Simple representation of the quantum entanglement of coupled harmonic oscillators in terms of the reflection coefficient

Yu V Tsykareva and D N Makarov
Department of Fundamental and Applied Physics, Northern Arctic Federal University named after M V Lomonosov, Severnaya Dvina Embankment 17, Arkhangelsk 163002, Russia
aisonoka@gmail.com

Abstract. Quantum entanglement of coupled harmonic oscillators is frequently applied in quantum and non-linear physics, molecular chemistry and biophysics, which is why its study is of a great interest for modern physics. In this work a quantum entanglement of a coupled harmonic oscillator in a simple form was found. This simple form is presented as a single parameter – reflection coefficient R. All parameters of the studied system are included in the R coefficient. It is shown that the derivation of the expression can have applications in quantum optics, in particular in quantum metrology.

1. Introduction
Studying the properties of coupled harmonic oscillators is a relevant area of modern physics. The interest towards it stems primarily from the fact that models of such systems can be encountered in many applications of quantum and non-linear physics, molecular chemistry and biophysics [1-2]. In quantum physics this interest generally originates from quantum entanglement for such a system. In particular, quantum communication protocols such as quantum cryptography, quantum dense coding, quantum calculation algorithms and teleportation of quantum states can be explained using entangled states. On the other hand, physical models of coupled harmonic oscillators are used in biophysics to explain the issue of photosynthesis [3] which is still considered an unsolved puzzle.

The simplest model capable of serving as a foundation for presenting multiple physical processes is the coupled harmonic oscillator model. Because of a large number of system parameters evaluation of quantum entanglement in a simple analytical form is problematic for this model. The goal of this research is to find simple expressions for evaluating the quantum entanglement of the harmonic oscillator in the case of a dynamic system that would only depend on reflection coefficient R. The Schmidt parameter [4] and the von Neumann entropy [5], both of which depend on a common parameter (the Schmidt mode) were considered as measures of quantum entanglement. In the expressions obtained as a result of the research the R coefficient incorporated all other system parameters, which allows the expressions of quantum entanglement evaluation to take simple analytical forms. It is worth noting that this approach can not only be used for evaluating the quantum entanglement, but also for other systems that can be presented in the form of linked harmonic oscillators.
2. Calculating methods

For this purpose we take a look at a system of two coupled harmonic oscillators with Hamiltonian (1)

\[ \hat{H} = \frac{1}{2} \left( \frac{1}{m_1} \hat{p}_1^2 + \frac{1}{m_2} \hat{p}_2^2 + A x_1^2 + B x_2^2 + C x_1 x_2 \right). \]  

Work [2] provides a solution for a nonstationary Schrödinger equation for this system under the condition that the system was not linked at zero time (C=0). However, for our case it is more convenient to present the creation and annihilation of quantum states via operators. In this case the Hamiltonian of the system will look as follows [6]

\[ \hat{H} = \sum_k \hbar \omega_k \left( \hat{a}_k^{\dagger} \hat{a}_k + \frac{1}{2} \right) + \hbar \Omega x_1 x_2, \]  

where \( \omega_1 = \sqrt{A/m_1} \); \( \omega_2 = \sqrt{B/m_2} \); \( \Omega = C / 2(m_1 m_2 AB)^{1/4} \); \( \hat{a}_k^{\dagger} = \frac{1}{\sqrt{2}} (x_k - \frac{\partial}{\partial x_k}) \); \( \hat{a}_k = \frac{1}{\sqrt{2}} (x_k + \frac{\partial}{\partial x_k}) \) – are the operators of annihilation and creation of quantum states \( s_k \) respectively. For the Hamiltonian in this form the following equation solution was found [6]

\[ |\psi(t)\rangle = e^{-i\Omega t} |s_1, s_2\rangle = \frac{1}{\sqrt{s_1! s_2!}} e^{-i\hat{a}_1^{\dagger} s_1 + i\hat{a}_2^{\dagger} s_2} |0,0\rangle. \]  

For further study we decompose the obtained wave function of the system according to the Schmidt theorem

\[ |\psi(t)\rangle = \sum_k \sqrt{\lambda_k} u_k(x_1, t) v_k(x_2, t), \]  

where \( u_k(x_1, t) \) and \( v_k(x_2, t) \) – are wave functions of a pure state of system’s oscillators, \( \lambda(t) \) – is the Schmidt mode. Some of the most well-known methods for measuring quantum entanglement are the Schmidt parameter (5) and the von Neuman entropy (6)

\[ K = \left( \sum_k \lambda_k \right)^{-\frac{1}{2}} \]  

\[ S_N = -\sum_k \lambda_k \ln(\lambda_k). \]  

For this task these measures of quantum entanglement are basically equal. All main patterns in these measures are identical, though different in absolute meaning. Some measures are sometimes more convenient to use in calculations; in our case it is the Schmidt parameter (5). Expressions become simpler and easier to interpret. As you can see, both methods depend on the Schmidt mode, therefore, the main issue during the measurement of quantum entanglement is the calculation of the Schmidt modes.

Furthermore, to move on to the reflection and transmission coefficients we used well-known transformations (7) for the linear beam splitter [7], where R is the reflection coefficient, T is the transmission coefficient, \( \phi \) is the phase shift

\[ R = \frac{\sin^2(\frac{\phi t}{2})}{1 + \epsilon^2}; \quad T = 1 - R; \]  

\[ \cos \phi = -\epsilon \sqrt{\frac{R}{T}}; \quad \epsilon = \frac{\omega_2 - \omega_1}{2\sqrt{\omega_1 \omega_2} \Omega}. \]  

As was demonstrated in work [6], the entire dependence on the multiparameter system for quantum entanglement can be presented as a single parameter – reflection coefficient R

\[ \lambda_k(R) = |c_{k,s_1+s_2-k}|^2, \]  

where \( c_{k,p} = \sum_{n=0}^{s_1+s_2} A_{p+n}^{s_1+s_2} A_{n}^{k} A_{n,s_1+s_2-n}^{k} e^{-2i n \arcsin(\sqrt{1-R} \sin \phi)}, \)
where $P$ are Jacobi polynomials, $k$ and $p$ are quantum numbers of the oscillator in the state $|k,p\rangle$, with $k + p = s_1 + s_2$. It is worth noting that $\varphi \in \left(0, \frac{\pi}{2}\right)$, yet the dependence on $\varphi$ is contained in trigonometrical functions. Therefore, the expression form of the dependence on $R$ alone is determined. Despite the fact that quantum entanglement can be calculated, its form is still difficult to interpret and can be compared to numerical computations. Indeed, in order to calculate quantum entanglement it is necessary to conduct a summation of rather complicated special functions. It is essential to use simple expressions that allow to analyze quantum entanglement rather easily. We found such expressions for certain states.

**Figure 1.** The dependence of the Schmidt parameter as a function of $R$ similar values of quantum number in set (a) and for various sets of quantum numbers (b). In the figures, the dependencies are presented for different initial values of quantum numbers $(s_1,s_2)$. 
3. Results and Discussion
The graphic interpretation of the Schmidt parameter depending on \( R \) for various pairs of quantum numbers \( s_1 \) and \( s_2 \) for harmonic oscillators is presented in Figure 1, where \( R \in (0,1) \).

The forms of expressions for the Schmidt parameter for various sets of quantum numbers \( s_1 \) and \( s_2 \):
For \( s_1 = 0 \) and \( s_2 = 2 \):
\[
K = \frac{1}{1-2R(1-R)(2-3R(1-R))}
\] (9)

For \( s_1 = 2 \) and \( s_2 = 2 \):
\[
K = \frac{1}{1-24R(1-R)(1-3R(1-R)(4-5R(1-R)(4-7R(1-R))))}
\] (10)

For \( s_1 = 1 \) and \( s_2 = 3 \):
\[
K = \frac{1}{1-2R(1-R)(10-R(1-R)(99-80R(1-R)(5-7R(1-R))))}
\] (11)

Using the expressions (9-11), it is possible to find such values of \( R \) for which the quantum entanglement of this pair of quantum numbers reaches its maximum. Table 1 presents the results of such a calculation. The expression for the Schmidt parameter for \( s_1 = 1 \) and \( s_2 = 1 \) is presented in work [6].

| \((s_1, s_2)\) | \(K\) | \(R\) |
|---------------|------|------|
| (1,1)         | 3    | 0.5 \(1 \pm 1/\sqrt{3}\) |
| (0,2)         | 2.67 | 0.5  |
| (2,2)         | 4.4  | 1/3 and 2/3 |
| (1,3)         | 4    | 0.5  |

This is quite an interesting result, since at first glance it seems that the maximum entanglement should be at \( R = 1/2 \). Figure 1 shows that quantum entanglement strongly depends on the reflection coefficient \( R \) but is always zero at \( R = 0; 1 \). The larger the quantum numbers \( s_1; s_2 \), the greater the quantum entanglement.

It should be added that the Eq. (8) can be used in quantum optics to calculate the quantum entanglement of photons in a beam splitter (with two input and output ports) with a reflection coefficient \( R \). In this case, \( \omega_1 \) and \( \omega_2 \) are the frequencies of 1 and 2 photons, respectively, and is some parameter characterizing the beam splitter. Combining the parameters \( \omega_1 \) and \( \omega_2 \); \( \Omega; \tau \), you can always choose the value of \( R \) with maximum quantum entanglement using the expressions (7).

4. Conclusions
It was shown that the quantum entanglement of two coupled harmonic oscillators is expressed as a single parameter, this is the reflection coefficient \( R \). It is shown that for certain values of \( R \) and a given pair of quantum numbers \( s_1 \) \& \( s_2 \) quantum entanglement has a maximum and can be large; for \( R = 0 \) and \( R = 1 \) there is no quantum entanglement; the equations for Schmidt mode presented in this work have the most simple analytical form; quantum entanglement, expressed as a single parameter (the reflection coefficient \( R \), was obtained without taking into account the environment that can significantly affect the quantum entanglement. It is worth adding that the search for simple expressions of quantum entanglement is a relevant task, as modern sources generally involve
numerical computations. This makes it difficult to see the pattern and even more so the fact that quantum entanglement can depend on a single parameter that includes all parameters of the studied system.

Acknowledgments
The work was supported by Russian Science Foundation grant № 20-72-10151.

References
[1] Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw-Hill)
[2] Makarov D N 2018 *Scientific Reports* **8** 8204
[3] Romero E, Augulis R, Novoderezhkin V I, Ferretti M, Thieme J, Zigmantas D and van Grondelle R 2014 *Nature Physics* **10** 676–682
[4] Ekert A and Knight P L 1995 *Amer.J.Phys.* **63** 415–423
[5] Casini H and Huerta M 1996 *J. Phys. A: Math.Theor.* **42** (50) 504007
[6] Makarov D N 2020 *Phys. Rev. E* **102** 052213
[7] Zhe-Yu Je Ou 2007 *Multi-Photon Quantum Interference* (New York: Springer)