Explicit matrix coefficients and test vectors for
discrete series representations

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Abstract

For the discrete series representations of GL(n) over a non-archimedean local field F, we define a notion of functions similar to ”zonal spherical functions” for the unramified principal series. We prove the existence of such functions in the level 0 case. As for unramified principal series, they give rise to explicit matrix matrix coefficients. We deduce a local proof of a known criterion of distinction of discrete series, in the level 0 case, for the Galois symmetric space GL(n, F)/GL(n, F_0), for any unramified quadratic extension F/F_0. We also exhibit explicit test vectors when these representations are distinguished.

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Introduction

In this introduction and throughout the article, F denotes a non-archimedean, non-discrete, locally compact field of any characteristic.
The aim of this work is twofold. First we show how to exhibit explicit matrix coefficients for level 0 discrete series representations of $GL(n,F)$. As an application, we obtain new results for the symmetric space $GL(n,F)/GL(n,F_0)$, for an unramified quadratic extension $F/F_0$. We give an entirely local proof of a known criterion of distinction for level 0 the discrete series representations of $GL(n,F)$, as well as explicit test vectors when these representations are distinguished.

The idea to obtain explicit matrix coefficients is to construct “zonal spherical functions” similar to classical Satake and Macdonald spherical functions attached to unramified principal series representations (see [MacD], [Sat], [Cass]). We first recall this classical framework.

Let $G$ be a connected reductive algebraic group defined over our base field $F$. For simplicity sake, let us assume that $G$ is split over $F$. Let $B$ be a Borel subgroup of $G = G(F)$ that we write $B = TU$, for a maximal $F$-split torus $T \subset B$ and where $U$ is the unipotent radical of $B$. Let $\chi$ be a regular smooth unramified character of $T$ and $\pi_\chi$ be the unramified principal series representation induced from $\chi$. Let $K$ be a good maximal compact subgroup of $G$. It is well known that the Hecke algebra $\mathcal{H}(G,K)$ is commutative and that the fixed vector space $\pi^K_\chi$ is 1-dimensional. In particular $\mathcal{H}(G,K)$ acts via a character $\psi_\chi$ on the line $\pi^K_\chi$. Let $c_\chi$ be a matrix coefficient of $\pi$ attached to non-zero $K$-invariant vectors in $\pi$ and its contragredient. Let $\mathcal{H}(G,K)$ be the $\mathbb{C}$-vector space of bi-$K$-invariant functions on $G$ (of any support). It is not an algebra, but nevertheless it is a left $\mathcal{H}(G,K)$-module for the action given by convolution. We let

$$\psi_\chi \mathcal{H}(G,K) = \{ f \in \mathcal{H}(G,K) ; \varphi \star f = \psi_\chi(\varphi).f, \ \varphi \in \mathcal{H}(G,K) \}.$$  

Then the space $\psi_\chi \mathcal{H}(G,K)$ is known to be 1-dimensional and the element $\Phi_\chi$ satisfying $\Phi_\chi(1) = 1$ is called the zonal spherical function attached to $\chi$. It turns out that $c_\pi$ is proportional to $\Phi_\chi$ and that an explicit formula for $\Phi_\chi$ is known in terms of the so called Macdonald’s polynomials.

The same kind of ideas adapt to the case of discrete series representations of $G = GL(n,F)$. Let $\mathcal{S}(G)$ denote the category of smooth complex representations of $G$. Let $(J,\lambda)$ be a simple type in $G$ in the sense of Bushnell and Kutzko [BK1]. In particular $J$ is an open compact subgroup of $G$ and $\lambda$ is an irreducible smooth representation of $J$. Let $\mathcal{S}_\lambda(G)$ denote the full subcategory of $G$ whose objects are those representations $(\pi,V)$ that are generated as $G$-modules by their $\lambda$-isotypic component $V^\lambda$. We denote by $\mathcal{H}_\lambda$ (resp. $\mathcal{H}_\lambda$) the spherical Hecke algebra (resp. the $\mathbb{C}$-vector space) of compactly supported functions (resp. functions of any support) $\psi : G \rightarrow \operatorname{End}_\mathbb{C}(W_\lambda)$.
such that
\[ \psi(j_1gj_2) = \tilde{\lambda}(j_1) \circ \psi(g) \circ \tilde{\lambda}(j_2), \quad g \in G, \quad j_1, j_2 \in J. \]

Here \(W_\lambda\) denotes the space of \(\lambda\) and \((\tilde{\lambda}, \tilde{W}_\lambda)\) denotes the contragredient of \((\lambda, W_\lambda)\). Note that \(\bar{H}_\lambda\) is naturally a left and right \(H_\lambda\)-module for the action by convolution. A basic result of type theory is that we have an equivalence of categories between \(S_\lambda(G)\) and the category \(H_\lambda\)-mod of left \(H_\lambda\)-modules. This equivalence maps a representation \((\pi, \mathcal{V})\) to the module \(M_\pi = \text{Hom}_J(\lambda, \mathcal{V})\).

Now let \((\pi, \mathcal{V})\) be an irreducible discrete series representation belonging to \(S_\lambda(G)\). Then the \(\mathbb{C}\)-vector space \(M_\pi\) is 1-dimensional so that \((\pi, \mathcal{V})\) gives rise to a character \(\chi_\pi\) of \(H_\lambda\). Let us denote by \(\chi_\pi \bar{H}_\lambda\) the \(\mathbb{C}\)-vector space formed of those functions \(\Psi \in \bar{H}_\lambda\) satisfying
\[ \varphi \ast \Psi = \chi_\pi(\varphi) \Psi, \quad \varphi \in H_\lambda. \]

In Proposition 3.3 we prove that \(\dim_{\mathbb{C}} \chi_\pi \bar{H}_\lambda \leq 1\). Moreover if \(\Psi \in \chi_\pi \bar{H}_\lambda, \ W \in W_\lambda, \ \tilde{w} \in \tilde{W}_\lambda\), then the function \(c_{w, \tilde{w}, \Psi} : G \to \mathbb{C}\) defined by
\[ c_{w, \tilde{w}, \Psi}(g) = \langle \Psi(g) \tilde{w}, w \rangle_\lambda, \quad g \in G, \]
is a matrix coefficient of \((\pi, \mathcal{V})\) (Proposition 3.4). Now in order to produce an explicit non-zero matrix coefficient of \((\pi, \mathcal{V})\), we have to produce a non-zero element of \(\chi_\pi \bar{H}_\lambda\) (and this will prove that \(\dim_{\mathbb{C}} \chi_\pi \bar{H}_\lambda = 1\)). In this goal we use the following recipe.

Assume first that \(J = I\) is an Iwahori subgroup and that \(\lambda\) is the trivial character of \(I\). In that case the irreducible discrete series representations lying in \(S_\lambda\) are twists of the Steinberg representation. As explained in the remark following Lemma 3.9 a very simple formula is known for the Iwahori spherical matrix coefficient of the Steinberg representation and from this it is easy to produce a non-zero element of \(\chi_\pi \bar{H}_\lambda\).

In the general case, it is a central result of \([\text{BK1}]\) that for any simple type \((J, \lambda)\) there exists a Hecke algebra isomorphism \(H_\lambda \sim H_{Iw}\), where \(H_{Iw}\) is the Iwahori-Hecke algebra of \(GL(m, L)\), for some divisor \(m\) of \(d\) and some finite field extension \(L\) of \(F\). Moreover such an isomorphism is made entirely explicit in \([\text{BH}]\) in the case of a level 0 discrete series representation. In the level 0 case, we use this explicit isomorphism to guess a formula for a non-zero \(\Psi_0\) in \(\chi_\pi \bar{H}_\lambda\) by copying the corresponding formula for the Steinberg representation of \(GL(m, L)\) (Theorem 3.6).

Having constructed our particular non-zero element \(\Psi_0 \in \chi_\pi \bar{H}_\lambda\) for a level 0 representation \(\pi\), if \(w \in W\) and \(\tilde{w} \in \tilde{W}\) both are non-zero, the matrix
coefficient \( c_{w,\tilde{w},\Psi_0} \) is non-zero. More precisely (Proposition 3.4) there exist \( J \)-embeddings \( W \subset V \) and \( \tilde{W} \subset \tilde{V} \) such that \( c_{w,\tilde{w},\Psi_0} \) is the matrix coefficient of \( \pi \) attached to the vector \( w \in V \) and to the linear form \( \tilde{w} \in \tilde{V} \).

In the second part of this work, we apply the previous construction to the theory of distinguished representations. Let us fix a quadratic unramified extension \( F/F_0 \) and set \( G_0 = \text{GL}(n,F_0) \). Recall that a representation \((\pi,V)\) of \( G \) is said to be \( G_0 \)-distinguished if the intertwining space \( \text{Hom}_{G_0}(V,\mathbb{C}) \) is non-trivial, where \( \mathbb{C} \) is acted upon by \( G_0 \) via the trivial character. Let us observe if \( \pi \) is \( G_0 \)-distinguished, its central character \( \omega_\pi \) is trivial on the center \( F_0 \times \overline{F_0} \) of \( G_0 \). A criterion is known ([An] Theorem 1.3 and [Ma] Corollary 4.2) to decide whether a discrete series representation of \( G \) is \( G_0 \)-distinguished. The proof of this criterion contains a global ingredient and the aim of this second part is to give a local proof in the case of level 0 discrete series representations.

In this aim we use the fact that the symmetric space \( G/G_0 \) is strongly discrete in the sense of Sakellaridis and Venkatesh [SV]. This means that if \( \pi \) an irreducible discrete series representation of \( G \) with central character trivial on \( F_0 \times \overline{F_0} \), then any matrix coefficient of \( \pi \) is integrable over \( G_0/F_0 \). From this it follows that if such a representation \( \pi \) has a matrix coefficient \( c \) satisfying
\[
\int_{G_0/F_0^\times} c(g) \, d\mu_{G_0/F_0^\times}(\dot{g}) \neq 0
\]
it has to be \( G_0 \)-distinguished (here \( \mu_{G_0/F_0^\times} \) is a fixed Haar measure on \( G_0/F_0^\times \)).

Let us fix an irreducible level 0 discrete series representation \((\pi,V)\) of \( G \) which satisfies the distinction criterion of [An] and [Ma]. Then one proves that there is a natural choice of simple type \((J,\lambda)\) for \( \pi \) such that \( J \) is \( \text{Gal}(F/F_0) \)-stable and such that \( \pi \simeq \pi \circ \sigma \), where \( \sigma \) is the generator of \( \text{Gal}(F/F_0) \). Setting \( J_0 = J \cap G_0 \), we then prove that \( \lambda \) is \( J_0 \)-distinguished with multiplicity one: \( \text{Hom}_{J_0}(\lambda,\mathbb{C}) \) is of dimension 1. Let us abbreviate \( W_\lambda = W \) and fix a non-zero element \( w \) (resp. \( \tilde{w} \)) of the line \( W_\lambda \) (resp. of the line \( \tilde{W}_\lambda \)). Let \( c_0 \) be the matrix coefficient \( c_{w,\tilde{w},\Psi_0} \) constructed in the first part of this work.

We prove the following formula (Proposition 5.4)
\[
\int_{G_0/F_0^\times} c_0(g) \, d\mu_{G_0/F_0^\times}(\dot{g}) = e \langle w,\tilde{w} \rangle_\lambda \mathcal{P}_{W_0}(\frac{-1}{q_0^f}) ,
\]
where \( e \) and \( f \) are integers depending on \( \pi \) (satisfying \( ef = n \)), \( q_0 \) is the size of the residue field of \( F_0 \), and \( \mathcal{P}_{W_0} \) is the Poincaré series of the affine Weyl group \( W_0 \) of \( G_0 \). Moreover we check that \( \langle w,\tilde{w} \rangle \neq 0 \) (Lemma 5.5) and
that $P_{W_0}$ does not vanishes at $-1/q_0^f$ (an explicit formula is known for $P_{W_0}$). As a consequence, this gives a local proof of distinction in our specific case (Theorem 5.2).

Now fix any $J$-embeddings $W \subset \mathcal{V}$ and $\tilde{W} \subset \tilde{\mathcal{V}}$. As a byproduct of the previous integral formula we obtain:

- an explicit local period for $\pi$, that is a non-zero element $\Phi$ of $\text{Hom}_{\mathcal{G}_0}(\pi, \mathbb{C})$ given by
  \[
  \Phi : \mathcal{V} \ni v \mapsto \int_{\mathcal{G}_0/F_0^\times} \langle \tilde{w}, v \rangle_\pi \, d\mu_{\mathcal{G}_0/F_0^\times} (\dot{g}),
  \]

- the fact that $w \in \mathcal{V}$ is a test vector for $\Phi : \Phi(w) \neq 0$.

A series of observations are needed at this point.

(1) The "zonal spherical functions" and attached matrix coefficients that we have defined for discrete series representations of $\text{GL}(n, F)$ may actually be defined for any type $(J, \lambda)$, in any reductive $p$-adic group $G$, as soon as the representation $\pi \in \mathcal{S}_\lambda(G)$ satisfies $\dim \text{Hom}_J(\lambda, \pi) = 1$, that is when $\lambda$ occurs in $\pi$ with multiplicity 1. On the other hand to construct a particular spherical function, one needs a description of the spherical Hecke algebra $\mathcal{H}(G, \lambda)$ in terms of an explicit Hecke algebra isomorphism. This description is not always available. For $\text{GL}(n)$, it is available for simple types of level 0 thanks to the explicit computations in [BH].

(2) Let $\pi$ be an irreducible discrete series representation of $\text{GL}(n, F)$ and $(J, \lambda)$ be a simple type for $\pi$. Bushnell and Kutzko attach to $\pi$ a certain extended affine Weyl group $W$ (cf. §2). It is a striking fact (Proposition 3.5) that the explicit matrix coefficient $c_0$ of $\pi$ constructed in this article has its support contained in $JWJ$.

(3) The Steinberg representation of $\text{GL}(n, F)$ is a particular level 0 discrete series representation. For certain Galois symmetric spaces $G/H$, the $H$-distinction of the Steinberg representation $\text{St}_G$ of $G$ was studied in [BC], where a conjecture of D. Prasad’s is proved. Moreover in [Br] I proved that if $\text{St}_G$ is $H$-distinguished, the value of a well chosen local period for $\text{St}_G$ at a well chosen Iwahori fixed vector of $\text{St}_G$ is given by a formula similar to that of Proposition 5.4, that is involving a Poincaré series.

(4) Let $\pi$ be a generic irreducible smooth representation of $\text{GL}(n, F)$ which is $\text{GL}(n, F_0)$-distinguished. Then Anandavardhanan and Matringe proved ([AM] Theorem 1.1) that the essential vector of Jacquet, Piatetski-Shapiro and Shalika is a test vector of $\pi$. It generally differs from the test vectors constructed in this paper.
The paper is organized as follows. The material needed from "type theory" is recalled in §1. Level 0 simple types are given in §2 where one recalls the structure of their Hecke algebras. The recipe to construct explicit matrix coefficients via "zonal spherical functions" is given in §3. The short section §4 is devoted to the notion of strongly discrete symmetric space and the main result of the article is proved in §5 by integrating the explicit matrix coefficient over $G_0$.

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1 Review of type theory

The references for this section are §4 of [BK1] and §§2-4 of [BK2].

Let $G$ be a connected reductive group defined over $F$. Fix a Haar measure $\mu_G$ on the locally compact totally disconnected group $G = \mathbb{G}(F)$. All representations of $G$ are over complex vector spaces. We denote by $S(G)$ the abelian category of smooth complex representations of $G$.

We consider pairs $(J, \lambda)$ formed of a compact open subgroup $J$ of $G$ and an irreducible complex smooth representation $(\lambda, W)$ of $J$. In particular the representation space $W$ is finite dimensional. If $(\pi, V)$ is an object of $S(G)$, we denote by $V^\lambda$ the $\lambda$-isotypic component of $V$. We let $S_{(J,\lambda)}(G)$ denote the full subcategory of $S(G)$ whose object are those representations $(\pi, V)$ that are generated by their $\lambda$-isotypic components. Following Bushnell and Kutzko [BK2], one says that $(J, \lambda)$ is a type if the category $S_{(J,\lambda)}(G)$ is stable by the operation of taking subquotients.

Recall that if $(K, \rho)$ is a pair in $G$ as above, one defines two kinds of induction. The smooth induced representation, denoted $\text{Ind}^G_K \rho$, acts by right translation on the space of smooth functions $f : G \to W_\rho$ satisfying $f(kg) = \rho(k)f(g)$, $g \in G$, $k \in K$, where $W_\rho$ denotes the space of $\rho$. The compactly induced representation $\text{ind}^G_K \rho$ acts by right translation on the space of functions $f : G \to W_\rho$, compactly supported modulo $H$, and satisfying $f(kg) = \rho(k)f(g)$, $g \in G$, $k \in K$. Later we shall need the basic fact that the smooth dual (or contragredient) of $\text{ind}^G_K \rho$ identifies canonically with $\text{Ind}^G_K \tilde{\rho}$, where $(\tilde{\rho}, \tilde{W}_\rho)$ denotes the contragredient of $(\rho, W_\rho)$. Indeed we have a $G$-invariant non-degenerate pairing

$$\text{ind}^G_K \rho \times \text{Ind}^G_K \tilde{\rho}, \quad (f, F) \mapsto \langle f, F \rangle$$

That is smooth under the right action of $G$. 

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where
\[ \langle f, F \rangle = \sum_{x \in K \setminus G} \langle f(x), F(x) \rangle_{\rho} \]
and \( \langle -, - \rangle_{\rho} \) denotes the natural \( K \)-invariant pairing on \( W \times \tilde{W}_{\rho} \).

With the notation as above fix a type \((J, \lambda)\) in \( G \). Recall that the spherical Hecke algebra attached to this pair is the space \( \mathcal{H}_{\lambda} \) of compactly supported functions \( f : G \rightarrow \text{End}_{C}(\tilde{W}) \) satisfying
\[ f(kgl) = \tilde{\lambda}(k) \circ f(g) \circ \tilde{\lambda}(l), \quad k, l \in J, \quad g \in G; \]
equipped with the convolution product
\[ (f_1 \ast f_2)(g) = \int_{G} f_1(x)f_2(x^{-1}g) d\mu_G(x), \quad f_1, f_2 \in \mathcal{H}_\lambda, \quad g \in G. \]

If \( a \in \text{End}_{C}(\tilde{W}) \), we denote by \( {}^t a \in \text{End}_{C}(W) \) its transpose endomorphism. For \( f \in \mathcal{H}_\lambda \), the function \( \hat{f} : g \mapsto {}^t f(g^{-1}) \) lies in \( \mathcal{H}_{\hat{\lambda}} \). The map \( f \mapsto \hat{f}, \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\hat{\lambda}} \), is an anti-isomorphism of \( C \)-algebras : \( \hat{f_1} \ast \hat{f_2} = \hat{f_1} \ast \hat{f_2}, \]

If \( \varphi \in \mathcal{H}_{\hat{\lambda}} \) and \( f \in \text{ind}^G_J \lambda \), the convolution production \( \varphi \ast f \), given by
\[ \varphi \ast f(g) = \int_{G} \varphi(x)f(x^{-1}g)d\mu_G(x) = \mu_G(J) \sum_{x \in G/K} \varphi(x)f(x^{-1}g), \quad g \in G, \]
is well defined and lies in \( \text{ind}^G_J \lambda \). It follows that \( \text{ind}^G_J \lambda \) is naturally a \( \mathcal{H}_{\lambda} \)-left module, whence a \( \mathcal{H}_{\lambda} \)-right module via \( (f, \varphi) \mapsto f \cdot \varphi = \hat{\varphi} \ast f, \quad f \in \text{ind}^G_J \lambda, \varphi \in \mathcal{H}_\lambda \).

Let \((\pi, V)\) be an object of \( S(G) \). Set \( V_\lambda := \text{Hom}_J(W, V) \). By Frobenius reciprocity for compact induction from an open compact subgroup, \( V_\lambda \) canonically identifies with \( \text{Hom}_G(\text{ind}^G_J \lambda, V) \). It follows that \( V_\lambda \) is naturally a \( \mathcal{H}_{\lambda} \)-module. Hence if \( \mathcal{H}_{\lambda} \text{-Mod} \) denotes the category of left unital \( \mathcal{H}_{\lambda} \)-modules, we have a well defined functor
\[ M_\lambda : S(G) \rightarrow \mathcal{H}_{\lambda} \text{-Mod}, \quad (\pi, V) \mapsto V_\lambda \]
On the other hand we have a functor \( V_\lambda : \mathcal{H}_{\lambda} \text{-Mod} \rightarrow S(G) \) given on objects by
\[ V_\lambda(M) = \text{ind}^G_J \lambda \otimes_{\mathcal{H}_{\lambda}} M \]
where the action of \( G \) on \( V_\lambda(M) \) comes from the (left) action of \( G \) on \( \text{ind}^G_J \lambda \).

One of the central results of Type Theory is the following result (cf. [BK2] Theorem (4.3)).

**Theorem 1.1** Let \((J, \lambda)\) be a type in \( G \). With the notation as above, the functors \( M_\lambda \) and \( V_\lambda \) restrict to equivalences of categories \( S_{\lambda}(G) \simeq \mathcal{H}_{\lambda} \text{-Mod} \) and are quasi-inverses of each other.
2 Types for the level 0 discrete series of $\text{GL}_n$

In this section $G$ is the reductive group $\text{GL}_n$, for some fixed integer $n \geq 2$, so that $G = \text{GL}_n(F)$. Recall that an irreducible smooth representation $(\pi, V)$ of $G$ is a discrete series representation if its central character is unitary and its matrix coefficients are square integrable modulo the center of $G$. Such a representation is obtained as follows. Fix a pair of positive integers $(e, f)$ such that $ef = n$ and a unitary irreducible supercuspidal representation $\tau$ of $\text{GL}_f(F)$. For $a \in \mathbb{C}$, set $\tau^a := \|\det\|^a \tau$, where $\det$ denotes the determinant map of $\text{GL}_f(F)$ and $|||$ the normalized absolute value of $F$. Then $\tau_M := \tau^{(1-e)/2} \otimes \tau^{(3-e)/2} \otimes \cdots \otimes \tau^{(e-1)/2}$ is a supercuspidal representation of the standard Levi subgroup $\text{GL}_f(F)^{\times e}$ of $G$. Define the generalized Steinberg representation $\text{St}_e(\tau)$ to be unique irreducible quotient of the representation of $G$ parabolically induced from $\tau_M$. It is a standard fact that any irreducible discrete series representation is of the form $\text{St}_e(\tau)$ for some pair $(e, f)$ and some unitary irreducible supercuspidal representation $\tau$ of $\text{GL}_f(F)$. Moreover $\text{St}_e(\tau)$ is of level 0 if and only if $\tau$ is. We fix once for all a pair $(e, f)$ and such a an irreducible level 0 supercuspidal representation $\tau$. We denote by $\omega_\tau$ its central character.

The aim of this section is to exhibit a type for $\text{St}_e(\tau)$, that is a pair $(J, \lambda)$ such that $\text{St}_e(\tau)$ is an object of the subcategory $\mathcal{S}_\lambda(G)$. We shall then compute the $\mathcal{H}_\lambda$-module $\text{M}_\lambda(\text{St}_e(\tau))$. We closely follow [BH].

The irreducible level 0 supercuspidal representations of $\text{GL}_f(F)$ are obtained as follows. Let $\lambda_0$ be an irreducible cuspidal representation of $\text{GL}_f(k_F)$. Recall that we have a canonical isomorphism of groups

$$\text{GL}_f(\mathfrak{o}_F)/(1 + \mathfrak{p}\text{M}_f(\mathfrak{o}_F)) \simeq \text{GL}_f(k_F)$$

so that $\lambda_0$ inflates to an irreducible smooth representation $\lambda_0$ of $\text{GL}_f(\mathfrak{o}_F)$. The group $U_F \subset \text{GL}_f(\mathfrak{o}_F)$ acts as a character $\theta_0$ on the representation space of $\lambda_0$. Fixing an extension $\omega_0$ of $\theta_0$ to the center $F^\times$ of $\text{GL}_f(F)$ determines an extension $\Lambda_0$ of $\lambda_0$ to $F^\times \text{GL}_f(\mathfrak{o}_F)$. Then $\text{ind}_{F^\times}^{\text{GL}_f(F)} \Lambda_0$ is irreducible and supercuspidal, has level 0, and any irreducible level 0 supercuspidal representation is obtained this way. We fix a representation $\lambda_0$ as above so that $\tau \simeq \text{ind}_{F^\times}^{\text{GL}_f(F)} \Lambda_0$, for some extension $\Lambda_0$ of $\lambda_0$. Let us remark that this extension is entirely determined by $\tau$ since one must have $\omega_0 = \omega_\tau$.

Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order of $\text{M}_n(F)$ whose elements are the $f \times f$ block matrices $(a_{ij})_{i,j=1,...,e}$, satisfying $a_{ij} \in \text{M}_f(\mathfrak{o}_F)$ if $j \geq i$, and $a_{ij} \in$ 

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2 Here we consider the normalized parabolic induction, taking unitary representations to unitary representations
\( p_F M_f(o_F) \) if \( j < i \). Its Jacobson radical \( \mathfrak{g} \) is the set of \( f \times f \) block matrices 
\((a_{ij})_{i,j=1,...,e}\), satisfying \( a_{ij} \in M_f(o_F) \) if \( j > i \), and \( a_{ij} \in p_F M_f(o_F) \) if \( j \leq i \). It is principal, generated by the element \( \Pi \) given in \( f \times f \) block form by:

\[
\Pi = \begin{pmatrix}
0_f & I_f & 0_f & \cdots & 0_f \\
: & 0_f & I_f & \cdots & 0_f \\
: & \vdots & \ddots & \ddots & \vdots \\
: & \vdots & \ddots & 0_f & \vdots \\
0_f & \cdots & \cdots & 0_f & I_f \\
\varpi_F I_f & 0_f & \cdots & \cdots & 0_F
\end{pmatrix}
\]

where \( I_f \) is the \( f \times f \) identity matrix.

We set \( J = \mathfrak{g}^e \subset G \) and \( J^1 = 1 + \mathfrak{g} \subset J \). Then \( J^1 \) is a normal subgroup of \( J \) and \( J/J^1 \cong \text{GL}_f(k_F)^{e^e} \). We denote by \( \lambda \) the inflation of \( \lambda^{e^e}_0 \) to \( J \).

**Proposition 2.1** The pair \((J, \lambda)\) is a type in \( G \) and \( \text{St}_c(\tau) \) is an object of the category \( S_\lambda(G) \).

**Proof.** The pair \((J, \lambda)\) is a simple type in the sense of \([BK1]\). The fact that simple types are types if proved in \([BK1]\) Theorem (8.4.2). The second assertion is proved in \([BH]\) §3 Lemma 7.

We now describe the Hecke algebra \( \mathcal{H}_\lambda \) of \((J, \lambda)\).

We let \( W_0 \subset \text{GL}_e(F) \) be the subgroup of monomial matrices with determinant 1 and entries powers of \( \varpi_F \). If \( w \in W_0 \) we denote by the same symbol \( w \) the \( f \times f \) block monomial matrix of \( \text{GL}_{ef}(F) \) given by the Kronecker product \( I_f \otimes w \). In this way we see \( W_0 \) as a subgroup of \( G \). The matrix \( \Pi \) normalizes \( W_0 \) and we denote by \( W \) the subgroup \( \Pi \varpi \times W_0 \) of \( G \) generated by \( W_0 \) and \( \Pi \). For \( i = 1, ..., e-1 \), we denote by \( s_i \) the permutation matrix in \( W_0 \) corresponding to the transposition \((i \ i + 1)\). Moreover we set \( s_0 := \Pi s_1 \Pi^{-1} \in W \).

We denote by \( X_0 \) the representation space of \( \lambda_0 \) so that \( X := X_0 \otimes \cdots \otimes X_0 \) is the representation space of \( \lambda \). We identify the dual \( \check{X} \) with \( \check{X}_0^{e^e} \). For \( i = 1, ..., e-1 \), we denote by \( \tilde{t}_i \) the endomorphism of \( \check{X} \) given by

\[
\tilde{t}_i(\check{v}_1 \otimes \cdots \otimes \check{v}_i \otimes \check{v}_{i+1} \otimes \cdots \otimes \check{v}_e) = \check{v}_1 \otimes \cdots \otimes \check{v}_{i+1} \otimes \check{v}_i \otimes \cdots \otimes \check{v}_e ,
\]

and we denote by \( \check{\Gamma} \) the endomorphism of \( \check{X} \) defined by

\[
\check{\Gamma}(\check{v}_1 \otimes \cdots \otimes \check{v}_e) = \check{v}_2 \otimes \cdots \otimes \check{v}_e \otimes \check{v}_1 .
\]
For \( i = 1, \ldots, e - 1 \), there exists a unique element of \( \mathcal{H}_\lambda \) with support \( J s_i J \) and whose value at \( s_i \) is \( \tilde{t}_i \). Similarly there exists a unique element \( \varphi_\Pi \) of \( \mathcal{H}_\lambda \) with support \( J \Pi J = \Pi J = J \Pi \) whose value at \( \Pi \) is \( \Gamma \).

In order to describe the algebra structure of \( \mathcal{H}_\lambda \), we need to introduce the standard affine Hecke algebra \( \mathcal{H}(e, q^f) \) of type \( \tilde{A}_e \). It has the following presentation. It is generated by elements \([s_i], \ i = 1, \ldots, e - 1, \ [\Pi] \) and \([\Pi^{-1}]\) with the following relations:

\begin{enumerate}
  \item \([\Pi][\Pi^{-1}] = [\Pi^{-1}][\Pi] = 1\)
  \item \([s_i] + 1)([s_i] - q^f) = 0\)
  \item \([\Pi][s_1] = [s_{e-1}][\Pi]^2\)
  \item \([\Pi][s_i] = [s_{i-1}][\Pi], \ 2 \leq i \leq e - 1\)
  \item \([s_i][s_{i+1}][s_i] = [s_{i+1}][s_i][s_{i+1}], \ 1 \leq i \leq e - 2\)
  \item \([s_i][s_j] = [s_j][s_i], \ 1 \leq i, j \leq e - 1, |i - j| \geq 2\)
\end{enumerate}

We set \([s_0] := [\Pi][s_1][\Pi]^{-1}\).

Recall that \((W_0, \{s_0, s_2, \ldots, s_{e-1}\})\) is a Coxeter system of type \( \tilde{A}_e \). We denote by \( l : W_0 \to \mathbb{Z}_{\geq 0} \) its length function. We extend it to \( W \) by setting \( l(\Pi w_0) = l(w_0), a \in \mathbb{Z}, w_0 \in W_0 \). If \( w = \Pi^k w_1 w_2 \cdots w_l, k \in \mathbb{Z}, w_i \in \{s_1, s_2, \ldots, s_{e-1}\}, i = 1, \ldots, l, \) is an element of \( W \), where \( w_1 w_2 \cdots w_l \) is a reduced expression in \( W_0 \), then the element \([\Pi]^k[w_1] \cdots[w_l]\) of \( \mathcal{H}(e, q^f) \) only depends on \( w \in W \). We denote it \([w]\). A basis of \( \mathcal{H}(e, q^f) \) as a \( \mathbb{C} \)-vector space is given by \(([w])_{w \in W}\).

The following classical result allows to perform calculations in \( \mathcal{H}(e, q^f) \).

**Proposition 2.2** For all \( i = 0, \ldots, e - 1 \), for all \( w \in W \), we have

\[
[s_i] * [w] = \begin{cases} 
[s_i w] & \text{if } l(s_i w) = l(w) + 1 \\
q[s_i w] + (q - 1)[w] & \text{if } l(s_i w) = l(w) - 1
\end{cases}
\]

Moreover \( l(s_i w) = l(w) + 1 \) if and only if for all reduced expression \( w = w_1 \cdots w_l, l = l(w), w_i \in \{s_0, \ldots, s_{e-1}\}, i = 1, \ldots, l \), we have \( w_1 \neq s_i \).

The following result describes most of the structure of \( \mathcal{H}_\lambda \).

**Theorem 2.3** ([BKL] Main Theorem (5.6.6)) a) There exists an isomorphism of \( \mathbb{C} \)-algebras

\[
\Upsilon : \mathcal{H}_\lambda \to \mathcal{H}(e, q^f)
\]

which preserves supports in that if \( \varphi \in \mathcal{H}_\lambda \) has support in \( J w J \), for some \( w \in W \), then \( \Upsilon(\varphi) \in \mathbb{C}[w] \).

b) If \( \Upsilon \) is such a preserving support isomorphism, then for \( i = 0, \ldots, e - 1 \), \( \Upsilon^{-1}(s_i) \) does not depends on the choice of \( \Upsilon \).

c) The \( \mathbb{C} \)-vector space formed of those \( \varphi \in \mathcal{H}_\lambda \) whose support is contained in \( J \Pi J = J \Pi \) is 1-dimensional.
A consequence of the previous result is that, for \( w \in W_0 \), the element \( \varphi_{[w]} := \Upsilon^{-1}([w]) \) of \( \mathcal{H}_\lambda \) does not depend on the choice of \( \Upsilon \).

By definition of the \( \varphi_{[w]} \), \( w \in W_0 \), a straightforward consequence of Proposition 2.2 is the following.

**Corollary 2.4** For all \( i = 0, \ldots, e - 1 \), for all \( w \in W_0 \), we have

\[
\varphi_{[s_i]} \ast \varphi_{[w]} = \begin{cases} 
\varphi_{[s_i,w]} & \text{if } l(s_i,w) = l(w) + 1 \\
q\varphi_{[s_i,w]} + (q - 1)\varphi_{[w]} & \text{if } l(s_i,w) = l(w) - 1 
\end{cases}
\]

The elements \( \varphi_{[s_i]} \), \( i = 1, \ldots, e - 1 \), were computed by Bushnell and Henniart.

**Proposition 2.5**

a) ([BH] §2 Proposition 3, relation (2.6.1)) For \( i = 1, \ldots, e - 1 \), the function \( \varphi_{[s_i]} \) has support \( Js_iJ \) and its value at \( s_i \) is \( \omega_\tau(-1)q^{-\frac{(f+1)}{2}} \tilde{t}_i \).

b) Define an endomorphism \( \tilde{t}_0 \) of \( X \) by

\[
\tilde{t}_0(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_e) = \tilde{v}_e \otimes \tilde{v}_2 \otimes \cdots \tilde{v}_{e-1} \otimes \tilde{v}_1
\]

Then \( \varphi_{[s_0]} \) has support \( Js_0J \) and its value at \( s_0 \) is \( \omega_\tau(-1)q^{-\frac{(f+1)}{2}} \tilde{t}_0 \).

**Proof.** Let \( \Upsilon : \mathcal{H}_\lambda \to \mathcal{H}(e, q^f) \) be any support preserving isomorphism. By definition we have \( \varphi_{[s_0]} = \Upsilon^{-1}(\Pi s_1 \Pi) \Upsilon^{-1}(\Pi) \ast \varphi_{[s_1]} \ast \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \). By Theorem 2.2(c), \( \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \ast \Upsilon^{-1}(\Pi) \) is proportional to \( \varphi_{[s_0]} \), so that \( \varphi_{[s_0]} \ast \varphi_{[s_1]} \ast \varphi_{[s_1]} \ast \varphi_{[s_1]} \ast \varphi_{[s_1]} \). This latter function has support \( J \Pi Js_1 \Pi^{-1}J = Js_0J \) and its value at \( s_0 \) is indeed \( \omega_\tau(-1)q^{-\frac{(f+1)}{2}} \tilde{t}_1 \tilde{t}_1^{-1} = \omega_\tau(-1)q^{-\frac{(f+1)}{2}} \tilde{t}_0 \).

We shall need later the expression of the images \( \tilde{\varphi}_{[s_i]} \), \( i = 0, \ldots, e - 1 \), \( \tilde{\varphi}_{[s_i]} \), of \( \varphi_{[s_i]} \), \( i = 0, \ldots, e - 1 \), and \( \varphi_{[s_i]} \) under the anti-isomorphism of algebras \( \mathcal{H}_\lambda \to \mathcal{H}_\lambda \), \( \varphi \to \tilde{\varphi} \).

**Lemma 2.6** For \( i = 1, \ldots, e - 1 \), define an element \( t_i \in \text{End}_C(X) \) by \( t_i(v_1 \otimes \cdots \otimes v_e) = v_1 \otimes \cdots \otimes v_{i+1} v_i \otimes \cdots \otimes v_e \). Moreover define \( t_0 \) and \( \Gamma \in \text{End}_C(X) \) by \( t_0(v_1 \otimes \cdots \otimes v_e) = v_e \otimes v_2 \otimes \cdots \otimes v_{e-1} v_1 \) and \( \Gamma(v_1 \otimes \cdots \otimes v_e) = v_e \otimes v_1 \otimes \cdots \otimes v_{e-1} \). Then for \( i = 0, \ldots, e - 1 \), \( \tilde{\varphi}_{[s_i]} \) is the unique element of \( \mathcal{H}_\lambda \) with support \( Js_iJ \) and taking value \( \omega_\tau(-1)q^{-\frac{(f+1)}{2}} t_i \) at \( s_i \). Similarly \( \tilde{\varphi}_{[s_i]} \) is the unique element of \( \mathcal{H}_\lambda \) with support \( J \Pi^{-1}J = J \Pi^{-1}J \) and taking value \( \Gamma \) at \( \Pi^{-1} \).

**Proof.** Straightforward calculations.

We now describe the \( \mathcal{H}_\lambda \)-module structure of \( M := M_A(\text{St}_e(\tau)) \).
Theorem 2.7 ([BH] §3 Proposition 4 and Proposition 6.(2).) a) The module $M$ is 1-dimensional so that $\mathcal{H}_\lambda$ acts on $M$ via a character $\chi$.
b) The character $\chi$ is given on a set of generators by the relations:

(i) $\chi(\varphi_{[s_i]}) = -1$, $i = 1, \ldots, e - 1$,
(ii) $\chi(\varphi_\Pi) = (-1)^{e-1} \omega_r((-1)^{e-1} \omega_F)$

Remarks. a) The expression giving $\chi(\varphi_\Pi)$ depends on the choice of the uniformizer. This is not a contradiction since $\Pi$ itself depends on the choice of $\omega_F$.
b) If $w \in W_0$ has reduced expression $w_1 w_2 \cdots w_l$, $w_i \in \{s_0, \ldots, s_{e-1}\}$. Then $\chi(\varphi_{[w]}) = (-1)^l$. In other words, we have the formula

$$\chi(\varphi_{[w]}) = (-1)^{l(w)}, \ w \in W_0$$

where we recall that $l$ is the length function of the Coxeter system $(W_0, \{s_0, \ldots, s_{e-1}\})$.

3 Explicit matrix coefficients

In the previous section we exhibited a type $(J, \lambda)$ for our fixed discrete series representation $(\pi, \mathcal{V}) = \text{St}_c(\tau)$, of $G = \text{GL}_n(F)$. The left $\mathcal{H}_\lambda$-module is 1-dimensional: we may identify it with the line $\mathbb{C} = C_\chi$, $\mathcal{H}_\lambda$ acting via the character $\chi$ that we described in Theorem 2.7.

Recall that we have an isomorphism of $G$-modules

$$\mathcal{V} \simeq \text{ind}_G^J \lambda \otimes_{\mathcal{H}_\lambda} \mathbb{C}_\chi$$

In other words $\mathcal{V}$ identifies with the quotient of $\mathcal{X} := \text{ind}_G^J \lambda$ by the sub-$G$-module

$$\mathcal{X}_\chi := \text{Span}_\mathbb{C} \{ \tilde{\phi} \star f - \chi(\varphi)f \ ; \ f \in \mathcal{X}, \ \varphi \in \mathcal{H}_\lambda \}$$

Hence the contragredient $\tilde{\mathcal{V}}$ of $\mathcal{V}$ is given by

$$\tilde{\mathcal{V}} \simeq \{ \Lambda \in \text{Ind}_G^J \lambda \ ; \ \Lambda(\tilde{\phi} \star f) = \chi(\varphi)\Lambda(f), \ f \in \mathcal{X}, \ \varphi \in \mathcal{H}_\lambda \}.$$
\[ \Lambda(\hat{\varphi} \ast f) = \sum_{x \in J \setminus G} \sum_{y \in J \setminus G} \langle \hat{\varphi}(xy^{-1}f(y)), \Lambda(x) \rangle_{\lambda} \]
\[ = \sum_{x \in J \setminus G} \sum_{y \in J \setminus G} \langle f(y), \varphi(yx^{-1})\Lambda(x) \rangle_{\hat{\lambda}} \]
\[ = \sum_{y \in J \setminus G} \langle f(y), \sum_{x \in J \setminus G} \varphi(yx^{-1})\Lambda(x) \rangle_{\hat{\lambda}} \]
\[ = (\varphi \ast \Lambda)(f) \]

where the convolution \( \varphi \ast \Lambda \) is defined by the finite sum:
\[ (\varphi \ast \Lambda)(g) = \sum_{x \in J \setminus G} \varphi(gx^{-1})\Lambda(x), \ g \in G. \]

We have proved:

**Lemma 3.1** We have a natural isomorphism of \( G \)-modules
\[ \hat{\mathcal{V}} \simeq \{ \Lambda \in \text{Ind}_J^G \hat{\lambda} : \varphi \ast \Lambda = \chi(\varphi)\Lambda, \ \varphi \in \mathcal{H}_\lambda \} \]

In order to produce an explicit matrix coefficient of \( \pi \), we have to choose two vectors in \( \mathcal{V} \) and \( \hat{\mathcal{V}} \) respectively. First fix a vector \( w \) in the space \( W \) of \( \lambda \). Let \( f_w \in \mathcal{X} = \text{ind}_J^G \lambda \) be the function with support \( J \), given by \( f_w(j) = \lambda(j).w, \ j \in J \). Its image \( \bar{f}_w \) in \( \mathcal{X}/\mathcal{X}_\chi \) is our favourite vector in \( \mathcal{V} \).

To produce vectors in \( \hat{\mathcal{V}} \) we proceed as follows. Let \( \bar{\mathcal{H}}_\lambda \) be the space of functions \( \Psi : G \to \text{End}_\mathbb{C}(\bar{W}) \) satisfying \( \Psi(j_1.gj_2) = \hat{\lambda}(j_1) \circ \Psi(g) \circ \hat{\lambda}(j_2), \ j_1, j_2 \in J, \ g \in G \) (with no condition on support). Let \( \bar{w} \in \bar{W} \) and \( \Psi \in \bar{\mathcal{H}}_\lambda \). Then the function \( \Lambda_{\Psi,\bar{w}} : G \to \bar{W} \) defined by \( \Lambda_{\Psi,\bar{w}}(g) = \Psi(g)(\bar{w}) \) is by construction an element of \( \text{Ind}_J^G \hat{\lambda} \).

For \( \varphi \in \mathcal{H}_\lambda \), and \( \Psi \in \bar{\mathcal{H}}_\lambda \) define the convolution \( \varphi \ast \Psi \) by the finite sum
\[ (\varphi \ast \Psi)(g) = \sum_{x \in J \setminus G} \varphi(gx^{-1})\Psi(x), \ g \in G. \]

This is indeed an element of \( \bar{\mathcal{H}}_\lambda \).

**Lemma 3.2** Let \( \Psi \in \bar{\mathcal{H}}_\lambda \) be a function satisfying \( \varphi \ast \Psi = \chi(\varphi)\Psi \), for all \( \varphi \in \mathcal{H}_\lambda \). Then, in the model of \( \hat{\mathcal{V}} \) given in Lemma 3.1 we have \( \Lambda_{\Psi,\bar{w}} \in \hat{\mathcal{V}} \).
Proof. If $\Psi \in \mathcal{H}_\lambda$ satisfies the assumption of the lemma, for $\varphi \in \mathcal{H}_\lambda$ and $g \in G$, we have

\[
(\varphi \star \Lambda_{\Psi,\tilde{w}})(g) = \sum_{x \in J \setminus G} \varphi(gx^{-1})\Lambda_{\Psi,\tilde{w}}(x)
= \sum_{x \in J \setminus G} \varphi(gx^{-1})\Psi(x)(\tilde{w})
= (\sum_{x \in J \setminus G} \varphi(gx^{-1})\Psi(x))(\tilde{w})
= (\varphi \star \Psi)(g)(\tilde{w})
= \chi(\varphi)\Lambda_{\Psi,\tilde{w}}(g)
\]

as required.

Let us write $\chi \mathcal{H}_\lambda$ for the $\mathbb{C}$-vector space given by

\[
\{ \Psi \in \mathcal{H}_\lambda ; \varphi \star \Psi = \chi(\varphi)\Psi, \varphi \in \mathcal{H}_\lambda \}
\]

One easily check that, for all $\Psi \in \chi \mathcal{H}_\lambda$, the map $L_{\psi} : \tilde{W} \ni w \mapsto \Lambda_{\psi,\tilde{w}} \in \tilde{V}$ is $J$-equivariant. We therefore have a linear map $L : \chi \mathcal{H}_\lambda \rightarrow \text{Hom}_J(\tilde{W}, \tilde{V})$. Moreover it is a straightforward consequence of the definition of $\Lambda_{\psi,\tilde{w}}$ that $L$ is injective.

**Proposition 3.3** We have $\dim_{\mathbb{C}} \chi \mathcal{H}_\lambda \leq 1$.

**Proof.** The spaces $\text{Hom}_J(\tilde{W}, \tilde{V})$, $\text{Hom}_J(\pi, \lambda)$ have the same dimension. But it is a standard fact (e.g. see [BH] §3, Lemma 7.(2)) that the simple type $\lambda$ occurs with multiplicity 1 in the discrete series representations $\pi$.

We shall see later that $\chi \mathcal{H}_\lambda$ is non-trivial.

**Proposition 3.4** Let $w \in W$, $\tilde{w} \in \tilde{W}$. Let $\Psi \in \mathcal{H}_\lambda$ satisfy $\varphi \star \Psi = \chi(\varphi)\Psi$, for all $\varphi \in \mathcal{H}_\lambda$. Let $c_{\Psi,\tilde{w}}$ be the matrix coefficient of $\pi$ attached to the vector $f_w \in V$ and to the linear form $\Lambda_{\Psi,\tilde{w}} \in \tilde{V}$.

(a) We have the formula:

\[
c_{\Psi,\tilde{w}}(g) = \langle w, \Psi(g^{-1})\tilde{w} \rangle_{\lambda}, \quad g \in G.
\]

(b) Assume $\Psi \neq 0$. There exist $J$-equivariant embeddings $W \subset V$ and $\tilde{W} \subset \tilde{V}$ such that $c_{\Psi,\tilde{w}}$ is the matrix coefficient of $\pi$ attached to the vector $w \in V$ and to the linear form $\tilde{w} \in \tilde{V}$. 

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Proof. (a) Let $g \in G$. By definition we have

$$c_{\Psi,w,\tilde{w}}(g) = \sum_{x \in J \setminus G} \langle f_w(xg), \Lambda_{\Psi,\tilde{w}}(x) \rangle \tilde{\lambda}$$

$$= \sum_{x \in J \setminus G} \langle f_w(xg), \Psi(x)(\tilde{w}) \rangle \tilde{\lambda}$$

Since the support of $f_w$ is contained in $J$, only the coset $Jg^{-1}$ may contribute to the sum, whence we obtain:

$$c_{\Psi,w,\tilde{w}}(g) = \langle f_w(1), \Psi(g^{-1})(\tilde{w}) \rangle \tilde{\lambda}$$

$$= \langle w, \Psi(g^{-1})(\tilde{w}) \rangle \lambda$$

as required.

(b) Indeed $W \ni w \mapsto \tilde{f}_w \in V$ and $\tilde{W} \ni \tilde{w} \mapsto \Lambda_{\Psi,\tilde{w}} \in \tilde{V}$ are non-zero and $J$-equivariant.

For $\Psi \in \mathcal{H}_\lambda$, the formula $\tilde{\Psi}(g) = \tilde{\Psi}(g^{-1})$, $g \in G$, defines an element of the $\mathbb{C}$-vector space $\mathcal{H}_\lambda$ of functions $\Psi' : G \to \text{End}_\mathbb{C}(W)$ satisfying $\Psi'(j_1gj_2) = \lambda(j_1) \circ \Psi'(g) \circ \lambda(j_2)$ for all $j_1, j_2 \in J$, $g \in G$. For $\Psi' \in \mathcal{H}_\lambda$ and $\varphi' \in \mathcal{H}_\lambda$, we define the convolution $\Psi' \ast \varphi$ by the finite sum

$$\Psi' \ast \varphi'(g) = \sum_{G/J} \Psi'(x)\varphi'(x^{-1}g)$$

It is straightforward to check that an element $\Psi$ of $\mathcal{H}_\lambda$ satisfies $\varphi \ast \Psi = \chi(\varphi)\Psi$, $\varphi \in \mathcal{H}_\lambda$, if, and only if, $\tilde{\Psi}$ satisfies $\tilde{\Psi} \ast \varphi' = \tilde{\chi}(\varphi')\tilde{\Psi}$, $\varphi' \in \mathcal{H}_\lambda$, where $\tilde{\chi}$ is the character of $\mathcal{H}_\lambda$ defined by $\tilde{\chi}(\tilde{\varphi}) = \chi(\varphi)$, $\varphi \in \mathcal{H}_\lambda$.

We can therefore reformulate Proposition 3.4 as follows.

**Proposition 3.5** Let $w \in W$ and $\tilde{w} \in \tilde{W}$. Let $\Psi$ be an element of $\mathcal{H}_\lambda$ satisfying $\Psi \ast \varphi = \tilde{\chi}(\varphi)\Psi$, for all $\varphi \in \mathcal{H}_\lambda$. Then the formula

$$c_{\Psi,w,\tilde{w}}(g) = \langle \Psi(g)(w), \tilde{w} \rangle \lambda, \ g \in G,$$

defines a matrix coefficient of $\pi$.

Moreover the support of that matrix coefficient is contained in $JWJ$.

Proof. Only the second assertion deserves to be proved. The pair $(J, \lambda)$ is a simple type in the sense of [BK1] Definition (5.5.10). By op. cit. Proposition (5.5.11), if $g \in G$ intertwine $(J, \lambda)$, that is if the intertwining space $\text{Hom}_{J \cap G}(\lambda, g\lambda)$ is not trivial, then $g$ must lie in $JWJ$. Moreover easily
adapting the proof of *loc. cit.* Proposition (4.1.1), if for $g \in G$, we have $\Psi(g) \neq 0$, then $g$ intertwines $(J, \lambda)$. As a consequence the support of $\Psi$, whence that of $c_{\Psi, w, \tilde{w}}$, is contained in $JWJ$.

The rest of the section is devoted to constructing a function $\Psi \in \bar{\mathcal{H}_\lambda}$ satisfying the hypothesis of Theorem 3.5.

Observe that any element $\varphi$ of $\bar{\mathcal{H}_\lambda}$ may be written

$$\varphi = \sum_{w \in W_0, k \in \mathbb{Z}} c_{w, k} \varphi_{\Pi}^k \star \varphi[w]$$

for some complex numbers $c_{w, k}$, $w \in W_0$, $k \in \mathbb{Z}$, and where the formal sum

$$\sum_{w \in W_0, k \in \mathbb{Z}} c_{w, k} \varphi_{\Pi}^k \star \varphi[w]$$

denotes the element of $\bar{\mathcal{H}_\lambda}$, with support in $JWJ$, and whose value at $\Pi^k w$, $k \in \mathbb{Z}$, $w \in W_0$, is $c_{w, k}(\varphi_{\Pi}^k \star \varphi[w])(\Pi^k w)$.

**Theorem 3.6** Define an element of $\mathcal{H}_\lambda$ by the formula

$$\Psi_0 = \sum_{w \in W_0, k \in \mathbb{Z}} \left(-\frac{1}{q_1}\right)^{l(w)} \chi(\varphi_{\Pi})^{-k} \varphi_{\Pi}^k \star \varphi[w]$$

where $q_1 = q^f$. Then $\Psi_0$ satisfies the assumption of Proposition 3.4:

$$\varphi \star \Psi_0 = \chi(\varphi)\Psi_0, \text{ for all } \varphi \in \mathcal{H}_\lambda \quad (1)$$

**Corollary 3.7** We have $\dim_{\mathbb{C}} \bar{\mathcal{H}_\lambda} = 1$.

*Proof.* Indeed by Proposition 3.3 we have $\dim_{\mathbb{C}} \bar{\mathcal{H}_\lambda} \leq 1$. On the other hand the previous theorem exhibits a non-zero element of $\dim_{\mathbb{C}} \bar{\mathcal{H}_\lambda} = 1$.

The proof of the theorem consists in several steps.

**Lemma 3.8** For all $\varphi \in \mathcal{H}_\lambda$, we have

$$\varphi \star \Psi_0 = \sum_{w \in W_0, k \in \mathbb{Z}} \left(-\frac{1}{q_1}\right)^{l(w)} \chi(\varphi_{\Pi})^{-k} \varphi \star \varphi_{\Pi}^k \star \varphi[w]$$

*Proof.* We have to prove that the LHS and the RHS both evaluated at $g \in G$ coincide for all $g \in JWJ$. But after evaluation at $g$, for any $g \in JWJ$, for a matter of support, only a finite number of terms may contribute to
the infinite formal sum, so that our result follows from the usual bilinear property of the convolution product in \( H_\lambda \).

It suffices to prove Equality \( \Box \) for \( \varphi = \varphi^l_l, \ l \in \mathbb{Z} \), and for \( \varphi = \varphi[i], \ i = 1, \ldots, e - 1 \).

We first deal with \( \varphi \) of the form \( \varphi^l_l \prod, \ l \in \mathbb{Z} \). Thanks to the last lemma, we have:

\[
\varphi^l_l \prod \star \Psi_0 = \sum_{w \in W_0, \ k \in \mathbb{Z}} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi \prod)^{-k} \varphi^{k+l} \prod \star \varphi[w]
\]

and our result follows by the change of variable \( k' = k + l \).

We now prove Equality \( \Box \) for \( \varphi = \varphi[i], \ i = 0, \ldots, e - 1 \). First recall that \( \chi(\varphi[i]) = -1 \) so that we must prove that \( \varphi[i] \star \Psi_0 = -\Psi_0 \).

We shall need:

**Lemma 3.9** We have the equality

\[
\Psi_0 = \sum_{w \in W_0, \ k \in \mathbb{Z}} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi \prod)^{-k} \varphi[w] \star \varphi_k \prod
\]

**Proof.** First, for \( k \in \mathbb{Z}, \ w \in W_0 \), we have \( \varphi^k_k \prod \varphi[w] \prod^{-k} = \varphi_{\prod_k \prod^{-k}} \). This can be proved by induction on \( l(w) \), reducing to the case \( w \in \{ s_0, \ldots, s_{e-1} \} \), where the equality follows from a straightforward calculation. Now our result follows using the fact that \( \prod_k \prod \) normalizes \( W_0 \) and by a suitable change of variable.

Let \( W_1 \) be the set of \( w \in W_0 \) such that \( l(s_i w) = l(w) + 1 \). Recall (Proposition 2.2) that \( w \in W_1 \) if and only if for any reduced expression \( w = w_1 w_2 \cdots w_l, \ l = l(w) \), we have \( w_1 \neq s_i \). Let \( W_2 \) be the complement \( W_0 \setminus W_1 \), so that we have \( W_2 = s_i W_1 \). We may write

\[
\varphi[i] \star \Psi_0 = \sum_{k \in \mathbb{Z}} \left( \sum_{w \in W_0} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi \prod)^{-k} \varphi[i] \star \varphi[w] \right) \star \varphi_k \prod
\]

For \( k \in \mathbb{Z} \), abbreviate:

\[
S_k = \sum_{w \in W_0} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi \prod)^{-k} \varphi[i] \star \varphi[w]
\]

Using the multiplication rules of Proposition 2.4 as well as suitable changes
of variable, we successively write:

\[ S_k = \sum_{w \in W_1} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[s_i] \ast \varphi[w] \]

\[ + \sum_{w \in W_2} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[s_i] \ast \varphi[w] \]

\[ = \sum_{w \in W_1} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[s_i] \ast \varphi[s_i] \]

\[ + \sum_{w' \in W_1} \left( -\frac{1}{q_1} \right)^{l(w')} \chi(\varphi_\Pi)^{-k} \varphi[s_i] \ast \varphi[s_i] \]

\[ = \sum_{w \in W_2} \left( -\frac{1}{q_1} \right)^{l(w)-1} \chi(\varphi_\Pi)^{-k} \varphi[w] \]

\[ + \sum_{w' \in W_1} \left( -\frac{1}{q_1} \right)^{l(s_i w')} \chi(\varphi_\Pi)^{-k} \varphi[w'] \]

\[ = -q_1 \sum_{w \in W_2} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[w] \]

\[ + q_1 \sum_{w' \in W_1} \left( -\frac{1}{q_1} \right)^{l(s_i w')} \chi(\varphi_\Pi)^{-k} \varphi[w'] \]

\[ + (q_1 - 1) \sum_{w' \in W_1} \left( -\frac{1}{q_1} \right)^{l(s_i w')} \chi(\varphi_\Pi)^{-k} \varphi[w'] \]

\[ = -q_1 \sum_{w \in W_2} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[w] \]

\[ - \sum_{w' \in W_1} \left( -\frac{1}{q_1} \right)^{l(w')} \chi(\varphi_\Pi)^{-k} \varphi[w'] \]

\[ + (q_1 - 1) \sum_{w \in W_2} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[w] \]

\[ = - \sum_{w \in W_0} \left( -\frac{1}{q_1} \right)^{l(w)} \chi(\varphi_\Pi)^{-k} \varphi[w] \]

and our result follows. This finishes the proof of Theorem 3.6.

Remark. Let us explain to the reader how we guessed the formula giving \( \Psi_0 \). Let \( \mathbb{H} \) be a connected semisimple group defined and split over \( F \). Assume for simplicity sake that \( \mathbb{H} \) is simply connected and that its relative root system is irreducible. Let \( T \) be a maximal split torus of \( T \), \( \mathcal{A} \) be the apartment of the affine building of \( \mathbb{H} \) attached to \( T \). Fix a chamber \( C \) of \( \mathcal{A} \) and let \( I \) be
the Iwahori subgroup of $\mathbb{H}(F)$ fixing $C$. Let $W$ be the affine Weyl group of $\mathbb{H}$ attached to $T$. Recall that we have the decomposition $\mathbb{H}(F) = \sqcup_{w \in W} IwI$. The chamber $C$ determines a generating set of involutions $S$ of $W$. The pair $(W, S)$ is a Coxeter system; let us denote by $l$ its length function. Let $St_H$ be the Steinberg representation of $H = \mathbb{H}(F)$. Finally let $c$ be the matrix coefficient of $St_H$ attached to an Iwahori fixed vector of $St_H$ and an Iwahori fixed linear form on $St_H$. Then $c$ is given by the following formula ([Bo] Equality (3) page 252, proof of Proposition 5.3):

$$c(k_1 w k_2) = C. \left(\frac{-1}{q}\right)^{l(w)}, \quad k_1, k_2 \in I, \ w \in W.$$ 

for some constant $C$.

Using Proposition 3.5, Theorem 3.6 may be rewritten as follows.

**Theorem 3.10** Define an element $\tilde{\Psi}_0$ of $\mathcal{H}_\lambda$ by

$$\tilde{\Psi}_0 = \sum_{w \in W_0, \ k \in \mathbb{Z}} \left(\frac{-1}{q^k}\right)^{l(w)} \chi(\tilde{\varphi}_[\Pi] - k \tilde{\varphi}_[\Pi] \ast \tilde{\varphi}_[\Pi]}$$

$Fiw u \in W$, $\tilde{u} \in \tilde{W}$. Then the formula

$$c_{u, \tilde{u}, \tilde{\Psi}_0}(g) = \langle \tilde{\Psi}_0(g)(u), \tilde{u}\rangle_\lambda, \ g \in G$$

defines a matrix coefficient of $St_e(\tau)$. Its support is contained in $JWJ$.

In fact most of the results of this section are not special to $GL_n$ and extend, with the same proof, to much more general situations.

Assume that $G$ is a connected reductive algebraic group defined over $F$. Fix a type $(J, \lambda)$ in $G = G(F)$ and, with the notation of the present section, an irreducible representation $\pi$ in $S_\lambda(G)$. Let us make the following assumption:

$(H)$ $M_\lambda(\pi) = \text{Hom}_J(\lambda, \pi)$ is $1-$dimensional.

Let $\chi$ be the character of $\mathcal{H}(G, \lambda)$ afforded by the module $M_\lambda(\pi)$. Define $\tilde{\mathcal{H}}_\lambda$ and $\chi\tilde{\mathcal{H}}_\lambda$ as before.

**Theorem 3.11** (i) The space $\chi\tilde{\mathcal{H}}_\lambda$ is of dimension $\leq 1$.

(ii) For any $\Psi \in \chi\tilde{\mathcal{H}}_\lambda$, $w \in \lambda$ and $\tilde{w} \in \tilde{\lambda}$, the formula

$$c_{\Psi, w, \tilde{w}}(g) = \langle \Psi(g)(w), \tilde{w}\rangle_\lambda, \ g \in G$$

defines a matrix coefficient of $\pi$.

**Remark.** If one moreover knows that $\mathcal{H}(G, \lambda)$ is isomorphic to the Iwahori-Hecke algebra of some other reductive group defined over $F$, in a support preserving way, then one can exhibit a non zero element of $\chi\tilde{\mathcal{H}}_\lambda$ using the method of this section. However we shall not push this further in this article.
4 Strongly discrete symmetric spaces and distinction

The notation in this section is independent of the previous ones.

Let $G/H$ be a $p$-adic reductive symmetric space: $G$ is the group of $F$-rational points of a connected reductive group $\mathbb{G}$ defined over $F$ and $H = \mathbb{H}(F)$, where $\mathbb{H}$ is the reductive group $\mathbb{G}^\theta$, for some $F$-rational involution $\theta$ of $\mathbb{G}$.

Let $\chi$ be a smooth abelian character of $H$. Recall that a smooth representation $(\pi, V)$ of $G$ is said to be $\chi$-distinguished if there is a non-zero linear form $l : V \to \mathbb{C}$ such that $l(\pi(g).v) = \chi(g)l(v)$, $g \in G$, $v \in V$.

If $\chi$ is the trivial character, one says that $\pi$ is $H$-distinguished.

Let $A_G$ be the maximal split torus in the center of $G$, $A_G^+$ be the connected component of $A_G \cap H$, and set $A_G^+ = A_G^+(F)$. Fix Haar measures $d\mu_H$ and $d\mu_A$ on $H$ and $A_G^+$ and let $d\mu_{H/A_G^+}$ be the quotient Haar measure on $H/A_G^+$.

Let $(\pi, V)$ be an irreducible smooth representation of $G$ whose central character $\omega_\pi$ is trivial on $A_G^+$. For $v \in V$, $\tilde{v} \in \tilde{V}$ one consider the matrix coefficient $c_{\pi,v,\tilde{v}}$ defines by $c_{\pi,v,\tilde{v}}(g) = \langle \pi(g)v, \tilde{v} \rangle$.

Following Sakellaridis and Venkatesh [SV], one says that $G/H$ is a strongly tempered (resp. strongly discrete) symmetric space if for any tempered (resp. discrete series) irreducible representation $(\pi, V)$ of $G$, whose central character is trivial on $A_G^+$, for all $v \in V$, for all $\tilde{v} \in \tilde{V}$, the matrix coefficient $c_{\pi,v,\tilde{v}}$ seen as a function on $H/A_G^+$ is $\mu_{H/A_G^+}$-integrable. Of course any strongly tempered symmetric space is strongly discrete.

In the following sections, we shall use the following fact.

**Theorem 4.1** [GO] Let $F/F_0$ be a quadratic extension. Then the symmetric space $GL_n(F)/GL_n(F_0)$ is strongly discrete.

We shall use this result to prove cases of distinctions. Indeed assume that $G/H$ is strongly discrete and let $\pi$ be an irreducible discrete series representation of $H$ satisfying $(\omega_\pi)|_{A_G^+} = 1$. Fix $\tilde{v}$ in $\tilde{V}$. Then the formula

$$l_{\tilde{v}} : v \in V \mapsto \int_{H/A_G^+} c_{\pi,v,\tilde{v}}(g) \, d\mu_{H/A_G^+}(g)$$

defines a $H$-invariant linear form on $V$, whence an element of $\text{Hom}_H(\pi, \mathbb{C})$. So we have the following criterion:
Criterion 4.2 If there exists a test vector for \( l_v \), that is a vector \( v \in V \) such that
\[
\int_{H/A} c_{\pi,v,\delta}(g) \, d\mu_{H/A}(\delta) \neq 0
\]
then \( \pi \) is \( H \)-distinguished.

5 The Galois symmetric space \( \text{GL}_n(F)/\text{GL}_n(F_0) \)

We use the notation of the previous sections.

Let \( F/F_0 \) be a quadratic unramified extension. We denote by \( \mathfrak{o}_{F_0}, \mathfrak{p}_{F_0}, k_{F_0}, \) and \( \varpi_{F_0} \) the ring of integers of \( F_0 \), its maximal ideal, its residue field and the choice of a uniformizer respectively. Set \( G = \text{GL}_n(F), G_0 = \text{GL}_n(F_0) \) so that \( G/G_0 \) is a reductive symmetric space. We denote by \( k_{F_0} \) the residue field of \( F_0 \) and set \( q_0 = \# k_{F_0} \), so that \( q = q_0^2 \). We set \( J_0 = J \cap G_0 \); this is the group of units of the principal hereditary order \( \mathfrak{A}_0 := \mathfrak{A}^\Gamma \) of \( M_n(F_0) \).

We choose the uniformizer \( \varpi_F \) so that \( \varpi_{F_0} = \varpi_{F_0} \in F_0 \). With that choice we have \( \Pi \in G_0 \) and \( W = \Pi^\varpi \preceq W_0 \subset G_0 \).

Irreducible discrete series representations of \( G \) distinguished by \( G_0 \) are classified ([An], [Ma]).

Theorem 5.1 ([An] Theorem 1.3, [Ma] Corollary 4.2) Let \( \eta_{F/F_0} \) be the quadratic character of \( F_0^\times \) attached to the extension \( F/F_0 \) via local class field theory. See \( \eta_{F/F_0} \) as a character of \( G_0 \) via \( \eta_{F/F_0}(g) = \eta_{F/F_0}(\det(g)), g \in G_0 \). Let \( \tau \) be an irreducible supercuspidal representation of \( \text{GL}_f(F_0) \). Then \( \text{St}_e(\tau) \) is distinguished by \( G_0 \) if and only if \( \tau \) is \( \eta_{F/F_0}^{-1} \)-distinguished.

The aim of this section is to give an entirely local proof of one implication of Matringe’s theorem when \( \tau \) has level 0. As a byproduct of the proof we shall exhibit explicit test vectors.

For simplicity sake we only deal with the case \( e \) odd. So we prove the following.

Theorem 5.2 With the notation as above, assume that \( \tau \) is of level 0 and that \( e \) is odd. Then if \( \tau \) is \( \text{GL}_f(F_0) \)-distinguished, \( \text{St}_e(\tau) \) is \( G_0 \)-distinguished.

In this aim, we shall use Criterion 4.2.

So let us fix an irreducible level 0 supercuspidal representation \( \tau \) of \( \text{GL}_f(F) \) as in §2. First we quote the following classical result without proof.

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Lemma 5.3 The representation $\tau$ is $GL_f(F_0)$-distinguished if and only if $(\omega_\tau)_{|F_0^\times} \equiv 1$ and $(\tilde{\lambda}_0, X_0)$ is $GL_f(k_{F_0})$-distinguished.

With the notation of the previous section, $A_G$ is the group $F^\times$ of scalars matrices in $G$, and $A_G^0 \simeq F_0^\times$ the group of scalar matrices in $G_0$. We fix a Haar measure $\mu_{G_0}$ on $G_0$ by imposing that $\mu_{G_0}(J_0) = 1$, and a Haar measure $\mu_{F_0^\times}$ on $F_0^\times$ normalized by imposing that the unit group $U(F_0)$ has measure 1. We let $\mu_{G_0/F_0^\times}$ denote the quotient measure on $G_0/F_0^\times$.

Let us assume that $\tau$ is $GL_f(F_0)$-distinguished. Fix $v_0 \in X_0$ (resp. $\tilde{v}_0 \in \tilde{X}_0$), a non-zero vector fixed by $GL_f(o_{F_0})$. Set $v := v_0 \otimes \cdots \otimes v_0 \in X$ and $\tilde{v} := \tilde{v}_0 \otimes \cdots \otimes \tilde{v}_0 \in \tilde{X}$. By construction both vectors are fixed by $J_0$; they are also fixed by the endomorphisms $t_i, i = 0, ..., e - 1$, and $\Gamma$. Let $c_{v,\tilde{v}}$ be the matrix coefficient of $St_e(\tau)$ constructed in Theorem 3.10:

$$c_{v,\tilde{v}}(g) = \langle \tilde{\Psi}_0(g).v, \tilde{v} \rangle_{\lambda}, \ g \in G$$

We are going to establish the following formula.

Proposition 5.4 We have

$$\int_{G_0/F_0^\times} c_{v,\tilde{v}}(g) \ d\mu_{G_0/F_0^\times}(\tilde{g}) = e \langle v, \tilde{v} \rangle_{\lambda} P_{W_0}(\frac{-1}{q_f})$$

where $P_{W_0}$ is the Poincaré series of the affine coxeter group $W_0$.

Recall that the Poincaré series $P_{W_0}$ is the formal series defined by

$$P_{W_0}(X) = \sum_{w_0 \in W_0} X^{l(w_0)} .$$

By [Bott, St] it is given by the formula

$$P_{W_0}(X) = \frac{1}{(1 - X)^{e-1}} \prod_{i=1}^{e-1} \frac{1 - X^{m_i}}{1 - X^{m_i - 1}}$$

where $m_1, m_2, ..., m_{e-1}$ are the exponent of the spherical Coxeter group $S_e$ (cf. [Bou], Chap. V, §6, définition 2). Indeed we have $m_i = i, i = 1, ..., e - 1$ (cf. [Bou], Planche I). In particular $P_{W_0}$ defines a non-vanishing function on the real open interval (-1,1).

Let us explain how Theorem 5.2 follows from the previous proposition. First, by the previous discussion, we have $P_{W_0}(-1/q_f^0) \neq 0$. We are thus reduced to proving that $\langle v, \tilde{v} \rangle_{\lambda} \neq 0$. First observe that

$$\langle v, \tilde{v} \rangle_{\lambda} = \langle v, \tilde{v} \rangle_{\tilde{\lambda}_0}$$

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where $\tilde{X}_0^\otimes$ is the representation of $\text{GL}_f(k_F)^{\times}$ in $X$. Since $\text{GL}_f(k_F)^{\times}/\text{GL}_f(k_{F_0})^{\times}$ is a Gelfand pair, we have

$$X^{\text{GL}_f(k_{F_0})^{\times}} = \mathbb{C} v \quad \text{and} \quad \tilde{X}^{\text{GL}_f(k_{F_0})^{\times}} = \mathbb{C} \tilde{v}$$

Now our claim follows from the following general result whose statement and proof were communicated to me by Dipendra Prasad.

**Lemma 5.5** (D. Prasad) Let $H/K$ be a Gelfand pair where $H$ and $K$ are finite groups. Let $(\pi, V)$ be an irreducible representation of $G$ such that $\pi$ and $\tilde{\pi}$ are $H$-distinguished. Let $v$ and $\tilde{v}$ be generators of the line $V^K$ and $\tilde{V}^K$ respectively. Then $\langle v, \tilde{v} \rangle \neq 0$.

**Proof.** The natural pairing $f : V \times \hat{V} \rightarrow \mathbb{C}$ is non-degenerate and $G$-invariant, whence $H$-invariant. Decompose $V$ as a direct sum of its isotypic components according to the action of $H$:

$$\mathcal{V} = \sum_{\rho \in \hat{H}} \mathcal{V}_\rho$$

where $\hat{H}$ denotes the dual of $H$. Here $\mathcal{V}_\rho$ is either trivial or isomorphic to $\rho$ as a $H$-module. Similarly we have $\hat{V} = \sum_{\rho \in \hat{H}} \hat{V}_\rho$. Hence we have the decomposition:

$$\mathcal{V} \times \hat{V} = \sum_{\rho, \tau \in \hat{H}} \mathcal{V}_\rho \times \hat{V}_\tau$$

Let $1_H$ be the trivial character of $H$. Then $f_{|\mathcal{V}_\rho \times \hat{V}_\tau}$ is trivial as soon as $\tau \not\sim 1_H$. Assume for a contradiction that $\langle v, \tilde{v} \rangle = 0$. Then $f_{|\mathcal{V}_\rho \times 1_H} = 0$. This implies that the orthogonal of $v$ with respect to $f$ is $\tilde{V}$, a contradiction.

The rest of this section is devoted to proving Proposition 5.4.

Recall that the support of $c_{v, \tilde{v}}$ is contained in $JWJ$. We first prove:

**Lemma 5.6** We have $JWJ \cap G_0 = J_0WJ_0$.

**Proof.** We need the following auxiliary lemma.

**Lemma 5.7** ([Stev] Lemma 2.1) Let $\Gamma$ be a finite group acting on a group $H$ by group automorphisms. Set $K$ be a $\Gamma$-invariant subgroup of $H$ and set $H_0 := HT$, $K_0 := KT$. Finally let $w \in H_0$. Then if the first cohomology set $H^1(\Gamma, K \cap wKw^{-1})$ is trivial, we have $( KwK ) \cap H_0 = K_0wK_0$.
So we are reduced to proving that for all \( w \in W \), we have

\[
H^1(\text{Gal}(F/F_0), J \cap wJw^{-1}) = \{1\}.
\]

Fix \( w \in W \). We have \( \mathfrak{A} = \mathfrak{A}_0 \otimes_{\mathfrak{o}_F} \mathfrak{o}_F \). So \( J \cap wJw^{-1} \) is the group of units of the \( \mathfrak{o}_F \)-order \((\mathfrak{A}_0 \cap w\mathfrak{A}_0w^{-1}) \otimes_{\mathfrak{o}_F} \mathfrak{o}_F \). Since \( F/F_0 \) is unramified, it is a classical fact in Galois Cohomology that if \( \mathfrak{M} \) is an \( \mathfrak{o}_F \)-order in \( M_n(F_0) \), then the following non-abelian cohomology set is trivial:

\[
H^1(\text{Gal}(F/F_0), (\mathfrak{M} \otimes_{\mathfrak{o}_F} \mathfrak{o}_F)^\times) = \{1\}.
\]

Let us write

\[
J_0WJ_0 = \bigsqcup_{k=0}^{e-1} \sum_{w_0 \in W_0} F_0^x J_0 \Pi^k w_0 J_0
\]

We observe the function \( c_{\nu, \tilde{\nu}} \) is bi-\( J_0 \)-invariant. Moreover since \( \tau \) is \( \text{GL}_f(F_0) \)-distinguished, its central character \( \omega \) is trivial on \( F_0^x \). Since \( \text{St}_c(\tau) \) and \( \tau \) share the same central character, it follows that any matrix coefficient of \( \text{St}_c(\tau) \) is \( F_0^x \)-invariant. As a consequence we may write:

\[
\int_{G_0/F_0^x} c_{\nu, \tilde{\nu}}(g) \, d\mu_{G_0/F_0^x}(g) = \sum_{k=0}^{e-1} \sum_{w_0 \in W_0} \int_{J_0w_0 \Pi^k J_0} c_{\nu, \tilde{\nu}}(g) \, d\mu_{G_0}(g)
\]

\[
= \sum_{k=0}^{e-1} \sum_{w_0 \in W_0} \mu_{G_0}(J_0 w_0 \Pi^k J_0) c_{\nu, \tilde{\nu}}(w_0 \Pi^k)
\]

**Lemma 5.8** For all \( w_0 \in W_0 \) and \( k \in \mathbb{Z} \), we have

\[
\mu_{G_0}(J_0 w_0 \Pi^k J_0) = q_0^{f_2(w_0)}.
\]

**Proof.** Since \( \Pi \) normalizes \( J_0 \), we have \( \mu_{G_0}(J_0 w_0 \Pi^k J_0) = \mu_{\mathfrak{o}_F}(J_0 w_0 J_0) \). Let \( I_0 \) be the standard Iwahori subgroup of \( \text{GL}_e(F_0) \) formed of those matrices in \( \text{GL}_e(\mathfrak{o}_F) \) that are upper triangular modulo \( \mathfrak{p}_F \). If one considers \( W_0 \) as a subgroup of \( \text{GL}_e(F_0) \) it coincides with the affine Weyl group of the diagonal torus. As a consequence, if \( \mu_0 \) denotes the Haar measure on \( \text{GL}_e(F_0) \) normalized by \( \mu_0(I_0) = 1 \), we have the Iwahori-Matsumoto formula

\[
\mu_0(I_0 w I_0) = q_0^{f(w)}, \quad w \in W_0
\]

([LM] Proposition 3.2). Considering \( J_0 \) as a block version of the Iwahori subgroup \( I_0 \), the formula of the lemma is a block version of Iwahori and Matsumoto’s formula, where one has to replace \( q_0 = |\mathfrak{o}_0/\mathfrak{w}_F| \) by \( q_0^2 = |M_F(\mathfrak{o}_0)/\mathfrak{w}_F M_F(\mathfrak{o}_0)| \). The proof is left to the reader.

We shall need a last lemma, whose proof we postpone to the end of that section.
Lemma 5.9 For all $k \in \mathbb{Z}$ and $w_0 \in W_0$, we have

$$c_{v,\tilde{v}}(w_0\Pi^k) = \left(\frac{-1}{q_1}\right)^{l(w_0)} q^{-\frac{f(k-1)}{2}l(w_0)} \langle v, \tilde{v} \rangle_\lambda.$$  

Proof. 

Thanks to the previous series of lemmas, we now may write:

$$\int_{G_0/F_0^\times} c_{v,\tilde{v}}(g) d\mu_{G_0/F_0^\times}(\tilde{g}) = \sum_{w_0 \in W_0} \mu_{G_0}(J_0\Pi^k w_0 J_0) c_{v,\tilde{v}}(\Pi^k w_0)$$

as required. This finishes the proof of Proposition 5.4.

Theorem 5.10 Assume that $e$ is odd. Fix any $J$-equivariant embedding $W \subset \mathcal{V} = \text{St}_e(\tau)$. Let $w \in W$ be a generator of $W^\times$. Then for all non-zero linear form $\Phi \in \text{Hom}_{\text{GL}_n(F_0)}(\text{St}_e(\tau), \mathbb{C})$, we have $\Phi(w) \neq 0$.

Proof. It is a result of Flicker's [F] that the pair $\text{GL}_n(F)/\text{GL}_n(F_0)$ has the following multiplicity 1 property: for any irreducible smooth representation $\sigma$ of $\text{GL}_n(F)$, one has

$$\dim_{\mathbb{C}} \text{Hom}_{\text{GL}_n(F_0)}(\sigma, \mathbb{C}) \leq 1.$$  

It follows that any linear form $\Phi \in \text{Hom}_{\text{GL}_n(F_0)}(\text{St}_e(\tau), \mathbb{C})$ is proportional to the form $\Phi_0$ given with the notation of §4 by

$$\Phi_0(u) = \int_{G_0/F_0^\times} c_{x,u,\tilde{v}}(g) d\mu_{G_0/F_0^\times}(\tilde{g}), \quad u \in \mathcal{V}.$$
Now the fact that \( w \) is a test vector for \( \Phi_0 \) follows from from Proposition 3.3(b) and from the formula of Proposition 5.4 together with Lemma 5.5.

**Proof of Lemma 5.9** By definition we have \( c_{\nu,\nu}(w_0\Pi^k) = \langle \tilde{\Psi}_0(w_0\Pi^k), v, \tilde{v} \rangle_\lambda \), with

\[
\tilde{\Psi}_0(w_0\Pi^k) = (-\frac{1}{q_1})^{l(w_0)} \chi(\tilde{\varphi}_\Pi) \left( \tilde{\varphi}_{[w_0]} * \tilde{\varphi}_\Pi^k \right)(w_0\Pi^k)
\]  

(2)

By Theorem 2.7, we have \( \chi(\tilde{\varphi}_\Pi) = (-1)^{e-1}\omega_e((-1)^{e-1}\varphi F_0) \). Since \( e \) is odd and \( \tau \) is GL(2,F)-distinguished, we obtain \( \chi(\tilde{\varphi}_\Pi) = 1 \).

A convolution calculation shows that \( \tilde{\varphi}_{[w_0]} * \tilde{\varphi}_\Pi^k(w_0\Pi^k) = \tilde{\varphi}_{[w_0]}(w_0)\Gamma^k \).

Therefore from (2), we deduce:

\[
c_{\nu,\nu}(w_0\Pi^k) = \left(-\frac{1}{q_1}\right)^{l(w_0)} \langle \tilde{\varphi}_{[w_0]}(w_0)\Gamma^k, v, \tilde{v} \rangle_\lambda
\]

(3)

\[
= \left(-\frac{1}{q_1}\right)^{l(w_0)} \langle \tilde{\varphi}_{[w_0]}(w_0), v, \tilde{v} \rangle_\lambda
\]

(4)

since \( v \) is \( \Gamma \)-invariant.

Set \( l = l(w_0) \) and let \( w_0 = s_{i_1} \cdots s_{i_l} \), \( i_k \in \{0, ..., e-1\} \), \( k = 1, ..., l \), be a reduced expression of \( w_0 \). We are going to prove that

\[
\tilde{\varphi}_{[w_0]}(w_0) = \left(q^{-\frac{(l-1)}{2}}\right)^{l(w_0)} t_{i_l} \circ \cdots \circ t_{i_l}
\]

(5)

Then Lemma 5.9 will follow since \( v \) is \( t_k \)-invariant, for \( k = 0, ..., e-1 \). We proceed by induction on the length \( l \) of \( w_0 \). If \( l = 1 \), then by Lemma 2.6, we have \( \tilde{\varphi}_{[w_0]}(w_0) = \omega_e(-1)q^{-\frac{l_0}{2}} t_{i_1} \), with \( \omega_e(-1) = 1 \), since \( \tau \) is GL(2,F)-distinguished, and the result follows. Assume that the result holds for elements of length \( l \), for some \( l \geq 1 \), and let \( w_0 \) be of length \( l + 1 \). Write \( w_0 = s_i w_0' \), with \( w_0' \) of length \( l \) and \( i \in \{0, ..., e-1\} \). It suffices to prove that

\[
\tilde{\varphi}_{[w_0]}(w_0) = q^{-\frac{l+1}{2}} t_i \circ \tilde{\varphi}_{[w_0]}(w_0')
\]

(6)

By Corollary 2.4, we have \( \tilde{\varphi}_{[w_0]} = \tilde{\varphi}_{[s_i]} \ast \tilde{\varphi}_{[w_0']} \). Therefore:

\[
\tilde{\varphi}_{[w_0]}(w_0) = \int_G \tilde{\varphi}_{[s_i]}(w_0 g^{-1}) \circ \tilde{\varphi}_{[w_0']} \circ \mu(g)
\]

(7)

\[
= \int_{J_{s_i}J_{w_0} \cap J_{w_0'} J} \tilde{\varphi}_{[s_i]}(w_0 g^{-1}) \circ \tilde{\varphi}_{[w_0']} \circ \mu(g)
\]

(8)

We have \( J_{s_i}J_{w_0} \cap J_{w_0'} J = J_{s_i}J_{s_i w_0'} \cap J_{w_0'} J \). We have the containment

\( J_{s_i}J_{s_i} \subset J \cup J_{s_i}J \)

26
which is a “block version” of a classical axiom of Tits systems. We deduce:

\[ J_{s_i}J_{s_j}w_0 \subset Jw_0' \cup J_{s_i}Jw_0' \]
\[ \subset Jw_0' \cup J_{s_i}Jw_0'J \]
\[ \subset Jw_0' \cup J_{s_i}w_0'J \]

where, once again, the equality \( J_{s_i}Jw_0'J = J_{s_i}w_0'J \) follows from \( l(s_iw_0') = l(w_0') + 1 \) and from a “block version of a classical axiom of Tits systems (cf. [BK1] Lemma (5.6.12) for a proof). Since \( J_{s_i}w_0'J \) and \( Jw_0'J \) are disjoint (cf. e.g. the proof of [BK1] Proposition (5.5.16)), we deduce that \( J_{s_i}Jw_0 \cap Jw_0'J = J_{s_i}w_0 \), and that \( J_{s_i}Jw_0 \cap Jw_0' = J_{s_i}w_0 \), since the containment \( Jw_0' \subset J_{s_i}Jw_0 \cap Jw_0' \) is obvious.

So from 8, we may write:

\[ \tilde{\varphi}_{[w_0]}(w_0) = \int_{J_{s_i}w_0'} \tilde{\varphi}_{[s_i]}(w_0g^{-1}) \circ \tilde{\varphi}_{[w_0']}(g) \, d\mu(g) \]
\[ = \tilde{\varphi}_{[s_i]}(s_i) \circ \tilde{\varphi}_{[w_0']}(w_0') \]

This finishes the proof of 6 and of Lemma 5.9.

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