A Rotating Kaluza-Klein Black Hole
with Squashed Horizons

Tower Wang

Institute of Theoretical Physics, Chinese Academy of Sciences,
P. O. Box 2735 Beijing 100080, China
wangtao218@itp.ac.cn

Abstract

We find a rotating Kaluza-Klein black hole solution with squashed $S^3$ horizons in five dimensions. This is a Kerr counterpart of the charged one found by Ishihara and Matsuno recently. The space-time is geodesic complete and free of naked singularities. Its asymptotic structure is a twisted $S^1$ fiber bundle over a four-dimensional Minkowski space-time. We also study the mass and thermodynamics of this black hole.
1 Introduction

Very recently, Ishihara and Matsuno [2] have found a black hole solution in the five-dimensional Einstein-Maxwell theory. It is a charged static black hole with a non-trivial asymptotic topology. Horizons of the black hole are in the shape of squashed $S^3$, and the space-time is asymptotically locally flat, approaching a twisted $S^1$ bundle over a four-dimensional Minkowski space-time. For simplicity, we will refer to this solution as an “Ishihara-Matsuno” black hole or an “IM” black hole for short, and use the name “Kerr-IM” for the Kerr counterpart with the same asymptotic geometry, etc..

1Maybe “RN-IM” is a more suitable name to adopt, but anyhow we will not do it here.
The Ishihara-Matsuno black holes deserve further investigations for several reasons.

First, string theory tells us that space-time has ten dimensions, and the idea of large extra dimensions suggests a possibility to produce black holes smaller than the extra dimensions at colliders [8, 9]. The Ishihara-Matsuno black holes bring us a new toy model to study. In this space-time, one extra dimension can be smaller than four uncompactified dimensions, but is much larger than other dimensions. The size of this large extra dimension is characterized by an adjustable parameter $r_\infty$ [2].

Second, it is a long-standing problem to define and compute conserved quantities in a space-time with a non-flat asymptotic structure or a non-trivial boundary topology. Some breakthroughs have been made [10, 11, 12], and more attention was attracted on this problem in recent years [14, 15, 13, 16, 17, 18]. Ishihara-Matsuno black holes have a non-trivial asymptotic structure, hence provide a new arena to study diverse techniques we have got. According to calculations performed by Cai et.al. [4], the boundary counter-term method [16] and the generalized Abbott-Deser method [14] result in the same mass (energy) for IM black holes, which also satisfies the first law of thermodynamics. However, as the the compactified dimension expands, such a mass increases, even in the absence of five-dimensional black holes. This is a little counterintuitive but may be explained by comparing IM black holes with Kakuza-Klein monopoles [1].

Third, from the discussion in [2], it is clear that horizons of Ishihara-Matsuno black holes are deformed owing to the non-trivial asymptotic structure. There is an interesting problem to ask: What will happen if we switch on angular momenta, the cosmological constant and Chern-Simons terms [5] or replace the black hole by a black ring [6]? We have tried to answer this question, but only bite a bit, that is, we obtain a Kerr-IM black hole solution with equal angular momenta in Einstein theory whose cosmological constant is zero. The Kerr-IM black hole with non-equal angular momenta is too complicated to struggle with. When looking for the metric of dS-IM black holes we lose our way. As far as we know, in five-dimensional asymptotically flat space-time, charged rotating black holes have not been obtained analytically in pure Einstein-Maxwell theory yet [7]. So a Kerr-Newman-IM black hole solution seems to be equally hard to get. The solution for Kaluza-Klein multi-black holes see [3].

The organization of this paper is as follows. We produce the Kerr-IM black hole and identify its asymptotic space-time in the next section. Then we prove in Section 3 that such a Kerr-IM space-time is geodesic complete and free of naked singularities. Section 4 and Section 5 are dedicated to computing the thermodynamic quantities
as well as checking the first law. In Section 5 we also analyze the results and make some comments on the relation between IM black holes and Kaluza-Klein monopoles. Finally, a brief summary is given in Section 6. Through the entire paper we exploit the abbreviations introduced at the beginning of this section.

2 Kerr-IM Black Holes

2.1 A Prescription to Squash Black Holes

By observing the Kaluza-Klein black hole with squashed horizons [2], and comparing it with the five-dimensional Reissner-Nordström black hole, we guess that there is a general prescription which transforms some known five-dimensional black hole into a new one with squashed horizons. We further conjecture a detailed “squashing transformation” as the following

1. Write the original black hole metric in terms of the left-invariant Maurer-Cartan 1-forms on $S^3$

$$
\begin{align*}
\sigma_1 & = -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\
\sigma_2 & = \cos\psi d\theta + \sin\psi \sin\theta d\phi \\
\sigma_3 & = d\psi + \cos\theta d\phi
\end{align*}
$$

where

$$
0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad 0 < \psi < 4\pi
$$

are Euler angles [3]. The Euler angles are related to ordinary angles $(\theta_o, \phi_o, \psi_o)$ via [5]

$$
\psi_o - \phi_o = \phi, \quad \psi_o + \phi_o = \psi, \quad \theta_o = \frac{1}{2} \theta
$$

2. Modify the metric as

$$
dr \rightarrow kdr, \quad \sigma_1 \rightarrow \sqrt{k}\sigma_1, \quad \sigma_2 \rightarrow \sqrt{k}\sigma_2
$$

while $k$ is a function of $r$ in the form

$$
k(r) = \frac{(r^2_\infty - r^2_+)(r^2_\infty - r^2_-)}{(r^2_\infty - r^2)^2}
$$

$r = r_+$ corresponds to the outer horizon and $r = r_-$ the inner horizon. $r_\infty$ characterizes the size of a $S^1$ fiber at infinity. We impose to them a constraint $0 \leq r_- \leq r_+ < r_\infty$. 

4
When we apply this procedure to Kerr black holes with non-equal angular momenta, it gives a metric too complicated to struggle with. So in the rest parts of this article, we focus on the squashed Kerr black holes with equal angular momenta, to which we sometimes refer as “Kerr-IM” black holes.

2.2 From Kerr to Squashed Kerr

In terms of Meurer-Cartan 1-forms, a five-dimensional Kerr black hole with two equal angular momenta takes the form

$$ds^2 = -dt^2 + \sum_{\Delta} dr^2 + \frac{r^2 + a^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{m}{r^2 + a^2}(dt - \frac{a}{2}\sigma_3)^2$$  \hspace{1cm} (6)

The parameters are given by

$$\Sigma = r^2(r^2 + a^2)$$
$$\Delta = (r^2 + a^2)^2 - mr^2$$ \hspace{1cm} (7)

Taking the second step of the “squashing transformation” we have described in Subsection 2.1 with

$$k(r) = \frac{(r^2 + a^2)^2 - mr^2}{(r^2 - r^2)^2}$$ \hspace{1cm} (8)

it follows that a Kerr black hole with squashed horizons is

$$ds^2 = -dt^2 + \sum_{\Delta} k^2 dr^2 + \frac{r^2 + a^2}{4}[k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2] + \frac{m}{r^2 + a^2}(dt - \frac{a}{2}\sigma_3)^2$$ \hspace{1cm} (9)

It is straightforward to check that metric (9) satisfies the vacuum Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$ \hspace{1cm} (10)

We set the order of parameters as $0 < r_- \leq r_+ < r_\infty$, with an outer horizon at $r = r_+$ and an inner horizon at $r = r_-$, and restrict $r$ within the range $0 < r < r_\infty$, then metric (9) is geodesic complete and has no naked singularity, as we will establish in Section 3. In particular, if we take the limit $r_\infty \to \infty$ and at the same time concentrate on a region with a finite value of $\frac{r_\infty - r}{r_\infty}$, the Kerr-IM metric (9) will reduce to a five-dimensional Kerr metric. This is similar to the IM black hole, which reduces to a five-dimensional Reissner-Nordström black hole in the same limit.
2.3 Asymptotic Structure

In order to see the asymptotic structure, we introduce a new radial coordinate \( \rho \) as

\[
\rho = \rho_0 \frac{r^2}{r^2_\infty - r^2}
\]

(11)

with

\[
\rho_0^2 = \frac{k_0}{4} (r^2_\infty + a^2)
\]

\[
k_0 = k(r = 0) = \frac{(r^2_\infty + a^2)^2 - m r^2_\infty}{r^4_\infty}
\]

(12)

and rewrite the metric (9) as

\[
ds^2 = -dt^2 + Ud\rho^2 + R^2 (\sigma_1^2 + \sigma_2^2) + W^2 \sigma_3^2 + V(dt - \frac{a}{2}\sigma_3)^2
\]

(13)

where \( K, V, W, R \) and \( U \) are functions of \( \rho \) in the form

\[
K^2 = \frac{\rho + \rho_0}{\rho + \frac{a^2}{r^2_\infty + a^2} \rho_0}
\]

\[
V = \frac{m}{r^2_\infty + a^2} K^2
\]

\[
W^2 = \frac{r^2_\infty + a^2}{4 K^2} = \frac{m}{4 V}
\]

\[
R^2 = \frac{(\rho + \rho_0)^2}{K^2}
\]

\[
U = \left(\frac{-r^2_\infty + a^2}{r^2_\infty + a^2}\right)^2 \times \frac{\rho_0^2}{W^2 - \frac{r^2_\infty + a^2}{4\rho \rho_0} V}
\]

(14)

In the limit \( \rho \to \infty \), i.e., \( r \to r_\infty \), it approaches

\[
ds^2 = -dt^2 + d\rho^2 + \rho^2 (\sigma_1^2 + \sigma_2^2) + \frac{r^2_\infty + a^2}{4} \sigma_3^2 + \frac{m}{r^2_\infty + a^2} (dt - \frac{a}{2}\sigma_3)^2
\]

(15)

At first sight, the cross-term between \( dt \) and \( \sigma_3 \) appears to imply a pathology of the space-time. Nevertheless, if we change the coordinates as

\[
\tilde{\psi} = \psi - \frac{2ma}{(r^2_\infty + a^2)^2 + ma^2} t
\]

\[
\tilde{t} = \sqrt{\frac{(r^2_\infty + a^2)^2 - m r^2_\infty}{(r^2_\infty + a^2)^2 + ma^2}} \ t
\]

(16)

and take the notation \( \tilde{\sigma}_3 = d\tilde{\psi} + \cos \theta d\phi \), the asymptotic space-time is actually all right

\[
ds^2 = -d\tilde{t}^2 + d\rho^2 + \rho^2 (\sigma_1^2 + \sigma_2^2) + \frac{(r^2_\infty + a^2)^2 + ma^2}{4(r^2_\infty + a^2)} \tilde{\sigma}_3^2
\]

(17)
This asymptotic topology is the same as that of the IM space-time \[2\]: a twisted \(S^1\) bundle over a four-dimensional Minkowski space-time. We have expected it because the asymptotic structure should not be affected by angular momenta or any other charges of black holes. From the same point of view, we predict that the asymptotic geometry ought to be a twisted \(S^1\) bundle over a four-dimensional Anti-de Sitter space-time for a AdS-IM black hole if it really exists.

At infinity, the size of the compactified dimension is controlled by \(r_\infty, m\) and \(a\) together rather than \(r_\infty\) alone, implied by metric (17). One should not be bothered by this illusion of parametrization. A better parameter may be obtained if we trade \(r_\infty^2\) for

\[
r'_\infty = \frac{(r_\infty^2 + a^2)^2 + ma^2}{r_\infty^2 + a^2}
\]

in which the size of the compactified dimension is apparently decoupled from \(m\) and \(a\), but that will only make our formulas more scattered. In our expressions, for brevity, we will always use the parameter \(r_\infty\) instead of \(r'_\infty\). But one should keep in mind that the geometric interpretation is clearer for \(r'_\infty\) than for \(r_\infty\).

We stress that the coordinate transformation (11) is valid only for a finite \(r_\infty\) value, i.e., \(r_\infty < \infty\). In the limit \(r_\infty \to \infty\), one merely finds \(\rho \to 0\). Thus we will carry out most computations based on the metric in the form (9) instead of (13). But the calculations of mass and angular momenta are inconvenient from (9), so we will in Section 5 resort to the coordinate \(\rho\) introduced in (11). We believe the calculations should always be right for a finite \(r_\infty\), given that the boundary counter-term method and the generalized Abbott-Deser method work.

3 Absence of Naked Singularities

3.1 Ingoing Eddington Coordinates

The metric (9) breaks down at \(\Delta = 0\), i.e., \(r = r_\pm\). These singularities can be removed by a coordinate transformation

\[
\begin{align*}
   dv &= dt + \frac{k(r^2 + a^2)^2}{\Delta} dr \\
   d\chi &= d\psi + \frac{2ak(r^2 + a^2)}{\Delta} dr
\end{align*}
\]

In the new coordinates \((v, r, \theta, \phi, \chi)\), metric (9) turns to

\[
ds^2 = -dv^2 + \frac{r^2 + a^2}{4} [k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2] + \frac{m}{r^2 + a^2} (dv - \frac{a}{2} \sigma_3')^2 + 2k(dv - \frac{a}{2} \sigma_3')dr
\]
in which we make use of a notation $\sigma'_3 = d\chi + \cos\theta$. The new coordinates are nothing but the ingoing Eddington coordinates. The metric is regular now at $r = r_{\pm}$.

In fact, $r = r_{\pm}$ are where Killing horizons sit. We will only discuss outer horizon $r = r_+$ in this article. By exchanging $r_+ \leftrightarrow r_-$ one immediately gets the results for inner horizon $r = r_-$. From (20) we can write down the Killing vector of the event horizon at $r = r_+$

$$\xi^\mu \partial_\mu = \kappa_0 (\partial_t + \Omega_H \partial_\chi)$$

$$\Omega_H = \frac{2a}{r_+^2 + a^2}$$

(21)

where $\kappa_0$ is a normalization constant to be determined in Section 4. In coordinates $(t, r, \theta, \phi, \psi)$, the Killing vector takes the form

$$\xi^\mu \partial_\mu = \kappa_0 (\partial_t + \Omega_H \partial_\psi)$$

(22)

This result may also be worked out from (9) directly.

### 3.2 Geodesic Completeness

In the following, we will prove that the space-time of a Kerr-IM black hole is geodesic complete. The same recipe will be efficacious for the IM black hole found by H. Ishihara et.al. [2].

The action of a particle with mass $m_{\text{part}}$ is given by [19]

$$I = -m_{\text{part}} \int_{-\infty}^{+\infty} d\lambda \left[ -g_{\mu\nu}(x(\lambda)) \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} \right]^{1/2}$$

(23)

Inserting (9), and taking $\lambda = t$, it becomes

$$I = \int_{-\infty}^{+\infty} dt \int d\tau \left[ 1 - \frac{\Sigma}{\Delta} k^2 r^2 - \frac{r^2 + a^2}{4} k \dot{\theta}^2 - \frac{r^2 + a^2}{4} k \sin^2 \theta \dot{\phi}^2 - \frac{r^2 + a^2}{4} (\ddot{\psi} + \cos \theta \dot{\phi})^2 - \frac{m}{r^2 + a^2} \left( 1 - \frac{a}{2} \ddot{\psi} - \frac{a}{2} \cos \theta \dot{\phi} \right)^2 \right]^{1/2}$$

(24)
in which a dot denotes a derivative with respect to $t$, and we have assumed $\dot{\theta} = 0$. As will be shown later, this assumption is consistent with geodesic equations.

In order to be brief, we introduce a notation

$$u = \left[ -g_{\mu\nu}(x(\lambda)) \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} \right]^{\frac{1}{2}}$$

$$= \left[ 1 - \frac{\Sigma}{\Delta} k^2 r^2 - \frac{r^2 + a^2}{4} k \sin^2 \theta \dot{\phi}^2 
- \frac{r^2 + a^2}{4} (\dot{\psi} + \cos \theta \dot{\phi})^2
- \frac{m}{r^2 + a^2} (1 - \frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi})^2 \right]^{\frac{1}{2}}$$

(25)

Then the Hamiltonian is

$$H = E = \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu - L$$

$$= \frac{m_{\text{part}}}{u} \left[ \frac{\Sigma}{\Delta} k^2 r^2 + \frac{r^2 + a^2}{4} k \sin^2 \theta \dot{\phi}^2 + \frac{r^2 + a^2}{4} (\dot{\psi} + \cos \theta \dot{\phi})^2 
+ \frac{m}{r^2 + a^2} (1 - \frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi})(-\frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi}) \right] + m_{\text{part}} u$$

$$= \frac{m_{\text{part}}}{u} \left[ 1 - \frac{m}{r^2 + a^2} (1 - \frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi}) \right]$$

(26)

For a stationary space-time the energy $E$ is a constant, and the first integrals of $\phi$ and $\psi$ are found to be

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}}$$

$$= \frac{m_{\text{part}}}{u} \left[ \frac{r^2 + a^2}{4} k \sin^2 \theta \dot{\phi} + \frac{r^2 + a^2}{4} \cos \theta (\dot{\psi} + \cos \theta \dot{\phi}) 
- \frac{m}{r^2 + a^2} \frac{a}{2} \cos \theta (1 - \frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi}) \right]$$

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}}$$

$$= \frac{m_{\text{part}}}{u} \left[ \frac{r^2 + a^2}{4} (\dot{\psi} + \cos \theta \dot{\phi}) - \frac{m}{r^2 + a^2} \frac{a}{2} (1 - \frac{a}{2} \dot{\psi} - \frac{a}{2} \cos \theta \dot{\phi}) \right]$$

(27)

We will assume $P_\phi = 0, P_\psi = 0$ and show the consistency a little later. Using these assumptions to solve (27), we get a clean result

$$\dot{\phi} = 0$$

$$\dot{\psi} = \frac{2ma}{(r^2 + a^2)^2 + ma^2}$$

(28)
Combining (26) and (28) yields

\[ u = \frac{m_{\text{part}}(r^2 + a^2)^2 - mr^2}{E(r^2 + a^2)^2 + ma^2} \]

\[ \dot{r} = \frac{1}{k} \frac{(r^2 + a^2)^2 - mr^2}{(r^2 + a^2)^2 + ma^2} \sqrt{\frac{(E^2 - m_{\text{part}}^2)(r^2 + a^2)^2 + m(m_{\text{part}}^2r^2 + E^2a^2)}{E^2r^2(r^2 + a^2)}} \]  

(29)

At last, the integration of the proper time \( \tau \) is [19]

\[ \int_{-\infty}^{+\infty} d\tau = \int_{-\infty}^{+\infty} d\lambda \left[ -g_{\mu \nu}(x(\lambda)) \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} \right]^{\frac{1}{2}} \]

\[ = \int_{0}^{r_{\infty}} \frac{dr}{r u} \]

\[ = \frac{m_{\text{part}}}{E} \int_{0}^{r_{\infty}} dr \frac{(r_{\infty}^2 + a^2)^2 - mr_{\infty}^2}{(r_{\infty}^2 - r^2)^2} \]

\[ \times \sqrt{\frac{E^2r^2(r^2 + a^2)}{(E^2 - m_{\text{part}}^2)(r^2 + a^2)^2 + m(m_{\text{part}}^2r^2 + E^2a^2)}} \]  

(30)

For \( E^2 > m_{\text{part}}^2 \) this integration is divergent when and only when \( r \) approaches \( r_{\infty} \), that is, for a particle with enough energy, their geodesics can be extended to all values of \( \tau \). Accordingly we can expect the Kerr-IM space-time is geodesic complete, and it is clear now that \( r \to r_{\infty} \) indeed corresponds to spatial infinity.

In the above, we have assumed \( \dot{\theta} = 0 \), \( P_\phi = 0 \) and \( P_\psi = 0 \). To be rigorous, we now demonstrate that these assumptions are consistent with geodesic equations. Christoffel connections are easy to derive from metric (9), and we find the expected vanishing components

\[ \Gamma^\theta_{\mu \nu} = \Gamma^\phi_{\mu \nu} = 0 \quad (\mu, \nu = t, r, \psi) \]  

(31)

This is consistent with our earlier assumptions \( \dot{\theta} = 0 \), \( P_\phi = 0 \), \( P_\psi = 0 \). Note that \( P_\phi - P_\psi \cos \theta = 0 \) is equivalent to \( \dot{\phi} = 0 \).

4 Some Thermodynamic Quantities

4.1 Temperature

To discuss the surface gravity and the temperature, it is convenient to rewrite metric (9) by defining coordinates \( \tilde{\psi} \) and \( \tilde{t} \) exactly as (16), then the Kerr-IM metric (9) and
Killing vector \( \xi \) have the form

\[
ds^2 = -\bar{g}_{00}d\bar{t}^2 + \sum_D k^2 dr^2 + \frac{r^2 + a^2}{4} k \left( \sigma_1^2 + \sigma_2^2 \right) + \frac{(r^2 + a^2)^2 + ma^2}{4(r^2 + a^2)} (\bar{\sigma}_3 - \bar{\omega}d\bar{t})^2
\]

\[
\xi^\mu \partial_\mu = \bar{\kappa}_0 (\partial_t + \bar{\Omega}_H \partial_{\bar{\psi}})
\]

and

\[
\bar{g}_{00} = \frac{(r^2 + a^2)^2 - mr_\infty^2}{(r^2 + a^2)^2 + ma^2} \times \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2}
\]

\[
\bar{\omega} = \left[ \frac{2ma}{(r^2 + a^2)^2 + ma^2} - \frac{2ma}{(r^2 + a^2)^2 + ma^2} \right] \times \sqrt{\frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2}}
\]

\[
\bar{\kappa}_0 = \kappa_0 \sqrt{\frac{(r_\infty^2 + a^2)^2 - mr_\infty^2}{(r_\infty^2 + a^2)^2 + ma^2}}
\]

\[
\bar{\Omega}_H = \left[ \Omega_H - \frac{2ma}{(r^2 + a^2)^2 + ma^2} \right] \times \sqrt{\frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2}}
\]

This form of metric has the merit that \( \bar{g}_{00} \to 1 \) and \( \bar{\omega} \to 0 \) at spatial infinity \( r \to r_\infty \).

Correspondingly the two-dimensional Euclidean Rindler space-time is

\[
ds_E^2 = \sum_D k^2 dr^2 + \bar{g}_{00}d\bar{t}^2
\]

It has a conical singularity at \( r = r_+ \) until we make a periodic identification \( \bar{t} \sim \bar{t} + \frac{2\pi}{\kappa} \), with

\[
\kappa = \frac{r_+ - r_-}{r_+^2 + a^2} \times \frac{1}{k(r_+)} \times \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2}
\]

\[
= \frac{(r_+ - r_-)}{r_+^2 + a^2} \frac{r_+ r_-}{r_+^2 + a^2} \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2} \times \frac{r_\infty^2 - r_+^2}{r_\infty^2 - r_-^2}
\]

The same result can also be derived from \([11]\) and \([22]\), using \( \xi \cdot \nabla \xi^\mu \big|_{r = r_\pm} = \kappa_\pm \xi^\mu \), and choosing a normalization to ensure \( \bar{g}_{00} \xi^i \xi_i \to 1 \) as \( r \to r_\infty \), that is

\[
\kappa_0 = \sqrt{\frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2}}
\]

Having the surface gravity on the outer horizon, one immediately obtains the temperature of the black hole

\[
T = \frac{\kappa}{2\pi}
\]

\[
= \frac{(r_+ - r_-)}{2\pi r_+ (r_+ + r_-)} \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2} \times \frac{r_\infty^2 - r_+^2}{r_\infty^2 - r_-^2}
\]
4.2 Angular Velocity and Entropy

Different angular-velocity-like quantities have appeared in Subsection 3.1 and 4.1: \( \Omega_H \), \( \tilde{\Omega}_H \) and \( \tilde{\omega} \). Which is the physical one? Notice that on the outer horizon \( \tilde{\omega} = \tilde{\Omega}_H \), so the problem to determine angular velocity on the outer horizon is reduced to a choice between \( \Omega_H \) and \( \tilde{\Omega}_H \). Their difference originates from different definitions of time and angular coordinates. As we have mentioned, the metric form (32) has better behaviors at spatial infinity, that is, a vanishing angular momentum and a satisfactory normalization of the temporal component. Hence we are sure now a meaningful angular velocity on the event horizon is \( \tilde{\Omega}_H \), given by (33), taking the form

\[
\tilde{\Omega}_H = \frac{2}{r_+ + r_-} \times \sqrt{\frac{r_-(r_\infty^2 - r_+^2)}{r_+(r_\infty^2 - r_-^2)}} \times \frac{r_\infty^2 + r_+^2 + 2r_+r_-}{\sqrt{(r_\infty^2 + r_+r_-)^2 + r_+(r_+ + r_-)^2}}
\] (38)

It is not surprising that we should take into account asymptotic properties of metric at infinity when computing thermodynamic quantities on the event horizon. The key point is that all thermodynamic quantities are “seen from infinity”. In [4] it has been found that the area formula of entropy still survives in spite of a deformation of the horizon. Hereby we calculate the entropy of Kerr-IM black hole according to the area formula. Starting with (9), (13), (20) or (32), one easily obtains a unique consequence

\[
A = 2\pi^2 r_+(r_+ + r_-)^2 \times \frac{r_\infty^2 - r_-^2}{r_\infty^2 - r_+^2}
\]

\[
S = \frac{A}{4} = \frac{\pi^2 r_+(r_+ + r_-)^2}{2} \times \frac{r_\infty^2 - r_-^2}{r_\infty^2 - r_+^2}
\] (39)

5 The First Law of Thermodynamics

5.1 Abbott-Deser Mass

In the previous section, we argue that when calculating thermodynamic quantities, the asymptotic behavior of metric should be taken into consideration. Especially, a suitable form of metric to derive those quantities must asymptotically normalize the temporal components and annihilate angular momenta. The metric form (32) has these properties, whereas a more convenient form would be an analogous one in terms of \( \rho \). It can be obtained from (13), after coordinate transformation (11), or
from (32), after coordinate transformation (11)

\[ ds^2 = -\tilde{g}_{00}dt^2 + Ud\rho^2 + R^2(\sigma_1^2 + \sigma_2^2) + \frac{(r_\infty^2 + a^2)^2 + ma^2K^4}{4(r_\infty^2 + a^2)K^2}((\sigma_3 - \tilde{\omega}dt)^2 \tag{40} \]

in which \( U, R \) and \( K \) are given by (14). \( \tilde{g}_{00} \) and \( \tilde{\omega} \) are the same as those in (33), also taking the form

\[ \tilde{g}_{00} = \frac{(r_\infty^2 + a^2)^2 - m(r_\infty^2 + a^2)K^2 + ma^2K^4}{(r_\infty^2 + a^2)^2 + ma^2K^4} \times \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)^2 - mr_\infty^2} \]

\[ \tilde{\omega} = \left[ \frac{2maK^4}{(r_\infty^2 + a^2)^2 + ma^2K^4} - \frac{2ma}{(r_\infty^2 + a^2)^2 + ma^2} \right] \times \sqrt{(r_\infty^2 + a^2)^2 + ma^2} \tag{41} \]

In Abbott and Deser’s definition, conserved charges are associated with isometries of the asymptotic geometry which is supposed to be the vacuum of the system [10].

In Kerr-IM space-time, corresponding to (40), the asymptotic geometry is described by (17).

Taking \( \bar{\xi}^\mu\partial_\mu = \partial_t \) as the canonically normalized time-like Killing vector, and repeating the computation done in [4], one is led to the generalized Abbott-Deser mass of the Kerr-IM black hole

\[ M_{\text{Kerr-IM}} = \frac{\pi}{4} \times \frac{(r_\infty^2 + a^2)^2 - ma^2}{\sqrt{(r_\infty^2 + a^2)^2 + ma^2}} \times \frac{(r_\infty^2 + a^2)^2 + ma^2}{(r_\infty^2 + a^2)\sqrt{(r_\infty^2 + a^2)^2 - mr_\infty^2}} \]

\[ = \frac{\pi}{4} \times \frac{(r_\infty^2 + r_+r_-)^2 - r_+r_-(r_+ + r_-)^2}{\sqrt{(r_\infty^2 + r_+r_-)^2 + r_+r_-(r_+ + r_-)^2}} \times \frac{(r_\infty^2 + 2r_+r_- + r_+^2)(r_\infty^2 + 2r_+r_- + r_-^2)}{(r_\infty^2 + r_+r_-)\sqrt{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}} \tag{42} \]

This mass and the mass of IM black hole obtained in [4] will be analyzed in Subsection 5.4.

5.2 Angular Momentum and the First Law

Starting with (40), after a few calculations, the Komar integrals give the angular momenta

\[ J_\phi = 0 \]

\[ J = J_\phi = \frac{\pi}{4}ma = \frac{\pi}{4}\sqrt{r_+r_-(r_+ + r_-)^2} \tag{43} \]

The first law of black hole thermodynamics in our case

\[ dM - TdS - \tilde{\Omega}_HdJ = 0 \tag{44} \]
involves five parameters. All of them have been obtained now. To check the first law, one might view \( r_+, r_- \) as variables while keep \( r_\infty \) as a constant. But that will break [14]. Remember that for Kerr-IM space-time the geometric parameter at infinity is not \( r_\infty \) but \( r'_{\infty} \), as we have explained in Subsection 2.3. In deed, if we fix \( r'_{\infty} \) rather than \( r_\infty \), the first law of black hole thermodynamics [14] is obeyed.

5.3 The Counter-term Method

For a space-time with an asymptotic boundary topology \( R \times S^1 \hookrightarrow S^2 \), a simple counter-term in gravitational action is introduced by Mann and Stelea [16]. As a matter of fact, asymptotic structures are the same for IM black holes and Kaluza-Klein monopoles, the latter of which are treated in [16]. As a counter-term method over the background subtraction method is that it is independent of reference backgrounds hence leads to a unique result. The stress tensor for the asymptotically flat space-times was proposed for the first time by Astefanesei and Radu in [13]. For different boundary topologies there exist different counter-terms, but the form of the stress tensor is generic.

We still start with [10]. After a substantial amount of calculations, we find the non-vanishing components of stress tensor

\[
8\pi GT^t_i = \frac{(r_\infty^2 + a^2)^4 + m(r_\infty^2 + a^2)^2 - m^2a^2(r_\infty^2 + a^2)^2 - 2m^2a^2(r_\infty^2 + a^2)}{4[(r_\infty^2 + a^2)^2 + ma^2](r_\infty^2 + a^2)^2} \cdot \frac{1}{r^2} + O\left(\frac{1}{r^3}\right)
\]

\[
8\pi GT^t_\phi = -\frac{ma}{4} \sqrt{\frac{r_\infty^2 + a^2}{(r_\infty^2 + a^2)^2 + ma^2}} \cdot \frac{1}{r^2} + O\left(\frac{1}{r^3}\right)
\]

\[
8\pi GT^\theta_i = \frac{(r_\infty^2 + a^2)^4 - m(r_\infty^2 + a^2)^2}{32[(r_\infty^2 + a^2)^3 + mr_\infty^2(r_\infty^2 + a^2)]} \cdot \frac{1}{r^3} + O\left(\frac{1}{r^4}\right)
\]

\[
8\pi GT^\phi_i = \frac{(r_\infty^2 + a^2)^4 - m(r_\infty^2 + a^2)^2}{32[(r_\infty^2 + a^2)^3 + mr_\infty^2(r_\infty^2 + a^2)]} \cdot \frac{1}{r^3} + O\left(\frac{1}{r^4}\right)
\]

\[
8\pi GT^\psi_i = \frac{ma \left[ (r_\infty^2 + a^2)^2 + ma^2 \right]^{\frac{3}{2}}}{(r_\infty^2 + a^2)^2 + ma^2} \cdot \frac{1}{r^2} + O\left(\frac{1}{r^3}\right)
\]

\[
8\pi GT^\psi_\phi = \frac{2(r_\infty^2 + a^2)^4 - m(r_\infty^2 + a^2)^2 + m^2a^2(r_\infty^2 + a^2)^2}{4[(r_\infty^2 + a^2)^2 + ma^2](r_\infty^2 + a^2)^2} \cdot \frac{\cos \theta}{r^2} + O\left(\frac{1}{r^3}\right)
\]

\[
8\pi GT^\psi_\psi = \frac{2(r_\infty^2 + a^2)^4 - m(r_\infty^2 + a^2)^2 + m^2a^2(r_\infty^2 + a^2)}{4[(r_\infty^2 + a^2)^2 + ma^2](r_\infty^2 + a^2)^2} \cdot \frac{1}{r^2} + O\left(\frac{1}{r^3}\right)
\]

Associated with the Killing vector \( \partial_t \), the conserved mass is the same as [12].
Corresponding to $\partial_{\phi}$ and $\partial_{\bar{\phi}}$ respectively, the angular momenta are given by (43).

5.4 Comments and Conjectures

In a recent paper [4], Cai et.al. have got an expression similar to (42). In their notations, the IM black hole mass is

$$M_{IM} = M_{CT} = M_{AD} = \frac{\pi \left( r_\infty^4 - 3r_+^2r_-^2 + r_\infty^2(r_+^2 + r_-^2) \right)}{4\sqrt{(r_\infty^2 - r_-^2)(r_\infty^2 - r_+^2)}}$$

(46)

Besides a background subtraction method, i.e., the generalized Abbott-Deser method [14], they also employed the counter-term method [16].

Both (42) and (46) grow as $r_\infty$ increases, that is, as the size of the compactified dimension increases. This is a little counterintuitive. More surprisingly, when the black hole disappears, i.e., in the “empty” limit $r_+ = r_- = 0$, both masses tend to

$$M_{mon} = \frac{\pi r_\infty^2}{4}$$

(47)

We observe that (47) is just the mass expression for Kaluza-Klein monopoles in four dimensions [1, 16]. Furthermore, if we set $r_+ = 0$ and $r_- = 0$, the metrics for both IM and Kerr-IM black holes simply reduce to the metric for Kaluza-Klein monopoles [1], whose geometry is perfectly regular in five dimensions. This indicates the physical meaning of (42) and (46): They express the masses of the Kaluza-Klein black holes in the sight of four dimensions. The Kaluza-Klein charges [1] account for their counterintuitive behaviors. In a five-dimensional point of view, there are no Kaluza-Klein charges, and (47) is the contribution of energy from background geometry. A further observation is that if we subtract (47) from (46), the first law of thermodynamics checked in [4] will not be affected, for they have fixed the value of $r_\infty$.

Strictly speaking, the expressions (42) and (46) for mass are reliable only in the case of a finite $r_\infty$ value, as we remarked in the end of Subsection 2.3. Yet we want to extrapolate the result to the limit $r \to r_\infty$. Then we propose the following conjectures for an arbitrary value of positive $r_\infty$.

1. The formula (46) represents the mass of IM black holes observed in four dimensions. Subtracting the monopole contribution (47), we conjecture the mass well-defined in five dimensions

$$M_{IM} = \frac{\pi \left( r_\infty^4 - 3r_+^2r_-^2 + r_\infty^2(r_+^2 + r_-^2) \right)}{4\sqrt{(r_\infty^2 - r_-^2)(r_\infty^2 - r_+^2)}} - \frac{\pi r_\infty^2}{4}$$

(48)
2. Similarly, the formula (42) represents the mass of Kerr-IM black holes observed in four dimensions. Subtracting the monopole contribution (47), we conjecture the mass well-defined in five dimensions

$$M_{Kerr-IM} = \frac{\pi}{4} \times \left[ \frac{(r_{\infty}^2 + a^2)^2 - ma^2}{\sqrt{(r_{\infty}^2 + a^2)^2 + ma^2}} \times \frac{(r_{\infty}^2 + a^2)^2 + m(r_{\infty}^2 + 2a^2)}{(r_{\infty}^2 + a^2)\sqrt{(r_{\infty}^2 + a^2)^2 - mr_{\infty}^2}} \right]$$

Here we have considered that the size of the compactified dimension is controlled by $r'_{\infty}$ instead of $r_{\infty}$.

Obviously, the masses $M_{IM}$ and $M_{Kerr-IM}$ also satisfy the first law of black hole thermodynamics in five dimensions. To inspect and improve our conjectures, one may check the law in four dimensions with $M_{IM}$ and $M_{Kerr-IM}$, by including monopole charges and allowing $r'_{\infty}$ variable. It is remarkable that in the “flat” limit $r \to r_{\infty}$, we find $M_{IM} \to \frac{3\pi(r_{\infty}^2 + r_{\infty}^2)}{8}$ and $M_{Kerr-IM} \to \frac{3\pi m}{8}$ respectively, which are the masses of ordinary five-dimensional Reissner-Nordström black holes and Kerr black holes respectively. In the “empty” limit $r_+ = r_- = 0$, both $M_{IM}$ and $M_{Kerr-IM}$ vanish. It would be interesting and important to look for a new background and a new counter-term to “fundamentally” produce the masses (48) and (49), and at the same time the angular momenta (43). But it is not the aim of this article.

6 Summary

In this paper, we have extended the recently found static Kaluza-Klein black holes with squashed horizons [2] to rotating counterparts. Our investigations are restricted to black holes with two equal angular momenta. These counterparts are geodesic complete and free of naked singularities. The thermodynamic quantities have been calculated and the first law has been checked. The angular momenta are computed using Komar integrals, then reproduced with the counter-term method. We calculated the mass by both background subtraction and counter-term method, and found a counterintuitive behavior as the compactified dimension expands. This behavior is also hidden in the mass of static black holes. We attributed it to Kaluza-Klein charges and gave some comments on it.

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