NEARLY SASAKIAN GEOMETRY AND $SU(2)$-STRUCTURES

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Abstract. We carry on a systematic study of nearly Sasakian manifolds. We prove that any nearly Sasakian manifold admits two types of integrable distributions with totally geodesic leaves which are, respectively, Sasakian and 5-dimensional nearly Sasakian manifolds. As a consequence, any nearly Sasakian manifold is a contact manifold. Focusing on the 5-dimensional case, we prove that there exists a one-to-one correspondence between nearly Sasakian structures and a special class of nearly hypo $SU(2)$-structures. By deforming such an $SU(2)$-structure one obtains in fact a Sasaki-Einstein structure. Further we prove that both nearly Sasakian and Sasaki-Einstein 5-manifolds are endowed with supplementary nearly cosymplectic structures. We show that there is a one-to-one correspondence between nearly cosymplectic structures and a special class of hypo $SU(2)$-structures which is again strictly related to Sasaki-Einstein structures. Furthermore, we study the orientable hypersurfaces of a nearly Kähler 6-manifold and, in the last part of the paper, we define canonical connections for nearly Sasakian manifolds, which play a role similar to the Gray connection in the context of nearly Kähler geometry. In dimension 5 we determine a connection which parallelizes all the nearly Sasakian $SU(2)$-structure as well as the torsion tensor field. An analogous result holds also for Sasaki-Einstein structures.

1. Introduction

Nearly Kähler manifolds were defined by Gray [15] as almost Hermitian manifolds $(M, J, g)$ such that the Levi-Civita connection satisfies
\[(\nabla_X J)Y + (\nabla_Y J)X = 0\]
for any vector fields $X$ and $Y$ on $M$. The development of nearly Kähler geometry is mainly due to the studies of Gray [15], [16], [17] and, more recently, to the work of Nagy [21], [22]. Nearly Sasakian manifolds where introduced by Blair, Yano and Showers in [4] as an odd dimensional counterpart of nearly Kähler manifolds, together with nearly cosymplectic manifolds, studied by Blair and Showers some years earlier [2], [3]. Namely, a smooth manifold $M$ endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$ is said to be nearly Sasakian or nearly cosymplectic if, respectively,
\[(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X,\]
\[(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0\]
for every vector fields $X$ and $Y$ on $M$. Since the foundational articles of Blair and his collaborators, these two classes of almost contact structures were studied by some authors and, later on, have played a role in the China-Gonzalez's classification of almost contact metric manifolds [8]. Recently, they naturally appeared in the study of harmonic almost contact structures (cf. [14], [18], [29]).

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Actually it is more difficult than expected to find relations between nearly Sasakian and nearly Kähler manifolds, like for Sasakian / Kähler geometry. For instance, it is known that, like Sasakian manifolds, the Reeb vector field $\xi$ of any nearly Sasakian manifold $M$ defines a Riemannian foliation. Then one would expect that the space of leaves of this foliation is nearly Kähler, but this happens if and only if $M$ is Sasakian, and in that case the space of leaves is Kähler. Moreover, it is not difficult to see that the cone over $M$ is nearly Kähler if and only if $M$ is Sasakian and, again, in this case the cone is Kähler. Similar results hold also in the nearly cosymplectic setting. For instance, if one applies the Morimoto’s construction [20] to the product $N$ of two nearly cosymplectic manifolds $M_1$ and $M_2$, one finds that $N$ is nearly Kähler if and only if both $M_1$ and $M_2$ are coKähler.

In the present paper we show in fact that there are many differences between nearly Kähler and nearly Sasakian manifolds, much more than in Kähler / Sasakian setting.

It is known that the structure $(1,1)$-tensor field $\phi$ of a Sasakian manifold is given by the opposite of the covariant derivative of the Reeb vector field. Thus in any nearly Sasakian manifold one is lead to define a tensor field $h$ by

$$\nabla \xi = -\phi + h.$$ 

This tensor field measures, somehow, the non-Sasakianity of the manifold and plays an important role in our study. Namely, first we prove that the eigenvalues of the symmetric operator $h^2$ are constants and its spectrum is of type

$$\text{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_r^2\}$$

with $\lambda_i \neq 0$ for each $i \in \{1, \ldots, r\}$. Then we prove the following theorem.

**Theorem 1.1.** Let $M$ be a (non-Sasakian) nearly Sasakian manifold with structure $(\phi, \xi, \eta, g)$. Then the tangent bundle of $M$ splits as the orthogonal sum

$$TM = D(0) \oplus D(-\lambda_1^2) \oplus \cdots \oplus D(-\lambda_r^2)$$

of the eigendistributions of $h^2$. Moreover,

a) the distribution $D(0)$ is integrable and defines a totally geodesic foliation of $M$ of dimension $2p + 1$. If $p > 0$ then the leaves of $D(0)$ are Sasakian manifolds;

b) each distribution $[\xi] \oplus D(-\lambda_i^2)$ is integrable and defines a totally geodesic foliation of $M$ whose leaves are 5-dimensional nearly Sasakian non-Sasakian manifolds.

Furthermore, if $p > 0$ the distribution $[\xi] \oplus D(-\lambda_1^2) \oplus \cdots \oplus D(-\lambda_r^2)$ is integrable and defines a Riemannian foliation with totally geodesic leaves, whose leaf space is Kähler.

As a consequence of Theorem 1.1 we shall prove that in every nearly Sasakian manifold the 1-form $\eta$ is a contact form. This establishes a sensible difference with respect to nearly Kähler geometry, since in any nearly Kähler manifold the Kähler form is symplectic if and only if the manifold is Kähler.

The point b) of Theorem 1.1 motivates us to further investigate 5-dimensional nearly Sasakian manifolds. Some early studies date back to Olszak [25] who proved that 5-dimensional nearly Sasakian non-Sasakian manifolds are Einstein and of scalar curvature $> 20$. In the present paper we characterize nearly Sasakian structures in terms of $SU(2)$-structures defined by a 1-form $\eta$ and a triple $(\omega_1, \omega_2, \omega_3)$ of 2-forms according to [9]. One of our main results is to prove that there exists a one-to-one correspondence between nearly Sasakian structures on a 5-manifold and $SU(2)$-structures satisfying the following equations

\[ d\eta = -2\omega_3 + 2\lambda \omega_1, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1 - 3\lambda \eta \wedge \omega_3, \]
for some real number \( \lambda \neq 0 \) which depends only on the geometry of the manifold via the formula \( s = 20(1 + \lambda^2) \), where \( s \) is the scalar curvature. By deforming \((\eta, \omega_1, \omega_2, \omega_3)\) we obtain a Sasaki-Einstein structure with the same underlying contact form (up to a multiplicative factor) and, conversely, each Sasaki-Einstein 5-manifold carries a nearly Sasakian structure (in fact, a 1-parameter family of nearly Sasakian structures).

In Section 5 we get analogous results in terms of SU(2)-structures for nearly cosymplectic 5-manifolds. In particular we prove that any nearly cosymplectic 5-manifold is Einstein with positive scalar curvature. We also show that nearly cosymplectic structures arise naturally both in nearly Sasakian and in Sasaki-Einstein 5-manifolds. In particular, it is known that any Sasaki-Einstein SU(2)-structure can be described by the data of three almost contact metric structures \((\phi_1, \xi, \eta, g)\), \((\phi_2, \xi, \eta, g)\), \((\phi_3, \xi, \eta, g)\), with the same Reeb vector field, satisfying the quaternionic-like relations

\[
\phi_i \phi_j = \phi_k = - \phi_j \phi_i
\]

for any even permutation \((i, j, k)\) of \((1, 2, 3)\) and such that \((\phi_3, \xi, \eta, g)\) is Sasakian with Einstein Riemannian metric \(g\). Actually we prove that \((\phi_1, \xi, \eta, g)\) and \((\phi_2, \xi, \eta, g)\) are both nearly cosymplectic.

In Section 6 we study the (orientable) hypersurfaces of a nearly Kähler 6-manifolds. In particular we study the SU(2)-structures induced on hypersurfaces whose second fundamental form is of type \( \sigma = \beta(\eta \otimes \eta)\nu \) or \( \sigma = (-g + \beta(\eta \otimes \eta))\nu \), for some function \( \beta \), where \( \nu \) denotes the unit normal vector field. In both cases we prove that the hypersurface carries a Sasaki-Einstein structure, thus generalizing a result of [11].

Finally, in the last section of the paper, we try to define a canonical connection for nearly Sasakian manifolds, which may play a role similar to the Gray connection in the context of nearly Kähler geometry, i.e. the unique Hermitian connection with totally skew-symmetric torsion. In [13] Friedrich and Ivanov provided necessary and sufficient conditions for an almost contact metric manifold to admit a (unique) connection with totally skew-symmetric torsion parallelizing all the structure tensors. One can easily deduce that a nearly Sasakian manifold admits such a connection if and only if it is Sasakian. Thus, weakening some hypotheses, we define a family of connections, parameterized by a real number \( r \), which parallelize the almost contact metric structure and such that the torsion is skew-symmetric on the contact distribution \( \ker(\eta) \). In particular, if \( M \) is a Sasakian manifold our connection coincides with the Okumura connection [23]. In dimension 5 the connection corresponding to the value \( r = \frac{1}{2} \) parallelizes all the tensors in the associated SU(2)-structure \((\eta, \omega_1, \omega_2, \omega_3)\), as well as the torsion tensor field. Then for Sasaki-Einstein 5-manifolds we prove that the Okumura connection corresponding to \( r = \frac{1}{2} \) parallelizes the whole SU(2)-structure.

All manifolds considered in this paper will be assumed to be smooth i.e. of the class \( C^\infty \), and connected. We use the convention that \( u \wedge v = u \otimes v - v \otimes u \). Unless in the last Section, we shall implicitly assume that all the nearly Sasakian (respectively, nearly cosymplectic) manifolds considered in the paper are non-Sasakian (respectively, non-coKähler).

### 2. Preliminaries

An almost contact metric manifold is a differentiable manifold \( M^{2n+1} \) endowed with a structure \((\phi, \xi, \eta, g)\), given by a tensor field \( \phi \) of type \((1,1)\), a vector field \( \xi \), a 1-form \( \eta \) and a Riemannian metric \( g \) satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]
for every vector fields $X, Y$ on $M$. From the definition it follows that $\phi \xi = 0$ and $\eta \circ \phi = 0$. Moreover one has that $g(X, \phi Y) = -g(\phi X, Y)$ so that the bilinear form $\Phi := g(-, \phi -)$ defines in fact a 2-form on $M$, called fundamental 2-form.

Two remarkable classes of almost contact metric manifolds are given by Sasakian and coKähler manifolds. An almost contact metric manifold is said to be Sasakian if tensor field $N_\phi := [\phi, \phi] + \eta \otimes \xi$ vanishes identically and $d\eta = 2\Phi$, coKähler if $N_\phi \equiv 0$ and $d\eta = 0$, $d\Phi = 0$. The Sasakian and coKähler conditions can be equivalently expressed in terms of the Levi-Civita connection by, respectively,

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

$\nabla \phi = 0.$

For further details on Sasakian and coKähler manifolds we refer to [4] [5] and [7], respectively.

An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called nearly Sasakian if the covariant derivative of $\phi$ with respect to the Levi-Civita connection $\nabla$ satisfies

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X$$

for every vector fields $X, Y$ on $M$, or equivalently,

$$(\nabla_X \phi)X = g(X, X)\xi - \eta(X)X$$

for every vector field $X$ on $M$. This notion was introduced in [4] in order to study an odd dimensional counterpart of nearly Kähler geometry, and then it was studied by other authors. One can easily check that (2) is also equivalent to

$$(3) \quad 3g((\nabla_X \phi)Y, Z) = -d\Phi(X, Y, Z) - 3\eta(Y)g(X, Z) + 3\eta(Z)g(X, Y).$$

We recall now some basic properties satisfied by nearly Sasakian structures which will be used in the following. We refer to [4] [24] [25] for the details.

It is known that the characteristic vector field $\xi$ is Killing and the Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$. One can define a tensor field $h$ of type $(1, 1)$ by putting

$$(4) \quad \nabla_X \xi = -\phi X + hX.$$

The operator $h$ is skew-symmetric and anticommutes with $\phi$. Moreover, $h\xi = 0$ and $\eta \circ h = 0$. The vanishing of $h$ provides a necessary and sufficient condition for a nearly Sasakian manifold to be Sasakian (24). Applying (2) and (4), one easily gets

$$(5) \quad \nabla_\xi \phi = \phi h.$$

We remark the circumstance that the operator $h$ is also related to the Lie derivative of $\phi$ with respect to $\xi$. Indeed, using (4) and (5), we get

$$(\mathcal{L}_\xi \phi)X = [\xi, \phi X] - \phi[\xi, X] = (\nabla_\xi \phi)X - \nabla_{\phi X} \xi + \phi(\nabla_X \xi) = 3\phi hX.$$ Denote by $R$ the Riemannian curvature tensor. Olszak proved the following formula in [24]:

$$(6) \quad R(\xi, X)Y = (\nabla_X \phi)Y - (\nabla_Y \phi)X = g(X - h^2X, Y)\xi - \eta(Y)(X - h^2X).$$

The above equation, together with (5), gives

$$(7) \quad \nabla_\xi h = \nabla_\xi \phi = \phi h.$$ Furthermore, taking $Y = \xi$ in (6), we obtain

$$R(X, \xi)\xi = -\eta(\xi)\xi + X - h^2X = -\phi^2X - h^2X$$

and the $\xi$-sectional curvatures for every unit vector field $X$ orthogonal to $\xi$ are

$$K(\xi, X) = g(R(X, \xi)\xi, X) = 1 + g(hX, hX) \geq 1.$$
Notice that (6) also implies that
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y - \eta(Y)h^2X + \eta(X)h^2Y. \]
Moreover, the Ricci curvature satisfies
\[ \text{Ric}(\phi X, \phi Y) = \text{Ric}(X, Y) - (2n - \text{tr}(h^2))\eta(X)\eta(Y). \]
In particular it follows that the Ricci operator commutes with \( \phi \). Finally, Olszak proved that the symmetric operator \( h^2 \) has constant trace and the covariant derivatives of \( \phi \) and \( h^2 \) satisfy the following relations:
\[ g((\nabla_X \phi)Y, hZ) = \eta(Y)g(h^2X, \phi Z) - \eta(X)g(h^2Y, \phi Z) + \eta(Y)g(hX, Z), \]
\[ (\nabla_X h^2)Y = \eta(Y)(\phi - h)h^2X + g((\phi - h)h^2X, Y)\xi. \]
We now recall some facts about nearly cosymplectic manifolds. A nearly cosymplectic manifold is an almost contact metric manifold \( (M, \phi, \xi, \eta, g) \) such that the covariant derivative of \( \phi \) with respect to the Levi-Civita connection \( \nabla \) satisfies
\[ (\nabla_X \phi)Y + (\nabla_Y \phi)X = 0 \]
for every vector fields \( X, Y \). The above condition is equivalent to \( (\nabla_X \phi)X = 0 \), or also to
\[ 3g((\nabla_X \phi)Y, Z) = -d\Phi(X, Y, Z) \]
for any \( X, Y, Z \in \mathfrak{X}(M) \). Also in this case we have that \( \xi \) is Killing, \( \nabla\xi = 0 \) and \( \nabla\eta = 0 \). The tensor field \( h \) of type \((1,1)\) defined by
\[ \nabla_X \xi = hX \]
is skew-symmetric and anticommutes with \( \phi \). It satisfies \( h\xi = 0, \eta \circ h = 0 \) and
\[ \nabla_\xi \phi = \phi h. \]
Furthermore, \( h \) is related to the Lie derivative of \( \phi \) in the direction of \( \xi \). Indeed,
\[ (L_\xi \phi)X = (\nabla_\xi \phi)X - \nabla_{\phi X}\xi + \phi(\nabla_X \xi) = 3\phi hX. \]
Finally, the following formulas hold (110):
\[ g((\nabla_X \phi)Y, hZ) = \eta(Y)g(h^2X, \phi Z) - \eta(X)g(h^2Y, \phi Z), \]
\[ (\nabla_X h^2)Y = g(h^2X, Y)\xi - \eta(Y)h^2X, \]
\[ \text{tr}(h^2) = \text{constant}. \]

3. The foliated structure of a nearly Sasakian manifold

In this section we show that any nearly Sasakian manifold is foliated by two types of foliations, whose leaves are respectively Sasakian or 5-dimensional nearly Sasakian non-Sasakian manifolds. An important role in this context is played by the symmetric operator \( h^2 \) and by its spectrum \( \text{Spec}(h^2) \). We recall the following result.

**Theorem 3.1** (24). If a nearly Sasakian manifold \( M \) satisfies the condition
\[ h^2 = \lambda(I - \eta \otimes \xi) \]
for some real number \( \lambda \), then \( \dim(M) = 5 \).

**Proposition 3.2.** The eigenvalues of the operator \( h^2 \) are constant.
Proof. Let \( \mu \) be an eigenvalue of \( h^2 \) and let \( Y \) be a local unit vector field orthogonal to \( \xi \) such that \( h^2 Y = \mu Y \). Applying (10) for any vector field \( X \), and taking \( Y = Z \) we get

\[
0 = g((\nabla_X h^2)Y, Y) = g(\nabla_X (h^2 Y), Y) - g(h^2 (\nabla_X Y), Y) = X(\mu) g(Y, Y) + \mu g(\nabla_X Y, Y) - g(\nabla_X Y, h^2 Y) = X(\mu) g(Y, Y)
\]

which implies that \( X(\mu) = 0 \).

Notice that 0 is an eigenvalue of \( h^2 \), since \( h\xi = 0 \). Furthermore, being \( h \) skew-symmetric, the non-vanishing eigenvalues of \( h^2 \) are negative, so that the spectrum of \( h^2 \) is of type

\[
\text{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_i^2\},
\]

\( \lambda_i \neq 0 \) and \( \lambda_i \neq \lambda_j \) for \( i \neq j \). Further, if \( X \) is an eigenvector of \( h^2 \) with eigenvalue \( -\lambda_i^2 \), then \( X, \phi X, hX, h\phi X \) are orthogonal eigenvectors of \( h^2 \) with eigenvalue \( -\lambda_i^2 \).

In the following we denote by \([\xi]\) the 1-dimensional distribution generated by \( \xi \), and by \( D(0) \) and \( D(-\lambda_i^2) \) the distributions of the eigenvectors 0 and \( -\lambda_i^2 \) respectively.

**Theorem 3.3.** Let \( M \) be a nearly Sasakian manifold with structure \((\phi, \xi, \eta, g)\) and let \( \text{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_i^2\} \) be the spectrum of \( h^2 \). Then the distributions \( D(0) \) and \([\xi]\) \( \oplus D(-\lambda_i^2) \) are integrable with totally geodesic leaves. In particular,

a) the eigenvalue 0 has multiplicity \( 2p + 1 \), \( p \geq 0 \). If \( p > 0 \), the leaves of \( D(0) \) are \((2p + 1)\)-dimensional nearly Sasakian manifolds;

b) each negative eigenvalue \( -\lambda_i^2 \) has multiplicity 4 and the leaves of the distribution \([\xi]\) \( \oplus D(-\lambda_i^2) \) are 5-dimensional nearly Sasakian (non-Sasakian) manifolds.

Therefore, the dimension of \( M \) is \( 1 + 2p + 4i \).

Proof. Consider an eigenvector \( X \) with eigenvalue \( \mu \). From (1) we deduce that \( \nabla_X \xi \) is an eigenvector with eigenvalue \( \mu \). On the other hand, (10) implies \( \nabla_\xi h^2 = 0 \), so that \( \nabla_\xi X \) is also an eigenvector with eigenvalue \( \mu \).

Now, if \( X, Y \) are eigenvectors with eigenvalue \( \mu \), orthogonal to \( \xi \), from (10), we get

\[
h^2(\nabla_X Y) = \mu \nabla_X Y - \mu g(\phi X - hX, Y)\xi.
\]

If \( \mu = 0 \), we immediately get that \( \nabla_X Y \in D(0) \). If \( \mu \neq 0 \), we have

\[
h^2(\phi^2 \nabla_X Y) = \phi^2(h^2 \nabla_X Y) = \mu \phi^2(\nabla_X Y)
\]

and thus \( \nabla_X Y = -\phi^2 \nabla_X Y + \eta(\nabla_X Y)\xi \) belongs to the distribution \([\xi]\) \( \oplus D(\mu) \). This proves the first part of the Theorem.

If \( X \) is an eigenvector of \( h^2 \) orthogonal to \( \xi \), with eigenvalue \( \mu \), also \( \phi X \) is an eigenvector with the same eigenvalue \( \mu \). Hence, the eigenvalue 0 has odd multiplicity \( 2p + 1 \), for some integer \( p \geq 0 \). If \( p > 0 \) the structure \((\phi, \xi, \eta, g)\) induces a nearly Sasakian structure on the leaves of the distribution \( D(0) \) whose associated tensor \( h \) vanishes. Therefore, the induced structure is Sasakian.

As regards b), since \( \phi \) preserves each distribution \( D(-\lambda_i^2) \), the structure \((\phi, \xi, \eta, g)\) induces a nearly Sasakian structure on the leaves of the distribution \([\xi]\) \( \oplus D(-\lambda_i^2) \), which we denote in the same manner. For such a structure the operator \( h \) satisfies

\[
h^2 = -\lambda_i^2 (I - \eta \otimes \xi).
\]

By Theorem 6.1 the leaves of this distribution are 5-dimensional, so that the multiplicity of the eigenvalue \( -\lambda_i^2 \) is 4. \( \square \)
Now using Theorem 3.3 we prove that every nearly Sasakian manifold is foliated by another foliation, which is both Riemannian and totally geodesic, and such that the leaf space is Kähler. Before we need the following preliminary result.

**Lemma 3.4.** Let \((M, \phi, \xi, \eta, g)\) be a nearly Sasakian manifold. For any \(X \in \mathcal{D}(\lambda^2_1)\), \(i \in \{1, \ldots, r\}\), and for any \(Z \in \mathcal{D}(0)\) one has that \(\nabla_Z X \in \mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\).

**Proof.** For any \(Z' \in \mathcal{D}(0)\) orthogonal to \(\xi\), since the distribution \(\mathcal{D}(0)\) is integrable with totally geodesic leaves, we have that \(g(\nabla_Z X, Z') = -g(\nabla_{Z'} X, Z) = 0\). \(\square\)

**Theorem 3.5.** With the notation of Theorem 3.3, assuming \(p > 0\), the distribution \(\mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\) is integrable and defines a transversely Kähler foliation with totally geodesic leaves.

**Proof.** We already know that each distribution \([\xi] \oplus \mathcal{D}(\lambda^2_1)\) is integrable with totally geodesic leaves. Moreover, by (10), one has for any \(X \in \mathcal{D}(\lambda^2_1), Y \in \mathcal{D}(\lambda^2_2)\) and \(Z \in \mathcal{D}(0)\) orthogonal to \(\xi\),

\[
g(\nabla_X Y, Z) = -\frac{1}{\lambda^2_1} g(\nabla_X h^2 Y, Z) - \frac{1}{\lambda^2_1} g((\nabla_X h^2)Y + h^2 \nabla_X Y, Z) = -\frac{1}{\lambda^2_1} g(\nabla_X Y, h^2 Z) = 0.
\]

Now we prove that \(\mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\) defines a Riemannian foliation. First, for any \(Z, Z' \in \mathcal{D}(0), (\mathcal{L}_g)(Z, Z') = 0\) since \(\xi\) is Killing. Next, by applying Lemma 3.4 we conclude that, for any \(X \in \mathcal{D}(\lambda^2_1), (\mathcal{L}_X g)(Z, Z') = 0\).

Now let us prove that also the tensor field \(\phi\) is projectable, i.e. it maps basic vector fields into basic vector fields. Let \(Z \in \mathcal{D}(0), Z\) orthogonal to \(\xi\), be a basic vector field, that is \([\xi, Z], [X, Z] \in \mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\) for any \(X \in \mathcal{D}(\lambda^2_1)\). Let us prove that \(g([X, \phi Z], Z') = 0\) for any \(Z' \in \mathcal{D}(0)\) orthogonal to \(\xi\).

By using (10) and Lemma 3.4 we get

\[
g([X, \phi Z], Z') = g(\nabla_X \phi Z, Z') - g(\nabla_{\phi Z} X, Z')
= g(\langle\nabla_X \phi\rangle Z, Z') + g(\phi \nabla_X Z, Z')
= -\frac{1}{3} d\Phi(X, Z, Z') - g(\nabla_X Z, \phi Z).
\]

Let us check that each summand in (18) vanishes. First notice that, since \(Z\) is basic and again by Lemma 3.4, one has \(g(\nabla_X Z, \phi Z') = 0\). Further, since the Riemannian metric \(g\) is bundle-like, and using Lemma 3.4 and (9), we have

\[
d\Phi(X, Z, Z') = X(\Phi(Z, Z')) - \Phi([X, Z], Z') - \Phi([Z, Z'], X) - \Phi([Z', X], Z)
= X(g(\nabla_Z Z', Z) - g([X, Z], \phi Z') - g([Z', X], \phi Z)
= g(\nabla_X \phi Z', Z) - g(\nabla_{\phi Z'} X, Z) - g(\nabla_{Z'} X, \phi Z) + g(\nabla_X Z, \phi Z')
= g(\nabla_X \phi Z', Z')
= \frac{1}{3} d\Phi(X, Z, Z'),
\]

from which it follows that \(d\Phi(X, Z, Z') = 0\). Therefore, in view of (18), we have \(g([X, \phi Z], Z') = 0\) for any \(Z' \in \mathcal{D}(0)\) orthogonal to \(\xi\), and thus we conclude that \(\phi Z\) is basic.

Thus we have proved that the Riemannian metric \(g\) and the tensor field \(\phi\) are projectable with respect to the foliation \(\mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\). Finally, from (10), the integrability of \(\mathcal{D}(0)\) and \(h\xi = 0\) it follows that \(\mathcal{D}(\lambda^2_1) \oplus \cdots \oplus \mathcal{D}(\lambda^2_r) \oplus [\xi]\) is transversely Kähler. \(\square\)
In view of Theorem 3.3 it becomes of great importance the study of 5-dimensional nearly Sasakian manifolds. This will be precisely the subject of the next Section.

4. Nearly Sasakian manifolds and SU(2)-structures

Let $M$ be a 5-dimensional manifold. An SU(2)-structure on $M$, that is an SU(2)-reduction of the bundle $L(M)$ of linear frames on $M$, is equivalent to the existence of three almost contact metric structures $(\phi_1, \xi, \eta, g)$, $(\phi_2, \xi, \eta, g)$, $(\phi_2, \xi, \eta, g)$ related by
\begin{equation}
\phi_i \phi_j = \phi_k = -\phi_j \phi_i
\end{equation}
for any even permutation $(i, j, k)$ of $(1, 2, 3)$. In [9] Conti and Salamon proved that, in the spirit of special geometries, such a structure is equivalently determined by a quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$, where $\eta$ is a 1-form and $\omega_i$, $i \in \{1, 2, 3\}$, are 2-forms, satisfying
\begin{equation}
\omega_i \wedge \omega_j = \delta_{ij} v
\end{equation}
for some 4-form $v$ with $v \wedge \eta \neq 0$, and
\begin{equation}
X \lrcorner \omega_1 = Y \lrcorner \omega_2 \implies \omega_3(X, Y) \geq 0.
\end{equation}
The endomorphisms $\phi_i$ of $TM$, the Riemannian metric $g$ and the 2-forms $\omega_i$ are related by
\begin{equation}
\omega_i(X, Y) = g(\phi_i X, Y),
\end{equation}
(see also [1]). A well-known class of SU(2)-structures on a 5-dimensional manifold is given by Sasaki-Einstein structures, characterized by the following differential equations:
\begin{equation}
d\eta = -2\omega_3, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1.
\end{equation}
For such a manifold the almost contact metric structure $(\phi_3, \xi, \eta, g)$ is Sasakian, with Einstein Riemannian metric $g$. A Sasaki-Einstein 5-manifold may be equivalently defined as a Riemannian manifold $(M, g)$ such that the product $M \times \mathbb{R}^+$ with the cone metric $dt^2 + t^2 g$ is Kähler and Ricci-flat (Calabi-Yau).

In [9], Conti and Salamon introduced hypo structures as a natural generalization of Sasaki-Einstein structures. Indeed, an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ is called hypo if
\begin{equation}
d\omega_3 = 0, \quad d(\eta \wedge \omega_1) = 0, \quad d(\eta \wedge \omega_2) = 0.
\end{equation}
These structures arise naturally on hypersurfaces of 6-manifolds endowed with an integrable SU(3)-structure. In [11] the authors introduced nearly hypo structures, defined as SU(2)-structures $(\eta, \omega_1, \omega_2, \omega_3)$ satisfying
\begin{equation}
d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1.
\end{equation}
Such structures arise on hypersurfaces of nearly Kähler SU(3)-manifolds.

We shall provide an equivalent notion of nearly Sasakian 5-manifolds in terms of SU(2)-structures. First we state the following lemmas.

**Lemma 4.1.** Let $M$ be a 5-manifold with an SU(2)-structure $\{(\phi_t, \xi, \eta, g)\}_{t \in \{1, 2, 3\}}$. Then for any even permutation $(i, j, k)$ of $(1, 2, 3)$, we have
\begin{equation}
g(N_{\phi_i}(X, Y), \phi_j Z) = -d\omega_j(X, Y, Z) + d\omega_j(\phi_i X, \phi_i Y, Z) + d\omega_k(\phi_i X, Y, Z) + d\omega_k(X, \phi_i Y, Z).
\end{equation}
Proof. A simple computation using the quaternionic identities (19) shows that
\[ \phi_i(\nabla_Z \phi_j) \phi_i = -\phi_i \nabla_Z \phi_k - \nabla_Z \phi_j + (\nabla_Z \phi_k) \phi_i. \]
Therefore
\[ (\nabla_Z \omega_j)(\phi_i X, \phi_i Y) = -g(\phi_i(\nabla_Z \phi_j) \phi_i X, Y) \]
\[ = -(\nabla_Z \omega_k)(X, \phi_i Y) + (\nabla_Z \omega_j)(X, Y) - (\nabla_Z \omega_k)(\phi_i X, Y). \]
The tensor field \( \phi_i \) can be written as
\[ N_{\phi_i}(X, Y) = (\nabla_{\phi_i} X) Y - (\nabla_X \phi_i) X + (\nabla_X \phi_i) Y - (\nabla_Y \phi_i) X + \eta(X) \nabla_Y \xi - \eta(Y) \nabla_X \xi \]
\[ = (\phi_i(\nabla_Y \phi_i) - \nabla_{\phi_i} \phi_i) X - (\phi_i(\nabla_X \phi_i) - \nabla_{\phi_i} \phi_i) Y + ((\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)) \xi. \]
It follows that
\[ \phi_j N_{\phi_i}(X, Y) = -\phi_k(\nabla_Y \phi_i) X - \phi_j(\nabla_{\phi_i} \phi_i) X + \phi_k(\nabla_X \phi_i) Y - \phi_j(\nabla_{\phi_i} \phi_i) Y. \]
Now, a straightforward computation shows that
\[ g(N_{\phi_i}(X, Y), \phi_j Z) = -d\omega_j(X, Y, Z) + d\omega_j(\phi_i X, \phi_i Y, Z) + d\omega_k(X, \phi_i Y, Z) + d\omega_k(\phi_i X, Y, Z) \]
\[ + (\nabla_Z \omega_j)(X, Y) - (\nabla_Z \omega_k)(\phi_i X, Y) - (\nabla_Z \omega_k)(\phi_i X, Y) \]
\[ - (\nabla_Z \omega_j)(\phi_i X, \phi_i Y). \]
Applying (26), we get (25). \( \square \)

Lemma 4.2. Let \( M \) be a 5-manifold endowed with an SU(2)-structure \( \{(\phi_i, \xi, \eta, g)\}_{i \in \{1, 2, 3\}} \).
Then for any even permutation \((i, j, k)\) of \((1, 2, 3)\) we have
\[ 2g((\nabla_X \phi_i) Y, Z) = -d\omega_j(X, \phi_i Y, \phi_i Z) + d\omega_j(X, \phi_i Y, Z) - d\omega_j(Y, Z, \phi_k X) \]
\[ + d\omega_j(\phi_i Y, \phi_i Z, \phi_k X) + d\omega_k(Y, \phi_i Z, \phi_k X) + d\omega_k(\phi_i Y, Z, \phi_k X) \]
\[ + d\eta(\phi_i Y, Z) \eta(X) - d\eta(\phi_i Z, Y) \eta(X) + d\eta(\phi_i Y, X) \eta(Z) - d\eta(\phi_i Z, X) \eta(Y). \]
Proof. The covariant derivative of \( \phi_i \) is given by (see [5 Lemma 6.1]):
\[ 2g((\nabla_X \phi_i) Y, Z) = -d\omega_j(X, \phi_i Y, \phi_i Z) + d\omega_j(X, \phi_i Y, Z) + g(N_{\phi_i}(Y, Z), \phi_i X) \]
\[ + d\eta(\phi_i Y, Z) \eta(X) - d\eta(\phi_i Z, Y) \eta(X) + d\eta(\phi_i Y, X) \eta(Z) - d\eta(\phi_i Z, X) \eta(Y). \]
Applying (25) to vector fields \( Y, Z \) and \( \phi_k X \), being \( \phi_j \phi_k = \phi_i \), we have
\[ g(N_{\phi_i}(Y, Z), \phi_i X) = -d\omega_j(Y, Z, \phi_k X) + d\omega_j(\phi_i Y, \phi_i Z, \phi_k X) \]
\[ + d\omega_k(Y, \phi_i Z, \phi_k X) + d\omega_k(\phi_i Y, Z, \phi_k X). \]
Combining (28) and (29), we get the result. \( \square \)

Theorem 4.3. Nearly Sasakian structures on a 5-dimensional manifold are in one-to-one correspondence with SU(2)-structures \((\eta, \omega_1, \omega_2, \omega_3)\) satisfying
\[ d\eta = -2\omega_3 + 2\lambda \omega_1, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1 - 3\lambda \eta \wedge \omega_3 \]
for some real number \( \lambda \neq 0 \). These SU(2)-structures are nearly hypo.
Proof. Let \((M, \phi, \xi, \eta, g)\) be a nearly Sasakian 5-manifold. The associated tensor \( h \) satisfies
\[ h^2 = -\lambda^2(I - \eta \otimes \xi), \]
for some non-vanishing constant \( \lambda \). Since \( h \) is skew-symmetric, anticommutes with \( \phi \) and satisfies \( h \xi = 0 \), the structure tensors \( \xi, \eta \) and \( g \), together with the \((1, 1)\)-tensor fields
\[ \phi_1 := \frac{1}{\lambda} h, \quad \phi_2 := \frac{1}{\lambda} \phi h, \quad \phi_3 := \phi, \]

and this completes the proof of the third equation in (30). Taking the 2-forms \( \omega_i, i \in \{1, 2, 3\} \), defined by \( \omega_i(X, Y) := g(\phi_i X, Y) \), we prove that the structure \((\eta, \omega_1, \omega_2, \omega_3)\) satisfies (30).

Using (4), we compute

\[
d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])
= g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)
= 2g(-\phi X + hX, Y)
= -2\omega_3(X, Y) + 2\lambda \omega_1(X, Y),
\]

which proves the first equation in (30). In particular we have \(\lambda d\omega_1 = \omega_1\). Now, by (3)

\[
d\omega_3(X, Y, Z) = 3g((\nabla_X \phi)Y, Z) + 3\eta(Y)g(X, Z) - 3\eta(Z)g(X, Y).
\]

For \(X = \xi\), applying (5), we get

\[
d\omega_3(\xi, Y, Z) = 3g((\nabla_\xi \phi)Y, Z) = 3g(\phi h Y, Z) = 3\lambda \omega_2(Y, Z).
\]

On the other hand, equation (6) implies that for every vector fields \(X, Y, Z\) orthogonal to \(\xi\), \(g((\nabla_X \phi)Y, Z) = 0\) and thus \(d\omega_3(X, Y, Z) = 0\). Therefore \(d\omega_3 = 3\eta \wedge \omega_2\). Being also \(d\omega_3 = \lambda d\omega_1\), we obtain the second equation in (30). Now, using the first two equations in (30), and (20), we have \(\eta \wedge d\omega_2 = d\eta \wedge \omega_2 = 0\), and thus, for every vector fields \(X, Y, Z\) orthogonal to \(\xi\),

\[
d\omega_2(X, Y, Z) = (\eta \wedge d\omega_2)(\xi, X, Y, Z) = 0.
\]

From (7), we get \(\nabla_\xi(\phi h) = -\lambda^2 \phi - h\). Hence, for every vector fields \(Y, Z\), using also (4), we compute

\[
\lambda d\omega_2(\xi, Y, Z) = g((\nabla_\xi \phi h)Y, Z) + g((\nabla_Y \phi h)Z, \xi) + g((\nabla_Z \phi h)\xi, Y)
= -3g(hY + \lambda^2 \phi Y, Z)
= -3\lambda \omega_1(Y, Z) - 3\lambda^2 \omega_3(Y, Z),
\]

and this completes the proof of the third equation in (30).

As for the converse, assume that \(M\) is a 5-manifold with an \(SU(2)\)-structure satisfying (30) for some non-vanishing real number \(\lambda\). Consider the associated almost contact metric structures \((\phi_i, \xi, \eta, g), i \in \{1, 2, 3\}\). Applying (27) and (30) we compute the covariant derivative of \(\phi_3\):

\[
2g((\nabla_X \phi_3)Y, Z) = -3\lambda \eta(X)\omega_2(\phi_3 Y, \phi_3 Z) + 3\lambda \eta(X)\omega_2(Y, Z) + 3\lambda \eta(Y)\omega_2(Z, X) + 3\lambda \eta(Z)\omega_2(X, Y)
+ 3\lambda \eta(Z)\omega_2(X, Y) - 3\eta(Y)\omega_2(X, \phi_2 X) - 3\eta(Z)\omega_2(\phi_2 X, Y)
- 3\lambda \eta(Z)\omega_3(\phi_2 X, \phi_3 Y) - 2\omega_3(\phi_3 Y, Z)\eta(X) + 2\omega_3(\phi_3 Y, Z)\eta(X)
+ 2\omega_3(\phi_3 Z, Y)\eta(X) - 2\omega_3(\phi_3 Z, Y)\eta(Z) - 2\omega_3(\phi_3 Y, X)\eta(Z)
+ 2\lambda \omega_1(\phi_3 Y, X)\eta(Z) + 2\lambda \omega_3(\phi_3 Z, X)\eta(Y) - 2\omega_3(\phi_3 Z, X)\eta(Y)
= \eta(X)\{3\lambda \eta(\phi_2 Y, Z) + 3\lambda \eta(\phi_2 Y, Z) - 2g(\phi_3^2 Y, Z) - 2\lambda \eta(\phi_2 Y, Z) + 2g(\phi_3^2 Y, Z)
+ 2\lambda \eta(\phi_2 Y, Z)\} + \eta(Y)\{3\lambda \eta(\phi_2 Z, X) - 3\eta(\phi_2 Z, X) + 3g(\phi_2 Z, \phi_2 X)
+ 3\lambda \eta(Z, \phi_2 X) + 2g(\phi_3^2 Z, X) + 2\lambda \eta(\phi_2 Z, X)\} + \eta(Z)\{3\lambda \eta(\phi_3 Y, X)
- 3g(\phi_3^2 Y, X) - 3g(\phi_3 Y, X) - 3\lambda \eta(\phi_2 X, Y)
- 2g(\phi_3^2 Y, X) - 2\lambda \eta(\phi_2 Y, X)\} = 2\eta(X)\omega_2(Y, Z) + 2\eta(Y)\omega_2(Z, X) + 2\lambda \eta(Z)\omega_2(X, Y) - 2\eta(Y)g(X, Z)
+ 2\lambda \eta(Z)g(X, Y)
\]
\[ = 2\lambda(\eta \wedge \omega_2)(X, Y, Z) - 2\eta(Y)g(X, Z) + 2\eta(Z)g(X, Y) \]
\[ = \frac{2}{3}d\omega_3(X, Y, Z) - 2\eta(Y)g(X, Z) + 2\eta(Z)g(X, Y) \]
\[ = -\frac{2}{3}d\Phi(X, Y, Z) - 2\eta(Y)g(X, Z) + 2\eta(Z)g(X, Y) \]

thus proving (39), so that \((\phi_3, \xi, \eta, g)\) is a nearly Sasakian structure. Now, considering the structure tensor field \(h = \nabla\xi + \phi_3\), we prove that \(h = \lambda\phi_1\). Indeed, by (35), \(\nabla\phi_1 = \phi_3 h\). On the other hand, using (3) and (30), we have

\[ g((\nabla\phi_3)Y, Z) = \frac{1}{3}d\omega_3(\xi, Y, Z) = \lambda(\eta \wedge \omega_2)(\xi, Y, Z) = \lambda g(\phi_2 Y, Z). \]

Therefore, \(\nabla\phi_3 = \lambda\phi_2 = \phi_3 h\), which implies that \(h = -\lambda\phi_3 \phi_2 = \lambda\phi_1\).

Finally, from (30) one gets \(d(\eta \wedge \omega_1) = -2\omega_1 \wedge \omega_1\), so that the \(SU(2)\)-structure \((\eta, \omega_1, \omega_2, \omega_3)\) is nearly hypo. \(\square\)

**Remark 4.4.** In [1] the authors determine explicit formulas for the scalar curvature and the Ricci tensor of the metric induced by an \(SU(2)\)-structure \((\eta, \omega_1, \omega_2, \omega_3)\) on a 5-manifold in terms of the intrinsic torsion. For an \(SU(2)\)-structure satisfying (30), the only non-vanishing torsion forms are \(f_2 = 2\lambda, f_3 = -2, f_{12} = 3\) and \(f_{23} = -3\lambda\). Therefore, from (3.2) and Theorem 3.8 in [1], it follows that \(\text{Ric} = 4(1 + \lambda^2)g\). We thus reacquire the result of Olszak ([25]) stating that each 5-dimensional nearly Sasakian manifold is Einstein and of scalar curvature \(s > 20\). In particular,

\[ s = 20(1 + \lambda^2) \]

implying that the constant \(\lambda\) in (30) is determined by the Riemannian geometry of the manifold.

Thus to any 5-dimensional nearly Sasakian manifold \((M, \phi, \xi, \eta, g)\) there are attached two other almost contact metric structures \((\phi_1, \xi, \eta, g)\) and \((\phi_2, \xi, \eta, g)\), with the same metric and characteristic vector field of \((\phi, \xi, \eta, g)\), such that the quaternionic relations (19) hold. In the following we investigate the class to which these two supplementary almost contact metric structures belong.

To begin with, we recall a slight generalization of nearly Sasakian manifolds. Namely, a nearly \(\alpha\)-Sasakian manifold is an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) satisfying the following relation

\[ (\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(X)Y - \eta(Y)X) \]
for some real number \(\alpha \neq 0\).

**Lemma 4.5.** Let \((M, \phi, \xi, \eta, g)\) be a 5-dimensional nearly Sasakian manifold. Then for all vector fields \(X, Y\) on \(M\) one has

\[ (\nabla_X \phi)Y = \eta(X)\phi hY - \eta(Y)(X + \phi hX) + g(X + \phi hX, Y)\xi, \]
\[ (\nabla_X h)Y = \eta(X)(\phi hY - \eta(Y)(h^2 X + \phi hX) + g(h^2 X + \phi hX, Y)\xi, \]
\[ (\nabla_X h)Y = g(\phi h^2 X - hX, Y)\xi + \eta(X)(\phi h^2 Y - hY) - \eta(Y)(\phi h^2 X - hX). \]

**Proof.** The first equation follows by a direct computation using (33), (30) and (32). Combining (31) and (34) one easily obtains (35). Finally, equations (34) and (35) imply (36). \(\square\)

Now, from (35) and (36) it follows that

\[ (\nabla_X h)Y + (\nabla_Y h)X = -\lambda^2(2g(X, Y)\xi - \eta(X)Y - \eta(Y)X) \]
\[ (\nabla_X \phi h)Y + (\nabla_Y \phi h)X = 0. \]
Thus we can state the following result.

**Theorem 4.6.** Let \((M, \phi, \xi, \eta, g)\) be a 5-dimensional nearly Sasakian manifold and let \((\phi_i, \xi, \eta, g)\), \(i \in \{1, 2, 3\}\), be the almost contact metric structures defined by the associated \(SU(2)\)-structure. Then \((\phi_2, \xi, \eta, g)\) is nearly cosymplectic and \((\phi_1, \xi, \eta, g)\) is nearly \(\alpha\)-Sasakian with \(\alpha = -\lambda\).

We now find some applications of Theorem 4.3, pointing out the relationship between nearly Sasakian geometry and Sasaki-Einstein manifolds.

**Corollary 4.7.** Each nearly Sasakian 5-dimensional manifold carries a Sasaki-Einstein structure. Conversely, each Sasaki-Einstein 5-manifold carries a 1-parameter family of nearly Sasakian structures.

**Proof.** Let \(M\) be a 5-dimensional manifold. Let \((\eta, \omega_1, \omega_2, \omega_3)\) be a nearly Sasakian \(SU(2)\)-structure on \(M\), i.e. \((\eta, \omega_1, \omega_2, \omega_3)\) is an \(SU(2)\)-structure satisfying (30) for some real number \(\lambda \neq 0\). Put
\[
\tilde{\eta} := \sqrt{1 + \lambda^2} \eta,
\]
\[
\tilde{\omega}_1 := \sqrt{1 + \lambda^2} (\omega_1 + \lambda \omega_3),
\]
\[
\tilde{\omega}_2 := (1 + \lambda^2) \omega_2,
\]
\[
\tilde{\omega}_3 := \sqrt{1 + \lambda^2} (\omega_3 - \lambda \omega_1).
\]
One can easily check that \(\tilde{\omega}_i \wedge \tilde{\omega}_j = \delta_{ij} \tilde{v}\), where \(\tilde{v} = (1 + \lambda^2) \omega_i \wedge \omega_i\), and \(\tilde{\eta} \wedge \tilde{v} \neq 0\). Furthermore, suppose that \(X \cdot \tilde{\omega}_1 = Y \cdot \tilde{\omega}_2\) for some vector fields \(X, Y\). Let \(\{(\phi_i, \xi, \eta, g)\}_{i \in \{1, 2, 3\}}\) be the almost contact metric structures associated to \((\eta, \omega_i)\). Then
\[
\phi_1 X + \lambda \phi_3 X = \sqrt{1 + \lambda^2} \phi_2 Y
\]
and applying \(\phi_2\), we have \(-\phi_3 X + \lambda \phi_1 X = \sqrt{1 + \lambda^2} (-Y + \eta(Y)\xi)\). Then,
\[
\tilde{\omega}_3(X, Y) = \sqrt{1 + \lambda^2} g(\phi_3 X - \lambda \phi_1 X, Y) = (1 + \lambda^2) g(Y, Y) - \eta(Y)\xi
\]
\[
= (1 + \lambda^2) g(\phi_1 Y, \phi_i Y) \geq 0.
\]
It is straightforward to verify that the \(SU(2)\)-structure \((\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)\) satisfies (22) and thus it is a Sasaki-Einstein structure.

Analogously, given a Sasaki-Einstein structure \((\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)\) on \(M\), for any real number \(\lambda \neq 0\), one can define the nearly Sasakian structure
\[
\eta := \frac{1}{\sqrt{1 + \lambda^2}} \tilde{\eta},
\]
\[
\omega_1 := \frac{1}{\sqrt{1 + \lambda^2 (1 + \lambda^2)}} (\tilde{\omega}_1 - \lambda \tilde{\omega}_3),
\]
\[
\omega_2 := \frac{1}{1 + \lambda^2} \tilde{\omega}_2,
\]
\[
\omega_3 := \frac{1}{\sqrt{1 + \lambda^2 (1 + \lambda^2)}} (\lambda \tilde{\omega}_1 + \tilde{\omega}_3).
\]
\[\square\]

Corollary 4.7 provides a way of finding new examples of nearly-Sasakian manifolds. In particular, each Sasaki-Einstein metric of the infinite family of Sasakian structures on \(S^2 \times S^3\) recently discovered in [19] gives examples of nearly Sasakian structures.
We point out that, in terms of almost contact metric structures, the Sasaki-Einstein structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ associated to any 5-dimensional nearly Sasakian manifold $(M, \phi, \xi, \eta, g)$ is given by

\begin{equation}
(41) \quad \tilde{\phi} = \frac{1}{\sqrt{1 + \lambda^2}} (\phi - h), \quad \tilde{\xi} = \frac{1}{\sqrt{1 + \lambda^2}} \xi, \quad \tilde{\eta} = \sqrt{1 + \lambda^2} \eta, \quad \tilde{g} = (1 + \lambda^2) g.
\end{equation}

The scalar curvatures $s$ and $\tilde{s}$ of $g$ and $\tilde{g}$, respectively, are related by $s = (1 + \lambda^2) \tilde{s}$, coherently with (33), since the scalar curvature of a 5-dimensional Sasaki-Einstein structure is $\tilde{s} = 20$.

**Remark 4.8.** One can find a more direct proof that the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ in (41) is Sasakian. Indeed,

$$d\eta(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) = 2g(X, (\phi - h)Y)$$

and thus $d\tilde{\eta}(X, Y) = 2\tilde{g}(X, \tilde{\phi}Y)$, implying that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact metric structure. Applying (8), a straightforward computation yields

$$\tilde{R}(X, Y)\xi = R(X, Y)\xi = \tilde{\eta}(Y)X - \tilde{\eta}(X)Y,$$

which ensures that the structure is Sasakian ([5, Proposition 7.6]).

**Remark 4.9.** Explicitly, the almost contact metric structures $(\tilde{\phi}_1, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ associated to the Sasaki-Einstein $SU(2)$-structure (39) is given by

\begin{align*}
\tilde{\phi}_1 &:= \frac{1}{\sqrt{1 + \lambda^2}} \left( \frac{1}{\lambda} h + \lambda \phi \right) = \frac{1}{3\lambda \sqrt{1 + \lambda^2}} \mathcal{L}_\xi \phi h, \\
\tilde{\phi}_2 &:= \frac{1}{\lambda} \phi h = \frac{1}{3\lambda} \mathcal{L}_\xi \phi, \\
\tilde{\phi}_3 &:= \frac{1}{\sqrt{1 + \lambda^2}} (\phi - h).
\end{align*}

Using Lemma 4.5 one can prove that $(\tilde{\phi}_1, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ and $(\tilde{\phi}_2, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are nearly cosymplectic. Actually we will see in Corollary 5.2 that this result holds for any Sasaki-Einstein $SU(2)$-structure.

**Remark 4.10.** The Sasaki-Einstein structure (39) defined on the nearly Sasakian manifold $M$ determines an integrable $SU(3)$-structure on $M \times \mathbb{R}_+$ which is given by the closed forms (see [9])

\begin{align*}
F &= \sqrt{1 + \lambda^2} \left\{ t^2 \omega_3 + t \eta \wedge dt - \lambda t^2 \omega_1 \right\}, \\
\Psi_+ &= (1 + \lambda^2) \left\{ t^2 (t \omega_1 \wedge \eta - \omega_2 \wedge dt) + \lambda t^3 \omega_3 \wedge \eta \right\}, \\
\Psi_- &= \sqrt{1 + \lambda^2} \left\{ t^2 (t \omega_2 \wedge \eta + \omega_1 \wedge dt) + \lambda t^2 \omega_3 \wedge dt + \lambda^2 t^3 \omega_2 \wedge \eta \right\}.
\end{align*}

In particular, the Kähler and Ricci-flat structure $(G, J)$ of the metric cone is given by

\begin{align*}
G &= dt^2 + (1 + \lambda^2) t^2 g, \\
JX &= \frac{1}{\sqrt{1 + \lambda^2}} (\phi X - hX) + \sqrt{1 + \lambda^2} \eta(X) T, \\
JY &= -\frac{1}{\sqrt{1 + \lambda^2}} \xi, \quad T = t \frac{\partial}{\partial t}.
\end{align*}
On the other hand, following [11, Theorem 3.7 and Corollary 3.8], one can define on the product $M \times [0, \pi]$ an $SU(3)$-structure which is nearly Kähler for $0 < t < \pi$:

$$F = \sqrt{1 + \lambda^2} \left\{ \sin^2 t (\sin t \omega_1 + \cos t \omega_3) + \sin t \eta \wedge dt + \lambda \sin^2 t (\sin t \omega_3 - \cos t \omega_1) \right\},$$

$$\Psi_+ = \sqrt{1 + \lambda^2} \left\{ \sin^3 t \eta \wedge \omega_2 + \sin^2 t (\cos t \omega_1 - \sin t \omega_3) \wedge dt + \lambda^2 \sin^2 t \eta \wedge \omega_2 + \lambda \sin^2 t (\cos t \omega_3 + \sin t \omega_1) \wedge dt \right\},$$

$$\Psi_- = (1 + \lambda^2) \left\{ \sin^3 t (-\cos t \omega_1 + \sin t \omega_3) \wedge \eta + \sin^2 t \omega_2 \wedge dt - \lambda \sin^3 t (\cos t \omega_3 + \sin t \omega_1) \wedge \eta + \lambda^2 \sin^2 t \omega_2 \wedge dt \right\}.$$  

In this case, the Riemannian metric and the almost complex structure are given by

$$G = dt^2 + (1 + \lambda^2) \sin^2 t g,$$

$$JX = \frac{1}{\sqrt{1 + \lambda^2}} \left\{ \sin t \left( \frac{1}{\lambda} hX + \lambda \phi X \right) + \cos t (\phi X - hX) \right\} + \sqrt{1 + \lambda^2} \eta(X) \Upsilon,$$

$$J\Upsilon = -\frac{1}{\sqrt{1 + \lambda^2}} \xi, \quad \Upsilon = \sin t \frac{\partial}{\partial t}.$$  

Corollary 4.11 together with Theorem 3.3 have an interesting application for a general nearly Sasakian manifold in any dimension.

**Corollary 4.11.** Every nearly Sasakian manifold is a contact manifold.

**Proof.** Let $M$ be a nearly Sasakian manifold of dimension $2n + 1$ with structure $(\phi, \xi, \eta, g)$. With the notation used in Section 3 preliminarily we prove that for any $X \in \mathcal{D}(-\lambda^2)$, $Y \in \mathcal{D}(-\lambda^2)$

$$d\eta(X, Y) = 0,$$

for each $i, j \in \{1, \ldots, r\}, i \neq j$. Indeed,

$$d\eta(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) = 2g(X, \phi Y) + 2g(hX, Y) = 0$$

since the operators $\phi$ and $h$ preserve $\mathcal{D}(-\lambda^2)$ and the distributions $\mathcal{D}(-\lambda^2)$ and $\mathcal{D}(-\lambda^2)$ are mutually orthogonal. In a similar way one can prove that for any $X \in \mathcal{D}(-\lambda^2)$ and $Z \in \mathcal{D}(0)$

$$d\eta(X, Z) = 0.$$

Now, fix a point $x \in M$. By a) in Theorem 3.3 there exists a basis $\{\xi_x, e_1, \ldots, e_{2p}\}$ of $\mathcal{D}_x(0)$ such that

$$\eta \wedge (d\eta)^p(\xi_x, e_1, \ldots, e_{2p}) \neq 0.$$  

By b) in Theorem 3.3 and Corollary 4.17 for each $i \in \{1, \ldots, r\}$ one can find a basis $\{\xi_x, v^1_i, v^2_i, v^3_i, v^4_i\}$ of $\mathcal{D}_x(-\lambda^2)$ such that

$$\eta \wedge (d\eta)^2(\xi_x, v^1_i, v^2_i, v^3_i, v^4_i) \neq 0.$$  

Then by (42), (43), (44) and (45) one has

$$\eta \wedge (d\eta)^n(\xi_x, e_1, \ldots, e_{2p}, v^1_i, v^2_i, v^3_i, v^4_i) \neq 0.$$  

Theorem 4.6 shows that any 5-dimensional nearly Sasakian manifold is naturally endowed with a nearly cosymplectic structure, via the nearly Sasakian $SU(2)$-structure (40). On the other hand, as pointed out in Remark 4.9 the deformed $SU(2)$-structure (39), which is Sasaki-Einstein, carries two other nearly cosymplectic structures. Thus we devote the next section to
further investigate nearly cosymplectic structures on 5-dimensional manifolds: we show that they are nothing but deformations of Sasaki-Einstein SU(2)-structures.

5. Sasaki-Einstein SU(2)-structures and nearly cosymplectic manifolds

First, we remark that in any 5-dimensional nearly cosymplectic manifold \((M, \phi, \xi, \eta, g)\) the vanishing of the operator \(h\) defined in \((13)\) provides a necessary and sufficient condition for the structure to be coKähler. Indeed, if \(h = 0\) then the distribution \(D\) orthogonal to \(\xi\) is integrable with totally geodesic leaves; the manifold \(M\) turns out to be locally isometric to the Riemannian product \(N \times \mathbb{R}\), where \(N\) is an integral submanifold of \(D = \ker(\eta)\) endowed with a nearly Kähler structure \((g, J)\) induced by the structure tensors \((g, \phi)\). On the other hand, it is known that 4-dimensional nearly Kähler manifolds are Kähler (see [17, Theorem 5.1]), and this implies that \((\phi, \xi, \eta, g)\) is a coKähler structure.

Let \((M, \phi, \xi, \eta, g)\) be a 5-dimensional nearly cosymplectic manifold. Let \(X\) be a local eigenvector field of the operator \(h^2\) with eigenvalue \(\mu \neq 0\). Then \(\{\xi, X, \phi X, hX, h\phi X\}\) is a local orthogonal frame, and \(hX, hX, h\phi X\) are eigenvector fields of \(h^2\) with the same eigenvalue \(\mu\). Then one has \(h^2 = \mu(I - \eta \otimes \xi)\) which, together with \((17)\), implies that \(\mu\) is constant. On the other hand, being \(h\) skew-symmetric, necessarily \(\mu < 0\). We put \(\mu = -\lambda^2, \lambda \neq 0\). In fact \(M\) is endowed with an \(SU(2)\)-structure, as described in the following theorem.

**Theorem 5.1.** A nearly cosymplectic structure on a 5-dimensional manifold is equivalent to an \(SU(2)\)-structure \((\eta, \omega_1, \omega_2, \omega_3)\) satisfying

\[
d\eta = -2\lambda \omega_1, \quad d\omega_1 = 3\lambda \eta \wedge \omega_2, \quad d\omega_2 = -3\lambda \eta \wedge \omega_1
\]

for some real number \(\lambda \neq 0\). These \(SU(2)\)-structures are hypo.

**Proof.** Let \((M, \phi, \xi, \eta, g)\) be a nearly cosymplectic 5-manifold. The operator \(h\) satisfies

\[
h^2 = -\lambda^2(I - \eta \otimes \xi),
\]

for some real number \(\lambda \neq 0\). Arguing as in Theorem 4.3 the tensor fields

\[
\phi_1 := -\frac{1}{\lambda} \phi h, \quad \phi_2 = \phi, \quad \phi_3 := -\frac{1}{\lambda} h
\]

determine an \(SU(2)\)-structure \((\eta, \omega_1, \omega_2, \omega_3)\), with \(\omega_i(X, Y) := g(\phi_i X, Y)\). We prove that this structure satisfies \((16)\). Indeed, using \((13)\), a simple computation shows that

\[
d\eta(X, Y) = 2g(hX, Y) = -2\lambda \omega_3(X, Y).
\]

By \((12)\), we have

\[
d\omega_2(X, Y, Z) = 3g((\nabla_X \phi)Y, Z).
\]

For \(X = \xi\), using \((14)\), we get

\[
d\omega_2(\xi, Y, Z) = 3g((\nabla_\xi \phi)Y, Z) = 3g(\phi h Y, Z) = -3\lambda \omega_1(Y, Z).
\]

Equation \((15)\) implies that for every vector fields \(X, Y, Z\) orthogonal to \(\xi\), \(g((\nabla_X \phi)Y, Z) = 0\) and thus \(d\omega_2(X, Y, Z) = 0\). Therefore \(d\omega_2 = -3\lambda \eta \wedge \omega_1\). In particular we get \(d(\eta \wedge \omega_1) = 0\) and hence, by \((20)\),

\[
\eta \wedge d\omega_1 = d\eta \wedge \omega_1 = 0.
\]

Therefore, for every vector fields \(X, Y, Z\) orthogonal to \(\xi\),

\[
d\omega_1(X, Y, Z) = (\eta \wedge d\omega_1)(\xi, X, Y, Z) = 0.
\]

Now, from \((16)\) we have \(\nabla_\xi h = 0\), and thus, by \((14)\),

\[
\nabla_\xi (\phi h) = (\nabla_\xi \phi) h = \phi h^2 = -\lambda^2 \phi.
\]
Hence, for every vector fields $Y$, $Z$, using also (13), we compute
\[ \lambda d\omega_1(\xi, Y, Z) = -g((\nabla_\xi \phi^h)Y, Z) - g((\nabla_Y \phi^h)Z, \xi) - g((\nabla_Z \phi^h)\xi, Y) = 3\lambda^2 g(\phi Y, Z) \]
which implies $d\omega_1(\xi, Y, Z) = 3\lambda\omega_2(Y, Z)$. Consequently, $d\omega_1 = 3\lambda\eta \wedge \omega_2$ and this completes the proof of (46).

As for the converse, assume that $M$ is a 5-manifold with an $SU(2)$-structure satisfying (46) for some real number $\lambda \neq 0$. Consider the associated almost contact metric structures $(\phi_i, \xi, \eta, g)$, $i \in \{1, 2, 3\}$. By using (27) and (46), a straightforward computation shows that the covariant derivative of $\phi_2$ is given by:
\[ g((\nabla_X \phi_2)Y, Z) = -\frac{1}{3} d\Phi(X, Y, Z) \]
so that $(\phi_2, \xi, \eta, g)$ is a nearly cosymplectic structure. The associated operator $h = \nabla \xi$ coincides with $-\lambda \phi_3$. Indeed, applying (46),
\[ g((\nabla_\xi \phi_2)Y, Z) = \frac{1}{3} d\omega_2(X, Y, Z) = -\lambda (\eta \wedge \omega_1)(\xi, Y, Z) = -\lambda g(\phi_1 Y, Z), \]
and thus $\nabla_\xi \phi_2 = -\lambda \phi_1$. On the other hand, by (14), $\nabla_\xi \phi_2 = \phi_2 h$. Hence, $h = \lambda \phi_2 \phi_1 = -\lambda \phi_3$.

Finally, form (46) the forms $\omega_3, \eta \wedge \omega_1, \eta \wedge \omega_2$ are closed so that the structure $(\eta, \omega_1, \omega_2, \omega_3)$ is hypo.

Note that if $(\eta, \omega_1, \omega_2, \omega_3)$ is an $SU(2)$-structure satisfying (46) and $(\phi_i, \xi, \eta, g)$, $i \in \{1, 2, 3\}$, are the associated almost contact metric structures, then applying (27) one can verify that also $(\phi_1, \xi, \eta, g)$ is a nearly cosymplectic structure, while the covariant derivative of $\phi_3$ is given by
\[ (\nabla_X \phi_3)Y = \lambda (g(X, Y)\xi - \eta(Y)X), \]
and thus $(\phi_3, \xi, \eta, g)$ is a $\lambda$-Sasakian structure. In particular, for $\lambda = 1$, equations (46) reduce to the equations of a Sasaki-Einstein structure, so that we deduce the following results.

**Corollary 5.2.** Let $(\eta, \omega_1, \omega_2, \omega_3)$ be an $SU(2)$-structure satisfying the Sasaki-Einstein equations (22). Let $(\phi_i, \xi, \eta, g)$, $i \in \{1, 2, 3\}$, be the associated almost contact metric structures. Then, for $i = 1, 2, (\phi_i, \xi, \eta, g)$ is a nearly cosymplectic structure.

**Corollary 5.3.** Each nearly cosymplectic 5-dimensional manifold carries a Sasaki-Einstein structure. Conversely, each Sasaki-Einstein 5-manifold carries a 1-parameter family of nearly cosymplectic structures.

**Proof.** Let $M$ be a 5-dimensional manifold and let $(\eta, \omega_1, \omega_2, \omega_3)$ be an $SU(2)$-structure satisfying (46) for some real number $\lambda \neq 0$. Put
\[ \tilde{\eta} := \lambda \eta, \quad \tilde{\omega}_1 := \lambda^2 \omega_1, \quad \tilde{\omega}_2 := \lambda^2 \omega_2, \quad \tilde{\omega}_3 := \lambda^2 \omega_3. \]
Obviously $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ is an $SU(2)$-structure and one can easily check that it satisfies (22). Conversely, given a Sasaki-Einstein structure $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ on $M$, for any real number $\lambda \neq 0$, one can define the $SU(2)$-structure
\[ \eta := \frac{1}{\lambda} \tilde{\eta}, \quad \omega_1 := \frac{1}{\lambda^2} \tilde{\omega}_1, \quad \omega_2 := \frac{1}{\lambda^2} \tilde{\omega}_2, \quad \omega_3 := \frac{1}{\lambda^2} \tilde{\omega}_3, \]
which satisfies (46). \( \square \)

In terms of almost contact metric structures, the Sasaki-Einstein structure $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ attached to any 5-dimensional nearly cosymplectic manifold $(M, \phi, \xi, \eta, g)$, stated by Corollary 5.3, is given by
\[ \hat{\phi} = -\frac{1}{\lambda} h, \quad \hat{\xi} = \frac{1}{\lambda} \xi, \quad \hat{\eta} = \lambda \eta, \quad \hat{g} = \lambda^2 g. \]
In particular, the scalar curvatures $s$ and $\tilde{s}$ of $g$ and $\tilde{g}$, respectively, are related by

$$s = \lambda^2 \tilde{s} = 20 \lambda^2.$$ 

Therefore we have the following

**Theorem 5.4.** Every nearly cosymplectic (non-coKähler) 5-dimensional manifold is Einstein with positive scalar curvature.

6. Hypersurfaces of nearly Kähler manifolds

Let $(N, J, \tilde{g})$ be an almost Hermitian manifold of dimension $2n+2$. Let $\iota : M \to N$ be a $C^\infty$ orientable hypersurface and $\nu$ a unit normal vector field. As it is known (see [3, Section 4.5.2]) on $M$ it is induced a natural almost contact metric structure $(\phi, \xi, \eta, g)$ given by

$$J\iota^*X = \iota^*\phi X + \eta(X)\nu, \quad J\nu = -\iota^*\xi, \quad g = \iota^*\tilde{g}.$$ 

We recall now the following fundamental results providing necessary and sufficient conditions for a hypersurface of a nearly Kähler manifold to be nearly cosymplectic or nearly Sasakian.

**Theorem 6.1** ([2]). Let $M$ be a hypersurface of a nearly Kähler manifold $(N, J, g')$. Then the induced almost contact metric manifold $(\phi, \xi, \eta, g)$ is nearly cosymplectic if and only if the second fundamental form is given by

$$\sigma = \beta(\eta \otimes \eta)\nu$$

for some function $\beta$.

**Theorem 6.2** ([4]). Let $M$ be a hypersurface of a nearly Kähler manifold $(N, J, g')$. Then the induced almost contact metric manifold $(\phi, \xi, \eta, g)$ is nearly Sasakian if and only if the second fundamental form is given by

$$\sigma = (-g + \beta(\eta \otimes \eta))\nu$$

for some function $\beta$.

Concerning 6-dimensional nearly Kähler manifolds, we shall further investigate the $SU(2)$-structure induced on hypersurfaces satisfying the conditions stated in Theorems 6.1 and 6.2.

First recall that, as proved in [17], any 6-dimensional nearly Kähler non-Kähler manifold $(N, J, g')$ is Einstein and of constant type, i.e. it satisfies

$$\|\nabla'_{X} JY\|^2 = \frac{s'}{30} \left(\|X\|^2 \cdot \|Y\|^2 - g'(X, Y)^2 - g'(X, JY)^2\right)$$

where $\nabla'$ is the Levi-Civita connection and $s' > 0$ is the scalar curvature of $g'$.

**Theorem 6.3.** Let $(N, J, g')$ be a 6-dimensional nearly Kähler non-Kähler manifold and let $M$ be a hypersurface such that the second fundamental form is given by $\sigma = \beta(\eta \otimes \eta)\nu$ for some function $\beta$. Let $(\phi, \xi, \eta, g)$ be the induced nearly cosymplectic structure on $M$ and $(\eta, \omega_1, \omega_2, \omega_3)$ the associated $SU(2)$-structure satisfying ([40]). Then the operator $h$ coincides with the covariant derivative $\nabla'_{\nu} J$ and the constant $\lambda$ satisfies

$$\lambda^2 = \frac{s'}{30}$$

Therefore, the scalar curvature of the Einstein Riemannian metric $g$ is $s = \frac{\lambda^2}{\lambda} s'$.

**Proof.** First notice that the hypothesis on the second fundamental form implies that, for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$\nabla'_{X} Y = \nabla'_{X} Y + \beta \eta(X)\eta(\nu), \quad \nabla'_{\nu} \nu = -\beta \eta(X)\xi.$$ 

Therefore,

$$(\nabla'_{\nu} J) X = -(\nabla'_{X} J) \nu$$

$$= \nabla'_{X} \xi + J(\nabla'_{X} \nu)$$

$$= \nabla'_{X} \xi + \beta \eta(X)\nu - \beta \eta(X) J \xi$$

$$= hX.$$
Now, taking a unit vector field \( X \) orthogonal to \( \xi \) and applying (49), we have
\[
\|hX\|^2 = \|(\nabla'_\nu J)X\|^2 = \frac{s'}{30}.
\]
On the other hand, being
\[
h^2 = -\lambda^2 (I - \eta \otimes \xi),
\]
then
\[
\|hX\|^2 = -g(h^2 X, X) = \lambda^2.
\]
The assertion on the scalar curvature is consequence of (33). \( \square \)

Under the hypothesis of the above theorem, applying the deformation (47) to the \( SU(2) \)-structure \( (\eta_\omega) \), one obtains a Sasaki-Einstein structure. Therefore,

Corollary 6.4. Every hypersurface of a 6-dimensional nearly Kähler non-Kähler manifold such that the second fundamental form is proportional to \( (\eta \otimes \eta)_\nu \) carries a Sasaki-Einstein structure.

The above Corollary generalizes Lemma 2.1 of [11] concerning totally geodesic hypersurfaces of nearly Kähler manifolds.

Analogously, we prove the following

Theorem 6.5. Let \( (N, J, g') \) be a 6-dimensional nearly Kähler non-Kähler manifold and let \( M \) be a hypersurface such that the second fundamental form is given by \( \sigma = (-g + \beta(\eta \otimes \eta))\nu \) for some function \( \beta \). Let \( (\phi, \xi, \eta, g) \) be the induced nearly Sasakian structure on \( M \) and \( (\eta, \omega_1, \omega_2, \omega_3) \) the associated \( SU(2) \)-structure satisfying (30). Then the operator \( h \) coincides with the covariant derivative \( \nabla'_\nu J \) and the constant \( \lambda \) satisfies
\[
\lambda^2 = \frac{s'}{30}.
\]
Therefore, the scalar curvature of the Einstein Riemannian metric \( g \) is
\[
s = 20 + \frac{2s'}{3}.
\]

Proof. For every vector fields \( X, Y \in \mathfrak{X}(M) \), we have
\[
\nabla'_X Y = \nabla_X Y - g(X, Y)\nu + \beta\eta(X)\eta(Y)\nu, \quad \nabla'_X \nu = X - \beta\eta(X)\xi.
\]
Therefore,
\[
(\nabla'_\nu J)X = -(\nabla'_X J)\nu
= \nabla'_X \xi + J(\nabla'_X \nu)
= \nabla_X \xi - \eta(X)\nu + \beta\eta(X)\nu + JX - \beta\eta(X)J\xi
= -\phi X + hX + \phi X
= hX.
\]
Taking a unit vector field \( X \) orthogonal to \( \xi \) and applying (49), we have \( \|hX\|^2 = \frac{s'}{30} \). On the other hand, \( \|hX\|^2 = -g(h^2 X, X) = \lambda^2 \). The assertion on the scalar curvature is consequence of (33). \( \square \)

In this case, applying the deformation (39) to the \( SU(2) \)-structure \( (\eta, \omega_1, \omega_2, \omega_3) \), we obtain a Sasaki-Einstein structure. Therefore,

Corollary 6.6. Every hypersurface of a 6-dimensional nearly Kähler non-Kähler manifold such that the second fundamental form is given by \( \sigma = (-g + \beta(\eta \otimes \eta))\nu \), for some function \( \beta \), carries a Sasaki-Einstein structure.

In particular the above Corollary holds for totally umbilical hypersurfaces of nearly Kähler manifolds with shape operator \( A = -I \).
Example 6.7. We recall two basic examples of 5-dimensional nearly cosymplectic and nearly Sasakian manifolds \([2, 4]\). First consider \(\mathbb{R}^7\) as the imaginary part of the Cayley numbers \(\mathbb{O}\), with the product vector \(\times\) induced by the Cayley product. Let \(S^6\) be the unit sphere in \(\mathbb{R}^7\) and \(N = \sum_{i=1}^{7} x_i \frac{\partial}{\partial x_i}\) the unit outer normal. One can define an almost complex structure \(J\) on \(S^6\) by \(JX = N \times X\). It is well known that this almost complex structure is nearly Kähler (non-Kähler) with respect to the induced Riemannian metric.

Consider \(S^5\) as a totally geodesic hypersurface of \(S^6\) defined by \(x^7 = 0\) with unit normal \(\nu = -\frac{\partial}{\partial x_7}\). Let \((\phi, \xi, \eta, g)\) be the induced almost contact metric structure on \(S^5\), with

\[
\xi = -J\nu = N \times \frac{\partial}{\partial x^7} = x^1 \frac{\partial}{\partial x^6} - x^2 \frac{\partial}{\partial x^5} - x^3 \frac{\partial}{\partial x^4} + x^4 \frac{\partial}{\partial x^3} + x^5 \frac{\partial}{\partial x^2} - x^6 \frac{\partial}{\partial x^1},
\]

and \(\eta\) given by the restriction of \(x^1dx^6 - x^6dx^1 + x^5dx^2 - x^2dx^5 + x^4dx^3 - x^3dx^4\) to \(S^5\). This almost contact metric structure is nearly cosymplectic non-coKähler. Considering the associated \(SU(2)\)-structure \((\eta, \omega_1, \omega_2, \omega_3)\) satisfying \([40]\), we have \(\lambda^2 = 1\) since the scalar curvature of \(S^6\) is \(s' = 30\). Coherently with Theorem \([6, 3]\) the scalar curvature of \(S^5\) is \(s = 20\).

Now consider \(S^5\) as a totally umbilical hypersurface of \(S^6\) defined by \(x^7 = \frac{x^2}{2}\), with unit normal at each point \(x\) given by \(\nu = x - \sqrt{2} \frac{\partial}{\partial x_7} = \sum_{i=1}^{6} x^i \frac{\partial}{\partial x^i} - \frac{x^7}{2} \frac{\partial}{\partial x_7}\), so that the shape operator is \(A = -I\). Let \((\phi, \xi, \eta, g)\) be the induced almost contact metric structure, where

\[
\xi = -J\nu = \sqrt{2} \left( x^1 \frac{\partial}{\partial x^6} - x^2 \frac{\partial}{\partial x^5} - x^3 \frac{\partial}{\partial x^4} + x^4 \frac{\partial}{\partial x^3} + x^5 \frac{\partial}{\partial x^2} - x^6 \frac{\partial}{\partial x^1} \right),
\]

and \(\eta\) given by the restriction of \(\sqrt{2} \left( x^1dx^6 - x^6dx^1 + x^5dx^2 - x^2dx^5 + x^4dx^3 - x^3dx^4 \right)\) to \(S^5\). This structure is nearly Sasakian, but not Sasakian and again, taking into account the associated \(SU(2)\)-structure satisfying \([30]\), the constant \(\lambda\) satisfies \(\lambda^2 = 1\). The scalar curvature of the hypersurface is \(40\), coherently with the fact that it has constant sectional curvature \(2\).

7. Canonical connections on nearly Sasakian manifolds

It is well known that nearly Kahler manifolds are endowed with a canonical Hermitian connection \(\nabla\), called Gray connection, defined by

\[
\nabla_X Y = \nabla_X Y + \frac{1}{2} (\nabla_X J) J Y,
\]

which is the unique Hermitian connection with totally skew-symmetric torsion. To the knowledge of the authors there does not exist any canonical connection, analogous to \(\nabla\), in the context of nearly Sasakian geometry. In particular, in \([13]\) Friedrich and Ivanov proved that an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) admits a (unique) linear connection with totally skew-symmetric torsion parallelizing all the structure tensors, if and only if \(\xi\) is Killing and the tensor \(N_\phi\) is totally skew-symmetric. Using this result, we prove the following

Proposition 7.1. A nearly Sasakian manifold \((M, \phi, \xi, \eta, g)\) admits a linear connection with totally skew-symmetric torsion parallelizing all the structure tensors if and only if it is Sasakian.

Proof. Recall that the tensor field \(N_\phi\) is also given by

\[
N_\phi(X, Y) = (\nabla_X \phi) Y - (\nabla_Y \phi) X + (\nabla_X \phi) \phi Y - (\nabla_Y \phi) \phi X + \eta(X) \nabla_X \xi - \eta(Y) \nabla_Y \xi.
\]

Setting \(N(X, Y, Z) := g(N_\phi(X, Y), Z)\), a straightforward computation using the above formula, \([3]\) and \([4]\), gives

\[
N(X, Y, \xi) + N(X, \xi, Y) = g(hX, Y).
\]

Hence, if \(N_\phi\) is totally-symmetric, then \(h = 0\) and the structure is Sasakian. \(\square\)
Thus it makes sense to find adapted connections which can be useful in the study of nearly Sasakian manifolds. We have the following theorem.

**Theorem 7.2.** Let \((M, \phi, \xi, \eta, g)\) be a nearly Sasakian manifold. Fix a real number \(r\). Then, there exists a unique linear connection \(\bar{\nabla}\) which parallelizes all the structure tensors and such that the torsion tensor \(\bar{T}\) of \(\bar{\nabla}\) satisfies the following conditions:

1) \(\bar{T}\) is totally skew-symmetric on \(\mathcal{D} = \ker(\eta)\),
2) the \((1,1)\)-tensor field \(\tau\) defined by

\[
\tau X = \bar{T}(\xi, X)
\]

satisfies

\[
\tau \phi + \phi \tau = -2(r + 1)\phi^2.
\]

This linear connection is given by:

\[
\bar{\nabla}_X Y = \nabla_X Y + H(X, Y)
\]

where

\[
H(X, Y) = \frac{1}{2}(\nabla_X \phi)\phi Y - r \eta(X)\phi Y + \eta(Y)(\phi - h)X - \frac{1}{2}g((\phi - h)X, Y)\xi.
\]

**Proof.** Let us consider the \((0,3)\)-tensors defined by \(H(X, Y, Z) := g(H(X, Y), Z)\) and \(\bar{T}(X, Y, Z) := g(\bar{T}(X, Y), Z)\). First, we prove that the linear connection defined by (51) and (52) parallelizes the structure. Notice that \(H(X, \xi) = \phi X - hX = -\nabla_X \xi\), and thus \(\nabla_X \xi = 0\).

The linear connection is metric if and only if

\[
H(X, Y, Z) + H(X, Z, Y) = 0.
\]

We compute,

\[
(\nabla_X \phi)\phi Y + \phi(\nabla_X \phi)Y = (\nabla_X \phi^2)Y
= (\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi
= g(Y, \nabla_X \xi) + \eta(Y)\nabla_X \xi
= -g(Y, \phi X - hX)\xi - \eta(Y)(\phi X - hX).
\]

A straightforward computation using (52) and (54) gives (53). Moreover, \(\bar{\nabla}\) satisfies \(\bar{\nabla}\phi = 0\) if and only if

\[
(\nabla_X \phi)Y + H(X, \phi Y) - \phi H(X, Y) = 0,
\]

which is proved again by a simple computation using (51). The torsion of \(\bar{\nabla}\) is given by

\[
\bar{T}(X, Y) = \frac{1}{2}((\nabla_X \phi)\phi Y - (\nabla_Y \phi)\phi X) - g((\phi - h)X, Y)\xi
- (r + 1)(\eta(X)\phi Y - \eta(Y)\phi X) + \eta(X)hY - \eta(Y)hX.
\]

Now, applying (52) and (55) we get

\[
(\nabla_Y \phi)\phi X = -\phi(\nabla_Y \phi)X - g(X, \phi Y - hY)\xi - \eta(X)(\phi Y - hY)
= \phi(\nabla_X \phi)Y + \eta(X)\phi Y + \eta(Y)\phi X + g(\phi X - hX, Y)\xi - \eta(X)(\phi Y - hY)
= -(\nabla_X \phi)\phi Y + \eta(X)hY + \eta(Y)hX.
\]
Therefore,
\[
\bar{T}(X,Y) = (\nabla_X\phi)\phi_Y - (r + 1)(\eta(X)\phi_Y - \eta(Y)\phi_X) \\
+ \frac{1}{2}\eta(X)h_Y - \frac{3}{2}\eta(Y)h_X - g((\phi - h)X,Y)\xi.
\]
In particular, for every \(X, Y, Z \in D\), applying (54), we have
\[
\bar{T}(X, Y, Z) + \bar{T}(X, Z, Y) = g(\nabla_X\phi\phi_Y + \phi(\nabla_X\phi)Y, Z) = 0
\]
which proves condition 1). Finally,
\[
\tau = (\nabla_\xi\phi)\phi - (r + 1)\phi + \frac{1}{2}h = \frac{3}{2}h - (r + 1)\phi,
\]
which implies (50).

We prove the uniqueness of the connection. Suppose that \(\bar{\nabla}\) is a linear connection parallelizing the structure and whose torsion satisfies 1) and 2). We determine the tensor \(H\) defined by (51). First we prove that for every \(X, Y, Z \in D\),
\[
H(X, Y, Z) = \frac{1}{2}g((\nabla_X\phi)\phi_Y, Z).
\]
Since \(\bar{\nabla}\) is a metric connection with totally skew-symmetric torsion on \(D\), for every \(X, Y, Z \in D\) we have
\[
\bar{T}(X, Y, Z) = \bar{T}(X, Y, Z) - \bar{T}(Y, Z, X) + \bar{T}(Z, X, Y)
= H(X, Y, Z) - H(Y, X, Z) - H(Y, Z, X)
+ H(Z, Y, X) + H(Z, X, Y) - H(X, Z, Y)
= 2H(X, Y, Z),
\]
and thus the tensor \(H\) is totally skew-symmetric on \(D\). Being \(\bar{\nabla}\phi = 0\), (53) holds. Hence
\[
H(X, Y, \phi Z) + H(X, \phi Y, Z) = -g((\nabla_X\phi)\phi_Y, Z).
\]
Now, we take the cycling permutation sum of the above formula. By the skew-symmetry of \(H\) and (2), we get
\[
2 \sum_{X,Y,Z} H(X, Y, \phi Z) = -3g((\nabla_X\phi)\phi_Y, Z).
\]
Substituting \(Y\) with \(\phi Y\), we have
\[
2H(X, \phi Y, \phi Z) + 2H(\phi Y, Z, \phi X) - 2H(Z, X, Y) = -3g((\nabla_X\phi)\phi_Y, Z).
\]
Now, applying (57) and (2),
\[
H(X, \phi Y, \phi Z) + H(\phi Y, Z, \phi X) = -H(\phi Y, X, \phi Z) + H(\phi Y, \phi X, Z)
= g((\nabla_{\phi Y}\phi)X, Z)
= -g((\nabla_X\phi)\phi Y, Z).
\]
Hence, substituting in (58), we get (56).

Now, being \(\bar{\nabla}\xi = 0\), for every vector field \(X\), we have \(H(X, \xi) = -\nabla_X\xi = \phi X - hX\). Moreover, since \(\bar{\nabla}\) is a metric connection, then \(H(X, Y, \xi) = -H(X, \xi, Y)\). Therefore, it remains to determine \(H(\xi, X)\). By \(\bar{\nabla}\phi = 0\), we have
\[
H(\xi, \phi X) - \phi H(\xi, X) = -\nabla_\xi(\phi X) = -\phi hX.
\]
We compute
\[(\tau \phi - \phi \tau)X = \bar{T}(\xi, \phi X) - \phi \bar{T}(\xi, X)\]
\[= H(\xi, \phi X) - H(\phi X, \xi) + \phi H(X, \xi)
\[= -2\phi hX - (\phi^2 X - h\phi X) + \phi(\phi X - hX)\]
\[= 3h\phi X.\]

Combining the above formula with condition 2), we obtain
\[2\tau \phi = 3h\phi - 2(r + 1)\phi^2.\]

Now, being \(\tau \xi = 0\), we get
\[\tau = \frac{3}{2}h - (r + 1)\phi.\]

It follows that
\[H(\xi, X) = \bar{T}(\xi, X) + H(X, \xi) = \frac{1}{2}hX - r\phi X.\]

This completes the proof that \(H\) coincides with the tensor defined in (52).

**Remark 7.3.** Suppose that \((M, \phi, \xi, \eta, g)\) is a Sasakian manifold. Recall that the covariant derivative of \(\phi\) is given by
\[\nabla_X \phi Y = g(X, Y)\xi - \eta(Y)X\]
(see [5, Theorem 6.3]). Then the tensor \(H\) in (52) becomes:
\[H(X, Y) = g(X, \phi Y)\xi - r\eta(X)\phi Y + \eta(Y)\phi X.\]

It follows that \(\nabla\) coincides with the linear connection defined by Okumura in [23] (see also [27]). In the case \(r = -1\), this is the Tanaka-Webster connection (cf. [28]). In the case \(r = 1\), this is the unique linear connection on the Sasakian manifold \(M\) parallelizing the structure and with totally skew-symmetric torsion defined in [19].

**Proposition 7.4.** Let \((M, \phi, \xi, \eta, g)\) be a 5-dimensional nearly Sasakian manifold. Let \(\bar{\nabla}\) be the canonical connection defined in (51) and (52). Then the structure tensor \(h\) is parallel with respect to \(\bar{\nabla}\) if and only if \(r = \frac{1}{2}\).

**Proof.** Using (52) and (34), we can compute
\[H(X, Y) = \frac{1}{2}\eta(X)hY - r\eta(X)\phi Y + \eta(Y)(\phi X - hX) - g(\phi X - hX, Y)\xi.\]

Now, using the above formula and (35), a straightforward computation gives
\[\nabla hY = (\nabla h)Y + H(X, hY) - hH(X, Y) = (1 - 2r)\eta(X)\phi hY\]
which proves our claim. \(\square\)

**Remark 7.5.** The canonical connection corresponding to \(r = \frac{1}{2}\) actually parallelizes the \(SU(2)\)-structure \(\{(\phi_i, \xi, \eta, g)\}_{i \in \{1, 2, 3\}}\), or equivalently \((\eta, \omega_1, \omega_2, \omega_3)\), associated to the nearly Sasakian non-Sasakian structure. Furthermore, the torsion of the canonical connection is given by
\[\bar{T}(X, Y) = \frac{3}{2}\eta(Y)(\phi X - hX) - \eta(X)(\phi Y - hY) - 2g(\phi X - hX, Y)\xi,\]
which turns out to satisfy \(\nabla h = 0\).

Now, if we apply the deformation (39), also the Sasaki-Einstein \(SU(2)\)-structure \((\bar{\eta}, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)\) is parallel with respect to the canonical connection \(\bar{\nabla}\). Furthermore, by (31) and (41), we obtain
\[H(X, Y) = \bar{g}(X, \phi Y)\bar{\xi} - \frac{1}{2}\bar{\eta}(X)\phi Y + \bar{\eta}(Y)\phi X.\]
Therefore, the canonical connection $\bar{\nabla}$ coincides with the Okumura connection associated to the Sasakian structure $(\phi, \xi, \eta, g)$ for $r = \frac{1}{2}$.

In general, for a Sasaki-Einstein 5-manifold we have the following

**Proposition 7.6.** Let $M$ be a Sasaki-Einstein 5-manifold with $SU(2)$-structure $(\eta, \omega_2, \omega_3)$. Then the Okumura connection corresponding to $r = \frac{1}{2}$ and associated to the Sasakian structure $(\phi_3, \xi, \eta, g)$ parallelizes the whole $SU(2)$-structure.

**Proof.** The Okumura connection corresponding to $r = \frac{1}{2}$ and associated to the Sasakian structure $(\phi_3, \xi, \eta, g)$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

where

$$H(X, Y) = g(X, \phi_3 Y)\xi - \frac{1}{2} \eta(X)\phi_3 Y + \eta(Y)\phi_3 X.$$

By Corollary 5.2, the almost contact metric structure $(\phi_2, \xi, \eta, g)$ is nearly cosymplectic, and thus

$$3g((\nabla_X \phi_2)Y, Z) = d\omega_2(X, Y, Z) = -3(\eta \wedge \omega_1)(X, Y, Z).$$

Therefore, an easy computation gives

$$(\nabla_X \phi_2)Y = g(X, \phi_1 Y)\xi - \eta(X)\phi_1 Y + \eta(Y)\phi_1 X.$$

Using the above equation, (60) and $\phi_2 \phi_3 = \phi_1 = -\phi_3 \phi_2$, we have

$$(\nabla_X \phi_2)Y = (\nabla_X \phi_3)Y + H(X, \phi_2 Y) - \phi_2 H(X, Y) = 0.$$

Hence, all the structure tensors $(\phi_i, \xi, \eta, g)$, $i \in \{1, 2, 3\}$, are parallel with respect to $\bar{\nabla}$.

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