Thermodynamics of Abelian Gauge Fields in Real Hyperbolic Spaces

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We work with $N$-dimensional compact real hyperbolic space $X_G$ with universal covering $M$ and fundamental group $\Gamma$. Therefore, $M$ is the symmetric space $G/K$, where $G = SO(N,1)$ and $K = SO(N)$. We regard $\Gamma$ as a discrete subgroup of $G$ acting isometrically on $M$, and we take $X_G$ to be the quotient space by that action: $X_G = \Gamma \backslash M = \Gamma \backslash G/K$. The natural Riemannian structure on $M$ (therefore on $X$) induced by the Killing form of $G$ gives rise to a connection $p$-form Laplacian $\mathcal{L}_p$ on the quotient vector bundle (associated with an irreducible representation of $K$). We study gauge theories based on abelian $p$-forms on the real compact hyperbolic manifold $X_G$. The spectral zeta function related to the operator $\mathcal{L}_p$, considering only the co-exact part of the $p$-forms and corresponding to the physical degrees of freedom, can be represented by the inverse Mellin transform of the heat kernel. The explicit thermodynamic functions related to skew-symmetric tensor fields are obtained by using the zeta-function regularization and the trace tensor kernel formula (which includes the identity and hyperbolic orbital integrals). Thermodynamic quantities in the high and low temperature expansions are calculated and new entropy/energy ratios established.

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I. INTRODUCTION

It is believed that the real universe is a form of Friedman-Robertson-Walker models. However, the time-dependent metrics of these models brings about at least two inconvenient problems: (i) the definition of vacuum state in a time-dependent background; the background starts producing particles in a continuous way, and therefore, vacuum states in the Minkowskian sense do not exist; (ii) the all-important thermodynamic investigation of the early universe becomes senseless due to the lack of a proper definition of equilibrium state. Einstein static universes can meet the inconveniences stated before. For these type of manifold, a vacuum state can be unambiguously defined both locally and globally. It can be shown that a closed universe has the same vacuum energy and pressure as a static Einstein universe. The radius of this universe is the instantaneous radius of the Friedman-Robertson-Walker universe. Finite-temperature field theory in curved backgrounds makes sense again. The thermodynamics of quantum fields in an Einstein universe for some radius is equivalent to that of an instantaneously static closed universe. The thermodynamics of positive curvature Einstein spaces was discussed by several authors before. In particular the so-called entropy bounds or entropy to thermal energy ratios were calculated and compared with known bounds such as the Bekenstein bound or the Cardy-Verlinde bound. For example, for a massless scalar field in $S^3$ space this was done in and for a massive scalar field in $\mathbb{R}^3$. Here we wish to extend the evaluation of those type of bounds to the case of skew-symmetric tensor fields in real compact hyperbolic spaces, a more delicate and involved task when compared with previous calculations in $S^3$ space, as we shall see. This paper is divided as follows: In Section II we set the relevant mathematical tools of the quantum dynamics of exterior forms of real hyperbolic spaces; in Section III we apply Fried trace formula to the tensor kernel and derive the identity and the hyperbolic contributions; in Section IV the spectral functions associated with exterior forms are obtained; in Sections V and VI we establish the pertinent high and low temperature expansions and calculate the entropy bounds for the problem at hand. Comparison with results obtained for a massive scalar field in $S^3$ and final remarks are left for the last Section. Throughout this paper we employ natural units, $\hbar = c = 1$; Boltzmann constant $k_B$ is also set equal to the unity.

II. QUANTUM DYNAMICS OF EXTERIOR FORMS OF REAL HYPERBOLIC SPACES

Let $X_G$ be a $N$-dimensional real compact hyperbolic space with universal covering $M$ and fundamental group
Γ. Then we can represent \( M = G/K \), where \( G = SO_1(N, 1) \) and \( K = SO(N) \) is a maximal compact subgroup of \( G \). We regard \( \Gamma \) as a discrete subgroup of \( G \) acting isometrically on \( M \), and we take \( X_\Gamma \) to be the quotient space by that action: \( X_\Gamma = \Gamma \backslash M = \Gamma \backslash G/K \). Let \( \tau \) be an irreducible representation of \( K \) on a complex vector space \( V_\tau \), and form the induced homogeneous vector bundle \( G \times_K V_\tau \) (the fiber product of \( G \) with \( V_\tau \) over \( K \)) over \( M \). Restricting the \( G \) action to \( \Gamma \) we obtain the quotient bundle \( E_\tau = \Gamma \backslash (G \times_K V_\tau) \to X_\Gamma \). The natural Riemannian structure on \( M \) (therefore on \( X_\Gamma \)) induced by the Killing form \( \langle \ , \ \rangle \) of \( G \) gives rise to a connection Laplacian \( \Delta \) acting isometrically on \( \Gamma \) we obtain the quotient bundle \( \Omega \) with \( \Omega \) operating on smooth sections \( \Gamma \backslash M \). If \( \Omega \) is even, and we shall specialize \( X_\Gamma \) to be the representation \( \lambda_\tau \) of \( \Gamma \) on \( M \) = \( \Omega \). For \( \lambda_\tau \) to be the identity representation of tensor fields or generalized abelian gauge fields. The Lagrangian associated with \( \lambda_\tau \), and \( \lambda_\tau \) (gauge field), and \( \lambda_\tau \) (co–gauge field). The Euler–Lagrangian equations, with the gauge, give: \( \Delta \lambda_\tau \omega_\lambda = 0 \), \( \delta \lambda_\tau = 0 \) (Lorentz gauge): \( \lambda_\tau \omega_\lambda = 0 \), \( \partial \lambda_\tau = 0 \) (co–Lorentz gauge). These Lagrangians provide a possible representation of tensor fields or generalized abelian gauge fields. The two representations of tensor fields are not completely independent, because of the well-known duality property of exterior calculus which gives a connection between star–conjugated gauge and co–gauge tensor fields. The gauge \( p \)-forms are mapped into the co–gauge \( (N - p) \)-forms under the action of the Hodge * operator. The vacuum–to–vacuum amplitude for the gauge \( p \)-form \( \omega_p \) becomes \( \mathcal{G} \):

\[
Z = N \int D\omega \exp \left[ -\langle \omega, \mathcal{L}_p \rangle \right] \times \prod_{j=1}^{p} \left( \text{Vol}_{p-j}(\det \mathcal{L}_{p-j}^{(j+1)/2})^{-1} \right)^{(-1)^{j+1}},
\]

where we need to factorize the divergent gauge group volume and integrate over the classes of gauge transformations \( (\omega \to \omega + d\phi) \).

### III. THE TRACE FORMULA APPLIED TO THE TENSOR KERNEL

The space of smooth sections \( \Gamma \backslash M \) is just the space of smooth \( p \)-forms on \( X \). We can therefore apply the version of the trace formula developed by Fried in \( \mathbb{G} \). First we set up some additional notation. For \( \sigma_p \) the natural representation of \( SO(2k-1) \) on \( \Lambda^p \mathbb{C}^{2k-1} \), we have the corresponding Harish–Chandra–Plancherel density given – for a suitable normalization of the Haar measure \( dx \) on \( G \) – by

\[
\mu_{\sigma_p}(r) = \frac{\pi}{24k-4|\Gamma(k)|^2} \left( \frac{2k-1}{p} \right) P_{\sigma_p}(r) \tanh(\pi r),
\]

for \( 0 \leq p \leq k-1 \), where

\[
P_{\sigma_p}(r) = \prod_{\ell=2}^{p+1} \left[ r^2 + \left( k - \ell + \frac{3}{2} \right)^2 \right]
\times \prod_{\ell=p+2}^{k} \left[ r^2 + \left( k - \ell + \frac{1}{2} \right)^2 \right]
\]

is an even polynomial of degree \( 2k - 2 \). We have that \( P_{\sigma_p}(r) = P_{\sigma_{2k-1-p}}(r) \) and \( \mu_{\sigma_p}(r) = \mu_{\sigma_{2k-1-p}}(r) \) for \( k \leq p \leq 2k-1 \). Define the Miatello coefficients \( \mathfrak{a}_{2\ell}^{(p)}(G = SO_1(2k+1, 1)) \) by

\[
P_p(r) = \sum_{\ell=0}^{k-1} a_{2\ell}^{(p)} r^{2\ell}, \quad 0 \leq p \leq 2k - 1.
\]

Let \( \text{Vol}(\Gamma \backslash G) \) denote the integral of the constant function \( 1 \) on \( \Gamma \backslash G \) with respect to the \( G \)-invariant measure on \( \Gamma \backslash G \) induced by \( dx \). For \( 0 \leq p \leq N-1 \) the Fried trace formula applied to kernel holds \( \mathfrak{a} \):

\[
\text{Tr} \left( e^{-\mu \lambda r} \right) = I_1^{(p)}(K) + I_1^{(p-1)}(K) + H_1^{(p)}(K) + H_1^{(p-1)}(K),
\]

where \( I_1^{(p)}(K), H_1^{(p)}(K) \) are the identity and hyperbolic orbital integral respectively. In the above formula

\[
I_1^{(p)}(K) \overset{\text{def}}{=} \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{4\pi} \int_{\mathbb{R}} d\mu_{\sigma_p}(r) \times e^{-\mu(r^2 + b^2) + (p\mu - \rho_0)^2},
\]

\[
H_1^{(p)}(K) \overset{\text{def}}{=} \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{4\pi} \int_{\mathbb{R}} d\mu_{\sigma_p}(r) \times e^{-\mu(r^2 + b^2) + (p\mu - \rho_0)^2},
\]

where \( \rho_0 \) is a constant determined by the normalization of \( \text{Vol}(\Gamma \backslash G) \).
where $C \subset \Gamma$ is a complete set of representations in $\Gamma$ of its conjugacy classes, $C(\gamma)$ is a well-defined function on $\Gamma \setminus \{1\}$ (for more details see Ref. [3]), $\rho_0 = (N - 1)/2$, $b^{(p)}$ are real constants, and $\chi_\sigma(m) = \text{trace}(\sigma(m))$ is the character $\sigma$ for $m \in SO(2k - 1)$.

For $p \geq 1$ there is a measure $\mu_\sigma(r)$ corresponding to a general irreducible representation $\sigma$. Let $\sigma_p$ be the standard representation of $SO(N - 1)$ on $\Lambda^p \mathbb{C}^{N-1}$. If $N = 2k$ is even then $\sigma_p$ is irreducible; if $N = 2k + 1$ then every $\sigma_p$ is irreducible except for $p = (N - 1)/2$, in which case $\sigma_p$ is the direct sum of two spin–(1/2) representations $\sigma^\pm$: $\sigma_p^+ = \sigma^+ \oplus \sigma^-$. For $p = k$ the representation $\tau_k$ of $K = SO(2k)$ on $\Lambda^k \mathbb{C}^{2k}$ is not irreducible: $\tau_k = \tau_k^+ \oplus \tau_k^-$ is the direct sum of two spin–(1/2) representations. In the case of the trivial representation ($p = 0$, i.e. for smooth functions or smooth vector bundle sections) the measure $\mu(r) \equiv \mu_0(r)$ corresponds to the trivial representation. Therefore, we take $I_0^{-1}(K_i) = H_0^{(p)}(K_i) = 0$. Since $\sigma_0$ is the trivial representation, we have $\chi_{\sigma_0}(m_\gamma) = 1$. In this case, formula (6) reduces exactly to the trace formula for $p = 0$.



III. THE SPECTRAL FUNCTIONS OF EXTERIOR FORMS

The spectral zeta function related to the Laplace operator $L_j$ can be represented by the inverse Mellin transform of the heat kernel $K_t = \text{Tr} \exp(-t \mathcal{L}_j)$. Using the Fried formula, we can write the zeta function as a sum of contributions:

$$
\zeta(s|\mathcal{L}_j) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{\zeta(\gamma)}{t } L_jC(\gamma)e^{-t(b^{(0)} + \rho_0^2)} - \frac{2}{\pi}, \tag{10}
$$

$$
H_1^{(p)}(K_i) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C \setminus \{1\}} \chi(\gamma) t\gamma C(\gamma)\chi_\sigma(m_\gamma)
\times e^{-t(b^{(p)} + (\rho_0 - \rho)^2) - t^2/4t},
\tag{11}
$$

$$
H_0^{(0)}(K_i) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C \setminus \{1\}} \chi(\gamma) t\gamma C(\gamma)e^{-t(b^{(0)} + \rho_0^2) - t^2/4t},
\tag{12}
$$

$$
\zeta(t^{(p)}|\mathcal{L}_j) \equiv \zeta^{(N)}(s, j) + \zeta^{(N)}(s, j - 1)
+ \zeta^{(N)}(s, j) + \zeta^{(N)}(s, j - 1).
\tag{13}
$$

For the identity component we have

$$
\zeta^{(N)}(s, j) = \frac{V_t}{\Gamma(s)} \int_0^\infty dt \frac{\zeta(\gamma)}{t } L_jC(\gamma)e^{-t(r^2 + \alpha_j^2)}
\tag{14}
$$

where $V_t = \chi(1)\text{Vol}(\Gamma \backslash G) / 4\pi$, and we define $\alpha_j^2 = b^{(j)} + (\rho_0 - j)^2$. Replacing the Harish-Chandra–Plancherel measure, we obtain two representations for $\zeta^{(N)}(s, j)$, which holds for the cases of odd and even dimension. Thus,

$$
\zeta^{(2k)}(s, j) = \frac{V_t C^{(j)}_{2k}}{\Gamma(s)} \sum_{\ell = 0}^{k-1} a^{(j)}_{2\ell, 2k}
\times \int_0^\infty dt \frac{1}{\Gamma(s)} \int_\mathbb{R} dr^{2\ell+1} \tanh(\pi r) e^{-t(r^2 + \alpha_j^2)}.
\tag{15}
$$

Using the identities

$$
\tanh(\pi r) = 1 - \frac{2}{1 + e^{2\pi r}},
\tag{16}
$$

$$
\int_0^\infty dr^{2\ell+1} \tanh(\pi r) e^{-t(r^2 + \alpha_j^2)} = \frac{(-1)^\ell (1 - 2^{-2\ell-2}) B_{2\ell}}{4\ell},
\tag{17}
$$

where $B_{\ell}$ is the $\ell$–th Bernoulli number, we get

$$
\zeta^{(2k)}(s, j) = \frac{V_t C^{(j)}_{2k}}{\Gamma(s)} \sum_{\ell = 0}^{k-1} a^{(j)}_{2\ell, 2k}
\times \left[ \Gamma(\ell + 1) \Gamma(s - \ell - 1) \alpha_j^{-2\ell+2} + \sum_{n = 0}^{\infty} \xi_n \Gamma(s + n) \alpha_j^{-2n-2n} \Gamma\right],
\tag{18}
$$

where we have defined

$$
\xi_n \equiv \frac{(-1)^{\ell+1} (1 - 2^{-2\ell-2-2n}) B_{2\ell+2n+2}}{n!(2\ell + 2n + 2)}.
\tag{19}
$$

In the odd dimensional case we get

$$
\zeta^{(2k+1)}(s, j) = \frac{V_t C^{(j)}_{2k+1}}{\Gamma(s)} \sum_{\ell = 0}^{k} a^{(j)}_{2\ell, 2k+1}
\times \int_0^\infty dt \frac{1}{\Gamma(s)} \int_\mathbb{R} dr^{2\ell+1} e^{-t(r^2 + \alpha_j^2)}
\times \Gamma(s - \ell - \frac{1}{2}) \alpha_j^{-2\ell+2+1}.
\tag{20}
$$

The hyperbolic component of the zeta function takes the form

$$
\zeta^{(N)}_{H}(s, j) = \sum_{\gamma \in C \setminus \{1\}} \chi(\gamma) t\gamma C(\gamma)\chi_\sigma(m_\gamma)
\frac{1}{\sqrt{4\pi t}} \int_0^\infty dt \frac{e^{-t(\alpha_j^2 + \gamma^2)}}{t^{s+2}}.
\tag{21}
$$

Using the McDonald function,

$$
K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty dt \frac{e^{-t - t^2/z}}{t^{\nu+1}},
\tag{22}
$$
where $|\arg z| < \pi/2$ and $\Re z^2 > 0$, we obtain

$$\zeta^{(N)}_H(s, j) = \sum_{\gamma \in C_{r=1}} \frac{\chi (\gamma) T^{(N)}_s (\gamma) \chi_{s+\frac{1}{2}} (\gamma) K_{s+\frac{1}{2}} (\gamma)}{\sqrt{\pi} I(s) j(\gamma)} (2\alpha s)^{s-\frac{1}{2}}. \quad (23)$$

V. THE HIGH TEMPERATURE EXPANSIONS

Using the Mellin representation for the zeta function, we can obtain useful formulas for the temperature dependent part of the identity and hyperbolic orbital components of the free energy (see for detail Refs. [10, 11, 12])

$$F^{(N)}_{I, H}(\beta, j) = -\frac{1}{2\pi i} \int_{\Re z = c} \frac{dz}{z^2} \zeta(z) \Gamma(z-1) \zeta_{I, H} \left( \frac{z-1}{2}, j \right), \quad (24)$$

where $\zeta(z)$ is the Riemann zeta function. A tedious calculation gives the following results:

$$F^{(2k)}_{I}(\beta, j) = -\frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k} a_{2k-2,2k} \Gamma(k) \zeta(2k+1) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k-1} - \frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k} a_{2k-2,2k} \Gamma(k) \zeta(2k-1) \Gamma \left( k - \frac{1}{2} \right) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k} - \frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k} a_{2k-2,2k} \Gamma(k) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k+1} + \mathcal{O}(-1)^j \beta^{-k+3}, \quad (25)$$

$$F^{(2k+1)}_{I}(\beta, j) = -\frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k+1} a_{2k+2k+1} \Gamma(k+1) \zeta(2k+2) \Gamma(k+1) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k-2} - \frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k+1} a_{2k+2k+1} \Gamma(k+1) \zeta(2k) \Gamma(k) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k+1} - \frac{V_T}{\sqrt{4\pi}} C^{(j)}_{2k+1} a_{2k+2k+1} \Gamma(k+1) \zeta(2k) \Gamma(k) \times \Gamma \left( k + \frac{1}{2} \right) \beta^{-2k} + \mathcal{O}(-1)^j \beta^{-2k+2}. \quad (26)$$

Note that the contribution associated to the hyperbolic orbital component is negligible small.

A. The thermodynamic functions and the entropy bound

In the context of the Hodge theory, the physical degrees of freedom are represented by the co-exact part of the $p$–form. For $0 \leq p \leq N - 1$ the Fried trace formula applied to the tensor kernel associated with co-exact forms has to be modified \[ \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \]

$$\text{Tr} \left( e^{-t_\lambda^p} \frac{\partial}{\partial t} \right) = \sum_{j=1}^{p} (-1)^j \left[ I^{(p-j)}_{I, K}(K_{\epsilon}) + I^{(p-j)}_{I, K}(K_{\epsilon}) \right]$$

$$+ \mathcal{H}^{(p-j)}_{I, K}(K_{\epsilon}) + H^{(p-j)}_{I, K}(K_{\epsilon}) - b_{p-j}, \quad (27)$$

where $b_j$ are the Betti numbers. Thus, the free energy becomes

$$\mathcal{F}^{(N)}(\beta) = \sum_{j=0}^{p} (-1)^j \left( \mathcal{F}^{(N)}_{I, K}(\beta, p-j) + \mathcal{F}^{(N)}_{I, K}(\beta, p-j-1) \right). \quad (28)$$

In the high temperature limit ($\beta \to 0$) we have

$$\mathcal{F}^{(N)}(\beta) = -A_1(N; \Gamma) \beta^{-N-1} + A_2(N; \Gamma) \beta^{-N+1} + O(\beta^{-N+3}), \quad (29)$$

where for the even dimensional case,

$$A_1(2k; \Gamma) = \frac{V_T}{\sqrt{4\pi}} \zeta(2k+1) \Gamma(k) \Gamma \left( k + \frac{1}{2} \right) \times C^{(j)}_{2k} a_{2k-2,2k}, \quad (30)$$

$$A_2(2k; \Gamma) = \frac{V_T}{\sqrt{4\pi}} \zeta(2k-1) \Gamma(k-1) \Gamma \left( k - \frac{1}{2} \right) \times C^{(j)}_{2k} a_{2k-2,2k} + (k-1) a_{2k-2,2k}, \quad (31)$$

and for the odd dimensional case,

$$A_1(2k+1; \Gamma) = \frac{V_T}{\sqrt{4\pi}} \zeta(2k+2) \Gamma(k+1) \Gamma \left( k + \frac{1}{2} \right) \times C^{(j)}_{2k+1} a_{2k+2k+1}, \quad (32)$$

$$A_2(2k+1; \Gamma) = \frac{V_T}{\sqrt{4\pi}} \zeta(2k+1) \Gamma(k) \Gamma \left( k - \frac{1}{2} \right) \times C^{(j)}_{2k+1} a_{2k+2k+1} - (k-1) a_{2k+2k+1} \beta^2. \quad (33)$$

The entropy and the total energy can be obtained by means of the following thermodynamic relations:

$$S^{(N)}(\beta) = \beta^2 \partial \mathcal{F}^{(N)}(\beta)/\partial \beta, \quad E^{(N)}(\beta) = \partial(\beta \mathcal{F}^{(N)}(\beta))/\partial \beta. \quad \text{Therefore,}$$

$$S^{(N)}(\beta) = (N+1) A_1(N; \Gamma) \beta^{-N} + (N-1) A_2(N; \Gamma) \beta^{-N+2} + \mathcal{O}(\beta^{-N+4}), \quad (34)$$

$$E^{(N)}(\beta) = -NA_1(N; \Gamma) \beta^{-N-1} - (N-2) A_2(N; \Gamma) \beta^{-N+1} + \mathcal{O}(\beta^{-N+3}), \quad (35)$$

Note that the contribution associated to the hyperbolic orbital component is negligible small.
The ratio entropy/energy becomes

\[
S^{(N)}(\beta) = \frac{N + 1}{N} - \frac{2}{N^2} A_2(N; \Gamma) \beta^3 + O(\beta^5). \tag{36}
\]

VI. THE LOW TEMPERATURE EXPANSIONS

In the low temperature limit we can use the following representation for the one-loop contribution to the free energy [10, 11]:

\[
\mathcal{F}^{(N)}(\beta) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^\infty dt \, t^{-3/2} e^{-n^2\beta^2/4t} \cosh^{-1}(c_E). \tag{37}
\]

For the identity contribution we have

\[
F_1^{(N)}(\beta, j) = \frac{V}{\pi} \sum_{n=1}^{\infty} \int_0^\infty dt \, t^{-3/2} \int_\mathcal{R} dr \, \mu_{\sigma_1}(r) \\
\times e^{-t(\sigma_j^2 + 7) - n^2\beta^2/4t}.
\]

Therefore, the following formulas hold

\[
F_2^{(2k)}(\beta, j) = \frac{2V}{\pi} C_N^{(j)} \sum_{n=1}^{\infty} \sum_{\ell=0}^{k-1} \frac{\Gamma(\ell + 1)}{\ell!} \left[ 2\beta^2 / (\ell + 1) \right] \\
\times \left( \frac{\alpha_j}{\alpha_j n^2} \right)^{\ell+3} K_{\ell+\frac{1}{2}}(\alpha_j n^2) \\
- (-1)^\ell \sum_{m=0}^{\infty} \frac{1}{m! (\ell + m + 1)} \frac{B_{2\ell + 2m + 2}}{2\ell + 2m + 2} \\
\times \alpha_j^{-2m+1} \left( \frac{\alpha_j n^2}{\alpha_j n^2} \right)^{\ell+1/2}, \tag{39}
\]

\[
F_1^{(k+1)}(\beta, j) = \frac{2V}{\pi} C_N^{(j)} \sum_{n=1}^{k} \sum_{\ell=0}^{k-1} \frac{\Gamma(\ell + 1)}{\ell!} \left[ 2\beta^2 / (\ell + 1) \right] \\
\times \left( \frac{\alpha_j}{\alpha_j n^2} \right)^{\ell+1} K_{\ell+1}(\alpha_j n^2) \\
= \frac{2V}{\pi} C_N^{(j)} \sum_{n=1}^{k} \sum_{\ell=0}^{k-1} \frac{\Gamma(\ell + 1)}{\ell!} \left[ 2\beta^2 / (\ell + 1) \right] \\
\times \left( \frac{\alpha_j}{\alpha_j n^2} \right)^{\ell+1} K_{\ell+1}(\alpha_j n^2) \\
\times \left( \frac{\alpha_j}{\alpha_j n^2} \right)^{\ell+1/2}, \tag{40}
\]

\[
F_{H}^{(N)}(\beta, j) = 2\alpha_j^2 \sum_{n=1}^{\infty} \sum_{\gamma \in C \setminus \{1\}} \frac{\chi(\gamma) t_1 C(\gamma) \chi_{\sigma_p}(m_\gamma)}{\pi j(\gamma)} \\
\times K_1 \left( \frac{n^2\beta^2 + \ell^2}{\alpha_j n^2} \right), \tag{41}
\]

Using the asymptotic expansion for the McDonald function [22]

\[
K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ \frac{1}{(2z)^{\nu + k/2}} \Gamma(\nu + k/2) \\
+ \frac{\theta \Gamma(\nu + k/2)}{(2z)^{\nu + k/2}} \right]. \tag{42}
\]

for \(\nu \in \mathbb{R}, \, z > 0, \, \ell > \nu - 1/2 (\ell = 1, 2, 3, \ldots), \, \|\theta\| \leq 1\) (see [18], page 963), in Eqs. [39], [40] and [41] we get the following result for the entropy

\[
S^{(N)}(\beta) = \frac{\tilde{A}_1(N, \Gamma) \beta^{1/2}}{\Gamma(\beta^{-3/2})}, \tag{43}
\]

where

\[
\tilde{A}_1(2k; \Gamma) = \sum_{\gamma \in C \setminus \{1\}} \frac{2\chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{3/2} \\
+ \sum_{\gamma \in C \setminus \{1\}} \frac{5 \chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{1/2} \\
+ \frac{3V}{\pi} \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (1 - 2 - 2\ell^{1/2}) B_{2\ell + 2}}{\ell + 1} \\
\times C_{2k}^{(p)(p)} a_{2k} a_p^{1/2}, \tag{44}
\]

\[
\tilde{A}_2(2k + 1; \Gamma) = \sum_{\gamma \in C \setminus \{1\}} \frac{2\chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{3/2} \\
+ \frac{\sqrt{V}}{\pi} C_{2k+1}^{(p)(p)} a_{0,2k+1} a_p^{3/2}, \tag{45}
\]

\[
\tilde{A}_3(2k; \Gamma) = \frac{\sqrt{V}}{\pi} C_{2k}^{(p)(p)} a_{0,2k} a_p^{1/2} \\
- \frac{7}{4} \sum_{\gamma \in C \setminus \{1\}} \frac{\chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{1/2} \\
+ \frac{\sqrt{V}}{4\pi} \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (1 - 2 - 2\ell^{1/2}) B_{2\ell + 2}}{\ell + 1} \\
\times C_{2k}^{(p)(p)} a_{2k,2k} a_p^{1/2}, \tag{46}
\]

The energy is given by

\[
E^{(N)}(\beta) = \frac{1}{\pi} \sum_{\gamma \in C \setminus \{1\}} \frac{\chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{3/2} \\
+ O(\beta^{-5/2}), \tag{48}
\]

where

\[
\tilde{A}_3(2k; \Gamma) = \frac{\sqrt{V}}{\pi} C_{2k}^{(p)(p)} a_{0,2k} a_p^{1/2} \\
- \frac{7}{4} \sum_{\gamma \in C \setminus \{1\}} \frac{\chi(\gamma) t_1 C(\gamma)}{\sqrt{2\pi j(\gamma)}} \chi_{\sigma_p}(m_\gamma) a_p^{1/2} \\
+ \frac{\sqrt{V}}{4\pi} \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (1 - 2 - 2\ell^{1/2}) B_{2\ell + 2}}{\ell + 1} \\
\times C_{2k}^{(p)(p)} a_{2k,2k} a_p^{1/2}, \tag{49}
\]
\[ \bar{A}_3(2k+1; \Gamma) = \sum_{\gamma \in C_1^{-1}} \frac{\chi(\gamma) T_4 C(\gamma)}{\sqrt{2\pi j} \gamma} \chi_{\sigma_p}(m_\gamma) \times (\alpha_p^{j/2} + 2 \alpha_p^3/2) \]

\[ + \frac{7V_T}{8\sqrt{2\pi}} C_{2k+1}(p_0,2k+1) \alpha_p^{1/2}. \] (50)

For the entropy/energy ratio in low temperature limit we get

\[ \frac{S^{(N)}(\beta)}{E^{(N)}(\beta)} = \beta - A_3(N; \Gamma) \left( \frac{A_2(N; \Gamma)}{A_1(N; \Gamma)} - \bar{A}_3(N; \Gamma) \right) - 1^{-1} \]

\[ + \mathcal{O}(\beta^{-2}). \] (51)

**VII. CONCLUSIONS**

We have considered gauge theories based on abelian $p$-forms in real compact hyperbolic manifolds. The explicit thermodynamic functions associated with skew-symmetric tensor fields are obtained by using zeta–function regularization and the trace tensor kernel formula. Thermodynamic quantities in the low and high temperature limits were calculated. We also have obtained the entropy/energy ratios (in both temperature limits). The dependence on the Miatello coefficients related to the structure of the Harish–Chandra–Plancherel measure stems from the second term of the expansion. In the case of scalar fields ($p = 0$) we have Eq. (30) with

\[ \frac{A_2(2k; \Gamma)}{A_1(2k; \Gamma)} = \frac{2}{2k-1} \frac{\zeta(2k-1)}{\zeta(2k+1)} \times \left( \frac{1}{-k} - \frac{a_2^{(0)} - 2a_2^{(0)} - a_0^2}{-k - 2a_2^{(0)} - a_0^2} \right), \] (52)

\[ A_2(2k+1; \Gamma) = \frac{1}{k} \frac{\zeta(2k)}{\zeta(2k+2)} \times \left( \frac{2}{2k-1} \frac{2a_2^{(0)} - 2a_2^{(0)} - a_0^2}{2a_2^{(0)} - a_0^2} \right), \] (53)

where $a_0^2 = \rho_0^2 + m^2$ ($a_0^2 = \rho_0^2$ for the massless case). For three-dimensional hyperbolic manifolds the Miatello coefficients read [17]: $a_0^{(0)} = a_2^{(0)} = 1$ and therefore

\[ \frac{S^{(3)}(\beta)}{E^{(3)}(\beta)} = \frac{4}{3} \beta + \frac{1}{3\pi^2} (2 - \alpha_0^2) \beta^3 + \mathcal{O}(\beta^5). \] (54)

This formula is in agreement with the result obtained in [4] where entropy bounds were calculated for spherical geometry and where the dependence on geometry of the backgrounds also stems from the second term of the expansion.

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