A symbolic method for \(k\)-statistics

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Abstract

Through the classical umbral calculus, we provide new, compact and easy to handle expressions of \(k\)-statistics, and more in general of \(U\)-statistics. In addition such a symbolic method can be naturally extended to multivariate case and to generalized \(k\)-statistics.

Keywords: umbral calculus, symmetric polynomials, \(U\)-statistics, \(k\)-statistics, joint cumulants.

1 Introduction

In 1929, Fisher [4] introduced the \(k\)-statistics as new symmetric functions of the random sample. The aim of Fisher was to estimate the cumulants by free-distribution methods without using moment estimators. He used only combinatorial techniques. \(k\)-statistics are related to the power sum symmetric polynomials whose variables are the random variables (r.v.'s) of the sample. The Fisher point of view is described with a wealth of details by Kendall and Stuart in [8]. The method is straightforward enough, however his execution leads to some intricate computations and some cumbersome expressions, except in very simple cases. This is why many authors tried to simplify the matter later.

One of the most relevant contributions has been given by Speed [7] in the Eighty. Speed resumed the Doubilet approach to symmetric functions [3], exploiting symmetric functions labelled by partition of a set rather than by partition of an integer. The entries in the transition matrices are computed via Möbius function generalizing and simplifying the presentation of \(k\)-statistics theory. Nevertheless, in order to extend such theory to generalized \(k\)-statistics, Speed must resort the tensor approach introduced by Kaplan in 1952.

We follow a different point of view by using the high computational potential of the classical umbral calculus. This symbolic method was introduced by Rota and Taylor in 1994 [1] and further developed in [1] and [2]. From a combinatorial perspective, we revisit the Fisher theory as exposed by Kendall and Stuart, taking into account the Doubilet approach to symmetric functions. The umbral calculus offers a nimble syntax method that allows both the computation without using the Möbius function and a natural extension to the multivariate case without bringing the tensor device into. What is more, this language clarifies the role played by the power sum symmetric polynomials in the expressions of \(k\)-statistics.

After recalling in Section 2 the strictly necessary umbral background, in Section 3 we put in umbral setting the four classical bases of symmetric polynomial algebra. Then we proceed in defining a general procedure to write down \(U\)-statistics. Such a procedure is given in full details in Section 5, where analogous formulae for \(k\)-statistics, \(h\)-statistics and multivariate \(k\)-statistics are given.

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2 The umbral calculus language

We start presenting the formal setting of the umbral calculus as introduced by Rota and Taylor in [3] and further developed in [1] and [2]. We shall confine our exposition to what is necessary to the aims of this paper.

The umbral calculus is a syntax consisting of the following data: an alphabet \( A = \{ \alpha, \beta, \ldots \} \) whose elements are named *umbrae*; a commutative integral domain \( R \) whose quotient field is of characteristic zero; a linear functional \( E \), called *evaluation*, defined on the polynomial ring \( R[A] \) and taking values in \( R \) and such that \( E[1] = 1 \), \( E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j] \cdots E[\gamma^k] \) for any set of distinct umbrae in \( A \) and for \( i, j, \ldots, k \) non-negative integers (uncorrelation property); an element \( \epsilon \in A \), called *augmentation*, such that \( E[\epsilon^n] = 0 \) for \( n \geq 1 \); an element \( u \in A \), called *unity* umbra, such that \( E[u^n] = 1 \), for \( n \geq 1 \).

The *support* of an umbral polynomial \( p \in R[A] \) is the set of all umbrae occurring in \( p \). Two umbral polynomials are said to be *uncorrelated* when their supports are disjoint.

An umbra carries the structure of a r.v., while making no reference to a probability space if the evaluation \( E \) is considered as the expectation operator. The *moments* of an umbra \( \alpha \) are the elements \( a_n \in R \) such that \( E[\alpha^n] = a_n \) for \( n \geq 1 \). We said that the umbra \( \alpha \) represents the sequence \( 1, a_1, a_2, \ldots \).

The *singleton umbra* \( \chi \) is the umbra whose moments are all zero, but the first equal to 1. The *factorial moments* of an umbra \( \alpha \) are the elements \( a_{(n)} \in R \) corresponding to the umbral polynomials \( (\alpha)_n = \alpha(\alpha - 1)\cdots(\alpha - n + 1) \), \( n \geq 1 \) via the evaluation \( E \), i.e. \( E[(\alpha)_n] = a_{(n)} \). The Bell umbra \( \beta \) is the umbra such that \( E[(\beta)_n] = 1 \), \( n \geq 1 \). The umbra \( \beta \) and the umbra \( \chi \) allow respectively a symbolic tool in order to handling composition and inversion of formal power series (for a detailed exposition of such two umbrae with their properties see [1] and [2]).

Two umbrae \( \alpha \) and \( \gamma \) are said to be *similar* when \( E[\alpha^n] = E[\gamma^n] \) for all \( n \geq 1 \), and we will write \( \alpha \equiv \gamma \). Then two similar umbrae represent the same moment sequence. Given a sequence \( 1, a_1, a_2, \ldots \) in \( R \) this is represented by infinitely many distinct and thus similar umbrae. Such a circumstance permits to deal with moment products by means of the uncorrelation property. Indeed, it is \( a_i a_j \neq E[\alpha^i \alpha^j] \) with \( a_i = E[\alpha^i] \) and \( a_j = E[\alpha^j] \), as well as \( a_i a_j = E[\alpha^i \alpha^j] \) with \( \alpha \equiv \alpha' \) and \( \alpha' \) uncorrelated with \( \alpha \). So, given \( n \in N \), the sequence

\[
\sum_{k=0}^{n} \binom{n}{k} a_{n-k} a_k
\]

gives the moments of \( \alpha + \alpha' \).

Let \( p \) and \( q \) be two umbral polynomials. We said that \( p \) is *umbrally equivalent* to \( q \) iff \( E[p] = E[q] \), in symbols \( p \simeq q \). This last equivalence relation turns out to be very useful in defining and handling umbra generating functions. We point out that all operations among umbrae correspond to analogous operation in the algebra of generating functions. Nevertheless, in the following we make no mention of generating functions (referring [3] to a formal exposition).

Thanks to the introduction of similarity notion, it is possible to extend the alphabet \( A \) with the so-called *auxiliary umbrae* derived from operations among similar umbrae. This leads to the construction of a *saturated umbral calculus* in which the auxiliary umbrae are treated as elements of the alphabet (cf. [4]). Let \( \alpha', \alpha'', \ldots, \alpha''' \) be \( n \) uncorrelated umbrae similar to an umbra \( \alpha \). The symbol \( n_\alpha \) denotes an auxiliary umbra similar to the sum \( \alpha' + \alpha'' + \cdots + \alpha''' \). The symbol \( \alpha^n \) denotes an auxiliary umbra similar to the product \( \alpha' \alpha'' \cdots \alpha''' \). Properties of such auxiliary umbrae are extensively described in [1] and they will...
be recalled whenever it is necessary. We will assume disjoint both the support of \( n.\alpha, m.\alpha \) and \( \alpha^n, \alpha^m \) whenever \( n \neq m \). If \( p \) and \( q \) are correlated umbral polynomials, then \( n.p \simeq p_1 + \cdots + p_n \) is correlated to \( n.q \simeq q_1 + \cdots + q_n \), and \( p_i \) is correlated to \( q_i \) but uncorrelated to \( q_j \) with \( i \neq j \). In [1], the following identity is stated:

\[
E[(n.\alpha)^i] = \sum_{k=1}^{i} (n)_k B_{i,k}(a_1, a_2, \ldots, a_{i-k+1}) \quad i \geq 1,
\]

(1)

where \( B_{i,k} \) are the (incomplete) exponential Bell polynomials and \( a_i \) is the \( i \)-th moment of \( \alpha \). Moreover, it is easy to verify that \( E[(\alpha^n)^i] = a^n_i \) for \( i \geq 0 \).

Two umbrae \( \alpha \) and \( \gamma \) are said to be inverse to each other when \( \alpha + \gamma = \epsilon \). The inverse of the umbra \( \alpha \) is denoted by \(-1.\alpha\). Note that, in dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two inverse umbrae of the same umbra are similar.

Replacing the integer \( n \) in \( n.\alpha \) with an umbra \( \gamma \), we obtain the auxiliary umbra \( \gamma.\alpha \) whose moments are

\[
E[(\gamma.\alpha)^i] = \sum_{k=1}^{i} g(k) B_{i,k}(a_1, a_2, \ldots, a_{i-k+1}) \quad i \geq 1,
\]

(2)

where \( g(k) \) are the factorial moments of \( \gamma \). In particular \( \beta.\alpha \) is called \( \alpha \)-partition umbra and its moments are the (complete) exponential Bell polynomials (cf. [1]). Moreover \( \chi.\alpha \) is called \( \alpha \)-cumulant umbra and \( \alpha.\chi \) is called \( \alpha \)-factorial umbra, with moments equal to the factorial moments of \( \alpha \) (cf. [2]). In particular it is

\[
\beta.\chi \equiv u \equiv \chi.\beta.
\]

(3)

Again, replacing the umbra \( \gamma \) in \( \gamma.\alpha \) with the umbra \( \gamma.\beta \), we obtain the composition umbra of \( \alpha \) and \( \gamma \), i.e. \( \gamma.\beta.\alpha \). The compositional inverse of an umbra \( \alpha \) is the umbra \( \alpha^{-1} \) such that \( \alpha^{-1}.\beta.\alpha \equiv \chi \equiv \alpha.\beta.\alpha^{-1} \). In particular it is

\[
u^{-1}.\beta \equiv \chi \equiv \nu^{-1}.\beta,
\]

(4)

where \( \nu^{-1} \) denotes the compositional inverse of \( \nu \). Via the umbral Lagrange inversion formula (cf. [1]), the moments of \( \nu^{-1} \) are \( E[(\nu^{-1})^n] = (-1)^n(n-1)! \). Finally it is

\[
\chi.\chi \equiv \nu^{-1} \quad -\chi.(-\chi) \equiv (-\nu)^{-1}.
\]

(5)

The disjoint sum of \( \alpha \) and \( \gamma \) is the umbra whose moments are the sum of \( n \)-th moments \( a_n \) and \( g_n \) of \( \alpha \) and \( \gamma \) respectively, in symbols \( (\alpha + \gamma)^n \simeq \alpha^n + \gamma^n \) (cf. [2]). For instance it turns out

\[
\chi.\alpha + \chi.\gamma \equiv \chi.(\alpha + \gamma)
\]

(6)

that is the well known additive property of cumulants. In the following, we denote by \( +_n \alpha \) the disjoint sum of \( n \) times the umbra \( \alpha \).

3 Umbral symmetric polynomials

A partition of an integer \( m \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \), where \( \lambda_i \) are weakly decreasing and \( \sum_{i=1}^{\ell} \lambda_i = m \). The integers \( \lambda_i \) are said parts of \( \lambda \). A different notation is \( \lambda = (r_1, r_2, \ldots) \), where \( r_i \) is the number of parts of \( \lambda \) equal to \( i \). The monomial symmetric polynomials in the variables \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are

\[
m_\lambda = \sum \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n},
\]

where the sum is over all distinct monomials having exponents \( \lambda_1, \ldots, \lambda_\ell \). When \( \lambda \)
ranges in the set of partition of the integer $m$, $m_\lambda$ is a bases for the algebra of the symmetric polynomials. There are different other bases; we recall just those necessary in the following: the $r$-th power sum symmetric polynomials $s_r = \sum_{i=1}^{n} \alpha_i^r$; the $k$-th elementary symmetric polynomials $e_k = \sum \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_k}$, where the sum is over $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, and the $r$-th complete homogeneous symmetric polynomials $h_r = \sum_{|\lambda|=r} m_\lambda$.

When the umbrae $\alpha_1, \cdots, \alpha_n$ are uncorrelated and similar to each other, these four classical bases of the symmetric polynomial algebra can be represented by means of the umbral polynomial $n.\chi^\alpha$ and its moments. Propositions 1, 3 and 4 are stated under this hypothesis. For the power sum symmetric polynomials it is

$$s_r \simeq n.\alpha^r \simeq n.(\chi^\alpha).$$

Note that $s_r$ are umbrally equivalent to the moments of $\hat{\chi}_n\alpha$ (cf. [2]).

**Proposition 1 (Umbral elementary polynomials)**

$$[n.(\chi\alpha)]^k \simeq \begin{cases} 
  k!e_k, & k = 1, 2, \ldots, n, \\
  0, & k = n + 1, n + 2, \ldots,
\end{cases}$$

(7)

**Proof.** For $k = 1, \ldots, n$ the result follows applying the evaluation $E$ to the multinomial expansion of $[n.(\chi\alpha)]^k \simeq (\chi_1\alpha_1 + \cdots + \chi_n\alpha_n)^k$ and observing that terms having powers of $\chi$ greater than 1 vanish. So just $k!$ monomials of the form $\chi_{j_1}\alpha_{j_1}\chi_{j_2}\alpha_{j_2}\cdots\chi_{j_k}\alpha_{j_k}$ has an evaluation not zero. Instead for $k = n+1, n+2, \ldots$ the result follows observing that at least one power of $\chi$ greater than 1 occurs in each monomial of the multinomial expansion.

**Proposition 2** It is

$$\chi.n.(\chi\alpha) \equiv u^{<-1>}(\hat{\chi}_n\alpha) \quad n.(\chi\alpha) \equiv \beta.\left[u^{<-1>}(\hat{\chi}_n\alpha)\right]$$

(8)

**Proof.** The first equivalence in (8) follows from (9) observing that

$$\chi.n.(\chi\alpha) \equiv \chi.(\chi_1\alpha_1 + \cdots + \chi_n\alpha_n) \equiv \hat{\chi}_n\chi.(\chi\alpha) \equiv \hat{\chi}_n u^{<-1>}\alpha.$$

The second one follows from the first, making the right dot-product with $\beta$ and recalling (8).

Equations (5) are the umbral version of the well-known relations between power sum symmetric polynomials $s_r$ and elementary symmetric polynomials $e_k$. Indeed $s_r$ is umbrally equivalent to the moments of $\hat{\chi}_n\alpha$ and $e_k$ is umbrally equivalent to the moments of $n.(\chi\alpha)$. The umbral expression of $m_\lambda$ requires the introduction of augmented monomial symmetric polynomials $\tilde{m}_\lambda$. Let $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ be a partition of the integer $m$, such polynomials are defined by

$$\tilde{m}_\lambda = \sum_{1 \leq j_1 \neq \cdots \neq j_{r_1} \neq j_{r_1+1} \neq \cdots \neq j_{r_1+r_2} \cdots \leq n} \alpha_{j_1} \cdots \alpha_{j_{r_1}} \alpha_{j_{r_1+1}}^2 \cdots \alpha_{j_{r_1+r_2}}^2 \cdots.$$  

(9)

The proof of the next proposition follows the same patterns of Proposition 1.

**Proposition 3** It is

$$\tilde{m}_\lambda \simeq [n.(\chi\alpha)]^{r_1} [n.(\chi\alpha^2)]^{r_2} \cdots.$$

The next corollary follows recalling that $m_\lambda = \tilde{m}_\lambda/ [r_1! r_2! \cdots]$. 
Corollary 3.1 (Umbral monomial polynomials)

\[ m_\lambda \simeq \frac{[n.(\chi \alpha)]^{r_1}}{r_1!} \frac{[n.(\chi \alpha^2)]^{r_2}}{r_2!} \ldots \]

Proposition 4 (Umbral complete polynomials)

\[-n.(\chi \alpha)^m \simeq m! h_m \quad m = 1, 2, \ldots \]

Proof. It is

\[-n.(\chi \alpha)^m \simeq -1.(\chi_1 \alpha_1) + \cdots + -1.(\chi_n \alpha_n)]^m \simeq m! \sum_{|\lambda|=m} [-1.(\chi_\lambda)]^{r_1} \frac{([-1.(\chi)]^{2})^{r_2} \ldots}{(1!)^{r_1} (2!)^{r_2} \ldots} \tilde{m}_\lambda \]

and the result follows from Corollary 3.1 being \([-1.(\chi)]^{r_1} \simeq (i!)^{r_1} \]

Note that equivalences (7) and (10) are the umbral version of the well-known identities:

\[ \sum_k e_k t^k = \prod_{i=1}^{n} (1 + \alpha_i t) \quad \sum_k h_k t^k = \frac{1}{\prod_{i=1}^{n} (1 - \alpha_i t)}. \]

Proposition 5

It is

\[-\chi n.(\chi \alpha) \equiv (-u)^{-1} > (\chi_\lambda \alpha) \quad -n.(\chi \alpha) \equiv \beta. \left[ (-u)^{-1} > (\chi_\lambda \alpha) \right] \]

Proof. The results follow from (8) replacing the umbra \( \chi \) with \(-\chi \) and recalling that \( u^{-1} \equiv \chi \cdot \chi \) must be replaced with \( (-u)^{-1} \equiv -\chi \cdot (-\chi) \).

Equations (11) are the umbral version of the well-known relations between power sum symmetric polynomials \( s_r \) and complete symmetric polynomials \( h_k \). Indeed \( s_r \) is umbrally equivalent to the moments of \( (\chi_\lambda \alpha) \) and \( h_k \) is umbrally equivalent to the moments of \(-n.(\chi \alpha)\).

4 \( U \)-statistics

Let \( X_1, X_2, \ldots, X_n \) be \( n \) independent r.v.'s. A statistic of the form

\[ U = \frac{1}{(n)_k} \sum \Phi(X_{j_1}, X_{j_2}, \ldots, X_{j_k}) \]

where the sum ranges in the set of all permutations \((j_1, j_2, \ldots, j_k)\) of \( k \) integers, \( 1 \leq j_i \leq n \), is called \( U \)-statistic \( \tilde{5} \). If \( X_1, X_2, \ldots, X_n \) have the same cumulative distribution function \( F(x) \), \( U \) is an unbiased estimator of the population character \( \theta(F) = \int \cdots \int \Phi(x_1, \cdots, x_k) dF(x_1) \cdots dF(x_k) \). In this case, the function \( \Phi \) may be assumed to be a symmetric function of its arguments. Often, in the applications, \( \Phi \) is a polynomial in \( X_i \)'s so that the \( U \)-statistic is a symmetric polynomial. By virtue of the fundamental theorem on symmetric polynomials, the \( U \)-statistic can be considered a polynomial in the elementary symmetric polynomials. The following theorem is an umbral reformulation of the above statement.

Theorem 4.1 (\( U \)-statistic) If \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \) is a partition of the integer \( m \leq n \) then

\[(\alpha_{1^{j_1}})^{r_1}(\alpha_{2^{j_2}})^{r_2} \cdots \simeq \frac{1}{(n)_k} [n.(\chi \alpha)]^{r_1} [n.(\chi \alpha^2)]^{r_2} \cdots \]

where \( j_i \in \{1, 2, \ldots, n\} \) and \( \sum_j r_j = k \).
Proof. The result follows observing that \([n.(\chi^\alpha)]_{\alpha \in \mathbb{N}} \simeq (n.\chi)^{r_1}(\alpha^{r_2})^{r_2} \text{ and } (n.\chi)^k \simeq (n)^k\). □

The formula \([\mathbb{E}]\) states how to estimate moment products by using only \(n\) information drawn out the population. Then the symmetric polynomial on the right side of \([\mathbb{E}]\) is the \(U\)-statistic of the uncorrelated and similar umbre \(\alpha_1, \alpha_2, \ldots, \alpha_n\) associated to \((\alpha_{j_1})^{r_1}((\alpha_{j_2})^{r_2})^{r_2} \cdots \).

**Example 4.1 Moment powers.** Set \(r_1 = 2\) and \(k = 2\), from \([\mathbb{E}]\) the \(U\)-statistic associated to \(\alpha^2 \simeq a_1^2\) is

\[
\alpha^2 \simeq \frac{1}{n(n-1)}[n.(\chi\alpha)]^2 \simeq \frac{1}{n(n-1)} \sum_{i \neq j} \alpha_i \alpha_j, \quad n \geq 2.
\]

**Example 4.2 \(h\)-statistics.** As it has been shown in \([\mathbb{E}]\), it is

\[
(a^\alpha)^r \simeq \sum_{k=0}^{r} \binom{r}{k} (-1)^k \alpha^k (\alpha')^{r-k} \simeq \sum_{k=0}^{r-2} \binom{r}{k} \sum_{k=0}^{r} \alpha^k (\alpha')^{r-k}
\]

where \(\alpha^a\) is the central umbra of \(\alpha\) about \(a_1 = E[\alpha]\) and \(\alpha' \equiv \alpha\) is an umbra uncorrelated with \(\alpha\). Replacing the product \(\alpha^k (\alpha')^{r-k}\) with the corresponding \(U\)-statistic \([\mathbb{E}]\), it results

\[
(a^\alpha)^r \simeq \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{[n.(\chi\alpha)]^k}{(n)^{k+1}} n. (\chi\alpha^{r-k}). \quad (13)
\]

When the products \([n.(\chi\alpha)]^k n. (\chi\alpha^{r-k})\) are expressed in terms of power sum symmetric polynomials, we have the so called \(h\)-statistics. On this matter, we will give more details in the next section.

### 5 \(k\)-statistics

The \(n\)-th \(k\)-statistic \(k_n\) is the unique symmetric unbiased estimator of the \(n\)-th cumulant \(\kappa_n\) of a given statistical distribution, i.e. \(E[k_n] = \kappa_n\). The \(k\)-statistics can be expressed in terms of the sums of the \(r\)-th powers of the data points. In this section we give an umbral syntax that provides a general computational method to generate such expressions. To this aim, we digress to introduce the exponential Bell umbral polynomials.

The most widespread expression of incomplete exponential Bell polynomials is referred to partition of an integer. Of course, it is also possible to express such polynomials referring them to partition of a set. Here we follow this last point of view. We denote by \(\Pi_{i,k}\) the set of all partitions of the set \([i] = \{1, 2, \ldots, i\}\) in \(k\) blocks. Let \(\pi = \{A_1, A_2, \ldots, A_k\}\) be an element of \(\Pi_{i,k}\). Then it is \(B_{i,k}(a_1, a_2, \ldots) = \sum_{\pi \in \Pi_{i,k}} a_{n_1} a_{n_2} \cdots a_{n_k}\) where \(|A_j| = n_j, j = 1, 2, \ldots, k\), as we will suppose from now on. Let us consider the following symmetric umbral polynomial:

\[
B_{i,k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum_{\pi \in \Pi_{i,k}} \alpha_1^{n_1} \alpha_2^{n_2} \cdots \alpha_k^{n_k}, \quad (14)
\]

where \(\alpha_1, \alpha_2, \ldots, \alpha_k\) are uncorrelated umbre similar to \(\alpha\) and \(a_{n_j} = E[\alpha^{n_j}], j = 1, 2, \ldots, k\). Obviously it is \(E[B_{i,k}] = B_{i,k}\), so that any expression containing the polynomials \(B_{i,k}\) could be replaced with an umbrally equivalent expression containing the polynomials \(B_{i,k}\). The polynomials \(B_{i,k}\) will be called (incomplete) umbral exponential Bell polynomials. The combinatorics underlain the polynomial \(B_{i,k}\) is the following: the set \([i]\) is partitioned in \(k\) blocks, to each of them one associates the umbra \(\alpha^{n_j}\) obtained firstly replacing the elements in the \(j\)-th block with the umbra \(\alpha\) and then labelling all blocks so that powers belonging
to different blocks result uncorrelated. Replacing in \( \text{(14)} \) the products \( \alpha_1^{n_1} \alpha_2^{n_2} \cdots \alpha_k^{n_k} \) with the umbrally corresponding \( U \)-statistic \( \text{(12)} \), we get for \( i \leq n \)

\[
\mathcal{B}_{i,k}(\alpha_1, \alpha_2, \cdots, \alpha_k) \simeq \sum_{\pi \in \Pi_{i,k}} \frac{1}{(n)_k} n. (\chi \alpha^{n_1}) n. (\chi \alpha^{n_2}) \cdots n. (\chi \alpha^{n_k}), \tag{15}
\]

by which we are able to give the umbral \( k \)-statistics. Indeed, the \( \alpha \)-cumulant umbra \( \kappa_\alpha \) is similar to \( \chi \alpha \), so that, being \( (\chi)_k \simeq (u^{<1>} k) \), from \( \text{(12)} \) and \( \text{(15)} \) it is

\[
(\chi \alpha)^i \simeq \sum_{k=1}^{i} (-1)^{k-1} \frac{(k-1)!}{(n)_k} \sum_{\pi \in \Pi_{i,k}} n. (\chi \alpha^{n_1}) n. (\chi \alpha^{n_2}) \cdots n. (\chi \alpha^{n_k}). \tag{16}
\]

Since \( n. (\chi \alpha^{n_k}) \) is umbrally equivalent to a symmetric power sum polynomial, equation \( \text{(16)} \) gives the moments of the \( \alpha \)-cumulant umbra in terms of power sum polynomials, i.e. the umbral form of \( k \)-statistics. Note that the symmetric power sum polynomials in \( \text{(16)} \) are correlated. So, in order to make formula \( \text{(16)} \) effective, we need a device by which to evaluate the product \( n. (\chi \alpha^{n_1}) n. (\chi \alpha^{n_2}) \cdots n. (\chi \alpha^{n_k}) \).

To this aim, by using the umbral exponential Bell polynomials \( \text{(14)} \), the moments of the umbra \( n. (\chi \alpha) \) can be evaluated from \( \text{(2)} \) recalling \( n. \alpha^{n_1} = (\pm \alpha)^{n_1} \) and the second equivalence in \( \text{(8)} \)

\[
[n. (\chi \alpha)]^i \simeq \sum_{k=1}^{i} \sum_{\pi \in \Pi_{i,k}} (u^{<1>} n_1) \cdots (u^{<1>} n_k) [n. \alpha^{m_1}] [n. \alpha^{m_2}] \cdots [n. \alpha^{m_k}]. \tag{17}
\]

The previous umbral equivalence is suitable to be generalized to the product of umbral polynomials \( [n. (\chi p_1)] [n. (\chi p_2)] \cdots [n. (\chi p_i)] \) with no disjoint support. Indeed, it results

\[
[n. (\chi p_1)] \cdots [n. (\chi p_i)] \simeq \sum_{k=1}^{i} \sum_{\pi \in \Pi_{i,k}} (u^{<1>} n_1) \cdots (u^{<1>} n_k) [n. P_{A_1}'] \cdots [n. P_{A_k}']. \tag{18}
\]

where \( P_{A_j} = \prod_{i=1}^{p_j} p_{j_i} \) and \( p_{j_i} \) are the polynomials indexed by the elements of the block \( A_j \), as we will suppose from now on. Equivalence \( \text{(18)} \) is the device required to generate \( k \)-statistics. Indeed, setting \( i = k \) and \( p_i = \alpha^{n_i} \) for \( t = 1, 2, \cdots, i \), it results:

\[
[n. (\chi \alpha^{n_1})] \cdots [n. (\chi \alpha^{n_k})] \simeq \sum_{j=1}^{k} \sum_{\pi \in \Pi_{k,j}} (u^{<1>} n_1) \cdots (u^{<1>} n_j) [n. \alpha^{m_1}] \cdots [n. \alpha^{m_j}]. \tag{19}
\]

where \( m_j = \sum_{i=1}^{n_j} n_{j_i} \) and \( n_{j_i} \) are indexed by the elements of the block \( A_j \). Note that the power sum polynomials on the right side of \( \text{(19)} \) are now uncorrelated so such equivalence gives augmented monomial symmetric polynomials in terms of power sum polynomials, translating Kendall and Stuart tables read downwards \( \text{(5)} \). Instead the following formula leads to a symbolic translation of the Kendall and Stuart tables read across (i.e. power sum polynomials in terms of augmented monomial ones):

\[
(n. p_1) \cdots (n. p_i) \simeq \sum_{k=1}^{i} \sum_{\pi \in \Pi_{i,k}} P_{A_1}' \cdots P_{A_k}' \simeq \sum_{k=1}^{i} \sum_{\pi \in \Pi_{i,k}} n. (\chi P_{A_1}) \cdots n. (\chi P_{A_k}). \tag{20}
\]

The first equivalence in \( \text{(20)} \) is obtained from \( \text{(11)} \) trough analogous considerations used to state \( \text{(18)} \); the second equivalence comes from \( \text{(12)} \) replacing \( \alpha \chi \) with the umbral polynomial \( p_i \), i.e.

\[
P_{A_1}' P_{A_2}' \cdots P_{A_k}' \simeq \frac{1}{(n)_k} n. (\chi P_{A_1}) n. (\chi P_{A_2}) \cdots n. (\chi P_{A_k}).
\]
Set in \( p_t = \alpha^n \) for \( t = 1, 2, \ldots, i \), we have

\[
(n.\alpha^n_1) \cdots (n.\alpha^n_i) \simeq \sum_{k=1}^{i} \sum_{\pi \in \Pi_{s,k}} (n.\chi\alpha^{m_1})(n.\chi\alpha^{m_2}) \cdots (n.\chi\alpha^{m_k})
\]

(21)

where \( m_j = \sum_{j=1}^{n_j} n_j \), and \( n_j \) are indexed by the elements of the block \( A_j \).

**Example 5.1** \( h \)-statistics. In \( \chi.P \) set \( n_1 = n - k \) and \( n_2 = \cdots = n_{k+1} = 1 \); from \( \chi.Q \) we get the umbral expression of \( h \)-statistics.

**Example 5.2** Joint cumulants. Let \( p_1, p_2, \ldots, p_i \) be umbral polynomials. Replacing \( n \) with \( \chi \) in the first equivalence in \( \chi.P \), we have

\[
(\chi.p_1)(\chi.p_2) \cdots (\chi.p_i) \simeq \sum_{k=1}^{i} (\chi)_k \sum_{\pi \in \Pi_{s,k}} P'_{A_1} \cdots P''_{A_k}.
\]

(22)

When the umbral polynomials \( p_i \) are interpreted as r.v.’s, equivalence \( \chi.P \) gives their joint cumulants. So we will call \( (\chi.p_1)(\chi.p_2) \cdots (\chi.p_i) \) the joint cumulant of \( p_1, \ldots, p_i \). Note that, setting \( p_i = \alpha \) for \( t = 1, 2, \ldots, i \), one has the \( i \)-th ordinary cumulant \( (\chi.\alpha)^i \). Through this equivalence it results \( \chi.(P + Q) \equiv \chi.p_1 + \cdots + \chi.p_i \). Now suppose to split the set \( \{p_1, p_2, \ldots, p_i\} \) in two subsets \( \{p_j, \ldots, p_k\} \) and \( \{p_k, \ldots, p_s\} \) with \( s + t = i \), such that polynomials belonging to different subsets are uncorrelated. Then it is

\[
(\chi.p_1)(\chi.p_2) \cdots (\chi.p_i) \simeq 0.
\]

(23)

Indeed, setting \( P = \sum_{i=1}^{t} p_{j_i} \) and \( Q = \sum_{s=1}^{t} p_{k_s} \), such polynomials are uncorrelated, so that \( \chi.(P + Q) \equiv \chi.P + \chi.Q \) from \( \chi.P \). Equivalence \( \chi.P \) follows observing that, due to the disjoint sum, products involving powers of \( \chi.P \) and \( \chi.Q \) vanish. When the umbral polynomials \( p_i \) are interpreted as r.v.’s, equivalence \( \chi.P \) states the following well-known result: if some of the r.v.’s are uncorrelated of all others, then their joint cumulant is zero.

**Example 5.3** Multivariate \( k \)-statistics. Equivalence \( \chi.P \) allows a compact expression of multivariate \( k \)-statistics. Replacing \( n \) with \( \chi \) in its second equivalence, we construct the \( U \)-statistic of the joint cumulant

\[
(\chi.p_1)(\chi.p_2) \cdots (\chi.p_i) \simeq \sum_{k=1}^{i} (\chi)_k \sum_{\pi \in \Pi_{s,k}} n.(\chi P_{A_1}) n.(\chi P_{A_2}) \cdots n.(\chi P_{A_k}).
\]

(24)

Again, in the product on the right side of \( \chi.P \) the umbral polynomials are correlated. In order to make effective the computation, it is necessary to rewrite \( \chi.P \) by using equivalence \( \chi.P \) with \( P_{A_i} \) instead of \( p_i \).

For instance, in order to express \( k_{21} \), set in \( \chi.P \) \( i = 3 \) and \( p_1 = p_2 = \alpha_1, p_3 = \alpha_2 \). It results

\[
(\chi.\alpha_1^2)(\chi.\alpha_2) \simeq \frac{\chi}{n} n.(\alpha_1^2 \alpha_2) + \frac{\chi}{(n)_2} \left\{ 2 n.(\alpha_1) n.(\alpha_1 \alpha_2) + n.(\alpha_1^2) n.(\alpha_2) \right\} \\
+ \frac{(\chi)_3}{(n)_3} \left\{ n.(\alpha_1)^2 n.(\alpha_2) \right\}
\]

(25)

Set \( s_{p,q} \simeq n.(\alpha_1^p \alpha_2^q) \). It is

\[
n.(\alpha_1) n.(\alpha_1 \alpha_2) \simeq (u^{-1})^2 n.(\alpha_1^2 \alpha_2) + (u^{-1})^2 n.\alpha_1' n.(\alpha_1 \alpha_2) \simeq -s_{2,1} + s_{1,0} s_{1,1}
\]

(26)

\[
n.(\alpha_2^2) n.(\alpha_2) \simeq (u^{-1})^2 n.(\alpha_1^2 \alpha_2) + (u^{-1})^2 n.\alpha_1^2 n.\alpha_2 \simeq -s_{2,1} + s_{2,0} s_{0,1}
\]

(27)

\[
n.(\alpha_1) n.(\alpha_1 \alpha_2) \simeq (u^{-1})^3 n.(\alpha_1^2 \alpha_2) + (u^{-1})^3 n.\alpha_1' n.\alpha_1 n.\alpha_2 \\
+ u^{-1} n.(u^{-1})^2 [n.\alpha_1^2 n.\alpha_2 + 2 n.\alpha_1 n.(\alpha_1 \alpha_2)] \\
\simeq 2 s_{2,1} - s_{2,0} s_{0,1} - 2 s_{1,0} s_{1,1} + s_{1,0}^2 s_{0,1}
\]

(28)
Equivalence (26) comes from (18) setting $i = 2$, $p_1 = \alpha_1$ and $p_2 = \alpha_1 \alpha_2$, equivalence (27) comes from (18) setting $i = 2$, $p_1 = \alpha_2$ and $p_2 = \alpha_2$, equivalence (28) comes from (18) setting $i = 3$, $p_1 = p_2 = \alpha_1$ and $p_3 = \alpha_2$. Substituting the above equivalences in (25) and rearranging the terms, we have the expression of $k_{21}$,

$$k_{21} \simeq (\chi \cdot \alpha_1)^2 (\chi \cdot \alpha_2) \simeq \frac{1}{(n)_3} \left[ n^2 s_{2,1} - 2 n s_{1,0} s_{1,1} - n s_{2,0} s_{0,1} + 2 s_{1,0}^2 s_{0,1} \right].$$

The expression of generalized $k$-statistics (as well as the multivariate ones) in terms of power sums comes from (22) replacing some of the umbrae $\chi$ with uncorrelated ones and then constructing the corresponding $U -$statistics.

References

[1] E. Di Nardo, D. Senato, ‘Umbral nature of the Poisson random variables’, In Algebraic combinatorics and computer science (eds. Crapo H. and Senato D.), Springer Italia (2001), 245–266.

[2] E. Di Nardo, D. Senato, ‘An umbral setting for cumulants and factorial moments’, Europ. Jour. Combinatorics, (2005) to appear.

[3] P. Doubilet, ‘On the foundations of combinatorial theory VII: Symmetric functions through the theory of distribution and occupancy’, Stud. Appl. Math. 11, (1972) 377–396.

[4] R.A. Fisher, ‘Moments and product moments of sampling distributions’, Proc. London Math. Soc. (2) 30, (1929) 199–238.

[5] W. Hoeffding, ‘A class of statistics with asymptotically normal distribution’, Ann. Math. Stat. 19, (1948) 293–325.

[6] G.-C. Rota, B.D. Taylor, ‘The classical umbral calculus’, SIAM J. Math. Anal. 25, (1994) 694–711.

[7] T. P. Speed, ‘Cumulants and partition lattices II: Generalized $k$-statistics’, J. Aust. Math. Soc., Ser. A 40, (1986) 34–53.

[8] A. Stuart, J.K. Ord, Kendall’s Advanced Theory of Statistics, Vol. 1, Charles Griffin and Company Limited, London, (1987).

[9] B.D. Taylor, ‘Difference equations via the classical umbral calculus’ In Mathematical Essays in Honor of Gian-Carlo Rota (eds. Sagan et al.), Birkhauser Boston (1998), 397–411.