On The Interaction of Gravitational Waves with Magnetic and Electric Fields

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Abstract

The existence of large–scale magnetic fields in the universe has led to the observation that if gravitational waves propagating in a cosmological environment encounter even a small magnetic field then electromagnetic radiation is produced. To study this phenomenon in more detail we take it out of the cosmological context and at the same time simplify the gravitational radiation to impulsive waves. Specifically, to illustrate our findings, we describe the following three physical situations: (1) a cylindrical impulsive gravitational wave propagating into a universe with a magnetic field, (2) an axially symmetric impulsive gravitational wave propagating into a universe with an electric field and (3) a ‘spherical’ impulsive gravitational wave propagating into a universe with a small magnetic field. In cases (1) and (3) electromagnetic radiation is produced behind the gravitational wave. In case (2) no electromagnetic radiation appears after the wave unless a current is established behind the wave breaking the Maxwell vacuum. In all three cases the presence of the magnetic or electric fields results in a modification of the amplitude of the incoming gravitational wave which is explicitly calculated using the Einstein–Maxwell vacuum field equations.

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1 Introduction

The existence of large scale magnetic fields in the universe has led to extensive studies of their behavior in cosmological models [1]–[5]. The observation by Marklund, Dunsby and Brodin [6] that gravity wave perturbations of Friedmann–Lemaître–Robertson–Walker cosmological models encountering weak magnetic test fields can produce electromagnetic waves is of particular significance. This phenomenon has recently been studied again in cosmology [7]. In the present paper we examine it in further detail by taking it out of the cosmological setting and by replacing the gravitational waves by a single impulsive wave in the following three illustrative situations: (1) a cylindrical impulsive gravitational wave propagating into a cylindrically symmetric universe containing an approximately uniform magnetic field (the Bonnor [8] universe, rediscovered by Melvin [9]), (2) an axially symmetric impulsive gravitational wave propagating into an axially symmetric universe containing an approximately uniform electric field (the Mc Vittie [10] universe; see also [11]) and (3) a ‘spherical’ impulsive gravitational wave propagating into a universe with no gravitational field but with a weak uniform magnetic test field. In each of these three cases the space–time to the future of the null hypersurface history of the impulsive gravitational wave (the model universe left behind by the wave) is calculated in a future neighborhood of the null hypersurface, using the Einstein–Maxwell vacuum field equations. In cases (1) and (3) we find that electromagnetic radiation is generated behind the gravitational wave. In case (2) no electromagnetic radiation appears after the wave unless a current is established behind the wave breaking the Maxwell vacuum. In all three cases the presence of the magnetic or electric fields in front of the gravitational wave modifies the amplitude of the gravitational wave and this modification is explicitly calculated using the Einstein–Maxwell vacuum field equations. The three cases are described in sections 2, 3 and 4 respectively of this paper followed by a discussion of the main features of our results in section 5. Some useful calculations pertinent to section 2 are given in appendix A.

2 The Cylindrically Symmetric Case

The cylindrically symmetric line–element can be written in the form [12]

\[ ds^2 = e^{2k-2U}(dt^2 - d\rho^2) - e^{-2U}\rho^2 d\phi^2 - e^{2U}dz^2 , \]  

(2.1)
where, in general, $k$ and $U$ are functions of $\rho$ and $t$. An example of a static model of a gravitational field having a magnetic field as origin is \cite{8}, \cite{9}

$$e^{2U} = e^k = f^2, \quad f = 1 + \frac{1}{4}B^2\rho^2,$$ (2.2)

with $B$ a real constant. The corresponding Maxwell field is given by the 2–form

$$F = B f^{-2} \rho \, d\rho \wedge d\phi.$$ (2.3)

Referred to an orthonormal tetrad this is a pure magnetic field with one physical component $Bf^{-2}$ and thus “is not a uniform field in the classical sense” \cite{8}. For a weak magnetic field, with terms of order $B^2$ neglected (more correctly, with dimensionless quantities of order $B^2 \rho^2$ neglected), the magnetic field (2.3) is approximately uniform. We wish to have an impulsive gravitational wave propagating into this universe. The history of such a wave is a null hypersurface. Respecting the cylindrical symmetry the simplest such null hypersurfaces in the space–time with line–element (2.1) have equations

$$u = t - \rho = \text{constant}.$$ (2.4)

Such a null hypersurface has the potential to be the history of a cylindrical wave. Changing to $u$ as a coordinate in place of $t$ according to (2.4) the line–element (2.1) reads

$$ds^2 = e^{2k-2U} \, du \, (du + 2 \, d\rho) - e^{-2U} \rho^2 \, d\phi^2 - e^{2U} \, dz^2,$$ (2.5)

with $k, U$ functions now of $\rho$ and $u$ in general but given by (2.2) for the magnetic universe above.

To construct a space–time model of a cylindrical impulsive gravitational wave propagating into the magnetic universe, with history $u = 0$ (say), and leaving behind a cylindrically symmetric Einstein–Maxwell vacuum we proceed as follows: We use coordinates labelled $x^\mu = (u, \rho, \phi, z)$ for $\mu = 1, 2, 3, 4$. The null hypersurface $\Sigma(u = 0)$ divides space–time into two halves $M_+(u > 0)$ and $M_-(u < 0)$. We take $M_-$ to be to the past of $\Sigma$ with line–element (2.5) with $U, k$ given by (2.2) and $M_+$ to be to the future of $\Sigma$ with line–element of the form (2.5) and with the as yet unknown functions $U, k$ denoted now by $U_+, k_+$. We assume that $\Sigma$ is singular, which means that the metric tensor of $M_- \cup M_+$ is only $C^0$ across $\Sigma$ and thus physically $\Sigma$ is in general the history of a cylindrically symmetric null shell and/or impulsive gravitational wave (see \cite{13} for a review of singular null hypersurfaces in general relativity). We seek to find the conditions on the functions $U, k, U_+, k_+$ so that $u = 0$ is the history of an impulsive gravitational wave and not a null shell. The
system of coordinates we are using is common to the two sides of Σ. Since
the metric tensor is $C^0$ the induced metrics on Σ from its embedding in $M_+$
and in $M_-$ must be identical and thus we shall have

$$U_+(u = 0, \rho) = \log f ,$$

with $f$ given by (2.2). The subset of coordinates $\xi^a = (\rho, \phi, z)$ with $a = 2, 3, 4$,
will be taken as intrinsic coordinates on Σ. Here $\rho$ is a parameter running
along the generators of Σ while $\theta^A = (\phi, z)$ with $A = 3, 4$ label the generators.
We denote by $e_{(a)} = \partial/\partial \xi^a$ the tangential basis vectors. Their scalar products
give the induced metric tensor $g_{ab}$ which is singular since Σ is null. The line–
element (2.5) restricted to Σ reads

$$ds^2|_\Sigma = g_{ab}d\xi^a d\xi^b = e_{(a)} \cdot e_{(b)} d\xi^a d\xi^b .$$

It is convenient to introduce a pseudo–inverse of $g_{ab}$ (see [13]) which we denote
by $g_{ab}^\ast$ and which is formed by the inverse $g^{AB}$ of $g_{AB}$ bordered by zeros. As
normal to Σ we take $n^\mu \partial/\partial x^\mu = \partial/\partial \rho = e_{(2)}$. This vector field is tangent to
Σ and in order to describe extrinsic properties of Σ we introduce a transversal
vector field $N^\mu$ on Σ which for convenience we take to be future–directed,
null and orthogonal to the two space–like vectors $e_{(A)}$ at each point of Σ. Thus we have

$$N_\mu n^\mu = 1 , \quad N_\mu N^\mu = 0 , \quad N_\mu e^\mu_{(A)} = 0 .$$

Thus $N_\mu = (\frac{1}{2}, 1, 0, 0)$. Following the algorithm developed in [13] we define
the transverse extrinsic curvature $K^-_{ab}$ on either side of Σ by

$$K^-_{ab} = -N_\mu (e^\mu_{(a)} + \pm \Gamma^\mu_{\lambda\sigma} e^\sigma_{(a)}) e^\lambda_{(b)} ,$$

with the comma denoting partial differentiation with respect to $x^\mu$ and $\pm \Gamma^\mu_{\lambda\sigma}$
the components of the Riemannian connection calculated on either side of Σ. The jump in the quantities (2.9) is defined by

$$\gamma_{ab} = 2 [K_{ab}] = 2 (K^+_{ab} - K^-_{ab}) .$$

We find in the present case that $\gamma_{ab} = 0$ except for

$$\gamma_{22} = -2 [\Gamma^2_{22}] ,$$
$$\gamma_{33} = -[\Gamma^3_{33}] - 2 [\Gamma^2_{33}] ,$$
$$\gamma_{44} = -[\Gamma^4_{44}] - 2 [\Gamma^2_{44}] .$$

The singular null hypersurface Σ can represent the history of a null shell
and/or an impulsive gravitational wave. The surface stress–energy tensor of

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the shell, if it exists, is calculated from $\gamma_{ab}$ and is given by (see Eq.(2.37) of \cite{13})

$$16\pi S^{ab} = \mu n^a n^b + P g_s^{ab}, \quad (2.14)$$

with the surface energy density $\mu$ and pressure $P$ defined by

$$16\pi \mu = -\gamma_{ab} g_s^{ab} = -\gamma_{AB} g^{AB}, \quad (2.15)$$

$$16\pi P = -\gamma^\dagger, \quad (2.16)$$

In (2.14) $n^a = (1, 0, 0)$ and in (2.16) $\gamma^\dagger = \gamma_{ab} n^a n^b = \gamma_{22}$. Hence the conditions for no shell read

$$\gamma^\dagger = \gamma_{22} = 0 \quad \text{and} \quad g_s^{ab} \gamma_{ab} = f^2 \gamma_{33} + f^{-2} \rho^2 \gamma_{44} = 0. \quad (2.17)$$

Using (2.11)–(2.13) and calculating the components of the Riemannian connection associated with the metric tensor given via the line–element (2.5) we find that the first of (2.17) requires that $[k_\rho] = 0$ (with the subscript denoting partial differentiation with respect $\rho$) and the second of (2.17) requires that $[k] = 0$. Hence the boundary condition for no shell is

$$[k] = 0, \quad (2.18)$$

expressing the continuity of the function $k(\rho, u)$ across $\Sigma(u = 0)$. The gravitational wave part of the signal with history $\Sigma$ is described by a part of $\gamma_{ab}$, denoted $\tilde{\gamma}_{ab}$, defined by Eq.(2.47) of \cite{13}:

$$\tilde{\gamma}_{ab} = \gamma_{ab} \frac{1}{2} g_{ab} g_c^{cd} \gamma_{cd} - 2 \gamma_{(a} N_{b)} + \gamma^\dagger N_a N_b, \quad (2.19)$$

with $N_a = (1, 0, 0)$. With (2.18) satisfied we find that $\tilde{\gamma}_{ab} = 0$ except

$$\tilde{\gamma}_{33} = \gamma_{33} = 2\rho^2 f^{-4} [U_u], \quad (2.20)$$

$$\tilde{\gamma}_{44} = \gamma_{44} = -2 [U_u]. \quad (2.21)$$

The fact that (2.20) and (2.21) are multiples of each other means that the gravitational wave here has only one degree of freedom. Hence we see that for an impulsive wave with history $\Sigma$ we must have

$$[U_u] \neq 0, \quad (2.22)$$

with the subscript denoting partial differentiation with respect to $u$.

The Maxwell field in $M_+ \cup M_-$ is given in general by the 2–form

$$F = w_\rho dz \wedge d\rho + w_u dz \wedge du + s_\rho d\rho \wedge d\phi + s_u du \wedge d\phi, \quad (2.23)$$
with \( w, s \) each functions of \( \rho, u \) and the subscripts as always denoting partial derivatives. In \( M_-(u < 0) \) we have
\[
  w = 0 \, , \quad s = -2 B^{-1} f^{-1} ,
\]
(2.24)
with \( f \) given by (2.2). Substitution of (2.24) into (2.23) yields (2.3).

To obtain the space–time \( M_+(u \geq 0) \) and the electromagnetic field for \( u \geq 0 \) we must satisfy the Einstein–Maxwell vacuum field equations in \( u \geq 0 \) with a line–element of the form (2.5) and a Maxwell 2–form of the form (2.23). These equations are listed in appendix A. For our purposes it is sufficient to solve these equations for small \( u > 0 \) (i.e. in a future neighborhood of \( \Sigma \)). The unknown functions of \( \rho, u \) are \( U, k, w, s \) with \( k \) continuous across \( u = 0 \) and \( U \) jumping across \( u = 0 \). Hence for small \( u \) we can write
\[
  U = \log f + u \theta(u) U_1 + O(u^2) ,
\]
(2.25)
\[
  k = 2 \log f + u \theta(u) k_1 + O(u^2) ,
\]
(2.26)
with \( f \) given by (2.2). Here \( \theta(u) \) is the Heaviside step function which is equal to unity if \( u > 0 \) and equal to zero if \( u < 0 \). For consistency with the expansions (2.25) and (2.26), and in the light of (2.24), we also assume that
\[
  s = -2 B^{-1} f^{-1} + u \theta(u) s_1 + O(u^2) ,
\]
(2.27)
\[
  w = u \theta(u) w_1 + O(u^2) .
\]
(2.28)
The unknown functions \( U_1, k_1, s_1, w_1 \) in (2.25)–(2.28) are functions of \( \rho \) only. Substituting (2.25)–(2.28) in Einstein’s equations (A-3)–(A-7) results in
\[
  w_1 = 0 ,
\]
(2.29)
and then
\[
  \frac{d}{d\rho} (\rho^{1/2} U_1) = B \rho^{-1/2} s_1 ,
\]
(2.30)
\[
  \frac{1}{\rho} \frac{dk_1}{d\rho} - 2 f^{-1} \frac{df}{d\rho} \frac{dU_1}{d\rho} = 2 \frac{B}{\rho} \frac{ds_1}{d\rho} + 2 B^2 f^{-2} U_1 ,
\]
(2.31)
\[
  \frac{dU_1}{d\rho} - U_1 f^{-1} \frac{df}{d\rho} + \frac{1}{2\rho} U_1 = \frac{dk_1}{d\rho} ,
\]
(2.32)
\[
  2 U_1 f^{-1} \frac{df}{d\rho} - 2 U_1^2 - \frac{1}{\rho} k_1 = \frac{2}{\rho^2} f^2 (s_1 - B \rho f^{-2}) s_1 .
\]
(2.33)
Maxwell’s equations (A-1) and (A-2) now provide just one extra relevant equation, namely,
\[
  \frac{d}{d\rho} (\rho^{-1/2} f s_1) = -B f^{-1} \rho^{1/2} U_1 .
\]
(2.34)
\[6\]
We observe that (2.30), (2.31) and (2.34) imply (2.32). Hence the strategy for solving these five equations is to first solve (2.30) and (2.34) for \( s_1, U_1 \), then substitute these solutions into (2.33) to obtain \( k_1 \) algebraically and then to check that (2.31) is automatically satisfied. Proceeding in this way we obtain the solutions

\[
U_1 = a_0 \rho^{-1/2} f^{-1} \left( 1 - \frac{1}{4} B^2 \rho^2 \right) + b_0 \rho^{1/2} B f^{-1},
\]

\[s_1 = -a_0 B \rho^{3/2} f^{-2} + b_0 \rho^{1/2} f^{-2} \left( 1 - \frac{1}{4} B^2 \rho^2 \right),\]

\[k_1 = -2 (a_0^2 + b_0^2) - a_0 B^2 \rho^{3/2} f^{-1} + 2 b_0 B \rho^{1/2} f^{-1},\]

where \( a_0, b_0 \) are real constants. A convenient way to interpret these results is to use them to obtain information about parts of the Weyl tensor and the Maxwell tensor in \( M_+ \cup M_- \) on a basis of 1–forms \( \theta^\mu (\mu = 1, 2, 3, 4) \) in terms of which the line–element (2.5) can be written

\[
ds^2 = 2 \theta^1 \theta^2 - (\theta^3)^2 - (\theta^4)^2.
\]

Such a basis is given by

\[
\theta^1 = e^{-2\rho u} (d\rho + \frac{1}{2} du), \quad \theta^2 = du, \quad \theta^3 = e^{-U} \rho d\phi, \quad \theta^4 = e^U dz.
\]

The Weyl tensor components on this basis, denoted \( C_{\mu\nu\lambda\sigma} \), for the space–time \( M_+ \cup M_- \) is dominated for small \( u \) by the tetrad component

\[
C_{2323} = \{ a_0 \rho^{-1/2} f^{-1} \left( 1 - \frac{1}{4} B^2 \rho^2 \right) + b_0 \rho^{1/2} B f^{-1} \} \delta(u),
\]

with all other terms and components at most \( O(u^0) \). Here \( \delta(u) \) is the Dirac delta function. This dominant part of the Weyl tensor of \( M_+ \cup M_- \) is type N in the Petrov classification with degenerate principal null direction \( \partial/\partial \rho \). It represents a gravitational wave. When \( B = 0 \) (\( f = 1 \)) we see a cylindrical impulsive gravitational wave which is singular on the axis \( \rho = 0 \) and having one degree of freedom manifested by the appearance of the real constant \( a_0 \).

The presence of the magnetic field \( B \) clearly modifies the amplitude of the gravitational wave. We will comment on this modification in section 5 when we can compare (2.40) with the examples described in the next two sections. The tetrad components of the Maxwell field in \( M_\pm \) will be denoted \( F_{\mu\nu}^\pm \). In general they jump across \( \Sigma \) with the jump given by

\[
f_{\mu\nu} = [F_{\mu\nu}] = F_{\mu\nu}^+ - F_{\mu\nu}^-.
\]
In the present case (2.27)–(2.29) and (2.36) mean that $f_{\mu \nu}$ vanishes except for
\[ f_{23} = f \rho^{-1} s_1. \] (2.42)

It thus follows that the bivector $f_{\mu \nu}$ is algebraically special (type N) in the classification of bivectors with $\partial/\partial \rho$ as degenerate principal null direction. Thus (2.42) indicates the presence of cylindrical electromagnetic waves behind the impulsive gravitational as it propagates through the universe with the magnetic field labelled by $B$.

3 The Axially Symmetric Case

We now construct a space–time model of an axially symmetric impulsive gravitational wave propagating into a static axially symmetric universe containing an electric field. The line–element of a simple such space–time is [10], [11]
\[ ds^2 = -W^{-3} dz^2 - W^{-1}(d\rho^2 + \rho^2 d\phi^2) + W dt^2, \] (3.1)
with $W = (1 + E z)^2$ and $E$ is a real constant. This is a solution of the Einstein–Maxwell vacuum field equations with the Maxwell 2–form given by
\[ F = E dz \wedge dt. \] (3.2)

This is clearly a pure electric field. When expressed on an orthonormal tetrad it has the one non–vanishing physical component $E W$ and so is not a uniform electric field in the classical sense. It is approximately uniform for small dimensionless parameter $E z$ however. A simple family of null hypersurfaces in this space–time is given by $u = \text{constant}$ with $u$ derived from
\[ du = dt - W^{-2} dz. \] (3.3)

Such null hypersurfaces can act as the histories of the wave–fronts of axially symmetric waves. Using $u$ instead of $t$ as a coordinate the line–element (3.1) reads
\[ ds^2 = W du^2 + 2 W^{-1} du dz - W^{-1}(d\rho^2 + \rho^2 d\phi^2). \] (3.4)

Here $z$ is a parameter running along the generators of the null hypersurfaces $u = \text{constant}$. It is convenient to work instead with an affine parameter $r$ along these generators which is related to $z$ by $1 + E z = (1 - E r)^{-1}$. Replacing $z$ by $r$ in (3.4) means the line–element now reads
\[ ds^2 = 2 du dr + (1 - E r)^{-2} du^2 - (1 - E r)^2 (d\rho^2 + \rho^2 d\phi^2). \] (3.5)
The Maxwell field (3.2) now takes the form

\[ F = E (1 - E r)^{-2} dr \wedge du . \] (3.6)

We now consider a space–time \( M_+ \cup M_- \) with \( M_-(u \leq 0) \) corresponding to the space–time with line–element (3.5) having as boundary the null hypersurface \( u = 0 \) and \( M_+(u \geq 0) \), with the same boundary \( u = 0 \), to be determined. To this latter end we solve the vacuum Einstein-Maxwell field equations for \( M_+ \cup M_- \) requiring that \( u = 0 \) is the history of an axially symmetric impulsive gravitational wave (and not a null shell). Our objective in doing this is to obtain the space–time \( M_+ \) with sufficient accuracy to determine the coefficient of \( \delta(u) \) in the Weyl tensor of \( M_+ \cup M_- \) and the jump, if it exists, in the Maxwell field across \( u = 0 \), in parallel with (2.40)–(2.42).

We find that the line–element of \( M_+ \cup M_- \), for small \( u \), can be written in the form (2.38) but with

\[
\theta^1 = dr + \frac{1}{2} \{ (1 - E r)^{-2} + u \theta(u) c_1 + O(u^2) \} \, du , \tag{3.7}
\]

\[
\theta^2 = du , \tag{3.8}
\]

\[
\theta^3 = (1 - E r) \{ (1 + u \theta(u) \alpha_1 + O(u^2)) \, d\rho + (u \theta(u) \beta_1 + O(u^2)) \, \rho \, d\phi \} , \tag{3.9}
\]

\[
\theta^4 = (1 - E r) \{ (u \theta(u) \beta_1 + O(u^2)) \, d\rho + (1 - u \theta(u) \alpha_1 + O(u^2)) \, \rho \, d\phi \} . \tag{3.10}
\]

The functions \( \alpha_1, \beta_1, c_1 \) are functions of \( r, \rho, \phi \). The field equations restrict the functions \( \alpha_1 \) and \( \beta_1 \) (they also determine the function \( c_1 \) but we will not require it here) according to

\[
\alpha_1 = \frac{\dot{\alpha}_1(\rho, \phi)}{1 - E r} , \quad \beta_1 = \frac{\dot{\beta}_1(\rho, \phi)}{1 - E r} , \tag{3.11}
\]

and the functions \( \dot{\alpha}_1, \dot{\beta}_1 \) must satisfy the equations

\[
\frac{\partial \dot{\alpha}_1}{\partial \phi} - \rho \frac{\partial \dot{\beta}_1}{\partial \rho} = 2 \dot{\beta}_1 , \tag{3.12}
\]

\[
\frac{\partial \dot{\beta}_1}{\partial \phi} + \rho \frac{\partial \dot{\alpha}_1}{\partial \rho} = -2 \dot{\alpha}_1 . \tag{3.13}
\]

Introducing the complex variable \( \zeta = \log \rho + i \phi \) we can integrate (3.12) and (3.13) to arrive at

\[
\dot{\alpha}_1 + i \dot{\beta}_1 = e^{-\zeta}H(\zeta) , \tag{3.14}
\]
where \( H \) is an arbitrary analytic function of \( \zeta \). In parallel with (2.40) and (2.42) we find in this case that

\[
C_{2323} - iC_{2324} = \frac{e^{-\zeta}H(\zeta)}{(1 - Er)} \delta(u) ,
\]

and

\[
f_{\mu\nu} = [F_{\mu\nu}] = F_{\mu\nu}^+ - F_{\mu\nu}^- = 0 .
\]

In (3.15) we see an axially symmetric impulsive gravitational wave propagating into the universe with the electric field labelled by the parameter \( E \). We also see that the presence of the electric field modifies the amplitude of the wave by the appearance of \( E \) in the coefficient of the delta function. The coefficient of the delta function in the Weyl tensor is type N in the Petrov classification with \( \partial/\partial r \) in this case as degenerate principal null direction (propagation direction in space–time). On account of (3.16) there is no electromagnetic radiation immediately behind the gravitational wave in this case. If we were to relax the Maxwell vacuum conditions in \( M_+ \) we can obtain a 4–current with tetrad components

\[
J_3 = \frac{2E}{1 - Er} f_{32} + O(u) ,
\]

\[
J_4 = \frac{2E}{1 - Er} f_{42} + O(u) .
\]

A bivector \( f_{\mu\nu} \) having only \( f_{32} \) and \( f_{42} \) non–zero is of radiative type with propagation direction \( \partial/\partial r \) and represents electromagnetic radiation.

### 4 The ‘Spherically’ Symmetric Case

Starting with the line–element of Minkowskian space–time in rectangular Cartesian coordinates and time \( X, Y, Z, T \) which reads

\[
\text{\( ds^2 = -dX^2 - dY^2 - dZ^2 + dT^2 \)},
\]

we make the coordinate transformation

\[
X + iY = r G^{1/2} e^{iy} , \quad Z = r x , \quad T = u + r ,
\]

with \( G = 1 - x^2 \) then (4.1) takes the form

\[
\text{\( ds^2 = 2 \, du \, dr + du^2 - r^2(G^{-1}dx^2 + G \, dy^2) \)}.
\]
Here \( u = \text{constant} \) are future null cones with vertices on the time-like geodesic \( r = 0 \), \( r \) is an affine parameter along the generators of the null cones and the generators are labelled by \( x, y \). Using (4.2) again we see that
\[
dX \wedge dY = r \, G \, dr \wedge dy - r^2 x \, dx \wedge dy .
\]
(4.4)

Thus in particular the Maxwell 2-form
\[
F = B \, r \, G \, dr \wedge dy - B \, r^2 x \, dx \wedge dy ,
\]
(4.5)

with \( B \) are real constant is a uniform magnetic field. We shall restrict considerations to a weak magnetic field in which squares and higher powers of \( B \) will be neglected. In this case Minkowskian space–time with the bivector (4.5) constitute an approximate solution of the Einstein–Maxwell vacuum field equations. To construct a model of a ‘spherical’ impulsive gravitational wave propagating into this universe we will take for \( M_-(u \leq 0) \) the space–time with line–element (4.3) and the future null cone \( u = 0 \) for the history of the wave. Since the future null cone is the history of a 2–sphere expanding with the speed of light we will refer to the gravitational wave with history \( u = 0 \) as a ‘spherical’ wave. The reason for the inverted commas is because such a wave will be found to have singular points on its spherical wave front, thus violating strict spherical symmetry (see below). Something like this is to be expected in general relativity on account of the Birkhoff theorem (see [13] section 1.2). Now the line–element of \( M_+ \cup M_- \), for small \( u \), can be written in the form (2.38) but with
\[
\begin{align*}
\theta^1 &= dr + \frac{1}{2} \left\{ 1 + u \, \theta(u) \, c_1 + O(u^2) \right\} \, du , \\
\theta^2 &= du , \\
\theta^3 &= r \, G^{-1/2}(1 + u \, \theta(u) \, \alpha_1 + O(u^2)) \, dx + r \, G^{1/2}(u \, \theta(u) \, \beta_1 + O(u^2)) \, dy , \\
\theta^4 &= r \, G^{-1/2}(u \, \theta(u) \, \beta_1 + O(u^2)) \, dx + r \, G^{1/2}(1 - u \, \theta(u) \, \alpha_1 + O(u^2)) \, dy .
\end{align*}
\]
(4.6–9)

Here the functions \( \alpha_1, \beta_1, c_1 \), along with the functions \( f_{\mu\nu} \), are functions for \( x, y, r \) and can be determined from the vacuum Einstein–Maxwell field equations (in particular ensuring by the vacuum conditions that no null shell can have \( u = 0 \) as history and also that there is no surface 4–current on \( u = 0 \). As in the example of section 3 we shall not require the function \( c_1 \) although it can be determined using the field equations of course. From Maxwell’s equations we find that
\[
\frac{\partial}{\partial r} (r \, f_{32}) = -B \, r \, G^{1/2} \beta_1 ,
\]
(4.10)
Neglecting $O(B^2 r^2)$-terms we conclude that
\begin{align*}
B r G^{1/2} f_{32} = K(x, y) \quad \text{and} \quad B r G^{1/2} f_{42} = L(x, y), \quad (4.12)
\end{align*}
with $K$ and $L$ arbitrary functions of $x, y$. Einstein’s equations with the electromagnetic energy–momentum tensor as source yield
\begin{align*}
\frac{\partial}{\partial r} (r \alpha_1) &= B r G^{1/2} f_{42}, \quad (4.13) \\
\frac{\partial}{\partial r} (r \beta_1) &= -B r G^{1/2} f_{32}, \quad (4.14)
\end{align*}
from which we conclude that, in the light of (4.12),
\begin{align*}
\alpha_1 &= L(x, y) + \frac{C(x, y)}{r}, \quad \beta_1 = -K(x, y) + \frac{D(x, y)}{r}, \quad (4.15)
\end{align*}
where $C$ and $D$ are arbitrary functions of $x, y$. Now defining
\begin{equation}
\zeta = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right) + iy, \quad (4.16)
\end{equation}
the remaining Einstein field equations give the single complex equation
\begin{equation}
\frac{\partial}{\partial \zeta} \left\{ G (\alpha_1 + i \beta_1) \right\} = -G x (L - i K), \quad (4.17)
\end{equation}
from which we conclude, using (4.15), that
\begin{equation}
G (C + i D) = \mathcal{F}(\bar{\zeta}) \quad \text{and} \quad L - i K = \mathcal{G}(\bar{\zeta}), \quad (4.18)
\end{equation}
where $\mathcal{F}, \mathcal{G}$ are arbitrary analytic functions. Thus (4.12) and (4.15) now read
\begin{align*}
f_{42} + i f_{32} &= B^{-1} G^{-1/2} \frac{1}{r} \mathcal{G}(\zeta), \quad (4.19)
\end{align*}
and
\begin{align*}
\alpha_1 - i \beta_1 &= \frac{1}{r} G^{-1} \mathcal{F}(\zeta) + \mathcal{G}(\zeta), \quad (4.20)
\end{align*}
respectively.
In this case the delta function part of the Weyl tensor is given by
\begin{equation}
C_{2323} - i C_{2324} = (\alpha_1 - i \beta_1) \delta(u) = \left\{ \frac{1}{r} G^{-1} \mathcal{F}(\zeta) + \mathcal{G}(\zeta) \right\} \delta(u). \quad (4.21)
\end{equation}
The first term in the coefficient of the delta function is the amplitude of a ‘spherical’ wave with the expected “directional” singularities at \(x = \pm 1\) (corresponding to \(G(x) = 0\), equivalently \(X = Y = 0\)) while the second term is the modification to the amplitude due to the wave encountering the weak magnetic field. In (4.19) we see the algebraically special jumps in the Maxwell field across \(u = 0\) which indicate the presence of electromagnetic radiation in the region \(M_+\) of space–time to the future of the history of the impulsive gravitational wave (i.e. behind the wave). This radiation is spherical–fronted \((u = \text{constant} > 0\) being the histories of the wave–fronts), singular at \(r = 0\) and also has directional singularities at \(x = \pm 1\).

5 Discussion

In sections 2, 3 and 4 above we have considered an impulsive gravitational wave propagating into a vacuum universe with a magnetic field present in the first and last cases and into a vacuum universe with an electric field present in the second case. If the vacuum is preserved after the wave has passed then in the region behind the wave electromagnetic radiation appears in the first and last cases but not in the second case. In addition we have found that in all three cases the amplitude of the impulsive gravitational wave is modified by the existence of the magnetic or electric field that it encounters. There is an interesting pattern to this modification when the magnetic and electric fields are weak in all three cases. In the cylindrically symmetric case, combining (2.40) and (2.42) with (2.36), we can write, approximately for small \(B\),

\[
C_{2323} = \{a_0 \rho^{-1/2} + \rho B f_{23}\} \delta(u), \tag{5.1}
\]

for the delta function part of the Weyl tensor. The coefficient of the delta function here is a sum of a cylindrical wave term and an interaction between the weak magnetic field and the electromagnetic radiation. For the axially symmetric case with a weak electric field we have from (3.15)

\[
C_{2323} - iC_{2324} = \{e^{-\xi} H(\zeta) + r E e^{-\xi} H(\zeta)\} \delta(u). \tag{5.2}
\]

In this case there is no electromagnetic radiation generated behind the gravitational wave but the coefficient of the delta function is the sum of an axially symmetric wave term and an interaction between the weak electric field and the gravitational radiation. Finally in the ‘spherical’ case we have (4.21) which with (4.19) can be written

\[
C_{2323} - iC_{2324} = \left\{\frac{1}{r} G^{-1} \mathcal{F}(\zeta) + r B G^{1/2} (f_{42} + i f_{32})\right\} \delta(u). \tag{5.3}
\]
The coefficient of the delta function here is a sum of a ‘spherical’ wave and an interaction between the weak magnetic field and the electromagnetic radiation.

When electromagnetic radiation appears above it takes the form of an electromagnetic shock wave accompanying the impulsive gravitational wave. The history of the electromagnetic shock wave is the null hypersurface \( u = 0 \). Should we wish to know the field in \( u > 0 \), to the future of the history of the wave, we would require, for example, the \( O(u^2) \)-terms in (2.25)–(2.28), (3.7)–(3.10) and (4.6)–(4.9). The examples given in this paper have motivated the development of a general, relativistically invariant, treatment of the interaction of impulsive gravitational waves with electromagnetic fields which will be described in a future publication.

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A Cylindrically Symmetric Einstein–Maxwell Equations

With a metric tensor given by the line–element (2.5) and a Maxwell tensor given by the 2–form (2.23) the Maxwell field equations (\(d^*F = 0\) with \(*F\) the dual of \(F\)) are given by (with subscripts denoting partial differentiation)

\[
\begin{align*}
\rho (e^{-2U} w_\rho)_u + (\rho e^{-2U}(w_u - w_\rho))_\rho &= 0 , \\
\rho^{-1} (e^{2U} s_\rho)_u + (\rho^{-1} e^{2U}(s_u - s_\rho))_\rho &= 0 .
\end{align*}
\]

(E.1)

(E.2)

Einstein’s equations \((R_{\mu\nu} = -2 E_{\mu\nu}\) with \(R_{\mu\nu}\) the Ricci tensor and \(E_{\mu\nu}\) the electromagnetic energy–momentum tensor) read

\[
\begin{align*}
k_{\rho\rho} - 2 k_{u\rho} + \frac{1}{\rho}(k_\rho - k_u) - U_{\rho\rho} + 2 U_{u\rho} - \frac{1}{\rho}(U_\rho - U_u) - 2 U_u^2 &= T_1 , \\
2 U_{u\rho} - U_{\rho\rho} + \frac{1}{\rho}(U_u - U_\rho) - 2 U_u U_\rho + k_{\rho\rho} - 2 k_{u\rho} + \frac{1}{\rho} k_\rho &= T_2 , \\
-U_\rho - \rho U_{\rho\rho} + U_u + 2 \rho U_{u\rho} &= T_3 ,
\end{align*}
\]

(A.3)

(A.4)

(A.5)

and

\[
\begin{align*}
w_\rho s_\rho &= s_u w_\rho + s_\rho w_u ,
\end{align*}
\]

(A.6)

with

\[
\begin{align*}
T_1 &= \frac{2}{\rho^2} e^{2U}(s_u^2 - s_u s_\rho + s_\rho^2) + 2 e^{-2U}(w_u^2 - w_u w_\rho + \frac{1}{2} w_\rho^2) , \\
T_2 &= \frac{1}{\rho^2} e^{2U}s_\rho^2 + e^{-2U} w_\rho^2 , \\
T_3 &= \frac{2}{\rho} e^{-2U} \left\{ s_\rho s_u - \frac{1}{2} s_\rho^2 - \rho^2 (w_\rho w_u - \frac{1}{2} w_\rho^2) \right\} .
\end{align*}
\]

(A.7)

(A.8)

(A.9)