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On \((\psi, \phi)\)-Rational Contractions

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Abstract: In this paper, we examine the notion of \((\psi, \phi)\)-contractions by involving rational forms in the context of complete metric spaces. We note that some well-known fixed point theorems for rational forms can be deduced from our main results. We also consider some examples to indicate the validity of the presented results.

Keywords: fixed point theorems; metric space; contraction mapping

1. Introduction and Preliminaries

Thousands of results have been published since Banach [1] proved the first fixed point theorem. Some of these results are equivalent to the results published previously, while others were understood to be a sub-result of the previous results. Therefore, recently, publications that collect and consolidate the results in the literature have started to appear.

Very recently, Proinov (2020) [2], to extend and unify many earlier results, proved that the fixed point theorem of Skof (1977) [3], in the setting of metric spaces, covers several existing results, including the attractive results of Wardowski (2012) [4] and Jleli-Samet (2014) [5]. He also proved that the analog of this observation holds true in the context of dislocated metric spaces.

On the other hand, starting from Das-Gupta (1975) [6] and Jaggi (1977) [7], rational expressions were used to prove fixed point theorems. Later, these approaches were modified for Boyd and Wong contractions, \(\phi\)-contractions, Geraghty contractions, Wardowski contractions, etc. We observe that the concerns of Proinov [2] are valid for fixed point theorems involving rational expression; that is, some published results are equivalent to earlier results or consequences.

In this paper, we prove that the analog of the fixed theorem of Skof [3] with rational expression unifies and extends several fixed point theorems in the literature.

To begin with, we recall the first main result of Proinov [2].

**Theorem 1.** [2] Let \((X, d)\) be a metric space and \(\tau : X \to X\) be a mapping such that:

\[ \psi(d(\tau x, \tau y)) \leq \phi(d(x, y)), \]

for all \(x, y \in X\) with \(d(\tau x, \tau y) > 0\), where the functions \(\psi, \phi : (0, \infty) \to \mathbb{R}\) are such that the following conditions are satisfied:

1. \(\psi\) is nondecreasing;
2. \(\phi(s) < \psi(s)\) for any \(s > 0\);
3. \(\limsup_{s \to s_0+} \phi(s) < \psi(s_0+)\) for any \(s_0 > 0\).

Then, \(\tau\) admits a unique fixed point.

We also recall the main results in which some rational expressions were studied in a contraction condition.
Theorem 2 ([6]). Let \((X, d)\) be a complete metric space and \(\mathcal{T} : X \to X\) be a mapping such that there exist \(\kappa_1, \kappa_2 \in [0, 1),\) with \(\kappa_1 + \kappa_2 < 1\) such that:

\[
d(\mathcal{T}x, \mathcal{T}y) \leq \kappa_1 \cdot d(y, \mathcal{T}y) \frac{1 + d(x, \mathcal{T}x)}{1 + d(x, y)} + \kappa_2 \cdot d(x, y) \tag{1}
\]

for all \(x, y \in X\). Then, \(\mathcal{T}\) has a unique fixed point \(v \in X\), and the sequence \(\{\mathcal{T}^n x\}\) converges to the fixed point \(v\) for all \(x \in X\).

Theorem 3 ([7]). Let \((X, d)\) be a complete metric space and \(\mathcal{T} : X \to X\) be a continuous mapping. If there exist \(\kappa_1, \kappa_2 \in [0, 1),\) with \(\kappa_1 + \kappa_2 < 1\) such that:

\[
d(\mathcal{T}x, \mathcal{T}y) \leq \kappa_1 \cdot \frac{d(x, \mathcal{T}x)d(y, \mathcal{T}y)}{d(x, y)} + \kappa_2 \cdot d(x, y), \tag{2}
\]

for all distinct \(x, y \in X\), then \(\mathcal{T}\) possesses a unique fixed point in \(X\).

We mention that over the last few years, many interesting and different generalizations for rational contractions have been provided (see, for example, [8–12]).

Finally, let us consider the next lemma (which can be found in many papers; see, e.g., [2]), which will be useful in the sequel.

Lemma 1 ([2]). Let \(\{x_n\}\) be a sequence in a metric space \((X, d)\) such that \(d(x_n, x_{n+1}) \to 0\) as \(n \to \infty\). If the sequence \(\{x_n\}\) is not Cauchy, then there exist \(\epsilon > 0\) and the subsequences \(\{q_k\}\) and \(\{r_k\}\) of positive integers such that:

\[
\lim_{k \to \infty} d(x_{q_k}, x_{r_k}) = \epsilon, \quad \lim_{k \to \infty} d(x_{q_k+1}, x_{r_k+1}) = \epsilon+. \tag{3}
\]

2. Main Results

Throughout this section, we will consider that \(\phi, \psi : (0, \infty) \to \mathbb{R}\) are two functions such that:

\((f_0)\) \quad \phi(s) < \psi(s), \text{ for all } s > 0.

Definition 1. Let \((X, d)\) be a complete metric space. A mapping \(\mathcal{T} : X \to X\) is a \((\psi, \phi)\)-rational contraction of Type 1 if for every distinct \(x, y \in X\) such that \(d(\mathcal{T}x, \mathcal{T}y) > 0\), the following inequality:

\[
\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \phi(\mathcal{M}_1(x, y)), \tag{4}
\]

holds, where \(\mathcal{M}_1\) is defined by:

\[
\mathcal{M}_1(x, y) = \max \left\{ d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}x)d(y, \mathcal{T}y)}{d(x, y)} \right\}. \tag{5}
\]

Theorem 4. Let \((X, d)\) be a complete metric space and \(\mathcal{T} : X \to X\) be a continuous \((\psi, \phi)\)-rational contraction of Type 1. Assume that:

\((f_1)\) \quad \lim_{s \to s_0} \psi(s) > -\infty, \text{ for any } s_0 > 0; \quad \lim_{s \to s_0} \phi(s) < \lim_{s \to s_0} \psi(s), \text{ for any } s_0 > 0; \quad \mathcal{T} \text{ is continuous.}

Then, \(\mathcal{T}\) admits exactly one fixed point.

Proof. Starting with a point \(x \in X\), we define the sequence \(\{x_n\}\) by:

\[
x_1 = \mathcal{T}x, x_2 = \mathcal{T}^2x, ... , x_n = \mathcal{T}^nx, \tag{6}
\]
with \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N} \) (indeed, on the contrary, if there exists \( j_n \in \mathbb{N} \) such that \( x_{j_n} = x_{j_n+1} = T x_{j_n} \), we get that \( x_{j_n} \) is a fixed point of \( T \)). Under this consideration, for \( x = x_{j_n} \) and \( y = x_{j_n} \), we have:

\[
M_1(x_{j_n-1}, x_{j_n}) = \max \left\{ \frac{d(x_{j_n-1}, x_{j_n}), d(x_{j_n-1}, T x_{j_n}), d(x_{j_n}, T x_{j_n})}{d(x_{j_n-1}, T x_{j_n})} \right\}
\]

\[
= \max \left\{ d(x_{j_n-1}, x_{j_n}), d(x_{j_n}, x_{j_n+1}) \frac{d(x_{j_n}, x_{j_n+1})}{d(x_{j_n-1}, x_{j_n})} \right\}
\]

and by \( (4) \) (since \( d(T x_{j_n-1}, T x_{j_n}) = d(x_{j_n}, x_{j_n+1}) > 0 \), we get:

\[
\psi(d(x_{j_n}, x_{j_n+1})) = \psi(d(T x_{j_n}, T x_{j_n})) \leq \phi(M_1(x_{j_n-1}, x_{j_n}))
\]

which is equivalent, denoting by \( \xi_n = d(x_{j_n-1}, x_{j_n}) \), to:

\[
\psi(\xi_{n+1}) \leq \phi(\max \{ \xi_n, \xi_{n+1} \}).
\]  \hfill (7)

(Of course, we can assume that \( \xi_n > 0 \), since on the contrary, we can find \( l \in \mathbb{N} \) such that \( d(x_{j_n-1}, x_{j_n}) = \xi_l = 0 \). Thus, \( x_{j_n+1} = T x_{j_n} \) and \( x_{j_n} \) is the fixed point of \( T \).) If there exists \( n \in \mathbb{N} \) such that \( \max \{ \xi_n, \xi_{n+1} \} = \xi_{n+1} \), then \( \psi(\xi_{n+1}) \leq \phi(\xi_{n+1}) \), which contradicts the assumption \( (f_0) \). Therefore, for all \( n > 0 \), we have \( \xi_n > \xi_{n+1} \), so that the sequence \( \{ \xi_n \} \) is decreasing, and since it is strictly positive, there exists \( \xi \geq 0 \) such that \( \lim_{n \to \infty} \xi_n = \xi \) and \( \xi_n > \xi \) for all \( n > 0 \). Supposing that \( \xi > 0 \), because \( M_1(x_{j_n-1}, x_{j_n}) = \xi_n \), replacing in \( (4) \) and taking into account \( (f_0) \), we have:

\[
\psi(\xi_{n+1}) \leq \phi(\xi_n) < \psi(\xi_n).
\]

It follows that the sequence \( \{ \psi(\xi_n) \} \) is strictly decreasing, and since it is bounded (below) (because \( \xi_n > \xi \) and due to the assumption \( (f_0) \)), we can conclude that \( \{ \psi(\xi_n) \} \) is a convergent sequence. Moreover, from the above inequality, the sequence \( \{ \phi(\xi_n) \} \) is also convergent as the same limit. Thus, keeping in mind \( (f_2) \),

\[
\liminf_{s \to \xi_+} \psi(s) \leq \lim_{n \to \infty} \psi(\xi_n) = \lim_{n \to \infty} \phi(\xi_n) \leq \limsup_{s \to \xi_+} \phi(s) < \liminf_{s \to \xi_+} \psi(s),
\]

which is a contradiction. Therefore, \( \xi = 0 \) and:

\[
\lim_{n \to \infty} \xi_n = \lim_{n \to \infty} d(x_{j_n-1}, x_{j_n}) = 0. \]  \hfill (8)

We claim that \( \{ x_n \} \) is a Cauchy sequence. Let us suppose by contradiction that the sequence \( \{ x_n \} \) defined by \( (6) \) is not Cauchy. Then, by Lemma 1, there exist \( \epsilon > 0 \) and two sequences of positive real numbers \( (q_k) \) and \( (r_k) \) such that:

\[
\lim_{k \to \infty} d(x_{q_k+1}, x_{r_k+1}) = \epsilon +, \quad \lim_{k \to \infty} d(x_{q_k}, x_{r_k}) = \epsilon.
\]  \hfill (9)

Furthermore, for all \( k \geq 1 \), we have \( d(x_{q_k+1}, x_{r_k+1}) > \epsilon \). Replacing \( x = x_{q_k+1} \) and \( y = x_{r_k+1} \) in \( (4) \) and taking into account \( (f_0) \), we have:

\[
\psi(d(x_{q_k+1}, x_{r_k+1})) \leq \phi(M_1(x_{q_k}, x_{r_k})) \leq \psi(M_1(x_{q_k}, x_{r_k})),
\]

which is a contradiction.
where:

\[ M_i(x_i, x_k) = \max \left\{ \frac{d(x_i, x_k), d(x_i, T(x_i)), d(x_i, T(x_k))}{d(x_k, x_k)} \right\} \]

\[ = \max \left\{ \frac{d(x_i, x_k), d(x_i, x_k + 1), d(x_i, x_k + 1)}{d(x_k, x_k)} \right\}. \]

Now, by (8) and (9), we have \( \lim_{k \to \infty} M_i(x_k, x_k) = \epsilon \), and it follows by (4) that:

\[ \lim_{s \to \epsilon^+} \psi(s) \leq \lim_{k \to \infty} \phi(d(x_k + 1, x_k + 1)) \leq \lim_{k \to \infty} \phi(M_i(x_k, x_k)) \leq \lim_{s \to \epsilon} \phi(s). \]

This contradicts (f2), and then, \( \{x_n\} \) is a Cauchy sequence on a complete metric space. Thus, the sequence converges to a point \( v \in X \), that is:

\[ \lim_{n \to \infty} d(x_n, v) = 0 \quad (10) \]

and since the mapping \( T \) is continuous, we have:

\[ v = \lim_{n \to \infty} x_n + 1 = \lim_{n \to \infty} T x_n = T(\lim_{n \to \infty} x_n) = T v \]

which shows that \( v \) is a fixed point of \( T \).

If there exists another fixed point of \( T \), \( \bar{v} \in X \), such that \( \bar{v} \neq v \), since \( d(T \bar{v}, T v) > 0 \), from (4), we have:

\[ \psi(d(\bar{v}, v)) = \psi(d(T \bar{v}, T v)) \leq \phi(M_i(\bar{v}, v)) \]

\[ = \phi(\max \left\{ d(\bar{v}, v), d(\bar{v}, T \bar{v}), d(v, T v) \right\} \frac{d(\bar{v}, v)}{d(v, T v)}) \]

\[ = \phi(\psi(\bar{v}, v)). \]

Therefore, from the above inequality together with (f6), we get:

\[ \psi(d(\bar{v}, v)) \leq \phi(d(\bar{v}, v)) < \psi(d(\bar{v}, v)) \]

which is a contradiction. This closes the proof. \( \square \)

**Example 1.** Let the set \( X = [0, 2] \) and \( d : X \to X \) be the distance defined as \( d(x, y) = |x - y| \) for every \( x, y \in X \). Let also \( T : X \to X \) be a self-mapping with \( T x = \frac{-x^2 + 2x + 4}{8} \) and two functions \( \psi, \phi : (0, \infty) \to \mathbb{R}, \psi(s) = \frac{s}{2} \) and \( \phi(s) = \frac{s}{4} \). Since the assumptions (f1)-(f3) are satisfied, it remains to check that \( T \) is a \((\psi, \phi)\)-rational contraction of Type 1. We have:

\[ d(T x, T y) = \left| \frac{-x^2 + 2x + 4}{8} - \frac{-y^2 + 2y + 4}{8} \right| = \frac{1}{8} |(x - y)(-x - y + 2)| = \frac{1}{8} |(x - y)| |(-x - y + 2)| \]

and since \(|(-x - y + 2)| < 4\) for every \( x, y \in [0, 1] \), we have:

\[ \psi(d(T x, T y)) = \frac{1}{16} |(x - y)| |(-x - y + 2)| \leq \frac{1}{4} |(x - y)| = \frac{1}{4} d(x, y) \leq \frac{1}{4} M_i(x, y), \]

which shows us that \( T \) is a \((\psi, \phi)\)-rational contraction of Type 1. Furthermore, by Theorem 4, we get that \( T \) has a unique fixed point in \( X \), that is \( x = 0.605551 \).

Next, we show that the continuity condition of the operator \( T \) can be replaced by the assumption of the continuity of only some iterations of \( T \).
Theorem 5. If in Theorem 4 the statement \((f'_5)\) is replaced by:

\[ T^m \text{ is continuous for some integer } m > 1, \]

then \( T \) has a unique fixed point.

**Proof.** Let \( \{x_n\} \) be the sequence defined by (6). By the proof of Theorem 4, we know that this sequence is convergent to some point \( v \in X \), which means that \( d(x_n, v) = 0 \). Let \( \{x_{n(j)}\} \) be a subsequence of \( \{x_n\} \), where \( n(j) = j \cdot m \) for all \( j \in \mathbb{N}_0 \) and \( m > 1 \) fixed. Moreover, assuming that \( T^0 \) is the identity map on \( x \), we have \( x_{n(j)} = T^m x_{n(j) - m} \). Then, since \( T^m \) is continuous,

\[
d(v, T^m v) = \lim_{j \to \infty} d(v, T^m x_{n(j) - m}) = \lim_{j \to \infty} d(v, x_{n(j)}) = d(v, v) = 0.
\]

This means that \( v \) is a fixed point of \( T^m \).

If we assume that \( v \neq T^m v \), we have for any \( j = 0, 1, ..., m - 1 \) that \( T^{m-j-1}v \neq T^{m-j}v \). By replacing \( x \) by \( T^{am-j-1}v \) and \( y \) by \( T^{am-j}v \), we have:

\[
M_1(T^{m-j-1}v, T^{m-j}v) = \max \left\{ \frac{d(T^{m-j-1}v, T^{m-j}v), d(T^{m-j}v, T^{m-j+1}v)}{d(T^{m-j-1}v, T^{m-j}v)}, \frac{d(T^{m-j}v, T^{m-j+1}v)}{d(T^{m-j-1}v, T^{m-j}v)} \right\} = \max \{d(T^{m-j-1}v, T^{m-j}v), d(T^{m-j}v, T^{m-j+1}v)\} \quad (11)
\]

and (4) becomes,

\[
\psi(d(T^{m-j}v, T^{m-j+1}v)) = \phi(M_1(T^{m-j}v, T^{m-j+1}v)) = \phi(\max \{d(T^{m-j}v, T^{m-j+1}v)\}). \quad (12)
\]

Taking into account \((f'_6)\), it follows that:

\[
\psi(d(T^{m-j}v, T^{m-j+1}v)) \leq \phi(\max \{d(T^{m-j}v, T^{m-j+1}v)\}) < \psi(\max \{d(T^{m-j}v, T^{m-j+1}v)\}).
\]

Now, since the function \( \psi \) is nondecreasing, we get:

\[
d(T^{m-j}v, T^{m-j+1}v) < \max \{d(T^{m-j}v, T^{m-j+1}v)\}
\]

This leads us to:

\[
d(T^{m-j}v, T^{m-j+1}v) < d(T^{m-k}v, T^{m-k}v),
\]

for every \( k = j, j + 1, ..., m - 1 \). Taking in the above inequality \( j = 0 \) and \( k = m - 1 \), we get:

\[
d(v, T^m v) = d(T^m v, T^{m+1}v) < d(v, T^m v).
\]

This is a contradiction. Consequently, \( T^m v = v \). \( \square \)

**Example 2.** Let the set \( X = [0, 2] \) be endowed with the usual distance \( d(x, y) = |x - y| \) for every \( x, y \in X \).

Let the mapping \( T : X \to X \) be defined by \( TX = \begin{cases} 0, & \text{if } x \in [0, 1] \\ 0.5, & \text{if } x \in (1, 2]. \end{cases} \) It is clear that the mapping \( T \) is not continuous and that Theorem 4 cannot be applied. However, we have that \( T^2x = 0 \) for any \( x \in X \), so the assumption \((f'_5)\) holds. Choosing, for example, the functions \( \psi, \phi : (0, \infty) \to \mathbb{R} \), where \( \psi(s) = e^s \) and \( \phi(s) = s + \frac{1}{2} \), we have that the assumptions \((f'_6)\) - \((f'_2)\) are also satisfied, and we need to check if the inequality (4) holds for all distinct \( x, y \in X \) with \( d(Tx, Ty) > 0 \).
Of course, since \( \phi(s) = s + 1 \) is an increasing function, for \( x \in [0, 1] \) and \( y \in (1, 2] \), we have:

\[
\psi(d(Tx, Ty)) = \psi \left( \frac{1}{2} \right) = \sqrt{2} < 1 + \frac{1}{2} < y + \frac{1}{2} = \phi \left( \left| y - \frac{1}{2} \right| \right) = \phi (d(y, Ty)) \leq \phi (M_2(x, y))
\]

so that all the assumptions of Theorem 5 are satisfied.

**Definition 2.** Let \((X, d)\) be a complete metric space. The mapping \( T : X \to X \) is said to be a \((\psi, \phi)\)-rational contraction of Type 2 if for all \( x, y \in X \) with \( d(Tx, Ty) > 0 \), the following condition is satisfied:

\[
\psi(d(Tx, Ty)) \leq \phi(M_2(x, y)), \quad (13)
\]

where \( M_2 \) is defined by:

\[
M_2(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Ty)(d(x, Tx) + 1)}{1 + d(x, y)} \right\}. \quad (14)
\]

**Theorem 6.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a \((\psi, \phi)\)-rational contraction of Type 2. Assume that:

1. \( (f'_1) \) \( \psi \) is non-decreasing and lower semi-continuous;
2. \( (f_4) \) \( \lim_{s \to s_0^+} \phi(s) < \psi(s_0+) \);

Then, \( T \) admits exactly one fixed point.

**Proof.** Let \( \{x_n\} \) be the sequence defined by (6). Thus, by similar reasoning, we have that \( \zeta_n = d(x_{n-1}, x_n) > 0 \) for every \( n \in \mathbb{N} \). Therefore, since \( d(Tx, Ty) > 0 \), for every \( n \in \mathbb{N} \), for \( x = x_{n-1} \) and \( y = x_n \), we have:

\[
M_2(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_n, Ty_n)(1+d(x_n, Ty_n))}{1+d(x_n, Ty_n)} \right\}
\]

\[
= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)(d(x_{n-1}, x_n) + 1)}{1+d(x_{n-1}, x_n)} \right\}
\]

\[
= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\}
\]

Consequently, by (13), we have:

\[
\psi(d(Tx_{n-1}, Ty_n)) \leq \phi(M_2(x_{n-1}, x_n)) = \phi(\max \{\zeta_n, \zeta_{n+1}\}),
\]

which, keeping in mind \( (f_4) \), is equivalent to:

\[
\psi(\zeta_{n+1}) \leq \phi(\max \{\zeta_n, \zeta_{n+1}\}) < \phi(\max \{\zeta_n, \zeta_{n+1}\}). \quad (15)
\]

Thus, due to the monotony of the function \( \psi \), \( \zeta_{n+1} < \max \{\zeta_n, \zeta_{n+1}\} \), so that \( 0 < \zeta_{n+1} < \zeta_n \), for each \( n \in \mathbb{N} \), then there exists \( \zeta \geq 0 \) such that \( \zeta_n \searrow \zeta \). We claim that \( \zeta = 0 \). If we assume by contradiction that \( \zeta > 0 \), we have:

\[
\psi(\zeta) \leq \psi(\zeta_{n+1}) \leq \phi(\zeta_n) < \psi(\zeta_n).
\]
which is a contradiction. Thus, we have:

\[
\zeta = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]  

(16)

Now, we claim that \( \{ x_n \} \) is a Cauchy sequence. Again, arguing by contradiction, by Lemma (1), we have that there exist \( \epsilon > 0 \) and the sequences of positive real numbers \( (q_k) \) and \( (r_k) \) such that:

\[
\lim_{k \to \infty} d(x_{q_k+1}, x_{q_k+1}) = \epsilon + \quad \text{and} \quad \lim_{k \to \infty} d(x_k, x_k) = \epsilon.
\]  

(17)

Thus, \( d(x_{q_k+1}, x_{q_k+1}) = d(Tx_{q_k}, Tx_{q_k}) > \epsilon > 0 \) for all \( k \geq 1 \), and from (13), together with (6) we have:

\[
\psi(d(x_{q_k+1}, x_{q_k+1})) \leq \psi(M_2(x_{q_k}, x_{q_k})) < \psi(M_2(x_{q_k}, x_{q_k})).
\]  

(18)

Since \( \psi \) is non-decreasing we get \( d(x_{q_k+1}, x_{q_k+1}) < M_2(x_{q_k}, x_{q_k}) \), for each \( k \geq 1 \), where:

\[
M_2(x_k, x_k) = \max \left\{ d(x_k, x_k), d(x_k, x_{k+1}), d(x_k, x_{k+1}), \frac{d(x_k, x_{k+1})(1 + d(x_k, x_{k+1}))}{1 + d(x_k, x_k)} \right\}
\]  

(19)

and taking into account (16) and (17):

\[
\lim_{k \to \infty} M_2(x_k, x_k) = \epsilon + .
\]

In this case, letting \( k \to \infty \) in (18), we have:

\[
\psi(\epsilon +) = \lim_{k \to \infty} \psi(d(x_{q_k+1}, x_{q_k+1})) \leq \limsup_{k \to \infty} \psi(M_2(x_{q_k}, x_{q_k})) \leq \limsup_{s \to \epsilon +} \psi(s) < \psi(\epsilon +),
\]

which is a contradiction. This shows that \( \{ x_n \} \) is a Cauchy sequence. By the completeness of the space \((x, d)\), the sequence \( \{ x_n \} \) converges to a point \( v \) in \( x \), that is:

\[
\lim_{n \to \infty} d(x_n, v) = 0.
\]  

(20)

We claim that \( v \) is a fixed point of \( T \). Supposing by contradiction that \( d(Tv, v) > 0 \) and using the same arguments as in the previous theorem, we have that there exists \( n_0 \in \mathbb{N} \) such that \( d(Tv, x_{n_0+1}) = d(Tv, Tx_{n_0}) > 0 \) for any \( n \geq n_0 \). Now, by (13) we have:

\[
\psi(d(Tv, Tx_{n_0})) \leq \psi(M_2(v, x_{n_0})),
\]  

(21)

where:

\[
M_2(v, x_{n_0}) = \max \left\{ d(v, x_{n_0}), d(v, Tv), d(x_{n_0}, x_{n_0+1}), \frac{d(x_{n_0}, x_{n_0+1})(1 + d(v, Tv))}{1 + d(v, x_{n_0})} \right\}.
\]

On the one hand, from (16) and (20), we get:

\[
M_2(v, x_{n_0}) = d(v, Tv), \quad \text{for } n \text{ sufficiently large}
\]  

(22)

and then:

\[
\psi(d(Tv, Tx_{n_0})) \leq \psi(d(v, Tv)) < \psi(d(v, Tv)).
\]
On the other hand, \( \lim_{n \to \infty} d(Tv, T\omega_0) = \lim_{n \to \infty} d(Tv, x_{n+1}) = d(Tv, v) \). Therefore, taking the inferior limit in (21) when \( n \to \infty \) and taking into account the lower semi-continuity of \( \psi \), we have:

\[
\liminf_{s \to d(Tv, v)} \psi(s) \leq \lim_{n \to \infty} \psi(Tv, T\omega_0) \leq \phi(d(v, Tv)) < \psi(d(v, Tv)) < \liminf_{s \to d(Tv, v)} \psi(s),
\]

which is a contradiction. Therefore, we have \( Tv = v \), and we claim that this is the unique fixed point of \( T \). If we suppose that \( \tilde{v} \) is also a fixed point of \( T \) such that \( d(Tv, T\tilde{v}) = d(v, \tilde{v}) > 0 \) and from (13), we have:

\[
\psi(d(Tv, T\tilde{v})) \leq \phi(M_2(v, \tilde{v})) = \phi(d(v, \tilde{v})) < \psi(d(v, \tilde{v})),
\]

which is a contradiction. \( \square \)

**Example 3.** Let \( X = \{\omega_1, \omega_2, \omega_3, \omega_4\} \) and \( d : X \times X \to [0, \infty) \) be a distance defined as follows:

\[
d(x, y) = d(y, x), \text{ for every } x, y \in X;
\]

\[
d(\omega_1, \omega_2) = 2, \; d(\omega_1, \omega_3) = 6, \; d(\omega_1, \omega_4) = 7;
\]

\[
d(\omega_2, \omega_3) = 4, \; d(\omega_2, \omega_4) = 5, \; d(\omega_3, \omega_4) = 1.
\]

Let the mapping \( T : X \to X \), with \( T\omega_1 = \omega_4 \), \( T\omega_2 = T\omega_3 = T\omega_4 = \omega_3 \). Letting \( \psi, \phi : (0, \infty) \to \mathbb{R} \), where \( \psi(s) = e^s \) and \( \phi(s) = 1 + \ln(1 + s) \), we have that the assumptions \((f_1), (f'_1), (f_4)\) are satisfied. We have to consider the following cases:

a. If \( x = \omega_1, \ y = \omega_2 \), then \( d(T\omega_1, T\omega_2) = d(\omega_4, \omega_3) = 1, \ d(\omega_1, \omega_2) = 2, \ d(\omega_1, T\omega_1) = 7, \ d(\omega_2, T\omega_2) = 4, \) and \( M_2(\omega_1, \omega_2) = \max \{2, 7, 4, \frac{32}{3}\} = \frac{32}{3} \):

\[
\psi(d(T\omega_1, T\omega_2)) = \psi(1) = e < 1 + \ln \frac{32}{3} < \phi(M_2(\omega_1, \omega_2)).
\]

b. If \( x = \omega_1, \ y = \omega_3 \), then \( d(T\omega_1, T\omega_3) = d(\omega_4, \omega_3) = 1, \ d(\omega_1, \omega_3) = 6, \ d(\omega_1, T\omega_1) = 7, \ d(\omega_3, T\omega_3) = 0, \) and \( M_2(\omega_1, \omega_3) = \max \{6, 7, 0, \frac{2}{3}\} = 7 \):

\[
\psi(d(T\omega_1, T\omega_3)) = \psi(1) = e < 1 + \ln 7 < \phi(M_2(\omega_1, \omega_3)).
\]

c. If \( x = \omega_1, \ y = \omega_4 \), then \( d(T\omega_1, T\omega_4) = d(\omega_4, \omega_3) = 1, \ d(\omega_1, \omega_4) = 7, \ d(\omega_1, T\omega_1) = 7, \ d(\omega_4, T\omega_4) = 1, \) and \( M_2(\omega_1, \omega_3) = \max \{7, 7, 1, \frac{14}{3}\} = 7 \):

\[
\psi(d(T\omega_1, T\omega_4)) = \psi(1) = e < 1 + \ln 7 < \phi(M_2(\omega_1, \omega_4)).
\]

Thus, all the assumptions of Theorem 6 hold, so that \( T \) has a unique fixed point.

**Theorem 7.** A \((\psi, \phi)\)-rational contraction of Type 2 on the complete metric space \((X, d)\) has a unique fixed point presuming that the following conditions are satisfied:

\((f_1)\) \( \lim_{s \to s_0} \psi(s) > -\infty \), for any \( s_0 > 0 \);

\((f'_1)\) \( \limsup_{s \to s_0} \phi(s) < \liminf_{s \to s_0} \psi(s) \);
we conclude that \( \varsigma \) is a contradiction, so that, for all \( n \in \mathbb{N} \), we have:
\[
\psi(\xi_{n+1}) \leq \phi(\xi_n) < \psi(\xi_n),
\]
for all \( n \in \mathbb{N} \). Then, the sequence \( \{\psi(\xi_n)\} \) is decreasing and also bounded (because \((f_i)\) and \(\xi_n > \varsigma\)). Therefore, the sequence \( \{\psi(\xi_n)\} \) is convergent, and moreover, by the above inequality, the sequence \( \{\phi(\xi_n)\} \) is also convergent to the same limit. Thus, keeping in mind \((f_2)\), we have:
\[
\liminf_{s \to \varsigma^+} \psi(s) = \lim_{n \to \infty} \psi(\xi_n) = \lim_{s \to \varsigma^+} \phi(s) < \liminf_{s \to \varsigma^+} \psi(s),
\]
which is a contradiction, so that,
\[
\varsigma = \lim_{n \to \infty} d(x_{n-1}, x_n) = 0. \tag{24}
\]

We will show that \( \{x_n\} \) is a Cauchy sequence. In order to prove that, arguing by contradiction, by Lemma 1, there exist \( \varepsilon > 0 \) and \((q_k), (r_k)\) two sequences of positive integers such that \((3)\) holds. Since \( \lim_{k \to \infty} d(x_{q_k+1}, x_{r_k+1}) = \varepsilon + \), we have that \( d(x_{q_k+1}, x_{r_k+1}) = d(Tq_k, Tr_k) > 0 \), and replacing in \((13)\), we get:
\[
\psi(d(x_{q_k+1}, x_{r_k+1})) = \psi(d(Tq_k, Tr_k)) < \phi(M_2(x_{q_k}, x_{r_k})).
\]

On the other hand, from the above inequality and \((f_2)\), we have:
\[
\liminf_{s \to \varepsilon} \psi(s) \leq \liminf_{k \to \infty} \psi(d(x_{q_k+1}, x_{r_k+1})) \leq \limsup_{k \to \infty} \phi(M_2) \leq \limsup_{s \to \varepsilon^+} \phi(s) < \liminf_{s \to \varepsilon^+} \psi(s).
\]

This is a contradiction, so that \( \{x_n\} \) is a Cauchy sequence, so it is convergent to some point \( v \in X \) (due to the completeness of the metric space \((X, d)\)). If we suppose that \( d(Tv, v) > 0 \), because \( d(Tv, Tr_n) \to d(Tv, v) \), we have that there exists \( n_0 \in \mathbb{N} \) such that \( d(Tv, Tn) > 0 \), for \( n \geq n_0 \). Then, from \((13)\),
\[
\psi(d(Tv, Tn)) \leq \phi(M_2(v, x_n)) = \phi \left( \max \{d(v, x_n), d(v, Tv), d(x_n, Tn), \frac{d(x_n, Tn)(1 + d(v, Tv))}{d(v, x_n)} \} \right)
\]
and moreover, taking into account \((22)\):
\[
\psi(d(Tv, Tn)) \leq \phi(d(v, Tv)).
\]

Taking the limit as inferior and using \((f_5)\), we obtain:
\[
\liminf_{s \to d(Tv, v)} \psi(s) \leq \liminf_{n \to \infty} \psi(d(Tv, x_n+1)) \leq \phi(d(v, Tv)) < \liminf_{s \to d(Tv, v)} \psi(s).
\]

This is a contradiction. Therefore, \( Tv = v \), that is \( v \) is a fixed point of \( T \), and using the same arguments as in Theorem 6, we have that, in fact, this fixed point is unique. \( \Box \)
Example 4. Let $X = [0, ∞)$ and $d$ be the usual distance on $X$. Let $T : X → X$, where $T x = \frac{1}{2} \ln(x^2 + x + 2)$ and $ψ, φ : (0, ∞) → \mathbb{R}$, $ψ(s) = e^s$ and $φ(s) = 1 + s$. We check that $T$ is a $(ψ, φ)$-rational contraction of Type 2. Indeed, if $x > y$ (and it is analogues for the case $x < y$), then:

$$d(Tx, Ty) = \frac{\ln(x^2 + x + 2) - \ln(y^2 + y + 2)}{2} = \frac{1}{2} \ln \frac{x^2 + x + 2}{y^2 + y + 2} = \ln \sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}}$$

On the other hand, since:

$$\sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}} \leq 1 + x - y ⇔ \frac{x^2 + x + 2}{y^2 + y + 2} \leq (1 + x - y)^2 ⇔ (1 + y^2)(x - y) + y^2 + xy + 3 \geq 0,$$

we obtain:

$$ψ(d(Tx, Ty)) = \sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}} \leq 1 + x - y = 1 + d(x, y) \leq 1 + M_2(x, y) = φ(M_2(x, y)).$$

Thus, (13) is satisfied, and by Theorem 7, we have that the mapping $T$ has a fixed point.

Definition 3. Let $(X, d)$ be a complete metric space. The mapping $T : X → X$ is said to be a $(ψ, φ)$-rational contraction of Type 3 if for all $x, y \in X$, when $\max \{d(x, Ty), d(y, Tx)\} \neq 0$, then $d(Tx, Ty) > 0$, and the following condition is satisfied:

$$ψ(d(Tx, Ty)) ≤ ψ(d(x, Ty)) + d(y, Ty) \frac{d(y, Tx)}{\max \{d(x, Ty), d(y, Tx)\}};$$

(25)

if $\max \{d(x, Ty), d(y, Tx)\} = 0$, then $d(Tx, Ty) = 0$.

Theorem 8. Let $(X, d)$ be a complete metric space and $T : X → X$ be a $(ψ, φ)$-rational contraction of Type 3. The mapping $T$ admits exactly one fixed point provided that:

$(f'_n)$ $ψ$ is non-decreasing and $\limsup_{n→∞} φ(s) < ψ(s_0+)$, for any $s_0 > 0$.

Proof. Let $\{x_n\}$ be the sequence defined by (6). Thus, by similar reasoning, we have that $x_n = d(x_{n-1}, x_0) > 0$ for every $n \in \mathbb{N}$. Therefore, since $d(Tx_{n-1}, Tx_n) > 0$, for every $n \in \mathbb{N}$, for $x = x_{n-1}$ and $y = x_0$, by (25), we have:

$$ψ(d(x_n, x_{n+1})) = ψ(d(Tx_{n-1}, Tx_n)) ≤ ψ(d(x_{n-1}, x_0)) = ψ(d(x_n, x_0))$$

which, keeping in mind $(f_0)$, gives us:

$$ψ(d(x_n, x_{n+1})) ≤ ψ(d(x_n, x_{n-1})) < ψ(d(x_0, x_1)).$$

(26)

Thus, from $(f'_n)$, $0 < d(x_0, x_{n+1}) < d(x_{n-1}, x_0)$ for each $n \in \mathbb{N}$, so the sequence $(d(x_n, x_{n+1}))$ is convergent to some $ζ ≥ 0$. We claim that $ζ = 0$. In the case that $ζ > 0$, from (25),

$$ψ(d(x_n, x_{n+1})) ≤ ψ(d(x_n, x_0)) < ψ(d(x_0, x_1)).$$
Taking the limit as superior in the above inequality and keeping in mind \( (f''_k) \), we get:

\[
\psi(\xi +) = \lim_{n \to \infty} \psi(d(x_0, x_{n+1})) \leq \limsup_{n \to \infty} \phi(d(x_{n-1}, x_0)) < \limsup_{n \to \infty} \psi(d(x_{n-1}, x_0)) < \psi(\xi +).
\]

This is a contradiction, and then, we have:

\[
\lim_{n \to \infty} d(x_0, x_{n+1}) = \xi = 0.
\] (27)

Now, we claim that \( \{x_0\} \) is a Cauchy sequence. Again, arguing by contradiction, by Lemma (1), we have that there exist \( \varepsilon > 0 \) and the sequences of positive real numbers \( (q_k) \) and \( (r_k) \) such that:

\[
\lim_{k \to \infty} d(x_{q_k+1}, x_{r_k+1}) = \varepsilon + \quad \text{and} \quad \lim_{k \to \infty} d(x_{q_k}, x_{r_k}) = \varepsilon.
\] (28)

Thus, it follows that \( d(x_{q_k+1}, x_{r_k+1}) = d(Tx_{q_k}, Tx_{r_k}) > \varepsilon > 0 \) for all \( k \geq 1 \), and from (25), together with (f0), we have:

\[
\psi(d(x_{q_k+1}, x_{r_k+1})) \leq \phi \left( \frac{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})}{\max\{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})\}} \right) < \phi \left( \frac{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})}{\max\{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})\}} \right)
\]

\[
= \phi \left( \frac{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})}{\max\{d(q_k, q_{k+1}) + d(q_{k+1}, q_{k+2}) + d(q_{k+2}, q_{k+3})\}} \right) = \phi \left( d(x_{q_k}, x_{q_{k+1}}) + d(x_{q_k}, x_{q_{k+1}}) \right) < \psi(d(x_{q_k}, x_{q_{k+1}}) + d(x_{q_k}, x_{q_{k+1}})).
\] (29)

Since \( \psi \) is non-decreasing, we get:

\[
d(x_{q_k+1}, x_{r_k+1}) < d(x_{q_k}, x_{q_{k+1}}) + d(x_{q_k}, x_{q_{k+1}}),
\]

for each \( k \geq 1 \).

Taking into account (27) and (28):

\[
0 < \varepsilon = \lim_{k \to \infty} d(x_{q_k+1}, x_{r_k+1}) < \lim_{k \to \infty} (d(x_{q_k}, x_{q_{k+1}}) + d(x_{q_k}, x_{q_{k+1}})) = 0.
\]

In this case, we get \( \varepsilon = 0 \), which shows us that \( \{x_0\} \) is a Cauchy sequence, and by the completeness of the space \( (X, d) \), \( (x_0) \) converges to a point \( v \) in \( x \), that is:

\[
\lim_{n \to \infty} d(x_0, v) = 0.
\] (30)

We claim that \( v \) is a fixed point of \( T \). Supposing by contradiction that \( d(Tv, v) > 0 \) and using the same arguments as in the previous theorem, we have that there exists \( n_0 \in \mathbb{N} \) such that \( d(Tv, x_{n+1}) = d(Tv, x_{q_k}) > 0 \) for any \( n \geq n_0 \). Now, by (25), we have:

\[
\psi(d(Tv, x_{n+1})) \leq \phi \left( \frac{d(v, Tv) + d(v, x_{n+1}) + d(x_{n+1}, x_{q_k})}{\max\{d(v, Tv) + d(v, x_{n+1}) + d(x_{n+1}, x_{q_k})\}} \right) < \phi \left( \frac{d(v, Tv) + d(v, x_{n+1}) + d(x_{n+1}, x_{q_k})}{\max\{d(v, Tv) + d(v, x_{n+1}) + d(x_{n+1}, x_{q_k})\}} \right)
\] (31)
Now, from \((f''_1)\), we have:

\[
0 < d(Tv, T\eta_0) < \frac{d(v, Tv)d(v, x_{n+1}) + d(x_n, x_{n+1})d(x_n, Tv)}{\max\{d(v, T\eta_0), d(x_n, Tv)\}}
\]

and letting \(n \to \infty\), we get \(0 < \lim_{n \to \infty} d(Tv, T\eta_0) < 0\), which is a contradiction. Therefore, we have \(Tv = v\). Finally, we claim that this is the unique fixed point of \(T\). If we suppose that \(\bar{v}\) is also a fixed point of \(T\) such that \(d(Tv, T\bar{v}) = d(v, \bar{v}) > 0\) and from (25): we have:

\[
\psi(d(Tv, T\bar{v})) = \psi\left(\frac{d(v, Tv)d(v, \bar{v}) + d(\bar{v}, T\bar{v})d(\bar{v}, Tv)}{\max\{d(v, T\bar{v}), d(\bar{v}, Tv)\}}\right)< \psi\left(\frac{d(\bar{v}, T\bar{v})d(\bar{v}, Tv)}{\max\{d(\bar{v}, Tv), d(T\bar{v}, Tv)\}}\right),
\]

Thus, by \((f''_1)\),

\[
0 < d(v, \bar{v}) < \frac{d(v, Tv)d(v, \bar{v}) + d(\bar{v}, T\bar{v})d(\bar{v}, Tv)}{\max\{d(v, T\bar{v}), d(\bar{v}, Tv)\}} = 0,
\]

which is a contradiction. \(\square\)

We can state many corollaries from our main results. For example, choosing \(\psi(s) = s\) and \(\phi(s) = \beta(s)\) in Theorem 4, we have:

**Corollary 1.** Let \((x, d)\) be a complete metric space and \(\beta : (0, \infty) \to (0, 1)\) be a function such that \(\limsup_{s \to s_0 +} \beta(s) < 1\) for every \(s_0 > 0\). A continuous mapping \(T : X \to X\) has a unique fixed point provided that:

\[
d(Tx, Ty) \leq \beta(M_1(x, y))M_1(x, y), \quad \text{for all } x, y \in X \text{ with } d(Tx, Ty) > 0.
\]

If in Theorem 7, we take \(\psi(s) = \kappa\psi(s)\), we get the following corollary.

**Corollary 2.** Let \((x, d)\) be a complete metric space and a self-mapping \(T\) on \(X\) such that for all \(x, y \in X\) with \(d(Tx, Ty) > 0\),

\[
\psi(d(Tx, Ty)) \leq \kappa\psi(M_2(x, y)),
\]

where \(\kappa \in [0, 1)\), \(\psi : (0, \infty) \to (0, \infty)\) is a nondecreasing and left-continuous function, and \(M_2\) is defined by (14). Then, \(T\) admits a unique fixed point.

Letting \(\psi(s) = \psi(s) - \tau\) in Theorem 8, we obtain an improvement of Theorem 3.1 in [12].

**Corollary 3.** Let \((x, d)\) be a complete metric space and a mapping \(T : X \to X\) such that there exist \(\tau > 0\) and a nondecreasing and left-continuous function \(\psi : (0, \infty) \to \mathbb{R}\) such that for all \(x, y \in X\) if \(\max\{d(x, Ty), d(Tx, y)\} \neq 0\), then \(d(x, y) > 0\):

\[
\tau + \psi(d(Tx, Ty)) \leq \psi\left(\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(y, Tx)\}}\right)
\]

and if \(\max\{d(x, Ty), d(Tx, y)\} \neq 0\), then \(d(x, y) = 0\). Then, \(T\) has a unique fixed point.

3. Conclusions

In this paper, we were interested in finding some conditions on the functions \(\psi\) and \(\phi\) that guarantee that \(T\) has a unique fixed point in terms of rational expression. Our main results offered improvements to known results by applying weaker conditions on the self-map of a complete metric space. Here we mentioned just one corollary for each type of \((\psi, \phi)\)-rational contraction by choosing
different functions $\psi$ and $\phi$, but it is clear that many similar consequences can be listed, consequences that actually represent independent results.

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