Multiparty Operational Quantum Mutual Information

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Mutual information is the reciprocal information that is common to or shared by two or more parties. Quantum mutual information (QMI) for bipartite quantum systems is non-negative, and bears the interpretation of total correlation between the two subsystems. This is, however, no longer true for three or more party quantum systems. It is possible that three-party QMI can be negative. Using the chain rule $H(X,Y) = H(X) + H(Y|X)$, the following expressions of CMI are equivalent:

$$K_1(A : B : C) = [H(A) + H(B) + H(C)] - [H(A,B) + H(A,C) + H(B,C)] + H(A,B,C).$$

$$K_2(A : B : C) = H(A,B) - H(B|A) - H(A|B) - H(A|C) - H(B|C) + H(A,B,C).$$

$$K_3(A : B : C) = [H(A) + H(B) + H(C)] - [H(A,B) + H(A,C) + H(A|B,C)].$$

The above definitions of CMI can be extended to the quantum domain. They are obtained by replacing the random variables by density matrices and Shannon entropies by von Neumann entropies, with appropriate measurements in the quantum conditional entropies. Hence,

$$I(A : B : C) = [S(A) + S(B) + S(C)] - [S(A,B) + S(A,C) + S(B,C)] + S(A,B,C),$$

$$J_1(A : B : C) = S(A,B) - S_M(B|A) - S_M(A|B) - S_M(A) - S_M(B) + S_M(A,B,C).$$

$$J_2(A : B : C) = [S(A) + S(B) + S(C)] - [S(A,B) + S(A,C) + S_M(A|B,C)],$$

where $S(X) \equiv S(\rho_X)$, and $S_M(X|Y) = S(\rho_{XM|Y})$ is the quantum conditional entropy obtained after some generalized measurement $\mathcal{M}$ has been performed on subsystem $Y$. It is asserted that the above quantum expressions are not equivalent as measurement assumes its role in the quantum conditional entropies. QMIs have certain drawbacks. First, surprisingly enough $I(A : B : C)$ is be negative. Using the chain rule $H(X,Y) = H(X) + H(Y|X)$, the following expressions of CMI are equivalent:

$$K_1(A : B : C) = [H(A) + H(B) + H(C)]$$

$$K_2(A : B : C) = H(A,B) - H(B|A) - H(A|B)$$

$$K_3(A : B : C) = [H(A) + H(B) + H(C)]$$

**I. INTRODUCTION**

Quantum correlations [1, 2] are essential ingredients in quantum information theory [3]. Various quantum correlations, different in nature and types, find huge applications in quantum information theory. Consequently, their characterization and quantification is a crucial task. Several non-classical correlation measures have been proposed for bipartite quantum systems, and some of them have been extended to multipartite settings. Mutual information, the reciprocal information that is common to or shared by two or more parties, has an authoritative stand in the arena. Quantum mutual information (QMI), whose definition is motivated by that of classical mutual information (CMI), is well defined for bipartite quantum systems. QMI of a bipartite quantum state $\rho_{AB}$ is defined as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$$= S(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0,$$  (1)

where $S(\rho) = -\text{tr}(\rho \log \rho)$ is von Neumann entropy and $S(\rho \| \sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$ is quantum relative entropy. It is non-negative, and bears the interpretation of total correlation between the two subsystems [4]. It is defined as the amount of work (noise) that is required to erase (destroy) the correlations completely. This is, however, no longer true for three or more party quantum systems. First, we consider three-variable CMI [5, 6] which is defined as

$$K(A : B : C) = K(A : B) - K(A,B,C),$$

where $K(A : B) = H(A) - H(A|B) = H(A) + H(B) - H(A,B)$ is two-variable CMI, $K(A,B,C) = H(A|C) + H(B|C) - H(A,B|C)$ is three-variable conditional mutual information, and $H(.)$ is Shannon entropy. Though both $K(A : B)$ and $K(A,B,C)$ are non-negative, the three-variable CMI can

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identically zero for arbitrary three-party pure quantum states [6] implying that mutual information among the subsystems of three-party pure quantum systems is zero. This is not true in the case of bipartite QMI. Second, $I(A : B : C)$ and other versions of QMI can be negative [6, 7]. How is this negative correlation useful for quantum information tasks? Though the existing definition of three-party QMI is argued to reveal the true nature of quantum correlations [6], the fact that QMI, being a measure of correlation, can assume negative value is confronting and unacceptable. This perplexing stance motivated us to propose an operational definition of multiparty QMI which is non-negative.

This paper is divided into four sections. In Sec. II, we provide operational definition of multiparty quantum mutual information, and compare with other mutual information. In Sec. III, we discuss multiparty quantum discord and present a few illustrations. We show that the symmetric version of quantum discord reveals the maximal quantumness. We finally conclude in Sec. IV.

II. OPERATIONAL QUANTUM MUTUAL INFORMATION

In this section, we propose an operational definition of multiparty quantum mutual information, henceforth operational quantum mutual information (OQMI). For an $n$-party quantum state $\rho_{A_1A_2\cdots A_n}$, our definition takes into account the shared information among $m$-parties, $2 \leq m \leq n$, and not only the common information among all parties. This is absolutely reasonable as information can be distributed or stored among $m$-parties. This can be understood readily using Venn diagram.

A. Two-party OQMI

From Fig. 1(a), we see that the only way the subsystems $A$ and $B$ interact with each other is region “$ab$”, i.e, $ab = A \cap B = A + B - A \cup B$. In entropy language, this translates as

$$I(A : B) = S(A) + S(B) - S(AB). \quad (9)$$

This is usual bipartite QMI.

B. Three-party OQMI

From Fig. 1(b), the possible ways the subsystems $A$, $B$ and $C$ interact among each other is region “$abc$” which is common to all three, and regions “$ab$, $ac$, $bc$” which are pairwise common. Taking just the region “$abc$”, i.e, $abc = A \cap B \cap C = A + B + C - (A \cup B + A \cup C + B \cup$

$$C)$ + $A \cup B \cup C$, this “common information” translates into

$$I_c(A : B : C) := [abc] = [S(A) + S(B) + S(C)]$$

$$- [S(AB) + S(AC) + S(BC)] + S(ABC). \quad (10)$$

Note that in doing so we have discarded pairwise interactions. However, a priori, there is no reason to throw them away. Moreover, they can provide important information when examined together. Then “two-party shared information” reads as

$$I_{s2}(A : B : C) := [ab + ac + bc]. \quad (11)$$

Thus, the total three-party QMI is sum of the common information and the pairwise shared information:

$$I(A : B : C) = I_c(A : B : C) + I_{s2}(A : B : C)$$

$$:= [A \cup B \cup C - (a + b + c)]. \quad (12)$$

Eq. (12) guarantees that $I(A : B : C)$ is not identically zero for arbitrary three-party pure quantum systems. After simple algebra, $I(A : B : C)$ can be expressed in entropy language, thereby in bipartite QMI, as

$$I(A : B : C) = S(AB) + S(AC) + S(BC) - 2S(ABC)$$

$$= \frac{1}{3} \left[ 2 \sum I(X_1 : X_2X_3) - \sum I(X_1 : X_2) \right]$$

$$\geq 0, \quad (13)$$

where $X_i \in \{A, B, C\}$. The last inequality follows since $I(X_1 : X_2X_3) \geq I(X_1 : X_2)$. Thus, by construction, three-party OQMI is non-negative. Similarly, one can define OQMI for four- and higher party quantum systems. Below we define four-party OQMI.

C. Four-party OQMI

Fig. 2(a) does not represent the true Venn diagram of four variables as pairwise interacting regions “$ab$” and “$bc$” are missing. The correct four-variable Venn diagram
is represented in Fig. 2(b). The total four-party QMI is then defined as
\[ I(A : B : C : D) := [A \cup B \cup C \cup D - (a + b + c + d)]. \tag{14} \]

Again, after some simple algebra, \( I(A : B : C : D) \) can be expressed as [8]
\[ I(A : B : C : D) = \sum_{X_1, X_2, X_3} S(X_1 X_2 X_3) - 3S(ABCD), \tag{15} \]
where \( X_i \in \{A, B, C, D\} \). We list in Table I the values of common information \( (I_c) \) and QOMI \( (I) \) of some typical states.

| State     | \( I_c \) | \( I \) |
|-----------|-----------|--------|
| GHZ_2     | 2         | 2      |
| GHZ_3     | 2         | 3      |
| D\(|\psi_{ua}\rangle\) | 0.275489 | 4.75489 |
| D\(|\psi_{ua}\rangle\) | 2.490225 | 3.24511 |
| C\(|\psi_{ua}\rangle\) | 2         | 4      |

TABLE I. Values of common information \( (I_c) \) and QOMI \( (I) \) of \( \text{GHZ}_n \), \( \text{GHZ}_3 \), D\(|\psi_{ua}\rangle\), D\(|\psi_{ua}\rangle\), C\(|\psi_{ua}\rangle\). For pure states, \( I = I_c \), as defined in Eq. (20).

An \( n \)-party QMI can be analogously defined as
\[ I(A_1 : A_2 : \cdots : A_n) := [A_1 \cup A_2 \cup \cdots \cup A_n - (a_1 + a_2 + \cdots + a_n)] \]
\[ = \sum S(X_{k_1} X_{k_2} \cdots X_{k_{n-1}}) - (n - 1)S(A_1 A_2 \cdots A_n). \tag{16} \]

**Theorem 1**: QOMI is non-negative.

**Proof.** Let us denote \( S_X \equiv S(\rho_X) \). First, we will prove the theorem for three- and four parties, and then generalize our argument to arbitrary \( n \)-party case. We will extensively use a variant of strong subadditivity relation, \( S_{XY} + S_Y \leq S_{XY} + S_{YZ} \), which states that conditioning reduces entropy, i.e., \( S_{X|Y} \leq S_{X|Y} \).

**Three-party case.**
\[ I(A_1 : A_2 : A_3) = S_{12} + S_{13} + S_{23} - 2S_{123} \]
\[ = S_{12} - S_{1|23} - S_{2|13} \]
\[ \geq S_{12} - S_{1|2} - S_{2|1} \]
\[ = S_1 + S_2 - S_{12} \geq 0. \tag{17} \]

**Four-party case.**
\[ I(A_1 : A_2 : A_3 : A_4) = S_{123} + S_{124} + S_{134} + S_{234} - 3S_{1234} \]
\[ = S_{12} - S_{1|23} - S_{2|13} - S_{3|12} \]
\[ \geq S_{12} - S_{1|2} - S_{2|1} \]
\[ = S_1 + S_2 - S_{12} \geq 0. \tag{18} \]

**n-party case.**
\[ I(A_1 : A_2 : \cdots : A_n) = S_{12\cdots(n-1)} + S_{12\cdots(n-2)} + \cdots + S_{23\cdots n} - (n - 1)S_{12\cdots n} \]
\[ \geq S_{12\cdots(n-1)} - S_{1|23\cdots(n-1)} \]
\[ \geq S_{12\cdots(n-1)} - S_{1|23\cdots(n-1)} - S_{2|13\cdots(n-1)} \]
\[ \cdots - S_{n-1|12\cdots(n-2)} \]
\[ \geq S_{12\cdots(n-1)} - S_{1|23\cdots(n-1)} - S_{2|13\cdots(n-1)} \]
\[ \cdots - S_{n-1|12\cdots(n-2)} \geq 0. \tag{19} \]

Hence, the theorem is proved.

The operational QMI then, by contrast, bears the interpretation of total correlation of a multiparty quantum system. Further, we can obtain generalized QOMI by replacing von Neumann entropy, \( S(\rho) = -\text{tr}(\rho \log \rho) \), with generalized entropies like Renyi entropy, \( S_q^R(\rho) = \frac{1}{1-q} \log \text{tr}(\rho^q) \) [13], Tsallis entropy, \( S_q^T(\rho) = \frac{1}{1-q} \text{tr}(\rho^q) - 1 \) [14], and (smooth) min-max entropies [15]. Both Renyi and Tsallis entropies reduce to von Neumann entropy in the limit \( q \to 1 \).
D. Comparison with other QMI

Often, in literature, multipartite mutual information is defined as

\[
I_x(A_1 : A_2 : \cdots : A_n) = \sum_{k=1}^{n} S(A_k) - S(A_1A_2 \cdots A_n)
\]

\[
= S(\rho_{A_1A_2\cdots A_n}) \leq 0.
\]

This is straightforward generalization of bipartite mutual information. Though it equals OQMI for pure states, it does not have any operational interpretation as such. In following theorem, we obtain lower and upper bounds on OQMI in terms of the conventional QMI

\[
I_x \equiv I_x(A_1 : A_2 : \cdots : A_n).
\]

\textbf{Theorem 2:} \( I_x - (n - 2)S_{12\cdots n} \leq I(A_1 : A_2 : \cdots : A_n) \leq I_x + 2S_{12\cdots n}. \)

\textbf{Proof.} Using strong subadditivity entropic relation, \( S_X + S_Y \leq S_{XZ} + S_{YZ}, \) and Araki-Lieb inequality, \( S_X - S_Y \leq S_{XY} \Rightarrow S_X - S_{XY} \leq S_Y, \) we can, respectively, obtain

\[
\sum_{i=1}^{n} S_{A_i} \leq \sum_{i=1}^{n} S_{A_kA_{k+1} \cdots A_{k+n-1}}
\]

and

\[
\sum_{i=1}^{n} S_{A_i} - nS_{A_1A_2 \cdots A_n} \leq \sum_{i=1}^{n} S_{A_i}.
\]

Therefore,

\[
I(A_1 : A_2 : \cdots : A_n) = \sum_{i=1}^{n} S_{A_kA_{k+1} \cdots A_{k+n-1}}
\]

\[
- (n - 1)S_{A_1A_2 \cdots A_n}
\]

\[
\geq \sum_{i=1}^{n} S_{A_i} - (n - 1)S_{A_1A_2 \cdots A_n}
\]

\[
= I_x(A_1 : A_2 : \cdots : A_n)
\]

\[
- (n - 2)S_{A_1A_2 \cdots A_n}
\]

\[
\leq \sum_{i=1}^{n} S_{A_i} + S_{A_1A_2 \cdots A_n}
\]

\[
= I_x(A_1 : A_2 : \cdots : A_n)
\]

\[
+ 2S_{A_1A_2 \cdots A_n}.
\]

Hence, the proof.

![Graphs](image.png)

\textbf{FIG. 3.} (Color online) Plots of logarithmic-negativity \( E_N(A : BC) \) [16, 17], common information \( I_c(A : B : C); I_s(A : B : C), \) and operational QMI \( I(A : B : C) \) against the white noise parameter \( p \), of three-party (a) GHZ state \( |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \) (b) W state \( |W\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle + |100\rangle), \) and (c) totally antisymmetric state \( |\psi_{\text{as}}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle). \) In all the three cases, while \( I_c(A : B : C) \) vanishes at \( p = 0, 1 \) and is negative at intermediate values, OQMI \( I(A : B : C) \) vanishes at \( p = 0 \) only and is positive for other values. OQMI is greater than or equal to \( I_x(A : B : C) \).

\section{III. MULTIPARTY QUANTUM DISCORD}

In this section, we extend the definition of bipartite quantum discord [18, 19] to multipartite setting. Quan-
as
\[ D(\rho_{AB}) = I(\rho_{AB}) - \max_M J(\rho_{AB}), \]
where
\[ I(\rho_{AB}) = I(A : B) = S(A) + S(B) - S(AB), \]
and
\[ J(\rho_{AB}) = S(B) - S_M(B|A). \]

Here measurement is performed on subsystem \( A \) with a rank-one projection-valued measurement \( \{A_i\} \), producing the states \( \rho_{Bi} = \frac{1}{p_i} \text{tr}_A((A_i \otimes I_B)\rho(A_i \otimes I_B)) \), with probability \( p_i = \text{tr}_A((A_i \otimes I_B)\rho(A_i \otimes I_B)) \). \( I \) is the identity operator on the Hilbert space of \( B \). Hence, the conditional entropy of \( \rho_{AB} \) is given by
\[ S_M(B|A) = \sum_i p_i S(\rho_{Bi}). \]

Three-party generalized quantum discord can then be defined, when measurement is performed on subsystem \( A \), subsystem \( AB \) and the whole system, as follows
\[ D_A(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_A} I(\Phi_A(\rho_{ABC})) \]
\[ D_{AB}(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_{AB}} I(\Phi_{AB}(\rho_{ABC})) \]
and
\[ D_{ABC}(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_{ABC}} I(\Phi_{ABC}(\rho_{ABC})) \]

where
\[ I(\sigma_{XYZ}) = I(X : Y : Z) \]
\[ \Phi_A(\rho_{ABC}) = \sum_i \Phi_{A_i}\rho_{ABC}\Phi_{A_i} \]
\[ \Phi_{AB}(\rho_{ABC}) = \sum_{i,j} \Phi_{A_iB_j}\rho_{ABC}\Phi_{A_iB_j} \]
\[ \Phi_{ABC}(\rho_{ABC}) = \sum_{i,j,k} \Phi_{A_iB_jC_k}\rho_{ABC}\Phi_{A_iB_jC_k} \]

with \( \Phi_{A_i} = \pi_i \otimes I \otimes I, \Phi_{A_iB_j} = \pi_i \otimes \pi_j \otimes I, \) and \( \Phi_{A_iB_jC_k} = \pi_i \otimes \pi_j \otimes \pi_k \). Eq. (31) is the symmetric version of quantum discord or global quantum discord (GQD) [20].

The operational quantum discord, employing von Neumann entropy, of three-party GHZ state and W state mixed with white noise is shown in Fig. 4. Quite unexpectedly, we find that \( D_A \leq D_{AB} \leq D_{ABC} \), that is, quantumness increases when measurement is performed on larger number of subsystems. This contradicts the interpretation of measured mutual information as classical information because measuring more than one subsystem should yield more classical information and hence less quantum discord.

A natural question then arises: whether the above observation is a property of the operational quantum discord, i.e., quantum discord obtained using OQMI? Fig. 5 displays conventional quantum discord (that is, quantum discord obtained employing conventional QMI \( I_x \)) of three-party GHZ state and W state mixed with white noise. Here also quantum discord increases when measurement is performed on larger number of subsystems. Thus above observation seems to be independent of the definition of QMI. Since, the symmetric version of quantum discord requires measurement on all the parties, it reveals the maximal quantumness.

IV. CONCLUSION

To sum up, we have introduced the notion of operational quantum mutual information (OQMI) and generalized it to multipartite setting. It is non-negative, and has the interpretation of total correlation. We then employed OQMI to define multiparty quantum discord. Surprisingly, we found that more quantumness can be
harnessed by performing measurements on a larger number of parties which is quite counter-intuitive. The symmetric version of quantum discord reveals the maximal quantumness. This suggests that measured mutual information should not be interpreted as classical correlation. We believe that our work will provide further insights in understanding the nature of non-classical correlations.

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