\[\mathfrak{sl}_3\]-Foams and the Khovanov-Lauda categorification of quantum \(\mathfrak{sl}_k\).

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Abstract

In this paper I define certain interesting 2-functors from the Khovanov-Lauda 2-category which categorifies quantum \(\mathfrak{sl}_k\), for any \(k > 1\), to a 2-category of universal \(\mathfrak{sl}_3\) foams with corners. For want of a better name I use the term foamation to indicate those 2-functors. I conjecture the existence of similar 2-functors to the 2-category of \(\mathfrak{sl}_n\) foams with corners, for any \(n > 1\).

1 Introduction

In this paper I relate the Khovanov-Lauda (KL) categorification of quantum \(\mathfrak{sl}_k\), for \(k > 1\), defined in [2], to the universal \(\mathfrak{sl}_3\)-foams, defined in [5]. Since both can be defined in terms of 2-categories, the most natural formulation of the relation is as a 2-functor, which I call foamation, because it produces a foam for each KL-diagram. As a matter of fact there are several such 2-functors, all very similar, depending on a finite parameter.

Unfortunately there are some signs which do not work out well under foamation, so I have defined everything over \(\mathbb{F}_2\). Hopefully this problem can be solved in the future. If not, one wonders if foamation leads to a slightly different categorification of quantum \(\mathfrak{sl}_k\).

With \(\mathfrak{sl}_2\)-foams, defined by Bar-Natan [1], foamation also works, but there is not much to check, since the image of the KL-diagrams is mostly zero. I conjecture that foamation also works with \(\mathfrak{sl}_n\)-foams, for \(n \geq 4\), using the Kapustin-Li formula as in [4]. I have checked some relations for \(n = 4\) and \(n = 5\) and they are preserved by foamation. To generalize this to higher \(n\) one would have to compute the Kapustin-Li formula for foams with facets with arbitrary thickness, which is a computational challenge in its own right.

The foamation 2-functors should be related to the representation 2-functors \(\Gamma_N\) in [2]. If foamation exists for all \(n \geq 2\), then it should be a faithful family of 2-functors, meaning that for \(n\) big enough foamation is faithful on all homogeneous elements of a fixed degree. The latter conjecture also shows an interesting feature of foamation: if true, it would allow one, in principal, to find all the relations in the categorification of quantum \(\mathfrak{sl}_k\) just from the Kapustin-Li formula. Since it is also the Kapustin-Li formula which, in a certain sense, gives rise to the \(\mathfrak{sl}_n\) link homologies [4], a nice unifying picture seems to arise.
The 2-category \( \mathcal{U}_k \)

In this section we recall the definition of Khovanov and Lauda’s categorification \( \mathcal{U}_k \) of the quantum groups \( U_q(sl_k) \), for any \( k > 1 \). For motivation and more details we refer to their paper [2]. The 2-category \( \mathcal{U}_k \) has the structure of an additive \( F_2 \)-linear 2-category. Thus between any two objects there is an \( F_2 \)-linear Hom category, and composition and identities are given by additive \( F_2 \)-linear functors.

From now on let \( k \in \mathbb{N}_{>1} \) be arbitrary but fixed. In the sequel we use signed sequences \( i = (\pm i_1, \ldots, \pm i_m) \), for any \( m \in \mathbb{N} \) and any \( i_j \in \{1, \ldots, k-1\} \). The set of signed sequences we denote \( \text{SSeq} \). For \( i = (\varepsilon_1 i_1, \ldots, \varepsilon_m i_m) \in \text{SSeq} \) we define \( i_X := \varepsilon_1(i_1)_X + \cdots + \varepsilon_m(i_m)_X \), where

\[ (i_j)_X = (0, 0, \ldots, -1, 2, -1, 0, \ldots, 0), \]

such that the vector starts with \( i_j - 1 \) and ends with \( k - 2 - i_j \) zeros. We also define the symmetric \( \mathbb{Z} \)-valued bilinear form on \( \{1, \ldots, k-1\} \) by \( i \cdot i = 2, i \cdot (i + 1) = -1 \) and \( i \cdot j = 0 \), for \( |i - j| > 1 \).

**Definition 2.1.** \( \mathcal{U}_k \) is an additive \( F_2 \)-linear 2-category. The 2-category \( \mathcal{U}_k \) consists of

- objects: \( \lambda = (\lambda_1, \ldots, \lambda_{k-1}) \), with \( \lambda_i \in \mathbb{Z} \).

The homs \( \mathcal{U}_k(\lambda, \lambda') \) between two objects \( \lambda, \lambda' \) are additive \( F_2 \)-linear categories consisting of:

- objects\(^{\dagger} \) of \( \mathcal{U}_k(\lambda, \lambda') \): a 1-morphism in \( \mathcal{U}_k \) from \( \lambda \) to \( \lambda' \) is a formal finite direct sum of 1-morphisms

  \[ \varepsilon_i 1_{\lambda} \{t\} = 1_{\lambda'} \varepsilon_i 1_{\lambda} \{t\} \]

  for any \( t \in \mathbb{Z} \) and signed sequence \( i \in \text{SSeq} \) such that \( \lambda' = \lambda + i_X \).

- morphisms of \( \mathcal{U}_k(\lambda, \lambda') \): for 1-morphisms \( \varepsilon_i 1_{\lambda} \{t\}, \varepsilon_j 1_{\lambda} \{t'\} \in \mathcal{U}_k \), hom sets \( \mathcal{U}_k(\varepsilon_i 1_{\lambda} \{t\}, \varepsilon_j 1_{\lambda} \{t'\}) \) of \( \mathcal{U}_k(\lambda, \lambda') \) are graded \( F_2 \)-vector spaces given by linear combinations of degree \( t - t' \) diagrams, modulo certain relations, built from composites of:

  i) Degree zero identity 2-morphisms \( 1_x \) for each 1-morphism \( x \) in \( \mathcal{U}_k \); the identity 2-morphisms \( 1_{\varepsilon_i 1_{\lambda}} \{t\} \) and \( 1_{\varepsilon_{-i} 1_{\lambda}} \{t\} \), for \( i \in I \), are represented graphically by

  \[
  \begin{array}{c}
  1_{\varepsilon_i 1_{\lambda}} \{t\} \\
  \varepsilon_i 1_{\lambda} \{t\} \\
  \lambda + i_X \\
  \lambda \end{array}
  \]

\[
\begin{array}{c}
1_{\varepsilon_{-i} 1_{\lambda}} \{t\} \\
\varepsilon_{-i} 1_{\lambda} \{t\} \\
\lambda - i_X \\
\lambda \end{array}
\]

\[ \deg 0 \]

\[ \deg 0 \]

\(^{\dagger}\)We refer to objects of the category \( \mathcal{U}_k(\lambda, \lambda') \) as 1-morphisms of \( \mathcal{U}_k \). Likewise, the morphisms of \( \mathcal{U}_k(\lambda, \lambda') \) are called 2-morphisms in \( \mathcal{U}_k \).
and more generally, for a signed sequence \( i = \varepsilon_1 i_1 \varepsilon_2 i_2 \ldots \varepsilon_m i_m \), the identity \( 1_{\varepsilon_1 i_1 \varepsilon_2 i_2 \ldots \varepsilon_m i_m} \) 2-morphism is represented as

\[
\begin{array}{c|c|c}
\lambda + iX & i_1 & i_2 \ldots i_m \\
\hline
i_1 & i_2 & \ldots & \lambda \\
i_1 & i_2 & \ldots & i_m
\end{array}
\]

where the strand labelled \( i_\alpha \) is oriented up if \( \varepsilon_\alpha = + \) and oriented down if \( \varepsilon_\alpha = - \). We will often place labels with no sign on the side of a strand and omit the labels at the top and bottom. The signs can be recovered from the orientations on the strands.

ii) For each \( \lambda \in X \) the 2-morphisms

| Notation: | \( \uparrow_{i,\lambda} \) | \( \downarrow_{i,\lambda} \) | \( \llcorner_{i,j,\lambda} \) | \( \lrcorner_{i,j,\lambda} \) |
|---|---|---|---|---|
| 2-morphism: | \( \lambda + iX \) | \( \lambda + \lambda + iX \) | \( \lambda \) | \( \lambda \) |
| Degree: | \( i \cdot i \) | \( i \cdot i \) | \( -i \cdot j \) | \( -i \cdot j \) |

| Notation: | \( \cup_{i,\lambda} \) | \( \cap_{i,\lambda} \) | \( \llcorner_{i,\lambda} \) | \( \lrcorner_{i,\lambda} \) |
|---|---|---|---|---|
| 2-morphism: | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) |
| Degree: | \( 1 + \lambda_i \) | \( 1 - \lambda_i \) | \( 1 + \lambda_i \) | \( 1 - \lambda_i \) |

- The \( \mathfrak{sl}_2 \) relations:
  i) \( 1_{\lambda + iX} \varepsilon^+_{i+1} 1_{\lambda} \) and \( 1_{\lambda} \varepsilon^-_{i-1} 1_{\lambda + iX} \) are biadjoint, up to grading shifts:

\[
\begin{align*}
\lambda + iX \cup_{i,\lambda} = \lambda + iX \quad & \quad \lambda \cup_{i,\lambda} = \lambda \quad & \quad \lambda + iX \cap_{i,\lambda} = \lambda + iX \\
\lambda + iX \cap_{i,\lambda} = \lambda + iX \quad & \quad \lambda \cap_{i,\lambda} = \lambda \quad & \quad \lambda + iX \cup_{i,\lambda} = \lambda + iX.
\end{align*}
\]

(2.1)

(2.2)
ii) All dotted bubbles of negative degree are zero. That is,

\[
\begin{align*}
\lambda_i \lambda_{i} & = \lambda + i_X \quad \text{and} \quad \lambda_i \lambda_{i} = \lambda + i_X \\
\end{align*}
\]

(2.3)

iii) All dotted bubbles of negative degree are zero. That is,

\[
\begin{align*}
\lambda_i \lambda_{-i} = 0 & \quad \text{if } \alpha < \lambda_i - 1 \\
\lambda_i \lambda_{-i} = 0 & \quad \text{if } \alpha < -\lambda_i - 1 \\
\end{align*}
\]

(2.4)

for all \(\alpha \in \mathbb{Z}_+\), where a dot carrying a label \(\alpha\) denotes the \(\alpha\)-fold iterated vertical composite of \(\uparrow_{i,\lambda}\) or \(\downarrow_{i,\lambda}\) depending on the orientation. A dotted bubble of degree zero equals 1:

\[
\begin{align*}
\lambda_i \lambda_{i} = 1 & \quad \text{for } \lambda_i \geq 1, \\
\lambda_i \lambda_{-i} = 1 & \quad \text{for } \lambda_i \leq -1. \\
\end{align*}
\]

iv) For the following relations we employ the convention that all summations are increasing, so that \(\sum_{f=0}^{\alpha} f\) is zero if \(\alpha < 0\).

\[
\begin{align*}
\lambda_i \lambda_{i} & = \sum_{f=0}^{\lambda_i} \lambda_{i-1+f} \quad \text{and} \quad \lambda_i \lambda_{i} = \sum_{f=0}^{\lambda_i} \lambda_{i-1+f} \\
\lambda_i \lambda_{-i} & = \sum_{f=0}^{\lambda_i} \lambda_{-i-1+g} \\
\end{align*}
\]

(2.5)

\[
\begin{align*}
\lambda_i \lambda_{i} & = \lambda_i \lambda_{i} + \sum_{f=0}^{\lambda_i} \sum_{g=0}^{f} \lambda_{i-1-f} \\
\lambda_i \lambda_{-i} & = \sum_{f=0}^{\lambda_i} \sum_{g=0}^{f} \lambda_{-i-1+g} \\
\end{align*}
\]

(2.6)

for all \(\lambda \in X\). Notice that for some values of \(\lambda\) the dotted bubbles appearing above have negative labels. A composite of \(\uparrow_{i,\lambda}\) or \(\downarrow_{i,\lambda}\) with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called fake bubbles, are formal symbols inductively defined by the equation.
(2.7)

and the additional condition

\[
\begin{pmatrix}
\begin{array}{c}
\lambda_i
\end{array}
\end{pmatrix}
\begin{pmatrix}
-1
\end{pmatrix}
\begin{pmatrix}
\lambda_i
\end{pmatrix}
= 1 \quad \text{if } \lambda_i = 0.
\]

Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles. See [3] for more details.

v) NilHecke relations:

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
-1
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
= 0,
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
-1
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
-1
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
= \lambda
\]

(2.8)

(2.9)

We will also include (2.10) for \( i = j \) as an \( \mathfrak{s}\mathfrak{l}_2 \)-relation.

- All 2-morphisms are cyclic\(^3\) with respect to the above biadjoint structure. This is ensured by the relations (2.3), and the relations

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
j
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
i
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
j
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
i
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
j
\end{pmatrix}
= \lambda
\]

\[
\begin{pmatrix}
\begin{array}{c}
\lambda
\end{array}
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}
\begin{pmatrix}
i
\end{pmatrix}
= \lambda
\]

(2.10)

The cyclic condition on 2-morphisms expressed by (2.3) and (2.10) ensures that diagrams related by isotopy represent the same 2-morphism in \( \mathcal{U}_k \).

\(^3\)See [3] and the references therein for the definition of a cyclic 2-morphism with respect to a biadjoint structure.
It will be convenient to introduce degree zero 2-morphisms:

\[
\begin{align*}
\lambda_{i \to j}^\lambda & := \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

where the second equality in (2.11) and (2.12) follow from (2.10).

• For \(i \neq j\)

\[
\begin{align*}
\lambda_{i \to j}^\lambda & = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
(2.13)
\end{align*}
\]

• The \(R(\nu)\)-relations:

i) For \(i \neq j\)

\[
\begin{align*}
\lambda_{i \to j}^\lambda & = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

\[
\begin{align*}
\lambda_{i \to j}^\lambda & = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

\[
\begin{align*}
\lambda_{i \to j}^\lambda & = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

ii) Unless \(i = k\) and \(i \cdot j = -1\)

\[
\begin{align*}
\lambda_{i \to j}^\lambda & = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

For \(i \cdot j = -1\)

\[
\begin{align*}
\lambda_{i \to j}^\lambda & + \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]

\[
\begin{align*}
\lambda_{i \to j}^\lambda & + \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\lambda \\
i
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\lambda \\
j
\end{array}
\end{array} \\
\lambda_{i \to j}^\lambda
\end{align*}
\]
For example, for any shift \( t \) there are 2-morphisms

\[
\begin{align*}
\lambda & : \mathcal{E}_{i+1} \mathbf{1}_\lambda \{t\} \Rightarrow \mathcal{E}_{i+1} \mathbf{1}_\lambda \{t-2\} \\
\lambda & : \mathcal{E}_{i+j} \mathbf{1}_\lambda \{t\} \Rightarrow \mathcal{E}_{i+j} \mathbf{1}_\lambda \{t-i \cdot j\} \\
\lambda & : \mathbf{1}_\lambda \{t\} \Rightarrow \mathcal{E}_{i} \mathbf{1}_\lambda \{t-c_{i,\lambda}\} \\
\lambda & : \mathcal{E}_{-i} \mathbf{1}_\lambda \{t\} \Rightarrow \mathcal{E}_{-i} \mathbf{1}_\lambda \{t-c_{-i,\lambda}\}
\end{align*}
\]

in \( \mathcal{U}_k \), and the diagrammatic relation

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (0.5,0.5) node {\( \lambda \)};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (0.5,0.5) node {\( \lambda \)};
\end{tikzpicture}
\end{array}
\]

\]

\text{gives rise to relations in} \( \mathcal{U}_k \left( \mathcal{E}_{i} \mathbf{1}_\lambda \{t\}, \mathcal{E}_{ii} \mathbf{1}_\lambda \{t+3i \cdot i\} \right) \) for all \( t \in \mathbb{Z} \).

- the additive \( \mathbb{Z} \)-linear composition functor \( \mathcal{U}_k(\lambda, \lambda') \times \mathcal{U}_k(\lambda', \lambda'') \to \mathcal{U}_k(\lambda, \lambda'') \) is given on 1-morphisms of \( \mathcal{U}_k \) by

\[
\mathcal{E}_{i} \mathbf{1}_\lambda \{t\} \Rightarrow \mathcal{E}_{i} \mathbf{1}_\lambda \{t+t'\}
\]

\text{for} \( \delta X = \lambda - \lambda' \), and on 2-morphisms of \( \mathcal{U}_k \) by juxtaposition of diagrams

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (0.5,0.5) node {\( \lambda'' \)};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (0.5,0.5) node {\( \lambda' \)};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (0.5,0.5) node {\( \lambda \)};
\end{tikzpicture}
\end{array}
\end{array}
\]

\( \mathcal{U}_k \) has graded 2-homs defined by

\[
\text{HOM}_{\mathcal{U}_k}(x,y) := \bigoplus_{t \in \mathbb{Z}} \text{Hom}_{\mathcal{U}_k}(x\{t\},y).
\]

\[
(2.19)
\]

3 \textbf{The 2-category of universal } \mathfrak{sl}_3\textbf{-foams}

In this section we define a 2-category of foams related to the universal \( \mathfrak{sl}_3 \)-foams defined in \cite{5}.
3.1 The category Foam/ℓ

First let us recall the definition and some basic facts about the category of universal sl₃ foams Foam/ℓ from [5]. A closed web is an oriented trivalent graph such that the edges meeting at a trivalent vertex are all oriented away from the vertex or are all oriented towards it. A foam is a cobordism with singular arcs between two closed webs. A singular arc in a foam f is the set of points of f that have a neighborhood homeomorphic to the letter Y times an interval (see the examples in Figure 1).

Interpreted as morphisms, we read foams from bottom to top by convention, and the orientation of the singular arcs is by convention as in Figure 1. Foams can have dots that can move freely on the facet to which they belong, but are not allowed to cross singular arcs. Let \( \mathbb{F}_2[a, b, c] \) be the ring of polynomials in \( a, b, c \) with coefficients in \( \mathbb{F}_2 \).

**Definition 3.1.** Foam is the category whose objects are closed webs and whose morphisms are \( \mathbb{F}_2[a, b, c] \)-linear combinations of isotopy classes of foams.

Foam is an additive category.

In order to construct the universal theory we divide Foam by the local relations \( \ell = (3D, CN, S, \Theta) \) below.

\[
\begin{align*}
\begin{array}{c}
\includegraphics[scale=0.5]{3D.png} \\
\end{array}
\end{align*}
\]

(3D)

\[
\begin{align*}
\begin{array}{c}
\includegraphics[scale=0.5]{CN.png} \\
\end{array}
\end{align*}
\]

(CN)

\[
\begin{align*}
\begin{array}{c}
\includegraphics[scale=0.5]{S.png} \\
\end{array}
\end{align*}
\]

(S)

Theta-foams are obtained by gluing three oriented disks along their boundaries (their orientations must coincide), as shown on the right. Note the orientation of the singular circle. Let \( \alpha, \beta, \gamma \) denote the number of dots on each facet. The (\( \Theta \)) relation says that for \( \alpha, \beta \) or \( \gamma \leq 2 \)

\[
\theta(\alpha, \beta, \gamma) = \begin{cases} 
1 & (\alpha, \beta, \gamma) = (1, 2, 0) \text{ or any permutation} \\
0 & \text{else}
\end{cases} \quad (\Theta)
\]
Reversing the orientation of the singular circle gives the same values for \( \theta(\alpha, \beta, \gamma) \). Note that when we have three or more dots on a facet of a foam we can use the (3D) relation to reduce to the case where it has less than three dots.

A closed foam \( f \) can be viewed as a morphism from the empty web to itself which by the relations \((3D, \, CN, \, S, \, \Theta)\) is an element of \( \mathbb{F}_2[a, b, c] \). It can be checked that this set of relations is consistent and determines uniquely the evaluation of every closed foam \( f \), denoted \( C(f) \). Define a \( q \)-grading on \( \mathbb{F}_2[a, b, c] \) as \( q(1) = 0, \, q(a) = 2, \, q(b) = 4 \) and \( q(c) = 6 \). We define the \( q \)-degree of a foam \( f \) with \( d \) dots by

\[
q(f) = -2\chi(f) + \chi(\partial f) + 2d,
\]

where \( \chi \) denotes the Euler characteristic and \( \partial f \) the boundary of \( f \).

**Definition 3.2.** Foam \( /\ell \) is the quotient of the category Foam by the local relations \( \ell \). For webs \( \Gamma, \, \Gamma' \) and for families \( f_i \in \text{Hom}_{\text{Foam}}(\Gamma, \, \Gamma') \) and \( c_i \in \mathbb{F}_2[a, b, c] \) we impose \( \sum_i c_i f_i = 0 \) if and only if \( \sum_i c_i C(g' f_i g) = 0 \) holds, for all \( g \in \text{Hom}_{\text{Foam}}(\emptyset, \, \Gamma) \) and \( g' \in \text{Hom}_{\text{Foam}}(\Gamma', \, \emptyset) \).

**Lemma 3.3.** We have the following relations in Foam \( /\ell \):

\[
(4C) \quad \quad + \quad = \quad + \quad +
\]

\[
(\text{RD}) \quad = \quad +
\]

\[
(\text{DR}) \quad = \quad +
\]

\[
(\text{SqR}) \quad = \quad +
\]

In Figure 2 we also have a set of useful identities, called *dot migration*, which establish the way we can exchange dots between faces. These identities can be used for the simplification of foams and are an immediate consequence of the relations in \( \ell \).

One can generalize Foam \( /\ell \) to open webs and foams with corners. The local relations are the same and a linear combination of foams with corners is zero if and only if for any way of closing the foams we get a linear combination of closed foams which is equal to zero. The grading formula has to be slightly changed, as we will show in the next section, where we define a 2-category of foams with corners more precisely.
3.2 The 2-category $2\text{Foam}_{/\ell}$

In order to define foamation as a 2-functor we have to define a 2-category of $\mathfrak{sl}_3$-foams closely related to $\text{Foam}_{/\ell}$.

**Definition 3.4.** Let $2\text{Foam}_{/\ell}$ be the 2-category whose objects are finite sequences of points. We take the points to lie along the y-axis with a fixed distance between them, e.g. each point has coordinates $(0, n, 0)$ for some $n \in C_6$.

Let the 1-morphisms be finite sequences of vertical line segments possibly linked by horizontal line segments. Each vertical line segment is allowed to be a finite disjoint union of smaller subsegments of the form $\{(0, n, z) : a \leq z \leq b\}$ for some fixed $n \in C_6$ and $0 \leq a < b \leq 1$. Except for the bottom and top endpoints, which have $z$-coordinate equal to 0 and 1 respectively, each endpoint of a vertical line subsegment is also the endpoint of a horizontal line segment. Furthermore there might be points in the interior of a subsegment which are endpoints of horizontal line segments. We consider such a line segment to be a 1-morphism from its intersection with the $x = z = 0$ axis to its intersection with the line $\{(0, y, 1) : y \in \mathbb{R}\}$ (after adjusting the $z$-variable of the latter points we get an object as defined above). Composition is defined by vertical glueing and rescaling. If one vertical line ends with a hole and the other starts with a hole, the composite will be a line with a hole at the place of the two old holes, i.e. the glueing does not close the holes. If one vertical line ends with hole and another starts without one, the two are non-composable.

A 2-morphism is a foam whose boundary consists of a finite sequence of squares whose vertices are of the form $\{(x, n, z) : x \in \{-1, 0\}, z \in \{0, 1\}\}$, for some $n \in C_6$, and oriented horizontal line segments between their vertical parts. The vertical parts of the squares are allowed to have holes in them, the horizontal ones are not. The foams are required to be such that when we take the intersection with the $x = 0$ and $x = -1$ planes, the source and target of the 2-morphism, we get 1-morphisms as described above. As in the previous subsection the foams are subject to the relations in $\ell$ and two open foams are equivalent if and only if all their possible closures are the same modulo $\ell$. See Figure 3 for an example. Horizontal composition is defined by horizontal glueing and rescaling and vertical composition by vertical glueing and rescaling. As above “holes can only be glued to holes”.

Note that $2\text{Foam}_{/\ell}$ is a strict 2-category. We also define a grading on $2\text{Foam}_{/\ell}$, i.e. the 2-
morphisms have degrees. This grading corresponds to the one given in the previous section with one extra term because the foams have corners.

**Definition 3.5.** The degree of a 2-morphism \( f \) in \( \text{2Foam}/\ell \) is given by the formula

\[
q(f) = -2\chi(f) + \chi(\partial f) + 2d + \beta,
\]

where \( \beta \) is the number of vertical boundary components of \( f \).

Note that \( q \) is additive under vertical and horizontal composition. Note also that relation \((S)\) implies the following result, which we will need in the following section.

**Lemma 3.6.** Dotted spheres of negative degree are zero.

## 4 Foamation

In this section we define 2-functors \( \mathcal{H}_s : \mathcal{U}_k \to \text{2Foam}/\ell \), for \( s \in \{0, 1, 2, 3\} \). The main idea is that the image of a 2D-picture in \( \mathcal{U}_k \) should be a 3D-foam in \( \text{2Foam}/\ell \) such that vertical cross sections of that foam correspond to the initial 2D-picture. How to do that precisely is the content of the following definition. In the sequel we keep \( k \) fixed in the definition of \( \mathcal{U}_k \).

**Definition 4.1.** On objects we define \( \mathcal{H}_s(\lambda) \) to be a subset of \( \{(0, n, 0) : n = 0, \ldots, k-1\} \). To see which points appear in \( \mathcal{H}_s(\lambda) \) label the last point, i.e. \((0, k-1, 0)\), by \( s \). Label \((0, k-2, 0)\) by \( s - \lambda_1 \), \((0, k-3, 0)\) by \( s - \lambda_1 - \lambda_2 \) etc. If one of the points has a label outside the range \( \{0, 1, 2, 3\} \), we define the image of \( \lambda \) to be zero. Otherwise, remove the points labelled 0 or 3. The image of \( \lambda \) is given by the remaining points.

On 1-morphisms we define \( \mathcal{H}_s(\varepsilon_1 1, \lambda) \) as follows: first take \( k \) vertical line segments \( v_n \), where \( n = 0, \ldots, k-1 \), with lower vertices \( \{(0, k-1-n, 0) : n = 0, \ldots, k-1\} \) and upper vertices \( \{(0, k-1-n, 1) : n = 0, \ldots, k-1\} \). Suppose \( \varepsilon_i = \varepsilon_{e_i 1, i} \cdots \varepsilon_{e_{m} 1, m} \). Let \( h_j = 1 - j/(m+1) \). For each \( \varepsilon_{e_{j} 1, j} \), add a horizontal line segment with height \( h_j \) between \( v_{i_j-1} \) and \( v_{i_j} \). If \( \varepsilon_j = + \) orient the horizontal line segment to the right, if not orient it to the left. Then proceed as follows: label the lower vertex of \( v_0 \) by \( s_0 = s \) and the lower vertex of \( v_n \) by \( s_n = s - \lambda_1 - \cdots - \lambda_n \) for \( n = 1, \ldots, k-1 \). On each \( v_n \) this label remains constant until we meet a point where \( v_n \) is glued to a horizontal line segment (when moving upwards). When we pass such a glueing point the label changes by +1 if the horizontal line segment is oriented towards \( v_n \) and by -1 if not. This way all line subsegments of the \( v_n \) are labelled. If there is a subsegment which has a label outside the range \( \{0, 1, 2, 3\} \) we define \( \mathcal{H}_s(\varepsilon_1 1, \lambda) = 0 \). If not, remove the subsegments with label 0 or 3 and take \( \mathcal{H}_s(\varepsilon_1 1, \lambda) \) to be the remaining graph.
Finally, on 2-morphisms we define $H_s$ as follows:

We have indicated the labels of the facets, with the $s_n$ as defined above.

Suppose $\mathbf{c}_i = \mathbf{c}_{e_{1i_1}} \cdots e_{e_{ni_n}}$. Let $h_j = 1 - (j/m + 1)$. We define $H_s(1_{e_{1i_1}})$ as above, but with more horizontal sheets. To be precise, for each $j$, we add a horizontal sheet with height $h_j$ between the $i_j - 1$-th and the $v_i$-th vertical sheets. If $e_j = +$ we orient the horizontal sheet as shown in the picture above, if not we give it the opposite orientation. The labellings of the facets are as indicated above, and depend on the orientation of the horizontal sheets. If one of the labels is outside the range of $\{0, 1, 2, 3\}$, then the image of $1_{e_{1i_1}}$ is zero. If all labels are within range, remove the facets labelled 0 or 3. The image of $1_{e_{1i_1}}$ is the remaining foam. Let us do one concrete example to clarify our definition. Suppose $k = 3$, $s = 2$ and $\mathbf{c}_{(1,-2)} = \mathbf{c}_{+1} \mathbf{c}_{-2}$ and $\lambda = (0,0)$. Then $H_2(1_{e_{(1,-2)}}1_{(0,0)})$ is given by the foam obtained from

by removing the facets labelled 0 and 3. Note that

$$\mathbf{c}_{(+1,-2)}1_{(0,0)} = 1_{(3,-3)} \mathbf{c}_{+1}1_{(1,-2)} \mathbf{c}_{-2}1_{(0,0)}$$

and that the pairs $(3,-3), (1,-2)$ and $(0,0)$ correspond precisely to the differences between the labels on the right and left vertical sheets on the level above, between and under the two horizontal sheets respectively.

Next we define $H_s$ for the other generating 2-morphisms. For shortness we omit the vertical sheets which are unimportant in the pictures. The labels are calculated as indicated in the pictures, and labels outside the indicated range give rise to a zero image. If all labels are within
range, the facets labelled 0 or 3 have to be removed to obtain the final image under $H_s$. Dots on the 2-morphisms are mapped to dots on the corresponding facets. We only give the images of the 2-morphisms with one particular orientation. For the other orientations just reorient the foams and relabel the facets accordingly.

We define \( \xrightarrow{i,i,\lambda} \) similarly. If \( j > i + 1 \) we define

The images of the cups and caps are defined as follows:
We are now ready to prove the main theorem of our paper.

**Theorem 4.2.** \( \mathcal{H}_s : \mathcal{U}_k \to \text{2Foam}_/\ell \) is a degree preserving strict \( \mathbb{F}_2 \)-linear 2-functor, for any \( s \in \{0, 1, 2, 3\} \).

**Proof.** Clearly the composition rules are preserved by \( \mathcal{H}_s \), for any \( s \in \mathbb{N} \). It is also not hard to check that \( \mathcal{H}_s \) preserves the degrees. Note that one has to be careful and remove facets labelled 0 or 3 before computing the degree.

What is left to prove is that \( \mathcal{H}_s \) respects the relations in \( \mathcal{U}_k \). We first prove this for the \( sl_2 \) relations. Relations 2.1, 2.2 and 2.3 are preserved because \( \mathcal{H}_s \) maps the left and right sides of each relation to isotopic foams.

To see that relations 2.4 are preserved we have to realize that the image of a bubble with dots is given by \( k - 1 \) vertical sheets with a tube with dots between two of them, possibly with some vertical facets removed because they are labelled 0 or 3. By the relations (CN) and (RD) in \( \text{2Foam}_/\ell \) such a foam is equivalent to a \( \mathbb{F}_2[a, b, c] \)-linear combination of foams which are given by vertical sheets with dots and without holes, and spheres with dots. Since \( \mathcal{H}_s \) preserves degrees and \( \mathbb{F}_2[a, b, c] \) has non-negative grading only, each of these foams in the linear combination has to have negative degree. Vertical sheets with dots have non-negative degree, so the spheres with dots have to have negative degree. Therefore, by Lemma 3.6 everything has to be zero.

The same line of reasoning shows that the image of a degree zero bubble equals the identity. In this case there is only one non-zero term in the linear combination, which is the term with dotless vertical sheets and degree zero sphere with dots. Therefore we get plus or minus the identity.

For the next relations we have to determine the image of fake bubbles. Note that the fake bubbles are uniquely defined in terms of ordinary bubbles by the infinite Grassmannian relation 2.7 and the additional condition which says that the degree zero fake bubbles are equal to one. Since we know the image of the ordinary bubbles, all we have to do is indicate foams which satisfy the infinite Grassmannian relation and satisfy the degree zero condition in \( \text{2Foam}_/\ell \). It suffices to do this for \( k = 2 \), since these relations are essentially \( sl_2 \) relations. For \( \lambda \geq 0 \) and \( j < \lambda + 1 \), we put where a dot to the power \( r \) means \( r \) dots. Note that the labels of the interiors of the rightmost and leftmost discs are equal to \( s - 1 \) and \( s - \lambda + 1 \) respectively. If one of these labels is equal to 0 or 3 we have to remove the corresponding disc, so we cannot put dots on that disc. To avoid contradictions we simply assume that terms with a disc labelled 0 or 3 and a positive number of dots on that disc are equal to zero in the sum. By straightforward calculations one can check that
with this definition the images of the fake bubbles satisfy the infinite Grassmannian relation and the degree zero condition. The images of the fake bubbles with the opposite orientation are defined likewise.

Next we prove the right relation in 2.5. The left one can be shown using similar arguments. Again, since this is an \( sl_2 \) relation we can assume that \( k = 2 \) without loss of generality. First suppose that \( \lambda < 0 \). Then the sum on the right-hand side of the equation is equal to zero, so we have to show that the image of the left-hand side is equal to zero as well. That image is given by

Note that the left and right inner discs are labelled \( s - \lambda \) and \( s + 2 \) respectively. Since we are assuming that \( \lambda < 0 \) we see that this foam is zero for \( s \neq 0 \) because \( s \) or \( s - \lambda + 2 \) will be outside the range \( \{0, 1, 2, 3\} \). For the same reason the foam is zero when \( s = 0 \) and \( \lambda < -1 \). If \( s = 0 \) and \( \lambda = -1 \) the foam we get is a facet with a bubble without dots (after removing the facets labelled 0 and 3). By (CN) and (\( \Theta \)) such a foam is equal to zero.

If \( \lambda \geq 0 \), we can apply (DR) to the singular tube in the picture. After removing possible bubbles with dots we get a linear combination of foams like

with some dots.
The foam corresponding to the right-hand side of relation 2.5 is a linear combination of foams of the form

\[
\begin{align*}
&\alpha & &\beta & &\gamma \\
&s - \lambda + 1 & &s + 1 & &s
\end{align*}
\]

for certain non-negative integers \(\alpha, \beta, \gamma\) and \(\delta\). After applying (RD) to both internal discs and removing the bubbles and the spheres we again obtain a linear combination of foams as pictured in the second last figure with some dots. All these calculations are straightforward and one can check that in all cases the right relation in 2.5 is preserved by \(\mathcal{H}_s\) after moving the dots around using dot migration.

Let us now show that relation 2.6 is preserved. We show this for the first relation, the second being similar. We can assume that \(\lambda\) is a non-negative integer (instead of a sequence of integers). It suffices to consider the cases \(s = 2, 3\) for \(\lambda = 0, 1, 2, 3\). For \(s = 1, 2\) the same arguments work because the foams in that case can be obtained from those for \(s = 2, 3\) by applying a symmetry in the \(xz\)-plane which slices the foams in the middle. Therefore, let \(s = 2, 3\). The left-hand side of relation 2.6 is mapped to

\[
\begin{align*}
&\alpha & &\beta & &\gamma \\
&s - \lambda & &s
\end{align*}
\]

The right-hand side is mapped to

\[
\begin{align*}
\lambda - 1 - f & &1 - \alpha \\
&+ \sum_{f=0}^{\lambda-1} \sum_{g=0}^{\lambda+1} \sum_{\alpha=0}^{\lambda+1} f - g
\end{align*}
\]
In the figure above we have omitted some orientations to avoid the cluttering of lines. In the first foam the orientations can be obtained by the rule which says that at each trivalent vertex the edges are all oriented inwards or all outwards. The circles in the vertical sheets in the second foam are oriented such that neighboring arcs have opposite orientations. We have also omitted the labels of some facets. From the orientations of the edges and the labels which we have given one can easily compute the missing ones. To see that the first relation in 2.6 is preserved one first has to use (RD) on all discs in the tube in the second foam above. Then use (3D) and (S) and, if necessary, dot migration. After doing that we get the following: for \( s = 2, \lambda = 0 \) and \( s = 3, \lambda = 1 \) relation 2.6 corresponds to an isotopy between foams. For \( s = 2, \lambda = 1 \) it corresponds to the (SqR) relation on foams. For \( s = 2, \lambda = 2 \) and \( s = 3, \lambda = 2 \) it corresponds to (DR). For \( s = 3, \lambda = 3 \) it corresponds to (CN). For all other values of \( s \) and \( \lambda \) the functor \( H_s \) maps all foams to zero.

The last type of \( \mathfrak{sl}_2 \) relations we have to check are the NilHecke relations 2.8 and 2.9. Let us first have a look at the first relation in 2.8. The diagram is mapped to

The curves in the middle of each vertical sheet have the same orientation as the line segments on that sheet. We have omitted two labels, which can be easily computed using the orientations. It is easy to see that this foam is always equal to zero. Either it is mapped to zero, because some label is out of range, or we get a foam with a facet with a bubble without dots, which is equal to zero by (RD) and (S). Therefore the first NilHecke relation is always preserved by \( H_s \).

The second NilHecke relation in 2.8 is always preserved by \( H_s \) because the two sides of the relation are mapped to isotopic foams.

The third NilHecke relation, which is the one in 2.9 corresponds to the (DR) relation for all values of \( s \) and \( \lambda \) such that \( H_s \) is non-zero. The foams can be easily read off from the definition of \( H_s \) above.

The next relations we have to check are the ones in 2.13. Consider the first one, the second being analogous. Suppose \( k = 3 \) and \((i, j) = (1, 2)\).
The left-hand side of the equation is mapped to

\[ s - \lambda_1 - \lambda_2 + 1 \quad s - \lambda_1 - 2 \quad s + 1 \]

\[ s - \lambda_1 - \lambda_2 \quad s - \lambda_1 \quad s \]

Note that to be completely accurate one would have to draw the crossings in the middle vertical sheet as little dumbbells, but that would make the picture much harder to read. Note that in that middle sheet the regions between the arcs are labelled \( s - \lambda_1 - 1 \). Suppose that all labels are within the allowed range. Then either \( s - \lambda_1 = 3 \) or \( s - \lambda_1 - 2 = 0 \) has to hold. Now look closely at the singular curve in the middle sheet and you see that the foam can be isotoped to

The case \( j = i - 1 \) is similar. If \( |i - j| > 1 \) the isotopy is straightforward, because the horizontal sheets are glued to different vertical sheets.

Next we prove that the \( R(v) \)-relations are preserved. The first \( R(v) \)-relation is not too hard. First suppose \( k = 3 \) and \((i, j) = (1, 2)\). The left-hand side of relation [2.14] is mapped to a foam like the one in Figure [1] except that both horizontal sheets have the same orientation. Note that in the middle sheet the two arcs have opposite orientation. There are now two possible non-zero cases. The middle region of the middle sheet is labelled 0 or 3, and therefore corresponds to a hole, or it is labelled 1 or 2, in which case it is a disc bounded by a singular curve. In the latter case one can apply the (RD) relation to the disc and obtain the desired sum of identity foams with dots. In the former case there are two singular curves formed by the arcs that do not bound the hole. One can apply relation (DR) to obtain the desired sum of identity foams with dots. If \( i \cdot j = 0 \), the horizontal sheets in the foam are glued to different vertical sheets and we can apply a simple isotopy.

The second set of \( R(v) \) relations, given in [2.15] is preserved by \( \mathcal{G} \) because the \( i \)-strands before and after the crossing are mapped to the same facet in the corresponding foam and so are
the $j$-strands, since $i \neq j$. Therefore the dots before and after the crossing both live on the same facet and the relation is preserved because dots can freely move on a facet.

Finally we have to prove relations 2.16 and 2.17. For $i \neq k$ and $i \cdot j = 0$, relation 2.16 clearly corresponds to an isotopy of the corresponding foams, because the horizontal sheet corresponding to the $j$-strand is not attached to the same vertical sheet as the one corresponding to the $i$-strand.

The proof of relation 2.17 is slightly harder. Suppose $i = k$ and $i \cdot j \neq 0$. Note that the case $i = j = k$ was already proved above. Here we only prove the case $j = i + 1$, the case $j = i - 1$ being similar. Without loss of generality we assume that $k = 2$ and $i = 1$. In the figure below we show the images under $\mathcal{N}_q$ of the left and the right-hand sides of relation 2.17 respectively, which gives an equation of foams of the form $A - B = C$.

We only show the labels at the bottom. The other labels can be easily computed using the orientations of the edges. Note that the square regions in the middle of the middle vertical sheet of $A$ and $B$ have label $s - \lambda_1 - 2$ and $s - \lambda_1 + 1$ respectively. The label at the bottom of that sheet is $s - \lambda_1$. Taking into account the other labels as well, we see that there are essentially three distinct non-trivial cases: $s - \lambda_1 = 1$, $s - \lambda_1 = 2$ and $s - \lambda_1 = 3$ (cases 1,2 and 3). In the first case $s - \lambda_1 - 2 = -1$, so $A = 0$. To see that $B$ is isotopic to $C$ one only has to follow the singular curves. The third case is similar, $B$ being zero and $A$ being isotopic to $C$. In the second case both $A$ and $B$ can be non-zero. In that case we get the (SqR) relation, which again can be seen from looking carefully at the singular curves of $A$, $B$ and $C$. 

\[ \square \]
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