The renormalisation group and nuclear forces

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I give an outline of recent applications of the renormalisation group to effective theories of nuclear forces, focussing on the use of a Wilsonian approach to analyse systems of two or three nonrelativistic particles.

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1. Introduction

The last twenty years have seen a major shift in how we describe the forces between nucleons. This was triggered by the suggestion of Weinberg (1990, 1991) that the ideas of effective field theory (EFT) could be applied to these strongly interacting systems. This approach offers the possibility of a systematic, model-independent treatment which, ultimately, could form a bridge to the underlying theory of the strong interaction, Quantum Chromodynamics (QCD).

Effective field theories are built out of fields corresponding to the appropriate low-energy degrees of freedom and contain all possible terms consistent with the symmetries of the underlying dynamics. To have predictive power, such a theory must be expandable in powers of ratios of low-energy scales to those of the underlying physics. For nuclear physics, the low-energy scales, denoted generically by $Q$, include the momenta of the nucleons and the mass of the pion. The underlying scales of QCD, denoted by $\Lambda_0$, include $4\pi F_\pi$ (the scale associated with the hidden chiral symmetry) as well as the masses of the nucleons and $\rho$, $\omega$ and other mesons.

Since Weinberg made his original proposal, debate has raged within the community over the appropriate power counting to use in organising this expansion in powers of low-energy scales. Extensive reviews can be found in the articles by Beane et al. (2001), Bedaque & van Kolck (2002) and Epelbaum et al. (2009), and more recent summaries from two different points of view have been given by Epelbaum and Gegelia (2009) and Birse (2009). The main bones of contention have been: how do we renormalise nonperturbative systems consistently, and which pieces of the interaction should we iterate to all orders?

To get unambiguous answers to these questions we need a rigorous tool to analyse the scale dependences of physical systems, and this is provided by the renormalisation group (RG). I concentrate here on the version developed in Manchester (Birse et al. 1999; Barford & Birse, 2003, 2005; Birse, 2006a, 2006b) which uses a Wilsonian approach to construct functional RG equations. These can be solved exactly, at least in simple cases. However many of the results discussed here were first obtained from more heuristic RG equations with momentum-space cutoffs (Lepage 1997; Bedaque & van Kolck 1998; van Kolck 1999; Bedaque et al. 1999a, 1999b,
by using dimensional regularisation with subtraction of power-law divergences (Kaplan et al. 1998a, 1998b; Phillips et al. 1999; Kong & Ravndal 1999, 2000; Ando & Birse 2008), or with a Bogolubov-Parasiuk-Hepp-Zimmermann subtractive renormalisation scheme (Gegelia, 1999a, 1999b).

Other closely related approaches that are being explored include extensions of the subtractive renormalisation scheme (Frederico et al. 1999, 2000; Hammer & Mehen 2001; Afnan & Phillips 2004; Timóteo et al. 2005, 2010; Yang et al. 2008, 2009a, 2009b) and the use of a radial cutoff in coordinate space (Pavón Valderrama & Ruiz Arriola 2004a, 2004b, 2006a, 2006b, 2008, 2009).

In addition, there is a powerful functional RG based on the Legendre-transformed effective action (Wetterich 1993; Berges et al. 2002). This has recently been applied to few-body (as well as many-body) systems of strongly interacting, non-relativistic particles (Birse et al. 2005, 2010a, 2010b; Diehl et al. 2007, 2008; Birse 2008; Floerchinger et al. 2009; Moroz et al. 2009; Schmidt & Moroz 2010). As generally implemented, this is less rigorous than the versions of the RG mentioned above because it relies on truncations of the effective action that are not based on any systematic power counting. None-the-less it is proving very useful in situations where exact solutions of other RG equations cannot be found, such as four-body systems and dense matter.

Lastly, I should mention the approach known as $V_{\text{low-k}}$ (Bogner et al. 2003a, 2003b) and related applications of the similarity renormalisation group to nuclear forces (Bogner et al. 2007a, 2007b; Jurgenson et al. 2008, 2009). Both of these lead to effective interactions that evolve according to RG equations. However the structures of the resulting equations are more complicated than those of the approaches mentioned above and, at least so far, it has not been possible to analyse their scaling behaviours.

Whatever regulator or subtraction scheme we choose, we should first identify fixed-point solutions of the RG equations. If we expand a general solution around one of these points, we can use the linearised RG equations to classify the perturbations as relevant, marginal or irrelevant. The eigenvalues of the linear equations give the anomalous dimensions of the corresponding operators and hence can be used to construct a power-counting scheme. This counting is what makes it possible to define a systematic expansion of the corresponding EFT.

In systems whose particles interact very strongly at low energies, the terms in the potential can have large anomalous dimensions. Their scaling behaviour, and hence the power counting for the corresponding EFT, is then quite different from naive dimensional analysis. This is particularly true of systems close to the “unitary limit”, that is, with very large two-body scattering lengths. Nuclear forces are a prime example of this, since the nucleon-nucleon scattering lengths are of the order of 5–20 fm, much larger than the range of the interactions.

In such cases, the leading two-body interaction can become a relevant term, and other two-body operators are strongly promoted compared to naive expectations. The two-body sectors of the resulting effective theories are really just versions of the much older effective-range expansions (Bethe 1949; Blatt & Jackson 1949; van Haeringen & Kok 1982; Badalyan et al. 1982). The benefit of the modern field-theoretic framework is that it can provide consistent effective current operators as well as three- and more-body forces.
The first application of RG methods to three-body forces in theories with two-body contact interactions was to channels of three fermions with mixed-symmetry spatial wave functions. These give rise to a repulsive “particle-exchange” force between an interacting pair and the third particle (Bedaque & van Kolck, 1998; Bedaque, Hammer & van Kolck, 1998). This work showed that three-body forces were not required to describe the low-energy physics in these systems, implying that there are no large, negative anomalous dimensions. Subsequent, more detailed RG analyses in momentum space (Griesshammer 2005) and in coordinate space (Birse 2006b) have determined the exact anomalous dimensions for these repulsive three-body channels.

In systems of three bosons or three fermions with a fully symmetric spatial wave function, the particle-exchange force is attractive. This can have dramatic consequences, most notably the Efimov effect (Efimov 1971, 1979): a tower of bound states with energies in a geometric sequence. The RG flow in these systems displays a limit-cycle behaviour (Bedaque, Hammer & van Kolck, 1999a, 1999b, 2000; Glazek and Wilson, 2004; Barford & Birse, 2005; Mohr et al., 2006), which corresponds to anomalous breaking of scale invariance to a discrete remnant of the symmetry. The leading three-body force in these systems is a marginal term, corresponding to the starting point on the limit cycle. This means that one piece of three-body physics is sufficient to determine their low-energy behaviour, as has long been known from the Phillips line (Phillips 1968, 1977) which shows a correlation between the triton binding energy and *nd* scattering length for forces fitted to the two-body scattering data.

In contrast, four-body systems show no evidence for relevant or marginal forces, even in cases where the three-body subsystems display Efimov behaviour (Platter, Hammer & Meissner, 2004, 2005; Platter & Hammer, 2007). This means that their low-energy observables are determined purely by two- and three-body physics, providing an explanation for the “Tjon line” correlation between $^4$He and triton binding energies (Tjon 1975). No exact results are available for the scaling behaviour in four-body systems, but functional RG methods are now being used to estimate anomalous dimensions for both bosonic and fermionic systems (Schmidt & Moroz, 2010; Birse et al., 2010b).

All of the RG techniques mentioned above are applicable to any system of strongly interacting, nonrelativistic particles, where they can be used to determine the pertinent power counting and hence to set up a consistent EFT description. Other areas where these ideas are now being applied within nuclear and hadron physics are weakly bound “halo” nuclei (Bertulani et al. 2002; Bedaque et al. 2003b; Higa et al. 2008) and the interactions of mesons containing heavy quarks (Braaten & Kusonoki 2004; Fleming et al. 2007; Braaten et al. 2010; Hagen et al. 2010). Looking further afield, they are being used increasingly in studies of ultracold atomic systems (see, for example: Braaten & Hammer 2006).

2. Scales and scaling

Underpinning any viable EFT is an expansion in powers of the low-energy scales for a system. The RG can help to elucidate this scale dependence but, first, we need to identify all the relevant scales, $Q$. The most obvious ones for any low-energy theory are particles’ momenta, both on-shell and off-shell.
Typical momenta in nuclear systems are often of the order of 100 or 200 MeV, which is comparable to the mass of the pions. These mesons are not only the approximate Goldstone bosons of QCD, they also give rise to the longest-range forces between nucleons. If we wish to describe physics on this scale, we need to include pions in our EFT, and to count their mass as one of our low-energy scales.

So far, this list of scales is just the same as in chiral perturbation theory (ChPT), Weinberg’s original EFT for low-energy meson physics (Weinberg, 1979). However, in contrast to the case of pions, where the hidden chiral symmetry ensures that their interactions are weak at low energies, nucleons interact strongly, forming bound states (nuclei).

![Figure 1. Loop diagram for two-body scattering](image)

To see why this can lead to problems with extending ChPT to two or more nucleons, we should look at the nonrelativistic loop diagram for NN scattering in figure 1. For contact interactions, this has the form

\[
\frac{M}{(2\pi)^3} \int \frac{d^3q}{p^2 - q^2 + i\epsilon} = -i M p \frac{p}{4\pi} + \text{analytic in } p^2. \tag{2.1}
\]

As noted by Weinberg (1990, 1991), this is enhanced to order \(Q\), instead of \(Q^2\) as in the relativistic case. Nonetheless, the leading terms of the potential are of order \(Q^0\) (OPE and the simplest contact interaction) and so each iteration is suppressed by a power of \(Q/\Lambda_0\). The theory is therefore still perturbative, provided \(Q < \Lambda_0\).

In fact the analytic part of the integral (2.1) is linearly divergent and so we need to either cut it off or subtract it at some scale \(q = \Lambda\). Iterating the potential then leads to contributions with powers of \(\Lambda/\Lambda_0\). These will again be perturbative provided we keep our cutoff within the domain of our EFT, \(\Lambda < \Lambda_0\), and so they cannot generate bound states.

We therefore need to identify further low-energy scales. Of particular interest are any that promote some of the interactions to order \(Q^{-1}\) (making them marginal terms in RG language) since these can, and indeed must, be treated nonperturbatively. The first examples to be identified in NN scattering were provided by the \(S\)-wave scattering lengths (Bedaque & van Kolck 1998; van Kolck 1999; Kaplan et al. 1998a, 1998b; Birse et al. 1999).

In addition, there are scales associated with long-range interactions. A simple example is provided by the Coulomb potential between two charged particles, such as two protons. Here, after we scale the nucleon mass \(M_N\) out of the Hamiltonian,
the strength of interaction can be expressed in terms of the inverse Bohr radius,

\[ \kappa = \frac{\alpha M_N}{2} \approx 3.4 \text{ MeV}. \]  

The long-range pion-exchange forces can be expanded using the methods of ChPT. The leading piece is one-pion exchange (OPE) whose strength can be expressed in terms of the momentum scale

\[ \lambda_{\pi NN} = \frac{16\pi F^2_\pi}{g_A^2 M_N} \approx 290 \text{ MeV}. \]  

This is built out of high-energy scales in chiral perturbation theory, \(4\pi F_\pi\) and \(M_N\). Counting it as a high-energy scale, would lead us to a perturbative treatment of OPE, as developed by Kaplan et al. (1998a, 1998b). Numerically, however, the value of \(\lambda_{\pi NN}\) is only about twice \(m_\pi\), suggesting that it may be better viewed as a low-energy scale. If we do so, OPE is promoted to order \(Q^{-1}\), implying that it should be iterated.

Following a Wilsonian approach, the next step in the RG is to cut off our EFT at some arbitrary scale \(\Lambda\), lying above the low-energy scales \(Q\) but below the scale \(\Lambda_0\) of the underlying physics, as in figure 2. (This assumes good separation of these scales, as required for an EFT with a convergent expansion.)

\[ \begin{array}{c}
\hline
E \\
\hline
\Lambda_0
\end{array} \begin{array}{c}
\hline
\Lambda \\
\hline
Q
\end{array} \]

Figure 2. The running cutoff \(\Lambda\).

Then we can follow the evolution of our theory as we “integrate out” more and more of the physics by lowering \(\Lambda\). As we vary our arbitrary cutoff, we demand that physics (for example, the scattering matrix) be independent of \(\Lambda\). This means that the couplings in our EFT must run with \(\Lambda\) to compensate for the physics we are integrating out. Ultimately, for \(\Lambda \ll \Lambda_0\), we lose all memory of the underlying physics and the only scale left is \(\Lambda\).

Finally, we rescale the theory by expressing all dimensioned quantities in units of \(\Lambda\). At one of the end points of the RG flow, all couplings are then just numbers independent of \(\Lambda\). We have arrived at a fixed point of the RG: a theory that describes a scale-free system. Two are shown in figure 3. The one on the left is stable: any nearby theory will flow towards it as the the cut-off is lowered. In contrast, the one on the right has an unstable direction: the flow can take theories away from the fixed point unless they lie on the “critical surface”.

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Close to a fixed point, we can find perturbations that scale with definite powers of $\Lambda$. They can be classified into three types:

- $\Lambda^{-\nu}$: relevant (super-renormalisable in the language of particle physics), for example mass terms in quantum field theories like QED;
- $\Lambda^0$: marginal (renormalisable), for example the couplings familiar in gauge theories like the Standard Model (typically these show a log $\Lambda$ dependence on the cut-off);
- $\Lambda^{+\nu}$: irrelevant (nonrenormalisable), for example the interactions in mesonic ChPT.

We now can expand our EFT around one of the fixed points using these perturbations. Their RG scaling maps directly onto the order in the usual power counting (Weinberg, 1979), an interaction running as $\Lambda^\nu$ corresponding to a term in the EFT of order $Q^d$ where $d = \nu - 1$.

3. Short-range forces

To illustrate how RG methods can be applied to scattering of nonrelativistic particles, let us look at a system of two particles at energies where the range of the forces is not resolved (for example, two nucleons with an energy below about 10 MeV). This can be described by an effective Lagrangian with two-body contact interactions or, equivalently, a Hamiltonian with $\delta$-function potentials. In momentum space, the $S$-wave potential can be written

$$V(k', k, p) = b_{00} + b_{20}(k^2 + k'^2) + b_{02} p^2 \cdots,$$

where $k$ and $k'$ denote the initial and final relative momenta and the energy-dependence is expressed in terms of the on-shell momentum $p = \sqrt{ME}$.

Scattering can be described by the reactance matrix, defined similarly to the scattering matrix but with standing-wave boundary conditions. This has the advantage that it is real below the particle-production threshold. For $S$-wave scattering, it satisfies the Lippmann-Schwinger equation

$$K(k', k, p) = V(k', k, p) + M \frac{2\pi}{2\pi^2} P \int_0^{\Lambda} q^2 dq \frac{V(k', q, p) K(q, k, p)}{p^2 - q^2},$$

where $P$ denotes the principal value. This integral equation sums chains of the bubble diagrams in figure 1 to all orders.
With contact interactions, the integral over the momentum $q$ of the virtual states is divergent and so we need to regulate it. Here I follow the method developed by Birse et al. (1999) and simply cut the integral off at $q = \Lambda$. Demanding that the off-shell $K$-matrix be independent of $\Lambda$,

$$\dot{K} \equiv \frac{\partial K}{\partial \Lambda} = 0,$$

ensures that scattering observables will be independent of the arbitrary cut-off. If we write the integral equation for $K$ in the schematic form

$$K = V + VGK,$$  

(3.4)

differentiating it gives

$$0 = \dot{V} + \dot{V}GK + V\dot{G}K,$$

(3.5)

where $\dot{G}$ implies differentiation with respect to the cut-off on the integral. Since this involves the off-shell $K$ matrix, we can use the integral equation for $K$ to convert it into the form

$$\dot{V} = -V\dot{G}V.$$  

(3.6)

This describes the evolution of the potential as the cutoff is lowered and states are integrated out of the effective theory. Written out explicitly, it is

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda).$$

(3.7)

Note that the use of the fully off-shell $K$ matrix was essential to obtaining an equation involving only the potential. The corresponding equation for the evolution of $V_{\text{low-k}}$ (Bogner et al. 2003a, 2003b) is based on the half-off-shell $T$ matrix and so still involves the scattering matrix.

Our equation for the cutoff dependence of the effective potential is still not quite an RG equation: the final step is to express all dimensioned quantities in units of $\Lambda$. Rescaled momentum variables (denoted with hats) are defined by $\hat{k} = k/\Lambda$ etc., and a rescaled potential by

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{MA}{2\pi^2} V(\Lambda\hat{k}', \Lambda\hat{k}, \Lambda\hat{p}, \Lambda).$$

(3.8)

(The factor $M$ in this corresponds to dividing an overall factor of $1/M$ out of the Schrödinger equation.) This satisfies the RG equation

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V}$$

$$+ \hat{V}(\hat{k}', 1, \hat{p}, \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}, \Lambda).$$

(3.9)

The sum of logarithmic derivatives is similar to the structure of analogous RG equations used in other areas of physics; it counts the powers of low-energy scales present in the potential. The boundary conditions on its solutions are that they should be analytic functions of $\hat{k}^2$, $\hat{k}'^2$ and $\hat{p}^2$ (since they should arise from an effective Lagrangian constructed out of $\partial / \partial t$ and $\nabla^2$).

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Having constructed our RG equation, the first thing we need to do is to look for its fixed points – solutions that are independent of $\Lambda$. Let us start with the obvious one, the trivial fixed point:

$$\hat{V} = 0. \quad (3.10)$$

Since this gives no scattering, it obviously describes a scale-free system.

To describe more interesting physics, we need to expand around the fixed point, looking for perturbations that scale with definite powers of $\Lambda$. These are eigenfunctions of the linearised RG equation. They have the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}), \quad (3.11)$$

and they satisfy the eigenvalue equation

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi = \nu \phi. \quad (3.12)$$

Its solutions are

$$\phi(\hat{k}', \hat{k}, \hat{p}) = C \hat{k}^{2l} \hat{p}^{2m}, \quad (3.13)$$

with $k, l, m \geq 0$ since only non-negative, even powers satisfy the boundary condition. The corresponding eigenvalues are

$$\nu = 2(l + m + n) + 1. \quad (3.14)$$

These are all positive and so the fixed point is stable. The eigenvalues $\nu$ simply count the powers of low-energy scales and can be written $\nu = d + 1$ where $d$ is the “engineering dimension” of an operator in the potential.

In addition, there are various nontrivial fixed points, all of which are unstable. The most interesting one is purely energy-dependent. To study it, let us focus on potentials of the form $V(p, \Lambda)$. The RG equation for these can be rewritten in the form of linear equation for $1/\hat{V}(\hat{p}, \Lambda)$,

$$\Lambda \frac{\partial}{\partial \Lambda} \left( \frac{1}{\hat{V}} \right) = \hat{p} \frac{\partial}{\partial \hat{p}} \left( \frac{1}{\hat{V}} \right) - \frac{1}{\hat{V}} - \frac{1}{1 - \hat{p}^2}. \quad (3.15)$$

To find its fixed point, we set the LHS of this equation to zero. The resulting ODE can then be integrated easily. The solution that satisfies our boundary condition is

$$\frac{1}{V_0(\hat{p})} = -1 + \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}}. \quad (3.16)$$

The precise form of this is regulator-dependent, but the presence of a negative constant of order unity is universal.

Since this potential has no momentum dependence, the integral equation for $K$ simplifies to an algebraic equation. In rescaled, dimensionless form, it can be written

$$\frac{1}{K(\hat{p})} = \frac{1}{V_0(\hat{p})} \int_0^1 \frac{\hat{q}^2 \text{d}\hat{q}}{\hat{p}^2 - \hat{q}^2}. \quad (3.17)$$

The integral here is, up to a sign, the same as the one in $1/V_0$ itself and hence we get

$$\frac{1}{K(\hat{p})} = 0. \quad (3.18)$$
The corresponding $T$ matrix has a pole at $p = 0$ and so the fixed-point describes a system with a bound state at exactly zero energy. This is often called the “unitary limit” of two-body scattering and it forms another example of a scale-free system.

More general systems can be described by perturbing around the fixed point. In particular, energy-dependent perturbations can be found by substituting

$$\frac{1}{V(\hat{p}, \Lambda)} = \frac{1}{V_0(\hat{p})} + \Lambda^n \phi(\hat{p})$$

(3.19)

into the RG equation. The functions $\phi(\hat{p})$ satisfy the eigenvalue equation

$$\hat{p} \frac{\partial \phi}{\partial \hat{p}} - \phi = \nu \phi.$$

(3.20)

The solutions to this are powers of the energy,

$$\phi(\hat{p}) = C \hat{p}^{2n},$$

(3.21)

with eigenvalues

$$\nu = 2n - 1.$$  

(3.22)

The RG eigenvalues for these perturbations have been shifted by $-2$ compared to the simple “engineering” power counting. There is one negative eigenvalue and so the fixed point is unstable.

A slice through the RG flow for equation (3.9) is shown in figure 4. The two fixed points can be seen, as well as the critical line through the nontrivial one. As in the generic flow in figure 3, potentials close to this line initially flow towards the fixed point as we lower the cut-off but are then diverted away from it. A potential to the right of the line is not quite strong enough to produce a bound state. As $\Lambda$ passes through the scale associated with the virtual state, the flow turns to approach the trivial fixed point from the weakly attractive side. In contrast, a potential to the left of the critical line generates a finite-energy bound state. This state drops out of our low-energy effective theory when the cut-off reaches the corresponding momentum scale. As this happens, the RG flow takes the potential to infinity and it then reappears from the right, ultimately approaching the trivial fixed point from the weakly repulsive side.

Physical observables are given by the on-shell $K$-matrix. Returning to physical units, this is

$$\frac{1}{K(p)} = \frac{M}{2\pi^2} \sum_{n=0}^{\infty} C_n p^{2n},$$

(3.23)

where the $C_n$ are the coefficients of the RG eigenfunctions in $1/\hat{V}$. Comparing this with

$$\frac{1}{K(p)} = -\frac{Mp}{4\pi} \left(-\frac{1}{a} + 1 + \frac{1}{2} r_e p^2 + \cdots\right),$$

(3.24)

we see that this expansion is just the effective-range expansion (Bethe 1949; Blatt & Jackson 1949). Note that the terms in the expansion of our effective theory correspond directly to scattering observables.

The discussion above deals only with energy-dependent perturbations around the nontrivial fixed point. There are also RG eigenfunctions that depend on the
off-shell momenta. However, in contrast to the expansion around the trivial point, these do not appear at the same orders as the corresponding on-shell terms (Birse et al. 1999). The momentum-dependent terms have larger scaling dimensions and so appear at higher orders in the EFT.

It is worth noting that the promotion of terms in the expansion around the unitary limit can be understood from the form of the wave functions at short distances. Two particles in the unitary limit are described by irregular solutions of the Schrödinger equation. At small radii (in $S$ waves) these behave as $\psi(r) \propto r^{-1}$. Any cutoff smears a contact interaction over range $R \sim \Lambda^{-1}$. If we require observables to be independent of $\Lambda$, we therefore need the extra factor of $\Lambda^{-2}$ in the interaction to cancel the cutoff dependence from $|\psi(R)|^2 \propto \Lambda^2$ in its matrix elements. This provides the “anomalous dimension” of $-2$ for the on-shell contact interactions.

4. Long-range forces

Exactly the same techniques can be applied to renormalise short-range interactions in systems with known long-range forces. Details can be found in the papers of Barford & Birse (2003, 2005) and Birse (2006a, 2006b). The parameters of the resulting EFT again have a direct connection to scattering observables, either via a distorted-wave Born expansion (for weakly interacting systems) or via a distorted-wave effective-range expansion (for strong short-range interactions) (Bethe, 1949; van Haeringen & Kok 1982; Badalyan et al. 1982).

The simplest example of these is the $1/r^2$ centrifugal barrier in partial waves.

Figure 4. RG flow of the potential $\hat{V}(\hat{p}, \Lambda) = b_0(\Lambda) + b_2(\Lambda) \hat{p}^2 + \cdots$. 

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with nonzero angular momentum \( L \) (Barford & Birse, 2003). This leads to wave functions that behave as \( r^L \) for small \( r \) and so, near the trivial fixed point, short-range interactions scale with an additional power of \( \Lambda^{2L} \) compared to \( S \) waves. This matches exactly with the number of derivatives needed to produce a contact interaction with a nonzero contribution in these waves.

The RG flow in these higher partial waves can also end up at fixed points with multiple unstable directions. However these turn out not to be physically realizable, even with enough fine tuning: either the wave functions are not normalisable (Kaplan et al., 2009) or causality is not respected (Hammer & Lee, 2009, 2010).

In three-body systems with short-range pairwise interactions, long-range forces are generated by exchange of one particle between an interacting pair and the third particle. Close to the unitary limit, these can lead to scaling behaviour of three-body interactions that is quite different from what naive dimensional analysis would suggest. The reasons for this can be seen most clearly in position space if we work in hyperspherical coordinates. At short distances the ‘particle-exchange” potential has a \( 1/R^2 \) form, where \( R \) is the hyperradius (the radial coordinate that is zero when all three particles coincide).

In three-body channels where this long-range potential is repulsive, it acts just like a centrifugal term but with a noninteger “angular momentum”. This leads to wave functions that vanish as powers of the hyperradius as it tends to zero and hence to irrational anomalous dimensions for the three-body contact interactions (Birse, 2006b). The values for these have also been obtained from a momentum-space treatment (Griesshammer, 2005) and by Werner and Castin (2006a, 2006b) from the eigenvalues of three-body systems trapped in a harmonic oscillator potential.

Systems of three bosons or three distinct fermions with a completely symmetric spatial wave function behave very differently. Close to the unitary limit, these display the towers of geometrically spaced bound states known as the Efimov effect (Efimov, 1971, 1979). This is a consequence of the attractive \( 1/R^2 \) potential in these systems, which leads to wave functions with the form \( \psi(R) \propto R^{-2.506} \), with \( s_0 \simeq 1.006 \) as \( R \to 0 \). As a result, the leading three-body force is promoted to a marginal term, of order \( Q^{-1} \). The oscillatory behaviour associated with the imaginary part of the exponent is the origin of the Efimov effect. It causes the RG flow in these systems to tend to a limit cycle instead of a fixed point (Bedaque, Hammer & van Kolck, 1999a, 1999b, 2000; Glazek and Wilson, 2004; Barford & Birse, 2005).

In the presence of the Coulomb potential, the scaling behaviour of the short-range interactions is the same as that described in the previous section for systems without long-range forces (Barford & Birse, 2003, Ando & Birse, 2008). This is because the \( 1/r \) singularity of the long-range potential is not strong enough to change the power-law behaviour of the wave functions at short distances. For strongly interacting systems, such as \( pp \) scattering, the resulting EFT embodies the Coulomb distorted-wave effective-range expansion (Kong & Ravndal, 1999, 2000).

Perhaps the most long-range important potential for nuclear physics is one-pion exchange (OPE). Although formally of order \( Q^0 \) within the framework of ChPT, the large value of the pion-nucleon coupling means that it plays a central role in nuclear forces. The unnaturally small value of the scale \( \lambda_{\pi NN} \) associated with OPE suggests that it should be added to our list of low-energy scales. This implies that OPE should be treated nonperturbatively.

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The central piece of OPE is the only one that contributes to scattering in spin-singlet waves. Like the Coulomb potential, this has a $1/r$ singularity, and so it does not alter the power-law forms of the wave functions at small $r$. The scattering in singlet waves with $L \geq 1$ is weak, and so the corresponding effective potential can be expanded using naive dimensional analysis. In contrast, the $^1S_0$ channel has a low-energy virtual state and so the expansion of its short-range potential is like the one around the unitary fixed point.

The tensor piece of OPE is important in spin-triplet waves. It has a much stronger, $1/r^3$, singularity at the origin. The resulting short-distance wave functions have the form $\psi(r) \propto r^{-1/4}$, multiplied by either a sine or an exponential of $(\lambda_{\pi NN} r)^{-1/2}$. As a result, short-range interactions are strongly promoted in these waves and so a new power counting needed, as observed by Nogga et al. (2005). An RG analysis shows that the leading contact interaction is of order $Q^{-1/2}$ in waves with $L = 1$ or 2 (Birse, 2006a). In the $^3S_1$ wave, there is a further enhancement of the short-range interactions, analogous to that in the $^1S_0$ wave, associated with the deuteron bound state.

Turning now to three-nucleon forces, two-pion exchange interactions are purely long-range and so are not renormalised by the effects discussed here. In contrast, other interactions, involving two- or three-nucleon contact operators, are affected by the short-distance behaviour of the wave functions. For example, one-pion exchange terms of the type represented by figure 5, contain two-nucleon-one-pion contact operators and these are substantially enhanced if either the initial or final pair is in an $S$ wave. The most important of these operators is one that couples the

![Figure 5. A three-body OPE interaction.](image)

$^1S_0$ and $^3S_1$ NN channels. This is promoted to order $Q^{5/4}$ by the nonperturbative treatment of the two-body forces in these channels. In addition, there can be strong promotion of operators that couple $S$ and $P$ waves, and of those that couple various combinations of $P$ and $D$ waves. The effect of tensor OPE on three-body contact interactions is currently unknown. The leading term is expected to be promoted, albeit less dramatically than in the case of pure short-range forces. Determining the anomalous dimension for this force will entail solving the three-body problem with $1/r^3$ two-body potentials. I have recently summarised the results obtained from a range of RG studies of two- and three-nucleon forces (Birse, 2009).

Finally I should mention four-body systems. So far at least, these have not yielded to a detailed RG analysis of the sort outlined here. Numerical treatments of these systems have found no signs of promotion of four-body forces to relevant or marginal terms (Platter, Hammer & Meissner, 2004, 2005; Platter & Hammer, 2007) but the anomalous dimensions of these forces remain to be determined. In this
context, the RG for the Legendre-transformed effective action (Wetterich 1993) may provide some help. Applications of this rely on truncations of the effective action to a small number of local terms and, while less rigorous than truncations based on a specific power counting, these have proved remarkably successful for a wide variety of systems (Berges et al. 2002). Recently, the first applications have been made to few-body systems that make it possible to estimate the scaling dimensions of three- and four-body forces in both bosonic and fermionic systems close to the unitary limit (Schmidt & Moroz, 2010; Birse et al., 2010b).

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