Testing Reactive Probabilistic Processes

Sonja Georgievska and Suzana Andova
Department of Mathematics and Computer Science
Eindhoven University of Technology
The Netherlands

s.georgievska@tue.nl, s.andova@tue.nl

We define a testing equivalence in the spirit of De Nicola and Hennessy for reactive probabilistic processes, i.e. for processes where the internal nondeterminism is due to random behaviour. We characterize the testing equivalence in terms of ready-traces. From the characterization it follows that the equivalence is insensitive to the exact moment in time in which an internal probabilistic choice occurs, which is inherent from the original testing equivalence of De Nicola and Hennessy. We also show decidability of the testing equivalence for finite systems for which the complete model may not be known.

1 Introduction

A central paradigm behind process semantics based on observability (e.g. [18]) is that the exact moment an internal nondeterministic choice is resolved is unobservable. This is because an observer does not have insight into the internal structure of a process but only in its externally visible actions. Unobservability of internal choice has been also achieved by the testing theory [7, 17], where two processes are treated equivalent iff they can not be distinguished when interacting with their environment (which is an arbitrary process itself). It is natural, therefore, for this property to hold when internal choice is quantified with probabilities. It turned out, however, that it was not trivial to achieve unobservability of internal probabilistic choice in probabilistic testing theory. The following example illustrates some points that cause this problem.

Consider a system consisting of a machine and a user, that communicate via a menu of two buttons “head” and “tail” positioned at the machine. The machine makes a fair choice whether to give a prize if “head” is chosen or if “tail” is chosen. The user can choose “head” or “tail” by pressing the appropriate button. If the user chooses the right outcome, a prize follows. Note that by no means the machine’s choice could have been revealed beforehand to the user. The machine can be modeled by the process graph $s$ in Fig. 1. That is, in half of the machine runs, it offers a prize after the “head” button has been pressed (out of the two-button menu “head” and “tail”), while in the other half of the runs it offer a prize after the “tail” button has been pressed (out of the two-button menu “head” and “tail”). The user can be modeled by process $u$ in Fig. 1. Sometimes she would press “head” and sometimes “tail”; however, her goal is to win a prize, denoted by action $\triangleright$, and be “happy” afterwards, denoted by action $\triangleright\smiley$.

Let the user and the machine interact, i.e. let them synchronize on all actions, except on the “user happiness” reporting action $\triangleright\smiley$. In terms of testing theory [7], process $s$ is tested with test $u$. It can be computed, by means of the probability theory, that the probability with which the user has guessed the machine choice is $\frac{1}{2}$. That is, the probability of a $\triangleright\smiley$ action being reported is $\frac{1}{2}$. However, most of the existing approaches for probabilistic testing, in particular probabilistic may/must testing [8, 19, 28, 30],

As shown in [26] the process semantics based on [18] and [7] coincide for a broad class of processes.
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Consider the synchronization $s \parallel u$ represented by the graph in Fig. 1 where actions are hidden after they have synchronized. A scheduler resolves the choices of actions in the two states reachable in one probabilistic step from the initial state of the graph $s \parallel u$, thus yielding a fully probabilistic system. For $s \parallel u$ in Fig. 1 there are four possible schedulers. They yield the following set of probabilities with which $s$ passes the test $u$: $\{0, \frac{1}{3}, 1\}$. We can see that, because the power of the schedulers is unrestricted, nonviable upper and lower bounds for the probability are obtained. Observe that this happens due to the effect of “cloning” the action choice of $h$ and $t$ (the choice between $h$ and $t$ has been “cloned” in both futures after the probabilistic choice in $s \parallel u$), and allowing a scheduler to schedule differently in the two “clones”. This, in fact, corresponds to a model where the user is given power to see the result of the probabilistic choice made by the machine before she makes her guess. However, this is not the model we had initially in mind when the separate components, the machine and the user, were specified.

Consider now process $\bar{s}$ in Fig. 1. To the user this process may as well represent the behaviour of the machine – the user cannot see whether the machine makes the choice before or after making the “head or tail” offers. According to the user, the machine acts as specified as long as she is able to guess the result in half of the cases. In fact, both schedulers, obtained by methods in [8, 19, 28, 30, 32], when applied to $\bar{s} \parallel u$ yield exactly probability $\frac{1}{3}$ of reporting action $\otimes$. Consequently, none of the approaches in [8, 19, 28, 30, 32] equate processes $s$ and $\bar{s}$: when tested with test $u$, they produce different bounds for the probabilities of reporting $\otimes$. On the other hand, if the probabilities are ignored and the probabilistic choice is treated as an internal choice, processes $s$ and $\bar{s}$ are equivalent by the testing equivalence of [7].

Being able to equate $s$ and $\bar{s}$ means allowing distribution of external choice over internal probabilistic choice [18]. Actually, distribution of external choice over internal choice is closely related to distribution of action prefix over internal choice. If distribution of external choice over internal probabilistic choice is not allowed, then distribution of action prefix over internal probabilistic choice is questioned too, otherwise the congruence properties of asynchronous or concurrent parallel composition [18] (where processes synchronize on their common actions while interleave on the other actions) would not hold. For instance, we would not be able to equate processes $e.a.(b \oplus \frac{1}{2}c)$ and $e.((a.b) \oplus \frac{1}{2}(a.c))$. (The operator “.” stands for prefixing and the operator “$\oplus$” stands for a probabilistic choice.) Running each of these two processes concurrently with process $e.d$, yield processes that, unless distribution of external choice over internal probabilistic choice is allowed, cannot be equated. If we are not able to relate processes that differ only in the moment internal probabilistic choice is resolved, before or after an action execution...
Motivated by the previous observations, in [12] we propose a testing preorder which can deal with this problem. According to this testing semantics, the probability with which process $s$ passes test $u$ (Fig. 1) is exactly $\frac{1}{2}$. The model considered in [12] is rather general and allows probabilistic as well as internal non-deterministic choice, in addition to action choice. Moreover, the testing preorder is given a characterization in terms of a probabilistic ready-trace preorder (a ready-trace is an alternating sequence of “action menus” and executed actions). From this characterization it follows that the underlying equivalence equates processes $s$ and $\bar{s}$.

Since the tests in the model of [12] have internal transitions, in general, infinitely many tests need to be considered to determine equivalence between two finite processes. Therefore, the decidability of the testing equivalence for the general model at the moment relies on the characterization of the equivalence in terms of ready-traces. However, in practice, if we aim at testing whether the system is equivalent to the model, we may not have access to the ready-traces and the internal transitions of the system that are necessary to establish the equivalence. It is, therefore, of practical interest to investigate for which type of systems there exists a procedure to decide testing equivalence based only on testing itself (see also [10, 23, 33] for similar discussions).

In this paper we investigate decidability for systems of the testing equivalence of [12] for reactive probabilistic systems [21], where all internal nondeterminism is due to random behaviour. We first point out that, under the condition that a test “knows” the current set of actions on which it can synchronize with the system (i.e. the menu of actions-candidates for synchronization), there exists a statistical procedure to estimate the result of testing a system with a given test. We then show that the set of tests necessary to determine equivalence of two finite systems is finite, from which the decidability result follows directly.

More concretely, we prove that deterministic (i.e. non-probabilistic) tests suffice for distinguishing between finite processes. This result follows from the proof that the testing equivalence coincides with the probabilistic ready-trace equivalence. In this paper we also present the characterization proof, which is technically much more involved than the corresponding proof in [12], due to tests having “less power” than in [12]. From this characterization it also follows that the testing equivalence, when applied to the model of reactive probabilistic processes, preserves the previously mentioned desirable properties: it is insensitive to the exact moment of occurrence of an internal probabilistic choice and it refines the equivalence for the non-probabilistic case proposed in [7].

Structure of the paper In Sec. 2 we define some notions needed for the rest of the paper. In Sec. 3 we recall the definition of probabilistic ready trace equivalence from [12]. In Sec. 4 we define a testing equivalence for the reactive probabilistic processes. In Sec. 5 we prove that the equivalences defined in sections 3 and 4 coincide. In Sec. 6 we show the decidability results for the testing equivalence. Sec. 7 ends with discussion of related work, other than [12], and concluding remarks.

2 Preliminaries

We define some preliminary notions needed for the rest of the paper.
2.1 Bayesian probability

For a set $A$, $2^A$ denotes its power-set. The following definitions are taken from [22].

We consider a sample space, $\Omega$, consisting of points called elementary events. Selection of a particular $a \in \Omega$ is referred to as an "$a$ has occurred". An event is a set of elementary events. $A,B,C$ range over events. An event $A$ has occurred iff, for some $a \in A$, $a$ has occurred. Let $A_1, A_2, \ldots$ be a sequence of events and $C$ be an event. The members of the sequence are exclusive given $C$, if whenever $C$ has occurred no two of them can occur together, that is, if $A_i \cap A_j \cap C = \emptyset$ whenever $i \neq j$. $C$ is called a conditioning event. If the conditioning event is $\Omega$, then "given $\Omega$" is omitted.

For certain pairs of events $A$ and $B$, a real number $P(A|B)$ is defined and called the probability of $A$ given $B$. These numbers satisfy the following axioms:

A1: $0 \leq P(A|B) \leq 1$ and $P(A|A) = 1$.

A2: If the events in $\{A_i\}_{i=1}^{\infty}$ are exclusive given $B$, then $P(\bigcup_{i=1}^{\infty} A_i | B) = \sum_{i=1}^{\infty} P(A_i | B)$.

A3: $P(C|A \cap B) \cdot P(A|B) = P(A \cap C|B)$.

For $P(A|\Omega)$ we simply write $P(A)$.

2.2 Probabilistic transition systems

In a probabilistic transition system (PTS) there are two types of transitions, viz. action and probabilistic transitions; a state can either perform action transitions only (action state) or (unobservable) probabilistic transitions only (probabilistic state). To simplify, we assume that probabilistic transitions lead to action states. In action states the choice is between a set of actions, but once the action is chosen, the next state is determined. The outgoing transitions of a probabilistic state $s$ define probability over the power-set of the set of action states.

We give a formal definition of a PTS. Presuppose a finite set of actions $\mathcal{A}$.

Definition 2.1 (Probabilistic Transition System (PTS)) A PTS is a tuple $\mathcal{P} = (S_n, S_p, \rightarrow, \rightarrow\rightarrow)$, where

- $S_n$ and $S_p$ are finite disjoint sets of action and probabilistic states, resp.,
- $\rightarrow \subseteq S_n \times \mathcal{A} \times S_n \cup S_p$ is an action transition relation such that $(s,a,t) \in \rightarrow$ and $(s,a,t') \in \rightarrow$ implies $t = t'$, and
- $\rightarrow\rightarrow \subseteq S_p \times (0,1] \times S_n$ is a probabilistic transition relation such that, for all $s \in S_p$, $\sum_{(s,\pi,t)\in \rightarrow\rightarrow} \pi = 1$.

We denote $S_n \cup S_p$ by $S$. We write $s \overset{a}{\rightarrow} t$ rather than $(s,a,t) \in \rightarrow$, and $s \overset{\pi}{\rightarrow\rightarrow} t$ rather than $(s,\pi,t) \in \rightarrow\rightarrow$ (or $s \rightarrow\rightarrow t$ if the value of $\pi$ is irrelevant in the context). We write $s \overset{a}{\rightarrow}$ to denote that there exists an action transition $s \overset{a}{\rightarrow} s'$ for some $s' \in S$. We agree that a state without outgoing transitions belongs to $S_n$.

Given a process $s$ and action $a \in \mathcal{A}$, denote by $s_a$ the process, if it exists, for which $s \overset{a}{\rightarrow} s_a$. Given a PTS $\mathcal{P} = (S_n, S_p, \rightarrow, \rightarrow\rightarrow)$, let $I: S_n \rightarrow 2^{\mathcal{A}}$ be a function such that, for all $a \in \mathcal{A}, s \in S_n$, it holds $a \in I(s)$ iff $s \overset{a}{\rightarrow}$. $I(s)$ is called the menu of $s$. Intuitively, for $s \in S_n$, $I(s)$ is the set of actions that the process $s$ can perform initially.

As standard, we define a process graph (or simply process) to be a state $s \in S$ together with all states reachable from $s$, and the transitions between them. A process graph is usually named by its root state, in this case $s$. 
3 Probabilistic ready trace semantics

In this section we recall the ready-trace equivalence for reactive probabilistic processes defined in [12].

**Definition 3.1 (Ready trace)** A ready trace of length n is a sequence \((M_1, a_1, M_2, a_2, \ldots, M_{n-1}, a_{n-1}, M_n)\) where \(M_i \in 2^S\) for all \(i \in \{1, 2, \ldots, n\}\) and \(a_i \in M_i\) for all \(i \in \{1, 2, \ldots, n-1\}\).

We assume that the observer has the ability to observe the actions that the process performs, together with the menus out of which actions are chosen. Intuitively, a ready trace \(\mathcal{O} = (M_1, a_1, M_2, a_2, \ldots, M_{n-1}, a_{n-1}, M_n)\) can be observed if the initial menu is \(M_1\), then action \(a_1 \in M_1\) is performed, then the next menu is \(M_2\), then action \(a_2 \in M_2\) is performed and so on, until the observing ends at a point when the menu is \(M_n\). It is essential that, since the probabilistic transitions are not observable, the observer cannot infer where exactly they happen in the ready trace.

Clearly the probability of observing a ready trace \(\{(a, b), a, \{c\}\}\) is conditioned on choosing the action \(a\) from the menu \(\{a, b\}\). This suggests that, when defining probabilities on ready traces, the Bayesian definition of probability is more appropriate than the measure-theoretic definition that is usually taken.

Next, given a process \(s\), we define a process \(s_{(M,a)}\). Intuitively, \(s_{(M,a)}\) is the process that \(s\) becomes, assuming that menu \(M\) was offered to \(s\) and action \(a\) was performed.

**Definition 3.2** Let \(s\) be a process graph. Let \(M \subseteq \omega', a \in M\) be such that \(I(s) = M\) if \(s \in S_n\) or otherwise there exists a transition \(s \rightarrow s'\) such that \(I(s') = M\). The process graph \(s_{(M,a)}\) is obtained from \(s\) in the following way:

- if \(s \in S_n\) then the root of \(s_{(M,a)}\) is the state \(s'\) such that \(s \xrightarrow{a} s'\), and
- if \(s \in S_p\) then a new state \(s_{(M,a)}\) is created. Let \(\pi = \sum_{s \rightarrow s_i, I(s_i) = M} \pi_i\). For all \(s'_i\) such that \(s \xrightarrow{\pi} s_i \xrightarrow{a} s'_i\) and \(I(s_i) = M\):
  - if \(s'_i \rightarrow\rightarrow\) then an edge \(s_{(M,a)} \xrightarrow{\pi/\pi} s'_i\) is created;
  - for all transitions \(s'_i \xrightarrow{\pi_i} s''_i\), an edge \(s_{(M,a)} \xrightarrow{\pi_i \pi/\pi} s''_i\) is created.

**Definition 3.3** Let \((M_1, a_1, M_2, a_2, \ldots, M_{n-1}, a_{n-1}, M_n)\) be a ready trace of length \(n\) and \(s\) be a process graph. Functions \(P^1_s(M)\) and \(P^n_s(M_n|M_1, a_1, \ldots, M_{n-1}, a_{n-1})\) (for \(n > 1\)) are defined in the following way:

\[
P^1_s(M) = \begin{cases} \sum_{s \rightarrow s'} \pi \cdot P^1_s(M) & \text{if } s \in S_p, \\ 1 & \text{if } s \in S_n, I(s) = M, \\ 0 & \text{otherwise}. \end{cases}
\]

\[
P^2_s(M_1, a_1) = \begin{cases} P^1_s(M_1) & \text{if } P^1_s(M_1) > 0, \\ \text{undefined} & \text{otherwise}. \end{cases}
\]

\[
P^n_s(M_n|M_1, a_1, \ldots, M_{n-1}, a_{n-1}) = \begin{cases} P^{n-1}_{s_{(M_1,a_1)}}(M_n|M_2, a_2, \ldots, M_{n-1}, a_{n-1}) & \text{if } P^1_s(M_1) > 0, \\ \text{undefined} & \text{otherwise}. \end{cases}
\]

Let the sample space consist of all possible menus and \(s \in S\). Function \(P^1_s(M)\) can be interpreted as the probability that the menu \(M\) is observed initially when process \(s\) starts executing. Let the sample
space consist of all ready traces of length $n$ and let $s \in S$. The function $P^n_s(M_n|M_1, a_1, \ldots, a_{n-1})$ can be interpreted as the probability of the event $\{(M_1, a_1, \ldots, M_{n-1}, a_{n-1}, M_n) : X \in 2^\mathcal{A}\}$, given the event $\{(M_1, a_1, \ldots, M_{n-1}, a_{n-1}, X) : X \in 2^\mathcal{A}\}$, if observing ready traces of process $s$. It can be checked that these probabilities are well defined, i.e., they satisfy the axioms A1-A3 of Section 2.

**Definition 3.4 (Probabilistic ready trace equivalence)** Two processes $s$ and $\bar{s}$ are probabilistically ready trace equivalent, notation $s \approx_{\text{rt}} \bar{s}$, iff:

- for all $M$ in $2^\mathcal{A}$, $P^n_s(M) = P^n_{\bar{s}}(M)$ and
- for all $n > 1$, $P^n_s(M_n|M_1, a_1, \ldots, M_{n-1}, a_{n-1})$ is defined if and only if $P^n_{\bar{s}}(M_n|M_1, a_1, \ldots, M_{n-1}, a_{n-1})$ is defined, and in case they are both defined, they are equal.

Informally, two processes $s$ and $\bar{s}$ are ready-trace equivalent iff for every $n$ and every ready trace $(M_1, a_1, M_2, a_2, \ldots, M_n)$, the probability to observe $M_n$, under condition that previously the sequence $(M_1, a_1, M_2, a_2, \ldots, a_{n-1})$ was observed, is defined at the same time for both $s$ and $\bar{s}$; moreover, in case both probabilities are defined, they coincide. Note that it is straightforward to construct a testing scenario in the lines of [3,15] for this ready-trace equivalence. Namely, in [15] a ready trace machine is described, that allows for the ready traces to be observed. To *estimate* the conditional probabilities of the ready traces of length $n$, only basic statistical analysis needs to be applied to the set of all ready traces obtained from the ready-trace machine.

**Example** For processes $s$ and $\bar{s}$ in Fig. 1 it holds $s \approx_{\text{rt}} \bar{s}$.

## 4 Testing equivalence

In this section we define a testing equivalence in the style of [7] for reactive probabilistic processes.

Recall that a division of two polynomials is called a *rational function*. For example, $\frac{2x}{x+1}$ is a rational function with arguments $x$ and $y$. A possible domain for this function is $(0, \infty) \times (0, \infty)$. We exploit a subset $\mathcal{R}$ of the rational functions whose argument names belong to the action labels $\mathcal{A}$, which is generated by the following grammar:

$$\mathcal{R} := \alpha \mid a \mid \mathcal{R} + \mathcal{R} \mid \mathcal{R} \cdot \mathcal{R} \mid \frac{\mathcal{R}}{\mathcal{R}}.$$

where $\alpha$ is a non-negative scalar, $a \in \mathcal{A}$, and $+, \cdot$, and $\div$ are ordinary algebraic addition, multiplication and fraction, resp. Brackets are used in the standard way to change the priority of the operators. For our purposes, we assume that the arguments $a, b, \ldots$ can only take positive values, i.e. the domain of every function in $\mathcal{R}$ is $(0, \infty)^n$, where $n$ is the size of the action set. Therefore, two rational functions in $\mathcal{R}$ are equal iff they can be transformed to equal terms using the standard transformations that preserve equivalence (e.g. for $a, b \in \mathcal{A}$, $\frac{1}{2} \cdot \frac{a}{a+b} + \frac{1}{2} \cdot \frac{b}{a+b} = \frac{1}{2} \cdot \frac{(a+b)}{2 \cdot (a+b)} = \frac{1}{2}$).

A test $T$, as standard, is a finite process $\bar{\mathcal{T}}$ such that, for a symbol $\omega \notin \mathcal{A}$, there may exist transitions $s \xrightarrow{\omega} \bar{s}$ for some states $s$ of $T$, denoting success. Denote the set of all tests by $\mathcal{T}$. Next, we define the result of testing a process with a given test. The informal explanation follows afterwards.

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2We emphasize the word “estimate”, as it is common knowledge that statistics provides only estimations of the probabilities.

3For now we restrict to non-recursive tests, as the characterization proof in Sec 5 is already involved; however, it is not uncommon to restrict to non-recursive tests in probabilistic testing initially, for the sake of clear presentation (see e.g. [8,9]). In fact, usually recursive tests do not increase the distinguishing power of the finite tests [17,30,33], since infinite paths in tests cannot report success.
Definition 4.1 The function $R: S \times \mathcal{T} \mapsto \mathcal{R}$ that gives the result of testing a process $s$ with a test $T$ is defined as follows:

$$R(s, T) = \begin{cases} 1, & \text{if } T \xrightarrow{\omega_0}, \\ \sum_{i \in I} \pi_i \cdot R(s_i, T), & \text{if } s \xrightarrow{\pi_i} s_i \text{ for } i \in I, T \xrightarrow{\pi_i} \\ \sum_{i \in I} \pi_i \cdot R(s, T_i), & \text{if } T \xrightarrow{\pi_i} T_i \text{ for } i \in I, s \not\xrightarrow{\pi_i} \\ \sum_{a \in K} \frac{a}{\sum_{b \in K}} R(s_a, T_a), & \text{for } K = I(s) \cap I(T), \text{otherwise.} \end{cases}$$

As usual, the result of testing a process with a success test is 1. The result of testing a process with a probabilistic state as a root (i.e., initially probabilistic process) is a weighted sum of the results of testing the subsequent processes with the same test. Similarly when the test is initially probabilistic. Non-standard, however, is in the result of testing a process $s$ with a test $T$ that can initially perform actions from $\mathcal{A}$ only. Namely, when the process and the test synchronize on an action, the resulting transition is labeled with a “weighting factor”, containing information about the way this synchronization happened. This information has form of a rational function, the numerator of which represents the synchronized action itself, while the denominator is the sum of the common initial actions of $s$ and $T$, i.e., all actions on which $s$ and $T$ could have synchronized at the current step. In order to compute the final result of the testing, the rational function is temporarily treated as “symbolic” probability. The final result is again a rational function in $\mathcal{R}$.

For example, it is easy to compute that the result of testing either $s$ or $\bar{s}$ with $u$ (given in Fig. 1) is equal to $\frac{1}{2}$, which establishes one of our goals set in Sec. 1. However, in many cases the result is a non-scalar rational function.

Definition 4.2 Two processes $s$ and $\bar{s}$ are testing equivalent, notation $s \approx T \bar{s}$, iff $R(s, T)$ and $R(\bar{s}, T)$ are equal functions for every test $T$.

Obviously, comparing two results boils down to comparing two polynomials, after both rational functions have been transformed to equal denominators.

Remark In [12], in order to keep the probabilities in a composed system, the actions resulting from synchronization have a label containing information about the present and the history of synchronization – i.e., a sequence of previous menus of actions-candidates for synchronizing and the actual synchronized actions. This is because (i) we would like to denote that both choices in $s || u$ (Fig. 1) are resolved in the same way and (ii) the history of resolution of choices, as usual, can play a role in the current resolution. In the present paper one of our main goals is to prove that the testing equivalence coincides with the ready-trace equivalence for the model of reactive probabilistic processes. It turns out that, in order to achieve this goal, we can simplify the notation for the label of a synchronized action. Here the label of a resulting synchronized action contains only information about the current circumstances of synchronization in the form of a rational expression and the result of testing remains a rational expression.

(The rational function is a suitable form of “remembering” the information, because “in the world of rational expressions” commutativity and distributivity laws hold, analogous to those we try to achieve “in the world of processes”.) Besides simplifying the notation, this labeling enables us to present the proof of Theorem 5.2 (Sec. 5) in a much more concise way.

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4In the setting without internal nondeterminism, preorder relations become superfluous, since in [12], as usual, a process implements another one iff the former contains less internal nondeterminism.
5 Relationship between $\approx_{\mathcal{F}}$ and $\approx_{\mathcal{O}}$

We establish our main result, namely that the testing equivalence $\approx_{\mathcal{F}}$ coincides with the probabilistic ready-trace equivalence $\approx_{\mathcal{O}}$. In [12], given that two processes are not ready-trace equivalent, we provide a procedure on how to construct a test that distinguishes between the processes. The procedure heavily relies on the fact that tests can perform internal transitions (which can be manipulated based on the synchronization history). In the present case, the internal transitions of the tests, as those of the processes, are fully random (both the tests and the processes belong to the same model, in the spirit of [12]). Because of this, the present characterization proof is rather based on contradiction and is much more technically involved.

As an intermediate result, we prove that the probabilistic transitions do not add distinguishing power to the tests.

The following lemma, which considers the determinant of a certain type of an almost-triangular matrix, shall be needed in the proof of Theorem 5.2.

Lemma 5.1 Let $Q$ be a square $n \times n$ matrix with elements $q_{ij}$, for $1 \leq i \leq n$ and $1 \leq j \leq n$. Suppose $q_{ij} \in \{0, 1\}$ for $i > j$, $q_{ij} = 1$ for $i = j + 1$, $q_{ij} = 0$ for $i > j + 1$, and $q_{ij} = \frac{Q_j^1}{Q_i^1}$ for $1 \leq j \leq n$, where $Q_1^1, Q_2^1, \ldots, Q_n^1$ are irreducible, mutually prime polynomials with positive variables, and of non-zero degrees. Then the determinant of $Q$ is a non-zero rational function.

Proof The determinant $\text{Det}(Q)$ of matrix $Q$ can be obtained from the general recursive formula $\text{Det}(Q) = \sum_{j=1}^{n} (-1)^{i+j} q_{ij} \text{Det}(Q_{ij})$, where $Q_{ij}$ is the matrix obtained by deleting the first row and the $j$-th column of $Q$. Observe that $Q_{1n}$ is an upper-triangular matrix, the diagonal elements of which are all equal to one. Since the determinant of a triangular matrix is equal to the product of its diagonal elements, we have $\text{Det}(Q_{1n}) = 1$. Therefore, the coefficient in front of the rational function $\frac{Q_1^1}{Q_n^1}$ in $\text{Det}(Q)$ is equal to 1. Suppose $\text{Det}(Q)$ is a zero-function. Then, the rational function $\frac{1}{Q_n^1}$ is equal to a linear combination of $\frac{1}{Q_1^1}, \ldots, \frac{1}{Q_{n-1}^1}$. This means that the rational function $\frac{Q_1^1 \cdot Q_2^1 \cdots Q_{n-1}^1}{Q_n^1}$ is a polynomial. The last is impossible, since, by assumption, the denominator is an irreducible polynomial of non-zero degree and is not contained in the numerator. Therefore, $\text{Det}(Q)$ is not a zero-function.

Theorem 5.2 Let $s$ and $t$ be two processes such that $s \not\approx_{\mathcal{O}} t$. There exists a test $T$ that has no probabilistic transitions such that $R(s, T) \neq R(t, T)$.

Proof We prove the theorem by induction on the minimal length $m$ of a ready-trace that distinguishes between $s$ and $t$. For $m = 1$, we prove that the test $T = \sum_{a \in M} a \cdot \omega$, where $M$ is a menu with a minimal possible number of actions such that $P_s^1(M) \neq P_t^1(M)$, distinguishes between $s$ and $t$. For $m > 1$ the proof goes as follows. If $P_s^1(M) = P_t^1(M)$ for every menu $M$, then by the inductive assumption it follows that there exists a test $T_1$, menu $M_1$ and action $a_1 \in M_1$ such that $R(s(M_1, a_1), T_1) \neq R(t(M_1, a_1), T_1)$. We show that there exists a subset of the action set, say $\text{Act}$, such that the test $T = a_1 \cdot T_1 + \sum_{b \in \text{Act}} b \cdot \omega$ distinguishes between $s$ and $t$. To prove this, we take $M_1$ to be the menu containing a minimal possible number of actions such that $P_s^1(M_1) > 0$, $a_1 \in M_1$, and $R(s(M_1, a_1), T_1) \neq R(t(M_1, a_1), T_1)$. Then we take the set $\text{Act}'$ to consist of the actions that can be initially performed by $s$ but do not belong to menu $M_1$. Then, we show that there must exist a subset $\text{Act}'$ of $\text{Act}'$ such that the test $T = a_1 \cdot T_1 + \sum_{b \in \text{Act}} b \cdot \omega$ distinguishes between $s$ and $t$ (otherwise, we obtain that $R(s(M_1, a_1), T_1) = R(t(M_1, a_1), T_1)$, which contradicts our assumption).

We now proceed with a detailed presentation of the proof.
From $s \not\approx_{\mathcal{O}} t$ and by Def. 3.4 there must exist a ready-trace $(M_1, a_1, \ldots, M_m)$ such that

\[ \text{Det}(Q) = \sum_{j=1}^{n} (-1)^{i+j} q_{ij} \text{Det}(Q_{ij}) \]
\[ P_s^m(M_m|a_1, a_2, \ldots, a_{m-1}) \neq P_t^m(M_m|a_1, a_2, \ldots, a_{m-1}). \] The proof is by induction on \( m \).

**Case 1** \((m = 1)\) Suppose first that there exists a menu \( M \) such that \( P_s^1(M) \neq P_t^1(M) \). Let \( M \) be a menu with a minimal possible number of actions such that \( P_s^1(M) \neq P_t^1(M) \). Take \( T = \sum s \in \mathcal{M} \cdot \omega \). We have \( R(s, T) = 1 - \sum_{M \subseteq M} P_s^1(M') \), because the actions of \( s \) and \( T \) will fail to synchronize if and only if the random choice decides that menu \( M \) or some menu \( M' \subset M \) is offered to process \( s \) initially. Similarly, \( R(t, T) = 1 - \sum_{M \subseteq M} P_t^1(M') \). Now, suppose that \( R(s, T) = R(t, T) \). We have \( \sum_{M \subseteq M} P_s^1(M') = \sum_{M \subseteq M} P_t^1(M') \). From this and \( P_s^1(M) \neq P_t^1(M) \), it follows that there exists a menu \( M' \subset M \) such that also \( P_s^1(M') \neq P_t^1(M') \). But this contradicts the assumption that \( M \) is a menu with a minimal possible number of actions such that \( P_s^1(M) \neq P_t^1(M) \).

**Case 2** \((m > 1)\) Suppose now that \( P_s^1(M) = P_t^1(M) \) for every menu \( M \). Let \( (M_1, a_1, \ldots, M_m) \) be a ready-trace such that \( P_s^m(M_m|a_1, a_2, \ldots, a_{m-1}) \neq P_t^m(M_m|a_1, a_2, \ldots, a_{m-1}) \). From \( P_s^1(M_1) = P_t^1(M_1) \), and from Definitions 3.2 and 3.3 it follows that \( P_s^{m-1}(M_m|a_1, a_2, \ldots, a_{m-1}) \neq P_t^{m-1}(M_m|a_1, a_2, \ldots, a_{m-1}) \) (in case \( m = 2 \), \( P_s^1(M_2) \neq P_t^1(M_2) \)). Now, by the inductive assumption, there exists a non-probabilistic test \( T_1 \) such that \( R(s(M_1, a_1), T_1) \neq R(t(M_1, a_1), T_1) \).

**Case 2.1** Suppose first that \( a_1 \) does not belong to any first-level menu of \( s \) other than \( M_1 \), i.e. that for every menu \( M \), \( P_s^1(M) > 0 \) and \( a_1 \in M \) implies \( M = M_1 \). Then the test \( T = a_1, T_1 \) distinguishes between \( s \) and \( t \).

**Case 2.2** Suppose now that \( a_1 \) belongs to at least one first-level menu of \( s \) other than \( M_1 \), i.e. there exists at least one menu \( M \neq M_1 \) such that \( P_s^1(M) > 0 \) and \( a_1 \in M \). Without loss of generality, assume that \( M_1 \) is a menu with a minimal possible number of actions such that \( P_s^1(M_1) > 0 \), \( a_1 \in M_1 \), and \( R(s(M_1, a_1), T_1) \neq R(t(M_1, a_1), T_1) \). Let \( \{b_j\}_{j \in J} \) be the set of actions that appear in the first level of \( s \) (and therefore \( t \)) but not in \( M_1 \), i.e. \( b \in \{b_j\}_{j \in J} \) if and only if \( b \not\in M_1 \) and there exists a menu \( M \) such that \( P_s^1(M) > 0 \), \( b \in M \). We shall prove that there exists \( J' \subseteq J \) such that the test \( T = a_1, T_1 + \sum_{j \in J} b_j, \omega \) distinguishes between \( s \) and \( t \). More concretely, we shall prove that, assuming the opposite, it follows that \( R(s(M_1, a_1), T_1) = R(t(M_1, a_1), T_1) \), thus obtaining contradiction.

**Case 2.2.a** Suppose first that \( \{b_j\}_{j \in J} = \emptyset \). This means that there are no actions other than those in \( M_1 \), that appear in the first level of \( s \). Therefore, all menus \( M \) for which \( P_s^1(M) > 0 \) satisfy \( M \subseteq M_1 \). We prove that the test \( T = a_1, T_1 \) distinguishes between \( s \) and \( t \). Assume that \( R(s, T) = R(t, T) \). From the last and from Def. 4.1 we obtain

\[ \sum_{M: P_s^1(M) > 0, a_1 \in M \subseteq M_1} (R(s(M_1, a_1), T_1) - R(t(M_1, a_1), T_1)) = 0. \]

By assumption, for every \( M \subset M_1 \) such that \( a_1 \in M \) it holds \( R(s(M_1, a_1), T_1) = R(t(M_1, a_1), T_1) \). Therefore, from \( (1) \) we obtain \( R(s(M_1, a_1), T_1) = R(t(M_1, a_1), T_1) \), which contradicts the assumption \( R(s(M_1, a_1), T_1) \neq R(t(M_1, a_1), T_1) \).

**Case 2.2.b** Suppose now that \( \{b_j\}_{j \in J} \neq \emptyset \). Given action \( b_1 \in \{b_j\}_{j \in J} \), denote by \( \mathcal{M}_1 \) the set of all first-level menus of \( s \) that contain \( b_1 \) and \( a_1 \), i.e. \( M \in \mathcal{M}_1 \) iff \( P_s^1(M) > 0 \) and \( b_1, a_1 \in M \); denote by \( \mathcal{M}_1^{C} \) the set of all first-level menus of \( s \) that do not contain \( b_j \) but have \( a_1 \), i.e. \( M \in \mathcal{M}_1^{C} \) iff \( P_s^1(M) > 0 \), \( b_1 \not\in M \) and \( a_1 \in M \).
Let $T = a_1.T_1 + \sum_{j \in J} b_j.\omega$ for some $J' = \{1, 2, \ldots n\} \subseteq J$ and suppose $R(s, T) = R(t, T)$. Since $P^i_1(M) = P^i_1(M)$ for every menu $M$, observe that only if action $a_1$ is performed initially, it is possible for the test $T = a_1.T_1 + \sum_{j \in J} b_j.\omega$ to make a difference between $s$ and $t$. Because of this and by Definitions [4,1] and [3,2] it follows that

$$
\sum_{M \in \mathcal{M}_J \cap \mathcal{M}_n} \frac{a_1}{a_1} P^1_1(M) \sigma(M) + \sum_{M \in \mathcal{M}_J \cap \mathcal{M}_n} \frac{a_1}{a_1 + b_1} P^1_1(M) \sigma(M) + \cdots
$$

$$
+ \sum_{M \in \mathcal{M}_J \cap \mathcal{M}_n} \frac{a_1}{a_1 + \sum_{j=1}^n b_j} P^1_1(M) \sigma(M) = 0, \tag{2}
$$

where by $\sigma(M)$ we denote $R(s_{(M_a1)}, T_1) - R(t_{(M_a1)}, T_1)$. Each intersection appearing under the $\sum$-operators of (2) can be mapped bijectively to a binary number of $n$ digits – the $i$-th digit being 0 if the intersection contains $\mathcal{M}_{n+1-i}$, and 1 if the intersection contains $\mathcal{M}_{n+1-i}$. (For reasons that will become clear later, the order of the indexing is reversed.)

Suppose $R(s, T) = R(t, T)$ for every test $T = a_1.T_1 + \sum_{j \in J} b_j.\omega$, where $J' \subseteq J$. We shall prove that, in this case, every sum $\sum \sigma(M)$ that appears in (2) when $J' = J$ is equal to a zero-function. In particular, the equality

$$
\sum_{M \in \mathcal{M}_J \cap \mathcal{M}_J} \sigma(M) = 0 \tag{3}
$$

will hold. Note that the set $\bigcap_{j \in J} \mathcal{M}_j$ contains all first-level menus of $s$ that have the action $a_1$ but do not have any other action that does not appear in $M_1$. Therefore, $\bigcap_{j \in J} \mathcal{M}_j$ consists of the subsets of $M_1$ that contain $a_1$. Thus, the equation (3) is equivalent to the equation (1) which leads to $R(s_{(M_a1)}, T_1) = R(t_{(M_a1)}, T_1)$, i.e. to contradiction. This would complete the proof of the theorem.

We now proceed with proving the above stated claim. We prove a more general result, namely that for $J' \subseteq J$, under assumption that $R(s, T) = R(t, T)$ for every test $T = a_1.T_1 + \sum_{j \in J} b_j.\omega$ such that $J'' \subseteq J$ and $|J''| \leq |J'|$, it holds that every sum $\sum \sigma(M)$ that appears in (2) is equal to zero.

Suppose first that $|J'| = 1$, i.e. $J' = \{1\}$. Assume that

$$
R(s, a_1.T_1) = R(t, a_1.T_1) \tag{4}
$$

and

$$
R(s, a_1.T_1 + b_1.\omega) = R(t, a_1.T_1 + b_1.\omega). \tag{5}
$$

From (4), Def. [4,1] and because $P^i_1(M) = P^i_1(M)$ for every menu $M$, we obtain

$$
\sum_{M \in \mathcal{M}_{i+1} \cap \mathcal{M}_1} \frac{a_1}{a_1} P^1_1(M)(R(s_{(M,a1)}, T_1) - R(t_{(M,a1)}, T_1)) = 0. \tag{6}
$$

The equation (2) turns into

$$
\sum_{M \in \mathcal{M}_1} \frac{a_1}{a_1} P^1_1(M) \sigma(M) + \sum_{M \in \mathcal{M}_1} \frac{a_1}{a_1 + b_1} P^1_1(M) \sigma(M) = 0. \tag{7}
$$
Denote $\sum_{M \in \mathcal{M}^C} P^1_0(M)\sigma(M)$ by $x_0$ and $\sum_{M \in \mathcal{M}_1} P^1_1(M)\sigma(M)$ by $x_1$. Our goal is to show that $x_0 = 0$ and $x_1 = 0$, i.e., that they are zero-functions. From (6) and (7) we obtain the system of equations for the unknowns $x_0$ and $x_1$

$$Q_1 x = 0,$$

where

$$Q_1 = \begin{pmatrix} a_1 & a_1 \\ a_1 & a_1 + b_1 \\ 1 & 1 \end{pmatrix}, x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \text{ and } 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the matrix $Q_1$ is not a zero-function, it follows that $x_0 = 0$ and $x_1 = 0$ is the only solution of the system.

To present a better intuition on the proof in the general case, we shall also consider separately the case $|J| = 2$. Let $J' = \{1, 2\}$ and assume that $R(s, T) = R(t, T)$ for every test $T = a_1 T_1 + \sum_{i \in J'} b_i \omega$ such that $J'' \subseteq J$ and $|J''| \leq |J|$. The equation (2) turns into

$$\sum_{M \in \mathcal{M}^C \cap \mathcal{M}_2} \frac{a_1}{a_1 + b_1} P^1_s(M)\sigma(M) + \sum_{M \in \mathcal{M}_1 \cap \mathcal{M}_2} \frac{a_1}{a_1 + b_2} P^1_s(M)\sigma(M) + \sum_{M \in \mathcal{M}_2 \cap \mathcal{M}_1} \frac{a_1}{a_1 + b_1 + b_2} P^1_s(M)\sigma(M) = 0. \quad (8)$$

Denoting $\sum_{M \in \mathcal{M}^C \cap \mathcal{M}_2} P^1_s(M)\sigma(M)$ by $x_{00}$ and so on, (8) turns into

$$\frac{a_1}{a_1} x_{00} + \frac{a_1}{a_1 + b_1} x_{01} + \frac{a_1}{a_1 + b_2} x_{10} + \frac{a_1}{a_1 + b_1 + b_2} x_{11} = 0. \quad (9)$$

From $\sum_{M \in \mathcal{M}^C} P^1_1(M)\sigma(M) = 0$ we obtain $x_{00} + x_{01} = 0$, and from $\sum_{M \in \mathcal{M}_2} P^1_1(M)\sigma(M) = 0$ we obtain $x_{10} + x_{11} = 0$. Similarly, from $\sum_{M \in \mathcal{M}_1} P^1_s(M)\sigma(M) = 0$ we obtain that $x_{01} + x_{11} = 0$. Therefore, we have the system $Q_2 x = 0$, where

$$Q_2 = \begin{pmatrix} a_1 & a_1 & a_1 & a_1 \\ a_1 & a_1 + b_1 & a_1 + b_2 & a_1 + b_1 + b_2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

By Lemma 5.1 Det($Q_2$) is not a zero-function, which implies that the vector of zero-functions is the only solution of the above system of equations.

We now present how each matrix $Q_{n+1}$ can be obtained from the matrix $Q_n$.

In general, for $\mathcal{M}_1^* \in \{\mathcal{M}_1, \mathcal{M}_1^C\}$, it holds

$$\sum_{M \in \mathcal{M}^C \cap \mathcal{M}_1^*} P^1_1(M)\sigma(M) + \sum_{M \in \mathcal{M}_1^* \cap \mathcal{M}_2} P^1_1(M)\sigma(M) = \sum_{M \in \mathcal{M}_2 \cap \mathcal{M}_1^*} P^1_1(M)\sigma(M). \quad (10)$$

This means that, in the general case, each solution $x_{i_1 i_2 \ldots i_n} = 0$ of the system $Q_n x = 0$ generates the following equations for the next system:

$$x_{i_1 i_2 \ldots i_k 0 i_{k+1} \ldots i_n} + x_{i_1 i_2 \ldots i_k 1 i_{k+1} \ldots i_n} = 0,$$
for every $0 \leq k \leq n$. Note that each row of $Q_2$, except the first one, contains exactly two 1’s, at positions whose binary representations differ in exactly one place.

Informally, the general algorithm for obtaining the elements $q_{n+1}^{ij}$ of a $2^{n+1} \times 2^{n+1}$ matrix $Q_{n+1}$ from matrix $Q_n$, assuming $Q_n$ is non-singular, is as follows. First, initialize all elements of $Q_{n+1}$ to zero. Then, copy $Q_n$ into the upper left corner of $Q_{n+1}$. Then, copy $Q_n$, excluding the first row, into the lower right corner of $Q_{n+1}$. Then, assign 1 to $q_{n+1}^{ij}$ for $i = 2^n + 1$ and $j \in \{2^n, 2^n+1\}$. Finally, add the appropriate new rational fractions in the second half of the first row of $Q_{n+1}$. The key observation is that in this way, we obtain again a matrix such that each row, except the first one, contains exactly two 1’s, at positions whose binary representations differ in exactly one place. Formally,

\[
q_{n+1}^{ij} = \begin{cases} 
q_n^{ij} & \text{if } 1 \leq i \leq 2^n \text{ and } j \leq 2^n, \\
1 & \text{if } i = 2^n + 1 \text{ and } j \in \{2^n, 2^n+1\}, \\
q_n^{ij} & \text{if } 2^n + 1 < i \text{ and } 2^n < j, \\
\frac{a_1}{a_1 + \sum_{k \in K} b_k + b_{n+1}} & \text{if } i = 1, j > 2^n, \\
\text{and } q_n^{(i)(j-2^n)} = \frac{a_1}{a_1 + \sum_{k \in K} b_k} & \text{otherwise.} 
\end{cases}
\]

Assuming matrix $Q_n$ satisfies the conditions of Lemma 5.1, it easily follows that matrix $Q_{n+1}$ also satisfies the conditions of Lemma 5.1. Therefore, its determinant is not a zero function. This means that the system $Q_{n+1}x = 0$ has only zero-functions as solutions, which we were aiming to prove. Therefore, the proof of the theorem is complete.

**Theorem 5.3** Let $s$ and $t$ be two processes. If $s \approx \eta t$ then $s \approx \not\approx t$.

**Proof** Straightforward: see [13].

From Theorems 5.3 and 5.2 the following statements directly follow.

**Theorem 5.4** For arbitrary processes $s$ and $t$, $s \approx \not\approx t$ if and only if $s \approx \eta t$.

**Theorem 5.5** For arbitrary processes $s$ and $t$, $s \not\approx \not\approx t$ if and only if there exists a test $T$ without probabilistic transitions such that $R(s, T) \neq R(t, T)$.

**Remark** It is interesting to note that, while in the non-probabilistic case the may/must testing equivalence can be characterized with the failure equivalence [26], in the probabilistic case we obtain a bit finer characterization. However, this is not unusual in the probabilistic case, due to the “effect” of the probabilities – e.g. the same phenomenon appears also in the fully probabilistic case [27].

### 6 Testing systems and decidability

In this section we outline how testing can be applied to systems for which only partial information may be known, and we show that the testing equivalence is decidable for finite systems or up to a certain depth of the systems.

So far we have discussed testing “processes”, i.e. models of systems. In practice, to test a system with a given test, the probabilistic transitions of the system need not be known. Namely, assume that when the system and the test are ready to synchronize on an action, the test can “see” the actions-candidates
for synchronization. If the system is tested with the test exhausting all possible ways of synchronization and sufficiently many times, then the result shall be a set of rational functions without scalars; a standard statistical analysis will give an estimation of the probability distribution over the rational functions. (A detailed description of the procedure is beyond the goals of the current paper.) Two systems would not be distinguished under a given test iff the resulting distributions are the same. The assumption that the test can see the actions-candidates for synchronization, on the other hand, corresponds to the user (e.g. \textit{u} in Fig. 1) being able to see the menu that the machine (e.g. \textit{s} in Fig. 1) offers. Indeed, this assumption does not exist in the standard non-probabilistic testing theory [7]. However, in real-life systems this is not unusual. Moreover, this assumption is mild with respect to the probabilistic may/must testing approaches discussed in Sec. 1 where one needs to have insight into the internal structure of the composed system in order to determine the possible schedulers.

From Theorem 5.5 it follows that non-probabilistic, i.e. deterministic tests suffice to distinguish between two processes. Therefore, since the action set is finite, an algorithm for deciding equivalence on finite processes, or up to a certain length, can be easily constructed. Namely, in this case the characteristic set of tests of a given length is finite. In case the length of the processes is unknown, the procedure stops when, for a certain length of the tests, the testing yields result 0 for every test of that length and every tested process (meaning that the maximal length of the processes has been exceeded).

**Proposition 6.1** There exists an algorithm that decides \( \approx \) for finite processes.

### 7 Related work and conclusion

There is a plethora of equivalences defined on probabilistic processes in the last two decades (e.g. [3, 5, 6, 10, 11, 21, 25, 29, 32]). However, we think that closely related to ours are the research reports that face the challenge of allowing unobservability of the internal probabilistic choice, but still not allowing more identifications than the standard must-testing [7, 17], if probabilistic choice is treated as a kind of internal choice.

Testing equivalences in the style of [7] for processes with external choice and internal probabilistic choice, that allow unobservability of the probabilistic choice, i.e. distribution of prefix over probabilistic choice, have been also defined in [1, 2, 25]. Of these, only [2], under certain conditions, equate processes \( s \) and \( \bar{s} \) of Fig. 1. In [2] process states are enriched with labels, and a testing equivalence is defined by means of schedulers that synchronize with processes on the labels. While in our work processes \( s \) and \( \bar{s} \) in Fig. 1 are equivalent, in [2] these two processes can be equated iff the labeling is right.

Probabilistic equivalences in ready-trace style have been defined in [24] and [16], also for processes where the internal nondeterminism has been quantified with probabilities. However, in contrast to our approach, these definitions do not imply testing scenarios that can characterize the equivalences, as the one given in [15] for the non-probabilistic ready-trace equivalence.

Other equivalences, that also allow distributivity of prefix over probabilistic choice, but are not closely related to ours, include trace-style equivalences ( [3, 4, 11, 29, 31]) and button-pushing testing equivalences ( [20], [23]). Of these, only [31], [11] and [23] also allow distribution of external choice over probabilistic choice. However, in these approaches the environment is not a process itself, but rather a sequence of actions. In other words, their motivation does not include sensitivity to deadlock.

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\(^5\)See [15] for the properties that the must-testing equivalence preserves, but are not preserved by a (completed) trace equivalence.
and branching structure – e.g. they also identify processes $c.a \oplus_\frac{1}{2} c.b$ and $c.(a+b) \oplus_\frac{1}{2} c$ (“+” being the operator for external choice).

The present paper is also related to the newer research in [2][4][14], in the sense that it restricts the power of the schedulers that resolve the nondeterminism in a parallel composition. Contrary to [2][4][14], the “schedulers” in the present paper do not use information about the state in which a process is. We believe that this approach is more appropriate when defining a testing equivalence on processes, as it is closer in nature to the work in [7].

Finally, so far, none of the proposals of testing equivalences in the style of [7] for probabilistic processes having “external nondeterminism” deal with the problem of deciding equivalence based on the testing semantics itself. We refer the reader also to [33] for a survey of the testing equivalences on probabilistic processes and decidability results.

To conclude, we have proposed a testing equivalence in the style of [7] for processes where the internal nondeterminism is quantified with probabilities (e.g. [21][23]). We showed that it can be characterized as a probabilistic ready-trace equivalence. From the characterization it follows that: (i) the testing equivalence is insensitive to the exact moment of occurrence of an internal probabilistic choice, (ii) it equates no more processes than the equivalence of [7] when probabilities are not treated, and (iii) a decidability procedure exists for determining if two finite processes are testing equivalent, or if two infinite processes are testing equivalent up to a certain depth. Moreover, the testing semantics provides a way to compute the testing outcomes in practice, without requiring access to the internal structure of the system other than the actions-candidates for synchronization between the system and the test. To our knowledge, this is the first equivalence that accomplishes all of the above stated goals.

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