O(N) VECTOR MODELS IN THE LIMIT $g \to g_c$ AND FINITE TEMPERATURE$^*$

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ABSTRACT

In the limit where $N \to \infty$ and the coupling constant $g \to g_c$ in a correlated manner, O(N) symmetric vector models represent filamentary surfaces. The purpose of these studies is to gain intuition for the long lasting search for a possible description of quantum field theory in terms of extended objects. It is shown here that a certain limiting procedure has to be followed in order to avoid several difficulties in establishing the theory at a critical negative coupling constant. It is also argued that at finite temperature a certain metastable-false vacuum disappears as the temperature is increased.

From a phenomenological viewpoint it is desirable to be able to describe short and long distance hadronic phenomena by very distinct descriptions of quantum field theory: (1) local quanta (2) extended objects, respectively. The understanding of quantum field theories in terms of extended objects is, however, a long lasting problem in elementary particles theory. In the limit, where $N \to \infty$ and the coupling constant $g \to g_c$ in a correlated manner, O(N) symmetric vector models represent filamentary surfaces$^{1-3}$ randomly branched polymers. This is in the same manner as matrix models provide representations of dynamically triangulated random surfaces in their double scaling limit$^4$. The surfaces and polymer chains are represented by the Feynman graphs of the matrix and vector model, respectively. The double scaling limit in O(N) vector quantum field theories reveals an interesting new phase structure, as was argued also in the case of matrix models. Though matrix theories attract most attention, a detailed understanding of the extended objects description of these theories exists, to some extent, only for dimensions $d \leq 2$. On the other hand, in many cases, the O(N) vector models can be successfully studied also in dimensions $d \geq 2$ and thus gaining intuition for the search for a possible description of quantum field theory in terms of extended objects in four dimensions (see for example Ref.$^5$).

At zero temperature, the self interacting scalar O(N) symmetric vector model in $d$ Euclidian dimensions is defined by the functional integral

$$Z_N = \int \mathcal{D}\Phi \exp\left\{-\int d^d x \left\{ \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{\rho_0^2}{2} \Phi^2 + \frac{\lambda_0}{4} (\Phi^2)^2 \right\} \right\}$$

where $\Phi$ is an N-component real scalar field, and we denote $g_0 \equiv \lambda_0 N$.

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In the conventional treatment of the large N limit Eq. [1] is expressed by:

\[ Z_N = \int D\Phi \int D\sigma \int Dm^2 \exp \left\{ -\int d^d x \left\{ \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{\mu_0^2}{2} \sigma + \frac{\lambda_0}{4} \sigma^2 - \frac{m^2}{2} (\sigma - \Phi^2) \right\} \right\}. \] [2]

The double scaling limit, \( N \to \infty \) and \( g_0 \to g_c \), of the O(N) vector models results in an expansion in inverse powers of N.

\[ \ln Z_N = \sum_{G,b} N^{(1-G)} \left( \frac{N}{\beta} \right)^b F_b \] [3]

There is a clear classification of the Feynman graphs contributing to each power of \( N^{-G} \) and the expansion resembles an expansion of randomly branched polymers\(^{1-3}\). Following here the suitable scaling procedure, Eq. [3] turns into an expansion in a "polymer" coupling constant, where its criticality is due to the potential and the centrifugal barrier in Eq.[1].

Eq.[3] is the analog of the genus expansion in matrix models. The double scaling limit in large N matrix models in zero dimensions provides a nonperturbative treatment of string theory \(^4\). This limit is taken in the calculation of the partition function

\[ Z_N(g) = \int D\Phi e^{-\beta Tr(U(\Phi))} \] [4]

where \( \Phi \) is an NxN Hermitian matrix , \( U(\Phi) \) is the potential, depending on the coupling constant(s) \( g \). The measure \( D\Phi \) can be written in the form

\[ D\Phi = \prod_i^N d\phi_i \prod_{i<j} d\text{Re}\phi_{ij} d\text{Im}\phi_{ij} = \prod_i^N d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 d\omega \] [5]

After performing the integration on the U(N) angle variables \( d\omega \), one is left with the integration on the eigenvalues \( \lambda_i \)

\[ Z_N(g) = \Omega_N \int \prod_i^N d\lambda_i \exp \left\{ 2 \sum_{i,j} \ln |\lambda_i - \lambda_j| - \beta \sum_i U(\lambda_i) \right\}. \] [6]

In Eq. [6] one notes a Pauli repulsion between the eigenvalues, and a critical point \( g = g_c \) is found, when the Fermi level reaches the extremum of the potential \(^4\). The genus expansion of the fixed area partition function is:

\[ \ln Z_N = a + b \ln \beta + \sum_{G,S} N^{2(1-G)} \left( \frac{N}{\beta} \right)^A F_S \] [7]

Following the suitable scaling procedure in the double scaling limit; \( N=\beta \to \infty \) and \( g \to g_c \) in a correlated way, Eq. [7] turns into an expansion in the string coupling constant since every genus is relevant now.

Returning now to the O(N) vector model, the singularity structure in the coupling constant \( g_0 \) of each term in the expansion is determined from the leading \( \frac{1}{N} \)
term. In order to calculate the leading term in the effective action one integrates out first $\tilde{\phi}(x)$, (where $\Phi(x) = \phi_c + \tilde{\phi}(x)$) and $\sigma(x)$. The $m^2(x)$ integration is carried out by a saddle point integration. The effective action $S_{eff}(\phi_c)$ is then obtained in the large $N$ limit. $L^{-d}S_{eff}(\phi_c)$ is proportional to the free energy per unit volume$^6$ at fixed $\phi_c = L^{-d}\int d^d x \Phi(x)$.

One finds:

$$e^{-S_{eff}(\phi_c)} = C e^{-\frac{N^2 d}{4} \left\{ \int \frac{d^d p}{(2\pi)^d} \ln \left( \frac{\pi^2}{2N\lambda_0} \right) - (2N\lambda_0)^{-1}(m^2 - \mu_0^2)^2 + m^2 \left( \frac{\phi_c^2}{2} \right) \right\}}$$

$$\int D\alpha(x) e^{-\frac{N}{2} \left\{ \int d^d x \alpha(x) \left( \int \frac{d^d p}{(2\pi)^d} \frac{\alpha(\alpha(\alpha(x)))}{p^2 + m^2} - (N\lambda_0)^{-1}(m^2 - \mu_0^2)^2 + \left( \frac{\phi_c^2}{2} \right) \right) \right\}} \left[ \int d^d y \alpha(x) \int \frac{d^d p}{(2\pi)^d} e^{i p(x-y)} \left( \Sigma(p) + \frac{1}{N\lambda_0} \right) \right] + O(\alpha^3)$$

where $m^2(x) = m^2 + \alpha(x)$ and $m^2 = m^2(\tilde{\phi}_c^2)$ is the solution of the gap equation-saddle point condition:

$$m^2 = \mu_0^2 + \lambda_0 N \left\{ \int \frac{d^d k}{(2\pi)^d} \left( \frac{1}{k^2 + m^2} \right) + \frac{\phi_c^2}{2} \right\}$$

where $\tilde{\phi}_c$ has been rescaled by a factor of $N$.

The double scaling limit$^{1-3}$ is reached in the limit at which $N \to \infty$ is correlated with the limit $\lambda_0 \to \lambda_{oc}$, where $\lambda_{oc}$ is the value of the coupling constant at which the $O(\alpha^2)$ term at $p=0$ in Eq.[8] vanishes (This can be formulated either in terms of the unrenormalized theory or in the renormalized theory, where $\lambda_{Renorm.} \equiv \lambda \to \lambda_c$).

Namely,

$$\Sigma(p = 0) + \frac{1}{N\lambda_{oc}} = 0$$

where

$$\Sigma(p) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k-p)^2 + m^2][k^2 + m^2]}$$

In physical terms, the double scaling limit is reached when the self coupling $\lambda_0$ of $\Phi$ reaches the value $\lambda_{oc}$, at which the strength of the force between the quanta of the $\Phi$ fields, the fundamental scalars in the theory, binds them together strongly enough to create a massless scalar, $O(N)$ singlet, $\Phi \cdot \Phi$ bound state. One also notes that the ground state is $O(N)$ symmetric and the physical mass $m^2$ of the $\Phi$ scalars is the solution of the gap equation at $\phi_c^2 = 0$, namely, $m^2 = m^2(\phi_c^2) = 0$ (The minimum of the effective action is at $\phi_c^2 = 0$).

Following the above procedure, it has been shown in Ref.7 that in $d=4$ the effective potential is complex and the critical vector model cannot be consistently defined. This property is closely related to the triviality of $\lambda \Phi^4$ in $d=4$ in the large $N$ limit$^8$. Namely, at large $N$, the positively coupled theory ($\lambda_0 > 0$) is trivial, while the negatively coupled theory ($\lambda_0 < 0$) is inconsistent for large enough $|\lambda_0|$. It has been also shown$^9$ that in $d=2$ a similar problem occurs, associated with the absence of massless bound states in two dimensions.

In fact, it can be shown that for any $d$ there is no consistent theory at $\lambda_0 \leq \lambda_{oc} < 0$. For this purpose it is enough to consider the unrenormalized theory where the gap
equation and the criticality conditions are:

\[ m^2 = \mu_0^2 + g_0 \left[ I(m^2) + \phi_c^2 \right] \quad \text{and} \quad \Sigma(p = 0) = \frac{1}{g_0} \]  \[ 11 \]

where \( I(m^2) \) denotes (at finite ultraviolet cutoff):

\[ I(m^2) = \int \frac{d^d k}{(2\pi)^d} \left\{ \frac{1}{k^2 + m^2} \right\} . \]  \[ 12 \]

We have \( \frac{1}{g_0} = \left. \frac{\partial I(m^2)}{\partial m^2} \right|_{m^2 = m^2(0)} \) and \( \mu_0^2 = m^2(0) - \left[ I(m^2)/2\partial I(m^2)/\partial m^2 \right]_{m^2 = m^2(0)} \), where \( m^2(0) \) is the solution of the gap equation at \( \phi_c^2 = 0 \).

Inserting \( g_0 \) and \( \mu_0 \) into the gap equation one finds:

\[ [m^2(\phi_c^2) - m^2(0)] [1 + \left( \frac{I(m^2(\phi_c^2)) - I(m^2(0))}{m^2(\phi_c^2) - m^2(0)} \right) \left. \frac{1}{\partial I(m^2)/\partial m^2} \right|_{m^2 = m^2(0)}] = \left. \frac{\partial I(m^2)}{\partial m^2} \right|_{m^2 = m^2(0)} \phi_c^2 \]  \[ 13 \]

One notes however that for any \( d \) and fixed ultraviolet cutoff we have:

\[ I(m^2) > 0 \ , \ \frac{\partial I(m^2)}{\partial m^2} < 0 \ , \ \frac{\partial^2 I(m^2)}{\partial^2 m^2} < 0 \ etc. \]  \[ 14 \]

and thus, there is no real solution for \( m^2(\phi_c^2) \) in Eq[13]. A similar result is obtained for the renormalized theory at \( \lambda = \lambda_c \).

The emerging physical picture for the double scaling limit is clear. In this limit, the force between the fundamental quanta is set to the value at which the mass of their bound state vanishes. This requires, however, tuning the coupling constant to a negative value at which the theory is inconsistent. Vacuum fluctuations avoid giving a physical meaning to the large \( N \) theory at the value of the coupling constant \( \lambda_0 = \lambda_{0c} < 0 \). In the scalar theory the instability of this limit is manifested by the absence of a real solution to the relevant equations, namely, the absence of a real effective action. In the case of asymptotically free theories (in \( d=2 \) and probably at \( d=4 \)) the double scaling critical condition is not satisfied for any value of the coupling constant. In any case, a formal analytical continuation is required in order to give a physical meaning to the double scaling limit\(^{10}\).

A possible definition of the double scaling limit can be accepted starting with \( \lambda_0 > \lambda_{0c} \) and large \( N \). In this region of parameters, a metastable ("false vacuum") exists whose lifetime\(^{8,11}\) is of the order of \( \exp[O(N)] \). A suitable limit can be now taken at which the theory is well defined at any large but finite \( N \) and at any finite \( \epsilon = \lambda_0 - \lambda_{0c} \). The double scaling limit is now taken while \( N \to \infty \) and \( \lambda_0 \to \lambda_{0c} \) in a correlated manner (namely, taking the cutoff to infinity and \( N \to \infty \) in a correlated manner\(^5\)).

At finite temperature the above procedure requires extra attention due to the presence of the negative coupling constant. \( I(m^2) \) in Eq.[12] is replaced by:

\[ I(m^2, T) = \int \frac{d^d k}{(2\pi)^d} \left\{ \frac{1}{k^2 + m^2} + \frac{2\pi \delta(k^2 + m^2)}{\epsilon^{k^2/T} - 1} \right\} . \]  \[ 15 \]
One finds now\textsuperscript{12} that even if we start with $0 > \lambda_0 > \lambda_{0c}$ at some given temperature, namely, in a metastable vacuum as in the $T=0$ case, the metastable vacuum disappears as the temperature is increased\textsuperscript{8} beyond some $T = T_c$. The decay of the false vacuum as the temperature is raised can be traced to the fact that $\lambda_0 < 0$.

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