Lie symmetries of two-dimensional shallow water equations with variable bottom topography

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We carry out the group classification of the class of two-dimensional shallow water equations with variable bottom topography using an optimized version of the method of furcate splitting. The equivalence group of this class is found by the algebraic method. Using algebraic techniques, we construct additional point equivalences between some of the listed cases of Lie symmetry extensions, which are inequivalent up to transformations from the equivalence group.

1 Introduction

The shallow water equations are among the most studied models in geophysical fluid dynamics. Due to the ability of modeling both slow moving (Rossby) and fast moving (gravity) waves, the shallow water equations provide an ideal test bed for the development of new numerical approaches to be used for future numerical models in geophysical fluid dynamics, see e.g. [1, 13, 14, 20, 21, 23, 38].

Besides playing an important role as an intermediate-complexity model for designing new numerical approaches for weather and climate modeling, the shallow water equations are still routinely used in research and operational tsunami propagation models [14, 42, 43, 44]. A main challenge arising in the application of the shallow water equations in ocean wave propagation is the need to incorporate a variable bottom topography, since the height of the water column over the bottom of the ocean basin determines the phase speed of gravity waves.

Owing to the considerable interest in the shallow water equations, there is a large body of literature devoted to finding exact solutions and conservation laws for these equations. Both are important for numerical considerations since exact solutions can be used for benchmarking numerical methods and conservation laws can be applied to checking the relevance of new numerical schemes. The construction of exact solutions and conservation laws is already a challenging problem without considering variable bottom topographies. With variable bottom topography, exact solutions are known mostly for simple profiles, such as linear slopes [16] or parabolic bowls [41].

Below, we briefly review some of existing works on the one- and the two-dimensional shallow water equations within the framework of group analysis of differential equations, which are related to the present paper and include the computation of exact solutions and conservation laws. The main point of division in the various studies in this regard is whether Lagrangian or Eulerian coordinates are used.

The two-dimensional shallow water equations (as well as the semi-geostrophic equations that arise in meteorology and oceanography) in Lagrangian coordinates over flat bottom topography were considered from the point of view of the group analysis of differential equations, e.g., in [9],
where Lie symmetries and certain potential and variational symmetries were computed, and variational symmetries were used for finding first-order conservation laws according to Noether’s theorem. The one-dimensional shallow water equations over flat bottom topography were considered in [39] in both Lagrangian and Eulerian coordinates as a limit case of the Green–Naghdi model, and their first-order conservation laws in Lagrangian coordinates were found that have no counterparts among conservation laws in Eulerian variables. Special first-order conservation laws for the one-dimensional case over variable bottom topography in Lagrangian coordinates were constructed in [4] using their relation to hydrodynamic conservation laws of a potential system for the shallow water equations in Eulerian variables.

In Eulerian variables, Lie symmetries of the two-dimensional shallow water equations with parabolic bottom topography were computed in [27] and then used for finding exact solutions via classifying classical Lie reductions. It was shown in [19] that in one dimension, the shallow water equations with a linear bottom topography can be mapped to the shallow water equations with flat bottom topography by a point transformation. Classical symmetry analysis of a modified system of one-dimensional shallow water equations, including the construction of the maximal Lie invariance algebra of this system and the classification of invariant solutions, was carried out in [40]. Lie symmetries and zeroth-order conservation laws in the one-dimensional case with variable bottom topography were described in [3], and the geometric structure of self-similar solutions of the second kind for the flat bottom topography was studied in [15]. Therein an excellent comprehensive review of previous results related to the symmetry analysis of the latter model was presented. The two-dimensional shallow water equations with constant Coriolis force were investigated in [17], where Lie symmetries were used to find a transformation relating this case to the shallow water equations in a resting reference frame. This result was generalized in [18] via finding a point transformation mapping the shallow water equations over a constantly rotating parabolic basin to the shallow water equations in a resting reference frame over a flat bottom topography.

Non-canonical Hamiltonian structures and generalized Hamiltonian structures of the shallow water equations were considered in [38], see also [37]. In [38] these Hamiltonian structures were used to construct conservative numerical schemes for the shallow water equations, a subject which was continued in [49], where conservation law characteristics were used for the same purpose. Numerical schemes preserving Lie symmetries of the shallow water equations in Lagrangian and Eulerian coordinates were considered in [8].

In the present paper, we carry out the complete group classification of the class of systems of two-dimensional shallow water equations with variable bottom topography, which are of the form

\[ \begin{align*}
  u_t + uu_x + vu_y + h_x &= b_x, \\
  v_t + uv_x + vv_y + h_y &= b_y, \\
  h_t + (uh)_x + (vh)_y &= 0.
\end{align*} \tag{1} \]

Here \((u, v)\) is the horizontal fluid velocity averaged over the height of the fluid column, \(h\) is the thickness of a fluid column and \(b = b(x, y)\) is a parameter function that is the bottom topography measured downward with respect to a fixed reference level. In this class, \((t, x, y)\) is the tuple of independent variables, \((u, v, h)\) is the tuple of the dependent variables and \(b\) is considered to be the arbitrary element of the class. These values are graphically represented in Figure 1.

We classify cases of Lie-symmetry extensions for systems from the class (1) up to equivalence generated by the equivalence group \(G^\sim\) of this class. Then we find additional equivalences among listed \(G^\sim\)-inequivalent cases of Lie-symmetry extensions. These additional equivalences are induced by admissible point transformations within the class (1) that are not generated jointly by elements of \(G^\sim\) and by point symmetry groups of systems from the class (1).

We solve the group classification problem within the framework of the infinitesimal approach using an optimized version of the method of furcate splitting. This method was suggested in [29]
in the course of the group classification of the class of nonlinear Schrödinger equations of the form \( i\psi_t + \Delta \psi + F(\psi, \psi^*) = 0 \) with an arbitrary number \( n \) of space variables. Here \( \psi \) is an unknown complex-valued function of real variables \( (t, x_1, \ldots, x_n) \), and \( F \) is an arbitrary sufficiently smooth function of \( (\psi, \psi^*) \), which is the arbitrary element of this class. Subsequently, the method of furcate splitting was applied to the group classification of various classes of \((1+1)\)-dimensional variable-coefficient reaction–convection–diffusion equations, where arbitrary elements depend on single but possibly different arguments \([25, 33, 46, 47, 48]\). Therefore, the present paper gives only the second example of solving the group classification problem for a class of (systems of) differential equations with arbitrary elements depending on two arguments. In the course of applying the method of furcate splitting, we obtained a set of template-form equations, which are inhomogeneous first-order quasilinear partial differential equations with respect to the arbitrary element \( b \) with two independent variables \( x \) and \( y \). Each of these equation is canonically associated with a vector field in the space with the coordinates \( (x, y, b) \). Optimizing the computation within the method of furcate splitting, we show that the set of such vector field is a Lie algebra with respect to the Lie bracket of vector fields.

The further organization of the paper is the following. The equivalence group \( G^\sim \) of the class \((1)\) is computed in Section 2 by the algebraic method. Section 3 contains the preliminary analysis of determining equations and the statement of classification results. The proof of the group classification is presented in Section 4. As the class \((1)\) is not semi-normalized, additional equivalences between equations from the class have to be studied. This is done in Section 5. In the final Section 6 we summarize the findings of the paper and discuss possible future research directions.

### 2 Equivalence group

According to the interpretation of \( b \) as a varying arbitrary element or a fixed function, we will refer to \((1)\) as to a class of systems of differential equations or to a fixed system. The complete system of auxiliary equations for the arbitrary element \( b \) of the class \((1)\) consists of the equations

\[
\begin{align*}
b_u &= b_{ut} = b_{ux} = b_{ux} = 0, \quad b_v = b_{vt} = b_{vx} = b_{vy} = 0, \\
b_h &= b_{ht} = b_{hx} = b_{hy} = 0, \quad b_t = 0,
\end{align*}
\]

Note that there are no auxiliary inequalities for the arbitrary element \( b \).
The arbitrary element $b$ depends only on independent variables. Therefore, we can treat it as one more dependent variable and consider the extended system

\begin{align}
  u_t + uu_x + vu_y + h_x &= b_x, \\
v_t + w_x + vu_y + h_y &= b_y, \\
h_t + (uh)_x + (vh)_y &= 0, \\
b_t &= 0.
\end{align}

(2)

Here we also use the fact that the arbitrary element $b$ does not depend on $t$ as well.

Since the arbitrary element $b$ does not involve derivative of dependent variables, the generalized equivalence group $G^\sim$ of the class (1) can be assumed to act in the space with the coordinates $(t, x, y, u, v, h)$ and thus to coincide with the point symmetry group $G$ of the system (2). Analogously, the generalized equivalence algebra $g^\sim$ of the class (1) can be identified with the maximal Lie invariance algebra $g$ of the system (2). This is why it suffices to find $g$ and $G$ instead of $g^\sim$ and $G^\sim$ respectively.

To construct the group $G$, we invoke the algebraic method, which was suggested in [24] and further developed in [7, 5, 22]. For this, we need first to compute the algebra $g$, and the infinitesimal method [10, 11, 30, 34] is relevant here. The algebra $g$ consists of the infinitesimal generators of one-parameter point symmetry groups of the system (2), which are vector fields in the space with coordinates $(t, x, y, u, v, h, b)$,

\[ v = \tau \partial_t + \xi^1 \partial_x + \xi^2 \partial_y + \eta^1 \partial_u + \eta^2 \partial_v + \eta^3 \partial_h + \eta^4 \partial_b, \]

where the components $\tau$, $\xi^1$, $\xi^2$ and $\eta^i$, $i = 1, 2, 3, 4$, are smooth functions of these coordinates. For convenience, hereafter we simultaneously use the notation $(w^1, w^2, w^3, w^4)$ for $(u, v, h, b)$.

The infinitesimal invariance criterion implies that

\begin{align}
  \text{pr}^{(1)} v(w_1^1 + w^1 w_x^1 + w^2 w_y^1 + w^3 - w_x^1) &= 0, \\
  \text{pr}^{(1)} v(w_2^1 + w^1 w_x^2 + w^2 w_y^2 + w^3 - w_y^1) &= 0, \\
  \text{pr}^{(1)} v(w_3^1 + (w^1 w^3)_x + (w^2 w^3)_y) &= 0, \\
  \text{pr}^{(1)} v(w_4^1) &= 0,
\end{align}

(3)

whenever the system (2) holds. Here $\text{pr}^{(1)} v$ is the first order prolongation of the vector field $v$,

\[ \text{pr}^{(1)} v = v + \sum_{i=1}^{4} (\eta^{1i} \partial_{w^1} + \eta^{2i} \partial_{w^2} + \eta^{3i} \partial_{w^3}), \]

where $\eta^{1i} = D_t(\eta^i - \tau w^i + \xi^1 w_x^i - \xi^2 w_y^i) + \tau w_{tt}^i + \xi^1 w_{tx}^i + \xi^2 w_{ty}^i$, and similarly for $\eta^{2i}$ and $\eta^{3i}$.

We substitute the expressions for $w_{ti}^i$, $i = 1, \ldots, 4$, in view of the system (2) into the expanded equations (3), and then split them with respect to the derivatives $w_x^i$ and $w_y^i$, $i = 1, \ldots, 4$. This procedure results in the system of differential equations on the components $\tau$, $\xi^1$, $\xi^2$ and $\eta^i$, $i = 1, \ldots, 4$, of the vector field $v$, which are called the determining equations. Integrating this system, we derive the explicit form of the vector field components,

\[ \tau = (c_5 - c_7)t + c_1, \quad \xi^1 = c_5 x + c_6 y + c_2, \quad \xi^2 = -c_6 x + c_5 y + c_3, \]
\[ \eta^1 = c_7 u + c_6 v, \quad \eta^2 = -c_6 u + c_7 v, \quad \eta^3 = 2c_7 h, \quad \eta^4 = 2c_7 b + c_4, \]

where $c_1, \ldots, c_7$ are arbitrary real constants.
Thus, the maximal Lie invariance algebra $g$ of the system (2) is spanned by the seven vector fields\footnote{The components of vector fields from $g$ that correspond to the independent variables $(t, x, y)$ and dependent variables $(u, v, h)$ of the system (1) do not depend on the arbitrary element $b$. Interpreting this result in terms of equivalence algebras, we obtain that the generalized equivalence algebra $g^-$ of the class (1) coincides with its usual equivalence algebra.}

\[
P^t = \partial_t, \quad P^x = \partial_x, \quad P^y = \partial_y, \quad P^b = \partial_b, \quad D^1 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h + 2b\partial_b, \\
D^2 = t\partial_t - u\partial_u - v\partial_v - 2h\partial_h - 2b\partial_b, \quad J = x\partial_y - y\partial_x + u\partial_v - v\partial_u.
\]

Let us fix the basis $\mathcal{B} = (P^t, P^x, P^y, P^b, D^1, D^2, J)$ of the Lie algebra $g$. Up to anticommutativity of the Lie bracket of vector fields, the only nonzero commutation relations between the basis elements are

\[
[P^x, D^1] = P^x, \quad [P^y, D^1] = P^y, \quad [P^b, D^1] = 2P^b, \\
[P^t, D^2] = P^t, \quad [P^b, D^2] = -2P^b, \quad [P^x, J] = P^y, \quad [P^y, J] = -P^x.
\]

In other words, the complete list of nonzero structure constants of the Lie algebra $g$ in the basis $\mathcal{B}$ is exhausted, up to permutation of subscripts, by

\[
c_{25}^2 = 1, \quad c_{35}^2 = 1, \quad c_{45}^2 = 2, \quad c_{16}^1 = 1, \quad c_{46}^4 = -2, \quad c_{27}^3 = 1, \quad c_{37}^2 = -1.
\]

The general form $A = (a_j^{ij})_{i,j=1}^7$ of automorphism matrices of the algebra $g$ in the basis $\mathcal{B}$ can be found via solving the system of algebraic equations

\[
c_{ij}^k a_i^* a_j^* = c_{ij}^k a_j^* a_i^*, \quad i, j = 1, \ldots, 7, \tag{4}
\]

under the condition $\det A \neq 0$. Here we assume the summation over the repeated indices. As a result, we obtain that the automorphism group $\text{Aut}(g)$ of $g$ can be identified with the matrix group that consists of the matrices of the general form

\[
A = \begin{pmatrix}
a_1^1 & 0 & 0 & 0 & 0 & a_6^1 & 0 \\
0 & a_2^2 & -\varepsilon a_3^3 & 0 & a_5^2 & 0 & a_7^2 \\
0 & a_2^2 & \varepsilon a_3^3 & 0 & -\varepsilon a_5^2 & 0 & \varepsilon a_7^2 \\
0 & 0 & 0 & a_4^4 & -a_6^4 & a_6^4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \varepsilon
\end{pmatrix},
\]

where $\varepsilon = \pm 1$, and the remaining parameters $a_j^i$’s are arbitrary real constants with

\[
a_1^1 ((a_2^2)^2 + (a_3^3)^2) a_4^4 \neq 0.
\]

**Theorem 1.** A complete list of discrete symmetry transformations of the extended system (2) that are independent up to combining with each other and with continuous symmetry transformations of this system is exhausted by two transformations alternating signs of variables,

\[
(t, x, y, u, v, h, b) \mapsto (-t, x, y, -u, -v, h, b), \\
(t, x, y, u, v, h, b) \mapsto (t, x, -y, u, -v, h, b).
\]

**Proof.** The maximal Lie invariance algebra $g$ of the system (2) is finite-dimensional and non-trivial. The complete automorphism group $\text{Aut}(g)$ of $g$ is computed above. It is not much wider.
The general solution of the system is

\[
\begin{pmatrix}
  e^{-\theta_6} 0 & 0 & 0 & 0 & \theta_1 & 0 \\
 0 & e^{-\theta_6} \cos \theta_7 & e^{-\theta_6} \sin \theta_7 & 0 & \theta_2 & 0 & -\theta_3 \\
 0 & -e^{-\theta_6} \sin \theta_7 & e^{-\theta_6} \cos \theta_7 & 0 & \theta_3 & 0 & \theta_2 \\
 0 & 0 & 0 & e^{2\theta_6 - 2\theta_5} & 2\theta_4 & -2\theta_4 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where the parameters \( \theta_1, \ldots, \theta_7 \) are arbitrary constants. Continuous point symmetries of the system (1) can be easily found by composing elements of one-parameter groups generated by basis elements \( \mathfrak{g} \). Moreover, such symmetries constitute the connected component of the identity transformation in the group \( \text{Aut}(\mathfrak{g}) \), which induces the entire group \( \text{Inn}(\mathfrak{g}) \). This is why it suffices to look only for discrete symmetry transformations, and in the course of the related computation within the framework of the algebraic method one can factor out inner automorphisms. The quotient group \( \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g}) \) can be identified with the matrix group consisting of the diagonal matrices of the form \( \text{diag} (\varepsilon', 1, \varepsilon, \varepsilon'', 1, 1, \varepsilon) \), where \( \varepsilon, \varepsilon', \varepsilon'' = \pm 1 \). Suppose that the push-forward \( \mathcal{T}_s \) of vector fields in the space with the coordinates \((t, x, y, u, v, h, b)\) by a point transformation

\[
\mathcal{T}: \ (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{h}, \tilde{b}) = (T, X, Y, U, V, H, B)(t, x, y, u, v, h, b)
\]
generates the automorphism of \( \mathfrak{g} \) with the matrix \( \text{diag}(\varepsilon', 1, \varepsilon, \varepsilon'', 1, 1, \varepsilon) \), i.e.,

\[
\begin{align*}
\mathcal{T}_s P^t &= \varepsilon' \tilde{P}^t, \quad \mathcal{T}_s P^x = \tilde{P}^x, \quad \mathcal{T}_s P^y = \varepsilon \tilde{P}^y, \quad \mathcal{T}_s P^b = \varepsilon'' \tilde{P}^b, \\
\mathcal{T}_s D^1 &= \tilde{D}^1, \quad \mathcal{T}_s D^2 = \tilde{D}^2, \quad \mathcal{T}_s J = \varepsilon \tilde{J}.
\end{align*}
\]

Here tildes over vector fields means that these vector fields are given in the new coordinates. The above conditions for \( \mathcal{T}_s \) imply a system of differential equations for the components of \( \mathcal{T} \),

\[
\begin{align*}
T_t &= \varepsilon', & T_t &= tT_t, & T_x &= T_y = T_u = T_v = T_h = T_b = 0, \\
X_x &= 1, & X_x &= xX_x, & X_t &= X_y &= X_u &= X_v &= X_h &= X_b = 0, \\
Y_y &= \varepsilon, & Y &= yY_y, & Y_t &= Y_x &= Y_u &= Y_v &= Y_h &= Y_b = 0, \\
U_t &= U_x = U_y = U_b = 0, & V_t &= V_x &= V_y &= V_b = 0, \\
vU_u - uU_v &= \varepsilon V, & uU_u + vU_v + 2hU_h &= U, \\
vV_u - uV_v &= -\varepsilon U, & uV_u + vV_v + 2hV_h &= V, \\
H_t &= H_x = H_y = H_b = 0, & vH_u - uH_v &= 0, & uH_u + vH_v + 2hH_h &= 2H, \\
B_b &= \varepsilon'' b, & B_t &= B_x = B_y = 0, & vB_u - uB_v &= 0, \\
uB_u + vB_v + 2hB_h &= 2B - 2\varepsilon'' b.
\end{align*}
\]

The general solution of the system is

\[
\begin{align*}
T &= \varepsilon' t, & X &= x, & Y &= \varepsilon y, \\
U &= uF_1 \left( \frac{h}{u^2 + v^2} \right) + \varepsilon vF_2 \left( \frac{h}{u^2 + v^2} \right), \\
V &= -uF_2 \left( \frac{h}{u^2 + v^2} \right) + \varepsilon vF_1 \left( \frac{h}{u^2 + v^2} \right), \\
H &= (u^2 + v^2)F_3 \left( \frac{h}{u^2 + v^2} \right), & B &= \varepsilon'' b + (u^2 + v^2)F_4 \left( \frac{h}{u^2 + v^2} \right),
\end{align*}
\]

where \( F_1, F_2, F_3 \) and \( F_4 \) are arbitrary smooth functions of \( h/(u^2 + v^2) \).
We continue the computations within the framework of the direct method in order to complete the system of constraints for $T$. Using the chain rule, we express all required transformed derivatives $\tilde{w}_i^T$, $\tilde{w}_x^T$, $\tilde{w}_y^T$, $i = 1, \ldots, 4$, in terms of the initial coordinates. Then we substitute the obtained expressions into the copy of the system (2) in the new coordinates. The expanded system should identically be satisfied by each solution of the system (2). This condition implies that

$$T = \varepsilon' t, \quad X = x, \quad Y = \varepsilon y, \quad U = \varepsilon'u, \quad V = \varepsilon\varepsilon' v, \quad H = h, \quad B = b.$$ 

Therefore, discrete symmetries of the equation (2) are exhausted, up to combining with continuous symmetries and with each other, by the two involution

$$(t, x, y, u, v, h, b) \mapsto (-t, x, y, -u, -v, h, b),$$

$$(t, x, y, u, v, h, b) \mapsto (t, x, -y, u, -v, h, b),$$

which are associated with the values $(\varepsilon', \varepsilon) = (-1, 1)$ and $(\varepsilon', \varepsilon) = (1, -1)$, respectively. □

**Corollary 2.** The factor group of the complete point symmetry group $G$ of the extended system (2), with respect to its identity component is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The complete point symmetry group $G$ of the extended system (2) is generated by one-parameter point transformation groups associated with vector fields from the algebra $g$ and two discrete transformations given in Theorem 1.

**Corollary 3.** The complete point symmetry group $G$ of the extended system (2) consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x - \delta_4 y + \delta_5, \quad \tilde{y} = \delta_4 x + \varepsilon\delta_3 y + \delta_6,$n

$$\tilde{u} = \frac{\delta_3}{\delta_1} u - \frac{\delta_4}{\delta_1} v, \quad \tilde{v} = \frac{\delta_4}{\delta_1} u + \varepsilon\frac{\delta_3}{\delta_1} v, \quad \tilde{h} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2} h, \quad \tilde{b} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2} b + \delta_7,$$

where $\varepsilon = \pm 1$ and the parameters $\delta_i, i = 1, \ldots, 7$, are arbitrary constants with $\delta_1 (\delta_3^2 + \delta_4^2) \neq 0$.

Since the generalized equivalence group $G^\sim$ of the class of two-dimensional shallow water equations (1) coincides with the complete point symmetry group $G$ of the system (2), we can rephrase Theorem 1 and Corollary 3 in terms of equivalence transformations of the class (1).

**Theorem 4.** A complete list of discrete equivalence transformations of the class of two-dimensional shallow water equations (1) that are independent up to combining with each other and with continuous equivalence transformations of this class is exhausted by two transformations alternating signs of variables,

$$(t, x, y, u, v, h, b) \mapsto (-t, x, y, -u, -v, h, b),$$

$$(t, x, y, u, v, h, b) \mapsto (t, x, -y, u, -v, h, b).$$

**Theorem 5.** The generalized equivalence group $G^\sim$ of the class of two-dimensional systems of shallow water equations (1) coincides with the usual equivalence group of this class and consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x - \delta_4 y + \delta_5, \quad \tilde{y} = \delta_4 x + \varepsilon\delta_3 y + \delta_6,$n

$$\tilde{u} = \frac{\delta_3}{\delta_1} u - \frac{\delta_4}{\delta_1} v, \quad \tilde{v} = \frac{\delta_4}{\delta_1} u + \varepsilon\frac{\delta_3}{\delta_1} v, \quad \tilde{h} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2} h, \quad \tilde{b} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2} b + \delta_7,$$

where $\varepsilon = \pm 1$ and the parameters $\delta_i, i = 1, \ldots, 7$, are arbitrary constants with $\delta_1 (\delta_3^2 + \delta_4^2) \neq 0$. 


3 Preliminary analysis and classification result

Let \( \mathcal{L}_b \) be a system from the class (1) with a fixed value of the arbitrary element \( b \) and suppose that a vector field \( \mathbf{v} \) of the general form

\[
\mathbf{v} = \tau(t, x, y, u, v, h) \partial_t + \xi^1(t, x, y, u, v, h) \partial_x + \xi^2(t, x, y, u, v, h) \partial_y \\
+ \eta^1(t, x, y, u, v, h) \partial_u + \eta^2(t, x, y, u, v, h) \partial_v + \eta^3(t, x, y, u, v, h) \partial_h
\]

defined in the space with the coordinates \((t, x, y, u, v, h)\) be the infinitesimal generator of a one-parameter Lie symmetry group for the system \( \mathcal{L}_b \). The set of such vector fields is the maximal Lie invariance algebra \( \mathfrak{g}_b \) of the system \( \mathcal{L}_b \).

The infinitesimal invariance criterion requires that

\[
\text{pr}^{(1)} \mathbf{v}(\mathcal{L}_b)|_{\mathcal{L}_b} = 0. \tag{5}
\]

The first prolongation \( \text{pr}^{(1)} \mathbf{v} \) of the vector field \( \mathbf{v} \) is computed similarly to the previous section. We expand the condition (5) and confine it on the manifold defined by \( \mathcal{L}_b \) in the corresponding first-order jet space, assuming the first-order derivatives of the dependent variables \((u, v, h)\) with respect to \( t \) as the leading ones and substituting for these derivatives in view of the system \( \mathcal{L}_b \),

\[
\begin{align*}
    u_t &= -uu_x - vv_y - h_x + b_x, \\
    v_t &= -uw_x - vv_y - h_y + b_y, \\
    h_t &= -(uh)_x - (vh)_y.
\end{align*}
\]

Then we split the obtained equations with respect to the first-order parametric derivatives, which are the first-order derivatives of the dependent variables \((u, v, h)\) with respect \( x \) and \( y \). After an additional rearrangement and excluding equations which are differential consequences of the other, we derive the system of determining equations for the components of the vector field \( \mathbf{v} \),

\[
\begin{align*}
    \tau_x &= \tau_y = \tau_u = \tau_v = \tau_h = 0, \\
    \xi^1_u &= \xi^1_v = \xi^2_h = 0, \quad \xi^2_u = \xi^2_v = 0, \quad \xi^1_x = \xi^2_y, \quad \xi^1 + \xi^2_x = 0, \\
    \eta^1 &= (\xi^1_x - \tau_t)u + \xi^2_y v + \xi^1_t, \\
    \eta^2 &= \xi^2_x u + (\xi^2_y - \tau_t)v + \xi^2_t, \\
    \eta^3 &= 2(\xi^1_x - \tau_t)h, \\
    \eta^1_1 + uu^1_x + vv^1_y + hh^3 + (\eta^1_u - \tau_t)bx + \eta^1_vby &= \xi^1_bxx + \xi^2_bxy, \\
    \eta^2_2 + uu^2_x + vv^2_y + hh^3 + (\eta^2_u - \tau_t)bx + (\eta^2_v - \tau_t)by &= \xi^1_bxy + \xi^2_byy, \\
    \eta^3_3 + uu^3_x + vv^3_y + hh^3 + hh^3_x + hh^3_y &= 0.
\end{align*}
\] \tag{6}

Integrating the subsystem of the system (6) that consists of the equations not containing the arbitrary element \( b \), we get the following form of the components of the vector field \( \mathbf{v} \)

\[
\begin{align*}
    \tau &= 2F^1 - c_1 t, \\
    \xi^1 &= F^1_1 x + F^0_0 y + F^2, \\
    \xi^2 &= -F^0_0 x + F^1_1 y + F^3, \\
    \eta^1 &= (-F^1_1 + c_1)u + F^0_0 v + F^1_1 x + F^2, \\
    \eta^2 &= -F^0_0 u + (-F^1_1 + c_1)v + F^1_1 y + F^3, \\
    \eta^3 &= 2(-F^1_1 + c_1)h.
\end{align*}
\] \tag{7}
where \( F_i, i = 1, 2, 3, 4, \) are sufficiently smooth functions of \( t, \) and \( c_1 \) is a constant. From the last two equations of the system (6), we derive as a differential consequence that \( F_1^0 = 0. \) Thus, \( F_0 \) is a constant, and we will denote \( c_2 := F^0. \) In other words, for any \( b \)

\[
g_b \subset g(1) := \langle D(F^1), D^t J, P(F^2, F^3) \rangle
\]

where the parameters \( F^1, F^2 \) and \( F^3 \) runs through the set of smooth functions of \( t, \)

\[
D(F^1) := F^1 \partial_t + \frac{1}{2} F^1 x \partial_x + \frac{1}{2} F^1 y \partial_y - \frac{1}{2} (F^1 u - F^1 x) \partial_u - \frac{1}{2} (F^1 v - F^1 y) \partial_v - F^1 h \partial_h,
\]

\[
D^t := t \partial_t - u \partial_u - v \partial_v - 2h \partial_h, \quad J := x \partial_y - y \partial_x + u \partial_u - v \partial_v,
\]

\[
P(F^2, F^3) := F^2 \partial_x + F^3 \partial_y + F^2 \partial_u + F^3 \partial_v.
\]

It is convenient to denote \( D^t := 2D(t) - 2D_x = x \partial_x + y \partial_y + u \partial_u + v \partial_v + 2h \partial_h. \)

The parameters \( F^1, F^2, F^3, c_1 \) and \( c_2 \) additionally satisfy two equations implied by the last two equations from (6), which explicitly involve the arbitrary element \( b \) and thus are the classifying equations for the class (1). They can be integrated to the single equation

\[
(F^3_x + c_2 y + F^2) b_x + (-c_2 x + F^1 y + F^3) b_y + 2(F^3 - c_1) b
\]

\[
- F^1 \frac{x^2 + y^2}{2} - F^2 x - F^3 y - F^4 = 0,
\]

(8)

where \( F^4 \) is one more sufficiently smooth parameter function of \( t. \) The equation (8) can be considered as the only classifying equation instead the above ones. Thus, the group classification problem for the class (1) reduces to solving the equation (8) up to \( G^\sim \)-equivalence with respect to the arbitrary element \( b \) and the parameters \( F^1, \ldots, F^4, c_1 \) and \( c_2. \)

In the next theorem and in Section 4, it is convenient to use, simultaneously with \( (x, y), \) the polar coordinates \( (r, \varphi) \) on the \( (x, y) \)-plane,

\[
r := \sqrt{x^2 + y^2}, \quad \varphi := \arctan \frac{y}{x}.
\]

**Theorem 6.** The kernel Lie invariance algebra of systems from the class (1) is \( g(1) = (D(1)). \) A complete list of \( G^\sim \)-inequivalent Lie symmetry extensions within the class (1) is exhausted by the following cases, where \( f \) denotes an arbitrary smooth function of a single argument, \( \alpha, \beta, \mu \) and \( \nu \) are arbitrary constants with \( \alpha \geq 0 \) mod \( G^\sim, \beta > 0 \) and additional constraints indicated in the corresponding cases, \( \varepsilon = \pm 1 \) mod \( G^\sim \) and \( \delta \in \{0, 1\} \) mod \( G^\sim. \)

1. \( b = r^{\nu} f(\varphi + \alpha \ln r), \nu \notin \{-2, 0\}: \quad g_b = \langle D(1), 4D(t) - (\nu + 2)D^t - 2\alpha J \rangle; \)

2. \( b = f(\varphi + \alpha \ln r) + \nu \ln r: \quad g_b = \langle D(1), 2D(t) - D^t - \alpha J \rangle; \)

3. \( b = f(r) + \alpha \varphi : \quad g_b = \langle D(1), J \rangle; \)

4. \( b = f(r) e^{\beta \varphi}: \quad g_b = \langle D(1), 2J - \beta D^t \rangle; \)

5. \( b = f(y) e^{\nu} : \quad g_b = \langle D(1), D^t - P(2, 0) \rangle; \)

6. \( (a) \) \( b = r^{-2} f(\varphi): \quad g_b = \langle D(1), D(t), D(t^2) \rangle; \)

\( (b) \) \( b = r^{-2} f(\varphi) + \frac{1}{2} r^2: \quad g_b = \langle D(1), D(e^{2t}), D(e^{-2t}) \rangle; \)

\( (c) \) \( b = r^{-2} f(\varphi) - \frac{1}{2} r^2: \quad g_b = \langle D(1), D(\cos 2t), D(\sin 2t) \rangle; \)

7. \( b = f(y) + \delta x: \quad g_b = \langle D(1), P(1, 0), P(t, 0) \rangle; \)

8. \( b = f(y) + \frac{1}{2} x^2: \quad g_b = \langle D(1), P(e^0, 0), P(e^{-t}, 0) \rangle; \)

9. \( b = f(y) - \frac{1}{2} x^2: \quad g_b = \langle D(1), P(\cos t, 0), P(\sin t, 0) \rangle; \)
10. \( b = \varphi - \nu \ln r, \nu = \pm 1 \mod G^\sim \) if \( \delta = 0 \): \( \mathfrak{g}_b = \langle D(1), 2D(t) - D^t, J \rangle \);

11. \( b = \nu e^{\alpha \varphi}, \nu \neq -2, (\alpha, \nu) \notin \{(0, 0), (0, 2)\} \): \( \mathfrak{g}_b = \langle D(1), 4D(t) - (\nu + 2)D^t, 2J - \alpha D^t \rangle \);

12. (a) \( b = \nu e^{\alpha \varphi} \): \( \mathfrak{g}_b = \langle D(1), D(t), D(t^2), \alpha D^x + 4J \rangle \);
    (b) \( b = \nu e^{\alpha \varphi} + \frac{1}{2} t^2 \): \( \mathfrak{g}_b = \langle D(1), D(e^{2t}), D(e^{-2t}), \alpha D^x + 4J \rangle \);
    (c) \( b = \nu e^{\alpha \varphi} - \frac{1}{2} t^2 \): \( \mathfrak{g}_b = \langle D(1), D(\cos 2t), D(\sin 2t), \alpha D^x + 4J \rangle \);

13. \( b = \nu e^{\alpha \varphi} - \delta x, \nu \notin \{-2, 0, 2\} \): \( \mathfrak{g}_b = \langle D(1), 4D(t) - (\nu + 2)D^t - \delta (\nu - 1)P(t^2, 0), P(1, 0), P(t, 0) \rangle \);

14. \( b = \nu e^{\alpha \varphi} + \delta x \): \( \mathfrak{g}_b = \langle D(1), D^t - D^t - \frac{1}{2} \delta P(t^2, 0), P(1, 0), P(t, 0) \rangle \);

15. \( b = \nu e^{\alpha \varphi} - \frac{1}{2} t^2 \): \( \mathfrak{g}_b = \langle D(1), D(t), D^2, \alpha D^x + 4J \rangle \);

16. (a) \( b = \nu e^{\alpha \varphi} \): \( \mathfrak{g}_b = \langle D(1), D(t), D(e^{2t}), D(e^{-2t}), \alpha D^x + 4J \rangle \);
    (b) \( b = \nu e^{\alpha \varphi} + \frac{1}{2} t^2 \): \( \mathfrak{g}_b = \langle D(1), D(\cos 2t), D(\sin 2t), \alpha D^x + 4J \rangle \);
    (c) \( b = \nu e^{\alpha \varphi} - \frac{1}{2} t^2 \): \( \mathfrak{g}_b = \langle D(1), D(\cos 2t), D(\sin 2t), \alpha D^x + 4J \rangle \);

17. \( b = \frac{1}{2} x^2 + \frac{1}{2} \beta y^2, 0 < \beta < 1 \): \( \mathfrak{g}_b = \langle D(1), D^x, P(e^t, 0), P(e^{-t}, 0), P(0, e^{\beta t}), P(0, e^{-\beta t}) \rangle \);

18. \( b = \frac{1}{2} x^2 + \frac{1}{2} \beta y^2, 0 < \beta < 1 \): \( \mathfrak{g}_b = \langle D(1), D^x, P(e^t, 0), P(e^{-t}, 0), P(0, 1), P(0, t) \rangle \);

19. \( b = \frac{1}{2} x^2 - \frac{1}{2} \beta y^2, \beta > 0 \): \( \mathfrak{g}_b = \langle D(1), D^x, P(e^t, 0), P(e^{-t}, 0), P(0, \cos \beta t), P(0, \sin \beta t) \rangle \);

20. \( b = \frac{1}{2} x^2 + \frac{1}{2} \beta y^2, 0 < \beta < 1 \): \( \mathfrak{g}_b = \langle D(1), D^x, P(e^t, 0), P(e^{-t}, 0), P(0, 1), P(0, t) \rangle \);

21. \( b = \frac{1}{2} x^2 - \frac{1}{2} \beta y^2, 0 < \beta < 1 \): \( \mathfrak{g}_b = \langle D(1), D^x, P(\cos t, 0), P(\sin t, 0), P(0, \cos \beta t), P(0, \sin \beta t) \rangle \);

22. (a) \( b = 0 \): \( \mathfrak{g}_b = \langle D(1), D(t), D(t^2), D^x, J, P(1, 0), P(t, 0), P(0, 1), P(0, t) \rangle \);
    (b) \( b = x \): \( \mathfrak{g}_b = \langle D(1), D(t), D(t^2), D^x, J, P(1, 0), P(t, 0), P(0, 1), P(0, t) \rangle \);
    (c) \( b = \frac{1}{2} x^2 \): \( \mathfrak{g}_b = \langle D(1), D(e^{2t}), D(\cos 2t), D^x, J, P(e^t, 0), P(e^{-t}, 0), P(0, e^t), P(0, e^{-t}) \rangle \);
    (d) \( b = \frac{1}{2} x^2 \): \( \mathfrak{g}_b = \langle D(1), D(\cos 2t), D(\sin 2t), D^x, J, P(\cos t, 0), P(\sin t, 0), P(0, \cos t), P(0, \sin t) \rangle \).

**Remark 7.** For Cases 1–9 really present maximal Lie symmetry extensions, the parameter function \( f \) should take only values for which the corresponding values of the arbitrary element \( b \) are not \( G^\sim \)-equivalent to ones from the other listed cases.

**Corollary 8.** The dimension of the maximal Lie invariance algebra of any system from the class (1) is not greater than nine. More specifically, \( \dim \mathfrak{g}_b \in \{1, 2, 3, 4, 5, 6, 9\} \) for any \( b = b(x, y) \).

**Corollary 9.** A system from the class (1) is invariant with respect to a six-dimensional Lie algebra if and only if the corresponding value of the arbitrary element \( b \) is at most a quadratic polynomial in \( (x, y) \).
4 Proof of the classification

According to the method of furcate splitting, we fix an arbitrary value of the variable $t$ in the classifying equation (8) and obtain the following template form of equations for the arbitrary element $b$:

$$a_1(xb_x + yb_y) + a_2(yb_x - xb_y) + a_3xb_x + a_4yb_y + a_5b$$
$$+ a_6\frac{x^2 + y^2}{2} + a_7x + a_8y + a_9 = 0,$$

(9)

where $a_1, \ldots, a_9$ are constants. For each value of the arbitrary element $b$, we denote by $k = k(b)$ the maximal number of template-form equations with linearly independent coefficient tuples $\vec{a}^i = (a_1^i, \ldots, a_9^i)$, $i = 1, \ldots, k$, that are satisfied by this value of $b$. It is obvious that $0 \leq k \leq 9$. Moreover, for the system of template-form equations

$$a_1^i(xb_x + yb_y) + a_2^i(yb_x - xb_y) + a_3^ibx + a_4^iby + a_5^ib$$
$$+ a_6^i\frac{x^2 + y^2}{2} + a_7^ix + a_8^iy + a_9^i = 0, \quad i = 1, \ldots, k,$$

(10)

with rank $A = k$ to be consistent with respect to $b$, it is required that $k \leq 5$. Here

$$A := (a_j^i)_{i,j = 1, \ldots, 9}, \quad A_l := (a_j^i)_{i = 1, \ldots, l}$$

are the matrix of coefficients of the system (10) and its submatrix constituted by the first $l$ columns of $A$, respectively. We also have rank $A_5 = \text{rank} A = k$, and, if $k < 5$, rank $A_4 = k$ as well. Indeed, if the last condition is not satisfied, the system (10) has an algebraic consequence of the form $b = R(x,y) := \beta_4(x^2 + y^2) + \beta_1x + \beta_2y + \beta_0$, where $\beta_0$, $\beta_1$, $\beta_2$ and $\beta_3$ are constants, and such values of $b$ satisfy five independent template-form equations (see the case $k = 5$ below),

$$xb_x + yb_y = xR_x + yR_y, \quad yb_x - xb_y = yR_x - xR_y, \quad b_x = R_x, \quad b_y = R_y, \quad b = R.$$

For checking the consistency of the system (10) with $k > 1$, to the $i$th equation of this system for each $i = 1, \ldots, k$ we associate the vector field

$$v_i = (a_1^i x + a_2^iy + a_3^i) \partial_x + (a_4^iy - a_5^ix + a_4^i) \partial_y$$
$$- (a_7^i + a_9^i)(x^2 + y^2) + a_7^ix + a_8^iy + a_9^i) \partial_b.$$

(11)

Note that $v_1, \ldots, v_k \in a$, where

$$a := \langle x\partial_x + y\partial_y, -x\partial_y + y\partial_x, \partial_x, \partial_y, b\partial_b, (x^2 + y^2)\partial_b, x\partial_b, y\partial_b, \partial_b \rangle \subset a, \quad i, i' = 1, \ldots, k.$$

The span $a$ is closed with respect to the Lie bracket of vector fields, i.e., it is a Lie algebra, and thus $[v_i, v_{i'}] \in a, \quad i, i' = 1, \ldots, k$. More specifically,

$$[v_i, v_{i'}] \subset \langle \partial_x, \partial_y, b\partial_b, (x^2 + y^2)\partial_b, x\partial_b, y\partial_b, \partial_b \rangle \subset a, \quad i, i' = 1, \ldots, k.$$

In other words, the equation on $b$ that is associated with $[v_i, v_{i'}]$ is a differential consequence of the system (10) and is of the same template form (9). By its definition, the number $k = k(b)$ is equal to the maximal number of linearly independent vector fields associated with template-form equations for the corresponding value of the arbitrary element $b$. Therefore,

$$[v_i, v_{i'}] \subset \langle v_1, \ldots, v_k \rangle, \quad i, i' = 1, \ldots, k.$$

(12)

We can also use the counterpart of the condition (12) for the projections $\hat{v}_1, \ldots, \hat{v}_k$ of the vector fields $v_1, \ldots, v_k$ to the space with the coordinates $(x, y)$,

$$[\hat{v}_i, \hat{v}_{i'}] \subset \langle \hat{v}_1, \ldots, \hat{v}_k \rangle, \quad i, i' = 1, \ldots, k.$$

(13)
To simplify the computation, we can gauge coefficients of the system (10) by linearly combining its equations and using transformations from \( G^\sim \). In particular, we can set \( a_j^i = 1 \), dividing the entire \( i \)th equation of (10) by \( a_j^i \) if \( a_j^i \neq 0 \). In the case \((a_1^1, a_2^1) \neq (0, 0)\), we can make \( a_3^1 = a_4^1 = 0 \) with point equivalence transformations of simultaneous shifts with respect to \( x \) and \( y \). Another possibility is to use these shifts for setting \( a_7^i = a_8^i = 0 \) if \( a_i^6 \neq 0 \). Similarly, if \( a_5^i \neq 0 \), then we can shift \( b \) to set \( a_5^i = 0 \).

The case with \( k = 0 \) corresponds to the kernel Lie invariance algebra \( g^\cap \) of systems from the class (1), which is also the Lie invariance algebra for a general value of \( b \). For elements of \( g^\cap \), the classifying equation (8) is identically satisfied by \( b \). Thus, we can successively split it with respect to \( b \) and its derivatives and with respect to \( x \) and \( y \) to obtain \( F_1^1 = F_2^2 = F_3^3 = 0 \) and \( c_2 = c_1 = 0 \), i.e., \( \tau = \text{const.} \), \( \xi^1 = \xi^2 = \eta^1 = \eta^2 = \eta^3 = 0 \). In other words, the algebra \( g^\cap \) is one-dimensional and spanned by the only basis element \( \partial_t \).

\[
g^\cap = \langle \partial_t \rangle.
\]

In the next sections, we separately consider the cases \( k = 1, \ldots, k = 5 \). For each of these cases, we make the following steps, splitting the consideration into subcases depending on values of the parameters \( a_i \)’s:

- find the values of the parameters \( a_i \)’s for which the corresponding system (10) is compatible and follow from the equation (8),
- gauge, if possible, some of the parameters \( a_i \)’s by recombining template-form equations and by transformations from the group \( G^\sim \) and re-denote the remaining parameters \( a_i \)’s,
- integrate the system (10) with respect to the arbitrary element \( b \),
- gauge, if possible, the integration constant by transformations from the group \( G^\sim \),
- solving the system of determining equations with respect to the parameters \( c_1, c_2, F_1^1, F_2^2 \) and \( F_3^3 \), construct the maximal Lie invariance algebras, \( g_k \), of systems from the class (1) with obtained values of \( b \).

The order of steps can be varied, and steps can intertwine.

### 4.1 One independent template-form equation

In the case \( k = 1 \), the right-hand side of the equation (8) is proportional to the right-hand side of the single equation (10) with the proportionality coefficient \( \lambda \) that is a sufficiently smooth function of \( t \),

\[
F_1^1(xb_x + yb_y) + c_2(yb_x - xb_y) + F_2^2 b_x + F_3^3 b_y + 2(F_1^1 - c_1)b
\]

\[
-
\frac{F_{11}^1 x^2 + y^2}{2} - F_{12}^1 x + F_{13}^3 y - F_4^4
\]

\[
= \lambda \left( a_1^1(xb_x + yb_y) + a_2^1(yb_x - xb_y) + a_3^1 b_x + a_4^1 b_y + a_5^1 b \right.
\]

\[
+ a_6^1 x^2 + y^2 \left. \right( + a_7^1 x + a_8^1 y + a_0^1 \right).
\]

The function \( \lambda \) does not vanish for any vector field from the complement of \( g^\cap \) in \( g_k \). We can split the last equation with respect to derivatives of \( b \), including \( b \) itself, and the independent variables \( x \) and \( y \). As a result, we obtain the system

\[
F_1^1 = a_1^1 \lambda, \quad a_2^1 \lambda = c_2, \quad F_2^2 = a_3^1 \lambda, \quad F_3^3 = a_4^1 \lambda, \quad 2(F_1^1 - c_1) = a_5^1 \lambda,
\]

\[
F_{11}^1 = -a_6^1 \lambda, \quad F_{12}^1 = -a_7^1 \lambda, \quad F_{13}^3 = -a_8^1 \lambda, \quad F_4^4 = -a_0^1 \lambda.
\]

(14)
The condition rank $A_4 = k = 1$ means here that $(a_1^1, a_2^1, a_3^1, a_4^1) \neq (0, 0, 0, 0)$. Therefore, the further consideration splits into three cases,
\[
\begin{align*}
  a_1^1 &\neq 0: \quad a_1^1 = 0, \quad a_2^1 \neq 0: \quad a_1^1 = a_2^1 = 0, \quad (a_3^1, a_4^1) \neq (0, 0).
\end{align*}
\]

$a_1^1 \neq 0$. We can set $a_1^1 = 1$ by rescaling the entire equation (10). To simplify the computation we gauge other coefficients of (10) by transformations from $G^\sim$. Thus, we can make $a_3^1 = a_4^1 = 0$ with point equivalence transformations of simultaneous shifts with respect to $x$ and $y$. Consequently, we have that $F^2 = 0$ and $F^3 = 0$. Then the system (14) implies that $a_1^1 = a_3^1 = 0$. The equation (10) reduces in the polar coordinates $(r, \varphi)$ to the form
\[
rb_r - a_2^1x a_3^1 + b + \frac{1}{2}a_4^1r^2 + a_5^1 = 0.
\]

The integration of the above equation depends on the values of the parameters $a_j^1$, $j = 2, 5, 6, 9$.

If $(a_2^1, a_3^1) = (0, 2)$, then we can set $a_5^1 = 0$ by shifts with respect to $b$ and $a_3^1 \in \{0, -4, 4\}$ by scaling equivalence transformations, which leads to Cases 6a, 6b and 6c of Theorem 6, respectively.

If $(a_2^1, a_3^1) \neq (0, 2)$, in view of the system (14) we get that $c_1 = (1 - a_3^1/2)\lambda$, $c_2 = a_3^1\lambda$. Thus, $\lambda$ is a constant, which yields that $F_1^1$ is also constant, and therefore $a_2^1 = 0$. Depending on whether $a_3^1 \neq 0$ or $a_3^1 = 0$, we get Cases 1 and 2, respectively. In the former case, we additionally set $a_3^1 = 0$ by shifts with respect to $b$.

$a_1^1 = 0$, $a_2^1 \neq 0$. Rescaling the equation (10) and using shifts with respect to $x$ and $y$, we can set $a_2^1 = 1$ and $a_4^1 = a_5^1 = 0$. In view of the system (14), the above conditions for $a_4^1$ imply $F_1^1 = 0$, $F_2^1 = F_3^1 = 0$, $\lambda = c_2$, and thus $a_2^1 = a_3^1 = a_5^1 = 0$. In the polar coordinates $(r, \varphi)$, the equation (10) takes the form $b_\varphi = a_2^1b + a_5^1$. Integrating this equation separately for $a_2^1 = 0$ and $a_5^1 \neq 0$ and additionally setting $a_5^1 = 0$ by shifts with respect to $b$ under the second condition, we respectively get Cases 3 and 4.

$a_1^1 = a_2^1 = 0$, $(a_3^1, a_4^1) \neq (0, 0)$. Due to rotation equivalence transformations and the possibility of scaling the entire equation (10), we can rotate and scale the vector $(a_3^1, a_4^1)$ to set $a_3^1 = 1$ and $a_4^1 = 0$. From the system (14), we derive that $F_1^1 = F_3^1 = 0$, $c_3 = 0$, $2c_1 = -a_1^1\lambda$, $F_2^1 = \lambda$ and thus $a_1^1 = a_2^1 = 0$ and $F_2^1 = -a_4^1 F_2^1$. The template-form equation (10) reduces to $b_\varphi = a_2^1 b + a_5^1 = 0$.

If $a_5^1 = 0$, then we can set $a_5^1 \in \{-1, 0, 1\}$ by using scaling equivalence transformations, which leads to Cases 7, 8 and 9. Note that $a_3^1 = 0 \mod G^\sim$ if $a_7^1 \neq 0$ and $a_5^1 \in \{0, -1\} \mod G^\sim$ if $a_3^1 = 0$.

If $a_3^1 \neq 0$, then $\lambda$ is a constant, and thus $a_1^1 = 0$. We can again set $a_3^1 = 0$ by shifts with respect to $b$ as well as $a_5^1 = 1$ up to scaling equivalence transformations and alternating signs of $(x, u)$. This leads to Case 5.

### 4.2 Two independent template-form equations

For $k = 2$, the right-hand side of the equation (8) is a linear combination of right-hand sides of the first and the second equations of the system (10) with coefficients $\lambda^1$ and $\lambda^2$ that depend on $t$,
\[
\begin{align*}
F^1_t(xb_x + yb_y) + c_2(yb_x - xb_y) + F^2_b x + F^3 b_y + 2(F^1_t - c_1) b \\
- F^1_t x^2 + y^2 - F^2_t x - F^3_t y - F^4 \\
= \sum_{i=1}^2 \lambda^i \left( a_i^1 (xb_x + yb_y) + a_i^2 (yb_x - xb_y) + a_i^3 b_x + a_i^4 b_y + a_i^5 b \\
+ a_i^6 x^2 + y^2 + a_i^7 x + a_i^8 y + a_i^9 \right).
\end{align*}
\]
**Remark 10.** The coefficients $\lambda^1$ and $\lambda^2$ are not proportional with the same constant multiplier for all vector fields from $\mathfrak{g}_b$, since otherwise there is no additional Lie symmetry extension in comparison with the more general case of Lie symmetry extension with $k = 1$, where the corresponding linear combination of equations of the system (10) plays the role of a single template-form equation. In particular, both these coefficients do not vanish identically for some vector fields from $\mathfrak{g}_b$.

Splitting the resulting condition with respect to $b$ and its derivatives $b_x$ and $b_y$, we derive the system

$$F'_1 = a_1^1 \lambda^1 + a_1^2 \lambda^2, \quad c_2 = a_2^1 \lambda^1 + a_2^2 \lambda^2, \quad F^2 = a_4^1 \lambda^1 + a_4^2 \lambda^2, \quad F^3 = a_5^1 \lambda^1 + a_5^2 \lambda^2,$$

$$2(F'_1 - c_1) = a_6^1 \lambda^1 + a_6^2 \lambda^2, \quad F'_2 = -a_7^1 \lambda^1 - a_7^2 \lambda^2,$$

$$F'_3 = -a_8^1 \lambda^1 - a_8^2 \lambda^2, \quad F^4 = -a_9^1 \lambda^1 - a_9^2 \lambda^2.$$

(15)

The further consideration for $k = 2$ is partitioned into different cases depending on the rank of the submatrix $A_2$ that is constituted by the first two columns of $A$. Since $\text{rank } A_2 \leq 2$, we have the cases $\text{rank } A_2 = 2$, $\text{rank } A_2 = 1$ and $\text{rank } A_2 = 0$. 

**rank $A_2 = 2$.** Linearly re-combining equations of the system (10), we can set the matrix $A_2$ to be the identity matrix, i.e., $a_1^1 = a_2^1 = 1$ and $a_2^2 = a_2^2 = 0$. To further simplify the form of the system (10), we set $a_2^1 = a_2^1 = 0$ by equivalence transformations of shifts with respect to $x$ and $y$. In view of the condition (12), the vector fields $v_1$ and $v_2$, which are associated with the first and the second equations of the reduced system (10), respectively, commute. This yields the system of algebraic equations with respect to the coefficients $a_j^i$,

$$a_2^2 = a_2^2 = 0, \quad a_5^2 - a_5^1 a_2^2 + a_5^1 a_3^2 = 0, \quad a_5^2 - a_5^1 a_2^2 + a_5^1 a_3^2 = 0,$$

$$2a_6^2 - a_6^1 a_2^2 + a_6^1 a_5^2 = 0, \quad a_7^2 - a_7^1 a_2^2 - a_7^1 a_3^2 = 0.$$  

(16)

The reduced form of the system (15) is

$$F'_1 = \lambda^1, \quad \lambda^2 = c_2, \quad F^2 = 0, \quad F^3 = 0, \quad (a_5^1 - 2)\lambda^1 = -2c_1 - a_5^2 c_2,$$

$$\lambda_1^1 + a_6^1 \lambda^1 + a_6^2 \lambda^2 = 0, \quad a_7^1 \lambda^1 + a_7^2 \lambda^2 = 0, \quad a_7^1 \lambda^1 + a_7^2 \lambda^2 = 0, \quad F^4 = -a_9^1 \lambda^1 - a_9^2 \lambda^2.$$  

(17)

In view of Remark 10, the seventh and the eighth equations of the system (17) imply $a_7^1 = a_7^2 = 0$ and $a_9^1 = a_9^2 = 0$, respectively. The reduced form of the system (10) in the polar coordinates $(r, \varphi)$ is $rb_r = a_7^1 b + b^1 a_7^1 r^2 + a_7^1$, $b_r = a_7^2 b + b^2 a_7^2 r^2 + a_7^2$, the last equation of (16), up to equivalence transformations with respect to $b$ we can set $a_7^1 = a_7^2 = 0$ if $(a_5^1, a_5^2) \neq (0, 0)$.

The further consideration depends on whether or not the parameter $a_5^1$ is equal to 2 and, in the latter case, whether or not the parameter $a_5^2$ is zero.

If $a_5^1 = 2$, then the parameter $a_5^1$ can be assume to belong to $\{-4, 0, 4\}$, and $a_5^2 = a_5^2 a_6^1/4$. Integrating the corresponding system (10), we find the general form of the arbitrary element $b$,

$$b = b_0 r^{-2} \exp(a_2^2 \varphi) - \frac{a_6^1}{8} r^2,$$

where the integration constant $b_0$ is nonzero since otherwise this value of $b$ is associated with the value $k = 5$. This is why we can scale $b_0$ by an equivalence transformation to $\varepsilon = \pm 1$, which leads, depending on the value of $a_6^1$, to Cases 12a, 12b and 12c of Theorem 6.

If $a_5^1 \neq 2$, then $\lambda^1$ is a constant. Then the sixth equation of the system (17) takes the form $a_6^1 \lambda^1 + a_6^2 \lambda^2 = 0$, implying according to Remark 10 that $a_6^1 = a_6^2 = 0$. Depending on whether $(a_5^1, a_5^2) \neq (0, 0)$ or $a_5^1 = a_5^2 = 0$, we obtain Cases 10 and 11 of Theorem 6, respectively.

**rank $A_2 = 1$.** Linearly re-combining equations of the system (10), we reduce the matrix $A_2$ to the form

$$A_2 = \begin{pmatrix} a_1^1 & a_2^1 \\ 0 & 0 \end{pmatrix},$$
i.e., $a_1^2 = a_2^2 = 0$ and $(a_1, a_1^2) \neq (0, 0)$. We also have $(a_3^2, a_4^2) \neq (0, 0)$ since rank $A_4 = 2$. Hence we can set $a_3^2 = 1$, $a_4^2 = 0$ by a rotation equivalence transformation and re-scaling the second equation as well as $a_1^2 = a_4^2 = 0$ by shifts of $x$ and $y$. The condition (12) implies that $a_1^2 = 0$ and hence $a_1^4 \neq 0$. This is why we can set $a_1^4 = 1$ by rescaling of the first equation. Then the condition (12) is equivalent to the commutation relation $[v_1, v_2] = -v_2$, yielding the following system of algebraic equations on the remaining coefficients $a_i^a$:

$$
a^2_5 = 0, \quad a^2_6(a^1_5 + 3) = 0, \quad a^2_8(a^1_5 + 2) = 0, \quad a^2_7(a^1_5 + 2) = a^6_5, \quad a^2_5(a^1_5 + 1) = a^1_7. \quad (18)
$$

The system (15) is simplified to

$$
F^1_t = \lambda^1, \quad c^2 = 0, \quad F^2 = \lambda^2, \quad F^3 = 0, \quad 2c_1 = (2 - a^4_1)\lambda^1, \\
F_{tt} = -a^4_1\lambda^1 - a^2_6\lambda^2, \quad F^2 = -a^1_4\lambda^1 - a^2_2\lambda^2, \quad a^1_4\lambda^1 + a^2_8\lambda^2 = 0, \quad F^4 = -a^4_1\lambda^1 - a^2_6\lambda^2. \quad (19)
$$

In view of Remark 10, the eighth equation of the system (19) imply $a^6_3 = a^2_5 = 0$.

Supposing that $a^2_6 \neq 0$, we successively derive from the second equation of (18) and the system (19) that $a^3_5 = -3$, $\lambda^1$ is a constant, $a^1_6\lambda^1 + a^2_8\lambda^2 = 0$ and, according to Remark 10, $a^1_6 = a^2_8 = 0$, which is a contradiction. Hence $a^2_6 = 0$.

It is obvious from the second equation of (18), $2c_1 = (2 - a^4_1)\lambda^1$, that the value $a^4_1 = 2$ is special. If $a^2_6 \neq 2$, repeating the argumentation with constant $\lambda^1$, we derive $a^4_1 = 0$, and thus the fourth equation of the system (18) takes the form $(a^1_6 + 2)a^2_7 = 0$, implying $a^2_7 = 0$ if $a^2_6 \neq 2$. Therefore, the value $a^4_1 = -2$ is special as well. In the course of integrating the system (10) the value $a^4_1 = 0$ is additionally singled out. Moreover, if $\lambda^1 \neq 0$, we can make $a^2_6 = 0$ using a shift of $b$. As a result, we need to separately consider each of the above values $2, 0, -2$ of $a^4_1$ and the case $a^4_1 \notin \{-2, 0, 2\}$.

1. $a^4_1 = 2$. Shifting $b$, we set $a^1_6 = 0$. The system (18) reduces to the equations $a^{2}_{7} = a^{6}_{5}/4$ and $a^{2}_{5} = a^{1/3}_{7}$.

Let $a^4_1 = 0$. Then shifting $x$ and recombining the equations of the system (10), we also set $a^1_7$ and consequently $a^2_7 = 0$. The general solution to the system (10) is $b = b_0 y^{-2} - \frac{1}{2}a^5_6(x^2 + y^2)$, where the integration constant $b_0$ is nonzero since otherwise this value of the arbitrary element $b$ in associated with $k = 5$. Using scaling equivalence transformations, we can set $b_0, a^1_5/4 \in \{-1, 1\}$.

Depending on the sign of $a^1_5$, we obtain Cases 16b and 16c of Theorem 6.

If $a^2_6 = 0$, then $a^2_7$ is also zero. Integrating (10), we get $b = b_0 y^{-2} - a^2_6 x$, where again the integration constant $b_0$ is nonzero since otherwise this value of the arbitrary element $b$ is associated with $k = 5$. Using scaling equivalence transformations and alternating the signs of $(x, u)$, we can set $b_0 \in \{-1, 1\}$ and $a^3_7/2 \in \{0, -1\}$, which gives Case 16a.

2. $a^4_1 = 0$. Then $a^2_7 = 0$ and $a^1_5 = a^2_5$. The system (10) integrates to $b = -a^4_1 \ln |y| - a^3_6 x + b_0$, where the parameter $a^4_1$ is nonzero since otherwise $k = 5$. The integration constant $b_0$ can be set to zero by shifts of $b$, as well as $a^1_5 \in \{-1, 1\}$ and $a^3_7 \in \{-1, 0\}$ up to scaling equivalence transformations and alternating the signs of $(x, u)$. This leads to Case 14.

3. $a^4_1 = -2$. Then $a^1_7 = -a^3_7$. We shift $b$ for setting $a^1_5 = 0$. The general solution of the system (10) is $b = b_0 y^{-2} - \frac{1}{2}a^3_5(x^2 + y^2) - a^2_7 x$ but this value of the arbitrary element $b$ is associated with $k > 2$.

4. $a^4_1 \notin \{-2, 0, 2\}$. Solving the system (10), we obtain $b = b_0 |y| - a^2_6 x$, where the integration constant $b_0$ should be nonzero for $k$ to be equal two. Setting $b_0 \in \{-1, 1\}$ and $a^3_7 \in \{-1, 0\}$ by scaling equivalence transformations and alternating the signs of $(x, u)$ results in Case 13.

**rank $A_2 = 0$.** This means that $a^1_5 = a^2_2 = a^3_4 = a^2_2 = 0$. Since rank $A_4 = 2$, we can linearly recombine equations of the system (10) to set $a^3_5 = a^4_2 = 1$ and $a^1_5 = a^2_2 = 0$. The compatibility condition (12) means that the vector fields $v_1$ and $v_2$ associated to equations of the reduced system (10) commutes, $[v_1, v_2] = 0$, which results to the system

$$
\begin{align*}
a^2_5 - a^2_4 a^2_5 - a^4_6 a^2_5 = 0, & \quad a^2_4 a^2_6 - a^5_6 a^2_5 = 0, \\
a^1_6 + a^3_8 a^2_5 - a^1_5 a^2_8 = 0, & \quad a^1_6 - a^2_7 + a^9_6 a^2_5 - a^5_6 a^2_8 = 0.
\end{align*} \quad (20)
$$
Suppose that \( a_1^1 = a_2^3 = 0 \). The system (20) then reduces to \( a_6^1 = a_6^2 = 0 \) and \( a_4^1 = a_4^2 \). In view of the last equation, we can set \( a_8^1 = a_7^2 = 0 \) by rotation equivalence transformations. The general solution of the system (10) is

\[
b = -\frac{a_1^1}{2}x^2 - \frac{a_2^2}{2}y^2 - a_4^1 x - a_2^2 y + b_0,
\]

where \( b_0 \) is an integration constant. This form of the arbitrary element \( b \) is related to the value \( k = 3 \) if \( a_1^1 \neq a_2^2 \) and to the value \( k = 5 \) if \( a_1^1 = a_2^2 \), which contradicts the supposition \( k = 2 \).

This is why \( (a_3^1, a_3^2) \neq (0, 0) \), and we set \( a_4^1 = 0 \) and \( a_2^2 = -1 \) by equivalence transformations of rotations, scalings and alternating signs. Then the first equation of the system (20) implies \( a_6^1 = 0 \), and thus the first six equations of the system (15) takes the form \( F_1^1 = 0 \), \( c_2 = 0 \), \( F_2 = \lambda^1 \), \( F_3 = \lambda^2 = 2c_1 \) and \( a_6^2 \lambda^2 = 0 \). In view of the last equation, we get \( a_6^2 = 0 \). Therefore, the system (20) is equivalent to \( a_1^1 = a_6^1 = 0 \) and \( a_2^2 = -a_1^1 \). The eighth equation of (15) gives \( a_2^1 \lambda^2 = 0 \), i.e., \( a_2^1 = 0 \). We can set \( a_3^2 = 0 \) up to equivalence transformations of shifts of \( b \). The system (10) integrates to \( b = b_0 e^{a^3} + a_2^2 x \), where the integration constant \( b_0 \) is nonzero since otherwise \( b \) is a linear function, for which \( k = 5 \). Equivalence transformations of scalings and alternating the signs of \((x, u)\) allow us to set \( b_0 = \pm 1 \) and \( a_2^1 \in \{0, 1\} \). This results in Case 15 of Theorem 6.

### 4.3 More independent template-form equations

We show below that \( k > 2 \) if and only if \( b \) is an most quadratic polynomial in \((x, y)\).

**k = 3.** Since \( \text{rank } A_4 = \text{rank } A = k = 3 \), we have that \( \text{rank } A_2 > 0 \).

Suppose that the submatrix \( A_2 \) is of rank two. Recombining equations of the system (10), we can be reduce this submatrix to the form

\[
A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then the projections \( \hat{v}_1, \hat{v}_2 \) and \( \hat{v}_3 \) of the vector fields \( v_1, v_2 \) and \( v_3 \) to the space with the coordinates \((x, y)\) are

\[
\hat{v}_1 = (x + a_3^1) \partial_x + (y + a_3^1) \partial_y, \quad \hat{v}_2 = (y + a_3^1) \partial_x - (x - a_3^2) \partial_y, \quad \hat{v}_3 = a_3^1 \partial_x + a_3^2 \partial_y. \tag{21}
\]

In view of the condition (13), the commutator \( [\hat{v}_2, \hat{v}_3] = -a_3^2 \partial_x + a_3^2 \partial_y \) should belong to the span \( \langle \hat{v}_1, \hat{v}_2, \hat{v}_3 \rangle \) but this is not the case, which leads to a contradiction.

Therefore, \( \text{rank } A_2 = 1 \), and thus the matrix \( A_4 \) can be reduced to the form

\[
A_4 = \begin{pmatrix} a_1^1 & a_1^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } (a_1^1, a_2^1) \neq (0, 0).
\]

Then the compatibility condition (12) is equivalent to the commutation relations

\[
[v_1, v_2] = -a_1^1 v_2 + a_2^1 v_3, \quad [v_1, v_3] = -a_1^1 v_3 - a_2^1 v_2, \quad [v_2, v_3] = 0.
\]

From the first two commutation relations, we obtain \( a_1^1 a_2^2 - a_2^1 a_2^2 = 0, a_2^2 a_2^2 + a_1^1 a_3^2 = 0 \), and thus \( a_2^2 = a_3^2 = 0 \) since \( (a_1^1, a_2^1) \neq (0, 0) \). Then the last commutation relation yields \( a_2^2 = a_3^2 = 0 \) and \( a_2^2 = a_3^2 \). Up to rotation equivalence transformations, we can set \( a_2^2 = a_3^2 = 0 \). Under this gauge, from the former commutation relations we get the system

\[
(2a_1^1 + a_1^1)(a_2^2 - a_3^2) = 0, \quad a_1^1(a_2^2 - a_3^2) = 0,
\]

\[
a_6^1 = (2a_1^1 + a_1^1)a_7^2, \quad a_4^1 = (a_1^1 + a_3^1)a_5^2 - a_2^1 a_3^2, \quad a_5^1 = (a_1^1 + a_3^1)a_3^2 + a_2^1 a_5^2.
\]

(22)
Since the arbitrary element $b$ satisfies the equations $b_x + a_7^2 x + a_3^3 = 0$ and $b_y + a_3^3 y + a_3^3 = 0$, it is a quadratic function of $(x, y)$. More specifically,

$$b = \frac{1}{2} a_3^2 x^2 - \frac{1}{2} a_3^2 y^2 - a_3^3 x - a_3^3 y$$

(23)

up to equivalence transformations of shifts with respect to $b$. Hence $a_7^2 \neq a_3^3$ since otherwise $k = 5$ for this value of $b$, which contradicts the supposition $k = 4$. Then the system (22) reduces to

$$a_1^4 = 0, \quad a_2^1 = -2a_1^1, \quad a_3^1 = 0, \quad a_4^1 = -a_1^4 a_2^3, \quad a_5^1 = -a_1^4 a_2^3$$

and guaranties that the above value of $b$ satisfies the entire corresponding system (10).

Modulo $G^\sim$-equivalence, we can assume that $a_7^2 = \pm 1$, $a_3^3 = 0$; $|a_3^3| < |a_7^2|$ if $a_7^2 a_3^3 > 0$; $a_3^3 = 0$ if $a_7^2 \neq 0$; $a_3^3 \in \{-1, 0\}$ if $a_7^2 = 0$. $G^\sim$-inequivalent values of $b$ of the form (23) with the associated maximal Lie invariance algebras are listed in Cases 17–21 of Theorem 6.

$k = 4$. Since rank $A_4 = \text{rank} A = 4$, linearly re-combining the equations (10), we can set $A_4$ to be the $4 \times 4$ identity matrix. In view of the form of the vector fields $v_1, \ldots, v_4$ associated to the equations of the reduced system (10), the compatibility condition (12) implies the commutation relations

$$[v_2, v_3] = v_4, \quad [v_2, v_4] = -v_3, \quad [v_3, v_4] = 0.$$  

From the first two commutation relations, we get that $a_1^3 = a_3^3 = 0$. The last commutation relation together with the previous restrictions on the coefficients $a_j$ yields $a_6^3 = a_8^3 = 0$ and $a_5^3 = a_3^3$. Returning to the first two commutation relations, we obtain the equations

$$a_1^2 = -2a_3^1 + 2a_3^2 + a_2^3 a_3^2 = 2a_3^5 + a_5^3 a_8^3, \quad a_7^2 = a_3^3 a_8^3 = 0$$

implying $a_3^3 = 0$ and $a_3^3 = a_3^3$. Since the arbitrary element $b$ satisfies the equations $b_x + a_7^3 x + a_3^3 = 0$ and $b_y + a_3^3 y + a_3^3 = 0$, it is a quadratic function of $(x, y)$ with the same coefficients of $x^2$ and of $y^2$. This means that in fact $k = 5$, which contradicts the supposition $k = 4$.

$k = 5$. The 5×5 matrix $A$ of the coefficients of the system (10) is of rank 5 and, up to recombining the equations of this system, can be assumed to be the 5×5 identity matrix. Then the last equation of the system (10) implies that $b$ is the specific quadratic polynomial of $(x, y)$,

$$b = -\frac{1}{2} a_4^5 (x^2 + y^2) - a_5^5 x - a_6^5 y - a_9^5.$$

There are four $G^\sim$-equivalent values of the arbitrary element $b$ among such quadratic polynomials, $b = 0$, $b = x$, $b = \frac{1}{2} (x^2 + y^2)$ and $b = -\frac{1}{2} (x^2 + y^2)$, which correspond to Cases 22a, 22b, 22c and 22d of Theorem 6, respectively.

5 Additional equivalence transformations and modified classification result

It is obvious that the class (1) is not normalized. In other words, it possesses admissible transformations (e.g., those related to Lie symmetries of systems from this class) that are not generated by elements of $G^\sim$. See [6, 35, 36] for definitions. Moreover, we will show below that the class (1) is not even semi-normalized since some of its admissible transformations cannot be presented as compositions of admissible transformations generated by elements of $G^\sim$ and by point symmetries of systems from this class as well. Such admissible transformations may lead to additional point equivalences among classification cases listed in Theorem 6.

Since we do not have the complete description of the equivalence groupoid of the class (1), in the course of looking for the above additional equivalences we need to use algebraic tools that do
not involve this description. If two systems of differential equations are similar with respect to a point transformation, then the corresponding maximal Lie invariance algebras are isomorphic in the sense of abstract Lie algebras. Moreover, these maximal Lie invariance algebras are similar as realizations of Lie algebras by vector fields with respect to the same point transformation. This gives necessary conditions of similarity for systems of differential equations with respect to point transformations.

Lie algebras of different dimensions are nonisomorphic. Hence we categorize the Lie algebras presented in Theorem 6 according to their dimensions to distinguish the cases that are definitely not equivalent to each other with respect to point transformations,

- \(\dim g_b = 2\): Cases 1, 2, 3, 4 and 5;
- \(\dim g_b = 3\): Cases 6a, 6b, 6c, 7, 8, 9, 10 and 11;
- \(\dim g_b = 4\): Cases 12a, 12b, 12c, 13, 14 and 15;
- \(\dim g_b = 5\): Cases 16a, 16b and 16c;
- \(\dim g_b = 6\): Cases 17, 18, 19, 20 and 21;
- \(\dim g_b = 9\): Cases 22a, 22b, 22c and 22d.

However, the same dimension of algebras does not ensure their isomorphism.

Finding a pair of classification cases with isomorphic maximal Lie invariance algebras and fixing bases of these algebras that are concordant under the found algebra isomorphism, we aim to obtain a point transformation that respectively maps the basis elements of the first algebra to the basis elements of the second one. The existence of such a point transformation hints that the two classification cases may be equivalent with respect to the same point transformation.

In this way, we find three families of \(G^\sim\)-inequivalent admissible transformations for the class (1) that are not induced by equivalence transformations of this class and whose target arbitrary elements differ from their source arbitrary elements. In each of these families, the source arbitrary elements are parameterized by an arbitrary function of a single argument. We present these families jointly with the corresponding induced additional equivalences among classification cases of Theorem 6:

1. \(b = r^{-2}f(\varphi) - \frac{1}{2}t^2\), \(\tilde{b} = \tilde{r}^{-2}f(\tilde{\varphi})\),
   \[
   \tilde{t} = \tan t, \quad \tilde{x} = x \sec t, \quad \tilde{y} = y \sec t, \quad \tilde{u} = u \cos t + x \sin t, \quad \tilde{v} = v \cos t + y \sin t, \quad \tilde{h} = h \cos^2 t, \\
   6c \rightarrow 6a, \quad 12c \rightarrow 12a, \quad 16b \rightarrow 16a_{\delta=0}, \quad 22d \rightarrow 22a.
   \]

2. \(b = r^{-2}f(\varphi) + \frac{1}{2}t^2\), \(\tilde{b} = \tilde{r}^{-2}f(\tilde{\varphi})\),
   \[
   \tilde{t} = \frac{1}{2}e^{2t}, \quad \tilde{x} = e^t x, \quad \tilde{y} = e^t y, \quad \tilde{u} = e^{-t} (u + x), \quad \tilde{v} = e^{-t} (v + y), \quad \tilde{h} = e^{-2t} h, \\
   6b \rightarrow 6a, \quad 12b \rightarrow 12a, \quad 16c \rightarrow 16a_{\delta=0}, \quad 22c \rightarrow 22a.
   \]

3. \(b = f(y) + x\), \(\tilde{b} = f(\tilde{y})\),
   \[
   \tilde{t} = t, \quad \tilde{x} = x + \frac{1}{2}t^2, \quad \tilde{y} = y, \quad \tilde{u} = u + t, \quad \tilde{v} = v, \quad \tilde{h} = h, \\
   22b \rightarrow 22a, \quad 7, \quad 13, \quad 14, \quad 15, \quad 16a, \quad 18, \quad 20; \quad \delta = 1 \rightarrow \delta = 0.
   \]

Both the first and second families of admissible transformations can be generalized to the rotating reference frame. The generalization of the transformation with \(f = 0\) from the first family to the rotating reference frame was found for the first time for the shallow water equations in cylindrical coordinates in [18, Theorem 1].

\(^2\)For Cases 18 and 20, we should compose the corresponding point transformation with the permutation \((x, u) \leftrightarrow (y, v)\), which is an equivalence transformations of the class (1).
Each of the above admissible transformations is $G$-equivalent to no admissible transformations generated by point symmetries of systems from this class. Therefore, the class (1) is not semi-normalized.

As by-product, we also prove the following assertions.

**Proposition 11.** Any system from the class (1) that is invariant with respect to a nine-dimensional Lie algebra of vector fields is equivalent, up to point transformations, to the system from the same class with $b = 0$, which is the system of shallow water equations with flat bottom topography.

**Proposition 12.** Any system from the class (1) with five-dimensional maximal Lie invariance algebra is reduced by a point transformation to the system from the same class with $b = \pm y^{-2}$.

For the other possible dimensions of maximal Lie invariance algebras of systems from the class (1), we prove the inequivalence of the remaining classification cases whenever it is possible to do so using the algebraic technique based on Mubarakzianov’s classification of Lie algebras up to dimension four [28] and Turkowski’s classification of six-dimensional solvable Lie algebras with four-dimensional nilradicals [45]. For convenience, we take these classifications in the form given in [12], where the notation of algebras from Mubarakzianov’s classification was modified, in particular, by indicating parameters for families of algebras and where the basis elements of the algebras from Turkowski’s classification were renumbered in order to have bases in $K$-canonical forms. For each abstract Lie algebra appearing in the consideration, we present all the nonzero commutation relations among basis elements up to antisymmetry.

Unfortunately, the algebraic criterion of inequivalence with respect to point transformations is not sufficiently powerful for systems with two-dimensional maximal Lie invariance algebra since there are only two nonisomorphic two-dimensional Lie algebras $2A_1$ and $A_{2,1}$, the abelian and the non-abelian ones. Applying this criterion, we can only partition the corresponding classification cases into two sets, Cases $1_{\nu=2}$ and 3 with abelian two-dimensional Lie invariance algebras and Cases $1_{\nu\neq2}$, 2, 4 and 5 with non-abelian two-dimensional Lie invariance algebras. There are definitely no point transformations between cases that belong to different sets. Comparing the dimensions of the associated spaces of zeroth-order conservation laws [2], we can additionally conclude that Case $1_{\nu=2}$ and Case 3 are inequivalent.

For the classification cases with maximal Lie invariance algebras of greater dimensions, the algebraic criterion is more advantageous. Thus, the maximal Lie invariance algebras in Cases 6a, 7a=0, 8, 9, 10 and 11 of Theorem 6 are isomorphic to the three-dimensional Lie algebras $\text{sl}(2, \mathbb{R})$, $A_{3,1}$, $A_{3,1}^\perp$, $A_{3,5}^0$, $A_{2,1} \oplus A_1$ and $A_{2,1} \oplus A_1$, respectively. These Lie algebras are defined by the following commutation relations:

\[
\begin{align*}
\text{sl}(2, \mathbb{R}): & \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = -2e_2; \\
A_{3,1}^\perp: & \quad [e_2, e_3] = e_1; \\
A_{3,1}^\perp: & \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2; \\
A_{3,5}^0: & \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1; \\
A_{2,1} \oplus A_1: & \quad [e_1, e_2] = e_1,
\end{align*}
\]

and they are well known to be non-isomorphic to each other. This implies the pairwise inequivalence of the above cases of Lie-symmetry extensions with respect to point transformations, except the pair of the last two cases. Their inequivalence follows from different dimensions of the spaces of zeroth-order conservation laws for the corresponding systems of the form (1), three for Case 10 and two for Case 11, see [2].

In a similar way, we prove that the maximal Lie invariance algebras associated with Cases 12a, 13d=0, 14d=0 and 15i=0, are also not isomorphic to each other although all of them are four-dimensional. The corresponding abstract Lie algebras are $\text{sl}(2, \mathbb{R}) \oplus A_1$ (Case 12a) as well as $A_{4,8}$.
with $a = \nu/(2 - \nu)$ if $\nu < 1$ and with $a = (2 - \nu)/\nu$ if $\nu \geq 1$ (Case 13), with $a = 0$ (Case 14) and with $a = -1$ (Case 15), where

$$A_{4,18}^a, |a| \leq 1: \quad [e_2, e_3] = e_1, [e_1, e_4] = (1 + a)e_1, [e_2, e_4] = e_2, [e_3, e_4] = ae_3.$$ 

As a result, Cases 12a, 13, 14 and 15 of Theorem 6 are equivalent neither to each other nor to other cases of this theorem with respect to point transformations. Moreover, the parameter $\nu$ in Case 13 cannot be gauged by point transformations.

In Cases 17, 18, 19, 20, and 21, the corresponding maximal Lie invariance algebras are six-dimensional solvable Lie algebras with four-dimensional abelian nilradicals that are isomorphic to the algebras $N^{abc_1}_{6,1}$ with $(a, b, c, d) = \frac{1}{2}(\beta - 1, 1 + \beta, 1 + \beta, 1 - \beta)$, $N^{-1,1,1}_{6,2}$, $N^{-1/3,1/3,1}_{6,15}$, $N^{0,1}_{6,16}$, $N^{0,3}_{6,18}$ from Turkowski’s classification, respectively. Here the canonical commutation relations are the following:

$$N^{abc_1}_{6,1} \quad a \neq 0$$

$$a^2 + b^2 \neq 0$$

$$[e_1, e_5] = ae_1, \quad [e_2, e_5] = be_2, \quad [e_4, e_5] = e_4,$$

$$[e_1, e_6] = ce_1, \quad [e_2, e_6] = de_2, \quad [e_3, e_6] = e_3$$

$$N^{abc_1}_{6,2} \quad a^2 + b^2 \neq 0$$

$$[e_1, e_5] = ae_1, \quad [e_2, e_5] = e_2, \quad [e_4, e_5] = e_3,$$

$$[e_1, e_6] = be_1, \quad [e_2, e_6] = ce_2, \quad [e_3, e_6] = e_3, \quad [e_4, e_6] = e_4$$

$$N^{abc_1}_{6,13} \quad a^2 + b^2 \neq 0$$

$$[e_1, e_5] = ae_1, \quad [e_2, e_5] = be_2, \quad [e_3, e_5] = e_4, \quad [e_4, e_5] = -e_3,$$

$$[e_1, e_6] = ce_1, \quad [e_2, e_6] = de_2, \quad [e_3, e_6] = e_3, \quad [e_4, e_6] = e_4$$

$$N^{abc}_{6,16} \quad a \neq 0$$

$$[e_2, e_5] = e_1, \quad [e_3, e_5] = ae_3 + e_4, \quad [e_4, e_5] = -e_3 + ae_4,$$

$$[e_1, e_6] = e_1, \quad [e_2, e_6] = e_2, \quad [e_3, e_6] = be_3, \quad [e_4, e_6] = be_4$$

$$N^{abc}_{6,18} \quad b \neq 0$$

$$[e_1, e_5] = e_2, \quad [e_2, e_5] = -e_1, \quad [e_3, e_5] = ae_3 + be_4, \quad [e_4, e_5] = -be_3 + ae_4,$$

$$[e_1, e_6] = e_1, \quad [e_2, e_6] = e_2, \quad [e_3, e_6] = ce_3, \quad [e_4, e_6] = ce_4$$

Since the above isomorphisms are not as obvious as for algebras of lower dimensions, we present the necessary basis changes to $(e_1, e_2, e_3, e_4, e_5, e_6)$:

$$17: \quad (P(0, e^{\beta t}), P(0, e^{-\beta t}), P(e^t, 0), P(e^{-t}, 0), \frac{1}{2}D^x + \frac{1}{2}D(1), \frac{1}{2}D^x - \frac{1}{2}D(1))$$

$$18: \quad (P(e^t, 0), P(e^{-t}, 0), (0, 1), P(0, 0), -D(1), D^x) \quad \text{for} \quad \delta = 0$$

$$19: \quad (P(e^t, 0), P(e^{-t}, 0), P(0, \cos \beta t), P(0, \sin \beta t), \beta^{-1}D(1), D^x)$$

$$20: \quad (P(0, 1), P(0, t), P(\sin t, 0), P(0, \cos t), -D(1), D^x) \quad \text{for} \quad \delta = 0$$

$$21: \quad (P(0, 0), P(0, t), P(0, \cos \beta t), P(0, \sin \beta t), D(1), D^x)$$

The Lie algebras from Turkowski’s classification appearing in the consideration are not isomorphic to each other, including the pairs of algebras from the same series with different values of the parameter $\beta$ within ranges indicated in the corresponding cases of Theorem 6. The claim on such pairs was checked by the direct computation in Maple. This is why Cases 17, 18, 20, and 21 are inequivalent with respect to point transformations. Moreover, the parameter $\beta$ in Cases 17, 19 and 21 cannot be gauged further.

Analyzing the classification cases listed in Theorem 6, the additional equivalences among them that are found in this section and the checked necessary algebraic condition for pairs of inequivalent cases, we can suppose that the following assertion holds.

**Conjecture 13.** A complete list of inequivalent (up to all admissible transformations) Lie symmetry extensions in the class (1) is exhausted by Cases 1, 2, 3, 4, 5, 6a, 7b, 8, 9, 10, 11, 12a, 13b, 14, 15, 16a, 17, 18, 19, 20, 21 and 22a of Theorem 6.
To prove this conjecture, we need to complete the verification of the inequivalence of cases within the set of cases with two-dimensional non-abelian maximal Lie invariance algebras as well as the impossibility of further gauging of parameters remaining in some cases. This can be done via the construction of the equivalence groupoid of the class (1), which is a nontrivial and cumbersome problem. It is quite difficult to prove even principal properties of admissible transformations of the class (1), which can be conjectured after analyzing the form of equivalence transformations of this class, of Lie symmetries of equations from this class and of the three obtained $G\sim$-inequivalent families of admissible transformations. These principal properties include the affineness with respect to the dependent variables, the fiber-preservation, i.e., the projectability to the space with the coordinates $(t, x, y)$, as well as the projectability to the space with the coordinate $t$. We can also conjecture the explicit structure of the equivalence groupoid of the class (1).

**Conjecture 14.** $G\sim$-inequivalent admissible transformations of the class (1) that are not generated by point symmetries of systems from this class are exhausted by the three families found in this section.

### 6 Conclusion

We solved the group classification problem for the class (1) of two-dimensional shallow water equations with variable bottom topography. The result is summarized in Theorem 6.

Applying the algebraic method, we first construct the generalized equivalence group $G\sim$ of the class (1), which is presented in Theorem 5 and is a necessary ingredient for solving the group classification problem. Note that the generalized equivalence group of the class (1) coincides with its usual equivalence group. The integration of the system of determining equations for the components of Lie symmetry vector fields, which is quite complicated in this case, required the application of the advanced method of furcate splitting. This method was additionally optimized via reducing the study of compatibility of template-form equations for the arbitrary element $b$ to checking whether the set of vector fields associated to these equations is closed with respect to the Lie bracket of vector fields. In the course of the classification, we continuously used transformations from the equivalence group $G\sim$ for gauging various constants involved in the specific values of the arbitrary element $b$, which leads to a significant simplification of computations. One more complication of the group classification for the class (1) is that this class is not normalized and even not semi-normalized. Therefore, this class possesses admissible point transformations which are not generated by its equivalence transformations and point symmetry transformations of systems belonging to it. Such admissible transformations give rise to additional point equivalences among the $G\sim$-inequivalent classification cases listed in Theorem 6. In Section 5 we found three families of $G\sim$-inequivalent admissible transformations of the above kind in the class (1) and presented the corresponding additional equivalences within the group-classification list for this class up to the $G\sim$-equivalence. Moreover, for all the pairs of listed cases that are possibly inequivalent to each other with respect to point transformations, we checked their inequivalences using algebraic techniques, except the inequivalences within the set of cases with two-dimensional non-abelian maximal Lie invariance algebras. This allowed us to conjecture the group classification of the class (1) up to its equivalence groupoid $G\sim$.

In future work, we plan to continue our study of shallow water equations to extend and generalize the obtained results. The first natural step in this study is to describe the equivalence groupoid $G\sim$ of the class (1), proving Conjecture 14. If this conjecture is proved, then the proof of Conjecture 13 will be straightforward. Moreover, the class (1) will then give, in addition to the class studied in [32, Section X] and in [31], one more example of a class that is not semi-normalized but in which all admissible transformations generated by no equivalence transformations are still related to Lie-symmetry extensions. An interesting question about the
structure of $\mathcal{G}^\sim$ is how many $G^\sim$-inequivalent maximal conditional equivalence groups of the class (1) exist; see related definitions in [35, 36].

The presented classification of Lie symmetries of two-dimensional shallow water equations with variable bottom topography provides a basis for finding exact solutions of these equations. Such solutions can be used for testing numerical schemes for this model. Moreover, Lie symmetries themselves may be applied for designing invariant numerical schemes. The detected additional point equivalences within the classification list are of great relevance here since they allow us to avoid the repeated construction of exact solutions for systems that are similar to simpler ones with respect to point transformations.

The zeroth-order conservations laws of systems from the class (1) were classified in [2] but this result needs an additional arrangement. In particular, generating sets [26] of zeroth-order conservations laws of these systems with respect to their Lie symmetries should be constructed.

The study of two-dimensional shallow water equations with variable bottom topography within the framework of symmetry analysis of differential equations can be further extended to classifications of generalized symmetries, cosymmetries, higher-order conservations laws and Hamiltonian structures for these equations.

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