Entropy theory for Delone sets in locally compact abelian groups

T. Hauser

February 15, 2019

Abstract

We extend the notion of patch counting entropy to Delone sets without finite local complexity (FLC). We show that for all Delone sets patch counting entropy equals the topological entropy of the corresponding Delone dynamical systems. Furthermore we present, that the classical definition can be obtained as a limit and is independent from the choice of a "uniformly growing" Van Hove net. The main tool of our investigation is the Ornstein-Weiss lemma. We will thus present our results for Delone sets in locally compact abelian groups for which a suitable version of this lemma holds, as for example $\mathbb{R}^d$ or any compactly generated locally compact abelian group.

Mathematics Subject Classification (2000): 37B40, 37A35, 52C23, 37B10.

Keywords: Patch counting entropy, Delone set, Entropy, Locally compact abelian group, Dynamical system, Ornstein-Weiss lemma.

1 Introduction

The study of structural properties of Delone sets is a key topic in the context of quasicrystals and aperiodic order. Such sets are defined as relatively dense and uniformly discrete subsets of $\mathbb{R}^d$ or more general groups, which are usually assumed to be locally compact abelian groups (LCAG). The concept of a Delone set can be seen as a mathematical abstraction for the positions of atoms in a solid state material, which in addition yields a physical motivation for this investigation. An important tool in the study of Delone sets is the patch counting entropy of a Delone set $\omega$. Patch counting entropy is classically defined as

$$\limsup_{n \to \infty} \frac{\log(|\text{Pat}_\omega(B_n)|)}{\mu(B_n)},$$

where $(B_n)_{n \in \mathbb{N}}$ is the sequence of centred closed balls of radius $n \in \mathbb{N}$ and

$$\text{Pat}_\omega(A) := \{ (\omega - g) \cap A : g \in \omega \}$$

is the set of all $A$-patches for a compact set $A$. See [Lag99], [LP03] or [BLR07] for reference. In [HR15] Definition 4.1] patch counting entropy is introduced for LCAG and the average is computed along Van Hove sequences, like $(B_n)_{n \in \mathbb{N}}$. It is remarked there that it is not clear under which assumptions this notion is independent from the choice of the Van Hove net. In Theorem 1.2 and Theorem 1.4 we will present a sufficient condition for the independence.

To every Delone set in a LCAG a dynamical system, called the Delone dynamical system, can be associated and geometrical properties of Delone sets translate as dynamical properties of the corresponding Delone dynamical system. The classical definition of patch counting entropy always gives the value infinity, whenever there is a compact set $A$ such that there are infinitely many $A$-patches. Delone sets with only finitely many $A$-patches for every compact $A$ are called Delone sets of finite local complexity. In [BLR07] it is shown that the patch counting entropy

$$\limsup_{n \to \infty} \frac{\log(|\text{Pat}_\omega(B_n)|)}{\mu(B_n)},$$

is independent from the choice of the Van Hove net.
corresponds to the topological entropy of the Delone dynamical system whenever $\omega \subseteq \mathbb{R}^d$ is of finite local complexity. We will present in Example 3.3 a Delone set without finite local complexity, hence infinite classical patch counting entropy for which the corresponding Delone dynamical system has zero topological entropy.

This indicates that the classical definition of patch counting entropy is not well-suited for Delone sets without finite local complexity. We thus give a definition that extends the one from the literature beyond sets with finite local complexity in such a way that it corresponds to topological entropy of the associated Delone dynamical system. During this paper we assume $G$ to be a LCAG for which a suitable version of the Ornstein-Weiss lemma is satisfied. We call such groups Ornstein-Weiss groups. See Subsection 2.4 for a precise definition. Examples of Ornstein-Weiss groups are all LCAG that contain a uniform lattice and in particular all compactly generated LCAG. For reference see Section 2 below. For subsets $\xi, \zeta \subseteq G$, $A \subseteq G$ compact and an open neighbourhood $V$ of the identity element we write

$$\xi \overset{A,V}{\approx} \zeta,$$

whenever $\xi$ and $\zeta$ agree in $A$ up to an error of $V$, i.e. if $\xi \cap A \subseteq \zeta + V$ and $\zeta \cap A \subseteq \xi + V$.

**Definition 1.1.** Let $G$ be an abelian Ornstein-Weiss group. For a Delone set $\omega \subseteq G$ we say that $F \subseteq \omega$ is an $A$-patch representation of scale $V$ for $\omega$, if for any $g \in \omega$ there is $f \in F$ s.t.

$$\omega - f A, V \overset{V}{\approx} \omega - g.$$

In Remark 5.7 we will present that there is always a finite $A$-patch representation of scale $V$ for $\omega$. We define $\text{pat}_A(A, V)$ as the minimal cardinality of an $A$-patch representation of scale $V$ for $\omega$. We define the *patch counting entropy* of the Delone set $\omega$ as

$$E_{pc}(\omega) := \sup_{V \in \mathcal{N}(G)} \sup_{K \in \mathcal{K}(G)} \limsup_{n \in I} \frac{\log(\text{pat}_A(A_n + K, V))}{\mu(A_n + K)}$$

for a point absorbing Van-Hove net $(A_n)_{n \in I}$ in $G$. In Remark 5.10 we will show that this notion is independent from the choice of a point absorbing Van-Hove net.

This definition can be simplified if we consider point absorbing and uniformly growing Van Hove sequences $(A_n)_{n \in \mathbb{N}}$. A Van Hove sequence is called uniformly growing, if for every compact $K \subseteq G$ there are $m \in \mathbb{N}$ and $C \subseteq G$ compact with $A_n + K \subseteq A_{n+m} + K \subseteq A_n + C$ for $n \in \mathbb{N}$. Important examples of a uniformly growing Van Hove sequences are the sequence of closed centred balls $(B_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$, the sequence of closed hypercubes $([-n, n]^d)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$ and the sequence $([-n, \cdots , n]^d)_{n \in \mathbb{N}}$ in $\mathbb{Z}^d$. Note that every uniformly growing Van Hove sequence in a compactly generated LCAG is point absorbing (see Remark 2.3).

**Theorem 1.2.** Let $G$ be an abelian Ornstein-Weiss group and $\omega \subseteq G$ be a Delone set. For every point absorbing and uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there holds

$$E_{pc}(\omega) = \sup_{V \in \mathcal{N}(G)} \limsup_{n \in \mathbb{N}} \frac{\log(\text{pat}_A(A_n, V))}{\mu(A_n)}.$$

A proof is given in Section 3. As Delone dynamical systems are naturally defined as actions of locally compact abelian groups acting on compact uniform spaces, we will extend the investigation of topological entropy of such actions from [Han19] in Section 3. This will allow us to show in Subsection 5.2 that the patch counting entropy of a Delone set $\omega$ equals the topological entropy $E(\pi_{\omega})$ of the corresponding Delone dynamical system $\pi_{\omega}$.

**Theorem 1.3.** Let $G$ be an abelian Ornstein-Weiss group. For every Delone set $\omega \subseteq G$ there holds $E_{pc}(\omega) = E(\pi_{\omega})$.

In Section 4 we will present that our definition extends the classical one. In [LP03] it is claimed that for Delone sets of finite local complexity in $\mathbb{R}^d$ the averaging along $(B_n)_{n \in \mathbb{N}}$ can be obtained as a limit. We present that this holds true in the following more general context.
Theorem 1.4. Let $G$ be a non-compact abelian Ornstein-Weiss group and $\omega$ be a Delone set of finite local complexity in $G$. For every point absorbing and uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there holds

$$E_{pc}(\omega) = \lim_{n \to \infty} \frac{\log(|\text{Pat}_\omega(A_n)|)}{\mu(A_n)}.$$

Finally we obtain that the patch counting entropy can always be computed with a fixed neighbourhood $V$, whenever $V$ is sufficiently small.

Theorem 1.5. Let $\omega$ be a Delone set of finite local complexity in a non-compact but compactly generated LCAG. For every uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there exists $V \in \mathcal{N}(G)$ such that for all $W \in \mathcal{N}(G)$ with $W \subseteq V$ there holds

$$E_{pc}(\omega) = \lim_{n \to \infty} \frac{\log(pat(A_n, W))}{\mu(A_n)}.$$

2 Preliminaries

In this section we provide notion and background on topological groups, uniform spaces, topological dynamical systems, Delone sets, amenable groups, Van Hove nets, uniform lattices and Ornstein-Weiss groups.

2.1 Topological groups

Consider a group $G$. We write $e_G$ for the neutral element in $G$. For subsets $A, B \subseteq G$ the Minkowski product is defined as $AB := \{ab; (a, b) \in A \times B\}$. For $A \subseteq G$ and $g \in G$ one also writes $Ag := A\{g\}$ and $gA := \{g\}A$. Furthermore we define the complement $A^c := G \setminus A$ and the inverse $A^{-1} := \{a^{-1}; a \in A\}$ of a subset $A \subseteq G$. We call a set $A \subseteq G$ symmetric, if $A = A^{-1}$. In order to omit brackets, we will use the convention, that the operation of taking the Minkowski product of two sets is stronger binding than set theoretic operations, except from taking the complement; and that the inverse and the complement are stronger binding than the Minkowski product. Note that the complement and the inverse commute, i.e. $(A^c)^{-1} = (A^{-1})^c$ for any $A \subseteq G$.

A topological group is a group $G$ equipped with a $T_0$-topology\footnote{A topology is called $T_0$, if for any two distinct points $g, g' \in G$ there is an open neighbourhood of $g$ that does not contain $g'$.}, such that the multiplication $\cdot: G \times G \to G$ and the inverse function $(-)^{-1}: G \to G$ are continuous. With our definition every topological group is regular, hence Hausdorff, as shown in [HR12] Theorem 4.8. An isomorphism of topological groups is a homeomorphism that is a group homomorphism as well. We write $\overline{A}$ for the closure of a subset $A \subseteq G$. We denote $K(G)$ for the set of all non-empty compact subsets of $G$, $\mathcal{A}(G)$ for the set of all closed subsets and $\mathcal{N}(G)$ for the set of all open neighbourhoods of $e_G$. Furthermore we abbreviate the term abelian locally compact group by LCAG. We will denote the action of a LCAG $G$ by $\cdot$, the inverse of $g \in G$ by $-g$ and the neutral element by 0. Therefore we write $A + B$ for the Minkowski product of subsets $A, B \subseteq G$ and similar notation. We furthermore define inductively for $n \in \mathbb{N}$ that $0 \odot A := \{0\}$ and $n \odot A := ((n-1) \odot A) + A$. We use the convention similar to natural numbers, that $\odot$ binds stronger than the Minkowski sum.

If $G$ is a locally compact group, a left (resp. right) Haar measure on $G$ is a non zero regular Borel measure $\mu$ on $G$, which satisfies $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) for all $g \in G$ and all Borel measurable subsets $A \subseteq G$. We call a measure on $G$ a Haar measure, if it is a right and a left Haar measure. A locally compact group that possesses a Haar measure is called a unimodular group. In particular all LCAG are unimodular. There is $\mu(U) > 0$ for all non empty open $U \subseteq G$ and $\mu(K) < \infty$ for all compact $K \subseteq G$. A Haar measure is unique up to scaling, i.e. if $\mu$ and $\nu$ are Haar measures on $G$, then there is $c > 0$ such that $\mu(A) = c\nu(A)$ for all Borel measurable sets $A \subseteq G$. If nothing else is mentioned, we denote a Haar measure of a topological group $G$ by $\mu$. If $G$ is a discrete group, then the counting
2.2 Uniform spaces

2.2.1 Binary relations

Let $X$ be a set. A binary relation on $X$ is a subset of $X \times X$. For binary relations $\eta$ and $\kappa$ on $X$ we denote the inverse $\eta^{-1} := \{(y, x); (x, y) \in \eta\}$ and the composition $\eta \kappa := \{(x, z) \in X : (x, z) \in \eta \text{ and } (z, y) \in \kappa\}$. A binary relation is called symmetric, if $\eta = \eta^{-1}$. For $\eta \subseteq X \times X$ and $x \in X$ we write $[x] \eta := \{y \in X; (x, y) \in \eta\}$ and $\eta[x] := [x] \eta^{-1}$. For $M \subseteq X$ we denote $[M] \eta := \bigcup_{x \in M} [x] \eta$ and $\eta[M] := [M] \eta^{-1}$.

2.2.2 Uniform spaces

A uniformity for a set $X$ is a non-empty family $\mathbb{U}_X$ of subsets of $X \times X$ such that

(a) each member of $\mathbb{U}_X$ contains the diagonal $\Delta_X$;
(b) if $\eta \in \mathbb{U}_X$, then $\eta^{-1} \in \mathbb{U}_X$;
(c) if $\eta \in \mathbb{U}_X$, then there is $\kappa \in \mathbb{U}_X$ such that $\kappa \kappa \subseteq \eta$;
(d) if $\eta$ and $\kappa$ are members of $\mathbb{U}_X$, then so is $\eta \cap \kappa$; and
(e) if $\eta \in \mathbb{U}_X$ and $\eta \subseteq \kappa \subseteq X \times X$, then $\kappa \in \mathbb{U}_X$.

The pair $(X, \mathbb{U}_X)$ is called a uniform space and the members of $\mathbb{U}_X$ are called entourages. An entourage $\eta \in \mathbb{U}_X$ is called open (or closed), whenever it is open (or closed) with respect to the product topology on $X \times X$. A subfamily $\mathbb{B}_X \subseteq \mathbb{U}_X$ is called a base for $\mathbb{U}_X$, if every entourage contains a member of $\mathbb{B}_X$. The family of all open and symmetric entourages form a base of the corresponding uniform space. If $(X, \mathbb{U}_X)$ is a uniform space the corresponding uniform topology $T_X$ consists of all subsets $U \subseteq X$ such that for each $x \in U$ there exists $\eta \in \mathbb{U}_X$ with $\eta[x] \subseteq U$. Topological terminology in the context of uniform spaces refers to this topology.

For a map $f : X \to Y$ we write $f \times f : X \times X \to Y \times Y$ for the map with $(f \times f)(x, y) := (f(x), f(y))$. A map $f : X \to Y$ between uniform spaces $(X, \mathbb{U}_X)$ and $(Y, \mathbb{U}_Y)$ is called uniformly continuous, if the preimage of every entourage of $Y$ under $f \times f$ is an entourage of $X$. Every uniformly continuous map between uniform spaces is continuous with respect to the corresponding uniform topologies. The reverse holds true, whenever the domain of the map is assumed to be compact, as shown in [Kel17, Theorem 6.31]. For further notions on uniform spaces, like the product of uniform spaces, see [Kel17].

Example 2.1. (i) If $(X, d)$ is a metric space we define for $\varepsilon > 0$

$$[d < \varepsilon] := \{(x, y) \in X \times X; d(x, y) < \varepsilon\}.$$ 

Then $\mathbb{B}_d := \{[d < \varepsilon]; \varepsilon > 0\}$ is a base for the uniformity $\mathbb{U}_X := \{\eta \subseteq X \times X; \exists \varepsilon > 0 : [d < \varepsilon] \subseteq \eta\}$.

The corresponding topology is the topology of open sets with respect to $d$.

(ii) Every compact Hausdorff space $X$ has a unique uniformity $\mathbb{U}_X$ consisting of all neighbourhoods of the diagonal $\Delta_X$ in $X \times X$. This can be obtained from the combination of [Kel17, Theorem 6.22] with [Mun00, Theorem 32.3].

For $\eta \in \mathbb{U}_X$ and $(x, y) \in \eta$, we say that $x$ is $\eta$-close to $y$. This notion is symmetric, if and only if $\eta$ is symmetric. If $x$ is $\eta$-close to $y$ and $y$ is $\kappa$-close to $z$, then $x$ is $\eta \kappa$-close to $z$. If $(X, d)$ is a metric space, then $x$ is $[d < \varepsilon]$-close to $y$, if and only if $d(x, y) < \varepsilon$. 

4
2.3 Actions of a group on a topological space

Let $G$ be a topological group and $X$ be a topological space. A continuous map $\pi: G \times X \to X$ is called an action of $G$ on $X$ (also dynamical system or flow), whenever $\pi(g, \cdot)$ is the identity on $X$ and for all $g, g' \in G$ there holds $\pi(g, \pi(g', \cdot)) = \pi(gg', \cdot)$. We write $\pi^g := \pi(g, \cdot) : X \to X$ for all $g \in G$. If $\pi$ and $\varphi$ are actions of a topological group $G$ on topological spaces $X$ and $Y$ respectively, we call a surjective continuous map $p: X \to Y$ a factor map, if $p \circ \pi^g = \varphi^g \circ p$ for all $g \in G$. We then refer to $\varphi$ as a factor of $\pi$ and write $\pi \to \varphi$. If $p$ is in addition a homeomorphism, then $p$ is called a topological conjugacy and we call $\pi$ and $\varphi$ topological conjugate.

2.4 Delone sets and Delone dynamical systems

Let $G$ be a LCAG. For $K \in \mathcal{K}(G)$ a subset $\omega \subseteq G$ is called $K$-dense in $G$, if $K + \omega = G$. Furthermore $\omega \subseteq G$ is called relatively dense, if there is $K \in \mathcal{K}(G)$ such that $\omega$ is $K$-dense in $G$. For $V \in \mathcal{N}(G)$ a subset $\omega \subseteq G$ is called $V$-discrete, if $\{V + g; g \in \omega\}$ is a disjoint family. Furthermore $\omega \subseteq G$ is called uniformly discrete, if it is $V$-discrete for some $V \in \mathcal{N}(G)$. A relatively dense and uniformly discrete subset $\omega \subseteq G$ is called a Delone set. For $K \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ we denote by $\mathcal{D}_{K,V}(G)$ the set of all $K$-dense and $V$-discrete subsets of $G$. Note that $G$ is assumed to be Hausdorff. Thus every $V$-discrete subset of $G$ is discrete and we get $\mathcal{D}_{K,V}(G) \subseteq \mathcal{A}(G)$. For $A \subseteq G$ compact and $g \in \omega$, we call $\omega - g \cap A$ an $A$-patch of $\omega \subseteq G$ and denote the set of all $A$-patches by $\text{Pat}_\omega(A)$. A Delone set is said to have finite local complexity (FLC), if $\text{Pat}_\omega(A)$ is finite for every compact set $A \subseteq G$. For $K \in \mathcal{K}(G)$, $V \in \mathcal{N}(G)$ and $\xi, \zeta \in \mathcal{A}(G)$ we denote

$$\xi \underset{K,V}{\approx} \zeta,$$

whenever there is $\xi \cap K \subseteq \zeta + V$ and $\xi \cap K \subseteq \xi + V$. Furthermore we define

$$\varepsilon(K,V) := \left\{ (\xi, \zeta) \in \mathcal{A}(G)^2; \xi \underset{K,V}{\approx} \zeta \right\}.$$

The set

$$\mathbb{B}_{\varepsilon} := \{\varepsilon(K,V); (K,V) \in \mathcal{K}(G) \times \mathcal{N}(G)\}$$

defines a base for a uniformity $U_{\mathcal{A}(G)}$ on $\mathcal{A}(G)$. We call this base the local rubber base. The uniformity is called the local rubber uniformity. In [HL04] Theorem 3 it is shown, that the corresponding topology, called local rubber topology, is a compact Hausdorff topology. For a Delone set $\omega \subseteq G$ we denote

$$D_\omega := \{\omega + g; g \in G\}$$

and $X_\omega$ for the closure of $D_\omega$ with respect to the local rubber topology. Then $X_\omega$ is a compact uniform space with base

$$\mathbb{B}_{\varepsilon}(\omega) := \{\varepsilon_\omega(K,V); (K,V) \in \mathcal{K}(G) \times \mathcal{N}(G)\},$$

where we denote $\varepsilon_\omega(K,V) := \varepsilon(K,V) \cap (X_\omega \times X_\omega)$ for the restricted entourages. We call $\mathbb{B}_{\varepsilon}(\omega)$ the (restricted) locally rubber base and define the Delone dynamical system

$$\pi_\omega: G \times X_\omega \to X_\omega$$

by $\pi_\omega(g, \xi) := \xi + g$. The continuity of this action is shown in [HL04]. If $\omega \subseteq G$ is a Delone set of finite local complexity there is another base of $U_{X_\omega}$ that allows more control over the considered sets. For $K \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ let

$$\eta(K,V) := \{(\xi, \zeta) \in \mathcal{K}(G)^2; \exists x, z \in V: (\xi + x) \cap K = (\zeta + z) \cap K\}.$$

Furthermore denote $\eta_\omega(K,V) := \eta(K,V) \cap (X_\omega \times X_\omega)$ for every Delone set $\omega \subseteq G$. 

5
Proposition 2.2. If $\omega \subseteq G$ is a Delone set with finite local complexity, then
\[ B_{\lfloor m \rfloor}(\omega) := \{ \eta_m(K,V); (K,V) \in K(G) \times N(G) \} \]
is a base of $U_{X_\omega}$. We will refer to this base as the local matching base of $U_{X_\omega}$.

Proof. It is shown in [BL04, Prop. 4.5] that the topology on $X_\omega$ induced by the uniformity generated by $B_{\lfloor m \rfloor}(\omega)$ is a compact Hausdorff topology and equals the topology induced by $U_{X_\omega}$. As compact Hausdorff spaces have a unique uniformity, the uniformities agree and $B_{\lfloor m \rfloor}(\omega)$ is a base for $U_{X_\omega}$.

2.5 Nets and convergence of nets

A partially ordered set $(I, \geq)$ is said to be directed, if $I$ is not empty and if every finite subset of $I$ has an upper bound. A map $f$ from a directed set $I$ to a set $X$ is called a net in $X$. We also write $x_i$ for $f(i)$ and $(x_i)_{i \in I}$ for $f$. A net $(x_i)_{i \in I}$ in a topological space $X$ is said to converge to $x \in X$, if for every open neighbourhood $U$ of $x$, there exists $j \in I$ such that $x_i \in U$ for all $i \geq j$. In this case we also write $\lim_{i \in I} x_i = x$. For a net $(x_i)_{i \in I}$ in $\mathbb{R} \cup \{-\infty, \infty\}$, we define $\limsup_{i \in I} x_i := \inf_{i \in I} \sup_{j \geq i} x_j$ and similarly $\liminf_{i \in I} x_i$. Note that $(x_i)_{i \in I}$ converges to $x \in \mathbb{R} \cup \{-\infty, \infty\}$, if and only if there holds $\lim_{i \in I} x_i = x = \liminf_{i \in I} x_i$. For more details, see [DS58] and [Kel17].

2.6 Amenable groups and Van Hove nets

Let $G$ be a unimodular group. For $K, A \subseteq G$ we define the $K$-boundary of $A$ as
\[ \partial_K A := K \overline{A} \cap \overline{K} \backslash A. \]

We use the convention, that the Minkowski product is stronger binding as the operation of taking the $K$-boundary and that the set theoretic operations, except from complementation, are weaker binding. From the definition we obtain that $K \mapsto \partial_K A$ is monotone. Note that $\partial_K A$ is the set of all elements $g \in G$ such that $K^{-1}g$ intersects both $\overline{A}$ and $\overline{K} \backslash A$.

A net $(A_i)_{i \in I}$ of measurable subsets of $G$ is called finally somewhere dense, if there is $j \in I$ such that for all $i \geq j$ the set $A_i$ is somewhere dense.\footnote{A subset of a topological space is called somewhere dense, if it has nonempty interior. This ensures $\mu(A_i) > 0.$} A finally somewhere dense net $(A_i)_{i \in I}$ of compact subsets of $G$ is called a Van Hove net, if for all $K \subseteq G$ compact, there holds
\[ \lim_{i \in I} \frac{\mu(\partial_K A_i)}{\mu(A_i)} = 0. \]

A unimodular group is called amenable, whenever there is a Van Hove net in $G$. A net $(A_i)_{i \in I}$ of subsets of $G$ we call point absorbing, whenever for every $g \in G$ there is some $j \in I$ such that for all $i \geq j$ there is $g \in A_i$. A Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ we call uniformly growing, if for every compact $K \subseteq G$ there are $m \in \mathbb{N}$ and $C \subseteq G$ compact with $A_n K \subseteq A_{n+m} \subseteq A_n C$ for every $n \in \mathbb{N}$. We refer to [Hau19] for the definition of Van Hove nets.

Remark 2.3. (i) The definition of Van Hove nets given above is equivalent to the concepts of Van Hove nets (and sequences respectively) given in [Tem], in [Sch99] and in [FGJOI18]. This is shown in [Hau19, Proposition 2.5].

(ii) A finally somewhere dense net $(A_i)_{i \in I}$ is called a Følner net, if for every $g \in G$ there holds
\[ \lim_{i \in I} \frac{\mu(A_i \Delta A g)}{\mu(A_i)} = 0. \]

Here $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of two sets. Every Van Hove net is a Følner net and the reverse holds true in discrete groups. In [Tem] Appendix; Ex. 3.4] an example of a Følner net in $\mathbb{R}^d$ is presented, that is not a Van Hove net. Nevertheless it can be shown that a unimodular group contains a Van Hove net if and only if it contains a Følner net. Thus our definition of amenability coincides with the definition given in the literature. See [Hau19] for reference.
(iii) Every LCAG is amenable, as shown in [Pie84, Proposition 12.2].

(iv) It is shown in [Hau19, Proposition 2.3] that for compact subsets $C, K \subseteq G$ and a Van Hove net $(A_i)_{i \in I}$ there holds
\[
\lim_{i \in I} \frac{\mu(C A_i)}{\mu(K A_i)} = 1.
\]

(v) If $(A_i)_{i \in I}$ is a point absorbing Van Hove net in $G$ and $G$ is not compact, then there holds $\lim_{i \in I} \mu(A_i) = \infty$. To see this, let $C \subseteq G$ be a compact neighbourhood of $0$. As $(A_i)_{i \in I}$ is a Van Hove net we obtain $\lim_{i \in I} \frac{\mu(C A_i)}{\mu(A_i)}$. For $K \subseteq G$ compact there is a finite set $F \subseteq G$ such that $K \subseteq CF$. Furthermore as $(A_i)_{i \in I}$ is point absorbing, there is $j \in I$ such that $F \subseteq A_j$ for $i \geq j$ and we obtain $K \subseteq CF \subseteq A_i$ for $i \geq j$. Thus $\mu(K) \leq \inf_{i \in I} \mu(C A_i) = \inf_{i \in I} \mu(A_i)$ for every compact set $K \subseteq G$. As $G$ is not compact there holds $\sup_{K \in \mathcal{K}(G)} \mu(K) = \infty$ and we obtain $\lim_{i \in I} \mu(A_i) = \infty$, hence the claimed statement.

(vi) Every uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ in a compactly generated LCAG is point absorbing. To see this choose a compact neighbourhood $K$ of $0$ and $C \in \mathcal{K}(G)$ and $m \in \mathbb{N}$ such that $A_n + K \subseteq A_{n+m} \subseteq A_n + C$. As $G$ is compactly generated for $g \in G$ there is $l \in \mathbb{N}$ such that for $n \leq m$ and $k \geq l$ there holds $g \in A_n + l \cdot K \subseteq A_n + k \cdot K \subseteq A_{n+km}$. This shows $g \in A_n$ for all $n \geq lm + 1$.

(vii) The sequence $(B_n)_{n \in \mathbb{N}}$ of centred closed balls $B_n \subseteq \mathbb{R}^d$ is uniformly growing. More generally the Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ with $A_n := [-n, n]^d \times \{-n, \ldots, n\}^m \times C$ is uniformly growing in $G = \mathbb{R}^d \times \mathbb{Z}^m \times C$, where $C$ is a compact group. As every compactly generated LCAG is algebraically and topologically isomorphic to such a group, we find every compactly generated LCAG to contain a uniformly growing Van Hove sequence.

(viii) The Van Hove sequence $([-n^2, n^2])_{n \in \mathbb{N}}$ in $\mathbb{R}$ is point absorbing but not uniformly growing.

### 2.7 Ornstein-Weiss groups

The Ornstein-Weiss lemma is the key tool in order to define entropy for amenable groups. We will thus introduce the following notion. A function $f: \mathcal{K}(G) \to \mathbb{R}$ is called **subadditive**, if for all $A, B \in \mathcal{K}(G)$ there holds
\[
f(A \cup B) \leq f(A) + f(B).
\]

Furthermore a mapping $f: \mathcal{K}(G) \to \mathbb{R}$ is said to be **right invariant**, if for all $A \in \mathcal{K}(G)$ and for all $g \in G$ there holds
\[
f(A g) = f(A).
\]

A function $f: \mathcal{K}(G) \to \mathbb{R}$ is called **monotone**, if for all $A, B \in \mathcal{K}(G)$ with $A \subseteq B$ there holds
\[
f(A) \leq f(B).
\]

An amenable group $G$ is called an **Ornstein-Weiss group**, if for any subadditive, right invariant and monotone function $f: \mathcal{K}(G) \to \mathbb{R}$ and for every Van Hove net $(A_i)_{i \in I}$ in $G$ the limit
\[
\lim_{i \in I} \frac{f(A_i)}{\mu(A_i)}
\]
exists, is finite and does not depend on the choice of the Van Hove net.
2.8 Examples

The following examples and further non-abelian examples are considered in [Hau19] Example 2.10.

**Example 2.4.** In [Hau19] Theorem 3.1 it is shown that every LCAG that contains a uniform lattice is an Ornstein-Weiss group. A uniform lattice is a discrete subgroup \( \Lambda \subseteq G \) such that there is a pre-compact and Borel measurable \( C \subseteq G \) that contains \( e_G \), satisfies \( 0 < \mu(C) \) and the following property. Each \( g \in G \) can be written uniquely as \( g = c z \) with \( c \in C \) and \( z \in \Lambda \). The set \( C \) is called a fundamental domain for \( \Lambda \) and satisfies \( 0 < \mu(C) \leq \mu(G) < \infty \).

**Example 2.5.** An amenable discrete group \( G \) contains the uniform lattice \( G \) with fundamental domain \( \{ e_G \} \). In particular all abelian discrete groups are Ornstein-Weiss groups. See also [Kri10] Theorem 1.1.] or [CSCK14] Theorem 1.1] for a direct proof that discrete amenable groups are Ornstein-Weiss groups.

**Example 2.6.** The Euclidean space \( \mathbb{R}^d \) is an Ornstein-Weiss group, as it contains the countable uniform lattice \( \mathbb{Z}^d \) with fundamental domain \([0,1]^d\).

**Example 2.7.** A compactly generated locally compact abelian group \( G \) is and Ornstein-Weiss group, as it contains a countable uniform lattice. See [Hau19] for reference.

2.9 Topological entropy for actions of Ornstein-Weiss groups on uniform spaces

As a Delone dynamical system acts naturally on compact uniform spaces we will now introduce topological entropy of actions of Ornstein-Weiss groups on compact uniform spaces. See [Hau19] for further reference.

2.9.1 Bowen action

For an action \( \pi: G \times X \rightarrow X \) on a compact uniform space we define the corresponding Bowen action \( \hat{\pi}: K(G) \times \mathcal{U}_X \rightarrow \mathcal{U}_X \) by \( \hat{\pi}(A, \eta) := \eta_A \), where

\[
\eta_A := \{ (x, y); \forall g \in A : (\pi^g(x), \pi^g(y)) \in \eta \} = \bigcap_{g \in A} (\pi^g \times \pi^g)^{-1}(\eta).
\]

It is shown in [Hau19] Lemma 4.2, that the continuity of \( \pi \) and the compactness of \( X \) yield that the image of the Bowen action is indeed contained in \( \mathcal{U}_X \). In order to omit brackets we will use the convention, that the Bowen action is a stronger operation than the product of entourages. The following rules are straightforward to prove. Note that (i) justifies that we can omit brackets and write \( \eta_{AB} \) for \( \eta_{(AB)} = (\eta_A)_B \).

**Remark 2.8.** For \( \eta, \kappa \in \mathcal{U}_X \) and \( A, B \subseteq G \) compact there holds

(i) \( \eta_{(AB)} = (\eta_A)_B \),

(ii) \( \eta_{A \cup B} = \eta_A \cap \eta_B \) and

(iii) \( \eta_{A \cap \kappa} \subseteq (\eta \kappa)_A \).

2.9.2 Topological entropy via small open covers

For \( \eta \in \mathcal{U}_X \) we say that a subset \( M \subseteq X \) is \( \eta \)-small, if \( M^2 \subseteq \eta \). We say, that a set \( \mathcal{U} \) of subsets of \( X \) is of scale \( \eta \), if \( U \) is \( \eta \)-small for every \( U \in \mathcal{U} \). We denote by \( \text{cov}_X(\eta) \) the minimum cardinality of an open cover of \( X \) of scale \( \eta \). This quantity is well defined by the compactness of \( X \). It is presented in [Hau19] Subsection 4.2] that the function \( K(G) \ni A \mapsto \log(\text{cov}_X(\eta_A)) \) is monotone, sub-additive and right invariant for every \( \eta \in \mathcal{U}_X \). If \( G \) is assumed to be an Ornstein-Weiss group, then the limit in the following definition of relative topological entropy exists and is independent from the choice of the Van Hove net.

\(^3\text{A subset } A \text{ of a topological space } X \text{ is called pre-compact, whenever the closure } \overline{A} \text{ is compact in } X.\)
Lemma 3.1. Let $\pi : G \times X \to X$ be an action of an Ornstein-Weiss group $G$ on a compact uniform space $X$. For any Van Hove net $(A_n)_{n \in \mathbb{N}}$ and $\eta \in \mathcal{U}_X$, we define the topological entropy of $\pi$ on scale $\eta$ as

$$E(\eta|\pi) := \lim_{n \to \infty} \frac{\log(\text{cov}_X(f|_{\eta A_n}))}{\mu(A_n)}.$$ 

We furthermore define the topological entropy of $\pi$ as

$$E(\pi) := \sup_{\eta \in \mathcal{U}_X} E_{\text{top}}(\eta, \pi).$$

3 Patch counting entropy via uniformly growing Van Hove sequences

In this section we show that the notion of patch counting entropy, defined in Definition 4.1, simplifies as claimed in Theorem 1.2.

Lemma 3.1. Let $(A_n)_{n \in \mathbb{N}}$ be a uniformly growing Van Hove sequence. Then for every $K \subseteq G$ compact there is $m \in \mathbb{N}$ such that $A_n + K \subseteq A_{n+m}$ and $\lim_{n \to \infty} \frac{\mu(A_{n+m})}{\mu(A_n + K)} = 1$.

Proof. Let $K \subseteq G$ be a compact subset. Then there is $m \in \mathbb{N}$ and $C \subseteq G$ with $A_n + K \subseteq A_{n+m} \subseteq A_n + C$ and we obtain by Remark 2.3(iv) that

$$1 = \frac{\mu(A_n + K)}{\mu(A_n + K)} \leq \frac{\mu(A_{n+m})}{\mu(A_n + K)} \leq \frac{\mu(A_n + C)}{\mu(A_n + K)} n \to \infty 1.$$

Proposition 3.2. Let $f : K(G) \to [0, \infty]$ be a monotone function. Then for every uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there holds

$$\limsup_{n \to \infty} \frac{f(A_n)}{\mu(A_n)} = \sup_{K \in K(G)} \limsup_{n \to \infty} \frac{f(A_n + K)}{\mu(A_n + K)}.$$

The statement remains valid, if the limit inferior instead of the limit superior is considered.

Proof. As $\{0\}$ is compact, we obtain

$$\limsup_{n \to \infty} \frac{f(A_n)}{\mu(A_n)} \leq \sup_{K \in K(G)} \limsup_{n \to \infty} \frac{f(A_n + K)}{\mu(A_n + K)}.$$

To show the other inequality, let $K \in K(G)$. By Lemma 3.1 there is $m \in \mathbb{N}$ such that $A_n + K \subseteq A_{n+m}$ and $\lim_{n \to \infty} \frac{\mu(A_{n+m})}{\mu(A_n + K)} = 1$. Thus

$$\limsup_{n \in \mathbb{N}} \frac{f(A_n + K)}{\mu(A_n + K)} = \sup_{n \in \mathbb{N}} \frac{f(A_n + K)}{\mu(A_n + m)} \leq \limsup_{n \in \mathbb{N}} \frac{f(A_{n+m})}{\mu(A_{n+m})} = \limsup_{n \in \mathbb{N}} \frac{f(A_n)}{\mu(A_n)}.$$

We obtain

$$\limsup_{n \to \infty} \frac{f(A_n)}{\mu(A_n)} \geq \sup_{K \in K(G)} \limsup_{n \to \infty} \frac{f(A_n + K)}{\mu(A_n + K)}$$

by taking the supremum over all $K \in K(G)$. A similar argument shows the statement for the limit inferior.

\[\square\]

As $K(G) \ni A \mapsto \text{pat}_\omega(A, V) \in [0, \infty]$ is monotone for every $V \in \mathcal{N}(G)$ we obtain Theorem 1.2 from Proposition 3.2.

Theorem 1.2. Let $G$ be an abelian Ornstein-Weiss group and $\omega \subseteq G$ be a Delone set. For every point absorbing and uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there holds

$$E_{\text{pc}}(\omega) = \sup_{V \in \mathcal{N}(G)} \limsup_{n \in \mathbb{N}} \frac{\log(\text{pat}_\omega(A_n, V))}{\mu(A_n)}.$$
Using Theorem 1.2 we present an example of a Delone set in $\mathbb{R}$ without finite local complexity but zero patch counting entropy.

**Example 3.3.** Let $G := \mathbb{R}$ and define

$$\omega := \{0\} \cup \left\{ n + \frac{1}{n} : n \in \mathbb{Z} \setminus \{0\} \right\}.$$  

Then $\omega$ is a Delone set. As $\text{Pat}_\omega\left([0, \frac{1}{2}]\right) = \left\{ \left\{ 0, \frac{2n - 1}{n^2 + n} \right\} : n \in \mathbb{N} \right\}$ this set is not of finite local complexity and there holds

$$\limsup_{n \in \mathbb{N}} \frac{\log(|\text{Pat}_\omega([-n,n])|)}{\mu([-n,n])} = \infty.$$  

We will now show that $E_{pc}(\omega) = 0$.

**Proof.** For every $\varepsilon > 0$ let $A_\varepsilon := [-\varepsilon, \varepsilon]$ and $V_\varepsilon := (-\varepsilon, \varepsilon)$. Then $(A_n)_{n \in \mathbb{N}}$ is a uniformly growing Van Hove net and by Theorem 1.2 we can obtain the patch counting entropy of $\omega$ as

$$E_{pc}(\omega) = \sup_{m \in \mathbb{N}} \limsup_{n \to \infty} \frac{\log(|\text{Pat}_\omega(A_n, V_{m-1})|)}{2n}.$$  

We claim that for $n, m \in \mathbb{N}$ with $n \geq 4m$ that the set $F_n := A_{4n} \cap \omega$ is an $A_{4n}$-patch representation of scale $V_{m-1}$. As $\omega$ is uniformly discrete there is a constant $c \in \mathbb{N}$ such $|F_n| \leq cn$ and we obtain from the claim that

$$\text{pat}_\omega(A_n, V_{m-1}) \leq |F_n| \leq cn.$$  

We calculate

$$0 \leq E_{pc}(\omega) = \sup_{m \in \mathbb{N}} \limsup_{n \to \infty} \frac{\log(|\text{Pat}_\omega(A_n, V_{m-1})|)}{2n} \leq \sup_{m \in \mathbb{N}} \limsup_{n \to \infty} \frac{\log(c) + \log(n)}{2n} = 0.$$  

It remains to show the claimed statement about $F_n$. To show this let $g \in \omega$ and consider first the case $g \in A_{4n}$. Then $g \in F_n$ and we can set $f := g$ to obtain

$$\omega - f \cong_{A_{4n}, V_{m-1}} \omega - g.$$  

If $g \notin A_{4n}$, we set $f := \max F_n$. As $f, g > 3n$ and $f, g \in \omega$, we obtain from the construction of $\omega$ that

$$(\omega - f) \cap A_n \subseteq (\mathbb{Z} + V_{2n-1}) \cap [-n,n]$$

$$\subseteq ((\omega - g) + V_{2n-1} + V_{2n-1}) \cap [-n,n]$$

$$\subseteq (\omega - g) + V_{m-1},$$

and similarly one shows $(\omega - g) \cap A_n \subseteq (\omega - f) + V_{m-1}$. This shows $F_n$ to be an $A_{4n}$-patch representation of scale $V_{m-1}$. \hfill $\square$

### 4 Topological entropy via dense subsets, open covers, separating sets and spanning sets

In Delone dynamical systems one knows the orbit $D_\omega$ to be dense in $X_\omega$. It is thus interesting, whether topological entropy can be calculated by considering open covers, separating subsets or spanning subsets of a dense subset instead of the whole space. During this section let $\pi$ be an action of an Ornstein-Weiss group $G$ on a compact uniform space $X$ and $D \subseteq X$ be a dense subset of $X$. Every open cover of $X$ restricts to an open cover of $D$. Thus for $\eta \in \cup_X$ the minimal cardinality of an open cover of $D$ of scale $\eta$ is dominated by $\text{cov}_X(\eta)$ and in particular there is a finite open cover of scale $\eta$. For the minimal cardinality of an open cover of $D$ of scale $\eta$ we denote $\text{cov}_D(\eta)$.  

10
Definition 4.1. A subset $S \subseteq X$ is called $\eta$-separated, if for every $s \in S$ there is no further element in $S$ that is $\eta$-close to $s$. We say that $S \subseteq X$ is $\eta$-spanning for $D$, if for all $d \in D$ there is $s \in S$ such that $s$ is $\eta$-close to $d$ or $d$ is $\eta$-close to $s$.

The following Lemma is shown in [Hau10 Lemma 4.16] and [Hau10 Lemma 4.17].

**Lemma 4.2.** For $\eta \in U_X$ the cardinality of an $\eta$-separated subset of $D$ is bounded from above by $\text{cov}_D(\eta) < \infty$ and every $\eta$-separated subset of $D$ of maximal cardinality is $\eta$-spanning for $D$.

In particular there are finite $\eta$-separated subsets of $D$ of maximal cardinality and finite subsets of $D$ that are $\eta$-spanning for $D$ of minimal cardinality.

**Definition 4.3.** We denote by $\text{sep}_D(\eta)$ the maximal cardinality of an $\eta$-separated subset of $D$. Furthermore $\text{spa}_D(\eta)$ is defined as the minimal cardinality of a subset of $D$ that is $\eta$-spanning for $D$.

The notions $\text{spa}_D$, $\text{sep}_D$ and $\text{cov}_D$ relate in the following way.

**Lemma 4.4.** For $\eta, \vartheta \in U_X$ such that $\vartheta$ is open and symmetric there holds

(i) $\text{spa}_D(\eta) \leq \text{sep}_D(\eta) \leq \text{cov}_D(\eta) \leq \text{cov}_X(\eta) < \infty$,

(ii) $\text{cov}_D(\vartheta \eta \vartheta) \leq \text{spa}_D(\eta)$, whenever $\eta$ is symmetric and

(iii) $\text{cov}_X(\vartheta \eta \vartheta) \leq \text{cov}_D(\eta)$.

**Proof.** From Lemma 4.2 we obtain (i). To show (ii) let $S \subseteq M$ be $\eta$-spanning for $M$. As $\vartheta$ is open and $\eta$ is symmetric we obtain $\{\vartheta \eta[s] \mid s \in S\}$ to be an open cover of $M$. It suffices to show that $\vartheta \eta[s]$ is $(\vartheta \eta \vartheta)$-small for any $s \in S$. For $x, y \in \vartheta \eta[s] = \vartheta^{-1} \eta^{-1}[s] = (\eta \vartheta)^{-1}[s]$ we know $x$ to be $\vartheta \eta$-close to $s$ and $s$ to be $\eta \vartheta$-close to $y$, hence $x$ to be $\vartheta \eta \vartheta$-close to $y$.

To obtain (ii), let $U$ be an open cover of $D$ by $\eta$-small sets. As $D$ is dense in $X$ we know that $V := \{\vartheta[U] \mid U \in \mathcal{U}\}$ is an open cover of $X$ and it suffices to show, that $\vartheta[U]$ is $\eta \vartheta$-small for every $U \in \mathcal{U}$. For $x, y \in \vartheta[U]$ there are $u_x, u_y \in U$ which are $\vartheta$-close to $x$ and $y$ respectively. As $U$ is $\eta$-small, $x$ is $\vartheta \eta \vartheta$-close to $y$ and we obtain $(\vartheta[U])^2 \subseteq \vartheta \eta \vartheta$. \hfill $\square$

The following yields the link between these notions and the Bowen action.

**Lemma 4.5.** For every entourage $\eta \in U_X$ there is an entourage $\vartheta \in U_X$ with $\vartheta \subseteq \eta$ such that for every compact $A \subseteq G$ there holds

(i) $\text{cov}_D(\eta_A) \leq \text{spa}_D(\vartheta_A)$ and

(ii) $\text{cov}_X(\eta_A) \leq \text{cov}_D(\vartheta_A)$.

**Proof.** To show (i) let $\vartheta \in U_X$ be symmetric and such that $\vartheta \vartheta \vartheta \subseteq \eta$. For $A \subseteq G$ compact we calculate $\vartheta_A \vartheta_A \vartheta_A = (\vartheta \vartheta \vartheta)_A \subseteq \eta_A$. Let now $\kappa \in U_X$ be open and symmetric such that $\kappa \subseteq \vartheta_A$. Thus Lemma 4.3(ii) yields

$$\text{cov}_D(\eta_A) \leq \text{cov}_D(\vartheta_A \vartheta_A \vartheta_A) \leq \text{cov}_D(\kappa \vartheta_A \kappa) \leq \text{spa}_D(\kappa \vartheta A)$$. 

Similarly, one obtains (ii) from Lemma 4.3(iii). \hfill $\square$

**Theorem 4.6.** Let $\pi$ be an action of an Ornstein-Weiss group $G$ on a compact uniform space $X$ and let $D \subseteq X$ be a dense subset. For any Van-Hove net $(A_i)_{i \in I}$ and any base $\mathcal{B}_X$ of $U_X$ holds

$$E(\pi) = \sup_{\eta \in \mathcal{B}_X} \limsup_{i \in I} \frac{\log(\text{cov}_D(\eta A_i))}{\mu(A_i)} = \sup_{\eta \in \mathcal{B}_X} \limsup_{i \in I} \frac{\log(\text{sep}_D(\eta A_i))}{\mu(A_i)} = \sup_{\eta \in \mathcal{B}_X} \limsup_{i \in I} \frac{\log(\text{spa}_D(\eta A_i))}{\mu(A_i)}$$.

The statement remains valid, if the limit inferior instead of the limit superior is considered.
Proof. As
\[ \eta \mapsto \limsup_{i \in I} \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \]
and the other similar terms are decreasing\(^4\), it suffices to show the statement for the base \( B_X = \mathbb{U}_X \). By Lemma 4.4(i) we obtain
\[ \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{spa}(\eta A_i))}{\mu(A_i)} \leq \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{sep}(\eta A_i))}{\mu(A_i)} \]
\[ = \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \leq \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{spa}(\theta A_i))}{\mu(A_i)} \]
By Lemma 4.5(i), we obtain
\[ \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \leq \sup_{\vartheta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\vartheta A_i))}{\mu(A_i)} \]
From Lemma 4.5(ii) it follows that
\[ E(\pi) = \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \leq \sup_{\vartheta \in \mathbb{U}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\vartheta A_i))}{\mu(A_i)} \]
An analogue argument shows the result for the limit inferior. \( \square \)

The question arises, whether we can replace the limit superior in the first line of Theorem 4.6 by a limit. As \( G \) is an Ornstein-Weiss group, we have to see, whether
\[ A \mapsto \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \]
is monotone, sub-additive and right invariant. Again monotonicity is a direct consequence of the definition and it is possible to show that this map is sub additive without further assumptions (see [Han19, Lemma 4.11]). Nevertheless this mapping is not necessarily right invariant. We need the following condition on \( D \).

**Definition 4.7.** Let \( \pi \) be an action of a group \( G \) on \( X \). A subset \( M \subseteq X \) is called \( \pi \)-invariant, if \( \pi^g(M) = M \) for all \( g \in G \).

Assume now \( D \) to be \( \pi \)-invariant and dense. For \( A \in \mathcal{K}(G) \), \( \eta, \vartheta \in \mathbb{U}_X \), an open cover \( U \) of \( X \) of scale \( \eta A \) and \( g \in G \) we obtain the set \( U_g = ((\pi g)^{-1}(U); U \in U) \) to be an open cover of \( D \) of scale \( \eta A_g \) and hence \( \text{cov}(\eta A_g) \leq \text{cov}(\eta A) \). This shows the mapping considered in (1) to be right invariant.

**Corollary 4.8.** Let \( \pi \) be an action of an Ornstein-Weiss group \( G \) on a compact uniform space \( X \) and let \( D \subseteq X \) be a \( \pi \)-invariant dense subset. For any Van-Hove net \( (A_i)_{i \in I} \) and any base \( \mathbb{B}_X \) of \( \mathbb{U}_X \) there holds
\[ E(\pi) = \sup_{\eta \in \mathbb{B}_X} \limsup_{i \in I} \frac{\log(\text{cov}(\eta A_i))}{\mu(A_i)} \]

5 Patch counting entropy for Delone sets without finite local complexity

In order to show that the patch counting entropy of a Delone set equals the topological entropy of the corresponding dynamical system, i.e. Theorem 1.3, we introduce intermediate concepts between \( A \)-patch representations and spanning sets in the corresponding Delone

\^4We call a map \( f: A \to B \) decreasing, whenever \( f \) is monotone after we reverse the ordering on \( A \). Here \( \mathbb{U}_X \) is ordered by set inclusion.
dynamical system. During this section we assume $G$ to be a non-compact abelian Ornstein-Weiss group and $\omega \subseteq G$ to be a Delone set. Let $A \subseteq G$ be a compact subset and $V$ be an open neighbourhood of 0. We say that $F \subseteq G$ is a non-centred $A$-patch representation of scale $V$ for $\omega$, if for any $g \in G$ there is $f \in F$ s.t.

$$\omega - g \sim A, V \omega - f.$$ 

Furthermore a subset $F \subseteq G$ we call an A-patch separation of scale $V$ for $\omega$, if for any $g, g' \in F$ with $g \neq g'$ we have that there does not hold

$$\omega - g \sim A, V \omega - g',$$

i.e. $(\omega - g) \cap A$ is not contained in $(\omega - g') + V$ or $(\omega - g') \cap A$ is not contained in $(\omega - g) + V$.

## 5.1 Non-centred patch counting and topological entropy

In order to establish the relationship between non-centred patch counting and topological entropy we first consider the entourages of the corresponding Delone dynamical system. A straightforward argument shows the following.

**Proposition 5.1.** Let $A \in \mathcal{K}(G), V \in \mathcal{N}(G)$ and $F \subseteq G$.

(i) $F$ is a non-centred $A$-patch representation of scale $V$, if and only if $\{\omega - f; f \in F\}$ is $\varepsilon_\omega(A, V)$-spanning for $D_\omega$.

(ii) $F$ is an A-patch separation of scale $V$, if and only if $\{\omega - f; f \in F\}$ is $\varepsilon_\omega(A, V)$-separated.

Combining Proposition 5.1 with Lemma 5.2 we obtain that there is a finite non-centred $A$-patch representation on scale $V$ for $\omega$ and that the cardinality of an $A$-patch separation on scale $V$ for $\omega$ is bounded.

**Definition 5.2.** We define $\text{npat}_\omega(A, V)$ as the minimal cardinality of a non-centred $A$-patch representation of scale $V$ for $\omega$. Furthermore we denote by $\text{sep}_\omega(A, V)$ the maximal cardinality of an $A$-patch separation of scale $V$ for $\omega$.

**Remark 5.3.** With this notion we obtain from Proposition 5.1 that for $A \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ there holds $\text{npat}_\omega(A, V) = \text{spa}_{D_\omega}(\varepsilon_\omega(A, V))$ and $\text{sep}_\omega(A, V) = \text{sep}_{D_\omega}(\varepsilon_\omega(A, V))$.

Next we present the link between the local rubber base and the Bowen action, which is the key to establish the equality of patch counting entropy and topological entropy of the corresponding dynamical system.

**Lemma 5.4.** For $A, K \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ there holds $\varepsilon_\omega(K - A, V) \subseteq \varepsilon_\omega(K - A, V)$.

**Proof.** To show $\varepsilon_\omega(K - A, V) \subseteq \varepsilon_\omega(K - A, V)$, let $(\xi, \zeta) \in \varepsilon_\omega(K - A, V)$. For $g \in A$ we obtain

$$\pi^g(\xi) \cap K = (\xi + g) \cap K \subseteq \zeta + V = \pi^g(\zeta) + V.$$

Similarly one shows $\pi^g(\zeta) \cap K \subseteq \pi^g(\zeta) + V$. This proves $(\pi^g(\xi), \pi^g(\zeta)) \in \varepsilon_\omega(K, V)$, i.e. $(\xi, \zeta) \in \varepsilon_\omega(K, V)_g$ for every $g \in A$. We thus obtain

$$\varepsilon_\omega(K - A, V) \subseteq \bigsqcup_{g \in A} \varepsilon_\omega(K, V)_g = \varepsilon_\omega(K, V)_A.$$

It remains to show $\varepsilon_\omega(K, V)_A \subseteq \varepsilon_\omega(K - A, V)$. For $(\xi, \zeta) \in \varepsilon_\omega(K, V)_A$ there holds $(\pi^g(\xi), \pi^g(\zeta)) \in \varepsilon_\omega(K, V)$ for every $g \in A$, hence

$$(\xi + g) \cap K = \pi^g(\xi) \cap K \subseteq \pi^g(\zeta) + V = \zeta + g + V.$$ 

We obtain $\xi \cap (K - g) \subseteq \zeta + V$ for all $g \in A$ and compute

$$\xi \cap (K - A) = \xi \cap \left( \bigcup_{g \in A} (K - g) \right) = \bigcup_{g \in A} (\xi \cap (K - g)) \subseteq \zeta + V.$$ 

As one shows similarly that $\zeta \cap (K - A) \subseteq \xi + V$, we conclude $(\xi, \zeta) \in \varepsilon_\omega(K - A, V)$. □
Proposition 5.5. For every Van Hove net \((A_i)_{i \in I}\) there holds
\[
E(\pi_\omega) = \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{npat}_\omega(K + A_i, V)) \mu(K + A_i).
\]
The statement remains valid, if the limit inferior instead of the limit superior and if \(\text{sep}_\omega\) instead of \(\text{npat}_\omega\) is considered.

Proof. Note first that \((-A_i)_{i \in I}\) is a Van Hove net. Furthermore by Remark 5.3(iv) there holds \(\frac{\mu(A_i)}{\mu(K + A_i)} = 1\) for every compact subset \(K \subseteq G\). Using Theorem 4.5, Lemma 5.1 and Remark 5.3 we compute
\[
E(\pi_\omega) = \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{spa}_{\omega}(\varepsilon(-A_i))) \mu(-A_i)
= \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{spa}_{\omega}(\varepsilon_\omega(K, V)(-A_i))) \mu(A_i)
= \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{spa}_{\omega}(\varepsilon_\omega(K + A_i, V))) \mu(A_i)
= \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{npat}_\omega(K + A_i, V)) \mu(A_i)
= \sup_{V \in \mathcal{N}(G)} \sup_{K \in K(G)} \limsup_{i \in I} \log(\text{npat}_\omega(K + A_i, V)) \mu(K + A_i).
\]
A similar argument yields the statement about \(\text{sep}_\omega\) and the limit inferior respectively. □

5.2 Centred and non-centred patch counting

We now establish the connection between non-centred and \(A\)-patch representations.

Proposition 5.6. Let \(K \in K(G)\) and \(V \in \mathcal{N}(G)\).

(i) If \(A \in K(G)\) satisfies \(0 \in A\) and \(F\) is a non-centred \(A\)-patch representation of scale \(V\), then \(F \cap (\omega + V)\) is an \(A\)-patch representation of scale \(V\), hence
\[
\text{pat}_\omega(A, V) \leq \text{npat}_\omega(A, V).
\]

(ii) Assume \(V\) to be symmetric and \(\omega\) to be \(K\)-dense. There is a finite set \(F_{K,V} \subseteq K\) such that for every \(A \in K(G)\) there holds the following. If \(F\) is an \((A + K)\)-patch representation of scale \(V\), then \(F + F_{K,V}\) is a non-centred \(A\)-patch representation of scale \(V + V\), hence there is a constant \(N_{K,V} \in \mathbb{N}\) that does not dependent on \(A\) with
\[
\text{npat}_\omega(A, V + V) \leq N_{K,V} \text{npat}_\omega(A + K, V).
\]

Proof. To show (i) assume \(0 \in A \in K(G)\). For every \(g \in \omega\) there is \(f \in F\) such that
\[
\omega - g \overset{A}{\approx} \omega - f.
\]
Hence \(0 \in (\omega - g) \cap A \subseteq (\omega - f) + V\) and we deduce \(f \in F \cap (\omega + V)\). To show (ii) assume \(V\) to be symmetric and \(\omega\) to be \(K\)-dense. As \(K\) is compact, there is a finite set \(F_{K,V} \subseteq K\) such that \(K \subseteq F_{K,V} + V\). Let \(g \in G\). As \(K + \omega = G\), there are \(e \in F_{K,V}\), \(v \in V\) and \(u \in \omega\) with \(e + v \in K\) and \(e + v + u = g\). In order to show \(F + F_{K,V}\) to be a non-centred \(A\)-patch representation of scale \(V + V\) it is sufficient to show that there is \(f \in F\) with
\[
\omega - g \overset{A + V + V}{\approx} \omega - (f + e).
\]
As \(F\) is an \(A + K\)-patch representation for \(\omega\) of scale \(V\) there is \(f \in F\) with
\[
\omega - f \overset{A + K,V}{\approx} \omega - u.
\]
As $e \in F_{K,V} \subseteq K$, we get $(\omega - f) \cap (A + e) \subseteq (\omega - u) + V$. We thus compute

$$(\omega - (f + e)) \cap A = ((\omega - f) \cap (A + e)) - e \subseteq (\omega - u) + V - e = (\omega - g) + (V + V).$$

As $e + v \in K$, we obtain that $(\omega - u) \cap (A + e + v) \subseteq (\omega - f) + V$. Thus $V = -V$ implies

$$(\omega - g) \cap A = (\omega - e - v - u) \cap A \subseteq (\omega - f) + V - e - v \subseteq (\omega - (f + e)) + (V + V).$$

This shows $\omega - g \approx \omega - (f + e)$. \hfill \Box

**Remark 5.7.** As every $A \in \mathcal{K}(G)$ satisfies $A \subseteq A \cup \{0\}$ Proposition 5.6(i) and Proposition 5.1(i) imply the existence of a finite $A$-part representation of scale $V$ for $V \in \mathcal{N}(G)$. In particular we obtain for $A \in \mathcal{K}(G)$ with $0 \in A$ and $V \in \mathcal{N}(G)$ that

$$\text{nat}_w(A,V) \leq \text{nat}_w(A,W) \leq \text{sep}_w(A,V) < \infty.$$

**Theorem 5.8.** For every point absorbing Van Hove net $(A_i)_{i \in I}$ there holds

$$E_{pc}(\omega) = \sup_{V \in \mathcal{N}(G)} \sup_{K \in \mathcal{K}(G)} \sup_{i \in I} \limsup \frac{\log(\text{nat}_w(A_i + K,V))}{\mu(A_i + K)}.$$

The statement remains valid, if the limit inferior instead of the limit superior is considered.

**Proof.** The inequality "$\leq" follows immediately from Proposition 5.6(i), as for every $K \subseteq G$ compact there is $j \in I$ such that $0 \in A_i + K$ for $i \geq j$. To show "$\geq$" let $K \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$. As $A \mapsto \text{nat}_w(A,V)$ is monotone we assume without loss of generality that $\omega$ is $K$-dense. Let $W \in \mathcal{N}(G)$ be symmetric with $W + W \subseteq V$. We obtain from Proposition 5.6(ii) the existence of a constant $N \in \mathbb{N}$ such that for all $A \in \mathcal{K}(G)$ there holds

$$\text{nat}_w(A,V) \leq \text{nat}_w(A,W + W) \leq N \text{nat}_w(A + K,W).$$

From Remark 2.3 we obtain $\lim_{i \in I} \mu(A_i + K) = \infty$ and $\lim_{i \in I} \frac{\mu(A_i + K + C)}{\mu(A_i + K)} = 1$ to compute

$$\limsup_{i \in I} \frac{\log(\text{nat}_w(A_i + K,V))}{\mu(A_i + K)} \leq \limsup_{i \in I} \left( \frac{\log(N)}{\mu(A_i + K)} + \frac{\log(\text{nat}_w(A_i + K + C,V))}{\mu(A_i + K)} \right) \leq \sup_{U \in \mathcal{N}(G)} \sup_{C \in \mathcal{K}(G)} \limsup_{i \in I} \frac{\log(\text{nat}_w(A_i + C, U))}{\mu(A_i + C)} = E_{pc}(\omega).$$

As similar argument yields the statement for the limit inferior. \hfill \Box

Using that $\mathcal{K}(G) \ni A \mapsto \text{nat}_w(A,V)$ is monotone for every $V \in \mathcal{N}(G)$ we apply Proposition 5.2 to obtain the following.

**Corollary 5.9.** For every uniformly growing and point absorbing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$ there holds

$$E_{pc}(\omega) = \sup_{V \in \mathcal{N}(G)} \limsup_{n \to \infty} \frac{\log(\text{nat}_w(A_n,V))}{\mu(A_n)}.$$
Theorem 6.3. Let $G$ be an abelian Ornstein-Weiss group. For every Delone set $\omega \subseteq G$ there holds $E_{pc}(\omega) = E(\pi_\omega)$.

Proof. The combination of Theorem 5.8 and Proposition 5.5 implies the statement. \hfill \Box

Remark 5.10. The combination of Theorem 5.8 and Proposition 5.5 also shows that the patch counting entropy is independent from the choice of a point absorbing Van Hove net.

6 Entropy of Delone sets with finite local complexity

We will show next that the classical definition yields the same value of patch counting entropy for all sets of finite local complexity. In this section we assume $G$ to be a non-compact abelian Ornstein-Weiss group and $\omega$ to be a Delone set of finite local complexity. For $A \subseteq G$ compact we call a subset $F \subseteq \omega$ an exact $A$-patch representation, if for all $g \in \omega$ there is $f \in F$ such that $|\omega - g) \cap A = (\omega - f) \cap A$. The minimal cardinality of an exact $A$-patch representation is $|\operatorname{Pat}_\omega(A)|$. Furthermore for every $V \in \mathcal{N}(G)$ we obtain that every exact $A$-patch representation is an $A$-patch representation of scale $V$. Thus for every $A \in \mathcal{K}(G)$ and every $V \in \mathcal{N}(G)$ there holds
\[
\operatorname{pat}_\omega(A,V) \leq |\operatorname{Pat}_\omega(A)|.
\]

We obtain that for every point absorbing Van Hove net $(A_i)_{i \in I}$ there holds
\[
E_{pc}(\omega) \leq \sup_{K \in \mathcal{K}(G)} \limsup_{i \in I} \log(|\operatorname{Pat}_\omega(A_i + K)|) \mu(A_i + K) \tag{2}
\]
and a similar statement for the limit inferior. As $\mathcal{K}(G) \ni A \mapsto \operatorname{Pat}_\omega(A)$ is monotone we furthermore get from Proposition 5.2 that for every point absorbing and uniformly growing Van Hove sequence there holds
\[
E_{pc}(\omega) \leq \limsup_{i \in I} \log(|\operatorname{Pat}_\omega(A_i)|) \mu(A_i) \tag{3}
\]
and a similar statement about the limit inferior. Note that in Example 6.3 it is shown that this inequality can be strict if $\omega$ is not of finite local complexity. We will now present that there holds equality in (2) and (3) whenever $\omega$ is of finite local complexity and if we assume in addition the Van Hove net to be finally compactly connected.

6.1 On compactly connected sets

We will need the concepts of compactly connected sets and finally compactly connected Van Hove nets in the following.

Definition 6.1. For a compact neighbourhood $K \subseteq G$ of 0 a subset $A \subseteq G$ is said to be $K$-connected, if for all $a, b \in A$ there are $a_0, \cdots, a_n \in A$ with $a_0 = a, a_n = b$ and $a_i - a_{i-1} \in K$ for every $i \in \{1, \cdots, n\}$. A Van Hove net $(A_i)_{i \in I}$ we call finally $K$-connected, if there is $j \in I$ such that $A_i$ is $K$-connected for every $i \geq J$. We say that a Van Hove net is finally compactly connected, if it is finally $K$-connected for some compact neighbourhood $K \subseteq G$ of 0.

Example 6.2. Every Van Hove net of path-connected sets is finally $K$-connected for every compact neighbourhood $K \subseteq G$ of 0. Thus the sequence $(B_n)_{n \in \mathbb{N}}$ of closed centred balls in $\mathbb{R}^d$ is finally compactly connected. In $\mathbb{Z}^d$ the sequence $\{-n, \cdots, n\}^d$ consists of $\{-1, 0, 1\}^d$-connected sets and is therefore finally compactly connected as well.

Lemma 6.3. Let $C$ be a compact neighbourhood of 0. If $A, B \in \mathcal{K}(G)$ are $C$-connected, then so is $A + B$. 

16
Proof. Let \( a + b, a' + b' \in A + B \) such that \( a, a' \in A \) and \( b, b' \in B \). As \( A \) and \( B \) are C-connected there are finite sequences \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_m \) in \( A \) and \( B \) respectively with \( a_0 = a, b_0 = b, a_n = a', b_m = b', a_i - a_{i-1} \in C \) for \( i \in \{1, \ldots, n\} \) and \( b_j - b_{j-1} \in C \) for \( j \in \{1, \ldots, m\} \). We define

\[
\begin{align*}
  c_k := \begin{cases}
    a_k + b_0 & , k \in \{0, \ldots, n\} \\
    a_n + b_{k-n} & , k \in \{n, \ldots, n + m\}
  \end{cases}
\end{align*}
\]

for \( k \in \{0, \ldots, n + m\} \). Then \( c_k - c_{k-1} \) for \( k \in \{1, \ldots, n + m\} \), \( a + b = a_0 + b_0 = c_0 \) and \( a' + b' = a_n + b_m = c_{n+m} \). We thus obtain \( A + B \) to be C-connected. \( \square \)

**Lemma 6.4.** If \( (A_i)_{i \in I} \) is a point absorbing and finally compactly connected Van Hove net and \( K \in K(G) \), then also \( (A_i + K)_{i \in I} \) is a point absorbing and finally compactly connected Van Hove net.

Proof. It is presented in Remark 2.3(iv) that \( (A_i + K)_{i \in I} \) is a Van Hove net. To show that this net is point absorbing let \( g \in G \). As \( (A_i)_{i \in I} \) is point absorbing for any \( k \in K \) there is \( j \in I \) such that \( g - k \in A_j \), hence \( g \in A_i + k \subseteq A_i + K \) for all \( i \geq j \). To show that \( (A_i + K)_{i \in I} \) is finally compactly connected we obtain the existence of \( j \in I \) and of a compact neighbourhood \( C \) of 0 such that \( A_j \) is C-connected for all \( i \geq j \). As \( C \cup (K - K) \) is compact we obtain \( C \cup (K + K) \) to be finally compactly connected. \( \square \)

**Lemma 6.5.** Let \( K \) be a compact and symmetric neighbourhood of 0. If \( A, B \subseteq G \) satisfy \( A \subseteq B \subseteq A + K \) and if \( A \) is K-connected, then \( B \) is K-connected.

Proof. Let \( b, b' \in B \). As \( B \subseteq A + K \) there are \( a, a' \in A \) such that \( b - a, b' - a' \in K \). As \( A \) is K-connected there is a finite sequence \( a_0, \ldots, a_n \) in \( A \subseteq B \) such that \( a = a_0, a'd = a_n \) and \( a_i - a_{i-1} \in K \) for \( i \in \{1, \ldots, n\} \). Set \( b_0 := b, b_n+2 := b' \) and \( b_i := a_{i+1} \) for \( i \in \{1, \ldots, n+1\} \). Then there holds \( b_j - b_{j-1} \in K \) for \( j \in \{1, \ldots, n+2\} \) and we obtain \( B \) to be K-connected. \( \square \)

**Lemma 6.6.** Every uniformly growing Van Hove sequence is finally compactly connected.

Proof. Let \( (A_n)_{n \in \mathbb{N}} \) be a uniformly growing Van Hove sequence. Let \( C \in K(G) \) and \( m \in \mathbb{N} \) such that \( A_n = A_n + \{0\} \subseteq A_{n+m} \subseteq A_n + C \) for all \( n \in \mathbb{N} \). Without loss of generality we assume \( C \) to be symmetric and \( A_n \) to be C-connected for \( n \leq m \). By Lemma 6.5 we obtain inductively \( A_n \) to be C-connected for all \( n \in \mathbb{N} \).

**Remark 6.7.** The Van Hove sequence \( ([-n^2, n^2])_{n \in \mathbb{N}} \) in \( \mathbb{R} \) is point absorbing and finally compactly connected but not uniformly growing. Furthermore \( ([0,n])_{n \in \mathbb{N}} \) is finally compactly connected but neither point absorbing nor uniformly growing. The Van hove sequence \( ([-n,n] \cup \{n^2\})_{n \in \mathbb{N}} \) is point absorbing but neither finally compactly connected nor uniformly growing.

### 6.2 Exact patches and patch separation

In order to establish the equality in Equation (2) for point absorbing and finally compactly connected Van Hove nets, we will use the local matching base \( \mathcal{B}_m(\omega) \) of \( X_{\mathcal{L}} \). Unfortunately there seems to hold no relationship between the Bowen action and \( \mathcal{B}_m(\omega) \) like established for the local rubber base \( \mathcal{B}_l(\omega) \) in Lemma 6.4. Nevertheless, we have the following.

**Lemma 6.8.** For \( K \in K(G) \), \( V \in \mathcal{N}(G) \) and \( g \in A \) there holds \( \eta_{\omega}(K, V)_g = \eta_{\omega}(K - g, V) \).

Proof. Note that \( (\xi, \zeta) \in \eta_{\omega}(K, V)_g \) if and only if there holds \( (\xi + g, \zeta + g) = (\pi^g(\xi), \pi^g(\zeta)) \in \eta_{\omega}(K, V) \). This is equivalent to the existence of \( x, z \in V \) with \( (\xi + g + x) \cap K = (\zeta + g + z) \cap K \), which reformulates as \( (\xi + x) \cap (K - g) = (\zeta + z) \cap (K - g) \). Such \( x, z \in V \) exist, if and only if \( (\xi, \zeta) \in \eta_{\omega}(K - g, V) \). \( \square \)
Lemma 6.9. Let $C \in \mathcal{K}(G)$ be symmetric and $V \in \mathcal{N}(G)$ be such that $V \subseteq C$. Let furthermore $x, a \in G$ such that $x - a \in C$. Then for all $V$-discrete $\xi, \zeta \subseteq G$ with $x \in \xi \cap \zeta$ and $(\xi, \zeta) \in \eta_{\omega}(3 \circ C + a, V)$ there holds $\xi \cap (C + x) = \zeta \cap (C + x)$.

Proof. As $(\xi, \zeta) \in \eta_{\omega}(3 \circ C + a, V)$, there are $y, z \in V$ with

$$\tag{4}(\xi + y) \cap (3 \circ C + a) = (\zeta + z) \cap (3 \circ C + a).$$

We obtain $x + y = (x - a) + a + y \in C + a + V \subseteq 3 \circ C + a$, hence

$$x + y \in (\xi + y) \cap (3 \circ C + a) \subseteq \zeta + z.$$

It follows that $x, x + (y - z) \in \zeta$. From $z \in V$ we see

$$x + y = z + x + y - z \in V + (x + (y - z))$$

and from $y \in V$ we obtain $x + y \in V + x$. Thus $V + x$ and $V + (x + (y - z))$ are not disjoint. As we assumed $\zeta$ to be $V$-discrete there holds $x = x + (y - z)$, hence $y = z$. It follows from (4) that

$$\xi \cap (3 \circ C + (a - y)) = \zeta \cap (3 \circ C + (a - y)).$$

From

$$C + x = C + (x - a) + y + (a - y) \subseteq C + C + V + (a - y) \subseteq 3 \circ C + (a - y)$$

we obtain the statement. \qed

Lemma 6.10. Let $C \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ such that $C$ is symmetric and such that $V \subseteq C$. Let $A \in \mathcal{K}(G)$ be $C$-connected and such that $0 \in A$. Then for all $\xi, \zeta \in \mathcal{D}_{C, V}(G)$ that satisfy $0 \in \xi \cap \zeta$ and $(\xi, \zeta) \in \eta_{\omega}(9 \circ C, V)_{-A}$ there holds $\xi \cap A = \zeta \cap A$.

Proof. Let $a \in \xi \cap A$. As $A$ is $C$-connected there are $a_0, \ldots, a_n \in A$ with $a_0 = 0$, $a_n = a$ and $a_i - a_{i-1} \in C$ for all $i \in \{1, \ldots, n\}$. Set $x_0 := 0$ and $x_n := a$. As $C + \xi = G$ there are $x_i \in \xi$ with $a_i - x_i \in C$. We will show $a = x_n \in \zeta$ by induction and thus obtain $a \in \zeta \cap A$. This shows $\xi \cap A \subseteq \zeta \cap A$. The other inclusion is shown analogously. It remains to show the induction. There holds $x_0 = 0 \in \xi$. Now assume $x_i \in \zeta$ for some $i \in \{0, \ldots, n - 1\}$. As $a_i \in A$ we obtain

$$(\xi, \zeta) \in \eta_{\omega}(9 \circ C, V)_{-A} \subseteq \eta_{\omega}(9 \circ C, V)_{(-a_i)} = \eta_{\omega}(3 \circ C + a_i, V).$$

Furthermore there holds $V \subseteq C \subseteq 3 \circ C$, $x_i - a_i \in C \subseteq 3 \circ C$ and $x_i \in \xi \cap \zeta$. Thus Lemma 6.9 yields

$$\xi \cap (3 \circ C + x_i) = \zeta \cap (3 \circ C + x_i).$$

From $x_{i+1} - x_i = (x_{i+1} - a_{i+1}) + (a_{i+1} - a_i) + (a_i - x_i) \in 3 \circ C$ we obtain $x_{i+1} \in \xi \cap (3 \circ C + x_i) = \zeta \cap (3 \circ C + x_i) \subseteq \zeta$. \qed

Lemma 6.11. Let $C \in \mathcal{K}(G)$ and $V \in \mathcal{N}(G)$ be such that $\omega \in \mathcal{D}_{C, V}(G)$ and assume $C$ to be symmetric and to satisfy $V \subseteq C$. For all $C$-connected $A \in \mathcal{K}(G)$ with $0 \in A$ there holds

$$|\text{Pat}_{\omega}(A)| \leq \text{sep}_{\omega}(\eta_{\omega}(9 \circ C, V)_{-A}).$$

Proof. Assume $F \subseteq G$ to be an exact $A$-patch representation of scale $V$ for $\omega$ of minimal cardinality. Then for distinct $x, y \in F$ with we have that $(\omega - x) \cap A \neq (\omega - y) \cap A$. As $F \subseteq \omega$ there holds $0 \in (\omega - x) \cap (\omega - y)$. From $\omega - x, \omega - y \in \mathcal{D}_{C, V}(G)$ we obtain by Lemma 6.10 that $(\omega - x, \omega - y) \notin \eta_{\omega}(9 \circ C, V)_{-A}$. This shows $\omega - f; f \in F$ to be an $\eta_{\omega}(9 \circ C, V)$-separated subset of $\omega$. \qed

Lemma 6.12. For every point absorbing and finally compactly connected Van Hove net $(A_i)_{i \in I}$ there holds

$$\limsup_{i \in I} \frac{\log(|\text{Pat}_{\omega}(A_i)|)}{\mu(A_i)} \leq E_{pc}(\omega).$$

The statement remains valid, if the limit inferior instead of the limit superior is considered.
Proof. There are \( V \in \mathcal{N}(G), j \in I \) and a compact neighbourhood \( C \) of 0 such that \( \omega \in \mathcal{P}_{C,V}(G) \), \( 0 \in A_i \) and \( A_i \) is \( C \)-connected for \( i \geq j \). As \( \eta_\omega(9 \odot C, V) \in U_X \) we obtain from Lemma 6.11, Theorem 4.6 and Theorem 1.3 that

\[
\limsup_{i \in I} \frac{\log(|\text{Pat}_\omega(A_i)|)}{\mu(A_i)} \leq \limsup_{i \in I} \frac{\log(\text{sep}_{D_\omega}(\eta_\omega(9 \odot C, V) - A_i))}{\mu(A_i)} \leq \sup_{\eta \in U_X} \limsup_{i \in I} \frac{\log(\text{sep}_{D_\omega}(\eta - A_i))}{\mu(A_i)} = E(\pi_\omega) = E_{pc}(\omega).
\]

A similar argument yields the statement about the limit inferior. \( \square \)

**Theorem 6.13.** Let \( G \) be a non-compact abelian Ornstein-Weiss group and \( \omega \) be a Delone set of finite local complexity in \( G \). For every point absorbing and finally compactly connected Van Hove net \((A_i)_{i \in I}\) there holds

\[
E_{pc}(\omega) = \sup_{K \in \mathcal{K}(G)} \limsup_{i \in I} \frac{\log(|\text{Pat}_\omega(A_i + K)|)}{\mu(A_i + K)}.
\]

The statement remains valid, if the limit inferior instead of the limit superior is considered.

**Proof.** By (2) it remains to show that for every \( K \in \mathcal{K}(G) \) there holds

\[
\limsup_{i \in I} \frac{\log(|\text{Pat}_\omega(A_i + K)|)}{\mu(A_i + K)} \leq E_{pc}(\omega).
\]

From Lemma 6.12 we obtain \((A_i + K)_{i \in I}\) to be a point absorbing and finally compactly connected Van Hove net. Thus Lemma 6.12 implies the statement. A similar argument shows the statement for the limit inferior. \( \square \)

**Theorem 1.4.** Let \( G \) be a non-compact abelian Ornstein-Weiss group and \( \omega \) be a Delone set of finite local complexity in \( G \). For every point absorbing and uniformly growing Van Hove sequence \((A_n)_{n \in \mathbb{N}}\) there holds

\[
E_{pc}(\omega) = \lim_{n \to \infty} \frac{\log(|\text{Pat}_\omega(A_n)|)}{\mu(A_n)}.
\]

**Proof.** As (3) holds for the limit inferior instead of the limit superior we obtain the statement from Lemma 6.12 and Lemma 6.6 by computing

\[
E_{pc}(\omega) \leq \liminf_{i \in I} \frac{\log(|\text{Pat}_\omega(A_n)|)}{\mu(A_n)} \leq \limsup_{i \in I} \frac{\log(|\text{Pat}_\omega(A_n)|)}{\mu(A_n)} \leq E_{pc}(\omega).
\]

**Remark 6.14.** In [BLR07] it is shown for Delone sets \( \omega \subseteq \mathbb{R}^d \) of finite local complexity that

\[
E(\pi_\omega) = \limsup_{n \to \infty} \frac{\log(|\text{Pat}_\omega(B_n)|)}{\mu(B_n)},
\]

where \((B_n)_{n \in \mathbb{N}}\) is the point absorbing and uniformly growing Van Hove sequence of the centred balls of radius \( n \in \mathbb{N} \). In that paper the question was raised, whether there is an overall factor of \( \frac{\mu(B_1)}{\mu(C_1)} \), if we replace the centred balls by centred cubes \( C_n \) of side length \( 2n \).

We obtain from Theorem 1.4 that this factor is 1.
6.3 Patch counting in compactly generated LCAG

In the remainder we assume in addition $G$ to be compactly generated, i.e. to be a non-compact compactly generated LCAG. In this subsection we present that Lemma 6.11 also yields that for every Delone set of finite local complexity in such a group there is a compact neighbourhood $V$ of 0 such that the supremum in the definition of patch counting entropy is attained for this $V$. We first need the following.

Lemma 6.15. For every compact neighbourhood $C \subseteq G$ of 0 there are $K_C \in \mathcal{K}(G)$, $V_C \in \mathcal{N}(G)$ and $N_C \in \mathbb{N}$, such that for all $C$-connected $A \in \mathcal{K}(G)$ with $0 \in A$ there holds

$$|\text{Pat}_\omega(A)| \leq N_C \text{pat}_\omega(A + K_C, V_C) \leq N_C \text{npat}_\omega(A + K_C, V_C) \leq N_C \text{sep}_\omega(A + K_C, V_C).$$

Proof. If we had shown the statement for some compact neighbourhood $C$ of 0, then it would be also valid for all compact neighbourhoods $C' \subseteq C$ of 0. We thus assume without loss of generality that $C$ is symmetric and that there is an open neighbourhood $V \subseteq C$ of 0 such that $\omega \in \mathcal{D}_{C,V}(G)$. The combination of Lemma 6.11 and Lemma 1.5(i) yields the existence of $\vartheta \in \mathbb{U}_{\mathcal{X}_\omega}$ such that for all $A \in \mathcal{K}(G)$ with $0 \in A$ there holds

$$\text{sep}_{D_\omega}(\eta_\omega(9 \circ C, V)_-A) \leq \text{spa}_{D_\omega}(\vartheta_-A).$$

As $\mathbb{B}_n(\omega)$ is a base of $\mathbb{U}_{\mathcal{X}_\omega}$ there are $\tilde{K}_C \in \mathcal{K}(G)$ with $0 \in \tilde{K}_C$ and $\tilde{V}_C \in \mathcal{N}(G)$ with $V_C \subseteq V$ such that $\varepsilon_\omega(\tilde{K}_C, \tilde{V}_C) \leq \vartheta$. From Lemma 6.11 and Remark 5.3 we obtain for all $A \in \mathcal{K}(G)$ with $0 \in A$ that

$$|\text{Pat}_\omega(A)| \leq \text{sep}_{D_\omega}(\eta_\omega(9 \circ C, V)_-A) \leq \text{spa}_{D_\omega}(\vartheta_-A)$$

$$\leq \text{npat}_\omega(A + \tilde{K}_C, \tilde{V}_C) = \text{pat}_\omega(A + \tilde{K}_C, \tilde{V}_C).$$

We set $K_C := \tilde{K}_C + C$ and choose $V_C \in \mathcal{N}(G)$ symmetric such that $V_C + V_C \subseteq \tilde{V}_C$. As $\omega$ is $C$-dense we can apply Proposition 5.6(ii) to obtain a constant $N_C \in \mathbb{N}$ such that for all $A \in \mathcal{K}(G)$ with $0 \in A$ there holds

$$|\text{Pat}_\omega(A)| \leq \text{npat}_\omega(A + \tilde{K}_C, \tilde{V}_C)$$

$$\leq \text{pat}_\omega(A + \tilde{K}_C, V_C + V_C)$$

$$\leq N_C \text{pat}_\omega(A + \tilde{K}_C + C, V_C)$$

$$= N_C \text{pat}_\omega(A + K_C, V_C).$$

The second and third inequality follow from Remark 5.3 as $0 \in A + K_C$.

Lemma 6.16. Let $C \subseteq G$ be a compact neighbourhood of 0. Denote by $\mathcal{K}_C^G(G)$ the set of all $C$-connected and compact subsets of $G$ that contain 0. If $f : \mathcal{K}(G) \mapsto [0, \infty]$ is a monotone map, then for every Van Hove net $(A_i)_{i \in I}$ there holds

$$\sup_{K \in \mathcal{K}(G)} \limsup_{i \in I} \frac{f(A_i + K)}{\mu(A_i + K)} = \sup_{K \in \mathcal{K}_C^G(G)} \limsup_{i \in I} \frac{f(A_i + K)}{\mu(A_i + K)}.$$

The statement remains valid, if the limit inferior instead of the limit superior is considered.

Proof. To show the non trivial inequality let $K \in \mathcal{K}(G)$. As $G$ is compactly generated we know that there is $n \in \mathbb{N}$ such that $K \subseteq n \circ C$. From Remark 2.3(iv) we obtain $\lim_{i \in I} \frac{\mu(A_i + n \circ C)}{\mu(A_i + K)} = 1$. As $C$ contains 0, $C$ is itself $C$-connected and Lemma 6.3 shows $n \circ C$ to be $C$-connected. Using the monotonicity of $f$ we compute

$$\limsup_{i \in I} \frac{f(A_i + K)}{\mu(A_i + K)} \leq \limsup_{i \in I} \frac{f(A_i + n \circ C)}{\mu(A_i + K)}$$

$$= \limsup_{i \in I} \frac{f(A_i + n \circ C)}{\mu(A_i + n \circ C)}$$

$$\leq \sup_{K \in \mathcal{K}_C^G(G)} \limsup_{i \in I} \frac{f(A_i + K)}{\mu(A_i + K)}.$$

The supremum over all $K \in \mathcal{K}(G)$ yields the claim.
Theorem 6.17. Let $\omega \subseteq G$ be a Delone set of finite local complexity in a non-compact countably generated locally compact abelian group $G$. For every compact neighbourhood $C \subseteq G$ of $0$ there is $V_C \in \mathcal{N}(G)$ such that for every $V \in \mathcal{N}(G)$ with $V \subseteq V_C$ the following statements hold true.

(i) For every point absorbing and finally $C$-connected Van Hove net $(A_i)_{i \in I}$, there holds

$$E_{pc}(\omega) = \sup_{K \in \mathcal{K}(G)} \limsup_{i \in I} \frac{\log(\text{pat}_{\omega}(A_i, V))}{\mu(A_i + K)}.$$

The statement remains valid, if the limit inferior instead of the limit superior is considered.

(ii) For every finally $C$-connected and uniformly growing Van Hove sequence $(A_n)_{n \in \mathbb{N}}$, there holds

$$E_{pc}(\omega) = \lim_{n \to \infty} \frac{\log(\text{pat}_{\omega}(A_n, V))}{\mu(A_n)}.$$

Furthermore the statements holds true, if pat$_{\omega}$ is replaced by $n$pat$_{\omega}$ or sep$_{\omega}$.

Proof. Let $j \in I$ be such that $0 \in A_i$ and such that $A_i$ is $C$-connected for some compact neighbourhood $C$ of $0$. Recall that we denote by $\mathcal{K}_0(C)$ the set of all $C$-connected and compact subsets that contain $0$. By Lemma 6.15 there are $K_C \in \mathcal{K}(G), V_C \in \mathcal{N}(G)$ and $N_C \in \mathbb{N}$ such that for every $A \in \mathcal{K}_0(C)$ there holds

$$|\text{Pat}_{\omega}(A)| \leq N_C \text{pat}_{\omega}(A + K_C, V_C).$$

As $(A_i)_{i \in I}$ is point absorbing and finally compactly connected there is $j \in I$ such that $A_j \in \mathcal{K}_0(C)$ for all $i \geq j$ and we obtain from Lemma 6.3 that there is $A_i, K \in \mathcal{K}_0(C)$ for $K \in \mathcal{K}_0(C)$ and $i \geq j$. Thus there holds

$$\sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \frac{\log(|\text{Pat}_{\omega}(A_i, K)|)}{\mu(A_i + K)} \leq \sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \frac{\log(N_C \text{pat}_{\omega}(A_i, K + K_C, V_C))}{\mu(A_i + K)}.$$

As $G$ is assumed to be non-compact, we obtain from Remark 2.3 that $\lim_{i \in I} \mu(A_i + K) = \infty$ and furthermore $\lim_{i \in I} \frac{\mu(A_i + K)}{\mu(A_i + K + K_C)}$ for every $K \in \mathcal{K}_0(C)$. We apply for $V \in \mathcal{N}(G)$ with $V \subseteq V_C$ Lemma 6.16 to $A \mapsto |\text{Pat}_{\omega}(A)|$ and $A \mapsto \text{Pat}_{\omega}(A, V)$ respectively and obtain from Theorem 1.4 the following

$$E_{pc}(\omega) = \sup_{K \in \mathcal{K}(G)} \limsup_{i \in I} \frac{\log(|\text{Pat}_{\omega}(A_i, K)|)}{\mu(A_i + K)} \leq \sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \frac{\log(N_C \text{pat}_{\omega}(A_i, K + K_C, V_C))}{\mu(A_i + K)} = \sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \left( \frac{\log(N_C)}{\mu(A_i + K)} + \frac{\log(\text{pat}_{\omega}(A_i + K + K_C, V_C))}{\mu(A_i + K)} \right) \leq \sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \frac{\log(\text{pat}_{\omega}(A_i + K', V))}{\mu(A_i + K')} = \sup_{K \in \mathcal{K}_0(C)} \limsup_{i \in I} \frac{\log(\text{pat}_{\omega}(A_i + K', V))}{\mu(A_i + K')} \leq E_{pc}(\omega).$$
This shows (i) for patω. Similarly one shows the statements for npatω, sepω and the limit inferior respectively. We obtain (ii) from Proposition 3.2 and (i) by
\[ E_{pc}(\omega) = \liminf_{n \to \infty} \frac{\log(pat_{\omega}(A_n, V))}{\mu(A_n)} \leq \limsup_{n \to \infty} \frac{\log(pat_{\omega}(A_n, V))}{\mu(A_n)} = E_{pc}(\omega). \]
Note that \((A_n)_{n \in \mathbb{N}}\) is point absorbing by Remark 2.3(vi).

**Theorem 1.5.** Let \(\omega\) be a Delone set of finite local complexity in a non-compact but compactly generated LCAG. For every uniformly growing Van Hove sequence \((A_n)_{n \in \mathbb{N}}\) there exists \(V \in \mathcal{N}(G)\) such that for all \(W \subseteq V\) there holds
\[ E_{pc}(\omega) = \lim_{n \to \infty} \frac{\log(pat_{\omega}(A_n, W))}{\mu(A_n)}. \]

**Proof.** \((A_n)_{n \in \mathbb{N}}\) is finally compactly connected by Lemma 6.6. Thus there is a compact neighbourhood \(C\) of 0 such that \((A_n)_{n \in \mathbb{N}}\) is finally \(C\)-connected and uniformly growing. We thus obtain the statement from Theorem 6.17(ii).

**References**

[BL04] Michael Baake and Daniel Lenz. Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra. *Ergodic Theory and Dynamical Systems*, 24(6):1867–1893, 2004.

[BLR07] Michael Baake, Daniel Lenz, and Christoph Richard. Pure point diffraction implies zero entropy for delone sets with uniform cluster frequencies. *Letters in Mathematical Physics*, 82(1):61–77, 2007.

[CSCK14] Tullio Ceccherini-Silberstein, Michel Coornaert, and Fabrice Krieger. An analogue of feketes lemma for subadditive functions on cancellative amenable semigroups. *Journal d’analyse mathémétique*, 124(1):59–81, 2014.

[DS58] Nelson Dunford and Jacob T Schwartz. *Linear operators part I: general theory*, volume 7. Interscience publishers New York, 1958.

[FJS18] Michael Baake, Daniel Lenz, and Christoph Richard. Pure point diffraction implies zero entropy for delone sets with uniform cluster frequencies. *Letters in Mathematical Physics*, 82(1):61–77, 2007.

[HR12] Edwin Hewitt and Kenneth A Ross. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*, volumes 115. Springer Science & Business Media, 2012.

[HR15] Christian Huck and Christoph Richard. On pattern entropy of weak model sets. *Discrete & Computational Geometry*, 54(3):741–757, 2015.

[Kel17] John L Kelley. *General topology*. Courier Dover Publications, 2017.

[Kri10] Fabrice Krieger. The ornstein–weiss lemma for discrete amenable groups. *Max Planck Institute for Mathematics Bonn, MPIM Preprint*, 48:2010, 2010.

[Lag99] Jeffrey C Lagarias. Geometric models for quasicrystals i. delone sets of finite type. *Discrete & Computational Geometry*, 21(2):161–191, 1999.

[LP03] Jeffrey C Lagarias and Peter AB Pleasants. Reptitive delone sets and quasicrystals. *Ergodic Theory and Dynamical Systems*, 23(3):831–867, 2003.

[Mun00] James R Munkres. *Topology*. Prentice Hall, 2000.

[Pie84] Jean-Paul Pier. *Amenable locally compact groups*. Wiley-Interscience, 1984.

[Sch99] Martin Schottmann. Generalized model sets and dynamical systems. In *CRM Monograph Series*. Citeeseer, 1999.

[Tem] Arkadii Tempelman. *Ergodic theorems for group actions: Informational and Thermodynamical Aspects.*