Partitioning a graph into degenerate subgraphs

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Abstract

Let $G = (V, E)$ be a graph with maximum degree $k \geq 3$ distinct from $K_{k+1}$. Given integers $s \geq 2$ and $p_1, \ldots, p_s \geq 0$, $G$ is said to be $(p_1, \ldots, p_s)$-partitionable if there exists a partition of $V$ into sets $V_1, \ldots, V_s$ such that $G[V_i]$ is $p_i$-degenerate for $i \in \{1, \ldots, s\}$. In this paper, we prove that we can find a $(p_1, \ldots, p_s)$-partition of $G$ in $O(|V| + |E|)$-time whenever $1 \geq p_1, \ldots, p_s \geq 0$ and $p_1 + \cdots + p_s \geq k - s$. This generalizes a result of Bonamy et al. (MFCS, 2017) and can be viewed as an algorithmic extension of Brooks’ theorem and several results on vertex arboricity of graphs of bounded maximum degree.

We also prove that deciding whether $G$ is $(p, q)$-partitionable is NP-complete for every $k \geq 5$ and pairs of non-negative integers $(p, q)$ such that $(p, q) \neq (1, 1)$ and $p + q = k - 3$. This resolves an open problem of Bonamy et al. (manuscript, 2017). Combined with results of Borodin, Kostochka and Toft (Discrete Mathematics, 2000), Yang and Yuan (Discrete Mathematics, 2006) and Wu, Yuan and Zhao (Journal of Mathematical Study, 1996), it also completely settles the complexity of deciding whether a graph with bounded maximum degree can be partitioned into two subgraphs of prescribed degeneracy.

1 Introduction

The concept of degenerate graphs introduced by Lick and White [11] in 1970 has since found a number of applications in graph theory, especially in graph

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partitioning and graph colouring problems. This is mainly because the class of degenerate graphs captures one of the earliest studied classes of graphs such as independent sets, forests and planar graphs. For example, results of Thomassen [14, 15] on decomposing the vertex set of a planar graph into degenerate subgraphs have lead to new proofs of the 5-colour theorem on planar graphs that do not use Euler’s formula. Another example is a result of Alon, Kahn and Seymour [1] that extends the well-known Turán’s theorem on the size of the largest independent set in a graph to the size of largest subgraph of any prescribed degeneracy.

In this paper, we shall investigate the complexity of partitioning the vertex set of a graph of bounded maximum degree into degenerate subgraphs. In order to make this statement more precise, we must first proceed with some definitions. Let $G = (V, E)$ be a graph, and let $k$ be a non-negative integer. We say that $G$ is $k$-degenerate if we can successively delete vertices of degree at most $k$ in $G$ until the empty graph is obtained. Expressed in another way, $G$ is $k$-degenerate if it admits a $k$-degenerate ordering – an ordering $x_1, \ldots, x_n$ of the vertices in $G$ such that $x_i$ has at most $k$ neighbours $x_j$ in $G$ with $j < i$. Given integers $s \geq 2$ and $p_1, \ldots, p_s \geq 0$, $G$ is said to be $(p_1, \ldots, p_s)$-partitionable if there exists a partition of $V$ into sets $V_1, \ldots, V_s$ such that $G[V_i]$ is $p_i$-degenerate for $i \in \{1, \ldots, s\}$.

We shall consider the following computational problem.

**Problem 1.** Given a graph $G$ and integers $s \geq 2$, $p_1, \ldots, p_s \geq 0$, determine the complexity of deciding whether $G$ is $(p_1, \ldots, p_s)$-partitionable.

We briefly review some existing results related to Problem 1. Let $G = (V, E)$ be a graph with maximum degree $k \geq 3$ distinct from $K_{k+1}$, and let $d$ be a non-negative integer. A $d$-colouring of $G$ is a function $f : V \rightarrow \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E$. Equivalently, $f$ is a $d$-colouring of $G$ if $f^{-1}(1), \ldots, f^{-1}(d)$ are pairwise vertex disjoint and each forms an independent set. The earliest result on Problem 1 is most likely the celebrated theorem of Brooks [6], which states that $G$ has a $d$-colouring for each $d \geq k$. Thus, given that an independent set is a 0-degenerate graph (and vice versa), Brooks’ theorem can be reformulated in the language of Problem 1 to state that $G$ is $(p_1, p_2, \ldots, p_s)$-partitionable for every $s \geq k$ and $p_1 = \cdots = p_s = 0$. Later on, Borodin, Kostochka and Toft [5] obtained a generalization of Brooks’ theorem by showing that $G$ remains $(p_1, \ldots, p_s)$-partitionable for every $s \geq 2$ and $p_1 + \cdots + p_s \geq k - s$. Observe that this result is algorithmic: Given a graph $G$ of maximum degree $k \geq 3$ and integers $s \geq 2$
and \( p_1, \ldots, p_s \geq 0 \) such that \( p_1 + \cdots + p_s \geq k - s \), one can check in polynomial time if \( G \) is \((p_1, \ldots, p_s)\)-partitionable, because the only computation needed is to verify whether \( G \) is isomorphic to \( K_{k+1} \). The question remains, however, whether one can find such a partition efficiently whenever it exists. In this direction, Bonamy et al. [3] have already considered the case \( s = 2 \) with \( p_1 = 0 \) and \( p_1 + p_2 \geq k - 2 \) by showing that one can find the partition in \( O(n + m) \)-time if \( k = 3 \) and in \( O(kn^2) \)-time if \( k \geq 4 \). In the first part of this paper, we generalize the first of these two results in the following theorem.

**Theorem 1.** Let \( G \) be a graph with maximum degree \( k \geq 3 \) distinct from \( K_{k+1} \). For every \( s \geq 2 \) and \( 1 \geq p_1, \ldots, p_s \geq 0 \) such that \( p_1 + \cdots + p_s \geq k - s \), a \((p_1, \ldots, p_s)\)-partition of \( G \) can be found in \( O(n + m) \)-time.

The proof of Theorem 1 appears in Section 2. We remark by our earlier discussion that Theorem 1 can be viewed as an algorithmic extension of Brooks’ theorem. (In fact, our approach differs from [3] but is instead a refinement of Lovász’ proof [12] of Brooks’ theorem – see [2] for an algorithmic analysis of [12].) We also remark that since the definition of forests coincides with the definition of 1-degenerate graphs, Theorem 1 can be viewed as an algorithmic counterpart to several results on vertex arboricity of graphs; see [7, 10] for some examples.

On a different tack, one might also ask what happens if the maximum degree of the graph exceeds \( s + \sum_{i=1}^{s} p_i \). In this direction, Yang and Yuan [17] have shown that the case \( s = 2 \) with \( p_1 = 0 \) and \( p_2 = 1 \) is \( \mathsf{NP} \)-complete for every \( k \geq 4 \). Wu, Yuan and Zhao [16] have also shown that the case \( s = 2 \) with \( p_1 = p_2 = 1 \) is \( \mathsf{NP} \)-complete for every \( k \geq 5 \). Extending Yang and Yuan’s result, Bonamy et al [4] have shown that the case \( s = 2 \) with \( p_1 = 0 \) and \( p_2 = t - 2 \) remains \( \mathsf{NP} \)-complete for every \( t \geq 3 \) and \( k \geq 2t - 2 \). They then posed as an open problem the case \( s = 2 \) with \( p_1 = 0 \) and \( p_2 = k - 3 \) for every \( k \geq 5 \). In the second part of this paper, we resolve this problem by proving, more generally, the following theorem.

**Theorem 2.** For every integer \( k \geq 5 \) and pairs of non-negative integers \((p, q)\) such that \((p, q) \neq (1, 1)\) and \( p + q = k - 3 \), deciding whether a graph with maximum degree \( k \) is \((p, q)\)-partitionable is \( \mathsf{NP} \)-complete.

The proof of Theorem 2 appears in Section 3. We remark that finding the least integer \( d \) such that a graph with maximum degree 3 is \( d \)-colourable can be done in polynomial time. Indeed, one can check in polynomial time
if $d = 1$ or $d = 2$ (for any arbitrary graph). If this is not the case, we check in polynomial time if the graph is isomorphic to $K_4$; if not, then we know by Brooks’ theorem that $d = 3$. Thus, combined with the aforementioned results in [5, 16, 17], Theorem 2 completely settles the complexity of deciding whether a graph with bounded maximum degree can be partitioned into two subgraphs with prescribed degeneracy. More formally, we now have the following complete solution to the case $s = 2$ of Problem 1.

**Corollary 1.1.** Given integers $p, q \geq 0$, deciding whether a graph with maximum degree $k \geq 3$ is $(p, q)$-partitionable is

(i) polynomial time solvable if $k = 3$ or $p + q \geq k$ or $p = q = 0$;

(ii) NP-complete otherwise.

## 2 A linear time algorithm

In this section, we prove Theorem 1. First, we need some standard definitions.

Throughout this section, let $k$ be a non-negative integer, and let $G$ be a graph with maximum degree $k$. Then $G$ is said to be $k$-regular if every vertex of $G$ has degree exactly $k$. A vertex $v$ of $G$ is called a cut vertex of $G$ if $G - \{v\}$ has more components than $G$. A block of $G$ is a maximal 2-connected subgraph of $G$, and an end block of $G$ is a block of $G$ that contains exactly one cut vertex of $G$. A forest partition of $G$ is a partition of $V$ into $k/2$ forests if $k$ is even and $\lfloor k/2 \rfloor$ forests and one independent set if $k$ is odd.

We require the next two lemmas. The proof of the first lemma is essentially the same as the proof of [8 Lemma 8] but with some minor adjustments.

**Lemma 1.** If $G$ is not $k$-regular, then a forest partition of $G$ can be found in $O(n + m)$ time.

**Proof.** If $G$ is not $k$-regular, then $G$ is $(k - 1)$-degenerate. We first compute a $(k - 1)$-degenerate ordering of the vertices of $G$ in $O(n + m)$ time as follows. We find a vertex $v$ of degree at most $k - 1$. Note that every neighbour of $v$ has degree at most $k - 1$ in $G - \{v\}$. So the algorithm that consists of first deleting $v$ and then, for each vertex deleted, deleting all of its neighbours until the empty graph is obtained, gives a degenerate ordering $v_1, v_2, \ldots, v_n$.
of $G$ in $O(n+m)$ time. We proceed to find a forest partition of $G$ in $O(n+m)$ time.

We shall only consider the case where $k$ is even since the case where $k$ is odd is similar. For $i = 1, \ldots, n$, we define $X_i = \{v_1, \ldots, v_i\}$. Then, by definition, $v_i$ has at most $k - 1$ neighbours in $X_{i-1}$. Let $r = \frac{k}{2}$. It suffices to show that, for $2 \leq i \leq n$, we can compute in $O(1)$ time a partition $\{Y_1, \ldots, Y_r\}$ of $X_i$, where $G[Y_s]$ is 1-degenerate for $s = 1, \ldots, r$, if we have as input such a partition of $X_{i-1}$. We note first that finding a partition of $X_1$ is trivial. Suppose $i > 1$ and let $\{Z_1, \ldots, Z_r\}$ be a partition of $X_{i-1}$ where $G[Z_s]$ is 1-degenerate for $s = 1, \ldots, r$. If $v_i$ has more than one neighbour in every $G[Z_s]$, then $v_i$ has at least $\sum_{s=1}^{r} 2 = k$ neighbours in $X_{i-1}$, a contradiction. Hence, $v_i$ has at most one neighbour in at least one set $Z_q$, which we can find in $O(1)$ time since we only need to check the neighbors of $v_i$ in $X_{i-1}$. We put $v_i$ into $Z_q$ to get the desired partition for $X_i$ in $O(1)$ time.

A pair of vertices $x, y$ in $G$ is called an eligible pair if $x$ and $y$ are at distance exactly two in $G$ and $G - \{x, y\}$ is connected. The proof of the next lemma makes use of the following result of Lovász.

**Lemma 2** ([11]). Let $G$ be a 2-connected graph that is not complete or a cycle. Then an eligible pair of $G$ can be found in $O(n+m)$ time.

**Lemma 3.** Let $k \geq 3$, and let $G \neq K_{k+1}$ be a 2-connected $k$-regular graph. Then a forest partition of $G$ can be found in $O(n+m)$ time.

**Proof.** By Lemma 2, we can find in $O(n+m)$ time an eligible pair of vertices $x, y$ in $G$. So there is a common neighbour of $x$ and $y$ in $G$ that we denote $v$. Let $G'$ be the graph obtained from $G$ by identifying $x$ and $y$ into a new vertex $z$, and let $z_1, \ldots, z_t$ denote the neighbours of $z$ distinct from $v$ that are common neighbours of $x$ and $y$ in $G$. We first demonstrate that a $(k-1)$-degenerate ordering of $G'$ starting at $z$ can be found in $O(n)$ time with the additional property that each $z_i$ has at most $k - 2$ neighbours earlier in the ordering. The proof of this statement is almost entirely contained in the proof [12], Lemma 9, but we repeat it for completeness.

By definition of an eligible pair, the graph $G^* = G' - \{z\}$ is connected. This means that, in $G^*$, there is a path that links $v$ to every neighbour of $z$. So, in $G'$, we start by successively deleting vertices of each path starting from $v$ towards the neighbour of $z$ on that path. Afterwards, we successively delete the remaining vertices distinct from $z$ of degree at most $k - 1$ until
is the only vertex left to be deleted. Because each \( z_i \) has degree precisely \( k - 1 \) in \( G' \), a routine check confirms that this procedure gives the claimed degenerate ordering.

We now proceed to find a forest partition \( \mathcal{F}' \) of \( G' \) in \( O(n+m) \) time with the property that \( z \) and \( z_i \) belong to different forests for each \( i = 1, \ldots, t \).

Define the sets \( X_i, Y_i \) and \( Z_i \) as in the proof of Lemma \( \mathcal{F} \). We remark that since the argument will be almost identical to the proof of Lemma \( \mathcal{F} \), we only highlight the differences. First, we put \( z \in Z_1 \). Second, each \( z_i \) has at most one neighbour in at least one \( Z_q \) for some \( q \geq 2 \) since otherwise \( z_i \) has at least \( k - 1 \) neighbours in \( X_i - 1 \). Thus, if we put \( z_i \in Z_q \), then we indeed obtain the desired partition \( \mathcal{F}' \).

Now, since every common neighbour of \( x \) and \( y \) in \( G \) is not a member of \( Z_1 \), the graph \( Z'_1 = Z_1 \cup \{x, y\} \setminus \{z\} \) is also a forest and, if \( k \) is odd, can be insured to be an independent set. Therefore, \( \mathcal{F} = (\mathcal{F}' \setminus Z_1) \cup Z'_1 \) is a forest partition of \( G \), which completes the proof of the lemma.

We are now able finish the proof of Theorem \( \mathcal{F} \).

**Proof of Theorem \( \mathcal{F} \).** Since a forest can be partitioned into two independent sets in \( O(n+m) \) time, the theorem will follow if we can show that we can find a forest partition of \( G \) in \( O(n+m) \) time. We first check in \( O(n+m) \) time whether \( G \) is \( k \)-regular. If \( G \) is not \( k \)-regular, we apply Lemma \( \mathcal{F} \). If \( G \) is \( k \)-regular, we compute in \( O(n+m) \) time a block decomposition of \( G \) (by using, for example, a depth-first search algorithm). If \( G \) is 2-connected (that is, \( G \) contains exactly one block), we can apply Lemma \( \mathcal{F} \).

So we can assume that \( G \) is \( k \)-regular and not 2-connected. We consider an end block \( B \) of \( G \), and let \( u \) be the cut vertex of \( G \) that is contained in \( B \). Let \( G' = G - B \), and let \( B' = B - \{v\} \). Note that \( G' \) and \( B' \) are not \( k \)-regular so we can apply Lemma \( \mathcal{F} \) to find forest partitions \( \mathcal{F}' \) and \( \mathcal{F}'' \) of \( G' \) and \( B' \), respectively. We first try to pair a forest \( F' \) in \( \mathcal{F}' \) with a forest \( F'' \) in \( \mathcal{F}'' \) in such a way that both \( F' \) and \( F'' \) contain some neighbour of \( v \) and \( F' \) and \( F'' \) are either both independent sets or both forests. If this is possible, we then arbitrarily pair the remaining forests in \( \mathcal{F}' \) with those in \( \mathcal{F}'' \) (of course, pairing the independent sets with one another). For each such pair \( (A, B) \), we let \( A \cup B \) be a forest in \( \mathcal{F}^* \). Suppose that \( \mathcal{F}^* \) contains a forest \( F^* \) that contains at most one neighbour \( u \) and at most one neighbour \( w \) of \( v \) such that \( u \in V(G') \) and \( w \in V(B') \). Then \( \mathcal{F} = (\mathcal{F}^* \setminus F^*) \cup (F^* \cup \{v\}) \) is a forest partition of \( G \). Similarly, if \( \mathcal{F}^* \) contains a forest that contains three
or more neighbours of $v$ or, if $k$ is odd, an independent set that contains two 
or more neighbours of $v$, then, by the bound on the number of forests and 
and independent sets, we can again find a forest in $F^*$ that contains at most one 
neighbours $u$ and at most one neighbour $w$ of $v$ such that $u \in V(G')$ and 
and $w \in V(B')$. Therefore, the theorem is proved as long as we can find $F'$ and 
and $F''$. So the only cause of difficulty is when $k$ is odd and $v$ has exactly one 
neighbour in $G'$ that, in fact, also belongs to the independent set of $F'$. In 
this case, we must try to find a more suitable forest partition $F''$ of $B'$ that 
we can combine with $F'$. 

We show that we can compute in $O(n + m)$ time a $(k - 1)$-degenerate 
ordering $\mathcal{O}$ of $B'$ starting at some neighbour of $v$, say $w$, in $B'$. Let $u$ be a 
vertex in $B \setminus (N(v) \cup \{v\})$. Since $B$ is an end block of $G$, it is 2-connected. 
Thus, by Menger's theorem, there are at least two internally disjoint paths in 
$B$ linking $u$ and $v$. Clearly, at least one of these paths contains some 
neighbour of $v$ distinct from $u$. Hence, we can successively delete vertices of 
degree at most $k - 1$ in $B'$, starting with every neighbour of $v$ in $B'$ distinct 
from $w$ towards every other vertex distinct from $w$ until, finally, $w$ is the last 
vertex deleted. This gives us the claimed ordering $\mathcal{O}$. Now, by an analogous 
as in the proof of Lemma 1 we can obtain from $\mathcal{O}$ a forest partition $F''$ of $B'$ such that $w$ belongs to the independent set of $F''$. As we have guaranteed 
that at least two vertices in the neighbourhood of $v$ belong to the independent 
set, the theorem follows. 

\section{Hardness for large maximum degree}

In this section, we prove Theorem 2. This will be done by exhibiting poly-
nomial time reductions from new variants of SAT, where each reduction 
“corresponds” to some combination of values of $p$ and $q$ in a $(p, q)$-partition 
of the graph. Let us first introduce these new variants of SAT.

Recall that an instance $(X, \mathcal{C})$ of SAT consists of a set of boolean vari-
ables $X = \{x_1, \ldots, x_n\}$ and a collection of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$, such 
that each clause is a disjunction of literals, where a literal is either $x_i$ or its 
negation $\neg x_i$ for some $x_i \in X$. A function $g : X \rightarrow \{\text{true, false}\}$ is called a 
\textit{satisfying truth assignment} if $\theta = C_1 \land \cdots \land C_m$ is evaluated to true under 
g. A SAT instance $(X, \mathcal{C})$ is called an RSAT instance if each clause is a 
disjunction of either exactly two literals or exactly four literals, and each 
literal appears at most twice in $\mathcal{C}$. A clause in $\mathcal{C}$ is called a $k$-\textit{clause} for
some positive integer $k$ if it contains exactly $k$ literals. We will reduce from the following variants of RSAT.

**NAE-RSAT**
- **Instance**: An instance $(X, C)$ of RSAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment with at least one true literal and at least one false literal per clause?

**EXACT-RSAT**
- **Instance**: An instance $(X, C)$ of RSAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment with exactly one true literal per clause?

**ALL-RSAT**
- **Instance**: An instance $(X, C)$ of RSAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment with at least one true literal per 4-clause and exactly one true literal per 2-clause?

**Lemma 4.** Each of the above variants of RSAT are $\text{NP}$-complete.

Before we prove the lemma, we require the following well-known $\text{NP}$-complete decision problems. Recall that an instance $(X, C)$ of SAT is a $k$-SAT instance if every clause in $C$ is a $k$-clause.

**$k$-SAT**
- **Instance**: An instance $(X, C)$ of $k$-SAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment?

**NAE $k$-SAT**
- **Instance**: An instance $(X, C)$ of $k$-SAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment with at least one true literal and at least one false literal per clause?

**1-in-$k$ SAT**
- **Instance**: An instance $(X, C)$ of $k$-SAT.
- **Question**: Does $(X, C)$ have a satisfying truth assignment with exactly one true literal per clause?

**Proof of Lemma** It is clear that each of the above variants of RSAT are in $\text{NP}$. We simultaneously demonstrate that they are $\text{NP}$-hard by exhibiting
a generic reduction from an instance \((X, \mathcal{C})\) of SAT in which every clause contains exactly four literals. We remark that our proof is identical to the proof that the variant of 3-SAT in which every literal appears in at most two clauses is \(\mathsf{NP}\)-hard. It will merely suffice to make a few additional observations.

Let \(\theta = C_1 \land \cdots \land C_m\). If a variable \(y \in X\) appears (as \(y\) or \(\neg y\)) in \(k > 1\) clauses, then we replace \(y\) with a set of new variables \(y_1, \ldots, y_k\) in the following way: we replace the first occurrence of \(y\) with \(y_1\), the second occurrence of \(y\) with \(y_2\), etc. If some of these occurrences are negated then we replace those occurrences with the negated versions of the new variables. We repeat this for each variable that appears in more than one clause. Next we link the new variables for \(y\) to each other with a set of clauses \((y_1 \lor \neg y_2), (y_2 \lor \neg y_3), \ldots, (y_k \lor \neg y_1)\). We denote by \((X', \mathcal{C}' = \{C'_1, \ldots, C'_m\})\) the resulting instance, and let \(\theta' = C'_1 \land \cdots \land C'_m\). Notice that every literal appears in at most two clauses of \(\mathcal{C}'\). Moreover, in every satisfying truth assignment \(g'\) of the variables in \(X'\), we have \(g'(y_1) = \cdots = g'(y_k)\) for every \(y \in X\).

Hence every 2-clause in \(\mathcal{C}'\) has exactly one true literal. Therefore, for every satisfying truth assignment to the variables in \(X\) and \(X'\),

- \(\theta\) has at least one true literal and at least one false literal per clause if and only if \(\theta'\) has at least one true literal and at least one false literal per clause;
- \(\theta\) has exactly one true literal per clause if and only if \(\theta'\) has exactly one true literal per clause;
- \(\theta\) has at least one true literal per clause if and only if \(\theta'\) has least one true literal per 4-clause and exactly one true literal per 2-clause.

Given that 4-SAT, NAE 4-SAT and 1-in-4 SAT are \(\mathsf{NP}\)-complete, it follows that ALL-RSAT, NAE-RSAT and EXACT-RSAT are \(\mathsf{NP}\)-hard. This completes the proof.

**Proof of Theorem 2.** The problem is clearly in \(\mathsf{NP}\). To show that it is \(\mathsf{NP}\)-hard, we simultaneously exhibit two reductions from a generic instance of RSAT.

Given an instance \((X, \mathcal{C})\) of RSAT we construct two graphs \(G\) and \(H\) of maximum degree \(k \geq 5\) in the following way.
Figure 1: The graph $S(x, 5)$.

Figure 2: The clause gadget that connects $S(x, 5)$ and $S(y, 5)$ via black edges, where literal $\ell$ corresponds to variable $x$ and literal $\ell'$ to variable $y$. Gray edges are edges of $S(x, 5)$ or $S(y, 5)$ and black edges are edges of $G$ and $H$. 
Figure 3: The clause gadget that connects $S(x_1, 5), \ldots, S(x_4, 5)$ via black and dashed edges, where literal $\ell_i$ corresponds to variable $x_i$ for $1 \leq i \leq 4$. Gray edges are edges of $S(x_i, 5)$ for $1 \leq i \leq 4$. Black edges are edges of $G$ and $H$ and dashed edges are edges of $H$.

First, let $f : \bigcup_{C \in \mathcal{C}} C \rightarrow \{1, 2, 3, 4\}$ be a function that associates an integer between 1 and 4 to every literal such that if literal $\ell$ is the $j$th occurrence of $y$ for some $y \in X$, then $f(\ell) = j$, while if $\ell$ is the $j$th occurrence of $\neg y$ for some $y \in X$, then $f(\ell) = 2 + j$. To each variable $x \in X$, we associate a variable gadget $S(x, k)$ as illustrated in Figure 1 for $S(x, 5)$. It is a graph with 12 vertices $a_{x,i}, a'_{x,i}, a^*_{x,i}$ for $i = 1, \ldots, 4$, six vertices $\hat{x}, \hat{x}', \hat{x}^*, \hat{x}, \hat{x}', \hat{x}^*$ and six copies $K(\hat{x}), K(\hat{x}'), K(1), K(2), K(3), K(4)$ of a complete graph on $k - 2$ vertices. It has edges

- $a'_{x,1} \hat{x}', a^*_{x,1} \hat{x}^*, a'_{x,2} \hat{x}, a^*_{x,2} \hat{x}', a'_{x,3} \hat{x}', a^*_{x,3} \hat{x}^*, a'_{x,4} \hat{x}, a^*_{x,4} \hat{x}', \hat{x} \hat{x}', \hat{x}' \hat{x}'$

and edges from

- each of $a_{x,i}, a'_{x,i}, a^*_{x,i}$ to every vertex of $K(i)$,
- each of $\hat{x}, \hat{x}', \hat{x}^*$ to every vertex of $K(\hat{x})$, and
- each of $\hat{x}, \hat{x}', \hat{x}^*$ to every vertex of $K(\hat{x}')$.

For each 2-clause in $\mathcal{C}$ with literals $\ell$ and $\ell'$ corresponding, respectively, to variables $x$ and $y$ for some $x, y \in X$, we construct a 2-clause gadget that
connects $S(x, k)$ and $S(y, k)$ by adding edges $a_{x,f(\ell)}a_{y,f(\ell')}$, $a'_{x,f(\ell)}a'_{y,f(\ell')}$ in both $G$ and $H$, as illustrated in Figure 3. We refer to $a_{x,f(\ell)}$ and $a_{y,f(\ell')}$ as the special vertices of the gadget.

For each 4-clause in $C$ with literals $\ell_1, \ldots, \ell_4$ corresponding, respectively, to variables $x_1, x_2, x_3, x_4$ for some $x_1, x_2, x_3, x_4 \in X$, we construct a 4-clause gadget that connects $S(x_1, k), \ldots, S(x_4, k)$ in the following way (see Figure 3 for an illustration):

- add edges $a_{x_4,f(\ell_4)}a_{x_1,f(\ell_1)}$ and $a_{x_i,f(\ell_i)}a_{x_{i+1},f(\ell_{i+1})}$ for $1 \leq i \leq 3$ in both $G$ and $H$, and
- add edges $a'_{x_1,f(\ell_1)}a'_{x_3,f(\ell_3)}a'_{x_2,f(\ell_2)}a_{x_4,f(\ell_4)}$ only in $H$.

We again refer to $a_{x_i,f(\ell_i)}$ as a special vertex of the gadget for $1 \leq i \leq 4$.

This completes the construction of $G$ and $H$. It can be readily verified that $G$ and $H$ have maximum degree $k \geq 5$. We require one more definition. Given a $(p, q)$-partition $(\mathcal{P}, \mathcal{D})$ of $G$ (or $H$), we say that a vertex $v$ of $G$ (or $H$) has colour $p$ if $v \in \mathcal{P}$ and colour $q$ if $v \in \mathcal{D}$.

**Claim 1.** In every $(p, q)$-partition $(\mathcal{P}, \mathcal{D})$ of $S(x, k)$, either

- $a_{x,1}, a_{x,2}$ have colour $p$ and $a_{x,3}, a_{x,4}$ have colour $q$, or
- $a_{x,1}, a_{x,2}$ have colour $q$ and $a_{x,3}, a_{x,4}$ have colour $p$.

**Proof of claim.** We must only show that if $a_{x,1}$ has colour $p$, then $a_{x,2}$ has colour $p$ and $a_{x,3}, a_{x,4}$ have colour $q$.

Suppose that $a_{x,1}$ has colour $p$. Recall that the set of neighbours of $a_{x,1}$ in $S(x, k)$ induce a complete graph $K^{(1)}$ on $k - 2$ vertices. Thus, exactly $p$ vertices of $K^{(1)}$ have colour $p$ and the other $q + 1$ vertices of $K^{(1)}$ have colour $q$, which, in turn, implies that $a'_{x,1}$ and $a^*_{x,1}$ have colour $p$.

If $\tilde{x}$ has colour $p$, then again we find that $\tilde{x}', \tilde{x}^*$ and exactly $p$ vertices of $K^{\tilde{x}}$ have colour $p$ while the other $q + 1$ of $K^{\tilde{x}}$ have colour $q$. But then vertices so far of colour $p$ contain a subgraph of minimum degree $p + 1$, which contradicts that $\mathcal{P}$ is $p$-degenerate. Thus, $\tilde{x}$ must have colour $q$. Reasoning as before, this implies that $\tilde{x}$ and $a_{x,2}$ have colour $p$, which, in turn, implies that $a_{x,3}$ and $a_{x,4}$ have colour $q$. The claim is proved.

We distinguish three cases depending on the values of $p$ and $q$.

**Case 1:** $p = 1$ and $q \geq 2$.  

We show that \((X, C)\) has a satisfying truth assignment with exactly one true literal per 2-clause if and only if \(G\) admits a \((p, q)\)-partition. This implies by Lemma \(4\) that deciding whether \(G\) has a \((p, q)\)-partition with \(p = 1\) and \(q \geq 2\) is \(\text{NP}\)-hard.

Suppose that \((X, C)\) has a satisfying truth assignment with exactly one true literal per 2-clause. For each \(x \in X\) and each literal \(\ell\) corresponding to \(x\), if \(\ell\) is set to true then we assign colour \(q\) to \(a_{x,f(\ell)}\); otherwise we assign colour \(p\) to \(a_{x,f(\ell)}\). Then, mindful of Claim \(1\) one can check that this partial colouring of \(G\) can be extended to a \((p, q)\)-partition of \(G\).

Conversely, suppose that \(G\) admits a \((p, q)\)-partition. For each \(x \in X\) and each literal \(\ell\) corresponding to \(x\), if \(a_{x,f(\ell)}\) has colour \(q\), we set \(\ell\) to true; otherwise we set \(\ell\) to false. By Claim \(1\) this is a valid truth assignment to the variable in \(X\). Notice that at least one special vertex of each 4-clause gadget has colour \(q\) given that \(p = 1\), and exactly one special vertex of each 2-clause gadget has colour \(q\) because otherwise a subgraph of colour \(q\) with minimum degree \(q + 1\) or a subgraph of colour \(p\) with minimum degree 2 can be found. This gives us the desired satisfying truth assignment of \((X, C)\) and completes Case 1.

**Case 2:** \(p = 0\) and \(q \geq 2\).

We show that \((X, C)\) has a satisfying truth assignment with exactly one true literal per clause if and only if \(H\) admits a \((p, q)\)-partition. This implies by Lemma \(4\) that deciding whether \(H\) has a \((p, q)\)-partition with \(p = 0\) and \(q \geq 2\) is \(\text{NP}\)-hard.

Suppose that \((X, C)\) has a satisfying truth assignment with exactly one true literal per clause. For each \(x \in X\) and each literal \(\ell\) corresponding to \(x\), if \(\ell\) is set to true then we assign colour \(p\) to \(a_{x,f(\ell)}\); otherwise we assign colour \(q\) to \(a_{x,f(\ell)}\). By Claim \(1\) one can check that this partial colouring of \(H\) can be extended to a \((p, q)\)-partition of every variable gadget and every 2-clause gadget. To see that it can also be extended to a \((p, q)\)-partition of every 4-clause gadget (and therefore to \((p, q)\)-partition of \(H\)), consider a 4-clause gadget with special vertices \(a_{x_1,f(\ell_1)}, \ldots, a_{x_4,f(\ell_4)}\) and suppose without loss of generality that \(a_{x_1,f(\ell_1)}\) has colour \(p\) and \(a_{x_2,f(\ell_2)}, a_{x_3,f(\ell_3)}, a_{x_4,f(\ell_4)}\) have colour \(q\). We extend this colouring to the rest of the clause gadget so that \(a'_{x_1,f(\ell_1)}\) has colour \(p\) and \(a'_{x_2,f(\ell_2)}, a'_{x_3,f(\ell_3)}, a'_{x_4,f(\ell_4)}\) have colour \(q\). It is clear that no two vertices of colour \(p\) are adjacent so it remains to show that vertices of colour \(q\) induce a \(q\)-degenerate subgraph. Notice that vertex \(a'_{x_4,f(\ell_4)}\) has \(q\) neighbours with colour \(q\) since its neighbours that are not in \(K^{(4)}\) have
colour $p$. So, if we first delete $a'_{x_4,f(\ell_4)}$, then delete its neighbours of colour $q$, then $a_{x_4,f(\ell_4)}$, $a_{x_2,f(\ell_2)}$ and $a_{x_3,f(\ell_3)}$ in this order, and so on, we eventually obtain a $q$-degenerate ordering of vertices of colour $q$, as needed.

Conversely, suppose that $H$ has a $(p,q)$-partition. For each $x \in X$ and each literal $\ell$ corresponding to $x$, if $a_{x,f(\ell)}$ has colour $p$, we set $\ell$ to true; otherwise we set $\ell$ to false. By Claim 1 this is a valid truth assignment to the variable in $X$. As in Case 1, exactly one special vertex of each 2-clause has colour $p$. Consider a 4-clause with special vertices $a_{x_1,f(\ell_1)}, \ldots, a_{x_4,f(\ell_4)}$. Exactly one of these special vertices has colour $p$:

- If there were more, then since $p = 0$ they cannot be adjacent; so it must be that $a_{x_i,f(\ell_i)}$ and $a_{x_{i+2},f(\ell_{i+2})}$ have colour $p$ (for some $i = 1, 2$), which implies that $a'_{x_i,f(\ell_i)}$ and $a'_{x_{i+2},f(\ell_{i+2})}$ have colour $p$, but this is impossible since these are adjacent.

- If every one of them has colour $q$, then, by considering edges of the gadget that are in $H$ but not in $G$, one can verify that there is a subgraph of colour $q$ with minimum degree $q + 1$, which is impossible.

This shows that we have a satisfying truth assignment of $(X, \mathcal{C})$ with exactly one true literal per clause. This completes Case 2.

Case 3: $p, q \geq 2$.

By similar arguments as in Cases 1 and 2, it is not difficult to show that a $(X, \mathcal{C})$ has a satisfying truth assignment with at least one false literal and at least one true literal per clause if and only if $H$ admits a $(p,q)$-partition. This implies by Lemma 4 that deciding whether $H$ has a $(p,q)$-partition with $p,q \geq 2$ is $\text{NP}$-hard. This completes Case 3 and thus the proof of the theorem.

4 Concluding remarks

We remark that a straightforward adaptation of the proof of Lemma 1 leads us to the following statement.

**Proposition 4.1.** Given integers $s \geq 2$ and $p_1, \ldots, p_s \geq 0$, a $(p_1, \ldots, p_s)$-partition of a graph $G$ with maximum degree $k \geq 3$ that is not $k$-regular can be found in $O(n + m)$-time as long as $\sum_{i=1}^{s} p_i \geq k - s$. 

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It is therefore unlikely that the time complexity increases by more than a factor of \( n \) in the outstanding case where \( G \) is \( k \)-regular.

**Conjecture 4.1.** Let \( G \) be a connected graph with maximum degree \( k \geq 3 \) distinct from \( K_{k+1} \). For every \( s \geq 2 \) and \( p_1, \ldots, p_s \geq 0 \) such that \( p_1 + \cdots + p_s \geq k - s \), a \((p_1, \ldots, p_s)\)-partition of \( G \) can be found in \( O(n^2) \) time.

We make a few remarks on the case \( s \geq 3 \) with \( \sum_{i=1}^{s} p_i < k - s \). A simple application of Proposition 4.1 leads us to the following statement.

**Proposition 4.2.** Given non-negative integers \( p, q, p_1, p_2, \ldots, p_t, q_1, \ldots, q_t \) such that \( \sum p_i = p - t \) and \( \sum q_i = q - t' \), if a graph \( G \) is \((p, q)\)-partitionable, then \( G \) is also \((p_1, \ldots, p_t, q_1, \ldots, q_t)\)-partitionable.

This statement can be understood to mean (although rather imprecisely) that the complexity of Problem 1 in the situation when \( \sum_{i=1}^{s} p_i < k - s \) does not increase as \( s \) increases. One would therefore hope that the problem is \( \text{NP} \)-complete whenever \( s \) is as large as possible (that is, when a partition into independent sets is sought) as this would suggest that it is most likely \( \text{NP} \)-complete for every \( s \geq 2 \). As it happens, however, when \( s \) is of maximum value, the problem is tractable as long as \( k \) is not too small and \((k - s) - \sum_{i=1}^{s} p_i \) is not very large [13]. This is perhaps indicative that determining the frontier between tractability and hardness for every value of \( s \) in Problem 1 will be a difficult task.

**Acknowledgements**

This work received support from the Research Council of Norway via the project CLASSIS.

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