SPECTRAL GAP ESTIMATES FOR SOME BLOCK MATRICES

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Abstract. We estimate the size of the spectral gap at zero for some Hermitian block matrices. Included are quasi-definite matrices, quasi-semidefinite matrices (the closure of the set of the quasi-definite matrices) and some related block matrices which need not belong to either of these classes. Matrices of such structure arise in quantum models of possibly disordered systems with supersymmetry or graphene like symmetry. Some of the results immediately extend to infinite dimension.

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A. Some auxiliary computations

Proof of equation (??)

Since the matrices of the type (??) appear to be the source of many illustrative examples we here give an explicit formula for their eigenvalues (which come in plus/minus pairs). We put

$$A = \begin{bmatrix} a_+ & a \\ \bar{a} & a_- \end{bmatrix}, \quad B = \begin{bmatrix} b_+ & b \\ \pm \bar{b} & b_- \end{bmatrix},$$

so that in the case of the minus sign in $B$ the diagonal elements $b_\pm$ are purely imaginary. A straightforward calculation gives

$$\lambda^2_{1,2} = \frac{s}{2} \pm \sqrt{s^2 - |a_+ a_- - b_+ b_-|^2 + |a|^2 + |b|^2 - |a_+ b_- - b_+ a_-| - 2\Re \bar{a}b|^2}.$$

with

$$s = \frac{a_+^2 + a_-^2 + |b_+|^2 + |b_-|^2}{2} + |a|^2 + |b|^2.$$

Proof of equation (??)

We derive the formula (??). Substituting $x_j = A \cos j\alpha + B \sin j\alpha$ in (??) we get

$$A \cos j\alpha \cos \alpha + A \sin j\alpha \sin \alpha B \sin j\alpha \cos \alpha - B \cos j\alpha \sin \alpha$$
$$+ \kappa (A \cos j\alpha + B \sin j\alpha)$$
$$+ A \cos j\alpha \cos \alpha - A \sin j\alpha \sin \alpha B \sin j\alpha \cos \alpha + B \cos j\alpha \sin \alpha$$
$$= 0$$

or

$$(A \cos j\alpha + B \sin j\alpha)(\cos \alpha + \cos \alpha + \kappa) = 0$$

thus implying

$$\kappa = -2 \cos \alpha.$$

The boundary conditions (??) yield

$$A = -A \cos \alpha - B \sin \alpha,$$

$$A \cos(m+1)\alpha + B \sin(m+1)\alpha = A \cos m\alpha + B \sin m\alpha,$$

which is a homogeneous linear system

$$(1 + \cos \alpha)A + B \sin \alpha = 0,$$

$$A(\cos(m+1)\alpha - \cos m\alpha) + B(\sin(m+1)\alpha - \sin m\alpha) = 0,$$

so its determinant must vanish:

$$(1 + \cos \alpha)2 \cos \frac{(2m+1)\alpha}{2} \sin \frac{\alpha}{2} + 2 \sin \alpha \sin \frac{(2m+1)\alpha}{2} \sin \frac{\alpha}{2} = 0,$$
or equivalently,

\[ 0 = \cos \frac{\alpha}{2} \cos \frac{(2m+1)\alpha}{2} + \sin \frac{\alpha}{2} \sin \frac{(2m+1)\alpha}{2} = \cos \left( \frac{\alpha}{2} - \frac{(2m+1)\alpha}{2} \right) = \cos m\alpha. \]

Hence

\[ \alpha = \alpha_k = \frac{2k + 1}{2m} \pi \]

and (??) follows.