A NOTE ON SPECIFICATION FOR ITERATED FUNCTION SYSTEMS

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Abstract. We introduce several notions of specification for iterated function systems and exhibit some of their dynamical properties. In particular, we show that topological entropy and algebraic pressure [4] of systems with specification are approximable by the corresponding expressions for finitely generated iterated function systems.

1. Introduction. Let $X$ be a compact metric space and $\mathcal{V}$ a family of homeomorphisms $v : D(v) \to v(D(v)) \subset X$ with closed domain $D(v) \subset X$. The pair $(X, \mathcal{V})$ is called a function system. It is called an iterated function system if there is a set $O \subset X$ such that,

$$\emptyset \neq \bigcup_{v \in \mathcal{V}} v(D(v) \cap O) \subset \bigcup_{v \in \mathcal{V}} D(v) \cap O.$$  

This notion has been studied in [4] where also some general motivation to investigate iterated function systems of this general type is discussed.

Fairly general conditions are given in [4] that a function system is in fact an iterated function system and self-homeomorphic in the sense that $X = \bigcup_{v \in \mathcal{V}} D(v)$. It then proceeds investigating some thermodynamic aspects of iterated function systems when $\mathcal{V}$ is countable. The algebraic pressure is defined as

$$P_{alg}(\phi) = \lim_{n \to \infty} \sum_{v_1, \ldots, v_n \in \mathcal{V}} \sup_x \prod_{i=1}^{n} \phi_{v_i}(v_{i-1} \ldots v_1(x)),$$

where $\phi_v : D(v) \to \mathbb{R}_+$ is a family of continuous functions and the supremum is taken over all $x$ in the domain of $v_n \circ v_{n-1} \circ \ldots \circ v_1$. The limit exists because the sequence is subadditive. One of the basic problems within the framework of its thermodynamic formalism is constructing conformal measures for a given family of potentials $\phi_v$ and the transition parameter $P_{alg}(\phi)$. This problem has been solved essentially in [4] reducing the problem to the existence of aperiodic points for a
potential ϕ of bounded variation (Theorem 3.17 in [4]). Constructing aperiodic points can be established using some type of specification property (see Section 2).

This is one of the reasons to study the specification property here in more detail. Another problem arises in calculating the algebraic pressure via finite iterated function systems (X, M) where M ⊂ Σ is finite. We derive a theorem of this type using specification.

In Section 2 we define the notion of specification and show that it implies topological mixing. We introduce a notion of topological pressure in Section 3 (in analogy to the classical definition (see [3, 7])) and show that topological entropy is strictly positive for an IFS with specification. The main result for IFS with weak specification appears in Section 5, that its algebraic pressure equals the supremum over the algebraic pressure for finite subsystems. The same type of result holds for the topological entropy. These results are in the same spirit as those for countable Markov chains in [5, 6, 2]. We finally add a result on the existence of iterated function systems for a function system with specification (see Section 6). This complements results in [4] where this type of problem was treated in general.

2. Specification. In this section we let (X, M) denote an iterated function system (IFS) as defined in [4]. This means that X is a metric space with metric d and M is a family of partially defined maps v : D(v) → X, where D(v) is non-empty and closed. Moreover, there exists a subset O ⊂ X such that (1) holds. We begin with a series of definitions.

Let Σ = {σ = (v1, v2, ..., vn, ...): vj ∈ M}. According to the definition of iterated functions systems (the condition (1)), there exists σ ∈ Σ and x ∈ X such that vσ(x) = vn ◦ vn−1 ◦ ... ◦ v1(x) is well defined for each n ≥ 1. We define for pairs (σ, x) ∈ Σ × X, vσ(x) = vn ◦ vn−1 ◦ ... ◦ v1(x) whenever it is well defined and the orbit of (x, σ) as

\[ O(x, σ) = \{ x, vσ(1)(x), vσ(2)(x), ..., vσ(n)(x), ... \}. \]

We also let v̄σ(n,m) = vn ◦ vn−1 ◦ ... ◦ vm+1, so v̄σ(n) = v̄σ(n,0). Define

\[ Σ_x = \{ σ ∈ Σ: v̄σ(n)(x) \text{ is well defined for all } n ∈ N \}. \]

Definition 2.1. A specification S = (τ, P, σ) consists of a finite collection τ = {I1, ..., Im} of finite disjoint intervals Ii = [ai, bi] ⊂ N, a map P : i=1 Im Ii → X and σ ∈ Σ such that for each ti, t2 ∈ I ∈ τ we have vσ(t2,t1)(P(t1)) = P(t2). The specification S is said to be M-spaced if ai+1 ≥ bi + M for all i ∈ {1, ..., m − 1}. Moreover, a specification S is ε-shadowed by x ∈ X if there is ϕ ∈ Σx such that d(vϕ(n)(x), P(n)) < ε for all n ∈ im Ii.

Definition 2.2. An iterated function system (X, M) has the specification property if for every ε > 0 there is M ∈ N such that every M-spaced specification is ε-shadowed by some x ∈ X.

If A ⊂ X and σ ∈ Σ define vσ(1)(A) = v1(D(v1) ∩ A) and, by induction, define vσ(n+1)(A) = vn+1(D(vn+1) ∩ vσ(n)(A)). We also let

\[ O(X) = \{ x ∈ X : Σ_x ≠ ∅ \}. \]

Note that for x ∈ O(X) and σ = (v1, v2, ...) ∈ Σx we have that v1(x) ∈ O(X) as well.
Definition 2.3. 1. An iterated function system \((X, \mathcal{V})\) is called weakly topologically mixing if for all nonempty, relatively open sets \(V \subset \bigcup_{v \in \mathcal{V}} v(D(v) \cap O(X))\) and \(U \subset O(X)\) there is \(N > 0\) such that for all \(n > N\) there is \(\sigma \in \Sigma\) such that \(v^{\sigma(n)}(U) \cap V \neq \emptyset\).

2. An iterated function system \((X, \mathcal{V})\) is called topologically mixing if for all nonempty, open sets \(V, U \subset X\) there is \(N > 0\) such that for all \(n > N\) there is \(\sigma \in \Sigma\) such that \(v^{\sigma(n)}(U) \cap V \neq \emptyset\).

3. An iterated function system \((X, \mathcal{V})\) is called topologically transitive if for all nonempty, open sets \(V, U \subset X\) there is \(\sigma \in \Sigma\) and \(n \in \mathbb{N}\) such that \(v^{\sigma(n)}(U) \cap V \neq \emptyset\).

Some elementary facts are contained in the first lemma.

Lemma 2.4. Let \((X, \mathcal{V})\) be an iterated function systems.

1. Assume that for all nonempty, open sets \(V \subset X\) and all nonempty, relatively open sets \(U \subset O(X)\) there are \(x \in U, \sigma \in \Sigma_x\) and \(n > 0\) such that \(v^{\sigma(n)}(x) \in V\). Then
   \[
   \bigcup_{v \in \mathcal{V}} v(D(v) \cap O(X)) = \bigcup_{v \in \mathcal{V}} D(v) \cap O(X) = \overline{O(X)} = X.
   \]

2. If \((X, \mathcal{V})\) has the specification property then
   \[
   \bigcup_{v \in \mathcal{V}} v(D(v) \cap O(X)) = \bigcup_{v \in \mathcal{V}} D(v) \cap O(X) = \overline{O(X)} = X.
   \]
   In particular, \((X, \mathcal{V})\) is locally self-homeomorphic (see [4], Definition 2.2).

3. If \((X, \mathcal{V})\) has the specification property then it is topologically transitive.

4. If \((X, \mathcal{V})\) has the specification property then it is topologically mixing.

5. If \(v : X \to X\) has the specification property then \(v\) is topologically mixing.

6. If \((X, \mathcal{V})\) is topologically transitive and satisfies
   \[
   \bigcup_{v \in \mathcal{V}} \text{int} D(v) = X,
   \]
   then there is \(x \in X\) such that for all \(y \in X\) and \(\epsilon > 0\) there is \(n \in \mathbb{N}\) and \(\sigma = (v_1, v_2, \ldots, v_n)\) with \(d(v^{\sigma(n)}(x), y) < \epsilon\).

Proof. 1. Let \(\emptyset \neq V \subset X\) be arbitrary. Choose a nonempty, relatively open set \(U \subset \bigcup_{v \in \mathcal{V}} D(v) \cap O\). For \(x \in U\), \(\sigma \in \Sigma_x\) satisfying \(v^{\sigma(n)}(x) \in V\) it follows that \(v^{\sigma(n-1)}(x), v^{\sigma(n)}(x) \in \bigcup_{v \in \mathcal{V}} D(v) \cap O(X)\).

2. This follows from 1. Let \(\emptyset \neq V \subset X\) be open and \(\emptyset \neq U \subset \bigcup_{v \in \mathcal{V}} D(v) \cap O(X)\) be relatively open. Choose points \(y, z\) and \(x\) in these sets and \(\epsilon > 0\) such that \(B(y, \epsilon) \subset V\) and \(B(z, \epsilon) \cap \bigcup_{v \in \mathcal{V}} D(v) \cap O(X) \subset U\). Take \(M > 0\) by specification to \(\epsilon > 0\). Then for each \(n > M\) let \(\tau = \{I_1, I_2\}\) with \(I_1 = \{0\}\) and \(I_2 = \{n\}\). Define \(P(0) = z\) and \(P(n) = y\). For any \(\sigma\) we have \((\tau, P, \sigma)\) is a \(M\)-spanned specification and by specification property we can find \(x = x_n\) and \(\phi = \phi_n \in \Sigma_x\) such that \(d(x, z) < \epsilon\) and \(d(v^{\sigma(n)}(x), y) < \epsilon\). Since \(x \in O(X)\), we have that \(x \in U \cap O(X) \cap \bigcup_{v \in \mathcal{V}} D(v)\), hence 1. applies.

3. and 4. This has been proved in 2.

5. is a special case of 4.

6. Let \(V_1, V_2, \ldots\) be a basis for the topology. Let \(x_1, x_2, \ldots\) be a dense set of points belonging to \(\bigcup_{v \in \mathcal{V}} \text{int} D(v)\). By transitivity, for \(i, j \in \mathbb{N}\) there is \(\sigma_{i,j} \in \Sigma\) and \(n \in \mathbb{N}\) such that \(v^{\sigma_{i,j}(n)}(\text{int}D(v)) \cap V_j \neq \emptyset\), where \(x_i \in \text{int} D(v)\). Since \(v^{\sigma_{i,j}(n)}\) is a homeomorphism, the inverse image \(U_{i,j}\) of \(v^{\sigma_{i,j}(n)}(\text{int}D(v)) \cap V_j\) is open. Thus
Example 2.5. 1. If \( v \in \mathfrak{W} \) is topologically transitive and \( D(v) \) is dense in \( X \), then the IFS \( (X, \mathfrak{W}) \) is topologically transitive. 

2. Let \( A = \{-1 + \frac{1}{2^n} | n \in \mathbb{N}\} \), \( B = \{1 - \frac{1}{2^n} | n \in \mathbb{N}\} \) and \( C = \{(1 - \frac{1}{2^n})i | n \in \mathbb{N}\} \). Define \( D(v_1) = D(v_2) = A \cup C \cup \{-1, 0, i\} \) and \( D(v_3) = B \cup C \cup \{0, 1, i\} \). If 

\[
\begin{align*}
v_1(-1) &= -1 & v_1(i) &= i, & v_1(0) &= -\frac{1}{2} & v_1(\frac{1}{2}i) &= 0 \\
v_1(-1 + \frac{1}{2^n}) &= -1 + \frac{1}{2n+1} & \text{if } n \in \mathbb{N} \\
v_1((1 - \frac{1}{2^n})i) &= (1 - \frac{1}{2n-1})i & \text{if } n \geq 2 \\
v_3(1) &= 1 & v_3(i) &= i & v_1(0) &= \frac{1}{2} & v_3(\frac{1}{2}i) &= 0 \\
v_3(1 - \frac{1}{2^n}) &= 1 - \frac{1}{2n+1} & \text{if } n \in \mathbb{N} \\
v_3((1 - \frac{1}{2^n})i) &= (1 - \frac{1}{2n-1})i & \text{if } n \geq 2 \\
v_2 &= v_1^{-1},
\end{align*}
\]

then \( (X, \mathfrak{W}) \) is an IFS, where \( X = A \cup B \cup C \cup \{-1, 0, i\} \subset \mathbb{C} \) and \( \mathfrak{W} = \{v_1, v_2, v_3\} \). \( (X, \mathfrak{W}) \) is not topologically transitive, but 0 is such that for all \( y \in X \), there is \( \sigma \in \Sigma_0 \) such that for all \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) with 

\[
d(v_{\sigma(n)}(0), y) < \epsilon.
\]

Recall from [4] that a point \( x \) is called aperiodic for the IFS \( (X, \mathfrak{W}) \) if for each \( m \geq 1 \) there is some \( n(m) = o(m) \) such that for every \( v = v_1 \circ v_2 \circ \ldots \circ v_m \) there exists \( k \leq n(m) \) and \( w = w_1 \circ w_2 \circ \ldots \circ w_k \) with 

\[
x \in D(w) \cap w^{-1}(D(v)).
\]

**Corollary 2.6.** Let \( (X, \mathfrak{W}) \) have the specification property and let \( int D(v) \neq \emptyset \) for all \( v \in \mathfrak{W} \). Then there exists \( x \in X \) such that for every \( v = v_1 \circ v_2 \circ \ldots \circ v_m \) there are \( k \in \mathbb{N} \) and \( w = w_1 \circ w_2 \circ \ldots \circ w_k \) with 

\[
x \in D(w) \cap w^{-1}(D(v)).
\]

**Proof.** We may assume that \( \bigcup_{v \in \mathfrak{W}} int D(v) = X \). Let \( x \) be chosen as in 6. of Lemma 2.4. Fix \( v = v_1 \circ v_2 \circ \ldots \circ v_m \). Then there is \( y \in D(v) \) and \( \epsilon > 0 \) such that \( B(y, \epsilon) \subset D(v) \). By choice of \( x \) exists \( w = w_1 \circ w_2 \circ \ldots \circ w_k \) such that \( w(x) \in B(y, \epsilon) \). Therefore, \( x \in D(w) \cap w^{-1}(D(v)) \). \( \square \)

**Remark 2.7.** Let \( v : \mathbb{T}^2 \to \mathbb{T}^2 \) a Anosov diffeomorphism and \( p_1 \neq p_2 \in \mathbb{T}^2 \) such that \( v(p_i) = p_i \) for \( i \in \{1, 2\} \). Define \( D(v_1) = \{v_1\} \) and \( v_i(p_i) = p_i \) for \( i \in \{1, 2\} \), and \( \mathfrak{W} = \{v, v_1, v_2\} \). Then the \( (X, \mathfrak{W}) \) is an IFS with the specification property, but the corollary is not true for any \( x \in \mathbb{T}^2 \).

**Remark 2.8.** If the IFS \( (X, \mathfrak{W}) \) has the specification property such that the spacing constant \( M \), depending on \( \epsilon \), satisfies 

\[
M(\eta_m) = o(m)
\]

where \( \eta_m \) is the largest radius of a ball centered at \( D(v) \) which is contained in \( D(v) \) for any \( v = v_1 \circ \ldots \circ v_m \), then \( (X, \mathfrak{W}) \) contains an aperiodic point.
3. **Topological entropy.** Let \((X, \mathcal{W})\) be an iterated function system. We begin with a modification of topological entropy for the case of an IFS (see [3, 7]). The basic idea is to count the minimal number of orbits from \((X, \mathcal{W})\) which are needed to shadow all orbits. This leads to the following definitions.

For a compact set \(K \subset X\) we call a set \(R \subset \Psi(K) = \{(x, \sigma) : x \in K, \sigma \in \Sigma_x\}\) an \((n, \epsilon)\)-spanning set of \(K\), if for all \((x, \sigma), (y, \varphi) \in \Psi(K)\) there is \((y, \varphi) \in R\), such that

\[
d(v^{\sigma(i)}(x), v^{\varphi(i)}(y)) < \epsilon, \quad \forall i \in \{1, ..., n-1\}.
\]

We denote the minimum cardinality of an \((n, \epsilon)\)-spanning set for \(K\) by \(r(n, \epsilon, K)\). Since we are not restricting the notion to the case of a finite family \(\mathcal{W}\), \(r(n, \epsilon, K)\) may be infinite.

We call a set \(S \subset \Psi(K)\) an \((n, \epsilon)\)-separated set for \(K\), if for all pairs \((x, \sigma), (y, \varphi) \in S\) there is some \(i \in \{1, ..., n-1\}\) satisfying

\[
d(v^{\sigma(i)}(x), v^{\varphi(i)}(y)) \geq \epsilon.
\]

We denote the maximum cardinality of an \((n, \epsilon)\)-separated set for \(K\) by \(s(n, \epsilon, K)\).

The **topological entropy** of a IFS \((X, \mathcal{W})\) is then given by

\[
h_{top}(X, \mathcal{W}) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, K)
\]

\[
= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, K).
\]

The existence of the limits and their equality follows from the following lemma.

**Lemma 3.1.**  
1. \(r(n, \epsilon, K) \leq s(n, \epsilon, K) \leq r(n, \frac{1}{2} \epsilon, K)\).

2. If \(\epsilon_1 < \epsilon_2\) then

\[
\limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon_1, K) \geq \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon_2, K) \quad \text{and}
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon_1, K) \leq \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon_2, K).
\]

**Proof.** Let \(S \subset \Psi(K)\) be a maximal \((n, \epsilon)\)-separated set of \(K\). We claim that for \((x, \sigma) \in \Psi(K)\) there is a \((y, \varphi) \in S\) such that

\[
d(v^{\sigma(i)}(x), v^{\varphi(i)}(y)) < \epsilon, \quad \forall i \in \{1, ..., n-1\}.
\]

If the claim would not hold, then for all \((y, \varphi) \in S\) there is some \(0 \leq i < n\) such that

\[
d(v^{\sigma(i)}(x), v^{\varphi(i)}(y)) \geq \epsilon,
\]

and \(S \cup \{(x, \sigma)\}\) is a separated set, strictly larger than \(S\). Therefore \(S\) is an \((n, \epsilon)\)-spanning set of \(K\), whence \(r(n, \epsilon, K) \leq s(n, \epsilon, K)\).

Suppose \(S \subset \Psi(K)\) is an \((n, \epsilon)\)-separated set for \(K\) and \(R \subset \Psi(K)\) is an \((n, \epsilon/2)\)-spanning set for \(K\). By definition of an \((n, \epsilon/2)\) spanning set, for each \((x, \sigma) \in \Psi(K)\) there is \((z, \varphi) \in R\) such that

\[
d(v^{\sigma(i)}(x), v^{\varphi(i)}(z)) < \epsilon, \quad \forall i \in \{0, ..., n-1\}.
\]

Thus for \((x, \sigma) \in S\) there is \((z, \varphi) \in R\) satisfying (2). Choose one of these elements \((z, \varphi)\) and denote it by \(G((x, \sigma))\). The map \(G : S \to R\) is then injective, since otherwise for \((x, \sigma), (y, \tau) \in S\) and \(G((x, \sigma)) = G((y, \tau)) = (z, \varphi)\) we obtain

\[
d(v^{\sigma(i)}(x), v^{\tau(i)}(y)) \leq d(v^{\sigma(i)}(x), v^{\varphi(i)}(z)) + d(v^{\varphi(i)}(z), v^{\tau(i)}(y)) < \epsilon
\]
for all \( i \in \{0, ..., n-1\} \), contradicting the fact that \( d(v^{\sigma(i)}(x), v^{\tau(i)}(y)) \geq \epsilon \) for some \( 0 \leq i < n \). It follows that \( \#R \geq \#S \) and \( s(n, \epsilon, K) \leq r(n, \frac{1}{2}\epsilon, K) \).

For \( (x, \sigma) \in \Psi(K) \) define \( \nu(x, \sigma, n) = \{v^{\sigma(i)}(x); 0 \leq i < n\} \) and \( \Xi'(K) = \{\nu(x, \sigma, n); (x, \sigma) \in \Psi(K)\} \). Let \( \Xi'_n(K) \) the set of all \( \nu \in \Xi'(K) \) with fixed \( n \) and define a pseudo-metric on \( \Xi'_n(K) \) by

\[
d'(\nu(x, \sigma, n), \nu(\varphi, \rho, n)) = \sup_{0 \leq i < n} d(v^{\sigma(i)}(x), v^{\rho(i)}(y)),
\]

where \( \nu(x, \sigma, n), \nu(\varphi, \rho, n) \in \Xi'_n(K) \). The relation \( \nu(x, \sigma, n) \approx \nu(\varphi, \rho, n) \) if

\[
d'(\nu(x, \sigma, n), \nu(\varphi, \rho, n)) = 0
\]

is, in fact, an equivalence relation and we let \( \Xi_n(K) \) be the set of the equivalence classes of \( \approx \). Hence,

\[
d_n([\nu(x, \sigma, n)], [\nu(\varphi, \rho, n)]) = d'(\nu(x, \sigma, n), \nu(\varphi, \rho, n))
\]

is a metric on \( \Xi_n(K) \). An IFS \( (X, \mathcal{U}) \) will be called orbit-compact if for all \( n \in \mathbb{N} \) we have \( \Xi_n := \Xi_n(X) \) is compact. Obviously if \( \mathcal{U} \) is finite then \( (X, \mathcal{U}) \) is orbit-compact.

**Remark 3.2.** Let \( K \subset X \) be a compact set, and for \( R \subset \Psi(K) \) define

\[
\overline{R}_n = \{[\nu(x, \sigma, n)] \in \Xi_n(K); (x, \sigma) \in R\}.
\]

Then \( R \) is a \((n, \epsilon)\)-spanning set of \( K \), if for all \([\nu(x, \sigma, n)] \in \Xi_n(K)\) there is \([\nu(\varphi, \rho, n)] \in \overline{R} \) such that

\[
d_n([\nu(x, \sigma, n)], [\nu(\varphi, \rho, n)]) < \epsilon.
\]

**Corollary 3.3.** If an IFS \( (X, \mathcal{U}) \) is orbit-compact then \( r(n, \epsilon, K) < \infty \).

**Theorem 3.4.** Let \( X \) have at least two points. If an iterated function system \((X, \mathcal{U})\) has the specification property, then its topological entropy is positive.

**Proof.** Let \( p \neq q \in X \) and \( 3\epsilon < d(p, q) \). The specification property assures there is some \( M > 0 \) such that any \( M \)-spaced specification is \( \epsilon \)-traced by some orbit. It follows that for a sequence \( a = (a_1, ..., a_n) \) with \( a_i \in \{p, q\} \) there is \( z(a) \) and \( \sigma_a \in \Sigma_{z(a)} \) such that

\[
d(v^{\sigma(1)}(z(a)), a_i) < \epsilon
\]

For \( a = (a_1, ..., a_n) \neq (b_1, ..., b_n) \) let \( i \) be any index such that \( a_i \neq b_i \). Since \( d(p, q) > 3\epsilon \),

\[
d(v^{\sigma(1)}(z(a)), v^{\sigma(1)}(z(b))) \\
\geq d(a_i, b_i) - d(a_i, v^{\sigma(1)}(z(a))) - d(b_i, v^{\sigma(1)}(z(b))) \\
\geq \epsilon.
\]

Hence there is a bijection between the sets \( \alpha_n = \{a = (a_1, ..., a_n); a_i \in \{p, q\}\} \) and \( S_n = \{(z(a), \sigma_a); a \in \alpha_n \} \subset \Psi(X) \). Since \( S_n \) is an \((nM, \epsilon)\)-separated set, \( s(nM, \epsilon, X) \geq 2^n \) and

\[
h_{top}(X, \mathcal{U}) \geq \lim_{n \to \infty} \frac{1}{nM} \log 2^n = \frac{\log 2}{M} > 0.
\]

\( \square \)
4. Weak specification. Let $(X, \mathcal{V})$ be an iterated function system. We first extend the definitions in Section 2.

A specification $S = (\tau, P, \sigma)$ in $\mathcal{V}^{(0)} \subset \mathcal{V}$ consists of a finite collection $\tau$ of finite disjoint intervals $J = [a_j, b_j] \subset \mathbb{N}$, a map $P : I = \bigcup_{J \in \tau} J \to X$ and $\sigma_j = (\sigma^j_{t})_{1 \leq i < |J|} \in (\mathcal{V}^{(0)})^{|J|-1}$ such that for each $J \in \tau$ and $t \in J$

$$\sigma^j_{t-a_j} \circ \sigma^j_{t-a_{j-1}} \circ ... \circ \sigma^j_{t}(P(a_j)) = P(t).$$

A specification in $\mathcal{V}$ agrees with the notion of a specification as introduced in Section 2. This means that a specification is a finite collection of orbit segments

$$P(t), \sigma_{t}(P(t)), ..., \sigma_{s-1} \circ ... \circ \sigma_{t}(P(t)) \quad J = [t, s] \in \tau$$

where the maps $\sigma$ belong to $\mathcal{V}^{(0)}$.

A specification in $\mathcal{V}^{(0)}$ is said to be $M$-spaced if the intervals $J \in \tau$ have a pairwise distance at least $M$.

A specification $S$ in $\mathcal{V}^{(0)}$ is $\epsilon$-shadowed by $x \in X$ within $\mathcal{V}^{(1)} \subset \mathcal{V}$ if there exist $\rho = (\rho_1, ..., \rho_L) \in (\mathcal{V}^{(1)})^L$, $L = \max J - 1$, such that for $J \in \tau$

$$d(\rho_1 \circ \rho_{t-1} \circ ... \circ \rho_1(x), P(t)) < \epsilon \quad \forall t \in J \in \tau$$

and

$$\rho_{a_j+i} = \sigma^j_{t} \quad \forall a_j + i \in J, \quad J \in \tau.$$ 

**Definition 4.1.** An iterated function system is said to have the **finite specification property** if for all $\eta > 0$ and all finite sets $\mathcal{V}^{(0)} \subset \mathcal{V}$ there is $M > 0$ and a finite set $\mathcal{V}^{(1)} \subset \mathcal{V}$ such that any $M$-spaced specification in $\mathcal{V}^{(0)}$ is $\eta$-shadowed within $\mathcal{V}^{(1)}$ by some point in $X$.

A weak specification is a specification where the intervals $J \in \tau$ are adjacent beginning at 1. Each specification $S = (\tau, P, \sigma)$ defines a weak specification $\tilde{S} = (\tilde{\tau}, \tilde{P}, \tilde{\sigma})$ by deleting the spacings, called the reduced specification: If the intervals in $\tau$ are denoted by $J_k = [t_k, t_k + \alpha_k]$ $(1 \leq k \leq q)$ in increasing order the new intervals are defined by

$$\tilde{J}_k = [\alpha_1 + ... + \alpha_{k-1} + 1, \alpha_1 + ... + \alpha_k] \quad \alpha_0 = 0, k = 1, ..., q$$

and $\tilde{P}$ and $\tilde{\sigma}$ by

$$\tilde{P}(t) = P(t+k - \alpha_1 - ... - \alpha_{k-1} - 1) \quad \tilde{\sigma}_t = \sigma^{J_k}_{t-t_k+1}$$

for $t \in \tilde{J}_k$, $k = 1, ..., q$. Given a weak specification $\tilde{S}$, any specification $S$ whose reduced specification is $\tilde{S}$ is called an associated specification.

**Definition 4.2.** An iterated function system is said to have the **finite weak specification property** if for all $\eta > 0$, and all finite sets $\mathcal{V}^{(0)} \subset \mathcal{V}$ there is a finite set $\mathcal{V}^{(1)} \subset \mathcal{V}$ and $M \geq 0$ such that any weak specification in $\mathcal{V}^{(0)}$ has an associated specification which is at most $M$-spaced and is $\eta$-shadowed within $\mathcal{V}^{(1)}$ by some point in $X$.

**Lemma 4.3.** If an iterated function system $(X, \mathcal{V})$ has the finite specification property then $(X, \mathcal{V})$ has the weak finite specification property.

**Proof.** Let $\mathcal{V}^{(0)} \subset \mathcal{V}$ be finite and $\eta > 0$. Take $\mathcal{V}^{(1)}$ and $M > 0$ as in the finite specification property. If $\tilde{S} = (\tilde{\tau}, \tilde{P}, \tilde{\sigma})$ is a weak specification, for each $\tilde{J}_k = \cdots$
Since the right hand side of (3) is clearly bounded by its left hand side it is left to show the converse inequality.

Let $\epsilon > 0$. Since $\log \phi$ is a family of uniformly continuous functions, there exists $\eta > 0$ such that for $\phi \in \Phi$ and $x, y \in D(v)$ with $d(x, y) < \eta$ we have $|\log \phi(x) - \log \phi(y)| < \epsilon$. Since $\log Z_n(\phi)$ is subadditive $\frac{1}{k} \log Z_k(\phi) \geq P_{alg}(\phi)$. Fix $k$ and choose a finite set $\mathcal{V}' \subset \mathcal{V}_k := \{v_1 \circ \ldots \circ v_k : v_i \in \mathcal{V}\}$ such that

$$\frac{1}{k} \log \sum_{v \in \mathcal{V}'} \sup_{x \in D(v)} \phi_v(x) \geq P_{alg}(\phi) - \epsilon.$$

Since $\mathcal{V}'$ is finite, there is a finite family $\mathcal{V}^{(0)} \subset \mathcal{V}$ such that $\mathcal{V}' \subset \mathcal{V}_k^{(0)}$, hence

$$\frac{1}{k} \log \sum_{v \in \mathcal{V}_k^{(0)}} \sup_{x \in D(v)} \phi_v(x) \geq P_{alg}(\phi) - \epsilon.$$
Given $\eta > 0$ and $\mathfrak{W}^{(0)}$, choose a subsystem $\mathfrak{W}^{(1)} \subset \mathfrak{W}$ and a spacing constant $L = L(\eta) \geq 0$ as in the finite weak specification property.

Let $\delta > 0$. For each $v \in \mathfrak{W}^{(0)}$ choose $x_v \in D(v)$ such that

$$\phi_v(x_v) \geq \sup_{y \in D(v)} \phi_v(y) - \delta.$$ 

Every ordered choice of $v^{(1)}, \ldots, v^{(s)} \in \mathfrak{W}^{(0)}$ defines a weak specification setting

$$\sigma = (v^{(1)}, \ldots, v^{(1)}_k, \ldots, v^{(s)}_k)$$

$$P(t_{j,k+1}) = x_{v(i)}.$$ 

By the finite weak specification property there is an orbit segment $z_0, \ldots, z_M$ such that

$$z_0 = x \quad \text{for some } x \in X,$$

$$M = m_1 + m_2 + \ldots + m_s + sk$$

$$M \leq s(k + L) \quad \text{and} \quad m_i \leq L \quad (1 \leq i \leq s)$$

$$z_{j+1} = w(z_j) \quad \text{for some } w \in \mathfrak{W}^{(1)} \quad 0 \leq j < M$$

$$z_{m_1+\ldots+m_i+(u-1)k+j+1} = v(z_{m_1+\ldots+m_i+(u-1)k+j+1}) \quad \text{for that } v \in \mathfrak{W}^{(0)}$$

$$d(z_{m_1+\ldots+m_i+(u-1)k+j+1}, P(t_{(u-1)k+j+1})) < \eta.$$ 

It follows that there is a map $\rho : (\mathfrak{W}^{(0)}_k)^s \rightarrow \bigcup_{s(k+L)}^{s(1)} (\mathfrak{W}^{(1)}_k)^M$ sending a weak specification to the associated $w$ as defined above. This map is clearly injective.

We now use the fact that for the pair $(z, w), z = (z_0, \ldots, z_M), w = (w_1, \ldots, w_{M-1})$, as above

$$\phi_w(z_0) = \prod_{j=0}^{M} \phi_{w,j}(w_{j-1} \ldots w_1(z_0))$$

$$\geq \left( \inf_{v \in \mathfrak{W}^{(1)}} \inf_{y \in D(v)} \phi_v(y) \right)^{M - sk} \prod_{i=1}^{s} \phi_{v(i)}(z_{m_1+\ldots+m_i+(i-1)k+1})$$

$$\geq \left( \inf_{v \in \mathfrak{W}^{(1)}} \inf_{y \in D(v)} \phi_v(y) \right)^{M - sk} \prod_{i=1}^{s} \phi_{v(i)}(x_{v(i)}) e^{-sk\epsilon}.$$ 

It follows that

$$\sum_{M=sk+1}^{s(k+L)} \sum_{w \in \mathfrak{W}^{(1)}_k} \sup_{x \in D(w)} \phi_w(x)$$

$$\geq \sum_{v=(v^{(1)}, \ldots, v^{(s)}) \in (\mathfrak{W}^{(0)}_k)^s} \prod_{i=1}^{s} \phi_{v(i)}(x_{v(i)}) \left( \inf_{w \in \mathfrak{W}^{(1)}} \inf_{y \in D(w)} \phi_u(y) \right)^{sL} e^{-sk\epsilon}.$$ 

Let $\gamma = \inf_{v \in \mathfrak{W}^{(1)}} \inf_{y \in D(v)} \phi_u(y)$ (we may assume that $\gamma < 1$). Taking the maximum on the right hand side we arrive at

$$sL \max_{sk+1 \leq M \leq s(k+L)} Z_M(\phi, \mathfrak{W}^{(1)}) \geq \gamma^{sL} e^{-sk\epsilon} \left( \sum_{v \in \mathfrak{W}^{(0)}_k} \phi_v(x_v) \right)^s$$

$$\geq \gamma^{sL} e^{-sk\epsilon} \left( \exp[k(P_{alg}(\phi) - \epsilon)] - \delta \mathfrak{W}^{(1)}_k \right)^s.$$
Since $\delta > 0$ is independent of all other quantities
\[ sL \max_{sk+1 \leq M \leq (k+L)} Z_M(\phi, \mathcal{V}^{(1)}) \geq \gamma^{sk}e^{-s\kappa} \exp[sk(P_{alg}(\phi) - \epsilon)]. \]

Finally, if the maximum is attained for $M = sk + a$
\[ \frac{1}{sk + a} \log Z_{sk+a}(\phi, \mathcal{V}^{(1)}) + \frac{1}{sk + a} \log(sL) \geq \frac{sk}{sk + a} P_{alg}(\Phi) - \frac{2sk}{sk + a} \epsilon + \frac{sL}{sk + a} \log \gamma. \]
Taking the limit as $s \to \infty$ we obtain using Lemma 5.1
\[ P_{alg}(\phi, \mathcal{V}^{(1)}) \geq P_{alg}(\phi) - 2\epsilon - \frac{L}{k} \log \gamma. \]
Note that $L$ only depends on $\eta$ and not on $k$, hence letting $k \to \infty$ and $\epsilon \to 0$ ends the proof. \hfill \Box

**Theorem 5.3.** If $(X, \mathcal{V})$ has the finite weak specification property, then
\[ h_{top}(X, \mathcal{V}) = \sup \{ h_{top}(X, \mathcal{V}^{(0)}) : \mathcal{V}^{(0)} \subset \mathcal{V} \text{ is finite} \}. \] (4)

The proof of this theorem is similar to the previous proof and therefore omitted.

6. **Existence of iterated function systems.** Let $(X, \mathcal{V})$ be a family of partially defined homeomorphisms $v : D(v) \to X$ where $D(v) \subset X$ is a closed subset; it is called a function system in [4]. The notions of specification can be defined for such systems in analogy with those for iterated function systems.

**Lemma 6.1.** Let $(X, \mathcal{V})$ be a function system which has the finite weak specification property and $X$ be compact. Then there is a subfamily $\mathcal{V}^{(1)} \subset \mathcal{V}$ and $\emptyset \neq E \subset X$ such that each $x \in E$ has an infinite orbit in $\mathcal{V}^{(1)}$.

**Proof.** Fix $v_* \in \mathcal{V}$ and let $\mathcal{V}^{(0)} = \{v_*\}$. For $\epsilon = 1$ take the spacing constant $L = L(\epsilon) > 0$ and $\mathcal{V}^{(1)}$ by the finite weak specification property. Define
\[ A_1 = \{ x \in \bigcup_{v \in \mathcal{V}^{(1)}} D(v) : \exists v \in \mathcal{V}^{(1)} \text{ such that } v(x) \in \bigcup_{v \in \mathcal{V}^{(1)}} D(v) \} \]
and by induction, define
\[ A_{n+1} := \{ x \in A_n : \exists v \in \mathcal{V}^{(1)} \text{ such that } v(x) \in A_n \} = \{ x \in X : \exists w \in (\mathcal{V}^{(1)})_{n+1} \text{ such that } x \in D(w) \}. \]
Then for each $n \in \mathbb{N}$ we have that $A_n$ is nonempty and closed. In fact, if $x \in D(v_*)$ we can define the following weak specification: $\tau = \{I_1, I_2, ..., I_n\}$, where $I_i = \{i\}$, $i \in \{1, ..., n\}$,
\[ P : \{1, 2, ..., n\} \to X \quad P(i) = x \]
and $\sigma = (v_*, v_*, v_*, ...).$ Then there is a specification at most $L$-spaced and $x_0 \in X$ such that $x_0$ shadows this specification within $\mathcal{V}^{(1)}$. Then, $x_0 \in A_n$. If $x_m \to x_0$ with $x_m \in A_n$, then there is $w_m \in \mathcal{V}^{(1)}_{m}$, such that $x_m \in D(w_m)$. As $\mathcal{V}^{(1)}$ is finite, there are a subsequence $m_k$ and $w \in \mathcal{V}^{(1)}_n$ such that $x_{m_k} \in D(w)$. Therefore, $x_0 \in D(w)$ and $A_n$ is closed.

Then, we have a nested sequence $\{A_n\}$ of compact and nonempty sets. Therefore, $E = \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$
If \( x \in \bigcup_{v \in \mathcal{V}(1)} v(D(v) \cap E) \) there exists \( a \in E \) and \( v \in \mathcal{V}(1) \) such that \( v(a) = x \). Then \( a \in A_n \) for all \( n \in \mathbb{N} \), therefore, \( v(a) \in A_{n+1} \) for all \( n \in \mathbb{N} \). As \( A_1 \subset A_2 \) \( v(a) \in E \subset \bigcup_{v \in \mathcal{V}(1)} D(v) \cap E \). As \( x \) is arbitrary, \( \bigcup_{v \in \mathcal{V}(1)} v(D(v) \cap E) \subset \bigcup_{v \in \mathcal{V}(1)} D(v) \subset E. \)

**Theorem 6.2.** Let \((X, \mathcal{V})\) be a function system which has the specification property and let \( X \) be compact. Then there is a nonempty closed subset \( X_0 \subset X \) and a subfamily \( \mathcal{V}(0) \subset \mathcal{V} \) such that \((X_0, \mathcal{V}(0))\) is an iterated function system.

**Proof.** We use Theorem 2.5 in [4]. There exists a pair \((\bar{K}, K)\) such that \( \emptyset \neq K, \bar{K} \subset K \), the pair is minimal in the sense that there is no smaller \( K \) and no larger \( \bar{K} \subset K \), and if \( \bar{K} \neq \emptyset \) then the claim holds. Hence we need to show that we can produce a pair with nonempty \( \bar{K} \).

Fix \( x_0 \in X \) and \( \eta > 0 \). By specification there are sequences \( w_n \in \mathcal{V}_n \) and \( x_n \in X \) such that \( w_{n+1} = vw_n \) with \( v \in \mathcal{V} \) and \( x_n \in D(w_1) \) such that
\[
d(w_n(x_{n+1}), w_n(x_n)) < \eta.
\]
In particular, \( w_n \) is a well defined map defined on \( \emptyset \neq D(w_n) \cap D(w_{n+1}) \). Therefore
\[
\bigcap_{n=1}^{\infty} D(w_n) \neq \emptyset
\]
by compactness and any point in this intersection permits an infinite orbit.

Now we can repeat the proof of Theorem 2.5 in [4] to finish the proof. \qed

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