UNBOUNDED ABSOLUTE WEAK CONVERGENCE IN BANACH LATTICES

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Abstract. Several recent papers investigated unbounded versions of order and norm convergences in Banach lattices. In this paper, we study the unbounded variant of weak convergence and its relationship with other convergences. In particular, we characterize order continuous Banach lattices and reflexive Banach lattices in terms of this convergence.

Throughout this paper, $E$ stands for a Banach lattice. A net $(x_\alpha)$ in $E$ is said to be unbounded order convergent (uo-convergent) to $x$ if for every $u \in E_+$ the net $|x_\alpha - x| \land u$ converges to zero in order. It is called unbounded norm convergent (un-convergent) to $x$ if $\| |x_\alpha - x| \land u \| \to 0$ for every $u \in E_+$. These concepts were investigated in [DOT17, GX14, G14, GTX17, KMT17]. We consider the unbounded version of weak convergence. We say that $(x_\alpha)$ is unbounded absolutely weakly convergent (uaw-convergent) to $x$ if $|x_\alpha - x| \land u$ converges to zero weakly for every $u \in E_+$; we write $x_\alpha \xrightarrow{\text{uaw}} x$. For undefined terminology, we refer the reader to [AB06, GTX17].

Lemma 1. (i) uaw-limits are unique;
(ii) If $x_\alpha \xrightarrow{\text{uaw}} x$ and $y_\beta \xrightarrow{\text{uaw}} y$, then $ax_\alpha + by_\beta \xrightarrow{\text{uaw}} ax + by$, for any scalars $a, b$;
(iii) If $x_\alpha \xrightarrow{\text{uaw}} x$, then $y_\beta \xrightarrow{\text{uaw}} x$, for every subnet $(y_\beta)$ of $(x_\alpha)$;
(iv) If $x_\alpha \xrightarrow{\text{uaw}} x$, then $|x_\alpha| \xrightarrow{\text{uaw}} |x|$;
(v) $x_\alpha \xrightarrow{\text{uaw}} x$ iff $(x_\alpha - x) \xrightarrow{\text{uaw}} 0$.

Proof. (i) Suppose that $x_\alpha \xrightarrow{\text{uaw}} x$ and $x_\alpha \xrightarrow{\text{uaw}} y$. Let $u \in E_+$. It follows from $|x - y| \land u \leq |x_\alpha - x| \land u + |x_\alpha - y| \land u$ that $f(|x - y| \land u) = 0$ for every $f \in E_+^*$ and, therefore, for every $f \in E^*$. It follows that $|x - y| \land u = 0$. Taking $u = |x - y|$, we conclude that $|x - y| = 0$, hence $x = y$.

The implications (ii) – (v) are straightforward. \qed
It is clear that every absolutely weakly convergent net is uaw-convergent; the converse is true for order bounded nets. The following example shows that in general the two convergences differ: let \( E = c_0 \) and \( x_n = n^2 e_n \), where \((e_n)\) stands for the standard basis of \( c_0 \). It is easy to see that \( x_n \xrightarrow{uaw} 0 \), yet it is not absolutely weakly null.

It was shown in [GTX17] that every disjoint net is \( uo \)-null. Let \((x_\alpha)\) be a disjoint net. For every \( u \in E_+ \), the net \((|x_\alpha| \wedge u)\) is disjoint and order bounded, hence weakly null. This yields the following.

**Lemma 2.** Every disjoint net is uaw-null.

The next result is analogous to [DOT17, Lemma 2.11]; the proof is similar.

**Lemma 3.** Let \( e \in E_+ \) be a quasi interior point. Then \( x_\alpha \xrightarrow{uaw} 0 \) iff \( |x_\alpha| \wedge e \xrightarrow{w} 0 \).

It was observed in Section 7 of [DOT17] that un-convergence is given by a topology, and sets of the form \( V_{u,\varepsilon} = \{ x \in E, \| |x| \wedge u \| < \varepsilon \} \); where \( u \in E_+ \) and \( \varepsilon > 0 \), form a base of zero neighborhoods for this topology. Using a similar argument, one can show that sets of the form

\[
V_{u,\varepsilon,f} = \{ x \in E : f(|x| \wedge u) < \varepsilon \},
\]

where \( u \in E_+ \), \( \varepsilon > 0 \), and \( f \in E^*_+ \) form a base of zero neighborhoods for a Hausdorff topology, and the convergence in this topology is exactly the uaw-convergence. Similarly to Lemmas 2.1 and 2.2 of [KMT17], for every \( u \in E_+ \), \( \varepsilon > 0 \), and \( f \in E^*_+ \), either \( V_{u,\varepsilon,f} \) is contained in \([-u,u]\), in which case \( u \) is a strong unit, or \( V_{u,\varepsilon,f} \) contains a non-trivial ideal. Now, a natural conjecture can arise as follows:

**Question.** Suppose \( E \) is infinite-dimensional. Does every uaw neighbourhood of zero contain a non-trivial ideal?

Clearly, un-convergence implies uaw-convergence. The converse is false in general; the standard unit sequence \((e_n)\) in \( \ell_\infty \) is uaw-null but not un-null.

**Theorem 4.** The following are equivalent:

(i) \( E \) is order continuous;

(ii) \( x_\alpha \xrightarrow{uaw} 0 \Leftrightarrow x_\alpha \xrightarrow{un} 0 \) for every net \((x_\alpha) \subseteq E\);

(iii) \( x_n \xrightarrow{uaw} 0 \Leftrightarrow x_n \xrightarrow{un} 0 \) for every sequence \((x_n) \subseteq E\).

**Proof.** (i) implies (ii) by [AB06, Theorem 4.17]. The implication (ii) \( \Rightarrow \) (iii) is trivial. To show that (iii) \( \Rightarrow \) (i), let \((x_n)\) be a disjoint order bounded sequence. By Lemma 2, it is uaw-null. By assumption, it is un-null. Since it is order bounded, it is norm null. It follows that \( E \) is order continuous. \( \square \)
Note that $uo$-convergence does not imply $uaw$-convergence, in general: consider $E = C([0,1])$. Define the sequence $(f_n) \subseteq E$ via $f_n(0) = 1$, $f_n(\frac{1}{n}) = f_n(1) = 0$, and linear in between. Then $(f_n)$ is $uo$-null but not $uaw$-null. The following result is analogous to Theorem 2.1 in [G14]; the proof is similar (cf. also Theorem 8.1 in [KMT17]). Note that sequences may be replaced by nets.

**Proposition 5.** The following are equivalent:

(i) $E$ is order continuous;
(ii) for every norm bounded sequence $(x_n^*) \subseteq E^*$, $x_n^* \xrightarrow{uaw} 0$ implies that $x_n^* \xrightarrow{w^*} 0$;
(iii) for every norm bounded sequence $(x_n^*) \subseteq E^*$, $x_n^* \xrightarrow{uaw} 0$ implies that $x_n^* \xrightarrow{\sigma(E^*,E)} 0$.

**Corollary 6.** Suppose $E$ is an order continuous Banach lattice. Then every norm bounded $uaw$-Cauchy net in $E^*$ is $w^*$-convergent.

**Proof.** Suppose $(x_n^*)$ is a norm bounded $uaw$-Cauchy net in $E^*$. By Proposition 5 $(x_n^*)$ is $w^*$-Cauchy, hence $w^*$-convergent by Alaoglu’s Theorem. □

Now, we characterize order continuity of the dual of a Banach lattice in term of $uaw$-convergence.

**Theorem 7.** The following are equivalent:

(i) $E^*$ is order continuous;
(ii) For every norm bounded net $(x_\alpha) \subseteq E$, $x_\alpha \xrightarrow{uaw} 0$ implies $x_\alpha \xrightarrow{w} 0$;
(iii) For every norm bounded sequence $(x_n) \subseteq E$, $x_n \xrightarrow{uaw} 0$ implies $x_n \xrightarrow{\sigma(E^*,E)} 0$.

**Proof.** The proof of (i) $\Rightarrow$ (ii) is similar to that of [DOT17, Theorem 6.4]. (ii) $\Rightarrow$ (iii) is trivial. To show (iii) $\Rightarrow$ (i), observe that every disjoint norm bounded sequence in $E$ is $uaw$-null by Lemma 2 hence weakly null. Now apply [AB06, Theorem 4.69]. □

In the following results, we characterize reflexive Banach lattices in terms of unbounded convergences.

**Theorem 8.** The following are equivalent:

(i) $E$ is reflexive;
(ii) Every norm bounded $uaw$-Cauchy net in $E$ is weakly convergent;
(iii) Every norm bounded $uaw$-Cauchy sequence in $E$ is weakly convergent.

**Proof.** (i) $\Rightarrow$ (ii) Let $(x_\alpha)$ be a bounded $uaw$-Cauchy net in $E$. Then $(x_\alpha)$ is weakly Cauchy by Theorem 7 hence weakly convergent by Alaoglu’s Theorem.
(ii) ⇒ (iii) Trivially.

(iii) ⇒ (i) It suffices to show that E contains no lattice copies of $c_0$ or $\ell_1$. Suppose that E contains a lattice copy of $\ell_1$. The unit vector basis of $\ell_1$ is disjoint and, therefore, $uaw$-null in E by Lemma. By assumption, it is weakly convergent in E and, therefore, in $\ell_1$; a contradiction. Suppose now that $c_0$ embeds into E as a sublattice. Let $(e_i)$ be the standard basis of $c_0$; put $x_n = \sum_{i=1}^{n} e_i$. Fix $x^* \in E_+^*$ and $u \in E_+$. Since $(x_n)$ is weakly Cauchy in $c_0$, we have $x^*(|x_m - x_n| \wedge u) \leq x^*(x_m - x_n) \to 0$ as $m, n \to \infty$ with $n < m$. Hence, $(x_n)$ is $uaw$-Cauchy, therefore, weakly convergent in E, hence in $c_0$; a contradiction.

Combining this with Theorem 4, we obtain the following.

**Corollary 9.** For an order continuous Banach lattice E, the following are equivalent:

(i) E is reflexive;
(ii) Every norm bounded un-Cauchy net in E is weakly convergent;
(iii) Every norm bounded un-Cauchy sequence in E is weakly convergent.

The order continuity assumption cannot be dropped: in $\ell_\infty$, every norm bounded un-Cauchy net is norm Cauchy and, therefore, weakly convergent.

Recall that a net $(x_\alpha)$ in a Banach lattice E is $uo$-Cauchy if the net $(x_\alpha - x_\beta)_{(\alpha, \beta)}$ is $uo$-null.

**Theorem 10.** The following are equivalent:

(i) E is reflexive;
(ii) Every norm bounded $uo$-Cauchy net in E is weakly convergent;
(iii) Every norm bounded $uo$-Cauchy sequence in E is weakly convergent.

**Proof.** The implication (i) ⇒ (ii) follows from Theorem because in an order continuous Banach lattice every $uo$-Cauchy net is $uaw$-Cauchy. (ii) ⇒ (iii) Trivially. The proof that (iii) ⇒ (i) is similar to that in Theorem. If E contains a lattice copy of $\ell_1$ then the standard basis of $\ell_1$ is disjoint, hence $uo$-null, so weakly null in E and, therefore, in $\ell_1$; a contradiction. Suppose that $c_0$ embeds into E as a sublattice; let $(e_i)$ be the standard basis of $c_0$; put $x_n = \sum_{i=1}^{n} e_i$. Then $(x_n)$ is $uo$-Cauchy in $c_0$. It follows from [GTX17, Corollary 3.3] that $(x_n)$ is $uo$-Cauchy in E. Hence, $(x_n)$ is weakly convergent in E and, therefore, in $c_0$; a contradiction.

[G14, Theorem 3.4] shows that every $w^*$-null net in $E^*$ is $uo$-null iff E is order continuous and atomic. [KMT17, Theorem 8.4] asserts that every $w^*$-null net in $E^*$ is
un-null iff $E^*$ is atomic and both $E$ and $E^*$ are order continuous. We show next that the latter result remains valid if "un-null" is replaced with "uaw-null".

**Proposition 11.** The following are equivalent:

(i) Every $w^*$-null net in $E^*$ is uaw-null;

(ii) $E^*$ is atomic and both $E$ and $E^*$ are order continuous.

*Proof.* $(ii) \Rightarrow (i)$ by [KMT17, Theorem 8.4]. Assume $(i)$ and suppose that $x_n^* \downarrow 0$ in $E^*$. It follows that $x_n^*(x) \downarrow 0$ for each $x \in E_+$, so that $x_n^* \overset{w^*}{\rightarrow} 0$. By assumption, $x_n^* \overset{uaw}{\rightarrow} 0$, hence $x_n^* \overset{w}{\rightarrow} 0$ and, furthermore, $\|x_n^*\| \rightarrow 0$ by Dini’s Theorem [AB06, Theorem 3.52]. It follows that $E^*$ is order continuous. By Theorem 4, every uaw-null net in $E^*$ is un-null. Now apply [KMT17, Theorem 8.4]. □

**Example 12.** Let $E = \ell_1$. $E$ is order continuous, $E^* = \ell_\infty$ is atomic but not order continuous. Define $(x_n^*)$ in $\ell_\infty$ via $x_n^* = (0, \ldots, 0, 1, \ldots)$, with $n$ zero terms. Then $x_n^* \overset{w}{\rightarrow} 0$, yet it is not weakly null, hence not uaw-null because it is contained in $[0, 1]$.

The following is a uaw variant of [KMT17, Proposition 6.6]; the proof follows easily from Theorem 7.

**Proposition 13.** Suppose that $E^*$ is order continuous. Then every norm closed convex norm bounded subset $C$ of $E$ is uaw-closed.

It is proved in [KMT17, Theorem 7.5] that the unit ball $B_E$ is un-compact iff $E$ is an atomic KB-space.

**Proposition 14.** $B_E$ is uaw-compact iff $E$ is an atomic KB-space.

*Proof.* It suffices to show that uaw-compactness of $B_E$ implies that $E$ is order continuous; the result then follows from Theorem 4 and [KMT17, Theorem 7.5]. Suppose that $B_E$ is uaw-compact. Since order intervals are uaw-closed by Lemma 1, they are uaw-compact, hence absolutely weakly compact, and, therefore, weakly compact. It follows that $E$ is order continuous. □

**Further remarks.** A preliminary version of this paper was posted on arXiv on August 6, 2016. In particular, it was proved there that in order continuous Banach lattices, uaw-convergence (and, therefore, un-convergence) is stable under passing to and from a sublattice; this was independently proved in [KMT17] as Corollary 4.6. Recently, there have been some results regarding unbounded convergence in locally solid vector lattices; see [DEM17, T17] for more details.
Acknowledgements. This note would not have existed without inspiring and worthwhile suggestions of V. G. Troitsky, my friend and my colleague. Thanks is also due to Niushan Gao and Foivos Xanthos for useful remarks. I would like to have a deep gratitude toward the referee for invaluable comments which improved the paper in the present form.

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