THE DEGREE OF $\mathbb{Q}$-FANO THREEFOLDS

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1. Introduction

In this paper a $\mathbb{Q}$-Fano variety is a normal projective variety $X$ with at worst $\mathbb{Q}$-factorial terminal singularities such that $-K_X$ is ample and Pic $X$ is of rank one. Fano varieties with terminal singularities form in important class because, according to the minimal model program, every variety of negative Kodaira dimension should be birationally equivalent to a fibration $Y \to Z$ whose general fibre $Y_\eta$ belong to this class. Moreover, in the case dim $Z = 0$, $Y_\eta = Y$ is of Picard number one, i.e., $Y$ is a $\mathbb{Q}$-Fano.

In dimension 2 the only $\mathbb{Q}$-Fano variety is the projective plane $\mathbb{P}^2$. In dimension 3 $\mathbb{Q}$-Fanos are bounded in the moduli sense by the following result of Kawamata:

(1.1) Theorem ([1]). There exist positive integers $r$ and $d$ such that for an arbitrary $\mathbb{Q}$-Fano threefold $X$ we have $-K_X^3 \leq d$ and $rK_X$ is Cartier.

Since the Weil divisor $-K_X$ gives a natural polarization of a $\mathbb{Q}$-Fano variety $X$, the rational number $-K_X^3$ is a very important invariant. It is called the degree of $X$. In this paper we find a sharp bound for $-K_X^3$:

(1.2) Theorem. Let $X$ be a $\mathbb{Q}$-Fano threefold. Assume that $X$ is not Gorenstein. Then $-K_X^3 \leq 125/2$ and the equality holds if and only if $X$ is isomorphic to the weighted projective space $\mathbb{P}(1,1,1,2)$.

Note that in the Gorenstein case we have the estimate $-K_X^3 \leq 64$ by the classification of Iskovskikh and Mori-Mukai and by Namikawa’s result [2].

The idea of the proof is as follows. In Sections 4 and 5 using Riemann-Roch formula for Weil divisors [3] and Kawamata’s estimates [1] we produce a short list of possibilities for singularities of $\mathbb{Q}$-Fanos of degree $\geq 125/2$. Here, to check a finite (but very huge) number of Diophantine conditions, we use a computer program (cf. [4]).
Section 6 we exclude all these possibilities except for $\mathbb{P}(1,1,1,2)$ by applying some birational transformations described in Section 3. The techniques used on this step is a very common in birational geometry (see [5], [6], [7]). It goes back to Fano-Iskovskikh “double projection method”. The present paper is a logical continuation of our previous papers [8], [9] where we studied effective bounds of degree for certain singular Fano threefolds.

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2. Preliminaries

Throughout this paper, we work over the complex number field $\mathbb{C}$.

(2.1) By $\text{Cl} X$ we denote the Weil divisor class group of a normal variety $X$ (modulo linear equivalence). There is a natural embedding $\text{Pic} X \hookrightarrow \text{Cl} X$. Let $X$ be a Fano variety with at worst log terminal singularities. It is well-known that both $\text{Pic} X$ and $\text{Cl} X$ are finitely generated and $\text{Pic} X$ is torsion free (see e.g. [10, §2.1]). Moreover, numerical equivalence of $\mathbb{Q}$-Cartier divisors coincides with $\mathbb{Q}$-linear one. Therefore one can define the following numbers:

\[
q_F(X) := \max\{q \mid -K_X \sim_q qH, \quad H \in \text{Pic} X\},
\]
\[
q_Q(X) := \max\{q \mid -K_X \sim_q qL, \quad L \in \text{Cl} X\},
\]
\[
q_W(X) := \max\{q \mid -K_X \sim qL, \quad L \in \text{Cl} X\}.
\]

By the above, all of them are positive, $q_Q(X), q_W(X) \in \mathbb{Z}$, and $q_F(X) \in \mathbb{Q}$. If $X$ is smooth all these numbers coincide with the Fano index of $X$. In general, we obviously have $q_Q(X) \geq q_F(X)$ and $q_Q(X) \geq q_W(X)$.

(2.1.1) Proposition (see e.g. [10, §2.1]). $q_F(X) \leq \dim X + 1$.

The index $q_W(X)$ was considered in [4]. In particular, it was proved that $q_W(X) \leq 19$ for any $\mathbb{Q}$-Fano threefold.

(2.2) Terminal singularities Let $(X, P)$ be a three-dimensional terminal singularity. It follows from the classification that there is a one-parameter deformation $\mathcal{X} \to \Delta \ni 0$ over a small disk $\Delta \subset \mathbb{C}$ such that the central fibre $\mathcal{X}_0$ is isomorphic to $X$ and the generic fibre $\mathcal{X}_\lambda$ has only cyclic quotient singularities $P_{\lambda,k}$ (see, e.g., [3]). Thus, to every threefold $X$ with terminal singularities, one can associate a collection $\mathcal{B} = \{(r_{P,k}, b_{P,k})\}$, where $P_{\lambda,k} \in \mathcal{X}_\lambda$ is a singularity of type...
\[ \frac{1}{r_{P,k}}(b_{P,k}, 1, -1), \ 1 \leq b_{P,k} \leq r_{P,k}/2, \ \gcd(r_{P,k}, b_{P,k}) = 1. \] This collection is uniquely determined by \( X \) and called the basket of singularities of \( X \). By abuse of notation, we also will write \( B = (r_{P,k}) \) instead of \( B = \{(r_{P,k}, b_{P,k})\} \). The index of \( P \) is the least common multiple of indices of points \( P_{\lambda,k} \).

**2.2.1 Lemma ([11 Corollary 5.2]).** Let \((X, P)\) be a three-dimensional terminal singularity of index \( r \) and let \( D \) be a Weil \( \mathbb{Q} \)-Cartier divisor on \( X \). There is an integer, \( i \) such that \( D \sim iK_X \) near \( P \). In particular, \( rD \) is Cartier.

**2.2.2 Corollary.** Let \( X \) be a Fano threefold with terminal singularities and let \( r \) be the Gorenstein index of \( X \). Then

(i) \( \gcd(r, qW(X)) = 1 \),

(ii) \( qF(X)r = qQ(X) \),

(iii) \( qW(X) \leq qQ(X) \leq 4r \).

**2.2.3** Let \((X, P)\) be a three-dimensional terminal singularity of index \( r \) and let \( D \) be a Weil \( \mathbb{Q} \)-Cartier divisor on \( X \). By Lemma (2.2.1) there is an integer \( i \) such that \( 0 \leq i < r \) and \( D \sim iK_X \) near \( P \). Deforming \( D \) with \((X, P)\) we obtain Weil divisors \( D_\lambda \) on \( X_\lambda \). Thus we have a collection of numbers \( i_k \) such that \( 0 \leq i_k < r_k \) and \( D_\lambda \sim i_kK_{X_\lambda} \) near \( P_{\lambda,k} \).

**2.3 Riemann-Roch formula [3].** Let \( X \) be a threefold with terminal singularities and let \( D \) be a Weil \( \mathbb{Q} \)-Cartier divisor on \( X \). Then

\[
\chi(D) = \frac{1}{12}D \cdot (D - K_X) \cdot (2D - K_X) + \frac{1}{12}D \cdot c_2 + \sum_{P \in B} c_P(D) + \chi(\mathcal{O}_X),
\]

where

\[
c_P(D) = -i_P \frac{r_P^2 - 1}{12r_P} + \sum_{j=1}^{i_P-1} \frac{b_{P,j}(r_P - b_{P,j})}{2r_P}.
\]

**2.4** Now let \( X \) be a Fano threefold with terminal singularities, let \( q := qQ(X) \), and let \( L \) be an ample Weil \( \mathbb{Q} \)-Cartier divisor on \( X \) such that \( -K_X \sim_q qL \). By (2.3.1) we have

\[
\chi(tL) = 1 + \frac{t(g + t)(g + 2t)}{12}L^3 + \frac{tL \cdot c_2}{12} + \sum_{P \in B} c_P(tL),
\]
\[ c_P(tL) = -i_P \frac{r_P^2 - 1}{12r_P} + \sum_{j=1}^{i_P} \frac{b_Pj(r_P - b_Pj)}{2r_P}. \]

If \( q > 2 \), then \( \chi(-L) = 0 \). Using this equality we obtain (see [4])

\[ L^3 = \frac{12}{(q-1)(q-2)} \left( 1 - \frac{L \cdot c_2}{12} + \sum_{P \in B} c_P(-L) \right). \]

\[ \text{(2.4.2)} \]

(2.5) In the above notation, applying (2.3.1), Serre duality and Kawamata-Viehweg vanishing to \( D = K_X \) we get the following important equality (see, e.g., [3]):

\[ 24 = -K_X \cdot c_2 + \sum_{P \in B} \left( r_P - \frac{1}{r_P} \right). \]

Similarly, for \( D = -K_X \) we have \( H^i(X, -K_X) = 0 \) for \( i > 0 \) and

\[ c_P(-K_X) = \frac{r_P^2 - 1}{12r_P} - \frac{b_P(r - b_P)}{2r_P}. \]

(see [5, §2]). Combining this with (2.5.1) we obtain

\[ \dim |-K_X| = -\frac{1}{2}K_X^3 + 2 - \sum_{P \in B} \frac{b_P(r_P - b_P)}{2r_P}. \]

In particular,

\[ \text{(2.5.3)} \]

\[ \text{(2.5.4) Theorem ([1], [12]).} \quad \text{In the above notation, } -K_X \cdot c_2(X) \geq 0. \]

As a corollary we have ([5, §2]):

\[ \dim |-K_X| \geq -\frac{1}{2}K_X^3 - 2. \]

\[ \text{(2.5.5)} \]

(2.5.6) Proposition ([5, §2]). Let \( X \) be a \( \mathbb{Q} \)-Fano threefold. If \( \dim |-K_X| \geq 2 \), then the linear system \( |-K_X| \) has no base components and is not composed of a pencil. (In particular, a general element of \( |-K_X| \) is reduced and irreducible.)

(2.6) Now let \( X \) be a \( \mathbb{Q} \)-Fano threefold, let \( q := q\mathbb{Q}(X) \), and let \( L \) be an ample Weil divisor on \( X \) that generates the group \( \text{Cl}X/Tors \). Let \( \mathcal{E} \) be the double dual to \( \Omega^1_X \). If \( \mathcal{E} \) is not semistable, there is a maximal destabilizing subsheaf \( \mathcal{F} \subset \mathcal{E} \). Clearly, \( c_1(\mathcal{F}) \equiv -pL \) for some \( p \in \mathbb{Z} \).
Put \( t := p/q \), so that \( c_1(F) \equiv tK_X \). According to [1] there are the following possibilities:

1. **(2.6.1) E is semistable.** Then \( -K_X^3 \leq -3K_X \cdot c_2(X) \).

2. **(2.6.2) E is not semistable and \( \text{rk} F = 2 \).** Then \( q \geq 2, \ 0 < t < 2/3 \), and
   \[
   t(4 - 3t)(-K_X^3) \leq -4K_X \cdot c_2(X).
   \]

3. **(2.6.3) E is not semistable, \( \text{rk} F = 1 \), and \((E/F)^*\) is semistable.** Then \( q \geq 4, \ 0 < t < 1/3 \), and
   \[
   (1 - t)(1 + 3t)(-K_X^3) \leq -4K_X \cdot c_2(X).
   \]

4. **(2.6.4) E is not semistable, \( \text{rk} F = 1 \), and \((E/F)^*\) is not semistable.** Then again \( q \geq 4 \) and \( 0 < t < 1/3 \). There exists an unstable reflexive sheaf \( F \subsetneq G \subsetneq E \). Write
   \[
   c_1(G/F) \equiv -p'L, \ p' \in \mathbb{Z}
   \]
   and put \( u := p'/q \), so that \( c_1(G/F) \equiv uK_X \). Then \( t < u \leq 1 - t - u \) and
   \[
   (tu + (t + u)(1 - t - u))(-K_X^3) \leq -K_X \cdot c_2(X),
   \]

**Corollary 2.7.** If \( qQ(X) = 1 \), then \( E \) is semistable. If \( qQ(X) \leq 3 \), then either \( E \) is semistable or we are in case (2.6.2).

### 3. Two birational constructions

1. **(3.1) Let \( X \) be a \( \mathbb{Q} \)-Fano threefold.** Throughout this paper we assume that the linear system \(| -K_X | \) is non-empty, has no fixed components, and is not composed of a pencil. Then a general member \( H \in | -K_X | \) is irreducible. By (2.5.5) and (2.5.6) this holds automatically when \( -K_X^3 \geq 8 \). Let \( q := qQ(X) \) and \( L \) be the ample Weil divisor that generates the group \( \text{Cl}X/\text{Tors} \). Thus we have \( -K_X \equiv qL \). Put \( \mathcal{H} := | -K_X | \). Let \( H \in \mathcal{H} \) be a general member.

2. **(3.2) Assume there is a diagram (Sarkisov link of type I or II)**

   \[
   \begin{array}{ccc}
   \tilde{X} & -\xrightarrow{X} & Y \\
   g \downarrow & & \downarrow f \\
   X & & Z
   \end{array}
   \]

   where \( \tilde{X} \) and \( Y \) have only \( \mathbb{Q} \)-factorial terminal singularities, \( \rho(\tilde{X}) = \rho(Y) = 2 \), \( g \) is a Mori extremal divisorial contraction, \( \tilde{X} \dashrightarrow Y \) is a sequence of log flips, and \( f \) is a Mori extremal contraction (either divisorial or fibre type). Thus one of the following holds: a) \( \dim Z = 1 \)
and $f$ is a $\mathbb{Q}$-del Pezzo fibration, b) $\dim Z = 2$ and $f$ is a $\mathbb{Q}$-conic bundle, or c) $\dim Z = 3$, $f$ is a divisorial contraction, and $Z$ is a $\mathbb{Q}$-Fano. Let $E$ be the $g$-exceptional divisor. We assume that the composition $f \circ \chi \circ g^{-1}$ is not an isomorphism. For a divisor $D$ on $X$, everywhere below $\tilde{D}$ and $D_Y$ denote strict birational transforms of $D$ on $\tilde{X}$ and $Y$, respectively. We also assume that the discrepancy $\alpha := a(E, X, \mathcal{H})$ is non-positive, i.e.,

\begin{equation}
0 \sim f^*(K_X + \mathcal{H}) = K_{\tilde{X}} + \tilde{\mathcal{H}} + \alpha E, \quad \alpha \in \mathbb{Z}, \quad \alpha \geq 0.
\end{equation}

By the above we have

\begin{equation}
\dim | - K_{\tilde{X}} | \geq \dim \tilde{\mathcal{H}} = \dim | - K_X |.
\end{equation}

(3.3) Similarly,

\begin{equation*}
0 \sim_q g^*(K_X + qL) \sim_q K_{\tilde{X}} + q\tilde{L} + \beta E.
\end{equation*}

Therefore,

\begin{equation}
K_Y + qL_Y + \beta E_Y \sim_q 0.
\end{equation}

If $q\mathbb{Q}(X) = qW(X)$, then $K_X + qL \sim 0$ and $\beta$ is an integer $\geq \alpha$.

Let $F = f^{-1}(pt)$ be a general fibre. Recall that $F$ is either $\mathbb{P}^1$ or a smooth del Pezzo surface. Restricting (3.3.1) to $F$ we get

\begin{equation}
K_F + qL_Y|_F + \beta E_Y|_F \sim 0.
\end{equation}

Here $-K_F$, $L_Y|_F$, and $E_Y|_F$ are proportional nef Cartier divisors. Moreover, $-K_F$ and $E_Y|_F$ are ample.

(3.4) We will use construction (3.2.1) in the following two situations:

(3.4.1) (see [6], [7]). Let $P \in X$ be a singularity of index $r$. Take $g$ to be a divisorial blowup of $P$ such that the discrepancy of the exceptional divisor $E$ is equal to $1/r$. Assume that the divisor $-K_{\tilde{X}}$ is nef, big and the linear system $| - nK_{\tilde{X}} |$ does not contract any divisors. Then the transformation in (3.2.1) is so-called “two rays game”. If $-K_{\tilde{X}}$ is ample, then $f \circ \chi$ is a composition of steps of the $K$-MMP. Otherwise, $f \circ \chi$ is a composition of a single flop followed by steps of the $K$-MMP. It is easy to see also that $f \circ \chi$ is an $-E$-MMP.

(3.4.2) (see [5]). The pair $(X, \mathcal{H})$ is not canonical. Let $c$ be the canonical threshold of $(X, \mathcal{H})$. Then $0 < c < 1$. Take $g$ to be an extremal divisorial $K_X + c\mathcal{H}$-crepant blowup. In this situation, $\alpha > 0$ and $f \circ \chi$ is an $K+c\mathcal{H}$-MMP. In particular, $f$ is an extremal $K_X + c\mathcal{H}$-negative contraction. The conditions of (3.2) are satisfied by [5].
(3.5) Properties of construction \((3.2)\).

(3.5.1) Claim.  \(E_Y\) is not contracted by \(f\).

Proof. Assume the converse, i.e., \(\dim f(E_Y) < \min(2, \dim Z)\). If \(f\) is birational, this implies that the map \(f \circ \chi \circ g^{-1} : X \rightarrow Z\) is an isomorphism in codimension one. Since both \(X\) and \(Z\) are Fano threefolds, this implies that \(f \circ \chi \circ g^{-1}\) is in fact an isomorphism. This contradicts our assumptions. If \(\dim Z \leq 2\), then \(E_Y\) is a pull-back of an ample Weil divisor on \(Z\). But then \(nE_Y\) is movable for some \(n > 0\). Again we derive a contradiction. \(\square\)

(3.5.2) Claim. For some \(n, m > 0\), there is a decomposition \(-nK_{\tilde{X}} \sim m\tilde{H} + M\), where \(|M|\) is a base point free linear system. In particular, \(|-nK_{\tilde{X}}|\) has no fixed components.

Proof. By \((3.2.2)\), for some \(0 < c \leq 1\), we have \(K_{\tilde{X}} + c\tilde{H} = g^*(K_X + cH)\). Hence we can take \(n, m > 0\) so that \(|-nK_{\tilde{X}} - m\tilde{H}|\) is base point free. \(\square\)

(3.5.3) Lemma \((13)\). If \(f\) is a \(\mathbb{Q}\)-conic bundle, then \(Z\) is a del Pezzo surface with at worst Du Val singularities of type \(A_n\) and \(\rho(Z) = 1\). Moreover, there is a natural embedding \(f^* : \text{Cl}\ Y \rightarrow \text{Cl}\ Z\).

Proof. The assertion about the base is an immediate consequence of the main result of \([13]\) and the fact that \(Z\) is uniruled. The last statement is obvious because both \(Y\) and \(Z\) have only isolated singularities and \(\text{Pic}(Z) = \mathbb{Z}\). \(\square\)

(3.5.4) Remark. (i) In the above notation the generic fibre of \(f\) is a smooth rational curve. The locus \(\Lambda := \{z \in Z \mid f\ \text{is smooth over} \ z\}\) is a closed subset of codimension \(\geq 1\) in \(Z\). The union of one-dimensional components of \(\Lambda\) is called the discriminant curve.

(ii) The classification of del Pezzo surfaces \(Z\) with Du Val singularities and \(\rho(Z) = 1\) is well-known. In particular, we always have \(K_Z^2 \leq 9\) and \(K_Z^2 \neq 7\). Moreover,

(i) if \(K_Z^2 = 9\), then \(Z \simeq \mathbb{P}^2\);
(ii) if \(K_Z^2 = 8\), then \(Z \simeq \mathbb{P}(1, 1, 2)\);
(iii) if \(K_Z^2 \leq 6\), then on \(Z\) there is a rational curve \(C\) such that \(-K_Z \cdot C = 1\).

(3.5.5) Lemma. Notation and assumptions as in \((3.2)\). Assume additionally that \(\dim |L| > 0\), \(q(X) \geq 4\) and \(f\) is not birational.
Then \( L_Y = f^* \Xi \) for some (integral) Weil divisor on \( Z \). Moreover, \( \dim |\Xi| = \dim |L| \) and the class of \( \Xi \) generates the group \( \mathrm{Cl} \, Z / \mathrm{Tors} \).

**Proof.** Since \( q \mathbb{Q}(X) \geq 4 \), relation (3.3.2) implies \( L_Y|_F = 0 \). Since \( f \) is a Mori contraction and \( Z \) is normal, \( L_Y = f^* \Xi \), where \( \Xi := f(L_Y) \). The rest is obvious. \( \square \)

(3.5.6) **Lemma.** Assume that \( (X, | - K_X|) \) is not canonical and we are applying construction (3.2). Further, assume that \( \dim Z = 2 \) and \( \alpha > 0 \). Then one of the following holds:

(i) \( \mathcal{H}_Y \) is \( f \)-ample. Then the discriminant curve of \( f \) is empty.

(ii) \( \mathcal{H}_Y \) is not \( f \)-ample. Then \( q \mathbb{Q}(X) \geq 7 \). Moreover, the equality \( \dim |K^2 \mathcal{H}_Y| = 35 \).

**Proof.** First we assume that \( \mathcal{H}_Y \) is \( f \)-ample. By (3.2.2) and Claim (3.5.1) \( E_Y \) and general elements of \( \mathcal{H}_Y \) are sections of \( f \). Hence \( f \) is smooth outside of a finite number of degenerate fibres.

Now we assume that \( \mathcal{H}_Y \) is not \( f \)-ample. Then \( \mathcal{H}_Y = f^* \mathcal{M} \), where \( \mathcal{M} \) is a linear system without fixed components. Let \( \Xi \) be an ample Weil divisor that generates \( \mathrm{Cl} \, Z / \mathrm{Tors} \). We can write \( \mathcal{M} \sim q \mathcal{X} \) and \( - K_Z \sim q' \Xi \), where \( q' := q \mathbb{Q}(Z) \), \( a \in \mathbb{Z} \). Clearly, \( q \mathbb{Q}(X) \geq a \).

By our assumption and by Reid’s Riemann-Roch formula [3] (9.1)],

\[
30 \leq \dim \mathcal{M} \leq \frac{1}{2} \mathcal{M} \cdot (\mathcal{M} - K_Z) + \sum c_P(\mathcal{M}) \leq \frac{a(a + q)}{2q} K_Z^2.
\]

Assume that \( a \leq 7 \). If \( K_Z^2 \leq 6 \), then \( q' = K_Z^2 \) by Remark (3.5.4). So, \( 60q' \leq a(a + q') \leq 49 + 7q' \), a contradiction. If \( K_Z^2 = 8 \), then \( q' = 4 \), so \( 120 \leq a(a + 4) \leq 77 \). Again we have a contradiction. Finally, let \( K_Z^2 = 9 \), i.e., \( Z \simeq \mathbb{P}^2 \). Then \( q' = 3 \), so \( 60 \leq a(a + 3) \leq 70 \). This inequality has only one solution: \( a = 7 \). But then \( q \mathbb{Q}(X) \leq 7 \). If \( q \mathbb{Q}(X) = 7 \), then \( a = 7 \), \( \mathcal{M} = |\mathcal{O}_{\mathbb{P}^2}(7)| \), and \( \dim \mathcal{M} = 35 \). \( \square \)

(3.5.7) **Lemma.** Notation and assumptions as in (3.2). Assume additionally that \( q \mathbb{Q}(X) = 1 \), \( Z \) is a surface, and the discriminant curve of \( f \) is empty. Then \( \dim |- K_X| < 30 \).

**Proof.** Suppose \( \dim |- K_X| \geq 30 \). Let \( \Gamma \subset Z \) is a smooth curve contained into the smooth locus of \( Z \). Then \( G := f^{-1}(\Gamma) \) is a smooth ruled surface over \( \Gamma \). We claim that \( \dim |- K_Y - G| \leq 0 \). Indeed, otherwise \( - K_Y \sim G + B \), where \( B \) is an integral effective divisor, \( \dim |B| \geq 1 \). Since \( q \mathbb{Q}(X) = 1 \), this gives a contradiction.

Now from (3.2.3) and from the exact sequence

\[
0 \rightarrow \mathcal{O}_Y(-K_Y - G) \rightarrow \mathcal{O}_Y(-K_Y) \rightarrow \mathcal{O}_G(-K_Y) \rightarrow 0
\]
we get \( h^0(\mathcal{O}_G(-K_Y)) \geq h^0(\mathcal{O}_Y(-K_Y)) - 1 \geq 30 \). It is easy to see that
\[
(-K_Y|_G)^2 = (-K_G + G|_G)^2 = K_G^2 - 2K_G \cdot G|_G = 8 - 8p_a(\Gamma) + 4\Gamma^2.
\]
By Claim (3.5.2) the linear system \(|-nK_Y|\) has no fixed components. Therefore we can take \( \Gamma \) so that \(|-nK_Y|_G\) has at worst isolated base points (in particular, it is nef). Moreover, \(|-nK_Y|_G\) is base point free for sufficiently large \( n \). If \(-K_Y|_G\) is ample, it is well-known that \( h^0(\mathcal{O}_G(-K_Y)) \leq (-K_Y|_G)^2 + 2 \) (see, e.g., [14]). If \(-K_Y|_G\) is not ample, we obtain the above inequality by applying the same arguments to \( \bar{G} \), where \( \bar{G} \) is the image of \( G \) under the birational contraction given by \(|-nK_Y|_G|\). In both cases we have
\[
8 - 8p_a(\Gamma) + 4\Gamma^2 = (-K_Y|_G)^2 \geq h^0(\mathcal{O}_G(-K_Y)) - 2 \geq 28.
\]
This gives us
\[
\Gamma^2 \geq 2p_a(\Gamma) + 5 = K_Z \cdot \Gamma + \Gamma^2 + 7, \quad -K_Z \cdot \Gamma \geq 7.
\]
If \( K_Z^2 < 8 \), then we can take \( \Gamma \) to be a general member of \(-K_Z\) and derive a contradiction. If \( K_Z^2 = 8 \) or 9, then we can take \( \Gamma \in |-\frac{1}{2}K_Z| \), or \(|-\frac{1}{3}K_Z|\), respectively. \( \square \)

(3.5.8) Lemma. If \( \dim Z = 1 \) and \( \dim |-K_X| \geq 30 \), then \( qQ(X) \geq 3 \).

Proof. Let \( F_1, F_2, F_3 \) be general Fibres. Then from the exact sequence
\[
0 \longrightarrow \mathcal{O}_Y(-K_Y - \sum F_i) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \bigoplus \mathcal{O}_{F_i}(-K_{F_i}) \longrightarrow 0
\]
we obtain
\[
h^0(-K_Y - \sum F_i) \geq h^0(-K_Y) - \sum h^0(-K_{F_i}).
\]
Since \( F_i \) are smooth del Pezzo surfaces, \( h^0(-K_{F_i}) = K_{F_i}^2 + 1 \leq 10 \). Hence, \( h^0(-K_Y - \sum F_i) > 0 \) by (2.5.5) and we have a decomposition \(-K_Y \sim \sum F_i + G\), where \( G \) is effective. Since \( F_i \) is movable, this gives us that \( qQ(X) \geq 3 \). \( \square \)

(3.6) Case: \( (X, |-K_X|) \) is canonical.

(3.6.1) Consider the case when \( (X, |-K_X| = \mathcal{H}) \) is canonical. According to [5] there is the following diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{g} & \bar{X} \\
 f & \downarrow & \\
 X & \rightarrow & Y \subset \mathbb{P}^n
\end{array}
\]
where \( g: (\tilde{X}, \tilde{\mathcal{H}}) \rightarrow (X, \mathcal{H}) \) is a terminal modification of \((X, \mathcal{H})\), \( n := \dim |-K_X| \), the morphism \( f \) is given by the (base point free) linear
system $\tilde{\mathcal{H}}$, $\dim Y = 2$ or 3, and $\tilde{X} \to \tilde{X} \to Y$ is the Stein factorization. We have

$$K_{\tilde{X}} + \tilde{\mathcal{H}} = g^*(K + \mathcal{H}) \sim 0.$$  

Since $(\tilde{X}, \tilde{\mathcal{H}})$ is terminal, a general member $\tilde{H} \in \tilde{\mathcal{H}}$ is a smooth K3 surface. From the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-K_{\tilde{X}}) \to \mathcal{O}_H(-K_{\tilde{X}}) \to 0$$

one can see that the restriction $f|_{\tilde{H}}$ is given by a complete linear system.

**3.6.2 Lemma.** Let $X$ be a $\mathbb{Q}$-Fano threefold. Assume that $(X, | - K_X| = \mathcal{H})$ is canonical and the image of the map given by $| - K_X|$ is a surface. If $\dim | - K_X| \geq 6$, then $2q\mathbb{Q}(X) \geq \dim | - K_X| - 1$.

**Proof.** We use notation of (3.6.1). By our assumption, $f(\tilde{H})$ is a curve. Thus $| - K_X|_{\tilde{H}}$ is a base point free elliptic pencil on $\tilde{H}$ and $f(\tilde{H}) \subset \mathbb{P}^n$ is a rational normal curve of degree $n - 1$. Hence $Y \subset \mathbb{P}^n$ is a surface of degree $n - 1$. Let $M$ be a hyperplane section of $Y$. It is well-known that in this situation one of the following holds (recall that $n \geq 6$):

(i) $Y$ is a rational scroll, $Y \simeq \mathbb{F}_e$, $M \sim \Sigma + al$, where $\Sigma$ and $l$ are the minimal section and a fibre of $\mathbb{F}_e$, respectively, and $a$ is an integer such that $a \geq e + 1$, $n - 1 = 2a - e$.

(ii) $Y$ is a cone over a rational normal curve of degree $n - 1$, $M \sim (n - 1)l$, where $l$ is a generator of the cone.

In case (i), $\tilde{\mathcal{H}} \sim f^*\Sigma + af^*l$. Here $|f^*l|$ is a linear system without fixed components and $f^*\Sigma$ is an effective divisor. So, $2q\mathbb{Q}(X) \geq 2a \geq n - 1$. In case (ii) we have $\tilde{\mathcal{H}} \sim f^*(n - 1)l$. Let $o \in Y$ be the vertex of the cone and let $G$ be the closure of $f^*l$ over $Y \setminus \{o\}$. Then $G$ is an integral Weil divisor and $\tilde{H} \sim_\mathbb{Q} (n - 1)G + T$, where $T$ is effective. Clearly, $g$ does not contract any component of $G$. This implies $q\mathbb{Q}(X) \geq n - 1$. \hfill \Box

Now assume that $\dim Y = 3$.

**3.6.3 Lemma (cf. [S] Corollary 1.8).** Let $X$ be a $\mathbb{Q}$-Fano threefold. Assume that $(X, | - K_X| = \mathcal{H})$ is canonical and the image of the map given by $| - K_X|$ is three-dimensional. Then $\dim | - K_X| \leq 37$. If moreover $q\mathbb{Q}(X) = 1$, then $\dim | - K_X| \leq 13$.

**Proof.** By the construction, $Y$ is a Fano threefold with canonical Gorenstein singularities and $Y \to Y \subset \mathbb{P}^N$ is the anticanonical map (see [S]). We have $\dim | - K_X| \leq \dim | - K_Y| \leq 38$ by the main result of [S]. Moreover, if $\dim | - K_X| = 38$, then $Y$ is isomorphic either $\mathbb{P}(3, 1, 1, 1)$ or $\mathbb{P}(6, 4, 1, 1)$. In particular, $Y$ is a toric variety. Since $\tilde{X}$ is a terminal modification of $Y$, it is also toric and so is $X$. By Lemma (3.6.4)
below \( \dim |-K_X| \leq \dim |-K_Y| \leq 33 \), a contradiction. If \( q(Q(X)) = 1 \), then \( -K_Y \) cannot be decomposed into a sum of two movable divisors. According to [15], \( \dim |-K_X| \leq \dim |-K_Y| \leq 13 \). \( \square \)

(3.6.4) Lemma. Let \( X \) be a toric \( \mathbb{Q} \)-Fano threefold. If \( X \not\cong \mathbb{P}^3 \), then \( -K_X^3 \leq 125/2 \) and \( \dim |-K_X| \leq 33 \).

Sketch of the proof. By considering cyclic covering tricks (cf. Proof of Proposition [5.3]) we reduce the question to the case \( \text{Cl}X \cong \mathbb{Z} \). For toric varieties this preserves the property \( \rho = 1 \). Then \( X \) is a weighted projective space. Using the fact that \( X \) has only terminal singularities we get the following cases: \( \mathbb{P}(1,1,1,2), \mathbb{P}(1,1,2,3), \mathbb{P}(1,2,3,5), \mathbb{P}(1,3,4,5), \mathbb{P}(2,3,5,7), \mathbb{P}(3,4,5,7) \). The lemma follows. \( \square \)

4. Case \( q(Q(X)) \leq 3 \)

In this section we consider the case \( q := q(Q(X)) \leq 3 \).

(4.1) Proposition. Let \( X \) be a \( \mathbb{Q} \)-Fano threefold. Assume that \( X \) is not Gorenstein, \( q := q(Q(X)) \leq 3 \) and \( -K_X^3 \geq 125/2 \). Then we have one of the following cases:

(4.1.1) \( q = 1, B = (2), -K_X^3 = 2g - 3/2, \dim |-K_X| = g + 1, 32 \leq g \leq 35; \)

(4.1.2) \( q = 1, B = (2,2), -K_X^3 = 63, \dim |-K_X| = 33; \)

(4.1.3) \( q = 1, B = (3), -K_X^3 = 188/3, \dim |-K_X| = 33; \)

(4.1.4) \( q = 2, B = (3), L^3 = 25/3, \dim |L| = 9, \dim |-K_X| = 35. \)

(4.2) Lemma. In notation of Proposition (4.1) we have \( -K_X \cdot c_2(X) \geq 125/8 \) and \( \sum_{P \in B} (r_P - 1/r_P) \leq 67/8 \). In particular, \( \sum r_P \leq 10 \).

Proof. By Corollary (2.7) we have cases (2.6.1) or (2.6.2). Hence,

\[
-K_X \cdot c_2(X) \geq \begin{cases} 
\frac{1}{3}(-K_X^3) \geq \frac{125}{6}, \\
\frac{1}{4}t(4 - 3t)(-K_X^3) \geq \frac{1}{4q} \left(4 - \frac{3}{q}\right) \frac{125}{2} \geq \frac{125}{8}.
\end{cases}
\]
(In the second line we used that \( t \geq 1/q \geq 1/3 \) and the function \( t(4-3t) \) is increasing for \( t \leq 2/3 \)). In both cases we have \(-K_X \cdot c_2(X) \geq 125/8\). Thus,

\[
\sum_{P \in B} \left( r_P - \frac{1}{r_P} \right) \leq 24 - \frac{125}{8} = \frac{67}{8}.
\]

Hence \( B \) contains at most 5 points and \( \sum r_P \leq \left\lfloor \frac{67}{8} + 5 \cdot \frac{1}{2} \right\rfloor \leq 10. \)

(4.3) Proposition. In notation of Proposition (4.1) we have \( \text{Cl} X \simeq \mathbb{Z} \).

Proof. Let \( T \) be an \( s \)-torsion element in the Weil divisor class group. By Riemann-Roch (2.3.1), Kawamata-Viehweg vanishing theorem and Serre duality we have

\[
0 = \chi(T) = 1 + \sum_P c_P(T),
\]

\[
0 = \chi(K_X + T) = 1 + \frac{1}{12} K_X \cdot c_2(X) + \sum_{P \in B} c_P(K_X + T).
\]

Subtracting we get

\[
0 = -\frac{1}{12} K_X \cdot c_2(X) + \sum_{P \in B} (c_P(T) - c_P(K_X + T)).
\]

Take \( i_{T,P} \) so that \( T \sim i_{T,P} K_X \) near \( P \in B \). Then \( si_{T,P} \equiv 0 \mod r_P \) and

\[
0 = -\frac{1}{12} K_X \cdot c_2(X) + \frac{1}{12} \sum_{P \in B} \left( r_P - \frac{1}{r_P} \right) - \sum_{P \in B} \frac{b_{p_iT,P}}{2r_P} \left( r_P - \frac{b_{p_iT,P}}{2r_P} \right).
\]

Therefore,

\[
2 = \sum_{P \in B} \frac{b_{p_iT,P}}{2r_P} \left( r_P - \frac{b_{p_iT,P}}{2r_P} \right).
\]

If \( i_{T,P} \not\equiv 0 \mod r_P \), we have

\[
\frac{b_{p_iT,P}}{2r_P} \left( r_P - \frac{b_{p_iT,P}}{2r_P} \right) \leq \frac{r_P}{8}.
\]

Combining the last two relations we get

\[
\sum_{P \in B'} r_P \geq 16,
\]

where the sum runs over all \( P \in B \) such that \( i_{T,P} \not\equiv 0 \mod r_P \). This contradicts Lemma (4.2) \( \square \).
Proof of Proposition (4.1). By Proposition (4.3) \(q = qQ(X) = qW(X)\). So, \(\gcd(q, r_P) = 1\) for all \(P \in B\).

(4.4) Case \(q = 3\). We will show that this case does not occur. By (2.4.2) we have

\[
- K_X^3 = q^3 L^3 = 162 + \frac{9}{2} K_X \cdot c_2(X) + 162 \sum_{P \in B} c_P(-L).
\]

By Lemma (4.2) \(- K_X \cdot c_2(X) \geq 125/8\) and \(- K_X^3 \geq 125/2\) by our assumptions. Combining this we obtain \(\sum_{P \in B} c_P(-L) \geq -467/2592\).

Again by Lemma (4.2) we have \(\sum (r_P - 1/r_P) \leq 67/8\). Assume that \(r_P = 2\) for all \(P \in B\). Note that \(c_P(L) = -1/8\) (because \(- K_X \sim L\) near each \(P\)). Hence \(B = (2)\). Then \(- K_X \cdot c_2(X) = 45/2\). By (4.4.1) we have \(- K_X^3 = 81/2 < 125/2\), a contradiction.

Thus we assume that at least one on the \(r_P\)'s is \(\geq 3\). Recall that \(\sum r_P \leq 10\), \(\sum (r_P - 1/r_P) \leq 67/8\) and \(3 \nmid r_P\). This gives us the following possibilities for \(B\):

\[
(4), (5), (7), (8), (2, 4), (2, 5), (2, 7), (2, 2, 4), (2, 2, 5), (4, 4), (2, 2, 2, 4).
\]

Take \(0 \leq i_P < r_P\) so that \(3i_P \equiv -1 \mod r_P\). Easy computations give us

| \(r_P\) | 2  | 4  | 5  | 7  | 8  |
|-------|----|----|----|----|----|
| \(i_P\) | 1  | 1  | 3  | 2  | 5  |
| \(c_P\) | -1/8 | -5/16 | -1/5 | -2/7, -3/7, -5/7 | -5/32 |

In all cases except for \(B = (8)\) we get a contradiction with \(\sum c_P(-L) \geq -467/2592\). Consider the case \(B = (8)\). Then by (4.4.1) we have

\[
-K_X^3 = 162 - \frac{9}{2} \times \frac{129}{8} - 162 \frac{5}{32} = \frac{513}{8}.
\]

Then by (2.5.2)

\[
\dim | - K_X | = 2 + \frac{513}{16} - \frac{b_p(8 - b_p)}{16} = 34 + \frac{1 - b_p(8 - b_p)}{16}.
\]

This number cannot be an integer, a contradiction.

(4.5) Case \(q = 1\). By (2.6.1) we have

\[
\sum_{P \in B} \left( r_P - \frac{1}{r_P} \right) = 24 + K_X \cdot c_2(X) \leq 24 + \frac{1}{2} K_X^3 \leq 24 - \frac{125}{6} = \frac{19}{6}.
\]
This gives the following possibilities: $B = (2), (3), \text{ or } (2, 2)$.

If $B = (2, 2)$, then $-K_X \cdot c_2(X) = 21$ and $-K_X^3 \leq 63$. On the other hand, $-K_X^3 \in \frac{1}{2}\mathbb{Z}$ (see [4, Lemma 1.2]). Hence $-K_X^3 = 63$ or $125/2$. Further, by (2.5.2)

$$\dim | -K_X | = -\frac{1}{2}K_X^3 + \frac{3}{2}. $$

Since this number should be an integer, the only possibility is $-K_X^3 = 63$ and $\dim | -K_X | = 33$.

If $B = (2)$, then $-K_X \cdot c_2(X) = 45/2$ and by (2.5.2)

$$\dim | -K_X | = -\frac{1}{2}K_X^3 + \frac{7}{4}. $$

Put $g := \dim | -K_X | - 1$. Then $-K_X^3 = 2g - 3/2$. We have

$$125/2 \leq -K_X^3 = 2g - 3/2 \leq 74 - \frac{9}{2}. $$

Hence $32 \leq g \leq 35$ and $-K_X^3 \in \{125/2, 129/2, 133/2, 137/2\}$.

Assume that $B = (3)$. Then $-K_X \cdot c_2(X) = 64/3$ and $-K_X^3 \leq 64$. As above,

$$\dim | -K_X | = -\frac{1}{2}K_X^3 + \frac{5}{3}. $$

We get only one possibility: $-K_X^3 = 188/3$ and $\dim | -K_X | = 33$.

(4.6) Case $q = 2$. If $E$ is semistable, then as above by (2.6.1) $B = (3)$. Otherwise we are in case (2.6.2) and as in the proof of Lemma (4.2) we have

$$\sum_{P \in \mathcal{B}} \left( r_P - \frac{1}{r_P} \right) = 24 + K_X \cdot c_2(X) \leq 24 + \frac{5}{16}K_X^3 \leq \frac{143}{32}. $$

Since gcd($r_P, q$) = 1, again we get the same possibility $B = (3)$.

Then $-K_X \cdot c_2(X) = 64/3$ and $L \cdot c_2(X) = 32/3$. Hence

$$5/4(-K_X^3) \leq t(4 - 3t)(-K_X^3) \leq 4 \cdot 64/3. $$

Thus $125/2 \leq -K_X^3 \leq 1024/15$ and $125/16 \leq L^3 \leq 128/15$. Since $3L^3 \in \mathbb{Z}$ (see [4, Lemma 1.2]), we have $L^3 = 8$ or $25/3$. As above the case $L^3 = 8$ is impossible by (2.5.2). Thus $L^3 = 25/3$. Then one can easily compute $h^0(L)$ and $h^0(-K_X)$ by (2.4.1).
5. Case $q\mathbb{Q}(X) \geq 4$

(5.1) **Proposition** Let $X$ be a $\mathbb{Q}$-Fano threefold. Assume that $X$ is not Gorenstein, $-K_X^3 \geq 125/2$, and $q := qW(X) = q\mathbb{Q}(X) \geq 4$. Then we have one of the following cases:

(5.1.1) $q = 4$, $B = (5)$, $-K_X^3 = 384/5$, $\dim |L| = 3$, $\dim |2L| = 10$, $\dim |-K_X| = 40$;

(5.1.2) $q = 4$, $B = (5, 5)$, $-K_X^3 = 64$, $\dim |L| = 2$, $\dim |2L| = 8$, $\dim |-K_X| = 33$;

(5.1.3) $q = 5$, $B = (2)$, $-K_X^3 = 125/2$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 33$;

(5.1.4) $q = 5$, $B = (2, 6)$, $-K_X^3 = 250/3$, $\dim |L| = 2$, $\dim |2L| = 7$, $\dim |-K_X| = 43$;

(5.1.5) $q = 5$, $B = (7)$, $-K_X^3 = 500/7$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 37$;

(5.1.6) $q = 5$, $B = (2, 2, 3, 6)$, $-K_X^3 = 125/2$, $\dim |L| = 1$, $\dim |2L| = 5$, $\dim |-K_X| = 32$;

(5.1.7) $q = 6$, $B = (5, 7)$, $-K_X^3 = 2592/35$, $\dim |L| = 1$, $\dim |2L| = 4$, $\dim |-K_X| = 38$;

(5.1.8) $q = 7$, $B = (3, 9)$, $-K_X^3 = 686/9$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |-K_X| = 39$;

(5.1.9) $q = 7$, $B = (2, 10)$, $-K_X^3 = 343/5$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |3L| = 6$, $\dim |-K_X| = 35$.

**Proof.** Let $L$ be a Weil divisor such that $-K_X \sim qL$. Since $qW(X) = q\mathbb{Q}(X)$, the group $\text{Cl}_X/\text{Tors}$ is generated by $L$. To get our cases we run a computer program. Below is the description of our algorithm.

1) By (2.5.1) and Theorem (2.5.4) we have $\sum_{P \in B}(1 - 1/r_P) \leq 24$. Hence there is only a finite (but very huge) number of possibilities for the basket $B$. In each case we know $-K_X \cdot c_2(X)$ from (2.5.1). Let $r := \text{lcm}(\{r_P\})$ be the Gorenstein index of $X$.

2) By Corollary (2.2.2) $q \leq 4r$ and $\gcd(q, r) = 1$. Hence we have only a finite number of possibilities for the index $q$.

3) In each case we compute $L^3$ and $-K_X^3 = q^3L^3$ by formula (2.4.2) and check the condition $-K_X^3 \geq 125/2$. Here, for $D = -L$, the number $i_P$ is uniquely determined by conditions $q_iP \equiv b_P \text{ mod } r_P$ and $0 \leq i_P < r_P$. 

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4) Next we check Kawamata’s inequalities \((2.6)\) i.e., we check that at least one of inequalities \((2.6.1) - (2.6.4)\) holds. In case \((2.6.2)\) we use the fact that the function \(t(4 - 3t)\) is increasing for \(t < 2/3\). Since \(t \geq 1/q\), we have \(\frac{1}{q}(4 - \frac{3}{q}) \leq t(4 - 3t)\) and
\[
\frac{1}{q} \left(4 - \frac{3}{q}\right) (-K_X^3) \leq -4K_X \cdot c_2(X).
\]
Similarly, in cases \((2.6.3)\) and \((2.6.4)\) we have, respectively,
\[
\left(1 - \frac{1}{q}\right) \left(1 + \frac{3}{q}\right) (-K_X^3) \leq -4K_X \cdot c_2(X),
\]
\[
\frac{1}{q} \left(2 - \frac{3}{q}\right) (-K_X^3) \leq -K_X \cdot c_2(X).
\]

5) Finally, by the Kawamata-Viehweg vanishing theorem we have \(\chi(tL) = h^0(tL) = 0\) for \(-q < t < 0\). We check this condition by using \((2.4.1)\).

At the end we get possibilities \((5.1.1)\) \(\square\)

\((5.2)\) Corollary (cf. \([4]\) Remark 2.14). Let \(X\) be a \(\mathbb{Q}\)-Fano threefold. If \(qW(X) = q\mathbb{Q}(X)\), then \(-K_X^3 \leq 250/3\).

Now we show that the condition \(qW(X) = q\mathbb{Q}(X)\) in Proposition \((5.1)\) is satisfied automatically.

\((5.3)\) Proposition. Let \(X\) be a \(\mathbb{Q}\)-Fano threefold. Assume that \(q := q\mathbb{Q}(X) > 3\) and \(-K_X^3 > 45\). Then \(\text{Cl}X \simeq \mathbb{Z}\).

Proof. Assume that the torsion part of \(\text{Cl}X\) is non-trivial for some \(X\) satisfying the conditions of Proposition \((5.1)\). Take \(X\) so that \(q\mathbb{Q}(X)\) is maximal. Write \(K_X + qL \sim_0 0\), where \(L\) is an (ample) integral Weil divisor. Since \(\text{Cl}X\) is finitely generated and by cyclic covering trick \((3.6)\), there is a finite étale in codimension one cover \(\pi: X' \to X\) such that \(\text{Cl}X'\) torsion free. Here \(K_{X'} + qL' \sim 0\), where \(L' := \pi^*L\). Note that \(X'\) has only terminal singularities. Hence \(X'\) is a Fano threefold with terminal singularities with \(qW(X') \geq q\). (It is possible however that \(X'\) is not \(\mathbb{Q}\)-factorial and \(\rho(X') > 1\). Denote \(n := \deg \pi\). Clearly,
\(-K_{X'}^3 = -nK_X^3 \geq -2K_X^3\). Hence \(\text{dim} \; -K_{X'} \geq -K_X^3 \geq 2 > 43\).

Let \(\sigma: X'' \to X'\) be a \(\mathbb{Q}\)-factorialization. (If \(X'\) is \(\mathbb{Q}\)-factorial, we take \(X'' = X'\)). Run \(K\)-MMP on \(X''\): \(\nu: X'' \dasharrow Y\). At the end we get a Mori-Fano fibre space \(f: Y \to Z\). Let \(L'' := \sigma^{-1}(L')\) and \(L_Y := \nu_*L''\). Then \(-K_Y \sim qL_Y\). If \(\text{dim} \; Z > 0\), then for a general fibre \(F := f^{-1}(o), o \in Z\), we have \(-K_F \sim qL_Y|_F\). This is impossible if \(q > 3\).

In the case \(\text{dim} \; Z = 0\), \(Y\) is a Fano with \(\rho(Y) = 1\) and \(qW(Y) \geq q\).

By our assumption of maximality of \(q = q\mathbb{Q}(X)\) we have \(q\mathbb{Q}(Y) = \ldots\)
\[ qW(Y) = q. \] Hence, \(-K_X^3 \leq 250/3\) by Corollary \([5.2]\). By \([2.5.3]\) we have \(\dim |-K_Y| \leq 43\). Using \([2.5.5]\) we obtain

\[ 43 \geq \dim |-K_Y| \geq \dim |-K_X'^\nu| \geq -\frac{1}{2}K_X'^3 - 2 \geq -K_X^3 - 2. \]

Thus \(-K_X^3 \leq 45\), a contradiction. \(\square\)

6. PROOF OF THE MAIN THEOREM

(6.1) To construct a Sarkisov link such as in \([3.2.1]\), we need the following result basically due to Ambro and Kawachi.

(6.1.1) Proposition (cf. \([6, \text{Th. 4.1}]\)). Let \(X\) be a Fano threefold with terminal singularities, and let \(S\) be an ample Cartier divisor proportional to \(-K_X\). Then the linear system \(|S|\) is non-empty and a general member of \(|S|\) is a reduced irreducible normal surface whose singularities are at worst log terminal of type T. Moreover, assume that \(K_X^2 \cdot S > 1\) and \(qF(X) \geq 1/2\). Then a general \(S \in |S|\) does not pass through non-Gorenstein points (and has at worst Du Val singularities).

Proof. According to \([16]\) the pair \((X, S)\) is plt for a general \(S \in |S|\). Then singularities of \(S\) are of type T by \([17]\). Note that the restriction map \(H^0(O_X(-K_X)) \to H^0(O_S(S))\) is surjective. Let \(P \in Bs|S|\) be a non-Gorenstein point of \(X\). Then \(P \in S\) is a log terminal non-Du Val singularity of type T.

Recall that Kawachi’s invariant of a normal surface singularity \((S, P)\) is defined as \(\delta_P := -((\Gamma - \Delta))^2\), where \(\Delta\) is the codiscrepancy divisor of \((S, P)\) on the minimal resolution \(\hat{S} \to S\) and \(\Gamma\) is the fundamental cycle on \(\hat{S}\) (see \([18]\)). If \((S, P)\) is a rational singularity, then \(\delta_P = \Gamma^2 - \Delta^2 + 4\). Hence in our case Kawachi’s invariant \(\delta_P\) is integral (because \(\Delta^2 \in \mathbb{Z}\), see \([17]\)). On the other hand, \(0 < \delta_P < 2\). Thus \(\delta_P = 1\). Now we apply the main result of \([18]\) to the linear system \(|S| = |K_S - K_X|\). It follows that there is a curve \(C\) on \(S\) passing through \(P\) and such that \(-K_X \cdot C < 1/2\). Since \(qF(X) \geq 1/2\), this is impossible. \(\square\)

(6.1.2) Proposition. In notation of Proposition \([6.1.1]\) assume additionally that \((2K_X + S)^2 \cdot S \geq 5\) and \(-(2K_X + S)\) is an ample divisor which is divisible in \(\text{Cl} X/\text{Tors}\). Then the linear system \(|-K_X|\) has only isolated base points.

Proof. Denote the restriction \(-K_X|S\) by \(D\). Since \(S\) does not pass through non-Gorenstein points, \(D\) is Cartier. By the Kawamata-Viehweg vanishing the map

\[ H^0(O_X(-K_X)) \to H^0(O_S(D)) \]
is surjective. Thus it is sufficient to show that the linear system $|D|$ is base point free. By the adjunction formula $D = K_S - (2K_X + S)|_S$. Let $\mu: \hat{S} \to S$ be the minimal resolution. Since $S$ has at worst Du Val singularities, $K_{\hat{S}} = \mu^*K_S$. Thus we can write $\mu^*D = K_{\hat{S}} + M$, where $M = \mu^*(-(2K_X + S)|_S)$ is nef. It is easy to see that $M^2 = (2K_X + S)^2 \cdot S \geq 5$ by our assumption. Suppose that the linear system $|\mu^*D| = |K_{\hat{S}} + M|$ has a base point $P$. By the main theorem of [19] there is an effective divisor $E$ on $\hat{S}$ passing through $P$ such that either $M \cdot E = 0$, $E^2 = -1$ or $M \cdot E = 1$, $E^2 = 0$. In the former case $E$ is contracted by $\mu$ and we get a contradiction by the genus formula. In the latter case we have $-(2K_X + S) \cdot \mu(E) = 1$. This is impossible because $-(2K_X + S)$ is divisible in $\text{Cl} X/\text{Tors}$ and $\mu(E)$ is contained in the Gorenstein locus of $X$. □

Since $qF(X) = q/r$, we have the following

(6.1.3) Corollary. Let $X$ be a $\mathbb{Q}$-Fano threefold, let $q := q\mathbb{Q}(X)$, and let $r$ be the Gorenstein index of $X$. Assume that $-K_X^3 \geq 125/2$. Then $X$ is such as in Propositions (4.1) or (5.1). In particular, $\dim |-K_X| \geq 32$. By Propositions (4.3) and (5.3) we also have $\text{Cl} X \simeq \mathbb{Z}$. We divide cases of (4.1) or (5.1) in four groups and treat these groups separately (see (6.3), (6.4), (6.5), (6.6)).

(6.2.1) Proposition. Notation and assumptions as in (6.2). If there exists a Sarkisov link (3.2.1) with birational $f$, then $-K_X^3 \geq 125/2$ except possibly for the following case

- $\dim |-K_Z| = \dim |-K_X| = 32$.

Proof. Assume the converse. Then $Z$ is a $\mathbb{Q}$-Fano with $\dim |-K_Z| \geq \dim |-K_X| \geq 32$ and $-K_X^3 < 125/2$. By (2.5.3)

$$\dim |-K_Z| + \frac{1}{2} \sum_{P \in \mathcal{B}_Z} \left(1 - \frac{1}{r_P}\right) \leq -\frac{1}{2}K_Z^3 + 2 < \frac{133}{4},$$

Therefore, $\dim |-K_Z| = 32$ or 33. Moreover, if $\dim |-K_Z| = 33$, then we have $r_P = 1$ for all $P \in \mathcal{B}_Z$, i.e., $Z$ is Gorenstein (and factorial). In
particular, \( q\mathbb{Q}(Z) = qF(Z) = qW(Z) \) and \( q\mathbb{Q}(Z)^3 \) divides \(-K_Z^2\). By Riemann-Roch, \(-K_Z^3 = 62\). Therefore, \( q\mathbb{Q}(Z) = 1 \). But then \(-K_Z\) cannot be decomposed into a sum of movable divisors. We derive a contradiction by \(\square\).

(6.3) Case (5.1.3)

(6.3.1) Proposition (see [20]). In case (5.1.3) \( X \simeq \mathbb{P}(1, 1, 1, 2) \).

Proof. Let \( S \in |2L| \) be a general member. Then \( S \) is Cartier and by Proposition (6.1.1) \( X \) is has at worst Du Val singularities. By the adjunction formula \( S \) is a del Pezzo surface of degree 9. It follows that \( S \) is smooth and \( S \simeq \mathbb{P}^2 \) (see Remark (3.5.4)). The restriction map \( H^0(X, \mathcal{O}_X(S)) \to H^0(S, \mathcal{O}_S(S)) \) is surjective. Hence the linear system \( |S| \) is base point free and determines a morphism \( \varphi: X \to \mathbb{P}^6 \). We have \( (\deg \varphi)(\deg \varphi(X)) = 5^3 = 4 \). So \( \varphi \) is birational and \( \varphi(X) \subset \mathbb{P}^6 \) is a variety of degree 4. A general hyperplane section \( \varphi(S) \subset \varphi(X) \) is a Veronese surface. It is well-known that in this situation \( \varphi(X) \) is a cone over \( \varphi(S) \), i.e., \( X \simeq \varphi(X) \simeq \mathbb{P}(1, 1, 1, 2) \). \(\square\)

(6.4) Cases (4.1.4) (5.1.1) (5.1.2) (5.1.4) (5.1.5) (5.1.6) (5.1.8) (5.1.9) We apply construction (3.4.1). Let \( r \) be the Gorenstein index of \( X \). First we construct a birational extremal extraction \( g: \tilde{X} \to X \) such that \( \tilde{X} \) has only terminal singularities and the exceptional divisor \( E \) of \( g \) has discrepancy 1/r.

(6.4.1) Claim. Either

(i) There is a cyclic quotient singularity \( P \in X \) of type \( \frac{1}{r}(b, -b, 1) \), where \( \gcd(r, b) = 1 \), or

(ii) we are in case (5.1.2) and there is a point \( P \in X \) of type \( cA/5 \) of axial weight 2.

Proof. Note that in all cases there is a basket point \( P \in \mathcal{B} \) of index \( r \). If this point is unique, it corresponds to a cyclic quotient singularity of \( X \). The point \( P \in \mathcal{B} \) of index \( r \) is not unique only in case (5.1.2). Then \( r = 5 \) and there are two points \( P_1, P_2 \in \mathcal{B} \) of index 5. They correspond either two cyclic quotient singularities of \( X \) or a point \( P \in X \) of type \( cA/5 \).

In case (i) the weighted blowup of \( P \in X \) with weights \( \frac{1}{r}(b, r - b, 1) \) gives us a desired contraction \( g \). Similarly, in case (ii) a suitable weighted blowup gives us a desired contraction \( g \) (see [21]).

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Further, \( r\mathcal{H} \) is the linear system of Cartier divisors. Hence we can write \( g^*\mathcal{H} = \mathcal{H} + \delta E \), where \( \delta \geq 1/r \). Thus,

\[
-K_{\tilde{X}} \sim q g^*(-K_X) - \frac{1}{r} E \sim q \mathcal{H} + (\delta - \frac{1}{r}) E.
\]

By Corollary \[ (6.1.3) \] the linear system \( \mathcal{H} \) has only isolated base points outside of \( E \). Hence, \( -K_{\tilde{X}} \) is nef.

If \( g(E) \) is a cyclic quotient singularity, then \( E \cong \mathbb{P}(b, r-b, 1) \), \( E|_E \sim \mathcal{O}_{\mathbb{P}(b, r-b, 1)}(-r) \), and \( E^3 = r^2/b(r-b) \). Therefore,

\[
-K_{\tilde{X}}^3 = -K_X^3 - \frac{1}{r} E^3 \geq \frac{125}{2} - \frac{r^2}{b(r-b)} > 0.
\]

This shows that \( -K_{\tilde{X}} \) is big. Similar computations shows that this fact also holds in case \((4.1.4)\), \((5.1.1)\), \((5.1.2)\), \((5.1.8)\), \((5.1.4)\), \((5.1.6)\), \((5.1.5)\), and \((5.1.9)\). Thus the linear system \( |-nK_{\tilde{X}}| \) does not contract any divisors.

Consider diagram \[ (3.2.1) \]. Since \( K_X + qL \sim 0 \), the constant \( \beta \) in \[ (3.3) \] is a non-negative integer. We can write

\[
K_{\tilde{X}} = g^*K_X + \frac{1}{r} E, \quad L = g^*L - \delta E,
\]

where \( \delta \in \mathbb{Q}, \delta > 0 \). Since \( rL \) is Cartier (see Lemma \[ (2.2.1) \]), \( \delta = k/r \) for some \( k \in \mathbb{Z}, k > 0 \). Therefore,

\[
\beta = -\frac{1}{r} + q\delta = \frac{qk - 1}{r}
\]

and the value of \( \beta \) is bounded from below as follows:

| case                  | \[ (4.1.4) \] | \[ (5.1.1) \] | \[ 5.1.2 \] | \[ 5.1.8 \] | \[ (5.1.4) \] | \[ 5.1.6 \] | \[ (5.1.5) \] | \[ 5.1.9 \] |
|-----------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \beta \)          | \( \geq 1 \) | \( \geq 3 \) | \( \geq 4 \) | \( \geq 4 \) | \( \geq 2 \) | \( \geq 2 \) | \( \geq 2 \) | \( \geq 2 \) |

First we assume that \( \dim Z = \dim X \). Then \( f \) is a divisorial contraction and \( Z \) is a \( \mathbb{Q} \)-Fano threefold. By \[ (3.3.1) \] we have \( K_Z + qL_Z + \beta E_Z \sim 0 \), where \( E_Z \) and \( L_Z \) are effective non-zero divisors. Hence, \( q\mathbb{Q}(Z) \geq q + \beta > 4 \). In particular, \( Z \) is not Gorenstein (see Corollary \[ (2.2.2) \]).

Assume that \( -K_Z^3 < 125/2 \). By Proposition \[ (6.2.1) \] \( \dim |-K_X| = \dim |-K_Z| = 32 \). Hence \( X \) is of type \[ (5.1.6) \]. By \[ (6.2.2) \] \( \dim |-K_Z| \geq 60 \) and by \[ (3.3.1) \] \( q\mathbb{Q}(Z) \geq 9 \). On the other hand, \( \text{discrep}(Z) \geq\)
discrep(\(\hat{X}\)) ≥ 1/5. Therefore the Gorenstein index of \(Z\) is at most 5 (see [21]). By Proposition (5.3) \(\text{Cl} Z \cong \mathbb{Z}\). Let \(L'\) be the ample generator of \(\text{Cl} Z \cong \mathbb{Z}\), let \(r' \leq 5\) be the Gorenstein index of \(Z\), and let \(S \in |r'L'|\) a general member. Then \(S\) be the ample generator of \(\text{Pic} Z\). By Proposition (6.1.1) \(S\) has at worst Du Val singularities. By the adjunction formula \(K_S = (r' - q\mathbb{Q}(Z))L'|_S\). Since \(L'|_S\) is a Cartier divisor, \(S\) is a del Pezzo surface with \(qF(S) \geq q\mathbb{Q}(Z) - r' \geq 4\). This is impossible (see (3.5.4)). Thus \(-K_Z^2 \geq 125/2\) and \(Z\) is such as in (5.1).

Now we consider possibilities for \(X\) case by case. In cases (5.1.4), (5.1.6), (5.1.8), and (5.1.9) we have \(q\mathbb{Q}(Z) \geq 9\), a contradiction. In cases (5.1.1), (5.1.2), and (5.1.5) we have \(q\mathbb{Q}(Z) = 7\). Hence \(Z\) is such as in (5.1.8) or (5.1.9). Then \(q + \beta = 7\). By (3.3.1) \(L_Z\) and \(E_Z\) are linear equivalent and they are generators of \(\text{Cl} Z\). On the other hand, \(\dim |L| \geq 2 > \dim |L_Z| = 1\), a contradiction.

In case (4.1.4) \(\hat{X}\) is of Gorenstein index 2. Hence, discrep(\(\hat{X}\)) = 1/2. On the other hand, \(f \circ \chi\) is a composition of a flop and steps of the \(\overline{K}\)-MMP. Therefore, discrep(\(Z\)) ≥ 1/2. This is possible only if \(Z\) of type (5.1.3). But then \(35 = \dim |-K_X| > \dim |-K_Z| = 33\), a contradiction.

(6.4.5) Thus we may assume that \(\dim Z < \dim X\). Let \(M \in |2L|\) be a general member. Note that by (6.4.3) \(q + \beta \geq 3\) and \(q + \beta = 3\) only in case (4.1.4). By (3.3.2) \(L_Y\) can be \(f\)-horizontal only in case (4.1.4) and if \(Z\) is a curve. By Lemma (3.5.8) we have a contradiction. Hence \(L_Y\) is \(f\)-vertical. As in Lemma (3.5.5) we have \(L_Y = f^*\Xi\) for some integral \(\text{Weil divisor} \ \Xi\) on \(Z\), \(\dim |\Xi| = \dim |L| \geq 1\), and \(\Xi\) is a generator of \(\text{Cl} Z/\text{Tors}\).

(6.4.6) Assume that \(Z\) is a surface. From (3.3.2) we get \(\beta \leq 2\). By (6.4.3) this is possible only in cases (4.1.4), (5.1.5) or (5.1.9). If \(K_Z^2 < 8\), we have \(\dim |\Xi| = 0\), a contradiction. Hence \(Z\) is either \(\mathbb{P}^2\) or \(\mathbb{P}(1, 1, 2)\). Consider the case \(Z \cong \mathbb{P}(1, 1, 2)\). Then \(\dim |\Xi| = 1\) and we are in case (5.1.9). Let \(M \in |3L|\) be a general member. We can write \(K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0\), where \(\gamma > 0\). This shows that \(M_Y\) is \(f\)-vertical. Thus \(M_Y \sim 3L_Y = 3f^*\Xi\) and \(\dim |M_Y| = \dim |3\Xi| = 4\), a contradiction.

Consider the case \(Z \cong \mathbb{P}^2\). Then \(\dim |\Xi| = 2\) and we are in case (5.1.5). Let \(M \in |2L|\) be a general member. We can write \(K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0\), where \(\gamma > 0\). This shows that \(\gamma = \beta = 2\) and \(M_Y\) is \(f\)-vertical. Thus \(M_Y \sim 2L_Y = f^*\Xi\) and \(\dim |M_Y| = \dim |2\Xi| = 5\), a contradiction.

(6.4.7) Assume that \(Z\) is a curve. Then \(Z \cong \mathbb{P}^1\). Since \(L_Y = f^*\Xi\) is not divisible in \(\text{Cl} Y\), \(\dim |\Xi| \leq 1\). So we are in cases (5.1.6), (5.1.8), (5.1.9)
or (5.1.9). Moreover, since \( \dim |L| > 0, \dim |\Xi| = 1 \). Case (5.1.6) is impossible because then \( \beta \geq 4 \). Let \( M \in |2L| \) be a general member. We can write \( K_Y + 3M_Y + L_Y + \gamma E_Y \sim 0 \), where \( \gamma > 0 \). This shows that \( M_Y \) is \( f \)-vertical. Thus \( M_Y \sim 2L_Y = 2f^*\Xi \) and \( \dim |M_Y| = \dim |2\Xi| = 2 \), a contradiction.

Now we consider case (5.1.7).

(6.5) Case (5.1.7). By Lemmas (3.6.2) and (3.6.3) the pair \((X, |−K_X|)\) is not canonical. Thus we apply the construction (3.2.1) in case (3.4.2). Then in (3.2.2) we have \( \alpha > 0 \). Assume that \( \dim Z = 3 \). Since \( \alpha > 0 \), and by Proposition (2.5.6) we have \( \dim |−K_Z| > \dim |−K_X| = 38 \). Then by Proposition (6.2.1) \( −K_Z^3 \geq 125/2 \). Hence \( Z \) is \( Q \)-Fano such as in Proposition (5.1). Moreover, by (3.3.1) we have \( qQ(Z) ≥ qQ(X) + \beta = 6 + \beta \). This implies that \( E_Z \sim L_Z \) is a generator of \( \text{Cl} Z \), \( qQ(Z) = 7 \), and \( \beta = 1 \). So, the variety \( Z \) is of type (5.1.8). Obviously, \( \dim |2L_Z| \geq \dim |2L| \). This contradicts Proposition (5.1).

Thus \( \dim Z = 1 \) or \( 2 \). If \( Z \) is a surface, then by Lemma (3.5.5) \( Z \simeq \mathbb{P}(1,1,2) \). Let \( M \in |2L| \) be a general member. We can write \( K_Y + 3M_Y + \gamma E_Y \sim 0 \), where \( \gamma > 0 \). Restricting to a general fibre we obtain that \( M_Y \) is \( f \)-vertical. Thus, \( M_Y \sim 2L_Y = 2f^*\Xi \) and \( \dim |M_Y| = \dim |2\Xi| \leq 3 \), a contradiction.

Finally we consider cases when \( qQ(X) = 1 \).

(6.6) Cases (4.1.1), (4.1.2), (4.1.3). By Lemmas (3.6.2) and (3.6.3) the pair \((X, |−K_X|)\) is not canonical. Thus we may apply construction (3.2) under assumptions (3.4.2).

Then in (3.2.2) we have \( \alpha > 0 \). Assume that \( \dim Z = 3 \). Similar to (6.5) \( \dim |−K_Z| > \dim |−K_X| \) and \( −K_Z^3 \geq 125/2 \). Hence \( Z \) is \( Q \)-Fano such as in Proposition (5.1) or (4.1) with \( qQ(Z) > 1 \). By (6.3) and (6.4) \( Z \) is of type (5.1.3) and \( Z \simeq \mathbb{P}(1,1,1,2) \). Then \( \dim |−K_X| < \dim |−K_Z| = 33 \), so \( X \) is of type (4.1.1) and \( \dim \mathcal{H}_Z \geq 32 \). Easy computations show that \( \mathcal{K}_Z \sim \mathcal{O}_{\mathbb{P}(1,1,1,2)}(n), \) with \( n \geq 5 \). On the other hand, \( −K_Z \sim \mathcal{K}_Z + \alpha E_Z \), where \( \alpha > 0 \), a contradiction.

Therefore, \( 1 \leq \dim Z \leq 2 \). If \( Z \) is a curve, we have a contradiction by Lemma (3.5.8). Thus \( Z \) is a surface. Then by Lemma (3.5.6) the fibration \( f \) has no discriminant curve. Hence by Lemma (3.5.7) we have \( \dim |−K_X| < 30 \), a contradiction.

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