THE WEIGHTS OF THE TAUTOLOGICAL CLASSES OF CHARACTER VARIETIES

VIVEK SHENDE

Abstract. I calculate the weights of the tautological classes of character varieties using the functorial mixed Hodge structure on simplicial schemes.

Let $X$ be a topological space of finite type and let $G$ be a linear algebraic group over $\mathbb{C}$. We write

$$\text{Loc}_G(X) = \text{Hom} (\Pi_1(X), G)/G$$

for the stack of locally constant principal $G$-bundles on $X$. It is algebraic of finite type since $\Pi_1(X)$ is finitely generated. There is a tautological $G$-bundle on $\text{Loc}_G(X) \times X$, given by taking the tautological flat bundle and forgetting the flat structure. This induces a classifying map

$$\text{Loc}_G(X) \times X \to BG$$

Passing to cohomology, we get a map $H^*(BG, \mathbb{Q}) \to H^*(\text{Loc}_G(X), \mathbb{Q}) \otimes H^*(X, \mathbb{Q})$. Taking the Kunneth components of the image classes on the first factor give the tautological classes of $H^*(\text{Loc}_G(X), \mathbb{Q})$. Given $C \in H^*(X, \mathbb{Q})^G$ and $\xi \in H^*(BG, \mathbb{Q})$, we write $f_{C, \xi}$ for the corresponding tautological class; of course in the case of usual interest when $X$ is a manifold, $C$ can be taken to be a cycle and the integral is an integral.

Since $\text{Loc}_G(X)$ is algebraic, its cohomology carries a mixed Hodge structure [D1, D2, D3]. That is, there is an increasing weight filtration $W_\ast$ and a decreasing Hodge filtration $F^\ast$ satisfying some axioms. We are particularly interested in the Hodge classes

$$k \text{Hdg}^n(Z) = F^k H^n(Z, \mathbb{C}) \cap F^{>k} H^n(Z, \mathbb{C}) \cap W_{2k} H^n(Z, \mathbb{Q})$$

Any maps on cohomology induced by algebraic maps preserve both filtrations, hence in particular $k \text{Hdg}^\ast$.

Deligne showed $H^\ast(BG, \mathbb{Q})$ carries a pure Hodge structure, and moreover $H^\ast(BG, \mathbb{Q}) = \bigoplus k \text{Hdg}^{2k}(BG)$ [D3]. However, the classifying map above is not algebraic in the second factor, and so need not preserve Hodge structures. In fact, it does not: for $G = \text{PGL}_r$, and $\Sigma$ a closed orientable 2-manifold, $\xi = c_k(T)$ the Chern class of the tautological bundle, Hausel and Rodriguez-Villegas showed [HRV]:

\begin{align*}
(0) \int_{\text{point}} c_k(T) & \in k \text{Hdg}^{2k} (\text{Loc}_{\text{PGL}_r}(\Sigma)) \\
(1) \int_{\text{curve}} c_k(T) & \in k \text{Hdg}^{2k-1} (\text{Loc}_{\text{PGL}_r}(\Sigma)) \\
(2) \int_{\Sigma} c_k(T) & \in k \text{Hdg}^{2k-2} (\text{Loc}_{\text{PGL}_r}(\Sigma))
\end{align*}

They moreover showed for $k = 2$ and conjectured in general that:

These classes are of particular interest in the case of $\text{PGL}_r$ bundles over a surface because, by a result of Markman – proven across the nonabelian Hodge correspondence on the Dolbeault model of the space of local systems, i.e., the moduli space of Higgs bundles – the tautological classes generate the cohomology [M]. Thus (0), (1), (2) together imply that in fact the mixed Hodge structure on $H^\ast(\text{Loc}_{\text{PGL}_r}(\Sigma), \mathbb{Q})$ is a direct sum of shifted pure Hodge structures of Tate type. Moreover, $H^\ast(\text{Loc}_{\text{PGL}_r}(\Sigma), \mathbb{Q}) = \bigoplus k \text{Hdg}^n (\text{Loc}_{\text{PGL}_r}(\Sigma))$ is a doubly graded ring – in particular, the relations are of homogenous weight. A clearer understanding of the generators may also shed light on the “curious Poincaré duality” and “curious Hard Lefschetz” conjectures of [HRV] and the “$P = W$” conjecture of [dCHM].

The purpose of this note is to prove (2), and in fact the analogous result for any $X, G, \xi, C$:

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Theorem. For $C \in \mathcal{H}^*(X, \mathbb{Q})$ and $\xi \in H^{2k}(BG, \mathbb{Q})$, we have

$$\int_C \xi \in k\text{Hdg}^{2k-1}(Loc_G(X))$$

Proof. Let $\Delta_X$ be a simplicial set with geometric realization homotopic to $X$. View it as a constant simplicial scheme.

There is an algebraic morphism $\text{Hom}_{\text{schemes}}(\Delta_X, BG) \xrightarrow{\sim} Loc_G(X)$, where the LHS is the internal hom in the category of simplicial schemes. It is a homotopy equivalence.

The identity map on the simplicial scheme $\text{Hom}_{\text{schemes}}(\Delta_X, BG)$ gives, under the usual adjunction, an algebraic evaluation map of simplicial schemes

$$\text{Hom}_{\text{schemes}}(\Delta_X, BG) \times \Delta_X \rightarrow BG$$

Recall that simplicial schemes carry mixed Hodge structures, functorial with respect to algebraic maps [D3]. Thus the induced map

$$H^*(BG, \mathbb{Q}) \rightarrow H^*(\text{Hom}_{\text{schemes}}(\Delta_X, BG), \mathbb{Q}) \otimes H^*(\Delta_X, \mathbb{Q})$$

respects the mixed Hodge structure. $H^*(\Delta_X, \mathbb{Q})$ is $H^*(X, \mathbb{Q})$ as a graded vector space, but carries the mixed Hodge structure in which everything has weight zero. Thus $\int_C$ decreases the cohomological degree by $\deg C$ but does not change the Hodge degrees. \(\square\)

Remark. A similar statement about Hodge degrees holds with $BG$ replaced by any simplicial scheme. However the fact that the mapping simplicial scheme $\text{Hom}_{\text{schemes}}(\Delta_X, BG)$ is a 1-stack is special to $BG$.

While I have made no use here of derived geometry, I learned to view the space of local systems as a mapping stack from $[\text{HAG2}, \text{PTVV}]$.

Appendix: review of simplicial terminology. The above constitutes a complete argument, but for those still seeking homotopical enlightenment, we recall the meanings of some of the words. Everything which follows is standard material, for which there are many references; we found [C] especially helpful.

We write $\Delta$ for the simplex category – its objects are the nonempty finite totally ordered sets, and its morphisms are order preserving maps. Evidently the objects are each uniquely isomorphic to some $[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$.

The maps are generated by the $n+1$ “include a face” maps $[n-1] \rightarrow [n]$ and the $n$ “degenerate an edge” maps $[n] \rightarrow [n-1]$, subject to some relations. That is, the category looks like this:

$$[0] \xleftarrow{\delta^0} [1] \xrightarrow{\delta^0} [2] \cdots$$

A “simplicial object” in a category $\mathcal{C}$ is just a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$. That is, it looks like this:

$$K_0 \xrightarrow{\delta^0} K_1 \xrightarrow{\delta^0} K_2 \cdots$$

The example which most concerns us here is the simplicial set (or space, or scheme) $BG$.

$$BG : pt \xrightarrow{pt} G \xleftarrow{pt} G \times G \cdots$$

That is, $BG_n = G^n$, the left-going maps are given by inclusion into a factor, and the right-going maps are projection or multiplication. For instance, the two maps $G \rightarrow G \times G$ are $g \mapsto (g, 1)$ and $g \mapsto (1, g)$, and the three maps $G \times G \rightarrow G$ are $(g, h) \mapsto g$, $(g, h) \mapsto gh$, and $(g, h) \mapsto h$.

A Cech cover $\{U_\alpha\}$ of a space $X$ determines a simplicial set $\mathfrak{U}_X$ by taking

$$\mathfrak{U}_{X,i} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \bigcap U_{\alpha_i} \neq \emptyset\}$$

The face and degeneracy maps are given by omitting and doubling indices.

A map of simplicial sets $E : \mathfrak{U}_X \rightarrow BG$, in degree one, corresponds to a specification of an element of $G$ for every double overlap $U_\alpha \cap U_\beta$. When $\alpha = \beta$, this element must be the identity, for compatibility with the degeneracy map from degree zero. In degree two, we should give an element $(g, h) \in G \times G$ for each triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$. The face maps assert that, in this case, $g$ should be the element assigned to $U_\alpha \cap U_\beta$, that $h$ should be the element assigned to $U_\beta \cap U_\gamma$, and $gh$ should be the element assigned to $U_\alpha \cap U_\gamma$. It turns out that associativity of $G$ then determines all higher morphisms. In other words, such a map determines a locally trivial
G-bundle on $X$, trivialized on each of the $U_{\alpha}$. If each $U_{\alpha}$ is small enough that all locally trivial $G$-bundles are trivial, this gives a bijection:

$$\text{Hom}_{\text{sets}}(\mathcal{U}_X, BG) \leftrightarrow \{\text{locally trivial } G\text{-bundles on } X, \text{ trivialized on each of the } U_{\alpha}\}$$

To more appropriately parameterize $G$ bundles, it is best to forget the trivialization. In fact, this happens automatically by promoting the LHS from the ordinary Hom, which is a set, to the internal Hom, which is a simplicial set. To describe this, let $\Delta_n$ be the "simplicial $n$-simplex", i.e., the functor on $\Delta$ given by

$$\Delta_n(\cdot) = \text{Hom}(\cdot, [n])$$

By Yoneda, this object has the feature that $\text{Hom}_{\text{sets}}(\Delta_n, X) = X([n]) = X_n$. This leads to the definition of a simplicial set $\text{Hom}_{\text{sets}}(X, Y)$ whose $n$-simplices are

$$\text{Hom}_{\text{sets}}(X, Y)_n = \text{Hom}_{\text{sets}}(\Delta_n, \text{Hom}_{\text{sets}}(X, Y)) = \text{Hom}_{\text{sets}}(\Delta_n \times X, Y)$$

The product of simplicial sets is the usual product of functors, i.e., $(X \times Y)_n = X_n \times Y_n$, and similarly on maps. Chasing a (fairly large) diagram shows that $\text{Hom}_{\text{sets}}(\mathcal{U}_X, BG)_1$ is the set of triples $(E, F, \phi)$ where $E, F \in \text{Hom}_{\text{sets}}(\mathcal{U}_X, BG)_1$, and $\phi$ is a map $\mathcal{U}_X \rightarrow G$ so that changing the trivialization accordingly carries $E$ to $F$. The two maps to $\text{Hom}_{\text{sets}}(\mathcal{U}_X, BG)_0$ are just the restrictions to $E$ and $F$. That is,

$$\text{Loc}_G(X) : \text{Hom}_{\text{sets}}(\mathcal{U}_X, BG)_0 \leftarrow \text{Hom}_{\text{sets}}(\mathcal{U}_X, BG)_1$$

is a usual presentation of the groupoid of local systems.

If we view $\mathcal{U}_X$ as a simplicial scheme by just viewing each element of $\mathcal{U}_X,i$ as a copy of $\text{Spec } \mathbb{C}$, we find the usual presentation of the (1-)stack of local systems:

$$\text{Loc}_G(X) : \text{Hom}_{\text{setschemes}}(\mathcal{U}_X, BG)_0 \leftarrow \text{Hom}_{\text{setschemes}}(\mathcal{U}_X, BG)_1$$

In fact, $\text{Hom}_{\text{setschemes}}(\Delta_X, BG)$ has vanishing higher homotopy groups, so mapping to the 1-truncation gives

$$\text{Hom}_{\text{setschemes}}(\Delta_X, BG) \cong \text{Loc}_G(X)$$

From a modern point of view this is rather the definition of $\text{Loc}_G(X)$, and then a calculation of homotopy groups reveals that it is a 1-stack rather than a higher stack. For a derived version of this (not needed here, but which specializes to the underived calculation), see [HAG2, Lemma 2.2.6.3].

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