A fixed-point action for the lattice Schwinger model

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We determine non-perturbatively a fixed-point (FP) action for fermions in the two-dimensional U(1) gauge (Schwinger) model. Our parameterization for the fermionic action has terms within a $7 \times 7$ square on the lattice, using compact link variables. With the Wilson fermion action as starting point we determine the FP-action by iterating a block spin transformation (BST) with a blocking factor of 2 in the background of non-compact gauge field configurations sampled according to the (perfect) Gaussian measure. We simulate the model at various values of $\beta$ and find excellent improvement for the studied observables.

1. INTRODUCTION

We are interested in a lattice representation of the continuum action for the 2D U(1) gauge theory with massless fermions \cite{2} (the Schwinger model). The lattice action should respect the basic symmetries and should have the correct (naive) classical limit. Also one has to take care for the fermion doubling problem. Beyond these requirements the form of the lattice action is largely arbitrary.

The traditional lattice actions are ultra-local but have corrections $O(a^2)$ (of the lattice spacing constant) for bosons and $O(a)$ for fermions. Improved actions have smaller corrections in powers $O(a^n)$ but introduce more terms. As long as the contributions are exponentially damped with regard to their extension in real space one calls the action local. Optimally, an improved action has no such corrections and thus no corrections to the leading critical behaviour. Such actions have been called “perfect”. More terms complicate the simulation and one has to find a compromise between efficiency and perfectness.

Various approaches to improvements are reviewed in \cite{2,3}. We follow the path inspired by scale transformations leading to FP-actions \cite{4}. We determine an optimal fermion action in the background of gauge field configurations sampled according to their (optimal) Gaussian measure. The resulting FP-action in this approximation defines a classically perfect action and for large $\beta$ we expect that it is a good approximation for the renormalized trajectory. In this limit the gauge field acts like a background field for the fermionic sector \cite{5}, and the fermionic action stays quadratic in the fermionic field variables. Further details may be found in \cite{6}. FP-actions for that model were also studied with the method of small fields \cite{7} and recently in a perturbative expansion \cite{8}.

2. DETERMINATION OF THE FP-ACTION

We denote the lattice action by
\begin{equation}
\beta S(A) - S_F(\bar{\Psi}, \Psi, A), \quad S_F = \bar{\Psi} M(A) \Psi,
\end{equation}
where $S(A)$ denotes the gauge field part, $M$ the lattice Dirac operator matrix, and $\beta = 1/g^2$ is the gauge field coupling.

We block from a so-called fine square lattice with sites $x \in Z_N \times Z_N$ to a coarse lattice organizing the fine lattice in $2 \times 2$ blocks which constitute the points $x'$ of the coarse lattice. The Grassmann fields are $\bar{\Psi}(x), \Psi(x)$ (respectively $\bar{\Psi}'(x'), \Psi'(x')$ on the coarse lattice); the real, non-compact gauge fields $A_\mu(x)$ live on the links. For the fermions we use anti-periodic boundary conditions and for the gauge field periodic ones.
The BST is defined as
\[ e^{-\beta S'(A') + S'(\bar{\Psi}', \Psi', A')} + c = \int \mathcal{D}f \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \] 
\[ \times e^{-\beta (S(A) + T(A, A')) + S(\bar{\Psi}, \Psi, A) + T(\bar{\Psi}, \bar{\Psi}', \Psi')} \]  
(2)

We fix the gauge within each block to the so-called fine gauge, where the plaquette field strength is distributed equally among the four link variables. We integrate only over gauge field configurations in the fine gauge, defining the path integral measure \( \mathcal{D}f \mathcal{D}A \).

The kernel of the BST for the fermions was taken from [2] with parameters suitable chosen in order to have maximum locality in the situation of free fermions. For the kernel of the gauge field we define an average over the four 2-link connections between corresponding sites in adjacent blocks. More details are discussed in [3]. The resulting action on the coarse lattice is gauge invariant, hermitian invariant, invariant under the charge conjugation and respects the lattice symmetry. It does violate chiral symmetry.

There exists a unique minimizing configuration of \( S(A) + T(A, A') \) which we denote by \( A_{\text{min}}(A') \). In our case it can be computed straightforwardly by solving a set of linear equations. For \( \beta \to \infty \) this saddle point \( A_{\text{min}}(A') \) dominates the path integral. The Grassmann integration with subsequent identification of all bilinear terms in the fermionic variables defines the fermionic block action and the FP equations (in the limit \( \beta \to \infty \)). In order to use the action also at moderate \( \beta \) values, we have to calculate the blocked action for strongly fluctuating configurations \( A' \), too.

**Gauge field FP-action:**
For our BST the ultra-local standard (non-compact) plaquette action is a FP, up to the wave function renormalization [4]. In \( d = 2 \) this action is \( S_p(A) = \frac{1}{4} \sum_x F(x)^2 \).

**Fermion FP-action:**
We parameterize \( S'(\bar{\Psi}', \Psi', A') \) with a finite number of coupling constants,
\[ S_F(\bar{\Psi}, \Psi, A) = \bar{\Psi} M_F(A) \Psi = \sum_{i=0}^3 \sum_{x,f} \rho_i(f) \bar{\Psi}(x) \sigma_i U(x, f) \Psi(x + \delta f) \]  
(3)

Here \( M_F(A) \) is the parameterized fermion matrix, \( f \) denotes a closed loop through \( x \) or a path from the lattice site \( x \) to \( x + \delta f \) (distance vector \( \delta f \)) and \( U(x, f) \) is the parallel transporter along this path. The \( \sigma_i \)-matrices denote the Pauli matrices for \( i = 1, 2, 3 \) and the unit matrix for \( i = 0 \).

The conditions \( \sum_f \rho_1(f)(\delta f)_1 = 1 \) and \( \sum_f \rho_0(f) = 0 \) guarantee, that \( \rho \) is normalized and \( S_F \) reproduces the action of the massless model in the naive continuum limit (\( (\delta f)_1 \) denotes the component in the 1-direction). We first considered terms connecting the central site \( x \) with any other site \( x + \delta f \) in a \( 7 \times 7 \) lattice. Invariances of the action under certain symmetries provide further reductions. In the iteration process we found that one may omit several of the original terms. Altogether we finally considered 33 different geometric shapes corresponding to 123 independent coupling constants [5].

The FP-action is determined in an iteration procedure, starting from the Wilson action for the massless Schwinger model (\( \kappa = 1/4 \)).

- We generate 50 gauge field configurations \( A' \) on the coarse \( 7 \times 7 \) lattice according to their probability distribution \( e^{-\beta S_F(A')} \). For each of these we then find the minimizing configuration \( A_{\text{min}}(A') \).
- With \( A_{\text{min}}(A') \) we construct the fermion matrix on the fine lattice and perform the BST (Grassmann integral) giving the \( (2 \cdot 7^2) \times (2 \cdot 7^2) \) fermion matrix \( M_{\text{BST}}(A_{\text{min}}) \) on the coarse lattice. This is done for all 50

![Figure 1](image-url). Logarithmic plot for the absolute values of the couplings \( |\rho_0| \) vs. their lattice extension.
gauge field configurations.

- The resulting fermion matrices are compared with the fermion matrices $M_F(A')$ for the coarse lattice. A new set of parameters according to (3) is determined by minimizing

$$\sum_{A'} \| M_{BST}(A_{\text{min}}(A')) - M_F(A') \|^2,$$

(4)

for the matrix norm $\| M \|^2 \equiv \sum_{i,j} |M_{ij}|^2$.

These steps are iterated until (typically after 10 iterations) the couplings remain stable within statistical fluctuations. We worked at $\beta' = 20$.

Part of the observed (small) fluctuations in the couplings may be due to cancellations of certain terms in the fermionic action (redundancies). In fig. 1 we demonstrate the locality of our FP-action. Our values are comparable or smaller than those obtained for the free fermion perfect action [9]. The complete set of couplings may be retrieved from [9].

3. TESTS FOR THE FP-ACTION

For our check we rely on direct simulations with the FP-action on lattices of size $16 \times 16$ (for details cf. [9]). At each value of $\beta$ considered we generated 10000 gauge field configurations with appropriate measure and performed the Grassmann integrals explicitly, i.e. by computing the corresponding determinant and inverse fermion matrix. Thus we obtain results for both situations, the 1-flavour and the 2-flavour model. For the error estimates we repeated the procedure several times. We studied propagators of various “mesonic” operators bilinear in the fermion fields. Comparing the propagators with analogous simulations for Wilson-fermions we find significantly improved rotational invariance, demonstrated both in real space and for the energy-momentum dispersion relations.

In the 2-flavour model one expects one massive mode and a massless flavour-triplet (fig. 2). We find non-vanishing $\pi$-masses only at small $\beta$, indicating deviation of our FP-action from the renormalized trajectory, i.e. a small signal of “imperfectness” to be expected. However, the overall scaling behaviour predicted from theory (for the 2-flavour model), $a(\beta) m_\eta = \sqrt{2/\pi \beta}$, is nicely recovered for moderately large values of $\beta > 3$.

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