Quantum search with interacting Bose-Einstein condensates

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One approach to the development of quantum search algorithms is the quantum walk. A spatial search can be effected by the continuous-time evolution of a single quantum particle on a graph containing a marked site. In many physical implementations, however, one might expect to have multiple particles. In interacting bosonic systems at zero temperature, the dynamics is well-described by a discrete nonlinear Schrödinger equation. We investigate the role of nonlinearity in determining the efficiency of the spatial search algorithm within the quantum walk model, for the complete graph. The analytical calculations reveal that the nonlinear search time scales with size of the search space $N$ like $\sqrt{N}$, equivalent to the linear case though with a different overall constant. The results indicate that interacting Bose-Einstein condensates at zero temperature could be natural systems for the implementation of the quantum search algorithm.

I. INTRODUCTION

Searching is arguably one of the most important problems in computer science. The search problem consists of finding a particular (marked) element in a set containing $N$ items. The simplest classical approach is to uniformly sample the set until the marked element is found, which on average occurs in a time $t = O(N)$. Though the sampling can be recast into the framework of Markov chains, the $O(N)$ scaling of the classical search time is optimal.

Grover [1, 2] constructed an algorithm based on quantum mechanics that can find the marked element of a set quadratically faster, in a time $t = O(\sqrt{N})$. This was proven to be optimal [3]. The quantum search algorithm can be recast in terms of quantum walks, the quantum mechanical extension to Markov chains. The origins of the discrete-time quantum walk can be traced to Meyer [4, 5], and the concept was further developed in 2001 by several others [6–8]. The continuous-time quantum walk, the quantum analog of a continuous-time Markov process, was introduced by Farhi and Gutmann [9] and extended by them and Childs [10]. Numerous algorithms based on the quantum walks were soon developed that were shown to be more efficient than the best-known classical algorithms [11–15]. Among these are quantum walk search algorithms, based both on the discrete-time quantum walk [16–20] and on the continuous-time quantum walk [21–23]. In these approaches, the set corresponds to the vertices of a graph, and the marked element is one distinguishable vertex. While the spatial search time attains the optimal scaling on most graphs, including the complete graph, complete bipartite graphs, $m$-partite graphs [20], the hypercube [16], the Johnson graph [24], and regular lattices with dimension equal to three or greater [22], it remains at best $t = O(\sqrt{N} \log N)$ for quantum walks on the two-dimensional square lattice despite much effort [18, 22, 23, 20], and $t = O(N)$ in one dimension.

Quantum walks have been realized using a variety of experimental approaches. Some of the earliest experiments were based on nuclear magnetic resonance [24, 25]. Three steps of a discrete-time quantum walk were realized with $^{25}$Mg$^+$ ions in a linear Paul trap [29]; longer quantum walks were effected more recently with $^{40}$Ca$^+$ ions [30]. Five steps of a quantum walk were implemented using passive optical elements [31, 32]. Quantum walks have also been implemented with single neutral $^{133}$Cs atoms [33] confined in optical lattices [34].

Because the behavior of quantum walks is governed by quantum interference, it is not necessary to restrict physical systems to single walkers. For example, a quantum walk with two identical photons was demonstrated using evanescently coupled waveguides [35, 36]. In fact, are several indications that quantum walks with multiple indistinguishable particles have unique properties. Non-classical correlations arise between two non-interacting photonic walkers [37, 38]. Two-photon quantum walks with conditioned interactions and strong non-linearities were recently reported [41]. Quantum walks have also been realized using Bose-Einstein condensates of $^{87}$Rb [42]. Some classically intractable problems, such as boson sampling, are efficiently solved using quantum walks with multiple indistinguishable bosons [43–45]. Theoretical work suggests that multiple indistinguishable walkers could help determine if two graphs are isomorphic, with interactions improving the power of the algorithm [45–47]. For suitably defined graphs, quantum walks with multiple interacting walkers are able to perform arbitrary quantum algorithms [48].

Large numbers of indistinguishable bosons at low temperatures can form a Bose-Einstein condensate (BEC). The implementation of a continuous-time quantum walk using BECs is equivalent to allowing the bosons to evolve under a governing lattice or graph Hamiltonian. In the presence of weak particle interactions, the resulting Gross-Pitaevskii (GP) equation of motion in the mean-field approximation is nonlinear [49]. In principle, the presence of nonlinearity in quantum dynamics can radically alter the performance of quantum algorithms [50], even allowing NP-complete problems to be solved in polynomial time. One might therefore conjecture that the timescale for the quantum search problem could be mod-
ified by the presence of nonlinearity. That said, the nonlinearity that appears in the GP equation has its origins in ordinary linear quantum mechanics, which would appear to rule out any improvement in the scaling of the quantum search time with \( N \) (though it could always be worse). In any case, it is important to know how the presence of nonlinearity in the governing equations would affect the performance of a quantum search. The influence of (a physically motivated) nonlinearity on the time to effect a quantum walk spatial search is the central question addressed in this work. The results indicate that interacting BECs can indeed implement a quantum spatial search algorithm.

We consider the quantum search algorithm on the complete graph using a continuous-time quantum walk based on the discrete GP equation. In the complete graph, each site or vertex is connected to every other; a boson at a given site can tunnel or hop to any other site with equal probability. While a physical lattice with the connectivity of the complete graph has not yet been realized experimentally, a recent theoretical proposal to simulate the hypercube graph with ultracold atoms [51] suggests that other graphs with unusual connectivity properties might be experimentally feasible in the future. In any case, the study of the complete graph offers several theoretical advantages. The search time obtained in the linear quantum walk algorithm has previously been shown to be optimal [21]. The neighborhood of every vertex corresponds to all other vertices, so that the quantum walk naturally mimics a uniform sampling of the set elements. Last, the symmetry of the complete graph allows the vertex set to be decomposed into two inequivalent elements: the marked vertex, and the set of unmarked vertices. This allows the \( N \)-dimensional Hilbert space to be reduced to two dimensions, greatly simplifying the analysis of the nonlinear problem.

This manuscript is organized as follows. In Sec. II we review the continuous-time quantum walk approach to the spatial search problem, and derive the associated nonlinear equation of motion for a quantum search based on interacting Bose-Einstein condensates. The analytical results are presented in Sec. III and the criteria for a complete search (unit output probability on the marked vertex in the limit of large \( N \)) and an incomplete search are derived. The performance of the algorithm in the presence of errors is analyzed numerically in this section. The results are summarized in Sec. IV.

II. BACKGROUND

A. Continuous-time quantum walk search algorithm

In the continuous-time quantum walk, the state of the quantum walker \( |\psi\rangle \) is evolved in time by the action of the Hamiltonian \( H_0 = -\gamma L \), where \( L \) is the Laplacian operator and \( \gamma \) is the transition amplitude. Given an \( N \)-dimensional graph \( G = (V, E) \) where \( V = \{1, 2, \ldots, N\} \) and \( E \) correspond to the set of vertices and edges respectively, one can define the Laplacian as \( L = A = D - A \), where \( A = A(G) \) is the adjacency matrix associated to the graph \( G \) and \( D \) is a diagonal matrix whose elements are \( D_{ii} = \sum_j A_{ij} = \deg(i) \), the degree of vertex \( i \) (the inclusion of the diagonal is not needed if the graph is regular). The adjacency matrix specifies the graph connectivity and its matrix elements are defined as

\[
A_{ij} = \begin{cases} 
1 & (i, j) \in E \\
0 & \text{otherwise.} 
\end{cases}
\]  

In the continuous-time quantum walk, one associates each vertex \( i \) to a basis vector \( |i\rangle \); the set of basis vectors spans the \( N \)-dimensional Hilbert space. The state of the quantum walker is \( |\psi(t)\rangle = \sum_i a_i(t) |i\rangle \), where \( a_i(t) \) are time-dependent complex coefficients. The quantum walk is then effected by performing the unitary transformation \( |\psi(t)\rangle = e^{-iH_0 t} |\psi(0)\rangle \) on the particle state vector \( (\hbar \text{ is set to unity in this work).} \]

In the continuous-time quantum walk search algorithm of Childs and Goldstone [21], one of the basis vectors \( |w\rangle \) is treated differently. This is accomplished by introducing a marking or oracle Hamiltonian:

\[
H_w = -|w\rangle \langle w|. 
\]  

The quantum state is then evolved according to the total Hamiltonian \( H = H_0 + H_w \), i.e. \( |\psi(t)\rangle = e^{-iH t} |\psi(0)\rangle \). It was shown that if the initial state is chosen to be the uniform superposition of all sites

\[
|\psi(0)\rangle = |S\rangle \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle, 
\]  

then there exists a time \( t_s \) and value of \( \gamma \) for which the probability on the marked site \( \langle w|\psi(t_s)\rangle^2 = |\psi_w(t_s)|^2 \) will attain unity. For the complete graph, \( t_s = \frac{\pi}{2} \sqrt{N} \) and \( \gamma = \frac{1}{N} \). For the hypercube and an \( m \)-dimensional square lattice for \( m > 4 \), the search time remains \( t_s = O(\sqrt{N}) \).

B. Discrete Gross-Pitaevskii equation

A convenient starting point for the description of interacting BECs is the Gross-Pitaevskii equation [49]

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \Psi(r, t) = \left[ -\frac{1}{2m} \nabla^2 + V(r) + U |\Psi(r)|^2 \right] \Psi(r, t), 
\]  

where \( V(r) \) is some time-independent external potential, \( U \) is the particle interaction strength, and \( \Psi(r, t) \) is the BEC wavefunction. All of the \( M \) bosons are assumed to be in the same single-particle state, so the normalization condition is \( \int dr |\Psi(r, t)|^2 = M \). It is convenient to define the BEC wavefunction in terms of a new wavefunction \( \Psi(r, t) = \sqrt{M} \psi(r, t) \) so that \( \int dr |\psi(r, t)|^2 = 1 \).
Consider functions $V(\mathbf{r})$ corresponding to a series of $N$ potential energy wells, each centered at $\mathbf{r}_j$ with $j = 1, \ldots, N$. A simple example in one dimension would be $V(x) = V_0 \cos^2(\pi x/a)$ in a box of length $L$, where $a$ is some arbitrary length scale and $0 \leq x \leq L$; if $L/a$ is an integer, the potential minima are found at $x/a = (2n + 1)/2$ with $n$ positive integers and $n_{\text{max}} = N = L/a$. If the confinement is sufficiently strong, the particles comprising the BEC will be completely confined to the sites of the potential $V(\mathbf{r})$; reducing the confinement would then allow tunneling between nearby sites. The BEC wavefunction can then be expanded in a basis of Wannier functions $w(\mathbf{r} - \mathbf{r}_j)$ localized to the sites, $\psi(\mathbf{r}) = \sum_j \psi_j w(\mathbf{r} - \mathbf{r}_j)$. Inserting this into Eq. (4), multiplying on the left by $w^* (\mathbf{r} - \mathbf{r}_k)$ and integrating over all space gives

$$i \frac{\partial}{\partial t} \psi_k = - \sum_j \gamma_{kj} \psi_j + g_k |\psi_k|^2 \psi_k,$$

where

$$\gamma_{kj} = - \int d\mathbf{r} w^* (\mathbf{r} - \mathbf{r}_k) \left[ - \frac{1}{2m} \nabla^2 + V(\mathbf{r}) \right] w(\mathbf{r} - \mathbf{r}_j)$$

is the amplitude to tunnel between sites centered at $\mathbf{r}_j$ and $\mathbf{r}_k$, and

$$g_k = MU \int d\mathbf{r} |w(\mathbf{r} - \mathbf{r}_k)|^4$$

is the on-site interaction strength. In deriving Eq. (5), the Wannier functions are assumed to be orthonormal, and to be so strongly localized that the spatial integrals of four Wannier functions are insignificant unless all their arguments are the same.

For all the analytical calculations in this work, we will make the further simplifying assumption that $\gamma_{kj} = \gamma > 0$ for all nearest neighbors $(j, k) \in E$ and that the particle interactions are site-independent $g_k = g$. For mean field theory to remain valid, one requires $g/\gamma \lesssim 5.8z$, where $z$ is the site coordination number (vertex degree) in the limit $z \to \infty$. For the complete graph $K_N$ with $N = |V|$ vertices investigated in the present work, each vertex has $z = N(N - 1)/2$ neighbors. Mean-field theory therefore requires $g/\gamma \lesssim N(N - 1)$ which is easy to satisfy for large $N$.

The discrete GP equation can then be written

$$i \frac{\partial}{\partial t} \psi_k = - \gamma A_{kj} \psi_j + g|\psi_k|^2 \psi_k,$$

where $A_{kj}$ are the matrix elements of the graph adjacency matrix $A$ defining the connectivity of the sites. For graphs with constant degree (i.e. site valency) $d$, the Laplacian is $L = A - (N - 1)I = J - NI$, where $J$ is the $N$-dimensional all-one matrix. In terms of the initial state $|S\rangle$ defined in Eq. (8), the Laplacian is $L = N|S\rangle \langle S| - NI$. Since $-NI$ corresponds to a constant energy shift it can not change the dynamics of the system. The Hamiltonian for the quantum walk search then becomes

$$H = -\gamma N|S\rangle \langle S| - |w\rangle \langle w| + g \sum_{i=1}^N |\psi_i|^2 |i\rangle \langle i|.$$
In the absence of the nonlinear term, this Hamiltonian corresponds to a two-level operator for the states $|S\rangle$ and $|w\rangle$. The Hamiltonian then rotates $|S\rangle$ into $|w\rangle$ in time $t_s \approx \sqrt{N}$ inversely proportional to the two states’ overlap $\langle S|w \rangle = 1/\sqrt{N}$.

It would be desirable to express the nonlinear Hamiltonian (11) as a two-level operator. For the complete graph, the initial condition (3) can be written as

$$|\psi(0)\rangle = |S\rangle = \frac{1}{\sqrt{N}} \left( |w\rangle + \sqrt{N-1} |\alpha\rangle \right),$$

where

$$|\alpha\rangle \equiv \frac{1}{\sqrt{N-1}} \sum_i' |i\rangle$$

labels the state orthogonal to $|w\rangle$ corresponding to the superposition of all unmarked sites, and the prime on the sum denotes the exclusion of $i = w$. The linear part of the Hamiltonian is therefore a two-level operator in $|w\rangle$ and $|\alpha\rangle$. The equation of motion for any site $k \neq w$ takes the form $\dot{\psi}_k = -\gamma \sum_m \psi_m + g|\psi_k|^2 \psi_k$, where $\psi_k \equiv \langle k|\psi \rangle$. Together with the fact that the initial amplitudes are the same on all sites, this means that the amplitudes on all unmarked sites are identical for all times. The nonlinear term can therefore be written

$$H_{NL} = g|\psi_w|^2 |w\rangle \langle w| + g|\psi_w|^2 \sum_i' |i\rangle \langle i|,$$

where $v$ denotes any vertex such that $v \neq w$. Because $\psi_\alpha = \sqrt{N-1}\psi_v$, the nonlinear term becomes

$$H_{NL} = g|\psi_w|^2 |w\rangle \langle w| + g|\psi_w|^2 \sum_i' |i\rangle \langle i|.$$

The equations of motion can be found using

$$\langle w| \frac{\partial}{\partial t} |\psi\rangle = \langle w| H |\psi\rangle; \quad \langle \alpha| \frac{\partial}{\partial t} |\psi\rangle = \langle \alpha| H |\psi\rangle.$$ (16)

After straightforward algebra, one obtains

$$i \frac{\partial}{\partial t} \psi_\alpha = -\gamma(N-1)\psi_\alpha - \gamma \sqrt{N-1} \psi_w + \frac{g}{N-1} |\psi_\alpha|^2 \psi_\alpha;$$

(17a)

$$i \frac{\partial}{\partial t} \psi_w = -\gamma \sqrt{N-1} \psi_\alpha - (1 + \gamma) \psi_w + g|\psi_w|^2 \psi_w,$$ (17b)

with the initial state

$$\begin{pmatrix} \psi_\alpha(0) \\ \psi_w(0) \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} \sqrt{N-1} \\ 1 \end{pmatrix}.$$ (18)

As hoped, the equations of motion for the complete graph have now been reduced to a two-level operator. For a large search space ($N \gg 1$), Eqs. (17) become

$$i \frac{\partial}{\partial t} \psi_\alpha \approx -\gamma N \psi_\alpha - \gamma \sqrt{N} \psi_w + \frac{g}{N} |\psi_\alpha|^2 \psi_\alpha;$$

(19a)

$$i \frac{\partial}{\partial t} \psi_w \approx -\gamma \sqrt{N} \psi_\alpha - (1 + \gamma) \psi_w + g|\psi_w|^2 \psi_w,$$ (19b)

with the initial state

$$\begin{pmatrix} \psi_\alpha(0) \\ \psi_w(0) \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ (20)

Since $\psi_\alpha$ and $\psi_w$ are complex variables, they can be represented as

$$\psi_\alpha \equiv \sqrt{N} \alpha e^{i\theta_\alpha}; \quad \psi_w \equiv \sqrt{N_w} e^{i\theta_w},$$ (21)

where $N_\alpha$ and $N_w$ are the populations of bosons in the states $|\alpha\rangle$ and $|w\rangle$, respectively. Eqs. (19) then correspond to four coupled nonlinear differential equations. To reduce these to two coupled equations, one can define new variables

$$\eta \equiv |\psi_w|^2 - |\psi_\alpha|^2 = N_w - N_\alpha;$$

(22a)

$$\phi \equiv \theta_w - \theta_\alpha;$$

(22b)

Eqs. (19) then become

$$\dot{\eta} = 2\gamma \sqrt{N} \sqrt{1 - \eta^2} \sin(\phi);$$

(23a)

$$\dot{\phi} = \frac{g}{2} \eta - 2\gamma \sqrt{N} \frac{\eta}{\sqrt{1 - \eta^2}} \cos(\phi),$$

(23b)

where $\delta$ is

$$\delta \equiv 1 - N \gamma - \frac{g}{2}.$$ (24)

Eqs. (23) are almost identical to those describing the Josephson dynamics of two weakly coupled Bose-Einstein condensates; see for example Eq. (2.6) in Ref. [60]. Note that taking the large-$N$ limit is not necessary; the general-$N$ case is recovered simply by replacing $N \to N - 1$. The initial conditions for these variables are

$$\eta(0), \phi(0) = \begin{pmatrix} -N - 2/N \\ 0 \end{pmatrix} = \begin{pmatrix} -1 + 2/N \\ 0 \end{pmatrix}.$$ (25)

### B. Complete search: $\delta = 0$

The complete search corresponds to the evolution of the state from the initial state (18) where $\{|\psi_{\alpha}(0)|^2, |\psi_w(0)|^2\} = \left\{ 1 - \frac{1}{N}, \frac{1}{N} \right\}$, corresponding to the maximum probability on the superposition of all unmarked sites, to the state with maximum probability on the marked site, i.e. $\{|\psi_{\alpha}|^2, |\psi_w|^2\} = \left\{ \frac{1}{N}, 1 - \frac{1}{N} \right\}$. In other words, the search is complete when $|\eta(t_s)| = |\eta(0)|$. To determine if this is possible, it is convenient to interpret Eqs. (23) as Hamilton’s equations of motion

$$\dot{\eta} = -\frac{\partial H_C}{\partial \phi}; \quad \dot{\phi} = \frac{\partial H_C}{\partial \eta}$$ (26)
for some classical Hamiltonian $H_C$. It is straightforward to verify that a Hamiltonian satisfying both Eqs. (26) and (23) is

$$H_C = \delta \eta - \frac{g}{4} \eta^2 + 2\gamma \sqrt{N} \sqrt{1 - \eta^2} \cos(\phi).$$  

(27)

This has the same form of classical Josephson Hamiltonian [61]. Since $\frac{\partial H}{\partial \eta} = 0$, this Hamiltonian is a constant of motion. The value of $H_C$ for a trajectory starting at the initial point $\{(\eta(0), \phi(0)) = \{-1 + \frac{2}{N}, 0\}$ must therefore be the same for the desired output $\{(\eta(t_s), \phi(t_s)) = \{1 - \frac{2}{N}, 0\}$. This is only possible if $2\delta (1 - \frac{2}{N}) = 0$ or $\delta = 0$, which corresponds to the homogeneous case. Setting $\delta = 0$ for a complete search specifies a critical value for $\gamma$:

$$\gamma^* = \frac{2 - g}{2N}$$  

(28)

Note that the hopping coefficient must be positive, which requires $g < 2$.

1. Fixed points

The fixed points are obtained by setting Eqs. (23) to zero. These are

$$\phi_1 = 2m\pi, \ m \in \mathbb{Z}; \ \eta_0 = 0;$$  

(29)

and

$$\phi_2 = (2m + 1)\pi; \ \begin{cases} \eta_+ = +\sqrt{1 - \frac{4g(2 - g)}{g^2N}}; \\ \eta_0 = 0; \\ \eta_- = -\sqrt{1 - \frac{4g(2 - g)}{g^2N}}. \end{cases}$$  

(30)

As shown in Fig. 1 $\eta_+$ and $\eta_-$ approach zero as $g$ decreases and they both vanish at $g = g^*$, where

$$g^* = \frac{4}{2 + \sqrt{N}}.$$  

(31)

To make further progress, one must identify the nature of the fixed points; a brief review of these concepts is given in the Appendix. From Eqs. (23) with $\delta = 0$, the functions appearing in the Jacobian (A.5) are

$$a(\eta, \phi) = 2\gamma \sqrt{N} \sqrt{1 - \eta^2} \sin(\phi);$$  

(32a)

$$b(\eta, \phi) = -\frac{g}{2} \eta - 2\gamma \sqrt{N} \frac{\eta}{\sqrt{1 - \eta^2}} \cos(\phi).$$  

(32b)

The Jacobian is therefore

$$J = \begin{pmatrix} \frac{2\gamma \sqrt{N}}{\sqrt{1 - \eta^2}} & -\eta \sin(\phi) \\ -2\frac{\sqrt{1 - \eta^2}}{4\gamma \sqrt{N}} + \frac{\cos(\phi)}{1 - \eta^2} & \eta \sin(\phi) \end{pmatrix}.$$  

(33)

For the first set of fixed points $(\eta, \phi) = (0, 2m\pi)$, the Jacobian matrix becomes

$$J_{(0,2m\pi)} = \begin{pmatrix} 2\gamma \sqrt{N} & -\frac{g}{2} \eta \\ -2\frac{\sqrt{1 - \eta^2}}{\sqrt{4\gamma \sqrt{N}}} & 0 \end{pmatrix},$$  

(34)

where the condition $\gamma^* = (2 - g)/2N$ has been applied. The eigenvalues are

$$\lambda_{\pm} = \pm i \sqrt{\frac{(2 - g)(4 + g(\sqrt{N} - 2))}{2N}}.$$  

(35)

Because $N \gg 1$ and $\gamma^* > 0$, the eigenvalues are strictly imaginary. The fixed points are therefore marginally stable, or centers. The orbit in the vicinity of the fixed point at $(\eta, \phi) = (0, 0)$ is counterclockwise, as shown in black in Fig. 2.

The second set of fixed points correspond to $(\eta, \phi) = \{ (\eta_\pm, 0), (2m + 1)\pi).$. Consider first the case $(\eta, \phi) = (0, (2m + 1)\pi)$. The eigenvalues of the Jacobian matrix are then

$$\lambda_{\pm} = \pm \sqrt{\frac{(2 - g)(-4 + g(\sqrt{N} + 2))}{2N}}.$$  

(36)

If $g > \frac{4}{2 + \sqrt{N}}$ these eigenvalues are both real. One is negative and the other positive, yielding an unstable saddle fixed point. The trajectories in the vicinity of these fixed points are depicted as dark blue vectors in Fig. 3. If $g < \frac{4}{2 + \sqrt{N}}$, then both eigenvalues are imaginary, yielding a marginally stable fixed point or center. The orbits in the vicinity of the fixed points $(\eta, \phi) = (0, \pm \pi)$ are clockwise, as shown in black in Fig. 2.

Consider next the fixed points $(\eta, \phi) = (\eta_\pm, (2m + 1)\pi)$. For both cases, the eigenvalues of the Jacobian matrix are

$$\lambda_{\pm} = \pm \frac{1}{2} \sqrt{\frac{4(2 - g)^2}{N} - g^2}.$$  

(37)

For $g < \frac{4}{2 + \sqrt{N}}$ the solutions $\eta_\pm$ do not exist. For $g > \frac{4}{2 + \sqrt{N}}$, both eigenvalues become imaginary, yielding a marginally stable fixed point or center. The only
difference between the two cases is that trajectories near the \( \eta_+ \) flow counterclockwise, opposite to the clockwise direction for those near \( \eta_- \); these are shown as green and red orbits in Fig. 3.

Fig. 2 clearly shows that all of the fixed points in the \( g \lesssim g^* \) regime are marginally stable or centers. Hence in this regime the trajectory starts from the initial point \((\eta(0), \phi(0)) = (-1 + \frac{2}{N}, 0)\), rotates around the origin, and reaches the final point of the search \((\eta(t_s), \phi(t_s)) = (1 - \frac{2}{N}, 0)\). This behavior is depicted as the light blue line in Fig. 2. Thus, a complete search is attainable in this regime.

In the other regime \( g > g^* \) there is a saddle fixed point \((\eta, \phi) = (\eta_-, \pi)\) near the initial point \((-1, 0)\). The trajectory will continue along the positive \( \phi \) and \( \eta \) will remain close to \( \eta_- \), so that it would never reach \( \eta = +1 \). As shown in Fig. 3, it is not likely to have a complete search in this regime. In principle, one might still have a complete search for \( g \gtrsim g^* \) because the linearization procedure is strictly valid only right at the fixed points, but the range is not likely to be extensive. In any case, for any \( g \in [0, g^*] \) a complete search is attainable. The value \( g = g^* \) will be chosen for the remainder of the calculations.

2. Search time

Now that the nonlinear quantum search on the complete graph has been shown to be successful for a range of interaction strengths \( g \), it is important to determine the scaling of the search time \( t_s \) with the number of sites for large \( N \). As shown in Fig. 2, the trajectory closely resembles a rectangle. Consider the dynamical equations (23) with \( \delta = 0 \) and the large-\( N \) values of \( g = g^* \approx 4/\sqrt{N} \) and \( \gamma = \gamma^* = (2 - g^*)/2N \approx 1/N \):

\[
\dot{\eta} = \frac{2}{\sqrt{N}} \sqrt{1 - \eta^2} \sin(\phi); \quad (38a)
\]

\[
\dot{\phi} = -\frac{2}{\sqrt{N}} \eta \left[ 1 + \frac{1}{\sqrt{1 - \eta^2}} \cos(\phi) \right], \quad (38b)
\]

For \( N \gg 1 \), the right hand sides of Eq. (38) approach zero, and the trajectories are approximated by \( \eta(t) \approx k_1 \) and \( \phi \approx k_2 \), with \( k_1 \) and \( k_2 \) constants. These can be approximately decomposed into the following steps:

(I) Constant \( \eta \): the initial point, \((\eta, \phi) \approx (-1, 0) \rightarrow (-1, \phi_c)\), where \( \phi_c \) is the intersection of the trajectory with the \( \phi \) axis;

(II) Constant \( \phi \): \((\eta, \phi) \approx (-1, \phi_c) \rightarrow (1, \phi_c)\);

(III) Constant \( \eta \): \((\eta, \phi) \approx (1, \phi_c) \rightarrow (1, 0)\).

The search time \( t_s \) can be written as

\[
t_s = \int_{(I)} dt + \int_{(II)} dt + \int_{(III)} dt. \quad (39)
\]

For the first and third steps \( \eta \) is approximately constant, just as \( \phi \) is approximately constant for the second step, so

\[
t_s \approx \int_0^{\phi_c} \frac{d\phi}{\phi} + \int_{-1}^1 \frac{d\eta}{\eta} + \int_{\phi_c}^0 \frac{d\phi}{\phi}. \quad (40)
\]

The initial condition (23) gives a trajectory \( \eta(t) \approx -1 \) and \( \phi \rightarrow \infty \) so that the first and third integrals make an insignificant contribution to \( t_s \). As shown in Fig. 4, the \( \eta \approx k_1 \) transition \( \phi = 0 \rightarrow \phi_c \) is much faster than the
\( \phi \approx k_2 \) transition \( \eta = -1 \to 1 \) (during which the phase hovers in the vicinity of \( \phi_c \)). This system is an example of a relaxation oscillator \[22\]. Therefore one can express \( t_s \) as

\[
t_s \approx \int_{-1}^{1} \frac{d\eta}{\eta} \approx \frac{\sqrt{N}}{2 \sin(\phi_c)} \int_{-1}^{1} \frac{d\eta}{1 - \eta^2} = \frac{\pi}{2} \left( \frac{1}{\sin(\phi_c)} \right) \sqrt{N} \tag{41}
\]

for large \( N \). Besides the factor of \( 1/\sin(\phi_c) \), this is the usual expression for the spatial search time.

Setting \( \delta = 0 \), \( g = g^* \), and \( \gamma = \gamma^* \), the classical Hamiltonian \( H_C \) in Eq. \[27\] takes the form:

\[
H_C = \frac{2\sqrt{1 - \eta^2} \cos(\phi) - \eta^2}{2 + \sqrt{N}}. \tag{42}
\]

For the initial condition \( \eta(0) = -1 + \frac{\delta}{1} \approx -1 \) for \( N \gg 1 \), the classical Hamiltonian becomes \( H_C \approx -\frac{1}{\sqrt{N}} \) and this value is preserved during the evolution. When the trajectory depicted in Fig.\[2\] crosses the \( \phi \) axis at the point \( (\eta, \phi) = (0, \phi_c) \), the Hamiltonian is approximately \( H_C \approx 2\cos(\phi_c)/\sqrt{N} \). The value of the phase at this point is therefore \( \phi_c = \cos^{-1}(-1/2) = 2\pi/3 \). This is consistent with the time-evolution of the phase shown in Fig.\[1\]. The time for the nonlinear search in the large-\( N \) limit, Eq. \[11\], is therefore

\[
t_s = \frac{\pi}{2} \frac{2}{\sqrt{3}} \sqrt{N} = \frac{\pi}{\sqrt{3}} \sqrt{N}, \tag{43}
\]

slower than the linear search by a constant factor of \( 2/\sqrt{3} \approx 1.155 \).

It is important to check that the linear search time is recovered in the case \( g \to 0 \). In this case one has \( \gamma^* = 1/N \) (valid for all \( N \)). The trajectories will then cross the \( \eta \) axis at the initial condition \( (\eta(0), \phi(0)) = (-1 + 3, 0) \) when the classical Hamiltonian \[27\] takes the value \( H_C \approx \frac{2}{\sqrt{N}} \), valid for \( N \gg 1 \). When the trajectory crosses the \( \phi \) axis at \( (\eta, \phi) = (0, \phi_c) \), the classical Hamiltonian becomes \( H_C = \frac{2}{\sqrt{N}} \cos(\phi_c) \). Because the Hamiltonian is a constant of the motion, one obtains \( \phi_c \approx \cos^{-1} \left( \frac{2}{N} \right) \approx \pi/2 \). The search time for the linear problem, Eq. \[11\], is then \( t_s = (\pi/2)\sqrt{N} \), consistent with expectations.

3. Errors

In the foregoing analysis, it has been assumed that the initial amplitudes on all sites are always identical, as are the amplitudes to hop from site to site. The derivation of the nonlinear Hamiltonian \[14\] is only valid under these conditions. Relaxing these assumptions prevents the reduction of the \( N \)-vertex system to a two-dimensional problem. Instead, one must solve \( N \) simultaneous coupled nonlinear differential equations of the form

\[
\frac{\partial}{\partial t} \psi_j = -\sum_{k=1}^{N} \gamma_{jk} \psi_k - \psi_j \delta_{jw} + g|\psi_j|^2 \psi_j, \tag{44}
\]

where \( j = 1, \ldots, N \) including the marked site \( j = w \).

Consider first the possibility that the initial state is not the uniform superposition of all sites, but instead some arbitrary input. Let the initial state be

\[
|\psi(0)\rangle = \frac{1}{\sqrt{N + N_s}} \sum_{j=1}^{N} \sum_{\epsilon} \sqrt{1 + \epsilon^2} e^{i\epsilon t_s} |j\rangle, \tag{45}
\]

where the uniform amplitudes of Eq. \[3\] on each site have now been deformed by a random real number \( |\epsilon| \leq \epsilon_{\max} \) as well as phases in the range \( \{-\epsilon_{\max}, \epsilon_{\max}\} \). Eqs. \[44\] were solved numerically in Mathematica with \( g = g^* = 4/\sqrt{N_{\gamma}} \), \( \gamma_{ij} = \gamma^* = (2 - g)/2N \), and \( t = t_s = \pi \sqrt{N}/3 \), and the resulting probability of finding the marked site \( |\psi_{w}(t_s)|^2 \) is plotted in Fig. \[5\] for the particular case \( N = 600 \). While the data show some fluctuations due to the randomization, the results clearly indicate that the nonlinear search can be found with probability exceeding 90% for errors \( \epsilon_{\max} \lesssim 0.15 \). Surprisingly, if the initial state is assumed to possess phase coherence (i.e. no randomized phases are included) then the marked site can be obtained with probability exceeding 90% for \( \epsilon_{\max} \lesssim 0.8 \), as shown in the inset of Fig. \[5\]. Note that the \( \epsilon_{\max} = 1.0 \) case corresponds to a completely random (but constant phase) initial state. These results indicate that the nonlinear quantum search is robust against initialization noise.

Consider second the possibility that the hopping amplitudes \( \gamma_{ij} \) can vary while the initial state is assumed to be uniform. The simplest case is to consider the effect of randomly deleting edges, i.e. to suppose that \( \gamma_{ij} = \gamma^* \) for some fraction \( \epsilon_{\min} \) of the edges, and is zero otherwise. Given a random variable \( \epsilon_{ij} \in \{0, 1\} \) then one can define

\[
\gamma_{ij} = \begin{cases} 
\gamma^* & \epsilon_{ij} > \epsilon_{\min} \\
0 & \text{otherwise},
\end{cases} \tag{46}
\]
decrease monotonically with $\epsilon$ requires to ensure 90% probability or better on the marked site. Consider for concreteness the time quantum walk [59]. If one adds the supplementary condition that $\gamma_{ij} > 2\gamma^* \epsilon_{\text{min}}$, the output probability does not exceed 90% can be obtained with approximately 12% of the complete graph’s edges deleted, compared with only approximately 2% in the unweighted case.

C. Incomplete search: $\delta \neq 0$

As discussed at the beginning of Sec. [III-B] a complete search is equivalent to finding a trajectory that makes the transition $\eta = -1 + \frac{2}{N} \rightarrow 1 - \frac{2}{N}$ possible. Because $H_C$ is a constant of the motion, however, a complete search requires $\delta = 0$. That said, it is conceivable that setting $\delta \neq 0$ could yield a time $t_s$ where the relative occupation of the marked site would instead be $\eta \lesssim 1 - \frac{2}{N}$. Consider again the fixed points of Eqs. (23). Evidently $\phi = m\pi$ ensures that the right hand side of Eq. (23a) is zero, but $\eta = 0$ no longer accomplishes this for the right hand side of Eq. (23a) because $\delta = 1 - N\gamma - g/2 \neq 0$ by assumption. Now that the search can not be complete, it is not necessary to keep the exact expressions for finite $N$, and one can work entirely in the limit $N \gg 1$. If one again sets $\gamma = \gamma^* = 1/N$, then $\delta = -g/2$. Choosing $g = \alpha/N^\beta$, where $\alpha$ and $\beta$ are both positive real numbers, gives $\delta \sim N^{-\beta}$ which vanishes for large $N$. Eqs. (23) then become

\begin{align}
\dot{\eta} &= \frac{2\sqrt{1-\eta^2}}{N} \sin(\phi); \\
\dot{\phi} &= -\frac{\alpha(1 + \eta)}{2N^\beta} \frac{2\eta}{\sqrt{N} \sqrt{1-\eta^2}} \cos(\phi).
\end{align}

To determine the maximum value reached by $\eta$, consider the classical Hamiltonian [27] which now has the
form
\[ H_C = -\frac{\alpha\eta(2 + \eta)}{4N^3} + 2\sqrt{1 - \eta^2} \cos(\phi). \] (48)

At the initial condition \((\eta, \phi) \approx (-1, 0)\), the classical Hamiltonian is \(H_C(0) = \alpha/4N^3\) which is a constant of the motion. Note that for the incomplete search, the classical Hamiltonian is now positive. The accessible values of \((\eta, \phi)\) are found by setting \(H_C = H_C(0)\). The \(N\)-dependence disappears if \(\beta = 1/2\) (i.e. \(g = \alpha/\sqrt{N}\)), and the only two real solutions correspond to \(\eta = -1\) and
\[ \eta_2 = -1 + \frac{4(3x)^{1/3}}{3\alpha^2} - \frac{16\cos^2(\phi)}{(3x)^{1/3}}, \] (49)
where
\[ x = \alpha^3\cos^2(\phi)\sqrt{3\sqrt{27\alpha^2 + 64\cos^2(\phi)}} + 9\alpha^4\cos^2(\phi). \] (50)

Though the solution \((49)\) is a bit unwieldy, a few observations can be immediately made. For small and large \(\alpha\), one obtains respectively
\[ \eta_2(\alpha \ll 1) \approx 1 - \frac{\alpha^2}{8} \sec^2(\phi); \] (51)
\[ \eta_2(\alpha \gg 1) \approx -1 + 4 \left( \frac{2\cos^2(\phi)}{\alpha^2} \right)^{1/3}. \] (52)

Note that only even powers of \(\alpha\) appear in these expansions, indicating that the dynamics is unaffected by the sign of the nonlinearity. Eq. (51) shows that a complete search is only possible for the non-interacting case \(\alpha = 0\), which is consistent with \(\delta = 0\). For any finite interaction strength in this \(\delta \neq 0\) regime, a complete search is not possible. The relative fraction on the marked site decreases with \(\alpha\), and Eq. (52) reveals that it is asymptotically zero for very large nonlinearities (though one still requires \(g < 2\)). Physically, the large interaction between atoms favors the initial state which is the superposition of occupying all sites of the graph. This is the dynamical self-trapping which has been noted previously for interacting Bose-Einstein condensates [63, 64].

Interestingly, \(\eta_2^{\text{max}}\) is independent of the size of the search problem (keep in mind however that \(g = \alpha/\sqrt{N}\) so that the strength of the nonlinearity decreases with \(N\)). Note also that the maximum probability is reached for \(\phi = 0\). For example, \(\eta_2^{\text{max}} = 0.99\) requires \(\alpha \approx 0.28\). The case \(\eta_2 = 0\) can be obtained directly from Eq. (49), and one obtains \(\alpha = \pm 8\cos(\phi)\). At this value of \(\alpha\), the probability of occupying the marked site exactly 1/2; for any smaller \(\alpha\) it is larger.

It remains to calculate the time for the incomplete search \(t_s\). As in the \(\delta = 0\) case, the right hand sides of Eqs. (47) approach zero, which means that the trajectories are approximated by constant lines. Again, the \(\phi\) evolution is much faster than the \(\eta\) evolution for the \(\eta \approx \eta_1 = -1\) trajectory. The \(\eta \approx \eta_2\) trajectory is not going to be as fast because the \(\sqrt{1 - \eta_2}\) term in Eq. (47) is no longer almost zero. The search time is then approximately
\[ t_s \approx \int_{-1}^{1} \frac{d\eta}{\eta} + \int_{0}^{\phi_c} \frac{d\phi}{\phi} = \frac{\pi\sqrt{N}}{2\sin(\phi_c)} + \int_{0}^{\phi_c} \frac{d\phi\sqrt{N}}{2\sin(\phi_c)} \frac{\alpha(1+\eta)}{2\eta^2 - 2\eta^2 \cos(\phi)}. \] (53)

The critical angle \(\phi_c\) is obtained by equating the classical Hamiltonian \((48)\) for \((\eta, \phi) = (0, \phi_c)\) with \(H_C(0)\); this gives \(\alpha/4\sqrt{N} = 2\cos(\phi_c)/\sqrt{N}\) or
\[ \phi_c = \cos^{-1} \left( \frac{\alpha}{8} \right) \approx \frac{\pi}{2} - \frac{\alpha}{8} \] (54)
for small \(\alpha\). For the linear case \((\alpha = 0)\), the critical angle coincides with that found in the previous section for the complete search. The first term in Eq. (53) is therefore \(\pi\sqrt{N}/2\sin(\phi_c) \approx (\pi\sqrt{N}/2)/\sqrt{1 - \alpha^2}/64\) which restricts the strength of the nonlinearity to \(|\alpha| < 8\) (recall that for \(|\alpha| > 8\) the probability on the marked state is less than one half).

The second term can be simplified by assuming that the value of \(\eta\), given in Eq. (49), remains approximately constant over the range of integration \(\{-\phi_c, 0\}\). In fact, plotting \(\eta\) for a range of \(\alpha\) shows that it varies from 0 to its maximum value, Eq. (51), but this increase occurs only over a very small region in the vicinity of \(\phi = \phi_c\). For small \(\alpha\), the time for the incomplete search, given by Eq. (53), can be approximated as
\[ t_s \approx \frac{\pi}{2} \sqrt{N} \left(1 + \frac{\alpha^2}{128}\right) - \sqrt{N} \int_{-\phi_c}^{\phi_c} \frac{d\phi}{\alpha + 4\cos(\phi)/\alpha} \approx \frac{\pi}{2} \sqrt{N} \left[1 + \frac{\alpha}{2\pi} \ln \left( \frac{\alpha}{16} + \frac{\alpha^2}{128} \right) \right]. \] (55)

Because the \(\alpha\)-dependent correction terms are negative in the regime \(0 < \alpha < 8\), the time for the incomplete search is generally shorter than that for the complete search by a small factor dependent on the strength of the nonlinearity.

IV. CONCLUSIONS

In this work we considered the spatial search algorithm on lattice with the topology of the complete graph, under the assumption that the continuous-time quantum walk is effected by a zero-temperature Bose-Einstein condensate. In the mean-field approximation, the equations of motion become nonlinear and correspond to the discrete Gross-Pitaevskii equation. The analytical results, using methods in nonlinear dynamics and numerical calculations, indicate that a complete spatial search remains possible even in the presence of nonlinearity.
For a successful search, the nonlinear coupling constant must decrease with the system size as $N^{-1/2}$ and the inter-site hopping amplitude decreases as $N^{-1}$, where $N$ is the number of sites. The latter condition coincides with the criterion for a complete search found for the linear search problem on the complete graph [21]. Under these circumstances, the search time is found to scale as $t_s \propto \sqrt{N}$, with an overall constant factor that depends weakly on the strength of the nonlinearity. The probability of success generically decreases with the strength of nonlinearity, but there are particular choices of parameters where the success probability can be made unity. The numerical results further indicate that random errors in the input state amplitudes and the hopping amplitudes are not deleterious for the performance of the algorithm, but that the inclusion of phase errors in the input state or edge deletions quickly erode the probability of finding the marked site. Overall, the quantum search is found to be robust to error under a variety of conditions. It would be interesting to explore the influence of non-unitary (thermal) noise on the performance of the nonlinear search. In the limit of zero nonlinearity, the present results in the absence of error completely recover those of Ref. [21].

The dynamics of the nonlinear system agree closely with those of the linear quantum walk. This indicates that nonlinearity, as long as its strength is kept bounded for a given system size, is no impediment to the implementation of a quantum spatial search. The results suggest that Bose-Einstein condensates consisting of huge numbers of particles can be candidates for the implementation of useful quantum algorithms. The hopping amplitude and the strength of the nonlinearity need to be adjusted for a given size of the search space. In ultracold atomic gases, for example, in principle the former could be accomplished by adjusting the depth or spacing of a lattice [34], and the latter through the use of Feshbach resonances [65], though constructing a lattice with the connectivity of the complete graph it is not currently feasible in these systems.

It would be preferable in practice to conduct the spatial search on a regular lattice, for example a square lattice in three or lower dimensions. Unfortunately, continuous-time quantum walks based on the ordinary discrete Schrödinger equation do not provide useful quantum speed-ups on these lattices over the classical search time, though discrete-time quantum walks can be constructed that do [13]. That said, if the particle dispersion relation is linear rather than quadratic, the full quantum speed-up is achievable on a three-dimensional square lattice [22]. One practical strategy to achieve this is to artificially induce Dirac fermions by suitably preparing an optical lattice [66]. More directly, the excitations of a zero-temperature weakly interacting BEC on a lattice are characterized by a linear dispersion relation, which is a hallmark of the underlying superfluidity in these systems [49]. It is therefore conceivable that the nonlinear mean-field dynamics of the BEC on a cubic lattice would yield the full quantum speed-up for the continuous-time spatial search problem. This possibility will be explored in future work.

During the preparation of this manuscript, we became aware of other work that investigated the same system addressed in the present study [67]. While their results are consistent with ours when there is overlap, in their work the strength of the nonlinear coupling is assumed to vary with time. Their methods and conclusions are therefore complementary to ours.

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Appendix: Characterizing fixed points

This discussion follows Ref. [62]. Consider a general two-dimensional nonlinear system of equations:

$$\dot{u} = a(u,v); \quad \dot{v} = b(u,v).$$

A fixed point or equilibrium point is defined by

$$\dot{w} = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0. \quad (A.1)$$

Suppose $(u^*, v^*)$ is a fixed point for this system, satisfying $a(u^*, v^*) = b(u^*, v^*) = 0$. Let $u' = u - u^*$ and $v' = v - v^*$. For small $u'$ and $v'$

$$\dot{u}' \approx a(u^*, v^*) + \frac{\partial a}{\partial u}(u^*, v^*)u' + \frac{\partial a}{\partial v}(u^*, v^*)v'; \quad \quad (A.2a)$$

$$\dot{v}' \approx b(u^*, v^*) + \frac{\partial b}{\partial u}(u^*, v^*)u' + \frac{\partial b}{\partial v}(u^*, v^*)v'. \quad (A.2b)$$

Since $(u^*, v^*)$ is a fixed point, $a(u^*, v^*) = b(u^*, v^*) = 0$. Close to the fixed points $(u' \ll 1, v' \ll 1)$, Eqs. (A.2) can be written as

$$\dot{w}' \approx J w', \quad (A.3)$$

where

$$w' = \begin{pmatrix} u' \\ v' \end{pmatrix}, \quad (A.4)$$

and $J$ is the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} \\ \frac{\partial b}{\partial u} & \frac{\partial b}{\partial v} \end{pmatrix}(u^*, v^*). \quad (A.5)$$
The general solution of Eq. (A.3) when $J$ is non-degenerate and invertible is

$$w(t) = c_1 z_1 e^{\lambda_1 t} + c_2 z_2 e^{\lambda_2 t},$$

(A.6)

where $\lambda_i$ and $z_i$ are the eigenvalues and eigenvectors of the Jacobian matrix $J$, respectively; $c_1, c_2$ are constants which are determined by the initial conditions. There are four possibilities for the stability of the fixed points:

1. $\lambda_1$ and $\lambda_2$ are both real:
   
   (a) **Stable fixed point**: If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $w' \to 0$ as $t \to \infty$.
   
   (b) **Unstable fixed point**: If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $w' \to \infty$ as $t \to \infty$.
   
   (c) **Unstable saddle point**: If $\lambda_1 < 0 < \lambda_2$, then if $w'(0)$ is a multiple of $z_1$ then $w' \to 0$ as $t \to \infty$ (stable along direction of $z_1$); alternatively if $w'(0)$ is a multiple of $z_2$ then $w' \to \infty$ as $t \to \infty$ (unstable along direction of $z_2$).

2. **Marginally stable fixed point / center**: $\lambda_1$ and $\lambda_2$ are both complex. If $\Re(\lambda_i) = 0$ then $|w'| \to \text{const}$ as $t \to \infty$. Trajectories circulate around the fixed point and eventually return to the initial point; these are closed orbits.

The Hartman-Grobman theorem states that the dynamics of the linearized system in the vicinity of hyperbolic fixed points, where $\Re(\lambda_i) \neq 0$, will be similar to that of the original nonlinear system. If one or both eigenvalues don’t satisfy this condition, the fixed point is non-hyperbolic and therefore the dynamics are fragile to the inclusion of nonlinearity. That said, suppose that $(w^*, v^*)$ is an isolated fixed point that is marginally stable, and there exists a conserved quantity $HC(u,v)$. If $(w^*, v^*)$ is a local minimum of $HC(u,v)$, then all the trajectories sufficiently close to $(w^*, v^*)$ are closed.
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