ON NON-COERCIVE MIXED PROBLEMS FOR PARAMETER-DEPENDENT ELLIPTIC OPERATORS

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Abstract. We consider a (generally, non-coercive) mixed boundary value problem in a bounded domain $D$ of $\mathbb{R}^n$ for a second order parameter-dependent elliptic differential operator $A(x, \partial, \lambda)$ with complex-valued essentially bounded measured coefficients and complex parameter $\lambda$. The differential operator is assumed to be of divergent form in $D$, the boundary operator $B(x, \partial)$ is of Robin type with possible pseudo-differential components on $\partial D$. The boundary of $D$ is assumed to be a Lipschitz surface. Under these assumptions the pair $(A(x, \partial, \lambda), B)$ induces a holomorphic family of Fredholm operators $L(\lambda) : H^+ (D) \to H^- (D)$ in suitable Hilbert spaces $H^+ (D), H^- (D)$ of Sobolev type. If the argument of the complex-valued multiplier of the parameter in $A(x, \partial, \lambda)$ is continuous and the coefficients related to second order derivatives of the operator are smooth then we prove that the operators $L(\lambda)$ are continuously invertible for all $\lambda$ with sufficiently large modulus $|\lambda|$ on each ray on the complex plane $\mathbb{C}$ where the differential operator $A(x, \partial, \lambda)$ is parameter-dependent elliptic. We also describe reasonable conditions for the system of root functions related to the family $L(\lambda)$ to be (doubly) complete in the spaces $H^+ (D), H^- (D)$ and the Lebesgue space $L^2 (D)$.

Introduction

The notion of a parameter-dependent elliptic operator provides a useful link between the theories of boundary value problems for parabolic and elliptic operators (see, for instance, \cite{1}). Investigating a boundary value problem for parameter-dependent elliptic operator $A(x, \partial, \lambda)$ on a ray in the complex plane, first one aims to prove the continuous invertibility in proper functional spaces $H^+ (D), H^- (D)$ of the corresponding family $L(\lambda) : H^+ (D) \to H^- (D)$ of the operators for all $\lambda$ with sufficiently large modulus $|\lambda|$ on the ray (see \cite{5}, \cite{6}, \cite{9}, \cite{21}). The next step is to prove the (multiple) completeness of the corresponding root functions associated with the parameter-dependent family (see for instance \cite{12}, \cite{16}, \cite{20}). Actually, this provides a justification for application of Galerkin type methods and numerical solution of the problem. For elliptic (coercive) problems the results of this type are well known. The investigation is usually based on the classical methods of functional analysis and the theory of partial differential equations (see \cite{7}, \cite{9}, \cite{10}, \cite{12}, \cite{16}, \cite{20} and many others). For domains with smooth boundaries, the standard Shapiro-Lopatinsky conditions with parameter and their generalizations

\textit{Date}: June 11, 2014.

\textit{2010 Mathematics Subject Classification.} Primary 35B25; Secondary 35P10.

\textit{Key words and phrases.} Mixed problems, non-coercive boundary conditions, parameter dependent elliptic operators, root functions.

\textit{THIS IS A PREPRINT VERSION OF THE PAPER PUBLISHED IN MATHEMATICAL COMMUNICATIONS, 20 (2015), 131-150}
are usually imposed (see [3], [8], [9]). It is appropriate to mention here that the spectral theory in non-smooth domains usually depends upon hard analysis near singularities on the boundary (see, for instance, [4], [24]).

Recently the classical approach was adapted for investigation of spectral properties of non-coercive mixed problems for strongly elliptic operators in Lipschitz domains (see [22], [23]). An essential part of the approach is the analysis in spaces of negative smoothness. We successfully apply this method for studying non-coercive boundary value problems for the parameter-dependent elliptic operators with complex coefficients in Lipschitz domains in the case where the argument of the complex-valued multiplier of the parameter in $A(x, \partial, \lambda)$ is continuous and the coefficients related to second order derivatives of the operator are smooth.

An example related to a non-coercive mixed problems for strongly elliptic two-dimensional Lamé system is considered.

1. A Fredholm holomorphic family of mixed problems

Let $D$ be a bounded domain in Euclidean space $\mathbb{R}^n$ with Lipschitz boundary $\partial D$, i.e. the surface $\partial D$ is locally the graph of a Lipschitz function.

We consider complex-valued functions defined in the domain $D$. We write $L^q(D)$ for the space of all (equivalence classes of) measurable functions $u$ in $D$, such that the Lebesgue integral of $|u|^q$ over $D$ is finite. As usual, this scale continues to include the case $q = \infty$, too. As usual, we denote by $H^1(D)$ the Sobolev space and by $H^s(D)$, $0 < s < 1$ the Sobolev-Slobodetskii spaces.

Consider a second order differential operator

$$A(x, \partial, \lambda)u = - \sum_{i,j=1}^n \partial_i (a_{i,j}(x) \partial_j u) + \sum_{j=1}^n a_j(x) \partial_j u + a_0(x)u + E(\lambda)u$$

of divergence form in the domain $D$ with a complex parameter $\lambda$; here $x = (x_1, \ldots, x_n)$ are the coordinates in $\mathbb{R}^n$ and $\partial_j = \frac{\partial}{\partial x_j}$ and

$$E(\lambda)u = \lambda \left( \sum_{j=1}^n a_j^{(1)}(x) \partial_j u + a_0^{(1)}(x)u \right) + \lambda^2 a_0^{(2)}(x)u.$$ 

The coefficients $a_{i,j}, a_j, a_j^{(1)}, a_0^{(1)}, a_0^{(2)}$ are assumed to be complex-valued functions of class $L^\infty(D)$.

We suppose that the matrix $A(x) = (a_{i,j}(x))_{i,j=1,\ldots,n}$ is Hermitian and satisfies

$$\sum_{i,j=1}^n a_{i,j}(x) \overline{w_i} w_j \geq 0 \text{ for all } (x, w) \in T \times \mathbb{C}^n, \quad (1.1)$$

$$\sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \geq m_0 |\xi|^2 \text{ for all } (x, \xi) \in T \times (\mathbb{R}^n \setminus \{0\}), \quad (1.2)$$

where $m_0$ is a positive constant independent of $x$ and $\xi$. Estimate (1.2) is nothing but the statement that the operator $A(x, \partial, 0)$ is strongly elliptic. It should be noted that, since the coefficients of the operator and the functions under consideration are complex-valued, the matrix $A(x)$ can be degenerate. In particular inequalities
\( (1.1) \) and \( (1.2) \) are weaker than the (strong) coerciveness of the Hermitian form, i.e. the existence of a constant \( m_0 \) such that
\[
\sum_{i,j=1}^n a_{i,j}(x) \overline{w}_i w_j \geq m_0 |w|^2
\] (1.3)
for all \((x, w) \in \mathcal{D} \times (\mathbb{C}^n \setminus \{0\})\).

We consider the following Robin type boundary operator
\[
B = b_1(x) \sum_{i,j=1}^n a_{i,j}(x) \nu_i \partial_j + \partial \tau + B_0
\]
where \( b_1 \) is a bounded function on \( \partial D \), \( \nu(x) = (\nu_1(x), \ldots, \nu_n(x)) \) is the unit outward normal vector of \( \partial D \) at \( x \in \partial D \), \( \partial \tau = \sum_{j=1}^n \tau_j(x) \partial_j \) is the tangential derivative with a tangential field \( \tau = (\tau_1, \ldots, \tau_n) \) on \( \partial D \) and \( B_0 \) is a densely defined linear operator in \( L^2(\partial D) \) of “order” does not exceeding 1. The function \( b_1(x) \) is allowed to vanish on an open connected subset \( S \) of \( \partial D \) with piecewise smooth boundary \( \partial S \) and the vector \( \tau \) vanishes identically on \( S \).

To specify the operator \( B_0 \), fix a number \( 0 \leq \rho \leq 1/2 \) and a bounded linear operator \( \Psi : H^\rho(\partial D) \to L^2(\partial D) \). The range of \( \rho \) is motivated by trace and duality arguments. We will consider operator \( B_0 \) of the following form
\[
B_0 = \chi_S u + b_1(\Psi^* \Psi(u) + \delta B_0)
\]
where \( \chi_S \) is the characteristic function of the set \( S \) on \( \partial D \), \( \Psi^* : L^2(\partial D) \to H^\rho(\partial D) \) is the adjoint operator for \( \Psi \) and \( \delta B_0 \) is a “low order” perturbation that we will describe later.

For \( \rho = 0 \) a typical operator \( \Psi \) is a zero order differential operator, i.e. it is given by \( \Psi u = \psi u \), where \( \psi \) is a function on \( \partial D \) locally bounded away from \( \partial S \). Then \( (\Psi^* \Psi u)(x) = |\psi(x)|^2 u(x) \) is invertible provided that \( |\psi(x)| \geq c > 0 \). If \( \partial D \) is \( C^2 \)-smooth then a model operator \( \Psi \) is \( \Psi = (1 + \Delta_{\partial D})^{\rho/2} \) where \( \Delta_{\partial D} \) is the Laplace-Beltrami operator on the boundary.

Consider the following family of boundary value problems. Given data \( f \) in \( D \) and \( u_0 \) on \( \partial D \), find a distribution \( u \) in \( D \) which satisfies
\[
\begin{aligned}
A(x, \partial, \lambda)u &= f \quad \text{in} \quad D, \\
B(x, \partial)u &= u_0 \quad \text{at} \quad \partial D.
\end{aligned}
\] (1.4)

If \( \lambda = 0 \) and \( \Psi \) is given by the multiplication on a function, this is a well known mixed problem of Zaremba type (see \[27\]). It can be handled in a standard way in Sobolev type spaces associated with Hermitian forms or in Hölder spaces and Sobolev spaces using the potential methods, (for the coercive case see \[27\], \[19\], \[17\], \[14\] and elsewhere). In the non-coercive case the methods should be more subtle (see, for instance, \[2\], \[23\]) because of the lack of regularity of its solutions near the boundary of the domain.

In \[23\] the method, involving non-negative Hermitian forms, was adopted to study problem \((1.4)\) in non-coercive cases with a zero order differential operator \( \Psi \). Namely, denote by \( C^1(\overline{D}, S) \) the subspace of \( C^1(\overline{D}) \) consisting of those functions whose restriction to the boundary vanishes on \( S \). Let \( H^1(D, S) \) be the closure of \( C^1(\overline{D}, S) \) in \( H^1(D) \). This space is Hilbert under the induced norm. Since on \( S \) the boundary operator reduces to \( B = \chi_S \) and \( \chi_S(x) \neq 0 \) for \( x \in S \), the functions of \( H^1(D) \) satisfying \( Bu = 0 \) on \( \partial D \) belong to \( H^1(D, S) \).
Split $a_0$ into two parts $a_0 = a_{0,0} + \delta a_0$, where $a_{0,0}$ is a non-negative bounded function in $D$. Then, under reasonable assumptions, the Hermitian form

$$(u,v) = \int_D \sum_{i,j=1}^n a_{i,j} \partial_i u \overline{\partial_j v} \, dx + (a_{0,0} u,v)_{L^2(D)} + (\psi(u),\psi(v))_{L^2(\partial D)}$$

defines a scalar product on $H^1(D)$. Denote by $H^+(D)$ the completion of the space $H^1(D)$ with respect to the corresponding norm $\| \cdot \|_+$. To study the problem (1.4) we need an embedding theorem for the space $H^+(D)$.

**Theorem 1.1.** Let the coefficients $a_{i,j}$ be $C^\infty$ in a neighbourhood $X$ of the closure of $D$, inequalities (1.7), (1.8) hold and there is a constant $c_1 > 0$, such that

$$\|\psi u\|_{L^2(\partial D)} \geq c_1 \|u\|_{H^r(\partial D)} \text{ for all } u \in H^1(\partial D,S).$$

If there is a positive constant $c_2$, such that

$$a_{0,0} \geq c_2 \text{ in } D$$

or the operator $A$ is strongly elliptic in a neighbourhood $X$ of $\partial D$ and

$$\int_X \sum_{i,j=1}^n a_{i,j} \partial_i u \partial_j u \, dx \geq m_1 \|u\|_{L^2(X)}^2$$

for all $u \in C^\infty_{\text{comp}}(X)$, with $m_1 > 0$ a constant independent of $u$ then the space $H^+(D)$ is continuously embedded into $H^+(D)$ where $s$ is given by

$$s = \begin{cases} 
1/2 - \epsilon & \text{with } \epsilon > 0, \text{ if } \rho = 0, \\
1/2, & \text{if } \rho = 0 \text{ and } \partial D \in C^2, \\
1/2 + \rho, & \text{if } 0 < \rho \leq 1/2.
\end{cases}$$

**Proof.** It is similar to the proof of [23, Theorem 2.5] corresponding to the case where $\rho = 0$ and $\psi$ is given by the multiplication on a function. \qed

Of course, under the coercive estimate (1.3), the space $H^+(D)$ is continuously embedded into $H^1(D)$. However, in general, the embedding, described in Theorem 1.1, is rather sharp (see [23, Remark 5.1] and [4] below). In particular, if $\rho = 0$ then it may happens that the space $H^+(D)$ can not be embedded into $H^{1/2+\epsilon}(D)$ with any $\epsilon > 0$. Thus the operator $\psi$ is introduced here in order to improve, if necessary, the smoothness of elements of $H^+(D)$ in the non-coercive case.

In order to pass to the generalized setting of the mixed problem we need that all the derivatives $\partial_j u$ belong to $L^2(D)$ for an element $u \in H^+(D)$, at least if $s \leq 1/2$ in Theorem 1.1. However if $0 < s < 1$ then the absence of coerciveness does not allow this. To cope with this difficulty we note that the operator $\sum_{i,j=1}^n \partial_i (a_{i,j} \partial_j \cdot)$ admits a factorisation, i.e. there is an $(m \times n)$-matrix $\mathcal{D}(x) = (\mathcal{D}_{i,j}(x))_{i=1,...,m}^{j=1,...,n}$ of bounded functions in $D$, such that

$$(\mathcal{D}(x))^* \mathcal{D}(x) = \mathcal{A}(x)$$

for almost all $x \in D$. For example, one could take the standard non-negative selfadjoint square root $\mathcal{D}(x) = \sqrt{\mathcal{A}(x)}$ of the matrix $\mathcal{A}(x)$. Then

$$\sum_{i,j=1}^n a_{i,j} \partial_i u \overline{\partial_j v} = (\mathcal{D} \nabla v)^* \mathcal{D} \nabla u = \sum_{k=1}^m \mathcal{D}_k v \mathcal{D}_k u,$$
for all smooth functions $u$ and $v$ in $\overline{D}$, where $\nabla u$ is thought of as an $n$-column with entries $\partial_1 u, \ldots, \partial_n u$, and $\mathcal{D} u := \sum_{k=1}^m \mathcal{D} k u(x) \partial_k u$, $k = 1, \ldots, m$. Then, by the definition of the space $H^+(D)$, any term $\alpha_k(x) \mathcal{D} u$, $k = 1, \ldots, m$, belongs to $L^2(D)$ if $u \in H^+(D)$ and $\alpha_k \in L^\infty(D)$. Thus, if $0 < s < 1$ then we may confine ourselves with first order summands of the form

$$\sum_{k=1}^m \alpha_k(x) \mathcal{D} k \text{ and } \sum_{k=1}^m \alpha_k^{(1)}(x) \mathcal{D} k$$

instead of $\sum_{j=1}^n a_j(x) \partial_j$ and $\sum_{k=1}^n a_j^{(1)}(x) \partial_j$. For this purpose, we fix a factorization $\mathcal{D}(x)$ of the matrix $\mathcal{A}(x)$ and functions $\alpha_k \in L^\infty(D)$, $\alpha_k^{(1)} \in L^\infty(D)$, $k = 1, \ldots, m$.

These considerations allow to handle problem (1.4) with the use of the standard tools of functional analysis. Indeed, let $H^-(D)$ stand for the completion of space $H^+(D)$ with respect to the norm

$$\|u\|_+ = \sup_{v \in H^+(D)} \frac{|\langle v, u \rangle_{L^2(D)}|}{\|v\|_+}$$

It is the dual space for the space $H^+(D)$ with respect to the pairing

$$\langle \cdot, \cdot \rangle : H^-(D) \times H^+(D) \to \mathbb{C}$$

induced by the scalar product $(\cdot, \cdot)_{L^2(D)}$,

$$\langle u, v \rangle = \lim_{\nu \to +\infty} \langle u, v \rangle_{L^2(D)}, \quad u \in H^-(D), v \in H^+(D)$$

where $\{u_\nu\} \subset H^+(D)$ converges to $u$ in $H^-(D)$, see [19]. Note that under hypothesis of Theorem 1.1, the natural embedding $i : H^+(D) \to L^2(D)$ is continuous; it is compact if (1.3) holds. Let $i' : L^2(D) \to H^+(D)$ stand for the adjoint map for $i$ with respect to the pairing $(\cdot, \cdot)$, i.e.

$$\langle i'u, v \rangle = \langle u, v \rangle_{L^2(D)} \text{ for all } u \in L^2(D), v \in H^+(D).$$

Now an integration by parts leads to a weak formulation of problem (1.4): given $f \in H^-(D)$, find $u \in H^+(D)$, such that

$$(u, v)_+ + \left( \sum_{j=k}^m \alpha_k \mathcal{D} k + \delta a_0 + E(\lambda) \right) u, v \rangle_{L^2(D)} + \langle (b_1^{-1} \partial_\tau + \delta B_0) u, v \rangle_{L^2(\partial D \setminus \mathcal{S})} = \langle f, v \rangle,$$

(1.10)

for all $v \in C^1(\overline{D}, \mathcal{S})$.

By the Cauchy inequality, if

$$\left| \langle (b_1^{-1} \partial_\tau + \delta B_0) u, v \rangle_{L^2(\partial D \setminus \mathcal{S})} \right| \leq c \|u\|_+ \|v\|_+$$

with a constant $c > 0$ independent on $u, v \in H^+(D)$, then (1.10) induces a holomorphic (with respect to $\lambda \in \mathbb{C}$) family $L(\lambda) : H^+(D) \to H^-(D)$ of bounded linear operators.

Denote by $L_0$ the operator $L(0)$ in the case where $\tau \equiv 0$, $\delta B_0 \equiv 0$, $\delta a_0 \equiv 0$, $\alpha_k \equiv 0$, $k = 1, \ldots, m$. According to [20] Lemma 2.6, the operator $L_0 : H^+(D) \to H^-(D)$ is continuously invertible and $\|L_0\| = \|L_0^{-1}\| = 1$. Then we can consider each operator $L(\lambda), \lambda \in \mathbb{C}$, as a perturbation of $L_0$.

Actually, it is convenient to endow the space $H^-(D)$ with the scalar product

$$\langle u, v \rangle_+ = (L_0^{-1} u, L_0^{-1} v)_+ = \langle L_0^{-1} u, v \rangle, \quad u, v \in H^-(D)$$

(1.11)
coherent with the norm \( \| \cdot \| \_ \) see, for instance \[23\] p. 3316 and formula (2.2).

We can provide more subtle properties of the family \( \{ L(\lambda) \}_{\lambda \in \mathbb{C}} \) under reasonable assumptions.

In the sequel \( L(H_1, H_2) \) stand for the space of bounded linear operators in Banach spaces \( H_1 \) and \( H_2 \).

**Lemma 1.2.** Under the hypothesis of Theorem 1.1 let \( 0 \leq \rho \leq 1/2 \). If \( \delta B_0 \) maps \( H^\rho(\partial D, S) \) continuously into \( H^{-\rho}(\partial D) \) then the term \( (\delta B_0 u, v)_{L^2(\partial D)} \) induces a bounded operator \( \delta L_B : H^+(D) \to H^-(D) \) and

\[
\| \delta L_B \|_{L^2(H^+(D), H^-(D))} \leq \| \delta B_0 \|_{L^2(H^\rho(\partial D, S), H^{-\rho}(\partial D))} \| \psi^{-1} \|^2.
\]

If \( \delta B_0 \) maps \( H^\rho(\partial D, S) \) compactly into \( H^{-\rho}(\partial D) \) then the operator \( \delta L_B \) is compact. In particular, if \( \delta B_0 \) is given by the multiplication on a function \( \delta b_0 \in L^\infty(\partial D \setminus S) \) then

1) \( \delta B_0 \) maps \( H^\rho(\partial D, S) \) compactly into \( H^{-\rho}(\partial D) \) for \( 0 < \rho \leq 1/2 \),

2) \( \delta B_0 \) maps \( L^2(\partial D, S) \) continuously into \( L^2(\partial D) \) for \( \rho = 0 \).

**Proof.** The proof is standard, cf. [23, Lemma 4.6]. \( \square \)

Clearly, the linear span of the vectors

\[ \tau_{i,j} = \bar{e}_i x_i - \bar{e}_j x_j, \quad i > j. \]

coincides with the tangential plan at each point \( x \in \partial D \) where it exists. Thus, without loss of generality, we may consider tangential partial differential operators of the following form:

\[
\partial_{\tau} = \sum_{i>j} k_{i,j}(x) \partial_{\tau_{i,j}}
\]

**Lemma 1.3.** Let \( H^+(D) \) be continuously embedded into \( H^1(D, S) \). If \( k_{i,j}/b_1 \) is of Hölder class \( C^{0,\lambda} \) in the closure of \( \partial D \setminus S \) for all \( i > j \), with \( \lambda > 1/2 \), then the “tangential” term \( (b_1^{-1} \partial_{\tau} u, v)_{L^2(\partial D \setminus S)} \) induces a bounded operator \( \delta L_\tau : H^+(D) \to H^-(D) \).

**Proof.** The statement was proved in [23, Lemma 6.6]. \( \square \)

**Theorem 1.4.** Under the hypothesis of Theorem 1.1 let \( \tau = 0 \) unless \( s = 1 \). If either the term \( (\delta B_0 u, v)_{L^2(\partial D)} \) induces a bounded operator \( \delta L_B \) from \( H^+(D) \) to \( H^-(D) \) with \( \| \delta L_B + \delta L_\tau \| < 1 \) or \( \| \delta L_\tau \| < 1 \) and the term \( (\delta B_0 u, v)_{L^2(\partial D)} \) induces a compact operator from \( H^+(D) \) to \( H^-(D) \) then \( \{ L(\lambda) \}_{\lambda \in \mathbb{C}} \) is a holomorphic family of Fredholm operators of zero index.

**Proof.** Follows from Lemmas 1.2 and 1.3 because \( H^+(D) \) is compactly embedded into \( L^2(D) \) under hypothesis of Theorem 1.1. \( \square \)

## 2. Mixed problems for parameter-dependent elliptic operators

To obtain the main theorem of this paper we invoke the notion of parameter-dependent ellipticity.

We recall that the operator \( A(x, \partial, \lambda) \) is parameter-dependent elliptic on a ray \( \Gamma = \{ \arg(\lambda) = \varphi_T \} \) on the complex plane \( \mathbb{C} \) if

\[
\sum_{i,j=1}^{n} a_{i,j}(x) \zeta_i \zeta_j + \lambda \sum_{j=1}^{n} a_j^{(1)}(x) \zeta_j + \lambda^2 a_0^{(2)}(x) \neq 0 \quad (2.1)
\]
for all \( x \in \overline{D} \) and all \((\lambda, \zeta) \in (\Gamma \times \mathbb{R}^n) \setminus \{0, 0\}\).

In particular, if the operator \( A(x, \partial, \lambda) \) is parameter-dependent elliptic on the ray \( \Gamma \) then taking \( \zeta = 0 \) and \( \lambda \neq 0 \) in (2.2) we obtain \( a_0^{(2)}(x) \neq 0 \) for all \( x \in D \).

In the sequel we consider the case where \( E(\lambda) = \lambda^2 a_0^{(2)}(x) \), the most common in applications. Then we prove that, under reasonable assumptions, the family \( L(\lambda) : H^+(D) \to H^-(D) \) is continuously invertible for all \( \lambda \) with sufficiently large modulus \(|\lambda|\) on the ray \( \Gamma \) where the operator \( A(x, \partial, \lambda) \) is parameter-dependent elliptic (cf. [3], [21]).

Let \( \varphi_0(x) = \arg(a_0^{(2)}(x)) \). Denote by \( C : H^+(D) \to H^-(D) \) the operator that is induced by the term \((a_0^{(2)}(x)u,v)_{L^2(D)}\).

**Lemma 2.1.** Let \( a_0^{(2)}(x) \neq 0 \) for almost all \( x \in D \). Then the operator \( C : H^+(D) \to H^-(D) \) is injective.

**Proof.** Indeed, if \( Cu = 0 \) then

\[
0 = (Cu, v) = \int_D a_0^{(2)}(x)u(x)|\mathbf{\U}(x)| \, dx \quad \text{for all} \quad v \in H^+(D).
\]

As the \( H^+(D) \) is dense in \( L^2(D) \) we see that \( a_0^{(2)}u = 0 \) almost everywhere in \( D \).

Finally, as \( a_0^{(2)}(x) \neq 0 \) for almost all \( x \in D \) we conclude that \( u = 0 \) almost everywhere in \( D \). \( \square \)

**Lemma 2.2.** Suppose that the matrix \( A(x) \) is Hermitian non-negative. If \( E(\lambda) = \lambda^2 a_0^{(2)} \) then the operator \( A(x, \partial, \lambda) \) is parameter-dependent elliptic on the ray \( \Gamma \) if and only if

\[
\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j > 0
\]

for all \((x, \xi) \in \overline{D} \times (\mathbb{R}^n \setminus \{0\})\),

\[
|a_0^{(2)}(x)| > 0 \quad \text{for all} \quad x \in \overline{D};
\]

\[
\cos(\varphi_0(x) + 2\varphi_\Gamma) > -1 \quad \text{for all} \quad x \in \overline{D}.
\]

**Proof.** Follows from the standard trigonometrical formulas. \( \square \)

Of course, if \( a_{i,j} \in C(\overline{D}) \) then (2.2) is equivalent to (2.2). If \(|a_0^{(2)}(x)| \in C(\overline{D})\) then (2.3) is equivalent to the following

\[
|a_0^{(2)}(x)| \geq \theta_0 > 0 \quad \text{for all} \quad x \in \overline{D};
\]

similarly, if \( \varphi_0(x) \in C(\overline{D}) \) then (2.4) is equivalent to the following

\[
\cos(\varphi_0(x) + 2\varphi_\Gamma) \geq \theta_1(\Gamma) = \theta_1 > -1 \quad \text{for all} \quad x \in \overline{D},
\]

where the constants \( \theta_0, \theta_1 \) do not depend on \( x \).

Clearly, under the hypothesis of Theorem 1.4 we can decompose

\[
L(\lambda) = L_0 + \delta_\zeta L + \delta_\partial L + \lambda^2 C
\]

where \( \delta_\zeta L : H^+(D) \to H^-(D) \) is a compact operator and \( \delta_\partial L : H^+(D) \to H^-(D) \) is a bounded one. Moreover, the family \( L(\lambda) \) is Fredholm if \(|\delta_\zeta L| < 1\).

Let \( \eta(\Gamma) = \max(0, -\theta_1) \).
Theorem 2.3. Let either $\Psi$ is given by the multiplication on a function $\psi \in L^\infty(\partial D)$ or $\partial D \in C^\infty$ and $\Psi$ is a pseudodifferential operator on $\partial D$. Let also $E(\lambda) = \lambda^2 a_0^{(2)}$, the hypothesis of Theorem 2.4 be fulfilled,

$$a_0^{(2)} \neq 0 \text{ almost everywhere in } D$$

and (2.6) hold true. If $\varphi \in C(\overline{D})$ and $\|\delta_s L\|^2 + \eta^2(\Gamma) < 1$ then

1) there is $\gamma_0 \in \Gamma$ such that the operators $L(\lambda) : H^+(D) \to H^-(D)$ are continuously invertible for all $\lambda \in \Gamma$ with $|\lambda| \geq |\gamma_0|$;

2) the operators $L(\lambda)$ are continuously invertible for all $\lambda \in \mathbb{C}$ except a discrete countable set $\{\lambda_r\}$ without limit points in $\mathbb{C}$.

Proof. We begin with the following lemma.

Lemma 2.4. Under the hypothesis of Theorem 2.3 there is $k_0 \in \mathbb{N}$ such that for all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$ we have

$$\|(L_0 + \delta_s L + \lambda^2 C)u\|_{-} \geq \left(\sqrt{1 - \eta^2(\Gamma)} - \|\delta_s L\|\right) \|u\|_+ \text{ for all } u \in H^+(D)$$

and there are positive constants $p_1 = p_1(\varphi_\Gamma), q_1 = q_1(\varphi_\Gamma)$ such that

$$\|(L_0 + \delta_s L + \lambda^2 C)u\|_{-} \geq p_1 \|u\|_+ + q_1 |\lambda|^2 \|Cu\|_{-} \quad (2.8)$$

for all $u \in H^+(D)$ and $\lambda \in \Gamma$ with $|\lambda| \geq k_0$.

Proof. Given any $u \in H^+(D)$ an easy computation with the use of formula (2.11) shows that

$$\lambda^2 \langle Cu, u \rangle = |\lambda|^2 \int_D \left|a_0^{(2)}(x)\right| |u(x)|^2 e^{\sqrt{-1}(\varphi_\Gamma + 2\varphi_\Gamma)} \, dx,$$

$$\int_D \left|a_0^{(2)}(x)\right| |u(x)|^2 e^{\sqrt{-1}(\varphi_\Gamma + 2\varphi_\Gamma)} \, dx,$$

$$\langle u, L_0 u \rangle + \lambda^2 \langle L_0^{-1} Cu, Cu \rangle + \lambda^2 \langle u, Cu \rangle + \lambda^2 \langle L_0^{-1} Cu, L_0 u \rangle =$$

$$\|u\|_+^2 + |\lambda|^4 \|Cu\|_+^2 + \overline{\lambda^2} \langle u, Cu \rangle + \lambda^2 \langle L_0^{-1} Cu, u \rangle_+ =$$

$$\|u\|_+^2 + |\lambda|^4 \|Cu\|_+^2 + \overline{\lambda^2} \langle u, Cu \rangle + \lambda^2 \langle Cu, u \rangle =$$

$$\|u\|_+^2 + |\lambda|^4 \|Cu\|_+^2 + 2 \mathbb{R} \left(\lambda^2 \langle Cu, u \rangle\right).$$

Clearly, for $\lambda \in \Gamma$,

$$\mathbb{R} \left(\lambda^2 \langle Cu, u \rangle\right) = |\lambda|^2 \int_D \left|a_0^{(2)}(x)\right| |u(x)|^2 \cos (\varphi_\Gamma + 2\varphi_\Gamma) \, dx. \quad (2.11)$$

If $\theta_1 \in [0, 1]$ then $\eta(\Gamma) = 0$ and we have immediately for all $u \in H^+(D)$:

$$\|(L_0 + \lambda^2 C)u\|_{-}^2 \geq \|u\|_+^2 + |\lambda|^4 \|Cu\|_+^2,$$

$$\|\alpha^2 L + \lambda^2 C)u\|_{-} \geq \|\|L_0 + \lambda^2 C\rangle u\|_{-} - \|\delta_s Lu\|_{-} \geq$$

$$\sqrt{\|u\|_+^2 + |\lambda|^4 \|Cu\|_+^2 - \|\delta_s Lu\|_{-}}.$$

Then, for $\alpha \in [0, \pi/2]$ and non-negative numbers $a, b$, we have

$$\sqrt{a + b} \geq \sqrt{a} \cos (\alpha) + \sqrt{b} \sin (\alpha). \quad (2.12)$$

As $\|\delta_s L\| < \sqrt{1 - \eta^2(\Gamma)} = 1$, there is $\alpha_0 \in (0, \pi/2)$ such that

$$\|\delta_s L\| < \cos (\alpha_0)$$
In particular, this means that for all \( u \in H^+(D) \) and all \( \lambda \in \Gamma \) we have:
\[
\|(L_0 + \delta_k L + \lambda^2 C)u\|_\infty \geq \|u\|_\infty - \|\delta_k L u\|_\infty \geq (1 - \|\delta_k L\|)\|u\|_\infty,
\]
\[
\|(L_0 + \delta_k L + \lambda^2 C)u\|_\infty \geq \cos (\alpha_0)\|u\|_\infty + \sin (\alpha_0)|\lambda|^2\|Cu\|_\infty - \|\delta_k L u\|_\infty \geq 
\]
\[
(\cos (\alpha_0) - \|\delta_k L\|)\|u\|_\infty + \sin (\alpha_0)|\lambda|^2\|Cu\|_\infty, 
\]
i.e. the desired inequalities are true if \( \theta_1 \in [0, 1] \).
Let \( \theta_1 \in (-1, 0) \) then, by (2.11) and (2.10),
\[
\Re \left( \lambda^2 \langle Cu, u \rangle \right) \geq -|\theta_1|\|\lambda\|^2 \int_D |a_0^{(2)}(x)||u(x)|^2 \, dx. \tag{2.13}
\]
Let us prove that for any \( \theta \in (-\theta_1, 1) \) and \( \gamma \in (0, 1) \) with \( \theta \sqrt{1 - \gamma} > -\theta_1 \) there is \( k_0 \in \mathbb{N} \) such that
\[
\|(L_0 + \lambda^2 C)u\|_\infty^2 \geq (1 - \theta^2) \|u\|_\infty^2 + \gamma |\lambda|^4\|Cu\|_\infty^2 \tag{2.14}
\]
for all \( u \in H^+(D) \) and all \( \lambda \in \Gamma \) with \( |\lambda| \geq k_0 \). Indeed, we argue by contradiction. Let there are \( \theta \in (-\theta_1, 1) \) and \( \gamma \in (0, 1) \) with \( \theta \sqrt{1 - \gamma} > |\theta_1| \) such that for each \( k \in \mathbb{N} \) there are \( u_k \in H^+(D) \) with \( \|u_k\|_\infty = 1 \), and a number \( \lambda_k \in \Gamma \) with \( |\lambda_k| \geq k \) such that
\[
\|(L_0 + \lambda_k^2 C)u_k\|_\infty^2 < 1 - \theta^2 + \gamma |\lambda_k|^4\|Cu_k\|_\infty^2.
\]
It follows from (2.10) and (2.11) that
\[
\theta^2 + |\lambda_k|^4\|Cu_k\|_\infty^2(1 - \gamma) + 2|\lambda_k|^2 \int_D \cos (\varphi_0 + 2\varphi_T)|a_0^{(2)}(x)||u_k(x)|^2 \, dx < 0,
\]
i.e.
\[
\left( \theta - \sqrt{1 - \gamma} \right)|\lambda_k|^2\|Cu_k\|_\infty^2 \tag{2.15}
\]
\[
2 \left( \theta \sqrt{1 - \gamma} + \frac{\int_D \cos (\varphi_0 + 2\varphi_T)|a_0^{(2)}(x)||u_k(x)|^2 \, dx}{\|Cu_k\|_\infty^2} \right) |\lambda_k|^2\|Cu_k\|_\infty^2 < 0,
\]
for all \( k \in \mathbb{N} \).
On the other hand, for all \( u \in H^+(D) \) with \( \|u\|_\infty = 1 \) we have
\[
\|Cu\|_\infty = \|e^{2\sqrt{1 - \gamma} \varphi_T} Cu\|_\infty \geq \left( e^{\sqrt{1 - (\varphi_0 + 2\varphi_T)}} \right) \|a_0^{(2)}(u, u)_{L^2(D)} \|, \tag{2.16}
\]
In particular, we have
\[
\left| \int_D \cos (\varphi_0 + 2\varphi_T)|a_0^{(2)}(x)||u_k(x)|^2 \, dx \right| \leq 1 \text{ for all } k \in \mathbb{N}.
\]
Now, if the sequence \( \{|\lambda_k|^2\|Cu_k\|_\infty^2\} \) is unbounded then extracting a subsequence \( \{|\lambda_k|^2\|Cu_k\|_\infty^2\} \) tending to \( +\infty \), dividing (2.15) by \( |\lambda_k|^4\|Cu_k\|_\infty^2 \) and passing to the limit with respect to \( k \to +\infty \) we obtain \( 1 \leq 0 \), a contradiction.
Let the sequence \( \{|\lambda_k|^2\|Cu_k\|_\infty^2\} \) be bounded. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence \( \{u_{k_j}\} \) weakly convergent to an element \( u_0 \) in the space \( H^+(D) \). Then \( \{Cu_{k_j}\} \) converges to \( Cu_0 \) in \( H^-(D) \) because \( C : H^+(D) \to H^-(D) \) is compact and \( \{u_{k_j}\} \) converges to \( u_0 \) in \( L^2(D) \) because the embedding \( \iota : H^+(D) \to L^2(D) \) is compact, too. Since the sequence \( \{\lambda_{k_j}^2 Cu_{k_j}\} \) is bounded in \( H^-(D) \) and \( |\lambda_k| \to +\infty \) we conclude that \( \{Cu_{k_j}\} \) converges to zero in \( H^-(D) \). This means that \( Cu_0 = 0 \) and then \( u_0 = 0 \) because \( a_0^{(2)}(x) \neq 0 \) if (2.7) is fulfilled on the ray \( \Gamma \) and then the operator \( C \) is injective (see Lemma 2.1).
According to compactness principle, we may consider the subsequences
\[
\{ |\lambda_{k_j}|^2 \|Cu_{j} \|_{-} \} \quad \text{and} \quad \left\{ - \int_{D} \cos(\varphi_0 + 2\varphi_{\mathcal{T}})|a_{0}^{(2)}(x)||u_{k_j}(x)|^2 \, dx \right\}
\]
as convergent to the limits \( \alpha \geq 0 \) and \( \beta \in [-1, 1] \) respectively. Now it follows from (2.16) that
\[
(\theta - \alpha)^2 + 2\alpha (\theta - \beta) \leq 0.
\]
If \( \alpha = 0 \) then we have a contradiction because \( \theta > 0 \). If \( \alpha > 0 \) and \( \beta \leq 0 \) then \( \theta - \beta > 0 \) and we again have a contradiction.

Let \( \alpha > 0 \) and \( \beta > 0 \). If \( \varphi_0 \in C(\overline{D}) \) then, according to Weierstraß Theorem, there is a polynomial sequence \( \{P_i(x)\} \) approximating \( \varphi_0(x) \) in this space. In particular, for each \( \varepsilon > 0 \), there is \( i_{\varepsilon} \in \mathbb{N} \) such that
\[
\max_{x \in \overline{D}} |1 - \cos(\varphi_0(x) - P_i(x))| < \varepsilon \quad \text{for all} \quad i \geq i_{\varepsilon}.
\]
Since \( u e^{\sqrt{-1}P_i(x)} \in H^{+}(D) \) we see that, for all \( i \geq i_{\varepsilon} \),
\[
\|Cu\|_{-} \geq \frac{|(e^{\sqrt{-1}(\varphi_0(x) - P_i(x))})|a_{0}^{(2)}|u, u|_{L^2(D)}|}{\|ue^{\sqrt{-1}P_i}\|_{+}} \geq \frac{|(\cos(\varphi_0(x) - P_i(x))|a_{0}^{(2)}|u, u|_{L^2(D)}|}{\|ue^{\sqrt{-1}P_i}\|_{+}} \geq \frac{(1 - \varepsilon)|a_{0}^{(2)}|u, u|_{L^2(D)}|}{\|ue^{\sqrt{-1}P_i}\|_{+}}.
\]
Hence if \( \varepsilon \in (0, 1) \) then
\[
\limsup_{k_{j} \to \infty} \frac{|(a_{0}^{(2)}|u_{k_j}, u_{k_j})|_{L^2(D)}|}{\|Cu_{k_j}\|_{-}} \leq \limsup_{k_{j} \to \infty} \frac{\|u_{k_j} e^{\sqrt{-1}P_i}\|_{+}}{1 - \varepsilon} \quad \text{for all} \quad i \geq i_{\varepsilon}.
\]
On the other hand, as \( |e^{\sqrt{-1}P_i}| = 1 \) we conclude that
\[
\|ue^{\sqrt{-1}P_i}\|_{L^2(D)}^2 = \|u\|_{+}^2 + \|(\mathcal{D}e^{\sqrt{-1}P_i})u\|_{L^2(D)}^2 + 2\mathcal{R}\left((\mathcal{D}e^{\sqrt{-1}P_i})u, e^{\sqrt{-1}P_i}\mathcal{D}u\right)_{L^2(D)} + \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)}^2 - \|\Psi(u)\|_{L^2(\partial D)}^2
\]
for all \( i \in \mathbb{N} \) and \( u \in H^{+}(D) \). If \( \Psi \) is given by the multiplication on a function \( \psi \in L^{\infty}(\partial D) \) then \( \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} = \|\Psi(u)\|_{L^2(\partial D)} \). If \( \partial D \in C^{\infty} \) and \( \Psi \) is a pseudodifferential operator of order \( \rho \) on \( \partial D \) then, as the multiplication on a smooth function is a pseudodifferential operator of order zero, we conclude that the commutator \( [\Psi, e^{\sqrt{-1}P_i}] = (\Psi \circ e^{\sqrt{-1}P_i} - e^{\sqrt{-1}P_i} \circ \Psi) \) is a pseudodifferential operator of order \( (\rho - 1) \) on \( \partial D \) (see for instance [15]). By the construction of \( \cdot \|_{+} \) and Theorem 1.1 the sequence \( \{u_{k_j}\} \) is bounded in \( H^{\rho}(\partial D) \) and then we can consider that the subsequence \( \{u_{k_j}\} \) converges weakly to zero in this space. Then
\[
\left| \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} - \|\Psi(u)\|_{L^2(\partial D)} \right| = \left| \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} - \|e^{\sqrt{-1}P_i}\Psi(u)\|_{L^2(\partial D)} \right| \leq \|[\Psi, e^{\sqrt{-1}P_i}](u)\|_{L^2(\partial D)}
\]
for all \( u \in H^{+}(D) \) and hence
\[
\lim_{k_{j} \to \infty} \left| \|\Psi(e^{\sqrt{-1}P_i}u_{k_j})\|_{L^2(\partial D)} - \|\Psi(u_{k_j})\|_{L^2(\partial D)} \right| = 0 \quad (2.18)
\]
because the operator $[\Psi, e^{\sqrt{-1}P_n}] : H^0(\partial D) \to L^2(\partial D)$ is compact by Rellich Theorem. Thus, as $u_{k_j} \to 0$ in $L^2(D)$ and $\|u_{k_j}\|_+ = 1$, it follows from (2.17) and (2.18) that

$$\limsup_{k_j \to \infty} \|u_{k_j} e^{\sqrt{-1}P_n}\|_+ = 1$$

for all $i \in \mathbb{N}$.

Therefore, if $\beta > 0$ then, by (2.13),

$$\beta = \lim_{k_j \to \infty} - \frac{\int_D \cos(\varphi_0 + 2\varphi_\Gamma)\phi_0(x)\|u_{k_j}(x)\|^2 \, dx}{\|Cu_{k_j}\|_+ - \|\bar{u}\|} \leq \limsup_{k_j \to \infty} \frac{\int_D \|\phi_0(x)\|u_{k_j}(x)\|^2 \, dx}{\|Cu_{k_j}\|_+ - \|\bar{u}\|} \leq \frac{\|\bar{u}\|}{1 - \varepsilon} \text{ for each } \varepsilon \in (0, 1).$$

This means that $\theta - \beta > 0$ if $\theta > |\theta_1|$ and we again have a contradiction with (2.16). Thus, (2.14) is fulfilled.

Finally, as $\|\delta_s L\| < 1 - \eta^2(\Gamma) = 1 - |\theta_1|^2$ we see that there are $\theta_2 \in (|\theta_1|, 1]$, $\gamma_0 \in [0, 1)$ with $\theta_2 \sqrt{1 - \gamma_0} > |\theta_1|$, $\alpha_2 \in (0, \pi/2)$ such that

$$\|\delta_s L\| < \cos(\alpha_2)\left(1 - \theta_2\right)^{1/2}.$$

Therefore, using (2.12), (2.14) we see that

$$\|(L_0 + \delta_s L + \lambda^2 C)u\|_+ \geq \sqrt{(1 - \theta_2^2)\|u\|_+^2 + \gamma_0|\lambda|^4\|Cu\|^2 - \|\delta_s Lu\|_+} \geq \cos(\alpha_2)\left(1 - \theta_2\right)^{1/2}\|u\|_+ + \sin(\alpha_2)\sqrt{\gamma_0}|\lambda|^2\|Cu\|_+ - \|\delta_s Lu\|_+ \geq \cos(\alpha_2)\left(1 - \theta_2\right)^{1/2}\|\delta_s L\|\|u\|_+ + \sin(\alpha_2)\sqrt{\gamma_0}|\lambda|^2\|Cu\|_+.$$

for all $u \in H^+(D)$ and all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$.

We continue with the proof of the property 1). For this purpose, using Lemma 2.4, we conclude that the operator $(L_0 + \delta_s L + \lambda^2 C)$ is continuously invertible if $\|\delta_s L\|^2 < 1 - \eta^2(\Gamma)$ and $\lambda \in \Gamma$ with $|\lambda| \geq k_0$. Hence we obtain

$$L(\lambda) = \left(I + \delta_s L(L_0 + \delta_s L + \lambda^2 C)^{-1}\right)(L_0 + \delta_s L + \lambda^2 C)$$

(2.19)

for all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$.

We will show that the operator $I + \delta_s L(L_0 + \delta_s L + \lambda^2 C)^{-1}$ is injective for all $\lambda \in \Gamma$ such that $|\lambda| \geq k_1$ with some $k_1 \in \mathbb{N}$ with $k_1 \geq k_0$. Indeed, we argue by contradiction. Suppose that for any $k \in \mathbb{N}$ there are $\lambda_k \in \Gamma$ with $|\lambda_k| \geq k$ and $f_k \in H^-(D)$, such that $\|f_k\|_+ = 1$ and

$$(I + \delta_s L(L_0 + \delta_s L + \lambda_k^2 C)^{-1})f_k = 0.$$ (2.20)

It follows from Lemma 2.4 that the sequence $u_k := (L_0 + \delta_s L + \lambda_k^2 C)^{-1}f_k$ is bounded in $H^+(D)$ for all $\lambda_k \in \Gamma$ with $|\lambda_k| \geq k_0$. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{f_{k_j}\}$ with the property that both $\{f_{k_j}\}$ and $\{u_{k_j}\}$ converge weakly in the spaces $H^-(D)$ and $H^+(D)$ to limits $f$ and $u$, respectively. Since $\delta_s L$ is compact, it follows that the sequence $\{\delta_s Lu_{k_j}\}$ converges to $\delta_s Lu$ in $H^-(D)$, and so $\{f_{k_j}\}$ converges to $f$ because of (2.20). Obviously,

$$\|f\|_+ = 1.$$
In particular, we conclude that the sequence \( \{ \delta_s L (L_0 + \delta_s L + \lambda^2 C)^{-1} f_{k_j} \} \) converges to \((-f)\) whence
\[
f = -\delta_s L u. \tag{2.21}
\]

Further, on passing to the weak limit in the equality \( f_{k_j} = (L_0 + \delta_s L + \lambda^2 C) u_{k_j} \)
we obtain
\[
f = L_0 u + \delta_s L u + \lim_{k_j \to \infty} \lambda^2_{k_j} C u_{k_j},
\]
for the continuous operator \( L_0 + \delta_s L : H^+(D) \to H^-(D) \) maps weakly convergent sequences to weakly convergent sequences.

As the operator \( C \) is compact, the sequence \( \{ C u_{k_j} \} \) converges to \( C u \) in the space \( H^-(D) \) and \( C u \neq 0 \) which is a consequence of \( (2.21) \) and the injectivity of \( C \) (see Lemma 2.1). This shows readily that the weak limit
\[
\lim_{k_j \to \infty} \lambda^2_{k_j} C u_{k_j} = f - L_0 u - \delta_s L u
\]
does not exist, a contradiction.

We have proved that the operator \( I + \delta_s L (L_0 + \delta_s L + \lambda^2 C)^{-1} \) is injective for all \( \lambda \in \Gamma \) with \( |\lambda| \geq k_1 \). Since this is a Fredholm operator of index zero, it is continuously invertible. Hence, the operators \( L(\lambda) \) are continuously invertible for all \( \lambda \in \Gamma \) with sufficiently large \( |\lambda| \).

Thus, \( \{ L^{-1}(\lambda) = (L_0 + \delta_s L + \delta_s L + \lambda^2 C)^{-1} \}_{\lambda \in \mathbb{C}} \) is a meromorphic family of Fredholm operators. In particular, since there is a point \( \gamma \) where \( L(\gamma) \) is continuously invertible, the operators \( L(\lambda) \) are continuously invertible for all \( \lambda \in \mathbb{C} \) except a discrete countable set \( \{ \lambda_{\nu} \} \) without limit points in \( \mathbb{C} \) (see, for instance, \cite{kato} or \cite{pol})

**Corollary 2.5.** Let either \( \Psi \) is given by the multiplication on a function \( \psi \in L^\infty(\partial D) \) or \( \partial D \in C^\infty \) and \( \Psi \) is a pseudodifferential operator on \( \partial D \). Let also \( (2.7) \) hold true, \( \varphi_0 \in C(\overline{D}) \) and
\[
\Phi = \sup_{x, y \in \overline{D}} (\varphi_0(x) - \varphi_0(y)) < 2\pi. \tag{2.22}
\]
Under the hypothesis of Theorem 2.7, for each compact operator \( \delta_s L : H^+(D) \to H^-(D) \) and each bounded operator \( \delta_s L : H^+(D) \to H^-(D) \) with
\[
\|\delta_s L\|^2 + (\max (0, -\cos(\Phi/2)))^2 < 1 \tag{2.23}
\]
the operators \( L(\lambda) = L_0 + \delta_s L + \lambda^2 C \) are continuously invertible for all \( \lambda \in \mathbb{C} \) except a countable number of the characteristic values \( \{\lambda_{\nu}\} \).

**Proof.** As \( \varphi_0 \in C(\overline{D}) \), the function admits maximal and minimal values
\[
\Phi_1 = \min_{x \in \overline{D}} \varphi_0(x), \quad \Phi_2 = \max_{x \in \overline{D}} \varphi_0(x),
\]
and \( \Phi = \Phi_2 - \Phi_1 \).

Then under \( (2.7) \) and \( (2.22) \) the operator \( A(x, \partial, \lambda) = (\mathcal{D} \nabla)^2 (\mathcal{D} \nabla) + \lambda^2 a_0^2(x) \)
satisfies conditions of Theorem 2.6 on the ray \( \Gamma_0 = \{ \arg(\lambda) = -(\Phi_2 + \Phi_1)/4 \} \)
because in this case we have
\[
-\pi < (\Phi_1 - \Phi_2)/2 \leq \varphi_0(x) + 2\varphi_{\Gamma_0} \leq (\Phi_2 - \Phi_1)/2 < \pi.
\]
For this particular ray we obtain
\[
\theta_1 = \min_{x \in \overline{D}} \cos (\varphi_0(x) + 2\varphi_{\Gamma_0}) \geq \cos(\Phi/2) > -1.
\]
Now Theorem 2.23 implies that if (2.23) is fulfilled then there is $\gamma_0 \in \Gamma_0$ such that the operator $L(\gamma_0)$ is continuously invertible. In particular, the operators $L(\lambda)$ are continuously invertible for all $\lambda \in \mathbb{C}$ except a countable number of the characteristic values $\{\lambda_0\}$.

3. ON THE COMPLETENESS OF ROOT FUNCTIONS

We are interested in studying the completeness of root functions related to the mixed problem in Sobolev type spaces $H^+(D)$, $H^-(D)$.

To this purpose we recall some basic definitions. Suppose $\lambda_0 \in \mathbb{C}$ and $F(\lambda)$ is a holomorphic function in a punctured neighbourhood of $\lambda_0$ which takes on its values in the space $\mathcal{L}(H_1, H_2)$ of bounded linear operators acting from a Hilbert space $H_1$ to a Hilbert space $H_2$. The point $\lambda_0$ is called a characteristic point of $F(\lambda)$ if there exists a holomorphic function $u(\lambda)$ in a neighborhood of $\lambda_0$ with values in $H_1$, such that $u(\lambda_0) \neq 0$ but $F(\lambda_0)u(\lambda)$ extends to a holomorphic function (with values in $H_2$) near the point $\lambda_0$ and vanishes at this point. Following [13], we call $u(\lambda)$ a root function of the family $F(\lambda)$ at $\lambda_0$.

If $N$ is the order of zero of the holomorphic function $F(\lambda)u(\lambda)$ at the point $\lambda_0$ then we have

$$\sum_{j=0}^{m} F_{m-j}u_j = 0 \text{ for all } m \in \mathbb{Z}_+ \text{ with } 0 \leq m \leq N - 1$$

(3.1)

where $u_j = \frac{1}{j!} \frac{d^j u}{dz^j}(\lambda_0) \in H_1$ and $F_j = \frac{1}{j!} \frac{d^j F}{dz^j}(\lambda_0) \in \mathcal{L}(H_1, H_2)$, $j \in \mathbb{N}$. The vector $u_0$ is called an eigenvector of the family $F(\lambda)$ at the point $\lambda_0$ and the vectors $u_j$, $1 \leq j \leq N - 1$, are said to be associated vectors for the eigenvector $u_0$. If the linear span of the set of all eigen- and associated vectors the family $F(\lambda)$ is dense in $H_1$ one says that the root functions of the family $F(\lambda)$ are complete in $H_1$.

However, the notion of root function of a holomorphic family is a generalization of the notion of a root vector of a linear operator. Namely, recall that a complex number $\mu \in \mathbb{C}$ is said to be an eigenvalue of a linear operator $T : H \to H$ in a Hilbert space $H$ if there is a non-zero element $u \in D_T$, such that $(T - \mu I)u = 0$, where $I$ is the identity operator in $H$. The element $u$ is called an eigenvector of $T$ corresponding to the eigenvalue $\mu$. A non-selfadjoint compact operator might have no eigenvalues. However, each non-zero eigenvalue (if exists) is of finite multiplicity, see for instance [10]. Similarly to the Jordan normal form of a linear operator on a finite-dimensional vector space one uses the more general concept of root vectors of operators.

More precisely, a non-zero element $u \in H$ is called a root vector of $T$ corresponding to an eigenvalue $\mu_0 \in \mathbb{C}$ if $u \in D(T-\mu_0 I)^k$, for all $1 \leq k \leq m$ and $(T-\mu_0 I)^m u = 0$ for some natural number $m$. The set of all root vectors corresponding to an eigenvalue $\mu_0$ (complemented by zero element) forms a vector subspace in $H$ whose dimension is called the (algebraic) multiplicity of $\mu_0$.

Note that under (2.7) the multiplication on the function $u_0(2) \in L^\infty(D)$ induces a bounded injective operator in the space $L^2(D)$; it is continuously invertible under (2.5). We will denote this operator by $C_0$. Then we can factorize $C = \iota'C_0\iota$.

Lemma 3.1. If (2.7) is fulfilled then, for the holomorphic Fredholm family $L(\lambda) = L(0) + \lambda^2 C : H^+(D) \to H^-(D)$ the set of all its root functions coincides with the set
of all the root vectors of one of the following closed densely defined linear operators:
\[ C^{-1}L(\gamma) : H^+(D) \to H^+(D) \text{ and } L(\gamma)C^{-1} : H^-(D) \to H^-(D), \]
where \( \gamma \in \mathbb{C} \) is an arbitrary point. Besides, if there is a point \( \gamma_0 \in \mathbb{C} \) where the operator \( L(\gamma_0) = L(0) + \lambda_0^2 C \) is continuously invertible, it also coincides with the set of all the root vectors of one of the following bounded linear operators:
\[ L^{-1}(\gamma_0)C : H^+(D) \to H^+(D) \quad CL^{-1}(\gamma_0) : H^-(D) \to H^-(D), \]
\[ \iota L^{-1}(\gamma_0)\iota' C_0 : L^2(D) \to L^2(D). \]

**Proof.** Follows immediately from (3.1). \( \square \)

To formulate the completeness results regarding to parameter-dependent elliptic operators we need the notion of a compact operator of finite order. If \( T : H \to H \) is compact, then the operator \( T^*T \) is compact, selfadjoint and non-negative. Hence it follows that \( T^*T \) possesses a unique non-negative selfadjoint compact square root \( (T^*T)^{1/2} \), often denoted by \( |T| \). By the Hilbert-Schmidt Theorem the operator \(|T|\) has countable system of non-negative eigenvalues \( s_\nu(T) \) which are called the \( s \)-numbers of \( T \). It is clear that if \( T \) is selfadjoint then \( s_\nu = |\mu_\nu| \), where \( \{\mu_\nu\} \) is the system of eigenvalues of \( T \). The operator \( T \) is said to belong to the Schatten class \( \mathfrak{S}_p \), with \( 0 < p < \infty \), if
\[ \sum |s_\nu(T)|^p < \infty. \]

After M.V. Keldysh a compact operator \( T \) is said to be of finite order if it belongs to a Schatten class \( \mathfrak{S}_p \). The infimum \( \text{ord} (T) \) of such numbers \( p \) is called the order of \( T \).

Let us denote by \( \mathfrak{C} : H^+(D) \to H^-(D) \) the linear bounded operator induced by the term \( (|a_0^{(2)}|^2 u, v)_{L^2(D)} \). Note that under \([2.4]\) the multiplication on the function \( |a_0^{(2)}| \in L^\infty(D) \) induces a bounded injective selfadjoint operator \( \mathfrak{C}_0 : L^2(D) \to L^2(D) \); it is continuously invertible under \([2.5]\).

In the following theorem \( h(\cdot, \cdot) \) stands for the Hermitian form
\[ h(u, v) = (|a_0^{(2)}|^2 u, v)_{L^2(D)}. \]

We note that, under \([2.4]\) it defines a scalar product on \( L^2(D) \); this Hilbert space we denote by \( L^2_2(D) \). The corresponding norm is not stronger than \( \| \cdot \|_{L^2(D)} \); it is equivalent to the original norm of this space if \([2.5]\) is fulfilled.

**Theorem 3.2.** Let \([2.7]\) hold true. Under the hypothesis of Theorem \([4]\) the operators
\[ L_0^{-1}\mathfrak{C} : H^+(D) \to H^+(D), \quad \mathfrak{C}L_0^{-1} : H^-(D) \to H^-(D), \quad \iota L_0^{-1}\iota' \mathfrak{C}_0 : L^2(D) \to L^2(D) \]
are compact and their orders are finite:
\[ \text{ord} (\mathfrak{C}L_0^{-1}) = \text{ord} (L_0^{-1}\mathfrak{C}) = \text{ord} (\iota L_0^{-1}\iota' \mathfrak{C}_0) = n/(2\rho + 1). \]

Moreover, the operators \( L_0^{-1}\mathfrak{C} \) and \( \mathfrak{C}L_0^{-1} \) are selfadjoint. Besides, the operators have the same systems of eigenvalues \( \{\mu_\nu\} \), the system \( \{b_\nu^{(+)}\} \) of eigenvectors of the operator \( L_0^{-1}\mathfrak{C} \) is complete in the spaces \( H^+(D), L^2(D) \) and \( H^-(D) \). Moreover, the system \( \{b_\nu^{(+)\iota}\} \) is an orthonormal basis in \( H^+(D) \), the system \( \{b_\nu^{(-)} = \mathfrak{C}b_\nu^{(+)\iota}\} \) of eigenvectors of the operator \( \mathfrak{C}L_0^{-1} \) is an orthogonal basis in \( H^-(D) \), the system \( \{b_\nu^{(0)} = \iota b_\nu^{(+)\iota}\} \) of eigenvectors of the operator \( \iota L_0^{-1}\iota' \mathfrak{C}_0 \) is an orthogonal basis in the
space $L^2_{h}(D)$ and the system $\{\sqrt{|a_0^{(2)}|} b^{(\nu)}_0\}$ is an orthogonal basis in $L^2(D)$. If, in addition, (2.35) holds then the operator $\iota L_{0}^{-1} \iota \mathcal{C}_0$ is selfadjoint in $L^2_{h}(D)$.

Proof. First of all we note that

$$L_{0}^{-1} \mathcal{C} = L_{0}^{-1} \iota \mathcal{C}_0 : H^+(D) \to H^+(D), \quad \mathcal{C}L_{0}^{-1} = \iota \mathcal{C}_0 L_{0}^{-1} : H^-(D) \to H^-(D).$$

Under the hypothesis of Theorem 1.1, $H^+(D)$ is continuously embedded to $H^s(D)$ and then, according to Rellich Theorem, the embedding $\iota : H^+(D) \to L^2(D)$ is compact. Hence the operators $L_{0}^{-1} \mathcal{C}, \mathcal{C}L_{0}^{-1}$ and $\iota L_{0}^{-1} \iota \mathcal{C}_0$ are compact.

Moreover,

$$(L_{0}^{-1} \mathcal{C} u, v)_+ \equiv \iota \mathcal{C}_0 u, v > = (\mathcal{C}_0, u, v)_{L^2(D)} = \int_D |a_0^{(2)}(x)| u(x) \overline{v(x)} \, dx, \quad (3.2)$$

$$(u, L_{0}^{-1} \mathcal{C} v)_+ = (L_{0}^{-1} \mathcal{C}_0 u, v)_+ = (\mathcal{C}_0, u, v)_{L^2(D)} = \int_D |a_0^{(2)}(x)| u(x) \overline{v(x)} \, dx$$

for all $u, v \in H^+(D)$, i.e. the operator $L_{0}^{-1} \mathcal{C}$ is selfadjoint. Then

$$(L_{0}^{-1} \mathcal{C} u, u)_+ = \int_D |a_0^{(2)}(x)| u(x)^2 \, dx \geq 0$$

for all $u \in H^+(D)$, see (3.2). Hence the operator $L_{0}^{-1} \mathcal{C}$ is non-negative and then it is positive because both $L_{0}^{-1}$ and $\mathcal{C}$ are injective.

According to [23, Corollary 3.5], the operator $\iota L_{0}^{-1} \iota : L^2(D) \to L^2(D)$ is compact selfadjoint and its order is finite:

$$\text{ord}(\iota L_{0}^{-1} \iota) = n/(2r + 1).$$

As $\mathcal{C}_0 : L^2(D) \to L^2(D)$ is bounded, the operators $\iota L_{0}^{-1} \iota$ and $\iota L_{0}^{-1} \iota \mathcal{C}_0$ have the same orders (see [12, Ch. 2, § 2], [10] or elsewhere).

As the operator $\iota$ is injective, we see that

$$(L_{0}^{-1} \iota \mathcal{C}_0 u - \mu I)u = 0$$

if and only if

$$(\iota L_{0}^{-1} \iota \mathcal{C}_0 - \mu I)u = 0.$$  

Therefore

$$(L_{0}^{-1} \iota \mathcal{C}_0 u - \mu I)^m u = 0$$

with some $m \in \mathbb{N}$ if and only if

$$(\iota L_{0}^{-1} \iota \mathcal{C}_0 - \mu I)^m u = 0.$$  

Thus, the sets of eigenvalues and root vectors of the operator $L_{0}^{-1} \mathcal{C}$ coincides with the sets of eigenvalues and root vectors of the operator $\iota L_{0}^{-1} \iota \mathcal{C}_0$. Besides the multiplicities of the eigenvalues coincide, too. Hence the orders of the operators $\iota L_{0}^{-1} \iota \mathcal{C}_0$ and $L_{0}^{-1} \mathcal{C}$ coincide.

By the Hilbert-Schmidt Theorem, there is an orthonormal basis $\{b^{(\nu)}_0\}$ in $H^+(D)$, consisting of the eigenvectors corresponding to the eigenvalues $\{\mu_\nu\}$ of the operator $L_{0}^{-1} \mathcal{C}$. Hence, by the discussion above, the vectors $\iota b^{(\nu)}_0 = b^{(0)}\nu \in L^2(D)$, $\nu \in \mathbb{N}$ are the eigenvectors of the operator $\iota L_{0}^{-1} \iota \mathcal{C}_0$ corresponding to the eigenvalues $\{\mu_\nu\}$ of the operator $L_{0}^{-1} \mathcal{C}$. As $H^+(D)$ is dense in $L^2(D)$, the system $\{ib^{(\nu)}_0\}$ is complete in $L^2(D)$. 
On the other hand,

\[ h(u,v) = \langle |a_0^{(2)}(x)|u,v \rangle_{L^2(D)} = \langle \mathcal{C}u,v \rangle_{L^2(D)} = (L_0^{-1}\mathcal{C}u, v)_{+} \]

for all \( u,v \in H^+(D) \). In particular, the system \( \{ib_{\nu}^{(+)} = b_{\nu}^{(0)}\} \) is orthogonal in \( L^2_{h}(D) \). It is complete in \( L^2_{h}(D) \) because the space can be considered as the completion of \( L^2(D) \) with respect to \( h(\cdot, \cdot) \). In particular, the system \( \{\sqrt{|a_0^{(2)}|}b_{\nu}^{(+)}\} \) is orthogonal in \( L^2(D) \). If a vector \( u \) from \( L^2(D) \) is orthogonal to the system \( \{\sqrt{|a_0^{(2)}|}b_{\nu}^{(+)}\} \) in \( L^2(D) \) then the vector \( \sqrt{|a_0^{(2)}|}u \in L^2(D) \) is orthogonal to the system \( \{ib_{\nu}^{(+)}\} \) in \( L^2(D) \). Since \( \{ib_{\nu}^{(+)}\} \) is complete in \( L^2(D) \) we see that \( \sqrt{|a_0^{(2)}|}u = 0 \) almost everywhere in \( D \) and then \( u = 0 \) because of (2.7). Hence the system \( \{\sqrt{|a_0^{(2)}|}b_{\nu}^{(+)}\} \) is an orthogonal basis in \( L^2(D) \).

Now, by the very construction, the space \( H^+(D) \) is dense in \( H^-(D) \) and hence the system \( \{b_{\nu}^{(+)}\} \) is complete \( H^-(D) \). Moreover,

\[
\langle a_0^{(2)}b_{\nu}^{(+)}(1), a_0^{(2)}b_{k}^{(+)}(1) \rangle = \langle L_0^{-1}\mathcal{C}b_{\nu}^{(+)}(1), L_0^{-1}\mathcal{C}b_{k}^{(+)}(1) \rangle = \mu_{\nu,\mu}^{k}_{\nu,k}
\]

i.e. \( \{\mathcal{C}b_{\nu}^{(+)}\} \) is orthogonal in \( H^-(D) \). It is complete because

\[
L_0^{-1}u = \sum_{\nu}(L_0^{-1}u, b_{\nu}^{(+)}b_{\nu}^{(+)} = \sum_{\nu}(L_0^{-1}u, b_{\nu}^{(+)}b_{\nu}^{(+)}
\]

for each \( u \in H^-(D) \) by the discussion above and then

\[
u(L_0^{-1}u, b_{\nu}^{(+)}b_{\nu}^{(+)} = \sum_{\nu}(L_0^{-1}u, b_{\nu}^{(+)}b_{\nu}^{(+)}.
\]

Since the operator \( L_0^{-1}\mathcal{C} : H^+(D) \rightarrow H^+(D) \) is selfadjoint, we have

\[
(\mathcal{C}L_0^{-1}u, v)_{-} = (L_0^{-1}\mathcal{C}L_0^{-1}u, L_0^{-1}v)_{+} = ((L_0^{-1}u, L_0^{-1}\mathcal{C}L_0^{-1}v)_{+} = (u, \mathcal{C}L_0^{-1}v)_{-}
\]

for all \( u,v \in H^-(D) \), i.e. the operator \( \mathcal{C}L_0^{-1} : H^-(D) \rightarrow H^-(D) \) is selfadjoint. Hence, by the Hilbert-Schmidt Theorem, there is an orthonormal basis \( \{b_{\nu}^{(-)}\} \) in \( H^-(D) \), consisting of the eigenvectors of the operator \( \mathcal{C}L_0^{-1} \).

On the other hand, as for all \( u \in H^+(D) \) the identity

\[
\mathcal{C}(L_0^{-1}\xi - \mu I)u = (\mathcal{C}L_0^{-1} - \mu I)\mathcal{C}u
\]

holds true and the operator \( \mathcal{C} \) is injective, we conclude that the systems of the eigenvalues of the operators \( L_0^{-1}\mathcal{C} \) and \( \mathcal{C}L_0^{-1} \) coincide. Moreover the eigenvalues has the same multiplicities and then \( \text{ord}(\mathcal{C}L_0^{-1}) = \text{ord}(L_0^{-1}\mathcal{C}) \). Therefore (3.3) implies that for each \( \nu \in \mathbb{N} \) the vector \( b_{\nu}^{(-)} = \mathcal{C}b_{\nu}^{(+)} \) is an eigenvector of the operator \( \mathcal{C}L_0^{-1} \) corresponding to the eigenvalue \( \mu_{\nu} \), too. In particular, we can consider the system \( \{\mathcal{C}b_{\nu}^{(+)}\} \) as an orthogonal basis in \( H^-(D) \), consisting of the eigenvectors of the operator \( \mathcal{C}L_0^{-1} \).

Finally, if (2.8) is fulfilled then \( L_2^2(D) \) is a Hilbert space coinciding with \( L^2(D) \) as the linear space and having an equivalent norm. Then

\[
h(\iota L_0^{-1}\iota^{-1}\mathcal{C}0u, v) = (\iota L_0^{-1}\iota^{-1}\mathcal{C}0u, |a_0^{(2)}(x)|v)_{L^2(D)} = \langle |a_0^{(2)}(x)|u, L_0^{-1}\iota^{-1}\mathcal{C}0v \rangle_{L^2(D)} = h(u, \iota L_0^{-1}\iota^{-1}\mathcal{C}0v)
\]

for all \( u, v \in L^2(D) \), i.e. the operator \( \iota L_0^{-1}\iota^{-1}\mathcal{C}0 : L^2_{h}(D) \rightarrow L^2_{h}(D) \) is selfadjoint. \( \square \)
Now we can use the famous Keldysh’ Theorem on the weak perturbation of compact selfadjoint operators (see, [16] or [12]).

**Corollary 3.3.** Let (2.7) hold true. Under the hypothesis of Theorem 1.1 for each compact operator \( \delta L : H^+(D) \to H^-(D) \) we have

1) for any \( \varepsilon > 0 \) all the characteristic values \( \lambda_\nu \) (except for a finite number) of the family \( L(\lambda) = L_0 + \delta \lambda L + \lambda^2 \mathcal{C} \) belong to the corners

\[
M_\varepsilon = \{|\arg(\lambda) - \pi/2| < \varepsilon\}, \quad M_-\varepsilon = \{|\arg(\lambda) + \pi/2| < \varepsilon\}
\]

(3.4)

and \( \lim_{\nu \to \infty} |\lambda_\nu| = +\infty \);

2) the system of root vectors of the family \( L(\lambda) = L_0 + \delta \lambda L + \lambda^2 \mathcal{C} \) is complete in the spaces \( H^+(D), L^2(D) \) and \( H^-(D) \).

**Proof.** Under the hypothesis of the theorem, \( \Phi = 0 \) and Corollary 2.5 implies that there is \( \gamma_0 \) on the ray \( L_0 = \{\arg(\lambda) = 0\} \) such that \( L(\gamma_0) \) is continuously invertible. Thus, according to Lemma 3.1 the proof of the statements 1) and 2) of the theorem can be reduced to the investigation of the properties of the compact operator \( L^{-1}(\gamma_0)\mathcal{C} \).

On the other hand, the operator

\[
\delta L = (L(\gamma_0) - L_0) = \delta \lambda L + \gamma_0^2 \mathcal{C}
\]

is compact and then the operator \( L^{-1}(\gamma_0)(\delta L) \) is compact, too. Easily, we obtain

\[
L^{-1}(\gamma_0) - L_0^{-1} = -L^{-1}(\gamma_0)(\delta L)L_0^{-1}.
\]

Hence the operator

\[
L^{-1}(\gamma_0)\mathcal{C} = L_0^{-1}\mathcal{C} - L^{-1}(\gamma_0)(\delta L)(L_0^{-1}\mathcal{C})
\]

can be considered as a weak perturbation of the selfadjoint operator \( L_0^{-1}\mathcal{C} \) (see [16] or [12]). Indeed, according to Theorem 3.2 the order of the operator \( L_0^{-1}\mathcal{C} \) is finite. As the operators \( L_0^{-1}\mathcal{C} \) and \( L^{-1}(\gamma_0)\mathcal{C} \) are injective, the statements the completeness of the root vectors \( \{b_\nu\} \) of the operator \( L^{-1}(\gamma_0)\mathcal{C} \) in the space \( H^+(D) \) follows from famous Keldysh’ Theorem (see [16] or [12]). Moreover, this theorem also implies that the sequence \( \{\mu_\nu\} \) of its eigenvalues converges to zero and belongs to the corner \( \{\arg(\mu) < \varepsilon\} \) (except for a finite number of its elements). Then, by the construction of the spaces, the system \( \{b_\nu\} \) is complete in \( L^2(D) \) and \( H^-(D) \), too.

As, \( \mu_\nu = (\gamma_0^2 - \lambda_\nu^2)^{-1}, \nu \in \mathbb{N} \), the property 1) of the theorem holds, too. \( \square \)

Finally, we may apply the method of rays of minimal growth of the resolvent to obtain the completeness of root vectors in the case of more general perturbations.

**Theorem 3.4.** Let either \( \Psi \) is given by the multiplication on a function \( \psi \in L^\infty(\partial D) \) or \( \partial D \subset C^\infty \) and \( \Psi \) is a pseudodifferential operator on \( \partial D \). Under the hypothesis of Theorem 1.1 let also (2.7) and

\[
\Phi = \sup_{x,y \in \partial D} (\varphi_0(x) - \varphi_0(y)) < \pi(2\rho + 1)/2n.
\]

(3.5)

hold true. If \( \varphi_0 \in C^{0,1}(\overline{D}) \) and

\[
\|\delta_x L\|^2 + (\max (0, -\cos((\pi(2\rho + 1) - 2n\Phi))/4n))^2 < 1
\]

(3.6)

then we have
1) for any \( \varepsilon > 0 \) all the characteristic values \( \lambda_\nu \) (except for a finite number) of the family \( L(\lambda) = L_0 + \delta_s L + \delta_c L + \lambda^2 C \) belong to the corners \( \{ |\arg(\lambda) \pm \pi/2| < \pi(2\rho + 1)/2n + \varepsilon \} \) and \( \lim_{\nu \to \infty} |\lambda_\nu| = +\infty \).

2) the system of root vectors of the family \( L(\lambda) = L_0 + \delta_c L + \delta_s L + \lambda^2 C \) is complete in the spaces \( H^+(D), H^-(D) \) and \( L^2(D) \).

**Proof.** First of all, we note (3.5) implies (2.22) and then Corollary 2.5 yields the existence of a number \( \gamma_0 \in \mathbb{C} \) such that \( L(\gamma) \) is continuously invertible. In particular, the operators \( L(\lambda) \) are continuously invertible for all \( \lambda \in \mathbb{C} \) except a countable number of the characteristic values \( \{ \lambda_\nu \} \).

As the operator \( \gamma_0 C : H^+(D) \to H^-(D) \) is compact, the family

\[
\tilde{L}(\lambda) = L_0 + \delta_s L + \delta_c L + \lambda^2 C
\]

with \( \delta_s L = \delta_c L + \gamma_0^2 C \) and \( \lambda^2 = \lambda^2 - \gamma_0^2 \) satisfies conditions of Theorem 3.4 too. Moreover, the operator \( \tilde{L}(0) = L(\gamma_0) \) is continuously invertible. Since the root functions and root vectors of the families \( \tilde{L}(\lambda) \) and \( L(\lambda) \) have obvious relations, we can replace the family \( L(\lambda) \) by the family \( \tilde{L}(\lambda) \). Thus without loss of generality we may consider that the operator \( L(0) \) is continuously invertible.

As \( 0 < (2\rho + 1)/2n \leq 1/2 \), it follows from (3.5) that there is \( 0 < \epsilon < \pi/2 \) such that

\[
\pi(2\rho + 1)/2n - \Phi = 2\epsilon.
\]

Then under (2.7) and (3.5) the operator \( A(x, \partial, \lambda) = (\nabla \nabla)^*(\nabla \nabla) + \lambda^2 u_0^2(x) \) satisfies conditions of Theorem 2.3 on any ray \( \Gamma \) with

\[
-(\pi + \Phi_1 - a\epsilon)/2 < \varphi_1 < (\pi - \Phi_2 - a\epsilon)/2
\]

where \( 0 < a < 1 \) is an arbitrary number. Indeed, in this case (3.5) implies that the interval \( (-\pi + \Phi_1 - a\epsilon, \pi - \Phi_2 - a\epsilon) \) is not empty and

\[
\varphi_0(x) + 2\varphi_1 \leq \Phi_2 + \pi - \Phi_2 - a\epsilon \leq \pi - a\epsilon < \pi,
\]

\[
\varphi_0(x) + 2\varphi_1 \geq \Phi_1 - \pi + a\epsilon \geq -\pi + a\epsilon > -\pi.
\]

For these rays we have

\[
\theta_1(\Gamma) = \min_{x \in \mathcal{D}} \cos \left( \varphi_0(x) + 2\varphi_1 \right) \geq \cos(\pi - a\epsilon) = -\cos a\epsilon > -1.
\]

Hence (3.7) implies that there is a number \( a \in (0, 1) \) such that we have

\[
\|\delta_s L\|^2 + \left( \max \left( 0, -\cos \left( \frac{a\epsilon}{2} \right) \right) \right)^2 < 1.
\]

Thus, according to Lemma 3.1 the proof of the statements 1) and 2) of the theorem can be reduced to the investigation of the properties of one of the operators \( L^{-1}(0)C \) and \( L(0)C^{-1} \).

If \( \varphi_0 \in C^{0,1}(\overline{D}) \) then the multiplication on the function \( e^{\sqrt{-1}t\varphi_0} \in C^{0,1}(\overline{D}) \) induces a bounded linear operator \( \delta_C : H^+(D) \to H^+(D) \). Hence the operator \( CL^{-1}(0) \) can be presented in the following form:

\[
CL^{-1}(0) = (CL_0^{-1}) L_0 \delta_C L^{-1}(0).
\]

It follows from Theorem 2.2 that the operator \( CL_0^{-1} \) belongs to the Schatten class \( \mathcal{S}_{n/(2\rho+1)+\epsilon} \) with any \( \epsilon > 0 \). As compositions with bounded operators preserve the
Consider the Lamé type system instead of the completeness (see Corollary 3.3 and Theorem 3.2 we should claim the multiple (double) completeness where $\Delta^2$ 

operators in $R$ the use of Phragmen-Lindelöf theorem which go back at least as far as [1].

instance, [11]); the vector function plays an essential role in the two-dimensional Linear Elasticity Theory (see, for instance, [1]). Actually, it follows from the reducing procedure of Lemma 3.1 that in § 2 we have 

\[ \| (L(0) - (\Delta^2)C)u \|_{-} \geq c_1 |\lambda|^2 \|Cu\|_{-} \] 

for all sufficiently large $\lambda$ on each of the rays. Hence the rays 

\[ a\epsilon - \Phi_1 < \arg (\mu) < 2\pi - \Phi_2 - a\epsilon \]

are the rays of the minimal growth of the resolvent of the closed operator $L(0)C^{-1}$, i.e. 

\[ \| (L(0)C^{-1} - \mu)^{-1}w \|_{-} \leq c_2 |\mu|^{-1} \|w\|_{-} \] 

for all sufficiently large $\mu$ on each of the rays, see, for instance, [1]. Moreover, it follows from (3.7) and (3.9) the angle between any two neighbouring rays of minimal growth are less than $\pi(2\rho + 1)/2n$ if $0 < \alpha < 1$.

Thus the statement of the theorem follows from the standard arguments with the use of Phragmen-Lindelöf theorem which go back at least as far as [1]. \hfill \Box

Remark 3.5. Actually, it follows from the reducing procedure of Lemma 3.1 that in Corollary 3.3 and Theorem 3.2 we should claim the multiple (double) completeness instead of the completeness (see [10, 20] and elsewhere). 

4. AN EXAMPLE

Consider an instructive example.

Let $n = 2$ and $A_0^{(2)}$ be a $(2 \times 2)$ matrix with real-valued entries of class $L^\infty(D)$. Consider the Lamé type system

\[ \tilde{A}(x, \vartheta, \lambda)V(x) = -\vartheta \Delta_2 I_2 V(x) - (\vartheta + \vartheta_1)\nabla_2 \text{div}_2 V(x) + \lambda^2 A_0^{(2)}(x)V(x) \]

where $V(x) = (V_1(x), V_2(x))$ is an unknown vector, $I_2$ is the identity $(2 \times 2)$-matrix, $\Delta_2$ the Laplace operator, $\nabla_2$ and $\text{div}_2$ are the gradient operator and the divergence operators in $\mathbb{R}^2$ respectively and $\vartheta, \vartheta_1$ are the Lamé parameters. This operator plays an essential role in the two-dimensional Linear Elasticity Theory (see, for instance, [11]); the vector function $V(x)$ represents the displacement of points of an elastic body. This operator can also be considered as a part of linearisation system of the stationary version of the two-dimensional Navier-Stokes type equations for viscous compressible fluid with known pressure and unknown velocity vector $V(x)$ (see [15, §15]); in this case the Lamé parameters represent viscosities. As it is know, the system is strongly elliptic and formally selfadjoint non-negative if $\vartheta > 0$, $2\vartheta + \vartheta_1 > 0$.

Let us consider a very special case where the first Lamé parameter $\vartheta_1$ is negative and $\vartheta_1 = -\vartheta$. Then $\tilde{A}(x, \vartheta, \lambda)$ reduces to

\[ \tilde{A}(x, \vartheta, \lambda) = -\vartheta \Delta_2 I_2 + \lambda^2 A_0^{(2)}(x). \] 

On the other hand,

\[ -\Delta_2 I_2 V = \text{rot}_2^2 \text{rot}_2 V + \text{div}_2^2 \text{div}_2 V \]

where $\text{rot}_2 V = (\partial_1 V_2 - \partial_2 V_1)$ is the rotation operator in $\mathbb{R}^2$ and $\text{rot}_2^2$, $\text{div}_2^2$ are the formal adjoint operators for $\text{rot}_2$, $\text{div}_2$ respectively.
Assume now that the matrix $A_0^{(2)}(x)$ has the following form

$$A_0^{(2)}(x) = \alpha(x)U(x)$$

where $\alpha(x) \in L^\infty(D)$ is a non-negative function and

$$U(x) = \begin{pmatrix} U_1(x) & -U_2(x) \\ U_2(x) & U_1(x) \end{pmatrix}$$

is an orthogonal matrix with entries $U_j \in L^\infty(D)$. Then, after the complexification

$$u(z) = V_1(z) + \sqrt{-1} V_2(z), \ z = x_1 + \sqrt{-1} x_2$$

system (4.1) with real-valued coefficients reduces to the following equation with complex-valued coefficients

$$A(x, \partial, \lambda)u = 4\vartheta \overline{\sigma} \vartheta u + \lambda^2 a_0^{(2)}(x) u$$

where $\overline{\sigma} = 1/2(\vartheta \frac{\partial}{\partial x_1} - \sqrt{-1} \vartheta \frac{\partial}{\partial x_2})$ is the Cauchy-Riemann operator, $\overline{\sigma}^* = -1/2(\vartheta \frac{\partial}{\partial x_1} - \sqrt{-1} \vartheta \frac{\partial}{\partial x_2})$ its formal adjoint and

$$a_0^{(2)}(x) = \alpha(x) (U_1(x) + \sqrt{-1} U_2(x)) .$$

Then, with a proper operator $\Psi : H^\rho(\partial D) \to L^2(\partial D)$, the Robin type operator $B$ has the form

$$B = 2\vartheta (\nu_1 - \sqrt{-1}\nu_2) \bar{\vartheta} + \Psi^* \Psi .$$

where $\nu_1, \nu_2$ is the unit normal vector field to $\partial D$. The boundary operators

$$\frac{\partial}{\partial \nu} = \nu_1 \partial_1 + \nu_2 \partial_2, \ \bar{\partial}_\nu = (\nu_1 - \sqrt{-1}\nu_2) \bar{\vartheta} = \frac{1}{2} \left( \frac{\partial}{\partial \nu} + \sqrt{-1}(\nu_1 \partial_2 - \nu_2 \partial_1) \right)$$

are known as the normal derivative and the complex normal derivative with respect to $\partial D$ respectively. Thus, we obtain a mixed problem of the type considered above:

$$\begin{cases}
-\vartheta \Delta_2 + \lambda^2 a_0^{(2)} u(z) = f \ \text{in} \ D, \\
2\vartheta \bar{\partial}_\nu + \Psi^* \Psi u(z) = 0 \ \text{at} \ \partial D.
\end{cases} \quad (4.2)$$

Note that the usual boundary conditions for the Navier-Stokes equations or the Lamé type operator are formulated by using the boundary stress tensor $\sigma$. In our particular case the tensor have the following components:

$$\sigma_{i,j} = \vartheta \left( \delta_{i,j} \frac{\partial}{\partial \nu} + \nu_j \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial x_j} \right), \ 1 \leq i, j \leq 2. \quad (4.3)$$

Hence, with the tangential operator $\partial_{\tau_0} = ((\nu(x) \text{div}_2)^T - \nu(x) \text{div}_2)$, we have

$$\sigma = \vartheta \left( \frac{\partial}{\partial \nu} I_2 + \partial_{\tau} \right) = \vartheta (\bar{\sigma} + 2\partial_{\tau_0}). \quad (4.4)$$

where the boundary tensor $\bar{\sigma}$ corresponds to the boundary operator $2\bar{\partial}_\nu$ after the decomplexification of the mixed problem (1.2), i.e. in the matrix form (1.2) reads as

$$\begin{cases}
-\vartheta \Delta_2 I_2 + \lambda^2 A_{0}^{(2)} V(x) = F \ \text{in} \ D, \\
((\sigma - 2\partial_{\tau_0}) + \Psi^* \Psi I_2) V(x) = 0 \ \text{at} \ \partial D.
\end{cases}$$

In Elasticity Theory, the boundary tensor $\bar{\sigma} = \vartheta^{-1} \sigma - 2\partial_{\tau_0}$ was discovered in [7].
We continue with the mixed problem (12). The corresponding scalar product of the space \( H^+ (D) \) related to the mixed problem has the form

\[
(u, v)_+ = 4 \partial \left( \bar{\partial} u, \bar{\partial} v \right)_{L^2 (\partial D)} + \left( \Psi u, \Psi v \right)_{L^2 (\partial D)}.
\]

Then, Theorem (1.1) grants the embedding of the space \( H^+ (D) \) into the Sobolev-Slobodetskii space \( H^s (D) \). However, for \( 0 < \rho < 1/2 \) each holomorphic function \( u \in H^{\rho + 1/2} (D) \) belongs to \( H^+ (D) \) but there is no reason for it to belong to \( H^1 (D) \), i.e. the embedding is sharp. For \( \rho = 0 \) the embedding described in Theorem (1.1) is sharp, too but the arguments become more subtle (see [22] or [23]).

In some cases we can obtain reasonable formulas for solutions to the problem. Let \( D \) be the unit circle \( B \) around the origin in \( \mathbb{C} \) and \( S = \emptyset \). We pass to polar coordinates \( z = r e^{\sqrt{-1} \phi} \) in \( \mathbb{R}^2 \), where \( r = |x| \) and \( \phi \in [0, 2\pi] \). The Laplace operator \( \Delta_2 \) takes the form

\[
\Delta_2 = \frac{1}{r^2} \left( \left( r \frac{\partial}{\partial r} \right)^2 + \frac{\partial^2}{\partial \phi^2} \right).
\]

Furthermore, since \( \partial D = \partial B \), we get

\[
\frac{\partial}{\partial \nu} = r \partial_r, \quad \bar{\partial}_\nu = \bar{z} \partial = \frac{1}{2} (r \partial_r + \sqrt{-1} \partial_\phi).
\]

As \( \partial B \) is smooth we may use powers of the Laplace-Beltrami operator on \( \partial B \) as \( \Psi^* \Psi \). For simplicity, we set

\[
\vartheta = 1, \quad \Psi^* \Psi = 2 \left( 1 - \frac{\partial^2}{\partial \phi^2} \right)^{\rho/2} a_0^{(2)} (z) = |z|^{2d}, \quad d \geq 0.
\]

If \( 0 < d \leq 1/2 \) then \( a_0^{(2)} \in C^{0, 2d} (\overline{D}) \).

To solve the homogeneous equation \( (-\Delta_2 + \lambda^2 |z|^{2d}) u = 0 \) we apply the Fourier method of separation of variables. Writing \( u(r, \phi) = g(r) h(\phi) \) we get two separate equations for \( g \) and \( h \), namely

\[
\left( - (r \partial_r)^2 + \lambda^2 r^{2(d+1)} \right) g = c g, \quad - \frac{\partial^2 h}{\partial \phi^2} = c h,
\]

where \( c \) is an arbitrary constant. The second equation possesses non-zero solutions if and only if \( c = k^2 \) and \( h_k = e^{\sqrt{-1} \kappa \phi}. \) In particular,

\[
\Psi^* \Psi h_k = 2 \left( 1 - \frac{\partial^2}{\partial \phi^2} \right)^{\rho/2} h_k = 2(1 + k^2)^{\rho/2} h_k, \quad \sqrt{-1} \partial_\phi h_k = -kh_k, \quad k \in \mathbb{Z}. \quad (4.6)
\]

Consider the Sturm-Liouville problem for the ordinary differential equation with respect to the variable \( r \) in the interval \((0, 1), \)

\[
(r \partial_r^2 + \partial_r - k^2 r^{-1} + \mu^2 r^{2d+1}) g = 0 \text{ in } (0, 1) \quad (4.7)
\]

\[
g \text{ is bounded at } 0, \quad (4.8)
\]

\[
(r \partial_r - k + (1+k^2)^{\rho/2}) g = 0 \text{ at } r = 1 \quad (4.9)
\]

see [24] Suppl. II, Introduction and P. 1, § 2]. Actually, as we have seen above \( \mu \) are non-negative real numbers (with \( \mu^2 = -\lambda^2 \)) and then (1.7) is a particular case of the Bessel equation. Its (real-valued) solution \( g(r) \) is a Bessel function defined on \((0, +\infty), \) and the space of all solutions is two-dimensional. For example, if \( \lambda^2 = 0 \) and \( d = 0 \) then \( g(r) = \alpha r^k + \beta r^{-k} \) with arbitrary constants \( \alpha \) and \( \beta \) is a general
solution to (4.7). In the general case the space of solutions to (4.7) contains a one-dimensional subspace of functions bounded at the point \( r = 0 \), cf. [25]:

\[
g_k(\mu) = \frac{J_1(\mu |k|)}{d+1},
\]

where \( J_{\mu}(t) \) are Bessel functions (see, for instance, [5]). As usual, for each \( k \in \mathbb{Z} \) the proper system of eigenvalues \( \{ \mu_{k}^{(\nu)} \}_{\nu \in \mathbb{N}} \) can be found as solutions to the transcendental equation

\[
\frac{\mu}{d+1} J'_1(\mu |k|) + \left( (1+k^2)^{\nu/2} - k \right) J_1(\mu |k|) = 0
\]

induced by (4.9) with \( g_k(\cdot, \mu) \) instead of \( g_k(\cdot, \mu) \).

For any \( k \in \mathbb{Z} \), fix a non-trivial solution \( g_{k}^{(\nu)}(r) \) of problem (4.7) corresponding to an eigenvalue \( \mu_{k}^{(\nu)} \). This system is an orthogonal basis in the weighted Lebesgue space \( L^2_{2d}(0, 1) \) with the scalar product

\[
h_{2d}(g, f) = \int_{0}^{1} r^{2d+1} g(r) f(r) \, dr,
\]

see [25, Suppl. II, Introduction and P. 1, § 2]. Then the function

\[
u_k^{(\nu)}(z) = g_k^{(\nu)}(r) e^{\sqrt{-1}k\phi}
\]

satisfies

\[
\begin{cases}
-\Delta u_{k}^{(\nu)}(z) = 0 & \text{in } C,

\partial_{\nu} + (1 - \partial^2_{P_{\phi}})^{\nu/2} u_{k}^{(\nu)}(z) = 0 & \text{at } \partial B.
\end{cases}
\]

(4.10)

where \( (\lambda_{k}^{(\nu)})^2 = - (\mu_{k}^{(\nu)})^2 \) Indeed, by (4.5), (4.7) and the discussion above we conclude that this equality holds in \( C \setminus \{ 0 \} \). We now use the fact that \( u_{k}^{(\nu)} \) is bounded at the origin to see that the differential equation of (4.10) holds in all of \( C \). On the other hand, the boundary condition (4.10) follows from (4.9) immediately, as already mentioned. Now, by the construction, the system \( \{ u_{k}^{(\nu)} \}_{k \in \mathbb{Z}, \nu \in \mathbb{N}} \) consists of eigenfunctions of the family \( L(\lambda) = L_0 + \lambda^2C \) in the case of the unit ball \( \mathbb{B} \) around the origin in \( C \). Obviously, it coincides with the system of all eigenvectors constructed in Theorem 3.2 if it is complete in the space \( L^2_{2d}(\mathbb{B}) \) with the scalar product

\[
h(u, v) = \int_{D} |z|^{2d} u(z) \overline{v}(z) \, dx.
\]

But \( \{ h_k \}_{k \in \mathbb{Z}} \) is an orthogonal basis in \( L^2(\partial \mathbb{B}) \) and \( \{ g_k^{(\nu)} \}_{k \in \mathbb{Z}, \nu \in \mathbb{N}} \) is an orthogonal basis in \( L^2_{2d}(0, 1) \) and hence Fubini Theorem implies that the system is orthogonal basis in the space \( L^2_{2d}(\mathbb{B}) \).

Acknowledgements The work was supported by the grant of the Russian Federation Government for scientific research under the supervision of leading scientist at the Siberian federal university, contract N. 14.Y26.31.0006, and by RFBR grant 14-01-00544.
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