal{H}_{W_{\varphi}} = \left\{ g \in \mathfrak{L}_2(\mathcal{M}, \mu) : \int_{\mathcal{M}} |g(p)|^2 |\varphi(p)|d\mu(p) < \infty \right\} = \text{dom}(W_{\varphi}^{1/2}) \subset \mathfrak{L}_2(\mathcal{M}, \mu),$$

which enables us to view $\mathcal{H}_{W_{\varphi}}$ as a linear subspace of $\mathfrak{L}_2(\mathcal{M}, \mu)$.

If $\text{ess inf}|\varphi| = 0$, we need to require $\mu(\{\varphi = 0\}) = 0$ so that $\ker W_{\varphi} = \{0\}$. Then

$$\mathcal{H}_{W_{\varphi}} = \left\{ g : \mathcal{M} \to \mathbb{C} : \text{measurable, } \int_{\mathcal{M}} |g(p)|^2 |\varphi(p)|d\mu(p) < \infty \right\},$$

but we cannot claim that $\mathcal{H}_{W_{\varphi}} \subset \mathfrak{L}_2(\mathcal{M}, \mu)$ since $0 \in \text{spec}(W_{\varphi})$. On the contrary, whenever $\text{ess sup}|\varphi| < \infty$, we have $\mathfrak{L}_2(\mathcal{M}, \mu) \subset \mathcal{H}_{W_{\varphi}}$.

Nevertheless, in any case, the positive inner product $[\cdot, \cdot]_{J_{\mathfrak{L}_2}}$ is given by

$$[f, g]_{J_{\mathfrak{L}_2}} = \int_{\mathcal{M}} \overline{f(p)}g(p)|\varphi(p)|d\mu(p), \quad f, g \in \mathcal{H}_{W_{\varphi}} = \mathfrak{L}_2(\mathcal{M}, |\varphi|\mu).$$

In [24, Proposition 4.1], it has been shown that any discrete frame for a Hilbert space $\mathcal{H}$ cannot be a frame for a non-regular Krein space $(\mathcal{H}_W, [\cdot, \cdot])$. The proof
remains valid for continuous frames \( \{ \eta_x^n, \ldots, \eta_x^n \}_{x \in M} \) of rank \( n \) for \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) and relies on the fact that, for all \( k \in \mathcal{H} \),

\[
a\|Wk\|^2 \leq \sum_{i=1}^{n} \int_{\mathcal{M}} |\langle \eta_x^i, Wk \rangle|^2 \, d|\nu|(x) = \sum_{i=1}^{n} \int_{\mathcal{M}} ||\eta_x^i, k||^2 \, d|\nu|(x) \leq b \|Wk\|^2,
\]

where \( a \leq b \) are frame bounds for the continuous frame in \((\mathcal{H}, \langle \cdot, \cdot \rangle)\). Therefore, if \( 0 \in \text{spec}(W) \) and \( 0 \neq k \in E(-\varepsilon, \varepsilon)\mathcal{H} \) for arbitrary small \( \varepsilon > 0 \), then \( Wk \neq 0 \) and

\[
\|Wk\|^2 = \langle \sqrt{|W|k}, |W| \sqrt{|W|k} \rangle \leq \varepsilon \langle \sqrt{|W|k}, \sqrt{|W|k} \rangle = \varepsilon \|k\|^2,
\]

so that a lower frame bound does not exist. Similarly, if \( W \) is an unbounded operator, we choose \( 0 \neq k \in E(R \setminus (-r, r))\mathcal{H} \) for arbitrary large \( r > 0 \) and obtain

\[
\|Wk\|^2 = \langle \sqrt{|W|k}, |W| \sqrt{|W|k} \rangle \geq r \langle \sqrt{|W|k}, \sqrt{|W|k} \rangle = r \|k\|^2,
\]

which proves that an upper frame bound cannot exist.

These observations show that, for a non-regular Krein space \((\mathcal{H}_{\mathcal{W}}, [\cdot, \cdot])\), a continuous frame for \((\mathcal{H}_{\mathcal{W}}, [\cdot, \cdot])\) will never be a continuous frame for \((\mathcal{H}_{\mathcal{W}}, [\cdot, \cdot])\). However, using the \(J\)-unitary operator \( U \) from Proposition 4.1(ii) and applying Proposition 3.11 makes it possible to transfer continuous frames back and forth between \(\mathcal{H} \) and \(\mathcal{H}_{\mathcal{W}}\). Note that a \(J\)-unitary operator does not change the frame bounds, and that \( U^{[n]} = U^{-1} = \sqrt{|W|^{-1}} \) on \( \text{dom}(\sqrt{|W|^{-1}}) \). We state the result in the following corollary.

**Corollary 4.2** Let \( W : \text{dom}(W) \rightarrow \mathcal{H} \) be a self-adjoint operator on the Hilbert space \(\mathcal{H} \) such that \( \ker(W) = \{0\} \), and let \( U : \mathcal{H}_{\mathcal{W}} \rightarrow \mathcal{H} \) denote the \(J\)-unitary operator from Proposition 4.1(ii). Then

(i) \( \{ \eta_x^1, \ldots, \eta_x^n \}_{x \in M} \) is a continuous frame of rank \( n \in \mathbb{N} \) for the Hilbert space \(\mathcal{H} \) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) and with frame bounds \( a \leq b \) if and only if \( \{ U^{-1} \eta_x^1, \ldots, U^{-1} \eta_x^n \}_{x \in M} \) is a continuous frame of rank \( n \) for the Krein space \(\mathcal{H}_{\mathcal{W}} \) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) and with frame bounds \( a \leq b \).

(ii) If \( \{ \eta_x^1, \ldots, \eta_x^n \}_{x \in M} \subset \text{dom}(\sqrt{|W|^{-1}}) \) is a continuous frame of rank \( n \in \mathbb{N} \) for the Hilbert space \(\mathcal{H} \) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\), then the continuous frame \( \{ U^{-1} \eta_x^1, \ldots, U^{-1} \eta_x^n \}_{x \in M} \) for the Krein space \(\mathcal{H}_{\mathcal{W}} \) with respect to \((\mathcal{M}, \mathcal{B}, \nu)\) is given by \( \{ \sqrt{|W|^{-1}} \eta_x^1, \ldots, \sqrt{|W|^{-1}} \eta_x^n \}_{x \in M} \).

In practice, it might be difficult to find an explicit expression for \( \sqrt{|W|^{-1}} \). If, however, \( W \) is given by a multiplication operator \( W_\varphi \) on a function space as in Example 5, then \( \sqrt{|W|^{-1}} \) can be expressed by \( W_{|\varphi|^{-1/2}} \). We illustrate this in our final example for \( \mathcal{H} = L_2(R, \mu) \). In a certain sense, this is the generic case since any self-adjoint operator on a separable Hilbert space is unitarily equivalent to a direct sum of multiplication operators, see e.g. [42, Theorem VII.3].
**Example 6** Let $\mu$ be a positive and $\Sigma$-finite Borel measure on $\mathcal{B}(\mathbb{R})$ and consider $\mathcal{H} := \mathcal{L}_2(\mathbb{R}, \mu)$. For a measurable real function $\varphi : \mathbb{R} \to \mathbb{R}$, define as in Example 5

$$(W_\varphi f)(t) := \varphi(t)f(t), \quad f \in \text{dom}(W_\varphi) := \left\{ g \in \mathcal{L}_2(\mathbb{R}, \mu) : \int_{\mathbb{R}} |g(t)|^2 \varphi(t)^2 \mathrm{d}\mu(t) < \infty \right\}.$$  

To ensure that $\ker W_\varphi = \{0\}$, we require that $\mu(\{ \varphi = 0 \}) = 0$. Then, as it has been shown in Example 5,

$${\mathcal{H}}_{W_\varphi} = \left\{ g : \mathbb{R} \to \mathbb{C} : \text{measurable, } \int_{\mathbb{R}} |g(t)|^2 |\varphi(t)| \mathrm{d}\mu(t) < \infty \right\}.$$  

From this, it follows immediately that $\sqrt{|\varphi|} g \in \mathcal{L}_2(\mathbb{R}, \mu)$ for all $g \in {\mathcal{H}}_{W_\varphi}$. Therefore the $J$-unitary operator $U$ in Proposition 4.1(ii) and its inverse $U^{-1}$ can be given by

$${\mathcal{H}} = W_{|\varphi|^{1/2}} : \mathcal{H}_{W_\varphi} \longrightarrow \mathcal{L}_2(\mathbb{R}, \mu), \quad (W_{|\varphi|^{1/2}} f)(t) := \sqrt{|\varphi(t)|} f(t),$$

$${U}^{-1} = W_{|\varphi|^{-1/2}} : \mathcal{L}_2(\mathbb{R}, \mu) \longrightarrow \mathcal{H}_{W_\varphi}, \quad (W_{|\varphi|^{-1/2}} f)(t) := \frac{1}{\sqrt{|\varphi(t)|}} f(t).$$

Now, according to Corollary 4.2(i), any continuous frame $\{\eta_x^1, \ldots, \eta_x^n\}_{x \in \mathcal{M}}$ of rank $n \in \mathbb{N}$ for $\mathcal{L}_2(\mathbb{R}, \mu)$ with respect to $({\mathcal{M}}, \mathcal{B}, \nu)$ determines a continuous frame

$$\left\{ \frac{1}{\sqrt{|\varphi|}} \eta_x^1, \ldots, \frac{1}{\sqrt{|\varphi|}} \eta_x^n \right\}_{x \in \mathcal{M}} \subset {\mathcal{H}}_{W_\varphi}$$

of rank $n$ for the Krein space $\mathcal{H}_{W_\varphi}$ with respect to $({\mathcal{M}}, \mathcal{B}, \nu)$. Note that we did not assume that $\{\eta_x^1, \ldots, \eta_x^n\}_{x \in \mathcal{M}} \subset \text{dom}(\sqrt{|W_\varphi|^{-1}})$ as it has been done in Corollary 4.2(ii).

**Acknowledgements** We thank the referees for valuable comments that helped to improve the paper. The second named author wishes to thank the Universidad de Caldas for financial support and hospitality. This is part of the second author’s project “Elementos aproximadamente invertibles en C*-álgebras y sus aplicaciones en teoría de operadores”. The first author acknowledges partial financial support from the CIC-UMSNH and from the CONACyT project A1-S-46784.

**References**

1. Ali, S.T., Antoine, J.P., Gazeau, J.P.: Coherent states, Wavelets, and their Generalizations, Theoretical and Mathematical Physics. Springer, New York (2014)
2. Ali, S.T., Antoine, J.P., Gazeau, J.P.: Continuous frames in Hilbert space. Ann. Phys. 222, 1–37 (1993)
3. Ali, S.T., Antoine, J.P., Gazeau, J.P.: Square integrability of group representations on homogeneous spaces. I. Reproducing triples and frames. Ann. Inst. Henri Poincaré 55, 829–855 (1991)
4. Ali, S.T., Antoine, J.P., Gazeau, J.P.: Square integrability of group representations on homogeneous spaces. II. Coherent and quasi-coherent states. The case of the Poincaré group. Ann. Inst. Henri Poincaré 55, 857–890 (1991)
5. Askari-Hemmat, A., Dehghan, M.A., Radjabalipour, M.: Generalized frames and their redundancy. Proc. Am. Math. Soc. 129, 1143–1147 (2001)
6. Azhini, M., Beheshti, M.: Some results on continuous frames for Hilbert spaces. Int. J. Ind. Math. 2, 37–42 (2010)
7. Azizov, T. Ya., Iokhvidov, I.S.: Linear Operator in Spaces with an Indefinite Metric. Wiley-Interscience, Chichester (1989)
8. Bedić, S., Kong, O.C.W., Ting, H.K.: Group theoretical approach to pseudo-Hermitian quantum mechanics with Lorentz covariance and → ∞ limit. Symmetry 13, 22 (2021)
9. Bender, C.M.: Introduction to PT-symmetric quantum theory. Contemp. Phys. 129, 277–292 (2005)
10. Bognár, J.: Indefinite Inner Product Spaces. Springer, Berlin (1974)
11. Casaza, P.G., Christensen, O.: Weyl Heisenberg frames for subspaces of $L_2(\mathbb{R})$. Proc. Am. Math. Soc. 129, 145–154 (2001)
12. Casaza, P.G.: The art of frame theory. Taiwan. J. Math. 4, 129–201 (2000)
13. Casaza, P.G., Leon, M.T.: Existence and construction of finite frames with a given frame operator. Int. J. Pure Appl. Math. 63, 149–157 (2010)
14. Christensen, O.: An Introduction to Frames and Riesz Bases. Birkhäuser, Boston (2003)
15. Christensen, O., Jensen, T.K.: An Introduction to the Theory of Bases, Frames, and Wavelets. Technical University of Denmark, Kongens Lyngby (1999)
16. Christofer, E., Walnut, D.: Continuous and discrete wavelet transforms. SIAM 31, 628–666 (1989)
17. Chui, C.K.: An Introduction to Wavelets. Academic Press, San Diego (1992)
18. Daubechies, I.: The wavelet transform, time-frequency localization and signal analysis. IEEE Trans. Inform. Theory 36, 961–1005 (1990)
19. Daubechies, I., Grossmann, A., Meyer, Y.: Painless nonorthogonal expansions. J. Math. Phys. 27, 1271–1283 (1986)
20. Deguang, H., Kornelson, K., Larson, D.R., Weber, E.: Frames for Undergraduates. American Mathematical Society, Providence (2007)
21. Díaz-Bautista, E., Oliva-Leyva, M., Concha-Sánchez, Y., Raya, A.: Coherent states in magnetized anisotropic 2D Dirac materials. J. Phys. A Math. Theor. 53, 105301 (2020)
22. Duffin, R.J., Schaeffer, A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. 72, 341–366 (1952)
23. Deguang, H., Larson, D.R.: Frames, Bases and Group Representations, Memoirs of the American Mathematical Society, Providence, Rhode Island (2000)
24. Esmeral, K., Ferrer, O., Wagner, E.: Frames in Krein spaces arising from a $W$-metric. Banach J. Math. Anal. 9, 1–16 (2015)
25. Escobar, G., Esmeral, K., Ferrer, O.: Construction and coupling of frames in Hilbert spaces with $W$-metrics. Revista Integración 34, 81–93 (2016)
26. Fornasier, M., Rauhut, H.: Continuous frames, function spaces, and the discretization. J. Fourier Anal. Appl. 11, 245–287 (2005)
27. Führ, H.: Abstract Harmonic Analysis of Continuous Wavelets Transform. Springer, Heidelberg (2005)
28. Gabardo, J.P., Han, D.: Frames associated with measurable spaces. Adv. Comput. Math. 18, 127–147 (2003)
29. Găvruţa, P.: On the duality of fusion frames. J. Math. Anal. Appl. 333, 871–879 (2007)
30. Giribet, J.I., Maestripiieri, A., Pería, F.M., Massey, P.: On frames for Krein spaces. J. Math. Anal. Appl. 393, 122–137 (2012)
31. Giribet, J.I., Maestripiieri, A., Pería, F.M.: Duality for frames in Krein spaces. Math. Nachr. 291, 879–896 (2018)
32. Kaiser, G.: A Friendly Guide to Wavelets. Birkhäuser, Boston (1994)
33. Kamuda, A., Kuzel, S.: On J-frames related to maximal definite subspaces. arXiv:1712.08050
34. Karmakar, S.: J-fusion frame and its application in Krein spaces. Poincare J. Anal. Appl. 2, 1–11 (2018)
35. Karmakar, S., Hossein, S.M., Paul, K.: Properties of J-fusion frames in Krein spaces. Adv. Operator Theory 2, 215–227 (2017)
36. Karmakar, S., Hossein, S.M.: Frames on Krein spaces. arXiv:1406.6205
37. Mostafazadeh, A.: Pseudo-Hermitian representation of quantum mechanics. Int. J. Geom. Methods Mod. Phys. 7, 1191–1306 (2010)
38. Oeckl, R.: Coherent states in fermionic Fock–Krein spaces and their amplitudes. In: Antoine, J.P., Bagarello, F., Gazeau, J.P. (eds.) Coherent States and Their Applications. Springer Proceedings in Physics, vol. 205. Springer, Cham (2018)
39. Peng, I., Waldron, S.: Signed frames and Hadamard products of Gram matrices. Linear Algebra Appl. 347, 131–157 (2002)
40. Perelomov, A.: Generalized Coherent States and Their Applications. Springer, Berlin (1986)
41. Rahimi, A., Najati, A., Dehghan, Y.N.: Continuous frames in Hilbert spaces. Methods Funct. Anal. Topol. 12, 170–182 (2006)
42. Reed, M., Barry, S.: Methods of Modern Mathematical Physics. Vol. I: Functional Analysis. Academic Press, San Diego (1980)
43. Schulz-Baldes, H., Villegas-Blas, C.: Krein signatures of transfer operators for half-space topological insulators. J. Phys. A Math. Theor. 49, 405201 (2016)
44. Sierra-Acosta, A.Y.: Estados coherentes de SU(1,1) en espacios de Krein. Master’s Thesis, Universidad Michoacana de San Nicolás de Hidalgo, Morelia (2021)
45. Thirulogasanthar, K., Bahsoun, W.: Continuous frames on Julia sets. J. Geom. Phys. 51, 183–194 (2004)