DUALIZABLE LINK HOMOLOGY

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Abstract. We modify our previous construction of link homology in order to include a natural duality functor \( \mathcal{F} \). To a link \( L \) we associate a triply-graded module \( H_{X,Y}^p(L) \) over the graded polynomial ring \( R(L) = \mathbb{C}[x_1, y_1, \ldots, x_\ell, y_\ell] \). The module has an involution \( \mathcal{F} \) that intertwines the Fourier transform on \( R(L) \), \( \mathcal{F}(x_i) = y_i, \mathcal{F}(y_i) = x_i \). In the case when \( \ell = 1 \) the module is free over \( R(L) \) and specialization to \( x = y = 0 \) matches with the triply-graded knot homology previously constructed by the authors. Thus we show that the corresponding super-polynomial satisfies the categorical version of \( q \rightarrow 1/q \) symmetry.

We also construct an isotopy invariant of the closure of a dichromatic braid and relate this invariant to \( H_{X,Y}^p(L) \).

1. Introduction

It is easy to see from the skein relations \cite{Jon87} that the HOMFLY-PT polynomial of a knot \( P_K(a, q) \in \mathbb{Z}[a, q^{\pm 1}] \) has a symmetry: \( P_K(a, q) = P_K(a, -1/q) \). In this paper we prove the conjecture that this symmetry lifts to the link homology.

Currently there are two triply-graded knot homologies \cite{KR08}, \cite{OR18d} whose doubly-graded Euler characteristic equals the HOMFLY-PT polynomial. It is expected that these homologies are equivalent, although this has not been proven yet. In this paper we study the knot homology of \cite{OR18d}.

The double-graded Poincare polynomial of the link homology \( P_K(a, q, t) \) is simply called the super-polynomial. By its definition, its specialization at \( t = -1 \) is the HOMFLY-PT polynomial: \( P_K(a, q, -1) = P_K(a, q) \). We prove that similar to the HOMFLY-PT polynomial, the super-polynomial is also palindromic.

Theorem 1.0.1. The super-polynomial of any knot \( K \) is palindromic:

\[
P_K(a, q, t) = P_K(a, tq^{-1}, t).
\]

In fact we upgrade the knot homology to an object of the derived category with a natural notion of Fourier transform and show that the object is preserved by the transform. The theorem above follows after we apply the derived functor of global section. We give an outline of the result below.

For the triply-graded homology \cite{KR08} the statement of the theorem was stated in the work of Dunfield, Gukov and Rasmussen \cite{DGR06} as a conjecture with a lot of numerical support. The equivalence of the homology theories \cite{KR08} and \cite{OR18d} would imply the original conjecture in \cite{DGR06}.
1.1. **An object-valued link invariant and its symmetry.** The following construction was motivated by the constructions of Batson-Seed [BS15], Cautis-Kamnitzer [CK16] and Gorsky-Hogancamp [GH17] and by an observation in [KR16] that the link homology of an $\ell$-component link can be promoted to an object in the derived category of modules over $\mathbb{C}[x_1, \ldots, x_\ell]$. The homology theory developed in [GH17] is particularly close to our theory, see the last section of the paper.

As usual, all our categories and vector spaces are equivariant with respect to the torus $T_{q,t} = \mathbb{C}_q^* \times \mathbb{C}_t^*$. For a positive integer $n$ we denote $x_n = x_1, \ldots, x_n$ and $y_n = y_1, \ldots, y_n$. Their $T_{q,t}$ weight are

\[
\deg x_i = q^2, \quad \deg y_i = q^{-2}t^2.
\]

In this paper $\ell$ denotes the number of components of a link $L$, and we consider an associated algebra $R(L) = \mathbb{C}[x_\ell, y_\ell]$. The derived category of $T_{q,t}$-equivariant (2-periodic) $R(L)$-modules $D_{\text{per}}^{2}(R(L))$ has a Fourier endofunctor $\mathcal{F}$ which transposes the variables and changes the generators of $T_{q,t}$:

\[
\mathcal{F}(x_i) = y_i, \quad \mathcal{F}(y_i) = -x_i, \quad \mathcal{F}(q) = tq^{-1}, \quad \mathcal{F}(t) = t.
\]

To a closure $L(\beta)$ of a braid $\beta$ we associate an object $E(L(\beta))$ of the category $D_{\text{per}}^{2}(R(L))$. By modifying argument of [OR18d] we prove

**Theorem 1.1.1.** The object $E(L(\beta))$ is invariant under the Markov moves, thus representing an invariant $E(L)$ of an oriented link $L$.

In contrast to the link homology defined [OR18d], the invariant $E(L)$ is symmetric with respect to the Fourier involution:

**Theorem 1.1.2.** The object $E(L)$ is invariant with respect to the Fourier involution:

\[
\mathcal{F}(E(L)) \cong E(L).
\]

The Fourier transform discussed here is related to several other Fourier transforms. It is conjectured that the homology of the algebraic knot are equal to the homology of the Hilbert scheme of points on the corresponding singular curve [ORS18]. In this algebro-geometric context the Fourier transform manifests itself as Serre duality for the stable pairs on the curve (see [ORS18]). There is also conjectural relation between the homology of the torus links and the representations of the rational Cherednik algebras [GORS14]. In this setting the Fourier transform becomes the Fourier transform for modules over the rational Cherednik algebras.

It is also easy to see that the object $E(L)$ is invariant with respect to the simultaneous reversal of orientation of all components of $L$:

\[
E(\bar{L}) \cong E(L),
\]

where $\bar{L}$ is the link with reversed orientation.
1.2. Dualizable homology. If we apply the functor of global sections to our invariant we obtain dualizable homology:

$$H_{XY}(L(\beta)) = R\Gamma(E(L(\beta))).$$

Thus we see that the extended homology have palindromic property:

**Theorem 1.2.1.** For any link $L$ we have:

$$\dim_{q,t,a} H_{XY}(L) = \dim_{q/t,a} H_{XY}(L).$$

The relation between the extended homology $H_{XY}(L)$ and the triply-graded homology $H(L)$ from [OR18c] is subtle and explained in the next subsection but in the case of knots all technicalities evaporate and we are left with

**Theorem 1.2.2.** For any knot $K$ the module $H_{XY}(K)$ is free and finite over $R(K) = \mathbb{C}[x,y]$ and

$$H_{XY}(K)|_{y=0} = H(K).$$

In particular the $\mathbb{C}[x]$-module $H(K)$ is free and there is a finite dimensional triply-graded vector space $\overline{\Pi}(K)$ such that:

$$(1.3) \quad H(K) = \overline{\Pi}(K) \otimes \mathbb{C}[x].$$

The graded dimension of the vector space $\overline{\Pi}(K)$ is what we call super-polynomial $P_{xy}(K)$ of the knot $K$. Thus the theorem [1.0.1] follows immediately from the previous statements.

In future work [OR19] we plan to explore a version of the constructions from [OR18c] for the homology theory $H_{XY}$ developed here. In particular, the sheaves related to the homology $H_{XY}$ have less singular support compare to the sheaves constructed in [OR18c] thus we expect that it would be possible to find a connection between the localization formalism for $H_{XY}$ homology theory and the conjectural theory of projectors proposed in [GRN16].

1.3. Invariant of the closure of a dichromatic braid. Generalizing the construction from [OR18d] we obtain a homomorphism from the groupoid of two-colored braids into a special monoidal category of matrix factorizations:

$$\Phi_{\text{dic}} : \mathcal{B}_{n}^{\text{dic}} \rightarrow \overline{\mathcal{MF}}[\bullet, \bullet].$$

Using this homomorphism we construct the trace functor from this groupoid:

$$(1.4) \quad E : \mathcal{B}_{n; \ell}^{\text{dic}} \rightarrow D_{\text{per}}^{\text{per}}(\mathcal{C}^\ell)$$

where $\mathcal{B}_{n; \ell}$ is the set of closable dichromatic braids with $\ell$ connected components of the closure. We show that the trace is actually an isotopy invariant of the closure.

Suppose $L = L(\beta)$, $\beta \in \mathcal{B}_{n; \ell}$ is a two-colored link. A subset $\mathcal{C} \subset \mathcal{C}_\ell = \{1, 2, \ldots, \ell\}$ determines a coloring of each component $L_i$ of $L$ by either an ‘$x$-box’ (if $i \notin \mathcal{C}$) or a ‘$y$-box’ (if $i \in \mathcal{C}$). Define a corresponding $R(L(\beta))$-module $R(L(\beta))_\mathcal{C}$ as a quotient of $R(L(\beta))$ by $\ell$ conditions: for each $i$ set $y_i = 0$ if $i \notin \mathcal{C}$ and $x_i = 0$ if $i \in \mathcal{C}$. The module $R(L(\beta))_\mathcal{C}$ is a ring and $\text{Spec}(R(\beta))$ is $\mathbb{C}^\ell$ appearing in (1.4).
When the braid has only one color, \( C = \emptyset \), then the space of derived global sections of the trace is the old invariant from the paper \([OR18d]\). The object \( E(L(\beta)) \) determines all colored objects through the derived restriction:

**Theorem 1.3.1.** For any \( \beta \in \mathfrak{B}_{n}^{\text{dic}} \) we have

\[
E(L(\beta)) = E(L(\beta)) \otimes_{R(L(\beta))} R(L(\beta)).
\]

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2. **Dualizable category**

2.1. **Notations.** In this text we do not discuss induction and restriction functors, so we can fix the size of our matrices to be \( n \) and use the standard notations \( g, b, n \) for Lie algebras associated with the group \( G = \text{GL}_n \). We also use notation \( \Delta : \mathfrak{b} \to \mathbb{C}^n \) for the linear map that extract the diagonal of the matrix. The same notation is used for linear map \( \Delta : \mathfrak{b} \to \mathfrak{h} \) when we identify \( \mathbb{C}^n \) with \( \mathfrak{h} \) in the most natural way.

The other set of notations are borrowed from our previous papers. In particular, \( X_+ \) denote upper-triangular and strictly upper-triangular parts of the matrix \( X \), so that \( X = X_- + X_{++} = X_{--} + X_+ \). The symmetry group \( S_n \) acts on upper-triangular matrices by permuting their diagonal entries:

\[
\sigma(X)_{ij} = X_{ij} - \delta_{i,j}(X_{ii} - X_{\sigma(i),\sigma(i)}).
\]

In particular \((\sigma \cdot X)_{ii} = X_{\sigma(i),\sigma(i)}\)

2.2. **Long categories.** Our basic category is the monodromic version of the category of \([OR18d]\). The monodromic categories have advantage over the category from \([OR18d]\): the Fourier transform acts on them as endofunctor. The monodromic category has two equivalent presentations, related by the Knörrer periodicity: the long one and the short one. The long category has an obvious monoidal structure, but the action of the Fourier endofunctor is obscured. The short category has an obvious Fourier symmetry, but the convolution looks unnatural and its Fourier symmetry requires a special proof.

The long category is the category of \( G \times \hat{B} \times B \)-equivariant matrix factorizations over the space

\[
\mathcal{F} = \mathfrak{g} \times G \times \mathfrak{b} \times G \times \mathfrak{b},
\]

\[
\mathcal{F} = \{ \Delta(Y_1) = \Delta(Y_2) \},
\]

\[
(g, b_1, b_2) \cdot (X, g_1, Y_1, g_2, Y_2) = (\text{Ad}_g(X), g_1 \cdot b_1, \text{Ad}_b Y_1, g_2 \cdot b_2, \text{Ad}_b Y_2).
\]

For a pair of permutations \( \sigma, \tau \) we define a \( G \times B^2 \)-invariant potential on \( \mathcal{F} \):

\[
W_{\sigma,\tau}(X, g_1, Y_1, g_2, Y_2) = \text{Tr}(X(\text{Ad}_{g_1}(\sigma \cdot Y_1) - \text{Ad}_{g_2}(\tau \cdot Y_2))).
\]
This potential is invariant with respect to the first factor of $T_{qt} = \mathbb{C}_q^* \times \mathbb{C}_t^*$ and has weight 2 with respect to the second factor:

$$(\lambda, \mu) \cdot (X, g_1, Y_1, g_2, Y_2) = (\lambda X, g_1, \lambda^{-1} \mu^2 Y_1, g_2, \lambda^{-1} \mu^2 Y_2).$$

Thus the long category is the category of $G \times B^2$ matrix factorization which have two-periodic differentials of $\mathbb{C}_q^*$ degree 1:

$$\text{MF}_{\sigma, \tau} := \text{MF}_{G \times B^2}(\mathcal{Z}_q, W_{\sigma, \tau}).$$

The convolution space $\mathcal{Z}_3 = g \times (G \times b)^3$ has three $G \times B^3$-equivariant projection $\pi_{ij} : \mathcal{Z}_3 \to \mathcal{Z}_2$ and we can define an associative convolution product between the categories:

$$\star : \text{MF}_{\sigma, \tau} \times \text{MF}_{\rho, \tau} \to \text{MF}_{\sigma, \rho},$$

(2.2)

$$F \star G := \pi_{134} (\text{CE}_{q}(\phi)(\pi_{12}^\tau(G) \otimes \pi_{23}^\tau(G))^T).$$

The smaller space $X = g \times G \times n \times G \times n$ naturally embeds into the big space

$$\xymatrix{ i : \mathcal{Z} \ar[r] & \mathcal{Z}_3.}$$

The pull-back $i^*(W_{\sigma, \tau})$ is independent of $\sigma, \tau$ and we denote this potential $W$. The corresponding matrix factorization category:

$$\text{MF} := \text{MF}_{G \times B^2}(\mathcal{Z}, W)$$

was studied in [OR18d]. In particular, the pull-back morphism $i^* : \text{MF}_{\bullet, \bullet} \to \text{MF}$ intertwines the above defined convolution product with the product from [OR18d]. In [OR18d] we define a homomorphism from the braid group $\mathfrak{Br}_n$ to $(\text{MF}, \star)$. It turns out that we can extend this homomorphism to the category $\text{MF}_{\bullet, \bullet}$. It is easier to explain this homomorphism with the short (that is, Knörrer-reduced) category $\text{MF}_{\sigma, \tau}$ which we introduce in the next subsection.

2.3. Short category. We define the short category as a category of $B^2$-equivariant matrix factorizations:

$$\text{MF}_{\sigma, \tau} := \text{MF}_{B^2}(\mathcal{Z}, W_{\sigma, \tau}), \quad \mathcal{Z} = b \times G \times b,$$

(2.3)

$$W_{\sigma, \tau}(X, g, Y) = \text{Tr}(X(\sigma \cdot Y - \text{Ad}_g(\tau \cdot Y))).$$

This category is related to the big category from the previous section by the Knorrer functor and the trivial extension of $G$-action. The argument is parallel to the argument from [OR18d]. First we introduce the intermediate space $\mathcal{Z}^\circ$ which is the $G$-quotient of the space $\mathcal{Z}$:

$$\mathcal{Z}^\circ \subset g \times G \times b^2, \quad \mathcal{Z}^\circ = \{ \Delta(Y_1) = \Delta(Y_2) \},$$

$$W_{\sigma, \tau}^\circ(X, g, Y_1, Y_2) = \text{Tr}(X(\sigma \cdot Y_1 - \text{Ad}_g(\tau \cdot Y_2))).$$

Since the $G$-action on $\mathcal{Z}$ is free, the quotient map is the equivalence between the categories $\text{MF}_{\sigma, \tau}$ and

$$\text{MF}_{\sigma, \tau}^\circ = \text{MF}_{B^2}(\mathcal{Z}, W_{\sigma, \tau}^\circ).$$
Next we observe that we can write the potential as a sum of two terms with the second term being quadratic:

\[ W_{\sigma,\tau}(X, g, Y_1, Y_2) = \text{Tr}(X_+(\sigma \cdot Y_1 - \text{Ad}_g(\tau \cdot Y_2))) + \text{Tr}(X_-(Y_1)_{++} + \text{Ad}_g(Y_2)_{++}) \]

Since \( \Delta(Y_1) = \Delta(Y_2) \) the first term is equal to \( \overline{W}_{\sigma,\tau} \) and the second term is quadratic with respect to the coordinates \( X_+ \) and \( (Y_1)_{++} + \text{Ad}_g(Y_2)_{++} \). Thus we can apply the Knorrer periodicity functor to establish the isomorphism of categories:

\[ \text{MF}(\overline{\mathcal{P}}^\circ, W_{\sigma,\tau}) \simeq \text{MF}(\overline{\mathcal{P}}, \overline{W}_{\sigma,\tau}). \]

To upgrade this relation to the level of \( B^2 \)-equivariant matrix factorizations we need to follow the method of [OR18d]. In more details we need to define an auxiliary subspace \( \widetilde{\mathcal{P}} \) of \( \mathcal{P}^\circ \):

\[ \mathcal{P}^\circ = \mathfrak{b} \times G \times \mathfrak{b} \times G \times \mathfrak{b}, \quad j^\#: \widetilde{\mathcal{P}} \to \mathcal{P}^\circ. \]

The auxiliary space \( \widetilde{\mathcal{P}} \) projects to the space \( \mathcal{P}^\circ \):

\[ \pi_y : \widetilde{\mathcal{P}} \to \mathcal{P}^\circ, \quad \pi_y(X, g_1, Y_1, g_2, Y_2) = (X, g_1^{-1}g_2, Y_2). \]

There is a unique \( B^2 \)-equivariant structure on \( \widetilde{\mathcal{P}} \) that makes map \( \pi_y \)-equivariant. On the other hand the embedding \( j^\# \) have enough \( B^2 \)-equivariant properties to have well-defined push-forward functor

\[ j^\#: \text{MF}_{B^2}(\mathcal{P}^\circ, \pi_y(\overline{W}_{\sigma,\tau})) \to \text{MF}_{G \times B^2}(\mathcal{P}, W_{\sigma,\tau}). \]

Thus we can define the equivariant Knorrer functor as composition:

\[ \Phi : \text{MF}_{G \times B^2}(\mathcal{P}, W_{\sigma,\tau}) \to \text{MF}_{B^2}(\mathcal{P}, \overline{W}_{\sigma,\tau}), \quad \Phi = j^\# \circ \pi_y#. \]

The adjoint functor \( \Psi = \pi_y# \circ j^{**} \) is the left inverse of \( \Phi \):

\[ (2.4) \quad \Psi \circ \Phi = 1. \]

2.4. Short convolutions. The short category has two convolution structures \( \overset{\bullet}{\cdot}, \overset{\ast}{\cdot} \) which we will prove to be the same. Both convolutions use the same \( B^3 \)-equivariant convolution space

\[ \mathcal{P}_3 = \mathfrak{b} \times G^3 \times \mathfrak{b}, \quad (b_1, b_2, b_3) \cdot (X, g_{12}, g_{23}, Y) = (\text{Ad}_{b_1}(X), g_{12}, g_{23}, \text{Ad}_{b_3}(Y)). \]

First we define the convolution that is transported from the big category by the Knorrer functor \( \Phi \). We define the projection maps \( \pi_{ij,\rho} : \mathcal{P}_3 \to \mathcal{P}^\circ \):

\[ \pi_{12,y,\rho}(X, g_{12}, g_{23}, Y) = (X, g_{12}, \text{Ad}_{g_{23}}(\rho \cdot Y)_{++} + \Delta(Y)), \]

\[ \pi_{23,y,\rho}(X, g_{12}, g_{23}, Y) = (\text{Ad}_{g_{12}}^{-1}(X)_+, g_{23}, Y), \]

\[ \pi_{13,y,\rho}(X, g_{12}, g_{23}, Y) = (X, g_{12}g_{23}, Y). \]

The construction of the convolution \( \overset{\bullet}{\cdot} \) is related to the convolution structure \( \overset{\ast}{\cdot} \) from [OR18d] by the pull-back \( \overset{\ast}{\cdot} \). In particular, the main potential is the sum

\[ \overline{W}_{\sigma,\tau}(X, g, Y) = \overline{W} + \delta \overline{W}_{\sigma,\tau}, \]
where
\[ W = -\text{Tr}(X\text{Ad}_g(Y_{++})), \quad \delta W_{\sigma,\tau} = \text{Tr}(X\sigma \cdot \Delta(Y)) - \text{Tr}(X\text{Ad}_g(\tau \cdot Y)). \]

**Proposition 2.4.1.** For any \( \sigma, \tau, \rho \) we have
\[ \bar{\pi}_{12, y, \rho}(\delta W_{\sigma,\tau}) + \bar{\pi}_{23, y, \rho}(\delta W_{\tau,\rho}) = \bar{\pi}_{13, y, \rho}(\delta W_{\sigma,\tau}). \]

**Proof.** The pull-back along \( \bar{i} \) of this equation yields a relation between the pull-backs of the potential \( \bar{W} \) and this relation is proven in proposition 5.3 of [OR18d]. Thus we only need to show the relation between the pull-backs of \( \delta W_{\sigma,\tau} \), which is verified by a direct computation:

\[
\begin{align*}
\text{Tr}(X\sigma \cdot \Delta(Y)) - \text{Tr}(X\text{Ad}_{g_{12}}(\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))_{++} + \tau \cdot \Delta(Y))) &+ \text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\tau \cdot \Delta(Y)) - \text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)(\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))_{++} + \tau \cdot \Delta(Y))) \ + \\
\text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\tau \cdot \Delta(Y)) &- \text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))) = \text{Tr}(X\sigma \cdot \Delta(Y)) - \\
\text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))_{++}) &- \text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))) = \text{Tr}(X\sigma \cdot \Delta(Y)) - \\
\text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)(\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y)) - \text{Ad}_{g_{23}}(\rho \cdot \Delta(Y)))) &- \text{Tr}(\text{Ad}_{g_{12}}^{-1}(X)\text{Ad}_{g_{23}}(\rho \cdot \Delta(Y))) \\
&= \bar{\pi}_{13, y, \rho}(\delta W_{\sigma,\tau}).
\end{align*}
\]

Thus for two matrix factorizations \( F \in \text{MF}_{\sigma,\tau} \), \( G \in \text{MF}_{\tau,\rho} \), the tensor product of the pull-backs \( \bar{\pi}_{12, y, \rho}(F) \) and \( \bar{\pi}_{23, y, \rho}(G) \) is a matrix factorization with the potential \( \bar{\pi}_{13, y, \rho}(\delta W_{\sigma,\tau}) \).

The convolution space \( \mathcal{F}_3 = b \times G^2 \times n \) from section 5 of [OR18d] embeds naturally inside the enlarged convolution space \( \mathcal{F}_3 \): \( \bar{i} : \mathcal{F}_3 \to \mathcal{F}_2 \).

The morphisms \( \bar{\pi}_{ij} : \mathcal{F}_3 \to \mathcal{F}_2 \) are intertwined by \( \bar{i} \):
\[
\bar{i} \circ \bar{\pi}_{ij} = \bar{\pi}_{ij} \circ \bar{i}.
\]

The maps \( \bar{\pi}_{ij, y, \rho} \) are not \( B^3 \)-equivariant but in section 5.4 of [OR18d] we presented a construction for \( B^3 \)-equivariant enrichment of the tensor product of \( \bar{\pi}_{12}(F') \) and \( \bar{\pi}_{23}(G') \):
\[
\bar{\pi}_{12}^*(F') \otimes_B \bar{\pi}_{23}^*(G') \in \text{MF}_{B^3}((\mathcal{F}_3, \bar{\pi}_{13}(W))), \quad F', G' \in \text{MF}.
\]

This construction allows us to define a binary operation [OR18d] on \( \text{MF} \):
\[
F' \star G' = \bar{\pi}_{13*}(\text{CE}_{n^2}(\bar{\pi}_{12}^*(F') \otimes_B \bar{\pi}_{23}^*(G'))^{\tau(\Phi)}).
\]

An almost verbatim repetition of the construction from the section 5.4 of [OR18d] yields the desired extension of the construction for the maps \( \pi_{ij, y, \rho} \):

**Proposition 2.4.2.** There is a \( B^3 \)-equivariant structure on the tensor product \( \bar{\pi}_{12, y, \rho}(F) \otimes \bar{\pi}_{23, y, \rho}(G) \), \( F \in \text{MF}_{\sigma,\tau} \), \( G \in \text{MF}_{\tau,\rho} \), which we denote
\[
\bar{\pi}_{12, y, \rho}(\mathcal{F}) \otimes_B \bar{\pi}_{23, y, \rho}(G) \in \text{MF}_{B^3}((\mathcal{F}_3, \bar{\pi}_{13, y, \rho}(\delta W_{\sigma,\rho}))),
\]
such that

\[(2.5) \quad i^*(\tilde{\pi}_{12,y,\rho}^*(F) \otimes_B \tilde{\pi}_{23,y,\rho}^*(G)) = \tilde{\pi}_{12}^*(\tilde{i}(F)) \otimes_B \tilde{\pi}_{23}^*(\tilde{i}(G)),\]

\[(2.6) \quad \Phi \circ \tilde{\pi}_{13,y,\rho,*}^*(C_{E_{n(2)}}(\tilde{\pi}_{12,y,\rho}^*(F) \otimes_B \tilde{\pi}_{23,y,\rho}^*(G))^T(2)) = \tilde{\pi}_{13,*}^*(C_{E_{n(2)}}(\pi_{12}^*(F') \otimes \pi_{23}^*(G'))^T(2)),\]

where $F' = \Phi(F)$, $G' = \Phi(G)$.

Using the previous proposition we define the binary operation:

\[(2.7) \quad F_y^x G = \tilde{\pi}_{13,y,\rho,*}^*(C_{E_{n(2)}}(\tilde{\pi}_{12,y,\rho}^*(F) \otimes_B \tilde{\pi}_{23,y,\rho}^*(G))^T(2)).\]

The equation (2.6) together with (2.4) implies that the operation is associative. On the other hand, equations (2.5) and (2.6) imply that $i^*$ and $\Phi$ are homomorphisms of the convolution algebras.

2.5. **Induction functors.** For a parabolic subgroup $P \subset G_n$ define auxiliary spaces:

\[\mathcal{F}_2^\circ(P) = \mathfrak{p} \times P \times \mathfrak{b}^2, \quad \mathcal{F}_2^\circ(P) = \mathfrak{b} \times P \times \mathfrak{b},\]

where $\mathfrak{p} = \text{Lie}(P)$. We denote $G_n = \text{GL}(n)$, and $P_k \subset G_n$ is the parabolic subgroup with Lie algebra generated by $\mathfrak{b}$ and $E_{k,1}$.

Respectively, we denote by $p_k$ the natural projection from $P_k$ to $G_k \times G_{n-k}$ and we use the same notation the corresponding projection of the Lie algebras. We also use notation $i_k$ for the embedding $P_k \to G_n$ and its Lie algebra cousins. These projections and inclusions induces the $B_n^2$-equivariant morphisms:

\[p_k : \mathcal{F}_2^\circ(P_k) \to \mathcal{F}_2^\circ(G_k) \times \mathcal{F}_2^\circ(G_{n-k}), \quad \hat{p}_k : \mathcal{F}_2^\circ(P_k) \to \mathcal{F}_2^\circ(G_k) \times \mathcal{F}_2^\circ(G_{n-k}),\]

\[i_k : \mathcal{F}_2^\circ(P_k) \to \mathcal{F}_2^\circ(G_n), \quad \hat{i}_k : \mathcal{F}_2^\circ(P_k) \to \mathcal{F}_2^\circ(G_n).\]

In the setting of symmetric group we have analogous notions of parabolic subgroups and inclusion homomorphism $i_k : S_k \times S_{n-k} \to S_n$. In close analogy to constructions from section 6 of [OR18d] we introduce the induction functors:

\[\text{ind}_k : \text{MF}_{B_n^2}(\mathcal{F}_2^\circ(G_k), W_{\sigma_1,\tau_1}) \times \text{MF}_{B_n^2}(\mathcal{F}_2^\circ(G_{n-k}), W_{\sigma_2,\tau_2}) \to \text{MF}_{B_n^2}(\mathcal{F}_2^\circ(G_n), W_{\sigma,\tau}),\]

\[\overline{\text{ind}}_k : \text{MF}_{B_n^2}(\overline{\mathcal{F}}_2(G_k), \overline{W}_{\sigma_1,\tau_1}) \times \text{MF}_{B_n^2}(\overline{\mathcal{F}}_2(G_{n-k}), \overline{W}_{\sigma_2,\tau_2}) \to \text{MF}_{B_n^2}(\overline{\mathcal{F}}_2(G_n), \overline{W}_{\sigma,\tau}),\]

where $\sigma = i_k(\sigma_1, \sigma_2)$, $\tau = i_k(\tau_1, \tau_2)$.

The arguments of section 6 of [OR18d] provide a proof of the key properties of the induction functors.

**Proposition 2.5.1.** We have

\[\text{ind}_k \circ (\Phi_k \times \Phi_{n-k}) = \Phi_n \circ \overline{\text{ind}}_k\]
Proposition 2.5.2. For any 
\[ \mathcal{F}_i, \mathcal{G}_i \in \text{MF}_{B^2_k}(\mathcal{P}_{\sigma_{1,1}}, W^\sigma_{\sigma_{1,1}}), \quad \mathcal{F}_2, \mathcal{G}_2 \in \text{MF}_{B^2_{n-k}}(\mathcal{P}_{\sigma_{2,2}}, W^\sigma_{\sigma_{2,2}}), \]
we have
\[ \text{ind}_k(\mathcal{F}_1, \mathcal{F}_2) \cdot \text{ind}_k(\mathcal{G}_1, \mathcal{G}_2) = \text{ind}_k(\mathcal{F}_1 \ast \mathcal{G}_1, \mathcal{G}_1 \ast \mathcal{G}_2) \]
where \( \tau_{1i} = \sigma_{2i}. \)

The combination of these two propositions also implies that the induction functor \( \text{ind}_k \) is a homomorphism of \( \mathcal{Y} \)-convolution algebras. We leave the detailed discussion of the induction functors for the forthcoming publication [OR19] where we explain the relation between the induction functors and the elliptic Hall algebra.

2.6. Units of convolution algebras. The potentials \( W_{\tau,\tau} \) and \( \overline{W}_{\tau,\tau} \) vanish on subvarieties \( \mathcal{Z}_2(B) \) and \( \overline{\mathcal{Z}}_2(B) \) thus we can define Koszul matrix factorizations:
\[ C| = i_{B*}(\mathcal{C}[\mathcal{Z}_2(B)]), \quad \overline{C}| = \overline{i}_{B*}(\mathcal{C}[\overline{\mathcal{Z}}_2]) \]
where \( i_B \) and \( \overline{i}_B \) are the corresponding embeddings.

The argument of proposition 7.1 and corollary 7.2 from [OR18d] implies that these matrix factorizations are units in the convolution algebras.

Proposition 2.6.1. The matrix factorizations \( C| \in \text{MF}_{\tau,\tau} \) and \( \overline{C}| \in \text{MF}_{\tau,\tau} \) are the units in the convolution algebras. That is for any \( \mathcal{F} \in \text{MF}_{\tau,\sigma}, \mathcal{G} \in \text{MF}_{\sigma,\tau}, \overline{\mathcal{F}} \in \text{MF}_{\sigma,\tau}, \overline{\mathcal{G}} \in \text{MF}_{\tau,\tau} \) we have:
\[ C| \ast \mathcal{F} = \mathcal{F}, \quad \mathcal{G} \ast C| = \mathcal{G}, \]
\[ \overline{C}| \ast \overline{\mathcal{F}} = \overline{\mathcal{F}}, \quad \overline{\mathcal{G}} \ast \overline{C}| = \overline{\mathcal{G}}. \]

Choose a basis in the group of characters \( B^\vee \) of \( B \subset \text{GL}_n: \chi_k(b) = b_{kk}, k = 1, \ldots, n. \) RFr a matrix \( B^2 \) factorization \( \mathcal{F} \) we denote by \( \mathcal{F}(\chi', \chi'') \) the matrix factorization with the twisted \( B^2 \)-action. In particular, we can consider the twisted version of the units. The same argument as before implies that these elements form a large commutative subalgebra related to the algebra of Jucys-Murphy elements [OR18b].

Proposition 2.6.2. The elements \( C|_{(\chi', \chi'')}, \overline{C}|_{(\chi', \chi'')}, \chi', \chi'' \in B^\vee \) mutually commute. The elements \( C|_{(1,0)}, \overline{C}|_{(1,0)} \) are the central elements of the convolution algebras.

Proposition 9.1 of [OR18d] implies that a twist of the unit by a pair of the characters \( \chi', \chi'' \) depends only on their sum:
\[ C|_{(\chi', \chi'')} \sim C|_{(\chi' + \beta, \chi'' - \beta)}, \quad \overline{C}|_{(\chi', \chi'')} \sim \overline{C}|_{(\chi' + \beta, \chi'' - \beta)}. \]
3. Braid group action

In this section we adjust the results of [OR18d] in order to obtain the homomorphisms from the braid group to the convolution algebras:

$$\Phi^{br}: \mathfrak{B}t_n \to \mathcal{MF}_{n,n}, \quad \Phi^{br}: \mathfrak{B}t_n \to \mathcal{MF}_{n,n}.$$

First, we construct the matrix factorizations for elementary two-strand braids, and then extend the homomorphism to arbitrary braids through the induction functor, verifying the braid relations on three-strand braids.

3.1. Braid groups. The braid group $\mathfrak{B}t_n$ is generated by elements $\sigma_i$, $i = 1, \ldots, n - 1$ with relations:

$$\sigma(i)\sigma(i + 1)\sigma(i) = \sigma(i + 1)\sigma(i)\sigma(i + 1), \quad i = 1, \ldots, n - 2.$$

There is a natural surjective homomorphism:

$$\Sigma: \mathfrak{B}t_n \to S_n$$

with kernel consisting of elements of the group of pure braids $\mathbb{P}\mathfrak{B}t_n$. Let us abbreviate the projection by $\Sigma(\tau) = \tau$.

It is convenient for us to work with labeled braids. These braids form a groupoid $\mathcal{L}\mathfrak{B}t_n$ which could be described as a subset of $S_n \times \mathfrak{B}t_n \times S_n$:

$$(s, \sigma, t) \in \mathcal{L}\mathfrak{B}t_n \text{ iff } \sigma t = s.$$

We use notation $s_\sigma t$ for $(s, \sigma, t) \in \mathcal{L}\mathfrak{B}t_n$. The elements $s_\sigma t$ and $r_\tau p$ are composable if $t = r$ and $s_\sigma t \cdot r_\tau p = s_\sigma r_\tau p$.

3.2. Two strand case. Let $\tau$ be a generator of $S_2$ and $\sigma_1$ is the positive generator of $\mathfrak{B}t_2$. Let us also fix the coordinates on the space $\mathcal{F}_2$:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{bmatrix}, \quad g = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Since the potential $W_{1,\tau} = W_{\tau,1}$ factors

$$\bar{W}_{1,\tau}(X, g, Y) = \bar{x}_0\bar{y}_0, \quad \bar{x}_0 = (x_{11} - x_{22})a_{11} + x_{12}a_{21}, \quad \bar{y}_0 = ((y_{11} - y_{22})a_{22} - y_{12}a_{21})/\det(g).$$

Thus we can define the Koszul matrix factorizations with the potentials $W_{\tau,1}, W_{\tau,1}$:

$$\underline{C}_+ = K^{W_{\tau,1}}(\bar{x}), \quad \underline{C}_+ = \Phi(\underline{C}_+).$$

We use same notation for the corresponding matrix factorizations from $\mathcal{MF}_{1,\tau}$ and $\mathcal{MF}_{1,\tau}$. Following blue-prints of [OR18d] we define the inverses of the above matrix factorizations:

$$\underline{C}_- = \underline{C}_+\langle -\chi_1, \chi_2 \rangle, \quad \underline{C}_- = \underline{C}_+\langle -\chi_1, \chi_2 \rangle.$$

The arguments of section 3.3 [OR18b] imply that the we can switch $\chi_1$ and $\chi_2$ in the last formula:

$$\underline{C}_- = \underline{C}_+\langle \chi_2, -\chi_1 \rangle, \quad \underline{C}_- = \underline{C}_+\langle \chi_2, -\chi_1 \rangle.$$
These matrix factorizations correspond to the positive and negative crossings on two strands:

**Proposition 3.2.1.** We have

\[ \mathcal{C}_+ \ast \mathcal{C}_- \sim \mathcal{C}_- \ast \mathcal{C}_+ \sim \mathcal{C}_\parallel, \quad \bar{\mathcal{C}}_+ \ast \bar{\mathcal{C}}_- \sim \bar{\mathcal{C}}_- \ast \bar{\mathcal{C}}_+ \sim \bar{\mathcal{C}}_\parallel. \]

### 3.3. Braid group generators.

Now we can describe the generators of the braid group and state our main result of the section. Indeed, let us define variations of the induction functors from the previous section:

\[
\text{ind}_{k,k+r} : \text{MF}(\mathcal{F}_2(G_{r+1}), W_{\sigma,\tau}) \rightarrow \text{MF}(\mathcal{F}_2(G_n), W_{\sigma,\tau}),
\]

\[
\text{ind}_{k,k+r} : \text{MF}(\mathcal{F}_2(G_{r+1}), \bar{W}_{\sigma,\tau}) \rightarrow \text{MF}(\mathcal{F}_2(G_n), \bar{W}_{\sigma,\tau}),
\]

as compositions:

\[
\text{ind}_{k,k+r}(\mathcal{F}) = \text{ind}_k(\mathcal{C}_\parallel, \text{ind}_r(\mathcal{F}; \mathcal{C}_\parallel)), \quad \bar{\text{ind}}_{k,k+r}(\mathcal{F}) = \bar{\text{ind}}_k(\bar{\mathcal{C}}_\parallel, \bar{\text{ind}}_r(\mathcal{F}; \bar{\mathcal{C}}_\parallel))
\]

With these functors we define the generators:

\[ \mathcal{C}^{(i)}_\pm = \text{ind}_{i,i+1}(\mathcal{C}_\pm), \quad \bar{\mathcal{C}}^{(i)}_\pm = \text{ind}_{i,i+1}(\bar{\mathcal{C}}_\pm). \]

The elements \( \tau_i \sigma(i)_1 \) and \( 1 \sigma(i)_\tau \) generate the groupoid \( \mathcal{L}B\mathcal{L}_n \), hence its representation is determined by their images:

**Theorem 3.3.1.** The assignments:

\[
\tau_i \sigma^{\pm 1}(i)_1 \mapsto \mathcal{C}^{(i)}_\pm, \quad 1 \sigma^{\pm 1}(i)_\tau \mapsto \mathcal{C}^{(i)}_\pm,
\]

\[
\tau_i \sigma^{\pm 1}(i)_1 \mapsto \bar{\mathcal{C}}^{(i)}_\pm, \quad 1 \sigma^{\pm 1}(i)_\tau \mapsto \bar{\mathcal{C}}^{(i)}_\pm,
\]

extend to groupoid homomorphisms:

\[
\Phi_{\text{brr}} : \mathcal{L}B\mathcal{L}_n \rightarrow \text{MF}_{\ast, \ast}, \quad \bar{\Phi}_{\text{brr}} : \mathcal{L}B\mathcal{L}_n \rightarrow \text{MF}_{\ast, \ast}.
\]

Recall a technical tool for computations with matrix factorizations from [OR18d]. Denote by \( G_S \subset G_n, S \subset \{1, \ldots, n\} \) the subgroup whose Lie algebra is generated by \( b \) and \( E_{ij}, i > j, i, j \in S \). Next, define smaller convolution spaces \( \mathcal{F}_2(G_S, G_T) = g \times G_S \times G_T \times b \), \( \mathcal{F}_2(G_S) = g \times G_S \times G \times b \). The transitivity of the induction functors implies that

\[
(3.1) \quad \mathcal{G}_y^* \text{ind}_{k,k+r}(\mathcal{F}) = \pi_{13,y,\rho^t}(\text{CE}_{n+1}^{(2)} (\pi_{12,y,\rho}(G) \otimes_{B_{r+1}} \pi_{23,y,\rho}(\mathcal{F}))).
\]

where the maps \( \pi_{13,y,\rho} \) and \( \pi_{12,y,\rho} \) are the restrictions of the corresponding maps to the space \( \mathcal{F}_2(G_{(k, \ldots, k+r)}; G) \) and \( \pi_{23,y,\rho} \) is the composition of \( \pi_{23,y,\rho} \) with a natural projection \( \mathcal{F}_2(G_{(k, \ldots, k+r)}; G) \rightarrow \mathcal{F}_2(G_r) \).

**Proof of theorem 3.3.1.** We have to show that the matrix factorization \( \mathcal{C}^{(i)}_\pm \) satisfy the braid relations for the three-stranded braids:

\[
(3.2) \quad \mathcal{C}^{(1)}_+ \ast \mathcal{C}^{(2)}_+ \ast \mathcal{C}^{(1)}_+ \sim \mathcal{C}^{(2)}_+ \bar{\mathcal{C}}^{(1)}_+ \ast \mathcal{C}^{(2)}_+
\]

together with the relation

\[
(3.3) \quad \mathcal{C}_+ \bar{\mathcal{C}}_- \sim \mathcal{C}_\parallel.
\]
which says that $\mathcal{C}_-$ is the inverse of $\mathcal{C}_+$ with respect to the convolution $\circ$. The relations in the multi-strand case follow from the properties of the induction functors.

First, we prove (3.3). Observe that both $\mathcal{C}_+$ and $\mathcal{C}_-$ are matrix factorizations that extend the Koszul complexes:

$$\mathcal{C}_+^+ = K[\bar{x}], \quad \mathcal{C}_-^+ = K[\bar{x}]\langle \chi_2, -\chi_1 \rangle.$$  

In particular, the differentials in these complexes do not depend on the entries of $Y$-matrix. Hence Theorem 9.6 from [OR18d] implies that

$$\pi_{13, g, \rho, \ast}(\mathrm{CE}_n(\pi_{12, g, \rho}(\mathcal{C}_+^+) \otimes R \pi_{23, g, \rho}(\mathcal{C}_-^+) )^{T(2)}) \sim K[g_{21}].$$

Finally, observe that since $g_{21}$ is an irreducible polynomial inside the structure ring, the matrix factorization $\mathcal{C}_+ = K\mathcal{W}[g_{21}]$ is a unique extension of $K[g_{21}]$ by Lemmas 3.1, 3.2 of [OR18d]. Thus the equation (3.3) follows.

In our discussion of braid relations we assume that both sides of (3.2) are objects of the category $\overline{\mathrm{MF}}_{1,c}$, $c = t_{12}t_{23}t_{12} = t_{23}t_{12}t_{23}$. To prove the braid relation we need to combine the previous construction with the formula (3.1). Indeed, first we argue that

$$\mathcal{C}_+^{(1)} \circ_\gamma \mathcal{C}_+^{(2)} \sim \mathcal{C}_+^{(12)}, \quad \mathcal{C}_+^{(2)} \circ_\gamma \mathcal{C}_+^{(1)} \sim \mathcal{C}_+^{(21)}.$$  

Next we observe that Lemma 10.4 of [OR18d] implies that

$$\mathcal{C}_+^{(1)} \circ_{\mathrm{ind}_{2,3}}(\mathcal{C}_+^+) \sim \mathcal{C}_+^{(12)},$$

where $\mathcal{C}_+^{(12)} = K[f, g, a_{31}]$. By proposition 10.2 the sequence $f, g, a_{31}$ is regular in the structure ring. Since $\mathcal{W}_{1, t_{12}t_{23}} \in (f, g, a_{31})$ by Lemmas 3.1, 3.2 there is a unique extension of the complex $\mathcal{C}_+^{(12)}$ to a matrix factorization with the corresponding potential and the equation (3.3) follows.

To complete our proof we need to use (3.1) again for $\mathcal{G} = \mathcal{C}_+^{(12)}$ and $\mathcal{F}_+ = \mathcal{C}_+^{(1)}$, and $k = 1, r = 1$. It is shown in Lemma 11.5 of [OR18d] that formula (3.1) can be used to show

$$\mathcal{C}_+^{(12)} \circ_{\gamma} \mathcal{C}_+^{(1)} \sim \mathcal{C}_+^{(121)},$$

where $\mathcal{C}_+^{(121)}$ is a complex that is quasi-isomorphic to a module in zeroth homological degree which is annihilated by the potential $\mathcal{W}_{1,c}$.

It is also shown in corollary 11.6 of [OR18d] that

$$\mathcal{C}_+^{(2)} \circ_{\gamma} \mathcal{C}_+^{(2)} \sim \mathcal{C}_+^{(2)}.$$
and $\mathcal{C}_{21}^+$ is isomorphic to $\mathcal{C}_{12}^+$ hence by the uniqueness of the extension lemma the braid relations follow.

In the rest of the paper we use notations $\alpha$, $\tilde{\alpha}$, $\alpha \in \mathcal{L} \mathfrak{B}_n$ for the corresponding matrix factorization.

4. Fourier transform

4.1. Specialization to unlabeled braids. The action of the group $S_n$ on the last factor of $\mathcal{D}_2$ induces the isomorphism of the categories:

$$\tau \cdot : \text{MF}_{\sigma,\rho} \to \text{MF}_{\sigma,\tau,\rho,\tau}.$$ 

Let us choose a special element in the orbit of the action:

$$\text{MF}_{\sigma} := \text{MF}_{\sigma,1}.$$ 

We denote by $\text{MF}$ the union of such categories. This union has a natural monoidal structure:

$$\hat{F} \hat{G} := \hat{F}(\tau^{-1} \cdot G), \quad G \in \text{MF}. $$

Theorem 3.3.1 implies that there is a homomorphism from the braid group to the last monoidal category:

**Corollary 4.1.1.** The assignments:

$$\sigma^\pm(i) \mapsto \mathcal{C}_\sigma^{(i)},$$

extend to the group homomorphism:

$$\Phi^{br} : \mathcal{B}_n \to (\text{MF}, \cdot, \hat{\ast}).$$

4.2. Fourier morphisms. There is a natural automorphism on the small space $\mathcal{D}_2$ defined by:

$$\hat{Y}(X, g, Y) \to (Y, g^{-1}, X).$$

This involution extends to the functor of the triangulated categories:

$$\hat{Y} : \text{MF} \to \text{MF}^{-1}.$$ 

The key observation of this paper is that the automorphism, which we call Fourier transform, respects the monoidal structure of the category $\text{MF}$:

**Theorem 4.2.1.** For any composable $\alpha, \beta \in \mathcal{B}_n$ we have:

$$\hat{Y} \left( \hat{Y}(\mathcal{C}_\alpha) \right) \sim \mathcal{C}_\beta \cdot \mathcal{C}_\alpha.$$
To start our argument we need to map out the morphisms that are used in both sides of the equation. In particular, the maps for the LHS fit into the diagram:

\[(4.1)\]

\[
\begin{array}{c}
(X, g_{12}, g_{23}, Y) \xrightarrow{\pi_{13,y,r}^{-1}} (X, g_{13}, Y) \xrightarrow{\tau^{-1}} (X, g_{13}, \tau \cdot Y) \\
(X, g_{12}, \text{Ad}_{g_{23}}(\tau \cdot Y)_{++} + \Delta(Y)) \xrightarrow{\pi_{12,y,r}^{-1} \circ \pi_{23,y,r}^{-1}} (X, g_{13}, g_{23}, Y) \xrightarrow{\tau^{-1}} (\tau \cdot Y, g_{13}^{-1}, X) \\
(X, g_{12}, g_{23}, \text{Ad}_{g_{23}}(\tau \cdot Y)_{++} + \Delta(Y), g_{23}^{-1}, X) \xrightarrow{\pi_{12,y,r}^{-1}} (\text{Ad}_{g_{23}}(\tau \cdot Y)_{++} + \Delta(Y), \text{Ad}_{g_{12}}^{-1}(X)_{++}, X) \xrightarrow{\tau^{-1}} (\tau \cdot Y, g_{23}^{-1}, \text{Ad}_{g_{12}}^{-1}(X)_{+})
\end{array}
\]

where \(g_{13} = g_{12}g_{23}\). To perform the convolution for LHS of the equation we need to place \(\tilde{\mathcal{C}}_\sigma \in \text{MF}_{\sigma} \) and \(\tilde{\mathcal{C}}_\tau \in \text{MF}_{\tau^{-1}} \) at the bottom of the diagram then pull-back along the solid arrows going up and push-forward along the dashed arrows. The same algorithm applies for the RHS of the equation but with the diagram:

\[(4.2)\]

\[
\begin{array}{c}
(X, g_{12}, g_{23}, Y) \xrightarrow{\pi_{13,y,\sigma^{-1}}^{-1}} (X, g_{13}, Y) \xrightarrow{\sigma^{-1}} (X, g_{13}, \sigma^{-1} \cdot Y) \\
(X, g_{12}, g_{23}, Y) \xrightarrow{\pi_{13,y,\sigma^{-1}}^{-1}} (X, g_{13}, \sigma^{-1} \cdot Y) \xrightarrow{\sigma^{-1}} (X, g_{13}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{13}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{13}, \text{Ad}_{g_{23}}(\sigma^{-1} \cdot Y)_{++} + \Delta(Y)) \\
(X, g_{12}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, \sigma^{-1} \cdot Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, \sigma^{-1} \cdot Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, Y) \xrightarrow{\sigma^{-1}} (X, g_{12}, g_{23}, Y)
\end{array}
\]

If we apply to the diagram \([4.1]\) the change of variable below we will get a diagram that is close to \([4.2]\):

\[(4.3)\]

\[
X \rightarrow \sigma^{-1} \cdot Y, \quad \tau \cdot Y \rightarrow X, \quad g_{12} \rightarrow g_{23}^{-1}, \quad g_{23} \rightarrow g_{12}^{-1}.
\]

The difference between the transformed diagram \([4.1]\) and \([4.2]\) is the compositions of the solid arrows. Let us fix notations for the composition of the solid arrows in diagram \(\text{[4.2]}\):

\[
\pi_{23,y,\sigma^{-1}}(X, g_{12}, g_{23}, Y) = (\text{Ad}_{g_{12}}^{-1}(X)_{+}, g_{23}, \sigma^{-1} \cdot Y),
\]

\[
\pi_{12,y,\sigma^{-1}}(X, g_{12}, g_{23}, X) = (X, g_{12}, \text{Ad}_{g_{23}}(\sigma^{-1} \cdot Y)_{++} + \Delta(Y)).
\]

Let us also fix notation \(\pi_{13,y,\sigma^{-1}}\) for composition of the dashed arrows. Respectively, the composition of the solid arrows in \(\text{[4.1]}\) after transformation \(\text{[4.3]}\) are:

\[
\pi_{23,x,\sigma^{-1}}(X, g_{12}, g_{23}, Y) = (\text{Ad}_{g_{12}}^{-1}(X)_{++} + \Delta(\tau^{-1} \cdot X), g_{23}, \sigma^{-1} \cdot Y)
\]

\[
\pi_{12,x,\sigma^{-1}}(X, g_{12}, g_{23}, Y) = (X, g_{12}, \text{Ad}_{g_{23}}(\sigma^{-1} \cdot Y)_{+}).
\]

Similarly, we denote by \(\tilde{\pi}_{13,x,\sigma^{-1}}\) the composition of the dashed arrows in the diagram \(\text{[4.1]}\).
Thus our theorem would follow from the equation
\[
\tilde{\pi}_{12,x,\sigma^{-1}}^*(\mathcal{F}) \otimes_B \tilde{\pi}_{23,x,\sigma^{-1}}^*(\mathcal{G}) \sim \tilde{\pi}_{12,y,\sigma^{-1}}^*(\mathcal{F}) \otimes_B \tilde{\pi}_{23,y,\sigma^{-1}}^*(\mathcal{G})
\]
for \( \mathcal{F} \in \overline{\mathcal{MF}}_x, \mathcal{G} \in \overline{\mathcal{MF}}_x \). The homotopy implies the relation between the two products:
\[
\mathcal{F}_y \mathcal{G} \sim \mathcal{F}_x \mathcal{G},
\]
where the product \( \hat{\cdot} \) is defined with the use of maps \( \tilde{\pi}_{ij,x,\cdot} \).

It is possible to prove the formula \( (4.4) \) in the full generality but the formula for the homotopy is rather complicated. The details will appear in the future publications and in this note we provide explanations for the simplest possible case of the formula, and explain why it is enough for our proof.

**Lemma 4.2.2.** Let us assume that \( \mathcal{F} = (M', D', \partial') \in \overline{\mathcal{MF}}_x, \mathcal{G} = (M'', D'', \partial'') \in \overline{\mathcal{MF}}_x \) and that the differentials \( D', \partial', D'', \partial'' \) are linear along the factors \( b \) in \( \mathcal{X}_2 \). Then there is a homotopy as in \( (4.4) \).

**Proof.** We can write an explicit formula for the homotopy. To simplify notations, we assume that all equivariant correction differentials vanish. We also identify \( \mathcal{F}, \mathcal{G} \) with the pull-backs \( \tilde{\pi}_{12,x,\sigma^{-1}}^*(\mathcal{F}), \tilde{\pi}_{23,x,\sigma^{-1}}^*(\mathcal{G}) \).

The convolution space \( \mathcal{X}_3 = b \times G \times G \times b \) has a standard coordinate system \( X, g_{12}, g_{23}, Y \) and two other coordinate systems pulled back along the maps \( \tilde{\pi}_{12,x,\sigma} \) and \( \tilde{\pi}_{23,x,\sigma} \). We denote the last two coordinate system by \( \mathcal{X}_1, g_{12}, g_{23}, Y \) and \( \mathcal{X}_2, g_{12}, g_{23}, Y \). For example \( Y_1 = \text{Ad}_{g_{12}}(\sigma^{-1} \cdot Y)_+ \).

By taking the derivatives of the potentials \( \overline{W}_{\sigma,1}, \overline{W}_{\tau,1} \) we get:
\[
[\hat{\partial}_{Y_1^{ij}} D', D'] = \tau^{-1} \cdot X_1^{ij} - (\text{Ad}_{g_{12}}^{-1} X')_{ii}, \quad [\hat{\partial}_{X_1^{ii}} D'', D''] = \sigma \cdot Y''_{ii} - (\text{Ad}_{g_{23}} Y'')_{ii}.
\]
Since all differentials and the potentials are \( b \)-linear by taking the second derivatives of the potentials along \( b \) we prove that operators
\[
u_i = \hat{\partial}_{Y_1^{ij}} D' \otimes \hat{\partial}_{X_1^{ii}} D'',
\]
satisfy relations
\[
u_i^2 = 0, \quad \nu_i \nu_j + \nu_j \nu_i = 0.
\]
Hence we can define the homotopy map by
\[
U = \exp(-\sum_{i=1}^n \nu_i).
\]
Combining all previous observations and taking into account \( X_1 = X, Y'' = \sigma^{-1} \cdot Y \) for \( (4.5) \) we arrive to
\[
U(D' \otimes I + I \otimes D'') U^{-1} = \tilde{D}' \otimes I + I \otimes \tilde{D}'',
\]
where \( \tilde{D}' \) and \( \tilde{D}'' \) are the differentials of the pull-backs \( \tilde{\pi}_{12,y,\cdot}^*(\mathcal{F}), \tilde{\pi}_{23,y,\cdot}^*(\mathcal{G}) \). \( \square \)
Proof of theorem 4.2.1. In the proof we omit the subindex from $S_n$ from the notations of the maps to simplify notations. Let $\omega = \sigma^{i_1}(i_1) \cdots \sigma^{i_l}(i_l)$ is a presentation of a braid as product of the elementary braids. We claim that there is a homotopy which implies the statement of theorem:

$$\tilde{C}^{(i_1)}_{x_1} \otimes \cdots \otimes \tilde{C}^{(i_l)}_{y_l} \sim \tilde{C}^{(i_1)}_{y_1} \otimes \cdots \otimes \tilde{C}^{(i_l)}_{y_l}.$$ 

Indeed, we can iterate formula (3.1) to obtain a formula for the long product:

$$G \ast \text{ind}_{i_1, i_1 + 1}(F_i) \ast \cdots \ast \text{ind}_{i_l, i_l + 1}(F_l) =$$

$$\tilde{\pi}_{l+2,y,i}((\text{CE}_{\ell}^\ast \circ \bigotimes_{\ell} \tilde{\pi}_{23,y,i}^\ast((F_1) \otimes \bigotimes_{\ell} \tilde{\pi}_{l+1,\ell+2,y,i}^\ast(F_i)))),$$

where $\tilde{\pi}_{l+2,y,i}$ is the map from

$$\overline{\mathcal{X}}(G, G_{i_1, i_1 + 1}, \ldots, G_{i_l, i_l + 1}) = b \times G_{i_1, i_1 + 1} \times \ldots \times G_{i_l, i_l + 1} \times b$$

to $\overline{\mathcal{X}}_2(G_n)$ given by

$$\tilde{\pi}_{l+2,y,i}(X, g_{12}, \ldots, g_{l+1,l+2}, Y) = (X, g_{12} \cdots g_{l+1,l+2}, Y)$$

and the maps $\tilde{\pi}_{k+1,y,i}$ are the corresponding twisted linear projections on $\overline{\mathcal{X}}_2(G_2)$.

The maps $\tilde{\pi}_{k+1,y,i}$ are obtained by a composition of the restriction of the map $\tilde{\pi}_{k,k+1,y,i}$ and a natural projection from $\overline{\mathcal{X}}_2(P_k)$ to $\overline{\mathcal{X}}_2(G_2)$. Thus we can apply lemma 4.2.2 and if we set $F_i = \tilde{C}_{e_i}$ and set $G = \tilde{C}_{e_l}$, we obtain the desired homotopy. \hfill \Box

5. Link invariant

5.1. Conjugacy class invariant. First we define conjugacy invariant of the braid. For that we observe the subvariety $b \times 1 \times b \subset \overline{\mathcal{X}}_2$ is invariant with respect to the adjoint $B$-action. Thus the corresponding embedding $j_c : b^2 \to \overline{\mathcal{X}}_2$ induces the functor:

$$j_c^! : \text{MF}_{*,\sigma} \to \text{MF}_B(b^2, Q_{\tau,\sigma}), \quad Q_{\tau,\sigma}(X, Y) = \text{Tr}(X(\tau \cdot Y - \sigma \cdot Y)).$$

The potential $Q_{\tau,\sigma}$ is a quadratic function of the diagonal elements, thus there is a Knorrer periodicity functor. Let us denote by $b^\alpha$, $\alpha \in S_n$ fixed by $\alpha$. Then the Knorrer periodicity functor provides us with a functor:

$$\overline{\text{KN}}_{\tau,\sigma} : \text{MF}_B(b^2, \bar{Q}_{\tau,\sigma}) \to D_B^\text{per}(b^\delta \times b^\delta),$$

where $\delta = \tau^{-1} \sigma$.

The composition of the above functors with $\text{CE}_n(\bullet)^T$ and push-forward along the projection $b^\delta \times b^\delta \to b^\delta \times b^\delta$ results into the categorical trace:

$$T_{\bar{\pi}}^0 : \text{MF}_{*,\sigma} \to D_B^\text{per}(b^\delta \times b^\delta).$$

There is a natural action of the symmetric group, which changes the labels. This action is the groupoid homomorphism:

$$\tau \cdot (\alpha \sigma \beta) = (\alpha \tau \sigma \beta \tau).$$

This automorphism is sent by $\overline{\text{DF}}$ to the functor induced by the $\tau \cdot$ action on $b$:

$$\tau : \text{MF}_{\alpha,\beta} \to \text{MF}_{\alpha \tau,\beta \tau}.$$
Now we are ready to state the main property of our trace functor:

**Proposition 5.1.1.** For any \( \mathcal{F} \in \text{MF}_{\gamma, \beta}, \mathcal{G} \in \text{MF}_{\beta, \gamma} \) we have:

\[
\mathcal{T} r_0(\mathcal{F} \otimes \mathcal{G}) = \mathcal{T} r_0(\tau \cdot \mathcal{G} \otimes \mathcal{F}),
\]

where \( \tau = \alpha \gamma^{-1} \).

It is easier to prove the trace property if we use the convolution in big category. So define a version of the trace functor that works for the convolution in big category, to be \( \tau \), where

\[
\text{B}
\]

The potential \( Q \) is independent of the last factor in \( C \). Then the pull-back morphism along \( j_i \) gives us a functor:

\[
\text{B}^* \Rightarrow \text{F}
\]

The potential \( Q_{\tau, \sigma} \) has a quadratic term \( \text{Tr}(X_{---}(Y_{---} - Y_{---})) \) thus there is Knorrer periodicity functor:

\[
\Phi : \text{MF}_{B}(\mathcal{Z}_{2}^\circ(\delta), Q_{\tau, \sigma}) \rightarrow \text{MF}_{B}(B \times B \times h^\delta, Q_{\tau, \sigma}).
\]

The Knorrer functors are intwiners: \( \Phi \circ j_e^* = j_e^* \circ \Phi \). Also since the potential \( Q_{\tau, \sigma} \) is independent of the last factor in \( B \times B \times h^\delta \) we can extend previously used functor to obtain:

\[
\text{B}^* \Rightarrow \text{F}
\]

We denote by \( O(\delta) \) the set of \( \delta \)-orbits inside \( \{1, \ldots, n\} \). There is a natural projection \( \pi_\delta : h \rightarrow h^\delta \), if \( x_o, o \in O(\delta) \) are coordinates on \( h^\delta \) then the projection is given by \( x_o = \sum_{i \in o} h_i \).

We use same notation for the projection \( \pi_\delta : B \rightarrow B^\delta \) as well as projection \( B^\delta \times B^\delta \times h^\delta \rightarrow h^\delta \times h^\delta \) which is an identity on the last factor and \( \pi_\delta \) on the second factor.

Let us fix coordinates \( (X, Y_1, Y_2, x) \) on the space \( \mathcal{Z}_{2}^\circ(\delta) \) and introduce Koszul complex on the space

\[
K^\delta = (R \otimes \Lambda^\bullet(\theta_1, \ldots, \theta_l), D^\delta), \quad D^\delta = \sum_{o \in O(\delta)} (x_o - \sum_{i \in o} X_{ii}) \frac{\partial}{\partial \theta_o},
\]

where \( l = |O(\delta)| \) and \( R = \mathbb{C}[\mathcal{Z}_{2}^\circ(\delta)] \). The complex \( K^\delta \) is not \( B \)-equivariant in strong sense but given \( \mathcal{C} = (M, D, \partial) \in \text{MF}_{B}(\mathcal{Z}_2(\delta), Q_{\tau, \sigma}) \) by lemma 3.6 of \[\text{OR18d}\] there is \( \partial' : M \otimes \Lambda^\bullet(\theta_\delta) \rightarrow M \otimes \Lambda^\bullet(\theta_\delta) \) such that

\[
(M \otimes \Lambda^\bullet(\theta_\delta), D + D^\delta, \partial + \partial') \in \text{MF}_{B}(\mathcal{Z}_2(\delta), Q_{\tau, \sigma}).
\]
This matrix factorization is unique up to homotopy (by Lemma 3.7 in [OR18d]) and we use notation $C \otimes_B K^\delta$ for it.

The trace map is defined by

$$\mathcal{T}r_0 : MF_{\sigma,\tau} \to D^\per_B (h^\delta \times h^\delta), \quad \mathcal{T}r_0(C) = \pi_{\delta*} \left( CE_n(KN_{r,\sigma} (\Phi (j_e^*(C) \otimes_B K^\delta)))^T \right).$$

**Proof of proposition 5.13.** From the construction we see that $\mathcal{T}r_0(\Phi(C)) = \mathcal{T}r_0(C)$. Thus we concentrate on the proof the statement for the trace in the big category.

If we only care about the structure of $C[h^\delta]$-module (not $C[h^\delta \times h^\delta]$) then the trace property follows from base changes in the commuting diagram:

$$\begin{array}{ccc}
\mathbb{Z}^\circ \times \mathbb{Z}^\circ & \xleftarrow{j_\Delta} & \mathbb{Z}^\circ \\
\downarrow_{\pi_{13}} & & \downarrow_\pi \\
\mathbb{Z}^\circ & \xleftarrow{j_e} & g \times b \times b \\
\downarrow_{j_{KN}} & & \downarrow_\tilde{\pi}_\delta \rightarrow h^\delta \times \tilde{h}^\delta
\end{array}
$$

(5.1)

here $\mathbb{Z} = g \times G \times b \times b$ and the new solid arrow maps are:

$$j_\Delta(X, Y_1, g_{12}, Y_2, g_{23}, Y_3) = (X, Y_1, g_{12}, Y_2) \times (Ad_{g_{12}}^{-1} X, Y_2, g_{23}, Y_3), \quad j_{KN}(X, Y) = (X, Y, Y),$$

$$j_e(X, Y_1, Y_2) = (X, Y_1, 1, Y_2),$$

and $\tilde{\pi}_\delta$ is the natural projection. The dashed arrows are constructed to make the diagram commute:

$$j(X, g, Y_1, Y_2) = (X, Y_1, g, Y_2, 1, Y_1), \quad \pi(X, g, Y_1, Y_2) = (X, \pi_\delta(Y_1)).$$

If we travel along the solid arrow from the upper-left corner to the lower-right corner we obtain a simplified version of our trace functor:

$$\pi_{\delta*} \circ j_{KN}^* \circ j_e^* \circ \pi_{13*} \circ j_\Delta^*(\mathcal{F} \boxtimes \mathcal{G}))^T = \pi_{\delta*} \left( CE_n(KN_{r,\sigma} (\Phi (j_e^*(\mathcal{F} \boxtimes \mathcal{G}))))^T \right).$$

On the other hand the base change tell us that

$$j_{KN}^* \circ j_e^* \circ \pi_{13*}^* = \pi_* \circ j_*^*.$$

The maps $j_\Delta \circ j$ and $\tilde{\pi}_\delta \circ \pi$ are equivariant with respect to the involution

$$sw_z(X, Y_1, Y_2) = (Ad_{g}^{-1} X, g^{-1}, Y_2, Y_1),$$

$$j_\Delta \circ j \circ sw_z = sw_z \circ j_\Delta \circ j, \quad \tilde{\pi}_\delta \circ \pi \circ sw_z = \tilde{\pi}_\delta \circ \pi,$$

where $sw_z$ swaps two copies of $\mathbb{Z}^\circ$. Thus the involution $sw_z$ induces the isomorphism of complexes of $C[b^\delta]$-modules:

$$\pi_{\delta*} \left( CE_n(KN_{r,\sigma} (\Phi (j_e^*(\mathcal{F} \boxtimes \mathcal{G}))))^T \right) \simeq \pi_{\delta*} \left( CE_n(KN_{r,\sigma} (\Phi (j_e^*(\mathcal{G} \boxtimes \mathcal{F}))))^T \right).$$

To upgrade this isomorphism to the statement about the trace $\mathcal{T}r_0$ we need to multiply the varieties of the diagram (5.1) by $h^\delta$ and observe that

$$\mathcal{T}r_0(\mathcal{F} \boxtimes \mathcal{G}) = CE_n(\tilde{\pi}_\delta^* \circ \pi_* (j_*^* \circ j_\Delta^*(\mathcal{F} \boxtimes \mathcal{G}) \otimes_B K^\delta))^T.$$
The only difficulty with applying the previous argument in this case is that the involution \( sw_z \) acts non-trivially on the complex \( K^\delta \). The pull-back of the complex \( sw_z^*(K^\delta) \) has differential:

\[
\tilde{D}^\delta = \sum_{o \in O(\delta)} \left( x_o - \sum_{i \in o} \Ad_g^{-1}(X)_{ii} \frac{\partial}{\partial \theta_o} \right).
\]

However, there is a homotopy connecting \( D^\delta \) and \( \tilde{D}^\delta \). If \( F = (M', D', \partial') \) and \( G = (M'', D'', \partial'') \) then the partial derivatives

\[
u'_o = \sum_{i \in o} \frac{\partial D'}{\partial Y_{ii}}, \quad u''_o = \sum_{i \in o} \frac{\partial D''}{\partial Y_{ii}},
\]

provide homotopies between \( \sum_{i \in o} X_{ii} \) and \( \sum_{i \in o} (\Ad_g X)_{ii} \) for \( o \in O(\delta) \). Thus the matrix \( M = \exp(\sum_{o \in O(\delta)} (u'_o \otimes 1 + 1 \otimes u''_o) \theta) \) provides us with the intertwiner:

\[
M \circ D^\delta \circ M^{-1} = \tilde{D}^\delta,
\]

and the proposition follows.

5.2. **Framed category and link invariants.** In this section we introduce a version of our short category with stability condition. This category has enough structure for defining the link invariant.

The framed version of the space \( \mathcal{X}_2 \) is a open subset of \( \mathcal{X}_2 \times V \), \( V = \mathbb{C}^n \) defined by the stability condition:

\[
(X, g, Y, v) \in \mathcal{X}_{2,fr} \iff \mathbb{C}\langle X, (\Ad g Y) \rangle v = V.
\]

The projection along the vector space \( V \) maps \( \mathcal{X}_{2,fr} \) to the open subset \( \mathcal{X}_{2,st} \subset \mathcal{X}_2 \). The pull-back along this projection induces the functor:

\[
\text{fgt} : \text{MF}_{st} \rightarrow \text{MF}_B(\mathcal{X}_{2,fr}, W_{\sigma,\tau}) = \text{MF}_{st}.
\]

Similarly, we define subspace \( \mathcal{X}_{3,fr} \) as stable locus of the product \( \mathcal{X}_3 \times V \). There is a natural extension of the maps \( \pi_{ij,\rho} \) to the maps between the spaces \( \mathcal{X}_{3,fr} \) and \( \mathcal{X}_{2,fr} \). Thus we can define the monoidal structure on the category \( \text{MF}_{st} \). Moreover, the argument of lemma 12.2 [OR18d] implies that the functor \( \text{fgt} \) is monoidal.

\[
\text{fgt}(\mathcal{F}_y \mathcal{G}) = \text{fgt}(\mathcal{F}_y) \text{fgt}(\mathcal{G}).
\]

The geometric trace functor from category \( \text{MF}_{st} \) lands into the category constructed on the space \( \widetilde{\mathbb{H}ilb}_{1,n} \) which is the space of triples \( (X, Y, v) \in \mathfrak{b}^2 \times V \) satisfying stability condition:

\[
\mathbb{C}\langle X, Y \rangle v = V.
\]

We denote by \( \widetilde{\mathbb{H}ilb}_{1,n} \) the \( B \)-quotient of \( \mathbb{H}ilb_{1,n} \). We also use notation \( j_e \) for embedding of \( \widetilde{\mathbb{H}ilb}_{1,n} \) inside \( \mathcal{X}_{2,fr} \). The pull-back along this map give us functor:

\[
j_e^* : \text{MF}_{st} \rightarrow \text{MF}_B(\mathbb{H}ilb_{1,n}^{free}, Q_{\sigma,\tau}).
\]
There is a natural projection $\chi: \mathcal{Hilb}_{1,n} \to \mathfrak{h} \times \mathfrak{h}$ and we define $\mathcal{Hilb}_{1,n}(\mathcal{Z}) := \chi^{-1}(\mathcal{Z})$, $\mathcal{Z} \subset \mathfrak{h} \times \mathfrak{h}$. We also use abbreviation $\mathcal{Hilb}_{1,n}(\delta)$, $\delta \in S_n$ for $\mathcal{Hilb}_{1,n}(\mathfrak{h}^\delta \times \mathfrak{h}^\delta)$. The Knorrer periodicity thus gives us functor:

$$\text{KN}_{\sigma,\tau} : \text{MF}_B(\mathcal{Hilb}_{1,n}^{free}, Q_{\sigma,\tau}) \to D_B^{per}(\mathcal{Hilb}_{1,n}(\sigma^{-1})).$$

Combining all the previous constructions we assign a two-periodic complex to a labeled braid $\beta \in \mathcal{LBr}_n$:

$$\mathbb{S}_\beta = \text{CE}_n(\text{KN}_{\sigma,\tau}(\text{fgt}(\Phi_{br}(\beta))))^T,$$

where $\sigma = s(\beta)$, $\tau = t(\beta)$. The same argument as above implies that this complex only depends on the conjugacy class of the element of the braid group.

The last component needed to define the knot homology is the tautological vector bundle $\mathcal{B}$ over $\mathcal{Hilb}_{1,n}(\delta)$ with fiber $V^\sigma$. We use same notation for vector bundle on the quotient $\mathcal{Hilb}_{1,n}(\delta)$. The hyper-cohomology

$$\mathbb{H}^k(\beta) = \mathbb{H}(\mathbb{S}_\beta \otimes \Lambda^k \mathcal{B})$$

is a doubly-graded module over the doubly-graded ring $R(\beta) = \mathbb{C}[X_{i1}, Y_{i1}, \ldots, X_{nn}, Y_{nn}]/I_{\sigma,\tau}$ where $I_{\sigma,\tau}$ is generated by the elements $X_{ii} - X_{\delta(i),\delta(i)}, Y_{ii} - Y_{\delta(i),\delta(i)}, \delta = \sigma\tau^{-1}, i = 1, \ldots, n$.

Denote $\ell(\beta)$ the number of connected components of the closure of $\beta$, which is also equal to the number of orbits of $\delta$ acting on $\{1, \ldots, n\}$. A choice of correspondence between the orbits and elements of $\{1, \ldots, \ell\}$, identifies $R(\beta)$ with $\mathbb{C}[x_1, y_1, \ldots, x_\ell, y_\ell]$. Let us also recall that there is a natural Fourier transform on $\mathfrak{S}$ on the ring $R(\beta)$ given by formula (1.1).

As in [OR18d] we use notation $q^{lt}$ for the shift of $qt$-grading. With this convention in mind the main result of the paper is the following:

**Theorem 5.2.1.** For any $\beta \in \mathcal{LBr}_n$ the triply-graded module over $R(\beta)$:

$$\text{HXY}(\beta) = \bigoplus_{k \in \mathbb{Z}} \mathbb{H}^k(\beta),$$

with $\mathbb{H}^k(\beta)$ defined by

$$\mathbb{H}^k(\beta) = q^{\text{wr}(\beta) + n} \cdot \mathbb{H}(\beta),$$

is an isotopy invariant of the closure $L(\beta)$, here $\text{wr}(\beta)$ is the wreath of the closure of $\beta$.

Moreover, there is an involution $\mathfrak{S}$ on $\text{HXY}(\beta)$ that intertwines the the Fourier transform on $R(\beta)$.

The theorem follows from more general statement. Using the projection $\chi$ we define:

$$\mathcal{T}_r_k(\beta) = \chi_*(\mathbb{S}_\beta \otimes \Lambda^k \mathcal{B}) \in D^{per}(\mathfrak{h}^\delta \times \mathfrak{h}^\delta), \quad \dot{\beta} = \delta.$$

Respectively, we denote by $\mathcal{T}$ the direct sum

$$\mathcal{T}_r(\beta) = \bigoplus_k \mathcal{T}_r_k(\beta).$$

Previously defined trace $\mathcal{T}_r_0$ could restricted to the stable part of the corresponding spaces to define:

$$\mathcal{T}_r_0 : \text{MF}_r^{st} \to D^{per}(\mathfrak{h}^\delta \times \mathfrak{h}^\delta), \quad \delta = \tau^{-1} \sigma.$$
This definition is consistent with the other construction of the trace since
\[(5.2) \quad \mathcal{T}r_0(\beta) = \mathcal{T}r_0(\mathcal{C}_\beta).\]

The first part of the theorem follows from the theorem below the second part is discussed in the next section.

**Theorem 5.2.2.** For any $\beta \in \mathfrak{B}_n$ the direct sum
\[\mathcal{E}(\beta) = q^{wr(\beta)+n} \cdot \mathcal{T}r(\beta)\]
is a link invariant of the closure $L(\beta)$.

**Proof.** We need to check that the trace satisfies the Markov move conditions. The first Markov move is about conjugacy invariance and it follows immediately from (5.2) and [5.11]. The second Markov move is equivalent to the pair of equations
\[(5.3) \quad \mathcal{T}r_k(\beta \cdot \sigma(1)) = \mathcal{T}r_k(\beta), \quad \mathcal{T}r_k(\beta \cdot \sigma(1)) = \mathcal{T}r_{k-1}(\beta),\]
where $\beta \in \mathfrak{B}_{n-1}$ is a braid on the strands $2, \ldots, n$.

These equations are implied by the same argument as theorem 13.3 of [OR18d]. Below we outline the steps of the proof. The reader may consult [Ob18] for a detailed account.

The images of the braids $\beta$ and $\beta \cdot \sigma(1)$ in the symmetric groups are $\delta' \in S_{n-1}$ and $\delta = \delta' \cdot t_{12} \in S_n$. Now we can use the nested nature of the scheme $\text{Hilb}_{1,n}$ to define the projection map:
\[\pi : \text{Hilb}_{1,n}^\text{free} \to \mathbb{C}_x \times \mathbb{C}_y \times \text{Hilb}_{1,n-1}^\text{free},\]
where the first two components of the map $\pi$ are $x_{11}, y_{11}$ and the last component is just forgetting of the first row and the first column of the matrices $X, Y$ and the first component of the vector $v$. Let us also fix notation for the line bundles on $\text{Hilb}_n^\text{free}$: we denote by $\mathcal{O}_k(-1)$ the line bundle induced from the twisted trivial bundle $\mathcal{O} \otimes \chi_k$. It is shown in proposition 13.1 of [OR18d] that the fibers of the map $\pi$ are projective spaces $\mathbb{P}^{n-1}$ and
\[(5.4) \quad \mathcal{B}_n/\pi^*(\mathcal{B}_{n-1}) = \mathcal{O}_n(-1), \quad \mathcal{O}_n(-1)_{\pi^{-1}(z)} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1).\]

We can combine the last proposition with the observation that the total homology $H^*(\mathbb{P}^{n-1}, \mathcal{O}(-l))$ vanish if $l \in (1, n-1)$ and is one-dimensional for $l = 0, n$ and obtain a conclusion of proposition 13.2 of [OR18d] For any $n$ we have:
- $\pi_*(\Lambda^kB_n) = \Lambda^kB_{n-1}$
- $\pi_*(\mathcal{O}_n(m) \otimes \Lambda^kB_n) = 0$ if $m \in [-n+2, -1]$.
- $\pi_*(\mathcal{O}_n(-n+1) \otimes \Lambda^kB_n) = \Lambda^{k-1}B_{n-1}[n]$

The main technical component of the proof is a careful analysis of matrix factorizations $\tilde{\mathcal{C}}_{\beta, \sigma(1)} \in \text{MF}(\mathcal{M}_n^{\text{st}}, \mathcal{W})$, see proof of theorem 13.3 in [OR18d]. The same argument is applicable for study of the curved complexes $\tilde{\mathcal{C}}_{\beta, \sigma(1)} \in \text{MF}_{\tau, \sigma}^{\text{st}}$. To state the version of the result we need to work with the space $\text{Hilb}_{1,n}(\mathfrak{h} \times \mathfrak{h}^{\sigma(1)})$, we denote the embedding map into $\text{Hilb}_{1,n}$ by $i_\Delta$. The image $\pi(\text{Hilb}_{1,n}(\mathfrak{h} \times \mathfrak{h}^{\sigma(1)}))$ is isomorphic to $\mathbb{C}_x \times \text{Hilb}_{1,n-1}$. Now
the version of the result from [OR13d] that we need show that curved complex $i_\Delta^*(\overline{C}_{\beta,\sigma(1)^r})$ has form:

$$\begin{array}{c}
C' \leftarrow & \cdots & C' \otimes V \leftarrow & \cdots & C' \otimes \Lambda^2 V \leftarrow & \cdots & C' \otimes \Lambda^3 V \leftarrow & \cdots & C' \otimes \Lambda^4 V \leftarrow & \cdots & \cdots
\end{array}$$

where $C' = \pi^*(\overline{C}_{\beta} \otimes K[x_{11}])$, $V = \mathbb{C}^{n-2}$, the dotted arrows are the differentials of the Koszul complex for the ideal $I = (g_{13}, \ldots, g_{1n})$ where $g_{ij}$ are the coordinates on the group inside the product $\mathbb{P}_n = b_n \times GL_n \times b_n$. Thus after the pull-back $j^*_e$ the dotted arrows of the curved complex vanish and we only left with the arrows going from the left to right.

Since the push-forward along $\pi$ intertwines the Knorrer functor:

$$CE_{n_{\epsilon}}(KN_{r,\sigma}(C)) = CE_{n_{\epsilon+1}}(KN_{r',\sigma'}(\pi_* \circ i_{\Delta}^*(C)))^{T_n}$$

to finish proof of the theorem we need to compute $\pi_*(i_{\Delta}^* \circ j^*_e(\overline{C}_{\beta,\sigma^1}) \otimes \Lambda^k B_n)$ and here we can apply the previous corollary. Thus if $\epsilon = 1$ then only the left extreme term of $j^*_e$ of the complex (5.5) survive the push-forward $\pi_*$. Since the non-trivial arrows of $j^*_e$ of (5.5) all are the solid arrows which are going the left to the right, the contraction of the $\pi_*$-acyclic terms do not lead to appearance of new correction arrows thus conclude that

$$\pi_*(i_{\Delta}^* \circ j^*_e(\overline{C}_{\beta,\sigma^1}) \otimes \Lambda^k B_n) = j^*_e(\overline{C}_{\beta} \otimes \Lambda^k B_{n-1}).$$

If $\epsilon = -1$ then only the right extreme term of $j^*_e$ of the complex (5.5) survive the push-forward $\pi_*$. Hence the similar argument as before implies:

$$\pi_*(i_{\Delta}^* \circ j^*_e(\overline{C}_{\beta,\sigma^{-1}}) \otimes \Lambda^k B_n) = j^*_e(\overline{C}_{\beta} \otimes \Lambda^k B_{n+1}).$$

5.3. Fourier transform for the trace. Let us use notation for the anti-involution inverting the direction of the braid:

$$\alpha \cdot \beta = \beta \cdot \alpha, \quad \sigma(i) = \sigma(i)$$

Let us also define an involution on the ring $R(\beta)$ by

$$\hat{\sigma}(x_i) = y_i, \quad \hat{\sigma}(y_i) = x_i.$$

**Theorem 5.3.1.** For any $\beta \in \mathfrak{B}_n$ we have

$$\hat{\sigma}(Tr(\beta)) = Tr(\overline{\beta})$$

**Proof.** The statement follows from the theorem [4.2.1] and the fact the properties of $\overline{C}_+$. Indeed, $\overline{C}_+ \in MF_{r,1}$ is the Koszul matrix factorization for the pair $\overline{x}_0$ and $\overline{y}_0$. The Fourier transform $\hat{\sigma}$ switches $\overline{x}_0$ and $\overline{y}_0$ hence:

$$\hat{\sigma}(\overline{C}_+) = \overline{C}_+$$

and the statement follows. \qed
Next we discuss the easy involution on our categories that manifests itself in the equation (1.2). We define an involution on the big space \( \mathcal{F} \) by:

\[
\mathcal{I}(X, g_1, Y_1, g_2, Y_2) = (X, g_2, -Y_2, g_1, -Y_1).
\]

This involution induces the isomorphism of categories:

**Proposition 5.3.2.** The involution \( \mathcal{I} \) induces the isomorphism's of categories:

\[
\mathcal{I}: \text{MF}_{\sigma, \tau} \rightarrow \text{MF}_{\tau, \sigma}
\]

such that

\[
\mathcal{I}(\mathcal{F} \ast \mathcal{G}) = \mathcal{I}(\mathcal{G}) \ast \mathcal{I}(\mathcal{F}).
\]

The corresponding involution on the ring \( R(\beta) \) is given by:

\[
\mathcal{I}(x_i) = x_i, \quad \mathcal{I}(y_i) = -y_i.
\]

The Fourier transform \( \mathcal{F} \) is the composition of these two involutions:

\[
\mathcal{F} = \mathcal{F} \circ \mathcal{I}.
\]

Thus combining theorem 5.3.1 and proposition 5.3.2 we derive that for any \( \beta \in \mathcal{LBr}_n \) we have:

\[
\mathcal{F}(\mathcal{T}r(\beta)) = \mathcal{T}r(\beta), \quad \text{HXY}(L(\beta)) = \mathcal{F}(\text{HXY}(L(\beta)))
\]

and Theorems 1.2.1, 5.2.1 and 1.1.2 follow.

**5.4. Relation with HOMFLYPT homology.** In our previous papers we worked with homology theory \( H(\beta) \) which categorifies the HOMFLYPT polynomial. In this section we explain how the old homology theory can be obtained from the new homology theory \( \text{HXY}(\beta) \).

Let us denote by \( R(\beta)_x \) the ring \( \mathbb{C}[x_1, \ldots, x_\ell] \) where \( \ell \) is number of connected components of the closure \( L(\beta) \). The ring \( R(\beta)_x \) is a \( R(\beta) \)-module and in the next section we show

**Theorem 5.4.1.** For any \( \beta \in \mathcal{LBr}_n \) we have the relation between the homology of the closure of the braid:

\[
\text{HXY}(\beta) \overset{\ell}{\otimes}_{R(\beta)} R(\beta)_x = H(\beta).
\]

The relation between the homology theories is especially simple in the case when \( L(\beta) \) is a knot, as it is stated in theorem 1.2.2.

**Proof of theorem 1.2.2.** From the construction of the trace \( \mathcal{T}r(\beta) \), \( \beta \mathcal{Br}_n \) \( K = L(\beta) \) we see that the variables \( x, y \) in \( R(K) = \mathbb{C}[x, y] \) are the sums of the diagonal elements \( x = \sum_{i=1}^n x_{ii}, \quad y = \sum_{i=1}^n y_{ii} \). Since these elements are not changed by a conjugation, the statement follows.

Thus we have shown (1.3) and the theorem 1.0.1 follows.
6. Two-colored link invariants

It is very likely that all known triply-graded categorifications of the knot homology coincide or differ by some trivial factor. However the case of links is more interesting, it is known that there are several link invariants that are equal HOMFLYPT polynomial on the knots but differ from HOMFLYPT on the general links. Thus we do not expect any kind of uniqueness of HOMFLYPT homology of links.

In this section we discuss the invariants of the links colored with two colors. If link is a knot the homology discussed here match with homology from the previous sections and homology from [OR18d]. For multi-component links we expect existence of the spectral sequence connecting these homology with the dualizable link homology.

6.1. Invariants of dichromatic links. Since we will work with the links colored with two colors, we are lead to a definition of the groupoid of dichromatic braids \( \mathcal{Br}^{dic}_{n} \). An element of the groupoid is a triple \( s, \beta, t \), \( s, t \subset \{1, \ldots, n\} \), \( \beta \in \mathcal{Br}_n \) and \( \beta(s) = t \). It is convenient to use notation \( \beta = (s, \beta, t) \) and \( s(\beta) = s, t(\beta) = t \). The braids \( \alpha \cdot \beta \) is defined if \( t(\alpha) = s(\beta) \).

Now we would like to construct an analogue of the homomorphisms \( \Phi^{br} \) and \( \bar{\Phi}^{br} \) for the dichromatic braids. We define \( \mathcal{X}^2[c, c'] = \mathfrak{g} \times G \times \mathfrak{b}[c] \times G \times \mathfrak{b}[c'] \), \( \overline{\mathcal{X}}^2[c, c] = \mathfrak{b}[c] \times G \times \mathfrak{b}[c'] \).

Respectively, we define the corresponding categories of matrix factorizations:

\[
\text{MF}(\sigma, c, \tau) = \text{MF}_{G \times \mathbb{Z}^2}(\mathcal{X}_2[\sigma^{-1}(c), \tau^{-1}(c)], W_{\sigma, \tau}), \quad \text{MF}_{\mathfrak{b}}(\sigma, c, \tau) = \text{MF}_{\mathbb{Z}^2}(\overline{\mathcal{X}}_2[\bar{c}, \tau^{-1}(c)], \overline{W}_{\sigma, \tau}).
\]

where \( W_{\sigma, \tau} \overline{W}_{\sigma, \tau} \) are given by the formulas (2.1) and (2.3).

We have the relation between the potentials:

\[
W_{\sigma, \tau}(X, g, Y_1, Y_2) = \overline{W}(X', g, Y_2') + \sum_{i \in c} X_i \delta(Y_1, \sigma(i), \sigma(i)) - \text{Ad}_g(Y_2)\tau(i), \tau(i)) \quad + \text{Tr}(X_\text{alt}((Y_1)_{++} + \text{Ad}_g(Y_2)_{++})),
\]

where \( X', Y_2' \) are projections of \( X, Y \) on \( \mathfrak{b}[\bar{c}] \) and \( \mathfrak{b}[\tau(c)] \) respectively. Thus just like in previous cases there is a Knorrer functors:

\[
\text{MF}(\sigma, c, \tau) \xrightarrow{\Phi} \overline{\text{MF}}[\sigma, c, \tau] \xleftarrow{\Psi} \text{MF}[\sigma, c, \tau]
\]

such that \( \Psi \circ \Phi = 1 \). We call the category \( \overline{\text{MF}}[\sigma, c, \tau] \) a short version of the category \( \text{MF}[\sigma, c, \tau] \).

The convolution space for the big category in this setting is \( \mathcal{X}^3[c] = \mathfrak{g} \times (G \times \mathfrak{b}[c])^3 \) and the monoidal structure \( \ast \) on the categories \( \text{MF}[\bullet, \bullet, \bullet] \) is defined by the formulas (2.2):

\[
\ast : \text{MF}(\sigma, c, \tau) \times \text{MF}(\tau', c, \rho) \to \text{MF}(\sigma, c, \rho).
\]
We can define the convolution structure \( \hat{\ast} \) on the category \( \text{MF}[\bullet, \bullet, \bullet] \) using Knorrer functors but we can also define the convolution in short category explicitly. To define the convolution few notations are needed. For a subset \( c \subset \{1, \ldots, n\} \) we generalize the upper-triangular truncation by

\[
X_{+c} = X_{++} + \Delta[c](X),
\]

where \( \Delta[c](X) \) is the diagonal matrix consisting of \( i \)-th, \( i \in c \) diagonal entries of \( c \).

The modified projection maps \( \pi_{12}[\sigma, \rho, c](X, g_{12}, g_{23}, Y) = (X, g_{12}, \text{Ad}_{g_{23}}(\rho \cdot Y)_{\sigma(c)+}) \),

\[
\pi_{23}[\sigma, \rho, c](X, g_{12}, g_{23}, Y) = (\text{Ad}_{g_{12}}^{-1}(X)_{c+}, g_{23}, Y),
\]

\[
\pi_{13}[\sigma, \rho, c](X, g_{12}, g_{23}, Y) = (X, g_{12}g_{23}, Y).
\]

Thus we define the convolution in the short category by

\[
\mathcal{F} \ast \mathcal{G} = \pi_{13*}[\sigma, \rho, c](\text{CE}_n(\pi_{12}^{*}[\sigma, \rho, c](\mathcal{F}) \otimes_B \pi_{23}^{*}[\sigma, \rho, c](\mathcal{G}))^{\text{Tr}(2)}) \in \text{MF}[\sigma, c, \rho],
\]

for \( \mathcal{F} \in \text{MF}[\sigma, c, \tau], \mathcal{G} \in \text{MF}[\tau, c, \rho] \).

Many of the categories \( \text{MF}[\bullet, \bullet, \bullet] \) are isomorphic since for any \( \alpha \in S_n \) there is a natural functor:

\[
\alpha : \text{MF}[\sigma, c, \tau] \to \text{MF}[\alpha \sigma, \alpha^{-1}(c), \tau \alpha].
\]

The the subset \( \mathcal{B}_{n,k}^{\text{dic}} \subset \mathcal{B}_n^{\text{dic}} \) of dichromatic braids \( \beta \) with \( |s(\beta)| = k \) is a sub-algebroid. For any \( c, |c| = k \) there is natural homomorphism the algebroids:

\[
\pi_c : \mathcal{L} \mathcal{B}_k \to \mathcal{B}_n^{\text{dic}},
\]

defined by \( \pi_c((s, \beta, t)) = (s(c), \beta, t(c)) \). Below we discuss the counter part of this homomorphism in convolution algebras of matrix factorizations.

The inclusion map \( i[c, c'] : \mathcal{F}_2[c, c'] \to \mathcal{F}_2 \) provides us the convolution algebra homomorphism. We will not need full generality of the intertwining statement, for our purposes it is enough to show the analog of lemma 4.2.2 in this setting.

**Proposition 6.1.1.** We have the commuting square of homomorphisms:

\[
\begin{array}{ccc}
\mathcal{L} \mathcal{B}_n & \xrightarrow{\Phi} & (\text{MF}[\bullet, \bullet, \bullet]) \\
\downarrow{\pi_c} & & \downarrow{i^*} \\
\mathcal{B}_{n,k}^{\text{dic}} & \xrightarrow{\Phi} & (\text{MF}[\bullet, c, \bullet], \bullet)
\end{array}
\]

where \( i \) is an abbreviation for \( i[\overline{c}, \bullet] \).

**Proof.** We only need to show that \( i^* \) is a homomorphism of the convolution algebras, the bottom arrow of the square is defined by commutativity of the square. We denote by \( i[\sigma, c] \) the natural embedding of \( \mathcal{F}_3[\rho, c] \) into \( \mathcal{F}_3 \) then lemma below and the argument of the proof of [1,2,2] implies the statement. \( \square \)
Lemma 6.1.2. Let us assume that $\mathcal{F} = (M', D', \tilde{c}') \in \underline{\text{MF}}_{\sigma, \tau}$, $\mathcal{G} = (M'', D'', \tilde{c}'') \in \underline{\text{MF}}_{\tau, \rho}$ and that the differentials $D', \tilde{c}', D'', \tilde{c}''$ are linear along the factors $b$ in $\mathcal{F}_{2}$, Then there is a homotopy

\[ i^*[\rho, c](\tilde{\pi}_{12, 9, \rho}^*(\mathcal{F}) \otimes B \tilde{\pi}_{23, 9, \rho}^*(\mathcal{G})) \sim \tilde{\pi}_{12}^*[\sigma, \rho, c](i^*[\tilde{c}, \sigma^{-1}(c)](\mathcal{F})) \otimes B \tilde{\pi}_{23}^*[\sigma, \rho, c](i^*[\tilde{c}, \rho^{-1}(c)](\mathcal{G})). \]

Proof. We can write an explicit formula for the homotopy. To simplify notations, we assume that all equivariant correction differentials vanish. Also we set $\tau = 1$, because of (6.1) this assumption is not restrictive.

We also identify $\mathcal{F}$, $\mathcal{G}$ with the pull-backs $\tilde{\pi}_{12, 9, \rho}^*(\mathcal{F}), \tilde{\pi}_{23, 9, \rho}^*(\mathcal{G})$ and we use notation $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ for the pull-backs $\tilde{\pi}_{12}[\sigma, \rho, c](i^*[\tilde{c}, \sigma^{-1}(c)](\mathcal{F}))$ and $\tilde{\pi}_{23}[\sigma, \rho, c](i^*[\tilde{c}, \rho^{-1}(c)](\mathcal{G}))$. The convolution spaces $\tilde{\mathcal{F}}_{3}$ and $\tilde{\mathcal{G}}_{3}[\rho, c]$ have standard coordinate systems $X, g_{12}, g_{23}, Y$ and two other coordinate systems pulled back along the maps $\tilde{\pi}_{12, 9, \rho}, \tilde{\pi}_{23, 9, \rho}$ and $\tilde{\pi}_{12}[\sigma, \rho, c]$, $\tilde{\pi}_{23}[\sigma, \rho, c]$. We denote the last two coordinate system by $(X', g_{12}, g_{23}, Y')$ and $(X'', g_{12}, g_{23}, Y'')$.

By taking the derivatives of the potentials $\tilde{\mathcal{F}}_{\sigma, 1}$, $\tilde{\mathcal{G}}_{1, \rho}$ we get:

\[ (6.2) \quad [\tilde{c}_{X_{ii}'} D', D'] = \sigma^{-1} \cdot X_{ii}' - (\text{Ad}_{g_{12}^{-1} X'})_{ii}, \quad [\tilde{c}_{X_{ii}''} D'', D''] = Y_{ii}'' - (\text{Ad}_{g_{23} \cdot Y''})_{ii}. \]

Since all differentials and the potentials are $b$-linear by taking the second derivatives of the potentials along $b$ we prove that operators

\[ u_{i} = \tilde{c}_{X_{ii}''} D' \otimes \tilde{c}_{X_{ii}''} D'', \]

satisfy relations $u_{i}^{2} = 0$, $u_{i} u_{j} + u_{j} u_{i} = 0$. Hence we can define the homotopy map by

\[ U = \exp(- \sum_{i \in \sigma(c)} u_{i}). \]

Finally, let us observe that in (6.2) $\sigma^{-1} \cdot X_{ii}' = 0$ for $i \in \sigma(c)$ and $(\text{Ad}_{g_{12}^{-1} X'})_{ii}, i \in \sigma(c)$ is sent to zero by the pull-back $i^*[\rho, c]$. Thus combining all previous observations we arrive to

\[ U(D' \otimes I + I \otimes D') U^{-1} = \tilde{D}' \otimes I + I \otimes \tilde{D}'', \]

where $\tilde{D}'$ and $\tilde{D}''$ are the differentials of the pull-backs $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$. \qed

6.2. Traces. There is natural trace functor on the categories $\underline{\text{MF}}[\sigma, c, \tau]$. Similar to the previous construction we define the embedding and the pull-back:

\[ j_{c}[c', c''] : b[c'] \times b[c''] \to \mathcal{F}_{2}[c', c''], \quad j^{*}_{c}[c', c''] : \text{MF}[\sigma, c, \tau] \to D_{B}^{\text{per}}(b[c] \times b[\tau^{-1}(c)]). \]

The set of diagonal elements of the product $b[c] \times b[\tau^{-1}(c)]$ naturally isomorphic to $\mathfrak{h}$. Thus by taking the $B$-invariants and tensoring with the exterior powers of the tautological bundle we arrive to the functor:

\[ \mathcal{T} r_{k} : \text{MF}[\sigma, c, \tau] \to D_{B}^{\text{per}}(\mathfrak{h}), \quad \mathcal{T} r_{k}(\mathcal{F}) = \text{CE}_{\mathfrak{n}}(j^{*}_{c}(\mathcal{F}) \otimes \Lambda^{k} \mathcal{B})^{T}. \]

The dichromatic braid $\beta$ is called closable if $s(\beta) = t(\beta)$. For a closable braid $\beta$ both sets $s(\beta)$ and $t(\beta)$ are unions of the orbits of the action of $\beta$. Let us assume that $\beta = \pi_{c}(\tau, \beta_{\alpha})$ then the identification between $\mathfrak{h}$ and $\mathfrak{h}[c] \times \mathfrak{h}[\tau^{-1}(c)]$ naturally extends to the identification between corresponding $\beta$-invariant pieces of the spaces. We denote by $\pi[\sigma, c, \tau]$ the
corresponding linear projection $h^\delta \to h$ and for the triples $\sigma, c, \tau$ such that $\sigma(c) = \tau(c)$ we define:

$$\mathcal{T}_{r_k} : \text{MF}[\sigma, c, \tau] \to D^\per(h^\delta), \quad \mathcal{T}_{r_k} = \pi_*[\sigma, c, \tau] \circ \mathcal{T}_{r'_k}.$$ 

where $\delta = \tau^{-1}\sigma$.

This last trace is compatible with the previously defined trace via pull-back morphism. Let $\check{j}[\sigma, c, \tau]$ be the embedding of $h^\delta = b[c]^\delta \times b[\tau(c)]^\delta$ inside $h^\delta \times h^\delta$, then we have

**Proposition 6.2.1.** For $\mathcal{F} \in \overline{\text{MF}}_{\sigma, \tau}$ and $c$ such that $\sigma(c) = \tau(c)$ we have the following relation between the traces:

$$\check{j}^*[\sigma, c, \tau] \circ \mathcal{T}_{r_k}(\mathcal{F}) = \mathcal{T}_{r_k}(\check{i}^*[c, \tau^{-1}(c)](\mathcal{F})).$$

**Proof.** We claim that both sides of the equations are two ways to travel from lower right corner of the commuting diagram to the upper left corner:

$$h^\delta \leftarrow_{\pi[\sigma, c, \tau]} h[c] \times h[\tau^{-1}(c)] \to h[c] \times h[\tau^{-1}(c)] \to \overline{\mathcal{F}}_2[c, \tau^{-1}(c)] \to \overline{\mathcal{F}}_2.$$ 

To be more precise to get both sides of the equation we need to pull-back along the labeled arrows, with exception of the arrows with labels $\pi[\sigma, c, \tau]$ and $\pi_\delta$, where we use push-forward, and apply the averaging functor $\text{CE}_n(\bullet)^T$ to in the reverse direction along the labels unmarked arrows.

Maps $j_{KN}$ is defined as an embedding of $h^\delta \times h$ into $h \times h$ defined with the help of identification between $b[c]^\delta \times b[\tau^{-1}(c)]^\delta$ and $h^\delta$ and $b[c] \times b[\tau^{-1}(c)]$. We use the same identifications to define the projection $\pi_\delta$. The dotted arrow is given by the projection of $h = b[c] \times b[\tau^{-1}(c)]$ onto $h^\delta = h^\delta \times 0 \subset h^\delta \times h$.

The Knorrer functor from the definition of the trace is the composition of the functors $\pi_\delta \circ j_{KN}^*$. Thus indeed the RHS of the equation is obtained by tracing the diagram the way we described. The fact that the LHS is obtain this way is immediate from the construction of the trace.

Thus to prove the equation we use the base change for the left commuting square of the diagram and commuting property of right commuting square of the diagram. \hfill \Box

**6.3. Relations between the invariants.** Using (6.1) we can define the monoidal structure on the union of categories $\overline{\text{MF}}[\sigma, c] := \overline{\text{MF}}[\sigma, c, 1]$ by

$$\mathcal{F} \ast \mathcal{G} := \mathcal{F}_\check{y}\mathcal{G}(\tau^{-1} \cdot \check{y}), \quad \mathcal{F} \in \overline{\text{MF}}[\sigma, c], \quad \mathcal{G} \in \overline{\text{MF}}[\tau, c'],$$

here $c = \tau(c')$. Let us also point out that

$$\overline{\text{MF}}[\sigma, c] = \text{MF}_{B}(b[c] \times G \times b[c], W)$$

does not depend on $\sigma$ but this index participate in the definition of the monoidal structure. As usual we use notation $\overline{\text{MF}}[\bullet, \bullet]$ for the union of these categories.

The proposition (6.1) implies that we have
Corollary 6.3.1. There is a groupoid homomorphism:
\[ \Phi_{\text{dic}} : \mathcal{B}_n \to \text{MF}[\bullet, \bullet] \]
such that if \( \pi_c(1) = \beta \) then
\[ \Phi_{\text{dic}}(\beta) = \iota^*[c, s(c)] \circ \Phi_{\text{br}}^{1}(1) \).

Using homomorphism from the corollary we define the trace on \( \mathcal{B}_n \) by
\[ \mathcal{T}_r : \mathcal{B}_n^{\text{dic}} \to D^{\text{per}}(\mathcal{C}) \quad \mathcal{T}_r(\beta) := \mathcal{T}_r(\Phi_{\text{br}}^{1}(1)) \],
where \( \pi_c(1) = \beta \) and \( \mathcal{B}_n^{\text{dic}} \) is the set of braids \( \beta \) with \( \ell \) connected components. The trace is defined only for closable braids.

For a closable braid \( \beta \in \mathcal{B}_n^{\text{dic}} \) we define the ring \( R(\beta) = \mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_{\ell-k}] \) where \( k \) is the number of \( \beta \)-orbits on the set \( s(\beta) \). The ring \( R(\beta) \) is a module over the ring \( R(\beta) = \mathbb{C}[x_1, \ldots, x_\ell, y_1, \ldots, y_{\ell}] \) and can summarize the results of this section in

Theorem 6.3.2. For any closable \( \beta \in \mathcal{B}_n^{\text{dic}} \) the element
\[ \mathcal{T}_r(\beta) \in D^{\text{per}}(\text{Spec}(R(\beta))) \]
is an isotopy invariant of the closure and
\[ \text{Tr}(\beta) = \text{Tr}(\beta) \otimes_{R(\beta)} R(\beta). \]

The groupoid \( \mathcal{B}_n^{\text{dic, 0}} \) consisting of the braid \( \beta \) with \( s(\beta) = \emptyset \) is equal to the braid group and the module of derived global sections of the trace is exactly the homology from [OR18d]:
\[ R\Gamma(\mathcal{T}_r(\beta)) = H(\beta). \]

Hence the theorem 5.4.1 follows.

7. Further directions and conjectures

7.1. Relation with y-fied homology. In the paper [GH17] the authors construct an isotopy invariant of the closure of a braid \( L(\beta) \), they use notation \( \text{HY}(\beta) \) for the invariant. The invariant \( \text{HY}(\beta) \) has many features of the homology \( \text{HXY}(\beta) \). In particular \( \text{HY}(\beta) \) is naturally a module over the ring \( \mathbb{C}[x_1, \ldots, x_\ell, y_1, \ldots, y_{\ell}] \) where \( \ell \) is the number of connected components of the closure.

The authors of [GH17] also conjecture that their invariant is symmetric with respect to the switch of \( x \) and \( y \)-variables. So it is natural to put forward

Conjecture 7.1.1.
\[ \text{HY}(\beta) = \text{HXY}(\beta). \]
7.2. Flag Hilbert schemes and pure braids. The pure braid group $\mathcal{PB}_n$ embeds inside $\mathcal{LB}_n$ as subset of elements $\beta$ with the property $s(\beta) = t(\beta) = 1$. In particular, we have homomorphism:

$$\Phi^{\text{pbr}} : \mathcal{PB}_n \rightarrow \mathcal{MF}_{1,1}.$$ 

The category $\mathcal{MF}_{1,1}$ is very close to the category dg category $\text{DG}(F\text{Hilb}_n)$ of the flag Hilbert scheme. Indeed, in this case the potential is equal:

$$\overline{W}_{1,1}(X, g, Y) = \text{Tr}(X(Y - \text{Ad}_g Y)).$$

In particular, if $g$ is in the first formal neighborhood of 1 then the potential becomes a standard potential for dg structure of $F\text{Hilb}$. This observation together with some explicit computations from [OR18a] motivate us to propose

**Conjecture 7.2.1.** There is an $A_\infty$-deformation $(\mathcal{A}_\infty(F\text{Hilb}_n), \tilde{\otimes})$ of $(\text{DG}(F\text{Hilb}_n), \otimes)$ such that

$$(\mathcal{A}_\infty(F\text{Hilb}_n), \tilde{\otimes}) \simeq (\mathcal{MF}_{1,1}, \tilde{\otimes}).$$

7.3. Quantization. The ring $R(\beta)$ has a natural symplectic structure $\omega$. In particular, $R(\beta)$ has a natural quantization $D(\beta)$ as ring of differential operators on $\mathbb{C}^\ell$. The invariants of two-colored links as we explain is a restriction of the invariant $\mathcal{E}(L(\beta))$ to the Lagrangian subvariety inside $\text{Spec}(R(\beta))$.

In theory nothing prevents us from restricting $\mathcal{E}(L(\beta))$ to any subvariety inside $\text{Spec}(R(\beta))$ but seems to us that only restriction to the Lagrangian (or isotropic) subvariety descends to the level of braids, not just closures of the braids. That motivates us to put forward somewhat speculative

**Conjecture 7.3.1.** There is a complex of filtered $D(\beta)$-modules $\tilde{\mathcal{E}}(L(\beta))$ such that the associated graded module recovers our invariant:

$$\text{gr}(\tilde{\mathcal{E}}(L(\beta))) = \mathcal{E}(L(\beta)).$$

In the case of torus links $T_{n,k}$ the triply-graded homology have another interpretation in terms of homology the homogeneous Hitchin fiber [OY16]. The homology of the homogeneous Hitchin fiber is a fiber of the complex of constructible sheaves on the Hitchin base. We expect that the complex of the $D$-modules $\tilde{\mathcal{E}}(T_{n,k})$ is the Riemann-Hilbert transform of the mentioned complex of constructible sheaves on the torus-fixed part of Hitchin base.

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