A NOTE ON THE J INVARIANT AND FOLIATIONS

OMEÑAR CALVO-ANDRADE AND FERNANDO CUKIERMAN

ABSTRACT. In this note we analyse the Exceptional Component of the space of integrable forms of degree two, introduced by Cerveau-Lins Neto, in terms of the geometry of Veronese curves and classical invariant theory.

1. Introduction

(1.1) Let $r$ and $d$ be natural numbers. Consider a differential 1-form in $\mathbb{C}^{r+1}$

$$\omega = \sum_{i=0}^{r} a_i dx_i$$

where the $a_i$ are homogeneous polynomials of degree $d + 1$ in variables $x_0, \ldots, x_r$, with complex coefficients. Let us assume that

$$\sum_{i=0}^{r} a_i x_i = 0$$

so that $\omega$ descends to the complex projective space $\mathbb{P}^r$ and defines a global section of the twisted sheaf of 1-forms $\Omega_{\mathbb{P}^r}^1 (d + 2)$.

We thank Jorge Vitório Pereira for some useful communications.
(1.2) For a $K$-vector space $V$ we denote $\mathbb{P}V = V - \{0\}/K^*$ the projective space of one-dimensional linear subspaces of $V$ and $\pi : V - \{0\} \to \mathbb{P}V$ the quotient map.

Consider the projective space $\mathbb{P}H^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d + 2))$ and the subset

$$\mathcal{F}(r, d) = \pi(\{\omega \in H^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d + 2)) - \{0\}/ \omega \wedge d\omega = 0\})$$

parametrizing 1-forms $\omega$ that satisfy the Frobenius integrability condition.

(1.3) It is clear that $\mathcal{F}(r, d)$ is an algebraic subset defined by quadratic equations. It is the space of degree $d$ foliations of codimension one on $\mathbb{P}^r$. One important problem of the area is to determine the irreducible components of $\mathcal{F}(r, d)$.

To fix notation, let us recall (see [CL]) the following families of irreducible components:

a) The rational components $\mathcal{R}(d_1, d_2) \subset \mathcal{F}(r, d)$ consisting of the 1-forms of type

$$\omega = p_1 F_2 \ dF_1 - p_2 F_1 \ dF_2$$

where $d + 2 = d_1 + d_2$ is a partition with $d_1, d_2$ natural numbers, $p_1, p_2$ are the unique coprime natural numbers such that $p_1d_1 = p_2d_2$ and $F_1, F_2$ are homogeneous polynomials of respective degrees $d_1, d_2$.

b) The logarithmic components $\mathcal{L}(d_1, \ldots, d_s) \subset \mathcal{F}(r, d)$ consisting of the 1-forms of type

$$\omega = \left( \prod_{j=1}^{s} F_j \right) \sum_{i=1}^{s} \lambda_i \ dF_i / F_i = \sum_{i=1}^{s} \left( \prod_{j \neq i} F_j \right) \lambda_i \ dF_i$$

where $s \geq 3$, the $d_i$ are integers such that $d + 2 = \sum_{i=1}^{s} d_i$, the $F_i$ are homogeneous polynomials of degree $d_i$ and $\lambda_i$ are complex numbers, not all zero, such that $\sum_{i=1}^{s} d_i \lambda_i = 0$.

c) The linear pull-back components $\mathcal{PBL}(d) \subset \mathcal{F}(r, d)$ consisting of the 1-forms of type

$$\omega = \pi^* \eta$$

where $\pi : \mathbb{C}^{r+1} \to \mathbb{C}^3$ is a non-degenerate linear map and $\eta$ is a global section of $\Omega^1_{\mathbb{P}^2}(d + 2)$. 
(1.4) The problem of determining the irreducible components of $\mathcal{F}(r, d)$ was solved by D. Cerveau and A. Lins Neto in [CL] for $d = 2$. These authors defined an irreducible component $\mathcal{E} \subset \mathcal{F}(3, 2)$, called "the exceptional component". The leaves of a typical foliation in $\mathcal{E}$ are the orbits of a linear action in $\mathbb{P}^3$ of the affine group in one variable. The main theorem of [CL] states that the irreducible components of $\mathcal{F}(r, 2)$ are the corresponding rational, logarithmic and linear pull-backs components and the component $\mathcal{E}_r$ obtained by linear pull-backs $\mathbb{P}^r \to \mathbb{P}^3$ of $\mathcal{E} = \mathcal{E}_3$.

(1.5) The purpose of this note is to give another description of $\mathcal{E}$, emphasizing the role of the $j$ invariant of a binary quartic; see also [CD], example (2.4.8), page 36. We shall consider the codimension one foliation on $\mathbb{P}^4$ induced by the natural action of $PGL(2, \mathbb{C})$ and will obtain $\mathcal{E}$ by restricting to a suitable hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$, namely, an osculating hyperplane of the Veronese curve.

(1.6) We introduce here some notation that will be useful later. Let $\omega \in H^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d + 2))$ as above. Denote

$$S(\omega)$$

the variety of zeros of $\omega$ and $S_k(\omega)$ the union of the irreducible components of $S(\omega)$ of dimension $k$. If $S_{r-1}(\omega)$ is non-empty (i.e. if $\omega$ vanishes in codimension one) then there exists a homogeneous polynomial $F$ of maximal degree $0 < e < d$ that divides $\omega$. We denote

$$\bar{\omega} = \omega / F \in H^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d - e + 2))$$

It is clear that $\bar{\omega}$ is well defined up to multiplicative constant and it does not vanish in codimension one.

2. Let us recall some known facts about the Veronese curves in $\mathbb{P}^4$, following [D] and [H].

(2.1) Our model for $\mathbb{P}^4$ will be the projective space of binary quartics.

Let $V$ be a two dimensional vector space over $\mathbb{C}$ and for each natural number $r$ denote

$$P(r) = \mathbb{P} \text{Sym}^r(V)$$

the projective space associated to the $r + 1$-dimensional vector space $\text{Sym}^r(V)$.
The general linear group $G = GL(V)$ acts on $V$ and hence naturally acts on $\text{Sym}^r(V)$ and on $P(r)$.

(2.2) Consider the Veronese map

$$\nu_r : P(1) \to P(r)$$

obtained by sending $v \in V$ to $v^r \in \text{Sym}^r(V)$. It is clear that $\nu_r$ is $G$-equivariant. The image $X_r = \nu_r(P(1))$ is called $r$-th Veronese curve. Notice that $G$ acts linearly on $P(r)$ and preserves $X_r$. Hence $G$ also preserves the tangential and secant varieties of $X_r$.

To write this out in coordinates, let $t_0, t_1$ denote a basis of $V$. Then \( \{ t_0^{r-i}t_1^i, \ i = 0, \ldots, r \} \) is a basis of $\text{Sym}^r(V)$ and a typical element of $\text{Sym}^r(V)$ is a binary form

$$F = \sum_{i=0}^{r} a_i t_0^{r-i}t_1^i$$

Thus $\nu_r$ is defined by

$$\nu_r(a_0t_0 + a_1t_1) = (a_0t_0 + a_1t_1)^r = \sum_{i=0}^{r} \binom{r}{i} a_0^{r-i}a_1^i t_0^{r-i}t_1^i$$

By assigning to a homogeneous polynomial $F \in \text{Sym}^r(V)$ its roots counted with multiplicity, we may conveniently think of $P(r)$ as the set of effective divisors of degree $r$ in $P(1)$.

In these terms, a typical point in $X_r$ is a divisor of the form $r.p$ for some $p \in P(1)$. The tangent line to $X_r$ at this point is the set of divisors of the form $(r-1)p+q$ for $q \in P(1)$. More generally, the osculating $k$-plane to $X_r$ at $r.p$ is the set of divisors of the form $(r-k)p+A$ where $A$ is any effective divisor of degree $k$ in $P(1)$.

(2.3) Let $r = 4$. The orbits of $G$ in $P(4)$ are:

a) $X_4 = \{ 4p/ p \in P(1) \} = \text{binary forms with a four-fold root} = \text{Veronese curve}$. It is the unique closed orbit.

b) $T = \{ 3p + q/ p, q \in P(1), p \neq q \} = \text{binary forms with a triple root}$. The closure $\bar{T}$ equals
the tangential surface (union of all tangent lines) of $X_4$.

c) $N = \{2p + 2q / p, q \in P(1), p \neq q\} =$ binary forms with two double roots = set of points of intersection of pairs of osculating 2-planes of $X_4$.

d) $\Delta = \{2p + q + r / p, q, r \text{ distinct points of } P(1)\} =$ binary forms with one double root. The closure $\bar{\Delta}$ is the discriminant hypersurface, consisting of binary forms that have a multiple root. It equals the union of all osculating 2-planes of $X_4$.

Notice that $\bar{T} \cup \bar{N} \subset \bar{\Delta}$ and $\bar{T} \cap \bar{N} = X_4$.

e) Denote $U = \{p + q + r + s / p, q, r, s \text{ distinct points of } P(1)\} = P(r) - \bar{\Delta}$ the open set of binary forms with simple roots. Then $U$ is a disjoint union of infinitely many $G$-orbits. More precisely, one has the classical function

$$j : U \to \mathbb{C}$$

such that the orbits are the sets $j^{-1}(t)$ for $t \in \mathbb{C}$. Thus, the foliation on $U$ with leaves the $G$-orbits is defined by the differential 1-form $dj$.

(2.4) As in [D], an explicit formula for $j$ may be written in the form

$$j = \frac{Q^3}{D}$$

where $Q, D$ are the homogeneous polynomials of respective degrees 2 and 6 given by :

$$Q = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$C = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3$$
$$D = Q^3 - 27C^2$$

Here $D$ is the discriminant of a binary quartic, so that $\bar{\Delta}$ is the set of zeros of $D$.

$Q$ is an equation for the unique $G$-invariant quadric containing $X_4$.

The cubic $C$ is called the catalecticant and is an equation for the secant variety Sec $X_4$ of
the Veronese curve.

It is also true that \( \{ Q, C \} \) generate the ring of invariants, but we will not use this fact here.

We may consider \( j \) as a rational function \( j : P(4) \to \mathbb{P}^1 \), regular in the complement of the base locus \( (C = Q = 0) \). In other terms, \( j \) is the rational map to \( \mathbb{P}^1 \) defined by the pencil of sextic hypersurfaces in \( P(4) \) spanned by \( \{ Q^3, D \} \) (or, equivalently, by \( \{ Q^3, C^2 \} \)). Notice that all the sextics of this linear system are singular along the base locus.

(2.5) There are three fibers of \( j \) that deserve special attention:

\[
\begin{align*}
    j^{-1}(0) &= (Q = 0) \\
    j^{-1}(\infty) &= (D = 0) = \bar{\Delta} \\
    j^{-1}(1728) &= (C = 0) = \text{Sec } X_4
\end{align*}
\]

Taking account of multiplicities and writing \( j^* \) for pull-back of divisors, we have:

\[
\begin{align*}
    j^*(0) &= 3(Q = 0) \\
    j^*(\infty) &= (D = 0) \\
    j^*(1728) &= 2(C = 0)
\end{align*}
\]

The fiber at \( \infty \) is reduced and irreducible, but it is singular in codimension one. In fact,

\[
\text{Sing } (\bar{\Delta}) = \bar{T} \cup \bar{N}
\]

and, more precisely, \( \bar{\Delta} \) is cuspidal along \( T \) and nodal along \( N \).

Since each orbit is smooth and irreducible and each fiber of \( j \) is a union of orbits, it follows from the description of the orbits in (2.3) that all other fibers of \( j \) are irreducible and smooth away from the base locus.

(2.6) Consider the codimension-one singular foliation \( F \) in \( P(4) \) with leaves the fibers of \( j \), that is, \( F \) is the singular foliation induced by the natural action of \( PGL(2, \mathbb{C}) \) on \( P(4) \). It will be convenient to consider the rational function

\[
    j' = \frac{j}{27(j - 1)} = \frac{Q^3}{C^2}
\]
Since \( j' \) and \( j \) differ only by an automorphism of \( \mathbb{P}^1 \), they define the same foliation. Therefore, the foliation \( F \) is defined by the differential form
\[
\omega = 3QdC - 2CdQ
\]
and hence belongs to the irreducible component \( \mathcal{R}(2, 3) \subset \mathcal{F}(4, 3) \).

In order to describe the zeros of \( \omega \), we need another fact about the geometry of \( X_4 \).

(2.7) Proposition: As in (2.3), let \( \tilde{T} \) denote the tangential surface of the Veronese curve \( X_4 \). Then \( \tilde{T} = (C = 0) \cap (Q = 0) \) and the intersection is generically transverse. In particular, the base locus of the pencil \( j \) is the tangential surface \( \tilde{T} \).

Proof: We have \( \tilde{T} \subset (C = 0) \) since in general the tangent variety is contained in the secant variety. The inclusion \( \tilde{T} \subset (Q = 0) \) follows from direct calculation with the formula for \( Q \) in (2.4) or by [FH], Ex. 11.32. Then, \( \tilde{T} \subset (C = 0) \cap (Q = 0) \). On the other hand, it is shown in [H], p. 245, that the tangential surface of the Veronese curve \( X_r \subset P(r) \) has degree \( 2r - 2 \). Therefore \( \tilde{T} \) has degree 6. Since \( (C = 0) \cap (Q = 0) \) also has degree 6, the desired equality and transversality hold.

(2.8) Proposition: \( S(\omega) = \tilde{T} \cup \tilde{N} \). In particular, all the irreducible components of \( S(\omega) \) are of codimension two in \( P(4) \).

Proof: The zeros of \( \omega \) consist of the base locus \( (C = 0) \cap (Q = 0) = \tilde{T} \) and of the singularities of the fibers of \( j \). We know from (2.5) that the fiber at \( \infty \) is singular along \( \tilde{N} \) and the fibers \( j^{-1}(t) \) are smooth away from the base locus for \( t \notin \{0, 1728, \infty\} \); this implies the result.

(2.9) Now we consider the restriction \( F_H \) of \( F \) to a hyperplane \( H \subset P(4) \). The singularities of \( F_H \) are:

a) the intersections with \( H \) of the singularities of \( F \), and

b) the tangencies of \( F \) and \( H \) (that is, the loci of contact of the leaves of \( F \) not transverse to \( H \)).

Denoting \( \omega_H \) the 1-form in \( H \) obtained by restriction of \( \omega \), the foliation \( F_H \) is defined by \( \omega_H \), with notation as in (1.6).
If $H$ is a general hyperplane then $\omega_H$ does not vanish in codimension one. Hence $F_H$ is defined by $\omega_H$ and is a rational foliation of type $\mathcal{R}(2,3)$ in $H \cong \mathbb{P}^3$. In particular $F_H$ is a degree 3 foliation.

(2.10) Now we analyze $F_H$ when $H$ is an osculating hyperplane to the Veronese curve. Let $p \in X_4$ be a point and consider the osculating flag of $X$ at $p$:

$$\mathbb{P}^1_p = \{3p+q, q \in \mathbb{P}^1\} \subset \mathbb{P}^2_p = \{2p+q+r, q, r \in \mathbb{P}^1\} \subset H = \mathbb{P}^3_p = \{p+q+r+s, q, r, s \in \mathbb{P}^1\}$$

Let us remark that the set

$$X_2 = \{2p+2q, q \in \mathbb{P}^1\} \subset \mathbb{P}^2_p$$

is a copy of a Veronese curve of degree two in $\mathbb{P}^2$, and

$$X_3 = \{p+3q, q \in \mathbb{P}^1\} \subset \mathbb{P}^3_p$$

is a copy of a Veronese curve of degree three in $\mathbb{P}^3$.

(2.11) Proposition: Let $\omega = 3QdC - 2CdQ$ denote as above the 1-form in $P(4)$ defining the foliation $F$ and $\omega_H$ its restriction to $H$. Then the zeros in codimension one of $\omega_H$ are

$$S_2(\omega_H) = \mathbb{P}^2_p \subset H$$

and the zeros of $\bar{\omega}_H$ (notation as in (1.6)) are

$$S(\bar{\omega}_H) = S_1(\bar{\omega}_H) = \mathbb{P}^1_p \cup X_2 \cup X_3 \subset H.$$

In particular the foliation induced by $\bar{\omega}_H$ has degree two.

Proof: Following (2.9)a), let us determine $S(\omega) \cap H = (\bar{T} \cup \bar{N}) \cap H$. We find, set theoretically,

$$\bar{T} \cap H = \{3r + s, r, s \in \mathbb{P}^1\} \cap H = \{3p + s, s \in \mathbb{P}^1\} \cup \{3r + p, r \in \mathbb{P}^1\} = \mathbb{P}^1_p \cup X_3$$

$$\bar{N} \cap H = \{2r + 2s, r, s \in \mathbb{P}^1\} \cap H = \{2p + 2s, s \in \mathbb{P}^1\} = X_2$$
Next, according to (2.9)b), let us look for tangencies. We claim that the leaf $j^{-1}(\infty) = \bar{\Delta}$ is tangent to $H$ along $\mathbb{P}_p^2$. In fact, the intersection $\bar{\Delta} \cap H$ has two irreducible components, namely

$$\bar{\Delta} \cap H = \{2q+r+s, q, r, s \in \mathbb{P}^1\} \cap H = \{2p+r+s, r, s \in \mathbb{P}^1\} \cup \{2q+r+p, q, r \in \mathbb{P}^1\} = \mathbb{P}_p^2 \cup \bar{\Delta}_H$$

where

$$\bar{\Delta}_H = \{2q+r+p, q, r \in \mathbb{P}^1\} \subset \mathbb{P}_p^3$$

is a copy of the discriminant hypersurface $\{2q + r, q, r \in \mathbb{P}^1\} \subset \mathbb{P}(3)$ consisting of singular cubic binary forms, and is hence an irreducible surface of degree four. (In general, the discriminant for homogeneous polynomials of degree $d$ in $n$ variables is an irreducible hypersurface of degree $n(d-1)^{n-1}$, see [GKZ]).

Since $\bar{\Delta}$ is the union of the osculating planes of $X_4$, it follows (see e.g. [H], Exercise (17.10)) that $\bar{\Delta}$ and $H$ are tangent to each other along $\mathbb{P}_p^2$, as claimed.

Notice that since $\bar{\Delta}$ is a sextic, we obtain the equality of divisors in $H$

$$\bar{\Delta}.H = 2\mathbb{P}_p^2 + \bar{\Delta}_H$$

In particular $\bar{\Delta}$ and $H$ are transverse generically along $\bar{\Delta}_H$ and therefore the only tangency contributed by the leaf $\bar{\Delta}$ is the one along $\mathbb{P}_p^2$. To finish we only need to see that there are no other leaves tangent to $H$. This follows from the fact that the orbits of binary cubics under the affine group are the same as before, Q.E.D.

3. We end this note with two proofs, alternative to the one given in [CL], of the fact that the closure of the orbit of $\bar{\omega}_H$, which we now denote $\mathcal{E}$, is an irreducible component of $\mathcal{F}(3,2)$. We denote $F$ the foliation defined by $\bar{\omega}_H$.

(3.1) Using the formulas in (2.4) we may write in appropriate coordinates

$$\bar{\omega}_H = x_3[(2x_1^2-3x_0x_2)dx_0 + (3x_2x_3-x_0x_1)dx_1 + (x_0^2-2x_1x_3)dx_2] - (x_0x_1^2-2x_0^2x_2+x_1x_2x_3)dx_3$$

as in [CL].
A straightforward computation shows that the singular set of $d(\omega_H)$ is a point, namely, the intersection point of the cubic, conic and line in $S(\omega_H)$. It follows from Corollary 1 (section 5.2) or Corollary 6.1 of [CP] that every foliation $F'$ sufficiently close to $F$ is induced by an action.

(3.2) It will be convenient to give explicit expressions for the vector fields on $\mathbb{P}^3$ inducing $F$. If we write the action of $\text{Aff}(\mathbb{C})$ on $\mathbb{C}^3[t]$ as $(at + b) \cdot p(t) = p(at + b)$ then the generators $x = (1 + \epsilon)t$ and $y = t + \epsilon$ of $\text{Aff}(\mathbb{C})$ act on basis elements $t^i$ as follows:

$$x \cdot t^i = ((1 + \epsilon)t)^i = t^i + i\epsilon t^i \mod \epsilon^2$$

It follows that the tangent sheaf of $F$ is generated by the vector fields

$$X = \sum_{i=0}^{3} iz_i \frac{\partial}{\partial z_i} \quad \text{and} \quad Y = \sum_{i=1}^{3} iz_{i-1} \frac{\partial}{\partial z_i}.$$  

After a change of coordinates of the form $(z_0, \ldots, z_3) \mapsto (\lambda_0 z_0, \ldots, \lambda_3 z_3)$ we can assume that

$$X = \sum_{i=0}^{3} iz_i \frac{\partial}{\partial z_i} \quad \text{and} \quad Y = \sum_{i=1}^{3} z_{i-1} \frac{\partial}{\partial z_i}.$$  

Notice that $[X, Y] = -Y$.

(3.3) Since the affine Lie algebra is rigid, the foliation $F'$ is induced by a 1-form $\omega'$ of the form $i_{X'}i_{Y'}i_R \Omega$ where $X'$, $Y'$ are close to $X$, $Y$ respectively and $[X', Y'] = -Y'$.

Consider the action of $X'$ on the space $V$ of linear forms on $\mathbb{C}^4$. Since the eigenvalues of $X$ are distinct so are the eigenvalues of $X'$. Thus $V$ decomposes as

$$V = \bigoplus_{i=0}^{3} V_{\lambda_i},$$

where $V_{\lambda_i}$ is the $\lambda_i$-eigenspace for the action of $X'$, i.e., $X'(v) = \lambda_i v$ for every $v \in V_{\lambda_i}$. If $v \in V_{\lambda_i}$ then

$$[X', Y'](v) = X'(Y'(v)) - Y'(\lambda_i v) = -Y'(v).$$

and consequently $X'(Y'(v)) = (\lambda_i - 1)Y'(v)$, i.e., when non zero $Y'(v)$ is an $(\lambda_i - 1)$-eigenvector of $X'$. 

By semicontinuity, the rank of $Y'$ is at least 3 and on the other hand the equation above implies that it is at most 3. Moreover we also have that $\ker Y'$ must be equal to one of the $V_{\lambda_i}$, say $V_{\lambda_k}$. After reordering, we obtain that $\lambda_i = i$.

At this point we have shown that $X'$ is conjugate to $X$. But $[X', Y'] = -Y'$ implies that

$$Y = \sum_{i=0}^{2} \lambda_i z_{i+1} \frac{\partial}{\partial z_i},$$

where $\lambda_i \in \mathbb{C}$. It is now evident that the Lie algebra generated by $X', Y'$ is conjugate to the one generated by $X, Y$ by an element in $\text{GL}(\mathbb{C}^4)$.

(3.4) The second alternative proof is the following. Let $\bar{\omega}_H$ be as in (3.1). By assigning to each differential form its singular set, the orbit of $\bar{\omega}_H$ under the automorphism group of $\mathbb{P}^3$ maps onto the space of pointed twisted cubics in $\mathbb{P}^3$ and hence has dimension at least equal to $\dim \text{Aut}(\mathbb{P}^3) - \dim \text{Aut}(\mathbb{P}^1) + 1 = 15 - 3 + 1 = 13$.

On the other hand, the tangent space to $\mathcal{F}(3, 2)$ at the point $\bar{\omega}_H$ is given by the forms $\eta \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(4))$ such that $(\bar{\omega}_H + \epsilon \eta) \wedge d(\bar{\omega}_H + \epsilon \eta) = 0$ (modulo $\epsilon^2$), that is:

$$\bar{\omega}_H \wedge d\eta + \eta \wedge d(\bar{\omega}_H) = 0$$

($\eta$ defined up to constant multiple of $\bar{\omega}_H$). One checks, by hand or by computer, that the space of solutions $\eta$ of this system of linear equations has dimension 13. It follows that $\mathcal{E}$ is an irreducible component of $\mathcal{F}(3, 2)$. Furthermore, the integrability condition provides $\mathcal{F}(3, 2)$ with a natural structure of scheme and the tangent space calculation above also implies that $\mathcal{E}$ is a reduced component.

(3.5) The component $\mathcal{E}$ considered in this article admits some direct generalizations; let us make some remarks about them.

a) For $r \geq 5$, the natural action of the group $PGL(2, \mathbb{C})$ on the projective space $\mathbb{P}^r = \mathbb{P}S^r(\mathbb{C}^2)$ of binary forms of degree $r$ induces a rigid foliation of dimension three and hence provides an irreducible component of the space $\mathcal{F}_{r-3}(r, 3)$ of foliations of codimension $r - 3$ and degree three in $\mathbb{P}^r$ ([CP], Example (6.6)).

Notice that the foliation induced by the action of $PGL(2, \mathbb{C})$ on binary forms of degree $r = 4$, considered in this article, is not rigid. This follows from a general fact proved in
[CP], Proposition (6.5), or may be seen directly as follows: in (2.6) we observed that the 1-form $\omega$ defining this foliation belongs to the component $R(2,3) \subset F(4,3)$ and in fact it is clear that the closure of the orbit of $\omega$ is a proper subvariety of $R(2,3)$ since they have different dimension.

b) Let $\text{Aff}(\mathbb{C}) \subset PGL(2,\mathbb{C})$ be the affine group in one variable. The action of $\text{Aff}(\mathbb{C})$ on $\mathbb{P}S^r(\mathbb{C}^2)$ obtained by restriction of the action of $PGL(2,\mathbb{C})$ considered in a) defines a foliation of dimension two. For $r \geq 4$ these are rigid and define irreducible components of $F_{r-2}(r,2)$ ([CP], Example (6.8)).

c) The component $E \subset F(3,2)$ is the first member of an infinite family of rigid components $E(n) \subset F(n, n-1)$ defined for $n \geq 3$ in Theorem 4 of [CP].

It would be interesting to carry out an analysis of these components similar to what was done here with the exceptional component $E$.

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