The Spectral Shift Function, Friedel Sum Rule, and Levinson Theorem

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We examine three trace formulas, the spectral shift function, Friedel sum rule and Levinson theorem in potential scattering theory. Although these formulas are closely related to each other, they have been often formulated in different settings so far. We propose a unified treatment for them. First we construct the spectral shift function in an alternative way, and then we prove that the spectral shift function thus constructed yields Friedel sum rule and Levinson theorem in arbitrary dimensions without assuming spherical symmetry of the potential.
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1 Introduction

In physical systems, universal nature often reflects the global, geometrical (topological) structure of the system. For example, Gauss’s law in classical electromagnetism is a consequence of the geometrical structure of the three-dimensional Euclidean space. It states that the flux, \( \Phi \), of the electric field, \( \mathbf{E} \), through any closed surface, \( \Sigma \), is proportional to the total charge, \( Q \), enclosed by the surface \( \Sigma \):

\[
\Phi = \int_{\Sigma} \mathbf{E} \cdot d\mathbf{a} = CQ
\]

with a constant \( C \).

In this paper, we study an analogue to Gauss’s law in scattering theory in quantum mechanics. Let us consider a metal with a single impurity at zero temperature. The impurity potential scatters the conduction electrons, and changes their charge distribution. For a fixed Fermi energy \( E_F \), the “excess charge”, \( Z(E_F) \), due to the impurity is defined to be the difference between the total numbers of levels in the Fermi sea with and without the impurity. Then the excess charge, \( Z(E_F) \), equals the total phase shifts, \( \theta(E_F) \), of the scattering matrix, \( S(E_F) \), for the impurity potential:

\[
\theta(E_F) = \frac{1}{2\pi i} \log \det S(E_F) = Z(E_F).
\]

This is known as Friedel sum rule [14] in solid state physics [25].

Since the excess charge, \( Z(E_F) \), is formally written in terms of the trace of the difference between the spectral projection operators with and without the impurity potential, it is closely related to the spectral shift function (SSF) which was initiated by Lifshitz [26]. We briefly describe the previous construction of the SSF. Let \( H \) and \( H_0 \) be a pair of self-adjoint operators. Then the SSF, \( \xi(\lambda) \), is defined as a function on \( \mathbb{R} \) satisfying the following property: If \( f \in C_0^\infty(\mathbb{R}) \), then

\[
\text{Tr} [f(H) - f(H_0)] = \int f'(\lambda)\xi(\lambda)d\lambda.
\]

Here we note that this formula fixes \( \xi(\lambda) \) up to an additive constant. The SSF is known to exist if, for example, \((H + i)^{-m} - (H_0 + i)^{-m}\) is a trace class operator with some \( m > 0 \) (see, e.g., Birman-Yafaev [4] or Yafaev [34]).

Formally, the SSF is written

\[
\xi(\lambda) = \text{Tr} [E_H(\lambda) - E_{H_0}(\lambda)],
\]

where \( E_A(\cdot) \) denotes the spectral projection of a self-adjoint operator \( A \). (This formal expression (2) is nothing but the excess charge!) It is well-known, however, \( E_H(\lambda) - E_{H_0}(\lambda) \) is not necessarily in the trace class, even when the above assumption is satisfied.
As is well known, there are two standard constructions of the SSF (see also Pushnitski [31] and references therein for a more sophisticated representation of the SSF). The first one is due to Krein who defines the SSF as a locally $L^1$ function on $\mathbb{R}$. This construction requires relatively weak assumptions, and the definition is global in $\lambda$. However, the existence of $\xi(\lambda)$ for a fixed $\lambda \in \mathbb{R}$ is not obvious in this construction. The other construction is to compute the difference of the spectral functions. Namely, under certain conditions, one can define

$$\xi'(\lambda) = \text{Tr} \left[ E'_H(\lambda) - E'_{H_0}(\lambda) \right]$$

for $\lambda$ in a “regular” energy region. This method is used widely in the semiclassical and microlocal study of the SSF (see Robert [33] and references therein). The advantage of this method is that one can study the behavior of $\xi(\lambda)$ in detail locally in $\lambda$. On the other hand, $\xi(\lambda)$ is not defined globally in $\lambda$, and the method requires slightly stronger assumption on the perturbation.

We also remark that the behavior of finite-volume spectral shift functions for a large volume is studied in ref. [24, 17, 18, 15, 16]. In particular, under a certain condition, a sequence of finite-volume spectral shift functions is shown to converge to the SSF in the infinite-volume limit [17, 18, 15, 16].

1.1 Main Results

We propose another construction of the spectral shift function, $\xi(\lambda) = \xi(\lambda; H, H_0)$, for a pair of Hamiltonians, $H = -\Delta + V$ and $H_0 = -\Delta$, on $L^2(\mathbb{R}^n)$. We assume that the potential $V$ satisfies

$$|V(x)| \leq C \langle x \rangle^{-\alpha}, \quad x \in \mathbb{R}^n$$

with some $\alpha > n + 3$ and some $C > 0$, where we have written $\langle x \rangle := \sqrt{1 + |x|^2}$. The idea for our construction is to show the existence of the boundary value of the perturbation determinant directly using the stationary scattering theory. This is a variation of Krein’s construction, but we can prove that $\xi(\lambda)$ is defined for each $\lambda \in (0, \infty) \setminus \sigma_{pp}(H)$ and continuous in the same region.

As an application, we consider Friedel sum rule. We first define the finite-volume excess charge, $Z_R(\lambda)$, due to the impurity potential, $V$, by

$$Z_R(\lambda) := \text{Tr} \left[ \vartheta_R(E_H(\lambda) - E_{H_0}(\lambda)) \vartheta_R \right],$$

where $\vartheta_R(x) = \vartheta(x/R)$ is a cutoff function with a large $R$ and with $\vartheta \in C^\infty_0(\mathbb{R}^n)$ satisfying $\vartheta = 1$ in a neighborhood of $x = 0$. Then we can prove

$$Z(\lambda) := \lim_{R \to \infty} Z_R(\lambda) = \xi(\lambda) \quad \text{for } \lambda \in (0, \infty) \setminus \sigma_{pp}(H).$$
Namely, the excess charge, $Z(\lambda)$, in the infinite-volume limit is equal to the SSF, $\xi(\lambda)$. On the other hand, the total phase shift, $\theta(\lambda)$, for the scattering matrix, $S(\lambda)$, is equal to $\xi(\lambda)$ from the Birman-Krein formula. From these, we obtain that Friedel sum rule (1) holds for $\mathcal{E}_F \in (0, \infty) \setminus \sigma_{pp}(H)$ in arbitrary dimensions.

We also show that the SSF thus obtained yields Levinson theorem: Let $n \geq 3$. Suppose that

$$|V(x)| \leq C(x)^{-\alpha}, \quad x \in \mathbb{R}^n$$

with some $\alpha > 2n$ and some $C > 0$. Then, we can prove

$$\theta(0) = \lim_{\lambda \to +0} \theta(\lambda) = N_{\leq} + q,$$

where $N_{\leq}$ is the number of the bound states of the Hamiltonian $H$ with an eigenvalues less than or equal to zero; the number $q$ is given as follows: In dimensions $n = 3, 4$,

$$q = \begin{cases} 1/2, & \text{for } n = 3; \\ 1, & \text{for } n = 4 \end{cases}$$

if the zero energy is a resonance for $H$, and $q = 0$ otherwise. In dimensions $n \geq 5$, $q = 0$ because of the absence of a resonance [20].

The result in three dimensions recovers several variants of Levinson theorem [30, 28, 12, 13, 9, 29]. Our normalization of the phase shift is not necessarily the same as the previous ones. As far as we know, the statement in the higher dimensions $n \geq 4$ is justified for the first time although it has been believed to be valid [6, 5]. We stress that our results of the total phase shift in one and two dimensions recover the previous results, too. We present their precise statements in Section 4 because they are more complicated than the results in dimensions $n \geq 3$. In consequence, the total phase shift in the zero energy limit is always quantized to an integer or a half-odd integer in arbitrary dimensions. Since the total phase shift equals the excess charge, this implies that the excess charge in the zero energy limit is quantized to the same value. We also stress that, since the previous approaches to Levinson theorem in dimensions $n \leq 3$ did not rely on the usefulness of the SSF, they suffered from the existence of exceptional points of the spectrum of the Hamiltonian $H$. In consequence, their statements are not necessarily expressed in a sophisticated form including all the exceptional cases. We refine the previous results in dimensions $n \leq 3$.

The present paper is organized as follows: In Section 2, we first describe our method to construct the SSF in three or lower dimensions, and then extend it to higher dimensions. In Section 3, we prove that the SSF is equal to the excess charge. In Section 4, we compute the total phase shifts in the zero energy limit.
2 Construction of the Spectral Shift Function

We construct the SSF for potential scattering theory. First, we describe our abstract scheme for the construction, and then prove the existence of the SSF. Consider a pair of Hamiltonians, 

$$H = H_0 + V, \quad H_0 = -\Delta \quad \text{on } L^2(\mathbb{R}^n).$$

We suppose the potential $V$ satisfies the bound (3) with $\alpha > n + 3$. We may allow $V$ to have some singularities, but assume that it is bounded for simplicity. By the invariance principle, we construct $\xi(\lambda)$ as

$$\xi(\lambda; H, H_0) = -\xi((\lambda + M)^{-\ell}; (H + M)^{-\ell}, (H_0 + M)^{-\ell})$$

with some integer $\ell > 0$ and a sufficiently large $M > 0$, where $\xi(\lambda; A, A_0)$ denotes the SSF for a pair $A$ and $A_0$. We recall the SSF is defined as

$$\xi(\lambda; A, A_0) = -\lim_{z \to \lambda + i0} \frac{1}{\pi} \text{Im} \log \Delta_{A/A_0}(z),$$

where $\Delta_{A/A_0}(z)$ denotes the perturbation determinant defined by

$$\Delta_{A/A_0}(z) = \det [(A - z)(A_0 - z)^{-1}] \quad \text{for } z \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(A_0)).$$

It is easy to see $\Delta_{A/A_0}(z)$ is well-defined if $A - A_0$ is trace class, and it is analytic in $z$. Moreover,

$$\Delta_{A/A_0}^{-1}(z)\Delta_{A/A_0}'(z) = -\text{Tr} [(A - z)^{-1} - (A_0 - z)^{-1}],$$

and $\xi(\lambda; A, A_0) = 0$ if $\lambda > \sup(\sigma(A) \cup \sigma(A_0))$ [or if $\lambda < \inf(\sigma(A) \cup \sigma(A_0))]$. Hence we have an expression of the SSF:

$$\xi(\lambda; A, A_0) = \lim_{z \to \lambda + i0} \frac{1}{\pi} \text{Im} \int_{\gamma_z} \text{Tr} [(A - w)^{-1} - (A_0 - w)^{-1}] \, dw,$$

where $\gamma_z$ denotes a contour in $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ such that $\gamma_z(0) = k > \sup(\sigma(A) \cup \sigma(A_0))$ and $\gamma_z(1) = z$. Note that this expression is consistent with the formal formula (2) by virtue of Stone formula.

2.1 Dimensions $n \leq 3$

First, we prove the existence of $\xi(\lambda)$ in dimensions $n \leq 3$. In the next section, we treat the case in dimensions $n \geq 4$. In this section, we set $\ell = 1$, namely,

$$A = (H + M)^{-1}, \quad A_0 = (H_0 + M)^{-1}$$

with a sufficiently large (fixed) $M > 0$. Then it is well-known $A - A_0 \in \mathcal{I}_1$, where $\mathcal{I}_p$ denotes the $p$-th trace ideal. Hence $\Delta_{A/A_0}(z)$ is well-defined, and the above definition applies. Now the key estimate of our construction is the following: We denote

$$\mu(z) = (z + M)^{-\ell} = (z + M)^{-1}.$$
Lemma 3. Let $\lambda \in (0, \infty) \setminus \sigma_{pp}(H)$. Then

$$\lim_{z \to \lambda+i0} \text{Tr} \left[ (A - \mu(z))^{-1} - (A_0 - \mu(z))^{-1} \right]$$

exists, and the limit is continuous in $(0, \infty) \setminus \sigma_{pp}(H)$.

Remark. We do not prove (or claim) $(A - \mu(\lambda+i0))^{-1} - (A_0 - \mu(\lambda+i0))^{-1} \in \mathcal{A}_1$. We only prove the existence of the limit of the trace.

Now combining Proposition 1 with the formula (4), we have the following result on the SSF:

Corollary 2. The SSF, $\xi(\lambda)$, exists for $\lambda \in (0, \infty) \setminus \sigma_{pp}(H)$, and $\xi(\lambda)$ is continuous in $(0, \infty) \setminus \sigma_{pp}(H)$.

Proof of Proposition 1. For $z \notin \sigma(H) \cup \sigma(H_0)$,

$$(A - \mu(z))^{-1} - (A_0 - \mu(z))^{-1} = -(A - \mu(z))^{-1}(A - A_0)(A_0 - \mu(z))^{-1} \in \mathcal{A}_1.$$ 

Let $3/2 < \beta < (\alpha - n)/2$. Then by the standard commutator computations, we see

$$(5) \quad W := \langle x \rangle^\beta (A - A_0) \langle x \rangle^\beta = -\langle x \rangle^\beta (H + M)^{-1}V(H_0 + M)^{-1}\langle x \rangle^\beta \in \mathcal{A}_1.$$ 

Then we write

$$(6) \quad \text{Tr} \left[ (A - \mu(z))^{-1} - (A_0 - \mu(z))^{-1} \right]$$

$$= -\text{Tr} \left[ (A - \mu(z))^{-1}\langle x \rangle^{-\beta}W\langle x \rangle^{-\beta}(A_0 - \mu(z))^{-1} \right]$$

$$= -\text{Tr} \left[ W\langle x \rangle^{-\beta}(A_0 - \mu(z))^{-1}(A - \mu(z))^{-1}\langle x \rangle^{-\beta} \right].$$

Now in order to complete the proof, it suffices to show the following lemma. 

Lemma 3. For $\lambda \in (0, \infty) \setminus \sigma_{pp}(H)$,

$$\lim_{z \to \lambda+i0} \langle x \rangle^{-\beta} (A_0 - \mu(z))^{-1}(A - \mu(z))^{-1}\langle x \rangle^{-\beta}$$

exists in $B(L^2(\mathbb{R}^n))$, and the limit is continuous in $(0, \infty) \setminus \sigma_{pp}(H)$.

Proof. We note

$$A_0 - \mu(z) = (H_0 + M)^{-1} - (z + M)^{-1} = -(z + M)^{-1}(H_0 - z)(H_0 + M)^{-1}$$

and hence

$$(A_0 - \mu(z))^{-1} = -(z + M)(H_0 + M)(H_0 - z)^{-1}$$

$$= -(z + M) - (z + M)^2(H_0 - z)^{-1}.$$
Similarly, we have

\[
(A - \mu(z))^{-1} = -(z + M) - (z + M)^2(H - z)^{-1} \\
= (z + M) - (z + M)(H_0 - z)^{-1} \\
+ (z + M)^2(H_0 - z)^{-1}V(H - z)^{-1}.
\]

Thus we have

\[
(A_0 - \mu(z))^{-1}(A - \mu(z))^{-1} = a_0(z) + a_1(z)(H_0 - z)^{-1} + a_2(z)(H_0 - z)^{-2}V(H - z)^{-1},
\]

where \(a_j(z)\) are polynomials in \(z\). Since

\[
\langle x \rangle^{-\gamma}(H_0 - z)^{-1}\langle x \rangle^{-\gamma}, \langle x \rangle^{-\beta}(H_0 - z)^{-2}\langle x \rangle^{-\beta}, \text{ and } \langle x \rangle^{-\gamma}(H - z)^{-1}\langle x \rangle^{-\gamma}
\]

(with \(\gamma > 1/2\)) are bounded and continuous in a complex neighborhood of \(\lambda\) in \(\mathbb{C}_+\) (see Agmon [1], Reed-Simon [32] Section XIII.8), we conclude the assertion.

\[\square\]

2.2 Dimensions \(n \geq 4\)

If \(n \geq 4\), we set

\[
A = (H + M)^{-\ell}, \quad A_0 = (H_0 + M)^{-\ell}
\]

with \(\ell \in \mathbb{Z}\) such that \(n/2 - 1 < \ell \leq n/2\). Then we have

\[
A - A_0 = -\sum_{j=1}^{\ell}(H + M)^{-j}V(H_0 + M)^{-\ell-1+j} \\
= -\sum_{j=1}^{\ell}(H_0 + M)^{-j}V(H_0 + M)^{-\ell-1+j} \\
+ \sum_{j=1}^{\ell}\sum_{k=1}^{j}(H + M)^{-k}V(H_0 + M)^{-j-1+k}V(H_0 + M)^{-\ell-1+j} \\
= \cdots.
\]

Iterating this procedure \(\ell\)-times, and using the fact \(V(H_0 + M)^{-j} \in \mathcal{J}_p\) for \(p > n/(2j)\), we learn \(A - A_0 \in \mathcal{J}_1\). Then the main part of the proof of Proposition 1 can be modified accordingly.

In order to modify the proof of Lemma 3, we use

\[
A_0 - \mu(z) = -\sum_{j=1}^{\ell}(z + M)^{-j}(H_0 - z)(H_0 + M)^{-\ell-1+j} \\
= -(H_0 - z)(H_0 + M)^{-1}L_0(z),
\]
where \( \mu(z) = (z + M)^{-\ell} \), and we have written

\[
L_0(z) = \sum_{j=1}^{\ell} (z + M)^{-j}(H_0 + M)^{-\ell+j}.
\]

Since \( \text{Re} (z + M) > M \) if \( z \sim \lambda \), \( L_0(z) \) is invertible, and we obtain

\[
(A_0 - \mu(z))^{-1} = -L_0^{-1}(z)(H_0 + M)(H_0 - z)^{-1} = -L_0^{-1}(z) \left[ 1 + (z + M)(H_0 - z)^{-1} \right].
\]

We also write

\[
L(z) = \sum_{j=1}^{\ell} (z + M)^{-j}(H + M)^{-\ell+j}.
\]

Then we have

\[
(A_0 - \mu(z))^{-1}(A - \mu(z))^{-1} = L_0^{-1}(z) \left\{ a_0(z) + a_1(z)(H_0 - z)^{-1} + a_2(z)(H_0 - z)^{-2} + a_3(z)(H_0 - z)^{-1}V(H - z)^{-1} + a_4(z)(H_0 - z)^{-2}V(H - z)^{-1} \right\} L^{-1}(z)
\]

with some polynomials \( a_j(z) \) in \( z \). Moreover, using the standard weight estimates,

\[
\langle x \rangle^\gamma (H_0 + M)^{-1} \langle x \rangle^{-\gamma}, \quad \langle x \rangle^\gamma (H + M)^{-1} \langle x \rangle^{-\gamma} \in B(L^2(\mathbb{R}^n)),
\]

we can carry out the same argument as in the proof of Lemma 3. Consequently, we have:

**Proposition 4.** Let \( \lambda \in (0, \infty) \setminus \sigma_{pp}(H) \). Then

\[
\lim_{z \to \lambda+0} \text{Tr} \left[ (A - \mu(z))^{-1} - (A_0 - \mu(z))^{-1} \right]
\]

exists, and the limit is continuous in \( (0, \infty) \setminus \sigma_{pp}(H) \). Moreover, the SSF, \( \xi(\lambda) \), exists for \( \lambda \in (0, \infty) \setminus \sigma_{pp}(H) \), and \( \xi(\lambda) \) is continuous in \( (0, \infty) \setminus \sigma_{pp}(H) \).

### 3 Friedel Sum Rule

In solid state physics [25], the difference of the number of the states given by the right-hand side of (2) has been often called the excess charge. In this section, we define the excess charge, and show that it is equivalent to the SSF. Besides, the SSF is equal to the total phase shift \( \theta(\lambda) \) which is given by

\[
e^{2\pi i \theta(\lambda)} = \det S(\lambda), \quad \lambda > 0,
\]
where $S(\lambda)$ is the scattering matrix. By the invariance principle and the Birman-Krein formula, we have

$$\theta(\lambda) = \xi(\lambda; H, H_0) = -\xi(\mu(\lambda); A, A_0)$$

with $\theta(\lambda) = 0$ for $\lambda < \sigma(H)$. Therefore, the excess charge is equal to the total phase shift. This is nothing but Friedel sum rule.

To begin with, we introduce a cutoff function $\vartheta_R(x) = \vartheta(x/R)$ with a large $R > 0$ and with $\vartheta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\vartheta = 1$ in a neighborhood of $x = 0$. Then the excess charge is defined by

$$Z(\lambda) := \lim_{R \to \infty} \text{Tr} \left[ \vartheta_R(E_H(\lambda) - E_{H_0}(\lambda)) \vartheta_R \right],$$

where $E_A(\lambda)$ denotes the spectral projection: $\chi_{(-\infty, \lambda]}(A)$. We want to show that the above limit exists, and that it is equivalent to the SSF under certain assumptions.

We denote $Z_R(\lambda) = \text{Tr} \left[ \vartheta_R(E_H(\lambda) - E_{H_0}(\lambda)) \vartheta_R \right]$ for $\lambda > 0$. Using the notation of Section 2, we recall

$$E_H(\lambda) = 1 - E_A(\mu(\lambda)) = -\lim_{z \to \mu(\lambda)+i0} \frac{1}{\pi} \int_{\gamma_\lambda} (A - w)^{-1} dw.$$ 

Hence we have

$$Z(\lambda) = -\lim_{z \to \mu(\lambda)+i0} \frac{1}{\pi} \int_{\gamma_\lambda} \text{Tr} \left[ \vartheta_R((A - w)^{-1} - (A_0 - w)^{-1}) \vartheta_R \right] dw$$

$$= \lim_{z \to \mu(\lambda)+i0} \frac{1}{\pi} \int_{\gamma_\lambda} \text{Tr} \left[ \vartheta_R(A - w)^{-1} \langle x \rangle^{-\beta} W \langle x \rangle^{-\beta} (A_0 - w)^{-1} \vartheta_R \right] dw,$$

since $\vartheta_R(A - w)^{-1} \langle x \rangle^{-\beta}$ and $\langle x \rangle^{-\beta} (A_0 - w)^{-1} \vartheta_R$ have norm limits as $w \to \mu(\lambda) + i0$. In particular, this implies the existence of $Z_R(\lambda)$. We show

**Theorem 5.** Let $\lambda \in (0, \infty) \setminus \sigma_{pp}(H)$. Then

$$\lim_{R \to \infty} Z_R(\lambda) = \xi(\lambda; H, H_0).$$

**Proof.** From (4), (6) and the above representation of $Z_R(\lambda)$, we have

$$Z_R(\lambda) - \xi(\lambda; H, H_0)$$

$$= Z_R(\lambda) + \xi(\mu(\lambda); A, A_0)$$

$$= \text{Im} \frac{1}{\pi} \int_{\gamma_\lambda} \text{Tr} \left[ W \langle x \rangle^{-\beta} (A_0 - w)^{-1} (\vartheta_R^2 - 1)(A - w)^{-1} \langle x \rangle^{-\beta} \right] dw.$$

Thus it suffices to show

$$\| \langle x \rangle^{-\beta} (A_0 - w)^{-1} (1 - \vartheta_R^2)(A - w)^{-1} \langle x \rangle^{-\beta} \| \to 0 \quad \text{as} \quad R \to \infty$$
uniformly in $w \in \gamma_{\lambda}$. If $w$ is away from $\sigma(A) \cup \sigma(A_0)$, then
\[
\|\langle x \rangle^{-\beta}(A_0 - w)^{-1}(1 - \vartheta^2_R)(A - w)^{-1} \langle x \rangle^{-\beta}\|
\]
\[
= \|\langle x \rangle^{-\beta}(A_0 - w)^{-1} (x)^{-\beta}(1 - \vartheta^2_R)(A - w)^{-1} \langle x \rangle^{-\beta}\|
\]
\[
\leq \|\langle x \rangle^{-\beta}(A_0 - w)^{-1} \langle x \rangle^{-\beta}\| \cdot \|\langle x \rangle^{-\beta}(1 - \vartheta^2_R)\| \cdot \|(A - w)^{-1}\| = O(R^{-\beta})
\]
locally uniformly in $w$. Thus it suffice to consider the case $w \sim \mu(\lambda) \pm i0$.

Analogously to the argument in Section 2, it suffices to show
\[
\|\langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)(H - z)^{-1} \langle x \rangle^{-\beta}\| \to 0 \quad \text{as } R \to \infty
\]
if $z \sim \lambda \pm i0$ in $\mathbb{C}_\pm = \{z \mid \pm \text{Im } z \geq 0\}$. Since
\[
\langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)(H - z)^{-1} \langle x \rangle^{-\beta}
\]
\[
= \langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)(H_0 - z)^{-1} \langle x \rangle^{-\beta}
\]
\[
- \langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)(H_0 - z)^{-1} \vartheta(H - z)^{-1} \langle x \rangle^{-\beta},
\]
Theorem 5 now follows from the next lemma.

**Lemma 6.** There exist $\mathcal{U}$: a neighborhood of $\lambda \pm i0$ in $\mathbb{C}_\pm$, $\varepsilon > 0$ and $C > 0$ such that
\[
\|\langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)(H_0 - z)^{-1} \langle x \rangle^{-\beta}\| \leq CR^{-\varepsilon}
\]
for $z \in \mathcal{U}$.

**Proof.** We consider the case $\lambda + i0$ only. The other case is similar. It is easy to observe that it suffices to show
\[
\|\langle x \rangle^{-\beta}(H_0 - z)^{-1}(1 - \vartheta^2_R)\eta(H_0 - z)^{-1} \langle x \rangle^{-\beta}\| \leq CR^{-\varepsilon},
\]
where $\eta \in C_0^\infty((0, \infty))$ such that $\eta = 1$ in a neighborhood of $\lambda$. In order to show this, we use a Mourre-type microlocal resolvent estimate of Isozaki-Kitada [19]. See also [27]. We use the free case only, which is considerably simpler than their general results. Let $\rho_\pm \in C^\infty([-1, 1])$ such that
\[
\rho_+(t) + \rho_-(t) = 1; \quad \rho_\pm(t) = 0 \quad \text{if } \pm t < -\frac{1}{2},
\]
We also set $\delta$ and $\varepsilon$ so that
\[
\frac{1}{2} < \delta < \beta - 1; \quad 0 < \varepsilon < \delta - \frac{1}{2}.
\]
We write
\[
p_\pm(x, \xi) = R^\varepsilon(x)^{-\varepsilon} \{1 - [\vartheta_R(x)]^2\} \rho_\pm(\hat{x} \cdot \hat{\xi})\eta(|\xi|^2),
\]
where \( \hat{\tau} = x/|x| \). We quantize \( p_{\pm} \) by the usual Kohn-Nirenberg pseudodifferential operator calculus:

\[
P_{\pm} f(x) = p_{\pm}(x, D_x) f(x) = (2\pi)^{-n/2} \int p_{\pm}(x, \xi) e^{ix \cdot \xi} \hat{f}(\xi) d\xi.
\]

Then we have

\[
(1 - \partial_x^2)\eta(H_0) = R^{-\varepsilon}\langle x \rangle^\varepsilon (P_+ + P_-),
\]

and \( p_{\pm}(x, \xi) \) satisfies the assumptions of Theorem 1 (or Theorem 1.2) of [19], uniformly in \( R > 1 \), and we have

\[
\| \langle x \rangle^\beta (H_0 - z)^{-1} \langle x \rangle^{-\beta} \| \leq C \quad \text{for} \ z \sim \lambda, z \in \mathbb{C}_+.
\]

Similarly, we can apply the same argument to \( (\langle x \rangle^\varepsilon P_+(x)^{-\varepsilon})^* \) instead of \( P_- \). In fact, \( (\langle x \rangle^\varepsilon P_+(x)^{-\varepsilon})^* \) is also a pseudodifferential operator, and its symbol can be computed by asymptotic expansions, up to an error of \( O(\langle x \rangle^{-\infty}) \).

Let \( \hat{p}_+(x, \xi) \) be the symbol of \( (\langle x \rangle^\varepsilon P_+(x)^{-\varepsilon})^* \). Then it has the same support property with \( p_+(x, \xi) \). In particular, we can show

\[
\hat{p}_+(x, \xi) = p_+(x, \xi) + i(\partial_x \cdot \partial_\xi) p_+(x, \xi) - i\partial_\xi p_+(x, \xi) \cdot (\varepsilon x / \langle x \rangle^2) + O(\langle x \rangle^{-2}).
\]

Thus we obtain

\[
\| \langle x \rangle^{-\beta} (H_0 - z)^{-1} \langle x \rangle^\varepsilon P_+(x)^\varepsilon \| \leq C \quad \text{for} \ z \sim \lambda, z \in \mathbb{C}_+,
\]

as well. Combining these, we have

\[
\begin{align*}
\| \langle x \rangle^{-\beta} (H_0 - z)^{-1}(1 - \partial_x^2)\eta(H_0)(H_0 - z)^{-1} \langle x \rangle^{-\beta} \|
&= R^{-\varepsilon}\| \langle x \rangle^{-\beta} (H_0 - z)^{-1} \langle x \rangle^\varepsilon (P_+ + P_-)(H_0 - z)^{-1} \langle x \rangle^{-\beta} \|
\leq R^{-\varepsilon} \left\{ \| \langle x \rangle^{-\beta} (H_0 - z)^{-1} \langle x \rangle^{-(\delta - \varepsilon)} \| \cdot \| \langle x \rangle^\delta P_- (H_0 - z)^{-1} \langle x \rangle^{-\beta} \| \\
&+ \| \langle x \rangle^{-\beta} (H_0 - z)^{-1} \langle x \rangle^\varepsilon P_+(x)^\varepsilon \| \cdot \| \langle x \rangle^{-(\delta - \varepsilon)}(H_0 - z)^{-1} \langle x \rangle^{-\beta} \| \right\}
\leq CR^{-\varepsilon}
\end{align*}
\]

for \( z \sim \lambda, z \in \mathbb{C}_+ \).

\[ \square \]

4 Levinson Theorem

In this section, we compute the SSF in the zero energy limit, i.e., the total phase shift at the zero energy. As a result, we show that the total phase shift at the zero energy is quantized to an integer or a half-odd integer. Our results are summarized as the following three theorems:
Theorem 7. Let \( n = 1 \). Suppose that the potential \( V \) satisfies
\[
|V(x)| \leq \text{Const.}\langle x\rangle^{-\alpha}, \quad x \in \mathbb{R}^n,
\]
with some \( \alpha > 5 \). Then, the total phase shift at the zero energy is given by
\[
\theta(0) = \lim_{\lambda \to +0} \theta(\lambda) = N_< - \frac{q}{2},
\]
where \( N_< \) is the number of the bound states of the Hamiltonian \( H \) with a strictly negative eigenvalue; \( q = 0 \) if the zero energy is a resonance for \( H \), and \( q = 1 \) otherwise.

The proof is given in Appendix A.

Remark. As is well known, there appears no bound state and at most a single resonance state at the zero energy in one dimension [7, 8]. The total phase shift \( \theta(0) \) at the zero energy in one dimension was derived by [7, 8]. The present assumption (7) on the potential \( V \) is slightly different from theirs [8].

Theorem 8. Let \( n = 2 \). Suppose that the potential \( V \) satisfies
\[
|V(x)| \leq \text{Const.}\langle x\rangle^{-\alpha}, \quad x \in \mathbb{R}^n,
\]
with some \( \alpha > 6 \). Then,
\[
\theta(0) = \lim_{\lambda \to +0} \theta(\lambda) = N_\leq + N_r,
\]
where \( N_\leq \) is the number of the bound states of the Hamiltonian \( H \) with the eigenvalue less than or equal to zero, and \( N_r \) is the number of the resonance states which show asymptotics,
\[
\psi \sim \frac{c \cdot x}{|x|^2},
\]
for a large \( |x| \). Here \( c \) is a two-component vector in \( \mathbb{C}^2 \). There appear at most two resonance states showing the above asymptotics because the number of linearly independent vectors is not beyond two in two dimensions.

The proof is given in Appendix B.

Remark. There appears another resonance state [6] which shows asymptotics, \( \psi \sim \text{Const.} \), for a large \( |x| \). Surprisingly, this state does not contribute to the total phase shift at the zero energy [6]. A Levinson-type theorem in two dimensions was derived by [6]. In the case that the zero energy is neither an eigenvalue nor a resonance for the Hamiltonian \( H \) for a class of potentials, Cheney [10] proved Levinson theorem in two dimensions.
Theorem 9. Let $n \geq 3$. Suppose that
\begin{equation}
|V(x)| \leq \text{Const.}(x)^{-\alpha}, \quad x \in \mathbb{R}^n
\end{equation}
with some $\alpha > 2n$. Then,
\begin{equation}
\theta(0) = \lim_{\lambda \to +0} \theta(\lambda) = N_\leq + q,
\end{equation}
where $N_\leq$ is the number of the bound states of the Hamiltonian $H$ with an
eigenvalues less than or equal to zero; the number $q$ is given as follows: In
dimensions $n = 3, 4$,
\begin{equation}
q = \begin{cases}
\frac{1}{2}, & \text{for } n = 3; \\
1, & \text{for } n = 4
\end{cases}
\end{equation}
if the zero energy is a resonance for $H$, and $q = 0$ otherwise. In dimensions
$n \geq 5$, $q = 0$ because of the absence of a resonance [20].

The proof is given in Sections 4.1, 4.2 and 4.3 in which we treat the cases
with $n = 3$, $n \geq 5$ and $n = 4$, respectively.

Remark. As is well known, there appears at most a single resonance state
at the zero energy in three [28] or four dimensions [21]. Under an assumption
on the existence of a time delay operator, a Levinson-type theorem was obtained by Osborn and Bollé [30], and Bollé and Wilk [9]. A
relation between the phase shifts at the zero and the high energy limits was
obtained by Newton [28, 29]. His derivation is based on the modified Fred-
holm determinant. In the case that the zero energy is neither an eigenvalue
nor a resonance for the Hamiltonian $H$ for a class of potentials, Dreyfus
[12, 13] proved that the total phase shift at the zero energy is equal to the
number of the bound states with a strictly negative eigenvalue under a sim-
ilar normalization condition to ours for the SSF. See also Theorem 1.14 in
Chap. 9 of [35].

We also remark that, quite recently, Bellissard and Schulz-Baldes [3]
obtain a Levinson-type theorem for tight-binding Hamiltonians with an im-
purity on $\mathbb{Z}^n$ with $n \geq 3$. For related works, see the references in [3].

In order to compare Theorem 9 with the result obtained by Newton [28],
we need the high energy asymptotics of the total phase shift $\theta(\lambda)$. Our
expression of the SSF also yields the well known asymptotics:

Theorem 10. Let $n = 3$. Suppose that in addition to the bound (9) in
Theorem 9, the potential $V$ satisfies
\begin{equation}
|\partial^\kappa_x V(x)| \leq \text{Const.}(x)^{-\alpha'}
\end{equation}
for all multi-indices $\kappa$ satisfying $|\kappa| = 1, 2$ and some $\alpha' > 3$. Then
\begin{equation}
\xi(\lambda) = \theta(\lambda) = -\frac{\lambda^{1/2}}{4\pi^2} \int d^3x V(x) + \frac{1}{16\pi^2 \lambda^{1/2}} \int d^3x V(x)^2 + o(\lambda^{-1/2})
\end{equation}
for a large $\lambda$. 

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The proof of Theorem 10 is given in Appendix C. Since one has
\[
\lim_{\lambda \to \infty} \left[ \theta(\lambda) + \frac{\lambda^{1/2}}{4\pi^2} \int d^3x V(x) \right] = 0
\]
immediately from Theorem 10, the result (10) for \( n = 3 \) coincides with that obtained by Newton [28]. Thus the result (10) is nothing but Levinson theorem in three dimensions. The high energy asymptotics (12) should be compared to the result by Jensen [22]. Since he treated the derivative \( \xi'(\lambda) \) of the SSF, he did not show that the constant term in the expansion (12) is vanishing [35]. His assumptions on the potential \( V \) are slightly different from ours. For related works, see the references in [22] and [35].

\section*{4.1 Proof of Theorem 9 in the Case of \( n = 3 \)}
As in the proof of Lemma 3, we have
\[
(A_0 - w)^{-1} = -(z + M) + \frac{(z + M)^2}{z - H_0}
\]
and
\[
(A - w)^{-1} = -(z + M) + \frac{(z + M)^2}{z - H}.
\]
Substituting these into the expression (4) of SSF, we have
\[
\xi(\lambda; H, H_0) = -\xi(\mu(\lambda); A, A_0)
\]
\[
= - \lim_{z \to \mu(\lambda) + i0} \frac{1}{\pi} \text{Im} \int_{\gamma_z} \text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right] \frac{(z + M)^2}{z - H_0} dw
\]
\[
= - \lim_{z \to \lambda + i0} \frac{1}{\pi} \text{Im} \int_{\gamma_z} \text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right] dz,
\]
where we have used \( w = (z + M)^{-1} \). In order to compute the total phase shift \( \theta(0) \) in the zero energy limit, we introduce the excess charge \( Z^0 \) at the zero energy as
\[
Z^0 = \lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int \gamma_{\varepsilon,\delta} dz \text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right],
\]
where the path \( \gamma_{\varepsilon,\delta} \) is parametrized as \( z = \varepsilon e^{i\phi} \) with \( \delta \leq \phi \leq 2\pi - \delta \). Then the total phase shift \( \theta(0) \) is given by
\[
\theta(0) = \lim_{\lambda \to 0} \theta(\lambda) = Z^0 + N_<,
\]
where \( N_< \) is the number of the bound states of \( H \) with a strictly negative eigenvalue, \( \lambda < 0 \).
The difference of the resolvents in the trace is written
\[
\frac{1}{z - H} - \frac{1}{z - H_0} = \frac{1}{z - H} V \frac{1}{z - H_0} = \frac{1}{z - H_0} (z - H_0) \frac{1}{z - H} V \frac{1}{z - H_0}.
\]

Write
\[(15)\quad u = |V|^{1/2} \text{sgn} V \quad \text{and} \quad v = |V|^{1/2}.
\]

Then, one has
\[(16)\quad (z - H_0)(z - H)^{-1} V = [1 + V(z - H)^{-1}] V
\]
\[= [1 + uv(z - H)^{-1}] uv
\]
\[= u [1 + v(z - H)^{-1} u] v
\]
for \(z \notin \sigma(H) \cup \sigma(H_0)\). These observations yield
\[
\text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right]
\]
\[= \text{Tr} \{ (z - H_0)^{-1} u [1 + v(z - H)^{-1} u] v(z - H_0)^{-1} \}
\]
\[= \text{Tr} \{ v(z - H_0)^{-2} u [1 + v(z - H)^{-1} u] \}.
\]

Here we have used the fact that the operators, \((z - H_0)^{-1} u\) and \(v(z - H_0)^{-1}\), are Hilbert-Schmidt class for \(z \notin \sigma(H_0)\). Further, using
\[(17)\quad [1 + v(z - H)^{-1} u] = [1 - v(z - H_0)^{-1} u]^{-1} \quad \text{for} \quad z \notin \sigma(H_0) \cup \sigma(H),
\]
one has
\[
\text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right]
\]
\[= \text{Tr} \{ v(z - H_0)^{-2} u [1 - v(z - H_0)^{-1} u]^{-1} \}.
\]

Write
\[
v \frac{1}{z - H_0} u = vR_0^< u + vR_0^> u,
\]
where \(R_0^<\) is the operator whose integral kernel is given by
\[
-\frac{1}{4\pi} \frac{e^{i\sqrt{\gamma} |x|}}{|x|} \chi_R^<(x)
\]
with the characteristic function
\[
\chi_R^<(x) = \begin{cases} 1, & \text{if } |x| \leq R; \\ 0, & \text{otherwise}, \end{cases}
\]

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Thus the statement follows from the assumption (9) for the potential $\chi$ with a small $\varepsilon > 0$ and the integral kernel of $R^\varepsilon$ is given by

$$-\frac{1}{4\pi} \frac{e^{i\sqrt{|x|}|x|}}{|x|} \chi_R^\varepsilon(x)$$

with $\chi_R^\varepsilon = 1 - \chi_R^\varepsilon$. We take $R = \varepsilon^{-\mu}$ with $2/5 < \mu < 1/2$, recalling $z = \varepsilon e^{i\phi}$ with a small $\varepsilon > 0$ and $\delta \leq \phi \leq 2\pi - \delta$.

**Lemma 11.** We have $v R_0^\varepsilon u \in \mathcal{J}_2$, and the Hilbert-Schmidt norm is bounded as

$$\|v R_0^\varepsilon u\|_2 \leq \text{Const.} \times \varepsilon^{1+\delta'}$$

for a small $\varepsilon$ with a positive constant $\delta'$.

**Proof.** Note that

$$\left| v(x) \frac{e^{i\sqrt{|z|}x-y}}{|x-y|} \chi_R^\varepsilon(x-y) u(y) \right|^2 \leq \frac{1}{R^2} \langle V(x) \rangle \langle v(x) \rangle \langle v(y) \rangle \langle x \rangle^{-3} \chi_R^\varepsilon(x-y) \langle y \rangle^{-3} \leq \frac{1}{R^2} \left( \frac{2}{R} \right)^3 \langle x \rangle^3 \langle V(x) \rangle \langle y \rangle^3 \langle V(y) \rangle.$$ 

Thus the statement follows from the assumption (9) for the potential $V$ and $R = \varepsilon^{-\mu}$ with $\mu > 2/5$. \hfill \square

The integral kernel of the operator $v R_0^\varepsilon u$ can be expanded as

$$v(x) \left( i \sqrt{|z|} x - y \right) \chi_R^\varepsilon(x-y) u(y)$$

(19) 

$$= -\frac{1}{4\pi} v(x) \frac{1}{|x-y|} \sum_{m=0}^{\infty} \frac{1}{m!} (i \sqrt{|z|} |x-y|)^m \chi_R^\varepsilon(x-y) u(y)$$

$$= -\frac{1}{4\pi} v(x) \frac{1}{|x-y|} \chi_R^\varepsilon(x-y) u(y) - i \frac{\sqrt{z}}{4\pi} v(x) \chi_R^\varepsilon(x-y) u(y)$$

$$+ \frac{\sqrt{z}}{8\pi} v(x) |x-y| \chi_R^\varepsilon(x-y) u(y)$$

$$- i \frac{\sqrt{z}}{4\pi} v(x) \sum_{m=3}^{\infty} \frac{1}{m!} (i \sqrt{|z|} |x-y|)^{m-1} \chi_R^\varepsilon(x-y) u(y).$$

Consider first the sum in the last line. Note that

$$v(x) (i \sqrt{z} |x-y|)^{m-1} \chi_R^\varepsilon(x-y) u(y)$$

$$= i \sqrt{z} (x)^{3/2} v(x) (y)^{3/2} u(y) (x)^{-3/2} (y)^{-3/2} (i \sqrt{z} |x-y|)^{m-2} \chi_R^\varepsilon(x-y) u(y).$$
for $m \geq 3$. The Hilbert-Schmidt norm of the operator of this right-hand side can be evaluated as $O(\varepsilon^{1/2+\delta'})$ with some positive $\delta'$, using
\[
\langle x \rangle^{-1} |x+y\rangle\langle y\rangle^{-1} \leq \langle x \rangle^{-1}(|x|+|y|)\langle y\rangle^{-1} \leq 2
\]
and $\varepsilon^{1/2} R = \varepsilon^{1/2-\mu}$ with $\mu < 1/2$. Therefore the Hilbert-Schmidt norm of the operator corresponding to the sum in the right-hand side of the second equality of (19) is estimated as $O(\varepsilon^{1+\delta'})$. The first term in the right-hand side of the second equality of (19) is written
\[
- \frac{1}{4\pi} v(x) \frac{1}{|x-y|} \chi_R(x-y) u(y) = - \frac{1}{4\pi} v(x) \frac{1}{|x-y|} u(y)
\]
\[
+ \frac{1}{4\pi} v(x) \frac{1}{|x-y|} \chi_R(x-y) u(y).
\]
In this right-hand side, the Hilbert-Schmidt norm of the operator corresponding to the second term can be estimated as $O(\varepsilon^{1+\delta'})$ in the same way. Further the second and third terms in the right-hand side of the second equality of (19) are written
\[
- i \sqrt{\frac{z}{4\pi}} v(x) \chi_R(x-y) u(y) + \frac{z}{8\pi} v(x) |x-y| \chi_R(x-y) u(y)
\]
\[
= - i \sqrt{\frac{z}{4\pi}} v(x) u(y) + \frac{z}{8\pi} v(x) |x-y| u(y) + O(\varepsilon^{1+\delta'})
\]
in the sense of Hilbert-Schmidt norm. Combining these observations and the preceding lemma, we have
\[
(20) \quad v \frac{1}{z - H_0} u = - \frac{1}{4\pi} v R_0(0) u - i \sqrt{\frac{z}{4\pi}} v R_0^{(1)} u + \frac{z}{8\pi} v R_0^{(2)} u + O(\varepsilon^{1+\delta'}),
\]
where $R_0^{(m)}$ are the operators whose integral kernel is given by $|x|^{m-1}$ for $m = 0, 1, 2, \ldots$.

Let us consider the case when there are a single resonance state with the zero energy and $N_0$ bound states with the zero energy because the other cases can be handled in much easier way.

Let $\varphi_{0,j}$ be the eigenvector of the operator $u R_0(0) v$ with the eigenvalue 1 for $j = 0, 1, \ldots, N_0$, where $R_0(z) = (z - H_0)^{-1}$. Namely,
\[
v R_0(0) u \varphi_{0,j} = \varphi_{0,j}, \quad j = 0, 1, \ldots, N_0.
\]
We take $\varphi_{0,0}$ to be the vector corresponding to the resonance state such that
\[
(u, \varphi_{0,0}) = \int d^3 x \ u(x) \varphi_{0,0}(x) \neq 0.
\]
Therefore the rest of $\varphi_{0,j}$ for $j = 1, \ldots, N_0$ can be taken to satisfy
\[
(21) \quad (u, \varphi_{0,j}) = 0 \quad \text{for} \quad j = 1, 2, \ldots, N_0.
\]
The left eigenvectors are given by

$$\hat{\varphi}_{0,j} = (\text{sgn } V) \varphi_{0,j} \quad \text{for } j = 0, 1, 2, \ldots, N_0.$$ 

As in the proof of Lemma 3.1 of [2], we can choose the vectors, $\varphi_{0,j}$, for $j = 0, 1, \ldots, N_0$, so that they satisfy

$$(\hat{\varphi}_{0,i}, \varphi_{0,j}) = \delta_{ij}.$$ 

We define two projections as

$$P_0 := \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots)$$

and

$$P_1 := \sum_{j=1}^{N_0} \varphi_{0,j}(\hat{\varphi}_{0,j}, \cdots).$$

Further we write $P = P_0 + P_1$ and $Q = 1 - P$. Using (20) and the properties of the eigenvectors $\varphi_{0,j}$, we have

$$1 - v \frac{1}{z - H_0} u = \left(1 - v \frac{1}{z - H_0} u\right) Q + \left(1 - v \frac{1}{z - H_0} u\right) P$$

$$= \left(1 - v \frac{1}{z - H_0} u \right) Q + i \sqrt{\frac{z}{4\pi}} v R_0^{(1)} u P - \frac{z}{8\pi} v R_0^{(2)} u P$$

$$+ O(\varepsilon^{1+\delta}).$$

$$= \left(1 - v \frac{1}{z - H_0} u \right) Q + i \sqrt{\frac{z}{4\pi}} v R_0^{(1)} u P_0 - \frac{z}{8\pi} v R_0^{(2)} u P_0$$

$$- \frac{z}{8\pi} v R_0^{(2)} u P_1 + O(\varepsilon^{1+\delta}).$$

We want to express the inverse of this operator in a similar expansion. Let $\Psi_{j,k}$ be the $N_0 \times N_0$ matrix which are defined by

$$\sum_{\ell=1}^{N_0} \Psi_{j,\ell}(\psi_{0,\ell}, \psi_{0,k}) = \delta_{j,k}$$

with

$$\psi_{0,j} := R_0(0) u \varphi_{0,j} \quad \text{for } j = 1, \ldots, N_0.$$ 

The functions $\psi_{0,j}$ are all in $L^2(\mathbb{R}^3)$ because of the condition (21). (See [28, 2].) These functions are nothing but the eigenvectors at the zero energy for the Hamiltonian $H$. We note that, as in the proof of Lemma 3.1 of [2], it can be easily shown that the operator $v R_0(0) u Q$ has no eigenvalue +1.
Therefore the operator $1 - vR_0(0)uQ$ is invertible. Relying on this fact, we introduce the following operators:

$$B^{(0)} = B_1^{(0)} + B_2^{(0)} + B_3^{(0)} + B_4^{(0)}$$

with

$$B_1^{(0)} = \left(1 - v \frac{1}{z - H_0} uQ\right)^{-1},$$

$$B_2^{(0)} = -\frac{1}{(u, \varphi_{0,0})} \varphi_{0,0}(u, \cdots)(1 - vR_0(0)uQ)^{-1},$$

$$B_3^{(0)} = -\frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j}\Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots)vR_0^{(2)} u \frac{\varphi_{0,0}(u, \cdots)}{(u, \varphi_{0,0})}(1 - vR_0(0)uQ)^{-1},$$

and

$$B_4^{(0)} = \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j}\Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots)vR_0^{(2)} u (1 - vR_0(0)uQ)^{-1}.$$ And

$$B^{(1)} = B_1^{(1)} + B_2^{(1)}$$

with

$$B_1^{(1)} := -\frac{4\pi i}{\sqrt{z}} \frac{1}{|(u, \varphi_{0,0})|^2} \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots)$$

and

$$B_2^{(1)} := -\frac{i}{2\sqrt{z}} \sum_{j,\ell=1}^{N_0} \varphi_{0,j}\Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots)vR_0^{(2)} u \frac{1}{|(u, \varphi_{0,0})|^2} \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots).$$

Further,

$$B^{(2)} := \frac{1}{z} \sum_{j,\ell=1}^{N_0} \varphi_{0,j}\Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots).$$

Using these operators, an approximate inverse for (22) can be written

$$B = B^{(0)} + B^{(1)} + B^{(2)}.$$ 

**Lemma 12.** The following relation holds:

$$\left(\hat{\varphi}_{0,j}, vR_0^{(2)} u \varphi_{0,k}\right) = -8\pi \left(\psi_{0,j}, \psi_{0,k}\right) \text{ for } j, k = 1, 2, \ldots, N_0.$$

**Proof.** Since

$$R_0(z)^2 = -\frac{\partial}{\partial z} R_0(z),$$
the integral kernel of $R_0(z)^2$ is given by

$$\frac{i}{8\pi \sqrt{z}} e^{i \sqrt{|z|}}.$$

Using this and $(u, \varphi_{0,j}) = 0$ for $j = 1, 2, \ldots, N_0$, one has

$$\left( \hat{\varphi}_{0,j}, v R_0^{(2)} u \varphi_{0,k} \right) = -8\pi \lim_{z \to 0} \left( \hat{\varphi}_{0,j}, v \left[ R_0(z)^2 - \frac{i}{8\pi \sqrt{z}} R_0^{(1)} \right] u \varphi_{0,k} \right)$$

$$= -8\pi \lim_{z \to 0} (\hat{\varphi}_{0,j}, v R_0(z)^2 u \varphi_{0,k})$$

$$= -8\pi \lim_{z \to 0} (R_0(z)^* v \hat{\varphi}_{0,j}, R_0(z) u \varphi_{0,k})$$

$$= -8\pi (\psi_{0,j}, \psi_{0,k})$$

From (20) and (22), one has

$$B^{(0)} \left( 1 - v \frac{1}{z - H_0} u \right) = B^{(0)} \left( 1 - v \frac{1}{z - H_0} u Q \right) + O(\varepsilon^{1/2})$$

$$= Q \frac{\varphi_{0,0}(u, \cdots)}{(u, \varphi_{0,0})} Q$$

$$- \frac{1}{8\pi} \sum_{j, \ell=1}^{N_0} \varphi_{0,j} \Psi_{j, \ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u \frac{\varphi_{0,0}(u, \cdots)}{(u, \varphi_{0,0})} Q$$

$$+ \frac{1}{8\pi} \sum_{j, \ell=1}^{N_0} \varphi_{0,j} \Psi_{j, \ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u Q + O(\varepsilon^{1/2}).$$
Similarly,

\[
B^{(1)} \left( 1 - v \frac{1}{z - H_0} u \right) = B^{(1)} \left( 1 - v \frac{1}{z - H_0} uQ \right) + B^{(1)} \left( \frac{1}{4\pi} \sqrt{\frac{z}{u}} v R_0^{(1)} u P_0 + O(\varepsilon^{1/2}) \right)
\]

\[
= \frac{1}{|(u, \varphi_{0,0})|^2} \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots) v R_0^{(1)} u Q
+ \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots) v R_0^{(1)} u P_0
\]

\[
+ O(\varepsilon^{1/2})
\]

\[
= \frac{\varphi_{0,0}(u, \cdots)}{(u, \varphi_{0,0})} Q + \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u \frac{\varphi_{0,0}(u, \cdots)}{(u, \varphi_{0,0})} Q
\]

\[
+ P_0 + \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u P_0 + O(\varepsilon^{1/2}).
\]

Further, we obtain

\[
B^{(2)} \left( 1 - v \frac{1}{z - H_0} u \right) = B^{(2)} \left( 1 - v \frac{1}{z - H_0} uQ \right) - \frac{z}{8\pi} v R_0^{(2)} u P_0
\]

\[
+ B^{(2)} \left( -\frac{z}{8\pi} v R_0^{(2)} u \right) P_1 + O(\varepsilon^\delta')
\]

\[
= -\frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u Q - \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u P_0
\]

\[
- \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u P_1 + O(\varepsilon^\delta')
\]

\[
= -\frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u Q - \frac{1}{8\pi} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) v R_0^{(2)} u P_0
\]

\[
+ P_1 + O(\varepsilon^\delta'),
\]

where we have used (23) and (24) for getting the last equality. From these
observations, we have
\[ B \left( 1 - v \frac{1}{z - H_0} u \right) = 1 + D \]
with an operator \( D \) having the norm,
\[ \|D\|_2 = O(\varepsilon\delta'), \]
with some positive constant \( \delta' \). Therefore we obtain
\[ (25) \quad \left( 1 - v \frac{1}{z - H_0} u \right)^{-1} = (1 + D)^{-1}B = B - (1 + D)^{-1}DB. \]
Substituting this into the right-hand side of (18), we have
\[ (26) \quad \text{Tr} \left[ (z - H)^{-1} - (z - H_0)^{-1} \right] \]
\[ = \text{Tr} \left\{ vR_0(z)^2u[B - (1 + D)^{-1}DB] \right\} \]
\[ = \text{Tr} vR_0(z)^2uB - \text{Tr} \left\{ vR_0(z)^2u(1 + D)^{-1}DB \right\}. \]
Consider first the first term in the right-hand side of the second equality. It can be written
\[ \text{Tr} vR_0(z)^2uB = \text{Tr} vR_0(z)^2uB^{(0)} + \text{Tr} vR_0(z)^2uB^{(1)} + \text{Tr} vR_0(z)^2uB^{(2)}. \]
Further,
\[ \text{Tr} vR_0(z)^2uB^{(0)} = \sum_{i=1}^{4} \text{Tr} vR_0(z)^2uB_i^{(0)}. \]
In the same way as in the proof of (20), we have
\[ (27) \quad vR_0(z)^2u = \frac{i}{8\pi \sqrt{z}}vR_0^{(1)}u - \frac{1}{8\pi}vR_0^{(2)}u + O(\varepsilon\delta'). \]
Combining this with the expression of \( B_i^{(0)} \), one can notice that their contributions to the excess charge of (14) is vanishing except for that for \( B_1^{(0)} \). From the definition of \( B_1^{(0)} \), we have
\[ \text{Tr} vR_0(z)^2uB_1^{(0)} = \text{Tr} vR_0(z)^2u \left( 1 - v \frac{1}{z - H_0} uQ \right)^{-1}. \]
Note that
\[ \left( 1 - v \frac{1}{z - H_0} uQ \right)^{-1} \]
\[ = \left( 1 - v \frac{1}{z - H_0} uQ \right)^{-1} - 1 + 1 \]
\[ = \left[ 1 - \left( 1 - v \frac{1}{z - H_0} uQ \right) \right] \left( 1 - v \frac{1}{z - H_0} uQ \right)^{-1} + 1 \]
\[ = \frac{1}{z - H_0} uQ \left( 1 - v \frac{1}{z - H_0} uQ \right)^{-1} + 1. \]
Therefore,
\[ \text{Tr} \ vR_0(z)^2 u B_1^{(0)} = \text{Tr} \ vR_0(z)^2 u + \text{Tr} \ vR_0(z)^2 uv(z - H_0)^{-1} u Q \left[ 1 - v(z - H_0)^{-1} u Q \right]^{-1} . \]

The contribution from the second term in the right-hand side to the excess charge can be shown to be vanishing by using the expansions, (20) and (27). The contribution from the first term is also vanishing, because one has
\[ \text{Tr} \ vR_0(z)^2 u = \frac{1}{z - H_0} V \frac{1}{z - H_0} = -\frac{1}{8\pi i \sqrt{z}} \int d^3 y V(y) \]
as in (88) in Appendix C.

Next consider \( vR_0(z)^2 u B^{(1)} \). From the condition (21) and the expansion (27), the nonvanishing contribution to the excess charge comes from
\[ \frac{i}{8\pi \sqrt{z}} \text{Tr} \ v R_0^{(1)} u B^{(1)}_1 = \frac{1}{2z} \frac{1}{|(u, \varphi_{0,0})|^2} \text{Tr} \ v R_0^{(1)} u \varphi_{0,0}(\hat{\varphi}_{0,0}, \ldots) = \frac{1}{2z}. \]
The corresponding contribution to the excess charge is 1/2.

In the same way, the nonvanishing contribution for \( vR_0(z)^2 u B^{(2)} \) comes from
\[ -\frac{1}{8\pi} \text{Tr} \ v R_0^{(2)} u B^{(2)}_2 = -\frac{1}{8\pi z} \sum_{j,\ell=1}^{N_0} \text{Tr} \ v R_0^{(2)} u \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,1}, \ldots) = \frac{1}{z} N_0, \]
where we have used (23) and (24). This gives \( N_0 \) for the excess charge.

Consider the second term in the right-hand side of the second equality of (26). Using the condition (21) and \( \|D\|_2 = O(\epsilon^\delta) \), we can show, in the same way, that the corresponding contribution to the excess charge is vanishing.

Combining these observations with (14) and (26), we obtain the excess charge,
\[ Z^0 = \frac{1}{2} + N_0, \]
at the zero energy. Consequently, the total phase shift in the zero energy limit is given by
\[ \theta(0) = \frac{1}{2} + N_0 + N_\prec. \]

### 4.2 Proof of Theorem 9 in the Case of \( n \geq 5 \)

In order to prove Theorem 9 in the case of \( n \geq 5 \), we choose \( \beta \) to satisfy \( n/2 < \beta < (\alpha - n)/2 \). Then \( W \) of (5) is a trace class operator. Using the polar decomposition for \( W \), we write
\[ A - A_0 = \langle x \rangle^{-\beta} W(x)^{-\beta} = uv \]
with \[ u = \langle x \rangle^{-\beta} |W|^{1/2} U \quad \text{and} \quad v = |W|^{1/2} \langle x \rangle^{-\beta}. \]

Here the unitary operator \( U \) satisfies \( U^* = U \) because \( A - A_0 \) is self-adjoint. The SSF (4) is expressed as

\[
\xi(\lambda; A, A_0) = \operatorname{Im} \frac{1}{\pi} \int_{\gamma_\lambda} dw \operatorname{Tr} \left[ (A - w)^{-1} - (A_0 - w)^{-1} \right] 
\]

\[
= -\operatorname{Im} \frac{1}{\pi} \int_{\gamma_\lambda} dw \operatorname{Tr} \left[ (A - w)^{-1} (A - A_0) (A_0 - w)^{-1} \right] 
\]

\[
= -\operatorname{Im} \frac{1}{\pi} \int_{\gamma_\lambda} dw \operatorname{Tr} \left[ (A - w)^{-1} uv (A_0 - w)^{-1} \right] 
\]

\[
= -\operatorname{Im} \frac{1}{\pi} \int_{\gamma_\lambda} dw \operatorname{Tr} \left[ v (A_0 - w)^{-1} (A - w)^{-1} u \right]. 
\]

One can easily check

\[
1 - v (A - w)^{-1} u = [1 - v (w - A_0)^{-1} u]^{-1}
\]

for \( w \notin \sigma(A_0) \cup \sigma(A) \). This relation yields

\[
(A_0 - w) (A - w)^{-1} u = [1 - (A - A_0) (A - w)^{-1}] u 
\]

\[
= [1 - uv (A - w)^{-1}] u 
\]

\[
= u [1 - v (A - w)^{-1} u] 
\]

\[
= u [1 - v (w - A_0)^{-1} u]^{-1}
\]

for \( w \notin \sigma(A_0) \cup \sigma(A) \). By using this and the invariance principle, the SSF can be written

\[
\xi(\lambda; H, H_0) = \operatorname{Im} \frac{1}{\pi} \int_{\gamma_\mu(\lambda)} dw \operatorname{Tr} \left\{ v (A_0 - w)^{-2} u [1 - v (w - A_0)^{-1} u]^{-1} \right\}. 
\]

First we want to rewrite \((w - A_0)^{-1}\). Note that

\[
w - A_0 = \left( \frac{1}{z + M} \right)^{\ell} - \left( \frac{1}{H_0 + M} \right)^{\ell} 
\]

\[
= \left( \frac{1}{z + M} - \frac{1}{H_0 + M} \right) \prod_{k=1}^{\ell-1} \left[ \frac{1}{\zeta_k(z + M)} - \frac{1}{H_0 + M} \right],
\]

where \( \zeta_\ell := \exp[2\pi i/\ell] \). Since one has

\[
\frac{1}{\zeta_k(z + M)} - \frac{1}{H_0 + M} = \frac{(H_0 + M) - \zeta_k(z + M)}{\zeta_k(z + M)(H_0 + M)},
\]

25
the inverse is given by
\[
\left[ \frac{1}{\zeta^k(z + M)} - \frac{1}{H_0 + M} \right]^{-1} = \zeta^k(z + M) \cdot \frac{H_0 + M}{(H_0 + M) - \zeta^k(z + M)}
\]
\[
= \zeta^k(z + M) \left[ 1 + \frac{\zeta^k(z + M)}{(H_0 + M) - \zeta^k(z + M)} \right].
\]

From these observations, we have the expression,
\[
(w - A_0)^{-1} = (z + M) \left[ 1 + \frac{z + M}{H_0 - z} \right]
\]
\[
\times \prod_{k=1}^{\ell-1} \zeta^k(z + M) \left[ 1 + \frac{\zeta^k(z + M)}{(H_0 + M) - \zeta^k(z + M)} \right].
\]

This yields
\[
\langle x \rangle^{-\beta} (w - A_0)^{-1} \langle x \rangle^{-\beta}
\]
\[
= (z + M) \left[ \langle x \rangle^{-2\beta} - (z + M) \langle x \rangle^{-\beta} (z - H_0)^{-1} \langle x \rangle^{-\beta} \right]
\]
\[
\times \prod_{k=1}^{\ell-1} \zeta^k(z + M) \left[ 1 + \langle x \rangle^\beta \frac{\zeta^k(z + M)}{(H_0 + M) - \zeta^k(z + M)} \langle x \rangle^{-\beta} \right].
\]

From the definitions of \( u \) and \( v \), we have
\[
v(w - A_0)^{-1} u = |W|^{1/2} \langle x \rangle^{-\beta} (w - A_0)^{-1} \langle x \rangle^{-\beta} |W|^{1/2} U.
\]

We write \( G_0(x; z) \) for the integral kernel of the resolvent \( R_0(z) = (z - H_0)^{-1} \). We decompose \( R_0(z) \) into two parts as \( R_0(z) = R_0^< (z) + R_0^> (z) \). Here the operator \( R_0^< (z) \) has the integral kernel \( G_0(x; z) \chi_R^<(x) \), and \( R_0^> (z) \) has the integral kernel \( G_0(x; z) \chi_R^>(x) \). In order to compute the excess charge \( Z^0 \) at zero energy, we take \( z = \varepsilon e^{i\phi} \) with a small \( \varepsilon > 0 \) and \( \delta < \phi < 2\pi - \delta \) with a small \( \delta > 0 \), and \( R = e^{-\mu} \) with \( 1/3 < \mu < 1/2 \).

Using the integral representation of \( G_0(x; z) \), one has
\[
\left| G_0(x; z) \chi_R^>(x) \right| \leq \text{Const.} \varepsilon^{1+\delta_1}, \quad x \in \mathbb{R}^n,
\]
with some positive \( \delta_1 \). Combining this with \( \beta > n/2 \), we obtain
\[
\left\| \langle x \rangle^{-\beta} R_0^> \langle x \rangle^{-\beta} \right\|_2 = O(\varepsilon^{1+\delta_1}).
\]

Using the expansion with respect to \( \sqrt{z|x-y|} \), one has
\[
\langle x \rangle^{-\beta} G_0(x - y; z) \chi_R^<(x - y) \langle y \rangle^{-\beta} = C_1 \langle x \rangle^{-\beta} \frac{1}{|x - y|^{n-2}} \chi_R^<(x - y) \langle y \rangle^{-\beta}
\]
\[
+ C_2 \langle x \rangle^{-\beta} \frac{z}{|x - y|^{n-4}} \chi_R^<(x - y) \langle y \rangle^{-\beta}
\]
\[
+ K_1(x, y),
\]
where $C_1$ and $C_2$ are some constants, and $K_1(x, y)$ is the integral kernel of a bounded operator. One can show that the norm of the operator for $K_1(x, y)$ is $O(\varepsilon^{1+\delta_2})$ with some positive $\delta_2$. From these observations, we have

$$\langle x \rangle^{-\beta}R_0(z)\langle x \rangle^{-\beta} = \langle x \rangle^{-\beta}R_0(0)\langle x \rangle^{-\beta} + \langle x \rangle^{-\beta}\frac{dR_0}{dz}(0)\langle x \rangle^{-\beta} \cdot z + O(\varepsilon^{1+\delta'})$$

with some positive $\delta'$. Combining this, (30) and (31), we obtain

\begin{equation}
(32) \quad v(w - A_0)^{-1}u = v[w(0) - A_0]^{-1}u - v[w(0) - A_0]^{-2}u\frac{dw}{dz}(0) \cdot z + O(\varepsilon^{1+\delta'})
\end{equation}

where we have written $w(z) = (z + M)^{-\ell}$.

Let $\varphi_{0,j}$ be the eigenvector of $v[w(0) - A_0]^{-1}u$ with the eigenvalue 1 as

$$v[w(0) - A_0]^{-1}u\varphi_{0,j} = \varphi_{0,j}, \quad j = 1, 2, \ldots, N_0.$$ 

The left eigenvectors are given by

$$\hat{\varphi}_{0,j} = U\varphi_{0,j}, \quad j = 1, 2, \ldots, N_0.$$ 

We choose these eigenvectors so that they satisfy

$$\left(\hat{\varphi}_{0,j}, \varphi_{0,k}\right) = \delta_{j,k}.$$ 

We define

$$P_1 := \sum_{j=1}^{N_0} \varphi_{0,j}(\hat{\varphi}_{0,j}, \cdots),$$

and write $Q = 1 - P_1$. Combining (32) with these properties of the eigenvectors, we have

\begin{equation}
(33) \quad 1 - v(w - A_0)^{-1}u = [1 - v(w - A_0)^{-1}u]Q + [1 - v(w - A_0)^{-1}u]P_1
\end{equation}

$$= \{1 - v[w(0) - A_0]^{-1}u\}Q - \frac{\ell z}{M^{\ell+1}}v[w(0) - A_0]^{-2}uQ$$

$$- \frac{\ell z}{M^{\ell+1}}v[w(0) - A_0]^{-2}uP_1 + O(\varepsilon^{1+\delta'}).$$

In order to express the inverse of this operator in a similar expansion, we introduce

$$\psi_{0,j} = [w(0) - A_0]^{-1}u\varphi_{0,j}, \quad j = 1, 2, \ldots, N_0.$$
One can check that $\psi_{0,j} \in L^2(\mathbb{R}^n)$, and $\psi_{0,j}$ are the eigenvectors of $A$ with the eigenvalue $w(0)$. This implies $H\psi_{0,j} = 0$. Let $\Psi_{j,\ell}$ be the $N_0 \times N_0$ matrix to satisfy
\[
\sum_{\ell=1}^{N_0} \Psi_{j,\ell}(\psi_{0,\ell},\psi_{0,k}) = \delta_{j,k}.
\]

We introduce
\[
B = \left\{ 1 - v[w(0) - A_0]^{-1}uQ \right\}^{-1}
\]
\[
- \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell},\cdots)v[w(0) - A_0]^{-2}u\{1 - v[w(0) - A_0]^{-1}uQ\}^{-1}
\]
\[
- \frac{M^{\ell+1}}{\ell z} \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell},\cdots).
\]

Then, from (33), we have
\[
B[1 - v(w - A_0)^{-1}u]
\]
\[
= Q + \sum_{j,\ell=1}^{N_0} \varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell},\cdots)v[w(0) - A_0]^{-2}uP_1 + O(\varepsilon^\delta).
\]

From the definitions of the projection $P_1$ and of the vectors $\psi_{0,j}$, one has
\[
(\hat{\varphi}_{0,\ell},\cdots)v[w(0) - A_0]^{-2}uP_1 = \sum_{k=1}^{N_0} (\hat{\varphi}_{0,\ell}, v[w(0) - A_0]^{-2}u\varphi_{0,k})(\hat{\varphi}_{0,k},\cdots)
\]
\[
= \sum_{k=1}^{N_0} (\psi_{0,\ell}, \psi_{0,k})(\hat{\varphi}_{0,k},\cdots).
\]

Therefore, we have
\[
B[1 - v(w - A_0)^{-1}u] = Q + P_1 + D = 1 + D,
\]
where $D$ is a bounded operator with the norm of $O(\varepsilon^\delta)$. Consequently, the desired inverse is given by
\[
[1 - v(w - A_0)^{-1}u]^{-1} = (1 + D)^{-1}B = B - (1 + D)^{-1}DB.
\]
From this and the expression (34) of the operator $B$, one notices that the nonvanishing contribution to the excess charge $Z^0$ at the zero energy comes
from the second sum in the right-hand side of (34). Actually, we have

\[
Z^0 = \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int dz \frac{dw}{dz} \text{Tr} \{ v(w - A_0)^{-2}u[1 - v(w - A_0)^{-1}u]^{-1} \}
\]

\[
= \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int dz \sum_{N_0} \text{Tr} \{ v[w(0) - A_0]^{-2}u\varphi_{0,j} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, \cdots) \}
\]

\[
= \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int dz \sum_{N_0} \Psi_{j,\ell}(\hat{\varphi}_{0,\ell}, u\varphi_{0,j})
\]

\[
= N_0
\]

from the definitions of the matrix \( \Psi_{j,\ell} \) and of the vectors \( \psi_{0,j} \). Adding this and the number of the eigenvalues below the zero energy, the desired result of Theorem 9 is obtained.

4.3 Proof of Theorem 9 in the Case of \( n = 4 \)

In the present case, we choose

\[ A = (H + M)^{-2}, \quad A_0 = (H_0 + M)^{-2} \]

with a large \( M > 0 \). We also choose \( \beta \) to satisfy \( 2 < \beta < (\alpha - 4)/2 \). Then \( W \) of (5) is a trace class operator. From the condition \( \beta > 2 \), we can write \( \beta = 2 + \delta \) with a small \( \delta > 0 \). We define \( u \) and \( v \) in the same way as in Section 4.2.

Since \( \ell = 2 \), we have

\[
(w - A_0)^{-1} = (z + M)^2 \left[ 1 - \frac{z + M}{z - H_0} \right] \left[ \frac{z + M}{z + 2M + H_0} - 1 \right]
\]

\[
= (z + M)^2 \left[ \frac{z + M}{z + 2M + H_0} - 1 \right] - \frac{(z + M)^3}{z - H_0} \left[ \frac{z + M}{z + 2M + H_0} - 1 \right]
\]

from (29). This yields

\[
\langle x \rangle^{-\beta}(w - A_0)^{-1}\langle x \rangle^{-\beta}
\]

\[
= (z + M)^2\langle x \rangle^{-\beta} \left[ \frac{z + M}{z + 2M + H_0} - 1 \right] \langle x \rangle^{-\beta}
\]

\[
- (z + M)^3 \langle x \rangle^{-\beta} \frac{1}{z - H_0} \langle x \rangle^{-\beta} \left[ \langle x \rangle^{-\beta} \frac{z + M}{z + 2M + H_0} \langle x \rangle^{-\beta} - 1 \right].
\]
In order to obtain the expansion of \((w - A_0)^{-1}\) with respect to a small \(|z|\), we want to obtain the expansion of the resolvent \(R_0(z) = (z - H_0)^{-1}\). We write \(G_0(x; z)\) for the integral kernel of \(R_0(z)\). We decompose \(R_0(z)\) into two parts as \(R_0(z) = R^>_0(z) + R^<_0(z)\). Here, the operator \(R^>_0(z)\) has the integral kernel \(G_0(x; z)\chi^>_R(x)\), and \(R^<_0(z)\) has \(G_0(x; z)\chi^<_R(x)\). We take \(z = \varepsilon e^{i\phi}\) with a small \(\varepsilon > 0\) and \(\delta < \phi < 2\pi - \delta\) with a small \(\delta > 0\). We also take \(R = \varepsilon^{-\mu}\) with \(\frac{3}{6 + \delta} < \mu < \frac{1}{2}\).

**Lemma 13.** We have

\[
\left\| \langle x \rangle^{-\beta} R^>_0(z) \langle x \rangle^{-\beta} \right\|_2 \leq \text{Const.} \varepsilon^{1+\delta'}
\]

with a small \(\delta' > 0\).

**Proof.** From the integral representation of the integral kernel \(G_0(x; z)\), one has

\[
\left| G_0(x; z)\chi^>_R(x) \right| \leq \text{Const.} \varepsilon^{3\mu-1/2}.
\]

Combining this with \(\beta = 2 + \delta\), we have

\[
\langle x \rangle^{-\beta} \left| G_0(x - y; z)\chi^>_R(x - y) \right| \langle y \rangle^{-\beta} \leq \text{Const.} \varepsilon^{3\mu-1/2 - \delta/2} \langle y \rangle^{-2\delta/2} R^{-\delta/2}.
\]

This yields

\[
\left\| \langle x \rangle^{-\beta} R^>_0(z) \langle x \rangle^{-\beta} \right\|_2 \leq \text{Const.} \varepsilon^{3\mu-1/2 + \delta\mu/2}.
\]

On the other hand, we have

\[
3\mu - 1/2 + \delta\mu/2 > 1
\]

from the condition \(\mu > 3/(6 + \delta)\). Combining this with the above bound yields the desired result. 

From the expansion of \(G_0(x; z)\) with respect to \(\sqrt{z}|x|\), we have

\[
G_0(x; z)\chi^>_R(x) = \left( \frac{1}{16\pi^2} z \log z + \kappa z + \frac{z}{8\pi^2} \log |x| - \frac{1}{4\pi^2} \frac{1}{|x|^2} \right) \chi^>_R(x)
+ O(\varepsilon^{1+\delta'})
\]

with the constant,

\[
\kappa = \frac{1}{8\pi^2} \left[ \frac{\pi}{2i} + \gamma - \log 2 - \frac{1}{2} \right],
\]
and with a small $\delta'$. Combining this with the above Lemma, we obtain

$$
\langle x \rangle^{-\beta} R_0(z) \langle x \rangle^{-\beta} = \frac{1}{16\pi^2} z \log z \langle x \rangle^{-\beta} R_0^{(1)}(z) \langle x \rangle^{-\beta} + \kappa z \langle x \rangle^{-\beta} + O(\varepsilon^{1+\delta'}),
$$

where the operators, $R_0^{(1)}$ and $\tilde{R}_0^{(1)}$, have the integral kernels, $1$ and $\log |x|$, respectively. Therefore, from this expansion, (31) and (36), we have

$$
v(w - A_0)^{-1}u = v[w(0) - A_0]^{-1}u + z \log z \tilde{A}_0 + z A_0 + O(\varepsilon^{1+\delta'}),
$$

where

$$
\tilde{A}_0 = \frac{M^3}{16\pi^2} v R_0^{(1)} \left[ 1 - \frac{M}{2M + H_0} \right] u
$$

and $A_0$ is a bounded operator. This yields

$$
1 - v(w - A_0)^{-1}u = 1 - v[w(0) - A_0]^{-1}u + K_1 + D_1,
$$

where

$$
K_1 = -z \log z \tilde{A}_0 - z A_0
$$

and the operator $D_1$ satisfies $\|D_1\|_2 = O(\varepsilon^{1+\delta'})$.

In order to obtain the inverse of $1 - v(w - A_0)^{-1}u$, let us consider the eigenvalue problem for the operator $v[w(0) - A_0]^{-1}u$. Let $\varphi$ be an eigenvector with the eigenvalue 1, i.e.,

$$
v[w(0) - A_0]^{-1}u \varphi = \varphi.
$$

The corresponding left-eigenvector is given by $\hat{\varphi} = U \varphi$. We set $\psi = [w(0) - A_0]^{-1}u \varphi$. Then,

$$
A \psi = \{[A_0 - w(0) + w(0)] + uv\} [w(0) - A_0]^{-1}u \varphi
= -w \varphi + w(0)[w(0) - A_0]^{-1}u \varphi + w \varphi
= w(0) \psi,
$$

where we have used $A = A_0 + (A - A_0) = A_0 + uv$. This implies

$$
(H + M)^{-2} \psi = M^{-2} \psi.
$$

From the expression of $(w - A_0)^{-1}$, one notices that, if $(1, u \varphi) \neq 0$, then $\psi \notin L^2(\mathbb{R}^4)$. This state $\psi$ is nothing but the resonance state at the zero energy.

On the other hand, when $(1, u \varphi) = 0$, $\psi \in L^2(\mathbb{R}^4)$. This $\psi$ is a bound state at the zero energy. In order to see this fact, we note the following: The integral kernel of $\Delta^{-1}$ is given by

$$
-\frac{1}{4\pi^2} \frac{1}{|x|^2}.
$$
For this integral kernel, we have
\[
\frac{1}{|x - y|^2} - \frac{1}{1 + |x|^2} \leq \frac{2|x||y| + 1 + |y|^2}{|x - y|^2(1 + |x|^2)}
\]
\[
= \frac{2|x||y|}{(1 + |x|^2)|x - y|^2} + \langle x \rangle^{-2} \frac{1}{|x - y|^2} \langle y \rangle^2
\]
\[
\leq 3 \frac{1}{1 + |x|} \frac{1}{|x - y|^2(1 + |y|)} + \langle x \rangle^{-2} \frac{1}{|x - y|^2} \langle y \rangle^2,
\]
where we have used the inequality,
\[
\frac{2|x|}{1 + |x|^2} \leq \frac{3}{1 + |x|},
\]
for getting the last inequality. From Lemma 2.3 of [20], both of two operators, \(\langle x \rangle^{-2} \Delta^{-1} \langle x \rangle^{-(\beta - 2)}\) and \(\langle x \rangle^{-1} \Delta^{-1} \langle x \rangle^{-(\beta - 1)}\), are a bounded operator. Combining these observations with the condition \((1, u\varphi_0) = 0\), one can show that \(\psi = [w(0) - A_0]^{-1} u\varphi_0 \in L^2(\mathbb{R}^4)\).

Thus, there appears at most a single resonance state at the zero energy in four dimensions [21]. In the following, we treat the case that there appear a single resonance state and \(N\) bound states at the zero energy because the rest of the cases can be handled in the same way.

Let \(\varphi_{0,j}, j = 0, 1, \ldots, N,\) satisfy the following conditions:
\[
v[w(0) - A_0]^{-1} u\varphi_{0,j} = \varphi_{0,j},
\]
\[(1, u\varphi_{0,0}) \neq 0,
\]
and
\[(39) \quad (1, u\varphi_{0,j}) = 0 \quad \text{for } j = 1, \ldots, N.
\]

The corresponding left-eigenvectors are given by
\[
\hat{\varphi}_{0,0} = N_{0,0} U \varphi_{0,0},
\]
and
\[
\hat{\varphi}_{0,j} = N_{0,j} U \varphi_{0,j}, \quad j = 1, \ldots, N,
\]
where \(N_{0,j}\) are normalization constants which we choose to satisfy
\[
(\hat{\varphi}_{0,i}, \varphi_{0,j}) = \delta_{ij}, \quad i, j = 0, 1, \ldots, N.
\]
We define
\[
P_0 = \varphi_{0,0}(\hat{\varphi}_{0,0}, \cdots)
\]
and
\[
P_1 = \sum_{j=1}^{N} \varphi_{0,j}(\hat{\varphi}_{0,j}, \cdots),
\]
and define \(P = P_0 + P_1\) and \(Q = 1 - P\).
Lemma 14. The following relation is valid:

\[(\hat{\varphi}_{0,k}, A_0 \varphi_{0,\ell}) = \frac{2N_{0,k}^*}{M^3} (\psi_{0,k}, \psi_{0,\ell}) \text{ for } k, \ell = 1, \ldots, N.\]

Proof. From the expansion (37), one has

\[
\frac{1}{z} (\hat{\varphi}_{0,k}, \{ v(w - A_0)^{-1}u - v[w(0) - A_0]^{-1}u \} \varphi_{0,\ell}) = \log z (\hat{\varphi}_{0,k}, \hat{A}_0 \varphi_{0,\ell}) + (\hat{\varphi}_{0,k}, A_0 \varphi_{0,\ell}) + O(\varepsilon')
\]

for \(k, \ell = 1, \ldots, N\), where we have used the conditions (39). The left-hand side is computed as

\[
\frac{1}{z} (\hat{\varphi}_{0,k}, \{ v(w - A_0)^{-1}u - v[w(0) - A_0]^{-1}u \} \varphi_{0,\ell}) = \frac{w(0) - w}{z} (\hat{\varphi}_{0,k}, A_0 \varphi_{0,\ell}) + O(\varepsilon')
\]

In the limit \(z \to 0\), this right-hand side goes to

\[
\frac{2N_{0,k}^*}{M^3} (\psi_{0,k}, \psi_{0,\ell})
\]

Relying on this lemma, one can find \(N \times N\) matrix \(\Phi\) such that

\[
\sum_{k=1}^{N} \Phi_{jk} (\hat{\varphi}_{0,k}, A_0 \varphi_{0,\ell}) = \delta_{j\ell}
\]

for \(j, \ell = 1, \ldots, N\).

Now, as an approximate inverse of the operator (38), we introduce

\[
B = (1 - \Pi K_1) \left\{ 1 - v[w(0) - A_0]^{-1}uQ \right\}^{-1} + \Pi
\]

with

\[
\Pi = -\frac{1}{z \log z M^3 N_{0,0}^*} \left[ \left( \varphi_{0,k} \Phi_{jk}(\hat{\varphi}_{0,k}, \cdot \cdot \cdot) \right)^2 P_0 \right] - \frac{1}{z} \sum_{j,k=1}^{N} \varphi_{0,j} \Phi_{jk}(\hat{\varphi}_{0,k}, \cdot \cdot \cdot)
\]

Using the conditions (39) and Lemma 14, we have

\[
\Pi K_1 P = P + \Xi + P_0 \times O(\varepsilon^{-1})
\]
with
\[ \Xi = \sum_{j,k=1}^{N} \varphi_{0,j} \Phi_{jk}(\hat{\varphi}_{0,k},A_0 \varphi_{0,0})(\hat{\varphi}_{0,0}, \cdots). \]

Combining this with the expression of the operator (38), we obtain
\[
B[1 - v(w - A_0)^{-1} u] \\
= (1 - \Pi K_1) Q + (1 - \Pi K_1) \left\{ 1 - v[w(0) - A_0]^{-1} u Q \right\}^{-1} K_1 \\
+ \Pi K_1 + O(\varepsilon^{d'}) \\
= Q + \Pi K_1 P + O(\varepsilon^{d'}) \\
= 1 + \Xi + P_0 \times O(|\log \varepsilon|^{-1}) + O(\varepsilon^{d'}). 
\]

In consequence,
\[
[1 - v(w - A_0)^{-1} u]^{-1} = \frac{1}{1 + D_2} (1 - \Xi) B, 
\]
where
\[ D_2 = P \times O(|\log \varepsilon|^{-1}) + O(\varepsilon^{d'}). \]

Let us calculate the excess charge \( Z^0 \) at the zero energy. In the same way as the expansion of (37), we have
\[
\frac{d}{dz} v(w - A_0)^{-1} u = -\frac{dw}{dz} v(w - A_0)^{-2} u \\
= (1 + \log z) \tilde{A}_0 + A_0 + O(\varepsilon^{d'}). 
\]

From the expression (35) for the excess charge \( Z^0 \), it is enough to calculate
\[- \int dz \text{Tr} \left[ (1 + \log z) \tilde{A}_0 + A_0 \right] \frac{1}{1 + D_2} (1 - \Xi) B. \]

Further, from the expression of the operator \( B \), it is enough to calculate the following two contributions:
\[
\frac{1}{z} \frac{32\pi^2}{M^3 N_0^* ||(1, u \varphi_{0,0})||^2} \text{Tr} \tilde{A}_0 \frac{1}{1 + D_2} (1 - \Xi) P_0 
\]
and
\[
\text{Tr} \left[ (1 + \log z) \tilde{A}_0 + A_0 \right] \frac{1}{1 + D_2} (1 - \Xi) \left[ \frac{1}{z} \sum_{j,k=1}^{N} \varphi_{0,j} \Phi_{jk}(\hat{\varphi}_{0,k}, \cdots) \right], 
\]
except for the integral about \( z \). The former is calculated as
\[
\frac{1}{z} + o(\varepsilon^{-1}) \]

in the same way as in the above. Therefore, this leads to the excess charge 1 from the resonance state. Using the conditions (39) and the definition (40) of the matrix Φ, the contribution from the latter becomes

$$\frac{N}{z} + o(\varepsilon^{-1}).$$

This implies that the $N$ bound states at the zero energy lead to the excess charge $N$.

**A Proof of Theorem 7**

Since $\alpha > 5$, one can find a small $\delta > 0$ such that $\alpha > 5 + \delta$. As in Section 4.1, we take $z = \varepsilon e^{i\phi}$ with $\delta \leq \phi \leq 2\pi - \delta$, and $R = \varepsilon^{-\mu}$ with $2/(4 + \delta) < \mu < 1/2$.

The integral kernel of the resolvent $(z - H_0)^{-1}$ is given by

$$G_0(x; z) = \frac{1}{2i\sqrt{z}}e^{i\sqrt{z}|x|}.$$

We decompose the resolvent $R_0(z) = (z - H_0)^{-1}$ as

$$R_0(z) = R_0^>(z) + R_0^<(z),$$

where the operators, $R_0^>(z)$ and $R_0^<(z)$, have the integral kernels, $G_0(x; z)\chi_R^>(x)$ and $G_0(x; z)\chi_R^<(x)$, respectively. We write $u = |V|^{1/2}\text{sgn}V$ and $v = |V|^{1/2}$.

**Lemma 15.** We have $vR_0^>(z)u \in J_2$, and the Hilbert-Schmidt norm is bounded as

$$\|vR_0^>(z)u\|_2 \leq \text{Const.} \times \varepsilon^{1/2 + \delta'}$$

for a small $\varepsilon$ with a positive constant $\delta'$.

**Proof.** We have

$$\left| |V(x)|^{1/2} \frac{1}{2i\sqrt{z}}e^{i\sqrt{z}|x-y|}\chi_R^>(x-y)|V(y)|^{1/2}\text{sgn}V(y) \right|^2 \leq \frac{1}{4\varepsilon}|V(x)||V(y)|\chi_R^>(x-y)$$

$$\leq \frac{1}{4\varepsilon}|V(x)||\langle x \rangle^{4+\delta}|\langle x \rangle^{-4-\delta}|V(y)||\langle y \rangle^{4+\delta}|\langle y \rangle^{-4-\delta}\chi_R^>(x-y)$$

$$\leq \frac{1}{4\varepsilon} \left( \frac{2}{R} \right)^{4+\delta}|V(x)||\langle x \rangle^{4+\delta}|V(y)||\langle y \rangle^{4+\delta}.$$
The assumption, \( \mu > 2/(4 + \hat{\delta}) \), yields
\[
-1/2 + (2 + \hat{\delta}/2)\mu > \frac{1}{2}.
\]
Therefore the desired bound holds. \( \square \)

As to \( R_0^<(z) \), we expand the integral kernel as
\[
\frac{1}{2i\sqrt{z}} e^{i\sqrt{z}|x-y|} \chi_R^<(x-y)
= \frac{1}{2i\sqrt{z}} \sum_{m=0}^{\infty} \frac{1}{m!} (i\sqrt{z}|x-y|)^m \chi_R^<(x-y)
= \frac{1}{2i\sqrt{z}} \left[ 1 + i\sqrt{z}|x-y| - \frac{1}{2}z|x-y|^2 + \sum_{m=3}^{\infty} \frac{1}{m!} (i\sqrt{z}|x-y|)^m \right] \chi_R^<(x-y).
\]
Note that
\[
\frac{1}{\sqrt{z}} \sum_{m=3}^{\infty} \frac{1}{m!} (i\sqrt{z}|x-y|)^m = \frac{1}{\sqrt{z}} \sum_{m=3}^{\infty} (-z)|x-y|^2 \frac{1}{m!} (i\sqrt{z}|x-y|)^{m-2}
= -\sqrt{z}|x-y|^2 \sum_{m=3}^{\infty} \frac{1}{m!} (i\sqrt{z}|x-y|)^{m-2},
\]
and
\[
|V(x)|^{1/2}|x-y|^2|V(y)|^{1/2} = |V(x)|^{1/2}\langle x \rangle^2 \langle x \rangle^{-2}|x-y|^2 \langle y \rangle^2 \langle y \rangle^{-2}|V(y)|^{1/2}
\leq 4|V(x)|^{1/2}\langle x \rangle^2 \langle y \rangle^2|V(y)|^{1/2}.
\]
From these observations, the integral kernel of \( vR_0^<(z)u \) is written
\[
\left[ \frac{1}{2i\sqrt{z}} v(x)u(y) + \frac{1}{2} v(x)|x-y|u(y) - \sqrt{z} \frac{v(x)|x-y|^2 u(y)}{4i} \right] \chi_R^<(x-y)
+ O(\varepsilon^{1/2+\hat{\delta}'})
\]
in the sense of Hilbert-Schmidt norm. Using the same argument as in the above, we can replace the cutoff \( \chi_R^<(x-y) \) with the identity. Combining this with Lemma 15, we obtain
\[
(41) \quad vR_0(z)u = \frac{1}{2i\sqrt{z}} vR_0^{(1)}u + \frac{1}{2} vR_0^{(2)}u - \sqrt{z} \frac{vR_0^{(3)}u}{4i} + O(\varepsilon^{1/2+\hat{\delta}'})
\]
where \( R_0^{(m)} \) are the operators whose integral kernel is given by \( |x|^{m-1} \).

Consider first the case with
\[
\int dx V(x) \neq 0.
\]
From (14) and (18), the excess charge at the zero energy is given by

\begin{equation}
Z^0 = \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\delta}} dz \text{Tr} \left\{ v(z - H_0)^{-2}u[1 - v(z - H_0)^{-1}u]^{-1} \right\}.
\end{equation}

We want to obtain the expression of the inverse of $1 - v(z - H_0)^{-1}u$ in the right-hand side.

Note that

$$1 - v(z - H_0)^{-1}u = 1 + \frac{i(u,v)}{2\sqrt{z}} \hat{P} - M,$$

where we have written

$$\hat{P} = (u,v)^{-1}v(u,\cdots),$$

and

\begin{equation}
M = \frac{1}{2} vR_0^{(2)} - \frac{\sqrt{z}}{4i} vR_0^{(3)} u + O(\varepsilon^{1/2} + \delta').
\end{equation}

The inverse of the right-hand side is formally written

$$\left[ 1 + \frac{i(u,v)}{2\sqrt{z}} \hat{P} - M \right]^{-1} = \left\{ 1 - \left[ 1 - \frac{i(u,v)}{2\sqrt{z} + i(u,v)} \hat{P} \right] M \right\}^{-1} \left[ 1 - \frac{i(u,v)}{2\sqrt{z} + i(u,v)} \hat{P} \right],$$

Note that

$$1 - \frac{i(u,v)}{2\sqrt{z} + i(u,v)} = 1 - \hat{P} + \left[ \frac{2\sqrt{z} + i(u,v)}{2\sqrt{z} + i(u,v)} - \frac{i(u,v)}{2\sqrt{z} + i(u,v)} \right] \hat{P}$$

$$= \hat{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u,v)} \hat{P},$$

where $\hat{Q} = 1 - \hat{P}$. Therefore, we have

\begin{equation}
\left[ 1 - v(z - H_0)^{-1}u \right]^{-1}
= \left[ 1 - \hat{Q}M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u,v)} \hat{P}M \right]^{-1} \left[ \hat{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u,v)} \hat{P} \right].
\end{equation}

Since we have

$$1 - \hat{Q}M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u,v)} \hat{P}M$$

$$= (1 - \hat{Q}M \hat{P}) \left[ 1 - \hat{Q}M \hat{Q} - (1 + \hat{Q}M \hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u,v)} \hat{P}M \right],$$
we obtain

\[
\begin{align*}
1 - \hat{Q} M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P} M &= \left[ 1 - \hat{Q} M \hat{Q} - (1 + \hat{Q} M \hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P} M \right]^{-1} (1 + \hat{Q} M \hat{P}).
\end{align*}
\]

Note that

\[
1 - \hat{Q} M \hat{Q} - (1 + \hat{Q} M \hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P} M = 1 - \hat{Q} M_0 \hat{Q} + \frac{\sqrt{z}}{4i} \hat{Q} v R_0^{(3)} u \hat{Q} - \frac{2\sqrt{z}}{i(u, v)} \left( 1 + \hat{Q} M_0 \hat{P} \right) \hat{P} M_0 + O(\varepsilon^{1/2 + \delta'}),
\]

where we have written

\[
M_0 = \frac{1}{2} v R_0^{(2)} u.
\]

In order to find the inverse of the operator of (46), consider the eigenvector with the eigenvalue +1 for the operator \(\hat{Q} M_0 \hat{Q}\). The following lemma is due to Bollé, Gesztesy and Wilk [8]:

**Lemma 16.** If the operator \(\hat{Q} M_0 \hat{Q}\) has the eigenvalue +1, then the eigenvalue must be simple and \(H\) correspondingly shows a zero-energy resonance.

**Proof.** Let \(\varphi_0\) be the eigenvector of \(\hat{Q} M_0 \hat{Q}\) with the eigenvalue +1, and let

\[
\psi = (u, v)^{-1}(u, v \Delta^{-1} u \varphi_0) - \Delta^{-1} u \varphi_0,
\]

where \(\Delta^{-1}\) is the inverse of the Laplacian \(\Delta\). The integral kernel of \(\Delta^{-1}\) is given by \(|x|/2\). Clearly, \(M_0 = v \Delta^{-1} u\). From \(\hat{Q} M_0 \hat{Q} \varphi_0 = \varphi_0\), we have \(\hat{P} \varphi_0 = 0\) and \((u, \varphi_0) = 0\). Using these, we have

\[
\hat{Q} M_0 \hat{Q} \varphi_0 = (1 - \hat{P}) M_0 \varphi_0 = M_0 \varphi_0 - \hat{P} M_0 \varphi_0 = M_0 \varphi_0 - (u, v)^{-1}(u, M_0 \varphi_0) v = \varphi_0.
\]

Therefore,

\[
(47) \quad v \psi = -\varphi_0.
\]

Substituting this into the expression of the vector \(\psi\), one has

\[
\psi = (u, v)^{-1}(u, v \Delta^{-1} u \varphi_0) + \Delta^{-1} V \psi.
\]

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Immediately, $\Delta \psi = V \psi$, i.e., $H \psi = 0$. Further, one obtains $(u, \varphi_0) = -(V, \psi) = 0$ and $(xu, \varphi_0) = -(xV, \psi)$ from (47). Using these relations, we have

$$\Delta^{-1}V \psi(x') = \frac{1}{2} \int_{x'}^\infty dy(y - x')V(y)\psi(y) + \frac{1}{2} \int_{-\infty}^{x'} dy(x' - y)V(y)\psi(y)$$

$$= -x' \int_{x'}^\infty dyV(y)\psi(y) + \int_{x'}^\infty dy'V(y')\psi(y') + \frac{1}{2}(xu, \varphi_0)$$

$$= \int_{x'}^\infty dy(y - x')V(y)\psi(y) + \frac{1}{2}(xu, \varphi_0).$$

Similarly,

$$\Delta^{-1}V \psi(x') = \int_{-\infty}^{x'} dy(y - y')V(y)\psi(y) - \frac{1}{2}(xu, \varphi_0).$$

These yield $\psi \in L^\infty(\mathbb{R})$.

In order to show that $\varphi_0$ is unique, let $\varphi'_0$ be another eigenvector with the eigenvalue +1, and let $\psi'$ be the corresponding resonance state of $H$. We choose their normalization so that they satisfy

$$(u, v)^{-1}(u, M_0(\varphi_0 - \varphi'_0)) + \frac{1}{2}(xu, (\varphi_0 - \varphi'_0)) = 0.$$

Then, we have

$$|\psi(x') - \psi'(x')| \leq \int_{x'}^\infty dy|y - x'||V(y)||\psi(y) - \psi'(y)|$$

$$\leq \int_{x'}^\infty dy(|y| + R)|V(y)||\psi(y) - \psi'(y)|$$

for $-R \leq x' < \infty$ with a large $R > 0$. Iterating this, we obtain

$$|\psi(x') - \psi'(x')| \leq \int_{x'}^\infty dy_1(|y_1| + R)|V(y_1)| \int_{y_1}^\infty dy_2(|y_2| + R)|V(y_2)|$$

$$\cdots \int_{y_{m-1}}^\infty dy_m(|y_m| + R)|V(y_m)||\psi(y_m) - \psi'(y_m)|$$

$$\leq \frac{||\psi - \psi'||_\infty}{m!} \left[ \int_{x'}^\infty dy(|y| + R)|V(y)| \right]^m.$$

This implies $\psi = \psi'$. Hence, $\varphi_0 = \varphi'_0$. \hfill \Box

In the following, we will treat only the case that $H$ shows a zero-energy resonance because the other case can be handled in a much easier way.

Let $\varphi_0$ be the eigenvector of $\tilde{Q} M_0 \tilde{Q}$ with the eigenvalue +1, i.e.,

$$\tilde{Q} M_0 \tilde{Q} \varphi_0 = \varphi_0.$$
Then the left eigenvector is given by \( \hat{\varphi}_0 = N_0(\text{sgn}V)\varphi_0 \). We choose the normalization \( N_0 \) so that \((\hat{\varphi}_0, \varphi_0) = 1\). We define the projection \( P_0 = \varphi_0(\hat{\varphi}_0, \cdots) \), and \( Q_0 = 1 - P_0 \). Clearly we have

\[
(u, \varphi_0) = 0
\]

and

\[
\hat{P}P_0 = P_0\hat{P} = 0.
\]

From these observations, we can write

\[
1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P}M = (1 - \hat{Q}M_0\hat{Q}Q_0)Q_0 + \sqrt{z}K_1 + D_1,
\]

where

\[
K_1 = \frac{1}{4i} \hat{Q}vR_0^{(3)}u\hat{Q} + \frac{2i}{(u, v)} \hat{P}M_0 + \frac{2i}{(u, v)} \hat{Q}M_0\hat{P}M_0,
\]

and the operator \( D_1 \) satisfies

\[
\|D_1\|_2 = O(\varepsilon^{1/2+\delta}).
\]

We introduce the approximate inverse,

\[
B = (1 + K_2)(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1} + \frac{N}{2i\sqrt{z}}P_0,
\]

for the operator of (48), where

\[
N = (N_0^*)^{-1}\left[(u, v)^{-2}|(u, M_0\varphi_0)|^2 + \frac{1}{4}|(xu, \varphi_0)|^2\right]^{-1}
\]

and

\[
K_2 = \frac{N}{8} P_0vR_0^{(3)}u\hat{Q} - \frac{N}{(u, v)} P_0M_0\hat{P}M_0.
\]

Note that

\[
\frac{N}{2i}P_0K_1 = P_0 - K_2Q_0.
\]

Combining this, (48) and (49), we have

\[
B \left[ 1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P}M \right] = 1 + D_2,
\]

where

\[
D_2 = \frac{N}{2i\sqrt{z}}P_0D_1 + \sqrt{z}(1 + K_2)(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}K_1 + O(\varepsilon^{1/2+\delta}).
\]
Therefore we obtain
\[
\left[ 1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P}M \right]^{-1} = \frac{1}{1 + D_2} B.
\]
Combining this, (44) and (45), we have
\[
[1 - v(z - H_0)^{-1}u]^{-1} = \frac{1}{1 + D_2} B(1 + \hat{Q}M\hat{P}) \left[ \hat{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \hat{P} \right].
\]

Let us compute the excess charge \( Z^0 \) of (42) at the zero energy. From the above result, we have
\[
\text{Tr} \left\{ v(z - H_0)^{-2}u[1 - v(z - H_0)^{-1}u]^{-1} \right\} = \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} B\hat{Q} \right\}
\]
\[
+ \frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} B(\hat{P} + \hat{Q}M\hat{P}) \right\}.
\]
The first term in the right-hand side is written
\[
\text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} B\hat{Q} \right\}
\]
\[
= \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} (1 + K_2)(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}\hat{Q} \right\}
\]
\[
+ \frac{N}{2i\sqrt{z}} \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} P_0 \right\}
\]
by using (49) and \( P_0\hat{P} = 0 \).

First, we will show that the first term in the right-hand side of (53) yields the vanishing contribution to the excess charge. To begin with, we note that we have
\[
v(z - H_0)^{-2}u = -\frac{i}{4} z^{-3/2}(u, v)\hat{P} - \frac{i}{8} z^{-1/2}vR_0^{(3)}u + O(\varepsilon^{-1/2} + \delta')
\]
in the same way as in (41). Formally substituting this expansion into the term, we obtain the desired result. But all of the terms so obtained cannot be expressed into a trace of a product of two Hilbert-Schmidt operators. By using the expansion and
\[
\frac{1}{1 + D_2} = 1 - D_2 \frac{1}{1 + D_2},
\]
the first term in the right-hand side of (53) is written
\[
\text{Tr} \left\{ v(z - H_0)^{-2}u \left[ 1 - D_2 \frac{1}{1 + D_2} + K_2 - D_2 \frac{1}{1 + D_2} K_2 \right] \right\}
\]
\[
\times (1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}\hat{Q} \right\}
\]
\[
= \text{Tr} \left\{ v(z - H_0)^{-2}u(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}\hat{Q} \right\} + O(\varepsilon^{-1/2}).
\]
Further, using

\[(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1} = 1 + (1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}\hat{Q}M_0\hat{Q}Q_0,\]

we obtain

\[
\text{Tr} \left\{ v(z - H_0)^{-2}u(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1}\hat{Q} \right\}
\]

\[
= \text{Tr} \left\{ v(z - H_0)^{-2}u\hat{Q} \right\} + O(\varepsilon^{-1/2})
\]

\[
= \text{Tr} v(z - H_0)^{-2}u - \text{Tr} v(z - H_0)^{-2}u\hat{P} + O(\varepsilon^{-1/2})
\]

\[
= -\frac{i}{4}z^{-3/2}(u, v) + \frac{i}{4}z^{-3/2}(u, v)\text{Tr} \hat{P} + O(\varepsilon^{-1/2}) = O(\varepsilon^{-1/2}).
\]

Thus the contribution is vanishing.

Similarly, the second term in the right-hand side of (53) is computed as

\[
\frac{N}{2\sqrt{z}} \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} P_0 \right\}
\]

\[
= -\frac{N}{16z} \text{Tr} vR_0^{(3)}u \frac{1}{1 + D_2} P_0 + O(\varepsilon^{-1+\delta})
\]

\[
= -\frac{N}{16z} (\hat{\varphi}_0, vR_0^{(3)}u\varphi_0) + O(\varepsilon^{-1+\delta})
\]

\[
= \frac{N}{8z} (\hat{\varphi}_0, xv)(xu, \varphi_0) + O(\varepsilon^{-1+\delta})
\]

\[
= \frac{NN_0}{8z} (xu, \varphi_0)^2 + O(\varepsilon^{-1+\delta}),
\]

where we have used \(\hat{P}_0\hat{P} = 0, (\hat{\varphi}_0, v) = (u, \varphi_0) = 0\)

Next, we treat the second term in the right-hand of (52). In the same way, we have

\[
\frac{2\sqrt{z}}{2\sqrt{z} + i(u, v)} \text{Tr} \left\{ v(z - H_0)^{-2}u \frac{1}{1 + D_2} B(\hat{P} + \hat{Q}M\hat{P}) \right\}
\]

\[
= \frac{1}{2\sqrt{z} + i(u, v)} \left( -\frac{i}{2} \right) \frac{(u, v)}{z} \text{Tr} \left\{ \hat{P} \frac{1}{1 + D_2} \right\}
\]

\[
\times \left[ (1 + K_2)(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1} + \frac{N}{2\sqrt{z}} P_0 \right] (\hat{P} + \hat{Q}M\hat{P}) \right\} + O(\varepsilon^{-1/2})
\]

\[
= -\frac{1}{2z} - \frac{1}{2\sqrt{z} + i(u, v)} \frac{N(u, v)}{4z^{3/2}} \text{Tr} \left\{ \hat{P} \frac{1}{1 + D_2} P_0 M\hat{P} \right\} + O(\varepsilon^{-1+\delta}),
\]

where we have used \(\hat{P}P_0 = 0, \hat{P}K_2 = 0\) and \(\hat{P}(1 - \hat{Q}M_0\hat{Q}Q_0)^{-1} = \hat{P}\). The
trace of the operator in the second term in the last line is computed as
\[
\text{Tr} \hat{P} \frac{1}{1 + D_2} P_0 M \hat{P} = \text{Tr} \hat{P} \left[ 1 - D_2 \frac{1}{1 + D_2} \right] P_0 M
\]
\[
= -\text{Tr} \hat{P} D_2 \frac{1}{1 + D_2} P_0 M
\]
\[
= -\sqrt{z} \text{Tr} \hat{P} K_1 \frac{1}{1 + D_2} P_0 M + O(\varepsilon^{1/2+\delta'})
\]
\[
= -2i(u, v)^{-1} \sqrt{z} \text{Tr} P_0 M_0 \hat{P} M + O(\varepsilon^{1/2+\delta'})
\]
\[
= -2i(u, v)^{-2} N_0^* \sqrt{z} |(u, M_0 \varphi_0)|^2 + O(\varepsilon^{1/2+\delta'}). 
\]
Substituting this into (56), the value becomes
\[
-\frac{1}{2z} + \frac{1}{2z}(u, v)^{-2} N_0^* |(u, M_0 \varphi_0)|^2 + O(\varepsilon^{-1+\delta'}). 
\]
Combining this, (50) and (55), all of the contributions turn out to cancel each other out. Thus the excess charge at the zero energy is vanishing.

Next consider the case with \( \int dx V(x) = 0 \).

We write
\[
\hat{P} = v(u, \cdots)
\]
with \( u = |V|^{1/2} \text{sgn} V \) and \( v = |V|^{1/2} \). Since we have
\[
v(z - H_0)^{-1} u = \frac{1}{2i \sqrt{z}} \hat{P} M,
\]
we obtain
\[
1 - v(z - H_0)^{-1} u = 1 + \frac{i}{2 \sqrt{z}} \hat{P} - M
\]
\[
= \left( 1 + \frac{i}{2 \sqrt{z}} \hat{P} \right) \left[ 1 - \left( 1 - \frac{i}{2 \sqrt{z}} \hat{P} \right) M \right].
\]
Here \( M \) is given by (43). Therefore,
\[
[1 - v(z - H_0)^{-1} u]^{-1} = \left[ 1 - M + \frac{i}{2 \sqrt{z}} \hat{P} M \right]^{-1} \left( 1 - \frac{i}{2 \sqrt{z}} \hat{P} \right).
\]
Since
\[
(u, M v) = (u, v \Delta^{-1} u v) + O(\varepsilon^{1/2}) = (V, \Delta^{-1} V) + O(\varepsilon^{1/2}),
\]
the matrix element \((u, M v)\) is nonvanishing for a small \( \varepsilon \). Relying on this, we define
\[
\hat{P} = (u, M v)^{-1} \hat{P} M,
\]
43
and 

\[ \tilde{Q} = 1 - \tilde{P}. \]

Then, one can easily show that \( \tilde{P}^2 = \tilde{P} \) and \( \tilde{Q} \tilde{P} = \tilde{P} \tilde{Q} = 0 \). From these observations, we have

\[
\begin{align*}
\left[ 1 - M + \frac{i(u, Mv)}{2\sqrt{z}} \tilde{P} \right]^{-1} &= \left[ 1 - M + \frac{i(u, Mv)}{2\sqrt{z}} \tilde{P} \right]^{-1} \\
&= \left\{ 1 - \left[ 1 - \frac{i(u, Mv)\tilde{P}}{2\sqrt{z} + i(u, Mv)} \right] M \right\}^{-1} \left[ 1 - \frac{i(u, Mv)\tilde{P}}{2\sqrt{z} + i(u, Mv)} \right]
\end{align*}
\]

in the same way as in the preceding case. Note that

\[
1 - \frac{i(u, Mv)\tilde{P}}{2\sqrt{z} + i(u, Mv)} = \tilde{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}.
\]

These yield

\[
\begin{align*}
[1 - v(z - H_0)^{-1}u]^{-1} &= \left\{ 1 - \left[ \tilde{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P} \right] M \right\}^{-1} \\
&\times \left[ \tilde{Q} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P} \right] \left( 1 - \frac{i}{2\sqrt{z}} \tilde{P} \right) \\
&= \left[ 1 - \tilde{Q}M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}M \right]^{-1} \\
&\times \left[ \tilde{Q} - \frac{i}{2\sqrt{z} + i(u, Mv)} \tilde{P} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P} \right],
\end{align*}
\]

where we have used \( \tilde{Q} \tilde{P} = 0 \) and \( \tilde{P} \tilde{P} = \tilde{P} \). Note that

\[
1 - \tilde{Q}M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}M = (1 - \tilde{Q}M \tilde{P}) \left[ 1 - \tilde{Q}M \tilde{Q} - (1 + \tilde{Q}M \tilde{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}M \right].
\]

Therefore, the inverse is

\[
\begin{align*}
&\left[ 1 - \tilde{Q}M - \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}M \right]^{-1} \\
&= \left[ 1 - \tilde{Q}M \tilde{Q} - (1 + \tilde{Q}M \tilde{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}M \right]^{-1} (1 + \tilde{Q}M \tilde{P}).
\end{align*}
\]
As a result, we obtain

\[(1 - v(z - H_0)^{-1}u)^{-1}\]

\[= \left[1 - \tilde{Q} M \tilde{Q} - (1 + \tilde{Q} M \tilde{P}) \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P} M\right]^{-1}\]

\[\times (1 + \tilde{Q} M \tilde{P}) \left[\tilde{Q} - \frac{i}{2\sqrt{z} + i(u, Mv)} \tilde{P} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)} \tilde{P}\right].\]

In order to obtain the explicit expression of the inverse of the operator in the right-hand side, we expand the operator in powers of $\sqrt{z}$.

To begin with, we note that

\[\tilde{Q} M \tilde{Q} = \tilde{Q}_0 M_0 \tilde{Q}_0 - \frac{\sqrt{z}}{4i} K_1 + O(\varepsilon^{1/2+\delta'})\]

where $K_1 = \frac{1}{4i} \left[K_0 M_0 \tilde{Q}_0 + \tilde{Q}_0 M_0 K_0 + \tilde{Q}_0 M_0 \tilde{Q}_0\right] + 2i \frac{u}{(u, M_0v)} \left[\tilde{P}_0 M_0 + \tilde{Q}_0 M_0 \tilde{P}_0 M_0\right]$, and the operator $D_1$ satisfies

\[\|D_1\|_2 = O(\varepsilon^{1/2+\delta'}).\]

The following lemma is due to Bollé, Gesztesy and Klaus [7]:

**Lemma 17.** If the operator $\tilde{Q}_0 M_0 \tilde{Q}_0$ has the eigenvalue $+1$, then the eigenvalue must be simple and $H$ correspondingly shows a zero-energy resonance.

**Proof.** Let $\varphi_1$ be the eigenvector of $\tilde{Q}_0 M_0 \tilde{Q}_0$ with the eigenvalue $+1$, and let

\[\psi = (u, M_0v)^{-1}(u, M_0^2 \varphi_1) - \Delta^{-1} u \varphi_1.\]
From $\tilde{Q}_0 M_0 \tilde{Q}_0 \varphi_1 = \varphi_1$, we have $\tilde{P}_0 \varphi_1 = 0$, and

$$\varphi_1 = (1 - \tilde{P}_0) M_0 \varphi_1 = M_0 \varphi_1 - (u, M_0 v)^{-1} v (u, \cdots) M_0^2 \varphi_1$$

$$= M_0 \varphi_1 - (u, M_0 v)^{-1} (u, M_0^2 \varphi_1) v.$$

Combining this with the definition of $\psi$, we obtain

$$v \psi = (u, M_0 v)^{-1} (u, M_0^2 \varphi_1) v - v \Delta^{-1} u \varphi_1$$

$$= (u, M_0 v)^{-1} (u, M_0^2 \varphi_1) v - M_0 \varphi_1 = -\varphi_1.$$

Therefore, we get the integral equation for $\psi$,

$$\psi = (u, M_0 v)^{-1} (u, M_0^2 \varphi_1) + \Delta^{-1} V \psi.$$

Clearly, $\Delta \psi = V \psi$, i.e., $H \psi = 0$. In the same way as in Lemma 16, the uniqueness of $\varphi_1$ can be proved with the help of the relation (59) below.

Let $\varphi_1$ be the eigenvector of $\tilde{Q}_0 M_0 \tilde{Q}_0$ with the eigenvalue +1. Then the left eigenvector is given by $\hat{\varphi}_1 = N_0 (\text{sgn} V) M_0 \varphi_1$, where $N_0$ is a normalization constant. Since one has $(u \Delta^{-1} u \varphi_1, \varphi_1) < 0$, we can choose $N_0$ to satisfy $(\hat{\varphi}_1, \varphi_1) = 1$.

In order to show that $\hat{\varphi}_1$ is the left eigenvector, we note the following: From $\tilde{Q}_0 M_0 \tilde{Q}_0 \varphi_1 = \varphi_1$, we have

$$0 = \tilde{P}_0 \varphi_1 = (u, M_0 v)^{-1} \tilde{P} M_0 \varphi_1$$

$$= (u, M_0 v)^{-1} \tilde{P} (\tilde{P}_0 + \tilde{Q}_0) M_0 \varphi_1$$

$$= (u, M_0 v)^{-1} \tilde{P} \tilde{Q}_0 M_0 \varphi_1$$

$$= (u, M_0 v)^{-1} \tilde{P} \varphi_1,$$

where we have used $\tilde{P} \tilde{P}_0 = 0$. This yields

$$(59) \quad (u, \varphi_1) = 0$$

and

$$(60) \quad (u, M_0 \varphi_1) = 0.$$

The latter relation yields

$$(61) \quad \tilde{P}_0^* \hat{\varphi}_1 = (u, M_0 v)^{-1} M_0^2 \hat{\varphi}_1$$

$$= (u, M_0 v)^{-1} u \Delta^{-1} v \cdot u (v, \cdots) (\text{sgn} V) M_0 \varphi_1$$

$$= (u, M_0 v)^{-1} u \Delta^{-1} v \cdot u (u, M_0 \varphi_1) = 0.$$
Note that

$$\tilde{Q}_0^*(\text{sgn} V)M_0 \tilde{P} = (\text{sgn} V)M_0 \tilde{P} - (u,M_0 v)^{-1}u \Delta^{-1}v \cdot u(v,\cdots)(\text{sgn} V)M_0 v(u,\cdots) = (\text{sgn} V)M_0 \tilde{P} - u\Delta^{-1}v \cdot u(u,\cdots) = 0.$$ 

Using these relations, we obtain

$$\tilde{Q}_0^*M_0 \tilde{Q}_0^* = \tilde{Q}_0^*M_0^2 \varphi_1$$

$$= \tilde{Q}_0^*(\text{sgn} V)[1 - (u,M_0 v)^{-1}M_0 \tilde{P}]M_0^2 \varphi_1$$

$$= \tilde{Q}_0^*(\text{sgn} V)M_0[1 - (u,M_0 v)^{-1}M_0 \tilde{P}]M_0 \varphi_1$$

$$= \tilde{Q}_0^*(\text{sgn} V)M_0 \tilde{Q}_0M_0 \varphi_1$$

$$= \tilde{Q}_0^*(\text{sgn} V)M_0 \varphi_1 = \varphi_1,$$

except for the normalization $N_0$ of $\varphi_1$.

We define $P_1 = \varphi_1(\varphi_1,\cdots)$, and $Q_1 = 1 - P_1$. As an approximate inverse for the operator of (58), we introduce

$$B = (1 + K_2)(1 - \tilde{Q}_0^*M_0 \tilde{Q}_0 Q_1)^{-1} + \frac{N}{2i\sqrt{z}}P_1,$$

where

$$N = (N_0^*)^{-1} \left[ (u,M_0 v)^{-2}||(u,M_0^2 \varphi_1)||^2 + \frac{1}{4}||(xu,\varphi_1)||^2 \right]^{-1}$$

and

$$K_2 = N \left[ \frac{1}{8}(P_1 M_0 K_0 + P_1 v R_0^{(3)} u \tilde{Q}_0) - (u,M_0 v)^{-1}P_1 M_0 \tilde{P}_0 M_0 \right].$$

From (60), we have

$$\langle \varphi_1,v \rangle = 0.$$

This yields

$$P_1 \tilde{P} = 0.$$

Further, from (61), we have

$$P_1 \tilde{P}_0 = 0.$$
Using these relations, we obtain $P_1 K_0 = 0$. From these observations, we have
\[
\frac{N}{2i} P_1 K_1 = \frac{N}{2i} \left[ \frac{1}{4i} (P_1 M_0 K_0 + P_1 v R_0^{(3)} u Q_0) + \frac{2i}{(u, M_0 v)} P_1 M_0 P_0 M_0 \right]
\]
\[
= -\frac{N}{8} \left[ (P_1 M_0 K_0 P_1 + P_1 v R_0^{(3)} u P_1) - (u, M_0 v)^{-1} P_1 M_0 P_0 M_0 P_1 \right] - K_2 Q_1.
\]
Note that
\[
P_1 M_0 K_0 P_1 = -(u, M_0 v)^{-1} P_1 M_0 \hat{P} v R_0^{(3)} u P_1
\]
\[
= -(u, M_0 v)^{-1} (\dot{\varphi}_1, M_0 v) (u, v R_0^{(3)} u \varphi_1) P_1
\]
\[
= 2(u, M_0 v)^{-1} N_0^* (M_0^2 \varphi_1, u) (u, x v)(x u, \varphi_1) P_1,
\]
where we have used $(M_0 v, \dot{\varphi}_1) = N_0 (u, M_0^2 \varphi_1)$, $(u, v) = 0$ and $(u, \varphi_1) = 0$.

Similarly,
\[
P_1 v R_0^{(3)} u P_1
\]
\[
= (\dot{\varphi}_1, v R_0^{(3)} u \varphi_1) P_1
\]
\[
= -2(\dot{\varphi}_1, x v)(x u, \varphi_1) P_1
\]
\[
= -2 N_0^* (M_0 \varphi_1, x v)(x u, \varphi_1) P_1
\]
\[
= -2 N_0^* (x u, \varphi_1)^2 P_1 - 2(u, M_0 v)^{-1} N_0^* (\hat{P} M_0^2 \varphi_1, x v)(x u, \varphi_1) P_1
\]
\[
= -2 N_0^* (x u, \varphi_1)^2 P_1 - 2(u, M_0 v)^{-1} N_0^* (M_0^2 \varphi_1, u)(v, x v)(x u, \varphi_1) P_1.
\]

Here, we have used
\[
(64) \quad M_0 \varphi_1 = (\bar{Q}_0 + \bar{P}_0) M_0 \varphi_1 = \varphi_1 + (u, M_0 v)^{-1} \hat{P} M_0^2 \varphi_1.
\]

Moreover, we have
\[
-(u, M_0 v)^{-1} P_1 M_0 \hat{P}_0 M_0 P_1 = -(u, M_0 v)^{-2} (\dot{\varphi}_1, M_0 v)(u, M_0^2 \varphi_1) P_1
\]
\[
= -(u, M_0 v)^{-2} N_0^* (u, M_0^2 \varphi_1)^2 P_1.
\]

Putting these together, we have
\[
\frac{N}{2i} P_1 K_1 = P_1 - K_2 Q_1.
\]

Immediately, this yields
\[
B \left[ 1 - \bar{Q} M \bar{Q} - (1 + \bar{Q} M \bar{P}) \frac{2 \sqrt{z}}{2 \sqrt{z} + i (u, M v)} \bar{P} M \right]
\]
\[
= B \left[ (1 - \bar{Q}_0 M_0 \bar{Q}_0 Q_1) Q_1 + \sqrt{z} K_1 + D_1 \right] = 1 + D_2,
\]
\[48\]
where
\[ D_2 = \frac{N}{2i\sqrt{z}}P_1D_1 + \sqrt{z}(1 + K_2)(1 - \tilde{Q}_0M_0\tilde{Q}_0Q_1)^{-1}K_1 + O(\varepsilon^{1/2+\delta'}). \]

Therefore, the desired expression of the inverse of the operator is
\[
\left[ 1 - \tilde{Q}M\tilde{Q} - (1 + \tilde{Q}M\tilde{P})\frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)}\tilde{P}M \right]^{-1} = \frac{1}{1 + D_2}B.
\]

Substituting this into (57), we obtain
\[ [1 - v(z - H_0)^{-1}u]^{-1} = \frac{1}{1 + D_2}B(1 + \tilde{Q}M\tilde{P})\left[ \tilde{Q} - \frac{i}{2\sqrt{z} + i(u, Mv)}\tilde{P} + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)}\tilde{P} \right]. \]

Using this expression, we have
\[ \text{Tr} \left\{ v(z - H_0)^{-2}u[1 - v(z - H_0)^{-1}u]^{-1} \right\} \]
\[ = \text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}B\tilde{Q} \]
\[ - \frac{i}{2\sqrt{z} + i(u, Mv)}\text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}B(\tilde{P} + \tilde{Q}M\tilde{P}) \]
\[ + \frac{2\sqrt{z}}{2\sqrt{z} + i(u, Mv)}\text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}B(\tilde{P} + \tilde{Q}M\tilde{P}), \]

where we have used \( \tilde{P}\tilde{P} = \tilde{P} \).

Consider first the first term in the right-hand side of (65). It is written
\[ \text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}B\tilde{Q} \]
\[ = \text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}(1 + K_2)(1 - \tilde{Q}_0M_0\tilde{Q}_0Q_1)^{-1}\tilde{Q} \]
\[ + \frac{N}{2i\sqrt{z}}\text{Tr} v(z - H_0)^{-2}u \frac{1}{1 + D_2}P_1\tilde{Q} \]

by using the expression (62) of \( B \). The expansion corresponding to (54) is given by
\[ v(z - H_0)^{-2}u = -i\frac{z^{-3/2}}{4}\tilde{P} - i\frac{z^{-1/2}}{8}vR_0^{(3)}u + O(\varepsilon^{-1/2+\delta'}). \]

In the same way as in the preceding case, it is enough to treat
\[ \text{Tr} v(z - H_0)^{-2}u\tilde{Q} \]
for the first term in the right-hand side of (66). Using \( \tilde{Q} = 1 - \tilde{P} \) and the above expansion, one can show that the corresponding contribution to the
excess charge is vanishing. Similarly, the second term in the right-hand side of (66) is computed as

\[
\frac{N}{2i\sqrt{z}} \text{Tr} v(z - H_0)^{-2} u \frac{1}{1 + D^2} P_1 \hat{Q}
\]

\[
= \frac{N}{2i\sqrt{z}} \left( \frac{i}{8} \right) z^{-1/2} \text{Tr} v R_0^{(3)} u \frac{1}{1 + D^2} P_1 \hat{Q} + O(\varepsilon^{-1+\delta'})
\]

\[
= -\frac{N}{16z} (\hat{\varphi}_1, v R_0^{(3)} u \varphi_1) + O(\varepsilon^{-1+\delta'})
\]

\[
= \frac{NN_0}{8z} [((u, \varphi_1))^2 + (u, M_0 v)^{-1}(M_0^2 \varphi_1, u)(v, xu)(xu, \varphi_1)] + O(\varepsilon^{-1+\delta'}),
\]

where we have used \( \hat{Q} \hat{P} = 0, P_1 \hat{Q}_0 = P_1, (\hat{\varphi}_1, v) = (u, \varphi_1) = 0 \) and (64).

In the same way, the second term in the right-hand side of (65) is computed as

\[
\frac{i}{2z} \text{Tr} v(z - H_0)^{-2} u \frac{1}{1 + D^2} B(\hat{P} + \hat{Q}M \hat{P})
\]

\[
= (u, M_0 v)^{-1} \frac{N}{16z} \text{Tr} v R_0^{(3)} u P_1 (1 + \hat{Q}_0 M_0) \hat{P} + O(\varepsilon^{-1+\delta'})
\]

\[
= (u, M_0 v)^{-1} \frac{N}{16z} (u, v R_0^{(3)} u \varphi_1)(\hat{\varphi}_1, M_0 v) + O(\varepsilon^{-1+\delta'})
\]

\[
= -\frac{NN_0}{8z} (u, M_0 v)^{-1}(u, xv)(xu, \varphi_1)(M_0^2 \varphi_1, u) + O(\varepsilon^{-1+\delta'}),
\]

where we have used \( \hat{P}^2 = 0, P_1 \hat{P} = 0, P_1 \hat{Q}_0 = P_1, (u, v) = 0 \) and \( (u, \varphi_1) = 0 \).

Similarly, the third term in the right-hand side of (65) is computed as

\[
\frac{2\sqrt{z}}{2z + i(u, M v)} \text{Tr} v(z - H_0)^{-2} u \frac{1}{1 + D^2} B(\hat{P} + \hat{Q}M \hat{P})
\]

\[
= -\frac{i}{2z} \frac{1}{2\sqrt{z} + i(u, M v)} \text{Tr} \hat{P} \left[ 1 - D^2 \frac{1}{1 + D^2} \right] B(1 + \hat{Q}M) + O(\varepsilon^{-1/2})
\]

\[
= -\frac{i}{2z} \frac{1}{2\sqrt{z} + i(u, M v)} \text{Tr} \hat{P} \left[ 1 - \frac{N}{2i} K_1 P_1 \right] (1 + \hat{Q}M) + O(\varepsilon^{-1+\delta'})
\]

\[
= -\frac{1}{2z} (u, M_0 v)^{-1} \text{Tr} \hat{P} M_0 - \frac{N}{2i} \text{Tr} \hat{P} K_1 P_1 M_0 + O(\varepsilon^{-1+\delta'})
\]

\[
= -\frac{1}{2z} + \frac{N}{4iz} \text{Tr} \hat{P} K_1 P_1 M_0 + O(\varepsilon^{-1+\delta'}),
\]

50
Combining this, (63), (67) and (68), we have the desired result, $R$ the resolvent right-hand side of (65) to the excess charge is

In consequence, the nonvanishing contribution from the third term in the last line of (69) is calculated as

$$\text{Tr} \hat{P}K_1P_1M_0$$

where we have used $\hat{P}\hat{P} = \hat{P}$, $\hat{P}P_1 = P_1\hat{P} = 0$, $(u,v) = 0$, $P_1\tilde{Q}_0 = P_1$, $\hat{P}\tilde{Q}_0 = \hat{P}$ and

$$\hat{P}(1 - \tilde{Q}_0M_0\tilde{Q}_0Q_1)^{-1} = \hat{P}.$$  

The second term in the last line of (69) is calculated as

$$\text{Tr} \hat{P}K_1P_1M_0$$

where we have used $\hat{P}K_0 = 0$, $\hat{P}\tilde{Q}_0 = P_1$, $\tilde{Q}_0P_1 = P_1$, $\tilde{P}_0\hat{P} = \hat{P}$ and $\tilde{P}_0\hat{P} = 0$. The first and the second terms in the second equality cancel each other out. The third term is written

$$2i(u, M_0v)^{-1}\text{Tr} \hat{P}M_0\tilde{Q}_0P_1M_0 = 2i\text{Tr} \tilde{P}_0M_0P_1M_0$$

$$= 2i\text{Tr} \tilde{P}_0M_0P_1M_0$$

$$= 2i(u, M_0v)^{-1}(u, M_0^2\varphi_1)(\varphi_1, M_0v)$$

$$= 2i(u, M_0v)^{-1}N_0^0(u, M_0^2\varphi_1)^2.$$  

Therefore, the corresponding contribution in (69) becomes

$$\frac{1}{2z}NN_0^0(u, M_0v)^{-2}|(u, M_0^2\varphi_1)|^2.$$  

In consequence, the nonvanishing contribution from the third term in the right-hand side of (65) to the excess charge is

$$-\frac{1}{2z} + \frac{NN_0^0}{2z}(u, M_0v)^{-2}|(u, M_0^2\varphi_1)|^2.$$  

Combining this, (63), (67) and (68), we have the desired result, $Z^0 = 0$, for the excess charge at the zero energy.

**B Proof of Theorem 8**

Since $\alpha > 6$, one can find a small $\hat{\delta} > 0$ such that $\alpha > 6 + \hat{\delta}$. As in Section 4.1, we take $z = \varepsilon e^{i\phi}$ with $\delta \leq \phi \leq 2\pi - \delta$ with $\delta > 0$, and $R = e^{-\mu}$ with $3/(6 + \hat{\delta}) < \mu < 1/2$. We write $G_0(x; z)$ for the integral kernel of the resolvent $R_0(z) = (z - H_0)^{-1}$. We decompose $R_0(z)$ into two parts as

$$R_0(z) = R_0^\geq(z) + R_0^\leq(z),$$

where $R_0^\geq(z)$ has the integral kernel $G_0(x; z)\chi_R^\geq(x)$, and $R_0^\leq(z)$ has $G_0(x; z)\chi_R^\leq(x)$. The integral representation of $G_0(x; z)$ yields

$$\left|G_0(x; z)\chi_R^\geq(x)\right| \leq \text{Const.}e^{\mu^{-1/2}}.$$  

We write $v = |V|^{1/2}$ and $u = |V|^{1/2}\text{sgn}V$.  

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Lemma 18. The following bound is valid:
\[
\left\| v R_0^>(z) u \right\| \leq \text{Const.} \varepsilon^{1+\delta'}
\]
with some positive \( \delta' \) for a small \( \varepsilon \).

**Proof.** Using the bound (70), one has

\[
\left| \left| V(x)^{1/2} G_0(x-y; z) \chi_R^>(x-y) \right| \right| \leq \text{Const.} \varepsilon^{2\mu-1}
\]

Combining this with the assumption on the decay of the potential \( V \) yields
\[
\left\| v R_0^>(z) u \right\| \leq \text{Const.} \varepsilon^{3\mu+\delta \mu/2-1/2}.
\]

Next, consider \( v R_0^>(z) u \). To begin with, we recall the expansion of \( G_0(x; z) \) with respect to \( \sqrt{z} |x| \) as

\[
G_0(x; z) = \left[ \kappa(z) + \frac{1}{2\pi} \log |x| \right] \times \left[ 1 - \frac{3}{4} |x|^2 + \sum_{\ell=2}^{\infty} c_{\ell} z^{\ell} |x|^{2\ell} \right] + \frac{1}{8\pi} z |x|^2 + \sum_{\ell=2}^{\infty} c'_{\ell} z^{\ell} |x|^{2\ell}
\]

with the coefficients, \( c_{\ell} \) and \( c'_{\ell} \), where

\[
\kappa(z) := \frac{1}{2\pi} \left( -\frac{\pi i}{2} + \gamma - \log 2 + \frac{1}{2} \log z \right)
\]

with Euler’s constant \( \gamma \). For \( \ell \geq 2 \), one has

\[
\left| v(x) z^\ell |x-y|^{2\ell} \chi_R^>(x-y) u(y) \right| \leq \varepsilon \left| V(x) \right| |x-y|^{2(\ell-1)} \chi_R^>(x-y) u(y) \leq \varepsilon \left| V(x) \right| |x|^{1/2} |y|^{1/2} \left| x-y \right|^{2\ell-2} |x-y|^{-2} \left| V(y) \right|^{1/2} \left| y \right|^{1/2} \left( \varepsilon^{1-2\mu} \right)^{\ell-1} \leq 4\varepsilon \left| V(x) \right| |x|^{1/2} |y|^{1/2} \left( \varepsilon^{1-2\mu} \right)^{\ell-1}.
\]

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Therefore the corresponding Hilbert-Schmidt norm becomes $O(\varepsilon^{1+\delta'})$ with some positive $\delta'$. This yields
\[
v(x)G_0(x-y; z)\chi_R^\leq(x-y)u(y) \\
= v(x) \left[ \kappa(z) + \frac{1}{2\pi} \log |x-y| \right] \times \left[ 1 - \frac{z}{4}|x-y|^2 \right] u(y)\chi_R^\leq(x-y) \\
+ \frac{1}{8\pi} z v(x)|x-y|^2\chi_R^\leq(x-y)u(y) + O(\varepsilon^{1+\delta'}).
\]
In the same way, one can replace the cutoff $\chi_R^\leq$ with 1 in this right-hand side. Combining the result with the above Lemma 18, we obtain
\[
vR_0(z)u = \kappa(z)v(u, \cdots) + v\Delta^{-1}u - \frac{1}{4} z \kappa(z)vR_0^{(3)}u \\
+ \frac{1}{8\pi} z vR_0^{(3)}u - \frac{1}{8\pi} z v\tilde{R}_0^{(3)}u + O(\varepsilon^{1+\delta'}),
\]
where the operators, $R_0^{(3)}$ and $\tilde{R}_0^{(3)}$, have the integral kernels, $|x|^2$ and $|x|^2 \log |x|$, respectively; $\Delta$ is the Laplacian.

Consider first the case with $\int d^2x V(x) = (u,v) \neq 0$. We define
\[
\hat{P} := (u,v)^{-1}v(u, \cdots)
\]
and $\hat{Q} := 1 - \hat{P}$. Write
\[
vR_0(z)u = (u,v)\kappa(z)\hat{P} + M
\]
with
\[
M := M_0 - \frac{z}{4} \kappa(z)vR_0^{(3)}u + \frac{z}{8\pi} vR_0^{(3)}u - \frac{z}{8\pi} v\tilde{R}_0^{(3)}u + O(\varepsilon^{1+\delta'}),
\]
where we have written $M_0 = v\Delta^{-1}u$. Note that
\[
\left[ 1 - (u,v)\kappa(z)\hat{P} \right]^{-1} = 1 + \frac{(u,v)}{\kappa^{-1}(z) - (u,v)} \hat{P} = \hat{Q} - \frac{\kappa^{-1}(z)}{(u,v) - \kappa^{-1}(z)} \hat{P}.
\]
This yields
\[
1 - v(z - H_0)^{-1}u = 1 - (u,v)\kappa(z)\hat{P} - M \\
= \left[ 1 - (u,v)\kappa(z)\hat{P} \right] \left[ 1 - \hat{Q}M + \frac{\kappa^{-1}(z)}{(u,v) - \kappa^{-1}(z)} \hat{P}M \right] \\
= \left[ 1 - (u,v)\kappa(z)\hat{P} \right] (1 - \hat{Q}M\hat{P}) \\
\times \left[ 1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P}) \frac{\kappa^{-1}(z)}{(u,v) - \kappa^{-1}(z)} \hat{P}M \right].
\]
Therefore the inverse is

\[
[1 - v(z - H_0)^{-1}u]^{-1} = \left[ 1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P})\frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)}\hat{P}M \right]^{-1} \\
\times (1 + \hat{Q}M\hat{P}) \left[ \hat{Q} - \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)}\hat{P} \right].
\]

Using the expansion (71) for \( M \), we have

\[
1 - \hat{Q}M\hat{Q} - (1 + \hat{Q}M\hat{P})\frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)}\hat{P}M = 1 - \hat{Q}M_0\hat{Q} + K_1 + D_1,
\]

where

\[
K_1 = \frac{z}{4}\kappa(z)\hat{Q}vR_0^{(3)}u\hat{Q} - \frac{z}{8\pi}\hat{Q}vR_0^{(3)}u\hat{Q} + \frac{z}{8\pi}\hat{Q}v\hat{R}_0^{(3)}u\hat{Q} \\
- \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)}(1 + \hat{Q}M_0\hat{P}) \left[ \hat{P}M_0 - \frac{z}{4}\kappa(z)\hat{P}vR_0^{(3)}u \right] \\
+ \frac{z}{4}(u, v)^{-1}\hat{Q}v\hat{R}_0^{(3)}u\hat{P}M_0,
\]

and the operator \( D_1 \) satisfies \( \|D_1\|_2 = O(\varepsilon|\log \varepsilon|^{-1}) \).

In order to obtain the inverse of the operator (73), we must consider the eigenvalue problem for the operator \( \hat{Q}M_0\hat{Q} \).

Let \( \varphi \) be an eigenvector of \( \hat{Q}M_0\hat{Q} \) with the eigenvalue 1, i.e., \( \hat{Q}M_0\hat{Q}\varphi = \varphi \). Clearly, one has \( \hat{P}\varphi = 0 \) and \( (u, \varphi) = 0 \). We note that

\[
\hat{Q}M_0\hat{Q}\varphi = (1 - \hat{P})M_0\varphi = M_0\varphi - \hat{P}M_0\varphi = M_0\varphi - (u, v)^{-1}(u, M_0\varphi)v = \varphi.
\]

The resonance state is given by [6]

\[
\psi = (u, v)^{-1}(u, M_0\varphi) - \Delta^{-1}u\varphi.
\]

Actually, one can check this as follows: Using this and the above relation, one has

\[
v\psi = (u, v)^{-1}(u, M_0\varphi)v - M_0\varphi = -\varphi.
\]

Substituting this into the expression of \( \psi \) yields

\[
\psi = (u, v)^{-1}(u, M_0\varphi) + \Delta^{-1}V\psi.
\]

Immediately, \( \Delta\psi = -V\psi \). Besides, the asymptotics becomes

\[
\psi \sim (u, v)^{-1}(u, M_0\varphi) - \frac{x}{2\pi|x|^2} \cdot \int d^2y \ yu(y)\varphi(y)
\]

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for a large $|x|$. Here, the leading term from the integral is vanishing because $(u, \varphi) = 0$. Thus, there appear at most three resonance states [6].

In the following, we will treat only the case that $H$ shows three resonance states at the zero energy and $N$ bound states at the zero energy because the other cases can be handled in a much easier way.

Let $\varphi_{0,i}$, $i = 1, 2, 3$, be the three zero-energy resonance states satisfying the following conditions:

\[
(u, M_0 \varphi_{0,1}) \neq 0,
\]
\[
(u, M_0 \varphi_{0,2}) = (u, M_0 \varphi_{0,3}) = 0,
\]

and

\[
(xu, \varphi_{0,2}) \neq 0, \quad (xu, \varphi_{0,3}) \neq 0.
\]

Here $x$ is the operator of the position. Therefore $x$ is a two-component vector, and the two vectors, $(xu, \varphi_{0,2})$ and $(xu, \varphi_{0,3})$ are linearly independent so that three zero-energy resonance states appear. Let $\phi_j$, $j = 1, 2, \ldots, N$, be the zero-energy bound states satisfying

\[
(u, M_0 \phi_j) = 0, \quad \text{and} \quad (xu, \phi_j) = 0
\]

for $j = 1, 2, \ldots, N$. Further we define the projection operators as follows:

\[
P_{0,i} := \varphi_{0,i}(\hat{\varphi}_{0,i}, \cdots) \quad \text{for } i = 1, 2, 3
\]

and

\[
P_1 := \sum_{j=1}^{N} \phi_j(\hat{\phi}_j, \cdots).
\]

Here $\hat{\varphi}_{0,i}$ and $\hat{\phi}_j$ are the corresponding left eigenvector for the operator $\hat{Q}M_0\hat{Q}$. They are given by $\hat{\varphi}_{0,i} = N_{0,i}(\text{sgn} V)\varphi_{0,i}$ and $\hat{\phi}_j = M_{0,j}(\text{sgn} V)\phi_j$.

Here, we have chosen the normalizations, $N_{0,i}$ and $M_{0,j}$, to satisfy

\[
(\hat{\varphi}_{0,i}, \varphi_{0,k}) = \delta_{ik},
\]
\[
(\hat{\phi}_j, \phi_\ell) = \delta_{j\ell},
\]

and

\[
(\hat{\varphi}_{0,i}, \phi_j) = (\hat{\phi}_j, \varphi_{0,i}) = 0.
\]

We also write $P_0 = P_{0,1} + P_{0,2} + P_{0,3}$, $P = P_0 + P_1$ and $Q = 1 - P$.

Since the two vector, $(xu, \varphi_{0,2})$ and $(xu, \varphi_{0,3})$ are linearly independent, there exists a matrix $\Phi_{ij}$ such that

\[
\sum_{j=2,3} \Phi_{ij}(\hat{\varphi}_{0,j}, xu) \cdot (xu, \varphi_{0,k}) = \delta_{i,k} \quad \text{for } i, k = 2, 3.
\]
Similarly, since the operator \( u\tilde{R}_0^{(3)} u \) is positive [6], there is a matrix \( \tilde{\Phi}_{ij} \) such that
\[
\sum_{j=1}^{N} \tilde{\Phi}_{ij}(\hat{\phi}_j, v\tilde{R}_0^{(3)} u\phi_k) = \delta_{i,k} \quad \text{for} \ i, k = 1, 2, \ldots, N.
\]

As an approximate inverse for the operator (73), we introduce
\[
B = (1 - \Pi K_1)(1 - \hat{Q}M_0\hat{Q})^{-1} + \Pi
\]
with
\[
\Pi = (\hat{\varphi}_{0,1}, K_1\varphi_{0,1})^{-1}P_{0,1} - \frac{2}{zK(z)} \sum_{i,j=2,3} \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots)
\]
\[
+ \frac{8\pi}{z} \sum_{k,\ell=1}^{N} \phi_k \tilde{\Phi}_{k,\ell}(\hat{\varphi}_\ell, \cdots).
\]

Using the properties of the eigenvectors of \( \hat{Q}M_0\hat{Q} \) with the eigenvalue 1, one can show
\[
\Pi K_1 = O(1).
\]

This yields
\[
B \left[ (1 - \hat{Q}M_0\hat{Q})Q + K_1 + D_1 \right] = (1 - \Pi K_1)Q + (1 - \Pi K_1)(1 - \hat{Q}M_0\hat{Q})^{-1}K_1 + \Pi K_1 + \Pi D_1 + o(\varepsilon)
\]
\[
= Q + \Pi K_1 P + (1 - \Pi K_1)(1 - \hat{Q}M_0\hat{Q})^{-1}K_1 + \Pi D_1 + o(\varepsilon).
\]

Similarly, the second term in the second line is calculated as
\[
\Pi K_1 P = P + \Xi + PD_2,
\]
where
\[
\Xi = \sum_{i,j=2,3} \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots) xv \cdot (xu, \cdots)P_{0,1} + \sum_{k,\ell=1}^{N} \phi_k \tilde{\Phi}_{k,\ell}(\hat{\varphi}_\ell, \cdots)v\tilde{R}_0^{(3)} uP_0
\]
\[
+ \frac{2\pi}{(u, v)} \sum_{k,\ell=1}^{N} \phi_k \tilde{\Phi}_{k,\ell}(\hat{\varphi}_\ell, \cdots)v\tilde{R}_0^{(3)} u\hat{P}M_0 P_{0,1}
\]
and the operator \( D_2 \) satisfies \( \|D_2\|_2 = o(1) \). Therefore we have
\[
B \left[ (1 - \hat{Q}M_0\hat{Q})Q + K_1 + D_1 \right] = 1 + \Xi + (1 - \hat{Q}M_0\hat{Q})^{-1}K_1 + PD_3 + o(\varepsilon),
\]
where the operator \( D_3 \) satisfies \( \|D_3\|_2 = o(1) \). In consequence, we obtain
\[
(74) \quad \left[ (1 - \hat{Q}M_0\hat{Q})Q + K_1 + D_1 \right]^{-1} = \frac{1}{1 + D_4}\Gamma B,
\]
where
\[ D_4 = (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} K_1 + PD_3' + o(\varepsilon) \]
with \( D_3' \) satisfies \( \|D_3'\|_2 = o(1) \), and
\[ \Gamma = 1 - \Xi + \Xi^2. \]
Here we have used
\[ (1 + \Xi)(1 - \Xi + \Xi^2) = 1 \]
which is derived from \( \Xi^3 = 0 \).

Consequently, from (72), (73) and (74), we have
\[
\left[ 1 - v(z - H_0)^{-1} u \right]^{-1} = \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) \times \left[ \hat{Q} - \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)} \hat{P} \right].
\]
From this expression, we have
\[
\text{Tr}\left\{ v(z - H_0)^{-2} u \left[ 1 - v(z - H_0)^{-1} u \right]^{-1} \right\} = \text{Tr}\left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) \right\} - \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)} \text{Tr}\left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) \right\}.
\]
First, we will show that the second term in the right-hand side of (75) does not contribute to the excess charge at the zero energy. To begin with, we note that one can obtain
\[
v(z - H_0)^{-2} u = -\frac{(u, v)}{4\pi z} \hat{P} + \frac{\kappa(z)}{4} - \frac{1}{16\pi} \text{Tr} R^{(3)}_0 u + \frac{1}{8\pi} v \tilde{R}^{(3)}_0 u + O(\varepsilon')
\]
in the same way as (71). Using this expansion, one has
\[
-\frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)} \text{Tr} \left[ v(z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) \right] = \frac{(u, v)}{4\pi z} \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)} \text{Tr} \hat{P} \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) \]
\[- \frac{\kappa^{-1}(z)}{(u, v) - \kappa^{-1}(z)} \frac{\kappa(z)}{4} - \frac{1}{16\pi} \text{Tr} \text{Tr} R^{(3)}_0 u \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M \hat{P}) + o(\varepsilon^{-1}).
\]
Using the properties of the vector \( \phi_j \), one can show that the contribution from this second term is vanishing. The first term in the right-hand side is
computed as

\[(77)\]
\[
\text{Tr} \left\{ \hat{P} \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M) \right\} = \text{Tr} \left\{ \hat{P} \left[ 1 - D_4 \frac{1}{1 + D_4} \right] \Gamma B (1 + \hat{Q} M) \right\}
\]
\[
= \text{Tr} \left\{ \hat{P} \Gamma B (1 + \hat{Q} M) \right\}
\]
\[
- \text{Tr} \left\{ \hat{P} D_4 \frac{1}{1 + D_4} \Gamma B (1 + \hat{Q} M) \right\}.
\]

Using \( \hat{P} \varphi_{0,i} = \hat{P} \phi_j = 0 \), the first term in the right-hand side of the second equality is calculated as

\[
\text{Tr} \hat{P} \Gamma B (1 + \hat{Q} M) = \text{Tr} \hat{P} B (1 + \hat{Q} M)
\]
\[
= \text{Tr} \hat{P} (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} (1 + \hat{Q} M)
\]
\[
= \text{Tr} \hat{P} (1 + \hat{Q} M) = \text{Tr} \hat{P} = 1.
\]

Here we have used \( \hat{P} (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} \hat{P} = \hat{P} \). As to the second term, we note that, in the same way, we have

\[
\hat{P} D_4 = \hat{P} K_1 + o(\varepsilon) = O(|\log \varepsilon|^{-1}),
\]

and

\[
B(1 + \hat{Q} M) \hat{P} = B \left[ (1 + \hat{Q} M_0) + \hat{Q} (M - M_0) \right] \hat{P}
\]
\[
= (1 - \Pi K_1) (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} (1 + \hat{Q} M_0) \hat{P}
\]
\[
+ \Pi (1 + \hat{Q} M_0) \hat{P}
\]
\[
+ \left[ (1 - \Pi K_1) (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} + \Pi \right] \hat{Q} (M - M_0) \hat{P}
\]
\[
= \Pi M_0 \hat{P} + \Pi (M - M_0) \hat{P} + O(1) = O(|\log \varepsilon|),
\]

where we have used \( (\hat{\varphi}_{0,i}, M_0 v) = 0 \) for \( i = 2, 3 \) and \( (\hat{\phi}_k, M_0 v) = 0 \) for \( k = 1, 2, \ldots, N \). Therefore the second term in the right-hand side of the second equality of \( (77) \) becomes \( O(1) \). Thus, the second term in the right-hand side of \( (75) \) does not contribute to the excess charge at the zero energy.

The first term in the right-hand side of \( (75) \) is written

\[(78)\]
\[
\text{Tr} \left\{ v (z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma B \hat{Q} \right\}
\]
\[
= \text{Tr} \left\{ v (z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma (1 - \Pi K_1) (1 - \hat{Q} M_0 \hat{Q} Q)^{-1} \hat{Q} \right\}
\]
\[
+ \text{Tr} \left\{ v (z - H_0)^{-2} u \frac{1}{1 + D_4} \Gamma \Pi \hat{Q} \right\}.
\]
by using the expression of $B$.

Consider the first term in the right-hand side of (78). If one can substitute the expansion (76) into the first term, it is vanishing. But this procedure is not justified as mentioned before. We have to treat $\text{Tr} v(z - H_0)^{-2}u \hat{Q}$. It is calculated as

$$
\text{Tr} v(z - H_0)^{-2}u \hat{Q} = \text{Tr} v(z - H_0)^{-2}u - \text{Tr} v(z - H_0)^{-2}u \hat{P} \\
= -\frac{(u,v)}{4\pi z} + \frac{(u,v)}{4\pi z} \text{Tr} \hat{P} + O(|\log \varepsilon|) \\
= O(|\log \varepsilon|),
$$

where we have used (76). Thus the contribution is vanishing.

Similarly, the second term in the right-hand side of (78) is computed as

$$
\text{Tr} v(z - H_0)^{-2}u = \text{Tr} v(z - H_0)^{-2}u^{(3)} + D_4 \Gamma \Pi \hat{Q} \\
= \left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_4} \Gamma \Pi \hat{Q} \\
+ \frac{1}{8\pi} \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_4} \Gamma \Pi \hat{Q} + O(\varepsilon^{-1+\delta}).
$$

Consider first the first term in the right-hand side of (79). Clearly, the nonvanishing contributions come from the second and third terms in $\Pi$ of $B$. Therefore we treat only the following two:

$$
\left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_4} \Gamma \left[ \frac{2}{z \kappa(z)} \sum_{i,j=2,3} \varphi_0,i \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots) \right] \hat{Q}
$$

and

$$
\left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_4} \Gamma \left[ \frac{8\pi}{z} \sum_{k,\ell=1}^{N} \phi_k \hat{\Phi}_{k\ell}(\hat{\phi}_\ell, \cdots) \right] \hat{Q}.
$$

The first one is written

$$
- \frac{1}{2z} \sum_{i,j=2,3} \text{Tr} \left( v R_0^{(3)} u \frac{1}{1 + D_4} \Gamma \varphi_0,i \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots) \right) + o(\varepsilon^{-1}) \\
= - \frac{1}{2z} \sum_{i,j=2,3} \text{Tr} \left( v R_0^{(3)} u \varphi_0,i \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots) \right) + o(\varepsilon^{-1}) \\
= - \frac{1}{2z} \sum_{i,j=2,3} \Phi_{ij}(\hat{\varphi}_{0,j}, v R_0^{(3)} u \varphi_0,i) + o(\varepsilon^{-1}) \\
= \frac{2}{z} + o(\varepsilon^{-1}).
$$
Thus the contribution to the excess charge becomes 2. This is nothing but the contribution from the two resonance states.

The other one is written

\[
\frac{8\pi}{z} \left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \sum_{k,\ell=1}^{N} \text{Tr} \left\{ vR_0^{(3)} u \frac{1}{1 + D_4} \Gamma \phi_k \tilde{\Phi}_{k\ell}(\hat{\phi}_\ell, \ldots) \right\}
\]

by using the properties of the vectors \( \phi_j \). The matrix elements are calculated as

\[
\langle \hat{\phi}_\ell, vR_0^{(3)} u \frac{1}{1 + D_4} \Gamma \phi_k \rangle
\]

\[
= \langle \hat{\phi}_\ell, vR_0^{(3)} u \left[ 1 - D_4 + D_4^2 \frac{1}{1 + D_4} \right] \phi_k \rangle
\]

\[
= \langle \hat{\phi}_\ell, vR_0^{(3)} u \left[ -D_4 + D_4^2 \frac{1}{1 + D_4} \right] \phi_k \rangle
\]

\[
= -(\hat{\phi}_\ell, vR_0^{(3)} u(1 - \hat{Q}M_0\hat{Q}Q)^{-1}K_1\phi_k) + o(|\log \varepsilon|^{-1})
\]

\[
= o(|\log \varepsilon|^{-1}).
\]

This implies that the corresponding contribution is vanishing.

Next, consider the second term in the right-hand side of (79). Clearly, the nonvanishing contribution is written

\[
\frac{1}{z} \sum_{k,\ell=1}^{N} \text{Tr} vR_0^{(3)} u \frac{1}{1 + D_4} \Gamma \phi_k \tilde{\Phi}_{k\ell}(\hat{\phi}_\ell, \ldots) \hat{Q}
\]

\[
= \frac{1}{z} \sum_{k,\ell=1}^{N} \tilde{\Phi}_{k\ell}(\hat{\phi}_\ell, vR_0^{(3)} u \phi_k) + o(\varepsilon^{-1}) = \frac{N}{z} + o(\varepsilon^{-1}).
\]

This implies that the corresponding contribution to the excess charge is equal to the number \( N \) of the bound states at the zero energy.

Next, consider the case with

\[
\int d^2 x V(x) = (u, v) = 0.
\]

We write

\[
\hat{P} = v(u, \ldots).
\]

Then, we have

\[
1 - v(z - H_0)^{-1} u = 1 - \kappa(z) \hat{P} - M
\]

\[
= [1 - \kappa(z) \hat{P}] \left[ 1 - (1 + \kappa(z) \hat{P})M \right]
\]

\[
= [1 - \kappa(z) \hat{P}] \left[ 1 - M - \kappa(z) \hat{P} M \right].
\]

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Since 
\[(v, Mu) = (u, v^{-1}uv) + O(\varepsilon \log |\varepsilon|),\]
the matrix element \((u, Mv)\) is nonvanishing for a small \(\varepsilon\). Relying on this fact, we introduce 
\[\tilde{P} := (u, Mv)^{-1} \hat{P} M,\]
and 
\[\tilde{Q} := 1 - \tilde{P}.\]
These satisfy \(\tilde{P}^2 = \tilde{P}\) and \(\tilde{Q} \tilde{P} = \tilde{P} \tilde{Q} = 0\). Note that 
\[\left[1 - (u, Mv)\kappa(z)\tilde{P}\right]^{-1} = 1 - \frac{(u, Mv)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} = \tilde{Q} - \frac{\kappa^{-1}(z)}{u, Mv) - \kappa^{-1}(z)} \tilde{P}.\]
Using this relation, one has 
\[1 - v(z - H_0)^{-1} u = [1 - \kappa(z) \hat{P}] \left[1 - (u, Mv)\kappa(z)\tilde{P} - M\right]\]
\[= [1 - \kappa(z) \hat{P}] [1 - (u, Mv)\kappa(z)\tilde{P}] \times \left[1 - \tilde{Q} M + \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right]\]
\[= [1 - \kappa(z) \hat{P}] [1 - (u, Mv)\kappa(z)\tilde{P}] (1 - \tilde{Q} M \tilde{P}) \times \left[1 - \tilde{Q} M \tilde{Q} + (1 + \tilde{Q} M \tilde{P}) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right].\]
Therefore, the inverse is given by 
\[(80) \quad [1 - v(z - H_0)^{-1} u]^{-1}\]
\[= \left[1 - \tilde{Q} M \tilde{Q} + (1 + \tilde{Q} M \tilde{P}) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right]^{-1}\]
\[\times (1 + \tilde{Q} M \tilde{P}) \left[\tilde{Q} - \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} + \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right]^{-1}\]
\[= \left[1 - \tilde{Q} M \tilde{Q} + (1 + \tilde{Q} M \tilde{P}) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right]^{-1}\]
\[\times (1 + \tilde{Q} M \tilde{P}) \left[\tilde{Q} - \frac{1}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} - \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P} M\right],\]
where we have used \(\tilde{P} \tilde{P} = \tilde{P}\) and \(\tilde{Q} \tilde{P} = 0\).
We write 
\[M = M_0 + M_1 + O(\varepsilon^{1+\delta'})\]
with 
\[M_1 = -\frac{z \kappa(z)}{4} v R_0^{(3)} u + \frac{z}{8\pi} v R_0^{(3)} u - \frac{z}{8\pi} v R_0^{(3)} u.\]
We have
\[ \tilde{P} = \tilde{P}_0 + K_0 + O(\varepsilon^{1+\delta'}) \]
with \( \tilde{P}_0 = (u, M_0v)^{-1}\tilde{P}M_0 \) and
\[ K_0 = (u, M_0v)^{-1} \left[ \tilde{P}M_1 - (u, M_1v)\tilde{P}_0 \right]. \]

From these expansions,
\[ \tilde{Q}M\tilde{Q} = \tilde{Q}_0M_0\tilde{Q}_0 - K_0M_0\tilde{Q}_0 + \tilde{Q}_0M_1\tilde{Q}_0 - \tilde{Q}_0M_0K_0 + O(\varepsilon^{1+\delta'}), \]
where \( \tilde{Q}_0 = 1 - \tilde{P}_0 \). In the same way,
\[
(1 + \tilde{Q}M\tilde{P}) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P}M
= \left(1 + \tilde{Q}_0M_0\tilde{P}_0\right) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P}_0M_0
+ \left[-K_0M_0\tilde{P}_0 + \tilde{Q}_0M_1\tilde{P}_0 + \tilde{Q}_0M_0K_0\right] \frac{\kappa^{-1}(z)}{(u, M_0v)} \tilde{P}_0M_0
+ \left(1 + \tilde{Q}_0M_0\tilde{P}_0\right) \frac{\kappa^{-1}(z)}{(u, M_0v)} \left[ \tilde{P}_0M_1 + K_0M_0 \right] + O(\varepsilon|\log \varepsilon|^{-1}).
\]

From these observations, we have
\[ (81) \quad 1 - \tilde{Q}M\tilde{Q} + (1 + \tilde{Q}M\tilde{P}) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P}M = 1 - \tilde{Q}_0M_0\tilde{Q}_0 + K_1 + D_1, \]
where
\[
K_1 = K_0M_0\tilde{Q}_0 - \tilde{Q}_0M_1\tilde{Q}_0 + \tilde{Q}_0M_0K_0
+ \left(1 + \tilde{Q}_0M_0\tilde{P}_0\right) \frac{\kappa^{-1}(z)}{(u, Mv) - \kappa^{-1}(z)} \tilde{P}_0M_0
+ \left[-K_0M_0\tilde{P}_0 + \tilde{Q}_0M_1\tilde{P}_0 + \tilde{Q}_0M_0K_0\right] \frac{\kappa^{-1}(z)}{(u, M_0v)} \tilde{P}_0M_0
+ \left(1 + \tilde{Q}_0M_0\tilde{P}_0\right) \frac{\kappa^{-1}(z)}{(u, M_0v)} \left[ \tilde{P}_0M_1 + K_0M_0 \right].
\]
and the operator \( D_1 \) satisfies \( \|D_1\|_2 = O(\varepsilon|\log \varepsilon|^{-1}) \).

In order to obtain the inverse of the operator (81), we must consider the eigenvalue problem for \( \tilde{Q}M_0\tilde{Q} \).

Let \( \varphi \) be an eigenvector of the operator with the eigenvalue 1, i.e.,
\[ \tilde{Q}M_0\tilde{Q}\varphi = \varphi. \]
In the same way as in the case in one dimension, the obvious relation, \( \tilde{P}_0\varphi = 0 \), yields \((u, \varphi) = 0\) and \((u, M_0\varphi) = 0\). The corresponding left eigenvector is given by \( \varphi = (\text{sgn}\,V)M_0\varphi. \)
The resonance state is given by
\[ \psi = (u, M_0 v)^{-1}(u, M_0^2 \varphi) - \Delta^{-1} u \varphi. \]

Using the relation,
\[ \varphi = (1 - \hat{P}_0)M_0 \varphi = M_0 \varphi - (u, M_0 v)^{-1} \hat{P} M_0^2 \varphi = M_0 \varphi - (u, M_0 v)^{-1}(u, M_0^2 \varphi) v, \]
we have
\[ v \psi = (u, M_0 v)^{-1}(u, M_0^2 \varphi) v - M_0 \varphi = -\varphi. \]

Therefore,
\[ \psi = (u, M_0 v)^{-1}(u, M_0^2 \varphi) + \Delta^{-1} V \psi. \]

Immediately, \( \Delta \psi = V \psi \). Besides, the asymptotics is given by
\[ \psi \sim (u, M_0 v)^{-1}(u, M_0^2 \varphi) - \frac{x}{2\pi |x|^2} \cdot \int d^2 y \, y u(y) \varphi(y) \]
for a large \( |x| \).

Similarly to the preceding case, we consider only the case that there are three resonance states and \( N \) bound states at the zero energy.

Let \( \varphi_{0,i}, i = 1, 2, 3, \) be the three zero-energy resonance states satisfying the following conditions:
\[ (u, M_0^2 \varphi_{0,1}) \neq 0, \]
\[ (u, M_0^2 \varphi_{0,2}) = (u, M_0^2 \varphi_{0,3}) = 0, \]
and
\[ (x u, \varphi_{0,2}) \neq 0, \quad (x u, \varphi_{0,3}) \neq 0. \]

Let \( \phi_j \) be the zero-energy bound states satisfying
\[ (u, M_0^2 \phi_j) = 0, \quad \text{and} \quad (x u, \phi_j) = 0 \]
for \( j = 1, 2, \ldots, N \). Moreover, we introduce the projection operators,
\[ P_{0,i} = \varphi_{0,i}(\hat{\varphi}_{0,i}, \ldots) \quad \text{for} \quad i = 1, 2, 3, \]
and
\[ P_1 = \sum_{j=1}^{N} \phi_j(\hat{\phi}_j, \ldots), \]
where \( \hat{\varphi}_{0,i} \) and \( \hat{\phi}_j \) are the left eigenvectors. By relying on the fact that
\[ (\hat{\varphi}, \varphi) = (u \Delta^{-1} u \varphi, \varphi) < 0, \]
we have chosen the normalizations as
\[ (\hat{\varphi}_{0,i}, \varphi_{0,j}) = \delta_{ij}, \]
\[ (\hat{\phi}_k, \phi_\ell) = \delta_{k,\ell}, \]
and

\[(\hat{\varphi}_{0,i}, \phi_j) = (\hat{\varphi}_j, \varphi_{0,i}) = 0.\]

We also write \(P_0 = P_{0,1} + P_{0,2} + P_{0,3},\) \(P = P_0 + P_1\) and \(Q = 1 - P.\)

Note that

\[M_0 \varphi_{0,i} = (\tilde{P}_0 + \tilde{Q}_0)M_0 \varphi_{0,i}\]
\[= \tilde{P}_0 M_0 \varphi_{0,i} + \varphi_{0,i}\]
\[= (u, M_0 v)^{-1} v(u, M_0^2 \varphi_{0,i}) + \varphi_{0,i} = \varphi_{0,i}\]

for \(i = 2, 3.\) Similarly, we have \(M_0 \phi_j = \phi_j\) for \(j = 1, 2, \ldots, N.\) Relying on these properties, we can find two matrices, \(\Phi_{ij}\) and \(\tilde{\Phi}_{k\ell},\) to satisfy

\[\sum_{j=2,3} \Phi_{ij}(\hat{\varphi}_{0,j}, xv) \cdot (xu, \varphi_{0,k}) = \delta_{ik}\] for \(i, k = 2, 3,\)

and

\[\sum_{j=1}^N \tilde{\Phi}_{ij}(\hat{\phi}_j, v \tilde{P}_0(3) u \phi_k) = \delta_{ik}\] for \(i, k = 1, 2, \ldots, N,\)

respectively.

As an approximate inverse for the operator (81), we introduce

\[B = (1 - \Pi K_1 + \Theta)(1 - \tilde{Q}_0 M_0 \tilde{Q}_0 Q)^{-1} + \Pi,\]

where

\[\Pi = (\hat{\varphi}_{0,1}, K_1 \varphi_{0,1})^{-1} P_{0,1} - \frac{2}{z \kappa(z)} \sum_{i,j=2,3} \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j}, \cdots)\]
\[+ \frac{8\pi}{z} \sum_{k,\ell=1}^N \phi_k \tilde{\Phi}_{k\ell}(\hat{\phi}_\ell, \cdots),\]

and

\[\Theta = \frac{2\pi}{(u, M_0 v)} \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_\ell, x^2 v)(u, M_0^2 \cdots).\]

Then, we have

\[B[(1 - \tilde{Q}_0 M_0 \tilde{Q}_0 Q)Q + K_1 + D_1]\]
\[= (1 - \Pi K_1 + \Theta)Q + (1 - \Pi K_1 + \Theta)(1 - \tilde{Q}_0 M_0 \tilde{Q}_0 Q)^{-1} K_1\]
\[+ \Pi K_1 + \Pi D_1 + P \times O(\varepsilon) + o(\varepsilon)\]
\[= Q + \Pi K_1 P + \Theta Q + (1 - \Pi K_1)(1 - \tilde{Q}_0 M_0 \tilde{Q}_0 Q)^{-1} K_1 + P \times o(1) + o(\varepsilon).\]

The second term in the last line is calculated as

\[\Pi K_1 P = P + \Xi + P \times o(1),\]
where

$$
\Xi = \sum_{i,j=2,3} \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j}, xv) \cdot (xu, \varphi_{0,1})(\hat{\varphi}_{0,1}, \cdots) \\
+ \sum_{k,\ell=1}^N \phi_k \tilde{\Phi}_{k\ell}(\hat{\phi}_{\ell}, \cdots) vR_0^{(3)} uP_0
$$

Using the properties of the eigenvectors of $\tilde{Q}_0M_0\tilde{Q}_0$ with the eigenvalue 1, we have

$$
\Pi K_1 = \frac{8\pi}{z} \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_{\ell}, \cdots) \frac{2\kappa(z)}{4} vR_0^{(3)} u\tilde{Q}_0 + O(1)
$$

(82)

$$
= 2\pi \kappa(z) \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_{\ell}, x^2 v)(u, \cdots) + O(1).
$$

Therefore,

$$
\Pi K_1 (1 - \tilde{Q}_0M_0\tilde{Q}_0) K_1
$$

$$
= 2\pi \kappa(z) \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_{\ell}, x^2 v)(u, \cdots)
$$

$$
\times (1 - \tilde{Q}_0M_0\tilde{Q}_0)^{-1}(1 + \tilde{Q}_0M_0\tilde{P}_0) \frac{\kappa^{-1}(z)}{(u, M_0 v)} \tilde{P}_0M_0 + o(1)
$$

$$
= \frac{2\pi}{(u, M_0 v)} \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_{\ell}, x^2 v)(u, \cdots) M_0 \tilde{P}_0M_0 + o(1)
$$

$$
= \frac{2\pi}{(u, M_0 v)} \sum_{k,\ell=1}^N \phi_k \Phi_{k\ell}(\hat{\phi}_{\ell}, x^2 v)(u, M_0^2 \cdots)(P_{0,1} + Q) + o(1)
$$

$$
= \Theta P_{0,1} + \Theta Q + o(1).
$$

where we have used

$$
(u, \cdots)(1 - \tilde{Q}_0M_0\tilde{Q}_0)^{-1} = (u, \cdots)
$$

for getting the second equality. From these observations, we have

$$
B[(1 - \tilde{Q}_0M_0\tilde{Q}_0)Q + K_1 + D_1]
$$

$$
= 1 + \Xi - \Theta P_{0,1} + (1 - \tilde{Q}_0M_0\tilde{Q}_0)^{-1} K_1 + P \times o(1) + o(\epsilon).
$$

The desired inverse is written

$$
[(1 - \tilde{Q}_0M_0\tilde{Q}_0)Q + K_1 + D_1]^{-1} = \frac{1}{1 + D_2} \Gamma B,
$$

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where
\[ D_2 = (1 - \tilde{Q}_0 M_0 \tilde{Q} Q)^{-1} K_1 + P \times o(1) + o(\varepsilon), \]
and
\[ \Gamma = 1 - \Xi + \Theta P_{0,1} + (\Xi - \Theta P_{0,1})^2. \]

From this, (80) and (81), we obtain
\[
[1 - v(z - H_0)^{-1}]^{-1} = \frac{1}{1 + D_2} \Gamma B (1 + \tilde{Q} M \tilde{P})
\times \left[ \tilde{Q} - \frac{\dot{\tilde{P}}}{(u, M v) - \kappa^{-1}(z)} - \frac{\kappa^{-1}(z)}{(u, M v) - \kappa^{-1}(z)} \hat{\tilde{P}} \right].
\]

Therefore, we have
\[
\text{Tr} \left\{ v(z - H_0)^{-2} u [1 - v(z - H_0)^{-1}]^{-1} \right\}
= \text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B \tilde{Q} \right\}
- \frac{1}{(u, M v) - \kappa^{-1}(z)} \text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B (1 + \tilde{Q} M \tilde{P}) \right\}
- \frac{\kappa^{-1}(z)}{(u, M v) - \kappa^{-1}(z)} \text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B (1 + \tilde{Q} M \tilde{P}) \right\},
\]
where we have used \( \tilde{P} \hat{\tilde{P}} = \tilde{P} \).

Consider first the first term in the right-hand side. Substituting the expression of \( B \) into it, one has
\[
\text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B \tilde{Q} \right\}
= \text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma (1 - \Pi K_1 + \Theta)(1 - \tilde{Q}_0 M_0 \tilde{Q} Q)^{-1} \hat{\tilde{Q}} \right\}
+ \text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma \Pi \hat{\tilde{Q}} \right\}.
\]

Note that
\[
v(z - H_0)^{-2} u = - \frac{\dot{\tilde{P}}}{4\pi z} + \left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] v R_0^{(3)} u + \frac{1}{8\pi} v \tilde{R}_0^{(3)} u + O(\varepsilon'^{2}).
\]

Since \( \tilde{Q} \dot{\tilde{P}} = 0 \), one formally obtains that the first term in the right-hand side of (84) does not contribute to the excess charge. Let us justify this fact. Because of the same reason as in the previous cases, it is sufficient to show that the contribution from \( \text{Tr} v(z - H_0)^{-2} u \tilde{Q} \) is vanishing. Using the above expansion and \( (u, v) = 0 \), we have
\[
\text{Tr} v(z - H_0)^{-2} u \tilde{Q} = \text{Tr} v(z - H_0)^{-2} u - \text{Tr} v(z - H_0)^{-2} u \dot{\tilde{P}}
= 0 + \frac{1}{4\pi z} \text{Tr} \dot{\tilde{P}} \dot{\tilde{P}} + O(\|\log \varepsilon\|) = O(\|\log \varepsilon\|).
\]
Thus the statement holds.

Similarly, the second term in the right-hand side of (84) is written

\[(85)\]
\[
\text{Tr} \left\{ v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma \Pi \hat{Q} \right\} \\
= \left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \text{Tr} \left\{ v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma \Pi \hat{Q} \right\} \\
+ \frac{1}{8\pi} \text{Tr} \left\{ v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma \Pi \hat{Q} \right\} + O(\varepsilon^{-1+\delta}).
\]

As to the first term in the right-hand side, the nonvanishing contributions come from the two summations in \(\Pi\) of \(B\). The first one is computed as

\[
\left[ \frac{\kappa(z)}{4} - \frac{1}{16\pi} \right] \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma \left[ -\frac{2}{z\kappa(z)} \sum_{i,j=2,3} \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j},\cdots) \right] \hat{Q} \\
= -\frac{1}{2z} \sum_{i,j=2,3} \text{Tr} v R_0^{(3)} u \Gamma \varphi_{0,i} \Phi_{ij}(\hat{\varphi}_{0,j},\cdots) + o(\varepsilon^{-1}) \\
= -\frac{1}{2z} \sum_{i,j=2,3} \Phi_{ij}(\hat{\varphi}_{0,j}, v R_0^{(3)} u \Gamma \varphi_{0,i}) + o(\varepsilon^{-1}) = \frac{2}{z} + o(\varepsilon^{-1}).
\]

This leads to the excess charge 2. As to the other one, we have

\[
\text{Tr} v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma \sum_{k,\ell=1}^N \hat{\Phi}_{k\ell}(\hat{\phi}_{\ell},\cdots) \hat{Q} \\
= \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma \sum_{k,\ell=1}^N \phi_k \hat{\Phi}_{k\ell}(\hat{\phi}_{\ell},\cdots) + O(\varepsilon |\log \varepsilon|) \\
= \sum_{k,\ell=1}^N \hat{\Phi}_{k\ell}(\hat{\phi}_{\ell}, v R_0^{(3)} u \frac{1}{1 + D_2} \phi_k) + O(\varepsilon |\log \varepsilon|),
\]

except for the prefactor which is \(O(\varepsilon^{-1} |\log \varepsilon|)\). The matrix elements in the last line is calculated as

\[
(\hat{\phi}_{\ell}, v R_0^{(3)} u \frac{1}{1 + D_2} \phi_k) \\
= (\hat{\phi}_{\ell}, v R_0^{(3)} u \left[ 1 - D_2 + D_2^2 \frac{1}{1 + D_2} \right] \phi_k) \\
= -(\hat{\phi}_{\ell}, v R_0^{(3)} u D_2 \phi_k) + o(|\log \varepsilon|^{-1}) \\
= o(|\log \varepsilon|^{-1}),
\]

where we have used the properties of \(\phi_j\). As a result, the contribution is vanishing.
Next, consider the second term in the right-hand side of (85). Clearly, the nonvanishing contribution is

\[
\frac{1}{8\pi} \text{Tr} v \tilde{P}_0^{(3)} u \frac{1}{1 + D_2} \Gamma x \frac{1}{z} \sum_{k, \ell=1}^N \phi_k \hat{\Phi}_{k\ell}(\hat{\phi}_\ell, \cdots) \hat{Q} = \frac{1}{z} \sum_{k, \ell=1}^N \hat{\Phi}_{k\ell}(\hat{\phi}_\ell, v \tilde{R}_0^{(3)} u \phi_k) + o(\varepsilon)
\]

\[
= \frac{N}{z} + o(\varepsilon).
\]

The corresponding contribution to the excess charge is equal to the number \(N\) of the bound states at the zero energy.

Finally, we will show that the second and third terms in the right-hand side of (83) do not contribute to the excess charge. The second term is written

\[
\text{Tr} v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P} = \left[ \frac{\kappa(z)}{4} \frac{1}{16\pi} \right] \text{Tr} v R_0^{(3)} u \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P} + \frac{1}{8\pi} \text{Tr} v \tilde{R}_0^{(3)} u \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P},
\]

except for the prefactor. In the same way as in the above, one can show that these yield a vanishing contribution.

Similarly, the third terms in the right-hand side of (83) is written

\[
\text{Tr} v(z - H_0)^{-2} u \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P} = -\frac{1}{4\pi z} \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P} + o(\varepsilon^{-1})
\]

except for the prefactor which is \(O(|\log \varepsilon|^{-1})\). Therefore, it is sufficient to show that the trace of the operator in the first term in the right-hand side is \(O(1)\). The trace of the operator is computed as

\[
\text{(86)} \quad \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B(1 + \tilde{Q}M) \hat{P} = \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B + \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B \tilde{Q}M = \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B + \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B \tilde{Q}M = \text{Tr} \hat{P} \frac{1}{1 + D_2} \Gamma B + \text{Tr} \hat{P} \Gamma B \tilde{Q}M - \text{Tr} \hat{P} D_2 \frac{1}{1 + D_2} \Gamma B \tilde{Q}M.
\]
Note that
\[
\hat{B}\hat{P} = \left[(1 - \Pi K_1 + \Theta)(1 - \hat{Q}_0 M_0 \hat{Q}_0 Q)^{-1} + \Pi\right] \hat{P} \\
= (1 - \Pi K_1 + \Theta)(1 - \hat{Q}_0 M_0 \hat{Q}_0 Q)^{-1} \hat{P} \\
= (1 - \Pi K_1 + \Theta) \hat{P} = O(1),
\]
where we have used (82). This implies that the first term in the right-hand side of the second equality of (86) is $O(1)$. The second term in the right-hand side of the second equality of (86) is calculated as
\[
\text{Tr} \hat{P} \Gamma B \tilde{Q} M = \text{Tr} \hat{P} B \tilde{Q} M = \text{Tr} \hat{P} (1 - \hat{Q}_0 M_0 \hat{Q}_0 Q)^{-1} \hat{Q} M = O(1).
\]
As to the third term, we obtain
\[
(u, \cdots) D_2 = (u, \cdots) K_1 + o(\varepsilon) = O(|\log \varepsilon|^{-1}),
\]
and
\[
B \tilde{Q} M v = B[\hat{Q}_0 - K_0][M_0 + O(\varepsilon |\log \varepsilon|)]v + O(\varepsilon^\delta)
\]
\[
= B \hat{Q}_0 M_0 v - B K_0 M_0 v + O(|\log \varepsilon|)
\]
\[
= (1 - \Pi K_1 + \Theta)(1 - \hat{Q}_0 M_0 \hat{Q}_0 Q)^{-1} \hat{Q}_0 M_0 v + \Pi M_0 v + O(|\log \varepsilon|)
\]
\[
= O(|\log \varepsilon|).
\]
These imply that the third term in the right-hand side of the second equality of (86) is $O(1)$, too.

\section*{C High Energy Asymptotics in Three Dimensions}

In order to give a proof of Theorem 10, we take the contour in the expression (13) of the SSF to be a circle which is parametrized by $z = \lambda e^{i\phi}$ with a large $\lambda$ and the angle $\phi$.

To begin with, we note that
\[
\frac{1}{z - H} - \frac{1}{z - H_0} = \frac{1}{z - H} \frac{1}{z - H_0}
\]
\[
= \frac{1}{z - H_0} \frac{1}{z - H_0} + \frac{1}{z - H} \frac{1}{z - H_0} \frac{1}{z - H_0}
\]
\[
= \frac{1}{z - H_0} \frac{1}{z - H_0} + \frac{1}{z - H_0} \frac{1}{z - H_0} \frac{1}{z - H_0}
\]
\[
+ \frac{1}{z - H} \frac{1}{z - H_0} \frac{1}{z - H_0} \frac{1}{z - H_0}.
\]
For $z \notin \sigma(H_0)$, one has

\begin{align}
\text{(88)} \quad \text{Tr} \frac{1}{z - H_0} V \frac{1}{z - H_0} &= \frac{1}{(4\pi)^2} \int d^3x \int d^3y \frac{e^{i\sqrt{|z|} |x-y|}}{|x-y|} V(y) \frac{e^{i\sqrt{|z|} |y-x|}}{|y-x|} \\
&= \frac{1}{(4\pi)^2} \int d^3x \int d^3y V(y) \frac{e^{2i\sqrt{|z|} |x-y|}}{|x-y|^2} \\
&= \frac{1}{(4\pi)^2} \int d^3y V(y) \cdot 4\pi \int_0^\infty r^2 dr \frac{e^{2i\sqrt{r}}}{r^2} \\
&= -\frac{1}{8\pi i\sqrt{z}} \int d^3y V(y).
\end{align}

Combining this with

\begin{align}
\frac{1}{2\pi i} \int \frac{dz}{\sqrt{z}} &= -\frac{2}{\pi i} \lambda^{1/2},
\end{align}

one obtains

\begin{align}
\text{(89)} \quad \frac{1}{2\pi i} \int dz \text{ Tr} \frac{1}{z - H_0} V \frac{1}{z - H_0} &= -\frac{\lambda^{1/2}}{4\pi^2} \int d^3y V(y).
\end{align}

Thus the leading term have been obtained.

For $z \notin \sigma(H_0)$,

\begin{align}
\text{Tr} \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} &= \text{Tr} V \frac{1}{z - H_0} V \left( \frac{1}{z - H_0} \right)^2 \\
&= \frac{1}{(4\pi)^2} \int d^3x \int d^3y V(x) \frac{e^{i\sqrt{|z|} |x-y|}}{|x-y|} V(y) \frac{1}{2i\sqrt{z}} e^{i\sqrt{|z|} |x-y|} \\
&= \frac{1}{32\pi^2 i\sqrt{z}} \int d^3p \int d^3q V(p) \frac{e^{2i\sqrt{|q|}|}}{|q|} V(p+q),
\end{align}

where we have changed the variables to $p = x, q = y - x$ for getting the last equality. Using the polar coordinate, $q = r\omega$ with $r = |q|$ and $\omega \in S^2$, and integrating by parts, one has

\begin{align}
\int d^3q \frac{e^{2i\sqrt{|q|}|}}{|q|} V(p+q) &= \int_{S^2} d\omega \int_0^\infty dr r^2 \frac{e^{2i\sqrt{r}}}{r} V(p+r\omega) \\
&= \int_{S^2} d\omega \frac{1}{2i\sqrt{z}} e^{2i\sqrt{r}} \cdot rV(p+r\omega) |_0^\infty \\
&- \int_{S^2} d\omega \frac{1}{2i\sqrt{z}} \int_0^\infty dr e^{2i\sqrt{r}} \frac{\partial}{\partial r} rV(p+r\omega) |_0^\infty \\
&= -\left( \frac{1}{2i\sqrt{z}} \right)^2 \int_{S^2} d\omega e^{2i\sqrt{r}} \frac{\partial}{\partial r} rV(p+r\omega) |_0^\infty \\
&+ \left( \frac{1}{2i\sqrt{z}} \right)^2 \int_{S^2} d\omega \int_0^\infty dr e^{2i\sqrt{r}} \frac{\partial^2}{\partial r^2} rV(p+r\omega).
\end{align}
The first term in the right-hand side of the third equality becomes $-\pi V(p)/z$ from the assumption (11) for the potential $V$. Therefore the contribution corresponding to the total phase shift $\theta(\lambda)$ is

$$
(90) \quad \frac{1}{16\pi^2 \lambda^{1/2}} \int d^3x V(x)^2
$$

by using

$$
\frac{1}{2\pi i} \int dz z^{-3/2} = \frac{2}{\pi i \lambda^{1/2}}.
$$

Let us show that the contribution of the second term becomes $O(\lambda^{-1/2})$. It is sufficient to show that

$$
\int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_0^\infty dr \left| e^{2i\sqrt{\pi}r} \frac{\partial^2}{\partial r^2} r V(p + r\omega) \right| \sim 0
$$

for $\lambda \sim \infty$. Write

$$
f(p, q) = \frac{\partial^2}{\partial r^2} r V(p + r\omega) = 2\frac{\partial}{\partial r} V(p + r\omega) + r \frac{\partial^2}{\partial r^2} V(p + r\omega).
$$

The left-hand side of (91) can be written

$$
(92) \quad \int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_0^\infty dr \exp \left[-2\lambda^{1/2} r \sin(\phi/2)\right] |f(p, q)|
$$

$$
= \int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_0^\epsilon dr \exp \left[-2\lambda^{1/2} r \sin(\phi/2)\right] |f(p, q)|
$$

$$
+ \int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_\epsilon^a dr \exp \left[-2\lambda^{1/2} r \sin(\phi/2)\right] |f(p, q)|
$$

$$
+ \int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_a^\infty dr \exp \left[-2\lambda^{1/2} r \sin(\phi/2)\right] |f(p, q)|.
$$

Here we take $\epsilon = \lambda^{-1/2}$ and take $a$ to be some positive cutoff. Clearly the first term in the right-hand side becomes $O(\lambda^{-1/2})$. The second term can be estimated as

$$
\int_0^{2\pi} d\phi \int_{\mathbb{S}^2} d\omega \int_\epsilon^a dr \exp \left[-2\lambda^{1/2} r \sin(\phi/2)\right] |f(p, q)|
$$

$$
\leq 16\pi \sup |f(p, q)| \int_\epsilon^a dr \int_0^{\pi/2} d\phi \exp[-2\lambda^{1/2} r \sin \phi]
$$

$$
\leq 16\pi \sup |f(p, q)| \int_\epsilon^a dr \int_0^{\pi/2} d\phi \exp[-4\lambda^{1/2} r \phi/\pi]
$$

$$
\leq 4\pi^2 \sup |f(p, q)| \lambda^{-1/2} \log(a\lambda^{1/2}).
$$
The third term can be written
\[
\int_0^{2\pi} d\phi \int_{S^2} d\omega \int_0^\infty dr \exp \left( -2\lambda^{1/2} r \sin(\phi/2) \right) |f(p, q)|
\]
\[
= \int_0^{2\pi} d\phi \int d^3 q \frac{1}{|q|^2} \chi_\omega(q) \exp \left( -2\lambda^{1/2} |q| \sin(\phi/2) \right) |f(p, q)|
\]
\[
= \int_0^{2\pi} d\phi \int d^3 y \frac{\chi_\omega(x - y)}{|x - y|^2} \exp \left( -2\lambda^{1/2} |x - y| \sin(\phi/2) \right) |f(x, y - x)|.
\]

This is also vanishing as \( \lambda \to \infty \) by using the same estimate and the assumption (11) on the potential \( V \). Consequently, we obtain

\[
\frac{1}{2\pi i} \int dz \, \text{Tr} \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} = \frac{1}{16\pi^2 \lambda^{1/2}} \int d^3 x V(x)^2 + o(\lambda^{-1/2})
\]

for a large \( \lambda \). Therefore it is enough to show that the contribution from the third term in the right-hand side of the third equality of (87) gives \( o(\lambda^{-1/2}) \).

Using \( u \) and \( v \) given by (15) and the relation (16), the third term in this right-hand side is written

\[
\text{Tr} \, v(z - H_0)^{-2} u \left[ 1 + v(z - H)^{-1} u \right] v \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} u.
\]

The operator \( uR_0(\lambda \pm 0)v \) is Hilbert-Schmidt class, and the following two bounds hold:

\[ \| vR_0(z)u \| \leq \text{Const.}|z|^{-1/2} \]

and

\[ \left\| \frac{\partial}{\partial z} R_0(z)u \right\| \leq \text{Const.}|z|^{-1} \]

for a large \( |z| \). (See Theorem 8.1 of [23].) Combining the bound about the resolvent \( R_0(z) \) with the relation (17), one notices that \( v(z - H)^{-1} u \) is uniformly bounded for a large \( \lambda \). From these observations, the contribution of the third term can be estimated as \( O(\lambda^{-1}) \).
Since both of the first and the second terms in the right-hand side of (94) can be estimated in the same way, we treat only the first term. Note that

\[
\text{Tr} \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0} V \frac{1}{z - H_0}
\]

implies

\[
\omega
\]

where we have written \( \Theta = \omega \).

Following [11], we change the variables as

\[
\begin{align*}
x_1 &= x_1 \\
x_2 &= x_1 + r \omega_2 \\
x_3 &= x_2 + r \omega_3 = x_1 + r(\omega_2 + \omega_3)
\end{align*}
\]

with \( r \geq 0 \) and with the constraint \( |\omega_2|^2 + |\omega_3|^2 = 1 \). Clearly, the constraint implies \( \omega := (\omega_2, \omega_3) \in S^5 \). Then we have

\[
\int d^3x_1 \int d^3x_2 \int d^3x_3 V(x_1) e^{i\sqrt{z}|x_1-x_2|} \frac{e^{i\sqrt{z}|x_2-x_3|}}{|x_1-x_2|} V(x_2) e^{i\sqrt{z}|x_3-x_1|} V(x_3) e^{i\sqrt{z}|x_3-x_1|} = \int d^3x_1 V(x_1) \int_{\mathbb{S}^5} d\omega \frac{1}{|\omega_2||\omega_3|} \int_0^\infty dr r^3 \exp[i\sqrt{z}r \Theta] \times V(x_1 + r \omega_2) V(x_1 + r(\omega_2 + \omega_3)),
\]

where we have written \( \Theta = |\omega_2| + |\omega_3| + |\omega_2 + \omega_3| \). We write

\[
f(r) = r^3 V(x_1 + r \omega_2) V(x_1 + r(\omega_2 + \omega_3)),
\]

and evaluate the integral about \( r \) as

\[
\int_0^\infty dr e^{i\sqrt{z}r \Theta} f(r) = \frac{1}{i\sqrt{z} \Theta} e^{i\sqrt{z}r \Theta} f(r) \bigg|_0^\infty - \frac{1}{i\sqrt{z} \Theta} \int_0^\infty dr e^{i\sqrt{z}r \Theta} \frac{\partial}{\partial r} f(r) = \frac{1}{\sqrt{z} \Theta^2} \int_0^\infty dr e^{i\sqrt{z}r \Theta} \frac{\partial^2}{\partial r^2} f(r)
\]

by integrating by parts repeatedly. In the same way as in the above, one can show that the contribution to the total phase shift becomes \( o(\lambda^{-1/2}) \).
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