A Feasibility Look to Two-stage Robust Optimization in Kidney Exchange

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Abstract

Kidney paired donation programs (KPDPs) are a way forward for patients who have a willing donor that is not a match for them. KPDPs match these willing donors to other patients with an assurance that their intended recipient will receive a kidney in return from a different donor. A patient and donor join a KPDP as a pair, known in the literature as patient-donor pairs. Pairs can then become vertices in a compatibility graph, where arcs represent compatible kidneys flowing from a donor in one pair to a patient in another. A challenge faced in real-world KPDPs is the possibility of a planned match being cancelled, e.g., due to late detection of organ incompatibility or patient-donor dropout. We therefore develop a two-stage robust optimization approach to the kidney exchange problem wherein (1) the first stage determines a kidney matching solution according to the original compatibility graph, and then (2) the second stage repairs the solution after observing transplant cancellations. We expand on the state of the art, which only considers homogeneous failure, by being the first to consider the homogeneous and non-homogeneous failure rate between vertices and arcs. To accomplish this, we develop solution algorithms with a feasibility-seeking master problem. Current solution methods are tied to the recourse policy in consideration. Instead, our solution framework solves in finitely many iterations the two-stage robust problem for recourse policies whose recourse solution set consists of a subset of the feasible cycles and chains of the compatibility graph, that contain at least one pair selected in the first stage. We tested our approach on publicly available instances and compared it to a state-of-the-art algorithm under homogeneous failure; the results show the superior performance of our methodology. Moreover, we provide insights on the scalability of our solution algorithms under non-homogeneous failure for two recourse policies and analyze their impact on highly-sensitized patients.
1 Introduction

Kidney paired donation programs (KPDPs) across the globe have facilitated living donor kidney transplants for patients experiencing renal failure who have a willing yet incompatible or sub-optimal donor. A patient in need of a kidney transplant registers for a KPDP with their incompatible donor (paired donor) as a pair. The patient then receives a compatible kidney from either the paired donor in another pair, who in turn receives a kidney from another donor, or from a singleton donor who does not expect a kidney in return (a.k.a. non-directed donor). The transplants are then made possible through the exchange of paired donors between patient-donor pairs. Since first discussed by Rapaport (1986), and put in practice for the first time in South Korea (Park et al., 1999), kidney exchanges performed through KPDPs have been introduced in several countries around the world, e.g., the United States (Saidman et al., 2006), the United Kingdom (Manlove and O’malley, 2015), Canada (?) and Australia (Cantwell et al., 2015), and the underlying matching of patients to donors has been the subject of study from multiple disciplines (see, e.g., Roth et al. (2005); Dickerson et al. (2016, 2019); Carvalho et al. (2020); Riascos-Álvarez et al. (2020)).

Despite this attention, KPDPs still face challenges from both a practical and theoretical point of view (Ashlagi and Roth, 2021). In this work, motivated by the high rate of exchanges that do not proceed to transplant (Bray et al., 2015; Dickerson et al., 2016; CBS, 2019), we provide a robust optimization framework that proposes a set of exchanges for transplant, observes failures in the transplant plan, and then repairs the affected exchanges provided that a recovery rule (i.e., recourse policy) is given by KPDP operators.

Kidney exchange, or the kidney exchange problem as it is known in the literature, can be modeled with a compatibility graph, (i.e., a digraph). Each vertex represents either a pair or a non-directed donor, and each arc indicates that the donor in the starting vertex is blood type and tissue type compatible with the patient in the ending vertex. Arcs may have an associated weight that represents the priority of that transplant. Exchanges then take the form of simple cycles and simple paths (a.k.a. chains).

Cycles consist of patient-donor pair exchanges only, whereas in a chain the first donation occurs from a non-directed donor to a patient-donor pair, and is followed by a sequence of donations from a paired donor to the patient in the next patient-donor pair. Since a donor/patient can withdraw from a KPDP at any time or late-detected medical issues can prevent a paired donor from donating in the future, cyclic transplants are performed simultaneously. Unlike cycles, a patient in a chain does not risk exchanging his/her paired donor without first receiving a kidney from another donor. Therefore, transplants in a chain can be executed sequentially, but depending on KPDP regulations they can also be performed simultaneously (Biró et al., 2021). Due to the simultaneity constraint for cycles, the maximum number of transplants in a cycle is limited by the logistical implications of arranging operating rooms and surgical teams. The maximum number of transplants in a chain can vary depending on whether the simultaneity constraint is relaxed and based on specific regulations. In the United States, the size of chains can be unbounded and theoretically infinite (Ashlagi et al., 2012; Dickerson et al., 2012b; Anderson et al., 2015; Ding et al., 2018; Dickerson et al., 2019) by allowing the donor in the last pair of a chain to become a bridge donor, i.e., a paired donor that acts as a non-directed donor in a future
algorithmic matching. However, in Canada (Malik and Cole, 2014) and in Europe (Biró et al., 2021; Carvalho et al., 2020) chains are limited in size since their KPDPs do not work with bridge donors. In this case, the donor in the last pair donates to a patient on the deceased donor’s waiting list, “closing up” a chain.

Operational and technological limitations cause uncertainty about the existence of vertices and arcs in the compatibility graph, and may be marked as infeasible when the exchanges go through a final checking process before the nephrectomies take place. Permanent or temporary withdrawals can also happen at any stage of the kidney exchange process. The availability of a patient or donor, just like the actual compatibility of an exchange, is confirmed only when a set of transplants has already been selected. This phenomenon is explained by changes/lack of accuracy on the “believed” information KPDP operators rely on when building the compatibility graph. There are multiple reasons a patient-donor pair or a non-directed donor selected for transplant may not be available, e.g., match offer rejection, already transplanted out of the KPDP, illness, pregnancy, reneging, etc. Thus, even if some of this information is captured prior to the matching process, there is still a chance of subsequent fallback. Additionally, tissue type compatibility is not known with certainty when the compatibility graph is built, unlike blood type compatibility. Tissue type compatibility is based on the result of a virtual crossmatch test, which typically has lower accuracy than a physical crossmatch test. Both tests try to determine if there are antibodies of importance in a patient that can lead to the rejection of a potential donor’s kidney. Physical crossmatch tests are challenging to perform, making them impractical, and thus unlikely to be performed between all patients and donors in real life (Carvalho et al., 2020). So, in the first stage, a “believed” compatibility graph is built according to the results of the virtual crossmatch test. Once the set of transplants has been proposed, those exchanges undergo a physical crossmatch test to confirm or rule out the viability of the transplants. After confirming infeasible transplants and depending on the KPDPs’ regulations, KPDP operators may attempt to repair the originally planned cycles and chains impacted by the non-existence of a vertex or an arc. We refer to these impacted cycles and chains as failed cycles and chains. A cycle fails completely if any of its elements (vertices or arcs) cease to exist, whereas a chain is cut short at the pair preceding the first failed transplant.

Failures in the graph, caused by the disappearance of a vertex or arc have a significant impact on the number of exchanges that actually proceed to transplant (Dickerson et al., 2019; CBS, 2019) and can even drive KPDP regulations (Carvalho et al., 2020). For instance, Dickerson et al. (2019) reported that for selected transplants from the UNOS program between 2010–2012, 93% did not proceed to transplant. Of those non-successful transplants, 44% had a failure reason (e.g., failed physical crossmatch result or patient/donor drop-out), which in turn caused the cancellation of the other 49% of non-successful transplants. In Canada, between 2009–2018, 62% of cycles and chains with six transplants failed, among which only 10% could be repaired. Half the cycles and chains with three or fewer transplants were successful, and approximately 30% of the total could not proceed to transplant (CBS, 2019).

The set of transplants that is proposed and later repaired by KPDPs does not account for subsequent failures, and thus the number of successful transplants performed is sub-optimal. There is a large body of work concerned with maximizing the expectation of proposed transplants (Awasthi and Sandholm, 2009; Dickerson
et al., 2014, 2019; Klimentova et al., 2016; Smeulders et al., 2022). From a practical perspective, maximizing expectation could increase the number of planned exchanges that become actual transplants. However, such a policy risks favoring patients that are likely to be compatible with multiple donors over highly-sensitized patients (Carvalho et al., 2020), i.e., patients for whom few kidney donors are available and whose associated exchanges tend to fail at a higher rate than non-sensitized patients. Furthermore, real data is limited and there is not currently a high enough understanding of the dynamics between patients and donors to derive a probability distribution of failures that could be generalized to most KPDPs. With this in mind, we model failure through an uncertainty set that does not target patient sensitization level or probabilistic knowledge, and aims to find a set of transplants that allows the biggest recovery under the worst-case failure scenario in the uncertainty set.

We develop a two-stage robust optimization (RO) approach to the kidney exchange problem wherein (1) the first stage determines a kidney matching solution according to the original compatibility graph, and then (2) the second stage repairs the solution after observing transplant cancellations. We extend the current state-of-the-art RO methodologies (Carvalho et al., 2020) by assuming that the failure rate of vertices and arcs is non-homogeneous—since failure reasons, such as a late-detected incompatibility, seem to be independent from patient/donor fallback and vice versa—and also considering the impact of scarce match possibilities for highly-sensitized patients. The contributions of this work are as follows:

1. We study a two-stage RO framework with non-homogeneous failure between vertices and arcs.
2. We present a novel general solution framework for any recourse policy whose recourse solution set is finite after selection of a first-stage set of transplants.
3. We are the first to introduce two feasibility-seeking reformulations of the second stage as opposed to optimality-based formulations (Carvalho et al., 2020; Blom et al., 2021). The number of decision variables grows linearly with the number of vertices and arcs. However, the price of this greater scalability, is the lack of a lower bound. We derive dominating scenarios and explore several second-stage solution algorithms to overcome the drawbacks and exploit the advantages.
4. We compare our framework to state-of-the-art algorithms, and show that ours results in significant computational and solution quality improvements.

The remainder of the paper is organized as follows. Section 2 presents a collection of related works. Section 3 establishes the problem we address. Sections 4 and 5 present the first and second-stage formulations, respectively. Section 6 presents the full algorithmic framework. Section 7 shows computational results. Lastly, Section 8 draws some conclusions and states a path for future work.

2 Related Work

Abraham et al. (2007) and Roth et al. (2007) introduced the first KPDP mixed-integer programming (MIP) formulations for maximizing the number/weighted sum of exchanges. These are the well-known edge formulation
and cycle formulation. The edge formulation uses arcs in the input graph to index decision variable; whereas the cycle formulation, which was initially proposed for cycles only, has a decision variable for every feasible cycle and chain in the input graph. Both formulations are of exponential size either in the number of constraints (edge formulation) or in the number of decision variables (cycle formulation). However, the cycle formulation, along with a subsequent formulation proposed by Dickerson et al. (2016), is the MIP formulation with the strongest linear relaxation. Due to its strength and natural adaptability, multiple works have designed branch-and-price algorithms employing the cycle formulation. The branch-and-price algorithm proposed in (Abraham et al., 2007) was effective for cycles of size up to three, while that of Lam and Mak-Hau (2020) solved the problem for long cycles, and Riascos-Álvarez et al. (2020) used decision diagrams to solve large instances with both long cycles and long chains for the first time. More recently, Omer et al. (2022) built on the work in (Riascos-Álvarez et al., 2020) by implementing a branch-and-price able to solve remarkably large instances (10000 pairs and 1000 altruists), opening the door to large-scale multi-hospital and multi-country efforts. Another trend has focused on new arc-based formulations (e.g., Constantino et al. (2013); Dickerson et al. (2016)) and arc-and-cycle-based formulations (e.g., Anderson et al. (2015); Dickerson et al. (2016)). Between these two approaches, arc-and-cycle-based formulations seem to outperform arc-based formulations (Dickerson et al., 2016), especially for instances with cycles with at most three exchanges.

The previously discussed studies do not consider uncertainty in the proposed exchanges. However, the high percentage of planned transplants that end up cancelled suggests a need to plan for uncertainty. There are two sources of uncertainty that have been studied in the literature: weight accuracy (e.g., McElfresh et al. (2019)) and vertex/arc existence (e.g., Dickerson et al. (2016); Klimentova et al. (2016); McElfresh et al. (2019); Carvalho et al. (2020); Smeulders et al. (2022)). Weight accuracy uncertainty assumes that the social benefit (weight) associated with an exchange can vary, e.g., due to changes in a patient’s health condition, and from the existence of multiple opinions from policy makers on the priority that should be given to each patient (McElfresh et al., 2019). Uncertainty in the existence of a vertex/arc, i.e., whether or not a patient or donor leaves the exchange or compatibility between a patient and donor changes, has received greater attention. There are three main approaches in the literature addressing vertex or arc existence as the source of uncertainty: (1) a maximum expected value approach; (2) an identification of exchanges for which a physical crossmatch test should be performed to maximize the expected number of realized transplants; and (3) a maximization of the number of transplants under the worst-case disruption of vertices and arcs. These are explained in detail in the following paragraphs.

The maximum expected value approach is the approach most investigated in the literature. It is concerned with finding the set of transplants with maximum expected value, i.e., a set of transplants that is most likely to yield either the maximum number of exchanges, or the maximum weighted sum of exchanges given some vertex/arc failure probabilities. This approach has mostly been modeled as a deterministic kidney exchange problem (KEP), where the objective function approximates the expected value of a matching using the given probabilities as objective coefficient multipliers of deterministic decisions.

Awasthi and Sandholm (2009) considered the failure of vertices in an online setting of the cycle-only ver-
sion for cycles with at most three exchanges. The authors generate sample trajectories on the arrival of patients/donors and patients survival, then use a REGRETS algorithm as a general framework to approximate the collection of cycles with maximum expectation. Dickerson et al. (2012a) proposed a heuristic method to learn the “potential” of structural elements (e.g., vertex), that quantifies the future expected usefulness of that element in a changing graph with new patient/donor arrivals and departures. Dickerson et al. (2013) considered arc failure probabilities and found a matching with maximum expected value, but solution repairs for failures are not considered. This work is extended in (Dickerson et al., 2019). Klimentova et al. (2016) studied the problem of computing the expected number of transplants for the cycle-only version while considering internal recourse and subset recourse to recover a solution in case of vertex or arc failure. Internal recourse, also known as back-arcs recourse (e.g., Carvalho et al. (2020)) allows surviving pairs to match among themselves, whereas subset recourse allows a wider subset of vertices to participate in the repaired solution. To compute the expectation, an enumeration tree is used for all possible failure patterns in a cycle and its extended subset. This subset consists of the additional vertices (for the subset recourse only) such that the pairs in the original cycle can form feasible cycles. To limit the size of the tree, the subset recourse is limited to a small subset of extra vertices and the internal recourse seems to scale for short cycles only. Alvelos et al. (2019) proposed to find the expected value for the cycle-only version while considering internal recourse through a branch-and-price algorithm, and they found that the overall run time grew rapidly with the size of the cycles.

To identify exchanges where a physical crossmatch test should be performed, Blum et al. (2013) modeled the KEP in an undirected graph representing pairwise exchanges only. They proposed to perform two physical crossmatch tests per patient-donor pair—one for every arc in a cycle of size two—before exchanges are selected with the goal of maximizing the expected number of transplants. They showed that their algorithm yields near-optimal solutions in polynomial time. Subsequent works (Assadi et al., 2019; Blum et al., 2020) evaluated adaptive and non-adaptive policies to query edges in the graph. In the same spirit, but for the general kidney exchange problem (with directed cycles and chains), Smeulders et al. (2022) formulated the maximization of the expected number of transplants as a two-stage stochastic integer programming problem with a limited budget on the number of arcs that can be tested in the first stage. Different algorithmic approaches are proposed, but scalability was a challenge.

In addressing worst-case vertex/arc disruption, McElfresh et al. (2019) found robust solutions with no recourse for budget failure of the number of arcs that fail in the graph. Carvalho et al. (2020) proposed a two-stage robust optimization model that allowed recovery of failed solutions through the back-arcs recourse and full recourse policies. The latter can be seen as a subset recourse policy (e.g., Klimentova et al. (2016)), in which all vertices that were not selected in the matching can be included in the repaired solution. Unlike our work, vertex and arc failure are treated as homogeneous, i.e., both elements fail with the same probability. Since in homogeneous failure there is a worst-case scenario in which all failures are vertex failures, the recourse policies are evaluated under vertex failure only. The back-arcs recourse policy only scales for instances with 20 vertices, whereas the full-recourse policy scales for instances up to 50 vertices. Blom et al. (2021) examined the general robust model for the full-recourse policy studied in Carvalho et al. (2020) and showed its structure
to be a defender-attacker-defender model. Two Benders-type approaches are proposed and tested using the same instances from Carvalho et al. (2020). The Benders-type approaches showed improved performance over the branch-and-bound algorithm proposed by Carvalho et al. (2020). This approach, however, is limited to homogeneous failure for the full-recourse policy. In this work, we allow for different failure rates between vertices and arcs, and present a solution scheme that can address recourse policies whose recourse solution set corresponds to a subset of the feasible cycles and chains in the compatibility graph, such as the full recourse, the back-arcs recourse and a new recourse policy we introduce later. Our solution method does require a robust MIP formulation adapted to a specific policy that can be solved iteratively as new failure scenarios are added. The second-stage problem is decomposed into a master problem and a subproblem. The master problem is formulated as the same feasibility problem regardless of the policy but it is implicit in the constraint set, whereas the subproblem (i.e., recourse problem) corresponds to a deterministic KEP where only non-failed cycles and chains contribute to the robust objective.

3 Preliminaries

In this section, we describe our two-stage robust model. We begin by formally describing the first-stage problem. While the most important notation used in this paper is presented in this section, we present a list with the most relevant notation used throughout the entire paper in Appendix E. Specifically, we define the compatibility graph and the feasible set for the first-stage decisions. We similarly define the second-stage problem and then introduce our two-stage robust problem. Finally, we define the uncertainty set and the recourse policies considered in this work.

First-stage compatibility graph. The KEP can be defined on a directed graph $D = (V,A)$, whose vertex set $V := P \cup N$ represents the set of patient-donor pairs, $P$, and the set of non-directed donors $N$. From this point onward, we will refer to patient-donor pairs simply as pairs. The arc set $A \subseteq V \times P$ contains arc $(u,v)$ if and only if the donor in vertex $u \in V$ is compatible with a patient in vertex $v \in P$. A matching of donors and patients in the KEP can take the form of simple cycles and simple chains (i.e., paths from the digraph). A cycle is feasible if it has no more than $K$ arcs, whereas a chain is feasible if it has no more than $L$ arcs ($L + 1$ vertices) and starts with a non-directed donor. The set of feasible cycles and chains is denoted by $C^K$ and $C^L$, respectively. Furthermore, let $V(\cdot)$ and $A(\cdot)$ be the set of vertices and arcs in $\cdot$, where $\cdot$ refers to a cycle, chain, etc.

Feasible set of first-stage decisions. A feasible solution to the KEP corresponds to a collection of vertex-disjoint feasible cycles and chains, referred to as a matching, i.e., $M \subseteq C^K \cup C^L$ such that $V(c) \cap V(c') = \emptyset$, for all $c,c' \in M$ with $c \neq c'$, noting that $c$ and $c'$ are entire cycles or chains in the matching. We let $M_D$ denote

\footnote{Note that since each vertex in the KEP compatibility graph corresponds to a patient-donor pair or a non-directed donor, a KEP matching—which is referred to as a matching in the literature and in this paper—is not actually a matching of the digraph. Instead it is a collection of simple cycles and paths in the digraph which leads to an actual underlying 1-1 matching of selected donors and patients.}
the set of all KEP matchings in graph $D$. Also, we define
\[ \mathcal{X} := \{x_M : M \in M_D\} \]
as the set of all binary vectors representing the selection of a feasible matching, where $x_M$ is the characteristic vector of matching $M$ in terms of the cycles/chains sets $C_K \cup C_L$. That is, $x_M \in \{0, 1\}^{|C_K \cup C_L|}$ with $x_{M,c} = 1$ if and only if $c \in M$, meaning that a patient in a pair obtains a transplant if it is in some cycle/chain $c$ selected in matching $M$.

**Transitory compatibility graph.** Upon selection of a solution $x \in \mathcal{X}$ but before uncertainty is revealed, there exists a *transitory* compatibility graph $D^\pi(x) = (V^\pi,x, A^\pi,x)$ with vertex set $V^\pi,x = \cup_{c \in C_K(x) \cup C_L(x)} V(c)$ and arc set $A^\pi,x = \cup_{c \in C_K(x) \cup C_L(x)} A(c)$. The set of feasible cycles $C^\pi_K(x)$ and feasible chains $C^\pi_L(x)$ in the transitory compatibility graph (i) are allowed under recourse policy $\pi \in \Pi$ and (ii) have at least one pair in $x$. Sections 3.1 and 3.2 provide details on uncertainty set $\Gamma$ and the types of recourse policies $\Pi$, respectively, and provide an explicit definition of condition (i). Condition (ii) enforces that each $c \in C^\pi_K(x) \cup C^\pi_L(x)$ satisfies $V(c) \cap V(x) \cap P \neq \emptyset$, where $V(x) = \cup_{c \in C_K \cup C_L : x,c = 1} V(c)$.

**Second-stage compatibility graph.** Once a failure scenario $\gamma \in \Gamma$ is observed in the first-stage compatibility graph, a digraph $D^\pi(x, \gamma)$ for the second-stage problem is induced in the transitory graph, thus revealing the cycles and chains with selected pairs from the first stage that were unaffected by $\gamma$. We refer to $V^\gamma$ and $A^\gamma$ as the set of vertices and the set of arcs that fail under scenario $\gamma$ in the first-stage compatibility graph, respectively. We then define a second-stage compatibility graph $D^\pi(x, \gamma) = (V^\pi,x \setminus \{V^\gamma \cap V\}, A^\pi,x \setminus \{A^\pi,x \cap A^\gamma\})$ as the sub-graph obtained by removing the vertices and arcs that fail under scenario $\gamma$ in the first-stage compatibility graph, and are also present in the transitory graph, from the transitory compatibility graph. Thus, we define $C^\pi_K(x, \gamma)$ and $C^\pi_L(x, \gamma)$ as the set of non-failed cycles and chains in the second-stage compatibility graph, respectively.

**Feasible set of second-stage decisions.** A solution to the second stage is referred to as a *recourse solution* in $D^\pi(x, \gamma)$ under some scenario $\gamma \in \Gamma$, leading to an alternative matching where pairs from the first-stage solution $x \in \mathcal{X}$ are re-arranged into non-failed cycles and chains, among those allowed under policy $\pi \in \Pi$. We can now define $M^\pi(x, \gamma) := \{M \subseteq C^\pi_K(x, \gamma) \cup C^\pi_L(x, \gamma) \mid V(c) \cap V(c') = \emptyset \text{ for all } c, c' \in M; c \neq c'\}$ as the set of allowed recovering matchings under policy $\pi$ such that every cycle/chain in $M^\pi(x, \gamma)$ contains at least one pair in $x$. In other words, a recourse solution must contain at least one pair from the first stage solution. Thus, similar to the first-stage characteristic vector $x_M$, let
\[ \mathcal{Y}^\pi(x, \gamma) := \{y_M : M \in M^\pi(x, \gamma)\} \]
be the set of all binary vectors representing the selection of a feasible matching in a second-stage compatibility graph with non-failed elements (vertices/arcs), under scenario \( \gamma \in \Gamma \) and policy \( \pi \in \Pi \) that contain at least one pair in \( x \).

**Two-stage RO problem.** A general two-stage RO problem for the KEP can then be defined as follows:

\[
\max_{x \in X} \min_{\gamma \in \Gamma} \max_{y \in \mathcal{Y}(x,\gamma)} f(x,\gamma,y) \tag{1}
\]

i.e., a set of transplants given by solution \( x \in X \) is selected in the first stage. Then, the uncertainty vector \( \gamma \in \Gamma \) is observed and a recourse solution \( y \in \mathcal{Y}(x,\gamma) \) is found to repair \( x \) according to recourse policy \( \pi \in \Pi \).

The second stage, established by the min-max problem \( (\max_{x \in X} \min_{\gamma \in \Gamma} \max_{y \in \mathcal{Y}(x,\gamma)} f(x,\gamma,y)) \), finds a recourse solution by solving the recourse problem (third optimization problem, \( \max_{y \in \mathcal{Y}(x,\gamma)} f(x,\gamma,y) \)) under failure scenario \( \gamma \in \Gamma \), but it is the lowest objective value among all failure scenarios. The scenario optimizing the second-stage problem is then referred to as the worst-case scenario for solution \( x \in X \).

The recourse objective function, \( f(x,\gamma,y) \), assigns weights to the cycles and chains of a recovered matching associated with a recourse solution \( y \) under failure scenario \( \gamma \), based on the number of pairs matched in the first stage solution \( x \). Thus, we define the recourse objective as

\[
f(x,\gamma,y) := \sum_{c \in C^R_{K}(x,\gamma) \cup C^R_{L}(x,\gamma)} w_c(x) y_c := w^\gamma(x)^\top y \tag{2}
\]

where \( w_c(x) := |V(c) \cap V(x) \cap P| \) is the weight of a cycle/chain \( c \in C^R_{K}(x,\gamma) \cup C^R_{L}(x,\gamma) \) corresponding to the number of pairs that were matched in the first stage by solution \( x \in X \), and can now also be matched in the second stage by recourse solution \( y \in \mathcal{Y}(x,\gamma) \), after failures are observed. As a result, the weight of every cycle and chain selected by \( y \) is the number of pairs present from the first-stage solution \( x \). Accordingly, an optimal KEP robust solution is a matching in the first stage which, under the worst-case scenario, has the fewest pairs disrupted by the recovery in the second stage. For the sake of a compact representation, and with a slight abuse of notation, we represent the recourse cost function as the inner product given on the right-hand side of (2), where the superscript \( \gamma \) on the weight vector determines its dimension and in turn makes it consistent with that of the recourse decision vector. Thus, our two-stage RO problem for the KEP is defined as follows:

\[
\max_{x \in X} \min_{\gamma \in \Gamma} \max_{y \in \mathcal{Y}(x,\gamma)} w^\gamma(x)^\top y \tag{3}
\]

### 3.1 Uncertainty set

Failures in a planned matching are observed after a final checking process. Failure reasons can include illness, late-detected incompatibilities, match offer rejection by a recipient, patient/donor dropout, etc. These failure reasons can lead to the removal of the affected vertices and arcs from the first-stage compatibility graph. The failure of a vertex/arc causes the failure of the entire cycle to which that element belongs and the shortening
of the chain at the last vertex before the first failure. In the literature, the uncertainty set has been defined as a polyhedron where vertices and arcs in the first-stage compatibility graph can fail at the same rate (homogeneously), and the total number of failures for both vertices and arcs are bounded by a predefined integer value (Blom et al., 2021; Carvalho et al., 2020). In homogeneous failure, there is no need to consider arc failures since there exists a worst-case scenario where all failures are vertex failures (Carvalho et al., 2020). We consider non-homogeneous failure by allowing vertices and arcs to fail at different rates; we define two failure budgets, one for vertices and one for arcs. In other words, we assume that there exist two unknown probability distributions causing vertices and arcs to fail independently from one another, rather than assuming that both vertices and arcs follow the same failure probability distribution. This approach can still model homogeneous failure since as it suffices to consider only vertex failures.

Traditionally, the uncertainty set in robust optimization does not depend on the first-stage decision. While we still consider vertex and arc failures as exogenous random variables, we define the uncertainty set $\Gamma$ in terms of all the uncertainty sets $\Gamma(x)$. Failure scenarios in $\Gamma(x)$ lead to a second-stage compatibility graph $D_\pi(x, \gamma)$ denoting the dependency between the selection of a first-stage decision and the ability to observe a failure. Recall that the checking process leading to the discovery of failures is based on the matching proposed for transplant. In other words, failures that involve vertices and arcs not present in the transitory graph cannot be discovered even if they occur since the set of alternative cycles and chains in the transitory graph that can be used to repair a first-stage solution $x$ are defined by policy $\pi$ regardless of the failure scenario $\gamma$. Therefore, we define the uncertainty set as follows:

$$\Gamma(x) := (\Gamma^v(x), \Gamma^a(x)) \text{ where,}$$

$$\Gamma^v(x) := \{\gamma^v \in \{0, 1\}^{|V|} | |V^{\pi,x} \cap V^{\gamma^v}| \leq \sum_{u \in V^\gamma^v} \gamma^v_u \leq r^v\}$$

$$\Gamma^a(x) := \{\gamma^a \in \{0, 1\}^{|A|} | |A^{\pi,x} \cap A^{\gamma^a}| \leq \sum_{(u,v) \in A^\gamma^a} \gamma^a_{uv} \leq r^a\}$$

$$\Gamma := \bigcup_{x \in X} \Gamma(x)$$

A failure scenario $\gamma = (\gamma^v, \gamma^a)$ is represented by binary vectors $\gamma^v$ and $\gamma^a$, where $\gamma^v$ refers to vertex failures and $\gamma^a$ refers to arc failures. A vertex $u \in V$ and arc $(u,v) \in A$ from the first-stage compatibility graph fail under a realized scenario if $\gamma^v_u = 1$ or $\gamma^a_{uv} = 1$, respectively. The total number of vertex and arc failures in the first-stage compatibility graph is controlled by user-defined parameters $r^v$ and $r^a$, respectively. Therefore, the number of vertex failures in the transitory compatibility graph $D^\pi(x)$ leading to a second-stage compatibility graph $D_\pi(x, \gamma)$ cannot exceed $r^v$. Likewise, the number of arc failures in $D^\pi(x)$ cannot exceed $r^a$. Thus, the uncertainty set $\Gamma$ is the union over all failure scenarios leading to a second-stage compatibility graph when a set of transplants in the first stage has been proposed for transplantation. Note that this uncertainty set definition only distinguishes failures by element type (vertex or arc), and does not distinguish between sensitized and non-sensitized patients.
3.2 Recourse policies

An important consideration made by KPDPs is the guideline according to which a selected matching is allowed to be repaired. Although re-optimizing the deterministic KEP with non-failed vertices/arcs is an option (De Klerk et al., 2005), some KPDPs opt for recovery strategies when failures in the matching given by the deterministic model are observed (CBS, 2019; Manlove and O’malley, 2015). Thus, it is reasonable to use those strategies as recourse policies when uncertainty is considered. We consider the full-recourse policy studied in Blom et al. (2021); Carvalho et al. (2020) and introduce a natural extension of this policy, which we refer to as the first-stage-only recourse policy.

3.2.1 Full recourse policy

Under the full-recourse policy, pairs that were selected in the first stage belonging to failed components are allowed to be re-arranged in the second stage within non-failed cycles and chains that may involve any other vertex (pair or non-directed donor), regardless of whether that vertex was selected or not in the first stage. Figure 1 shows an example of the full-recourse policy. The first-stage solution depicted with bold arcs has a total weight of eight, since there are eight exchanges. Suppose there is a scenario in which $\gamma_2 = 1$ and $\gamma_{56} = 1$. Assuming that $K = L = 4$, then the best recovery plan under this scenario, depicted by the recourse solution with shaded arcs, is to re-arrange vertices 3, 4 and 6 by bringing them together into a new cycle and include vertices 1 and 5 in a chain started by non-directed donor 8 (which was not selected in the first stage). Alternatively, vertices 1 and 5 could be selected along with vertex 7 to form a cycle. In both cases, the recourse solution involves only five pairs from the first stage.

3.2.2 First-stage-only recourse policy

We refer to the first-stage-only as a recourse policy in which only vertices selected in the first stage can be used to repair a first-stage solution, i.e., the new non-failed cycles and chains selected in the second stage must include vertices from that first-stage solution only. It is easy to see that the recourse solution set of the first-stage-only recourse policy corresponds to a subset of the full recourse policy. Although more conservative, KPDPs can opt for the first-stage-only since in the full recourse there can be vertices selected in the second stage that were not selected in the first stage, and thus have never been checked before by KPDP operators, adding uncertainty.
about the actual presence of vertices on a recourse solution under full recourse. The back-arcs recourse policy studied in Carvalho et al. (2020) also allows recovery of a first-stage solution with pairs that were selected in that solution only, but such a policy only allows recovery of a cycle (chain) if there exists other cycles (chains) nested within it, making the back-arcs recourse more conservative than the first-stage-only recourse. In Figure 1, the recourse solution under the first-stage-only recourse would include vertices 3, 4 and 6 in a cycle just as in the full-recourse policy, but chains started by vertex 8 and the cycle to which vertex 7 belongs would not be within the feasible recourse set. Thus, the recourse objective value corresponds to only three pairs from the first stage as opposed to five pairs.

4 Robust model and first stage

For a finite (yet possibly large) uncertainty set, the second optimization problem in Model (3) \( \min_{\gamma \in \Gamma} \) can be removed as an optimization level and included as a set of scenario constraints instead (Carvalho et al., 2020). Model (3) can be expressed as the following MIP robust formulation:

\[
\begin{align*}
\mathbf{P}(\pi, \Gamma) : \quad & \max_{x, Z_P} \quad Z_P \\
\text{subject to} \quad & Z_P \leq \max_{y \in Y^\pi(x, \gamma)} w^\gamma(x)^\top y \\
& \gamma \in \Gamma \\
& x \in \mathcal{X}
\end{align*}
\]

Observe that the optimization problem in (5b) is the recourse problem, which given a first-stage solution \( x \in \mathcal{X} \) and policy \( \pi \in \Pi \), it finds a recourse solution in the set of binary vectors \( Y^\pi(x, \gamma) \) whose cycles and chains under scenario \( \gamma \in \Gamma \) have the largest number of pairs from the first stage. Having the recourse problem in the constraints is challenging since it must be solved as many times as the number of failure scenarios \( \gamma \in \Gamma \) for any fixed decision \( x \), which can be prohibitively large. To reach a more tractable form, first observe that the direction of the external (Objective (5a)) and internal (Constraint (5b)) objectives in Model (5) is maximization.

We can then define a binary decision vector \( y^\gamma \) for every scenario \( \gamma \in \Gamma \) whose feasible space corresponds to a recourse solution in \( Y^\pi(x, \gamma) \). Thus, an equivalent model for the first-stage formulation (FSF) has the inner optimization problem removed and the set \( Y^\pi(x, \gamma) \) included as part of the constraints set for every failure scenario. The resulting model is as follows:

\[
\begin{align*}
\mathbf{P}(\pi, \Gamma) = \max_{x, y^\gamma, Z_P} \quad & Z_P \\
\text{subject to} \quad & Z_P \leq w^\gamma(x)^\top y \\
& y^\gamma \in Y^\pi(x, \gamma) \\
& \gamma \in \Gamma \\
& x \in \mathcal{X}
\end{align*}
\]

Observe that the optimization problem in (5b) is the recourse problem, which given a first-stage solution \( x \in \mathcal{X} \) and policy \( \pi \in \Pi \), it finds a recourse solution in the set of binary vectors \( Y^\pi(x, \gamma) \) whose cycles and chains under scenario \( \gamma \in \Gamma \) have the largest number of pairs from the first stage. Having the recourse problem in the constraints is challenging since it must be solved as many times as the number of failure scenarios \( \gamma \in \Gamma \) for any fixed decision \( x \), which can be prohibitively large. To reach a more tractable form, first observe that the direction of the external (Objective (5a)) and internal (Constraint (5b)) objectives in Model (5) is maximization.
Algorithm 1 Solving the robust KEP in Model (FSF)

**Input:** A policy $\pi \in \Pi$ and restricted set of scenarios $\tilde{\Gamma}$; $\tilde{\Gamma} := \emptyset$  
**Output:** Optimal robust solution $x^*$

1: repeat  
2: Solve $P(\pi, \tilde{\Gamma})$ and obtain optimal solution $\tilde{x}$  
3: if $\min_{\gamma \in \Gamma} \max_{y \in Y_{\pi}(\tilde{x}, \gamma)} w^\gamma(\tilde{x})^\top y < \tilde{Z}_P$ then  
4: $\Gamma \leftarrow \Gamma \cup \{\gamma^*\}$, add recourse decision vector $y_{\gamma^*}$, Constraints (6a), (6b) and go to Step 1  
5: until $\min_{\gamma \in \Gamma} \max_{y \in Y_{\pi}(\tilde{x}, \gamma)} w^\gamma(\tilde{x})^\top y \geq \tilde{Z}_P$  
6: $x^* \leftarrow x$  
7: return $x^*$

The first-stage solution and its corresponding recourse solution can be found in a single-level MIP formulation if the set of scenarios $\Gamma$ can be enumerated. Observe that once a scenario $\gamma \in \Gamma$ is fixed, $J^\pi(x, \gamma)$ can be expressed by a set of linear constraints and new decision variables, e.g., the position-indexed formulation for the robust KEP under full recourse proposed in Carvalho et al. (2020). This formulation follows the structure of Model (FSF). We present this formulation (Appendix A) and a variant of it that satisfies the first-stage-only recourse policy (Appendix B) in the Online Supplement.

To overcome the large size of $\Gamma$, we solve the robust KEP in Model (FSF) iteratively. This is accomplished by finding a new failure scenario whose associated optimal recourse solution $y^\gamma \in J^\pi(x, \gamma)$ recovers the maximum number of pairs from the first stage under that scenario in Model (FSF). We note that adding new failure scenarios worsens the optimal objective value in Model (FSF). The goal is to find a scenario that yields a recourse solution with an objective value lower than the current objective value of the robust KEP model, if such scenario exists. If a scenario is found, then a new set of recourse decision variables along with Constraints (6a) and (6b) are added to Model (FSF).

Algorithm 1 shows the iterative approach used to solve Model (FSF). It starts with an empty restricted set of scenarios $\tilde{\Gamma}$. First, $P(\pi, \tilde{\Gamma})$ is solved and its incumbent solution $\tilde{x}$ is retrieved. Then, the second-stage problem is solved to optimality for an incumbent solution $\tilde{x}$ under policy $\pi$ in line 3. The solution to the second-stage problem yields a failure scenario $\gamma^* \in \Gamma$ with the lowest optimal recourse objective value among all possible failure scenarios. We refer to that scenario as the worst-case scenario for a first-stage solution $\tilde{x} \in \mathcal{X}$. If such recourse objective value is lower than $\tilde{Z}_P$, then $\gamma^*$ is added to the restricted set of scenarios $\tilde{\Gamma}$. Then, the corresponding Constraints (6a) and (6b) are added to Model (FSF) along with decision vector $y_{\gamma^*}$, and Model (FSF) is re-optimized. We refer to Model (FSF) with a restricted set of scenarios $\tilde{\Gamma}$ as the first-stage problem or first-stage formulation to imply that the search for the optimal robust solution corresponding to some $\tilde{x} \in \mathcal{X}$ continues. In principle, as long as a failure scenario $\gamma \in \Gamma$ leads to an optimal recourse solution with an objective value lower than $\tilde{Z}_P$, that scenario could be added to Model (FSF). However, for benchmark purposes, at Step 2 the worst-case scenario is found for a given first-stage solution $\tilde{x}$ under policy $\pi$. If the optimal objective value of the second-stage problem equals $\tilde{Z}_P$, then an optimal robust solution is returned.

Algorithm 1 converges in a finite number of iterations due to the finiteness of $\mathcal{X}$ and $\Gamma$. Due to the large set of scenarios, Step 2 in Algorithm 1 is critical to efficiently solve the robust problem. In subsequent sections, we decompose the second-stage problem into a master problem yielding a failure scenario and a sub-problem (the
5 New second-stage decompositions

In this section, we present two decompositions of the second-stage problem that is solved at step 3 in Algorithm 1 as part of the iterative solution of the robust KEP. Both consist of a feasibility-seeking master problem that finds a failure scenario and a sub-problem that finds an alternative matching under that scenario. Each decomposition solves a recourse problem in either the second-stage compatibility graph (Appendix C.3, Model (14)) or the transitory graph (Appendix C.3, Model (16)). Such models use cycle-chain decision variables as proposed in Blom et al. (2021). Although the optimal solutions provided by both formulations are identical, we use the structure of each solution set to formulate the master problem of each decomposition accordingly.

5.1 An initial feasibility-seeking formulation for the second stage

In this section we introduce the first of the two new decompositions for the second-stage problem. We formulate the second-stage problem as a linear binary program whose optimal solution can be obtained through another linear binary program that finds a feasible failure scenario. The constraints in this feasibility-seeking formulation may be exponential as they require knowing an optimal recourse solution under all possible failure scenarios in advance. To efficiently solve such formulation, we decompose it into a master problem and sub-problem (recourse problem). Thus, every time a new failure scenario is found, a recourse solution is obtained and a new constraint is added to the master problem. Our solution framework is purely a cutting plane since no new decision variables need to be created when a new constraint is added to the master problem. From this point onwards our feasibility-seeking formulations are shown (and referred to) as master problems, since we assume that at the start of the cutting plane algorithm we have an empty set of failure scenarios $\hat{\Gamma}(\tilde{x}) \subseteq \Gamma(\tilde{x})$ for a given $\tilde{x} \in X$ rather than the full set. We show that once the master problem becomes infeasible, an optimal solution to the second-stage problem has been found. In this section, a recourse solution is found on the second-stage compatibility graph. Recall that a second-stage compatibility graph does not include cycles/chains that fail under the scenario that induced such graph in the transitory graph. Therefore, we refer to the master problem in the initial feasibility-seeking formulation as MasterSecond. In this section, we first present the recourse problem formulation on the second-stage problem, then we formally define MasterSecond and prove its validity. Finally, we present non-valid inequalities to strengthen the right-hand side of constraints in MasterSecond.

5.1.1 The recourse problem on the second-stage compatibility graph

Given a first-stage solution $\tilde{x} \in X$, let $\hat{\Gamma}(\tilde{x}) \subseteq \Gamma(\tilde{x})$ be a restricted set of failure scenarios such that $\hat{\gamma} \in \hat{\Gamma}(\tilde{x})$ induces a second-stage compatibility graph $D^\pi(\tilde{x}, \hat{\gamma})$ as defined in Section 3. The recourse problem consists of finding a matching of maximum weight in $D^\pi(\tilde{x}, \hat{\gamma})$, i.e., a matching with the largest number of pairs selected in the first stage. We define $R^\pi(\tilde{x}, \hat{\gamma})$ as the MIP formulation (Appendix C.3) for the recourse problem in the
second-stage compatibility graph $D\hat{\pi}(\hat{x}, \hat{\gamma})$, and define $y^{\star\hat{\gamma}}$ as an optimal recourse solution in $Y\hat{\pi}(\hat{x}, \hat{\gamma})$ with objective value $Z\hat{\pi}^{\star\hat{\gamma}}(\hat{x}, \hat{\gamma})$.

5.1.2 The master problem: MasterSecond

We continue to use $\gamma^v$ and $\gamma^a$ as binary vectors representing the failure of a vertex ($\gamma^v_u = 1 \forall u \in V$) and arc ($\gamma^a_{uv} = 1 \forall (u, v) \in A$), respectively. Thus, the master problem for the second-stage problem can be formulated as follows:

\begin{align*}
\text{MasterSecond}(\hat{x}): \quad & \text{Find } \gamma \quad \text{(MS)} \\
& \sum_{c \in C(\hat{x})} \left( \sum_{u \in V(c)} \gamma^v_u + \sum_{(u,v) \in A(c)} \gamma^a_{uv} \right) \geq 1 \quad (7a) \\
& \sum_{c \in C(\hat{x})} \left( \sum_{u \in V(c)} \gamma^v_u + \sum_{(u,v) \in A(c)} \gamma^a_{uv} \right) \geq 1 \quad \forall \hat{\gamma} \in \hat{\Gamma}(\hat{x}) \quad (7b) \\
& \gamma \in \hat{\Gamma}(\hat{x}) \quad (7c)
\end{align*}

where $C(\hat{x}) = C^K(\hat{x}, \hat{\gamma}) \cup C^L(\hat{x}, \hat{\gamma})$. Observe that because we start with an empty set of scenarios, Constraints (7a)-(7b) correspond to covering constraints in a set cover problem, one of Karp's 21 NP-complete problems (Karp, 1972).

At least one vertex or arc in every recourse solution $y^{\star\hat{\gamma}} \in Y\hat{\pi}(\hat{x}, \hat{\gamma})$ must fail before finding the optimal worst-case scenario $\gamma^* \in \Gamma$ for a first-stage solution $\hat{x} \in X$. Observe that at the start, when $\hat{\Gamma}(\hat{x}) = \emptyset$ the second-stage compatibility graph is equivalent to the transitory graph, and so the first recourse solution known is $\hat{x}$ itself. Intuitively, we can see that we can find a worse failure scenario if at least one element (vertex/arc) in $\hat{x}$ fails (Constraint (7a)). Otherwise, the optimal objective value of the second stage would equal $\hat{Z}_P$ and we would have found the optimal solution to the robust KEP in Algorithm 1. Suppose we “generate” a failure scenario $\hat{\gamma}$ affecting solution $\hat{x}$ which we use to induce a new second-stage compatibility graph and include that scenario in our restricted set $\hat{\Gamma}(\hat{x})$. We then need to find an optimal recourse solution $y^{\star\hat{\gamma}}$ under that scenario. However, unless we cause a failure in this new solution (Constraint (7b)) we will not succeed at finding a scenario that maximally decreases the value of $\hat{Z}_P$ in Model FSF. We repeat the same procedure until we realize that we can no longer generate a failure affecting the last recourse solution added to MasterSecond without violating the failure vertex budget $r^v$ and arc failure budget $r^a$ implicit in Constraint (7c). Therefore, when MasterSecond becomes infeasible, the worst-case scenario must be the one that led to the optimal recourse solution with the fewest number of pairs from the first stage.

Algorithm 2 generates new failure scenarios $\hat{\gamma} \in \hat{\Gamma}(\hat{x})$ until MS becomes infeasible, at which point the worst-case scenario, $\gamma^*$, and its associated optimal recourse solution value, i.e., the optimal objective value of the second-stage problem, $Z\hat{\pi}^{\star\hat{\gamma}}(\hat{x})$, have already been found. Algorithm 2 starts with an empty subset of scenarios $\hat{\Gamma}(\hat{x})$, and with an upper bound on the objective value of the second stage $\hat{Z}_P = \hat{Z}_P$. MS is solved for the first time to obtain a failure scenario $\hat{\gamma}$, which is then added to $\hat{\Gamma}(\hat{x})$. At Step 1, the recourse formulation $R\hat{\pi}(\hat{x}, \hat{\gamma})$ is
**Algorithm 2** Solving the second-stage problem (Model (3), Step 3 in Algorithm 1) using MS

**Input:** A recourse policy $\pi \in \Pi$, first-stage solution $\bar{x} \in X$ and current KEP robust objective $\bar{Z}_P$

**Output:** Optimal recovery plan value $Z^\pi_R(\bar{x})$ and worst-case scenario $\gamma^* \in \Gamma(\bar{x})$

1. $i = 1$; $\Gamma(\bar{x}) = \emptyset$; $Z^\pi_Q(\bar{x}) \leftarrow \bar{Z}_P$; Solve MS with Constraint (7a) to obtain scenario $\bar{\gamma}_i$; $\hat{\Gamma}(\bar{x}) \leftarrow \hat{\Gamma}(\bar{x}) \cup \{\bar{\gamma}_i\}$

2. Solve $R^\pi(\bar{x}, \bar{\gamma}_i)$ to obtain objective value $Z^\pi_R(\bar{x}, \bar{\gamma}_i)$ and recourse solution $y^*\bar{\gamma}_i$; create Constraint (7b)

3. if $Z^\pi_R(\bar{x}, \bar{\gamma}_i) < Z^\pi_Q(\bar{x})$ then

4. $Z^\pi_Q(\bar{x}) = Z^\pi_R(\bar{x}, \bar{\gamma}_i)$ and $\bar{\gamma}_i \leftarrow \bar{\gamma}_i$

5. $i \leftarrow i + 1$; Attempt to solve MasterSecond($\bar{x}$) to get a new candidate scenario $\bar{\gamma}_i$

6. if MasterSecond($\bar{x}$) is feasible then

7. Go to Step 1

8. $\gamma^* \leftarrow \bar{\gamma}_i$

9. $Z^\pi_Q(\bar{x}) \leftarrow Z^\pi_Q(\bar{x})$

10. **return** $\gamma^*$ and $Z^\pi_Q(\bar{x})$

Solved and an optimal recourse solution $y^*\gamma$ is obtained with objective value $Z^\pi_R(\bar{x}, \bar{\gamma}_i)$. The recourse solution is then added to $Y^\pi(\bar{x}, \bar{\gamma}_i)$ and a Constraint (7b) is created in MS. If the objective value of the recourse solution $Z^\pi_R(\bar{x}, \bar{\gamma}_i)$ is lower than the current upper bound $\bar{Z}^\pi_Q(\bar{x})$, then the upper bound is updated, along with the incumbent failure scenario $\gamma'$. At Step 2, a new iteration starts and an attempt is made to find a feasible failure scenario in MS. If such a scenario exists, then Step 1 is repeated, otherwise the algorithm goes to Step 3, and an optimal solution $(\gamma^*, Z^\pi_Q(\bar{x}))$ to the second stage is returned. To prove the validity of Algorithm 2, we state the following:

**Proposition 5.1.** Algorithm 2 returns the optimal objective value of the second stage $Z^\pi_Q(\bar{x})$ and worst-case scenario $\gamma^* \in \Gamma(\bar{x})$ for a first-stage decision $\bar{x} \in X$.

**Proof.** In the first part of the proof, we show that any optimal objective value of the recourse problem $Z^\pi_R(\bar{x}, \bar{\gamma}_i)$, regardless of the scenario $\bar{\gamma}_i \in \Gamma(\bar{x})$, is an upper bound on the optimal objective value of the second stage for a given $\bar{x} \in X$. In the second part, we show that the second-stage problem can be decomposed into an optimality-seeking non-linear binary program which can be linearized by the iterative addition of recourse solutions. We then show that constraints in the binary program have a one-to-one correspondence with those in MS.

Part I. Observe that given a candidate solution $\bar{x} \in X$ and policy $\pi \in \Pi$

\[
\max_{y \in \mathcal{Y}^\pi(\bar{x}, \bar{\gamma})} w^\top y \geq \min_{\bar{\gamma} \in \Gamma(\bar{x})} \max_{y \in \mathcal{Y}^\pi(\bar{x}, \bar{\gamma})} w^\top y \quad \bar{\gamma} \in \Gamma(\bar{x})
\]

\[
\sum_{c \in \mathcal{C}^L_{K,L}(\bar{x}, \bar{\gamma})} w_c(\bar{x}) \geq Z^\pi_Q(\bar{x}) \quad \bar{\gamma} \in \Gamma(\bar{x}); y^*\bar{\gamma}
\]

That is, the optimal objective value of a recourse solution, $\sum_{c \in \mathcal{C}_{K,L}(\bar{x}, \bar{\gamma})} w_c(\bar{x}) = Z^\pi_R(\bar{x}, \bar{\gamma})$, regardless of the scenario $\bar{\gamma} \in \Gamma(\bar{x})$ is at least as big as the smallest value that $Z^\pi_Q(\bar{x})$ can reach.

Part II. In what follows $1_{c,\gamma}$ is an indicating variable that takes on value one if cycle/chain $c \in \mathcal{C}^L_{K,L}(\bar{x}, \gamma)$ fails under scenario $\gamma \in \Gamma(\bar{x})$. Then, we define $\alpha_c$ as a binary decision variable for a cycle/chain $c \in \mathcal{C}^L_{K}(\bar{x}) \cup \mathcal{C}^L_{L}(\bar{x})$ in the transitory graph that takes on value one if such cycle/chain fails, and zero otherwise. We define $\mathcal{Y}^\pi_{\text{tr}}(\bar{x})$ as the set of binary vectors representing a recourse solution in the transitory graph and denote $\bar{y}$ to be a feasible recourse solution, as opposed to its decision vector counterpart $y$. In Appendix C we present a procedure to
find $C^*_K(\bar{x})$ and $C^*_L(\bar{x})$ given a first-stage solution $\bar{x} \in X$. Note that since all feasible chains from length 1 to $L$ are found, the shortening of a chain when a failure occurs is represented by some $\alpha_c$ taking on value zero and another one taking on value one. Then, the second-stage problem (SSP) can be reformulated as follows:

$$\begin{align*}
\text{SSP} = \min_{y \in \Gamma(\bar{x})} & \quad Z^*_Q(\bar{x}) \\
\text{s.t.} & \quad Z^*_Q(\bar{x}) \geq \max_{y \in \Gamma_2(\bar{x}, \gamma)} \sum_{c \in C^*_{K,L}(\bar{x}, \gamma)} (w_c(\bar{x}) - w_c(\bar{x}) \mathbb{1}_{c, \gamma}) y_c \\
& \quad y_c \in \mathcal{Y}(\bar{x}, \gamma)
\end{align*}$$

Note that all feasible failure scenarios $\gamma \in \Gamma(\bar{x})$ exist as a combination of vertex and arc failures in cycles/chains of the transitory graph, i.e., $c \in C^*_K(\bar{x}) \cup C^*_L(\bar{x})$. Therefore, we can model all possible values of the indicating variable $\mathbb{1}_{c, \gamma}$ by replacing it with a decision variable $\alpha_c$ that takes on value one if cycle/chain $c \in C^*_K(\bar{x}) \cup C^*_L(\bar{x})$ fails and zero otherwise. The idea is to link this new decision variable to a set of linear constraints such that whenever a vertex/arc in a cycle/chain fails so does the latter. To this end, we introduce the binary vectors $(\gamma^v, \gamma^a) \in \Gamma(\bar{x})$ as defined in Section 3.1. Thus, we include the minimization over the scenarios within the constraint set as follows:

$$\begin{align*}
\text{SSP} = \min & \quad Z^*_Q(\bar{x}) \\
\text{s.t.} & \quad Z^*_Q(\bar{x}) \geq \sum_{c \in C^*_K(\bar{x}) \cup C^*_L(\bar{x})} (w_c(\bar{x}) - w_c(\bar{x}) \mathbb{1}_{c, \gamma}) y_c \\
& \quad y_c \in \mathcal{Y}(\bar{x})
\end{align*}$$

Therefore, Constraints (8g) mandate that when a cycle/chain fails $\alpha_c$ takes on value one due to the minimization objective. Otherwise, $\alpha_c$ becomes zero and thus the weight of such cycle/chain is considered in Constraints (8f). Observe that once $\alpha_c$ is set to a value (either zero or one) Model 8e is equivalent to solving Models (8a) or (8c) when $\gamma = \hat{\gamma}$ and the indicating variables $\mathbb{1}_{c, \gamma}$ take their values accordingly for all $c \in C^*_K \cup C^*_L$. Lastly, Constraints (8h) require that the number of vertices and arcs that fail is not exceeded. Since there can exist an exponential number of Constraints (8f), a natural way to solve Model 8e is by assuming that the first recourse solution is $\bar{x}$, which can be obtained in the transitory graph. Then, using $\bar{y} = \bar{x}$ to create the first constraint of Constraints (8f). Afterwards, a feasible solution $\bar{\alpha}$ mapping to some $\hat{\gamma} \in \Gamma(\bar{x})$ can be found, allowing us to solve a recourse problem with optimal solution $y^*\hat{\gamma}$. Then, a new Constraint (8f) can be created using $y^*\hat{\gamma}$ and
a new failure scenario $\gamma$ can be found. So on and so forth until $Z^\pi_Q(\tilde{x}) = Z^\pi_R(\tilde{x}, \gamma)$. Thus, a formulation for SSP can be expressed as follows:

$$Q(\pi, \tilde{x}) = \min \ Z^\pi_Q(\tilde{x}) \quad (SSF)$$

$$Z^\pi_Q(\tilde{x}) \geq Z_P - \sum_{c \in C^\pi_K(\tilde{x})} w_c \alpha_c \tilde{x}_c \quad (8j)$$

$$Z^\pi_Q(\tilde{x}) \geq Z^\pi_R(\tilde{x}, \gamma) - \sum_{c \in C^\pi_K(\tilde{x}, \gamma)} w_c \alpha_c y^\gamma_c \quad \gamma \in \hat{\Gamma}(\tilde{x}); y^\gamma \quad (8k)$$

$$\alpha_c \leq \sum_{u \in V(c)} \gamma^u_c + \sum_{(u,v) \in A(c)} \gamma^a_{uv} c \in C^\pi_K(\tilde{x}) \cup C^\pi_L(\tilde{x}) \quad (8l)$$

$$\gamma \in \Gamma(\tilde{x}) \quad (8m)$$

Observe that since $Z_P \geq Z^\pi_R(\tilde{x}, \gamma) \geq Z^\pi_Q(\tilde{x})$, for $Z^\pi_Q(\tilde{x})$ to be as small as possible at least one cycle or chain, and thus, at least one vertex or arc must fail for some cycle(chain) in every matching associated with an optimal recourse solution (Constraints (8j) and (8k)). If it was not for the failure budget limiting the maximum number of failed vertices and arcs (Constraint (8m)), $Z^\pi_Q(\tilde{x})$ could reach zero. Note that if SSF is unable to cause the failure of a new optimal recourse solution $y^\star \gamma$ it is because in doing so Constraint (8m) would be violated. Thus, the worst-case scenario $\gamma^\star$ is found when there exists a Constraint (8k) associated to a recourse solution $y^\star \gamma$ with non failed cycles/chains or equivalently when $Z^\pi_Q(\tilde{x}) = Z^\pi_R(\tilde{x}, \gamma)$ for some $\gamma \in \hat{\Gamma}(\tilde{x})$. Thus, there is a one-to-one correspondence between (i) Constraint (7a) in SSF and Constraint (8j) in MS and (ii) Constraints (8k) in SSF and Constraints (7b) in MS. Therefore, the smallest value of $Z^\pi_R(\tilde{x}, \gamma)$ among all scenarios $\gamma \in \hat{\Gamma}(\tilde{x})$ before MS becomes infeasible is the optimal value to the second stage.

It is worth noting that a cycle-chain-based formulation similar to SSF was proposed by Blom et al. (2021) for the full-recourse policy under homogeneous failure, using the set of feasible cycles and chains in what we refer to as the first-stage compatibility graph. However, in the way we have defined our two-stage optimization problem, Model SSF can address multiple policies under homogeneous and non-homogeneous failure.

### 5.1.3 Lifting constraints

Unlike valid inequalities, the so-called non-valid inequalities cut off feasible solutions (Atamtürk et al., 2000; Hooker, 1994), and therefore are invalid in the standard sense. Although, non-valid inequalities remove some integer solutions, all optimal solutions are preserved. Next, we derive the first family of non-valid inequalities to narrow down the search of the worst-case scenario in MS.

Our goal is to strengthen the right-hand side of Constraints (7b) that have already been generated up to some iteration $i$ in Algorithm 2, by updating the minimum number of vertices/arcs that should fail in each of those constraints whenever a smaller value of $Z^\pi_Q(\tilde{x})$ is found at Step 3 of Algorithm 2.

For a recourse solution $y^\star \gamma$ and assuming that the failure of a vertex/arc completely causes the failure of a
cycle or chain, we sort cycle and chain weights $w_c(\tilde{x})$ with $c \in \mathcal{C}_{K,L}(\tilde{x}, \tilde{\gamma}) \forall y^{|\tilde{y}|}_c = 1$ in non-increasing order so that $w_c(\tilde{x})_1 \geq w_c(\tilde{x})_2 \geq \cdots \geq w_c(\tilde{x})_{|\mathcal{C}_{K,L}(\tilde{x}, \tilde{\gamma})|}$. The goal is to strengthen the right-hand side value of Constraint (7a) and Constraints (7b). We can now state the following:

**Proposition 5.2.** Constraints (7a) and (7b) with right-hand side value of $t$ is a non-valid inequality such that $t$ is the smallest index for which the following condition is true: $Z_{R^p}^\pi(\tilde{x}, \tilde{\gamma}) - \sum_{t=1}^{|\mathcal{C}_{K,L}(\tilde{x}, \tilde{\gamma})|} w_c(\tilde{x})_t < Z_Q^\pi(\tilde{x})$.

**Proof.** Observe that unless optimal $Z_{Q}^\pi(\tilde{x}) < Z_Q^\pi(\tilde{x}) \forall \tilde{\gamma} \in \hat{\Gamma}(\tilde{x})$ due to Part I of Proposition 5.1 and Step 4 of Algorithm 2. Thus, some cycles/chains must fail in each of the homologous constraints of (7a) and (7b) in Model SSF, i.e., Constraints (8j) and (8k). Also, observe that Constraints (8l) imply that whenever at least one vertex/arc fails, so does its associated cycle/chain. Therefore, finding the minimum number of cycles/chains that should fail in Constraints (8j) and (8k) to satisfy $Z_Q^\pi(\tilde{x}) < Z_Q^\pi(\tilde{x})$ implies finding the minimum number of vertices/arc fails that should fail in Constraints (7a) - (7b). It is easy to see that sorting the cycle/chain weight of every cycle/chain in non-increasing order, and then subtracting it in that order from $Z_{R^p}^\pi(\tilde{x}, \tilde{\gamma})$ until the subtraction is strictly lower than $Z_Q^\pi(\tilde{x})$, yields a valid lower bound on the number of cycles/chains that must fail. \qed

### 5.2 An expanded feasibility-seeking decomposition

Next, we introduce the second new decomposition for the second-stage problem.

#### 5.2.1 The recourse problem on the transitory graph

So far, we have solved the recourse problem using the second-stage compatibility graph $D^\pi(\tilde{x}, \tilde{\gamma})$. However, an optimal solution to the recourse problem $R^\pi(\tilde{x}, \tilde{\gamma})$ can also be found in the transitory graph $D^\pi(\tilde{x})$, by allowing failed cycles/chains into the optimal solution given by $R^\pi(\tilde{x}, \tilde{\gamma})$. Although this solution does not contribute to more recourse objective value, as we will show, it can prevent some dominated scenarios from being explored, and therefore help reduce the number of times the recourse problem is resolved in Algorithm 2. A failure scenario $\tilde{\gamma}' \in \hat{\Gamma}(\tilde{x})$ dominates another failure scenario $\tilde{\gamma} \in \hat{\Gamma}(\tilde{x})$ if $Z_R^\pi(\tilde{x}, \tilde{\gamma}') \leq Z_R^\pi(\tilde{x}, \tilde{\gamma})$. We refer to $R^\pi(\tilde{x}, \tilde{\gamma})$ (Appendix C.3) as the recourse problem solved in the transitory graph whose optimal recourse solutions are also optimal with respect to $R^\pi(\tilde{x}, \tilde{\gamma})$ but may allow some failed cycles/chains in the solution. We refer the reader to Appendix C.3 for proof. Thus, we let $\psi^{|\tilde{\gamma}|}_R(\tilde{x})$ be an optimal recourse solution to $R^\pi(\tilde{x}, \tilde{\gamma})$ in the transitory graph that is also optimal to $R^\pi(\tilde{x}, \tilde{\gamma})$ under scenario $\tilde{\gamma} \in \hat{\Gamma}(\tilde{x})$.

#### 5.2.2 The master problem: MasterTransitory

In the basic reformulation, we assumed the recourse solutions correspond to matchings with non-failed components only. In our expanded formulation we assume that recourse solutions can be expanded to fit some failed cycles/chains by solving the recourse problem in the transitory graph $D^\pi(\tilde{x})$. We let $\psi^{*\hat{\gamma}} \subseteq \psi^{*\tilde{\gamma}}$ be the subset of the optimal recourse solution $\psi^{*\tilde{\gamma}}$ that has no failed cycles/chains and thus corresponds to a feasible solution.
in $\mathcal{Y}^\gamma(\tilde{x}, \tilde{\gamma})$. Therefore, the expanded feasibility-seeking reformulation is expressed as follows:

\[
\text{MasterTransitory}(\tilde{x}): \quad \text{Find } \gamma \quad \text{(MT)}
\]

\[
\sum_{c \in \mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})} \left( \sum_{u \in V(c)} \gamma_u^c + \sum_{(u,v) \in A(c)} \gamma_{uv}^c \right) \geq 1 \quad (9a)
\]

\[
\sum_{c \in \mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})} \left( \sum_{u \in V(c)} \gamma_u^c + \sum_{(u,v) \in A(c)} \gamma_{uv}^c \right) \geq 1 \quad \tilde{\gamma} \in \hat{\Gamma}(\tilde{x}); \psi^* \tilde{\gamma} \quad (9b)
\]

\[
\sum_{c \in \mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})} \left( \sum_{u \in V(c)} \gamma_u^c + \sum_{(u,v) \in A(c)} \gamma_{uv}^c \right) \geq H^\gamma \quad \tilde{\gamma} \in \hat{\Gamma}(\tilde{x}); \psi^* \tilde{\gamma} \quad (9c)
\]

\[
\gamma \in \Gamma \quad (9d)
\]

Constraints (9a) and (9b) are equivalent to Constraints (7a) and (7b). Constraints (9c) require that a combination of at least $H^\gamma$ vertices and arcs fail in an expanded recourse solution $\psi^* \tilde{\gamma} \in \mathcal{Y}^\gamma(\tilde{x}, \tilde{\gamma})$. Proposition 5.3 defines the value of $H^\gamma$. The goal of MT is to enforce the failure of two different recourse solutions with identical objective values under scenario $\hat{\gamma} \in \hat{\Gamma}(\tilde{x})$, i.e. a recourse solution found in the second-stage compatibility graph $D^\gamma(\tilde{x}, \tilde{\gamma})$ and a solution found in the transitory graph $D^\gamma(\tilde{x})$.

For a recourse solution $\psi^* \tilde{\gamma} \in \mathcal{Y}^\gamma(\tilde{x}, \tilde{\gamma})$ again assuming that the failure of a vertex/arc completely causes the failure of a cycle or chain, we sort cycle and chain weights $w_c(\tilde{x}) \forall y_c = 1$ with $c \in \mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})$, in non-increasing order so that $w_c(\tilde{x})_1 \geq w_c(\tilde{x})_2 \geq \cdots \geq w_c(\tilde{x})_{|\mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})|}$. Thus, we can now state:

**Proposition 5.3.** $H^\gamma = t$ is a non-valid inequality such that $t$ is the smallest index for which the following condition is true $Z_{R^\gamma}^\gamma(\tilde{x}, \tilde{\gamma}) = \sum_{i=1}^{(|\mathcal{C}_K(\tilde{x}) \cup \mathcal{C}_L(\tilde{x})|)} w_c(\tilde{x})_i < Z_{Q^\gamma}^\gamma(\tilde{x})$.

**Proof.** It follows by the same arguments given for Proposition 5.2. \qed

In the following example we see that MT can yield failure scenarios that dominate those in MS.

**Example** Let us consider Figure 1 again under full recourse. Given a first-stage solution $\tilde{x} \in X$ involving eight pairs in Figure 1a, the first-stage compatibility graph also corresponds to the transitory graph $D^\gamma(\tilde{x})$. Once we observe failure scenario $\tilde{\gamma}_2 = 1$ and $\tilde{\gamma}_8 = 1$, the realization of the second-stage compatibility graph, $D^\gamma(\tilde{x}, \tilde{\gamma})$, is such that vertices 2, 9 and 10 do not belong to it. The optimal objective value to the recourse problem in $D^\gamma(\tilde{x}, \tilde{\gamma})$, i.e., $R^\gamma(\tilde{x}, \tilde{\gamma}) = Z_R^\gamma(\tilde{x}, \tilde{\gamma}) = 5$ with an optimal solution $\mathbf{1}^{\gamma \gamma}$ involving cycle $(3,4,6)$ and chain $(8,1,5)$. When we solve the recourse problem in the transitory graph $D^\gamma(\tilde{x})$, the optimal objective value is also 5, but its optimal recourse solution $\psi^* \tilde{\gamma} = \mathbf{y}^* \tilde{\gamma} \cup \{(2,9,10)\}$ involves in addition to cycle $(3,4,6)$ and chain $(8,1,5)$, the failed cycle $(2,9,10)$. A feasible scenario is highlighted in red for every iteration. Assuming $r^\gamma = 1$ and $r^a = 1$, the first two iterations for MS and MT are shown below:
Observe that in the second iteration, the failure scenario that is feasible for MS is not feasible for MT. The new failure scenario for MS would lead to an optimal recourse solution with again $Z_{R}^{\pi,*,(\bar{x}, \bar{\gamma})} = 5$, whereas the optimal recourse solution for MT would lead to $Z_{R}^{\pi,*(\bar{x}, \bar{\gamma})} = 3$. That is, in the third iteration the right-hand side of constraints MT will get updated with a new upper bound $\bar{Z}_{Q}(\bar{x}) = 3$, leading MT to infeasibility sooner. On the other hand, MS in the third iteration will need another attempt to “discover” a failure scenario that will bring down the ceiling of $\bar{Z}_{Q}(\bar{x})$.

5.2.3 Dominating scenarios
A failure scenario $\bar{\gamma}' \in \Gamma(\bar{x})$ dominates another failure scenario $\bar{\gamma} \in \Gamma(\bar{x})$ when the former implies the latter, i.e., $Z_{R}^{\pi,*(\bar{x}, \bar{\gamma})} \leq Z_{R}^{\pi,*(\bar{x}, \bar{\gamma}')}$, then $I(\bar{\gamma}')$ and $I(\bar{\gamma})$ be the number of failed vertices and arcs under their corresponding scenario. Moreover, let $C_{K,L}^{\pi}(\bar{x}, \bar{\gamma}') \subseteq C_{K,L}^{\pi}(\bar{x}) \cup C_{E}^{\pi}(\bar{x})$ and $C_{K,L}^{\pi}(\bar{x}, \bar{\gamma}) \subseteq C_{K,L}^{\pi}(\bar{x}) \cup C_{E}^{\pi}(\bar{x})$ be the set of feasible cycles and chains that fail in the transitory graph $D^{\pi}(\bar{x})$ under scenarios $\bar{\gamma}'$ and $\bar{\gamma}$, respectively. We then have the following result:

**Proposition 5.4.** If $C_{K,L}^{\pi}(\bar{x}, \bar{\gamma}') \subseteq C_{K,L}^{\pi}(\bar{x}, \bar{\gamma})$, $\bar{\gamma}'$ dominates $\bar{\gamma}$ and the following dominance inequality is valid for the second-stage problem.

$$
\sum_{u \in V, \gamma_u = 1} \gamma_u + \sum_{(u,v) \in A : \gamma_{uv} = 1} \gamma_{uv} \leq I(\bar{\gamma}) \left( I(\bar{\gamma}') - \sum_{u, \gamma_u = 1} \gamma_u - \sum_{(u,v), \gamma_{uv} = 1} \gamma_{uv} \right) \quad (10)
$$

**Proof.** By definition $C_{K,L}^{\pi}(\bar{x}, \bar{\gamma}') \subseteq C_{K,L}^{\pi}(\bar{x}, \bar{\gamma})$, thus, all cycles and chains that fail under $\bar{\gamma}$ also fail under $\bar{\gamma}'$, which means that $Z_{R}^{\pi,*(\bar{x}, \bar{\gamma}') \leq Z_{R}^{\pi,*(\bar{x}, \bar{\gamma})}$. Thus, it follows that $\bar{\gamma}'$ dominates scenario $\bar{\gamma}$. 

\end{proof}
Finding all $\tilde{\gamma}, \tilde{\gamma}' \in \Gamma(\tilde{x})$ satisfying Proposition 5.4 is not straightforward and there may be too many constraints of type (10) to feed into our master problem formulations. Therefore, we explore two alternatives to separate these dominating-scenario cuts. The first one consists of identifying a subset of scenarios that satisfy constraint (10) a priori. The second one consists of attempting to solve MS and MT via a heuristic and then “discovering” dominating scenarios on the fly. We refer to it as a heuristic because it may not always find a feasible failure scenario satisfying the constraints in MS or MT. Next, we present a subset of dominated scenarios following the two strategies.

**Adjacent-failure separation** A vertex failure in the transitory graph $D^\pi(x)$ is equivalent to the failure of that vertex and the failure of any arc adjacent to that vertex, since in both cases the failed cycles and chains are the same. Thus, for every vertex $u \in V$, we can build a scenario where the only non-zero value in vector $\tilde{\gamma}'_v$ is $\tilde{\gamma}'_{vu} = 1$. This dominates the scenario where all arcs either leave $u$, i.e., $\gamma^a_{uv} = 1$ or point towards it, i.e., $\tilde{\gamma}_{vu} = 1$. Note that the following constraints satisfy Proposition 5.4:

$$\sum_{(u,v) \in A} \gamma^a_{uv} + \sum_{(v,u) \in A} \gamma^a_{vu} \leq r^a \left(1 - \gamma^\pi_v\right), \quad u \in V$$

(11)

That is, if a vertex fails the arcs adjacent to it get disconnected from the graph. Therefore, the objective value of a recourse problem with a failure scenario where an arc adjacent to a failed vertex also fails is equivalent to the objective value obtained when we consider the scenario with only the vertex failure. Thus, Constraints (11) can be added to MS and MT before the start of Algorithm 2.

**Single-vertex-arc separation** Suppose we have a set of proposed failed arcs $A(\tilde{\gamma}^a)$ such that $\gamma^a_{uv} = 1 \forall (u, v) \in A(\tilde{\gamma}^a)$, and a set of proposed failed vertices, $V(\tilde{\gamma}^v)$, such that $\tilde{\gamma}^v_u = 1 \forall u \in V(\tilde{\gamma}^v)$. The second-stage compatibility graph corresponds to $D^\pi(\tilde{x}, \tilde{\gamma}) = (V^{\pi,a} \setminus V(\tilde{\gamma}^v), A^{\pi,a} \setminus A(\tilde{\gamma}^a))$. Then, the following two cases also satisfy Proposition 5.4:

1. Suppose there is a candidate pair $\tilde{v} \in V^{\pi,a} \setminus V(\tilde{\gamma}^v)$ and $C^{\pi,0}_{K,L}(\tilde{x})$ is the set of feasible cycles and chains in $D^\pi(\tilde{x}, \tilde{\gamma})$ that include vertex $\tilde{v}$. Then, if $C^{\pi,0}_{K,L}(\tilde{x}) = \emptyset$, scenario $\tilde{\gamma}$ dominates scenario $\tilde{\gamma} \cup \{\tilde{v}\}$ and thus for $Z^\pi_Q(\tilde{x})$ to decrease, another vertex $\tilde{v}$ should be proposed to fail.

2. Suppose there is a candidate arc $\tilde{a} \in A^{\pi,a} \setminus A(\tilde{\gamma}^a)$ and $C^{\pi,0}_{K,L}(\tilde{x})$ is the set of feasible cycles and chains in $D^\pi(\tilde{x}, \tilde{\gamma})$ that include arc $\tilde{a}$. Then, if $C^{\pi,0}_{K,L}(\tilde{x}) = \emptyset$, scenario $\tilde{\gamma}$ dominates scenario $\tilde{\gamma} \cup \{\tilde{a}\}$ and thus another arc $\tilde{a}$ should be proposed to fail if $Z^\pi_Q(\tilde{x})$ is to be lowered.

We perform the single-vertex-arc separation within a heuristic while attempting to find a feasible failure scenario satisfying the constraints in MS and MT, as detailed in the next section.
6 Solution algorithms for the second-stage problem

This section presents solution algorithms to solve the second stage. We refer to hybrid solution algorithms as the combination of a linear optimization solver and a heuristic attempting to exactly solve the second-stage problem. Our goal is to specialize the steps of Algorithm 2 to improve the efficiency of our final solution approaches.

6.1 A feasibility-based solution algorithm for MasterBasic

The first of our two feasibility-based solution algorithms has MS as master problem, and it is referred to as FBSA{MB}. FBSA{MB} requires a policy $\pi \in \Pi$ and a first-stage solution $\tilde{x}$ found at Step 0 of Algorithm 1. We refer to $\text{ToMng}(\cdot)$ as a function that “extracts” the matching from a recourse solution. This matching is then added to a set $\mathcal{M}$ containing all matchings associated with the recourse solutions found up to some iteration $i$ of the algorithm. At Step 0, $\mathcal{M}$ takes the matching associated with the first-stage solution $\tilde{x} \in \mathcal{X}$ as input. The procedure $\text{Heuristic} (\mathcal{M})$—whose algorithmic details we present shortly, attempts to find a failure scenario satisfying the constraints in MS. The heuristic returns a tuple $(\gamma', \text{cover})$ with two outputs. The first one corresponds to a candidate failure scenario $\gamma'$. The second output is $\text{cover}$, a boolean variable that indicates whether $\gamma'$ satisfies Constraints (7b). At Step 0, because there is only one matching in $\mathcal{M}$, it is trivial to heuristically choose one element (vertex or arc) that satisfies Constraint (7a) and thus the result of the boolean variable $\text{cover}$ is true. This failure scenario is then added to $\bar{\Gamma}(\tilde{x})$. At Step 1, we attempt to find the optimal recourse solution through a column generation (CG) algorithm, $\text{ColGen}(R^\pi(\tilde{x}, \tilde{\gamma}))$, for which we generate the sets $C_\pi^R(\tilde{x})$ and $C_\pi^L(\tilde{x})$ every time a new first-stage decision is made. For very large instances, large-scale decomposition algorithms such as the one proposed in (Riascos-Álvarez et al., 2020) can be used to find positive-price columns (cycles or chains) as opposed to searching through all cycles and chains. As usual in a CG algorithm, only positive-price columns are added to its master problem iteratively. Once the optimality of the CG master problem is proven, an upper bound $Z^{UB}_{cg}$ on the optimal value of the recourse problem is returned. Then, we take the decision variables from the master problem base and turn them into binary ones to obtain a feasible recourse solution $\tilde{y}$ with objective value $Z^*_cg$. If $Z^*_cg$ equals the upper bound $Z^{UB}_{cg}$, then $\tilde{y}$ is optimal to formulation $R^\pi(\tilde{x}, \tilde{\gamma})$. If that is the case, then at Step 1, $\tilde{y}$ becomes the optimal recourse solution $y^*$ and both $Z^*_R(\tilde{x}, \tilde{\gamma})$ and $\mathcal{M}$ are updated. Otherwise, we incur in the cost of solving $R^\pi(\tilde{x}, \tilde{\gamma})$ from scratch as a MIP instance, and update $Z^*_R(\tilde{x}, \tilde{\gamma})$ and $\mathcal{M}$, correspondingly. If the new recourse value $Z^*_R(\tilde{x}, \tilde{\gamma})$ is smaller than the upper bound on the objective value of the second stage, $Z^*_Q(\tilde{x})$, then we update both $Z^*_Q(\tilde{x})$ and the incumbent failure scenario $\gamma^*$ that led to that value. Since a new lower bound has been found, the right-hand side of Constraints (7a)-(7b) is updated following Proposition 5.2. At Step 2, a new iteration starts. We try to find a feasible failure scenario in $\text{MS+Constraints(11)}$ through either $\text{Heuristic}(\mathcal{M})$ or through solving $\text{MS+Constraints(11)}$ as a MIP instance. If a feasible failure scenario is found, then the recourse problem is solved again starting from Step 1. Otherwise, at Step 3, the algorithm ends and returns the optimal recovery plan $(Z^*_Q(\tilde{x}), \gamma^*)$. 

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Algorithm 3 FBSA_MB: A Feasibility-based Solution Algorithm for MS

Input: A recourse policy $\pi \in \Pi$ and first-stage solution $\bar{x} \in X$ and current KEP robust objective $\hat{Z}_P$

Output: Optimal recovery plan value $Z^{\pi,\star}_Q(\bar{x})$ and worst-case scenario $\gamma^* \in \Gamma(\bar{x})$

Step 0:
1: $i = 1$; $I \leftarrow I \cup \{i\}$; $Z^{\pi}_Q(\bar{x}) \leftarrow \hat{Z}_P$; $M \leftarrow \text{ToMng}(\bar{x})$
2: $(\gamma', \text{true}) \leftarrow \text{Heuristic}(M)$
3: $\tilde{\gamma} \leftarrow \gamma'$; $\hat{\Gamma}(\bar{x}) \leftarrow \hat{\Gamma}(\bar{x}) \cup \{\tilde{\gamma}\}$

Step 1:
4: $(Z_{cg}^*, Z_{cg}^{UB}, \tilde{\gamma}) \leftarrow \text{ColGen}(\overline{R}(\bar{x}, \tilde{\gamma}))$
5: if $Z_{cg}^* = Z_{cg}^{UB}$ then
6: $Z^*_R(\bar{x}, \tilde{\gamma}) \leftarrow Z_{cg}^*$
7: $y^{\tilde{\gamma}} \leftarrow \tilde{\gamma}$; $M \leftarrow M \cup \text{ToMng}(y^{\tilde{\gamma}})$
8: else
9: Solve $\overline{R}(\bar{x}, \tilde{\gamma})$ to obtain solution $y^{\tilde{\gamma}} \in \mathcal{Y}(\bar{x}, \tilde{\gamma})$ with objective value $Z^*_R(\bar{x}, \tilde{\gamma})$; $M \leftarrow M \cup \text{ToMng}(y^{\tilde{\gamma}})$
10: if $Z^*_R(\bar{x}, \tilde{\gamma}) < Z^*_Q(\bar{x})$ then
11: $Z^*_Q(\bar{x}) \leftarrow Z^*_R(\bar{x}, \tilde{\gamma})$; $\gamma^* \leftarrow \tilde{\gamma}$
12: else
13: Update right-hand side of constraints in MS, $\forall \tilde{\gamma} \in \hat{\Gamma}(\bar{x})$ following Proposition 5.2

Step 2:
14: $i \leftarrow i + 1$
15: $(\gamma', \text{cover}) \leftarrow \text{Heuristic}(M)$
16: if $\text{cover} = \text{true}$ then
17: $\tilde{\gamma} \leftarrow \gamma'$; $\hat{\Gamma}(\bar{x}) \leftarrow \hat{\Gamma}(\bar{x}) \cup \{\tilde{\gamma}\}$
18: Go to Step 1
19: else
20: Create (7b) for $y^{\tilde{\gamma}} \in \mathcal{Y}(\bar{x}, \tilde{\gamma})$. Attempt to solve MS+Consts.(11) to get new candidate scenario $\hat{\gamma}$
21: if MS is feasible then
22: $\hat{\Gamma}(\bar{x}) \leftarrow \hat{\Gamma}(\bar{x}) \cup \{\hat{\gamma}\}$
23: Go to Step 1
24: Return $\gamma^*$ and $Z^*_Q(\bar{x})$
6.2 A feasibility-based solution algorithm for MasterExp

Our second feasibility-based solution algorithm has MT as the master problem of the second stage, and we refer to it as FBSA_ME. FBSA_ME requires a recourse policy \( \pi \in \Pi \) and a first-stage solution \( \tilde{x} \in X \). In addition to the notation defined for FBSA_MB, we introduce new one. We refer to \( \mathcal{M}_{\text{exp}} \) as the set of matchings associated to the optimal recourse solutions when solving the recourse problem in the transitory graph \( D^\pi(\tilde{x}) \). The CG algorithm, \texttt{XColGen} \( (R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma})) \), returns a tuple with three values: \( Z^\text{UB}_{Xcg} \), \( Z^*_Xcg \) and \( \tilde{y} \). \( Z^\text{UB}_{Xcg} \) corresponds to the optimal objective value returned by the master problem of our CG algorithm; \( Z^*_Xcg \) is the feasible solution to the KEP after turning the decision variables of the optimal basis of the CG master problem into binaries. Lastly, \( \tilde{y} \) is the feasible recourse solution found by the CG algorithm with objective value \( Z^*_Xcg \). At the start, FBSA_ME takes the input the matching associated to the first-stage solution \( \tilde{x} \in X \) as input, and \texttt{Heuristic}(\mathcal{M}) returns the first failure scenario. At Step 1, we solve the linear relaxation of recourse problem \( R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}) \) through \texttt{XColGen} \( (R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma})) \). If \( Z^*_Xcg = Z^\text{UB}_{Xcg} \) then a function \texttt{TrueVal}(\mathcal{Z}_{Xcg}) returns the weighted sum of the cycles and chains in \( \tilde{y} \) that did not fail under the current failure scenario \( \tilde{\gamma} \in \widehat{\Gamma}(\tilde{x}) \). In case the feasible solution \( Z^*_Xcg \) has a lower value than \( Z^\text{UB}_{Xcg} \), then the original recourse problem \( R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}) \) is solved as a MIP instance. The reason for this decision is that \( R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}) \) includes a larger set of cycle-and-chain decision variables, possibly leading to scalability issues when the recourse problem is solved as a MIP instance. For iterations where this occurs, i.e., \( Z^*_Xcg < Z^\text{UB}_{Xcg} \), both \( \mathcal{M} \) and \( \mathcal{M}_{\text{exp}} \) take as input the matching associated to \( y^* \tilde{\gamma} \). If the upper bound \( Z^\pi_Q(\tilde{x}) \) is updated, so is the right-hand side of constraints in MT. At Step 2, a new iteration starts. A new attempt to find a feasible failure scenario is made first by \texttt{Heuristic}(\mathcal{M} \cup \mathcal{M}_{\text{exp}}) and then by solving MT+Constraints(11) as a MIP instance. If a feasible scenario is found, then a new recourse solution is found at Step 1. Otherwise, FBSA_ME returns an optimal solution to the second stage at Step 3.

6.3 Hybrid solution algorithms

The main difference between the following algorithms and the feasibility-based algorithms is the possibility of transitioning from the feasibility-seeking master problems to the optimality-seeking problem SSF after TR iterations. The goal is to obtain a lower bound on the objective value of the second stage, \( Z^\pi_Q(\tilde{x}) \), that can be compared to \( Z^\pi_Q(\tilde{x}) \) to prove optimality. We refer to these hybrid algorithms as HSA_MB and HSA_ME. The former has MS as master problem, whereas the latter has MT. For both hybrid algorithms we solve the recourse problem \( R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}) \) in the transitory graph, but in the case of HSA_MB whenever an attempt to find a failure scenario occurs, it is done with respect to MS+Constraints(11). The reason for this approach is to accumulate the recourse solutions given by \( R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}) \) and use them to solve SSF when the number of iterations exceeds TR. Algorithm 5 presents the changes required in Algorithms 3 and 4 to obtain HSA_MB and HSA_ME. Specifically, Algorithm 5 should be included in lines 12 and 14 of Algorithm 3 and Algorithm 4, respectively.
Algorithm 4 FBSA_ME: A Feasibility-based Solution Algorithm for MT

Input: A recourse policy \( \pi \in \Pi \), a first-stage solution \( \bar{x} \in \mathcal{X} \) and current KEP robust objective \( \bar{Z}_P \)

Output: Optimal recovery plan value \( Z_{\pi^*,\bar{\gamma}}(\bar{x}) \) and worst-case scenario \( \bar{\gamma}^* \in \Gamma(\bar{x}) \)

Step 0:
1. \( i = 1; \bar{Z} \bar{Q}(\bar{x}) \leftarrow \bar{Z}_P; \mathcal{M} \leftarrow \text{ToMng}(\bar{x}) \)
2. \( (\gamma', \text{true}) \leftarrow \text{Heuristic}(\mathcal{M}) \)
3. \( \bar{\gamma} \leftarrow \gamma'; \bar{\Gamma}(\bar{x}) \leftarrow \bar{\Gamma}(\bar{x}) \cup \{ \bar{\gamma} \} \)

Step 1:
4. \( (Z^*_{Xcg}, Z^*_{UB_{Xcg}}: \bar{\gamma}) \leftarrow \text{XColGen}(R^\pi_{Xcg}(\bar{x}, \bar{\gamma})) \)
5. if \( Z^*_{Xcg} = Z^*_{UB_{Xcg}} \) then
   6. \( \psi^*_{\bar{\gamma}} \leftarrow \bar{\gamma} \)
   7. \( Z^*_R(\bar{x}, \bar{\gamma}) \leftarrow \text{TrueVal}(Z^*_{Xcg}) \)
   8. \( \mathcal{M}_{\text{exp}} \leftarrow \mathcal{M}_{\text{exp}} \cup \text{ToMng}(\psi^*_{\bar{\gamma}}); \mathcal{M} \leftarrow \mathcal{M} \cup \text{ToMng}(\psi^*_{\bar{\gamma}}) \)
9. else
10. Solve \( R^\pi(\bar{x}, \bar{\gamma}) \) to obtain recourse objective value \( Z^*_R(\bar{x}, \bar{\gamma}) \) and recourse solution \( \bar{y}^*_{\bar{\gamma}} \);
   11. \( \mathcal{M}_{\text{exp}} \leftarrow \mathcal{M}_{\text{exp}} \cup \text{ToMng}(\bar{y}^*_{\bar{\gamma}}); \mathcal{M} \leftarrow \mathcal{M} \cup \text{ToMng}(\bar{y}^*_{\bar{\gamma}}) \)
12. if \( Z^*_R(\bar{x}, \bar{\gamma}) < Z^*_Q(\bar{x}) \) then
13. Update right-hand side of (9a) and (9b) following Proposition 5.2 and (9c) following Proposition 5.3 \( \forall \bar{\gamma} \in \bar{\Gamma}(\bar{x}) \).
14. \( i \leftarrow i + 1 \)
15. \( (\gamma', \text{cover}) \leftarrow \text{Heuristic}(\mathcal{M} \cup \mathcal{M}_{\text{exp}}) \)
16. if \( \text{cover} = \text{true} \) then
17. \( \bar{\gamma} \leftarrow \gamma' \)
18. Go to Step 1
19. else
20. Create Const.(9b) for \( \psi^*_{\bar{\gamma}} \in \mathcal{Y}^\pi(\bar{x}, \bar{\gamma}) \) and Const.(9c) for \( \psi^*_{\bar{\gamma}} \in \mathcal{Y}^\pi_{\text{tr}}(\bar{x}) \). Attempt to solve \( \text{MT} + \text{Consts.}(11) \) to get new candidate scenario \( \bar{\gamma} \);
21. if \( \text{MT} \) is feasible then
22. \( \bar{\Gamma}(\bar{x}) \leftarrow \bar{\Gamma}(\bar{x}) \cup \{ \bar{\gamma} \} \)
23. Go to Step 1
24. Step 3: \( Z^*_Q(\bar{x}) \leftarrow Z^*_Q(\bar{x}); \text{Return} \ \gamma^* \) and \( Z^*_Q(\bar{x}) \)

Algorithm 5 Additional steps for hybrid solution algorithms, HSA_MB and HSA_ME

Input: Parameter TR indicating iteration threshold at which SSF is solved

1. if \( i \geq \text{TR} \) then
2. Create one Const.(8k) per matching in \( \mathcal{M} \) or \( \mathcal{M}_{\text{exp}} \) and solve SSF to obtain \( Z^*_Q(\bar{x}) \) and failure scenario \( \bar{\gamma} \)
3. if \( Z^*_Q(\bar{x}) = Z^*_Q(\bar{x}) \) then
4. \( \gamma^* \leftarrow \bar{\gamma} \); Go to Step 3 in Algorithm 3 or Algorithm 4
5. else
6. Go to Step 1 in Algorithm 3 or Algorithm 4
6.4 Heuristic

We now discuss the heuristic (Algorithm 6), used in the feasibility-based and hybrid algorithms. Algorithm 6 requires a set of matchings $W$ as input. The heuristic returns a failure scenario $\gamma'$ and a boolean variable $\text{cover}$. If $\text{cover} = \text{true}$, $\gamma'$ satisfies either MS+Constraints(11) or MT+Constraints(11), accordingly. Although the heuristic cannot ensure that $\gamma'$ is always feasible with respect to the feasibility-seeking master problems, it does ensure (i) Constraints (11) are satisfied and (ii) $\gamma'$ is not dominated according to the two single-vertex-arc separation cases presented in Section 5.2.3.

We introduce some relevant notation, and then detail the steps of the heuristic. A function $\text{UniqueElms}(W)$ returns the set of unique vertices and arcs among all matchings in $W$, which are then kept in $E$. With some abuse of notation, we say that each vertex/arc $e_n \in E$ with $n = 1, \ldots, |E|$ has an associated boolean variable $s_n$ and a calculated weight $w_n$. The boolean variable $s_n$ becomes true if element $e_n$ has been checked, i.e., either it has been proposed for failure in $\gamma'$ or it has been proven that adding $e_n$ to the current failure scenario will result in a dominated scenario according to Section 5.2.3. Function $\text{Weight}(e_n, W)$ returns the weight $w_n$ corresponding to the number of times element $e_n$ is repeated in the set of matchings $W$. Moreover, a function $\text{IsNDD}(e_n^*)$ determines whether element $e_n^*$ is a non-directed donor. While it is true, a list of unique and non-checked elements $E$ is created as long as the vertex and arc budgets, $r^v$ and $r^a$ are not exceeded, respectively. If $E$ turns out to have no elements, the algorithm ends in line 10. Otherwise, the elements in $E$ are sorted in non-increasing order of their weights, and among the ones with the highest weight, an element $e_n^*$ is selected randomly. At line 14, the single-vertex-arc separation described in Section 5.2.3 is performed if (i) $e_n^*$ is a pair, and (ii) there is at least one arc or at least two vertices proposed for failure already. The reason we check that $e_n^*$ is a pair is because a non-directed donor does not belong to a cycle by definition, and when removed from the transitory graph or second-stage compatibility graph, its removal trivially causes the failure of all chains triggered by it. Between lines 15 and 24 we perform the two single-vertex-arc separation procedures using the same notation as in Section 5.2.3. A boolean variable $\text{ans}$ remains true if $e_n^*$ is a non-directed donor or a vertex/arc in a non-proven dominated scenario. Element $e_n^*$ is labeled as checked by setting $e_n^* = \text{true}$. Then, if $\text{ans} = \text{true}$, it is checked whether $e_n^*$ is a vertex or an arc, and the proposed failure scenario $\gamma$ is updated accordingly. If element $e_n^*$ is a vertex, at line 28 the set of arcs adjacent to $e_n^*$ is labeled as checked, so that $\gamma$ satisfies Constraints (11). We say that scenario $\gamma$ covers all matchings in $W$, if for every matching in that set, the number of vertices/arcs proposed for failure satisfies Proposition 5.2 (for Constraints (7a)-(7b) or Constraints (9a)-(9b)) and Proposition 5.3 (for Constraints (9c)), accordingly. If $\gamma$ covers all matchings in $W$, then $\text{cover} = \text{true}$. The heuristic is greedy in the sense that even when $\text{cover} = \text{true}$, it will attempt to use up the vertex and arc budgets as checked at line 34. If another vertex/arc can still be proposed for failure, then a new iteration in the While loop starts at line 3. Thus, the heuristic ends at either lines 10 or 35.
Algorithm 6 Heuristic

Input: A set of matchings $W$

Output: Failure scenario $\bar{\gamma}$ and a boolean variable $\text{cover}$ indicating whether $\bar{\gamma}$ covers matchings in $W$

Step 0:
1: $\text{cover} = \text{false}$; $E \leftarrow \text{UniqueElms}(W)$

Step 1:
2: while true do
3: $\mathcal{E} = \emptyset$
4: for $n = 1, \ldots, |E|$ do
5: if $e_n \in A$ and $s_n = \text{false}$ and $\sum_{u \in V} \bar{\gamma}_u^v < r^v$ then
6: $w_n = \text{Weight}(e_n, W)$; $\mathcal{E} \leftarrow \mathcal{E} \cup \{e_n\}$
7: else if $e_n \in V$ and $s_n = \text{false}$ and $\sum_{(u,v) \in A} \bar{\gamma}_{uv}^a < r^a$ then
8: $w_n = \text{Weight}(e_n, W)$; $\mathcal{E} \leftarrow \mathcal{E} \cup \{e_n\}$
9: if $\mathcal{E} = \emptyset$ then
10: Go to Step 2
11: Sort $\mathcal{E}$ in non-increasing order of values s.t. $w_1 \geq w_2 \geq \ldots \geq w_{|E|}$
12: Select $e^* \in \mathcal{E}$ randomly among elements whose weight equals $w_1$
13: $\text{ans} = \text{true}$ —Start single-vertex-arc separation 5.2.3—
14: if $(\sum_{u \in V} \gamma_u^v + \sum_{(u,v) \in A} \gamma_{uv}^a) \geq 1$ and $\text{IsNDD}(e^*_n) = \text{false}$ and $(\sum_{(u,v) \in A} \gamma_{uv}^a \geq 1$ or $\sum_{u \in V} \gamma_u^v \geq 2$) then
15: $V(\bar{\gamma}^v) \leftarrow V(\gamma^v)$; $A(\bar{\gamma}^a) \leftarrow A(\gamma^a)$
16: $\text{ans} = \text{false}$
17: if $e^*_n \in V$ then
18: $\bar{v} \leftarrow e^*_n$; $C_{K,L}^{\pi,\bar{v}}(\bar{x}) \leftarrow \text{Check Case 1}$
19: if $C_{K,L}^{\pi,\bar{v}}(\bar{x}) \neq \emptyset$ then
20: $\text{ans} = \text{true}$
21: else
22: $\bar{a} \leftarrow e^*_n$; $C_{K,L}^{\pi,\bar{a}}(\bar{x}) \leftarrow \text{Check Case 2}$
23: if $C_{K,L}^{\pi,\bar{a}}(\bar{x}) \neq \emptyset$ then
24: $\text{ans} = \text{true}$ —End separation—
25: $e^*_n = \text{true}$
26: if $\text{ans} = \text{true}$ then
27: if $e^*_n \in V$ then
28: $\gamma_{e_n}^v = 1$; $s_n = \text{true} \ \forall e_n \in \delta^+(e^*_n) \cup \delta^-(e^*_n)$
29: else
30: $\gamma_{e_n}^a = 1$
31: if $\bar{\gamma}$ covers all matchings in $W$ then
32: $\text{cover} = \text{true}$
33: if $\text{cover} = \text{true}$ then
34: if $\sum_{u \in V} \gamma_u^v = r^v$ and $\sum_{(u,v) \in A} \gamma_{uv}^a = r^a$ then
35: Go to Step 2
36: Step 2: Return $\bar{\gamma}$ and $\text{cover}$
7 Computational Experiments

In this section we present the results of our computational experiments. We used the same instances tested in Blom et al. (2021); Carvalho et al. (2020). There are three sets of 30 instances each with 20, 50 and 100 vertices. Our analysis focuses on the set with 100 vertices to study the performance under homogeneous failure since the other two sets are less challenging, and on the sets with 50 and 100 vertices for our policy-based analysis in the non-homogeneous case. There are 30 instances within every instance set. In the first part of this section, we compare the efficiency of our solution approaches to the state-of-the-art algorithm addressing the full-recourse policy under homogeneous failure, Benders-PICEF, proposed by Blom et al. (2021). Next, we analyze the performance and practical impacts of our approaches under non-homogeneous failure for the full-recourse and first-stage-only recourse policies presented in Section 3.2. Our implementations are coded in C++ using CPLEX 12.10, including the state-of-the-art algorithm we compare against, on a machine with Debian GNU/Linux as the operating system and a 3.60GHz processor Intel(R) Core(TM). A time limit of one hour is given to every run. The TR value for the HSA\textunderscore MB and HSA\textunderscore ME algorithms was 150 iterations for each run.

7.1 Homogeneous failure analysis

Figure 2 shows the performance profile for five algorithms: our lifted-constraint version of Benders-PICEF, FBSA\textunderscore MB, FBSA\textunderscore ME, HSA\textunderscore MB and HSA\textunderscore ME. Recall that FBSA\textunderscore MB and FBSA\textunderscore ME are feasibility-based algorithms without the optimization step. Figure 2 shows that solving the recourse problem in the transitory graph pays off for FBSA\textunderscore ME compared to FBSA\textunderscore MB. Although Benders-PICEF solves more instances in general than FBSA\textunderscore MB and FBSA\textunderscore ME, the performance of FBSA\textunderscore ME noticeably outperforms that of Benders-PICEF when the maximum length of cycles is four and that of chains is three and up to three vertices are allowed to fail. In the same settings, FBSA\textunderscore MB is comparable to Benders-PICEF. However, as soon as the feasibility-seeking master problems incorporate the optimization step (Algorithm 5), the performance of HSA\textunderscore MB and HSA\textunderscore ME is consistently ahead of all other algorithms. As we show shortly, in most cases HSA\textunderscore MB and HSA\textunderscore ME need a small percentage of iterations in the optimization version of the second-stage master problem to converge. Benders-PICEF is fast when cycles and chains of size up to three and four are considered, respectively; however, increased cycle length and budget failure result in worse performance of Benders-PICEF compared to the hybrid approaches.

Table 1 summarizes the computational performance details of FBSA\textunderscore ME, HSA\textunderscore ME and Benders-PICEF. On average, column 2SndS divided by column 1stS indicates the number of iterations that were needed per first-stage iteration to solve the second-stage problem. The average number of iterations per first-stage iteration can also be obtained by adding up columns Alg\textunderscore 6\textunderscore true, MT and SSF and then dividing the sum by column 1stS. The first observation is that the CG algorithm successfully finds optimal recourse solutions in most cases and it is responsible for most of the average total time for FBSA\textunderscore ME and HSA\textunderscore ME. On the other hand, the average total number of iterations needed by FBSA\textunderscore ME to converge was significantly higher than that needed by HSA\textunderscore ME and yet the average total time per iteration of the second stage with respect to the total time
(2ndS/total) is between 0.16 and 1.84 seconds.

FBSA_ME clearly generates scenarios that SSF would not explore due to its cycle-and-chain decision variables and minimization objective, both properties that FBSA_MB and FBSA_ME lack. However, also for the full recourse policy, Blom et al. (2021) tested a master problem analogous to SSF and showed that due to the large number of cycles and chains in the formulation, its scalability is limited as the size of cycles and chains grows. We find that solving the feasibility problem is much more efficient than solving SSF. In fact, (i) the heuristic Alg6, even when close to 1700 iterations, only accounts for about 26% of the CPU time, and (ii) even when the heuristic fails and MT is solved as a MIP, around 1000 iterations are needed to account for about 39% of the total time, while for SSF, <200 iterations are needed to account for about the same percentage.

Observe that in some cases the average total number of second-stage iterations spent by HSA_ME is a third of those spent by FBSA_ME, indicating that SSF helps convergence. However, the number of SSF iterations per first-stage solution (SSF/1stS) is on average 40.5 while the same statistics for the feasibility-seeking iterations are on average 132.5, thus, highlighting that MT helps reduce the number of iterations spent by SSF. As a result, the hybrid algorithms are able to converge quickly in all the tested settings. Thus, the feasibility-seeking master problems are able to find near-optimal failure scenarios within the first hundreds of iterations, but in the absence of a lower bound, they may require thousands of iterations to reach infeasibility and thus prove optimality.
Table 1: Computational performance details of FBSA_ME, HSA_ME and Benders-PICEF under homogeneous failure and full recourse for 30 instances with $|V| = 100$. MPBP and RPBP are the master problem and recourse problem presented in Blom et al. (2021), respectively. A column is left empty if it does not apply. Data includes both optimal and sub-optimal runs.

### FBSA_ME

| K | L | $r^*$ | opt | Avg. time (s) | Avg. total # iterations |
|---|---|---|---|---|---|
| 3 | 2 | 30 | 130.86 | 10.71 | 2.56 | 62.18 |
| 3 | 3 | 125.26 | 14.93 | 31.89 | 1.90 | 37.68 |
| 3 | 4 | 1804.25 | 26.32 | 39.55 | 1.52 | 24.91 |
| 3 | 2 | 1076.27 | 0.44 | 2.59 | 4.80 | 70.77 |
| 3 | 3 | 2226.59 | 3.71 | 7.72 | 4.79 | 65.04 |
| 3 | 4 | 2952.83 | 5.40 | 9.80 | 3.51 | 63.95 |
| 4 | 3 | 638.01 | 1.38 | 5.11 | 1.24 | 74.94 |
| 4 | 3 | 1681.14 | 8.01 | 20.28 | 0.84 | 62.28 |
| 4 | 4 | 2632.47 | 17.62 | 23.54 | 0.70 | 53.31 |
| 4 | 3 | 1385.25 | 0.29 | 1.56 | 2.49 | 81.29 |
| 4 | 3 | 2658.12 | 1.59 | 4.16 | 2.15 | 78.13 |
| 4 | 4 | 3197.12 | 4.37 | 5.91 | 2.11 | 75.68 |

### HSA_ME

| K | L | $r^*$ | opt | Avg. time (s) | Avg. total # iterations |
|---|---|---|---|---|---|
| 3 | 2 | 48.91 | 1.71 | 7.62 | 2.94 | 57.36 |
| 3 | 3 | 436.87 | 2.39 | 10.61 | 3.01 | 44.25 |
| 3 | 4 | 825.24 | 2.60 | 9.07 | 2.32 | 35.9 |
| 3 | 4 | 719.42 | 0.27 | 1.85 | 5.44 | 69.56 |
| 3 | 4 | 1409.03 | 0.40 | 1.73 | 3.57 | 63.41 |
| 4 | 3 | 264.44 | 0.69 | 2.94 | 1.27 | 73.33 |
| 4 | 3 | 473.65 | 1.81 | 2.97 | 1.08 | 68.36 |
| 4 | 3 | 1243.41 | 0.21 | 1.15 | 2.14 | 50.57 |
| 4 | 3 | 1554.62 | 0.46 | 0.89 | 1.79 | 73.24 |
| 4 | 3 | 1900.28 | 0.58 | 0.54 | 1.84 | 72.32 |

### Benders-PICEF

| K | L | $r^*$ | opt | Avg. time (s) | Avg. total # iterations |
|---|---|---|---|---|---|
| 3 | 2 | 26 | 271.82 | 6.64 | 78.39 |
| 3 | 3 | 28 | 722.85 | 23.65 | 66.11 |
| 3 | 4 | 854.58 | 10.61 | 3.01 | 44.25 |
| 3 | 4 | 135.46 | 6.70 | 80.54 |
| 3 | 4 | 867.87 | 18.64 | 75.15 |
| 3 | 4 | 1122.50 | 24.25 | 71.05 |
| 3 | 4 | 1801.11 | 0.73 | 92.42 |
| 3 | 4 | 2434.24 | 2.26 | 91.52 |
| 3 | 4 | 4181.72 | 3.73 | 91.97 |
| 3 | 4 | 2099.70 | 0.63 | 91.13 |
| 4 | 3 | 1544.62 | 0.46 | 0.89 | 1.79 |
| 4 | 3 | 2486.05 | 2.31 | 91.35 |
| 4 | 4 | 2659.81 | 2.81 | 93.09 |

### Notes

- **Avg. time**: Average time spent in each part of the optimization.
- **Avg. total # iterations**: Average total number of iterations.
- **1stS**: Total # of first-stage decisions required before finding the robust solution.
- **2ndS**: Total # of iterations to solve the second-stage problem for all the first-stage decisions.
- **Alg6-true**: Total # of iterations the heuristic found a feasible failure scenario.
- **CG-true**: Total # of iterations the CG algorithm found an optimal recourse solution.

### Definitions

- **MPBP**: Time solving MPBP as a MIP when the heuristic failed to find a feasible solution.
- **RPBP**: Time solving RPBP as a MIP when the CG algorithm failed to find an optimal recourse solution.
- **SSF**: Total # of iterations SSF was solved as a MIP when more than TR = 150 iterations passed without MT becoming infeasible.
- **1stS**: Total # of first-stage decisions required before finding the robust solution.
- **2ndS**: Total # of iterations to solve the second-stage problem for all the first-stage decisions.
- **Alg6-true**: Total # of iterations the heuristic found a feasible failure scenario.
- **CG-true**: Total # of iterations the CG algorithm found an optimal recourse solution.
7.2 Non-homogeneous failure analysis

In this part of the analysis, we focus on understanding both the scalability of the solution algorithms under non-homogeneous failure and the impact of the two recourse policies on the total number of pairs that can be re-arranged into new cycles and chains. We additionally examine the percent of re-arranged pairs that correspond to highly-sensitized patients, who are more impacted by non-homogenous failure rates due to their real-life higher failure rates. Pairs in the instances published by Carvalho et al. (2020) have an associated panel reactive antibody (PRA), which determines how likely a patient may reject a potential donor. The higher this value, the more sensitized a patient becomes and thus the patient becomes less likely to get a compatible donor. Typically, a patient with a PRA greater or equal to 90% is considered sensitized.

Table 2 presents computational details for HSA_MB with a maximum cycle length of three. A similar table for a maximum cycle length of four is presented in Appendix D. The average total number of dominated scenarios for instances in Table 1 was negligible and thus, not presented. The non-homogeneous case is more difficult to solve to based on the average total time, the average total number of first-stage iterations, and the average percentage of total time to solve the optimality problem SSF after 150 iterations. The average number of dominated scenarios is particularly high for the first-stage-only recourse. This behaviour may be explained because when a failure occurs, the transitory graph under this policy is smaller and thus more susceptible to having vertices and arcs that can no longer be part of cycles and chains. In terms of the robust objective, all instances that were able to be solved optimally under both policies, obtained the same objective. This interesting fact indicates that for the tested instances, there exists a set of transplants pairs that can be recovered amongst themselves, and there are as many as those that can be recovered by the full-recourse policy.

The percentage of highly-sensitized patients that can be recovered is 30% on average, with more variability in the first-stage-only recourse (range [12.50%, 51.89%]) than in the full recourse (range [20.25%, 48.88%]). It seems to increase for larger instances for the first-stage-only recourse. The length of chains does seem to have a positive effect on some settings but it is not a generalized effect. For larger networks (|V| = 100), when the maximum length of chains is increased from 3 to 4, and when the probability of arc failure increases, fewer instances are solved to optimality.

8 Conclusion

We presented a general solution framework for KPDPs to identify robust matchings that can be repaired after failure through a predefined recourse policy. We achieve this goal by solving a two-stage RO problem; we proposed two decompositions where the master problems seek feasibility to prove optimality of the second-stage problem. We also showed that the number of decision variables in our feasibility-seeking master problems grows polynomially with the vertices and arcs in the compatibility graph, which increases their scalability. The cost of this advantage is an inability to obtain a lower bound. To overcome this drawback, we proposed (i) separating a family of dominated scenarios using a heuristic that also attempts to find a feasible failure scenario for our master problems and (ii) solving the optimality-seeking counterpart of our master problems.
Table 2: Computational performance details of HSA MB under non-homogeneous failure for 30 instances with $K = 3$. Columns are mostly the same as in Table 1, with new columns defined below. Data includes both optimal and sub-optimal runs.*

| $\text{Full recourse}$ | $\text{Avg. total #}$ |
|------------------------|-----------------------|
| $L$ | $r^a$ | $r^b$ | HSP (%) | Total [s] | Alg1 (%) | MT(%) | RE (%) | CG (%) | SSF (%) | 1stS | 2ndS | Alg1-true | MT | RE | SSFtrue | DomS |
| | | | | \(\frac{V}{r}\) | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | |
| $|V| = 50$ | | | | | | | | | | | | | | | | | |
| 1 | 1.80 | 0.0 | 34.21 | 20.13 | 2.69 | 35.18 | 2.06 | 22.80 | 19.08 | 3.73 | 395.13 | 167.73 | 228.40 | 8.87 | 396.27 | 22.57 | 2.13 |
| 2 | 1.80 | 0.0 | 25.36 | 22.07 | 2.94 | 26.95 | 0.81 | 17.18 | 42.41 | 3.30 | 481.67 | 178.90 | 292.77 | 10.47 | 471.20 | 52.93 | 1.70 |
| 3 | 1.80 | 0.0 | 29.73 | 16.76 | 2.59 | 30.82 | 1.05 | 35.35 | 15.33 | 6.30 | 784.70 | 185.93 | 598.77 | 11.33 | 771.37 | 52.67 | 3.97 |
| 4 | 1.80 | 0.0 | 25.12 | 639.03 | 1.99 | 23.75 | 0.57 | 31.47 | 56.42 | 3.60 | 543.23 | 198.63 | 292.99 | 11.56 | 531.73 | 117.50 | 4.77 |
| 1 | 1.80 | 0.0 | 34.33 | 27.07 | 1.71 | 22.58 | 5.02 | 34.48 | 8.87 | 2.87 | 338.70 | 101.97 | 236.73 | 7.73 | 330.97 | 12.83 | 1.93 |
| 2 | 1.80 | 0.0 | 28.51 | 316.27 | 1.76 | 17.42 | 2.73 | 27.20 | 32.20 | 2.87 | 349.03 | 173.80 | 223.23 | 5.33 | 433.70 | 42.30 | 1.33 |
| 3 | 1.80 | 0.0 | 26.30 | 327.59 | 1.43 | 16.67 | 3.22 | 24.20 | 34.20 | 6.21 | 801.80 | 219.20 | 515.59 | 12.30 | 736.50 | 68.70 | 3.47 |
| 4 | 1.80 | 0.0 | 22.82 | 663.15 | 1.53 | 15.36 | 2.71 | 24.22 | 46.67 | 2.97 | 479.20 | 203.40 | 265.00 | 8.87 | 489.71 | 71.87 | 3.40 |

| $|V| = 100$ | | | | | | | | | | | | | | | | | |
| 1 | 3.27 | 16 | 48.88 | 2182.01 | 0.40 | 3.28 | 1.28 | 8.72 | 81.56 | 3.13 | 633.47 | 193.83 | 240.64 | 22.53 | 610.93 | 174.63 | 3.83 |
| 2 | 3.27 | 4 | 31.11 | 3201.20 | 0.33 | 2.35 | 0.39 | 4.17 | 89.23 | 1.93 | 376.23 | 236.60 | 122.63 | 10.47 | 365.77 | 87.10 | 1.30 |
| 3 | 5.90 | 3 | 30.11 | 3256.26 | 0.04 | 0.14 | 0.15 | 0.95 | 90.96 | 1.31 | 250.63 | 169.33 | 29.97 | 9.97 | 240.67 | 51.53 | 6.23 |
| 4 | 3.27 | 7 | 41.48 | 2951.24 | 0.14 | 1.20 | 7.80 | 34.43 | 44.43 | 2.37 | 475.50 | 214.67 | 133.80 | 11.80 | 463.70 | 127.03 | 1.60 |
| 5 | 3.27 | 4 | 37.62 | 3444.24 | 0.09 | 0.41 | 3.84 | 23.47 | 63.66 | 1.47 | 287.70 | 184.43 | 32.27 | 8.73 | 271.17 | 71.00 | 0.90 |
| 1 | 5.90 | 2 | 30.25 | 3417.89 | 0.05 | 0.22 | 4.08 | 18.12 | 70.91 | 1.57 | 320.27 | 163.17 | 46.47 | 8.73 | 311.53 | 88.63 | 2.20 |
| 2 | 5.90 | 1 | 27.27 | 3579.92 | 0.02 | 0.08 | 2.70 | 14.77 | 74.21 | 1.30 | 240.43 | 172.37 | 19.27 | 8.03 | 232.40 | 48.89 | 1.60 |

| $r^a$: Average number of arcs that fail, corresponding to either 5% or 10% of the arcs in the deterministic solution of an instance* | $r^b$: Average percentage of highly-sensitized patients re-matched in the second stage. Includes only instances solved to optimality. | $\text{DomS}$: Average total number of dominated scenarios that are found by the single-vertex-arc separation procedure (Section 5.2.3) |
as a finisher. Our final solution algorithms solve the two-stage robust problem exactly. We compared our solution methods with the state of the art under homogeneous failure. We could not compare to the state of the art under non-homogeneous failure, as our work is the first attempt to solve the two-stage robust problem in kidney exchange with non-homogeneous failure. Some of our algorithms outperform the state of the art when no lower bound is known. Our feasibility-based solution algorithms, consistently outperform the state-of-the-art algorithm when allowed to solving their optimality-seeking counterpart. Under non-homogeneous failure, we compared the efficiency and ability to re-assign highly-sensitized patients into new cycles and chains of the two policies studied in this paper. We find that while the percentage of highly-sensitized patients that can be re-arranged varies across runs, 30% seems to be the lower bound.

Our work can be slightly modified to define an uncertainty set where the failure of an arc/vertex has a non-unitary weight and thus the failure budget is measured in terms of the allowed weight for failures to occur. Likewise, the recourse function also allows for pairs to have different weights in case of prioritizing a specific set of patients. As future directions, we highlight that the definition of the uncertainty set can include the setting where the intervention of some arcs/vertices (e.g., immunotherapy, running a questionnaire on pairs) reduces the likelihood of an exchange failing if resources are allocated for that intervention. Moreover, we assumed that the failure of a vertex/arc completely removes it from the graph—which is indeed reasonable—but some vertices/arcs could experience failure in a different way, e.g., by arcs decreasing their arc weight or vertices becoming temporarily unavailable. Thus, a different meaning of failure is also a non yet explored research avenue.

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A Robust formulation for full recourse

Our solution approach supports the use of any MIP formulation that has the structure shown in Section 4. For the results presented in this paper, we adopt a variant presented by Carvalho et al. (2020) of the position-indexed cycle edge formulation (PICEF) proposed by Dickerson et al. (2016). Cycles are modeled through cycle variables $\gamma_c \forall c \in C_K$ for first-stage decisions and through $z^\gamma_c \forall c \in C_K$ for recourse cycles under scenario $\gamma \in \Gamma$. However, chains are modeled through first-stage decision variables $\delta_{uv\ell}$, indexed by arc $(u, v) \in A$ and the feasible position $\ell \in \mathcal{L}(u, v)$ of that arc within a chain. The set $\mathcal{L}(u, v) \subseteq \mathcal{L} = \{1, ..., L\}$ corresponds to the set of positions for which that arc is reached from some non-directed donor in a simple path with $\ell \leq L$ arcs. For vertices $u \in N$, the set of possible arc positions becomes $\mathcal{L}(u,v) = \{1\}$, since non-directed donors always start a chain. To identify $\mathcal{L}(u,v)$ for the other arcs, a shortest-path based search can be performed (Dickerson et al., 2016). Thus, we define $A_f = \{(u,v) \in A \mid \ell \in \mathcal{L}(u,v)\}$. Likewise, recourse chain decision variables for every scenario $\gamma \in \Gamma$ are denoted by $\delta^\gamma_{uv\ell}$. A binary decision variable $t^\gamma_c$ is also defined for every pair $v \in P$ and scenario $\gamma \in \Gamma$ to identify the pairs that are selected in both the first stage and in the second stage under some scenario $\gamma \in \Gamma$. Moreover, we denote by $C^\gamma_v$ and $C^\gamma_K$ the set of feasible cycles including vertex $v \in P$ and the set of feasible cycles including arc $(u, v) \in A$, respectively.

\[
\begin{align*}
\text{max} \quad & Z \\
Z - \sum_{v \in P} t^\gamma_v & \leq 0 \\
t^\gamma_v - \sum_{c \in C^\gamma_v} z_c - \sum_{\ell \in \mathcal{L}(u,v) \in A} \delta_{uv\ell} & \leq 0 & \gamma \in \Gamma, v \in P \\
t^\gamma_v - \sum_{c \in C^\gamma_v} z^\gamma_c - \sum_{\ell \in \mathcal{L}(u,v) \in A} \delta^\gamma_{uv\ell} & \leq 0 & \gamma \in \Gamma, v \in P \\
\sum_{u \in (u,v)} \delta^\gamma_{uv1} & \leq 1 - \gamma^v & \gamma \in \Gamma, v \in P \\
\sum_{c \in C^\gamma_v} z^\gamma_c - \sum_{\ell \in \mathcal{L}(u,v)} \delta^\gamma_{uv\ell} & \leq 1 - \gamma^v & \gamma \in \Gamma, (u,v) \in A \\
\sum_{(u,v) \in A_f} \delta^\gamma_{uv\ell} & \leq 0 & \gamma \in \Gamma, v \in P, \ell \in \mathcal{L} \setminus \{L - 1\} \\
\sum_{(u,v) \in A_f} \delta^\gamma_{uv\ell} & \leq 1 & v \in N \\
\sum_{c \in C^\gamma_K} \delta_{uv\ell} & \leq 1 & u \in P \\
\sum_{(u,v) \in A_f} \delta_{uv\ell} - \sum_{(u,v) \in A_f} \delta^\gamma_{uv(\ell+1)} & \leq 0 & \gamma \in \Gamma, (u,v) \in A, \ell \in \mathcal{L}(u,v) \\
\end{align*}
\]
Constraints (12b) are used to determine the scenario binding the number of patients that receive a transplant in both stages. Such scenario is the worst-case scenario. The objective (12a) is then equivalent to the maximum number of patients from the first stage that can be recovered in the second stage under the worst-case scenario. Constraints (12c) and (12d) assure that a pair \( v \) is counted as recovered in the objective if it is selected in the first-stage solution (Constraints (12c)) and it is also selected in the second stage under scenario \( \gamma \in \Gamma \) (Constraints (12d)). Constraints (12e) to Constraints (12g) guarantee that the solution obtained for every scenario \( \gamma \in \Gamma \) is a matching. Specifically, Constraints (12e) assure that if a non-directed donor fails under some scenario \( \gamma \in \Gamma \), i.e., \( \gamma_v = 1 \) then its corresponding arcs in position one cannot be used to trigger a chain. Similarly, Constraints (12f) guarantee that if a pair fails, then it cannot be present in either a cycle or a chain. Constraints (12g) ensure that when an arc \((u,v)\) fails under some scenario \( \gamma \in \Gamma \), it does not get involved in either a cycle or a chain. Constraints (12h) assure the continuity of a chain by selecting arcs in consecutive positions. Constraints (12i) to Constraints (12k) select a solution corresponding to a matching in the first stage. The remaining constraints correspond to the nature of the decision variables.

\[ \max \quad Z \]  
\[ (12b) - (12n) \]  
\[ \sum_{c \in C^K} z^\gamma_c - \sum_{\ell \in \mathcal{L}} \sum_{(u,v) \in A_{\ell}} \delta^\gamma_{uv\ell} \leq \sum_{c \in C^K} z_c - \sum_{\ell \in \mathcal{L}} \sum_{(u,v) \in A_{\ell}} \delta_{uv\ell} \quad \gamma \in \Gamma, v \in P \]  

C The recourse problem

We present in this section two algorithms to enumerate the feasible cycles and chains in \( C^\gamma_K(x) \) and \( C^\gamma_L(x) \), respectively, that lead to a transitory graph \( D^\gamma(x) \) and a realization of the second-stage compatibility graph \( D^\gamma(x, \tilde{\gamma}) \) under scenario \( \tilde{\gamma} \in \Gamma(x) \).

C.1 Cycles and chains for the full-recourse policy

We start by defining some notation. Let \( P(\tilde{x}) = V(\tilde{x}) \cap P \) be an auxiliary set, corresponding to the pairs selected in the first-stage solution. Moreover, we denote by \( C^u_K \) the set of feasible cycles that include vertex \( u \in P(\tilde{x}) \). Lastly, consider \( R \) as an auxiliary vertex set. Algorithm 7 iteratively builds \( C^\text{Full}_K(\tilde{x}) \). At the start, \( R \) and \( C^\text{Full}_K(\tilde{x}) \) are empty. Within the While loop, a pair \( u \) from the first-stage solution is selected. Then, in line 4, a deep search procedure, starting from vertex \( u \), is used to find \( C^u_K \) in graph \( \tilde{D} = (V \setminus R, A) \). Note that if \( R \neq \emptyset \), the new cycles are found in a graph where the previously selected vertices (the ones in set \( R \)) are removed, since otherwise, cycles already in \( C^u_K \) could be found again. The new cycles are then added to \( C^\text{Full}_K(\tilde{x}) \).
Algorithm 7 Obtaining $C^\pi_K(\tilde{x})$ with $\pi =$ Full

**Input:** Set with pairs from the first-stage solution, $P(\tilde{x})$

**Output:** Set $C^{\text{Full}}_K(\tilde{x})$

**Step 0:**
1: $R = \emptyset$; $C^{\text{Full}}_K(\tilde{x}) = \emptyset$

**Step 1:**
2: while $P(\tilde{x}) \neq \emptyset$ do
3: Select vertex $u \in P(\tilde{x})$
4: Find $C^u_K$ in graph $\tilde{D} = (V \setminus R, A)$ from vertex $u$
5: $C^{\text{Full}}_K(\tilde{x}) \leftarrow C^{\text{Full}}_K(\tilde{x}) \cup C^u_K$
6: $P(\tilde{x}) \leftarrow P(\tilde{x}) \setminus \{u\}$
7: $R \leftarrow R \cup \{u\}$

**Step 2:**
8: Return $C^{\text{Full}}_K(\tilde{x})$

Algorithm 8 Obtaining $C^\pi_L(\tilde{x})$ with $\pi =$ Full

**Input:** A first-stage solution $\tilde{x} \in \mathcal{X}$

**Output:** Set $C^{\text{Full}}_L(\tilde{x})$

**Step 0:**
1: $R = \emptyset$; $C^{\text{Full}}_L(\tilde{x}) = \emptyset$

**Step 1:**
2: while $\tilde{N} \neq \emptyset$ do
3: Select vertex $u \in \tilde{N}$
4: for all $1 \leq \ell \leq L$ do
5: Find $C^u_\ell$ in graph $D = (V, A)$ from vertex $u$
6: $C^{\text{Full}}_L(\tilde{x}) \leftarrow C^{\text{Full}}_L(\tilde{x}) \cup C^u_\ell$
7: $\tilde{N} \leftarrow \tilde{N} \setminus \{u\}$

**Step 2:**
8: Return $C^{\text{Full}}_L(\tilde{x})$

and vertex $u$ is removed from $P(\tilde{x})$. When no more vertices are left in that set, the algorithm ends.

The correctness of Algorithm 7 follows from the fact that only cycles/chains with at least one vertex from the first stage contribute to the weight of a recourse solution. Thus, it suffices to find such cycles per every vertex in $P(\tilde{x})$. Similar reasoning is used when finding chains. To this end, we include additional notation. Let $\tilde{N} = N$ be an auxiliary set that corresponds to the set of non-directed donors. Moreover, let $C^u_\ell$ be the set of chains that include at least one pair in $P(\tilde{x})$ and are triggered by vertex $u \in N$ with exactly $1 \leq \ell \leq L$ arcs. Algorithm 8 iteratively builds the set of chains that include at least one pair that is selected in the first stage.

C.2 Cycles and chains for the first-stage-only recourse policy

Algorithm 7 can be modified to accommodate the first-stage-only recourse for cycles. Specifically, we can replace $C^{\text{Full}}_K(\tilde{x})$ by $C^{1\text{stSO}}_K(\tilde{x})$, where the latter is the set of simple cycles with at least one vertex in $P(\tilde{x})$ satisfying the first-stage-only recourse policy. In line 4, the vertex set $V$ in graph $\tilde{D}$ is replaced by $P(\tilde{x})$ so that only pairs from the first stage can be part of the allowed cycles. The arc set of $\tilde{D}$ can then be defined such that every arc in it has a starting and terminal vertex in $P(\tilde{x})$. Likewise, Algorithm 8 can also support chains for the first-stage-only recourse. The only change in Algorithm 8 is to replace graph $D = (V, A)$ by graph $\tilde{D} = (V(\tilde{x}), \tilde{A})$ where every arc in the arc set $\tilde{A}$ has both extreme vertices in $V(\tilde{x})$.

The algorithms just described are used to obtain the cycles and chains that can participate in a recourse
solution when the recourse problem is solved as a sub-problem in the robust decomposition presented in the next section.

C.3 Formulations for the recourse problem

In this section we present the cycle-and-chain MIP formulations presented in (Blom et al., 2021) adapted to the problem we study. However, we note that a MIP formulation for the recourse problem does not require the explicit enumeration of cycles and chains. The advantage of enumeration is that different policies can be easily addressed, specifically, the full recourse and the first-stage-only recourse.

Recall that $\mathcal{C}_K(\bar{x}, \tilde{\gamma}) \cup \mathcal{C}_L(\bar{x}, \tilde{\gamma}) \subseteq \mathcal{C}_K(\bar{x}) \cup \mathcal{C}_L(\bar{x})$ is the set of non-failed cycles/chains in a second-stage compatibility graph under scenario $\tilde{\gamma} \in \Gamma$ and policy $\pi \in \Pi$, i.e., $\sum_{u \in V(c)} \gamma^u_u + \sum_{(u,v) \in A(c)} \gamma^a_{uv} = 0 \ \forall c \in \mathcal{C}_K(\bar{x}, \tilde{\gamma}) \cup \mathcal{C}_L(\bar{x}, \tilde{\gamma})$. We present the recourse problem based on the so-called cycle formulation (Abraham et al., 2007) as follows:

$$R^\pi(\bar{x}, \gamma) : \max_y \sum_{c \in \mathcal{C}_K(\bar{x}, \tilde{\gamma}) \cup \mathcal{C}_L(\bar{x}, \tilde{\gamma})} w_c(\bar{x}) y_c$$

$$\sum_{c : u \in V(c)} y_c \leq 1 \ \forall u \in V$$ (14a)

$$y_c \in \{0, 1\} \ \forall c \in \mathcal{C}_K(\bar{x}, \tilde{\gamma}) \cup \mathcal{C}_L(\bar{x}, \tilde{\gamma})$$ (14b)

Here, $y_c$ is a decision variable for every cycle/chain $c \in \mathcal{C}_K(\bar{x}, \tilde{\gamma}) \cup \mathcal{C}_L(\bar{x}, \tilde{\gamma})$, taking on value one if selected, and zero otherwise. Constraints (14a) ensure that a vertex belongs to at most one cycle/chain. An optimal solution to formulation $R^\pi(\bar{x}, \tilde{\gamma})$ finds a matching with the greatest number of matched pairs from the first stage after observing failure scenario $\tilde{\gamma}$. Thus, by solving formulation (14), a new Constraint (8f) can be created.

Blom et al. (2021) proposed to expand recourse solutions by including, in addition to the optimal non-failed cycles/chains found by formulation (14), failed cycles and chains while guaranteeing that the solution is still a matching. Although an expanded solution does not contribute to more recourse value under the failure scenario in consideration, it may imply other violated constraints. Consider two recourse solutions $\psi^{\ast \tilde{\gamma}} \in \mathcal{Y}_\Pi(\bar{x})$ and $y^{\ast \tilde{\gamma}} \in \mathcal{Y}(\bar{x}, \tilde{\gamma})$, one in the transitory compatibility graph and another one in the second-stage compatibility graph, respectively, and also assume that $y^{\ast \tilde{\gamma}} \subseteq \psi^{\ast \tilde{\gamma}}$. Then, constraint (8f) associated to $y^{\ast \tilde{\gamma}}$ is directly implied by that of $\psi^{\ast \tilde{\gamma}}$, i.e., if the constraint corresponding to $y^{\ast \tilde{\gamma}}$ is violated, so is the constraint corresponding to $\psi^{\ast \tilde{\gamma}}$.

We find expanded recourse solutions by solving a deterministic KEP in the transitory graph $D^\pi(\bar{x})$ instead of the second-stage graph $D^\pi(\bar{x}, \tilde{\gamma})$ in formulation (14), and by assigning new weights to all cycles/chains $c \in \mathcal{C}_K(\bar{x}) \cup \mathcal{C}_L(\bar{x})$ as in (Blom et al., 2021). The new weights, assigned to each cycle $c \in \mathcal{C}_K(\bar{x}) \cup \mathcal{C}_L(\bar{x})$, are given by

$$\hat{w}_c(\bar{x}, \tilde{\gamma}) = \begin{cases} w_c(\bar{x})|V| + 1 & \text{if } V(c) \subseteq V_\gamma \\ 1 & \text{otherwise} \end{cases}$$ (15)
We denote by $R^\pi_{\exp}(\bar{x}, \bar{\gamma})$ the resulting recourse problem with expanded solutions,

$$R^\pi_{\exp}(\bar{x}, \bar{\gamma}) : \max_{y} \sum_{c \in C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x})} \hat{w}_c(\bar{x}, \bar{\gamma}) y_c \tag{RE}$$

subject to

$$\sum_{c \in u \in V(c)} y_c \leq 1 \quad \forall c \in C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x})$$

$$y_c \in \{0, 1\} \quad \forall c \in C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x})$$

The following lemma, based on (Blom et al., 2021), states that the set of cycles/chains that do not fail in a recourse solution optimal to $R^\pi_{\exp}(\bar{x}, \bar{\gamma})$ is an optimal recourse solution to the original recourse problem $R^\pi(\bar{x}, \bar{\gamma})$.

**Lemma C.1.** For a recourse solution $\psi^{\pi} \in \mathcal{Y}^\pi_{\exp}(\bar{x})$ that is optimal to $R^\pi_{\exp}(\bar{x}, \bar{\gamma})$, its set of non-failed cycles/chains, $\psi^{\pi} \subseteq \psi^{\pi}$, under scenario $\hat{\gamma} \in \Gamma(\bar{x})$ is an optimal recourse solution to $R^\pi(\bar{x}, \bar{\gamma})$.

**Proof.** Let $Z^\pi_{\exp}(\bar{x}, \bar{\gamma})$ be the optimal objective value to solution $\psi^{\pi} \in \mathcal{Y}^\pi_{\exp}(\bar{x})$ and consider $|V|/2$ as the maximum number of cycles/chains that can originate in a feasible matching, i.e., the maximum number of decision variables for which $\psi^{\pi}_c = 1 \forall c \in C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x})$. Moreover, let $\mathbb{1}_c$ be an indicator variable that takes on value one if $\psi^{\pi}_c = 1 \forall c \notin C^\pi_K(\bar{x}, \bar{\gamma}) \cup C^\pi_L(\bar{x}, \bar{\gamma})$ and zero otherwise. Then, from (15) we know that

$$Z^\pi_{\exp}(\bar{x}, \bar{\gamma}) = \sum_{c \in C^\pi_K(\bar{x}, \bar{\gamma}) \cup C^\pi_L(\bar{x}, \bar{\gamma})} (w_c(\bar{x})|V| + 1) + \sum_{c \notin C^\pi_K(\bar{x}, \bar{\gamma}) \cup C^\pi_L(\bar{x}, \bar{\gamma})} \mathbb{1}_c \psi^{\pi}_c = 1$$

$$= |V| \sum_{c \in C^\pi_K(\bar{x}, \bar{\gamma}) \cup C^\pi_L(\bar{x}, \bar{\gamma})} \psi^{\pi}_c + \sum_{i=1}^{\lfloor V/2 \rfloor} 1 + \sum_{i=1}^{\lfloor V/2 \rfloor} 1$$

$$= z^*|V| + |V|$$

Now, suppose $\psi^{\pi}$ is not optimal to $R^\pi(\bar{x}, \bar{\gamma})$, therefore $\sum_{c \in C^\pi_K(\bar{x}, \bar{\gamma}) \cup C^\pi_L(\bar{x}, \bar{\gamma})} w_c(\bar{x})$ can be at most $z^* - 1$. If that is true, then, when replacing $z^*$ in (17c) by $z^* - 1$, we obtain that $Z^\pi_{\exp}(\bar{x}, \bar{\gamma}) = z^*|V| + |V| = z^*|V|$, which is a contradiction. Thus, it follows that $\psi^{\pi}$ must be optimal to $R^\pi(\bar{x}, \bar{\gamma})$. \hfill $\square$

It is worth noting that we formulated the recourse problem by means of cycle-and-chain decision variables, but since it is reduced to a deterministic KEP, the recourse problem can be solved via multiple formulations/algorithms from the literature, e.g., (Omer et al., 2022; Blom et al., 2021; Riascos-Álvarez et al., 2020).

**D Additional results for non-homogeneous failure**

The data in Table 3 includes all runs, optimal or not, except for column HSP(%) where only instances that were solved to optimality are included. The interpretation of the displayed columns correspond to those in Table 2.
Table 3: Policies comparison for $K = 4$ under HSA_MB and non-homogeneous failure.

| $V_r$ | $V_r = 503$ | $V_r = 1003$ |
|-------|--------------|--------------|
| $L_r$ | 1.83         | 1.83         |
| $r_r$ | 30           | 30           |
| $A_{opt}$ | 35.98       | 3.30         |
| HSP(%) | 21.98        | 45.42        |
| total | 35.98        | 45.42        |
| Alg6(%) | 35.98       | 45.42        |
| MT(%) | 0.19         | 3.30         |
| RE(%) | 14.41        | 7.23         |
| CG(%) | 0.72         | 0.13         |
| SSF(%) | 14.41        | 0.13         |
| 1stS | 35.98        | 45.42        |
| 2ndS | 14.41        | 7.23         |
| Alg6-true | 35.98 | 45.42 |
| MT | 0.19 | 3.30 |
| RE | 14.41 | 7.23 |
| CGtrue | 0.72 | 0.13 |
| SSF | 14.41 | 0.13 |
| DomS | 35.98 | 45.42 |

E Summary of Notation

The following is a list of the notation used throughout the paper and it is divided by sections. Although there is notation common to multiple sections, we present it in the section where it was introduced for the first time.

Preliminaries

- **$K$:** Maximum allowed length of cycles, user defined.
- **$L$:** Maximum allowed length of chains in terms of arcs, user defined.
- **$P$:** Set of patient-donor pairs or simply set of pairs.
- **$N$:** Set of non-directed donors.
- **$V := P \cup N$:** represents the set of patient-donor pairs, $P$, and the set of non-directed donors $N$.
- **$A \subseteq V \times P$:** Arc set containing arc $(u, v)$ if and only if the donor in vertex $u \in V$ is compatible with patient in vertex $v \in P$.
- **$D = (V, A)$:** First-stage compatibility graph.
- **$C_r$:** Set of feasible cycles (up to size $K$) in the first-stage compatibility graph.

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• $C_L$: Set of feasible chains (up to size $L$) in the first-stage compatibility graph.

• $V(\cdot)$ and $A(\cdot)$ be the set of vertices and arcs in $(\cdot)$, where $(\cdot)$ refers to a cycle, chain, etc.

• $M \subseteq C_K \cup C_L$: A feasible solution to the KEP, i.e., a collection of vertex-disjoint cycles and chains (referred to as a matching) if $V(c) \cap V(c') = \emptyset$, for all $c, c' \in M$ with $c \neq c'$, noting that $c$ and $c'$ are entire cycles or chains in the matching.

• $M_D$: Set of all KEP matchings in the first-stage compatibility graph $D$.

• $\mathcal{X} := \{x_M : M \in M_D\}$ is the set of all binary vectors representing the selection of a feasible matching in the first-stage compatibility graph, where $x_M$ is the characteristic vector of matching $M$ in terms of the cycles/chains sets $C_K \cup C_L$.

• $x \in \mathcal{X}$: A first-stage solution representing a matching in the first-stage compatibility graph $D$.

• $\pi \in \Pi$: Recourse policy (or simply policy) deciding on the cycles and chains that can be used to repair a first-stage decision $x \in \mathcal{X}$ after observing vertex/arc failures.

• $C_{\pi K}(x_M)$: Set of all cycles (up to size $K$) in the first-stage compatibility graph satisfying policy $\pi \in \Pi$ and including at least one pair in $x_M$.

• $C_{\pi L}(x_M)$: Set of all chains (up to size $L$) in the first-stage compatibility graph satisfying policy $\pi \in \Pi$ and including at least one pair in $x_M$.

• $D^\pi(x) = (V^\pi, x, A^\pi, x)$: Transitory graph with vertex set $V^\pi, x = \bigcup_{c \in C_{\pi K}(x_M) \cup C_{\pi L}(x_M)} V(c)$ and arc set $A^\pi, x = \bigcup_{c \in C_{\pi K}(x_M) \cup C_{\pi L}(x_M)} A(c)$.

• $V(x) = \bigcup_{c' \in C_K \cup C_L : x_{c'} = 1} V(c')$: Vertices (pairs and non-directed donors) selected by a first-stage solution $x \in \mathcal{X}$.

• $\gamma \in \Gamma$: A failure scenario represented as a binary vector in the uncertainty set $\Gamma$.

• $V^\gamma$ and $A^\gamma$: Set of vertices and set of arcs that fail under scenario $\gamma \in \Gamma$ in the first-stage compatibility graph, respectively.

• $D^\pi(x, \gamma) = (V^\pi, x \setminus \{V^\gamma \cap V(x)\}, A^\pi, x \setminus \{A^\gamma \cap A(x)\})$: A second-stage compatibility graph induced in the transitory graph $D^\pi(x)$ by a failure scenario $\gamma \in \Gamma$.

• $C_{\pi K}^\gamma(x, \gamma)$: Set of existing cycles (up to size $K$) in a second-stage compatibility graph as they do not fail under scenario $\gamma \in \Gamma$ and are allowed by policy $\pi \in \Pi$.

• $C_{\pi L}^\gamma(x, \gamma)$: Set of existing chains (up to size $L$) in a second-stage compatibility graph as they do not fail under scenario $\gamma \in \Gamma$ and are allowed by policy $\pi \in \Pi$. 
\[ M^\pi(x, \gamma) := \{ M \subseteq C^\pi_K(x, \gamma) \cup C^\pi_L(x, \gamma) \mid V(c) \cap V(c') = \emptyset \text{ for all } c, c' \in M; c \neq c' \} \]: Set of allowed recovering matchings under policy \( \pi \) such that every cycle/chain in \( M^\pi(x, \gamma) \) contains at least one pair in \( x \).

\[ Y^\pi(x, \gamma) := \{ y_M : M \in M^\pi(x, \gamma) \} \]: Set of all binary vectors representing the selection of a feasible matching in a second-stage compatibility graph \( D^\pi(x, \gamma) \) with non-failed elements (vertices/arc) under scenario \( \gamma \in \Gamma \) and policy \( \pi \in \Pi \) that contain at least one pair in \( x \). Here, \( y_M \) is the characteristic vector of matching \( M \) in terms of \( C^\pi_K(x, \gamma) \cup C^\pi_L(x, \gamma) \).

\( y \in Y^\pi(x, \gamma) \): A second-stage solution (a.k.a recourse solution) found in a second-stage compatibility graph.

\( f(x, \gamma, y) \): General robust objective function (a.k.a recourse objective function) that assigns weights to the cycles and chains of a recovered matching associated to a recourse solution \( y \) under failure scenario \( \gamma \), based on the number of pairs matched in the first stage solution \( x \).

\( w_c(x) := |V(c) \cap V(x) \cap P| \): weight of a cycle/chain \( c \in C^\pi_K(x, \gamma) \cup C^\pi_L(x, \gamma) \) corresponding to the number of pairs that having been matched in the first stage by solution \( x \in X \), can also be matched in the second stage by recourse solution \( y \in Y^\pi(x, \gamma) \), after failures are observed.

\( \Gamma(x) \in \Gamma \): Observable uncertainty set in the transitory graph when the first-stage solution is \( x \).

\( \gamma^v \in \{0, 1\}^{|V|} \): Binary vector representing a vertex failure scenario. If the entry corresponding to vertex \( u \in V \) in \( \gamma^v \) takes on value one then \( u \) fails, and on value zero otherwise.

\( \gamma^a \in \{0, 1\}^{|A|} \): Binary vector representing an arc failure scenario. If the entry corresponding to arc \( (u, v) \in A \) in \( \gamma^a \) takes on value one then \( (u, v) \) fails, and on value zero otherwise.

\( r^v \): User-defined number of vertices that can occur in the first-stage compatibility graph.

\( r^a \): User-defined number of arcs that can occur in the first-stage compatibility graph.

**Robust model and first stage**

- \( \tilde{\Gamma} \): Restricted set of scenarios for the first-stage problem FSF.

- \( \tilde{x} \in X \): Optimal first-stage solution to FSF in Algorithm 1 with restricted set of scenarios \( \Gamma \).

- \( \tilde{Z}_P \): Objective value of solution \( \tilde{x} \in X \).

**New second-stage decompositions**

- \( \tilde{\Gamma}(\tilde{x}) \subseteq \Gamma(\tilde{x}) \): Restricted set of failure scenarios observable in the transitory graph.

- \( \tilde{\gamma} \in \Gamma(\tilde{x}) \): A failure scenario that induces the second-stage compatibility graph \( D^\pi(\tilde{x}, \tilde{\gamma}) \).
• \( R^\pi(\bar{x}, \gamma) \): MIP formulation (R) for the recourse problem when solved in the second-stage compatibility graph \( D^\pi(\bar{x}, \gamma) \).

• \( C^\pi_{K,L}(\bar{x}, \gamma) = C^\pi_K(\bar{x}, \gamma) \cup C^\pi_L(\bar{x}, \gamma) \).

• \( y^\gamma \): Optimal recourse solution to the recourse problem \( R^\pi(\bar{x}, \gamma) \) with objective value \( Z^\pi_{R^\pi}(\bar{x}, \gamma) \).

• \( Z^\pi_{Q^\pi}(\bar{x}, \gamma) \): Optimal objective of the second-stage problem SSP.

• \( Z^\pi_{Q^\pi}(\bar{x}, \gamma) \): Decision variable indicating the objective value of the second-stage problem SSP.

• \( \bar{Z}^\pi_{Q^\pi}(\bar{x}) \): Upper bound on the objective value of the second-stage problem SSP.

• \( 1_{c, \gamma} \): Indicating variable that takes on value one if cycle/chain \( c \in C^\pi_{K,L}(\bar{x}, \gamma) \) fails under scenario \( \gamma \in \Gamma(\bar{x}) \).

• \( Y^\pi_{tr}(\bar{x}) \): Set of binary vectors representing a recourse solution in the transitory graph.

• \( R^\pi_{exp}(\bar{x}, \gamma) \): MIP formulation (RE) for the recourse problem solved in the transitory graph, whose optimal recourse solutions are also optimal to \( R^\pi(\bar{x}, \gamma) \) but may allow some failed cycles/chains in the solution.

• \( \psi^\gamma \in Y^\pi_{tr}(\bar{x}) \): Optimal recourse solution to \( R^\pi_{exp}(\bar{x}, \gamma) \) in the transitory graph that is also optimal to \( R^\pi(\bar{x}, \gamma) \) under scenario \( \gamma \in \hat{\Gamma}(\bar{x}) \).

• \( \psi^\gamma \subseteq \psi^\gamma \): Subset of the optimal recourse solution \( \psi^\gamma \) that has no failed cycles/chains and thus corresponds to a feasible solution in \( Y^\pi(\bar{x}, \gamma) \).

• \( I(\gamma) \): Number of failed vertices and arcs under scenario \( \gamma \).

• \( I(\gamma) \): Number of failed vertices and arcs under scenario \( \gamma \).

• \( C^\pi_{K,L}(\bar{x}, \gamma) \subseteq C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x}) \): Set of feasible cycles and chains that fail in the transitory graph \( D^\pi(\bar{x}) \) under scenario \( \gamma \).

• \( C^\pi_{K,L}(\bar{x}, \gamma) \subseteq C^\pi_K(\bar{x}) \cup C^\pi_L(\bar{x}) \): Set of feasible cycles and chains that fail in the transitory graph \( D^\pi(\bar{x}) \) under scenario \( \gamma \).

• \( C^\pi_{v,\bar{v}}(\bar{x}) \): Set of feasible cycles and chains in \( D^\pi(\bar{x}, \gamma) \) that include vertex \( \bar{v} \).

• \( C^\pi_{a,\bar{a}}(\bar{x}) \): Set of feasible cycles and chains in \( D^\pi(\bar{x}, \gamma) \) that include arc \( \bar{a} \).

Solution algorithms for the second-stage problem

• FBBSA \_MB: Algorithm 3.

• FBBSA \_ME: Algorithm 4.

• \( Z^\pi_{Q^\pi}(\bar{x}) \): Lower bound on the objective value of the second-stage problem SSP.

• HSA \_MB and HSA \_ME: Algorithms with additional steps needed for FBBSA \_MB and FBBSA \_ME to find \( Z^\pi_{Q^\pi}(\bar{x}) \), respectively. Such steps are given in Algorithm 5.
• $\mathcal{M}, \mathcal{M}_{\text{exp}}, W$: Set of matchings

• $\text{ToMng}(\cdot)$: Function that “extracts” the matching from a recourse solution ($\cdot$).

• $\text{Heuristic}(\cdot)$: Algorithm 6 which takes as input a set of matchings ($\cdot$) and returns a tuple ($\gamma', \text{cover}$).

• $\text{ColGen}(R^\pi(\tilde{x}, \tilde{\gamma}))$: Column generation algorithm used to solve the linear relaxation of $R^\pi(\tilde{x}, \tilde{\gamma})$ with optimal value $Z_{c\text{g}}^\text{UB}$.

• $Z_{c\text{g}}^\star$: Objective value of the feasible KEP solution obtained when the columns obtained by $\text{ColGen}(R^\pi(\tilde{x}, \tilde{\gamma}))$ are turned to binary.

• $\text{XColGen}(R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}))$: Column generation algorithm used to solve the linear relaxation of $R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma})$ with objective value $Z_{\text{Xcg}}^\text{UB}$.

• $Z_{\text{Xcg}}^\star$: Objective value of the feasible KEP solution obtained when the columns obtained by $\text{XColGen}(R^\pi_{\text{exp}}(\tilde{x}, \tilde{\gamma}))$ are turned to binary.

• $\text{TrueVal}(Z_{\text{Xcg}}^\star)$: A function that returns $w_c(x) := |V(c) \cap V(x) \cap P|$ for all cycles and chains $c$ in the feasible solution with objective value $Z_{\text{Xcg}}^\star$.

• $\text{UniqueElms}(W)$: Function that returns the set of unique vertices and arcs among all matchings in $W$, referred to as $E$.

• $e_n \in E$: Element, either vertex or arc in the set $E$.

• $\text{Weight}(e_n, W)$: Function that returns the weight $w_n$ of an element $e_n$ corresponding to the number of times $e_n$ is repeated in the set of matchings $W$.

• $\text{IsNDD}(e_n^\star)$: Function that returns true if element $e_n^\star$ is a non-directed donor, and false otherwise.