K-THEORY OF EQUIVARIANT QUANTIZATION

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ABSTRACT: Using an equivariant version of Connes’ Thom Isomorphism, we prove that equivariant K-theory is invariant under strict deformation quantization for a compact Lie group action.

1. INTRODUCTION

Let $\alpha$ be a strongly continuous action of $\mathbb{R}^n$ on a $C^*$-algebra $A$, and $J$ be a skew-symmetric matrix on $\mathbb{R}^n$. Rieffel [11] constructed a strict deformation quantization $A_J$ of $A$ via oscillatory integrals

$$a \times_J b := \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_J(u)\alpha_J(v)e^{2\pi i u \cdot v} du dv,$$

for $u, v \in \mathbb{R}^n$, and $a, b \in A^\infty$ (the smooth subalgebra of $A$ for $\alpha$). Such a construction gives rise to many interesting examples of noncommutative manifolds, e.g. quantum tori, $\theta$-deformation of $S^4$, etc. In [11], Rieffel proved that the $K$-theory of $A_J$ is equal to the $K$-theory of the original algebra $A$, by using Connes’ Thom isomorphism of $K$-theory.

In this paper, we are interested in examples that the algebra $A$ is also equipped with a strongly continuous action $\beta$ by a compact group $G$. When the two actions commute, the results in [11] naturally generalize to the equivariant setting. An easy observation is that, as the $G$-action $\beta$ commutes with the $R^n$-action $\alpha$, naturally $\alpha$ can be lifted to a strongly continuous action $\tilde{\alpha}$ on the crossed product algebra $A \rtimes_\beta G$. Rieffel’s construction [11] applies to the $\mathbb{R}^n$-action $\tilde{\alpha}$ on $A \rtimes_\beta G$, and defines a quantization algebra $(A \rtimes_\beta G)_J$. By the commutativity between $\alpha$ and $\beta$, we easily check that $\beta$ lifts to a strongly continuous action $\tilde{\beta}$ on $A_J$, and $A_J \rtimes_{\tilde{\beta}} G$ is isomorphic to $(A \rtimes_\beta G)_J$. Now by the results on the $K$-theory of strict deformation quantization [11], we conclude that

$$K_* (A \rtimes_\beta G) = K_* ((A \rtimes_\beta G)_J) = K_* (A_J \rtimes_{\tilde{\beta}} G).$$

In this paper, we generalize the above discussion of equivariant quantization to a situation where the actions $\alpha$ and $\beta$ do not commute. Define $GL(J)$ to be the group of invertible matrices $g$ such that $g^t J g = J$, and $SL_n(\mathbb{R}, J) := SL_n(\mathbb{R}) \cap GL(J)$. We remark that when $J$ is the standard skew-symmetric matrix on $\mathbb{R}^{2n}$, $GL(J)$ is the linear symplectic group. Let $\rho : G \to SL_n(\mathbb{R}, J)$ be a group homomorphism such that

$$\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g,$$

for any $g \in G, x \in \mathbb{R}^n$.

When $\rho$ is a trivial group homomorphism, the actions $\alpha$ and $\beta$ commute.

A natural example of such a system appears as follows.

Example 1.1. Let $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{R}^{2n}$ by reflection with respect to the origin. Let $\mathbb{Z}^{2n}$ be the integer lattice in $\mathbb{R}^{2n}$. The $2n$-torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ inherits an action of $\mathbb{Z}_2$ from the $\mathbb{Z}_2$ action on $\mathbb{R}^{2n}$. The group $\mathbb{R}^{2n}$ acts on $\mathbb{R}^{2n}$ by translation and descends to act on $T^{2n}$. Let $A$ be the $C^*$-algebra of continuous functions on $T^{2n}$, and $J$ be the standard symplectic matrix on $\mathbb{R}^{2n}$. The action $\alpha$ (and $\beta$) of $\mathbb{R}^{2n}$ (and $\mathbb{Z}_2$) on $A$ is the dual action of the corresponding
actions on $\mathbb{T}^{2n}$. We easily check that Eq. (2) holds in this case with $\rho$ being the natural inclusion $\mathbb{Z}_2 \to SL_{2n}(\mathbb{R}, J)$.

Different from the case where the actions $\alpha$ and $\beta$ commute, for a nontrivial $\rho : G \to SL_n(\mathbb{R}, J)$, the $\mathbb{R}^n$-action $\alpha$ on $A$ does not lift naturally to an action on $A \rtimes \beta G$. Therefore, we cannot apply Rieffel’s deformation construction to the algebra $A \rtimes \beta G$. Nevertheless, a simple calculation shows that

$$\beta_g(a \times J b) = \beta_g(a) \times J \beta_g(b), \quad \beta_g(a^*) = \beta_g(a)^*,$$

which shows that the $G$-action $\beta$ is still well-defined on $A_J$. Accordingly, we can consider the crossed product algebra $A_J \rtimes \beta G$. Applying this construction to Ex. 1.1, we obtain $A_J \rtimes \beta \mathbb{Z}_2$, which is well studied in literature, e.g. [5], [6], [8], and [14].

In this paper, we prove the following theorem about the $K$-theory groups of $A_J \rtimes \beta G$.

**Theorem 1.2.** If the actions $\alpha$, $\beta$ and the group homomorphism $\rho$ satisfy (2), then

$$K_\bullet(A_J \rtimes \beta G) \cong K_\bullet(A \rtimes \beta G), \quad \bullet = 0, 1.$$

The proof of this theorem will be presented in the next section. As applications of our theorem, we recover some results of [5] on the computation of the $K$-groups of $\mathbb{Z}_i$-quantum tori for $i = 2, 3, 4, 6$, and we also apply these results to the $\theta$-deformation [4] of $S^4$.

**Acknowledgments:** We would like to thank Professors S. Echterhoff and N. Higson for explaining the relationship between the equivariant Thom isomorphism theorem (Theorem 2.1) and the Connes-Kasparov conjecture. We also want to thank Professor H. Li for interesting discussions and comments which greatly helped us to improve the readability of the paper. We thank Professor H. Oyono-Oyono for helping us to remove a separability assumption in a previous version. And we are grateful to an anonymous referee for pointing out one mistake and various places to improve accuracy in a previous version of this paper. Tang’s research is partially supported by NSF grant 0900985. Tang would like to thank the School of Mathematical Sciences of Fudan University and the Max-Planck Institute for their warm hospitality of his visits. Yao’s research is partially supported by NSF grant 0903985 and NSFC grant 10901039 and 11231002.

## 2. Proof of the main theorem

Our proof of Theorem 1.2 is an equivariant generalization of Rieffel’s proof in [11]. We first prove the theorem under the assumption that $A$ is separable. Following [11], we will decompose our proof into 3 steps.

**Step I.** Following the notations in [11], we let $B^A$ be the space of smooth $A$-valued functions on $\mathbb{R}^n$ whose derivatives together with themselves are bounded on $\mathbb{R}^n$. Let $S^A$ be the space of $A$-valued Schwartz functions on $\mathbb{R}^n$. The integral

$$\langle f, g \rangle_A := \int f(x)^* g(x) dx$$

defines an $A$-valued inner product on $S^A$. Rieffel generalized the definition to $B^A$ by using oscillatory integrals. Namely, given $J$, we define a product on $B^A$ by

$$(F \times_J G)(x) := \int F(x + Ju) G(x + v) e^{2\pi i u \cdot v} dudv, \quad F, G \in B^A.$$  

Furthermore, $B^A$ acts on $S^A$ by

$$(L^J_F f)(x) := \int F(x + Ju) f(x + v) e^{2\pi i u \cdot v} dudv, \quad F \in B^A, \quad f \in S^A.$$
The above two integrals are both oscillatory ones. Via the $A$-valued inner product on $S^A$, we can equip $B^A_j$ with the operator norm $\| \cdot \|_J$, and obtain a pre-$C^*$-algebra $(B^A_j, \times, \| \cdot \|_J)$. Denote the corresponding $C^*$-algebra by $E^A_j$. Meanwhile, $S^A$ viewed as a $*$-ideal of $B^A_j$ (cf. Rieffel [10]), denoted by $S^A_j$, can be completed into $S^A_j$.

With an action $\alpha$ of $\mathbb{R}^n$ on $A$, Rieffel [11] Prop. 1.1 introduced a strongly continuous $\mathbb{R}^n$-action $\nu$ on $E^A_j$ and also on $S^A_j$ by

$$(\nu_t(F))(x) := \alpha_t(F(x - t)).$$

The fixed point subalgebra of this action $\nu$ is identified [11] Prop. 2.14 with the $C^*$-subalgebra of $E^A_j$ generated by elements

$$\tilde{a}(x) := \alpha_x(a), \quad a \in A^\infty,$$

which is exactly $A_J$.

In [11] Thm. 3.2, it is proved that $A_J$ is strongly Morita equivalent to $S^A_j \rtimes_\nu \mathbb{R}^n$. We will generalize this theorem to the equivariant setting with the $G$-action $\beta$. We introduce the $G$-action $\overline{\beta}$ on $B^A_j$ by

$$\overline{\beta}_g(F)(x) := \beta_g(F(\rho_{g^{-1}}(x))).$$

The exactly same arguments as in [11] Prop. 1.1 prove that the $G$-action $\overline{\beta}$ is strongly continuous on $S^A$, therefore so is it on $S^A_j$.

**Proposition 2.1.** The crossed product algebras $A_J \rtimes_\beta G$ and $(S^A_j \rtimes_\nu \mathbb{R}^n) \rtimes_\beta G$ are strongly Morita equivalent.

**Proof.** We will apply Combes’ theorem [1] Sec. 6] on equivariant Morita equivalence after proving that the $G$-actions $\beta$ and $\overline{\beta}$ are Morita equivalent, which will imply the Morita equivalence we seek.

According to [11], two $G$-actions $\beta^1, \beta^2$ on $A$ and $B$ are Morita equivalent if there is a strong Morita equivalence bimodule $X$ between $A$ and $B$ such that there is a $G$-action $\beta$ on $X$ satisfying

$$\beta_g(a\xi) = \beta^1_g(a)\beta_g(\xi), \quad \beta_g(\xi_b) = \beta_g(\xi)\beta^2_g(b),$$
$$A(\beta_g(\xi_1), \beta_g(\xi_2)) = \beta^1_g(A(\xi_1, \xi_2)), \quad (\beta_g(\xi_1), \beta_g(\xi_2))B = \beta^2_g((\xi_1, \xi_2)B).$$

for $\xi, \xi_1, \xi_2 \in X$.

Rieffel [11] constructed a Morita equivalence bimodule between $A_J$ and $S^A_j \rtimes_\nu \mathbb{R}^n$. We recall it now. Let $C_\infty(\mathbb{R}^n, A)$ be the $C^*$-algebra of $A$-valued functions on $\mathbb{R}^n$ that vanish at infinity. Let $\tau$ be the $\mathbb{R}^n$-action on $B^A_j$ by translation, $(\tau_tF)(x) = F(t + x)$, and $\mu$ be the action of $\mathbb{R}^n$ on $C_\infty(\mathbb{R}^n, A)$ by

$$\mu_s(f)(x) = e^{2\pi is \cdot x}f(x).$$

Define an action $\alpha$ of $\mathbb{R}^n$ on $S^A_j$ by

$$\alpha_t(F)(x) = \alpha_t(F(x)).$$

Both $\mu$ and $\tau$ act on $C_\infty(\mathbb{R}^n, A)$ and their combination gives an action of the Heisenberg group $H$ of dimension $2n + 1$ on $C_\infty(\mathbb{R}^n, A)$. This Heisenberg group action commutes with $\alpha$ and defines an $H \times \mathbb{R}^n$-action $\sigma$ on $C_\infty(\mathbb{R}^n, A)$. Define $X_0$ to be the subspace of $C_\infty(\mathbb{R}^n, A)$ of $\sigma$-smooth vectors. Rieffel [11] Prop. 2.2 proved that $X_0$ is a $*$-subalgebra of $S^A_j$ for any $J$, and a suitable completion $\overline{X}_0$ of $X_0$ serves as a strong Morita equivalence bimodule, which we refer to [11] for details.
Define a right $A_J$-module structure on $X_0$ by identifying $A_J$ with the subspace of $\nu$-invariant vectors in $B_j^n$, i.e.

$$f \cdot a := f \times_J \tilde{a}, \quad \text{for } a \in A^\infty$$

where $\tilde{a} \in C^\infty(\mathbb{R}^n, A)$ is defined by $\tilde{a}(x) = \alpha_x(a)$. The algebra $S_j^A \rtimes_\nu \mathbb{R}^n$ acts on $X_0$ by

$$\psi(f) := \int \psi(t) \times_J \nu_t(f) dt, \quad \psi \in S_j^A \rtimes_\nu \mathbb{R}^n, \ f \in X_0.$$ 

We define an $S_j^A \rtimes_\nu \mathbb{R}^n$-valued inner product on $X_0$ by

$$\langle f, g \rangle := \int \alpha_t(f^* \times_J g(-t)) dt, \quad f, g \in X_0.$$ 

We also know from [1] that $(X_0, (S_j^A \rtimes_\nu \mathbb{R}^n, (\cdot, \cdot)_{A_J}))$ is a strong Morita equivalence bimodule between $S_j^A \rtimes_\nu \mathbb{R}^n$ and $A_J$.

We easily check the following identities between the actions

$$\beta_g \alpha_t = \alpha_{\rho_g(t)} \beta_g, \quad \beta_g \nu_t = \tau_{\rho_g(t)} \beta_g, \quad \beta_g \nu_t = \mu_{(\rho_g)^{-1}}(t) \beta_g, \quad g \in G, \ t \in \mathbb{R}^n.$$ 

where $\rho_g^T$ is the transpose of $\rho_g$. These identities show that the $G$-action $\beta$ on $C^\infty(\mathbb{R}^n, A)$ preserves the subspace $X_0$ of $\sigma$-smooth vectors. Using the property that $\beta$ and $\overline{\beta}$ act strongly continuously on $S_j^A$, we can easily check that $\beta$ and $\overline{\beta}$ are Morita equivalent $G$-actions in the sense of Combes [1]. Therefore, $A_J \rtimes_\beta G$ is strongly Morita equivalent to $(S_j^A \rtimes_\nu \mathbb{R}^n) \rtimes_\overline{\beta} G$. □

As $A$ is separable, $A$ has a countable approximate identity. This implies that [1] Cor. 3.3] $A_J$ (and $S_j^A$) has a countable approximate identity. Accordingly, $A_J \rtimes_\beta G$ (and $(S_j^A \rtimes_\nu \mathbb{R}^n) \rtimes_\overline{\beta} G$) also has a countable approximate identity, and therefore has strictly positive elements. This together with the above Morita equivalence result shows that $A_J \rtimes_\beta G$ and $(S_j^A \rtimes_\nu \mathbb{R}^n) \rtimes_\overline{\beta} G$ are stably isomorphic. As stably isomorphic C*-algebras have isomorphic $K$-groups, we conclude that

$$K_\bullet(A_J \rtimes_\beta G) \cong K_\bullet((S_j^A \rtimes_\nu \mathbb{R}^n) \rtimes_\overline{\beta} G).$$ 

**Step II.** As we know, one powerful tool in dealing with the $K$-theory of C*-algebras is Connes’ Thom isomorphism, which remains to this day one of the few ways to prove isomorphism results of $K$-groups for crossed products. Let $\mathbb{C}_n$ be the complex Clifford algebra associated with $\mathbb{R}^n$. We first observe that the semidirect product group $\mathbb{R}^n \rtimes_\rho G$ is amenable, hence by Kasparov [4, §6, Thm. 2], we know that for a separable $G$-C*-algebra $B$, there exists an isomorphism from $KK^1(\mathbb{C}, B \times (\mathbb{R}^n \rtimes_\rho G))$ to $KK^1(\mathbb{C}, ((B \otimes \mathbb{C}_n) \times_\beta G)$. In other words, we need to use the following equivariant Thom isomorphism Theorem, which is a generalization of Connes’ Thom isomorphism Theorem [3]. This is a key ingredient of the whole approach.

**Theorem 2.1.** Let $\mathbb{R}^n$ and $G$ act strongly continuously on a separable C*-algebra $B$ with the actions denoted by $\alpha$ and $\beta$. Let $\rho : G \to GL(n, \mathbb{R})$. If the actions $\alpha$ and $\beta$ satisfy Equation (2), then

$$K_\bullet((B \times_\alpha \mathbb{R}^n) \rtimes_\beta G) \cong K_\bullet^G(B \times_\alpha \mathbb{R}^n) \cong K_\bullet^G(B \otimes \mathbb{C}_n) \cong K_\bullet((B \otimes \mathbb{C}_n) \rtimes_\beta G),$$

where $\mathbb{C}_n$ is the complex Clifford algebra associated with $\mathbb{R}^n$. \footnote{In [4, §6, Thm. 2], the connectivity of the group $G$ is assumed. But this assumption can be easily dropped using the same idea of the proof.}
Taking \( B = \mathfrak{S}_J^A \) in the above theorem which is separable (as \( A \) is separable), we conclude that
\[
K_\bullet((\mathfrak{S}_J^A \otimes \mathbb{C}_n) \rtimes_\beta G) \text{ is isomorphic to } K_\bullet((\mathfrak{S}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_\gamma G).
\]

**Step III.** Rieffel proved [10] Prop. 5.2 that there is an isomorphism

\[
\mathfrak{S}_J^A \cong A \otimes \mathcal{K} \otimes C_\infty(V_0),
\]

where \( \mathcal{K} \) is the algebra of compact operators on an infinite dimensional separable Hilbert space \( \mathcal{H} \), and \( V_0 \) is the kernel of \( J \) in \( \mathbb{R}^n \). Let \( U \) be the orthogonal complement of \( V_0 \) in \( \mathbb{R}^n \). It is easy to check that \( U \) is a \( J \)-invariant subspace, and both \( U \) and \( V_0 \) are \( G \)-invariant subspaces. As \( G \) is compact, there is a \( G \)-invariant complex structure on \( U \) compatible with \( J|_U \) (viewed a symplectic form on \( U \)). Without loss of generality, we will just assume that \( G \) preserves the standard Euclidean structure on \( U \). The key observation in the proof of [10] Prop. 5.2 is that when \( A \) is the trivial \( C^* \)-algebra \( \mathbb{C} \) and \( J \) invertible, \( \mathfrak{S}_J^C \) is naturally identified as the space of compact operators, still denoted by \( \mathcal{K} \), on the subspace \( \mathcal{H} \) of \( L^2(U) \) generated by elements

\[
g(\bar{z})e^{-\frac{|\bar{z}|^2}{2}},
\]

where \( g \) is an anti-holomorphic function. As \( \mathcal{H} \) is a \( G \)-invariant subspace, we can conclude that Rieffel’s isomorphism (3) is \( G \)-equivariant (note that \( G \) acts on \( \mathcal{K} \) by conjugation). By Combes’ result on \( G \)-equivariant Morita equivalence, \( (A \otimes \mathcal{K} \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_\beta G \) is strongly Morita equivalent to \((A \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_\beta G \).

Now we look at the decomposition of \( \mathbb{R}^n \) as \( V_0 \oplus U \). The Clifford algebra \( \mathbb{C}_n \) associated with \( \mathbb{R}^n \) is \( G \)-equivariantly isomorphic to \( C_\mathbb{V}_0 \otimes C_U \), where \( C_\mathbb{V}_0 \) and \( C_U \) are the complex Clifford algebras associated with \( V_0 \) and \( U \), respectively. Notice that \( J \) restricts to define a symplectic form on \( U \), and that the action of \( G \) preserves both the restricted \( J \) and the metric on \( U \). Therefore the \( G \)-action on \( U \) is \( \text{spin}^c \). Hence, the algebra \((A \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_\beta G \) is \( KK \)-equivalent to \((A \otimes C_\infty(V_0) \otimes \mathbb{C}_V) \rtimes_\beta G \). Again by the \( G \)-equivariant Thom isomorphism Thm. 2.1 for the trivial \( V_0 \) action on \( A \), we conclude that

\[
K_\bullet((\mathfrak{S}_J^A \otimes \mathbb{C}_n) \rtimes_\beta G) = K_\bullet((A \otimes \mathcal{K} \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_\beta G) = K_\bullet((A \otimes C_\infty(V_0) \otimes \mathbb{C}_V) \rtimes_\beta G) = K_\bullet(A \times_\beta G).
\]

Summarizing Step I-III, we have the following equality,

\[
K_\bullet(A_J \rtimes_\beta G) \overset{\text{Step I}}{=} K_\bullet((\mathfrak{S}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_\beta G) \overset{\text{Step II}}{=} K_\bullet((\mathfrak{S}_J^A \otimes \mathbb{C}_n) \rtimes_\beta G) \overset{\text{Step III}}{=} K_\bullet(A \times_\beta G).
\]

This completes the proof of Theorem 1.2 under the assumption that \( A \) is separable. For a general \( C^* \)-algebra \( A \), we can write \( A \) as an inductive limit of a net \( A^I \) of separable \( \mathbb{R}^n \rtimes_\rho \mathbb{G} \)-algebras. Then \( A_J \) is an inductive limit of the net \( A^I_J \) of separable \( \mathbb{G} \)-algebras. As \( K \)-groups commutes with inductive limit, we conclude that

\[
K_\bullet(A_J \rtimes_\beta G) = \lim_I K_\bullet(A^I_J \rtimes_\beta G) = \lim_I K_\bullet(A^I \rtimes_\beta G) = K_\bullet(A \times_\beta G).
\]

This completes the proof of Theorem 1.2 for general \( C^* \)-algebras.

### 3. Examples

In this section, we discuss some applications of Theorem 1.2.
3.1. Noncommutative toroidal orbifolds. We identify a 2-torus $T^2$ by $\mathbb{R}^2/\mathbb{Z}^2$. $\mathbb{R}^2$ acts on itself by translation and induces an action $\alpha$ on $T^2$. For $\theta \in \mathbb{R}$, we consider the symplectic form $J = \theta dx_1 \wedge dx_2$ on $\mathbb{R}^2$. The group $SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ preserving the lattice $\mathbb{Z}^2$ and therefore also acts on $T^2$, which is denoted by $\beta$. Inside $SL_2(\mathbb{Z})$, there are cyclic subgroups generated by

$$
\sigma_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},
$$

$$
\sigma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.
$$

The element $\sigma_i$ generates a cyclic subgroup $\mathbb{Z}_i$ of $SL_2(\mathbb{Z})$ of order $i = 2, 3, 4, 6$. In this example, the group $SL_2(\mathbb{R}, J)$ is identical to the group $SL_2(\mathbb{R})$. Define $\rho : \mathbb{Z}_i \rightarrow SL_2(\mathbb{R})$ to be the inclusion. And it is straightforward to check the actions $\alpha, \rho$ of $\mathbb{R}^2$ on $T^2$, $\rho$ of $\mathbb{Z}_i$ on $\mathbb{R}^2$, and $\alpha$ of $\mathbb{R}^2$ on $T^2$ satisfy Eq. (2). As is explained in Sec. 1 the group $\mathbb{Z}_i$ naturally acts on Rieffel’s deformation $A_J$, which is the quantum torus $A_\theta$, which is denoted by $T$. Theorem 1.2 states that

$$K_\bullet(A_J \rtimes \mathbb{Z}_i) = K_\bullet(A \rtimes \mathbb{Z}_i).$$

We recover with a completely different proof the result of [5, Cor. 2.2]. We have brought the question of computation of $K$-groups of these noncommutative orbifolds to a purely topological setting, and we refer to [5] and references therein for the explicit computation of the $K$-groups of the undeformed algebras $A \rtimes \mathbb{Z}_i$, $i = 2, 3, 4, 6$. For example, when $i = 2$, the $K$-groups of $A \rtimes \mathbb{Z}_2$ are

$$K_\bullet(A \rtimes \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}^6, & \bullet = 0, \\ 0, & \bullet = 1. \end{cases}$$

3.2. Theta deformation. Consider a 4-sphere $S^4$ centered at $(0, 0, 0, 0, 0)$ in $\mathbb{R}^5$ with radius 1. In coordinates, it is the set

$$\{(x_1, \cdots, x_5) | x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}.$$

Defines $T^2$-action on $S^4$ by, for $0 \leq t_1, t_2 < 2\pi$,

$$((t_1, t_2), (x_1, \cdots, x_5)) \rightarrow (x_1, \cdots, x_5) \begin{pmatrix} \cos(t_1) & \sin(t_1) & 0 & 0 & 0 \\ -\sin(t_1) & \cos(t_1) & 0 & 0 & 0 \\ 0 & 0 & \cos(t_2) & \sin(t_2) & 0 \\ 0 & 0 & -\sin(t_2) & \cos(t_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The same formula as above also defines an $\mathbb{R}^2$-action $\alpha$ on $S^4$. The action $\beta$ of $\mathbb{Z}_2$ on $S^4$ is by reflection

$$\sigma_2, (x_1, \cdots, x_5) \rightarrow (x_1, -x_2, x_3, -x_4, x_5).$$

The group $\mathbb{Z}_2$ also acts on $\mathbb{R}^2$ by reflection

$$\rho : \sigma_2 \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On $\mathbb{R}^2$, for $\theta \in \mathbb{R}$, consider the same symplectic form $J = \theta dx_1 \wedge dx_2$. It is easy to check that the actions $\alpha, \beta, \rho$ satisfy Eq. (2). Consider the algebra $C(S^4)$ of continuous functions on $S^4$. Rieffel’s construction defines a deformation $C(S^4_\theta)$ of $C(S^4)$ by $J$ and the action $\alpha$, which is the $\theta$-deformation [4] introduced by Connes and Landi. As is explained in Sec. 1 $\mathbb{Z}_2$ acts strongly continuously on $C(S^4_\theta)$. Theorem 1.2 states that

$$K_\bullet(C(S^4) \rtimes \mathbb{Z}_2) = K_\bullet(C(S^4_\theta) \rtimes \mathbb{Z}_2).$$
The $K$-theory of $C(S^4) \rtimes \mathbb{Z}_2$ can be computed \cite{Phillips3} topologically as the Grothendieck group of the monoid of all isomorphism classes of $\mathbb{Z}_2$-equivariant vector bundles on $S^4$.

Notice that the quotient $S^4/\mathbb{Z}_2$ is an orbifold homeomorphic to $S^4$. As an orbifold, $S^4/\mathbb{Z}_2$ \cite{Segal} has a good covering $\{U_i\}$ such that each $U_i$ and any none empty finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is a quotient of a finite group action on $\mathbb{R}^4$. Such a good covering allows to compute the topological $\mathbb{Z}_2$-equivariant $K$-theory of $S^4$ by the Čech cohomology on $S^4/\mathbb{Z}_2$ of the sheaf $K^\bullet_{\mathbb{Z}_2}$ introduced by Segal \cite{Segal}. The restriction of $K^\bullet_{\mathbb{Z}_2}$ to an open chart $U$ of $S^4/\mathbb{Z}_2$ is defined to be the $\mathbb{Z}_2$-equivariant $K$-theory of $\pi^{-1}(U)$ with $\pi$ the canonical projection $S^4 \to S^4/\mathbb{Z}_2$. Locally, when $U$ is sufficiently small, we can compute $K^\bullet_{\mathbb{Z}_2}(U)$ to be $K^\bullet(\pi^{-1}(U)^{\sigma_2}) \oplus K^\bullet(U)$, where $\pi^{-1}(U)^{\sigma_2}$ is the $\sigma_2$-fixed point submanifold. When $\bullet = 0$, it is equal to $Z|_{\pi^{-1}(U)^{\sigma_2}} \oplus Z_U$, and when $\bullet = 1$, it is zero. Gluing this local computation by the Mayer-Vietoris sequence, we conclude that $K_0(C(S^4_0) \rtimes \mathbb{Z}_2) = \mathbb{Z}^4$, $K_1(C(S^4_0) \rtimes \mathbb{Z}_2) = 0$.

**Remark 3.1.** We observe that in the above example, the group $\mathbb{Z}_2$ is not essential. Our computations generalize to $K^\bullet(C^\infty(S^4_0) \rtimes \mathbb{Z}_i)$, for $i = 3, 4, 6$.

**References**

\[\begin{align*}
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\end{align*}\]