PAPER

On a Pseudo-Formal Linearization Method Based on Fourier Series Expansion

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Abstract  This paper is concerned with a pseudo-formal linearization method using the Fourier series expansion. The given nonlinear dynamic system is piecewisely linearized into some augmented linear systems with respect to a linearization function that consists of the state variables and their trigonometric functions. The resulting linear systems are smoothly united into a single system. Its application to a nonlinear observer and numerical examples are also included.

Keywords: nonlinear system, pseudo-formal linearization, Fourier series expansion, trigonometric function, nonlinear observer, linearization function

1. Introduction

Most systems have inherently nonlinear characteristics and are naturally described by nonlinear differential equations. To adapt linear theories to them, one needs linearization methods for nonlinear systems. One of the most popular linearization approaches is Taylor expansion truncated at the first order [1],[4]. This is effective only in small oscillation or in almost linear systems. Geometric approaches [5]-[7] have also been investigated, but they are not easy to apply to practical systems. Next, a formal linearization approach [8],[9] has been considered, which has the big advantage of easier application to some nonlinear systems. Furthermore, a pseudo-formal linearization using Taylor series [10] and the orthogonal polynomials [11] has been studied.

In this study, we develop a pseudo-formal linearization method based on Fourier series expansion considering easy inversion [3],[8]. A formal linearization function that consists of the state variables and their trigonometric functions up to a higher order is introduced. Its nonlinear dynamic equation is piecewisely linearized with respect to the formal linearization function, where nonlinear equations excepting linear terms are approximated by Fourier series expansion. The resulting linear systems are smoothly united into a single pseudo-formal linear system.

A nonlinear observer is also synthesized as an application of the proposed method. The validity of this method is shown through numerical examples.

2. Statement of Problem

A nonlinear dynamic equation is described by

\[ \dot{x}(t) = f(x(t)) = A_L x(t) + f_{NL}(x(t)) \]  \hspace{1cm} (1)

where \( t \) denotes time, \( \cdot = d/dt \), \( x = [x_1, \ldots, x_n]^T \) is an \( n \)-dimensional state vector, \( A_L \) is an \( n \times n \) matrix, and \( f_{NL} = [f_{NL1}, \ldots, f_{NLn}]^T \) is a continuous nonlinear vector-valued function. \( D \) is a domain denoted by the Cartesian product

\[ D = \prod_{i=1}^{n} [m_i - p_i, m_i + p_i] \quad (m_i \in R, p_i > 0) \]

and is the limited domain expressed by these parameters \( m_i \) and \( p_i \). The problem is to translate these nonlinear systems into pseudo-formal linear systems by using Fourier series expansion.

3. Division of Domain

First, we introduce a vector-valued separable function to linearize the given nonlinear state differential
equation in Eq. (1) as
\[
C : \mathbb{D} \rightarrow \mathbb{R}^l \tag{2}
\]
Let \( C = [I : 0] \) \((I : L \times L \text{ unit matrix})\) for simplicity where \( L \) indicates the number of state variables that we choose to apply a formal linearization piecewisely. Considering the nonlinearity of the given nonlinear dynamic system and letting \( D \) be a domain of \( C^{-1} \), the domain \( D \) is divided into \((M + 1)\) subdomains:
\[
D = \bigcup_{k=0}^{M} D_k \tag{3}
\]
where
\[
D_M = D - \bigcup_{k=0}^{M-1} D_k
\]
and \( C^{-1}(D_0) \geq 0 \). \( D_k(0 \leq k \leq M - 1) \) endowed with a lexicographic order is the Cartesian product
\[
D_k = \prod_{j=1}^{L} [a_{kj}, b_{kj}) \quad (a_{kj} < b_{kj})
\]
We here introduce an automatic choosing function of the sigmoid type [10],
\[
I_k(\zeta) = \prod_{j=1}^{L} \left( 1 - \frac{1}{1 + \exp \left( 2\mu(\zeta_j - a_{kj}) \right)} \right) - \frac{1}{1 + \exp \left( -2\mu(\zeta_j - b_{kj}) \right)} \quad (0 \leq k \leq M - 1) \tag{4}
\]
so that
\[
\sum_{k=0}^{M} I_k(\zeta) = 1 \tag{5}
\]
where
\[
\zeta = [\zeta_1, \ldots, \zeta_L]^T = C(x)
\]
and \( \mu \) is a positive real value. \( I_k(\zeta) \) is analytic and almost unity on \( D_k \); otherwise, it is almost zero (see Fig. 1, which shows the case when the dimensions are set as \( n = 1 \) and \( L = 1 \)).

Secondly, the state vector \( x \) is changed into \( y \) so that we can use Fourier series expansion at each \( k \) \((0 \leq k \leq M) \). Let \( y \) be
\[
y = P^{(k)}^{-1}(x - M^{(k)}) \in D_0 \tag{7}
\]
where
\[
y = [y_1, \ldots, y_L, y_{L+1}, \ldots, y_n]^T
\]
\[
M^{(k)} = [m_1^{(k)}, \ldots, m_L^{(k)}, m_{L+1}, \ldots, m_n]_F
\]
\[
P^{(k)} = \text{diag}(p_1^{(k)}, \ldots, p_L^{(k)}, p_{L+1}, \ldots, p_n)
\]
Figure 1 Pseudo-formal linearization

\[
m_j^{(k)} = \frac{1}{2}(a_{kj} + b_{kj}), \quad p_j^{(k)} = \frac{1}{2\pi}(b_{kj} - a_{kj})
\]
so that \( y \) has the basic domain of the Fourier series expansion:
\[
\mathcal{D}_0 = \prod_{i=1}^{n} [-\pi, \pi] \tag{9}
\]

4. Fourier Series Expansion

The orthogonal system approximation is one of the most popular and important approaches for nonlinear functions [3],[11]. When we approximate a function \( G^{(k)}(y) \) using the Fourier series in terms of \( T_{q_i}(y_i) \), from the property of the Fourier series for orthogonal systems, even if truncated at any finite order \( N \), the resulting equation is always optimal in the sense of minimizing
\[
\int_{\mathcal{D}_0} \left\| G^{(k)}(y) - \sum_{q_1=0}^{2N} \cdots \sum_{q_n=0}^{2N} C_{(q_1, \ldots, q_n)}^{(k)} \prod_{i=1}^{n} T_{q_i}(y_i) \right\|^2 W(y)dy \tag{10}
\]
where
\[
\|X\|^2 = XT X, \quad W(y) = \prod_{i=1}^{n} W(y_i), \quad dy = dy_1 \cdots dy_n
\]
and \( W(y_i) = 1 \) is the weighting function. Namely, \( C_{(q_1, \ldots, q_n)}^{(k)} \) is the coefficient obtained by the least-squares method, and it yields [3]
\[
C_{(q_1, \ldots, q_n)}^{(k)} = \frac{1}{\pi^n} \int_{\mathcal{D}_0} C^{(k)}(y) \prod_{i=1}^{n} T_{q_i}(y_i)dy
\]
where
\[
\delta_{q_i} = \begin{cases} 1 & \text{if } q_i = 0 \\ 0 & \text{otherwise} \end{cases}
\]
In this paper, we define a formal linearization function that consists of the state variables and the trigonometric functions defined by

\[
\phi(x) \triangleq \left[ \phi^T_L(x), \phi^T_{SC}(y(x)) \right]^T \\
\phi_L(0 \cdots 0) (x), \phi_{SC}(0 \cdots 0) (y(x)),
\cdots, \phi_{SC}(0 \cdots 0) (y(x)),
\cdots, \phi_{SC}(2N \cdots 2N) (y(x)) \right]^T
\]

\[
= \left[ x_1, \cdots, x_n, \sin(y_1(x)), \cdots, \sin(y_n(x)), \cos(y_1(x)),
\cdots, \cos(y_n(x)), \cdots \right]^T
\]

\[
\in R^{n+2(2N+1)^n-1}
\]

Namely, \( \phi_{SC}(y) \) is a trigonometric function vector of \( y \) and can make the orthogonal systems up to the \( N \)-th order, and

\[
\phi_L(0 \cdots 0 \cdots 0 \cdots 0) (x) = x_i
\]

\[
\phi_{SC}(0 \cdots 0 \cdots 0 \cdots 0) (y) = T_{2S-1}(y_i)
\]

\[
\phi_{SC}(0 \cdots 0 \cdots 0 \cdots 0) (y) = T_{2S}(y_i)
\]

\[
\phi_{SC}(r \cdots r \cdots r \cdots r) (y) = \prod_{i=1}^n T_{r_i}(y_i)
\]

where

\[
T_0(y_i) = 1, T_{2S-1}(y_i) = \sin(Sy_i)
\]

\[
T_{2S}(y_i) = \cos(Sy_i) \quad (S = 1, \cdots, N)
\]

From Eqs. (1) and (11), the derivative of \( \phi \) is given as follows.

For \( r_1 + \cdots + r_n = 1, x \) is

\[
\dot{x} = \left[ \dot{\phi}_L(0 \cdots 0)(x), \cdots, \dot{\phi}_L(0 \cdots 0)(x),
\cdots, \dot{\phi}_L(0 \cdots 0)(x) \right]^T
\]

\[
= A_L x + f_{NL}(\theta + \mathcal{M}(k))
\]

\[
\triangleq A_L x + G^{(k)}_{r_1 \cdots r_n}(y)
\]

For \( r_1 + \cdots + r_n \geq 2 \), each element of \( \phi_{SC} \) is

\[
\dot{\phi}_{r_1 \cdots r_n}(x) = \frac{\partial}{\partial x} \phi_{r_1 \cdots r_n}(x) \cdot \dot{x}
\]

\[
= \frac{\partial}{\partial x} \phi_{r_1 \cdots r_n}(x) \cdot f(x) = \frac{\partial}{\partial y} \phi_{r_1 \cdots r_n}(y)
\]

\[
\times \phi_{r_1 \cdots r_n}(P^{(k)}y + \mathcal{M}(k)) \cdot f(P^{(k)}y + \mathcal{M}(k))
\]

\[
\triangleq G^{(k)}_{r_1 \cdots r_n}(y)
\]

or

\[
\phi_{SC}(y) = [G^{(k)}_{r_1 \cdots r_n}(y)]^2
\]

These \( \{G^{(k)}_{r_1 \cdots r_n}(y)\}_1, \{G^{(k)}_{r_1 \cdots r_n}(y)\}_2 \) are expanded by the Fourier series expansion of Eqs. (10) and (11) as

\[
G^{(k)}_{r_1 \cdots r_n}(y) = \sum_{q_1=0}^{2N} \cdots \sum_{q_n=0}^{2N} C^{(k)}_{(q_1 \cdots q_n)} \prod_{i=1}^n T_{q_i}(y_i)
\]

\[
+ R^{(k)}_{N+1}(r_1 \cdots r_n)
\]

\[
= [C^{(k)}_{(q_1 \cdots q_n)}] \phi_{SC}(y) + C^{(k)}_{(0 \cdots 0)}
\]

\[
+ R^{(k)}_{N+1}(r_1 \cdots r_n)
\]

where \( R^{(k)}_{N+1}(r_1 \cdots r_n) \) is the approximation error. Thus,

\[
\left[ \begin{array}{c}
\{C^{(k)}_{(r_1 \cdots r_n)}(y)\}_1 \\
\{C^{(k)}_{(r_1 \cdots r_n)}(y)\}_2
\end{array} \right]
\]

\[
\approx \left[ \begin{array}{c}
[C^{(k)}_{(q_1 \cdots q_n)}]_1 \\
[C^{(k)}_{(q_1 \cdots q_n)}]_2
\end{array} \right] \phi_{SC}(y) + \left[ \begin{array}{c}
C^{(k)}_{(0 \cdots 0)}_1 \\
C^{(k)}_{(0 \cdots 0)}_2
\end{array} \right]
\]

\[
\triangleq \left[ \begin{array}{c}
A^{(k)}_{12} \\
A^{(k)}_{22}
\end{array} \right] \phi_{SC}(y) + \left[ \begin{array}{c}
\tilde{b}^{(k)}_1 \\
\tilde{b}^{(k)}_2
\end{array} \right]
\]

Therefore, we have the following on subdomain \( D_k \),

\[
\dot{\phi}(x) = A^{(k)} \phi(x) + \tilde{b}^{(k)} + R^{(k)}_{N+1}(x)
\]

where

\[
A^{(k)} = \left[ \begin{array}{ccc}
A_{11} & A_{12}^T & b_1 \\
0 & A_{22} & b_2
\end{array} \right], \quad \tilde{b}^{(k)} = \left[ \begin{array}{c}
\tilde{b}^{(k)}_1 \\
\tilde{b}^{(k)}_2
\end{array} \right]
\]

\[
R^{(k)}_{N+1}(x) = \left[ r^{(k)}_{N+1}(r_1 \cdots r_n) (y(x)) \right]
\]

5. Pseudo-Formal Linearization

We unite \( (M + 1) \) linearized systems (Eq. 22)) on the subdomains into a single linear system on the whole domain using Eq. (5) as

\[
\dot{\phi}(x) = \sum_{k=0}^M \phi(x) I_k(\zeta)
\]

\[
= \sum_{k=0}^M (A^{(k)} \phi(x) + \tilde{b}^{(k)} + R^{(k)}_{N+1}(x)) I_k(\zeta)
\]

\[
= \tilde{A}(\zeta) \phi(x) + \tilde{b}(\zeta) + \tilde{R}_{N+1}(x, \zeta)
\]

where

\[
\tilde{A}(\zeta) = \sum_{k=0}^M A^{(k)} I_k(\zeta), \quad \tilde{b}(\zeta) = \sum_{k=0}^M \tilde{b}^{(k)} I_k(\zeta)
\]

\[
\tilde{R}_{N+1}(x, \zeta) = \sum_{k=0}^M R^{(k)}_{N+1}(x) I_k(\zeta)
\]
Finally a pseudo-formal linearization system is defined as
\[
\Sigma_2: \dot{\hat{x}}(t) = \hat{A}(\zeta)\hat{x}(t) + \hat{b}(\zeta), \quad \hat{x}(0) = \phi(x(0)) \tag{25}
\]
The resulting system (Eq. (25)) is a generalization of the standard formal linearization [8] because \(\hat{A}(\zeta)\) and \(\hat{b}(\zeta)\) are functions of \(\zeta\) in Eq. (4). From Eq. (12), its inversion is easily carried out by evaluating
\[
\dot{\hat{x}}(t) = [I, 0, \cdots, 0]z(t) \tag{26}
\]
as the approximate value of \(x(t)\), where \(I\) is the \(n \times n\) unit matrix.

For this approach, we have the following error bounds. Let \(\| \cdot \|\) be the Euclidean norm.

**Theorem 1**
An error bound when a nonlinear system is approximated by the pseudo-formal linearization is
\[
\varepsilon_{N+1} = \max_{k} \left\{ \sup_{x} \left\{ \| R_{N+1}^{(k)}(x) \| : x \in D_k \right\}, 0 \leq k \leq M \right\} \tag{27}
\]
where
\[
\| R_{N+1}^{(k)}(x) \| = \left[ \sum_{r_1=0}^{2N} \cdots \sum_{r_n=0}^{2N} (R_{N+1}^{(k)}(r_1, \cdots, r_n)(x))^2 \right]^{1/2} \tag{28}
\]

**Theorem 2**
An error bound of the pseudo-formal linearization for a nonlinear dynamic system is
\[
\| x(t) - \hat{x}(t) \| \leq \| \phi(x(t)) - z(t) \| \leq \varepsilon_{N+1} \max_{t} \left\{ \| A \|_{\text{max}} t \right\} \| \phi(x(0)) - z(0) \| + \varepsilon_{N+1} \max_{t} \left\{ \| A \|_{\text{max}} t \right\}
\]
where
\[
\| A \|_{\text{max}} = \max_{0 \leq k \leq M} \left\{ \| A^{(k)} \| : 0 \leq k \leq M \right\} \tag{29}
\]
(Proof. See Ref.[10])

6. Nonlinear Observer

We here synthesize a nonlinear observer as an application of the above pseudo-formal linearization [2].
We assume that the nonlinear dynamic system is the same as Eq. (1),
\[
\dot{x}(t) = f(x(t)) \tag{31}
\]
and the measurement equation is
\[
y(t) = h(x(t)) = H_L x(t) + h_{NL}(x(t)) \tag{32}
\]
where \(y \in \mathbb{R}^m\) is a measurement vector, \(H_L\) is an \(m \times n\) matrix, and \(h_{NL} = [h_1, \cdots, h_m]^T\) is a continuous smooth nonlinear vector-valued function.

The nonlinear dynamic system in Eq. (31) is transformed into the linear system in Eq. (25) by the pseudo-formal linearization in Sect. 5. To linearize the measurement equation with respect to the linearization function \(\phi\), we apply the Fourier series expansion shown in Eqs. (18) to (23) as follows.
\[
h_r(x) = h_r(P^{(k)}y + M^{(k)})
\]
\[
\approx \sum_{q_1=0}^{2N} \cdots \sum_{q_n=0}^{2N} C_{(q_1, \cdots, q_n)}^{(k)} r_1 \cdots r_n \tag{33}
\]
\[
= [C_{(0, \cdots, 0)}^{(k)}]\phi_{SC}(y) + C_{(0, \cdots, 0)}^{(k)} \tag{34}
\]

Therefore, we have the following approximation on subdomain \(D_k\).
\[
y \approx [H_L, H^{(k)}] \phi(x) + [d^{(k)}(r_1)] \tag{35}
\]
Applying Eq. (5) to Eq. (35) yields
\[
y \approx \sum_{k=0}^{M} (H^{(k)} \phi(x) + d^{(k)}) I_k(\zeta) \tag{36}
\]
Thus, a pseudo-formal linearization system for the measurement equation is approximately derived as
\[
y(t) = \tilde{H}(\zeta) z(t) + \tilde{d}(\zeta) \tag{37}
\]
where
\[
\tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \quad \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta)
\]
The linear observer theory [2] is applied to the linearized systems in Eqs. (25) and (37) and an identity observer is obtained as
\[
\dot{\hat{z}}(t) = \tilde{A}(\zeta) \hat{z}(t) + \tilde{b}(\zeta) + K(t) (y - \tilde{H}(\zeta) \hat{z}(t) - \tilde{d}(\zeta)) \tag{38}
\]
where \(\zeta = C(\hat{z})\). \(K^{(k)}(t)\) is the observer gain on subdomain \(D_k\) given by
\[
K^{(k)}(t) = \frac{1}{2} P^{(k)}(t) H^{(k)T} S^{(k)}(t)
\]

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\(P^{(k)}(t)\) satisfies the matrix Riccati differential equation,
\[
\dot{P}^{(k)}(t) = A^{(k)} P^{(k)}(t) + P^{(k)}(t) A^{(k)T} + Q^{(k)}(t)
\]
\[ - P^{(k)}(t) H^{(k)}T S^{(k)}(t) H^{(k)} P^{(k)}(t) \]  
(39)
where \(Q^{(k)}(t)\), \(S^{(k)}(t)\), and \(P^{(k)}(0)\) are arbitrary real symmetric positive definite matrices.

From Eq. (26), the estimate \(\hat{x}(t)\) of a nonlinear observer becomes
\[
\hat{x}(t) = [I, 0, \cdots, 0] \hat{z}(t)
\]  
(40)

### 7. Numerical Experiments

To show the effectiveness of the approach, numerical experiments on the presented pseudo-formal linearization and the nonlinear observer are illustrated.

#### 7.1 Pseudo-formal linearization

Consider the simple nonlinear dynamic system
\[
\dot{x} = x - x^2, \quad D = [0, 1.2] \subset R
\]  
(41)
The parameters are set as \(M = 5\), \(\mu = 80\), and \(\zeta = x\) in Eqs. (3) and (4). \(D\) is divided into \(D = \bigcup_{k=0} D_k\) where
\[
D_0 = [0, 0.2], D_1 = [0.2, 0.4], D_2 = [0.4, 0.6]
D_3 = [0.6, 0.8], D_4 = [0.8, 1.0], D_5 = [1, 1.2]
\]
The system parameters are set as
\[
\mathcal{M}^{(0)} = 0.1, \mathcal{M}^{(1)} = 0.3, \mathcal{M}^{(2)} = 0.5
\]
\[
\mathcal{M}^{(3)} = 0.7, \mathcal{M}^{(4)} = 0.9, \mathcal{M}^{(5)} = 1.1
\]  
(42)
\[
\mathcal{P}^{(k)} = \frac{1}{10\pi} \quad (k \geq 0)
\]
For example, the formal linearization function \(\phi(x)\) when \(N = 3\) is
\[
\phi(x) = [x, \sin(y(x)), \cos(y(x)), \sin(2y(x)), \cos(2y(x)),
\sin(3y(x)), \cos(3y(x))]^T
\]

Figure 2 shows the true value \(x(t)\) of Eq. (41) and the estimate \(\hat{x}(t)\) obtained by the pseudo-formal linearization of Eq. (26) when the order \(N\) is varied from 1 to 5. For comparison, \(\hat{x}(\text{Old})\) and \(\hat{x}(\text{Taylor})\) are indicated. \(\hat{x}(\text{Old})\) is the result obtained by a previously reported method [8] when the parameters are
\[
N = 3, \mathcal{M} = 0.4, \mathcal{P} = 0.35
\]
\[
\hat{x}(\text{Taylor})\] is the result obtained by the ordinary method [1] based on Taylor expansion truncated at the first order when the operating point is 0.2.

#### 7.2 Nonlinear observer

We simulate a nonlinear observer using the presented method in Eq. (40). Consider the nonlinear dynamic system
\[
\ddot{x} + a \dot{x} + b \sin \delta = c, \quad \delta \in R
\]  
(44)
When \(x_1 = \delta, x_2 = \dot{\delta}, a = 1, b = 0.4\), and \(c = 0\), the system yields
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} 
= 
\begin{bmatrix}
x_2(t) \\
-0.4 \sin(x_1(t)) - x_2(t)
\end{bmatrix} f(x(t))
\]  
(45)
Assume the measurement equation
\[ y = \sin(x_1(t)) = h(x(t)) \quad (46) \]
We divide the domain of \( x_1 \) into 4 subdomains:
\[ D_0 = [-0.2, 0.3), D_1 = [0.3, 0.8), D_2 = [0.8, 1.3) \]
\[ D_3 = [1.3, 1.8] \]
The system parameters are set as \( M = 3, \mu = 100, \zeta = x_1, \)
\[ \mathcal{M}^{(0)} = \begin{bmatrix} 0.05 & -0.165 \\ -0.165 & -0.165 \end{bmatrix}, \mathcal{M}^{(1)} = \begin{bmatrix} 0.55 \\ -0.165 \end{bmatrix} \]
\[ \mathcal{M}^{(2)} = \begin{bmatrix} 1.05 & -0.165 \\ -0.165 & -0.165 \end{bmatrix}, \mathcal{M}^{(3)} = \begin{bmatrix} 1.55 \\ -0.165 \end{bmatrix} \]
\[ \mathcal{P}^{(k)} = \begin{bmatrix} 0 & 0 \\ \frac{1}{4\pi} & 0 \\ 0 & \frac{1}{5\pi} \end{bmatrix}, \quad (k = 0, 1, 2, 3) \]
and the order of the formal linearization function is \( N = 2. \) The parameters for a nonlinear observer are set as
\[ x(0) = [1.5, 0]^T, \hat{x}(0) = [1.78, -0.3]^T \]
\[ Q^{(0)}(t) = P^{(0)}(0) = 10I, S^{(0)}(t) = 10 \]
\[ Q^{(k)}(t) = P^{(k)}(0) = 0.5I, S^{(k)}(t) = 0.5, (k = 1, 2, 3) \]
Figure 4 shows the true value \( x \) and the estimated \( \hat{x} \) in Eq. (40) obtained by the presented method. \( \hat{x} \) is the result obtained by the previous method [8] when the parameters are
\[ \mathcal{M} = \begin{bmatrix} 0.75 \\ -0.165 \end{bmatrix}, \mathcal{P} = \begin{bmatrix} 0.8 & 0 \\ 0 & \frac{1}{5\pi} \end{bmatrix} \]
\[ Q(t) = P(0) = 0.5I, \quad S(t) = 0.5, \quad \text{and the order of the formal linearization function is the same as} \quad N = 2. \]
\[ x(Taylor) \] is the result obtained by the ordinary method [1] based on Taylor expansion truncated at the first order when the operating point is \([0.75, 0]^T\). The parameters are set as \( Q(t) = P(0) = 0.5I \) and \( S(t) = 0.5 \).

Figure 5 shows the logarithmic integral square errors of the estimation
\[ J(t) = \int_0^t ||x(\tau) - \hat{x}(\tau)||^2 d\tau \]
for the various methods in these cases.

These results show that the performance of the presented pseudo-formal linearization is better than those of the previous methods.

8. Conclusions

We studied a pseudo-formal linearization method for nonlinear systems based on Fourier series expansion considering easy inversion. We also synthesized a nonlinear observer as an application of the method. From the results, the accuracy of this new method was found to be improved as the order of the linearization function increases and better than those of the previous methods. The application of this method to practical systems, such as electric power systems, is left for future studies.

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of the previous methods. The presented pseudo-formal linearization is better than those of the previous methods in these cases.

Figure 5 shows the logarithmic integral square error of the estimation (New) in Eq. (40) obtained by the presented method [8] when the parameters are set as $x(t) = 0.5, 1.5, 2, 0.5, 1, 0.5$ and the order of the formal linearization function is $N = 2$. Figure 4 shows the nonlinear observers (New) $\hat{x}(t)$ and (Old) $\hat{x}(t)$ for nonlinear systems based on Fourier series expansion. These results show that the performance of the presented method [8] is improved as the order of the linearization function increases and better than those of the previous methods [9].

Conclusions

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(Received July 31, 2020; revised September 27, 2020)