Quantum Matrix Models for Simple Current Orbifolds

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ABSTRACT: An algebraic formulation of the stringy geometry on simple current orbifolds of the WZW models of type $A_N$ is developed within the framework of Reflection Equation Algebras, $\text{REA}_q(A_N)$. It is demonstrated that $\text{REA}_q(A_N)$ has the same set of outer automorphisms as the corresponding current algebra $A^{(1)}_N$ which is crucial for the orbifold construction. The ensuing orbifold matrix models are shown to yield results on brane tensions and the algebra of functions in agreement with the exact BCFT data.

KEYWORDS: WZW models, simple current orbifolds, quantum groups, reflection equation algebras.

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1. Introduction.

The physics of D-branes has long been a subject of intense study\(^1\), driven by the motivation to obtain new insights into the structure of the moduli space of string theory proper and to better understand the emergence of an essentially stringy geometry and gauge dynamics within any field-theoretic or matrix model approach to the propagation of strings in curved gravitational backgrounds with fluxes.

An example of such a background is a compact Lie group \(G\) (\([3]\)), or a quotient thereof, known to support a nontrivial Kalb–Ramond field in a conformally invariant theory. The untwisted D-branes on \(G\) have been shown to localise stably (\([3]\)) around a discrete set of conjugacy classes (\([4]\)) and are enumerated by dominant integral affine weights \(\Lambda\) from the fundamental affine alcove \(P^\kappa_\mathfrak{g}(\mathfrak{g})\), i.e. in the case of interest we have:

\[
\text{(untwisted) D-branes on } A_N \sim \mathcal{R}e_{\text{reg.}}(A^{(1)}_N) := \bigoplus_{\Lambda \in P^\kappa_\mathfrak{g}(A_N)} R_\Lambda, \tag{1.1}
\]

with

\[
P^\kappa_+(A_N) := \left\{ \Lambda = \sum_{i=1}^N \lambda_i \Lambda^i \in P^\kappa(A_N), \quad \sum_{i=1}^N \lambda_i \leq \kappa \right\}. \tag{1.2}
\]

where \(P^\kappa(A_N)\) is the weight space of \(A_N\) and \(\Lambda^i\) are the fundamental weights.

Quantum groups \([6]\) with which we are going to work extensively here appear in BCFT at various points. It is widely known that the algebras: \(A^{(1)}_N\) and \(U_q(A_N)\) (with \(q := e^{\frac{i\pi \kappa}{\kappa_\mathfrak{g}}(\mathfrak{g})}\), in which \(\kappa(\mathfrak{g}) := \kappa + g^\vee(\mathfrak{g})\) and \(g^\vee\) - the dual Coxeter number) have numerous common features (\([7, 8, 9]\)): the respective representation theories coincide at the level of regular representations, the statistical dimensions of the boundary state modules (interpreted as D-brane tensions) are identical with the quantum dimensions of the corresponding \(U_q(A_N)\)-irreducible representations, the Operator Product Expansion (OPE) of the so-called primary boundary fields \(\Psi_{L_1,i_1}^{\Lambda_1,\Lambda_2}(x_1)\):

\[
\Psi_{L_1,i_1}^{\Lambda_1,\Lambda_2}(x_1) \Psi_{L_2,i_2}^{\Lambda_2,\Lambda_3}(x_2) \sim_{\text{OPE}} \sum_{L_3 \in P^\kappa_+(g)} \sum_{i_3=1}^{N_3} (x_1 - x_2)^{h_1 + h_2 - h_3} \Psi_{L_3,i_3}^{\Lambda_1,\Lambda_3}(x_2) F_{\lambda_2 \lambda_3} \left[ L_1 L_2 \right]_{\lambda_1}^{i_1,i_2;i_3} L_1 L_2 L_3 c_{i_1 i_2 i_3} \kappa, \tag{1.3}
\]

contains \(F_{\lambda_2 \lambda_3} \left[ L_1 L_2 \right]_{\lambda_1}^{i_1,i_2;i_3} \kappa\) as the so-called fusion matrix, a fundamental object in the BCFT approach \([10]\) and a strictly quantum one\(^3\).

In order to make contact with matrix models of branes we assign (cp \([11]\) and earlier papers on non-commutative stringy geometries):

\[
\Psi_{L_1,i_1}^{\Lambda_1,\Lambda_2}(x_1) \rightarrow \psi_{L_1,i_1}^{\Lambda_1,\Lambda_2}. \tag{1.4}
\]

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\(^1\)See \([1]\) for a detailed review and an exhaustive list of references.

\(^2\)Here, as anywhere else in this paper, we restrict our analysis of the pre-orbifolding BCFT to the diagonal case in which Cardy’s classification of boundary states applies \([3]\).

\(^3\)In the simplest nontrivial case, i.e. in the \(\mathfrak{su}(2)\)-model, its entries are the quantum \(6j\)-symbols of \(U_q(\mathfrak{su}(2))\).
so that the algebraic content of (1.3) is preserved,

\[ \psi_{L, i}^\Lambda_1 \Lambda_2 \psi_{L, i}^\Lambda_3 \sum_{L_3 \in P_s} \psi_{L, i}^\Lambda_1 \Lambda_3 \sum_{i_3 = 1} \psi_{L_3, i_3}^\Lambda_2 L_3 [\Lambda_1^L_2 \Lambda_2^L_3]^\kappa_{i_1 i_2 i_3} \psi_{L_1 L_2 L_3}^\Lambda_1 \Lambda_2 \Lambda_3, \]  

and only the worldsheet dependence is dropped. The new algebra, to be called the Boundary Algebra (BA) henceforth, turns out to be non-associative and so it has to be deformed to be embedded in a matrix algebra. The first attempt was made in [11] where the BA of $su_n(2)$ was reduced to the algebra of matrix harmonics of [12] and subsequently used as a basis of a semiclassical matrix model of D-brane dynamics on $SU(2)$. The model provides an attractive scheme for D-brane formation processes (cp [13]), driven by Renormalisation Group Flows, but otherwise gives a mere asymptotic picture of the nontrivial quantisation of the algebra of functions on $G$, encoded - according to [14] - in the boundary OPE algebra (1.3).

The goal of extracting the quantum (rather than classical) group symmetry hidden in (1.3) without drastically changing its general algebraic structure and, most importantly, without losing the crucial property of associativity was accomplished in a series of papers [14, 15] in which another deformation of the original BA was considered after [11] and [16], leading to the construction of a $\mathcal{U}_q(g)$-covariant coordinate algebra (see [17]), passing under the name of the Reflection Equation Algebra (REA) and denoted by $\text{REA}_q(g)$. Its structure is essentially determined by the so-called universal $\mathcal{R}$-matrix of the relevant quantum group via the defining Reflection Equation:

\[ R_{12} M_1 R_{21} M_2 = M_2 R_{12} M_1 R_{21}, \]  

and the additional size-setting $q$-determinant constraint ($\pi$ is the fundamental representation of the quantum algebra $\mathcal{U}_q(g)$):

\[ \det_q(M) = r^\dim\pi \approx (\alpha' \sqrt{\kappa})^{\dim\pi} I, \]  

imposed on the operator-valued matrix $M$ of generators of $\text{REA}_q(g)$. It ought to be emphasised that the above universal $\mathcal{R}$-matrix enters the BCFT analysis as well, under the guise of the braiding matrix ([8, 9, 10], cp also [17]).

Following [14] we expect the quantum algebra $\text{REA}_q(g)$ to have a central rôle to play in any specific $q$-algebraic description of the quantum WZW geometry and dynamics. It means that we can write down the most general form of the relevant matrix model ($T_0$ stands for the tension of an elementary brane), to be constructed out of certain $\mathcal{U}_q(g)$-covariant expressions in the fundamental $M$-matrix:

\[ S_{g, 0}^{eff, 0} = T_0 \text{Tr}_q \left[ I + \text{cov}_{\mathcal{U}_q(g)}(M) \right]. \]  

The model manages - unspecified as it is - to reproduce quite nicely the exact BCFT results on tensions of the so-called untwisted D-branes, represented by integrable modules of $\hat{g}_\kappa$ at the level of

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4 See [8, 9, 10].

5 For details on the conventions adopted, see: Sect. 3.
the BCFT and identified with the corresponding irreducible representations, $R_{\Lambda}$, of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ in the $q$-model framework,

$$S_{q,\mathfrak{g}}^{eff,0} \supset T_0 \text{Tr}_{q^I}|_{R_{\Lambda}} = \dim_q R_{\Lambda}.$$ (1.9)

Furthermore, it yields a promising picture of gauge fluctuations of the background $\mathcal{M}$ upon introducing in (1.8) terms mimicking the semiclassical matrix models of [11] and gives asymptotically correct results on masses of the lightest string states stretched between separate $q$-branes$^6$.

Altogether, the $q$-matrix model building should be regarded as an attempt to describe the quantum structure of the curved WZW geometries, taking as the input the quantum content of, say, (1.3) and deforming the latter in such a way as to turn it associative, whereupon it can be embedded in a matrix algebra. The algebra then turns out to be the above-mentioned REA, and the epithet "quantum" should be understood as a reflection of the non-commutative nature of the underlying geometry, as dictated by the string theory in the flux-permeated target space (cp [1], [11] and [21]).

At this stage we may already outline and assess the content of the present paper. In it, we extend the former construction of $q$-matrix models to a large class of new WZW backgrounds, namely - the so-called simple current orbifolds of the WZW geometries of type $A_N$. Simple currents are known to form an Abelian group, contained as a subgroup in the group of outer automorphisms of $A^{(1)}_N$ (we shall denote it by $\text{Out}(A^{(1)}_N)$). Thus orbifolding of the corresponding matrix model based on $\text{REA}_q(A_N)$ requires that the latter has the same group of outer automorphisms as the affine algebra. In Sec. 3. we show that this is indeed the case, shifting the details to the appendices. In the following section we apply our results in the construction of quantum orbifolds and discuss their properties in some detail. These two sections form the core of the paper.

The all-important details are contained in the rest of the paper which has the following structure: in Sect. 2. we introduce the necessary BCFT techniques and discuss the question of associativity of the orbifold OPE and its $q$-deformation; in Sect. 3. we explore the representation theory and the outer automorphism group of the relevant REA’s, whereby similarities with the affine counterparts are uncovered; Sect. 4. contains the most significant of our results: a construction of the $q$-orbifold algebra and a study of its (crossed-product-extended) proper subalgebras, giving rise to some essentially new quantum geometries (to be wrapped by fractional $q$-branes), and a direct calculation of orbifold $q$-brane tensions, providing strong evidence in favour of the model advanced; in the next section (Sect. 5.) we examine in detail the particularly simple example of the antipodal quotient of an equatorial quantum 2-sphere of the $\mathfrak{su}_k(2)$-model; in the final Sect. 6. we recapitulate our results. The reader is also urged to consult the appendices enclosed, in which important technical details and some conventions adopted in the present paper have been gathered.

$^6$Cp [13] and, e.g., [21].
2. The BCFT and its quantum deformation.

2.1 The orbifold OPE algebra.

The present paper focuses on the study of a class of quantum algebras and associated matrix models with the aim of encoding in them the physical content of stringy WZW orbifolds. It thus seems natural to begin our discussion with an exposition of some elements of the BCFT of simple current orbifolds relevant to the subsequent quantum algebraic analysis.

The orbifolding procedure for WZW models was laid out in [22], which we follow closely in this preparatory part, and leads to a formulation of string theory on quotient spaces $G/\Gamma$, with $\Gamma$ - a subgroup of the discrete group $sOut$ of strictly outer\textsuperscript{7} automorphisms of $\hat{g}_κ$. It consists in dividing out the action of $\Gamma$ which - at the level of the relevant OPE - is generated by simple currents, i.e. primaries with simple fusion rules with all other primaries. Denoting by $\Lambda_{g}$ the weight label of a simple current corresponding to $g \in \Gamma$ we have a fusion rule

\[ \exists !_{g \Lambda \in P_{\pm}(g)} : \Lambda_{g} \times \mathcal{F} \Lambda = g\Lambda \]  

for an arbitrary weight label $\Lambda \in P_{\pm}(g)$. Thus simpleness implies that there is a single summand on the right hand side of (1.3) whenever one of the factors is a simple current. The Abelian group formed by simple currents under fusion is known to be isomorphic with the group of strictly outer automorphisms.

The action of $\Gamma$ on the set of all primaries, labelled by weights $\Lambda \in P_{\pm}(g)$, decomposes the latter into orbits and we take $[\Lambda]$ to label all boundary conditions associated with $\Lambda$ by that action. Some of the orbits may have fixed points, i.e. there may be weights stabilised by subgroups of $\Gamma$ for which we reserve the symbol $S_{\Lambda} \subseteq \Gamma$. Among these there is a distinguished class of maximally stabilised weights with $S_{\Lambda} = \Gamma$. Specialising to the case $g = A_N$ and $\Gamma = sOut$ we shall call the corresponding fixed points central. An important property of the stabiliser subgroups $S_{\Lambda}$, used in the general BCFT construction, is their independence of a particular representative of the orbit $[\Lambda]$.

Another ingredient in the orbifolding recipe is the simple current charge $\hat{Q}_{g}(L) \in \mathbb{R}/2\mathbb{Z}$ of a given primary field $\Psi^{\Lambda_1,\Lambda_2}_{L,i}$ with respect to the simple current corresponding to $g$, determined by the so-called braiding matrix\textsuperscript{8} of the underlying CFT,

\[ (-1)^{\hat{Q}_{g}(L)} := B^{(+)}_{L \Lambda_{g}} \begin{bmatrix} L & \Lambda_{g} \\ 0 & g \Lambda \end{bmatrix}. \]  

A closely related object is the monodromy charge:

\[ Q_{g}(L) := \hat{Q}_{g}(L) \mod 1 \implies Q_{g}(L) = h_{L} + h_{\Lambda_{g}} - h_{gL} \mod 1, \]  

\textsuperscript{7}We borrow the terminology from [23] where it was used to distinguish the outer automorphisms of $\hat{g}_κ$ which are not automorphisms of the horizontal algebra $g$.

\textsuperscript{8}Cp [10].
constant on simple current orbits for $h_{\Lambda_g} \in \mathbb{Z}$, in which case its vanishing on $[\Lambda]$ for all $g \in \Gamma$ places the orbit among those to survive the $\Gamma$-orbifolding \((24)\).

The last piece of the BCFT machinery we need to deal with orbifold D-branes, in particular the fractional ones \((23)\), is a little group theory of the stabilisers. Indeed, according to the general theory we should have a unique D-brane species over $\Lambda$ for any of the inequivalent one-dimensional irreducible representations of $S_{[\Lambda]}$. We thus label the boundary states of the orbifold theory with the corresponding characters $e_a : S_{[\Lambda]} \to U(1)$. Upon introducing the numbers:

$$d^a_b(L) := \frac{1}{|S_{[\Lambda_1]} \cap S_{[\Lambda_2]}|} \sum_{h \in S_{[\Lambda_1]} \cap S_{[\Lambda_2]}} e_a(h)(-1)^{\hat{Q}_b(L)} e_b(h^{-1}) \quad (2.4)$$

for any pair of overlapping stabilisers $S_{[\Lambda_1]} \cap S_{[\Lambda_2]} \neq \emptyset$ and $L$ such that there exists a non-zero fusion rule: $N_{A_1A_2}^{gA_2} \neq 0$ for some $g \in \Gamma$, we then obtain the partition functions for stabiliser-resolved orbifold D-branes ($\tau$ is the standard modular parameter and $\cdot$ stands for the element-wise product of groups):

$$Z_{[\Lambda_1][\Lambda_2]}^{\text{orb}}(\tau) = \frac{1}{|S_{[\Lambda_1]}| \cdot |S_{[\Lambda_2]}|} \sum_{g \in \Gamma} \sum_{L \in P^\mu_{\Lambda_2}(g)} N_{A_1A_2}^{gA_2} L d^a_b(L) \chi_L(\tau), \quad (2.5)$$

summing up to the partition function of orbifold orbits (or unresolved D-branes):

$$Z_{[\Lambda_1][\Lambda_2]}^{\text{orb}}(\tau) = \sum_{g \in \Gamma} \sum_{L \in P^\mu_{\Lambda_2}(g)} N_{A_1A_2}^{gA_2} L \chi_L(\tau). \quad (2.6)$$

From \((2.5)\) we now readily derive the tensions of the fractional branes by specialising the formula to the case $S_{\Lambda_2} = \{\text{id}\}$ when it becomes

$$Z_{[\Lambda_1][\Lambda_2]}^{\text{orb}}(\tau) = \frac{1}{|S_{[\Lambda_1]}|} \sum_{g \in \Gamma} \sum_{L \in P^\mu_{\Lambda_2}(g)} N_{A_1A_2}^{gA_2} L \chi_L(\tau) = \frac{1}{|S_{[\Lambda_1]}|} Z_{[\Lambda_1][\Lambda_2]}^{\text{orb}}(\tau). \quad (2.7)$$

Hence the graviton coupling between the fractional D-brane carrying an arbitrary stabiliser label $a$ associated with $S_{\Lambda_1}$ and an off-fixed-point one contributes the fraction of $\frac{1}{|S_{[\Lambda_1]}|}$ to the overall graviton coupling between the (unresolved) D-branes, an intuitive result we shall demonstrate to be reproduced by the g-matrix model of Sect. 4.

Having introduced all the relevant formal instruments we may now define an action of $\Gamma$ on the primaries of the pre-orbifolding theory:

$$g \triangleright \Psi^A_{L,i} (x) := (-1)^{-\hat{Q}_b(L)} \Psi^A_{L,i} (x), \quad (2.8)$$

easily verified to be consistent with \(\text{(23)}\) due to the following property of the fusing matrix \((22)\):

$$\forall g \in \Gamma : \quad F_{g,l,k} \left[ \begin{array}{c} i \\ j \\ k \end{array} \right] \alpha_i \alpha_j \alpha_k = (-1)^{\hat{Q}_b(i) + \hat{Q}_b(j) - \hat{Q}_b(k)} F_{l,k} \left[ \begin{array}{c} i \\ j \\ k \end{array} \right] \alpha_i \alpha_j \alpha_k. \quad (2.9)$$

The definition \((2.8)\) provides us with a possibility to average over $\Gamma$ primaries interpolating between off-fixed-point boundary states, whereby the associated orbifold primaries are obtained:

$$\Psi^A_{L,i} (x) := \sum_{g' \in \Gamma} g' \triangleright \Psi^A_{L,i} (x). \quad (2.10)$$
Supplementing the above formula with its fixed-point counterpart\(^9\):

\[
\Psi_{L,i;g}^{[\Lambda_1\alpha][\Lambda_2\beta]} := \sum_{g_1 \in S_{\Lambda_1}} \sum_{g_2 \in S_{\Lambda_2}} \Psi_{L,i;g_1g_2}^{[\Lambda_1][\Lambda_2]} (-1)^{-\hat{Q}_{g_1}(L)} e_a(g_1)e_b(g_2^{-1}),
\]

with the simple current label \(g\) in both \((2.10)\) and \((2.11)\) such that there is a non-zero fusion rule: \(N_{\Lambda_1}^{\Lambda_2} L \neq 0\), we may finally write down the OPE of the stabiliser-resolved boundary primaries:

\[
\Psi_{L_1,i_1;g_{12}}^{[\Lambda_1\alpha_1][\Lambda_2\alpha_2]}(x_1) \Psi_{L_2,i_2;g_{23}}^{[\Lambda_2\alpha_3][\Lambda_3\alpha_4]}(x_2) = \delta_{\alpha_2\alpha_3} \sum_{g \in S_{\Lambda_2}} \sum_{L_3 \in \mathbb{P}_\kappa} \sum_{i_3=1}^{N_{\Lambda_3}^{L_3}} (x_1 - x_2)^{h_1 + h_2 - h_3} \Psi_{L_3,i_3;g_{123}g}^{[\Lambda_1\alpha_1][\Lambda_3\alpha_4]}(x_2)e_b(g) \times
\]

\[
\times (-1)^{-\hat{Q}_{g_{123}g}(L_2)} F_{g_{12}L_2,L_3} \left[ L_1 L_2 \right]^{i_1,i_2;\kappa}_{i_3} c_{i_1i_2i_3}^{L_1L_2L_3} \right] \sum_{i_1,i_2,i_3},
\]

where we have used the shorthand notation: \(g_{123} := g_{12}g_{23}\). An analogous formula for off-fixed-point boundary states can be obtained from \((2.12)\) by taking trivial stabiliser labels.

Prior to passing to the \(q\)-deformed OPE algebra we make, after \([22]\), one more significant remark: according to \([4]\) the OPE algebra of boundary primaries is a stringy deformation of the associative algebra of functions on the target geometry; the emergence of stabiliser resolution and the introduction of charge-weighted averages over \(\Gamma\) in the above OPE has - in this spirit - been considered to reflect the existence of an algebraic structure called the crossed product extension of the algebra of functions on the orbifold, present in the known matrix models of fixed-point geometries \((26)\). We shall have to say more about this issue in Sect. \([4]\).

### 2.2 The deformation.

The first step towards an algebraic description of WZW D-branes is taken at the level of the boundary OPE \((1.3)\). With the aim of extracting from it the D-brane geometry we make the usual assignment \((1.4)\). The BA thus defined is readily demonstrated to be non-associative, the non-associativity following from its hybrid quantum-classical structure. Indeed, while the three-point structure constants \(c_{i_1i_2i_3}^{L_1L_2L_3}\) are classical intertwiners (in the simplest case of the \(su(2)\) model they are just the ordinary Clebsch–Gordan coefficients of \(SU(2)\)) the fusion matrix is already a quantum entity\(^10\). The last observation leads us to the idea of deforming \((1.5)\),

\[
c_{i_1i_2i_3}^{L_1L_2L_3} \rightarrow \tilde{c}_{i_1i_2i_3}^{L_1L_2L_3},
\]

\((2.13)\)

\(^9\)A definition proved sensible in \([22]\).

\(^{10}\)The matrix enters the BCFT analysis at a rather abstract stage and both its rôle and uniqueness - a consequence of strict algebraic constraints imposed upon it - determine it as an already quantum(-group-theoretic) object. Cp \([4]\).
in such a way:

\[
\sum_{l,\alpha_i} F_{l,k} \left[ \begin{array}{ccc} i & j \\ k & l \end{array} \right]_{\alpha_i,\alpha_j,\alpha_l} \sum_{m,\alpha_j} F_{K,m} \left[ \begin{array}{ccc} i & j \\ k & l \end{array} \right]_{\alpha_i,\alpha_j,\alpha_l} c_{i j}^{\alpha_i,\alpha_j,\alpha_l} \alpha_{\alpha_k,\alpha_m} =
\]

\[
= \sum_{l,\alpha_i} F_{l,m} \left[ \begin{array}{ccc} i & j \\ l & k \end{array} \right]_{\alpha_i,\alpha_j,\alpha_l} F_{l,i} \left[ \begin{array}{ccc} j & k \\ i & l \end{array} \right]_{\alpha_i,\alpha_j,\alpha_l} c_{j i}^{\alpha_i,\alpha_j,\alpha_l} \alpha_{\alpha_j,\alpha_m}
\]

(2.14)

as to turn the latter purely quantum and associative\(^{11}\), so that

\[
\left( \psi_{L_1,i_1}^{A_1 A_2} \star_q \psi_{L_2,i_2}^{A_2 A_3} \right) \star_q \psi_{L_3,i_3}^{A_3 A_4} = \psi_{L_1,i_1}^{A_1 A_2} \star_q \left( \psi_{L_2,i_2}^{A_2 A_3} \star_q \psi_{L_3,i_3}^{A_3 A_4} \right)
\]

(2.15)

obtains for \( \star_q \) defined as \( \star \) in (1.5) but with the substitution (2.13). In the above-mentioned \( su_n(2) \) case the deformation boils down to replacing the classical Clebsch–Gordan coefficients with those of \( U_q(\mathfrak{su}(2)) \) and was given in \( [11] \), where it first appeared, an interpretation in terms of the so-called Drinfel'd twist. The idea was developed and exploited in \( [14] \), with essential emphasis on the indication, contained in the new boundary algebra, towards a quantum symmetry of the underlying D-brane geometry and, consequently, of an associated matrix model, built on the assumption of quantum group covariance. Thus the new algebra constitutes the basis, on the BCFT side, of the models developed in \( [14] \).

There is yet another feature of (2.13) which becomes particularly significant in our present context. The deformation (2.13)-(2.2) proves sufficient to turn the orbifold analogue of (1.5) associative. The proof of the last statement is presented in App. \( A \) and enables us to start the construction of the \( q \)-matrix model of the orbifold physics directly at the level of the original quantum matrix algebra of \( [14] \) and seek for the automorphisms of the latter corresponding to the elements of the simple current orbifold group. The ensuing quotient structure is expected to define the quantum geometry of untwisted D-branes wrapping the orbifold.

3. The Reflection Equation Algebras and their representation theory.

Similarities in the representation theory of the algebras: \( A_N^{(1)} \) and \( U_q(A_N) \) (\( q \) being a root of unity) are widely appreciated. There are, however, important differences as well. One of them is the outer automorphisms group\(^{12}\), \( \text{Out} = \text{Aut}/\text{Int} \). For \( A_N^{(1)} \) the latter reflects the symmetries of the appropriate Dynkin diagram, \( \text{Out} \left( A_N^{(1)} \right) = \mathbb{Z}_{N+1} \ltimes \mathbb{Z}_2 \), while for \( U_q(A_N) \) we have \( \text{Out}(U_q(A_N)) = \mathbb{Z}_N^2 \ltimes \mathbb{Z}_2 \). The difference is crucial in the context of simple current orbifolding since - according to \( [23] \) - the group generated by simple currents under the OPE is precisely the strictly outer factor of \( \text{Out} \left( A_N^{(1)} \right) \), that is - \( \mathbb{Z}_{N+1} \).

\(^{11}\)At least within a certain (truncated) range of its group-theoretic labels, cp \( [14] \).

\(^{12}\)\( \text{Aut} \) is the set of all automorphisms, with the identity adjoined to endow it with the group structure. \( \text{Int} \) is the set of inner automorphisms.
In this paper we investigate the so-called Reflection Equation Algebra ([18, 19]), closely related to a modification of $\mathcal{U}_q(A_N)$ named the extended quantum universal enveloping algebra and denoted as $\mathcal{U}_{q}^{\text{ext}}(A_N)$. As we explore its representation theory it shall become clear that, beside representations, $\text{REA}_q(A_N)$ shares with $A^{(1)}_N$ the set of outer automorphisms $\mathbb{Z}_{N+1} \rtimes \mathbb{Z}_2$, with - as indicated by our results - the same geometrical meaning of the $\mathbb{Z}_{N+1}$ factor as in the affine setup. The last property is crucial for constructing orbifold models based on $\text{REA}_q(A_N)$ in strict analogy with the WZW orbifolds discussed in the preceding section.

Recall that the Reflection Equation Algebra, $\text{REA}_q(A_N)$, is the algebra generated by the operator entries of the matrix $M$ determined by the celebrated Reflection Equation ([18]):

$$R_{12}M_1R_{21}M_2 = M_2R_{12}M_1R_{21}, \quad (3.1)$$

in which

$$M_1 = M \otimes I, \quad M_2 = I \otimes M \quad (3.2)$$

and

$$R_{12} = R^{(1)} \otimes R^{(2)} = R, \quad R_{21} = R^{(2)} \otimes R^{(1)} \quad (3.3)$$

is the universal $\mathcal{R}$-matrix of the Drinfel’d–Jimbo algebra $\mathcal{U}_q(A_N)$ in the bifundamental representation,

$$\mathcal{R} = (R_{\text{fund.}} \otimes R_{\text{fund.}})(\mathcal{R}) =: (\pi \otimes \pi)(\mathcal{R}). \quad (3.4)$$

The independent central terms of this algebra are well-known and given by

$$c_k := \text{tr}_q(M^k), \quad k \in \mathbb{1}, \mathbb{N}. \quad (3.5)$$

There is one more scalar\textsuperscript{13}, the quantum determinant, to be set to some specific value:

$$\det_q(M) \sim I. \quad (3.6)$$

As an immediate consequence of \textsuperscript{(3.1)} and \textsuperscript{(3.6)} we get

$$M \mapsto e^{2\pi i l} M, \quad l \in \mathbb{Z}_{N+1} \quad (3.7)$$

as automorphisms of $\text{REA}_q(A_N)$. In what follows we show that \textsuperscript{(3.7)} are in fact outer automorphisms for all $l \neq 0$. It is possible to construct a larger set of outer automorphisms by studying finite dimensional representations of $\text{REA}_q(A_N)$ induced through the homomorphisms:

$\text{REA}_q(A_N) \hookrightarrow \mathcal{U}_{q}^{\text{ext}}(A_N) \hookrightarrow \mathcal{U}_h(A_N)$. We shall demonstrate that the above homomorphisms generate $\text{Out}(\text{REA}_q(A_N)) = \mathbb{Z}_{N+1} \rtimes \mathbb{Z}_2$, with the $\mathbb{Z}_{N+1}$ factor realised analogously to $\mathbb{Z}_{N+1} \subset \text{Out} \left( A^{(1)}_N \right)$ and the remaining $\mathbb{Z}_2$ - the standard mirror symmetry of the Dynkin diagram of $A_N$.

\textsuperscript{13}For $q$ a root of unity there are additional scalar operators, cp Sect. \textsuperscript{8.2} and App. \textsuperscript{C.1}.

– 9 –
3.1 Relations between \( \text{REA}_q(A_N) \), \( \mathcal{U}_q^{\text{ext}}(A_N) \) and \( \mathcal{U}_h(A_N) \).

There is a set of algebra homomorphisms:

\[
\mathcal{U}_q(A_N) \to \mathcal{U}_q^{\text{ext}}(A_N) \to \mathcal{U}_h(A_N) \uparrow \tag{3.8}
\]

\( \text{REA}_q(A_N) \)

which we describe in the course of the paper. We denote the corresponding generators as:

- the quantum enveloping algebra \( \mathcal{U}_q(A_N) \): \( \{K_j, K_j^{-1}, E_j, F_j\}_{j \in \mathbb{N}} \);
- the extended quantum enveloping algebra \( \mathcal{U}_q^{\text{ext}}(A_N) \) (for details consult App. C): \( \{k_{\pm \epsilon_i}, E_j, F_j\}_{i \in \mathbb{N}+1, j \in \mathbb{N}} \);
- the \( h \)-adic Hopf algebra \( \mathcal{U}_h(A_N) \): \( \{H_j, E_j, F_j\}_{j \in \mathbb{N}} \).

The only nontrivial images of the generators, given by the homomorphisms (3.8), are

\[
\forall j \in \mathbb{N} : K_j \to k_{\epsilon_j} k_{-\epsilon_{j+1}}, \tag{3.9}
\]

\[
\forall i \in \mathbb{N}+1 : k_{\epsilon_i} \to q^{H_{\epsilon_i}}, \tag{3.10}
\]

where

\[
\forall i \in \mathbb{N}+1 : H_{\epsilon_i} := \sum_{j,l=1}^{N} (\epsilon_i, \alpha_j) A_{jl}^{-1} H_l. \tag{3.11}
\]

As for the representation theory \( \mathcal{R}ep(\text{REA}_q(A_N)) \), the most important is the vertical arrow which, in particular, indicates that for each representation of \( \mathcal{U}_q^{\text{ext}}(A_N) \) we have an associated representation of \( \text{REA}_q(A_N) \), a fact that shall be used quite heavily in this section. The corresponding homomorphism: \( \text{REA}_q(A_N) \hookrightarrow \mathcal{U}_q^{\text{ext}}(A_N) \) was originally discussed in [9, 27] (following the earlier results of [28]) and reads

\[
\mathbf{M} = \mathbf{L}^+ \mathbf{S} \mathbf{L}^- \in \text{Mat}((N+1) \times (N+1); \mathbb{C}) \otimes \mathcal{U}_q^{\text{ext}}(A_N). \tag{3.12}
\]

where

\[
\mathbf{L}^\pm = \sum_{i,j=1}^{N+1} e_{ij} \otimes L^\pm_{ij} \tag{3.13}
\]

are operator-valued matrices\(^{14}\). In general, the operators \( L^\pm_{ij} \in \mathcal{U}_q^{\text{ext}}(A_N) \) may have a complicated form so they are explicitly presented in App. D.1. Notice that with (3.12) we automatically have

\[
\det_q(\mathbf{M}) = I \tag{3.14}
\]

\(^{14}\) \( e_{ij} \) are the basis matrices \( (e_{ij})_{kl} = \delta_{ik} \delta_{jl} \).
3.2 $\text{Rep}(U_q^{\text{ext}}(A_N))$ and $\text{Out}(U_q^{\text{ext}}(A_N))$

The algebra $U_q^{\text{ext}}(A_N)$ possesses a rich representation theory, all the more so when $q$ is a root of unity. In what follows we focus on regular finite-dimensional highest weight representations exclusively. The latter can be obtained from those of $U_h(A_N)$ through the homomorphism (3.10) and a set of automorphisms discussed below. To start with, we recall that

$$\text{Rep}_{\text{r.h.w.}}(U_h(A_N)) = \bigoplus_{\Lambda \in P^+_\kappa(A_N)} R_{\Lambda},$$

(3.14)

where $P^+_\kappa(A_N)$ is the fundamental alcove of $A_N^{(1)}$, i.e. the set of dominant integral affine weights.

The automorphisms used to generate $\text{Rep}(U_q^{\text{ext}}(A_N))$ are of the phase-changing type:

$$(k_{\pm \epsilon_i}, E_j, F_j) \mapsto \left( e^{\pm \pi p_i} k_{\pm \epsilon_i}, E_j, e^{\pi (p_j - p_{j+1})} F_j \right), \quad i \in \overline{1,N+1}, \ j \in \overline{1,N},$$

(3.15)

$$2(p_j - p_{j+1}) = 0 \mod 2, \quad \sum_{l=1}^{N+1} p_l = 0 \mod 2.$$  

(3.16)

As the homomorphisms (3.8) identify the ladder generators $(E_j, F_j)$ of the algebras considered, their finite dimensional irreducible highest weight modules are isomorphic and have the same highest weight state (denoted as $V_{\Lambda}$). The only effect of the automorphisms (3.15)-(3.16) is a change of the phases of the eigenvalues of the Cartan generators $k_{\pm \epsilon_i}$.

One can show that the complete description of the $U_q^{\text{ext}}(A_N)$ representations thus obtained reads (see App. C.2): 

$$\text{Rep}_{\text{r.h.w.}}(U_q^{\text{ext}}(A_N)) = \bigoplus_{\Lambda \in P^+_\kappa(A_N)} \bigoplus_{\omega \in \mathbb{Z}_N} \bigoplus_{l \in \mathbb{Z}_{N+1}} R^{l, \omega}_{\Lambda} \cong \text{Rep}_{\text{r.h.w.}}(U_h(A_N)) \otimes (\mathbb{Z}_{N+1} \otimes \mathbb{Z}_N^2),$$

(3.17)

with each of the summands uniquely determined by the Cartan eigenvalues returned by the highest weight state,

$$\forall j \in \overline{1,N+1} : k_{\epsilon_j} \triangleright V_{\Lambda} =: k_j V_{\Lambda}.$$  

(3.18)

The latter eigenvalues are

$$k_1 = e^{-\pi L(l,\omega)_{N+1}} q^{(\epsilon_1,\Lambda)},$$

(3.19)

$$\forall j \in \overline{2,N+1} : k_j = e^{-\pi L(l,\omega)_{N+1}} e^{-\pi \sum_{m=1}^{l-1} \omega_m} q^{(\epsilon_j,\Lambda)},$$

(3.20)

where

$$L(l,\omega) = 2l + \sum_{m=1}^{N} (N+1-m)\omega_m, \quad l \in \mathbb{Z}_{N+1}.$$  

(3.21)
and the parameters \( l \in \mathbb{Z}_{N+1}, \omega \in \mathbb{Z}_2^N \) are the ones used in (3.17).

The associated group \( \text{Out}(\mathcal{U}_q^{\text{ext}}(A_N)) \) is

\[
\text{Out}(\mathcal{U}_q^{\text{ext}}(A_N)) = (\mathbb{Z}_2^N \otimes \mathbb{Z}_{N+1}) \rtimes \mathbb{Z}_2, \tag{3.22}
\]

where the distinguished \( \mathbb{Z}_2 \) factor corresponds to the classical mirror symmetry of the Dynkin diagram, present already in \( \text{Out}(\mathcal{U}_h(A_N)) \) (24).

3.3 \( \text{Rep}(\text{REA}_q(A_N)) \) and \( \text{Out}(\text{REA}_q(A_N)) \).

In what follows we take \( \text{Rep}(\text{REA}_q(A_N)) \) to be induced from \( \text{Rep}(\mathcal{U}_q^{\text{ext}}(A_N)) \) via \( (3.12) \) and seek for inequivalent representations obtained in this way. In so doing we consider the behaviour of the independent mutually diagonalisable operators of \( \text{REA}_q(A_N) \), i.e. the scalar operators \( 15 \) (3.5) and the Cartan generators \( M_{ii} \), under the action of the outer automorphisms \( (3.15)-(3.17) \) (mediated by the homomorphism map \( (3.12) \))

\[
c_n \to e^{-\frac{2\pi n L(l,\omega)}{N+1}} c_n, \tag{3.23}
\]

\[
M_{ii} \to e^{-\frac{2\pi n L(l,\omega)}{N+1}} M_{ii}, \tag{3.24}
\]

where \( L(l,\omega) \) is as in (3.21). Since \( l \) and \( \omega \) appear only in the combination \( L(l,\omega) \) above we may treat the latter as an independent quantity and take \( L \in \mathbb{Z}_{N+1} \). Generators of the latter we shall denote by \( \omega_{q,N} \). Thus

\[
\text{Out}(\text{REA}_q(A_N)) = \mathbb{Z}_{N+1} \rtimes \mathbb{Z}_2. \tag{3.25}
\]

The above form of \( \text{Out}(\text{REA}_q(A_N)) \) becomes reflected in the structure of the associated representation theory:

\[
\text{Rep}_{\text{ind.}}(\text{REA}_q(A_N)) = \bigoplus_{\Lambda \in P_+^\text{r.h.w.}(A_N) \cup \mathbb{Z}_{N+1}} \bigoplus_{L \in \mathbb{Z}_{N+1}} R^L_{\Lambda} \cong \text{Rep}_{\text{r.h.w.}}(\mathcal{U}_h(A_N)) \otimes \mathbb{Z}_{N+1}. \tag{3.26}
\]

In fact, we still need to demonstrate that there is no further reduction of the representation theory, or that the representations: \( R^L_{\Lambda} \), \( (\Lambda, L) \in P_+^\text{r.h.w.}(A_N) \times \mathbb{Z}_{N+1} \) are indeed pairwise inequivalent as representations of \( \text{REA}_q(A_N) \). The proof of this fact is rather technical and has therefore been moved to the App. D.2.

As an aside we remark at this point that the above-indicated behaviour of the \( M_{ii} \)'s under (3.15) is universal to all the entries of the \( M \)-matrix, that is - the matrix scales uniformly:

\[
\omega_{q,N}^L : M \to e^{-\frac{2\pi n L}{N+1}} M, \tag{3.27}
\]

under powers of the quantum \( \mathbb{Z}_{N+1} \)-automorphism \( \omega_{q,N} \), which sends us back to (3.7).

\(^{15}\)It ought to be noted that in the particular case of \( \text{Rep}(\mathcal{U}_q^{\text{ext}}(A_N)) \) the eigenvalues of all the nontrivial generators of \( Z_0, k_{N+1}^{2n} \), described in App. C.1 coincide on a given irreducible representation and so \( Z_0 \) is effectively generated by, say, \( k_{N+1}^{2n} \). The latter, however, is an element of \( \text{REA}_q(A_N) \), namely: \( k_{N+1}^{2n} \equiv M_{N+1,N+1}^{2n} \). This completes the analysis below.
3.4 Geometrical meaning of $Out$

The $Z_{N+1}$-factor in (3.25) has an interesting geometrical interpretation rooted in (3.23). It appears that its action on $c_n$ is the same as the induced action of the outer automorphism group $Z_{N+1} \subset Out\left(A_N^{(1)}\right)$. The latter action can be identified with a (rotational) symmetry of the fundamental affine alcove $P_{κ}^\kappa(A_N)$ and also with a cyclic symmetry of the extended Dynkin diagram. It generates cyclic permutations of the affine weights:

$$\varpi_N^\kappa : \left(\Lambda^1, \Lambda^2, \ldots, \Lambda^N, \Lambda^0 = κ - \sum_{j=1}^{N} \Lambda^j\right) \rightarrow \left(\Lambda^2, \Lambda^3, \ldots, \Lambda^N, \Lambda^0, \Lambda^1\right),$$

(3.28)

whereby the following map is induced within $P_{κ}^\kappa(A_N)$:

$$\omega_N^\kappa : [λ_1, λ_2, \ldots, λ_N] \rightarrow \left[λ_2, λ_3, \ldots, λ_N, κ - \sum_{i=1}^{N} λ_i\right].$$

(3.29)

The resulting action on the scalars is

$$c_n \rightarrow e^{\frac{2πi}{N+1}}c_n, \quad κ_{κ_{κ_{κ_{κ}}}^{κ_{κ}}}^{κ_{κ_{κ}}} \rightarrow e^{\frac{2πi}{N+1}}k_{κ_{κ_{κ}}}^{κ_{κ}},$$

(3.30)

which matches (3.23)-(3.24) exactly for $L(l, ω) = -1$.

The importance of the above result rests upon the deep geometrical meaning of the $Z_{N+1}$-factor of $Out\left(A_N^{(1)}\right)$ - it is isomorphic to the center of the $SU(N+1)$ group. In [14], string theory branes in the $SU(N+1)$ group manifold were associated with representations of $REA_q(A_N)$. In this picture the equations defining "positions" of the branes in the (quantum) group manifold were the Casimirs (3.5). Thus the (quantum) $SU(N+1)$ was "foliated" by the set of branes corresponding to all highest weight irreducible representations of $REA_q(A_N)$. The action of the elements of $Out(REA_q(A_N))$ corresponds to some discrete rotations of the branes inside the (quantum) $SU(N+1)$. It is worth noting that there are $N+1$ branes for a given set of "positions", (3.5). Equipped with this result one can now try to perform constructions of branes in some orbifold models of the compact group stringy background, e.g. on the manifold $SU(N+1)/Z_{N+1}$ in which case $Out(REA_q(A_N))$ is expected to mimic in the quantum setup the action of the simple current group $sOut\left(A_N^{(1)}\right) \cong Z_{N+1}$ of the relevant BCFT. We now turn to this issue.

4. The quantum orbifold.

The present section is central to our paper. It contains an explicit construction of essentially new quantum geometries, demonstrated to emerge from the original REA upon dividing out the action of the quantum automorphism $ω_{q,N}$ (3.27) studied in Sect. 3.3 and Sect. 3.4. The geometries are expected to describe fixed point D-branes on the simple current $Z_{N+1}$-orbifold of the WZW model of the type $A_N$. As the section develops we present direct evidence in favour of this identification.
4.1 New geometries in $\text{REA}_q(A_N) / \mathbb{Z}_{N+1}$.

According to the conclusions drawn in Sect. 3, the subalgebra of $\text{REA}_q(A_N)$ invariant under $s\text{Out} \left( A_N^{(1)} \right) \cong \mathbb{Z}_{N+1}$ has the form:

$$\text{REA}_q(A_N) / \mathbb{Z}_{N+1} = \text{span} \left\{ \prod_{i,j,l=1}^{N+1} M_{i,j,l} \right\} / I_{\text{det}_q}, \quad (4.1)$$

with $I_{\text{det}_q}$ - the ideal generated by the original reflection equation, (1.6), and the $q$-determinant constraint, (1.7). Its presence reduces the $\left((N+1)^{2(N+1)} + 1\right)$-element set of invariants to the minimal generating subset, or - the basis of the $q$-orbifold geometry.

In order to be able to assess the structure encoded in (4.1) we make the following observation: the set of $\mathbb{Z}_{N+1}$-invariants, (4.4), decomposes in restriction to a particular representation into (nontrivial) monomials of an arbitrary degree $0 < d \leq N+1$ determined by

$$d = N+1 : M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N+1}j_{N+1}},$$

$$d = N : c_1 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N}j_{N}},$$

$$d = N-1 : c_2 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N-1}j_{N-1}}, c_1^2 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N-1}j_{N-1}},$$

$$d = N-2 : c_3 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N-2}j_{N-2}}, c_2 c_1 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N-2}j_{N-2}}, c_1^3 M_{i_1j_1} M_{i_2j_2} \cdots M_{i_{N-2}j_{N-2}},$$

$$\vdots$$

$$d = 1 : c_N M_{i_1j_1}, c_{N-1} c_1 M_{i_1j_1}, c_{N-2} c_2 M_{i_1j_1}, c_{N-2} c_1^2 M_{i_1j_1}, \ldots, c_1^N M_{i_1j_1}. \quad (4.2)$$

The above decomposition shows that for a generic weight the $q$-orbifold geometry is that of the pre-orbifolding $q$-conjugacy class as its generating subalgebra coincides with the original algebra of $M_{i,j}$'s (modulo $I_{\text{det}_q}$, as usual). The non-generic features, on the other hand, come to the fore whenever we encounter a fixed point of the action of $\omega_N^\kappa$ or its higher powers on $P_+^\kappa(A_N)$, that is - whenever there is a classical weight stable under the relevant strictly outer automorphism. Among these there is a distinguished one: the one sitting in the geometric centre of the $N$-simplex of the fundamental affine alcove\textsuperscript{16}.

Let us be more specific about the admissible geometries emerging from (4.2). To these ends consider a restriction of the algebra of $\mathbb{Z}_{N+1}$-invariants to the representation of $\text{REA}_q(A_N)$ associated with a given weight $\Lambda$ and characterised by the following set of non-zero Casimir eigenvalues:

$$C_{n_1}, C_{n_2}, \ldots, C_{n_K} \neq 0, \quad \forall n \in \mathbb{N} \setminus \{n_1, n_2, \ldots, n_K\} : C_n = 0 \quad (4.3)$$

\textsuperscript{16}There is a classical weight at that point if and only if $\kappa \in (N+1)\mathbb{N}^*$ We also impose an extra physical condition to be satisfied by the associated fixed point WZW module, which is the vanishing of the associated monodromy charge. See: 24.
where \( C_{n_j} := c_{n_j} |_{R_A}, \ n_j \in \overline{1,N}, \ j \in \overline{1,K} \). We then claim that the corresponding coordinate subalgebra is isomorphic with the algebra of \( \mathbb{Z}_n \)-invariants, generated by independent monomials of the degree \( n \), given by

\[
  n = \gcd \left( N + 1, n_j \right)_{j \in \overline{1,K}}.
\]  

(4.4)

Thus, in particular, the \( \mathbb{Z}_n \) is a subgroup of the underlying \( \mathbb{Z}_{N+1} \), and the stabiliser of \( \Lambda \) within the latter. In the distinguished case of the central weight (and exclusively in that case) we recover the full \( \mathbb{Z}_{N+1} \) as the stabiliser of the associated fixed point conjugacy class, with the \( d = N + 1 \) subalgebra as the generating one (all Casimirs vanish on the central weight). For \( N + 1 \) prime, on the other hand, we end up with pre-orbifolding geometries over all non-central classical weights.

And here is the proof of our claim. Focus on the non-zero Casimir eigenvalue \( C_{n_j} \) multiplying all possible nontrivial monomials of the degree \( d_0^J := N + 1 - n_j \) in the operator variables \( M_{ij} \); we then verify, using the \( q \)-determinant constraint (1.7), that the corresponding subalgebra of the degree \( d_0^J \) reproduces upon multiplication a subalgebra of the degree \( d_{S_j}^L \) such that \( d_{S_j}^L | N + 1 \).

Indeed, assuming \( d_0^J \nmid N + 1 \) we may write\(^{17}\)

\[
  \exists w_0^J \geq 1 \exists r_0^J \leq a_0^J - 1 : \ N + 1 = w_0^J \cdot d_0^J + r_0^J.
\]  

(4.5)

Among the \((w_0^J + 1)\)-linear combinations of the generating monomials,

\[
  (w_0^J + 1) \cdot d_0^J = N + 1 + (d_0^J - r_0^J),
\]  

(4.6)

we find, in particular, arbitrary monomials of the degree \( d_1^J := d_0^J - r_0^J \) which can be written as

\[
  \det_q \mathbf{M} \cdot M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_d j_d}.
\]  

(4.7)

At this stage there are two possibilities: either \( d_1^J \mid N + 1 \), in which case we are done, with \( d_{S_j}^L := d_1^J \), or \( d_1^J \nmid N + 1 \) and we can proceed with the above \( q \)-determinant reduction, whereby - after a finite number \( S_j \) of steps - we do get a divisor (possibly trivial) of \( N + 1 \), i.e.

\[
  \exists w_{S_j}^J \geq 1 : \ N + 1 = w_{S_j}^J \cdot d_{S_j}^L.
\]  

(4.8)

The final subalgebra of the degree \( d_{S_j}^L \) does not yield any new structure via the \( q \)-determinant reduction. Furthermore, the original subalgebra of the degree \( d_0^J \) can be generated from the divisor subalgebra. The last property derives straightforwardly from \( 4.8 \) upon employing the decomposition with respect to the Casimir basis, \( 4.2 \). Indeed, among the \( w_{S_j}^J \)-linear combinations of the generating monomials of the degree \( d_{S_j}^L \) there are arbitrary monomials of the degree \( d_0^J \) which decompose as

\[
  c_{n_j} \cdot M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_d j_d}.
\]  

(4.9)

The next step consists in performing an analogous reduction with respect to \( c_{n_j} \), leading directly to a bigger subalgebra generated by monomials of the degree \( d_{S_j}^L \mid d_0^J \). After a finite number of

\(^{17}\)If \( d_0^J \mid N + 1 \) we simply set \( d_{S_j}^L := d_0^J \) and proceed further.
iterations of a successive application of the $q$-determinant reduction and the $c_{n_f}$ reduction we are
bound to obtain a subalgebra of monomials of the degree $d^J_{S_f \cap \Lambda} = \gcd(N + 1, n_f)$. Taking the
latter subalgebra as the starting point we can repeat the previous reasoning with respect to all
the non-zero Casimir eigenvalues, \[4.3\], and keep iterating the whole procedure until we ultimately
arrive at a subalgebra of the degree $n$, given by \[4.4\]. It ought to be stressed that the associated
group $\mathbb{Z}_n$ is the maximal common stabiliser of \[4.3\] and thus the stabiliser of $\Lambda$. The proof of the
reverse claim (about the existence of an appropriate subalgebra of \[4.1\] for any stabilised weight) is
trivial.

Before closing the present section let us once more emphasise its purport. We expect the
emergence of essentially new quantum geometries whenever for a given weight $\Lambda \in P^K_+(A_N)$ the
associated number \[4.4\] is greater than one. These quotient geometries are described by proper
subalgebras of $\text{REA}_q(A_N)$ and - as argued below - have a reduced representation theory. With this
result in mind we can proceed with our description of the $q$-orbifold geometry.

**4.2 The orbifold $q$-brane.**

Our starting point is the representation theory of the mother $\text{REA}$, \[3.26\]. So far we have been
dealing with pure algebra. In our search for further hints we need to reexamine the WZW models
we intend to describe with the $\text{REA}$’s at hand.

As has already been discussed, among untwisted WZW D-branes, to be described by represen-
tations of the $\text{REA}$, there are distinguished ones labelled by the so-called fixed point weights,
i.e. dominant integral affine weights which are invariant under $\omega_N^*$ or its higher powers. The D-
branes reside within the WZW manifold precisely at the fixed points of the orbifold action and
- consequently - give rise to fractional D-branes of the orbifold model. The action of the simple
current group on the original objects, which we intend to mimic in the $q$-algebraic setup, becomes
inner at these points and therefore enforces a reduction of the D-brane state space and produces
a stabiliser-charge degeneracy as we make the transition from the pre-orbifolding geometry to the
orbifold one. With the aim of reproducing the BCFT picture as closely as possible we propose to
take the specific $(N + 1)$-fold direct sums of irreducible representations:

$$
\mathcal{R}_\Lambda := \bigoplus_{L=0}^{N} R^L_{(\omega_N^*)L, \Lambda} 
$$

as fundamental objects in the $q$-matrix description of the effective stringy dynamics in the WZW
models of the type $A_N$. The irreducible representations in \[4.10\] were shown in Sect. \[3\] to share
the same set of scalar eigenvalues and are thus declared to describe a single untwisted $q$-brane. A
direct verification of this proposal is provided by the explicit calculation performed below of the
associated $q$-brane tensions. It yields results in perfect agreement with our earlier BCFT result,
\[2.7\]. Furthermore, it is not difficult to verify that the above $q$-brane configurations are stable
against decay.
As an aside, we note here that as we introduce (4.10) the requirement of consistency with the BCFT data for pre-orbifolding WZW models necessitates a reformulation of the original \( q \)-matrix model:

\[
S_{\text{eff},0}^{q,A_N} \rightarrow S_{\text{eff},0}^{q,A_N} := \frac{T_0}{N+1} \text{Tr}_q \left[ \mathbb{I} + \text{cov}_{U_q(A_N)}(M) \right],
\]

to be compared with (1.9). The action is to be evaluated on representations of the kind (4.10).

### 4.3 The fixed point resolution.

Now that we have elaborated on the representation theory of the quantum matrix model of the effective boundary state dynamics on \( A_N \) we witness - as desired - the emergence of a distinguished class of \( q \)-modules, namely - the fixed point modules \( \mathcal{R}_{\Lambda_{FP}} \), \( \mathcal{S}_{\Lambda_{FP}} \neq \{\text{id}\} \), on which a nontrivial subgroup \( \mathcal{S}_{\Lambda_{FP}} \) of the quantum \( \mathbb{Z}_{N+1} \), satisfying

\[
\forall \Lambda \in \mathbb{P}_+(A_N) \exists n \in \mathbb{Z}_{N+1}: (\mathcal{S}_\Lambda \cong \mathbb{Z}_n \land n \mid N + 1),
\]

acts in an inner way, i.e. it does not generate an overall representation transfer. The property means - in group-theoretic terms - that the automorphisms \( \left( \omega_{q,N}^D \right)^l \), with \( l \in \mathbb{Z}_n \) and \( D := \frac{N+1}{n} \), are realised on \( \text{REA}_q(A_N) \) as conjugations (i.e. as inner automorphisms). We denote the generator of the \( \mathbb{Z}_n \) discussed by \( \Theta_n \). The realisation then takes an obvious form on \( M_{ij} \), \( i,j \in \mathbb{I},N+1 \):

\[
\Theta_n \triangleright \mathcal{R}_{\Lambda_{FP}}(M_{ij}) = e^{\frac{2\pi i a}{n}} \mathcal{R}_{\Lambda_{FP}}(M_{ij})
\]

and is inherited by monomials of a higher degree.

An immediate consequence of the above is a reduction of the fixed point representation theory in orbifolding. The new object to be associated with an orbifold \( q \)-brane is the \( \mathbb{Z}_{N+1} \)-quotient of the set of conjugate representations:

\[
\mathcal{R}_\Lambda, \mathcal{R}_{\omega_N^D \Lambda}, \mathcal{R}_{(\omega_N^D)^2 \Lambda}, \ldots, \mathcal{R}_{(\omega_N^D)^{D-1} \Lambda},
\]

a conclusion consistent with the findings of Sect. 4.1. Thus the orbifold representations undergo the projection:

\[
\bigoplus_{k \in \mathbb{Z}_D} \mathcal{R}_{(\omega_N^D)^k \Lambda} \longrightarrow \mathcal{R}_{[\Lambda]} := \bigoplus_{L \in \mathbb{Z}_D} \mathcal{R}^L \left( \omega_N^D \right)^L \Lambda.
\]

Following further the BCFT arguments of [24] we are bound to allow for a separate boundary state (and therefore - an independent \( q \)-brane geometry) for each irreducible representation of the stabiliser \( \mathbb{Z}_n \). The latter group being Abelian, we have precisely \( n \) inequivalent such representations at our disposal, namely:

\[
\rho_a(\Theta_n) := e^{\frac{2\pi i a}{n}}, \quad a \in \mathbb{Z}_n.
\]
Hence we eventually obtain at an $S_{\Lambda F P} \cong \mathbb{Z}_n$ fixed point an $n$-tuple of $\text{REA}_q(A_N)$-modules, distinguished by their respective $\mathbb{Z}_n$-labels,

$$\text{an unresolved } q\text{-brane over } \Lambda_{FP} \sim \mathcal{R}_{[\Lambda_{FP}]}^n := \bigoplus_{a \in \mathbb{Z}_n} \mathcal{R}_{[\Lambda_{FP}]} \otimes \rho_a \cong \mathcal{R}_{[\Lambda_{FP}]} \otimes \mathbb{Z}_n; \quad (4.17)$$

the $a$-th fractional $q$-brane over $\Lambda_{FP} \sim \mathcal{R}_{[\Lambda_{FP}]}^a := \mathcal{R}_{[\Lambda_{FP}]} \otimes \rho_a. \quad (4.18)$

At the algebraic level all our observations can be put together in a compact structure known under the name of the crossed product algebra (in the present context introduced already in [22] in the classical limit), $\text{REA}_q(A_N) \rtimes_{w_{q,N}} \mathbb{Z}_n$, with $n$ varying as we move from one irreducible representation of the $\text{REA}$ to another. Upon quotienting it turns into $(\text{REA}_q(A_N)/\mathbb{Z}_{N+1}) \rtimes_{w_{q,N}} \mathbb{Z}_n$, representing the $q$-orbifold geometry. The latter structure is - as a vector space - a tensor product of the algebra $\text{REA}_q(A_N)/\mathbb{Z}_{N+1}$ of $\mathbb{Z}_{N+1}$-invariants $X_{1/2}$ and the stabiliser $\mathbb{Z}_n$, with the extra compatibility condition:

$$(X_1 \otimes \Theta_n^{m_1}) \cdot (X_2 \otimes \Theta_n^{m_2}) := X_1 \left[ (\Theta_n^{m_1})_{(1)} \triangleright X_2 \right] \otimes (\Theta_n^{m_1})_{(2)} \Theta_n^{m_2}, \quad (4.19)$$

where

$$\Delta \Theta_n = (\Theta_n)_{(1)} \otimes (\Theta_n)_{(2)} = \Theta_n \otimes \Theta_n \quad (4.20)$$

is the coproduct of the group-like element $\Theta_n$ of the standard Hopf algebra $\mathbb{Z}_n$ (see, e.g., [17]) in Sweedler’s notation, and the action $\triangleright$ is as in (4.13). It ought to be remarked at this point that the crossed product algebra has long been known to properly encode the algebraic data of fixed point matrix models of string theory on orbifold geometries ([26]).

Within the present framework the fixed point subalgebras discussed in Sect. 4.1. become $\mathbb{Z}_n$-degenerate. The isomorphic copies are generated by $^{18}$ - respectively -

$$X_{i_1j_1i_2j_2...i_nj_n}^{(a)} := M_{i_1j_1} M_{i_2j_2} \cdots M_{i_nj_n} \otimes P_{a}^{(n)}, \quad i_k, j_k \in \overline{1,N+1}, \quad k \in \overline{1,n}, \quad (4.21)$$

with the projectors:

$$P_{a}^{(n)} := \frac{1}{n} \sum_{l=0}^{n-1} e^{-\frac{2\pi a l}{n}} \Theta_n^{l}, \quad (4.22)$$

naturally associated with the $n$ irreducible representations of $\mathbb{Z}_n$,

$$\rho_a \left( P_{b}^{(n)} \right) = \delta_{ab}. \quad (4.23)$$

Eventually, we arrive at the $q$-algebraic description of the geometry of the (unresolved) fixed point $q$-brane:

$$\mathcal{R}_{[\Lambda_{FP}]} \left( \text{REA}_q(A_N)/\mathbb{Z}_{N+1} \right) \rtimes_{w_{q,N}} \mathbb{Z}_n = \text{span} \left\{ \mathcal{R}_{[\Lambda_{FP}]} \left( X_{i_1j_1i_2j_2...i_nj_n}^{(a)} \right) \right\}_{a \in \mathbb{Z}_n, \; i_k, j_k \in \overline{1,N+1}, \; k \in \overline{1,n}} / I_{\text{RE, det}_q}, \quad (4.24)$$

$^{18}$One still has to impose the RE and the $q$-determinant constraint in order to obtain a minimal generating set.
matching nicely the concluding remarks of Sect. 2.1. (compare the example of Sect. 3.)

An issue one is compelled to address next is the \( q \)-matrix description of the off-fixed-point geometries. A first step towards this goal consists in identifying an appropriate subset of primary boundary fields of the orbifold BCFT, closed under the OPE and of conformal dimensions vanishing in the classical limit:

\[
\begin{aligned}
\kappa \to \infty \\
\alpha' \to 0 \\
\alpha' \sqrt{\kappa} \to \infty
\end{aligned}
\]  
(4.25)

The subset is expected to correspond - in the sense of [4] - to a quantisation of the algebra of functions on the worldvolume of the orbifold D-branes. We thus consider the \( \mathbb{Z}_{N+1} \)-averaged orbifold primaries (2.10) of [22]. Amongst them, we readily distinguish two classes of fields associated with a single orbifold D-brane, namely - the "diagonal" ones:

\[
\Psi_{L,L,i}^{\Lambda}(x) := \sum_{g \in \mathbb{Z}_{N+1}} \Psi_{L,L,i}^{g \Lambda g}(x),
\]  
(4.26)

and the "off-diagonal" ones:

\[
\Psi_{L,L,i;g}^{\Lambda}(x) := \sum_{h \in \mathbb{Z}_{N+1}} \Psi_{L,L,i}^{h \Lambda h g \Lambda}(x), \quad g \neq 1.
\]  
(4.27)

Clearly, the operators (4.26) are to be associated with \( \mathbb{Z}_{N+1} \)-related \((N+1)\)-tuples of open strings attached each to one and only one of the \( N+1 \) D-branes on the covering space identified upon orbifolding, whereas (4.27) describe states of strings stretched between pairs of simple-current-conjugate branes. In spite of joining mutually identified subspaces of the orbifold geometry the latter strings have a non-vanishing length and thence also a non-vanishing mass surplus with respect to the former, given roughly by their geodesic length measured in the closed string metric\(^{19}\). In consequence, they decouple from the diagonal set (4.26) and can be excluded from our \( q \)-geometric picture. Thus, after the dust has cleared, we are left with just the right number of distinguished primaries to fill up a single pre-orbifolding module for any off-fixed-point weight \( \Lambda \), or the associated D-brane. Furthermore, their OPE is identical with that of the pre-orbifolding boundary state so that we end up with a geometry isomorphic with the original one.

Looking at the relevant algebra, we note - on the one hand - the triviality of the reduction of the representation theory over \( \Lambda \),

\[
\bigoplus_{k \in \mathbb{Z}_{N+1}} \mathcal{R}(\omega_N)^k \Lambda \rightarrow \mathcal{R}[\Lambda] \cong \mathcal{R}_A
\]  
(4.28)

and - on the other - an unchanged coordinate \( q \)-algebra for this off-fixed-point geometry, which is simply the entire \( \text{REA}_q(A_N) \). All that is precisely as dictated by the BCFT. The last result completes our exposition of the quantum orbifold algebra. We may now turn to the physics of the model.

\(^{19}\)See: [20].
4.4 Orbifold $q$-brane tensions.

An important and nontrivial test of the construction presented above is the computation of tensions of the orbifold $q$-branes within the framework developed. In so doing we follow the scheme advanced in the original papers, [14], that is - we compute the tension of a $q$-brane labelled by the weight $\Lambda \in P^+_\kappa(A_N)$ by evaluating the first term of the effective action (4.11) on the associated representation, (4.15). Beside giving rise to the aforementioned crossed product extension of the fixed point algebra of $\mathbb{Z}_{N+1}$-invariants, the latter yields the correct result for the tension of the resolved fixed point $q$-branes (as compared with the pre-orbifolding value):

$$E_{\text{Orb}_{q,N}}[\Lambda] = \frac{T_0}{N+1} \text{Tr} q_{R_\Lambda} = \frac{1}{N+1} \cdot D \cdot \text{Tr} q_{R_\Lambda} = \frac{1}{n} \mathcal{E}_\Lambda, \quad a \in \mathbb{Z}_n, \quad (4.29)$$

falling in perfect agreement with the BCFT data (cp (2.7)). It is supplemented by an equally satisfactory result for the generic $q$-branes:

$$E_{\text{Orb}_{q,N}}[\Lambda] = \frac{T_0}{N+1} \text{Tr} q_{R_\Lambda} = \frac{T_0}{N+1} \cdot (N+1) \cdot \text{Tr} q_{R_\Lambda} = \mathcal{E}_\Lambda, \quad (4.30)$$

an immediate consequence of the lack of an internal mechanism of spectrum reduction in these off-fixed-point cases.

It should be added that there is an additional piece of data hiding in (4.15). Namely, upon quotienting the pre-orbifolding representation theory we restrict the range of admissible affine weight labels $\Lambda$ to a fundamental region cut out of $P^+_\kappa(A_N)$ by the generating quantum automorphism, $\omega_{q,N}$. It has a shape of a hyperdeltoid $\mathcal{D}$ spanned by the vectors ($\Lambda^{(i)}$ are the fundamental weights):

$$\mathcal{D} : \left\{ 0, \frac{\kappa}{N+1} \sum_{i=1}^{N} \Lambda^{(i)}, \frac{\kappa}{N+1} \Lambda^{(i)} \right\}. \quad (4.31)$$

5. An example: $q$-branes on $\mathbb{R}P^2_q$.

Below we detail the particularly simple example of the antipodal $\mathbb{Z}_2$-orbifold of the quantum matrix model for $\mathfrak{su}_\kappa(2)$, $\kappa \in 4\mathbb{N}$. The model is exemplary in that it develops precisely along the lines discussed in the previous sections, hence we restrict ourselves here to its $q$-geometric interpretation, relating our results to some well-established mathematical constructs of [30] and [31]. The BCFT properties of the corresponding branes were discussed in [22, 32].

5.1 The setup.

Our starting point shall be the REA suggested in [14] to give a plausible compact description of the quantum D-brane geometry on WZW group manifolds. Since we aim at describing the stringy geometry of the antipodal orbifold of $SU(2)$ we will focus on the REA generated by the operator entries of a matrix $\mathbf{M}$ subject to the reflection equation (1.6) in which $\mathbf{R}$ is the standard universal

\[ \text{Cp [17].} \]
\( R \)-matrix of \( \mathcal{U}_q(\mathfrak{su}(2)) \),

\[
R = \sum_{i,j,k,l=1}^2 R^i_{kl} e_{ik} \otimes e_{jl} = q e_{11} \otimes e_{11} + q e_{22} \otimes e_{22} + e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + \lambda e_{12} \otimes e_{21}. \tag{5.1}
\]

Upon expanding the generator matrix in the \( q \)-Pauli basis\(^{21}\)

\[
M = \sum_{\mu \in \{0,1,-1\}} M_\mu \sigma_\mu = \begin{pmatrix} M_4 - iq^{-2} M_0 & iq^{-\frac{3}{2}} \sqrt{2|q|} M_{-1} \\ -iq^{-\frac{3}{2}} \sqrt{2|q|} M_1 & M_4 + i M_0 \end{pmatrix}, \tag{5.2}
\]

we may write down the additional Casimir constraint:

\[
r^2 \equiv \det_q(M) = M_4^2 + \sum_{i,j \in \{0,1,-1\}} (g_q^{(1)})_{ij} M_i M_j = M_4^2 + M_0^2 - q^{-1} M_1 M_{-1} - q M_{-1} M_1, \tag{5.3}
\]

in a natural way, with \( g_q^{(1)} \) - the quantum metric\(^{22}\) on \( S^2_q \) and \( r^2 \) interpreted as the radius squared of the \( q \)-group manifold, \( S^3_q \), which we set roughly proportional to the level of the underlying WZW model (cf \([14]\)),

\[
r \approx \alpha' \sqrt{\kappa_2}. \tag{5.4}
\]

With \( M \) parametrised as in (5.2) the RE (1.6) takes a manageable (component) form:

\[
[M_4, M_i] = 0, \quad i \in \{0,1,-1\} \tag{5.5}
\]

\[
M_0 M_{-1} = q M_{-1} (q M_0 - i \lambda M_4), \tag{5.6}
\]

\[
M_1 M_0 = q (q M_0 - i \lambda M_4) M_1, \tag{5.7}
\]

\[
M_1 M_{-1} - M_{-1} M_1 = \lambda M_0 (M_0 - i M_4). \tag{5.8}
\]

The above algebra, supplemented by (5.3), is easily seen to reproduce (upon diagonalising the central element \( M_4 \) and some trivial rescalings) the celebrated Podleś’ spheres, \( S^2_{q,c} \), with the parameter \( c \) essentially determined by the value of \( M_4 \) (cf \([30]\)). Indeed, choosing for the given non-zero value of the Casimir, \( M_4 = rm_4 \neq 0 \), the new generators: \( (e_0, e_{\pm}) := \frac{1}{r m_4} \left( -M_0, \frac{1}{|q|^2} M_{\pm 1} \right) \)

we obtain the equilateral Podleś’ sphere algebra, \( \mathcal{O}(S^2_{q,c}) \), with \( c = -\frac{1}{|q|^2 m_4^2} \) (or, more adequately, a generalisation thereof for \( q \) a phase). Analogously, for \( M_4 = 0 \), we readily verify that the particular choice of rescaled generators: \( (\bar{e}_0, \bar{e}_{\pm}) := \frac{|q|}{r} \left( M_0, -\frac{1}{|q|^2} M_{\mp 1} \right) \) yields a generalisation of the equatorial Podleś’ sphere algebra, \( \mathcal{O}(S^2_{q,\infty}) \). Thus, consistently with standard mathematical results, it is natural to give to the \( M_i \)'s the meaning of local coordinates on a particular \( q \)-sphere

\(^{21}\)We choose the convention: \( \sigma_\mu \to (1, iq^{-1} \sigma_1)_{\mu \in \{0,\pm 1\}} \), with \( \sigma_1 \) as in \([14]\). \( \)

\(^{22}\)Cp \([16]\).
whose location in the mother manifold, $S^3_q$, is determined by the "transverse" coordinate, $M_4$.

Bearing in mind the clear geometric picture of the REA, we should expect that the general $q$-automorphism (3.7) corresponds to the standard antipodal identification of "points" on $S^3_q$ in that picture as indeed it does. In order to verify that we consider the embedding (3.12) in the case at hand,

$$M = r \begin{pmatrix} q^H + q^{-1} \lambda_2 F E & q^{-1} \lambda F \\ -\lambda q^{-H} E & q^{-H} \end{pmatrix},$$

(5.9)

localising $q$-branes through

$$M_4 |_{R^\Lambda} = \frac{1}{2} \text{tr}_q(M) |_{R^\Lambda} = r \frac{\cos(\Lambda+1)\pi}{\cos(\frac{\pi}{2})} \mathbb{I},$$

(5.10)

as dictated by the relevant BCFT. Specialising (3.7) to the case $N = 1$ we end up with the generating quantum $\mathbb{Z}_2$-automorphism:

$$\omega_{q,1} : M_\mu \mapsto -M_\mu, \quad \mu \in \{4, 0, 1, -1\},$$

(5.11)

just as required for the automorphism to have the interpretation of an antipodal map on $S^3_q$. We are now in a position to explicitly construct the $q$-matrix model for the $\mathbb{Z}_2$-antipod.-orbifold of the $su(2)$ WZW model at $\kappa \in 4\mathbb{N}^*$.

### 5.2 The $q$-orbifold.

In the light of the general results of Sect. 4.3, the coordinate algebra of the $\mathbb{Z}_2$-antipod.-orbifold of the stringy $q$-manifold $S^3_q$ is the algebra of quadratic monomials in the generators $M_\mu$ of $\text{REA}_q(\mathfrak{su}(2))$, $\mathbb{Z}_2$-extended - in the case of $\kappa \in 4\mathbb{N}^*$ - at the unique central fixed point, $\Lambda_{FP} = \frac{\pi}{2}$. The algebra of $\mathbb{Z}_2$-invariants, defining a quantum manifold which could be called the real quantum projective 3-plane, $\mathbb{R}P^3_q$, is easily verified to be generated by the mutually independent operators:

$$\mathcal{O}(\mathbb{R}P^3_q) = \text{span} \{ \mathbb{I}, M_0^2, M_1^2, M_{-1}^2, M_0 M_1, M_{-1} M_0, M_1^4, M_4 M_0, M_4 M_1, M_4 M_{-1} \} / I,$$

(5.12)

where $I$ is the ideal defined by a set of relations, deriving directly from (5.5). The relations are not very illuminating and shall therefore be left out.

An important feature of the ensuing algebra is the central character of $M_4^2$ which is used to label inequivalent irreducible representations. Thus for $M_4^2 \neq 0$, or $\Lambda \neq \frac{\pi}{2}$, we recover quantum 2-spheres, as discussed in Sect. 4.1, and indicated by the BCFT. At the fixed point, on the other
hand, where the original REA and (5.3) simplify,

\[ M_0 M_{-1} = q^2 M_{-1} M_0, \tag{5.13} \]

\[ M_1 M_0 = q^2 M_0 M_1, \tag{5.14} \]

\[ M_1 M_{-1} - M_{-1} M_1 = \lambda M_0^2, \tag{5.15} \]

\[ M_0^2 - q^{-1} M_{1-1} M_{-1} = q M_{1-1} M_1 = \frac{1}{r^2}, \tag{5.16} \]

the orbifold algebra reduces - upon choosing the rescaled generators:

\[ (P, R, \tilde{R}, T, \tilde{T}) := \left( \frac{[2]}{q^{2 \pi^2}} \left( \frac{1}{[2]} q^2 M_0^2, q^2 M_1^2, q^2 M_{-1}^2, \frac{q}{\sqrt{[2]}} M_0 M_1, -\frac{q}{\sqrt{[2]}} M_{-1} M_0 \right) \right), \tag{5.17} \]

to the \( \mathbb{R}P^2 \) algebra found by Hajac et al. in [31], generalized here to \( q \) a phase and no longer real.

**The cutoff - a technicality.**

Here we perform more scrupulous a comparison of the two descriptions of the fixed point geometry: the BCFT and the \( q \)-algebraic one. Our aim is to count the respective numbers of the geometric degrees of freedom, described by the boundary primaries on the BCFT side and by independent monomials in the generators (5.17) realised in the representation of the weight \( \frac{\lambda}{2} \) on the \( q \)-model side.

Thus we begin with the basis of the \( \mathbb{Z}_2 \)-invariant subalgebra of the equatorial BA (\( \pm \) is the stabiliser label, \( I \) is the natural spin label and \( i \) is the associated ”magnetic” label):

\[ X_{I,i}^{\pm \pm} := \begin{cases} \frac{1}{2} \psi_{I,i;0}^{[\Lambda_{FP}] [\Lambda_{FP}] \pm} \equiv \frac{1}{2} \left( \psi_{I,i;0}^{[\Lambda_{FP}] [\Lambda_{FP}] \pm} \pm \psi_{I,i;1}^{[\Lambda_{FP}] [\Lambda_{FP}] \pm} \right) & \text{for } I \in 2\mathbb{N}, \\ 0 & \text{for } I \in 2\mathbb{N} + 1, \end{cases} \tag{5.18} \]

satisfying \( (\Lambda_{FP} = \frac{\lambda}{2}) \)

\[ X_{I,i}^{\pm \pm} \ast_q X_{J,j}^{\pm \pm} = \sum_{K,k} X_{K,k}^{\pm \pm} \begin{bmatrix} I & J \cr \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} J \cr K \end{bmatrix}_q, \tag{5.19} \]

\[ X_{I,i}^{\pm \pm} \ast_q X_{J,j}^{\mp \mp} = 0, \tag{5.20} \]

with \( \begin{bmatrix} I & J \cr \frac{1}{2} & \frac{1}{2} \end{bmatrix}_q \) - the 6j-symbols of \( U_q(\mathfrak{su}(2)) \) as the fusion matrix, and \( \begin{bmatrix} I \cr i \cr j \cr k \end{bmatrix}_q \) - the corresponding Clebsch-Gordan coefficients as the \( q \)-deformed three-point correlation functions of the appropriate CVO’s. The spin label of these operators is bounded by the limiting ”regular” value \( \frac{\lambda}{2} \) and hence
the stringy geometry decomposes as \((\kappa \in 4\mathbb{N}^*)\):

\[
\text{span}\{X^\pm_{I,I-i}\}_{I\in \mathbb{N}, i \in \mathbb{N}/2} \equiv \bigoplus_{K=0}^{\kappa/4} (2K + 1).
\]  

(5.21)

In the \(q\)-matrix model, on the other hand, we are free to take arbitrary monomials in the generators \(P, R, \bar{R}, T, \bar{T}\), with the (even integer) spin of any such monomial determined by its overall degree in the original coordinates \(M_j, j \in \{0, 1, -1\}\). Restricting to the basis monomials \((\mathbb{B})\), ordered according to their spin \(2s \in 2\mathbb{N}\),

\[
P^n R^{s-m}, \quad P^n \bar{R}^{s-n}, \quad P^l T R^{s-n-1}, \quad P^n \bar{T} \bar{R}^{s-n-1}, \quad m \in 0, s, \quad n \in 0, s-1,
\]

(5.22)

we should obtain as many independent operators as there are even-spinned primaries, which - clearly - is not the case\(^{23}\) unless we truncate the spin of the monomial generators of the equatorial \(\mathbb{R}P^2\) algebra as \(s \leq \frac{q}{4}\). The dimension of the space of monomials reduces accordingly,

\[
\sum_{s \in \mathbb{N}} (4s + 1) \rightarrow \sum_{s=0}^{\frac{q}{4}} (4s + 1) = \frac{1}{2} \left(\frac{\kappa}{2} + 1\right) \left(\frac{\kappa}{2} + 2\right),
\]

(5.23)

to yield precisely the BCFT-dictated number of the geometric degrees of freedom on the side of the \(q\)-matrix model. With this last observation we conclude our exposition of the simple example of quantum orbifold geometry.

6. Summary.

In summary, we wish to emphasise the simplicity of our algebraic approach to the stringy geometry of quotient WZW manifolds and the naturalness of its representation-theoretic structure, \((4.10)\), proposed in the present paper.

Firstly, we remark that the operation of composing quantum coordinate algebras of several substructures, represented independently of one another, has long been known in the mathematical literature in the case of \(q \in \mathbb{R}\), passing under the name of gluing of operator algebras\(^{24}\). Even though the present case of \(q / \notin \mathbb{R}\) differs substantially from the former one, in particular at the level of the relevant representation theory, it feels natural to have similar patterns in both setups.

Secondly, we point out that it is precisely the structure \((1.10)\) of the Ansatz discussed that enforces the desired inner-automorphic reduction of the pre-orbifolding model and leads to a successful

\(^{23}\)The origin of the mismatch traces back to the failure of the original Drinfel’d twisting procedure to render the ensuing \(q\)-deformed OPE algebra associative in the entire range of spin parameters of the theory. It turns out that - as was remarked already in \((1.4)\) - the associativity breaks down as early as \(\Lambda = E\left(\frac{q}{4}\right)\), half-way between the north pole of \(S^3_q\) and its equator.

\(^{24}\)Consult \((1.13)\) where, in particular, the case of a quantum 2-sphere algebra, made up of two disc algebras corresponding to its upper and lower hemispheres, was worked out.
implementation of the crossed product extension. The latter property provides strong evidence for the orbifolding scenario advanced and thus also for the pre-orbifolding model under consideration.

The last conclusion brings us to the crucial element of our reasoning which is the agreement of the results obtained from our \( q \)-algebraic analysis with those provided by the relevant BCFT, assumed to be the ultimate foundation of all our constructions. It is perhaps worth mentioning that - in addition to the calculations carried out explicitly in the paper - we are able within the present framework to test the quantum stability of the \( q \)-brane configurations (4.10) as well as examine the ensuing inter-brane excitations using the techniques developed in [15]. So far our results seem to indicate towards stability against decay and reproduce an asymptotically correct picture of the lightest open strings stretched between distant \( q \)-branes.

On the more formal side, we draw the reader’s attention to the attractive pattern in the (regular) representation theory of \( \text{REA}_q(A_N) \) uncovered in this paper, admitting a straightforward explanation in reference to the purely geometrical symmetries of the associated affine structure \( A_N^{(1)} \). An immediate consequence of its presence is the construction of an entire class of new quantum geometries, wrapped by the fractional \( q \)-branes of the matrix model.

Altogether, the arguments in favour of the proposal (4.10)-(4.11) and the associated orbifolding process are purely formal as well as physical in nature, turning our description of both the pre- and post-orbifolding \( q \)-geometry into a viable candidate for a quantum matrix model of (simple current orbifold) WZW geometry. At the same time, we must stress that there are still pieces of the complete BCFT picture that we are unable to derive from our \( q \)-algebraic approach, hoping that the understanding comes soon.

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Appendices.

A. Proof of the associativity of the \(q\)-deformed boundary algebra.

In this appendix we explicitly prove the useful fact: given \(2.13\), the \(q\)-deformation of the orbifold Boundary Algebra,

\[
\psi_{L_1,i_1:g_1}^{[\Lambda_1]a_1[\Lambda_2]a_2} \ast_q \psi_{L_2,i_2:g_2}^{[\Lambda_2]a_3[\Lambda_3]a_4} =
\]

\[
= \delta_{a_2,a_3} \sum_{g \in S_{\Lambda_2}} \sum_{L_3,i_3} \psi_{L_3,i_3:ggg_{12}}^{[\Lambda_1]a_1[\Lambda_3]a_4} e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_3} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_3}^{i_1,i_2,i_3} \right)_{\Lambda_2 g g_{12} \Lambda_3},
\]

(A.1)

renders the latter associative (we retrieve the unresolved case from \([A.1]\) upon setting \(g = \text{id} = g'\):

\[
\left( \psi_{L_1,i_1:g_1}^{[\Lambda_1]a_1[\Lambda_2]a_2} \ast_q \psi_{L_2,i_2:g_2}^{[\Lambda_2]a_3[\Lambda_3]a_4} \right) \ast_q \psi_{L_3,i_3:g_3}^{[\Lambda_3]a_5[\Lambda_4]a_6} =
\]

\[
= \delta_{a_2,a_3} \sum_{g \in S_{\Lambda_2}} \sum_{L_4,i_4} \psi_{L_4,i_4:ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_4} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \delta_{a_2,a_3} \delta_{a_4,a_5} \sum_{g \in S_{\Lambda_2}} \sum_{g' \in S_{\Lambda_4}} \psi_{L_4,g':ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} \times
\]

\[
\times F_{g_1\Lambda_2,L_4} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \delta_{a_2,a_3} \delta_{a_4,a_5} \sum_{g \in S_{\Lambda_2}} \sum_{g' \in S_{\Lambda_4}} \psi_{L_4,g':ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_4} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} \times
\]

\[
\times F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \delta_{a_2,a_3} \delta_{a_4,a_5} \sum_{g \in S_{\Lambda_2}} \sum_{g' \in S_{\Lambda_4}} \psi_{L_4,g':ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} \times
\]

\[
\times F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \delta_{a_2,a_3} \delta_{a_4,a_5} \sum_{g \in S_{\Lambda_2}} \sum_{g' \in S_{\Lambda_4}} \psi_{L_4,g':ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_4} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} \times
\]

\[
\times F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \delta_{a_2,a_3} \delta_{a_4,a_5} \sum_{g \in S_{\Lambda_2}} \sum_{g' \in S_{\Lambda_4}} \psi_{L_4,g':ggg_{12}}^{[\Lambda_1]a_1[\Lambda_4]a_6} e_{a_2}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{ggg_1}(L_2)} F_{g_1\Lambda_2,L_4} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} \times
\]

\[
\times F_{g_1\Lambda_2,L_5} \left[ L_1 L_2 \right]_{\Lambda_1 g g_{12} \Lambda_3} c_{L_1 L_2 L_4}^{i_1,i_2,i_4} \right)_{\Lambda_2 g g_{12} \Lambda_3} L_4 L_5
\]

\[
= \psi_{L_1,i_1:g_1}^{[\Lambda_1]a_1[\Lambda_2]a_2} \ast_q \left( \psi_{L_2,i_2:g_2}^{[\Lambda_2]a_3[\Lambda_3]a_4} \ast_q \psi_{L_3,i_3:g_3}^{[\Lambda_3]a_5[\Lambda_4]a_6} \right) \square.
\]

(A.2)
B. A useful feature of the representation theory of $A_N$.

We begin by recalling the simple relation between the highest weight, $\Lambda_{\text{max}}$, and the lowest weight, $\Lambda_{\text{min}}$, of any highest weight module of $A_N$:

$$\Lambda_{\text{max}} = [\lambda_1, \lambda_2, \ldots, \lambda_N] \implies \Lambda_{\text{min}} = [-\lambda_N, -\lambda_{N-1}, \ldots, -\lambda_1]. \tag{B.1}$$

Let the lowest weight state $v_\Lambda$ of the module $\mathcal{H}_\Lambda$ of $U_q(A_N)$, associated with the highest weight $\Lambda$, be written symbolically\(^{25}\) as

$$v_\Lambda \sim_{\text{symb.}} F_1^{M_1} F_2^{M_2} \ldots F_N^{M_N} \triangleright V_\Lambda, \tag{B.2}$$

with $V_\Lambda$ - the highest weight state of the module and the integers $M_j$, $j \in 1, N$. The equation (B.1) sets the following constraints:

$$\forall j \in 1, N : M_{j-1} - 2M_j + M_{j+1} = -(\lambda_j + \lambda_{N+1-j}), \quad M_0 := 0 =: M_{N+1}. \tag{B.3}$$

After some elementary algebra these are seen to yield

$$M_1 = \sum_{j=1}^{N} \lambda_j = M_N, \tag{B.4}$$

$$\forall k \in 2, \ldots, \left(\frac{N+1}{2}\right) : M_k = k \sum_{j=1}^{N} \lambda_j - \sum_{l=1}^{k-1} (k - l) (\lambda_l + \lambda_{N+1-l}) = M_{N+1-k}. \tag{B.5}$$

An important consequence of (B.4) and (B.3) is that for any $m_N$ satisfying the condition:

$$0 \leq m_N \leq M_N \tag{B.6}$$

there exists in the module considered at least one non-zero state $v^{(m)}$ such that

$$v^{(m)} \sim_{\text{symb.}} F_1^{m_1} F_2^{m_2} \ldots F_N^{m_N} \triangleright V_\Lambda \neq 0 \tag{B.7}$$

for some $\{m_1, m_2, \ldots, m_{N-1}\}$ satisfying

$$\forall i \in 1, N-1 : 0 \leq m_i \leq M_i. \tag{B.8}$$

In order to verify the last statement, which we shall need later on, we recall two elementary facts from the representation theory of $U_h(A_N)^{26}$. Firstly, every state in a highest weight module of $U_h(A_N)$ belongs simultaneously to all irreducible $U_h(\mathfrak{su}(2))$-submodules $\mathcal{H}_\alpha$ associated with positive roots, $\alpha \in P_+(A_N)$. Secondly, each of these modules is Weyl-symmetric.

---

\(^{25}\)Here we are not concerned with the relative ordering of the lowering operators, only their net exponents.

\(^{26}\)They follow directly from their classical counterparts upon adjaucing the argument of structural correspondence between highest weight modules of classical Lie algebras and those of quantum enveloping algebras. See, e.g., \[29\]. Furthermore, they admit a trivial extension to the related quantum algebras: $U_q(A_N)$ and $U^{ext}_q(A_N)$. 
that is - a state of the module of the weight $\lambda$ with respect to the appropriate Cartan operator, $H_\alpha$, is paired with its unique conjugate having the opposite weight, $\lambda^* = -\lambda$. We are now in a position to prove our claim. To start with, consider the following subset\(^{27}\) of positive roots:

\[ P_N := \{\alpha_N, \alpha_{N-1N}, \alpha_{N-2N-1N}, \ldots, \alpha_{12N} \} \subset P_+(A_N), \quad (B.9) \]

with

\[ \alpha_{kk+1\ldots N} := \sum_{i=k}^N \alpha_i, \quad (B.10) \]

and further recall that the corresponding $U_h(\mathfrak{su}(2))$-subalgebras of $U_h(A_N)$ are generated by the so-called root vectors obtained from the distinguished simple root ladder operators $\{E_j, F_j\}_{j \in 1,N}$ by means of the q-Weyl maps, described, e.g., in [29]. The root vectors depend on the choice of the reduced decomposition $\delta$ of $s_0$ but we shall suppress that dependence in our notation which reads: $(E_\alpha, F_\alpha)$. We also adjoin the relevant Cartan generators: $H_\alpha$, independent of $\delta$. The crucial property of $E_\alpha$ (resp. $F_\alpha$) with $\alpha \in P_N$ is that it depends linearly on each of the operators $E_j$ (resp. $F_j$) corresponding to simple roots appearing in the decomposition of $\alpha$ in the simple root basis as in (B.10).

Define for convenience the Cartan eigenvalues:

\[ H_\alpha \triangleright \nu =: h_\alpha(\nu)\nu \quad (B.11) \]

and the states:

\[ \nu^{\lambda N}_{\alpha} := F^\lambda_{\alpha_{N-1N}} F^\lambda_{\alpha_{N-2N-1N}} \cdots F^\lambda_{\alpha_{1NN}} \triangleright V_\lambda. \quad (B.12) \]

The next thing to be noted is that the highest weight state $V_\lambda$ returns the following $H_\alpha$-eigenvalues:

\[ h_\alpha (V_\lambda) = \lambda_N, \quad \forall k \in 1,N-1 : \quad h_{\alpha_{kk+1\ldots N}} (V_\lambda) = \sum_{i=k}^N \lambda_i. \quad (B.13) \]

Upon inspecting their behaviour under the action of the lowering operator $F_N$,

\[ h_{\alpha_N} (\nu^N_\lambda) = -\lambda_N, \quad h_{\alpha_{N-1N}} (\nu^N_\lambda) = \lambda_{N-1}, \]

\[ \forall k \in 1,N-2 : \quad h_{\alpha_{kk+1\ldots N}} (\nu^N_\lambda) = \sum_{l=k}^{N-1} \lambda_l, \quad (B.14) \]

we find out that we have been carried all the way from the top of the $U_h(\mathfrak{su}(2))_{\alpha_N}$-submodule, or the highest weight state , to its bottom, or the lowest weight state. Clearly, the final state

---

\(^{27}\)The subset is directly related to the ordering of the positive roots of $A_N$ associated to a reduced decomposition of the maximal element of the Weyl group of $A_N$, $s_0$, in terms of elementary Weyl reflections: $s_0 = s_N \circ s_{N-1} \circ \cdots \circ s_1 \circ \eta$, in which $\eta$ is an arbitrary completion of the decomposition.
does exist and so do all the states of the form $F^{p_N}_{\alpha N} \triangleright V_\Lambda$ with $0 \leq p_N \leq \lambda_N$. It is no less obvious - from what we have said - that also $u_{\Lambda}^{N-1N}$ is non-zero. Indeed, we obtain for this state\textsuperscript{28}:

$$h_{\alpha N-1N} (u_{\Lambda}^{N-1N}) = -\lambda_{N-1}, \quad h_{\alpha N-2N-1N} (u_{\Lambda}^{N-1N}) = \lambda_{N-2},$$

$$\forall k \in \mathbb{N} : h_{\alpha kk+1...N} (u_{\Lambda}^{N-1N}) = \sum_{l=k}^{N-2} \lambda_l.$$ \hspace{0.5cm} (B.15)

Hence $u_{\Lambda}^{N-1N}$ is the Weyl-conjugate of $u_{\Lambda}^N$ within the corresponding $U_h(\mathfrak{su}(2))_{\alpha N-1N}$-submodule and its nontriviality is granted by the Weyl symmetry of the entire module. Along the very same lines we could descend all the way down to the state $u_{\Lambda}^{12...N}$, convincing ourselves that its nontriviality as well as the nontriviality of its predecessors of the particular form:

$$u_{\Lambda}^{l+1l+2...N} := F^{p_l}_{\alpha l+1...N} \triangleright u_{\Lambda}^{l+1l+2...N},$$ \hspace{0.5cm} (B.16)

with

$$0 \leq p_l \leq \lambda_l$$ \hspace{0.5cm} (B.17)

are protected by the chain of Weyl symmetries of the irreducible module of $U_h(A_N)$. Bearing in mind the multilinear character of the root vectors in (B.16) we may next compute the net exponent of the lowering operator $F_N \equiv F_{\alpha N}$ in any such state, whereby we verify that the above provides an explicit construction of the states sought for, \textsuperscript{(B.7)}. □

C. The algebra $U_q^{\text{ext}}(A_N)$.

We begin with the definition of $U_q^{\text{ext}}(A_N)$ which we take to be generated by the elements:

$$\{k_{\pm \epsilon_i}, E_j, F_j\}_{i \in 1, N+1, j \in 1, N},$$ \hspace{0.5cm} (C.1)

subject to the relations\textsuperscript{29}:

$$k_{\epsilon_i} k_{\epsilon_j} = k_{\epsilon_j} k_{\epsilon_i}, \quad k_{\epsilon_i} k_{-\epsilon_i} = \mathbb{I} = k_{-\epsilon_i} k_{\epsilon_i},$$ \hspace{0.5cm} (C.2)

$$k_{\epsilon_1} k_{\epsilon_2} \cdots k_{\epsilon_{N+1}} = \mathbb{I},$$ \hspace{0.5cm} (C.3)

$$k_{\epsilon_i} E_j k_{-\epsilon_i} = q^{\delta_{ij}-\delta_{i-1,j}} E_j, \quad k_{\epsilon_i} F_j k_{-\epsilon_i} = q^{-\delta_{ij}+\delta_{i-1,j}} F_j.$$ \hspace{0.5cm} (C.4)

\hspace{0.5cm}

\textsuperscript{28}The $H_{\alpha N}$-eigenvalue is irrelevant at this stage of the analysis.

\textsuperscript{29}We denote $\lambda := q - q^{-1}$, $[2] := q + q^{-1}$ and $k_{\pm \epsilon_i} := k_{\pm \epsilon_i}^n$, $n \in \mathbb{N}$.
\[ [E_i, F_j] = \frac{k_{\epsilon_i} k_{-\epsilon_{i+1}} - k_{-\epsilon_i} k_{\epsilon_{i+1}}}{\lambda}, \quad (C.5) \]

\[ E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 \quad \text{for } |i-j| = 1, \quad (C.6) \]

\[ E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \quad \text{for } |i-j| > 1. \quad (C.7) \]

The Cartan generators \( k_{\pm\epsilon_j}, j \in \overline{1, N+1} \) are defined in direct reference to the standard embedding of the root space of the classical algebra, \( P^*(A_N) \), in \( \mathbb{R}^{N+1} \). We thus have the well-known transformation between the simple-root basis, \( \alpha_i \), and the orthonormal (Cartesian) one, \( \epsilon_i \),

\[ \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in \overline{1, N}, \quad (\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (C.8) \]

with the orthogonality condition on weights \( \Lambda \):

\[ \sum_{i=1}^{N+1} (\Lambda, \epsilon_i) = 0 \quad (C.9) \]

Clearly, the orthogonality condition is encoded in (C.3) at the level of the algebra.

The algebra is endowed with a Hopf structure. In particular, it has the antipode:

\[ Sk_{\pm\epsilon_i} = k_{\pm\epsilon_i}, \quad SE_i = -k_{-\epsilon_i} k_{\epsilon_{i+1}} E_i, \quad SF_i = -F_i k_{\epsilon_i} k_{-\epsilon_{i+1}}, \quad (C.10) \]

employed frequently in the sequel.

\section*{C.1 The center of \( \mathcal{U}_q^{\text{ext}}(A_N) \).}

Another interesting aspect of the general theory of the extended quantum enveloping algebras is the structure of their center, \( Z_q(A_N) \), playing a crucial role in any representation-theoretic analysis. In the case of the deformation parameter \( q \) being the \( 2\kappa_N \)-th primitive root of unity the center is known ([34]) to be generated by the scalar operators:

\[ Z_0 = \text{span} \left\langle k_{\epsilon_i}^{2\kappa_N}, \epsilon_{ij}^{2\kappa_N}, f_{ij}^{2\kappa_N} \right\rangle_{i,j \in \overline{1, N+1}}, \quad (C.11) \]

peculiar to the root-of-unity case, and the Casimir (scalar) operators:

\[ Z_1 = \text{span} \left\langle C_n \right\rangle_{n \in \overline{1, N}}, \quad (C.12) \]
expressible in terms of the so-called $L^\pm$-operators$^{30}$,

$$C_n = \frac{q^{n(N+1-n)}}{\mathcal{N}_n (q^{-2}) \mathcal{N}_{N+1-n} (q^{-2})} \times$$

$$\times \sum_{\sigma,\sigma'} (-q^{-1})^{l(\sigma)+l(\sigma')} l^{(+)}_{\sigma(1)\sigma'(1)} l^{(+)}_{\sigma(2)\sigma'(2)} \cdots l^{(+)}_{\sigma(n)\sigma'(n)} l^{(-)}_{\sigma(n+1)\sigma'(n+1)} l^{(-)}_{\sigma(n+2)\sigma'(n+2)} \cdots l^{(-)}_{\sigma(N+1)\sigma'(N+1)},$$

(C.13)

where $\sigma, \sigma'$ are elements of the permutation group $\mathfrak{S}_{N+1}$, $l(\sigma)$ is the length of the permutation $\sigma$ and:

$$l^{(+)}_{\sigma i} = k_{\epsilon_\sigma} = \left(l^{(-)}_{\sigma i}\right)^{-1},$$

(C.14)

$$l^{(+)}_{ij} = 0 = l^{(-)}_{ji} \text{ for } i > j,$$

(C.15)

$$l^{(+)}_{ij} = (-1)^{j-i+1} \lambda f_{ij} k_{\epsilon_\sigma} \quad , \quad l^{(-)}_{ji} = (-1)^{i-j} \lambda k_{-\epsilon_\sigma} \tilde{e}_{ij} \text{ for } i < j,$$

(C.16)

$$\mathcal{N}_n(x) = \prod_{l=1}^n \frac{1-2^l}{1-x}.$$  

(C.17)

In writing the expressions (C.14)-(C.16), as well as (C.11), the following notation, defined recursively, was used:

$$\forall_{i,j \in \Omega, N} : \begin{cases} e_{i,i+1} = \tilde{e}_{i,i+1} = E_i, \\ e_{i,j+1} = e_{ij} E_j - q^{-1} E_j e_{ij} \text{ for } i < j, \\ \tilde{e}_{i,j+1} = \tilde{e}_{ij} E_j - q E_j \tilde{e}_{ij} \text{ for } i < j, \\ f_{i,j+1} = \tilde{f}_{i,j+1} = F_i, \\ f_{i,j+1} = f_{ij} F_j - q^{-1} F_j f_{ij} \text{ for } i < j, \\ \tilde{f}_{i,j+1} = \tilde{f}_{ij} F_j - q F_j \tilde{f}_{ij} \text{ for } i < j. \end{cases}$$  

(C.18)

$^{30}$A slightly different convention on the $L^\pm$-operators is being used throughout the paper, after [35]. The present convention has been rewritten from [34] only for the sake of compactness of the formulæ appearing in our paper, hence no attempt has been made to unify the $L^\pm$-notation.
The explicit form (C.13) of the basis Casimir operators leads us to the useful conclusion: the operators may well be identified with symmetric polynomials in the operator variables \( k_i \). Indeed, upon employing some simple algebra and keeping track of the permutations in (C.13) one can readily verify that, up to an irrelevant representation-independent constant \( \alpha_k(N) \), the Casimir operators restricted to a highest weight state take the following form:

\[
Z_1 \ni C_n | \bigcap_{i \in 1, N} \ker E_i = \alpha_n(N)c_n \left( q^N k_{2e_1}, q^{N-2} k_{2e_2}, \ldots, q^{-N} k_{2e_{N+1}} \right),
\]

in which

\[
c_n(x_1, x_2, \ldots, x_{N+1}) := \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq N+1} x_{j_1}x_{j_2} \cdots x_{j_n}
\]

represents the \( n \)-th elementary symmetric polynomial in \( N + 1 \) variables. The above formula can further be resolved upon specifying the representation theory to be considered. We will make use of that fact presently.

C.2 A proof of (3.17).

Denote the Cartan eigenvalues on the highest weight state \( V_\Lambda \) of \( \mathcal{U}_q^{\text{ext}}(A_N) \) as

\[
\forall j \in 1, N+1 \ : \ k_{ej} \triangleright V_\Lambda =: k_j V_\Lambda.
\]

Then, according to the representation theory of \( \mathcal{U}_q(A_N) \) (Ch.7 of [29]) and (3.9),

\[
\forall j \in 1, N \ : \ k_j k_j^{-1} = e^{i\pi \omega_j q^\Lambda_j}, \quad \omega := (\omega_1, \omega_2, \ldots, \omega_N) \in \mathbb{Z}^N_2.
\]

Imposing (C.3) we further constrain the parameters in (C.22):

\[
\forall j \in 2, N+1 \ : \ k_j \overset{!}{=} e^{-i\pi \sum_{m=1}^{j-1} \omega_m q^{-\sum_{n=1}^{j-1} \lambda_n} k_1},
\]

\[
k_1^{N+1} \overset{!}{=} e^{i\pi \sum_{m=1}^N (N+1-m) \omega_m q^{\sum_{n=1}^N (N+1-n) \lambda_n}},
\]

which means that we have an \((N+1)\)-tuplet of irreducible representations:

\[
k_1 = e^{-i\pi L(l, \omega)_{N+1} \over N+1} q^{(\epsilon_1, \Lambda)},
\]

\[
\forall j \in 2, N+1 \ : \ k_j = e^{-i\pi L(l, \omega)_{N+1} \over N+1} e^{-i\pi \sum_{m=1}^{j-1} \omega_m q^{(\epsilon_j, \Lambda)}},
\]

where

\[
L(l, \omega) = 2l + \sum_{m=1}^N (N + 1 - m) \omega_m, \quad l \in \mathbb{Z}_{N+1}.
\]

The latter provide also the most general solution to (3.16) upon a straightforward identification of phases \( p_i \).
What remains to be done is a verification of mutual inequivalence of the irreducible representations found. Due to (3.8) and (3.9) representations with $(\Lambda, \omega) \neq (\Lambda', \omega')$ are inequivalent\footnote{Compare [29].} as they induce inequivalent representations of $U_q(A_N)$. Next, we consider the eigenvalues of the Casimir operators (C.19) on the highest weight state of the irreducible module $R^l,\omega$:

$$C_n|_{R^l,\omega} = \alpha_n(N)c_n \left(q^Ne^{-\frac{2\pi il(L\omega)}{N+1}}q^{2(\epsilon_1,\Lambda)}, q^{-2}e^{-\frac{2\pi il(L\omega)}{N+1}}q^{2(\epsilon_2,\Lambda)}, \ldots, q^{-N}e^{-\frac{2\pi il(L\omega)}{N+1}}q^{2(\epsilon_{N+1},\Lambda)}\right),$$  

(C.27)

with

$$\forall i \in \{1,N+1\}: (\epsilon_i, \Lambda) = \sum_{j,l=1}^{N}(\epsilon_i, \alpha_j)A_{jl}^{-1}(\alpha_l, \Lambda).$$  

(C.28)

Thus

$$C_n|_{R^l,\omega} = e \frac{2\pi ik}{N+1}C_n|_{R^{l'},\omega},$$  

(C.29)

and by (C.23)-(C.25) also

$$k_{\epsilon_j}^{2\kappa_N}|_{R^l,\omega} = e \frac{2\pi ik}{N+1}k_{\epsilon_j}^{2\kappa_N}|_{R^{l'},\omega}.$$  

(C.30)

Hence, if $2l \neq 2l'$ mod $N+1$ the representations are inequivalent. Finally, we have to resolve the nontrivial $2l' = 2l + N + 1$ case (assuming $l' > l$). This can be done by noting that the potential equivalence would mean there exists within the carrier space $\mathcal{H}_\Lambda^{0,0}$ spanned by

$$\mathcal{H}_\Lambda^{0,0} \ni \nu^{(m)}_\Lambda \sim \text{symb.} F_1^{m_1}F_2^{m_2} \ldots F_N^{m_N} \triangleright V_\Lambda,$$

(C.31)

such that

$$k_{\epsilon_j} \triangleright \nu^{(m)}_\Lambda = q^{\sum_{i=1}^{N}(\epsilon_j,\alpha_i)m_i}q^{(\epsilon_j,\Lambda)}\nu^{(m)}_\Lambda,$$

(C.32)

a pairing among states which takes the following form (compare: (3.6) and the subsequent remarks):

$$R^{l,\omega} \sim R^{l+N+1,\omega} \implies \forall m \in M_\Lambda \exists m' \in M_\Lambda \forall j \in \{1,N+1\} : q^{\sum_{i=1}^{N}(\epsilon_j,\alpha_i)m_i} = -q^{\sum_{i=1}^{N}(\epsilon_j,\alpha_i)m'_i}.$$  

(C.33)

However, for $j = 1$ above we get

$$q^{m'_1-m_1} = -1$$

(C.34)

which is contradicted by the following chain of inequalities:

$$-\kappa_N < -\kappa < -\sum_{i=1}^{N}\lambda_i \leq -m_1 \leq (m'_1 - m_1) \leq m'_1 \leq \sum_{i=1}^{N}\lambda_i \leq \kappa < \kappa_N.$$  

(C.35)

This ultimately disproves the equivalence assumed in (C.33) and leaves us with the complete description (3.17) of the regular highest weight representations of $U^\text{ext}_q(A_N)$.
D. The algebra $\text{REA}_q(A_N)$.

D.1 The $L^\pm$-operators.

Below we explicitly list the entries of the $L^\pm$-operators obtained from the standard universal $R$-matrices of $\mathcal{U}_q(A_N)$ by means of the algorithm of Faddeev, Reshetikhin and Takhtajan (\[35\]):

\[
\forall i, j \in 1, N+1 : \begin{cases}
L_{ii}^+ &= k_{\epsilon_i} = (L_{ii})^{-1}, \\
L_{ij}^+ &= 0 = L_{ji}^- & \text{for } i > j, \\
L_{ij}^+ &= \lambda k_{\epsilon_j} E_{ji} & \text{for } i < j, \\
L_{ij}^- &= -\lambda E_{ji} k_{-\epsilon_j} & \text{for } i > j,
\end{cases}
\]

where we have introduced the recursively defined symbols:

\[
\forall i, j \in 1, N : \begin{cases}
E_{i,i+1} &= E_i, \\
E_{i,j+1} &= E_{ij} E_j - q E_j E_{ij} & \text{for } i < j, \\
E_{i+1,i} &= F_i, \\
E_{i+1,j} &= E_{i+1,j+1} F_j - q^{-1} F_j E_{i+1,j+1} & \text{for } i > j.
\end{cases}
\]

The above $L^\pm$-operators are known (\[35\]) to satisfy the following relations:

\[
R_{12}^+ L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R_{12}^+, \quad (D.3)
\]

\[
L_1^+ R_{12}^+ S L_2^- = S L_2^+ R_{12}^+ L_1^+, \quad (D.4)
\]

and hence also

\[
R_{21}^+ S L_1^\pm S L_2^- = S L_2^\pm S L_1^\pm R_{21}^+, \quad (D.5)
\]

\[
L_2^+ R_{21}^+ S L_1^- = S L_1^- R_{21}^+ L_2^+, \quad (D.6)
\]

with

\[
R^+ = \sum_{1 \leq i, j \leq N+1} q^{{\delta}_{ij}} e_{ii} \otimes e_{jj} + \lambda \sum_{1 \leq i < j \leq N+1} e_{ij} \otimes e_{ji}. \quad (D.7)
\]
At this stage it is a matter of an elementary algebra to verify that the fundamental \( M \)-matrix, (3.12), with the operator entries:

\[
M_{ij} = \sum_{k=1}^{N+1} L_{ik}^+ S L_{kj}^-,
\]

(B.8)
does indeed satisfy (3.1) with \( R = R^+ \).

Finally, upon substituting (D.1) in the above formula and rearranging the resulting expressions we derive

\[
M_{ij} \bigg|_{i>j} = (-1)^{j+1-i} \lambda k_{2\epsilon_i} \left[ \tilde{E}_{ji} + q^{-2} \lambda \sum_{i<k \leq N+1} (-1)^{i-k} k_{-\epsilon_i} E_{ki} \tilde{E}_{jk} \right],
\]

(D.9)

\[
M_{ij} \bigg|_{i<j} = q^{-1} \lambda k_{\epsilon_i} k_{\epsilon_j} \left[ E_{ji} + \lambda \sum_{j<k \leq N+1} (-1)^{k+1-j} k_{-\epsilon_j} E_{ki} \tilde{E}_{jk} \right],
\]

(D.10)

\[
M_{ii} = k_{2\epsilon_i} \left[ 1 + q^{-1} \lambda^2 \sum_{i<k \leq N+1} (-1)^{k+1-i} k_{-\epsilon_i} E_{ki} \tilde{E}_{ik} \right],
\]

(D.11)

with the operators \( \tilde{E}_{ij} \) defined in analogy to (D.2),

\[
\forall_{i,j} \in \overline{1,N} : \begin{cases} 
\tilde{E}_{i,i+1} = E_i, \\
\tilde{E}_{i,j+1} = q^{-1} E_j \tilde{E}_{ij} - \tilde{E}_{ij} E_j \text{ for } i < j,
\end{cases}
\]

(D.12)

and further related to the \( E_{ij} \)'s through

\[
\forall_{i,j} \in \overline{1,N+1}, i<j : SE_{ij} = (-1)^{j-i} k_{-\epsilon_i} E_{ij} \tilde{E}_{ij}.
\]

(D.13)

Amongst the \((N+1)^2\) entries of the \( M \)-matrix there is a distinguished group of the diagonal ones, \( M_{ii} \), of which \( N \) can be chosen to commute with one another and therefore span the Cartan subalgebra of \( \text{REA}_q(A_N) \). We shall have need for them in the sequel.

The convention on the \( L^\pm \)-operators just displayed proves exceptionally convenient for the analysis to come. Finally, let us also note that the Casimir operators of the underlying \( \text{REA}_q(A_N) \) translate naturally into Casimir operators of \( \mathcal{U}^\text{ext}_q(A_N) \) described by means of (D.9)-(D.11),

\[
\mathfrak{c}_n \to \mathcal{C}_n.
\]

(D.14)
D.2 Inequivalence among irreducible representations of \( \text{REA}_q(A_N) \).

The aim of the present section is to prove mutual inequivalence of the following irreducible representations:

\[
\mathcal{R}_{\text{ind.}}(\text{REA}_q(A_N)) = \bigoplus_{\Lambda \in P^+(A_N)} R^L_{\Lambda}, \quad R^L_{\Lambda} \sim R^L_{\Lambda} |_{L(l, \omega) = l \mod 2}. \quad (D.15)
\]

By (3.23) and (3.30) all \( R^l_{(\omega_N)^l \Lambda} \) for \( l \in \mathbb{Z}_{N+1} \) have equal scalar operators. It is therefore the latter that we focus on in the sequel, further reducing our problem by making the following observation:

\[
\exists \Lambda \in P^+(A_N) \exists l_1, l_2 \in \mathbb{Z}_{N+1} : R^{l_1}_{(\omega_N)^{l_1} \Lambda} \sim R^{l_2}_{(\omega_N)^{l_2} \Lambda} \implies R^{l_2 - l_1}_{(\omega_N)^{l_2} \Lambda} \sim R^{0}_{(\omega_N)^{l_1} \Lambda}, \quad (D.16)
\]

manifestly true in view of, e.g., the cyclicity of the elementary automorphism \( \omega_{q,N} \). Accordingly, we next compare the eigenvalues of

\[
M_{N+1,N+1} = k_{2\epsilon_{N+1}} \quad (D.17)
\]
on \( R^0_{\Lambda} \) and \( R^l_{(\omega_N)^l \Lambda} \), \( l \neq 0 \). To these ends we take a general state of the module \( H_{\Lambda} \) - respectively - of the module \( H_{(\omega_N)^l \Lambda} \) (\( V_0, V_l \) are the corresponding highest weight states),

\[
H_{\Lambda} \ni u_{(m^0)} \sim F_{1}^{m^0} \ldots F_{N}^{m^0} > V_0, \quad H_{(\omega_N)^l \Lambda} \ni u_{(m^l)} \sim F_{1}^{m_l} \ldots F_{N}^{m_l} > V_l, \quad (D.18)
\]

with the \( m^0_j \)'s and the \( m^l_j \)'s constrained as in (B.6). Using (3.20) and (C.28) we then verify that

\[
M_{N+1,N+1} > u_{(m^n)} = q^{h(n)} u_{(m^n)}, \quad n \in \{0, l\} \quad (D.19)
\]
for

\[
h(n) := 2m^n_N - \frac{2\kappa_N n}{N + 1} - \frac{2s_n(\Lambda)}{N + 1}, \quad (D.20)
\]
in which

\[
s_0(\Lambda) := \sum_{k=1}^{N} k\lambda_k, \quad (D.21)
\]
\[
s_l(\Lambda) := \sum_{k=1}^{N} k\omega_N^l(\lambda_k) = s_0(\Lambda) - (N + 1) \sum_{k=1}^{N} \lambda_k + (N + 1 - l)\kappa. \quad (D.22)
\]

In the present notation the condition of equivalence of the two irreducible representations boils down - in view of (B.3) - to the following statement:

\[
\forall_{0 \leq m^0_N \leq s_0(\Lambda)} \exists_{0 \leq m^l_N \leq s_l(\Lambda)} : m^l_N - m^0_N + \frac{s_0(\Lambda) - s_l(\Lambda)}{N + 1} - \frac{l\kappa_N}{N + 1} = 0 \mod \kappa_N, \quad (D.23)
\]
where
\[ s'_0(\Lambda) = \sum_{k=1}^{N} \lambda_k , \quad s'_l(\Lambda) := \sum_{k=1}^{N} (a_k, (\omega^*_N)^l \Lambda) = \kappa - \lambda_l. \] (D.24)

An easy computation then shows the left hand side of (D.23) to be
\[ m^l_N - m^0_N + \sum_{k=1}^{N} \lambda_k - l - \kappa \] (D.25)
and so, choosing\(^{32}\) \(m^0_N := \sum_{k=1}^{N} \lambda_k \leq s'_0(\Lambda)\) for an arbitrary \(l \in \mathbb{1, N}\) and using \(m^l_N \leq s'_l(\Lambda) \leq \kappa\), we arrive at the inequality:
\[ -\kappa - l \leq \text{(D.25)} < 0. \] (D.26)
Thus (D.23) is showed to be false which completes our proof of mutual inequivalence of the irreducible representations (D.15). \(\square\)

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\(^{32}\)According to the discussion under (B.7), there exists a nontrivial state for any such choice.
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