GEOMETRY OF HERMITIAN SYMMETRIC SPACES UNDER 
THE ACTION OF A MAXIMAL UNIPOTENT GROUP

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Abstract. Let $G/K$ be a non-compact irreducible Hermitian symmetric space of rank $r$ and let $NAK$ be an Iwasawa decomposition of $G$. By the polydisc theorem, $AK/K$ can be regarded as the base of an $r$-dimensional tube domain holomorphically embedded in $G/K$. As every $N$-orbit in $G/K$ intersects $AK/K$ in a single point, there is a one-to-one correspondence between $N$-invariant domains in $G/K$ and tube domains in the product of $r$ copies of the upper half-plane in $\mathbb{C}$. In this setting we prove a generalization of Bochner’s tube theorem. Namely, an $N$-invariant domain $D$ in $G/K$ is Stein if and only if the base $\Omega$ of the associated tube domain is convex and “cone invariant”. We also obtain a precise description of the envelope of holomorphy of an arbitrary holomorphically separable $N$-invariant domain over $G/K$.

An important ingredient for the above results is the characterization of several classes of $N$-invariant plurisubharmonic functions on $D$ in terms of the corresponding classes of convex functions on $\Omega$. This also leads to an explicit Lie group theoretical description of all $N$-invariant potentials of the Killing metric on $G/K$.

1. Introduction

The classical Bochner’s tube theorem states that the envelope of holomorphy of a tube domain $\mathbb{R}^n + i\Omega$ in $\mathbb{C}^n$ is univalent and coincides with the convex envelope $\mathbb{R}^n + i\text{conv}(\Omega)$. Moreover, there is a one-to-one correspondence between the class of $\mathbb{R}^n$-invariant plurisubharmonic functions on a Stein tube domain in $\mathbb{C}^n$ and the class of convex functions on its base in $\mathbb{R}^n$ (cf. [Gun90]).

Here our goal is to obtain analogous results in the setting of an irreducible Hermitian symmetric space of the non-compact type, under the action of a maximal unipotent group of holomorphic automorphisms.

Any such space can be realized as a quotient $G/K$, where $G$ is a non-compact real simple Lie group and $K$ is a maximal compact subgroup of $G$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be an Iwasawa decomposition of $\mathfrak{g}$, where $\mathfrak{n}$ is a maximal nilpotent
subalgebra, $\mathfrak{a}$ is a maximally split abelian subalgebra and $\mathfrak{k}$ is the Lie algebra of $K$. The integer $r := \dim \mathfrak{a}$ is by definition the rank of $G/K$.

Let $\text{NAK}$ be the corresponding Iwasawa decomposition of $G$, where $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. The group $N$ acts on $G/K$ by biholomorphisms and every $N$-orbit in $G/K$ intersects the smooth, real $r$-dimensional submanifold $A \cdot eK$ transversally in a single point.

As the space $G/K$ is Hermitian symmetric, $G$ contains $r$ pairwise commuting subgroups isomorphic to $SL_2(R)$. The orbit of the base point $eK \in G/K$ under the product of such subgroups is a closed complex submanifold of $G/K$ which contains $A \cdot eK$ and is biholomorphic to $\mathbb{H}^r$, the product of $r$ copies of the upper half-plane in $\mathbb{C}$. Moreover, every $N$-orbit in $G/K$ intersects $\mathbb{H}^r$ in an $R^r$-orbit.

This fact is an analogue of the polydisk theorem and determines a one-to-one correspondence between $N$-invariant domains in $G/K$ and tube domains in $\mathbb{H}^r$ (cf. Prop. 4.1 and Cor. 4.3). If $D$ is an $N$-invariant domain in $G/K$, then it is in terms of the base $\Omega$ of the associated tube domain in $\mathbb{H}^r$ that the properties of $N$-invariant objects on $D$ can be best described.

Define the cone

$$C := \begin{cases} (\mathbb{R}^{>0})^r, & \text{in the non-tube case}, \\ (\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case}. \end{cases}$$

A set $\Omega \subset \mathbb{R}^r$ is $C$-invariant if $y \in \Omega$ implies $y + v \in \Omega$, for all $v \in C$. Our generalization of Bochner’s tube theorem is as follows.

**Theorem 4.9.** Let $G/K$ be a non-compact irreducible Hermitian symmetric space of rank $r$. Let $D$ be an $N$-invariant domain in $G/K$ and let $\mathbb{R}^r + i\Omega$ be the associated $r$-dimensional tube domain. Then $D$ is Stein if and only if $\Omega$ is convex and $C$-invariant.

We also show that a holomorphically separable, $N$-equivariant, Riemann domain over $G/K$ is necessarily univalent (cf. Prop. 4.13). This implies the following corollary.

**Corollary 4.14.** The envelope of holomorphy $\hat{D}$ of an $N$-invariant domain $D$ in $G/K$ is the smallest Stein domain in $G/K$ containing $D$. The base $\hat{\Omega}$ of the $r$-dimensional tube domain associated to $\hat{D}$ is the convex, $C$-invariant hull of $\Omega$.

One approach to the proof of the above theorem uses smooth $N$-invariant functions. There is a one-to-one correspondence between $N$-invariant functions on $D$ and functions on $\Omega$, and such correspondence preserves regularity. An important ingredient is the computation of the Levi form of a smooth $N$-invariant function $f: D \to \mathbb{R}$ in terms of the Hessian and the gradient of the corresponding function $\hat{f}: \hat{\Omega} \to \mathbb{R}$. To this end, a simple pluripotential argument enables us to exploit the restricted root decomposition of $\mathfrak{n}$ (cf. Prop. 3.1 and Prop. 4.3).
Then, in the smooth case, the proof of Theorem 4.9 is carried out by showing that $D$ is Levi pseudoconvex, and therefore Stein, if and only if the base $\Omega$ of the associated tube domain is convex and $C$-invariant.

The general case follows from the smooth case by exhausting $D$ with an increasing sequence of Stein, $N$-invariant domains with smooth boundary. For this we adapt a classical approximation method for convex functions on convex domains to our $C$-invariant context.

In Section 6, an alternative proof of Theorem 4.9 is carried out by realizing $G/K$ as a Siegel domain and by combining some results from the theory of normal $J$-algebras with some convexity arguments.

The aforementioned computation of the Levi form leads to a characterization of smooth $N$-invariant plurisubharmonic functions on $N$-invariant domains in $G/K$ in terms of the corresponding functions on $\Omega$. By classical approximation methods, a similar characterization is obtained for arbitrary $N$-invariant (strictly) plurisubharmonic functions on $D$. In order to formulate such results we need the following definition.

Let $\hat{f}: \Omega \rightarrow \mathbb{R}$ be a function defined on a $C$-invariant domain in $(\mathbb{R}_{>0})^r$ and let $\overline{C}$ be the closure of the cone $C$. Then $\hat{f}$ is $\overline{C}$-decreasing if for every $y \in \Omega$ and $v \in \overline{C}$ the restriction of $\hat{f}$ to the half-line $\{y + tv : t \geq 0\}$ is decreasing.

**Theorem.** (see Thm. 5.5) Let $D$ be a Stein, $N$-invariant domain in a non-compact, irreducible Hermitian symmetric space $G/K$ of rank $r$ and let $\Omega$ be the base of the associated $r$-dimensional tube domain.

An $N$-invariant function $f : D \rightarrow \mathbb{R}$ is (strictly) plurisubharmonic if and only if the corresponding function $\hat{f} : \Omega \rightarrow \mathbb{R}$ is (stably) convex and $\overline{C}$-decreasing.

It follows that every $N$-invariant plurisubharmonic function on $D$ is continuous.

In fact, the above theorem holds true both in the smooth and non-smooth context, and can be regarded as a generalization of the well known result for $\mathbb{R}^n$-invariant plurisubharmonic functions on tube domains in $\mathbb{C}^n$ (see Sect. 5 for precise definitions and statements).

In the appendix, as an application of our methods we explicitly determine all the $N$-invariant potentials of the Killing metric on $G/K$ in a Lie group theoretical fashion.

2. **Preliminaries**

Let $G/K$ be an irreducible Hermitian symmetric space, where $G$ is a real non-compact semisimple Lie group and $K$ is a maximal compact subgroup of $G$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, with Cartan involution $\theta$. Denote by $B(\cdot, \cdot)$ both the Killing form of $\mathfrak{g}$ and its $\mathbb{C}$-linear extension to $\mathfrak{g}^C$ (which coincides with the Killing form of $\mathfrak{g}^C$).
Let \( a \) be a maximal abelian subspace in \( p \). The dimension of \( a \) is by definition the rank \( r \) of \( G/K \). Let \( g = m \oplus a \oplus \bigoplus_{\alpha \in \Sigma} g^\alpha \) be the restricted root decomposition of \( g \) determined by the adjoint action of \( a \), where \( m \) denotes the centralizer of \( a \) in \( k \). For a simple Lie algebra of Hermitian type \( g \), the restricted root system is either of type \( C_r \) (if \( G/K \) is of tube type) or of type \( B_{2r} \) (if \( G/K \) is not of tube type), i.e. there exists a basis \( \{ e_1, \ldots, e_r \} \) of \( a^* \) for which a positive system \( \Sigma^+ \) is given by

\[
\Sigma^+ = \{ 2e_j, 1 \leq j \leq r, e_k \pm e_l, 1 \leq k < l \leq r \}, \quad \text{for type } C_r, \\
\Sigma^+ = \{ e_j, 2e_j, 1 \leq j \leq r, e_k \pm e_l, 1 \leq k < l \leq r \}, \quad \text{for type } B_{2r}.
\]

The roots \( 2e_1, \ldots, 2e_r \) form a maximal set of long strongly orthogonal positive restricted roots. The root spaces \( g^{2e_1}, \ldots, g^{2e_r} \) are one-dimensional and one can choose generators \( E^j \in g^{2e_j} \) such that the \( sl(2) \)-triples \( \{ E^j, \theta E^j, A_j := [\theta E^j, E^j] \} \) are normalized as follows

\[
[A_j, E^l] = \delta_{jl}2E^l, \quad \text{for } j, l = 1, \ldots, r.
\]

Denote by \( I_0 \) the \( G \)-invariant complex structure of \( G/K \). We assume that \( I_0(E^j - \theta E^j) = A_j \). By the strong orthogonality of \( 2e_1, \ldots, 2e_r \), the vectors \( A_1, \ldots, A_r \) form a \( B \)-orthogonal basis of \( a \), dual to \( e_1, \ldots, e_r \) of \( a^* \), and the associated \( sl(2) \)-triples pairwise commute.

Let \( g = n \oplus a \oplus k \) be the Iwasawa decomposition subordinated to \( \Sigma^+ \), where \( n = \bigoplus_{\alpha \in \Sigma^+} g^\alpha \), and let \( G = NAK \) be the corresponding Iwasawa decomposition of \( G \). Then \( S = NA \) is a real split solvable group acting freely and transitively on \( G/K \). In particular, the tangent space to \( G/K \) at the base point \( eK \) can be identified with the Lie algebra \( s = n \oplus a \).

The map \( \phi: s \to p \), given by \( \phi(X) := \frac{1}{2}(X - \theta X) \), is an isomorphism of vector spaces. As a consequence,

\[
\langle X, Y \rangle := B(\phi(X), \phi(Y)) = -\frac{1}{2}B(X, \theta Y),
\]

for \( X, Y \in s \), defines a positive definite symmetric bilinear form on \( s \). Moreover, the map \( J: s \to s \), given by

\[
JX := \phi^{-1} \circ I_0 \circ \phi(X),
\]

defines a complex structure on \( s \), such that \( \phi(JX) = I_0\phi(X) \). The complex structure \( J \) permutes the restricted root spaces of \( s \) (cf. \([RoVe73]\)), namely

\[
Ja = \bigoplus_{j=1}^r g^{2e_j}, \quad Jg^{e_j - e_l} = g^{e_j + e_l}, \quad Jg^{e_j} = g^{e_j}.
\]

In order to obtain a precise description of \( J \) on \( s \), we recall a few more facts. Let \( g^C = h^C \oplus \bigoplus_{\mu \in \Delta} g^\mu \) be the root decomposition of \( g^C \) with respect to a maximally split Cartan subalgebra \( h = b \oplus a \) of \( g \), where \( b \) is an abelian subalgebra of \( m \). Let \( \sigma \) be the conjugation of \( g^C \) with respect to \( g \). Let \( \theta \) denote also the \( \mathbb{C} \)-linear extension of \( \theta \) to \( g^C \). One has \( \theta \sigma = \sigma \theta \). Write \( Z := \sigma Z \), for \( Z \in g^C \).
As $\sigma$ and $\theta$ stabilize $\mathfrak{h}$, they induce actions on $\Delta$, defined by $\overline{\mu}(H) := \overline{\mu(H)}$ and $\theta\mu(H) := \mu(\theta(H))$, for $H \in \mathfrak{h}$, respectively. Fix a positive root system $\Delta^+$ compatible with $\Sigma^+$, meaning that $\mu|_{\mathfrak{a}} = Re(\mu) \in \Sigma^+$ implies $\mu \in \Delta^+$. Then $\sigma\Delta^+ = \Delta^+$.

Given a restricted root $\alpha \in \Sigma$, the corresponding restricted root space $\mathfrak{g}^\alpha$ decomposes into the direct sum of ordinary root spaces with respect to the Cartan subalgebra $\mathfrak{h}$ as follows

$$
\mathfrak{g}^\alpha = \left( \bigoplus_{\mu \in \Delta, \mu + \overline{\mu}} \mathfrak{g}^{\mu} \oplus \mathfrak{g}^{\overline{\mu}} \oplus \mathfrak{g}^\lambda \right) \cap \mathfrak{g},
$$

where $\lambda \in \Delta$ is possibly a root satisfying $\lambda = \overline{\lambda} = \alpha$. The next lemma is obtained by combining Lemma 2.2 in [Gel21] with (3).

**Lemma 2.1. (the complex structure $J$ on $\mathfrak{s}$).**

(a) For $j = 1, \ldots, r$, let $A_j \in \mathfrak{a}$ and $E_j \in \mathfrak{g}^{2\epsilon_j}$ be elements normalized as in (7). Then $JE_j = \frac{1}{2}A_j$ and $JA_j = -2E_j$.

(b) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{\epsilon_j + \epsilon_l}$, where $\mu \in \Delta^+$ is a root satisfying $Re(\mu) = \epsilon_j - \epsilon_l$ and $Z^\mu \in \mathfrak{g}^{\mu}$ (if $\overline{\mu} = \mu$, we may assume $Z^\mu = \overline{Z^\mu}$ and set $X = Z^\mu$). Then $JX = [E^l, X] \in \mathfrak{g}^{\epsilon_j - \epsilon_l}$.

(c) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{\epsilon_j}$, where $\mu$ is a root in $\Delta^+$ satisfying $Re(\mu) = \epsilon_j$ and $Z^\mu \in \mathfrak{g}^{\mu}$ (as $\dim \mathfrak{g}^{\epsilon_j}$ is even, one necessarily has $\overline{\mu} = \mu$). Then $JX = iZ^\mu + i\overline{Z^\mu} \in \mathfrak{g}^{\epsilon_j}$.

**Remark 2.2. (a $J$-stable basis of $\mathfrak{s}$)** In view of Lemma 2.1, one can choose a $J$-stable basis of $\mathfrak{s}$, compatible with the restricted root decomposition.

(a) As a basis of $\mathfrak{a} \oplus J\mathfrak{a}$, take pairs of elements $A_j$, $JA_j = -2E_j$, for $j = 1, \ldots, r$, normalized as in (7).

(b) As a basis of $\mathfrak{g}^{\epsilon_j - \epsilon_l} \oplus \mathfrak{g}^{\epsilon_j + \epsilon_l}$, take $4$-tuples of elements

$$
X = Z^\mu + \overline{Z^\mu}, \quad X' = iZ^\mu + i\overline{Z^\mu}, \quad JX = [E^l, X], \quad JX' = [E^l, X'],
$$

parametrized by the pairs of roots $\mu \neq \overline{\mu} \in \Delta^+$ satisfying $Re(\mu) = \epsilon_j - \epsilon_l$ (with no repetition), with $Z^\mu$ a root vector in $\mathfrak{g}^{\mu}$. For $\mu = \overline{\mu}$, one may assume $Z^\mu = \overline{Z^\mu}$ and take the pair $X = Z^\mu$, $JX = [E^l, X]$.

(c) As a basis of $\mathfrak{g}^{\epsilon_j}$ (non-tube case), take pairs of elements

$$
X = Z^\mu + \overline{Z^\mu}, \quad JX = iZ^\mu + i\overline{Z^\mu},
$$

parametrized by the pairs of roots $\mu \neq \overline{\mu} \in \Delta^+$ satisfying $Re(\mu) = \epsilon_j$ (with no repetition), with $Z^\mu \in \mathfrak{g}^{\mu}$. 

The next lemma contains some identities which are needed in Section 3. Its proof is essentially contained in [GeIa21], Lemma 2.4.

**Lemma 2.3.** Let \( \mu \in \Delta^+ \) be a root satisfying \( \text{Re}(\mu) = e_j - e_l \) and let \( Z^\mu \) a root vector in \( \mathfrak{g}^\mu \). Let \( X = Z^\mu + Z^\mu \in \mathfrak{g}^{e_j - e_l} \) and \( JX = [E^j, X] \in \mathfrak{g}^{e_j + e_l} \). If \( \bar{\mu} \neq \mu \), let \( X' = iZ^\mu + i\bar{Z}^\mu \). If \( \bar{\mu} = \mu \), let \( X = iZ^\mu + i\bar{Z}^\mu \). If \( \bar{\mu} = \mu \), let \( X' = iZ^\mu + i\bar{Z}^\mu \). Then

(a) \( [JX, X] = [JX', X'] = sE^j \), for some \( s \in \mathbb{R}, s \neq 0 \);
(b) \( [JX', X] = 0 \).

Let \( \mu \) be a root in \( \Delta^+ \), with \( \text{Re}(\mu) = e_j \) (non-tube case) and let \( Z^\mu \) a root vector in \( \mathfrak{g}^\mu \). Let \( X = Z^\mu + Z^\mu \) and \( JX = iZ^\mu + i\bar{Z}^\mu \). Then

(c) \( [JX, X] = tE^j \), for some \( t \in \mathbb{R}, t \neq 0 \).

3. The Levi form of an \( N \)-invariant function on \( G/K \)

Let \( G/K \) be a non-compact, irreducible Hermitian symmetric space of rank \( r \), and let \( G = N \exp(a)K \) be an Iwasawa decomposition of \( G \). Let \( D \) be an \( N \)-invariant domain in \( G/K \). Then \( D \) is uniquely determined by a domain \( D \) in \( \mathfrak{a} \) by

\[
D := N \exp(D) \cdot eK. 
\]

Similarly, an \( N \)-invariant function \( f : D \rightarrow \mathbb{R} \) is uniquely determined by the function \( \tilde{f} : \mathcal{D} \rightarrow \mathbb{R} \), defined by

\[
\tilde{f}(H) := f(\exp(H)K).
\]

The goal of this section is to express the Levi form, i.e. the real symmetric \( J \)-invariant bilinear form

\[
h_f(\cdot, \cdot) := -dd^c f(\cdot, J \cdot),
\]

of a smooth \( N \)-invariant function \( f \) on \( D \), in terms of the first and second derivatives of the corresponding function \( \tilde{f} \) on \( \mathcal{D} \). This will enable us to characterize smooth \( N \)-invariant strictly plurisubharmonic functions on a Stein \( N \)-invariant domain \( D \) in \( G/K \) by appropriate conditions on the corresponding functions on \( \mathcal{D} \) (Prop. 3.1). As \( f \) is \( N \)-invariant, \( h_f \) is \( N \)-invariant as well. Therefore it will be sufficient to carry out the computation along the slice \( \exp(\mathcal{D}) \cdot eK \), which meets all \( N \)-orbits.

For \( X \in \mathfrak{g} \), denote by \( \tilde{X} \) the vector field on \( G/K \) induced by the left \( G \)-action. Its value at \( z \in G/K \) is given by

\[
\tilde{X}_z := \frac{d}{ds}|_{s=0} \exp{sX} \cdot z.
\]

Let \( X \in \mathfrak{g}^\alpha \), for \( \alpha \in \Sigma^+ \cup \{0\} \) (here \( X \in \mathfrak{a} \), when \( \alpha = 0 \)). If \( z = aK \), with \( a = \exp{H} \) and \( H \in \mathfrak{a} \), then the vector field \( \tilde{X} \) can also be expressed as

\[
\tilde{X}_z = e^{-\alpha(H)}a_*X.
\]
Set

\[ b := B(A_1, A_1) = \ldots = B(A_r, A_r), \tag{11} \]

which is a real positive constant only depending on the Lie algebra \( \mathfrak{g} \).

**Proposition 3.1.** Let \( D \) be an \( N \)-invariant domain in \( G/K \) and let \( f : D \to \mathbb{R} \) be a smooth \( N \)-invariant function. Fix \( a = \exp H \), with \( H = \sum a_j A_j \in \mathcal{D} \). Then, in the basis of \( \mathfrak{s} \) defined in Remark 2.2, the form \( h_f \) at \( z = aK \in D \) is given as follows.

(i) The spaces \( a_* \mathfrak{a}, a_* \mathfrak{J} \mathfrak{a}, a_* \mathfrak{e}^{e_j \cdot e_i}, a_* \mathfrak{e}^{e_j + e_i} \) and \( a_* \mathfrak{g}^{e_j} \) are pairwise \( h_f \)-orthogonal.

(ii) For \( A_j, A_l \in \mathfrak{a} \) one has

\[ h_f(a_* A_j, a_* A_l) = -2 \delta_{j,l} \frac{\delta^j_l}{a_{aj, a_{al}}}(H) + \frac{\delta_{j,l}^2}{a_{aj, a_{al}}}(H). \]

On the blocks \( a_* \mathfrak{e}^{e_j - e_i} \) and \( a_* \mathfrak{g}^{e_j} \) the restriction of \( h_f \) is diagonal and the only non-zero entries are given as follows.

(iii) For \( X, X' \in \mathfrak{g}^{e_j - e_i} \) as in Remark 2.2(b), one has

\[ h_f(a_* X, a_* X) = -2 \frac{|X|^2}{b} \frac{\delta^j_i}{a_{aj}}(H), \quad h_f(a_* X', a_* X) = -2 \frac{|X'|^2}{b} \frac{\delta^j_i}{a_{aj}}(H). \]

(iv) (non-tube case) For \( X \in \mathfrak{g}^{e_i} \) as in Remark 2.2(c), one has

\[ h_f(a_* X, a_* X) = -2 \frac{|X|^2}{b} \frac{\delta^j_i}{a_{aj}}(H). \]

On the remaining blocks \( h_f \) is determined by (4), the \( J \)-invariance of \( h_f \), (i) and (iii) above.

**Proof.** Let \( f : G/K \to \mathbb{R} \) be a smooth \( N \)-invariant function. The computation of \( h_f \) uses the fact that, for \( X \in \mathfrak{n} \), the function \( \mu^X : G/K \to \mathbb{R} \), given by \( \mu^X(z) := d^c f(\widetilde{X}_z) \), satisfies the identity

\[ d\mu^X = - X dd^c f, \tag{12} \]

where \( d^c f := df \circ J \) (see [HeSc07], Lemma 7.1 and [Gel21], Sect. 2). We begin by determining \( d^c f(\widetilde{X}_z) \), for \( X \in \mathfrak{n} \) and \( z \in G/K \). By the \( N \)-invariance of \( f \) and of \( J \) one has

\[ d^c f(\widetilde{X}_{z,n}) = d^c f(\Ad_{n^{-1}} X_z), \tag{13} \]

for every \( z \in G/K \) and \( n \in N \). Thus it is sufficient to take \( z = aK \in \exp(\mathcal{D}) \cdot eK \). Let \( H = \sum a_j A_j \in \mathcal{D} \) and \( a = \exp H \). Then

\[ d^c f(\widetilde{X}_z) = \begin{cases} \frac{1}{2} e^{-2a_j} \frac{\delta^j_l}{a_{aj}}(H), & \text{for } X = E^j \in \mathfrak{g}^{2e_j} \\ 0, & \text{for } X \in \mathfrak{g}^\alpha, \text{with } \alpha \in \Sigma^+ \setminus \{2e_1, \ldots, 2e_r\}. \end{cases} \tag{14} \]
The first part of equation (14) follows from (10) and Lemma 2.1(a):
\[ df\left((\widetilde{E}_j)_z\right) = e^{-2e_j(H)} df\left(a_\ast J E_j\right) = \frac{1}{2} e^{-2e_j} \frac{d}{d\tau}|_{\tau=0} \tilde{f}(H+sA_j) = \frac{1}{2} e^{-2e_j} \frac{\partial \tilde{f}}{\partial e_j}(H). \]
For the second part, let \( X \in g^\alpha \), with \( \alpha \in \Sigma^+ \backslash \{ 2e_1, \ldots, 2e_r \} \). Then \( JX \in g^\beta \), with \( \beta \in \Sigma^+ \). By (10) and the \( N \)-invariance of \( f \), one obtains the desired result
\[ df\left(\widetilde{X}_z\right) = e^{-\alpha(H) + \beta(H)} df\left(J \widetilde{X}_z\right) = 0. \]

(i) Orthogonality of the blocks. Let \( X \in g^\alpha \) and \( Y \in g^\gamma \), where \( \alpha \in \Sigma^+ \) and \( \gamma \in \{0\} \cup (\Sigma^+ \backslash \{ 2e_1, \ldots, 2e_r \}) \) are distinct restricted roots (here \( Y \in a \), when \( \gamma = 0 \)). Then \( JY \in g^\beta \), for some \( \beta \in \Sigma^+ \). By (10) and (12), one has
\[ h_f(a_\ast X, a_\ast Y) = -dd^c f(a_\ast X, a_\ast J Y) = -e^{(H)+\beta(H)} dd^c f(\widetilde{X}_z, \widetilde{JY}_z) \]
\[ = e^{(H)+\beta(H)} d\mu^X(\widetilde{JY}_z) = e^{(H)+\beta(H)} \frac{d}{ds}|_{s=0} \mu^X(\exp s \widetilde{JY} \cdot z) \]
\[ = e^{(H)+\beta(H)} \frac{d}{ds}|_{s=0} \mu^X(\widetilde{X}_z) = e^{(H)+\beta(H)} \frac{d}{ds}|_{s=0} d^c f(\widetilde{X}_z - s[\widetilde{JY}, \widetilde{X}]_z + o(s^2)) \]
\[ = -e^{(H)+\beta(H)} d^c f([\widetilde{JY}, \widetilde{X}]_z). \]

The brackets \([\widetilde{JY}, \widetilde{X}] \) lie in \( g^{\alpha+\beta} \). Since \( \alpha \neq \gamma \), one sees that \( \alpha + \beta = 2e_1, \ldots, 2e_r \). Then, by (14), the expression (15) vanishes, proving the orthogonality of \( a_\ast g^\alpha \) and \( a_\ast g^\gamma \), for all \( \alpha \) and \( \gamma \) as above. The \( J \)-invariance of \( h_f \) implies that \( a_\ast a \) is orthogonal to \( a_\ast g^\beta \), for all \( \beta \in \Sigma^+ \), and concludes the proof of (i).

Next we determine the form \( h_f \) on the essential blocks.

(ii) The form \( h_f \) on \( a_\ast a \).
Let \( A_j, A_t \in a \). Since \( J A_t = -2E^t \), one has
\[ h_f(a_\ast A_j, a_\ast A_t) = -2dd^c f(a_\ast E^t, a_\ast A_j) = -2e^{2e_i(H)} dd^c f((\widetilde{E}_i)_z, (\widetilde{A}_j)_z) \]
\[ = 2e^{2e_i(H)} d\mu^{E^t}((\widetilde{A}_j)_z) = 2e^{2e_i(H)} \frac{d}{dt}|_{t=0} \mu^{E^t}(\exp tA_j \cdot z) \]
\[ = 2e^{2e_i(H)} \frac{d}{dt}|_{t=0} d^c f((\widetilde{E}_i)_{\exp tA_j \cdot z}), \]
which, by (14), becomes
\[ = 2e^{2e_i(H)} \frac{d}{dt}|_{t=0} \frac{1}{2} e^{-2e_i(H+tA_j)} \frac{\partial \tilde{f}}{\partial e_i}(H + tA_j) = -2 \frac{\partial \tilde{f}}{\partial e_i}(H) \delta_{ij} + \frac{\partial^2 \tilde{f}}{\partial e_i \partial e_j}(H). \]
This concludes the proof of (ii).

(iii) The form \( h_f \) on \( a_\ast g^{e_j-e_l} \).
Let \( X, X' \in g^{e_j-e_l} \) be elements of the basis given in Remark 2.2(b). Then \( JX, JX' \in g^{e_j+e_l} \). From (15), (14) and Lemma 2.3(a) one has
\[ h_f(a_\ast X, a_\ast X) = -dd^c f(a_\ast X, a_\ast J X) \]
\[ = -e^{(e_j+e_l)(H)} e^{(e_j-e_l)(H)} d^c f([JX, X]_z) \]
\[ = -e^{2e_j(H)} \left( s d^c f((\widetilde{E}_j)_z) \right) = -\frac{s^2 \tilde{f}}{2 e_j}(H), \]
for some \( s \in \mathbb{R}\setminus\{0\} \). By Remark \( \text{6.4} \), one has \( s > 0 \). By the comparison of \( (16) \) with the formula obtained in Remark \( \text{7.2} \), one deduces the exact value of \( s \), namely \( s = \frac{4|X|^2}{b} \). Therefore, one has

\[
h_f(a_*X, a_*X) = -2\frac{|X|^2}{b} \frac{\delta f}{\partial X_j}(H), \quad h_f(a_*X, a_*X') = -2\frac{|X|^2}{b} \frac{\delta f}{\partial X_j}(H),
\]
as stated. From \( (15) \) and Lemma \( \text{2.3} \), one obtains \( h_f(a_*X, a_*X') = 0 \). From \( (15) \), the skew symmetry of \( dd^c f \) and the fact that \( 2(e_j - e_i) \notin \Sigma^+ \), one obtains \( h_f(a_*X, a_*JX) = h_f(a_*X, a_*JX') = 0 \), respectively. Finally, let \( X = Z^\mu + \overline{Z^\mu} \), and \( Y = Z^\nu + \overline{Z^\nu} \) be elements of the basis of \( g_{e_j - e_i} \) given in Remark \( \text{2.2} \), for \( \mu, \nu \in \Delta^+ \) distinct roots satisfying \( \nu \neq \mu, \bar{\mu} \). Then, by \( (15) \) and Lemma \( \text{2.1} \), one has

\[
h_f(a_*X, a_*Y) = -e^{2c_j(H)} d^c f([JY, X], z) = 0,
\]
since no non-real roots in \( \Delta \) have real part equal to \( 2c_j \). This completes the proof of (iii).

(iv) The Hermitian form \( h_f \) on \( a_*g_{e_j} \).

Let \( X = Z^\mu + \overline{Z^\mu} \) and \( JX = iZ^\mu + i\overline{Z^\mu} \) be elements of the basis of \( g_{e_j} \) given in Remark \( \text{2.2} \). Then, from \( (15) \) and Lemma \( \text{2.3} \), one obtains

\[
h_f(a_*X, a_*X) = -e^{2c_j(H)} d^c f([JX, X], z) = -e^{2c_j(H)} d^c f([\overline{E^j}], z) = -\frac{t}{2\epsilon_{a_j}}(H), \quad (17)
\]
for some \( t \in \mathbb{R}\setminus\{0\} \). By Remark \( \text{6.4} \), one has \( t > 0 \). By the comparison of \( (17) \) with the formula obtained in Remark \( \text{7.2} \), one deduces the exact value of \( t \), namely \( t = \frac{4|X|^2}{b} \) and

\[
h_f(a_*X, a_*X) = h_f(a_*JX, a_*X) = -2\frac{|X|^2}{b} \frac{\delta f}{\partial X_j}(H).
\]
Finally, let \( X = Z^\mu + \overline{Z^\mu} \) and \( Y = Z^\nu + \overline{Z^\nu} \) be elements of the basis of \( g_{e_j} \) given in Remark \( \text{2.2} \), for \( \mu, \nu \in \Delta^+ \) distinct roots satisfying \( \nu \neq \mu, \bar{\mu} \). Then, by \( (15) \) and Lemma \( \text{2.1} \) one has \( h_f(a_*X, a_*Y) = 0 \). This concludes the proof of (iv) and of the proposition. \( \square \)

Remark. The usual Levi form \( L_f^C \) of \( f \) is given by \( L_f^C(Z, W) = 2(h_f(X, Y) + ih_f(X, JY)) \), where \( Z = X - iJX \) and \( W = Y - iJY \) are elements of type \((1, 0)\). One easily sees that \( L_f^C \) is (strictly) positive definite if and only if \( h_f \) is (strictly) positive definite.
The main goal of this section is to characterize the Stein \( N \)-invariant domains \( D \) in \( G/K \) in terms of an associated \( r \)-dimensional tube domain. We show that \( D \) is Stein if and only if the base of the associated tube domain is convex and satisfies an additional geometric condition, arising from the features of the \( N \)-invariant plurisubharmonic functions on \( D \).

At the end of the section we also prove a univalence result for \( N \)-equivariant Riemann domains over \( G/K \). As a by-product, a precise description of the envelope of holomorphy of \( N \)-invariant domains in \( G/K \) follows.

Resume the notation introduced in Section 2. Denote by \( R := \exp ( \bigoplus \mathfrak{g}^{2e_j} ) \) the unipotent abelian subgroup of \( G \), isomorphic to \( \mathbb{R}^r \). The orbit of the base point \( eK \in G/K \) under the product of the \( r \) commuting \( SL_2(\mathbb{R})'s \) contained in \( G \) is the \( r \)-dimensional \( R \)-invariant closed complex submanifold of \( G/K \)

\[
R \exp(a) \cdot eK.
\]

By the Iwasawa decomposition of \( G \), such manifold intersects all \( N \)-orbits in \( G/K \). Equivalently,

\[
N \cdot (R \exp(a) \cdot eK) = G/K.
\]

The above facts together with the next proposition can be regarded as an analogue, for the \( N \)-action, of the polydisk theorem (cf. [Wol72], p. 280). Denote by \( \mathbb{H} \) the upper half-plane in \( \mathbb{C} \), with the usual \( \mathbb{R} \)-action by translations.

\textbf{Proposition 4.1.} \textit{The map} \( \mathcal{L} : \mathbb{H}^r \to R \exp a \cdot eK \), \textit{defined by}

\[
(x_1 + iy_1, \ldots, x_r + iy_r) \mapsto \exp(\sum_j x_j E^j) \exp(\frac{1}{2} \sum_j \ln(y_j) A_j) K,
\]

\textit{is an equivariant biholomorphism.}

\textbf{Proof.} The map is clearly bijective and equivariant. To prove that it is holomorphic, it is sufficient to consider the rank-1 case. Computing separately

\[
d\mathcal{L}_z \frac{d}{dx} \bigg|_z = d\mathcal{L}_z \frac{d}{dy} \bigg|_z = \frac{d}{dt} \bigg|_{t=0} \mathcal{L}(x + i(y + t)) = d \bigg|_{t=0} \exp(xE) \exp(\frac{1}{2} \ln(y + t) A) K
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \exp(xE) \exp((\frac{1}{2} \ln y + \frac{t}{2} y + o(t^2)) A) K = (\exp(xE) \exp(\frac{1}{2} \ln y A)) \ast \frac{1}{2} A
\]

and

\[
J \mathcal{L}_z \frac{d}{dx} \bigg|_z = J \frac{d}{dx} \bigg|_{t=0} \mathcal{L}(x + t + iy) = J \frac{d}{dt} \bigg|_{t=0} \exp((x + t) E) \exp(\frac{1}{2} \ln y A) K
\]

\[
= J \frac{d}{dt} \bigg|_{t=0} \exp(xE) \exp(t E) \exp(\frac{1}{2} \ln y A) K
\]

\[
= J \frac{d}{dt} \bigg|_{t=0} \exp(xE) \exp(\frac{1}{2} \ln y A) \exp(t Ad \exp(-\frac{1}{2} \ln y A) E) K
\]

\[
= J \exp(xE) \ast \exp(\frac{1}{2} \ln y A) \ast \frac{1}{2} E = (\exp(xE) \exp(\frac{1}{2} \ln y A)) \ast \frac{1}{2} A
\]

we obtain the desired identity \( d\mathcal{L}_z J \frac{d}{dx} \bigg|_z = J d\mathcal{L}_z \frac{d}{dx} \bigg|_z \), for all \( z \in \mathbb{H} \). \( \square \)
Remark 4.2. The closed complex submanifold $R \exp(a) \cdot eK$ can also be regarded as the local orbit of $eK$ under the universal complexification $R^C$ of $R$. Up to a traslation, $\mathcal{L}$ is the local $R^C$-orbit map through $eK$.

As a consequence of the above biholomorphism we obtain a one-to-one correspondence between $\mathbb{R}^r$-invariant tube domains in $\mathbb{H}^r$ and $N$-invariant domains in $G/K$. Denote by $L : \mathbb{R} > 0 \times \ldots \times \mathbb{R} > 0 \to a$ the diffeomorphism determined by $\mathcal{L}$

$$L(y_1, \ldots, y_r) := \frac{1}{2} \sum_j \ln(y_j) A_j.$$  \hspace{1cm} (18)

Corollary 4.3. ($N$-invariant domains in $G/K$ and tube domains in $\mathbb{C}^r$).

(i) Let $D = N \exp(D) \cdot eK$ be an $N$-invariant domain in $G/K$ and let $R \exp(D) \cdot eK$ be its intersection with the closed complex submanifold $R \exp(a) \cdot eK$. Then the $r$-dimensional tube domain associated to $D$ is by definition the preimage of $R \exp(a) \cdot eK$ under $\mathcal{L}$, namely

$$\mathbb{R}^r + i\Omega, \quad \text{where } \Omega := L^{-1}(D).$$

(ii) Conversely, a tube domain $\mathbb{R}^r + i\Omega$ in $\mathbb{H}^r$ determines a unique $N$-invariant domain

$$D = N \exp(D) \cdot eK, \quad \text{where } D = L(\Omega).$$

Remark 4.4. If $D$ is Stein, then the associated tube domain $\mathbb{R}^r + i\Omega \subset \mathbb{C}^r$ is Stein, being biholomorphic to the Stein closed complex submanifold $R \exp(D) \cdot eK$ of $D$. In particular, the base $\Omega$ is an open convex set in $(\mathbb{R} > 0)^r$.

On the other hand, already in the case of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, with $n > 1$, one can see that the base $\Omega$ of an $N$-invariant Stein subdomain $D$ must be an entire half-line, and cannot be just an arbitrary convex subset of $\mathbb{R} > 0$.

The main goal of this section is to give a precise characterization of the convex sets $\Omega \subset (\mathbb{R} > 0)^r$ arising from $N$-invariant Stein domains $D$ in $G/K$. As we shall see, their shape is determined by the particular features of the Levi form of the $N$-invariant functions on $D$, which involve both the Hessian and the gradient of $f$ (cf. Prop. 3.1).

Let $f : D \to \mathbb{R}$ be an $N$-invariant plurisubharmonic function. Then $f$ is uniquely determined by the function $\tilde{f}(H) := f(\exp H \cdot eK)$ on $D$ (cf. (7)) and also by the function

$$\tilde{f}(\mathbf{y}) := f(\exp(L(y))K) = \tilde{f}(L(y))$$  \hspace{1cm} (19)
defined for $y \in \Omega$, as shown by the following commutative diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\hat{f}} & \mathbb{R} \\
\downarrow{\text{exp}} & & \downarrow{\text{exp}} \\
D & \xrightarrow{f} & D
\end{array}
\]

Since the $N$-action on $D$ is proper and every $N$-orbit intersects transversally the smooth slice $\exp(L(\Omega)) \cdot eK$ in a single point, it is easy to check that the map $f \rightarrow \hat{f}$ is a bijection from the class $C^0(D)^N$ of continuous $N$-invariant functions on $D$ and the class $C^0(\Omega)$ of continuous functions on $\Omega$. By Theorem 4.1 in [Ple78], such a map is also a bijection between $C^\infty(D)^N$ and $C^\infty(\Omega)$. Analogous statements hold true for the map $f \rightarrow \hat{r}$.

Given a non-compact irreducible Hermitian symmetric space, define the cone

\[ C := \begin{cases} 
(\mathbb{R}^{>0})^r, & \text{in the non-tube case,} \\
(\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case.}
\end{cases} \quad (20) \]

The next lemma characterizes the plurisubharmonicity of a smooth $N$-invariant function $f$ in terms of the corresponding functions $\hat{f}$ and $\hat{r}$.

**Proposition 4.5.** Let $D$ be an $N$-invariant domain in $G/K$ and let $f : D \to \mathbb{R}$ be a smooth, $N$-invariant, plurisubharmonic function. Then the following conditions are equivalent:

(i) $f$ is plurisubharmonic (resp. strictly plurisubharmonic) at $z = aK$, with $a = \exp(H)$ and $H \in D$;

(ii) the form

\[
-2\delta_{\text{H}} \frac{\partial^2}{\partial a_j \partial a_l}(H) \big|_{j,l=1,\ldots,r} \quad (21)
\]

in Proposition 3.1(ii) is positive semidefinite (resp. positive definite) and $\nabla \hat{f}(H) \cdot v \leq 0$ (resp. $< 0$), for all $v \in \mathbb{C}\setminus\{0\}$;

(iii) the Hessian of $\hat{f}$ is positive semidefinite (resp. positive definite) at $y = (y_1, \ldots, y_r) = L^{-1}(H)$ and

\[
\nabla \hat{f}(y) \cdot v \leq 0 \quad (\text{resp.} \quad < 0), \quad \text{for all} \quad v \in \mathbb{C}\setminus\{0\}. \quad (22)
\]

**Proof.** The equivalence $(i) \Leftrightarrow (ii)$ follows directly from Proposition 3.1. $(ii) \Leftrightarrow (iii)$ Since $L(y_1, \ldots, y_r) = (\frac{1}{2} \ln(y_1), \ldots, \frac{1}{2} \ln(y_r))$ (see [15]), one has

\[
\hat{f}(a_1, \ldots, a_r) = \hat{f}(e^{2a_1}, \ldots, e^{2a_r}).
\]

Therefore

\[
\frac{\partial^2}{\partial a_j \partial a_l} \hat{f}(a_1, \ldots, a_r) = 2 \frac{\partial^2}{\partial y_j \partial y_l} \hat{f}(e^{2a_1}, \ldots, e^{2a_r})e^{2a_j} \quad (23)
\]
\[
\frac{\partial^2 \hat{f}}{\partial a_j \partial a_l}(H) = 4 \frac{\partial^2 \hat{f}}{\partial y_j \partial y_l}(e^{2a_1}, \ldots, e^{2a_r}) e^{2a_j} e^{2a_l} + 4 \frac{\partial^2 \hat{f}}{\partial y_j \partial a_l}(e^{2a_1}, \ldots, e^{2a_r}) e^{2a_j} \delta_{jl}.
\]

(24)

By combining formulas (23) and (24) one obtains

\[
(4 \frac{\partial^2 \hat{f}}{\partial y_j \partial y_l}(e^{2a_j}) j,l - 2 \frac{\partial^2 \hat{f}}{\partial a_j \partial a_l}(e^{2a_j}) j,l) \hat{f}. \]

(25)

Also, by (23), the same monotonicity conditions hold both for \( \hat{f} \) and for \( \hat{f} \). □

Definition 4.6. A smooth function \( g : \mathbb{R}^r \to \mathbb{R} \) is convex (resp. stably convex) if its Hessian is semidefinite (positive definite).

Remark 4.7. The above lemma shows that the function \( \hat{f} \) corresponding to a smooth \( N \)-invariant plurisubharmonic function is not just an arbitrary smooth convex function, but it must satisfy the additional monotonicity conditions (22). (cf. Rem. 5.2).

Definition 4.8. A set \( \Omega \subset \mathbb{R}^r \) is \( C \)-invariant if \( y \in \Omega \) implies \( y + C \subset \Omega \)

Equivalently, if \( y \in \Omega \) implies \( y + \overline{C} \subset \Omega \), where \( \overline{C} \) denotes the closure of \( C \).

Theorem 4.9. Let \( G/K \) be a non-compact irreducible Hermitian symmetric space and let \( D \) be an \( N \)-invariant domain in \( G/K \). Then \( D \) is Stein if and only if the base \( \Omega \) of the associated tube domain is convex and \( C \)-invariant.

The proof of the above theorem is divided into two parts. If \( D \) has smooth boundary, then the argument relies on the computation of the Levi form of smooth, \( N \)-invariant functions on \( D \) (Prop. 3.1) and some elementary convex-geometric properties of \( \Omega \).

In the general case, the proof of the theorem is obtained by realizing \( D \) as an increasing union of Stein, \( N \)-invariant domains with smooth boundary.

Proof of Theorem 4.9 (the smooth case). The rank-1 tube case is trivial, since every \( \mathbb{R} \)-invariant domain in the upper half-plane \( \mathbb{H} \) is Stein. So we deal with the remaining cases: the rank-one non-tube case and the higher rank cases.

We use the notation \( y = (y_1, \ldots, y_r) \), for elements in \( \mathbb{R}^r \). Let \( D \subset G/K \) be a Stein, \( N \)-invariant domain with smooth boundary and let \( \mathbb{R}^r + i\Omega \subset \mathbb{C}^r \) be its associated tube domain. Then \( \Omega \) is a convex set with smooth boundary (cf. Rem. 4.4). Assume by contradiction that \( \Omega \) is not \( C \)-invariant, i.e. there exist \( y \in \Omega \) and \( z \in (y + C) \cap \partial \Omega \). By the convexity of \( \Omega \), the open segment from \( y \) to \( z \) is contained in \( \Omega \). In addition, the vector \( v = z - y \in C \) is transversal to the tangent hyperplane \( T_y \partial \Omega \) and points outwards. Therefore, given a smooth local defining function \( \hat{f} \) of \( \partial \Omega \) near \( z \), one has

\[
\frac{\partial \hat{f}}{\partial v}(z) = \text{grad} \hat{f}(z) \cdot v > 0.
\]
In the tube case, the above inequality and (23) imply that \( \frac{\partial^2 f}{\partial y_j^2}(H) > 0 \), for some \( j \in \{1, \ldots, r - 1\} \). Then, by Proposition 3.1(iii), the Levi form of the corresponding \( N \)-invariant function \( f \) is negative definite on the \( J \)-invariant subspace \( a_s g^{e_j} \in \mathfrak{g}(\partial D) \), the tangent space to \( \partial D \) in \( aK \). In the non-tube case, one has \( \frac{\partial^2 f}{\partial y_j^2}(H) > 0 \), for some \( j \in \{1, \ldots, r\} \). By Proposition 3.1(iv), the Levi form of the corresponding \( N \)-invariant function \( f \) is negative definite on the \( J \)-invariant subspace \( a_s g^{e_j} \) of \( T_{aK}(\partial D) \). This contradicts the fact that \( f \) is a defining function of the Stein \( N \)-invariant domain \( D \) and proves that \( \Omega \) is \( C \)-invariant.

Conversely, assume that \( \Omega \) is convex and \( C \)-invariant. We prove that \( D \) is Stein by showing that it is Levi-pseudoconvex, i.e. for all points \( aK \in \partial D \) and local defining functions \( f \) of \( D \) near \( aK \), one has \( h_f(X, X) \geq 0 \), for every tangent vector \( X \in T_{aK}\partial D \cap JT_{aK}\partial D \), the complex tangent space to \( \partial D \) at \( aK \).

Let \( z \in \partial \Omega \) and let \( aK = L(z) \). Denote by \( W := T_z\partial \Omega \) the tangent space to \( \partial \Omega \) in \( z \). One can verify that the complex tangent space to \( \partial D \) at \( aK \) is given by

\[
a_s(\bigoplus g^{e_j} \oplus \bigoplus g^{e_j} \oplus (L_s)_z W \oplus J(L_s)_z W.
\]

Let \( v = (v_1, \ldots, v_r) \) be an outer normal vector to \( W \) in \( \mathbb{R}^r \). The \( C \)-invariance and the convexity of \( \Omega \) imply that \( v_j \leq 0 \), for \( j = 1, \ldots, r \) in the non-tube case, and \( v_j = 0 \), for \( j = 1, \ldots, r - 1 \) in the tube case. Otherwise the space \( W \) would intersect \( y + C \), for every \( y \in \Omega \), yielding a contradiction.

Let \( \tilde{f} \) be a smooth local defining function of \( \Omega \) near \( z \). By the convexity of \( \Omega \), the Hessian \( Hess(\tilde{f})(z) \) is positive definite on \( W \). Moreover, as the gradient \( \text{grad} \tilde{f}(z) \) is a positive multiple of \( v \), one has \( \frac{\partial \tilde{f}}{\partial y_j}(z) \leq 0 \), for all \( j = 1, \ldots, r \), in the non-tube case, and \( \frac{\partial \tilde{f}}{\partial y_j}(z) \leq 0 \), for all \( j = 1, \ldots, r - 1 \), in the tube case.

Let \( f \) be the corresponding \( N \)-invariant local defining function of \( D \) near \( aK = \exp L(z)K \). By Proposition 4.5, the Levi form of \( f \) is positive definite on \( (L_s)_z W \oplus J(L_s)_z W \subset a_s a \oplus a_s J a \).

In addition, by (23) and Proposition 3.1 the Levi form of \( f \) is positive definite on \( a_s(\bigoplus g^{e_j} \oplus \bigoplus g^{e_j}) \). As a result, \( D \) is Levi pseudoconvex in \( aK = \exp L(z)K \). Since \( aK \) is an arbitrary point in \( \partial D \cap \exp a \cdot eK \) and both \( D \) and \( f \) are \( N \)-invariant, the domain \( D \) is Levi-pseudoconvex and therefore Stein, as desired.

In order to prove Theorem 4.9 in the non-smooth case, we need some preliminary Lemmas.

**Lemma 4.10.** Let \( D \) be a domain in a Stein manifold, let \( D' \subset D \) be a subdomain with smooth boundary and let \( z \in \partial D \cap \partial D' \). If \( D' \) is not Levi pseudoconvex in \( z \), then \( D \) is not Stein.

**Proof.** Under our assumption, there exists a one dimensional complex submanifold \( M \) through \( z \) in \( X \) with \( M \setminus \{z\} \subset D' \) ([Ran86], proof of Thm. 2.11, p. 56).
This implies that \( D \) is not Hartogs pseudoconvex ([Ran86], Thm. 2.9, p. 54) and in particular it is not Stein.

For a domain \( \Omega \) in \( \mathbb{R}^r \), denote by \( d_\Omega : \Omega \to \mathbb{R} \) the distance function from the boundary (if \( z \in \Omega \), then \( d_\Omega(z) \) is by definition the radius of the largest ball centered in \( z \) and contained in \( \Omega \)). The next lemma is a known characterization of convex domains.

**Lemma 4.11.** A proper subdomain \( \Omega \) of \( \mathbb{R}^r \) is convex if and only if the function \(- \ln d_\Omega : \Omega \to \mathbb{R} \) is convex.

In what follows, for a fixed domain \( \Omega \) in \( \mathbb{R}^r \), we denote

\[
    u := - \ln d_\Omega.
\]

Denote by \( B_\rho(y) \) the open ball of center \( y = (y_1, \ldots, y_r) \in \mathbb{R}^r \) and radius \( \rho \). Fix a smooth, positive, radial function \( \sigma : \mathbb{R}^r \to \mathbb{R} \) (only depending on \( R^2 = |w|^2 \)), with support in \( B_1(0) \), such that \( \sigma'(R^2) < 0 \) and \( \int_{\mathbb{R}^r} \sigma(w)dw = 1 \). For \( \varepsilon > 0 \), define \( \Omega_{\varepsilon} := \{ y \in \Omega : d_\Omega(y) > \varepsilon \} \) and \( u_\varepsilon : \Omega_{\varepsilon} \to \mathbb{R} \) by

\[
    u_\varepsilon(y) := \frac{1}{\varepsilon} \int_{\mathbb{R}^r} u(z)\sigma(\frac{y - z}{\varepsilon})dz = \int_{\mathbb{R}^r} u(y + \varepsilon w)\sigma(w)dw.
\]

The functions \( u_\varepsilon \) are clearly smooth. Let \( \nu : (\mathbb{R}^r_{>0})^r \to \mathbb{R}^r_{>0} \) be the stably convex positive function given by \( \nu(y) := \sum_j \frac{1}{y_j} \). Define \( v_\varepsilon : \Omega_{\varepsilon} \to \mathbb{R} \) by

\[
    v_\varepsilon(y) := u_\varepsilon(y) + \varepsilon \nu(y).
\]

**Lemma 4.12.** Let \( \Omega \) be a convex, \( C \)-invariant domain in \( (\mathbb{R}^r_{>0})^r \). Then the following facts hold true:

(i) The domain \( \Omega_{\varepsilon} \) is convex and \( C \)-invariant for every \( \varepsilon > 0 \).

(ii) The smooth functions \( v_\varepsilon \) are stably convex and, for \( \varepsilon \searrow 0 \), they decrease to \( u \) uniformly on the compact subsets of \( \Omega \).

(iii) Let \( \delta_\varepsilon := - \ln 3\varepsilon \). The sublevel set \( \Omega_{\varepsilon} := \{ y \in \Omega_{\varepsilon} : v_\varepsilon(y) < \delta_\varepsilon \} \) is convex and \( C \)-invariant.

(iv) The boundary of \( \Omega_{\varepsilon} \) in \( (\mathbb{R}^r_{>0})^r \) coincides with \( \{ y \in \Omega_{\varepsilon} : v_\varepsilon(y) = \delta_\varepsilon \} \) and it is smooth.

(v) As \( n \in \mathbb{N} \) increases, the sequence of convex, \( C \)-invariant subdomains with smooth boundary \( \Omega_{1/n} \) exhausts \( \Omega \).

**Proof.** (i) Let \( y \) and \( y + v \) be elements of \( \Omega_{\varepsilon} \). Then \( B_\varepsilon(y) \) and \( B_\varepsilon(y + v) \) are contained in \( \Omega \) and, by the convexity of \( \Omega \), the same is true for \( B_\varepsilon(y + tv) \), for every \( t \in [0, 1] \). This shows that \( \Omega_{\varepsilon} \) is convex. Moreover, as \( \Omega \) is \( C \)-invariant, if \( B_\varepsilon(y) \) is contained in \( \Omega \) and \( v \) is an element of the cone \( C \), then also the open ball \( B_\varepsilon(y + v) \) is contained in \( \Omega \). This shows that \( \Omega_{\varepsilon} \) is \( C \)-invariant.
(ii) As $u$ is convex, for $y, y + v \in \Omega$ and $t \in [0,1]$, one has

$$u_\varepsilon(y + tv) := \int_{\mathbb{R}^r} u(y + tv + \varepsilon w)\sigma(w)dw$$

$$\leq \int_{\mathbb{R}^r} ((1-t)u(y + \varepsilon w) + tu(y + \varepsilon w + v))\sigma(w)dw = (1-t)u_\varepsilon(y) + tu_\varepsilon(y + v),$$

showing that the smooth function $u_\varepsilon$ is convex. Since $\nu$ is smooth and stably convex, it follows that $v_\varepsilon := u_\varepsilon + \varepsilon \nu$ is smooth and stably convex. Moreover, as convexity implies subharmonicity, then the last part of statement (ii) follows from [Hör94], Thm 3.2.3(ii), p.143.

(iii) Since the function $v_\varepsilon$ is convex, then the domain $\tilde{\Omega}_\varepsilon$ is convex. In order to show that $\tilde{\Omega}_\varepsilon$ is $C$-invariant, we prove that

$$v_\varepsilon(y + v) < v_\varepsilon(y),$$

for every $y \in \Omega_\varepsilon$ and $v \in C$. Since $\Omega$ is $C$-invariant, if for some $y \in \Omega$ the ball $B_\varepsilon(y)$ is contained in $\Omega$, then also the ball $B_\varepsilon(y + v)$ is contained in $\Omega$, for all $v \in C$. It follows that $d_{\Omega}(y) \leq d_{\Omega}(y + v)$ and consequently $u(y + v + \varepsilon w) \leq u(y + \varepsilon w)$, for all $v \in C$, and $w \in B_\varepsilon(0)$. One deduces that

$$u_\varepsilon(y + v) = \int_{\mathbb{R}^r} u(y + v + \varepsilon w)\sigma(w)dw \leq \int_{\mathbb{R}^r} u(y + \varepsilon w)\sigma(w)dw = u_\varepsilon(y),$$

for every $y \in \Omega_\varepsilon$, $v \in C$. Since $\nu(y + v) < \nu(y)$, one concludes that $v_\varepsilon(y + v) < v_\varepsilon(y)$, and $\tilde{\Omega}_\varepsilon$ is $C$-invariant, as desired.

(iv) For $y$ close to $\partial \Omega_\varepsilon = \{z \in \Omega : d_{\Omega}(z) = \varepsilon\}$, a rough estimate shows that $d_{\Omega}(y + \varepsilon w) < 3\varepsilon$, for every $w \in B_\varepsilon(0)$. Therefore $v_\varepsilon(y) > u_\varepsilon(y) > -\ln 3\varepsilon$, implying that the boundary of $\tilde{\Omega}_\varepsilon$ is contained in $\Omega_\varepsilon$ and it is given by $\partial \tilde{\Omega}_\varepsilon = \{y \in \Omega_\varepsilon : v_\varepsilon(y) = \delta_\varepsilon\}$. Concerning the smoothness of $\partial \tilde{\Omega}_\varepsilon$, the rank one case is trivial. So assume $r > 1$.

Let $\hat{y} \in \partial \tilde{\Omega}_\varepsilon$. Set $v := (1, \ldots, 1)$, in the non-tube case, and $v := (1, \ldots, 1, 0)$, in the tube case. Since $v$ lies in the cone $C$, the inequality (26) implies that for $\gamma$ small enough the real function $g : (-\gamma, \gamma) \to \mathbb{R}$, defined by $g(t) := v_\varepsilon(\hat{y} + tv)$, is strictly decreasing. By the stable convexity of $v_\varepsilon$, it is also strictly convex and $g'(0) < 0$. As $g'(0)$ is a directional derivative of $v_\varepsilon$ in $\hat{y}$, the differential $dv_\varepsilon|_\hat{y}$ does not vanish and the boundary of $\tilde{\Omega}_\varepsilon$ is smooth.

(v) For $m > n$, the inclusion $\Omega_{1/n} \subset \Omega_{1/m}$ and the inequality $v_{1/n} > v_{1/m}$ imply that $\tilde{\Omega}_{1/n} \subset \tilde{\Omega}_{1/m}$. This concludes the proof of the lemma.

Proof of Theorem 4.9: the general case. Let $D$ be an arbitrary Stein, $N$-invariant domain in $G/K$. By Remark 4.4, the base $\Omega$ of the associated tube domain is necessarily convex. Assume by contradiction that $\Omega$ is not $C$-invariant (cf. Def 4.8 and (20)), i.e. there exist $y \in \Omega$ and $z \in (y + C) \cap \partial \Omega$. By the convexity of $\Omega$, the open segment from $y$ to $z$ is contained in $\Omega$. Moreover, the
vector $v = z - y$ lies in the cone $C$ and points to the exterior of $\Omega$. Let $B_\varepsilon(y)$ be a relatively compact ball in $\Omega$ and define

$$t_{\max} := \max\{ t > 0 : B_\varepsilon(y + tv) \subset \Omega \}.$$ 

Then there exists $w \in \partial B_\varepsilon(y + t_{\max}v) \cap \partial \Omega$, and by construction

$$\langle w - (y + t_{\max}v), v \rangle > 0.$$ 

This implies that the outer normal $n := w - (y + t_{\max}v)$ to $B_\varepsilon(y + t_{\max}v)$ satisfies $n_j > 0$, for some $j \in \{1, \ldots, r\}$ in the non-tube case (resp. $n_j > 0$, for some $j \in \{1, \ldots, r-1\}$, in the tube case). From the result of the theorem in the smooth case, it follows that the $N$-invariant subdomain $N \exp(L(B_\varepsilon(y + t_{\max}v))) \cdot eK$, with smooth boundary, is not Levi pseudoconvex in $\exp(L(w))K$. Then Lemma 4.10 implies that $D$ is not Stein, contradicting the assumption.

Conversely, assume that $\Omega$ is convex and $C$-invariant. By Lemma 4.12, the domain $D$ can be realised as the increasing union of $N$-invariant domains $D_{1/n} := N \exp(L(\tilde{\Omega}_{1/n})) \cdot eK$, where the open sets $\tilde{\Omega}_{1/n} \subset \mathbb{R}^r$ are convex, $C$-invariant and have smooth boundary. By the result of the theorem in the smooth case, the domains $D_{1/n}$ are Stein and so is their increasing union $D$. This completes the proof of the theorem. $\square$

We conclude this section with a univalence result for Stein, $N$-equivariant, Riemann domains over $G/K$.

**Proposition 4.13.** Any holomorphically separable, $N$-equivariant, Riemann domain over $G/K$ is univalent.

**Proof.** Let $Z$ be a holomorphically separable, $N$-equivariant, Riemann domain over $G/K$. By [Ros63], $Z$ admits an holomorphic, $N$-equivariant open embedding into its envelope of holomorphy, which is a Stein $N$-equivariant, Riemann domain over $G/K$. Hence, without loss of generality, we may assume that $Z$ is Stein.

Denote by $\pi : Z \to G/K$ the $N$-equivariant projection and let $\pi(Z) = N \exp(L(\Omega)) \cdot eK$ be the image of $Z$ under $\pi$. Define $\Sigma := \exp(L(\Omega)) \cdot eK$ and $\tilde{\Sigma} := \pi^{-1}(\Sigma)$. Note that $\tilde{\Sigma}$ is a closed submanifold of $Z$.

**Claim.** The map $\tilde{\phi} : N \times \tilde{\Sigma} \to Z$, given by $(n, x) \to n \cdot x$, is a diffeomorphism.

**Proof of the claim.** Since $\Sigma = \pi(Z) \cap \exp(\mathfrak{a}) \cdot eK$ is a closed real submanifold of $\pi(Z)$ and $\pi$ is a local biholomorphism, the restriction $\pi|_\Sigma : \tilde{\Sigma} \to \Sigma$ is a local diffeomorphism. Moreover one has the commutative diagram

$$\begin{array}{ccc}
N \times \tilde{\Sigma} & \xrightarrow{\tilde{\phi}} & Z \\
\downarrow{\text{Id} \times (\pi|_\Sigma)} & & \downarrow{\pi} \\
N \times \Sigma & \xrightarrow{\phi} & N \exp L(\Omega) \cdot eK
\end{array}$$
where the maps $Id \times (\pi|_{\Sigma})$, $\phi$ and $\pi$ are local diffeomorphisms. Hence so is the map $\tilde{\phi}$.

To prove that $\tilde{\phi}$ is surjective, let $z \in Z$ and note that $\pi(z) = n \exp(L(y))K$, for some $n \in N$ and $y \in \Omega$. Then the element $w := n^{-1} \cdot z \in \tilde{\Sigma}$ satisfies $n \cdot w = z$, implying the surjectivity of $\tilde{\phi}$.

To prove that $\tilde{\phi}$ is injective, assume that $n \cdot w = n' \cdot w'$, for some $n, n' \in N$ and $w, w' \in \tilde{\Sigma}$. From the equivariance of $\pi$ it follows that $n \cdot \pi(w) = n' \cdot \pi(w')$. As $\phi$ is bijective, it follows that $n = n'$ and $\pi(w) = \pi(w')$. Thus $w = (n^{-1}n') \cdot w' = w'$, implying the injectivity of $\tilde{\phi}$ and concluding the proof of the claim.

Now, in order to prove the univalence of $\pi$, it is sufficient to show that the restriction $\pi|_{\Sigma}: \tilde{\Sigma} \to \Sigma$ of $\pi$ to $\tilde{\Sigma}$ is injective. For this, consider the closed complex submanifold $R \cdot \tilde{\Sigma} = \pi^{-1}(R \cdot \Sigma)$ of $Z$. As $Z$ is Stein, so is $R \cdot \tilde{\Sigma}$. Hence the restriction $\pi|_{R \cdot \tilde{\Sigma}} : R \cdot \tilde{\Sigma} \to R \cdot \Sigma$ defines an $R$-equivariant, Stein, Riemann domain over the Stein tube $R \cdot \Sigma$. As $R$ is isomorphic to $\mathbb{R}^r$, from [CoLo86] it follows that $\pi|_{R \cdot \tilde{\Sigma}}$ is injective. Hence the same is true for $\pi|_{\Sigma}$ and $\pi$, as wished.

**Corollary 4.14.** The envelope of holomorphy $\hat{D}$ of an $N$-invariant domain $D$ in $G/K$ is the smallest Stein domain in $G/K$ containing $D$. More precisely, $\hat{D}$ is the tube domain with base $\hat{\Omega}$, the convex $C$-invariant hull of $\Omega$.

### 5. $N$-invariant psh functions vs. cvxdec functions

Let $D$ be a Stein, $N$-invariant domain in a non-compact, irreducible Hermitian symmetric space $G/K$ of rank $r$ and let $\Omega$ be the base of the associated $r$-dimensional tube domain. Then $\Omega$ is a convex, $C$-invariant domain in $(\mathbb{R}^{\geq 0})^r$ (Thm. 4.9). From Proposition 4.5 it follows that there is a one-to-one correspondence between the class of smooth $N$-invariant plurisubharmonic functions on $D$ and the class of smooth convex functions on $\Omega$ satisfying an additional monotonicity condition (cf. Rem. 4.7 and Rem. 5.2). In this section we obtain an analogous result in the non-smooth context.

Let $\overline{C}$ be the closure of the cone defined in (21).

**Definition 5.1.** A function $\hat{f} : \Omega \to \mathbb{R}$ is (strictly) $\overline{C}$-decreasing if for every $y \in \Omega$ and $v \in \overline{C}\backslash\{0\}$ the restriction of $\hat{f}$ to the half-line $\{y + tv : t \geq 0\}$ is (strictly) decreasing.

**Remark 5.2.** (i) A smooth function $\hat{f} : \Omega \to \mathbb{R}$ is $\overline{C}$-decreasing if and only if $\grad f(y) \cdot v \leq 0$ for every $y \in \Omega$ and $v \in \overline{C}\backslash\{0\}$. 
(ii) A smooth, stably convex (cf. Def. 4.6) function \( \hat{f} : \Omega \to \mathbb{R} \) is \( \mathbb{C} \)-decreasing if and only if \( \text{grad} f(y) \cdot v < 0 \), for every \( y \in \Omega \) and \( v \in \mathbb{C} \setminus \{0\} \). This follows from the fact that the directional derivatives \( \text{grad} f(y) \cdot v \) of a stably convex, \( \mathbb{C} \)-decreasing function \( \hat{f} \) never vanish. In particular \( \hat{f} \) is automatically strictly \( \mathbb{C} \)-decreasing.

In view of the above observations, we define the following classes of functions:

- \( \text{ConvDec}^{\infty,+}(\Omega) \): smooth, stably convex, \( \mathbb{C} \)-decreasing functions on \( \Omega \),
- \( \text{ConvDec}^{\infty}(\Omega) \): smooth, convex, \( \mathbb{C} \)-decreasing functions on \( \Omega \),
- \( Psh^{\infty,+}(D)^N \): smooth, \( N \)-invariant, strictly plurisubharmonic functions on \( D \),
- \( Psh^{\infty}(D)^N \): smooth, \( N \)-invariant, plurisubharmonic functions on \( D \).

Proposition 4.5 established a one-to-one correspondence between \( \text{ConvDec}^{\infty,+}(\Omega) \) and \( Psh^{\infty,+}(D)^N \), as well as between \( \text{ConvDec}^{\infty}(\Omega) \) and \( Psh^{\infty}(D)^N \). The next goal is to extend such correspondences beyond the smooth context.

Let \( \hat{h} : \Omega \to \mathbb{R} \) be the smooth, stably convex, strictly \( \mathbb{C} \)-decreasing function
\[
\hat{h}(y) := \sum_j \frac{1}{y_j}, \quad \text{for } y = (y_1, \ldots, y_r) \in \Omega,
\]
and let \( h \) be the \( N \)-invariant strictly plurisubharmonic function on \( D \) associated to \( \hat{h} \).

**Definition 5.3.** A function \( \hat{f} : \Omega \to \mathbb{R} \) is stably convex and \( \mathbb{C} \)-decreasing if every point in \( \Omega \) admits a convex \( \mathbb{C} \)-invariant neighborhood \( W \) and \( \varepsilon > 0 \) such that \( \hat{f} - \varepsilon \hat{h} \) is a convex, \( \mathbb{C} \)-decreasing function on \( W \).

**Definition 5.4.** An \( N \)-invariant function \( f : D \to \mathbb{R} \) is strictly plurisubharmonic if every point in \( D \) admits an \( N \)-invariant neighborhood \( U \) and \( \varepsilon > 0 \) such that \( f - \varepsilon h \) is a \( N \)-invariant plurisubharmonic function on \( U \) (see also [Gun90], Vol. 1, Def. 1, p. 118).

In the smooth context the above notions coincide with the ones introduced earlier. Denote by

- \( \text{ConvDec}^{+}(\Omega) \): stably convex and \( \mathbb{C} \)-decreasing functions on \( \Omega \),
- \( \text{ConvDec}(\Omega) \): convex, \( \mathbb{C} \)-decreasing functions on \( \Omega \),
- \( Psh^{+}(D)^N \): strictly plurisubharmonic, \( N \)-invariant functions on \( D \),
- \( Psh(D)^N \): plurisubharmonic, \( N \)-invariant functions on \( D \).

The next theorem summarizes our results.

**Theorem 5.5.** Let \( D \) be a Stein \( N \)-invariant domain in a non-compact, irreducible Hermitian symmetric space \( G/K \) of rank \( r \). The map \( f \to \hat{f} \) is a bijection between the following classes of functions.
(i) \(Psh^{x,+}(D)^N\) and \(\text{ConvDec}^{x,+}(\Omega)\),
(ii) \(Psh^{x}(D)^N\) and \(\text{ConvDec}^{x}(\Omega)\),
(iii) \(Psh(D)^N\) and \(\text{ConvDec}(\Omega)\),
(iv) \(Psh^{+}(D)^N\) and \(\text{ConvDec}^{+}(\Omega)\).

In particular, \(N\)-invariant plurisubharmonic functions on \(D\) are necessarily continuous.

**Proof.** (i) and (ii) follow from Proposition 4.5 and Remark 5.2.
(iii) Let \(f\) be a function in \(Psh(D)^N\). Since the restriction of \(f\) to the embedded \(r\)-dimensional Stein tube domain \(R\exp(L(\Omega)) \cdot eK \cong \mathbb{R}^r \times i\Omega\) (cf. Cor. 4.3) is plurisubharmonic and \(R\)-invariant, then \(\hat{f}\) is necessarily convex. Assume by contradiction that \(\hat{f}\) is not \(\mathbb{C}\)-decreasing. Then there exists \(s \in \mathbb{R}\) such that the sublevel set \(\{\hat{f} < s\}\) is not \(\mathbb{C}\)-invariant. By Theorem 4.9, the corresponding \(N\)-invariant domain \(\{f < s\}\) is not Stein. Since \(G/K\) is biholomorphic to a Stein domain in \(\mathbb{C}^n\) and \(f\) is plurisubharmonic, this contradicts \[\text{Car73}, \text{Thm. B}, \text{p. 419}.\] Hence \(\hat{f}\) belongs to \(\text{ConvDec}(\Omega)\), as claimed.

In order to prove the converse, as in the previous section, for \(\varepsilon > 0\) consider the convex \(C\)-invariant set \(\Omega_\varepsilon := \{y \in \Omega : d_{\Omega}(y) > \varepsilon\}\). For \(\hat{f}\) in \(\text{ConvDec}(\Omega)\), let \(\hat{f}_\varepsilon : \Omega_\varepsilon \to \mathbb{R}\) be the function

\[
\hat{f}_\varepsilon(y) := \int_{\mathbb{R}^r} \hat{f}(y + \varepsilon w) \hat{\sigma}(w) dw + \varepsilon \hat{h},
\]

where \(\hat{h}\) is the function given in (27) and \(\hat{\sigma} : \mathbb{R}^r \to \mathbb{R}\) is a smooth, positive, radial function (only depending on \(R^2 = \|w\|^2\)), with support in \(B_1(0)\), such that \(\hat{\sigma}'(R^2) < 0\) and \(\int_{\mathbb{R}^r} \hat{\sigma}(w) dw = 1\). Arguments analogous to those used in Lemma 4.12 show that the functions \(\hat{f}_\varepsilon\) are in \(\text{ConvDec}^{x,+}(\Omega_\varepsilon)\). Then (i) implies that the corresponding functions \(f_\varepsilon\) belong to \(Psh^{x,+}(D)^N\) and consequently \(f\) belongs to \(Psh(D)^N\).

(iv) follows directly from the definition of \(Psh^{+}(D)^N\) and of \(\text{ConvDec}^{+}(\Omega)\).

Finally, from the inclusions

\[
\text{ConvDec}^{+}(\Omega) \subset \text{ConvDec}(\Omega) \subset C^0(\Omega)
\]

\[
\text{ConvDec}^{x,+}(\Omega) \subset \text{ConvDec}^{x}(\Omega)
\]

it follows that all the above functions on \(\Omega\) are continuous, and so are the corresponding \(N\)-invariant plurisubharmonic functions on \(D\).

\[\square\]

### 6. The Siegel domain point of view

The goal of this section is to present an alternative characterization of Stein \(N\)-invariant domains in an irreducible Hermitian symmetric space \(G/K\), realized as a Siegel domain.
Denote by $S = NA$ the real split solvable group arising from the Iwasawa decomposition of $G$ subordinated to $\Sigma^+$. With the complex structure $J$ described in (2) and the linear form $f_0 \in s^*$ defined by $f_0(X) := B(X, Z_0)$, where $Z_0 \in Z(\frakt)$ is the element inducing the complex structure on $p$, the Lie algebra $s = n \oplus a$ of $S$ has the structure of a normal $J$-algebra (see [GPSV68] and [RoVe73], Sect. 5, A).

This means in particular that $\omega(X, Y) := -f_0([X, Y])$ is a non-degenerate skew-symmetric bilinear form on $s$ and that the symmetric bilinear form $\langle X, Y \rangle := -f_0([JX, Y])$ is the $J$-invariant positive definite inner product on $s$ defined in (2).

The adjoint action of $a$ on $s$ decomposes $s$ into the orthogonal direct sum of the restricted root spaces. Moreover, the adjoint action of the element $A_0 = \frac{1}{2} \sum j A_j \in a$ decomposes $s$ and $n$ as

$$s = s_0 \oplus s_{1/2} \oplus s_1, \quad n_j = n \cap s_j$$

where

$$s_0 = a \oplus \bigoplus_{1 \leq j < l \leq r} g^{e_j - e_l}, \quad s_{1/2} = \bigoplus_{1 \leq j \leq r} g^{e_j}, \quad s_1 = \bigoplus_{1 \leq j \leq r} g^{2e_j} \oplus \bigoplus_{1 \leq j < l \leq r} g^{e_j + e_l}. \quad (28)$$

Let $E_0 := \sum E^j$. The orbit

$$V := Ad_{exp s_0} E_0$$

is a sharp convex homogeneous selfadjoint cone in $s_1$ and

$$F: s_{1/2} \times s_{1/2} \rightarrow s_1 + i s_1, \quad F(W, W') = \frac{1}{4}([JW', W] - i[W', W]),$$

is a $V$-valued Hermitian form, i.e. it is sesquilinear and $F(W, W) \in \overline{V}$, for all $W \in s_{1/2}$. The Hermitian symmetric space $G/K$ is realized as a Siegel domain in $s_1^0 \oplus s_{1/2}$ as follows

$$D(V, F) = \{(Z, W) \in s_1 \oplus is_1 \oplus s_{1/2} \mid Im(Z) - F(W, W) \in V\}.$$  

If $s_{1/2} = \{0\}$ then $G/K$ is of tube type, otherwise it is of non-tube type. The group $S$ acts on $D(V, F)$ by the affine transformations

$$(Z, W) \mapsto (Ad_{s} Z + a + 2i F(Ad_{s} W, b) + i F(b, b), Ad_{s} W + b), \quad (30)$$

where $s \in exp s_0$, $a \in s_1$, and $b \in s_{1/2}$. Recall that $Ja = \oplus_j g^{2e_j}$, (cf. [4]) and denote by $Ja^+$ the positive octant in $Ja$. One easily verifies that if $E \in Ja^+$, then $Ad_{exp a} E = Ja^+$. This and the fact that $S$ acts freely and transitively on $D(V, F)$ imply that every $N$-orbit meets the set $Ja^+$ is a unique point.

Let $D$ be an $N$-invariant domain in a symmetric Siegel domain. Then

$$D = \{(Z, W) \in D(V, F) \mid Im(Z) - F(W, W) \in V_D\},$$

where $V_D$ is an $Ad_{exp s_0}$-invariant open subset in $V$, determined by

$$i V_D := D \cap i V.$$  

The $r$-dimensional set

$$v_D := V_D \cap Ja^+,$$
intersects every $N$-orbit of $D$ in a unique point, and it is the base of an $r$-
-dimensional tube domain in $Ja \oplus iJa$. The map $\exp a \cdot eK \rightarrow \exp a \cdot (iE_0, 0)$
\[
\exp(\sum_j x_j E^j) \exp(\frac{1}{2} \sum_k \ln(y_k) A_k)K \mapsto (i\Ad_{\exp(\frac{1}{2} \sum_k \ln(y_k) A_k)} E_0 + \sum_j x_j E^j, 0)
\]
is the inverse of the map $\mathcal{L}$ of Proposition 4.1 (cf. Cor. 4.3).

Let $C$ be the cone defined in (20). Then the characterization of $N$-invariant
Stein domains in a symmetric Siegel domain can be formulated as follows.

**Proposition 6.1.** Let $D$ be an $N$-invariant domain in an irreducible symmetric
Siegel domain. Then $D$ is Stein if and only if $V_D$ is convex and $C$-invariant.

In order to prove the above proposition, we need some preliminary results. For
this we separate the tube and the non-tube case.

**The tube case.** Denote by $\text{conv}(V_D)$ the convex hull of $V_D$ in $s_1$. Since $V_D$
is $Ad_{\exp n_0}$-invariant and the action is linear, then also $\text{conv}(V_D)$ is $Ad_{\exp n_0}$-invariant.

Denote by $p: s_1 \rightarrow Ja$ the projection onto $Ja$, parallel to $\oplus g^{\delta_{ij} + \varepsilon_l}$. Denote by
\[
(E^1)^* \ldots (E^r)^*
\]
the elements in the dual $n^*$ of $n$, with the property that $(E^1)^*(E^l) = \delta_{jl}$ and
$(E^j)^*(X^\alpha) = 0$, for all $X^\alpha \in g^\alpha$, with $\alpha \in \Sigma^+ \setminus \{2\varepsilon_1, \ldots, 2\varepsilon_r\}$.

**Lemma 6.2.** One has

(i) Let $E = \sum x_k E^k \in Ja^+$, where $x_k \in \mathbb{R}^{>0}$. Then
\[
p(\Ad_{\exp n_0} E) = E + C_{r-1}.
\]
In particular, $(E^l)^*(\Ad_{\exp tX} E) = x_r$, for all $X \in n_0$ and $t \in \mathbb{R}$.

(ii) Let $X \in g^{\delta_{ij} - \varepsilon_i}$. Then $[[E^l, X], X] = sE^j$, for some $s \in \mathbb{R}^{>0}$.

(iii) One has $p(\text{conv}(V_D)) = \text{conv}(p(V_D))$.

**Proof.** (i) Let $E \in Ja^+$ and let $h_0 \in \exp n_0$, where $n_0 = \oplus_{1 \leq i < j \leq r} g^{\delta_{ij} - \varepsilon_i}$. By
Theorem 4.10 in [RoVe73], for every $1 \leq i < j \leq r$ there exists a basis $\{E^p_{ij}\}$ of
$g^{\delta_{ij} - \varepsilon_i}$, with coordinates $\{x^p_{ij}\}$, such that
\[
(E^l)^*(\Ad_{h_0} E) = x_i (1 + \sum_{p, j > i} (x^p_{ij})^2)
\]
(formula (4.13) in [RoVe73]). Since $i < r$, one has $p(\Ad_{\exp X} E) = E + C_{r-1}$, as
claimed. In particular the $i^{th}$ coordinate of $E$ does not vary under the $Ad_{\exp n_0}$
action.

(ii) Let $X \in g^{\delta_{ij} - \varepsilon_i}$. Then $\exp tX \in \exp n_0$ and the curve
\[
\Ad_{\exp tX} E_0 = \exp ad_{tX} (E_0) = E_0 + t[X, E^l] + \frac{t^2}{2} [X, [X, E^l]], \ t \in \mathbb{R},
\]
is contained in $V$. By Lemma 2.3(a), its projection onto $Ja$ is given by
\[
p(\Ad_{\exp tX} E_0) = (E^l)^*(\Ad_{\exp tX} E_0) E^j = (1 + \frac{t^2}{2}s) E^j,
\]
for some $s \in \mathbb{R}$, $s \neq 0$. Now (i) implies that $1 + \frac{t^2}{2}s > 0$, for all $t \in \mathbb{R}$. Therefore $s > 0$, as claimed.

(iii) We prove the two inclusions. By the linearity of $p$, the set $p(\text{conv}(V_D))$ is convex and contains $p(V_D)$. Hence, $p(\text{conv}(V_D)) \supset \text{conv}(p(V_D))$. Conversely, let $z \in \text{conv}(V_D)$. Then there exist $t_0 \in (0, 1)$ and $x, y \in V_D$ such that $z = t_0x + (1 - t_0)y$. Since $p(z) = t_0p(x) + (1 - t_0)p(y)$, one has $p(\text{conv}(V_D)) \supset \text{conv}(p(V_D))$. \qed

The non-tube case. Denote by $\widetilde{p} : s_1^C \oplus s_{1/2} \to iJa$ the projection onto $iJa$ parallel to $s_1 \oplus i(\oplus g^{e_j + \epsilon_l}) \oplus s_{1/2}$.

Lemma 6.3. Let $E \in Ja^+$. Then $\widetilde{p}(N \cdot (iE, 0)) = i(E + \overline{C_r})$.

Proof. The $N$-orbit of the point $(iE, 0) \in s_1^C \oplus s_{1/2}$ is given by

$$N \cdot (iE, 0) = S_{1/2}a_1Ad_{\exp_{0E}}(iE, 0) = (a + i(Ad_{\exp_{0E}} + F(b, b)), b),$$

where $a \in s_1$ and $b \in s_{1/2}$. By (32) and Lemma 6.2 (i), one has $\widetilde{p}(N \cdot (iE, 0)) = i(E + C_{r-1} + \overline{p}(F(s_{1/2}, s_{1/2})))$. Since in the symmetric case $\{[b, b], b \in s_{1/2} \} = \overline{Ja}^+$, it follows that $\widetilde{p}(N \cdot (iE, 0)) = i(E + \overline{C_r})$, as claimed. \qed

Remark 6.4. (a) Statement (i) in Lemma 6.2 explains why in Prop 3.1 (iii) no conditions appear on $\frac{\partial f}{\partial a}$.

(b) Statement (ii) in Lemma 6.2 and the fact that $F(b, b) = [Jb, b]$, for $b \in s_{1/2}$, takes values in $\overline{Ja}^+$, explain why the real constants $s$ and $t$ in Lemma 2.3 (a)(b) and later in Proposition 6.1 (iii)(iv) are strictly positive.

Proof of Proposition 6.1. The tube case. An $N$-invariant domain $D$ in a symmetric tube domain $D(V)$ is itself a tube domain with base the $Ad_{\exp_{0E}}$-invariant set $V_D$. Hence all we have to prove is that $V_D$ is convex if and only if $v_D$ is convex and $v_D + C_{r-1} \subset v_D$.

Assume that $V_D$ is convex. Then $v_D$ is convex, being the intersection of $V_D$ with the positive octant $Ja^+$. To prove that $v_D$ is $C$-invariant, let $E = \sum_j x_j E_j \in v_D$, where $x_j > 0$, and let $X \in g_{e_j - \epsilon_l}$ be a non-zero element. For every $t \in \mathbb{R}$,

$$Ad_{\exp tX}E = E + tx_l[X, E'] + \frac{1}{2}t^2x_l[X, [X, E']]$$

lies in $V_D$ and, by the convexity assumption, so does $E + \frac{1}{2}t^2x_l[X, [X, E']] = E + i^2sxtE_j$, where $s > 0$ (cf. Lemma 6.2 (ii)). This argument applied to all $j = 1, \ldots, r-1$ and the convexity of $v_D$ show that $v_D + C_{r-1} \subset v_D$, as desired.

Conversely, assume that $v_D$ convex and $C$-invariant. We prove the convexity of $V_D$ by showing that $\text{conv}(V_D) \subset V_D$. From Lemma 6.2 (ii) and the $C$-invariance of $v_D$, one has

$$p(V_D) = p(Ad_{\exp_{0E}}v_D) = v_D + C_{r-1} \subset v_D.$$
Moreover, from Lemma 6.2 (iii), the above inclusion and the convexity of \( V_D \), one has
\[
\text{conv}(V_D) \cap J\mathfrak{a} \subset p(\text{conv}(V_D)) = \text{conv}(p(V_D)) \subset V_D.
\]
Finally, from the \( \text{Ad}_{\exp n_0} \)-invariance of \( \text{conv}(V_D) \) it follows that
\[
\text{conv}(V_D) = \text{Ad}_{\exp n_0}(\text{conv}(V_D) \cap J\mathfrak{a}) \subset \text{Ad}_{\exp n_0}V_D = V_D.
\]
This completes the proof of the proposition in the tube case.

The non-tube case. Let \( D \) be an \( N \)-invariant domain in a Siegel domain \( D(V,F) \). Denote by \( \text{conv}(D) \) the convex hull of \( D \) in \( \mathfrak{s}_1^C \oplus \mathfrak{s}_{1/2} \). As \( N \) acts on \( D \) by affine transformations, also \( \text{conv}(D) \) is \( N \)-invariant.

If \( D \) is Stein, then \( D \cap \{ W = 0 \} \) is a Stein tube domain in \( \mathfrak{s}_1^C \) with base \( V_D \).

By the result for the tube case and Lemma 6.3, \( V_D \) is convex and \( \mathfrak{C}_r \subset V_D \).

Conversely, assume that \( V_D \) is convex and \( \mathcal{C} \)-invariant, i.e. \( \mathfrak{v}_D + \mathfrak{C}_r \subset V_D \) (see Def. 4.8). We are going to prove that \( D \) is convex. By Lemma 6.3 one has
\[
\tilde{\mathfrak{p}}(D) = \tilde{\mathfrak{p}}(N \cdot \mathfrak{v}_D) = i(\mathfrak{v}_D + \mathfrak{C}_r) \subset i\mathfrak{v}_D.
\]
Moreover,
\[
\text{conv}(D) \cap iJ\mathfrak{a} \subset \tilde{\mathfrak{p}}(\text{conv}(D)) = \text{conv}(\tilde{\mathfrak{p}}(D)) \subset i\mathfrak{v}_D.
\]
By the \( N \)-invariance of \( \text{conv}(D) \), one obtains
\[
\text{conv}(D) = N \cdot (\text{conv}(D) \cap iJ\mathfrak{a}) \subset N \cdot i\mathfrak{v}_D = D.
\]
Hence \( D \) is convex and therefore Stein (cf. [Gun90], Vol.1, Thm.10, p. 67). This concludes the proof of the proposition.

Remark. The assumption \( \mathfrak{v}_D + \mathfrak{C}_r \subset \mathfrak{v}_D \) implies \( \mathfrak{v}_D + \mathfrak{C}_{r-1} \subset \mathfrak{v}_D \) and in particular \( V_D \) is convex. This means that if \( D \subset D(V,F) \) is Stein, then the tube domain \( D \cap \{ W = 0 \} \) is Stein. The converse may not hold true, as \( V_D = \text{Ad}_{\exp n_0} \mathfrak{v}_D \) convex does not imply \( \mathfrak{v}_D + \mathfrak{C}_r \subset \mathfrak{v}_D \).

7. Appendix: \( N \)-invariant potentials for the Killing metric.

Let \( G/K \) be a non-compact, irreducible Hermitian symmetric space. The Killing form \( B \) of \( \mathfrak{g} \), restricted to \( \mathfrak{p} \), induces a \( G \)-invariant Kähler metric on \( G/K \), which we refer to as the Killing metric. In this section we exhibit an \( N \)-invariant potential of the Killing metric and the associated moment map in a Lie theoretical fashion. All the \( N \)-invariant potentials of the Killing metric are determined in Remark 7.5.

Let \( f : G/K \to \mathbb{R} \) be a smooth \( N \)-invariant function. The map \( \mu : G/K \to \mathfrak{n}^* \), defined by
\[
\mu_f(z)(X) := d^c f(\tilde{X}_z), \quad (33)
\]
for \( X \in \mathfrak{n} \), is \( N \)-equivariant (cf. [13]). If \( f \) is strictly plurisubharmonic, then it is referred to as the moment map associated with \( f \).
Proposition 7.1. Let \( z = naK \in G/K \), where \( n \in N \), \( a = \exp H \in A \) and \( H = \sum_j a_j A_j \in \mathfrak{a} \). Let \( b \) be the constant defined in (11).

(i) The \( N \)-invariant function \( \rho : G/K \to \mathbb{R} \) defined by
\[
\rho(naK) := -\frac{b}{2} \sum_{j=1}^r B(H, A_j) = -\frac{b}{2}(a_1 + \cdots + a_r),
\]
is a potential of the Killing metric.

(ii) The moment map \( \mu_{\rho} : G/K \to \mathfrak{n}^* \) associated with \( \rho \) is given by
\[
\mu_{\rho}(naK)(X) = -\frac{b}{4} \sum_{j=1}^r e^{-2a_j}(E^j)^*(\text{Ad}_{n^{-1}}X) = B(\text{Ad}_{n^{-1}}X, \text{Ad}_a Z_0),
\]
where \( X \in \mathfrak{n} \), and the \((E^j)^*\) are defined in (31).

Proof. (i) Let \( naK \in G/K \), where \( a = \exp H \) and \( H = \sum_j a_j A_j \). The function \( \tilde{\rho} : \mathfrak{a} \to \mathbb{R} \) associated to \( \rho \) is given by \( \tilde{\rho}(H) = -\frac{b}{2} \sum_{j=1}^r a_j B(A_j, A_j) \) (cf. (7)). In order to obtain (i), we first prove the identities (34). By (33) and (14), one has
\[
\mu_{\rho}(aK)(X) = d^c \rho(\tilde{X}_aK) = -\frac{b}{4} \sum_{j=1}^r e^{-2a_j}(E^j)^*(X).
\]
By (2), one has
\[
(E^j)^*(X) = B(X, \theta E^j)/B(E^j, \theta E^j) = 2B(X, \frac{1}{2}(E^j + \theta E^j))/B(E^j, \theta E^j).
\]
Since
\[
b := B(A_j, A_j) = B(I_0A_j, I_0A_j) = B(E^j - \theta E^j, E^j - \theta E^j) = -2B(E^j, \theta E^j)
\]
and \( Z_0 = S_0 + \frac{1}{2} \sum_j E^j + \theta E^j \), for some \( S_0 \in \mathfrak{m} \) (cf. [GeIa21], Sect. 2), one obtains
\[
-\frac{b}{4} \sum_{j=1}^r e^{-2a_j}(E^j)^*(X) = -\frac{b}{2} \sum_{j=1}^r e^{-2a_j} B(X, \frac{1}{2}(E^j + \theta E^j))/B(E^j, \theta E^j)
\]
\[
= \sum_{j=1}^r B(X, \text{Ad}_a \frac{1}{2}(E^j + \theta E^j)) = B(X, \text{Ad}_a Z_0),
\]
and (34) follows from the \( N \)-equivariance of \( \mu_{\rho} \).

Next we are going to show that on \( \mathfrak{p} \times \mathfrak{p} \) one has
\[
h_{\rho}(a_{\ast \cdot}, a_{\ast \cdot}) = B(\cdot, \cdot).
\]
Every \( X \in \mathfrak{s} \) decomposes as \( X = (X - \phi(X)) + \phi(X) \in \mathfrak{k} \oplus \mathfrak{p} \) (see Sect. 2). Since the projection \( \phi : \mathfrak{s} \to \mathfrak{p} \) is a linear isomorphism, (36) is equivalent to
\[
h_{\rho}(a_{\ast X}, a_{\ast Y}) = h_{\rho}(a_{\ast \phi(X)}, a_{\ast \phi(Y)}) = B(\phi(X), \phi(Y)) = -\frac{1}{2} B(X, \theta Y),
\]
for all \( X, Y \) in \( \mathfrak{s} \). By Proposition 3.1(i), it is sufficient to consider \( X, Y \) both in the same block \( a_{\ast \mathfrak{a}} \), \( a_{\ast \mathfrak{g}^{\alpha - \epsilon_i}} \), and \( a_{\ast \mathfrak{g}^{\beta}} \).

Let \( A_j, A_l \in \mathfrak{a}, \) be as in (11). Then, by (ii) of Proposition 3.1 one has
\[
h_{\rho}(a_{\ast A_j}, a_{\ast A_l}) = \delta_{jl} B(A_l, A_l) = B(A_j, A_l).
\]
Let \( X, Y \in \mathfrak{g}^\alpha \), with \( \alpha = e_j - e_i \) or \( \alpha = e_j \). Then \( JY \in \mathfrak{g}^\beta \), for \( \beta = e_j + e_i \) or \( \beta = e_j \), respectively. From (15) and (i) one obtains
\[
h_{\rho}(a_{\ast X}, a_{\ast Y}) = -e^{(H) + (H)} d^c \rho([JY, X]_z)
\]
\[ e^{-e^{\alpha(H) + \beta(H)}} B([JY, X], Ad_aZ_0). \quad (38) \]

From the invariance properties of the Killing form \( B \), the decomposition of \( X \) and \( JY \) in \( \mathfrak{g} \oplus \mathfrak{p} \) and the identity \( \phi(JY) = I_0\phi(J\cdot) \) (cf. (33)), one has

\[
B([JY, X], Ad_aZ_0) = B(Ad_{a^{-1}}[JY, X], Z_0) = e^{-(\alpha(H) + \beta(H))} B([JY, X], Z_0) \\
e^{-\alpha(H) + \beta(H))} (B([JY - \phi(JY), X - \phi(X)], Z_0) + B([\phi(JY), \phi(X)], Z_0)) \\
e^{-\alpha(H) + \beta(H)} B([Z_0, \phi(Y)], \phi(X)], Z_0) = e^{-\alpha(H) + \beta(H)} B(\phi(X), [Z_0, \phi(Y)]) \\
= -e^{-\alpha(H) + \beta(H)} B(\phi(X), \phi(Y)) = \frac{1}{2} e^{-(\alpha(H) + \beta(H))} B(X, \theta Y).
\]

It follows that
\[ h_\rho(a_sX, a_sY) = -\frac{1}{2} B(X, \theta Y), \quad (39) \]
as desired. This concludes the proof of (i).

(ii) The identity (39) implies that the \( N \)-invariant function \( \rho \) is strictly plurisub-harmonic. Hence \( \mu_\rho \) is the moment map associated to \( \rho \). \( \square \)

**Remark 7.2.** Combining (10) and (17) in Proposition 7.1 with (37), we obtain the exact value of the positive quantities \( s \) and \( t \)

\[ s = \frac{4|x|^2}{b}, \quad \text{for} \quad X \in \mathfrak{g}^{e_j - e_i}, \quad \text{and} \quad t = \frac{4|x|^2}{b}, \quad \text{for} \quad X \in \mathfrak{g}^{2e_j}. \]

**Remark 7.3.** The map \( \mu_G : G/K \rightarrow \mathfrak{g}^* \) given by \( \mu_G(gK)(\cdot) = B(Ad_{g^{-1}} \cdot, Z_0) \) is a moment map for the \( G \)-action on \( G/K \). The moment map \( \mu_\rho \) in (ii) of Proposition 7.7 can be obtained by restricting \( \mu_G(\mathfrak{n}aK) \) to \( \mathfrak{n} \). Namely, for \( X \in \mathfrak{n} \) and \( \mathfrak{n}aK \in G/K \) one has

\[ \mu_\rho(\mathfrak{n}aK)(X) = \mu_G(\mathfrak{n}aK)(X) = B(Ad_{(\mathfrak{n}a)^{-1}} X, Z_0). \]

In the next remark, all possible \( N \)-invariant potentials of the Killing metric are determined.

**Remark 7.4.** Let \( \rho : G/K \rightarrow \mathbb{R} \) be the potential of the Killing metric given in Proposition 7.7 and let \( \sigma \) be another \( N \)-invariant potential. Let \( \hat{\rho} \) and \( \hat{\sigma} \) be the corresponding functions on \( (\mathbb{R}^{>0})^r \) defined in (19).

(a) In the non-tube case, one has \( \hat{\sigma} = \hat{\rho} + d \), and therefore \( \sigma = \rho + d \), for some \( d \in \mathbb{R} \);

(b) In the tube case, one has \( \hat{\sigma}(y) = \hat{\rho}(y) + cy + d \), for \( c, d \in \mathbb{R} \). In particular

\[ \sigma(n \exp(L(y))K) = \rho(n \exp(L(y))K) + cy + d, \]

where \( n \in N \), \( y = (y_1, \ldots, y_r) \in (\mathbb{R}^{>0})^r \), and \( c, d \in \mathbb{R} \).
Proof. Let \( f := \sigma - \rho \) be the difference of the two potentials. Then \( f \) is a smooth \( N \)-invariant function on \( G/K \) such that \( dd^c f(\cdot, J \cdot) \equiv 0 \). Let \( \hat{f} : \Omega \to \mathbb{R} \) be the associated function.

(a) In the non-tube case, by Proposition 3.1 (iv) and (23), the function \( \hat{f} \) satisfies \( \frac{\partial \hat{f}}{\partial y_j} \equiv 0 \), for all \( j = 1, \ldots, r \). Hence \( \hat{f} \) is constant on \( (\mathbb{R}^\geq)^r \) and \( f \) is constant on \( G/K \).

(b) In the tube case, from Proposition 3.1, (25) and (23), it follows that \( \frac{\partial \hat{f}}{\partial y_j} \equiv 0 \), for all \( j = 1, \ldots, r-1 \), and \( \frac{\partial^2 \hat{f}}{\partial y_r^2} \equiv 0 \). Hence \( \hat{f} \) is an affine function of the variable \( y_r \). Equivalently, \( \sigma(y) - \rho(y) = cy_r + d \), for \( c, d \in \mathbb{R} \), as claimed. \( \square \)

Remark 7.5. Let \( D(V, F) \) be a symmetric Siegel domain. Then the Bergman kernel function \( K(z, z) \) is \( N \)-invariant and \( \ln K(z, z) \) is a potential of the Bergman metric. As both the Killing and the Bergman metric are \( G \)-invariant, they differ by a multiplicative constant. It follows that \( \ln K(z, z) \) is a multiple of one of the \( N \)-invariant potentials of the Killing metric described in the above remark.

Example 7.6. As an application of Remark 7.5, we compute all \( N \)-invariant potentials of the Killing metric for the upper half-plane in \( \mathbb{C} \) and for the Siegel upper half-plane of rank 2.

(a) Let \( G = \text{SL}(2, \mathbb{R}) \) and let \( G/K \) be the corresponding Hermitian symmetric space. Fix an Iwasawa decomposition \( NAK \) of \( G \). Since \( b = 8 \) and \( r = 1 \), then the potential of the Killing metric given in Proposition 7.1 is \( \rho(naK) = -4a_1 \) and \( \hat{\rho}(y_1) = \rho(\exp(y_1)K) = \ln \frac{1}{y_1} \).

Realize \( G/K \) as the upper half-plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \), i.e. the orbit of \( i \in \mathbb{C} \) under the \( SL(2, \mathbb{R}) \)-action by linear fractional transformations. Fix

\[
N = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{R} \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{-a_1} \end{pmatrix} : a_1 \in \mathbb{R} \right\},
\]

and let \( \{ x_1 + iy_1 \in \mathbb{C} : y_1 > 0 \} \) be the tube associated to \( G/K \). Since

\[
x_1 + iy_1 \to \exp(x_1 E^1) \exp(\frac{1}{2} \ln y_1 A_1) \cdot i = x_1 + iy_1
\]

(cf. Prop. 4.1), then the potential \( \rho \) on \( \mathbb{H} \) reads as \( \rho(z) = \ln \frac{1}{(\text{Im}z)^2} \).

If \( \sigma : \mathbb{H} \to \mathbb{R} \) is an arbitrary \( N \)-invariant potential of the Killing metric, then by Remark 7.5

\[
\sigma(z) = \ln \frac{1}{(\text{Im}z)^2} + c\text{Im}z + d, \quad c, d \in \mathbb{R}.
\]

(b) The Siegel upper half-plane of rank 2

\[
\mathcal{P} = \{ W = S + iT \in M(2,2, \mathbb{C}) : \im W = W, \ T > 0 \},
\]
of $2 \times 2$ complex symmetric matrices with positive definite imaginary part, is the orbit of $iI_2$ under the action by linear fractional transformations of the real symplectic group $Sp(2, \mathbb{R})$. Fix the Iwasawa decomposition such that

$$N = \left\{ \begin{pmatrix} n & m \\ 0 & n^{-1} \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\},$$

where $n$ is unipotent, $n^t m$ is symmetric and $a = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$, with $a_1, a_2$ coordinates in $a$ with respect to the basis defined in Lemma 2.2.

As $b = 12$, the potential of the Killing metric defined in Proposition 7.1 is given by

$$\rho(naK) = -6(a_1 + a_2) \quad \text{and} \quad \tilde{\rho}(y_1, y_2) = \rho(\exp L(y_1, y_2)K) = \ln \frac{1}{(y_1 y_2)^{3/2}}.$$ 

A matrix $S + iT \in \mathcal{P}$ can be expressed in a unique way as

$$na \cdot iI_2 = n \cdot \begin{pmatrix} ie^{2a_1} & 0 \\ 0 & ie^{2a_2} \end{pmatrix}.$$

If $T = \begin{pmatrix} t_1 & t_3 \\ t_3 & t_2 \end{pmatrix}$, a simple computation shows that $e^{2a_1} = t_1 - t_3^2/t_2$ and $e^{2a_2} = t_2$.

Hence $y_1 = t_1 - t_3^2/t_2$, $y_2 = t_2$ and $\rho(S + iT) = \ln \frac{1}{(t_1, t_2 - t_3^2)}$.

If $\sigma$ is an arbitrary $N$-invariant potential of the Killing form, then by Remark 7.5

$$\sigma(S + iT) = \ln \frac{1}{(t_1, t_2 - t_3^2)} + ct_2 + d, \quad \text{for some } c, d \in \mathbb{R}.$$

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