Resurgent trans-series for generalized Hastings–McLeod solutions

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Received 5 March 2020, revised 10 June 2020
Accepted for publication 24 June 2020
Published 12 August 2020

Abstract
We show that the physical Hastings–McLeod solution of the integrable Painlevé II equation generalizes in a natural way to a class of non-integrable equations, in a way that preserves many of the significant qualitative properties. The Hastings–McLeod solution of Painlevé II is an important and universal example of resurgent relations between perturbative and non-perturbative physics. We derive the trans-series structure of the generalized Hastings–McLeod solutions, demonstrating that integrability is not essential for the resurgent asymptotic properties of the solutions.

Keywords: resurgence, trans-series, Painleve, asymptotics

(Some figures may appear in colour only in the online journal)

1. Introduction

Painlevé equations arise in a wide range of applications in physics, as a class of nonlinear special functions [1–6]. They are integrable in the sense that their moveable singularities are only poles [7, 8]. They have also been widely studied in the context of resurgent asymptotics [9–20], illustrating the deep connections between perturbative and non-perturbative expansions, for both asymptotic and convergent expansions. For example, the Painlevé II equation describes the double-scaling limit of the Gross–Witten–Wadia unitary matrix model [14, 19, 21–23], which has a third order phase transition in the infinite $N$ limit, the immediate vicinity of which is characterized by the Hastings–McLeod solution of the Painlevé II equation [24–26]. As nonlinear equations, the Painlevé equations exhibit exponential sensitivity to boundary conditions, leading to intricate patterns of separatrices [8, 24, 25, 27]. In this paper, we investigate these phenomena of resurgence and exponential sensitivity in non-integrable deformations of the Painlevé II equation, using a combination of numerical and analytic methods. We choose to analyze deformations of Painlevé II because of its universal appearance in physical...
applications. Many of the physical examples of resurgence in quantum mechanics, matrix models and quantum field theory have involved certain features of integrability. Here our motivation is to gain more understanding of how the resurgent connections between perturbative and non-perturbative ‘instanton’ physics arise in models without special integrability properties. We find the full trans-series structure of the generalized Hastings–McLeod solutions, and confirm the general mathematical result that the resurgent structure of nonlinear differential equations does not rely on integrability [11, 13].

The Painlevé II equation reads

\[ y''(x) = 2y^3(x) + xy(x) + \alpha \]  

(1)

With vanishing parameter \( \alpha = 0 \), this equation has a unique real solution, known as the Hastings–McLeod solution [25], satisfying the following boundary conditions at \( x = \pm \infty \):

\[ y(x) \sim Ai(x), \quad x \to +\infty \]  

(2)

\[ y(x) \sim \left( \frac{-x}{2} \right)^{1/3}, \quad x \to -\infty \]  

(3)

The power-law behavior as \( x \to -\infty \) follows from a balance of the \( y^3 \) and \( xy \) terms (when \( \alpha = 0 \)). The Airy behavior as \( x \to +\infty \) follows from the linearization of the differential equation (1) (with \( \alpha = 0 \)), but this linearization does not fix the coefficient of the Airy function solution: \( y(x) \sim kAi(x) \). The non-trivial result for the Hastings–McLeod solution is that the coefficient of the Airy function in (2) must be exactly 1 in order to match smoothly with the power-law behavior as \( x \to -\infty \). If \( k > 1 \) the solution diverges for \( x < 0 \), while if \( k < 1 \) the solution oscillates for \( x < 0 \) [24, 25]. See figure 2 and [8] for graphics. The full asymptotics of this Hastings–McLeod solution is expressed as a trans-series, including both (‘perturbative’) power-series terms as well as exponentially-suppressed (‘non-perturbative’) terms, whose structures are entwined by resurgent relations [10, 13, 14, 19]. The form of this trans-series is very different as \( x \to +\infty \) and as \( x \to -\infty \), changing its structure as we cross the phase transition at \( x = 0 \) [14, 19]; this is an explicit realization of the physical phenomenon of instanton condensation [23], in which all orders of the instanton expansion must be resummed near the phase transition. The Hastings–McLeod solution also provides a clear example of the Lee–Yang phenomenon: the phase transition of the Gross–Witten–Wadia model appears as the \( N \to \infty \) coalescence of complex zeros of the finite \( N \) partition function, since in the double-scaling limit this is described by the Hastings–McLeod solution, which has singularities only in two wedges of the complex plane, pinching the real axis at the phase transition at \( x = 0 \) [20, 28, 29].

In this paper we study the resurgent trans-series structure of a class of non-integrable deformations of the Painlevé II equation, in which we deform the power of the non-linear term away from the integrable value of 3:

\[ y''(x) = 2y^N(x) + xy(x), \quad N \geq 2, \quad N \in \mathbb{Z}^+ \]  

(4)

The Hastings–McLeod boundary conditions (2) and (3) generalize naturally to the following:

\[ yy(x) \sim k_N Ai(x), \quad x \to +\infty \]  

(5)

\[ yy(x) \sim \left( \frac{-x}{2} \right)^{1/3}, \quad x \to -\infty \]  

(6)
Figure 1. Numerical solution to (4) and (5), with $N = 2$. The red dashed and blue dotted curves correspond to $k_2^+ = 0.671232$ and $k_2^- = 0.671231$, respectively. The solid black line is the $x < 0$ asymptotic behavior in (6). The critical separatrix value of $k_2$ lies between $k_2^+$ and $k_2^-$. See table 1.

Figure 2. Numerical solution to (4) and (5), with $N = 3$. This is the Hastings–McLeod Painlevé II case [24, 25]. The red dashed and blue dotted curves correspond to $k_3^+ = 1.00000001$ and $k_3^- = 0.9999999$, respectively. The solid black line is the $x < 0$ asymptotic behavior in (6). The critical separatrix value, $k_3 = 1$, lies between $k_3^+$ and $k_3^-$. Here $k_N$ is a real constant whose value depends on the integer-valued non-linearity parameter $N$. We show that for each $N \geq 2$ there is a unique choice of the boundary condition parameter $k_N$ such that the solution, satisfying both (5) and (6), is real. We denote this generalized Hastings–McLeod solution as $y_N(x)$.

When $N = 3$, equations (4)–(6) reduce to the Hastings–McLeod case for the Painlevé II equation, with $k_3 = 1$, which is integrable: in the vicinity of a moveable singularity (i.e. one related to the boundary conditions) at $x_0$, the solution can be expanded with only pole singularities [7]:

$$y_3(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0(x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + \ldots$$

(7)

This solution is completely characterized by the pole location $x_0$ and the coefficient $h_0$ of the $(x - x_0)^3$ term: all other expansion coefficients are expressed as polynomials in $x_0$ and $h_0$. 

Figure 3. Numerical solution to (4) and (5), with $N = 4$. The red dashed and blue dotted curves correspond to $k_4^+ = 1.191125$ and $k_4^- = 1.191124$, respectively. The solid black line is the $x < 0$ asymptotic behavior in (6). The critical separatrix value of $k_4$ lies between $k_4^+$ and $k_4^-$. See table 1.

Figure 4. Numerical solution to (4) and (5), with $N = 10$. The red dashed and blue dotted curves correspond to $k_{10}^+ = 1.577095$ and $k_{10}^- = 1.577094$, respectively. The solid black line is the $x < 0$ asymptotic behavior in (6). The critical separatrix value of $k_{10}$ lies between $k_{10}^+$ and $k_{10}^-$. See table 1.

For $N \neq 3$ such an expansion is not possible. In this paper we study the effect of this non-integrability on the resurgent trans-series structure of the solutions.

2. Numerical computation of the $k_N$ parameter

To begin, we study numerically the boundary condition parameter $k_N$ in (5), as a function of the non-linearity parameter $N$. This computation was made using an explicit Runge–Kutta method in Mathematica 12, with 32-digit working precision. For each of the various $N$ values we use a shooting method, starting at a large positive value of $x$ with the boundary condition (5), to tune the respective $k_N$ to match the smoothly to the behavior in (6) in the $x \to -\infty$ limit. Just as for the Hastings–McLeod solution of Painlevé II, we find that there is a unique separatrix value of $k_N$ that is sensitive to the tuning such that less than one part in $10^6$ difference results in a drastic change in the asymptotic behavior in the $x \to -\infty$ regime. An initial condition less than the critical $k_N$ value results in oscillation about the negative $x$ axis, while an initial condition greater than the critical value results in a divergence of $y_N$ as $x \to -\infty$. This behavior
Table 1. Numerically computed \( k_N \) values for selected \( N \) values. For each \( N \), \( k_N \) is the boundary condition parameter such that both (5) and (6) are satisfied.

| \( N \) | \( k_N \) |
|-------|--------|
| 2     | 0.671 2312 |
| 3     | 1.000 0000 |
| 4     | 1.191 1248 |
| 5     | 1.313 7885 |
| 6     | 1.398 8704 |
| 7     | 1.461 2750 |
| 8     | 1.508 9818 |
| 9     | 1.546 6298 |
| 10    | 1.577 0949 |
| 11    | 1.602 2534 |
| 12    | 1.623 3808 |
| 13    | 1.641 3746 |
| 14    | 1.656 8841 |
| 15    | 1.670 3910 |
| 16    | 1.682 2598 |
| 17    | 1.692 7718 |
| 18    | 1.702 1474 |
| 19    | 1.710 5614 |
| 20    | 1.718 1548 |
| 21    | 1.725 0422 |
| 22    | 1.731 3178 |
| 23    | 1.737 0596 |
| 24    | 1.742 3332 |
| 25    | 1.747 1936 |
| 26    | 1.751 6877 |
| 27    | 1.755 8553 |
| 28    | 1.759 7308 |
| 29    | 1.763 3440 |
| 30    | 1.766 7206 |
| 50    | 1.806 2107 |
| 100   | 1.836 2658 |
| 500   | 1.860 6755 |
| 1000  | 1.863 7629 |
| 10 000| 1.866 5555 |

is illustrated in figures 1–4, for \( N = 2, 3, 4, 10 \). Table 1 lists numerical values for the critical \( k_N \) for further values of \( N \).

We obtain some analytic understanding of the \( N \) dependence of \( k_N \) from the following argument. As the numerical solution with initial condition (5) enters the negative \( x \) regime, the Airy function behavior continues in the form of an oscillatory solution if \( k_N \) is less than the critical separatrix value. This means that there is a first maximum of this numerically integrated solution. For all \( N \) we observe that this first maximum lies below the eventual asymptotic form, \( y_N(x) \sim (\frac{-x}{2})^{\frac{3}{2}} \), so we obtain an upper bound

\[
k_N \leq \frac{1}{\text{Ai}(x_0)} \left( \frac{-x_0}{2} \right)^\frac{1}{3} \tag{8}\n\]
where \( x_0 = -1.018792972 \ldots \) is the first zero of \( \text{Ai}'(x) \). In the \( N \to \infty \) limit this implies an estimate

\[
\lim_{N \to \infty} k_N = \frac{1}{\text{Ai}(x_0)} \approx 1.866867495 \ldots
\]  

(9)

This bound is supported by the large \( N \) values listed in table 1 and plotted in figure 5.

A complementary approach to the separatrix parameter \( k_N \) is to recast the differential equations (4)–(6) as an integral equation in the \( x \geq 0 \) region:

\[
y(x) = k_N \text{Ai}(x) + 2\pi \int_x^\infty dz y^N(z) [\text{Ai}(x) \text{Bi}(z) - \text{Ai}(z) \text{Bi}(x)]
\]

(10)

It is straightforward to verify that this is equivalent to the differential equation (4), with boundary condition (5). The integral equation (10) can be iterated, resulting in an expansion of \( y(x) \) in powers of \( k_N \). Once again we find that the parameter \( k_N \) must be tuned with high precision in order to match the smooth non-oscillatory behavior in (6) as \( x \to -\infty \). This procedure is comparable in precision to precise numerical integration of the ODE, but gives an alternative perspective which will be useful below.

3. Trans-series solution for positive \( x \)

In the \( x \to +\infty \) region, the boundary condition in (5) has the familiar asymptotic expansion of the Airy function:

\[
y_N(x) \sim k_N e^{\frac{2}{3} x^{3/2}} \left( \frac{x}{\sqrt{\pi}} \right)^{1/4} \sum_{m=0}^\infty \frac{(-1)^m \Gamma \left( m + \frac{1}{2} \right) \Gamma \left( m + \frac{3}{2} \right)}{2 \pi \left( \frac{4}{3} x^{3/2} \right)^m m!} x^{-m/2}, \quad x \to +\infty
\]

(11)

To determine the basic form of the trans-series solution as \( x \to +\infty \) we insert an ansatz

\[
y_N(x) \sim \sum_{n=0}^\infty k_N^n Y_{[n]}(x)
\]

(12)

where \( Y_{[0]}(x) \equiv \text{Ai}(x) \). Matching powers of the trans-series parameter \( k_N \) leads to a tower of linear equations for the higher instanton terms \( Y_{[n]} \). Because the nonlinearity of the ODE depends on \( N \), not all powers of \( k_N \) appear in this expansion, and so we find a refined trans-series expansion of the form:

\[
y_N(x) \sim \sum_{n=0}^\infty \left( k_N e^{\frac{2}{3} x^{3/2}} \frac{x}{\sqrt{\pi}} \right)^{(N-1)n+1} \mathcal{F}_{[(N-1)n+1]}(x)
\]

(13)

This trans-series involves specific powers (depending on the parameter \( N \)) of the Airy exponential factor, multiplied by a fluctuation series of the form

\[
\mathcal{F}_{[n]}(x) \sim \frac{1}{x^n} \sum_{m=0}^\infty \frac{d_m^{(n)}}{x^{3m/2}}
\]

(14)

The trans-series expansion in (13) has the characteristic structure of a ‘multi-instanton’ expansion in physics, as studied for the Painlevé II case \( N = 3 \) in [14, 17, 19] for applications to the strong-coupling region of the Gross–Witten–Wadia unitary matrix model. This identifies the boundary condition parameter \( k_N \) as the trans-series parameter [13, 14, 19], which must be
tuned to a special unique value in order for this trans-series to match smoothly across the phase transition at $x = 0$ to the $x \to -\infty$ behavior in (6).

Inserting the trans-series ansatz (13) and (14) into the differential equation (4), we obtain a tower of non-linear recursion formulas for the fluctuation coefficients $d_m^{(n)}$, one for each non-perturbative sector labeled by the index $n$. For example, in the $n = 1$ sector, the first non-trivial ‘instanton’ sector, we find the recursion relation for the $d_m^{(1)}$ fluctuation coefficients (for $m = 0, 1, 2, \ldots$):

$$d_m^{(1)} + \frac{N}{2}(N + 6m - 3)d_{m-1}^{(1)} + \frac{1}{16}(N + 6m - 8)(N + 6m - 4)d_{m-2}^{(1)} = \frac{b_{m+1}}{(N^2 - 1)}$$

with initial conditions $d_0^{(1)} = d_1^{(1)} = 0$. Here the $b_m$ are the expansion coefficients obtained by raising the Airy function asymptotic expansion to the $N$th power:

$$2\left(\sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(l + \frac{1}{6}) \Gamma(l + \frac{5}{6})}{2\pi (\frac{4}{3} x^{3/2})^l l!}\right)^N \equiv \sum_{m=0}^{\infty} \frac{b_m}{x^{3m/2}}$$

The recursion relation (15) can be used to generate the ‘one-instanton’ fluctuation coefficients $d_m^{(1)}$ to very high order, for various values of the non-linearity parameter $N$. For all $N$, these coefficients alternate in sign and grow factorially in magnitude. Using the generated coefficients, combined with Richardson acceleration [3], we have found the following results for the leading and sub-leading asymptotics:

$$N = 4 : d_m^{(1)} \sim (-1)^m \frac{1}{\pi} \left(\frac{3}{4}\right)^{m-1} \Gamma(m) \left[1 + \frac{61}{18} \left(\frac{3}{4}\right)^m + \ldots\right], \quad m \to \infty$$

$$N = 5 : d_m^{(1)} \sim (-1)^m \frac{15}{32\pi} \left(\frac{3}{4}\right)^{m-1} \Gamma(m) \left[1 + \frac{29}{12} \left(\frac{3}{4}\right)^m + \ldots\right], \quad m \to \infty$$
\[ N = 6 : \quad a_m^{(1)} \sim (-1)^n \left(\frac{3}{10\pi} \left(\frac{3}{4}\right)^{m-1}\Gamma(m) \left[ 1 + \frac{97}{75} \frac{1}{(m-1)} + \ldots \right], \quad m \to \infty \] (19)

\[ N = 7 : \quad a_m^{(1)} \sim (-1)^n \left(\frac{7}{32\pi} \left(\frac{3}{4}\right)^{m-1}\Gamma(m) \left[ 1 + \frac{25}{12} \frac{1}{(m-1)} + \ldots \right], \quad m \to \infty \] (20)

These expansions can also be confirmed using the integral equation (10). Inserting the refined trans-series ansatz (13) for \( y_N(x) \), yields a tower of linear equations with solutions:

\[ y_N(x) \sim \sum_{n=0}^{\infty} d_{N-1}^{(1)} N Y_{[1]}(x), \quad x \to +\infty \] (21)

and expanding in powers of the trans-series parameter \( k_N \), yields a tower of linear equations with solutions:

\[ Y_{[1]}(x) = Ai(x) \] (22)

\[ Y_N(x) = 2\pi \left( Ai(x) \int_{x}^{\infty} (Y_{[1]}(z))^N Bi(z) \, dz - Bi(x) \int_{x}^{\infty} (Y_{[1]}(z))^N Ai(z) \, dz \right) \] (23)

\[ Y_{[2N-1]}(x) = 2N\pi \left( Ai(x) \int_{x}^{\infty} Y_N(z) (Y_{[1]}(z))^{N-1} Bi(z) \, dz 
- Bi(x) \int_{x}^{\infty} Y_N(z) (Y_{[1]}(z))^{N-1} Ai(z) \, dz \right) \] (24)

The corresponding asymptotic expansions can therefore be generated straightforwardly from those of the Airy functions. This confirms the trans-series structure in (13) and (14), and identifies \( y_{(N-1)n+1}(x) \) with the fluctuation functions \( c_{(N-1)n+1}(x) \) in (14), multiplied by the \( (N-1)n+1 \)th power of the Airy exponential instanton factor. This trans-series structure in (22)–(24) generalizes that of the \( N = 3 \) case, the Painlevé II equation [19]. The \( N = 2 \) case is special: see section 5 below.

4. Trans-series solution for negative \( x \)

In the \( x \to -\infty \) limit the trans-series has a completely different structure. As \( x \to -\infty \), the boundary condition in (6) naively generalizes to a formal series expansion of the form

\[ y_N(x) \sim \left(\frac{-x}{2}\right)^{\frac{1}{3}} \sum_{l=0}^{\infty} \frac{e_l}{(-x)^l}, \quad x \to -\infty \] (25)
Here \( c_0 \equiv 1 \). The higher expansion coefficients \( c_l \) can be generated recursively by inserting this ansatz into the differential equation (4). For example, we find

\[
\begin{align*}
c_1 &= -\frac{(N - 2)}{(N - 1)^2}, \\
c_2 &= -\frac{(N - 2)(25N^2 - 64N + 40)}{2(N - 1)^6}
\end{align*}
\] (26)

[The \( N = 2 \) case is special, as discussed below in section 5.] The remaining terms can be generated from a recursion relation. First note that for the expansion (25), we find that

\[
\frac{d^2 y_N}{dx^2} \sim \frac{2}{(N - 1)^2} \left( -\frac{x}{2} \right)^N \sum_{l=1}^{\infty} \frac{|((3l - 3)N - (3l - 2))((3l - 2)N - (3l - 1))| c_{l-1}}{(-x)^{3l}}
\] (27)

To satisfy the differential equation we see that the expansion coefficients are given by

\[
c_{l-1} = \frac{(N - 1)^2}{[((3l - 3)N - (3l - 2))((3l - 2)N - (3l - 1))] g_l, \quad l = 1, 2, 3, \ldots
\] (28)

where the \( g_l \) are the coefficients of the expansion:

\[
\left( \sum_{l=0}^{\infty} \frac{c_l}{(-x)^{3l}} \right)^N = \left( \sum_{l=0}^{\infty} \frac{c_l}{(-x)^{3l}} \right) \equiv \sum_{l=1}^{\infty} \frac{g_l}{(-x)^{3l}}
\] (29)

However, it is clear that the expansion (25) cannot be the full expansion, because there is no boundary condition parameter: all the \( c_l \) coefficients are completely determined recursively. The ‘missing’ boundary condition parameter enters because the expansion (25) is an asymptotic formal series which must be completed with an infinite sum of exponentially small non-perturbative terms, resulting in a trans-series expansion:

\[
y_N(x) \sim \left( \frac{-x}{2} \right)^N \sum_{n=0}^{\infty} \sigma_n \ W_{[n]}(x), \quad x \to -\infty
\] (30)

where \( W_{[0]} \) is the formal perturbative series solution in (25). Expanding in powers of the trans-series parameter \( \sigma_n \) produces a tower of linear equations for \( W_{[n]}(x) \) for \( n \geq 1 \). The first such equation is:

\[
W_{[0]}'' = 2NW_{[n]-1} W_{[1]} + x \ W_{[1]}
\] (31)

We can solve this equation by postulating the form \( W_{[1]}(x) \sim (-x)^{3l} e^{-\sqrt[3]{(-x)^{3/2}} \left( 1 + (-x)^{3/2} + \ldots \right)} \). Using the expansion of \( W_{[0]}(x) \) given in (25) and (26), we determine the parameters \( \alpha, \beta, \gamma \), and so deduce that the first non-trivial trans-series term has the form as \( x \to -\infty \):

\[
W_{[1]}(x) \sim \left( \frac{x}{2} \right)^{\frac{1}{N}} e^{-\sqrt[3]{N^{-1}(-x)^{3/2}}} \left( 1 - \frac{21(N - 1)^2 - 16}{48(N - 1)^{5/2}} \frac{1}{(-x)^{3/2}} + \ldots \right)
\] (32)
The further fluctuation terms can be generated recursively. This pattern extends to higher orders in a straightforward way, leading to a full trans-series expansion\(^2\):

\[
y_N(x) \sim \left( \frac{-x}{2} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left( \frac{\sigma_N}{(-x)^{N+3/(4N-1)}} \right)^n \sum_{l=0}^{\infty} \left( \frac{c_l}{(-x)^{3l/2}} \right), \quad x \to -\infty
\]

(33)

The trans-series parameter \(\sigma_N\) characterizes this family of asymptotic solutions, all having the same leading behavior (6) as \(x \to -\infty\). This \(x \to -\infty\) trans-series parameter \(\sigma_N\) must be simultaneously tuned, together with the trans-series parameter \(k_N\) in the \(x \to +\infty\) trans-series (13), in order to obtain the unique real solution matching both the \(x \to \pm \infty\) boundary conditions in (5) and (6).

Notice that the exponent of the instanton factor in (33) differs from that in the \(x \to +\infty\) trans-series (13) by more than just \(x \to -x\); there is an additional factor of \(\sqrt{N-1}\). Physically, this means that as the phase transition (at \(x = 0\)) is approached from either side, all instanton orders of the appropriate trans-series are required, and a highly non-trivial instanton condensation phenomenon occurs, resumming all instanton terms into a different instanton factor, as has been studied for the Painlevé II equation [26]. This is an example of the non-linear Stokes phenomenon, characteristic of non-linear ODEs, and very different from the familiar linear Stokes phenomenon. We comment that if we were interested in the most general solution to the ODE (4), we would consider a double trans-series, with two trans-series parameters (since it is a second order ODE), including also powers of an exponential term that grows as \(x \to -\infty\).

This violates the boundary condition (6) for the generalized Hastings–McLeod solution, so we can exclude such terms in the trans-series ansatz (33). However, both factors are crucial for a full understanding of the Stokes phenomenon and the connection problem [17, 26]. This is a surprisingly subtle problem even for the integrable \((N = 3)\) case of Painlevé II where one has access to the additional technical machinery of isomonodromic deformation [26]. For the non-integrable deformations, with \(N \neq 3\), the isomonodromy methods are not applicable. Such an analysis is beyond the scope of this paper.

Information about the trans-series structure in (33) can also be deduced from a study of the large order behavior of the perturbative expansion coefficients \(c_l\) of the formal perturbative series in (25). This complementary approach highlights some of the resurgent properties of the trans-series expansions. The expansion coefficients \(c_l\) are determined from the recursion formula (28), so it is straightforward to generate many coefficients. Numerically, we find the large-order behavior:

\[
N = 3: \quad c_l \sim -(0.1466323 \ldots) \times \left( \frac{9}{8} \right)^{l} \Gamma \left( 2l - \frac{1}{2} \right) \left( 1 - \frac{17}{2(2l - \frac{1}{2} - 1)} + \ldots \right), \quad l \to \infty
\]

(34)

\[
N = 4: \quad c_l \sim -(0.12986376 \ldots) \times \left( \frac{9}{12} \right)^{l} \Gamma \left( 2l - \frac{7}{18} \right) \left( 1 - \frac{173}{2(2l - \frac{7}{18} - 1)} + \ldots \right), \quad l \to \infty
\]

(35)

\(^2\) Note that for \(n = 0\), the fluctuation expansion in (25) is actually in powers of \(1/(-x)^3\) rather than \(1/(-x)^{3/2}\).
\[ N = 5 : c_l \sim -(0.108 \, 6460 \ldots) \times \left( \frac{9}{16} \right)^l \Gamma \left( 2l - \frac{1}{3} \right) \left( 1 - \frac{5}{2(2l - \frac{1}{3} - 1)} + \ldots \right) , \quad l \to \infty \] (36)

These results exhibit resurgent features of the trans-series. To see this, we rewrite these large-order expressions for general \( N \) as

\[ c_l^{(N)} \sim -(\text{constant}_N) \times \frac{\Gamma(2l - \nu_N)}{S_N^{2l}} \left( 1 + S_N \frac{\delta_N}{(2l - \nu_N - 1)} + \ldots \right) \] (37)

where

\[ S_N = \frac{2}{3} \sqrt{N - 1} \] (38)

\[ \nu_N = \frac{2}{3} \left( \frac{N + 3}{4(N - 1)} \right) \] (39)

\[ \delta_N = - \left( \frac{21(N - 1)^2 - 16}{48(N - 1)^{3/2}} \right) \] (40)

This shows that the \( S_N^{2l} \) factor in the large order growth (37) of the coefficients of the formal perturbative series (25) is directly related to the exponent in the non-perturbative instanton exponential factor in the higher-order terms in the trans-series (33). \( S_N \) is raised to the power \( 2l \), rather than \( l \), because of the \((2l)!\) factorial growth (rather than \( l! \) factorial growth). (This double-factorial growth phenomenon is related to the existence of the aforementioned exponentially growing solutions: they affect the large-order growth even when excluded from the actual trans-series by boundary conditions [17].) Furthermore, the offset \( \nu_N \) of the gamma function factor in (37) is directly related to the power of the prefactor of the exponential instanton term in (33). Finally, the coefficient \( \delta_N \) in the first power-law subleading correction in the large order growth (37) coincides with the first coefficient of the fluctuation about the leading instanton in \( W_{(1)}(x) \) shown in (32). This resurgent pattern continues to higher orders. These non-trivial resurgence relations, valid for general \( N \), are clear indications of the generic large-order/low-order perturbative/non-perturbative relations found in resurgent trans-series [10, 30]. Note that these resurgent relations occur beyond the integrable \( N = 3 \) (Painlevé II) case. However, one special feature of integrability is that for \( N = 3 \) we recognize the (numerically determined) overall coefficient \( 0.146 \, 6323 \ldots \) in (34) as \( \frac{1}{\pi} \sqrt{\frac{2}{5\pi}} \), expressed in terms of the closed-form Stokes constant of Painlevé II. For \( N > 3 \), even though we can determine these prefactor coefficients to extremely high precision using high order Richardson extrapolation, we have not been able to identify any closed-form expressions for these coefficients. This is consistent with the notion that closed-form Stokes constants are only expected in integrable systems. This question deserves further study.

The perturbative coefficients \( c_l \) diverge factorially in magnitude (for \( N \geq 3 \)), and do not alternate in sign. Therefore, if we wish the trans-series expansion (33) to represent a real solution to the equation, the trans-series parameter \( \sigma_N \) must have an imaginary part, in order for the higher-order exponential terms to cancel the imaginary terms generated by Borel summation of the factorially divergent non-sign-alternating perturbative expansion [30]. This mimics the well-known behavior of the Painlevé II trans-series in the \( x \to -\infty \) region, and is a significant feature of the deep resurgent connection between perturbative and non-perturbative physics [12–14].
5. Trans-series for the $N = 2$ case

The $N = 2$ case is interestingly different from the other cases, for $N \geq 3$. We demonstrate in this section that this is because for $N = 2$ the original nonlinear ODE (4) has a novel additional duality symmetry relating $x \leftrightarrow -x$.

The large-order behavior of the first fluctuation terms in the $x \rightarrow +\infty$ region displays a different growth rate [contrast with (17)–(20)]:

$$N = 2 : d_m^{(1)} \sim (4.9161362 \ldots) \times (-1)^m \left(\frac{3}{2}\right)^{m-1} \left(m - \frac{1}{6}\right)!, \quad m \rightarrow \infty$$

(41)

However, this leads to a qualitatively similar trans-series structure in the $x \rightarrow +\infty$ region, since the coefficients still diverge factorially and alternate in sign. The biggest difference arises in the $x \rightarrow -\infty$ region, where the leading asymptotic form in (6), $y_2(x) \sim -\frac{x}{2}$, is in fact an exact solution of the nonlinear differential equation (4) when $N = 2$. Thus, all the coefficients $c_l$ in the expansion (25) vanish for $l \geq 1$. This is consistent with (26). This fact has important consequences for the matched solution. Even though $y_2(x) = -\frac{x}{2}$ is an exact solution to the ODE, and manifestly satisfies the $x \rightarrow -\infty$ boundary condition (6), it does not match smoothly with the $x \rightarrow +\infty$ asymptotics in (5). There is another solution to the ODE, which has an infinite tower of exponentially small terms in the negative $x$ region, and it is this solution that matches smoothly to the Airy-like solution in the positive $x$ region. This is the $N = 2$ analog of the $N = 3$ Hastings–McLeod behavior. To see this, it is simplest to study the iterated integral equation form of the problem as in (10).

For the $x \rightarrow +\infty$ region this is given by the trans-series expansion in (21)–(24) for $N = 2$:

$$y(x) \sim k_2 Y_{[1]}(x) + k_2^2 Y_{[2]}(x) + k_2^3 Y_{[3]}(x) + O(k_2^4), \quad x \rightarrow +\infty$$

(42)

where

$$Y_{[1]}(x) = \text{Ai}(x)$$

(43)

$$Y_{[2]}(x) = 2\pi \left(\text{Ai}(x) \int_x^\infty \text{Ai}^2(z) \text{Bi}(z) \, dz - \text{Bi}(x) \int_x^\infty \text{Ai}^3(z) \, dz\right)$$

(44)

For $N = 2$, we can write an iterated integral equation form as $x \rightarrow -\infty$:

$$y(x) \sim -\frac{x}{2} + \sigma_2 W_{[1]}(x) + \sigma_2^2 W_{[2]}(x) + \sigma_2^3 W_{[3]}(x) + O(\sigma_2^4), \quad x \rightarrow -\infty$$

(45)

where

$$W_{[1]}(x) = \text{Ai}(-x)$$

(46)

$$W_{[2]}(x) = 2\pi \left(\text{Ai}(-x) \int_{-x}^\infty \text{Ai}^2(z) \text{Bi}(z) \, dz - \text{Bi}(-x) \int_{-x}^\infty \text{Ai}^3(z) \, dz\right)$$

(47)

The higher trans-series terms in the $x \rightarrow \pm\infty$ expansions are identical, up to $x \rightarrow -x$. This remarkable correspondence can be attributed to the following special duality symmetry. Define
Figure 6. The dashed curve is the $x \to -\infty$ asymptotics $y(x) \sim -\frac{x}{2}$, which is an exact solution for $N = 2$. The black curve is the exact numerical solution to (4)–(6) for $N = 2$. The blue-dashed and red curves denote the $x < 0$ trans-series solution in (45), including the first two non-trivial trans-series contributions, $W_{[1]}(x)$ and $W_{[2]}(x)$, respectively, and with trans-series parameter $\sigma_2 = k_2 = 0.6712312$. This agrees very well with the positive $x$ solution.

Then it follows that $w_2(x)$ satisfies the nonlinear ODE

$$w_2''(x) = 2w_2(x)^2 - x w_2(x)$$

which differs from the original ODE in (4) only by $x \to -x$. This explains the duality symmetry between the $x \to +\infty$ and $x \to -\infty$ trans-series expansions in (42)–(44) and (45)–(47), respectively. Therefore, in order to match the two expansions at $x = 0$, we see that we require the two trans-series parameters to be equal: $k_2 = \sigma_2$. Using the numerically computed value of $k_2 = 0.6712312$, we plot in figure 6 the two trans-series expressions developed at $x = \pm\infty$. We observe that even with just the first exponentially suppressed terms added the agreement is extremely good, improving even further with the next term. Thus we see that even though $y_2(x) = -\frac{x}{2}$ is an exact solution, there is a non-perturbative completion, in the form of the trans-series (45), which is the function that smoothly connects with the Airy-like behavior as $x \to +\infty$. This is a further indication of the exponential sensitivity of the generalized Hastings–McLeod solution in (4)–(6).

6. Conclusions

We have shown that the non-integrable generalizations (4) of the Painlevé II equation, with boundary conditions (5) and (6), have unique real solutions with many qualitative features in common with the physical Hastings–McLeod solution of Painlevé II. For each power $N$, there is a unique real boundary condition parameter $k_N$ in (5) such that the $x \to +\infty$ behavior matches smoothly to the $x \to -\infty$ behavior. We find the full trans-series structure of these solutions in both asymptotic regions, indicating a nonlinear Stokes transition between the two regimes. We showed that these solutions exhibit standard large-order/low-order perturbative/non-perturbative resurgence relations for general $N$, not only in the integrable $N = 3$ case. These results demonstrate explicitly that the integrability of the Painlevé
equations is not essential for many of the well-known results regarding trans-series and resurgent asymptotics for nonlinear differential equations.

Acknowledgments

This material is based upon work supported by the US Department of Energy, Office of Science, Office of High Energy Physics under Award Number DE-SC0010339 (GD). This work was part of an undergraduate research project (NC).

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