Quantum Critical Behavior of the Cluster Glass Phase

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In disordered itinerant magnets with arbitrary symmetry of the order parameter, the conventional quantum critical point between the ordered phase and the paramagnetic Fermi-liquid (PMFL) is destroyed due to the formation of an intervening cluster glass (CG) phase. In this Letter we discuss the quantum critical behavior at the CG-PMFL transition for systems with continuous symmetry. We show that fluctuations due to quantum Griffiths anomalies induce a first-order transition from the PMFL at T=0, while at higher temperatures a conventional continuous transition is restored. This behavior is a generic consequence of enhanced non-Ohmic dissipation caused by a broad distribution of energy scales within any quantum Griffiths phase in itinerant systems.

In magnetic systems with disorder, it is always possible to find isolated, defect-free regions which act as finite-size copies of a clean, bulk sample. Fluctuations of these rare regions (or droplets) lead to singularities of the free energy in what is known as the Griffiths phase \[1\]. While these singularities are extremely weak (essential) singularities in classical systems \[2\], the situation can change dramatically near (T=0) quantum phase transitions \[5\]. Recently, there has been significant progress made in understanding the quantum Griffiths phase by focusing on the dynamics of the rare regions \[10\]. By considering the rare regions as independent droplets described by a local order parameter, an elegant classification of quantum Griffiths phenomena was proposed based only on general symmetry principles \[8\].

In the independent droplet picture, the system is assumed to be correlated over the length scale of the individual droplets, each of which has dynamics in the imaginary time direction related to the energy cost of coherently flipping a large volume of spins. The local droplet degrees of freedom therefore map to classical, one-dimensional ferromagnetic spin chains in the imaginary-time direction, with their dynamics corresponding to interactions in this single dimension. The critical behavior of the bulk system can then be understood through the behavior of the independent spin chains which depends only on their dynamics and the spin symmetry \[11\]. For example, in the case of itinerant Ising magnets, independent spin chains are effectively above their lower critical dimension and undergo a phase transition in the universality class of the Kosterlitz-Thouless transition \[11\]. Thus, if sufficiently large, these droplets freeze over, leading to their eventual ordering at low enough temperature. Therefore, at zero temperature, the system is always in its ordered phase and the quantum phase transition present in the clean magnet becomes destroyed or rounded due to the presence of disorder \[5\]. In contrast, independent droplets in itinerant Heisenberg magnets are exactly at their lower critical dimension \[11\], corresponding to weaker dissipation. Within the independent droplet picture, this behavior leads to quantum Griffiths effects, and the quantum phase transition to the magnetically ordered phase is conjectured to be in the universality class of the infinite randomness fixed point (IRFP) \[8\]. These general issues have attracted much interest in recent years due to the possible role of such quantum Griffiths phase phenomena as a driving force for the disorder-driven non-Fermi liquid behavior in correlated electron systems (for a recent review see Ref. \[12\]).

In realistic systems, however, the droplets are not independent and interactions must be considered, providing a qualitatively different mechanism for dissipation. Such interaction effects were recently \[9\] shown to represent a singular perturbation within any quantum Griffiths phase, leading to enhanced non-Ohmic dissipation. This mechanism leads to the freezing of sufficiently large droplets and the generic emergence of a cluster glass (CG) phase intervening between the uniformly ordered phase and the paramagnetic Fermi liquid (PMFL). (For a schematic phase diagram of an itinerant magnet with and without interactions, we refer the reader to Ref. \[3\].)

In this Letter, we examine the nature of the quantum phase transition to the cluster glass phase. To study the critical behavior at the quantum CG-PMFL transition, we focus on the continuous spin symmetry case for which the interaction effects produce a sharp phase transition at \(T = 0\). By studying the transition both from the CG and from the PMFL, we show that a region of coexistence of these two phases exists, indicating that the \(T = 0\) quantum phase transition is first-order. At higher temperatures, a tricritical point is found above which a conventional second-order transition is restored. In contrast, we show that a collection of identical droplets displays a conventional (continuous) phase transition even down to the lowest temperatures. From this we conclude that fluctuations due to Griffiths phase anomalies are responsible for driving the transition first-order.

**Self-consistency conditions.**— Our approach here mirrors that of previous works \[5,6\]. In the presence of disorder, defect-free regions can form with near uniform ordering while the bulk system is in the non-ordered phase. These rare regions (droplets) can be expressed as a fluc-
tuating order parameter field in the spirit of Ginzburg-Landau with effective action 
\[ S = \sum_i S_{L,i} + \sum_{ij} S_{i,j}. \]
For the case of an itinerant antiferromagnet \[ L \], the local action (for a single droplet) is
\[ S_{L,i} = \sum_{\omega_n} \phi_i(\omega_n)(r_i + |\omega_n|)\phi_i(-\omega_n) \] (1)
\[ + \frac{u}{2N} \int_0^\beta d\tau \phi_i(\tau) \phi_i(\tau), \]
where \( \phi_i \) is the \( N \)-component order parameter field for the \( i \)th droplet and \( r_i \) is its bare mass, selected from the distribution \( P(r_i) \). This local action has been studied previously and is well understood \[ 9 \]. The novelty arises when we consider interactions between droplets, mediated by the weakly correlated metal in the bulk. These take the form of the RKKY interaction
\[ S_{i,j} = \frac{J_{ij}}{(R_{ij})^d} \int_0^\beta d\tau \phi_i(\tau) \phi_j(\tau), \] (2)
with \( J_{ij} \) random, of zero mean and having variance \( \langle J_{ij}^2 \rangle = J^2 \). Very recent work has suggested that even infinitesimally weak RKKY interactions destabilize the Griffiths phase found in the non-interacting droplet theory, leading to the cluster glass phase \[ 9 \]. These results demonstrated that the nontrivial physics of non-Ohmic dissipation persists even when the problem is solved at the saddle-point level, which is formally justified in the large-\( N \) limit of the model \[ 14 \].

We proceed in the usual way by averaging over disorder using the replica method and decoupling the resultant quartic term by introducing an auxiliary Hubbard-Stratonovich field \[ 8, 13 \]. This new field describes the dynamics of spin fluctuations caused by the long-ranged RKKY interactions, and contributes an additional dissipative term to the action. Its saddle-point value is \[ 9 \]
\[ \chi(\omega_n) = \frac{1}{2} \int d\tau \frac{P(r_i)}{r_i + \lambda_i + |\omega_n| - \tilde{g}_\chi(\omega_n)} \] (3)
where \( \tilde{g} \equiv J^2 \sum_i (R_{ij})^{-2d} \) is the RKKY coupling. \( \lambda_i \) is a Hubbard-Stratonovich field decoupling the quartic interaction in the single-site action and satisfies the local self-consistency condition \[ 14 \]
\[ \lambda_i = \frac{uT}{2} \sum_{\omega_n} \frac{1}{r_i + \lambda_i + |\omega_n| - \tilde{g}_\chi(\omega_n)}. \] (4)

Our original action now becomes a single-site problem with effective action
\[ S_{\text{eff}} = \sum_{i,\omega_n} \phi_i(\omega_n)(r_i + \lambda_i + |\omega_n| - \tilde{g}_\chi(\omega_n))\phi_i(\omega_n). \] (5)
Physically, this corresponds to independent droplets in the presence of additional dissipation arising due to interactions with all other droplet degrees of freedom and represented by the \( \omega \) dependence of the local spin susceptibility \( \chi(\omega_n) \).

**Quantum criticality for uniform droplets.**— An interesting limit of this problem is revealed if all the droplets are assumed to have the same size \( P(r_i) = \delta(r_i - \hat{r}) \) for all sites \( i \). In this case, the coupled set of equations \[ 9 \] and \[ 11 \] are identical to those describing a metallic spin glass \[ 10 \] and, at large enough \( \tilde{g} \), we expect a zero temperature transition from a paramagnet to a spin glass. The self-consistency condition \[ 3 \] for \( \chi(\omega_n) \) reduces to a simple algebraic equation and is easily solved:
\[ \chi_u(\omega_n) = \frac{1}{2\tilde{g}} \left( \hat{r} + \hat{\lambda} + |\omega_n| \right) \] (6)
\[ \pm \sqrt{(\hat{r} + \hat{\lambda} + |\omega_n|)^2 - 2\tilde{g}} \]
where \( \hat{\lambda} \equiv \lambda(\hat{r}) \). Clearly, the paramagnetic solution becomes unstable when \( \hat{r} + \hat{\lambda} = \sqrt{2\tilde{g}} \) and we define an energy scale \[ 17 \] \( \Delta = \hat{r} + \hat{\lambda} - \sqrt{2\tilde{g}} \). The phase boundary can be determined by setting \( \Delta = 0 \) and solving the self-consistency condition \[ 4 \] for \( \hat{r}(T) \).

The phase for \( \Delta > 0 \) (\( \hat{r} > \hat{r}_c \)) is characterized by the frequency dependence of \( \chi_u(\omega_n) \). From the solution \[ 6 \], we find three separate regimes,
\[ \chi_u(0) - \chi_u(\omega) \sim \begin{cases} \omega & , \omega < \Delta \ll \sqrt{2\tilde{g}} \\ \sqrt{\omega} & , \Delta < \omega \ll \sqrt{2\tilde{g}} \\ \chi_u(0) - \omega^{-1} & , \omega \gg \sqrt{2\tilde{g}} \end{cases} \] (7)

The linear, low-\( \omega \) behavior of \( \chi_u(\omega) \) is characteristic of a Fermi liquid, while, right at the transition, \( \chi_u \sim \sqrt{\omega} \), characteristic of non-Fermi liquid behavior.

The transition at \( T = 0 \) can also be studied from the magnetically ordered side. In this case, the quantum critical point is identified when the mean-field stability criterion for the glass phase vanishes \[ 18 \] i.e. \( \tilde{\lambda}_{SG} \equiv 1 - \sqrt{2\tilde{g}}(0) = 0 \). Again, using the solution \[ 6 \] for \( \chi_u \), we find \( \tilde{\lambda}_{SG} \sim \sqrt{T} \) which vanishes at \( \Delta = 0 \).

Thus, examining the transition through the instability of either the paramagnetic Fermi liquid or the spin glass phase yields a quantum critical point at \( \Delta = 0 \), consistent with a conventional second-order transition in this limit of the problem.

In the following we consider the model with a distribution of droplet sizes, and show that the two approaches do not coincide at the transition, revealing the singular effects of droplet fluctuations.

**Griffiths phase behavior of distributed droplets.**— For dilute impurities, the defect-free regions assume a Poisson distribution, which can be written in terms of the local coupling constant as
\[ P(r_i) = \frac{2\pi\alpha}{u} e^{2\pi\alpha(\hat{r} - \hat{r}_c)/u}, \quad r_i \leq \hat{r} < 0. \] (8)
The offset \( \hat{r} \) is used to tune through the transition, \( \alpha \) is related to the droplet density \( \rho \) by \( \alpha = \rho u/2\pi\delta \) and \( \delta < 0 \).
is the coupling constant of the clean magnet. In this way, the distribution is smoothly connected to the uniform-$r_i$ limit as $\alpha \to \infty$. We can therefore reasonably apply the uniform solution \([6]\) as a zeroth order approximation and iterate equations \([3]\) and \([4]\) until self-consistency is achieved. By accounting for droplets of all sizes, we are also including their fluctuations which are associated with the Griffiths behavior.

The instability of the Griffiths phase at $T = 0$ is already apparent at one iteration loop (1IL). Physically, this corresponds to the regime where very large droplets are very dilute, but the typical droplets (in their local environment) are approaching quantum criticality of the uniform droplet model. Integrating expression \([4]\) and identifying the local droplet energy $\epsilon_i = r_i + \lambda_i - \tilde{g}\chi(0)$, we find the relation between droplet energy and coupling strength $\epsilon_i \sim \exp(2\pi f^{-1}r_i/u)$ where we have defined

$$f = \frac{2\sqrt{\Delta^2 + 2\sqrt{2g}\Delta}}{\Delta + \sqrt{2g} + \sqrt{\Delta^2 + 2\sqrt{2g}\Delta}}.$$  

We can now switch from integration over local coupling constants to integration over energies via the replacement

$$\int \mathcal{P}(r_i)(\cdots)dr_i \to \int \mathcal{P}(\epsilon_i)(\cdots)d\epsilon_i$$

where the distribution of droplet energies is a power-law $\mathcal{P}(\epsilon) \sim \epsilon^{\alpha' - 1}$, $\alpha' \equiv f\alpha$. From Eq. (11) we conclude that the “renormalized Griffiths exponent” $\alpha'$ decreases as the critical point is approached. Scaling arguments \([3]\) show that such a distribution leads to strong Griffiths behavior in the isolated droplet case for $\alpha' < 1$.

For the case of interacting droplets, however, integration of expression \([3]\) produces $\chi(\omega) - \chi(0) \sim -|\omega|^\alpha + O(|\omega|)$ and non-Ohmic dissipation obtains for $\alpha' < 2$. As argued previously \([3]\), this qualitatively changes the dynamics of the droplets which then find themselves above their lower critical dimension. Sufficiently large droplets ($r_i > r_c$) can therefore order (freeze) in imaginary time before the Griffiths phase is reached, and this new phase has been dubbed the “cluster glass”.

**IRFP of isolated droplets at quantum criticality.**— It should be pointed out that our 1IL expression for $\alpha'$ underscores the close analogy between the present study and the physics of infinite-randomness fixed points (IRFP). Most remarkably, from Eq. (11) we conclude that, within such 1IL calculation, the “renormalized Griffiths exponent” $\alpha'$ vanishes precisely at the uniform droplet quantum critical point as

$$\alpha' \sim 1/\Delta.$$  

This result should be contrasted with the theory of noninteracting droplets \([8]\) where the “bare” Griffiths exponent $\alpha$ is a non-universal function of parameters.

The dynamical critical exponent governing an IRFP is commonly defined \([10]\) as $z' = d/\alpha'$ ($d$ is the spatial dimensionality), diverging right at the transition. Equation \([9]\) provides us with an explicit expression for $z'$ which captures both the quantum Griffiths phase and the behavior reminiscent of an IRFP at $\Delta = 0$. Of course, this expression is not fully self-consistent, and will not be appropriate at the lowest temperatures where the system freezes into the cluster glass state. However, quantum Griffiths and IRFP behavior obtained from the 1IL approximation is expected to hold at temperatures above the cluster glass transition, as anticipated previously \([8]\).

**Fluctuation induced first-order transition.**— In the paramagnetic phase, it is clear that the leading order behavior of $\chi(\omega)$ is linear in $\omega$ and it is not difficult to show that the slope $m_\chi = \partial \chi/\partial \omega|_{\omega=0}$ determines the renormalized exponent $\alpha' = f\alpha$ through the factor $f = (1 - \tilde{g}m_\chi)^{-1}$, as can be checked at the 1IL level. This relationship can be further illuminated by calculating $m_\chi$ self-consistently from equation \([3]\): \(m_\chi = -2(1 - \tilde{g}m_\chi)\chi(0)^2\). (11)

Solving for the slope, we find the relationship

$$f = (1 - \tilde{g}m_\chi)^{-1} = (1 - 2\tilde{g}\chi^2(0)).$$  

What is striking about this result is that the term in parentheses on the right-hand side of \([12]\) is the exact analogue of the spin glass stability criterion $\lambda_{\text{SG}}$ for droplets with distributed site-energies \([18]\); in this case, the stability criterion is $\lambda_{\text{SG}} = 1 - 2\tilde{g}\chi^2(0)$. The fully self-consistent value of $\alpha'$ is then $\alpha' = f\alpha = \lambda_{\text{SG}}\alpha$, implying that $\alpha' = 0$ at the transition determined by the stability of the CG phase. On the other hand, approaching from the PMFL side, the transition is still given by $\alpha' = 2$ where non-Ohmic dissipation obtains. So, unlike the uniform droplet case, the stability criteria for the PMFL and the CG do not coincide at a unique point, implying that the quantum phase transition in the distributed droplet case is not a conventional second-order transition.

To gain further insight into the nature of the transition, we recast the problem as an eigenvalue analysis of the constitutive free energy for which the solution to \([3]\) is a minimum. The relevant contribution to the free energy can be expressed near the minimum as

$$\mathcal{F}_\chi = \int_{\omega,\omega'} (\chi(\omega) - \chi(0)\Gamma(\omega,\omega')\chi(\omega') - \chi(0))$$  

so that the eigenvalue determining the stability of the solution $\chi_0$ satisfies

$$\lambda_\chi \leq \frac{\int_{\omega,\omega'} (\chi(\omega) - \chi(0)\Gamma(\omega,\omega')\chi(\omega') - \chi(0))}{\int_{\omega}(\chi(\omega) - \chi(0))^2}.$$  

To implement this iteratively, we can approximate the difference $(\chi - \chi_0) \approx (\chi^{(n-1)} - \chi^{(n)})$, where $\chi^{(n)}$ is the value of $\chi$ at the nth iteration step. To determine how $\Gamma$ operates on $\chi^{(n)}$, note that the free energy functional is constructed such that $\delta \mathcal{F}/\delta \chi = 0$.
The CG and PMFL phases where
which indicates that there is a region of coexistence of
finite and positive (i.e. both solutions are stable). The
restored. The boundary given by
two criteria merge and the usual continuous transition is
shown in the upper panel of Figure 1. Plotted in both
panels, open squares correspond to first-order transitions, full
circles to second-order transitions. The tricritical point is
indicated by the solid star.

yields the self-consistency condition (4) for $\chi$. Thus,
\[
\int d\omega T_\chi(\omega,\omega')\chi^{(n)}(\omega') = \chi^{(n)}(\omega) - \chi^{(n+1)}(\omega).
\]
We can now rewrite the defining relation (14) for the eigenvalue
valid at each iteration step
\[
\lambda^{(n)} = 1 - \frac{\int d\omega (\chi^{(n-1)} - \chi^{(n)})(\chi^{(n)} - \chi^{(n+1)})}{\int d\omega (\chi^{(n-1)} - \chi^{(n)})^2}
\]
and proceed by iterating numerically. The instability of
the PMFL is then given by $\lambda = 0$ at $n \to \infty$.

The results of this procedure at finite temperature are
shown in the upper panel of Figure 1. Plotted in both
full and empty symbols is the temperature $T_\chi$ at which
$\lambda = 0$ for given values of the tuning parameter $\hat{r}$. Plotted
in the lower panel is the value of $\lambda_{SG}$ at $T_\chi(\hat{r})$.
We can see that $\lambda_{SG} > \lambda = 0$ for $-0.65 \leq \hat{r} \leq \hat{r}_c(T = 0)$
which indicates that there is a region of coexistence
of the CG and PMFL phases where $\lambda = 0$ and $\lambda_{SG}$ are both
finite and positive (i.e. both solutions are stable). The
transition for $\hat{r} > -0.65$ is therefore of first-order, so the
open squares represent the spinodal line; the other spinodal
corresponds to $\lambda_{SG} = 0, \lambda > 0$. For $\hat{r} < -0.65$, the
two criteria merge and the usual continuous transition is
restored. The boundary given by $\lambda = 0$ in this regime is
then the true phase boundary which we can fit using the
form $\log(T_c) \sim (\hat{r} - \hat{r}_c)^{-1}$, as shown by the full line in the
upper panel, reflecting the fact that the critical temperature
is proportional to the number of frozen droplets.

Thus, open squares represent a discontinuous transition for $\hat{r} > -0.65$ while the full circles represent a continuous transition for $\hat{r} < -0.65$; the tricritical point at $\hat{r} = -0.65$ is shown by the solid star.

In summary, we examined quantum criticality of the
cluster glass phase which typically emerges as disorder
is introduced in itinerant magnets. We have shown how
coupling between droplets enhances the quantum Griffiths effects, leading to singularly enhanced dissipation in the vicinity of the quantum critical point. As a result, any system with widely distributed droplet sizes cannot display conventional quantum criticality, but instead features a first-order transition at $T = 0$. In contrast, if the effects of the largest droplets are eliminated, conventional criticality is restored, as we found in a related model with a bounded distribution of droplet sizes. Similar behavior is also found at sufficiently high temperatures, where thermal fluctuations impose a cutoff on the Griffiths phase anomalies. This implies that fluctuations due to these Griffiths phase anomalies are responsible for driving the transition first-order. Our results could be of direct relevance to experiments on disordered superconductors where singular corrections due to rare regions have recently been demonstrated.

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\footnotesize
\begin{itemize}
  \item Electronic address: case@magnet.fsu.edu
  \item [1] R. B. Griffiths, Phys. Rev. Lett. \textbf{23}, 17 (1969).
  \item [2] M. Wortis, Phys. Rev. B \textbf{10}, 4665 (1974).
  \item [3] D. S. Fisher, Phys. Rev. Lett. \textbf{69}, 534 (1992).
  \item [4] D. S. Fisher, Phys. Rev. B \textbf{51}, 6411 (1995).
  \item [5] O. Motrunich \textit{et al.}, Phys. Rev. B \textbf{61}, 1160 (2000).
  \item [6] A. J. Millis \textit{et al.}, Phys. Rev. B \textbf{66}, 174433 (2002).
  \item [7] T. Vojta, Phys. Rev. Lett. \textbf{90}, 107202 (2003).
  \item [8] T. Vojta and J. Schmalian, Phys. Rev. B \textbf{72}, 045438 (2005).
  \item [9] V. Dobrosavljević and E. Miranda, Phys. Rev. Lett. \textbf{94},
187203 (2005).
  \item [10] T. Vojta, J. Phys. A \textbf{39}, R143 (2006).
  \item [11] J. M. Kosterlitz, Phys. Rev. Lett. \textbf{37}, 1577 (1976).
  \item [12] E. Miranda and V. Dobrosavljević, Rep. Prog. Phys. \textbf{68},
2337 (2005).
  \item [13] J. A. Hertz, Phys. Rev. B \textbf{14}, 1165 (1976).
  \item [14] J. Zinn-Justin, \textit{Quantum Field Theory and Critical Phe-
nomena} (Cambridge University Press, 1996).
  \item [15] K. Binder and A. P. Young, Rev. Mod. Phys. \textbf{58}, 801 (1986).
  \item [16] S. Sachdev, \textit{Quantum Phase Transitions} (Cambridge
University Press, 1999).
  \item [17] J. Ye \textit{et al.}, Phys. Rev. Lett. \textbf{70}, 4011 (1993).
  \item [18] A. A. Pastor and V. Dobrosavljević, Phys. Rev. Lett. \textbf{83},
4642 (1999).
  \item [19] A. P. Young, Phys. Rev. B \textbf{56}, 11691 (1997).
  \item [20] O. Vafek \textit{et al.}, cond-mat/0505688 (unpublished).
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