ON THE WIDTH OF VERBAL SUBGROUPS
OF THE GROUPS OF TRIANGULAR MATRICES
OVER A FIELD OF ARBITRARY CHARACTERISTIC

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Abstract. The width $\text{wid}(G, W)$ of the verbal subgroup $v(G, W)$ of a group $G$ defined by a collection of group words $W$ is the smallest number $m$ in $\mathbb{N} \cup \{+\infty\}$ such that every element of $v(G, W)$ is the product of at most $m$ words in $W$ evaluated on $G$ and their inverses.

Recall that every verbal subgroup of the group $T_n(K)$ of triangular matrices over an arbitrary field $K$ can be defined by just one word: an outer commutator word or a power word. We prove that for every outer commutator word $w$ the equality $\text{wid}(T_n(K), w) = 1$ holds on $T_n(K)$ and that if $w = x^s$ then $\text{wid}(T_n(K), w) = 1$ except in two cases:

1. The field $K$ is finite and $s$ is divisible by the characteristic $p$ of $K$ but not by $|K| - 1$;
2. The field $K$ is finite and $s = p^r(|K| - 1)^u$ for $r, t, u \in \mathbb{N}$ with $n \geq p^t + 3$, while $r$ not divisible by $p$.

In these cases the width equals 2. For finitary triangular groups the situation is similar, but in the second case the restriction $n \geq p^t + 3$ is superfluous.

Keywords: The group of triangular matrices; verbal subgroup; the width of a verbal subgroup.

1. Introduction and Main Results

Verbal subgroups are widely used in group theory. Given a group $G$, we may regard the verbal subgroup $v(G, W)$ defined by a certain collection of group words $W$ as a measure of distance from $G$ to the groups of the variety described by the collection $W$. It is equally important that verbal subgroups are endomorphically admissible, and so they are widely used as we pass to quotient groups. To describe the set of all verbal subgroups of a known class of groups is a current problem. A complete description of the set of verbal subgroups of the groups $T_n(K)$, $FT(K)$, $UT_n(K)$, and $FUT(K)$ of upper triangular, finitary upper triangular, upper unitriangular, and finitary upper unitriangular matrices over an arbitrary field $K$ of characteristic 0 is obtained in [4]. It turns out that the verbal subgroups of $UT_n(K)$ and $FUT(K)$ are precisely the members of the lower central series of these groups. In $T_n(K)$ and $FT(K)$ the list includes in addition the subgroups $UT_n(K)(D_n(K))^s$ and $FUT(K)(FD(K))^s$ for $s \in \mathbb{N}$ respectively. A similar result for fields of arbitrary characteristic is announced in [13] and proved in [5]. The description of verbal subgroups implies that every verbal subgroup in these groups is defined by just one word: a power word or an outer commutator word. Recall that an outer commutator word is a group word obtained from a finite sequence of distinct variables $x_1, x_2, \ldots, x_r$ using only group commutators. For instance, $[[x_1, x_2], [x_3, x_4]]$ is an outer commutator word, while $[x_1, x_2][x_3, x_4]$ and...
every element of the verbal subgroup means that every element of the commutant of \( w \). However, if \( wid(G, w) \) is divisible by \( p \) then \( wid(G, w) \) is not divisible by \( p + 2 \) and \( wid(G, w) \) is not divisible by \( p \) and \( |K| - 1 \). For the group \( FUT(K) \) the situation regarding the width of verbal subgroups is even simpler. Indeed, \( wid(FUT(K), w) = 1 \) when \( w \) is an outer commutator word or \( w = x^s \) and \( s \) is not divisible by \( |K| - 1 \). However, if \( s \) is divisible by the characteristic of \( K \) then \( wid(FUT(K), x^s) = 2 \).

In this article we prove the following theorems.

Theorem 1. Take a field \( K \) of characteristic \( p \geq 0 \) with at least three elements and arbitrary integers \( n \geq 2 \) and \( s \geq 1 \). The width of the verbal subgroup of the group \( T_n(K) \) of triangular matrices defined by the power word \( x^s \) satisfies the following equalities:

(a) \( wid(T_n(K), x^s) = 1 \) when \( s \) is not divisible by \( p \);
(b) \( wid(T_n(K), x^s) = 2 \) when \( s \) is divisible by \( p \) and not divisible by \( |K| - 1 \) in the case that \( K \) is finite;
(c) if \( K \) is finite and \( s = p^t(|K| - 1)^ru \) for some \( r, t, u \in \mathbb{N} \), where \( r \) is co-prime to \( p \) and to \( |K| - 1 \), then \( wid(T_n(K), x^s) = 1 \) for \( n \leq p^t + 2 \) and \( wid(T_n(K), x^s) = 2 \) for \( n \geq p^t + 3 \).

Theorem 2. Take a field \( K \) of characteristic \( p \geq 0 \) with at least three elements and a positive integer \( s \). The width of the verbal subgroup of the group \( FT(K) \) of finitary triangular matrices defined by the power word \( x^s \) satisfies the following equalities:
(a) \( \text{wid}(FT(K), x^s) = 1 \) when \( s \) is not divisible by \( p \);
(b) \( \text{wid}(FT(K), x^s) = 2 \) when \( s \) is divisible by \( p \).

**Theorem 3.** The equality \( \text{wid}(T_n(K), w) = 1 \) holds in the group \( T_n(K) \) of triangular matrices for every field \( K \), every outer commutator word \( w \), and every integer \( n \geq 2 \).

**Theorem 4.** The equality \( \text{wid}(FT(K), w) = 1 \) holds in the group \( FT(K) \) of finitary triangular matrices for every field \( K \) and every outer commutator word \( w \).

The description of verbal subgroups in \( T_n(K) \) and \( FT(K) \) implies that all of them can be defined by just one word \( w \), which is an outer commutator word or a power word. Thus, Theorems 1–4 enable us to compute the width of all possible verbal subgroups of \( T_n(K) \) and \( FT(K) \) with respect to the natural sets of generators: the values of \( w \). Observe that in case (c) of Theorem 1 the verbal subgroup \( v(T_n(K), x^s) \) can be defined also by the outer commutator word \([x_1, x_2], [x_3, x_4], \ldots, [x_{2p-1}, x_{2p}][\ldots]\); furthermore, the width of this verbal subgroup with respect to this word always equals 1 independently of \( n \).

### 2. Proofs of the Theorems

**Proof of Theorem 1.** We prove claim (a) of Theorem 1 by induction on \( n \). For \( n = 2 \), given elements \( \alpha, \beta, \gamma, \) and \( \delta \) of \( K \) with \( \alpha \neq 0 \) and \( \gamma \neq 0 \), the equality

\[
\begin{pmatrix}
\alpha & \beta \\
0 & \gamma
\end{pmatrix}^s = \begin{pmatrix}
\alpha^s & (\alpha^{s-1} + \alpha^{s-2}\beta + \cdots + \alpha\gamma^{s-2} + \gamma^{s-1})\beta \\
0 & \gamma^s
\end{pmatrix}
\]

shows that we can obtain every matrix

\[
\begin{pmatrix}
\alpha^s & \delta \\
0 & \gamma^s
\end{pmatrix}
\]

in \( v(T_2(K), x^s) \) by raising to power \( s \) some matrix in \( T_2(K) \) found by choosing suitable \( \beta \) provided that \( \lambda = \alpha^{s-1} + \alpha^{s-2}\gamma + \cdots + \alpha\gamma^{s-2} + \gamma^{s-1} \) is nonzero. However, if \( \lambda = 0 \) then \( \alpha^s = \gamma^s \) and \( \alpha/\gamma \) is a degree \( s \) root of unity. Replacing \( \gamma \) with \( \alpha \), we arrive at \( \lambda = \alpha^{s-1} \) and \( \lambda \neq 0 \) since \( s \) is not divisible by \( p \).

Suppose that we can obtain every matrix \( a \) in \( v(T_n(K), x^s) \) by raising to power \( s \) some matrix \( c \) in \( T_n(K) \) and that either the diagonal entries of \( c \) are equal or their ratios are not degree \( s \) roots of unity. Represent an arbitrary matrix \( \bar{a} \) in \( T_{n+1}(K) \) as \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) for some matrix \( a \) in \( T_n(K) \), a column vector \( b \) of height \( n \), and an element \( \gamma \) of \( K \). If \( \bar{a} \) lies in \( v(T_{n+1}(K), x^s) \) then \( a \) lies in \( v(T_n(K), x^s) \) and \( \gamma = \delta^s \) for some \( \delta \) in \( K \). By the inductive assumption, there exists a matrix \( \gamma \) in \( T_n(K) \) such that \( a = c^s \) and the ratio of every pair of unequal diagonal entries of \( c \) is not a degree \( s \) root of unity. The equality

\[
\begin{pmatrix}
c & d \\
0 & \delta
\end{pmatrix}^s = \begin{pmatrix}
c^s & (c^{s-1} + \delta c^{s-2} + \cdots + \delta^{s-2}c + \delta^{s-1}e)d \\
0 & \delta^s
\end{pmatrix},
\]

where \( e \) is the identity matrix of size \( n \), shows that we can obtain every matrix in \( v(T_{n+1}(K), x^s) \) by raising to power \( s \) some matrix in \( T_{n+1}(K) \) found by choosing a suitable vector \( d \) provided that \( f = c^{s-1} + \delta c^{s-2} + \cdots + \delta^{s-2}c + \delta^{s-1}e \) is an invertible matrix. However, if \( f \) is not invertible then it has a zero main diagonal entry. As in the case \( n = 2 \), if at least two diagonal entries of \( f \) vanish then by the inductive assumption all corresponding diagonal entries of \( c \) are equal. Indeed, if
where \( f_{ii} = f_{jj} = 0 \) with \( i \neq j \) then \( c_{ii}/\delta \) and \( c_{jj}/\delta \) are degree \( s \) roots of unity, and so is \( c_{ii}/c_{jj} \). Therefore, \( c_{ii} = c_{jj} \). Replacing \( \delta \) with \( c_{ii} \), we obtain an invertible matrix \( f \).

(b) Suppose now that the exponent \( s \) is divisible by the characteristic \( p \) of the field \( K \) and, in the case of finite \( K \), that \( s \) is not divisible by \( |K| - 1 \). The relation

\[
\begin{pmatrix}
\alpha & \beta \\
0 & \alpha
\end{pmatrix}^p = \begin{pmatrix}
\alpha^p & p\alpha^{p-1}\beta \\
0 & \alpha^p
\end{pmatrix} = \begin{pmatrix}
\alpha^p & 0 \\
0 & \alpha^p
\end{pmatrix},
\]

the uniqueness of degree \( p \) roots in \( K \), and the description of the set of verbal subgroups of the groups of triangular matrices show that the width of the verbal subgroup \( v(T_2(K), x^p) \) with respect to the set of generators equal to the \( p \)th powers of the matrices in \( T_2(K) \) is greater than 1. Similar arguments are valid for every \( s \) dividing \( p \) and every \( n \geq 2 \).

Take the identity matrix \( e \) and an arbitrary matrix \( a \) in \( T_{n-1}(K) \), column vectors \( b \) and \( c \) of height \( n - 1 \), and nonzero elements \( \alpha \) and \( \beta \) of the field \( K \). If a nonzero element \( \beta \) of \( K \) is not a degree \( s \) root of unity then the relation

\[
\begin{pmatrix}
\beta^{-1}a & 0 \\
0 & \alpha
\end{pmatrix} \begin{pmatrix}
\beta e & c \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\alpha^s & (\beta^{s-1} + \beta^{s-2} + \cdots + \beta + 1)\beta^{-s}a^s c \\
0 & \alpha^s
\end{pmatrix}
\]

shows that by choosing a suitable vector \( c \) we can express an arbitrary matrix

\[
\begin{pmatrix}
a^s & b \\
0 & \alpha^s
\end{pmatrix}
\]

in \( v(T_n(K), x^s) \) as the product of two matrices which are the \( s \)th powers of matrices in \( T_n(K) \). Consequently, claim (b) of Theorem 1 is established as well.

(c) Since in this case the multiplicative group of the field \( K \) is a cyclic group of order \( |K| - 1 \), it follows that \( a^{|K|-1} \) lies in \( UT_n(K) \) for every \( a \in T_n(K) \). Since \( UT_n(K) \) is a \( p \)group and \( (|K|-1)^{\alpha r} \) is coprime to \( p \), every matrix in \( UT_n(K) \) equals itself raised to power \((|K|-1)^{\alpha r}\). Consequently, \( v(T_n(K), x^{(|K|-1)^{\alpha r}}) = UT_n(K) \) and \( \text{wid}(T_n(K), x^{(|K|-1)^{\alpha r}}) = 1 \). Then

\[
v(T_n(K), x^s) = v(T_n(K), x^{(|K|-1)^{\alpha r}}), x^{p^f}) = v(UT_n(K), x^{p^f}).
\]

It is shown in [5] that \( \text{wid}(UT_n(K), x^{p^f}) = 1 \) for \( n \leq p^f + 2 \) and \( \text{wid}(UT_n(K), x^{p^f}) = 2 \) for \( n \geq p^f + 3 \). This suffices to complete the proof of claim (c) of Theorem 1.

Since the group \( FT(K) \) consists of the invertible infinite triangular matrices over the field \( K \) which differ from the identity matrix only in finitely many entries, Theorem 2 easily follows from Theorem 1.

**Proof of Theorem 3.** Recall that for every field \( K \) with more than two elements and all positive integers \( r \) and \( s \) we have

\[
[UT^r_n(K), T_n(K)] = UT^r_n(K),
\]

\[
[UT^r_n(K), UT^s_n(K)] = UT^{r+s}_n(K),
\]

where \( UT^r_n(K) \) is a subgroup of \( UT_n(K) \) consisting of the matrices with \( r - 1 \) zero diagonals above the main diagonal (see [4, p. 40] for instance). Thus, in order to prove Theorem 3 by induction on the number of commutators in the word \( w \) it suffices to show that every matrix in the group appearing on the right-hand side of each of these relations is the commutator of two matrices in the groups appearing on the left-hand sides. Consider two cases corresponding to these relations.
1. Take an arbitrary matrix $c$ in $UT_n^r(K)$ and find a matrix $a$ in $UT_n^r(K)$ and a matrix $b$ in $T_n(K)$ so that $[a, b] = c$. Since the field $K$ contains at least three elements, we can choose a matrix $a = e + \sum_{i=1}^{n-r} a_{i,i+r} e_{i,i+r}$ satisfying $a_{i,i+r} \neq 0$ and $a_{i,i+r} + c_{i,i+r} \neq 0$ for $i = 1, \ldots, n - r$. Here, as usual, $e$ stands for the identity matrix, and $e_{i,i+r}$ is a matrix unit of size $n$. Expanding the matrix equality $ab = bac$ entrywise, we obtain

\begin{align}
(1) & \quad a_{i,i+r} b_{i+r,i+r} = b_{i,i}(a_{i,i+r} + c_{i,i+r}), \quad i = 1, 2, \ldots, n - r, \\
(2) & \quad a_{i,i+r} b_{i+r,j+i+r} = f_{i,j}, \quad i = 1, 2, \ldots, n - r, \quad j = i + 1, \ldots, n - r,
\end{align}

in $K$, where $f_{i,j}$ are linear functions of the entries in row $i$ of $b$ with coefficients obtained from the entries of $a$ and $c$ using addition and multiplication. If we assign arbitrary values in $K$ to the entries in rows $1$ through $r$ of $b$ so that the diagonal entries are nonzero then, using (1), we can determine the diagonal entries $b_{i+1,i+1}, \ldots, b_{n,n}$ of $b$, which moreover are all nonzero, and using (2), we can successively determine the values of the off-diagonal entries in rows $r + 1$ through $n$ of $b$.

2. Take now an arbitrary matrix $c$ in $UT_n^{r+s}(K)$ and find a matrix $a$ in $UT_n^r(K)$ and a matrix $b$ in $UT_n^s(K)$ satisfying $[a, b] = c$. Put $a = e + \sum_{i=1}^{n-r} c_{i,i+r}$. Expanding the equality $ab = bac$ entrywise, we obtain

\begin{align}
(3) & \quad b_{i+r,j+i+s} = f_{i,j}, \quad i = 1, \ldots, n - r - s, \quad j = i + 1, \ldots, n - r - s,
\end{align}

where $f_{i,j}$ are linear functions of the entries in row $i$ of $b$ with coefficients obtained from the entries of $c$ using addition and multiplication. If we assign arbitrary values to the off-diagonal entries in rows $1$ through $r$ of $b$ then (3) enables us to successively determine the entries in rows $(r + 1)$ through $n$ of $b$. \hfill \Box

Theorem 4 follows easily from Theorem 3.

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