On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus

Salvador Addas Zanata

Department of Mathematics
Princeton University
Fine Hall-Washington Road, Princeton, NJ 08544-1000, USA

Abstract

We prove that for a large and important class of $C^1$ twist maps of the torus periodic and quasi-periodic orbits of a new type exist, provided that there are no rotational invariant circles (R.I.C’s). These orbits have a non-zero "vertical rotation number" (V.R.N.), in contrast to what happens to Birkhoff periodic orbits and Aubry-Mather sets. The V.R.N. is rational for a periodic orbit and irrational for a quasi-periodic. We also prove that the existence of an orbit with a $V.R.N = a > 0$, implies the existence of orbits with $V.R.N = b$, for all $0 < b < a$. In this way, related to a generalized definition of rotation number, we characterize all kinds of periodic and quasi-periodic orbits a twist map of the torus can have. And as a consequence of the previous results we obtain that a twist map of the torus with no R.I.C’s has positive topological entropy, which is a very classical result. In the end of the paper we present some examples, like the Standard map, such that our results apply.

Key words: twist maps, rotational invariant circles, topological methods, vertical rotation number, Nielsen-Thurston theory

E-mail: szanata@math.princeton.edu

supported by CNPq, grant number: 200564/00-5 (part of this work was done while the author was under support by FAPESP, grant number: 96/08981-3)
1 Introduction and statements of the principal results

Twist maps are $C^1$ diffeomorphisms of the cylinder (or annulus, torus) onto itself that have the following property: The angular component of the image of a point increases as the radial component of the point increases (more precise definitions will be given below). Such maps were first studied in connection with the three body problem by Poincaré and later it was found that first return maps for many problems in Hamiltonian dynamics are actually twist maps. Although they have been extensively studied, there are still many open questions about their dynamics. A great progress has been achieved in the nearly integrable case, by means of KAM theory (see [23]) and many important results have been proved in the general case, concerning the existence of periodic and quasi-periodic orbits (Aubry-Mather sets), see [19], [3], [14]. In this work a result is proved associating the non-existence of rotational invariant circles to the appearance of periodic and quasi-periodic orbits of a new type for an important class of twist maps of the torus.

Notation and definitions:

0) Let $(\phi, I)$ denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where $\phi$ is defined modulo 1. Let $(\tilde{\phi}, \tilde{I})$ denote the coordinates for the universal cover of the cylinder, $\mathbb{R}^2$. For all maps $\tilde{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ we define

$$(\phi', I') = \tilde{f}(\phi, I) \\
(\tilde{\phi}', \tilde{I}') = f(\tilde{\phi}, \tilde{I})$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $\tilde{f}$.

1) $D_1^1(\mathbb{R}^2) = \{ f : \mathbb{R}^2 \to \mathbb{R}^2 / f \text{ is a } C^1 \text{ diffeomorphism of the plane, } \tilde{f}'(\tilde{\phi}, \tilde{I}) \to \pm \infty \tilde{\phi} > 0 \text{ (twist to the right), } \tilde{f}'(\tilde{\phi}, \tilde{I}) \to \pm \infty \tilde{\phi} > 0 \text{ and } f \text{ is the lift of a } C^1 \text{ diffeomorphism } \tilde{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} \}$.

2) $Diff_1^{1}(S^1 \times \mathbb{R}) = \{ \tilde{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} / \tilde{f} \text{ is induced by an element of }$
\[ D^1(\mathbb{R}^2) \].

3) Let \( p_1 : \mathbb{R}^2 \to \mathbb{R} \) and \( p_2 : \mathbb{R}^2 \to \mathbb{R} \) be the standard projections, respectively in the \( \tilde{\phi} \) and \( \tilde{I} \) coordinates \((p_1(\tilde{\phi}, \tilde{I}) = \tilde{\phi} \) and \( p_2(\tilde{\phi}, \tilde{I}) = \tilde{I} \). We also use \( p_1 \) and \( p_2 \) for the standard projections of the cylinder.

4) Given a measure \( \mu \) on the cylinder that is positive on open sets, absolutely continuous with respect to the Lebesgue measure and a map \( \hat{T} \in \text{Diff}^1(S^1 \times \mathbb{R}) \) we say that \( \hat{T} \) is \( \mu \)-exact if \( \mu \) is invariant under \( \hat{T} \) and for any open set \( A \) homeomorphic to the cylinder we have:

\[ \mu(\hat{T}(A) \setminus A) = \mu(A \setminus \hat{T}(A)) \]  

For an area-preserving map \( \hat{T} \in \text{Diff}^1(S^1 \times \mathbb{R}) \), there is a simple criteria to know if it is exact or not. \( \hat{T} \) is exact if and only if its generating function \( h(\tilde{\phi}, \tilde{\phi}') \), defined on \( \mathbb{R}^2 \), satisfies \( h(\tilde{\phi} + 1, \tilde{\phi}' + 1) = h(\tilde{\phi}, \tilde{\phi}') \) (see [20]).

5) Let \( TQ \subset D^1(\mathbb{R}^2) \) be such that for all \( T \in TQ \) we have:

\[
T : \begin{cases}
\tilde{\phi}' = T_\phi(\tilde{\phi}, \tilde{I}) \\
\tilde{I}' = T_I(\tilde{\phi}, \tilde{I})
\end{cases}, \text{ with } \partial_\tilde{\phi}' \tilde{\phi} = \partial_I T_\phi(\tilde{\phi}, \tilde{I}) > 0 \text{ and:}
\]

\[
\begin{align*}
T_\phi(\tilde{\phi}, \tilde{I} + 1) &= T_\phi(\tilde{\phi}, \tilde{I}) \\
T_I(\tilde{\phi} + 1, \tilde{I}) &= T_I(\tilde{\phi}, \tilde{I}) \\
T_\phi(\tilde{\phi} + 1, \tilde{I} + 1) &= T_\phi(\tilde{\phi}, \tilde{I} + 1) \\
T_I(\tilde{\phi}, \tilde{I} + 1) &= T_I(\tilde{\phi}, \tilde{I}) + 1 \\
T_\phi(\tilde{\phi}, \tilde{I} + 1) &= T_\phi(\tilde{\phi}, \tilde{I}) + 1
\end{align*}
\]

(2)

Every \( T \in TQ \) induces a map \( \hat{T} \in \text{Diff}^1(S^1 \times \mathbb{R}) \) and a map \( \hat{T} : T^2 \to T^2 \), where \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) is the 2-torus. Let \( p : \mathbb{R}^2 \to T^2 \) be the associated covering map.

6) Given \( T \in TQ \), we say that \( \beta \in ]0, \pi/2[ \) is a uniform angle of deviation for \( T \) if

\[
DT \bigg|_x \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in C_I(\beta) \text{ and } DT^{-1} \bigg|_z \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in C_{II}(\beta),
\]

for all \( x, z \in \mathbb{R}^2 \), where \( C_I(\beta) \) and \( C_{II}(\beta) \) are the angular sectors:

\[
C_I(\beta) = \{ (\phi, I) \in \mathbb{R}^2 : \phi > 0 \text{ and } -\cotan(\beta) \cdot \phi \leq I \leq \cotan(\beta) \cdot \phi \}
\]

\[
C_{II}(\beta) = \{ (\phi, I) \in \mathbb{R}^2 : \phi < 0 \text{ and } \cotan(\beta) \cdot \phi \leq I \leq -\cotan(\beta) \cdot \phi \}
\]

2
7) Let \( \pi : \mathbb{R}^2 \to S^1 \times \mathbb{R} \) be the following covering map:

\[
\pi(\tilde{\phi}, \tilde{I}) = \left( \tilde{\phi} \ (\text{mod} \ 1), \ \tilde{I} \right)
\]  

8) For maps of the torus we can generalize the notion of rotation number, originally defined for circle homeomorphisms, as follows:

Given a map \( \tilde{f} : T^2 \to T^2 \) and \( x \in T^2 \), let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( \tilde{f} \) and \( \tilde{x} \in p^{-1}(x) \). The rotation vector \( \rho(x, f) \) is defined as (let \( \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \))

\[
\rho(x, f) = \lim_{n \to \infty} \frac{f^n(\tilde{x}) - \tilde{x}}{n} \in \overline{\mathbb{R}}^2 , \text{ if the limit exists.} \quad (4)
\]

Of course for a map \( \tilde{f} : T^2 \to T^2 \) that is not homotopic to the identity, the above limit may depend on the choice of \( \tilde{x} \in p^{-1}(x) \). So, for all \( T \in TQ \) we have to modify a little the above definition for rotation vector. The following lemma shows what type of changes must be done:

**Lemma 1**: Given \( T \in TQ \) and \( \tilde{z} \in \mathbb{R}^2 \) such that \( \rho(\tilde{z}, T) = \lim_{n \to \infty} \frac{T^n(\tilde{z}) - \tilde{z}}{n} \) exists, we have two possibilities:

1) \( \rho(\tilde{z}, T) = (\omega, 0) \), with \( \omega \in \mathbb{R} \),
2) \( \rho(\tilde{z}, T) = (+\infty, \omega) \) or \( (-\infty, \omega) \), with \( \omega \in \mathbb{R} \).

Given \( i, j \in \mathbb{Z} \), we have:

- \( \rho(\tilde{z}, T) = (\omega, 0) \Rightarrow \rho(\tilde{z} + (i, j), T) = (\omega + j, 0) \)
- \( \rho(\tilde{z}, T) = (+\infty, \omega) \Rightarrow \rho(\tilde{z} + (i, j), T) = (+\infty, \omega) \)

So, for all \( \tilde{x} \in \mathbb{R}^2 \) such that \( \rho(\tilde{x}, T) \) exists, there are 2 different cases:

- Case 1: \( p_1 \circ \rho(\tilde{x}, T) \in \mathbb{R} \). In this case we shall define the rotation vector of \( x = p(\tilde{x}) \in T^2 \) as follows:

\[
\rho(x, T) = \left( \lim_{n \to \infty} \frac{p_1 \circ T^n(\tilde{x}) - p_1(\tilde{x})}{n} \ (\text{mod} \ 1), \ 0 \right) \quad (5)
\]
• Case 2: \( |p_1 \circ \rho(x, T)| = \infty \). In this case \( \rho(x, T) \) is defined as in (4):

\[
\rho(x, T) = \lim_{n \to \infty} \frac{T^n(x) - \bar{x}}{n}, \quad \text{where } x = p(\bar{x})
\]  

(6)

We just remark that even in this case \( p_2 \circ \rho(x, T) \) may be zero.

When \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) induces a map \( \overline{f} : T^2 \to T^2 \) homotopic to the identity map (in this case the rotation vector is given by expression (4)), in many situations (see [10], [18] and [21]) we can guarantee the existence of a convex open set \( B \subset \mathbb{R}^2 \), such that for all \( v \in B \), \( \exists x \in T^2 \) such that \( \rho(x, f) = v \) and if \( v = (\frac{r}{q}, \frac{s}{q}) \), then \( x \in T^2 \) can be chosen such that \( f^q(x) = x + (r, s) \). A major difference in the situation studied here is that given \( T \in TQ \), as we have already said, the diffeomorphism \( \overline{T} : T^2 \to T^2 \) induced by \( T \) is not homotopic to the identity. In fact, it is homotopic to the following linear map (where \( \phi \) and \( I \) are taken mod 1):

\[
\begin{pmatrix}
\phi' \\
I'
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
\phi \\
I
\end{pmatrix}
\]  

(7)

Before presenting the first theorem we still need more definitions.

**Definitions** : Given \( T \in TQ \), let \( \overline{T} : T^2 \to T^2 \) be the torus diffeomorphism induced by \( T \).

• We say that \( x \in T^2 \) belongs to a \( n \)-periodic orbit (or set), if for some \( n \in \mathbb{N}^* \) we have \( \overline{T}^n(x) = x \) and for all \( m \in \mathbb{N}^* \), \( 0 < m < n \), \( \overline{T}^m(x) \neq x \). So the periodic orbit which \( x \) belongs is \( O_x = \{ x, \overline{T}(x), ..., \overline{T}^{n-1}(x) \} \). In this case we have the following implications (now we consider the standard projections \( p_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2 \)):

\[
\begin{cases}
p_2 \circ \rho(x, T) = 0 \Rightarrow p_1 \circ \rho(x, T) \text{ is a rational number}, \\
p_1 \circ \rho(x, T) = \pm \infty \Rightarrow p_2 \circ \rho(x, T) \text{ is a non-zero rational number}.
\end{cases}
\]

• We say that \( Q \subset T^2 \) is a quasi-periodic set for \( \overline{T} \) in the following cases:

1) \( Q \) is the projection of an Aubry-Mather set \( \tilde{Q} \subset S^1 \times \mathbb{R} \). In this case \( p_1 \circ \rho(z, T) \in [0, 1[ \) is an irrational number which does not depend on the choice of \( z \in Q \).
2) $Q$ is a compact $\mathcal{T} - invariant$ set such that $p_2 \circ \rho(z, T)$ is an irrational number which does not depend on the choice of $z \in Q$.

As before, let $T \in TQ$ and $\mathcal{T} : T^2 \to T^2$ be the diffeomorphism induced by $T$. So as a simple consequence of lemma (1) we have the following classification theorem:

**Theorem 1**: Let $x \in T^2$ belong to a periodic or a quasi-periodic set. Then there are two different situations:

1) $\exists C > 0$ such that $|p_2 \circ T^n(\bar{x}) - p_2(\bar{x})| < C$, for all $n > 0$ and $\bar{x} \in p^{-1}(x) \Rightarrow \rho(x, T) = (\omega, 0)$ for some $\omega \in [0, 1]$

2) $p_2 \circ T^n(\bar{x}) \xrightarrow{n \to \infty} \pm \infty$, for all $\bar{x} \in p^{-1}(x) \Rightarrow \rho(x, T) = (\pm \infty, \omega)$ for some $\omega \in \mathbb{R}^*$.

Remarks:

- If we call $\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ the map of the cylinder induced by $T$, it is easy to see that case 1 above corresponds to periodic and quasi-periodic orbits for $\hat{T}$. These are the standard periodic and quasi-periodic orbits, whose existence is assured by theorem (2) (see [13], [1], [14], [16], [17]).

- Case 2 corresponds to orbits for $\hat{T}$ that either go up or down on the cylinder, depending on the sign of $\omega \in \mathbb{R}^*$. If $\omega > 0$ ($< 0$) then $\rho(x, T) = (+\infty(-\infty), \omega)$.

- As we said, there may be a point $x \in T^2$ such that $\rho(x, T) = (\pm \infty, 0)$. It is clear that $x$ is not periodic because its $\mathcal{T}$-orbit can not be finite and $x$ does not belong to a quasi-periodic set, because any component of $\rho(x, T)$ is irrational.

A periodic or quasi-periodic orbit $O$ for $\mathcal{T}$ that belongs to case 2 in theorem (1) can (as the orbits belonging to case 1) be characterized by a single number, the ”vertical rotation number”, which is defined in the following way:

$$\rho_V(O) = p_2 \circ \rho(x, T) = \lim_{n \to \infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n}, \text{ for any } x \in O.$$  \hspace{1cm} (8)

As we want to characterize all kinds of periodic and quasi-periodic orbits a twist map can have, we recall a well-known result:
Theorem 2 : Given a map $T \in TQ$ such that $\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is $\mu$-exact for some measure $\mu$ we have: For every $\omega \in \mathbb{R}$ there is a $\hat{T}$-periodic or quasi-periodic orbit $O \subset S^1 \times \mathbb{R}$, respectively, for rational and irrational values of $\omega$, such that $\rho(O) = \omega$.

See [19], [4], [14], [16], [17] for different proofs. And we have the following corollary:

Corollary 1 : Given a map $T \in TQ$ such that $\hat{T}$ is $\mu$-exact for some measure $\mu$ we have: For every $\omega \in [0,1]$ there is a $\hat{T}$-periodic or quasi-periodic orbit $O \subset \mathbb{T}^2$, respectively, for rational and irrational values of $\omega$, such that $\rho(x,T) = (\omega,0)$, for all $x \in O$.

So the first type of orbit that appears in theorem (1) always exists.

Before presenting the next results we need another definition.

Definition : Given a map $T \in TQ$ such that $\hat{T}$ is $\mu$-exact we say that $C$ is a rotational invariant circle (R.I.C.) for $T$ if $C$ is a homotopically non-trivial simple closed curve on the cylinder and $\hat{T}(C) = C$.

By a theorem essentially due to Birkhoff, $C$ is the graph of some Lipschitz function $\psi : S^1 \to \mathbb{R}$ (see [15], page 430).

The following theorems are the main results of this paper:

Theorem 3 : Let $T \in TQ$ be such that $\hat{T}$ is $\mu$-exact. Then:

- given $k \in \mathbb{Z}^*$, $\exists N > 0$, such that $\hat{T}$ has a periodic orbit with $\rho_V$ (vertical rotation number) = $\frac{k}{N}$, if and only if, $T$ does not have R.I.C’s.

The next theorem shows how these periodic orbits appear:

Theorem 4 : Again, for all $T \in TQ$ such that $\hat{T}$ is $\mu$-exact, if $\hat{T}$ has a periodic orbit with $\rho_V = \frac{k}{N}$, then for every pair $(k',N') \in \mathbb{Z}^* \times \mathbb{N}^*$, such that $0 < \left| \frac{k'}{N'} \right| < \frac{1}{N}$ and $k.k' > 0$, $\hat{T}$ has at least 2 periodic orbits with vertical rotation number $\rho'_V = \frac{k'}{N'}$. 

6
About the quasi-periodic orbits we have the following:

**Theorem 5**: For all $T \in TQ$ such that $\hat{T}$ is $\mu$-exact we have:

If $T$ has an orbit with $\rho_V = \omega$, then for all $\omega' \in \mathbb{R}\setminus\mathbb{Q}$ such that $0 < |\omega'| < |\omega|$ and $\omega.\omega' > 0$, $T$ has a quasi-periodic set with vertical rotation number $\rho'_V = \omega'$.

As a consequence of the proof of theorem (5) we prove the following classical result:

**Theorem 6**: Every $T \in TQ$ without R.I.C’s such that $\hat{T}$ is $\mu$-exact induces a map $T : T^2 \to T^2$ such that $h(T) > 0$, where $h(T)$ is the topological entropy of $T$.

Theorem (1) is an immediate consequence of lemma (1), which is proved using simple ideas and the structure of the set $TQ$. Theorems (3), (4) and (5) are proved using topological ideas, essentially due to the twist condition and some results due to Le Calvez (see [16], [17] and the next section). In the proofs of theorems (3) and (5) we also use some results from the Nielsen-Thurston theory of classification of homeomorphisms of surfaces up to isotopy, to isotope the map to a pseudo-Anosov one and then some results due to M. Handel, to prove the existence of quasi-periodic orbits with irrational vertical rotation number.

## 2 Basic tools

First we recall some topological results for twist maps essentially due to Le Calvez (see [16] and [17]). Let $\tilde{f} \in Diff^1(S^1 \times \mathbb{R})$ and $f \in D^1_1(\mathbb{R}^2)$ be its lifting. For every pair $(p, q)$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets:

$$\bar{K}(p, q) = \left\{ (\bar{\phi}, \bar{I}) \in \mathbb{R}^2 : p_1 \circ f^q(\bar{\phi}, \bar{I}) = \bar{\phi} + p \right\}$$

and

$$K(p, q) = \pi \circ \bar{K}(p, q)$$

Then we have the following:
Lemma 2: For every $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(p,q) \supset C(p,q)$, a connected compact set that separates the cylinder.

Lemma 3: Let $\hat{f} \in Diff^1_r(S^1 \times \mathbb{R})$ be a $\mu$-exact map. Then the following intersection holds: $\hat{f}(C(p,q)) \cap C(p,q) \neq \emptyset$.

Now we need a few definitions:

For every $q \geq 1$ and $\phi \in \mathbb{R}$ let

$$\mu_q(t) = f^q(\phi, t), \text{ for } t \in \mathbb{R} \quad (10)$$

We say that the first encounter between $\mu_q$ and the vertical line through some $\phi_0 \in \mathbb{R}$ is for:

$t_F \in \mathbb{R}$ such that

$$t_F = \min\{t \in \mathbb{R}: p_1 \circ \mu_q(t) = \phi_0\}$$

And the last encounter is defined in the same way:

$$t_L \in \mathbb{R} \text{ such that } t_L = \max\{t \in \mathbb{R}: p_1 \circ \mu_q(t) = \phi_0\}$$

Of course we have $t_F \leq t_L$.

Lemma 4: For all $\phi_0, \phi \in \mathbb{R}$, let $\mu_q(t) = f^q(\phi, t)$, as in (10). So we have the following inequalities: $p_2 \circ \mu_q(t_L) \leq p_2 \circ \mu_q(t) \leq p_2 \circ \mu_q(t_F)$, for all $\overline{t} \in \mathbb{R}$ such that $p_1 \circ \mu_q(\overline{t}) = \phi_0$.

For all $s \in \mathbb{Z}$ and $N \in \mathbb{N}^*$ we can define the following functions on $S^1$:

$$\mu^-(\phi) = \min\{p_2(Q): Q \in K(s, N) \text{ and } p_1(Q) = \phi\}$$

$$\mu^+(\phi) = \max\{p_2(Q): Q \in K(s, N) \text{ and } p_1(Q) = \phi\}$$

And we can define similar functions for $\hat{f}^N(K(s, N))$:

$$\nu^-(\phi) = \min\{p_2(Q): Q \in \hat{f}^N \circ K(s, N) \text{ and } p_1(Q) = \phi\}$$

$$\nu^+(\phi) = \max\{p_2(Q): Q \in \hat{f}^N \circ K(s, N) \text{ and } p_1(Q) = \phi\}$$
Lemma 5 : Defining Graph\(\{\mu^\pm\}\)=\{((\phi,\mu^\pm(\phi)) : \phi \in S^1)\} we have:

\[\text{Graph}\{\mu^-\} \cup \text{Graph}\{\mu^+\} \subset C(s,N)\]

So for all \(\phi \in S^1\) we have \((\phi,\mu^\pm(\phi)) \in C(s,N)\).

And we have the following simple corollary to lemma (4):

Corollary 2 : \(\hat{f}^N(\phi,\mu^-(\phi)) = (\phi,\nu^+(\phi))\) and \(\hat{f}^N(\phi,\mu^+(\phi)) = (\phi,\nu^-(\phi))\).

Now we are going to present a lemma due to M. Casdagli (see [6]), that together with lemma (3) guarantees the existence of periodic orbits with all rational rotation numbers, for all \(\mu\)-exact \(\hat{f} \in \text{Diff}_r^1(S^1 \times \mathbb{R})\).

Lemma 6 : If \(z \in C(s,N) \cap \hat{f}(C(s,N)) \Rightarrow z\) is \((s,N)\) periodic for \(\hat{f}\).

We say that \(z\) is \((s,N)\) periodic for \(\hat{f}\) if

\[\hat{f}^N(z) = z\] and \[\frac{p_1 \circ f^N(z) - p_1(z)}{N} = \frac{s}{N},\]

where \(f : \mathbb{R}^2 \to \mathbb{R}^2\) is a lift of \(\hat{f}\) and \(\tilde{z} \in \pi^{-1}(z)\).

For proofs of all the previous results see Le Calvez [16] and [17].

The following is another classical result (due to Birkhoff) with some small changes:

Theorem 7 : Given \(T \in TQ\) without R.I.C’s such that \(\hat{T}\) is \(\mu\)-exact we have:

For all \(s,l \in \mathbb{Z}, s > 0\) and \(l < 0\), \(\exists P,Q \in S^1 \times [0,1]\) and numbers

\[1 < n_P,n_Q \in \mathbb{N} \text{ such that } \begin{cases} p_2 \circ \hat{T}^{n_P}(P) > s \\ p_2 \circ \hat{T}^{n_Q}(Q) < l. \end{cases}\]

For a proof see [15].

As we have already said, in the proof of theorems (3) and (4) we use some results from the Nielsen-Thurston theory of classification of homeomorphisms of surfaces up to isotopy and some results due to M. Handel.

The following is a brief summary of these results, taken from [18]. For more information and proofs see [22], [19] and [21].
Let $M$ be a compact, connected oriented surface possibly with boundary, and $f : M \to M$ be a homeomorphism. Two homeomorphisms are said to be isotopic if they are homotopic via homeomorphisms. In fact, for closed orientable surfaces, all homotopic pairs of homeomorphisms are isotopic.

There are two basic types of homeomorphisms which appear in the Nielsen-Thurston classification: the finite order homeomorphisms and the pseudo-Anosov ones.

A homeomorphism $f$ is said to be of finite order if $f^n = id$ for some $n \in \mathbb{N}$. The least such $n$ is called the order of $f$. Finite order homeomorphisms have topological entropy zero.

A homeomorphism $f$ is said to be pseudo-Anosov if there is a real number $\lambda > 1$ and a pair of transverse measured foliations $F^S$ and $F^U$ such that $f(F^S) = \lambda^{-1}F^S$ and $f(F^U) = \lambda F^U$. Pseudo-Anosov homeomorphisms are topologically transitive, have positive topological entropy, and have Markov partitions.

A homeomorphism $f$ is said to be reducible by a system

$$C = \bigcup_{i=1}^n C_i$$

of disjoint simple closed curves $C_1, ..., C_n$ (called reducing curves) if

1. $\forall i, C_i$ is not homotopic to a point, nor to a component of $\partial M$,
2. $\forall i \neq j, C_i$ is not homotopic to $C_j$,
3. $C$ is invariant under $f$.

**Theorem 8**: If the Euler characteristic $\chi(M) < 0$, then every homeomorphism $f : M \to M$ is isotopic to a homeomorphism $F : M \to M$ such that either

(a) $F$ is of finite order,

(b) $F$ is pseudo-Anosov, or

(c) $F$ is reducible by a system of curves $C$.

Homeomorphisms $F$ as in theorem (8) are called Thurston canonical forms for $f$.

**Theorem 9**: If $f$ is pseudo-Anosov and $g$ is isotopic to $f$, then $h(g) \geq h(f)$. 


Some results due to M. Handel can be trivially adapted to the situation studied here. To be more precise, we can change in propositions 1.1 and 1.2 of [12], annulus homeomorphisms by torus homeomorphisms homotopic to the map \( LM : T^2 \to T^2 \), which is the torus map induced by the following linear map of the plane:

\[
\begin{pmatrix}
\tilde{\phi}' \\
\tilde{I}'
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi} \\ \tilde{I} \end{pmatrix}
\] (11)

In our case we also have to present appropriate definitions for rotation number and rotation set.

Given a homeomorphism \( \overline{f} : T^2 \to T^2 \) that is homotopic to \( LM \) and a lift of \( \overline{f} \) to the cylinder, \( \hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} \), we define the vertical rotation set as

\[ \rho_V(\hat{f}) = \cup \rho_V(\hat{f}, z) \]

where the union is taken over all \( z \in T^2 \) such that the vertical rotation number \( (\hat{\varepsilon} \in S^1 \times \mathbb{R} \) is any lift of \( z \in T^2 \))

\[ \rho_V(\hat{f}, z) = \lim_{n \to \infty} \frac{p_2 \circ \hat{f}^n(\hat{\varepsilon}) - p_2(\hat{\varepsilon})}{n} \]

exists.

We say that \( \overline{f} : T^2 \to T^2 \) is pseudo-Anosov relative to a finite invariant set \( Q \subset T^2 \) if it satisfies all of the properties of a pseudo-Anosov homeomorphism except that the associated stable and unstable foliations may have 1-prolonged singularities at points in \( Q \). As a last definition, for every set \( A \subset T^2 \) let \( \hat{A} \subset S^1 \times \mathbb{R} \) be the full (cylinder) pre-image of \( A \). Now we are ready to state the modified versions of propositions 1.1 and 1.2 of [12]:

**Proposition 1 (modified 1.1):** If \( \overline{f} : T^2 \to T^2 \) homotopic to \( LM \), is pseudo-Anosov relative to some finite invariant set \( Q \), then \( \rho_V(\hat{f}) \) is a closed interval. For each \( \omega \in \rho_V(\hat{f}) \), there is a compact invariant set \( E_\omega \subset T^2 \) such that \( \rho_V(\hat{f}, z) = \omega \) for all \( z \in E_\omega \). Moreover, if \( \omega \in \text{int} \left( \rho_V(\hat{f}) \right) \), then we may choose \( E_\omega \subset T^2 \setminus Q \).
Proof. As in [12]. ■

Proposition 2 (modified 1.2): Suppose that \( \widetilde{f} : T^2 \rightarrow T^2 \) is pseudo-Anosov relative to a finite invariant set \( Q \) and that \( \widetilde{T} : T^2 \rightarrow T^2 \) (induced by some element of \( TQ \)) is homotopic to \( \widetilde{f} \) relative to \( Q \). If \( \hat{f} : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R} \) and \( \hat{T} : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R} \) are lifts that are equivariantly homotopic relative to \( \hat{Q} \), then \( \rho_V(\hat{T}) \supset \text{int}(\rho_V(\hat{f})). \) Moreover, for each \( \omega \in \text{int}(\rho_V(\hat{f})) \), there is a compact \( T^{-}\)-invariant set \( E_\omega \subset T^2 \) such that \( \rho_V(\hat{T},z) = \omega \) for all \( z \in E_\omega \).

Proof. Also as in [12]. ■

3 Proofs

From up to now, for simplicity, we will omit the \( \hat{\ } \) in the coordinates \( (\hat{\phi},\hat{I}) \) of the plane. First of all we prove lemma (1). This lemma is a trivial consequence of the following result:

Lemma 7 : Let \( T \in TQ \) and \( z, w_+, w_- \in \mathbb{R}^2 \) be points such that:

\[ \begin{align*}
  i) & \quad |p_2 \circ T^n(z)| < C, \text{ for all } n \geq 0 \text{ and some constant } C > 0 \\
  ii) & \quad |p_2 \circ T^n(w_\pm) - p_1(w_\pm)| \xrightarrow{n \rightarrow \infty} \pm \infty
\]

So we have:

\[ \begin{align*}
  i) & \quad \exists K > 0, \text{ such that for all } n > 0, \quad \left| \frac{p_1 \circ T^n(z) - p_1(z)}{n} \right| < K \\
  ii) & \quad \frac{p_1 \circ T^n(w_\pm) - p_1(w_\pm)}{n} \xrightarrow{n \rightarrow \infty} \pm \infty
\]

Proof.

As the proofs for \( w_+ \) and \( w_- \) are equal, we only analyze \( w_+ \), which will be called just \( w \). For all \( n > 0 \) we define \( z = (\phi^0_z, I^0_z), \phi^n_z = p_1 \circ T^n(z), I^n_z = p_2 \circ T^n(z) \) and \( w = (\phi^0_w, I^0_w), \phi^n_w = p_1 \circ T^n(w), I^n_w = p_2 \circ T^n(w) \). From the initial hypothesis, \( |I^j_z| < C \) for all \( j > 0 \), so defining the following \( \phi \)-periodic function \( \widetilde{T}_\phi(\phi,I) = T_\phi(\phi,I) - \phi \), there is a constant \( K > 0 \) such that:

\[ \left| \frac{p_1 \circ T^n(z) - p_1(z)}{n} \right| \leq \frac{1}{n} \sum_{j=0}^{n-1} \left| \widetilde{T}_\phi(\phi^j_z, I^j_z) \right| < \frac{n.K}{n} = K. \]
Now we write $I_w^n = I_{w_0}^n + k_n$, with $I_{w_0}^n \in [0, 1)$ and $k_n \in \mathbb{Z}$. Of course the hypothesis in the lemma implies that $k_n \to \infty$, because $p_2 \circ T^n(w) \to \infty$. As above $\exists \overline{K} > 0$ such that for all $j > 0$, $|T_{\phi}^j(\phi_w, I_{w_0}^j)| \leq \max_{\phi, I \in [0, 1]^2} |T_{\phi}(\phi, I)| < \overline{K}$.

So for all $n > 0$:

$$\frac{p_1 \circ T^n(w) - p_1(w)}{n} = \frac{\sum_{j=0}^{n-1} T_{\phi}^j(\phi_w, I_{w_0}^j) + \sum_{j=0}^{n-1} k_j}{n} > -\overline{K} + \frac{\sum_{j=0}^{n-1} k_j}{n}.$$

In order to finish the proof that $\lim_{n \to \infty} \frac{p_1 \circ T^n(w) - p_1(w)}{n} = \infty$, we remember the Cesaro theorem, which says that $\lim_{j \to \infty} \frac{n-1}{n} k_j = \infty \Rightarrow \lim_{n \to \infty} \frac{n-1}{n} = \infty$.

The following is a very important lemma.

**Lemma 8**: Given $T \in TQ$ such that $\tilde{T}$ is $\mu$-exact and $T$ does not have R.I.C’s we have:

For all $k \in \mathbb{N}^*$, $\exists N > 0$ and $\tilde{P} = (\phi_{\tilde{P}}, I_{\tilde{P}}) \in [0, 1]^2$, such that $T^N(\tilde{P}) = T^N(\phi_{\tilde{P}}, I_{\tilde{P}}) = (\phi_{\tilde{P}} + s, I^N_{\tilde{P}})$ for some $s \in \mathbb{Z}$, with $I^N_{\tilde{P}} > I_{\tilde{P}} + k$.

**Proof.**

The proof will be done by contradiction. Suppose that exists $k_0 \geq 1$, such that $\forall N > 0$, there is no $\tilde{P} \in [0, 1]^2$ such that $T^N(\tilde{P}) = T^N(\phi_{\tilde{P}}, I_{\tilde{P}}) = (\phi_{\tilde{P}} + s, I^N_{\tilde{P}})$ for some $s \in \mathbb{Z}$, with $I^N_{\tilde{P}} > 1 + k_0 \geq I_{\tilde{P}} + k_0$.

First let us note that given a map $T \in TQ$, $\exists a > 0$, such that $\forall Q \in \mathbb{R}^2$, $p_2 \circ (T(Q) - Q) > -a$. In fact, from the definition of the set $TQ$, we just have to take $a > -\inf_{Q \in [0, 1]^2} p_2 \circ (T(Q) - Q)$, because as $[0, 1]^2$ is compact, $a < \infty$.

All $T \in TQ$ can be written in the following way,

$$T : \begin{cases} \phi' = T_\phi(\phi, I) \\ I' = T_I(\phi, I) \end{cases}$$

and for all $(\phi, I) \in \mathbb{R}^2$ we have the following estimates:
\[ \exists b > 0, \text{ such that } \left| \frac{\partial T_\phi}{\partial \phi} \right| < b \quad (12) \]

\[ \exists K > 0, \text{ such that } \frac{\partial T_\phi}{\partial I} \geq K \text{ (twist condition)} \quad (13) \]

As \( T \) does not have R.I.C’s, theorem (7) implies that

\[ \exists P = (\phi_P, I_P) \in [0, 1]^2 \text{ and } N_1 > 1 \text{ such that:} \]

\[ p_2 \circ T^{N_1}(P) > \left( (k_0 + 2) + \frac{(2 + b)}{K} \right) + a \]

A very natural thing is to look for a point \( \tilde{P} \) as described above, in the line segment \( r = \{(\phi, I) \in [0, 1]^2 : \phi = \phi_P \} \).

First, let us define

\[ \text{Max.H.L}(T^n(r)) = \sup_{x, y \in [0, 1]} |p_1 \circ T^n(\phi_P, x) - p_1 \circ T^n(\phi_P, y)|. \quad (14) \]

It is clear that

\[ \text{Max.H.L}(T^n(r)) \geq |p_1 \circ T^n(\phi_P, 0) - p_1 \circ T^n(\phi_P, 1)| = n \xrightarrow{n \to \infty} \infty. \quad (15) \]

So, for all \( n > 1 \), \( \exists \) at least one \( s \in \mathbb{Z} \), such that \( \phi_P + s \in p_1(T^n(r)) \).

The hypothesis we want to contradict implies that for all \( n > 0 \) and \( Q \in r \), such that

\[ p_1 \circ T^n(Q) = \phi_P \text{ (mod 1)}, \quad (16) \]

we have:

\[ p_2 \circ T^n(Q) \leq (k_0 + 1) \quad (17) \]

As \( p_2 \circ T^{N_1}(P) > \left( (k_0 + 2) + \frac{(2 + b)}{K} \right) + a \), \( \exists P_1 \in r \) such that:
\[ p_2 \circ T^{N_1}(P_1) = (k_0 + 2) + a \]

and

\[ \forall Q \in \mathbb{P}P_1 \subset r, \]

\[ p_2 \circ T^{N_1}(Q) \geq (k_0 + 2) + a \]

The reason why such a point \( P_1 \) exists is the following: As \( N_1 > 1 \), \( \exists \) at least one \( s \in \mathbb{Z} \) such that \( \phi_P + s \in p_1 \left(T^{N_1}(r)\right) \). Thus, from (16) and (17), \( T^{N_1}(r) \) must cross the line \( l \) given by: \( l = \{ (\phi, (k_0 + 2) + a), \text{ with } \phi \in \mathbb{R} \} \)

Also from (16) and (17) we have that:

\[ \sup_{Q, R \in \mathbb{P}P_1} |p_1 \circ T^{N_1}(R) - p_1 \circ T^{N_1}(Q)| < 1 \]

Now let \( \gamma_{N_1} : J \to \mathbb{R}^2 \) be the following curve:

\[ \gamma_{N_1}(t) = T^{N_1}(\phi_P, t), \quad t \in J = \text{interval whose extremes are } I_P \text{ and } I_{P_1} \quad (18) \]

It is clear that it satisfies the following inequalities:

\[ p_2 \circ \gamma_{N_1}(I_P) - p_2 \circ \gamma_{N_1}(I_{P_1}) > \frac{(2+b)}{K} \]

\[ \sup_{t, s \in J} |p_1 \circ \gamma_{N_1}(t) - p_1 \circ \gamma_{N_1}(s)| < 1 \]

**Claim 1**: Given a continuous curve \( \gamma : J = [\alpha, \beta] \to \mathbb{R}^2, \) with

\[ \sup_{t, s \in J} |p_1 \circ \gamma(t) - p_1 \circ \gamma(s)| < 1 \quad (19) \]

\[ |p_2 \circ \gamma(\beta) - p_2 \circ \gamma(\alpha)| > \frac{(2+b)}{K} \quad (20) \]

Then \( \exists \ s \in \mathbb{Z}, \) such that \( \phi_P + s \in p_1 \left(T \circ \gamma(J)\right). \)

**Proof.**

\[ \sup_{t, s \in J} |p_1 \circ T \circ \gamma(t) - p_1 \circ T \circ \gamma(s)| = \]

\[ = \sup_{t, s \in J} |T_{\phi} \circ \gamma(t) - T_{\phi} \circ \gamma(s)| \geq |T_{\phi} \circ \gamma(\beta) - T_{\phi} \circ \gamma(\alpha)| \geq \]

\[ \geq -b + K \cdot \frac{(2+b)}{K} = 2 \]

So the claim is proved.  

15
\(\gamma_{N_1}(t)\) (see [15]) satisfies the claim hypothesis, by construction. So \(\exists s \in \mathbb{Z}\) such that \(\phi_P + s \in p_1(T \circ \gamma_{N_1}(J)) = p_1(T^{N_1+1}(PP))\).

As \(\inf_{t \in J} p_2(\gamma_{N_1}(t)) = p_2(\gamma_{N_1}(I_P)) = (k_0 + 2) + a\), from the choice of \(a > 0\) we get that \(\inf_{t \in J} p_2(T \circ \gamma_{N_1}(t)) > (k_0 + 2)\).

So there is \(\mathbf{7} \in J\) and \(\mathbf{P} = (\phi_P, \mathbf{7}) \in r\) such that:

\[
\begin{align*}
p_1 \circ T^{N_1+1}(\mathbf{P}) &= \phi_P \pmod{1} \\
p_2 \circ T^{N_1+1}(\mathbf{P}) &= (k_0 + 2)
\end{align*}
\]

And this contradicts (16) and (17). So for all \(k \geq 1\), \(\exists N > 0\) and \(\tilde{P} \in r\), such that \(T^N(\tilde{P}) = T^N(\phi_P, I_P) = (\phi_P + s, I_{N_P})\) for some \(s \in \mathbb{Z}\), with \(I_{N_P} > I_P + k\).

\[\blacksquare\]

Remark:

• Of course for all \(k \leq -1\), \(k \in \mathbb{Z}\), there are also \(N > 0\) and \(\tilde{Q} = (\phi_{Q}, I_{Q}) \in [0,1]^2\), such that \(T^N(\tilde{Q}) = T^N(\phi_{Q}, I_{Q}) = (\phi_{Q} + s, I_{N_Q})\) for some \(s \in \mathbb{Z}\), with \(I_{N_Q} < I_{Q} + (k - 1)\). The proof in this case is completely similar to the above one, because as \(T\) does not have R.I.C’s, again by theorem (16) for all \(l < 0\) there exist \(Q = (\phi_Q, I_Q) \in [0,1]^2\) and \(n_Q > 1\) such that \(p_2 \circ T^{n_Q}(Q) < l\).

Below we prove the main results of this paper.

\textit{Proof. of theorem (3)}

As the 2 cases, \(k > 0\) and \(k < 0\) are completely similar, let us fix \(k > 0\).

\((\Rightarrow)\)

If \(\mathbf{T}\) has a periodic point \(P\), with \(\rho_{\nu}(P) = \frac{k}{N}\), for some \(k > 0\) and \(N > 0\), then \(p_2 \circ \mathbf{T}^n(P) \rightarrow (0, k)\), which implies that there can be no R.I.C.

\((\Leftarrow)\)

To prove the existence of a periodic orbit with \(\rho_{\nu} = \frac{k}{N}\), for a given \(k > 0\) and some \(N > 0\) sufficiently large, it is enough to show that there exists a point \(P \in S^1 \times \mathbb{R}\) such that:

\[\mathbf{T}^N(P) = P + (0, k)\]  

(21)
As \( T \in TQ \), for each \((s,l) \in \mathbb{Z}^2 \) and \( N > 0 \) the sets \( C(s,N) \), defined in lemma (2), satisfy: \( C(s + l,N,N) = C(s,N) \uparrow (0,l) \)

So, for each fixed \( N > 0 \), there are only \( N \) distinct sets of this type: \( C(0,N), C(1,N), \ldots, C(N - 1,N) \)

The others are just integer vertical translations of them. Another trivial remark about the sets \( C(s,N) \): \( C(s,N) \cap C(r,N) = \emptyset \), if \( s \neq r \)

For all \( s \in \mathbb{Z} \), we get from lemma (3) that \( \hat{T}(C(s,N)) \cap C(s,N) \neq \emptyset \). So we can apply lemma (6) and conclude that \( \exists \, P_s \in C(s,N) \) such that \( \hat{T}^N(P_s) = P_s \).

From lemma (5), for the given \( k > 0 \), \( \exists \, N > 0 \) and \( P = (\phi_P^-, I_P^-) \in S^1 \times \mathbb{R} \), such that \( \hat{T}^N(P) = \hat{T}^N(\phi_P^-, I_P^-) = (\phi_P^+, I_P^+) \), with \( I_P^+ > I_P^- + k \). So \( P \in K(\tilde{s}, N) \) (see expression (4)), for a certain \( \tilde{s} \in \mathbb{Z} \) and \( p_2 \circ \hat{T}^N(P) - p_2(P) > k \).

From corollary (2) we get that \( \hat{T}^N(\phi_P^-, \mu^-(\phi_P^-)) = (\phi_P^-, \nu^+(\phi_P^-)) \), so defining \( \tilde{P} = (\phi_P^-, \mu^-(\phi_P^-)) \in C(\tilde{s}, N) \) (see lemma (3)) we have: \( p_2 \circ \hat{T}^N(P) - p_2(P) > k \).

And as we proved above, \( \exists \, T_P \in C(\tilde{s}, N) \) such that \( p_2 \circ \hat{T}^N(T_P) - p_2(T_P) = 0 \).

So as \( C(\tilde{s}, N) \) is connected, \( \exists \, P \in C(\tilde{s}, N) \) such that:

\[ p_2 \circ \hat{T}^N(P) = p_2(P) + k \]

And the theorem is proved. Now we present an alternative proof, suggested by a referee, which is much shorter. We decided to maintain the original proof because it is based on lemma (8), which will be used in future works, so we wanted to keep it in the present paper.

For a given \( k > 0 \), we are going to prove the existence of a point \( P \in C(0, N) \) such that \( \hat{T}^N(P) = P + (0,k) \), for a sufficiently large \( N \). As \( \hat{T} \) is \( \mu \)-exact, we get that there is a point \( P_0 = (\phi_0, I_0) \in C(0,1) \) such that \( \hat{T}(P_0) = P_0 \).

For any given \( N > 0 \), let \( \mu^-, \mu^+, \nu^-, \nu^+ \) be the maps associated to \( C(0,N) \).

From the choice of \( P_0 \) we get that \( \mu^-(\phi_0) \leq I_0 \leq \nu^+(\phi_0) \). In the proof of lemma (4) (see [4]), the following property for the lift of the map \( \mu^- \) to \( \mathbb{R} \) is obtained (we are denoting the lift also by \( \mu^- \)): for any \( n \in \{1,2,\ldots,N\} \), the
point \( T^n(\phi, \mu^-(\phi)) \) is the first point where the image of \( \phi \times \mathbb{R} \) by \( T^n \) meets the vertical passing through \( T^n(\phi, \mu^-(\phi)) \), and for the same reasons, we get that 
\[ p_1 \circ T^n(\phi', \mu^-(\phi')) < p_1 \circ T^n(\phi, \mu^-(\phi)) \] 
if \( \phi' < \phi \). So the image by \( T^n \) of the graph of \( \mu^- \) is also a graph and the order given by \( p_1 \) is preserved. Moreover, as \( T \) is a twist map, we can prove that (see lemma 13.1.1, page 424 of [15]) if \( \phi > \phi' \), then \( \mu^-(\phi) - \mu^-(\phi') \geq -\cotan(\beta).(\phi - \phi') \), where \( \beta \) is a uniform angle of deviation for \( T \). By periodicity of \( \mu^- \) we get that \( \max \mu^- - \min \mu^- \leq \cotan(\beta) \) and analogous inequalities for the other maps.

As in the above proof, we know that \( p_2 \circ \hat{T}^N - p_2 \) vanishes on \( C(0, N) \). Suppose that this map does not take the value \( k \) on \( C(0, N) \). Then as \( C(0, N) \) is compact, it is strictly smaller and we have \( \nu^+(\phi) - \mu^-(\phi) < k \), for all \( \phi \in S^1 \).

So for any \( \phi \in S^1 \), we get the following estimates:

\[
\mu^-(\phi) = \mu^-(\phi) - \mu^-(\phi_0) + \mu^-(\phi_0) - \nu^+(\phi_0) + \nu^+(\phi_0) > -\cotan(\beta) - k + I_0 \\
\nu^+(\phi) = \nu^+(\phi) - \nu^+(\phi_0) + \nu^+(\phi_0) - \mu^-(\phi_0) + \mu^-(\phi_0) < \cotan(\beta) + k + I_0 
\]

And the above inequalities imply that 
\[
\hat{T}^N(S^1 \times ]-\infty, -\cotan(\beta) - k + I_0[) \subset S^1 \times ]-\infty, \cotan(\beta) + k + I_0[, 
\]

which can not hold for all \( N > 0 \) by theorem (7).

\[\blacksquare\]

**Proof. of theorem (6)**

Again we fix \( k > 0 \Rightarrow k' > 0 \). The case \( k < 0 \) is completely similar.

By contradiction, suppose that for some \( 0 < \frac{k'}{N'} < \frac{k}{N} \) and any fixed \( s \in \mathbb{Z} : \)

\[ p_2 \circ \hat{T}^{N'}(Q) - p_2(Q) - k' \leq 0, \forall Q \in C(s, N'). \]

So in particular, we have: \( \nu^+(\phi) - \mu^-(\phi) - k' \leq 0 \) for all \( \phi \in S^1 \). This means that the unbounded connected component of \( C(s, N') \), which is below \( C(s, N') \) and we denote by \( U \), satisfies the following equation: \( \hat{T}^{N'}(U) - (0, k') \subset U \), so \( \hat{T}^{1,N'}(U) - (0, i,k') \subset U \), for all \( i > 0 \). Now let us choose a point \( P \in U \), such that

\[
\lim_{n \to \infty} \frac{p_2 \circ \hat{T}^n(P) - p_2(P)}{n} = \frac{k}{N} \tag{22}
\]
So we get that for all $i > 0$, $\left[p_2 \circ \hat{T}^i_{N'}(P) - p_2(P) - i \cdot k'\right] \leq C - p_2(P)$,
where $C = \sup\{p_2(x) : x \in C(s, N')\}$. And this implies that:

$$
\lim_{i \to \infty} \frac{p_2 \circ \hat{T}^i_{N'}(P) - p_2(P)}{i \cdot N'} \leq \frac{k'}{N'} \Rightarrow \lim_{n \to \infty} \frac{p_2 \circ \hat{T}^n(P) - p_2(P)}{n} \leq \frac{k'}{N'}
$$

which contradicts (22). So there is a point $P_1 = (\phi_1, I_1) \in C(s, N')$ such that $p_2 \circ \hat{T}^N_{N'}(P_1) - p_2(P_1) - k' > 0$ and from the $\mu$-exactness of $\hat{T}$, $\exists P_0 = (\phi_0, I_0) \in C(s, N')$ such that $p_2 \circ \hat{T}^N_{N'}(P_0) - p_2(P_0) < 0$. Now let $\Delta_0$ and $\Delta_1$ be the proper simple arcs given by:

$$
\Delta_0 = \{\phi_0\} \times [\mu^+(\phi_0), +\infty] \cup \hat{T}^{-N'}(\{\phi_0\} \times -\infty, \nu^-(\phi_0)]
\Delta_1 = \{\phi_1\} \times -\infty, \mu^-(\phi_1)] \cup \hat{T}^{-N'}(\{\phi_1\} \times [\nu^+(\phi_1), +\infty[
$$

It is easy to see that $\Delta_0 \cap C(s, N') = (\phi_0, \mu^+(\phi_0)), \Delta_1 \cap C(s, N') = (\phi_1, \mu^-(\phi_1))$, and that $(\Delta_0 \cup \Delta_1)^c$ is an open set that divides $C(s, N')$ into 2 connected components, $C_1$ and $C_2$ ($C(s, N') = C_1 \cup C_2$), such that $C_1 \cap C_2 = (\phi_0, \mu^+(\phi_0)) \cup (\phi_1, \mu^-(\phi_1))$. Therefore the function $p_2 \circ \hat{T}^N_{N'} - p_2 - k'$ has at least one zero in each $C_i$.

**Proof. of theorem (7)**

The proof will be divided into 2 cases (as before we fix $\omega > 0 \Rightarrow \omega' > 0$):

Case 1) $\omega \in \mathbb{Q}$.

As $\omega' \in \mathbb{R} \setminus \mathbb{Q}$, there is a sequence

$$
\frac{p_i}{q_i} \xrightarrow{i \to \infty} \omega', \text{ with } 0 < \frac{p_i}{q_i} < \omega, \ \forall i > 0
$$

and (from theorem (3)) a family of periodic orbits

$$
E_i = \{P_{i1}, P_{i2}, ..., P_{iq_i}\} \subset T^2, \text{ with } \rho_V(E_i) = \frac{p_i}{q_i}.
$$

So in the Hausdorff topology there is a subsequence $E_{i_n} \xrightarrow{n \to \infty} E \subset T^2$ that for simplicity we will call $E_n$. The convergence in the Hausdorff topology means that: Given $\epsilon > 0, \exists n_0 \in \mathbb{N}$, such that for all $n \geq n_0, E_n \subset B_\epsilon(E)$ and $E \subset B_\epsilon(E_n)$, where $B_\epsilon(\bullet)$ is the $\epsilon$ neighborhood of the given set.
In this way, for all \( z \in E \) there is a sequence \( z_n \to z \), such that \( z_n \in E_n \). But there is still a problem to obtain that for all \( z \in E \), \( \rho_V(z) = \omega' \). Because we do not have any control under the uniformity of the vertical rotation numbers of the family of orbits \( E_n \). In the Aubry-Mather case, the periodic orbits, whose limit in the Hausdorff topology is a quasi-periodic set, have a very strong uniformity condition; they are of Birkhoff type (see for instance [14]). Indeed, if we knew that given \( \epsilon > 0 \), \( \exists \tilde{i}(\epsilon) > 0 \) (\( \tilde{i} \) independent of \( n \)), such that \( \forall n > 0 \) and \( \forall z_n \in E_n \)
\[
\left| \frac{p_2 \circ T^{\tilde{i}}(z_n) - p_2(z_n)}{\tilde{i}} - \frac{p_n}{q_n} \right| < \epsilon, \text{ for all } i > \tilde{i},
\]
then the problem would be solved. In order to overcome this problem we use the following important lemma that is a consequence of propositions (1), (2), some ideas from [18] and some results from the Nielsen-Thurston theory:

**Lemma 9**: Under the hypothesis of theorem (5), for all \( \omega' \in (0, \omega) \setminus Q \), there is a quasi-periodic set \( E \), such that \( \rho_V(E) = \omega' \).

**Proof.** See the end of section 3. \( \blacksquare \)

Case II) \( \omega \notin Q \).

From case I), we just have to prove that for all \( 0 < \frac{p}{q} < \omega \) there is a \( q \)-periodic orbit with \( \rho_V = \frac{p}{q} \). The proof of this fact is identical to the proof of theorem (4), so we omit it. \( \blacksquare \)

We still have to prove lemma (1) and theorem (6). The following are auxiliary results that are important in these proofs.

**Lemma 10**: Let \( \overline{T} : T^2 \to T^2 \) be a homeomorphism homotopic to \( LM \) (see (14)), and let \( C \subset T^2 \) be a homotopically non-trivial simple closed curve, \( \overline{T}^s \)-invariant, for some \( s > 0 \). Then \( C \) is a rotational simple closed curve on the cylinder \( S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \). Moreover, \( [C] \) (homotopy class of \( C \)) is the only homotopy class of simple closed curves on the torus that is preserved by iterates of \( \overline{T} \).
Proof.

The action of $T$ on $\pi_1(T^2)$ is given by

$$T_*([C]) = T_*(c_\phi, c_I) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi \\ c_I \end{pmatrix},$$

and the eigenvector corresponding to the eigenvalue 1 is $(1, 0)$.

Lemma 11: Let $f : T^2 \to T^2$ be a homeomorphism isotopic to $LM$. If $\exists \lambda > 0$ such that $f^\lambda$ has a rotational invariant curve $\gamma$ with $[\gamma] = x_\phi = (1, 0)$, then $f$ cannot have a periodic orbit with vertical rotation number $\rho_V \neq 0$ and another with $\rho_V = 0$.

Proof.

Let $\hat{F}_0$ be a lift of $f^\lambda$ to the cylinder which fixes $\hat{\gamma}$, a lift of $\gamma \subset T^2$. This implies that the vertical rotation number for $\hat{F}_0$ of every point is zero. So, given any lift $\hat{F}$ of $f^\lambda$, the vertical rotation number of every point is equal. In particular, given a lift $\hat{f}$ of $f$, the vertical rotation number of every point is the same, which is what we wanted to prove.

We have already seen that all $T \in TQ$ such that $\hat{T}$ is $\mu$-exact and $T$ does not have R.I.C’s induces a map $\hat{T}$ defined on the torus that has periodic orbits with non-zero vertical rotation numbers. Suppose that $\hat{T}$ has a periodic orbit with $\rho_V = \omega > 0$. Given $\omega' \in \mathbb{R} \setminus \mathbb{Q}$, with $0 < \omega' < \omega$, let us choose irreducible fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$, such that

$$0 < \frac{a_1}{b_1} < \frac{a_2}{b_2} \leq \omega,$$

and periodic orbits $Q_1$ and $Q_2$ with $\rho_V(Q_i) = \frac{a_i}{b_i}$ and $\#\{Q_i\} = b_i$, for $i = 1, 2$ (this is possible by theorem (6)).

As $T \in TQ$ and $\hat{T}$ is $\mu$-exact, from lemmas (8) and (9) it is clear that $\exists R \in T^2$ such that $\hat{T}(R) = R$ and

$$p_2 \circ T(\hat{R}) = p_2(\hat{R}) \quad p_1 \circ T(\hat{R}) = p_1(\hat{R}) \pmod{1},$$

for any $\hat{R} \in \hat{p}^{-1}(R)$.
Let $Q = Q_1 \cup Q_2 \cup R$. Now we blow-up each $x \in Q$ to a circle $S_x$. Let $T^2_Q$ be the compact manifold (with boundary) thereby obtained; $T^2_Q$ is the compactification of $T^2 \setminus Q$, where $S_x$ is a boundary component where $x$ was deleted. Now we extend $\mathcal{T} : T^2 \setminus Q \to T^2 \setminus Q$ to $\mathcal{T}_Q : T^2_Q \to T^2_Q$ by defining $\mathcal{T}_Q : S_x \to S_x$ via the derivative; we just have to think of $S_x$ as the unit circle in $T_x T^2$ and define

$$T_Q(v) = \frac{D\mathcal{T}_x(v)}{\|D\mathcal{T}_x(v)\|}, \text{ for } v \in S_x.$$ 

$T_Q$ is continuous on $T^2_Q$ because $\mathcal{T}$ is $C^1$ on $T^2$. Let $b : T^2_Q \to T^2$ be the map that collapses each $S_x$ onto $x$. Then $\mathcal{T} \circ b = b \circ T_Q$. This gives $h(T_Q) \geq h(\mathcal{T})$ (see [15], page 111). Actually $h(T_Q) = h(\mathcal{T})$, because each fibre $b^{-1}(y)$ is a simple point or an $S_x$ and the entropy of $\mathcal{T}$ on any of these fibres is 0 (the map on the circle induced from any linear map has entropy 0). This construction is due to Bowen (see [5]).

Now we have the following:

**Theorem 10**: The map $T_Q : T^2_Q \to T^2_Q$ is isotopic to a pseudo-Anosov homeomorphism of $T^2_Q$.

*Proof.*

By theorem (8), $T_Q$ is isotopic to a homeomorphism $F_Q : T^2_Q \to T^2_Q$ (Thurston canonical form) such that either:

i) $F_Q$ has finite order,

ii) $F_Q$ is pseudo-Anosov,

iii) $F_Q$ is reducible by a system of curves $C$.

We must think of $T^2_Q$ as a torus with round disks removed, all of the same size, each one centered in a point $x \in Q$. Let $F : T^2 \to T^2$ be the completion of $F_Q$, i.e. the homeomorphism obtained by radially extending $F_Q$ into all the holes (see [8]).

It is easy to see that $F_Q$ does not have finite order, because there are points with different rotation numbers (by construction of $Q$).
We say that a simple closed curve $\gamma$ on a torus with holes is rotational if after filling in the holes, $\gamma$ is homotopically non-trivial. Suppose that $F_Q$ has a rotational reducing curve $\gamma$ and let $[\gamma] \in \pi_1(T^2)$ be its homotopy class in the torus without holes. Then, for some $n > 0$, we have:

$$F^n_Q(\gamma) = \gamma \Rightarrow F^n(\gamma) = \gamma.$$  

And, as $F_Q$ is isotopic to $T_Q$, $F$ is isotopic to $LM$. In this way, from lemma (10) the homotopy class of $\gamma$ in the torus $T^2 = S^1 \times S^1$ is $[\gamma] = x_\phi = (1, 0)$ ($\gamma$ is a rotational simple closed curve in the cylinder $S^1 \times \mathbb{R}$). So, from the existence of the periodic orbits $Q_1$ (or $Q_2$) and $R$, applying lemma (11) we conclude that, $F$ and thus $F_Q$ do not have any rotational reducing curves.

And if $\gamma$ is a non-rotational reducing curve, then $\gamma$ must surround at least 2 holes (because $\gamma$ is not homotopic to a component of $\partial T^2_Q$). These holes must have the same rotation number and this is impossible, because $\rho_V(Q_1) \neq \rho_V(Q_2) \neq \rho_V(R) = 0$ and 2 points from the same orbit can not be surrounded by the same curve (by construction of $Q_1$ and $Q_2$).

So $F_Q : T^2_Q \to T^2_Q$ is a pseudo-Anosov homeomorphism.  

Now we prove theorem (6):

Proof. of theorem (6)

We just have to see that after all the previous work, the map $T_Q : T^2_Q \to T^2_Q$ is isotopic to a pseudo-Anosov homeomorphism of $T^2_Q$. Then:

$$h(T) = h(T_Q) \text{ and } h(T_Q) > 0, \text{ by theorem (11)}.  

And finally we prove lemma (9):

Proof. of lemma (9)

By theorem (10), $T_Q : T^2_Q \to T^2_Q$ is isotopic to a pseudo-Anosov homeomorphism, $F_Q : T^2_Q \to T^2_Q$. So, as $Q$ is an invariant and finite set we just have to apply propositions (1) and (2).

4. Examples and applications

We conclude by giving some examples.
1) It is obvious that the well-known Standard map \( S_M : T^2 \to T^2 \) given by

\[
S_M : \begin{cases} 
\phi' = \phi + I' \pmod{1} \\
I' = I - \frac{k}{2\pi} \sin(2\pi \phi) \pmod{1}
\end{cases}
\]

is induced by an element of \( TQ \).

Also, it is easy to see that its generating function is:

\[
h_{S_M}(\phi, \phi') = \frac{(\phi' - \phi)^2}{2} + \frac{k}{4\pi^2} \cos(2\pi \phi) \Rightarrow h_{S_M}(\phi + 1, \phi' + 1) = h_{S_M}(\phi, \phi'),
\]

so \( S_M \) is an exact map. In this way, as we know that for sufficiently large \( k > 0 \), \( S_M \) does not have R.I.C’s, we can apply our previous results to this family of maps. In fact, theorems (3) and (4) can be used to produce a new criteria to obtain estimates for the parameter value \( k_{cr} \), which is defined in the following way: if \( k > k_{cr} \), then \( S_M \) does not have R.I.C and for \( k \leq k_{cr} \) there is at least one R.I.C. This happens because for each \( \frac{1}{n}, n \in \mathbb{N}^* \), there is a number \( k_n \), such that for \( k \geq k_n \), \( S_M \) has a \( n \)-periodic orbit with \( \rho_v = \frac{1}{n} \) and for \( k < k_n \) it does not have such an orbit. From the theorems cited above, if \( n > m \) then \( k_n \leq k_m \) and \( \lim_{n \to \infty} k_n = k_{cr} \). In a future work, we will try to obtain estimates for \( k_{cr} \) using this method.

2) In [1] the dynamics near a homoclinic loop to a saddle-center equilibrium of a 2-degrees of freedom Hamiltonian system was studied by means of an approximation of a certain Poincaré map. In an appropriate coordinate system this map is given by:

\[
\tilde{F} : S^1 \times [0, +\infty[ \to S^1 \times [0, +\infty[,
\]

where \( \tilde{F} : \begin{cases} 
\phi' = \mu(\phi) + \gamma \log(I') \pmod{\pi} \\
I' = J(\phi)I
\end{cases} \)

and

\[
J(\phi) = \alpha^2 \cos^2(\phi) + \alpha^{-2} \sin^2(\phi) \\
\mu(\phi) = \arctan\left(\frac{\tan(\phi)}{\alpha}\right), \mu(0) = 0
\]

So \( J(\phi) \) is \( \pi \)-periodic and \( \mu(\phi + \pi) = \mu(\phi) + \pi \). In this case \( S^1 \) will be identified with \( \mathbb{R}/(\pi \mathbb{Z}) \).

A direct calculation shows that:

\[
h_{\tilde{F}}(\phi, \phi') = \gamma \exp\left(\frac{\phi'}{\mu(\phi)}\right)
\]

24
And so
\[ h_{\tilde{F}}(\phi + \pi, \phi' + \pi) = h_{\tilde{F}}(\phi, \phi'), \] because \(\mu(\phi + \pi) = \mu(\phi) + \pi.\)

Thus \(\tilde{F}\) is also exact. Applying the following coordinate change
\[
\begin{cases}
\tilde{\phi} = \phi \\
\tilde{I} = \gamma \log(I)
\end{cases}
\]
we get (omitting the \(^\sim\)):
\[
\tilde{F} : S^1 \times \mathbb{R} \rightarrow : \begin{cases}
\phi' = F_\phi(I, \phi) = \mu(\phi) + I' \quad (\text{mod } \pi) \\
I' = F_I(I, \phi) = \gamma \log(J(\phi)) + I
\end{cases}
\]

It is obvious that in these coordinates \(\tilde{F}\) is \(\mu\)-exact and \(\mu\) is given by:
\[
\mu(A) = \int_A e^{\frac{1}{\gamma}d\phi}dI
\]

It is also easy to see that \(\tilde{F}\) induces a map \(\tilde{F} : T^2 \rightarrow T^2 \) (\(T^2 = \mathbb{R}^2/(\pi\mathbb{Z})^2\)) given by
\[
\tilde{F} : \begin{cases}
\phi' = F_\phi(I, \phi) = \mu(\phi) + I' \quad (\text{mod } \pi) \\
I' = F_I(I, \phi) = \gamma \log(J(\phi)) + I \quad (\text{mod } \pi)
\end{cases}
\]

that is also induced by an element of \(TQ\).

And from \(\text{[1]}\), \(\exists \alpha_{\text{crit}}(\gamma)\) such that for \(\alpha > \alpha_{\text{crit}}(\gamma)\), \(\tilde{F}\) does not have R.I.C’s.

In this case, we can apply the same criteria explained for the standard map. But as there are 2 parameters, we do not obtain a critical value, we obtain a critical set in the \((\gamma, \alpha)\) plane. Another important application of this theory is to obtain properties about the structure of the unstable set of the above mentioned homoclinic loop (to the saddle-center equilibrium), when the former is unstable. The periodic orbits given by theorem \(\text{[3]}\) were analyzed in \(\text{[2]}\) and it was proved that for every vertical rotation number \(\frac{m}{n} > 0\), there is an open set in the parameter space with a \(\frac{m}{n}\)-periodic orbit which is topologically a sink. In particular, it can be proved that, for a fixed value of \(\gamma > 0\), given an \(\epsilon > \alpha_{\text{cr}}(\gamma) > 1\), where \(\alpha_{\text{cr}}(\gamma)\) is analogous to the constant \(k_{\text{cr}}\) defined for the standard map, there is a number \(\frac{m}{n} > 0\) and an open interval \(I_{\frac{m}{n}} \subset (\alpha_{\text{cr}}(\gamma), \epsilon)\), such that for \(\alpha \in I_{\frac{m}{n}}\), \(\tilde{F}\) has a vertical periodic orbit with \(\rho_V = \frac{m}{n}\) which is also
a topological sink. So we can say that one of the mechanisms that cause the lost of stability of the homoclinic loop is the creation of periodic sinks for $\overline{F}$. And in [3] it was proved that the existence of a topological sink for $\overline{F}$ implies many interesting properties on the topology of the set of orbits that have the saddle-center loop as their $\alpha$-limit set (a set analogous to the unstable manifold of a hyperbolic periodic orbit). More precisely, in this case, given an arbitrary neighborhood of the original homoclinic loop, a set of positive measure contained in this neighborhood escapes from it following (or clustering around) a finite set of orbits that in a certain sense, correspond to the topological sinks for $\overline{F}$.

In a forthcoming paper, we will analyze the following function:

$$\rho_{V}^{\text{max}}(\gamma, \alpha) = \sup_{P \in \mathbb{T}^2} \rho_{V}(P) = \sup_{P \in \mathbb{T}^2} \left[ \lim_{n \to \infty} \frac{p_2 \circ F^n(P) - p_2(P)}{n} \right],$$

where the supremum is taken over all $P \in \mathbb{T}^2$ such that $\rho_{V}(P)$ exists. Using a method developed in [2] and results from [22], we plan to prove the density of periodic sinks in the subset of the parameter space $(\gamma, \alpha)$ where $\overline{F}$ does not have R.I.C’s.

3) Given a $C^2$ circle diffeomorphism $f : S^1 \to S^1$ ($f(\phi + 1) = f(\phi) + 1$) we can define the following generating function:

$$h_f(\phi, \phi') = \exp(\phi' - f(\phi))$$

As $h_f(\phi + 1, \phi' + 1) = h_f(\phi, \phi')$ the associated twist map $\hat{T}_f : S^1 \times [0, +\infty[ \to S^1 \times [0, +\infty[\to$ is exact:

$$\hat{T}_f : \begin{cases} \phi' = f(\phi) + \log(I') \pmod{1} \\ I' = \frac{1}{f'(\phi)} I \end{cases}$$

By the same coordinate change applied to $\hat{F}$

$$\begin{cases} \tilde{\phi} = \phi \\ \tilde{I} = \log(I) \end{cases}$$

we can write $\hat{T}_f$ in the following way:

$$\begin{cases} \phi' = f(\phi) + I' \pmod{1} \\ I' = \log\left(\frac{1}{f'(\phi)}\right) + I \end{cases}$$
As above, in these coordinates $\hat{T}_f$ is $\mu$-exact for the following measure:

$$\mu(A) = \int_A e^I d\phi dI$$

And $\hat{T}_f$ induces a torus map

$$\hat{T}_f: \begin{cases} 
\phi' = f(\phi) + I' \pmod{1} \\
I' = \log\left(\frac{1}{f'(\phi)}\right) + I \pmod{1}
\end{cases}$$

such that our results apply.

**Acknowledgements:** I am very grateful to C. Grotta Ragazzo for listening to oral expositions of these results, reading the first manuscripts and for many discussions, comments and all his support, to J. Mather for all his support and to the referees for a very careful reading of the paper, for the suggestion of a new proof of theorem (3), for comments on how to obtain 2 periodic orbits in theorem (4) in the general case and for all their other remarks that improved the text.

**References**

[1] Addas Zanata S. (2000): *On the dynamics of twist maps of the torus.* Doctoral Thesis. In Portuguese. IMEUSP.

[2] Addas Zanata S. and Grotta Ragazzo C. (2001): On the stability of some periodic orbits of a new type for twist maps. *preprint*

[3] Addas Zanata S. and Grotta Ragazzo C. (2001): Conservative dynamics: unstable sets for saddle-center loops. *preprint*

[4] Aubry S. and Le Daeron P. (1983): The discrete Frenkel-Kontorova model and its extensions. *Physica D 8,* 381-422

[5] Bowen R. (1978): Entropy and the Fundamental Group. *Springer Lec. Notes in Math.* 668, 21-29
[6] Casdagli M. (1987): Periodic orbits for dissipative twist maps. *Ergod. Th. & Dynam. Sys.* 7, 165-173

[7] Epstein D. (1966): Curves on 2-manifolds and isotopies. *Acta Math.* 115, 83-107

[8] Epstein D. (1981): Pointwise periodic homeomorphisms. *Proc. London Math. Soc.* 42, 415-460

[9] Fathi A., Laudenbach F. and Poenaru V. (1979): Travaux de Thurston sur les surfaces. *Astérisque.* 66-67

[10] Franks J. (1989): Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.* 1, 107-115

[11] Grotta Ragazzo C. (1997): On the Stability of Double Homoclinic Loops. *Comm. Math. Phys.* 184, 251-272

[12] Handel M. (1990): The Rotation Set of a Homeomorphism of the Annulus is Closed. *Comm. Math. Phys.* 127, 339-349

[13] Handel M. and Thurston W. (1985): New proofs of some Results of Nielsen. *Adv. Math.* 56, 173-191

[14] Katok A. (1982): Some remarks on the Birkhoff and Mather twist map theorems. *Ergod. Th. & Dynam. Sys.* 2, 185-194

[15] Katok A. and Hasselblatt B. (1995): *Introduction to Modern Theory of Dynamical Systems.* Cambridge University Press

[16] Le Calvez P. (1986): Existence d’orbites quasi-périodiques dans les attracteurs de Birkhoff. *Comm. Math. Phys.* 106, 383-394

[17] Le Calvez P. (1991): Propriétés Dynamiques des Difféomorphismes de L’Anneau et du Tore. *Astérisque* 204
[18] Llibre J. and Mackay R. (1991): Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergod. Th. & Dynam. Sys.* **11**, 115-128

[19] Mather J. N. (1982): Existence of quasi-periodic orbits for twist homeomorphisms of the annulus. *Topology.* **21 No. 4**, 457-467

[20] Meiss J. D. (1992): Symplectic maps, variational principles, and transport. *Rev. Mod. Phys.* **64**, 795-848.

[21] Misiurewicz M. and Ziemian K. (1989): Rotation Sets for Maps of Tori. *J. London Math. Soc. (2)*** **40**, 490-506

[22] Misiurewicz M. and Ziemian K. (1991): Rotation sets and ergodic measures for torus homeomorphisms. *Fund. Math.* **137**, 44-52

[23] Moser J. (1973): *Stable and Random Motions in Dynamical Systems.* Princeton: Princeton University Press.

[24] Nielsen J. (1944): Surface transformation classes of algebraically finite type. *Danske Vid. Selsk. Math. Phys. Medd.* **68**

[25] Thurston W. (1988): On the Geometry and Dynamics of Diffeomorphisms of Surfaces. *Bull. Amer. Math. Society.* **19 (2)** 417-431