Higher categorical aspects of Hall Algebras

T. Dyckerhoff

March 5, 2022

Abstract

These are extended notes for a series of lectures on Hall algebras given at the CRM Barcelona in February 2015. The basic idea of the theory of Hall algebras is that the collection of flags in an exact category encodes an associative multiplication law. While introduced by Steinitz and Hall for the category of abelian p-groups, it has since become clear that the original construction can be applied in much greater generality and admits numerous useful variations. These notes focus on higher categorical aspects based on the relation between Hall algebras and Waldhausen’s S-construction.

Contents

Introduction ........................................... 2

1 Classical Hall algebras .............................. 5
  1.1 Hall’s algebra of partitions ....................... 5
  1.2 Proto-abelian categories ......................... 6
  1.3 A first example .................................. 8
     1.3.1 The categories Vect_{F_q} and Vect_{F_1} ... 8
     1.3.2 Statistical interpretation of q-analogs ...... 10
  1.4 Hall’s algebra of partitions and symmetric functions .... 11
     1.4.1 The Hall algebra of F_1[[t]] ................. 11
     1.4.2 Relation to symmetric functions ............... 13
     1.4.3 Zelevinsky’s statistic ....................... 15

2 Hall algebras via groupoids ......................... 20
  2.1 Groupoids of flags ................................ 20
  2.2 Spans of groupoids and the abstract Hall algebra .... 23
  2.3 Groupoid functions ................................ 24
  2.4 Green’s theorem .................................. 28
     2.4.1 Squares, frames, and crosses ................. 28
     2.4.2 Yoneda’s theory of extensions ............... 32
     2.4.3 Proof of Green’s theorem ..................... 36
     2.4.4 Example .................................... 38
  2.5 Hall monoidal categories ......................... 40
     2.5.1 Hall⊗(Vect_{F_1}) .......................... 43
     2.5.2 Hall⊗(Vect_{F_q}) ......................... 44
Introduction

While studying the combinatorics of flags

\[ M = M_0 \supset M_1 \supset \cdots \supset M_n = 0 \]

of abelian \( p \)-groups, Hall [Hal59] (and, in fact, more than 50 years earlier Steinitz [Ste01]) had the striking insight that the numbers obtained by counting flags in the various abelian \( p \)-groups \( M \) form the structure constants of an associative algebra: the Hall algebra. The combinatorics of these structure constants is quite subtle and relates beautifully to the theory of symmetric functions [Mac95].

Ringel [Rin90] noticed that Hall’s construction can be applied to the category of representations of a quiver over a finite field \( \mathbb{F}_q \). He showed that, in the case of a simply laced Dynkin quiver, the resulting associative algebra realizes the upper triangular part of the quantum group classified by the graph underlying the quiver. In the same context, Lusztig [Lus91] defined a geometric variant of Hall’s algebra using perverse sheaves on moduli spaces of flags. These groundbreaking discoveries created a surge of activity providing new perspectives on the theory of quantum groups (see the survey papers [Rin93], [Sch06], [Hub], [Rou07]). More recently, a host of variants of Hall algebras have been used by various authors to study counting invariants: Reineke [Rei03] [Rei06] used Ringel’s Hall algebra to count rational points and compute Betti numbers of quiver moduli, Joyce [Joy07] introduced motivic Hall algebras to study generalized Donaldson-Thomas invariants, Kontsevich-Soibelman [KS11] introduce cohomological Hall algebras for a similar purpose. Toën [Toë06] has introduced Hall algebras for derived categories.

The point of view taken in this course is based on the observation that the collections \( S_n \) of flags of varying length \( n \geq 0 \) naturally organize into a simplicial object

\[ S_\bullet : \Delta^{op} \to \mathcal{D}, \ [n] \mapsto S_n \]

where the target category \( \mathcal{D} \) depends on the context. The above variants of Hall algebras \( H \) can be obtained as various specializations of the simplicial object \( S_\bullet \):

1. Ringel’s Hall algebra: \( S_\bullet \) is a simplicial groupoid and \( H \) is obtained by passing to functions on isomorphism classes

2. Lusztig’s geometric Hall algebra: \( S_\bullet \) is a simplicial stack and \( H \) is obtained by passing to the Grothendieck group of perverse sheaves
Joyce’s motivic Hall algebra: $S_\bullet$ is a simplicial stack and $H$ is obtained by passing to relative Grothendieck groups

Kontsevich-Soibelman’s cohomological Hall algebra: $S_\bullet$ is a simplicial stack and $H$ is obtained by passing to equivariant cohomology

Toën’s derived Hall algebra: $S_\bullet$ is a simplicial space and $H$ is obtained by passing to locally constant functions

Remarkably, the simplicial object $S_\bullet$ also plays a central role in the work of Waldhausen [Wal85] on algebraic K-theory where it is simply called the $S_\bullet$-construction. We adopt this terminology even though the relation to algebraic K-theory will not play any role in these notes.

The following question will serve as a guide for this course:

**Question 0.1.** To what extent can we study Hall algebras “universally” in terms of the simplicial object $S_\bullet$ without passing to any of the above specializations?

A first answer is that associativity of the Hall algebra can be seen on a universal level: it corresponds to the fact that $S_\bullet$ satisfies certain natural conditions called 2-Segal conditions in [DK12].

We outline the contents of these notes:

In Section 1, after giving the historical definition of Hall’s algebra of partitions, we define Hall algebras for finitary proto-abelian categories. This allows us to treat categories linear over $F_q$ and categories “linear over $F_1$” on the same footing. Via two examples, we demonstrate the idea of [Szc11] to interpret, in some cases, the Hall algebra of an $F_q$-linear category as a $q$-analog of the Hall algebra of a suitably defined $F_1$-linear category. The methods used in this section pay tribute to combinatorial aspects such as statistics. The main reference is [Mac95].

The remaining sections are devoted to the analysis of Question 0.1 for various target categories $D$:

Section 2: $D = \{\text{groupoids}\}$. We introduce the simplicial groupoid $S_\bullet$ of flags in a proto-abelian category $C$ and establish the 2-Segal conditions. We construct from $S_\bullet$ an abstract Hall algebra in the monoidal category of spans of groupoids. This is an instance of the Baez-Hoffnung-Walker groupoidification program [BHW10] [Wal13]. We attempt to advertise the benefits of this point of view by lifting the main part of the proof of Green’s theorem [Gre95] [Rin96] into this language, following proposals of Baez and Kapranov. The Hall algebra of Section 1 can, for finitary $C$, be obtained from the abstract Hall algebra by passing to groupoid functions as indicated in (1). We conclude with the observation that the simplicial 2-Segal groupoid $S_\bullet$ encodes structure of higher categorical nature which is lost by passing to functions: we define Hall monoidal categories via a construction which fits into Day’s [Day74] theory of monoidal convolution. For the category $\text{Vect}_{F_1}$, we recover a classical monoidal category: Schur’s [Sch01] category of polynomial functors. For $\text{Vect}_{F_q}$, we obtain Joyal-Street’s [JS95] category of representations of the general linear groupoid over a finite field.

Section 3: $D = \{\infty\text{-groupoids}\}$. We introduce the Waldhausen $S_\bullet$-construction of a pretriangulated differential graded category and establish the 2-Segal conditions. We explain how to obtain Toën’s derived Hall algebra by passing to locally constant functions. This is a reformulation of the results of [Toë06] using the language of $\infty$-categories which makes the constructions
entirely parallel to the ones of Section 1. The material in this section is taken from [DK12].

Section 4, $\mathcal{D} = \{\text{differential } \mathbb{Z}/(2)\text{-graded categories}\}$. One of the motivations for formulating Question 0.1 is as follows: the 2-Segal simplicial space $S_\bullet$ from Section 3 can be constructed for any stable $\infty$-category $\mathcal{C}$ -- finiteness conditions on $\mathcal{C}$ are only needed when passing to functions. For example, the $S_\bullet$-construction of a 2-periodic dg category makes perfect sense while it is not clear how to pass to functions since the finiteness conditions are violated. In this example, we give an answer to Question 0.1 which can only be seen universally: the $S_\bullet$-construction has a canonical cyclic structure. Observed heuristically in [DK12], this statement has since been established in [Nad15, DK13, Lur14].

The cyclic structure should be regarded as a symmetric Frobenius structure on the abstract Hall algebra of a 2-periodic dg category. Instead of making this statement precise, we give an application: using a categorification of the state sum formalism from 2-dimensional topological field theory, we obtain invariants of oriented surfaces ([DK13]). Remarkably, a universal variant of this construction recovers Kontsevich’s construction [Kon09] of a version of the Fukaya category of a noncompact Riemann surface.

A nonlinear generalization of the results discussed in this section, and beyond, has been given by Lurie [Lur14] who also provides an interpretation in terms of a rotation invariance statement for algebraic $K$-theory.

Throughout this text, we consider categories which are small without explicitly mentioning this. For example, when we speak about the category of all sets then we really mean the category of $U$-small sets for a chosen Grothendieck universe which we leave implicit. Other examples of categories like groupoids, vector spaces, etc, are defined as small categories in a similar way.

Acknowledgements. I would like to thank the organizers of the “Advanced Course: (Re)emerging methods in commutative algebra and representation theory” at the CRM Barcelona for the invitation to give this lecture series. I thank Mikhail Kapranov for the many inspiring discussions on the subject. Much of the material presented in these notes originates either directly or indirectly from our joint work. Many thanks to Pranav Pandit for influential comments and for leading the working sessions. Further, I would like to thank Joachim Kock for interesting discussions. Last but not least, I thank Malte Leip for correcting many typos in a first draft of these notes.
1 Classical Hall algebras

1.1 Hall’s algebra of partitions

We outline the original context in which Hall algebras first appeared. Let \( p \) be a prime number, and let \( M \) be a finite abelian \( p \)-group. By the classification theorem for finitely generated abelian groups the group \( M \) decomposes into a direct sum of cyclic \( p \)-groups. Therefore, we have

\[ M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(p^{\lambda_i}) \]

where we may assume \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \) so that the sequence

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots) \]

is a partition, i.e., a weakly decreasing sequence of natural numbers with finitely many nonzero components. We call the partition \( \lambda \) the type of \( M \). The association

\[ M \mapsto \text{type of } M \]

provides a bijective correspondence between isomorphism classes of finite abelian \( p \)-groups and partitions.

Given partitions \( \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(s)}, \lambda \), we define

\[ g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(s)}}(p) \]

to be the number of flags

\[ M = M_0 \supset M_1 \supset \cdots \supset M_{s-1} \supset M_s = 0 \]

such that \( M_{i-1}/M_i \) has type \( \mu^{(i)} \) where \( M \) is a fixed group of type \( \lambda \). In this context, Hall had the following insight:

**Theorem 1.1.** The numbers \( g_{\mu\nu}^\lambda(p) \) form the structure constants of a unital associative algebra with basis \( \{u_\lambda\} \) labelled by the set of all partitions. More precisely, the \( \mathbb{Z} \)-linear extension of the formula

\[ u_\mu u_\nu = \sum_\lambda g_{\mu\nu}^\lambda(p)u_\lambda \]

defines a unital associative multiplication on the abelian group \( \bigoplus_\lambda \mathbb{Z}u_\lambda \).

**Proof.** One shows that the product

\[ u_\mu u_\nu u_\lambda, \]

with any chosen bracketing, is equal to

\[ \sum_\pi g_{\mu\nu\lambda}^\pi(p)u_\pi. \]

We will provide a proof of this statement in greater generality in Section 2 so that, at this point, we leave the details as an exercise.

The resulting associative algebra is called Hall’s algebra of partitions. We compute some examples of products \( u_\mu u_\nu \). Fixing an abelian \( p \)-group \( M \) of type \( \lambda \), the number \( g_{\mu\nu}^\lambda(p) \) is the number of subgroups \( N \subset M \) such that \( N \) has type \( \nu \) and \( M/N \) has type \( \mu \). In particular, we obtain that \( g_{\mu\nu}^\lambda(p) \) is nonzero if and only if \( M \) is an extension of a \( p \)-group \( N' \) of type \( \mu \) by a \( p \)-group \( N \) of type \( \nu \).
(1) We compute 
\[ u_{(1)}u_{(1)} = g_{(1)(1)}^{(1,1)}(p)u_{(1,1)} + g_{(1)(1)}^{(2)}(p)u_{(2)}. \]
Further, \( g_{(1)(1)}^{(1,1)}(p) \) is the number of subgroups 
\[ N \subset M = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \]
such that \( N \cong \mathbb{Z}/(p) \) and \( M/N \cong \mathbb{Z}/(p) \). This coincides with the number of 1-dimensional subspaces in the \( \mathbb{F}_p \)-vector space \( (\mathbb{F}_p)^2 \) (here \( \mathbb{F}_p \) denotes the field with \( p \) elements) of which there are \( p + 1 \). The number \( g_{(1)(1)}^{(2)}(p) \) is the number of subgroups 
\[ N \subset M = \mathbb{Z}/(p^2) \]
such that \( N \cong \mathbb{Z}/(p) \) and \( M/N \cong \mathbb{Z}/(p) \). Any such \( N \) must lie in the \( p \)-torsion subgroup of \( M \) which is \( p\mathbb{Z}/(p^2) \). But \( p\mathbb{Z}/(p^2) \cong \mathbb{Z}/(p) \) and so \( N = p\mathbb{Z}/(p^2) \) which implies \( g_{(1)(1)}^{(2)}(p) = 1 \). In conclusion, we have 
\[ u_{(1)}u_{(1)} = (p + 1)u_{(1,1)} + u_{(2)}. \]

(2) We compute 
\[ u_{(1,1)}u_{(1)} = g_{(1,1)(1)}^{(1,1)}(p)u_{(1,1,1)} + g_{(1,1)(1)}^{(2,1)}(p)u_{(2,1)}. \]
To compute \( g_{(1,1)(1)}^{(1,1)}(p) \) we have to determine the number of subgroups 
\[ N \subset M = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \]
such that \( N \cong \mathbb{Z}/(p) \) and \( M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \). This is equivalent to counting 1-dimensional subspaces of \( \mathbb{F}_p^3 \) of which there are \( p^2 + p + 1 \). Further, the number \( g_{(1,1)(1)}^{(2,1)}(p) \) is the number of subgroups 
\[ N \subset M = \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p) \]
such that \( N \cong \mathbb{Z}/(p) \) and \( M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \). As above, the subgroup \( N \) must be contained in the \( p \)-torsion subgroup of \( M \) which is \( p\mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p) \). There are \( p + 1 \) such subgroups, but only one of them satisfies the condition \( M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \): \( N = p\mathbb{Z}/(p^2) \), contained in the first summand of \( M \). Therefore, we have \( g_{(1,1)(1)}^{(2,1)}(p) = 1 \) so that 
\[ u_{(1,1)}u_{(1)} = (p^2 + p + 1)u_{(1,1,1)} + u_{(2,1)}. \]

1.2 Proto-abelian categories

Our goal in this section will be to introduce a certain class of categories to which Hall’s associative multiplication law can be generalized.

**Definition 1.2.** A category \( \mathcal{C} \) is called proto-abelian if the following conditions hold.

1. The category \( \mathcal{C} \) is pointed.
2. (a) Every diagram in \( \mathcal{C} \) of the form

\[
\begin{array}{c}
A \\
\downarrow \\
C
\end{array} \rightarrow \begin{array}{c}
B
\end{array}
\]

is pointed.
can be completed to a pushout square of the form
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D.
\end{array}
\]

(b) Every diagram in \( \mathcal{C} \) of the form
\[
\begin{array}{ccc}
B & \longrightarrow & C \\
\downarrow & & \downarrow \\
D & \longrightarrow & D
\end{array}
\]

can be completed to a pullback square of the form
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D.
\end{array}
\]

(3) A commutative square in \( \mathcal{C} \) of the form
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

is a pushout square if and only if it is a pullback square. We also call such a square \textit{biCartesian}.

**Example 1.3.** A pushout diagram of the form
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A'
\end{array}
\]
is called a \textit{short exact sequence}, or, an \textit{extension} of \( A' \) by \( A \).

**Example 1.4.**

(1) Abelian categories are proto-abelian.

(2) We introduce a category \( \textbf{Vect}_{F_1} \) of \textit{finite dimensional vector spaces over} \( F_1 \) as follows. An object of \( \textbf{Vect}_{F_1} \) is a finite set \( K \) equipped with a marked point \( * \in K \) (a \textit{pointed set}). A morphism \( (K,*) \to (L,*) \) is a map \( f : K \to L \) of underlying sets such that \( f(*) = * \) and the restriction \( f|_{K \setminus f^{-1}(*)} \) is injective. The category \( \textbf{Vect}_{F_1} \) is proto-abelian.

(3) Let \( \mathcal{C} \) be a proto-abelian category. Then the opposite category \( \mathcal{C}^{\text{op}} \) is proto-abelian. Given a category \( I \), the category \( \text{Fun}(I, \mathcal{C}) \) of \( I \)-diagrams in \( \mathcal{C} \) is proto-abelian.

To define the Hall algebra of a proto-abelian category \( \mathcal{C} \), we have to impose additional finiteness conditions. Two extensions \( A \hookrightarrow B \to A' \) and \( A \hookrightarrow C \to A' \) of \( A' \) by \( A \) are called equivalent if there exists a commutative diagram
\[
\begin{array}{ccc}
A' & \longrightarrow & B \\
\downarrow & \searrow & \downarrow \\
A' & \longrightarrow & C
\end{array}
\]
\[
\begin{array}{ccc}
B & \longrightarrow & A' \\
\downarrow & \nearrow & \downarrow \\
A' & \longrightarrow & A'
\end{array}
\]

We denote by \( \text{Ext}_{\mathcal{C}}(A', A) \) the set of equivalence classes of extensions of \( A' \) by \( A \).
Definition 1.5. A proto-abelian category $\mathcal{C}$ is called finitary if, for every pair of objects $A, A'$, the sets $\text{Hom}_{\mathcal{C}}(A', A)$ and $\text{Ext}_{\mathcal{C}}(A', A)$ have finite cardinality.

Theorem 1.6. Let $\mathcal{C}$ be a finitary proto-abelian category. Consider the free abelian group

$$\text{Hall}(\mathcal{C}) = \bigoplus_{[M] \in \text{iso}(\mathcal{C})} \mathbb{Z}[M]$$

on the set of isomorphism classes of objects in $\mathcal{C}$. Then the bilinear extension of the formula

$$[N] \cdot [L] = \sum_{[M] \in \text{iso}(\mathcal{C})} g_{N,L}^M [M]$$

where $g_{N,L}^M$ denotes the number of subobjects $A \subset M$ such that $A \cong L$ and $M/A \cong N$ defines a unital associative multiplication law on $\text{Hall}(\mathcal{C})$.

Proof. The axioms for a proto-abelian category are chosen so that the argument for Theorem 1.1 generalizes verbatim. □

1.3 A first example

1.3.1 The categories $\text{Vect}_{\mathbb{F}_q}$ and $\text{Vect}_{\mathbb{F}_1}$

Let $\text{Vect}_{\mathbb{F}_q}$ denote the category of finite dimensional vector spaces over the field $\mathbb{F}_q$ where $q$ is some prime power. The category $\text{Vect}_{\mathbb{F}_q}$ is finitary proto-abelian, and we have

$$\text{Hall}(\text{Vect}_{\mathbb{F}_q}) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[\mathbb{F}_q^n]$$

with multiplication given by

$$[\mathbb{F}_q^n][\mathbb{F}_q^m] = g_{n,m}^{n+m}(q)[\mathbb{F}_q^{n+m}]$$

where

$$g_{n,m}^{n+m}(q) = \frac{|\{V \subset \mathbb{F}_q^{n+m} \mid V \cong \mathbb{F}_q^m, \mathbb{F}_q^{n+m}/V \cong \mathbb{F}_q^n\}|}{|\text{GL}_n(\mathbb{F}_q)|}$$

$$= \frac{(q^{n+m} - 1)(q^{n+m} - q)\cdots(q^{n+m} - q^{m-1})}{(q^m - 1)(q^m - q)\cdots(q^m - q^{m-1})}$$

$$= \frac{(q^{n+m} - 1)(q^{n+m-1} - 1)\cdots(q^{n+1} - 1)}{(q^m - 1)(q^{m-1} - 1)\cdots(q - 1)}$$

$$= \left[ \begin{array}{c} n+m \\ m \end{array} \right]_q.$$
and we finally set
\[
\begin{bmatrix}
  n + m \\
  m
\end{bmatrix}_q = \frac{[n + m]_q!}{[m]_q! [n]_q!}.
\]

We have an isomorphism of \(\mathbb{Z}\)-algebras
\[
\text{Hall}(\text{Vect}_{\mathbb{F}_q}) \cong \mathbb{Z}[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots] \subset \mathbb{Q}[x],
\]
\[
[F^n_q] \mapsto \frac{x^n}{[n]_q!}.
\]

On the other hand, consider the category \(\text{Vect}_{\mathbb{F}_1}\) which is easily seen to be finitary and proto-abelian. We have
\[
\text{Hall}(\text{Vect}_{\mathbb{F}_1}) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[\{* \cup \mathbb{N}, \{1, 2, \ldots, n\}]
\]
with multiplication
\[
\left[\{\{*, 1, 2, \ldots, n\}\} \{\{*, 1, 2, \ldots, m\}\} \right] = \lambda_{n,m}^{n+m} n,m [\{*, 1, 2, \ldots, n+m\}]
\]
where
\[
\lambda_{n,m}^{n+m} = \frac{|\{\{*, 1, 2, \ldots, m\} \mapsto \{*, 1, 2, \ldots, n+m\}|}{|S_m|} = \binom{n+m}{m}.
\]

We have an isomorphism of algebras
\[
\text{Hall}(\text{Vect}_{\mathbb{F}_1}) \cong \mathbb{Z}[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots] \subset \mathbb{Q}[x],
\]
\[
[F^n_q] \mapsto \frac{x^n}{n!}.
\]

Therefore, the algebra \(\text{Hall}(\text{Vect}_{\mathbb{F}_1})\) is the free divided power algebra on one generator. Note that, interpreting \(q\) as a formal variable, we have
\[
\lambda_{n,m}^{n+m} = \delta_{n,m}^{n+m}(1)
\]
so that \(\text{Hall}(\text{Vect}_{\mathbb{F}_q})\) can be regarded as a \(q\)-analog of \(\text{Hall}(\text{Vect}_{\mathbb{F}_1})\).

**Remark 1.8.** We can give a natural explanation for the commutativity of both Hall algebras computed in this section: Assume that a finitary proto-abelian category \(\mathcal{C}\) is equipped with an exact equivalence \(D : \mathcal{C}^{\text{op}} \to \mathcal{C}\) such that, for every object \(M\), we have \(D(M) \cong M\). Then \(\text{Hall}(\mathcal{C})\) is a commutative algebra. Now we have:

1. The category \(\text{Vect}_{\mathbb{F}_q}\) is equipped with the exact duality \(V \mapsto V^* = \text{Hom}_{\text{Vect}_{\mathbb{F}_q}}(V, F_q)\) which satisfies \(V^* \cong V\).

2. The category \(\text{Vect}_{\mathbb{F}_1}\) is equipped with the exact duality \(K \mapsto K^* = \text{Hom}_{\text{Vect}_{\mathbb{F}_1}}(K, \{1, *\})\). Curiously, in contrast to \(\text{Vect}_{\mathbb{F}_q}\), the dual \(K^*\) can be canonically identified with \(K\).
1.3.2 Statistical interpretation of $q$-analogs

We introduce terminology to discuss the phenomenon of equation (1.7) somewhat more systematically. Let $S$ be a finite set. A statistic on $S$ is a function

$$f : S \rightarrow \mathbb{N}.$$ 

Given a statistic $f$, we define the corresponding partition function to be

$$Z(q) = \sum_{s \in S} q^{f(s)}.$$ 

**Remark 1.9.** Evaluation of the partition function at $q = 1$ yields the cardinality of the set $S$ so that $Z(q)$ can be interpreted as a $q$-analog of $|S|$. Note that, any $q$-analog obtained in this way from a statistic will therefore, by construction, be polynomial in $q$.

**Example 1.10.**

1. Consider the set $S = \{1, \ldots, n\}$. We define a statistic on $S$ via

$$f : S \rightarrow \mathbb{N}, \ i \mapsto i - 1.$$ 

The corresponding partition function is

$$Z(q) = 1 + q + \cdots + q^{n-1} = [n]_q.$$ 

2. Consider the set $S_n$ underlying the symmetric group on $n$ letters. We define the inversion statistic on $S_n$ as

$$\text{inv} : S_n \rightarrow \mathbb{N}, \ \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix} \mapsto |\{(i,j) \mid i < j, \ \sigma_i > \sigma_j\}|$$

We claim that we have

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]_q !.$$ 

To show this, we interpret the summands in the expansion of the product

$$[n]_q ! = 1(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).$$ 

Given a summand $q^a$ of the product, it must arise as a product $q^{i_1}q^{i_2} \cdots q^{i_n}$ with $0 \leq i_k \leq k - 1$. We produce a corresponding permutation $\sigma$ by providing an algorithm to write the list $\sigma_1, \ldots, \sigma_n$. In step 1, we start by writing the number 1. In step 2, we write the number 2 to the left of 1 if $i_2 = 1$ or to the right of 1 if $i_2 = 0$. At step $k$, there will be $k - 1$ numbers and we label the gaps between the numbers by $0, \ldots, k - 1$ from right to left. We fill in the number $k$ into the gap with label $i_k$. This algorithm produces a permutation $\sigma$ with $\text{inv}(\sigma) = a$. The claim follows immediately from this construction.

Finally, we would like to find a statistical interpretation of the $q$-binomial coefficient. We will achieve this by defining a statistic on the set $P(m,n+m)$ of subsets of $\{1, \ldots, n+m\}$ of cardinality $m$. A lattice path in the rectangle of size $(n,m)$ is a path in $\mathbb{R}^2$ which begins at $(0,0)$, ends at $(n,m)$, and is obtained by sequence of steps moving either one integer step to the east or to the north. Given $K \in P(m,n+m)$, we can construct a lattice path by the following rule: at step $i$, we move east if $i \notin K$, and north if $i \in K$. It is immediate that this construction provides a bijective correspondence between $P(m,n+m)$ and lattice paths in the rectangle of size $(n,m)$. We now define the statistic as

$$a : P(m,n+m) \rightarrow \mathbb{N}, \ K \mapsto a(K)$$

where $a(K)$ denotes the area of the part of the rectangle of size $(n,m)$ which lies above the lattice path corresponding to $K$. 

10
Proposition 1.11. We have
\[ \sum_{K \in P(m,n+m)} q^{a(K)} = \binom{n + m}{m}_q. \]

Proof. We prove this equality by showing that both sides satisfy the recursion
\[ Q(m,n)(q) = q^n Q(m-1,n)(q) + Q(m,n-1). \]
On the left hand side, the term \( q^n Q(m-1,n)(q) \) is the contribution from subsets \( K \) such that \( n + m \in K \), the term \( Q(m,n-1) \) is the contribution from subsets such that \( n + m \not\in K \). The right hand side satisfies the recursion by a straightforward calculation.

In conclusion, we obtain that the structure constants of the Hall algebra of \( \text{Vect}_{\mathbb{F}_q} \) have a statistical interpretation and are hence polynomial in \( q \). The value at \( q = 1 \) is realized by the structure constants of the Hall algebra of \( \text{Vect}_{\mathbb{F}_1} \).

The phenomenon observed seems to be a general feature: the Hall algebra of a finitary \( \mathbb{F}_q \)-linear abelian category often describes a \( q \)-analog of an algebra of more classical nature obtained by specializing at \( q = 1 \). The idea to describe this latter algebra as the Hall algebra of a category of combinatorial nature is quite recent and due to M. Szczesny [Szc11].

In the next section, we will discuss this phenomenon in the classical context: We will see that Hall’s algebra of partitions is isomorphic to the ring \( \Lambda \) of symmetric functions and equips it with an interesting \( \mathbb{Z} \)-basis given, up to some renormalization, by the so-called Hall-Littlewood symmetric functions. This basis is a \( q \)-analog of the classical basis of \( \Lambda \) given by the monomial symmetric functions which can be obtained via the Hall algebra of a proto-abelian category of combinatorial nature: the category of vector spaces over \( \mathbb{F}_1 \) equipped with a nilpotent endomorphism.

1.4 Hall’s algebra of partitions and symmetric functions

Let \( R \) be a discrete valuation ring with (a principal ideal domain with exactly one nonzero maximal ideal \( m \)) with residue field of finite cardinality \( q \). It turns out that the Hall algebra of the category of finite \( R \)-modules only depends on \( q \). For the \( p \)-adic integers \( \mathbb{Z}_p \), we recover the category of abelian \( p \)-groups and thus Hall’s classical algebra of partitions. For \( R = \mathbb{F}_q[[t]] \) we obtain another construction of the same Hall algebra.

We will analyze the algebra \( \text{Hall}(\mathbb{F}_q[[t]]) \) by establishing an interpretation similar to the one provided in Section 1.3. The case \( q = 1 \) corresponds to the Hall algebra of a proto-abelian category of combinatorial nature and \( \text{Hall}(R) \) is studied as a \( q \)-analog by giving a statistical interpretation of (some of) its structure constants.

1.4.1 The Hall algebra of \( \mathbb{F}_1[[t]] \)

Definition 1.12. A finite \( \mathbb{F}_1[[t]] \)-module is an object of \( \text{Vect}_{\mathbb{F}_1} \) equipped with a nilpotent endomorphism.

The isomorphism classes of finite \( \mathbb{F}_1[[t]] \)-modules are naturally labelled by the set of all partitions: a pointed set \( K \) equipped with a nilpotent endomorphism \( T \), corresponds to rooted tree with vertices given by the elements of \( K \) and an edge from \( k \) to \( k' \) if \( T(k) = k' \). We obtain a partition by reordering the tuple of lengths of the branches.
Example 1.13. The finite $F[[t]]$-module corresponding to the partition $(3, 2, 2, 1, 0, \ldots)$ is represented by the rooted tree

```
  *  \
 /\ 
.overlay  
/\  
/  
.overlay
```

We write

$$\text{Hall}(F[[t]]) = \bigoplus_{\lambda} \mathbb{Z} u_{\lambda}$$

for the Hall algebra of the category of finite $F[[t]]$-modules where $\lambda$ runs over all partitions.

Example 1.14. An example of a short exact sequence of finite $F[[t]]$-modules is given by

```
  *  \
 /\ 
.overlay  
/\  
/  
.overlay
```

Given a finite $F[[t]]$-module $K$ with nilpotent endomorphism $T$, the dual $D(K) = \text{Hom}(K, \{1, *\})$ is naturally equipped with a nilpotent endomorphism obtained by precomposing with $T$. The corresponding rooted tree $\Gamma_{D(K)}$ is obtained from $\Gamma_K$ by removing $\ast$, reversing the orientation of all edges, and adding $\ast$ as a root. This description implies that $D(K)$ has the same branch lengths and is hence isomorphic to $K$. We deduce that $\text{Hall}(F[[t]])$ is commutative.

Proposition 1.15. Let $\lambda$ be a partition of length $s$. Then, in $\text{Hall}(F[[t]])$, we have

$$u_{(1^{\lambda_1})} u_{(1^{\lambda_2})} \cdots u_{(1^{\lambda_s})} = \sum_{\mu} a_{\lambda \mu} u_{\mu}$$

where $a_{\lambda \mu}$ denotes the number of $N$-by-$N$ matrices with entries in $\{0, 1\}$ with column sums $\lambda$ and row sums $\mu$.

Proof. By definition, the coefficient $a_{\lambda \mu}$ is the number of flags

$$K = K_0 \supset K_1 \supset \cdots \supset K_{s-1} \supset K_s = \{\ast\}$$

where $K$ is fixed of type $\mu$ and $K_{i-1}/K_i$ has type $(1^{\lambda_i})$. We represent $K$ as an oriented graph with branches labelled by $1, \ldots, s$. Let $K_1 \subset K$ be a submodule such that $K/K_1$ has type $(1^{\lambda_1})$. Then the set $K \setminus K_1$ consists of exactly $\lambda_1$ elements which form the tips of pairwise disjoint branches. We may encode this in a vector $v_1$ in $\{0, 1\}^s$ where we mark those branches of $K$ which contain a point in $K \setminus K_1$ by 1 and all remaining branches by 0. Note that the sum over all entries in $v_1$ equals $\lambda_1$. We repeat this construction for each $K_i \subset K_{i-1}$ and organize the resulting vectors $v_1, v_2, \ldots, v_s$ as the columns of a matrix. By construction, this matrix has column sums $\lambda$ and row sums $\mu$. This construction establishes a bijection between flags of the above type and $\{0, 1\}$-matrices with column sums $\lambda$ and row sums $\mu$. □
To analyze the nature of the matrix \((a_{\lambda\mu})\) indexed by the set of all partitions, we introduce some terminology: We define two orders on the set \(P_n\) of partitions of a natural number \(n\):

1. The **lexicographic order**: \(\mu \leq l \lambda\) if \(\mu = \lambda\) or the first nonzero difference \(\mu_i - \lambda_i\) is negative.

2. The **dominance order**: \(\mu \leq d \lambda\) if, for all \(i \geq 1\),
   \[\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i.\]

Observe that \(\mu \leq d \lambda\) implies \(\mu \leq l \lambda\).

**Proposition 1.17.** Define \(\tilde{a}_{\lambda\mu} = a_{\lambda'\mu}\). Then, we have \(\tilde{a}_{\lambda\lambda} = 1\), and \(\tilde{a}_{\lambda\mu} = 0\) unless \(|\lambda| = |\mu|\) and \(\mu \leq d \lambda\).

**Proof.** First assume that \((A_{ij})\) is a \([0,1]\)-matrix with column sums \(\lambda'\) and row sums \(\mu\) so that \((A_{ij})\) has no gaps. Here a gap is a 0 entry in a column which is followed below by an entry 1 in the same column.

The condition that \((A_{ij})\) has no gaps means that the 1-entries of the matrix constitute a Young diagram of the partition \(\mu\) whose transpose is \(\lambda'\). This implies \(\lambda = \mu\). Vice versa, it is easy to see that this “Young” matrix is the unique \([0,1]\)-matrix with column sums \(\lambda'\) and row sums \(\lambda\). Therefore, we obtain \(\tilde{a}_{\lambda\lambda} = 1\). Now suppose that \((A_{ij})\) is a \([0,1]\)-matrix with column sums \(\lambda'\) and row sums \(\mu\) which has a gap. Pick a column with a gap and swap the 0 forming the gap with the lowest 1 in the same column, thus obtaining a new matrix \((\tilde{A}_{ij})\).

The column sums of the new matrix have not changed. The sequence of row sums \(\alpha = (\alpha_1, \alpha_2, \ldots)\) has changed, in particular, it may not form a partition. But, enlarging the definition of the dominance order from partitions to arbitrary sequences with entries in \(\mathbb{N}\), it is immediate to verify \(\alpha \geq d \mu\). We obtain a modified matrix \((\tilde{A}_{ij})\) with less gaps, column sums \(\lambda'\), and row sums given by \(\alpha \geq d \mu\). If the matrix \((\tilde{A}_{ij})\) has no gaps, then the above argument shows that \(\alpha = \lambda\). Otherwise we iterate, producing a totally ordered chain of sequences in \(\mathbb{N}\)

\[\mu \leq d \alpha \leq d \cdots \leq d \lambda\]

Showing that \(\mu \leq d \lambda\). \(\square\)

The proposition implies that the matrix \((\tilde{a}_{\lambda\mu})\) is upper unitriangular with respect to the lexicographic order so that it is invertible over \(\mathbb{Z}\). We deduce the invertibility of the matrix \((a_{\lambda\mu})\) and conclude:

**Corollary 1.18.** The set \(\{u(r)\mid r > 0\}\) is algebraically independent and generates \(\text{Hall}(\mathbb{F}_1[[t]])\) as a \(\mathbb{Z}\)-algebra. In other words,

\[\text{Hall}(\mathbb{F}_1[[t]]) = \mathbb{Z}[u(1), u(2), \ldots]\]

is a polynomial ring in countably many variables.

### 1.4.2 Relation to symmetric functions

Let \(\mathbb{Z}[x_1, \ldots, x_n]\) denote the ring of polynomials with integer coefficients. The symmetric group \(S_n\) acts by permuting the variables and the polynomials which are invariant under this action are called **symmetric polynomials**. They form a ring which we denote by

\[A_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}\.]
The ring $\Lambda_n$ is graded by total degree so that we have
$$\Lambda_n^k = \bigoplus_{k \geq 0} \Lambda_n^k.$$ 

Given a tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we obtain a monomial
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$ 

For a partition $\lambda$ of length $l(\lambda) \leq n$, we define
$$m_\lambda(x_1, \ldots, x_n) = \sum_{\alpha} x^\alpha$$
where the sum ranges over all distinct permutations $\alpha$ of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. For example, we have
$$m_{(1,1,\ldots,1)}(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$$
$$m_{(1,1,0,\ldots,0)}(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j$$
$$m_{(k,0,0,\ldots,0)}(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k.$$ 

It is immediate that the collection of polynomials $\{m_\lambda(x_1, \ldots, x_n) | l(\lambda) \leq n\}$ forms a $\mathbb{Z}$-basis of $\Lambda_n$. In particular, the set $\{m_\lambda(x_1, \ldots, x_n) | l(\lambda) \leq n, |\lambda| = k\}$ forms a $\mathbb{Z}$-basis of $\Lambda_n^k$.

Many statements and formulas involving symmetric polynomials hold independently of the number of variables $n$. This can be naturally incorporated by introducing symmetric functions which formalize the notion of symmetric polynomials in countably many variables. Fix $k \geq 0$ and consider the projective system of abelian groups
$$\cdots \to \Lambda_n^{k+1} \xrightarrow{\rho_n} \Lambda_n^k \xrightarrow{\rho_n} \Lambda_n^{k-1} \to \cdots$$
where the map $\rho_{n+1} : \Lambda_n^{k+1} \to \Lambda_n^k$ is obtained by sending $x_{n+1}$ to 0. We denote the inverse limit of the projective system by
$$\Lambda^k = \lim_{\leftarrow} \Lambda_n^k.$$ 

**Example 1.19.** Let $\lambda$ be a partition and let $n > l(\lambda)$. Then we have
$$m_\lambda(x_1, \ldots, x_{n-1}, 0) = m_\lambda(x_1, \ldots, x_{n-1}).$$ 

Therefore, the sequence $\{m_\lambda(x_1, \ldots, x_n) | n > l(\lambda)\}$ defines an element of $\Lambda^k$ which we denote by $m_\lambda$. We call the symmetric functions $\{m_\lambda\}$ the monomial symmetric functions.

We finally define
$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$
called the ring of symmetric functions. There is an apparent bilinear multiplication map $\Lambda^k \times \Lambda^k' \to \Lambda^{k+k'}$ which makes $\Lambda$ a graded ring. The set $\{m_\lambda\}$ where $\lambda$ ranges over all partitions forms a basis of the ring $\Lambda$ of symmetric functions.

We introduce another family of symmetric functions. For $r > 0$, let
$$e_r = m_{(1^r)} = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$
and for a partition $\lambda$, we let
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_s}$$
where $s = l(\lambda)$. The symmetric functions $\{e_\lambda\}$ are called elementary symmetric functions.
Proposition 1.20. Let \( \lambda \) be a partition. We have
\[
e_\lambda = \sum_{\mu} a_{\lambda \mu} m_{\mu}
\]
where, as above, \( a_{\lambda \mu} \) denotes the number of \( \{0, 1\} \)-matrices with column sums \( \lambda \) and row sums \( \mu \).

**Proof.** Let \( x^\mu \), where \( \mu \) is a partition, be a monomial which appears in the product expansion of
\[
e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_s}.
\]
This means that \( x^\mu \) must be of the form
\[
x^\mu = y_1 y_2 \cdots y_s \tag{1.21}
\]
where \( y_j \) is a monomial term of \( e_{\lambda_j} \). We write each monomial \( y_j \) as
\[
y_j = \prod_i x_i^{A_{ij}}
\]
with \( A_{ij} \in \mathbb{N} \). We then observe that the condition that \( y_j \) be a monomial term of \( e_{\lambda_j} \) simply translates into the condition that the \( j \)th column of the matrix \( (A_{ij}) \) has entries in \( \{0, 1\} \) and satisfies \( \sum_i A_{ij} = \lambda_j \). Similarly, the condition that equation (1.21) holds translates into the condition that, for every \( i \), the \( i \)th row of the matrix \( (A_{ij}) \) sums to \( \mu_i \). This implies that the number of terms \( x^\mu \) (and hence, since the result is symmetric, the number of terms \( m_{\mu} \)) in \( e_\lambda \) is given by \( a_{\lambda \mu} \).

We obtain immediate corollaries:

**Corollary 1.22** ([Szc11]). There is a \( \mathbb{Z} \)-algebra isomorphism
\[
\varphi : \text{Hall}(\mathbb{F}_1[[t]]) \xrightarrow{\cong} \Lambda
\]
determined by \( \varphi(u_{(1^r)}) = e_r \). We further have \( \varphi(u_\lambda) = m_\lambda \) for an arbitrary partition \( \lambda \).

**Corollary 1.23.** The set \( \{e_\lambda\} \) forms a \( \mathbb{Z} \)-basis of \( \Lambda \). In particular, the set \( \{e_1, e_2, \ldots\} \) is algebraically independent and generates \( \Lambda \) as a ring so that
\[
\Lambda = \mathbb{Z}[e_1, e_2, \ldots]
\]
is a polynomial ring in countably many variables.

The statement of this last corollary is known as the **fundamental theorem on symmetric functions**.

### 1.4.3 Zelevinsky’s statistic

Let \( R \) is a discrete valuation ring with residue field of cardinality \( q \). Our discussion of \( \text{Hall}(R) \) will be based on \( q \)-analogs of the arguments in Section 1.4.1 which are obtained via a statistic introduced by Zelevinsky.

We start with some preparatory remarks. Recall that the length \( l(M) \) of an \( R \)-module \( M \) is defined to be the length of a composition series of \( M \). For example, the cyclic module \( R/m^n \) has composition series
\[
R/m^n \supset m/m^n \supset m^2/m^n \supset \cdots \supset m^{n-1}/m^n \supset 0
\]
so that we have \( l(R/m^n) = n \). From this, we deduce that a finite module \( M \) of type \( \lambda \) has length given by \( l(M) = |\lambda| \). Further, given a finite module \( M \), the sequence \((\mu_1, \mu_2, \ldots)\) with 

\[ \mu_i = \dim_k(m^{i-1}M/m^iM) \]

is the conjugate partition of \( \lambda \). Finally, recall that the length is additive in short exact sequences so that, for a submodule \( N \subset M \), we have \( l(M) = l(N) + l(M/N) \).

Let \( \lambda \) be a partition of length \( s \). We introduce numbers \( b_{\lambda \mu} \) via the equation

\[ u_{(1^\lambda_1)}u_{(1^\lambda_2)} \cdots u_{(1^\lambda_s)} = \sum_\mu b_{\lambda \mu}u_\mu. \]

We will interpret the numbers \( b_{\lambda \mu} \) as \( q \)-analog of the numbers \( a_{\lambda \mu} \) from Proposition 1.15 by introducing a statistic on the set \( \mathcal{A}_{\lambda \mu} \) of \( \{0,1\} \)-matrices with column sums \( \lambda \) and row sums \( \mu \). To this end, we have to introduce some terminology. For compositions \( \alpha, \beta \), we define an array \( A \) of shape \( \alpha \) and weight \( \beta \) to be a labelling of the squares of the diagram of \( \alpha \) such that the number \( i \) appears \( \beta_i \) times. We express an array as a function 

\[ A : \alpha \rightarrow \mathbb{N}^+ \]

where we consider (the diagram of) \( \alpha \) as a subset of \( \mathbb{N}^+ \times \mathbb{N}^+ \). We will extend \( A \) to all of \( \mathbb{N}^+ \times \mathbb{N}^+ \) by letting elements in the complement of \( \alpha \) have value \( \infty \). For \( x = (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \), we denote \( x^- = (i,j+1) \). We call an array row-ordered (resp. row-strict) if, for every \( x \in \alpha \), \( A(x^-) \geq A(x) \) (resp. \( A(x^-) > A(x) \)). Analogously, we define column-ordered and column-strict arrays.

**Remark 1.24.** Given a row-strict array of shape \( \mu \) and weight \( \lambda \), we introduce a \( \{0,1\} \)-matrix which has entry 1 at the positions \( \{(i,A(x))\} \) where \( x = (i,j) \) runs over all \( x \in \alpha \), and entry 0 elsewhere. It is immediate that this construction provides a bijection between row-strict arrays of shape \( \mu \) and weight \( \lambda \) and the set \( \mathcal{A}_{\lambda \mu} \) of \( \{0,1\} \)-matrices with column sums \( \lambda \) and row sums \( \mu \).

We define a lexicographic order on \( \mathbb{N}^+ \times \mathbb{N}^+ \) via

\[ (i,j) < (i',j') \iff \text{either } j < j', \text{ or } j = j' \text{ and } i > i'. \]

For a row-strict array \( A \) of shape \( \alpha \), we define

\[ d(A) = |\{(x,y) \in \alpha \times \alpha \mid y < x \text{ and } A(x) < A(y) < A(x^-)\}|. \]

By the above remark, we obtain a statistic

\[ d : \mathcal{A}_{\lambda \mu} \rightarrow \mathbb{N}, A \mapsto d(A) \]

on the set of \( \{0,1\} \)-matrices with column sums \( \lambda \) and row sums \( \mu \) which we call Zelevinsky’s statistic. It provides the following statistical interpretation of the coefficients \( b_{\lambda \mu} \):

**Theorem 1.25 (Zelevinsky).** We have

\[ b_{\lambda \mu} = \sum_A q^{d(A)} \]

where \( A \) ranges over all row-strict arrays of shape \( \mu \) and weight \( \lambda \).
Before giving a proof, we analyze consequences of the theorem. We introduce the polynomials
\[ a_{\lambda \mu}(t) = \sum A t^{d(A)} \] so that \( b_{\lambda \mu} = a_{\lambda \mu}(q) \). We have:

1. The polynomial \( a_{\lambda \mu}(t) \) has nonnegative integral coefficients.
2. \( a_{\lambda \mu}(1) \) is the number of \( \{0,1\} \)-matrices with column sums \( \lambda \) and row sums \( \mu \).
3. \( a_{\lambda \mu}(t) = 0 \) unless \( \mu \leq d^{\lambda} \). Moreover, \( a_{\lambda \lambda}(t) = 1 \).

The statements of (1) and (2) are immediate. To see (3), note that we have \( a_{\lambda \mu}(1) = 0 \) unless \( \mu \leq d^{\lambda} \) which implies the first statement. \( a_{\lambda \lambda}(t) = 1 \) follows from direct computation, using the fact that there is precisely one \( \{0,1\} \)-matrix with column sums \( \lambda \) and row sums \( \lambda \) and the corresponding array \( A \) satisfies \( d(A) = 0 \).

We deduce that the matrix \( (a_{\lambda \mu}(t)) \) has an inverse over \( \mathbb{Z}[t] \) and have the following immediate consequences.

**Theorem 1.26.** Let \( R \) be a discrete valuation ring with residue field of cardinality \( q \). Then the following hold:

1. The set \( \{ u_{(1^r)} | r > 0 \} \) is algebraically independent and generates \( \text{Hall}(R) \) as a \( \mathbb{Z} \)-algebra. In particular, there is a \( \mathbb{Z} \)-algebra isomorphism
   \[ \psi : \text{Hall}(R) \cong \Lambda \]
   determined by \( \psi(u_{(1^r)}) := m_{(1^r)} \).
2. The structure constants of \( \text{Hall}(R) \) are polynomial in \( q \) so that there exist polynomials \( g^\lambda_{\mu \nu}(t) \) such that
   \[ u_{\mu} u_{\nu} = \sum_{\lambda} g^\lambda_{\mu \nu}(q) u_{\lambda} \].

   The numbers \( g^\lambda_{\mu \nu}(1) \) are the structure constants of \( \text{Hall}(\mathbb{F}_1[[t]]) \) and, hence, the structure constants of the \( \mathbb{Z} \)-basis of \( \Lambda \) given by the monomial symmetric functions.

   From the theorem, we deduce that we have constructed a family of \( \mathbb{Z} \)-bases \( \{ \psi(u_{\lambda}) \} \) of \( \Lambda \) which varies polynomially in \( q \) and specializes for \( q = 1 \) to the basis given by the monomial symmetric functions. Up to some renormalization, the symmetric functions comprising this family are known as the Hall-Littlewood symmetric functions. Remarkably, this family of bases also relates to another natural basis of \( \Lambda \): the Schur functions. This can be seen from a more refined analysis of Zelevinsky’s statistic relating it to Kostka numbers ([Mac95 Appendix A]).

   We only mention the final result of this discussion which leads to the following augmentation of Theorem 1.26:

3. The polynomial \( g^\lambda_{\mu \nu}(t) \) has degree \( \leq n(\lambda) - n(\mu) - n(\nu) \) and the various coefficients of \( t^{n(\lambda) - n(\mu) - n(\nu)} \) are the structure constants of \( \Lambda \) with respect to the Schur basis \( \{ s_{\lambda} \} \).

The results of Theorem 1.26 have been known independently to Hall (1959) and Steinitz (1901).

We conclude with a proof of Zelevinsky’s theorem:

**Proof.** We prove a slightly stronger version which will allow for an inductive argument: Let \( \mu, \lambda \) be partitions and let \( \alpha \sim \mu \) be a composition which is a permutation of \( \mu \). Then we will show that
\[ b_{\lambda \mu} = \sum_A q^{d(A)} \]
where $A$ ranges over all arrays of shape $\alpha$ and weight $\lambda$.

**Step 1.** We reformulate the formula in terms of sequences of compositions: For $\alpha$, $\beta$ compositions, we write $\beta \vdash \alpha$ if, for all $i$, $\alpha_i - 1 \leq \beta_i \leq \alpha_i$. For $\beta \vdash \alpha$, we define

$$d(\alpha, \beta) = |\{(i, j) \mid \beta_i = \alpha_i, \beta_j = \alpha_j - 1, \text{ and } (j, \alpha_j) < (i, \alpha_i)\}|$$

We then observe that, given $\alpha$ composition and $\lambda$ partition of length $\leq s$, we have a natural bijection between

$$\{\text{row-strict arrays } A \text{ of shape } \alpha \text{ and weight } \lambda\}$$

and

$$\left\{\text{sequences } 0 = \alpha^{(0)} \vdash \alpha^{(1)} \vdash \cdots \vdash \alpha^{(s)} = \alpha \text{ with } |\alpha^{(i)}| - |\alpha^{(i-1)}| = \lambda_i\right\}.$$

Under this correspondence, we have

$$d(A) = \sum_{i \geq 0} d(\alpha^{(i)}, \alpha^{(i-1)}).$$

**Step 2.** Induction: Assume that

$$u_{(\lambda_1)} \cdots u_{(\lambda_s)} = \sum_\nu \left(\sum_{0=\beta^{(0)}=\cdots=\beta^{(s-1)}=\beta} \prod_{i \geq 1} q^{d(\beta^{(i)}, \beta^{(i-1)})}\right) u_\nu$$

where, for each $\nu$, $\beta$ is a fixed permutation of $\nu$. Then to show that

$$u_{(\lambda_1)} \cdots u_{(\lambda_s)} = \sum_\mu \left(\sum_{0=\alpha^{(0)}=\cdots=\alpha^{(s)}=\alpha} \prod_{i \geq 1} q^{d(\alpha^{(i)}, \alpha^{(i-1)})}\right) u_\mu$$

where, for each $\mu$, $\alpha$ is a fixed permutation of $\mu$, it suffices to show

$$u_\nu u_{(1^r)} = \sum_\mu \left(\sum_{\beta \vdash \alpha, |\alpha| - |\beta| = r} q^{d(\alpha, \beta)}\right) u_\mu$$

where, for each $\mu$, $\alpha$ is any fixed permutation of $\mu$.

**Step 3.** In other words, we have to verify for the structure constant $g^\mu_{\nu(1^r)}$ of Hall($R$), the formula

$$g^\mu_{\nu(1^r)} = \sum_{\beta} q^{d(\alpha, \beta)}$$

where $\alpha$ is a fixed permutation of $\mu$ and $\beta$ runs through all permutations of $\nu$ such that $\beta \vdash \alpha$, and $|\alpha| - |\beta| = r$. Recall that, for a fixed $R$-module $M$ of type $\mu$ the number $g^\mu_{\nu(1^r)}$ is the number of submodules $N \subset M$ such that $N$ has type $(1^r)$ and $M/N$ has type $\nu$. In particular, we have $mN = 0$, so that $N$ is a $r$-dimensional $k$-subspace of the $k$-vector space given by the socle $S = \{x \in M \mid m x = 0\}$ of $M$. We denote by $G_r(S)$ the set of all $r$-dimensional subspaces of $S$. For every choice of a basis $\{v_i \mid i \in I\}$ of $S$ together with a total order of $I$, the set $G_r(S)$ is the disjoint union of *Schubert cells* defined as follows: We have one Schubert cell $C_J$ for every $r$-subset $J$ of $I$. The elements of $C_J$ have coordinates $(c_{ij} \in k \mid j \in J, i \in I \setminus J, i > j)$ where the subspace of $S$ corresponding to a coordinate $(c_{ij})$ has basis $\{v_j + \sum_i c_{ij} v_i\}_{j \in J}$.

Therefore, we have

$$|C_J| = q^{d(J)}$$

18
where \( d(J) \) is the number of pairs \((i, j)\) such that \( j \in J, \ i \in J \setminus I, \ i > j \). Suppose

\[
M \cong \bigoplus_{i \in I} R/m^{\alpha_i}
\]

We order \( I \) so that \( j < i \) iff \((j, \alpha_j) < (i, \alpha_i)\). Then we have a bijective correspondence between subsets \( J \subset I \) and compositions \( \beta + \alpha \) (where \( \beta_i = \alpha_{i-1} \) iff \( i \in J \)). Under this correspondence, we have \( d(J) = d(\alpha, \beta) \). Further, for all \( k\)-subspaces \( N \subset S \subset M \) which lie in a fixed Schubert cell \( C_J \), the quotient \( M/N \) has the same type \( \lambda \sim \beta \). Therefore, only those Schubert cells so that \( M/N \) has type \( \nu \) contribute to the count and we obtain precisely the claimed formula. \( \square \)
2 Hall algebras via groupoids

2.1 Groupoids of flags

A groupoid is a category in which all morphisms are invertible.

Example 2.1. (1) Let $G$ be a group. Then we can define a groupoid $BG$ which has one object $*$ and $\text{Hom}(*,*) = G$ where the composition of morphisms is given by the group law.

(2) Let $X$ be a topological space. Then we can define the fundamental groupoid $\Pi(X)$ whose objects are the points of $X$ and a morphism between $x$ and $y$ is a homotopy class of continuous paths connecting $x$ to $y$.

(3) Let $\mathcal{C}$ be a category. Then we can form the maximal groupoid $\mathcal{C}^{\circ\circ}$ in $\mathcal{C}$ by simply discarding all noninvertible morphisms in $\mathcal{C}$.

We introduce a family of groupoids which will be of central relevance for this section. Let $\mathcal{C}$ be a proto-abelian category and let $S_n = S_n(\mathcal{C})$ denote the maximal groupoid in the category of diagrams of the form

\[
\begin{array}{cccc}
0 & \rightarrow & A_{0,1} & \rightarrow & A_{0,2} & \rightarrow & \cdots & \rightarrow & A_{0,n-1} & \rightarrow & A_{0,n} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
0 & \rightarrow & A_{1,2} & \rightarrow & \cdots & \rightarrow & A_{1,n-1} & \rightarrow & A_{1,n} \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
0 & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & A_{n-2,n-1} & \rightarrow & A_{n-2,n} \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A_{n-1,n} \\
\downarrow & & \downarrow \\
0 \\
\end{array}
\]

in $\mathcal{C}$ where $0$ is a fixed zero object in $\mathcal{C}$ and all squares are required to be biCartesian. The crucial observation is that the various groupoids $S_\bullet$ are related to one another: for every $0 \leq k \leq n$, we have a functor

\[ \partial_k : S_n \rightarrow S_{n-1} \]

obtained by omitting in the diagram the objects in the $k$th row and $k$th column and forming the composite of the remaining morphisms. Similarly, for every $0 \leq k \leq n$, we have functors

\[ \sigma_k : S_n \rightarrow S_{n+1} \]

given by replacing the $k$th row by two rows connected via identity maps and replacing the $k$th column by two columns connected via identity maps.

Recall the definition of the simplex category $\Delta$. The objects are given by the standard ordinals $[n] = \{0, 1, \ldots, n\}, n \geq 0$, and a morphism from $[m]$ to $[n]$ is a map

\[ \{0, 1, \ldots, m\} \rightarrow \{0, 1, \ldots, n\} \]
of underlying sets which preserves $\leq$.

**Proposition 2.3.** The collection of groupoids $S_\bullet$ naturally forms a simplicial groupoid, i.e., a functor

$$S_\bullet : \Delta^{op} \to \text{Grpd}$$

with values in the category of groupoids.

**Remark 2.4.** For convenience reasons, we sometimes implicitly replace the simplex category by the larger category of all finite nonempty linearly ordered sets. This is harmless since every such a set is isomorphic to some standard ordinal $[n]$ via a unique isomorphism. For example, we write $S_{\{0,1,2\}} \to S_{\{0,2\}}$ to refer to the face map $\partial_1 : S_2 \to S_1$ leaving the natural inclusion $\{0,2\} \to \{0,1,2\}$ and the identification with $[1] \to [2]$ implicit.

Given a diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{C} & \xrightarrow{G} & \mathcal{B} \\
\end{array}
$$

of groupoids, we introduce the 2-pullback to be the groupoid

$$
\mathcal{A} \times^{(2)} \mathcal{B}
$$

with objects given by triples $(a, b, \varphi)$ where $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\varphi : F(a) \cong G(b)$. A morphism $(a, b, \varphi) \to (a', b', \varphi')$ is given by a pair of morphisms $f : a \to a'$, $g : b \to b'$ such that the diagram

$$
\begin{array}{ccc}
F(a) & \xrightarrow{\varphi} & G(b) \\
\downarrow{F(f)} & & \downarrow{G(g)} \\
F(a') & \xrightarrow{\varphi'} & G(b')
\end{array}
$$

commutes. We call a diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\
\downarrow{G'} & \nearrow{\eta} & \downarrow{G} \\
\mathcal{A} & \xrightarrow{F} & \mathcal{C}
\end{array}
$$

where $\eta : F \circ G' \Rightarrow G \circ F'$ is a natural isomorphism, a 2-pullback diagram if the functor

$$
\mathcal{X} \to \mathcal{A} \times^{(2)} \mathcal{B}, x \mapsto (G'(x), F'(x), \eta(x))
$$

is an equivalence. The property which makes 2-pullbacks better-behaved than ordinary ones is that an equivalence of diagrams of groupoids induces an equivalence of their 2-pullbacks.

It turns out that in some cases, ordinary pullback squares of categories are actually 2-pullback squares: A functor $F : \mathcal{A} \to \mathcal{B}$ is called isofibration if, for every object $a \in \mathcal{A}$ and every isomorphism $\varphi : F(a) \cong b$ in $\mathcal{B}$, there exists an isomorphism $\tilde{\varphi} : a \to a'$ in $\mathcal{A}$ such that $F(\tilde{\varphi}) = \varphi$.

**Proposition 2.5.** Let

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{F} & \mathcal{C}
\end{array}
$$

be a commutative square of categories and assume that $F$ is an isofibration. Then the square is a 2-pullback square.
We come to a key property of the simplicial groupoid \( S \): it satisfies the 2-Segal conditions which we now introduce. Let \( X \) be a simplicial object in \( \text{Grpd} \).

1. Consider a planar \( n+1 \)-gon \( P \) with vertices labelled cyclically by the set \( \{0, 1, \ldots, n\} \). Let \( i < j \) be the vertices of a diagonal of \( P \) which subdivides the polygon into two polygons with labels \( \{0, 1, \ldots, i, j, j+1, \ldots, n\} \) and \( \{i, i+1, \ldots, j\} \). We obtain a corresponding commutative square of groupoids

\[
\begin{array}{ccc}
X_{\{0,1,\ldots,n\}} & \longrightarrow & X_{\{0,1,\ldots,i,j,\ldots,n\}} \\
\downarrow & & \downarrow \\
X_{\{i,i+1,\ldots,j\}} & \longrightarrow & X_{\{i,j\}}.
\end{array}
\] (2.6)

2. For every \( 0 \leq i < n \), there is commutative square

\[
\begin{array}{ccc}
X_{\{0,1,\ldots,n-1\}} & \longrightarrow & X_{\{i\}} \\
\sigma_i & & \sigma_i \\
X_{\{0,1,\ldots,n\}} & \longrightarrow & X_{\{i,i+1\}}
\end{array}
\] (2.7)

where \( \sigma_i \) denotes the \( i \)th degeneracy map.

The following definition was introduced in [DK12] (an equivalent condition was independently discovered and studied in [GCKT14]).

**Definition 2.8.** A simplicial object \( X \) in \( \text{Grpd} \) is called 2-Segal if,

1. for every polygonal subdivision, the corresponding square (2.6) is a 2-pullback square,
2. all squares (2.7) are 2-pullback squares.

**Remark 2.9.** To explain the terminology, recall that a simplicial set \( K \) is called Segal if, for every \( 0 < k < n \), the square

\[
\begin{array}{ccc}
K_{\{0,1,\ldots,n\}} & \longrightarrow & K_{\{0,1,\ldots,k\}} \\
\downarrow & & \downarrow \\
K_{\{k,k+1,\ldots,n\}} & \longrightarrow & K_{\{k\}}
\end{array}
\]

is a pullback square. We can interpret the datum \( 0 < k < n \) as a subdivision of the interval \( [0,n] \) into intervals \( [0,k] \) and \( [k,n] \) so that the 2-Segal condition is a 2-dimensional analog of the Segal condition.

**Theorem 2.10.** The simplicial groupoid \( S \) of flags in a proto-abelian category \( \mathcal{C} \) is 2-Segal.

**Proof.** We note that all maps in the diagram (2.0) are isofibrations so that it suffices to check that the squares are ordinary pullback squares. The functor

\[
S_{\{0,1,\ldots,n\}} \longrightarrow S_{\{0,1,\ldots,i,j,\ldots,n\}} \times S_{\{i,j\}} S_{\{i,i+1,\ldots,j\}}
\]

is a forgetful functor which forgets those objects in the diagram (2.2) whose indices \((x,y)\) correspond to diagonals of \( P_n \) which cross the diagonal \((i,j)\). But these objects can be filled back in by forming pullbacks or pushouts, using the axioms of a proto-abelian category. We leave the verification of (2.7) to the reader. \( \square \)

Below, we will interpret the lowest 2-Segal conditions (2.6) and (2.7) as associativity and unitality, respectively, of the abstract Hall algebra of a proto-abelian category \( \mathcal{C} \).
2.2 Spans of groupoids and the abstract Hall algebra

We introduce the category Span(Grpd) of spans in groupoids. The objects are given by groupoids. The set of morphisms between groupoids $A, B$, is defined to be

$$\text{Hom}(A, B) = \left\{ \begin{array}{c} \chi \\ A \rightarrow B \end{array} \right\} / \sim$$

where two spans $A \leftarrow \chi \rightarrow B$ and $A \leftarrow \chi' \rightarrow B$ are considered equivalent if there exists a diagram

$$\begin{array}{ccc} & \chi & \\
A & \leftarrow & \chi' \\
& F & \\
\downarrow & & \downarrow \\
A & \leftarrow & B \end{array}$$

where $F$ is an equivalence. The composition of morphisms is given by forming 2-pullbacks: Given morphisms $f : A \to B$ and $g : B \to C$ in Span(Grpd), we represent them by spans $A \leftarrow \chi \rightarrow B$ and $B \leftarrow \gamma \rightarrow C$ and form the diagram

$$\begin{array}{ccc} & \chi & \\
X & \leftarrow & \gamma \\
& F & \\
\downarrow & & \downarrow \\
X & \leftarrow & \chi' \end{array}$$

where the square is a 2-pullback square. We then define the composite $g \circ f$ to be the morphism from $A$ to $C$ represented by the span

$$\begin{array}{ccc} & \chi & \\
A & \leftarrow & \chi' \end{array}$$

It follows from the invariance properties of 2-pullbacks that this operation is well-defined.

The category Span(Grpd) has a natural monoidal structure: The tensor product is defined on objects via $A \otimes B = A \times B$ and on morphisms via

$$\begin{array}{ccc} & \chi & \\
A & \otimes & B \\
& \otimes & \\
A' & \leftarrow & \chi' \\
& \otimes & \\
B' & \leftarrow & B \times B' \end{array}$$

**Theorem 2.11.** Let $\mathcal{C}$ be a proto-abelian category and let $S_\bullet$ be the corresponding groupoids of flags in $\mathcal{C}$. The morphisms in Span(Grpd) represented by the spans

$$\begin{array}{ccc} & \chi & \\
S_{\{0,1\}} & \leftarrow & S_{\{1,2\}} \\
& \mu & \\
\otimes & & \otimes \\
S_{\{0,1\}} \times S_{\{1,2\}} & \leftarrow & S_{\{0,2\}} \end{array}$$
and

\[
\begin{array}{c}
e: \\
\begin{array}{c}
\text{id} \\
S_0
\end{array} \\
\begin{array}{c}
\text{id} \\
\rightarrow \rightarrow
\end{array} \\
\begin{array}{c}
S_0 \\
\rightarrow \\
S_1
\end{array}
\end{array}
\]

make \( S_1 \) an algebra object in \( \text{Span}(\text{Grpd}) \). We call \((S_1, \mu, e)\) the abstract Hall algebra of \( C \).

**Proof.** We have to verify \( \mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu) \). To compute the left-hand side, we consider the commutative diagram

\[
\begin{array}{c}
S_{\{0,1,2,3\}} \\
\downarrow \\
S_{\{0,1,2\}} \times S_{\{2,3\}} \\
\downarrow \\
S_{\{0,1\}} \times S_{\{1,2\}} \times S_{\{2,3\}}
\end{array}
\]

\[
\begin{array}{c}
\times \text{id} \\
\times \text{id}
\end{array}
\]

We claim that the left-hand square is a pullback square: There is a sequence of natural functors

\[
S_{\{0,1,2,3\}} \rightarrow (S_{\{0,1,2\}} \times S_{\{2,3\}}) \xrightarrow{(2)} S_{\{0,2,3\}} \rightarrow S_{\{0,1,2\}} \times S_{\{2,3\}}
\]

where the composite is an equivalence by the 2-Segal condition (2.6) corresponding to the triangulation \( \{0, 1, 2\}, \{0, 2, 3\} \) of the square, and the second functor is an equivalence by direct verification. It follows that the first functor is an equivalence which shows the claim. Therefore, we have

\[
\mu \circ (\mu \otimes \text{id}) = S_{\{0,1,2,3\}} \\
\downarrow \\
S_{\{0,3\}} \times S_{\{0,1\}} \times S_{\{1,2\}} \\
\downarrow \\
S_{\{2,3\}}
\]

Similarly, using the 2-Segal condition for the second triangulation of the square, we obtain that

\[
\mu \circ (\text{id} \otimes \mu) = S_3 \\
\downarrow \\
S_1 \times S_1 \times S_1 \\
\downarrow \\
S_1
\]

so that we deduce associativity. Unitality follows similarly from the lowest two conditions (2.7).

### 2.3 Groupoid functions

We explain how, assuming \( C \) to be finitary, we can obtain the usual Hall algebra from the abstract one by passing to groupoid functions.

Let \( K, S \) be finite sets. We have

\[
|K \amalg S| = |K| + |S|
\]

and

\[
|K \times S| = |K||S|
\]

so that the categorical operations \( \amalg \) and \( \times \) yield upon application of \(|-|\) the numerical operations of addition and multiplication. One may wonder if there is a categorical analog of division (or subtraction).
Example 2.12. Consider the set $K = \{1, 2, 3, 4\}$ equipped with the action of the cyclic group $C_2 = \langle \tau \rangle$ of order 2 where $\tau$ acts via the permutation $(14)(23)$. Then we can form the orbit set $K/C_2$ and have
\[
|K/C_2| = 2 = |K|/|C_2|
\]
so that categorical construction of forming the quotient yields division by 2 upon application of $|\cdot|$. However, this interpretation fails as soon as the group action has nontrivial stabilizers: Letting the group $C_2$ act on the set $S = \{1, 2, 3\}$ via the permutation $(13)$, we obtain a quotient $S/C_2$ of cardinality $2 \neq \frac{3}{2}$.

We will now define a notion of cardinality for groupoids which solves the issue of the example. From now on we use the notation $\pi_0(A)$ for the set of isomorphism classes of objects in $A$. A groupoid $A$ is called finite if

1. the set $\pi_0(A)$ of isomorphism classes of objects is finite,
2. for every object $a \in A$, the set of automorphisms of $a$ is finite.

Given a finite groupoid $A$, we introduce the groupoid cardinality
\[
|A| = \sum_{\left[a\right] \in \pi_0(A)} \frac{1}{|\text{Aut}(a)|}.
\]

Remark 2.13. It is immediate from the definition, that groupoid cardinality is invariant under equivalences of finite groupoids.

Example 2.14. Any finite set $K$ can be interpreted as a discrete groupoid with $K$ as its set of objects and morphisms given by identity morphisms only. The groupoid cardinality of the discrete groupoid associated with $K$ agrees with the cardinality of the set $K$.

Let $K$ be a finite set equipped with a right action of a finite group $G$. We define the action groupoid $K//G$ to have $K$ as its set of objects and morphisms between two elements $k$ and $k'$ given by elements $g \in G$ such that $k.g = k'$.

Proposition 2.15. We have
\[
|K//G| = |K|/|G|.
\]

Proof. The set of isomorphism classes $\pi_0(K//G)$ can be identified with the set of orbits of the action of $G$ on $K$. The automorphism group of an object $k$ of $K//G$ coincides with the stabilizer group $G_k = \{g \in G | k.g = k\} \subset G$. The orbit of an element $k$ under $G$ can be identified with the quotient set $G/G_k$ of cardinality $|G|/|G_k|$. We compute
\[
|K//G| = \sum_{[k] \in \pi_0(K//G)} \frac{1}{|G_k|} = \frac{1}{|G|} \sum_{[k] \in \pi_0(K//G)} \frac{|G|}{|G_k|} = \frac{|K|}{|G|},
\]
where the last equality follows since the disjoint union of the orbits yields the set $K$.  

25
Given a set $K$ and a function $\varphi : K \to \mathbb{Q}$ with finite support, we can introduce the integral

$$\int_K \varphi = \sum_{k \in K} \varphi(k).$$

If $K$ is finite, then we have $\int_K 1 = |K|$ where $1$ denotes the constant function on $K$ with value 1. We give a generalization to groupoids.

Given a groupoid $A$, we define $\mathcal{F}(A)$ to be the $\mathbb{Q}$-vector space of functions $\varphi : \text{ob} A \to \mathbb{Q}$ which are

1. constant on isomorphism classes,
2. nonzero on only finitely many isomorphism classes.

We call $A$ locally finite if every connected component $A(a)$ is finite. Given a locally finite groupoid $A$ and $\varphi \in \mathcal{F}(A)$, we define the groupoid integral

$$\int_A \varphi = \sum_{[a] \in \pi_0(A)} \frac{\varphi(a)}{|\text{Aut}(a)|}.$$  

Note that, if $A$ is finite, then we have $\int_A 1 = |A|$.

We further introduce a relative version of the groupoid integral given by integration along the fibers: A functor $F : A \to \mathcal{B}$ of groupoids is called

1. finite if every 2-fiber of $F$ is finite,
2. locally finite if, for every $a \in A$, the restriction of $F$ to $A(a)$ is finite,
3. $\pi_0$-finite if the induced map of sets $\pi_0(A) \to \pi_0(\mathcal{B})$ has finite fibers.

Given a locally finite functor $F : A \to \mathcal{B}$ and a function $\varphi \in \mathcal{F}(A)$, we define the pushforward $F_! \varphi \in \mathcal{F}(\mathcal{B})$ via

$$F_! \varphi(b) := \int_{A_b} \varphi|_{A_b}$$

where $A_b$ is the 2-fiber of $F$ at $b$ and $\varphi|_{A_b}$ denotes the pullback of $\varphi$ along the natural functor $A_b \to A$. We obtain a $\mathbb{Q}$-linear map

$$F_! : \mathcal{F}(A) \to \mathcal{F}(\mathcal{B}).$$

**Example 2.16.** For a locally finite groupoid $A$ the constant functor $F : A \to \{ * \}$ is locally finite and we have $F_! \varphi(*) = \int_A \varphi$.

Given a $\pi_0$-finite functor $F : A \to \mathcal{B}$ and a function $\varphi \in \mathcal{F}(\mathcal{B})$, we define the pullback $F^* \varphi \in \mathcal{F}(A)$ via

$$F^* \varphi(a) := \varphi(F(a)).$$

We obtain a $\mathbb{Q}$-linear map

$$F^* : \mathcal{F}(\mathcal{B}) \to \mathcal{F}(A).$$

The central properties of the pullback and pushforward operations are captured in the following Proposition.

**Proposition 2.17.**  (1) **Functoriality.**
(a) Let $F : A \to B$ and $G : B \to C$ be $\pi_0$-finite functors of groupoids. Then the composite $G \circ F$ is $\pi_0$-finite and we have

$$(G \circ F)^* = F^* \circ G^*.$$ 

(b) Let $F : A \to B$ and $G : B \to C$ be locally finite functors of groupoids. Then the composite $G \circ F$ is locally finite and we have

$$(G \circ F)! = G! \circ F!.$$ 

(2) **Base change.** Let

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\
G' & \downarrow & \mathcal{G} \\
A & \xrightarrow{F} & \mathcal{C}
\end{array}$$

be a 2-pullback square with $F$ locally finite and $G$ $\pi_0$-finite. Then $F'$ is locally finite, $G'$ is $\pi_0$-finite, and we have

$$(F')! \circ (G')^* = G^* \circ F!.$$ 

We define a subcategory of $\text{Span}^f(\text{Grpd}) \subset \text{Span}(\text{Grpd})$ with the same objects but morphisms given by spans

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{L} & \mathcal{A} \\
& \searrow R & \mathcal{B}
\end{array}$$

such that $R$ locally finite and $L$ $\pi_0$-finite. The composition of such spans is well-defined by Proposition 2.17.

**Proposition 2.19.** Let $\mathcal{C}$ be a finitary proto-abelian category.

(1) The abstract Hall algebra $(S_1, \mu, e)$ defines an algebra object in the monoidal category $\text{Span}^f(\text{Grpd}) \subset \text{Span}(\text{Grpd})$.

(2) The association

$$\begin{array}{ccc}
\mathcal{A} & \mapsto & \mathcal{F}(\mathcal{A}) \\
\mathcal{X} & \xrightarrow{L} & \mathcal{A} \\
& \searrow R & \mathcal{B} \\
\mathcal{X} & \xrightarrow{R \circ L^*} & \mathcal{F}(\mathcal{B})
\end{array}$$

defines a monoidal functor

$$\mathcal{F} : \text{Span}^f(\text{Grpd}) \to \text{Vect}_\mathbb{Q}.$$ 

The resulting algebra object $\mathcal{F}(S_1, \mu, e)$ in $\text{Vect}_\mathbb{Q}$ is isomorphic to the (opposite) Hall algebra of $\mathcal{C}$. 

27
2.4 Green’s theorem

Green’s theorem (cf. Gre95) states that, under certain assumptions on an abelian category \( \mathcal{C} \), we can introduce a coproduct on the Hall algebra Hall(\( \mathcal{C} \)) making it a bialgebra up to a certain twist. In this section, we will use the abstract Hall algebra introduced in Section 2.2 to provide a proof of this statement.

Let \( \mathcal{C} \) be a proto-abelian category and \( S_* \) the corresponding groupoids of flags. Instead of considering the span

\[
\mu : \quad G \quad \rightarrow \quad S_2 \quad F \\
S_1 \times S_1 \quad \downarrow \quad S_1
\]

which represents the multiplication on the abstract Hall algebra, we may form the reverse span

\[
\Delta : \quad F \quad \rightarrow \quad S_2 \quad G \\
S_1 \quad \downarrow \quad S_1 \times S_1
\]

Also taking into account the reverse \( c \) of the unit morphism \( e \), it is immediate that \( (S_1, \Delta, c) \) forms a coalgebra object in Span(Grpd). The proof consists of reading all span diagrams involved in the proof of Theorem 2.11 in reverse direction.

Given an associative \( k \)-algebra \( A \), equipped with a coproduct \( \Delta : A \rightarrow A \otimes A \), we may ask if multiplication and comultiplication are compatible in the sense

\[
\Delta(ab) = \Delta(a)\Delta(b). \tag{2.22}
\]

In other words, introducing on \( A \otimes A \) the algebra structure \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \), we ask if the coproduct \( \Delta \) is a homomorphism of algebras.

**Example 2.23.** Let \( G \) be a group and let

\[
k[G] = \bigoplus_{g \in G} kg
\]
denote the group algebra over the field \( k \). The \( k \)-linear extension of the formula \( \Delta(g) = g \otimes g \) defines a coproduct

\[
\Delta : k[G] \rightarrow k[G] \otimes k[G]
\]
on \( k[G] \). It is immediate to verify (2.22).

We address the analogous compatibility question for the abstract Hall algebra.

2.4.1 Squares, frames, and crosses

To analyze whether or not the equation (2.22) holds for the abstract Hall algebra, we explicitly compute both sides using (2.20) and (2.21). The left-hand side is given by the composite

\[
+ \quad \rightarrow \quad S_2 \quad \rightarrow \quad S_1 \times S_1 \\
\downarrow \quad \downarrow \quad F \\
S_1 \times S_1
\]
which yields

\[
\begin{array}{c}
\text{+} \\
L & \text{+} & R \\
S_1 \times S_1 & \text{+} & S_1 \times S_1
\end{array}
\]

where + denotes the groupoid of exact crosses in \( \mathcal{C} \): diagrams

\[
\begin{array}{ccc}
B & \downarrow & \\
A' & \to & B' \\
& \downarrow & \\
B'' & \to & C'
\end{array}
\]

consisting of two exact sequences \( \mathcal{C} \) with common middle term. The functor \( L \) associates to such a cross the pair of objects \((B, B'')\), the functor \( R \) assigns the pair \((A', C')\). Note that we can compute the 2-pullback + as an ordinary pullback since the functor \( F \) is an isofibration. The right-hand side of (2.22) is given by

\[
\begin{array}{ccc}
\square & \to & S_2 \times S_2 \\
& \downarrow & \downarrow \\
S_2 \times S_2 & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
S_1 \times S_1 & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& F \times F & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& P & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& F \times F & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& G \times G & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& \square & \to & S_2 \times S_2 \\
& \downarrow & \downarrow \\
& S_2 \times S_2 & \to & S_1 \times S_1
\end{array}
\]

where the functor \( P \) assigns to a pair \((A \to A' \to A'', C \to C' \to C'')\) of short exact sequences the 4-tupel \((A, C, A'', C'')\) of objects in \( S_1 \). The composite is represented by the span

\[
\begin{array}{ccc}
\square & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
S_1 \times S_1 & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& M & \to & S_1 \times S_1 \\
& \downarrow & \downarrow \\
& N & \to & S_1 \times S_1
\end{array}
\]

where \( \square \) denotes the groupoid of exact frames in \( \mathcal{C} \): diagrams

\[
\begin{array}{ccc}
& A & \to & B & \to & C \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& A' & \to & B' & \to & C' \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& A'' & \to & B'' & \to & C''
\end{array}
\]

where the two complete rows and columns are short exact sequences. The functor \( M \) assigns to such a frame the pair \((B, B'')\) while the functor \( N \) assigns the pair \((A', C')\). To compare the groupoids + and \( \square \), we introduce another groupoid: the groupoid \( \Box \) of exact 3-by-3 squares.
given by commutative diagrams in $C$ of the form

\[
\begin{array}{ccc}
A & \\ \downarrow & & \downarrow \\
B & \\ \downarrow & & \downarrow \\
C & \\
\end{array}
\begin{array}{ccc}
A' & \\ \downarrow & & \downarrow \\
B' & \\ \downarrow & & \downarrow \\
C' & \\
\end{array}
\begin{array}{ccc}
A'' & \\ \downarrow & & \downarrow \\
B'' & \\ \downarrow & & \downarrow \\
C'' & \\
\end{array}
\]

where all rows and columns are required to be short exact sequences. We obtain a commutative diagram

\[
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\]

of groupoids. In what follows, we will assume that the category $C$ is abelian.

**Lemma 2.26.** The forgetful functor $F : \boxplus \rightarrow +$ is an equivalence of groupoids.

**Proof.** Fully faithfulness is clear since the missing objects needed to complete a cross to square are characterized by universal properties. To show essential surjectivity, one shows that one may always complete a cross to a square by first forming the top-left corner as a pullback, then the bottom-right corner as a pushout. The remaining objects are obtained by forming kernels/cokernels. A few applications of the snake lemma are needed to verify that the resulting 3-by-3 square is exact. 

Using Lemma 2.26, we may replace (2.24) by the span

\[
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\]

which represents the same morphism in $\text{Span}(\text{Grpd})$. We obtain a diagram

\[
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\begin{array}{ccc}
& S_1 \times S_1 & \\
\downarrow & & \downarrow \\
\square & & \square \\
\end{array}
\]

of groupoids. If the forgetful functor $\pi : \square \rightarrow \square$ were an equivalence, this would imply the compatibility of the multiplication and comultiplication for the abstract Hall algebra. As it turns out, this is not the case – the situation is more subtle. We analyze the discrepancy of the functor $\pi$ from being an equivalence by calculating its 2-fibers. Note that, $\pi$ being an
isofibration, we may calculate the 2-fibers as ordinary fibers of \( \pi \). Suppose the frame \( f \) is given by the diagram

\[
\begin{array}{ccccccc}
A & \rightarrow & B & \rightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
A' & & C' \\
\downarrow & & \downarrow \\
A'' & \rightarrow & B'' & \rightarrow & C''.
\end{array}
\]

The fiber \( \boxplus f \) is the groupoid of diagrams of the form

\[
\begin{array}{ccccccc}
A & \rightarrow & B & \rightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
A' & \rightarrow & Y & \rightarrow & C' \\
\downarrow & & \downarrow & & \downarrow \\
A'' & \rightarrow & B'' & \rightarrow & C''.
\end{array}
\]

with morphisms inducing the identity on the fixed outer frame \( f \). Note that associated to the frame \( f \), there is a long exact sequence

\[
\xi_f : 0 \rightarrow A \rightarrow A' \amalg_A B \rightarrow B'' \times_{C''} C' \rightarrow C'' \rightarrow 0
\]

given as the Baer sum of the two outer long exact sequences of the frame.

**Definition 2.28.** Let

\[
\xi : 0 \rightarrow Q \rightarrow R \rightarrow S \rightarrow T \rightarrow 0
\]

be an exact sequence in an abelian category \( A \). We introduce a corresponding groupoid \( \mathcal{T}riv(\xi) \) of commutative diagrams of the form

\[
\begin{array}{ccc}
Q & \rightarrow & R \\
\downarrow & & \downarrow \\
Y & \rightarrow & S \rightarrow T
\end{array}
\]

such that the natural maps \( Y/Q \rightarrow S \) and \( Y/R \rightarrow T \) are isomorphisms. The morphisms in \( \mathcal{T}riv(\xi) \) are given by isomorphisms of diagrams which induce the identity on \( \xi \).

**Lemma 2.30.** There is an equivalence of groupoids

\[
\boxplus f \rightarrow \mathcal{T}riv(\xi_f).
\]

**Proof.** It is clear that, given an exact 3-by-3 square, we obtain a canonical diagram of the form \([2.29]\). Various applications of the snake lemma show that this association yields a well-defined functor which is an equivalence. \( \square \)

It turns out that the groupoid \( \mathcal{T}riv(\xi) \) has a beautiful interpretation in the context of Yoneda’s theory of extensions. We review some aspects of this theory.
2.4.2 Yoneda’s theory of extensions

Let $\mathcal{C}$ be an abelian category and let $A, B$ be objects in $\mathcal{C}$. An $n$-extension of $B$ by $A$ is an exact sequence

$$\xi : 0 \rightarrow A \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \ldots \rightarrow X_0 \rightarrow B \rightarrow 0$$

in $\mathcal{C}$. Given another extension

$$\xi' : 0 \rightarrow A \rightarrow X'_{n-1} \rightarrow X'_{n-2} \rightarrow \ldots \rightarrow X'_0 \rightarrow B \rightarrow 0$$

we say that $\xi$ and $\xi'$ are Yoneda equivalent if there exists a commutative diagram

with exact rows. Yoneda equivalence in fact defines an equivalence relation and we denote by $\text{Ext}^n(B, A)$ the set of equivalence classes of $n$-extensions of $B$ by $A$. The association

$$(B, A) \mapsto \text{Ext}^n(B, A)$$

is functorial in both arguments: Given a morphism $f : A \rightarrow A'$, and an extension $\xi$ of $B$ by $A$ as above, we obtain an $n$-extension

$$f_*\xi : 0 \rightarrow A' \rightarrow X'_{n-1} \Pi_A A' \rightarrow X'_{n-2} \rightarrow \ldots \rightarrow X'_0 \rightarrow B \rightarrow 0$$

of $B$ by $A'$ called the Yoneda pushout of $\xi$ along $f$. Dually, given a morphism $g : B' \rightarrow B$, we obtain an $n$-extension

$$g^*\xi : 0 \rightarrow A \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \ldots \rightarrow X_0 \times_B B' \rightarrow B' \rightarrow 0$$

of $B'$ by $A$ called the Yoneda pullback of $\xi$ along $g$. Further, the set $\text{Ext}^n(B, A)$ is equipped with an addition law: Given two extensions $\xi$ and $\xi'$ as above, we first define

$$\xi \oplus \xi' : 0 \rightarrow A \oplus A \rightarrow X_{n-1} \oplus X'_{n-1} \rightarrow \ldots \rightarrow X_0 \oplus X'_0 \rightarrow B \oplus B \rightarrow 0$$

and then the Baer sum

$$\xi + \xi' = (\Delta_B)^*((\nabla_A)_*(\xi \oplus \xi'))$$

where $\Delta_B : B \rightarrow B \oplus B$ and $\nabla_A : A \oplus A \rightarrow A$ denote diagonal and codiagonal, respectively. The Baer sum defines an abelian group structure on the set $\text{Ext}^n(B, A)$.

Example 2.32. In the case $n = 2$, we obtain

$$\xi + \xi' : 0 \rightarrow A \rightarrow X_1 \Pi_A X'_1 \rightarrow X_0 \times_B X'_0 \rightarrow B \rightarrow 0$$

which is the operation used to produce the exact sequence (2.27).
Due to the complicated nature of the equivalence relation \((2.31)\), it is hard to decide whether two given extensions \(\xi\) and \(\xi'\) are equivalent, let alone to compute the group \(\text{Ext}^n(B, A)\). The situation simplifies greatly if the abelian category \(\mathcal{A}\) has enough projectives which we assume from now on (what follows can alternatively be done via dual arguments assuming that \(\mathcal{A}\) has enough injectives).

Given an extension
\[
\xi : 0 \rightarrow A \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \ldots \rightarrow X_0 \rightarrow B \rightarrow 0
\]
we choose a projective resolution
\[
\ldots \rightarrow P_1 \rightarrow P_0 \rightarrow B
\]
of \(B\). Using the projectivity of the objects \(P_i\) we can construct a lift of the identity map \(B \rightarrow B\) to a morphism of complexes
\[
\ldots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow B
\]
so that we obtain an element \(f_n \in \text{Hom}(P_n, A)\) satisfying \(f_n \circ d = 0\). This element therefore defines a \(n\)-cocycle in the complex \(\text{Hom}(P_{\bullet}, A)\).

**Lemma 2.34.** The association \(\xi \mapsto f_n\) defines an isomorphism of abelian groups
\[
\text{Ext}^n(B, A) \leftrightarrow H^n(\text{Hom}(P_{\bullet}, A)).
\]

Since \(\text{Ext}^n(B, A)\) forms an abelian group, there exists a distinguished equivalence class of \(n\)-extensions which are trivial in the sense that they represent the neutral element. We will now provide a detailed study of trivial extensions in the cases \(n = 1\) and \(n = 2\).

Let
\[
\xi : 0 \rightarrow A \rightarrow X_0 \rightarrow B \rightarrow 0
\]
be an extension of \(B\) by \(A\). A splitting of \(\xi\) is a morphism \(s : X_0 \rightarrow A\) such that \(si = \text{id}_A\).

We denote by \(\text{Split}(\xi)\) the set of splittings of \(\xi\) which we will now analyze explicitly: Fix a projective resolution \(P_{\bullet}\) of \(B\), and a lift of \(\text{id} : B \rightarrow B\) as in \((2.33)\). In particular, we obtain a corresponding cocycle \(f_1 \in \text{Hom}(P_1, A)\). Consider the differential
\[
d : \text{Hom}(P_0, A) \rightarrow \text{Hom}(P_1, A).
\]

**Proposition 2.35.** There is a canonical bijection of sets
\[
d^{-1}(f_1) \leftrightarrow \text{Split}(\xi).
\]

In particular,

1. A splitting exists if and only if the class of \(\xi\) in \(\text{Ext}^1(B, A)\) is trivial.

2. If the class of \(\xi\) is trivial, then the set of different splittings admits a simply transitive action of the abelian group \(\text{Hom}(B, A)\).
Proof. The diagram (2.33) induces a commutative diagram

\[
\begin{array}{c}
0 & \rightarrow & P_1 / \text{im} P_2 & \rightarrow & P_0 & \rightarrow & B & \rightarrow & 0 \\
0 & \rightarrow & A & \rightarrow & X_0 & \rightarrow & B & \rightarrow & 0 \\
\end{array}
\]

with exact rows. Forming the pushout of the top-left square, we obtain a commutative diagram

\[
\begin{array}{c}
0 & \rightarrow & A & \rightarrow & A \amalg P_1 P_0 & \rightarrow & B & \rightarrow & 0 \\
0 & \rightarrow & A & \rightarrow & X_0 & \rightarrow & B & \rightarrow & 0 \\
\end{array}
\]

with exact rows so that, by the snake lemma, the morphism \(g\) is an isomorphism. Therefore, splittings of \(\xi\) are canonically identified with splittings of the short exact sequence

\[
0 \rightarrow A \rightarrow A \amalg P_1 P_0 \rightarrow B \rightarrow 0.
\]

(2.36)

We now provide the claimed bijection. Let \(\varphi \in \text{Hom}(P_0, A)\) such that \(\varphi \circ d = f_1\). Then we obtain a morphism

\(A \amalg P_1 P_0 \rightarrow A, (a, p) \mapsto a + \varphi(p)\)

which defines a splitting of (2.36). Vice versa, given a splitting \(s\), we pull back via the canonical morphism \(P_0 \rightarrow A \amalg P_1 P_0\) to obtain a morphism \(\varphi : P_0 \rightarrow A\) which, by the relations defining the pushout, satisfies \(\varphi \circ d = f_1\). It is immediate to verify that these two assignments define inverse maps.

We will now describe an analogous point of view on trivial 2-extensions. Let \(\xi : 0 \rightarrow A \rightarrow X_1 \rightarrow X_0 \rightarrow B \rightarrow 0\) be a 2-extension of \(B\) by \(A\). A trivialization of \(\xi\) is a commutative diagram

\[
\begin{array}{c}
A & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & B \\
\downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
Y & & & & & & &
\end{array}
\]

(2.37)

such that the natural maps \(Y/A \rightarrow S\) and \(Y/X_1 \rightarrow B\) are isomorphisms. Note that, in contrast to the case \(n = 1\), where the collection of splittings forms a set, the collection of trivializations naturally organizes into a groupoid: the groupoid \(\text{Triv}(\xi)\) introduced in Definition 2.28. We will now argue that the groupoid \(\text{Triv}(\xi)\) is the \(n = 2\) analog of the set \(\text{Split}(\xi)\): Fix a projective resolution \(P_\bullet\) of \(B\), and a lift of \(\text{id} : B \rightarrow B\) as in (2.33). We obtain a corresponding cocycle \(f_2 \in \text{Hom}(P_2, A)\) and consider the complex

\[
\text{Hom}(P_0, A) \xrightarrow{d} \text{Hom}(P_1, A) \xrightarrow{d} \text{Hom}(P_2, A).
\]

Proposition 2.38. There is a canonical equivalence of groupoids

\[
T : d^{-1}(f_2)//\text{Hom}(P_0, A) \xrightarrow{\sim} \text{Triv}(\xi)
\]

where the left-hand side denotes the action groupoid corresponding to the action of the abelian group \(\text{Hom}(P_0, A)\) on \(d^{-1}(f_2)\) via \((t, g) \mapsto t + g \circ d\). In particular,
A trivialization exists, i.e., $\text{Triv}(\xi) \neq \emptyset$, if and only if the class of $\xi$ in $\text{Ext}^2(B, A)$ is trivial.

Assume that the class of $\xi$ is trivial. Then

(i) The set of isomorphism classes $\pi_0(\text{Triv}(\xi))$ is acted upon simply transitively by the group $\text{Ext}^1(B, A)$.

(ii) The automorphism group of any object of $\text{Triv}(\xi)$ is isomorphic to the abelian group $\text{Hom}(B, A)$.

**Proof.** Let $t$ be an object of the action groupoid, i.e., an element $t \in \text{Hom}(P_1, A)$ such that $t \circ d = f_2$. From the chosen diagram

\[
\begin{array}{ccccccccc}
P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & B
\end{array}
\]

we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & P_1/\text{im } P_2 & \longrightarrow & P_0 & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}
\]

with exact rows. We form the pushout

$Y_t := X_1 \amalg_{P_1/\text{im } P_2} P_0 \cong X_1 \amalg_{P_1} P_0$

to obtain a commutative diagram

\[
\begin{array}{cccccc}
& & & & Y_t & & \\
& & & \downarrow & & \downarrow & & \\
A & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & B.
\end{array}
\]

The sequence

\[
0 \longrightarrow X_1 \longrightarrow Y_t \longrightarrow B \longrightarrow 0
\]

is a Yoneda pushout of the top exact sequence in (2.39) and therefore exact. We further obtain from (2.40) a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & X_1/A & \longrightarrow & Y_t/A & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1/A & \longrightarrow & X_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}
\]

where the top row is exact by the third isomorphism theorem, and the bottom row is trivially exact. The snake lemma implies that $Y_t/A \rightarrow X_0$ is an isomorphism so that the diagram (2.40) defines an object of $\text{Triv}(\xi)$. This defines the functor $T$ on objects. Given a morphism between objects $t$ and $t'$, i.e., an element $g \in \text{Hom}(P_0, A)$ such that $t' = t + g \circ d$, we obtain an induced morphism $Y_t \rightarrow Y_t'$ via the formula

\[
X_1 \amalg_{P_1} P_0 \rightarrow X_1 \amalg_{P_1} P_0, \ (x, p) \mapsto (x + i(g(p)), p)
\]

This association defines $T$ on morphisms. Fully faithfulness and essential surjectivity are straightforward to verify. \qed
### 2.4.3 Proof of Green’s theorem

In the previous sections we have seen that the compatibility
\[ \Delta(ab) = \Delta(a)\Delta(b) \quad (2.41) \]
of multiplication and comultiplication fails for the abstract Hall algebra since the forgetful functor
\[ \pi : \boxplus \to \square \]
from exact 3-by-3 squares to exact frames is not an equivalence. However, the language of groupoids gives us a precise measure for the failure of (2.41): the 2-fibers of the functor \( \pi \).

As we have seen in Lemma 2.30, the 2-fiber \( \boxplus_f \) over a fixed frame \( f \) is given by the groupoid \( \text{Triv}(\xi_f) \) of trivializations of the 2-extension \( \xi_f \) obtained as the Baer sum of the two 2-extensions of \( C'' \) by \( A \) which form the frame \( f \).

We will now make sufficient assumptions on the category \( C \) so that we can work around the fact that \( \pi \) is not an equivalence after passing from groupoids to functions by means of the monoidal functor
\[ \mathcal{F} : \text{Span}^f(\text{Grpd}) \to \text{Vect}_Q. \]
Namely, we will assume that the abelian category \( C \) is

1. **finitary** in the sense of Definition 1.5,
2. **cofinitary**: every object of \( C \) has only finitely many subobjects,
3. **hereditary**: for every pair of objects \( A, B \) of \( C \), we have \( \text{Ext}^i(A, B) \cong 0 \) for \( i > 1 \).

We have seen that the condition on \( C \) to be finitary implies that the abstract Hall algebra defines an algebra object in \( \text{Span}^f(\text{Grpd}) \subset \text{Span}^f(\text{Grpd}) \).

**Proposition 2.42.** Let \( C \) be a finitary abelian category.

1. The condition on \( C \) to be cofinitary implies that the object \( (S_1, \Delta, c) \) defines a coalgebra object in \( \text{Span}^f(\text{Grpd}) \).
2. The condition on \( C \) to be hereditary implies that all 2-fibers of \( \pi \) are nonempty.

Consider the commutative diagram

![Commutative Diagram](image)

The failure of the equality (2.41) after passing to functions is given by
\[ (R')^!(L')^* \neq (R)^!(L)^*. \]
We compute the left-hand side explicitly: letting
\[ \varphi = \|_{(A \to B \to C, A'' \to B'' \to C'')} \in F(S_2 \times S_2), \]
we have
\[ (R')_!(L')^*(\varphi) = R_! \pi_! \pi'^* L^*(\varphi) = \frac{| \text{Ext}^1(C'', A)|}{| \text{Hom}(C'', A)|} R_! L^*(\varphi). \]  
\[ (2.43) \]
The last equality follows from Lemma 2.44 below since, using Proposition 2.38 together with the assumption on \( C \) to be hereditary, we have
\[ | \boxplus \delta f | = | \text{Triv}(\delta f) | = \frac{| \text{Ext}^1(C'', A)|}{| \text{Hom}(C'', A)|}. \]

**Lemma 2.44.** Let \( F : A \to B \) be a functor which is both \( \pi_0 \)-finite and locally finite. Let \( b \) be an object of \( B \). Then, for every \( \varphi \in F(B) \), we have
\[ (F! F^* \varphi)(b) = |A_b| \varphi(b) \]
so that the effect of \( F! F^* \) on the function \( \varphi \) is given by rescaling with the groupoid cardinalities of the 2-fibers of \( F \).

**Proof.** This follows immediately from the definitions. \( \square \)

To compensate for the rescaling factor
\[ \frac{| \text{Ext}^1(C'', A)|}{| \text{Hom}(C'', A)|} \]
appearing in (2.43), we use the following modifications:

1. Instead of the comultiplication \( \Delta \), we use the comultiplication represented by the span
\[ \Delta' : \begin{array}{ccc}
F & \to & G' \\
S_2 & \leftarrow & S_1 \times S_1
\end{array} \]
where \( G' \) assigns to a short exact sequence \( A \to B \to C \) the pair of objects \( (C, A) \) (instead of \( (A, C) \)).

2. Letting \( H = F(S_1, \mu, e) \) we define on \( H \otimes H \) the twisted algebra structure
\[ \mu^t : (H \otimes H) \otimes (H \otimes H) \to H \otimes H \]
given by setting
\[ (1_A \otimes 1_B)(1_A' \otimes 1_B') := \frac{| \text{Ext}^1(A', B)|}{| \text{Hom}(A', B)|} (1_A 1_{A'}) \otimes (1_B 1_{B'}). \]

We arrive at the main result of this section.

**Theorem 2.45** (Green). Let \( C \) be a finitary, cofinitary, and hereditary abelian category and consider the datum \( H = F(S_1, \mu, e, \Delta', c) \). Then the diagram
\[ \begin{array}{ccc}
H \otimes H & \xrightarrow{\mu} & H \\
\downarrow{\Delta' \otimes \Delta'} & & \downarrow{\Delta'} \\
(H \otimes H) \otimes (H \otimes H) & \xrightarrow{\mu^t} & H \otimes H
\end{array} \]
commutes.
2.4.4 Example

Note, that we have investigated the compatibility of product and coproduct without determining an explicit formula for the coproduct. We now provide a formula and analyze the compatibility of multiplication and comultiplication of product and coproduct for the category $\text{Vect}_{\mathbb{F}_q}$ of finite dimensional $\mathbb{F}_q$-vector spaces.

Let $\mathcal{C}$ be a finitary and cofinitary abelian category. For objects $A, A'$ in $\mathcal{C}$, we introduce the groupoid $\mathcal{E}xt(A', A)$ with objects given by short exact sequences

$$0 \to A \to X \to A' \to 0$$

in $\mathcal{C}$ and morphisms given by diagrams

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \cong \\
A & \longrightarrow & X'
\end{array}
\quad \text{id} \\
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
A & \longrightarrow & X'
\end{array}
$$

We denote by $\mathcal{E}xt(A', A)^B$ the full subgroupoid of $\mathcal{E}xt(A', A)$ consisting of those short exact sequences where $X \cong B$. With this notation, we have

$$\Delta'(1_B) = (G^*)F^*(1_B) = \sum_{[A'], [A]} |\mathcal{E}xt(A', A)^B|1_{[A']} \otimes 1_{[A]}.$$ 

**Example 2.46.** For the category of finite dimensional $\mathbb{F}_q$-vector spaces we have

$$\mathcal{F}(S_1) \cong \bigoplus_{n \geq 0} \mathbb{Q}1_n$$

where we set $1_n := 1_{[\mathbb{F}_q^n]}$. Further, we have seen

$$1_n1_m = \left[ \begin{array}{cc} n + m \\ m \end{array} \right]_q 1_{m+n}.$$ 

We compute

$$\Delta'(1_n) = \sum_{k+l=n} |\mathcal{E}xt(\mathbb{F}_q^k, \mathbb{F}_q^l)\mathbb{F}_q^n|1_k \otimes 1_l$$

$$= \sum_{k+l=n} q^{-kl}1_k \otimes 1_l.$$ 

Thus, we have

$$\Delta'(1_m1_n) = \left[ \begin{array}{cc} n + m \\ n \end{array} \right] \sum_{x+y=n+m} q^{-xy}1_x \otimes 1_y$$

and

$$\Delta'(1_m)\Delta'(1_n) = \left( \sum_{k+l=m} q^{-kl}1_k \otimes 1_l \right) \left( \sum_{r+s=n} q^{-rs}1_r \otimes 1_s \right)$$

$$= \sum_{n=r+s, m=k+l} q^{-kl-rs-l}1_{k+r} \otimes 1_{l+s}$$

$$= \sum_{n+m=x+y} \sum_{k \leq n} q^{k(n-k)} \left[ \begin{array}{cc} x \\ k \end{array} \right] q^{n-k} \left[ \begin{array}{cc} y \\ n-k \end{array} \right].$$
The compatibility of product and coproduct up to twist therefore amounts to the formula

\[
\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \sum_{k \leq n} q^{k(n-k)} \begin{bmatrix} x \\ k \end{bmatrix}_q \begin{bmatrix} y \\ n-k \end{bmatrix}_q
\]

where \( x + y = n + m \).
2.5 Hall monoidal categories

The construction of the abstract Hall algebra in Section 2.2 only makes use of the groupoids $S_{\leq 3}$ and the 2-Segal conditions involving them. In this section, we use functors instead of functions to define the Hall monoidal category of a finitary proto-abelian category $\mathcal{C}$. This construction utilizes the 2-Segal conditions involving $S_{\leq 4}$.

Given a groupoid $\mathcal{A}$, we denote by $\text{Fun}(\mathcal{A})$ the category of functors from $\mathcal{A}$ to the category $\text{Vect}_\mathbb{C}$ of finite dimensional complex vector spaces which are nonzero on only finitely many isomorphism classes of $\mathcal{A}$. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of groupoids. We have:

- if $F$ is $\pi_0$-finite, then we have a corresponding pullback functor
  \[ F^* : \text{Fun}(\mathcal{B}) \to \text{Fun}(\mathcal{A}), \varphi \mapsto \varphi \circ F. \]

- if $F$ is locally finite, then we have a pushforward functor
  \[ F_! : \text{Fun}(\mathcal{A}) \to \text{Fun}(\mathcal{B}) \]
  which is defined as a left Kan extension functor. By the pointwise formula for Kan extensions, we have
  \[ F_!(\varphi)(b) = \colim_{A_b} \varphi|_{A_b} \]
  where $A_b$ denotes the 2-fiber of $F$ over $b$.

These operations satisfy the following compatibility conditions (in analogy to Proposition 2.17).

**Proposition 2.47.** (1) **Functoriality.**

(a) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be $\pi_0$-finite functors of groupoids. Then we have
  \[ (G \circ F)^* = F^* \circ G^*. \]

(b) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be locally finite functors of groupoids. Then we have a canonical isomorphism
  \[ (G \circ F)_! \cong G_! \circ F_! . \]

(2) **Base change.** Let

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\
\downarrow{G'} & \nearrow \mathcal{G} & \downarrow{G} \\
\mathcal{A} & \xleftarrow{F} & \mathcal{C}
\end{array}
\]

be a 2-pullback square with $F$ locally finite and $G$ $\pi_0$-finite. Then we have a canonical isomorphism
  \[ (F')_! \circ (G')^* \cong G^* \circ F_! . \]

Let $\mathcal{C}$ be a finitary proto-abelian category. Given a span of groupoids

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{L} & \mathcal{A} \\
\downarrow{R} & \nearrow & \downarrow{F} \\
\mathcal{B} & \xleftarrow{D} & \mathcal{C}
\end{array}
\]

40
with $L_0$-finite and $R$ locally finite, we obtain a corresponding functor

$$R_! \circ L^* : \text{Fun}(\mathcal{A}) \to \text{Fun}(\mathcal{B}).$$

Applying this to the span

$$\begin{array}{ccc}
S_2 & \xrightarrow{G} & S_1 \\
\downarrow & & \downarrow \\
S_1 \times S_1 & \xrightarrow{F} & S_1
\end{array}$$

yields a functor $\text{Fun}(S_1 \times S_1) \to \text{Fun}(S_1)$ which we precompose with the pointwise tensor product $\text{Fun}(S_1) \times \text{Fun}(S_1) \to \text{Fun}(S_1 \times S_1)$ to obtain

$$\otimes : \text{Fun}(S_1) \times \text{Fun}(S_1) \to \text{Fun}(S_1).$$

Further, from the span

$$\begin{array}{ccc}
S_0 & \xrightarrow{\text{id}} & S_0 \\
\downarrow & & \downarrow \\
S_0 & \xrightarrow{\sigma_0} & S_1
\end{array}$$

we obtain a functor $\text{Fun}(S_0) \to \text{Fun}(S_1)$ which we evaluate on $k$ to obtain

$$I \in \text{Fun}(S_1).$$

**Theorem 2.49.** Let $\mathcal{C}$ be a finitary proto-abelian category. The datum $(\text{Fun}(S_1), \otimes, I)$ naturally extends to a monoidal structure on the category $\text{Fun}(S_1)$.

**Proof.** We sketch the basic idea of the proof, the main point being the derivation of MacLane’s pentagon. There are five 2-Segal conditions involving $S_4$, corresponding to the five possible subdivisions of a planar pentagon. We denote by $P_{ij}$ the subdivision $0 \leq i < j \leq 4$ of the pentagon. We further introduce the notation

$$S(P_{ij}) = S\{0,\ldots,i,j,\ldots,4\} \times_{S\{i,j\}} S\{i,\ldots,j\}$$

for the corresponding pullback so that, for every subdivision, we have, by the 2-Segal property, an equivalence

$$S_4 \to S(P_{ij}).$$

For example, we have

$$S(P_{13}) = S\{0,1,3,4\} \times_{S\{1,3\}} S\{1,2,3\} \cong S_3 \times_{S_1} S_2$$

Similarly, we label the five different triangulations $T_{ij,kl}$ of the pentagon via their internal edges $i \to j$ and $k \to l$. We use the notation $S(T_{ij,kl})$ analogous to (2.50) so that we have, for example,

$$S(T_{13,14}) = S\{0,1,4\} \times_{S\{1,4\}} S\{1,3,4\} \times_{S\{1,3\}} S\{1,2,3\} \cong S_2 \times_{S_1} S_2 \times_{S_1} S_2.$$
We obtain a commutative diagram of groupoids

\[
\begin{array}{c}
S(T_{02,24}) \\
\downarrow \\
S(P_{24}) \\
\downarrow \\
S(T_{14,24}) \\
\downarrow \\
S(P_{14}) \\
\downarrow \\
S(T_{13,14}) \\
\downarrow \\
S(P_{13}) \\
\downarrow \\
S(T_{03,13}) \\
\downarrow \\
S(P_{03}) \\
\downarrow \\
S(T_{02,03}) \\
\end{array}
\]

in which, by the various 2-Segal conditions, all functors are equivalences.

The basic idea of the argument is now as follows: Let \( \varphi, \psi, \xi \) be objects in \( \text{Fun}(S_1) \). We have a diagram of groupoids

\[
S\{0,1,2\} \times S\{0,2,3\} \longrightarrow S\{0,1,2,3\} \longrightarrow S\{0,1,3\} \times S\{1,2,3\}
\]

where, by the lowest 2-Segal conditions corresponding to the two triangulations of a square, both functors are equivalences. They are responsible for isomorphisms

\[
(\varphi \otimes \psi) \otimes \xi \overset{\alpha_1}{\longrightarrow} \varphi \otimes (\psi \otimes \xi) \overset{\alpha_2}{\longrightarrow} \varphi \otimes (\psi \otimes \xi)
\]

which we use to define the associator of the monoidal structure as \( \alpha = \alpha_2 \circ \alpha_1^{-1} \). Here, the middle term \( \varphi \otimes \psi \otimes \xi \) is defined as a pull-push along the span

\[
S\{0,1\} \times S\{1,2\} \times S\{2,3\} \overset{\longrightarrow}{\longrightarrow} S\{0,1,2,3\} \overset{\longrightarrow}{\longrightarrow} S\{0,3\}.
\]

Given four objects \( \varphi, \psi, \xi, \varepsilon \), the commutative diagram (2.51) is responsible for a commutative
Along the boundary of (2.52), we extract the commutative MacLane pentagon for the tensor product on Fun($S_1$). To make this argument formally precise is somewhat tedious: it is best done by interpreting monoidal structures in terms of Grothendieck fibrations over the category $\Delta^{op}$.

We call the monoidal category $(\text{Fun}(S_1), \otimes, I)$ the **Hall monoidal category** $\text{Hall}^{\otimes}(\mathcal{C})$ of $\mathcal{C}$.

**Remark 2.53.** The construction of the Hall monoidal category can be understood as an instance of Day convolution [Day74]: From the simplicial groupoid of flags we can construct a promonoidal structure on the groupoid $S_1$ which is then turned into a monoidal one by passing to functors.

We discuss two examples:

**2.5.1 Hall$^{\otimes}(\text{Vect}_{S_1})$**

We restrict attention to the skeleton $\mathcal{C} \subset \text{Vect}_{S_1}$ consisting of the standard pointed sets $\{*, 1, \ldots, n\}, n \geq 0$. Thus, we have

$$S_1(\mathcal{C}) \cong \coprod_{n \geq 0} BS_n$$

so that an object of $\text{Hall}^{\otimes}(\mathcal{C})$ is given by a sequence $(\rho_n)_{n \geq 0}$ of representations of $S_n$ in $\text{Vect}_{\mathbb{C}}$ where only finitely many representations are nonzero. We have

$$(\rho_n)_{n \geq 0} \cong \bigoplus_{n \geq 0} \rho_n$$

where $\rho_n$ is interpreted as an object of $\text{Hall}^{\otimes}(\mathcal{C})$ which is zero on all groupoids $BS_m, m \neq n$. The tensor product of $\text{Hall}^{\otimes}(\mathcal{C})$ is additive so that it suffices to describe $\rho_n \otimes \rho_m$. This is obtained by pull-push along the span of groupoids

$$S_1 \times S_1 \leftarrow S_2 \rightarrow S_1$$
which factors through the pull-push along the span

\[ BS_n \times BS_m \xrightarrow{\cong} B(S_n \times S_m) \longrightarrow BS_{n+m}. \]

This is obtained by restricting to the subgroupoid of \( S_2 \) spanned by a fixed chosen short exact sequence

\[ \{*, 1, \ldots, n\} \hookrightarrow \{*, 1, \ldots, n+m\} \twoheadrightarrow \{*, 1, \ldots, m\}. \tag{2.54} \]

and noting that this choice determines an embedding of the automorphism group \( S_n \times S_m \) of \( \{*, 1, \ldots, n+m\} \) into the automorphism group \( S_{n+m} \) of \( \{*, 1, \ldots, n+m\} \). The tensor product \( \rho_n \otimes \rho_m \) in \( \text{Hall}^\otimes(\mathcal{C}) \) is therefore given by the induced representation of the external tensor product \( \rho_n \boxtimes \rho_m \) along the embedding \( S_n \times S_m \subset S_{n+m} \).

The resulting monoidal category plays an important role in classical representation theory. It is canonically monoidally equivalent to the category of polynomial functors: functors \( F : \text{Vect}_\mathbb{C} \to \text{Vect}_\mathbb{C} \) satisfying the following condition

- for every collection of morphisms \( f_i : V \to W, \ 1 \leq i \leq n \), between fixed vector spaces, the expression \( F(\lambda_1 f_1 + \cdots + \lambda_r f_r), \lambda_i \in \mathbb{C} \), is a function polynomial with coefficients in \( \text{Hom}(F(V), F(W)) \).

The polynomial functor corresponding to the representation \( \rho_n \) is given by

\[ F : \text{Vect}_\mathbb{C} \to \text{Vect}_\mathbb{C}, V \mapsto (X_n \otimes V^\otimes n)^{S_n}. \]

The Hall monoidal structure induces the structure of an associative algebra on the Grothendieck group \( K_0(\text{Hall}^\otimes(\mathcal{C})) \). Using the interpretation via polynomial functors we can canonically identify \( K_0(\text{Hall}^\otimes(\mathcal{C})) \) with the algebra \( \Lambda \) of symmetric functions. Under this identification, the basis given by isomorphism classes of irreducible representations gets identified with the basis of \( \Lambda \) given by the Schur functions. Therefore, we obtain yet another Hall algebraic construction of the algebra of symmetric functions which naturally exhibits an interesting basis. For a detailed exposition of this theory (due to Schur) we refer the reader to [Mac95].

### 2.5.2 Hall\(^\otimes(\text{Vect}_F)\)

We consider the skeleton \( \mathcal{C} \subset \text{Vect}_F \) consisting of the standard objects \( F_q^n \) so that we have

\[ S_1(\mathcal{C}) \simeq \bigsqcup_{n \geq 0} B\text{GL}_n(F_q). \]

An object of \( \text{Hall}^\otimes(\mathcal{C}) \) is therefore given by a sequence \((\rho_n)_{n \geq 0}\) of representations of \( \text{GL}_n(F_q) \) in \( \text{Vect}_\mathbb{C} \) with only finitely many nonzero components. The tensor product \( \rho_n \otimes \rho_m \) is obtained by pull-push along the span of groupoids

\[ S_1 \times S_1 \leftarrow S_2 \twoheadrightarrow S_1 \]

which factors through the pull-push along the span

\[ B\text{GL}_n(F_q) \times B\text{GL}_m(F_q) \xlongleftarrow{\cong} B P_{n,m}(F_q) \longrightarrow B\text{GL}_{n+m}(F_q) \]

where \( P_{n,m} \) is the parabolic subgroup of \( \text{GL}_{n+m} \) given by the automorphism group of a fixed short exact sequence

\[ F_q^n \hookrightarrow F_q^{n+m} \twoheadrightarrow F_q^m. \]
Therefore, the tensor product $\rho_n \otimes \rho_m$ in $\text{Hall}^\otimes(\mathcal{C})$ is given by first pulling back the representation of $\text{GL}_n(\mathbb{F}_q) \times \text{GL}_m(\mathbb{F}_q)$ along $P_{n,m} \to \text{GL}_n(\mathbb{F}_q) \times \text{GL}_m(\mathbb{F}_q)$ and then forming the induced representation along $P_{n,m} \subset \text{GL}_{n+m}(\mathbb{F}_q)$.

Green [Gre55, Mac95] has developed a $q$-analog of Schur’s theory which uses the associative algebra given by the Grothendieck group of $\text{Hall}^\otimes(\text{Vect}_{\mathbb{F}_q})$ to construct all irreducible characters of the groups $\text{GL}_n(\mathbb{F}_q)$. The monoidal category $\text{Hall}^\otimes(\text{Vect}_{\mathbb{F}_q})$ itself features in the work of Joyal-Street [JS95] who explain that the commutativity of Green’s algebra comes from a (partial) braided structure.
3 Derived Hall algebras via ∞-groupoids

The idea of constructing the Hall algebra via the simplicial groupoid of flags $S_\bullet$ is a very flexible one. We explain how it can be adopted to construct Hall algebras of derived categories or, more generally, stable ∞-categories. This section is a translation of [Toè06] (also cf. [Ber13]) into the language of ∞-categories which makes the analogy to proto-abelian categories immediate. We use [Lur09] as a standard reference.

3.1 Coherent diagrams in differential graded categories

An ∞-category $C$ is a simplicial set such that, for every $0 < i < n$ and every $\Lambda_n^i \to C$, there exists a commutative diagram

$$\begin{array}{ccc}
\Lambda_n^i & \longrightarrow & C \\
\downarrow & \cong & \\
\Delta^n & \longrightarrow & C
\end{array}$$

The arrow $\Lambda_n^i \to C$ represents the boundary of an $n$-simplex with $i$th face removed, called an inner horn in $C$, and the condition asks that it can be filled to a full $n$-simplex in $C$.

Example 3.1. The nerve of a small category provides an example of an ∞-category where every inner horn has a unique filling. This corresponds to the fact that every $n$-tupel of composable morphisms has a unique composite. In a general ∞-category the composite is not required to be unique. However, the totality of all horn filling conditions encodes that it is unique up to a coherent system of homotopies.

Example 3.2. Given ∞-categories $C$, $D$, we define the simplicial set $\text{Fun}(C, D)$ of functors from $C$ to $D$ given by the internal hom in the category of simplicial sets. Then $\text{Fun}(C, D)$ is an ∞-category.

Example 3.3. A differential graded (dg) category $T$ is a category enriched over the monoidal category of complexes of abelian groups. The collection of dg categories organizes into a category $\mathbf{dgcat}$ with morphisms given by enriched functors. Following [Lur11], we associate to $T$ an ∞-category called the dg nerve of $T$.

We associate to the $n$-simplex $\Delta^n$ a dg category $\text{dg}(\Delta^n)$ with objects given by the set $\{0, 1, \ldots, n\}$. The graded Z-linear category underlying $\text{dg}(\Delta^n)$ is freely generated by the morphisms

$$f_I \in \text{dg}(\Delta^n)(i_-, i_+)^{-m}$$

where $I$ runs over the subsets $\{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subset \{0, 1, \ldots, n\}$, $m \geq 0$. On these generators, the differential is given by the formula

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_I \setminus \{i_j\} - f_{\{i_j < \cdots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \cdots < i_j\}}$$

and extended to all morphisms by the Z-linear Leibniz rule. We have $d^2 = 0$ on generators and therefore on all morphisms. The dg categories $\text{dg}(\Delta^n)$, $n \geq 0$, assemble to form a cosimplicial object in $\mathbf{dgcat}$ which allows us to define the dg nerve of $T$

$$\text{N}_\text{dg}(T) = \text{Hom}_{\mathbf{dgcat}}(\text{dg}(\Delta^\bullet), T).$$

It is shown in [Lur11 1.3.1.10] that $\text{N}_\text{dg}(T)$ is in fact an ∞-category.

It is instructive to analyze the low-dimensional simplices of the dg nerve $\text{N}_\text{dg}(T)$:
The 0-simplices are the objects of $T$.

A 1-simplex in $N_{dg}(T)$ is a morphism $f_{\{0,1\}}: a_0 \to a_1$ of degree 0 which is closed, i.e., $df = 0$.

A 2-simplex in $N_{dg}(T)$ is given by objects $a_0, a_1, a_2$, closed morphisms $f_{\{0,1\}}: a_0 \to a_1$, $f_{\{1,2\}}: a_1 \to a_2$, $f_{\{0,2\}}: a_0 \to a_2$, and a morphism $f_{\{0,1,2\}}: a_0 \to a_2$ of degree $-1$ which satisfies

$$df_{\{0,1,2\}} = f_{\{0,2\}} - f_{\{1,2\}} \circ f_{\{0,1\}}$$

so that we obtain a triangle in $T$ which commutes up to the chosen homotopy $f_{\{0,1,2\}}$. A key point here is that we do not simply require the triangle to commute up to homotopy, but the homotopy is part of the data forming the triangle.

A 3-simplex in $N_{dg}(T)$ involves the data of the four boundary 2-simplices as above and, in addition, a morphism $f_{\{0,1,2,3\}}: a_0 \to a_3$ of degree $-2$ such that

$$df_{\{0,1,2,3\}} = f_{\{0,1,3\}} - f_{\{2,3\}} \circ f_{\{0,1,2\}} - f_{\{0,2,3\}} + f_{\{1,2,3\}} \circ f_{\{0,1\}}.$$

We can interpret this data as follows: The boundary of a 3-simplex in $N_{dg}(T)$ encodes two homotopies between $f_{\{0,3\}}$ and the composite $f_{\{2,3\}} \circ f_{\{1,2\}} \circ f_{\{0,1\}}$ given by $f_{\{0,1,3\}} + f_{\{1,2,3\}} \circ f_{\{0,1\}}$ and $f_{\{0,2,3\}} + f_{\{2,3\}} \circ f_{\{0,1,2\}}$, respectively. To obtain a full 3-simplex in $N_{dg}(T)$ we have to provide the homotopy $f_{\{0,1,2,3\}}$ between these homotopies.

The passage from a dg category $T$ to the $\infty$-category $N_{dg}(T)$ allows us (and forces us) to systematically consider diagrams in $T$ which commute up to specified coherent homotopy: Let $I$ be a category and $N(I)$ its nerve. We define a coherent $I$-diagram in $T$ to be a functor, i.e., a map of simplicial sets

$$N(I) \to N_{dg}(T).$$

**Example 3.4.** Consider the category $I$ given by the universal commutative square: $I$ has four objects $1, 2, 3, 4$, morphisms $f_1 : 1 \to 2$, $f_2 : 2 \to 4$, $f_3 : 1 \to 3$, $f_4 : 3 \to 4$ subject to the relation $f_2 \circ f_1 = f_4 \circ f_3$. An $I$-coherent diagram in $T$ consists of

$$\begin{array}{ccc}
  a_1 & \stackrel{f_1}{\longrightarrow} & a_2 \\
  f_3 & \downarrow & f_2 \\
  a_3 & \stackrel{g}{\longrightarrow} & a_4 \\
  & \downarrow_{h_1} & \\
  & f_4 &
\end{array}$$

where the morphisms $f_1, f_2, f_3, f_4$ and $g$ are closed of degree 0, and we have $dh_1 = g - f_2 \circ f_1$, $dh_2 = g - f_4 \circ f_3$.

One of the main advantage of homotopy coherent diagrams over homotopy commutative ones is the existence of a good theory of limits. We give an example.

**Example 3.5.** Let $\mathcal{A}$ be an abelian category with enough projectives and consider the dg category $\text{Ch}^{-}(\mathcal{A}_{\text{proj}})$ of bounded-above cochain complexes of projective objects in $\mathcal{A}$. We define the bounded-above derived $\infty$-category of $\mathcal{A}$ as the dg nerve $\mathcal{D}^{-}(\mathcal{A}) := N_{dg}(\text{Ch}^{-}(\mathcal{A}_{\text{proj}}))$. The ordinary bounded-above derived category is obtained by passing to the homotopy category $\text{h}(\mathcal{D}^{-}(\mathcal{A}))$ which is defined as the ordinary category obtained by identifying homotopic
morphisms. Consider an edge \( f : X \to Y \) in \( \mathcal{D}^-(A) \), i.e., a morphism between bounded-above complexes of projectives \( X \) and \( Y \). Consider the cone of \( f \), i.e., the complex with

\[
\text{cone}(f)_n = X^{n+1} \oplus Y^n
\]

and differential given by

\[
d = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}.
\]

We obtain a coherent square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
0 & \xrightarrow{i} & \text{cone}(f)
\end{array}
\]

in \( \mathcal{D}^-(A) \) where

\[
i : Y \to \text{cone}(f), \, y \mapsto (0, y)
\]

and

\[
h : X \to \text{cone}(f), \, x \mapsto (-x, 0)
\]

so that we have \( dh = 0 - i \circ f \). The key fact is that the diagram (3.6) is a pushout diagram in the \( \infty \)-category \( \mathcal{D}^-(A) \) so that the cone is characterized by a universal property. This statement becomes wrong if we pass from \( \mathcal{D}^-(A) \) to the homotopy category \( h(\mathcal{D}^-(A)) \): the image of the square (3.6) commutes up to unspecified homotopy, but this data is, in general, insufficient to characterize \( \text{cone}(f) \) by a universal property. As an extreme case, consider the cone of the zero morphism \( X \to 0 \) which is the translation \( X[1] \). The coherent square

\[
\begin{array}{ccc}
X & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{i} & X[1]
\end{array}
\]

involves the map

\[
h : X \to X[1], \, x \mapsto x
\]

considered as a self-homotopy of 0. For precise definitions and proofs, we refer the reader to [Lur11, 1.3.2].

### 3.2 Stable \( \infty \)-categories

We take for granted the existence of a theory of limits for \( \infty \)-categories (cf. [Lur09]).

**Definition 3.7.** An \( \infty \)-category \( \mathcal{C} \) is called **stable** if the following conditions hold:

1. The \( \infty \)-category \( \mathcal{C} \) is pointed.
2. (a) Every diagram in \( \mathcal{C} \) of the form

\[
\begin{array}{ccc}
A & \xrightarrow{0} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{0} & C
\end{array}
\]

is a pullback diagram.
can be completed to a pushout square of the form

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

(b) Every diagram in \( \mathcal{C} \) of the form

\[
\begin{array}{ccc}
& B \\
C & \rightarrow & D
\end{array}
\]

can be completed to a pullback square of the form

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

(3) A square in \( \mathcal{C} \) of the form

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

is a pushout square if and only if it is a pullback square.

**Remark 3.8.** Note that these axioms correspond precisely to the axioms defining a proto-abelian category, except that we drop the conditions on horizontal (resp. vertical) morphisms to be monic (resp. epic).

**Example 3.9.** The bounded-above derived \( \infty \)-category \( \mathcal{D}^- (A) \) of an abelian category with enough projectives is stable.

**Remark 3.10.** It can be shown (cf. [Lur11]) that the homotopy category of a stable \( \infty \)-category \( \mathcal{C} \) has a canonical triangulated structure. The stable \( \infty \)-category \( \mathcal{C} \) can be regarded as an *enhancement* of \( \text{h}\mathcal{C} \) with better properties such as the existence of functorial cones.

To generalize the simplicial groupoid of flags \( S_\bullet \) to stable \( \infty \)-categories we need to understand what the \( \infty \)-categorical analog of a groupoid is: An \( \infty \)-groupoid is an \( \infty \)-category \( \mathcal{C} \) whose homotopy category is a groupoid.

**Example 3.11.**

1. The nerve of a groupoid is an \( \infty \)-groupoid.

2. Given an \( \infty \)-category \( \mathcal{C} \), let \( \mathcal{C}^\simeq \) denote the simplicial subset of \( \mathcal{C} \) consisting of only those simplices whose edges become isomorphisms in the homotopy category \( \text{h}\mathcal{C} \). Then \( \mathcal{C}^\simeq \) is an \( \infty \)-groupoid called the *maximal \( \infty \)-groupoid in \( \mathcal{C} \).*

3. Let \( X \) be a topological space. We define the singular simplicial set \( \text{Sing}(X) \) with \( n \)-simplices

\[
\text{Sing}(X)_n := \text{Hom}_{\text{Top}}(|\Delta^n|, X)
\]

given by continuous morphisms from a geometric \( n \)-simplex to \( X \). Then \( \text{Sing}(X) \) is an \( \infty \)-groupoid. The functor \( \text{Sing} \) from topological spaces to simplicial sets has a left adjoint given by the geometric realization \( |K| \) of a simplicial set. The pair of functors \( (- | - , \text{Sing}) \) defines a Quillen equivalence between suitably defined model categories of topological spaces and \( \infty \)-groupoids so that, from the point of view of homotopy theory, the two concepts are equivalent (see, e.g., [Lur09]).
3.3 The $S$-construction and the derived Hall algebra

With these concepts at hand, we can adapt the theory of Section 2 from proto-abelian categories to stable $\infty$-categories. We define the analog of the simplicial groupoid of flags $S_\bullet$ for a stable $\infty$-category $\mathcal{C}$ as follows: Let $n \geq 0$. We introduce the category $T^n = \text{Fun}([1],[n])$ where we interpret the linearly ordered sets $[1]$ and $[n]$ as categories. A functor $N(T^n) \to \mathcal{C}$ is simply a coherent version of a triangular diagram of shape (2.2). We define $S_n \subset \text{Fun}(N(T^n), \mathcal{C}) \cong$ to be the $\infty$-groupoid of coherent diagrams so that

1. the diagonal objects are 0,
2. all squares are pushout squares and hence, by stability, also pullback squares.

Using Example 3.11(3), we typically interpret the resulting simplicial $\infty$-groupoid $S_\bullet$ as a simplicial space.

Remark 3.12. In other words, besides the idea to use coherent diagrams, the only substantial modification in comparison to the case when $\mathcal{C}$ is proto-abelian is to allow arbitrary chains of morphisms as opposed to flags given by chains of monomorphisms.

Theorem 3.13 (DK12). Let $\mathcal{C}$ be a stable $\infty$-category. The simplicial space $S_\bullet(\mathcal{C})$ is 2-Segal.

Remark 3.14. For the simplicial groupoids of flags in proto-abelian categories, the pullback conditions on the squares (2.6) and (2.7) have to be interpreted in the 2-category of groupoids: the squares have to be 2-pullback squares. In the context of Theorem 3.13, the pullback conditions have to be interpreted in the $\infty$-category of $\infty$-groupoids. Using the equivalence between $\infty$-groupoids and topological spaces this can be made quite explicit: the squares have to be homotopy pullback squares.

The construction of the Hall algebra of a stable $\infty$-category from $S_\bullet(\mathcal{C})$ can also be adapted to our new context: given a topological space $X$, we pass to the vector space $\mathcal{F}(X)$ of functions $\varphi : X \to \mathbb{Q}$ which are constant along connected components and only supported on finitely many connected components. All definitions of Section 2.3 admit natural generalizations to this context. The central idea is to replace the groupoid cardinality

$$|A| = \sum_{[a] \in \pi_0(A)} \frac{1}{|\text{Aut}(a)|}$$

by the homotopy cardinality

$$|X| = \sum_{[x] \in \pi_0(X)} \frac{1}{\pi_1(X,x)} \frac{|\pi_2(X,x)|}{1} \frac{1}{|\pi_3(X,x)|} \cdots$$

as introduced by Baez-Dolan [BD01], and 2-pullbacks by homotopy pullbacks. In particular, we obtain natural pushforward and pullback operations for maps $X \to Y$ of topological spaces satisfying suitable finiteness conditions. Assume that the stable $\infty$-category $\mathcal{C}$ is finitary: for every pair of objects $X,Y$, the groups $\text{Hom}(X,Y[i])$ of morphisms in the homotopy category are finite and non-zero for only finitely many $i$. Then we may apply the constructions of Section 2.3 adapted to the current situation, to obtain the Hall algebra of $\mathcal{C}$. 






Example 3.15. Let $\mathcal{A}$ be a finitary abelian category of finite global dimension with enough projective objects. Then the dg nerve $D^b(\mathcal{A})$ of the full dg subcategory of $\text{Ch}^{-}(\mathcal{A}_{proj})$ consisting of those complexes with bounded cohomology objects is finitary. In this example, we obtain the derived Hall algebra as defined in [Toë06].

In complete analogy to [Toë06], we obtain an explicit description of the structure constants of the derived Hall algebra of a stable $\infty$-category $\mathcal{C}$. Given objects $X, Y, Z$, we have

$$g_{X,Y}^Z = \frac{|\text{Hom}(X, Z)_Y| \prod_{i > 0} |\text{Hom}(X[i], Z)|^{(-1)^i}}{|\text{Aut}(X)| \prod_{i > 0} |\text{Hom}(X[i], X)|^{(-1)^i}}$$

where $\text{Hom}$ denotes the morphisms in the homotopy category of $h\mathcal{C}$ of $\mathcal{C}$ and $\text{Hom}(X, Z)_Y$ denotes the subset given by those morphisms whose cone is isomorphic to $Y$. Note that the structure constants only depend on the triangulated category $h\mathcal{C}$. 

51
4 Triangulated surfaces in triangulated categories

The abstract Hall algebra construction of Section 2.2 can be generalized to simplicial 2-Segal spaces and allows us to interpret the lowest 2-Segal conditions involving \( S_{\leq 3} \) as associativity and unitality. There are several ways to express the relevance of the remaining higher 2-Segal conditions: One can generalize the construction of the Hall monoidal category of Section 2 to define a monoidal \( \infty \)-category. Alternatively, one constructs a variant of the abstract Hall algebra as an algebra object in a monoidal \( \infty \)-category of spans of spaces (cf. [DK12]).

In this section, we provide another construction where the higher 2-Segal constraints are crucial: We exhibit a cyclic symmetry on the \( S_* \)-construction of a pretriangulated differential \( \mathbb{Z}/(2) \)-graded category and show how, in combination with the 2-Segal conditions, this can be utilized to construct invariants of marked oriented surfaces. More details are contained in [DK13].

4.1 State sums in associative algebras

We sketch a combinatorial construction of surface invariants which form what is known as a 2-dimensional open oriented topological field theory ([Laz01, Moo01]). These can be abstractly defined as monoidal functors from a certain 2-dimensional noncompact oriented bordism category into the category of vector spaces. The central result of the theory classifies such functors: they correspond to symmetric Frobenius algebras. While there are much more elegant and intrinsic constructions of the topological field theory associated to such an algebra, the one we describe is the closest analog to the generalization given in Section 4.2 below.

Let \( \mathbf{k} \) be a field, and let \( A \) be an associative finite dimensional \( \mathbf{k} \)-algebra with chosen basis \( E = \{ e_1, e_2, \ldots, e_r \} \). The multiplication law of \( A \) is numerically encoded in the structure constants \( \lambda_{ij}^k \in \mathbf{k} \) defined via \( e_i e_j = \sum_k \lambda_{ij}^k e_k \). Associativity is then expressed by the equations

\[
\sum_t \lambda_{ij}^t \lambda_{ik}^t = \lambda_{ijk}^t = \sum_t \lambda_{it}^t \lambda_{jk}^t
\]

where the generalized structure constants \( \{ \lambda_{ijk}^t \} \) are given by \( e_i e_j e_k = \sum_t \lambda_{ijk}^t e_l \). We can think of the numbers \( \{ \lambda_{ij}^k \} \) and \( \{ \lambda_{ijk}^l \} \) as numerical invariants attached to triangles and squares, respectively, where the set of vertices is ordered and the edges are labeled by \( E \) as illustrated in

\[
\begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{k} \\
\text{j} \\
\text{0}
\end{array} \\
\begin{array}{c}
\text{1}
\end{array} \\
\begin{array}{c}
\text{2}
\end{array}
\end{array}
\quad \rightarrow \quad 
\begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{k} \\
\text{j} \\
\text{0}
\end{array} \\
\begin{array}{c}
\text{1}
\end{array} \\
\begin{array}{c}
\text{2}
\end{array}
\end{array}
\rightleftharpoons \lambda_{ij}^k,
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{k} \\
\text{j} \\
\text{l}
\end{array} \\
\begin{array}{c}
\text{1}
\end{array} \\
\begin{array}{c}
\text{2}
\end{array} \quad \rightarrow \quad 
\begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{k} \\
\text{j} \\
\text{l}
\end{array} \\
\begin{array}{c}
\text{1}
\end{array} \\
\begin{array}{c}
\text{3}
\end{array}
\end{array}
\rightleftharpoons \lambda_{ijk}^l.
\]

Equation (4.1) is then geometrically reflected by the fact that \( \{ \lambda_{ijk}^l \} \) can be computed in terms of \( \{ \lambda_{ij}^k \} \) via two different formulas corresponding to the two possible triangulations of the square. Similarly, this observation extends to yield numerical invariants of planar convex polygons with ordered vertices and \( E \)-labeled edges which can be computed in terms of \( \{ \lambda_{ij}^k \} \) via any chosen triangulation.

Assume now that \( A \) carries a Frobenius structure: a \( \mathbf{k} \)-linear map \( \text{tr} : A \to \mathbf{k} \), called trace, such that

1. \( \text{tr} \) is non-degenerate: the association \( a \mapsto \text{tr}(a-) \) defines an isomorphism \( A \to A^* \),
2. \( \text{tr} \) is symmetric: for every \( a, b \in A \), we have \( \text{tr}(ab) = \text{tr}(ba) \).
Then we can introduce a dual basis $E^* = \{e_1^*, e_2^*, \ldots, e_r^*\}$ of $A$ which is defined by the requirement

$$\text{tr}(e_i e_j^*) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else}. \end{cases}$$

This allows us to enlarge the range of definition of the above system of invariants to include planar polygons with oriented $E$-labeled edges such as

$$\begin{array}{c}
  \begin{array}{c}
    m \\
    l \\
    \circ \\
    k
  \end{array} \\
  \mapsto \text{tr}(e_m e_i^* e_j e_k^* e_l)
\end{array}$$

where, due to the cyclic invariance of the trace expression, no linear ordering of the vertices is needed as long as we remember the orientation of the polygon. Again, these invariants can be computed by choosing any triangulation involving the vertices of the polygon.

With this new flexibility at hand, we can enlarge the range of definition of our invariants quite drastically: Let $S$ be a compact oriented surface, possibly with boundary, equipped with a finite nonempty set $M$ of marked points where we assume that there is at least one marked point on every boundary component. Unless the pair $(S, M)$ is one of the unstable cases

1. $S$ is a sphere and $|M| < 3$,
2. $S$ is a boundary marked disk with $|M| < 3$

it is always possible to find a triangulation of the surface $S$ with $M$ as vertices. Further, it is known that any two triangulations of $(S, M)$ are related by a sequence of Pachner moves given by replacing the diagonal of a chosen local square by the opposite diagonal.

Let $(S, M)$ be a stable compact oriented marked surface. Fix an orientation and an $E$-labelling of all boundary arcs between the boundary marked points. We associate a number to this datum as follows:

1. Choose a triangulation $\Delta(S, M)$ of $(S, M)$ and an orientation of all internal edges.
2. Given an $E$-labelling $L$ of all internal edges, we define $\lambda_l$ to be the product of the numerical invariants of all $E$-labelled triangles involved in the triangulation.
3. The invariant is given by the state sum

$$\sum_l \lambda_l$$

where $l$ ranges over all possible $E$-labellings of the internal edges of $\Delta(S, M)$.

From the above discussion it is straightforward to show that the resulting number is independent of the auxiliary data chosen to compute it – it only depends on $(S, M)$ as well as the orientation and $E$-labelling of the boundary arcs. Varying over all possible $E$-labels of the boundary arcs, these numbers form the matrix entries of a $k$-linear map

$$A^{\otimes i} \rightarrow A^{\otimes o}$$

where $i$ denotes the number of boundary arcs whose orientation is compatible with the surface orientation, and $o$ denotes the number of remaining boundary arcs. The resulting $k$-linear maps assemble to provide a functor from a suitably defined 2-dimensional bordism category into the category of vector spaces – the topological field theory corresponding to $A$.  

53
4.2 State sums in stable $\infty$-categories

The goal of this section is to show that certain symmetries present in 2-periodic pretriangulated differential graded categories can be exploited to define invariants of oriented surfaces via a formalism similar to the one explained in Section 4.1. The conceptual explanation for why this is possible is that, in a certain sense, the abstract Hall algebra associated to a 2-periodic pretriangulated differential graded category is a Frobenius algebra.

We begin with a heuristic version of the construction in the context of triangulated categories which is completely parallel to the discussion in Section 4.1. Let $\mathcal{T}$ be a triangulated category with set of objects $E = \{A, B, \ldots, A', B', \ldots\}$. We associate to an $E$-labeled triangle the collection of all distinguished triangles in $\mathcal{T}$ involving the objects determined by the edge labels. To a triangulated square with $E$-labeled edges, we attach the following collections of diagrams

\[
\begin{align*}
A & \quad \Rightarrow \quad \{A \xrightarrow{+1} A'\} \\
A' & \quad \Rightarrow \quad \{A \xrightarrow{+1} A', B \xrightarrow{+1} A''\} \\
A'' & \quad \Rightarrow \quad \{A \xrightarrow{+1} A', B \xrightarrow{+1} A''\}
\end{align*}
\]

(4.3)

where the $\ast$-marked triangles are distinguished, the unmarked triangles commute, and the objects $B$ and $B'$ are allowed to vary. The two types of diagrams correspond to the upper and lower cap of an octahedron. On a heuristic level, the role of the associativity in Equation (4.1) will now be played by the octahedral axiom which allows us to pass from one triangulation of the square to the other. More generally, to an $E$-labeled polygon with a chosen triangulation we associate the collection of certain Postnikov systems $[\text{GM03}]$ such as

\[
\begin{align*}
A' & \quad \Rightarrow \quad \{A \xrightarrow{+1} A'\} \\
A'' & \quad \Rightarrow \quad \{A \xrightarrow{+1} A', B \xrightarrow{+1} A'', D \xrightarrow{+1} A'''\}
\end{align*}
\]

(4.4)

The analog of the Frobenius structure in Section 4.1 turns out to be a 2-periodic structure on $\mathcal{T}$: an isomorphism of functors $\Sigma^2 \simeq \text{id}$. This structure allows us to rewrite any distinguished triangle as

\[
\begin{align*}
A & \quad \Rightarrow \quad \{A \xrightarrow{+1} A'\} \\
B & \quad \Rightarrow \quad \{A \xrightarrow{+1} A', B \xrightarrow{+1} A''\}
\end{align*}
\]

where the right-hand form exhibits a cyclic symmetry analogous to the symmetry of the trace expression $\text{tr}(e_i e_j e_k^\ast)$ from Section 4.1.

These heuristics suggest the existence of invariants of marked oriented surfaces associated with any 2-periodic triangulated category $\mathcal{T}$. Further, state sum formulas should lead to a description of these invariants in terms of surface Postnikov systems: collections of distinguished triangles in $\mathcal{T}$ parametrized by a chosen triangulation of the surface.

We have already defined the concepts needed to provide a rigorous version of the above heuristics:
(1) to address the issues with the octahedral axiom, we should assume that the triangulated category comes with an enhancement: it is the homotopy category of a differential graded category $T$,

(2) the invariant associated to a polygon $P_n$ is the space $S_n$ of $n$-simplices in the $S_\bullet$-construction of the dg nerve of $T$,

(3) the 2-Segal property allows us to identify the space $S_n$ with the space of Postnikov systems corresponding to a chosen triangulation of $P_n$.

The only missing ingredient is an interpretation of 2-periodicity. One of the main points of this section is, that it can be captured in terms of Connes’ [Con94] cyclic category $\Lambda$. The category $\Lambda$ has an object $\langle n \rangle$ for every $n \geq 0$. Let $S^1$ denote the unit circle in $\mathbb{C}$ and $\mu_n \subset S^1$ the subset of $n$th roots of unity. The morphisms from $\langle m \rangle$ to $\langle n \rangle$ are then given by homotopy classes of monotone degree 1 maps $\varphi : S^1 \to S^1$ such that $\varphi(\mu_{n+1}) \subset \mu_{m+1}$. There is a natural embedding $\Delta \subset \Lambda$ so that every morphism in $\Lambda$ can be uniquely expressed as the composite of a cyclic rotation and a morphism in $\Delta$. A cyclic structure on a simplicial object $X : \Delta \to \mathcal{C}$ is a lift

$$\Delta \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\sigma} \Lambda$$

**Theorem 4.5** ([DK13]). Let $T$ be a pretriangulated differential $\mathbb{Z}/(2)$-graded category. Denote by $S_\bullet(T)$ the simplicial space given by Waldhausen’s $S_\bullet$-construction of the dg nerve of $T$. Then $S_\bullet(T)$ admits a canonical cyclic structure.

**Proof.** Let $A^n$ denote the $k$-linear envelope of the linearly ordered set $\{1, 2, \ldots, n\}$. We may consider $A^n$ as a differential $\mathbb{Z}/(2)$-graded category which is concentrated in even degrees. Given a differential $\mathbb{Z}/(2)$-graded category $T$, one can construct an equivalence of spaces

$$S_n(N_{\text{dg}}(T)) \xrightarrow{\cong} \text{Map}(A^n, T)$$

where Map denotes the mapping space of the localization of the category $\text{dgcat}^{(2)}$ of differential $\mathbb{Z}/(2)$-graded categories along Morita equivalences.

The key of the proof is to replace $A^n$ by a Morita equivalent model in which the cyclic functoriality is more apparent: the differential $\mathbb{Z}/(2)$-graded category

$$\text{MF}_{\mathbb{Z}/(n+1)}(k[z], z^{n+1})$$

of $\mathbb{Z}/(n+1)$-graded matrix factorizations of the polynomial $z^{n+1}$. More precisely, we focus on the full dg subcategory $F^n \subset \text{MF}(k[z], z^{n+1})$ spanned by the objects

$$[i, j] := k[z]^i \frac{z^{i-j}}{z^{j-i}} k[z]^j.$$  

Here, $i$ and $j$ range over $\mathbb{Z}/(n+1)$ and, in the symbols $z^{i-j}$ and $z^{j-i}$, we replace the exponent by its representative in $\{0, 1, \ldots, n\}$. There is a dg functor

$$A^n \to F^n, i \mapsto [0, i]$$

which is a Morita equivalence. The dg categories $\mathcal{G}^\bullet$ naturally organize into a cocyclic object

$$\mathcal{G}^\bullet : \Lambda \longrightarrow \text{dgcat}^{(2)}$$
whose underlying cosimplicial object is 2-Segal (where the pullback conditions become pushout conditions) after localizing along Morita equivalences. The cyclic structure on $S_\bullet(N_{dg}(T))$ is now obtained via

$$S_\bullet(N_{dg}(T)) \simeq \text{Map}(\mathcal{F}^\bullet, T).$$

The following version of a result of [DK13] shows that the expected surface invariants can indeed be defined and computed in terms of a limit which should be regarded as the analog of the state sum [1.2].

**Theorem 4.6.** Let $\mathcal{C}$ be an $\infty$-category with limits and let $X$ be a 2-Segal object in $\mathcal{C}$ equipped with a cyclic structure. Let $(S, M)$ be a stable marked oriented surface. Then there exists an object $X_{(S, M)}$ in $\mathcal{C}$ which, for every triangulation $\Delta(S, M)$ of $(S, M)$, comes equipped with canonical isomorphism

$$X_{(S, M)} \xrightarrow{\simeq} \lim_{\Lambda^n \to \Delta(S, M)} X_n.$$ 

Further, the mapping class group of $(S, M)$ acts coherently on $X_{(S, M)}$ via equivalences in $\mathcal{C}$.

An application of the theorem to the $S_\bullet$-construction of a pretriangulated differential $\mathbb{Z}/(2)$-graded category $T$ yields the surface invariants predicted heuristically in Section 1.2. As a remarkable feature of the proof, note that we can give a universal variant of this construction by applying the theorem to the cocyclic object $\mathcal{F}^\bullet$. As a result, we obtain, for every stable oriented marked surface $(S, M)$ a dg category

$$\mathcal{F}^{(S, M)}$$

which, for every triangulation $\Delta(S, M)$, comes equipped with a universal surface Postnikov system. We give some examples:

### 4.3 Examples

#### 4.3.1 Boundary-marked disk

Consider the marked surface

$$(S, M) = \begin{tikzpicture}
\node (N1) at (0,0) {$0$};
\node (N2) at (1,0) {$1$};
\node (N3) at (2,0) {$n$};
\draw (N1) edge[->] node[below] {$1$} (N2);
\draw (N2) edge[->] node[below] {$2$} (N3);
\end{tikzpicture}$$

given by a disk with $n$ marked points on its boundary so that, by construction, we have

$$\mathcal{F}^{(S, M)} \simeq \mathcal{F}^n \subset \text{MF}_{\mathbb{Z}/(n+1)}(k[z], z^{n+1}).$$

The universal Postnikov system in $\mathcal{F}^{(S, M)}$ corresponding to the triangulation with all edges starting in 0 (cf. [1.4]) is given by

$$\begin{array}{cccccccccccccc}
[0, 1] & \xrightarrow{+1} & [0, 2] & \xrightarrow{+1} & [0, 3] & \cdots & [0, n - 1] & \xrightarrow{+1} & [0, n] \\
[1, 2] & \xleftarrow{+1} & [2, 3] & \xleftarrow{+1} & [n - 1, n] & \end{array}$$

The action of the mapping class group $\text{Mod}(S, M) \cong \mathbb{Z}/(n+1)$ is given by shifting the $\mathbb{Z}/(n+1)$-grading and is part of the cocyclic structure on $\mathcal{F}^\bullet$. 

56
4.3.2 Disk with two marked points

\[(S, M) = \begin{array}{c}
\quad \circ \\
\quad 0 \\
\quad 1 \\
\quad 2 \\
\end{array} \quad \rightsquigarrow \quad \mathcal{F}^{(S, M)} = D^b(\text{coh } \mathbb{A}^1)^{(2)}\]

The universal Postnikov system in \(\mathcal{F}^{(S, M)}\), corresponding to the indicated triangulation, is given by

\[
\begin{array}{c}
X \quad \to \quad k \quad \leftarrow \quad k[x] \\
0 \quad 1 \quad 2 \\
\end{array}
\]

The mapping class group \(\text{Mod}(S, M)\) is trivial.

4.3.3 Annulus with two marked points

\[(S, M) = \begin{array}{c}
\quad \circ \\
\quad 0 \quad 1 \quad 0' \\
\quad 1' \\
\quad 2 \\
\end{array} \quad \rightsquigarrow \quad \mathcal{F}^{(S, M)} = D^b(\text{coh } \mathbb{P}^1)^{(2)}\]

The universal Postnikov system in \(\mathcal{F}^{(S, M)}\), corresponding to the indicated triangulation, is given by

\[
\begin{array}{c}
X \quad Z' \quad Y \\
0' \quad 1' \quad 2' \\
\end{array}
\quad \rightsquigarrow \quad \mathcal{O} \quad \mathcal{O}(1)
\]

The generator of \(\text{Mod}(S, M) \cong \mathbb{Z}\) acts via \(- \otimes \mathcal{O}(1)\).

4.3.4 Sphere with 3 marked points

\[(S, M) = \begin{array}{c}
\quad \circ \\
\quad 0 \quad 1 \quad 0' \\
\quad 1' \\
\quad 2 \\
\end{array} \quad \rightsquigarrow \quad \mathcal{F}^{(S, M)} = D^b(\text{coh } \mathbb{C}[x, y]/(xy))^{(2)}\]

The universal Postnikov system is given by

\[
\begin{array}{c}
X \quad Z \quad \Sigma Z \\
0' \quad 1 \quad 0' \quad 2 \\
\end{array}
\quad \rightsquigarrow \quad S/(x) \quad S/(y)
\]

There is an action of \(\text{Mod}(S, M) \cong S_3\) on \(\mathcal{F}^{(S, M)}\) which permutes the objects \(\Sigma S, S/(x)\) and \(S/(y)\).
Remark 4.7. As a final remark, we conclude by mentioning an interpretation of the dg categories $F^{(S,M)}$ as purely topological Fukaya categories which is due to Kontsevich [Kon09]. In this context, the state sum formula given by the limit in Theorem 4.6 can then be regarded as implementing a 2-dimensional instance of Kontsevich’s proposal on localizing the Fukaya category along a singular Lagrangian spine (given in our context as the dual graph of the chosen triangulation).

References

[BD01] John C. Baez and James Dolan. From finite sets to Feynman diagrams. In Mathematics unlimited—2001 and beyond, pages 29–50. Springer, Berlin, 2001.

[Ber13] Julia E. Bergner. Derived Hall algebras for stable homotopy theories. Cah. Topol. Géom. Différ. Catég., 54(1):28–55, 2013.

[BHW10] John C Baez, Alexander E Hoffnung, and Christopher D Walker. Higher dimensional algebra vii: groupoidification. Theory Appl. Categ, 24(18):489–553, 2010.

[Con94] A. Connes. Noncommutative Geometry. Academic Press, San Diego, New York, London, 1994.

[Day74] Brian Day. An embedding theorem for closed categories. In Category Seminar, pages 55–64. Springer, 1974.

[DK12] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces I. arXiv preprint arXiv:1212.3563, 2012.

[DK13] Tobias Dyckerhoff and Mikhail Kapranov. Triangulated surfaces in triangulated categories. arXiv preprint arXiv:1306.2545, 2013.

[GCKT14] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition Spaces, Incidence Algebras and Möbius Inversion. arXiv preprint arXiv:1404.3202, 2014.

[GM03] Sergei Izrailevich Gelfand and Yuri Manin. Methods of homological algebra: Springer monographs in mathematics. Springer, 2003.

[Gre55] J. A. Green. The characters of the finite general linear groups. Trans. Amer. Math. Soc., 80:402–447, 1955.

[Gre95] James A. Green. Hall algebras, hereditary algebras and quantum groups. Invent. Math., 120(2):361–377, 1995.

[Hal59] Philip Hall. The algebra of partitions. In Proceedings of the 4th Canadian mathematical congress, Banff, page 147159, 1959.

[Hub] Andrew W Hubery. Ringel–Hall algebras. preprint, available at the author’s homepage.

[Joy07] Dominic Joyce. Configurations in abelian categories. II. Ringel-Hall algebras. Adv. Math., 210(2):635–706, 2007.

[JS95] André Joyal and Ross Street. The category of representations of the general linear groups over a finite field. Journal of Algebra, 176(3):908–946, 1995.
[Kon09] Maxim Kontsevich. Symplectic geometry of homological algebra. *preprint*, 2009.

[KS11] Maxim Kontsevich and Yan Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun. Number Theory Phys.*, 5(2):231–352, 2011.

[Laz01] Calin-Iuliu Lazaroiu. On the structure of open–closed topological field theory in two dimensions. *Nuclear Physics B*, 603(3):497–530, 2001.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur11] Jacob Lurie. Higher Algebra. *preprint*, May 2011, available at the author’s homepage.

[Lur14] Jacob Lurie. Rotation Invariance in Algebraic K-Theory. *preprint*, 2014, available at the author’s homepage.

[Lus91] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Moo01] Gregory Moore. Some comments on branes, G-flux, and K-theory. *International Journal of Modern Physics A*, 16(05):936–944, 2001.

[Nad15] David Nadler. Cyclic symmetries of $A_n$-quiver representations. *Adv. Math.*, 269:346–363, 2015.

[Rei03] Markus Reineke. The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli. *Invent. Math.*, 152(2):349–368, 2003.

[Rei06] Markus Reineke. Counting rational points of quiver moduli. *Int. Math. Res. Not.*, pages Art. ID 70456, 19, 2006.

[Rin90] Claus Michael Ringel. Hall algebras and quantum groups. *Invent. Math.*, 101(3):583–591, 1990.

[Rin93] Claus Michael Ringel. *The Hall algebra approach to quantum groups*. Sonderforschungsbereich 343, 1993.

[Rin96] Claus Michael Ringel. Green’s theorem on Hall algebras. In *Representation theory of algebras and related topics (Mexico City, 1994)*, volume 19 of *CMS Conf. Proc.*, pages 185–245. Amer. Math. Soc., Providence, RI, 1996.

[Rou07] Raphaël Rouquier. Hall algebras, 2007.

[Sch01] Issai Schur. *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*. Friedrich Wilhelms Universität, Berlin., 1901.

[Sch06] Olivier Schiffmann. Lectures on Hall algebras. *arXiv preprint math/0611617*, 2006.

[Ste01] E. Steinitz. Zur Theorie der Abel’schen Gruppen. In *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 9, pages 80–85, 1901.
[Szc11] Matt Szczesny. Representations of Quivers Over F1 and Hall Algebras. *International Mathematics Research Notices*, page 113, 2011.

[Toē06] Bertrand Toën. Derived Hall algebras. *Duke Mathematical Journal*, 135(3):587–615, 2006.

[Wal85] F. Waldhausen. Algebraic k-theory of spaces. *Algebraic and geometric topology*, pages 318–419, 1985.

[Wal13] Christopher Walker. A Categorification of Hall Algebras. *arXiv preprint arXiv:1304.0219*, 2013.