Representability of Hom implies flatness

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Abstract

Let \( X \) be a projective scheme over a noetherian base scheme \( S \), and let \( \mathcal{F} \) be a coherent sheaf on \( X \). For any coherent sheaf \( \mathcal{E} \) on \( X \), consider the set-valued contravariant functor \( \mathcal{H}om(\mathcal{E}, \mathcal{F}) \) on \( S \)-schemes, defined by \( \mathcal{H}om(\mathcal{E}, \mathcal{F})(T) = \text{Hom}(\mathcal{E}_T, \mathcal{F}_T) \) where \( \mathcal{E}_T \) and \( \mathcal{F}_T \) are the pull-backs of \( \mathcal{E} \) and \( \mathcal{F} \) to \( X_T = X \times_S T \). A basic result of Grothendieck ([EGA] III 7.7.8, 7.7.9) says that if \( \mathcal{F} \) is flat over \( S \) then \( \mathcal{H}om(\mathcal{E}, \mathcal{F}) \) is representable for all \( \mathcal{E} \).

We prove the converse of the above, in fact, we show that if \( L \) is a relatively ample line bundle on \( X \) over \( S \) such that the functor \( \mathcal{H}om(L^{-n}, \mathcal{F}) \) is representable for infinitely many positive integers \( n \), then \( \mathcal{F} \) is flat over \( S \). As a corollary, taking \( X = S \), it follows that if \( \mathcal{F} \) is a coherent sheaf on \( S \) then the functor \( T \mapsto H^0(T, \mathcal{F}_T) \) on the category of \( S \)-schemes is representable if and only if \( \mathcal{F} \) is locally free on \( S \). This answers a question posed by Angelo Vistoli.

The techniques we use involve the proof of flattening stratification, together with the methods used in proving the author’s earlier result (see [N1]) that the automorphism group functor of a coherent sheaf on \( S \) is representable if and only if the sheaf is locally free.

Let \( S \) be a noetherian scheme, and let \( X \) be a projective scheme over \( S \). If \( \mathcal{E} \) and \( \mathcal{F} \) are coherent sheaves on \( X \), consider the contravariant functor \( \mathcal{H}om(\mathcal{E}, \mathcal{F}) \) from the category of schemes over \( S \) to the category of sets which is defined by putting

\[
\mathcal{H}om(\mathcal{E}, \mathcal{F})(T) = \text{Hom}(\mathcal{E}_T, \mathcal{F}_T)
\]

for any \( S \)-scheme \( T \to S \), where \( X_T = X \times_S T \), and \( \mathcal{E}_T \) and \( \mathcal{F}_T \) denote the pull-backs of \( \mathcal{E} \) and \( \mathcal{F} \) under the projection \( X_T \to X \). This functor is clearly a sheaf in the fpqc topology on \( \text{Sch}/S \). It was proved by Grothendieck that if \( \mathcal{F} \) is flat over \( S \) then the above functor is representable (see [EGA] III 7.7.8, 7.7.9).

Our main theorem is as follows, which is a converse to the above.

**Theorem 1** Let \( S \) be a noetherian scheme, \( X \) a projective scheme over \( S \), and \( L \) a relatively very ample line bundle on \( X \) over \( S \). Let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then the following three statements are equivalent:

1. The sheaf \( \mathcal{F} \) is flat over \( S \).
(2) For any coherent sheaf \( \mathcal{E} \) on \( X \), the set-valued contravariant functor \( \text{Hom}(\mathcal{E}, \mathcal{F}) \) on \( S \)-schemes, defined by \( \text{Hom}(\mathcal{E}, \mathcal{F})(T) = \text{Hom}_X(\mathcal{E}_T, \mathcal{F}_T) \), is representable.

(3) There exist infinitely many positive integers \( r \) such that the set-valued contravariant functor \( \mathcal{G}^{(r)} \) on \( S \)-schemes, defined by \( \mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r}) \), is representable.

In particular, taking \( X = S \) and \( L = \mathcal{O}_X \), we get the following corollary.

**Corollary 2** Let \( S \) be a noetherian scheme, and \( \mathcal{F} \) a coherent sheaf on \( S \). Consider the contravariant functor \( F \) from \( S \)-schemes to sets, which is defined by putting \( F(T) = H^0(T, f^*\mathcal{F}) \) for any \( S \)-scheme \( f : T \to S \). This functor (which is a sheaf in the fpqc topology) is representable if and only if \( F \) is locally free as an \( \mathcal{O}_S \)-module.

Note that the affine line \( \mathbb{A}^1_S \) over a base \( S \) admits a ring-scheme structure over \( S \) in the obvious way. A **linear scheme** over a scheme \( S \) will mean a module-scheme \( V \to S \) under the ring-scheme \( \mathbb{A}^1_S \). This means \( V \) is a commutative group-scheme over \( S \) together with a ‘scalar-multiplication’ morphism \( \mu : \mathbb{A}^1_S \times_S V \to V \) over \( S \), such that the module axioms (in diagrammatic terms) are satisfied.

A **linear functor** \( F \) on \( S \)-schemes will mean a contravariant functor from \( S \)-schemes to sets together with the structure of an \( H^0(T, \mathcal{O}_T) \)-module on \( F(T) \) for each \( S \)-scheme \( T \), which is well-behaved under any morphism \( f : U \to T \) of \( S \)-schemes in the following sense: \( F(f) : F(T) \to F(U) \) is a homomorphism of the underlying additive groups, and \( F(f)(a \cdot v) = f^*(a) \cdot (F(f)v) \) for any \( a \in H^0(T, \mathcal{O}_T) \) and \( v \in F(T) \). In particular note that the kernel of \( F(f) \) will be an \( H^0(T, \mathcal{O}_T) \)-submodule of \( F(T) \).

The functor of points of a linear scheme is naturally a linear functor. Conversely, it follows by the Yoneda lemma that if a linear functor \( F \) on \( S \)-schemes is representable, then the representing scheme \( V \) is naturally a linear scheme over \( S \).

For example, the linear functor \( T \mapsto H^0(T, \mathcal{O}_T)^n \) (where \( n \geq 0 \)) is represented by the affine space \( \mathbb{A}^n_S \) over \( \text{Spec } \mathbb{Z} \), with its usual linear-scheme structure. More generally, for any coherent sheaf \( \mathcal{Q} \) on \( S \), the scheme \( \text{Spec } \text{Sym}(\mathcal{Q}) \) is naturally a linear-scheme over \( S \), where \( \text{Sym}(\mathcal{Q}) \) denotes the symmetric algebra of \( \mathcal{Q} \) over \( \mathcal{O}_S \). It represents the linear functor \( F(T) = \text{Hom}(\mathcal{Q}_T, \mathcal{O}_T) \) where \( \mathcal{Q}_T \) denotes the pull-back of \( \mathcal{Q} \) under \( T \to S \).

With this terminology, the functor \( \mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r}) \) of Theorem 1(3) is a linear functor. Therefore, if a representing scheme \( \mathcal{G}^{(r)} \) exists, it will naturally be a linear scheme. Note that each \( \mathcal{G}^{(r)} \) is obviously a sheaf in the fpqc topology.

The proof of Theorem 1 is by a combination of the result of Grothendieck on the existence of a flattening stratification ([TDTE -IV]) together with the techniques which were employed in [N1] to prove the following result.

**Theorem 3 (Representability of the functor \( GL_E \))** Let \( S \) be a noetherian scheme, and \( E \) a coherent \( \mathcal{O}_S \)-module. Let \( GL_E \) denote the contravariant functor on \( S \)-schemes which associates to any \( S \)-scheme \( f : T \to S \) the group of all \( \mathcal{O}_T \)-linear
automorphisms of the pull-back $E_T = f^* E$ (this functor is a sheaf in the fpqc topology). Then $GL_E$ is representable by a group scheme over $S$ if and only if $E$ is locally free.

We re-state Grothendieck’s result (see [TDTE IV]) on the existence of a flattening stratification in the following form, which emphasises the role of the direct images $\pi_*(\mathcal{F}(r))$. For an exposition of flattening stratification, see Mumford [M] or [N2].

**Theorem 4 (Grothendieck)** Let $S$ be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_S$ where $n \geq 0$. There exists an integer $m$, and a collection of locally closed subschemes $S_f \subset S$ indexed by polynomials $f \in \mathbb{Q}[\lambda]$, with the following properties.

(i) The underlying set of $S_f$ consists of all $s \in S$ such that the Hilbert polynomial of $\mathcal{F}_s$ is $f$, where $\mathcal{F}_s$ denotes the pull-back of $\mathcal{F}$ to the schematic fibre $\mathbf{P}^n_s$ over $s$ of the projection $\pi : \mathbf{P}^n_S \to S$. All but finitely many $S_f$ are empty (only finitely many Hilbert polynomials occur). In particular, the $S_f$ are mutually disjoint, and their set-theoretic union is $S$.

(ii) For each $r \geq m$, the higher direct images $R^j \pi_*(\mathcal{F}(r))$ are zero for $j \geq 1$ and the subschemes $S_f$ give the flattening stratification for the direct image $\pi_*(\mathcal{F}(r))$, that is, the morphism $i : \coprod_f S_f \to S$ induced by the locally closed embeddings $S_f \hookrightarrow S$ has the universal property that for any morphism $g : T \to S$, the sheaf $g^* \pi_*(\mathcal{F}(r))$ is locally free on $T$ if and only if $g$ factors via $i : \coprod_f S_f \to S$.

(iii) The subschemes $S_f$ give the flattening stratification for $\mathcal{F}$, that is, for any morphism $g : T \to S$, the sheaf $\mathcal{F}_T = (1 \times g)^* \mathcal{F}$ on $\mathbf{P}^n_T$ is flat over $T$ if and only if $g$ factors via $i : \coprod_f S_f \to S$. In particular, $\mathcal{F}$ is flat over $S$ if and only if each $S_f$ is an open subscheme of $S$.

(iv) Let $\mathbb{Q}[\lambda]$ be totally ordered by putting $f_1 < f_2$ if $f_1(p) < f_2(p)$ for all $p \gg 0$. Then the closure of $S_f$ in $S$ is set-theoretically contained in $\bigcup_{g \geq f} S_g$. Moreover, whenever $S_f$ and $S_g$ are non-empty, we have $f < g$ if and only if $f(p) < g(p)$ for all $p \geq m$.

The following elementary lemma of Grothendieck on base-change does not need any flatness hypothesis. The price paid is that the integer $r_0$ may depend on $\phi$. (See [N2] for a cohomological proof.)

**Lemma 5** Let $\phi : T \to S$ be a morphism of noetherian schemes, let $\mathcal{F}$ a coherent sheaf on $\mathbf{P}^n_S$, and let $\mathcal{F}_T$ denote its pull-back under the induced morphism $\mathbf{P}^n_T \to \mathbf{P}^n_S$. Let $\pi_S : \mathbf{P}^n_S \to S$ and $\pi_T : \mathbf{P}^n_T \to T$ denote the projections. Then there exists an integer $r_0$ such that the base-change homomorphism $\phi^* \pi_{S*} \mathcal{F}(r) \to \pi_{T*} \mathcal{F}_T(r)$ is an isomorphism for all $r \geq r_0$.

**Proof of Theorem 1** The implication (1) $\Rightarrow$ (2) follows by [EGA] III 7.7.8, 7.7.9, while the implication (2) $\Rightarrow$ (3) follows by taking $\mathcal{E} = L^{\otimes -r}$. Therefore it now remains to show the implication (3) $\Rightarrow$ (1). This we do in a number of steps.
Step 1: Reduction to $S = \text{Spec } R$ with $R$ local, $X = P^n_S$ and $L = \mathcal{O}_{P^n_S}(1)$

Suppose that $\mathcal{F}$ is not flat over $S$, but the linear functor $\mathcal{G}^{(r)}$ on $S$-schemes, defined by $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by a linear scheme $G^{(r)}$ over $S$ for arbitrarily large integers $r$. As $\mathcal{F}$ is not flat, by definition there exists some $x \in X$ such that the stalk $\mathcal{F}_x$ is not a flat module over the local ring $\mathcal{O}_{S, \pi(x)}$ where $\pi : X \to S$ is the projection. Let $U = \text{Spec } \mathcal{O}_{S, \pi(x)}$, let $\mathcal{F}_U$ be the pull-back of $\mathcal{F}$ to $X_U = X \times_S U$ and let $G^{(r)}_U$ denote the pull-back of $G^{(r)}$ to $U$. Then $\mathcal{F}_U$ is not flat over $U$ but for any integer $m$, there exists an integer $r \geq m$ such that the functor $\mathcal{G}^{(r)}_U$ on $U$-schemes, defined by $\mathcal{G}^{(r)}_U(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by the $U$-scheme $G^{(r)}_U$.

Therefore, by replacing $S$ by $U$, we can assume that $S$ is of the form $\text{Spec } R$ where $R$ is a noetherian local ring. Let $i : X \hookrightarrow P^n_S$ be the embedding given by $L$. Then replacing $\mathcal{F}$ by $i_* \mathcal{F}$, we can further assume that $X = P^n_S$ and $L = \mathcal{O}_{P^n_S}(1)$.

Step 2: Flattening stratification of $\text{Spec } R$ There exists an integer $m$ as asserted by Theorem 11 such that for any $r \geq m$, the flattening stratification of $S$ for the sheaf $\pi_* \mathcal{F}(r)$ on $S$ is the same as the flattening stratification of $S$ for the sheaf $\mathcal{F}$ on $P^n_S$. Let $r \geq m$ be any integer. As $\mathcal{F}$ is not flat over $S = \text{Spec } R$, the sheaf $\pi_* \mathcal{F}(r)$ is not flat. Let $M_r = H^0(S, \pi_* \mathcal{F}(r))$, which is a finite $R$-module. Let $m \subset R$ be the maximal ideal, and let $k = R/m$ the residue field. Let $s \in S = \text{Spec } R$ be the closed point, and let $d = \dim_k(M_r/mM_r)$. Then there exists a right-exact sequence of $R$-modules of the form

$$R^d \xrightarrow{\psi} R^d \to M_r \to 0$$

Let $I \subset R$ be the ideal formed by the matrix entries of the $(d \times d)$-matrix $\psi$. Then $I$ defines a closed subscheme $S' \subset S$ which is the flattening stratification of $S$ for $M_r$. As $M_r$ is not flat by assumption, $I$ is a non-zero proper ideal in $R$.

It follows from Theorem 11 that $I$ is independent of $r$ as long as $r \geq m$.

Step 3: Reduction to artin local case with principal $I$ with $mI = 0$ Let $I = (a_1, \ldots, a_t)$ where $a_1, \ldots, a_t$ is a minimal set of generators of $I$. Let $J \subset R$ be the ideal defined by

$$J = (a_2, \ldots, a_t) + mI$$

Then note that $J \subset I \subset m$, and the quotient $R' = R/J$ is an artin local $R$-algebra with maximal ideal $m' = m/J$, and $I' = I/J$ is a non-zero principal ideal which satisfies $m'I' = 0$. For the base-change under $f : \text{Spec } R' \to \text{Spec } R$, the flattening stratification $f^* \pi_* \mathcal{F}(r)$ is defined by the ideal $I'$ for $r \geq m$. Let $\mathcal{F}'$ denote the pull-back of $\mathcal{F}$ to $P^n_{R'}$, and let $\pi'_F : P^n_{R'} \to \text{Spec } R'$ the projection. As $f$ is a morphism of noetherian schemes, by Lemma 11 there exists some integer $m'$ such that the base-change homomorphism $f^* \pi_* \mathcal{F}(r) \to \pi'_F \mathcal{F}(r)$ is an isomorphism whenever $r \geq m'$. Choosing some $m' \geq m$ with this property, and replacing $R$ by $R'$, $\mathcal{F}$ by $\mathcal{F}'$ and $m$ by $m'$, we can assume that $R$ is artin local, and $I$ is a non-zero principal ideal with $mI = 0$, which defines the flattening stratification for $\pi_* \mathcal{F}(r)$ for all $r \geq m$. 

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Step 4: Decomposition of $\pi_*F(r)$ via lemma of Srinivas

Lemma (Srinivas) Let $R$ be an artin local ring with maximal ideal $m$, and let $E$ be any finite $R$ module whose flattening stratification is defined by an ideal $I$ which is a non-zero proper principal ideal with $mI = 0$. Then there exist integers $i \geq 0$ and $j > 0$ such that $E$ is isomorphic to the direct sum $R^i \oplus (R/I)^j$.

Proof See Lemma 4 in [N1].

We apply the above lemma to the $R$-module $M_r = H^0(S, \pi_*F(r))$, which has flattening stratification defined by the principal ideal $I$ with $mI = 0$, to conclude that (up to isomorphism) $M_r$ has the form

$$M_r = R^{i(r)} \oplus (R/I)^{j(r)}$$

for non-negative integers $i(r)$ and $j(r)$ with $j(r) > 0$.

Note that $i(r) + j(r) = \Phi(r)$ where $\Phi$ is the Hilbert polynomial of $F$.

Step 5: Structure of the hypothetical representing scheme $G^{(r)}$ Let $\phi : \text{Spec}(R/I) \rightarrow \text{Spec} R$ denote the inclusion and $F'$ denote the pull-back of $F$ under $P^n_{R/I} \hookrightarrow P^n_R$. The sheaf $F'$ is flat over $R/I$, and the functor $G^{(r)}_{R/I}$, which is the restriction of $G^{(r)}$, is represented by the linear scheme $\mathbb{A}^d_{R/I} = \text{Spec}(R/I)[y_1, \ldots, y_d]$ over $R/I$, where $d = \Phi(r)$ where $\Phi$ is the Hilbert polynomial of $F$. Hence, the pull-back of the hypothetical representing scheme $G^{(r)}$ to $R/I$ is the linear scheme $\mathbb{A}^d_{R/I}$. We now use the following fact (see Lemma 6 and Lemma 7 of [N1] for a proof).

Lemma Let $R$ be a ring and $I$ a nilpotent ideal ($I^n = 0$ for some $n \geq 1$). Let $X$ be a scheme over $\text{Spec} R$, such that the closed subscheme $Y = X \otimes_R (R/I)$ is isomorphic over $R/I$ to $\text{Spec} B$ where $B$ is a finite-type $R/I$-algebra. Let $b_1, \ldots, b_d \in B$ be a set of algebra generators for $B$ over $R/I$. Then $X$ is isomorphic over $R$ with $\text{Spec} A$ where $A$ is a finite-type $R$-algebra. Moreover, there exists a set of $R$-algebra generators $a_1, \ldots, a_d$ for $A$, such that each $a_i$ restricts modulo $I$ to $b_i \in B$ over $R/I$. Let $R[x_1, \ldots, x_d]$ be a polynomial ring in $d$ variables over $R$, and consider the surjective $R$-algebra homomorphism $R[x_1, \ldots, x_d] \rightarrow A$ defined by sending each $x_i$ to $a_i$, and let $J$ be its kernel. Then $J \subset IR[x_1, \ldots, x_d]$.

It follows from the above lemma that $G^{(r)}$ is affine of finite type over $R$, and its co-ordinate ring $A$ as an $R$ algebra is of the form

$$A = R[a_1, \ldots, a_d] = R[x_1, \ldots, x_d]/J$$

where $a_i$ is the residue of $x_i$, and $a_1, \ldots, a_d$ restrict over $R/I$ to the linear coordinates $y_1, \ldots, y_d$ on the linear scheme $\mathbb{A}^d_{R/I}$, and $J$ is an ideal with $J \subset I \cdot R[x_1, \ldots, x_d]$.

Being an additive group-scheme, $G^{(r)}$ has its zero section $\sigma : \text{Spec} R \rightarrow G^{(r)}$, and this corresponds to an $R$-algebra homomorphism $\sigma^* : A \rightarrow R$. Modulo $I$, the section $\sigma$ restricts to the zero section of $\mathbb{A}^d_{R/I}$ over $\text{Spec}(R/I)$, therefore $\sigma^*(a_i) \in I$ for all $i = 1, \ldots, d$. Let $x'_i = x_i - \sigma^*(a_i) \in R[x_1, \ldots, x_d]$ and $a'_i = a_i - \sigma^*(a_i) \in A$ be its residue modulo $J$. Then $R[x_1, \ldots, x_d] = R[x'_1, \ldots, x'_d]$, the elements $a'_1, \ldots, a'_d$
generate $A$ as an $R$-algebra, and moreover the $a'_i$ restrict over $R/I$ to the linear coordinates $y_i$ on the linear scheme $A^d_{R/I}$. Therefore, by replacing the $x_i$ by the $x'_i$ and the $a_i$ by the $a'_i$, we can assume that for each $i$, we have

$$\sigma^*(a_i) = 0$$

Next, consider any element $f(x_1,\ldots,x_d) \in J$. Then $f(a_1,\ldots,a_d) = 0$ in $A$, so $\sigma^* f(a_1,\ldots,a_d) = 0 \in R$, which shows that the constant coefficient of $f$ is zero, as $\sigma^*(a_i) = 0$. As we already know that $J \subset I \cdot R[x_1,\ldots,x_d]$, the vanishing of the constant term of any element of $J$ now establishes that

$$J \subset I \cdot (x_1,\ldots,x_d)$$

From the above, using $I^2 = 0$, it follows that for any $(b_1,\ldots,b_d) \in I^d$, we have a well-defined $R$-algebra homomorphism

$$\Psi_{(b_1,\ldots,b_d)} : A \rightarrow R : a_i \mapsto b_i$$

We now express the linear-scheme structure of $G^{(r)}$ in terms of the ring $A$, using the fact that each $a_i$ restricts to $y_i$ modulo $I$, and $G^{(r)}_{R/I}$ is the standard linear-scheme $A^d_{R/I}$ with linear co-ordinates $y_i$. Note that the vector addition morphism $A^d_{R/I} \times_{R/I} A^d_{R/I} \rightarrow A^d_{R/I}$ corresponds to the $R/I$-algebra homomorphism

$$(R/I)[y_1,\ldots,y_d] \rightarrow (R/I)[y_1,\ldots,y_d] \otimes_{R/I} (R/I)[y_1,\ldots,y_d] : y_i \mapsto y_i \otimes 1 + 1 \otimes y_i$$

while the scalar-multiplication morphism $A^1_{R/I} \times_{R/I} A^d_{R/I} \rightarrow A^d_{R/I}$ corresponds to the $R/I$-algebra homomorphism

$$(R/I)[y_1,\ldots,y_d] \rightarrow (R/I)[t,y_1,\ldots,y_d] = (R/I)[t] \otimes_{R/I} (R/I)[y_1,\ldots,y_d] : y_i \mapsto ty_i$$

It follows that the addition morphism $\alpha : G^{(r)} \times_R G^{(r)} \rightarrow G^{(r)}$ corresponds to an algebra homomorphism $\alpha^* : A \rightarrow A \otimes_R A$ which has the form

$$a_i \mapsto a_i \otimes 1 + 1 \otimes a_i + u_i \text{ where } u_i \in I(A \otimes_R A).$$

Let the element $u_i$ in the above equation for $\alpha^*(a_i)$ be written as a polynomial expression

$$u_i = f_i(a_1 \otimes 1,\ldots,a_d \otimes 1,1 \otimes a_1,\ldots,1 \otimes a_d)$$

with coefficients in $I$. The additive identity $0$ of $G^{(r)}(R)$ corresponds to $\sigma^* : A \rightarrow R$ with $\sigma^*(a_i) = 0$, and we have $0+0 = 0$ in $G^{(r)}(R)$. This implies that $f_i(0,\ldots,0) = 0$, and so the constant term of $f_i$ is zero. From this, using $I^2 = 0$, we get the important consequence that

$$f_i(w_1,\ldots,w_{2d}) = 0 \text{ for all } w_1,\ldots,w_{2d} \in I$$

The scalar-multiplication morphism $\mu : A^1_{R} \times_R G^{(r)} \rightarrow G^{(r)}$ prolongs the standard scalar multiplication on $A^d_{R/I}$, and so $\mu$ corresponds to an algebra homomorphism $\mu^* : A \rightarrow A[t] = R[t] \otimes_R A$ which has the form

$$a_i \mapsto ta_i + v_i \text{ where } v_i \in IA[t].$$
Let \( v_i \) be expressed as a polynomial \( v_i = g_i(t, a_1, \ldots, a_d) \) with coefficients in \( I \). As multiplication by the scalar 0 is the zero morphism on \( G^{(r)} \), it follows by specialising under \( t \mapsto 0 \) that \( g_i(0, a_1, \ldots, a_d) = 0 \). This means \( v_i = g_i(t, a_1, \ldots, a_d) \) can be expanded as a finite sum

\[
v_i = \sum_{j \geq 1} t^j h_{i,j}(a_1, \ldots, a_d)
\]

where the \( h_{i,j}(a_1, \ldots, a_d) \) are polynomial expressions with coefficients in \( I \). As the zero vector times any scalar is zero, it follows by specialising under \( \sigma^* \) that \( g_i(t, 0, \ldots, 0) = 0 \). It follows that the constant term of each \( h_{i,j} \) is zero. From this, and the fact that \( I^2 = 0 \), we get the important consequence that

\[
g_i(t, b_1, \ldots, b_d) = 0 \quad \text{for all } b_1, \ldots, b_d \in I
\]

Step 6: The kernel of the map \( G^{(r)}(R) \to G^{(r)}(R/I) \)

Lemma Let \( \Psi_{(b_1, \ldots, b_d)} : A \to R \) be the \( R \)-algebra homomorphism defined in terms of the generators by \( \Psi_{(b_1, \ldots, b_d)}(a_k) = b_k \). Let \( \Psi : I^d \to \text{Hom}_{R-alg}(A, R) \) be the set-map defined by \( (b_1, \ldots, b_d) \mapsto (\Psi_{(b_1, \ldots, b_d)} : A \to R) \). Then \( \Psi \) is a homomorphism of \( R \)-modules, where the \( R \)-module structure on \( \text{Hom}_{R-alg}(A, R) \) is defined by its identification with the \( R \)-module \( G^{(r)}(R) \).

The map \( \Psi \) is injective, and its image is the \( R \)-submodule \( \ker G^{(r)}(\phi) \subset G^{(r)}(R) \), where \( \phi : \text{Spec}(R/I) \to \text{Spec} R \) is the inclusion.

Proof For any \((b_1, \ldots, b_d)\) and \((c_1, \ldots, c_d)\) in \( I^d \), we have

\[
(\Psi_{(b_1, \ldots, b_d)} + \Psi_{(c_1, \ldots, c_d)})(a_i) = (\Psi_{(b_1, \ldots, b_d)} \otimes \Psi_{(c_1, \ldots, c_d)})(\alpha^*(a_i)) = b_i + c_i + f_i(b_1, \ldots, b_d, c_1, \ldots, c_d) \quad \text{by substituting for } \alpha^*(a_i)
\]

This shows the equality \( \Psi_{(b_1, \ldots, b_d)} + \Psi_{(c_1, \ldots, c_d)} = \Psi_{(b_1, \ldots, b_d) + (c_1, \ldots, c_d)} \), which means the map \( \Psi : I^d \to G^{(r)}(R) \) is additive.

For any \( \lambda \in R \), let \( f_\lambda : R[t] \to R \) be the \( R \)-algebra homomorphism defined by \( f_\lambda(t) = \lambda \). Then for any \((b_1, \ldots, b_d)\) in \( I^d \) we have

\[
(\lambda \cdot \Psi_{(b_1, \ldots, b_d)})(a_i) = (f_\lambda \otimes \Psi_{(b_1, \ldots, b_d)})(\mu^*(a_i)) = (f_\lambda \otimes \Psi_{(b_1, \ldots, b_d)})(ta_i + g_i(t, a_1, \ldots, a_d)) = \lambda b_i + g_i(\lambda, b_1, \ldots, b_d) = \lambda b_i \quad \text{as } b_i \in I
\]

This shows the equality \( \lambda \cdot \Psi_{(b_1, \ldots, b_d)} = \Psi_{\lambda \cdot (b_1, \ldots, b_d)} \), hence the map \( \Psi : I^d \to G^{(r)}(R) \) preserves scalar multiplication. This completes the proof that \( \Psi : I^d \to G^{(r)}(R) \) is a homomorphism of \( R \)-modules.
The map $\Psi$ is clearly injective. The map $G^{(r)}(\phi) : G^{(r)}(R) \to G^{(r)}(R/I)$ is in algebraic terms the map $\text{Hom}_{R-\text{alg}}(A, R) \to \text{Hom}_{R-\text{alg}}(A, R/I)$ induced by the quotient $R \to R/I$. An element $g \in \text{Hom}_{R-\text{alg}}(A, R/I)$ represents the zero element of $G^{(r)}(R/I)$ exactly when $g(a_i) = 0 \in R/I$ for the generators $a_i$ of $A$. Therefore $f \in \text{Hom}_{R-\text{alg}}(A, R)$ is in the kernel of $G^{(r)}(\phi)$ precisely when $f(a_i) \in I$ for the generators $a_i$. Putting $b_i = f(a_i)$, we see that such an $f$ is the same as $\Psi(b_1, ..., b_d)$.

This completes the proof of the Lemma that $\ker G^{(r)}(\phi) = I^d$.

In particular, as $mI = 0$, it follows from the above Lemma that $\ker G^{(r)}(\phi)$ is annihilated by $m$, so it is a vector space over $R/m$, and its dimension as a vector space over $R/m$ is $d = \Phi(r)$, as by assumption $I$ is a non-zero principal ideal.

The above determination of the dimension over $R/m$ of the kernel of $G^{(r)}(\phi)$ will contradict a more direct functorial description, which is as follows.

Step 7: Functorial description of kernel of $G^{(r)}(R) \to G^{(r)}(R/I)$ As $\mathcal{F}_{R/I}(r)$ is flat over $R/I$, and as for $r \geq m$ all higher direct images of $\mathcal{F}(r)$ vanish, $G^{(r)}(R/I)$ is isomorphic to the $R/I$-module $(R/I)^d$ where $d = \Phi(r)$. By Lemma 5 there exists $m'' \geq m$ such that for $r \geq m''$ the inclusion $\phi : \text{Spec}(R/I) \hookrightarrow \text{Spec} R$ induces an isomorphism $\phi^* \pi_* \mathcal{F}(r) \to \pi'_* \mathcal{F}'(r)$ where $\pi' : P^n_{R/I} \to \text{Spec}(R/I)$ is the projection and $\mathcal{F}'$ is the pull-back of $\mathcal{F}$ under $P^n_{R/I} \hookrightarrow P^n_R$. Note that $G^{(r)}(R) = R^{i(r)} \oplus (R/I)^{j(r)}$, and so for $r \geq m''$ we get an induced decomposition

$$G^{(r)}(R/I) = (R/I)^{i(r)} \oplus (R/I)^{j(r)}$$

such that the map $G^{(r)}(\phi) : G^{(r)}(R) \to G^{(r)}(R/I)$ is the map

$$(q, 1) : R^{i(r)} \oplus (R/I)^{j(r)} \to (R/I)^{i(r)} \oplus (R/I)^{j(r)}$$

where $q$ is the quotient map modulo $I$. It follows that the kernel of $G^{(r)}(\phi)$ is the $R$-module $I^{i(r)} \oplus 0 \subset R^{i(r)} \oplus (R/I)^{j(r)} = G^{(r)}(R)$. This is a vector space over $R/m$ of dimension $i(r) < i(r) + j(r) = \Phi(r)$.

We thus obtain two different values for the dimension of the same vector space $\ker G^{(r)}(\phi) = \ker G^{(r)}(\phi)$, which shows that our assumption that $G^{(r)}$ is representable for arbitrarily large values of $r$ is false. This completes the proof of the Theorem.

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