On Borel Equivalence Relations Related To Self-Adjoint Operators

Hiroshi Ando@ Yasumichi Matsuzawa

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Abstract

In a recent work, we initiated the study of Borel equivalence relations defined on the Polish space \(SA(H)\) of self-adjoint operators on a Hilbert space \(H\), focusing on the difference between bounded and unbounded operators. In this paper, we extend the analysis and show how the difficulty of specifying the domains of self-adjoint operators is reflected in Borel complexity of associated equivalence relations. More precisely, we show that the equality of domains, regarded as an equivalence relation on \(SA(H)\) is continuously bireducible with the orbit equivalence relation of the standard Borel group \(\ell^\infty(N)\) on \(\mathbb{R}^N\). Moreover, we show that generic self-adjoint operators have purely singular continuous spectrum equal to \(\mathbb{R}\).

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1 Introduction

In the recent paper [AM14], the authors have studied Borel complexity of various equivalence relations defined on the space \(SA(H)\) of all (not necessarily bounded) self-adjoint operators on a separable and infinite-dimensional Hilbert space \(H\) equipped with the strong resolvent topology (SRT). One major difference between bounded and unbounded operators is that due to the domain problems, \(SA(H)\) is not even a vector space: recall that the sum of self-adjoint operators \(A,B\) is defined as the operator \(C\) with \(\text{dom}(C) = \text{dom}(A) \cap \text{dom}(B)\) and \(C\xi := A\xi + B\xi, \xi \in \text{dom}(C)\). In general, there is no reason to expect that \(C\) is densely defined even if \(\text{dom}(A), \text{dom}(B)\) are dense. In fact, Israel [Isr04] has shown that if \(A \in SA(H)\) has empty essential spectrum, then the set of all unitaries \(u\) satisfying \(\text{dom}(A) \cap u \cdot \text{dom}(A) = \{0\}\) forms a norm dense \(G_\delta\) subset of the unitary group \(U(H)\). Thus \(\text{dom}(A + uAu^*) = \{0\}\) for norm-generic \(u\). Therefore, it is natural to expect that the domain equivalence relation

\[
AE_{\text{dom}}^{SA(H)} B \iff \text{dom}(A) = \text{dom}(B)
\]

has a high degree of complexity. In this paper, we determine its exact Borel complexity by showing that \(E_{\text{dom}}^{SA(H)}\) is an \(F_\sigma\) (but not \(K_\sigma\)) equivalence relation, and that it is continuously bireducible (see §2 for the definition) with the \(\ell^\infty(N,\mathbb{R})\)-orbit equivalence relation \(E_{\ell^\infty}^{\mathbb{R}^N}\) defined on \(\mathbb{R}^N\) by

\[
(a_n)_{n=1}^\infty E_{\ell^\infty}^{\mathbb{R}^N} (b_n)_{n=1}^\infty \iff \sup_{n \in \mathbb{N}} |a_n - b_n| < \infty, \quad (a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \in \mathbb{R}^N.
\]
Since Rosendal [Ros05, Proposition 19] has shown that \( E^{\mathbb{R}^n}_{\infty} \) is universal for \( K_\sigma \)-equivalence relations, \( E^{\mathbb{SA}(H)}_{\dom} \) also enjoys this property. Moreover, since by this universality the notorious \( K_\sigma \) equivalence relation \( E_1 \) (see §3) is Borel reducible to \( E^{\mathbb{SA}(H)}_{\dom} \), \( E^{\mathbb{SA}(H)}_{\dom} \) is not Borel reducible to any orbit equivalence relation of a Borel action of a Polish group, by the Kechris-Louveau Theorem [KL97, Theorem 4.2]. Moreover, we show that the related equivalence relation \( E^{\mathbb{SA}(H)}_{\dom,u} \) (unitary equivalence of domains) given by

\[
AE^{\mathbb{SA}(H)}_{\dom,u} B \Leftrightarrow \exists u \text{ unitary } [u \cdot \dom(A) = \dom(B)]
\]

is Borel reducible to a \( K_\sigma \) equivalence relation, whence \( E^{\mathbb{SA}(H)}_{\dom,u} \leq_B E^{\mathbb{SA}(H)}_{\dom} \) as a corollary.

Finally, we strengthen our previous genericity result [AM14, Theorem 3.17 (1)] that elements in \( \mathbb{SA}(H) \) which have essential spectrum \( \mathbb{R} \), form a dense \( G_\delta \) set. Namely we prove that elements in \( \mathbb{SA}(H) \) which have purely singular continuous spectrum \( \mathbb{R} \), forms a dense \( G_\delta \) set in \( \mathbb{SA}(H) \). This shows that although every self-adjoint operator can be approximated by diagonal operators (Weyl-von Neumann Theorem), generic self-adjoint operators have rather pathological spectral properties (cf. [CN98, LPS05]). The proof is based on Simon’s Wonderland Theorem [Sim95].

2 Preliminaries

We refer the reader to [AM14, §2] for relevant definitions and notation. Basic facts about operator theory (resp. descriptive set theory) can be found in [Sch10] (resp. in [Gao09, Hjo00, Kec96]). Below we give some definitions here for convenience. Let \( H \) be a separable infinite-dimensional Hilbert space.

**Definition 2.1.** The strong resolvent topology (SRT) on the space \( \mathbb{SA}(H) \) of all self-adjoint operators on \( H \) is the coarsest topology which makes the map \( \mathbb{SA}(H) \ni A \mapsto (A - i)^{-1} \in \mathbb{B}(H) \) continuous with respect to the strong operator topology (SOT).

\( \mathbb{SA}(H) \) is Polish with respect to SRT. The domain of \( A \in \mathbb{SA}(H) \) is written as \( \dom(A) \).

**Definition 2.2.** Let \( E \) (resp. \( F \)) be equivalence relations on a Polish space \( X \) (resp. \( Y \)). We say that \( E \) is Borel (resp. continuously reducible to \( F \), denoted \( E \leq_B F \) (resp. \( E \leq_c F \)), if there is a Borel (resp. continuous) map \( f : X \to Y \) which is a reduction of \( E \) to \( F \) (i.e., \( xEy \leftrightarrow f(x)Ff(y) \) holds for \( x, y \in X \)). If moreover \( f \) is injective, we say that \( E \) is Borel (resp. continuously embeddable into \( F \), denoted \( E \subseteq_B F \) (resp. \( E \subseteq_c F \)). We say that \( E \) is Borel (resp. continuously bireducible with \( F \), if \( E \leq_B F \) and \( F \leq_B E \) (resp. \( E \leq_c F \) and \( F \leq_c E \)) hold. In this case we write \( E \sim_B F \) (resp. \( E \sim_c F \)).

In the next section we consider the following three equivalence relations.

**Definition 2.3.** We define \( E^{\mathbb{R}^n}_{\infty} \), \( E^{\mathbb{SA}(H)}_{\dom} \) and \( E^{\mathbb{SA}(H)}_{\dom,u} \) by:

1. The equivalence relation \( E^{\mathbb{R}^n}_{\infty} \) on the Polish space \( \mathbb{R}^n \) is the orbit equivalence relation of the action of the standard Borel group \( \ell^\infty = \ell^\infty(\mathbb{N}) \) on \( \mathbb{R}^n \) by addition. In other words, we have \((a_n)_{n=1}^{\infty}E^{\mathbb{R}^n}_{\infty} (b_n)_{n=1}^{\infty} \Leftrightarrow \sup_{n\in\mathbb{N}} |a_n - b_n| < \infty \) for \((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \in \mathbb{R}^n \).

2. The equivalence relation \( E^{\mathbb{SA}(H)}_{\dom} \) on \( \mathbb{SA}(H) \) is given by \( AE^{\mathbb{SA}(H)}_{\dom} B \Leftrightarrow \dom(A) = \dom(B) \).
(3) The equivalence relation $E_{dom,u}^{SA(H)}$ on $SA(H)$ is given by $AE_{dom,u}^{SA(H)} B \iff \exists u \in \mathcal{U}(H) \ [u \cdot \ \text{dom}(A) = \text{dom}(B)]$.

We also recall a result on operator ranges. Recall that a subspace $\mathcal{R} \subset H$ is an operator range in $H$, if $\mathcal{R}$ is equal to the range $\text{Ran}(T)$ for some $T \in \mathcal{B}(H)$. We may choose $T$ to be self-adjoint with $0 \leq T \leq 1$. In this case, we set $H_n := E_T(\{2^{-n-1}, 2^{-n}\})H$ ($n = 0, 1, \cdots$). Then $H_n$ are mutually orthogonal closed subspaces of $H$ with $H = \bigoplus_{n=0}^{\infty} H_n$ (by the density of $\mathcal{R}$). $\{H_n\}_{n=0}^{\infty}$ are called the associated subspaces for $T$ (see [FW71, §3] for details). Since we are only concerned with dense operator ranges, we state the following result [FW71, Theorem 3.3] for dense operator ranges (in this case the condition (1) of the cited theorem is automatic).

**Theorem 2.4 (Köthe, Fillmore-Williams).** Let $\mathcal{R}$ and $\mathcal{S}$ be dense operator ranges in $H$ with associated subspaces $\{H_n\}_{n=0}^{\infty}$ and $\{K_n\}_{n=0}^{\infty}$, respectively. Then there exists $u \in \mathcal{U}(H)$ such that $u\mathcal{R} = \mathcal{S}$, if and only if there exists $k \geq 0$ such that for each $n \geq 0$ and $l \geq 0$, one has

$$\dim(H_n \oplus \cdots \oplus H_{n+l}) \leq \dim(K_{n-k} \oplus \cdots \oplus K_{n+l+k}),$$

$$\dim(K_n \oplus \cdots \oplus K_{n+l}) \leq \dim(H_{n-k} \oplus \cdots \oplus H_{n+l+k}),$$

where we use the convention $H_m = K_m = \{0\}$ for $m < 0$.

Finally, for $A \in SA(H)$, we denote by $\sigma_p(A)$, $\sigma_{ac}(A)$ and $\sigma_{sc}(A)$ the set of eigenvalues, absolutely continuous spectrum, and singular continuous spectrum of $A$, respectively (see [RNS1, §VII 2]). We put $\sigma_{ac}(A) = \emptyset$ (resp. $\sigma_{sc}(A) = \emptyset$) if there is no absolutely continuous part (resp. singular continuous part) of $A$, and we say that $A$ has purely singular continuous spectrum, if $\sigma_p(A) = \emptyset = \sigma_{ac}(A)$ holds.

### 3 Main Results

Now we state the main result.

**Theorem 3.1.** $E_{dom}^{SA(H)}$ is an $F_\sigma$ equivalence relation which is continuously bireducible with $E_{\ell^\infty}^{\mathbb{R}}$.

Before going to the proof, let us state an immediate corollary. We need two important results. Recall that a subspace of a topological space is called $K_\sigma$ or $\sigma$-compact, if it is a countable union of compact subsets. First, Rosendal [Ros95, Proposition 19] has shown that

**Theorem 3.2 (Rosendal).** $E_{\ell^\infty}^{\mathbb{R}}$ is universal for $K_\sigma$ equivalence relations in the sense that any $K_\sigma$ equivalence relation on a Polish space is Borel reducible to $E_{\ell^\infty}^{\mathbb{R}}$.

Secondly, recall the $K_\sigma$ equivalence relation $E_1$ on $\mathcal{C}^N$ (where $\mathcal{C} = \mathcal{O}^N$) defined by

$$(a_n)_{n=1}^{\infty} E_1 (b_n)_{n=1}^{\infty} \iff \exists N \in \mathbb{N} \ \forall n \geq N \ [a_n = b_n].$$

Since $\mathcal{C}$ and $\mathbb{R}$ are Borel isomorphic, $E_1$ may alternatively be defined (when talking about Borel reducibility) as the tail equivalence relation on $\mathbb{R}^N$. Kechris-Louveau [KL97, Theorem 4.2] has shown that $E_1$ is an obstruction for a given equivalence relation to be Borel reducible to orbit equivalence:

**Theorem 3.3 (Kechris-Louveau).** $E_1 \not\leq_B E_G^N$ for any Polish group $G$ and Polish $G$-space $X$.  

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Here, $E^X_{\mathcal{K}}$ stands for the orbit equivalence relation associated with the Borel $G$-action. Since there are many orbit equivalence relations that are turbulent (in the sense of [Hjo00]) and Borel reducible to $E^\mathbb{R}_N$, (e.g. $P^\mathbb{R}(1 \leq p < \infty)$ actions on $\mathbb{R}^\mathbb{N}$), Theorems 3.1, 3.2 and 3.3 imply that:

**Corollary 3.4.** $E^\text{SA}(H)_{\text{dom}}$ is universal for $K_\sigma$-equivalence relations. In particular, it is unclassifiable by countable structures, not Borel reducible to orbit equivalence relation of any Polish group action.

Now we prove Theorem 3.1 in few steps.

**Proposition 3.5.** $E^\text{SA}(H)_{\text{dom}}$ is an $F_\sigma$ equivalence relation which is not $K_\sigma$.

The proof relies on Douglas’ range inclusion Theorem [Don66] (cf. [FW71, Theorem 2.1]).

**Theorem 3.6** (Douglas). Let $A, B \in \mathbb{B}(H)$. Then $\text{Ran}(A) \subset \text{Ran}(B)$ holds if and only if there exists $\lambda > 0$ such that $AA^* \leq \lambda B B^*$.

**Proof of Proposition 3.5.** It is clear that $\tau : \text{SA}(H)^2 \ni (A, B) \mapsto (B, A) \in \text{SA}(H)^2$ is a homeomorphism. Define $S := \{(A, B) \in \text{SA}(H)^2; \text{dom}(A) \subset \text{dom}(B)\}$. Since $E^\text{SA}(H)_{\text{dom}} = S \cap \tau(S)$, it suffices to show that $S$ is $F_\sigma$ in $\text{SA}(H)^2$. For $A, B \in \text{SA}(H)$, we have $\text{dom}(A) = \text{Ran}((|A| + 1)^{-1})$, $\text{dom}(B) = \text{Ran}((|B| + 1)^{-1})$. Therefore Theorem 3.6 shows that

$$\text{dom}(A) \subset \text{dom}(B) \iff \exists \lambda > 0 \ [ (|A| + 1)^{-2} \leq \lambda(|B| + 1)^{-2} ]$$

$$\iff \exists k \in \mathbb{N} \forall \xi \in H \ [ \langle \xi, (|A| + 1)^{-2}\xi \rangle \leq k \langle \xi, (|B| + 1)^{-2}\xi \rangle ] .$$

Therefore $S = \bigcup_{k \in \mathbb{N}} \bigcap_{\xi \in H} S_{k, \xi}$, where $S_{k, \xi} := \{(A, B); \langle \xi, (|A| + 1)^{-2}\xi \rangle \leq k \langle \xi, (|B| + 1)^{-2}\xi \rangle \}$.

It is easy to see that $\text{SA}(H) \ni A \mapsto (|A| + 1)^{-2} \in \mathbb{B}(H)$ is SRT-SOT continuous, hence each $S_{k, \xi}$ is SRT-closed. Therefore $S$ is $F_\sigma$. The last assertion follows from the fact that $\text{SA}(H)$ is not $K_\sigma$ (it contains a homeomorphic copy of $\mathbb{R}^\mathbb{N}$) and a well-known fact: note that if an equivalence relation $E$ on a Polish space $X$ is $K_\sigma$, then $X$ must be $K_\sigma$. \hfill $\blacksquare$

**Proof of Theorem 3.1** $E^\text{SA}(H)_{\text{dom}}$ is $F_\sigma$ but not $K_\sigma$ by Proposition 3.5. We show that $E^\text{SA}(H)_{\text{dom}}$ is continuously bireducible with $E^\mathbb{R}_N$. We first show that $E^\text{SA}(H)_{\text{dom}} \leq_c E^\mathbb{R}_N$. Fix a dense countable subset $\{\xi_n\}_{n=1}^\infty$ of $H$. Given $A \in \text{SA}(H)$, define $T_A := (|A| + 1)^{-2}$. Since $T_A$ is positive and 0 is not an eigenvalue for $T_A$, $\langle \xi_n, T_A \xi_n \rangle > 0$ for every $n \in \mathbb{N}$. Moreover, $A \mapsto T_A$ is SRT-SOT continuous by functional calculus. Therefore we may define a continuous map $\varphi : \text{SA}(H) \to \mathbb{R}^\mathbb{N}$ by

$$\varphi(A) := (a_n(A))_{n=1}^\infty, \quad a_n(A) := \log(\langle \xi_n, T_A \xi_n \rangle), \quad A \in \text{SA}(H), \ n \in \mathbb{N}.$$

We show that $\varphi$ is a reduction map. Let $A, B \in \text{SA}(H)$. By the proof of Proposition 3.5, we have

$$\text{dom}(A) = \text{dom}(B) \iff \exists C_1 > 0 \exists C_2 > 0 \ [ C_1 T_B \leq T_A \leq C_2 T_B ]$$

$$\iff \exists C_1 > 0 \exists C_2 > 0 \forall n \in \mathbb{N} \ [ C_1 \langle \xi_n, T_B \xi_n \rangle \leq \langle \xi_n, T_A \xi_n \rangle \leq C_2 \langle \xi_n, T_B \xi_n \rangle ]$$

$$\iff \exists C_1 > 0 \exists C_2 > 0 \forall n \in \mathbb{N} \ [ \log C_1 \leq a_n(A) - a_n(B) \leq \log C_2 ]$$

$$\iff \sup_{n \in \mathbb{N}} |a_n(A) - a_n(B)| < \infty$$

$$\iff \varphi(A) E^\mathbb{R}_N \varphi(B).$$
which shows that $E_{\text{dom}}^{SA(H)} \leq c E_{\text{dom}}^{R^N}$.

Next we show that $E_{\text{dom}}^{R^N} \leq c E_{\text{dom}}^{SA(H)}$. The proof is similar to the first part. Fix a complete orthonormal system (CONS) $\{\eta_n\}_{n=1}^\infty$ for $H$. For each $(x_n)_{n=1}^\infty \in \mathbb{R}^N$, define $(\tilde{x}_n)_{n=1}^\infty \in \mathbb{R}_{\geq 0}^N$ by

$$(\tilde{x}_{2n-1}, \tilde{x}_{2n}) = \begin{cases} (|x_n|, 0) & (x_n \geq 0) \\ (0, |x_n|) & (x_n < 0) \end{cases}, \quad n \in \mathbb{N}.$$ Thus $(1, \frac{1}{2}, 4, 0, \cdots)$ is mapped to $(0, 0, \frac{1}{2}, 4, 0, 0, \cdots)$, etc. It is easy to see that $\mathbb{R}^N \ni (x_n)_{n=1}^\infty \mapsto (\tilde{x}_n) \in \mathbb{R}^N_{\geq 0}$ is an injective continuous map satisfying

$$\sup_{n \in \mathbb{N}} |x_n - y_n| < \infty \iff \sup_{n \in \mathbb{N}} |\tilde{x}_n - \tilde{y}_n| < \infty, \quad (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in \mathbb{R}^N. \quad (1)$$

We define $\psi: \mathbb{R}^N \to SA(H)$ by

$$\psi(\alpha) := \sum_{n=1}^\infty \{\exp(\frac{1}{2} \tilde{x}_n) - 1\} (\eta_n, \cdot) \eta_n, \quad \alpha = (x_n)_{n=1}^\infty \in \mathbb{R}^N.$$ It is easy to see that $\psi$ is continuous, and

$$T_\psi(\alpha) = (\psi(\alpha) + 1)^{-2} = \sum_{n=1}^\infty \exp(-\tilde{x}_n) (\eta_n, \cdot) \eta_n, \quad \alpha = (x_n)_{n=1}^\infty \in \mathbb{R}^N.$$ We show that $\psi$ is a reduction map. Given $\alpha = (x_n)_{n=1}^\infty, \beta = (y_n)_{n=1}^\infty \in \mathbb{R}^N$, we have (by (1))

$$\text{dom}(\psi(\alpha)) = \text{dom}(\psi(\beta)) \iff \exists C_1 > 0 \exists C_2 > 0 [ C_1 T_\psi(\alpha) \leq T_\psi(\beta) \leq C_2 T_\psi(\beta) ]$$

$$\iff \exists C_1 > 0 \exists C_2 > 0 \forall n \in \mathbb{N} [ C_1 \exp(-\tilde{y}_n) \leq \exp(-\tilde{x}_n) \leq C_2 \exp(-\tilde{y}_n) ]$$

$$\iff \sup_{n \in \mathbb{N}} |\tilde{y}_n - \tilde{x}_n| < \infty$$

$$\iff \alpha E_{\text{dom}}^{R^N} \beta,$$ whence $E_{\text{dom}}^{R^N} \leq c E_{\text{dom}}^{SA(H)}$. This shows that $E_{\text{dom}}^{R^N}$ is continuously bireducible with $E_{\text{dom}}^{SA(H)}$. \qed

As another corollary to Theorem 3.1 we prove that $E_{\text{dom},u}^{SA(H)} \leq_B E_{\text{dom}}^{SA(H)}$. This is done by showing that $E_{\text{dom},u}^{SA(H)}$ is Borel reducible to a $K_\sigma$ equivalence relation. Regard $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ as a one-point compactification of $\mathbb{N} = \{1, 2, \cdots\}$. Thus $\mathbb{N}^*$ is homeomorphic to $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ by $n \mapsto \frac{1}{n}$ ($n \in \mathbb{N}$) and $\infty \mapsto 0$. Consider the compact Polish space $X := \prod_{n=0}^\infty (\mathbb{N}^* \cup \{0\})$, and define $X_0 := \{(a_n)_{n=0}^\infty \in X ; \sum_{n=0}^\infty a_n = \infty\}$. Then $X_0$ is a (dense) $G_\delta$ subspace of $X$, whence Polish.

**Definition 3.7.** Define an equivalence relation $E_\Sigma$ on $X$ by $(a_n)_{n=0}^\infty E_\Sigma (b_n)_{n=0}^\infty$ if and only if there exists $k \geq 0$ such that for each $l \geq 0$ and $n \geq 0$,

$$\sum_{i=0}^{l} a_{n+i} \leq \sum_{j=-k}^{l+k} b_{n+j} \quad \text{and} \quad \sum_{i=0}^{l} b_{n+i} \leq \sum_{j=-k}^{l+k} a_{n+j}.$$ Here, we regard $a_n = b_n = 0$ ($n < 0$) and $\infty + n = n + \infty = \infty + \infty = \infty$ ($n \in \mathbb{N}$).
Proposition 3.8. \( E_\Sigma \) is a \( K_\sigma \) equivalence relation, and \( E_{\text{dom},u}^{SA(H)} \sim_B E_\Sigma \upharpoonright X_0 \ (\leq_B E_\Sigma) \). In particular, \( E_{\text{dom},u}^{SA(H)} \) is Borel reducible to a \( K_\sigma \) equivalence relation.

We omit the proof of the next easy lemma.

Lemma 3.9. For \( n,m \in \mathbb{N} \cup \{0\} \{n \leq m\} \), the map \( X \ni (a_k)_{k=0}^\infty \mapsto \sum_{k=n}^m a_k \in \mathbb{N}^* \) is continuous.

Lemma 3.10. Let \( a,b \in \mathbb{R}, a < b \), and let \( I = (a,b), [a,b) \) or \( (a,b) \). Then the map \( \text{SA}(H) \ni A \mapsto \text{rank}(E_A(I)) \in \mathbb{N}^* \) is Borel.

Proof. We show the case of \( I = [a,b) \). Let \( S_n := \{A \in \text{SA}(H); \text{rank}(E_A([a,b))) \leq n\} \ (n \in \mathbb{N} \cup \{0\}) \). Then by a similar argument to the proof of [AM14 Proposition 3.18] (especially that \( S_n \) is \( \text{SRT}\)-closed), we can show that \( S_n \) is \( \text{SRT}\)-closed. Therefore \( \{A \in \text{SA}(H); \text{rank}(E_A((a,b))) = n\} = S_n \setminus S_{n-1} \ (n \geq 1) \) and \( S_0 \) are Borel. Then \( S_\infty = \text{SA}(H) \setminus \bigcup_{n \geq 0} S_n \) is Borel too. Thus the map \( A \mapsto \text{rank}(E_A(I)) \) is Borel.

Proof of Proposition 3.8. It is easy to see that \( \text{dom}(A) = \text{dom}([|A|+1) \) for every \( A \in \text{SA}(H) \), and \( \text{dom}(A) = \text{Ran}((|A|+1)^{-1}) \). The associated subspaces for \( T_A = (|A|+1)^{-1} \) are

\[ H_n(T_A) = E_{T_A}((2^{-n-1}, 2^n]), \quad n \geq 0. \]

Note that for \( \lambda \in \sigma(A) \),

\[ (|\lambda| + 1)^{-1} \in (2^{-n-1}, 2^n) \Leftrightarrow \lambda \in (1 - 2^{n+1}, 1 - 2^n) \cup [2^n - 1, 2^{n+1} - 1) \]

Let \( d_0(A) := \text{rank}(E_A(-1,1)) \) and \( d_n(A) := \dim H_n(T_A) = \text{rank}(E_A(I_n)) + \text{rank}(E_A(J_n)) \ (n \geq 1) \). By Lemma 3.10, \( d_n: \text{SA}(H) \to \mathbb{N}^* \) is Borel for each \( n \geq 0 \).

Now, note that \( E_\Sigma = \bigcup_{k=0}^\infty E_k \), where

\[ E_k := \bigcap_{l,n=0}^\infty \left\{ (a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty; \sum_{i=0}^l a_{n+i} \leq \sum_{j=-k}^{l+k} b_{n+j} \quad \text{and} \quad \sum_{i=0}^l b_{n+i} \leq \sum_{j=-k}^{l+k} a_{n+j} \right\}. \]

It is immediate to see that \( E_\Sigma \) is \( K_\sigma \) because each \( E_k \) is a closed subset of the compact space \( X \times X \) by Lemma 3.9. Define a Borel map \( \varphi: \text{SA}(H) \to X_0 \) by \( \varphi(A) := (d_n(A))_{n=0}^\infty \). Since \( H \) is infinite-dimensional, \( \varphi(A) \in X_0 \). Moreover, \( A E_{\text{dom},u}^{SA(H)} B \) if and only if \( \varphi(A) E_\Sigma \varphi(B) \) by Theorem 2.4. Therefore \( E_{\text{dom},u}^{SA(H)} \leq_B E_\Sigma \upharpoonright X_0 \leq_B E_\Sigma \). To show \( E_\Sigma \upharpoonright X_0 \leq_B E_{\text{dom},u}^{SA(H)} \), let

\[ X_{0,k} := \{(a_n)_{n=0}^\infty \in X_0; \sharp\{n \in \mathbb{N} \cup \{0\}; a_n = \infty\} = k\}, \quad k \in \mathbb{N}^* \cup \{0\}. \]

Note that each \( X_{0,k} \) is a Borel subset of \( X_0 \): it is enough to show that \( X_{0,k} \) is a closed subset of \( X_0 \). But if \( a_i = (a_n)_{n=0}^\infty \in X_{0,k} \) tends to \( \alpha = (a_n)_{n=0}^\infty \in X_0 \), then if \( a_{n_1} = \cdots = a_{n_p} = \infty (n_1 < n_2 < \cdots < n_p) \), then by assumption there exists \( k \) such that for each \( i \geq k \)

\[ a_{n_i} = \cdots = a_{n_p} = \infty, \quad \text{so} \quad p \leq k. \]

Therefore \( \alpha \in X_{0,k} \), and \( X_{0,k} \) is closed.

Now define for each \( k \in \mathbb{N}^* \cup \{0\} \) a Borel map \( \psi_k: X_{0,k} \to \text{SA}(H) \) by the following:

Case \( k = 0 \). Fix a CONS \( \{\xi_n\}_{n=1}^{\infty} \) for \( H \). For \( \alpha = (a_n)_{n=0}^\infty \in X_{0,0} \), define

\[ \psi_0(\alpha) := \sum_{n=0}^{\infty} (2^\xi_n - 1) e_n(\alpha), \]

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where the projection $e_{n,0}(\alpha)$ is inductively defined as follows: $e_{0,0}(\alpha)$ is the projection onto $\text{span}\{\xi_1, \cdots, \xi_n\}$ (if $a_0 \geq 1$) and $e_{0,0}(\alpha) = 0$ otherwise, and for $k \geq 0$,

$$e_{k+1,0}(\alpha) := \text{projection onto span}\{\xi_{a_0+\cdots+a_k+1}, \xi_{a_0+\cdots+a_k+a_{k+1}}\} \text{ if } a_{k+1} \geq 1,$$

and $e_{k+1,0}(\alpha) := 0$ otherwise. Then it is easy to see that $\psi_0 : X_{0,0} \to \text{SA}(H)$ is continuous, and $T_{\psi_0}(\alpha) = \sum_{n=0}^{\infty} 2^{-n} e_{n,0}(\alpha)$. In particular, the rank of the associated subspace for $T_{\psi_0}(\alpha)$ is $d_n(\psi(\alpha)) = a_n$ ($n \geq 0$).

**Case 1** $1 \leq k \leq \infty$.

Let $\alpha = (a_n)_{n=0}^{\infty} \in X_{0,k}$, and suppose that $a_{n_1} = \cdots = a_{n_k} = \infty (n_1 < \cdots < n_k)$ (for $k = \infty$ case this means that $n_1 < n_2 < \cdots$ is an infinite sequence) and $a_n < \infty$ ($n \notin \{n_1, \cdots, n_k\}$).

Fix another CONS $\{\eta_n, \zeta_{p,n}; n \geq 1, 1 \leq p \leq k\}$ for $H$, and define $\psi_k(\alpha) \in \text{SA}(H)$ by

$$\psi_k(\alpha) := \sum_{n=0}^{\infty} (2^k - 1) e_{n,k}(\alpha),$$

where the projection $e_{n,k}(\alpha)$ is defined as follows: define $(b_n)_{n=0}^{\infty} \in X_0$ inductively by

$$b_0 := \begin{cases} a_0 & (a_0 < \infty) \\ 0 & (a_0 = \infty) \end{cases}, \quad b_{k+1} := \begin{cases} b_k + a_{k+1} & (a_{k+1} < \infty) \\ b_k & (a_{k+1} = \infty) \end{cases}, \quad k \geq 0,$$

and then put $e_{0,k}(\alpha) = \text{projection onto span}\{\eta_1, \cdots, \eta_n\}$ if $a_0 < \infty$, and $e_{0,k}(\alpha) := \text{projection onto span}\{\xi_{i,j}\}_{i=1}^{n} \text{ if } a_0 = \infty$. For $n \geq 1$, put

$$e_{n,k}(\alpha) := \begin{cases} \text{projection onto span}\{\eta_{b_{n-1}+1}, \cdots, \eta_b\} & (0 < a_n < \infty) \\ \text{projection onto span}\{\zeta_{p,i}\}_{i=1}^{n} & (n = n_p) \end{cases}.$$  

Again $\psi_k : X_{0,k} \to \text{SA}(H)$ is continuous, and $d_n(\psi_k(\alpha)) = a_n$ ($n \geq 0$).

Finally define $\psi : X_0 \to \text{SA}(H)$ by $\psi | X_{0,k} := \psi_k$. Then since each $X_{0,k}$ is Borel and $\psi_k$ is continuous on $X_{0,k}$, $\psi$ is Borel. Moreover, since $d_n(\psi(\alpha)) = a_n (n \geq 0)$ for every $\alpha = (a_n)_{n=0}^{\infty} \in X_0$, it follows that $a \Sigma_\beta \Leftrightarrow \psi(\alpha) E_{\text{SA}(H)}^{\text{SA}(H)} \psi(\beta)$ for $\alpha, \beta \in X_0$. This shows that $E_{\Sigma} | X_0 \leq_B E_{\text{dom},u}^{\text{SA}(H)}$. Therefore $E_{\Sigma} | X_0 \sim_B E_{\text{dom},u}^{\text{SA}(H)}$ holds.

**Corollary 3.11.** $E_{\text{dom},u}^{\text{SA}(H)} \leq_B E_{\text{dom}}^{\text{SA}(H)}$ holds.

**Proof.** By Proposition 3.8 Theorems 3.1 and 3.2 it holds that $E_{\text{dom},u}^{\text{SA}(H)} \leq_B E_{\text{dom}}^{\text{SA}(H)}$.

**Remark 3.12.** It is not clear whether $E_{\text{dom}}^{\text{SA}(H)} \leq_B E_{\text{dom},u}^{\text{SA}(H)}$ holds.

4 **Generic $A$ has purely singular continuous spectrum $\mathbb{R}$**

In [AMT14, Theorem 3.17 (1)], we have shown a genericity result that the set $\{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \mathbb{R}\}$ is dense $G_\delta$ in $\text{SA}(H)$. In this last section, we show that generic self-adjoint operators in fact have much more pathological spectral property:

**Theorem 4.1.** The set $\mathcal{G} := \{A \in \text{SA}(H); \sigma_p(A) = \sigma_{\text{ac}}(A) = \emptyset, \sigma_{\text{sc}}(A) = \mathbb{R}\}$ is dense $G_\delta$ in $\text{SA}(H)$.
The proof relies on the surprising theorem of Simon (which he calls “Wonderland Theorem”).

**Definition 4.2.** ([Sim95]) Let \((X,d)\) be a metric space of self-adjoint operators on \(H\). \(X\) is called a regular metric space, if \(d\) is complete and generates a topology stronger than or equal to SRT.

**Theorem 4.3** (Simon’s Wonderland Theorem). Let \((X,d)\) be a regular metric space of self-adjoint operators on \(H\). Suppose that for some open interval \((a,b)\),

1. \(\{A \in X; A\) has purely continuous spectrum on \((a,b)\}\) is dense in \(X\).
2. \(\{A \in X; A\) has purely singular spectrum on \((a,b)\}\) is dense in \(X\).
3. \(\{A \in X; A\) has \((a,b)\) in its spectrum\} is dense in \(X\).

Then \(\{A \in X; (a,b) \subset \sigma_{ac}(A), (a,b) \cap \sigma_p(A) = \emptyset, (a,b) \cap \sigma_{ac}(A) = \emptyset\}\) is dense \(G_\delta\) in \(X\).

First we prove the density.

**Proposition 4.4.** The set \(\{A \in \text{SA}(H); \sigma_p(A) = \sigma_{ac}(A) = \emptyset\}\) is dense in \(\text{SA}(H)\).

**Lemma 4.5.** Let \(H\) be an infinite-dimensional separable Hilbert space. There exists a sequence \(\{A_n\}_{n=1}^\infty \subset \text{SA}(H)\) with purely singular continuous spectrum, such that \(A_n \xrightarrow{\text{SRT}} 1_H\).

**Proof.** Let \(\mu\) be a singular continuous probability measure on \(\mathbb{R}\). We identify \(H = L^2(\mathbb{R}, \mu)\), and define \(A_n\) to be the multiplication by \(f_n\), where \(f_n(x) := \frac{1}{n} x + 1 \ (x \in \mathbb{R}, n \in \mathbb{N})\). Then each \(A_n\) has purely singular continuous spectrum, and \(A_n \xrightarrow{\text{SRT}} 1_H\) by Lebesgue Dominated Convergence Theorem. \(\square\)

**Proof of Proposition 4.4.** Let \(A \in \text{SA}(H)\) and let \(V\) be an SRT-open neighborhood of \(A\). By Weyl-von Neumann Theorem, there exists \(A_0 \in V\) of the form \(A_0 = \sum_{n=1}^\infty a_n \langle \xi_n, \cdot \rangle \xi_n\), where \(\{a_n\}_{n=1}^\infty \subset \mathbb{R}\) and \(\{\xi_n\}_{n=1}^\infty\) is an orthonormal basis for \(H\). Let \(e_n\) be the orthogonal projection of \(H\) onto \(\mathbb{C} \xi_n\) \((n \in \mathbb{N})\). Let \(k \in \mathbb{N}\). Choose a sequence of disjoint subsets \(I_1^{(k)}, I_2^{(k)}, \ldots, I_k^{(k)}\) of \(\mathbb{N} \setminus \{1, 2, \ldots, k\}\) such that \(|I_1^{(k)}| = |I_2^{(k)}| = \cdots = |I_k^{(k)}| = \infty\) and \(\mathbb{N} \setminus \{1, 2, \ldots, k\} = \bigsqcup_{i=1}^k I_i^{(k)}\). Then for each \(1 \leq i \leq k\), let \(e_i^{(k)}\) be the projection of \(H\) onto the closed linear span of \(\{\xi_m; m \in I_i^{(k)}\}\), which is of infinite-rank. Define a new operator \(A_k \in \text{SA}(H)\) by \(A_k := \sum_{n=1}^k a_n e_n + \sum_{n=1}^k a_n e_n^{(k)}\). Then \(A_k \xrightarrow{k \to \infty} A_0\) (SRT), so that there exists \(k_0 \in \mathbb{N}\) such that \(A_k \in V\) holds. Now let \(H_i\) \((1 \leq i \leq k_0)\) be the range of \(e_i + e_i^{(k_0)}\), which is infinite-dimensional. Thus by Lemma 4.3 we may find a sequence \(\{A_{i,m}\}_{m=1}^\infty \subset \text{SA}(H_i)\) with \(\sigma_p(A_{i,m}) = \sigma_{ac}(A_{i,m}) = \emptyset \ (m \in \mathbb{N})\) such that \(A_{i,m} \xrightarrow{m \to \infty} a_i 1_{H_i}\) (SRT) for each \(1 \leq i \leq k_0\). Let \(A_m := \bigoplus_{i=1}^{k_0} A_{i,m} \in \text{SA}(H)\) \((m \in \mathbb{N})\). It follows that \(A_m \xrightarrow{m \to \infty} A_{k_0} \in \mathcal{V}\) (SRT), so that there exists \(m_0 \in \mathbb{N}\) such that \(A_{m_0} \in V\). Since \(\sigma_p(A_{m_0}) = \sigma_{ac}(A_{m_0}) = \emptyset\) and \(V\) is arbitrary, the claim follows. \(\square\)

**Proof of Theorem 4.3.** For each \(n \in \mathbb{N}\) define

\[
G_n := \{A \in \text{SA}(H); \sigma_p(A) \cap (-n,n) = \sigma_{ac}(A) \cap (-n,n) = \emptyset, (-n,n) \subset \sigma_{ac}(A)\}.
\]

Since \(\mathcal{G} = \bigcap_{n \in \mathbb{N}} G_n\), it suffices to show that each \(G_n\) is dense \(G_\delta\) in \(\text{SA}(H)\). We see that assumptions of Theorem 4.3 are satisfied for \(X = \text{SA}(H)\) with \((a,b) = (-n,n)\):

1. and (2): the sets
\[
\{A \in \text{SA}(H); A\) has purely continuous spectrum on \((-n,n)\}\]
and
\[
\{ A \in \text{SA}(H) : A \text{ has purely singular spectrum on } (-n, n) \}
\]
are dense in \( \text{SA}(H) \), by Proposition 4.3

(3): By [AM14, Theorem 3.17 (1)], the set \( \text{SA}_{\text{full}}(H) = \{ A \in \text{SA}(H) : \sigma_{\text{ess}}(A) = \mathbb{R} \} \) is a dense \( G_\delta \) subset of \( \text{SA}(H) \). In particular, \( \{ A \in \text{SA}(H) : (-n, n) \subset \sigma(A) \} \) is dense in \( \text{SA}(H) \).
Therefore By Theorem 4.3, \( G_n \) is dense \( G_\delta \) in \( \text{SA}(H) \) for every \( n \in \mathbb{N} \), which finishes the proof.

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Hiroshi Ando  
Department of Mathematical Sciences,  
University of Copenhagen  
Universitetsparken 5  
2100 Copenhagen O Denmark  
ando@math.ku.dk  
[http://andonuts.miraiserver.com/index.html](http://andonuts.miraiserver.com/index.html)

Yasumichi Matsuzawa  
Department of Mathematics, Faculty of Education, Shinshu University  
6-Ro, Nishi-nagano, Nagano, 380-8544, Japan  
myasu@shinshu-u.ac.jp  
[https://sites.google.com/site/yasumichimatsuzawa/home](https://sites.google.com/site/yasumichimatsuzawa/home)