Fatou property, representations, and extensions of law-invariant risk measures on general Orlicz spaces

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Abstract We provide a variety of results for quasiconvex, law-invariant functionals defined on a general Orlicz space, which extend well-known results from the setting of bounded random variables. First, we show that Delbaen’s representation of convex functionals with the Fatou property, which fails in a general Orlicz space, can always be achieved under the assumption of law-invariance. Second, we identify the class of Orlicz spaces where the characterization of the Fatou property in terms of norm-lower semicontinuity by Jouini, Schachermayer and Touzi continues to hold. Third, we extend Kusuoka’s representation to a general Orlicz space. Finally, we prove a version of the extension result by Filipović and Svindland by replacing norm-lower semicontinuity with the (generally non-equivalent) Fatou property. Our results have natural applications to the theory of risk measures.

Keywords Risk measures · Law-invariance · Fatou property · Dual representations · Conditional expectations · Orlicz spaces

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1 Introduction

The theory of risk measures is a well-established and still fruitful research area in the growing field of mathematical finance. In essence, a risk measure can be viewed as a rule to assign a certain indicator of risk—typically a capital requirement—to a given financial position—typically the net capital position (assets net of liabilities) of a financial institution. Originally articulated in the context of a finite probability space in the landmark paper by Artzner et al. [2], the theory was later extended to general probability spaces by Delbaen [7]. In the general case, one faces the problem of choosing a suitable model space for the underlying positions. The standard theory was developed for bounded positions, and a comprehensive account of the main results in this setting can be found in Föllmer and Schied [14, Chap. 4]. However, most realistic models in finance and insurance involve unbounded positions, and this calls for a mathematical extension beyond the bounded setting. A possible extension to the entire set of random variables was already discussed in Delbaen [7] and then again in Delbaen [8]. The extension to Lebesgue spaces is presented in Svindland [28] and Kaina and Rüschendorf [20], and the more general extension to the setting of Orlicz spaces was first investigated by Biagini and Frittelli [5] and Cheridito and Li [6]. Further results in Orlicz spaces have been obtained by Orihuela and Ruiz Galán [25], Krätschmer et al. [22], Gao and Xanthos [18], Gao et al. [17], and Delbaen and Owari [9]. A treatment of risk measures in the context of abstract spaces is provided in Frittelli and Rosazza Gianin [15], Drapeau and Kupper [10], and Farkas et al. [12].

In this paper, we work in the context of a nonatomic probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and provide a variety of representation and extension results for quasiconvex, law-invariant risk measures defined on a general Orlicz space $L^\Phi$. In particular, we do not assume that $\Phi$ satisfies the so-called $\Delta_2$ condition, under which $L^\Phi$ coincides with its Orlicz heart $H^\Phi$. The assumption of quasiconvexity is standard in the literature and reflects the diversification principle according to which the risk of an aggregated position should be controllable by the risk of the individual positions. The assumption of law-invariance, which stipulates that a risk measure depends solely on the distribution of the underlying position, is also standard and motivated by the ubiquitous use of time series analysis in finance and insurance practice.

Our main contribution can be broken down into the following results.

**Fatou property and dual representations.** It is known since Delbaen [7] that for a proper convex functional $\rho : L^\infty \to (-\infty, \infty]$, the following are equivalent:

1. $\rho$ is $\sigma(L^\infty, L^1)$-lower semicontinuous.
2. $\rho$ has the Fatou property, i.e.,

$$X_n \to X \text{ a.s., } |X_n| \leq Y \text{ for some } Y \in L^\infty \implies \rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

In this case, one can always represent $\rho$ in dual terms as

$$\rho(X) = \sup_{Z \in L^1} \left( \mathbb{E}[ZX] - \rho^*(Z) \right), \quad X \in L^\infty,$$
where $\rho^* : L^1 \to (-\infty, \infty]$ is defined by

$$\rho^*(Z) = \sup_{X \in L^\infty} (\mathbb{E}[ZX] - \rho(X)), \quad Z \in L^1.$$ 

The preceding result shows that once the Fatou property is fulfilled, the functional $\rho$ admits a “nice” dual representation, where the corresponding dual elements belong to a tractable subspace of the topological dual. In particular, if $\rho$ is a cash-additive risk measure, the dual elements can be identified with probability measures that are absolutely continuous with respect to $\mathbb{P}$. The appealing feature of the above result is that many risk measures on $L^\infty$ do satisfy the Fatou property. Most notably, as established by Jouini et al. [19], all convex cash-additive risk measures on $L^\infty$ that are law-invariant have the Fatou property.

It has been an open question since Biagini and Frittelli [5] and Owari [26] whether the above equivalence could be established in the context of a general Orlicz space $L^\Phi$, where $L^\Phi$ plays the role of $L^\infty$ and $L^\Psi$, with $\Psi$ being the conjugate of $\Phi$, the role of $L^1$. A positive result was obtained in Delbaen and Owari [9] for a special class of Orlicz spaces. A definitive answer has been finally provided in Gao et al. [17], where the authors proved that the above equivalence holds if and only if either the Orlicz function $\Phi$ or its conjugate $\Psi$ is $\Delta_2$.

It is therefore natural to wonder whether one can still establish the same equivalence in the context of a general Orlicz space by imposing suitable additional assumptions on the underlying functionals. The paper contributes to this line of research by showing that under the assumption of law-invariance, one can indeed prove the above equivalence without any restriction on the reference Orlicz space. More specifically, we prove the following result.

**Theorem 1.1** Let $\rho : L^\Phi \to (-\infty, \infty]$ be a proper, quasiconvex, law-invariant functional. Then the following statements are equivalent:

1. $\rho$ is $\sigma(L^\Phi, L^\infty)$-lower semicontinuous.
2. $\rho$ is $\sigma(L^\Phi, H^\Psi)$-lower semicontinuous.
3. $\rho$ is $\sigma(L^\Phi, L^\Psi)$-lower semicontinuous.
4. $\rho$ has the Fatou property.

A far-reaching result by Jouini et al. [19] established that for a proper convex functional $\rho : L^\infty \to (-\infty, \infty]$ that is additionally assumed to be law-invariant, the Fatou property is automatically implied by the (generally weaker) property of norm-lower semicontinuity. The result was obtained in the context of a standard nonatomic probability space and was later extended to arbitrary nonatomic probability spaces in Svindland [29]. Since every cash-additive risk measure on $L^\infty$ is norm-continuous, it follows that a convex cash-additive risk measure on $L^\infty$ has the Fatou property whenever it is law-invariant.

We extend the result in Jouini et al. [19] by characterizing the class of Orlicz spaces $L^\Phi$ where the above implication remains true for every proper, convex, law-invariant functional. In particular, we show that norm-lower semicontinuity no longer automatically implies the Fatou property unless $\Phi$ satisfies the $\Delta_2$ condition. This is the content of our second main result.
Theorem 1.2 The following statements are equivalent:

1. Any proper, quasiconvex, law-invariant functional \( \rho : L^\Phi \to (-\infty, \infty] \) that is norm-lower semicontinuous has the Fatou property.
2. \( \Phi \) is \( \Delta_2 \).

In addition to the above results, we extend to Orlicz spaces the representation for law-invariant risk measures obtained by Kusuoka [23] in the coherent case and generalized by Frittelli and Rosazza Gianin [16] to the convex case; see also Shapiro [27] and Belomestny and Krätschmer [3, 4]. Here, we denote by \( \text{ES}_\alpha(X) \) the expected shortfall of a random variable \( X \) at the level \( \alpha \in (0, 1) \). Moreover, we denote by \( \mathcal{P}((0,1]) \) the set of all probability measures over \((0,1]\).

Theorem 1.3 Let \( \rho : L^\Phi \to (-\infty, \infty] \) be a convex, law-invariant, cash-additive risk measure with the Fatou property. Then there exists a proper convex functional \( \gamma : \mathcal{P}((0,1]) \to (-\infty, \infty] \) such that

\[
\rho(X) = \sup_{\mu \in \mathcal{P}((0,1])} \left( \int_{(0,1]} \text{ES}_\alpha(X) \, d\mu(\alpha) - \gamma(\mu) \right), \quad X \in L^\Phi.
\]

We also show that the above Kusuoka representation fails if the Fatou property is replaced by the weaker property of norm-lower semicontinuity.

Extensions. In Filipović and Svindland [13], it was shown that every proper, convex, law-invariant, norm-lower semicontinuous functional \( \rho : L^\infty \to (-\infty, \infty] \) can be uniquely extended to a convex, law-invariant functional on \( L^p, 1 \leq p < \infty \), that is also norm-lower semicontinuous. This extension result played a fundamental role in the study of robustness properties of risk measures as discussed in Krätschmer et al. [22]; see also Koch-Medina and Munari [21]. We show that a similar extension result still holds in the context of a general Orlicz space if one replaces norm-lower semicontinuity by the Fatou property.

Theorem 1.4 Any proper, convex, law-invariant functional \( \rho : L^\infty \to (-\infty, \infty] \) with the Fatou property (or, equivalently, norm-lower semicontinuous) admits a unique proper, convex, law-invariant extension to \( L^\Phi \) with the Fatou property.

We also show that \( \rho \) cannot be extended in a unique way to an Orlicz space if one wants to preserve norm-lower semicontinuity only.

Structure of the paper. The paper is structured as follows. In Sect. 2, we recall some fundamental facts about Orlicz spaces and risk measures. In Sect. 3, we establish some properties of conditional expectations on Orlicz spaces. In Sect. 4, we study law-invariant sets in Orlicz spaces. In Sect. 5, we provide proofs of the main results together with some related corollaries.
2 Orlicz spaces and risk measures

Throughout the paper, we use standard notation from measure theory and functional analysis as can be found e.g. in Aliprantis and Border [1]. We refer to Edgar and Sucheston [11, Chap. 2] for a comprehensive account on Orlicz spaces. Recall that a function \( \Phi : [0, \infty) \rightarrow [0, \infty) \) is called an Orlicz function if it is convex, increasing, and \( \Phi(0) = 0 \). Define the conjugate function of \( \Phi \) by

\[
\Psi(s) = \sup\{ts - \Phi(t) : t \geq 0\}, \quad s \geq 0.
\]

If \( \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty \) (or, equivalently, \( \Psi \) is finite-valued), then \( \Psi \) is also an Orlicz function, and its conjugate is \( \Phi \). Throughout this paper, \((\Phi, \Psi)\) stands for a fixed Orlicz pair satisfying \( \Phi(t) > 0 \) for \( t > 0 \) and \( \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty \). Note that our restrictions on \( \Phi \) are minor as they only eliminate the case where \( L^\Phi \) coincides with \( L^1 \) or \( L^\infty \), in which cases our main results are either trivial or known.

Fix a nonatomic probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). In the sequel, we freely use the fact that for any event \( A \in \mathcal{F} \) and any \( p_1, \ldots, p_k \geq 0 \) with \( \sum_{i=1}^k p_i \leq \mathbb{P}[A] \), there exist disjoint measurable subsets \( A_1, \ldots, A_k \) of \( A \) such that \( \mathbb{P}[A_i] = p_i \) for \( 1 \leq i \leq k \); see e.g. [1, Sect. 13.9].

The Orlicz space \( L^\Phi := L^\Phi(\Omega, \mathcal{F}, \mathbb{P}) \) is the Banach lattice of all random variables \( X \) (modulo a.s. equality under \( \mathbb{P} \)) such that

\[
\|X\|_\Phi := \inf\left\{ \lambda > 0 : \mathbb{E}\left[ \Phi\left( \frac{|X|}{\lambda} \right) \right] \leq 1 \right\} < \infty.
\]

The norm \( \| \cdot \|_\Phi \) is called the Luxemburg norm. The subspace of \( L^\Phi \) consisting of all \( X \in L^\Phi \) such that

\[
\mathbb{E}\left[ \Phi\left( \frac{|X|}{\lambda} \right) \right] < \infty \quad \text{for all } \lambda > 0
\]

is conventionally called the Orlicz heart of \( L^\Phi \) and is denoted by \( H^\Phi \). It is well known that \( L^\infty \subseteq H^\Phi \subseteq L^\Phi \subseteq L^1 \) and that \( H^\Phi \) is a norm-closed subspace of \( L^\Phi \). Moreover, \( L^\Phi = H^\Phi \) if and only if the Orlicz function \( \Phi \) is \( \Delta_2 \), i.e., there exist \( t_0 \in (0, \infty) \) and \( k \in \mathbb{R} \) such that \( \Phi(2t) < k\Phi(t) \) for all \( t \geq t_0 \). We endow the conjugate Orlicz space \( L^\Psi \) with the Orlicz norm

\[
\|Y\|_\Psi := \sup_{X \in L^\Phi, \|X\|_\Phi \leq 1} |\mathbb{E}[XY]|, \quad Y \in L^\Psi,
\]

which is equivalent to the Luxemburg norm on \( L^\Psi \). Under the canonical duality induced by the pairing \((X, Y) := \mathbb{E}[XY]\) for \( X \in L^\Phi \) and \( Y \in L^\Psi \), the space \( L^\Psi \) can be identified with the order continuous dual \((L^\Phi)^\sim\) of \( L^\Phi \), which is a subspace of the norm dual \((L^\Phi)^*\) of \( L^\Phi \). Moreover, \( L^\Psi = (L^\Phi)^\psi \) if and only if the function \( \Phi \) is \( \Delta_2 \).

A net \((X_\alpha)\) in \( L^\Phi \) is said to order converge to \( X \in L^\Phi \), denoted \( X_\alpha \rightharpoonup X \) in \( L^\Phi \), if there exists a net \((Y_\alpha)\) in \( L^\Phi \) such that \( Y_\alpha \downarrow 0 \) in \( L^\Phi \) and \( |X_\alpha - X| \leq Y_\alpha \) for any \( \alpha \). For a sequence \((X_n)\) in \( L^\Phi \) and \( X \in L^\Phi \), one can easily verify that \( X_n \rightharpoonup X \) is equivalent to dominated almost sure convergence, i.e., \( X_n \rightarrow X \) a.s. and \( |X_n| \leq Y \) for some...
A set $C \subseteq L/\Phi_1$ is order closed in $L/\Phi_1$ if it contains the limit of every order convergent net with elements in $C$. It is well known that if $X_\alpha \overset{o}{\to} X$ in $L/\Phi_1$, then there exists a sequence $(\alpha_n)$ such that $X_{\alpha_n} \overset{o}{\to} X$. Thus $C$ is order closed in $L/\Phi_1$ whenever it contains the limit of every order convergent sequence with elements in $C$. Note that every order closed set $C$ is automatically norm-closed. Indeed, if $(X_n) \subseteq C$ converges in norm to $X$, then a subsequence $(X_{nk})$ order converges to $X$, so that $X \in C$. (This uses the classic fact that any Cauchy sequence in $L/\Phi_1$ contains a subsequence which converges almost surely and is dominated by a majorant in $L/\Phi_1$.)

A proper (i.e., not identically $\infty$) functional $\rho : L/\Phi_1 \to (-\infty, \infty]$ is said to have the Fatou property whenever

$$X_n \to X \text{ a.s., } |X_n| \leq Y, \forall n \in \mathbb{N}, \text{ for some } Y \in L/\Phi_1 \implies \rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

We say that $\rho$ is order lower semicontinuous if the sublevel set

$$\{\rho \leq \lambda\} := \{X \in L/\Phi_1 : \rho(X) \leq \lambda\}$$

is order closed for all $\lambda \in \mathbb{R}$. This is equivalent to

$$X_n \overset{o}{\to} X \text{ in } L/\Phi_1 \implies \rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

In other words, as remarked in Biagini and Frittelli [5], the Fatou property is equivalent to order lower semicontinuity. As a result, it follows that a functional with the Fatou property is automatically norm-lower semicontinuous. If $\rho$ is additionally assumed to be monotone (decreasing), i.e., $\rho(X) \leq \rho(Y)$ for any $X, Y \in L/\Phi_1$ with $X \geq Y$, then the Fatou property is also equivalent to continuity from above, i.e., $\rho(X_n) \to \rho(X)$ whenever $X_n \downarrow X$ in $L/\Phi_1$.

A proper functional $\rho : L/\Phi_1 \to (-\infty, \infty]$ is called convex if it satisfies that

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$$

for all $X, Y \in L/\Phi_1$ and $\lambda \in [0, 1]$, and quasiconvex if the sublevel set $\{\rho \leq \lambda\}$ is convex for every $\lambda \in \mathbb{R}$. Moreover, we say that $\rho$ is positively homogeneous if $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in L/\Phi_1$ and $\lambda \in [0, \infty)$. The functional is called law-invariant if $\rho(X) = \rho(Y)$ whenever $X, Y \in L/\Phi_1$ have the same law.

In this paper, we use cash-additive risk measures to illustrate our general results on law-invariant functionals defined on Orlicz spaces. Recall that $\rho$ is a cash-additive risk measure if it is monotone and satisfies

$$\rho(X + m1) = \rho(X) - m$$

for any $X \in L/\Phi_1$ and $m \in \mathbb{R}$. A cash-additive risk measure is coherent if it is convex and positively homogeneous. Two prominent cash-additive risk measures are the Value-at-Risk at level $\alpha \in (0, 1)$, which is defined by setting

$$\text{VaR}_\alpha(X) := \inf\{m \in \mathbb{R} : \mathbb{P}[X + m < 0] \leq \alpha\}, \quad X \in L/\Phi_1,$$

and the expected shortfall at level $\alpha \in (0, 1]$, which is given by

$$\text{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) \, d\beta, \quad X \in L/\Phi_1.$$
We refer to [14, Chap. 4] for more information about risk measures and their financial applications.

3 Conditional expectations on Orlicz spaces

As emerges from Jouini et al. [19] and Svindland [29], conditional expectations play an important role in the study of law-invariant risk measures on $L^\infty$. However, some key properties of conditional expectations fail once we abandon the setting of bounded positions. This section is devoted to collecting a variety of useful properties of conditional expectations on $L^\Phi$ that will allow us to overcome this failure.

Recall that conditional expectations are contractions on Orlicz spaces, i.e.,

$$\|\mathbb{E}[X|\mathcal{G}]\|_\Phi \leq \|X\|_\Phi$$

for any $X \in L^\Phi$ and any $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$ (see [11, Corollary 2.3.11]). In the sequel, we write $\pi$ to denote a finite measurable partition of $\Omega$ whose members have nonzero probabilities, and denote by $\sigma(\pi)$ the finite $\sigma$-algebra generated by $\pi$. For convenience, we always write

$$\mathbb{E}[X|\pi] := \mathbb{E}[X|\sigma(\pi)].$$

The collection $\Pi$ of all such $\pi$ is directed by refinement and we write $\pi' \geq \pi$ whenever $\pi'$ is a refinement of $\pi$. In particular, the family of conditional expectations $(\mathbb{E}[X|\pi])$ becomes a net with directed set $\Pi$. A fundamental result used in the $L^\infty$-case (see [19] and [29]), is recorded in the following lemma; this uses the standard notation for convergence of nets with respect to a topology.

**Lemma 3.1** For every $X \in L^\infty$, we have

$$\mathbb{E}[X|\pi] \xrightarrow{\|\|_{\infty}} X.$$ (3.1)

Indeed, for any $\varepsilon > 0$, by partitioning $[-\|X\|_\infty, \|X\|_\infty]$ into intervals of length at most $\varepsilon$ and considering the corresponding preimages under $X$, we obtain a partition $\pi_0 = \{A_1, \ldots, A_n\} \in \Pi$ such that the oscillation of $X$ on each $A_i$, $1 \leq i \leq n$, is at most $\varepsilon$. Then it is easily seen that $\|\mathbb{E}[X|\pi] - X\|_{\infty} \leq \varepsilon$ for all $\pi \geq \pi_0$. This result, however, fails on Orlicz spaces in general. Indeed, for $X \in L^\Phi$, we easily see that $
\mathbb{E}[X|\pi] \in L^\infty \subseteq H^\Phi$ for all $\pi \in \Pi$ and therefore

$$\mathbb{E}[X|\pi] \xrightarrow{\|\|_\Phi} X \iff X \in H^\Phi.$$ 

In particular, condition (3.1) can be extended to $L^\Phi$ if and only if we have $L^\Phi = H^\Phi$ or, equivalently, $\Phi$ is $\Delta_2$. The right reformulation of (3.1) in the context of a general Orlicz space is as follows.

**Proposition 3.2** For every $X \in L^\Phi$ and every $Y \in L_{+,\Phi}^\Phi$, we have

$$\mathbb{E}[Y|\mathbb{E}[X|\pi] - X] \longrightarrow 0.$$
Proof Assume first that \( X \geq 0 \). We claim that
\[
\lim_{k \to \infty} \sup_{\pi \in \Pi} \mathbb{E}\left[ \mathbb{E}[X|\pi] Y \mathbb{I}_{A_k} \right] = 0, \tag{3.2}
\]
where \( A_k = \{ X > k \} \) for \( k \in \mathbb{N} \). To show this, assume that (3.2) does not hold so that we find \( \varepsilon > 0 \), \( k_n \to \infty \) and \( (\pi_n) \subseteq \Pi \) such that \( A_{k_n} \in \sigma(\pi_n) \) for each \( n \in \mathbb{N} \) and satisfying
\[
\mathbb{E}\left[ \mathbb{E}[X|\pi_n] Y \mathbb{I}_{A_{k_n}} \right] > \varepsilon \quad \text{for all } n \in \mathbb{N}.
\]
Pick any \( n \in \mathbb{N} \) and suppose that \( A_{k_n} = B_{1,n} \cup \cdots \cup B_{\ell_n,n} \) with \( B_{i,n} \in \pi_n \) for \( 1 \leq i \leq \ell_n \). Since \( \mathbb{P}[A_k] \to 0 \) as \( k \to \infty \) and \( X \in L^1 \), one can use dominated convergence to infer that \( \mathbb{E}[X \mathbb{I}_{B_{i,n} \cap A_k}] \to 0 \) as \( k \to \infty \), so that \( \mathbb{E}[X \mathbb{I}_{B_{i,n} \cap A_k}] \to \mathbb{E}[X \mathbb{I}_{B_{i,n}}] \) as \( k \to \infty \). The same holds for \( Y \). Thus, by passing to a convenient subsequence of \( (k_n) \), we may assume without loss of generality that each \( k_{n+1} \) is large enough so that
\[
\mathbb{E}[X \mathbb{I}_{B_{i,n} \cap A_{k_{n+1}}}] \geq \frac{1}{2} \mathbb{E}[X \mathbb{I}_{B_{i,n}}] \quad \text{and} \quad \mathbb{E}[Y \mathbb{I}_{B_{i,n} \cap A_{k_{n+1}}}] \geq \frac{1}{2} \mathbb{E}[Y \mathbb{I}_{B_{i,n}}]
\]
for all \( 1 \leq i \leq \ell_n \). Write \( C_{i,n} = B_{i,n} \setminus A_{k_{n+1}} \) for all \( 1 \leq i \leq \ell_n \). Then
\[
\sum_{i=1}^{\ell_n} \mathbb{E}[X|C_{i,n}] \mathbb{E}[Y \mathbb{I}_{C_{i,n}}] \geq \frac{1}{4} \sum_{i=1}^{\ell_n} \mathbb{E}[X|B_{i,n}] \mathbb{E}[Y \mathbb{I}_{B_{i,n}}] = \frac{1}{4} \mathbb{E}[X|\pi_n] \mathbb{E}[Y \mathbb{I}_{A_{k_n}}] \geq \frac{\varepsilon}{4}.
\]
Note that \( C_n = \{ C_{1,n}, \ldots, C_{\ell_n,n} \} \) is a measurable partition of \( A_{k_n} \setminus A_{k_{n+1}} \) for all \( n \in \mathbb{N} \). Thus \( \{ A_{k_n}^{\varepsilon} \} \cup \bigcup_n C_n \) is a measurable partition of \( \Omega \). Let \( \mathcal{G} \subseteq \mathcal{F} \) be the generated \( \sigma \)-algebra. It follows from the preceding inequality that
\[
\infty > \| \mathbb{E}[X|\mathcal{G}] \|_\Phi \| Y \|_\Psi \geq \mathbb{E}[\mathbb{E}[X|\mathcal{G}] Y] \geq \sum_{n \in \mathbb{N}} \sum_{i=1}^{\ell_n} \mathbb{E}[X|C_{i,n}] \mathbb{E}[Y \mathbb{I}_{C_{i,n}}] = \infty.
\]
This contradiction completes the proof of (3.2). As a result, for any \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) large enough such that \( \mathbb{E}[\mathbb{E}[X|\pi] Y \mathbb{I}_{A_k}] < \varepsilon \) for any \( \pi \in \Pi \) with \( A_k \in \sigma(\pi) \). Since \( XY \in L^1 \), we may take \( k \) so large to ensure \( \mathbb{E}[XY \mathbb{I}_{A_k}] < \varepsilon \). By the norm-convergence of conditional expectations on \( L^\infty \), we find a finite measurable partition \( \pi_0 \in \Pi \) such that \( A_k \in \sigma(\pi_0) \) and \( \| \mathbb{E}[X \mathbb{I}_{A_k^{\varepsilon}}|\pi] - X \mathbb{I}_{A_k^{\varepsilon}} \|_\infty < \varepsilon \) for any \( \pi \in \Pi \) refining \( \pi_0 \). Take now any such \( \pi \) and note that since \( A_k \in \sigma(\pi) \), we have
\[
\mathbb{E}[Y|\mathbb{E}[X|\pi] - X] = \mathbb{E}[Y \mathbb{I}_{A_k}|\mathbb{E}[X|\pi] - X] + \mathbb{E}[Y \mathbb{I}_{A_k^{\varepsilon}}|\mathbb{E}[X|\pi] - X] \\
\leq \mathbb{E}[\mathbb{E}[X|\pi] Y \mathbb{I}_{A_k}] + \mathbb{E}[XY \mathbb{I}_{A_k}]
\]
\[
+ \| \mathbb{E}[X \mathbb{I}_{A_k^{\varepsilon}}|\pi] - X \mathbb{I}_{A_k^{\varepsilon}} \|_\infty \| Y \|_1
\]
\[
< 2\varepsilon + \varepsilon \| Y \|_1.
\]
This establishes the assertion for \( X \geq 0 \). To conclude, take now an arbitrary \( X \in L^\Phi \) and note that
\[
\mathbb{E}[Y \mid \mathbb{E}[X \mid \pi] - X] \leq \mathbb{E}[Y \mid \mathbb{E}[X^+ \mid \pi] - X^+] + \mathbb{E}[Y \mid \mathbb{E}[X^- \mid \pi] - X^-] \to 0
\]
by what we have just established.

In light of the link between the Fatou property and order lower semicontinuity, it would be desirable that conditional expectations converge in order. However, this is also not true in a general Orlicz space.

**Proposition 3.3** Let \( X \in L^\Phi \). Then \( \mathbb{E}[X \mid \pi] \xrightarrow{\sigma} X \) in \( L^\Phi \) if and only if \( X \in L^\infty \).

**Proof** Suppose that \( X \in L^\infty \). For each \( \pi \in \Pi \), set \( \lambda_\pi = \sup_{\pi' \geq \pi} \| \mathbb{E}[X \mid \pi'] - X \|_\infty \) and note that \( \lambda_\pi \downarrow 0 \) by Lemma 3.1. Then \( \| \mathbb{E}[X \mid \pi - \pi] - X \|_\infty \) implies \( \mathbb{E}[X \mid \pi] \xrightarrow{\sigma} X \) in \( L^\Phi \).

Conversely, suppose that \( X \notin L^\infty \). Without loss of generality, assume that we have \( \mathbb{P}[X > k] > 0 \) for all \( k \in \mathbb{N} \) and let \( \pi_0 = \{A_1, \ldots, A_n\} \) be an arbitrary member of \( \Pi \). It is easy to see that there exists some \( A_i, 1 \leq i \leq n \), such that \( \mathbb{P}[A_i \cap \{X > k\}] > 0 \) for all \( k \in \mathbb{N} \). By renumbering, say \( A_1 \) satisfies this. Fix now an arbitrary \( k \in \mathbb{N} \). If \( \mathbb{P}[A_1 \cap \{X \leq k\}] = 0 \), then we immediately see that \( \mathbb{E}[X \mid \pi_0] \geq k \) a.s. on \( A_1 \).

Otherwise, we must have \( \mathbb{P}[A_1 \setminus \{X > k\}] > 0 \). Set \( c = \mathbb{P}[A_1 \setminus \{X > k\}] \). Since \( X \in L^1 \), there exists \( 0 < \varepsilon < c \) such that \( \mathbb{E}[|X |_{B}] < k c \) whenever \( B \in \mathcal{F} \) satisfies \( \mathbb{P}[B] \leq \varepsilon \). By nonatomicity, we can take finitely many measurable subsets \( B_1, \ldots, B_j \) of \( A_1 \setminus \{X > k\} \) such that \( \mathbb{P}[B_i] < \varepsilon \) for each \( 1 \leq i \leq j \) and \( \bigcup_{i=1}^j B_i = A_1 \setminus \{X > k\} \).

Now for any \( 1 \leq i \leq j \), the set \( C_i = (A_1 \cap \{X > k\}) \cup B_i \subseteq A_1 \) satisfies \( \mathbb{P}[C_i] \leq 2 c \).

Take a refinement \( \pi_i \geq \pi_0 \) such that \( C_i \in \pi_i \). Then we have
\[
\sup_{\pi \geq \pi_0} \mathbb{E}[X \mid \pi] \geq \mathbb{E}[X \mid \pi_i] = \frac{1}{\mathbb{P}[C_i]} \mathbb{E}[X 1_{A_1 \cap \{X > k\}} + X 1_{B_i}] \geq \frac{1}{2 c} (\mathbb{E}[X 1_{A_1 \cap \{X > k\}}] - \mathbb{E}[|X |_{B_i}]) \geq \frac{1}{2 c} \left( k c - \frac{k c}{2} \right) = \frac{k}{4} \quad \text{a.s. on } C_i.
\]
Since \( \bigcup_{i=1}^j C_i = A_1 \), we infer that
\[
\sup_{\pi \geq \pi_0} \mathbb{E}[X \mid \pi] \geq \frac{k}{4} \quad \text{a.s. on } A_1.
\]
Since $k$ was arbitrary, it follows that
\[ \sup_{\pi \geq \pi_0} \mathbb{E}[X|\pi] = \infty \quad \text{a.s. on } A_1. \]

This implies that the net $(\mathbb{E}[X|\pi])$ has no order bounded tail and thus does not order converge. In particular, $(\mathbb{E}[X|\pi])$ does not order converge to $X$. □

In spite of the preceding negative result, for every random variable $X \in L^\Phi$, we can always select a sequence of partitions with respect to which the conditional expectations of $X$ do order converge to $X$ itself.

**Proposition 3.4** For every $X \in L^\Phi$, there exists a sequence $(\pi_n) \subseteq \Pi$ such that
\[ \mathbb{E}[X|\pi_n] \overset{o}{\to} X \quad \text{in } L^\Phi. \]

**Proof** Without loss of generality, assume that $\|X\|_\Phi \leq \frac{1}{2}$ so that $\mathbb{E}[\Phi(2|X|)] < \infty$. For each $n \in \mathbb{N}$, take $k_n \in \mathbb{N}$ large enough such that
\[ \mathbb{E}[\Phi(2|X|)\mathbb{1}_{A_n}] \leq \frac{1}{2^n}, \quad (3.3) \]
where $A_n = \{|X| \geq k_n\}$. Then take $\pi_n \in \Pi$ satisfying $A_n \in \sigma(\pi_n)$ and
\[ \|X\mathbb{1}_{A_n} - \mathbb{E}[X\mathbb{1}_{A_n}|\pi_n]\|_\infty \leq \frac{1}{2^n}. \quad (3.4) \]

We claim that $\mathbb{E}[X|\pi_n] \overset{o}{\to} X$ in $L^\Phi$. To prove this, fix $n \in \mathbb{N}$ and note first that
\[ \mathbb{E}\left[\Phi(\mathbb{E}[2|X|\mathbb{1}_{A_n}])\mathbb{1}_{A_n}\right] \leq \mathbb{E}\left[\mathbb{E}[\Phi(2|X|)\mathbb{1}_{A_n}]\mathbb{1}_{A_n}\right] = \mathbb{E}[\Phi(2|X|)\mathbb{1}_{A_n}] \leq \frac{1}{2^n} \quad (3.5) \]
by the conditional version of Jensen’s inequality. Moreover, assumption (3.4) ensures that
\[ \|X\mathbb{1}_{A_n} - \mathbb{E}[X\mathbb{1}_{A_n}|\pi_n]\|_\Phi \leq \frac{\|\|\|_\Phi}{2^n}. \quad (3.6) \]

Finally, set
\[ Y_n = X\mathbb{1}_{A_n} - \mathbb{E}[X\mathbb{1}_{A_n}|\pi_n] \quad \text{and} \quad Z_n = X\mathbb{1}_{A_n^c} - \mathbb{E}[X\mathbb{1}_{A_n^c}|\pi_n] \]
and note that
\[ X - \mathbb{E}[X|\pi_n] = Y_n + Z_n. \]

It is clear that by setting
\[ X_0 = \sup_{n \in \mathbb{N}} |Y_n| + \sum_{n \in \mathbb{N}} |Z_n|, \]
we have $|X - \mathbb{E}[X|\pi_n]| \leq X_0$ for all $n \in \mathbb{N}$. We claim that $X_0 \in L^\Phi$. To show this, note first that

$$||Z_n||_\Phi \leq \frac{1}{2^n}$$

by (3.6). Hence $(\sum_{n=1}^\infty |Z_n|)_m$ is a Cauchy sequence with respect to $||\cdot||_\Phi$, and thus its limit $\sum_{n \in \mathbb{N}} |Z_n|$ belongs to $L^\Phi$. By (3.3) and (3.5) and by convexity of $\Phi$, we have

$$\mathbb{E}[\Phi(|Y_n|)] \leq \frac{1}{2} \mathbb{E}[\Phi(2|X|1_{A_n})] + \frac{1}{2} \mathbb{E}\left[\Phi(\mathbb{E}[2|X|1_{A_n}])\right] \leq \frac{1}{2^n}$$

for every $n \in \mathbb{N}$. Hence the continuity and strict monotonicity of $\Phi$ yield

$$\mathbb{E}\left[\Phi\left(\sup_{n \in \mathbb{N}} |Y_n|\right)\right] = \mathbb{E}\left[\sup_{n \in \mathbb{N}} \Phi(|Y_n|)\right] \leq \sum_{n \in \mathbb{N}} \mathbb{E}[\Phi(|Y_n|)] \leq \sum_{n \in \mathbb{N}} \frac{1}{2^n} = 1.$$

This shows that $\sup_{n \in \mathbb{N}} |Y_n| \in L^\Phi$ as well, so that $X_0 \in L^\Phi$. Now by Markov’s inequality, we have

$$\Phi(\varepsilon)\mathbb{P}[|Y_n| > \varepsilon] \leq \mathbb{E}[\Phi(|Y_n|)] \leq \frac{1}{2^n}$$

for any $\varepsilon > 0$ and $n \in \mathbb{N}$. It follows that $(Y_n)$ converges to 0 in probability. Since $\|Z_n\|_\Phi \to 0$, [11, Corollary 2.1.10] implies that $(Z_n)$ also converges to 0 in probability. As a result, the sequence $(X - \mathbb{E}[X|\pi_n])$ converges to 0 in probability. A subsequence of it converges to 0 a.s., and thus in order, since even the whole sequence $(X - \mathbb{E}[X|\pi_n])$ is order bounded by $X_0$. The sequence of partitions corresponding to this special subsequence, which we still denote by $(\pi_n)$, is then easily seen to satisfy $\mathbb{E}[X|\pi_n] \overset{\phi}{\to} X$ in $L^\Phi$. \[\square\]

4 Law-invariant sets in Orlicz spaces

In this section, we establish a key result on law-invariant sets, which is later applied to sublevel sets of law-invariant functionals defined on Orlicz spaces. Here, we say that a subset $C$ of $L^\Phi$ is law-invariant if $X \in C$ for any $X \in L^\Phi$ that has the same law as some element of $C$.

We start with the following observation. For any $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$, consider the “localized” probability space $(A, \mathcal{F}_{\mid A}, \mathbb{P}_{\mid A})$, where $\mathcal{F}_{\mid A} := \{B \in \mathcal{F} : B \subseteq A\}$ and $\mathbb{P}_{\mid A} : \mathcal{F}_{\mid A} \to [0, 1]$ is defined by $\mathbb{P}_{\mid A}(B) := \mathbb{P}(B|A)$. Note that $(A, \mathcal{F}_{\mid A}, \mathbb{P}_{\mid A})$ is also nonatomic. For any $X \in L^\Phi$, we denote by $X_{\mid A}$ the random variable on $(A, \mathcal{F}_{\mid A}, \mathbb{P}_{\mid A})$ obtained by restricting $X$ to $A$. Let now $(\Omega', \mathcal{F}', \mathbb{P}')$ be any nonatomic probability space and recall that by applying any quantile function of $X$ to a uniform random variable over $(\Omega', \mathcal{F}', \mathbb{P}')$, we obtain a random variable over $(\Omega', \mathcal{F}', \mathbb{P}')$ that has the same law as $X$. Working at a “localized” level, we can use the same idea to show that given two sets $A, B \in \mathcal{F}$ with $\mathbb{P}[A] = \mathbb{P}[B]$, we always find a random variable $Z \in L^\Phi$ such that $X_{\mid A}$ has the same law as $Z_{\mid B}$.
Lemma 4.1 Let $X \in L^\Phi$, $\varepsilon > 0$ and $\pi \in \Pi$ be fixed. Then there exist $B \in \mathcal{F}$ with $\mathbb{P}[B] \leq \varepsilon$ and $X_1, \ldots, X_N \in L^\Phi$ with the same law as $X$ such that setting $Y = \frac{1}{N} \sum_{i=1}^{N} X_i$, we have

$$\mathbb{E}[Y|\pi] = \mathbb{E}[X|\pi], \quad \|Y \perp \Phi \| \leq \varepsilon, \quad Y \perp B^c \in L^\infty.$$ 

Proof Without loss of generality, we assume that $\varepsilon < 1$ and $\|X\|_\Phi \leq 1$ so that $\mathbb{E}[\Phi(|X|)] \leq 1$.

Let $A \in \mathcal{F}$ be such that $\mathbb{P}[A] > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \frac{2}{\varepsilon}$ and $c > 0$ such that $\mathbb{P}[|X| < c] \cap A > 0$. Moreover, choose $c' > 0$ large enough to ensure that

$$c' > (N-1)c,$$

$$\mathbb{P}[|X| > c'] \cap A \leq \frac{1}{N} \min \left( P[|X| < c] \cap A, \varepsilon \right),$$

$$\mathbb{E}[\Phi(|X|) \mathbb{1}_{|X| > c}] \leq \frac{1}{N}.$$

Set $C = \{|X| > c'\} \cap A$ and note that $N \mathbb{P}[C] \leq \mathbb{P}[|X| < c] \cap A]$. By nonatomicity, there exist pairwise disjoint measurable subsets $B_1, \ldots, B_N$ of $\{|X| < c\} \cap A$ such that $\mathbb{P}[B_i] = \mathbb{P}[C]$ for all $1 \leq i \leq N$. Observe that $B_i$ is disjoint from $C$ for all $1 \leq i \leq N$ because $c' > c$. Now for any fixed $i \in \{1, \ldots, N\}$, we may use the above observation to ensure the existence of $Z_i, Z'_i \in L^\Phi$ such that $Z_i \perp B_i$ has the same law as $X \perp C$ and $Z'_i \perp C$ has the same law as $X \perp B_i$. Setting $X_i = Z_i \perp B_i + Z'_i \perp C + X \perp D_i$ where $D_i = A \setminus (C \cup B_i)$, one clearly sees that $X_i = X_i \perp A$ has the same law as $X \perp A$. Define now $Y = \frac{1}{N} \sum_{i=1}^{N} X_i$. Moreover, set $B = \bigcup_{i=1}^{N} B_i \subseteq A$ and note that $\mathbb{P}[B] \leq \varepsilon$. Since for each $1 \leq i \leq n$, the random variable $Z'_i \perp C$ has the same law as $X \perp B_i$ and $|X| < c$ a.s. on $B_i$, we see that $|Z'_i| < c$ a.s. on $C$. This together with the inclusion $D_i \subseteq A \setminus C \subseteq \{|X| \leq c'\}$ implies that

$$|X_i \perp B^c| = |Z'_i \perp C \cap B^c| + |X \perp D_i \cap B^c| \leq (c + c') \mathbb{1}.$$ 

This shows that $Y \perp B^c \in L^\infty$.

Next, we claim that $\|Y \perp B\|_\Phi \leq \varepsilon$. To prove this, fix $j \in \{1, \ldots, N\}$ and note first that $|X_j \perp B_j| = |X \perp D_j \cap B_j| \leq c \perp B_j$ for all $1 \leq i \leq N$ with $i \neq j$. In addition, since $Z_j \perp B_j$ has the same law as $X \perp C$ and $|X| > c'$ a.s. on $C$, we easily see that we have $|X_j| = |Z_j| > c'$ a.s. on $B_j$. As a result, we obtain

$$N|Y \perp B_j| \leq \sum_{i=1}^{N} |X_i \perp B_j| + |X_j \perp B_j| \leq (N - 1)c \perp B_j + |X_j \perp B_j|$$

$$\leq c' \perp B_j + |X_j \perp B_j| \leq 2|X_j \perp B_j|. \quad \Box$$
Since $|X_i| > c'$ a.s. on $B_i$ and $X_i$ has the same law as $X 1_{A_i}$ for all $1 \leq i \leq N$, it follows that

$$\mathbb{E} \left[ \Phi \left( \frac{N}{2} |Y| \right) 1_{B_i} \right] = \sum_{i=1}^{N} \mathbb{E} \left[ \Phi \left( \frac{N}{2} |Y| \right) 1_{B_i} \right]$$

$$\leq \sum_{i=1}^{N} \mathbb{E} \left[ \Phi ( |X_i| ) 1_{B_i} \right]$$

$$\leq \sum_{i=1}^{N} \mathbb{E} \left[ \Phi ( |X_i| ) 1_{|X_i| > c'} \right]$$

$$= N \mathbb{E} \left[ \Phi ( |X| ) 1_{|X| > c'} \right]$$

$$\leq N \mathbb{E} \left[ \Phi ( |X| ) 1_{|X| > c'} \right] \leq 1.$$  

This yields $\|Y 1_{B_i}\| \Phi \leq \frac{\varepsilon}{n} \leq \varepsilon$ and concludes the proof of the claim.

Suppose now that $\pi = \{A_1, \ldots, A_n\}$ and fix $k \in \{1, \ldots, n\}$. Applying the preceding argument to $A_k$, we can ensure the existence of a measurable set $B_k \subseteq A_k$ with $\mathbb{P}[B_k] \leq \frac{\varepsilon}{n}$ as well as of random variables $X_{ki} \in L^\Phi$, $1 \leq i \leq N_k$, such that $X_{ki}$ is zero a.s. on $A_k^c$ and has the same law as $X 1_{A_k}$ and such that setting $Y_k = \frac{1}{N_k} \sum_{i=1}^{N_k} X_{ki}$, we have

$$\|Y_k 1_{B_k}\| \Phi \leq \frac{\varepsilon}{n} \quad \text{and} \quad Y_k 1_{B_k^c} \in L^\infty.$$  

Now set $Y = \sum_{k=1}^{n} Y_k$ and $B = \bigcup_{k=1}^{n} B_k$. Then $\mathbb{P}[B] \leq \varepsilon$ and $Y 1_{B^c} \in L^\infty$. Moreover, since $Y_k$ vanishes on $A_j$ for $j \neq k$, we have $Y 1_B = \sum_{k=1}^{n} Y_k 1_{B_k}$ and hence $\|Y 1_B\| \Phi \leq \varepsilon$. It also follows that $\mathbb{E}[Y 1_{A_k}] = \mathbb{E}[Y_k 1_{A_k}] = \mathbb{E}[X 1_{A_k}]$ for every $1 \leq k \leq n$, so that $\mathbb{E}[Y |\pi] = \mathbb{E}[X |\pi]$. Setting $N = \prod_{k=1}^{n} N_k$ and for every $1 \leq k \leq n$, repeating each $X_{ki}$, $1 \leq i \leq N_k$, $N_k$ times, we can write $Y_k = \frac{1}{N} \sum_{i=1}^{N_k} X_{ki}^{(k)}$, where each $X_{ki}^{(k)}$ is zero a.s. on $A_k^c$ and has the same law as $X 1_{A_k}$. Then $Y = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{n} X_{ki}^{(k)}$, and clearly each $\sum_{k=1}^{n} X_{ki}^{(k)}$ has the same law as $X$.

To establish our key result on law-invariant sets, we also need to use the following result, which is contained in Step 2 in the proof of [29, Lemma 1.3].

**Lemma 4.2** Let $X \in L^\infty$, $\varepsilon > 0$ and $\pi \in \Pi$. Then there exist $X_1, \ldots, X_N \in L^\infty$ which have the same law as $X$ and satisfy

$$\left\| \frac{1}{N} \sum_{i=1}^{N} X_i - \mathbb{E}[X |\pi] \right\|_\infty \leq \varepsilon.$$  

**Proposition 4.3** Let $C$ be a convex, norm-closed, law-invariant set in $L^\Phi$. Then $\mathbb{E}[X |\pi] \in C$ for every $X \in C$ and every $\pi \in \Pi$.  

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Proof Let $X \in C$, $\pi = \{A_1, \ldots, A_n\} \in \Pi$ and fix an arbitrary $\varepsilon > 0$. For any $B \in \mathcal{F}$ and $Y \in L^\Phi$, we have

$$
\|\mathbb{E}[Y|A_i \cap B^c]|1_{A_i \cap B^c} - \mathbb{E}[Y|A_i]|1_{A_i}\|_\Phi \\
\leq \|(\mathbb{E}[Y|A_i \cap B^c] - \mathbb{E}[Y|A_i])|1_{A_i \cap B^c}\|_\Phi + \|\mathbb{E}[Y|A_i]|1_{A_i \cap B}\|_\Phi \\
= \|\mathbb{P}[B|A_i](\mathbb{E}[Y|A_i \cap B^c] - \mathbb{E}[Y|A_i \cap B])|1_{A_i \cap B^c}\|_\Phi + \|\mathbb{E}[Y|A_i]|1_{A_i \cap B}\|_\Phi \\
\leq \left(\frac{\mathbb{P}[B]\mathbb{E}[|Y|]}{\mathbb{P}[A_i]|\mathbb{P}[A_i|B^c]} + \|\mathbb{E}[Y]\|_\Phi\|1\|_\Psi\right)\|1\|_\Phi + \frac{\|\mathbb{E}[|Y|]\|_\Phi}{\mathbb{P}[A_i]}\|1\|_\Phi
$$

for every $1 \leq i \leq n$. As a result, there exists $0 < \delta < \varepsilon$ such that whenever $\mathbb{P}[B] < \delta$, $\|Y|_B\|_\Phi < \delta$ and $\mathbb{E}[|Y|] \leq \mathbb{E}[|X|]$, we have

$$
\left\|\sum_{i=1}^n \mathbb{E}[Y|A_i \cap B^c]|1_{A_i \cap B^c} - \mathbb{E}[Y]\right\|_\Phi < \varepsilon. \quad (4.1)
$$

Here we have used the fact that $\|1_A\|_\Phi \to 0$ as $\mathbb{P}[A] \to 0$. Applying Lemma 4.1, we obtain $B \in \mathcal{F}$ with $\mathbb{P}[B] < \delta$ and $X_1, \ldots, X_N \in L^\Phi$ which have the same law as $X$ and satisfy

$$
\mathbb{E}[Y] = \mathbb{E}[X], \quad \|Y|_B\|_\Phi < \delta, \quad Y|_B^\infty \in L^\infty,
$$

where we set $Y = \frac{1}{N} \sum_{i=1}^N X_i$. In particular, note that $Y \in C$ by convexity and law-invariance of $C$. Moreover, it follows from $|Y| \leq \frac{1}{N} \sum_{i=1}^N |X_i|$ that $\mathbb{E}[|Y|] \leq \mathbb{E}[|X|]$ so that (4.1) holds. Now consider the nonatomic probability space $(B^c, \mathcal{F}|_{B^c}, \mathbb{P}|_{B^c})$. Applying Lemma 4.2 to $Y|_{B^c}$ and the partition $\{A_1 \cap B^c, \ldots, A_n \cap B^c\}$ of the state space $B^c$, we obtain $Y_1, \ldots, Y_M \in L^\infty(B^c, \mathcal{F}|_{B^c}, \mathbb{P}|_{B^c})$ such that $Y_j$ has the same law as $Y|_{B^c}$ for all $1 \leq j \leq M$ and

$$
\left|\sum_{i=1}^n \mathbb{E}|_{B^c}[Y|_{B^c}|A_i \cap B^c]|1_{A_i \cap B^c} - \frac{1}{M} \sum_{j=1}^M Y'_j\right| \leq \varepsilon \quad \mathbb{P}|_{B^c}$-a.s. on $B^c$,
$$

where $\mathbb{E}|_{B^c}$ denotes the expectation under $\mathbb{P}|_{B^c}$. A direct computation shows that for $1 \leq i \leq n$, we have $\mathbb{E}[Y|A_i \cap B^c] = \mathbb{E}|_{B^c}[Y|_{B^c}|A_i \cap B^c]$, so that

$$
\left|\sum_{i=1}^n \mathbb{E}[Y|A_i \cap B^c]|1_{A_i \cap B^c} - \frac{1}{M} \sum_{j=1}^M Y'_j\right| \leq \varepsilon \quad \mathbb{P}|_{B^c}$-a.s. on $B^c$.

Set $Y_j = Y|_B + Y'_j|_{B^c}$ for $1 \leq j \leq M$. Then $Y_j$ has the same law as $Y$, and hence $Y_j \in C$ by law-invariance for every $1 \leq j \leq M$. Note that $\frac{1}{M} \sum_{j=1}^M Y_j \in C$ by con-
vexity of $C$ and that

\[
\left\| \sum_{i=1}^{n} \mathbb{E}[Y | A_i \cap B^c] \mathbb{1}_{A_i \cap B^c} - \frac{1}{M} \sum_{j=1}^{M} Y_j \right\|_{\Phi} \\
= \left\| \sum_{i=1}^{n} \mathbb{E}[Y | A_i \cap B^c] \mathbb{1}_{A_i \cap B^c} - \frac{1}{M} \sum_{j=1}^{M} Y_j \mathbb{1}_{B^c} - Y \mathbb{1}_B \right\|_{\Phi} \\
\leq \varepsilon \left\| \mathbb{1}_{B^c} \right\|_{\Phi} + \left\| Y \mathbb{1}_B \right\|_{\Phi} \leq \varepsilon \left\| \mathbb{1} \right\|_{\Phi} + \varepsilon.
\]

(4.2)

Since $\mathbb{E}[Y | \pi] = \mathbb{E}[X | \pi]$, we can easily combine (4.1) and (4.2) to obtain

\[
\left\| \mathbb{E}[X | \pi] - \frac{1}{M} \sum_{j=1}^{M} Y_j \right\|_{\Phi} \leq (2 + \left\| \mathbb{1} \right\|_{\Phi})\varepsilon.
\]

Finally, by norm-closedness of $C$, we infer that $\mathbb{E}[X | \pi] \in C$. □

**Remark 4.4** Note that in order to obtain the above proposition, one cannot directly truncate an arbitrary $X \in L^{\Phi}$ and then apply Lemma 4.2 because the tail of $X$ need not have small norm. In fact, $\left\| X \mathbb{1}_{(|X| > k)} \right\|_{\Phi} \to 0$ as $k \to \infty$ precisely when $X \in H^{\Phi}$.

Recall that every order closed set in $L^{\Phi}$ is automatically norm-closed. This, together with Proposition 3.4 and Proposition 4.3, immediately implies the following characterization of the elements of a law-invariant set in terms of their conditional expectations.

**Corollary 4.5** Let $C$ be a convex, law-invariant, order closed set in $L^{\Phi}$ and take $X \in L^{\Phi}$. Then we have $X \in C$ if and only if $\mathbb{E}[X | \pi] \in C$ for every $\pi \in \Pi$.

We are now in a position to derive the main result of this section, which shows that for a law-invariant set in $L^{\Phi}$, order closedness and $\sigma(L^{\Phi}, L^{\psi})$-closedness are equivalent. When applied to sublevel sets of convex, law-invariant functionals on $L^{\Phi}$, this equivalence immediately yields the desired characterization of the Fatou property in terms of $\sigma(L^{\Phi}, L^{\psi})$-lower semicontinuity.

**Corollary 4.6** For a convex law-invariant set $C$ in $L^{\Phi}$, the following are equivalent:

1. $C$ is order closed.
2. $C$ is $\sigma(L^{\Phi}, L^{\psi})$-closed.
3. $C$ is $\sigma(L^{\Phi}, H^{\psi})$-closed.
4. $C$ is $\sigma(L^{\Phi}, L^{\infty})$-closed.

**Proof** Clearly, we only need to prove that (1) implies (4). To this effect, assume that $C$ is order closed and consider a net $(X_\alpha) \subseteq C$ and $X \in L^{\Phi}$ such that

$X_\alpha \xrightarrow{\sigma(L^{\Phi}, L^{\infty})} X.$
Moreover, fix a partition \( \pi = \{ A_1, \ldots, A_n \} \in \Pi \). Then for any norm-continuous linear functional \( \phi : L^\Phi \to \mathbb{R} \), we have

\[
\phi(\mathbb{E}[X_\alpha | \pi]) = \mathbb{E} \left[ X_\alpha \sum_{i=1}^{n} \frac{\phi(\mathbb{1}_{A_i})}{\mathbb{P}[A_i]} \mathbb{1}_{A_i} \right] \to \mathbb{E} \left[ X \sum_{i=1}^{n} \frac{\phi(\mathbb{1}_{A_i})}{\mathbb{P}[A_i]} \mathbb{1}_{A_i} \right] = \phi(\mathbb{E}[X | \pi]).
\]

Thus \( (\mathbb{E}[X_\alpha | \pi]) \) converges weakly to \( \mathbb{E}[X | \pi] \). Since \( \mathbb{E}[X_\alpha | \pi] \in C \) for any index \( \alpha \) by Corollary 4.5 and since, being order closed and thus norm-closed, the convex set \( C \) is weakly closed, we infer that \( \mathbb{E}[X | \pi] \in C \). In light of Corollary 4.5, this yields \( X \in C \) and proves that \( C \) is \( \sigma(L^\Phi, L^\infty) \)-closed. \( \square \)

5 Proofs of the main results

In this final section, we prove the results stated in the introduction and derive a variety of corollaries for functionals and risk measures defined on Orlicz spaces.

**Proof of Theorem 1.1** Recall from Sect. 2 that \( \rho \) has the Fatou property if and only if the sublevel set \( \{ \rho \leq \lambda \} \) is order closed for any \( \lambda \in \mathbb{R} \). Recall also that \( \rho \) is \( \sigma(L^\Phi, L^\Psi) \)- (respectively, \( \sigma(L^\Phi, H^\Psi) \)- or \( \sigma(L^\Phi, L^\infty) \))-lower semicontinuous if and only if each sublevel set is \( \sigma(L^\Phi, L^\Psi) \)- (respectively, \( \sigma(L^\Phi, H^\Psi) \)- or \( \sigma(L^\Phi, L^\infty) \))-closed. Since each sublevel set is convex and law-invariant, the equivalence follows directly from Corollary 4.6. \( \square \)

The following dual representation of functionals with the Fatou property is an immediate consequence of Theorem 1.1 and of the Fenchel–Moreau duality.

**Corollary 5.1** Let \( \rho : L^\Phi \to (-\infty, \infty] \) be a proper, convex, law-invariant functional with the Fatou property. Then we have

\[
\rho(X) = \sup_{Z \in L^\Psi} \left( \mathbb{E}[ZX] - \rho^*(Z) \right), \quad X \in L^\Phi,
\]

where

\[
\rho^*(Z) = \sup_{X \in L^\Phi} \left( \mathbb{E}[ZX] - \rho(X) \right), \quad Z \in L^\Psi.
\]

The first supremum can be equivalently taken over \( H^\Psi \) or \( L^\infty \).

We specialize the preceding theorem to cash-additive risk measures. Here, we denote by \( \mathcal{M}(L^\Psi) \) (respectively, \( \mathcal{M}(H^\Psi) \) and \( \mathcal{M}(L^\infty) \)) the set of all probability measures \( Q \) over \( \Omega \) that are absolutely continuous with respect to \( P \) and such that \( \frac{dQ}{dP} \) belongs to \( L^\Psi \) (respectively, \( H^\Psi \) and \( L^\infty \)).

**Corollary 5.2** Let \( \rho : L^\Phi \to (-\infty, \infty] \) be a proper, convex, law-invariant, cash-additive risk measure with the Fatou property. Then we have

\[
\rho(X) = \sup_{Q \in \mathcal{M}(L^\Psi)} \left( \mathbb{E}_Q[-X] - \rho^*(Q) \right), \quad X \in L^\Phi,
\]
where
\[ \rho^*(\mathbb{Q}) = \sup_{X \in L^\Phi} \left( \mathbb{E}_\mathbb{Q}[-X] - \rho(X) \right), \quad \mathbb{Q} \in \mathcal{M}(L^\Phi). \]

The first supremum can be equivalently taken over \( \mathcal{M}(H^\Phi) \) or \( \mathcal{M}(L^\infty) \).

Proof of Theorem 1.2 First assume that \( \Phi \) is \( \Delta_2 \). Then \( L^\Phi = H^\Phi \) and every order convergent sequence is also norm-convergent to the same limit by [11, Theorem 2.1.14]. In particular, every norm-closed set is also order closed. Let \( \rho : L^\Phi \to (-\infty, \infty] \) be a proper, quasiconvex, law-invariant functional that is norm-lower semicontinuous. Since every sublevel set of \( \rho \) is norm-closed and hence order closed, it follows that \( \rho \) is order lower semicontinuous or, equivalently, has the Fatou property. This shows that (2) implies (1). To prove the converse implication, assume that \( \Phi \) is not \( \Delta_2 \) so that \( L^\Phi \neq H^\Phi \) and set
\[ C = \{ X \in L^\Phi : X^- \in H^\Phi, \mathbb{E}[X] \geq 0 \}. \]

It is clear that \( C \) is a law-invariant cone. Moreover, it is convex since for any \( X, Y \in C \) and \( \lambda \in [0, 1] \), we have
\[ 0 \leq (\lambda X + (1 - \lambda)Y)^- \leq \lambda X^- + (1 - \lambda)Y^- \in H^\Phi, \]
showing that \( (\lambda X + (1 - \lambda)Y)^- \in H^\Phi \). The set \( C \) is also easily seen to be monotone. Indeed, for any \( X \in C \) and \( Y \in L^\Phi \) with \( Y \geq X \), it holds
\[ 0 \leq Y^- \leq X^- \in H^\Phi, \]
implying that \( Y \in C \). We claim that \( C \) is norm-closed but not order closed. To establish norm-closedness, take \( (X_n) \subseteq C \) and \( X \in L^\Phi \) satisfying \( X_n \xrightarrow{\|\cdot\|_\Phi} X \). Since this implies \( X_n \xrightarrow{\|\cdot\|_1} X \), we see that \( \mathbb{E}[X] \geq 0 \). Moreover, since \( H^\Phi \) is norm-closed, it follows from
\[ \|X_n^- - X^-\|_\Phi \leq \|X_n - X\|_\Phi \longrightarrow 0 \]
that \( X^- \in H^\Phi \). Therefore we conclude that \( X \in C \), showing that \( C \) is norm-closed. To prove that \( C \) is not order closed, take a positive non-zero \( Y \in L^\Phi \setminus H^\Phi \). Moreover, take any \( \lambda \geq \mathbb{E}[Y] \) and set
\[ X_n = \lambda \mathbbm{1} - \min(Y, n \mathbbm{1}), \quad n \in \mathbb{N}, \quad \text{and} \quad X = \lambda \mathbbm{1} - Y. \]

Note that \( (X_n) \subseteq L^\infty \subseteq H^\Phi \) so that \( (X_n^-) \subseteq H^\Phi \) and \( \mathbb{E}[X_n] \geq \lambda - \mathbb{E}[Y] \geq 0 \) for every \( n \in \mathbb{N} \), showing that \( (X_n) \subseteq C \). Clearly, we have \( X_n \downarrow X \) so that \( X_n \xrightarrow{o} X \) in \( L^\Phi \). However, \( X \) does not belong to \( C \), for otherwise \( X^- \in H^\Phi \) and \( 0 \leq X^+ \leq \lambda \mathbbm{1} \in H^\Phi \) would imply \( X \in H^\Phi \) and thus \( Y \in H^\Phi \). This shows that \( C \) is not order closed. It follows from what we established above that \( C \) is a law-invariant, coherent, monotone subset of \( L^\Phi \) that is norm-closed, but fails to be order closed. Consider now the law-invariant, coherent, cash-additive risk measure \( \rho : L^\Phi \to [-\infty, \infty] \) defined by
\[ \rho(X) := \inf\{m \in \mathbb{R} : X + m \mathbbm{1} \in C\}, \quad X \in L^\Phi. \]
It is immediate to see that $\rho$ does not attain the value $-\infty$ and is proper. Moreover, since $C$ is norm-closed, $\rho$ is norm-lower semicontinuous by [24, Corollary 3.3.8]. However, since $\{\rho \leq 0\} = C$ by [24, Proposition 3.2.7], it follows that $\rho$ is not order lower semicontinuous or, equivalently, fails to have the Fatou property. This proves that (1) implies (2) and concludes the proof. □

Remark 5.3 The functional defined in (5.1) provides an explicit example of a law-invariant, coherent, cash-additive risk measure on $L^\Phi$ that is norm-lower semicontinuous, but fails to satisfy the Fatou property when $\Phi$ is not $D_2$.

The last two results announced in the introduction, namely the generalization of Kusuoka’s representation and the Fatou-property-preserving extension result, will be derived from the following “localization” lemma. This result is of independent interest once we recall that in a general Orlicz space, the space $L^\infty$ need not be norm-dense in $L^\Phi$.

Lemma 5.4 Let $\rho_1, \rho_2 : L^\Phi \rightarrow (-\infty, \infty]$ be proper, quasiconvex, law-invariant functionals with the Fatou property. Then we have $\rho_1 = \rho_2$ whenever $\rho_1$ and $\rho_2$ coincide on $L^\infty$.

Proof Fix any $X \in L^\Phi$ and take a sequence $(\pi_n) \subseteq \Pi$ such that $\mathbb{E}[X|\pi_n] \rightarrow X$. This is always possible in view of Proposition 3.4. Since $\rho_1$ is order lower semicontinuous, we have

$$\rho_1(X) \leq \liminf_{n \rightarrow \infty} \rho_1(\mathbb{E}[X|\pi_n]).$$

The set $C = \{Y \in L^\Phi : \rho_1(Y) \leq \rho_1(X)\}$ is convex, law-invariant, order closed and clearly contains $X$. Hence by Corollary 4.5, we have $\mathbb{E}[X|\pi_n] \in C$ for every $n \in \mathbb{N}$, so that

$$\limsup_{n \rightarrow \infty} \rho_1(\mathbb{E}[X|\pi_n]) \leq \rho_1(X).$$

As a consequence, we infer that $\rho_1(\mathbb{E}[X|\pi_n]) \rightarrow \rho_1(X)$. The same conclusion holds for $\rho_2$ as well. Since $\mathbb{E}[X|\pi_n] \in L^\infty$ for every $n \in \mathbb{N}$, it follows from our assumption that $\rho_1(X) = \rho_2(X)$. □

The following result will be also needed, in addition to the preceding lemma, to establish the generalization of Kusuoka’s representation.

Lemma 5.5 Let $\rho : L^\Phi \rightarrow (-\infty, \infty]$ be a proper, quasiconvex, law-invariant functional with the Fatou property. Then its restriction to $L^\infty$ is also proper and has the Fatou property.

Proof Denote by $\rho_{|L^\infty}$ the restriction of $\rho$ to $L^\infty$ and take any $X_0 \in L^\Phi$ such that $\rho(X_0) < \infty$. Then since the sublevel set $C = \{Y \in L^\Phi : \rho(Y) \leq \rho(X_0)\}$ is convex, law-invariant, order closed and contains $X_0$, it follows from Corollary 4.5 that $\mathbb{E}[X_0] \subseteq C$, so that $\rho(\mathbb{E}[X_0]) < \infty$. This proves that $\rho_{|L^\infty}$ is proper. Take now a
sequence \((X_n) \subseteq L^\Phi\) and \(X \in L^\Phi\) such that \(X_n \to X\) a.s. and \(|X_n| \leq Y\) for some \(Y \in L^\infty\) and all \(n \in \mathbb{N}\). Since \(Y \in L^\Phi\), it follows that \(X_n \overset{a.s.}{\to} X\) in \(L^\Phi\). Thus we infer that \(\rho_{|L^\infty}(X) \leq \liminf_{n \to \infty} \rho_{|L^\infty}(X_n)\), and this shows that \(\rho_{|L^\infty}\) has the Fatou property. \(\square\)

**Proof of Theorem 1.3** Denote by \(\rho_{|L^\infty}\) the restriction of \(\rho\) to \(L^\infty\). It follows from Lemma 5.5 that \(\rho_{|L^\infty}\) is a convex, law-invariant, cash-additive risk measure satisfying the Fatou property. Then [14, Theorem 4.62] implies that

\[
\rho_{|L^\infty}(X) = \sup_{\mu \in \mathcal{P}((0,1])} \left( \int_{(0,1]} \text{ES}_\alpha(X) \, d\mu(\alpha) - \gamma(\mu) \right), \quad X \in L^\infty,
\]

where

\[
\gamma(\mu) = \sup_{X \in L^\infty, \mu(X) = 0} \int_{(0,1]} \text{ES}_\alpha(X) \, d\mu(\alpha), \quad \mu \in \mathcal{P}((0,1]).
\]

Now define \(\rho' : L^\Phi \to (-\infty, \infty]\) by setting

\[
\rho'(X) = \sup_{\mu \in \mathcal{P}((0,1])} \left( \int_{(0,1]} \text{ES}_\alpha(X) \, d\mu(\alpha) - \gamma(\mu) \right), \quad X \in L^\Phi.
\]

Clearly, \(\rho'\) is a convex, law-invariant, cash-additive risk measure satisfying the Fatou property. Moreover, since \(\rho\) and \(\rho'\) coincide on \(L^\infty\), it follows that \(\rho = \rho'\) by Lemma 5.4. This shows that \(\rho\) has the desired representation. \(\square\)

**Proof of Theorem 1.4** By [13, Theorem 2.2], \(\rho\) admits a proper, convex, law-invariant extension \(\overline{\rho}\) to \(L^1\) that is norm-lower semicontinuous. Denote by \(\rho'\) the restriction of \(\overline{\rho}\) to \(L^\Phi\) and note that \(\rho'\) has the Fatou property. To see this, consider a sequence \((X_n) \subseteq L^\Phi\) and \(X \in L^\Phi\) such that \(X_n \to X\) a.s. and \(|X_n| \leq Y\) for some \(Y \in L^\Phi\) and all \(n \geq 1\). Since \(Y \in L^1\), the dominated convergence theorem implies that \(X_n \overset{\|\|_1}{\to} X\) and therefore \(\rho'(X) \leq \liminf_{n \to \infty} \rho'(X_n)\) by norm-lower semicontinuity of \(\overline{\rho}\). The uniqueness follows from Lemma 5.4. \(\square\)

**Remark 5.6** Differently from the case of bounded positions, Theorems 1.3 and 1.4 no longer hold if we replace the Fatou property, or, equivalently, order lower semicontinuity, by norm-lower semicontinuity. Indeed, assume that \(\Phi\) is not \(\Delta_2\) and let \(\rho\) be the coherent, law-invariant, norm-lower semicontinuous, cash-additive risk measure constructed in (5.1). Then \(\rho\) does not admit a Kusuoka-type representation because it would otherwise satisfy the Fatou property. Moreover, note that being cash-additive, the restriction of \(\rho\) to \(L^\infty\), denoted by \(\rho_{|L^\infty}\), is norm-continuous. Applying Theorem 1.4, we obtain a convex, law-invariant extension \(\rho'\) of \(\rho_{|L^\infty}\) to the whole of \(L^\Phi\) that satisfies the Fatou property. Now \(\rho\) and \(\rho'\) coincide on \(L^\infty\), but are not equal since one of them has the Fatou property whereas the other does not.
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