ON AN EXTENSION OF JUMP-TYPE SYMMETRIC DIRICHLET FORMS

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Abstract
We show that any element from the $(L^2)$-maximal domain of a jump-type symmetric Dirichlet form can be approximated by test functions under some conditions. This gives us a direct proof of the fact that the test functions is dense in Bessel potential spaces.

1 Introduction
In this note, we are concerned with the following symmetric quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined on $L^2(\mathbb{R}^d)$:

$$
\begin{align*}
\mathcal{E}(u, v) &:= \frac{1}{2} \int_{x \neq y} (u(x) - u(y))(v(x) - v(y)) n(x, y) \, dx \, dy, \\
\mathcal{D}(\mathcal{E}) &:= \{ u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty \},
\end{align*}
$$

(1)

where $n(x, y)$ is a positive measurable function on $x \neq y$.

In order that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ makes sense, we assume that the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : n(x, y) = \infty \}$ is a Lebesgue null set. In fact, under this condition, we have already shown that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mathbb{R}^d)$ in the wide sense (see [11] and [6]). Moreover if we set $C^0_{0,1}(\mathbb{R}^d)$ the totality of all uniformly Lipschitz continuous functions defined on $\mathbb{R}^d$ with compact support, then $\mathcal{D}(\mathcal{E}) \supset C^0_{0,1}(\mathbb{R}^d)$ if and only if the following conditions are satisfied (see [12], [6] and also [3, Example 1.2.4]): For some $\varepsilon > 0$,

$$
\begin{align*}
\Phi_\varepsilon(\cdot) &:= \int_{|h| \leq \varepsilon} |h|^2 j(\cdot, \cdot + h) \, dh \in L^1_{\text{loc}}(\mathbb{R}^d), \\
\Psi_\varepsilon(\cdot) &:= \int_{|h| > \varepsilon} j(\cdot, \cdot + h) \, dh \in L^1_{\text{loc}}(\mathbb{R}^d),
\end{align*}
$$

(A) (B)
where \( j(x, y) = n(x, y) + n(y, x) \). Then under (A) and (B), the quadratic form \( (\mathcal{E}, \mathcal{F}) \) becomes a regular symmetric Dirichlet form on \( L^2(\mathbb{R}^d) \), where \( \mathcal{F} \) is the closure of \( C_0^{0,1}(\mathbb{R}^d) \) with respect to the norm \( \sqrt{\mathcal{E}(\cdot, \cdot) + ||\cdot||^2_{L^2}} \). Note that, from the integral representation of the form \( \mathcal{E} \), we can adopt the test functions, \( \mathcal{C}_\infty^0(\mathbb{R}^d) \), as a core instead of \( \mathcal{C}_0^1(\mathbb{R}^d) \) under the conditions (A) and (B). We now give some examples (see e.g., [11, 12]):

**Example 1**

1. (symmetric \( \alpha \)-stable process) Let
   \[
   n(x, y) = c|x - y|^{-\alpha - d}, \quad x \neq y.
   \]
   Then (A) and (B) hold if and only if \( 0 < \alpha < 2 \) and \( c > 0 \). This is nothing but the Dirichlet form corresponding to a symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \).

2. (symmetric stable-like process) For a measurable function \( \alpha(x) \) defined on \( \mathbb{R}^d \), set
   \[
   n(x, y) = |x - y|^{-\alpha(x) - d}, \quad x \neq y.
   \]
   Then (A) and (B) hold if and only if the following three conditions are satisfied:
   
   (i) \( 0 < \alpha(x) < 2 \) a.e.,
   
   (ii) \( 1/\alpha, 1/(2 - \alpha) \in L^1_{\text{loc}}(\mathbb{R}^d) \),
   
   (iii) for some compact set \( K \), \( \int_K |x|^{-d-\alpha(x)}dx < \infty \).

3. (symmetric Lévy process) For a positive measurable function \( \tilde{n} \) defined on \( \mathbb{R}^d - \{0\} \) satisfying \( \tilde{n}(x) = \tilde{n}(-x) \) for any \( x \neq 0 \), set
   \[
   n(x, y) = \tilde{n}(x - y), \quad x \neq y.
   \]
   (A) and (B) are satisfied if and only if \( \int_{h \neq 0} (1 \wedge |h|^2)\tilde{n}(h)dh < \infty \).

In general, we do not know whether the set \( \mathcal{F} \) coincides with \( \mathcal{D}(\mathcal{E}) \). Determining the domains of the Dirichlet form corresponds, in some sense, to solve the boundary problem of the associated Markov processes. This analytic structure was investigated first by Silverstein in [7] and [8], and then by Chen [1] and Kuwae [5].

### 2 Identification of the domains

In order to classify the domains of the forms, we will consider the following conditions: there exists a positive constant \( C > 0 \) such that

\[
\Phi_1 \in L^1_{\text{loc}}(\mathbb{R}^d), \quad j(x + z, y + z) \leq Cj(x, y), \quad |x - y| \leq 1, \quad |z| \leq 1 \quad (A')
\]

or

\[
\Phi_1 \in L^\infty(\mathbb{R}^d), \quad j(x + z, y + z) \leq Cj(x, y), \quad |x - y| \leq 1, \quad |z| \leq 1 \quad (A'')
\]

and

\[
\Psi_1(\cdot) = \int_{|h| > 1} j(\cdot, \cdot + h)dh \in L^\infty(\mathbb{R}^d). \quad (B')
\]

Note that \( (A'') \Rightarrow (A') \Rightarrow (A) \) and \( (B') \Rightarrow (B) \).
Theorem 1 Assume that (A') and (B') hold. Then we can show
\[ \mathcal{D}(\mathcal{E}) = \{ u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty \} = \mathcal{F}, \]
that is, any element in \( \mathcal{D}(\mathcal{E}) \) can be approximated from elements of \( C_0^\infty(\mathbb{R}^d) \) with respect to \( \mathcal{E}_1 \).

Proof: Take \( \rho \in C_0^\infty(\mathbb{R}^d) \) satisfying
\[ \rho(x) \geq 0, \quad \rho(x) = \rho(-x), \quad x \in \mathbb{R}^d, \quad \text{supp}[\rho] = \overline{B_0(1)}, \quad \int_{\mathbb{R}^d} \rho(x)dx = 1. \]

For any \( \varepsilon > 0 \), define \( \rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon) \) so that \( \int_{\mathbb{R}^d} \rho_\varepsilon dx = 1. \)

For \( u \in \mathcal{D}(\mathcal{E}) \), set the convolution of \( u \) and \( \rho_{1/n} \):
\[ w_n(x) := J_{1/n}(u)(x) := \rho_{1/n} * u(x) = \int_{\mathbb{R}^d} \rho_{1/n}(x-z)u(z)dz, \quad x \in \mathbb{R}^d. \]

Since \( u \in L^2(\mathbb{R}^d) \), \( w_n \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and
\[ \|w_n\|_{L^2} \leq \|\rho_{1/n}\|_{L^1}\|u\|_{L^2} = \|u\|_{L^2} \text{ and } \|w_n - u\|_{L^2} \to 0 \text{ as } n \to \infty. \]

Let \( \psi_n(t), \ t \geq 0, \) be non-negative \( C^\infty \)-functions such that
\[ \psi_n(t) = 1, \ 0 \leq t \leq n, \quad \psi_n(t) = 0, \quad t \geq n + 2, \quad -1 \leq \psi_n'(t) \leq 0, \quad t \leq n + 2. \]

We put \( v_n(x) = \psi_n(|x|), \ x \in \mathbb{R}^d. \) Then \( v_n \in C_0^\infty(\mathbb{R}^d) \) and
\[ V_n(x) := \int_{|x-y|<1} (v_n(x) - v_n(y))^2 j(x, y)dy, \quad x \in \mathbb{R}^d \]
satisfies the following inequality:
\[ V_n(x) \leq d \int_{|x-y|<1} |x-y|^2 j(x, y)dy = d\Phi_1(x), \quad x \in \mathbb{R}^d. \quad (2) \]

Then we see that
\[ v_n(x) \not\to 1, \quad x \in \mathbb{R}^d \quad \text{and} \quad M := \sup_{n} \sup_{x \in \mathbb{R}^d} V_n(x) \leq d\|\Phi_1\|_{\infty} < \infty \]
and
\[ \|w_n v_n - u\|_{L^2} \leq \|w_n v_n - uv_n\|_{L^2} + \|uv_n - u\|_{L^2} \]
\[ \leq \|w_n - u\|_{L^2} \|v_n - u\|_{L^2} \to 0 \text{ as } n \to \infty. \]

Now we estimate \( \mathcal{E}(w_n v_n, w_n v_n) \):
\[ \mathcal{E}(w_n v_n, w_n v_n) = \int_{\mathbb{R}^d} (w_n(x)v_n(x) - w_n(y)v_n(y))^2 j(x, y)dx \]
\[ = \left( \int_{|x-y|<1} + \int_{|x-y|\geq1} \right) (w_n(x)v_n(x) - w_n(y)v_n(y))^2 j(x, y)dx \]
\[ =: (I) + (II). \]
Now we estimate (I).

\[(I) = \iint_{|x-y|<1} (w_n(x)v_n(x) - w_n(y)v_n(y))^2 j(x,y)dxdy\]
\[\leq 2 \iint_{|x-y|<1} (w_n(x))^2 (v_n(x))^2 j(x,y)dxdy\]
\[+ 2 \iint_{|x-y|<1} (v_n(x) - v_n(y))^2 (w_n(y))^2 j(x,y)dxdy\]
\[= 2 |\Pi_{1}|_{L^\infty} \int_{\mathbb{R}^d} (J_{1/n}(u)(x))^2 dx \leq 4 |\Pi_{1}|_{L^\infty} |u|_{L^2}^2.\]

Now we estimate (II).

\[(II) = 4 \int_{|x-y|\geq 1} (w_n(x))^2 j(x,y)dxdy\]
\[= 4 \int_{\mathbb{R}^d} (J_{1/n}(u)(x))^2 dx \int_{|x-y|\geq 1} j(x,y)dy\]
\[\leq 4 |\Pi_{1}|_{L^\infty} \int_{\mathbb{R}^d} (J_{1/n}(u)(x))^2 dx \leq 4 |\Pi_{1}|_{L^\infty} |u|_{L^2}^2.\]

Since \(\text{supp}[\rho_{1/n}] \subset B_{1/n}(0) \subset B_{1}(0)\) for \(n \in \mathbb{N}\), we see

\[(I-1) \leq \iint_{|x-y|<1} \left( \int_{\mathbb{R}^d} (u(x-z) - u(y-z))^2 \rho_{1/n}(z)dz \right) j(x,y)dxdy\]
\[= \int_{\mathbb{R}^d} \left( \iint_{|x-y|<1} (u(x-z) - u(y-z))^2 j(x,y)dxdy \right) \rho_{1/n}(z)dz\]
\[= \int_{\mathbb{R}^d} \left( \iint_{|x-y|<1} (u(x) - u(y))^2 j(x+z, y+z)dxdy \right) \rho_{1/n}(z)dz\]
\[\leq \int_{\mathbb{R}^d} \left( \iint_{|x-y|<1} (u(x) - u(y))^2 Cj(x,y)dxdy \right) \rho_{1/n}(z)dz\]
\[= C \iint_{|x-y|<1} (u(x) - u(y))^2 j(x,y)dxdy \leq CE(u,u) < \infty.\]
In the first inequality we used the Jensen inequality for the measure \( \rho_{1/n}(z)dz \), while the second is from the Fubini theorem, the third is by translation and the fourth is obtained by the assumption \((A^\gamma)\).

\[
(I-2) \quad \begin{align*}
&= \int_{\mathbb{R}^d} (w_n(y))^2 \int_{|x-y|<1} (v_n(x) - v_n(y))^2 j(x,y) dxdy \\
&= \int_{\mathbb{R}^d} (w_n(y))^2 V_n(y) dy \leq M \int_{\mathbb{R}^d} (w_n(y))^2 dy \leq M \|u\|_{L^2}^2.
\end{align*}
\]

Summarizing the calculus done above, we see

\[
\mathcal{E}(w_n v_n, w_n v_n) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (w_n(x) v_n(x) - w_n(y) v_n(y))^2 j(x,y) dxdy \leq 4\|\Psi_1\|_{L^\infty} \|u\|_{L^2}^2 + 2C\mathcal{E}(u,u) + 2M\|u\|_{L^2}^2.
\]

\[
= 2 \left( C\mathcal{E}(u,u) + (2\|\Psi_1\|_{L^\infty} + M)\|u\|_{L^2}^2 \right) < \infty.
\]

That is, \( \mathcal{E}(w_n v_n, w_n v_n) \) are uniformly bounded. Moreover we have seen that \( \|w_n v_n\|_{L^2} \) are also uniformly bounded and \( w_n v_n \) converges to \( u \) in \( L^2(\mathbb{R}^d) \). Thus the Cesàro means of a subsequence of \( \{w_n v_n\} \) are \( \mathcal{E}_1 \)-Cauchy and convergent to \( u \) a.e. Hence \( u \in \mathcal{F} \). Thus

\[
\{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty \} = \mathcal{F} := C_{0,\infty}(\mathbb{R}^d)/(\mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2}^2)^1/2.
\]

\[
\square
\]

**Example 2**

(1) Let \( n(x,y) = c|x-y|^{-d-\alpha}, \) \( x \neq y \) for some \( 0 < \alpha < 2 \) and \( c > 0 \). For this \( n \), we can easily see that the conditions \((A^\gamma)\) and \((B')\) hold. In this case, the \( L^2\)-maximal domain \( \mathcal{D}(\mathcal{E}) \) is nothing but the “Bessel potential space” \( L^2_{\alpha/2}(\mathbb{R}^d) \) (see Proposition V. 4 in [9]).

(2) For \( 0 < \alpha < 2 \) and \( c_i > 0 \) \((i = 1, 2)\), we assume

\[
c_1|x-y|^{-d-\alpha} \leq n(x,y) \leq c_2|x-y|^{-d-\alpha}, \quad 0 < |x-y| \leq 1.
\]

and

\[
\sup_{x} \int_{|x-y|\geq 1} (n(x,y) + n(y,x)) dy < \infty.
\]

Then this satisfies the conditions \((A^\gamma)\) and \((B')\). A Markov process corresponding to the Dirichlet form \( (\mathcal{E},\mathcal{D}(\mathcal{E})) \) is called “stable-like process” by Chen-Kumagai [2].

For a subclass \( \mathcal{B} \) of all measurable functions on \( \mathbb{R}^d \), we denote by \( \mathcal{B}_0 \) the bounded functions in \( \mathcal{B} \). In the following, we always assume that \((A)\) and \((B)\) hold. Then a symmetric Dirichlet form \((\eta,\mathcal{D}(\eta))\) on \( L^2(\mathbb{R}^d) \) is said to be an extension of the Dirichlet form \((\mathcal{E},\mathcal{F})\) if \( \mathcal{D}(\eta) \supset \mathcal{F} \) and \( \eta(u,u) = \mathcal{E}(u,u) \) whenever \( u \in \mathcal{F} \). Denote by \( \mathcal{A}(\mathcal{E},\mathcal{F}) \) the totality of the extensions of \((\mathcal{E},\mathcal{F})\). By this definition, \((\mathcal{E},\mathcal{D}(\mathcal{E}))\) is an element of \( \mathcal{A}(\mathcal{E},\mathcal{F}) \). An element \((\eta,\mathcal{D}(\eta))\) of \( \mathcal{A}(\mathcal{E},\mathcal{F}) \) is called a Silverstein extension if \( \mathcal{F}_0 \) is an algebraic ideal in \( \mathcal{D}(\eta)_0 \). For the probabilistic counterpart or an application of Silverstein extensions, see for example, [5], [10] and [4].
**Theorem 2** Suppose that (A') and (B) hold. Then the Dirichlet form \((E, D(E))\) is a Silverstein extension of the form \((E, F)\). That is, \(F_b\) is an ideal of \(D(E)_b\).

**Proof:** It is enough to show that \(u \cdot f \in F_b\) whenever \(u \in D(E)_b\) and \(f \in C_0^\infty(\mathbb{R}^d)\). Let \(\rho\) and \(\rho_{1/n}\) be the same functions in the proof of the preceding theorem. Take the convolution of functions \(uf\) and \(\rho_{1/n} : w_n = \rho_{1/n} * (uf)\). Then \(w_n \in C_0^\infty(\mathbb{R}^d)\), \(w_n\) converges to \(uf\) in the \(L^2\)-space and the inequality \(||w_n||_{L^\infty} \leq ||uf||_{L^\infty}\) holds.

Denote by \(K\) the support of the function \(f\). As in the proof of the preceding theorem, we estimate \(E(w_n, w_n)\) as follows:

\[
E(w_n, w_n) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (w_n(x) - w_n(y))^2 j(x, y) dx dy
\]

\[
= \left( \iint_{|x-y| < 1} + \iint_{|x-y| \geq 1} \right) (w_n(x) - w_n(y))^2 j(x, y) dx dy
\]

\[
=: \ (I) + \ (II).
\]

\[
(II) \leq 2 \iint_{|x-y| \geq 1} (w_n(x))^2 + (w_n(y))^2 j(x, y) dx dy.
\]

Since \(j(x, y) = j(y, x)\), we see

\[
(II) \leq 4 \iint_{|x-y| \geq 1} (w_n(x))^2 j(x, y) dx dy
\]

\[
= 4 \int_{\mathbb{R}^d} (w_n(x))^2 dx \int_{|x-y| \geq 1} j(x, y) dy
\]

\[
= 4 \int_{K_n} (w_n(x))^2 \Psi_1(x) dx
\]

\[
\leq 4 \|w_n\|_{L^\infty}^2 \int_{K_1} \Psi_1(x) dx \leq 4 \|uf\|_{L^\infty}^2 \|\Psi_1 1_{K_1}\|_{L^1},
\]

where \(K_n = \{x + y \in \mathbb{R}^d : x \in K, y \in B(0, 1/n)\}\).

Now we estimate (I).

\[
(I) = \iint_{|x-y| < 1} (w_n(x) - w_n(y))^2 j(x, y) dx dy
\]

\[
= \iint_{|x-y| < 1} \left( \int_{\mathbb{R}^d} \rho_{1/n}(z) ((uf)(x-z) - (uf)(y-z)) dz \right)^2 j(x, y) dx dy
\]

\[
\leq \iint_{|x-y| < 1} \left( \int_{\mathbb{R}^d} ((uf)(x-z) - (uf)(y))^2 \rho_{1/n}(z) dz \right) j(x, y) dx dy
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{|x-y| < 1} ((uf)(x) - (uf)(y))^2 j(x+z, y+z) dx dy \right) \rho_{1/n}(z) dz
\]
\[
\begin{align*}
&\leq C \int_{|x-y|<1} ((uf)(x) - (uf)(y))^2 f(x,y) dx dy \int_{\mathbb{R}^d} \rho_{1/n}(z) dz \\
&\leq 2C (||u||^2_{L^\infty} \mathcal{E}(f,f) + ||f||^2_{L^\infty} \mathcal{E}(u,u)).
\end{align*}
\]

Combining the estimates (II) and (I), we have
\[
\mathcal{E}(w_n, w_n) \leq 2C (||u||^2_{L^\infty} \mathcal{E}(f,f) + ||f||^2_{L^\infty} \mathcal{E}(u,u)) + 4||u||_{L^\infty} ||\Psi_1 K_i||_{L^1} < \infty.
\]

So \(\mathcal{E}(w_n, w_n)\) are uniformly bounded. We have already known that \(w_n \in C_0^\infty(\mathbb{R}^d)\) converges to \(uf\) in \(L^2\). Then by making use of the Banach-Saks theorem, the Cesàro means of a subsequence of \(\{w_n\}\) are \(\mathcal{E}\)-Cauchy and converges to \(uf\) a.e. Hence \(uf \in \mathcal{F}\). This shows that \(\mathcal{F}_b\) is an ideal of \(\mathcal{D}(\mathcal{E})_b\), whence \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is a Silverstein extension of \((\mathcal{E}, \mathcal{F})\). \(\square\)

**Remark 1** If the form \((\mathcal{E}, \mathcal{F})\) is moreover conservative, then, using a theorem from [6], we can show that the Silverstein extension is unique. Hence this implies that \(\mathcal{F} = \mathcal{D}(\mathcal{E})\). In [7], we showed that under some conditions (which includes the condition \((B')\)), the form \((\mathcal{E}, \mathcal{F})\) is conservative. So, we have an alternative proof of Theorem 1 under \((A')\) and \((B')\).

In the following, we consider ‘the homogeneous’ Dirichlet space:
\[
D_0(\mathcal{E}) = \{u \in L^0(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\},
\]
where \(\mathcal{E}\) is defined in §1 and \(L^0(\mathbb{R}^d)\) is the family of all measurable functions on \(\mathbb{R}^d\). We assume \((A)\) and \((B)\) hold. Since \(\mathcal{E}\) is defined as an integral form, we can easily see that \(D_0(\mathcal{E}) \cap L^\infty(\mathbb{R}^d) =: D_\infty(\mathcal{E})\) is dense in \(D_0(\mathcal{E})\) with respect to quasi-norm \(\mathcal{E}\).

We now want to consider when any function in \(D_\infty(\mathcal{E})\) (hence, in \(D_0(\mathcal{E})\)) can be approximated from a sequence of the test functions with respect to \(\mathcal{E}\). Of course, this relates the notion of ‘the extended Dirichlet space’ \(\mathcal{F}_e\). In general,
\[
D_0(\mathcal{E}) \supset \mathcal{F}_e \supset \mathcal{F} := C_0^\infty(\mathbb{R}^d)^\mathcal{E}_1.
\]

If the form \((\mathcal{E}, \mathcal{F})\) is transient, then \(\mathcal{F} = \mathcal{F}_e \cap L^2(\mathbb{R}^d)\) (see Theorem 1.5.2(iii) in [3]). It is not easy to see whether the ‘homogeneous’ domain \(D_0(\mathcal{E})\) coincides with \(\mathcal{F}_e\) except the special cases. In order to consider this, we introduce a little bit stronger condition as follows: there exists a positive function \(\bar{n}(x)\) defined on \(\mathbb{R}^d - \{0\}\) satisfying the condition in Example 1 (3) so that for some constants \(c_i > 0\) \((i = 1, 2)\),
\[
c_1 \bar{n}(x-y) \leq n(x,y) \leq c_2 \bar{n}(x-y), \quad x \neq y.
\]

**Proposition 1** Suppose that \((C)\) holds. Moreover, we assume the Dirichlet form \((\mathcal{E}, \mathcal{F})\) is recurrent. Then any element in \(D_\infty(\mathbb{R}^d)\) (hence, in \(D_0(\mathcal{E})\)) can be approximated from the test functions with respect to \(\mathcal{E}\). That is, \(D_0(\mathcal{E}) = \mathcal{F}_e\).

**Proof:** First note that a similar argument developed in the proof of Theorem 2 gives us that \(\varphi \cdot u \in D_0(\mathcal{E})\) provided that \(u \in D_0(\mathcal{E})\) and \(\varphi \in C_0^\infty(\mathbb{R}^d)\). Take the test function \(\rho\) defined in the proof of Theorem 1. And also consider the function \(\rho_{1/n}\) for each \(n\). Then considering the
In the first inequality, we used the Schwarz inequality, and the second follows from (C). Accordingly, we see that the sequence \( \{u_n\} \) is \( E \)-bounded. Since \( \|u_n - \varphi u\|_{L^2} \) converges to 0, a subsequence of \( u_n \) converges to \( \varphi u \) almost everywhere. So we can find the Cesaro mean \( \{\bar{u}_{n_k}\} \) of some subsequence from \( \{u_n\} \) so that \( E(\bar{u}_{n_k} - u, \bar{u}_{n_k} - u) \) converges to 0 and \( \bar{u}_{n_k} \to \varphi u \) a.e. This means that there exists a sequence from test functions which converges to \( \varphi u \) with respect to \( E \) and with respect to almost everywhere convergence.

On the other hand, the Dirichlet form \( (E, F) \) is recurrent, we can construct a sequence \( \{\varphi_k\} \subset C_0^\infty(\mathbb{R}^d) \) satisfying

\[
0 \leq \varphi_k \to 1 \quad \text{a.e.,} \quad \|\varphi_k\|_{L^\infty} \leq 1 \quad \text{and} \quad E(\varphi_k, \varphi_k) \to 0.
\]

Note that \( \varphi_k \cdot u \in D(E) \cap L^2(\mathbb{R}^d) \) for each \( k \) because \( \varphi_k \in C_0^\infty(\mathbb{R}^d) \). Similarly, noting the following estimates and the property of \( \varphi_k \), we can see that the cesaro means \( \bar{\varphi}_{n_k}u \) of some subsequence of \( \{\varphi_ku\} \) converges to \( u \) with respect to \( E \) and with respect to almost everywhere convergence:

\[
E(\varphi_k u, \varphi_k u) \leq 2E(u, u) + 2\|u\|^2_{L^\infty}E(\varphi_k, \varphi_k).
\]

Now for each \( k \), take \( f_k \in C_0^\infty(\mathbb{R}^d) \) so that \( E(\bar{\varphi}_{n_k} u - f_k, \bar{\varphi}_{n_k} u - f_k) < 1/k \), Then we see

\[
E(f_k - u, f_k - u)^{1/2} \leq E(f_k - \bar{\varphi}_{n_k} u, f_k - \bar{\varphi}_{n_k} u)^{1/2} + E(\bar{\varphi}_{n_k} u - u, \bar{\varphi}_{n_k} u - u)^{1/2} \\
\leq 1/k + E(\bar{\varphi}_{n_k} u - u, \bar{\varphi}_{n_k} u - u)^{1/2}.
\]

So, taking \( k \to \infty \), we see that \( f_k \) converges to \( u \) with respect to the quasi-norm \( E \). This concludes the proof. \( \square \)
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