Finite-Size Effects and Critical Behavior of the Deconfinement Phase Transition

M. Ladrem, A. Ait-El-Djoudi and G. Yezza

Laboratoire de Physique des Particules et Physique Statistique. Department of Physics, Ecole Normale Supérieure-Kouba, BP92, Vieux-Kouba, 16050, Algiers, Algeria.

Abstract

We study the finite-size effects on the deconfinement phase transition (DPT) of hot and / or dense hadronic matter, using a simple thermodynamic model based on the assumption of coexistence of confined and deconfined phases in a finite volume, with the two-phases matter Equations of State (EoS). For the QGP, we consider a partition function (PF) with the exact color-singletness requirement. A problem arises in the limit of small QGP volumes when using the usual color-singlet partition function (CSPF) derived in the saddle point approximation. To avoid this problem, we have then proposed a method for calculating a suitable CSPF which allows us to accurately calculate physical quantities describing well the DPT at finite volumes like the order parameter, the energy density and the entropy density. We show that in the limit of infinite volume, these thermodynamic quantities exhibit a discontinuity at a critical temperature $T_c(\infty)$ if the transition is temperature driven, or chemical potential $\mu_c(\infty)$ if it is density driven. In a finite size system, all singularities are smoothed out over a broadened critical region, shifted from the critical point position in the thermodynamic limit. At the level of the first derivatives of these thermodynamic quantities, the specific heat and the susceptibility show delta function singularities at the critical point in the thermodynamic limit. In a finite volume, these delta singularities are smeared into finite peaks of widths $\delta T(V)$ or $\delta \mu(V)$, with the maxima of the peaks occurring at pseudo-critical points $T_c(V)$ or $\mu_c(V)$. For the temperature driven DPT, an analysis of the finite size scaling behavior at criticality of these maxima as well as of the width of the transition region and the shift of the transition temperature allows us to determine the critical exponents characterizing the deconfinement phase transition. The obtained results are in good agreement with those predicted by other studies for a first-order phase transition.
I. INTRODUCTION

Phase transitions in statistical physics are known to be infinitely sharp only in the thermodynamic limit ($V \rightarrow \infty$). Only in this limit is the thermodynamical potential or any of its derivatives singular at the critical point. However, real systems and the systems we simulate are finite. For instance, the reaction zone in which the formation of a quark-gluon plasma (QGP) in a relativistic heavy ion collision is assumed to take place is limited, with a width of about 1 fermi, and a transverse area of a radius comparable to the radii of the colliding nuclei, i.e., $V \simeq A^{2/3} \text{fm}^3$. Also, lattice QCD calculations are necessarily performed in finite lattices.

In general, finite size effects lead to a mixed phases system and a rounding of the transition. The main care is then how to sign a possible phase transition in a finite system. It turns out that we can extract the true critical behavior of infinite systems from calculations on finite systems, by studying how some characteristic thermodynamical quantities vary with the size of the system, namely by a Finite Size Scaling (FSS) analysis. Studies in statistical physics have shown that despite the apparent diversity in the underlying structure of systems undergoing phase transitions, these latter take place with some universal global behavior, depending only on the range of interaction of the forces at play and the dimensionality of the problem. This means the singular behavior of some characteristic observables near criticality is identical for many systems when appropriately scaled. This universal behavior of finite systems at criticality is characterized by indices called critical exponents. The determination of critical exponents has long been one of the main interests for both analytical calculations and numerical simulations.

In the present work, we study the finite size effects on the deconfinement phase transition (DPT) when temperature driven and when density driven, within a simple thermodynamical model used in [1], based on the assumption of the coexistence of confined and deconfined phases in a finite volume, and using the equations of state (EoS) of the two phases. For the QGP, we consider a partition function with the exact color-singletness requirement. This latter is usually derived using the saddle point approximation, and gaussian approximation which break down at $VT^3 < < 1$ [2, 3, 4]. To avoid the problem at small temperature and / or volumes, we have then proposed a method for calculating a suitable CSPF, expanded in a power series of the QGP volume fraction, which allows us to accurately calculate physical
quantities describing well the DPT at finite volumes like the order parameter, the energy density and the entropy density. We show that in the limit of infinite volume, these thermodynamic quantities exhibit a discontinuity at a critical temperature $T_c(\infty)$ for a temperature driven DPT. In a finite size system, no singularity occurs and the variations of these quantities are perfectly smooth over a broadened critical region, shifted from the critical point position in the thermodynamic limit.

At the level of the first derivatives of these thermodynamic quantities, the specific heat and the susceptibility show delta function singularities at the critical temperature in the thermodynamic limit. In finite volumes, these delta singularities are smeared into finite peaks of widths $\delta T(V)$, with the maxima of the peaks occurring at pseudo-critical temperatures $T_c(V)$. An analysis of the finite size scaling behavior at criticality of these maxima as well as of the width of the transition region and the shift of the effective transition temperature $T_c(V)$ relative to the true one $T_c(\infty)$ allows us to determine the critical exponents characterizing the deconfinement phase transition.

II. COLOR-SINGLET PARTITION FUNCTION OF THE QGP

The partition function for a color-singlet quark-gluon plasma contained in a volume $V_{QGP}$, at temperature $T$ and quark chemical potential $\mu$, is determined by [2, 3, 4]:

$$Z(T, V_{QGP}, \mu) = \frac{8}{3\pi^2} e^{-\frac{BV_{QGP}}{T}} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} d\left(\frac{\varphi}{2}\right) d\left(\frac{\psi}{3}\right) M(\varphi, \psi) \tilde{Z}(T, V_{QGP}, \mu; \varphi, \psi),$$

where $M(\varphi, \psi)$ is the weight function (Haar measure) given by:

$$M(\varphi, \psi) = \left(\sin\left(\frac{1}{2}(\psi + \varphi)\right) \sin\left(\frac{\varphi}{2}\right) \sin\left(\frac{1}{2}(\psi - \varphi)\right)\right)^2,$$

and $\tilde{Z}$ the generating function defined by:

$$\tilde{Z}(T, V_{QGP}, \mu; \varphi, \psi) = Tr \left[ \exp \left( -\beta \left( \hat{H}_0 - \mu \left( \hat{N}_q - \hat{N}_\bar{q} \right) \right) + i\varphi \hat{I}_3 + i\psi \hat{Y}_8 \right) \right]$$

where $\beta = \frac{1}{T}$ with the units chosen as: $k_B = h = c = 1$, $\hat{H}_0$ is the free quark-gluon Hamiltonian, $\hat{N}_q$ ($\hat{N}_\bar{q}$) denotes the (anti-) quark number operator, and $\hat{I}_3$ and $\hat{Y}_8$ are the color “isospin” and “hypercharge” operators respectively.
The generating function \( \tilde{Z} (T, V_{\text{QGP}}, \mu; \varphi, \psi) \) can be factorized into the quark contribution and the glue contribution as:

\[
\tilde{Z} (T, V_{\text{QGP}}, \mu; \varphi, \psi) = \tilde{Z}_{\text{quark}} (T, V_{\text{QGP}}, \mu; \varphi, \psi) \tilde{Z}_{\text{glue}} (T, V_{\text{QGP}}; \varphi, \psi),
\]

where the quark contribution is given by:

\[
\tilde{Z}_{\text{quark}} (T, V_{\text{QGP}}, \mu; \varphi, \psi) = \exp \left[ \frac{\pi^2}{12} T^3 V_{\text{QGP}} d_Q \sum_{q=r,b} \left( \frac{7}{30} - \left( \frac{\alpha_q - i \frac{\mu}{T}}{\pi} \right)^2 + \frac{1}{2} \left( \frac{\alpha_q - i \frac{\mu}{T}}{\pi} \right)^4 \right) \right],
\]

with \( q = r, b \) the color indices, \( d_Q = 2N_f \) counts the spin-isospin degeneracy of quarks and the angles \( \alpha_q \) being determined by the eigenvalues of the color charge operators in eq. (3):

\[
\begin{align*}
\alpha_r &= \frac{\varphi}{2} + \frac{\psi}{3}, \\
\alpha_g &= -\frac{\varphi}{2} + \frac{\psi}{3}, \\
\alpha_b &= -\frac{2\psi}{3},
\end{align*}
\]

and the glue contribution is given by:

\[
\tilde{Z}_{\text{glue}} (T, V_{\text{QGP}}; \varphi, \psi) = \exp \left[ \frac{\pi^2}{12} T^3 V_{\text{QGP}} d_G \sum_{g=1}^{4} \left( -\frac{7}{30} + \left( \frac{\alpha_g - \pi}{\pi} \right)^2 - \frac{1}{2} \left( \frac{\alpha_g - \pi}{\pi} \right)^4 \right) \right],
\]

with \( d_G = 2 \) the degeneracy factor of gluons and \( \alpha_g \) (\( g = 1, \ldots, 4 \)) being:

\[
\alpha_1 = \alpha_r - \alpha_g, \quad \alpha_2 = \alpha_g - \alpha_b, \quad \alpha_3 = \alpha_b - \alpha_r, \quad \alpha_4 = 0.
\]

Let us write the generating function as:

\[
\tilde{Z} (T, V_{\text{QGP}}, \mu; \varphi, \psi) = e^{V_{\text{QGP}} T^3 g(\varphi, \psi, \mu)},
\]

with:

\[
g(\varphi, \psi, \mu) = \frac{\pi^2}{12} \left( \frac{21}{30} d_Q + \frac{16}{15} d_G \right) + \frac{\pi^2 d_Q}{12} \sum_{g=r,b} \left( -1 + \left( \frac{\alpha_g - i \frac{\mu}{T}}{\pi} \right)^2 \right) \sum_{q=r,b} \left( \frac{\alpha_q - \frac{\mu}{T}}{\pi} \right)^2
\]

\[
- \frac{\pi^2 d_G}{12} \sum_{g=1}^{4} \left( \frac{\alpha_g - \pi}{\pi} \right)^2 - 1 \right)^2,
\]

then eq. (1) can be written on the form:

\[
Z(T, \mu, V_{\text{QGP}}) = \frac{8}{3\pi^2} e^{xV(T^3 g_0(\mu) - \frac{\mu}{T})} \int_{-\pi}^{+\pi} d(\varphi/2) d(\psi/3) M(\varphi, \psi) e^{g(\varphi, \psi, \mu) - g_0(\mu)} x VT^3,
\]
where \( g_0 \left( \frac{\mu}{T} \right) \) is the maximum of \( g(\varphi, \psi, \frac{\mu}{T}) \) for \( \varphi, \psi \in [-\pi, +\pi] \), given as:

\[
g_0 \left( \frac{\mu}{T} \right) = \frac{\pi^2}{12} \left( \frac{21}{30} d_g + \frac{16}{15} d_g \right) + \frac{\pi^2 d_g}{12} \left( \frac{3\mu^2}{\pi^2 T^2} + \frac{3\mu^4}{2\pi^4 T^4} \right),
\]

(12)

and \( x \) the QGP volume fraction defined by: \( x = 1 - h \), since assuming the phases coexistence model, the hadronic gas and QGP phases occupy fractional volumes \( V_{HG} = h V \) and \( V_{QGP} = (1 - h) V \), characterized by the parameter \( h \) lying in the range \( 0 - 1 \), where \( h = 1 \) corresponds to a pure hadron phase and \( h = 0 \) to a pure QGP phase.

When \( VT^3 \gg 1 \), we can evaluate the integral in eq. (11) by the saddle point approximation around the maximum of the integrand at \((0, 0)\). After some calculation, eq. (11) can be put on the form [3]:

\[
Z(T, \mu, Vx) \approx 4 e^{xV(T^3g_0(\frac{\mu}{T}) - \frac{\mu}{T})} \frac{e^{VT^3a(\frac{\mu}{T})}}{9\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} d\varphi d\psi M^{(0,0)}(\varphi, \psi)e^{xg^{(0,0)}(\varphi, \psi; VT^3, \#)};
\]

(13)

with: \( a(\#) = \left( \frac{d_g}{16} \left( \frac{3\mu^2}{\pi^2 T^2} + 1 \right) + \frac{3}{8} d_g \right) \), \( M^{(0,0)} \) and \( g^{(0,0)} \) the expansions of \( M \) and \( (g - g_0) \) respectively, around \((\varphi, \psi) = (0, 0)\), given by:

\[
\begin{align*}
M^{(0,0)}(\varphi, \psi) &= \frac{1}{64}\varphi^2 \left( \psi^2 - \frac{\varphi^2}{4} \right)^2 \\
g^{(0,0)}(\varphi, \psi; VT^3, \frac{\mu}{T}) &= -\frac{2}{3} \left( \varphi^2 + \frac{4}{3} \psi^2 \right) + \frac{1}{\pi (a(\frac{\mu}{T}))^{3/2} \sqrt{VT^3}} \left( \frac{\varphi^3}{4} - \varphi^2 \right) \\
&\quad - \frac{7}{12\pi^2 (a(\frac{\mu}{T}))^2 VT^3} \left( \frac{\varphi^4}{8} + \frac{2\psi^4}{9} + \frac{\varphi^2 \psi^2}{3} \right).
\end{align*}
\]

(14)

Let us write the exponential in the integral of eq. (13) for \( VT^3 \gg 1 \) as:

\[
e^{xg^{(0,0)}(\varphi, \psi; VT^3, \#)} = e^{-\frac{x}{2}(\varphi^2 + \frac{4}{3} \psi^2)} \left( 1 + \frac{x}{\pi (a(\frac{\mu}{T}))^{3/2} \sqrt{VT^3}} \left( \frac{\varphi^3}{4} - \varphi^2 \right) \\
&\quad - \frac{7x}{12\pi^2 (a(\frac{\mu}{T}))^2 VT^3} \left( \frac{\varphi^4}{8} + \frac{2\psi^4}{9} + \frac{\varphi^2 \psi^2}{3} \right) \right),
\]

(15)

and do the expansion:

\[
e^{-\frac{x}{2}(\varphi^2 + \frac{4}{3} \psi^2)} = \sum_{j=0}^{\infty} \left( -\frac{2}{3} \left( \varphi^2 + \frac{4}{3} \psi^2 \right) \right)^j \frac{x^j}{j!};
\]

(16)

then we obtain for the CSPF of the QGP:

\[
Z_{QGP}(T, \mu, Vx) = \frac{4}{9\pi^2} \left( \frac{e^{VT^3a(\frac{\mu}{T})}}{a(\frac{\mu}{T}) VT^3} \right)^4 \left( \sum_{j=0}^{\infty} \alpha_j x^j - \frac{7}{12\pi^2 (a(\frac{\mu}{T}))^2 VT^3} \sum_{j=0}^{\infty} \beta_j x^{j+1} \right),
\]

(17)
such as the coefficients $\alpha_j$ and $\beta_j$ of the two series are given by:

$$
\begin{align*}
\alpha_j &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\varphi d\psi M^{(0,0)}(\varphi, \psi) \left( -\frac{2}{3} \left( \varphi^2 + \frac{4}{3} \psi^2 \right) \right)^j \frac{1}{j!}, \\
\beta_j &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\varphi d\psi M^{(0,0)}(\varphi, \psi) \left( \varphi^4 + \frac{2}{9} \psi^4 + \frac{\varphi^2 \psi^2}{3} \right) \frac{1}{j!} \left( -\frac{2}{3} \left( \varphi^2 + \frac{4}{3} \psi^2 \right) \right)^j.
\end{align*}
$$

(18)

It can easily be shown that the series in eq. (17) are convergent, and can be truncated at a rank $j$ above which the contribution of the terms $\alpha_j x^j$ to the sums is negligible.

III. FINITE SIZE EFFECTS

To study the effects of volume finiteness on the DPT within the phases coexistence model (PCM), let us examine the behavior of some thermodynamic quantities in the vicinity of the critical point. The quantities of interest are the order parameter, which is simply in this case the hadronic volume fraction, the energy density and the entropy density. The mean value of any intensive thermodynamic quantity of the system can be calculated by:

$$
\langle A(T, \mu, V) \rangle = \frac{\int_0^1 ((1 - h) A_Q ((1 - h)V) + h A_H (hV)) Z (h) \, dh}{\int_0^1 Z (h) \, dh},
$$

(19)

where $A_{QGP} (h)$ and $A_{HG} (h)$ are contributions relative to the individual QGP and hadronic gas phases, respectively, and $Z (h)$ the total PF of the mixed system, which factorizes into the QGP and hadronic gas partition functions as: $Z (h) = Z_{QGP} (h) Z_{HG} (h)$.

For the hadronic gas phase, let us consider the partition function of a pion gas given by:

$$
Z_{HG} = e^{-\frac{\pi^2}{30} T^3 V_{HG}}.
$$

(20)

The mean value of the order parameter is then given by:

$$
< h(T, \mu, V) > = 1 - \frac{\int_0^1 dx \exp \left( \left( \left( \frac{\pi^2}{30} + g_0 \left( \frac{\mu}{T} \right) \right) T^3 - \frac{B}{T} \right) xV \right) \left( \frac{7}{12 \pi^2} \sum_{j=0}^{\infty} \alpha_j x^{j+2} - \sum_{j=0}^{\infty} \beta_j x^{j+1} \right)}{\int_0^1 dx \exp \left( \left( \left( \frac{\pi^2}{30} + g_0 \left( \frac{\mu}{T} \right) \right) T^3 - \frac{B}{T} \right) xV \right) \left( \sum_{j=0}^{\infty} \alpha_j x^j - \sum_{j=0}^{\infty} \beta_j x^{j+1} \right)}.
$$

(21)
and those of the energy and entropy densities can be written, respectively, as:

\[
\langle \varepsilon(T, \mu, V) \rangle = e_{HG} + e_0 + (e_{QGP} - e_{HG}) \langle x(T, \mu, V) \rangle \tag{22}
\]
\[
\langle s(T, \mu, V) \rangle = s_{HG} + s_0 + (s_{QGP} - s_{HG}) \langle x(T, \mu, V) \rangle, \tag{23}
\]

with:

\[
\begin{align*}
\langle x(T, \mu, V) \rangle &= 1 - \langle \eta(T, \mu, V) \rangle \\
e_{HG} &= \frac{\pi^2}{30} T^4, \quad e_0 = -12 \frac{T}{V}, \quad e_{QGP} = \frac{\pi^2}{3} \left( \frac{7N_f}{3} + \frac{32}{15} \right) T^4 - \frac{N_f}{3} \mu^4 + \frac{N_f}{2} T^2 \mu^2 + B \\
s_{HG} &= \frac{2\pi^2}{15} T^3, \quad s_0 = -\frac{12}{V} - \frac{2}{T} \ln \left( VT^3 a \left( \frac{\mu}{T} \right) \right), \quad s_{QGP} = \frac{\pi^2}{3} \left( \frac{7N_f}{3} + \frac{32}{15} \right) T^3 + N_f \mu^2 T.
\end{align*}
\tag{24}
\]

As it is known, the deconfined quark matter state can be obtained at extreme conditions of temperature and/or density, i.e., either by raising temperature or chemical potential. The deconfinement phase transition is then temperature driven or density driven.

We’ll examine in the following the FSE for a temperature driven DPT, at a vanishing chemical potential (\(\mu = 0\)), and for a density driven DPT at a fixed temperature (\(T = 100\, \text{MeV}\)), considering the two lightest quarks \(u\) and \(d\) (\(N_f = 2\)), and using \(B^{1/4} = 145\, \text{MeV}\) for the bag constant.

Fig.(1-a) illustrates the variations of the order parameter, the energy density normalized by \(T^4\), and the entropy density normalized by \(T^3\), versus temperature for different system sizes. The curves show that in the limit of an infinite volume, the three quantities exhibit a sharp discontinuity at a transition temperature \(T_c(\infty)\), reflecting the first order character of the transition. For small size systems, the transition is smoothed out over a range of temperature \(\delta T(V)\), around a pseudo-critical temperature \(T_c(V)\), shifted from the true transition temperature \(T_c(\infty)\). The broadening of the transition region, as well as the shift of the critical temperature get larger, smaller is the volume.

In the same way as for temperature, we can deal with the density driven phase transition, at a fixed temperature \((T = 100\, \text{MeV})\). The variations of the order parameter, the energy density and the entropy density versus chemical potential are presented for various system sizes on Fig. (1-b). The first-order character of the transition can, also in this case, clearly be seen at the large volumes limit. In small systems, the transition is perfectly smooth over a region of chemical potential of width \(\delta \mu(V)\), and the effective critical chemical potential \(\mu_c(V)\) is shifted away from the true one \(\mu_c(\infty)\).
IV. FINITE-SIZE SCALING ANALYSIS

A. Finite size scaling and critical exponents

In the thermodynamic limit, phase transitions are characterized by the appearance of singularities in some second derivatives of the thermodynamic potential, such as the susceptibility $\chi$ and the specific heat $c$. For a first order phase transition, the divergences are typically $\delta$-function singularities, corresponding to the discontinuities in the first derivatives of the thermodynamic potential.

For temperature driven phase transitions, in finite volumes, the singularities in $\chi(T, V)$ and $c(T, V)$ are rounded over a range of temperature $\delta T(V)$. The peaks occurring at a pseudo-critical temperature $T_c(V)$ may be shifted away from the true critical temperature $T_c(\infty)$. The width of the transition region, the shift of the critical temperature and the maxima of the peaks of the specific heat and the susceptibility, represented in Fig. (2), show a scaling behavior at criticality, characterized by critical exponents as:

\[
\begin{align*}
\delta T(V) & \sim V^{-\theta_T} \\
\tau_T(V) & = T_c(V) - T_c(\infty) \sim V^{-\lambda_T} \\
c^{\max}_T(V) & \sim V^{\alpha_T} \\
\chi^{\max}_T(V) & \sim V^{\gamma_T}.
\end{align*}
\]

(25)

The finite size scaling (FSS) analysis is used to recover the critical exponents, and it has been shown \[6\] that for a first order phase transition, the finite size quantities $\delta T(V)$, $\tau_T(V)$ scale as: $V^{-1}$ and the maxima $c^{\max}_T(V)$ and $\chi^{\max}_T(V)$ scale as: $V$. The critical exponents $\theta_T$, $\lambda_T$, $\alpha_T$ and $\gamma_T$ are then all equal to the dimensionality.

B. Numerical determination of the critical exponents

1. The susceptibility critical exponent

For the temperature driven phase transition, the susceptibility is defined to be the first derivative of the order parameter with respect to temperature, i.e.,

\[
\chi(T, V) = \frac{\partial \langle h(T, V) \rangle}{\partial T}.
\]

(26)
It is clear in this case from Fig. (3-left) that the delta function singularity of the susceptibility in the thermodynamic limit is smeared, in a finite volume, into a finite peak of width $\delta T(V)$, with the maximum of the peak $|\chi_T|^{\text{max}}(V)$ occurring at the pseudo-critical temperature $T_c(V)$. The plot of the maxima $|\chi_T|^{\text{max}}(V)$ versus volume is illustrated in Fig. (3-right), and the linearity of the data with $V$ can clearly be noted. A numerical parametrization with the power-law form: $|\chi_T|^{\text{max}}(V) \sim V^{\gamma_T}$, gives the value of the susceptibility critical exponent $\gamma_T$. The determination of the location of the maxima being done using a numerical method with an error on the temperature and on the maxima $|\chi_T|^{\text{max}}(V)$, this yields a systematic error on the exponent $\gamma_T$. The systematic error is given for all the critical exponents determined in the following. The result for the susceptibility critical exponent and the estimated systematic error are then: $\gamma_T = 0.99 \pm 0.04$.

2. The specific heat critical exponent

The specific heat density representing the smeared delta function of the latent heat is defined as:

$$c(T, V) = \frac{\partial \langle \varepsilon(T, V) \rangle}{\partial T},$$

(27)

$\varepsilon$ being the energy density of the system. The variations of the specific heat density with temperature are presented for different sizes in Fig. (4-left), which shows the rounding of the delta function singularity of $c(T, V)$ in finite systems into a finite peak of width $\delta T(V)$ and height $c_T^{\text{max}}(V)$. For decreasing volume, the width get larger while the height of the peak decreases. The data of the maxima of the specific heat $c_T^{\text{max}}(V)$ are fitted to the power-law form: $c_T^{\text{max}}(V) \sim V^{\alpha_T}$ in Fig. (4-right), and the obtained specific heat critical exponent is: $\alpha_T = 0.99 \pm 0.04$.

3. The shift critical exponent

For the study of the shift of the transition temperature $\tau_T(V) = T_c(V) - T_c(\infty)$, we need to have the effective transition temperature in a finite volume. It is usually defined to be the temperature at which the rounded peaks of the susceptibility and the specific heat reach their maxima.
Fig. (5) illustrates the results for the shift of the critical temperature plotted versus inverted volume, and shows the linear character of the variations. The shift critical exponent obtained from a fit to the form: $\tau_T(V) \sim V^{-\lambda_T}$, is: $\lambda_T = 1.0085 \pm 0.0009$.

4. The smearing critical exponent

The width of the transition region can be defined by the gap: $\delta T(V) = T_2(V) - T_1(V)$ with $T_1(V)$ and $T_2(V)$ the temperatures at which the second derivative of the order parameter reaches its maxima, or in other terms the temperatures at which the third derivative of the order parameter vanishes, i.e.,

$$\frac{\partial^3}{\partial T^3} \langle h(T,V) \rangle \bigg|_{T_1(V),T_2(V)} = 0. \quad (28)$$

Fig. (6-left) illustrates the variations of the second derivative of the order parameter with temperature for various sizes, and shows that the gap between the two extrema, which represents the broadening of the transition region, decreases with increasing volume. The results for the widths $\delta T(V)$, plotted in Fig. (6-right) vs volume, were fitted to the power law form: $\delta T(V) \sim V^{-\theta_T}$, and the obtained smearing critical exponent is: $\theta_T = 1.016 \pm 0.007$.

V. CONCLUSION

Our work has shown the influence of the finiteness of the system size on the behavior of thermodynamical quantities near criticality. The sharp transition observed in the thermodynamic limit, signaled by discontinuities in the first derivatives of the thermodynamic potential at a critical temperature or chemical potential, is rounded off in finite volume, and the variations of the thermodynamic quantities are perfectly smooth on the whole range of temperature or chemical potential. For the temperature driven DPT, a finite-size scaling analysis of the behavior of the width of the transition region $\delta T(V)$, the shift of the pseudo-critical temperature relative to the true one $\tau_T(V) = T_c(V) - T_c(\infty)$, and the maxima of the rounded peaks of the susceptibility $\chi_T^{\max}(V)$ and the specific heat $c_T^{\max}(V)$ near criticality,
shows their power-law variations with the volume:

\[
\begin{align*}
\delta T (V) & \sim V^{-\theta_T} \\
\tau_T (V) & = T_c (V) - T_c (\infty) \sim V^{-\lambda_T} \\
c_T^{\text{max}} (V) & \sim V^{\alpha_T} \\
\chi_T^{\text{max}} (V) & \sim V^{\gamma_T}
\end{align*}
\]

characterized by the scaling critical exponents $\theta_T$, $\lambda_T$, $\alpha_T$ and $\gamma_T$. Numerical results for these critical exponents have been obtained, and the associated systematic errors, resulting from the numerical method of calculating the maxima $\chi_T^{\text{max}} (V)$ and $c_T^{\text{max}} (V)$ and their localizations as well as the gap $\delta T (V) = T_2 (V) - T_1 (V)$, have been estimated. Our results for the critical exponents are in good agreement with the analytical ones: $\theta_T = \lambda_T = \alpha_T = \gamma_T = 1$ found in a parallel work [4]. These results are characteristic of the first order phase transition, as predicted by the FSS theory.

A further FSS analysis is viewed using an exact color-singlet partition function of the QGP, derived without any approximation, with an appropriate numerical treatment. Also, the density driven DPT can be studied, and similar critical exponents associated to the shift of the critical chemical potential $[\mu_c (V) - \mu_c (\infty)]$, the width of the transition region $\delta \mu (V)$ and the maxima of the rounded peaks of the susceptibility $\chi_\mu^{\text{max}} (V)$ and the specific heat $c_\mu^{\text{max}} (V)$ (relative to the derivatives with respect to $\mu$) can be determined.

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Figure 1: Plots of (a) order parameter (bottom), energy density normalized by $T^4$ (middle) and entropy density normalized by $T^3$ (top) versus temperature at $\mu = 0$, and of (b) order parameter (bottom), energy density (middle) and entropy density (top) versus chemical potential at $T = 100\text{MeV}$, for different system sizes.
Figure 2: Illustration of the critical behavior of the susceptibility $\chi(T, V)$, the specific heat $c(T, V)$ and the second derivative of the order parameter $\partial^2 \chi / \partial T^2$. 
Figure 3: (Left) Susceptibility $\chi(T, V)$ as a function of temperature for varying volume, and (Right) linear fit of the results for the maxima of the susceptibility vs volume.
Figure 4: Variations of (left) the specific heat density with temperature for different system volumes and of (right) the maxima of the specific heat density with the volume.
\[ \tau_T(V) \propto V^{\lambda_T} \]

\[ \lambda_T = 1.0085 \pm 0.0009 \]

Figure 5: Plot of the shift of the critical temperature vs inversed volume.
Figure 6: (Left) Illustration of the second derivative of the order parameter \( \frac{\partial^2 \langle h \rangle \langle h \rangle}{\partial T^2} \) vs temperature, taken at different volumes. (Right) Data of the width of the temperature region over which the transition is smeared, fitted to a power-law of the inversed volume.