On real center singularities of complex vector fields on surfaces

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ABSTRACT

One of the various versions of the classical Lyapunov-Poincaré center theorem states that a nondegenerate real analytic center type planar vector field singularity admits an analytic first integral. In a more proof of this result, R. Moussu establishes important connection between this result and the theory of singularities of holomorphic foliations [R. Moussu, Une démonstration géométrique d’un théorème de Lyapunov-Poincaré, Astérisque 98–99 (1982), pp. 216–223]. In this paper we consider generalizations for two main frameworks: (i) planar real analytic vector fields with ‘many’ periodic orbits near the singularity and (ii) germs of holomorphic foliations having a suitable singularity in dimension two. In this paper we prove versions of Poincaré-Lyapunov center theorem, including for the case of holomorphic vector fields. We also give some applications, hinting that there is much more to be explored in this framework.

1. Introduction and main results

Let $X$ be a smooth vector field defined in an open set $U \subset \mathbb{R}^2$. An isolated singularity $p \in \text{sing}(X) \subset U$ of $X$ is a center if there is a neighborhood $p \in V \subset U$ such that $V \setminus \{p\}$ consists only of periodic orbits of $X$. For sake of simplicity, in the case of local study, we may assume that $p = O \in U \subset \mathbb{R}^2$. If the origin is a center we say that it is a nondegenerate center if the linear part $DX(O) \in \text{Lin}(\mathbb{R}^2, \mathbb{R}^2)$ is nonsingular. The classical Poincaré-Lyapunov center theorem reads as follows:

**Theorem 1.1 (Poincaré-Lyapunov [7,10]):** If $X$ is a real analytic vector field having a nondegenerate center at the origin $O \in \mathbb{R}^2$ then $X$ admits a real analytic first integral of Morse type in a neighborhood of the origin.

There are some equivalent statements in terms of differential one-forms ([6]). A quite geometrical proof was given by Moussu ([9]). In his paper he makes use of the complexification of the 1-form, obtaining therefore a holomorphic 1-form with a suitable singularity at the origin $O \in \mathbb{C}^2$. To this complex 1-form it is applied Mattei-Moussu theorem ([8]) which assures the existence of a holomorphic first integral near the singular point ([8], Theorem B).
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The authors prove if each point $p$ of a normal space $\omega$ immersed connected submanifold $V$ is a multicenter at the origin. Given two germs of holomorphic function $g$ and $f$, the hypotheses on the existence of a neighborhood consisting only of period orbits. In order to compensate this loss we require a resonance condition on the linear part of the vector field, as follows. Let $X$ be an analytic vector field in a neighborhood $U$ of the origin $O \in \mathbb{R}^{2n}$ and such that $X(O) = O$. According to Ref. [2] we say that the linear part $DX(O)$ of $X$ generates a multirotation if all the orbits of the one-parameter group $\exp(tDX(O)), t \in \mathbb{R}$, outside the origin of $\mathbb{R}^{2n}$, are periodic and nontrivial. This corresponds to the fact that the eigenvalues of $DX(O)$ are purely imaginary of the form $\pm i\omega_j$ where $\omega_j \in \mathbb{R} \setminus \{0\}, j = 1, \ldots, n$ and $\omega_k/\omega_j \in \mathbb{Q}, j, k = 1, \ldots, n$ where $i^2 = -1$. In particular $DX(O)$ is nonsingular. The singular point is called a multicenter if there is a neighborhood $U$ of the origin such that all orbits in $U$ are periodic. In this case we have a period function $T: U \to (0, +\infty)$. In Ref. [14] the authors prove if $X$ is an analytic vector field in a neighborhood $U$ of $O \in \mathbb{R}^{2n}$, with a multicenter at the origin, if $DX(O)$ generates a multirotation and if the period function $T$ is bounded in some punctured neighborhood $U \setminus \{O\}$ of the origin then $X$ is $C^\infty$ orbitally-equivalent to its linear part $DX(O)$ in some neighborhood of the origin (cf. This result was improved by Brunella-Villarini as follows ([2] Theorem (2.4), page 157): If $X$ is analytic in some neighborhood of the origin $O = X(O) \in \mathbb{R}^{2n}$, with $DX(O)$ generating a multirotation then $X$ has a multicenter at the origin provided that (i) there is a sequence of periodic orbits $\gamma_v \subset U \setminus \{O\}$ such that $\gamma_v \to O$ as $v \to \infty$ (in the sense of Hausdorff topology) and (ii) the sequence of periods $T(\gamma_v)$ of the orbits $(\gamma_v)$ is bounded.

In this paper we address some variants of the above situation, for real analytic vector fields in dimension two and for integrable analytic (real and complex) one-forms in dimension $n \geq 2$. Before stating our main results we must introduce some useful notions relating the real analytic and the complex holomorphic frameworks.

### 1.1. Complex analytic foliations and real singularities

In what follows, by a germ of a holomorphic foliation at the origin $O \in \mathbb{C}^2$ we shall mean a germ of a holomorphic foliation by curves, with an isolated singularity at the origin $O \in \mathbb{C}^2$.

We recall that a submanifold $V$ of a complex surface $M$ is called totally real if the complex structure $J: TM \to TM$ of $M$ maps each tangent space $T_pV \subset T_pM$ of $V$ into the normal space $(T_pV)^\perp \subset T_pM$. We refer to [1] for a detailed exposition about totally real manifolds.

Two irreducible and reduced germs of holomorphic function $f, g \in \mathcal{O}_2$ with $f(O) = g(O) = 0$ are in general position if the analytic curves $(f = 0)$ and $(g = 0)$ meet transversely at the origin. Given two germs of holomorphic function $f, g: \mathbb{C}^2 \to \mathbb{C}$ in general position and vanishing at $O \in \mathbb{C}^2$ then the intersection $V^2 = (\text{Re}(f) = \text{Re}(g)) \cap (\text{Im}(f) = -\text{Im}(g))$ is a germ of a totally real surface at the origin $O \in \mathbb{C}^2$.

Given a real foliation $\mathcal{F}$ of codimension $k$ in a differentiable manifold $M$ and an immersed connected submanifold $V \subset M$, the contact order of $\mathcal{F}$ with $V$ at a point $p \in V$ is the dimension of the intersection $T_p(V) \cap T_p(\mathcal{F}) \subset T_p(M)$ as linear subspaces of the tangent space $T_p(M)$. We say that $\mathcal{F}$ has contact order $r$ with $V$ if their contact order is $r$ at each point $p \in V$. In the case where $\mathcal{F}$ is a holomorphic foliation of (complex) codimension
one in an open subset $U \subset \mathbb{C}^2$ with $\text{sing}(\mathcal{F}) = \{O\} \subset U$, and $V^2 \subset U$ is a real surface, we have:

- $V^2$ is transverse to $\mathcal{F}$ off the origin iff $V^2 \setminus \{O\}$ and $\mathcal{F}$ have contact order equal to zero.
- $V^2$ is $\mathcal{F}$ invariant iff $V^2 \setminus \{O\}$ and $\mathcal{F}$ have contact order equal to 2.
- $V^2 \setminus \{O\}$ has contact order with $\mathcal{F}$ equal to 1 iff $V^2 \setminus \{O\}$ is a totally real submanifold not invariant by $\mathcal{F}$.

For our next results it is better to state a definition:

**Definition 1.2 (real singularity):** Let $\mathcal{F}$ be a one-dimension holomorphic foliation with singularities on a Stein surface $N^2$. An isolated singularity $p \in \text{sing}(\mathcal{F}) \subset N$ is said to be real if there is a germ of $p$ of an analytic dimension two totally real submanifold $V^2 \subset N^2$, having contact order one with $\mathcal{F}$ outside of $p \in V$. Since $V^2 \subset N^2$ is totally real, we conclude that for some neighborhood $U \subset N^2$ of $p$, the restriction $\mathcal{F}|_U$ is the corresponding complexification of the foliation $\mathcal{F}|_{V^2}$ with respect to $V^2$. We shall refer to the pair $(\mathcal{F}|_V, V)$ (or sometimes just to the foliation $\mathcal{F}|_V$) as a real model of (the germ of) $\mathcal{F}$ at $p$. The submanifold $V^2$ is then called real section of the real singularity. The definitions above can be easily extended to higher dimension. Let $\omega$ be an integrable holomorphic one-form in a complex manifold $N^n$. A singularity $p \in \text{sing}(\omega)$ where the singular set of $\omega$ has codimension $\geq 2$ will be called real if: there is a germ of totally real submanifold $V^n \subset N^n$ with $p \in V$, having contact order $n - 1$ with $\mathcal{F}$ and such that the foliation $\omega = 0$ coincides in a neighborhood of $p$ in $N^n$ with the complexification of the foliation given by the restriction $\omega|_{V^n}$. This foliation $\omega|_{V^n}$ will be called the real model of $\omega$ at $p$ with respect to $V^n$.

Our first result then reads as follows:

**Theorem A:** Let $\mathcal{F}$ be a germ of a (Siegel resonant type) holomorphic foliation at the origin $O \in \mathbb{C}^2$ given by $\omega = 0$ where $\omega = x \, dy + y \, dx + \tilde{\omega}$, where $\tilde{\omega}$ is holomorphic and has order $\geq 2$ at the origin. Assume that the origin is a real singularity for $\mathcal{F}$ and that $\mathcal{F}$ admits a sequence of closed leaves $L_v \subset U$ in a neighborhood $U$ of the origin $O \in \mathbb{C}^2$ such that $L_v \to O$ as $v \to \infty$. Then $\mathcal{F}$ admits a holomorphic first integral of the form $\tilde{x}\tilde{y}$ for some suitable holomorphic coordinates $(\tilde{x}, \tilde{y})$ at $O \in \mathbb{C}^2$.

Conversely, if $\mathcal{F} : \omega = x \, dy + y \, dx + \tilde{\omega} = 0$ admits such a holomorphic first integral of type $\tilde{x}\tilde{y}$ as above then the origin is a real singularity with a quadratic center as real model.

As a corollary of the proof we obtain:

**Corollary 1.3:** Let $\mathcal{F}$ be a germ of a (Siegel resonant type) holomorphic foliation at the origin $O \in \mathbb{C}^2$ given by $\omega = 0$ where $\omega = x \, dy + y \, dx + \tilde{\omega}$, where $\tilde{\omega}$ has jet of order one equal to zero. Assume that:

(i) The origin is a real singularity.
(ii) The corresponding real model admits a sequence of compact leaves converging to the origin.
Then $\mathcal{F}$ admits a holomorphic first integral of the form $fg$ for irreducible germs $f, g \in \mathcal{O}_2$ in general position at $O$.

Now we consider higher dimensional versions of the above statements:

**Theorem B:** Let $\omega = d(fg) + fg\tilde{\omega}$ be an integrable holomorphic one-form defined in a neighborhood $U$ of the origin $O \in \mathbb{C}^n$, $n \geq 2$, where $\tilde{\omega}$ and $f, g : U \to \mathbb{C}$ are holomorphic, vanish at $O$ and $\tilde{f}, \tilde{g}$ intersect transversely at the origin. Assume that:

(i) The foliation $\mathcal{F} : \omega = 0$ in $U$ admits a sequence of closed leaves $L_\nu \subset U$ such that $L_\nu \to O \in \mathbb{C}^n$ as $\nu \to \infty$ in the sense of Hausdorff topology.

(ii) There is a germ of a totally real analytic submanifold $V^n \subset \mathbb{C}^n$ with $O \in V^n$, with respect to which $\mathcal{F}$ is the complexification.

Then $\mathcal{F}$ admits a holomorphic first integral of the form $\tilde{f}\tilde{g}$ for irreducible germs $\tilde{f}, \tilde{g} \in \mathcal{O}_2$ in general position at $O$.

Similarly to what happens to Theorem A above, we have a variant of the complex case.

**Corollary 1.4:** Let $\omega = d(x_1^2 + x_2^2) + (x_1^2 + x_2^2)\tilde{\omega}$, be an integrable real analytic one-form defined in a neighborhood $U$ of the origin $O \in \mathbb{R}^n$, $n \geq 2$, in coordinates $(x_1, x_2, \ldots, x_n)$ where $\tilde{\omega}$ is analytic and vanishes at the origin. Assume that the real analytic foliation $\mathcal{F} : \omega = 0$ in $U$ admits a sequence of closed leaves $L_\nu \subset U$ such that $L_\nu \to O \in \mathbb{R}^n$ as $\nu \to \infty$ in the sense of Hausdorff topology. Then $\mathcal{F}$ admits a real analytic first integral of the form $(\tilde{x}_1)^2 + (\tilde{x}_2)^2$ in suitable coordinates $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ at $O \in \mathbb{R}^n$.

Now we obtain an application of our result to the case of complete vector fields. Recall that a complex vector field is complete if its local flow is actually defined for all complex time ([5,12]). In this case the vector field is induced by a holomorphic action of the complex additive group $(\mathbb{C}, +)$. Holomorphic flows on Stein surfaces exhibit a quite rich list of properties (see the work of M. Suzuki ([12,13]). In our case, our techniques apply to the case such a flow admits a real singularity of center type.

**Theorem C:** Let $Z$ be a complete holomorphic vector field on a Stein surface $N^2$. Assume that there is a real singularity $p \in \text{sing}(Z)$ such that the corresponding real model is a nondegenerate analytic center. Then $Z$ has a periodic flow, and admits a holomorphic first integral $f : N \to \mathbb{C}$.

**Remark 1.5:** (i) The conclusion of Theorem C remains valid if instead of assuming (a) we assume that: (a)$'$ $Z$ is of the form $Z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + \text{h. o. t.}$ in a neighborhood of $p$ and the real model admits a sequence of compact leaves $L_\nu \subset V^2 \setminus \{O\}$ such that $L_\nu \to p$ as $\nu \to \infty$.

(ii) The case of a complete holomorphic vector field $Z$ in a Stein surface, having a holomorphic first integral and a linearizable singularity of type $xy = \text{constant}$ has been described in Ref. [5] in full details.
2. Basic lemmas

Let us first state a few lemmas we shall need. First we recall that given a topological space $X$, a point $p \in X$ and $h: U \to h(U) \subset X$ a homeomorphism between $U$ and $h(U)$ open subsets of $X$, such that $h(p) = p$, we can define the pseudo-orbit of a point $q \in U$ as the set of all possible iterates $h^n(q) \in U$, $n \in \mathbb{Z}$. We shall say that the pseudo-orbit of $q \in U$ is closed in $U$ if its a closed subset of $U$ in the classical sense of topology. This means either of the following. There are only finitely many possible iterates of $q$ or if any point $z \in U$ which is a limit of a sequence of iterates $z = \lim_{k \to \infty} h^k(q)$ of some point $q \in U$, with $k_j \in \mathbb{N}$ and $\lim_{k \to \infty} k_j = \infty$ then $z$ belongs to the pseudo-orbit. We shall say that the orbit of $p$ is periodic (or, that $p$ is a periodic point) of period $k \geq 1$, with respect to $h$, if $f^\ell(p) = p$, $\forall \ell \in \mathbb{N}$, $f^k(p) = p$ and $f^\ell(p) \neq p$, $\forall \ell = 0, \ldots, p - 1$. Using representatives we shall state similar notions for germs of homeomorphisms with a fixed point. For the case of a complex diffeomorphism map germ we have:

**Lemma 2.1:** Let $f \in \Diff(\mathbb{C}, 0)$ be a germ of a holomorphic diffeomorphism such that:

1. $f'(0) \in \mathbb{C}$ is a root of the unit.
2. There is a sequence of points $p_\nu \neq 0$ such that $p_\nu$ is periodic with respect to $f$ and $p_\nu \to 0$ as $\nu \to \infty$.

Then $f$ has finite order, i.e. $f^k = \Id$ for some $k \in \mathbb{N} \setminus \{0\}$.

**Proof:** Since $f$ has a periodic linear part, there is a smaller positive integer $k \in \mathbb{N} \setminus \{0\}$ such that $f^k(z) = z + (\cdots)$. If $f$ does not have finite order then we must have $f^k(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \cdots$ for some $a_{k+1} \neq 0$ and $\ell > 0$. By replacing $f$ by $f^k$ we may then assume that $f$ is tangent to the identity $f''(0) = 1$ but $f \neq \Id$. Using now Camacho theorem ([3]) we conclude that no point $p \neq 0$ has a periodic orbit, contradiction. $lacksquare$

An extension of the above lemma, with a similar proof is:

**Lemma 2.2:** Let $f \in \Diff(\mathbb{C}, 0)$ be a germ of a holomorphic diffeomorphism such that:

1. $f'(0) \in \mathbb{C}$ is a root of the unit.
2. There is a sequence of points $p_\nu \neq 0$ such that the pseudo-orbit of $p_\nu$ is finite with uniformly bounded order, with respect to $f$, and $p_\nu \to 0$ as $\nu \to \infty$.

Then $f$ has finite order, i.e. $f^k = \Id$ for some $k \in \mathbb{N} \setminus \{0\}$.

**Proof:** As in the preceding proof we may assume that $f(z) = z + a_{k+1}z^{k+1} + \text{h. o. t.}$, $a_{k+1} \neq 0$ and look for a contradiction. The topological description of the orbits of $f$ ([3]) then shows that the closer to the origin, the more different iterates a point will have and, given any $k \in \mathbb{N}$, and no matter which neighborhood $0 \in V \subset \mathbb{C}$ we choose, there is a small closed disk $0 \in D \subset V$ centered at the origin, such that for every point $p \in D \setminus \{0\}$ the orbit of $p$ has order at least $k^2$. This gives the desired contradiction. $lacksquare$
3. The real analytic case and complexification of foliations

We shall first recall some classical facts. Let $X$ be a smooth vector field in a real surface $V^2$. We say that a point $p \in V^2$ is an elliptic point for $X$ if it is a singular point with complex eigenvalues. In this case, as it is well-known, there are two possibilities: the singularity is either a focus, or it is a center-type point. An elliptic singular point is non-degenerate. Let $\Sigma' \simeq [0, e')$, be a smooth interval embedded in $V^2$ with the end point $O$ at $p$ and transversal to $X$ off $p$. Then we have:

Lemma 3.1 (R. Roussarie [11] Lemma 7 page 52): Let $\epsilon \in (0, e')$ with $e' > 0$ small enough and put $\Sigma \simeq [0, \epsilon)$. Then the return map for $X$, $h: \Sigma \rightarrow \Sigma'$ is well defined. This map, extended by $h(O) = 0$, is analytic if $X$ is analytic.

Once we have the above result, we may state the following definition:

Definition 3.2: Given a vector field $X$ in a neighborhood $U$ of the origin $O \in \mathbb{R}^2$ we shall say that $X$ admits a transverse segment if there is a continuous injective map $\phi: [0, \epsilon) \rightarrow \mathbb{R}^2$ such that: (i) $\phi(0) = O$; (ii) $\phi|_{(0, \epsilon)}: (0, \epsilon) \rightarrow \mathbb{R}^2$ is a smooth immersion and (iii) $\phi 2$ is transverse to $X$ at any point off the origin. In this case we shall simply say that $\Sigma = \phi(0, \epsilon)$ is a transverse segment to $X$. Given a point $p \in \Sigma$ it has a bounded order by $k \in \mathbb{N}$ if the corresponding trajectory $\gamma_p$ of $X$ satisfies $\sharp(\gamma_p \cap \Sigma) \leq k$.

This is clearly the case when we have a center type singularity. There are other examples: take a cusp singularity $x^2 - y^3 = cte$ and the hamiltonian $X = 3y^2 \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}$ and consider the vertical axes.

Let $\mathcal{F}$ be a real analytic codimension one foliation with singularities in a neighborhood of the origin $0 \in \mathbb{R}^n$. This means that $\mathcal{F}$ is defined by a real analytic 1-form $\omega = \sum_{j=1}^n a_j(x) \, dx_j$, defined in a neighborhood of the origin, and satisfying the integrability condition $\omega \wedge d\omega = 0$. We consider the complexification of $\mathcal{F}$ which we denote by $\mathcal{F}_\mathbb{C}$. This is a codimension one holomorphic foliation with singularity, defined in a neighborhood of the origin $O \in \mathbb{C}^n$ by the complexification $\omega_C$ of the form $\omega$. In complex coordinates $(z_1, \ldots, z_n)$ we can write $z_j = x_j + iy_j$ and $\omega_C = d(\sum_{j=1}^n z_j^2) + \omega_C$ for some 1-form $\omega_C$ with zero first jet at the origin. Now we consider the real space $\mathbb{R}^n \subset \mathbb{C}^n$ given by $y_j = 0, j = 1, \ldots, n$.

The next result is a well-known easy to prove lemma:

Lemma 3.3: Let $\mathcal{F}$ be a real analytic foliation in a neighborhood of the origin $0 \in \mathbb{R}^n$ whose complexification $\mathcal{F}_\mathbb{C}$ admits a holomorphic first integral. Then $\mathcal{F}$ admits a real analytic first integral, defined in some neighborhood of the origin. Indeed, there is a real analytic first integral $f$ for $\mathcal{F}$ such that the complexification $f_\mathbb{C}$ off $f$ is a holomorphic first integral for $\mathcal{F}_\mathbb{C}$.

The main point is the following pair of results:

Proposition 3.4: Let $X$ be a real analytic vector field in a neighborhood $U$ of the origin $O \in \mathbb{R}^2$, such that $X(O) = O$ and $DX(O)$ generates a rotation. Assume also that there exists a sequence of periodic orbits $\gamma_\nu \subset U \setminus \{O\}$ of $X$ converging to $O$ (in the classical sense of
Hausdorff topology). Then the complexification $X_C$ of $X$ admits a holomorphic first integral. Indeed, the origin is a quadratic center type singularity for $X$.

**Proof:** The complexification $X_C$ of $X$ is a complex analytic vector field defined in a neighborhood of the origin $O \in \mathbb{C}^2$. As we have remarked before, we may assume that the linear part $DX_C(O)$ has eigenvalues given by $\pm i$. We may therefore find suitable complex coordinates $(x, y) \in V \subset \mathbb{C}^2$ such that $X_C = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + X_2$ where $X_2$ has a zero order one jet at the origin. Then $X_C$ generates a holomorphic foliation $\mathcal{F}_C$ with an isolated Siegel type singularity at the origin, of the form $y \, dx + x \, dy + \ldots = 0$. The germ of foliation $\mathcal{F}_C$ is in the Siegel domain and we may assume that the coordinate axes are invariant ([8]). In this case the quadratic blow-up $\pi: \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ at the origin $O \in \mathbb{C}^2$, induces a foliation $\tilde{\mathcal{F}} := (\mathcal{F}_C)^*$ in a neighborhood of the exceptional divisor $\pi^{-1}(O) = E$ in the blow-up space $\mathbb{C}^2_0$. The foliation $\tilde{\mathcal{F}}$ leaves invariant the exceptional divisor $E \simeq \mathbb{CP}(1)$ and has exactly two singularities, the north and south poles, in $E$, both of Siegel resonant type (indeed, these singularities and the structure of the restriction of $\tilde{\mathcal{F}}$ to $E$ are determined by the linear part $ny \, dx + mx \, dy = 0 \text{ of } \mathcal{F}$). Let us study this a bit more thoroughly. Given complex coordinates $(x, y) \in \mathbb{C}^2$, we consider the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$ as given by $\text{Im}(x) = \text{Im}(y) = 0$. The inverse image of the real plane in the blow-up $\mathbb{C}^2_0$ corresponds to a Moebius band $\mathcal{M}^2 \subset \tilde{\mathbb{C}}^2$, intersecting the exceptional divisor transversely at the equator $E$ of the exceptional divisor $E$. The pull-back foliation $\tilde{\mathcal{F}}$ in $\mathbb{C}^2_0$ leaves invariant this Moebius band. Now we consider the projective holonomy group of the exceptional divisor $E$. This means the holonomy group ([4]) of the leaf $E \setminus \text{sing}(\tilde{\mathcal{F}})$ for the foliation $\tilde{\mathcal{F}}$. From what we have seen above, this foliation has exactly two singularities in $E$, corresponding to the north and south poles of $E$. Thus the holonomy group above mentioned is generated by a simple loop around the equator, i.e. this is a cyclic group. Let us denote by $h$ a generator of this group obtained as follows. Choose a point $p \in E$ and a local transverse disc $\Sigma$ to $E$ centered at $p$. Then denote by $H: (\Sigma, p) \to (\Sigma, p)$ the holonomy map corresponding to the equator $\gamma = \mathcal{M}^2 \cap E$. Notice that, since $E$ is invariant by $\tilde{\mathcal{F}}$, the equator $\Sigma$ corresponds to a compact leaf (periodic orbit) of the induced foliation in $\mathcal{M}^2$. A transverse open segment $\sigma \simeq (-1, 1) \subset \mathcal{M}^2$, transverse to $E$ at a base point $p \in E$ will then induce an associate first return map (Poincaré map) $h: (\sigma, p) \to (\sigma, p)$ corresponding to the periodic orbit $E$. Moreover, each closed (i.e. periodic) orbit $\gamma$ of $X$ in $\mathbb{R}^2$ lifts into a closed compact curve $\tilde{\gamma}$ in the Moebius band $\mathcal{M}^2$. This closed curve $\tilde{\gamma}$ then corresponds to a periodic orbit for the holonomy map $h$, this orbit consisting of only two elements, except for the case of $\gamma = E$ which gives the fixed point of $h$. Thus we have a germ of a real analytic map $h \in \text{Diff}^w(\mathcal{E}, p) \simeq \text{Diff}^w(\mathbb{R}, 0)$. This map has a sequence of periodic points (of period 2) accumulating at the origin. By the identity principle, $h$ must be periodic of period 2, i.e. $h^2 = \text{Id}$. This already implies that the $\tilde{\mathcal{F}}$-holonomy map $H$ admits a real analytic curve $\gamma \cap \Sigma$ where its orbits are periodic of period $\leq 2$. Since $\gamma$ contains the origin, $H$ is a periodic map of period 2. This implies, by standard methods described in Ref. [9] and from Mattei-Moussu’s theorem ([8] page 473) the foliation $\mathcal{F}_C$ admits a holomorphic first integral in a neighborhood of the origin $O \in \mathbb{C}^2$. From Lemma 3.3 we conclude that the vector field $X$ admits an analytic first integral. Let us denote by $f: U, O \to \mathbb{R}, 0$ an analytic first integral of $X$. This means that $X(f) = O$, i.e. $f$ is constant on each orbit of $X$ in $V$. Thanks to the linear part of $X$ we
may assume that \( f(x_1, x_2) = x_1^2 + x_2^2 + \) higher order terms and thanks to Morse lemma we conclude that the origin is a center singularity for \( X \).

Using the notation and construction above we have:

**Lemma 3.5:** Let \( X \) be a real analytic vector field in a neighborhood \( U \) of the origin \( O \in \mathbb{R}^2 \), such that \( X(O) = O \) and \( DX(O) \) is a nonsingular matrix. Assume also that there exists a sequence of periodic orbits \( \gamma_v \subset U \setminus \{O\} \) of \( X \) converging to \( O \) (in the classical sense of Hausdorff topology). Then \( DX(O) \) generates a rotation.

**Proof:** We first observe that the complexification \( X_C \) of \( X \) is a complex analytic vector field defined in a neighborhood of the origin \( O \in \mathbb{C}^2 \). This vector field has a nondegenerate singularity at the origin. After a suitable change of coordinates in \((\mathbb{C}^2, O)\) we may assume that, up to multiplication by a unit, \( X_C(x, y) = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y} + h \) o.t.. Such a change of coordinates does not necessarily preserve the original real plane, and we do not need that. Let \( q_1 \) and \( q_2 \) have periodic holonomy maps of order two. Since these singularities are nondegenerate. By the hypothesis in the real picture, there is an invariant surface \( N^2 \) diffeomorphic to the Moebius band, intersecting the exceptional divisor in a closed simple curve \( N^2 \cap \mathbb{E} = \gamma \) isotopic to the equator of \( \mathbb{E} \) and such that each connected component of \( \mathbb{E} \setminus \gamma \) contains one of the two singularities above mentioned. Again the holonomy map of the curve \( \gamma \) identifies to a holomorphic diffeomorphism \( h: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0) \), leaving invariant a germ of real analytic curve \( C \) diffeomorphic to \((−1, 1)\), and having a sequence of order two periodic points \( q_v \in C \) such that \( q_v \rightarrow 0 \in C \) as \( v \rightarrow \infty \). By the identity principle (applied to the holomorphic map \( h \)) we conclude that \( h \) is periodic of period 2. Now we observe that the homotopy group of \( \mathbb{E} \setminus \{q_1, q_2\} \) is generated by the loop \( C \) and therefore we may assume that the local holonomy maps of the singularities \( q_1 \) and \( q_2 \) are given by \( h \) and \( h^{-1} \). This implies that the singularities \( q_1 \) and \( q_2 \) have periodic holonomy maps of order two. Since these singularities are nondegenerate this implies that the germs of foliations admit local holomorphic first integrals and as in the preceding periodic case, there is a holomorphic first integral for the foliation in a neighborhood of the origin in \( \mathbb{C}^2 \).

Combining the above results we obtain our next result. Though similar to the classical ones, for dimension 2, we have no restrictions on the period function and we do not require that the singularity is a center. We ask for a sequence of periodic orbits converging to the singularity, which is assumed to be nondegenerate.

**Proposition 3.6:** Let \( X \) be an analytic vector field in a neighborhood \( U \) of the origin \( O \in \mathbb{R}^2 \). Assume that \( X(O) = O \) and \( DX(O) \) is nonsingular. Then the origin is a center singularity for \( X \) provided that there is a sequence of periodic orbits \( \gamma_v \subset U \setminus \{O\}, v \in \mathbb{N} \), of \( X \) such that \( \gamma_v \rightarrow O \) as \( v \rightarrow \infty \) (in the sense of the Hausdorff topology). In this case, \( X \) admits an analytic first integral in the strong sense in a neighborhood of the origin.
We recall that the nondegeneracy condition cannot be dropped even in the case of two-dimension centers ([9] Remark (iii) page 217).

**Proof of Proposition 3.6:** According to Lemma 3.5 and Proposition 3.4 the origin is a center singularity. Evoking then Lyapunov-Poincaré theorem (Theorem 1.1) we conclude that \( \mathcal{F} \) admits a real analytic first integral.

\[ \blacksquare \]

In the course of the proof of Proposition 3.6 we have indeed obtained:

**Proposition 3.7:** Let \( X \) be an analytic vector field in a neighborhood \( U \) of the origin \( O \in \mathbb{R}^2 \). Assume that \( X(O) = O \) and \( DX(O) \) is nondegenerate. Assume that there is a sequence of points \( \{p_\nu\}_{\nu \in \mathbb{N}} \subset U \setminus \{O\} \) converging to the origin such that \( p_\nu \) has uniformly bounded order by some \( k \in \mathbb{N} \), with respect to some transverse segment \( \Sigma \). Then \( X \) has a center type singularity at the origin. In particular, \( X \) admits a real analytic first integral in a neighborhood of the origin.

**Proof of Proposition 3.7:** We proceed similarly to Proposition 3.6. Indeed, Lemma 2.1 is now replaced by Lemma 2.2 in order to, with the use of an easy adaptation of Lemma 3.5 and Proposition 3.4, conclude that the singularity is a center. The remaining part follows as usual.

\[ \blacksquare \]

4. Holomorphic foliations: proof of Theorems A and C

Let us now consider the case of complex foliations. We shall first prove Theorem A.

**Proof of Theorem A:** Let us start with the final part (the converse) part. We proceed as in Ref. [6]. Assume that \( \mathcal{F} \) admits a holomorphic first integral of the form \( fg \) with \( f, g \in \mathcal{O} \), \( f(O) = g(O) = 0 \), \( f \) and \( g \) (being germs reduced and irreducible and) in general position.

We consider the analytic varieties of real codimension one \( R : (\text{Re} f = \text{Re} g) \subset \mathbb{R}^4 \) and \( I : (\text{Im} f = -\text{Im} g) \subset \mathbb{R}^4 \). Since \( f \) and \( g \) are in general position the intersection \( R \cap I = V^2 \) is a two-dimensional analytic submanifold. Also \( O \in V^2 \) because \( f \) and \( g \) vanish at the origin. Let us now put \( X = \frac{f + g}{2} \) and \( Y = \frac{f - g}{2i} \). Then \( f = X + iY \) and \( g = X - iY \) and therefore \( fg = X^2 + Y^2 \). Moreover, in the submanifold \( V^2 \) we have \( X = \text{Re}(f) = \text{Re}(g) \) and \( Y = \text{Im}(f) = -\text{Im}(g) \) so that, restricted to \( V^2 \) we have \( fg = ||f||^2 = ||g||^2 \). This shows that the restriction to \( V^2 \) of the foliation \( \mathcal{F} \) is a real analytic foliation by curves which are closed.

In particular, the contact order of \( \mathcal{F} \) with \( V^2 \) is one. Indeed the restriction \( \mathcal{F}|_{V^2} \) gives an analytic center type singularity at the origin \( O \in V^2 \). Finally, since \( \mathcal{F} \) is holomorphic and has contact order equal to one with \( V^2 \) it follows that \( V^2 \) is a totally real submanifold. This proves the converse part in Theorem A.

Let us now prove the first part of the statement. By hypothesis the holomorphic foliation \( \mathcal{F} \) is defined in a neighborhood of the origin \( O \in \mathbb{C}^2 \) by a 1-form \( \omega = d(xy) + \tilde{\omega} \) where \( \tilde{\omega} \) has zero jet of order one at the origin. Since it is well-known that a Siegel type singularity admits exactly two separatrices and these are transverse, we shall rewrite \( \omega = x \; dy + y \; dx + xy \omega_1 \) for some holomorphic one-form \( \omega_1 \). This makes the coordinate axes as these separatrices.
We are also assuming that there is a germ of a totally real analytic submanifold \( V^2 \subset \mathbb{C}^2 \) having contact order one with \( F \) and such that the restriction of \( F \) to \( V^2 \) has a nondegenerate linear part singularity at the origin in \( V^2 \) which is the limit \( p = \lim_{v \to \infty} L_v \) of a sequence of compact leaves \( L_v \) of the restriction \( F|_V \). Up to an analytic change of coordinates in \( \mathbb{C}^2 \) we may then assume, without changing the linear part \( x \, dy + y \, dx \) of \( \omega \), that \( V^2 \subset \mathbb{C}^2 \) corresponds to the totally real space \( \mathbb{R}^2 \subset \mathbb{C}^2 \), i.e. in suitable local coordinates \((x,y) \in \mathbb{C}^2 \) we have \( V^2 : (\text{Im}(x) = \text{Im}(y) = 0) \).

Now we claim:

**Claim 4.1:** In these coordinates \((x,y) , F \) is the complexification of a real analytic foliation \( F_{\mathbb{R}} \) which has a center type singularity at the origin \( O \in \mathbb{R}^2 \).

**Proof of Claim.** \( F \) has contact order one with the real space \( \mathbb{R}^2 \subset \mathbb{C}^2 \) and its restriction to this space exhibits a nondegenerate linear part singularity at the origin \( O \in \mathbb{R}^2 \). Recall that the real space above is given by \( \text{Im}(x) = \text{Im}(y) = 0 \) where \((x,y) \in \mathbb{C}^2 \) are affine coordinates in \( \mathbb{C}^2 \). Moreover, still by hypothesis, the foliation \( F^\ast \) in \( \mathcal{M}^2 \) admits a sequence of leaves \( L_v \), \( v \in \mathbb{N} \) such that each \( L_v \) is compact and \( L_v \to \gamma \) as \( v \to \infty \). By Proposition 3.4 this real foliation has a center singularity at the origin, this singularity admitting an analytic first integral. Also the complex foliation \( F \) admits a holomorphic first integral.

As we have seen above, \( F \) admits a holomorphic first integral. It is not difficult to use the linear part of \( F \) to conclude that this first integral is of the form \( fg \) where \( f, g \in \mathcal{O}_2 \) are irreducible and reduced and, up to reordering \( f \) and \( g \), we must have \( x|f \) and \( y|g \) in \( \mathcal{O}_2 \).

The proof of Theorem B is now based on the Extension Lemma in Ref. [8] page 507.

**Proof of Theorem B:** We consider any imbedding \( \varphi : (\mathbb{C}^2, O) \to (\mathbb{C}^n, O) \) which is general position with respect to \( F \) (in the sense of Ref. [8] page 507, i.e. if \( \text{codim} \text{Sing} \varphi^\ast(\omega) \geq 2 \)). The pull-back foliation \( \mathcal{F}^* = \varphi^\ast(F) \) is then given by a holomorphic one-form \( \omega^\ast = x \, dy + y \, dx + \omega \) as in Theorem A. Indeed, since \( V^n \subset \mathbb{C}^n \) is totally real, the induced foliation germ \( \mathcal{F}^\ast \) at \( O \in \mathbb{C}^2 \) admits a real singularity having as real section the intersection \( V^2_{\varphi} := \varphi^{-1}(V^n \cap \varphi(\mathbb{C}^2, O)) = \varphi^{-1}(V^n) \subset (\mathbb{C}^n, O) \). The fact that \( \mathcal{F} \) admits a sequence of closed leaves \( L_v \) in \( U \) with \( L_v \to O \) as \( v \to \infty \) in the Hausdorff topology, implies that \( \mathcal{F}^\ast \) satisfies the second hypothesis of Theorem A. By Theorem A \( \mathcal{F}^\ast \) admits a holomorphic first integral of type \( f \tilde{g} \) with \( \tilde{f}, \tilde{g} \in \mathcal{O}_2 \) in general position. Since the imbedding \( \varphi : (\mathbb{C}^2, O) \to (\mathbb{C}^n, O) \) is in general position with respect to \( F \), this implies by Ref. [8] (Theorem 1 page 507) that \( \mathcal{F} \) admits a holomorphic first integral. It is not difficult to conclude that \( \mathcal{F} \) admits a first integral of same type than \( \mathcal{F}^\ast \), i.e. of the form \( FG \) with \( F, G \in \mathcal{O}_n \) in general position.

Finally, we present the proof of Theorem C as a consequence of combining our results with the work of M. Suzuki:

**Proof of Theorem C:** First we recall that the main reference for holomorphic flows on Stein spaces is the work of M. Suzuki ([12,13]). In particular, from Suzuki’s work we know that there is a typical orbit for the vector field in the following sense: there is a zero logarithmic capacity subset \( \sigma \subset N \) such that \( \sigma \) is invariant, and every orbit of \( Z \) off \( \sigma \) is
diffeomorphic to $\mathbb{R}$ where $\mathbb{R}$ is a Riemann surface belonging to the following list $\{\mathbb{C}, \mathbb{C}^*\}$. Moreover, still according to Suzuki, in case the typical orbit is diffeomorphic to $\mathbb{C}^*$ the foliation $\mathcal{F}(Z)$ admits a meromorphic first integral $f: N^2 \to \overline{\mathbb{C}}$.

Now we observe that, from our hypotheses and from the proof of Theorem A, the real model induces a complexification that admits a holomorphic first integral in a neighborhood of $p$ which is a singularity of the form $x \, dy + y \, dx + h \, o. \, t. = 0$ in suitable holomorphic coordinates $(x, y) \in U \subset N^2$, centered at $p$. This singularity admits a holomorphic first integral and it is therefore analytically linearizable. This implies by Ref. [5] that the flow is periodic, having therefore typical orbit diffeomorphic to $\mathbb{C}^*$ and, by Suzuki, it admits a meromorphic first integral $f: N^2 \to \overline{\mathbb{C}}$. ■

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**References**

[1] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild, *Real Submanifolds in Complex Space and Their Mappings*, Princeton Mathematical Series; Vol. 47, Princeton University Press, Princeton, NJ, 1999.

[2] M. Brunella and M. Villarini, On the poincaré-Lyapunov center theorem, *Bol. Soc. Mat. Mex.* (3) 5 (1999), pp. 155–161.

[3] C. Camacho, On the local structure of conformal maps and holomorphic vector fields in $\mathbb{C}^2$, *Astérisque* 59(60) (1978), pp. 83–94. Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978, 3), Soc. Math. France, Paris.

[4] C. Camacho and Lins Neto A., *Geometric theory of foliations*, Translated from the Portuguese by Sue E. Goodman. Birkhauser Boston, Inc., Boston, MA 1985.

[5] C. Camacho and B. Scárdua, Nondicritical $\mathbb{C}^*$-actions on two-dimensional Stein manifolds, *Manuscripta Math.* 129(1) (2009), pp. 91–98.

[6] V. León and B. Scárdua, On a theorem of lyapunov-Poincaré in higher dimensions, *Arnold Math. J.* 7(4) (2021), pp. 561–571. https://doi.org/10.1007/s40598-021-00183-x.

[7] A. Lyapunov, Étude d’un cas particulier du problème de la stabilité du mouvement, *Mat. Sb.* 17 (1893), pp. 252–333. (Russe).

[8] J.F. Mattei and R. Moussu, Holonomie et intégrales premières, *Ann. Sci. École Norm. Sup.* 13(4) (1980), pp. 469–523.

[9] R. Moussu, Une démonstration géométrique d’un théorème de Lyapunov-Poincaré, *Astérisque* 98(99) (1982), pp. 216–223. tome.

[10] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle (I), *J. Math. Pures Appl.* 7 (1881), pp. 375–422. 3e série, tome.

[11] Roussarie Robert H., *Bifurcation of Planar Vector Fields and Hilbert’s Sixteenth Problem Volume 164 de Progress in Inflammation Research*, Progress in mathematics; Vol. 164, Ed. Birkhäuser, 1998. ISSN 0743-1643.
[12] M. Suzuki, Sur les opérations holomorphes de \( \mathbb{C} \) et de \( \mathbb{C}^* \) sur un espace de Stein, *Séminaire Norguet Springer Lect. Notes* 670 (1977), pp. 80–88.

[13] M. Suzuki, Sur les opérations holomorphes du groupe additif complexe sur l’espace de deux variables complexes, *Ann. Sci. Éc. Norm. Sup.* 10(4) (1977), pp. 517–546.

[14] Urabe Minoru and Sibuya Yasutaka, On center of higher dimensions, *J. Sci. Hiroshima Univ. Ser. A* 19(1) (1955), pp. 87–100.