Random Walks on Directed Networks: the Case of PageRank

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Abstract

PageRank, the prestige measure for Web pages used by Google, is the stationary probability of a peculiar random walk on directed graphs, which interpolates between a pure random walk and a process where all nodes have the same probability of being visited. We give some exact results on the distribution of PageRank in the cases in which the damping factor $q$ approaches the two limit values 0 and 1. When $q \to 0$ and for several classes of graphs the distribution is a power law with exponent 2, regardless of the in-degree distribution. When $q \to 1$ it can always be derived from the in-degree distribution of the underlying graph, if the out-degree is the same for all nodes.

1 Introduction

Since the letter of Pearson\textsuperscript{[Pearson, 1905]}, published on Nature in 1905, random walk has become a central concept in many branches of the physical sciences. The number of applications and studies dedicated to the subject is in fact so large that to give even a very partial list of references is an overwhelming enterprise (see Hughes\textsuperscript{[Hughes, 1995]} for a recent and fairly complete review). Most of the attention, so far, has been devoted to the study of random walks and related stochastic processes on $d$-dimensional euclidean spaces and regular lattices, for their obvious relevance to physical problems. To extend the definition of random walk to an arbitrary graph is trivial, but its study is relatively less developed. In this paper we address the issue of the stationary probability of a random walk on a directed scale-free graph.

The specific application we have in mind is the study of Pagerank (PR), the prestige measure that the search engine Google (and several other search engines) employs to measure the prestige of Web pages. When a user submits a query, the hits returned by Google are ranked according to their PR values. As it will be clear in a moment, such a measure is the stationary probability of a random walk on the Web graph, where each node represents a Web page and edges represent the hyperlinks (naturally directed) connecting the pages.

Let us consider an arbitrary undirected graph and a random walker moving on it. At any (discrete) time step the walker jumps from the node where it is sitting on to one of its neighbors chosen with equal probability. It is trivial to show that, at stationarity, the probability of each node to be visited is proportional to its degree, i.e. the number of neighbors of the node. If the graph is directed we have to distinguish (see Fig. \textsuperscript{[0]}) the links adjacent to a node in incoming (those that point to the node) and outgoing (those that point away from it). If the random walker is allowed to follow only the outgoing links from the node where it presently is, the problem of finding the stationary probability is far more complicated. Such probability will in general depend on the overall topological organization of the graph.
itself, and cannot be expressed in terms of simple topological quantities like the degree of a node. In fact, due to the directedness of the links, the graph may have regions that the walker can enter in but not escape from. The stationary probability will be trivially concentrated in these regions. In order to prevent this from happening we will consider a modified (directed) random walker whose behavior is defined by the following two rules:

- With probability $1 - q$ the walker follows any outgoing link of $i$, chosen with equal probability;
- With probability $q$ it moves to a generic node of the network (including $i$), chosen with equal probability.

This will suffice to ensure a non-zero stationary probability on every node. When considered in the context of the Web graph, the process described above could be thought as a rough modelization of a Web surfer that occasionally (with probability $q$) decides to interrupt his/her browsing and to restart it from a randomly chosen page. The stationary probability of this process is exactly $PR$. To adhere to the computer science terminology, we will refer to the probability $q$ as to the damping factor. The damping factor adopted in real applications is generally small ($q \sim 0.15$).

A brave analogy with the undirected case could lead to the hypothesis that $PR$ is roughly proportional to the in-degree of a node (number of incoming links), modulo corrections due to the small damping factor. Such a view could be further supported by the observation that the distribution of $PR$ for the real Web has a power law decay [Pandurangan et al., 2002] characterized by an exponent 2.1 (see Fig. 2), like the distribution of the in-degree [Albert et al., 2000] (note that, when referring to the Web and unless otherwise specified, we always assume a damping factor of $q \sim 0.15$). A direct measure of $PR$ versus in-degree on two large samples of the Web graph is shown in Fig. 3, where the value of $PR$ has been averaged over nodes with the same in-degree. The plot exhibits an almost linear behavior with deviations at small degrees, when the effect of the damping factor is more relevant. Mean field calculations show that there is a positive correlation between $PR$ and in-degree [Fortunato et al., 2005] and a linear relation between in-degree and the mean $PR$ for nodes of equal in-degree can be safely assumed if the degree correlations between adjacent nodes are weak. On a generic directed graph, the linear relationship between $PR$ (even if considered on average) and the in-degree is not granted and it depends on the global organization of the graph itself. To address the issue of $PR$ distribution for an arbitrary graph and a generic $q$ would therefore require a case by case study. In this paper, therefore, we concentrate on the two interesting limits, i.e. $q \rightarrow 0$ and $q \rightarrow 1$, that show some degree of universality. In these two limits it is possible to derive analytical expressions for the distribution of $PR$. For small $q$-values, a master equation approach allows us to solve the problem for special classes of networks. For $q \rightarrow 1$, it is possible to establish a
Figure 2: PR distribution for a large sample of the Web graph, produced by the WebBase collaboration in 2003 (www-diglib.stanford.edu/~testbed/doc2/WebBase/). The damping factor is \( q = 0.15 \).

one-to-one correspondence between the distribution of PR and that of in-degree, as long as the number of outgoing links from each node (out-degree) is the same. Further, to have a better control on the topological characteristics of the graph and how they correlate with the PR distribution we work with graphs generated by random processes or processes of growth.

2 PageRank

Let us consider a generic directed network with \( n \) nodes. Let \( p(i) \) be the PR of node \( i \). The vector \( p \) satisfies the following self-consistent system of relations:

\[
p(i) = q \frac{1}{n} + (1 - q) \sum_{j \rightarrow i} \frac{p(j)}{k_{out}(j)}, \quad i = 1, 2, \ldots, n
\]  

(1)

where \( j \rightarrow i \) indicates a link from \( j \) to \( i \) and \( k_{out}(j) \) is the out-degree of node \( j \). In the following we always assume that each node has at least one outgoing link, and therefore Eq. (1) is well defined. To compute \( p \) amounts to solve the eigenvalue problem for the transition matrix \( M \), whose element \( M_{ij} \) is given by the expression:

\[
M_{ij} = q \frac{1}{n} + (1 - q) \frac{1}{k_{out}(j)} A_{ji},
\]

(2)

and where \( A \) is the adjacency matrix of the graph (\( A_{ji} = 1 \) if there is a link from \( j \) to \( i \), otherwise \( A_{ji} = 0 \)).

The stationary probability of the process described by \( M \) is given by its principal eigenvector. Its calculation is a standard problem of numerical analysis and can be achieved by repeatedly applying the matrix \( M \) to a generic vector \( p_0 \) not orthogonal to \( p \). It is easy to show, in fact, that \( 1 = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_n \) (\( \lambda \)'s being the eigenvalues of \( M \)), and therefore \( \lim_{l \rightarrow \infty} M^l p_0 = p \). The powers of the matrix \( M \) introduce powers of the eigenvalues \( \lambda \) in the decomposition of \( p_0 \) in eigenvectors of \( M \).
calculating $p$ with an accuracy $\epsilon$, the vector $M^t p_0$ delivers $p$ with corrections at most of the order $\lambda_1$, so we can safely stop the procedure when $\lambda_1 \sim \epsilon$, i.e. when $l \sim \log(\epsilon)/\log(\lambda_1)$. In this way, the method converges rather quickly: in practical applications, it turns out that less than one hundred iterations suffice to calculate the PR of a network with $10^7 - 10^8$ vertices.

PR, and therefore its distribution, depends on the damping factor $q$ in a non-trivial way. A first rigorous investigation of this problem was presented in [Boldi et al., 2005] with focus on how the ranking of pages is influenced by changing $q$ and where some close expressions for derivatives of PR with respect to $q$ were derived. The damping factor can be considered as an interpolation parameter between a simple random walk and a pure scattering process. When $q = 0$, the process reduces to a simple random walk, and one may end up with a trivial invariant measure concentrated on a small subset of nodes. When $q = 1$, the walker can jump to any node at each step, with probability $1/n$. The PR of all nodes is then the same, and equals $1/n$, as one can see by setting $q = 1$ in Eq. (1). The distribution of PR is therefore a Dirac $\delta$ function centered at $1/n$. For $0 < q < 1$ the distribution is not trivial, and in general it strongly depends on the underlying graph. On the other hand, in the two limits $q \to 0$ and $q \to 1$, Eq. (1) assumes forms which lend themselves to simple analytical derivations and the PR distribution can be exactly determined for a large set of graphs. It is worth remarking that the limit of small $q$ is the relevant one for Web applications.

3 The general case of a direct-loopless graph

Given a generic directed graph, the PR of a specific node depends on the overall arrangement of the graph and cannot be calculated on the basis of local properties only. In the following we focus on a wide, although more restricted class of networks for which analytical solutions are possible. To this class belong networks obtained through a growth process that are particularly important for real world applications.

Let us label the nodes of the network $1, 2, ..., n$. We assume that if an oriented path from node $i$ to node $j$ exists, there is no path from $j$ to $i$ (in other words, it is impossible to get back to a given starting
point following an oriented path of the graph). Networks that result from a growth process, where new nodes are introduced at discrete time steps together with their new oriented links, obviously belong to this class. In fact, we can label nodes according to their age (node 1 being the oldest) so that a directed link between $i$ and $j$ may exist only if $i > j$.

It is easy to verify that the PR $p(i)$ of a generic node $i$ of a graph in this class can be written as follows:

$$p(i) = \frac{q}{n} \left[ 1 + \sum_{j=1}^{n} \sum_{l^{(j)} \in L_{ij}} \frac{(1-q)^{d(l^{(j)})}}{k(l^{(j)})} \right]$$

where the first sum runs over all nodes in the graph and the second over all paths from a generic node $j$ to node $i$. Each path in the second sum is weighted by as many factors $(1 - q)$ as links along the path ($d(l^{(j)})$ is the length of path $l^{(j)}$) and is also weighted by the inverse of the degree ($k(l^{(j)})$) of each node $s$ encountered along the path. Although correct for any $q$, the formula above is not very transparent. In order to get some understanding on the expected distribution of Pagerank in a graph, we specialize Eq. (3) to the case in which each node has a fixed number $m$ of outgoing links. We further focus on the limit $q \to 0$ that has an immediate interpretation in terms of walks over the graph. Under the assumptions above, the expression for the PR of a node $i$ simplifies to

$$p(i) = \frac{q}{n} \left[ 1 + \sum_{j=1}^{n} \sum_{l^{(j)} \in L_{ij}} \left( \frac{1 - q}{m} \right)^{d(l^{(j)})} \right]$$

(4)

In the following we show that if the graph is grown according to preferential attachment or copying mechanism and $q$ is sufficiently small, we should expect an algebraic distribution of PR characterized by an exponent 2, for any value of $m$. We will give the proof in the case $m = 1$ and a hint to a general proof in Sec. 3.4.

### 3.1 The limit $q \to 0$

Let us suppose that $q$ is very small ($q \sim 0$) and can be treated as an infinitesimal. Eq. (1), to the first order in $q$, reads:

$$p(i) \sim \frac{q}{n} \left[ 1 + \sum_{j \rightarrow i} \frac{p(j)}{k_{\text{out}}(j)} \right] \quad i = 1, 2, \ldots, n$$

(5)

where we have made the approximation $1 - q \sim 1$. The general expression in Eq. (3) grants that this approximation leads to the exact result. Since $m = 1$ there cannot be more than one path between two nodes and the network is an oriented tree. Under the assumption that nodes have out-degree 1, Eq. (5) reads:

$$p(i) \sim \frac{q}{n} \left[ 1 + \sum_{j \rightarrow i} p(j) \right] \quad i = 1, 2, \ldots, n.$$  

(6)

meaning that the PR of a node is the sum of a constant term ($q/n$) and the PR of its in-neighbors. In Fig. 4 we show a subgraph of a tree. Node $A$ is the root of the subgraph. A random walker moving from any node in the subtree and constrained by the directions of the links will necessarily reach $A$. We call therefore the nodes in the subtree predecessors of $A$ (we include $A$ among its predecessors). The three empty circles are “leaves” of the subgraph, as they have no incoming links. Starting from the leaves, and using Eq. (6) recursively, it is possible to calculate the PR of all nodes of the diagram. The values are reported next to the nodes. The figure shows that

- all PR values are multiples of the elementary unit $q/n$;
- PR increases if one moves from a node to another by following a link;
Figure 4: Subgraph of a tree. A node A is shown together with all its predecessors.

- the PR of each node $i$, in units of $q/n$, equals the number of its predecessors.

In the following, PR is measured in units of $q/n$, and, accordingly, the probability distribution is written as $P_{PR}(l)$, with $l = 1, 2, ..., n$. When a new node $N$ gets connected to a generic node of the subgraph of Fig. (4), the PR of node $A$ increases by $q/n$. Further, all the nodes on the path between $N$ and $A$ count $N$ as a predecessor and therefore they similarly increase their PR by $q/n$ (Fig. (5)). In the next subsections we specialize the above to networks grown by a linear preferential attachment mechanism, either explicitly (Barabási-Albert [Albert & Barabási, 1999] and Dorogovtsev et al. [Dorogovtsev & Mendes, 2000]), or implicitly (Copying model [Kleinberg et al., 1999]).

3.2 Explicit preferential attachment

In the model of Dorogovtsev et al. (DMS) [Dorogovtsev & Mendes, 2000], adapted to a directed graph, the probability that a new node $i$ attaches its link to a node $j$ (with in-degree $k_j$) is

$$\Pi(k_j, a) = \frac{a + k_j}{\sum_{i=1}^{\infty} (a + k_i)}$$

i.e. it only depends on the in-degree of the target node and on a real constant $a > 0$. Eq. (7) is a generalization of the linking probability of the Barabási and Albert (BA) model [Albert & Barabási].
Figure 5: If a new node $N$ gives its link to any node of the subgraph, the PR of the uppermost node $A$ will increase by $q/n$.

It is known that the DMS model leads to a scale-free in-degree distribution with exponent $\gamma = 2 + a$. We start from a network with $n$ nodes. The probability distribution of PR is, initially, $P_{PR}^n(l)$. In order to write a master equation that relates $P_{PR}^{n+1}(l)$ to $P_{PR}^n(l)$, one notes that the addition of node $n+1$ increases by $q/n$ the PR of all nodes in the path between $n+1$ and 1, while the others remain unaffected.

In this way, among the nodes of the path, PR $l-1$ will become $l$, whereas PR $l$ will become $l+1$. Let us consider a generic node $i$ with PR equal to $l$. The probability $\Pi^n_i$ that the new link will change the PR of $i$ from $l$ to $l+1$ is equal to the probability that the link is received by any predecessor of $i$ (including $i$), i.e.

$$\Pi^n_i = \sum_{j \rightarrow i} \frac{a + k_j}{\sum_{i=1}^{m}(a + k_i)}$$

where $j \rightarrow i$ indicates that $j$ is a predecessor of $i$. Note that even if other predecessors of $i$ (besides $i$ itself) increase their PR due to the attachment of the new node, they cannot reach the value $l+1$, as their initial values are necessarily smaller than $l$. Since all nodes have out-degree $m = 1$, the total
number of links of a network with \( n \) nodes is \( n - 1 \) (we assume that the first node does not create links) and the denominator of Eq. (8) takes the simple form

\[
\sum_{i=1}^{n} (a + k_i) = an + n - 1 = (a + 1)n - 1. \tag{9}
\]

The number of predecessors of \( i \) (see Fig. (4)). One finally obtains:

\[
\Pi^n = \sum_{j=1}^{\infty} \frac{a + k_j}{(a + 1)n - 1} = \frac{(a + 1)l - 1}{(a + 1)n - 1}. \tag{10}
\]

The probability \( \Pi^n(l) \) that the new link will alter the value of any node in the “PR class” \( l \) is then:

\[
\Pi^n(l) = nP^n_{PR}(l)\Pi^n = \frac{(a + 1)l - 1}{(a + 1) - 1/n} P^n_{PR}(l). \tag{11}
\]

The master equation then reads:

\[
(n + 1)P^n_{PR}(l) - nP^n_{PR}(l) = \Pi^n(l - 1) - \Pi^n(l). \tag{12}
\]

Eq. (12) is a balance equation: the left-hand side expresses the variation of the number of nodes in the “PR class” \( l \), after the addition of the \((n + 1)\)th node. The first term of the right-hand side is the probability that the introduction of the new node increases the number of nodes in the “PR class” \( l \) by one, the other term instead is the probability that a node leaves that class because its PR increases by one unit. Since a single link is added at each iteration, only one node can make either transition, so the right-hand side represents the expected variation in the population of nodes in the “PR class” \( l \), i.e. exactly what we have on the left-hand side of Eq. (12).

Note that Eq. (12) holds if \( l > 1 \). When \( l = 1 \), it must be modified, because there are no nodes with zero PR and the first term on the right-hand-side would be ill-defined. The modification, however, is simple. The new node \( n + 1 \) is a “leaf”, and it has PR 1. At each iteration, therefore, the population of “PR class” 1 is increased by one. We have

\[
(n + 1)P^n_{PR}(1) - nP^n_{PR}(1) = 1 - \Pi^n(1). \tag{13}
\]

We are interested in the stationary solutions of Eqs. (12) and (13), which can be derived by setting

\[
P^n_{PR}(l) = P^n_{PR}(l) = P_{PR}(l) \tag{valid in the limit when } n \to \infty). \]

In this limit, one can safely neglect \( 1/n \) in Eq. (11). After rearranging terms we obtain:

\[
P_{PR}(l) = \begin{cases} \frac{(a + 1)l - a - 2}{(a + 1)l + a} P_{PR}(l - 1), & \text{if } l > 1; \\ \frac{a}{2a + 1}, & \text{if } l = 1. \end{cases} \tag{14}
\]

which leads to:

\[
P_{PR}(l) = \frac{a(a + 1)}{((a + 1)l + a)[(a + 1)l - 1]} \sim \frac{1}{l^2}, \text{for } l \gg 1. \tag{15}
\]

The probability distribution of PR for a network built according to the DMS model has a power law tail with exponent \( \beta = 2 \), independently of \( a \). Fig. (6) shows PR distributions obtained from numerical simulations. They refer to three DMS networks, with parameter \( a = 1/2, 3, \text{ and } 10^5 \), respectively. The number of nodes is \( n = 10^6 \) and \( q = 0.001 \). The tails of the three curves are straight lines in the double-logarithmic scale of the plot, indicating a power law decay, and they are parallel. The continuous line has the slope of the predicted trend, showing an excellent agreement.

As noted above, our analytical result and the simulation for \( a = 10^5 \) shows that \( \beta \) is independent of the parameter \( a \), surprisingly in contrast with what happens for the in-degree, that, in the limit \( a \to \infty \), turns out to have an exponential distribution. The networks whose PR distributions are shown in the plot have been generated with \( m = 3 \). Fig. (6) then confirms that our result holds even when \( m > 1 \).
3.3 Implicit preferential attachment: the Copying model

The Copying model (CM) [Kleinberg et al., 1999; Krapivsky & Redner, 2001] was originally introduced to model the growth of the Web graph. It is based on the reasonable assumption that Web administrators, in creating a new page, often “copy” hyperlinks of pages they know. In this framework, a newly created node $i$ is a copy of a randomly chosen existing node $j$. This implies that $i$ sets links to all the neighbors of $j$. Then, with probability $\alpha$, those links are rewired to other nodes, again chosen at random. The model produces a scale-free network with a power law in-degree distribution characterized by an exponent $\gamma = (2-\alpha)/(1-\alpha)$.

Although the linking mechanism is apparently unrelated to the degree of the target node, a closer inspection reveals that the copying mechanism implies an effective linear preferential attachment [Pastor-Satorras & Vespignani, 2004]. To derive the PR distribution, we follow closely the strategy of the previous subsection.

In order to affect the PR of a node $i$, the link set by the new node $n+1$ must again attach to a predecessor of $i$. It is useful to distinguish between the “copying” phase and the “rewiring” phase of the linking process.

In the copying phase, to affect the PR in $i$, the target node has to be a predecessor of $i$, excluding $i$ itself. After the rewiring phase, the node $i$ will avail itself of a new contribution in PR if the new link is untouched by the rewiring or rewired to another predecessor of $i$ (this time including $i$ itself). Let’s assume that node $i$ is originally in “PR class” $l$. The probability to pick at random a predecessor of $i$ is $l/n$, if we include $i$, or $(l-1)/n$, if we exclude $i$. So, the probability $\Pi^a_l$ that the new link will change the PR of $i$ is:

$$\Pi^a_l = (1-\alpha) \frac{l-1}{n} + \alpha \frac{l}{n} = \frac{l + \alpha - 1}{n}. \quad (16)$$

The $\alpha$-dependent terms express the probability to have copying $(1-\alpha)$ and rewiring $(\alpha)$. From Eq. (16), one can extend the result to all nodes with PR $l$, like in Eq. (17):

$$\Pi^a(l) = nP^a_{PR}(l)\Pi^a_i = (l + \alpha - 1)P^a_{PR}(l). \quad (17)$$

Plugging the expression of $\Pi^a(l)$ in the balance equations (12) and (13), one obtains the following sta-
tionary solutions

\[ P_{PR}(l) = \begin{cases} \frac{l+\alpha-2}{l+\alpha} P_{PR}(l-1), & \text{if } l > 1; \\ 1, & \text{if } l = 1. \end{cases} \]  

(18)

From the recursive relation of Eq. (18) the final expression for the PR distribution follows

\[ P_{PR}(l) = \frac{\alpha}{(l+\alpha)(l+\alpha-1)} \sim \frac{1}{l^2}, \text{for } l \gg 1. \]  

(19)

The result is analogous to the one obtained in the previous section. Since, as mentioned above, the linking mechanism of the CM hides an effective linear preferential attachment, the result is not totally unexpected.

A numerical test of the prediction in Eq. (19) can be found in Fig. 7, where the PR distributions for three networks built with the CM, with \( \alpha \) equal to 0.1, 1/2 and 1, respectively, are shown. The other relevant parameters are \( m = 3, n = 10^6 \) and \( q = 0.001 \). All the curves show the same slope (with exponent 2) in a double-logarithmic plot. Note that the CM with \( \alpha = 1 \) generates a network with an exponential in-degree distribution, analogously to the DMS model in the limit \( a \to \infty \). Again, this fact does not affect the PR distribution.

3.4 Hint to a general proof

We now hint to the possibility to extend the proof presented above to the case \( m > 1 \). Let us work in the preferential attachment framework. Starting from Eq. (14), we need to introduce the quantity

\[ p_{ij} = \sum_{l \in L_{ij}} \frac{1-q}{l+\alpha} d(l') \]  

i.e. the contribution of node \( j \) to the PR in node \( i \). This quantity, obviously, does not change in time. The addition of a new node at time \( t \) (therefore the node is labelled \( t \)) contributes, on average, to the PR in \( i \)

\[ p_{it} = \sum_{j \in n,n,i} p_{ij} \frac{k_j(t)}{2mt} (1-q) \]  

(20)
where the sum runs over the nearest neighbors of the new node $t$ and $k_j(t)$ is the degree of node $j$ at time $t$. Taking the average over all realizations of the process of growth and the limit for continuous time, one arrives to the following equation:

$$p(t_0, t) = \int_{t_0}^{t} p(t_0, s)k(s, t)\frac{(1-q)}{2mt}ds$$  \hspace{1cm} (21)$$

where $p(t_0, t)$ is the average contribution to the PR of a node born at time $t_0$ from a node born at time $t$, and $k(s, t)$ is the average degree at time $t$ of a node born at time $s$. If $p(t_0, t)$ can be explicitly found, then $p_T(t_0)$, the average PR at a generic time $T$ of a node born at time $t_0$ can be easily calculated as $p_T(t_0) = \int_{t_0}^{T} p(t_0, t)dt$. To compute $p(t_0, t)$ we need $k(s, t)$ first. In the context of preferential attachment $k(s, t)$ is found to be $k(s, t) = m(t/t_0)^{1/2}$ (this easily follows from $dk/dt = k/2mt$ and $k(t_0, t_0) = m$). This expression provides the kernel for the integral equation (21). Once Eq. (21) is solved (taking into account the correct boundary condition $p(t_0, t_0) = 1$) and the result properly integrated, it gives, in the limit $T >> t_0$ and $q \to 0$, $p_T(t_0) \propto T^{3/2}/t_0$, which in turn gives the expected result: PR is algebraically distributed with an exponent equal to 2 independently of $m$. In general the kernel $k(s, t)/2tm$ in (21) needs to be replaced by that appropriate to the growth model under consideration.

3.5 Beyond preferential attachment

We have seen that the PR distribution for special networks has a power law tail with exponent 2, independently of the in-degree distribution of the network, which needs not even be a power law (e.g. DMS model for $a \to \infty$, CM for $\alpha \to 1$). This evidence, together with the observation that the PR distribution for the real Web (where a relatively small $q$ is usually employed) has also a power law distribution with exponent close to 2, may erroneously lead to the conclusion that the above result applies to a general graph.

A numerical test on a random graph a la Erdős-Rényi [Erdős & Rényi, 1959] shows the limits of the validity of our result. An Erdős-Rényi graph is built starting from a set of $n$ nodes, and setting a link independently and with a probability $r$ between any pair of nodes. The resulting network has a Poissonian degree distribution, with mean $rn$. In order to make the graph directed, we orient the link $i \to j$ with equal probability from $i$ to $j$ or from $j$ to $i$. There is no “center” and no PR flux towards a core of nodes, unlike the networks we have studied above. All nodes will thus have equal rights, and we expect little differences in their PR values. Fig. (3) shows the PR distribution for a random graph with 50000 nodes and $r = 0.0002$; the damping factor $q$ is 0.01. The distribution appears to be a Poissonian, like that of in-degree.

It would be interesting to understand whether the result presented in this paper holds for all networks in which random walkers stream towards a core of nodes. We expect the PR distribution to be a power law quite generally, but we have no arguments hinting to a universal occurrence of the exponent 2. Numerical evidences suggest, in fact, that other exponents are possible. In Fig. (4) we show the small-$q$ PR distribution for a citation network of U.S. patents ($q = 0.001$). Citation networks are practical examples of the directed trees we have analyzed so far, as a new paper must necessarily cite older papers. The data [Hall et al., 2001] refer to over 3 million U.S. patents granted between January 1963 and December 1999, and comprise all citations made to these patents between 1975 and 1999. The PR distribution is skewed, as expected, but the slope of the tail is quite different from 2, being close to 3.

4 The limit $q \to 1$

When $q = 1$ all nodes have the same PR value $1/n$. In the following we study the limit $q \to 1$ but $q \neq 1$. In our Eq. (1), the constant $q/n \sim 1/n$ is now much larger than the sum on the right-hand-side (we treat
1 − q as an infinitesimal). The PR distribution will then be very narrow and squeezed towards $q/n$, which is not interesting. However, the sum over the neighbors in Eq. (14) determines the variable contribution to PR, which is responsible for the differences in PR between the nodes. Therefore, we isolate this piece, and call it reduced PageRank (RPR). So, the RPR $p_r(i)$ of a node $i$ is defined as

$$p_r(i) = p(i) - \frac{q}{n}$$

(22)

The RPR is the probability that, during the PR process, a node is visited by a walker coming through any of its incoming links. One can show that the distribution of RPR coincides with the in-degree distribution on every graph, provided the out-degree is a constant $m$. In this case, in fact, when we replace PR with RPR through the relation (22), Eq. (1) assumes the following form

$$p_r(i) = \frac{1 - q}{m} \sum_{j: j \rightarrow i} [p_r(j) + q/n] = \frac{q(1 - q)}{mn} k_{in}(i) + \frac{1 - q}{m} \sum_{j: j \rightarrow i} p_r(j),$$

(23)

where $k_{in}(i)$ is the in-degree of $i$. From Eq. (19) it follows that the RPR of a node is of order $1 - q$. All terms coming from the sum are of order $(1 - q)^2$ and can be safely neglected. Finally,

$$p_r(i) \sim \frac{q(1 - q)}{mn} k_{in}(i), \quad i = 1, 2, \ldots, n.$$

(24)

The RPR of a node is then proportional to its in-degree, and the corresponding distributions coincide, under no assumptions other than the out-degree is a constant. Therefore, the result has a wide generality. It is also intuitive how to extend it to the case in which the out-degree is not constant but approximately the same for all nodes. Out-degree distributions concentrated about some value, like Gaussians, Poissonians, exponentials, etc., should not change the result.

Figure 8: Small-q PR distribution for an Erdős-Rényi random graph.
Figure 9: Small-q PR distribution for a citation network of U. S. patents. The continuous line is a power law fit of the tail.

Fig. (10) shows a test of Eq. (24). Each of the four plots is a scatter plot relative to a different network; three of them are scale-free and one has an exponential in-degree distribution (bottom right), as it has been generated with a CM process for $\alpha = 1$. The RPR of a generic node is compared with the right-hand-side of Eq. (24). The continuous line represents the equality of the two variables. The comparison with the data points is excellent in all cases.

After the submission of the paper we realized that the result of this section had already been derived and presented in [Chen et al., 2006]. We apologize with Chen and colleagues for the unfortunate accident.

5 Conclusions

Since the birth of Google, PR has attracted a lot of interest from the scientific community, but the deep reasons behind its capacity to capture the “quality” better than other and more used topological descriptors (e.g. in-degree) are not yet clear. We studied PR in a more general framework than its original field of application (the Web graph). We derived some exact results for PR distributions in the limit when the damping factor $q$ approaches the two extreme values 0 and 1. When $q \to 0$, for networks without directed loops and where walkers stream towards a central core of nodes (roots), PR can be in principle calculated in a single sweep over the nodes, starting from the leaves and converging shell-wise towards the center. This feature allowed us to calculate exactly the distribution of PR for networks built according to some peculiar linking strategies, like that of the DMS model (which includes the BA model as a special case) and of the CM. In these cases, the PR distribution has a power law tail with exponent 2, for any choice of the model parameters, that, on the contrary, strongly affect the in-degree distribution. This possibly suggests that the PR process allows to diversify the roles of the different nodes much more than in-degree, and it is a better criterion to rank nodes. Many networks have the features that grant, on a first approximation, the applicability of our results. Networks grown about one or more centers, with new nodes pointing mostly to older nodes belong to this class. The Web itself could be taken as an example of this kind of networks. The PR distribution of the Web graph is usually calculated for $q = 0.15$, which is quite close to zero, showing an exponent indeed very close to 2 (see Fig. (24)). Work is in progress to determine what are the broadest conditions that yield this “universal” behavior.
In the limit $q \to 1$, PR is a linear function of in-degree, as long as the out-degree of the nodes is fixed. The relation holds at the level of the single node, and not merely in the statistical sense. We plan to investigate how general this result is by relaxing the assumption of constant out-degree and trying various distributions.

To summarize, the PR distribution strongly depends on the value of the damping factor $q$, is in general “uncorrelated” from the corresponding in-degree distribution, but depends on the overall topological organization of the graph. This is not in contradiction with the findings of Fortunato et al. [2005], where a correlation between the two variables was observed, because the correlation involves the in-degree and the mean PR-value of all nodes with that in-degree. Within each in-degree class PR has large fluctuations.

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