Confluence of singularities of non-linear differential equations via Borel–Laplace transformations

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Abstract

Borel summable divergent series usually appear when studying solutions of analytic ODE near a multiple singular point. Their sum, uniquely defined in certain sectors of the complex plane, is obtained via the Borel–Laplace transformation. This article shows how to generalize the Borel–Laplace transformation in order to investigate bounded solutions of parameter dependent non-linear differential systems with two simple (regular) singular points unfolding a double (irregular) singularity. We construct parametric solutions on domains attached to both singularities, that converge locally uniformly to the sectoral Borel sums. Our approach provides a unified treatment for all values of the complex parameter.

1 Introduction

When studying formal solutions of complex analytic ODE near a multiple singular point, it is the general rule to find divergent series. However, one can always construct true analytic solutions, defined on certain sectors attached to the singularity, which are asymptotic to the formal solution, and which are in some sense unique. In general, the solutions on different sectors do not coincide, and if extended to larger sectors, they may drastically change their asymptotic behavior due to the presence of hidden exponentially small terms, known as the (non-linear) Stokes phenomenon. In case where the singularity is a generic double point such sectorial solutions are obtained, using the Borel and Laplace transformations, as the Borel sums of the formal one. It is now understood, that the divergence of the asymptotic series is caused by singularities of its Borel transform, which also encode information on the geometry of the singularity. Another way how to understand the Stokes phenomena is by considering generic parameter depending deformations which split the multiple (irregular) singular point into several simple (regular) singularities: the local analytic solutions at each singular point of the deformed equation do not match, thus explaining why solutions with nice asymptotic behavior at the limit when the singular points coalesce only exist in sectors.

Key words: Irregular singularity, unfolding, Borel summation, parametric differential equations, saddle–node, confluence, non-linear Stokes phenomenon.
When investigating families of analytic systems of ODEs depending on a complex parameter, that unfold a multiple singularity, one is faced with the problem that the Borel method of summation of formal series does not allow to deal with several singularities and their confluence.

The goal of this article is to show how one can generalize (unfold) the corresponding Borel–Laplace transformations, in order to study bounded solutions of first order non-linear parametric systems with an unfolded double singularity of the form

\[(x^2 - \epsilon) \frac{dy}{dx} = M(\epsilon)y + f(x, y, \epsilon), \quad (x, y, \epsilon) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C},\]

with \(M(0)\) an invertible \(m \times m\)-matrix and \(f(x, y, \epsilon) = O(\|y\|^2) + xO(\|y\|) + (x^2 - \epsilon)O(1)\). Such solutions correspond to ramified center manifolds of an unfolded codimension 1 saddle-node singularity in a family of complex vector fields

\[
\dot{x} = x^2 - \epsilon, \quad \dot{y} = M(\epsilon)y + f(x, y, \epsilon).
\]

It is well known that for generic (non-resonant) values of the parameter \(\epsilon \neq 0\), there exists a local analytic solution on a neighborhood of each simple singularity \(x = \pm \sqrt{\epsilon}\). Previous studies of the confluence phenomenon (see [SS], [G]) have focused at the limit behavior of these local solutions when \(\epsilon \to 0\). Because the resonant values of \(\epsilon\) accumulate at 0 in a finite number of directions, these directions of resonance in the parameter space could not be covered in those studies.

We will construct a new kind of parametric solutions of systems (1) which are defined and bounded on certain ramified domains attached to both singularities \(x = \pm \sqrt{\epsilon}\) (at which they possess a limit) in a spiraling manner. They depend analytically on the parameter \(\epsilon\) taken from a ramified sector of opening \(> 2\pi\) (or \(\sqrt{\epsilon}\) from a sector of opening \(> \pi\)), thus covering a full neighborhood of the origin in the parameter space (including those parameters values for which the unfolded system is resonant), and they converge uniformly when \(\epsilon\) tends radially to 0 to a pair of the classical sectoral solutions: Borel sums of the formal power series solution of the limit system, defined on two sectors covering a full neighborhood of the double singularity at the origin. In fact, each such pair of the sectoral Borel sums for \(\epsilon = 0\) unfolds to a unique above mentioned parametric solution. We state these results in Section 2.2 and illustrate them in Section 2.3 on the problem of existence of normalizing transformations for families of linear differential systems unfolding a non-resonant irregular singularity of Poincaré rank 1.

While these solutions can also be obtained by other methods, the advantage of our approach is that it provides a unified treatment for all values of the parameter \(\epsilon\) and elucidates the form of natural domains on which the solutions exist and are bounded. Most importantly, it gives an insight to intrinsic properties of the singularity and to the source of the divergence similar to that provided by the classical Borel–Laplace approach.

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2 Statement of results

Notation. Throughout the text \((a,b)\) (resp. \([a,b]\)) denotes the open (resp. closed) oriented segment between two points \(a,b \in \mathbb{C}\); \(e^{i\alpha}\mathbb{R}^+ = [0, +\infty e^{i\alpha})\) is an oriented ray, and \(c + e^{i\alpha}\mathbb{R} = (c - \infty e^{i\alpha}, c + \infty e^{i\alpha})\), with \(\alpha \in \mathbb{R}\), \(c \in \mathbb{C}\), is an oriented line. The symbol \(\mathbb{N} = \{0, 1, 2, \ldots\}\) denotes the set of non-negative integers, and \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\).

2.1 Borel–Laplace transformations and their unfolding

The Borel method of summation of (1-summable) divergent series is used to construct their sectoral Borel sums: unique analytic functions that are asymptotic to the series in certain sectors of opening \(> \pi\) at the singular point and satisfy the same differential relations.

Let \(\hat{y}(x) = \sum_{k=1}^{+\infty} y_k x^k\) be a formal power series. Using the Euler formula for the \(\Gamma\)-function: \(\Gamma(k) = \int_{0}^{+\infty} z^{k-1} e^{-z} dz\), which is equal to \((k-1)!\) if \(k \in \mathbb{N}^*\), one can write \(x^k = \int_{0}^{+\infty} e^{i\alpha 0} \xi^{k-1} \cdot e^{-\xi x} d\xi\), hence

\[
\hat{y}(x) = \sum_{k=1}^{+\infty} y_k x^k = \sum_{k=1}^{+\infty} \int_{0}^{+\infty} e^{i\alpha} \frac{y_k}{(k-1)!} \xi^{k-1} \cdot e^{-\xi x} d\xi.
\]

The formal Borel transform of \(\hat{y}\) is the series

\[
\hat{B}[\hat{y}](\xi) = \sum_{k=1}^{+\infty} y_k \frac{\xi^{k-1}}{(k-1)!}.
\]  

If the coefficients of \(\hat{y}(x)\) have at most factorial growth (\(|y_k| \leq c^k k!\) for some \(c > 0\)), then the series \(\hat{B}[\hat{y}](\xi)\) is convergent on a neighborhood of 0 with a sum \(\phi(\xi)\). If moreover \(\phi\) has an analytic extension to a half-line \(e^{i\alpha}\mathbb{R}^+\) and has at most exponential growth there (\(|\phi(\xi)| \leq K e^{\Lambda |\xi|}, \xi \in e^{i\alpha}\mathbb{R}^+, \) for some \(K, \Lambda > 0\)), then its Laplace transform in the direction \(\alpha\)

\[
L_\alpha[\phi](x) = \int_{0}^{+\infty} \phi(\xi) \cdot e^{-\xi x} d\xi
\]  

is convergent for \(x\) in a small open disc of diameter \(\frac{1}{\Lambda}\) centered at \(\frac{\pi \alpha}{2}\) and extends to 0 (which lies on the boundary of the disc), defining there the Borel sum of \(\hat{y}(x)\) in direction \(\alpha\). Such Borel sum of \(\hat{y}(x)\) is asymptotic to the formal series \(\hat{y}(x)\) at the origin, and most importantly, if \(\hat{y}(x)\) is a formal solution to some analytic differential equation, then so are the Borel sums.

A series \(\hat{y}[x]\) is called Borel summable (or 1-summable) if its Borel sum exists in all but finitely many directions \(0 \leq \alpha < 2\pi\). When varying continuously the direction in which the series is summable, the corresponding Borel sums are analytic extensions one of the other, yielding a function defined on a sector of opening \(> \pi\). Let us remark that \(\hat{y}[x]\) is convergent if and only if it is Borel summable in all directions. This means that the Borel sums of divergent series can only exist on sectors. This is also known as the Stokes phenomenon.

More detailed information on the Borel summability can be found, for example, in [MR2] and [M].
A typical source of Borel summable power series are formal solutions of generic ODEs at a double irregular singular point.

Example 1. A non-linear analytic ODE with a double singularity at the origin

\[ \frac{d^2y}{dx^2} = y + f(x,y), \quad (x,y) \in \mathbb{C} \times \mathbb{C}, \]  

where \( f(x,y) = O(x) + O(||y||^2) \) is a germ of analytic function, possesses a unique formal solution \( \hat{y}(x) \). Generically, this series is divergent (for instance if \( f(x,y) = -x \) then \( \hat{y}(x) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n} x^n \) is the Euler series). The reason for the divergence of \( \hat{y}(x) \) is materialized by the singularity of the Borel transform \( \hat{B}[\hat{y}](\xi) \) at \( \xi = 1 \). The Borel sum \( y(x) = L^\alpha[\hat{B}[\hat{y}]](x) \) is well defined in a ramified sector \( \arg x \in (-\frac{\pi}{2}, \frac{5\pi}{2}) \). The set \( (x,y(x)) \) is a center manifold of a saddle-node singularity of the vector field

\[ \dot{x} = x^2, \quad \dot{y} = y + f(x,y). \]

Hence, this example shows that in general an analytic center manifold does not exist, but instead there are “sectoral center manifolds”.

The analytic Borel transformation in direction \( \alpha \) of a germ of function \( y(x) \), which is analytic in a closed sector of opening \( \geq \pi \) bisected by \( e^{i\alpha} \mathbb{R}^+ \) and vanishes at 0 as \( O(x^\lambda) \) uniformly in the sector for some \( \lambda > 0 \), is defined as the “Cauchy principal value” (V.P.) of the integral

\[ B_\alpha[y](\xi) = \frac{1}{2\pi i} \text{V.P.} \int_\gamma y(x) e^{\xi x} \frac{dx}{x^2}, \quad \text{for} \quad \xi \in e^{i\alpha} \mathbb{R}, \]  

over a circle \( \gamma = \{ \Re\left(e^{i\alpha} x \right) = C \}, \quad C > 0 \), inside the sector. The plane of \( \xi \) is also called the Borel plane.

The formal Borel transform \( \hat{b}[y] \) of an analytic germ \( y \) vanishing at 0 is related to the analytic one by

\[ B_\alpha[y](\xi) = \chi_\alpha^+(\xi) \cdot \hat{B}[\hat{b}](\xi), \quad \text{for} \quad \xi \in e^{i\alpha} \mathbb{R}, \]  

where

\[ \chi_\alpha^+(\xi) = \begin{cases} 
1, & \text{if} \quad \xi \in (0, +\infty e^{i\alpha}), \\
0, & \text{if} \quad \xi \in (-\infty e^{i\alpha}, 0). 
\end{cases} \]

The idea of unfolding the Borel–Laplace operators in order to generalize the methods of Borel summability and resurgent analysis to systems with several confluent singularities was initially brought up by Sternin and Shatalov in [SS]. The key lies in appropriate unfolding of the “kernels” \( e^{\xi x} \frac{dx}{x^2} \) and \( e^{-\xi x} \frac{d\xi}{x^2} \) of the transformations \( \hat{b} \) and \( \hat{B} \), and in right determination of the paths of integration. The Borel transformation is designed so that it converts the derivation \( x^2 \frac{d}{dx} \) to multiplication by \( \xi \), and we will want to preserve this property.

The complex vector field \( x^2 \frac{\partial}{\partial x} \) with a double singularity at the origin is naturally (and universally) unfolded as

\[ (x^2 - \epsilon) \frac{\partial}{\partial x}, \quad \epsilon \in \mathbb{C}. \]
We will associate to it the unfolded Borel and Laplace transformations

\[ B_\alpha^+ [y](\xi, \sqrt{\epsilon}) = \frac{1}{2\pi i} V.P. \int_{\text{Re} e^{i\pi t(x, \epsilon) = C}}^{} y(x) e^{t(x, \epsilon)\xi} \, dt(x), \quad 0 < C < \text{Re} (e^{i\alpha \pi i \sqrt{\epsilon}}), \tag{8} \]

\[ L_\alpha [\phi](x, \sqrt{\epsilon}) = \int_{-\infty}^{+\infty} \phi(\xi) e^{-t(x, \epsilon)\xi} \, d\xi, \]

where \( t(x, \epsilon) \) is the negative complex time of the vector field \( (7) \),

\[ \frac{dx}{dt} = -(x^2 - \epsilon). \tag{9} \]

Let us remark that the (unilateral) Laplace transformation \( L_\alpha [\phi] \) is equal to the (bilateral) Laplace transformation \( L_\alpha [\phi] \) with \( \epsilon = 0 \) and \( t(x, 0) = \frac{1}{x} \), if one extends the integrand by 0 for \( \xi \in (-\infty, 0) \):

\[ L_\alpha [\phi](x) = L_\alpha [\chi^+_\alpha \phi](x, 0). \]

In Sections 3 and 4 we will establish some general properties of these transformations based on the classical theory of Fourier and Laplace integrals, and in Section 5 we will apply them to study solutions of \( (1) \) in the vicinity of the singular points.

### 2.2 Center manifold of an unfolded codimension 1 saddle–node type singularity

An isolated singular point of a holomorphic vector field in \( \mathbb{C}^{m+1} \) is of saddle–node type if its linearization matrix has exactly one zero eigenvalue; it is of codimension 1 if the multiplicity of the singular point is 2. In convenient coordinates, such singularity can be written as

\[ \dot{x} = x^2, \quad \dot{y} = M_0 y + f_0(x, y), \quad (x, y) \in (\mathbb{C} \times \mathbb{C}^m, 0), \tag{10} \]

with \( M_0 \) an invertible \( m \times m \)-matrix and \( f_0(x, y) = O(x) + O(\|y\|^2) \) a germ of analytic vector function. We consider a generic family of vector fields in \( \mathbb{C}^{m+1} \) depending analytically on a parameter \( \epsilon \in (\mathbb{C}, 0) \) unfolding \( (10) \). Such a family is locally orbitally analytically equivalent to a family of vector fields

\[ \dot{x} = x^2 - \epsilon, \quad \dot{y} = M(\epsilon) y + f(x, y, \epsilon), \quad (x, y, \epsilon) \in (\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}, 0), \tag{11} \]

with \( M(0) = M_0 \) invertible and \( f(x, y, \epsilon) = O(\|y\|^2) + x O(\|y\|) + (x^2 - \epsilon) O(1) \) a germ of analytic vector function at the origin of \( \mathbb{C} \times \mathbb{C}^m \times \mathbb{C} \), \( f(x, y, 0) = f_0(x, y) \).

The vector field \( (10) \) possesses a ramified 1-dimensional center manifold consisting of several sectoral pieces tangent to the \( x \)-axis. We will study its parametric unfolding in the family \( (11) \): It is given as a graph of a function \( y = y(x, \sqrt{\epsilon}) \), ramified at \( x = \pm \sqrt{\epsilon} \), satisfying the singular non-linear system of \( m \) ordinary differential equations

\[ (x^2 - \epsilon) \frac{dy}{dx} = M(\epsilon) y + f(x, y, \epsilon), \quad (x, y, \epsilon) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}. \tag{12} \]

\footnote{See [RT], Proposition 3.1; it is stated and proved for planar vector fields \( (m = 1) \), but it stays valid for any dimension \( m \geq 1 \).}
**Proposition 2** (Formal solution). The system \[[12]\] possesses a unique solution in terms of a formal power series in \((x, \epsilon)\):

\[
\hat{y}(x, \epsilon) = (x^2 - \epsilon) \sum_{k,j=0}^{+\infty} y_{kj} x^k \epsilon^j, \quad y_{kj} \in \mathbb{C}^m. \tag{13}
\]

**Proof.** Write \(\hat{y}(x, \epsilon) = (x^2 - \epsilon) \sum_{k,j} y_{kj} x^k \epsilon^j\) and \(f(x, y, \epsilon) = (x^2 - \epsilon) \sum_{k,j} f_{0,kj} x^k \epsilon^j + \sum_{|l|\geq 1} \sum_{k,j} f_{l,kj} x^k \epsilon^j y^l\), with \(y_{kj}, f_{l,kj} \in \mathbb{C}^m\), and \(M(\epsilon) = \sum_j M_j \epsilon^j\). Substituting \(\hat{y}(x, \epsilon)\) for \(y\) in \(f\) and writing \(\frac{dy}{dx} = \sum_{k,j} (k+1)(y_{k-1,j} - y_{k+1,j-1}) x^k \epsilon^j\) in \[[12]\], one can then divide by \((x^2 - \epsilon)\) and compare the coefficients of \(x^k \epsilon^j\), obtaining a set of equations

\[
M_0 y_{kj} = -f_{0,kj} + P_{k,j} \{y_{k',j'} \mid k' \leq k, j' \leq j, k' + j' \leq k + j - 1\} - (k+1)y_{k+1,j+1},
\]

where \(P_{k,j}\) is a polynomial in \(y_{k,j}\) without constant term whose coefficients are linear combinations of columns of \(M_j m\) and \(f_{l,k',j''}, j'' \leq j, k'' \leq k, k'' + 2|l| \leq k + 2j\). Recursively with respect to the linear ordering of the indices \((k, j)\) given by:

\[
(k', j') < (k, j) \quad \text{if} \quad k' + j' < k + j \quad \text{or} \quad k' + j' = k + j \quad \text{and} \quad j' < j
\]

this uniquely determines all the coefficient vectors \(y_{kj}\).

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**Sectoral center manifold and its unfolding.**

For \(\epsilon = 0\) it is known that the equation \[[12]\] has a unique solution in terms of a 1-summable formal power series \(\hat{y}_0(x) = \hat{y}(x, 0)\) (cf. [BB], or [MR1] for \(m = 1\)). Its formal Borel transform \(\hat{B}[\hat{y}_0](\xi)\) extends analytically on \(\Xi_0 := \mathbb{C} \setminus \bigcup_{\lambda \in \text{Spec } M_0} \lambda + \infty \lambda\) with singularities at the eigenvalues of \(M_0\). The series \(\hat{y}_0(x)\) is Borel 1-summable in each direction \(\alpha\) with \(e^{i\alpha \mathbb{R}^+} \subset \Xi_0\). To each connected component \(\Omega\) of \(\mathbb{C} \setminus \bigcup_{\lambda \in \text{Spec } M_0} \lambda \mathbb{R}^+\) in the Borel plane (Figure [I]) corresponds a unique Borel sum of \(\hat{y}_0(x)\), a solution of the equation, defined on a sector in the \(x\)-plane of opening \(\pi\) asymptotic to \(\hat{y}_0(x)\) (cf. [MII]). For each two opposite components \(\Omega^+, \Omega^-\) of the Borel plane (i.e. such that \(\Omega^+ \cup \Omega^- \cup \{0\}\) contains some straight line \(e^{i\alpha \mathbb{R}}\)), the two corresponding sectors of summability form a covering of a neighborhood of the origin in the \(x\)-plane.

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**Figure 1:** The rays \(\lambda \mathbb{R}^+, \lambda \in \text{Spec } M(0)\) divide the Borel plane in sectors. The integration path \(e^{i\alpha \mathbb{R}^+}\) of the Laplace transform \(L_\alpha[\hat{B}[\hat{y}_0]]\) varies in such sectors.
Theorem 3 below shows that each such covering pair of sectors \((Z^+, Z^-)\) unfolds for \(\epsilon \neq 0\) to a single ramified domain \(Z(\sqrt{\epsilon})\), adherent to both singular points \(x = \pm \sqrt{\epsilon}\) (see Figure 2), on which there exists a unique bounded solution \(y(x, \sqrt{\epsilon})\) of (12), depending analytically on \(\sqrt{\epsilon}\) taken from a sector \(S\) of opening \(> \pi\), that converge uniformly to the two respective Borel sums of \(\hat{y}_0(x)\) on \(Z^+(0), Z^-(0)\), when \(\sqrt{\epsilon} \to 0\).

**Theorem 3.** Consider a system \([12]\) with \(M(\epsilon)\) a germ of an invertible \(m \times m\)-matrix and \(f(x, y, \epsilon) = O(\|y\|^2) + x O(\|y\|) + (x^2 - \epsilon) O(1)\) a germ of an analytic function at the origin of \(\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}\).

(i) To each pair \(\Omega^+, \Omega^-\) of opposite sectoral components of \(\mathbb{C} \setminus \bigcup_{\lambda \in \text{Spec}(M(0))} \lambda \mathbb{R}^+\) (i.e. such that \(\Omega^+ \cup \Omega^- \cup \{0\}\) contains some straight line \(e^{i\alpha} \mathbb{R}\)), there exists an associated family of ramified domains \(Z(\sqrt{\epsilon})\) parametrized by \(\sqrt{\epsilon}\) from a sector \(S\) of opening \(> \pi\) (see Figure 2), and a unique bounded analytic solution \(y(x, \sqrt{\epsilon})\) to (12) that is uniformly continuous on

\[
Z = \{(x, \sqrt{\epsilon}) \mid x \in Z(\sqrt{\epsilon})\}
\]

and analytic on the interior of \(Z\) and vanishes (is uniformly \(O(x^2 - \epsilon)\)) at the singular points. To be more precise:

- \(S\) is a simply connected sectoral domain such that \(\nu S \subseteq S\) for any \(\nu \in [0, 1]\), and \(S \setminus \{0\}\) is open.
- Each \(Z(\sqrt{\epsilon})\) is a simply connected ramified set in the \(x\)-plane, whose ramification points \(\pm \sqrt{\epsilon}\) belong to \(Z(\sqrt{\epsilon})\) and are approached from within its interior \(Z(\sqrt{\epsilon}) \setminus \{\sqrt{\epsilon}, -\sqrt{\epsilon}\}\) following a logarithmic spiral. The domain \(Z(0)\) is composed of two opposing sectoral domains \(Z^\pm(0)\) of opening \(> \pi\) intersecting at the origin.
- The domains \(Z(\sqrt{\epsilon})\) depend continuously on \(\sqrt{\epsilon} \in S \setminus \{0\}\) and converge radially to a sub-domain \(\lim_{\nu \to 0^+} Z(\sqrt{\epsilon}) \subseteq Z(0)\), while \(Z(0)\) is the union of these radial limits.
- There exists a fixed neighborhood of the origin in the \(x\)-plane covered by each domain \(Z(\sqrt{\epsilon})\), for each \(\sqrt{\epsilon}\) small enough.
- When \(\sqrt{\epsilon}\) tends radially to 0, the solution \(y(x, \sqrt{\epsilon})\) converges to \(y(x, 0)\) uniformly on compact sets of the sub-domain \(\lim_{\nu \to 0^+} Z(\sqrt{\epsilon})\) of \(Z(0)\). The restriction of \(y(x, 0)\) to \(Z^\pm(0)\) is the Borel sum of the formal series \(\hat{y}(x, 0)\) \((13)\) in directions \(\alpha\) with \(e^{i\alpha} \mathbb{R}^+ \subset \Omega^+ \cup \Omega^- \cup \{0\}\).

In the time \(t\)-coordinate \(\mathbf{[9]}\), the domains \(Z(\sqrt{\epsilon})\) are simply unions of slanted strips of continuously varying directions \(-\alpha + \frac{\pi}{\sqrt{\epsilon}}, \alpha + \frac{\pi}{\sqrt{\epsilon}}\) with \(\alpha \in (\beta_1, \beta_2) \cap (\arg \sqrt{\epsilon} + \eta, \arg \sqrt{\epsilon} + \pi - \eta)\), for \(\eta > 0\) arbitrarily small and \(\beta_1 < \beta_2\) such that the cone \(\bigcup_{\beta \in (\beta_1, \beta_2)} e^{i\beta} \mathbb{R}\) is contained in \(\Omega^+ \cup \Omega^- \cup \{0\}\), that pass in between closed discs centered at the points \(k \frac{\pi}{\sqrt{\epsilon}}, k \in \mathbb{Z}\), whose radius is independent of \(\sqrt{\epsilon}\). See Figure 3.

The solution \(y(x, \sqrt{\epsilon})\), and its domain \(Z\), associated to each pair \(\{\Omega^+, \Omega^-\}\) are unique up to the reflection \((x, \sqrt{\epsilon}) \to (x, -\sqrt{\epsilon})\), and an analytic extension.

(ii) If, moreover, the spectrum of the matrix \(M(0)\) is of Poincaré type (the convex hull of \(\text{Spec} M(0)\) does not contain \(0\) inside or on the boundary), i.e. if there exists a
Figure 2: Example of the spiraling domains $Z(\sqrt{\epsilon})$ of Theorem 3(i) according to $\sqrt{\epsilon} \in S$.

Figure 3: The domains $Z(\sqrt{\epsilon})$ of Theorem 3(i) in the time $t$-coordinate. They are obtained as unions of strips of convergence of the unfolded Laplace transforms $\mathcal{L}_\alpha[y^+](x(t), \sqrt{\epsilon})$ with varying $\alpha \in (\beta_1, \beta_2) \cap (\arg \sqrt{\epsilon} + \eta, \arg \sqrt{\epsilon} + \pi - \eta)$ (here $\beta_1 \sim \frac{\pi}{4}$, $\beta_2 \sim \frac{3\pi}{4}$).
Figure 4: Example of the spiraling domains $Z_1(\sqrt{\epsilon})$ of Theorem 3 (ii) according to $\sqrt{\epsilon} \in S_1$. 

(unique) component $\Omega_1$ of $C \setminus \bigcup_{\lambda \in \text{Spec}(M(0))} \lambda \mathbb{R}^+$ of opening $> \pi$, then the solution $y_1(x, \sqrt{\epsilon})$ on the domain $Z_1(\sqrt{\epsilon})$, $\sqrt{\epsilon} \in S_1$, associated to the pair $\{\Omega_1, \Omega_1\}$ is ramified only at one of the singular points, and analytic at the other (see Figure 4). Such is the case in dimension $m = 1$.

The solutions $y(x, \sqrt{\epsilon})$ will be constructed in Section 5.

Remark 4 (Hadamard–Perron interpretation for $\epsilon \neq 0$). The linearization of the vector field (11) at $x = \pm \sqrt{\epsilon}$ is equal to

$$
\dot{x} = \pm 2\sqrt{\epsilon} \cdot (x \mp \sqrt{\epsilon}), \quad \dot{y} = M(\epsilon) \cdot y. \tag{14}
$$

(i) Let a line $e^{i\alpha}\mathbb{R}$ separate the point $2\sqrt{\epsilon}$ and $k$ of the eigenvalues of $M(\epsilon)$ from the point $-2\sqrt{\epsilon}$ and the other $m-k$ eigenvalues ($0 \leq k \leq m$), see Figure 5. Then by the Hadamard–Perron theorem the vector field (11) has a unique $(k+1)$-dimensional local invariant manifold at $(\sqrt{\epsilon}, 0)$, tangent to the $x$-axis and the corresponding $k$ eigenvectors, and a unique $(m-k+1)$-dimensional local invariant manifold at $(-\sqrt{\epsilon}, 0)$, tangent to the $x$-axis and the corresponding $m-k$ eigenvectors. They intersect transversally as the graph of the solution $y(x, \sqrt{\epsilon})$ of Theorem 3. Since the root parameter $\sqrt{\epsilon}$ can vary within the half-plane bounded by the line $e^{i\alpha}\mathbb{R}$, whose angle $\alpha$ can also vary a bit, this gives a sector $S$ of opening $> \pi$. We see that one cannot continue this description in $\sqrt{\epsilon}$ beyond such maximal sector $S$.

(ii) If all the eigenvalues of $M(\epsilon)$ are in a same open sector of opening $< \pi$ (i.e. Spec$(M(0))$ is of Poincaré type), and $-2\sqrt{\epsilon}$ lies in the interior of the complementary
sector of opening $> \pi$, one obtains the solution $y_1(x, \sqrt{\epsilon})$ from Theorem 3 as a continuation of the local analytic solution at $x = -\sqrt{\epsilon}$ (i.e. of the local invariant manifold of (11), tangent to the $x$-axis, provided by the Hadamard–Perron theorem) to the domain $Z_1(\sqrt{\epsilon})$.

While the Hadamard–Perron approach explains where do the solutions of Theorem 3 come from, it does not provide their natural domain on which they are bounded. One should however notice the similarities between the description provided by the Hadamard–Perron theorem for $\epsilon \neq 0$ (Figure 5) and that of the Borel summation for $\epsilon = 0$ (Figure 1). In Section 5 we will unify the two of them using the unfolded Borel–Laplace transformations (8).

Remark 5 (Local invariant manifolds for non-resonant $\epsilon \neq 0$ and their convergence). If the simple singular point of (11) at $x = \sqrt{\epsilon} \neq 0$ satisfies the following non-resonance condition

$$2 \sqrt{\epsilon} \mathbb{N} \cap \text{Spec} M(\epsilon) = \emptyset,$$

then it is known that the equation (12) possesses a unique convergent formal solution near $x = \sqrt{\epsilon}$, i.e. the vector field (11) has a 1-dimensional local analytic invariant manifold tangent to the $x$-axis at the singularity. The resonant values $\sqrt{\epsilon} = \frac{\lambda}{2n}$, $\lambda \in \text{Spec} M(\epsilon)$, $n \in \mathbb{N}^*$, accumulate at the origin along the rays $\lambda \mathbb{R}^+$, $\lambda \in \text{Spec} M(\epsilon)$, dividing the $\sqrt{\epsilon}$-plane in a finite number of sectors (Figure 6). It has been shown\(^2\) that if $\sqrt{\epsilon} \neq 0$ lies in one of these sectors (i.e. $\sqrt{\epsilon} \mathbb{R}^+ \cap \text{Spec} M(0) = \emptyset$), then the local analytic solution at $x = \sqrt{\epsilon}$ converges, when $\sqrt{\epsilon}$ tends radially to 0, to the sectoral Borel sum $L_\alpha [\hat{B}(\hat{y}_0)](x)$ of the formal solution of the limit system (cf. Figure 1), where $\alpha = \arg \sqrt{\epsilon}$ is the direction on which lies the eigenvalue $2 \sqrt{\epsilon}$ of the linearization (14) at $x = \sqrt{\epsilon}$. Unless the spectrum of $M(0)$ is of Poincaré type, these sectors in the $\sqrt{\epsilon}$-plane, on which the convergence happens, are of opening $< \pi$.

\(^2\)In [G] for planar vector fields, $m = 1$, and in [SS] for linear systems; the method of latter can be generalized also for non-linear systems (12).
Figure 6: The resonant values of $\sqrt{\epsilon} = \frac{\lambda}{2\pi}$, $\lambda \in \text{Spec} M(\epsilon)$, $n \in \mathbb{N}^*$, accumulate along the rays $\lambda \mathbb{R}^+$, dividing the $\sqrt{\epsilon}$-plane in sectors on which the local analytic solutions near $x = \sqrt{\epsilon} \neq 0$ converge as $\sqrt{\epsilon} \to 0$ to sectoral solutions.

2.3 Sectoral normalization of families of non-resonant linear differential systems

An application of Theorem 3 interesting on its own, is the problem of existence of normalizing transformations for linear differential systems near an unfolded non-resonant irregular singularity of Poincaré rank 1. We will show that this problem can be reduced to a system (12) of $m = n(n-1)$ Ricatti equations (where $n$ is the dimension of the system), providing thus a simple proof of a sectoral normalization theorem by Lambert and Rousseau [LR].

Consider a parametric family of linear systems $\Delta(x, \epsilon) y = 0$ given by

$$\Delta(x, \epsilon) = (x^2 - \epsilon) \frac{d}{dx} - A(x, \epsilon), \quad (x, \epsilon) \in (\mathbb{C} \times \mathbb{C}, 0)$$

(15)

where $y(x, \epsilon) \in \mathbb{C}^n$, $A(x, \epsilon)$ is analytic, and assume that the eigenvalues $\lambda_i^{(0)}(0)$, $i = 1, \ldots, n$, of the matrix $A(0, 0)$ are distinct. Let $\lambda_i(x, \epsilon) = \lambda_i^{(0)}(\epsilon) + x\lambda_i^{(1)}(\epsilon)$, $i = 1, \ldots, n$, be the eigenvalues of $A(x, \epsilon)$ modulo $O(x^2 - \epsilon)$, and define

$$\tilde{\Delta}(x, \epsilon) = (x^2 - \epsilon) \frac{d}{dx} - \Lambda(x, \epsilon), \quad \Lambda(x, \epsilon) = \text{Diag}(\lambda_1(x, \epsilon), \ldots, \lambda_n(x, \epsilon)),$$

(16)

the formal normal form for $\Delta$. The problem we address, is to find a bounded invertible linear transformation $y = T(x, \sqrt{\epsilon}) u$ between the two systems $\Delta y = 0$ and $\tilde{\Delta} u = 0$. Such $T$ is a solution of the equation

$$(x^2 - \epsilon) \frac{dT}{dx} = AT - TA.$$ 

(17)

Note that if $V(x, \epsilon)$ is an analytic matrix of eigenvectors of $A(x, \epsilon)$ then the transformation $y = V(x, \epsilon) y_1$ brings the system $\Delta y = 0$ to $\Delta_1 y_1 = 0$, whose matrix is written as $A_1(x, \epsilon) = \Lambda(x, \epsilon) + (x^2 - \epsilon) R(x, \epsilon)$, with $R = -V^{-1} \frac{dV}{dx}$ (see [LR]). Hence we can suppose that system (15) is already in such form.

Theorem 6 ([LR], Theorem 4.21). Let $\Delta(x, \epsilon)$ be a non-resonant system (15) with $A(x, \epsilon) = \Lambda(x, \epsilon) + (x^2 - \epsilon) R(x, \epsilon)$ for some analytic germ $R(x, \epsilon)$, and let $\tilde{\Delta}(x, \epsilon)$ be its formal normal form (16). Then there exists a family of ramified “spiraling”
domains $Z(\sqrt{\epsilon})$, $\sqrt{\epsilon} \in S$, as in Theorem 3(i). Figure 3, on which there exists a normalizing transformation $T(x, \sqrt{\epsilon})$, solution to the equation \(17\), which is uniformly continuous on

$$Z = \{(x, \sqrt{\epsilon}) \mid x \in Z(\sqrt{\epsilon})\}$$

and analytic on its interior, and such that $T(\pm \sqrt{\epsilon}, \sqrt{\epsilon}) = I + O(\sqrt{\epsilon})$ is diagonal. This transformation $T$ on $Z$ is unique modulo right multiplication by an invertible diagonal matrix constant in $x$.

**Proof.** Write $T(x, \sqrt{\epsilon}) = (I + U(x, \sqrt{\epsilon})) \cdot T_D(x, \sqrt{\epsilon})$, where $T_D(x, \sqrt{\epsilon})$ is the diagonal of $T$, and the matrix $U(x, \sqrt{\epsilon}) = O(x^2 - \epsilon)$ has only zeros on the diagonal. We search for $U(x, \sqrt{\epsilon})$, such that $y_D = (I + U(x, \sqrt{\epsilon}))^{-1} y$ satisfies

$$\left(x^2 - \epsilon \right) \frac{dy_D}{dx} - \left(\Lambda(x, \epsilon) + (x^2 - \epsilon)D(x, \sqrt{\epsilon})\right)y_D = 0,$$

for some diagonal matrix $D(x, \sqrt{\epsilon})$, and set

$$T_D(x, \sqrt{\epsilon}) = e^\int_{x}^{\sqrt{\epsilon}} D(s, \sqrt{\epsilon}) \, ds.$$

The matrix $U(x, \sqrt{\epsilon})$ is solution to

$$\left(x^2 - \epsilon \right) \frac{dU}{dx} = \Lambda U - U \Lambda + (x^2 - \epsilon) \left( R(I + U) - (I + U) D \right),$$

where one must set $D$ to be equal to the diagonal of $R(I + U)$. Therefore, $U = (u_{ij})_{i,j=1}^n$ is solution to the system of $n(n-1)$ equations

$$\left(x^2 - \epsilon \right) \frac{du_{ij}}{dx} = (\lambda_i - \lambda_j) u_{ij} + (x^2 - \epsilon) \left( r_{ij} + \sum_{k \neq j} r_{ik} u_{kj} - u_{ij} r_{jj} - u_{ij} \sum_{k \neq j} r_{jk} u_{kj} \right), \quad i \neq j,$$

and one can apply Theorem 3. \qed

### 3 Preliminaries on Fourier–Laplace transformations

We will recall some basic elements of the classical theory of Fourier–Laplace transformations on a line in the complex plane. The book [D] can serve as a good reference.

For an angle $\alpha \in \mathbb{R}$ and a locally integrable function $\phi : e^{i\alpha} \mathbb{R} \to \mathbb{C}$, one defines its two-sided Laplace transform

$$\mathcal{L}_\alpha[\phi](t) = \int_{-\infty}^{+\infty} e^{i\alpha \xi} \phi(\xi) e^{-t\xi} d\xi$$

whenever it exists. Later on, in Section 4, we will replace the variable $t$ by the time variable $t(x, \epsilon)$ (24) of the vector field (7).

**Definition 7.** Let $A, B \in \mathbb{C}$ be such that $\text{Re}(e^{i\alpha} A) < \text{Re}(e^{i\alpha} B)$. Let us introduce the two following norms on locally integrable functions $\phi : e^{i\alpha} \mathbb{R} \to \mathbb{C}$:

$$|\phi|_{e^{i\alpha} \mathbb{R}}^{A,B} = \sup_{\xi \in e^{i\alpha} \mathbb{R}} |\phi(\xi)| \cdot \left( |e^{-A\xi}| + |e^{-B\xi}| \right),$$

$$||\phi||_{e^{i\alpha} \mathbb{R}}^{A,B} = \int_{-\infty}^{+\infty} |\phi(\xi)| \cdot \left( |e^{-A\xi}| + |e^{-B\xi}| \right) d\xi \cdot e^{-i\alpha}.$$
Proposition 8. If \( \|\phi\|_{C_{\alpha}^1}^A < +\infty \), then the Laplace transform \( L_{\alpha}[\phi](t) \) converges absolutely and is analytic for \( t \) in the closed strip
\[
T_{\alpha}^{A,B} := \{ t \in \mathbb{C} \mid \Re(e^{i\alpha}t) \leq \Re(e^{i\alpha}t) \leq \Re(e^{i\alpha}B) \}.
\]
Moreover, \( L_{\alpha}[\phi](t) \) tends uniformly to 0 as \( t \to \infty \) in \( T_{\alpha}^{A,B} \).

Proof. The integral \( \int_{-\infty}^{0} \phi(\xi) e^{-i\xi} d\xi \) converges absolutely in the closed half-plane \( \Re(e^{i\alpha}t) \leq \Re(e^{i\alpha}B) \), while the integral \( \int_{0}^{+\infty} \phi(\xi) e^{-i\xi} d\xi \) converges absolutely in the closed half-plane \( \Re(e^{i\alpha}t) \geq \Re(e^{i\alpha}A) \). For the second statement see [D], Theorem 23.6.

Lemma 9. If \( A, B, D \in \mathbb{C} \) are such that \( 0 < \Re(e^{i\alpha}D) < \frac{1}{2} \Re(e^{i\alpha}(B - A)) \), then for any function \( \phi : e^{i\alpha}\mathbb{R} \to \mathbb{C} \),
\[
\| \phi \|_{C_{\alpha}^1}^{A+D,B-D} \leq \frac{4}{\Re(e^{-i\alpha}D)} |\phi|_{C_{\alpha}^1}^{A,B}.
\]

Proof.
\[
\int_{-\infty}^{0} |\phi(\xi)| \left( |e^{-(A+D)\xi}| + |e^{-(B-D)\xi}| \right) d\xi \cdot e^{-i\alpha} \\
\leq \int_{-\infty}^{0} |e^{i\alpha}D| \sup_{\xi \in \mathbb{C}} |\phi(\xi)| \left( |e^{-A\xi} - 2D\xi| + |e^{-B\xi}| \right) \\
\leq \frac{1}{\Re(e^{-i\alpha}D)} \cdot 2|\phi|_{C_{\alpha}^1}^{A,B},
\]

since \( |e^{-A\xi} - 2D\xi| \leq |e^{-B\xi}| \leq |e^{-A\xi}| + |e^{-B\xi}| \), for \( \xi \in (-\infty, 0] \). The same kind of estimate is obtained also for \( \int_{0}^{+\infty} e^{i\alpha}d\xi \).

Corollary 10. If \( |\phi|_{C_{\alpha}^1}^{A,B} < +\infty \), then the Laplace transform \( L_{\alpha}[\phi](t) \) converges absolutely and is analytic for \( t \) in the open strip
\[
T_{\alpha}^{A,B} := \{ t \in \mathbb{C} \mid \Re(e^{i\alpha}A) < \Re(e^{i\alpha}t) < \Re(e^{i\alpha}B) \}.
\]
Moreover, \( L_{\alpha}[\phi](t) \) tends to 0 as \( t \to \infty \) uniformly in each \( T_{\alpha}^{A_1,B_1} \subseteq T_{\alpha}^{A,B} \).

Definition 11. The Borel transformation is defined for any function \( f \) analytic on some open strip \( T_{\alpha}^{A,B} \), that vanishes at infinity uniformly in each closed substrip \( T_{\alpha}^{A_1,B_1} \subseteq T_{\alpha}^{A,B} \), by
\[
\tilde{f}(\xi) = B_{\alpha}[f](\xi) = \frac{1}{2\pi i} \int_{C+\infty e^{-i\alpha}}^{C+\infty e^{-i\alpha}} f(t) e^{i\xi t} dt, \quad \text{for} \ \xi \in e^{i\alpha}\mathbb{R}, \quad (19)
\]
where \( \lim_{N \to +\infty} \int_{C-\infty e^{-i\alpha}}^{C+\infty e^{-i\alpha}} f(t) e^{i\xi t} dt \) stands for the “Cauchy principal value” \( \lim_{N \to +\infty} \int_{C-\infty e^{-i\alpha}}^{C+\infty e^{-i\alpha}} f(t) e^{i\xi t} dt \), and \( C \in T_{\alpha}^{A,B} \).

The (two-sided) Laplace transformation [18] and the Borel transformation (19) of analytic functions are inverse one to the other when defined. We will only need the following particular statement.
Theorem 12.
1) Let $f \in O(T^{A,B}_\alpha)$ be absolutely integrable on each line $C+ie^{-\alpha}R \subseteq T^{A,B}_\alpha$ and vanishing at infinity uniformly in each closed sub-strip of $T^{A,B}_\alpha$. Then the Borel transform $\tilde{f}(\xi) = B_{\alpha}[f](\xi)$ is absolutely convergent and continuous for all $\xi \in e^{i\alpha}R$,

$$|\tilde{f}|^{A_1,B_1}_{e^{i\alpha}R} \leq \frac{1}{2\pi} \sup_{C \in T^{A_1,B_1}_\alpha} \int_{C+ie^{-\alpha}R} |f(t)| \, dt \quad \text{for} \quad T^{A_1,B_1}_\alpha \subseteq T^{A,B}_\alpha,$$

and $f(t) = L_{\alpha}[\tilde{f}](t)$ for all $t \in T^{A,B}_\alpha$.

2) Let $f$ be as in 1) with $B = B_1 = +\infty e^{-\alpha}$, the strips being replaced by half-planes. Then the Borel transform $\tilde{f}(\xi) = B_{\alpha}[f](\xi)$ is absolutely convergent and continuous on $e^{i\alpha}R$, and $\tilde{f}(\xi) = 0$ for $\xi \in (-\infty e^{i\alpha}, 0)$,

$$|\tilde{f}|^{A_1,+.\infty e^{-i\alpha}}_{e^{i\alpha}R} = \sup_{\xi \in (0,+.\infty) e^{i\alpha}} |\tilde{f}(\xi)e^{-A_1\xi}| \leq \frac{1}{2\pi} \sup_{\text{Re}(e^{i\alpha}C) \geq \text{Re}(e^{i\alpha}A_1)} \int_{C-\infty e^{-i\alpha}} |f(t)| \, dt,$$

and

$$f(t) = L_{\alpha}[\tilde{f}](t) = \int_{0}^{+.\infty e^{i\alpha}} \tilde{f}(\xi) e^{-t\xi} d\xi$$

is the one-sided Laplace transform of $\tilde{f}$ in the direction $\alpha$.

Proof. See [D], Theorems 28.1 and 28.2.

Under the assumptions of Theorem 12 the Borel transformation converts derivative to multiplication by $-\xi$:

$$B_{\alpha}[\frac{d\tilde{f}}{d\xi}](\xi) = -\xi \cdot B_{\alpha}[f](\xi),$$

which can be seen by integration by parts. It also converts the product to the convolution:

$$B_{\alpha}[f_1 \cdot f_2](\xi) = \tilde{f}_1 \ast \tilde{f}_2|_{\alpha}(\xi),$$

defined by

$$[\phi \ast \psi]_{\alpha}(\xi) = [\psi \ast \phi]_{\alpha}(\xi) := \int_{-\infty e^{i\alpha}}^{+.\infty e^{i\alpha}} \phi(s) \psi(\xi - s) \, ds. \quad (20)$$

Indeed, we have $L_{\alpha}[f_1 \ast f_2](t) = L_{\alpha}[f_1](t) \cdot L_{\alpha}[f_2](t) = f_1(t) \cdot f_2(t)$ using Fubini theorem and Theorem 12 and the assertion is obtained by the inversion theorem of the Laplace transform: $B_{\alpha}[L_{\alpha}[\phi]](\xi) = \frac{1}{2\pi} \lim_{\nu \to 0^+} \phi(\xi + e^{i\alpha} \nu) + \phi(\xi - e^{i\alpha} \nu)$ (cf. [D], Theorem 24.3), using the continuity of $[f_1 \ast f_2]|_{\alpha}(\xi)$.

Lemma 13 (Young’s inequality).

$$|\phi \ast \psi|^{A,B}_{e^{i\alpha}R} \leq |\phi|^{A,B}_{e^{i\alpha}R} \cdot ||\psi||^{A,B}_{e^{i\alpha}R} \quad \text{and} \quad \leq ||\phi||^{A,B}_{e^{i\alpha}R} \cdot ||\psi||^{A,B}_{e^{i\alpha}R}.$$ 

Proof. Observe that

$$(|e^{-A\xi}| + |e^{-B\xi}|) \leq (|e^{-A\xi}| + |e^{-B\xi}|) \cdot (|e^{-A(\xi-s)}| + |e^{-B(\xi-s)}|),$$

the rest follows easily.
### 3.0.1 Convolution of analytic functions on open strips

In the subsequent text, rather then dealing with functions on a single line $e^{i\alpha}\mathbb{R}$, one will work with functions which are analytic on some open strips in the $\xi$-plane (also called the Borel plane), or on more general regions obtained as a connected union of open strips of varying directions $\alpha$.

If $\Omega$ is a non-empty open strip in direction $\alpha$, then for two constants $A, B \in \mathbb{C}$, with $\text{Re}(e^{i\alpha}A) < \text{Re}(e^{i\alpha}B)$, define the norm of analytic functions $\phi \in \mathcal{O}(\Omega)$,

$$
|\phi|_{\Omega}^{A,B} = \sup_{c+e^{i\alpha}r \in \Omega} |\phi|_{c+e^{i\alpha}\mathbb{R}},
$$

$$
||\phi||_{\Omega}^{A,B} = \sup_{c+e^{i\alpha}r \in \Omega} ||\phi||_{c+e^{i\alpha}\mathbb{R}}.
$$

Similarly for more general domains. For any two strips $\Omega_j, \ j = 1, 2$, of the same direction $\alpha$, and two analytic functions $\phi_j \in \mathcal{O}(\Omega_j)$ of bounded $|| \cdot ||_{\Omega}^{A,B}$-norm, their convolution

$$
(\phi_1 * \phi_2)(\xi) = \int_{c_1-\infty}^{c_1+\infty} \phi_1(s) \phi_2(\xi - s) \, ds, \quad \xi \in c_1 + c_2 + e^{i\alpha}\mathbb{R}, \ c_j \in \Omega_j
$$

is well defined and analytic on the strip $\Omega_1 + \Omega_2$. The Lemma [13] is easily generalized as

$$
|\phi_1 * \phi_2|_{\Omega_1 + \Omega_2}^{A,B} \leq \min \left\{ |\phi_1|_{\Omega_1}^{A,B} \cdot |\phi_2|_{\Omega_2}^{A,B}, \ |\phi_1|_{\Omega_1}^{A,B} \cdot |\phi_2|_{\Omega_2}^{A,B} \right\}
$$

(22)

$$
||\phi_1 * \phi_2||_{\Omega_1 + \Omega_2}^{A,B} \leq ||\phi_1||_{\Omega_1}^{A,B} \cdot ||\phi_2||_{\Omega_2}^{A,B}.
$$

(23)

### 3.0.2 Dirac distributions in the Borel plane

It is convenient to introduce for each $\alpha \in \mathbb{C}$ the Dirac mass distribution $\delta_\alpha(\xi)$, acting on the $\xi$-plane as shift operators $\xi \mapsto \xi - \alpha$: If $\phi(\xi)$ is an analytic function on some strip $\Omega$ in a direction $\alpha$ one defines

$$
\left[ \delta_\alpha * \phi \right](\xi) := \phi(\xi - \alpha),
$$

its translation to the strip $\Omega - \alpha$. With this definition, the operator $\delta_0$ plays the role of the unity of convolution. One can represent each $\delta_\alpha$ as a “boundary value” of the function $\frac{1}{2\pi i (\xi - a)}$ (cf. [3]): Let

$$
\delta_\alpha^B(\xi) := \frac{1}{2\pi i (\xi - a)} \, \text{V.P.} \int_{\xi - a - \infty e^{i\alpha}}^{\xi - a + \infty e^{i\alpha}} \phi(\xi - s) \, ds,
$$

$$
\delta_\alpha^A(\xi) := \frac{1}{2\pi i (\xi - a)} \, \text{V.P.} \int_{\xi - a + \infty e^{i\alpha}}^{\xi - a - \infty e^{i\alpha}} \phi(\xi - s) \, ds,
$$

be its restrictions to the two cut regions (see Figure [7]). One then writes

$$
\delta_\alpha(\xi) = \delta_\alpha^B(\xi) - \delta_\alpha^A(\xi),
$$

and defines the convolution and the Laplace transform involving $\delta_\alpha$ by integrating each term $\delta_\alpha^B$ (resp. $\delta_\alpha^A$) along deformed paths $\gamma_\alpha^B$ (resp. $\gamma_\alpha^A$) of direction $\alpha$ in their respective domains as in Figure [7]

$$
\left[ \delta_\alpha * \phi \right](\xi) = \text{V.P.} \int_{\gamma_\alpha^B - \gamma_\alpha^A} \frac{1}{2\pi i (s - a)} \phi(\xi - s) \, ds = \phi(\xi - \alpha),
$$

$$
\mathcal{L}_\alpha[\delta_\alpha](t) = \text{V.P.} \int_{\gamma_\alpha^B - \gamma_\alpha^A} \frac{1}{2\pi i (\xi - a)} e^{-it\xi} \, d\xi = e^{-at}.
$$
Figure 7: The domains of definition of $\delta_a^\downarrow$ (resp. $\delta_a^\uparrow$) together with the deformed integration paths $\gamma_a^\downarrow$ (resp. $\gamma_a^\uparrow$).

4 The unfolded Borel and Laplace transformations associated to the vector field $(x^2 - \epsilon) \frac{\partial}{\partial x}$

In this section we define the unfolded Borel and Laplace transformations $B_\alpha, L_\alpha$ and summarize their basic properties. We need to specify:

- the time function $t(x, \epsilon)$ of the kernel,
- the paths of integration,
- the domains in $x$-space and $\xi$-space where the transformations live,
- sufficient conditions on functions for which the transformations exist.

We provide these depending analytically on a root parameter $\sqrt{\epsilon} \in \mathbb{C}$. Here $\sqrt{\epsilon}$ is to be interpreted simply as a symbol for a new parameter (a coordinate on the "$\sqrt{\epsilon}$-plane"), that naturally projects on the original parameter $\epsilon = (\sqrt{\epsilon})^2$.

Let $t(x, \epsilon) = -\int \frac{dx}{x^2 - \epsilon} := \begin{cases} \frac{1}{2\sqrt{\epsilon}} \log \frac{x - \sqrt{\epsilon}}{x + \sqrt{\epsilon}}, & \text{if } \epsilon \neq 0, \\ \frac{1}{x}, & \text{if } \epsilon = 0, \end{cases}$ \hspace{1cm} (24)

with $t(\infty, \epsilon) = 0$, be the complex time of the vector field $-(x^2 - \epsilon) \frac{\partial}{\partial x}$, well defined for $x \in \mathbb{CP}^1 \setminus [-\sqrt{\epsilon}, \sqrt{\epsilon}]$. And let $\mathcal{X}(\sqrt{\epsilon})$ denote the Riemann surface of the analytic continuation of $t(\cdot, \epsilon)$. Let us remark that the limit surface $\lim_{\sqrt{\epsilon} \to 0} \mathcal{X}(\sqrt{\epsilon})$ is composed of $\mathbb{Z}$-many complex planes identified at the origin, but the surface $\mathcal{X}(0)$ is just the $x$-plane in the middle.

**Definition 14.** For $0 \leq \Lambda < \frac{\pi}{2\sqrt{|\epsilon|}}$, denote

$$X(\Lambda, \sqrt{\epsilon}) := \{ x \in \mathbb{C} \mid |t(x, \epsilon) - k \frac{\pi i}{\sqrt{\epsilon}}| > \Lambda, \ k \in \mathbb{Z} \}$$

an open neighborhood of the origin in the $x$-plane (of radius $\sim \frac{1}{\Lambda}$ when $\epsilon$ is small) containing the roots $\pm \sqrt{\epsilon}$.

If $\alpha$ is a direction, assuming that $\Lambda$ satisfies $0 \leq 2\Lambda < -\text{Re}(\frac{e^{i\alpha \pi i}}{\sqrt{\epsilon}})$, denote

$$\mathcal{X}_\alpha^+(\Lambda, \sqrt{\epsilon}) := \{ x \in \mathcal{X}(\sqrt{\epsilon}) \mid \Lambda < \text{Re}(e^{i\alpha t(x, \epsilon)}) < -\text{Re}(\frac{e^{i\alpha \pi i}}{\sqrt{\epsilon}}) - \Lambda \},$$

$$\mathcal{X}_\alpha^- (\Lambda, \sqrt{\epsilon}) := \{ x \in \mathcal{X}(\sqrt{\epsilon}) \mid -\Lambda > \text{Re}(e^{i\alpha t(x, \epsilon)}) > \text{Re}(\frac{e^{i\alpha \pi i}}{\sqrt{\epsilon}}) + \Lambda \},$$
open domains of the ramified surface $\tilde{X}(\sqrt{\epsilon})$, corresponding to slanted strips of direction $-\alpha + \frac{\pi}{2}$ in the $t$-coordinate passing between two discs of radius $\Lambda$ centered at $0$ and $\mp \frac{\pi}{\sqrt{\epsilon}}$ (see Figures 8 and 9). Their projection to the $x$-plane is contained inside the neighborhood $X(\Lambda, \sqrt{\epsilon})$. Let us remark that the radial limits $\lim_{\nu \to 0} \tilde{X}_\nu^\pm(\Lambda, \nu \sqrt{\epsilon})$ split each into two opposed discs of radius $\frac{1}{2\Lambda}$ tangent at the origin, of which only one lies inside the surface $\tilde{X}(0)$ (i.e. the $x$-plane): $\tilde{X}_\alpha^+(\Lambda, 0)$ is a disc centered at $e^{i\alpha} \frac{1}{2\Lambda}$, and $\tilde{X}_\alpha^-(\Lambda, 0)$ is a disc centered at $-e^{i\alpha} \frac{1}{2\Lambda}$ (Figure 9 (b)).

$$\begin{align*}
\text{(a) } \sqrt{\epsilon} \neq 0 & \quad \begin{array}{c}
\tilde{t}(X_{\alpha}(\Lambda, \sqrt{\epsilon}))
\end{array} \\
& \quad \begin{array}{c}
\tilde{t}(X_{\alpha}(\Lambda, \sqrt{\epsilon}))
\end{array} \\
& \quad \begin{array}{c}
\tilde{t}(X_{\alpha}(\Lambda, \sqrt{\epsilon}))
\end{array}
\end{align*}$$

$$\begin{align*}
\text{(b) } \sqrt{\epsilon} = 0 & \quad \begin{array}{c}
\tilde{t}(X_{\alpha}(\Lambda, \sqrt{\epsilon}))
\end{array} \\
& \quad \begin{array}{c}
\tilde{t}(X_{\alpha}(\Lambda, \sqrt{\epsilon}))
\end{array}
\end{align*}$$

Figure 8: The domains $\tilde{X}_\alpha^\pm(\Lambda, \sqrt{\epsilon})$ in the time coordinate $t$ with the integration paths of the Borel transformation for $\alpha = \frac{\pi}{2}$.

$$\begin{align*}
\text{(a) } \sqrt{\epsilon} \neq 0 & \quad \begin{array}{c}
X_{\alpha}(\Lambda, \sqrt{\epsilon})
\end{array} \\
& \quad \begin{array}{c}
X_{\alpha}(\Lambda, \sqrt{\epsilon})
\end{array} \\
& \quad \begin{array}{c}
X_{\alpha}(\Lambda, \sqrt{\epsilon})
\end{array}
\end{align*}$$

$$\begin{align*}
\text{(b) } \sqrt{\epsilon} = 0 & \quad \begin{array}{c}
X_{\alpha}(\Lambda, \sqrt{\epsilon})
\end{array} \\
& \quad \begin{array}{c}
X_{\alpha}(\Lambda, \sqrt{\epsilon})
\end{array}
\end{align*}$$

Figure 9: The domains $X^\pm_\alpha(\Lambda, \sqrt{\epsilon})$ projected to the $x$-plane for $\alpha = \frac{\pi}{2}$. The integration paths $\gamma^\pm_\alpha$ are projections of the paths $c^\pm - ie^{\pm i\alpha} R$ in the $t$-coordinate (which have opposite direction than those in Figure 8).

In order to apply the Borel transformation in a direction $\alpha$ to a function $f$ analytic on the neighborhood $X(\Lambda, \sqrt{\epsilon})$, one may choose to lift $f$ either to $\tilde{X}_\alpha^+(\Lambda, \sqrt{\epsilon})$ or to $\tilde{X}_\alpha^-(\Lambda, \sqrt{\epsilon})$ giving rise to two different transforms $B^+_\alpha[f]$ and $B^-_\alpha[f]$:

**Definition 15.** Assume that $X^\pm_\alpha(\Lambda, \sqrt{\epsilon}) \neq \emptyset$, $\alpha \in (\arg \sqrt{\epsilon}, \arg \sqrt{\epsilon} + \pi)$, and let $f \in O(\tilde{X}^\pm_\alpha(\Lambda, \sqrt{\epsilon}))$ vanish at both points $\sqrt{\epsilon}, -\sqrt{\epsilon}$. The *unfolded Borel transforms* $B^\pm_\alpha[f]$ are defined as:

$$B^\pm_\alpha[f](\xi, \sqrt{\epsilon}) = \frac{1}{2\pi i} \int_{c^{\pm} - \infty e^{-i\alpha}}^{c^{\pm} + \infty e^{-i\alpha}} f(x(t, \epsilon)) e^{i\xi t} dt, \quad c^{\pm} \in t \left[ X^\pm_\alpha(\Lambda, \sqrt{\epsilon}), \epsilon \right].$$
For $\sqrt{\epsilon} \neq 0$: If $x \in \tilde{X}_\alpha^\pm(\Lambda, \sqrt{\epsilon})$ respectively, then
\[
t(x, \epsilon) = -\frac{1}{2\sqrt{\epsilon}} \left( \log \frac{\sqrt{\epsilon-x}}{\sqrt{\epsilon+x}} \pm \pi i \right),
\]
\[
B_\alpha^\pm [f](\xi, \sqrt{\epsilon}) = e^{\pm \frac{\xi i}{2\sqrt{\epsilon}}} \cdot \frac{1}{2\pi i} \int_{\gamma_\alpha^\pm} \frac{f(x)}{x^2} (\sqrt{\epsilon-x}/\sqrt{\epsilon+x})^{-\frac{\xi}{2\sqrt{\epsilon}}} \, dx,
\]
(25)
where the integration path $\gamma_\alpha^\pm$ (see Figure 9) follows a real time trajectory of the vector field $ie^{-ia}(x^2-\epsilon)\frac{\partial}{\partial x}$ inside $\tilde{X}_\alpha^\pm(\Lambda, \sqrt{\epsilon})$. Hence
\[
B_\alpha^- [f](\xi, \sqrt{\epsilon}) = e^{\frac{\xi i}{2\epsilon}} \cdot B_\alpha^+ [f](\xi, \sqrt{\epsilon})
\]
(26)
\[
= -B_\alpha^+ [f](\xi, e^{\pi i} \sqrt{\epsilon}),
\]
(27)
as $\tilde{X}_\alpha^-(\Lambda, \sqrt{\epsilon}) = \tilde{X}_\alpha^+(\Lambda, e^{\pi i} \sqrt{\epsilon})$.

For $\sqrt{\epsilon} = 0$:
\[
B_\alpha^\pm [f](\xi, 0) = \frac{1}{2\pi} \int_{\gamma_\alpha^\pm} \frac{f(x)}{x^2} e^{\frac{\xi}{\epsilon}} \, dx,
\]
(28)
where $\gamma_\alpha^\pm$ is a real time trajectory of the vector field $ie^{-ia}x^2\frac{\partial}{\partial x}$ inside $\tilde{X}_\alpha^\pm(\Lambda, 0)$.

It is the radial limit of the precedent case as $\sqrt{\epsilon} \to 0$,
\[
B_\alpha^\pm [f](\xi, 0) = \lim_{\nu \to 0^+} B_\alpha^\pm [f](\xi, \nu \sqrt{\epsilon}).
\]
The transformation $B_\alpha^+ [f](\xi, 0)$ is the standard analytic Borel transform in direction $\alpha$, and
\[
B_\alpha^- [f](\xi, 0) = -B_\alpha^+ [f](\xi, 0).
\]
(29)
If $f = f(x, \epsilon)$ depends analytically on $\epsilon$, we define $B_\alpha^\pm [f](\xi, \sqrt{\epsilon}) := B_\alpha^\pm [f(\cdot, \epsilon)](\xi, \sqrt{\epsilon})$.

The following proposition summarizes some basic proprieties of these unfolded Borel transformations.

**Proposition 16.** Let $\alpha$ be a direction, and suppose that $\arg \sqrt{\epsilon} \in (\alpha - \pi, \alpha)$ if $\epsilon \neq 0$.

1) If $\sqrt{\epsilon} \neq 0$, let a function $f \in O(\tilde{X}_\alpha^\pm(\Lambda, \sqrt{\epsilon}))$, be uniformly $O(|x-\sqrt{\epsilon}|^a |x+\sqrt{\epsilon}|^b)$ at the points $\pm \sqrt{\epsilon}$, for some $a, b \in \mathbb{R}$ with $a + b > 0$. Then the transforms $B_\alpha^\pm [f](\xi, \sqrt{\epsilon})$ converge absolutely for $\xi$ in the strip
\[
\Omega_\alpha = \{ -\text{Im}(e^{-ia}2b\sqrt{\epsilon}) > \text{Im}(e^{-ia}\xi) > \text{Im}(e^{-ia}2a\sqrt{\epsilon}) \},
\]
(30)
see Figure 10, and are analytic extensions of each other for varying $\alpha$. Moreover for any $\Lambda < \Lambda_1 < -\text{Re}(\frac{e^{a+b}i}{2\sqrt{\epsilon}})$ and $A = e^{-ia}\Lambda_1$, $B = -\frac{e^{ia}b}{\sqrt{\epsilon}} - e^{-ia}\Lambda_1$, they are of bounded norm $|B_\alpha^\pm [f]|_{A+B}^{c+e^{a}R}$ on any line $c + e^{a}R \subseteq \Omega_\alpha$.

2) If $\sqrt{\epsilon} \neq 0$ and $a + b > 0$, then for $\xi \in \Omega_\alpha$ (defined in 30)
\[
B_\alpha^+ [(x-\sqrt{\epsilon})^a (x+\sqrt{\epsilon})^b](\xi, \sqrt{\epsilon}) = e^{-\frac{\xi}{2\epsilon}+a\pi i} \cdot (2\sqrt{\epsilon})^{a+b-1} \cdot \frac{1}{2\pi} B(a - \frac{\xi}{2\epsilon}, b + \frac{\xi}{2\epsilon}),
\]
where $B$ is the Beta function.
3) In particular, for a positive integer \( n \), and \( \xi \) in the strip in between 0 and 2n\( \sqrt{\epsilon} \),
\[
B_{\alpha}^\pm [(x-\sqrt{\epsilon})^n](\xi, \sqrt{\epsilon}) = \chi_{\alpha}^\pm(\xi, \sqrt{\epsilon}) \cdot ((\frac{\xi}{n-1}) - 2\sqrt{\epsilon}) \cdot ((\frac{\xi}{m-2}) - 2\sqrt{\epsilon}) \cdot \ldots \cdot (\frac{\xi}{\pi} - 2\sqrt{\epsilon}),
\]
where for \( \sqrt{\epsilon} \neq 0 \) and \( \alpha \in (\arg \sqrt{\epsilon}, \arg \sqrt{\epsilon} + \pi) \)
\[
\chi_{\alpha}^+(\xi, \sqrt{\epsilon}) := \frac{1}{1 - e^{\frac{\pi i}{\sqrt{\epsilon}}}}, \quad \chi_{\alpha}^-(\xi, \sqrt{\epsilon}) := \frac{-1}{1 - e^{-\frac{\pi i}{\sqrt{\epsilon}}}},
\]
and for \( \sqrt{\epsilon} = 0 \)
\[
\chi_{\alpha}^+(\xi, 0) := \begin{cases} 
1, & \text{if } \xi \in (0, +\infty e^{i\alpha}), \\
\frac{1}{2}, & \text{if } \xi = 0, \\
0, & \text{if } \xi \in (-\infty e^{i\alpha}, 0),
\end{cases} \quad \chi_{\alpha}^-(\xi, 0) := \chi_{\alpha}^+(\xi, 0) - 1.
\]
Let us remark that \( \chi_{\alpha}^+(\xi, \nu\sqrt{\epsilon}) \xrightarrow{\nu \to 0} \chi_{\alpha}^+(\xi, 0) \) for \( \xi \in e^{i\alpha} \mathbb{R} \setminus \{0\} \).

4) If \( f(x) \) is analytic on an open disc of radius \( r > 2\sqrt{|\epsilon|} \) centered at \( x_0 = -\sqrt{\epsilon} \) (or \( x_0 = \sqrt{\epsilon} \)) and \( f(x_0) = 0 \), then
\[
B_{\alpha}^\pm [f](\xi, \sqrt{\epsilon}) = \chi_{\alpha}^\pm(\xi, \sqrt{\epsilon}) \cdot \phi(\xi)
\]
where \( \phi \) is an entire function with at most exponential growth at infinity \( \leq e^{\frac{\pi |\xi|}{\sqrt{\epsilon}} \cdot O(\sqrt{|\xi|}) \) for any \( 2\sqrt{|\epsilon|} < R < r \) (where the big \( O \) is uniform for \( (\xi, \sqrt{\epsilon}) \to (\infty, 0) \)).

5) For \( \sqrt{\epsilon} \neq 0 \), \( c \in \mathbb{C} \), the Borel Transform \( B_{\alpha}^\pm \left[ \left( \frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}} \right)^c \right] (\xi, \sqrt{\epsilon}) = \delta_{2c\sqrt{\epsilon}}(\xi) \) acts as translation operator on the Borel plane by \( \xi \mapsto \xi - 2c\sqrt{\epsilon} \):
\[
B_{\alpha}^\pm \left[ \left( \frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}} \right)^c \cdot f \right] (\xi, \sqrt{\epsilon}) = B_{\alpha}^\pm [f](\xi - 2c\sqrt{\epsilon}, \sqrt{\epsilon}).
\]

Remark 17. Although in 1) and 2) of Proposition 16 the function \( f = O((x-\sqrt{\epsilon})^a(x+\sqrt{\epsilon})^b) \), \( a + b > 0 \), might not vanish at both points \( \pm \sqrt{\epsilon} \) as demanded in Definition 15 one can write
\[
(x-\sqrt{\epsilon})^a(x+\sqrt{\epsilon})^b = \left( \frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}} \right)^c (x-\sqrt{\epsilon})^{a-c}(x+\sqrt{\epsilon})^{b+c}, \quad \text{for any } -b < c < a,
\]
hence, using 5) of Proposition 16, the Borel transform $B_\alpha^\pm[f]$ is well defined as the translation by $2c\sqrt{\epsilon}$ of the Borel transform of the function $f \cdot \left( \frac{2\sqrt{\epsilon}}{n+\sqrt{\epsilon}} \right)^{-c}$, this time vanishing at both points:

$$B_\alpha^\pm[f](\xi, \sqrt{\epsilon}) = B_\alpha^\pm \left[ f \cdot \left( \frac{2\sqrt{\epsilon}}{n+\sqrt{\epsilon}} \right)^{-c} \right](\xi - 2c\sqrt{\epsilon}, \sqrt{\epsilon}).$$

Proof of Proposition 16. 1) For $\sqrt{\epsilon} \neq 0$, one can express

$$x - \sqrt{\epsilon} = 2\sqrt{\epsilon} \frac{e^{-2\sqrt{\epsilon}t}}{1 - e^{-2\sqrt{\epsilon}t}}, \quad x + \sqrt{\epsilon} = 2\sqrt{\epsilon} \frac{1}{1 - e^{-2\sqrt{\epsilon}t}}.$$

If $\xi$ is in the strip $\Omega_\alpha$, $\xi \in 2c\sqrt{\epsilon} + e^{i\alpha}\mathbb{R}$ for some $\epsilon [-b, a]$, one writes

$$B_\alpha^\pm[(x - \sqrt{\epsilon})^a(x + \sqrt{\epsilon})^b](\xi, \sqrt{\epsilon}) = \frac{1}{2\pi i} \int_{C^\pm + e^{-\epsilon i}a} (x - \sqrt{\epsilon})^{a-c}(x + \sqrt{\epsilon})^{b+c}e^{i\epsilon(-2c\sqrt{\epsilon})t} dt$$

$$= (2\sqrt{\epsilon})^{a+b} \frac{1}{2\pi i} \int_{C^\pm + e^{-\epsilon i}a} \left( \frac{e^{-2\sqrt{\epsilon}t}}{1 - e^{-2\sqrt{\epsilon}t}} \right)^{a-c} e^{i\epsilon(-2c\sqrt{\epsilon})t} dt.$$  

The term $e^{i\epsilon(-2c\sqrt{\epsilon})t}$ stays bounded along the integration path, while the term $\frac{e^{-2\sqrt{\epsilon}t}}{1 - e^{-2\sqrt{\epsilon}t}}$ decreases exponentially fast as $t - C^\pm \rightarrow +\infty ie^{-i\alpha}$ and $t - C^\pm \rightarrow -\infty ie^{-i\alpha}$, if $\alpha \notin \arg \sqrt{\epsilon} + \pi \mathbb{Z}$.

2) From (25) and (26), one writes

$$B_\alpha^\pm[(x - \sqrt{\epsilon})^a(x + \sqrt{\epsilon})^b](\xi, \sqrt{\epsilon}) = -e^{-\frac{\xi}{2\sqrt{\epsilon}}} \frac{1}{2\pi i} \int_{\gamma_\alpha^+} (\sqrt{\epsilon} - x)^{a-\frac{\xi}{\sqrt{\epsilon}}}(\sqrt{\epsilon} + x)^{b-\frac{\xi}{\sqrt{\epsilon}}} dx$$

$$= -e^{-\frac{\xi}{2\sqrt{\epsilon}}} \left( 2\sqrt{\epsilon} \right)^{a+b-1} \cdot \frac{1}{2\pi i} \int_0^1 (1 - s)^{a-\frac{\xi}{\sqrt{\epsilon}}} s^{b-\frac{\xi}{\sqrt{\epsilon}}} ds,$$

substituting $s = \frac{\sqrt{\epsilon} + x}{2\sqrt{\epsilon}}$. For $\alpha = \arg \sqrt{\epsilon} + \frac{\pi}{2}$, the integration path $\gamma_\alpha^+ (= a$ real trajectory of the vector field $e^{-i\epsilon}\sqrt{\epsilon}(x^2 - \epsilon)\frac{\partial}{\partial x}$) can be chosen as the straight oriented segment $(\sqrt{\epsilon}, -\sqrt{\epsilon})$. The result follows.

3) From 2) using standard formulas.

4) For $x_0 = -\sqrt{\epsilon}$, one can write $f(x)$ as a convergent series $f(x) = \sum_{n=1}^{+\infty} a_n (x + \sqrt{\epsilon})^n$ with $|a_n| \leq CK^n$ for some $C > 0$ and $\frac{1}{2} < K < \frac{1}{2}$. Hence

$$(1 - e^{-\frac{\xi}{\sqrt{\epsilon}}}) \cdot B_\alpha^+[f](\xi, \sqrt{\epsilon}) = \sum_{n=1}^{+\infty} a_n \left( \frac{\xi}{\sqrt{\epsilon}} - 2\sqrt{\epsilon} \right) \cdots \left( \frac{\xi}{\sqrt{\epsilon}} - 2\sqrt{\epsilon} \right) = \sum_{n=1}^{+\infty} b_n(\xi, \sqrt{\epsilon}),$$

where the series on the right is absolutely convergent for any $\xi \in \mathbb{C}$. Indeed, let $N = N(\xi, \sqrt{\epsilon})$ be the positive integer such that

$$\frac{\vert \xi \vert}{N + 1} \leq R - 2\sqrt{\vert \xi \vert} < \frac{\vert \xi \vert}{N},$$

then

$$20$$
for \( n \geq N + 1 \): 
\[
K \cdot \left( \frac{|\xi|}{n} + 2\sqrt{|\xi|} \right) \leq RK,
\]

for \( n \leq N \): 
\[
2\sqrt{|\xi|} < \frac{2\sqrt{|\xi|}}{R - 2\sqrt{|\xi|}} \cdot \frac{|\xi|}{n} \quad \text{and hence} \quad K \cdot \left( \frac{|\xi|}{n} + 2\sqrt{|\xi|} \right) \leq K \frac{|\xi|}{n} \left( 1 + \frac{2\sqrt{|\xi|}}{R - 2\sqrt{|\xi|}} \right) \leq \frac{1}{n} \cdot \frac{|\xi|}{R - 2\sqrt{|\xi|}}.
\]

\[
\sum_{n=1}^{+\infty} |b_n(\xi, \sqrt{\epsilon})| = \sum_{n=0}^{N-1} |b_{n+1}(\xi, \sqrt{\epsilon})| + \sum_{n=N}^{+\infty} |b_n(\xi, \sqrt{\epsilon})|
\]

\[
\leq \sum_{n=0}^{N-1} CK \frac{1}{n!} \left( \frac{|\xi|}{R - 2\sqrt{|\xi|}} \right)^n + CK \frac{1}{N!} \left( \frac{|\xi|}{R - 2\sqrt{|\xi|}} \right)^N \cdot \sum_{n=N}^{+\infty} (RK)^{n-N}
\]

\[
\leq CK e^{-\frac{2\sqrt{|\xi|}}{R}} + CK \cdot \Gamma \left( \frac{|\xi|}{R - 2\sqrt{|\xi|}} \right)^{-1} \left( \frac{|\xi|}{R - 2\sqrt{|\xi|}} \right)^{N+\infty} \cdot \frac{1}{1-RK}
\]

\[
= e^{-\frac{2\sqrt{|\xi|}}{R}} + CK \cdot \frac{1}{1-RK} \sqrt{\frac{|\xi|}{2\pi(R - 2\sqrt{|\xi|})}} + O \left( \sqrt{\frac{|\xi|}{2\pi(R - 2\sqrt{|\xi|})}} \right),
\]

using \([32]\) and the Stirling formula: 
\[
\Gamma(z)^{-1} = \left( \frac{z}{\pi} \right)^{\frac{1}{2}} \left( \sqrt{\frac{2\pi}{z}} + O \left( \frac{1}{\sqrt{z}} \right) \right), \quad z \to +\infty.
\]

5) From the definition. \(\square\)

There is also a converse statement to point 1) of Proposition \([16]\).

**Proposition 18.** Let \( \epsilon \neq 0 \) and \( \alpha \in (\arcsin, \arcsin + \pi) \). If \( \phi(\xi) \) is an analytic function in a strip \( \Omega_\alpha(30) \), with \( a + b > 0 \), such that it has a finite norm \( |\phi|_{2\sqrt{\epsilon}+i\alpha R} \) on each line \( 2c\sqrt{\epsilon} + e^{i\alpha R} \subseteq \Omega_\alpha \), for some \( 0 \leq \Lambda < -\Re \left( \frac{\epsilon^{\alpha}}{2\sqrt{\epsilon}} \right) \) and \( A = e^{-i\alpha} \Lambda \), \( B = -\frac{e^{i\alpha}+1}{\sqrt{\epsilon}} - e^{-i\alpha} \Lambda \), then the unfolded Laplace transform of \( \phi \)

\[
L_\alpha[\phi](x, \sqrt{\epsilon}) = \int_{2c\sqrt{\epsilon}+e^{i\alpha}a}^{2c\sqrt{\epsilon}+e^{i\alpha}a} \phi(\xi)e^{-t(x,\xi)} d\xi, \quad c \in (-b, a)
\]

is analytic on the domain \( \hat{X}_\alpha^\pm(\Lambda, \sqrt{\epsilon}) \), and is uniformly \( o(|x-\sqrt{\epsilon}|^{a_1} |x+\sqrt{\epsilon}|^{b_1}) \) for any \( a_1 < a, b_1 < b \), on any sub-domain \( \hat{X}_\alpha^\pm(\Lambda_1, \sqrt{\epsilon}) \), \( \Lambda_1 > \Lambda \).

**Proof.** This is a reformulation of Corollary \([10]\) which also implies that \( L_\alpha[\phi] \) is \( o \left( \left| \frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}} \right| \right) \) for any \( -b < c < a \). \(\square\)

**Definition 19** (Borel transform of \( x \)). We know form Proposition \([16]\) that for \( \sqrt{\epsilon} \neq 0 \), \( B_\alpha^+[x + \sqrt{\epsilon}] = \chi_\alpha^+ \) in the strip in between \( -2\sqrt{\epsilon} \) and \( 0 \), while \( B_\alpha^+[x - \sqrt{\epsilon}] = \chi_\alpha^- \) in the strip in between \( 0 \) and \( 2\sqrt{\epsilon} \), and the function \( \chi_\alpha^\pm \) has a simple pole at \( 0 \) with residue \( \text{Res}_0 \chi_\alpha^\pm \). Therefore

\[
B_\alpha^+[x + \sqrt{\epsilon}] - B_\alpha^+[x - \sqrt{\epsilon}] = 2\sqrt{\epsilon} \delta_0
\]

in the sense of distributions (see section \([3.0.2]\), where \( \delta_0 \) is the Dirac distribution (identity of convolution). Hence one can define the distribution

\[
B_\alpha^+[x] := B_\alpha^+[x - \sqrt{\epsilon}] + \sqrt{\epsilon} \delta_0 = B_\alpha^+[x + \sqrt{\epsilon}] - \sqrt{\epsilon} \delta_0.
\]
Correspondingly, the convolution of \( B^\pm_\alpha[x] \) with a function \( \phi \), analytic on an open strip containing the line \( e^{i\alpha}R \), is then defined as

\[
[B^\pm_\alpha[x] * \phi]_\alpha(\xi, \sqrt{\epsilon}) = \int_{c_1 + e^{i\alpha}R} \phi(\xi - s) \chi^\pm_\alpha(s, \sqrt{\epsilon}) \, ds + \sqrt{\epsilon} \phi(\xi), \quad c_1 \in (0, 2\sqrt{\epsilon})
\]

\[
= \int_{c_2 + e^{i\alpha}R} \phi(\xi - s) \chi^\pm_\alpha(s, \sqrt{\epsilon}) \, ds - \sqrt{\epsilon} \phi(\xi), \quad c_2 \in (-2\sqrt{\epsilon}, 0).
\]

### 4.1 Remark on Fourier expansions

For \( \sqrt{\epsilon} \neq 0 \), we have defined the Borel transformations \( B^\pm_\alpha \) for directions transverse to \( \sqrt{\epsilon}R \): in fact, we have restricted ourselves to \( \alpha \in (\arg \sqrt{\epsilon}, \arg \sqrt{\epsilon} + \pi) \). Let us now take a look at the direction \( \arg \sqrt{\epsilon} \). So instead of integrating on a line \( c^\pm + ie^{-i\alpha}R \) in the \( t \)-coordinate as in Figure 3, this time we shall consider an integrating path \( c_R + \frac{1}{\sqrt{\epsilon}}R \) in the half plane \( \Re(e^{i\arg \sqrt{\epsilon}t}) > 0 \) (resp. \( c_L + \frac{1}{\sqrt{\epsilon}}R \) in the half plane \( \Re(e^{i\arg \sqrt{\epsilon}t}) < 0 \), see Figure 11). If \( f \) is analytic on a neighborhood of \( x = \sqrt{\epsilon} \) (resp. \( x = -\sqrt{\epsilon} \)), then the lifting of \( f \) to the time coordinate, \( f(x(t, \epsilon)) \), is \( \frac{1}{\sqrt{\epsilon}} \)-periodic in the half-plane \( \Re(e^{i\arg \sqrt{\epsilon}t}) > \Lambda \) (resp. \( \Re(e^{i\arg \sqrt{\epsilon}t}) < -\Lambda \)) for \( \Lambda \) large enough, and can be written as a sum of its Fourier series expansion:

\[
f(x) = \sum_{n=0}^{+\infty} a_n^R e^{-2n\sqrt{\epsilon}t(x)} = \sum_{n=0}^{+\infty} a_n^R \cdot \left( \frac{x - \sqrt{\epsilon}}{x + \sqrt{\epsilon}} \right)^n,
\]

resp.

\[
f(x) = \sum_{n=0}^{+\infty} a_n^L e^{-2n\sqrt{\epsilon}t(x)} = \sum_{n=0}^{+\infty} a_n^L \cdot \left( \frac{x + \sqrt{\epsilon}}{x - \sqrt{\epsilon}} \right)^n.
\]

The Borel transform \( B^R[f](\xi, \sqrt{\epsilon}) \) of \( f(x(t, \epsilon)) \) on the line \( c_R + \frac{1}{\sqrt{\epsilon}}R \) (resp. \( c_L + \frac{1}{\sqrt{\epsilon}}R \)) is equal to the formal sum of distributions

\[
B^R[f](\xi, \sqrt{\epsilon}) := \frac{1}{2\pi i} \int_{c_R + \frac{1}{\sqrt{\epsilon}}R} f(x(t, \epsilon)) \, e^{t\xi} dt = \sum_{n=0}^{+\infty} a_n^R \delta_{2n\sqrt{\epsilon}}(\xi),
\]

resp.

\[
B^L[f](\xi, \sqrt{\epsilon}) := \frac{1}{2\pi i} \int_{c_L - \frac{1}{\sqrt{\epsilon}}R} f(x(t, \epsilon)) \, e^{t\xi} dt = \sum_{n=0}^{+\infty} a_n^L \delta_{-2n\sqrt{\epsilon}}(\xi).
\]

These transformations were studied by Sternin and Shatalov in [SS]. Let us remark that one can connect the coefficients \( a_n^\bullet \) of these expansions to residues of the unfolded Borel transforms \( B^\pm_\alpha \), arg \( \sqrt{\epsilon} < \alpha < \arg \sqrt{\epsilon} + \pi \),

\[
a_0^R = f(\sqrt{\epsilon}), \quad a_n^R = 2\pi i \text{Res}_{2n\sqrt{\epsilon}} B^R_\alpha[f], \quad n \in \mathbb{N}^*,
\]

\[
a_0^L = f(-\sqrt{\epsilon}), \quad a_n^L = 2\pi i \text{Res}_{-2n\sqrt{\epsilon}} B^L_\alpha[f], \quad n \in \mathbb{N}^*.
\]

The residues of \( B^+_\alpha[f] \) and \( B^-_\alpha[f] \) at the points \( \xi \in 2\sqrt{\epsilon} \mathbb{Z} \) are equal.

**Remark 20.** Without providing details, let us remark that one could apply these Borel transformations \( B^R \) (resp. \( B^L \)) to the system \( \text{[12]} \) to show the convergence of its unique local analytic solution at \( x = \sqrt{\epsilon} \neq 0 \) (resp. \( x = -\sqrt{\epsilon} \neq 0 \)) to a Borel sum in direction \( \arg \sqrt{\epsilon} \) of the formal solution \( \hat{y}_0(x) \) of the limit system, when \( \sqrt{\epsilon} \to 0 \) radially in a sector not containing any eigenvalue of \( M(\epsilon) \), as mentioned in Remark 5.
5 Solution to the equation (12) in the Borel plane

We will use the unfolded Borel transformation $B^\pm_\alpha$ to transform the equation
\[(12) : \quad (x^2 - \epsilon) \frac{dy}{dx} = M(\epsilon) y + f(x, y, \epsilon)\]
to a convolution equation in the Borel plane (= the $\xi$-plane), and study its solutions there. We write the function $f(x, y, \epsilon) = O(\|y\|^2) + x O(\|y\|) + (x^2 - \epsilon) O(1)$ as
\[f(x, y, \epsilon) = \sum_{|l| \geq 2} m_l(\epsilon) \tilde{y}^l + x \cdot \sum_{|l| \geq 1} a_l(\epsilon) \tilde{y}^l + (x^2 - \epsilon) \cdot \sum_{|l| \geq 0} g_l(x, \epsilon) \tilde{y}^l, \tag{34}\]
where $\tilde{y}^l := \tilde{y}^l_1 \cdots \tilde{y}^l_m$ for each multi-index $l = (l_1, \ldots, l_m) \in \mathbb{N}^m$, and $|l| = l_1 + \ldots + l_m$.

Let a vector variable $\tilde{y} = \tilde{y}(\xi, \sqrt{\epsilon})$ correspond to the Borel transform $B^\pm_\alpha[y](\xi, \sqrt{\epsilon})$, with $\alpha \in (\arg \sqrt{\epsilon}, \arg \sqrt{\epsilon} + \pi)$ if $\sqrt{\epsilon} \neq 0$. Then the equation (12) is transformed to a convolution equation in the Borel plane
\[\xi \tilde{y} = M(\epsilon) \tilde{y} + \sum_{|l| \geq 2} m_l(\epsilon) \tilde{y}^l + \tilde{h}^0 + \sum_{|l| \geq 1} (a_l \tilde{x}^l + \tilde{h}^l) \ast \tilde{y}^l, \tag{35}\]
where $\tilde{y}^l := \tilde{y}^l_1 \ast \ldots \ast \tilde{y}^l_m$ is the convolution product of components of $\tilde{y}$, each taken $l_i$-times,
\[\tilde{h}^l(\xi, \sqrt{\epsilon}) = B^\pm_\alpha[(x^2 - \epsilon) g_l](\xi, \sqrt{\epsilon}), \]
and $\tilde{x}^l = B^\pm_\alpha[x]$ is the distribution of Definition 19. The convolutions are taken in the direction $\alpha$. Let us remark that by 1) of Proposition 16 the functions $\tilde{h}_l(\xi, \sqrt{\epsilon})$ are analytic in the $\xi$-plane in strips passing in between the points $-2\sqrt{\epsilon}$ and $2\sqrt{\epsilon}$. In Proposition 22 we will find a unique analytic solution $\tilde{y}^\pm(\xi, \sqrt{\epsilon})$ of the convolution equation (35) as a fixed point of the operator
\[G^\pm[\tilde{y}](\xi, \sqrt{\epsilon}) := (\xi I - M(\epsilon))^{-1} \cdot \left( \sum_{|l| \geq 2} m_l \tilde{y}^l + \tilde{h}^0 + \sum_{|l| \geq 1} (a_l \tilde{x}^l + \tilde{h}^l) \ast \tilde{y}^l \right) \tag{36}\]
on a domain $\Omega(\sqrt{\epsilon})$ in the $\xi$-plane, obtained as union of (a bit more narrow) strips $\Omega_\alpha(\sqrt{\epsilon})$ of continuously varying direction $\alpha$, that stay away from the eigenvalues of

Figure 11: The integration paths $c_\bullet + \frac{1}{\sqrt{\epsilon}} R \ (\bullet = L, R)$ in the time $t$-coordinate.
the matrix $M(\epsilon)$ as well as from all the points $\pm 2\sqrt{\epsilon}N^* \ (N^* = \mathbb{N} \setminus \{0\})$; see Figure 12.

In general, several ways of choosing such a domain $\Omega(\sqrt{\epsilon})$ are possible, depending on its position relative with respect to the eigenvalues of $M(\epsilon)$. Different choices of the domain $\Omega(\sqrt{\epsilon})$ will, in general, lead to different solutions $\tilde{y}^\pm(x, \sqrt{\epsilon})$ of (35), as shown in Example 26 below.

**Figure 12:** The regions $\Omega(\sqrt{\epsilon})$ and the eigenvalues $\lambda_1, \ldots, \lambda_m$ (here $m = 3$) of $M(\epsilon)$ in the $\xi$-plane according to $\sqrt{\epsilon} \in S$, together with integration paths $e^{i\alpha}R$ of the Laplace transformation $L_\alpha$.

**Family of regions $\Omega(\sqrt{\epsilon})$ in the Borel plane, parametrized by $\sqrt{\epsilon} \in S$.**

Let $\rho > 0$ be small enough, and let $\beta_1 < \beta_2$ be two directions, such that for $|\sqrt{\epsilon}| < \rho$ none of the closed strips

$$\Omega_\alpha(\sqrt{\epsilon}) = \bigcup_{c \in [-\frac{3}{2}\sqrt{\epsilon}, \frac{3}{2}\sqrt{\epsilon}]} c + e^{i\alpha}\mathbb{R},$$

with $\alpha \in (\beta_1, \beta_2)$, contains any eigenvalue of $M(\epsilon)$. Let $0 < \eta < \frac{1}{2}(\beta_2 - \beta_1) \leq \frac{\pi}{2}$ be an arbitrarliy small angle and define a family of regions $\Omega(\sqrt{\epsilon})$ in the $\xi$-plane depending parametrically on $\sqrt{\epsilon} \in S$ as

$$\Omega(\sqrt{\epsilon}) := \bigcup_\alpha \Omega_\alpha(\sqrt{\epsilon}),$$

where

$$\max\{\arg \sqrt{\epsilon} + \eta, \beta_1\} < \alpha < \min\{\beta_2, \arg \sqrt{\epsilon} + \pi - \eta\},$$
and where $S$ is a sector at the origin in the $\sqrt{\epsilon}$-plane of opening $\pi$, determined by (38),
\[
S = \{ \sqrt{\epsilon} \in \mathbb{C} \mid \arg \sqrt{\epsilon} \in (\beta_1 - \pi + \eta, \beta_2 - \eta), \ |\sqrt{\epsilon}| < \rho \} \cup \{0\}.
\] (39)

We denote $\Omega$ the union of the $\Omega(\sqrt{\epsilon})$, $\sqrt{\epsilon} \in S$, in the $(\xi, \sqrt{\epsilon})$-space
\[
\Omega := \{(\xi, \sqrt{\epsilon}) \mid \xi \in \Omega(\sqrt{\epsilon})\}.
\] (40)

**Definition 21.** Let $\Omega$ be as above, with some $\rho, \eta > 0$, and let $0 \leq \Lambda < \frac{\pi \sin \eta}{2\rho}$. For a vector function $\phi = (\phi_1, \ldots, \phi_m) : \Omega \to \mathbb{C}^m$, we say that it is analytic on $\Omega$, if it is continuous on $\Omega$, analytic on the interior of $\Omega$, and $\phi(\cdot, \sqrt{\epsilon})$ is analytic on $\Omega(\sqrt{\epsilon})$ for all $\sqrt{\epsilon} \in S$. We define the norms
\[
|\phi|^A_{\Omega} := \max_i \sup_{\xi, \alpha} |\phi_i|_{\Omega, \sqrt{\epsilon}(\xi)}, \quad \|\phi\|^A_{\Omega} := \max_i \sup_{\xi, \alpha} \|\phi_i\|_{A, \sqrt{\epsilon}(\xi), \phi},
\]
where $\sqrt{\epsilon} \in S$ and $\alpha$ as in (38), i.e. such that $\Omega(\sqrt{\epsilon}) \subset \Omega(\sqrt{\epsilon})$, and $A_\alpha = e^{-i\alpha} \Lambda$, $B_\alpha = -\pi \epsilon - e^{-i\alpha} \Lambda$.

Let us remark that the convolution of two analytic functions $\phi, \psi$ on $\Omega(\sqrt{\epsilon})$ does not depend on the direction $\alpha$ (38), and that the norms $|\phi * \psi|_{\Omega}^A$, $\|\phi * \psi\|^A_{\Omega}$ satisfy the Young’s inequalities (22) and (23):
\[
|\phi * \psi|_{\Omega}^A \leq \min \{ |\phi|_{\Omega}^A \cdot |\psi|_{\Omega}^A, \ |\phi|_{\Omega}^A \cdot |\psi|_{\Omega}^A \},
\] (41)
\[
\|\phi * \psi\|^A_{\Omega} \leq \|\phi\|^A_{\Omega} \cdot \|\psi\|^A_{\Omega}.
\] (42)

**Proposition 22.** Suppose that the matrix $M(\epsilon)$ and the vector function $f(x, y, \epsilon)$ in the equation (12) are analytic for
\[
x \in X(\Lambda_1, \sqrt{\epsilon}), \quad \sum_{i=1}^m |y_i| < \frac{1}{\Lambda_1}, \quad |\sqrt{\epsilon}| < \rho_1, \quad \text{for } \Lambda_1, L_1, \rho_1 > 0.
\]

Then there exists $\Lambda > \Lambda_1$, $0 < \rho < \rho_1$, and a constant $c > 0$, such that the operator $G^+ : \phi(\xi, \sqrt{\epsilon}) \mapsto G^+[\phi](\xi, \sqrt{\epsilon})$ (36) is well-defined and contractive on the space
\[
\{ \phi : \Omega \to \mathbb{C}^m \mid \phi \text{ is analytic on } \Omega, \ |\phi|_{\Omega}^A \leq c, \ |\phi|_{\Omega}^A < +\infty \}
\]
with respect to both the $|\cdot|_{\Omega}^A$-norm and the $\|\cdot\|_{\Omega}$-norm. Hence the equation $G^+[\ddot{\gamma}^+](\xi, \sqrt{\epsilon})$ possesses a unique analytic solution $\ddot{\gamma}^+(\xi, \sqrt{\epsilon})$ on $\Omega$, satisfying $\|\ddot{\gamma}^+\|_{\Omega}^A \leq c$ and $|\ddot{\gamma}^+|_{\Omega}^A < +\infty$. Similarly, the vector function $\ddot{\gamma}^-(\xi, \sqrt{\epsilon}) := e^{\frac{\xi \epsilon}{2}} \cdot \ddot{\gamma}^+(\xi, \sqrt{\epsilon})$ is a unique analytic solution of the equation $G^-[\ddot{\gamma}^-](\xi, \sqrt{\epsilon}) = \ddot{\gamma}^- \in \Omega$.

To prove this proposition we will need the following technical lemmas which will allow us to estimate the norms of $G^+[\phi]$.

**Lemma 23.** There exists a constant $C = C(\Lambda_1, \eta) > 0$ such that, if $f \in \mathcal{O}(X(\Lambda_1, \sqrt{\epsilon}))$, $|\sqrt{\epsilon}| < \rho$, and $\Lambda_1 < \Lambda < \frac{\pi \sin \eta}{2\rho}$ (where $\eta, \rho > 0$ are as in (38), (39)), then
\[
|B_\alpha^+[(x^2 - \epsilon) f]|_{\Omega}^A \leq C \rho \sup_{x \in X(\Lambda_1, \sqrt{\epsilon})} |f(x)|.
\]

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The statement now follows from the convolution inequalities (41) (resp. (42)).

Proof. By a straightforward estimation. Essentially, we need to estimate the integral
\[ \int_{\mathbb{R}(e^{\alpha}t)=\Lambda} \left| \frac{\sqrt{\pi} e^{\alpha t}}{\sqrt{1+t^2}} \right| \, dx, \]
with \( c \in \left[ -\frac{3}{4}, \frac{3}{4} \right] \) and \( \alpha \in (\arg \sqrt{\epsilon} + \eta, \arg \sqrt{\epsilon} + \pi - \eta). \)

Lemma 24. Let \( \phi \) be an analytic function on \( \Omega \) with a finite \( ||\phi||^A_{\Omega} \) (resp. \( ||\phi||^A_{\Omega} \)). Then its convolution with the distribution \( \tilde{x}^\pm \) (Definition 19) is again an analytic function on \( \Omega \) whose norm satisfies

\[ ||\tilde{x}^\pm \ast \phi||^A_{\Omega} \leq ||\phi||^A_{\Omega} \cdot (\rho + ||\chi^\pm||^A_{\Omega}), \tag{43} \]

resp.

\[ ||\tilde{x}^\pm \ast \phi||^A_{\Omega} \leq ||\phi||^A_{\Omega} \cdot (\rho + ||\chi^\pm||^A_{\Omega}), \tag{44} \]

where \( \chi^\pm \) is given in (31), \( \rho \) is the radius of \( S \), and

\[ \Omega_L(\sqrt{\epsilon}) = \Omega(\sqrt{\epsilon}) \cap (\Omega(\sqrt{\epsilon}) - 2\sqrt{\epsilon}), \quad \text{for each } \sqrt{\epsilon} \in S. \tag{45} \]

Proof. It follows from Definition 19 and 2,\( \sqrt{\epsilon} \)-periodicity of \( \chi^\pm \).

Lemma 25. If \( \phi, \psi : \Omega \to \mathbb{C}^m \) are analytic vector functions such that \( ||\phi||^A_{\Omega}, ||\psi||^A_{\Omega} \leq a \), then for any multi-index \( l \in \mathbb{N}^m, |l| \geq 1, \)

\[ ||\phi^\ast l - \psi^\ast l||^A_{\Omega} \leq |l| \cdot a^{|l|-1} \cdot ||\phi - \psi||^A_{\Omega}. \]

The same holds for the \( || \cdot ||^A_{\Omega} \)-norm as well.

Proof. Writing \( \phi^\ast l = \phi_{i_1} \ast \ldots \ast \phi_{i_{|l|}}, i_j \in \{1, \ldots, m\} \), we have

\[ \phi^\ast l - \psi^\ast l = (\phi_{i_1} - \psi_{i_1}) \ast \ldots \ast (\phi_{i_{|l|}} - \psi_{i_{|l|}}) + \psi_{i_1} \ast (\phi_{i_2} - \psi_{i_2}) \ast \ldots \ast \phi_{i_{|l|}} + \ldots + \psi_{i_{|l|-1}} \ast (\phi_{i_{|l|}} - \psi_{i_{|l|}}). \]

The statement now follows from the convolution inequalities (41) (resp. (42)).

Proof of Proposition 22. Let \( m_1(\epsilon), a_1(\epsilon), g_l(x, \epsilon) \) be as in (34). If \( L > m \cdot L_1 \), then there exists \( K > 0 \) such that for each multi-index \( l \in \mathbb{N}^m \)

\[ \max\{||m_1(\epsilon)||, ||a_1(\epsilon)||, ||g_l(x, \epsilon)||\} \leq K \cdot (\frac{|l|}{l}) L^{|l|}, \]

where for \( y \in \mathbb{C}^m, ||y|| = \sum_{i=1}^m |y_i|, \) and where \( (\frac{|l|}{l}) \) are the multinomial coefficients given by \( (y_1 + \ldots + y_m)^k = \sum_{|l|=k} (\frac{|l|}{l}) y^l, \) satisfying

\[ \sum_{|l|=k} (\frac{|l|}{l}) = m^k. \]

It follows from Lemma 23, Lemma 24 and Lemma 9 that if \( \Lambda > \Lambda_1 \), then the terms of (35) can be bounded by

\[ ||\tilde{h}^\pm \ast l||^A_{\Omega} \leq K_1, \quad ||a_i \tilde{x}^\pm + \tilde{h}^\pm \ast l||^A_{\Omega} \leq K_1 \cdot (\frac{|l|}{l}) L^{|l|}, \]

for some \( K_1 > 0 \). Moreover, if we take \( \Lambda \) sufficiently large and \( \rho \) sufficiently small, then we can make the constant \( K_1 \) small enough so that it satisfies (46) below.

Let

\[ \delta = \max_{(\xi, \sqrt{\epsilon}) \in \Omega} \left\| (I - M(\epsilon))^{-1} \left( \begin{array}{c} 1 \\ \frac{1}{l} \end{array} \right) \right\| + \frac{1}{10mKL}. \]

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then $\delta < +\infty$ if the radius $\rho$ of $S$ is small, and let

$$c = \frac{1}{50m^2KL^2} \leq \frac{1}{5mL}, \quad \text{and} \quad K_1 \leq (5cL)^2K \leq (5cL)K. \quad (46)$$

First we show that $\|\phi\|_\Omega \leq c$ implies $\|G^+[\phi]\|_\Omega \leq c$:

$$\|G^+[\phi]\|_\Omega \leq \delta \cdot \left( \sum_{k=2}^{+\infty} \sum_{|l|=k} K(l) L|l| c^{|l|} + K_1 \sum_{k=2}^{+\infty} \sum_{|l|=k} K_1(l) L^{|l|} c^{|l|} \right)$$

$$\leq \delta \cdot \left( K \sum_{k=2}^{+\infty} m^k L^k c^k + K_1 \sum_{k=0}^{+\infty} m^k L^k c^k \right)$$

$$\leq \delta \cdot \left( (5cL)^2 \sum_{k=2}^{+\infty} \frac{1}{5^k} + K_1 \sum_{k=0}^{+\infty} \frac{1}{5^k} \right) \leq c \cdot \left( \frac{1}{2} + \frac{5}{4} \right) \leq c,$$

using (23) and (46); in the first sum of the second line, we have $c^k \leq (5cL)^2$ since $k \geq 2$. Similarly, $\|G^+[\phi]\|_\Omega \leq \max\{c, \|\phi\|_\Omega\}$ if $\|\phi\|_\Omega \leq c$.

Now we show that $|G^+[\phi] - G^+[\psi]|_\Omega \leq \frac{1}{2}\|\phi - \psi\|_\Omega$ if $\|\phi\|_\Omega, \|\psi\|_\Omega \leq c$. Using Lemma 25 (46) and the convolution inequality (41), we can write

$$\frac{|G^+[\phi] - G^+[\psi]|_\Omega}{\|\phi - \psi\|_\Omega} \leq \delta \cdot \left( \sum_{k=2}^{+\infty} \sum_{|l|=k} K(l) L^k \cdot k \cdot c^{k-1} + \sum_{k=1}^{+\infty} \sum_{|l|=k} K_1(l) L^k \cdot k \cdot c^{k-1} \right)$$

$$\leq \delta \cdot \left( (5cL)KmL \sum_{k=2}^{+\infty} \frac{k}{5^{k-1}} + K_1 mL \sum_{k=1}^{+\infty} \frac{k}{5^{k-1}} \right)$$

$$\leq 5cm^2 \delta KL^2 \cdot \left( \frac{9}{16} + \frac{25}{16} \right) \leq \frac{1}{2}.$$ 

The same holds for the $\| \cdot \|_\Omega$-norm. Hence the operator $G^+$ is $\| \cdot \|_\Omega$-contractive, and the sequence $(G^+[\phi])^n[0]$ converges, as $n \to +\infty$, $\| \cdot \|_\Omega$-uniformly to an analytic function $\tilde{y}^+$ satisfying $G^+[\tilde{y}^+] = \tilde{y}^+$.

From (26) it follows that $\tilde{h}^- = e^{\frac{\varepsilon u}{x}} \cdot \tilde{h}^+$ and $\tilde{x}^- = e^{\frac{\varepsilon u}{x}} \cdot \tilde{x}^+$, hence $G^-[\tilde{y}^-] = G^-[e^{\frac{\varepsilon u}{x}} \tilde{y}^+] = e^{\frac{\varepsilon u}{x}} \cdot G^+[\tilde{y}^+] = e^{\frac{\varepsilon u}{x}} \cdot \tilde{y}^+$ is a fixed point of $G^-$.

The following example shows that the solutions $\tilde{y}^+ \pm$ of the convolution equation (35) in the Borel plane depend on the choice of the domain $\Omega$.

**Example 26.** Let $u$ satisfy

$$(x^2 - \varepsilon) \frac{du}{dx} = u + (x^2 - \varepsilon), \quad (47)$$

and let $y = (x^2 - \varepsilon)u$. It satisfies a differential equation

$$(x^2 - \varepsilon) \frac{dy}{dx} = y + 2xy + (x^2 - \varepsilon)^2. \quad (48)$$

The Borel transform of the equation (47) is

$$\xi \tilde{u}^+_\alpha = \tilde{u}^+\alpha + \xi \cdot \chi^+\alpha, \quad 27$$
therefore \( \tilde{u}_\alpha^+(\xi, \sqrt{\epsilon}) = \xi^{-1} \chi_\alpha^+(\xi, \sqrt{\epsilon}) \), which is independent of the direction \( \alpha \). This is no longer true for the solution \( \tilde{y}_\alpha^+ = \tilde{u}_\alpha^+ * B^\pm_\alpha [x^2 - \epsilon] \) of the Borel transform of the equation (45)

\[
\xi \tilde{y}_\alpha^+ = \tilde{y}_\alpha^+ + 2x^\pm \tilde{y}_\alpha^+ + \chi_\alpha^+ (\xi^3 - 4\epsilon \xi).
\]

If, for instance, \( \text{Im}(\sqrt{\epsilon}) < 0 \), and \( \arg \sqrt{\epsilon} < \alpha_1 < 0 < \alpha_2 < \arg \sqrt{\epsilon} + \pi \), then the strips \( \Omega_{\alpha_1}(\sqrt{\epsilon}), \Omega_{\alpha_2}(\sqrt{\epsilon}) \) in directions \( \alpha_1, \alpha_2 \), are separated by the point \( \xi = 1 \), and one easily calculates that for \( \xi \in \Omega_{\alpha_1}(\sqrt{\epsilon}) \cap \Omega_{\alpha_2}(\sqrt{\epsilon}) \)

\[
\tilde{y}_{\alpha_1}^+ (\xi, \sqrt{\epsilon}) - \tilde{y}_{\alpha_2}^+ (\xi, \sqrt{\epsilon}) = (\xi - 1) \chi_\alpha^+ (1, \sqrt{\epsilon}) \chi_\alpha^+ (\xi - 1, \sqrt{\epsilon}),
\]

i.e. the two solutions \( \tilde{y}_{\alpha_1}^+, \tilde{y}_{\alpha_2}^+ \) differ near \( \xi = 0 \) by a term that is exponentially flat in \( \sqrt{\epsilon} \).

**Proposition 27.** If the spectrum of \( M(0) \) is of Poincaré type, i.e. if it is contained in a sector of opening \( < \pi \), then, for small \( \sqrt{\epsilon} \), the region \( \Omega(\sqrt{\epsilon}) \) may be chosen so that it has all the eigenvalues of \( M(\epsilon) \) on the same side—let’s say the side where \( 2\sqrt{\epsilon} \) is. In such case, let \( \Omega_1(\sqrt{\epsilon}) \) be the extension of \( \Omega(\sqrt{\epsilon}) \) to the whole region on the opposite side (see Figure 13). The solutions \( \tilde{y}^\pm(\xi, \sqrt{\epsilon}) \) of Proposition 22 can be analytically extended to \( \Omega_1(\sqrt{\epsilon}) \setminus (-2\sqrt{\epsilon})N^* \) with at most simple poles at the points \(-2\sqrt{\epsilon}N^* \) (where \( N^* = N \setminus \{0\} \)). The function \( \tilde{y}_\alpha^+ \) is analytic in \( \Omega_1 \) and has at most exponential growth \( < Ce^{\Lambda|\xi|} \) for some \( \Lambda, C > 0 \) independent of \( \sqrt{\epsilon} \).

**Proof.** The solution \( \tilde{y}^+ \) is constructed as a limit of the iterative sequence of functions \( (G^+)^n[0] \), \( n \to +\infty \). We will show by induction that for each \( n \), the function \( (G^+)^n[0] \) is analytic on \( \Omega_1 \setminus \{\xi \in -2\sqrt{\epsilon}N^*\} \) and has at most simple poles at the points \( \xi \in -2\sqrt{\epsilon}N^* \), and that the sequence converges uniformly to \( \tilde{y}^+ \) with respect to the norm

\[
\| \phi \|_{\Omega_1} := \sup_{(\xi, \sqrt{\epsilon}) \in \Omega_1} |\phi(\xi, \sqrt{\epsilon})| e^{-\Lambda|\xi|}.
\]

To do so we will introduce another norm \( | \cdot |_{\Omega_1} \), defined in (53) below, such that the two norms satisfy convolution inequalities similar to those satisfied by \( | \cdot |_{\Omega_1}^A \) and \( \| \cdot \|_{\Omega_1} \) (Lemma 28 below). Then one can simply replicate the proof of Proposition 22 with the norm \( | \cdot |_{\Omega_1}^A \) in place of \( | \cdot |_{\Omega_1} \) and the norm \( \| \cdot \|_{\Omega_1}^A \) in place of \( \| \cdot \|_{\Omega_1} \).

Let us first show that if \( \phi, \psi \) are two functions analytic on \( \Omega_1(\sqrt{\epsilon}) \setminus (-2\sqrt{\epsilon})N^* \), then so is their convolution \( \phi * \psi \). If \( \xi \in \Omega_1(\sqrt{\epsilon}) \setminus \sqrt{\epsilon}R \), then the analytic continuation of \( \phi * \psi \) at the point \( \xi \) is given by the integral

\[
(\phi * \psi)(\xi) = \int_{\Gamma_\xi} \phi(s) \psi(\xi - s) \, ds
\]

with \( \Gamma_\xi \) a symmetric path with respect to the point \( \frac{\xi}{2} \) passing through the segments \([-3\sqrt{\epsilon}, \frac{3\sqrt{\epsilon}}{2}] \) and \([\xi - 3\sqrt{\epsilon}, \xi + \frac{3\sqrt{\epsilon}}{2}] \), as in Figure 14. Note that when \( \xi \) approaches a point on \((-\infty, -2\sqrt{\epsilon}) \setminus (-2\sqrt{\epsilon}N^*) \) from one side or another, the values of the two integrals are identical, since both paths \( \Gamma_\xi \) pass in between the same singularities.

Suppose now that \( \phi, \psi \) have at most simple poles at the points \(-2\sqrt{\epsilon}N^* \). If \( \xi \) is in \( \Omega(\sqrt{\epsilon}) \cup 2\Omega_L(\sqrt{\epsilon}) \) (\( \Omega_L \) is defined in (45)), then \( \Gamma_\xi = c + e^{iaR} \) for some \( c \in
Figure 13: The extended regions $\Omega_1(\sqrt{\epsilon})$ in the Borel plane, together with the modified integration path $\Gamma$ of the Laplace transform (compare with Figure 12). The limit region $\Omega_1(0) := \bigcup_{\sqrt{\epsilon} \in S} \Omega_1(\sqrt{\epsilon}) \cap (-2\sqrt{\epsilon})N^*$ is composed of two sectors connected at the origin; the solution $\tilde{y}^+(\xi, 0)$ vanishes on the lower sector, while the solution $\tilde{y}^-(\xi, 0)$ vanishes on the upper one.

$[-\frac{3}{2}\sqrt{\epsilon}, -\frac{1}{2}\sqrt{\epsilon}] \subset \Omega_L(\sqrt{\epsilon})$. Else $\xi \in 2\Omega_L(\sqrt{\epsilon}) - 2k\sqrt{\epsilon}$ for some $k \in \mathbb{N}^*$, and one can express the convolution as

$$
(\phi \ast \psi)(\xi) = \int_{c - 2k\sqrt{\epsilon} + e^{iaR}} \phi(s) \psi(\xi - s) \, ds + 2\pi i \sum_{j = 1}^{k} \text{Res}_{-2j\sqrt{\epsilon}} \phi \cdot \psi(\xi + 2j\sqrt{\epsilon})
$$

$$
= \int_{c + e^{iaR}} \phi(t - 2k\sqrt{\epsilon}) \psi(\xi_0 - t) \, dt - 2\sqrt{\epsilon} \sum_{j = 0}^{k - 1} \frac{\phi}{\chi^+(-2(k - j)\sqrt{\epsilon})} \cdot \psi(\xi_0 - 2j\sqrt{\epsilon}),
$$

(50)

where $c \in [-\frac{3}{2}\sqrt{\epsilon}, -\frac{1}{2}\sqrt{\epsilon}] \subset \Omega_L(\sqrt{\epsilon})$ and $\xi_0 = \xi + 2k\sqrt{\epsilon} \in c + \Omega_L(\sqrt{\epsilon})$, i.e. $\xi - s \in \Omega_L(\sqrt{\epsilon})$, see Figure 14. We will use this formula to obtain an estimate for the norm $\|\phi \ast \psi\|_{\Omega_1}^\Lambda$, $\Lambda \geq 0$. Since $\left|\frac{1}{\chi^+}(\xi, \sqrt{\epsilon})\right| \leq (1 + |e^{\frac{\pi i}{\sqrt{\epsilon}}}|)(1 + |e^{\frac{\pi i}{\sqrt{\epsilon} + \sqrt{\epsilon}}}|)$, cf. (21), we have

$$
\left|\frac{\phi \ast \psi}{\chi^+}(\xi)\right| e^{-\Lambda |\xi|} \leq \sup_{s \in \Omega_L(\sqrt{\epsilon}) - 2k\sqrt{\epsilon}} |\phi(s)|(1 + |e^{\frac{\pi i}{\sqrt{\epsilon} + \sqrt{\epsilon}}}|) e^{-\Lambda |s|} \cdot \left|\psi\|_{\Omega_L(\sqrt{\epsilon})}^\Lambda
$$

$$
+ 2\sqrt{\epsilon} \cdot \|\phi\|_{\Omega_1}^\Lambda \sum_{j = 0}^{k - 1} \left|\frac{\psi}{\chi^+}(\xi_0 - 2j\sqrt{\epsilon})\right| e^{-\Lambda |\xi_0 - 2j\sqrt{\epsilon}|},
$$

(51)

due to the $2\sqrt{\epsilon}$-periodicity of $\chi^+$. 

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Let $\mu \geq 1$ be such that
\[ 1 + |e^{s\sqrt{\epsilon}}| \leq \mu \frac{1}{\chi^r(s, \sqrt{\epsilon})} \quad \text{for all } s \in \Omega_L(\sqrt{\epsilon}) \quad (52) \]
and define
\[
|\psi|^\Lambda_{\Omega_1} := \mu \|\psi\|_{\Omega_1} + \sup_{\sqrt{\epsilon} \in S} 2\sqrt{|\epsilon|} \sum_{k=0}^{+\infty} \frac{|\psi|}{\chi}(\xi - 2k\sqrt{\epsilon}) |e^{-\Lambda|\xi - 2k\sqrt{\epsilon}|}| \quad (53)
\]
Then \[(51)\] implies that
\[
\|\phi \ast \psi\|^\Lambda_{\Omega_1} \leq \|\phi\|_{\Omega_1} \cdot |\psi|^\Lambda_{\Omega_1}.
\]
Note that by 4) of Proposition 16, if $f(\frac{x}{\sqrt{\epsilon}})$ is analytic on $\{(|x^2 - \epsilon| < r^2) \times \{|\epsilon| < \rho^2\}$ for some $r > 2\rho > 0$, then for any $\Lambda > \frac{1}{\sqrt{r - 2\rho}}$
\[
\|B_\Lambda[f]\|_{\Omega_1} \leq +\infty, \quad \|B_\Lambda^+[f]\|_{\Omega_1} \leq +\infty,
\]
and one can see that $|B_\Lambda[f]|_{\Omega_1}$ can be made arbitrarily small taking $\Lambda$ sufficiently large (cf. Lemma 9).

**Lemma 28.** Let $\frac{\phi}{\sqrt{\epsilon}}, \frac{\psi}{\sqrt{\epsilon}}$ be analytic functions on $\Omega_1$ such that $\frac{\phi}{\sqrt{\epsilon}}(0, \sqrt{\epsilon}) = \frac{\psi}{\sqrt{\epsilon}}(0, \sqrt{\epsilon}) = 0$. Then
\[
|\phi \ast \psi|^\Lambda_{\Omega_1} \leq \|\phi\|^\Lambda_{\Omega_1} \cdot |\psi|^\Lambda_{\Omega_1},
\]
\[
|\phi \ast \psi|^\Lambda_{\Omega_1} \leq \|\phi\|^\Lambda_{\Omega_1} \cdot |\psi|^\Lambda_{\Omega_1}.
\]

**Proof.** The first inequality is given in the proof of Lemma 27. We need to prove the second one. By definition
\[
|\phi \ast \psi|^\Lambda_{\Omega_1} = \mu \|\phi \ast \psi\|_{\Omega_1} + \sup_{\sqrt{\epsilon} \in S} 2\sqrt{|\epsilon|} \sum_{k=0}^{+\infty} \frac{|\phi \ast \psi|}{\chi}(\xi - 2k\sqrt{\epsilon}) |e^{-\Lambda|\xi - 2k\sqrt{\epsilon}|}|.
\]
The first term is smaller than
\[ \mu \|\phi\|_\Omega^A \|\psi\|_\Omega^A \leq \mu^2 \|\phi\|_\Omega^A \|\psi\|_\Omega^A \quad \text{since} \quad \mu \geq 1. \]

For the second term, using (50), (52) and (35) of Proposition 22 on \( \Omega \) with bounded \( \|\cdot\| \) indicated in Figure 13, and use the Cauchy formula to express

\[ \sum_{k=0}^{+\infty} |\phi_{x^k}(\xi - 2k\sqrt{\epsilon})| e^{-\Lambda|\xi - 2k\sqrt{\epsilon}|} \]
\[ \leq \int_{c+te^{i\alpha}} e^{-\Lambda|t-2k\sqrt{\epsilon}|} \cdot \left| |\psi(\xi - t)| \right| (1 + |e^{\frac{(\xi-t)\epsilon}{\sqrt{\epsilon}}}|) e^{-\Lambda|\xi - t|} \, dt + \]
\[ + 2\sqrt{|\epsilon|} \sum_{k=0}^{+\infty} \sum_{j=1}^{+\infty} |\psi_{x^j}(\xi - 2j\sqrt{\epsilon})| e^{-\Lambda|2j\sqrt{\epsilon}|} \cdot |\psi_{x^j}(\xi - 2k - j\sqrt{\epsilon})| e^{-\Lambda|\xi - 2(k - j)\sqrt{\epsilon}|} \]
\[ \leq \sup_{\xi \in \Omega \Upsilon(\sqrt{\epsilon})} \sum_{k=0}^{+\infty} |\phi_{x^k}(\xi - 2k\sqrt{\epsilon})| e^{-\Lambda|\xi - 2k\sqrt{\epsilon}|} \cdot \left( \mu \|\psi\|_\Omega^A + \sup_{\xi \in \Omega \Upsilon(\sqrt{\epsilon})} 2\sqrt{|\epsilon|} \sum_{j=0}^{+\infty} |\psi_{x^j}(\xi - 2j\sqrt{\epsilon})| e^{-\Lambda|\xi - 2j\sqrt{\epsilon}|} \right). \]

\[ \square \]

**Proof of Theorem 3.** i) Let \( \tilde{y}^\pm(\xi, \sqrt{\epsilon}) \) be the solution of the convolution equation (35) of Proposition 22 on \( \Omega \) with bounded \( \|\cdot\|_\Omega^A \)-norm. Its Laplace transform

\[ y^\pm(x, \sqrt{\epsilon}) := L[y^\pm](x, \sqrt{\epsilon}) = \int_{-\infty}^{+\infty e^{i\alpha}} \tilde{y}^\pm(\xi, \sqrt{\epsilon}) e^{-t(x,\epsilon)\xi} \, d\xi, \quad (54) \]

where \( \alpha \) can vary as in (38), is a solution of (34) defined for \( x \in \bigcup \alpha \mathcal{A}_\alpha^\pm(\Lambda, \sqrt{\epsilon}), \) (see Figure 3 for the domain of convergence in the time \( t(x) \)-coordinate). Both \( y^+ \) and \( y^- \) project to the same ramified solution on a domain \( Z(\sqrt{\epsilon}) \) in the \( x \)-plane (Figure 2).

ii) If the spectrum of \( M(0) \) is of Poincaré type and \( \tilde{y}^\pm(\xi, \sqrt{\epsilon}) \) is defined on \( \Omega_1 \) as in Proposition 27 with \( \|\tilde{y}^\pm\|_{\Omega_1} < +\infty, \) then, for \( x \in Z_1(\sqrt{\epsilon}) \cap \{\text{Re}(e^{i\arg\sqrt{\epsilon}t(x,\epsilon)}) < -\Lambda\}, \) one may deform the integration path of the Laplace transform (54) to \( \Gamma, \) indicated in Figure 13 and use the Cauchy formula to express \( y^\pm(x, \sqrt{\epsilon}), \) for \( \sqrt{\epsilon} \neq 0, \)

\[ y^\pm(x, \sqrt{\epsilon}) = \int_{\Gamma} \tilde{y}^\pm(\xi, \sqrt{\epsilon}) e^{-t(x,\epsilon)\xi} \, d\xi = 2\pi i \sum_{k=1}^{\infty} \text{Res}_{-2k\sqrt{\epsilon}} \tilde{y}^\pm \cdot \left( \frac{x + \sqrt{\epsilon}}{x - \sqrt{\epsilon}} \right)^k \]
\[ = -2\sqrt{\epsilon} \sum_{k=1}^{\infty} \left( \frac{y^\pm}{x^k} \right)(-2k\sqrt{\epsilon}, \sqrt{\epsilon}) \cdot \left( \frac{x + \sqrt{\epsilon}}{x - \sqrt{\epsilon}} \right)^k. \quad (55) \]

This series is convergent for \( \|\frac{x + \sqrt{\epsilon}}{x - \sqrt{\epsilon}}\| < e^{-2\sqrt{\epsilon}\Lambda}, \) and its coefficients are the same in both cases \( \tilde{y}^+ \) and \( \tilde{y}^- \). It defines a solution \( y_1(x, \sqrt{\epsilon}) \) of (12) on a domain \( Z_1(\sqrt{\epsilon}), \) analytic at \( x = -\sqrt{\epsilon} \) and ramified at \( x = \sqrt{\epsilon} \) (Figure 4). \( \square \)
5.1 Concluding remark

The article shows how to unfold the analytic Borel and Laplace transformations and to apply them to study the center manifold of an unfolded codimension 1 complex saddle-node singularity. It allows to generalize some of the classical Borel–Laplace methods to the unfolded parametric situation. But there is also an important difference between the two situations, that yet needs to be better understood. In the classical (non-unfolded) case, for \( \epsilon = 0 \), the transformed equation (35) in the Borel plane has a unique analytic solution: its germ at \( \xi = 0 \) is the Borel transform of the unique formal solution \( \hat{y}_0(x) = \hat{y}(x, 0) \) (13), which extends analytically to a ramified solution on the whole Borel plane. On the other hand, when one unfolds, this is no longer true: the analytic solution \( \tilde{y}^\pm \) of (35) on the region \( \Omega(\sqrt{\epsilon}) \) depends on the position of the region with respect to the singular points \( \xi \in \text{Spec } M(\epsilon) \). The question of what is the relation between the solutions on different domains \( \Omega \) remains yet to be answered. (We can expect that the difference of their germs at \( \xi = 0 \) is exponentially flat in \( \sqrt{\epsilon} \).)

Another related problem that deserves a further investigation is: In what sense can the unfolded sectoral solutions provided by Theorem 3 be considered as “sums” of the divergent formal series \( \hat{y}(x, \epsilon) \) (13)?

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