IN Variant DISTRIBUTIONS FOR HOMOGENEOUS FLOWS
AND AFFINE TRANSFORMATIONS

LIVIO FLAMINIO, GIOVANNI FORNI AND FEDERICO RODRIGUEZ HERTZ
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ABSTRACT. We prove that every homogeneous flow on a finite-volume homogeneous manifold has countably many independent invariant distributions unless it is conjugate to a linear flow on a torus. We also prove that the same conclusion holds for every affine transformation of a homogeneous space which is not conjugate to a toral translation. As a part of the proof, we have that any smooth partially hyperbolic flow on any compact manifold has countably many distinct minimal sets, hence countably many distinct ergodic probability measures. As a consequence, the Katok and Greenfield-Wallach conjectures hold in all of the above cases.

1. INTRODUCTION

A smooth flow \( \phi_t \) generated by a smooth vector field \( X \) on a compact manifold \( M \) is called \textit{stable} if the range of the Lie derivative \( \mathcal{L}_X : C^\infty(M) \to C^\infty(M) \) is closed, and it is called \textit{cohomology-free} or \textit{rigid} if it is stable and the range of the Lie derivative operator has codimension one. For a smooth diffeomorphism \( f \) on \( M \), stability and rigidity are analogously defined by considering the range of the operator \( f^* - I : C^\infty(M) \to C^\infty(M) \). The properties of stability and rigidity are easily seen to be invariant under smooth conjugacies, in the case of either flows or diffeomorphisms, and under smooth reparametrizations in the case of flows.

The Katok (or Katok-Hurder) conjecture [26, 27, 24] for flows states that every cohomology-free smooth flow is smoothly conjugate to a linear flow on a torus with Diophantine frequencies. It is not hard to prove that all cohomology-free flows are volume preserving and uniquely ergodic (see for instance [19]). An analogous conjecture can be stated for smooth diffeomorphisms.

We also recall that the Katok conjecture for flows is equivalent to the Greenfield-Wallach conjecture [23] stating that every globally hypoelliptic vector field is smoothly conjugate to a Diophantine linear flow (see [19]). A smooth vector field \( X \) is called \textit{globally hypoelliptic} if any 0-dimensional current \( U \) on \( M \) is smooth under the condition that the current \( \mathcal{L}_X U \) is smooth. Greenfield and Wallach in [23] proved this conjecture for smooth flows on compact surfaces, for homogeneous flows in dimension 3, and for homogeneous flows on compact
Lie groups in all dimensions. (The equivalence of the Katok and Greenfield-Wallach conjectures was essentially proved already in [7] as noted by the third author of this paper. The details of the proof can be found in [19]).

The best general result to date in the direction of a proof is the joint paper of the third author [40], where it is proved that every cohomology-free vector field has a factor smoothly conjugate to a Diophantine linear flow on a torus of dimension equal to the first Betti number of the manifold $M$. This result has been developed independently by several authors [19, 30, 34] to give a complete proof of the conjecture in dimension 3 and by the first author in the joint paper [20] to prove that every cohomology-free flow can be embedded continuously as a linear flow in a possibly non-separated Abelian group.

From the definition, it is clear that there are two main mechanisms which may prevent a smooth flow from being cohomology-free: it can happen that the flow is not stable or it can happen that the closure of its range has codimension higher than one (or both).

Examples of stable flows or diffeomorphisms with range of infinite codimension have been known for a long time: geodesic flows on manifolds of negative curvature and in general transitive Anosov flows and diffeomorphisms are perhaps the oldest. In this case, by the Livsic theorem [32], the joint kernel of all invariant measures carried by periodic orbits coincides with the closure of the range of the Lie derivative operator in the H"older category. The Livsic theorem has been generalized to the smooth or analytic category in several cases [8, 21, 13]. For partially hyperbolic diffeomorphisms A. Katok and A. Kono- nenko [28] introduced a family of continuous functionals on $C^\alpha$ functions, the Periodic Cycle Functionals (PCF’s). They showed that under a local accessibility condition a $C^\alpha$ function is a coboundary, modulo constants, if and only if it belongs to the kernel of all PCF’s. A. Wilkinson [45] generalized this result to accessible partially hyperbolic systems. Hence these systems are $C^\alpha$ stable. To the best of our knowledge there is no proof in the literature that PCF’s span an infinite dimensional space or even that they are always non-trivial. By an elegant argument suggested by A. Katok in a personal communication, it is possible to derive that PCF’s span an infinite dimensional space of distributions under the condition that the system is not uniquely ergodic and that finite linear combinations of PCF’s are never measures. In section 2 we show that by Wilkinson’s work [45] under the accessibility condition, Theorem 2.1 implies as an easy corollary that the Periodic Cycle Functionals span an infinite dimensional space of continuous functionals on $C^\alpha(M)$ (see Corollary 2.3) for all partially hyperbolic $C^1$ diffeomorphisms. A similar statement holds for flows (see Corollary 2.5). The proof depends on an extension of Wilkinson’s theorem on solutions of the cohomological equation to flows. To the best of our knowledge such an extension is not in the literature; however, it follows quite easily from Wilkinson’s results for maps (see Theorem 2.4).

We also note that generalizations of Livsic theorem in the non-accessible case are due to Veech [44] and Dolgopyat [14].
Among non-hyperbolic systems, in fact among systems of parabolic type, linear toral skew-shifts [26], translation flows [18, 35], horocycle flows [16] and nilflows [17] are in general uniquely ergodic and stable but have range of countable codimension in the smooth category. Translation flows are a special case as the range closure is finite dimensional in all spaces of finitely differentiable functions [18].

Flows and diffeomorphisms on compact manifolds with range closure of codimension one in the space of smooth functions are called distributionally uniquely ergodic (DUE) (see, for instance, [1]). The motivation for this terminology comes from the fact that DUE flows or diffeomorphisms can also be defined by the condition that they are uniquely ergodic and that the space of all invariant distributions is spanned by the unique invariant probability measure. We recall that an invariant distribution for a flow or a diffeomorphism is a distribution (in the sense of S. Sobolev and L. Schwartz) that is invariant under the natural action by push-forward under the flow or diffeomorphism on the space of distributions. In the case of flows an equivalent definition requires that the Lie derivative of the distribution along the flow vanishes in the sense of distributions.

Linear flows on tori with Liouvillean frequencies are examples of non-stable DUE flows; it is remarkable that distributional unique ergodicity may coexist with a “chaotic” property such as weak-mixing. In fact B. Fayad [15] has constructed examples of mixing smooth time-changes of Liouvillean linear flows on tori.

Until recently there were no examples of DUE systems, except for toral systems derived from linear Liouvillean systems. In the past few years several new examples of this kind have been found by A. Avila and collaborators. Avila and A. Kocsard [3] have proved that all smooth circle diffeomorphisms with irrational rotation number are DUE. Recently, Avila, Fayad, and Kocsard [1] have constructed examples of DUE flows on certain higher dimensional compact manifolds, not diffeomorphic to tori, which admit a non-singular smooth circle action (hence Conjecture 6.1 of [19] does not hold).

The goal of this paper is to prove that DUE examples do not appear among non-toral homogeneous flows, so that a non-toral homogeneous flow always fails to be cohomology-free already because the closure of its range has codimension higher than one. In fact, we prove that for any homogeneous flow on a finite-volume homogeneous manifold $M$, except for the case of flows smoothly isomorphic to linear toral flows, the closure of the range of the Lie derivative operator on the space of smooth functions has countable codimension, or, in other terms, the space of invariant distributions for the flow has countable dimension. In particular, the Katok and Greenfield-Wallach conjectures hold for general homogeneous flows on finite-volume homogeneous manifolds. Our main result can be stated as follows.

**Theorem 1.1.** Let $G/D$ be a connected finite volume homogeneous space. A homogeneous flow $(G/D, \phi_R)$ is either smoothly isomorphic to a linear flow on a...
torus or it has countably many independent invariant distributions of bounded order (at most \( \dim(G/D) - 1 \) in the Sobolev sense, at most \( 1/2 \) in the Hölder sense).

We can also prove an analogous theorem for affine diffeomorphisms.

**Theorem 1.2.** Let \( G/D \) be a connected finite volume homogeneous space. An affine diffeomorphism \((G/D, \psi)\) is either smoothly isomorphic to an ergodic translation on a torus or it has countably many independent invariant distributions of bounded order (at most \( \dim(G/D) \) in the Sobolev sense, at most \( 1/2 \) in the Hölder sense).

An important feature of our argument is that in the case of partially hyperbolic flows we prove the stronger and more general result that any partially hyperbolic flow or diffeomorphism on any compact manifold, not necessarily homogeneous, has infinitely many distinct minimal sets (see Theorem 2.1). In particular, we have a proof of the Katok and Greenfield-Wallach conjectures in this case. We are not able to generalize this result to the finite-volume case. However, we can still prove that a partially hyperbolic homogeneous flow or an affine diffeomorphism on a finite-volume manifold has countably many ergodic probability measures (see Proposition 5.3).

In the non-partially hyperbolic homogeneous case, that is, in the quasi-unipotent case, by the Levi decomposition we are able to reduce the problem to flows on semi-simple and solvable manifolds. The semi-simple case is reduced to the case of \( \mathrm{SL}_2(\mathbb{R}) \) by an application of the Jacobson-Morozov Lemma, which states that any nilpotent element of a semi-simple Lie algebra can be embedded in an \( \mathfrak{sl}_2(\mathbb{R}) \)-triple. The solvable case can be reduced to the nilpotent case for which our main result was already proved by the first two authors in [17]. In both these cases the construction of invariant distributions is based on the theory of unitary representations for the relevant Lie group (Bargmann’s classification for \( \mathrm{SL}(2, \mathbb{R}) \) and Kirillov’s theory for nilpotent Lie groups).

The paper is organized as follows. In Section 2 we deal with partially hyperbolic flows on compact manifolds. In Section 3 give the background on homogeneous flows that allows us to reduce the analysis to the solvable and semi-simple cases. A further reduction is to consider quasi-unipotent flows (Section 4) and partially hyperbolic flows (Section 5) on finite-volume non-compact manifolds; then the main theorem follows easily (Section 6). Finally, in Section 7 we state a general conjecture on the stability of homogeneous flows and a couple of more general related open problems.

### 2. Partially Hyperbolic Flows and Diffeomorphisms on Compact Manifolds

The goal of this section is to prove the following theorem.

**Theorem 2.1.** Let \( M \) be a compact connected manifold, \( \phi_t \), \( t \in \mathbb{R} \) or \( t \in \mathbb{Z} \), an \( \mathbb{R} \)-action (a flow) or a \( \mathbb{Z} \)-action (a diffeomorphism) on \( M \), and assume \( \phi_t \) leaves invariant a foliation \( \mathcal{F} \) with smooth leaves and continuous tangent bundle, e.g.,
the unstable foliation of a partially hyperbolic flow. Assume also that the action \( \phi_t \) expands the norm of the vectors tangent to \( \mathcal{F} \) uniformly. Then there are infinitely many different \( \phi_t \)-minimal sets.

The existence of at least one non-trivial (i.e., different from the whole manifold) minimal sets goes back G. Margulis (see [43, 29, 31]) and Dani [11, 12]. A similar idea was already used by R. Mañé in [33] and more recently by A. Starkov [43], and F. & J. Rodriguez Hertz and R. Ures [41, Lemma A.4.2 (Keep-away Lemma)], in different contexts.

Theorem 2.1 for diffeomorphisms can be derived from the result for flows by passing to a suspension, and for flows it will follow almost immediately from the next lemma.

**Lemma 2.2.** Let \( \phi_t \) be a flow like in Theorem 2.1. For any \( k \)-tuple \( p_1, \ldots, p_k \in M \) of points in different orbits and for any open set \( W \subset M \), there exist \( \epsilon > 0 \) and \( q \in W \) such that \( d(\phi_t(q), p_i) \geq \epsilon \) for all \( t \geq 0 \) and for all \( i = 1, \ldots, k \).

Let us show how Theorem 2.1 follows from Lemma 2.2.

**Proof of Theorem 2.1.** Since \( M \) is compact there is a minimal set \( K \). Assume now by induction that there are \( K_1, \ldots, K_k \) different minimal set, then we will show that there is a minimal set \( K_{k+1} \) disjoint from the previous ones. Let \( p_i \in K_i \), \( i = 1, \ldots, k \) be \( k \) points and take \( q \) and \( \epsilon > 0 \) from Lemma 2.2. Since \( d(\phi_t(q), p_i) \geq \epsilon \) for any \( t \geq 0 \) and for \( i = 1, \ldots, k \) we have that for any \( i = 1, \ldots, k \), \( p_i \notin \omega(q) \), the omega-limit set of \( q \). Since the \( K_i \)'s are minimal, this implies that \( K_i \cap \omega(q) = \emptyset \) for \( i = 1, \ldots, k \). Take now a minimal subset of \( \omega(q) \) and call it \( K_{k+1} \).

Let \( F = T\mathcal{F} \) be the tangent bundle to the foliation with fiber at \( x \in M \) given by \( F(x) = T_x\mathcal{F}(x) \). We denote by \( d \) the distance on \( M \) induced by some Riemannian metric. Let \( X \) be the generator of the flow \( \phi_t \). Let also \( E(x) = (F(x) \oplus \langle X(x) \rangle)^\perp \) be the orthogonal bundle and \( \mathcal{E}_r(x) = \exp_x(B^r_f(x)) \) be the image of the \( r \) ball in \( E(x) \) by the exponential map. Let \( f \) be the dimension of the foliation \( \mathcal{F} \) and \( m \) the dimension of \( M \). By the compactness of the manifold \( M \), there exists \( r_0 > 0 \) such that for all \( r \leq r_0 \) the union \( \mathcal{E} := \bigsqcup_{x \in M} \mathcal{E}_r(x) \) is disjoint and forms a \((m - f - 1)\)-dimensional continuous disc bundle over \( M \). Denote with \( d_{\mathcal{F}} \) and \( d_\mathcal{E} \) the distances along the leaves of \( \mathcal{F} \) and \( \mathcal{E} \), and let

\[
\mathcal{F}_r(x) = \{ y \in \mathcal{F}(x) \mid d_{\mathcal{F}}(y, x) \leq r \} \subset \mathcal{F}(x)
\]

be the \( f \)-dimensional closed disc centered at \( x \) and of radius \( r > 0 \). Clearly \( d \leq d_{\mathcal{F}} \) and \( d \leq d_\mathcal{E} \).

We may assume that the Riemannian metric on \( M \) is adapted so that \( \mathcal{F}_r(x) \supset \phi_{-t}\mathcal{F}_r(\phi_t x) \) for all \( x \in M \) and \( r, t \geq 0 \). In fact if \( g \) is a Riemannian metric such that \( \| (\phi_t)_*v \|_g \geq C\lambda^t \| v \|_g \) for all \( v \in F(x) \), all \( x \in M \), and all \( t \geq 0 \) (where \( \lambda > 1 \)), then setting \( \tilde{g} = I^{T_0}_r(\phi_t)_* g \) \( \frac{d}{dt} \), \( T_0 = -\log_\lambda(C/2) \), we have that, for all \( v \in F(x) \) and \( x \in M \), the function \( \| (\phi_t)_*v \|_{\tilde{g}} \) is strictly increasing with \( t \).
We may choose \( r_1 < r_0 \) such that if \( r \leq r_1 \) then, for all \( x \in M \), we have \( d_{\mathcal{F}}(y, z) \leq 2d(y, z) \) for any \( y, z \in \mathcal{F}(x) \) and \( d_{\mathcal{F}}(y, z) \leq 2d(y, z) \) for any \( y, z \in \mathcal{F}(x) \).

For \( x \in M \), let

\[
V_{\delta, r}(x) = \bigcup_{z \in \delta(x)} \mathcal{F}(z).
\]

There exists \( r_2 \leq r_1 \) such that, if \( r \) and \( \delta \) are both less than \( r_2 \), then \( V_{\delta, Ar}(x) \) is homeomorphic to a disc of dimension \((m - 1)\) transverse to the flow.

**Normalization assumption:** After a constant rescaling of \( X \) we may assume that given any \( x \in M, z, y \in \mathcal{F}(x) \) and \( t \geq 1 \) we have \( d_{\mathcal{F}}(\phi_t(z), \phi_t(y)) \geq 4 d_{\mathcal{F}}(z, y) \).

Henceforth, in this section, we shall tacitly make this assumption.

**Proof of Lemma 2.2.** Let \( p_1, \ldots, p_k \in M \) be points belonging to different orbits and let \( W \subset M \) be an open set. We shall find \( r > 0 \) and a point \( x_0 \in W \) with \( \mathcal{F}(x_0) \subset W \) and then construct, by induction, a sequence of points \( x_n \in M \) and iterates \( \tau_n \geq 1 \) satisfying, for some \( \delta > 0 \), the following conditions:

\[
(A_{n+1}) \quad \phi^{-\tau_n}(\mathcal{F}(x_{n+1})) \subset \mathcal{F}(x_n), \quad \text{for all } n \geq 0,
\]

and

\[
(B_n) \quad \phi_{\tau_n} x \in \mathcal{F}(x_n) \implies \phi_t(x) \notin \bigcup_i V_{\delta, 2r}(p_i), \quad \text{for all } t \in [0, T_{n+1}).
\]

Here we have set \( T_n := \sum_{k=0}^{n-1} \tau_k \). Then defining \( D_n := \phi^{-\tau_n} \mathcal{F}(x_n) \) we have \( D_{n+1} \subset D_n \subset \mathcal{F}(x_0) \) and any point \( q \in \bigcap_n D_n \subset \mathcal{F}(x_0) \subset W \) will satisfy the statement of the lemma.

By the choice of an adapted metric we have

\[
d_{\mathcal{F}}(\phi_t(x), \phi_t(p)) \geq d_{\mathcal{F}}(x, p), \quad \text{for all } p \in M \text{ and all } x \in \mathcal{F}(p).
\]

This implies that for all \( i = 1, \ldots, k \) any \( r > 0 \) and any \( t \geq 0 \)

\[
(1) \quad x \in \mathcal{F}_{4r}(p_i) \sim \mathcal{F}_{2r}(p_i) \implies \phi_t(x) \notin \mathcal{F}_{2r}(\phi_t(p_i)).
\]

Hence there exists \( \delta_0 < r_2 \) such that for all \( \delta < \delta_0 \) and all \( r \leq r_2 \) we have:

(i) for all \( i \in \{1, \ldots, k\} \),

\[
\phi_{[0,1]} \left( V_{\delta, Ar}(p_i) \sim V_{\delta, 2r}(p_i) \right) \cap V_{\delta, r}(p_i) = \emptyset.
\]

The above assertion follows immediately by continuity if \( p_i \) is not periodic of minimal period less or equal to 1. If \( p_i \) is periodic of period less or equal to 1, then it follows by continuity from formula (1). As the orbits of \( p_1, \ldots, p_k \) are all distinct and the set \( W \) is open, we may choose a point \( x_0 \in W \) and positive real numbers \( r, \delta < \delta_0 \) so that the following conditions are also satisfied:

(ii) \( \mathcal{F}(x_0) \subset W; \)

(iii) for all \( i, j \in \{1, \ldots, k\} \), with \( i \neq j \),

\[
\phi_{[0,1]} \left( V_{\delta, Ar}(p_j) \right) \cap \phi_{[0,1]} \left( V_{\delta, 2r}(p_i) \right) = \phi_{[0,1]} \left( V_{\delta, Ar}(p_i) \right) \cap \bigcup_{r \in [0,1]} \mathcal{F}(\phi_t x_0) = \emptyset.
\]
If for all \( t > 0 \) we have \( \mathcal{F}_r(\phi_r(x_0)) \cap \bigcup_i V_{\delta,r}(p_i) = \emptyset \), then \( d(\phi_r(x_0), p_i) > r \) for all \( i = 1, \ldots k \) and all \( t > 0 \), proving the lemma with \( q = x_0 \) and \( \epsilon = r \). Thus we may assume that

\[
\tau_0 := \inf \left\{ t > 0 : \mathcal{F}_r(\phi_r(x_0)) \cap \bigcup_i V_{\delta,r}(p_i) \neq \emptyset \right\} < \infty
\]

and define

\[
\hat{x}_0 = \phi_{\tau_0}(x_0).
\]

The above condition (iii) implies that \( \tau_0 \geq 1 \), hence by the normalization assumption it follows that

\[
(2) \quad \mathcal{F}_{4r}(\hat{x}_0) \subset \phi_{\tau_0}(\mathcal{F}_r(x_0)).
\]

Assume, by induction, that points \( x_k \in M \) and iterates \( \tau_k \geq 1 \) satisfying the conditions \((A_n)\) and \((B_n)\) have been constructed for all \( k \in \{0, \ldots, n\} \), and assume that the point \( \hat{x}_n := \phi_{\tau_n}(x_n) \in M \) is such that \( \mathcal{F}_r(\hat{x}_n) \) intersects non-trivially some disc \( V_{\delta,r}(p_i) \). Since \( V_{\delta,r}(p_i) \) is saturated by \( \mathcal{F}_r \), it follows that \( \mathcal{F}_{2r}(\hat{x}_n) \cap B_{\delta,r}(p_i) \) consists of a unique point \( z_n \) with \( d_\mathcal{F}(z_n, \hat{x}_n) \leq 2r \); we define \( x_{n+1} \in \mathcal{F}_r(\hat{x}_n) \) as the point at distance \( 3r \) on the geodesic ray in \( \mathcal{F}(\hat{x}_n) \) going from \( z_n \) to \( \hat{x}_n \) (or any point on the geodesic ray issued from \( z_n \). Then we have

\[
(3) \quad \mathcal{F}_r(x_{n+1}) \subset \mathcal{F}_{4r}(\hat{x}_n) \cap V_{\delta,4r}(p_i) \sim V_{\delta,2r}(p_i).
\]

By the disjointness conditions (i) and (iii), it follows that, for all \( t \in (0, 1) \),

\[
\mathcal{F}_r(\phi_{t}x_{n+1}) \cap \bigcup_{i=1}^{k} V_{\delta,r}(p_i) = \emptyset.
\]

It follows that if we define

\[
\tau_{n+1} := \inf \left\{ t > 0 : \mathcal{F}_r(\phi_{t}x_{n+1}) \cap \bigcup_i V_{\delta,r}(p_i) \neq \emptyset \right\}, \quad \hat{x}_{n+1} = \phi_{\tau_{n+1}}(x_{n+1})
\]

(assuming \( \tau_{n+1} < +\infty \)), then \( \tau_{n+1} \geq 1 \), and by the normalization assumption and by the inclusion in formula (3) we have

\[
\mathcal{F}_r(x_{n+1}) \subset \mathcal{F}_{4r}(\hat{x}_n) = \mathcal{F}_{4r}(\phi_{\tau_n}x_n) \subset \phi_{\tau_n}(\mathcal{F}_r(x_n))
\]

and by construction, having set \( T_{n+2} := \sum_{k=0}^{n+1} \tau_k \), we also have

\[
x \in D_{n+1} := \phi_{-\tau_{n+1}}(\mathcal{F}_r(x_{n+1})) \implies \phi_{t}(x) \in \bigcup_i V_{\delta,2r}(p_i), \quad \text{for all } t \in [0, T_{n+2}).
\]

The inductive construction is thus completed. As we explained above we have that \( (D_n) \) is a decreasing sequence of closed sub-intervals of \( \mathcal{F}_r(x_0) \) and that any point \( q \in \cap_n D_n \) satisfies \( \phi_t(q) \in \bigcup_i V_{\delta,r}(p_i) \) for all \( t \geq 0 \).

The above inductive construction may fail if at some stage \( n \geq 0 \) we have \( \tau_n = +\infty \). In this case let \( q \) be any point in \( \phi_{-\tau_n}(\mathcal{F}_r(x_n)) \). Again such a point \( q \in W \) satisfies the statement of the lemma, hence the proof is completed. \( \square \)
Corollary 2.3. Let $\psi$ be a partially hyperbolic $C^1$ diffeomorphism on a compact connected manifold $M$. If the map $\psi$ satisfies the accessibility condition, there exists $\alpha > 0$ such that every Hölder invariant distribution of order at most $\alpha$ belongs to the closure (in the space $\mathcal{D}'(M)$ of all distributions on $M$) of the linear space spanned by an invariant measure and by the family of Periodic Cycle Functionals. In particular, the space spanned by all Periodic Cycle Functionals is infinite dimensional.

Proof. By [45] there exist $\alpha, \beta \in (0, 1)$ with $\beta \geq \alpha$ such that a function $f \in C^\beta(M)$ belongs to the joint kernel of all Periodic Cycle Functionals (PCFs) and of a $\psi$-invariant measure $m$ if and only if $f$ is a coboundary with a continuous, hence $C^\alpha$, primitive (i.e., transfer function). It follows that $f$ belongs to the kernel of all Hölder invariant distributions of order at most $\alpha > 0$. Since the space $C^\infty(M)$ is reflexive, by the Hahn-Banach theorem if a distribution $D \in \mathcal{D}'(M)$ does not belong to the closure in $\mathcal{D}'(M)$ of the linear space spanned by the invariant measure $m$ and by the family of all PCFs, then there exists a function $f \in C^\infty(M)$ such that $f$ has zero average with respect to $m$ and belongs to the kernel of all PCFs but $D(f) \neq 0$. However, $f$ is a coboundary with $C^\alpha$ transfer function, which implies that $D(f) = 0$. This contradiction implies that the space $\mathcal{D}_\psi^\alpha(M)$ of all Hölder invariant distributions of order at most $\alpha$ is a subset of the closure in the topology of $\mathcal{D}'(M)$ of the linear space spanned by the invariant measure $m$ and by the family of Periodic Cycle Functionals. Since by Theorem 2.1 the partially hyperbolic diffeomorphism $\psi$ has infinitely many distinct minimal sets (it follows by considering a suspension flow), the space $\mathcal{D}_\psi^\alpha(M)$ is infinite dimensional for any $\alpha \geq 0$, hence the argument is concluded.

In order to prove an analogous result for partially hyperbolic flows, we extend Wilkinson’s theorem on solutions of the cohomological equation to the case of flows. As we shall see, it is a simple corollary of the theorem for maps.

Theorem 2.4. Let $\phi_t$ be a $C^1$ partially hyperbolic flow generated by a vector field $X$ on a compact manifold $M$. If the flow $\phi_t$ satisfies the accessibility conditions, then for any $\beta \in (0, 1)$ there exists $\alpha \in (0, 1)$ such that any Hölder function $f \in C^\beta(M)$ which belongs to the joint kernel of all Periodic Cycle Functionals is cohomologous to a constant over the flow $\phi_t$ with a Hölder transfer function in $C^\alpha(M)$, that is, there exists a Hölder (transfer) function $u \in C^\alpha(M)$ and a constant $c \in \mathbb{C}$ such that the following cohomological equation holds (in the distributional sense)

$$Xu = f - c.$$ 

Conversely, if for any function $f \in C^\beta(M)$ there exists a continuous function $u \in C^0(M)$ and a constant $c \in \mathbb{C}$ such that the above cohomological equation holds, then $u \in C^\alpha(M)$, hence $f$ belongs to the kernel of all Periodic Cycle Functionals.

Proof. For any $t > 0$, let $\psi_{(t)}$ denote the time-$t$ map of the flow $\phi_t$. The $C^1$ diffeomorphism $\psi_{(t)}$ is partially hyperbolic and its stable and unstable foliations coincide with the stable and unstable foliations of the flow. It follows that $\psi_{(t)}$
has the accessibility property and that the set of its stable-unstable paths coincides with the set of stable-unstable paths for the flow. Let \( f \in C^\beta(M) \) belong to the kernel of all Periodic Cycle Functionals (PCF’s) for the flow. By the above remark and by the definition of the PCF’s, it follows that for any \( t > 0 \) the function

\[
\tilde{f}_t := \int_0^t f \circ \phi_s \, ds
\]

belongs to the kernel of all PCF’s for the time-\( t \) map \( \psi(t) \). Let \( m \) be any invariant measure for the flow \( \phi_t \), hence for all its time-\( t \) maps. By Wilkinson’s theorem [45], there exists a unique function \( u_t \in C^0(M) \), of zero average with respect to \( m \), and a constant \( c_t \in \mathbb{C} \) such that

\[
f_t - c_t = u_t \circ \psi(t) - u_t.
\]

We claim that for all \( t > 0 \) we have \( u_{2t} = u_t \) and \( c_{2t} = 2c_t \). In fact,

\[
\tilde{f}_{2t} = f_t + f_t \circ \psi(t) = u_t \circ \psi(t) - u_t + c_t + (u_t \circ \psi(t) - u_t + c_t) \circ \psi(t)
\]

\[= u_t \circ \psi(t) - u_t + (u_t \circ \psi(t) - u_t) \circ \psi(t) + 2c_t = u_t \circ \psi(2t) - u_t + 2c_t.
\]

It follows by the uniqueness of the solution that \( u_{2t} = u_t \) and \( c_{2t} = 2c_t \) as claimed.

By the above claim it follows that for all \( n \geq 0 \) we have

\[
u_{1/2^n} = u_1 \quad \text{and} \quad c_{1/2^n} = c_1/2^n.
\]

We can therefore write, after multiplying on both sides by the factor \( 2^n \),

\[
2^n \int_0^{1/2^n} f \circ \phi_s \, ds - c_1 = 2^n(u_1 \circ \psi(1/2^n) - u_1).
\]

By taking the limit as \( n \to +\infty \) (in the sense of distributions), we finally derive the equation

\[
f - c_1 = Xu_1.
\]

We have thus proved that the cohomological equation for the flow has a solution \((u,c) = (u_1,c_1)\) with a continuous transfer function \( u \in C^0(M) \). In order to prove that the transfer function is in fact Hölder, we argue that under the assumption that the function \( f \in C^\beta(M) \), the transfer function \( u \in C^\alpha(M) \). In fact, the above cohomological equation for the flow implies, by integration along the flow up to time \( t = 1 \), the following cohomological equation for the time-1 map:

\[
f_1 - c = u \circ \psi(1) - u.
\]

It follows then by Wilkinson’s theorem [45], since the time-1 map is partially hyperbolic and satisfies the accessibility condition, the function \( f \in C^\beta(M) \), hence the integrated function \( f_1 \in C^\beta(M) \) as well, and the transfer function \( u \in C^\alpha(M) \), that in fact \( u \in C^\alpha(M) \). Finally, we note that if the above cohomological equation for the flow has a Hölder solution, then the function \( f \in C^\beta(M) \) belongs to the kernel of all PCF’s (for the flow) since all PCF’s are invariant functionals, bounded on any Hölder space, which vanish on constant functions. The argument is therefore completed.

\[\square\]

From the above theorem and from Theorem 2.1 we can then derive the following corollary, whose proof is entirely analogous to that of Corollary 2.3.
Corollary 2.5. Let $\phi_t$ be a partially hyperbolic $C^1$ flow on a compact connected manifold $M$. If the flow $\phi_t$ satisfies the accessibility condition, there exists $\alpha > 0$ such that every Hölder invariant distribution of order at most $\alpha$ belongs to the closure (in the space $\mathcal{D}(M)$ of all distributions on $M$) of the linear space spanned by an invariant measure and by the family of all Periodic Cycle Functionals. In particular, the space spanned by all Periodic Cycle Functionals is infinite dimensional.

3. Homogeneous flows and affine diffeomorphisms

Henceforth $G$ will be a connected Lie group and $G/D$ a finite volume space; this means that $D$ is a closed subgroup of $G$ and that $G/D$ has a finite $G$-invariant (smooth) measure. The group $D$ is called the isotropy group of the space $M = G/D$. As we are only interested in the quotient space $M$, we may assume that $G$ is simply connected and that the isotropy group $D$ is a quasi-lattice, i.e., that the largest connected normal subgroup of $D$ is reduced to the identity.

Let $\mathfrak{g}$ be the Lie algebra of $G$. The exponential map $\exp: \mathfrak{g} \to G$ sets up a bijective correspondence between elements $X \in \mathfrak{g}$ and one-parameter subgroups $(\phi_t = \exp tX)_{t \in \mathbb{R}}$. We denote a one-parameter subgroup $(\phi_t)_{t \in \mathbb{R}}$ of $G$ by $\phi_{\mathbb{R}}$. The flow generated by left translations by this one-parameter subgroup on the finite volume space $G/D$ will be denoted $(G/D, \phi_{\mathbb{R}})$ or simply $\phi_{\mathbb{R}}$.

If $G$ is simply connected, then the group $\text{Aut}(G)$ of Lie group automorphisms of $G$ is identified with the algebraic group $\text{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$, via the map associating to an automorphism its differential at the identity. In this case we shall not make a distinction between these groups.

For any group $G$ we denote its center by $Z(G)$. For any element $x$ of a group $G$ we let $\text{Int}(x) \in \text{Aut}(G)$ be the inner automorphism given by the conjugation by $x$. For any normal subgroup $H \triangleleft G$, we denote by $\text{Int}_H(x)$ the automorphism of $H$ given by the conjugation by $x \in G$. The center of $Z(G)$ is precisely the kernel of the map $\text{Int}: G \to \text{Aut}(G)$. The adjoint representation $\text{Ad}: G \to \text{Aut}(\mathfrak{g})$ is defined by letting $\text{Ad}(g)$ be the differential of the inner automorphism $\text{Int}(g)$ at the identity of $G$.

An affine map of a Lie group $G$ is the composition of a continuous automorphism $A$ of $G$ and a (left) translation by an element of $G$. We denote by $\psi = uA$ the affine map defined by $\psi(x) = uA(x)$ for all $x \in G$, where $u \in G$ and $A \in \text{Aut}(G)$. Affine maps form a group under composition which may be be identified to the semi-direct product $\text{Aff}(G) := \text{Aut}(G) \times G$. As $G$ is a normal subgroup of $\text{Aff}(G)$ conjugation by the affine map $\psi = uA$ yields an automorphism of $G$, which is easily seen to be given by $\text{Int}_G(\psi) := \text{Int}(u) \circ A$.

An affine map $\psi = uA$ induces a smooth quotient map of $G/D$ if and only if $A(D) \subset D$, and it induces a diffeomorphism of $G/D$ if and only if the equality $A(D) = D$ holds true. We call an affine diffeomorphism of $G/D$ a diffeomorphism of $G/D$ induced by an affine map of $G$ and denote by $\text{Aff}(G/D)$ the group of such diffeomorphisms. The group $\text{Aff}(G/D)$ of affine diffeomorphisms of $G/D$ is a quotient group of the group $\text{Aff}(G)$ of affine maps of $G$. 

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**Lemma 3.1.** Let \( G/D \) be a finite volume space with \( D \) a quasi-lattice. Let \( \psi = uA \) be an affine map of \( G \) projecting to an affine diffeomorphism \( \bar{\psi} \) of \( G/D \) and let \( \phi_R \) be a one-parameter subgroup of \( G \). Then

- The flow by left translations by \( \phi_R \) on \( G/D \) commutes with the map \( \bar{\psi} \) if and only if the subgroup \( \phi_R \) is fixed by the automorphism \( \text{Int}(u) \circ A \in \text{Aut}(G) \).
- The affine diffeomorphism \( \psi \) is the identity on \( G/D \) if and only if the map \( \bar{\psi} \) is the right translation by an element in \( D \), i.e., if and only if \( u = \gamma \) and \( A = \text{Int}(\gamma^{-1}) \), for \( \gamma \in D \). Thus the map
  \[
  \psi \in \text{Aff}(G/D) \rightarrow \text{Int}_G(\psi) = \text{Int}(u) \circ A \in \text{Aut}(G)
  \]
  is a well-defined group homomorphism, as the right hand side does not depend on the choice of the affine map \( \psi = uA \in \text{Aff}(G) \) projecting to \( \bar{\psi} \).

**Proof.** We have \( \phi_t uA(x)D = uA(\phi_t x)D \) for all \( x \in G \) and all \( t \in \mathbb{R} \), if and only if \( A(\phi_{-t})u^{-1}\phi_t u \in yD\gamma^{-1} \) for all \( y \in G \) and all \( t \in \mathbb{R} \). But \( D \) is a quasi-lattice of \( G \), hence it does not have any non-trivial connected normal subgroups. The first statement of the lemma follows.

The map \( \bar{\psi} \) is the identity on \( G/D \) if and only if it commutes with every one-parameter subgroup of \( G \). By the previous statement, this condition is equivalent to the identity \( A = \text{Int}(u^{-1}) \). Thus \( xD = \bar{\psi}(xD) = xuD \) for all \( x \in G \). This implies that \( u \in D \).

The conclusion of the above lemma may be stated by saying that, if \( D \) is a quasi-lattice, the group \( \text{Aff}(G/D) \) of affine diffeomorphisms of \( G/D \) is isomorphic to the “adjoint group” \( \{ \text{Int}_G(\psi) \in \text{Aut}(G) \mid \psi \in \text{Aff}(G) \} \). Hence we obtain

**Corollary 3.2.** Let \( G/D \) be a finite volume space with \( D \) a quasi-lattice. Any finite subgroup of affine map of the homogenous space \( G/D \), acts as a finite group of automorphisms on the Lie algebra of \( G \).

Our results for affine diffeomorphisms will be derived from the corresponding results for flows by the following method. By a modification of a construction of S. G. Dani [9] it is possible to reduce the general case of affine diffeomorphisms to the case of homogeneous flows up to the action of a finite subgroup.

Invariant distributions for affine maps are then obtained from invariant distributions for homogenous flows. This reduction is based on the fundamental well-known fact that the spaces of invariant distributions for a diffeomorphism and for its suspension flow are isomorphic. We include a proof below for the convenience of the reader.

**Definition 3.3.** The suspension flow of a diffeomorphism \( \psi \) on a smooth manifold \( M \) is a flow \( \{ \phi_t \} \) on a manifold \( \Sigma M \) such that \( M \subset \Sigma M \) is transverse to the flow and the map \( \psi \) coincides with the first return map to \( M \) and with the time-1 map of the flow \( \{ \phi_t \} \) on \( \Sigma M \) (in other words the return time function is constant and equal to 1).

The suspension flow is unique up to diffeomorphism and can be constructed as follows. Let \( \Sigma M \) denote the quotient of the product \( M \times \mathbb{R} \) with respect to the
equivalence relation defined as
\[(x, r) \sim_\psi \psi^{-1}(x), r + 1), \quad \text{for all } (x, r) \in M \times \mathbb{R}.
\]
Let \(\{\phi_t\}\) be the projection to \(\Sigma M = M \times \mathbb{R}/\sim_\psi\) of the ‘vertical’ flow on \(M \times \mathbb{R}/\sim_\psi\), that is, of the flow \(\{\phi_t\}\) defined as
\[\phi_t(x, r) = (x, r + t), \quad \text{for all } t \in \mathbb{R} \text{ and } (x, r) \in M \times \mathbb{R}.
\]
The flow \(\{\phi_t\}\) is generated by the vector field \(V\) on \(\Sigma M\) obtained by projection of the ‘vertical’ vector field \((0, \partial/\partial r)\) on \(M \times \mathbb{R}\). The manifold \(M\) is diffeomorphic to the manifold \(M \times \{0\}/\sim_\psi\), by the composition of the inclusion \(M \to M \times \{0\}\) and the projection \(M \times \mathbb{R} \to \Sigma M\), and the flow \(\{\phi_t\}\) is the suspension flow of the diffeomorphism \(f\) as a map on \(M \times \{0\}/\sim_\psi\).

**Proposition 3.4.** Let \((\Sigma M, \phi_t)\) be the suspension flow of a diffeomorphism \(\psi\) of a smooth manifold \(M\). Then the space of \(\psi\)-invariant distributions on \(M\) of a given Sobolev or Hölder order (resp. of \(\psi\)-invariant measures on \(M\)) is isomorphic to the space of \(\phi_t\)-invariant distributions on \(\Sigma M\) of the same Sobolev or Hölder order (resp. of \(\phi_t\)-invariant measures on \(\Sigma M\)).

The statement about measures in the above Proposition follows immediately from the above construction of the suspension flow. The complete proof of the proposition will follow from Lemma 3.5 below.

Let \(M\) be a manifold and let \(\Sigma M\) be the suspension space as defined above. For any function \(\theta \in C^\infty_0((-1/2, 1/2))\), defined on \(\mathbb{R}\), such that
\[
\int_{-1/2}^{1/2} \theta(r) dr = 1,
\]
we define a continuous linear operator \(E_\theta : C^\infty_0(M) \to C^\infty_0(\Sigma M)\) as follows. For any \(f \in C^\infty_0(M)\), let
\[
E_\theta(f)(x, r) = \sum_{n \in \mathbb{Z}} f \circ \psi^{-n}(x) \theta(r + n), \quad \text{for all } (x, r) \in M \times \mathbb{R}.
\]
Since \(\theta \in C_0((-1/2, 1/2))\) the above sum is finitely supported and defines a function \(E_\theta(f) \in C^\infty(M \times \mathbb{R})\), which is constant on all equivalence classes of the relation \(\sim_\psi\), hence descends to a function \(E_\theta(f) \in C^\infty(\Sigma M)\). It is immediate from the definition that the function \(E_\theta(f)\) has compact support and that
\[
E_\theta : C^\infty_0(M) \to C^\infty_0(\Sigma M)
\]
is a continuous linear operator. By the above definition, it is also clear that \(E_\theta\) extends to a linear continuous operator from Sobolev and Hölder spaces of functions on \(M\) to Sobolev, respectively Hölder, spaces of functions on \(\Sigma M\) of the same order.

Let \(E^*_\theta : \mathcal{D}'(\Sigma M) \to \mathcal{D}'(M)\) denote the dual operator on distributions, which is defined as follows:
\[
E^*_\theta(D)(f) = D(E_\theta(f)), \quad \text{for all } f \in C^\infty_0(M).
\]
Since the operator \(E_\theta\) extends to a linear continuous operator on Sobolev and Hölder spaces of functions and it preserves the order, its dual \(E^*_\theta\) maps Sobolev
and Hölder spaces of distributions on $\Sigma M$ to Sobolev, respectively Hölder, spaces of distributions on $M$ of the same order.

**Lemma 3.5.** Let $\mathcal{F}(\Sigma M) \subset \mathcal{D}'(\Sigma M)$ be the subspace of distributions invariant under the suspension flow $\{\phi_1\}$. The restriction of the operator $E_\theta^*$ to $\mathcal{F}(\Sigma M)$ is a continuous operator $E^*$ which does not depend on the choice of the function $\theta \in \mathcal{C}_0^\infty(-1/2, 1/2)$ satisfying the conditions in formula (4). Moreover, the operator $E^* : \mathcal{D}'(\Sigma M) \rightarrow \mathcal{D}'(M)$ induces an isomorphism between the space $\mathcal{F}(\Sigma M)$ and the subspace $\mathcal{F}_\psi(M)$ of distributions invariant under the diffeomorphism $\psi$ on $M$, which preserves the Sobolev as well as the Hölder order of invariant distributions.

**Proof.** Let $\theta_1, \theta_2 \in \mathcal{C}_0^\infty(-1/2, 1/2)$ be any two functions such that

$$\int_{-1/2}^{1/2} \theta_1(r) dr = \int_{-1/2}^{1/2} \theta_2(r) dr = 1.$$  

We claim that, for any $f \in \mathcal{C}_0^\infty(M)$, the function $E_{\theta_1}(f) - E_{\theta_2}(f)$ is a smooth coboundary for the suspension flow. In fact, since $\theta_1 - \theta_2$ has zero average on the interval $(-1/2, 1/2)$ there exists a smooth function $\chi \in \mathcal{C}_0^\infty(-1/2, 1/2)$ such that

$$\theta_1 - \theta_2 = \frac{d\chi}{dr}.$$  

Let $\tilde{F} \in \mathcal{C}_0^\infty(M \times \mathbb{R})$ be the function defined as

$$\tilde{F}(x, r) := \sum_{n \in \mathbb{Z}} f \circ \psi^{-n}(x) \chi(r + n), \quad \text{for all } (x, r) \in M \times \mathbb{R}.$$  

The function $\tilde{F}$ is well-defined and it projects to a smooth function $F \in \mathcal{C}_0^\infty(\Sigma M)$. In addition, from the identity

$$\tilde{E}_{\theta_1}(f)(x, r) - \tilde{E}_{\theta_2}(f)(x, r) = \frac{d\tilde{F}}{dr}(x, r), \quad \text{for all } (x, r) \in M \times \mathbb{R},$$  

it follows by projection that $E_{\theta_1}(f) - E_{\theta_2}(f) = VF$, as claimed. For any distribution $D \in \mathcal{D}'(\Sigma M)$ invariant under the suspension flow and for all $f \in \mathcal{C}_0^\infty(M)$ we then have

$$\left(E_{\theta_1}^*(D) - E_{\theta_2}^*(D)\right)(f) = D \left(E_{\theta_1}(f) - E_{\theta_2}(f)\right) = D(VF) = 0.$$  

We have thus proved that the restriction of $E_\theta^*$ to the subspace of invariant distributions for the suspension flow is a continuous linear operator $E^*$ independent of the choice of the function $\theta \in \mathcal{C}_0^\infty(-1/2, 1/2)$ with integral equal to 1.

Next we prove that the linear operator $E_\theta^*$ maps the subspace $\mathcal{F}_\psi(M)$ of invariant distributions for the suspension flow $\phi_1$ into the subspace $\mathcal{F}_\psi(M)$ of invariant distributions for the diffeomorphism $\psi$. By construction, for any $f \in \mathcal{C}_0^\infty(M)$ we have the identity

$$\tilde{E}_\theta(f \circ \psi)(x, r) = \tilde{E}_\theta(f)(x, r + 1), \quad \text{for all } (x, r) \in M \times \mathbb{R},$$  

which, under projection on $\Sigma M$, implies the following identity:

$$E_\theta(f \circ \psi) = E_\theta(f) \circ \phi_1.$$
It follows that, for all $D \in \mathcal{A}_\mathcal{V}(\Sigma M)$ and all $f \in C_0^\infty(M)$, we have

$$E_\theta^*(D) = D\left(E_\theta(f)\right) = D\left(E_\theta(f) \circ \phi_1\right) = D\left(E_\theta(f)\right) = E_\theta^*(D).$$

Hence $E_\theta^*(D) \in \mathcal{A}_\mathcal{V}(M)$.

That the continuous linear operator $E_\theta^*$ is an isomorphism of $\mathcal{A}_\mathcal{V}(\Sigma M)$ onto $\mathcal{A}_\mathcal{V}(M)$ follows from the construction of an inverse operator.

Let $I : C_0^\infty(\Sigma M) \to C_0^\infty(M)$ be the continuous linear operator defined as follows. For any $F \in C_0^\infty(\Sigma M)$, the function $I(F) \in C_0^\infty(M)$ is defined as

$$(6) \quad I(F)(x) = \int_{-1/2}^{1/2} F \circ \phi_1(x, t) \, dt, \quad \text{for all } x \in M.$$

By the above definition, it is clear that the operator $I$ extends to a linear continuous operator from Sobolev and Hölder spaces of functions on $\Sigma M$ to Sobolev, respectively Hölder, spaces of functions on $M$ of the same order.

It is immediate from the construction that

$$(I \circ E_\theta)(f) = f, \quad \text{for all } f \in C_0^\infty(M),$$

hence the dual operator $I^* : \mathcal{D}(M) \to \mathcal{D}(\Sigma M)$ is a right inverse of the operator $E_\theta^*$:

$$(E_\theta^* \circ I^*)(D) = D, \quad \text{for all } D \in \mathcal{D}(M).$$

We claim that for all $F \in C_0^\infty(\Sigma M)$, the function $(E_\theta \circ I)(F) - F$ is a smooth coboundary for the suspension flow. Let $\tilde{F} \in C^\infty(M \times \mathbb{R})$ be any lift of the function $F$ on $\Sigma M$ to $M \times \mathbb{R}$. By construction

$$\int_0^1 [(\tilde{E}_\theta \circ I)(F) - \tilde{F}](x, n + t) \, dt = 0, \quad \text{for all } (x, n) \in M \times \mathbb{Z}.$$

It follows that the smooth function $\tilde{U}$ on $M \times \mathbb{R}$ defined as

$$\tilde{U}(x, r) = \int_0^r [(\tilde{E}_\theta \circ I)(F) - \tilde{F}](x, 0) \, dt, \quad \text{for all } x \in M,$$

has a well-defined projection $U \in C_0^\infty(M)$ to the quotient $\Sigma M = M \times \mathbb{R} / \sim$, since the function $(\tilde{E}_\theta \circ I)(F) - \tilde{F}$ has a well-defined projection and

$$\tilde{U}(x, n) = 0, \quad \text{for all } (x, n) \in M \times \mathbb{Z}.$$

In addition, again by construction we have

$$\frac{d}{dr} \tilde{U}(x, r) = [(\tilde{E}_\theta \circ I)(F) - \tilde{F}](x, r), \quad \text{for all } (x, r) \in M \times \mathbb{R},$$

hence $VU = (E_\theta \circ I)(F) - F$, that is, the function $(E_\theta \circ I)(F) - F$ is a coboundary for the suspension flow, as claimed. It follows that for any invariant distribution $D \in \mathcal{A}_\mathcal{V}(\Sigma M)$ we have

$$D[(E_\theta \circ I)(F)] = D(F), \quad \text{for all } F \in C_0^\infty(\Sigma M),$$

hence the dual operator $I^* : \mathcal{A}_\mathcal{V}(M) \to \mathcal{A}_\mathcal{V}(\Sigma M)$ is a left inverse of the operator $E_\theta^*$, that is,

$$(I^* \circ E_\theta^*)(D) = D, \quad \text{for all } D \in \mathcal{A}_\mathcal{V}(\Sigma M).$$
We thus conclude that the continuous linear operator $E^* : \mathcal{I}_V(\Sigma M) \to \mathcal{I}_W(M)$ is an isomorphism as its inverse $I^* : \mathcal{I}_W(M) \to \mathcal{I}_V(\Sigma M)$ is well-defined and continuous. Since the operator $I$ extends to a linear continuous operator from Sobolev and Hölder spaces of functions on $\Sigma M$ to Sobolev, respectively Hölder, spaces of functions on $M$ of the same order, its dual $I^*$ maps Sobolev and Hölder spaces of distributions on $M$ to Sobolev, respectively Hölder, spaces of distributions on $\Sigma M$ of the same order. It follows that the isomorphism $E^* : \mathcal{I}_V(\Sigma M) \to \mathcal{I}_W(M)$ preserves the Sobolev as well as the Hölder order of invariant distributions.  

When a group $\mathcal{H}$ operates on a group $\mathcal{A}$ by automorphisms of $\mathcal{A}$, we denote by $\text{Aff}_{\mathcal{H}}(\mathcal{A})$ the group of affine maps $\psi = uA$ of $\mathcal{A}$ such that $A \in \mathcal{H}$. We also denote by $\text{Aff}_{\mathcal{H}}(\mathcal{A}/\mathcal{D})$ the group of diffeomorphisms of $\mathcal{A}/\mathcal{D}$ induced by affine maps in $\text{Aff}_{\mathcal{H}}(\mathcal{A})$.

**Lemma 3.6.** Let $\mathcal{A}$ and $\mathcal{H}$ be linear algebraic groups and assume that $\mathcal{H}$ acts on $\mathcal{A}$ by rational automorphisms. Let $\mathcal{N} \subset \mathcal{A}$ be an $\mathcal{H}$-invariant normal subgroup of $\mathcal{A}$ and let $\mathcal{D}$ be a quasi-lattice in $\mathcal{N}$. Let $\psi \in \text{Aff}_{\mathcal{H}}(\mathcal{N}/\mathcal{D})$ be an affine diffeomorphism.

There exist a connected Lie group $\mathcal{N}''$ such that $\mathcal{N} \subset \mathcal{N}''$, a quasi-lattice $\mathcal{D}'$ of $\mathcal{N}''$, a non-trivial one-parameter subgroup $\phi_R \subset \mathcal{N}''$ and an affine map $F$ of $\mathcal{N}$ with the following properties:

1. The map $F$ is induces a periodic affine map $F_m$ of $\mathcal{N}/\mathcal{D}$. Let $\mathcal{F}$ be the group generated by $F_m$.
2. the map $F$ commutes with the action of $\phi_R$ by left translations on $\mathcal{N}$; hence we obtain a quotient flow $(\mathcal{F}/\mathcal{N}/\mathcal{D}, \phi_R)$ on the double coset space $\mathcal{F}/\mathcal{N}/\mathcal{D}$.
3. the flow $(\mathcal{F}/\mathcal{N}/\mathcal{D}, \phi_R)$ is smoothly conjugate to a suspension of the affine map $\psi$ on $\mathcal{N}/\mathcal{D}$.

The map $F$ yields an affine map of the Levi factor $\mathcal{L}$ of $\mathcal{N}$.

**Proof.** The argument is adapted from the initial step of the proof of Theorem 7.1 in Dani’s paper [9].

Since $\text{Aff}_{\mathcal{H}}(\mathcal{N}/\mathcal{D}) < \text{Aff}_{\mathcal{H}}(\mathcal{A})$, we may consider the algebraic hull $\langle \psi \rangle^Z$ of the subgroup generated by $\psi$ in the algebraic group $\text{Aff}_{\mathcal{H}}(\mathcal{A})$. The group $\langle \psi \rangle^Z$ is Abelian with finitely many connected components, hence there exists an element $f \in \langle \psi \rangle^Z$ of order $m$ and a one-parameter group $\tilde{\phi}_R = (\tilde{\phi}_1)_{t \in \mathbb{R}} < \langle \psi \rangle^Z$ commuting with $f$ such that $\psi = f \circ \tilde{\phi}_1$. This implies that $\psi^m = \tilde{\phi}_m$. (Here we used the symbol $\circ$ for the product in $\text{Aff}_{\mathcal{H}}(\mathcal{A})$ as we think of it as a group of maps of $\mathcal{A}$.)

The group $\tilde{\phi}_R$ is non-trivial, otherwise $\psi = f$ and $\psi^m = \text{id}$, but by hypothesis the order of $\psi$ is infinite.

Let us embed the group $\mathcal{A}$ (and its subgroups) into $\text{Aff}_{\mathcal{H}}(\mathcal{A})$ by identifying elements of the group with the associated left translations. Since the subgroup $\mathcal{N}$ is normal in $\mathcal{A}$ and is $\mathcal{H}$ invariant, it follows that it embeds as a normal subgroup of $\text{Aff}_{\mathcal{H}}(\mathcal{A})$. The subgroup $\mathcal{N}$ of $\text{Aff}_{\mathcal{H}}(\mathcal{A})$ generated by the subgroups
\( \hat{\psi} \) and \( \mathcal{N} \) may fail to be a topological group. For this reason we will then define \( \hat{\mathcal{N}} := \mathcal{N} \times \hat{\phi}_R \) to be the external semi-direct product of the universal cover \( \hat{\phi}_R \) of \( \hat{\psi} \) which acts on the normal subgroup \( \mathcal{N} \) by inner automorphisms. We begin by explaining the algebraic outline of the construction, ignoring, at first, the topological difficulty mentioned above.

Since \( \psi \in \text{Aff}_\mathcal{A}(\mathcal{N}/\mathcal{D}) \), its automorphism part \( A_\psi := \text{Aut}(\psi) \) belongs to \( \mathcal{A} \) and maps the quasi-lattice \( \mathcal{D} \) into itself. We claim that the product \( \hat{\mathcal{D}} := \mathcal{D} \cdot (A_\psi^m)_{m \in \mathbb{Z}} \) is a subgroup of \( \hat{\mathcal{N}} \). In fact, since \( A_\psi^m = \text{Aut}(\hat{\phi}_m) \) and the translation part of \( \hat{\phi}_m = \psi^m \) belongs to \( \mathcal{N} \), it follows that \( A_\psi^m \in \hat{\mathcal{N}} \). Since \( \mathcal{D} \) is a subgroup in \( \mathcal{N} \), it follows that \( \hat{\mathcal{D}} \) is a subgroup of \( \hat{\mathcal{N}} = \phi_R \circ \mathcal{N} \). Let \( \bar{F} \) be defined by

\[
\bar{F}(x) = f \circ x \circ A_\psi^{-1}, \quad \text{for every } x \in \hat{\mathcal{N}}.
\]

The map \( \bar{F} \) is a diffeomorphism of \( \hat{\mathcal{N}} \) onto itself. In fact, by construction, the affine map \( f \) coincides with \( \phi_{-1} \circ \psi \) and commutes with the group \( \hat{\phi}_R \). Furthermore \( \psi \circ n = \psi(n) \circ A_\psi \), for all \( n \in \mathcal{N} \). Hence for all \( (t, n) \in \mathbb{R} \times \mathcal{N} \), we have

\[
\begin{align*}
 f \circ \phi_t \circ n \circ A_\psi^{-1} &= \phi_t \circ f \circ n \circ A_\psi^{-1} = \phi_t \circ \phi_{-1} \circ \psi \circ n \circ A_\psi^{-1} \\
 &= \phi_{t-1} \circ \psi(n) \in \hat{\mathcal{N}}.
\end{align*}
\]

Since the diffeomorphism \( \bar{F} \) commutes with the action of the subgroup \( \hat{\mathcal{D}} \) by multiplication on the right, since \( f \) has finite order \( m \in \mathbb{N} \) in the group \( \text{Aff}_\mathcal{A}(\mathcal{A}) \) and since \( A_\psi^m \in \hat{\mathcal{D}} \), it follows that the diffeomorphism \( \bar{F} \) induces a periodic diffeomorphism \( \bar{F}_m \) of order \( m \) of the quotient space \( \mathcal{N}/\hat{\mathcal{D}} \).

Let \( \mathcal{F} \approx \mathbb{Z} / m \mathbb{Z} \) be the group of diffeomorphisms of \( \mathcal{N}/\hat{\mathcal{D}} \) generated by \( F_m \). The one-parameter group \( \hat{\phi}_R \subset \mathcal{N} \) acts by left translations on the quotient \( \mathcal{F} \setminus \mathcal{N}/\hat{\mathcal{D}} \) since the map \( \bar{F} \) commutes with the left multiplication by the one-parameter group \( \hat{\phi}_R \). By the above formula, it is also clear that the left translation by the element \( \hat{\phi}_1 \) on \( \mathcal{F} \setminus \mathcal{N}/\hat{\mathcal{D}} \), restricted to the subset \( \mathcal{N}/\hat{\mathcal{D}} \), coincides with the map \( \psi \) on \( \mathcal{N}/\hat{\mathcal{D}} \).

The map \( \bar{F} \) is an affine diffeomorphisms of \( \mathcal{N} \). In fact, the inner automorphism \( \text{Int}(\psi) \) determined by \( \psi \) on \( \text{Aff}_\mathcal{A}(\mathcal{A}) \) induces an automorphism of \( \mathcal{N} \) as shown by the following formula. For \( j \in \mathbb{Z} \), let \( n_{\psi^j} \in \mathcal{N} \) be the translation part of the affine map \( \psi^j \) which we write as \( \psi^j = n_{\psi^j} \circ A_{\psi^j} \). For all \( (t, n) \in \mathbb{R} \times \mathcal{N} \) we have

\[
\text{Int}(\psi)(\phi_t \circ n) = \psi \circ \phi_t \circ n \circ \psi^{-1} = \phi_t \circ \psi \circ n \circ \psi^{-1} = \phi_t \circ \psi(n) \circ n_{\psi^j}^{-1} \in \mathcal{N}.
\]

It follows that the map \( \bar{F} \) is the composition of an automorphism of \( \mathcal{N} \) (restriction of the inner automorphism \( \text{Int}(\psi) \) of \( \text{Aff}_\mathcal{A}(\mathcal{A}) \) given by \( \psi \)), followed by a left translation (by the element \( \phi_{-1} \in \mathcal{N} \)) and by a right translation (by the element \( n_{\psi} \in \mathcal{N} \)), that is, for all \( x \in \mathcal{N} \),

\[
(8) \quad \bar{F}(x) := \phi_{-1} \circ \text{Int}(\psi)(x) \circ n_{\psi}.
\]

As any right translation is an affine map given by an inner automorphism followed by a left translation, we have proved that the diffeomorphism \( \bar{F} \) is affine.
Let us now take care of the topological part of the construction. As the universal cover of the group \( \hat{\phi}_R \) is isomorphic to \( \mathbb{R} \), we define the group \( \hat{\mathcal{N}} := \mathcal{N} \rtimes \mathbb{R} \) to be the external semi-direct product of the group \( \mathbb{R} \) which acts on the normal subgroup \( \mathcal{N} \) via the action of the one-parameter group \( \hat{\phi}_R \) by inner automorphisms. In other terms, by definition the group \( \hat{\mathcal{N}} \) is the Cartesian product \( \mathcal{N} \times \mathbb{R} \) endowed with the following product law: for all \( (n_1, t_1), (n_2, t_2) \in \mathcal{N} \times \mathbb{R} \), we let

\[
(n_1, t_1) \ast (n_2, t_2) = (n_1 \circ \hat{\phi}_{t_1} \circ n_2 \circ \hat{\phi}_{t_1}^{-1}, t_1 + t_2).
\]

The group \( \hat{\mathcal{N}} \) endowed with the product topology is a connected Lie group. By construction there exists a group epimorphism \( \pi : (n, t) \in \hat{\mathcal{N}} \rightarrow n \circ \hat{\phi}_t \in \mathcal{N} \). Let \( \hat{\mathcal{D}} := \pi^{-1}(\mathcal{D}) \) be the inverse image of the subgroup \( \hat{\mathcal{D}} \subset \hat{\mathcal{N}} \). By definition of the subgroup \( \hat{\mathcal{D}} \), the group \( \hat{\mathcal{D}} \) is generated by \( \mathcal{D} \times \{0\} \) and by the elements \( A_{\gamma} = n_{\gamma}^{-1} \circ \psi^m \circ n_{\gamma} \circ \hat{\phi}_m \) hence is given by following formula:

\[
\hat{\mathcal{D}} = \{ (\gamma n_{\gamma}^{-1} m) | k \in \mathbb{Z}, \gamma \in D \}.
\]

It follows that the subgroup \( \hat{\mathcal{D}} \) is a quasi-lattice in \( \hat{\mathcal{N}} \) as its projection on the second coordinate is the lattice \( m\mathbb{Z} \) in \( \mathbb{R} \), and for each \( t \in \mathbb{R} \) the fiber over \( t \) of \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \) is the finite volume space \( \mathcal{N} / \mathcal{D} \).

The one-parameter group \( \hat{\phi}_R \) defined by \( \hat{\phi}_t = (1, t) \), for all \( t \in \mathbb{R} \), acts by left multiplication on \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \), hence defining a flow on \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \) denoted by the same letter. For all \( (1, t), (n, 0) \in \hat{\mathcal{N}} \), let

\[
F((1, t), (n, 0)) := (1, t - 1) \ast (\psi(n), 0).
\]

It is clear, by the definition, that \( F : \hat{\mathcal{N}} \rightarrow \hat{\mathcal{N}} \) is a diffeomorphism commuting with the flow of left translations by \( \hat{\phi}_R \). Furthermore, formula (8) shows that the map \( F \) projects, via the epimorphism \( \pi : \hat{\mathcal{N}} \rightarrow \mathcal{N} \), to the map \( \tilde{F} : \hat{\mathcal{N}} \rightarrow \hat{\mathcal{N}} \).

We claim that the map \( F \) induces a periodic diffeomorphism \( F_m \) of order \( m \) of the quotient space \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \) and we denote by \( \mathcal{F} = \mathbb{Z} / m\mathbb{Z} \) the finite group of affine diffeomorphisms of \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \) generated by \( F_m \). To see this, it suffices to verify that \( F_m \) induces the identity map on \( \hat{\mathcal{N}} / \hat{\mathcal{D}} \). In fact, for all \( (1, t), (n, 0) \in \mathcal{N} \times \mathbb{R} \), we have

\[
F^m((1, t), (n, 0)) \ast \hat{\mathcal{D}} = (1, t - m) \ast (\psi^m(n), 0) \ast (n^{-1}_{\psi^m}, m) \ast \hat{\mathcal{D}} = (1, t) \ast (1, -m) \ast (\psi^m(n)n^{-1}_{\psi^m}, 0) \ast (1, m) \ast \hat{\mathcal{D}} = (1, t) \ast (1, -m) \ast (\text{Int}(\psi^m(n), 0)) \ast (1, m) \ast \hat{\mathcal{D}} = (1, t) \ast (n, 0) \ast \hat{\mathcal{D}}.
\]

It follows from formula (10) that the points \( (1, t, (n, 0)) = \psi_t((n, 0)) \) and \( \psi_{t-1}((\psi(n), 0)) \) are identified by the map \( F \), and therefore by the group \( F \) of affine diffeomorphisms generated by \( F \). Thus, in the quotient space \( \mathcal{F} \backslash \mathcal{N} \), the first return map of the flow \( \phi_t \) to the transverse section \( \mathcal{N} \times \{0\} \approx \mathcal{N} \) coincides with the time-1 map of the flow and is conjugated to the affine map \( \psi \) on \( \mathcal{N} \) in
the sense that
\[ F(\phi_1(n,0)) = (\psi(n),0) \quad \text{for all } n \in \mathcal{N}. \]
By passing to the quotient by \( \mathcal{D} \), it follows that the return map of the flow \( \phi_R \) to the transverse section \( (\mathcal{N} \times \{0\})/\mathcal{D} \approx \mathcal{N}/\mathcal{D} \) in the double-coset space \( \mathcal{F}/\mathcal{N}/\mathcal{D} \) coincides with the time-1 map and is conjugated to the affine map \( \psi \) on \( \mathcal{N}/\mathcal{D} \).

Let \( \mathcal{L} \) be a Levi subgroup of the group \( \mathcal{N} \) and let \( q: \mathcal{N} \to \mathcal{L} \) be the projection of group \( \mathcal{N} \) onto \( \mathcal{L} \), with kernel the radical of \( \mathcal{N} \). As the radical is a characteristic subgroup of \( \mathcal{N} \), any affine map of \( \mathcal{N} \) projects, via the homomorphism \( q \), to an affine map of the Levi subgroup \( \mathcal{L} \). In particular the affine diffeomorphism \( F \) and the one-parameter group \( \phi_R \) project to an affine diffeomorphism \( q(F) \) of \( \mathcal{L}/q(\mathcal{D}) \) and to a one-parameter group \( q(\phi_R) < \mathcal{L} \), inducing commuting actions on \( \mathcal{L}/q(\mathcal{D}) \).

The argument is therefore completed.

**Remark 3.7.** To prove Theorem 1.1, we may limit ourselves to consider the case where the flow \( \phi_R \) is ergodic on \( G/D \) with respect to a finite \( G \)-invariant measure. This is due to the fact that the ergodic components of the flow \( \phi_R \) are closed subsets of \( G/D \) (see [43, Thm. 2.5]). Since \( G/D \) is connected, if the flow \( \phi_R \) is not ergodic, then it has infinitely many ergodic components, in which case Theorem 1.1 follows. The proof of Theorem 1.2 relies, after some technical reductions, on Theorem 1.1 and Lemma 3.6.

Let us explain the main difficulty in applying Lemma 3.6. We recall that, by Proposition 3.4, the space of invariant distribution for a suspension flow \( (\Sigma M, \phi_R) \) of a diffeomorphism \( (M, \psi) \) is isomorphic to the space of invariant distribution for the diffeomorphism \( \psi \). However, the flow \( (\mathcal{N}/\mathcal{D}, \phi_R) \) constructed in Lemma 3.5 is the suspension of a power of the affine diffeomorphism \( \psi \) on \( \mathcal{N}/\mathcal{D} \). The actual suspension of the diffeomorphism \( \psi \) is given by the projected flow \( (\mathcal{F}/\mathcal{N}/\mathcal{D}, \phi_R) \). Now, invariant distributions for the flow \( (\mathcal{N}/\mathcal{D}, \phi_R) \), produced, for example, by Theorem 1.1, may vanish when projected on the space \( \mathcal{F}/\mathcal{N}/\mathcal{D} \), and special care must be taken to avoid this problem. We remark, however, that this difficulty does not arise when the invariant distributions for the flow \( (\mathcal{N}/\mathcal{D}, \phi_R) \) are given by measures, since the projections of ergodic probability measures of a flow under a finite-to-one projection are ergodic probability measures for the projected flow. Thus, in applying Lemma 3.6, we may suppose that the homogeneous flow \( (\mathcal{N}/\mathcal{D}, \phi_R) \) there constructed is ergodic, by the same argument applied at the beginning of this remark.

Henceforth we shall consider ergodic flows. Whenever convenient we may also assume that \( G \) is simply connected by pulling back the isotropy group \( D \) to the universal cover of \( G \). We remark that if \( D \) is a quasi-lattice this pull-back is also a quasi-lattice.

Let \( G = L \times R \) be the Levi decomposition of a simply connected Lie group \( G \) and let \( G_\infty \) be the smallest connected normal subgroup of \( G \) containing the Levi factor \( L \). Let \( q: G \to L \) be the projection onto the Levi factor. We shall use the following result:
Theorem 3.8 ([9, 42], [43, Lemma 9.4, Thm. 9.5]). If G is a simply connected Lie group and the flow \((G/D, \phi_G)\) on the finite volume space \(G/D\) is ergodic, then

- The group \(R \cap D\) is a quasi-lattice in \(R\).
- It follows that \(R/R \cap D\) is compact, that \(q(D)\) is closed in \(L\) and that \(G/D\) factors onto \(L/q(D) \cong G/RD\) with fiber \(R/R \cap D\).
- The semi-simple flow \((L/q(D), q(\phi_G))\) and the solvable flow \((G/G_\infty \cap D, \phi_G)\) are ergodic.

By Theorem 3.8 it is possible to reduce the analysis of the general case to that of the semi-simple and solvable cases. In fact, the following basic result holds.

Lemma 3.9. Let \(p : G \to G^{(1)}\) be an epimorphism with \(D \subset p^{-1}(D^{(1)})\) and let \(\hat{\phi} : G/D \to G^{(1)}/D^{(1)}\) be the induced quotient map. Let \(\psi : G/D \to G/D\) and \(\psi^{(1)} : G^{(1)}/D^{(1)} \to G^{(1)}/D^{(1)}\) be smooth maps intertwined by \(\hat{\phi}\). Then the dimension of the space of \(\psi\)-invariant distributions on \(G/D\) of Sobolev order at most \(s\) is greater than or equal to the dimension of the space of \(\psi^{(1)}\)-invariant distributions \(\mathcal{D}'(G^{(1)}/D^{(1)})\) of Sobolev order at most \(s\).

In particular, if the flow \((G/D, \phi_G)\) of \(G/D\) projects onto a flow \((G^{(1)}/D^{(1)}, \phi_{G^{(1)}})\) via the epimorphism \(p\), then the existence of countably many independent invariant distributions for the flow \((G^{(1)}/D^{(1)}, \phi_{G^{(1)}})\) implies the existence of countably many independent invariant distributions for the flow \((G/D, \phi_G)\). An analogous statement is valid for an affine map \(\psi\) projecting, via \(p\), onto an affine map \(\psi^{(1)}\) of \(G^{(1)}/D^{(1)}\).

Proof. Let \(H = L^2(G^{(1)}/D^{(1)})\) and let \(\hat{\phi} : G/D \to G^{(1)}/D^{(1)}\) be the map induced by the epimorphism \(p\). The map \(\hat{\phi}\) is \(G\)-equivariant for the natural left action of \(G\) on \(G/D\) and \(G^{(1)}/D^{(1)}\) and measure preserving for the \(G\)-invariant probability measures on these spaces. It follow that the pull-back map \(\hat{\phi}^*\) is a \(G\)-equivariant isometry of \(H\) onto the \(G\)-invariant closed subspace \(\hat{\phi}^*(H) \subset L^2(G/D)\). We therefore define the push-forward map \(\hat{\phi}_*: \hat{\phi}^*(H) \to H\) as the inverse of \(\hat{\phi}^*\).

Since the orthogonal decomposition \(L^2(G/D) = \hat{\phi}^*(H) \oplus \hat{\phi}^*(H)^\perp\) is \(G\)-invariant, for any smooth function \(f\) on \(G/D\) its components in this orthogonal decomposition are smooth. It follows that the push-forward map \(\hat{\phi}_*\) is a linear map of the space of compactly supported smooth functions \(C_0^\infty(G/D)\) onto a dense subspace of \(C_0^\infty(G^{(1)}/D^{(1)})\). Setting, for any \(D \in \mathcal{D}'(G^{(1)}/D^{(1)})\) and any \(f \in C_0^\infty(G/D)\),
\[
\hat{\phi}^*(D)(f) = D(\hat{\phi}_*(f)),
\]
we obtain a linear continuous injection of \(\mathcal{D}'(G^{(1)}/D^{(1)})\) into \(\mathcal{D}'(G/D)\). This map preserves the Sobolev order of distributions because it is \(G\)-equivariant.

For any pair of smooth maps \(\psi\) and \(\psi^{(1)}\) of \(G/D\) and \(G^{(1)}/D^{(1)}\) respectively, such that \(\hat{\phi} \circ \psi = \psi^{(1)}\), if \(D \in \mathcal{D}'(G^{(1)}/D^{(1)})\) is \(\psi\)-invariant, the image distribution \(\hat{\phi}^*(D)\) is \(\psi^{(1)}\) invariant. Thus the dimension of the space of \(\psi\)-invariant distributions in \(\mathcal{D}'(G/D)\) is greater than or equal to the dimension of the space of \(\psi^{(1)}\)-invariant distributions in \(\mathcal{D}'(G^{(1)}/D^{(1)})\).

In dealing with solvable groups it is useful to recall the theorem by Mostow (see [43, Theorem E.3]).
**Theorem 3.10** (Mostow). If $G$ is a solvable Lie group, then $G/D$ is of finite volume if and only if $G/D$ is compact.

When $G$ is semi-simple, in proving Theorem 1.1, we may suppose that $G$ has finite center and that the isotropy group $D$ is a lattice. This is the consequence of the following proposition.

**Proposition 3.11.** Let $G$ be a connected semi-simple group and let $G/D$ be a finite volume space. If there exists an ergodic flow on $G/D$, then the connected component of the identity of $D$ in $G$ is normal in $G$. Hence we may assume that $G$ has finite center and that $D$ is discrete.

*Proof.* We have a decomposition $G = K \cdot S$ of $G$ as the almost-direct product of a compact semi-simple normal subgroup $K$ and of a totally non-compact normal semi-simple group $S$. Let $\rho : G \to K^{(1)} := G/S$ be the projection of $G$ onto the semi-simple compact connected group $K^{(1)}$. Let $\phi_t$ be the flow generated by $\tilde{X} = \rho_t X$ on the connected, compact, Hausdorff space $Y := K^{(1)}/p(D)$. As $Y$ is a homogeneous space of a compact semi-simple group, the fundamental group of $Y$ is finite. The closure of the one-parameter group $(\exp t \tilde{X})_{t \in \mathbb{R}}$ in $K^{(1)}$ is a torus subgroup $T < K^{(1)}$; it follows that the closures of the orbits of $\phi_t$ on $Y$ are the compact tori $Tk\bar{p}(D)$, $(k \in K^{(1)})$, homeomorphic to $T/T \cap k\bar{p}(D)k^{-1}$.

Let $\phi_\mathbb{R}$ be an ergodic flow on $G/D$, generated by $X \in \mathfrak{g}_0$. Since $\phi_t$ acts ergodically on $Y$, the action of $T$ on $Y$ is transitive. In this case we have $Y = T/T \cap \bar{p}(D)$, and since the space $Y$ is a torus with finite fundamental group, it is reduced to a point. It follows that $T \leq \bar{p}(D) = K^{(1)}$. Thus $p(D)$ is dense in $K^{(1)} = G/S$ and $SD$ is dense in $G$.

Let $\bar{D}^Z$ denote the Zariski closure of $\text{Ad}(D)$ in $\text{Ad}(G)$ (we refer to Remark 1.6 in [10]). By Borel Density Theorem (see [10, Thm. 4.1, Cor. 4.2] and [36]) the hypothesis that $G/D$ is a finite volume space implies that $\bar{D}^Z$ contains all hyperbolic elements and unipotent elements in $\text{Ad}(G)$. As these elements generate $\text{Ad}(S)$, we have $\text{Ad}(S) < \bar{D}^Z$, and the density of $SD$ in $G$ implies $\text{Ad}(G) = \bar{D}^Z$. Since the group of $\text{Ad}(g) \in \text{Ad}(G)$ such that $\text{Ad}(g)(\text{Lie}(D)) = \text{Lie}(D)$ is a Zariski-closed subgroup of $\text{Ad}(G)$ containing $\bar{D}^Z$, we obtain that the identity component $D^0$ of $D$ is a normal subgroup of $G$ and $G/D \cong (G/D^0)/(D/D^0)$. We have thus proved that we can assume that $D$ is discrete. We can also assume that $G$ has finite center since $D$ is a lattice in $G$ and therefore it meets the center of $G$ in a finite index subgroup of the center. This concludes the proof. \hfill \Box

Our proof of Theorem 1.1 considers separately the cases of quasi-unipotent and partially hyperbolic flows. We recall the relevant definitions.

Let $X$ be the generator of the one-parameter subgroup $\phi_\mathbb{R}$ and let $\mathfrak{g}^\mu$ denote the generalized eigenspaces of eigenvalue $\mu$ of $\text{ad}(X)$ on $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. The Lie algebra $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}^\mu$ and we have $[\mathfrak{g}^\mu, \mathfrak{g}^\nu] \subset \mathfrak{g}^{\mu+\nu}$. Let

$$p^0 = \sum_{\Re\mu = 0} \mathfrak{g}^\mu, \quad p^+ = \sum_{\Re\mu > 0} \mathfrak{g}^\mu, \quad p^- = \sum_{\Re\mu < 0} \mathfrak{g}^\mu.$$
**Definition 3.12.** A flow $\phi_R$ on $G/D$ is called *quasi-unipotent* if $g = p^0$, and it is *partially hyperbolic* otherwise. Thus the flow subgroup $\phi_R$ is quasi-unipotent or partially hyperbolic according to whether the spectrum of the group $\text{Ad}(\phi_t)$ acting on $g$ is contained in $U(1)$ or not.

4. The quasi-unipotent case

4.1. The semi-simple quasi-unipotent case. In this subsection we assume that the group $G$ is semi-simple and that the one-parameter subgroup $\phi_R$ is quasi-unipotent.

**Definition 4.1.** An $\text{sl}_2(\mathbb{R})$ triple $(a, n^+, n^-)$ in a Lie algebra $g_0$ is a non-zero triple satisfying the commutation relations

$$[a, n^\pm] = \pm n^\pm, \quad [n^+, n^-] = 2a.$$

We prove a generalized version of the Jacobson-Morozov Lemma [25].

**Lemma 4.2** (Jacobson-Morozov Lemma). Let $n^+$ be a nilpotent element in a semi-simple Lie algebra $g_0$. Assume that $n^+$ is invariant under the action of a compact subgroup $\mathcal{F}$ of automorphisms of the Lie algebra $g_0$. Then we can find a semi-simple element $a \in g_0$ and a nilpotent element $n^-_1$ such that $(a, n^+, n^-)$ is an $\text{sl}_2(\mathbb{R})$ triple invariant under the action of $\mathcal{F}$.

**Proof.** By the Jacobson-Morozov Lemma there exists a semi-simple element $a_0$ and a nilpotent element $n^-_0$ such that $(a_0, n^+, n^-_0)$ is an $\text{sl}_2(\mathbb{R})$ triple. Let $m$ be the probability Haar measure on $\mathcal{F}$. We define

$$a := \int_{\mathcal{F}} f(a_0) \, dm(f)$$

and

$$n^-_1 := \int_{\mathcal{F}} f(n^-_0) \, dm(f).$$

By construction the element $a$ is $\mathcal{F}$-invariant. Since $n^+$ is also $\mathcal{F}$-invariant and $(a_0, n^+, n^-_0)$ is an $\text{sl}_2(\mathbb{R})$ triple it follows that

$$[a, n^+] = \int_{\mathcal{F}} [f(a_0), n^+] \, dm(f) = \int_{\mathcal{F}} f([a_0, n^+]) \, dm(f) = \int_{\mathcal{F}} f(n^+) \, dm(f) = n^+;$$

$$[n^+, n^-_1] = \int_{\mathcal{F}} [n^+, f(n^-_0)] \, dm(f) = \int_{\mathcal{F}} f([n^+, n^-_0]) \, dm(f) = 2\int_{\mathcal{F}} f(a_0) \, dm(f) = 2a.$$

By Morozov’s Lemma, there is an element $n^-_2$ such that $(a, n^+, n^-_2)$ is a $\text{sl}_2(\mathbb{R})$ triple. Let us then define

$$n^- := \int_{\mathcal{F}} f(n^-_2) \, dm(f).$$
Then \( a, n^+ \) and \( n^- \) are \( \mathcal{F} \)-invariant. Moreover,
\[
[n^+, n^-] = \int_{\mathcal{F}} [n^+, f(n^-)] \, dm(f) = \int_{\mathcal{F}} f((n^+, n^-)) \, dm(f) \\
= 2 \int_{\mathcal{F}} f(a) \, dm(f) = 2a;
\]
\[
[a, n^-] = \int_{\mathcal{F}} [a, f(n^-)] \, dm(f) = \int_{\mathcal{F}} f([a, n^-]) \, dm(f) \\
= -\int_{\mathcal{F}} f(n^-) \, dm(f) = -n^-.
\]
In conclusion, the elements \((a, n^+, n^-)\) form an \( sl_2(\mathbb{R}) \) triple (in particular \( n^- \) is a non-trivial nilpotent element), invariant under the action of the compact subgroup \( \mathcal{F} \subset \text{Aut}(\mathfrak{g}_0) \).

Given a unitary representation of \((\pi, H)\) of a Lie group on a Hilbert space \( H \), we denote by \( H^\infty \) the subspace of \( C^\infty \)-vectors of \( H \) endowed with the \( C^\infty \) topology, and by \((H^\infty)'\) its topological dual.

**Lemma 4.3** ([16]). Let \( U_t \) be a unipotent subgroup of \( \text{PSL}_2(\mathbb{R}) \). For each non-trivial irreducible unitary representation \((\pi, H)\) of \( \text{PSL}_2(\mathbb{R}) \) there exists a distribution, i.e., an element of \( D \in (H^\infty)' \), such that \( U_t D = D \).

**Proposition 4.4.** Let \( G \) be a semi-simple group and let \( G/D \) be a finite volume space. Suppose that \( \phi_\mathbb{R} \) is a quasi-unipotent subgroup of \( G \) such that the flow of \( \phi_\mathbb{R} \) on \( G/D \) is ergodic and commutes with the action of a finite Abelian group \( \mathcal{F} \) of affine diffeomorphisms of \( G/D \). Then, there exists infinitely many independent \( \phi_\mathbb{R} \)-invariant distributions on \( C^\infty(G/D) \) of Sobolev order \( 1/2 \) which are also \( \mathcal{F} \)-invariant.

**Proof.** By Proposition 3.11 we may assume that \( G \) has finite center and that \( D \) is a lattice in \( G \). By the Jordan decomposition we can write \( \phi_t = c_t \times u_t \) where \( c_t \) is semi-simple and \( u_t \) is unipotent with \( c_t, u_t \) commuting one-parameter subgroups of \( G \). Since \( c_t \) is semi-simple and quasi-unipotent, its closure in \( G \) is a torus \( T \). By Lemma 3.1, the group of affine diffeomorphisms \( \mathcal{F} \) acts by conjugation by automorphism of the Lie algebra \( \mathfrak{g} \) of \( G \) fixing the subgroup \( \phi_\mathbb{R} \). By the uniqueness of the Jordan decomposition of \( \phi_\mathbb{R} \), the action by conjugation of \( \mathcal{F} \) fixes the subgroups \( c_t \) and \( u_t \), and consequently it fixes the torus \( T \), closure of \( c_t \) in \( G \). Let \( \mathcal{F} \) be the compact Abelian group of affine diffeomorphisms of \( G/D \) generated by \( \mathcal{F} \) and left translations by elements of torus \( T \).

By the generalized version of the Jacobson-Morozov Lemma 4.2 we can find an \( sl_2(\mathbb{R}) \) triple \((a, n^+, n^-)\) invariant under \( \mathcal{F} \). Thus the action by left translations on \( G/D \) of the analytic group \( S \) generated by the triple \((a, n^+, n^-)\) commutes with the action of \( \mathcal{F} \).

It is well known that the center \( Z(S) \) of \( S \) is finite and that, consequently, there exists a maximal compact subgroup \( K \approx S^1 \) of \( S \) containing \( Z(S) \). (Indeed, the adjoint representation \( \text{Ad}_G|S \) of \( S \) on the Lie algebra of \( G \), as a finite dimensional representation of \( S \), factors through \( SL(2,\mathbb{R}) \), a double cover of \( S/Z(S) \). The
kernel of \(\text{Ad}_G|S\) is contained in \(Z(G)\), because \(G\) is connected. Since \(Z(S)\) is monogenic, we have that \(Z(G)\) is a subgroup of index one or two of \(Z(G)Z(S)\). It follows that \(Z(S)\) is finite.

The group \(T^{(0)} = \hat{\mathcal{F}} \cdot K\) is a compact, Abelian group of affine transformation of \(G/D\) whose connected component of identity is a torus. It follows that the double coset space \(T^{(0)} \backslash G/D\) is a non-trivial orbifold and that the space \(H^{(0)}\) of \(L^2\) functions on \(G/D\) which are invariant under the action of \(T^{(0)}\) has infinite dimension. The space \(H^{(0)}\) is contained in the space \(H_\mathcal{F}\) of \(L^2\) functions on \(G/D\) invariant under \(Z(S)\) and \(\mathcal{F}\), space on which the group \(S\) acts unitarily. As the center \(Z(S)\) of \(S\) acts trivially on \(H_\mathcal{F}\), the Hilbert space \(H_\mathcal{F}\) decompose as a direct integral \(\int H_\alpha d\nu_\alpha\) of irreducible unitary representations \(H_\alpha\) of \(\text{PSL}_2(\mathbb{R})\), where \(\nu_\alpha\) is a measure on the unitary dual of \(\text{PSL}_2(\mathbb{R})\). Since every irreducible unitary representation of \(\text{PSL}_2(\mathbb{R})\) contains at most one \(K\)-invariant vector and since the space \(H^{(0)} \subset H_\mathcal{F}\) of \(K\) invariant vectors is infinite dimensional, we deduce that the measure \(\nu_\alpha\) has an infinite support, i.e., that the space \(H_\mathcal{F}\) is not a finite sum of irreducible unitary representations of \(\text{PSL}_2(\mathbb{R})\).

By the previous lemma each unitary irreducible representation \(H_\alpha\) of \(\text{PSL}_2(\mathbb{R})\) occurring in the support of \(\nu_\alpha\) contains a distribution \(D_\alpha \in (H^{\infty})'\) of Sobolev order 1/2 which is \(u_t\)-invariant. Since \(H_\mathcal{F}\) consists of functions that are \(\hat{\mathcal{F}}\)-invariant, this distribution is also invariant by translations \(\phi_t = c_t u_t\) and by the affine maps in \(\mathcal{F}\). Since the space \(H^\infty_\mathcal{F}\) coincides with the Fréchet space of \(C^\infty\) functions on \(G/D\) which are \(T \cdot Z(S)\) invariant as well as \(\mathcal{F}\)-invariant, the proposition is proved. \(\square\)

### 4.2. The solvable quasi-unipotent case.

In this subsection we assume that the group \(G = R\) is solvable and the one-parameter subgroup \(\phi_\theta\) is quasi-unipotent and ergodic on the finite volume space \(R/D\).

We recall the following definition.

**Definition 4.5.** A solvable group \(R\) is called a class (l) group if, for every \(g \in R\), the spectrum of \(\text{Ad}(g)\) is contained in the unit circle \(U(1) = \{z \in \mathbb{C} | |z| = 1\}\). It will also be useful remark that if \(R\) is solvable and \(R/D\) is a finite measure space, then we may assume that \(R\) is simply connected and that \(D\) is a quasi-lattice (in the language of Auslander and Mostow, the space \(R/D\) is then a presentation); in fact, if \(\tilde{R}\) is the universal covering group of \(R\) and \(\tilde{D}\) is the pull-back of \(D\) to \(\tilde{R}\), then the connected component of the identity \(\tilde{D}_0\) of \(\tilde{D}\) is simply connected [37, Them. 3.4]; hence \(R/D \approx \tilde{R}/\tilde{D} \approx R'/D'\), with \(R' = \tilde{R}/\tilde{D}_0\) solvable, connected and simply connected and with \(D' = \tilde{D}/\tilde{D}_0\) a quasi-lattice.

We also recall the construction, originally due to Malcev and generalized by Auslander et al., of the semi-simple or Malcev splitting of a simply connected, connected solvable group \(R\) (see [2, 5, 6, 22]).

A solvable Lie group \(G\) is split if \(G = N_G \rtimes T\) where \(N_G\) is the nilradical of \(G\) and \(T\) is an Abelian group acting on \(G\) faithfully by semi-simple automorphisms.
A semi-simple or Malcev splitting of a connected simply connected solvable Lie group $R$ is a split exact sequence

$$0 \to R \xrightarrow{m} M(R) \xrightarrow{\tau} T \to 0$$

embedding $R$ into a split connected solvable Lie group $M(R) = N_{M(R)} \times T$ such that $M(R) = N_{M(R)} \cdot m(R)$; here $N_{M(R)}$ and $T$ are as before. The image $m(R)$ of $R$ is normal and closed in $M(R)$ and it will be identified with $R$.

The semi-simple splitting of a connected simply connected solvable Lie group $R$ is unique up to an automorphism fixing $R$.

Let $\text{Aut}(\tau) \approx \text{Aut}(R)$ be automorphism group of the Lie algebra $\tau$ of $R$. The adjoint representation $\text{Ad}$ maps the group $R$ to the solvable subgroup $\text{Ad}(R) < \text{Aut}(\tau)$; since $\text{Aut}(\tau)$ is an algebraic group we may consider the Zariski closure $\text{Ad}(R)^*$ of $\text{Ad}(R)$. The group $\text{Ad}(R)^*$ is algebraic and solvable, since it’s the algebraic closure of the solvable group $\text{Ad}(R)$. It follows that $\text{Ad}(R)^*$ has a Levi-Chevalley decomposition $\text{Ad}(R)^* = U^* \rtimes T^*$, with $T^*$ an Abelian group of semi-simple automorphisms of $\tau$ and $U^*$ the maximal subgroup of unipotent elements of $\text{Ad}(R)^*$.

Let $T$ be the image of $\text{Ad}(R)$ into $T^*$ by the natural projection $\text{Ad}(R)^* \to T^*$. Since $T$ is a group of automorphisms of $R$, we may form the semi-direct product $M(R) = R \rtimes T$. By definition we have a split sequence

$$0 \to R \to M(R) \xrightarrow{\tau} T \to 0.$$

It can be proved that $M(R)$ is a split connected solvable group $N_{M(R)} \rtimes T$ and it is the semi-simple splitting of $R$ (see loc. cit.). We remark that the splitting $M(R) = N_{M(R)} \rtimes T$ yields two projection maps $\tau: M(R) \to T$ and $\pi: M(R) \to N_{M(R)}$ defined, for any $g \in M(R)$, by

$$g = \tau(g)\pi(g), \quad \tau(g) \in T, \quad \pi(g) \in N_{M(R)}.$$

The projection $\tau$ an epimorphism. Composing $\tau$ with the inclusion $R \to M(R)$ we obtain a surjective homomorphism $p: R \to T$.

The proof of the following easy lemma is omitted.

**Lemma 4.6.** Let $G$ be a Lie group, $\mathcal{H}$ a subgroup of automorphisms of $G$ and let $\hat{G}$ be the semi-direct product $G \rtimes \mathcal{H}$. Then any group $\mathcal{F}$ of automorphisms $A \in \text{Aut}(G)$ normalizing $\mathcal{H}$ extends to a group of automorphisms $\hat{A} \in \text{Aut}(\hat{G})$ by setting

$$\hat{A}(g \cdot H) = A(g) \cdot (AHA^{-1}), \quad \text{for all } A \in \mathcal{F}.$$ 

Hence every group $\mathcal{F}$ of affine maps $\psi = uA$ of $G$, such that $A$ normalizes $\mathcal{H}$ for all $\psi \in \mathcal{F}$, extends to a group of affine maps $\hat{\psi}$ of $\hat{G}$, defined by $\hat{\psi} = u \cdot \hat{A}$.

**Proposition 4.7.** Let $\mathcal{F}$ be a finite group of automorphisms of a connected, simply connected, solvable Lie group $R$. For any semi-simple splitting $M(R)$ of $R$ the group $\mathcal{F}$ extends to a group of automorphisms of $M(R)$, that is, there is a homomorphism $A \in \mathcal{F} \to \hat{A} \in \text{Aut}(M(R))$ such that $\hat{A} = A$ on $R$. 


Proof. By the previous lemma, it suffices to show that, if \( M(R) = R \rtimes T \), the
group \( T \) operates on \( R \) by a group of automorphisms normalized by \( \mathcal{F} \).

Since, for any \( A \in \text{Aut}(R) \), we have \( A\text{Ad}(r)^{-1} = \text{Ad}(A(r)) \), the group \( \text{Ad}(R) \)
and its Zariski closure \( \text{Ad}(R)^* \) are normal in \( \text{Aut}(R) \). Let us show that if \( \text{Ad}(R)^* = U^* \rtimes T^* \) is a Levi-Chevalley decomposition of \( \text{Ad}(R)^* \), then the torus \( T^* \) is normal-
ized by any finite group \( \mathcal{F} \) of automorphisms of \( R \).

In fact since any two tori in \( \text{Ad}(R)^* \) are conjugate in \( \text{Ad}(R)^* \), for any \( A \in \mathcal{F} \)
there exists an element \( u_A \in U^* \) such that \( AT^* A^{-1} = u_A T^* u_A^{-1} \). Thus, if we
denote by \( \text{Norm}_{U^*}(T^*) \) the normalizer of \( T^* \) in \( U^* \) we obtain a homomorphism
\( \mathcal{F} \to U^*/\text{Norm}_{U^*}(T^*) \). Since the group \( \mathcal{F} \) is finite and \( U^*/\text{Norm}_{U^*}(T^*) \) is a
real unipotent algebraic group, this homomorphism is trivial. Thus, for any \( A \in \mathcal{F} \), we have \( AT^* A^{-1} = T^* \).

By definition, for any \( r \in R \) such that \( \text{Ad}(r) = u \cdot t \), with \( u \in U^* \) and \( t \in T^* \),
the projection of \( \text{Ad}(r) \) in \( T^* \) is equal to the element \( t \). For all \( A \in \mathcal{F} \), we have \( A\text{Ad}(r)^{-1} = AuA^{-1} \cdot ATA^{-1} \). Since \( AT A^{-1} \in T^* \), we derive that the projection of \( A\text{Ad}(r)^{-1} \) in \( T^* \) is equal to \( AT A^{-1} \). We have therefore proved that \( AT A^{-1} \in T \),
for all \( t \in T \) and all \( A \in \mathcal{F} \), concluding the proof. \( \square \)

It is useful to recall a part of Mostow’s structure theorem for solvmanifolds,
as reformulated by Auslander [5, IV.3] and [6, p. 271]:

**Theorem 4.8** (Mostow, Auslander). Let \( D \) be a quasi-lattice in a simply connected,
connected, solvable Lie group \( R \), and let \( M(R) = R \rtimes T \) be a semi-simple
splitting of \( R \). Then \( T \) is a closed subgroup of \( \text{Aut}(R) \) and the projection \( \tau(D) \) of
\( D \) in \( T \) is a lattice of \( T \). If \( R \) is class (I), the semi-simple splitting \( M(R) \) of \( R \)
admits a structure of real algebraic group, with \( N_{M(R)} \) its unipotent radical and \( T \) a
maximal torus acting on \( N_{M(R)} \) by semi-simple automorphisms.

The following theorem was first proved in [2, Thm. 4.4] under the hypothesis
that \( D \cap N_{M(R)} = D \). This amounts to supposing that \( D \) is nilpotent, which is
the case when \( R/D \) supports a minimal flow, as it is proved in [6, Thm. C]. A
simplification of the latter proof under the hypothesis that \( R/D \) carries an ergo-
dic flow appears in [43, Theorem 7.1].

**Theorem 4.9** (Auslander, Starkov). Let \( \phi_R \) be an ergodic flow on a class (I) com-
 pact solvable manifold \( R/D \). There exists a semi-simple splitting \( M(R) = R \rtimes T = N_{M(R)} \rtimes T \) of \( R \) such that \( D < N_{M(R)} \) and the projection map \( \pi : M(R) \to N_{M(R)} \)
duces a diffeomorphism of \( R/D \) onto the compact nilmanifold \( N_{M(R)}/D \) conju-
gating the flow \( \phi_R \) to a nilflow.

In the following lemma we show that, for our purposes, we may assume that \( R \) is a class (I) solvable group and give a new proof that \( D \) is a subgroup of the
unipotent radical of the algebraic splitting.

**Lemma 4.10.** If the flow \( \phi_R \) is ergodic and quasi-unipotent on the finite volume
solvmanifold \( R/D \), then the group \( R \) is of class (I).
The orbits of by the one-parameter group that the affine map \( \hat{\rho} \) of the solvable group, the torus \( T \) tion \( M \) of semi-simple and unipotent elements. Since every semi-

\[ \text{Proof.} \] We may assume \( R \) simply connected and connected and \( D \) a quasi-lattice. Let \( M(R) = N_{M(R)} \times T = R \times T \) be the semi-simple splitting of \( R \). Since the one-

\[ \phi_R < R \] be a one-parameter group and let \( \mathcal{F} \) be a finite group of affine maps of \( R/(D) \) commuting with \( \phi_R \). Assume that the flow of \( \phi_R \) is ergodic and quasi-unipotent. Then there exists a semi-simple splitting \( M(R) = R \times T = N_{M(R)} \times T \) of \( R \) with the following properties:

- We have \( D < N_{M(R)} \). Hence the restriction to \( D \) of the projection map \( \pi: M(R) \to N_{M(R)} \), defined by (11), is the identity group isomorphism of \( D \).
- The map \( \pi \) induces a diffeomorphism of \( R/D \) onto the compact nilmanifold \( N_{M(R)}/D \) conjugating the flow \( \phi_R \) to a nilflow \( u_R \) on \( N_{M(R)}/D \).
- The group \( \mathcal{F} \) projects via \( \pi \) to a finite group \( \overline{\mathcal{F}} \) of affine maps on \( N_{M(R)}/D \) commuting with \( u_R \).

\[ \text{Proof.} \] The flow \( \phi_R \) is ergodic and quasi-unipotent on the finite volume solv-

manifold \( R/D \) hence, by Lemma 4.10, the group \( R \) is of class (I). The theorem of Mostow and Auslander 4.8 states that \( R \) embeds into the algebraic solvable group \( M(R) \). Thus, we may consider, for all \( t \in \mathbb{R} \), the Jordan decomposition \( \phi_t = a_t u_t \) of the element \( \phi_t \in M(R) \): here \( a_t \) and \( u_t \) are commuting one-parameter groups, respectively, of semi-simple and unipotent elements. Since every semi-

simple element is included in a torus of \( M(R) \), the Levi-Chevalley decomposi-
tion \( M(R) = N_{M(R)} \rtimes T \), may be chosen so that \( a_t < T \). Since \( R \) is a class (I) solvable group, the torus \( T \) is the closure of the one-parameter group \( a_R \).

For any \( \psi = g A \in \mathcal{F} \) set \( A_{\psi} := A \). By Proposition 4.7, the finite group of automorphisms of \( R \) defined by \( \{ A_{\psi} \in \text{Aut}(R) \mid \psi \in \mathcal{F} \} \) extends to a finite group \( \{ \hat{A}_{\psi} \in \text{Aut}(M(R)) \mid \psi \in \mathcal{F} \} \).

Let \( \psi = g A \in \mathcal{F} \). As the map \( \psi \) commutes with the one-parameter group \( \phi_R \) for all \( t \in \mathbb{R} \) we have the identity \( g A(\phi_t)g^{-1} = \phi_t = g \hat{A}(\phi_t)g^{-1} \), which implies that the affine map \( \hat{\psi} := \hat{g} \hat{A} \) of \( M(R) \) commutes with the flow of left translation by the one-parameter group \( \phi_R \) on \( M(R) \). This identity also implies, by the
uniqueness to the Jordan decomposition, the identities
\[ g \hat{A}(a) g^{-1} = a, \quad g \hat{A}(u) g^{-1} = u, \]
and therefore
\[ g \hat{A}(z) = zg, \quad \text{for all } z \in T. \]
Since the affine maps \( \psi = g A \in \mathcal{F} \) induce affine maps of \( R/D \) we have \( A(D) = D, \)
and therefore \( \hat{A}(D) = D. \) Thus the affine map \( \hat{\psi} \), passes to the quotient \( M(R)/D. \) Note
also that, since \( N_{M(R)} \) is the unipotent radical of \( M(R) \), any automorphism of \( M(R) \) maps \( N_{M(R)} \) to itself.

It is now easy to prove the Auslander and Green theorem: the map \( p: R \to N_{M(R)} \) induces a diffeomorphism \( \hat{p}: R/D \to N_{M(R)}/D \) intertwining the flow of
left translation by \( \phi_R \) with the flow of left translations by \( u_R. \) Let us show that the
diffeomorphism \( \hat{p} \) also intertwines the group of affine maps \( \mathcal{F} \) with a group of
affine maps \( \mathcal{F} : = \{ \psi = \pi(g) \hat{A} | g A \in \mathcal{F} \} \) of \( N_{M(R)}/D. \) Let \( \psi = g A \in \mathcal{F} \)
and set \( \overline{\psi} = \pi(g) \hat{A}; \) for any \( h \in R \) we have
\[
\hat{p}(\psi(hD)) = \pi(gA(hD)) = \pi(gA(\tau(h)\pi(h))D) \\
= \pi(\hat{A}(\tau(h))g \hat{A}(\pi(h))D) = \pi(g) \hat{A}(\pi(h)D) = \overline{\psi}(\hat{p}(hD)).
\]
Since the action by left translation by the one-parameter group \( \phi_R < R \) on \( R/D \)
commutes with the action of the group \( \mathcal{F}, \) and since these actions are mapped
by \( \hat{p}, \) respectively, to the action by left translation by the one-parameter group \( u_R < R \) and to the action of the group \( \mathcal{F} \) of affine maps of \( N_{M(R)}/D, \) the proof
is completed.

**Lemma 4.12.** Any finite order affine diffeomorphism \( \psi \) of a nilmanifold which
commutes with an ergodic flow is a translation by an element of the center.

**Proof.** Let \( \psi = g A \) be an affine map of a nilmanifold \( N/D \) whichcommutes with an ergodic flow \( \phi_R \) on \( N/D. \) Let \( \hat{N} : = N/[N,N] \) and \( \hat{D} = [D,D]. \) Since
\( [D,D] \) and \( [N,N] \) are characteristic groups of \( D \) and \( N, \) respectively, the affine
map \( \psi \) yields, by projection, an affine map \( \hat{\psi} \) of \( \hat{N}/\hat{D}. \) The map \( \hat{\psi} \) on the torus
\( \hat{N}/\hat{D} \) commutes with the ergodic flow \( \phi_R \) on \( \hat{N}/\hat{D}, \) projection of the flow \( \phi_R \)
on \( N/D. \) If an affine map of a torus commutes with an ergodic flow, then its
automorphism part is the identity, since a toral automorphism fixing a vector
with rationally independent coordinates is the identity.

Let \( A_\ast \in \text{Aut}(n) \) denote the automorphism of the Lie algebra \( n \) of \( N \) induced
by \( A \in \text{Aut}(N). \)

Let \( \{ n^{(k)} \} \) denote the descending central series of \( n \) defined by induction as
\( n^{(0)} = n \) and \( n^{(k+1)} = [n^{(k)}, n] \) for all \( k \in \mathbb{N}. \)

Since \( A_\ast \) projects to the identity map on the Abelianized Lie algebra \( n/[n,n], \)
we can write \( A_\ast = I + L_1 \) for some linear map \( L_1: n \to n^{(1)} \). Assume, by recurrence
on \( i, \) that we have \( A_\ast = I + L_i \) where \( L_i \) is a linear map from \( n \) to \( n^{(i)} \). Then, for
\( x = [y,z] \) with \( y \in n^{(i-1)} \) and \( z \in n, \) we have
\[
A_\ast x = [A_\ast y, A_\ast z] = [y + L_i y, z + L_i z] = x + x'.
\]
with $x' \in n^{(i+1)}$. It follows that $A_* = I + L_{i+1}$ where $L_{i+1}$ is a linear map from $n$ to $n^{(i+1)}$. For $i$ equal to the degree of nilpotency of $N$, we conclude that $A_*$ is the identity automorphism of $n$ and that the affine map $\psi$ is a translation.

We claim that any finite order translation of a nilmanifold is a translation by an element of the center. In fact, a translation of a nilmanifold $N/D$ by an element $m \in N$ is equal to the identity if and only if

$$n^{-1}mn \in D, \quad \text{for all } n \in N.$$ 

It follows that $m \in D \cap Z(N)$, where $Z(N)$ is the center of $N$.

Thus if the $k$-th power of a translation by $m \in N$ is the identity of $N/D$ we have that $m^k \in D \cap Z(N)$. Since the exponential map is onto, we conclude that $m$ belongs to the center $Z(N)$ as claimed.

The first two authors have proved that the main theorem holds for general nilflows, that is, that the following result holds:

**Theorem 4.13** ([17]). An ergodic nilflow which is not toral has countably many independent invariant distributions of Sobolev order $1/2$.

**Lemma 4.14.** Any translation of a non-toral nilmanifold has infinitely many invariant distributions of Sobolev order $1/2$.

**Proof.** Since the exponential map of any nilpotent group is surjective, every translation of a nilmanifold is the return map (with constant return time) of a nilflow. Since the suspension of a non-toral nilmanifold by a translation is non-toral, the suspension flow, hence its return map, has infinitely many independent invariant distributions of Sobolev order $1/2$. 

**Lemma 4.15.** Let $M$ be a closed connected submanifold of a torus $\mathbb{T}^d$ transverse to a linear minimal flow $(\phi_t)$ such that the return time of the flow to $M$ is everywhere constant (assume is equal to 1). Then $M$ is a subtorus and the map $\phi_1$ is a constant translation on this subtorus.

**Proof.** Let $x \in M$ and let $x_n = \phi_n(x)$. Since the translation $\phi_n$ maps $M$ into itself, we have that $T_xM = T_{x_n}M \subset \mathbb{R}^d$. Since the set $x_n$ is dense in $M$, by continuity ($M$ needs to be at least $C^1$) we have that $T_yM = E \subset \mathbb{R}^d$ is a constant space $E$ for all $y \in M$. It follows that $M$ coincides locally with a translate of the projection of $E$ to the torus, hence there exits a translate $T'$ of the projection of $E$ to the torus such that the set $T' \cap M$ non-empty, open and closed. Since $M$ is connected it follows that $M = T'$ and since $M$ is closed, it is a subtorus.

In conclusion we have

**Proposition 4.16.** An ergodic quasi-unipotent flow on a finite volume solv-manifold is either smoothly conjugate to a linear toral flow or it admits countably many independent invariant distributions of Sobolev order $1/2$. 

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5. Partially hyperbolic homogeneous flows

In the non-compact, finite volume case, by applying results of D. Kleinbock and G. Margulis we are able to generalize Theorem 2.1 to flows on semi-simple manifolds. We think that it is very likely that a general partially hyperbolic flow on any finite volume manifold has infinitely many different minimal sets, but we were not able to prove such a general statement.

For non-compact finite volume we recall the following result by D. Kleinbock and G. Margulis [29] and its immediate corollary.

**Theorem 5.1** (Kleinbock and Margulis). Let $G$ be a connected semi-simple Lie group of dimension $n$ without compact factors, $\Gamma$ an irreducible lattice in $G$. For any partially hyperbolic homogeneous flow $\phi_R$ on $G/\Gamma$, for any closed invariant set $Z \subset G/\Gamma$ of (Haar) measure zero and for any nonempty open subset $W$ of $G/\Gamma$, we have that

$$\dim_H(\{x \in W \mid \phi_Rx \text{ is bounded and } \overline{\phi_Rx \cap Z = \emptyset} \}) = n.$$  

Here $\dim_H$ denotes the Hausdorff dimension.

Observe that if the flow $\phi_R$ is ergodic, it is enough to assume that the closed invariant set $Z \subset G/\Gamma$ be proper.

**Corollary 5.2.** Under the conditions of Theorem 5.1 the flow $(G/\Gamma, \phi_R)$ has infinitely many different compact minimal invariant sets.

**Proposition 5.3.** Let $G$ be a connected semi-simple Lie group and $G/D$ a finite volume space. Assume that the flow $\phi_R$ on $G/D$ is partially hyperbolic. Then the flow $(G/D, \phi_R)$ has infinitely many distinct minimal invariant sets supporting infinitely many $g_R$-invariant and mutually singular ergodic probability measures.

**Proof.** Let $G = K \times S$ be the decomposition of $G$ as the almost-direct product of a compact semi-simple subgroup $K$ and a totally non-compact semi-simple group $S$, with $K$ and $S$ connected normal subgroups. Since the flow $(G/\Gamma, \phi_R)$ is partially hyperbolic $S$ is not trivial. Since $K$ is compact and normal, then $D' = DK = KD \subset G$ is a closed subgroup, and since $D \subset KD$, then $G/KD$ is of finite volume. Moreover, $(G/K)/DK ~ G/KD$ is of finite volume and $G' = G/K ~ S/S \cap K$ is semi-simple without compact factor with $D' \subset G'$ a closed subgroup with $G'/D'$ of finite volume and a projection $p : G/D \to G'/D'$. Thus we may assume that $G$ is totally non-compact, and by Proposition 3.11, that the center of $G$ is finite and that $D$ is a lattice.

If $D$ is irreducible, then the statement follows immediately from Corollary 5.2. Otherwise, by [39, Theorem 5.22], the group $G$ is the product of connected normal semi-simple subgroups $G_i < G$, ($i \in \{1, \ldots, l\}$), such that, for each $i$ in $\{1, \ldots, l\}$, $G_i \cap \prod_{j \neq i} G_j$ is discrete, the subgroups $\Gamma_i := \Gamma \cap G_i$ are irreducible lattices in $G_i$, and $\Gamma_0 = \prod_i \Gamma_i$ is a finite index normal subgroup of $\Gamma$. Observe that $G/\Gamma_0 \sim \prod_i G_i/\Gamma_i$. Let $p : G/\Gamma_0 \to G/\Gamma$ be the finite-to-one covering and let $p_i : G/\Gamma_0 \to G_i/\Gamma_i$ be the projections onto the factors. Let $\phi_R^{(0)}$ be the flow induced by the one-parameter group $\phi_R$ on $G/\Gamma_0$ and let $\phi_R^{(i)}$ be the projected
flow on \( G_i/\Gamma_i \), for all \( i \in \{1, \ldots, l\} \). Since \( \Gamma_i \) is an irreducible lattice in \( G_i \), whenever \( \phi_R^{(i)} \) is partially hyperbolic we can apply Corollary 5.2. Since \( \phi_R^{(0)} \) is partially hyperbolic there is at least one \( j \in \{1, \ldots, l\} \) such that \( \phi_R^{(j)} \) is partially hyperbolic. By Corollary 5.2, the flow \( \phi_R^{(j)} \) has a countable family \( \{K_n|n \in \mathbb{N}\} \) of distinct minimal subsets of \( G_j/\Gamma_j \) such that each \( K_n \) supports an invariant probability measure \( \eta_n \). For all \( n \in \mathbb{N} \), let us define \( \hat{\mu}_n := \eta_n \times \text{Leb} \) on \( G/\Gamma_0 \). By construction the measures \( \hat{\mu}_n \) are invariant, for all \( n \in \mathbb{N} \), and have mutually disjoint supports. Finally, since the map \( p : G/\Gamma_0 \to G/\Gamma \) is finite-to-one, it follows that the family of sets \( \{p(K_n \times \prod_{i \neq j} G_i/\Gamma_i)|n \in \mathbb{N}\} \) consists of countably many disjoint closed sets supporting invariant measures \( \mu_n := p_* \hat{\mu}_n \). The proof of the proposition is therefore complete.

\[ \square \]

6. THE GENERAL CASE

We may now prove our main theorem. We shall consider separately the two cases of a flow and of an affine map.

**Proof of Theorem 1.1.** By Remark 3.7 we may suppose that the flow \( (G/D, \phi_R) \) is ergodic. Let us also assume that \( G \) is simply connected, by possibly pulling back \( D \) to the universal cover of \( G \). Recall that by Theorem 3.8 the ergodic flow \( (G/D, \phi_R) \) projects onto the ergodic flow \( (L/q(D), q(\phi_R)) \), where \( L \) is the Levi factor of \( G \) and \( q : G \to L \) the projection of \( G \) onto this factor. Assume that the finite measure space \( (L/q(D)) \) is not trivial. Then the statement of the theorem follows from Proposition 4.4 if the flow \( (L/q(D), q(\phi_R)) \) is quasi-unipotent and by Proposition 5.3 if it is partially hyperbolic.

If the finite measure space \( (L/q(D)) \) is reduced to a point, then, using again Theorem 3.8, we have \( G/D \approx R/R \cap D \), where \( R \) is the radical of \( G \). We obtain in this way that our original flow is diffeomorphic to an ergodic flow on a finite volume solvmanifold. By Mostow’s Theorem (see Theorem 3.10), a finite volume solvmanifold is compact. Hence the statement of the theorem follows from Theorem 2.1 if the projected flow is partially hyperbolic and by Proposition 4.16 if it is quasi-unipotent. The proof is therefore complete.

\[ \square \]

**Proof of Theorem 1.2.** In the course of the proof we shall use many times Lemmata 3.5 and 3.9, which allow us to say that, whenever a quotient map or a suspension of an affine diffeomorphism has an infinite dimensional space of invariant distribution of a given order, so does the affine diffeomorphism. The same statement applies to measures. In the sequel, the term “by standard arguments” will refer to the application of this line of reasoning to infer that an affine diffeomorphism has an infinite dimensional space of invariant distribution (or measures).

Let \( \psi_0 = gA \) be an affine map of \( G/D \). Let \( \mathcal{A} = \mathcal{H} = \text{Aut}(G) \approx \text{Aut}(g) \) and let \( \mathcal{H} \) act on \( \mathcal{A} \) by inner automorphisms (i.e., by conjugation). The groups \( \mathcal{A} \) and \( \mathcal{H} \) are real algebraic groups and \( \mathcal{H} \) acts rationally on \( \mathcal{A} \). Let \( \mathcal{N} = \text{Ad}_G(G) < \mathcal{A} \). Since for all \( A \in \mathcal{A} \) and all \( x \in G \) we have \( A \circ \text{Ad}_G(x) \circ A^{-1} = \text{Ad}_G(Ax) \), the group \( \mathcal{N} \) is normal in \( \mathcal{A} \) and stable under the action of \( \mathcal{H} \) on \( \mathcal{A} \).
The epimorphism \( \text{Ad}_G: G \to \mathcal{N} \approx G/Z(G) \) maps the closed subgroup \( D \) to the subgroup \( DZ(G) \). Let \( \mathcal{D} = DZ(G) \). Then, the map \( \text{Ad}_G \) induces a smooth submersion of \( G/D \) onto the finite volume space \( \mathcal{N}/\mathcal{D} \approx G/DZ(G) \). This submersion intertwines the affine map \( \psi_0 = gA \) of \( G/D \) with the affine map \( \psi = \text{Ad}_G(g) \circ \text{Int}_{\text{Aut}(G)}(A) \in \text{Aff}(\hat{\mathcal{H}}) \), with \( \text{Int}_{\text{Aut}(G)}(A) \in \hat{\mathcal{H}} \) the conjugation by \( A \in \text{Aut}(G) \).

By standard arguments, if the space of \( \psi \)-invariant measures on \( \mathcal{N}/\mathcal{D} \) (respectively, the space of \( \psi \)-invariant distributions on \( \mathcal{N}/\mathcal{D} \) of a given order) has infinite dimension, so does the space of \( \psi_0 \)-invariant measures on \( G/D \) (respectively, the space of \( \psi_0 \)-invariant distributions on \( G/D \) of the same order).

By Lemma 3.6, there exist a connected Lie group \( \hat{\mathcal{N}} \) such that \( \mathcal{N} < \hat{\mathcal{N}} \), a quasi-lattice \( \hat{\mathcal{D}} \) of \( \hat{\mathcal{N}} \) containing \( \mathcal{D} \), a non-trivial one-parameter subgroup \( \phi_\mathcal{R} \subset \hat{\mathcal{N}} \) and a finite cyclic group \( \mathcal{F} \) of affine diffeomorphisms of \( \hat{\mathcal{N}}/\hat{\mathcal{D}} \) with the following properties:

1. The group \( \mathcal{F} \) commutes with the flow of the one-parameter group \( \phi_\mathcal{R} \) by left translations on \( \hat{\mathcal{N}} \); hence we obtain a quotient flow \( (\mathcal{F}\setminus\hat{\mathcal{N}})/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) on the double coset space \( \mathcal{F}\setminus\hat{\mathcal{N}}/\hat{\mathcal{D}} \).
2. The flow \( (\mathcal{F}\setminus\hat{\mathcal{N}}/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) is smoothly conjugate to a suspension of the affine map \( \psi \) on \( \mathcal{N}/\mathcal{D} \).

The structure of the proof is now analogous to the proof of Theorem 1.1 and proceeds by analyzing the different cases for the dynamics of the flow \( \phi_\mathcal{R} \) on the homogeneous space \( \hat{\mathcal{N}}/\hat{\mathcal{D}} \) and then deriving the consequences for the affine map \( \psi \) of \( \mathcal{N}/\mathcal{D} \) and finally for the original map \( \psi_0 \) of \( G/D \).

First we notice that we may assume that the flow \( (\hat{\mathcal{N}}/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) is ergodic: in fact the ergodic decomposition of this flow yields an infinite dimensional space of \( \phi_\mathcal{R} \)-invariant signed measures, which may be averaged under the action of \( \mathcal{F} \), yielding again an infinite dimensional space of \( \phi_\mathcal{R} \)-invariant and \( \mathcal{F} \)-invariant signed measures; then we conclude, by standard arguments, that the affine map \( \psi_0 \) preserves infinitely many ergodic mutually singular invariant probability measures. Thus we assume that the flow \( (\hat{\mathcal{N}}/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) is ergodic.

By Theorem 3.8, the ergodic flow \( (\hat{\mathcal{N}}/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) projects onto the ergodic flow \( (\mathcal{L}/q(\hat{\mathcal{D}}), q(\phi_\mathcal{R})) \), where \( \mathcal{L} \) is the Levi factor of \( \hat{\mathcal{N}} \) and \( q: \hat{\mathcal{N}} \to \mathcal{L} \) the projection of \( \hat{\mathcal{N}} \) onto this factor. The group of affine diffeomorphisms \( \mathcal{F} \) of \( \hat{\mathcal{N}}/\hat{\mathcal{D}} \) projects under \( q \) to a quotient finite cyclic group of affine diffeomorphisms \( \mathcal{F} \circ q \) of the quotient space \( \mathcal{L}/q(\hat{\mathcal{D}}) \) commuting with the ergodic flow \( q(\phi_\mathcal{R}) \).

**Case A: non-trivial Levi factor \( \mathcal{L} \).** Suppose that the finite measure space \( \mathcal{L}/q(\hat{\mathcal{D}}) \) is not trivial. We distinguish two cases according to whether the flow \( (\mathcal{L}/q(\hat{\mathcal{D}}), q(\phi_\mathcal{R})) \) is partially hyperbolic or quasi-unipotent.

**Partially hyperbolic flow on the Levi factor.** By Proposition 5.3, the flow \( q(\phi_\mathcal{R}) \) has infinitely many distinct compact invariant sets supporting infinitely many invariant and mutually singular ergodic probability measure. Then, by standard arguments, the covering flow \( (\hat{\mathcal{N}}/\hat{\mathcal{D}}, \phi_\mathcal{R}) \) has infinitely many \( \phi_\mathcal{R} \)-invariant and
mutually singular ergodic probability measures \((\mu_i)_{i \in I}\). By averaging this collection of measures under the action of the finite cyclic group \(F\) we obtain an infinite sub-collection of probability measures on the quotient space \(\mathcal{F} \setminus \mathcal{N} / \mathcal{D}\) which are invariant and ergodic for the quotient flow \(\phi\). Then, by standard arguments, we conclude that the affine diffeomorphism \(\psi\), first, and the affine diffeomorphism \(\psi_0\), next, have an infinite set of probability invariant measures.

**Quasi unipotent flow on the Levi factor.** If the flow \((\mathcal{L} / q(\mathcal{D}), q(\phi))\) is quasi-unipotent, by Proposition 4.4, there exists infinitely many independent \(q(\phi)\)-invariant distributions on the space \(\mathcal{L} / q(\mathcal{D})\) of Sobolev order 1/2 which are also invariant under the finite cyclic group \(\mathcal{F} \circ q\). By standard arguments, we obtain that the dimension of the space of distributions of Sobolev order 1/2 on \(\mathcal{N} / \mathcal{D}\) which are simultaneously \(\phi_\mathcal{D}\)-invariant and invariant under \(F\) is infinite. This is the same as saying that the space of distributions of Sobolev order 1/2 on \(\mathcal{F} \setminus \mathcal{N} / \mathcal{D}\) which are invariant under the flow \(\phi\) is infinite. Then, by standard arguments, we conclude that the affine diffeomorphism \(\psi\), first, and the affine diffeomorphism \(\psi_0\), next, have an infinite dimensional space of invariant distributions of Sobolev order 1/2.

**Case B: Solvable \(\mathcal{N}\).** Thus the theorem is proved if the finite measure space \(\mathcal{L} / q(\mathcal{D})\) is not trivial. In the opposite case, by Theorem 3.8 we may and will assume that \(\mathcal{N} / \mathcal{D}\) is a finite volume solvmanifold.

Then the same is true of the manifold \(\mathcal{N} / \mathcal{D}\), since \(\mathcal{N}\) is a subgroup of \(\mathcal{N}\) hence solvable, and of the manifold \(G / \mathcal{D}\), since \(\mathcal{N} \approx G / Z(G)\). By Mostow’s Theorem 3.10, a finite volume solvmanifold is compact.

If the flow \((\mathcal{N} / \mathcal{D}, \phi_\mathcal{D})\) is partially hyperbolic, so are the maps \(\psi\) and \(\psi_0\). Hence, in this case by Theorem 2.1, the map \(\psi_0\) admits infinitely many minimal sets and independent invariant ergodic probability measures.

If, on the contrary, the flow \((\mathcal{N} / \mathcal{D}, \phi_\mathcal{D})\) is quasi-unipotent, then by Proposition 4.11 there exists a diffeomorphism \(h: \mathcal{N} \rightarrow N\) onto a nilpotent Lie group \(N\). The diffeomorphism \(h\) satisfies the following properties:

1. It induces a quotient diffeomorphism \(h: \mathcal{N} / \mathcal{D} \rightarrow N / \Delta\) onto a compact nilmanifold \(N / \Delta\) conjugating the flow \(\phi_\mathcal{D}\) on \(\mathcal{N} / \mathcal{D}\) with a nilflow \(u_\mathcal{D}\) on \(N / \Delta\).
2. Restricted to \(\mathcal{D}\), the diffeomorphism \(h\) is a group isomorphism of \(\mathcal{D}\) onto \(\Delta\).
3. Thus we may identify \(\widehat{\mathcal{D}} \approx \Delta\).

It follows, in particular, that \(\widehat{\mathcal{D}}\) is a (co-compact) lattice in \(\mathcal{N}\).

By Lemma 4.12, the group \(\mathcal{F}'\) is generated by a translation by an element of the center of \(N\). This fact has several consequences. First, the group \(\mathcal{F}' / \Delta\) is discrete and the quotient space \(\mathcal{F}' \setminus N / \Delta\) coincides with the compact nilmanifold \(N / \mathcal{F}' / \Delta\). Second, as the group \(\mathcal{F}'\) operates without fixed points on \(N / \Delta\), so does the group \(\mathcal{F}\) on \(\mathcal{N} / \mathcal{D}\). Thus the quotient space \(\mathcal{F} \setminus \mathcal{N} / \mathcal{D}\) is a smooth manifold\(^1\).

\(^1\)This is actually true, by construction of \(\mathcal{F}\), in a more general situation, e.g., whenever we may assume that the flow \(\phi_\mathcal{D}\) commuting with \(\mathcal{F}\) is minimal.
Third, the group $\mathcal{F}$ commutes with the lattice $\hat{\mathcal{D}}$ and the diffeomorphism $\tilde{h}$ is an isomorphism of the group $\mathcal{F}\hat{\mathcal{D}}$ onto the group $\mathcal{F}\Delta$. In particular we may regard the diffeomorphism $h$ as a diffeomorphism of the double coset quotient space $\mathcal{F}\backslash\hat{\mathcal{N}}\hat{\mathcal{D}}$ onto $N/\mathcal{F}\Delta$, conjugating the flow $\phi_\mathcal{R}$ with the flow $t_\mathcal{R}$.

**Suppose $N$ is not Abelian.** If the connected nilpotent group $N$ is not Abelian, then, by the results of [17], the flow $u_\mathcal{R}$ on the compact nilmanifold $N/\mathcal{F}\Delta$ admits infinitely many independent invariant distributions of Sobolev order $1/2$. It follows that the same is true for the flow $(\mathcal{F}\backslash\hat{\mathcal{N}}\hat{\mathcal{D}},\phi_\mathcal{R})$ and by standard arguments for the affine map $\psi_0$ on $G/D$.

**Suppose $N$ is Abelian.** We are left to consider the case where the connected nilpotent group $N$ is Abelian. Then the compact nilmanifolds $N/\Delta$ and $N/\mathcal{F}\Delta$ are tori. The flow of the one-parameter group $u_\mathcal{R}$ on these manifolds is conjugate, respectively, to the flows $(\hat{\mathcal{N}}\hat{\mathcal{D}},\phi_\mathcal{R})$ and $(\mathcal{F}\backslash\hat{\mathcal{N}}\hat{\mathcal{D}},\phi_\mathcal{R})$. In particular the manifolds $\hat{\mathcal{N}}\hat{\mathcal{D}}$ and $\mathcal{F}\backslash\hat{\mathcal{N}}\hat{\mathcal{D}}$ are tori, up to a diffeomorphism.

Recall that the flow $(\mathcal{F}\backslash\hat{\mathcal{N}}\hat{\mathcal{D}},\phi_\mathcal{R})$ is the suspension of the affine diffeomorphism $(\mathcal{N}\backslash\mathcal{D},\psi)$. Thus the manifold $M := h(\mathcal{N}\backslash\mathcal{D})$ is a submanifold of the torus $N/\mathcal{F}\Delta$, such that the linear flow $u_\mathcal{R}$ has constant return time to $M$. By Lemma 4.15 the manifold $M$ is a subtorus $\mathcal{T}^k$ of $N/\mathcal{F}\Delta$ and the return map to $M = \mathcal{T}^k$ is a translation on this torus. Thus $(\mathcal{N}\backslash\mathcal{D},\psi)$ is diffeomorphically conjugate to a torus translation $(\mathcal{T}^k,\tau)$. As $\hat{\mathcal{D}}$ is a lattice and $\hat{\mathcal{N}}\hat{\mathcal{D}}$ a compact torus (up to a diffeomorphism), by construction of the groups $\hat{\mathcal{N}}$ and $\hat{\mathcal{D}}$, we have that the subgroup $\hat{\mathcal{D}} < \hat{\mathcal{D}}$ is a lattice in the subgroup $\mathcal{N} < \hat{\mathcal{N}}$.

By definition we have $\mathcal{N} = G/Z(G)$ and $\hat{\mathcal{D}} = DZ(G)$. However, as $\mathcal{D}$ is, by construction, a subgroup of the discrete group $\hat{\mathcal{D}}$, a lattice in $\mathcal{N}$ we obtain that $\mathcal{D} = DZ(G)$, that is, $DZ(G)$ is a closed subgroup of the solvable group $G$. Thus, in the present situation, we have an affine diffeomorphism $\psi_0$ of the compact solvmanifold $G/D$ inducing, via the submersion $Ad_G: G/D \rightarrow G/DZ(G)$, an affine map $\psi$ of the quotient space $G/DZ(G)$ which is conjugated by a diffeomorphism $F: G/DZ(G) \rightarrow \mathcal{T}^d$ to an ergodic translation of the torus $\mathcal{T}^d$.

The minimal sets of $\psi_0$ are compact subsets of $G/D$ surjecting onto $G/DZ(G)$, hence carrying a unique $\psi_0$-invariant measure. Since $G/D$ is connected, either $\psi_0$ is a minimal diffeomorphism of $G/D$ or there are infinitely many disjoint minimal sets of $\psi_0$, and in particular infinitely $\psi_0$-invariant independent probability measure on $G/D$. Thus we may assume that the diffeomorphism $\psi_0$ is (uniquely) ergodic. This implies by Theorem 2.1 that the diffeomorphism $\psi_0$ is quasi-unipotent.

By Corollary 6.4 below, the diffeomorphism $\psi_0$ either admits infinitely many invariant independent distributions or is smoothly diffeomorphic to a translation on a torus. The proof is therefore complete. 

For any manifold $M$, let $\chi(M)$ denote the space of vector fields on $M$. In what follows we identify the Lie algebra $\mathfrak{g}$ of any Lie group $G$ with the space $\chi(M)^G$ of right invariant vector fields on $G$. For any homogeneous space $G/D,$
the Lie algebra $\mathfrak{g}$ is identified with the subspace $\chi(G/D)^G \subset \chi(G/D)$ given by the projections on $G/D$ of the right invariant vector fields on $G$.

**Lemma 6.1.** Let $G$ be a connected, simply connected, solvable Lie group, let $D < G$ be a lattice in $G$, and let $\psi_0 = u_0 \Lambda_0$ be an ergodic affine quasi-unipotent diffeomorphism of $G/D$. Assume that $Z(G)D$ is a closed subgroup of $G$. Let $F: G/Z(G)D \to \mathbb{T}^d$ be a diffeomorphism conjugating the map $\psi$, induced by $\psi_0$ on $G/Z(G)D$, with an ergodic translation $\tau$ of $\mathbb{T}^d$.

Let $G/Z(G)D \to G/D$.

Then there exists a structure of nilmanifold on $G/D$ (of degree of nilpotency at most 2) with respect to which the map $\psi_0$ is affine and unipotent. More precisely, there exists a connected, simply connected, nilpotent Lie group $N$, a lattice $\Gamma < N$, a diffeomorphism $F_0: G/D \to N/\Gamma$ and an unipotent ergodic affine diffeomorphism $\tau_0: N/\Gamma \to N/\Gamma$ such that $F_0 \circ \tau_0 = \tau \circ F_0$.

**Proof.** Step 1. Let $Z_0(G)$ the connected component of the identity of the center $Z(G)$ of $G$. The covering $G/Z(G)_0D \to G/Z(G)D$ is finite since $G/D$ is a compact solvmanifold, by Mostow’s Theorem, which covers $G/Z(G)_0D$. Hence $G/Z(G)_0D$ is diffeomorphic to a torus. Thus, with no loss of generality, we shall assume that $Z(G)$ is connected.

Let $Z = Z(G)/(Z(G) \cap D)$. The group $Z$ acts freely on $G/D$ by left translations. Since the orbit space $G/Z(G)D$ is Hausdorff, and $G/D$ is compact, the orbits of $Z$ are compact. Hence $Z$ is a compact connected Abelian group acting freely on $G/D$. Composing the projection $\pi: G/D \to G/Z(G)D$ with the diffeomorphism $F: G/Z(G)D \to \mathbb{T}^d$, the map $p: G/D \to \mathbb{T}^d$ so obtained endows the space $G/D$ with the structure of a principal $Z$-bundle over $\mathbb{T}^d$.

Fix a connection $\omega$ for the principal bundle $(G/D, p)$.

Denote by $\mathfrak{z} \approx \mathbb{R}^d$ the Lie algebra of the group $Z$. The Lie algebra $\mathfrak{z}$ may be identified to the *fundamental vertical vector fields* on $G/D$, generators of the left action of $Z(G)$ on $G/D$.

For any “constant” (i.e., invariant under group translation) vector field $X$ on $\mathbb{T}^d$, let $X^*$ be its horizontal lift for the connection $\omega$. We call such lifted vector fields $X^*$, the *fundamental horizontal vector fields* (for the connection $\omega$). Let $X_1^*, \ldots, X_d^*$ be a basis of fundamental horizontal vector fields for the connection.
we conclude that the vector fields $X_i$. As the $X_i$’s commute, we have
$$[X_i^*, X_j^*] = -\Omega(X_i^*, X_j^*),$$
where $\Omega$ is the 3-valued curvature two-form of the connection $\omega$. We recall that, since $Z$ is Abelian, the curvature form of any connection on $M$ is simply the differential of the 3-valued connection form.

Let $\tilde{\Omega}$ be the 3-valued two-form on $T^d$ defined by $\tilde{\Omega}(X_i, X_j) = \Omega(X_i^*, X_j^*)$, so that $\Omega = p^* \tilde{\Omega}$. The two-form $\tilde{\Omega}$ is closed (since $p^*$ is injective and $d\Omega = d^2 \omega = 0$). Thus $\tilde{\Omega}$ is cohomologous to a constant 3-valued two-form $\Omega_0$ on $T^d$, that is, there is a 3-valued one-form $\lambda$ on $T^d$ such that $\tilde{\Omega} = \Omega_0 + d\lambda$. Let us define a new connection by $\omega' = \omega + p^* \lambda$. If $X_i', \ldots, X_d'$ are the horizontal lifts of the constant fields $X_i$’s for the connection $\omega'$, since $p^* X_i = p^* X_i^* = X_i$ for all $i = 1, \ldots, d$, we have
$$[X_i', X_j'] = -\Omega'(X_i', X_j') = -d\omega'(X_i', X_j')$$
$$= -d\omega(X_i', X_j') + dp^* \lambda(X_i', X_j')$$
$$= -\Omega(X_i^*, X_j^*) + p^* d\lambda(X_i^*, X_j^*)$$
$$= -\tilde{\Omega}(X_i, X_j) + d\lambda(X_i, X_j) = -\Omega_0(X_i, X_j).$$

Let $(V^*_a)$ be a basis of fundamental vertical vector fields associated to a basis of $\tilde{\Omega}$ denoted by the same letters. Then for all $\alpha = 1, \ldots, n$, as the the group $Z$ is Abelian, we have
$$[V^*_a, X_j'] = 0.$$

We conclude that the vector fields $X_i', \ldots, X_d'$, and $(V^*_a)$ on $G/D$ generate a $(d+s)$-dimensional nilpotent Lie algebra $\mathfrak{n}$ of degree of nilpotency 2 at most. Let $N$ be the simply connected, connected nilpotent group of Lie algebra $\mathfrak{n}$. The group $N$ operates locally faithfully on $G/D$ via an action $\alpha: N \times G/D \to G/D$, whose generators are the vector fields $X_i', \ldots, X_d'$ and $(V^*_a)$. As the sub-algebra $\mathfrak{z}$ is contained in $\mathfrak{n}$, the universal cover $\tilde{Z}$ of the group $Z$ is contained in $N$. The group $N/\tilde{Z}$ is isomorphic to $\mathbb{R}^d$ via a mapping sending the generators $X_i' + \tilde{\mathfrak{z}}$ to standard generators of $\mathbb{R}^d$. Since we have covering homomorphisms $\tilde{Z} \to Z(G) \to Z$, the $\tilde{Z}$-orbits on $G/D$ coincide with the $Z(G)$-orbits; in particular they are closed. It follows that the action $\alpha$ of $N$ on $G/D$ induces a quotient action
$$\tilde{\alpha}: N/\tilde{Z} \times G/Z(G)D \to G/Z(G)D$$
of the Abelian group $N/\tilde{Z} \approx \mathbb{R}^d$ on $G/Z(G)D$. It is plain that the composition with the diffeomorphism $F: G/Z(G)D \to \mathbb{T}^d$ yields an action of $N/\tilde{Z} \approx \mathbb{R}^d$ on $\mathbb{T}^d$ which is simply the action of $\mathbb{R}^d$ on $\mathbb{T}^d$ with generators $X_1, \ldots, X_d$, i.e., the plain action $\mathbb{R}^d$ on $\mathbb{T}^d$ by translations. It follows that the action of $N$ (or $N/\tilde{Z}$) on the compact space $G/Z(G)D$ is transitive. Since the $\tilde{Z}$-orbits are compact, we conclude that the action on $G/D$ is transitive.

Fixing $x_0 \in G/D$ and defining $\Gamma$ as the isotropy group $\{ n \in N | \alpha(n, x_0) = x_0 \}$ of the point $x_0$, we obtain a diffeomorphism $N/\Gamma \to G/D$ whose inverse will be denoted $F_0: G/D \to N/\Gamma$. We leave to the reader the easy verification that
the induced quotient map $G/Z(G)D \to N/\tilde{Z}T$ coincides with the given diffeomorphism $F$, via the identification $N/Z = \mathbb{R}^d$ defined above. In particular the map $G/D \to N/\tilde{Z}T = \mathbb{T}^d$ coincides with the principal bundle projection $p$. We summarize the above construction with the following diagram:

\[
\begin{array}{ccc}
G/D & \xrightarrow{F_0} & N/\Gamma \\
\downarrow \pi & & \downarrow \\
G/Z(G)D & \xrightarrow{F} & N/\tilde{Z}\Gamma \approx \mathbb{T}^d
\end{array}
\]

with

\begin{equation}
(13) \quad zF_0(xD) = F_0(zxD), \quad \forall z \in Z(G), \quad \forall xD \in G/D.
\end{equation}

Henceforth the nilmanifold $N/\Gamma$ will be endowed with the above defined connection $\omega'$ having (constant) curvature $\Omega_0$. To simplify notations these forms will be renamed $\omega$ and $\Omega$. We recall that for any fundamental horizontal fields $X^*, Y^*$ projecting to constant fields $X$ and $Y$ and any fundamental vertical vector field $V^*$ we have $[X^*, Y^*] = -\Omega(X, Y)$ and $[X^*, V^*] = 0$.

**Remark.** The construction above depends on the arbitrary choice of a primitive $\lambda$ of the exact form $\Omega - \Omega_0$. Clearly $\lambda$ is determined up to a closed one-form, i.e., up to a form $\lambda_0 + df$, with $\lambda_0$ and $f$, respectively, a constant one-form and a smooth function on the torus $\mathbb{T}^d$. The effect of adding a constant one-form to $\lambda$ consists in composing the map $F_0$ with a diffeomorphism $N/\Gamma \to N/\Gamma'$ induced by an automorphism of $N$ which projects to the identity automorphism of $N/\tilde{Z}$. Adding an exact one-form $df$ to $\lambda$ results into composing the diffeomorphism $F_0$ with the fiber-wise diffeomorphism $x\Gamma \to (\exp f(x\tilde{Z}\Gamma)) x\Gamma$.

Thus the group structure of $N$ is only determined up to these ambiguities.

**Step 2.** Recall that the Lie algebra $\mathfrak{g}$ is identified with the space $\chi(G/D)^G$ of vector fields generating the left action of $G$ on $G/D$. The push-forward map of vector fields $\psi_{\theta_0} = (d\psi_0) \circ \psi_0^{-1}$ induced by the affine diffeomorphism $\psi_0 = u_0 A_0$ maps the Lie algebra $\chi(G/D)^G \approx \mathfrak{g}$ onto itself and it is easily identified with the automorphism of $\mathfrak{g}$ defined by $B_0 = \text{Ad}_{C}(u_0) \circ A_0$. By hypothesis the automorphism $B_0$ is quasi-unipotent.

The center $Z(G)$ is a characteristic subgroup of $G$, hence any automorphism of $G$ restricts to an automorphism of $Z(G)$. Furthermore, the restriction of the automorphism $B_0$ to the sub-algebra $\mathfrak{z}$ coincides with $A_0$. Since $A_0(Z(G)) = Z(G)$ and $A_0(D) = D$ the automorphism $A_0$ defines a quasi-unipotent automorphism of the torus $Z(G)/Z(G) \cap D$. It follows that the spectrum of $B_0$ restricted to $\mathfrak{z}$ consists of roots of the unity. Equivalently the spectrum of $\psi_{\theta_0}$, restricted to the space of fundamental vertical vector fields, consists of roots of the unity.

**Step 3.** Let $\tau_0 = F_0 \circ \psi_0 \circ F_0^{-1}$. The map $\tau_0: N/\Gamma \to N/\Gamma$ induces an ergodic translation $\tau$ on the quotient torus $N/\tilde{Z}\Gamma$.
Since for any \( z \in Z(G) \) and any \( xD \in G/D \) we have \( \psi_0(zxD) = A_0(z)\psi_0(xD) \), and since by formula (13) the diffeomorphism \( F_0 \) intertwines the actions of \( Z(G) \) on the spaces \( G/D \) and \( N/\Gamma \), we obtain a similar identity for the diffeomorphism \( \tau_0 \): 

\[
\tau_0(x\Gamma) = A_0(z)\tau_0(x\Gamma), \quad \forall z \in Z(G), \quad \forall x\Gamma \in N/\Gamma,
\]

or, equivalently,

\[
(\tau_0)_* V^* = A_0(V^*),
\]

for any fundamental vector field \( V^* \) on \( N/\Gamma \).

By definition, constant vector fields \( X \) on \( N/Z(G)D \) are vector fields invariant by all translations, hence satisfying

\[
(\tau)_* X = X.
\]

Thus, for any fundamental horizontal vector field \( X^* \) projecting to a constant vector field \( X \), we have

\[
(\tau_0)_* X^* = X^* + \mu(X),
\]

with \( \mu \) a smooth one-form on \( N/Z(G)D \) with values in \( \mathfrak{z} \). From this identity it follows that, for any two fundamental horizontal vector fields \( X_1^* \) and \( X_2^* \) projecting to constant vector fields \( X_1 \) and \( X_2 \), we have

\[
(\tau_0)_*[X_1^*, X_2^*] = [X_1 + \mu(X_1), X_2 + \mu(X_2)] = [X_1^*, X_2^*] + d\mu(X_1, X_2).
\]

Using the identity (14) and considering that \( [X_1^*, X_2^*] \) equals the fundamental vertical vector field \( -\Omega(X_1, X_2) \) we obtain the identity

\[
d\mu = \Omega - A_0 \circ \Omega.
\]

As the right hand side is a constant two-form on \( N/\hat{\Gamma} \) and the left hand side is an exact two-form, both terms are zero. Hence

1. the one-form \( \mu \) is closed, and
2. \( A_0(V^*) = V^* \) for all \( V^* \in [n,n] \).

**Step 4.** Having studied the spectrum of the automorphism \( B_0 = u_0 A_0 u^{-1} \in \text{Aut}(g) \), restricted to fundamental vertical vector fields (i.e., to \( \mathfrak{z} \)), we now proceed to consider the spectrum of the automorphism \( B \in \text{Aut}(g/\mathfrak{z}) \) induced by \( B_0 \) on \( g/\mathfrak{z} \). As by previous remarks, the automorphism \( B \) will be identified with the restriction of the push-forward map \( \psi_* \) to the vector fields arising from the left action of \( G/Z(G) \) on \( G/Z(G)D \).
The Lie algebras \( g \cong \chi(G/D)^G \) and \( g/\mathfrak{z} \cong \chi(G/Z(G))^{G/Z(G)} \) are mapped by the push-forward maps \( F_{0*} \) and \( F_* \) respectively to isomorphic sub-algebras \( F_{0*}(g) \subset \chi(N/\Gamma) \) and \( F_*(g/\mathfrak{z}) \subset \chi(N/\tilde{\Gamma}) \) of vector fields on \( N/\Gamma \) and \( N/\tilde{\Gamma} \) according to the following diagram

\[
\begin{array}{c}
Y' \in \mathfrak{g} \\
\downarrow \pi_* \downarrow p_* \\
\hat{Y} \in g/\mathfrak{z} \\
\downarrow F_* \downarrow \\
Y \in \chi(N/\tilde{\Gamma})
\end{array}
\]

The automorphism \( B \in \text{Aut}(g/\mathfrak{z}) \) is conjugated by the map \( F_* \) to an automorphism of the sub-algebras \( F_*(g/\mathfrak{z}) \), still denoted by \( B \), and coinciding with the push forward map \( \tau_* \):

\[
\begin{array}{c}
\mathfrak{g}/\mathfrak{z} \\
\downarrow \psi_* = B \\
\mathfrak{g}/\mathfrak{z} \\
\downarrow F_* \\
\mathfrak{g}/\mathfrak{z} \\
\downarrow \tau_* = B \\
\mathfrak{g}/\mathfrak{z}
\end{array}
\]

Thus we have, for any vector field \( Y \in F_*(g/\mathfrak{z}) \),

\[
(\tau)_* Y = B(Y).
\]

Let \( y = (y_1, \ldots, y_d) \) be “linear coordinates” on the torus \( N/\tilde{\Gamma} \), for which the lattice \( \tilde{\Gamma} \) is the lattice \( \mathbb{Z}^d \) in \( \mathbb{R}^d \) and the (ergodic) translation \( \tau \) is the translation modulo \( \mathbb{Z}^d \) by the irrational vector \( \alpha \in \mathbb{R}^d \). In these coordinates the differential of the \( \tau \) is the identity and we obtain, for all vector fields \( Y \) on \( N/\tilde{\Gamma} \) and all \( y \in N/\tilde{\Gamma} \)

\[
(\tau)_* Y(y) = Y(\tau^{-1}(y)).
\]

Thus, by the identity (16), we obtain that for all \( Y \in F_*(g/\mathfrak{z}) \) we have

\[
(17) \quad Y(\tau^{-1}(y)) = B(Y)(y), \quad \text{for all } y \in N/\tilde{\Gamma}.
\]

Let \( Y_1, \ldots, Y_d \) be a basis of \( F_*(g/\mathfrak{z}) \) and let \( X_1, \ldots, X_d \) be a basis of constant vector fields on \( N/\tilde{\Gamma} \). Then we can write

\[
(18) \quad Y_i = \sum_j H_{ij} X_j, \quad \text{for all } i = 1, \ldots, d.
\]

with \( H = (H_{ij}) : N/\tilde{\Gamma} \to \text{Gl}(\mathbb{R}^d) \) a smooth function on \( N/\tilde{\Gamma} \). Writing the matrix valued function \( H \) in Fourier series with respect to the coordinates \( (y_i) \), we have

\[
(19) \quad H(y) = \sum_{n \in \mathbb{Z}^d} h_n e_n(y),
\]

with \( e_n(y) = \exp(2\pi i n \cdot y) \). Denoting the matrix on the automorphism \( B \) with respect to the basis \( (Y_i) \) by the same letter \( B \), the equation (17) reads

\[
(B - e_n(-\alpha)I) h_n = 0
\]

or

\[
h_n^T (B^T - e_n(-\alpha)I) = 0.
\]
This identity shows that the matrix $h_n^T$ vanishes on the range of $(B^T - e_n(-\alpha)I)$. Thus, in Fourier series of $H(y)$, there are at most $d$ coefficients $h_n$ which do not vanish; they correspond to the indices $n$ for which $e_n(-\alpha)$ is an eigenvalues of the matrix $B$. Let $E_{n_1}, \ldots, E_{n_k}$ be the generalized eigenspaces of $B^T$ corresponding to the eigenvalues $e_{n_1}(-\alpha) \ldots e_{n_k}(-\alpha)$. Then $\bigoplus_{\ell=1}^k E_{n_\ell} = \mathbb{C}^d$ and $\ker h_n^T \supset \bigoplus_{m \neq \ell} E_{n_m}$. Since the matrix $H(y)$ is invertible for all $y \in N/\tilde{Z}\Gamma$, the kernel of $h_n^T$ cannot be larger than $\bigoplus_{m \neq \ell} E_{n_m}$ and consequently we have the identity $\ker h_n^T = \bigoplus_{m \neq \ell} E_{n_m}$. It follows that the matrix $h_n$ is a linear surjective map of $\mathbb{C}^d$ onto $E_{n_\ell}$ and a linear isomorphism when restricted to $E_{n_\ell}$. Hence $B|E_{n_\ell} = e_{n_\ell}(-\alpha)I$. We have proved that the automorphism $B$ of $F_*(\mathfrak{g}/\mathfrak{z})$ (or equivalently the automorphism $B$ of $\mathfrak{g}/\mathfrak{z}$) is semi-simple with spectrum $\{e_{n_\ell}(-\alpha)|\ell = 1, \ldots, k\}$ corresponding to eigenspaces $E_{n_\ell} \subset F_*(\mathfrak{g}/\mathfrak{z})$, $\ell = 1, \ldots, k$.

Since $\alpha$ is a irrational number, none of the numbers $e_{n_\ell}(-\alpha)$ occurring in the spectrum of $B$ is a root of unity, if $n_\ell \neq 0$. From the results of Step 2, we deduce that every eigenspace $E_{n_\ell} \subset \mathfrak{g}/\mathfrak{z}$ with $n_\ell \neq 0$ lift to an eigenspace $E'_{n_\ell}$ of the automorphism $B_0$ of $\mathfrak{g}$. The subspace $E_0 \subset \mathfrak{g}/\mathfrak{z}$ is an Abelian sub-algebra of $\mathfrak{g}/\mathfrak{z}$, mapped to constant vector fields by the conjugation $F_*$.

Chasing definitions, we conclude that the identity

$$\tau_0 \bar{Y} = e_n(-\alpha) \bar{Y},$$

holds true for all $\bar{Y} \in F_0\ast(E_{n_\ell})$.

**Step 5.** Let $(Y_\ell)_{\ell=1,\ldots,d}$ be a basis of the sub-algebra $F_*(\mathfrak{g}/\mathfrak{z})$ of eigenvectors of the automorphism $B \approx \tau_\ast$ with eigenvalues $\lambda_\ell = e_{n_\ell}(-\alpha)$, $\ell = 1, \ldots, d$. The elements $(Y_\ell)$ with $n_\ell \neq 0$ come in conjugate pairs. We may assume that elements $(Y_\ell)$ with $n_\ell = 0$ are real. Let $(Y_\ell^\ast)_{\ell=1,\ldots,d}$ be the horizontal lifts of the fields $(Y_\ell)_{\ell=1,\ldots,d}$ (then this set of elements of $\mathfrak{g}$ is closed under conjugation).

Let $I_0$ be the set of indices $\ell = 1, \ldots, d$ such that $n_\ell = 0$, and let $I_0^\ell$ be its complement in the integer interval $[1, d]$. For $\ell \in I_0^\ell$ let $\bar{Y}_\ell \in F_0\ast(E'_{n_\ell})$ be an eigenvector $\tau_0 \ast$ projecting to $Y_\ell$, so that

$$\tau_0 \ast \bar{Y}_\ell = e_{n_\ell}(-\alpha) \bar{Y}_\ell, \quad \forall \ell \in I_0^\ell;$$

for $\ell \in I_0$, let $\bar{Y}_\ell$ be a generalised eigenvector of eigenvalue 1 for $\tau_0 \ast$, projecting to the (constant) field $Y_\ell$; then there exist vertical fields so that $V_{\ell,1}, V_{\ell,2}, \ldots, V_{\ell,j_\ell}$, such that

$$\tau_0 \ast \bar{Y}_\ell = k V_{\ell,1} + k V_{\ell,2} + \cdots + k V_{\ell,j_\ell}, \quad \forall \ell \in I_0^\ell \forall k \in \mathbb{N}. $$

Choose fundamental horizontal vector fields $(X'_{\ell})_{\ell=1,\ldots,d}$ so that at the point $\Gamma \in N/\Gamma$ they coincide with the horizontal vectors $(Y_\ell^\ast(\Gamma))_{\ell=1,\ldots,d}$. Then the vector fields $(X'_{\ell})_{\ell=1,\ldots,d}$ project to constant vector fields $(X_{\ell})_{\ell=1,\ldots,d}$ which at the point $\tilde{Z}\Gamma \in N/\tilde{Z}\Gamma$ coincide with the vectors $(Y_\ell(\tilde{Z}\Gamma))$, i.e., we have $Y_\ell(\tilde{Z}\Gamma) = X_\ell$, for all $\ell = 1, \ldots, d$.

With these choices, the formula (18), in virtue of the definition (19), becomes

$$Y_\ell = e_{n_\ell} X_\ell, \quad \ell = 1, \ldots, d,$$
and, consequently, we have
\[ Y_\ell^* = e_{n_\ell} X_\ell^* \quad \ell = 1, \ldots, d. \]

From the identity (15) we obtain
\[ (\tau_0)_* Y_\ell^* = e_{n_\ell}(-\alpha)\left( Y_\ell^* + \mu(Y_\ell) \right), \]
for all \( \ell = 1, \ldots, d. \)

Let \( \nu \) be the \( 3 \)-valued one-form on the torus \( N/\tilde{\Gamma} \) defined by
\[ \tilde{Y}_\ell = Y_\ell^* + \nu(Y_\ell), \quad \forall \ell = 1, \ldots, d. \]

For \( Y = Y_\ell \), using (20) and the fact that \( (\tau_0)_* \) restricted to vertical vectors coincides with the automorphism \( A_0 \), we obtain the identity
\[ (\tau_0)_*(\nu(Y))(x) = (\text{d}r_0)_{\gamma_{\nu}^{-1}(x)}\left( \nu_{\gamma_{\nu}^{-1}(x)}(Y_{\gamma_{\nu}^{-1}(x)}) \right) = A_0\left( \nu_{\gamma_{\nu}^{-1}(x)}(Y_{\gamma_{\nu}^{-1}(x)}) \right) \]
\[ = A_0\left( (\tau_\nu \nu)(\tau_\nu Y)_x \right) = e_{n_\nu}(-\alpha)A_0\left( (\tau_\nu \nu)(Y)_x \right), \]
where, as usual, the push-forward of a one-form by a diffeomorphism is defined as the pull-back by the inverse diffeomorphism: for any scalar one-form \( \beta \) on the torus \( N/\tilde{\Gamma} \) we define \( \tau_\nu \beta = (\tau^{-1})^* \beta \) (so that \( \beta_\nu(X_\ell) = \beta_{\nu^{-1}}(X_\ell) \) for all \( \ell = 1, \ldots, d \)). Hence, from (23) and the above definition (24), we derive
\[ (\tau_0)_* \tilde{Y}_\ell = e_{n_\nu}(-\alpha)\left[ Y_\ell^* + \mu(Y_\ell) + (A_0 \circ \tau_\nu \nu)(Y_\ell) \right], \quad \forall \ell = 1, \ldots, d. \]

On the other hand, for \( \ell \in I_0^c \), the same definition (24) together with the identity (21) yields
\[ (\tau_0)_* \tilde{Y}_\ell = e_{n_\nu}(-\alpha)\left[ Y_\ell^* + \nu(Y_\ell) \right], \quad \forall \ell \in I_0^c. \]

Comparing the two expressions above we obtain, for all \( \ell \in I_0^c \), the identity
\[ \mu(Y_\ell) = \nu(Y_\ell) - (A_0 \circ \tau_\nu \nu)(Y_\ell), \quad \text{or, equivalently,} \]
\[ \mu(X_\ell) = \nu(X_\ell) - (A_0 \circ \tau_\nu \nu)(X_\ell), \quad \forall \ell \in I_0^c. \]

For the case \( \ell \in I_0 \), we consider the following generalizations of the formulas (23) and (25): for all \( \ell = 1, \ldots, d \), we have
\[ (\tau_0)_*^k Y_\ell^* = e_{n_\nu}(-k\alpha)(Y_\ell^* + \sum_{j=1}^k (A_0^{j-1} \circ \tau_\nu^{j-1} \mu)(Y_\ell)) \]
and
\[ (\tau_0)_*^k \tilde{Y}_\ell = e_{n_\nu}(-k\alpha)\left[ Y_\ell^* + \sum_{j=1}^k (A_0^{j-1} \circ \tau_\nu^{j-1} \mu)(Y_\ell) + (A_0^k \circ \tau_\nu^k \nu)(Y_\ell) \right]. \]

For \( \ell \in I_0 \), the definition (24) and the formula (22) give
\[ (\tau_0)_*^k \tilde{Y}_\ell = \tilde{Y}_\ell^* + \nu(Y_\ell) + kV_{\ell,1} + \left( \frac{k}{2} \right)V_{\ell,2} + \cdots + \left( \frac{k}{j} \right)V_{\ell,j}, \quad \forall \ell \in I_0^c. \]

Taking differences and considering that for \( \ell \in I_0 \) we have \( Y_\ell = X_\ell \) we obtain
\[ \sum_{j=1}^k (A_0^{j-1} \circ \tau_\nu^{j-1} \mu)(X_\ell) = \nu(X_\ell) - (A_0^k \tau_\nu^k \nu)(X_\ell) + kV_{\ell,1} + \left( \frac{k}{2} \right)V_{\ell,2} + \cdots + \left( \frac{k}{j} \right)V_{\ell,j}. \]
We have therefore shown that the diffeomorphism \( \tau \)

\[ \text{as all the eigenvalues of the automorphism } A \text{ form the basis of a nilpotent Lie algebra isomorphic to } n \]

\[ \text{and that } (A_0 \circ \tau_\ast v)(X) = X^* + \mu(X) + (A_0 \circ \tau_\ast v)(X). \]

**Step 6.** Define a new connection \( \omega_0 \) on \( N/\Gamma \) by setting

\[ \omega_0 = \omega - \nu. \]

The horizontal lifts of a constant vector field \( X \) with respect to this connection is now given by the formula

\[ \tilde{X}^* = X^* + \nu(X). \]

From the identity (15) we obtain

\[ (\tau_0)_\ast (\tilde{X}^*) = (\tau_0)_\ast (X^*) + (A_0 \circ \tau_\ast v)(X) = X^* + \mu(X) + (A_0 \circ \tau_\ast v)(X). \]

For \( \ell \in I_0 \), by the identity (26) we obtain

\[ (\tau_0)_\ast (\tilde{X}^*_\ell) = X^* + \mu(X_\ell) + A_0 \tau_\ast v(X_\ell) = X^*_\ell + \nu(X_\ell) = \tilde{X}^*_\ell. \]

For \( \ell \in I_0 \), the identity (27), yields by a similar computation,

\[ (\tau_0)_\ast (\tilde{X}^*_\ell) = X^*_\ell + \nu(X_\ell). \]

We have therefore shown that the diffeomorphism \( \tau_0 \) has a constant Jacobian matrix with respect to the basis of vector fields \( \{ \tilde{X}^*_i, V^*_{a_i} \mid \ell = 1, \ldots, d, a = 1, \ldots s \} \)

and that \( \tau_0 \circ (\tilde{X}^*_\ell) = \tilde{X}^*_\ell + \sum c_{\ell,a} V^*_{a_i}. \)

**Step 7.** The \( 3 \)-valued one-form \( \nu \) is \( Z(G) \) invariant, hence for any constant vector field \( X \) on the torus \( N/\tilde{\Gamma} \) and every fundamental vertical vector field \( V^* \) we have \( [V^*, \nu(X)] = 0 \) and therefore \( [V^*, \tilde{X}^*] = [V^*, X^* + \nu(X)] = 0. \) It follows that

\[ (\tau_0)_\ast [\tilde{X}^*_i, \tilde{X}^*_j] = [\tilde{X}^*_i, \tilde{X}^*_j]. \]

On the other hand we have

\[ [\tilde{X}^*_i, \tilde{X}^*_j] = -\Omega(X_i, X_j) + d\nu(X_i, X_j) = C(X_i, X_j). \]

Hence

\[ (\tau_0)_\ast [\tilde{X}^*_i, \tilde{X}^*_j] = (A_0 \circ \tau_\ast C)(X_i, X_j) = C(X_i, X_j) \]

that is,

\[ A_0 \circ \tau_\ast C = C. \]

As all the eigenvalues of the automorphism \( A_0 \) are roots of unity by taking a suitable power of the automorphism \( A_0 \), we obtain the identity \( A_0^k \circ \tau_\ast C = C \), with \( A_0^k \) unipotent. It follows, by the ergodicity of all the iterates of \( \tau \), that the two-form \( C \) is constant, i.e., that \( d\nu = 0 \). We observe that the vector fields \( \{ \tilde{X}^*_i, V^*_a \} \)

form the basis of a nilpotent Lie algebra isomorphic to \( n \), since the only non-trivial commutation relations are given by

\[ [\tilde{X}^*_i, \tilde{X}^*_j] = -\Omega(X_i, X_j). \]
Summarizing, by the remark at the end of Step 1, we may compose the diffeomorphism \( F_0 \) by a diffeomorphism \( h \) of \( N \) projecting to the identity diffeomorphism of \( N/\Gamma \), so that, after lifting the vector fields \( X_i, \tilde{X}_i^* \) and \( V_a^* \) to \( N \), we have \( h_* X_i = \tilde{X}_i^* \) and \( h_* V_a = V_a^* \). The vector fields \( \{\tilde{X}_i^*, V_a^*\} \) define a new Lie group structure on \( N \), with Lie algebra \( \mathfrak{n} \), with respect to which the diffeomorphism \( \tau_0 \) has “constant Jacobian matrix” since

\[
(\tau_0)_* V_a^* = A_0(V_a^*)
\]

and since there exist \( V_i \in \mathfrak{z} \)

\[
(\tau_0)_* \tilde{X}_i^* = \tilde{X}_i^* + V_i, \quad i = 1, \ldots, d.
\]

Thus, up to replacing \( F_0 \) with \( h \circ F_0 \) and the lattice \( \Gamma \) by \( h(\Gamma) \) and renaming the \( \tilde{X}_i^* \) as \( X_i^* \), we have proved that there exists

- a connected, simply connected, nilpotent Lie group \( N \),
- a lattice \( \Gamma < N \),
- a diffeomorphism \( F_0: G/D \to N/\Gamma \),
- a diffeomorphism \( \tau_0: N/\Gamma \to N/\Gamma \),
- and a basis \( \{X_i^*, V_a^* | i = 1, \ldots, d, \alpha = 1, \ldots, s\} \) of the Lie algebra \( \mathfrak{n} \) of \( N \) such that

(i) The only non-trivial commutation relations in \( \mathfrak{n} \) are given by

\[
[X_i^*, X_j^*] = V_{ij} \quad \text{with} \ V_{ij} \in \mathfrak{z} := \text{span}(V_a^*).
\]

(ii) The diffeomorphism \( F_0 \) intertwines the maps \( \psi_0 \) and \( \tau_0 \):

\[
F_0 \circ \psi_0 = \tau_0 \circ F_0.
\]

(iii) The diffeomorphism \( \tau_0 \) has “constant Jacobian matrix” given by

\[
(\tau_0)_* V_a^* = A_0(V_a^*), \quad (\tau_0)_* \tilde{X}_i^* = \tilde{X}_i^* + V_i, \quad i = 1, \ldots, d, \quad \alpha = 1, \ldots, s,
\]

with \( A_0 \) a quasi-unipotent automorphism of the Abelian subalgebra \( \mathfrak{z} \) and \( V_i \in \mathfrak{z} \).

It is immediate to deduce that \( \tau_0 \) is an affine map of \( N \), hence concluding the proof. In fact let \( B_0 \) be the quasi-unipotent automorphism of \( \mathfrak{n} \) defined by \( (\tau_0)_* \).

Let \( \tilde{\tau}_0 \) be the lift to \( N \) of the diffeomorphism \( \tau_0 \) and set \( \tilde{\tau}_0(e_N) = x_0 \). The map \( \sigma: x \in N \mapsto x^{-1}_0 \tilde{\tau}_0(x) \in N \) fixes the neutral element \( e_N \in N \). Its tangent map is the automorphism of \( \mathfrak{n} \) given by \( B_1 := \text{Int}(x_0) \circ B_0 \), which is easily verified to be quasi-unipotent. Thus the map \( B^{-1}_1 \circ \sigma \) is a diffeomorphism of \( N \) fixing the neutral element \( e_N \) and whose tangent map induces the identity map \( \mathfrak{n} \). It follows that \( \tilde{\tau}_0(x) = x_0 B_1(x) \) for all \( x \in N \). Hence the map \( \tau_0 \) is an affine quasi-unipotent diffeomorphism of \( N/\Gamma \). As the diffeomorphism \( \psi_0 \) is ergodic, so is the diffeomorphism \( \tau_0 \). By Corollary 1 of Parry’s paper [38], a quasi-unipotent ergodic affine diffeomorphism of a nilmanifold is unipotent. This concludes the proof.

\[\square\]

**Lemma 6.2.** Let \( N \) be a connected, simply connected, nilpotent Lie group, let \( \Gamma < N \) be a lattice in \( N \) and \( \psi: N/\Gamma \to N/\Gamma \) a \( \text{Ad} \)-unipotent affine transformation,
with $\psi(x\Gamma) = uA(x)\Gamma$. Then the automorphism $A$ is unipotent with a rational generalized Jordan basis.

By a rational basis, we mean a basis $\mathcal{B}$ of $\mathfrak{n}$ such that every linear form on $\mathfrak{n}$ which is integral on $\log \Gamma$ takes rational values on every $v \in \mathcal{B}$. A rational basis $\mathcal{B}$ is a generalized Jordan basis for the automorphism $A \in \text{Aut}(N)$ if its matrix with respect to this basis is upper (or lower) triangular.

**Proof.** The Lemma is immediate if $N$ is Abelian, since in this case $d\psi = A$ and $A(\Gamma) = \Gamma$.

Suppose, by induction, that the lemma is true for all nilmanifolds $N/\Gamma$ such that the degree of nilpotency of $N$ is less than $n$. Let the degree of nilpotency of $N$ be equal to $n$.

Let $\{n^{(k)}\}$ denote the descending central series of the Lie algebra $\mathfrak{n}$ of $N$, defined by induction by $n^{(0)} = \mathfrak{n}$ and $n^{(k+1)} = [n^{(k)}, n]$ for all $k \in \mathbb{N}$, and let $N^{(k)}$ be the corresponding analytic groups.

Then $N/N^{(n-1)}$ is a nilpotent group of degree of nilpotency $(n-1)$. Since the induced affine diffeomorphism $\bar{\psi}$ of the nilmanifold $N/\Gamma N^{(n-1)}$ is unipotent, by the induction hypothesis, the induced automorphism $\bar{A} \in \text{Aut}(n/n^{(n-1)})$ is unipotent and has a generalized rational Jordan basis, that is, a basis

$\mathcal{B}_{n-1} = \{\bar{X}_{1,1}, \ldots, \bar{X}_{1,\ell_1}, \bar{X}_{2,1}, \ldots, \bar{X}_{2,\ell_2}, \ldots, \bar{X}_{k,1}, \ldots, \bar{X}_{k,\ell_k}\}$,

such that

$$\bar{A}\bar{X}_{i,j} = \bar{X}_{i,j} + \sum_{k=1}^{j-1} c_{i,k} \bar{X}_{i,k},$$

with $c_{i,k} \in \mathbb{Q}$ for all possible values of $i$ and $k$.

Let $\mathcal{B}_{n-1} = \{X_{1,1}, \ldots, X_{1,\ell_1}, X_{2,1}, \ldots, X_{2,\ell_2}, \ldots, X_{k,1}, \ldots, X_{k,\ell_k}\}$ a system of rational vectors projecting to $\mathcal{B}_{n-1}$ under the natural map $n \to n/n^{(n-1)}$. Since the differential $d\psi$ of the $\text{Ad}$-unipotent affine transformation $\psi$ preserves the center $N^{n-1}$ and it coincides with the automorphism $A$, there exists a rational Jordan basis $\mathcal{C}_{n-1}$ of $n^{n-1}$ for the restriction of the automorphism $A$ to $N^{n-1}$.

It is now immediate to check that $\mathcal{B}_n = \mathcal{B}_{n-1} \cup \mathcal{C}_{n-1}$ is a generalized rational Jordan basis for the automorphism $A$, and that $A$ is unipotent. \qed

**Lemma 6.3.** Let $N$ be a connected, simply connected, nilpotent Lie group, $\Gamma < N$ a lattice in $N$, and $\psi: N/\Gamma \to N/\Gamma$ a unipotent ergodic affine transformation. Then either the diffeomorphism $\psi$ admits infinitely many invariant independent distributions of Sobolev order $1/2$ or is smoothly diffeomorphic to a toral translation.

**Proof.** Recall that $N$ is an algebraic affine group. The exponential and logarithm map $\exp: n \to N$ and $\log: N \to n$ are polynomial maps, hence rational. By Lemma 6.2, if $\psi(x\Gamma) = uA(x)\Gamma$, then the automorphism $A$ is unipotent with a rational Jordan basis. It follows that there exists a one-parameter group $B = \{B^t\}_{t \in \mathbb{R}}$ of automorphisms of $N$ such that $B^1 = A$. With respect to a rational Jordan basis of $n$, the automorphisms $\{B^t\}$ are polynomial in the variable $t$, hence they operate rationally on $N$. It follows that the semi-direct product $H = N \rtimes B$ is an algebraic
nilpotent group. The set \( \{(u, A)^n \mid n \in \mathbb{Z}\} \) is a subgroup of \( H \); let \( T \) be its Zariski closure in \( H \). As \( A \) is unipotent and \( N \) connected and simply connected, the Abelian group \( T \) is connected for the Hausdorff topology. It follows that the group \( T \) is a one-parameter group of \( H \), necessarily of the form \( \{(u(t), B^1)\}_{t \in T} \), such that \( (u(1), B^1) = (u, A) \).

Define \( \Lambda = \{(\gamma, B^\gamma) \mid n \in \mathbb{Z}, \gamma \in \Gamma\} \), since \( B^1 = A \) and \( A(\Gamma) = \Gamma \), the set \( \Lambda \) is a discrete subgroup of \( H \). In fact the space \( H/\Lambda \) is a compact nilmanifold. We consider the nilmanifold \( N/\Gamma \) as a submanifold of \( H/\Lambda \) via the embedding given by the mapping \( x\Gamma \rightarrow (x, 1)\Lambda \).

Let \( \langle \phi_t \rangle_{t \in \mathbb{R}} \) be the flow on \( H/\Lambda \) given by left translation by the one-parameter group \( \langle B^1 \rangle_{t \in \mathbb{R}} \). It is immediate to check that the first return of the flow \( \langle \phi_t \rangle_{t \in \mathbb{R}} \) to the submanifold \( N/\Gamma \) occurs at time \( t = 1 \) and the first return map coincides with the affine map \( \psi \). Hence the flow \( \langle \phi_t \rangle_{t \in \mathbb{R}} \) on the compact nilmanifold \( H/\Lambda \) is the suspension of the affine diffeomorphism \( \psi \) and in particular it is ergodic.

By Theorem 4.13, the flow \( \langle \phi_t \rangle_{t \in \mathbb{R}} \) either admits infinitely many invariant independent distributions of Sobolev order 1/2 or is diffeomorphic to an ergodic translation flow on a torus. This latter possibility occurs only if the nilmanifold \( H/\Lambda \) is a torus; this is the case only if the automorphism \( A \) is trivial and the group \( N \) is Abelian, that is, if the affine map \( \psi \) is a translation and on a torus. Thus, unless the diffeomorphism \( \psi \) is a translation on a torus, the diffeomorphism \( \psi \) admits infinitely many invariant independent distributions of Sobolev order 1/2.

**Corollary 6.4.** Let \( G \) be a connected, simply connected, solvable Lie group, let \( D < G \) be a lattice in \( G \) and let \( \psi_0 = u_0A_0 \) be an ergodic affine quasi-unipotent diffeomorphism of \( G/D \). Assume that \( Z(G)D \) is a closed subgroup of \( G \). Let \( F: G/Z(G)D \rightarrow \mathbb{T}^d \) be a diffeomorphism conjugating the map \( \psi \), induced by \( \psi_0 \) on \( G/Z(G)D \), with an ergodic translation \( \tau \) of \( \mathbb{T}^d \). Then either the diffeomorphism \( \psi_0 \) admits infinitely many invariant independent distributions of Sobolev order 1/2 or is smoothly diffeomorphic to a toral translation.

**Proof.** By Lemma 6.1 there exists a connected, simply connected, nilpotent Lie group \( N \), a lattice \( \Gamma < N \), a diffeomorphism \( F_0: G/D \rightarrow N/\Gamma \) and an unipotent ergodic affine diffeomorphism \( \tau_0: N/\Gamma \rightarrow N/\Gamma \) such that \( F_0 \circ \psi_0 = \tau_0 \circ F_0 \).

Thus the statement follows from Lemma 6.3.

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7. Open problems

We conclude the paper by stating some (mostly well-known) open problems and conjectures on the stability and the codimension of smooth flows.

**Conjecture 7.1.** (A. Katok) Every homogeneous flow (on a compact homogeneous space) which fails to be stable (in the sense that the range of the Lie derivative on the space of smooth functions is not closed) projects onto a Liouvillean linear flow on a torus. In this case, the flow is still stable on the orthogonal complement of the subspace of toral functions (in other words, the subspace of all functions with zero average along each fiber of the projection).
As mentioned in the Introduction, hyperbolic and partially hyperbolic, central isometric (or more generally with uniform sub-exponential central growth), accessible systems are stable.

In the unipotent case, it is proved in [16] that $SL(2, \mathbb{R})$ unipotent flows (horocycles) on finite volume homogeneous spaces are stable and in [17] that the above conjecture holds for nilflows.

**Problem 7.1.** Classify all compact manifolds which admit uniquely ergodic flows with (a) a unique invariant distribution (equal to the unique invariant measure) up to normalization; (b) a finite dimensional space of invariant distributions.

Example of manifolds (and flows) of type (a) have been found by A. Avila, B. Fayad and A. Kocsard [1]. Note that the Katok (Greenfield-Wallach) conjecture implies that in all non-toral examples of type (a) the flow cannot be stable. Recently A. Avila and A. Kocsard [4] have announced that they have constructed maps on the two-torus having a space of invariant distributions of arbitrary odd dimension. It is unclear whether examples of this type can be stable:

**Problem 7.2.** (M. Herman) Does there exists a stable flow with finitely many invariant distributions which is not smoothly conjugate to a Diophantine linear flow on a torus?

The only known example which comes close to an affirmative answer to this problem is given by generic area-preserving flows on compact higher genus surfaces [18, 35]. Such flows are generically stable and have a finite dimensional space of invariant distributions in every finite differentiability class (but not in the class of infinitely differentiable functions).

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Livio Flaminio <livio.flaminio@math.univ-lille1.fr>: Unité Mixte de Recherche CNRS 8524, Unité de Formation et Recherche de Mathématiques, Université de Lille, F59655 Villeneuve d’Asq CEDEX, France

Giovanni Forni <gforni@math.umd.edu>: Department of Mathematics, University of Maryland, College Park, MD 20742, USA

Federico Rodriguez Hertz <hertz@math.psu.edu>: Mathematics Department, The Pennsylvania State University, University Park, PA 16802, USA