Schultz and Modified Schultz Polynomials for Vertex – Identification Chain and Ring – for Hexagon Graphs

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ABSTRACT

The aim of this paper is to find polynomials related to Schultz, and modified Schultz indices of vertex identification chain and ring for hexagonal rings (6 – cycles). Also to find index and average index of all of them.

Keywords: Schultz, modified Schultz, vertex identification chain and ring.

1. INTRODUCTION:

We will let all graphs in this paper to be connected, finite, undirected and simple, which means empty from loops and multiple edges. Let \( G = (V, E) \) be a connected simple graph, and \( V = V(G) \) and \( E = E(G) \) denote the sets of vertices and edges, respectively, of \( G \).

In any graph \( G \) represent the number of vertices the order of \( G \) and denoted that by symbol \( p = p(G) = |V(G)| \), and we called the number of edges the size of \( G \), and denoted that by symbol \( q = q(G) = |E(G)| \). We say for any two vertices \( u, v \) in \( G \) adjacent in \( G \) if there exists edge between them, and we write \( e = uv \), as well as we say the edge incident on \( u \) and \( v \). We called the degree of vertex \( u \) as the number of edges incident on it and denoted that by \( degu \) as such that for vertex \( v \) in \( G \) [5].

Now, we define the distance between any two vertices \( u, v \) in \( G \). The distance is the length of a shortest path that join between \( u \) and \( v \) in \( G \) which is denoted by \( d_G(u, v) \) or \( d(u, v) \). We called the maximum distance between any two vertices \( u \) and \( v \) in \( G \) the diameter and denoted that by \( diamG \) [4]. In 2005, Gutman introduced the graph...
polynomials related to the Schultz and modified Schultz indices [12], and in 2011, Behmaram et al. found the Schultz polynomials of some graph operation [3]. Farahani [9], gave Schultz and modified Schultz polynomials of some Harary graphs in 2013. Ahmed and Haitham studied Schultz and modified Schultz polynomials, indices, and index average for two Gutman’s operations [1]. Also they found general formulas for Schultz and modified Schultz polynomials, indices, and index average of cog-special graphs [2]. Also there are many studies about their applications ([6,7,8,10, 11]).

Schultz had introduced and studied in 1989 Schultz index (molecular topological index) [18]. Then, in 1997 Klavžar and Gutman introduced the modified Schultz index [17].

They have defined Schultz and modified Schultz indices, respectively, as:

\[ Sc(G) = \sum_{(u,v) \in V(G)} (deg v + deg u) \ d(u, v). \]
\[ Sc^*(G) = \sum_{(u,v) \in V(G)} (deg v \cdot deg u) \ d(u, v). \]

Schultz and modified Schultz polynomials are considered very important polynomials through studying some properties of their coefficients. Schultz and modified Schultz polynomials are defined, respectively, as:

\[ Sc(G; x) = \sum_{(u,v) \in V(G)} (deg v + deg u) \ x^{d(u,v)}. \]
\[ Sc^*(G; x) = \sum_{(u,v) \in V(G)} (deg v \cdot deg u) \ x^{d(u,v)}. \]

We can obtain the indices of Schultz and modified Schultz by taking derivative of them with respect to \( x \) at \( x = 1 \), as explained below.

\[ Sc(G) = \frac{d}{dx} (Sc(G; x))|_{x=1} \text{ and } Sc^*(G) = \frac{d}{dx} (Sc^*(G; x))|_{x=1}. \]

While we can obtain the average of the Schultz and modified Schultz indices for connected graph \( G \) with order \( p(G) \) that are defined as:

\[ \overline{Sc}(G) = 2Sc(G)/p(G) \ (p(G) - 1) \text{ and } \overline{Sc^*}(G) = 2Sc^*(G)/p(G) \ (p(G) - 1). \]

In any connected graph \( G \), we refer to the set of unordered pairs of vertices which are distance \( k \) apart by the symbol \( D_k(G) \) and let \( |D_k(G)| = D(G,k) \).

Now let that \( D_k(r,n) \) be the set of all unordered pairs of vertices \( u,v \) in \( G \), which are of distance \( k \) and of \( deg u = r, \ deg v = h \).

It is obvious that \( \sum_{k=1}^{diam(G)} |D_k(G)| = p(G)(p(G) - 1)/2, \) where \( D(G,k) = \overline{|D_k(G)|}. \)

Finally, Schultz indices are considered very interesting to determine some properties of chemical structures, see more ([13,14,15,16]).

2. Main Results:

2.1. The Vertex – Identification Chain (VIC) – Graphs:

Let \( \{G_1, G_2, \ldots, G_n\} \) be a set of pairwise disjoint graphs with vertices \( u_i, v_i \in V(G_i), i = 1,2,\ldots, n, n \geq 2 \), then the vertex-identification chain graph \( C_v(G_1, G_2, \ldots, G_n) \equiv C_v(G_1, G_2, \ldots, G_n; v_1 \cdot u_2; v_2 \cdot u_3; \ldots; v_{n-1} \cdot u_n) \) of \( \{G_i\}_{i=1}^{n} \) with respect to the vertices \( \{v_i, u_{i+1}\}_{i=1}^{n-1} \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_n \) by identifying the vertex \( v_i \) with the vertex \( u_{i+1} \) for all \( i = 1,2,\ldots, n - 1 \). (See Fig. 2-1)
Some Properties of Graph $C_v(G_1, G_2, ..., G_n)$:

1. $p(C_v(G_1, G_2, ..., G_n)) = \sum_{i=1}^{n} p(G_i) - (n - 1)$.
2. $q(C_v(G_1, G_2, ..., G_n)) = \sum_{i=1}^{n} q(G_i)$.
3. $n \leq diam(C_v(G_1, G_2, ..., G_n)) \leq \sum_{i=1}^{n} diam(G_i)$.

The equality of both bounds are satisfied at complete graphs, but the upper bound is satisfied at path graphs in which $v_i$ are end-vertices of $G_i$ for $i = 1, 2, ..., n$. If $G_i \equiv H_p$, for all $1 \leq i \leq n$, where $H_p$ is a connected graph of order $p$, we denoted $C_v(H_p, H_p, ..., H_p)$ by $C_v(H_p)_n$.

Schultz and modified Schultz of $C_v(C_6)_{p/2}$

From Fig. 2-1-2, we note that $p\left(C_v(C_6)_{p/2}\right) = \frac{5p}{2} + 1$, $q(C_v(C_6)_{p/2}) = 3p$ and $diam\left(C_v(C_6)_{p/2}\right) = \frac{3p}{2}$. For all $1 \leq i, j \leq p$, $i \neq j$ and $2 \leq m, h \leq \frac{p}{2}, m \neq h$ we have:

| $\times$ | $deg_u_i = 2$ | $deg_v_i = 2$ | $degw_1 = 2$ | $degw_{p/2+1}$ | $degw_m = 4$ |
|--------|----------------|----------------|--------------|----------------|--------------|
| $deg_j = 2$ | 4              | 4              | 4            | 4              | 6            |
| $degv_j = 2$ | 4              | 4              | 4            | 4              | 6            |
| $degw_1 = 2$ | 4              | 4              | 4            | 4              | 6            |
| $degw_{p/2+1}$ | 4              | 4              | 4            | 4              | 6            |
| $degw_h = 4$ | 4              | 4              | 4            | 4              | 6            |

Table 2.1
Theorem 2.1.1: For \( p \geq 4 \), then:

1. \( Sc \left( C_p(C_6)_{\frac{p}{2}}; x \right) = 8(2p - 1)x + 24(p - 1)x^2 + 12(2p - 3)x^3 + \frac{20}{3}\sum_{k=4}^{3p} (3p - 2k)x^k + \frac{4}{3}x(3x^2 + 2x + 4) \sum_{k=1}^{\frac{p}{2}}x^k \).

2. \( Sc^* \left( C_p(C_6)_{\frac{p}{2}}; x \right) = 4(5p - 4)x + 4(7p - 8)x^2 + 4(7p - 12)x^3 + \frac{3p}{2}\sum_{k=4}^{3p} (24p - 16)x^k + 4x^3 \frac{3p}{2} \).

Proof: For all \( p \geq 8 \) and every two vertices \( u, v \in V \left( C_p(C_6)_{\frac{p}{2}} \right) \), there is \( d(u, v) = k \).

1 \leq k \leq \frac{3p}{2} \), we will have ten partitions for proof:

P1. If \( d(u, v) = 1 \), then \( |D_1| = 3p = q \left( C_p(C_6)_{\frac{p}{2}} \right) \) and we have two subsets of the edge set:

P1.1 \( |D_1(2,2)| = \left\{ (u_{2i-1}, u_{2i}), (v_{2i-1}, v_{2i}) : 1 \leq i \leq \frac{p}{2} \right\} \cup \{(w_1, u_1), (w_1, v_1), \left( u_{\frac{p}{2}+1}, w_p \right), \left( w_{\frac{p}{2}+1}, v_p \right) \} = p + 4.

P1.2 \( |D_1(2,4)| = \left\{ (u_{2i-1}, w_{2i+1}), (v_{2i-1}, w_{2i+1}), (u_{2i-1}, w_{i+1}), (v_{2i}, w_{i+1}) : 1 \leq i \leq \frac{p}{2} - 1 \right\} = 2p - 4.

P2. If \( d(u, v) = 2 \), then, we have two subsets of \( D_2 \):

P2.1 \( |D_2(2,2)| = \left\{ (u_{2i}, u_{2i+1}), (v_{2i}, v_{2i+1}), (u_{2i}, v_{2i+1}), (v_{2i}, u_{2i+1}) : 1 \leq i \leq \frac{p}{2} - 1 \right\} \cup \{(w_1, u_2), (w_1, v_2), \left( u_{\frac{p}{2}+1}, u_{p-1} \right), \left( w_{\frac{p}{2}+1}, v_{p-1} \right) \} = 3p.

P2.2 \( |D_2(2,4)| = \left\{ (u_{2i-1}, w_{2i+1}), (v_{2i-1}, w_{i+1}), (u_{2i+2}, w_{i+1}), (v_{2i+2}, w_{i+1}) : 1 \leq i \leq \frac{p}{2} - 1 \right\} = 2p - 4.

Therefore, \( |D_2| = 5p - 4 \).

P3. If \( d(u, v) = 3 \), then, we have three subsets of \( D_3 \):

P3.1 \( |D_3(2,2)| = \left\{ (u_i, u_{i+2}), (v_i, v_{i+2}), (u_i, v_{i+2}), (v_i, u_{i+2}) : 1 \leq i \leq p - 2 \right\} \cup \{(u_{2i-1}, v_{2i}), (v_{2i-1}, u_{2i}) : 1 \leq i \leq \frac{p}{2} \} = 5p - 8.

P3.2 \( |D_3(2,4)| = \left\{ (w_1, w_2), \left( u_{\frac{p}{2}+1}, w_{\frac{p}{2}} \right) \right\} = 2.

P3.3 \( |D_3(4,4)| = \left\{ (w_{i+1}, w_{i+2}) : 1 \leq i \leq \frac{p}{2} - 2 \right\} = \frac{p}{2} - 2.

Therefore, \( |D_3| = \frac{11p}{2} - 8 \).

P4. If \( d(u, v) = k \), when \( k = 3j + 4, j = 0, 1, ..., \frac{p}{2} - 3 \), then, we have two subsets of \( D_k \):

P4.1 \( |D_k(2,2)| = \left\{ (u_{2i-1}, u_{2i+\frac{k+2}{3}}), (v_{2i-1}, v_{2i+\frac{k+2}{3}}), (u_{2i-1}, v_{2i+\frac{k+2}{3}}), (v_{2i-1}, u_{2i+\frac{k+2}{3}}) : 1 \leq i \leq \frac{p}{2} - \frac{k-1}{3} \right\} \cup \{(w_1, u_{\frac{k+1}{3}}), (w_1, v_{\frac{k+1}{3}}), \left( u_{\frac{p}{2}+1}, u_{\frac{k+1}{3}} \right), \left( w_{\frac{p}{2}+1}, v_{\frac{k+1}{3}} \right) \} = 2p - \frac{4(k-4)}{3}.

P4.2 \( |D_k(2,4)| = \left\{ (u_{2i}, w_{\frac{k+2}{3}}), (v_{2i}, w_{\frac{k+2}{3}}), (u_{2i}, v_{\frac{k+2}{3}}), (v_{2i}, u_{\frac{k+2}{3}}) : 1 \leq i \leq \frac{p}{2} - \frac{k+2}{3} \right\} = 2p - \frac{4(k+2)}{3}.

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Therefore $|D_k| = 4p - \frac{8}{3}(k - 1)$, for $k = 3j + 4, j = 0, 1, ..., \frac{p}{2} - 3$.

**P5.** If $d(u, v) = k$, when $k = 3j + 5, j = 0, 1, ..., \frac{p}{2} - 3$, then, we have two subset of $D_k$:

**P5.1** $|D_k(2, 2)| = |\left\{\left(\frac{u_{2i}, u_{i+2k}}{2}, \frac{v_{2i}, v_{i+2k}}{2}, \frac{u_{2i}, u_{i+2k}}{2}, \frac{v_{2i}, v_{i+2k}}{2}\right): 1 \leq i \leq \frac{p}{2} - \frac{k+1}{3}\right\} \cup \left\{\left(\frac{w_1, u_{2i+1}}, \frac{w_{2i+1}, u_{i+2k}}{2}, \frac{w_{2i+1}, u_{i+2k}}{2}, \frac{w_{2i+1}, u_{i+2k}}{2}\right)\right\}| = 2p - \frac{4(k-2)}{3}.

**P5.2** $|D_k(2, 4)| = |\left\{\left(\frac{u_{2i-1}, w_{i+2k}}{2}, \frac{v_{2i-1}, w_{i+2k}}{2}, \frac{u_{2i+2k}, w_{i+2k}}{2}, \frac{v_{2i+2k}, w_{i+2k}}{2}\right): 1 \leq i \leq \frac{p}{2} - \frac{k+1}{3}\right\}| = 2p - \frac{4(k+1)}{3}.

Thus $|D_k| = 4p - \frac{4}{3}(2k - 1)$ for $k = 3j + 5, j = 0, 1, ..., \frac{p}{2} - 3$.

**P6.** If $d(u, v) = k$, when $k = 3j + 6, j = 0, 1, ..., \frac{p}{2} - 4$, then, we have three subsets of $D_k$:

**P6.1** $|D_k(2, 2)| = |\left\{\left(\frac{u_{i}, u_{i+2k}}{2}, \frac{v_{i}, v_{i+2k}}{2}, \frac{u_{i}, u_{i+2k}}{2}, \frac{v_{i}, v_{i+2k}}{2}\right): 1 \leq i \leq p - \frac{2k}{3}\right\}| = 4p - \frac{8k}{3}.

**P6.2** $|D_k(2, 4)| = |\left\{\left(\frac{w_1, w_{i+2k}}{2}, \frac{w_{p+1} + k + 1, w_{i+2k}}{2}\right): 1 \leq i \leq \frac{p}{2} - \frac{k+1}{3}\right\}| = 2.

**P6.3** $|D_k(4, 4)| = |\left\{\left(\frac{w_{i+1}, w_{i+2k}}{2}, \frac{w_{p+1} + k + 1}{2}\right): 1 \leq i \leq \frac{p}{2} - \frac{k+1}{3}\right\}| = \frac{p}{2} - \frac{k}{3} - 1.

Thus $|D_k| = 9p - 3k + 1$ for $k = 3j + 6, j = 0, 1, ..., \frac{p}{2} - 4$.

**P7.** If $d(u, v) = \frac{3p}{2} - 3$, then, we have two subsets of $D_{\frac{3p}{2} - 3}$:

**P7.1** $\left|D_{\frac{3p}{2} - 3}(2, 2)\right| = |\left\{\left(\frac{u_{i}, u_{i+p+i-2}}, \frac{v_{i}, v_{i+p+i-2}}, \frac{u_{i}, u_{i+p+i-2}}, \frac{v_{i}, u_{i+p+i-2}}{2}\right): i = 1, 2\right\}| = 8.

**P7.2** $\left|D_{\frac{3p}{2} - 3}(2, 4)\right| = |\left\{\left(\frac{w_1, w_{p+1}}, \frac{w_{p+1}, w_{p+1}}{2}\right)\right\}| = 2.

Therefore $\left|D_{\frac{3p}{2} - 3}\right| = 10$.

**P8.** If $d(u, v) = \frac{3p}{2} - 2$, then $\left|D_{\frac{3p}{2} - 2}\right| = 8$, because:

$\left|D_{\frac{3p}{2} - 2}(2, 2)\right| = |\left\{\left(\frac{u_{i}, u_{i+p+i-2}}, \frac{v_{i}, v_{i+p+i-2}}, \frac{u_{2i}, u_{i+p+i-2}}, \frac{v_{2i}, u_{i+p+i-2}}{2}\right): i = 1, 2\right\}| = 8.

**P9.** If $d(u, v) = \frac{3p}{2} - 1$, then $\left|D_{\frac{3p}{2} - 1}\right| = 4$, because:

$\left|D_{\frac{3p}{2} - 1}(2, 2)\right| = |\left\{\left(\frac{u_1, w_{p+1}}, \frac{u_1, w_{p+1}}, \frac{v_1, w_{p+1}}, \frac{v_1, w_{p+1}}{2}\right): 1\right\}| = 4.

**P10.** If $d(u, v) = \frac{3p}{2}$, then $\left|D_{\frac{3p}{2}}\right| = 1$, since $\left|D_{\frac{3p}{2}}(2, 2)\right| = |\left\{\left(\frac{w_1, w_{p+1}}{2}\right)\right\}| = 1$.

From $P_1$ to $P_{10}$ and Table 2.1.1, we have:

$Sc\left(C_e(C_9)_{\frac{p}{2}}x\right) = \{4(p + 4) + 6(2p - 4)\}x + \{4(3p) + 6(2p - 4)\}x^2$

$+ \{4(5p - 8) + 6(2) + 8\left(\frac{p}{2} - 2\right)\}x^3$
Now, we find modified Shultz polynomial:

\[ \text{Sc}^* \left( C_v(C_6)_{\frac{p}{2}}; x \right) = \left\{ 4(p + 4) + 8(2p - 4) \right\} x + \left\{ 4(3p) + 8(2p - 4) \right\} x^2 \]
\[ + \left\{ 4(5p - 8) + 8(2) + 16 \left( \frac{p}{2} - 2 \right) \right\} x^3 \]
\[ + \sum_{k=4,7,10,...}^{3p-5} \left\{ 4 \left( 2p - \frac{4(k-4)}{3} \right) + 8(2p - \frac{4(k+2)}{3}) \right\} x^k \]
\[ + \sum_{k=5,8,11,...}^{3p-4} \left\{ 4 \left( 2p - \frac{4(k-2)}{3} \right) + 8(2p - \frac{4(k+1)}{3}) \right\} x^k \]
\[ + \sum_{k=6,9,12,...}^{3p-6} \left\{ 4 \left( 4p - \frac{8k}{3} \right) + 8(2) + 16 \left( \frac{p}{2} - \frac{k}{3} - 1 \right) \right\} x^k \]
\[ + \left\{ 4(8) + 8(2) \right\} x^{\frac{3p-3}{2}} + \left\{ 4(8) \right\} x^{\frac{3p-2}{2}} + \left\{ 4(4) \right\} x^{\frac{3p-1}{2}} + \left\{ 4(1) \right\} x^\frac{3p}{2}. \]
\[ = 4(5p - 4)x + 4(7p - 8)x^2 + 4(7p - 12)x^3 \]
\[ + \sum_{k=4,7,10,...}^{3p-2} (24p - 16k)x^k \]
\[ + \sum_{k=5,8,11,...}^{3p-1} (24p - 16k)x^k \]
\[ + \sum_{k=6,9,12,...}^{3p} (24p - 16k)x^k \]
\[ = 4(5p - 4)x + 4(7p - 8)x^2 + 4(7p - 12)x^3 \]
\[ + \sum_{k=4,7,10,...}^{3p-2} (24p - 16k)x^k + 4x^{\frac{3p}{2}}. \]

Remark:

1. \( \text{Sc}(C_v(C_6)_{\frac{p}{2}}; x) = 56x + 72x^2 + 60x^3 + 32x^4 + 16x^5 + 4x^6. \)
2. \( \text{Sc}^*(C_v(C_6)_{\frac{p}{2}}; x) = 64x + 80x^2 + 64x^3 + 32x^4 + 16x^5 + 4x^6. \)
3. \( \text{Sc}\left(C_v(C_6)_{\frac{p}{2}}; x \right) = 104x + 136x^2 + 120x^3 + 80x^4 + 64x^5 + 48x^6 + 32x^7 + 16x^8 + 4x^9. \)

Corollary 2.1.2: For \( p \geq 4 \), then we have:

1. \( \text{Sc}(C_v(C_6)_{\frac{p}{2}}) = \frac{3p}{2}(5p^2 + 3p + 10). \)
2. \( \text{Sc}^*(C_v(C_6)_{\frac{p}{2}}) = 9p(p^2 + 2). \)

Corollary 2.1.3: If \( n \) is the number of cycles \( C_6 \) in the graph \( C_v(C_6)_n \), \( n \geq 2 \), then

1. \( \text{Sc}(C_v(C_6)_n) = 6n(10n^2 + 3n + 5). \)
2. \( Sc^*(C_9(C_6)_n) = 36n(2n^2 + 1) \)

**Corollary 2.1.4:** For \( p \geq 4 \), then we have:

1. \( \overline{Sc}(C_9(C_6)_2) = \frac{12}{25} (5p + 1 + \frac{48}{5p+2}) \).
2. \( \overline{Sc}^*(C_9(C_6)_{p-1}) = \frac{72}{125} (5p - 2 + \frac{54}{5p+2}) \).

2.2. The Vertex – Identification Ring (VIR) – Graph:

Let \( \{G_1, G_2, \ldots, G_n\} \) be a set of pairwise disjoint graphs with vertices \( u_i, v_i \in V(G_i), i = 1, 2, \ldots, n \geq 3 \), then the vertex-identification Ring graph \( R_v(G_1, G_2, \ldots, G_n) \equiv R_v(G_1, G_2, \ldots, G_n; v_1 \cdot u_2; v_2 \cdot u_3; \ldots; v_{n-1} \cdot u_n; v_n \cdot u_1) \) of \( \{G_i\}_{i=1}^n \) with respect to the vertices \( \{v_i, u_i\}_{i=1}^n \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_n \) by identifying the vertex \( v_i \) with the vertex \( u_{i+1} \) for all \( i = 1, 2, \ldots, n \). (See Fig. 2-2) where \( u_{n+1} \equiv u_1 \).

![Fig. 2-2-1. \( R_v(G_1, G_2, \ldots, G_n) \)](image)

Some Properties of the graph \( R_v(G_1, G_2, \ldots, G_n) \):

1. \( p(R_v(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^n p(G_i) - n \).
2. \( q(R_v(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^n q(G_i) \).
3. \( \left\lfloor \frac{n-1}{2} \right\rfloor \leq diam(R_v(G_1, G_2, \ldots, G_n)) \leq \frac{\sum_{i=1}^n diam(G_i)}{2} \).

The equality of both bounds are satisfied at complete graphs but the upper bound is satisfied at path graphs in which \( v_i, u_i \) are end-vertices of \( G_i \) for \( i = 1, 2, \ldots, n \).

If \( G_i \equiv H_p \), for all \( 1 \leq i \leq n \), where \( H_p \) is a connected graph of order \( p \), we denoted \( R_v(H_p, H_p, \ldots, H_p) \) by \( R_v(H_p)_n \).

**Schultz and modified Schultz of \( R_v(C_6)_{p/2} \):**

![Fig. 2-2-2. The Graph \( R_v(C_6)_{p/2} \). \( p \geq 6 \), even \( p \).](image)
From Fig. 2-2-2, we note that \( p(R_v(C_6)_{\frac{p}{2}}) = \frac{5p}{2}, q(R_v(C_6)_{\frac{p}{2}}) = 3p \) and 
\[ \operatorname{diam}(R_v(C_6)_{\frac{p}{2}}) = \frac{p}{2} + \left\lceil \frac{p-2}{4} \right\rceil. \] 
For all \( 1 \leq i, j \leq p, i \neq j \), then we have:

| \( + \) | \( \times \) | \( \text{deg}_{u_i} = 2 \) | \( \text{deg}_{v_i} = 2 \) | \( \text{deg}_{w_i} = 4 \) |
| --- | --- | --- | --- | --- |
| \( \text{deg}_{u_j} = 2 \) | 4 | 4 | 6 |
| \( \text{deg}_{v_j} = 2 \) | 4 | 4 | 6 |
| \( \text{deg}_{w_j} = 4 \) | 8 | 8 | 16 |

**Table 2.1.1**

**Theorem 2.1.2:** For \( p \geq 8 \), then we have:

1. \( \text{Sc} \left( R_v(C_6)_{\frac{p}{2}}; x \right) = 16px + 24px^2 + 24px^3 \)

\[
+ \left\{ 20p \sum_{k=4,5,6,...}^{\frac{p}{2}+\left\lceil \frac{p-2}{4} \right\rceil} x^k + 10px^2 \left\lceil \frac{p-2}{4} \right\rceil, \text{when } p = 12, 16, 20, \ldots \right. \]

2. \( \text{Sc}^* \left( R_v(C_6)_{\frac{p}{2}}; x \right) = 20px + 28px^2 + 28px^3 \)

\[
+ \left\{ 24p \sum_{k=4,5,6,...}^{\frac{p}{2}+\left\lceil \frac{p-2}{4} \right\rceil} x^k + 12px^2 \left\lceil \frac{p-2}{4} \right\rceil, \text{when } p = 12, 16, 20, \ldots \right. \]

**Proof:** For all \( p \geq 12 \), and every two vertices \( u, v \in V(R_v(C_6)_{\frac{p}{2}}) \), there is \( d(u, v) = k \), 
\( 1 \leq k \leq \frac{3p}{2} \), we will have seven partitions for proof:

**P1.** If \( d(u, v) = 1 \), then \( |D_1| = 3p = q \left( R_v(C_6)_{\frac{p}{2}} \right) \) and we have two subsets of the edge set:

**P1.1** \( |D_1(2, 2)| = \left\| \left\{ (u_{2i-1}, u_{2i}), (v_{2i-1}, v_{2i}); 1 \leq i \leq \frac{p}{2} \right\} \right\| = p \).

**P1.2** \( |D_1(2, 4)| = \left\| \left\{ (u_{2i-1}, w_i), (v_{2i-1}, w_i), (u_{2i}, w_{i+1}), (v_{2i}, w_{i+1}); 1 \leq i \leq \frac{p}{2} \right\} \right\| = 2p \),
where \( w_{\frac{p}{2}+1} \equiv w_1 \).

**P2.** If \( d(u, v) = 2 \), then, we have two subsets of \( D_2 \):

**P2.1** \( |D_2(2, 2)| = \left\| \left\{ (u_{2i}, u_{2i+1}), (v_{2i}, v_{2i+1}), (u_{2i+1}, v_{2i+1}), (v_{2i+1}, u_{2i}); 1 \leq i \leq \frac{p}{2} \right\} \cup \left\{ (u_i, v_i); 1 \leq i \leq p \right\} \right\| = 3p \), where \( u_{p+1} \equiv u_1 \) and \( v_{p+1} \equiv v_1 \).

**P2.2** \( |D_2(2, 4)| = \left\| \left\{ (u_{2i-1}, w_{i+1}), (v_{2i-1}, w_{i+1}), (u_{2i}, w_i), (v_{2i}, w_i); 1 \leq i \leq \frac{p}{2} \right\} \right\| = 2p \),
where \( w_{\frac{p}{2}+1} \equiv w_1 \).

Thus \( |D_2| = 5p \).

**P3.** If \( d(u, v) = 3 \), then, we have three subsets of \( D_3 \):

**P3.1** \( |D_3(2, 2)| = \left\| \left\{ (u_i, u_{i+2}), (v_i, v_{i+2}), (u_i, v_{i+2}), (v_i, u_{i+2}); 1 \leq i \leq p \right\} \cup \right\| = p \),
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\[ \{(u_{2i}, v_{2i-1}), (v_{2i}, u_{2i-1}) : 1 \leq i \leq \frac{p}{2}\} = 5p, \]

where \( u_{p+a} \equiv u_a \) and \( v_{p+a} \equiv v_a, a = 1, 2. \)

**P3.2** \(|D_3(4, 4)| = \left| \{(w_i, w_{i+1}) : 1 \leq i \leq \frac{p}{2}\} \right| = \frac{p}{2}, \) where \( w_{\frac{p}{2}+1} \equiv w_1. \)

Thus \(|D_3| = \frac{11p}{2}.\)

**P4.** If \( d(u, v) = k, \) when \( k = 3j + 4, \) and \( p = 12, 16, 20, \ldots, j = 0, 1, 2, ..., \frac{p}{4} - 2, \) and when \( p = 14, 18, 22, \ldots, j = 0, 1, ..., \frac{p-2}{4} - 2, \) then, we have two subsets of such \((u, v)\) pairs of \( D_k:\)

**P4.1** \(|D_k(2, 2)| = \left| \{(u_{2i-1}, u_{2i+2(k-1)/3}), (v_{2i-1}, v_{2i+2(k-1)/3}), (u_{2i-1}, v_{2i+2(k-1)/3}), (v_{2i}, u_{2i+2(k-1)/3}) : 1 \leq i \leq \frac{p}{2}\} \right| = 2p, \)

where \( u_{p+a} \equiv u_a \) and \( v_{p+a} \equiv v_a, a = 2, 4, 6, ..., \frac{2(k-1)}{3}. \)

**P4.2** \(|D_k(2, 4)| = \left| \{(u_{2i}, w_{i+k+2/3}), (v_{2i}, w_{i+k+2/3}), (u_{2i}, v_{i+k+2/3}), (v_{2i}, u_{i+k+2/3}) : 1 \leq i \leq \frac{p}{2}\} \right| = 2p, \)

where \( w_{p+b} \equiv w_b, b = 1, 2, 3, ..., \frac{k+2}{3}. \)

Thus \(|D_k| = 4p, \) \( k = 3j + 4. \)

**P5.** If \( d(u, v) = k, \) when \( k = 3j + 4, \) and \( p = 12, 16, 20, \ldots, j = 0, 1, 2, ..., \frac{p}{4} - 2, \) and when \( p = 14, 18, 22, \ldots, j = 0, 1, 2, ..., \frac{p-2}{4} - 2, \) then, we have two subsets of such \((u, v)\) pairs of \( D_k:\)

**P5.1** \(|D_k(2, 2)| = \left| \{(u_{2i}, u_{2i+2(k-1)/3}), (v_{2i}, v_{2i+2(k-1)/3}), (u_{2i}, v_{2i+2(k-1)/3}), (v_{2i}, u_{2i+2(k-1)/3}) : 1 \leq i \leq \frac{p}{2}\} \right| = 2p, \)

where \( u_{p+a} \equiv u_a \) and \( v_{p+a} \equiv v_a, a = 1, 2, 3, ..., \frac{2k-1}{3}. \)

**P5.2** \(|D_k(2, 4)| = \left| \{(u_{2i-1}, w_{i+k+1/3}), (v_{2i-1}, w_{i+k+1/3}), (u_{2i-1}, v_{i+k+1/3}), (v_{2i-1}, u_{i+k+1/3}) : 1 \leq i \leq \frac{p}{2}\} \right| = 2p, \)

where \( u_{p+a} \equiv u_a, v_{p+a} \equiv v_a, a = 2, 4, 6, ..., \frac{2(k-2)}{3} \) and \( w_{p+b} \equiv w_b, b = 1, 2, 3, ..., \frac{k+1}{3}. \)

Thus \(|D_k| = 4p, k = 3j + 5, \) for \( j = 0, 1, 2, ..., \frac{p-2}{4} - 2. \)

**P6.** If \( d(u, v) = k, \) when \( k = 3j + 6, \) and when \( p = 12, 16, 20, \ldots, j = 0, 1, 2, ..., \frac{p}{4} - 3, \)

and when \( p = 14, 18, 22, \ldots, j = 0, 1, ..., \frac{p-2}{4} - 2, \) then, we have three subsets of such \((u, v)\) pairs of \( D_k:\)

**P6.1** \(|D_k(2, 2)| = \left| \{(u_i, u_{i+2k/3}), (v_i, v_{i+2k/3}), (u_i, v_{i+2k/3}), (v_i, u_{i+2k/3}) : 1 \leq i \leq p\} \right| = 4p, \)

where \( u_{p+a} \equiv u_a \) and \( v_{p+a} \equiv v_a \) \( a = 1, 2, 3, ..., \frac{2k}{3}. \)

**P6.2** \(|D_k(4, 4)| = \left| \{(w_i, w_{i+k/3}) : 1 \leq i \leq \frac{p}{2}\} \right| = \frac{p}{2}, \)

where \( w_{p+b} \equiv w_b, b = 1, 2, 3, ..., \frac{2k-3}{3}. \)

Thus \(|D_k| = \frac{9p}{2}, k = 3j + 6, \) for \( j = 0, 1, 2, ..., \frac{p}{4} - 3. \)
P7. If \( d(u, v) = \frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor \), then we have:

- If \( p = 12, 16, 20, \ldots \), then, we have two subsets of \( D_{\frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor} \):

  \[
  P7.1 \left\lfloor \frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor \right\rfloor (2, 2) = \left\lfloor \left( u_i, u_{i+\frac{p}{2}} \right), \left( v_i, v_{i+\frac{p}{2}} \right), \left( u_i, v_{i+\frac{p}{2}} \right), \left( v_i, u_{i+\frac{p}{2}} \right) : 1 \leq i \leq \frac{p}{2} \right\rfloor = 2p.
  \]

  Thus \( \left\lfloor \frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor \right\rfloor = \frac{p}{2} \), for even \( \frac{p}{2} \).

- If \( p = 14, 18, 22, \ldots \) then, we have two subsets of \( D_{\frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor} \):

  \[
  P7.2 \left\lfloor \frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor \right\rfloor (4, 4) = \left\lfloor \left( w_i, w_{i+\frac{p}{2}} \right) : 1 \leq i \leq \frac{p}{4} \right\rfloor = \frac{p}{4}.
  \]

  Thus \( \left\lfloor \frac{p}{2} + \left\lfloor \frac{p-2}{4} \right\rfloor \right\rfloor = \frac{p}{4} \), for odd \( \frac{p}{2} \).

From P1 to P7 and Table 2.1.2, we have:

\[
Sc (R_2(G_2)) = \left\{ 4(p) + 6(2p) \right\} x + \left\{ 4(3p) + 6(2p - 4) \right\} x^2 + \left\{ 4(5p) + 8\left( \frac{p}{4} \right) \right\} x^3
\]

\[
\sum_{k=4,7,10,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 2} \left\{ 4(2p) + 6(2p) \right\} x^k + \sum_{k=5,8,11,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 1} \left\{ 4(2p) + 6(2p) \right\} x^k
\]

\[
\sum_{k=6,9,12,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 3} \left\{ 4(4p) + 8\left( \frac{p}{4} \right) \right\} x^k + \left\{ 4(2p) + 8\left( \frac{p}{4} \right) \right\} x^{2\left( \frac{p}{4} \right)}
\]

\[
\sum_{k=4,7,10,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 2} \left\{ 4(2p) + 6(2p) \right\} x^k + \sum_{k=5,8,11,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 2} \left\{ 4(2p) + 6(2p) \right\} x^k
\]

\[
\sum_{k=6,9,12,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 3} \left\{ 4(4p) + 8\left( \frac{p}{4} \right) \right\} x^k + \left\{ 4(2p) + 6(2p) \right\} x^{2\left( \frac{p}{4} \right)}
\]

\[
\sum_{k=4,7,10,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 2} \left\{ 4(2p) + 6(2p) \right\} x^k + \sum_{k=5,8,11,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 1} \left\{ 4(2p) + 6(2p) \right\} x^k
\]

\[
\sum_{k=6,9,12,\ldots}^{\left\lfloor \frac{p}{4} \right\rfloor - 3} \left\{ 4(4p) + 8\left( \frac{p}{4} \right) \right\} x^k + \left\{ 4(2p) + 6(2p) \right\} x^{2\left( \frac{p}{4} \right)}
\]

Now, we find modified Shultz polynomial:
\( Sc^* \left( R_v(C_6)_{p/2} | x \right) = \left\{ 4(p) + 8(2p) \right\} x + \left\{ 4(3p) + 8(2p - 4) \right\} x^2 + \left\{ 4(5p) + 16 \left( \frac{p}{2} \right) \right\} x^3 \)

\[
\begin{align*}
&\sum_{k=4,7,10,...}^{p+\left\lfloor \frac{p-2}{4} \right\rfloor -2} \{ 4(2p) + 8(2p) \} x^k + \sum_{k=5,8,11,...}^{p+\left\lfloor \frac{p-2}{4} \right\rfloor -1} \{ 4(2p) + 8(2p) \} x^k \\
&+ \sum_{k=6,9,12,...}^{\left\lfloor \frac{p-2}{4} \right\rfloor} \left\{ 4(4p) + 16 \left( \frac{p}{2} \right) \right\} x^k + \left\{ 4(2p) + 16 \left( \frac{p}{4} \right) \right\} x^2, \text{ when } p = 12,16,20, ... \\
&+ \sum_{k=4,7,10,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -2} \{ 4(2p) + 8(2p) \} x^k + \sum_{k=5,8,11,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -1} \{ 4(2p) + 8(2p) \} x^k \\
&+ \sum_{k=6,9,12,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor} \left\{ 4(4p) + 16 \left( \frac{p}{2} \right) \right\} x^k + \left\{ 4(2p) + 8(2p) \right\} x^2, \text{ when } p = 14,18,22, ... \\
= 20px + 28px^2 + 28px^3 \\
&+ \frac{p}{4} \sum_{k=4,7,10,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -2} \left\{ 4(2p) + 8(2p) \right\} x^k + \frac{p}{4} \sum_{k=5,8,11,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -1} \left\{ 4(2p) + 8(2p) \right\} x^k \\
&+ 12px^2, \text{ when } p = 12,16,20, ... \\
&+ \frac{p}{4} \sum_{k=4,7,10,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -3} \left\{ 4(2p) + 8(2p) \right\} x^k + \frac{p}{4} \sum_{k=5,8,11,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -2} \left\{ 4(2p) + 8(2p) \right\} x^k \\
&+ 24px^2, \text{ when } p = 14,18,22, ... \\
&+ 20px + 28px^2 + 28px^3 \\
&+ \frac{p}{4} \sum_{k=4,7,10,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor -1} \left\{ 4(2p) + 8(2p) \right\} x^k + \frac{p}{4} \sum_{k=5,8,11,...}^{3p+\left\lfloor \frac{p-2}{4} \right\rfloor} \left\{ 4(2p) + 8(2p) \right\} x^k, \text{ when } p = 14,18,22, ...
\end{align*}
\]

By simply, we can calculate:

1. \( Sc(R_v(C_6)_4 | x) = 128x + 192x^2 + 192x^3 + 160x^4 + 160x^5 + 80x^6 \).
2. \( Sc^*(R_v(C_6)_4 | x) = 160x + 224x^2 + 224x^3 + 192x^4 + 192x^5 + 96x^6 \).
3. \( Sc(R_v(C_6)_5 | x) = 160x + 240x^2 + 240x^3 + 200x^4 + 200x^5 + 200x^6 + 200x^7 \).
4. \( Sc^*(R_v(C_6)_5 | x) = 200x + 280x^2 + 280x^3 + 240x^4 + 240x^5 + 240x^6 + 240x^7 \).

**Remark:**

1. \( Sc(R_v(C_6)_2 | x) = 64x + 96x^2 + 56x^3 \).
2. \( Sc^*(R_v(C_6)_2 | x) = 80x + 112x^2 + 64x^3 \).
3. \( Sc(R_v(C_6)_3 | x) = 96x + 144x^2 + 144x^3 + 120x^4 \).
4. \( Sc^*(R_v(C_6)_3 | x) = 120x + 168x^2 + 168x^3 + 144x^4 \).

**Corollary 2.1.2:** For \( p \geq 4 \), then we have:

1. \( Sc \left( R_v(C_6)_{p/2} \right) = \left\{ \frac{p}{8} \left( 45p^2 + 128 \right) \right\}, \text{ when } p = 4,8,12, ... \)
2. \( Sc^* \left( R_v(C_6)_{p/2} \right) = \left\{ \frac{p}{4} \left( 27p^2 + 64 \right) \right\}, \text{ when } p = 4,8,12, ... \)

**Corollary 2.1.3:** If \( n \) is the number of cycles \( C_6 \) in the graph \( R_v(C_6)_n \), \( n \geq 2 \), then we have:
1. $\text{Sc}(R_v(C_6)_n) = \begin{cases} n(45n^2 + 32), & \text{when } n = 2, 4, 6, \ldots \\ 9n(5n^2 + 3), & \text{when } n = 3, 5, 7, \ldots \end{cases}$

2. $\text{Sc}^*(R_v(C_6)_n) = \begin{cases} 2n(27n^2 + 16), & \text{when } n = 2, 4, 6, \ldots \\ 2n(27n^2 + 13), & \text{when } n = 3, 5, 7, \ldots \end{cases}$

**Corollary 2.1.4:** For $p \geq 4$, then we have:

1. $\overline{\text{Sc}}(R_v(C_6)_p) = \begin{cases} \frac{1}{5} \left(9p + \frac{18}{5} + \frac{676}{5(p-2)}\right), & \text{when } p = 4, 8, 12, \ldots \\ \frac{3}{5} \left(3p + \frac{6}{5} + \frac{162}{5(p-2)}\right), & \text{when } p = 6, 10, 14, \ldots \end{cases}$

2. $\overline{\text{Sc}}^*(C_v(C_6)_p) = \begin{cases} \frac{2}{5} \left(27p + \frac{54}{5} + \frac{1708}{5(p-2)}\right), & \text{when } p = 4, 8, 12, \ldots \\ \frac{2}{25} \left(27p + \frac{54}{5} + \frac{1408}{5(p-2)}\right), & \text{when } p = 6, 10, 14, \ldots \end{cases}$
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