On positive hypergraphs

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Abstract
Camarena, Csóka, Hubai, Lippner, and Lovász introduced the notion of positive graphs. This notion naturally extends to \( r \)-uniform hypergraphs. In the case when \( r \) is odd, we prove that a hypergraph is positive if and only if its Levi graph is positive. As an application, we show that the 1-subdivision of \( K_{r,r} \) is not a positive graph when \( r \) is odd.

Keywords: positive graph, homomorphism density, Levi graph, subdivision, grid hypergraph

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1. Introduction

For a (hyper-)graph \( G \), we denote by \( V(G) \) and \( E(G) \) its vertex-set and edge-set, respectively. Let \( v(G) := |V(G)| \) and \( e(G) := |E(G)| \). Let \( \mathcal{F} \) denote the class of all bounded measurable functions \( f : [0, 1]^2 \to \mathbb{R} \). Let \( \mathcal{F}_{sym}^r \) denote the class of all bounded measurable symmetric \( r \)-variate functions \( f : [0, 1]^r \to \mathbb{R} \). In particular, \( \mathcal{F}_{sym}^2 \) is a subclass of \( \mathcal{F} \).

For \( f \in \mathcal{F}_{sym}^2 \), the homomorphism density from a graph \( G \) to \( f \) is defined as

\[
t_G(f) := \int \prod_{\{u,v\} \in E(G)} f(x_u, x_v) \, d\mu^{v(G)},
\]

where \( \mu \) is the Lebesgue measure on \([0, 1]\) (see [6]).

Let \( \pi \) be an automorphism of \( G \) such that \( \pi = \pi^{-1} \) (that is, \( \pi \) is an involution). Then \( \pi \) defines a partition of \( V(G) \) into three subsets, \( V_0, V_+, V_- \), where \( V_0 = \{ v \in V(G) : \pi(v) = v \} \) and \( \pi(V_+) = V_- \). Following the terminology of [3], we call \( \pi \) a stable involution if there are no edges with both ends in \( V_0 \), nor edges with one end in \( V_+ \) and the other in \( V_- \).

A graph \( G \) is called positive if \( t_G(f) \geq 0 \) for all \( f \in \mathcal{F}_{sym}^2 \). It is not hard to see that if \( G \) has a stable involution, then \( G \) is positive. Camarena, Csóka, Hubai, Lippner, and Lovász [1] conjectured that every positive graph must have a stable involution (in the language of [1], such graphs are called “symmetric”). This is known as the positive graph conjecture.

The 1-subdivision of \( K_{r,r} \), where \( r \) is even, has a stable involution, and hence, is positive. When \( r \geq 3 \) is odd, the 1-subdivision of \( K_{r,r} \) does not have a
stable involution and is one of the simplest graphs for which the positive graph conjecture is still open.

If \( G \) is a connected bipartite graph, then its vertices are partitioned into two sets, and definition (1.1) naturally extends to asymmetric functions \( f \in \mathcal{F} \). Our first result is

**Theorem 1.1.** If \( G \) is a positive bipartite graph, then \( t_G(f) \geq 0 \) for any asymmetric \( f \in \mathcal{F} \).

We define positivity for \( r \)-graphs (which are \( r \)-uniform hypergraphs) in the same way as it was done for simple graphs. Let \( H \) be an \( r \)-graph (possibly, with multiple edges). For \( f \in \mathcal{F}^{\text{sym}}_r \), we define

\[
t_H(f) := \int \prod_{\{v_1, v_2, \ldots, v_r\} \in E(H)} f(x_{v_1}, x_{v_2}, \ldots, x_{v_r}) \, d\mu^r(H).
\]

We say that \( H \) is **positive** if \( t_H(f) \geq 0 \) for all \( f \in \mathcal{F}^{\text{sym}}_r \).

The **Levi graph** (or incidence graph) of \( r \)-graph \( H \) is a bipartite graph \( L(H) \) with the bipartition \( (V(H), E(H)) \), where vertices \( v \in V(H) \) and \( e \in E(H) \) are adjacent if and only if \( v \) is incident to \( e \) in \( H \). Hence, the adjacency matrix of \( L(H) \) is the incidence matrix of \( H \). We use Theorem 1.1 to prove

**Theorem 1.2.** Let \( r \geq 3 \) be odd. Then an \( r \)-graph is positive if and only if its Levi graph is positive.

The requirement for \( r \) to be odd in Theorem 1.2 is essential. Indeed, consider an \( r \)-graph \( H \) that consists of a single edge. Obviously, \( H \) is not positive. However, \( L(H) = K_{r,1} \) is positive when \( r \) is even.

A **homomorphism** from an \( r \)-graph \( H \) to an \( r \)-graph \( H' \) is a map from \( V(H) \) to \( V(H') \) which maps edges to edges. An **endomorphism** of \( H \) is a homomorphism from \( H \) to itself. An **automorphism** of \( H \) is an endomorphism that is a bijection. We call an edge \( e \in E(H) \) odd if for any endomorphism \( \pi \) of \( H \), \( e \) is the image of exactly one of the edges, that is, \( |\{e' \in E(H) : \pi(e') = e\}| = 1 \). In particular, if every pair of vertices is contained in at least one of the edges, then every endomorphism of \( H \) is an automorphism, so each edge is odd.

We now give a couple of examples to illustrate new opportunities which Theorem 1.2 provides.

**Proposition 1.3.** An \( r \)-graph that has an odd edge is not positive.

Let \( n \geq m > k \). The **set-inclusion graph** \( I(n, m, k) \) is a bipartite graph whose vertices are subsets of \( [n] := \{1, 2, \ldots, n\} \) of sizes \( m \) and \( k \), and edges are pairs \( X, Y \subseteq [n] \) where \( |X| = m, |Y| = k, Y \subset X \). If \( m \geq 2k \), and \( r = \binom{m}{k} \), then \( I(n, m, k) \) is the Levi graph of an \( r \)-graph where for every two vertices there is an edge that contains both of them.

**Corollary 1.4.** If \( \binom{m}{k} \) is odd and \( n \geq m \geq 2k \), then \( I(n, m, k) \) is not positive.
Let $r$-graph $H$ be $r$-partite, that is, there exists a homomorphism from $H$ to a single edge. Then $H$ has no odd edges. Let $H_r$ denote an $r$-graph with $r^2$ vertices which are the nodes of an $r \times r$ grid, and with $2r$ edges which are formed by the horizontal and vertical lines of that grid. It is easy to see that $H_r$ is $r$-partite. Observe that $L(H_r)$ is the 1-subdivision of $K_{r,r}$.

**Proposition 1.5.** The grid hypergraph $H_r$ is not positive for odd $r \geq 3$.

**Corollary 1.6.** The 1-subdivision of $K_{r,r}$ is not positive for odd $r \geq 3$.

2. Proofs

An $r$-dimensional symmetric tensor of size $n$ is an array of real values $[a_{i_1,i_2,\ldots,i_r}]$ with $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, n\}$ such that $a_{i_1,i_2,\ldots,i_r} = a_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(r)}}$ for any permutation $\sigma$ of $1, 2, \ldots, r$. Every symmetric tensor of size $n$ and dimension $r$ may be uniquely associated with a homogeneous polynomial of degree $r$ in $n$ variables. It is a classical result known from the XIX century that there always exists a symmetric decomposition

$$a_{i_1,i_2,\ldots,i_r} = \sum_{j=1}^{N} \lambda_j b_{i_1,j} b_{i_2,j} \cdots b_{i_r,j} \quad \text{for} \quad i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, n\}, \quad (2.1)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are reals and $[b_{ij}]$ is a real $n \times N$ matrix. Lemma 4.2 of [2] proves a similar statement for complex tensors, and this proof can be repeated verbatim for real tensors. The basic idea is to translate symmetric tensors in polynomials, and then the non-existence of a decomposition for a tensor translates in the existence of a nonzero polynomial that vanishes everywhere on $\mathbb{R}^n$, giving a contradiction.

We call a symmetric $r$-variate function $h \in \mathcal{F}_r^{sym}$ a step function of size $n$ if there exists a partition $[0,1] = X_1 \cup X_2 \cup \ldots \cup X_n$ with $\mu(X_1) = \mu(X_2) = \cdots = \mu(X_n) = 1/n$ and a symmetric $r$-dimensional tensor $[a_{i_1,i_2,\ldots,i_r}]$ of size $n$ such that $h$ is equal to $a_{i_1,i_2,\ldots,i_r}$ everywhere on $X_{i_1} \times X_{i_2} \times \cdots \times X_{i_r}$.

Let $\pi$ be a homomorphism from $r$-graph $H$ to $r$-graph $G$. The homomorphic image of $H$ (under $\pi$) is the subgraph of $G$ whose vertices (edges) are images of all vertices (edges) of $H$. Let $w: E(G) \to \mathbb{R}$ define weights of the edges of $G$. Then the weight of $\pi$ is $W(w, \pi) := \prod_{e \in H} w(\pi(e))$. If the sum of $W(w, \pi)$ over all homomorphisms $\pi$ from $H$ to $G$ is negative, then $H$ is not positive. Indeed, we may assume that $V(G) = \{1, 2, \ldots, N\}$. Define a symmetric tensor $[a_{i_1,i_2,\ldots,i_r}]$ with $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, N\}$ as follows. If $e = \{i_1, i_2, \ldots, i_r\}$ is an edge of $G$, we set $a_{i_1,i_2,\ldots,i_r} = w(e)$, otherwise set $a_{i_1,i_2,\ldots,i_r} = 0$. Then for a step-function $h \in \mathcal{F}_r^{sym}$ associated with this tensor, we get $t_{H_r}(h) < 0$.

**Proof of Theorem 1.1.** If $G_1, G_2, \ldots, G_k$ are connected components of $G$, then $t_G(f) = \prod_{i=1}^{k} t_{G_i}(f)$. Thus, it is sufficient to consider the case when $G$ is
connected. Consider an asymmetric function \( f \in \mathcal{F} \). Define its “transpose” \( f^T \) as \( f^T(x, y) := f(y, x) \). Define symmetric function \( g \in \mathcal{F}_2^{sym} \) as follows:

\[
g(x, y) := \begin{cases} 
0 & \text{if } 0 \leq x, y < 1/2; \\
f(2x, 2y - 1) & \text{if } 0 \leq x < 1/2 \leq y \leq 1; \\
f(2y, 2x - 1) & \text{if } 0 \leq y < 1/2 \leq x \leq 1; \\
0 & \text{if } 1/2 \leq x, y \leq 1.
\end{cases}
\]

As \( G \) is positive and \( g \) is symmetric, \( t_G(g) \geq 0 \). As \( G \) is connected,

\[
t_G(g) = 2^{-v(G)}(t_G(f) + t_G(f^T)).
\]

Hence, \( t_G(f) + t_G(f^T) \geq 0 \).

Let \( h \) be the tensor product of \( f \) and \( f^T \), that is, a function on \([0, 1]^4\) defined by \( h((x, y), (z, w)) := f(x, z) f^T(y, w) \). Since there is a measure-preserving bijection from \([0, 1]^2\) onto \([0, 1]^2\), we may regard \( h \) as a function on \([0, 1]^2\).

Then \( t_G(h) = t_G(f) \cdot t_G(f^T) \). As \( G \) is positive and \( h \) is symmetric, we get \( t_G(f) \cdot t_G(f^T) \geq 0 \). Since both the sum and the product of \( t_G(f) \), \( t_G(f^T) \) are nonnegative, \( t_G(f) \) itself must be nonnegative.

**Proof of Theorem 1.2.** Suppose first that \( r \)-graph \( H \) is positive. For \( f \in \mathcal{F}_2^{sym} \), define \( h \in \mathcal{F}_r^{sym} \) by

\[
h(x_1, x_2, \ldots, x_r) := \int f(x_1, y) f(x_2, y) \cdots f(x_r, y) \, dy.
\]

Then \( t_{L(H)}(f) = t_H(h) \geq 0 \), so \( L(H) \) is also positive.

Now suppose that \( H \) is not positive. Then there exists a function \( g \in \mathcal{F}_r^{sym} \) such that \( t_H(g) < 0 \). Since the functional \( t_H(\cdot) \) is continuous in the \( L_1 \) metric, there exists an integer \( n \) and a step function \( h \in \mathcal{F}_r^{sym} \) of size \( n \) such that \( t_H(h) < 0 \). Let \([a_{i_1, i_2, \ldots, i_r}]\) be a symmetric tensor associated with the values of \( h \). As \( r \) is odd, we can get rid of coefficients \( \lambda_j \) in representation (2.1) by rescaling \( c_{i_1} = (N \lambda_j)^{1/r} b_{i_1} \):

\[
a_{i_1, i_2, \ldots, i_r} = \frac{1}{N} \sum_{j=1}^{N} c_{i_1} c_{i_2} \cdots c_{i_r} \quad \text{for } i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, n\}.
\]

Consider a partition \([0, 1] = Y_1 \cup Y_2 \cup \ldots \cup Y_N \) where \( \mu(Y_1) = \mu(Y_2) = \cdots = \mu(Y_N) = 1/N \) and an asymmetric function \( f \in \mathcal{F} \) which is equal to \( c_{ij} \) everywhere on \( X_i \times Y_j \) for \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, N \). Then

\[
h(x_1, x_2, \ldots, x_r) = \int f(x_1, y) f(x_2, y) \cdots f(x_r, y) \, dy,
\]

and \( t_{L(H)}(f) = t_H(h) < 0 \). Therefore, by Theorem 1.1, \( L(H) \) is not positive.

**Proof of Proposition 1.3.** Let \( e \) be an odd edge in \( r \)-graph \( H \). Set \( w(e) = -1 \) and \( w(e') = +1 \) for edges \( e' \neq e \). Then for every endomorphism \( \pi \) of \( H \), we have \( W(w, \pi) = -1 \). Hence, \( H \) is not positive.
In order to prove Proposition 1.5, we need a few auxiliary results. An \( r \)-graph is called linear if \( |e' \cap e''| \leq 1 \) for any pair of distinct edges \( e', e'' \). An \( r \)-graph is called a triangle if it consists of 3 edges \( e', e'', e''' \) such that \( |e' \cap e'''| = |e'' \cap e'''| = 1 \) and \( e' \cap e'' \cap e''' = \emptyset \).

**Lemma 2.1.** Let \( r \geq 3 \). Let \( \pi \) be a homomorphism from \( H_r \) to a linear \( r \)-graph \( G \). Then the image of \( H_r \) under \( \pi \) is either isomorphic to \( H_r \), or is a single edge, or contains a triangle.

**Proof.** We denote the vertices of \( H_r \) by \( (i, j) \) where \( i, j = 1, 2, \ldots, r \). The edges are rows \( R_i = \{(i, 1), (i, 2), \ldots, (i, r)\} \) and columns \( C_j = \{(1, j), (2, j), \ldots, (r, j)\} \). If the image of \( H_r \) has \( r^2 \) vertices, then it is isomorphic to \( H_r \). Thus, we may assume that there are two vertices which have the same image. Such two vertices can not belong to the same row or column. Without limiting generality, we may assume that \( \pi(1, 2) = \pi(2, 1) \). Notice that the image of \( R_1 \) and the image of \( C_1 \) have two common vertices \( \pi(1, 1) \) and \( \pi(1, 2) = \pi(2, 1) \) where \( \pi(1, 1) \neq \pi(1, 2) \). Since \( G \) is linear, then \( \pi(1, 2) = \pi(2, 1) \). Notice that the image of \( R_1 \) and the image of \( C_1 \) must coincide. Similarly, the images of \( R_2 \) and \( C_2 \) must coincide, too. We now may renumber rows \( R_i \) with \( i \geq 3 \) and columns \( C_j \) with \( j \geq 3 \) in such a way that \( \pi(i, i) = \pi(i, 1) \) for all \( i \geq 3 \). Then the images of \( C_i \) and \( R_i \) for \( i \geq 3 \) have two common vertices \( \pi(i, i) \) and \( \pi(1, 1) \neq \pi(1, i) \). By a similar argument, we conclude that the images of \( R_i \) and \( C_i \) coincide.

We claim that if two rows \( R_i \) and \( R_j \) \( (i \neq j) \) have the same image, then all rows share the same image, so the image of \( H_r \) is a single edge. Indeed, consider a row \( R_k \) where \( k \neq i, j \). As \( R_i, R_j, C_i, C_j \) have the same image, the image of \( R_k \) shares with them two common vertices, \( \pi(k, i) \) and \( \pi(k, j) \). As \( G \) is linear, the image of \( R_k \) must be the same as the image of \( R_i \) and \( R_j \).

Now assume that all rows have distinct images. Since the image of \( C_j \) is the same as the image of \( R_j \), the images of rows \( R_i \) and \( R_j \) have \( \pi(i, j) \) as a common vertex. As \( G \) is linear, then \( |\pi(R_i) \cap \pi(R_j)| = 1 \) for \( i \neq j \). Then the images of \( R_1, R_2, R_3 \) either form a triangle, or \( |\pi(R_1) \cap \pi(R_2) \cap \pi(R_3)| = 1 \). To finish the proof, we need to exclude the latter case. Suppose, \( \pi(1, j_1) = \pi(2, j_2) = \pi(3, j_3) \). Then by an argument similar to the one we used before, the image of \( C_{j_1} \) must be the same as the image of \( R_2 \), and simultaneously, the image of \( C_{j_3} \) must be the same as the image of \( R_3 \). Hence rows \( R_2 \) and \( R_3 \) have the same image, which contradicts our assumption.

\( \square \)

**Lemma 2.2 ([5, Section 1.4]).** Let \( r \geq 3 \). For all sufficiently large \( n \), there exists a linear \( r \)-graph \( G_{r,n} \) with \( n \) vertices and at least \( n^{3/2} \) edges that contains neither a triangle nor \( H_r \) as a subgraph.

**Proof of Proposition 1.5.** Let \( G_{r,n} \) be the \( r \)-graph from Lemma 2.2. By Lemma 2.1, every homomorphism from \( H_r \) to \( G_{r,n} \) maps \( H_r \) to a single edge of \( G_{r,n} \). We construct the "box product" \( r \)-graph \( G_{r,n} \boxtimes G_{r,n} \) whose vertex set is \( V(G_{r,n}) \times V(G_{r,n}) \). It has two types of edges. The "horizontal" ones are \( \{(x_1, y_1), (x_2, y_2), \ldots, (x_r, y_r)\} \) where \( \{x_1, x_2, \ldots, x_r\} \) is an edge in \( G_{r,n} \). The "vertical" edges are of form \( \{(x, y_1), (x, y_2), \ldots, (x, y_r)\} \) where \( \{y_1, y_2, \ldots, y_r\} \) is an edge in \( G_{r,n} \). Then \( e(G_{r,n} \boxtimes G_{r,n}) = 2ne(G_{r,n}) \).
We claim that any homomorphism from $H_r$ to $G_{r,n} \square G_{r,n}$ falls to one of two types. A homomorphism of the first type maps $H_r$ to a single edge of $G_{r,n} \square G_{r,n}$ (horizontal or vertical). A homomorphism of the second type maps the vertices of $H_r$ to $r^2$ vertices $(x_i, y_j)$ of $G_{r,n} \square G_{r,n}$ $(i, j = 1, 2, \ldots, r)$ where \{x_1, x_2, \ldots, x_r\} and \{y_1, y_2, \ldots, y_r\} are edges in $G_{r,n}$. In a homomorphism of the second type, the images of the edges of $H_r$ are $r$ horizontal and $r$ vertical edges of $G_{r,n} \square G_{r,n}$. Indeed, let $\pi$ be a homomorphism from $H_r$ to $G_{r,n} \square G_{r,n}$. Suppose, there are two horizontal edges $e', e''$ in $H_r$ such that $\pi(e')$ is a horizontal edge in $G_{r,n} \square G_{r,n}$ and $\pi(e'')$ is a vertical edge. Notice that $|\pi(e') \cap \pi(e'')| \leq 1$, and for any $z' \in \pi(e') - \pi(e'')$ and $z'' \in \pi(e'') - \pi(e')$ there is no edge in $G_{r,n} \square G_{r,n}$ that would contain both $z'$ and $z''$. In this case, $\pi$ would not be able to map vertical edges of $H_r$ to the edges of $G_{r,n} \square G_{r,n}$. Thus, we conclude that parallel edges of $H_r$ must be mapped to parallel edges of $G_{r,n} \square G_{r,n}$. Suppose $\pi$ maps all edges of $H_r$ to horizontal edges of $G_{r,n} \square G_{r,n}$. As $H_r$ is connected, each vertex $v$ of $H_r$ must be mapped to vertex $(x_v, y_0)$ with fixed $y_0$. Then, by the construction of $G_{r,n}$, $\pi$ maps all edges of $H_r$ to a single horizontal edge of $G_{r,n} \square G_{r,n}$. Similarly, if all edges of $H_r$ are mapped to vertical edges of $G_{r,n} \square G_{r,n}$, then the edges of $H_r$ are mapped to a single vertical edge. Hence, if $\pi$ does not belong to the first type, it must map all horizontal (vertical) edges of $H_r$ to horizontal (vertical) edges of $G_{r,n} \square G_{r,n}$, or vice versa. In this case, it is a homomorphism of a second type.

The number of homomorphisms of the first type is $c_r \cdot e(G_{r,n} \square G_{r,n}) = 2^{r+1} \cdot e(G_{r,n})$. The number of homomorphisms of the second type is $C_r \cdot e(G_{r,n})^2$. Thus, for sufficiently large $n$, the number of homomorphisms of the second type is greater. We assign weight $+1$ to each horizontal edge of $G_{r,n} \square G_{r,n}$ and $-1$ to each vertical edge. Then the weight of a homomorphism of the first type is equal to $+1$. As $r$ is odd, the weight of a homomorphism of the second type is equal to $-1$. Thus, for sufficiently large $n$, the sum of the weights of all homomorphisms is negative. Therefore, $H_r$ is not positive.  

3. Concluding remarks

The notion of stable involution can be generalized to $r$-graphs. Let $\pi$ be an involution of $V(H)$. Then $\pi$ defines a partition of $V(G)$ into three subsets, $V_0, V_+, V_-$, where $V_0 = \{v \in V(G) : \pi(v) = v\}$ and $\pi(V_+) = V_-$. We call $\pi$ a stable involution if it preserves the edges, and there are no edges which are contained in $V_0$ entirely, nor edges which intersect both $V_+$ and $V_-$. It is not hard to show that if such an involution exists then $H$ is positive.

**Conjecture 3.1.** An $r$-graph is positive if and only if it has a stable involution.

Theorem 1.2 demonstrates that Conjecture 3.1 for odd $r$ would follow from the positive graph conjecture.

Conlon and Lee [4] proved that every positive graph must contain at least one vertex of even degree. It implies that a positive $r$-graph with odd $r$ also must have a vertex of even degree.
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