Unloading Behavior of Elastic-power-law Strain Hardening Materials Indented by Elastic Indenter

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Unloading behavior of elastic-power-law strain hardening materials indented by elastic indenter

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Abstract

Both strain hardening and indenter elastic deformation usually cannot be neglected in engineering contacts. By the finite element (FE) method, this paper investigates the unloading behavior of elastic-power-law strain-hardening half-space frictionlessly indented by elastic sphere for systematic materials. The effects of strain hardening and indenter elasticity on the unloading curve, cavity profile during unloading and residual indentation are analyzed. The unloading curve is observed to follow a power-law relationship, whose exponent is sensitive to strain hardening but independent upon indenter elastic deformation. Based on the power-law relationship of the unloading curve and the expression of the residual indentation fitted from the FE data, an explicit theoretical unloading law is developed. Its suitability is validated numerically and experimentally by strain hardening materials contacted by elastic indenter or rigid flat.

Keywords: Contact mechanics, Unloading, Elastic-power-law strain-hardening, Finite element method
1. Introduction

Fundamental to all assembled systems, contact mechanics is integral to mechanical design. This is evident in various material processes and engineering applications, such as hardness measurement [1, 2], particle and powder interactions [3, 4], particle erosion [5], thermal spray [6-8], electrical contact [9-11], biomechanics [12-15] and additive manufacturing [16, 17]. Surface curvature or roughness often causes the extremely small contact area, resulting highly local deformation and stress concentrations. Most local contact events are characterized by plastic deformation [18, 19] that reduces the contact force, produces a permanent indentation and makes the unloading path to be different from the loading path. Due to the simplicity and motive by the hardness tests, most studies focus on the deformation analysis of the contact of elastic-plastic material by rigid indenter or rigid flat.

A complete elastic-plastic contact cycle consists of both the loading and unloading phases. For elastic contact problem, Hertz contact model [20] is widely applied. Using the methodology originated from Hertz’s analysis of the contact between two non-conforming elastic solids [20], two main types of elastic-plastic contact models consisting of both the loading and unloading laws were developed, including the indenting models [18, 21-28] and the flattening models [29-35]. These contact models are derived from the theoretical analyses or by curve-fitting approach from the numerical results. Recently, both the deformations of the sphere and the half-space are considered by Dong, et al. [36], Ghaednia and Jackson [37], Ghaednia, Dan [38] and Cermik, Ghaednia [39]. Although the loading law is comprehensively studied, the
unloading law is investigated insufficiently.

To predict the elastic-plastic contact behavior accurately, the unloading phase is as important as the loading phase [40-44]. The measurement of material properties, like yield strength, and elastic modulus is always relied on the analysis of the unloading phase [45]. However, the analysis of unloading phase is cumbersome and difficult due to the complicated deformation behavior, such as shallowing [46], pile-up [47-51], re-yielding in the surface region just outside the contact area [47, 51] and high residual stress [51]. Few studies have been devoted to the unloading behavior of elastic-plastic contact. The previous studies mainly aimed at the unloading behavior of the elastic-perfectly plastic materials.

For the elastic-perfectly plastic materials, the unloading is generally assumed to be elastic [24-26, 36, 43, 52, 53]. The unloading law is usually characterized by the indentation \( \delta_m \) and contact force \( F_m \) at the beginning of unloading, and residual indentation \( \delta_r \) and the residual radius of contact curvature \( R^{*e} \) after fully unloading. \( \delta_m \) and \( F_m \) are determined by the loading phase. \( \delta_r \) and \( R^{*e} \) are determined by the aid of the assumptions accounting for the plastic deformation effects. Generally, the unloading laws for elastic-perfectly plastic materials can be divided into two types: Hertzian-type and non-Hertzian-type. For Hertzian-type unloading laws, one of \( \delta_r \) and \( R^{*e} \) should be determined in advance and then another one can be calculated according to continuity condition between loading phase and unloading phase.

For some Hertzian-type unloading laws, the expression of \( \delta_r \) is fitted from the FE data or from the limited experimental data [22, 38, 54]. For other Hertzian-type
unloading laws, $R^e$ is solved by some theoretical assumptions [25, 26, 28, 30-32, 36, 43, 52, 53, 55-58]. Du and Wang [55] enforced $R^e$ to be the effective contact radius $R^e$. Thornton [30] assumed a continuity of a defined contact area to calculate $R^e$. Li, Wu [52] solved $R^e$ from the assumed parabola relation between indentation and contact area. Vu-Quoc, Zhang [32] calculated $R^e$ by a defined relationship of $R^e$ with contact force. Stronge [25] solved $R^e$ by assuming a geometry similarity. Dong, Yin [36] and Chen, Yin [43] revised Strong’s geometry similarity to calculate $R^e$ by introducing a scaling factor. Different solving equations for $R^e$ were suggested by Brake [53] and Big-Alabo, Harrison [28] for unloading during elastic-plastic and fully plastic loading phases, respectively.

For non-Hertzian-type unloading laws, Li and Gu [44] has presented the empirical relation of contact force to indentation by curve-fitting approach from the FE and experimental results. Christoforou, Yigit [59], Christoforou, Yigit [60] and Ibrahim and Yigit [61] simplified the unloading law by the linearizing approach.

In fact, most engineering materials exhibit strain hardening during inelastic deformation, which causes the yield strength to increase with the evolution of plasticity [62]. The investigations of unloading behavior are mainly devoted to two types of strain hardening behavior: linear strain hardening and elastic-power-law strain hardening. For the linear strain hardening elastic-plastic material, some unloading laws have been proposed by Etsion, Kligerman [63], Kadin, Kligerman [64], Biplab and Prasanta [65], Chatterjee and Sahoo [66], Chatterjee and Sahoo [67] and Shankar, Arjun [68]. For 2% linear hardening elastic-plastic materials in contact with a rigid flat, Etsion, Kligerman
suggested two exponential unloading relations of contact force to indentation and contact area to indentation. The exponent in the unloading law does not equal to 1.5 for the Hertzian-type unloading law. It is dependent upon $\delta_m / \delta_y$, where $\delta_y$ is initial yield indentation. For the elastic-power-law strain-hardening materials, some unloading behaviors have been studied by Kral, Komvopoulos [47], Song and Komvopoulos [62], Zhao, Nagao [34] and Zhao, Zhang [35]. For the unloading behavior of half-space indented by rigid sphere, Kral, Komvopoulos [47] studied the effect of strain hardening by using the FE method. It is found that after an initial linear response, the unloading curve tends to a nonlinearity which becomes more pronounced with increasing strain hardening. For the material with the strain hardening exponent 0.5, the residual indentation at the contact center is about one-half of that for the nonhardening material. It illustrates that the strain hardening plays a significant role in the permanent deformation. By using FE method, Zhao, Nagao [34] and Zhao, Zhang [35] investigated the unloading of an elastic-power-law strain-hardening sphere compressed by a rigid flat under the frictionless and stick conditions, respectively. By using the form of the unloading law suggested by Etsion, Kligerman [63], they fitted the exponents from the FE data. Despite of the dependence on $\delta_m / \delta_y$, the exponents are found to depend on the power-law strain-hardening exponent for the frictionless contact by Zhao, Nagao [34], or depend on both the power-law strain-hardening exponent and Passion’s ratio for the stick contact by Zhao, Zhang [35], but be independent upon the yield strain. However, Song and Komvopoulos [62] found that the residual indentation is dependent on both the power-law strain-hardening exponent and yield strain.
Till now, most studies are focused on the unloading behavior of the elastic-perfectly plastic materials or the strain hardening elastic-plastic materials contacted by a rigid sphere or a rigid flat. Taljat, et al. [45, 69] found that there are discernible or large differences between the unloading force-indentation curves obtained by the experimental test and calculated by the FE method when the indenter is modeled as rigid. They pointed out that even though the ratio of elastic moduli between the indenter and indented materials is at least three times, the indenter compliance cannot be neglected. Rodriguez, Alcala [70] showed that there are 26% and 17% permanent indentation errors for the spherical indenter and for the conical and Berkovich indenters, respectively, even though the reduce modulus are used in the conventional model to take into account the indenter elastic deformation. For the unloading behavior of elastic-perfectly plastic materials, the indenter elastic deformation has been considered and found to have significant influence recently by Dong, Yin [36] and Chen, Yin [43]. A comprehensive study of unloading behavior for power-law strain hardening materials accounting for the indenter elasticity is still lacked.

The objective of this paper is to study comprehensively the unloading behavior of the frictionless contacts for a wide range of material combinations between elastic sphere and elastic-plastic half-space by considering material strain hardening effect. Abundant information on unloading is provided by intensive FE simulations. Based on the FE results, the effects of both strain hardening and indenter elasticity on the unloading curve, cavity profile during unloading and residual indentation are analyzed. Analytical expression of the residual indentation is fitted from the FE data. Then the
unloading law is developed based on the analysis of the unloading curve and the expression of the residual indentation. The unloading law is sufficiently validated numerically and experimentally. It demonstrates that the present unloading law is accuracy and can be suitable for elastic-plastic materials with or without strain hardening contacted by elastic indenter or rigid flat ranging from small to large deformations.

2. Analysis method

2.1 Problem description and elastic unloading law

As a discriminating feature for the contact problem, the profile of the indented surface is characterized by pile-up or sink-in depending on the hardening properties of material [71, 72]. By considering the hardening strain effect, Fig. 1 depicts a schematic of an elastic-plastic half-space indented by an elastic sphere in both pile-up and sink-in deformation cases. In the figure, the radius of the sphere is $R$. The dotted line represents the permanent indent profile after unloading.

![Fig. 1 A schematic illustration of pile-up and sink-in deformations.](image)
During the loading phase, the sphere penetrates the half-space to a total indentation depth $\delta$ with an indentation of the half-space $\delta_1$, an indentation of the sphere $\delta_2 = \delta - \delta_1$ and a contact radius $a$. Static indentation loading was simulated by incrementally advancing the elastic sphere into the half-space to the total maximum indentation depth $\delta_m$ at the maximum force $F_m$, accompanying the indentation of the half-space $\delta_{im}$ and the maximum contact radius $a_m$. Then, an unloading is performed by incrementally retracting the elastic sphere from the half-space in a similar manner as for the loading. $\delta_m$ is very small compared to the radius of the sphere, and the analyses is carried out based on the small strain theory. After the fully unloading, the plastic deformation during loading yields a residual impression depth $\delta_r$, called residual indentation. For an elastic-perfectly plastic contact, the residual cavity profile is usually assumed to be a spherical surface with a uniform radius $R_{res}$ [25, 26, 43, 53].

For elastic-perfectly plastic contact, the force-indentation relationship during unloading is usually assumed to follow the Hertz contact law, but changing the effective contact radius $R^*$ to the residual radius of contact curvature $R^{*e}$ [25, 26, 36, 38, 53, 54]. The Hertz-type unloading law follows

$$F = 4E^* \sqrt{R^{*e}} (\delta - \delta_r)^{3/2}$$  \hspace{1cm} (1)

where $E^*$, $R^{*e}$, $\alpha = 1.5$ and $\delta_r$ are the effective elastic modulus, residual radius of contact curvature, exponent and permanent indentation,

$$\frac{1}{E^*} = \frac{1}{E_1} + \frac{1}{E_2} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

where $E_1^*$, $E_2^*$, $E_1$, $E_2$, $\nu_1$ and $\nu_2$ denote the plane strain elastic modulus, elastic
modulus and Poisson’s ratio for the elastic-plastic half-space and the elastic sphere, respectively. The critical contact force, indentation and contact radius at the inception of contact yield, $F_y$, $\delta_y$ and $a_y$, are usually applied to normalize $\delta$, $F$ and $a$, respectively,

\[
\delta_y = (3\pi / 4)^2 (\sigma_y \sigma_y / E^*)^2 R^*
\]

\[
F_y = 4E^* R^{*0.5} \delta_y^{1.5} / 3
\]

\[
a_y = (\delta_y R^*)^{0.5}
\]

where $1/R^* = 1/R_1 + 1/R$, $\sigma_y$ is a constant coefficient 1.1 for spherical indenter [18], and the yield strength $\sigma_y$ [53, 73, 74] is

\[
\sigma_y = \min[\sigma_{y1}, \sigma_{y2}]
\]

where $\sigma_{y1}$ and $\sigma_{y2}$ are the yield strength of the half-space and sphere, respectively. For elastic-perfectly plastic contact, a residual cavity with a uniform radius $R_{res}$ [25, 26, 53] remains on half-space after unloading. Similar to the definition of the effective contact radius $R^*$, the residual radius of contact curvature $R^{*e}$ can be defined as

\[
1/R^{*e} = 1/R - 1/R_{res}
\]

In this paper, to investigate more general unloading law, the normalized contact variables $F / F_y$, $F / F_m$, $a / a_y$, $\delta / \delta_m$ and $\delta / \delta_y$ are introduced.

2.2 FE model

In this section, the FE model of an elastic-plastic half-space indented by an elastic sphere is built to consider the effects of both strain hardening and indenter elasticity on unloading behavior. The half-space material is assumed to be elastic-plastic, whose plastic behavior obeys the $J_2$ flow theory and satisfies a power-law strain hardening
law reconstructed by the classical Ramberg-Osgood curve [75]. The deformation gradient is decomposed into elastic and plastic components, and the equivalent plastic strain $\varepsilon_{eq}$ is defined as

$$
\varepsilon_{eq} = \int_\Omega \left[ 2d\varepsilon^p_{ij}d\varepsilon^p_{ij} / 3 \right]^{0.5}
$$

(7)

where the integration is carried out over the strain path $\Omega$, and $d\varepsilon^p_{ij}$ denotes the components of the plastic strain increment. The volume variation under plastic deformation is negligible. Elastic constitutive equation is used when $\sigma_{eq} < \sigma_y$, in which $\sigma_{eq}$ is the von Mises equivalent stress.

The relation of the strain $\varepsilon$ to the stress $\sigma$ for an elastic-power-law strain-hardening material is given as

$$
\sigma = \begin{cases} 
E_i \varepsilon, & \sigma \leq \sigma_y \\
\sigma_y ((E_i / \sigma_y) \varepsilon)^n, & \sigma > \sigma_y 
\end{cases}
$$

(8)

where $n$ is the strain hardening exponent. $n = 0$ indicates the elastic-perfectly plastic case, which can be considered as a special strain hardening case.

Fig. 2 shows the finite element model used in the present study. Since a large number of simulations are carried out, the symmetry 2-D contact is modeled with all symmetries considered to improve upon the efficiency of computation by utilizing the commercial program ANSYS. Normal contact between the elastic sphere and elastic-plastic medium is assumed to be frictionless. The elastic sphere is modeled as a circular of the radius $R$. The half-space is modeled as a square of the length of side $L$. The contact area is much localized and the contact radius is much smaller than all other dimensions. The remote boundaries do not affect the solution [33] because of the Saint Venant’s principle. The size of the square is selected to be large enough in order to
eliminate the size effect on contact, $L/R > 22$, according to the convergence tests. The sphere is pressed into contact with the cylinder fixed at the bottom plane. The half-space and sphere are meshed by axisymmetric, isoperimetric elements. Nodes on the bottom surface of the sphere are displaced uniformly by imposing a downward vertical displacement, which is also the indentation characterizing the contact deformation. Nodes on the symmetry axis of the half-space are constrained against displacement in the horizontal direction, and those on the bottom boundary are constrained against displacement in both the horizontal and vertical directions.

**Fig. 2** Meshes used for contact simulating in (a) global zone, (b) contact zone and (c) transition zone.
To ensure the computational accuracy, the contact area is meshed regularly and densely as shown in Fig. 2 (b). A coarse mesh is selected in the low stress zone far away from the contact area, so as to improve the computation efficiency. In order to avoid numerical errors caused by a sudden change in the mesh size, the element size gradually increases in a radial gradient for outside high stress zone as shown in Fig. 2 (c). The value of stiffness for the contact element and the tolerance of the current work are set according to the mesh convergence tests. The contact area is controlled to be meshed at least 80 contact elements in contact for each applied indentation according to the convergence tests. As the indentation increases, the contact zone evolves from the small deformation in early stage to large deformation. To achieve the balance at computations in the full indentation range, the sequential finite element sub-models are designed in order to resolve indentations ranging from small to large deformation. In the present study, \( a / R \) is designed to range from 0.002 to 0.42. For the fast modeling, the finite element sub-models are designed parametrically utilizing the APDL language and created automatically by ANSYS software. For different sub-models, the half-space consists of 10434~13074 elements with a total of 10552~13342 nodes, and the sphere consists of 7200~10800 elements with a total of 7411~11101 nodes.

To verify the accuracy of the FE model, the half-space (\( L = 800 \text{mm} \)) indented by the sphere (\( R = 35 \text{mm} \)) is simulated for both loading and unloading phases, where \( \sigma_{r_1} = \sigma_{r_2} = 2550 \text{ MPa}, \quad \nu_1 = \nu_2 = 0.3 \) and \( E_1 = E_2 = 200 \text{ GPa} \). In the simulation, no plastic deformation takes place and the contact is pure elastic. Fig. 3 shows the comparisons of the simulated curves with the Hertz solution. It shows that the FE results
of the contact force and contact radius are in good agreement with the Hertz solution.

The loading and unloading curves are almost the same. The simulation errors of the maximum contact force and maximum contact radius are 1.2% and 2.2%, respectively.

The close agreements between the FE simulation results and the Hertz solutions validate the modeling assumptions and the suitability of the FE mesh in Fig. 2.

Fig. 3 FE model verifications: (a) contact force-indentation curve and (b) contact radius.

Table 1 Material combinations.

| Material ID | $E_1^*/E_2^*$ | $\sigma_y$ (MPa) | $E_1^*/\sigma_y$ | $E_1^*/\sigma_y$ |
|-------------|---------------|------------------|-----------------|-----------------|
| 1           | 0.252         | 50               | 921.30          | 1153.84         |
| 2           | 0.505         | 50               | 1533.54         | 2307.69         |
| 3           | 1.010         | 50               | 2296.65         | 4615.38         |
| 4           | 2.019         | 50               | 1528.66         | 4615.38         |
| 5           | 4.038         | 50               | 916.03          | 4615.38         |
| 6           | 0.252         | 120              | 383.87          | 480.76          |
| 7           | 0.505         | 120              | 638.97          | 961.53          |
| 8           | 1.010         | 120              | 956.93          | 1923.07         |
| 9           | 2.019         | 120              | 636.94          | 1923.07         |
| 10          | 4.038         | 120              | 381.67          | 1923.07         |
| 11          | 0.252         | 345              | 133.52          | 167.22          |
| 12          | 0.505         | 345              | 222.25          | 334.44          |
| 13          | 1.010         | 345              | 1533.54         | 2307.69         |
| 14          | 2.019         | 345              | 221.54          | 668.89          |
| 15          | 4.038         | 345              | 132.75          | 668.89          |
| 16          | 0.252         | 550              | 167.51          | 209.79          |
| 17          | 0.505         | 550              | 139.41          | 209.79          |
| 18          | 1.010         | 550              | 208.78          | 419.58          |
| 19          | 2.019         | 550              | 138.96          | 419.58          |
To comprehensively study the unloading behavior of elastic-plastic strain hardening materials, 1525 FE simulations are performed for a wide range of materials. The radius of the sphere $R$ is 35 mm and length of the half-space $L$ is 800 mm. The Poisson’s ratios of the half-space $\nu_1$ and sphere $\nu_2$ are 0.3, respectively (a common value for metallic materials). Twenty kinds of sphere and half-space material combinations and five levels of strain hardening exponent $n$ (0, 0.2, 0.3, 0.5, 0.6) are selected for the simulations. Hence, a total of 100 sets of mechanical properties are simulated, where the twenty kinds of material combinations are listed in Table 1. The material yield strain covers from $E^*/\sigma_y=138.96$ to $4615.38$, where $E^*/\sigma_y=$ 132.75 to 2296.65 and $E_i^*/E_2^*=0.252$ to 4.038. $E_i^*/E_2^*=0.252$ indicates the sphere is almost to be rigid and $E_i^*/E_2^*=4.038$ indicates the sphere is relative soft.

3. Results and Discussion

By means of the FE models developed above, the unloading behavior of elastic-plastic contact is investigated for the elastic-power-law strain-hardening materials contacted by elastic indenter. The unloading indentation $\delta_m$ is selected through a wide range of deformations from $1.2 \delta_y$ to $218 \delta_y$. For post-yield deformations, the unloading behavior is strongly associated with the initial unloading, unloading curve and residual deformation, which are crucial to characterize the unloading behavior. The initial unloading is regard as the maximum indentation of both deformable objects during loading. The unloading curve represents the relationship of unloading contact force to indentation. The residual deformation contains a cavity profile and a residual
indentation $\delta_i$. The effect of both strain hardening and indenter elasticity on the unloading behavior is discussed in the bellows. Based on the analyses of the unloading behavior, a new unloading model is presented by fitting the FE data to consider the effects of both strain hardening and indenter elasticity.

### 3.1 Strain hardening effect

Most engineering metals and alloys exhibit strain hardening. Strain hardening increases the capacity of the material to accumulate plastic deformation at stress levels above the initial yield strength. In this section, the effect of strain hardening on the unloading behavior is investigated.

#### 3.1.1 Unloading curve

To examine the effect of strain hardening on the unloading curves, Fig. 4 (a) and (b) show the simulated unloading curves for Mats. 11 and 14 with five stain hardening exponents when $\delta_m / \delta_y = 197.77$ and $\delta_m / \delta_y = 198.18$, respectively. For the sake of comparison, the normalized indentations have been shifted laterally by subtracting the residual indentation $\delta_r$ from the total indentation $\delta$, in order to make the normalized curves to pass through a common origin. The careful examinations observe that none of the curves is linear. Each unloading curve is slightly concave up over its entire span.

As mentioned by Oliver and Pharr [76], unloading curves can be well described by a power-law relation like

$$F = \alpha \delta^\gamma$$

(9)

Therefore, in Fig. 4, the power-law fitting curves are plotted and values of $\gamma$ are calculated as listed in the plots. $\gamma$ is found to vary with the strain hardening exponents...
As can be seen from Fig. 4, for Mat. 11 and Mat. 14 with \( n = 0 \) (elastic-perfectly plastic material), \( \gamma = 1.501 \) and 1.502. They are nearly as the same as \( \gamma = 1.5 \) for the Hertzian-type unloading law [43]. However, when \( n > 0 \), \( \gamma \) is distinctly smaller than 1.5. It varies from 1.327 to 1.402. It demonstrates that for strain hardening materials the unloading curve no longer meets the Hertzian-type unloading law, but it still meets a power-law relationship.

Fig. 4 Effect of strain hardening on unloading curve: (a) Mat. 11 \((\delta_n / \delta_y = 197.77)\) and (b) Mat. 14 \((\delta_n / \delta_y = 198.18)\).

### 3.1.2 Residual indentation

Fig. 5 shows the effect of strain hardening on \( \delta_{im} / \delta_m \). The simulation data for the same \( E_1^* / E_2^* \) and the same strain hardening exponent \( n \) are found to coalesce to a single master curve. For elastic-perfectly plastic materials \((n = 0)\), \( \delta_{im} / \delta_m \) has the largest value. For strain hardening materials contacted by elastic intender, \( \delta_{im} / \delta_m < 1 \).

Because the strain hardening will reduce the plastic indentation of the half-space, \( \delta_{im} / \delta_m \) will decrease with the increasing \( n \) as shown in Fig. 5.
Fig. 5 Effect of strain hardening on maximum indentation of half-space: (a) $E'_1 / E'_2 = 0.252$, (b) $E'_1 / E'_2 = 0.505$, (c) $E'_1 / E'_2 = 1.010$, (d) $E'_1 / E'_2 = 2.019$ and (e) $E'_1 / E'_2 = 4.038$.

Fig. 6 shows the effect of strain hardening on the residual indentation. The simulation data for the same $E'_1 / E'_2$ and the same strain hardening exponent $n$ are found to coalesce to a single master curve. For elastic-perfectly plastic materials ($n = 0$), $\delta_r / \delta_m$ has the largest value. For strain hardening materials, $\delta_r / \delta_m$ decreases with the increasing $n$. The variation feature of $\delta_r / \delta_m$ is found to be similar to $\delta_{im} / \delta_m$. 
Fig. 6 Effect of strain hardening on residual indentation (solid lines are fitting curves): (a) $E'_1/E'_2 = 0.252$, (b) $E'_1/E'_2 = 0.505$, (c) $E'_1/E'_2 = 1.010$, (d) $E'_1/E'_2 = 2.019$ and (e) $E'_1/E'_2 = 4.038$.

3.1.3 Pile-up deformation and cavity profile

For unloading, the spherical surface assumption is generally applied to the indentation of elastic-perfectly plastic materials by rigid sphere [25, 26, 53]. The shape of the residual cavity is assumed to be a spherical surface with a uniform radius $R_{res}$ after the indenter is unloaded and the material elastically recovers. This assumption was
validated by Chen, Yin [43] for the elastic-perfectly plastic materials indented by elastic sphere. The importance of this assumption is that the elastic contact solution exists during the unloading of spherical indentation. For the indentation of elastic-power-law strain-hardening materials by elastic spherical indenter, the spherical assumption should be validated.

Fig. 7 shows the simulated residual cavity profiles for Mat. 14 with five strain hardening exponents under $\delta_m / \delta_y = 198.18$. The profiles are normalized by the sphere radius $R$ to make the solution universal. As can be seen from Fig. 7, the strain hardening plays a significant role on the residual cavity profile. If the residual cavity is fitted by a sphere surface, the sphere surface can be in perfect agreement with the residual cavity profile for the elastic-perfectly plastic material ($n = 0$). However, for the strain hardening materials ($n > 0$), the fitting circles deviate from the simulated profiles. The aspheric surface of the residual cavity profile becomes more significant for the stronger strain hardening with larger $n$. The spherical surface assumption for the residual cavity profiles is less of validity. Hence, the residual cavity profiles can be divided two types according to the types of indented materials. For elastic-perfectly plastic materials, the residual cavity profile is spherical. For strain hardening materials, the residual cavity profile is aspheric.
As can be seen from Fig. 7, the strain hardening also plays a significant role in the pile-up deformation. The residual pile-up deformation occurs outside the contact region for all the contact cases. For elastic-perfectly plastic materials \((n = 0)\), the residual deformation exhibits the largest pile-up deformation and the sharpest transition from the contact region to the free boundary. For elastic-power-law strain-hardening materials \((n > 0)\), the residual pile-up deformation becomes smaller as \(n\) increases. For a rigid sphere indenting into elastic-power-law strain-hardening materials, Kral, Komvopoulos [47] obtained the same result. The residual pile-up is caused by the residual compressive stress, which is the greatest for the nonhardening materials and becomes smaller with increasing strain hardening [47].

To analyze the effect of strain hardening on the cavity profile during unloading process, the cavity profiles at four unloading indentations, \(\delta = \delta_m, \; \delta = 2/3\delta_m, \; \delta = 1/3\delta_m\), and \(\delta = \delta_r\) are extracted from the simulated results. For Mat. 10 \((E_1/E_2 = 4.038)\) with \(n = 0\) and \(n = 0.6\), Fig. 8 shows the cavity profiles at the four unloading
indentations when $\delta_m / \delta_y = 200$, where the contact area is marked by the green color.

As can be seen from Fig. 8 (a), for the elastic-perfectly plastic material ($n = 0$), the fitting spherical surface can always be in perfect agreement with the whole cavity profile including both the contact area and outside region of the cavity, even though the contact area decreases to zero. It illustrates that the whole cavity profile maintains a spherical surface during the entire unloading process.

As can be seen from Fig. 8 (b), for the strain hardening material ($n = 0.6$), the fitting surface can only be in coincident with the part of the contact area. The outside cavity profile cannot maintain the same spherical surface. After the fully unloading, the whole cavity profile becomes entirely aspheric.

![Fig. 8](image)

**Fig. 8** Normalized residual cavity profile (○) and circle curve fitted (—) during unloading process: (a) $n = 0$ and (b) $n = 0.6$ ( green represents contact area).

### 3.2 Indenter elasticity effect

The indenter deformation cannot be neglected for the contact by a softer indenter. The level of indenter elasticity can be represented by $E^e / E^s$. In this section, the effect of indenter elasticity on the unloading behavior is investigated.
3.2.1 Unloading curve

Fig. 9 shows the effect of elastic deformation of indenter on the unloading curve. Under $\delta_m/\delta_y=200$, the simulated unloading curves are plotted for Mats. 6-10 with $n=0$ and $n=0.6$. As can be seen from Fig. 9 (a), for the elastic-perfectly plastic materials ($n=0$), all the normalized unloading curves for different values of $E'_1/E'_2$ coalesce to a single master power-law curve with the exponent $\gamma=1.5$. The Hertzian-type unloading law can be applied. The elastic deformation of indenter has almost no influence on the unloading curve.

As can be seen from Fig. 9 (b), for the strain hardening materials ($n=0.6$), the normalized unloading curves for different values of $E'_1/E'_2$ cannot coalesce to a single master curve. They are scattered and dependent on the elastic deformation of indenter. The power-law fitting curves have different values of $\alpha$ for different elastic modulus ratios $E'_1/E'_2$, but have almost the same exponent $\gamma=1.401$. It means that the elastic deformation of indenter significantly influences the unloading curve, but does not affect the power-law exponent of the unloading curve.

Fig. 9 Effect of indenter elasticity on unloading curve: (a) $n=0$ and (b) $n=0.6$. 
3.2.2 Residual indentation

Fig. 10 shows the effect of elastic deformation of indenter on indentation deformation of the half-space $\delta_{im}/\delta_m$. For each $n$, the simulation data for the same $E'_1/E'_2$ and are found to coalesce to a single master curve. $\delta_{im}/\delta_m$ has the largest value for $E'_1/E'_2=0.252$ and decreases with the increasing $E'_1/E'_2$. It is reasonable because the softer spherical indenter has larger elastic deformation. The increasing rate of $\delta_{im}/\delta_m$ decreases as $\delta_m/\delta_l$ increases due to the increase of the plastic deformation. The increasing rate for the softer spherical indenter is larger than the harder spherical indenter. For the hardest spherical indenter $E'_1/E'_2=0.252$, $\delta_{im}/\delta_m$ approaches to a constant after $\delta_m/\delta_l=50$. 

![Graphs showing the effect of elastic deformation of indenter on indentation deformation of the half-space.](image)
Fig. 10 Effect of indenter elasticity on maximum indentation of half-space: (a) $n = 0$, (b) $n = 0.2$, (c) $n = 0.3$, (d) $n = 0.5$ and (e) $n = 0.6$.

Fig. 11 shows the effect of elastic deformation of indenter on the residual deformation. For each $n$, the simulation data for the same $E_1^*/E_2^*$ and are found to coalesce to a single master curve. $\delta_r/\delta_m$ increases with the decreasing $E_1^*/E_2^*$. Because the softer spherical indentation has larger elastic deformation and smaller plastic deformation of the half-space, $\delta_r/\delta_m$ has the largest value for the hardest spherical indenter ($E_1^*/E_2^* = 0.252$) and decreases as the spherical indenter become softer. The increasing rate of $\delta_r/\delta_m$ decreases as $\delta_m/\delta_r$ increases due to the increase of the plastic deformation. The increasing rate for the softer spherical indenter is larger than the harder spherical indenter. For the elastic-perfectly plastic materials ($n = 0$), the elastic deformation of indenter has very slight affection. The $\delta_r/\delta_m$ curves for different $E_1^*/E_2^*$ almost coalesce to a same master curve. For the strain hardening materials ($n > 0$), the $\delta_r/\delta_m$ curves become more and more separate as the strain hardening increases. It indicates that the elastic deformation of indenter has more significant effect on the residual indentation for strain hardening materials, while nearly has no influence on $\delta_r/\delta_m$ for elastic-perfectly plastic materials.
Fig. 11 Effect of indenter elasticity on residual indentation: (a) \( n = 0 \), (b) \( n = 0.2 \), (c) \( n = 0.3 \), (d) \( n = 0.5 \) and (e) \( n = 0.6 \).

3.2.3 Cavity profile

To consider the effect of indenter elasticity on the residual cavity, Fig. 12 shows the effect of elastic deformation of indenter on the residual cavity profile. The simulated cavity profiles of the five material combinations when \( \delta_m / \delta_y = 200 \) are plotted. The five material combinations have the same half-space material with different elastic
modulus of the sphere indenter. The three material combinations are Mat. 8 \( \frac{E_1^s}{E_2^s} = 1.010 \), Mat. 9 \( \frac{E_1^s}{E_2^s} = 2.019 \) and Mat. 10 \( \frac{E_1^s}{E_2^s} = 4.038 \). The other two material combinations Mat. 21 and Mat. 22 are not listed in Table 1. Their sphere elastic moduli are chosen according to \( \frac{E_1^s}{E_2^s} = 0.252 \) and 0.505, respectively. For each material combination, the FE simulations are implemented for the elastic-perfectly plastic material \( (n = 0) \) and the strain hardening material \( (n = 0.6) \).

As shown in Fig. 12 (a), if the residual cavity is fitted by a sphere surface, the sphere surface can be in perfect agreement with the residual cavity profile for the elastic-perfectly plastic materials \( (n = 0) \). All the residual cavity profiles maintain perfectly spherical surface. The spherical shape of the residual cavity is not influenced by elastic deformation of indenter. As shown in Fig. 12 (b), the fitting circles deviate from the simulated profiles for the strain hardening materials. The residual cavity profiles are aspheric surface. The aspheric surface of the residual cavity profile becomes more significant for higher \( \frac{E_1^s}{E_2^s} \) (softer spherical indenter). It means that the elastic indenter deformation significantly affects the residual cavity profile for strain hardening materials, while has no influence on the spherical shape of the residual cavity for elastic-perfectly plastic materials.
Fig. 12 Normalized residual cavity profile (○) and circle curve fitted (——) for simulations with different effective elastic modulus ratios under $\delta_1 / \delta_2 = 200$: (a) $n = 0$ and (b) $n = 0.6$.

To analyze the effect of indenter elasticity on the cavity profile during unloading process, the cavity profiles at the unloading indentations $\delta = \delta_m$, $\delta = 2/3 \delta_m$, $\delta = 1/3 \delta_m$ and $\delta = \delta$, are extracted from the simulated results. The simulated material combination is Mat. 21 with $E_1' / E_2' = 0.252$ as used in Fig. 12. The half-space materials, $\delta_m / \delta_y$ and $n$ are the same as those in Fig. 8. Fig. 13 shows the cavity profiles at the four unloading indentations when $\delta_m / \delta_y = 200$. The contact area is marked by the green color. Compared with the spherical indenter with $E_1' / E_2' = 4.038$ used in Fig. 8, the spherical indenter with $E_1' / E_2' = 0.252$ used in Fig. 13 is harder. To compare the results for hard spherical indenter in Fig. 13 with those for soft spherical indenter in Fig. 8 can examine the effect deformation of indenter on the cavity profile.

By comparing the cavity profiles in Fig. 13 (a) with those in Fig. 8 (a), all the cavity profiles are spherical surfaces for the elastic-perfectly plastic materials ($n = 0$). The spherical shape of the cavity profile has not changed for the hard and soft spherical indenters. By comparing of the cavity profiles in Fig. 13 (b) with those in Fig. 8 (b), all
the cavity profiles are aspheric surfaces for the stain hardening materials \((n = 0.6)\), the aspheric cavity profile has not changed for the hard and soft spherical indenters. It can be concluded that the indenter elastic deformation does not change the spherical or aspheric feature of the residual cavity during unloading for both elastic-perfectly plastic and strain hardening materials.

![Fig. 13 Normalized residual cavity profile (○) and circle curve fitted (—) during unloading process: (a) \(n = 0\) and (b) \(n = 0.6\) (green represents contact area).](image)

### 3.3 Unloading law

In this section, according to the analyses of unloading behaviors as mentioned in sections 3.1 and 3.2, a new unloading model is presented by fitting the FE data to consider the effect of both strain hardening and indenter elasticity.

#### 3.3.1 Determination of \(\delta_r\)

As discussed in sections 3.1 and 3.2, the variations of \(\delta_r / \delta_m\) with respect to \(\delta_m / \delta_y\) for all the simulation data can be categorized by strain hardening exponent \(n\) or \(E'_1 / E'_2\). It illustrates that the relationship of \(\delta_r / \delta_m\) with \(\delta_m / \delta_y\) is dependent on both \(n\) and \(E'_1 / E'_2\). As can be seen from Fig. 6 and Fig. 11, for a given \(n\) or
$E_1' / E_2'$, the curves of $\delta_i / \delta_m$ versus $\delta_m / \delta_r$ have the same variation trend. Therefore, each curve can be fitted with the same function. The empirical expression of $\delta_i / \delta_m$ is fitted from the FE data. The expression is selected by modifying the formula used by Etsion, Kligerman [63], Zhao, Nagao [34], Song and Komvopoulos [62] and Zhao, Zhang [35],

$$\frac{\delta_i}{\delta_m} = \left(1 - \frac{1}{\left(\frac{\delta_m}{\delta_r}\right)^4}\right)^2$$  \hspace{1cm} (10)

where $A$ is a coupling term between $n$ and $E_1' / E_2'$, and fitted from the FE data

$$A = A_1 + B_1 n$$  \hspace{1cm} (11)

$$A_1 = 0.06754 \ast (E_1' / E_2' + 1)^{-1.0538} + 0.36253$$

$$B_1 = -0.0234 \ast E_1' / E_2' - 0.345$$

If the spherical indenter is rigid, $E_2' = +\infty$, $A_1$, $B_1$ are the constants. Then $A$ is only a linear function of $n$, as the same form suggested by Zhao, Nagao [34].

By comparing with the FE results shown in Fig. 6, the prediction by Eq. (10) shows a good agreement. The maximum fitting errors of $\delta_i$ and $\delta_m - \delta_r$ are 13.22% and 7.45% and the average fitting errors of $\delta_i$ and $\delta_m - \delta_r$ are 3.54% and 1.78%, respectively.

### 3.3.2 Unloading model

The unloading curve is a key unloading behavior and represents the relationship of contact force with indentation. Recently, Zhao, Nagao [34] studied the unloading curve of the contact between a power-law hardening material with a rigid flat. The equations governing the unloading curve of frictionless indentation of an elastic-power-law
strain-hardening material by an elastic sphere will be presented in this section.

The unloading behavior is strongly affected by the maximum loading indentation \( \delta_m \) [43, 63]. To analyze the unloading behavior of an indentation of a material combination, the unloading curves at different indentations need to be simulated. Hence, a total of 1525 unloading curves are simulated for the twenty material combinations listed in Table 1, to reveal the effect of strain hardening and indenter elasticity on unloading. For example, Fig. 14 shows the unloading curves for Mat. 14 with \( n = 0 \) and \( n = 0.6 \) when \( \delta_m / \delta_y = 118.35 \) and 198.18, respectively.

![Fig. 14 Unloading curves of \( F/F_y \) versus \( \delta/\delta_y \) for \( n = 0 \) and \( n = 0.6 \).](image)

For strain-hardening elastic-plastic materials, as discussed in the sections 3.1 and 3.2, the unloading curves can be described by a power-law relation, but the power-law exponents are not 1.5 as the same for the Hertz contact law. For the 2% linear strain hardening elastic plastic sphere compressed by rigid flat, Etsion, Kligerman [63] proposed a unloading law,

\[
F / F_y = F_m / F_y (\delta / \delta_y - \delta_m / \delta_y) / (\delta_m / \delta_y - \delta_i / \delta_y))^m
\]

\[
m = 1.5(\delta_m / \delta_y)^e
\]

where the coefficient \( e \) was fitted from the FE data. This unloading law was applied
by Song and Komvopoulos [62], Zhao, Nagao [34] and Zhao, Zhang [35] for power-law strain hardening materials indented by rigid sphere or compressed by rigid flat.

As analyzed above, under the same $\frac{\delta_m}{\delta_r}$, the exponents of the power-law relation $m$ is merely dependent upon the strain hardening exponent $n$. By detailed observations, the exponent $m$ is also related to the maximum indentation as reported by Etsion, Kligerman [63], Zhao, Nagao [34] and Zhao, Zhang [35]. In the present study, the fundamental form of the unloading law (12) is adopted to express the unloading relation for elastic-power-law strain-hardening materials. By curve-fitting approach, the coefficient $e$ is fitted from the 1525 unloading curves. $e$ is merely dependent upon the strain hardening exponent and can be expressed as:

$$e = \begin{cases} 
\ln(1.5 - 0.57526n) / 5.29831 - 0.076527, & 0 \leq n < 0.3 \\
\ln(1.25346 + 0.24654n) / 5.29831 - 0.076527, & 0.3 \leq n < 1
\end{cases}$$

(13)

The expression (12) with the coefficient $e$ (13) is also suitable for elastic-perfectly plastic materials, because $e = 1.5$ as $n = 0$.

For the new unloading law (12), the residual indentation $\delta_r$ and the coefficient $e$ are solved by new Eqs. (10) and (13), respectively. For example, Fig. 14 shows that the unloading curves predicted by the present unloading law are perfectly coincident with the FE curves for both the elastic-perfectly plastic material and the elastic-power-law strain-hardening materials indented by elastic sphere over a large range of $\frac{\delta_m}{\delta_r}$.

4. Validations

4.1 Numerical verifications

To validate the present unloading law numerically, the existing FE results obtained
by Zhao, Nagao [34] and some new contact simulations based on the real engineering materials are used for comparisons.

By using a FE model, Zhao, Nagao [34] studied the power-law hardening elastic-plastic spheres frictionlessly compressed by rigid flat. For comparisons, the residual indentations and eight unloading curves for $\frac{\delta_m}{\delta_Y} = 60$ and $110$ of strain hardening contacts ($n=0.1, 0.3, 0.5$ and $0.8$) are extracted. Fig. 15 shows that the predictions of residual indentations and unloading curves by the present unloading law are both in good agreement with the numerical results by Zhao, Nagao [34]. Although the present unloading law is obtained from the simulations within $n = 0.6$, it predicts accurately the residual indentations and unloading curves for $n = 0.8$. The comparisons illustrate that the present unloading law can be applied to strain hardening materials compressed by rigid flat.

![Fig. 15](image.png)

**Fig. 15** Numerical comparisons for contacts by rigid flat: (a) residual indentation and (b) unloading curve.

For comparisons, the indentation simulations of some real engineering materials are performed at $\frac{\delta_m}{\delta_Y} = 50, 100, 150$ and $200$, respectively. The FE models are similar to those in section 2.2. The properties of the five engineering materials are listed
in Table 2. The three elastic-perfectly plastic materials with high yield stress of materials 1-3 are selected as the sphere materials. The two power-law strain hardening materials with low yield stress of materials 4-5 are selected as the half-space materials. To ensure the purely elastic deformations of the spheres, the ratio of the sphere yield stress to the half-space yield stress should be larger than 2.5 as suggested by Tabor [46] and 2.0 as suggested by Larsson and Olsson [74]. Four contact cases as listed in Table 3 are simulated. For all contact cases, the ratio of the sphere yield stress to the half-space yield stress is larger than 3.5.

**Table 2** Mechanical properties of the sphere and half-space.

| ID | Material                                | E (GPa) | σ_y (MPa) | ν   | n  |
|----|-----------------------------------------|---------|-----------|-----|----|
| 1  | 25Mn                                    | 210     | 295       | 0.28| 0  |
| 2  | Ti-6Al-2Sn-2Zr-2Mo-2Cr-0.25Si (SS)      | 123     | 1070      | 0.33| 0  |
| 3  | AL-1                                    | 69.6    | 685       | 0.31| 0  |
| 4  | Annealed copper Cu                      | 110     | 33        | 0.3 | 0.5|
| 5  | AISI 1100 steel                         | 210     | 195       | 0.3 | 0.233|

**Table 3** Contact cases.

| Case | Half-space  | Sphere     | E'_s / E'_y | E'_y / σ_y |
|------|------------|------------|-------------|-----------|
| 1    | Material 4 | Material 1 | 0.52        | 2393.36   |
| 2    | Material 4 | Material 3 | 1.58        | 1408.52   |
| 3    | Material 5 | Material 2 | 1.71        | 433.64    |
| 4    | Material 5 | Material 3 | 3.02        | 291.79    |

Fig. 16 and Fig. 17 show respectively the residual indentations and unloading curves solved by the FE models, the present unloading law and the suggested unloading law by Zhao, Nagao [34]. The FE data of the contact force and indentation at the inception of unloading are used for the theoretical predictions. As can be seen from Fig. 16, for Cases 1-4, the maximum prediction errors of the residual indentation predicted
by the present unloading law are 0.9%, 1.64%, 1.95% and 2.0%, respectively. However, the errors predicted by the unloading law suggested by Zhao, Nagao [34] are 14.4%, 41.2%, 16.4 and 26.1%, respectively. These large errors might be due to the neglecting of indenter elasticity effect. As can be seen from Fig. 17, the present unloading law fits the FE curves better than the suggested unloading law by Zhao, Nagao [34]. The comparisons illustrate that the effects of both strain hardening and indenter elasticity on unloading behavior are unneglectable.

![Graph](image1.png)

**Fig. 16** Numerical comparisons for residual indentation of power-law strain hardening materials:
(a) Cases1-2 and (b) Cases 3-4.

![Graph](image2.png)

**Fig. 17** Numerical comparisons for unloading curve of power-law strain hardening materials: (a) Case1 and (b) Case 3.
4.2 Experimental validation

Six compliance curves including loading and unloading phases shown in Fig. 18 were measured by Bartier, Hernot [77]. The experiment used a tungsten carbide sphere to indent an AISI 1065 steel plate. The material properties are listed in Table 4. By the use of the measured force and indentation at the inception of unloading, the present unloading law predicts the residual indentations and unloading curves as shown in Fig. 18. It can be seen that the predictions agree perfectly with the experimental data without any calibration parameters. The maximum predicting error of the residual indentation is 7.0% and the mean error is only 3.79%. Even for the large contact deformation with $\delta_m$ up to 3969.9 $\delta_i$, the prediction is still accurate.

Table 4 Material and geometric properties for the experiment [77].

| Material             | Elastic Modulus (GPa) | Poisson’s ratio | Radius (mm) | Yield strength (MPa) | Strain hardening exponent |
|----------------------|-----------------------|-----------------|-------------|----------------------|--------------------------|
| AISI 1065 Steel      | 210                   | 0.30            | $\infty$   | 285                  | 0.2                      |
| Tungsten Carbide     | 600                   | 0.28            | 1.25        |                      |                          |

Fig. 18 Experimental comparisons for unloadings: (a) residual indentation and (b) unloading curve.
5. Conclusions

In the present study, the unloading behavior of elastic-power-law strain-hardening half-space indented without friction by elastic spherical indenter is investigated comprehensively in the light of FE method. All 1525 FE simulations are performed for twenty material combinations covering a wide range of strain hardening exponents, yield strengths and indenter elasticity. The intensive FE results show that both the strain hardening and the indenter elastic deformation have significant influences on unloading behaviors. The main contributions of the present study are concluded as the followings:

(1) Unloading curves can be well described by a power-law relation. The power-law exponent of unloading curve is strongly dependent on strain hardening while independent on $\frac{E_1^e}{E_2^e}$. For elastic-perfectly plastic materials, the exponents are always 1.5 and the unloading curve follows Hertzian-type unloading law. For strain hardening materials, the exponents are distinctly smaller than 1.5, implying the inadequate of the Hertzian-type unloading law.

(2) Based on the FE analyses, both strain hardening and indenter elasticity have strong influences on the residual indentation $\delta_r / \delta_m$. The relationship of $\delta_r / \delta_m$ to $\delta_m / \delta_y$ are dependent upon $E_1^e / E_2^e$, strain hardening exponent $n$ and $\delta_m / \delta_y$, but nearly independent upon other parameters, such as $E_1^e / \sigma_y$ and $E^e / \sigma_y$. The dependence of $\delta_r / \delta_m$ on $E_1^e / E_2^e$ becomes stronger for higher strain hardening materials.

(3) During unloading, there are two types of cavity profiles, spherical and aspheric. For the elastic-perfectly plastic materials, the cavity profiles always maintain
spherical during the entire unloading process and after fully unloading. For the strain hardening materials, the cavity profile merely remains spherical within the contact area and is aspheric outside the contact area during an unloading process. The cavity profile is sensitive to strain hardening. For high strain hardening materials, the residual cavity profile becomes aspheric more significantly. However, the elastic deformation is found to not affect the type of the cavity profile during unloading and the residual cavity after fully unloading. It only affects the aspheric degree of cavity profile.

(4) By curve-fitting approach, a new unloading law is presented to consider the effects of both strain hardening and indenter elastic deformation. The new unloading law is suitable for both the elastic-perfectly plastic and strain hardening materials.

(5) The present unloading law is verified numerically and experimentally. For strain-hardening materials compressed by rigid flat, the present unloading law still can be applied with high prediction accuracy. For four engineering strain-hardening material combinations, the FE simulations demonstrate that the present unloading law has high prediction accuracy. The experimental verifications of AISI 1065 steel plate (\( n = 0.2 \)) indented by a tungsten carbide sphere show that the present unloading law has high accuracy, even when the finite indentation deformation reaches 3969.9 \( \delta_f \). Hence, the present unloading law can be suitable for elastic-plastic materials with strain hardening contacted by elastic indenter or rigid flat ranging from small to large deformations.
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Declarations

The authors declare that the manuscript is an original paper, which has not been published before and has not been submitted to other journals simultaneously.

All authors named on the manuscript have made a significant contribution to the manuscript. Each author's contributions are listed as follows:

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Xiaochun Yin: Conceptualization, Methodology, Writing-review & editing
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