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The exact solution for the Dirac equation with the Cornell potential

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Abstract An analytical solution of the Dirac equation with a Cornell potential, with identical scalar and vectorial parts, is presented. The solution is obtained by using the linear potential solution, related to Airy functions, multiplied by another function to be determined. The energy levels are obtained and we notice that they obey a band structure.

Keywords Cornell Potential · Dirac equation · Airy Functions · interquark forces · confining · energy levels

1 Introduction

The Cornell potential [1] was proposed to describe the interaction of quarks, containing a confining linear term and a Coulomb interaction. To our best knowledge, the potential does not possess exact solutions within the common equations of quantum mechanics, i.e., the nonrelativistic Schrödinger equation, relativistic Dirac, Klein-Gordon, Proca, and Duffin-Kemmer-Petiau (DKP) equations. Here, we focus on Dirac equation, with equal scalar and vectorial components.

2 The Solution for the Cornell Potential

We consider each quark with mass $m_\alpha$ described by a confining potential $V(r)$ in the Dirac equation, which is given by

\[
\left[ \mathbf{\alpha} \cdot \mathbf{p} + \beta m_\alpha + \frac{1}{2}(1 + \beta)V(r) \right] \Psi(x) = E_{nn}\Psi(x),
\]

where $\mathbf{\alpha}$ and $\beta$ are the usual Dirac matrices, and the index $\alpha$ refers to the kind of particle ($u$, $d$, or $s$ quarks, in the present case). By decomposing the above equation in spherical coordinates, we have the component wave function

\[
\Psi(x) = \begin{pmatrix} \chi(x) \\ \varphi(x) \end{pmatrix} = \begin{pmatrix} f(r)\Omega^m_{\alpha\alpha}(\theta, \phi) \\ ig(r)\Omega^m_{\beta\alpha}(\theta, \phi) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} u(r)\Omega^m_{\alpha\alpha}(\theta, \phi) \\ iv(r)\Omega^m_{\beta\alpha}(\theta, \phi) \end{pmatrix},
\]

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such that, after separating the radial parts, we can get the following two radial equations

\[
\frac{du(r)}{dr} = - \frac{\kappa}{r} u(r) + \left[E_{\alpha n} + m_\alpha\right] v(r),
\]

\[
\frac{dv(r)}{dr} = - \frac{\sigma}{r} v(r) - \left[E_{\alpha n} - m_\alpha - V(r)\right] u(r),
\]

where \(\kappa = -(j + 1/2) = -(l + 1)\) if \(l = j - 1/2\) and \(\kappa = j + 1/2 = l\) if \(l = j + 1/2\) is the Dirac kappa.

The system may be reduced to a Schrödinger like equation

\[
\frac{d^2u(r)}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} u(r) + \left[E_{\alpha n} + m_\alpha\right] \left[E_{\alpha n} - m_\alpha - V(r)\right] u(r) = 0.
\]

The solution for the case of a linear potential, \(V(r) = \lambda r\), and \(\kappa = -1\) is described below (see details in [3] and [4]): The unnormalized solution of (4) is

\[
u(r) = \text{Ai}(\lambda^{1/3} r + a_\alpha)\]

where \(\text{Ai}(\lambda^{1/3} r + a_\alpha)\) is the Airy function and \(a_\alpha\) are its roots. The energy levels are given by: \(E_{\alpha n} = m_\alpha - \lambda a_\alpha / K_\alpha\), where \(K_\alpha \equiv \sqrt[3]{\lambda(E_{\alpha n} + m_\alpha)}\). The above form avoids singularities at origin.

For the Cornell potential, \(V(r) = \lambda r - \sigma / r\), we suppose a solution such as \(u(r) = \text{Ai}(\lambda^{1/3} r + a_\alpha) F(r)\).

The energy levels are renamed to \(E'_{\alpha n'} + \delta_{nn'}\), with the \(\delta_{nn'}\) term to be determined in a consistent way with the energy levels [2]. The relation between \(\delta_{nn'}\) and \(E'_{\alpha n'}\) is:

\[
\frac{(E'_{\alpha n'}^2 - m_\alpha^2)}{[E'_{\alpha n'} + m_\alpha + \delta_{nn'}]^2} = a_\alpha.
\]

By replacing \(u(r)\) in Eq. [8], we notice that it can be separated in a part that is the Airy equation, and another that, in the limit \(r \to 0\), has the same form of that of the hydrogen atom (the part where the potential \(-\sigma / r\) dominates). Therefore, the solution has a known form: \(F(r) = P_{n'}(r)e^{-\sigma r^2}\) and (if \(l = 0\))

\[
\eta^2 + \delta_{nn'}(2E'_{\alpha n'} + \delta_{nn'}) = 0,
\]

\[-2q(1 + n') + (E'_{\alpha n'} + \delta_{nn'} + m_\alpha)\sigma = 0,
\]

where \(n'\) is the quantum number for the resulting equation for \(F(r)\).

3 Results and Conclusion

Note that there are two quantum numbers, \(n\) and \(n'\), due to the relation among equations [5] and [6]. In this manner, the energy spectrum has a band structure, and if \(n' \to \infty\), \(E'_{\alpha n'} + \delta_{nn'} \to E_{\alpha n}\). The Table shows the numerical calculated energy values for the linear potential and also compares with the numerical energy values for the Cornell potential obtained thorough the first-order perturbation correction taking the potential \(-\sigma / r\) as a perturbation. The results shown are for \(\lambda = 400MeV/fm\) and \(\sigma = 100MeV/fm\) and \(m_\alpha = 0\). A more detailed work [5] will be presented soon.

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