Lower Bounds on the Running Time of Two-Way Quantum Finite Automata and Sublogarithmic-Space Quantum Turing Machines

Zachary Remscrim
Department of Mathematics
MIT
remscrimsmit.edu

Abstract

The two-way finite automaton with quantum and classical states (2QCFA), defined by Ambainis and Watrous, is a model of quantum computation whose quantum part is extremely limited; however, as they showed, 2QCFA are surprisingly powerful: a 2QCFA with only a single-qubit can recognize the language $L_{\text{pal}} = \{ w \in \{a,b\}^* : w \text{ is a palindrome} \}$ with bounded error in expected time $2^{O(n)}$, on inputs of length $n$.

We prove that their result essentially cannot be improved upon: a 2QCFA (of any size) cannot recognize $L_{\text{pal}}$ with bounded error in expected time $2^{o(n)}$. To our knowledge, this is the first example of a language that can be recognized with bounded error by a 2QCFA in exponential time but not in subexponential time. Moreover, we prove that a quantum Turing machine (QTM) running in space $o(\log n)$ and expected time $2^{n^{1-o(1)}}$ cannot recognize $L_{\text{pal}}$ with bounded error; again, this is the first lower bound of its kind.

Far more generally, we establish a lower bound on the running time of any 2QCFA or $o(\log n)$-space QTM that recognizes any language $L$ in terms of a natural “hardness measure” of $L$. This allows us to exhibit a large family of languages for which we have asymptotically matching lower and upper bounds on the running time of any such 2QCFA or QTM recognizer.

1 Introduction

Quantum algorithms, such as Shor’s quantum polynomial time integer factorization algorithm [38], Grover’s algorithm for unstructured search [15], and the linear system solver of Harrow, Hassidim, and Lloyd [16], provide examples of natural problems on which quantum computers seem to have an advantage over their classical counterparts. However, these algorithms are designed to be run on a quantum computer that has the full power of a quantum Turing machine, whereas current experimental quantum computers only possess a rather limited quantum part.

This naturally motivates the study of models of quantum computation that are far weaker than a polynomial time quantum Turing machine, such as the two-way finite automaton with quantum and classical states (2QCFA), originally defined by Ambainis and Watrous [2]. Informally, a 2QCFA is a two-way deterministic finite automaton (2DFA) that has been augmented by a quantum register of constant size; we define the 2QCFA model formally in Section 2.2. 2QCFA are surprisingly powerful, as originally demonstrated by Ambainis and Watrous, who showed that a 2QCFA, with only a single-qubit quantum register, can recognize, with bounded error, the language $L_{\text{eq}} = \{ a^m b^m : m \in \mathbb{N} \}$ in expected time $O(n^4)$ and the language $L_{\text{pal}} = \{ w \in \{a,b\}^* : w \text{ is a palindrome} \}$ in expected time $2^{O(n)}$. In a recent paper [32], we presented further evidence of the power of few qubits by showing that 2QCFA are capable of recognizing many group word problems with bounded error.
It is known that 2QCFA are more powerful than 2DFA and two-way probabilistic finite automata (2PFA). A 2DFA can only recognize regular languages [31]. A 2PFA can recognize some nonregular languages with bounded error, given sufficient running time: in particular, a 2PFA can recognize $L_{eq}$ with bounded error in expected time $2^{O(n)}$ [11]. However, a 2PFA cannot recognize $L_{eq}$ with bounded error in expected time $2^{o(n)}$, by a result of Greenberg and Weiss [12]; moreover, a 2PFA cannot recognize $L_{pal}$ with bounded error in any time bound [10]. More generally, the landmark result of Dwork and Stockmeyer [9] showed that a 2PFA cannot recognize any nonregular language in expected time $2^{n^{o(1)}}$. In order to prove this statement, they defined a particular “hardness measure” $D_L : N \to N$ of a language $L$. They showed that, if a 2PFA recognizes some language $L$ with bounded error in expected time at most $T(n)$ on all inputs of length at most $n$, then there is a positive real number $a$ (that depends only on the number of states of the 2PFA), such that $T(n) = \Omega \left( 2^{D_L(n)^a}\right)$ [9, Lemma 4.3]; we will refer to this statement as the “Dwork-Stockmeyer lemma.”

Very little was known about the limitations of 2QCFA. Are there any languages that a single-qubit 2QCFA can recognize with bounded error in expected exponential time but not in expected subexponential time? In particular, is it possible for a single-qubit 2QCFA to recognize $L_{pal}$ with bounded error in expected subexponential time, or perhaps even in expected polynomial time? More generally, are there any languages that a 2QCFA (that is allowed to have a quantum register of any constant size) can recognize with bounded error in expected exponential time but not in expected subexponential time? These natural questions, to our knowledge, were all open (see, for instance, [2,3,46] for previous discussions of these questions).

In this paper, we answer these and other related questions. In particular, we show that a 2QCFA (of any size) cannot recognize $L_{pal}$ with bounded error in expected time $2^{o(n)}$. Far more generally, we prove an analogue of the Dwork-Stockmeyer lemma for 2QCFA: if a 2QCFA recognizes some language $L$ with bounded error in expected time at most $T(n)$ on all inputs of length at most $n$, then there a positive real number $a$ (that depends only on the number of states of the 2QCFA), such that $T(n) = \Omega \left( D_L(n)^a\right)$. We note that, while our lower bound on the running time of a 2QCFA is exponentially weaker than the lower bound on the running time of a 2PFA provided by the Dwork-Stockmeyer lemma, both lower bounds are in fact (asymptotically) tight; the exponential difference provides yet another example of a situation in which quantum computers have an exponential advantage over their classical counterparts. We also establish a lower bound on the expected running time of a 2QCFA recognizer of $L$ in terms of the one-way deterministic communication complexity of testing membership in $L$.

Furthermore, we show that the class of languages recognizable with bounded error by a 2QCFA in expected polynomial time is contained in $L/poly$. This result, which shows that the class of languages recognizable by a particular quantum model is contained in the class of languages recognizable by a particular classical model, is a type of dequantumization result. It is (qualitatively) similar to the Adleman-type [1] derandomization result $BPL \subseteq L/poly$, where $BPL$ denotes the class of languages recognizable with bounded error by a probabilistic Turing machine (PTM) that uses $O(\log n)$ space and runs in expected polynomial time. The only previous dequantization result that we are aware of was of a very different type: the class of languages recognizable by a 2QCFA, or more generally a quantum Turing machine (QTM) that uses $O(\log n)$ space, with algebraic number transition amplitudes (even with unbounded error and with no time bound), is contained in $\text{DSPACE}(O(\log^2 n))$ [42]. This dequantization results is analogous to the derandomization result: the class of languages recognizable by a PTM that uses $O(\log n)$ space (even with unbounded error and with no time bound), is contained in $\text{DSPACE}(O(\log^2 n))$ [6].

We then generalize our results to prove a lower bound on the expected running time $T(n)$ of a
QTM that uses sublogarithmic space (i.e., $o(\log n)$ space) and recognizes a language $L$ with bounded error, where this lower bound is also in terms of $D_L(n)$. In particular, we show that $L_{pal}$ cannot be recognized with bounded error by a QTM that uses sublogarithmic space and runs in expected time $2^{\Omega(1)}$. This result is particularly intriguing as $L_{pal}$ can be recognized by a deterministic TM in $O(\log n)$ space (and, trivially, polynomial time); therefore, $L_{pal}$ provides an example of a natural problem for which polynomial time quantum TMs have no (asymptotic) advantage over polynomial time deterministic TMs in terms of the needed amount of space.

We also investigate which group word problems can be recognized by 2QCFA, or sublogarithmic-space QTM, with particular resource bounds. Informally, the word problem of a finitely generated group is the problem of determining if the product of a sequence of elements of that group is equal to the identity element. There is a deep connection between the algebraic properties of a finitely generated group $G$ and the complexity of its word problem $W_G$, as has been demonstrated by many famous results; for example, $W_G \in \text{REG} \iff G$ is finite [4], $W_G \in \text{CFL} \iff G$ is virtually free [8, 28], $W_G \in \text{NP} \iff G$ is a subgroup of a finitely presented group with polynomial Dehn function [5]. We have recently shown that if $G$ is virtually abelian, then $W_G$ may be recognized with bounded error by a single-qubit 2QCFA in expected polynomial time, and that, for any group $G$ in a certain broad class of groups of exponential growth, $W_G$ may be recognized with bounded error by a 2QCFA (in many cases a single-qubit 2QCFA) in expected time $2^{O(n)}$ [32].

We now show that, if $G$ has exponential growth, then $W_G$ cannot be recognized by a 2QCFA with bounded error in expected time $2^{o(n)}$, thereby providing a broad and natural class of languages that may be recognized with bounded error by a 2QCFA in expected time $2^{O(n)}$ but not $2^{o(n)}$. We also show that, if $W_G$ is recognizable by a 2QCFA with bounded error in expected polynomial time, then $G$ must be virtually nilpotent (i.e., $G$ must have polynomial growth), thereby obtaining progress towards an exact classification of those word problems recognizable by a 2QCFA in expected polynomial time. Furthermore, we show analogous results for sublogarithmic-space QTMs.

One of the key tools used in our proof is a quantum version of Hennie’s [17] notion of a crossing sequence, which may be of independent interest. Crossing sequences played an important role in the aforementioned 2PFA results of Dwork and Stockmeyer [9] and of Greenberg and Weiss [12]. In particular, we show that the computation of a 2QCFA on a particular portion of the input string can be modeled by an operator that is, in fact, a quantum channel. This allows us to bring the tools of quantum information theory to bear to analyze the behavior of a 2QCFA.

The remainder of this paper is organized as follows. In Section 2, we briefly recall the fundamentals of quantum computation and the definition of 2QCFA. In Section 3, we develop our notion of a quantum crossing sequence. The Dwork-Stockmeyer hardness measure $D_L$ of a language $L$, as well as several other related hardness measures of $L$, play a key role in our lower bounds; we recall the definitions of these hardness measures in Section 4.1. Then, in Section 4.2, using our notion of a quantum crossing sequence, we prove an analogue of the Dwork-Stockmeyer lemma for 2QCFA. Using this lemma, in Section 4.3, we establish various lower bounds on the expected running time of 2QCFA for particular languages and prove certain complexity class separations and inclusions. In Section 5, we establish lower bounds on the expected running time of sublogarithmic-space QTMs. In Section 6, we study group word problems and establish lower bounds on the expected running time of 2QCFA and sublogarithmic-space QTMs that recognize certain word problems. Finally, in Section 7, we discuss several interesting open problems related to our work.
2 Preliminaries

2.1 Quantum Computation

In this section, we briefly recall the fundamentals of quantum computation needed in this paper (see, for instance, [23, 30, 44] for a more detailed presentation of the material in this section). We begin by establishing some notation. Let $V$ denote a finite-dimensional complex Hilbert space with inner product $(\cdot, \cdot) : V \times V \to \mathbb{C}$. The dual space $V^*$ of $V$ is the $\mathbb{C}$-vector space consisting of all linear functionals on $V$ (i.e., all $\mathbb{C}$-linear maps of the form $f : V \to \mathbb{C}$). We use the standard Dirac bra-ket notation throughout this paper. We denote elements of $V$ by kets: $|\psi\rangle$, $|\varphi\rangle$, $|q\rangle$, etc. For the ket $|\psi\rangle \in V$, we define the corresponding bra $\langle \psi | \in V^*$ to be the linear functional on $V$ given by $\langle \psi | : V \to \mathbb{C}$ (i.e., for any $|\varphi\rangle \in V$, we have $\langle \psi | (|\varphi\rangle) = \langle |\psi\rangle | \varphi\rangle$). For notational clarity and brevity, we write $\langle \psi | \varphi \rangle$ in place of $\langle \psi | (|\varphi\rangle)$.

Let $L(V)$ denote the $\mathbb{C}$-vector space consisting of all $\mathbb{C}$-linear maps of the form $A : V \to V$. For $|\psi\rangle, |\varphi\rangle \in V$, we define $|\psi\rangle \langle \varphi | \in L(V)$ in the natural way: for $|\rho\rangle \in V$, $|\psi\rangle \langle \varphi | (|\rho\rangle) = |\psi\rangle \langle \psi | (|\varphi\rangle) = \langle |\varphi\rangle | \rho\rangle$. For $A, A' \in L(V)$ and $|\psi\rangle \in V$, we, again for the sake of notational clarity and brevity, write $AA'$ to denote the element $A(|\psi\rangle) \in V$ obtained by applying the map $A$ to the element $|\psi\rangle$ and write $A'$ to denote the composition $A \circ A'$. Let $1_V \in L(V)$ denote the identity operator on $V$ (i.e., $1_A|\psi\rangle = |\psi\rangle$, $\forall|\psi\rangle \in V$) and let $0_V \in L(V)$ denote the zero operator on $V$ (i.e., $0_V|\psi\rangle = 0$ (the zero vector in $V$), $\forall|\psi\rangle \in V$). For $A \in L(V)$, we define $A^\dagger \in L(V)$, the Hermitian transpose of $A$, to be the unique element of $L(V)$ such that $\langle A|\psi_1\rangle, |\psi_2\rangle = \langle |\psi_1\rangle, A^\dagger |\psi_2\rangle$, $\forall|\psi_1\rangle, |\psi_2\rangle \in V$. Let $\text{Herm}(V) = \{A \in L(V) : A = A^\dagger\}$ denote the set of Hermitian operators on $V$, let $\text{Pos}(V) = \{A^\dagger A : A \in L(V)\} \subseteq \text{Herm}(V)$ denote the set of positive semi-definite operators on $V$, let $\text{Proj}(V) = \{A \in \text{Pos}(V) : A^2 = A\}$ denote the set of projection operators on $V$, let $U(V) = \{A \in L(V) : AA^\dagger = 1_V\}$ denote the set of unitary operators on $V$, and let $\text{Den}(V) = \{A \in \text{Pos}(V) : \text{Tr}(A) = 1\}$ denote the set of density operators on $V$.

A quantum register is specified by a finite set of quantum basis states $Q = \{q_0, \ldots, q_{k-1}\}$. Corresponding to these $k$ quantum basis states is an orthonormal basis $\{|q_0\rangle, \ldots, |q_{k-1}\rangle\}$ of the finite-dimensional complex Hilbert space $\mathbb{C}^k$. The quantum register stores a superposition $|\psi\rangle = \sum_q \alpha_q |q\rangle \in \mathbb{C}^k$, where each $\alpha_q \in \mathbb{C}$ and $\sum_q |\alpha_q|^2 = 1$; in other words, a superposition $|\psi\rangle$ is simply an element of $\mathbb{C}^k$ of norm 1. Let $\mathbb{C}^Q$ denote the $\mathbb{C}$-vector space consisting of all functions from $Q$ to $\mathbb{C}$. Of course, $\mathbb{C}^Q \cong \mathbb{C}^k$; it will often be more convenient to think of superpositions as being elements of $\mathbb{C}^Q$ of norm 1.

Following the original definition of Ambainis and Watrous [2], a 2QCFA may only interact with its quantum register in two ways: by applying a unitary transformation or performing a quantum measurement of a certain simple type. If the quantum register is currently in the superposition $|\psi\rangle \in \mathbb{C}^Q$, then after applying the unitary transformation $T \in U(\mathbb{C}^Q)$, the quantum register will be in the superposition $T|\psi\rangle$. A von Neumann measurement is specified by some $P_1, \ldots, P_l \in \text{Proj}(\mathbb{C}^Q)$, such that $P_i P_j = 0_{\mathbb{C}^Q}$, $\forall i, j$ with $i \neq j$, and $\sum_j P_j = 1_{\mathbb{C}^Q}$. Quantum measurement is a probabilistic process where, if the quantum register is currently in the superposition $|\psi\rangle$, then the result of the measurement has the value $r \in \{1, \ldots, l\}$ with probability $\|P_r |\psi\rangle\|^2$; if the result is $r$, then the quantum register collapses to the superposition $\frac{P_r |\psi\rangle}{\|P_r |\psi\rangle\|^2} \cdot P_r |\psi\rangle$. We emphasize that performing a quantum measurement changes the state of the quantum register.

An ensemble of pure states of the quantum register is a set $\{(p_i, |\psi_i\rangle) : i \in I\}$, for some index set $I$, where $p_i \in [0, 1]$ denotes the probability of the quantum register being in the superposition $|\psi_i\rangle \in \Psi$, and $\sum_i p_i = 1$. This ensemble corresponds to the density operator $A = \sum_i p_i |\psi_i\rangle \langle \psi_i | \in \text{Den}(\mathbb{C}^Q)$. Of course, many distinct ensembles correspond to the density operator $A$; however, all ensembles that correspond to a particular density operator will behave the same, for our purposes.
(see, for instance, [30, Section 2.4] for a detailed discussion of this phenomenon, and of the following claims). That is to say, for any ensemble described by a density operator $A \in \text{Den}(\mathbb{C}^Q)$, applying the transformation $T \in \text{U}(\mathbb{C}^Q)$ produces an ensemble described by the density operator $TAT^\dagger$. Similarly, consider the von Neumann measurement specified by some $P_1, \ldots, P_l \in \text{Proj}(\mathbb{C}^Q)$. Then for any ensemble described by the density operator $A$, the probability that the result of this measurement is $r$ is given by $\text{Tr}(P_r AP_r^\dagger)$, and if the result is $r$ then the ensemble collapses to an ensemble described by the density operator $\frac{1}{\text{Tr}(P_r AP_r^\dagger)}P_r AP_r^\dagger$.

Let $V$ and $V'$ denote a pair of finite-dimensional complex Hilbert spaces. Let $T(V, V')$ denote the $\mathbb{C}$-vector space consisting of all $\mathbb{C}$-linear maps (i.e., operators) of the form $\Phi : L(V) \rightarrow L(V')$. Define $T(V) = T(V, V)$ and let $\mathbb{1}_{L(V)} \in T(V)$ denote the identity operator. Consider some $\Phi \in T(V, V')$. We say that $\Phi$ is positive if, $\forall A \in \text{Pos}(V)$, we have $\Phi(A) \in \text{Pos}(V')$. We say that $\Phi$ is completely-positive if, for every finite-dimensional complex Hilbert space $W$, $\Phi \otimes \mathbb{1}_{L(W)}$ is positive, where $\otimes$ denotes the tensor product. We say that $\Phi$ is trace-preserving if, $\forall A \in L(V)$, we have $\text{Tr}(\Phi(A)) = \text{Tr}(A)$. If $\Phi$ is both completely-positive and trace-preserving, then we say $\Phi$ is a quantum channel. Let $\text{Chan}(V, V') = \{ \Phi \in T(V, V') : \Phi$ is a quantum channel$\}$ denote the set of all such channels, and define $\text{Chan}(V) = \text{Chan}(V, V)$.

As we wish for our lower bound to be as strong as possible, we wish to consider a variant of the 2QCFA model that is as strong as possible; in particular, we will allow a 2QCFA to perform any physically realizable quantum operation on its quantum register. Following Watrous [42], a selective quantum operation $\mathcal{E}$ is specified by a set of operators $\{ E_{r,j} : r \in R, j \in \{1, \ldots, l\} \} \subseteq L(\mathbb{C}^Q)$, where $R$ is a finite set and $l \in \mathbb{N}_{\geq 1}$ (throughout the paper, we write $\mathbb{N}_{\geq 1}$ to denote the positive natural numbers, $\mathbb{R}_{\geq 0}$ to denote the nonnegative real numbers, and so on), such that $\sum_{r,j} E_{r,j}^\dagger E_{r,j} = \mathbb{1}_{\mathbb{C}^Q}$. For $r \in R$, we define $\Phi_r \in T(\mathbb{C}^Q)$ such that, $\Phi_r(A) = \sum_j E_{r,j} A E_{r,j}^\dagger$, $\forall A \in L(V)$. Then, if the quantum register is described by some density operator $A \in \text{Den}(\mathbb{C}^Q)$, applying the selective quantum operation $\mathcal{E}$ will have result $r \in R$ with probability $\text{Tr}(\Phi_r(A))$; if the result is $r$ (which requires $\text{Tr}(\Phi_r(A)) > 0$), then the quantum register is described by density operator $\frac{1}{\text{Tr}(\Phi_r(A))} \Phi_r(A)$. Observe that both unitary transformations and von Neumann measurements are special cases of selective quantum operations. By [44, Theorem 2.22], for any selective quantum operation $\mathcal{E}$, each $\Phi_r$ is completely-positive; moreover, one may always obtain a family of operators that represent $\mathcal{E}$ with $l \leq |Q|^2$, and therefore with $l = |Q|^2$ (by defining any extraneous operators to be $0_{\mathbb{C}^Q}$). Let $\text{QuantOp}(\mathbb{C}^Q, R)$ denote the set of all selective quantum operations specified by some $\{ E_{r,j} : r \in R, j \in \{1, \ldots, |Q|^2\} \} \subseteq L(\mathbb{C}^Q)$.

### 2.2 Definition of the 2QCFA Model

In this section, we define two-way finite automata with quantum and classical states (2QCFA), essentially following the original definition given by Ambainis and Watrous [2], with a few alterations that (potentially) make the model stronger. Again, we wish to define the 2QCFA model to be as strong as possible so that our lower bounds against this model are as general as possible. Of course, all of our results would apply to the (potentially) weaker original definition of a 2QCFA.

Informally, a 2QCFA is a two-way DFA that has been augmented with a quantum register of constant size; the machine may apply unitary transformations to the quantum register and perform (perhaps many) measurements of its quantum register during its computation. Formally, a 2QCFA is a 10-tuple,

$$N = (Q, C, \Sigma, R, \theta, \delta, q_{\text{start}}, c_{\text{start}}, c_{\text{acc}}, c_{\text{rej}}),$$

where $Q$ is a finite set of quantum basis states, $C$ is a finite set of classical states, $\Sigma$ is a finite input alphabet, $R$ is a finite set that specifies the possible results of selective quantum operations,
\(\theta\) and \(\delta\) are the quantum and classical parts of the transition function, \(q_{\text{start}} \in Q\) is the quantum start state, \(c_{\text{start}} \in C\) is the classical start state, and \(c_{\text{acc}}, c_{\text{rej}} \in C\), with \(c_{\text{acc}} \neq c_{\text{rej}}\), specify the classical accept and reject states, respectively. We define \(#_L, #_R \notin \Sigma\), with \(#_L \neq #_R\), to be special symbols that serve as a left and right end-marker, respectively; we then define the tape alphabet \(\Sigma_+ = \Sigma \cup \{#_L, #_R\}\). Let \(\hat{C} = C \setminus \{c_{\text{acc}}, c_{\text{rej}}\}\) denote the non-halting classical states. The components of the transition function are as follows: \(\theta : \hat{C} \times \Sigma_+ \to \text{QuantOp}(C^Q, R)\) specifies the selective quantum operation that is to be performed on the quantum register and \(\delta : \hat{C} \times \Sigma_+ \times R \to C \times \{-1, 0, 1\}\) specifies how the classical state and (classical) head position evolve.

On an input \(w = w_1 \cdots w_n \in \Sigma^*\), with each \(w_i \in \Sigma\), the 2QCFA \(N\) operates as follows. The machine has a read-only tape that contains the string \(#_L w_1 \cdots w_n #_R\). Initially, the classic state of \(N\) is \(c_{\text{start}}\), the quantum register is in the superposition \(|q_{\text{start}}\rangle\) (which is described by the density operator \(|q_{\text{start}}\rangle\langle q_{\text{start}}| \in \text{Den}(C^Q)\)), and the head is at the left end of the tape, over the left end-marker \(#_L\). On each step of the computation, if the classic state is currently \(c \in \hat{C}\) and the head is over the symbol \(\sigma \in \Sigma_+\), \(N\) behaves as follows. First, the selective quantum operation \(\theta(c, \sigma)\) is performed on the quantum register producing some result \(r \in R\). If the result was \(r\), and \(\delta(c, \sigma, r) = (c', d)\), where \(c' \in C\) and \(d \in \{-1, 0, 1\}\), then the classical state becomes \(c'\) and the head moves left (resp. stays put, moves right) if \(d = -1\) (resp. \(d = 0, d = 1\)).

Due to the fact that applying a selective quantum operation is a probabilistic process, the computation of \(N\) on an input \(w\) is probabilistic. For any language \(L\) and any \(\epsilon \in [0, \frac{1}{2})\), we say that a 2QCFA \(N\) recognizes \(L\) with two-sided bounded error \(\epsilon\) if, \(\forall w \in L\), \(\Pr[\text{\(N\) accepts \(w\)}] \geq 1 - \epsilon\), and, \(\forall w \notin L\), \(\Pr[\text{\(N\) accepts \(w\)}] \leq \epsilon\). Then, for any function \(T : \mathbb{N} \to \mathbb{N}\), we define \(\text{B2QCFA}(k, d, T(n), \epsilon)\) as the class of languages \(L\) for which there is a 2QCFA, with at most \(k\) quantum basis states and at most \(d\) classical states, that recognizes \(L\) with two-sided bounded error \(\epsilon\), and has expected running time at most \(T(n)\) on all inputs of length at most \(n\).

In order to make our lower bound as strong as possible, we do not require \(N\) to halt with probability 1 on all \(w \in \Sigma^*\) (i.e., we permit \(N\) to reject an input by looping) and we permit language recognition under the more relaxed condition of two-sided bounded error. The bounds that we show for this 2QCFA model of course also apply to the 2QCFA model as originally defined by Ambainis and Watrous [2], which required \(N\) to halt with probability 1 on all inputs and operated under the more restrictive negative one-sided bounded error recognition condition.

### 3 2QCFA Crossing Sequences

In this section, we develop a generalization of Hennie’s [17] notion of crossing sequences to 2QCFA, in which we make use of several ideas from the 2PFA results of Dwork and Stockmeyer [9] and Greenberg and Weiss [12]. This notion will play a key role in our proof of a lower bound on the expected running time of a 2QCFA.

Consider a 2QCFA \(N = (Q, C, \Sigma, R, \theta, \delta, q_{\text{start}}, c_{\text{start}}, c_{\text{acc}}, c_{\text{rej}})\) and an input \(w = w_1 \cdots w_n \in \Sigma^*\), where each \(w_i \in \Sigma\). When \(N\) is run on input \(w\), the tape consists of \(#_L w_1 \cdots w_n #_R\); for convenience, we define \(w_0 = #_L\) and \(w_{n+1} = #_R\). One may describe the total configuration of a single probabilistic branch of \(N\) at any particular point in time by a triple \((A, c, h)\), where \(A \in \text{Den}(C^Q)\) describes the current state of the quantum register, \(c \in C\) is the current classical state, and \(h \in \{0, \ldots, n+1\}\) is the current head position. To clarify, each step of the computation of \(N\) involves applying a selective quantum operation, which is a probabilistic process that produces a particular result \(r \in R\) with a certain probability (depending on the particular operation that is performed and the state of the quantum register); that is to say, the 2QCFA probabilistically
branches, with a child for each \( r \in R \).

We partition the input as \( u = xy \), in some manner to be specified later. We then imagine running \( N \) beginning in the configuration \((A,c,|x\rangle)\), for some \( A \in \text{Den}(\mathbb{C}^Q) \) and \( c \in \hat{C} = C \setminus \{c_{\text{acc}},c_{\text{rej}}\} \), where \(|x\rangle\) denotes the length of the string \( x \) (i.e., the head is initially over the rightmost symbol of \(#_Lx\)). We wish to describe the configuration (or, more accurately, ensemble of configurations) that \( N \) will be in when it “finishes computing” on the prefix \(#_Lx\), either by “leaving” the string \(#_Lx\) (where here we say that \( N \) “leaves” \(#_Lx\) if \( N \) moves its head right when over the rightmost symbol of \(#_Lx\)), or by accepting or rejecting its input. Of course, \( N \) may leave \(#_Lx\), then later reenter \(#_Lx\), then later leave \(#_Lx\) again, and so on, which will naturally lead to our notion of a crossing sequence. Note that the particular choice of the string \( y \) does not affect this subcomputation as it occurs entirely within the prefix \(#_Lx\).

More generally, we consider the case in which \( N \) is run on the prefix \(#_Lx\), where \( N \) starts in some ensemble of configurations \( \{\langle p_i, (A_i,c_i,|x\rangle) \rangle : i \in I \} \), where the probability of being in configuration \((A_i,c_i,|x\rangle)\) is given by \( p_i \) (note that the head position in each configuration is over the rightmost symbol of \(#_Lx\)); we call this ensemble a starting ensemble. We then wish to describe the ensemble of configurations that \( N \) will be in when it “finishes computing” on the prefix \(#_Lx\), (essentially) as defined above; we call this ensemble a stopping ensemble\(^1\). Much as it was the case that an ensemble of pure states of a quantum register can be described by a density operator, we may also describe an ensemble of configurations of a 2QCFA using density operators. This will greatly simplify our definition and analysis of the crossing sequence of a 2QCFA.

### 3.1 Describing Ensembles of Configurations of 2QCFA

Let \( \hat{H}_x = \{0, \ldots, |x\rangle \} \) denote the head positions corresponding to the prefix \(#_Lx\), and let \( H_x = \{0, \ldots, |x\rangle + 1 \} \) denote the set of possible positions the head of \( N \) may be in when \( N \) is run on the prefix \(#_Lx\) until \( N \) “finishes computing” on the prefix \(#_Lx\). The 2QCFA \( N \) possesses both a constant-sized quantum register, that is described by some density operator at any particular point in time, and a constant-sized classical register, that stores a classical state \( c \in C \). We can naturally interpret each \( c \in C \) as an element \(|c\rangle \in \mathbb{C}^C\), of a special type; that is to say, each classical state \( c \) corresponds to some element \(|c\rangle\) in the natural orthonormal basis of \( \mathbb{C}^C \) (whereas each superposition \(|\psi\rangle\) of the quantum register corresponds to an element of \( \mathbb{C}^Q \) of norm 1). One may also view \( N \) as possessing a head register that stores a (classical) head position \( h \in H_x \) (when computing on the prefix \(#_Lx\)); of course, the size of this pseudo-register grows with the input prefix \( x \). We analogously interpret a head position \( h \in H_x \) as being the “classical” element \(|h\rangle \in \mathbb{C}^{H_x}\), in the same way as we have done for the classical state \( c \in C \). A configuration \((A,c,h) \in \text{Den}(\mathbb{C}^Q) \times C \times H_x\) of \( N \) is then simply a state of the combined register, which consists of the quantum, classical, and head registers. Let \( \text{Den}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \) denote the set of all density operators on the combined space \( \mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x} \).

We then consider an ensemble of configurations \( \{\langle p_i, (A_i,c_i,h_i) \rangle : i \in I \} \), for some index set \( I \), where \( p_i \in [0,1] \) denotes the probability of being in configuration \((A_i,c_i,h_i) \in \text{Den}(\mathbb{C}^Q) \times C \times H_x\), and \( \sum_{i} p_i = 1 \). We represent this ensemble (non-uniquely) by the density operator \( Z = \sum_{i} \langle p_i A_i \otimes |c_i\rangle \rangle \langle c_i | \otimes |h_i\rangle \rangle \langle h_i | \rangle \in \text{Den}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \). Let \( \hat{c}(c,h) = \{i \in I : (c_i,h_i) = (c,h)\} \) denote the indices of those configurations in classical state \( c \) and with head position \( h \). Then

\[
Z = \sum_{c \in \mathbb{C}^C} \sum_{h \in \mathbb{C}^{H_x}} \sum_{i \in \hat{c}(c,h)} \langle p_i A_i \otimes |c_i\rangle \rangle \langle c_i | \otimes |h_i\rangle \rangle \langle h_i | \rangle = \sum_{c \in \mathbb{C}^C} \left( \sum_{h \in \hat{c}(c,h)} p_i A_i \right) \otimes |c\rangle \rangle \langle c | \otimes |h\rangle \rangle \langle h | \rangle.
\]

\(^1\)We use the terms “starting ensemble” and “stopping ensemble” to make clear the similarity to the notion of a “starting condition” and of a “stopping condition” used by Dwork and Stockmeyer [9] in their 2PFA result.
We then define \( p : C \times H_x \to [0,1] \) such that \( p(c,h) = \sum_{i \in \{c,h\}} p_i \) is the total probability (over the ensemble in question) of being in classical state \( c \) and having head position \( h \), and \( \sum_{c,h} p(c,h) = 1 \). Moreover, we define \( A : C \times H_x \to \text{Den}(\mathbb{C}^Q) \) such that, if \( p(c,h) \neq 0 \), then \( A(c,h) = \sum_{i \in \{c,h\}} \hat{p}_i A_i \) is the density operator obtained by “merging” all density operators \( A_i \) that come from configurations \( (A_i,c_i,h_i) \) with classical state \( c_i = c \) and head position \( h_i = h \); if \( p(c,h) = 0 \), then we define \( A(c,h) \) arbitrarily (say \( A(c,h) = \langle \psi_{\text{start}} | \psi_{\text{start}} \rangle \)). Then \( Z = \sum_{c,h} (p(c,h)A(c,h) \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|) \). Let \( \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \) denote the set of all density operators given by some \( Z \) of the above form (i.e., those density operators that respect the fact that both the classical state and head position are classical).

Remark. As noted in Section 2.1, many distinct ensembles of pure states of a quantum register correspond to the same density operator \( A \in \text{Den}(\mathbb{C}^Q) \), however they all behave the same when applying a unitary transformation or a quantum measurement (or, more generally, a selective quantum operation). In much that same way, many distinct ensembles of configurations of a 2QCFA are described by the same density operator \( Z \in \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \), but, as we will see, they all behave the same for our purposes.

We also consider the case in which the head position does not need to be recorded and we are only interested in the combined state of the quantum register and classical register. We then analogously describe an ensemble \( \{p_i,(A_i,c_i) : i \in I\} \) by the density operator \( Z = \sum_i (p_i A_i \otimes |c_i\rangle \langle c_i|) \in \text{Den}(\mathbb{C}^Q \otimes \mathbb{C}^C) \), and we define \( \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C) \) to be the set of all density operators that describe a valid ensemble of states of the quantum register and classical register.

In a starting ensemble, as defined above, all configurations have the same head position: \( |x| \). We define the operator \( I_x \in \text{T}(\mathbb{C}^Q \otimes \mathbb{C}^C,\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \) such that \( I_x(Z) = Z \otimes ||x|| \langle|x|\rangle \), \( \forall Z \in \text{L}(\mathbb{C}^Q \otimes \mathbb{C}^C) \). Notice that, for any \( Z \in \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C) \), if \( \{p_i,(A_i,c_i) : i \in I\} \) is any ensemble of states of the quantum register and classical register of \( N \) that is described by \( Z \), then the ensemble \( \{(p_i,(A_i,c_i,|x|)) : i \in I\} \) of configurations of \( N \) is described by \( I_x(Z) \in \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \).

Similarly, in a stopping ensemble, all configurations either have head position \( |x| + 1 \) or are accepting or rejecting configurations (in which the head position is not relevant). Let \( \text{Tr}_{\mathbb{C}^{H_x}} = 1_{\text{L}(\mathbb{C}^Q \otimes \mathbb{C}^C)} \otimes \text{Tr} \in \text{T}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x},\mathbb{C}^Q \otimes \mathbb{C}^C) \) denote the partial trace with respect to \( \mathbb{C}^{H_x} \) (i.e., it is the unique operator that satisfies \( \text{Tr}_{\mathbb{C}^{H_x}}(Z_{QC} \otimes Z_H) = \text{Tr}(Z_H)Z_{QC}, \forall Z_{QC} \in \text{L}(\mathbb{C}^Q \otimes \mathbb{C}^C), \forall Z_H \in \text{L}(\mathbb{C}^{H_x}) \)). Notice that, for any \( Z \in \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \), if \( \{p_i,(A_i,c_i,h_i) : i \in I\} \) is any ensemble of configurations of \( N \) described by \( Z \), then the ensemble \( \{(p_i,(A_i,c_i,|x|)) : i \in I\} \) of states of the quantum register and classical register of \( N \) is described by \( \text{Tr}_{\mathbb{C}^{H_x}}(Z) \in \widetilde{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C) \).

### 3.2 Overview of 2QCFA Crossing Sequences

We now sketch our definition of the **crossing sequence** of the 2QCFA \( N \) on the partitioned input \( xy \). Consider running \( N \) on the prefix \#\_L\_x beginning in some starting ensemble \( \{(p_i,(A_i,c_i,|x|)) : i \in I\} \). To avoid unnecessary cases later, we also allow \( N \) to start in a configuration of the form \( (A_i,c_i,|x|) \), where \( c_i \in \{c_{\text{acc}},c_{\text{rej}}\} \), where we adopt the convention that in such a case \( N \) immediately leaves \#\_L\_x in the configuration \( (A_i,c_i,|x| + 1) \). For any \( m \in \mathbb{N} \), we define the m-truncated stopping ensemble as the ensemble of configurations (of the quantum register and classical register, we ignore the head position here) that \( N \) will be in when it “finishes computing” on \#\_L\_x, as defined above, with the modification that if any particular branch of \( N \) attempts to run for more than \( m \) steps, the computation of that branch will be “interrupted” immediately before it attempts to perform the \( m + 1 \) step and instead immediately reject. To be clear, this truncation of branches occurs only in the analysis of \( N \); we do not modify the 2QCFA \( N \).
We then define the \textit{m-truncated transfer operator} \(N_{x,m}^\leq \in T(C^Q \otimes C^C)\) such that, for any \(Z \in \overline{\text{Den}}(C^Q \otimes C^C)\), if \(N\) is run on the prefix \(#_Lx\) beginning in an ensemble of configurations described by \(I_x(Z)\), then the \(m\)-truncated stopping ensemble will be described by \(N_{x,m}^\leq(Z)\). For \(m\) sufficiently large, with respect to the expected running time of \(N\) on the (total) input \(xy\), this operator accurately describes the behavior of \(N\) when computing on the prefix \(#_Lx\), as truncating branches that run for an extremely long time, will have a negligible impact on the behavior of \(N\). Symmetrically, we define the operator \(N_{y,m}^\leq \in T(C^Q \otimes C^C)\) that describes the behavior of \(N\) when computing on the suffix \(y\#_R\). The \textit{m-truncated crossing sequence} will then consist of the sequence of density operators obtained by beginning with the simple density operator that describes the ensemble of configurations of (a slightly modified version of) \(N\) when it first crosses between \(#_Lx\) and \(y\#_R\), and then alternately applying the operators \(N_{x,m}^\leq\) and \(N_{y,m}^\leq\) in an infinite sequence.

\textbf{Remark}. We emphasize that the density operators that appear in the crossing sequence of a 2QCFA \(N\) do not describe the configurations of \(N\) at a particular points in time, but rather at particular events: the \(j\)th time the head of \(N\) crosses between \(#_Lx\) and \(y\#_R\), for each \(j \in \mathbb{N}_1\).

Crucially, we will observe that \(N_{x,m}^\leq, N_{y,m}^\leq \in \text{Chan}(C^Q \otimes C^C), \forall x, y \in \Sigma^*, \forall m \in \mathbb{N}\). This will allow us to make use of the machinery of quantum information theory to analyze the behavior of a 2QCFA. In fact, the analysis that we perform on the \(m\)-truncated transfer operators, which allows us to exhibit a lower bound on the expected running time of a 2QCFA, only requires a somewhat weaker property than being a quantum channel; we prove this stronger property as these notions of transfer operators and crossing sequences may be of use in proving other properties of 2QCFA in the future. In the following section, we formally define \(N_{x,m}^\leq\) and \(N_{y,m}^\leq\) and show that they are, in fact, quantum channels. A reader that is willing to believe these claims may feel free to skip the next section on an initial reading, and proceed directly to Section 4, where quantum crossing sequences are used to prove a lower bound on the running time of 2QCFA.

\section{Definition and Properties of 2QCFA Crossing Sequences}

We now formally define the notion of a crossing sequence of a 2QCFA, sketched in the previous section, and prove certain needed properties. We begin by establishing some notation that will better allow us to describe the parts of the transition function of \(N\).

\textbf{Definition 3.1}. Consider a 2QCFA \(N = (Q, C, \Sigma, R, \theta, \delta, q_{\text{start}}, c_{\text{acc}}, c_{\text{rej}})\). Suppose \(c \in \widehat{C} = C \setminus \{c_{\text{acc}}, c_{\text{rej}}\}, \sigma \in \Sigma_+ = \Sigma \cup \{\#_L, \#_R\}, \) and \(r \in \mathbb{R}\). Let \(J = \{1, \ldots, |Q|^2\}\).

(i) For \(j \in J\), define \(E_{c,\sigma,r,j} \in L(C^Q)\) such that the selective quantum operation \(\theta(c, \sigma) \in \text{QuantOp}(C^Q, R)\) is described by \(\{E_{c,\sigma,r,j} : r \in \mathbb{R}, j \in J\}\), where \(\sum_{r,j} E_{c,\sigma,r,j}^\dagger E_{c,\sigma,r,j} = \mathbbm{1}_{C^Q}\).

(ii) Let \(\Phi_{c,\sigma,r} \in T(C^Q)\) denote the operator corresponding to result \(r \in R\) in \(\theta(c, \sigma)\); i.e., \(\Phi_{c,\sigma,r}(A) = \sum_{j} E_{c,\sigma,r,j}^\dagger A E_{c,\sigma,r,j}, \forall A \in L(C^Q)\).

(iii) Let \(\gamma_{c,\sigma,r} \in C\) and \(d_{c,\sigma,r} \in \{-1, 0, 1\}\) denote, respectively, the new classical state and the motion of the head, if the result of applying \(\theta(c, \sigma)\) is \(r\); i.e., \(\delta(c, \sigma, r) = (\gamma_{c,\sigma,r}, d_{c,\sigma,r})\).

Consider some \(x \in \Sigma^*\). As before, \(\widehat{H}_x = \{0, \ldots, |x|\}\) denotes the head positions corresponding to \(\#_Lx\), and \(H_x = \{0, \ldots, |x|+1\}\) denotes the possible head positions of \(N\) until it “finishes computing” on \(\#_Lx\). A configuration \((A, c, h) \in \text{Den}(C^Q) \times C \times H_x\) of \(N\) is described by the density operator \(A \otimes |c\rangle \langle c| \otimes |h\rangle \langle h| \in \overline{\text{Den}}(C^Q \otimes C^C \otimes C^{H_x})\). We define an operator \(S_x \in T(C^Q \otimes C^C \otimes C^{H_x})\) that describes a single step of the computation of \(N\) on \(\#_Lx\), as follows. If \((c, h) \in \widehat{C} \times \widehat{H}_x\), then
$S_x(A \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|)$ describes the ensemble of configurations of $N$ after running $N$ for a single step beginning in the configuration $(A, c, h)$; otherwise, $S_x(A \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|)$ = $A \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|$ (i.e., if $c \in \{c_{\text{acc}}, c_{\text{ref}}\}$ or $h = |x|$, which means $N$ has “finished computing” on $\#_L x$, then $S_x$ leaves the configuration unchanged). We will then observe that $S_x$ correctly describes the behavior of $N$ on an ensemble of configurations, and that $S_x$ is a quantum channel.

**Definition 3.2.** Using the notation of Definition 3.1, consider a 2QCFA $N$ and a string $x \in \Sigma^*$. Let $x_h \in \Sigma$ denote the symbol of $x$ at position $h$, and let $x_0 = \#_L$ denote the left end-marker.

(i) For $c \in C$, $h \in H_x$, $r \in R$, and $j \in J$, we define $\tilde{E}_{x,c,h,r,j} \in L(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x})$ as follows.

$$\tilde{E}_{x,c,h,r,j} = \begin{cases} 
E_{c,x_h,r,j} \otimes |\gamma_{c,x_h,r}\rangle \langle \gamma_{c,x_h,r}| \otimes |h + d_{c,x_h,r}\rangle \langle h + d_{c,x_h,r}|, & \text{if } (c, h) \in \hat{C} \times \hat{H} \\
\frac{1}{\sqrt{|R||J|}} \mathbb{1}_{\mathbb{C}^Q} \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|, & \text{otherwise.}
\end{cases}$$

(ii) Define $S_x \in T(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x})$ such that

$$S_x(Z) = \sum_{(c,h,r,j) \in C \times H_x \times R \times J} \tilde{E}_{x,c,h,r,j} Z \tilde{E}_{x,c,h,r,j}^\dagger, \quad \forall Z \in L(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}).$$

We next observe that $S_x$ has the desired properties.

**Lemma 3.3.** Using the notation of Definition 3.2, for any $x \in \Sigma^*$, consider any $(A, \hat{c}, \hat{h}) \in \text{Den}(\mathbb{C}^Q) \times \hat{C} \times \hat{H}_x$, and let $\hat{Z} = A \otimes |\hat{c}\rangle \langle \hat{c}| \otimes |\hat{h}\rangle \langle \hat{h}|$. Then $S_x(\hat{Z})$ describes the ensemble of configurations obtained after running $N$ for a single step, beginning in the configuration $(A, \hat{c}, \hat{h})$, on input prefix $\#_L x$.

**Proof.** Let $\tilde{R}_{x,\hat{c},\hat{h},A} = \{r \in R : \text{Tr}(\Phi_{\tilde{c},x_h,r}(A)) \neq 0\}$. After running $N$ as described, it is in an ensemble of configurations

$$\left\{ \left( \text{Tr}(\Phi_{\tilde{c},x_h,r}(A)), \frac{1}{\text{Tr}(\Phi_{\tilde{c},x_h,r}(A))} \Phi_{\tilde{c},x_h,r}(A) \otimes |\gamma_{\tilde{c},x_h,r}\rangle \langle \gamma_{\tilde{c},x_h,r}| \otimes |\hat{h} + d_{\tilde{c},x_h,r}\rangle \langle \hat{h} + d_{\tilde{c},x_h,r}| \right) : r \in \tilde{R}_{x,\hat{c},\hat{h},A} \right\}.$$

Note that $A \in \text{Den}(\mathbb{C}^Q) \subseteq \text{Pos}(\mathbb{C}^Q)$, which implies $\Phi_{\tilde{c},x_h,r}(A) \in \text{Pos}(\mathbb{C}^Q)$; therefore, we have $\text{Tr}(\Phi_{\tilde{c},x_h,r}(A)) = 0$ precisely when $\Phi_{\tilde{c},x_h,r}(A) = 0_{\mathbb{C}^Q}$. The above ensemble of configurations is described by the density operator $\hat{Z}'$ given by

$$\hat{Z}' = \sum_{r \in \tilde{R}_{x,\hat{c},\hat{h},A}} \left( \frac{\text{Tr}(\Phi_{\tilde{c},x_h,r}(A))}{\text{Tr}(\Phi_{\tilde{c},x_h,r}(A))} \Phi_{\tilde{c},x_h,r}(A) \otimes |\gamma_{\tilde{c},x_h,r}\rangle \langle \gamma_{\tilde{c},x_h,r}| \otimes |\hat{h} + d_{\tilde{c},x_h,r}\rangle \langle \hat{h} + d_{\tilde{c},x_h,r}| \right)$$

$$= \sum_{r \in \tilde{R}} \left( \Phi_{\tilde{c},x_h,r}(A) \otimes |\gamma_{\tilde{c},x_h,r}\rangle \langle \gamma_{\tilde{c},x_h,r}| \otimes |\hat{h} + d_{\tilde{c},x_h,r}\rangle \langle \hat{h} + d_{\tilde{c},x_h,r}| \right).$$

Let $B_{x,\hat{c},\hat{h},r} = |\gamma_{\tilde{c},x_h,r}\rangle \langle \gamma_{\tilde{c},x_h,r}| \otimes |\hat{h} + d_{\tilde{c},x_h,r}\rangle \langle \hat{h} + d_{\tilde{c},x_h,r}|$. For any $(c, h, r, j) \in \hat{C} \times \hat{H}_x \times R \times J$,

$$\tilde{E}_{x,c,h,r,j} \tilde{E}_{x,c,h,r,j}^\dagger = \tilde{E}_{x,c,h,r,j} \left( A \otimes |\tilde{c}\rangle \langle \tilde{c}| \otimes |\tilde{h}\rangle \langle \tilde{h}| \right) \tilde{E}_{x,c,h,r,j}^\dagger$$

$$= (E_{c,x_h,r,j} \otimes |\gamma_{c,x_h,r}\rangle \langle \gamma_{c,x_h,r}| \otimes |h + d_{c,x_h,r}\rangle \langle h + d_{c,x_h,r}|)(E_{c,x_h,r,j}^\dagger \otimes |\gamma_{c,x_h,r}\rangle \langle \gamma_{c,x_h,r}| \otimes |h + d_{c,x_h,r}|)$$
Lemma 3.5. Configuration \( (x,c,h,r,j) \). By assumption, \( \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} E_{x,c,h,r,j} \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} Z_{x,c,h,r,j} \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} \Phi_{c,x,h,r,j}(A) \otimes B_{x,c,h,r} = \hat{Z}'.

Similarly, if \((c,h) \notin \hat{C} \times \hat{H}_x\) and \((r,j) \in R \times J\), then \(\tilde{E}_{x,c,h,r,j} \hat{Z} E_{x,c,h,r,j} = \emptyset_{CQ \otimes C^C \otimes C^{\hat{H}_x}}\). Therefore

\[
S_x(\hat{Z}) = \sum_{(r,j) \in R \times J} \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} \tilde{E}_{x,c,h,r,j} \hat{Z} E_{x,c,h,r,j} = \sum_{(r,j) \in R \times J} \left( E_{x,c,h,r,j} \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} Z_{x,c,h,r,j} \sum_{(c,h) \in \mathcal{C} \times \hat{H}_x} \Phi_{c,x,h,r,j}(A) \otimes B_{x,c,h,r} \right)
\]

Lemma 3.4. Using the notation of Definition 3.2, consider any \(x \in \Sigma^*\). For any \(Z \in \text{Den}(CQ \otimes C^C \otimes C^{\hat{H}_x})\), if \(\{(p_i,A_i,c_i,h_i) : i \in I\}\) is any ensemble of configurations described by \(Z\), then let \(\{(p'_k,(A'_k',c'_k,h'_k')) : k \in K\}\) denote the ensemble of configurations of \(N\) obtained by replacing each configuration \((A_i,c_i,h_i)\), where \((c_i,h_i) \in (\hat{C} \times \hat{H}_x)\), by the ensemble (scaled by \(p_i\)) of configurations obtained by running \(N\) for a single-step beginning in the configuration \((A_i,c_i,h_i)\), and leaving each configuration \((c_i,h_i) \notin (\hat{C} \times \hat{H}_x)\) unchanged. Then \(\{(p'_k,(A'_k',c'_k,h'_k')) : k \in K\}\) is described by \(S_x(Z)\).

Proof. By assumption, \(Z = \sum_{i \in I} (p_i A_i \otimes |c_i\rangle \langle c_i| \otimes |h_i\rangle \langle h_i|)\). Therefore,

\[
S_x(Z) = S_x \left( \sum_{i \in I} (p_i A_i \otimes |c_i\rangle \langle c_i| \otimes |h_i\rangle \langle h_i|) \right) = \sum_{i \in I} p_i S_x (A_i \otimes |c_i\rangle \langle c_i| \otimes |h_i\rangle \langle h_i|).
\]

For any \((A,c',h') \in \text{Den}(CQ) \times (C \times H_x)\), let \(Z' = A \otimes |c'\rangle \langle c'| \otimes |h'\rangle \langle h'|\). Then, by inspection, \(S_x(Z') = Z'\); that is to say, any configuration in which \(N\) has “finished computing” on \#Lx is left unchanged by \(S_x\). The claim then follows by Lemma 3.3 and linearity.

Lemma 3.5. Using the notation of Definition 3.2, \(S_x \in \text{Chan}(CQ \otimes C^C \otimes C^{\hat{H}_x})\), \(\forall x \in \Sigma^*\).

Proof. The family \(\{\tilde{E}_{x,c,h,r,j} : c \in C, h \in H_x, r \in R, j \in J\}\) is a Kraus representation of the operator \(S_x\) (see, for instance, [44, Section 2.2] for a formal definition); therefore, \(S_x \in \text{Chan}(CQ \otimes C^C \otimes C^{\hat{H}_x})\) if an only if \(\sum_{c,h,r,j} \tilde{E}_{x,c,h,r,j} \tilde{E}_{x,c,h,r,j} = 1_{CQ} \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|, \forall (c,h) \in C \times H_x\). First, suppose \((c,h) \in \hat{C} \times \hat{H}_x\); we then have

\[
\tilde{E}_{x,c,h,r,j} \tilde{E}_{x,c,h,r,j} = (E_{x,c,h,r,j} \otimes |c\rangle \langle c| \otimes |h\rangle \langle h| + d_{c,x,h,r}) (E_{x,c,h,r,j} \otimes |c\rangle \langle c| \otimes |h\rangle \langle h| + d_{c,x,h,r})
\]

This implies,

\[
\sum_{(r,j) \in R \times J} \tilde{E}_{x,c,h,r,j} \tilde{E}_{x,c,h,r,j} = \left( \sum_{(r,j) \in R \times J} E_{x,c,h,r,j} \right) \otimes |c\rangle \langle c| \otimes |h\rangle \langle h| = 1_{CQ} \otimes |c\rangle \langle c| \otimes |h\rangle \langle h|. \]
If, instead, \((c, h) \notin \hat{C} \times \hat{H}_x\), then
\[
\tilde{E}^\dagger_{x, c, h, r, j} E_{x, c, h, r, j} = \left( \frac{1}{\sqrt{|R||J|}} \mathbb{1}_C \otimes |c\rangle \langle h| \right)^\dagger \left( \frac{1}{\sqrt{|R||J|}} \mathbb{1}_C \otimes |c\rangle \langle h| \right) = \frac{1}{|R||J|} \mathbb{1}_C \otimes |c\rangle \langle h| \langle h|.
\]
This implies
\[
\sum_{(r, j) \in R \times J} \tilde{E}^\dagger_{x, c, h, r, j} E_{x, c, h, r, j} = \mathbb{1}_C \otimes |c\rangle \langle h| \langle h|.
\]
We then have
\[
\sum_{(c, h) \in C \times H_x} \sum_{(r, j) \in R \times J} \tilde{E}^\dagger_{x, c, h, r, j} E_{x, c, h, r, j} = \sum_{(c, h) \in C \times H_x} \left( \mathbb{1}_C \otimes |c\rangle \langle h| \langle h| \right) = \mathbb{1}_C \otimes C \otimes \mathbb{C}^{|H_x|}.
\]

We next define a truncation operator \(T_x\) that terminates all branches on which \(N\) has not yet “finished computing” on the prefix \(#_Lx\).

**Definition 3.6.** Consider a 2QCFA \(N = (Q, C, \Sigma, R, \theta, q_{\text{start}}, c_{\text{start}}, c_{\text{acc}}, c_{\text{rej}})\) and input prefix \(x \in \Sigma^*\).

(i) For each \(c \in C\) and each \(h \in H_x\), we define \(\hat{E}_{x, c, h} \in L(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|})\) as follows.
\[
\hat{E}_{x, c, h} = \begin{cases} 
\mathbb{1}_C \otimes |c_{\text{rej}}\rangle \langle \text{rej}| \langle h|, & \text{if } (c, h) \in \hat{C} \times \hat{H}_x \\
\mathbb{1}_C \otimes |c\rangle \langle h| \langle h|, & \text{otherwise.}
\end{cases}
\]

(ii) We then define the operator \(T_x \in T(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|})\) such that
\[
T_x(Z) = \sum_{(c, h) \in C \times H_x} \hat{E}_{x, c, h} Z \hat{E}^\dagger_{x, c, h}, \quad \forall Z \in L(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|}).
\]

**Lemma 3.7.** Using the notation of Definition 3.6, the following statements hold.

(i) For any \(Z \in \hat{\text{Den}}(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|})\), if \(\{(p_i, (A_i, c_i, h_i)) : i \in I\}\) is any ensemble of configurations described by \(Z\), then \(T_x(Z)\) describes an ensemble of configurations for which each configuration with \((c_i, h_i) \in \hat{C} \times \hat{H}_x\) is replaced by the configuration \((A_i, c_{\text{rej}}, h_i)\) and all other configurations are left unchanged. In other words, all configurations that correspond to the case in which \(N\) has “finished computing” on \(#_Lx\) are left unchanged, and all other configurations become rejecting configurations.

(ii) We have \(T_x \in \text{Chan}(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|})\).

**Proof.** (i) Immediate from definitions.

(ii) As in the proof of Lemma 3.5, we may straightforwardly show \(\sum_{c, h} \hat{E}^\dagger_{x, c, h} \hat{E}_{x, c, h} = \mathbb{1}_{\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|}}\), which implies \(T_x \in \text{Chan}(\mathbb{C}^Q \otimes C \otimes \mathbb{C}^{|H_x|})\) [44, Corollary 2.27].
We now formally define the notion of a \textit{m-truncated transfer operator} and of a \textit{m-truncated crossing sequence}. Firstly, given a 2QCFA $N$, we produce an equivalent $N'$ of a certain convenient form, in much the same way that Dwork and Stockmeyer \cite{Dwork:1983} converted a 2PFA to an equivalent 2PFA of a convenient form. The 2QCFA $N'$ is identical to $N$, except for the addition of two new classical states: $c'_{\text{start}}$ and $c'$, where $c'_{\text{start}}$ will be the classical start state of $N'$. On any input $w$, $N'$ will move its head continuously to the right until it reaches $#_R$, remaining in state $c'_{\text{start}}$ and performing the trivial transformation to its quantum register along the way. When the head reaches $#_R$, $N'$ will enter $c'$ and perform the trivial transformation to its quantum register; then, $N'$ will move its head continuously to the left until it reaches $#_L$, remaining in state $c'$ and performing the trivial transformation to its quantum register along the way. When the head reaches $#_L$, $N'$ will enter the original classical start state $c_{\text{start}}$ and perform the trivial transformation to its quantum register. After this point, $N'$ behaves identically to $N$. For the remainder of the paper, we assume all 2QCFA under consideration have this form.

**Definition 3.8.** Consider a 2QCFA $N = (Q, C, \Sigma, R, \theta, \delta, q_{\text{start}}, c_{\text{start}}, c_{\text{acc}}, c_{\text{rej}})$.

(i) For any $x \in \Sigma^*$, define $I_x \in T(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_1} \otimes \mathbb{C}^{H_2}, \mathbb{C}^Q \otimes \mathbb{C}^C)$ as in Section 3.1; also define $S_x, T_x \in T(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_1} \otimes \mathbb{C}^{H_2}, \mathbb{C}^Q \otimes \mathbb{C}^C)$ as in Definition 3.2 and Definition 3.6, respectively. For each $m \in \mathbb{N}$, we define the \textit{m-truncated transfer operator} $N_{x,m}^\triangledown = \text{Tr}_{C,H_x} \circ T_x \circ S_x^m \circ I_x \in T(\mathbb{C}^Q \otimes \mathbb{C}^C)$.

(ii) For any $y \in \Sigma^*$, we next consider the “dual case” of running $N$ on the suffix $y#_R$ beginning in some ensemble of configurations \{(p_i, (A_i, c_i, |x| + 1)) : i \in I\} (i.e., the head position of every configuration is over the leftmost symbol of $y#_R$). We define the notion of an m-truncated stopping ensemble, and all other notions, symmetrically. That is to say, a branch of $N$ “finishes computing” on $y#_R$ when it either “leaves” $y#_R$ (by moving its head left from the leftmost symbol of $y#_R$), or accepts or rejects the input, or runs for more than $m$ steps. We then define $N_{y,m}^\uparrow \in T(\mathbb{C}^Q \otimes \mathbb{C}^C)$ as the corresponding “dual” m-truncated transfer operator for $y$.

(iii) For any $x, y \in \Sigma^*$ and any $m \in \mathbb{N}$, we then define the \textit{m-truncated crossing sequence} of $N$ with respect to the (partitioned) input $xy$ to be the sequence $Z_1, Z_2, \ldots \in \overline{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C)$, defined as follows. The density operator $Z_1$ describes the ensemble consisting of the single configuration (of the quantum register and classical register) $(|q_{\text{start}}\rangle, c_{\text{start}}\rangle)$ that $N$ is in when it first crosses from $#_L x$ into $y#_R$, which is of this simple form due to the assumed form of $N$. The sequence $Z_1, Z_2, \ldots$ is then obtained by starting with $Z_1$ and alternately applying $N_{y,m}^\triangledown$ and $N_{x,m}^\triangledown$. To be precise,

$$Z_i = \begin{cases} |q_{\text{start}}\rangle \langle q_{\text{start}}| \otimes |c_{\text{start}}\rangle \langle c_{\text{start}}|, & i = 1 \\ N_{y,m}^\triangledown(Z_{i-1}), & i > 1, i \text{ is even} \\ N_{x,m}^\triangledown(Z_{i-1}), & i > 1, i \text{ is odd} \end{cases}$$

**Lemma 3.9.** Using the notation of Definition 3.8, the following statements hold.

(i) For any $Z \in \overline{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C)$, if $N$ is run on the prefix $#_L x$ beginning in any ensemble of configurations described by $I_x(Z)$ (i.e., the head position of every configuration is over the rightmost symbol of $#_L x$), then the m-truncated stopping ensemble is described by $N_{x,m}^\triangledown(Z)$. 

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(ii) Symmetrically, for any \( Z \in \widehat{\text{Den}}(\mathbb{C}^Q \otimes \mathbb{C}^C) \), if \( N \) is run on the suffix \( y\#_R \) beginning in any ensemble of configurations described by \( \overline{I}_y(Z) \) (i.e., the head position of every configuration is over the leftmost symbol of \( y\#_R \)), then the \( m \)-truncated stopping ensemble is described by \( N_{y,m}^\rightarrow(Z) \).

(iii) We have \( N_{x,m}^\rightarrow, N_{y,m}^\rightarrow \in \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C), \forall x, y \in \Sigma^*, \forall m \in \mathbb{N} \).

Proof. (i) Immediate by Definition 3.8(i), Lemma 3.4, and Lemma 3.7(i).

(ii) Immediate by Definition 3.8(ii), and analogous versions of Lemma 3.4, and Lemma 3.7(i).

(iii) By definition, \( N_{x,m}^\rightarrow = \text{Tr}_{C^H_x} \circ T_x \circ S_x^m \circ I_x \). By Lemma 3.5 and Lemma 3.7(ii), we have \( S_x, T_x \in \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x}) \) and \( \text{Tr}_{C^H_x} \in \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C \otimes \mathbb{C}^{H_x} \otimes \mathbb{C}^Q \otimes \mathbb{C}^C) \) and that the composition of quantum channels is a quantum channel (see, for instance, [44, Section 2.2]). The claim for \( N_{y,m}^\rightarrow \) follows by an analogous argument. 

Remark. Note that the \( \{Z_i\} \) that comprise a crossing sequence do not describe the ensemble of configurations of \( N \) at particular points in time during its computation on the input \( xy \); instead, \( Z_i \) describes the ensemble of configurations of the set of all the probabilistic branches of \( N \) at the \( i \)th time each branch crosses between \( \#_Lx \) and \( y\#_R \) (with the convention stated above of considering a branch that has accepting or rejected its input to “cross” in classic state \( c_{\text{acc}} \) or \( c_{\text{rej}} \), respectively, indefinitely; as well as the convention that if a given branch of \( N \) attempts to run for more than \( m \) steps within the prefix \( \#_Lx \) or within the suffix \( y\#_R \), that branch is interrupted and immediately forced to reject). Of course, a given branch may not cross between \( \#_Lx \) and \( y\#_R \) more than \( i \) times within the first \( i \) steps of the computation; this will allow us to use such crossing sequences to prove a lower bound on the expected running-time of \( N \).

Remark. Moreover, while the \( m \)-truncated crossing operator \( N_{x,m}^\rightarrow \) completely suffices for our analysis, one could also define a non-truncated transfer operator \( N_{x}^\rightarrow \in \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C) \) as an accumulation point of the sequence \( (N_{x,m}^\rightarrow)_{m \in \mathbb{N}} \); such an accumulation point exists due to the fact that \( \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C) \) is compact [44, Proposition 2.28]. Using \( N_{x}^\rightarrow \) and the symmetrically defined \( N_{y}^\rightarrow \), one could then define the non-truncated crossing sequence of \( N \) on \( xy \). The resulting analyses of these two types of crossing sequences would essentially be identical, and so we do not consider this definition further here; however, the (somewhat cleaner) non-truncated crossing sequence may be more useful in other applications.

4 Lower Bounds on the Running Time of 2QCFA

Dwork and Stockmeyer proved a lower bound [9, Lemma 4.3] on the expected running time \( T(n) \) of any 2PFA that recognizes any language \( L \) with bounded error, where the lower bound is in terms of their hardness measure \( D_L(n) \). We prove that an analogous claim holds for any 2QCFA. The preceding quantum generalization of a crossing sequence plays a key role in the proof, essentially taking the place of the Markov chains used both in the aforementioned result of Dwork and Stockmeyer and in the earlier result of Greenberg and Weiss [12] that showed that a 2PFA cannot recognize \( L_{eq} = \{a^mb^m : m \in \mathbb{N} \} \) with bounded error in subexponential time.
4.1 Nonregularity, Automaticity, and Similar Hardness Measures

For any language $L$, Dwork and Stockmeyer [9] defined a particular “hardness measure” $D_L : \mathbb{N} \to \mathbb{N}$, which they called the nonregularity of $L$. We begin by recalling this definition. Let $\Sigma$ be a finite alphabet, $L \subseteq \Sigma^*$ a language, and $n \in \mathbb{N}$. Let $\Sigma^{\leq n} = \{w \in \Sigma^* : |w| \leq n\}$ denote the set of all strings over $\Sigma$ of length at most $n$ and consider some $x, x' \in \Sigma^{\leq n}$. We say that $x$ and $x'$ are $(L, n)$-dissimilar, which we denote by $x \not\sim_{L,n} x'$, if $\exists y \in \Sigma^{\leq n}$, where $n' = n - \max(|x|, |x'|)$, such that $xy \in L \iff x'y \notin L$. Recall the classic Myhill-Nerode inequivalence relation, in which $x, x' \in \Sigma^*$ are $L$-dissimilar if $\exists y \in \Sigma^*$, such that $xy \in L \iff x'y \notin L$. Then $x, x' \in \Sigma^{\leq n}$ are $(L, n)$-dissimilar precisely when they are $L$-dissimilar, and the dissimilarity is witnessed by a “short” string $y$. We then define the function $D_L : \mathbb{N} \to \mathbb{N}$ such that $D_L(n)$ is the largest $h \in \mathbb{N}$ such that $\exists x_1, \ldots, x_h \in \Sigma^{\leq n}$ that are pairwise $(L, n)$-dissimilar (i.e., $x_i \not\sim_{L,n} x_j$, $\forall i, j$ with $i \neq j$).

In fact, the hardness measure $D_L$ of a language $L$ has been defined by many authors, both before and after Dwork and Stockmeyer, who gave many different names to $D_L$ and who (repeatedly) rediscovered certain basic facts about $D_L$; we refer the reader to the excellent paper of Shallit and Breitbart [35] for a detailed history of the study of $D_L$ and related hardness measures. In the remainder of this section, we briefly recall two crucial equivalent definitions of $D_L$, as well as the definition of a certain related (inequivalent) hardness measure, which we will need in order to prove our various lower bounds in their full generality.

For some DFA (one-way deterministic finite automaton) $M$, let $|M|$ denote the number of states of $M$ and let $L(M)$ denote the language of $M$ (i.e., the set of strings accepted by $M$). The earliest definition of a hardness measure equivalent to Dwork-Stockmeyer nonregularity was given by Karp [22], who defined $A_L(n) = \min\{|M| : M \text{ is a DFA and } L(M) \cap \Sigma^{\leq n} = L \cap \Sigma^{\leq n}\}$ to be the minimum number of states of a DFA that agrees with $L$ on all strings of length at most $n$; Shallit and Breitbart use the term deterministic automaticity to refer to $A_L$. For any language $L$, it is immediately obvious that $A_L(n) \geq D_L(n), \forall n$; somewhat less obviously, $A_L(n) = D_L(n), \forall n [21, 22, 35]$, and so the notions of nonregularity and deterministic automaticity coincide.

Consider a language $L \subseteq \Sigma^*$ and two communicating parties: Alice, who knows some string $x \in \Sigma^*$, and Bob, who knows some string $y \in \Sigma^*$. Alice sends some message $A(x) \in \{0, 1\}^*$ to Bob, after which Bob must be able to determine, using $A(x)$ and $y$, if the string $w = xy$ is in $L$. Let $C_L(n)$ denote the maximum, taken over all $x, y \in \Sigma^*$ such that $|xy| \leq n$, of the number of bits sent from Alice to Bob by the optimal such (deterministic one-way) protocol. This quantity, the one-way deterministic communication complexity of testing membership in $L$, is related to the nonregularity of $L$; in particular, $C_L(n) = \log D_L(n), \forall n [7]$.

Lastly, we recall the definition of a related (but inequivalent) hardness measure used by Ibarra and Ravikumar [20] in their study of non-uniform small-space DTMs (deterministic Turing machines). Let $\Sigma^n = \{w \in \Sigma^* : |w| = n\}$. We then consider 2DFA (two-way deterministic finite automata), and use the same notation as was used above for DFA. For a language $L$, define $A_{L,=}^{2\text{DFA}}(n) = \min\{|M| : M \text{ is a 2DFA and } L(M) \cap \Sigma^n = L \cap \Sigma^n\}$ to be the minimum number of states of a 2DFA that agrees with $L$ on all strings of length exactly $n$. Clearly, for any language $L$, $A_{L,=}^{2\text{DFA}}(n) \leq A_L(n), \forall n$. They then defined NUDSPACE($O(S(n))$) (non-uniform deterministic space $O(S(n))$) to be the class of languages $L$ such that $A_{L,=}^{2\text{DFA}}(n) = 2^{O(S(n))}$. Note that NUDSPACE($O(S(n))$) = DSPACE($O(S(n))$)/2^{O(S(n))}, the class of languages recognizable by a DTM that, on any input $w$, uses space $O(S(|w|))$, and has access to an “advice” string $y_{|w|}$, which depends only on the length $|w|$ of the input and is itself of length $|y_n| = 2^{O(S(n))}$. In particular, $L/{\text{poly}} := \text{DSPACE}(O(\log n))/2^{O(\log n)} = \text{NUDSPACE}(O(\log n)) = \{L : A_{L,=}^{2\text{DFA}}(n) = n^{O(1)}\}$. 

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4.2 A 2QCFA Analogue of the Dwork-Stockmeyer Lemma

Dwork and Stockmeyer showed that, if a 2PFA recognizes some language \( L \) with bounded error in expected time at most \( T(n) \) on all inputs of length at most \( n \), then \( \exists a \in \mathbb{R}_{>0} \) (that depends only on the number of states of the 2PFA), such that \( T(n) = \Omega \left( 2^{D_L(n)} \right) \) [9, Lemma 4.3]. We now prove that an analogous claim holds for 2QCFA: if a 2QCFA recognizes some language \( L \) with bounded error in expected time at most \( T(n) \) on all inputs of length at most \( n \), then \( \exists a \in \mathbb{R}_{>0} \) (that depends only on the number of states of the 2QCFA), such that \( T(n) = \Omega \left( D_L(n)^n \right) \). We note that, while there is an exponential difference between our lower bound for 2QCFA and the Dwork-Stockmeyer lower bound for 2PFA, both lower bounds are (asymptotically) tight.

The main idea is as follows. Suppose the 2QCFA \( \mathcal{N} \) recognizes \( L \subseteq \Sigma^* \), with two-sided bounded error \( \epsilon \in \mathbb{R}_{>0} \), in expected time at most \( T(n) \). We show that, if \( D_L(n) \) is “large,” then, for any \( m \in \mathbb{N} \), we can find \( x, x' \in \Sigma^* \) such that \( x \not\sim_{L,n} x' \) and the distance between the corresponding \( m \)-truncated transfer operators \( N_{x,m}^\Sigma \) and \( N_{x',m}^\Sigma \) is “small.” By definition, \( \exists y \in \Sigma^\leq n \), where \( n' = n - \max(|x|,|x'|) \), such that \( xy \in L \iff x'y \notin L \); note that \( xy, x'y \in \Sigma^\leq n \). Without loss of generality, we assume \( xy \in L \), and hence \( x'y \notin L \). We also show that, for \( m \) sufficiently large, if the distance between \( N_{x,m}^\Sigma \) and \( N_{x',m}^\Sigma \) is “small,” then the behavior of \( \mathcal{N} \) on the partitioned inputs \( xy \) and \( x'y \) will be similar; in particular, if \( T(n) \) is “small,” then \( \Pr[\mathcal{N} \text{ accepts } xy] \approx \Pr[\mathcal{N} \text{ accepts } x'y] \). However, as \( xy \in L \), we must have \( \Pr[\mathcal{N} \text{ accepts } xy] \geq 1 - \epsilon \), and as \( x'y \notin L \), we must have \( \Pr[\mathcal{N} \text{ accepts } x'y] \leq \epsilon \), which is impossible. This contradiction allows us to establish a lower bound on \( T(n) \) in terms of \( D_L(n) \). In this section, we formalize this idea.

For \( p \in \mathbb{N}_{\geq 1} \), we define the Schatten \( p \)-norm \( \|\cdot\|_p : L(V) \to \mathbb{R}_{>0} \), where \( \|Z\|_p = (\text{Tr}((Z^\dagger Z)^{\frac{p}{2}}))^{\frac{2}{p}} \), \( \forall Z \in L(V) \). Observe that the Schatten \( p \)-norm is indeed a norm, for every \( p \). We also use the term trace norm to refer to the Schatten 1-norm. We define the induced trace norm \( \|\cdot\|_1 : T(V,V') \to \mathbb{R}_{>0} \), where \( \|\Phi\|_1 = \sup\{\|\Phi(Z)\|_1 : Z \in L(V), \|Z\|_1 \leq 1\} \), for any \( \Phi \in T(V,V') \). Observe that the induced trace norm is also a norm.

For density operators \( Z, Z' \in L(C^Q \otimes C^C) \), we use \( \|Z - Z'\|_1 \), the distance metric induced by the trace norm, to measure the distance between \( Z \) and \( Z' \). For \( x, x' \in \Sigma^* \) and \( m \in \mathbb{N} \), we use \( \|N_{x,m}^\Sigma - N_{x',m}^\Sigma\|_1 \), the distance metric induced by the induced trace norm, to measure the distance between \( N_{x,m}^\Sigma \) and \( N_{x',m}^\Sigma \). Suppose \( \mathcal{N} \) is run on two distinct partitioned inputs \( xy \) and \( x'y \), producing two distinct \( m \)-truncated crossing sequences, following Definition 3.8(iii). We first show that if \( \|N_{x,m}^\Sigma - N_{x',m}^\Sigma\|_1 \) is “small”, then these crossing sequences are similar.

**Lemma 4.1.** Consider a 2QCFA \( \mathcal{N} \) with quantum basis states \( Q \), classical states \( C \), and input alphabet \( \Sigma \). For \( x, x', y \in \Sigma^* \) and \( m \in \mathbb{N} \), let \( Z_1, Z_2, \ldots \in \text{Den}(C^Q \otimes C^C) \) (resp. \( Z'_1, Z'_2, \ldots \in \text{Den}(C^Q \otimes C^C) \)) denote the \( m \)-truncated crossing sequence obtained when \( \mathcal{N} \) is run on \( xy \) (resp. \( x'y \)). Then \( \|Z_i - Z'_i\|_1 \leq \left( \frac{1}{2^m} \right) \|N_{x,m}^\Sigma - N_{x',m}^\Sigma\|_1 \), \( \forall i \in \mathbb{N}_{\geq 1} \).

**Proof.** Note that \( \|\Phi(Z)\|_1 \leq \|Z\|_1 \), \( \forall Z \in L(C^Q \otimes C^C) \), \( \forall \Phi \in \text{Chan}(C^Q \otimes C^C) \) [44, Corollary 3.40]. Therefore, for any \( \Phi \in \text{Chan}(C^Q \otimes C^C) \) and any \( Z, Z' \in L(C^Q \otimes C^C) \), we have
\[
\|\Phi(Z) - \Phi(Z')\|_1 = \|\Phi(Z - Z')\|_1 \leq \|Z - Z'\|_1.
\]

That is to say, the distance metric on \( L(C^Q \otimes C^C) \) induced by the trace norm is **contractive** under any map \( \Phi \in \text{Chan}(C^Q \otimes C^C) \). By Lemma 3.9(iii), \( N_{x,m}^\Sigma, N_{x',m}^\Sigma, N_{y,m}^\Sigma \in \text{Chan}(C^Q \otimes C^C) \).

By definition, \( Z_1 = \rho_{\text{start}} \otimes \rho_{\text{start}} \) and \( Z'_1 = Z_1 \), and so \( \|Z_1 - Z'_1\|_1 = 0 \). For \( i \) even, we have, by definition, \( Z_i = N_{y,m}^\Sigma(Z_{i-1}) \) and \( Z'_i = N_{y,m}^\Sigma(Z'_{i-1}) \). By the above observation concerning
the contractivity of the distance metric, we then have

\[ \|Z_i - Z'_i\|_1 = \|N_{x,m}(Z_{i-1}) - N_{x,m}(Z'_{i-1})\|_1 \leq \|Z_{i-1} - Z'_{i-1}\|_1, \text{ if } i \text{ is even.} \]

For \( i \) odd, with \( i > 1 \), we have, by definition \( Z_i = N_{x,m}(Z_{i-1}) \) and \( Z'_i = N_{x',m}(Z'_{i-1}) \). Note that, for any \( Z \in \text{Den}(C^Q \otimes C^C) \), we have \( \|Z\|_1 = 1 \), which implies \( \|\Phi(Z)\|_1 \leq \|\Phi\|_1, \forall \Phi \in T(C^Q \otimes C^C) \); of course, \( N_{x,m} - N_{x',m} \in T(C^Q \otimes C^C) \). By this observation and the earlier observation concerning the contractivity of the distance metric, we have

\[ \|Z_i - Z'_i\|_1 = \|N_{x,m}(Z_{i-1}) - N_{x',m}(Z'_{i-1})\|_1 \leq \|N_{x,m}(Z_{i-1}) - N_{x,m}(Z'_{i-1})\|_1 + \|N_{x,m}(Z'_{i-1}) - N_{x',m}(Z'_{i-1})\|_1 \]

\[ = \|N_{x,m}(Z_{i-1} - Z'_{i-1})\|_1 + \|N_{x,m} - N_{x',m}\|_1 \leq \|Z_{i-1} - Z'_{i-1}\|_1 + \|N_{x,m} - N_{x',m}\|_1, \text{ if } i \text{ odd, } i > 1. \]

The claim then follows by induction on \( i \in \mathbb{N}_1 \).

**Lemma 4.2.** Consider a language \( L \) over some finite alphabet \( \Sigma \). Suppose \( L \in \text{B2QCFA}(k,d,T(n),\epsilon) \), for some \( k,d \in \mathbb{N}_{\geq 2}, T : \mathbb{N} \rightarrow \mathbb{N}, \) and \( \epsilon \in (0, \frac{1}{2}) \). If, for some \( n \in \mathbb{N} \), \( \exists x,x' \in \Sigma^n \) such that \( x \neq_{L,n} x' \), then \( T(n) \geq \frac{(1 - 2\epsilon)^2}{2} \|N_{x,m} - N_{x',m}\|^{-1}_1, \forall m \geq \left\lceil \frac{1}{2\epsilon} T(n) \right\rceil \).

**Proof.** By definition, \( x \neq_{L,n} x' \) precisely when \( \exists y \in \Sigma^* \) such that \( xy, x'y \in \Sigma^n \), and \( xy \in L \iff x'y \notin L \). Fix such a \( y \), and assume, without loss of generality, that \( xy \in L \) (and hence \( x'y \notin L \)). For \( m \in \mathbb{N} \), suppose that, when \( N \) is run on the partitioned input \( xy \) (resp. \( x'y \)), we obtain the \( m \)-truncated crossing sequence \( Z_{m,1}, Z_{m,2}, \ldots \in \text{Den}(C^Q \otimes C^C) \) (resp. \( Z'_{m,1}, Z'_{m,2}, \ldots \in \text{Den}(C^Q \otimes C^C) \)). For \( c \in C \), let \( E_c = \mathbb{1}_{C^Q} \otimes |c\rangle\langle c| \in L(C^Q \otimes C^C) \). For \( s \in \mathbb{N}_{\geq 1} \), define \( p_{m,s}, p'_{m,s} : C \rightarrow [0,1] \) such that \( p_{m,s}(c) = \text{Tr}(E_c Z_{m,s} E_c^\dagger) \) and \( p'_{m,s}(c) = \text{Tr}(E_c Z'_{m,s} E_c^\dagger) \). Then, for any \( c \in C \),

\[ |p_{m,s}(c) - p'_{m,s}(c)| = |\text{Tr}(E_c Z_{m,s} E_c^\dagger) - \text{Tr}(E_c Z'_{m,s} E_c^\dagger)| = |\text{Tr}(E_c (Z_{m,s} - Z'_{m,s}) E_c^\dagger)| \leq \|Z_{m,s} - Z'_{m,s}\|_1. \]

By Lemma 4.1, \( \|Z_{m,s} - Z'_{m,s}\|_1 \leq \underline{s}^{-1} \frac{1}{2} \|N_{x,m} - N_{x',m}\|_1, \forall s \in \mathbb{N}_{\geq 1} \), and so we conclude

\[ |p_{m,s}(c) - p'_{m,s}(c)| \leq \frac{s - 1}{2} \|N_{x,m} - N_{x',m}\|_1. \]

Notice that \( p_{m,s}(c_{\text{acc}}) \) (resp. \( p'_{m,s}(c_{\text{acc}}) \)) is the probability that \( N \) accepts \( xy \) (resp. \( x'y \)) within the first \( s \) times (on a given branch of the computation) the head of \( N \) crosses the boundary between \( x \) (resp. \( x' \)) and \( y \), where any branch that runs for more than \( m \) steps between consecutive boundary crossings is forced to halt and reject immediately before attempting to perform the \( m + 1 \)st such step. Let \( p_N(w) \) denote the probability that \( N \) accepts an input \( w \in \Sigma^* \), let \( p_N(w,s) \) denote the probability that \( N \) accepts \( w \) within \( s \) steps, and let \( h_N(w,s) \) denote the probability that \( N \) halts on input \( w \) within \( s \) steps.

Note that \( x'y \notin L \) implies \( p_N(x'y) \leq \epsilon \). Clearly, \( p'_{m,s}(c_{\text{acc}}) \leq p_N(x'y) \), for any \( m \) and \( s \), as all branches that attempt to perform more than \( m \) steps (between consecutive crossings) are considered to reject the input in the \( m \)-truncated crossing sequence. Suppose \( s \leq m \). Any branch that runs for a total of at most \( s \) steps before halting is unaffected by \( m \)-truncation. Moreover, if a branch halts (and accepts) within \( s \) steps, it will certainly halt (and accept) within \( s \) crossings between \( \#_Lx \) and \( y\#_R \). This implies \( p_N(xy,s) \leq p_{m,s}(c_{\text{acc}}) \). Therefore, if \( s \leq m \),

\[ p_N(xy,s) \leq p_{m,s}(c_{\text{acc}}) \leq p'_{m,s}(c_{\text{acc}}) + |p_{m,s}(c_{\text{acc}}) - p'_{m,s}(c_{\text{acc}})| \leq \epsilon + \frac{s - 1}{2} \|N_{x,m} - N_{x',m}\|_1. \]
By definition, the expected running time of $N$ on input $xy$ is at most $T(|xy|)$; therefore, by Markov’s inequality, $1 - h_N(xy, s) \leq \frac{T(|xy|)}{s}$. Note that $xy \in L$ implies $p_N(xy) \geq 1 - \epsilon$. Therefore, for any $m \geq s \geq 1$, we have

$$1 - \epsilon \leq p_N(xy) \leq p_N(xy, s) + (1 - h_N(xy, s)) \leq \epsilon + \frac{s - 1}{2} \left| N_{x,m} - N_{x',m} \right| + \frac{T(|xy|)}{s}.$$ 

Set $s = \left\lceil \frac{2}{1 - 2\epsilon} T(n) \right\rceil$, and notice that $|xy| \leq n$ implies $T(|xy|) \leq T(n)$. For any $m \geq s$, we then have

$$1 - 2\epsilon \leq \left[ \frac{2}{1 - 2\epsilon} T(n) \right] - 1 \leq \frac{2}{1 - 2\epsilon} \left| N_{x,m} - N_{x',m} \right| + \frac{T(n)}{1 - 2\epsilon} \left| N_{x,m} - N_{x',m} \right| + 1 - 2\epsilon.$$

Therefore,

$$T(n) \geq \frac{(1 - 2\epsilon)^2}{2} \left| N_{x,m} - N_{x',m} \right|^{-1}, \quad \forall m \geq \left[ \frac{2}{1 - 2\epsilon} T(n) \right]$$

The following lemma shows that any “large” set of input prefixes contains a pair of input prefixes $x, x'$ such that $N_{x,m}^{\subset}$ and $N_{x',m}^{\subset}$ are at “small” distance from one another, for all $m \in \mathbb{N}$.

**Lemma 4.3.** Consider a 2QCFA $N$ with quantum basis states $Q$, classical states $C$, and input alphabet $\Sigma$; let $k = |Q|$ and $d = |C|$. Further, consider any finite $X \subseteq \Sigma^*$ such that $|X| \geq 2$. Then $\forall m \in \mathbb{N}, \exists x, x' \in X$ such that $x \neq x'$ and $\left| N_{x,m}^{\subset} - N_{x',m}^{\subset} \right| < 4\sqrt{2}k^2d^2 \left( |X|^\frac{1}{4} - 1 \right)^{-1}$.

**Proof.** For $q, q' \in Q$ and $c, c' \in C$, let $F_{q,q',c,c'} = |q\rangle\langle q'| \otimes |c\rangle\langle c'| \in L(\mathbb{C}Q \otimes \mathbb{C}C)$. Let $J : T(\mathbb{C}Q \otimes \mathbb{C}C) \to L(\mathbb{C}Q \otimes \mathbb{C}C \otimes \mathbb{C}Q \otimes \mathbb{C}C)$ denote the Choi isomorphism, which is given by

$$J(\Phi) = \sum_{(q,q',c,c') \in Q^2 \times C^2} F_{q,q',c,c'} \otimes \Phi(F_{q,q',c,c'}), \quad \forall \Phi \in T(\mathbb{C}Q \otimes \mathbb{C}C).$$

Consider any $x \in \Sigma^*$ and $m \in \mathbb{N}$. We will show that $J(N_{x,m}^{\subset})$ is of a special form. We first show that, if $(c_1, c_2) \neq (c'_1, c'_2)$, then $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle = 0$ (where $\langle q_2 c_2 |$ denotes $\langle q_2 | \otimes |c_2\rangle$). To see this, recall that, by Definition 3.8(i), $N_{x,m}^{\subset} = T_{C,m} \circ T_{x} \circ S_{x}^{m} \circ I_{x}$. If $c_1 \neq c'_1$, then, by inspection, $N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) = 0_{\mathbb{C}Q \otimes \mathbb{C}C}$, which implies $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle = 0$. If $c_2 \neq c'_2$, then $\forall Z \in L(\mathbb{C}Q \otimes \mathbb{C}C \otimes \mathbb{C}H_z)$, $\langle q_2 c_2 | T_{C,m} \circ T_{x} \circ S_{x}^{m} \circ I_{x} | q_2' c_2' \rangle = 0$, which implies $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle = 0$.

By Lemma 3.9(iii), $N_{x,m}^{\subset} \in \text{Chan}(\mathbb{C}Q \otimes \mathbb{C}C)$, which implies $J(N_{x,m}^{\subset}) \in \text{Pos}(\mathbb{C}Q \otimes \mathbb{C}C \otimes \mathbb{C}Q \otimes \mathbb{C}C)$ [44, Corollary 2.27]. Therefore, for any $q_1, q_2, q'_1, q'_2 \in Q$ and for any $c_1, c_2, c'_1, c'_2 \in C$, we have $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle = \langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2 c_2 \rangle$ and $\langle q_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2 \rangle \in \mathbb{R}$.

Therefore, for any $m \in \mathbb{N}$ and any $x \in \Sigma^*$, $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle$ is only potentially non-zero at the $k^2d^2$ elements where $(c_1, c_2) = (c'_1, c'_2)$; moreover, the elements where $(q_1, q_2) \neq (q'_1, q'_2)$ come in conjugate pairs. We define the function $g_{N,m} : \Sigma^* \to \mathbb{R}^{k^2d^2}$ such that $g_{N,m}(x)$ encodes all the potentially non-zero $\langle q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2' c_2' \rangle$, without redundancy (only encoding one element of a conjugate pair). To be precise, the first $k^2d^2$ entries of $g_{N,m}(x)$ are given by $\{(q_2 c_2 | N_{x,m}^{\subset}(F_{q_1,q'_1,c_1,c'_1}) | q_2 c_2) : q_1, q_2 \in Q, c_1, c_2 \in C \} \subseteq \mathbb{R}$. Establish some total order $\geq$ on $Q$, and let $Q^2 = \{(q_1, q'_1, q_2, q'_2) \in Q^4 : q'_1 > q_1 \text{ or } (q'_1 = q_1 \text{ and } q'_2 > q_2)\}$. The remaining $k^4d^2 - k^2d^2$ entries are given by encoding each of the $\frac{1}{2}(k^4d^2 - k^2d^2)$ potentially non-zero entries
\(\{q_2c_2|N_{x,m}^\circ(F_{q_1,q_2,c_1,c_2})|q_2c_2\} : (q_1,q_2,q_\tilde{1},q_\tilde{2}) \in \tilde{Q}^4, c_1,c_2 \in C\} \subseteq \mathbb{C} \) as the pair of real numbers that comprise their real and imaginary parts.

Let \(h = k^4d^2\). In the following, in addition to the notation for the norms of operators established earlier in this section, we write \(\|\| : \mathbb{R}^h \to \mathbb{R}_{\geq 0}\) to denote the usual Euclidean 2-norm on \(\mathbb{R}^h\). It is straightforward to show that \(\|\Phi\|_1 \leq \|\Phi(J)\|_1, \forall \Phi \in T(\mathbb{C}^Q \otimes \mathbb{C}^C)\) (see, for instance [44, Section 3.4] or [29, Remark 4]). Therefore, for any \(x,x' \in \Sigma^*\), we have

\[
\|N_{x,m}^\circ - N_{x',m}^\circ\|_1 \leq \|J(N_{x,m}^\circ - N_{x',m}^\circ)\|_1 \leq \sqrt{\text{rank}(J(N_{x,m}^\circ - N_{x',m}^\circ))}\|J(N_{x,m}^\circ - N_{x',m}^\circ)\|_2 \\
\leq \sqrt{h}\|J(N_{x,m}^\circ) - J(N_{x',m}^\circ)\|_2 \leq \sqrt{2h}\|g_{N,m}(x) - g_{N,m}(x')\|.
\]

Note that \(N_{x,m}^\circ \in \text{Chan}(\mathbb{C}^Q \otimes \mathbb{C}^C)\), which implies \(\|N_{x,m}^\circ\|_1 = 1\) [44, Corollary 3.40]. Then, \(\forall q,q' \in Q, \forall c \in C\), we have \(\|F_{q,q',c,c}\|_1 = 1\), which implies \(\|N_{x,m}^\circ(F_{q,q',c,c})\|_1 \leq 1\). Therefore, for any \(x \in \Sigma^*\), we have

\[
g_{N,m}(x) \leq \|J(N_{x,m}^\circ)\|_2 \leq \|J(N_{x,m}^\circ)\|_1 \leq \sum_{q,q' \in Q, c \in C} \|N_{x,m}^\circ(F_{q,q',c,c})\|_1 \leq \sum_{q,q' \in Q, c \in C} 1 = k^2d = \sqrt{h}.
\]

To complete the proof, for \(v_0 \in \mathbb{R}^h\) and \(r \in \mathbb{R}_{>0}\), let \(B(v_0,r) = \{v \in \mathbb{R}^h : \|v_0 - v\| \leq r\}\) denote the closed ball centered at \(v_0\) of radius \(r\) in \(\mathbb{R}^h\). There is a constant \(c_h \in \mathbb{R}_{>0}\) such that \(B(v_0,r)\) has volume \(\text{vol}(B(v_0,r)) = c_hr^h\). For any \(x \in \Sigma^*\), \(\|g_{N,m}(x)\| \leq \sqrt{h}\), which implies that, for any \(\delta \in \mathbb{R}_{>0}\), \(B(g_{N,m}(x),\delta)\) is contained in \(B(0,\sqrt{h} + \delta)\). Suppose \(\forall x,x' \in X\) with \(x \neq x'\), we have \(B(g_{N,m}(x),\delta) \cap B(g_{N,m}(x'),\delta) = \emptyset\). Then \(\cup_{x \in X} B(g_{N,m}(x),\delta) \subseteq B(0,\sqrt{h} + \delta)\), which implies \(|X|c_h\delta^h \leq c_h(\sqrt{h} + \delta)^h\). Set \(\delta = \frac{2\sqrt{h}}{|X|^{1/h} - 1}\). Then \(\exists x,x' \in X\), with \(x \neq x'\), such that \(B(g_{N,m}(x),\delta) \cap B(g_{N,m}(x'),\delta) \neq \emptyset\), which implies \(\|g_{N,m}(x) - g_{N,m}(x')\| \leq 2\delta\). Therefore,

\[
\|N_{x,m}^\circ - N_{x',m}^\circ\|_1 \leq \sqrt{2h}\|g_{N,m}(x) - g_{N,m}(x')\| \leq \sqrt{2h}\frac{4\sqrt{h}}{|X|^{1/h} - 1} \leq 4\sqrt{2k^4d^2}\left(|X|\frac{1}{\epsilon^{4d^2}} - 1\right)^{-1}.
\]

We now prove the main technical result of this section: a 2QCFA analogue of the Dwork-Stockmeyer lemma [9, Lemma 4.3].

**Theorem 4.4.** If \(L \in \text{B2QCFA}(k,d,T(n),\epsilon)\), for some \(k,d \in \mathbb{N}_{\geq 2}, T : \mathbb{N} \to \mathbb{N}\), and \(\epsilon \in [0,\frac{1}{2})\), then \(\exists N_0 \in \mathbb{N}\) such that \(T(n) \geq \frac{(1-2\epsilon)^2}{16\sqrt{2k^4d^2}}D_L(n)^{\frac{1}{4d^2\epsilon}}\), \(\forall n \geq N_0\).

**Proof.** Consider some language \(L\) over some finite alphabet \(\Sigma\). By [9, Lemma 3.1], \(L \in \text{REG}\) if and only if \(\exists b \in \mathbb{N}_{\geq 1}\) such that \(D_L(n) \leq b\), \(\forall n \in \mathbb{N}\). Therefore, if \(L \in \text{REG}\), the claim is immediate (recall that \(T(n) \geq n\)); for the remainder of the proof, we assume \(L \notin \text{REG}\).

Suppose \(L \in \text{B2QCFA}(k,d,T(n),\epsilon)\). For each \(n \in \mathbb{N}\), we define \(X_n = \{x_1, \ldots, x_{D_L(n)}\} \subseteq \Sigma^n\) such that the \(x_i\) are pairwise \((L,n)\)-dissimilar. As \(D_L(n)\) is not bounded above by any constant, \(\exists N_0 \in \mathbb{N}\) such that \(D_L(N_0) \geq 2k^4d^2\). Then, \(\forall n \geq N_0\), we have \(|X_n| = D_L(n) \geq D_L(N_0) \geq 2k^4d^2\). Fix \(n \geq N_0\) and set \(m = \left\lceil \frac{1-2\epsilon}{2\epsilon}T(n) \right\rceil\). By Lemma 4.3, \(\exists x,x' \in X_n\) such that \(x \neq x'\) and

\[
\|N_{x,m}^\circ - N_{x',m}^\circ\|_1 \leq 4\sqrt{2k^4d^2}\left(|X_n|\frac{1}{\epsilon^{4d^2}} - 1\right)^{-1} \leq 8\sqrt{2k^4d^2}|X_n|^{-\frac{1}{4d^2\epsilon}} = 8\sqrt{2k^4d^2}D_L(n)^{-\frac{1}{4d^2\epsilon}}.
\]

Fix such a pair \(x,x'\), and note that \(x \not\sim_{L,n} x'\), by construction. By Lemma 4.2,

\[
T(n) \geq \frac{(1-2\epsilon)^2}{2}\|N_{x,m}^\circ - N_{x',m}^\circ\|_1^{-1} \geq \frac{(1-2\epsilon)^2}{16\sqrt{2k^4d^2}}D_L(n)^{\frac{1}{4d^2\epsilon}}.
\]

\(\square\)
4.3 2QCFA Running Time Lower Bounds and Complexity Class Separations

Theorem 4.4 has several significant implications on the power of 2QCFA. To allow us to properly state our results, as well as to better enable us to discuss existing results, we now define a collection of complexity classes that capture the power of 2QCFA with particular resource bounds. We first define $B2QCFA(T(n)) = \bigcup_{k,d \in \mathbb{N}, \epsilon \in [0,1]} B2QCFA(k, d, T(n), \epsilon)$ to be the class of languages recognizable with two-sided bounded error by a 2QCFA with any constant number of quantum basis states and any constant number of classical states, in expected time at most $T(n)$ on all inputs of length at most $n$. We use the standard big O, little o, $\Omega$, etc. notation to denote the asymptotic behavior of functions. For a family $T$ of functions of the form $T : \mathbb{N} \to \mathbb{N}$, let $B2QCFA(T) = \bigcup_{T \in \mathcal{T}} B2QCFA(T(n))$. We then write, for example, $B2QCFA(2^{o(n)})$ to denote the union, taken over every function $T : \mathbb{N} \to \mathbb{N}$ such that $T(n) = 2^{o(n)}$, of $B2QCFA(T(n))$. See Section 4.1 for the definition of $D_L$ and related hardness measures. We immediately obtain the following corollaries of Theorem 4.4.

**Corollary 4.4.1.** If $L \in B2QCFA(T(n))$, then $D_L(n) = T(n)^{O(1)}$ and $C_L(n) = O(\log T(n))$.

**Corollary 4.4.2.** If a language $L$ satisfies $D_L(n) = 2^{\Omega(n)}$, then $L \notin B2QCFA(2^{o(n)})$.

Notice that $D_L(n) = 2^{O(n)}$, for any language $L$. We next exhibit a language for which $D_L(n) = 2^{\Omega(n)}$, thereby yielding a strong lower bound on the running time of any 2QCFA that recognizes $L$. For $w = w_1 \cdots w_n \in \Sigma^*$, where each $w_i \in \Sigma$, let $w^\text{rev} = w_n \cdots w_1$ denote the reversal of the string $w$. We consider the language $L_{pal} = \{ w \in \{a,b\}^* : w = w^\text{rev} \}$ consisting of all palindromes over the alphabet $\{a,b\}$.

**Corollary 4.4.3.** $L_{pal} \notin B2QCFA(2^{o(n)})$.

**Proof.** For each $n \in \mathbb{N}$, let $W_n = \{ w \in \{a,b\}^* : |w| = n \}$ denote all words over the alphabet $\{a,b\}$ of length $n$. For any $w, w' \in W_n$, with $w \neq w'$, we have $|ww^\text{rev}| = 2n = |w'w^\text{rev}|$, $ww^\text{rev} \in L_{pal}$, and $w'w^\text{rev} \notin L_{pal}$; therefore, by definition, $w \neq L_{pal}, 2n$, $w', \forall w, w' \in W_n$ such that $w \neq w'$. This implies that $D_{L_{pal}}(2n) \geq |W_n| = 2^n$. Corollary 4.4.2 then implies $L_{pal} \notin B2QCFA(T(n))$. \hfill $\square$

We define $BQE2QCFA = B2QCFA(2^{O(n)})$ to be the class of languages recognizable with two-sided bounded error in expected exponential time (with linear exponent) by a 2QCFA. Next, we say that a 2QCFA $N$ recognizes a language $L$ with negative one-sided bounded error $\epsilon \in \mathbb{R}_{>0}$ if, $\forall w \in L$, $\Pr[N \text{ accepts } w] = 1$, and, $\forall w \notin L$, $\Pr[N \text{ accepts } w] \leq \epsilon$. We define $\text{coR2QCFA}(k, d, T(n), \epsilon)$ as the class of languages recognizable with negative one-sided bounded error $\epsilon$ by a 2QCFA, with at most $k$ quantum basis states and at most $d$ classical states, that has expected running time at most $T(n)$ on all inputs of length at most $n$. We define $\text{coR2QCFA}(T(n))$ and $\text{coRQE2QCFA}$ analogously to the two-sided bounded error case.

Ambainis and Watrous [2] showed that $L_{pal} \in \text{coRQE2QCFA}$; in fact, their 2QCFA recognizer for $L_{pal}$ has only a single-qubit (i.e., $k = 2$ quantum basis states). Clearly, $\text{coR2QCFA}(T(n)) \subseteq B2QCFA(T(n))$, for any $T$, and $\text{coRQE2QCFA} \subseteq BQE2QCFA$. Therefore, the class of languages recognizable by a 2QCFA with bounded error in expected subexponential time is properly contained in the class of languages recognizable by a 2QCFA in expected exponential time, as formalized by the following corollary.

**Corollary 4.4.4.** We have $B2QCFA(2^{o(n)}) \subseteq BQE2QCFA$ and $\text{coR2QCFA}(2^{o(n)}) \subseteq \text{coRQE2QCFA}$.

We next define $BQP2QCFA = B2QCFA(n^{O(1)})$ to be the class of languages recognizable with two-sided bounded error in expected polynomial time by a 2QCFA.
Corollary 4.4.5. If $L \in \text{BQP2QCFA}$, then $D_L(n) = n^{O(1)}$ and $C_L(n) = O(\log n)$. Therefore, BQP2QCFA $\subseteq L/poly$.

Proof. The first statement is a special case of Corollary 4.4.1. To see that BQP2QCFA $\subseteq L/poly$, recall that, as noted in Section 4.1, $L/poly = \{ L : A_{L=\sharp}^{2DF}(n) = n^{O(1)} \}$; clearly, for any language $L$ and any $n \in \mathbb{N}$, $A_{L=\sharp}^{2DF}(n) \leq A_L(n) = D_L(n)$.

Of course, there are many languages $L$ for which one can establish a strong lower bound on $D_L(n)$, and thereby establish a strong lower bound on the expected running time $T(n)$ of any 2QCFA that recognizes $L$. In Section 6, we consider the case in which $L$ is the word problem of a group, and we show that very strong lower bounds can be established on $D_L(n)$. In the current section, we consider two especially interesting languages; the relevance of these languages was brought to our attention by Richard Lipton (personal communication). For a number $p \in \mathbb{N}$, let $\langle p \rangle_2 \in \{0,1\}^*$ denote the binary representation of $p$; let $L_{\text{primes}} = \{ \langle p \rangle_2 : p \text{ is prime} \}$. Note that $D_{L_{\text{primes}}}(n) = 2^{\Omega(n)}$ [34], which immediately implies the following.

Corollary 4.4.6. $L_{\text{primes}} \not\subseteq \text{B2QCFA}(2^{o(n)})$.

Say a string $w = w_1 \cdots w_n \in \{0,1\}^n$ has a length-3 arithmetic progression (3AP) if $\exists i,j,k \in \mathbb{N}$ such that $1 \leq i < j < k \leq n$, $j - i = k - j$, and $w_i = w_j = w_k = 1$; let $L_{\text{3ap}} = \{ w \in \{0,1\}^* : w \text{ has a 3AP} \}$. It is straightforward to show the lower bound $D_{L_{\text{3ap}}}(n) = 2^{n^{1-o(1)}}$, as well as the upper bound $D_{L_{\text{3ap}}}(n) = 2^{n^{\Omega(1)}}$. Therefore, one obtains the following lower bound on the running time of a 2QCFA that recognizes $L_{\text{3ap}}$, which, while still quite strong, is not as strong as that of $L_{\text{pal}}$ or $L_{\text{primes}}$.

Corollary 4.4.7. $L_{\text{3ap}} \not\subseteq \text{B2QCFA}\left(2^{n^{1-O(1)}}\right)$.

Remark. While $L_{\text{primes}}$ and $L_{\text{3ap}}$ provide two more examples of natural languages for which our method yields strong lower bound on the running time of any 2QCFA recognizer, they also suggest the potential of proving a stronger lower bound for certain languages. That is to say, for $L_{\text{pal}}$, one has (essentially) matching lower and upper bounds on the running time of any 2QCFA recognizer; this is certainly not the case for $L_{\text{primes}}$ and $L_{\text{3ap}}$. In fact, we currently do not know if either $L_{\text{primes}}$ or $L_{\text{3ap}}$ can be recognized by a 2QCFA with bounded error at all (i.e., regardless of time bound).

4.4 Transition Amplitudes of 2QCFA

As in Definition 3.1(i), for some 2QCFA $N = (Q, C, \Sigma, R, \theta, \delta, q_{\text{start}}, c_{\text{start}}, c_{\text{acc}}, c_{\text{rej}})$, let $\{ E_{c,\sigma,r,j} : r \in R, j \in J \} \subseteq L(\mathbb{C}^{Q})$ denote the set of operators that describe the selective quantum operation $\theta(c, \sigma) \in \text{QuantOp}(\mathbb{C}^{Q}, R)$ that is applied to the quantum register when the classical state of $N$ is $c \in \hat{C}$ and the head of $N$ is over the symbol $\sigma \in \Sigma_+$. The transition amplitudes of $N$ are the set of numbers $\{ \langle q \mid E_{c,\sigma,r,j}, q' \rangle : c \in \hat{C}, \sigma \in \Sigma_+, r \in R, j \in J, q, q' \in Q \} \subseteq \mathbb{C}$.

While other types of finite automata are often defined without any restriction on their transition amplitudes, for 2QCFA, and other types of QFA, the allowed class of transition amplitudes strongly affects the power of the model. For example, using non-computable transition amplitudes, a 2QCFA can recognize certain undecidable languages with bounded error in expected polynomial time [33]. Our lower bound holds even in this setting of unrestricted transition amplitudes. For $\mathbb{F} \subseteq \mathbb{C}$, we define complexity classes $\text{coR2QCFAG}(k, d, T(n), \epsilon)$, $\text{coRQE2QCFAG}$, etc., that are variants of the corresponding complexity class in which the 2QCFA are restricted to have transition amplitudes.
Using our terminology, Ambainis and Watrous [2] showed that \( L_{\text{pal}} \in \text{coRQE2QCFA} \), where \( \mathbb{Q} \) denotes the algebraic numbers, which are, arguably, the natural choice for the permitted class of transition amplitudes of a quantum model of computation. Therefore, \( L_{\text{pal}} \) can be recognized with negative one-sided bounded error by a single-qubit 2QCFA with transition amplitudes that are all algebraic numbers in expected exponential time; however, \( L_{\text{pal}} \) cannot be recognized with two-sided bounded error (and, therefore, not with one-sided bounded error) by a 2QCFA (of any constant size) in expected subexponential time, regardless of the permitted transition amplitudes.

5 Lower Bounds on the Running Time of Small-Space QTMs

In this section, we show that our technique also yields a lower bound on the expected running time of a quantum Turing machine (QTM) that uses sublogarithmic space (i.e., \( o(\log n) \) space) and recognizes a language \( L \) with bounded error. The key idea is that a QTM \( M \) that runs in expected time at most \( T(n) \) and uses space at most \( S(n) \) can be viewed as a sequence \( (M_n)_{n \in \mathbb{N}} \) of 2QCFA, where \( M_n \) has \( 2^{O(S(n))} \) (classical and quantum) states and \( M_n \) simulates \( M \) on all inputs of length at most \( n \) (therefore, \( M_n \) and \( M \) have the same probability of acceptance and the same expected running time on any such input). The techniques of the previous section apply to 2QCFA with a sufficiently slowly growing number of states. See, for instance, [9,22] for examples of this approach for classical TMs.

We consider the \textit{classically controlled} space-bounded QTM model that allows \textit{intermediate measurements}, following the definition of Watrous [42]. While several such QTM models have been defined, we focus on this model as we wish to prove our lower bound in the greatest generality possible. We note that the definitions of such QTM models by, for instance, Ta-Shma [39], Watrous [43, Section VII.2], and (essentially, without the use of random access) van Melkebeek and Watson [26] are special cases of the QTM model that we consider.

A QTM has three tapes: (1) a classical read-only input tape, where each cell stores a symbol from the input alphabet (with special end-markers at the left and right ends), (2) a classical one-way infinite work tape, where each cell stores a symbol from some potentially larger (finite) alphabet, and (3) a one-way infinite quantum work tape, where each cell contains a single qubit. Each tape has a bidirectional (classical) head. A QTM also has a finite set of classical states that serve as its finite control, and a finite-size quantum register.

The computation of a QTM is entirely \textit{classically controlled}. Each step of the computation consists of a \textit{quantum phase} followed by a \textit{classical phase}. In the quantum phase, depending on the current classical state and the symbols currently under the heads of the input tape and of the classical work tape, a QTM performs a selective quantum operation on the combined register consisting of its internal quantum register and the single qubit currently under the head of the quantum work tape. In the classical phase, depending on the current classical state, the symbols currently under the heads of the input tape and of the classical work tape, and the result of the operation performed in the quantum phase, a QTM updates its configuration as follows: a new classical state is entered, a symbol is written on the cell of the classical work tape under the head, and the heads of all tapes move at most one cell in either direction.

A (branch of the computation of a) QTM halts and accepts/rejects its input by entering a special classical accept/reject state. As we wish to make our lower bound as strong as possible, we wish to be as generous as possible with the rejecting criteria of a QTM, and so we allow a QTM to also reject by looping (as we did with 2QCFA); similarly, no restriction is placed on the transition amplitudes of the QTM (see the discussion in Section 4.4). Let \( \text{BQTISP}_\epsilon(T(n),S(n)) \) denote the class of languages recognizable with two-sided bounded error \( \epsilon \in [0,1/2) \) by a QTM that runs in
at most $T(n)$ expected time, and uses at most $S(n)$ space, on all inputs of length at most $n$; of course, only the space used on the (classical and quantum) work tapes is counted. Furthermore, let $\text{BQTISP}(T(n), S(n)) = \cup_{\epsilon \in [0,1/2]} \text{BQTISP}_\epsilon(T(n), S(n))$.

Remark. We emphasize that the QTM model that we consider permits intermediate measurements. In the case of time-bounded quantum computation, allowing a QTM to perform intermediate measurements provably does not increase the power of the model. This is due to the principle of safe storage, which allows all measurements to be deferred until the end of a computation without affecting the running time; however, the standard technique for deferring measurements may cause a significant increase in the required space. It remains an open question whether or not allowing a space-bounded QTM to perform intermediate measurements increases the power of the model. Again, as we want our lower bound to be as strong as possible, we allow intermediate measurements. We also note that the consideration of classically controlled QTMs is natural, as the clear separation of quantum and classical parts accurately reflects the design of current and near-term experimental quantum computers. We direct the reader to [26, Section 2] for a detailed discussion of the various models of space-bounded QTMs.

As noted at the beginning of this section, we may view a QTM $M$ that operates in space $S(n)$ as a sequence of 2QCFA with a growing number of states. This yields the following analogue of Theorem 4.4 for sublogarithmic-space QTMs.

**Theorem 5.1.** Suppose $L \in \text{BQTISP}(T(n), S(n))$, and suppose further that $S(n) = o(\log \log D_L(n))$. Then $\exists \theta_0 \in \mathbb{R}_{>0}$ such that, $T(n) = \Omega(2^{-b_0 S(n)} D_L(n)^{2^{-b_0 S(n)}})$.

Proof. By definition, there is some QTM $M$ that recognizes $L$ with two-sided bounded error $\epsilon$, for some $\epsilon \in [0,1/2)$, where $M$ runs in expected time at most $T(n)$, and uses at most $S(n)$ space, on all inputs of length at most $n$. Let $F$ (resp. $P$) denote the finite set of classical states (resp. quantum basis states) of $M$, and let $\Sigma$ (resp. $\Gamma$) denote the finite input alphabet (resp. classical work tape alphabet) of $M$.

For each $n \in \mathbb{N}$, we define a 2QCFA $M_n$ that correctly simulates $M$ on any $w \in \Sigma^{\leq n}$, in the obvious way. The (only) head of the 2QCFA $M_n$ (on its read-only input tape) directly simulates the head of the QTM $M$ on its read-only input tape. $M_n$ uses its classical states $C_n$ to keep track of the state $f \in F$ of the finite control of $M$, the string $y \in \Gamma^{S(n)}$ that appears in the first $S(n)$ cells of the classical work tape, and the positions $h_{c-\text{work}}, h_{q-\text{work}} \in \{1, \ldots, S(n)\}$ of the heads on the (classical and quantum) work tapes. $M_n$ uses its quantum register, which has quantum basis states $Q_n$, to store the first $S(n)$ qubits of the quantum work tape and the $\log |P|$ qubits of the internal quantum register. The transition function of $M_n$ is defined such that, if $M_n$ is in a classic state $c \in C_n$ which (along with the head position on the input tape) completely specifies the classical part of a configuration of $M$, then $M_n$ performs the same quantum phase and classical phase that $M$ would in this configuration. Clearly, for any $w \in \Sigma^{\leq n}$, $M_n$ and $M$ have the same probability of acceptance and expected running time.

Let $k_n = |Q_n| = |P|2^{S(n)}$ denote the number of quantum basis states of $M_n$ and let $d_n = |C_n| = |F||\Gamma|^{S(n)} S(n)^2$ denote the number of classical states of $M_n$. Then, $\exists \theta_0 \in \mathbb{R}_{>0}, \exists \tilde{N}_0 \in \mathbb{N}$ such that, $\forall n \geq \tilde{N}_0$, we have $k_n^2 d_n^2 \leq 2^{b_0 S(n)}$. Moreover, as $S(n) = o(\log \log D_L(n))$, $\exists \tilde{N}_0 \in \mathbb{N}$ such that, $\forall n \geq \tilde{N}_0$, $D_L(n)^{2^{-b_0 S(n)}} \geq 2$. Set $N_0 = \max(\tilde{N}_0, \tilde{N}_0)$. For any $n \geq N_0$, we may then construct $X_n \subseteq \Sigma^{\leq n}$ such that $|X_n| = D_L(n) \geq 2$ and the elements of $X_n$ are pairwise $(L,n)$-dissimilar. By Lemma 4.3, $\exists x, x' \in X_n$ such that $x \neq x'$ and

$$\|N_{x,m} - N_{x',m}\|_1 \leq 4\sqrt{2}k_n^2 d_n^2 \left(D_L(n)^{\frac{1}{\log D_L(n)}} - 1\right)^{-1} \leq (4\sqrt{2})2^{b_0 S(n)} \left(D_L(n)^{2^{-b_0 S(n)}} - 1\right)^{-1}.$$
Let \( a_\epsilon = \frac{(1-2\epsilon)^2}{2} \in \mathbb{R}_{>0} \). By Lemma 4.2,
\[
T(n) \geq a_\epsilon \|N_{x,m} - N_{x,m}^{-1}\|^{-1} \geq \frac{a_\epsilon}{4\sqrt{2}} 2^{-b_0 S(n)} \left(D_L(n)^{2^{-b_0 S(n)}} - 1\right) \geq \frac{a_\epsilon}{8\sqrt{2}} 2^{-b_0 S(n)} D_L(n)^{2^{-b_0 S(n)}} 
\]
\[\blacksquare\]

Remark. Recall that, for any language \( L \), \( D_L(n) = 2^{O(n)} \); therefore, the supposition of the above theorem that \( S(n) = o(\log \log D_L(n)) \) implies \( S(n) = o(\log n) \), and so this theorem only applies to QTMs that use sublogarithmic space. Moreover, this requirement also implies that \( D_L(n) = \omega(1) \), and hence \( L \not\in \text{REG} \) [9, Lemma 3.1]; of course, for any \( L \in \text{REG} \), we trivially have \( L \in \text{BQTISP}(n, O(1)) \).

Note that, if \( S(n) = o(\log n) \), then for any constants \( b_1, b_2 \in \mathbb{R}_{>0} \), \( 2^{-b_1 S(n)} \geq n^{-b_2} \), for all sufficiently large \( n \). We therefore obtain the following corollary.

**Corollary 5.1.1.** If a language \( L \) satisfies \( D_L(n) = 2^{\Omega(n)} \), then \( L \not\in \text{BQTISP} \left(2^{n^{-1-\Omega(1)}}, o(\log n)\right) \).

In particular, \( L_{\text{pal}} \not\in \text{BQTISP} \left(2^{n^{-1-\Omega(1)}}, o(\log n)\right) \).

Remark. Of course, \( L_{\text{pal}} \) can be recognized by a deterministic Turing machine in \( O(\log n) \) space (and, trivially, polynomial time). Therefore, the previous corollary exhibits a natural problem for which polynomial time quantum Turing machines cannot (asymptotically) outperform polynomial time deterministic Turing machines in terms of the amount of space used.

### 6 The Word Problem of a Group

We begin by formally defining the word problem of a group; for further background, see, for instance [25]. For a set \( S \), let \( F(S) \) denote the free group on \( S \). Fix sets \( S, R \) such that \( R \subseteq F(S) \), let \( N \) denote the normal closure of \( R \) in \( F(S) \); for a group \( G \), if \( G \cong F(S)/N \), then we say that \( G \) has presentation \( \langle S|R \rangle \), which we denote by writing \( G = \langle S|R \rangle \).

Suppose \( G = \langle S|R \rangle \), with \( S \) finite; we now define \( W_G = \langle S|R \rangle \), the word problem of \( G \) with respect to the presentation \( \langle S|R \rangle \). We define the set of formal inverses \( S^{-1} \), such that, for each \( s \in S \), there is a unique corresponding \( s^{-1} \in S^{-1} \), and \( S \cap S^{-1} = \emptyset \). Let \( \Sigma = S \sqcup S^{-1} \), let \( \Sigma^* \) denote the free monoid over \( \Sigma \), and let \( \phi : \Sigma^* \rightarrow G \) be the natural (monoid) homomorphism that takes each string in \( \Sigma^* \) to the element of \( G \) that it represents. We use \( 1_G \) to denote the identity element of \( G \). Then \( W_G = \langle S|R \rangle = \phi^{-1}(1_G) \).

We say that \( G \) is finitely generated if it has a presentation \( \langle S|R \rangle \) where \( S \) is finite. Note that the word problem of \( G \) is only defined when \( G \) is finitely generated and that the definition of the word problem does depend on the particular presentation. However, it is well-known (see, for instance, [18]) that if \( L \) is any complexity class that is closed under inverse homomorphism, then if \( \langle S|R \rangle \) and \( \langle S'|R' \rangle \) are both presentations of some group \( G \), and \( S \) and \( S' \) are both finite, then \( W_G = \langle S|R \rangle \in L \Leftrightarrow W_{G'} = \langle S'|R' \rangle \in L \). As all complexity classes considered in this paper are easily seen to be closed under inverse homomorphism, we will simply write \( W_G \in L \) to mean that \( W_G = \langle S|R \rangle \in L, \) for every presentation \( G = \langle S|R \rangle \), with \( S \) finite.

We note that the languages \( L_{\text{pal}} \) and \( L_{\text{eq}} \), which Ambainis and Watrous [2] showed satisfy \( L_{\text{pal}} \in \text{coRQE2QCFA} \) and \( L_{\text{eq}} \in \text{BQP2QCFA} \), are closely related to the word problems of the groups \( F_2 \) and \( \mathbb{Z} \), respectively (see [32] for a full discussion of this correspondence).

In this section, we consider the (quantum) computational complexity of the word problem \( W_G \) corresponding to a finitely generated group \( G \). We will show that there is a close correspondence
between $D_{WG}$ and the growth rate of the group $G$, which will enable us to exhibit a strong lower bound on the expected running time of a 2QCFA that recognizes a word problem from a particular class of groups. By combining these lower bounds with a recent result of ours [32] that showed that 2QCFA can recognize certain wide classes of group word problems within particular time bounds, we obtain a natural class of languages that 2QCFA can recognize with bounded error in expected exponential time, but not in expected subexponential time, as well as strong statements about the class of group word problems that a 2QCFA can recognize with bounded error in expected polynomial time.

6.1 The Growth Rate of a Group and Nonregularity

Consider a group $G = \langle S | R \rangle$, with $S$ finite. As above, let $\Sigma = S \cup S^{-1}$, and let $\phi : \Sigma^* \to G$ denote the natural map that takes each string in $\Sigma^*$ to the element of $G$ that it represents. For $g \in G$, we define the length of $g$ with respect to $S$, which we denote by $l_S(g)$, as the smallest $m \in \mathbb{N}$ such that $\exists \sigma_1, \ldots, \sigma_m \in \Sigma$ such that $g = \phi(\sigma_1 \cdots \sigma_m)$. For $n \in \mathbb{N}$, we define $B_{G,S}(n) = \{g \in G : l_S(g) \leq n\}$ and we further define $\beta_{G,S}(n) = |B_{G,S}(n)|$, which we call the growth rate of $G$ with respect to $S$. The following straightforward lemma demonstrates an important relationship between $\beta_{G,S}$ and $D_{WG=\langle S | R \rangle}$.

**Lemma 6.1.** Suppose $G = \langle S | R \rangle$ with $S$ finite. Using the notation established above, let $W_G := W_{G=\langle S | R \rangle} = \phi^{-1}(1_G)$ denote the word problem of $G$ with respect to this presentation. Then, $\forall n \in \mathbb{N}$, $D_{WG}(2n) \geq \beta_{G,S}(n)$.

**Proof.** Fix $n \in \mathbb{N}$, let $k = \beta_{G,S}(n)$, and let $B_{G,S}(n) = \{g_1, \ldots, g_k\}$. For a string $x = x_1 \cdots x_m \in \Sigma^*$, where each $x_j \in \Sigma$, let $|x| = m$ denote the (string) length of $x$ and define $x^{-1} = x_m^{-1} \cdots x_1^{-1}$. Note that, $\forall g \in G$, $l_S(g) = \min_{w \in \phi^{-1}(g)} |w|$. Therefore, for each $i \in \{1, \ldots, k\}$ we may define $w_i = \phi^{-1}(g_i)$ such that $|w_i| = l_S(g_i)$. Observe that $w_i w_i^{-1} \in W_G$ and $|w_i w_i^{-1}| = 2|w_i| = 2l_S(g_i) \leq 2n$; moreover, for each $j \neq i$, we have $w_j w_i^{-1} \notin W_G$ and $|w_j w_i^{-1}| = |w_j| + |w_i| = l_S(g_j) + l_S(g_i) \leq 2n$. Therefore, $w_1, \ldots, w_k$ are pairwise $(W_G, 2n)$-dissimilar, which implies $D_{WG}(2n) \geq k = \beta_{G,S}(n)$. \qed

**Remark.** In fact, one may also easily show that $D_{WG}(2n) \leq \beta_{G,S}(n) + 1$, though we do not need this here. Essentially, $\beta_{G,S}(n)$ is (another) equivalent characterization of the nonregularity $D_{WG}(2n)$ (see Section 4.1 for a discussion of many such characterizations of nonregularity).

While $\beta_{G,S}$ does depend on the particular choice of the generating set $S$, the dependence is quite minor, in a sense that we now clarify. For a pair of non-decreasing functions $f_1, f_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we write $f_1 \prec f_2$ if $\exists C_1, C_2 \in \mathbb{R}_{>0}$ such that $\forall r \in \mathbb{R}_{\geq 0}$, $f_1(r) \leq C_1 f_2(C_1 r + C_2) + C_2$; if both $f_1 \prec f_2$ and $f_2 \prec f_1$, then we say that $f_1$ is quasi-equivalent to $f_2$, which we denote by $f_1 \sim f_2$. We extend a growth function $\beta_{G,S} : \mathbb{N} \to \mathbb{N}$ to $\beta_{G,S} : \mathbb{R}_{\geq 0} \to \mathbb{N}$ by defining $\beta_{G,S}(r) = \beta_{G,S}([r])$, $\forall r \in \mathbb{R}_{\geq 0}$. Suppose $G = \langle S' | R' \rangle$, where $S'$ is finite. It is straightforward to show that $\beta_{G,S}$ and $\beta_{G,S'}$ are non-decreasing, and that $\beta_{G,S} \sim \beta_{G,S'}$ (see, for instance, [25, Proposition 6.2.4]). For this reason, we will often omit $S$ and simply write $\beta_G$ to denote the growth rate of $G$, when we only care about the growth rate up to quasi-equivalence. We then make the following definition.

**Definition 6.2.** Suppose $G$ is a finitely generated group.

(i) If $\beta_G \sim (n \mapsto e^n)$, we say $G$ has exponential growth.

(ii) If $\exists c \in \mathbb{R}_{\geq 0}$ such that $\beta_G \prec (n \mapsto n^c)$, we say $G$ has polynomial growth.

(iii) If $G$ has neither polynomial growth nor exponential growth, we say $G$ has intermediate growth.

Note that, for any finitely generated group $G$, we have $\beta_G \prec (n \mapsto e^n)$, and so the term “intermediate” growth is justified.
6.2 Word Problems Recognizable by 2QCFA and Small-Space QTMs

By making use of two very powerful results in group theory, the Tits’ Alternative [41] and Gromov’s theorem on groups of polynomial growth [14], we exhibit useful lower bounds on $D_{W_G}$, which in turn allows us to show a strong lower bound on the expected running time of a 2QCFA that recognizes $W_G$. In the following, we use the notation for complexity classes established in Section 4.3. As previously noted, the membership of $W_G$ in any of the complexity classes in question does not depend on the particular choice of presentation, and so we write, for example, $W_G \in \text{BQP2QCFA}$ to mean $W_G=\langle S|R \rangle \in \text{BQP2QCFA}$ for some (equivalently every) presentation $G=\langle S|R \rangle$, with $S$ finite.

**Theorem 6.3.** For any finitely generated group $G$, the following statements hold.

(i) If $W_G \in \text{B2QCFA}(k,d,T(n),\epsilon)$, then $\beta_G \prec (n \mapsto T(n)^{k^2d^2})$.

(ii) If $G$ has exponential growth, then $W_G \notin \text{B2QCFA}(2^{o(n)})$.

(iii) If $G$ is a linear group over a field of characteristic 0, and $G$ is not virtually nilpotent, then $W_G \notin \text{B2QCFA}(2^{o(n)})$.

(iv) If $W_G \in \text{BQP2QCFA}$, then $G$ is virtually nilpotent.

**Proof.** (i) Follows immediately from Lemma 6.1 and Corollary 4.4.1.

(ii) Follows immediately from Definition 6.2(i) and part (i) of this theorem.

(iii) As a consequence of the famous Tits’ Alternative [41], every finitely generated linear group over a field of characteristic 0 either has polynomial growth or exponential growth, and has polynomial growth precisely when it is virtually nilpotent ( [41, Corollary 1], [45]). The claim then follows by part (ii) of this theorem.

(iv) If $W_G \in \text{BQP2QCFA}$, then $W_G \in \text{B2QCFA}(k,d,n^c,\epsilon)$ for some $k,d,c \in \mathbb{N}_{\geq 1}, \epsilon \in [0,\frac{1}{2})$.

By part (i) of this theorem, $\beta_G \prec (n \mapsto n^{ck^2d^2})$, which implies $G$ has polynomial growth. By Gromov’s theorem on groups of polynomial growth [14], a finitely generated group has polynomial growth precisely when it is virtually nilpotent. 

\[ \square \]

**Remark.** We note that, while finitely generated groups of intermediate growth provably exist [13], all known groups of intermediate growth have growth rate quasi-equivalent to $(n \mapsto e^{n^c})$, for some $c \in (1/2, 1)$. Therefore, if $W_G$ is the word problem for one of these known groups of intermediate growth, a strong lower bound may be established on $D_{W_G}$, which in turn allows a strong lower bound to be established on the running time of any 2QCFA that recognizes $W_G$ for one of these known groups of intermediate growth. We also note that one may show that the conclusion of Theorem 6.3(iv) still holds even if $W_G$ is only assumed to be recognized in slightly super-polynomial time. In particular, by a quantitative version of Gromov’s theorem due to Shalom and Tal [36, Corollary 1.10], $\exists c \in \mathbb{R}_{>0}$ such that if $\beta_{G,S}(n) \leq n^{c(\log \log n)^{\frac{1}{c}}}$, for some $n > 1/c$, then $G$ is virtually nilpotent.

Let $\mathcal{G}_{v\text{Ab}}$ (resp. $\mathcal{G}_{v\text{Nilp}}$) denote the collection of all finitely generated virtually abelian (resp. nilpotent) groups. Let $\overline{\mathbb{Q}}$ denote the algebraic numbers and let $U(k,\overline{\mathbb{Q}})$ denote the group of $k \times k$ unitary matrices with entries in $\overline{\mathbb{Q}}$, and let $\mathcal{U}$ denote the family of finitely generated groups $G$ such that $G$ is isomorphic to a subgroup of $U(k,\overline{\mathbb{Q}})$, for some $k$. We have recently shown that if $G \in \mathcal{U}$, then $W_G \in \text{coRQE2QCFA}_{\mathbb{Q}}$ [32, Corollary 1.4.1]. Observe that $\mathcal{G}_{v\text{Ab}} \subseteq \mathcal{U}$ and that all groups in $\mathcal{U}$
are finitely generated linear groups over a field of characteristic zero. Moreover, \( \mathcal{U} \cap \mathcal{G}_{vNilp} = \mathcal{G}_{vAb} \) (see, for instance, [40, Proposition 2.2]). We therefore immediately obtain the following corollary of Theorem 6.3(iii), which exhibits a broad and natural class of languages that a 2QCFA can recognize with bounded error in expected exponential time, but not in expected subexponential time. We note that \( \mathcal{U} \setminus \mathcal{G}_{vAb} \) is a rather wide class of groups, see [32] for a full discussion and related results.

**Corollary 6.3.1.** For any \( G \in \mathcal{U} \setminus \mathcal{G}_{vAb} \) and for any \( T : \mathbb{N} \to \mathbb{N} \) such that \( T(n) = 2^{o(n)} \), we have \( W_G \in \text{coRQE2QCFA}_{\mathbb{Q}} \) but \( W_G \not\in \text{B2QCFA}(T(n)) \).

Let \( \text{coRQP2QCFA}_{\mathbb{Q}}(2) \) denote the class of languages recognizable with negative one-side bounded error by a 2QCFA, with a single-qubit quantum register and algebraic number transition amplitudes, in expected polynomial time. We have also recently shown that \( W_G \in \text{coRQP2QCFA}_{\mathbb{Q}}(2) \subseteq \text{BQP2QCFA} \), \( \forall G \in \mathcal{G}_{vAb} \) [32, Theorem 1.2]. By Theorem 6.3(iv), if \( W_G \in \text{BQP2QCFA} \), then \( G \in \mathcal{G}_{vNilp} \). This naturally raises the question of whether or not there is some \( G \in \mathcal{G}_{vNilp} \setminus \mathcal{G}_{vAb} \) such that \( W_G \in \text{BQP2QCFA} \). In particular, consider the (three-dimensional discrete) Heisenberg group \( H = \langle x, y, z | [x, y], [x, z] = [y, z] = 1 \rangle \) (where \( [x, y] = x^{-1}y^{-1}xy \) denotes the commutator of \( x \) and \( y \) and we have expressed the relators as equations, rather than words in \( F(x, y, z) \), for convenience). The word problem \( W_H \) of the Heisenberg group \( H \) is a natural choice for a potential “hard” word problem for 2QCFA, due to the lack of faithful finite-dimensional unitary representations of \( H \) (see [32] for further discussion). In fact, it is possible, and perhaps plausible, that \( W_H \) cannot be recognized with bounded error by a 2QCFA in any time bound. We next show that if \( W_H \not\in \text{BQP2QCFA} \), then we have a complete classification of those word problems recognizable by 2QCFA in expected polynomial time.

**Proposition 6.4.** If \( W_H \not\in \text{BQP2QCFA} \), where \( H \) is the Heisenberg group, then for any finitely generated group \( G \), \( W_G \in \text{BQP2QCFA} \iff W_G \in \text{coRQP2QCFA}_{\mathbb{Q}}(2) \iff G \in \mathcal{G}_{vAb} \).

**Proof.** By the above discussion, it suffices to show the following claim: if \( W_G \in \text{BQP2QCFA} \), for some \( G \in \mathcal{G}_{vNilp} \setminus \mathcal{G}_{vAb} \), then \( W_H \in \text{BQP2QCFA} \). Begin by noting that BQP2QCFA is easily seen to be closed under inverse homomorphism and intersection with regular languages. Suppose \( G \) and \( G' \) are finitely generated groups such that \( G' \) is (isomorphic to) a subgroup of \( G \), if \( W_G \in \text{BQP2QCFA} \), then \( W_{G'} \in \text{BQP2QCFA} \) (see, for instance, [19, Lemma 2]). It is well-known that \( H \in \mathcal{G}_{vNilp} \setminus \mathcal{G}_{vAb} \) and, \( \forall G \in \mathcal{G}_{vNilp} \setminus \mathcal{G}_{vAb} \), \( G \) has a subgroup isomorphic to \( H \) (see, for instance, [19, Theorem 12] for these facts, as well as for their application towards understanding the computational complexity of the group word problem).

We next obtain the following analogue of Theorem 6.3 for small-space QTM.

**Theorem 6.5.** For any finitely generated group \( G \), the following statements hold.

(i) If \( G \) has exponential growth, then \( W_G \not\in \text{BQTISP}(2^{n^{1-o(1)}}, o(\log n)) \).

(ii) If \( G \) is a linear group over a field of characteristic 0, and \( G \) is not virtually nilpotent, then \( W_G \not\in \text{BQTISP}(2^{n^{1-o(1)}}, o(\log n)) \).

(iii) If \( W_G \in \text{BQTISP}(n^{O(1)}, o(\log \log \log n)) \), then \( G \) is virtually nilpotent.

**Proof.** (i) Follows immediately from Definition 6.2(i), Corollary 5.1.1, and Lemma 6.1.

(ii) The claim follows from the Tits’ Alternative [41] and part (i) of this theorem.

(iii) If \( W_G \in \text{BQTISP}(n^{O(1)}, o(\log \log \log n)) \), then \( \forall c \in \mathbb{R}_{>0} \) and for all sufficiently large \( n \) we have, by Theorem 5.1, \( D_L(n) \leq n^{c(\log \log n)} \). By Lemma 6.1 and the quantitative version of Gromov’s theorem due to Shalom and Tal [36, Corollary 1.10], \( G \) is virtually nilpotent.
7 Discussion

In this paper, we established strong lower bounds on the expected running time of 2QCFA, or sublogarithmic-space QTM, that recognize particular languages with bounded error. In particular, the language \( L_{\text{pal}} \) had been shown by Ambainis and Watrous [2] to be recognizable with bounded error by a single-qubit 2QCFA in expected time \( 2^{O(n)} \). We have given a matching lower bound: no 2QCFA (of any size) can recognize \( L_{\text{pal}} \) with bounded error in expected time \( 2^{o(n)} \). Moreover, we have shown that no QTM, that runs in expected time \( 2^{n-1-O(1)} \) and uses space \( o(\log n) \), can recognize \( L_{\text{pal}} \) with bounded error. This latter results is especially interesting, as a deterministic TM can recognize \( L_{\text{pal}} \) using space \( O(\log n) \) (and, of course, polynomial time); therefore, polynomial time quantum TMs have no (asymptotic) advantage over polynomial time deterministic TMs in terms of the amount of space needed to recognize \( L_{\text{pal}} \).

Our main technical result, Theorem 4.4, showed that, if a language \( L \) is recognized with bounded error by a 2QCFA in expected time \( T(n) \), then \( \exists a \in \mathbb{R}_{>0} \) (that depends only on the number of states of the 2QCFA) such that \( T(n) = \Omega(D_L(n)^a) \), where \( D_L \) is the Dwork-Stockmeyer nonregularity of \( L \). This result is extremely (qualitatively) similar to the landmark result of Dwork and Stockmeyer [9, Lemma 4.3], which showed that, if a language \( L \) is recognized with bounded error by a 2PFA in expected time \( T(n) \), then \( \exists a \in \mathbb{R}_{>0} \) (that depends only on the number of states of the 2PFA) such that \( T(n) = \Omega(2^{D_L(n)^a}) \). We again note that both of these lower bounds are tight.

We conclude by stating a few interesting open problems. While our lower bound on the expected running time \( T(n) \), of a 2QCFA that recognizes a language \( L \), in terms of \( D_L(n) \) cannot be improved, it is natural to ask if one could establish a lower bound on \( T(n) \) in terms of a different hardness measure of \( L \) that would be stronger for certain languages. Generalizing the definitions made in Section 4.1, let \( \mathcal{F} \) denote a class of finite automata (e.g., DFA, NFA, 2DFA, etc.), let \( L \) be a language over some alphabet \( \Sigma \), and let \( A_{\mathcal{F}, \leq}^L(n) = \min\{|M| : M \in \mathcal{F} \text{ and } L(M) \cap \Sigma \leq n = L \cap \Sigma \leq n \} \) denote the smallest number of states of an automaton of type \( \mathcal{F} \) that agrees with \( L \) on all strings of length at most \( n \). As discussed earlier, \( A_{\text{DFA}, \leq}^L(n) = D_L(n) \), for any language \( L \) and for any \( n \in \mathbb{N} \). Recall that DFA and 2DFA both recognize precisely the regular languages [31], but for some \( \hat{L} \in \text{REG} \), the smallest 2DFA that recognizes \( \hat{L} \) might have many fewer states than the smallest DFA that recognizes \( \hat{L} \). In fact, there is a sequence of regular languages \( (L_k)_{k \in \mathbb{N}} \) such that \( L_k \) can be recognized by a \( 5k + 5 \)-state 2DFA, but any DFA that recognizes \( L_k \) requires at least \( k^k \) states [27]; however, this is (essentially) the largest succinctness advantage possible, as any language recognizable by a \( d \)-state 2DFA is recognizable by a \( (d+2)^d+1 \)-state DFA [37]. Of course, for any language \( L \), we have \( A^2_{\text{DFA}, \leq}(n) \leq A^D_{\text{DFA}, \leq}(n) \), \( \forall n \). For certain languages \( L \), we have \( A^2_{\text{DFA}, \leq}(n) \ll A^D_{\text{DFA}, \leq}(n) \), \( \forall n \); most significantly, this holds for the languages \( L_{\text{pal}} \) and \( L_{\text{eq}} \) shown by Ambainis and Watrous [2] to be recognizable with bounded error by 2QCFA in, respectively, expected exponential time and expected polynomial time. In particular, it is easy to show that \( A^D_{\text{DFA}, \leq}(n) = 2^\Theta(n) \), \( A^D_{\text{DFA}, \leq}(n) = \Theta(n) \), and \( A^D_{\text{DFA}, \leq}(n) = n^{\Theta(1)} \); moreover, \( A^2_{\text{DFA}, \leq}(n) = \log^{\Theta(1)}(n) \) [20, Theorem 3 and Corollary 4]. In fact, this same phenomenon occurs for all the group word problems that we can show [32] are recognized by 2QCFA. Might this be true for all languages recognizable by 2QCFA?

**Open Problem 7.1.** If a language \( L \) is recognizable with bounded error by a 2QCFA in expected time \( T(n) \), does a stronger lower bound than \( T(n) = (A^2_{\text{DFA}, \leq}(n))^{\Omega(1)} \) hold?

We have shown that the class of languages recognizable with bounded error by a 2QCFA in expected polynomial time is contained in \( \text{L/poly} \). This type of dequantumization result, which shows that the class of languages recognizable by a particular quantum model is contained in the
class of languages recognizable by a particular classical model, is analogous to the Adleman-type \[1\] derandomization result \( \text{BPL} \subseteq \text{L/poly} \). It is natural to ask if our dequantumization result might be extended, either to 2QCFA that run in a larger time bound, or to small-space QTM. Note that \( \text{L/poly} = \{ L : A^{2DFA}_{L \leq n}(n) = n^{O(1)} \} \supseteq \{ L : A^{DFA}_{L \leq n}(n) = n^{O(1)} \} \). This further demonstrates the value of the preceding open problem, as any improvement in the lower bound on \( T(n) \) in terms of \( A^{2DFA}_{L \leq n}(n) \) would directly translate into an improved dequantumization result.

The seminal paper of Lipton and Zalcstein \[24\] showed that, if a finitely generated group \( G \) has a faithful finite-dimensional (linear) representation over a field of characteristic 0, then \( W_G \in \text{L} \) (deterministic logspace). We \[32\] recently adapted their technique to show that 2QCFA can recognize the word problem \( W_G \) of any group \( G \) that belongs to a certain (proper) subset of the set of groups to which their result applies: any group \( G \) that has a faithful finite-dimensional unitary representation of a certain special type. The requirement, imposed by the laws of quantum mechanics, that the state of the quantum register of a 2QCFA must evolve unitarily, prevents a 2QCFA from (directly) implementing the Lipton-Zalcstein algorithm for any other groups; on the other hand, for those groups \( G \) that do have such a representation, these same laws allow a 2QCFA to recognize \( W_G \) using only a constant amount of space. The word problem \( W_G \) of any group \( G \) that lacks such a representation (for example, all \( G \in \mathcal{G}_{\text{vNilp}} \setminus \mathcal{G}_{\text{vAb}} \), or any infinite Kazhdan group, or any group of intermediate growth) seems to be a plausible candidate for a hard problem for 2QCFA (see \[32\] for further discussion).

**Open Problem 7.2.** Is there a finitely generated group \( G \) that does not have a faithful finite-dimensional projective unitary representation for which \( W_G \in \text{BQE2QCFA} \)?

Concerning those groups with word problem recognizable by a 2QCFA in expected polynomial time, we have shown that, if \( G \in \mathcal{G}_{\text{vAb}} \), then \( W_G \in \text{coRQP2QCFA}_{O(2)} \subseteq \text{BQP2QCFA} \) \[32, Theorem 1.2\]; moreover, if \( W_G \in \text{BQP2QCFA} \), then \( G \in \mathcal{G}_{\text{vNilp}} \) (Theorem 6.3(iv)). We have also shown, if \( W_H \notin \text{BQP2QCFA} \), where \( H \in \mathcal{G}_{\text{vNilp}} \) is the (three-dimensional discrete) Heisenberg group, then the classification of those groups whose word problem is recognizable by a 2QCFA in expected polynomial time would be complete; in particular, we would have \( W_G \in \text{BQP2QCFA} \Leftrightarrow G \in \mathcal{G}_{\text{vAb}} \) (Proposition 6.4). This naturally raises the following question.

**Open Problem 7.3.** Is there a group \( G \in \mathcal{G}_{\text{vNilp}} \setminus \mathcal{G}_{\text{vAb}} \) such that \( W_G \in \text{BQP2QCFA} \)? In particular, is \( W_H \in \text{BQP2QCFA} \), where \( H \) is the Heisenberg group?

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