GLOBAL EXISTENCE AND BLOWUP FOR CHOQUARD EQUATIONS WITH AN INVERSE-SQUARE POTENTIAL

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Abstract. In this paper, the Choquard equation with an inverse-square potential and both focusing and defocusing nonlinearities in the energy-subcritical regime is investigated. For all the cases, the local well-posedness result in $H^1(\mathbb{R}^N)$ is established. Moreover, the global existence result for arbitrary initial values is proved in the defocusing case while a global existence/blowup dichotomy below the ground state is established in the focusing case.

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1. Introduction and main results

In this paper, we consider the Cauchy problem for the Choquard equation with an inverse-square potential

$$\begin{cases}
i\partial_t u - \mathcal{L}_b u = -a(I_\alpha * |u|^p)|u|^{p-2}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N, 
\end{cases}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, $\alpha \in (0, N)$, $(N + \alpha)/N < p < (N + \alpha)/(N - 2)$, $I_\alpha$ is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{N}{2})\pi^{N/2}2^{\alpha}|x|^{N-\alpha}} := \frac{A}{|x|^{N-\alpha}}$$

with $\Gamma$ denoting the Gamma function (see [21]), $a = 1$ (focusing case) or $-1$ (defocusing case), and $\mathcal{L}_b$ is an inverse-square potential. More precisely,

$$\mathcal{L}_b := -\Delta + b|x|^{-2}$$

with $b > -(N - 2)^2/4$, and we consider the Friedrichs extension of the quadratic form $Q$ defined on $C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ via

$$Q(v) := \int_{\mathbb{R}^N} |\nabla v(x)|^2 + \frac{b}{|x|^2}|v(x)|^2 \, dx.$$

The choice of the Friedrichs extension is natural from a physical point of view; furthermore, when $b = 0$, $\mathcal{L}_b$ reduces to the standard Laplacian $-\Delta$. For more details, see for example [13].

The restriction on $b$ guarantees the positivity of $\mathcal{L}_b$. In fact, by using the sharp Hardy inequality

$$\frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla v(x)|^2 \, dx$$

for any $v \in H^1(\mathbb{R}^N)$.

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we obtain that

\[ Q(v) = \| \sqrt{\mathcal{L}} v \|_{L^2}^2 \sim \| \nabla v \|_{L^2}^2 \text{ for any } b > -(N - 2)^2/4. \]

In particular, the Sobolev space \( H^1(\mathbb{R}^N) \) is isomorphic to the space \( H_b^1(\mathbb{R}^N) \) defined in terms of \( \mathcal{L}_b \) and we denote

\[ \| v \|_{H_b^1}^2 := Q(v). \]

Solutions to \((\text{CH}_b)\) conserve the mass and energy (see Section 3), defined respectively by

\[ M(u(t)) := \int_{\mathbb{R}^N} |u(t,x)|^2 \, dx \]

and

\[ E_b(u(t)) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u(t,x)|^2 + b|x|^{-2}|u(t,x)|^2) \, dx - \frac{a}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u(t,x)|^p)|u(t,x)|^p \, dx. \]

Equation \((\text{CH}_b)\) enjoys the scaling invariance

\[ u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2 + \alpha}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \]

Direct calculation gives that

\[ \| u_\lambda(0,x) \|_{\dot{H}^\gamma} = \lambda^{\gamma - \frac{N}{2} + \frac{2 + \alpha}{2p}} \| u_0(x) \|_{\dot{H}^\gamma}. \]

This gives the critical Sobolev exponent

\[ \gamma_b = \frac{N}{2} - \frac{2 + \alpha}{2p - 2}. \]

The mass-critical case corresponds to \( \gamma_b = 0 \) (or \( p = p_b := 1 + (2 + \alpha)/N \)); The energy-critical case corresponds to \( \gamma_b = 1 \) (or \( p = p_b := (N + \alpha)/(N - 2) \)); And the inter-critical case corresponds to \( \gamma_b \in (0,1) \) (or \( p \in (p_b, p_b^\alpha) \)).

Equation \((\text{CH}_b)\) is a nonlocal counterpart of the Schrödinger equation

\[ \begin{cases}
  i\partial_t u - \mathcal{L}_b u = -a|u|^q u, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
  u(0,x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases} \tag{\text{NLS}_b} \]

which has been studied extensively, see [8] and [17] and the references therein. For equation \((\text{NLS}_b)\), the mass-critical case corresponds to \( q = q_b \) and the energy-critical case corresponds to \( q = q^b \), where

\[ q_b := \frac{4}{N} \quad \text{and} \quad q^b := \begin{cases}
  \frac{4}{N - 2}, & \text{if } N \geq 3, \\
  \infty, & \text{if } N = 1,2.
\end{cases} \]

Define

\[ \tilde{E}_b(v) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + b|x|^{-2}|v|^2) \, dx - \frac{a}{q + 2} \int_{\mathbb{R}^N} |v|^{q+2} \, dx, \]

\[ \tilde{\sigma} := \frac{4 - (N - 2)q}{Nq - 4}, \]

\[ \tilde{H}(b) := \tilde{E}_{b,0}(\tilde{Q}_{b,0})M(\tilde{Q}_{b,0})^{\tilde{\sigma}}, \quad \tilde{K}(b) := \| \tilde{Q}_{b,0} \|_{H_{b,0}^1} \| \tilde{Q}_{b,0} \|_{L^2}^2, \]

where \( \tilde{Q}_b \) with \(-(N - 2)^2/4 < b \leq 0 \) is a radial ground state to the elliptic equation

\[ \mathcal{L}_b Q + Q = Q^{q+1}. \]
We summarize parts of the results for (NLS$_b$) in the following theorem.

**Theorem A** ([8, 20]). Let $N \geq 3, b > -(N - 2)^2/4, a = \pm 1$ and $u_0 \in H^1(\mathbb{R}^N)$.

1. If $0 \leq q < q^b$, then (NLS$_b$) is local well-posed.
2. If $a = -1$ and $0 \leq q < q^b$, then (NLS$_b$) is global well-posed.
3. If $a = 1$ and $0 \leq q < q^b$, then the solution to (NLS$_b$) exists globally.
4. Assume that $a = 1$ and $q = q_0$. If $\|u_0\|_{L^2} < \|\hat{Q}_{b,0}\|_{L^2}$, then the solution $u$ exists globally and $\sup_{t \in \mathbb{R}} \|u\|_{H^1} < \infty$; If $\dot{E}_b(u_0) < 0$ and either $xu_0 \in L^2(\mathbb{R}^N)$ or $u_0$ is radial, then the solution blows up in finite time.
5. Assume that $a = 1, q_0 < q < q^b$ and $\dot{E}_b(u_0)M(u_0)^q < \dot{H}(b)$. If $\|u_0\|_{\dot{H}^1} \|u_0\|_{L^2}^{\frac{q}{2}} < \tilde{K}(b)$, then the solution $u$ exists globally and
   \[ \|u(t)\|_{\dot{H}^1} \|u(t)\|_{L^2}^{\frac{q}{2}} < \tilde{K}(b) \]
for any $t \in \mathbb{R}$; If $\|u_0\|_{\dot{H}^1} \|u_0\|_{L^2}^{\frac{q}{2}} > \tilde{K}(b)$ and either $xu_0 \in L^2(\mathbb{R}^N)$ or $u_0$ is radial, then the solution $u$ blows up in finite time and
   \[ \|u(t)\|_{\dot{H}^1} \|u(t)\|_{L^2}^{\frac{q}{2}} > \tilde{K}(b) \]
for any $t$ in the existence time.

For equation (CH$_b$), when $b = 0$, (CH$_0$) is space-translation invariant. When $p = 2$, (CH$_0$) is called the Hartree equation. In this case, the local well-posedness and asymptotic behavior of the solutions were established in [5] and [11]. The global well-posedness and scattering for the defocusing energy-critical problem were discussed by Miao et al. [18]. The dynamics of the blowup solutions with minimal mass for the focusing mass-critical problem were investigated by Miao et al. [19]. When $N \geq 3, \alpha = 2, 2 \leq p < (N + \alpha)/(N - 2)$, Genev and Venkov [10] studied the local and global well-posedness, the existence of standing waves, the existence of blowup solutions, and the dynamics of the blowup solutions in the mass-critical case. For the general case $0 < \alpha < N$ and $2 \leq p < (N + \alpha)/(N - 2)$, Chen and Guo [6] studied the existence of blowup solutions and the strong instability of standing waves. Bonanno et al. [1] investigated the soliton dynamics. Feng and Yuan [9] studied the local and global well-posedness, finite time blowup and the dynamics of blowup solutions. More precisely, [9] obtained the following result.

**Theorem B.** Let $N \geq 3, (N - 4)_+ < \alpha < N, 2 \leq p < (N + \alpha)/(N - 2)$, $a = \pm 1$ and $u_0 \in H^1(\mathbb{R}^N)$.

1. (CH$_0$) is local well-posed.
2. If one of the following cases hold, (i) $a = -1$; (ii) $a = 1$ and $2 \leq p < p_b$; (iii) $a = 1, p = p_0$ and $\|u_0\|_{L^2} < \|Q_{0}\|_{L^2}$, where $Q_0$ is a radial ground state to (1.4) with $b = 0$: Then the solution to (CH$_0$) exists globally.
3. If $\max\{p_b, 2\} \leq p < p_b$, $xu_0 \in L^2(\mathbb{R}^N)$ and one of the following cases hold, (i) $E_0(u_0) < 0$; (ii) $\dot{E}_0(u_0) = 0$ and $\text{Im} \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla u_0 dx < 0$; (iii) $E_0(u_0) > 0$ and $\text{Im} \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla u_0 dx < -2E_0(u_0)\|xu_0\|_{L^2}$; Then the solution to (CH$_0$) blows up in finite time.

When $b \neq 0$, equation (CH$_b$) is not space-translation invariant anymore. It is known that $\dot{W}^{\gamma,q}_b(\mathbb{R}^N)$ is not equivalent to $\dot{W}^{\gamma,q}(\mathbb{R}^N)$ for general $\gamma$ and $q$ (see [15]), which restricts the application of Strichartz estimates on the study of the local well-posedness and scattering of global solutions (see [17] for the study of the Schrödinger equation with an inverse-square potential). Fortunately, Okazawa et al. [20] formulated an improved energy method to treat equation with an inverse-square potential. Based of the abstract theorem established in [20], there has been
great progress for equation (CH$_b$) with $p = 2$. For example, Okazawa et al. [20] obtained the local well-posedness result when $(N - 4)_+ \leq \alpha < N$ and $a = \pm 1$, and further the global existence result when $a = -1$ or $a = 1$ and $\alpha \geq (N - 2)_+$. Suzuki studied the scattering of the global solution when $a = -1$ and $\alpha \in (N - 2, N - 1)$ in [24] and the blowup result for initial value $u_0 \in H^1(\mathbb{R}^N)$ with $\|xu_0\|_{L^2(\mathbb{R}^N)}$ and $E_b(u_0) < 0$ in [23]. To our knowledge, there are not any results for (CH$_b$) with general exponent $p \neq 2$.

By comparing the results for (CH$_b$) and (NLS$_b$), we see that the whole picture of (CH$_b$) is far to be completed, even in the case $b = 0$. For example, the sharp global existence/blowup dichotomy in the inter-critical case and the blowup result for radial initial values remain open. So in this paper, we study the local well-posedness, global existence and blowup dichotomy for equation (CH$_b$) with general $p$ and $\alpha$, and expect to obtain similar results to (NLS$_b$).

Before stating our main results, we make some notations. Define

$$
\sigma := \frac{N + \alpha - Np + 2p}{Np - N - \alpha - 2},
$$

$$
H(b) := E_b(a(Q_{b,0}))\|Q_{b,0}\|^2_{L^2}, \quad K(b) := \|Q_{b,0}\|_{H^1_{b,0}}\|Q_{b,0}\|^2_{L^2},
$$

$$
H(b, \text{rad}) := E_b(Q_{b,\text{rad}})\|Q_{b,\text{rad}}\|^2_{L^2}, \quad K(b, \text{rad}) := \|Q_{b,\text{rad}}\|_{H^1_{b,\text{rad}}}\|Q_{b,\text{rad}}\|^2_{L^2},
$$

where $Q_b$ with $-(N - 2)/4 < b \leq 0$ is the radial ground state to the elliptic equation

$$
\mathcal{L}_bQ + Q = (I_\alpha * |Q|^p)|Q|^{p-2}Q
$$

and $Q_{b,\text{rad}}$ with $b > 0$ is the radial solution to (1.4) obtained in Section 4.

We begin by defining solutions to (CH$_b$).

**Definition 1.1.** Let $I \subset \mathbb{R}$ be an open interval containing 0 and $u_0 \in H^1(\mathbb{R}^N)$. We call $u : I \times \mathbb{R}^N \to \mathbb{C}$ a weak solution to (CH$_b$) if it belongs to $L^\infty(K, H^1(\mathbb{R}^N)) \cap W^{1,\infty}(K, H^{-1}(\mathbb{R}^N))$ and satisfies (CH$_b$) in the sense of $L^\infty(K, H^{-1}(\mathbb{R}^N))$ for any compact $K \subset I$. Moreover, if $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$, we call it a solution to (CH$_b$). We call $I$ the lifespan of $u$. We call $u$ a maximal-lifespan solution if it cannot be extended to a strictly larger interval. We call $u$ global if $I = \mathbb{R}$ and blowup in finite time if $I \neq \mathbb{R}$.

The main results of this paper are as follows.

**Theorem 1.2.** Let $N \geq 3$, $b > -(N - 2)/4$ and $b \neq 0$, $\alpha \in ((N - 4)_+, N)$, $2 \leq p < p^b$ and $a = \pm 1$. Then for any $u_0 \in H^1(\mathbb{R}^N)$, there exists a unique maximal-lifespan solution $u \in C(I; H^1(\mathbb{R}^N)) \cap C^1(I; H^{-1}(\mathbb{R}^N))$ to (CH$_b$). Moreover, $u$ satisfies the conservation laws

$$
M(u(t)) = M(u_0), \quad E_b(u(t)) = E_b(u_0), \text{ for any } t \in I,
$$

where $M$ and $E_b$ are defined in (1.2) and (1.3), respectively.

**Theorem 1.3.** Let $N \geq 3$, $b > -(N - 2)/4$ and $b \neq 0$, $\alpha \in ((N - 4)_+, N)$ and $u \in C(I; H^1(\mathbb{R}^N)) \cap C^1(I; H^{-1}(\mathbb{R}^N))$ be the maximal-lifespan solution to (CH$_b$) with initial value $u_0 \in H^1(\mathbb{R}^N)$. If one of the following conditions hold:

(i) $a = -1$ and $2 \leq p < p^b$;
(ii) $a = 1$ and $2 \leq p < p_6$;

Then $u$ exists globally and $\sup_{t \in \mathbb{R}} \|u\|_{H^1} < \infty$. 

Theorem 1.4. Let $N \geq 3$, $b > -(N - 2)^2/4$ and $b \neq 0$, $\alpha \in ((N - 4)_+, (N), a = 1$, 
$p = p_b$ and $u \in C(I; H^1(\mathbb{R}^N)) \cap C^1(I; H^{-1}(\mathbb{R}^N))$ be the maximal-lifespan solution to (CH$_b$) with initial value $u_0 \in H^1(\mathbb{R}^N)$.

(i) If $\|u_0\|_{L^2} < \|Q_{b_0}\|_{L^2}$, then $u$ exists globally and $\sup_{t \in \mathbb{R}} \|u\|_{H^1} < \infty$;
(ii) If $E_b(u_0) < 0$ and $x u_0 \in L^2(\mathbb{R}^N)$, then $u$ blows up in finite time.

Remark 1.5. (1). The condition $E_b(u_0) < 0$ in Theorem 1.4 (ii) is a sufficient but not necessary condition, see the proof in Section 5.

(2). Since we cannot estimate the nonlocal nonlinearity in the local virial identity, the case $E_b(u_0) < 0$ and $u_0 \in H^1(\mathbb{R}^N)$ in Theorem 1.4 (ii) is left open.

(3). If $-(N-2)^2/4 < b < 0$, then
$$u_T(t, x) = \frac{1}{(T-t)^{\frac{N}{2}}} e^{\frac{-(N-2)^2 t}{4}} Q_b \left( \frac{x}{T-t} \right), \quad T > 0$$
is a solution to (CH$_b$) which blows up at time $T$ and $\|u_T(0)\|_{L^2} = \|Q_b\|_{L^2}$.

(4). In the case $b > 0$, in view of the radial sharp Gagliardo-Nirenberg inequality, similarly to the proof of Theorem 1.4 (i), we can show that if $u_0 \in H^1(\mathbb{R}^N)$ and $\|u_0\|_{L^2} < \|Q_{b, \text{rad}}\|_{L^2}$, then the solution to (CH$_b$) exists globally. Moreover, it is easy to show that
$$u_T(t, x) = \frac{1}{(T-t)^{\frac{N}{2}}} e^{\frac{(N-2)^2 t}{4}} Q_{b, \text{rad}} \left( \frac{x}{T-t} \right), \quad T > 0$$is a solution to (CH$_b$) which blows up at time $T$ and $\|u_T(0)\|_{L^2} = \|Q_{b, \text{rad}}\|_{L^2}$.

Theorem 1.6. Let $N \geq 3$, $b > -(N - 2)^2/4$ and $b \neq 0$, $\alpha \in ((N - 4)_+, (N), a = 1$, 
$p_b < p < p^b$ and $u \in C(I; H^1(\mathbb{R}^N)) \cap C^1(I; H^{-1}(\mathbb{R}^N))$ be the maximal-lifespan solution to (CH$_b$) with initial value $u_0 \in H^1(\mathbb{R}^N)$ and $E_b(u_0)\|u_0\|_{H^1}^2 < H(b)$.

(i) If $\|u_0\|_{H^1}^2 \|u_0\|_{L^2}^2 < K(b)$, then $u$ exists globally and
$$\|u(t)\|_{H^1}^2 \|u(t)\|_{L^2}^2 < K(b) \quad \text{for any} \ t \in \mathbb{R};$$
(ii) If $\|u_0\|_{H^1}^2 \|u_0\|_{L^2}^2 > K(b)$, and either $x u_0 \in L^2(\mathbb{R}^N)$ or $u_0$ is radial (in this case, we further assume that $p < 2\frac{N+6}{N+1}$), then $u$ blows up in finite time and
$$\|u(t)\|_{H^1}^2 \|u(t)\|_{L^2}^2 > K(b) \quad \text{for any} \ t \in I.$$

Remark 1.7. In Theorem 1.6 (ii), when $u_0 \in H^1(\mathbb{R}^N)$, the restriction $p < 2\frac{N+6}{N+1}$ is added for the following reason: Roughly speaking, the nonlocal nonlinearity is of order $|u|^{2p}$, so it can be controlled by $\|u\|_{H^1}^2$ only if $p$ is in a subset of $2 \leq p < p^b$, see (5.43) and (5.44).

In view of the radial sharp Gagliardo-Nirenberg inequality, similarly to the proof of Theorem 1.6, we obtain the following result.

Theorem 1.8. Let $N \geq 3$, $\alpha \in ((N - 4)_+, (N), p_b < p < p^b$, $b > 0$, $a = 1$ and 
$u \in C(I; H^1(\mathbb{R}^N)) \cap C^1(I; H^{-1}(\mathbb{R}^N))$ be the maximal-lifespan solution to (CH$_b$) with initial value $u_0 \in H^1(\mathbb{R}^N)$ and $E_b(u_0)\|u_0\|_{H^1}^2 < H(b, \text{rad})$.

(i) If $\|u_0\|_{H^1}^2 \|u_0\|_{L^2}^2 < K(b, \text{rad})$, then $u$ exists globally and
$$\|u(t)\|_{H^1}^2 \|u(t)\|_{L^2}^2 < K(b, \text{rad}) \quad \text{for any} \ t \in \mathbb{R};$$
(ii) If $\|u_0\|_{H^1}^2 \|u_0\|_{L^2}^2 > K(b, \text{rad})$ and $p_b < p < 2\frac{N+6}{N+1}$, then $u$ blows up in finite time and
$$\|u(t)\|_{H^1}^2 \|u(t)\|_{L^2}^2 > K(b, \text{rad}) \quad \text{for any} \ t \in I.$$
Remark 1.9. Since $C_{GN}(b, \text{rad}) < C_{GN}(b)$ for $b > 0$, we see from Section 4 that $H(b) < H(b, \text{rad})$ and $K(b) < K(b, \text{rad})$. This shows that the class of radial solution enjoys strictly larger thresholds for the global existence/blowup dichotomy.

Our arguments parallel those of [8], where the Schrödinger equation with an inverse-square potential was considered. New technical obstructions appear in our arguments, due to the nonlocal nonlinearity and the fast decay of the potential. The first difficulty we face is the local well-posedness. The usual ways to show the local well-posedness in $H^1(\mathbb{R}^N)$ are the Kato’s method and the energy method. In the presence of the singular potential $b|x|^{-2}$, for the homogeneous Sobolev spaces $W^{q,\gamma}_b(\mathbb{R}^N)$ and the usual ones $W^{q,\gamma}(\mathbb{R}^N)$ are equivalent only in a certain range of $\gamma$ and $q$, the Kato’s method does not allow us to study $(CH_b)$ in the energy space with the full range $b > -\frac{(N-2)^2}{4}$. Moreover, Okazawa-Suzuki-Yokota [20] pointed out that the energy method developed by Cazenave is not enough to study $(CH_b)$ in the energy space. So they formulated an improved energy method to treat equation with an inverse-square potential and established an abstract theorem. Based of which, [22] studied a nonlocal equation, taking $(CH_b)$ with $p = 2$ as a special example. Motivated by [20] and [22], in this paper, we further use this method to study $(CH_b)$ with general $p$, which needs much more complicated calculations and an important inequality from [4]. The global existence is a direct result of the local well-posedness and the sharp Gagliardo-Nirenberg inequality. To show the blowup phenomenon the virial identity plays an important role, which has not been proved in the presence of the fast decay potential. This is the second difficulty we encounter. By examining the proof of the virial identity in Proposition 6.5.1 in [5], the $H^2(\mathbb{R}^N)$ regularity of the solution is important. However, we only have obtained the $H^1(\mathbb{R}^N)$ solution by using the improved energy method. In order to improve the regularity, the equivalence of $W^{q,\gamma}_b(\mathbb{R}^N)$ and $W^{q,\gamma}(\mathbb{R}^N)$ is needed, so we can only obtain the result in parts of the range $b > -\frac{(N-2)^2}{4}$. So this method is not effective. Motivated by [23], we consider a proximation problem $(CH^\delta_b)$ of $(CH_b)$. By using the results and the methods of [5], we can obtain the virial identity for the solution $u_\delta$ to $(CH^\delta_b)$, and then by letting $\delta \to 0$, we obtain the virial identity for the solution $u$ to $(CH_b)$ avoiding the $H^2$ regularity of $u$. We should point that the proof in [23] depends heavily on $p = 2$. Our proof is a modification of the methods of [5] and [23].

This paper is organized as follows. In Section 2, we recall some preliminary results related to equation $(CH_b)$. In Section 3, we study the local well-posedness for equation $(CH_b)$ in the energy-subcritical case. In Section 4, we first study the sharp Gagliardo-Nirenberg inequality by variational methods, and then, based of which, we give the proofs of global existence results. In Section 5, we establish the virial identities and prove the blowup results.

Notations. The notation $A \lesssim B$ means that $A \leq CB$ for some constant $C > 0$. If $A \lesssim B \lesssim A$, we write $A \sim B$. We write $A \wedge B = \min\{A, B\}$, $A \vee B = \max\{A, B\}$. We use $C, C_1, C_2, \cdots$ to denote various constant which may change from line to line. We use $L^q(I, L^r(\mathbb{R}^N))$ time-space norms defined via

$$
\|u\|_{L^q(I, L^r(\mathbb{R}^N))} := \left( \int_I \|u(t)\|_{L^r(\mathbb{R}^N)}^q \right)^{\frac{1}{q}}
$$
Lemma 2.3. \( p \) independent of \( u \).

Remark 2.2. (1). By the Hardy-Littlewood-Sobolev inequality above, for any \( \alpha \), we obtain
\[
K = C(N, \alpha, p, r) \frac{\pi^{\frac{N-2}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)} \right\}^{-\frac{N}{2}}.
\]

Remark 2.2. (2). By the Hardy-Littlewood-Sobolev inequality above and the Sobolev embedding theorem, we obtain
\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \leq C \left( \int_{\mathbb{R}^N} |u|^{2p/N} \right)^{\frac{N}{2p}} \leq C \|u\|_{L^p}^{2p},
\]
for any \( p \in \left[ \frac{N+\alpha}{N}, \frac{N+\alpha}{N-2} \right] \), where \( C > 0 \) is a constant depending only on \( N \), \( \alpha \) and \( p \).

Lemma 2.3. Let \( N \geq 3 \), \( \alpha \in (0, N) \) and \( p \in \left[ \frac{N+\alpha}{N}, \frac{N+\alpha}{N-2} \right] \). Assume that \( \{w_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N) \) satisfying \( w_n \to w \) weakly in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \), then
\[
(I_\alpha * |w_n|^p)|w_n|^{p-2}w_n \to (I_\alpha * |w|^p)|w|^{p-2}w \text{ weakly in } H^{-1}(\mathbb{R}^N) \text{ as } n \to \infty.
\]
Proof. By the Rellich theorem, \( w_n \to w \) strongly in \( L^r_{\text{loc}}(\mathbb{R}^N) \) with \( r \in [1, 2^*) \). By using the Hardy-Littlewood-Sobolev inequality, for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} (I_n * |w_n|^p)|w_n|^{p-2}w_n\varphi dx - \int_{\mathbb{R}^N} (I_n * |w|^p)|w|^{p-2}w\varphi dx
\]

\[
= \int_{\mathbb{R}^N} (I_n * (|w_n|^p - |w|^p))|w|^{p-2}w\varphi dx
\]

\[
+ \int_{\mathbb{R}^N} (I_n * |w_n|^p)(|w_n|^{p-2}w_n - |w|^{p-2}w)\varphi dx
\]

\[
= \int_{\mathbb{R}^N} (I_n * (|w|^{p-2}w\varphi))(|w_n|^p - |w|^p)dx
\]

\[
+ \int_{\mathbb{R}^N} (I_n * |w_n|^p)(|w_n|^{p-2}w_n - |w|^{p-2}w)\varphi dx
\]

\[
\to 0
\]

as \( n \to \infty \). By the dense of \( C_0^\infty(\mathbb{R}^N) \) in \( H^1(\mathbb{R}^N) \), we complete the proof. \( \square \)

We recall the following radial Sobolev embedding, see [7].

**Lemma 2.4.** Let \( N \geq 2 \) and \( 1/2 \leq s < 1 \). Then for any \( u \in H^1_s(\mathbb{R}^N) \),

\[
\sup_{x \in \mathbb{R}^N \setminus \{0\}} |x|^{-\frac{N+2s}{p}}|u(x)| \leq C(N, s)\|u\|_{L^p}^{-s}\|u\|_{H^1_s(\mathbb{R}^N)}^{s}.
\]

Moreover, the above inequality also holds for \( N \geq 3 \) and \( s = 1 \).

We recall the Strichartz estimates for the Schrödinger operator with an inverse-square potential (see [2] and [3]). We begin by introducing the notion of admissible pair.

**Definition 2.5.** We say that a pair \((p, q)\) is Schrödinger admissible, for short \((p, q) \in S\), if

\[
\frac{2}{p} + \frac{N}{q} = \frac{N}{2}, \quad p, q \in [2, \infty] \quad \text{and} \quad (p, q, N) \neq (2, \infty, 2).
\]

**Proposition 2.6.** Let \( N \geq 3 \) and \( b > -(N - 2)^2/4 \). Then for any \((p, q)\), \((a, b) \in S\), the following inequalities hold:

\[
\|e^{itL_b} \varphi(x)\|_{L^p(\mathbb{R}^N)} \lesssim \|\varphi(x)\|_{L^2},
\]

\[
\left\| \int_0^t e^{i(t-s)L_b} F(s, x) ds \right\|_{L^p(\mathbb{R}^N)} \lesssim \|F(t, x)\|_{L^{p'}(\mathbb{R}^N)}.
\]

Here, \((a, a')\) and \((b, b')\) are Hölder dual pairs.

We recall the convergence of the operators \( \{L^n_b\}_{n=1}^\infty \) defined below arising from the lack of translation symmetry for \( L_b \).

**Definition 2.7.** For \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^N \), define

\[
L^n_b := -\Delta + \frac{b}{|x + x_n|^2}, \quad L^\infty_b := \left\{ \begin{array}{ll} -\Delta + \frac{b}{|x + x_n|^2}, & \text{if } x_n \to x_\infty \in \mathbb{R}^N, \\ -\Delta, & \text{if } |x_n| \to \infty. \end{array} \right.
\]

By definition, we have \( L_b[u(x - x_n)] = [L^n_b u](x - x_n) \). The operator \( L^\infty_b \) appears as a limit of the operators \( L^n_b \) in the following senses, see [14].
Lemma 2.8. Let $N \geq 3$ and $b > - (N - 2)^2 / 4$. Suppose $\{ t_n \}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies $t_n \to t_\infty \in \mathbb{R}$ and $\{ x_n \}_{n=1}^{\infty} \subset \mathbb{R}^N$ satisfies $x_n \to x_\infty \in \mathbb{R}^N$ or $|x_n| \to \infty$. Then
\[
\lim_{n \to \infty} \| L_b^n u - L_b^\infty u \|_{\dot{H}^{-1}} = 0 \quad \text{for any } u \in \dot{H}^1(\mathbb{R}^N),
\]
\[
\lim_{n \to \infty} \| e^{-it_n L_b^n} u - e^{-it \infty L_b^\infty} u \|_{\dot{H}^{-1}} = 0 \quad \text{for any } u \in \dot{H}^{-1}(\mathbb{R}^N),
\]
\[
\lim_{n \to \infty} \| \sqrt{L_b^n} u - \sqrt{L_b^\infty} u \|_{L^2} = 0 \quad \text{for any } u \in \dot{H}^1(\mathbb{R}^N).
\]
Furthermore, for any $(p, q) \in S$ with $p \neq 2$,
\[
\lim_{n \to \infty} \| e^{-it_n L_b^n} u - e^{-it \infty L_b^\infty} u \|_{L^p(\mathbb{R}, L^q)} = 0 \quad \text{for any } u \in L^2(\mathbb{R}^N).
\]

3. Local well-posedness

In this section, we study the local well-posedness for equation (CHb) by using the abstract theory established in [20] by using an improved energy method.

3.1. Abstract theory of nonlinear Schrödinger equations. Let $S$ be a nonnegative selfadjoint operator in a complex Hilbert space $X$. Set $X_S := D(S^{1/2})$. Then we have the usual triplet: $X_S \subset X = X^* \subset X_S^*$, where $*$ denotes conjugate space. Under this setting $S$ can be extended to a nonnegative selfadjoint operator in $X_S^*$ with domain $X_S$. Now consider

\[
\begin{cases}
  i\partial_t u = Su + g(u), & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\]

where $g : X_S \to X_S^*$ is a nonlinear operator satisfying the following conditions.

(G1) Existence of energy functional: there exists $G \in C^1(X_S; \mathbb{R})$ such that $G' = g$, that is, given $u \in X_S$, for any $\epsilon > 0$ there exists $\delta = \delta(u, \epsilon) > 0$ such that
\[
|G(u + v) - G(u) - \text{Re}(g(u), v)_{X_S^*, X_S}| \leq \epsilon \|v\|_{X_S}
\]
for any $v \in X_S$ with $\|v\|_{X_S} < \delta$;

(G2) Local Lipschitz continuity: for any $M > 0$ there exists $C(M) > 0$ such that
\[
\|g(u) - g(v)\|_{X_S^*} \leq C(M) \|u - v\|_{X_S}
\]
for any $u, v \in X_S$ with $\|u\|_{X_S}, \|v\|_{X_S} \leq M$;

(G3) Hölder-like continuity of energy functional: given $M > 0$, for any $\delta > 0$ there exists a constant $C_\delta(M) > 0$ such that
\[
|G(u) - G(v)| \leq \delta + C_\delta(M) \|u - v\|_X
\]
for any $u, v \in X_S$ with $\|u\|_{X_S}, \|v\|_{X_S} \leq M$;

(G4) Gauge type condition: for any $u \in X_S$,
\[
\text{Im}(g(u), u)_{X_S^*, X_S} = 0;
\]

(G5) Closedness type condition: let $I \subset \mathbb{R}$ be a bounded open interval and $\{u_n\}_{n=1}^{\infty}$ be any bounded sequence in $L^\infty(I; X_S)$ such that
\[
\begin{align*}
  u_n(t) &\to u(t) \quad \text{weakly in } X_S \text{ as } n \to \infty \text{ for almost all } t \in I, \\
g(w_n) &\to f \quad \text{weakly* in } L^\infty(I; X_S^*) \text{ as } n \to \infty,
\end{align*}
\]
then
\[
\text{Im} \int_I \langle f(t), w(t) \rangle_{X_S^*, X_S} dt = \lim_{n \to \infty} \text{Im} \int_I \langle g(w_n(t)), w_n(t) \rangle_{X_S^*, X_S} dt.
\]

Under the above assumptions on $g$, the authors in [20] established the following local well-posedness result for (3.1).
Theorem 3.1. Assume that \(g : X_S \to X_S^*\) satisfies (G1)-(G5). Then for any \(u_0 \in X_S\) with \(\|u_0\|_{X_S} \leq M\) there exists an interval \(I_M \subset \mathbb{R}\) containing 0 such that (3.1) admits a local weak solution \(u \in L^\infty(I_M, X_S) \cap W^{1,\infty}(I_M, X_S^*)\). Moreover, \(u \in C_w(I_M, X_S)\) and \(\|u(t)\|_X = \|u_0\|_X\) for any \(t \in I_M\). Further assume the uniqueness of local weak solutions to (3.1). Then
\[
u \in C(I_M, X_S) \cap C^1(I_M, X_S^*)\]
and the conservation law holds:
\[
E(u(t)) = E(u_0), \quad \text{for any } t \in I_M,
\]
where \(E\) is the energy of equation (3.1) defined by
\[
E(\varphi) := \frac{1}{2}\|S^{1/2}\varphi\|_X^2 + G(\varphi), \quad \varphi \in X_S.
\]

3.2. Local well-posedness to (CH\(_0\)). In this subsection, we use Theorem 3.1 to prove Theorem 1.2. To this end, we first give some lemmas.

Lemma 3.2. ([22]) Assume that \(\alpha_1, \beta_1, \gamma_1, \rho_1 \in [1, \infty]\), \(k(x, y) \in L^{\beta_1}L^{\gamma_1} \cap L^{\beta_1}L^{\gamma_1}\), and
\[
\alpha_1 \leq \rho_1 \leq \beta_1, \quad \frac{1}{\alpha_1} + \frac{1}{\beta_1} + \frac{1}{\gamma_1} = 1 + \frac{1}{\rho_1}.
\]
Then the operator defined by
\[
Kf(x) := \int_{\mathbb{R}^N} k(x, y)f(y)dy
\]
is linear and bounded from \(L^{\gamma_1}(\mathbb{R}^N)\) to \(L^{\rho_1}(\mathbb{R}^N)\). Moreover,
\[
\|Kf\|_{L^{\rho_1}} \leq \left(\|k\|_{L^{\beta_1}L^{\alpha_1}} \vee \|k\|_{L^{\beta_1}L^{\alpha_1}}\right)\|f\|_{L^{\gamma_1}} \quad \text{for any } f \in L^{\gamma_1}(\mathbb{R}^N).
\]

Lemma 3.3. Assume that \(\alpha_1, \beta_1 \in [1, \infty]\), \(\alpha_1 \leq \beta_1\) and \(1/\alpha_1 + 1/\beta_1 \leq 4/N\). Set
\[
\frac{1}{\gamma_1} := 1 - \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right).
\]
Assume that \(k(x, y) \in L^{\beta_1}L^{\gamma_1}\) is symmetric, that is, \(k(x, y) = k(y, x)\) for any \(x, y \in \mathbb{R}^N\). Then, for any \(f, g \in L^{\gamma_1}(\mathbb{R}^N)\),
\[
\|gK(f)\|_{L^1} \leq \|k\|_{L^{\beta_1}L^{\gamma_1}}\|f\|_{L^{\gamma_1}}\|g\|_{L^{\gamma_1}}
\]
and
\[
\|K(f)\|_{L^{\rho_1}} \leq \|k\|_{L^{\beta_1}L^{\alpha_1}}\|f\|_{L^{\gamma_1}}.
\]

Proof. Applying Lemma 3.2 with \(\rho_1 = \gamma_1\) and by using the Hölder inequality, we obtain that
\[
\|gK(f)\|_{L^1} \leq \|K(f)\|_{\rho_1}\|g\|_{\rho_1} \leq \|k\|_{L^{\beta_1}L^{\alpha_1}}\|f\|_{L^{\gamma_1}}\|g\|_{L^{\gamma_1}},
\]
which completes the proof. \(\Box\)

Lemma 3.4. Let \(N \geq 3\), \(\alpha \in ((N-4)_+, N)\), \(2 \leq p < (N+\alpha)/(N-2)\) and \(a = \pm 1\). Define
\[
g(u) := -a(I_{\alpha} \ast |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N)
\]
and
\[
G(u) := -\frac{a}{2p}\int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^pdx, \quad u \in H^1(\mathbb{R}^N).
\]
Then \( g \) and \( G \) satisfy conditions (G1)-(G5) as in subsection 3.1 with \( S = \mathcal{L}_b \), \( X = L^2(\mathbb{R}^N) \), \( X_S = H^1(\mathbb{R}^N) \) and \( X_S^* = H^{-1}(\mathbb{R}^N) \).

**Proof.** When \( p = 2 \), the lemma is proved in [22]. So in the following, we assume that \( p > 2 \). By the definition of \( g \), (G4) holds.

We verify (G1). By using the following inequality from [4]

\[
||\tilde{a} + \tilde{b}|^m - |\tilde{a}|^m - m|\tilde{a}|^{m-2}\tilde{b}| \leq C(|\tilde{a}|^{m-m^*} |\tilde{b}|^{m^*} + |\tilde{b}|^m),
\]

\( \tilde{a}, \tilde{b} \in \mathbb{R}, m > 1, m^* = \min\{m, 2\}, C \) is independent of \( \tilde{a}, \tilde{b} \),

and the Young inequality

\[
\tilde{a}^\frac{p}{r} + \tilde{b}^\frac{p}{r'} \leq \frac{\tilde{a}}{r} + \frac{\tilde{b}}{r'},
\]

we have, for \( p/2 \geq 2 \),

\[
(|u|^2 + 2\text{Re}(uv) + |v|^2)^{p/2} = |u|^p + p|u|^{p-2}\text{Re}(uv) + p/2|u|^{p-2}|v|^2
\]

\[
+ O(|u|^{p-4}(2\text{Re}(uv) + |v|^2)^2 + (2\text{Re}(uv) + |v|^2)^{p/2})
\]

\[
= |u|^p + p|u|^{p-2}\text{Re}(uv) + O(|u|^{p-2}|v|^2 + |v|^p)
\]

\[
:= |u|^p + p|u|^{p-2}\text{Re}(uv) + h(u, v),
\]

and for \( 1 < p/2 < 2 \),

\[
(|u|^2 + 2\text{Re}(uv) + |v|^2)^{p/2} = |u|^p + p|u|^{p-2}\text{Re}(uv) + p/2|u|^{p-2}|v|^2
\]

\[
+ O(2\text{Re}(uv) + |v|^2)^{p/2}
\]

\[
= |u|^p + p|u|^{p-2}\text{Re}(uv) + O(|u|^{p/2}|v|^{p/2} + |v|^p)
\]

\[
:= |u|^p + p|u|^{p-2}\text{Re}(uv) + h(u, v).
\]

Note that we can write \( h(u, v) \) in a uniform form for all \( p > 2 \),

\[
h(u, v) = |u|^{p-r}|v|^r + |v|^p \quad \text{for some } 1 < r < p.
\]

By using (3.4), (3.5), the Hardy-Littlewood-Sobolev inequality, the Young inequality, the Hölder inequality, the Sobolev imbedding theorem, and the equality

\[
|u + v|^p = (|u + v|^2)^{p/2} = (|u|^2 + 2\text{Re}(uv) + |v|^2)^{p/2},
\]
for any given $u \in H^1(\mathbb{R}^N)$, we obtain that
\[
|G(u + v) - G(u) - \text{Re}(g(u), v)_{H^{-1}, H^1}| \\
= \left| \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u + v|^p)|u + v|^p dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \right| \\
- \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}\text{Re}(u\bar{v})dx \\
= \left| \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * (|u|^p + p|u|^{p-2}\text{Re}(u\bar{v}) + h(u, v))) \\
\times (|u|^p + p|u|^{p-2}\text{Re}(u\bar{v}) + h(u, v)) dx \right| \\
- \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}\text{Re}(u\bar{v})dx \\
\leq \int_{\mathbb{R}^N} (I_\alpha * h(u, v))(|u|^p + p|u|^{p-2}\text{Re}(u\bar{v}) + h(u, v)) dx \\
+ \int_{\mathbb{R}^N} (I_\alpha * (p|u|^{p-2}\text{Re}(u\bar{v})))(|u|^p + p|u|^{p-2}\text{Re}(u\bar{v}) + h(u, v)) dx \\
\leq \int_{\mathbb{R}^N} (I_\alpha * (|v|^p + |u|^{p-1}|v|))(|u|^p + |v|^p) dx \\
+ \int_{\mathbb{R}^N} (I_\alpha * (|u|^{p-1}|v|))(|u|^p + |v|^p) dx \\
\lesssim \|v\|^2_{H^1} + \|v\|_{H^1}^2,
\]
which implies that (G1) holds.

We verify (G2). By using the following inequalities
\[
|u|^p - |v|^p \lesssim (|u| + |v|)^{p-1}|u - v|,
\]
the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and the Sobolev imbedding theorem, we have, for any $\varphi \in H^1(\mathbb{R}^N)$,
\[
|\langle g(u) - g(v), \varphi \rangle_{H^{-1}, H^1}| \\
= \left| \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}u\bar{\varphi} dx - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^{p-2}v\bar{\varphi} dx \right| \\
\leq \left| \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(|u|^{p-2}u - |v|^{p-2}v)\bar{\varphi} dx \right| \\
+ \left| \int_{\mathbb{R}^N} (I_\alpha * (|u|^p - |v|^p))|v|^{p-2}v\bar{\varphi} dx \right| \\
\lesssim \|u\|^p_{H^1}||u|^{p-2} + ||v|^{p-2}||u - v||_{H^1}||\varphi||_{H^1} \\
+ \|v\|^{p-1}_{H^1}||u||^{p-1}_{H^1} + ||v||^{p-1}_{H^1}||u - v||_{H^1}||\varphi||_{H^1},
\]
which implies that
\[
||g(u) - g(v)||_{H^{-1}} \lesssim C(M)||u - v||_{H^1}.
\]
for \( \|u\|_{H^1}, \|v\|_{H^1} \leq M \). Thus, (G2) holds.

We verify (G3). Firstly, we define \( k(x, y) := \frac{A}{|x-y|^{N-\alpha}} \) and for any \( R > 0 \), define

\[
k_R(x, y) := \begin{cases} 
  k(x, y), & k(x, y) \leq R, \\
  R, & k(x, y) > R
\end{cases}
\]

and

\[
l_R(x, y) := k(x, y) - k_R(x, y) = \begin{cases} 
  k(x, y) - R, & |x-y| < \left(\frac{A}{R}\right)^{1/(N-\alpha)}, \\
  0, & |x-y| \geq \left(\frac{A}{R}\right)^{1/(N-\alpha)}.
\end{cases}
\]

It is easy to see that the following facts hold:

(i) \( k_R(x, y) = k_R(y, x) \), \( l_R(x, y) = l_R(y, x) \);

(ii) \( k_R(x, y) \in L_y^\infty L_x^{\alpha_1} \);

(iii) For any \( y \in \mathbb{R}^N \) and \( 1 \leq \alpha_1 < N/(N-\alpha) \),

\[
\int_{\mathbb{R}^N} |l_R(x, y)|^{\alpha_1} dx = \int_{|x-y| < \left(\frac{A}{R}\right)^{1/(N-\alpha)}} |l_R(x, y)|^{\alpha_1} dx \\
\leq \int_{|x-y| < \left(\frac{A}{R}\right)^{1/(N-\alpha)}} \frac{1}{|x-y|^{(N-\alpha)\alpha_1}} dx \\
\leq \int_0 \left(\frac{A}{R}\right)^{N/(N-\alpha)} r^{-(N-\alpha)\alpha_1} r^{N-1} dr \\
\leq \left(\frac{A}{R}\right)^{N/(N-\alpha)}
\]

which implies that \( l_R(x, y) \in L_y^\infty L_x^{\alpha_1} \) and \( l_R(x, y) \to 0 \) in \( L_y^\infty L_x^{\alpha_1} \) as \( R \to \infty \) for any \( 1 \leq \alpha_1 < N/(N-\alpha) \).

For any given \( M > 0 \), and any \( u, v \in H^1(\mathbb{R}^N) \) with \( \|u\|_{H^1}, \|v\|_{H^1} \leq M \), we calculate

\[
|G(u) - G(v)| = \left| \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p dx \right| \\
= \left| \int_{\mathbb{R}^N} (I_\alpha * (|u|^p + |v|^p))(|u|^p - |v|^p) dx \right| \\
= \int_{\mathbb{R}^N} k_R(x, y)(|u(y)|^p + |v(y)|^p)(|u(x)|^p - |v(x)|^p) dx dy \\
+ \int_{\mathbb{R}^N} l_R(x, y)(|u(y)|^p + |v(y)|^p)(|u(x)|^p - |v(x)|^p) dx dy \\
:= I + II.
\]

In the following, we calculate \( I \) and \( II \). Since \( p < \frac{N+\alpha}{N-2} \), we can choose \( \alpha_1 \in [1, N/(N-\alpha)) \) such that \( 1/\alpha_1 \leq 2(1-p/2^*) \). For such choice of \( \alpha_1 \), we define

\[
\frac{1}{\gamma_1} := 1 - \frac{1}{2} \left( \frac{1}{\alpha_1} + \frac{1}{\infty} \right),
\]
then $2 \leq p\gamma_1 \leq 2^*$. For any $\delta > 0$, we use Lemma 3.3 and the Sobolev imbedding theorem to have
\begin{align}
II & \lesssim \|\mathcal{L}_G \|_{L^p_w L^\infty_v} \|u|^p + |v|^p\|_{L^\gamma_1} \|u|^p - |v|^p\|_{L^\gamma_3} \\
& \lesssim \|\mathcal{L}_G \|_{L^p_w L^\infty_v} (\|u\|_{L^{2p,\gamma_3}}^2 + \|v\|_{L^{2p,\gamma_3}}^2) \\
& \lesssim \|\mathcal{L}_G \|_{L^p_w L^\infty_v} (\|u\|_{H^1}^2 + \|v\|_{H^1}^2) \\
& \lesssim \delta
\end{align}
(3.8)
by choosing $R$ large enough. For such choice of $R$, by using Lemma 3.3, the Hölder inequality, the Sobolev imbedding theorem, (3.6) and the Young inequality
\begin{align}
\epsilon > 0 \text{ is an arbitrary real number, we obtain that} \\
& I \leq \|kR\|_{L^\infty_w L^\infty_v} \|u|^p + |v|^p\|_{L^1} \|u|^p - |v|^p\|_{L^1} \\
& \lesssim R\|(u)|_{L^p}^p + \|v|^p\|_{L^p} (\|u|_{L^{p-1}}^{p-1} + \|v|_{L^{p-1}}^{p-1}) \|u - v\|_{L^p} \\
& \lesssim R\|M^{2p-1} \|u - v\|_{L^2} + \|u - v\|_{L^2}^\theta \|u - v\|_{L^2}^\theta \\
& \lesssim \epsilon \|u - v\|_{L^2} + \epsilon^{-\theta/(1-\theta)} C(M) \|u - v\|_{L^2} \\
& \leq \delta + C\epsilon(M) \|u - v\|_{L^2},
\end{align}
(3.9)
by choosing $\epsilon > 0$ small enough, where $\theta$ is defined by
\[
\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2^*}.
\]
Inserting (3.8) and (3.10) into (3.7), we complete the proof of (G3).

We verify (G5). Let $I \subset \mathbb{R}$ be a bounded open interval, $\{w_n\}_{n=1}^\infty$ be any bounded sequence in $L^\infty(I; H^1(\mathbb{R}^N))$ satisfying
\begin{equation}
\begin{cases}
\phi_n(t) \to w(t) \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty \text{ for almost all } t \in I, \\
g(w_n(t)) \to f(t) \text{ weakly } \ast \text{ in } L^\infty(I; H^{-1}(\mathbb{R}^N)) \text{ as } n \to \infty.
\end{cases}
\end{equation}
(3.11)
We will show that $f(t) = g(w(t))$, and thus (G5) follows in view of (G4).

It follows from (3.11) that $\{g(w_n)\}$ is bounded in $L^\infty(I; H^{-1}(\mathbb{R}^N))$. Hence, there exists $C > 0$ such that
\begin{equation}
\|g(w_n(t))\|_{H^{-1}} \leq C \text{ for almost all } t \in I \text{ and any } n \in \mathbb{N}.
\end{equation}
(3.12)
Consequently, for almost all $t \in I$, there exists a subsequence $g(w_{n_j}(t))$ and $\tilde{f}(t) \in H^{-1}(\mathbb{R}^N)$ such that
\begin{equation}
g(w_{n_j}(t)) \to \tilde{f}(t) \text{ weakly in } H^{-1}(\mathbb{R}^N) \text{ as } j \to \infty.
\end{equation}
(3.13)
By (3.11), (3.13), Lemma 2.3 and the uniqueness of limit, we have $\tilde{f}(t) = g(w(t))$. Hence,
\begin{equation}
g(w_n(t)) \to g(w(t)) \text{ weakly in } H^{-1}(\mathbb{R}^N) \text{ for almost all } t \in I.
\end{equation}
(3.14)
For any $\phi(t) \in L^1(I, H^1(\mathbb{R}^N))$, we have $\phi(t) \in H^1(\mathbb{R}^N)$ for almost all $t \in I$. Hence, by (3.14),
\begin{equation}
\langle g(w_n(t)), \phi(t) \rangle_{H^{-1},H^1} \to \langle g(w(t)), \phi(t) \rangle_{H^{-1},H^1} \text{ for almost all } t \in I.
\end{equation}
By (3.12), we have
\[ \left| \langle g(w_{n}(t)), \varphi(t) \rangle_{H^{-1}, H^{1}} \right| \leq \| g(w_{n}(t)) \|_{H^{-1}} \| \varphi(t) \|_{H^{1}} \leq C \| \varphi(t) \|_{H^{1}} \in L^{1}(\mathbb{R}^{N}). \]

Thus, the Lebesgue’s dominated convergence theorem implies that
\[ \int_{I} \langle g(w_{n}(t)), \varphi(t) \rangle_{H^{-1}, H^{1}} dt \to \int_{I} \langle g(w(t)), \varphi(t) \rangle_{H^{-1}, H^{1}} dt \]
as \( n \to \infty \). That is \( g(w_{n}(t)) \to g(w(t)) \) weakly * in \( L^{\infty}(I, H^{-1}(\mathbb{R}^{N})) \). Hence, \( f(t) = g(w(t)) \). The proof of (G5) is complete. \( \square \)

**Lemma 3.5.** Let \( N \geq 3, \alpha \in ((N - 4), N), 2 \leq p < (N + \alpha)/(N - 2), \)
a = \pm 1, \( g \) and \( G \) be as in Lemma 3.4. Assume that \( u_{j} \in L^{\infty}((-T, T), H^{1}(\mathbb{R}^{N})) \cap \)
\( W^{1,\infty}((-T, T), H^{-1}(\mathbb{R}^{N})) (j = 1, 2) \) are two local weak solutions to \((CH_{b})\) on \((-T, T)\) with initial values \( u_{j}(0) = u_{j,0} \). Then for any \((p_{2}, q_{2}) \in S,\)
\[ \| u_{1}(t) - u_{2}(t) \|_{L^{p_{2}}((-T, T), L^{q_{2}})} \leq C \| u_{1,0} - u_{2,0} \|_{L^{2}}, \]
where \( C \) is a constant depending on \( \| u_{1} \|_{L^{\infty}((-T, T), H^{1})} \) and \( \| u_{2} \|_{L^{\infty}((-T, T), H^{1})}. \)

**Proof.** It follows from the integral expressions of \( u_{j} (j = 1, 2) \)
\[ u_{j}(t) = e^{-it\mathcal{L}_{b}}u_{j,0} - i \int_{0}^{t} e^{-i(t-s)\mathcal{L}_{b}}g(u_{j}(s))ds \]
that \( v(t) := u_{1}(t) - u_{2}(t) \) satisfies
\[ v(t) = e^{-it\mathcal{L}_{b}}(u_{1,0} - u_{2,0}) - i \int_{0}^{t} e^{-i(t-s)\mathcal{L}_{b}}(g(u_{1}(s)) - g(u_{2}(s)))ds. \]

Let \( \gamma = \frac{2Np}{N + \alpha} \). By the Hardy-Littlewood-Sobolev inequality, for any \( u, v, \varphi \in H^{1}(\mathbb{R}^{N}), \)
\[ \langle g(u) - g(v), \varphi \rangle \leq \left( \| u \|_{L^{\gamma}}^{2Np/(N + \alpha)} \vee \| v \|_{L^{\gamma}}^{2Np/(N + \alpha)} \right)^{2p-2} \| u - v \|_{L^{\gamma}}^{2Np/(N + \alpha)} \| \varphi \|_{L^{\gamma}}^{2Np/(N + \alpha)}, \]
which implies that
\[ \| g(u) - g(v) \|_{L^{\gamma'}} \leq \left( \| u \|_{L^{\gamma}}^{2Np/(N + \alpha)} \vee \| v \|_{L^{\gamma}}^{2Np/(N + \alpha)} \right)^{2p-2} \| u - v \|_{L^{\gamma}}^{2Np/(N + \alpha)}. \]

Set
\[ (p_{1}, q_{1}) = \left( \frac{4p}{Np - (N + \alpha)}, \frac{2Np}{N + \alpha} \right) \in S. \]
For any \((p_2, q_2) \in S\) and \(T_0 \in (0, T]\), it follows from Proposition 2.6, the Hölder inequality and (3.16) that
\[
\left\| \int_0^t e^{-i(t-s)L_0} (g(u_1(s)) - g(u_2(s))) ds \right\|_{L^{p_2}((-T_0, T_0), L^{q_2})} \\
\lesssim \left\| g(u_1(t)) - g(u_2(t)) \right\|_{L^{p_1}((-T_0, T_0), L^{q_1})} \\
\lesssim \left( \int_{-T_0}^{T_0} \left( \| u_1 \|_{L^{q_1}} \vee \| u_2 \|_{L^{q_1}} \right)^{(2p-2)p_1' \left\| v \right\|_{L^{p_1} L^{q_1}} \| u \|_{L^{p_1} L^{q_1}}} \right)^{1/p_1'} \\
\lesssim \left( \int_{-T_0}^{T_0} \left( \| u_1 \|_{L^{q_1}} \vee \| u_2 \|_{L^{q_1}} \right)^{4p(p-1)/2} \| u \|_{L^{p_1} L^{q_1}} \| u \|_{L^{p_1} L^{q_1}} \right) \\
\lesssim \left( \| u_1 \|_{L^{\infty}((-T_0, T_0), L^{q_1})} + \| u_2 \|_{L^{\infty}((-T_0, T_0), L^{q_1})} \right)^{2(p-1)} \\
\times T_0^{(2-N)p_1p_2(N+a_1)/2p} \left\| v \right\|_{L^{p_1}((-T_0, T_0), L^{q_1})}.
\] (3.17)

By (3.15), (3.17) and Proposition 2.6, we obtain that
\[
\| v \|_{L^{p_2}((-T_0, T_0), L^{q_2})} \leq C \| u_{1,0} - u_{2,0} \|_{L^2} \\
+ CM^{2(p-1)}T_0^{(2-N)p_1p_2(N+a_1)/2p} \left\| v \right\|_{L^{p_1}((-T_0, T_0), L^{q_1})},
\] (3.18)
where \( M = \| u_1 \|_{L^\infty((-T,T), H^1)} \vee \| u_2 \|_{L^\infty((-T,T), H^1)}. \)

Since \( p < \frac{N+1}{N+a_1} \), by choosing \((p_2, q_2) = (p_1, q_1)\) and \( T_0 \in (0, T] \) such that
\[
CM^{2(p-1)}T_0^{(2-N)p_1p_2(N+a_1)/2p} \leq 1/2,
\]
we obtain from (3.18) that
\[
\| v \|_{L^{p_1}((-T_0, T_0), L^{q_1})} \leq 2C \| u_{1,0} - u_{2,0} \|_{L^2},
\]
and from (3.18) again that
\[
\| v \|_{L^{p_2}((-T_0, T_0), L^{q_2})} \leq 2C \| u_{1,0} - u_{2,0} \|_{L^2}
\]
for any \((p_2, q_2) \in S\).

Extending the interval for finite steps, similarly to the above arguments, we obtain that
\[
\| v \|_{L^{p_1}((-T,T), L^{q_1})} \leq C \| u_{1,0} - u_{2,0} \|_{L^2}
\]
and
\[
\| v \|_{L^{p_2}((-T,T), L^{q_2})} \leq C \| u_{1,0} - u_{2,0} \|_{L^2}
\]
for any \((p_2, q_2) \in S\). The proof is complete.

\textbf{Proof of Theorem 1.2.} It is a direct result of Theorem 3.1, Lemmas 3.4 and 3.5.
4. Sharp Gagliardo-Nirenberg inequality and global existence

In this section, we derive the sharp Gagliardo-Nirenberg inequality and some useful identities with respect to problem (CH). Based of which and the local well-posedness result established in Section 3, we prove the global existence results in various cases.

4.1. Sharp Gagliardo-Nirenberg inequality. We consider the sharp Gagliardo-Nirenberg inequality

\[ \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u|^p \, dx \leq C_{GN}(b) \|u\|_{L^2}^{N+\alpha-Np+2p} \|u\|_{H^b}^{Np-N-\alpha}, \tag{4.1} \]

where the sharp constant \(C_{GN}(b)\) is defined by

\[ C_{GN}^{-1}(b) = \inf \{ J_b(u) : u \in H^b_0(\mathbb{R}^N) \setminus \{0\} \} \tag{4.2} \]

and

\[ J_b(u) = \frac{\|u\|_{L^2}^{N+\alpha-Np+2p} \|u\|_{H^b}^{Np-N-\alpha}}{\int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u|^p \, dx}. \tag{4.3} \]

We also consider the sharp radial Gagliardo-Nirenberg inequality

\[ \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u|^p \, dx \leq C_{GN}(b, \text{rad}) \|u\|_{L^2}^{N+\alpha-Np+2p} \|u\|_{H^b}^{Np-N-\alpha}, \tag{4.4} \]

where the sharp constant \(C_{GN}(b, \text{rad})\) is defined by

\[ C_{GN}^{-1}(b, \text{rad}) = \inf \{ J_b(u) : u \in H^b_0(\mathbb{R}^N) \setminus \{0\}, \ u \text{ is radial} \}. \tag{4.5} \]

By combining the proofs of Theorem 4.1 in [8] and Theorem 2.3 in [9], we obtain the following result. For the completeness, we give the proof here.

**Lemma 4.1.** Let \(N \geq 3, \alpha \in (0, N), (N + \alpha)/N < p < (N + \alpha)/(N - 2), b > -(N - 2)^2/4\). Then \(C_{GN}(b) \in (0, \infty)\) and

(1) If \(-(N - 2)^2/4 - b \leq 0\), then the equality in (4.1) is attained by a nontrivial radial ground state \(Q_b \in H^b_0(\mathbb{R}^N)\) to equation (1.4). Moreover,

\[ C_{GN}(b) = \frac{2p}{N + \alpha - (N - 2)p} \left( \frac{N + \alpha - (N - 2)p}{Np - N - \alpha} \right)^{\frac{Np - N - \alpha}{2}} \|Q_b\|_{L^2}^{2-2p}, \tag{4.6} \]

\[ \int_{\mathbb{R}^N} (I_\alpha \ast |Q_b|^p) |Q_b|^p \, dx = \frac{2p}{N + \alpha - (N - 2)p} \|Q_b\|_{L^2}^2, \tag{4.7} \]

and

\[ \|Q_b\|_{H^b}^2 = \frac{Np - N - \alpha}{N + \alpha - (N - 2)p} \|Q_b\|_{L^2}^2. \tag{4.8} \]

(2) If \(b > 0\), then \(C_{GN}(b) = C_{GN}(0)\) and the equality in (4.1) is never attained. However, \(C_{GN}(b, \text{rad})\) is attained by a radial solution \(Q_{b, \text{rad}}\) to equation (1.4). Moreover, the same identities as in (4.6)-(4.8) hold true with \(Q_{b, \text{rad}}, C_{GN}(b, \text{rad})\) in place of \(Q_b, C_{GN}(b)\) respectively.

**Proof.** The case \(b = 0\) is already considered in [9], so we assume that \(b \neq 0\). By using the Hardy-Littlewood-Sobolev inequality, the interpolation inequality, the Sobolev
imbedding theorem and the equivalence of \( \| \cdot \|_{H^1_b} \) and \( \| \cdot \|_{H^1} \), for any \( u \in H^1_b(\mathbb{R}^N) \), we obtain that
\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p dx \leq C \| u \|_{H^1_b}^{2p} \leq C \| u \|_{H^1}^{Np-N} \| u \|_{L^2}^{Np-N+2p},
\]
which implies that \( 0 < C_{GN}^{-1}(b) < \infty \).

**Case 1** \((-\frac{(N-2)^2}{4} < b < 0)\). Let \( \{u_n\}_{n=1}^\infty \subset H^1_b(\mathbb{R}^N) \setminus \{0\} \) be a minimizing sequence of \( J_b \), that is, \( \lim_{n \to \infty} J_b(u_n) = C_{GN}^{-1}(b) \). Denote by \( u_n^* \) the Schwartz symmetrization of \( u_n \). From Chapter 3 in [16], we have
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p dx \leq \int_{\mathbb{R}^N} (I_\alpha * |u_n^*|^p)|u_n^*|^p dx,
\]
\[
\int_{\mathbb{R}^N} |u_n|^2 dx = \int_{\mathbb{R}^N} |u_n^*|^2 dx,
\]
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u_n^*|^2 dx,
\]
and
\[
\int_{\mathbb{R}^N} |x|^{-2}|u_n|^2 dx \leq \int_{\mathbb{R}^N} |x|^{-2}|u_n^*|^2 dx.
\]
The above inequalities and \( b < 0 \) imply that \( J_b(u_n^*) \leq J_b(u_n) \). Hence, we may assume that \( \{u_n\}_{n=1}^\infty \) is radial.

Direct calculation shows that \( J_b \) is invariant under the scaling
\( u_{\lambda, \mu}(x) = \lambda u(\mu x), \lambda, \mu > 0 \),
that is, \( J_b(u_{\lambda, \mu}) = J_b(u) \). For the sake, we set \( v_n = \lambda_n u_n (\mu_n x) \) with
\[
\lambda_n = \frac{\| u_n \|_{L^2}^{N/2-1}}{\| u_n \|_{H^1_b}^{N/2-1}} \quad \text{and} \quad \mu_n = \frac{\| u_n \|_{L^2}}{\| u_n \|_{H^1_b}}.
\]
Then \( \| v_n \|_{L^2} = \| v_n \|_{H^1_b} = 1 \). That is, \( \{v_n\}_{n=1}^\infty \subset H^1_b(\mathbb{R}^N) \setminus \{0\} \) is a bounded radial symmetric minimizing sequence of \( J_b \). Hence, there exists \( v \in H^1_b(\mathbb{R}^N) \) such that
\( v_n \to v \) weakly in \( H^1_b(\mathbb{R}^N) \),
\( v_n \to v \) strongly in \( L^r(\mathbb{R}^N) \) with \( r \in (2, 2^*) \),
which combine with the Hardy-Littlewood-Sobolev inequality and the lower semi-continuity of the norm imply that
\[
\| v \|_{L^2} \leq \liminf_{n \to \infty} \| v_n \|_{L^2}, \quad \| v \|_{H^1_b} \leq \liminf_{n \to \infty} \| v_n \|_{H^1_b},
\]
\[
\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p dx.
\]
Hence,
\[
J_b(v) \leq \liminf_{n \to \infty} J_b(v_n) = C_{GN}^{-1}(b).
\]
By the definition of \( C_{GN}^{-1}(b) \), we obtain that
\[
J_b(v) = \lim_{n \to \infty} J_b(v_n) = \inf_{u \in H^1_b(\mathbb{R}^N) \setminus \{0\}} J_b(u) = C_{GN}^{-1}(b).
\]
and \(\|v\|_{L^2} = \|v\|_{H^1} = 1\). In particular, \(v\) satisfies
\[
\frac{d}{d\epsilon} J_b(v + \epsilon \varphi) |_{\epsilon=0} = 0 \text{ for any } \varphi \in H^1_b(\mathbb{R}^N).
\]

Consequently, \(v\) satisfies the elliptic equation
\[
\frac{CG_N(b)(NP - N - \alpha)}{2p} L_b v + \frac{CG_N(b)(N + \alpha - NP + 2p)}{2p} v = (I_\alpha * |v|^p)|v|^{p-2} v.
\]

Set \(v(x) = \lambda Q_b(\mu x)\) with
\[
\lambda = \left(\frac{(N + \alpha - NP + 2p)^{\alpha/2 + 1} CG_N(b)}{2p(NP - N - \alpha)^{\alpha/2}}\right)^{1/(2p-2)}
\]
and
\[
\mu = \left(\frac{N + \alpha - NP + 2p}{NP - N - \alpha}\right)^{1/2},
\]
then \(Q_b(x)\) satisfies (1.4).

Multiplying (1.4) by \(Q_b\) and \(x \cdot \nabla Q_b\) and integrating on \(\mathbb{R}^N\), respectively, we obtain that
\[
\|Q_b\|_{H^1_b}^2 + \|Q_b\|_{L^2}^2 = \int_{\mathbb{R}^N} (I_\alpha * |Q_b|^p)|Q_b|^p dx
\]
and
\[
(N - 2)\|Q_b\|_{H^1_b}^2 + N\|Q_b\|_{L^2}^2 = \frac{N + \alpha}{p} \int_{\mathbb{R}^N} (I_\alpha * |Q_b|^p)|Q_b|^p dx,
\]
which implies (4.7) and (4.8). Since \(C_G(b) = J_b(Q_b)\), we obtain (4.6). The functional of (1.4) is defined by
\[
\dot{E}(Q) = \frac{1}{2}\|Q\|_{H^1_b}^2 + \frac{1}{2}\|Q\|_{L^2}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p dx
\]
\[
= \frac{p - 1}{N + \alpha - (N - 2)p} \|Q\|_{L^2}^2,
\]
which implies that \(Q_b\) is a ground state of (1.4).

Case 2 \((b > 0)\). Choose a sequence \(\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N\) with \(|x_n| \to \infty\). Let \(Q_0\) be a positive radial ground state to
\[
L_0 Q + Q = (I_\alpha * |Q|^p)|Q|^{p-2} Q.
\]
Then by Lemma 2.8,
\[
J_b(Q_0(\cdot - x_n)) \to J_0(Q_0) = C^{\alpha}_{GN}(0),
\]
which implies that \(C^{\alpha}_{GN}(b) \leq C^{\alpha}_{GN}(0)\).

On the other hand, since \(b > 0\), \(\|u\|_{H^1} < \|u\|_{H^1_b}\) for any \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\). The sharp Gagliardo-Nirenberg inequality for \(b = 0\) implies that
\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq C_{GN}(0)\|u\|_{L^2}^{N+\alpha-Np+2p}\|u\|_{H^1_b}^{Np-N-\alpha}
\]
\[
< C_{GN}(0)\|u\|_{L^2}^{N+\alpha-Np+2p}\|u\|_{H^1_b}^{Np-N-\alpha}.
\]
Hence, \(J_b(u) > C^{\alpha}_{GN}(0)\) for any \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\). Since \(H^1(\mathbb{R}^N)\) is equivalent to \(H^1_b(\mathbb{R}^N)\), we obtain that \(C^{\alpha}_{GN}(b) \geq C^{\alpha}_{GN}(0)\). Thus, \(C_{GN}(b) = C_{GN}(0)\). The inequality (4.9) also implies that the equality in (4.1) is never attained.

If we only consider radial functions, the result follows exactly as case 1. The proof is complete.
Remark 4.2. (1). When \(-\frac{(N-2)^2}{4} < b \leq 0\), from Lemma 4.1 and the definitions of \(H(b)\) and \(K(b)\), we obtain that

\[
K(b) = \left( \frac{2p}{(Np - N - \alpha)C_{GN}(b)} \right)^{1/(Np - N - \alpha - 2)}
\]

and

\[
H(b) = \frac{Np - N - \alpha - 2}{2(Np - N - \alpha)} \left( \frac{2p}{(Np - N - \alpha)C_{GN}(b)} \right)^{2/(Np - N - \alpha - 2)}.
\]

(2). Similarly, when \(b > 0\), the same identities as in (4.10) and (4.11) hold true with \(K(b, \text{rad})\), \(H(b, \text{rad})\), \(C_{GN}(b, \text{rad})\) in place of \(K(b)\), \(H(b)\), \(C_{GN}(b)\) respectively.

4.2. Global existence.

Proof of the global existence parts in Theorems 1.3, 1.4 and 1.6. Let 
\(u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))\) be the weak solution obtained in Theorem 1.2 with initial value \(u_0(x) \in H^1(\mathbb{R}^N)\). In view of the conservation laws, we just need to bound \(\|u(t)\|_{\tilde{H}^1_b}\) for any \(t \in I\). It is trivial for the defocusing case \(a = -1\) from the conservation laws. In the following, we consider the focusing case \(a = 1\).

Case 1 \((2 \leq p < 1 + (2 + \alpha)/N)\). By using the sharp Gagliardo-Nirenberg inequality and the conservation laws, we obtain that

\[
\|u(t)\|_{\tilde{H}^1_b}^2 = \int_{\mathbb{R}^N} (|\nabla u(t)|^2 + b|x|^{-2}|u(t)|^2)dx
\]

\[
= 2E_b(u(t)) + \frac{1}{p} \int_{\mathbb{R}^N} (I_{a} * |u|^p)|u|^p dx
\]

\[
\leq 2E_b(u(t)) + \frac{1}{p} C_{GN}(b)\|u(t)\|_{L^2}^{N+\alpha-Np+2p}\|u(t)\|_{\tilde{H}^1_b}^{Np-N-\alpha}
\]

\[
= 2E_b(u_0) + \frac{1}{p} C_{GN}(b)\|u_0\|_{L^2}^{N+\alpha-Np+2p}\|u(t)\|_{\tilde{H}^1_b}^{Np-N-\alpha}.
\]

Since \(p < 1 + (2 + \alpha)/N\), we have \(Np - N - \alpha < 2\) and thus, by using the Young inequality, \(\|u(t)\|_{\tilde{H}^1_b}\) is bounded for any \(t \in I\). This proves the global existence result in Theorem 1.3.

Case 2 \((p = 1 + (2 + \alpha)/N)\). In this case, we have

\[
N + \alpha - Np + 2p = 2(2 + \alpha)/N, \quad Np - N - \alpha = 2,
\]

\[
2 - 2p = -2(2 + \alpha)/N, \quad C_{GN}(b) = p\|Q_b\|_{L^2}^{-2p}.
\]

Similarly to case 1, we obtain that

\[
\|u(t)\|_{\tilde{H}^1_b}^2 = 2E_b(u(t)) + \frac{1}{p} \int_{\mathbb{R}^N} (I_{a} * |u|^p)|u|^p dx
\]

\[
\leq 2E_b(u(t)) + \frac{1}{p} C_{GN}(b)\|u(t)\|_{L^2}^{N+\alpha-Np+2p}\|u(t)\|_{\tilde{H}^1_b}^{Np-N-\alpha}
\]

\[
= 2E_b(u_0) + \left( \frac{\|u_0\|_{L^2}}{\|Q_{b^0}\|_{L^2}} \right)^{2(2+\alpha)/N}\|u(t)\|_{\tilde{H}^1_b}^2.
\]

Since \(\|u_0\|_{L^2} < \|Q_{b^0}\|_{L^2}\), we obtain that \(\|u(t)\|_{\tilde{H}^1_b}\) is bounded for any \(t \in I\). This proves the global existence result (i) in Theorem 1.4.
Case 3 \((1 + (2 + \alpha))/N < p < (N + \alpha)/(N - 2)\). Multiplying both sides of \(E_b(u(t))\) by \(\|u(t)\|_{L^2}^{2\sigma}\) with \(\sigma = \frac{Np - N - \alpha + 2}{\sqrt{2Np - N - \alpha}}\) and by using the sharp Gagliardo-Nirenberg inequality, we obtain that

\[
E_b(u(t))\|u(t)\|_{L^2}^{2\sigma} \geq \frac{1}{2}(\|u\|_{H^1_b}^2 \|u(t)\|_{L^2}^{2\sigma} - \frac{1}{2p} C_{GN}(b) \|u(t)\|_{L^2}^{N + \alpha - Np + 2p + 2\sigma} \|u\|_{H^1_b}^{Np - N - \alpha})
\]

\[
= \frac{1}{2}(\|u\|_{H^1_b}^2 \|u(t)\|_{L^2}^{2\sigma} - \frac{C_{GN}(b)}{2p} (\|u\|_{H^1_b} \|u(t)\|_{L^2}^{2\sigma})^{Np - N - \alpha}) = f(\|u\|_{H^1_b}^2 \|u(t)\|_{L^2}^{2\sigma}),
\]

where

\[
f(s) := \frac{1}{2} s^2 - \frac{C_{GN}(b)}{2p} s^{Np - N - \alpha}, \quad s \in [0, \infty).
\]

Direct calculation shows that \(f\) has a unique critical point \(s^*\) which corresponds to its maximum point, where

\[
s^* = \left(\frac{2p}{(Np - N - \alpha)C_{GN}(b)}\right)^{1/(Np - N - \alpha - 2)}
\]

and

\[
f(s^*) = \frac{Np - N - \alpha - 2}{2(Np - N - \alpha)} \left(\frac{2p}{(Np - N - \alpha)C_{GN}(b)}\right)^{2/(Np - N - \alpha - 2)}.
\]

From Remark 4.2, we know \(K(b) = s^*\) and \(H(b) = f(s^*) = f(K(b))\).

By (4.12), the conservation laws, and the assumptions in Theorem 1.6, we know

\[
f(\|u(t)\|_{H^1_b} \|u(t)\|_{L^2}^{2\sigma}) \leq E_b(u(t))\|u(t)\|_{L^2}^{2\sigma}
\]

\[
= E_b(u_0)\|u_0\|_{L^2}^{2\sigma} < H(b)
\]

for any \(t \in I\). Since \(\|u_0\|_{H^1_b} \|u_0\|_{L^2}^{2\sigma} < K(b)\), by the continuity argument, we have \(\|u(t)\|_{H^1_b} \|u(t)\|_{L^2}^{2\sigma} < K(b)\) for any \(t \in I\). This proves the global existence result (i) in Theorem 1.6.

5. Virial identities and blowup

In this section, we first establish the virial identities for the solution to \((\text{CH}_b)\), and then, based on which, we prove the blowup results in the main theorems. So throughout this section, we assume \(a = 1\) in \((\text{CH}_b)\).

5.1. Virial identities.

For any \(\delta > 0\), we consider the following approximation problem of \((\text{CH}_b)\):

\[
(\text{CH}_b^\delta) \quad \begin{cases}
    i\partial_t u + \Delta u = \frac{b}{|x|^{n+2}} u - (I_\alpha * |u|^p)|u|^{p-2} u := -g_\delta(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
    u(0, x) = u_0(x), \quad x \in \mathbb{R}^N.
\end{cases}
\]
When $\delta = 0$, we refer $(\text{CH}_b^\delta)$ to $(\text{CH}_b)$. It follows from (1.1) that for any $b > -\frac{(N-2)^2}{4}$, $\delta > 0$ and $\varphi \in H^1(\mathbb{R}^N)$, the inequality

$$
\left(1 - \frac{4b_+}{(N-2)^2}\right) \|\nabla \varphi\|_{L^2}^2 \leq \|\nabla \varphi\|_{L^2}^2 + \int_{\mathbb{R}^N} \frac{b|\varphi|^2}{|x|^2 + \delta} \, dx 
\leq \left(1 + \frac{4b_+}{(N-2)^2}\right) \|\nabla \varphi\|_{L^2}^2
$$

(5.1)

holds. Moreover, by the Lebesgue dominated convergence theorem,

$$
(-\Delta + \frac{b}{|x|^2 + \delta}) \varphi \rightarrow (-\Delta + \frac{b}{|x|^2}) \varphi \text{ strongly in } H^{-1}(\mathbb{R}^N) \text{ as } \delta \rightarrow 0.
$$

(5.2)

Hence, $(\text{CH}_b^\delta)$ is a good approximation of $(\text{CH}_b)$.

For the Cauchy problem $(\text{CH}_b^\delta)$, we have the following facts:

**Proposition 5.1.** Let $N \geq 3$, $\alpha \in ((N-4)+, N)$, $p \in [2, \frac{N+\alpha}{N-2})$, $b > -\frac{(N-2)^2}{4}$. Then for any $\delta > 0$ and $u_0 \in H^1(\mathbb{R}^N)$, there exists a unique maximal-lifespan solution $u \in C^1(I, H^1(\mathbb{R}^N)) \cap C(I, H^{-1}(\mathbb{R}^N))$ to $(\text{CH}_b^\delta)$, and the conservation laws hold:

$$
\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_{b,\delta}(u(t)) = E_{b,\delta}(u_0), \quad \text{for any } t \in I,
$$

where

$$
E_{b,\delta}(u(t)) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u(t, x)|^2 + \frac{b}{|x|^2 + \delta}|u(t, x)|^2) \, dx 
- \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u(t, x)|^p)|u(t, x)|^p \, dx.
$$

See Theorem 4.3.1 in [5]. Moreover, the solution depends continuously on the initial value, that is, if $u_{n_0} \rightarrow u_0$ strongly in $H^1(\mathbb{R}^N)$, then $u_n \rightarrow u$ in $C([-T_1, T_2], H^1(\mathbb{R}^N))$ as $n \rightarrow \infty$ for any $[-T_1, T_2] \subset I$. See Theorem 3.3.9 in [5]. Moreover, if $u_0 \in H^2(\mathbb{R}^N)$, then $u(t) \in C(I, H^2(\mathbb{R}^N)) \cap C^1(I, L^2(\mathbb{R}^N))$, see Theorem 4.8.1 in [5].

In the following, for any $\delta > 0$, we obtain the virial identity for the solution $u_\delta$ to $(\text{CH}_b^\delta)$, and then by letting $\delta \rightarrow 0$, we obtain the virial identity for the solution $u$ to $(\text{CH}_b)$. Let $u$ be the solution to $(\text{CH}_b^\delta)$ with $\delta \geq 0$ and $w(x) = |x|^2$ or $w(x) = \varphi_R(x)$, where $\varphi_R(x)$ is defined in (5.31). We define

$$
V_w(t) := \int_{\mathbb{R}^N} w(x)|u(t, x)|^2 \, dx.
$$

Then we have the following virial identity.

**Lemma 5.2.** Let $N \geq 3$, $\alpha \in ((N-4)+, N)$, $p \in [2, \frac{N+\alpha}{N-2})$, $b > -\frac{(N-2)^2}{4}$, $u_0 \in H^1(\mathbb{R}^N)$ with $\sqrt{w(x)}u_0 \in L^2(\mathbb{R}^N)$, $\delta > 0$. Assume $u(t) \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ is the maximal-lifespan solution to $(\text{CH}_b^\delta)$ with initial value $u_0$, then $\sqrt{w(x)}u(t, x) \in C(I, L^2(\mathbb{R}^N))$. Moreover, $V_w(t) \in C^2(I, \mathbb{R})$,

$$
\frac{d}{dt} V_w(t) = 2\text{Im} \int_{\mathbb{R}^N} \bar{u} \nabla w \cdot \nabla u \, dx
$$

(5.3)


and

$$\frac{d^2}{dt^2} V_w(t) = 4\text{Re} \int_{\mathbb{R}^N} \partial_i u \partial_j \bar{u} \partial_{ij} w dx - \int_{\mathbb{R}^N} |u|^2 \Delta^2 w dx + 4b \int_{\mathbb{R}^N} \frac{x \cdot \nabla |u|^2}{(|x|^2 + \delta)^2} |u|^2 dx$$

$$- 2A(N - \alpha) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (x - y) \cdot \left( \nabla w(x) - \nabla w(y) \right) |u(x)|^p |u(y)|^p dxdy$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^p \Delta w dx$$

for any $t \in I$.

Proof. We follow the proof of Proposition 6.5.1 in [5]. By Lemma 6.5.2 in [5] and Proposition 5.1, we obtain that $\sqrt{w(x)u(t, x)} \in C(I, L^2(\mathbb{R}^N))$, $V_w(t) \in C^1(I, \mathbb{R})$ and (5.3) holds for all $t \in I$. It remains to show that $V_w(t) \in C^2(I, \mathbb{R})$ and (5.4) holds. The proof we give below is based on two regularizations. Therefore, we proceed in two steps.

Step 1. The case $u_0 \in H^2(\mathbb{R}^N)$. By the $H^2$ regularity (Proposition 5.1), $u(t) \in C(I, H^2(\mathbb{R}^N)) \cap C^1(I, L^2(\mathbb{R}^N))$. For any $\epsilon > 0$, we define $\theta_\epsilon(x) = e^{-\epsilon \cdot w(x)}$ and

$$h_\epsilon(t) = \text{Im} \int_{\mathbb{R}^N} \theta_\epsilon(x) \bar{u} \nabla w \cdot \nabla u dx, \text{ for any } t \in I.$$

Then $h_\epsilon \in C^1(I, \mathbb{R})$ and

$$h_\epsilon'(t) = \text{Re} \int_{\mathbb{R}^N} iu_\epsilon(2\theta_\epsilon \nabla \bar{u} \cdot \nabla w + \theta_\epsilon \bar{u} \Delta w + \bar{u} \nabla \theta_\epsilon \cdot \nabla w) dx$$

(5.5)

$$= -\text{Re} \int_{\mathbb{R}^N} (\Delta u + g_\beta(u))(2\theta_\epsilon \nabla \bar{u} \cdot \nabla w + \theta_\epsilon \bar{u} \Delta w + \bar{u} \nabla \theta_\epsilon \cdot \nabla w) dx.$$ 

Here, we have used the Green’s formula and equation (CH_5).

For $u \in H^2(\mathbb{R}^N)$, by using the Green’s formula and elementary calculation, we know

$$\text{Re} \int_{\mathbb{R}^N} \Delta u \theta_\epsilon \nabla \bar{u} \cdot \nabla w dx = -\text{Re} \int_{\mathbb{R}^N} (2\nabla u \cdot \nabla \theta_\epsilon \nabla \bar{u} \cdot \nabla w + 2\theta_\epsilon \partial_j u \partial_i \bar{u} \partial_{ij} w$$

$$- |\nabla u|^2 \nabla \theta_\epsilon \cdot \nabla w - |\nabla u|^2 \theta_\epsilon \Delta w) dx,$$

$$\text{Re} \int_{\mathbb{R}^N} \Delta u \bar{u} \Delta w dx = -\text{Re} \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \theta_\epsilon \bar{u} \Delta w + \theta_\epsilon |\nabla u|^2 \Delta w$$

$$- \frac{1}{2} |\nabla \theta_\epsilon^2 \nabla (\Delta w) - \frac{1}{2} |\nabla \theta_\epsilon \Delta w|^2) dx,$$

$$\text{Re} \int_{\mathbb{R}^N} \Delta u \bar{u} \nabla \theta_\epsilon \cdot \nabla w dx$$

(5.8)

$$= -\text{Re} \int_{\mathbb{R}^N} (|\nabla u|^2 \nabla \theta_\epsilon \cdot \nabla w + \bar{u} \partial_j u \partial_i \bar{u} \theta_\epsilon \partial_{ij} w + \bar{u} \partial_j u \partial_i \theta_\epsilon \partial_{ij} w) dx,$$

$$\text{Re} \int_{\mathbb{R}^N} \frac{b}{|x|^2 + \delta} u \theta_\epsilon \nabla \bar{u} \cdot \nabla w dx$$

(5.9)

$$= \text{Re} \int_{\mathbb{R}^N} \frac{b}{|x|^2 + \delta} \theta_\epsilon \nabla (|u|^2) \cdot \nabla w dx$$

$$= \int_{\mathbb{R}^N} \left( \frac{2b \theta_\epsilon \cdot \nabla w}{(|x|^2 + \delta)^2} |u|^2 - \frac{b \nabla \theta_\epsilon \cdot \nabla w}{|x|^2 + \delta} |u|^2 - \frac{b \theta_\epsilon \Delta w}{|x|^2 + \delta} |u|^2 \right) dx.$$
and

\[
\text{Re} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}u \theta \nabla \bar{u} \cdot \nabla w \, dx \\
= \int_{\mathbb{R}^N} \theta (I_\alpha * |u|^p)|u|^{p-2}(u \nabla \bar{u} + \bar{u} \nabla u) \cdot \nabla w \, dx \\
= \frac{2}{p} \int_{\mathbb{R}^N} \theta (I_\alpha * |u|^p) \nabla (|u|^p) \cdot \nabla w \, dx \\
= -\frac{2}{p} \int_{\mathbb{R}^N} [(I_\alpha * |u|^p)|u|^p \nabla \theta \cdot \nabla w + (I_\alpha * |u|^p)|u|^p \theta \Delta w \\
\quad + \theta |u|^p \nabla (I_\alpha * |u|^p) \cdot \nabla w] \, dx,
\]

where

\[
\int_{\mathbb{R}^N} \theta |u|^p \nabla (I_\alpha * |u|^p) \cdot \nabla w \, dx \\
= A \int_{\mathbb{R}^N} \nabla \left( \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \, dy \right) \cdot \nabla w(x)|u(x)|^p \theta(x) \, dx \\
= A \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\alpha - N)(x-y) \cdot \nabla w(x) \theta(x)|u(x)|^p|u(y)|^p \, dy \, dx \\
= \frac{(\alpha - N)A}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot \nabla w(x) \theta(x)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha+2}} \, dy \, dx \\
\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(y-x) \cdot \nabla w(y) \theta(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha+2}} \, dy \, dx \right) \\
= \frac{(\alpha - N)A}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot (\theta \nabla w(x) - \theta \nabla w(y))|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha+2}} \, dy \, dx.
\]

Inserting (5.11) into (5.10), we obtain that

\[
\text{Re} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}u \theta \nabla \bar{u} \cdot \nabla w \, dx \\
= -\frac{2}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p (\nabla \theta \cdot \nabla w + \theta \Delta w) \, dx \\
\quad + \frac{(N - \alpha)A}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot (\theta \nabla w(x) - \theta \nabla w(y))|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha+2}} \, dy \, dx.
\]
Inserting (5.6)-(5.9) and (5.12) into (5.5), we obtain that

\[ h'_\epsilon(t) = \Re \int_{\mathbb{R}^N} (2\nabla u \cdot \nabla \theta \nabla \bar{u} \cdot \nabla w + 2\theta \partial_i u \partial_j \bar{u} \partial_j \partial_i w + \nabla u \cdot \nabla \theta \bar{u} \Delta w - \frac{1}{2} |u|^2 \nabla \theta \cdot \nabla (\Delta w) - \frac{1}{2} |u|^2 \theta \Delta^2 w + \bar{u} \partial_j u \partial_j \theta \partial_i w + \frac{2b \theta \epsilon x \cdot \nabla w}{(|x|^2 + \delta)^2} |u|^2) \, dx \]

\[ + \left( \frac{2}{p} - 1 \right) \int_{\mathbb{R}^N} (I_\alpha |u|^p \nabla \theta \cdot \nabla w + \theta \Delta w) \, dx \]

\[ - \frac{(N - \alpha)A}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x - y) \cdot (\theta(x) \nabla w(x) - \theta(y) \nabla w(y)) |u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha + 2}} \, dy \, dx. \]

Note that \( \theta_\epsilon, \partial_t \theta_\epsilon, \partial_j \theta_\epsilon \) are bounded with respect to both \( x \) and \( \epsilon \), and by the mean value theorem,

\[ (x - y) \cdot (\theta(x) \nabla w(x) - \theta(y) \nabla w(y)) \lesssim |x - y|^2. \]

Furthermore, \( \theta_\epsilon \to 1, \partial_t \theta_\epsilon \to 0, \partial_j \theta_\epsilon \to 0 \) and \( \epsilon \to 0 \). On the other hand, for any \( t \in I \), we have \( u(t) \in H^2(\mathbb{R}^N) \) and \( \sqrt{w(x)} u(t) \in L^2(\mathbb{R}^N) \), so by the choices of \( w(x) \) and \( \theta_\epsilon(x) \) and by the Lebesgue dominated convergence theorem, we obtain from (5.13) that

\[ \lim_{\epsilon \to 0} h'_\epsilon(t) = 2\Re \int_{\mathbb{R}^N} \partial_i u \partial_j \bar{u} \partial_j \partial_i w \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Delta^2 w \, dx + 2b \int_{\mathbb{R}^N} \frac{x \cdot \nabla w}{(|x|^2 + \delta)^2} |u|^2 \, dx \]

\[ - \frac{A(N - \alpha)}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x - y) \cdot (\nabla w(x) - \nabla w(y)) |u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha + 2}} \, dy \, dx \]

\[ + \left( \frac{2}{p} - 1 \right) \int_{\mathbb{R}^N} (I_\alpha |u|^p |u|^p \Delta w) \, dx. \]

Since

\[ \lim_{\epsilon \to 0} 2h_\epsilon(t) = 2\Im \int_{\mathbb{R}^N} \bar{u} \nabla w \cdot \nabla u \, dx = \frac{d}{dt} V_w(t), \]

we see that \( V_w(t) \in C^2(I, \mathbb{R}) \) and (5.4) holds.

Step 2. Let \( \{u_{n0}\}_{n=1}^\infty \subset H^2(\mathbb{R}^N) \) be such that \( u_{n0} \to u_0 \) strongly in \( H^1(\mathbb{R}^N) \) and \( \sqrt{w(x)} u_{n0} \to \sqrt{w(x)} u_0 \) strongly in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \), and let \( u_n \) be the corresponding solution to (CH) with initial value \( u_{n0} \). Let \( \Phi(t) \) denote the right-hand side of (5.4) and let \( \Phi_n(t) \) denote the right-hand side of (5.4) corresponding to the solution \( u_n \). It follows from Step 1 that

\[ \| \sqrt{w(x)} u_n(t) \|^2_{L^2} = \| \sqrt{w(x)} u_n \|^2_{L^2} + 2t \Im \int_{\mathbb{R}^N} \bar{u}_{n0} \nabla w \cdot \nabla u_{n0} \, dx + \int_0^t \int_0^s \Phi_n(\tau) \, d\tau \, ds. \]

By the continuous dependence (Proposition 5.1) and Corollary 6.5.3 in [5], we may let \( n \to \infty \) in (5.14) and obtain

\[ \| \sqrt{w(x)} u(t) \|^2_{L^2} = \| \sqrt{w(x)} u_0 \|^2_{L^2} + 2t \Im \int_{\mathbb{R}^N} \bar{u}_0 \nabla w \cdot \nabla u_0 \, dx + \int_0^t \int_0^s \Phi(\tau) \, d\tau \, ds, \]

which implies (5.4).
Lemma 5.3. Let $N \geq 3$, $\alpha \in ((N-4)_+, N)$, $b > -\frac{(N-2)^2}{4}$ and $b \neq 0$, $2 \leq p < \frac{N+2}{N+4}$. If $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ is a maximal-lifespan solution to (CH$_b$) with initial value $u_0 \in H^1(\mathbb{R}^N)$ and $\sqrt{w(x)}u_0 \in L^2(\mathbb{R}^N)$, then for any $t \in I$,

$$\frac{d}{dt} V_w(t) = 2 \text{Im} \int_{\mathbb{R}^N} \bar{u} \nabla w \cdot \nabla u dx$$

and

$$\frac{d^2}{dt^2} V_w(t) = 4 \text{Re} \int_{\mathbb{R}^N} \partial_i u \partial_j \bar{u} \partial_i \partial_j w dx - \int_{\mathbb{R}^N} |u|^2 \Delta^2 w dx + 4b \int_{\mathbb{R}^N} |x|^{-4} |u|^2 x \cdot \nabla w dx - 2A(N-\alpha) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot (\nabla w(x) - \nabla w(y)) |u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha+2}} dx dy$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \Delta w dx.$$

Proof. Inspired by the proof of (3.1) in [23], we prove this lemma in two steps. For any $\delta > 0$, let $u_\delta \in C(I_\delta, H^1(\mathbb{R}^N)) \cap C^1(I_\delta, H^{-1}(\mathbb{R}^N))$ be the maximal-lifespan solution to (CH$_b$) with initial value $u_0$.

Step 1. We claim that there exists $M_0 > 0$ and $T > 0$ such that

$$(5.16) \quad \|u_\delta(t)\|_{H^1} \leq M_0 \text{ for any } \delta > 0 \text{ and } t \in [-T, T].$$

Indeed, we define

$$M := 2 \sqrt{\left( 1 + \frac{4b_+}{(N-2)^2} \right) / \left( 1 + \frac{4b_-}{(N-2)^2} \right)} \|u_0\|_{H^1}$$

and

$$\tau_\delta := \sup_{T > 0} \{ \|u_\delta(t)\|_{H^1} \leq M, \ t \in [-T, T] \}.$$

If $\tau_\delta = \infty$, the claim is proved. Thus, we assume $\tau_\delta < \infty$. Since $u_\delta \in C(I_\delta, H^1(\mathbb{R}^N))$, $\tau_\delta$ satisfies

$$(5.17) \quad \|u_\delta(\tau_\delta)\|_{H^1} = M \text{ or } \|u_\delta(-\tau_\delta)\|_{H^1} = M.$$

It follows from the conservation laws, (G3) and (5.1) that

$$\left( 1 - \frac{4b_-}{(N-2)^2} \right) \|u_\delta\|^2_{H^1} - \left( 1 + \frac{4b_+}{(N-2)^2} \right) \|u_0\|^2_{H^1}$$

$$\leq \int_{\mathbb{R}^N} (|\nabla u_\delta|^2 + \frac{b}{|x|^2 + \delta} |u_\delta|^2 + |u_\delta|^2) dx$$

$$- \int_{\mathbb{R}^N} (|\nabla u_0|^2 + \frac{b}{|x|^2 + \delta} |u_0|^2 + |u_0|^2) dx$$

$$= \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_\delta|^p) |u_\delta|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^p) |u_0|^p dx$$

$$\leq \epsilon + C_\epsilon(M) \|u_\delta - u_0\|_{L^2}$$

for any $t \in [-\tau_\delta, \tau_\delta]$ and $\epsilon > 0$. On the other hand, in view of (G2) and the fact that $u_\delta$ satisfies equation (CH$_b$), we know

$$(5.19) \quad \|\partial_t u_\delta\|_{H^{-1}} \leq \| - \Delta u_\delta + \frac{b}{|x|^2 + \delta} u_\delta\|_{H^{-1}} + \|(I_\alpha * |u_\delta|^p) |u_\delta|^p - 2u_\delta\|_{H^{-1}}$$

$$\leq C(M)$$
for any \( t \in [-\tau_\delta, \tau_\delta] \). Applying Lemma 3.3.6 in [5], we obtain that
\[
\|u_\delta(t) - u_\delta(s)\|_{L^2} \leq C(M)|t - s|^\frac{1}{4} \quad \text{for any } t, s \in [-\tau_\delta, \tau_\delta].
\]
Inserting (5.20) with \( s = 0 \) into (5.18), we have
\[
\left(1 - \frac{4b_-(N-2)^2}{(N-2)^2}\right)\|u_\delta\|^2_{H^1} - \left(1 + \frac{4b_+(N-2)^2}{(N-2)^2}\right)\|u_0\|^2_{H^1} \leq \epsilon + C_\epsilon(M)C(M)|t|^\frac{1}{4}.
\]
Letting \( t = \tau_\delta \) or \(-\tau_\delta\) and applying (5.17), we have
\[
3\left(1 + \frac{4b_+(N-2)^2}{(N-2)^2}\right)\|u_0\|^2_{H^1} \leq \epsilon + C_\epsilon(M)C(M)|\tau_\delta|^\frac{1}{4},
\]
which implies that \( \tau_\delta \) has a positive lower bound by choosing \( \epsilon > 0 \) small enough.

The proof of the claim is complete.

Step 2. In view of (5.16) and (5.19), by Proposition 1.1.2 in [5], there exists \( \{\delta_j\}^{\infty}_{j=1} \subset (0, \infty) \) with \( \delta_j \to 0 \) as \( j \to \infty \) and \( v \in C_w([-T, T], H^1(\mathbb{R}^N)) \cap W^{1, \infty}([-T, T], H^{-1}(\mathbb{R}^N)) \) such that
\[
u_{\delta_j}(t) \to v(t) \text{ weakly in } H^1(\mathbb{R}^N) \text{ for any } t \in [-T, T]
\]
and
\[
\partial_t u_{\delta_j}(t) \to \partial_t v(t) \text{ weakly } * \text{ in } L^\infty([-T, T], H^{-1}(\mathbb{R}^N)).
\]
By (5.2) and (5.21), we have
\[
\left(-\Delta + \frac{b}{|x|^2 + \delta_j}\right)u_{\delta_j}(t) \to \left(-\Delta + \frac{b}{|x|^2}\right)v(t) \text{ weakly in } H^{-1}(\mathbb{R}^N)
\]
as \( j \to \infty \) for any \( t \in [-T, T] \). By (5.16), (5.22), (5.23) and the fact that \( u_{\delta_j} \) satisfies equation \((\text{CH}_b)\), there exists \( f \) such that
\[
-(I_\alpha * |u_{\delta_j}|^p)|u_{\delta_j}|^{p-2}u_{\delta_j} = i\partial_t u_{\delta_j} - \left(-\Delta + \frac{b}{|x|^2 + \delta_j}\right)u_{\delta_j}
\]
\[
\to i\partial_t v - \mathcal{L}_b v =: f \text{ weakly } * \text{ in } L^\infty([-T, T], H^{-1}(\mathbb{R}^N)).
\]
By (G5), (5.21) and (5.24), we have
\[
\text{Im} \int_0^t \langle f(s), v(s) \rangle_{H^{-1}, H^1} ds
\]
\[
= \lim_{n \to \infty} \text{Im} \int_0^t \langle -(I_\alpha * |u_{\delta_j}(s)|^p)|u_{\delta_j}(s)|^{p-2}u_{\delta_j}(s), u_{\delta_j}(s) \rangle_{H^{-1}, H^1} ds
\]
\[
= 0, \text{ for any } t \in [-T, T].
\]
By (5.24) and (5.25), we obtain the conservation of mass of \( v \). Hence, it follows from (5.21), the conservation laws of \( u_{\delta_j} \) and \( u_{\delta_j}(0) = u_0 \) that
\[
\|v(t)\|_{L^2} = \|v(0)\|_{L^2} = \|u_0\|_{L^2} = \|u_{\delta_j}\|_{L^2}, \text{ for any } t \in [-T, T].
\]
Hence we see from (5.21) and (5.26) that
\[
u_{\delta_j}(t) \to v(t) \text{ strongly in } L^2(\mathbb{R}^N) \text{ for any } t \in [-T, T] \text{ as } j \to \infty,
\]
which combines with (5.16) and the Sobolev imbedding theorem shows that
\[
u_{\delta_j}(t) \to v(t) \text{ strongly in } L^r(\mathbb{R}^N) \text{ for any } r \in [2, 2^*) \text{ and } t \in [-T, T].
\]
By (5.16), (5.24) and (5.27), we have \( f = -(I_\alpha * |v|^p)|v|^{p-2}v \). Thus, \( v \) satisfies
\[
\begin{cases}
  i\partial_t v - L_b v = -(I_\alpha * |v|^p)|v|^{p-2}v, & \text{in } L^\infty([-T, T], H^{-1}(\mathbb{R}^N)), \\
  v(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\]
On the other hand, there exists a unique weak solution to (5.28), see the proof of Theorem 1.2. Hence, \( v = u \) and \( u \) satisfies the conservation laws
\[
(5.29) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_b(u(t)) = E_b(u_0), \quad \text{for any } t \in [-T, T].
\]
By (5.27), (5.29), the conservation laws of \( u_\delta \), we have
\[
\|\nabla u_\delta\|^2_{L^2} + b \left( \frac{u_\delta}{|x|^{2+\delta_j}} \right) = 2E_{b, \delta_j}(u_\delta(t)) + \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_\delta|^p)|u_\delta|^p dx - 2E_b(u_0) + \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \to 2E_{b, \delta_j}(u_0) + \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \quad \text{as } \delta \to 0.
\]
By Lemma 2.2 in [23], we obtain \( \nabla u_\delta \to \nabla u \) strongly in \( L^2(\mathbb{R}^N) \). Hence,
\[
(5.30) \quad u_\delta \to u \text{ strongly in } H^1(\mathbb{R}^N) \text{ for any } t \in [-T, T].
\]
On the other hand, Lemma 6.5.2 in [5] implies
\[
\int_{\mathbb{R}^N} w|u|^2 dx = \int_{\mathbb{R}^N} w|u_0|^2 dx + 2\text{Im} \int_0^t \int_{\mathbb{R}^N} \bar{u} \nabla w \cdot \nabla u dx.
\]
Thus Lemma 2.3 in [23] with (5.16) and (5.30) give that
\[
\sqrt{w(x)} u_\delta(t) \to \sqrt{w(x)} u(t) \text{ strongly in } L^2(\mathbb{R}^N) \text{ for any } t \in [-T, T].
\]
Replacing \( u \) by \( u_\delta \) in (5.15) and letting \( \delta \to 0 \), we complete the proof. \( \square \)

Let \( w = |x|^2 \) in Lemma 5.3, then
\[
\nabla w = 2x, \quad \Delta w = 2N, \quad \partial_j w = 2\delta_{ij}, \quad \Delta^2 w = 0, \quad \partial_i \partial_j w = 2|\nabla u|^2, \quad x \cdot \nabla w = 2|x|^2, \quad (x - y) \cdot (\nabla w(x) - \nabla w(y)) = 2|x - y|^2.
\]
Hence, we have the following standard virial identity.

**Lemma 5.4.** Let \( N \geq 3, \alpha \in ((N-4)^+ , N), b > \frac{(N-2)^2}{4} \) and \( b \neq 0, 2 \leq p < \frac{N+2}{N-2} \), \( u_0 \in H^1(\mathbb{R}^N) \) with \( xu_0 \in L^2(\mathbb{R}^N) \). Assume that \( u(x, t) \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N)) \) is the maximal-lifespan solution to (CH\( _b \)). Then \( xu \in C(I, L^2(\mathbb{R}^N)) \) and for any \( t \in I \),
\[
\frac{d^2}{dt^2} \|xu(t)\|^2_{L^2} = 8\|u\|^2_{H^1_b} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx.
\]
Let \( \psi : [0, \infty) \to [0, \infty) \) be a smooth function satisfying
\[
\psi(r) = \begin{cases} r^2, & \text{for } 0 \leq r \leq 1, \\
\text{constant,} & \text{for } r \geq 10, \\
\psi'(r) \leq 2r, \quad \psi''(r) \leq 2 & \text{for any } r \geq 0.
\end{cases}
\]
For any \( R > 1 \), we define
\[
(5.31) \quad \psi_R(r) = R^2 \psi \left( \frac{r}{R} \right), \quad \varphi_R(x) = \psi_R(|x|).
\]
By letting \( w(x) = \varphi_R(x) \) in Lemma 5.3, we have the following local virial identity.

**Lemma 5.5.** Let \( N \geq 3 \), \( \alpha \in ((N-4)_+, N) \), \( b > -\frac{(N-2)^2}{2} \) and \( b \neq 0 \), \( 2 \leq p < p^* \), \( u_0 \in H^1_x(\mathbb{R}^N) \) and \( u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N)) \) be the maximal-lifespan radial solution to (CH\(_b\)). Then for any \( t \in I \),

\[
\frac{d^2}{dt^2} \|\varphi_R(x)u\|_{L^2}^2 = 8\|u\|_{H^1}^2 + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \\
+ O(R^{-2}) + O(R^{-(N-1)(p-\frac{N+\alpha}{N})}) \|u\|_{H^1}^{\frac{N+\alpha}{N}} \\
+ O(R^{-(N-\alpha+N-1)(p-2)}) \|u\|_{H^1}^{\frac{N+1}{N-1}(p-2)}).
\]

**Proof.** By Lemma 5.3, we have

\[
\frac{d^2}{dt^2} \|\varphi_R(x)u\|_{L^2}^2 \\
= 4\text{Re} \int_{\mathbb{R}^N} \partial_t u \partial_j \bar{u} \partial_t \partial_j \varphi_R dx - \int_{\mathbb{R}^N} |u|^2 \Delta^2 \varphi_R dx + 4b \int_{\mathbb{R}^N} |x|^{-4} |u|^2 x \cdot \nabla \varphi_R dx \\
- \frac{2A(N-\alpha)}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) |u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha+2}} dxdy \\
+ \left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \Delta \varphi_R dx.
\]

Direct calculation gives that

\[
\frac{d^2}{dt^2} \|\varphi_R(x)u\|_{L^2}^2 = 4\text{Re} \int_{\mathbb{R}^N} \partial_t u \partial_j \bar{u} \partial_t \partial_j (|x|^2) dx \\
+ 4\text{Re} \int_{|x|>R} \partial_t u \partial_j \bar{u} \left( \partial_t \partial_j \varphi_R - \partial_t (|x|^2) \right) dx,
\]

\[
\int_{\mathbb{R}^N} |u|^2 \Delta^2 \varphi_R dx = \int_{\mathbb{R}^N} |u|^2 \Delta^2 (|x|^2) dx + \int_{|x|>R} |u|^2 (\Delta^2 \varphi_R - \Delta^2 (|x|^2)) dx,
\]

\[
4b \int_{\mathbb{R}^N} |x|^{-4} |u|^2 x \cdot \nabla \varphi_R dx = 4b \int_{\mathbb{R}^N} |x|^{-4} |u|^2 x \cdot \nabla (|x|^2) dx \\
+ 4b \int_{|x|>R} |x|^{-4} |u|^2 (x \cdot \nabla \varphi_R - x \cdot \nabla (|x|^2)) dx,
\]

\[
\left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \Delta \varphi_R dx \\
= \left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \Delta (|x|^2) dx \\
+ \left( \frac{4}{p} - 2 \right) \int_{|x|>R} (I_\alpha * |u|^p)|u|^p (\Delta \varphi_R - \Delta (|x|^2)) dx
\]

\[
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and

\[
\begin{align*}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|u(x)|^p|u(y)|^p dxdy \\
= &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (x - y) \cdot (|x|^2 - |y|^2)|u(x)|^p|u(y)|^p dxdy \\
- &\int_{|x|>R} \int_{|y|>R} (x - y) \cdot (|x|^2 - |y|^2)|u(x)|^p|u(y)|^p dxdy \\
+ &\int_{|x|<R} \int_{|y|>R} (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|u(x)|^p|u(y)|^p dxdy \\
+ &\int_{|x|>R} \int_{|y|<R} (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|u(x)|^p|u(y)|^p dxdy.
\end{align*}
\]

(5.37)

Inserting (5.33)-(5.37) into (5.32), we obtain that

\[
\frac{d^2}{dt^2}\|\varphi_R(x)u\|^2_{L^2} = 8\|u\|^2_{H^2(\mathbb{R}^N)} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \\
- 8\|u\|^2_{H^1(\mathbb{B}_R)} - \frac{4\alpha + 4N - 4Np}{p} \int_{|x|>R} (I_\alpha * |u|^p)|u|^p dx \\
+ 4\text{Re} \int_{|x|>R} \partial_i u \partial_j \bar{u} \partial_i \varphi_R dx - \int_{|x|>R} |u|^2 \Delta^2 \varphi_R dx \\
+ 4b \int_{|x|>R} |x|^{-4}|u|^2 \cdot \nabla \varphi_R dx - \left( \frac{4}{p} - 2 \right) \int_{|x|>R} (I_\alpha * |u|^p)|u|^p \Delta \varphi_R dx \\
+ \frac{2A(\alpha - N)}{p} \int_{|x|>R} \int_{\mathbb{R}^N} (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|u(x)|^p|u(y)|^p dxdy \\
- \frac{2A(\alpha - N)}{p} \int_{|x|<R} \int_{|y|>R} (x - y) \cdot (|x|^2 - |y|^2)|u(x)|^p|u(y)|^p dxdy \\
+ \frac{2A(\alpha - N)}{p} \int_{|x|<R} \int_{|y|<R} (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))|u(x)|^p|u(y)|^p dxdy.
\]

(5.38)

By direct calculation, we have

\[
\psi_R'(r) = R\psi'(\frac{r}{R}), \quad \psi_R''(r) = \psi''(\frac{r}{R}),
\]

\[
2 - \psi_R''(r) \geq 0, \quad 2 - \frac{\psi_R'(r)}{r} = 2 - \frac{R}{r} \psi'(\frac{r}{R}) \geq 0,
\]

\[
2N - \Delta \varphi_R(x) = 2N - \left( \psi_R'(|x|) + \frac{N - 1}{|x|} \psi_R'(|x|) \right) \geq 0.
\]
\[
\partial_t \varphi_R(x) = R \psi'(\frac{|x|}{R}) \frac{x}{|x|}, \quad x \cdot \nabla \varphi_R(x) = R |x| \psi'(\frac{|x|}{R}),
\]
\[
\partial_{ij} \varphi_R(x) = \psi''(\frac{|x|}{R}) \frac{x_i x_j}{|x|^2} + R \psi'(\frac{|x|}{R}) \frac{\delta_{ij}}{|x|} - R \psi'(\frac{|x|}{R}) \frac{x_i x_j}{|x|^3} \lesssim 1,
\]
\[
\Delta \varphi_R(x) = \psi''(\frac{|x|}{R}) + (N - 1) \frac{R}{|x|} \psi'(\frac{|x|}{R}) \lesssim 1, \quad \Delta^2 \varphi_R(x) \lesssim R^{-2},
\]
\[
\partial_t u \partial_j \vec{u} \partial_{ij} \varphi_R(x) = |\partial_r u|^2 \left( \psi''(\frac{|x|}{R}) + \frac{R}{|x|} \psi'(\frac{|x|}{R}) - \frac{R}{|x|} \psi'(\frac{|x|}{R}) \right) = |\nabla u|^2 \psi''(\frac{|x|}{R}),
\]
and
\[
(x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \lesssim |x - y|^2.
\]

The above estimates and the conservation laws imply that
\[
(5.39) \quad \int_{|x| > R} |u|^2 \Delta^2 \varphi_R dx \lesssim \int_{|x| > R} |u|^2 R^{-2} dx = O(R^{-2}),
\]
\[
(5.40) \quad 4 \Re \int_{|x| > R} \partial_t u \partial_j \vec{u} \partial_{ij} \varphi_R dx + 4b \int_{|x| > R} |x|^{-4} |u|^2 x \cdot \nabla \varphi_R dx - 8 \|u\|_{H^1_0(B_R)}^2 \lesssim R^{-2}
\]
\[
\int_{|x| > R} \left( I_{\alpha} * |u|^p \right) |u|^p \Delta \varphi_R dx + \int_{|x| > R} \int_{\mathbb{R}^N} \frac{(x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) |u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha + 2}} dx dy
\]
\[
+ \int_{|x| < R} \int_{|y| > R} \frac{(x - y) \cdot (\nabla |x|^2 - \nabla |y|^2) |u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha + 2}} dx dy
\]
\[
+ \int_{|x| < R} \int_{|y| > R} \frac{(x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) |u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha + 2}} dx dy
\]
\[
\lesssim \int_{|x| > R} \left( I_{\alpha} * |u|^p \right) |u|^p dx.
\]

We use the Hardy-Littlewood-Sobolev inequality, the interpolation inequality, the conservation laws, the Sobolev imbedding theorem and Lemma 2.4 to estimate
\[
(5.42) \quad \int_{|x| > R} \left( I_{\alpha} * |u|^p \right) |u|^p dx
\]
\[
= A \int_{|x| > R} \int_{|y| > \frac{R}{2}} \frac{|u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha}} dy dx + A \int_{|x| > R} \int_{|y| < \frac{R}{2}} \frac{|u(x)|^p |u(y)|^p}{|x - y|^{N - \alpha}} dy dx
\]
\[
:= I + II,
\]
where

\[ I \lesssim \|u^p \chi_{B_{R/2}'}\|_{L^{\frac{2N}{N+\alpha}}} \|u^p \chi_{B_{R}}\|_{L^{\frac{2N}{N+\alpha}}} \]

\[ \lesssim \|u\|_{L^\infty}^{\frac{2}{N+\alpha}} \|u\|^p_{L^2} \|u\|^p_{L^\infty(|x| > \frac{R}{2})} \|u\|^p_{L^\infty(|x| > R)} \]

\[ \lesssim R^{-(N-1)(p-\frac{N+\alpha}{2})} \|u\|^p_{H^1_b} \]

and

\[ II \lesssim R^{-(N-\alpha)} \int_{|y| < \frac{R}{2}} |u(y)|^p dy \int_{|x| > R} |u(x)|^p dx \]

\[ \lesssim R^{-(N-\alpha)} \|u\|^{(1-\sigma)p}_{L^2} \|u\|_{L^\infty}^{\frac{\sigma}{2}p} \|u\|^p_{L^2} \|u\|^p_{L^\infty(|x| > R)} \]

\[ \lesssim R^{-(N-\alpha)} \|u\|^{\frac{\sigma}{2}p}_{H^1_b} R^{\frac{N-2\sigma}{2}(p-2)} \left( \sup_{|x| > R} |x|^{\frac{N-2\sigma}{2}} |u(x)| \right)^{p-2} \]

\[ \lesssim R^{-(N-\alpha) - \frac{N-2\sigma}{2}(p-2)} \|u\|^{\frac{\sigma}{2}p}_{H^1_b} \|u\|^{\frac{N+1}{2}(p-2)}_{H^1_b} \]

with \( s = \frac{1}{2} \) and \( \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2} \). Inserting (5.39)-(5.44) into (5.38), we complete the proof. \( \square \)

5.2. Blowup.

Proof of the blowup part (ii) in Theorem 1.4. By using the standard virial identity (Lemma 5.4) and the conservation laws, we have

\[ \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 16E_b(u(t)) = 16E_b(u_0) < 0. \]

By the standard convexity arguments (see [12]), we know that \( u \) blows up in finite time in both time directions.

Proof of the blowup part (ii) in Theorem 1.6. We proceed as in the proof of Theorem 1.6 (i). It follows from the assumption

\[ E_b(u_0) \|u_0\|_{L^2}^{2\gamma} < H(b) \]

that there exists \( \delta_1 > 0 \) small enough such that

\[ E_b(u_0) \|u_0\|_{L^2}^{2\gamma} < (1 - \delta_1)H(b), \]

which combines with (4.12) and the conservation laws implies that

\[ f(\|u(t)\|_{H^1_b}^{\frac{2}{\gamma}}, u(t)\|_{L^2}^2) \leq E_b(u(t)) \|u(t)\|_{L^2}^{2\gamma} < (1 - \delta_1)H(b) \]

for any \( t \in I \).

Since \( f(K(b)) = H(b), \|u_0\|_{H^1_b}^{\frac{2}{\gamma}}, u_0\|_{L^2}^2 > K(b) \), in view of (5.47), and the continuity argument, there exists \( \delta_2 > 0 \) depending on \( \delta_1 \) such that

\[ \|u(t)\|_{H^1_b}^{\frac{2}{\gamma}}, u(t)\|_{L^2}^2 > (1 + \delta_2)K(b) \]

for any \( t \in I \).

Next we claim that for \( \epsilon > 0 \) small enough, there exists \( c > 0 \) such that

\[ (8 + \epsilon)\|u(t)\|_{H^1_b}^{\frac{2}{\gamma}} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq -c \]
for any $t \in I$. Indeed, multiplying the left side of (5.49) by $\|u(t)\|^2_{L^2}$ and using (5.47), (5.48), the conservation laws and $H(b) = \frac{Np - N - \alpha - 2}{2Np - N - \alpha} K^2(b)$, we get that

$$\text{LHS}(5.49) \times \|u(t)\|^2_{L^2} = 8(Np - N - \alpha)E_b(u)\|u(t)\|^2_{L^2} + (8 + 4\alpha + 4N - 4Np + \epsilon)\|u(t)\|^2_{\dot{H}^1_t} \leq 8(Np - N - \alpha)(1 - \delta_1)H(b) + (8 + 4\alpha + 4N - 4Np + \epsilon)(1 + \delta_2)^2 K(b)^2$$

by choosing $\epsilon > 0$ small enough. Hence, the claim holds.

Case 1 ($xu_0 \in L^2(\mathbb{R}^N)$). By using the standard virial identity (Lemma 5.4) and (5.49), we have

$$\frac{d^2}{dt^2} \|xu(t)\|^2_{L^2} = 8\|u(t)\|^2_{\dot{H}^1_t} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq -c,$$

which implies that $u$ blows up in finite time.

Case 2 ($u_0 \in H^1(\mathbb{R}^N)$). Since $p < \min\{p_b, \frac{2N+6}{4N-2}\}$, we have $p - \frac{N+\alpha}{N} < 2$ and $\frac{N+\alpha}{p-2} < 2$. By using the local virial identity (Lemma 5.5), the Young inequality (3.9) and (5.49), for any $\epsilon > 0$, we have

$$\frac{d^2}{dt^2} \|\psi_R(x)u(t)\|^2_{L^2} \leq 8\|u\|^2_{\dot{H}^1} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$$

$$+ O(R^{-2}) + O(R^{-\left(N-1\right)(p-\frac{N+\alpha}{N})})\|u\|^p_{\dot{H}^1}$$

$$+ O(R^{-\left(N-\alpha + \frac{N-1}{2}(p-2)\right)} \|u\|^p_{\dot{H}^1})$$

$$\leq (8 + \epsilon)\|u(t)\|^2_{\dot{H}^1} + \frac{4\alpha + 4N - 4Np}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$$

$$+ O(R^{-2}) + O(\epsilon) \left(\frac{N+\alpha}{2(p-2)\frac{N+\alpha}{N}}\right) - \left(N-1\right)(p-\frac{N+\alpha}{N}) R^{-\frac{2}{2-\frac{N+\alpha}{N}}} \right)$$

$$+ O(\epsilon) \left(\frac{N+\alpha}{2(p-2)\frac{N+\alpha}{N}}\right) - \left(N-\alpha + \frac{N-1}{2}(p-2)\right) R^{-\frac{2}{2-\frac{N+\alpha}{N}}} \right)$$

$$\leq -c/2$$

by choosing $\epsilon > 0$ small enough and by choosing $R > 1$ large enough depending on $\epsilon$. Hence, the solution $u$ blows up in finite time.

**Proof of (1) in Remark 1.5.** Let $E_b > 0$, we find initial value $u_0 \in H^1$ with $E_b(u_0) = E_b$ such that the corresponding solution $u$ blows up in finite time. We follow the standard argument (see Remark 6.5.8 in [5]). Using the virial identity with $p = p_b$, we have

$$\frac{d^2}{dt^2} \|xu(t)\|^2_{L^2} = 16E_b(u_0).$$

Hence,

$$\|xu(t)\|^2_{L^2} = \|xu_0\|^2_{L^2} + 4t \text{Im} \int_{\mathbb{R}^N} \overline{u_0} x \cdot \nabla u_0 dx + 8t^2 E_b(u_0) := f(t).$$
Note that if \( f \) takes negative values, then the solution \( u \) must blow up in finite time. In order to make \( f \) takes negative values, we need

\[
(5.50) \quad \left( \text{Im} \int_{\mathbb{R}^N} u_0 x \cdot \nabla u_0 \, dx \right)^2 > 2E_b(u_0)\|xu_0\|^2_{L^2}.
\]

Now fix \( \theta \in C_0^\infty(\mathbb{R}^N) \) a real-valued function and set \( \psi(x) = e^{-i|x|^2} \theta(x) \). We see that \( \psi \in C_0^\infty(\mathbb{R}^N) \) and

\[
\text{Im} \int_{\mathbb{R}^N} \overline{\psi} x \cdot \nabla \psi \, dx = -2 \int_{\mathbb{R}^N} |x|^2 |\theta|^2 \, dx < 0.
\]

Denote

\[
A = \frac{1}{2} \|\psi\|^2_{H^1_b}, \quad B = \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\alpha} * |\psi|^p)|\psi|^p \, dx,
\]
\[
C = \|x\psi\|^2_{L^2}, \quad D = -\text{Im} \int_{\mathbb{R}^N} \overline{\psi} x \cdot \nabla \psi \, dx.
\]

Then \( A, B, C, D > 0 \). For \( \lambda, \mu > 0 \), set \( u_0(x) = \lambda \psi(\mu x) \). By direct calculation, we have

\[
E_b(u_0) = \lambda^2 \mu^{2-N} A - \lambda^{2p} \mu^{-N-\alpha} B = \lambda^2 \mu^{2-N} \left( A - \frac{\lambda^{2p-2}}{\mu^{2+\alpha}} B \right),
\]
\[
\|xu_0\|^2_{L^2} = \lambda^2 \mu^{-2-N} C,
\]
\[
\text{Im} \int_{\mathbb{R}^N} u_0 x \cdot \nabla u_0 \, dx = -\lambda^2 \mu^{-N} D.
\]

Next, we choose \( \lambda \) and \( \mu \) such that \( E_b(u_0) = E_b \) and (5.50) holds. Hence,

\[
(5.51) \quad \lambda^2 \mu^{2-N} \left( A - \frac{\lambda^{2p-2}}{\mu^{2+\alpha}} B \right) = E_b
\]

and

\[
(5.52) \quad \frac{D^2}{C} > 2 \left( A - \frac{\lambda^{2p-2}}{\mu^{2+\alpha}} B \right).
\]

Fix \( 0 < \epsilon < \min\{A, \frac{D^2}{C}\} \) and choose

\[
(5.53) \quad \frac{\lambda^{2p-2}}{\mu^{2+\alpha}} B = A - \epsilon.
\]

It is obvious that (5.52) is satisfied. Condition (5.51) and (5.53) imply

\[
(5.54) \quad \lambda^2 \mu^{2-N} \epsilon = E_b.
\]

So we can solve \( \mu \) and \( \lambda \) from (5.53) and (5.54). The proof is complete.

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