Thermodynamic geometry of the RN-AdS black hole and non-local observables

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Abstract
This paper studies the thermodynamic geometry of the Reissner–Nordström-anti-de Sitter (RN-AdS) black hole via detection of the non-local observables in the dual field theory, including the entanglement entropy and equal-time two-point correlation function. With the dimensional analysis, we construct the principle of corresponding states of black hole thermodynamics. As a result, our findings can be applied to black holes with different AdS backgrounds. In this sense, the probe of the thermodynamic geometry of the RN-AdS black hole though the non-local observables in dual field theory has been confirmed numerically.

Keywords: black hole thermodynamic, thermodynamic geometry, phase transition, non-local observables

(Some figures may appear in colour only in the online journal)

1. Introduction
The pioneering work by Hawking and Bekenstein on black hole temperature and entropy [1, 2] made people realize that black holes are thermodynamic systems that exhibit phase transition behavior. The first black hole phase transition is the Hawking–Page phase transition [3], which is related to the confinement/deconfinement phase transition [4] in the dual field theory, and makes the anti-de Sitter (AdS) spacetime more charming. Next, the discovery of the van der Waals-like phase transition in charged AdS black holes [5] implied that the black hole will undergo second-order and first-order phase transition successively before it reaches the stable phase, which revealed the connection between black holes and ordinary thermodynamic systems.

By treating the negative cosmological constant \( \Lambda = - (d - 1)(d - 2)/\ell^2 \) as the thermodynamic pressure \( P = - \Lambda/8\pi G \) [6–8], the extended phase space of the AdS black hole is established, in which the van der Waals-like phase transition was reconstructed [9]. Within this framework, the phase transition and critical behavior of black holes have been extensively studied in [10–18]. However, due to the importance of the AdS black hole in the AdS/conformal field theory (AdS/CFT) duality, which describes the fact that the correspondence between quantum gravity and gauge field theory resides on the boundary [19, 20], the holographic interpretations of the variation of \( \Lambda \) have received significant attention.

Several works in [21–23] have proposed that the variation of \( \Lambda \) corresponds to varying the number of color \( N \); alternatively, the number of degrees of freedom \( N^2 \) of the dual field theory. In CFT, the number of degrees of freedom is denoted by the central charge with \( C = l^{d-2}/G_d \) [24–26], where \( G_d \) is the Newtonian constant in \( d \)-dimensional spacetime. Therefore, the change in \( \Lambda \) corresponds to varying the central charge in CFT. In a fixed CFT, it is suggested that the central charge \( C \) should be constant [27]. Recently, Visser derived the holographic thermodynamics in the dual field theory by applying the central charge as a thermal variable, which plays a similar role to the particle number in the thermodynamic system [28]. Inspired by this, considerable work has been carried out to reveal the properties of the dual
field theory in this context [27, 29, 30]. Going a step further, in [31–35], by fixing the AdS radius, the extensive black hole thermodynamics have been developed.

While the development of black hole thermodynamics is in full swing, considerable interest has been put to probing their thermodynamic behavior. To address this, holographic entanglement entropy (HEE) [36, 37] in the dual field theory was employed to investigate the thermodynamic behavior of the Reissner–Nordström-anti-de Sitter (RN-AdS) black hole in [38]. The results demonstrated the existence of oscillating behavior in the temperature–HEE plane, resembling the van der Waals phase transition. Furthermore, the critical behavior and the Maxwell equal area law [39, 40] were examined and found to be fulfilled. Further study suggested that the oscillating behavior was also observed in the coordinate space organized by the Hawking temperature and geodesic length on the AdS boundary, which is related to the equal-time two-point correlation function and Wilson loop [41, 42].

The phase transition in classical thermodynamics originates from intermolecular interaction, but the microstructure of black holes is still a mystery. The introduction of the Ruppeiner geometry [43–45] provides some insight into that. By considering the fluctuation theory, line elements are proposed to measure the distance between fluctuation states, which take the form of

\[ \Delta l^2 = -\frac{\partial^2 S}{\partial X^\mu \partial X^\nu} \Delta X^\mu \Delta X^\nu. \]  

(1)

Here, \( S \) is the entropy of the system, and \( X^\mu \) are the thermodynamic coordinates depending on the choice of the thermodynamic potential. The curvature scalar calculated from equation (1) describes the interaction within the system. Specifically, a positive (negative) value of the curvature scalar indicates a repulsive (attractive) interaction within the black hole domain, and the noninteracting system corresponding to the flat Ruppeiner metric [46–58]. In this sense, the application of the Ruppeiner geometry is the reverse process of statistical physics, i.e. detecting the microstructure of a system with its thermodynamic behavior. It has also been observed that the divergent point of the curvature scalar corresponds to the phase transition point [59], which implied the phase structure will be exposed by the curvature scalar. In [60], the author constructed a new formalism of thermodynamic geometry (NTG) by changing the coordinates of the Ruppeiner metric equation (1) using Jacobian matrices. Therefore, the line element turns to

\[ \text{d}l_{\text{NTG}}^2 = -\frac{1}{T} \left( \eta'_{ij} \frac{\partial^2 \Xi}{\partial X^i \partial X^j} \right) \text{d}X^i \text{d}X^j, \]  

(2)

where \( \eta'_{ij} = \text{diag}(-1, 1, \ldots, 1) \), and \( \Xi \) is the thermodynamic potential. This framework showed that the geometrothermodynamics [61] is related to their new formalism of the thermodynamic geometry with the use of a singular conformal transformation. More importantly, this new approach established a one-to-one correspondence between the phase transitions and the singularities of the scalar curvature. This method was employed to investigate phase transitions and the underlying microstructure of the black hole in pure Lovelock gravity [62].

Considering that the non-local variables have the same oscillating behavior as the black hole phase transition, it is natural to ask whether the information of the underlying microstructure depicted by the thermodynamic geometry can be read in a given CFT. Inspired by this, we explore the connection between the black hole entropy and the non-local observables in dual field theory, including the HEE and the equal-time two-point correlation function; the results specified that the observation of the quantities in CFT will expose the information of the black hole phase structure and thermodynamic geometry.

The outline of this paper is as follows. In section 2, we review the phase structure and the thermodynamic geometry of the RN-AdS black hole. The numerical results to describe the thermodynamic geometry of the RN-AdS black hole with the HEE and two-point correlation function will be investigated in section 3. We end this paper with a conclusion in section 4. Throughout this paper, we adopt the units \( h = c = k_b = G = 1 \) for convenience.

2. RN-AdS black hole

The RN-AdS black hole in four-dimensional spacetime is characterized by the action in the form of

\[ I = \frac{1}{16\pi} \int \sqrt{-g} \, \text{d}x^4 \left( R - F_{\mu \nu} F^{\mu \nu} + \frac{6}{l^2} \right), \]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu = \left( \frac{Q}{r}, 0, 0, 0 \right). \]  

(3)

where \( l \) is the AdS radius, and the equation of motion is

\[ \text{d}s^2 = -f(r) \text{d}t^2 + \frac{1}{f(r)} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2). \]

The metric function \( f(r) \) can be obtained easily

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^4}{l^4}. \]  

(4)

The parameters \( M \) and \( Q \) in the metric function are the mass and charge of the black hole, respectively. The temperature and entropy are given by

\[ T = \frac{1}{4\pi r_h} \left( 1 + \frac{3r_h^2}{l^2} - \frac{Q^2}{r_h^3} \right), \]

\[ S = \pi r_h^2, \]  

(5)

(6)

where \( r_h \) is the position of the outer event horizon. The mass of the black hole can be deduced with the condition \( f(r_h) = 0 \)

\[ M = \frac{r_h}{2} - \frac{\Phi Q}{2} + \frac{r_h^3}{2l^2}, \]  

(7)

where \( \Phi = Q/r_h \) is the electric potential difference between the horizon and infinity.

In classical thermodynamics, the critical point of a van der Waals fluid is denoted by the model parameters, and the
equation of state with reduced parameters is applicable for the system with different components, which is called the principle of corresponding states. In an AdS background, the cosmological constant plays the role of model parameter as that of van der Waals fluids. Consequently, by considering the dimensional analysis of the thermodynamic variables, the dimensionless thermal parameters are introduced as

$$s = \frac{S}{l^2}, \quad q = \frac{Q}{l}, \quad t = \frac{t}{l}. \quad \text{(8)}$$

In this way, the equation of state with the dimensionless parameters will hold for charged black holes in different AdS backgrounds, which is the law of corresponding states. The equation of state can be redefined as follows

$$t = \frac{3s^2 + \pi s - \pi^2 q^2}{4(\pi s)^{5/2}}, \quad \text{(9)}$$

in which the AdS radius $l$ can be set as an arbitrary constant.

The critical point decided by $\left(\frac{\partial t}{\partial s}\right)_q = 0$ is marked as

$$s_c = \frac{\pi}{6}, \quad q_c = \frac{1}{6}, \quad t_c = \frac{\sqrt{6}}{3\pi}. \quad \text{(10)}$$

It is clear that the critical point is universal for the black hole in different AdS spacetime. After obtaining the principle of corresponding states, we would like to depict the van der Waals-like phase transition of the RN-AdS black hole. Based on equation (9), the iso-q process is shown in figure 1.

In the figure, the charge was set as $q = 0.6q_c$, $q_c$ and $2q_c$ from top to bottom. The box formed by the black dashed line corresponds to the Maxwell equal area law is constructed by the black dashed line.

While the charge is less than $q_c$, the iso-q curve exhibits similar oscillatory behavior to that of the van der Waals phase transition. According to thermodynamic theory, a small stable state will directly transform into a large stable one when the temperature of the black hole exceeds $t^*$. Together with the local extremal point in purple, the iso-q curve was divided into five segments. The red solid lines correspond to the small (SBH) and large black holes (LBH), which are thermodynamically stable. The black hole on the purple dashed line is unstable (SBH+ LBH). The green solid lines are the metastable curves of the black hole, and they separate the stable and unstable states of the black hole, with the left part being the superheated SBH (SHSBH), while the right part corresponds to the supercooled LBH (SCLBH). As the charge reaches the critical value, these two local extremal points merge into one at the inflection point on the orange curve in figure 1. Upon further increasing the value of $q$, the curve is monotonic, and the black hole is in the supercritical phase (SCBH).

To reveal the information of the underlying microstructure of the black hole, we adopt the Ruppeiner geometry of the RN-AdS black hole. In the $\{S, Q\}$ space, from equation (2) with $\Xi = M$, the thermodynamic metric comes to

$$g_{ss} = \frac{3s^2}{\pi^2} + 3\pi^2 q^2 - \pi S, \quad \text{and} \quad g_{qq} = -\frac{4\pi^2 S}{3\pi^2 + \pi S - \pi^2 Q^2}. \quad \text{(11)}$$

Therefore, the thermodynamic curvature scalar arises

$$\mathcal{R} = \frac{(Q^2 - r_s^2)^2 + 3n_r^2(10Q^4 - 9Q^2r_s^2 + 3r_s^4)/l^2 + 18n_s^4(3Q^2 - r_s^2)/l^4}{\pi(n_s^6 - Q^2 + 3n_r^2/l^2)(3Q^2 - r_s^2 + 3n_s^4/l^4)}. \quad \text{(12)}$$

and with the dimension $[\mathcal{R}] = 1/l^2$. We combine the equations in equations (5) and (6) to demonstrate the variation of the dimensionless curvature scalar $\tilde{\mathcal{R}} = \mathcal{R}/l^2$ in terms of entropy $s$ with $q < q_c$. $q = q_c$ and $q > q_c$ in figure 2, respectively.

From equation (5), we know that the scalar curvature starts from negative infinity caused by $t = 0$, marked by the red dotted line, which describes the extremal black holes. When $q < q_c$, the scalar curvature is divided into three parts by two divergent points. These parts are the stable small, unstable and stable large phases from left to right. With the increase in charge, these divergent points get closer and merge into one at $q = q_c$, and the black hole undergoes a second-order phase transition. When the charge keeps increasing, the curve will be continuous and the black hole is in the supercritical phase. The figure also suggests that the scalar curvature is always negative, implying that the interaction in the black hole is an attractive domain, until entropy is large enough.

As discussed above, the vanishing point and divergent point of $\tilde{\mathcal{R}}$ are helpful for us to learn about the underlying microstructure and the phase structure of the black hole.
checking the characteristic curve, which includes the sign-changing curve \( t_{sc} \) and the divergent curve \( t_{di} \), of scalar curvature is important. With equations (5), (6) and (12), the characteristic curves can be deduced as

\[
t_{sc} = \frac{1 + s}{2\sqrt{s}}, \\
t_{di} = \frac{3(13\sqrt{s} + 13s^{3/2} - 9s^{3} - 14s^{2} - 55s - 32)}{16(1 + 5s)},
\]

which are plotted in figure 3.

The blue solid curve in the figure is the coexistence curve based on equations (10) and (11), which is divided into the saturated small phase and saturated large phase using the extreme point. The orange and red curves are the divergence curve and the sign-changing curve of \( \tilde{K} \), respectively. The area above the \( t_{sc} \) curve that is painted in pink corresponds to the positive value of the curvature scalar, and the interaction is a repulsive domain. Meanwhile, other areas with \( \tilde{K} < 0 \) implied the interaction between the parts of the black hole is an attractive domain. The figure tells us that only the large black hole in low temperatures would show the repulsive interaction. The phase structure was also exhibited in the figure. Above the blue dashed line, the black hole is in the supercritical phase.

3. Thermodynamic geometry and non-local variables

As we have reviewed the thermodynamic geometry of the RN-AdS black hole, we will further investigate its relation to the non-local observables in a given CFT, including the (HEE) and the two-point correlation function. We would like to start this topic with HEE, which has been proven to show similar oscillation behavior as that of the van der Waals phase transition. In this sense, we will explore whether HEE can reflect the thermodynamic geometry of a black hole.

The entanglement entropy (EE) denotes the relationship between two subsystems of a quantum system, which is denoted by \( A \) and \( A^c \). When the quantum system lives in a CFT, the EE can be computed by the Ryu-Takayanagi (RT) recipe \[36, 37\]

\[
S_A = \frac{\text{Area}(\Gamma_A)}{4G}.
\]

Here, the \( \Gamma_A \) is a codimension-2 minimal surface in bulk AdS space that has the same boundary conditions as \( A \) on the boundary CFT. According to the metric function equations (4) and (13), the area of the minimal surface \( \Gamma_A \) can be calculated as

\[
S = \frac{\pi}{2} \int_{0}^{\theta_0} \mathcal{L}(r(\theta), \theta) d\theta, \\
\mathcal{L} = r \sin \theta \sqrt{(\frac{r'}{r})^2 + 1},
\]

where \( r' = dr/d\theta \) with \( \theta_0 \) as the boundary condition of HEE in the \( \theta \) direction. Here, \( \mathcal{L} \) is now introduced as the Lagrangian with \( \theta \). The only analytical solution of \( r(\theta) \) is pure AdS spacetime in the bulk. In the study of HEE, we will solve the equation of motion in ordinary spacetime numerically with the conditions that

\[
r'(0) = 0, \quad r(0) = r_0.
\]
Notice that for the UV-divergent of the EE, we should regulate it by subtracting the area of the minimal surface in pure AdS, which we denote as $\delta S$. Calculating the EE with the RT formula, we ask $\theta_0$ to be a small value to make sure the minimal surface can return to the subsystem continuously. Therefore, we set $\theta_0 = 0.1$ and the UV-cutoff in CFT with $r(0.099)$. To check whether the HEE can be exploited to reflect the curvature scalar of the black hole, we study the relationship between the Ruppeiner geometry and HEE. The equation for HEE in equation (14) indicated the relation in $r_s$ and $S_\chi$, with which we can establish a one-to-one correspondence between the black hole phase transition and the oscillating behavior of non-local observables in the given CFT. The numerical results of the curvature scalar in terms of HEE with different charges is shown in figure 4. The behavior of the curvature scalar with respect to the dimensionless parameter $\delta S/\ell^2$ with $q < q_c$, $q = q_c$, and $q > q_c$ is shown from left to right. With the numerical method, there is a point-to-point correspondence to that in figure 2. As with the black hole entropy, the EE also reveals the phase structure of the RN-AdS black hole with a fixed charge, which is identical to that of the van der Waals fluid. Furthermore, with a glimpse of the value of the HEE, we can assert the inter-action within the corresponding black hole.

To show the relation between the thermodynamic properties of the RN-AdS black hole and the HEE, we plot the characteristic curves for the Hawking temperature and EE of CFT in figure 5. The blue dashed line denotes the critical value of $t$, beyond which the corresponding black hole is undergoing a first-order phase transition. When the temperature is at the critical value, the black hole will show a sec-ond-order phase transition. Furthermore, the black hole is in the superficial phase and $1/t$ exceeds the blue dashed line. The blue solid curve is the coexistence curve, which separates the saturated small and large black hole phases through the critical point. The EE $\delta S$ with a small or large value indicates that the black hole is thermodynamically stable. The orange curve is the divergent curve of the scalar curvature, which distinguishes the unstable phase from the metastable phase. The red solid curve is the sign-changing curve, and the top area painted in red represents $R > 0$. Moreover, the correspondence of these characteristic curves between the black hole entropy and HEE announces that the thermodynamic information of the former can be read from the dual field theory.

Now, let us focus on the two-point correlation function in CFT. The AdS/CFT correspondence implies that the equal-time two-point correlation function with a large conformal dimension $\Delta$ of the scalar operator $O$ in the dual field theory is holographically approximated as [63]

$$\langle O(t_0, x_i)O(t_0, x_j) \rangle \approx e^{-\Delta L},$$

where $L$ is the length between the points $(t_0, x_i)$ and $(t_0, x_j)$ on the AdS boundary measured by the metric of the bulk geodesic. Due to the spacetime symmetry, we can let $x_i = \theta$, and the boundary is marked as $\theta_0$. Therefore, the proper length can be parameterized as

$$L = \int_{\theta_0}^{\theta} L(r(\theta), r) d\theta, \quad L \equiv \sqrt{(i)^2 + r^2}, \quad (16)$$

where the dot denotes $i = dr/d\theta$. The equation of motion is obtained from the Euler–Lagrange equation with the Lagrangian $L$ with respect to $\theta$. By applying the boundary condition equation (15), $r(\theta)$ can be deduced by solving equation (16). We apply the numerical method to calculate the geodesic length, which is difficult to obtain in analytical
form. Due to the divergence of the geodesic length at the boundary $\theta_0$, it should be regularized by subtracting the geodesic length in pure AdS with the same boundary conditions, denoted as $\delta L$. For that purpose, we choose $\theta_0 = 0.1$ and the UV-cutoff in the dual field theory as $r(0.099)$. The relationship between $\tilde{\chi}$ and $\delta L/l$ is shown in figure 6.

Figure 6. The dimensionless curvature scalar for the RN-AdS black hole in terms of $\delta L/l$. Here, we set (a) $q < q_c$, (b) $q = q_c$ and (c) $q > q_c$ from left to right. The entropy on the red line corresponds to the vanishing value of the reduced temperature $t$.

Figure 7. The characteristic curve of the curvature scalar in coordinate space $\{\delta L/l, 1/t\}$. The orange and red solid lines are the divergence curve and the variable sign curve of the curvature scalar, respectively. The blue solid line is the coexistence curve. The blue dashed curve in the figure corresponds to the critical point temperature.

4. Conclusion

In this paper, we investigate the probe of the thermodynamic geometry of the RN-AdS black hole by the non-local observables in a given CFT. Through dimensional analysis, we introduce dimensionless thermodynamic variables and establish the corresponding state laws. This principle suggests that charged black holes in different AdS backgrounds behave similarly as long as they are in the same state described by the same dimensionless parameters. By considering the black hole mass, we obtain the thermodynamic scalar curvature equation (12) and show the thermodynamic behavior of the black hole, which implies that $\tilde{\mathcal{R}}$ can be exploited to display the phase transition of the black hole, and the underlying microstructure. We then investigate the thermodynamic geometry of the black hole with the non-local observables in the given CFT. Interestingly, the HEE and equal-time two-point correlation function also show similar behavior in the $\tilde{\mathcal{R}} - \delta S/l^2$ and $\tilde{\mathcal{R}} - \delta L/l$ planes, respectively, to the black hole. These results suggest that the HEE and equal-time two-point correlation function in the given CFT can serve as good probes of the black hole’s thermodynamic geometry.

In view of the invisibility of the black hole microstructure, if the holographic form of the thermodynamic geometry is established by the AdS/CFT dual theory, it may disclose the microscopic mechanism of the black hole thermodynamic behavior. We will focus on this issue in our future work.

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