Monopole and Dyon Spectra in $N=2$ SYM with Higher Rank Gauge Groups

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Abstract: We derive parts of the monopole and dyon spectra for $N=2$ super-Yang–Mills theories in four dimensions with gauge groups $G$ of rank $r \geq 2$ and matter multiplets. Special emphasis is put on $G = SU(3)$ and those matter contents that yield perturbatively finite theories. There is no direct interpretation of the soliton spectra in terms of naive selfduality under strong–weak coupling and exchange of electric and magnetic charges. We argue that, in general, the standard procedure of finding the dyon spectrum will not give results that support a conventional selfduality hypothesis — the $SU(2)$ theory with four fundamental hypermultiplets seems to be an exception. Possible interpretations of the results are discussed.

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1. Introduction

The last years have seen a tremendous progress in the understanding of nonperturbative aspects of four-dimensional field theory. New techniques [1,2,3,4,5] enable calculation of exact results valid beyond the perturbative level. It was long ago conjectured [6,7] that the $N = 4$ supersymmetric Yang–Mills (SYM) theories should possess some kind of strong–weak coupling duality. These theories are perturbatively finite [8,9,10,11,12,13], and actually exactly finite [14]. Actual calculations of dyon spectra in these theories [15,16,17,18], and also other tests [19] give strong support for the duality hypothesis. There are also an infinite number of theories, possessing $N = 2$, but not $N = 4$, supersymmetry that are one-loop, and thus perturbatively, finite. The only one of these theories that has undergone closer examination with respect to duality properties is the $SU(2)$ model with four hypermultiplets in the fundamental representation. There, all results confirm duality, and it is tempting to conclude that the same is true for all perturbatively finite $N = 2$ SYM theories. Since all explicit calculations of BPS states in $N = 4$ theories and the finite $N = 2$ $SU(2)$ theory so far are in excellent agreement with predictions from duality, it is natural to continue this program and include also the other perturbatively finite $N = 2$ theories. The aim of this paper is to do this by calculating part of the dyon spectra for such theories. As we will demonstrate, a number of problems arise. They are partly associated with the lattice structures of electric and magnetic charges, and also with the inaccessibility of monopole–anti-monopole configurations.

In sections 2 and 3, basic properties about monopoles and their moduli spaces are reviewed. Section 4 applies an index theorem to find the dimensions of bundles of zero-modes of the various fields in the theories over moduli space. Section 5 contains a discussion on the lattice properties of electric and magnetic charges, giving a general argument against naïve duality. In section 6, the effective action for the monopoles is derived from the field theory, and some aspects of its quantization are discussed. Section 7 applies this quantization to some specific examples, and derives the corresponding dyon spectra. They do not support naïve duality. In section 8, the implications of the results are discussed.

2. Monopoles — Symmetry Breaking and Topology

In this section, we will give a quick review of the concept of Bogomolnyi–Prasad–Sommerfield (BPS) monopoles [20,21] and their topological properties, aiming at a topological description suited for the index calculations of section 4. A BPS (multi-)monopole is a static configuration of the Yang–Mills–Higgs (YMH) system that due to its topological character has a relation between mass (energy) and magnetic charge. Consider the hamiltonian of the YMH system with gauge group $G$ (the Higgs field is in the adjoint representation):

$$H = \frac{1}{2} \int d^3x \text{Tr}(B_i B_i + D_i \Phi D_i \Phi) = \frac{1}{4} \int d^3x \text{Tr}\left\{(B_i + D_i \Phi)^2 + (B_i - D_i \Phi)^2\right\}.$$  \hspace{1cm} (2.1)

If the Bogomolny equation

$$B_i = \pm D_i \Phi$$  \hspace{1cm} (2.2)

is imposed (note that this equation alone implies that the equations of motion are satisfied), the
energy becomes topological:

\[ H = \pm \int d^3x \text{Tr} \, B_i D_i \Phi = \int_{\mathbb{R}^3} \text{Tr} \, F D \Phi = \int_{\mathbb{R}^3} \text{Tr} \, D (F \Phi) = \int_{S^2_\infty} \text{Tr} \, F \Phi , \quad (2.3) \]

and can be related to the topological magnetic charges of the field configuration (see below).

The topological information of the BPS configuration resides entirely in the asymptotic behaviour of the Higgs field. Let us denote the Higgs field at the two-sphere \( S^2_\infty \) at spatial infinity by \( \phi(x) \). By a gauge transformation, it can always (locally) be brought to an element in the Cartan subalgebra (CSA) of \( g \), the Lie algebra of \( G \), and furthermore, by Weyl reflections, into a fundamental Weyl chamber. The equations of motion then imply that this element is constant on \( S^2_\infty \). We thus have

\[ \psi = g^{-1}(x) \phi(x) g(x) , \quad (2.4) \]

where \( \psi \) is a constant element in the CSA. The group element \( g(x) \) is not globally defined on \( S^2_\infty \), though \( \phi \) and \( \psi \) are. If \( g \) is defined patchwise on the two hemispheres, the difference on the equator is an element in \( H \subset G \), the stability group of \( \phi \). We will only consider the generic case of maximal symmetry breaking, when \( H \) is the maximal torus of \( G \). This occurs as long as the diagonalized Higgs field \( \psi \) does not happen to be orthogonal to any of the roots. \( H = (U(1))^r \) is the unbroken gauge group, where \( r \) is the rank of \( G \). In the light of equation (2.4), the Higgs field on \( S^2_\infty \) may be viewed as a map from \( S^2_\infty \) to the homogeneous space \( G/H \), and all the topological information now lies in the gauge transformation \( g(x) \). The relevant classification is \( \pi_2(G/H) \), which (for semisimple \( G \)) is isomorphic to \( \pi_1(H) \). For the case at hand, this group is \( \mathbb{Z}^r \), i.e. there are \( r \) magnetic charges. It is straightforward to calculate the vector \( k \) of magnetic charges. The gauge transformation (2.4) induces a connection \( \omega = g^{-1}dg \) with field strength \( f = d\omega + \omega^2 = 0 \) locally but not globally (with the two patches defined above, \( f \) has distributional support on the equator), the magnetic charges of which can be expressed as

\[ k \cdot T = \frac{1}{2\pi i} \int_{S^2} f = \frac{1}{2\pi i} \int_{S^1} (\omega_{\text{north}} - \omega_{\text{south}}) \quad (2.5) \]

(the last integral is evaluated at the equator of \( S^2 \) where the two patches of the connection meet). The mass of the configuration is expressed in terms of \( k \) using equation (2.3):

\[ m = \pm \int_{S^2} \text{Tr} \, f \psi = 2\pi |h \cdot k| \quad (2.6) \]

where \( \psi \) is expressed in terms of the vector \( h \) as \( \psi = h \cdot T \in \text{CSA} \). In section 4, we will use the gauge transformation \( g \) in order to calculate indices of Dirac operators in a monopole background, yielding the number of zero-modes of certain fields in the presence of a monopole.

The magnetic charge vector obtained from equation (2.5) lies on the coroot lattice \( \Lambda^\vee_r \) of \( G \). This agrees with the generalized Dirac quantization condition on electric and magnetic charges, that

\[ e \cdot k \in \mathbb{Z} \quad (2.7) \]

for any charge vectors \( e \) and \( k \). Since \( e \) must lie on the weight lattice \( \Lambda_w \) of \( G \), \( k \) must lie on the dual lattice of the weight lattice, i.e. the coroot lattice. We should comment on our choice of
normalization for the magnetic charges. It means that the scale of the coroot lattice is chosen so that the coroots are
\[ \Lambda_\vee \ni \alpha_\vee = 2 \frac{\alpha}{|\alpha|^2}, \]
and coincide with the roots for simply laced groups.

An elegant and convenient way of treating the YMH system in a unified way is to consider the Higgs field as the fourth component of a euclidean four-dimensional gauge connection. We thus let \( A_4 = \Phi \), and demand that no fields depend on \( x^4 \). It is useful to go to a quaternionic formalism, where the gauge connection sits in a quaternion \( A = A_\mu e_\mu \in H \), \( e_4 = 1 \) being the quaternionic unit element and \( e_i, i = 1, 2, 3 \) the imaginary unit quaternions: \( e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \). The Bogomolnyi equation (2.2) now becomes an (anti-)self-duality equation for the field strength \( F_{\mu\nu} \):
\[ F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \]
and the topological character of the solutions becomes even more obvious. We will use the fact that a selfdual antisymmetric tensor can be expressed as an imaginary quaternion, and is formed from two vectors as \( f^+ = \text{Im}(vw^*) \). An anti-selfdual tensor is formed as \( f^- = \text{Im}(v^*w) \). Spinors of both chiralities come as quaternions. The Weyl equations are for the s chirality \( D_* s = 0 \) and for the c chirality \( D_c = 0 \). For a more detailed discussion of the quaternionic formalism, transformation properties etc., see e.g. reference [22].

3. MODULI SPACES AND ZERO-MODES

A monopole solution is not an isolated phenomenon. There are always deformations of the field configuration that do not modify the energy. These always continue to satisfy the Bogomolnyi equation (2.2, 2.9). Deformations of the YMH system alone define tangent directions in the moduli space of monopole solutions at given magnetic charge \( k \). One obvious set of such deformations is given by simply translating the (localized) solution. Therefore, the moduli space always contains a factor \( \mathbb{R}^3 \), but there are in general more possible moduli. Also, when other fields are present, as in the \( N=2 \) models we consider, these may also possess zero-modes in the BPS monopole background. These zero-modes also have to be considered in the low energy treatment we will make.

We will first give a resumé of some of the geometric aspects of the geometry of the moduli spaces (following reference [23], but in the quaternionic formalism of [22]), and then move on to the full \( N=2 \) model.

Suppose we search for a deformation \( \delta A \) of the gauge connection (in a quaternionic form, containing the Higgs field). The linearized version of the Bogomolnyi equation (with the plus sign — the anti-selfdual case is analogous) is \( \text{Im}(D^* \delta A) = 0 \), where the rule for formation of an anti-selfdual tensor from from two vectors has been used. Denote the tangent directions by an index \( m \). The natural metric is induced from the kinetic term in the action,
\[ g_{mn} = \int d^3 x \text{Tr} (\delta_m A_\mu \delta_n A_\mu) = \int d^3 x \text{Tr} \text{Re} (\delta_m A^* \delta_n A) \equiv <\delta_m A, \delta_n A> \]
the deformations $\delta_m A$ are collected in

$$D^* \delta_m A = 0 \ .$$

(3.2)

We note that this equation is formally identical to a Weyl equation for one of the four-dimensional spinor chiralities. It is also straightforward to show that the Weyl equation for the other chirality never can have $L^2$ solutions, simply because the background field strength is selfdual. The dimension of a moduli space at given $k$ can therefore be calculated as the $L^2$ index of the Dirac operator on $\mathbb{R}^3$ in a known BPS background. As we will see, the only essential information that goes into the index calculation is the asymptotic behaviour of the Higgs field. This calculation will yield the complex dimension of the moduli space, provided some selfdual solution with this asymptotic behaviour exists.

All moduli spaces are known to be hyperKähler. The action of the complex structures on the tangent vectors is easily understood. If a tangent vector $\delta_m A$ satisfies equation (3.2), then also $\delta_m A e_i$ satisfy the same equation. The three complex structures act as

$$J^{(i)}_m n \delta_n A = \delta_m A e_i \ .$$

(3.3)

They can be shown to be covariantly constant with respect to the connection derived from (3.1).

A parallel transport in the tangent directions of moduli space on the space of zero-modes should preserve the condition that tangent vectors are orthogonal to gauge modes. In order to achieve this, one introduces the gauge parameters $\varepsilon_m(x)$ and writes

$$\delta_m A = \partial_m A - D\varepsilon_m \ .$$

(3.4)

Parallel transport is generated by the covariant derivative $s_m = \partial_m + \text{ad} \varepsilon_m$ (more generally, $\varepsilon_m$ acts in the appropriate representation of the gauge group), with the property $[s_m, D] = \delta_m A$.

This implies that $D_{A+dt^m \delta_m A} (\varphi + dt^m s_m \varphi) = 0$ for zero-modes in any representation, so that $s_m$ provides a good parallel transport of all zero-modes. It is straightforward to calculate the Christoffel connection of the metric (3.1),

$$\Gamma^{m np} = g^{mq} \int d^3 x \ Tr \delta_q A_p s_n \delta_p A_m = g^{mq} \int d^3 x \ Tr \text{Re} (\delta_q A^* s_n \delta_p A) = g^{mq} <\delta_q A, s_n \delta_p A> \ ,$$

(3.5)

and the riemannian curvature [22],

$$R_{mnpq} = <\delta_p A, [s_m, s_n] \delta_q A> - <s_m \delta_p A, \Pi_+ s_n \delta_q A> - <s_n \delta_p A, \Pi_+ s_m \delta_q A> - 4 P_{+pq}^{rs} <\delta_m A, [s_r, s_s] \delta_q A> \ ,$$

(3.6)

where $\Pi_+ = D(D^*D)^{-1} D^*$ is the projection operator on higher modes and $P_{+pq}^{rs} = \frac{1}{4} J^{(a)}_{[p} J^{(a)}_{q]} J^{(s)}_{r} J^{(t)}_{s}$ is the projection operator on the part of an antisymmetric tensor that commutes with the complex structures, i.e. the $Sp(n)$ part, $4n$ being the real dimension ($J^{(k)}$ is defined as the unit matrix). The curvature is a (1,1)-form with respect to all three complex structures, which is equivalent to $Sp(n)$ holonomy, i.e. "selfduality".
The action for our \( N=2 \) super-Yang–Mills theory with matter is most conveniently formulated as the dimensional reduction of an \( N=1 \) theory in \( D=(1,5) \). The six-dimensional action reads:

\[
L = -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} Re(\lambda^i \Sigma^M D_M \lambda) - \frac{1}{2} D_M q_f^* D^M q_f + \frac{1}{2} Re(\psi_f^i \tilde{\Sigma}^M D_M \psi_f) + Re(\psi_f^i \lambda q_f^*) + \frac{1}{8} (q_f^* \times q_f)^2 .
\] (3.7)

Here, representation indices and traces have been suppressed for clarity. In addition to the gauge potential and its superpartner \( \lambda \) in the adjoint representation, there are the matter bosons \( q \) and fermions \( \psi \). The subscript \( f \) labels the matter multiplets. A dagger denotes quaternionic conjugation and transposition, and, if the representations of \( G \) are complex, also complex conjugation. The matrices \( \Sigma \) and \( \tilde{\Sigma} \) are six-dimensional quaternionic sigma matrices, and the cross product in the last term denotes Clebsh–Gordan coefficients for formation of an element in the adjoint representation. The fermions \( \lambda \) and \( \psi \) are two-component quaternionic spinors of opposite six-dimensional chiralities, and the matter boson \( q \) is a scalar quaternion.

The supersymmetry transformations are:

\[
\begin{align*}
\delta A_M &= Re(\varepsilon^i \Sigma_M \lambda) , \\
\delta \lambda &= -\frac{1}{2} F_{MN} \Sigma^M \varepsilon + \frac{1}{2} \varepsilon (q_f^* \times q_f) , \\
\delta q_f &= \psi_f^i \varepsilon , \\
\delta \psi_f &= \Sigma^M \varepsilon D_M q_f^* .
\end{align*}
\] (3.8)

It is clear from the transformation of \( \lambda \) that a BPS background, obeying (2.2), breaks half the supersymmetry.

The Higgs field comes as one of the components (\( A_4 \), say) of the six-dimensional gauge connection. The euclidean four-dimensional formulation automatically comes out on reduction to four euclidean dimensions, upon which a spinor (of any six-dimensional chirality) splits into a pair of quaternionic spinors of opposite four-dimensional chiralities.

In order to examine which fields carry zero-modes in the BPS background, and go into a low energy expansion, we give the moduli parameters a slow time dependence and expand the equations of motion in the parameter \( n = \#(\frac{d}{dt}) + \frac{1}{2} \#(\text{fermions}) \). At \( n=0 \) one only has the background fields \( A \) with residual field strength. At \( n=\frac{1}{2} \), there are the Weyl equations for the upper (\( s \) chirality) components of \( \lambda \) and \( \psi \), which we denote \( \alpha \) and \( \beta \), respectively. Their lower (\( c \) chirality) components vanish to this order. The time dependence of the bosonic moduli is modeled so that \( A = A(x, X(t)) \). Then the equations at order \( n=1 \) imply, using \( A = \dot{X}^m (\delta_m A + D \varepsilon_m) \),

\[
\begin{align*}
A_0 &= \dot{X}^m \varepsilon_m + (D^* D)^{-1} (-\alpha^* \alpha + \frac{1}{2} \beta^* \times \beta_f) , \\
A_5 &= (D^* D)^{-1} (\alpha^* \alpha + \frac{1}{2} \beta_f^* \times \beta_f) , \\nq_f^* &= -(D^* D)^{-1} (\alpha^* \beta_f) .
\end{align*}
\] (3.9)

We see that the only fields that carry zero-modes, apart from the tangent directions to moduli space itself in the YMH system, are the fermions, both in the vector multiplet and the matter multiplets.

In order to get information about the number of fermionic zero-modes in the BPS background, we have to apply the index theorem of Callias [24] to the appropriate representations of \( \lambda \) (the adjoint) and \( \psi \). We have already seen that the equation for tangent vectors to the moduli space is
equivalent to a Weyl equation, so that the zero-modes of $\lambda$ will come in the tangent bundle over moduli space, whose dimension is given by the index theorem. The zero-modes of $\psi$ will come in some other index bundles with some connections. These connections and their curvatures are derived analogously to the riemannian curvature above. if the mode functions are denoted $\varrho_\alpha$ and normalized so that $\alpha$ is the fiber index of an orthonormal bundle, the connection is

$$\omega_{m\alpha\beta} = \langle \varrho_\alpha, s_m \varrho_\beta \rangle ,$$

and the curvature

$$F_{m\alpha\beta} = \langle \varrho_\alpha, [s_m, s_n] \varrho_\beta \rangle + \langle s_m \varrho_\alpha, \Pi_+ s_n \varrho_\beta \rangle - \langle s_n \varrho_\alpha, \Pi_+ s_m \varrho_\beta \rangle .$$

4. Dimensions of Moduli Spaces and Index Bundles

Callias [24] has given an index theorem for the Dirac operator on $\mathbb{R}^{2n-1}$ in the presence of a gauge connection and a scalar matrix valued hermitian (Higgs) field that takes some nonzero values at spatial infinity. This index theorem is applicable precisely to the situation at hand. The index only depends on the (topological) behaviour of the Higgs field $\Phi$ at infinity. Callias theorem states that the $L^2$ index of the Dirac operator on $\mathbb{R}^3$ in the representation $\varrho$ is given as

$$\text{index} D / \varrho = -\frac{1}{16\pi i} \int_{S^2_\infty} \text{Tr}_\varrho (UdUDU) .$$

Here, the matrix $U$ is defined as $U = (\phi^2)^{-1/2} \phi$. Callias postulates that $\phi$ should have no zero eigenvalues, so that $U$ is well defined. This assumption is directly related to the Dirac operator being Fredholm. If it does not have this property, there is a continuous spectrum around zero that, depending on the behaviour of the density of states, may contribute to the index calculation and give an incorrect result. Actually, in the case we are interested in, there are zero eigenvalues, corresponding to the fields that remain massless after the symmetry breaking. E. Weinberg [25] has shown that the massless vector bosons of the generic maximal symmetry breaking pattern do not contribute in the index calculation. On the other hand, for nonmaximal breaking to a nonabelian group $H$, one has to be more careful, and examine the exact contribution due to the roots orthogonal to the Higgs field. The same is true for some special values of the Higgs field that becomes orthogonal to some weight in a representation for the matter fields (see below). In the generic case, though, all one has to do is to replace the matrix $\phi$ by its restriction to the subspace spanned by the eigenvectors with nonzero eigenvalues. The corresponding restricted Dirac operator will have the desired Fredholm property.

The actual computation of the index is conveniently performed using the gauge transformation $g$ of section 2. After the gauge transformation has been performed, the Higgs field has changed to the diagonalized Higgs field $\psi \in \text{CSA}$, and the derivative simply becomes the commutator with the induced connection $\omega$, since $d\psi = 0$. We thus have

$$\text{index} D / \varrho = -\frac{1}{16\pi i} \int_{S^2_\infty} \text{Tr}_\varrho (V[\omega, V]^2) = \frac{1}{4\pi i} \int_{S^2_\infty} \text{Tr}_\varrho (Vd\omega) ,$$

where $V = (\psi^2)^{-1/2} \psi$, and we have used $V^2 = 1$ and $d\omega = -\omega^2$. Taking the trace in the representation $\rho$ gives the result, using equation (2.5) for the magnetic charge vector,

$$\text{index}\mathcal{D}_\rho = k \cdot \Lambda,$$

$$\Lambda = \frac{1}{2} \sum_{\lambda \in \rho} \lambda \text{sign}(h \cdot \lambda),$$

(4.3)

the sum being performed over the weights of the representation $\rho$. It is clear that the index stays constant as long as $h$ does not become orthogonal to some weight, in which case the index changes discontinuously.

The expression (4.3) enables us to calculate the index explicitly for any magnetic charge and any representation of $G$. We will now turn to some examples that will be of use later. We first define the simple roots with respect to the value of the diagonalized Higgs field in the Cartan subalgebra. The vector $h$ can always be chosen in the fundamental Weyl chamber so that its scalar product with all simple roots is positive. This is illustrated for $SU(3)$ in figure 1.

![Figure 1. The fundamental Weyl chamber and the simple roots for SU(3).](image)

Starting with the adjoint representation, it may be verified that when the magnetic coroot vector is expressed as a linear combination of the simple coroots (with the normalization (2.8)) as

$$k = k_1 \alpha_1^\vee + k_2 \alpha_2^\vee + \ldots + k_r \alpha_r^\vee,$$

(4.4)

the index for the Dirac operator is

$$\text{index}\mathcal{D}_{\text{adj}} = 2 (k_1 + k_2 + \ldots + k_r)$$

(4.5)

for any semisimple Lie group (and maximal symmetry breaking).

In order to translate this result into the complex dimension of a moduli space at magnetic charge $k$, some care has to be taken — it is only true provided that some selfdual configuration with the corresponding asymptotic behaviour of the Higgs field actually exists (or anti-selfdual, so that the dimension is minus the index). The result indicates that the real dimension of a moduli space
for \( k \) a simple coroot is 4. This can be verified — such selfdual solutions exist, and are described by embeddings of the ‘t Hooft–Polyakov \([26,27]\) \( SO(3) \) monopole. We denote these simple monopoles. According to the interpretation of E. Weinberg \([25]\), any multi-monopole at a \( k \) given by \((4.4)\) with only positive coefficients \( k_i \) can in an asymptotic region be approximated by a superposition of well separated simple monopoles, and analogously for anti-monopoles. This agrees with the linearity of the index in \( k \). A magnetic coroot formed as \((4.4)\) with both positive and negative coefficients would asymptotically correspond to a field configuration that is approximately selfdual in some regions and anti-selfdual in others. Such a configuration can not be static, since the magnetic and Higgs forces between a monopole and an anti-monopole do not cancel. Either such configurations do not exist, or they are simply inaccessible to us at our present understanding. This is of course a problem already with gauge group \( SU(2) \), but there it does not manifest itself in terms of allowed and disallowed sectors in the coroot lattice, as it does for higher rank gauge groups, merely as a lack of understanding of the interaction between monopoles and anti-monopoles. If one doubts the above argument, it is illuminating to consider the points in the coroot lattice where the index \((4.5)\) vanishes. Since the dimensionality of a moduli space can not be zero (translations are always moduli), it becomes clear that no static BPS configurations with these magnetic charges can exist. The allowed sectors for magnetic charges in an \( SU(3) \) theory are shown in figure 2, where unfilled roots indicate forbidden magnetic charges.

Another representation of special interest is the fundamental representation of \( SU(N_c) \). Beginning with \( SU(3) \), and ordering the (co)roots by \( h \cdot \alpha_1 > h \cdot \alpha_2 \), the index becomes \( \text{index} \mathcal{D}_{3(SU(3))} = k_1 \). This is the complex dimension of the fiber of the index bundle of zero-modes in the fundamental representation for allowed positive magnetic charges. We note that when the Higgs field aligns with the root \( \alpha_1 + \alpha_2 \) in the middle of the fundamental Weyl chamber, a quark and an antiquark become massless, and the index formula of Callias may give the wrong result. In fact, when \( h \) crosses this line, the zero-mode at \( k = \alpha_1 \) disappears and a new zero-mode instead appears at \( k = \alpha_2 \). The index formula gives a result in between, which clearly is nonsense. The Dirac operator is not Fredholm in the fundamental representation in this case. It is possible, though, to follow the asymptotic behaviour of the solutions to the Dirac equation. For generic \( h \) the normalizable solutions decay exponentially with the radius, while for a degenerate case as this one there is a power law behaviour. One may check that these solutions have a leading term proportional to \( r^{-1/2} \), so they are not \( L^2 \). For this special direction of the Higgs field there are thus no zero-modes in the fundamental representation. A similar situation occurs at the boundary of the fundamental Weyl chamber, where the symmetry breaking pattern changes to \( H = SU(2) \times U(1) \) as some vector bosons become massless. We do not consider this nonmaximal symmetry breaking in this paper.

The indices in the fundamental representations of other \( SU(N_c) \) groups behave in a similar way. We can illustrate by looking at \( SU(4) \), where we have the simple roots \( \alpha_1,2,3 \) with \( \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 2 \), \( \alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -1 \), \( \alpha_1 \cdot \alpha_3 = 0 \). In the interior of the fundamental Weyl chamber the symmetry breaking pattern is the maximal one, \( SU(4) \to U(1) \times U(1) \times U(1) \). At the three planes forming the boundary, \( SU(4) \) is broken to \( SU(2) \times U(1) \times U(1) \), and where the planes intersect to \( SU(2) \times SU(2) \times U(1) \) (one line) or \( SU(3) \times U(1) \) (two lines). The weights in the representation 4, specified by their scalar products with the simple roots, are \( \lambda(1,0,0), \lambda(-1,1,0), \lambda(0,-1,1) \) and \( \lambda(0,0,-1) \). The fundamental Weyl chamber divides in two parts, related by the \( \mathbb{Z}_2 \) of outer automorphisms, and we choose to stay in the region where \( h \cdot \alpha_1 > h \cdot \alpha_3 \). On the boundary there are massless quarks. This also happens when \( h \cdot \lambda(-1,1,0) = 0 \). This plane divides the half fundamental Weyl chamber in two parts. In the region where \( h \cdot \lambda(-1,1,0) > 0 \) the index is \( \text{index} \mathcal{D}_{4(SU(4))} = k_2 \) and when \( h \cdot \lambda(-1,1,0) < 0 \) it is \( \text{index} \mathcal{D}_{4(SU(4))} = k_1 \). Similar statements hold for higher \( SU(N_c) \) groups. The index in the fundamental representation depends only on one of the simple magnetic charges.
The representation 6 of $SU(3)$ is interesting because it is contained in one of the perturbatively finite models. The index is $\text{index}_{6(SU(3))} = 3k_1$, with the same choice of ordering of the roots as above. The number of complex zero-modes for the representations 3, 6 and 8 of $SU(3)$ in the allowed sector of positive $k$ are shown in figure 2 (negative $k$ are obtained by reflection in the origin).

Figure 2. The allowed positive magnetic charges and number of zero-modes in 3, 6 and 8 of $SU(3)$.

5. LATTICES OF ELECTRIC AND MAGNETIC CHARGES — DUALITY?

In this section, we will comment on the possibilities for dual theories from the viewpoint of electric and magnetic charge lattices. In the original Goddard–Nuyts–Olive (GNO) duality conjecture [7], which generalizes the Montonen–Olive conjecture [6] of $SO(3)$ to higher rank gauge groups (both applying to $N = 4$ SYM), it is noted that the magnetic charges lie on the coroot lattice $\Lambda^\vee$, which is the root lattice of the “dual group”, i.e. the group where long and short roots are interchanged. It should be noted, for clarity, that even if the spectrum of electric charges of the elementary excitations of the theory does not span the entire weight lattice (of $SU(N_c)$, say), this does not leave us with more choices for the magnetic charges, as one might suspect from the generalized Dirac condition (2.7). Indeed, the magnetic charges still are constrained to the coroot lattice, as seen from equation (2.5), disregarding of the matter content of the theory. So, in the case of $N = 4$ SYM with gauge group $SU(N_c)$, where all fields come in the adjoint representation, i.e. only in one of the $N_c$ conjugacy classes of the weight lattice, so the actual gauge group is $SU(N_c)/\mathbb{Z}_{N_c}$, the magnetic charges still lie on the coroot lattice of $SU(N_c)$, i.e. on the weight lattice of $SU(N_c)/\mathbb{Z}_{N_c}$. The GNO conjecture states that an $N = 4$ SYM theory is $\mathbb{Z}_2$ dual to the $N = 4$ SYM theory with the dual gauge group. The validity of the conjecture has been partially vindicated by actual calculation of parts of the dyon spectra [15,16,17,18].

When we consider $N = 2$ models with a matter content that makes the theory perturbatively
finite, the GNO interpretation of the coroot lattice must be revised. For example in the $SU(2)$ theory with four fundamental hypermultiplets, the coroot lattice of $SU(2)$ is reinterpreted as the weight lattice of $SU(2)$ instead of the root lattice. This simply amounts to a rescaling by a factor 2. The $\mathbb{Z}_2$ pictures of the quarks now reside at $k = \pm \alpha^\vee$, where in the $N=4$ theory the duals of the massive vector bosons were found. This is of course possible due to the simple fact that the root and weight lattices of $SU(2)$ are isomorphic up to an overall scale. Some of the dyonic states with low magnetic charges have been found, and support the duality hypothesis [2,28,29].

When we move to more general gauge groups, the picture is less clear. As a first example, we have the two perturbatively finite $SU(3)$ theories, one with six hypermultiplets in the fundamental representation, the other with one fundamental multiplet and one in the representation 6. The elementary excitations now carry electric charges in all three conjugacy classes of $SU(3)$, so we want also the magnetic charges to fill out the entire weight lattice of $SU(3)$, if $\mathbb{Z}_2$ duality is supposed to hold. This reinterpretation of the coroot lattice is indeed possible, since the root and weight lattices of $SU(3)$ are isomorphic up to a scale. It is therefore meaningful to examine the actual spectrum of monopoles and dyons in these two models in order to find signs for or against strong–weak coupling duality. As we will see later, the dyon spectra do not support naïve selfduality.

In general, already considering the lattices seems to contradict naïve duality. The coroot lattice, being the root lattice of the dual group, is generically not isomorphic to a weight lattice containing representations that allow matter multiplets in other conjugacy classes than the trivial one. Take $SU(4)$ as an example. The (co)root lattice of magnetic charges is the fcc lattice, while the weight lattice (dual to the (co)root lattice) of electric charges is the bcc lattice. With the GNO interpretation of the coroot lattice, the dual gauge group is $SU(4)/\mathbb{Z}_4$ and there is no room for matter in nontrivial conjugacy classes. The only possible matter content is in the adjoint representation, yielding the $N=4$ theory. One might look for a dyon spectrum that only contains states on some sublattice of the coroot lattice, isomorphic to the relevant part of the weight lattice [30]. Such sublattices exist, but as we will show explicitly (with $SU(4)$ as an example), the dyon spectrum is not confined to such a sublattice. Again, we recognize no signs of selfduality in the dyon spectrum.

Another point, already touched upon in section 4, is that even if the isomorphism between the root and weight lattices for $SU(3)$ is used as above, or if one tries to pick out a sublattice isomorphic to (part of) the weight lattice, one is immediately led to considering states in “forbidden sectors”, asymptotically consisting of superpositions of monopoles and anti-monopoles. Such configurations are not included in the present treatment. Whether this is a fundamental impossibility or an incompleteness of the semi-classical procedure is not clear to us (there might exist non-static configurations that are possible to interpret as bound states of monopoles and anti-monopoles, although it is unclear to us how such states could saturate a Bogomolnyi bound).

6. The Effective Action — Quantization

The procedure we follow in order to find the soliton spectrum of the full quantum field theory is to make a low energy approximation of the theory in a BPS background. Due to the mass gap, corresponding to the Fredholm property of section 4, the number of degrees of freedom in this approximation becomes finite. The field configuration moves adiabatically on the moduli space, and the behaviour of the model is that of a supersymmetric quantum mechanical model with the moduli space as target space. The number of supersymmetries is half the number of
supersymmetries in the original field theory. The reason why this low energy approximation gives reasonable information about the spectrum of the full theory is that if we find BPS saturated states at low energy, these will come in a short multiplet of the $N=2$ supersymmetry algebra, and will necessarily continue to do so at any scale [31]. If the theory is finite, the mass formula of the adiabatic approximation will be exact, while, if the theory is renormalizable, it is renormalized.

In order to find the supersymmetric quantum mechanical model corresponding to the actual theory we are interested in, we only keep the zero-modes of sections 3 and 4 as dynamical variables. Concretely, we derive the low energy action by solving for all fields to order $n=1$ as in (3.9), plug the solutions back into the field theory action (3.7), and keep terms of order $n=2$. We then integrate over three-space, using the expressions for metrics, connections and curvatures of section 3. The resulting lagrangian was calculated in [22], and reads

$$\mathcal{L} = -2\pi h \cdot k + \frac{1}{2}g_{mn}\dot{X}^m \dot{X}^n + \frac{1}{2}g_{mn}\lambda^m D_t \lambda^n + \frac{1}{2}\psi^\alpha D_t \psi^\alpha - \frac{1}{4}F_{mn\alpha\beta}\lambda^m \lambda^n \psi^\alpha \psi^\beta ,$$

(6.1)

Here, we have denoted the fermionic variables, in sections of appropriate bundles over moduli space, with the same letters that were used in the field theory action. The covariant derivatives used on the fermions are defined as $D_t \lambda^m = \dot{\lambda}^m + \Gamma^m_{np}\dot{X}^n \lambda^p$ and $D_t \psi^\alpha = \dot{\psi}^\alpha + \omega^\alpha_\beta \dot{X}^m \psi^\beta$. If one has $N=4$ supersymmetry, also the $\psi$'s come in the tangent bundle, and the field strength $F$ is the riemannian curvature. The lagrangian (6.1) has “$N=\frac{1}{2} \times 4$” supersymmetry, meaning that there are four real supersymmetry generators. They take the form

$$Q(a) = \lambda^m f^{(a)}_m \psi^n V_n ,$$

(6.2)

where $V_n$ is the velocity $g_{mn}\dot{X}^n$. It is essential, and a necessary consequence of the existence of these supersymmetries, that $F$ is selfdual, i.e. a $(1,1)$-form with respect to all three complex structures.

When quantizing the supersymmetric quantum mechanical system given by (6.1), we look for “ground states”, i.e states that continue to saturate a Bogomolnyi bound. These are zero energy states for the system given by the lagrangian without the first term, at least when the electric charge vanishes. The electric charges modify equation (2.6) to

$$m^2 = (h \cdot e)^2 + \left(\frac{2\pi}{g^2} h \cdot k\right)^2 ,$$

(6.3)

where the coupling constant has been reinstated explicitly (this relation follows from the form of the extension of the $N=2$ supersymmetry algebra). Consider the solutions (3.9) to the field equations. We can use them to derive an explicit expression for the electric charge density:

$$D_t E_i = \dot{X}^m D_t \delta_m A_i + \alpha^* \alpha - \frac{1}{2} \beta^* \times \beta .$$

(6.4)

Integrating this over three-space gives a “topological” electric charge from the first term, which is the momentum on the $S^1$ of the moduli space. Here, the contribution to the charge density is $\dot{X}^m D_t D_t \Phi$, and the electric field is proportional to the magnetic field, with the proportionality constant being the velocity on $S^1$, so that electric charges that arise this way are collinear with the magnetic ones. The second term does not contribute. Using $\alpha = \delta_m A^m \lambda^m$, where $\lambda^m$ are real fermionic oscillators, it gives after integration $\lambda^\beta \lambda^\alpha <\delta_m A^\alpha \delta_m A^\beta >= 0$. Using $\beta = \theta_\alpha \psi^\alpha$, the last term becomes $\psi^\alpha \psi^\beta <\theta_\alpha \times \theta_\beta>$. For a complex representation, this may contain an element in the
Cartan subalgebra. A straightforward calculation, using the orthogonality relations for the zero-modes of the fundamental representation of $SU(3)$ and magnetic charge $\alpha_1$, shows that it indeed is $Q\bar{\psi}\psi$, where $Q$ is the $U(1)$ charge of the representation 2 in the decomposition $3 \rightarrow 2_{1/6} \oplus 1_{-1/3}$ under $SU(3) \rightarrow SU(2) \times U(1)$, the $SU(2)$ being defined by $\alpha_1$ as in the following section.

A comment on the mass–charge relation: When we find a quantum mechanical state using the low energy action, we can not expect to find the exact expression for the mass from the corresponding hamiltonian. What we see is a low energy approximation. For the $S^1$ momenta, it gives the first term in the series expansion for low velocity on the circle. For the “orthogonal” charges from the matter fermions, there is no continuous classical analogue, and these electric charges are not seen in the low energy hamiltonian. However, we can deduce from the fact that the states come in short multiplets that they must be BPS saturated.

One has to divide the fermionic variables into creation and annihilation operators. Using the Kähler property, we can take $\lambda^\mu$ as creation operators and $\lambda^\mu$ as annihilation operators, where $\mu$ is a complex index. We then have two equivalent pictures: either the states are forms with anti-holomorphic indices, or we view $\lambda^m$ as gamma matrices as in the quantization of the spinning string, and the states are Dirac spinors. The equivalence is easily seen from a representation point of view — when the full holonomy $SO(4n)$ is reduced to $SU(2n)$, the two spinor chiralities decompose into even and odd forms. Zero energy states are harmonic forms, or spinors satisfying the Dirac equation.

The presence of the $\psi$'s means that the forms/spinors have to be harmonic with respect to the connection $\omega$, and also carry antisymmetric indices coming from the creation operator part of $\psi$ (or a spinor index). In the case of $N=4$, the fermions together come in the complexified tangent bundle, so that ground states are any harmonic forms.

The general pattern is that the part of the $\lambda$'s belonging to the $\mathbb{R}^3 \times S^1$ part of moduli space generates the appropriate number of states of a short multiplet of the space-time supersymmetry algebra. We thus only have to consider the internal space (and only count singlets under the discrete group that is divided out) in order to find the number of multiplets. When the dimension of the internal space is four, one can use the selfduality of the field strength for a vanishing theorem, completely analogous to the one used in space-time: all the solutions to the Dirac equation have to come in only one of the spinor chiralities. This reduces the problem of identifying the ground states to that of calculating the index of the Dirac operator. For higher-dimensional moduli spaces, there is a priori no such vanishing theorem, and it seems like one has to resort to calculating the $L^2$ cohomology, which of course is a much harder problem, of which little seems to be known.

7. Dyon Spectra for Low Magnetic Charges

The moduli spaces for the magnetic charge being any simple coroot is identical to the monopole moduli space in the $SU(2)$ theory. Also, when $k$ is a multiple of a simple coroot, the moduli space is identical to the corresponding $SU(2)$ moduli space. The new ingredient for higher rank gauge groups comes when $k$ is a linear combination of different simple coroots. If $k$ is a linear combination of orthogonal simple coroots, the moduli space factorizes metrically into the product of $SU(2)$ moduli spaces. The only nontrivial example that is accessible so far is the space at $k = \alpha^\vee + \beta^\vee$, where $\alpha^\vee \cdot \beta^\vee < 0$. As is pointed out in [17,18], a very general argument tells us that the isometry group of the inner moduli space has to be $SU(2) \times U(1)$ (the “extra” $U(1)$ isometry is associated with local conservation of the “relative” magnetic charge). The unique
regular hyperKähler manifold with this isometry is Taub–NUT with positive mass parameter, and global considerations (see Appendix A) lead to to the moduli space
\[ \mathcal{M} = \mathbb{R}^3 \times \frac{S^1 \times \text{Taub–NUT}}{\mathbb{Z}_2}. \]

Appendix B contains some basic facts about Taub–NUT space.

We also would like to find explicit expressions for the connections and curvatures of the index bundles associated with the various matter fermions. Starting with the fundamental representation of \( SU(3) \) as a model example, we consider the magnetic charge \( k = \alpha_1 \).

![Figure 3. The representation 3 of SU(3).](image)

It is clarifying to calculate the index for the Dirac operator using a decomposition into \( SU(2) \times U(1) \), where the \( SU(2) \) is defined by the root \( \alpha_1 \). The decomposition of the representation 3 is \( 3 \rightarrow 2_{1/6} \oplus 1_{-1/3} \). Only the 2 of \( SU(2) \) (containing the weights \( \lambda_1 \) and \( \lambda_2 \) of figure 3) has a zero-mode, so that the zero-modes carry a \( U(1) \) electric charge \( 1/6 \). The \( S^1 \) in the moduli space is generated by gauge transformations with (the \( SU(2) \) part of) the Higgs field as gauge parameter. Already when this transformation arrives at the group element \( \exp(\pi i \alpha_1 \cdot T) \), the nontrivial element in the center of \( SU(2) \), it acts as the identity in the adjoint representation. In the fundamental representation of \( SU(2) \), on the other hand, this element acts as minus the identity, which means that the index bundle has a \( \mathbb{Z}_2 \) twist around the \( S^1 \) [32]. This is true also here. If one imposes single-valuedness of the wave function, this implies that there is a correlation between the \( S^1 \) momentum, which is the electric charge in the \( \alpha_1 \) direction and the excitation number of the \( \psi \)'s, carrying electric charge in the direction orthogonal to \( \alpha_1 \). The result of these considerations is that the electric charges are constrained to lie on the weight lattice, which is of course expected. The electric spectrum at \( k = \alpha_1 \) for the theory with six fundamental hypermultiplets is indicated in figure 4. The numbers denote representations under the flavour \( SU(6) \).

The representation 6 is treated similarly. It decomposes as \( 6 \rightarrow 3_{-1/3} \oplus 2_{1/6} \oplus 1_{2/3} \). The 3 carries two zero-modes (in the tangent bundle) and the 2 one. This last zero-mode again has a \( \mathbb{Z}_2 \) twist around \( S^1 \). The electric spectrum at \( k = \alpha_1 \) for the model with one fundamental hypermultiplet and one in the representation 6 is depicted in figure 5, where the number of multiplets at each lattice point is indicated.
At $k = \alpha_2$, there are no matter zero-modes. We just get one multiplet of states at electric charges that are multiples of $\alpha_2$. The same statement holds true in the presence of matter in the representation 6.

At $k = \alpha_1 + \alpha_2$, there is one zero-mode in the fundamental representation. We have to find the connection of the index bundle over the Taub–NUT space. It has to be a $U(1)$ connection with selfdual field strength. It is a well known fact that there exists only one (linearly independent) selfdual harmonic two-form on Taub–NUT space, to which the field strength then has to be proportional (see appendix B). We have to determine the normalization factor $c$ in front of the connection. This can be done by considering the holonomy in the region of moduli space where the monopoles are well separated, i.e. at large $r$. If we move around the circle generated by $\frac{\partial}{\partial \psi}$, the first time we should get back to the original configuration is after completing the whole circle.
Integrating along a curve $C_\gamma : 0 \leq \psi \leq \gamma$ at constant $r$ gives $\int_{C_\gamma} \omega = \gamma c \frac{r-M}{r+M}$. Thus, the smallest value of $\gamma$ for which $\exp(i \int_{C_\gamma} \omega) = 1$ at infinite radius should be $4\pi$. This gives $\int_{C_4} \omega = 2\pi$, and $c = \frac{1}{2}$. We then need to find the index of the Dirac operator for fields of various charges with respect to the $U(1)$ connection. This is completely analogous to the calculation performed in [28,29] for the Atiyah–Hitchin manifold. One can use the Atiyah–Patodi–Singer index theorem and push the boundary to infinity. An additional issue here is that if we want to know the spectrum of the electric charge orthogonal to $\alpha_1 + \alpha_2$, we must investigate how the solutions depend on the coordinate $\psi$. Luckily enough, both the index and the explicit expressions for the mode functions are known [33]. If we call the charge of one creation operator for the matter fermions 1, the states will come with charges $q$ which are the “vacuum charge” plus $n$, where $n$ is the number of creation operators applied. When the number of matter multiplets is even (we consider self-conjugate electric spectra) these charges will be integers. Pope [33] showed that the number of zero-modes of the Dirac operator for positive charge $q$ is $\frac{1}{2}q(q+1)$ and that they depend on the $\psi$ coordinate as $\exp(-\frac{1}{2}i\nu\psi)$, $\nu = 1 \ldots q$, the number of states at each value of $\nu$ being $\nu$, together with an analogous statement for negative $q$. Taub–NUT space is simply connected, so the charges are a priori not restricted by any quantization rule, and the results in [33] contain this more general case. The value of $\nu$ is related to the electric charge in the direction orthogonal to $\alpha_1 + \alpha_2$ by $Q = \nu/6$ with the normalization for the $U(1)$ charge used earlier. The $\mathbb{Z}_2$ identification of the moduli space produces a correlation between $\nu$ and the $S^1$ momentum. The spectrum of electric charges for $k = \alpha_1 + \alpha_2$ in the $SU(3)$ model with six fundamental hypermultiplets is depicted in figure 6, where the numbers indicate $SU(6)$ representations.

![Figure 6](image-url)

Figure 6. The electric spectrum at $k = \alpha_1 + \alpha_2$ for six multiplets in $3$ of $SU(3)$.

It is probably reasonably straightforward to derive the spectrum at $k = \alpha_1 + \alpha_2$ also in the presence of matter in the representation 6. We have not done this.

The results for the $SU(3)$ theory with six fundamental hypermultiplets are summarized in figures 4 and 6, together with the electric spectrum at $k = \alpha_2$, which just consists of one multiplet at any integer multiple of $\alpha_2$. It is also straightforward to extend the dyon spectrum to $k = 2\alpha_1$
and $k = 2\alpha_2$. We have not been able to find the states at the magnetic charges where $\mathbb{Z}_2$ duals of the vector bosons would be expected to reside. These are either outside of the allowed sectors or have eight-dimensional inner moduli spaces, whose metrics are not known. If we examine the electrically uncharged states at the magnetic charges where the dual quarks were expected, we find, instead of six multiplet at each lattice point, twenty, one and zero multiplets at charge $\alpha_1$, $\alpha_2$ and $\alpha_1 + \alpha_2$ respectively.

For theories with $N = 4$, the calculations are simpler. As shown in section 6, there are no fermion contributions to the electric charges, so the electric charge aligns with the magnetic charge. As already mentioned, ground states correspond to any (normalizable) harmonic forms on the internal moduli space. For simple magnetic coroots, the moduli space is $\mathbb{R}^3 \times S^1$, and there is only one short multiplet for electric charges at integer multiples of the corresponding root (note that the integer in Dirac’s quantization condition (2.7) is even). For $k$ at twice a coroot, there is, as demonstrated by Sen [15], a unique selfdual harmonic two-form on the Atiyah–Hitchin manifold, corresponding to one multiplet at $e$ being any odd multiple of the root (the selection of odd multiples comes from the $\mathbb{Z}_2$ divided out in the definition of the moduli space). At $k$ being the sum of two simple coroots with negative scalar product, one has, as noted in [17,18], again the unique selfdual harmonic two-form mentioned earlier, that now gives one multiplet at any integer multiple of the corresponding root. Porrati [16] has presented convincing evidence for the existence of all states predicted by $Sl(2;\mathbb{Z})$ duality for the $N = 4$ $SU(2)$ model. Note that the $Sl(2;\mathbb{Z})$ duals of the massive vector bosons in any $N = 4$ theory always lie in the allowed sectors for the magnetic charges.

When we continue this discussion to higher rank gauge groups, nothing changes in principle. Part of the above discussion applies to moduli spaces at simple coroots or sums of two simple coroots for any gauge group. We have also seen (for the $SU(N_c)$ groups) that the matter in the fundamental representation behaves very similarly to what it does in $SU(3)$. This means that we can not hope to find dyon spectra with magnetic charges confined to a sublattice isomorphic to the weight lattice. There will always be some states at the simple coroots, which will not be in such a sublattice.

8. Conclusions and Outlook

The results of this paper are essentially the following. In spite of the success of the procedure applied here in finding the (low lying) dyon states predicted by $Sl(2;\mathbb{Z})$ duality for the $N = 4$ models and the $N = 2$ $SU(2)$ model with four fundamental hypermultiplets, the picture we see for higher rank gauge groups and matter content making the theory perturbatively finite is much less clear. We have for example not been able to identify the purely magnetically charged states in the quantum theory with the elementary excitations of some “dual” finite $N = 2$ theory. There are also sectors of the magnetic charge lattices that are inaccessible due to our inability of treating systems containing monopoles and anti-monopoles, and this seems to exclude the treatment of states needed for duality. This is no problem for the $N = 4$ theories, since the states needed for duality align with the roots, and are always found in the allowed sectors, but renders the situation problematic for $N = 2$ models with gauge groups of rank $r \geq 2$.

As we see it, there are a couple of possible interpretations of the results of this paper. One is that the procedure in some way is incomplete. We saw that some of the magnetic charges we would need for a duality conjecture lie in forbidden sectors, that would correspond to superpositions of
monopoles and anti-monopoles, something that is not accessible even in the SU(2) models. We do not know how to describe scattering of monopoles and anti-monopoles, unless we move to a dual picture. On the other hand, if such configurations were relevant, they would enter at any magnetic charge, and they would probably modify the successful calculations supporting duality for the finite SU(2) model. We find it unlikely that this could explain any shortcomings. In addition, we have seen that the lattice structures have problems that such a modification hardly could overcome.

A very drastic explanation of the results would be that the theories under consideration are not finite — that there would be instanton corrections to the β function, although one has perturbative finiteness. This sounds very strange and quite unlikely to us, but to our knowledge instanton contributions have not been calculated. On the other hand, the methods of [1, 2] have been applied to the case of SU(N_c) with fundamental matter [34,35,36,37], and these results, support exact finiteness (although some of the statements are conflicting). It should be possible to perform at least a one-instanton calculation in order to verify that these models also are nonperturbatively finite.

A last possibility, which seems most likely, is that there is some kind of modified version of duality that does not include the Z_2 of strong–weak coupling. A consideration that might give a clue is the following. The duality group has been conjectured to be not only SL(2;Z), but Sp(r;Z), where r is the rank of the gauge group. When we examine the dyon spectrum of the N = 4 theories, on the other hand, we only find electric charge vectors aligned with the magnetic ones (this is a direct consequence of the properties of monopole configurations at a multiple of a coroot, being embedded SU(2) monopoles), so that we see only SL(2;Z) pictures of the elementary excitations. When we move to N = 2 theories with higher rank groups, the “off-diagonal” part of the Sp(r;Z) matrices, i.e. the one exchanging electric and magnetic charge, consists of a tensor in Λ∨_r ⊗ Λ∨_r and one in Λ_w ⊗ Λ_w. Of course, in a suitable basis, these just become matrices with integer entries, but when the basis vectors for the two lattices are not aligned (which they in general are not, since the lattices are different) such a basis is not natural, in view of the mass formula (6.3). This means that in general, and even for SU(3), there is no “natural” way of choosing an SL(2;Z) subgroup of Sp(r;Z), where the tensors mentioned above would become diagonal. A supposed Z_2 duality would in turn lie in such an SL(2;Z) subgroup. One might then speculate in some kind of “duality” for higher rank gauge groups that actually does not include a Z_2 of electric–magnetic exchange. We find this issue interesting to pursue. In connection it is also worth mentioning that peculiar lattice properties of the charges in higher rank gauge groups have been found earlier. In reference [38], the existence of simultaneously massless dyons with nonvanishing Sp(r;Z) product was demonstrated (for gauge group SU(3)), so that there should exist vacua where elementary excitations couple both electrically and magnetically to the gauge field. The evidence points towards a quite rich and interesting structure of these theories.

In conclusion, the results of this paper, rather than giving definite answers, raises a number of questions we find it urgent to investigate.

Note added: After correspondence with the authors of reference [37], we realize that for gauge groups of rank 2, and only then, there is a “natural” Z_2 transformation, namely where the above mentioned tensors are the “epsilon tensors” α'_γ ⊗ α'_δ - α'_δ ⊗ α'_γ and λ_1 ⊗ λ_2 - λ_2 ⊗ λ_1. In an orthonormal basis these become antisymmetric matrices, and do not depend on the choice of simple coroots or weights. Since they relate two different vector spaces, they can be thought of as unit matrices. Such a transformation maps the electric charges of the fundamental representation of SU(3) on coroots, so we do not find support for duality under this Z_2 group. In [37], subgroups
of $Sp(r;\mathbb{Z})$ are considered that preserve the scalar products between roots of $SU(N_c)$ (up to a scale), so that the transformation of the “coupling matrix” only consists of a transformation of the complex coupling constant. We hope to return to a closer examination of subgroups that might explain parts of the spectrum we observe (though it is difficult to conceive how the entire spectra could be generated). Our attention has also been drawn to reference [39], where some of the arguments and results are very close to ours.

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Appendix A: Topology of the Moduli Space at $k = \alpha_1 + \alpha_2$

As we have seen, the moduli space at $k = \alpha_1 + \alpha_2$ is actually completely determined just by considering its isometries together with the hyperKähler property. In this appendix, we will use the correspondence between moduli spaces and spaces of rational holomorphic maps to get direct information about the topology of this space, and support the indirect arguments. This procedure could in principle be continued along the lines of [40] to obtain also the metric.

For $SU(2)$ monopoles, there is an isomorphism between the moduli space at charge $k$ and the space of rational holomorphic maps $S^2 \to S^2$, due to Donaldson [41]. This result was extended to more general groups by Hurtubise [42], where the case of maximal breaking was considered, and the moduli spaces shown to be isomorphic to spaces of rational holomorphic maps from $S^2$ to $G/H$ ("the broken gauge group"). The target space of the holomorphic map is a "flag manifold", i.e. a space of nested vector subspaces $\mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^N$. This makes it quite straightforward to write down explicit coordinates for these manifolds as $\mathbb{C}P^1$ bundles over $\mathbb{C}P^2$ bundles over $\ldots$ over $\mathbb{C}P^{N-1}$.

We will examine the case of $SU(3)/(U(1) \times U(1))$, i.e. the manifold of complex lines in a complex plane in $\mathbb{C}^3$. This clearly implies an $S^2$ bundle over $\mathbb{C}P^2$. Explicit parametrization of the plane and the line, and some minor redefinition in order to make things as symmetric as possible, gives the coordinates $(x_i, y_i; \zeta_i)$ in patch $i$, $i = 1, 2, 3$, with the overlaps

$$(x_{i+1}, y_{i+1}; \zeta_{i+1}) = (y^{-1}_i, x_i y^{-1}_i; -\alpha x_i - \alpha^{-1} y_i \zeta^{-1}_i) ,$$

(A.1)

where $\alpha = e^{2\pi i/3}$ and $3+1$ is understood as 1. Here, the $\zeta$ coordinates are fiber coordinates for $S^2$. We have not cared to write two separate patches for the fiber, since the one-point compactification of $\mathbb{C}$ is trivial. The coordinates for the base manifold are the standard ones on $\mathbb{C}P^2$. If one instead considers the flag "turned inside out", i.e. consider the complementary (normal) vector subspaces, one is led to an alternative fibration, given by the coordinate transformations

$$(\tilde{x}_i, \tilde{y}_i; \tilde{\zeta}_i) = (\zeta_{i+1}^{-1}, \zeta_{i+2}; y_{i+1}) .$$

(A.2)

These coordinates have identical overlap relations as the original ones. The transformation corresponds to the action of the nontrivial element in the $\mathbb{Z}_2$ of outer automorphisms of $SU(3)$.

We now want to examine some simple rational holomorphic maps from $S^2$ to this manifold. These maps should be "based". We choose the base point condition $(x, y, \zeta)(\infty) = (1, 1; 1)$, which is the same in all patches. It is easy to find a basis for the second homotopy. The fiber $S^2$ of course has second homotopy $\mathbb{Z}$, and so has the base manifold $\mathbb{C}P^2$, being $S^5/U(1)$. The holomorphic maps corresponding to one winding on the fiber, i.e. one of the simple magnetic charges, say $\alpha_1$, are easily written down:

$$
\begin{pmatrix}
  x_1 & y_1 & \zeta_1 \\
  x_2 & y_2 & \zeta_2 \\
  x_3 & y_3 & \zeta_3 \\
\end{pmatrix}
( z ) =
\begin{pmatrix}
  1 & 1 & \frac{z + A}{z + B} \\
  1 & 1 & \frac{z + B}{z + C} \\
  1 & 1 & \frac{z + C}{z + A}
\end{pmatrix}
,$$

(A.3)

where $A + \alpha B + \alpha^2 C = 0$. The easiest way of finding the maps corresponding to the other simple root $\alpha_2$ is to apply the coordinate transformation (A.2) to the right hand side of (A.3) to obtain

$$
\begin{pmatrix}
  x_1 & y_1 & \zeta_1 \\
  x_2 & y_2 & \zeta_2 \\
  x_3 & y_3 & \zeta_3 \\
\end{pmatrix}
( z ) =
\begin{pmatrix}
  \frac{z + A}{z + C} & \frac{z + B}{z + C} & \frac{z + A}{z + B} \\
  \frac{z + B}{z + A} & \frac{z + C}{z + A} & \frac{z + B}{z + C} \\
  \frac{z + C}{z + B} & \frac{z + A}{z + B} & \frac{z + C}{z + A}
\end{pmatrix}
( z ) .
$$

(A.4)
The corresponding monopoles are the embedded 't Hooft–Polyakov solutions, and it is easy to
deduce that the topology of these moduli spaces is $\mathbb{C} \times \mathbb{C}^* \cong \mathbb{R}^3 \times S^1$. A more interesting case is
the magnetic charge $\alpha_1 + \alpha_2$. This map winds once around each of the primitive cycles. We write
down the most general ansatz possible, and then derive constraints on the parameters that enter:

$$
\begin{pmatrix}
x_1 & y_1 & \zeta_1 \\
x_2 & y_2 & \zeta_2 \\
x_3 & y_3 & \zeta_3 
\end{pmatrix}
(\zeta) = \begin{pmatrix}
\frac{2z + A}{z + A} & \frac{2z + B}{z + A} & \frac{2z + D}{z + E} \\
\frac{2z + C}{z + B} & \frac{2z + C}{z + B} & \frac{2z + C}{z + E} \\
\frac{2z + A}{z + A} & \frac{2z + A}{z + A} & \frac{2z + A}{z + A}
\end{pmatrix}.
$$
(A.5)

The outer automorphisms act as $(A, B, C) \mapsto (D, E, F)$. Using the overlap functions we arrive at
the constraints between the six complex parameters:

$$
A + D + \alpha(B + E) + \alpha^2(C + F) = 0,
$$
$$
AD + \alpha BE + \alpha^2 CF = 0,
$$
(A.6)

so that we arrive at the counting of section 4 for the dimension of this moduli space — it has real
dimension eight.

When we investigate the topology, it is useful to consider holomorphic vector fields on the flag
manifold. Some of these will generate holomorphic isometries on the moduli space. The regular
vector fields we consider take the same form in all three patches (they are the only ones with this
property):

$$
\begin{aligned}
\mathcal{Y}^{(1)} &= (1 - xy) \frac{\partial}{\partial x} + (x - y^2) \frac{\partial}{\partial y} - (\alpha + y\zeta + \alpha^{-1}x\zeta^2) \frac{\partial}{\partial \zeta}, \\
\mathcal{Y}^{(2)} &= (y - x^2) \frac{\partial}{\partial x} + (1 - xy) \frac{\partial}{\partial y} + (\alpha y + x\zeta + \alpha^{-1}\zeta^2) \frac{\partial}{\partial \zeta}.
\end{aligned}
$$
(A.7)

There are also the translations on $S^2$, inducing the vector field $\mathcal{Y}^{(3)} = x'(z) \frac{\partial}{\partial x} + y'(z) \frac{\partial}{\partial y} + \zeta'(z) \frac{\partial}{\partial \zeta}$.
All of these transformations commute. The transformations induce transformations of the parameters $A, \ldots, F$. These are better expressed in a basis where the vector fields act diagonally,

$$
\begin{aligned}
a &= A + B + C, & d &= D + E + F, \\
b &= A + \alpha B + \alpha^2 C, & e &= D + \alpha E + \alpha^2 F, \\
c &= A + \alpha^2 B + \alpha C, & f &= D + \alpha^2 E + \alpha F.
\end{aligned}
$$
(A.8)

Then $\mathcal{Y}^{(3)}$ only acts on $a$ and $d$ as translation, while, if we denote the induced action of $\frac{i}{\sqrt{3}}(\alpha \mathcal{Y}^{(1)} - \alpha^{-1} \mathcal{Y}^{(2)})$ by $\delta_+$ and that of $\frac{i}{\sqrt{3}}(\alpha \mathcal{Y}^{(1)} + \alpha^{-1} \mathcal{Y}^{(2)})$ by $\delta_-$, the action on the moduli parameters is

$$
\begin{aligned}
\delta_+ b &= 2b, & \delta_+ e &= 2e, \\
\delta_+ c &= c, & \delta_+ f &= f, \\
\delta_- b &= 0, & \delta_- e &= 0, \\
\delta_- c &= c, & \delta_- f &= -f,
\end{aligned}
$$
(A.9)

while $a$ and $d$ are inert. The constraints are

$$
\begin{aligned}
b + e &= 0, \\
\alpha c + \alpha d + c f &= 0.
\end{aligned}
$$
(A.10)
and they are preserved by all the transformations. The transformation $\delta_+$ generates the $\mathbb{C}^*$ that together with the $\mathbb{C}$ of $\gamma^{(3)}$ forms $\mathbb{R}^3 \times S^1$. The imaginary part of $\delta_-$ is a $U(1)$ isometry. We can choose a location $\theta$ on the $S^1$ by a finite action $\text{exp}(i\theta \text{Im}\delta_+)$ on some given base point. The parameters $c$ and $f$ are coordinates for the “inner part” of the moduli space. By considering the action of this translation on the total moduli space, we conclude that the topology is

$$ \mathcal{M} \cong \mathbb{R}^3 \times \frac{S^1 \times \mathbb{R}^4}{\mathbb{Z}_2}. $$

(A.11)

The “inner” or “relative” moduli space is topologically $\mathbb{R}^4$. This is the topology of Taub–NUT space with positive mass parameter.

**APPENDIX B: TAUB–NUT SPACE — METRIC AND CONNECTIONS**

This appendix contains a short summary about Taub–NUT space (see e.g. reference [43] for more detailed discussions). Taub–NUT space is a member of a very restricted family of four-dimensional regular hyperKähler manifolds with $SO(3)$ isometry [43], that also includes the Atiyah–Hitchin manifold (contained in the moduli space for magnetic charge twice a simple coroot), and the Eguchi–Hansson manifold. The properties obtained from simple physical considerations, that the metric asymptotically approaches $\mathbb{R}^3 \times S^1$ and that the isometry is $SU(2) \times U(1)$, singles out Taub–NUT as the internal moduli space for magnetic charges that are the sum of two simple coroots with negative scalar product.

The metric may be written

$$ g = \frac{r + M}{r - M} \, dr \otimes dr + (r^2 - M^2)(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + 4M^2 \frac{r - M}{r + M} \, \sigma_3 \otimes \sigma_3, \quad (B.1) $$

where the ranges of the coordinates are $M \leq r$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 4\pi$, and the $\sigma_i$ are left-invariant one-forms on $S^3 \cong SU(2)$:

$$ \begin{align*}
\sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\sigma_2 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \\
\sigma_3 &= d\psi + \cos \theta d\phi,
\end{align*} \quad (B.2) $$

with the dual vector fields $v_i$, $v_i(\sigma_j) = \delta_{ij}$:

$$ \begin{align*}
v_1 &= \cos \psi \frac{\partial}{\partial \theta} + \sin \psi \frac{\partial}{\sin \theta \partial \phi} - \cot \theta \sin \psi \frac{\partial}{\partial \psi}, \\
v_2 &= -\sin \psi \frac{\partial}{\partial \theta} + \cos \psi \frac{\partial}{\sin \theta \partial \phi} - \cot \theta \cos \psi \frac{\partial}{\partial \psi}, \\
v_3 &= \frac{\partial}{\partial \psi}.
\end{align*} \quad (B.3) $$

If we write the vierbein one-forms as $e_r = f \, dr$, $e_i = c_i \sigma_i$, the functions $f$, $c_i$ satisfy (prime denotes differentiation with respect to $r$)

$$ \frac{c_1'}{f} = \frac{c_2^2 - (c_2 - c_3)^2}{2c_2c_3} \quad \text{and cyclic.} \quad (B.4) $$
This equation enables us to calculate the curvature quite easily:

\[ R_{0i} = \frac{1}{2} \varepsilon_{ijk} R_{jk} \]
\[ R_{01} = -k'_1 dr \wedge \sigma_1 + (-k_1 + k_2 + k_3 - 2k_2k_3) \sigma_2 \wedge \sigma_3 \quad \text{and cyclic}, \]

where \( k_i = \frac{\partial}{\partial x_i} \). The first equation states that \( R \) is selfdual. The curvature may then be used in the calculation of the index of the Dirac operator \([33]\), using the Atiyah–Patodi–Singer index theorem \([44]\) and pushing the boundary to infinite radius.

When we consider matter zero-modes, we will need a U(1) connection on Taub–NUT space with selfdual field strength. There is exactly one selfdual harmonic two-form (up to normalization). It is

\[ F = c \left( \frac{2M}{(r+M)^2} dr \wedge \sigma_3 - \frac{r-M}{r+M} \sigma_1 \wedge \sigma_2 \right). \quad (8.1) \]

The corresponding potential is

\[ \omega = c \frac{r-M}{r+M} \sigma_3. \quad (8.2) \]

The coefficient \( c \) is determined by physical considerations.