A novel method for global minimization combined filled function method and dimensionality reduction technique

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Abstract. This paper introduces a new filled function method based on the dimensionality reduction technique for the global minimization problem. By utilizing the so called $\alpha$–dense curves, we first transform the original function of n-variables into a single variable function, and then minimize the transformed function with the filled function method. The constructed filled function contains just one parameter which can be adjusted readily during the iterative process. The theoretical properties of the filled function are discussed, and the filled function algorithm is given. At last, a few numerical experiments are included.

1. Introduction

Lots of real world problems in Economics and Engineering can be modeled as nonlinear global minimization problems, and this motivates a growing attention in search for global solutions of nonlinear optimization problems. In the last decades, lots of algorithmic approaches, either stochastic or deterministic, have been developed. For general global minimization problems with no specific structure, filled function method, one category of efficient deterministic methods, could be applied. The filled function method was original proposed by Ge [1] for dealing with the global optimization problem $(P)$: $\min_{x\in X} f(x)$, where $X$ is a box set. And later, the filled function method was further improved in [2,3,4,5,6]. Recently, dimensionality reduction technique was proposed and successfully applied in global optimization in literature [7,8,9]. Through the constructed dimensionality reduction transformation, an n-variable global optimization problem could be converted into an one-variable one. In this paper, we combine the dimensionality reduction technique and filled function algorithm and propose a new method for global optimization. The idea of our method is that we first transform the n-variable problem into an one-variable problem using the constructed dimensionality reduction transformation, then we use filled function method to search for the global minimizer of the transformed problem. The filled function method generally consists of two stages. The first stage searches for one of local minimizers of the original problem, and the second stage tries to find a better starting point (with lower objective function value) for stage 1. Two stages are performed repeatedly, until no improved optimizer can be found.
Generally speaking, global optimization faces two difficulties. The first one is how to escape from the current optimizer to locate a better optimizer, and the other is how to check the current optimizer is a global one. This paper focuses only on the former issue.

This paper is organized as follows: Following this introduction, in Section 2, we will give basic knowledge about the transformation for dimensionality reduction. In Section 3, we will introduce a new filled function and discuss its properties. In Section 4, we will give the filled function algorithm. And last, in Section 5, a few numerical examples are included.

2. Transformation for Dimensionality Reduction and $\alpha$–dense

In this paper, we consider the following box-constrained global minimization problem $(P)$:

$$\min_{x \in X} f(x), \quad X = \prod_{k=1}^{n} [a_k, b_k]$$

To begin with, we make the following assumptions:

**Assumption 1.** The function $f(x)$ is Lipschitz continuous on $X$ with rank $L$.

**Assumption 2.** The number of the different minimal values of $f(x)$ is finite.

By using the dimensionality reduction transformation [6]:

$$x_i = h_i(\theta), \quad i = 1, 2, ..., n$$

we can convert an $n$-variables function $(P)$ into an one-variable function $F(\theta)$.

The definition of $\alpha$–dense is given as follows:

**Definition 1.** If the subspace $S \subseteq \mathbb{R}^n$ has the following property, then $S$ is called $\alpha$–dense.

For any given point $Q$, there exists one point $Q' \in S$, such that $d(Q, Q') \leq \alpha$, where $d(.)$ is Euclidean distance, and $\alpha > 0$ is small enough. For given $\varepsilon > 0$, let $\alpha = \frac{1}{\sqrt{n-1}} \frac{\varepsilon}{2L}$.

The corollary 1 of the theorem 2 in [8] shows that the subspace $S = \{(x_1, x_2, ..., x_n): x_i = h_i(\theta), i = 1, 2, ..., n\}$ is $\alpha$–dense.

**Theorem 1[10]:** If $S = \{(x_1, x_2, ..., x_n): x_i = h_i(\theta), i = 1, 2, ..., n\}$ is $\alpha$–dense, then global minimizer of the function $F(\theta)$ can be approximated by the global solution of the function of $F(\theta)$.

By performing the transformation: $x_i = h_i(\theta), \quad i = 1, 2, ..., n$, the original problem $(P)$ is turned into the following problem:

$$(TP) \min_{x \in D - [0, x]} F(\theta) = f(h_1(\theta), h_2(\theta), ..., h_n(\theta)).$$

According to Theorem 1 and the definition of $\alpha$–dense, for a given local minimizer $x^*$ of $f(x)$, and a given $\varepsilon > 0$, there exists one point $\theta^* \in [0, \pi]$ such that $|f(x^*) - F(\theta^*)| < \varepsilon$. Therefore, we can obtain the $\varepsilon$–approximate global solution of the $f(x)$ via solving one-variable function $F(\theta)$.

In the next section, we will propose a filled function algorithm to solve the problem $(TP)$.

3. A Filled Function and Its Properties

For simplicity, we denote the set of local minimizers of problem $(TP)$ by $L(TP)$,

$S_1 = \{\theta \in [0, \pi]: F(\theta) \geq F(\theta^*), \theta \neq \theta^*\}, \quad S_2 = \{\theta \in [0, \pi]: F(\theta) < F(\theta^*)\},$ and let $\theta^* \in L(TP)$.

Now, we give the definition of filled function for the global minimization problem $(TP)$.

**Definition 2.** A function $P(\theta, \theta^*)$ is said to be a filled function of problem $(TP)$ at $\theta^* \in L(TP)$, if it satisfies following conditions:
(1) \( \theta^* \) is a strictly maximizer of \( P(\theta, \theta^*) \) on the interval \([0, \pi]\).

(2) \( \frac{dP(\theta, \theta^*)}{d\theta} \neq 0 \) for any point on the \( S_1 \).

(3) Assume that \( \theta^* \in L(TP) \), but it is not a global minimizer, then there exists at least one point \( x_0 \in S_2 \) which is the minimizer of the function \( P(\theta, \theta^*) \) over the set \([0, \pi]\).

Define

\[
g_r(t) = \begin{cases} 
0 & t \geq 0 \\
r\arctan(t^3) + t^3 & t < 0.
\end{cases}
\] (2)

Then, it can be verified that the above function is continuously differentiable with its derivative given as follows:

\[
g_r'(t) = \begin{cases} 
0 & t \geq 0 \\
3rt^2 \left(1 + \frac{1}{1+r^6}\right) & t < 0.
\end{cases}
\] (3)

Now we establish a new filled function as follows:

\[
P(\theta, \theta^*, r) = \frac{1}{1 + |\theta - \theta^*|} + g_r \left( F(\theta) - F(\theta^*) \right).
\] (4)

The following theorems show that \( P(\theta, \theta^*, r) \) is a filled function.

**Theorem 1.** \( \theta^* \) is a strict local maximizer of \( P(\theta, \theta^*, r) \) over \([0, \pi]\).

**Proof.** For \( \theta^* \in L(P) \), we consider the following three cases: \( \theta^* = 0, \theta^* = \pi, \text{ and } \theta^* = (0, \pi) \).

Case I: \( \theta^* = 0 \). In this case, there exists a suitable small constant \( \sigma_1 > 0 \), such that \( F(\theta) \geq F(\theta^*) \) for all \( \theta \in (0, \sigma_1) \). Thus, we have that

\[
g_r \left( F(\theta) - F(\theta^*) \right) = 0, \quad P(\theta, \theta^*, r) = \frac{1}{1 + \theta} < 1 = P(\theta^*, \theta^*, r).
\]

Case II: \( \theta^* = \pi \). In this case, there exists a suitable small constant \( \sigma_2 > 0 \), such that \( F(\theta) \geq F(\theta^*) \) for all \( \theta \in (\pi - \sigma_2, \pi) \). Thus, we have that

\[
g_r \left( F(\theta) - F(\theta^*) \right) = 0, \quad P(\theta, \theta^*, r) = \frac{1}{1 + \pi - \theta} < 1 = P(\theta^*, \theta^*, r).
\]

Case III: \( \theta^* \in (0, \pi) \). In this case, there exists a suitable small constant \( \sigma_3 > 0 \), such that \( F(\theta) \geq F(\theta^*) \) for all \( \theta \in (\theta^* - \sigma_3, \theta^* + \sigma_3), \theta \neq \theta^* \). Thus, we have that

\[
g_r \left( F(\theta) - F(\theta^*) \right) = 0, \quad P(\theta, \theta^*, r) = \frac{1}{1 + \theta - \theta^*} < 1 = P(\theta^*, \theta^*, r).
\]

The results of the above three cases show that \( \theta^* \) is a strict local maximizer of \( P(\theta, \theta^*, r) \) over \([0, \pi]\).

**Theorem 2.** The function \( P(\theta, \theta^*, r) \) has no stationary points over \( S_1 \) except for \( \theta = 0, \text{ and } \theta = \pi \).

**Proof.** For \( \theta^* \in L(P) \), we consider the following three cases: \( \theta^* = 0, \theta^* = \pi, \text{ and } \theta^* = (0, \pi) \).

Case I: \( \theta^* = 0 \). In this case, we have \( F(\theta) \geq F(\theta^*) \) for all \( \theta \in S_1 \). Thus, for \( \theta \in S_1 \), it holds that
\[ P(\theta, \theta^*, r) = \frac{1}{1 + \theta^*}, \quad \frac{dP(\theta, \theta^*, r)}{d\theta} = -\frac{1}{(1 + \theta^*)^2} < 0. \]

Case II: \( \theta^* = \pi \). In this case, we have that
\[ g_r \left( F(\theta) - F(\theta^*) \right) = 0, \quad P(\theta, \theta^*, r) = \frac{1}{1 + \pi - \theta^*}, \quad \frac{dP(\theta, \theta^*, r)}{d\theta} = -\frac{1}{(1 + \pi - \theta^*)^2} > 0. \]

Case III: \( \theta^* \in (0, \pi) \). In this case, we have that
\[ g_r \left( F(\theta) - F(\theta^*) \right) = 0, \quad P(\theta, \theta^*, r) = \frac{1}{1 + |\theta - \theta^*|}, \quad \frac{dP(\theta, \theta^*, r)}{d\theta} = -\frac{1}{(1 + |\theta - \theta^*|)^2} |\theta - \theta^*| \neq 0. \]

Therefore, \( \frac{dP(\theta, \theta^*, r)}{d\theta} \neq 0 \) for any point on the \( S_i \).

**Theorem 3.** Assume that \( \theta^* \in L(TP) \), but it is not a global minimizer, then there exists at least one stationary point of the function \( P(\theta, \theta^*, r) \) over the set \( S_2 \).

**Proof.** By the given conditions, there exists \( \theta^*_i \in L(TP) \), such that \( F(\theta^*_i) < F(\theta^*) \). Let \( r > 0 \) be suitable large such that
\[ r > \frac{1}{\left(1 + |\theta^* - \theta^*_i|\right) \left( \arctan \left( F(\theta^*) - F(\theta^*_i) \right)^3 + \left( F(\theta^*) - F(\theta^*_i) \right)^3 \right)}, \]
then, we have that
\[ P(\theta^*_i, \theta^*, r) = \frac{1}{\left(1 + |\theta^* - \theta^*_i|\right)} - r \left( \arctan \left( F(\theta^*) - F(\theta^*_i) \right)^3 + \left( F(\theta^*) - F(\theta^*_i) \right)^3 \right) < 0. \]

Let \( \theta_0 \) be the global minimizer of the function \( P(\theta, \theta^*, r) \) on the interval \( [0, \pi] \), then in holds that
\[ P(\theta_0, \theta^*, r) \leq P(\theta^*_i, \theta^*, r) < 0, \]
which implies that
\[ F(\theta_0) < F(\theta^*). \]

Thus, there exists at least one point \( x_0 \in S_2 \) which is the minimizer of the function \( P(\theta, \theta^*, r) \) over the set \( [0, \pi] \).

That completes the proof of the theorem.

**4. Filled Function Algorithm**

In the above section, we discussed some properties of the filled function. Now, we give a filled function algorithm below.

**Filled function algorithm**

**Initialization step:**
Select the upper bound \( r_0 \) of the parameter \( r > 0 \), the initial point \( \theta_0 \), the positive integer number \( m > 0 \), and let \( k = 1 \). Transform the objective function \( f(x) \) into \( F(\theta) \), and go to the main step.

**Main step:**
0. Starting \( \theta_0 \), from minimize \((TP)\) by any local minimization procedure and find its local minimizer \( \theta^*_i \), and go to 2.
1. Set \( r = 1 \).
2. Set \( i = 1 \).
3. Construct a filled function $P(\theta, \theta^*, r)$ and go to 4.

4. select a set $\{\theta^*_i : i = 1, 2, ..., m\}$ for initial points.

5. If $k > m$, then go to 8; otherwise, set $\theta = \theta^*_k$, and take $\theta$ as an initial point to find a local minimizer $\theta_k$ of the following problem: $\min_{y \in [0,\pi]} P(y, \theta^*, r)$.

6. If $\theta_k \notin [0,\pi]$, then set $i = i + 1$, and go to 4; otherwise, go to 7.

7. If $F(\theta_k) < F(\theta^*)$, then, (a) set $\theta = \theta_k$, $k = 1$. (b) Use $\theta$ as a new initial point and minimize $(TP)$ to find its another local minimizer $\theta^*_2$ with $F(\theta^*_2) < F(\theta^*)$. (c) Set $\theta^* = \theta^*_2$ and $k = k + 1$, go to 1; Else if $F(\theta_k) \geq F(\theta^*)$ then go to 8.

8. Increase $r$ by setting $r = 10r$. If $r < r_u$, then set $k = 1$, and go to 2; otherwise, take $\theta^*_1$ as a global minimizer, and the algorithm stops.

5. Numerical Experiment

In this section, we carry out some numerical tests to show the effectiveness of the proposed filled function. All tests are coded in Fortran 95. The computational results are listed in the table, and the symbols used in the table are given as follows:

- $\theta_0$: the initial point used for $\min_{[0,\pi]} F(\theta)$ by some local optimization algorithm.
- Approximate global minimum: the approximate global minimum of the original problem obtained by the filled function algorithm.
- Global minimizing value: the global minimum given by [8,9,10].

Problem 1 [8,9]. $\min 100(x_2 - x_1^2)^2 + (1 - x_1)^2, |x_1| \leq 5, |x_2| \leq 5$.

Problem 2 [8,9]. $\min x_1^2 + x_2^2 - \frac{1}{10}(\cos(5\pi x_1) + \cos(5\pi x_2)), |x_1| \leq 1, |x_2| \leq 1$.

Problem 3 [8,9]. $\min \sum_{i=1}^{6} \left(x_i^2 - \frac{1}{10}\cos(5\pi x_i)\right), |x_i| \leq 1, i = 1, 2, ..., 6$.

Problem 4 [9,10]. $\min \sum_{i=1}^{10} \left(x_i^2 - \frac{1}{10}\cos(5\pi x_i)\right), |x_i| \leq 1, i = 1, 2, ..., 10$.

| Problem | $\theta_0$ | Approximate global minimum | Global minimizing value |
|---------|-----------|---------------------------|-------------------------|
| 1       | 2.13230   | 0.000085                  | 0                       |
| 2       | 2.55900   | -0.200000                 | -0.2                    |
| 3       | 1.78540   | -0.594030                 | -0.6                    |
| 4       | 1.39940   | -0.988400                 | -1                      |

6. Conclusion

In this paper, we developed a novel filled function method for box-constrained global optimization. The constructed filled function contains just one parameter and it can be adjusted easily during the iterative process. The proposed filled function method combined the novel filled function and the dimensionality reduction technique. By utilizing the so called $\alpha$ - dense curves, the original function of $n$-variables is first turned into a single variable function, and then is minimized by the filled function method. We also investigated the theoretical properties of the filled function and designed a filled function algorithm. Finally, a few numerical experiments are included and numerical results.
showed that the proposed method is efficient. However, how to apply this method to the real optimal problem will be studied in the near future.

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