Calculation of a variable cross-section beam on an elastic foundation with two coefficients of compliance

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Abstract. In this paper, the universal Kurdyumov-Kochanov method with approximation of the solution by a power function is used to calculate a beam of variable cross section located on an elastic foundation, characterized by two coefficients of compliance (bed coefficients). It is shown that this method is characterized by simplicity and efficiency in the study of beams not only with continuous, but also with piecewise-continuous laws of change in the dimensions of sections and external load. In addition, it can also be used to analyze the behavior of beams with discrete laws for changing the stiffness of an elastic base and loads acting on local sections of the beam. The effectiveness of the implementation of the method is confirmed by a number of test cases.

1. Introduction

Mathematical modeling of objects of different nature at the design stage allows effectively assessing their state and behavior when exposed to external factors. In recent years, thanks to the rapid development of computing, the methods of computer mathematics have gained widespread use in the study of mathematical models. However, the improvement of practical methods for solving complex mathematical problems is still relevant. In particular, the development of new or actualization of already known methods is necessary for the solution, which has already become classical, of a problem related to modeling the behavior of a beam on an elastic foundation.

The calculation of beams on an elastic foundation has a long history, the presentation of which is the subject of separate studies. There are many attempts to find exact solutions to the problem, among which the works are important of K. Hayashi [1], A.N. Krilov [2] and many others. Such solutions were found under various laws of load variation for beams located on the classical (with one bed coefficient) and modified (with two bed coefficients) Winkler foundation, including one characterized by variable stiffness [3]. However, for beams of variable section, as noted A.N. Krilov [2], to find such solutions is possible only in some special cases.

One example of calculating a beam of variable cross section on a classical elastic Winkler foundation is presented in [4]. In this case, the law of changing the cross-sectional shape of the beam in the considered example was pre-set. In the work [5], a solution was obtained to the problem of the behavior of a beam located on a modified Winkler elastic foundation, which is already characterized by two bed coefficients. However, in contrast to the case described above, the solution to the problem was obtained only for a beam of constant cross section.
In the work [6], to study the behavior of a tapered rod (screw) on an elastic foundation, characterized by two bed coefficients, assuming that \( \frac{dI(x)}{dx} \approx 0 \), a fourth-order differential equation with variable coefficients was used

\[
EI(x) \frac{d^4v}{dx^4} - 2t \frac{d^2v}{dx^2} + kv = q(x)
\]

where \( v \) is the deflection, \( EI(x) \) is the flexural rigidity of the beam \( (E \) is the Young's modulus of the material, \( I(x) \) is the axial moment of inertia of the cross section), \( k, t \) are the first and second bed coefficients of the elastic foundation, respectively, \( q(x) \) is the external load distributed over the length of the beam. Thus, the solution obtained by the authors actually described the behavior of the beam of constant cross section on a modified elastic foundation with variable stiffness.

The above and many other examples allow noting that the simulation of beams on an elastic foundation, described by a fourth-order differential equation with variable coefficients, has remained relevant for many decades. In addition, the methods of solving such equations are of interest in themselves. The purpose of this research is to evaluate the capabilities of the universal Kurdyumov-Kochanov method for studying beams of variable cross section on an elastic modified Winkler foundation with two bed coefficients, including variable stiffness.

2. Materials and methods

Consider a beam located on an elastic (not necessarily solid) modified Winkler foundation with two bed coefficients, the deformation of which in the general case can be described by a differential equation

\[
\frac{d^2}{dx^2} \left( EI(x) \frac{d^2v}{dx^2} \right) - 2t \frac{d^2v}{dx^2} + kv = q(x),
\]

whose analytical solution in the general case cannot be found [2]. In this regard, the approximate solution (2) will be sought in the form of some function

\[
v(x) \approx u(x, a_0, a_1, ..., a_n),
\]

with coefficients \( a_0, a_1, ..., a_n \). In this case, the problem can be reduced to an analytical or numerical finding of such coefficients for which the residual

\[
\epsilon(x) = \frac{d^2}{dx^2} \left( EI(x) \frac{d^2u}{dx^2} \right) - 2t \frac{d^2u}{dx^2} + ku - q(x)
\]

will be minimal \( \forall x \in [0, l] \).

To assess the quality of the function \( u \), selected to approximate the solution of equation (2), the functional can be used

\[
\Phi_1 = \int_0^l \epsilon(x)^2 \, dx.
\]

Because the required function \( v(x) \) is an elastic line of the neutral axis of the beam, it is appropriate to approximate it by a power function with a sufficiently large number of unknown coefficients \( (n \geq 4) \):

\[
u(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i,
\]

satisfying the boundary conditions of the problem.

In this study, solutions of equation (2) were obtained taking into account the fact that the beam relies only on an elastic foundation and, therefore, the boundary conditions were set as follows [2,4]:

\[
EI(x) \frac{d^2u(x)}{dx} \Big|_{x=0} = 0, \quad \frac{d}{dx} \left( EI(x) \frac{d^2u(x)}{dx} \right) \Big|_{x=0} = 0,
\]

\[
EI(x) \frac{d^2u(x)}{dx} \Big|_{x=l} = 0, \quad \frac{d}{dx} \left( EI(x) \frac{d^2u(x)}{dx} \right) \Big|_{x=l} = 0.
\]
Many methods are known for numerically solving equations of the form (2), by approximating it with function (6), but each of them has not only its own advantages, but also certain disadvantages.

For example, when using the collocation method [7], after substituting (6) into (2), one comes to a system of linear algebraic equations, for given (in the general case, arbitrarily) \( n + 1 \) values \( x_k, k = 1, n + 1 \).

In various optimization methods, the unknown coefficients \( a_i, i = 0, n \) take as variable parameters, and the sum of squares of residuals calculated for given values of \( x_k, k = 1, N \) (where \( N \) is a sufficiently large number) - as an objective function

\[
\Phi_2 = \sum_{k=1}^{n+1} \epsilon(x)^2 \int_0^1 \epsilon(x_k)^2. \tag{9}
\]

Obviously, in the case of a search for a solution by the collocation method, its error can be quite significant, since equality of the left and right sides of equation (2) will be ensured not at the entire length of the beam, but only at certain points. Therefore, such a decision in individual cases may turn out to be erroneous and will not allow estimating the real deformation of the beam, which, in turn, will lead to the design errors of the real object.

When using optimization methods, depending on the degree \( n \) of function (6) and, accordingly, on the number of unknown coefficients, relatively large expenditures of computational resources may be required. In addition, as a result of the implementation of this method, the probability of obtaining a solution corresponding to the local extremum of the objective function (9) is quite high and the optimal (real) solution of the problem in this case may not be detected.

The elimination of the above disadvantages is possible when implementing the combined method, in which the initial approximations of the desired coefficients are determined, for example, by the collocation method, and the solution is specified by one of the optimization methods. However, there is reason to believe that this method can also be effective only when solving a certain class of problems.

The analysis of solutions of a set of practical problems suggests that as a universal method, which allows easily finding approximate solutions of an equation of the form (2) with different values of parameters, it is possible to consider the Kurdyumov-Kochanov method, which, as is known, is a modification of the Bubnov-Galerkin variation method [7]. Its main advantage compared with other methods is the ability to satisfy complex boundary conditions, including those determined by expressions (7), (8).

It should also be noted that for beams with a constant bending stiffness and a simple form of the law of change of the external load, solutions of equation (2) can be easily obtained using one of the computer algebra systems, for example, Maple, Mathematica, Matcad, Matlab, etc. However, in general, the algorithms embedded in these systems, such as \texttt{rkf45}, \texttt{ck45}, \texttt{dverk78}, \texttt{rosenbrock} and others need to be modified, since their capabilities are limited by the complexity of the problem being solved. This study compares the Kurdyumov-Kochanov method with a numerical algorithm for solving \texttt{numeric.bvp}, designed to solve linear boundary value problems in the Maple system of analytical calculations.

3. Implementation and evaluation of numerical methods for studying a mathematical model of a beam located on an elastic foundation

The derivatives of order \( k \) of function (6) have the following form:

\[
\frac{d^k u}{d x^k} = \sum_{i=0}^{n} t(i - 1)(i - 2) \ldots (i - k + 1) a_i x^{i-k}, \tag{10}
\]

where \( \forall i = 0, n, \ k \leq n; \text{if } i < k; i - k = 0 \).

As an example for finding solutions to differential equation (2), consider the use of a power function (6) of degree \( n = 12 \). Then, taking into account (3), the authors substitute (10) into (2) and after forming the factors \( c_i(x) \) with the required coefficients \( a_i, i = 0, 12 \), the researchers will have:

\[
\sum_{i=0}^{12} c_i(x) a_i = q(x). \tag{11}
\]
The authors choose (generally arbitrarily) the coordinates of the beam sections \( x_k, k = 1,13 \), and substitute them into (11). As a result, after calculating the values of \( c_i(x) \) and \( q(x) \) at each point \( x_k \), the system of nine linear algebraic equations with unknowns \( a_i, i = 0,12 \) is obtained:

\[
\sum_{i=0}^9 c_i(x_k) a_i = q(x_k), \ k = 1,5. \tag{12}
\]

Besides, in addition to (12), it is also necessary to specify a system of four linearly independent functions in order to satisfy the boundary conditions (7), (8). In the simplest case, such a system can be directly included in (12). It is clear that in order to increase the accuracy of the desired solution, one should choose a function (6) of a higher degree.

As an example, the collocation method was used to analyze the deformations of a wedge-shaped concrete beam of rectangular cross section located on a solid elastic foundation with a length of \( l_1 = 6 \) m, taking into account its operation on compression and shear. In this case, the characteristic parameter values for this problem were used: beam dimensions: length \( l = 6 \) m, width \( b = 0.6 \) m, initial height \( h_0 = 0.6 \) m; modulus of elasticity of the beam material is \( E = 2 \times 10^5 \) Pa; coefficients of compliance (bed coefficients) of an elastic foundation are \( k = 145.8 \times 10^5 \) Pa, \( r = 189.6 \times 10^5 \) N. The law of variation of the axial moment of inertia of the beam section was given as follows:

\[
l(x) = \frac{b}{12} \left[ h_0 \left(1 - \frac{x}{l}\right) + h_1 \right]^3.
\]

The results of calculations in graphical form are shown in Figure 1 for two variants of external load:

a) The maximum modulus load is applied at the center of the beam: \( q(x) = -q_0 \sin \left(\frac{\pi x}{l}\right) \).

b) The maximum modulus load is applied at the ends of the beam: \( q(x) = -q_0 e^{-\sin \left(\frac{\pi x}{l}\right)} \).

The solution of equation (2) with the parameters adopted in the example described above was also performed by the Kurdyumov-Kochanov method and using the numeric.bvp algorithm of which is implemented in Maple. The results of the solutions in graphical form are shown in Figure 2.

The comparison of the charts in Figures 1 and 2 shows that the collocation method makes it possible to obtain a very approximate solution of equation (2) with relatively simple loading laws. However, the error of the solution increases significantly in cases where the load applied to the beam varies according to a rather complex law.

Similar conclusions can be drawn from the analysis of solutions obtained by one of the optimization methods, as well as by the combined method. But in these cases, the errors of the solutions are determined mainly by the choice of initial approximations.

The solutions of equation (2) under the conditions of the considered example, obtained by the methods of Kurdyumov-Kochanov (with \( n = 8 \)) and numeric.bvp (with absolute error \( \delta = 10^{-5} \)) with the load \( q(x) = -q_0 \sin \left(\frac{\pi x}{l}\right) \), turned out to be quite close. But with load \( q(x) = -q_0 e^{-\sin \left(\frac{\pi x}{l}\right)} \), acting on the beam, it was not possible to obtain solutions using the numeric.bvp method even with significantly lower accuracy.

This suggests that the Kurdyumov-Kochanov method with sufficient accuracy for practice can be used to study mathematical models of beams on an elastic foundation under significantly more complicated conditions. In particular, these can be models with piecewise continuous or discretely varying loads, as well as models with variable stiffness of an elastic base.

The representation of piecewise-continuous and discrete functions in mathematical models of beams on an elastic foundation is possible by using generalized functions: Dirac \( \delta \)-functions and Heaviside functions. As is well known, these functions can be defined analytically and have continuous derivatives. Therefore, their use to represent the parameters of equation (2) will not affect the nature of its solution.
Figure 1. Solutions of equation (2) by the collocation method with different laws of change in the external load: a) \( q(x) = q_0 \sin \left( \pi \frac{x}{l} \right) \); b) \( q(x) = q_0 e^{-\sin(\pi x)} \)

Figure 2. Solutions of equation (2) by the Kurdyumov-Kochanov method: a) with external load \( q(x) = -q_0 \sin \left( \pi \frac{x}{l} \right) \); b) \( q(x) = q_0 e^{-\sin(\pi x)} \); c) by Kurdyumov-Kochanov method with external load \( q(x) = q_0 e^{-\sin(\pi x)} \)

Figure 3 shows a chart of the piecewise-continuous load function.

\[
q(x) = q_0 \left\{ \sin \left( \pi \frac{x}{l} \right) [H(x - 2) - 1] - H(x - 4) \right\}
\]

where \( q_0 = 20000 \frac{N}{m} \), \( H(*) \) is the generalized Heaviside function and the corresponding chart of the dependence \( v(x) = \sum_{i=0}^{12} a_i x^i \), obtained by the Kurdyumov-Kochanov method.
Figure 3. Chats of the dependencies: a) piecewise-continuous function of external load $q(x)$ and b) of the corresponding function of deflection $v(x)$

Figure 4 shows the chart of the discrete function of the external load.

$$q(x) = -q_0\left[1 - H(x - 0.2)\right] + 2\left[H(x - 5.8) - H(x - 6)\right]$$

and the corresponding chart of the dependence $v(x) = \sum_{i=0}^{12} c_i x^i$, also obtained by the Kurdyumov-Kochanov method.

Figure 4. Graphs of dependences: a) a discrete function of external load $q(x)$ and b) of the corresponding function of the deflection $v(x)$

Practically important are also models of beams located on a continuous elastic foundation. In this case, the bed coefficients are also represented as functions of the form $k = k(x)$ and $t = t(x)$, which can be continuous, piecewise-continuous, or discrete.

Assessing the effect of changes in the performances of an elastic base on the behavior of a beam is a fairly voluminous problem, therefore, within the framework of this paper, only the possibility of obtaining an approximate solution using the Kurdyumov-Kochanov method for an elastic foundation, characterized by power laws for changing bed coefficients, which were specified using the Heaviside function, was studied:

$$k(x) = k_0 H(x - 2), t(x) = t_0 H(x - 2),$$

where $k_0 = 189.6 \cdot 10^5$ Pa, $t_0 = 145.8 \cdot 10^5$ N.

Such a case may correspond, for example, to a model of a beam resting on an elastic base not along its entire length.
Figure 5 shows the charts of the laws of the change in the bed coefficient \( k(x) \) and the corresponding deflection of the beam \( v(x) \), obtained for the values of the parameters that were used in the previous examples.

![Graph of the dependences](image)

**Figure 5.** Graphs of the dependences: a) the discrete law of change of the bed coefficient \( k(x) \) and b) of the corresponding to it function of the deflection \( v(x) \)

### 4. Conclusion

The analysis of the solutions obtained in this paper makes it possible to note that the study of mathematical models of a beam with a variable cross-section along length, located on an elastic foundation (not always solid), characterized by two bed coefficients, can be effectively performed using the classical Galerkin method and in particular, its modifications - the Kurdyumov-Kochanov method. At the same time, approximate solutions with sufficient accuracy for practice can be easily obtained for models of a beam with continuous, piecewise-continuous and discrete laws of variation of parameters, as well as for elastic foundations with variable stiffness.

The representation of the laws of variation of the beam parameters and the elastic foundation using the generalized Dirac and Heaviside functions does not affect the nature of the problem solution and does not require a greater degree of the desired function than in the case of continuously changing parameters. However, with other numerical methods, the required accuracy of the solution can always be achieved by increasing the degree of the approximation function.

The possibility of obtaining reliable solutions using a method tested in solving a multitude of practically important problems allows expanding the class of these problems, including applications from various fields of human activity.

For example, currently in medicine, implants of various designs are widely used, intended to replace biological organs or their parts. Once installed in a biological system, implants interact with living tissues. In the first approximation, we can assume that the mathematical model of this interaction will be represented by a differential equation of the form \( (2) \), whose solution can be reliably obtained using the Kurdyumov-Kochanov method studied in this paper.

It is clear that for solving specific problems, special methods, directly developed for their solution, have greater efficiency. However in the case of the study of problems formulated in the classical formulation, the Kurdyumov-Kochanov method can be considered universal.

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