Generalized Mixtures of Exponential Distribution and Associated Inference

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Abstract: A new generalization of the exponential distribution, namely the generalized mixture of exponential distribution, is introduced. Some of its basic properties, such as hazard function, moments, order statistics, mean deviation, measures of uncertainly, and reliability probability, are studied. Three different estimation methods are investigated by the maximum likelihood estimator, least-square estimator, and weighted least-square estimator. The performances of the estimators are assessed by simulation studies. Real-world applications of the proposed distribution are explored, and data fitting results show that the new distribution performs better than its competitors.

Keywords: generalized mixture of exponential distribution; reliability probability; maximum likelihood estimator; weighted least-square estimator

1. Introduction

Among the parametric models, exponential distribution is perhaps the most widely applied statistical distribution in several fields and plays an important role in the statistical theory of reliability and lifetime analysis. Based on this reason, statisticians have been interested in defining new classes of univariate distributions by adding one or more shape parameters to provide greater flexibility in modeling real data in many applied fields. Gupta and Kundu [1] studied the generalized exponential distribution and used it as an alternative to gamma or Weibull distribution in many situations. Gupta and Kundu [2] used the idea of Azzalini [3], introducing a new class of weighted exponential (WE) distributions, and Kharazmi et al. [4] extended it into the generalized weighted exponential (GWE) distribution. Nadarajah and Haghighi [5] discussed a new two-parameter generalization of the exponential distribution, which had its mode at zero and allowed increasing, decreasing, and constant hazard rates.

On the other hand, generalizations of exponentiated type distributions can be obtained from the class of generalized beta distributions, in particular after the works of Eugene et al. [6] Nadarajah and Kotz [7] introduced the beta exponential distribution, generated from the logit of a beta random variable. Barreto-Souza et al. [8] discussed beta generalized exponential distribution, which includes the beta exponential and generalized exponential distributions as special cases. A generalization of the exponentiated Frenchet distribution, called the beta Frenchet distribution, was studied by Barreto-Souza et al. [9]. Ristic and Balakrishnan [10] proposed the gamma exponential distribution generated by gamma random variables. Being of a similar methodology, many X-family exponential distributions were investigated recently. These include the Weibull exponential (WED) distributions which were introduced by Oguntunde et al. [11], Marshall–Olkin generalized exponential distributions which were defined by Ristic and Kundu [12], Kumaraswamy Marshall–Olkin exponential distributions which were given by George and Thobias [13].
and generalized extended exponential-Weibull (GExtEW) distributions which were proposed by Shakhatreh et al. [14].

In this paper, a new class of generalized mixture exponential (GME) distribution is introduced, which has the exponential and WE distributions as its submodels. In order to motivate interest, let us first present the definition of the generalized skew normal distribution introduced by Kumar and Anusree [15]. A random variable $Z$ is said to have a generalized skew normal distribution if its probability density function (pdf) is of the following form,

$$h(z, \lambda, \beta) = \frac{2}{2 + \beta} f(z) (1 + \beta F(\lambda z)),$$

where $f(z) = \phi(z)$, $F(\lambda z) = \Phi(\lambda z)$, $\lambda \in \mathbb{R}$ and $\beta > -2$. In fact, the correct values of $\beta$ should be $\beta \geq -1$, which has been discussed in Tian et al. [16].

The rest of the article is organized as follows. The GME distribution is introduced in Section 2. Some important properties of GME distributions, such as cumulative distribution function (cdf), hazard function, mean deviations, order statistics, measure of uncertainly, and reliability probability, are discussed in Section 3. Three different estimation methods are studied in Section 4. Simulations are conducted to investigate and compare the performances of the proposed estimation methods in Section 5. Two real data sets are analyzed for illustrating the usefulness of the proposed GME distribution in Section 6. Some conclusions are presented in Section 7.

2. Generalized Mixture Exponential Distribution

The GME distribution offers more flexible distributions with applications in lifetime modeling, which is defined as follows.

**Definition 1.** A random variable $X$ is said to have a GME distribution if its pdf is of the following form,

$$f(x; \lambda, \alpha, \beta) = \frac{(\alpha + 1)\lambda}{\alpha + 1 + a\beta} e^{-\lambda x} [1 + \beta(1 - e^{-\alpha \lambda x})], \quad x > 0,$$

where $\alpha > 0$ is the scale parameter, $\lambda > 0$ and $\beta \geq -1$ are the shape parameters, and we denote it as $X \sim \text{GME}(\lambda, \alpha, \beta)$.

**Remark 1.**

(i) For $\beta = 0$, or $\alpha \to 0$, or $\alpha \to \infty$, $\text{GME}(\lambda, \alpha, \beta)$ is reduced into exponential distribution with parameter $\lambda$ namely, $E(\lambda)$.

(ii) For $\beta = -1$, $\text{GME}(\lambda, \alpha, \beta)$ is reduced into $E(\lambda(\alpha + 1))$.

(iii) For $\beta \to \infty$, $\text{GME}(\lambda, \alpha, \beta)$ is reduced into $WE(\lambda, \alpha)$.

For different values of $\lambda, \alpha, \beta$, the pdfs of $\text{GME}(\lambda, \alpha, \beta)$ are presented in the Figure 1, which indicate that the GME distribution can generate distributions with various shapes.

**Proposition 1.** The cdf, the survival function and the hazard function of $X \sim \text{GME}(\lambda, \alpha, \beta)$ are given by

$$F(x; \lambda, \alpha, \beta) = 1 + \frac{\beta e^{-(\alpha + 1)\lambda x}}{\alpha + 1 + a\beta} - \frac{(\alpha + 1)(\beta + 1)e^{-\lambda x}}{\alpha + 1 + a\beta},$$

$$h(x; \lambda, \alpha, \beta) = \frac{(\alpha + 1)\lambda[1 + \beta(1 - e^{-\alpha \lambda x})]}{(\alpha + 1 + a\beta) + \beta(1 - e^{-\lambda x})},$$

$$S(x; \lambda, \alpha, \beta) = \frac{(\alpha + 1)(\beta + 1)e^{-\lambda x}}{\alpha + 1 + a\beta} - \frac{\beta e^{-(\alpha + 1)\lambda x}}{\alpha + 1 + a\beta}.$$
Proof of Proposition 1. According to the Equation (1), we have

\[ F(x; \lambda, \alpha, \beta) = \frac{\lambda(\alpha + 1)}{\alpha + 1 + \alpha \beta} \left[ (1 + \beta) \int_0^x e^{-\lambda t} dt - \beta \int_0^x e^{-\lambda \alpha t} dt \right] = 1 - e^{-\lambda x} + \frac{\beta}{\alpha + 1 + \alpha \beta} \left[ e^{-\lambda(1+\alpha)x} - e^{-\lambda x} \right]. \]

Therefore,

\[ h(x; \lambda, \alpha, \beta) = \frac{f(x; \lambda, \alpha, \beta)}{1 - F(x; \lambda, \alpha, \beta)} = \frac{(\alpha + 1)\lambda \left[ 1 + \beta \left( 1 - e^{-\lambda x} \right) \right]}{(\alpha + 1 + \alpha \beta) + \beta \left( 1 - e^{-\lambda x} \right)}. \]
\[ S(x; \lambda, \alpha, \beta) = 1 - F(x; \lambda, \alpha, \beta) = \frac{(\alpha + 1)(\beta + 1)e^{-\lambda x}}{\alpha + 1 + \alpha \beta} - \frac{\beta e^{-(\alpha + 1)\lambda x}}{\alpha + 1 + \alpha \beta}. \]

This ends the proof of Proposition 1. \qed

Figure 2 shows that the GME distribution produces flexible hazard rate shapes, such as decreasing, increasing, and stable.

![Figure 2](image-url)

**Figure 2.** The hazard function curves for different values of parameter in \( \text{GME}(\lambda, \alpha, \beta) \).

### 3. General Properties of the GME Distribution

In what follows, we discuss various properties associated with the proposed distribution.

**Proposition 2.** The shapes of density function of \( X \sim \text{GME}(\lambda, \alpha, \beta) \) can be characterized as follows,

(i) \( f(x) \) is monotone decreasing, if \( -1 \leq \beta \leq 0 \) or \( 0 < \alpha \beta \leq 1 \),

(ii) \( f(x) \) is unimodal, if \( \alpha \beta > 1 \).

**Proof of Proposition 2.** The derivatives of \( f(x) \) are obtained by Equation (1),

\[ f'(x) = \lambda^2 (\alpha + 1) e^{-\lambda x} \left( \frac{\beta (\alpha + 1)}{\beta + 1} e^{-a \lambda x} - 1 \right). \]

(i) if \( -1 \leq \beta \leq 0 \), we have \( \frac{\lambda^2 (\alpha + 1) e^{-\lambda x}}{\alpha + 1 + \alpha \beta} > 0 \), and \( (\alpha + 1)\beta \leq 0 \), thus,

\[ \frac{\lambda^2 (\alpha + 1) e^{-\lambda x}}{\alpha + 1 + \alpha \beta} \left( \frac{\beta (\alpha + 1)}{\beta + 1} e^{-a \lambda x} - 1 \right) < 0; \]

if \( 0 < \alpha \beta \leq 1 \), we get \( e^{-a \lambda x} < 1 \) and \( \frac{\beta (\alpha + 1)}{\beta + 1} < 1 \), thus, \( \left[ \frac{\beta (\alpha + 1)}{\beta + 1} e^{-a \lambda x} - 1 \right] < 0. \)

(ii) Setting \( f'(x) = 0 \), we have \( x_0 = -\frac{1}{\alpha \lambda} \log \frac{\beta + 1}{\beta + \alpha \beta} \), and \( f''(x_0) > 0 \). Thus, if \( \alpha \beta > 1 \), we have \( f(x) \) is monotone increasing on \( 0 < x < x_0 \) and monotone decreasing on \( x > x_0 \). Thus, \( f(x) \) is unimodal.
Let $X$ be a random variable. We study the entropy measures for $X$ in the fields of communication theory, statistical physics, and probability theory. In particular, we consider the two properties:

Proposition 2. The two properties in Proposition 2 are actually exclusive.

Proof of Proposition 2. According to Equation (1) and the definition of moment generating function,

$$
M_X(t) = \frac{\lambda(a+1)}{\alpha+1+\alpha\beta} \left(1 + \frac{\beta}{\lambda - t} - \frac{\beta}{\lambda \alpha + \lambda - t}\right), \quad t < \lambda.
$$

This ends the proof of Proposition 2.

Corollary 1. Let $X \sim \text{GME}(\lambda, \alpha, \beta)$, the first four moments of $X$ are

$$
E[X] = \frac{(1+a)^2(1+\beta) - \beta}{\lambda(a+1+\alpha\beta)(1+\alpha)}, \quad E[X^2] = \frac{2(1+a)^3(1+\beta) - 2\beta}{\lambda^2(a+1+\alpha\beta)(1+\alpha)^2},
$$

$$
E[X^3] = \frac{6(1+a)^4(1+\beta) - 6\beta}{\lambda^3(a+1+\alpha\beta)(1+\alpha)^3}, \quad E[X^4] = \frac{24(1+a)^5(1+\beta) - 24\beta}{\lambda^4(a+1+\alpha\beta)(1+\alpha)^4}.
$$

Proposition 4. Let $X \sim \text{GME}(\lambda, \alpha, \beta)$ and $\mu = E[X]$, then the mean deviation about the mean of $X$ is given by

$$
D(\mu) = \frac{2(a+1+\alpha\beta+\beta)}{\lambda(a+1+\alpha\beta)} e^{-\lambda\mu} - \frac{2\beta}{\lambda(a+1+\alpha\beta)(1+\alpha)} e^{-\lambda\mu(1+\alpha)}.
$$

Proof of Proposition 4. According to Equation (1) and $D(\mu) = E[|X - \mu|]$, with $\mu = E[X]$, we have

$$
D(\mu) = \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x - \mu)f(x)dx
$$

$$
= \mu \left( \int_0^\mu f(x)dx - \int_\mu^\infty f(x)dx \right) + \left( - \int_0^\mu x f(x)dx + \int_\mu^\infty x f(x)dx \right)
$$

$$
= \mu (2F(\mu) - 1) + \left( - \int_0^\mu x f(x)dx \right)
$$

$$
= \frac{2\beta}{\lambda(a+1+\alpha\beta)(1+\alpha)} \left(1 - e^{-\lambda\mu(1+\alpha)}\right) - \frac{2(a+1+\alpha\beta+\beta)}{\lambda(a+1+\alpha\beta)} - (1 - e^{-\lambda\mu}) + 2\mu
$$

$$
= \frac{2(a+1+\alpha\beta+\beta)}{\lambda(a+1+\alpha\beta)} e^{-\lambda\mu} - \frac{2\beta}{\lambda(a+1+\alpha\beta)(1+\alpha)} e^{-\lambda\mu(1+\alpha)}.
$$

This ends the proof of Proposition 4.

The entropy of a random variable is a measure of uncertainty, which is an important topic in the fields of communication theory, statistical physics, and probability theory. In the following, we study the entropy measures for $X \sim \text{GME}(\lambda, \alpha, \beta)$. 

This ends the proof of Proposition 2. □

Remark 2. The two properties in Proposition 2 are actually exclusive.
Proposition 5. Let $X \sim \text{GME}(\lambda, \alpha, \beta)$, then the Shannon entropy, $S(x)$, and Renyi entropy, $R(x)$, of $X$ are given by

$$S(x) = -\log \left( \frac{\lambda(a+1)}{a+1+a\beta} \right) - \left( 1 + \frac{1}{a} \right) \log(1 + \beta) + \frac{\log(\beta)}{\alpha} - \frac{\log(u)}{\alpha} - \log(1 - u),$$

$$R_\gamma(x) = \frac{1}{1 - \gamma} \log \left[ \int_0^\infty f(x; \lambda, \alpha, \beta) dx \right] - \frac{1}{1 - \gamma} \log \left( \frac{\lambda(a+1)}{a+1+a\beta} \right) \int_0^\infty e^{-\lambda x} (1 + \beta e^{-\lambda x}) \gamma \beta dx$$

$$= \frac{1}{1 - \gamma} \log \left[ \int_0^\infty f(x; \lambda, \alpha, \beta) dx \right] - \frac{1}{1 - \gamma} \log \left( \frac{\lambda(a+1)}{a+1+a\beta} \right) \int_0^\infty e^{-\lambda x} (1 + \beta e^{-\lambda x}) \gamma \beta dx$$

$$\times \int_0^\infty \left( \frac{\beta(1 + \beta)}{1 + \beta} e^{-\lambda x} \right)^2 \left( 1 - \frac{\beta(1 + \beta)}{1 + \beta} e^{-\lambda x} \right) \gamma \beta dx$$

$$= \frac{1}{1 - \gamma} \log \left( \frac{\lambda(a+1)}{a+1+a\beta} \right) + \left( 1 + \frac{\gamma}{\alpha} \right) \log(1 + \beta) - \frac{\gamma}{\alpha} \log(\beta) - \log(\lambda a)$$

$$+ \log \left( B \left( \frac{\beta}{1 + \beta}, \frac{\gamma}{\alpha}, \gamma + 1 \right) \right).$$

The Shannon entropy $S(x)$ is the limiting value of $R_\gamma(x)$ as $\gamma \to 1$ and, thus, the results are obtained.

This ends the proof of Proposition 5. \(\square\)

In the next proposition, we study the probability that one of the two independent GME random variables exceeds the other, which is named as the reliability probability.

Proposition 6. Suppose two independent random variables $X$ and $Y$ follow $\text{GME}(\lambda, \alpha, \beta)$, then the reliability probability is given by

$$P(X > Y) = \frac{(1 + a)^2(1 + \beta)^2}{2(1 + a + a\beta)^2} + \frac{\lambda \beta^2}{2(1 + a + a\beta)^2} - \frac{\beta(1 + \beta)(1 + a)^2}{(2 + a)(1 + a + a\beta)^2} - \frac{\beta(1 + \beta)(1 + a)}{(2 + a)(1 + a + a\beta)^2}.\]$$

Proof of Proposition 6. Let $Z = Y - X$ and $X = Z$, the joint density function of $X$ and $Z$ is obtained as

$$f(x, z; \lambda, \beta, \alpha) = \frac{\lambda^2(1 + a)^2(1 + \beta)^2 e^{-\lambda(2z + x)}}{(1 + a + a\beta)^2} - \frac{\lambda^2 \beta(1 + \beta)(1 + a)^2 e^{-\lambda(1 + a)z + (2 + a)x}}{(1 + a + a\beta)^2} - \frac{\lambda^2 \beta(1 + \beta)(1 + a)^2 e^{-\lambda z + (2 + a)x}}{(1 + a + a\beta)^2} + \frac{\lambda^2 \beta^2(1 + a)^2 e^{-\lambda(1 + a)z + (2 + a)x}}{(1 + a + a\beta)^2}.$$
Therefore, the marginal density function of \( Z \) is
\[
f(z; \lambda, \beta, \alpha) = \frac{(\alpha + 1)^2(1 + \beta)^2}{(1 + \alpha + \alpha \beta)^2} e^{\lambda z} + \frac{\lambda \beta^2}{2(1 + \alpha + \alpha \beta)^2} e^{\lambda(1 + \alpha) z} - \frac{\beta(1 + \beta)(1 + \alpha)^2}{(2 + \alpha)(1 + \alpha + \alpha \beta)^2} e^{\lambda z} - \frac{\beta(1 + \beta)(1 + \alpha)}{(2 + \alpha)(1 + \alpha + \alpha \beta)^2} e^{\lambda(1 + \alpha) z}.
\]

Thus, the result is obtained by \( P(X > Y) = P(Z < 0) \). This ends the proof of Proposition 6. \( \square \)

Order statistics are fundamental tools in non-parametric statistics and inference. In what follows, we derive an expression for the density function of the \( r^{th} \) order statistic in a random sample size \( n \geq r \) from the GME distribution.

**Proposition 7.** Suppose \( X_1, X_2, \cdots, X_n \) is a random sample from GME(\( \lambda, \alpha, \beta \)). Let \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) denote the corresponding order statistics. Then the pdf and cdf of \( r^{th} \) order statistic, \( X_{r:n} \), \( 1 \leq r \leq n \), are respectively,
\[
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \left[ \frac{(1 + \alpha) \lambda}{\alpha + 1 + \alpha \beta} (e^{-\lambda x} - e^{-(\lambda(1+\alpha)) x}) - e^{-\lambda x} + 1 \right]^{r-1} \times \left[ \frac{(1 + \alpha) \lambda}{\alpha + 1 + \alpha \beta} (e^{-\lambda x} - e^{-(\lambda(1+\alpha)) x}) + e^{-\lambda x} \right]^{n-r} (\alpha + 1) \lambda \alpha e^{-\lambda x} [1 + \beta(1 - e^{-\lambda x})],
\]
\[
F_{r:n}(x) = \sum_{i=r}^{n} \sum_{u=0}^{i-r} (-1)^u \binom{i}{u} \binom{n-r}{i-r} \left[ \frac{(1 + \alpha) \lambda}{\alpha + 1 + \alpha \beta} (e^{-\lambda x} - e^{-(\lambda(1+\alpha)) x}) - e^{-\lambda x} + 1 \right]^{i+u}.
\]

**Proof of Proposition 7.** It is well known that the pdf and cdf of \( X_{r:n} \), \( 1 \leq r \leq n \) are given by
\[
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x),
\]
\[
F_{r:n}(x) = \sum_{i=r}^{n} \binom{n-r}{i-r} [F(x)]^{i-r} [1 - F(x)]^{n-i},
\]
respectively. Thus, the result is obtained directly from Equation (1) and Proposition 1. This ends the proof of Proposition 7. \( \square \)

**Proposition 8.** Let \( X \sim \text{GME}(\lambda, \alpha, \beta) \), the quantile function of GME distribution, \( x_q \), wherein \( 0 < q < 1 \), can be obtained by solving the following equation,
\[
\beta e^{-(\alpha+1)\lambda x_q} - (\alpha + 1)(\beta + 1)e^{-\lambda x_q} = (q - 1)(1 + \alpha + \alpha \beta).
\] (2)

We can see from Equation (2) that there is no closed form of the solution in \( x_q \) and, thus, we have to use numerical techniques to obtain the quantile.

The mean residual life (MRL) function plays a very important role in reliability engineering, survival analysis and many other fields. It represents the period from time \( t \) till the time of failure, and the MRL also represents the expected additional life length for a unit.

**Proposition 9.** Let \( X \sim \text{GME}(\lambda, \alpha, \beta) \), the MRL function of GME distribution, defined as \( \mu_X(t) \), is given by
\[
\mu_X(t) = \frac{(1 + \alpha)^2(1 + \beta)e^{-\lambda t} - \beta e^{-(1+\alpha)\lambda t}}{(1 + \alpha)^2(1 + \beta)\lambda e^{-\lambda t} - (1 + \alpha)\lambda \beta e^{-(1+\alpha)\lambda t}} t > 0.
\]
Proof of Proposition 9. For $t > 0$, we have

$$
\mu_X(t) = E(X - t|X > t) = \int_t^\infty S(x; \lambda, \alpha, \beta)dx
$$

where $S(\cdot)$ is survival function of GME distribution. We know that

$$
\int_t^\infty \left[ \frac{(a + 1)(b + 1)e^{-\lambda x}}{\alpha + 1 + a\beta} - \frac{\beta e^{-(a+1)\lambda x}}{\alpha + 1 + a\beta} \right]dx
$$

Thus, the result is obtained.

This ends the proof of Proposition 9. $\square$

4. Methods of Estimation

In this section, we consider the methods of maximum likelihood, least squares, and weighted least squares to estimate the unknown parameters, $\theta = (\lambda, \alpha, \beta)$, of the GME distribution. Suppose $x_1, x_2, \cdots, x_n$ is a random sample from GME($\lambda, \alpha, \beta$).

4.1. Maximum Likelihood Estimator

The method of maximum likelihood is the most frequently used method for parameter estimation. According to the Equation (1), the likelihood function is calculated as

$$
L(\lambda, \alpha, \beta|x_1, \cdots, x_n) = \left( \frac{(a + 1)}{\alpha + 1 + a\beta} \right)^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} [1 + \beta(1 - e^{-\alpha x_i})].
$$

The log-likelihood function is given by

$$
\ell(\lambda, \alpha, \beta|x_1, \cdots, x_n) = n[\log(a + 1) + \log(\lambda) - \log(a + 1 + a\beta)] - \lambda \sum_{i=1}^{n} x_i
$$

$$
+ \sum_{i=1}^{n} \log[1 + \beta(1 - e^{-\alpha x_i})].
$$

We denote the first partial derivatives of (3) by $\ell_\lambda$, $\ell_\alpha$ and $\ell_\beta$. Setting $\ell_\lambda = 0$, $\ell_\alpha = 0$, and $\ell_\beta = 0$, we have

$$
\ell_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\beta a x_i e^{-\alpha x_i}}{1 + \beta(1 - e^{-\alpha x_i})} = 0,
$$

$$
\ell_\alpha = \frac{n}{\alpha} + \frac{n(1 + \beta)}{\alpha + 1 + a\beta} + \sum_{i=1}^{n} \frac{\beta x_i e^{-\alpha x_i}}{1 + \beta(1 - e^{-\alpha x_i})} = 0,
$$

$$
\ell_\beta = -\frac{n\alpha}{\alpha + 1 + a\beta} + \sum_{i=1}^{n} \frac{1 - e^{-\alpha x_i}}{1 + \beta(1 - e^{-\alpha x_i})} = 0.
$$

The maximum likelihood estimator (MLE) $\hat{\theta}$ of the unknown parameters $\theta$ can be obtained by optimizing the log-likelihood function with respect to the involved parameters. Due to the non-linearity of these equations, the MLEs of parameters can be obtained.
numerically. These estimators can be easily obtained by using the functions from the statistical software R. Fisher information is helpful to get the reference priors for the model parameters. In the following, we observe that the Fisher information is given by

$$I(\theta) = -E \begin{bmatrix} \ell_{\lambda\lambda} & \ell_{\lambda\alpha} & \ell_{\lambda\beta} \\ \ell_{\alpha\lambda} & \ell_{\alpha\alpha} & \ell_{\alpha\beta} \\ \ell_{\beta\lambda} & \ell_{\beta\alpha} & \ell_{\beta\beta} \end{bmatrix},$$

where

$$\ell_{\lambda\lambda} = \frac{n}{\lambda^2} - \sum_{i=1}^{n} \frac{\beta(1 + \beta) {x_i}^2 e^{-2a\lambda x_i}}{[1 + \beta(1 - e^{-a\lambda x_i})]^2},$$

$$\ell_{\alpha\alpha} = -\frac{1}{(\alpha + 1)^2} + \frac{n(1 + \beta)^2}{(1 + \alpha + \alpha \beta)^2} - \sum_{i=1}^{n} \frac{\beta(1 + \beta) \lambda^2 {x_i}^2 e^{-2a\lambda x_i}}{[1 + \beta(1 - e^{-a\lambda x_i})]^2},$$

$$\ell_{\beta\lambda} = \sum_{i=1}^{n} \frac{\alpha x_i e^{-a\lambda x_i}}{[1 + \beta(1 - e^{-a\lambda x_i})]^2} = \ell_{\lambda\beta},$$

$$\ell_{\beta\alpha} = \sum_{i=1}^{n} \frac{\lambda x_i e^{-a\lambda x_i}}{[1 + \beta(1 - e^{-a\lambda x_i})]^2} = \ell_{\alpha\beta},$$

$$\ell_{\alpha\alpha} = \sum_{i=1}^{n} \frac{\beta(1 + \beta)(1 - a\lambda x_i)x_i e^{-a\lambda x_i} - \beta^2 x_i e^{-2a\lambda x_i}}{[1 + \beta(1 - e^{-a\lambda x_i})]^2} = \ell_{\alpha\alpha},$$

$$\ell_{\beta\beta} = \frac{n\alpha^2}{(1 + \alpha + \alpha \beta)^2} - \sum_{i=1}^{n} \frac{(1 - e^{-a\lambda x_i})^2}{[1 + \beta(1 - e^{-a\lambda x_i})]^2}.$$

### 4.2. Least-Square Estimator

Suppose \( F(x_{(i)}) \) denotes the distribution function of the ordered random variables \( x_{(1)} < \cdots < x_{(n)} \). Denote the following function

$$h(\lambda, \alpha, \beta) = \sum_{i=1}^{n} \left[ F(x_{(i)}; \lambda, \alpha, \beta) - \frac{i}{n + 1} \right]^2,$$  \( (4) \)

where \( F(x; \lambda, \alpha, \beta) = \frac{\beta}{\alpha + 1 + \alpha \beta} \left[ e^{-\lambda(1 + \alpha)x} - e^{-\lambda x} \right] - e^{-\lambda x} + 1 \), and the least-square estimator (LS) of \( \theta \) can be obtained by minimizing \( h(\lambda, \alpha, \beta) \). Therefore, \( \hat{\theta} \) can be obtained by solving the following equations,

$$\frac{\partial h(\lambda, \alpha, \beta)}{\partial \lambda} = \sum_{i=1}^{n} \left\{ 2Q_i \left[ \frac{\beta}{\alpha + 1 + \alpha \beta} \left[ - (1 + \alpha) C_i + B_i \right] + B_i \right] \right\} = 0,$$

$$\frac{\partial h(\lambda, \alpha, \beta)}{\partial \alpha} = \sum_{i=1}^{n} \left\{ 2Q_i \left[ \frac{-1 + \beta}{(\alpha + 1 + \alpha \beta)^2} [e^{-\lambda(1 + \alpha)X_{(i)}} - e^{-\lambda x_{(i)}}] - \frac{\beta \lambda}{\alpha + 1 + \alpha \beta} C_i \right] \right\} = 0,$$

$$\frac{\partial h(\lambda, \alpha, \beta)}{\partial \beta} = \sum_{i=1}^{n} \left\{ 2Q_i \left[ \frac{1 - \alpha}{(\alpha + 1 + \alpha \beta)^2} [e^{-\lambda(1 + \alpha)X_{(i)}} - e^{-\lambda x_{(i)}}] \right] \right\} = 0,$$

where \( Q_i = \frac{\beta}{\alpha + 1 + \alpha \beta} \left[ e^{-\lambda(1 + \alpha)x_{(i)}} - e^{-\lambda x_{(i)}} \right] - e^{-\lambda x_{(i)}} + 1 - \frac{i}{n + 1}, B_i = x_{(i)} e^{-\lambda x_{(i)}} \) and \( C_i = x_{(i)} e^{-\lambda(1 + \alpha)x_{(i)}} \).
4.3. Weighted Least-Square Estimator

The weighted least-square estimator (WLS) is an extension of LS and proposed by Swain et al. [17], which studied the WLS is obtained by minimizing the function,

\[ W(\lambda, \alpha, \beta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ F(x_i; \lambda, \alpha, \beta) - \frac{i}{n+1} \right]^2, \]

where \( F(\cdot) \) function has been given in Equation (4). Therefore, the WLS of \( \theta \) can be obtained by

\[
\frac{\partial W(\lambda, \alpha, \beta)}{\partial \lambda} = \sum_{i=1}^{n} \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{\beta}{\alpha + 1 + a\beta} \left[ - (1 + \alpha)C_i + B_i \right] + B_i \right\} \right\} = 0,
\]

\[
\frac{\partial W(\lambda, \alpha, \beta)}{\partial \alpha} = \sum_{i=1}^{n} \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{1}{(\alpha + 1 + a\beta)^2} \left[ e^{-\lambda(1+a)x(0)} - e^{-\lambda x(0)} \right] \right\} \right\} = 0,
\]

\[
\frac{\partial W(\lambda, \alpha, \beta)}{\partial \beta} = \sum_{i=1}^{n} \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{\beta \lambda}{\alpha + 1 + a\beta} C_i \right\} \right\} = 0,
\]

where \( Q_i, B_i \) and \( C_i, i = 1, \ldots, n \) are defined as above.

5. Simulation Studies

In this section, we assess the performance of the estimation methods proposed in the previous section by conducting several simulations for different sample sizes and values of the parameter. As indicated in Proposition 1, the \( F(x; \lambda, \alpha, \beta) \) is used to generate pseudo-random numbers from the GME distribution. This technique is called the inverse transform method, which consists of the following steps:

(i) Generate a random number \( u \) from the standard uniform distribution in the interval [0,1].

(ii) Apply the numerical techniques to solve the equation \( F(x) = u \) with given \( \lambda, \alpha, \beta \).

We take the sample size \( n = 50, 100, 200, 300, 400, 500, 1000 \) for each simulation, and each sample was replicated \( N = 1000 \) times. The values of parameter \( \theta = (1, 5, -0.5), (2, 1, 1.5), \) and \( (3, 10, 5) \) are considered, respectively. All the results were computed using the R programming. The evaluation of the estimators are performed on the average bias and the standard error (SE) for each single parameter, where \( Bias(\hat{\theta}_i) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i^{(i)} - \theta_i) \), \( SE(\hat{\theta}_i) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i^{(i)} - \theta_i)^2 \), and \( \theta_j \) is the \( j \)th component of \( \theta \). Moreover, the overall bias and mean squared error (MSE) of \( \hat{\theta} \) are also considered, where \( Bias(\hat{\theta}) = \sum_{j=1}^{3} Bias(\hat{\theta}_j) \), \( MSE(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} ||\hat{\theta}^{(i)} - \theta||^2 \), and \( || \cdot || \) is the Euclidean norm. The simulation results for different scenarios are given in the Tables 1–3.
Table 1. \( \lambda = 1, \alpha = 5, \beta = -0.5 \).

| Sample Size | Method | \( \hat{\lambda} \) (SE) | \( \hat{\alpha} \) (SE) | \( \hat{\beta} \) (SE) | MSE |
|-------------|--------|----------------|----------------|----------------|-----|
| n = 50      | MLE    | 0.9337 (0.3743) | 5.0836 (2.4378) | -0.1080 (1.7229) | 1.5273 |
|             | LS     | 0.9747 (0.2605) | 4.7614 (2.3440) | -0.4964 (0.2255) | 0.9491 |
|             | WLS    | 0.9162 (0.2344) | 5.0376 (1.1337) | -0.4444 (0.7600) | 0.7204 |
| n = 100     | MLE    | 0.8927 (0.3087) | 5.7791 (3.1581) | -0.5004 (0.8806) | 1.4852 |
|             | LS     | 1.0136 (0.2050) | 4.6547 (2.2373) | -0.4550 (0.2004) | 0.8934 |
|             | WLS    | 0.9433 (0.1481) | 4.6418 (0.9328) | -0.4858 (0.4072) | 0.5307 |
| n = 200     | MLE    | 0.9407 (0.2054) | 5.3528 (2.7663) | -0.5606 (0.2314) | 1.0774 |
|             | LS     | 0.9954 (0.1574) | 4.7508 (2.2406) | -0.4717 (0.1863) | 0.8692 |
|             | WLS    | 0.9646 (0.1087) | 4.9761 (0.9436) | -0.49608 (0.1881) | 0.4350 |
| n = 300     | MLE    | 0.9780 (0.1157) | 4.9851 (2.1748) | -0.5229 (0.1454) | 0.8116 |
|             | LS     | 1.0224 (0.1291) | 5.4474 (2.1971) | -0.4555 (0.1605) | 0.8504 |
|             | WLS    | 0.9654 (0.0982) | 5.0504 (0.8171) | -0.5127 (0.1298) | 0.4120 |
| n = 400     | MLE    | 0.9731 (0.1137) | 4.0055 (1.4628) | -0.5137 (0.1432) | 0.6742 |
|             | LS     | 1.0158 (0.1180) | 5.1742 (1.9422) | -0.4639 (0.1519) | 0.7472 |
|             | WLS    | 0.9980 (0.0733) | 5.0675 (0.9835) | -0.4948 (0.1078) | 0.4018 |
| n = 500     | MLE    | 0.9841 (0.0970) | 4.8382 (1.9489) | -0.5135 (0.1202) | 0.7242 |
|             | LS     | 1.0023 (0.1017) | 4.9641 (1.7818) | -0.4779 (0.1345) | 0.6818 |
|             | WLS    | 0.9821 (0.0610) | 4.9304 (0.9001) | -0.5062 (0.0849) | 0.3783 |
| n = 1000    | MLE    | 0.9938 (0.0705) | 4.9315 (1.6172) | -0.5020 (0.1432) | 0.5937 |
|             | LS     | 1.0060 (0.0763) | 4.9750 (1.7033) | -0.4797 (0.1029) | 0.6367 |
|             | WLS    | 0.9894 (0.0501) | 4.8743 (0.8206) | -0.5019 (0.0635) | 0.3545 |

Table 2. \( \lambda = 2, \alpha = 1, \beta = 1.5 \).

| Sample Size | Method | \( \hat{\lambda} \) (SE) | \( \hat{\alpha} \) (SE) | \( \hat{\beta} \) (SE) | MSE |
|-------------|--------|----------------|----------------|----------------|-----|
| n = 50      | MLE    | 1.4458 (0.9097) | 1.2917 (0.5589) | 1.6460 (3.2067) | 1.6314 |
|             | LS     | 1.8709 (0.3747) | 1.4399 (0.5717) | 2.0283 (2.1736) | 1.1183 |
|             | WLS    | 1.9537 (0.3217) | 0.9440 (0.6706) | 2.2223 (1.8281) | 0.9885 |
| n = 100     | MLE    | 1.7026 (0.7886) | 1.3034 (0.5729) | 1.9781 (2.6490) | 1.3911 |
|             | LS     | 1.9554 (0.2942) | 1.2025 (0.6341) | 2.3221 (2.0580) | 0.9073 |
|             | WLS    | 1.9519 (0.2279) | 0.9008 (0.6066) | 1.9902 (1.6569) | 0.8606 |
| n = 200     | MLE    | 1.7535 (0.6525) | 1.3555 (0.5795) | 1.6087 (1.8850) | 1.0782 |
|             | LS     | 1.9669 (0.2506) | 1.1475 (0.6229) | 1.9986 (1.5482) | 0.7287 |
|             | WLS    | 1.9578 (0.1269) | 1.0596 (0.4735) | 1.5661 (0.6906) | 0.4392 |
| n = 300     | MLE    | 1.8438 (0.5467) | 1.2813 (0.5962) | 1.6082 (1.5177) | 0.9151 |
|             | LS     | 1.8927 (0.1585) | 1.6745 (0.4603) | 1.3887 (0.8766) | 0.5793 |
|             | WLS    | 1.9835 (0.1121) | 1.0563 (0.4295) | 1.5963 (0.6744) | 0.4141 |
| n = 400     | MLE    | 1.8684 (0.4775) | 1.3238 (0.5647) | 1.5057 (1.2785) | 0.8075 |
|             | LS     | 1.9413 (0.1308) | 1.3760 (0.5588) | 1.5542 (0.8849) | 0.5718 |
|             | WLS    | 1.9638 (0.0916) | 1.1273 (0.3971) | 1.4569 (0.3923) | 0.3095 |
| n = 500     | MLE    | 1.9002 (0.4561) | 1.2733 (0.5697) | 1.6020 (1.1967) | 0.7662 |
|             | LS     | 2.0143 (0.1602) | 1.0437 (0.5819) | 1.9070 (0.9159) | 0.5142 |
|             | WLS    | 1.9573 (0.0739) | 1.0416 (0.3624) | 1.5810 (0.5662) | 0.3489 |
| n = 1000    | MLE    | 1.9189 (0.3525) | 1.2974 (0.5674) | 1.5166 (0.8654) | 0.6228 |
|             | LS     | 1.9928 (0.1392) | 1.1979 (0.6193) | 1.7116 (0.7494) | 0.5268 |
|             | WLS    | 1.9700 (0.0613) | 1.1250 (0.2917) | 1.4208 (0.2599) | 0.2280 |
Table 3. \( \lambda = 3, \alpha = 10, \beta = 5 \).

| Sample Size | Method | \( \hat{\lambda} (SE) \) | \( \hat{\alpha} (SE) \) | \( \hat{\beta} (SE) \) | MSE |
|-------------|--------|-----------------|-----------------|-----------------|-----|
| n = 50      | MLE    | 3.0424 (0.4181) | 9.7778 (2.4919) | 5.6122 (3.8162) | 2.2299 |
|             | LS     | 3.0549 (0.5084) | 9.9521 (2.3594) | 4.837 (4.1413)  | 2.3605 |
|             | WLS    | 3.0281 (0.4395) | 10.1081 (2.0774) | 5.0283 (3.9031) | 2.1545 |
| n = 100     | MLE    | 3.0380 (0.3025) | 9.8275 (2.4712) | 5.5335 (3.4914) | 2.0887 |
|             | LS     | 3.0416 (0.3537) | 9.8668 (2.2719) | 5.0004 (4.0233) | 2.2335 |
|             | WLS    | 2.9993 (0.3392) | 10.1385 (2.0807) | 4.6623 (3.5851) | 2.0216 |
| n = 200     | MLE    | 3.0172 (0.2143) | 9.5225 (2.1802) | 5.4294 (3.2236) | 1.9055 |
|             | LS     | 3.0208 (0.2368) | 9.7251 (2.2322) | 4.9982 (3.6205) | 2.0550 |
|             | WLS    | 3.0072 (0.2337) | 10.3121 (2.2217) | 4.6433 (2.9485) | 1.8293 |
| n = 300     | MLE    | 3.0239 (0.1718) | 9.5911 (2.2102) | 5.5706 (3.1488) | 1.8680 |
|             | LS     | 3.0251 (0.2052) | 9.5669 (1.9457) | 5.3041 (3.7496) | 2.0098 |
|             | WLS    | 3.0066 (0.1789) | 10.052 (2.0334) | 4.6414 (2.7485) | 1.7685 |
| n = 400     | MLE    | 3.0222 (0.1455) | 9.3730 (1.9607) | 5.4066 (2.9740) | 1.7391 |
|             | LS     | 3.0285 (0.1623) | 9.5468 (1.307)  | 5.1461 (3.3433) | 1.9214 |
|             | WLS    | 3.0104 (0.1522) | 9.9920 (1.9927) | 4.6140 (2.4915) | 1.5730 |
| n = 500     | MLE    | 3.0166 (0.1405) | 9.3607 (1.9663) | 5.3163 (2.8919) | 1.6961 |
|             | LS     | 3.0163 (0.1561) | 9.4042 (1.9839) | 4.9983 (3.4092) | 1.9053 |
|             | WLS    | 3.0071 (0.1374) | 10.0114 (1.9581) | 4.7762 (2.4242) | 1.5292 |
| n = 1000    | MLE    | 3.0139 (0.0984) | 9.3293 (1.1715) | 5.1518 (2.4282) | 1.4799 |
|             | LS     | 3.0235 (0.1120) | 9.2747 (1.8501) | 5.1674 (3.4079) | 1.8668 |
|             | WLS    | 3.0136 (0.1032) | 9.7242 (1.6997) | 4.9825 (2.2380) | 1.3759 |

From Tables 1–3, we find that the SE of all three estimator decrease as the sample size \( n \) increases and all estimators will tend to more accuracy when \( n \) is large. In addition, the estimated values obtained by the three estimator are close to the true values. Furthermore, the plots of bias of the simulated estimators of \( \lambda, \alpha, \) and \( \beta \), corresponding with different sample size \( n \), are shown in Figures 3–5, respectively.

Figure 3. Cont.
From Figures 3–5, we observe that the magnitude of bias of all estimators tends to zero as \( n \) grows, which means these estimators are asymptotically unbiased and consistent for the parameters. Thus, these estimator techniques perform well for estimating the parameters in the GME distribution. For further studying, we draw the plots of overall bias and MSE for \( \hat{\theta} \) in Figures 6 and 7.

Figures 6 and 7 show the bias and MSE of \( \hat{\theta} \), and we can find that as \( n \) increases, the bias of \( \hat{\theta} \) towards to zero, and the WLS always has the smallest value of MSE. The LS estimators have the largest MSE among the three considered estimators. Thus, we can conclude that WLS can be chosen as a more reliable estimator for the GME distribution.
Figure 4. Bias of estimator $\hat{\alpha}$ versus sample size $n$ for different scenarios.
Figure 5. Bias of estimator $\hat{\beta}$ versus sample size $n$ for different scenarios.
Figure 6. Bias of estimator $\hat{\theta}$ versus sample size $n$ for different scenarios.
6. Real Data Analysis

In this section, we use the weighted least-square estimator to analyze two real data sets for investigating the advantage of proposed GME distribution and compare it with some other distributions, including the exponential distribution distribution, WED distribution, GExtEW distribution, and GWE distribution, where the pdfs are given as follows.

(1) Exponential distribution: \( E(\lambda) \)

\[ f_E(x) = \lambda e^{-\lambda x}, \ x \geq 0, \lambda > 0. \]

(2) Weibull exponential distribution: \( WED(\lambda, \alpha, \beta) \)

\[ f_{WED}(x) = \alpha \beta (\lambda e^{-\lambda x}) \left[ \frac{(1 - e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}} \right] \exp \left\{ -\alpha \left[ \frac{1 - e^{-\lambda x}}{e^{-\lambda x}} \right]^{\beta} \right\}, \ x > 0, \alpha, \beta, \lambda > 0. \]
(3) Generalized extended exponential-Weibull distribution: \( G\text{ExtEW}(\lambda, \alpha, \beta, r, c) \)

\[
f_{G\text{ExtEW}}(x) = c\alpha(r\beta x^{r-1} + \lambda)(\beta x^r + \lambda x)^{-1} e^{-(\beta x^r + \lambda x)/c}, x > 0, c, \beta, \lambda > 0, r \in (0, \infty) \setminus \{1\}. \]

(4) Generalized weighted exponential distribution: \( G\text{WE}(\lambda, \alpha, k) \)

\[
f_{G\text{WE}}(x) = \frac{\alpha}{B(1/\alpha, k + 1)} \lambda e^{-\lambda x} (1 - e^{-\lambda x})^k, x > 0, \alpha, \lambda > 0, k \in \mathbb{Z}^+. \]

6.1. Data Set 1: Waiting Times

This data set represents the waiting times (in minutes) before the service of 100 bank customers, which has been previously used by Ghitany et al. [18]. It can be seen in Appendix A.1 of Appendix A. Table 4 shows the parameter estimator results of the GME, E, WED, GExtEW, and GWE distributions for these data. The corresponding minus log-likelihood, Akaike information criterion (AIC), and Bayesian information criterion (BIC) are also presented. From Table 4, we find that the GME has the smallest values of all criteria for comparing all other distributions.

| GME(\( \lambda, \alpha, \beta \)) | E(\( \lambda \)) | WED(\( \lambda, \alpha, \beta \)) | GExtEW(\( \lambda, \alpha, \beta, r, c \)) | GWE(\( \lambda, \alpha, k \)) |
|---|---|---|---|---|
| \( \lambda \) | 0.1554 | 0.0883 | 0.0234 | 0.0807 | 0.1284 |
| \( \alpha \) | 0.8202 | - | 5.3703 | 3.9509 | 6.2681 |
| \( \beta \) | 124.9639 | - | 1.3486 | 0.0812 | - |
| \( r \) | - | - | - | 1.7231 | - |
| \( c \) | - | - | - | 0.4711 | - |
| \( k \) | - | - | - | 4 | - |
| -loglike | 317.3592 | 329.9063 | 321.8420 | 317.1620 | 318.4602 |
| AIC | 640.7184 | 661.8126 | 649.6839 | 644.3241 | 642.9205 |
| BIC | 648.5339 | 664.4177 | 657.4995 | 657.3499 | 650.7360 |

Figure 8 shows the fitted models for data set 1. The first subgraph of Figure 8 shows the fitted densities to the data set histogram and some estimated distributions, and the second subgraph displays the empirical distribution function for the data set and the estimated distributions. Both figures reveal that the GME distribution provides a qualified fit for the data set.
6.2. Date Set 2: Survival Times

The second data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. This data set has recently been studied by Lee [19] and Tahir et al. [20]. The data set can be seen in Appendix A.2 of Appendix A. We compare the GME distribution with the E, WED, GExtEW, and GWE distributions. The estimated value of the parameters, AIC and BIC statistics of these distributions are listed in Table 5. It can be seen that GME distribution provides the best fit among these competing models. Figure 9 displays the fitted pdfs and cdfs of
the GME, E, WED, GExtEW, and GWE distributions for data set 2, and suggests that the fit of the GME distribution is reasonable.

Table 5. Data set 2: Comparison between the GME, E, WED, GExtEW, and GWE by using different criteria.

|                | GME(λ, α, β) | E(λ) | WED(λ, α, β) | GExtEW(λ, α, β, r, c) | GWE(λ, α, k) |
|----------------|--------------|------|--------------|-----------------------|--------------|
| λ              | 0.0311       | 0.0197 | 0.0114       | 0.0280               | 0.0237       |
| α              | 0.6384       | -     | 1.2263       | 1.8338               | 11.8562      |
| β              | 7.5142       | -     | 1.0035       | 0.0069               | -            |
| r              | -            | -     | -            | 0.8196               | -            |
| c              | -            | -     | -            | 0.9454               | -            |
| k              | -            | -     | -            | -                    | 3            |
| -loglike       | 578.9263     | 585.5995 | 580.1301     | 580.6417             | 590.0420     |
| AIC            | 1163.8530    | 1173.1990 | 1166.2600    | 1171.2830            | 1186.0850    |
| BIC            | 1172.2400    | 1175.9950 | 1174.6480    | 1185.2630            | 1194.4720    |

Figure 9. Fitted pdfs and the relative histogram, empirical and fitted cdfs.

7. Conclusions

In this paper, we introduce a new lifetime distribution, GME distribution, and propose several statistical properties of it. As it is not feasible to compare these methods theoretically, we have studied several simulations to identify the most efficient estimation method for GME distribution. The simulation results show that weighted least-square estimator (WLS) is the best performing estimator in terms of MSE. That is, the weighted least-square estimation method is more feasible for estimating parameters in the GME distribution. Finally, two real data sets were analyzed to indicate the importance and flexibility of GME distribution in comparison to some existing lifetime distributions. In the future, the development of properties and proper estimation procedure of the bivariate model and multivariate generalization will be of interest, and more work is needed along that direction.

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Abbreviations
The following abbreviations are used in this manuscript:
WE weighted exponential distribution
GWE generalized weighted exponential distribution
WED Weibull exponential distribution
GExtEW generalized extented exponential-Weibull distributions
GME generalized mixture exponential distribution
MLE maximum likelihood estimator
LS least-square estimator
WLS weighted least-square estimator

Appendix A. Data Set

Appendix A.1. Data Set 1

0.8 0.8 3.3 3.4 6.4 4.7 6.2 6.2 7.7 8.9 7.9 8.12 5.12 9.17 3.1 27.31 1.3 3.3 5 4 7.6 2 8.2 10.7
13 18.2 33.1 1.5 1.8 1.9 3.6 4.1 4.8 4.9 4.9 6.5 6.7 6.9 8.6 8.6 8.6 10.9 11 11 13 13.3 13.6 18.4
18.9 19 38.5 19 2.1 2.6 4.2 4.2 4.3 5 5.3 5.5 7.1 7.1 8.8 8.8 8.9 11 11 12 11.2 13.7 13.9 14.1
19 9 20.6 21 3.2 2.9 3.1 4.3 4.4 4.4 5.7 5.7 6.1 7.1 7.4 7.6 8.9 9.5 9.6 11.5 11.9 12.4 15.4 14.7 3.3
21.4 21.9 23.

Appendix A.2. Data Set 2

0.3 0.3 5.0 5.6 6.2 6.3 6.6 6.8 7.4 7.5 8.4 8.4 10.3 11.0 11.8 12.2 12.3 13.5 14.4 14.4 14.8
15.5 15.7 16.2 16.3 16.5 16.8 17.2 17.3 17.5 17.9 19.8 20.4 20.9 21.0 21.0 21.1 23.0 23.4 23.6 24.0
24.0 27.9 28.2 29.1 30.0 31.0 31.0 32.0 32.0 35.0 35.0 37.0 37.0 38.0 38.0 38.0 38.0 39.0 39.0 40.0 40.0
40.0 41.0 41.0 42.0 43.0 43.0 43.0 43.0 45.0 46.0 47.0 48.0 49.0 51.0 51.0 51.0 52.0
54.0 55.0 56.0 57.0 58.0 59.0 60.0 60.0 61.0 62.0 65.0 67.0 67.0 68.0 69.0 78.0 80.0 83.0
88.0 89.0 90.0 93.0 96.0 103.0 105.0 109.0 109.0 111.0 115.0 117.0 125.0 126.0 127.0 129.0 129.0
139.0 154.0.

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