The Complexity of the Proper Orientation Number

A. Ahadi\footnote{E-mail addresses: arash.ahadi@mehr.sharif.edu, ali.dehghan16@aut.ac.ir.}, A. Dehghan\footnote{E-mail addresses: arash.ahadi@mehr.sharif.edu, ali.dehghan16@aut.ac.ir.}

\textit{Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran}

\textit{Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran}

Abstract

Graph orientation is a well-studied area of graph theory. A proper orientation of a graph $G = (V, E)$ is an orientation $D$ of $E(G)$ such that for every two adjacent vertices $v$ and $u$, $d_D^*(v) \neq d_D^*(u)$ where $d_D^*(v)$ is the number of edges with head $v$ in $D$. The proper orientation number of $G$ is defined as $\chi^*(G) = \min D \in \Gamma \max v \in V(G) d_D^*(v)$ where $\Gamma$ is the set of proper orientations of $G$. We have $\chi(G) - 1 \leq \chi^*(G) \leq \Delta(G)$. We show that, it is \textbf{NP}-complete to decide whether $\chi^*(G) = 2$, for a given planar graph $G$. Also, we prove that there is a polynomial time algorithm for determining the proper orientation number of 3-regular graphs. In sharp contrast, we will prove that this problem is \textbf{NP}-hard for 4-regular graphs.

\textbf{Key words:} Proper orientation; Vertex coloring; \textbf{NP}-completeness; Planar 3-SAT (type 2); Graph orientation; Polynomial algorithms.

1 Introduction

Graph orientation is a well-studied area of graph theory, that provides a connection between directed and undirected graphs \cite{17}. There are several problems concerned with orienting the edges of an undirected graph in order to minimize some measures in the resulting directed graph, for instance see \cite{9, 12}. On the other hand, there are many ways to color the vertices of graphs properly. A proper vertex coloring of a digraph $D$ is defined, simply a vertex coloring of its underlying graph $G$. The chromatic number of a digraph...
provides interesting information about its subdigraphs. For instance, a theorem of Gallai proves that digraphs with high chromatic number always have long directed paths [13].

Venkateswaran [21] initiated the study of the problem of orienting the edges of a given simple graph so that the maximum indegree of vertices is minimized. Afterwards, Asahiro et al. in [4] generalized this problem for weighted graphs. It was proved that, this problem can be solved in polynomial-time if all the edge weights are identical [4, 19, 21], but it is NP-hard in general [4]. Furthermore, the problem can be solved in polynomial-time if the input graph is a tree, but for planar bipartite graphs it is NP-hard [4]. For more information about the recent results about this problem see [3].

On the other hand, in 2004 Karoński, Łuczak and Thomason initiated the study of proper labeling [16]. They introduced an edge-labeling which is additive vertex-coloring that means for every edge uv, the sum of labels of the edges incident to u is different from the sum of labels of the edges incident to v [16]. Also, it is conjectured that three labels \{1, 2, 3\} are sufficient for every connected graph, except \(K_2\) (1, 2, 3-Conjecture, see [16]). This labeling have been studied extensively by several authors, for instance see [1, 2, 5, 11, 15]. Afterwards, Borowiecki et al. consider the directed version of this problem. Let \(D\) be a simple directed graph and suppose that each edge of \(D\) is assigned an integer label. For a vertex \(v\) of \(D\), let \(q^+(v)\) and \(q^-(v)\) be the sum of labels lying on the arcs outgoing form \(v\) and incoming to \(v\), respectively. Let \(q(v) = q^+(v) - q^-(v)\). Borowiecki et al. proved that there is always a labeling from \{1, 2\}, such that \(q(v)\) is a proper coloring of \(D\) [7], see also [18].

Furthermore, Borowiecki et al. consider another version of above problems and they show that every undirected graph \(G\) can be oriented so that adjacent vertices have different in-degrees [7]. In this work, we consider the problem of orienting the edges of an undirected graph such that adjacent vertices have different in-degrees and the maximum indegree of vertices is minimized in the resulting directed graph.

The proper orientations of \(G\) which are orientations \(D\) of \(G\) such that for every two adjacent vertices \(v\) and \(u\), \(d_D^-(v) \neq d_D^-(u)\). The proper orientation number of \(G\) is defined as \(\overrightarrow{\chi}(G) = \min_{D \in \Gamma} \max_{v \in V(G)} d_D^-(v)\), where \(\Gamma\) is the set of proper orientations of \(G\).

The proper orientation number is well-defined and every proper orientation of graph introduces a proper vertex coloring for its vertices. Thus, \(\chi(G) - 1 \leq \overrightarrow{\chi}(G)\). On the other hand \(\overrightarrow{\chi}(G) \leq \Delta(G)\). Consequently,

\[\chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G).\]  \hfill (1)

In this work, we focus on regular graphs and planar graphs. We show that there is
a polynomial-time algorithm for determining the proper orientation number of 3-regular graphs. But it is NP-complete to decide whether the proper orientation number of a given 4-regular graph is 3 or 4.

**Theorem 1** Determining the proper orientation number of a given 4-regular graph is NP-hard; but there is a polynomial-time algorithm to determine the proper orientation number for 3-regular graphs.

It is easy to see that $\overrightarrow{\chi}(G) = 1$ if and only if every connected component of $G$ is a star. But for $\overrightarrow{\chi}(G) = 2$, we have the following:

**Theorem 2** It is NP-complete to decide $\overrightarrow{\chi}(G) = 2$, for a given planar graph $G$.

In this paper we consider simple graphs and we refer to [6] for standard notation and concepts. For a graph $G$, we use $n$ and $m$ to denote its numbers of vertices and edges, respectively. Also, for every $v \in V(G)$, $d(v)$ and $N_G(v)$ denote the degree of $v$ and the neighbor set of $v$, respectively. A spanning subgraph of a graph $G$ is a subgraph whose vertex set is $V(G)$. We say that a set of vertices are independent if there is no edge between these vertices. The independence number, $\alpha(G)$ of a graph $G$ is the size of a largest independent set of $G$. Also a clique of a graph is a set of mutually adjacent vertices. We denote the maximum degree of a graph $G$ by $\Delta(G)$. A directed graph $G$ is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, with an incidence function $D$ that associates with each edge of $G$ an ordered pair of vertices of $G$. If $e = uv$ is an edge and $G(e) = u \rightarrow v$, then $e$ is from $u$ to $v$. The vertex $u$ is the tail of $e$, and the vertex $v$ its head. Note that every orientation $D$ of a graph, introduced a digraph. The indegree $d^-(v)$ of a vertex $v$ in $D$ is the number of edges with head $v$ of $v$. For $k \in \mathbb{N}$, a proper vertex $k$-coloring of $G$ is a function $c: V(G) \rightarrow \{1, \ldots, k\}$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. The smallest integer $k$ such that $G$ has a proper vertex $k$-coloring is called the chromatic number of $G$ and denoted by $\chi(G)$. Similarly, for $k \in \mathbb{N}$, a proper edge $k$-coloring of $G$ is a function $c: E(G) \rightarrow \{1, \ldots, k\}$, such that if $e, e' \in E(G)$ share a common endpoint, then $c(e)$ and $c(e')$ are different. The smallest integer $k$ such that $G$ has a proper edge $k$-coloring is called the edge chromatic number of $G$ and denoted by $\chi'(G)$. By Vizing’s theorem [22], the edge chromatic number of a graph $G$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$. Those graphs $G$ for which $\chi'(G) = \Delta(G)$ are said to belong to Class 1, and the others to Class 2. For a graph $G = (V, E)$, the line graph $G$ is denoted by $L(G)$, is a graph with the set of vertices $E(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in $G$. For simplicity and with a slight abuse of notation, we also denote by $D$ the digraph resulting from an orientation $D$ of the graph $G$. 

3
2 Regular graphs

Let $G$ be an $r$-regular graph and suppose that $D$ is a proper orientation of $G$ with maximum indegree $\chi^r(G)$. We have $\chi^r(G) > \frac{1}{n} \sum_{v \in V(G)} d_D^-(v) = \frac{rn/2}{n}$. So we have the following simple observation about regular graphs.

**Observation 1** For every $r$-regular graph $G$ with $r \neq 0$, $\chi^r(G) \geq \lceil \frac{r+1}{2} \rceil$.

Here, we present a lemma about $(2k + 1)$-regular graphs, then we use it to prove Theorem 1.

**Lemma 1** For every $(2k + 1)$-regular graph $G$, $k \in \mathbb{N}$,

(i) $\chi(L(G)) = 3k$ if and only if $G$ belongs to Class 1.

(ii) $\chi(G) = k + 1$ if and only if $\chi(G) = 2$.

**Proof** (i) First, let $G$ be a $(2k + 1)$-regular graph belonging to Class 1. $G$ has a proper edge coloring $c$, such that $c : E(G) \to \{1, \ldots, 2k+1\}$. Let $\{e_1, \ldots, e_m\}$ be the set of edges of $G$ and $V(L(G)) = E(G)$. Now, we present a proper orientation for $L(G)$ with maximum indegree $3k$. For every edge $e_i e_j \in E(L(G))$, such that $c(e_j) - c(e_i) > k$, orient $e_i e_j$ from $e_i$ to $e_j$. Also for every two numbers $p, q \in \{1, \ldots, 2k + 1\}$, such that $|p - q| \leq k$, let $H_{p,q}$ be the induced subgraph on the vertices $c^{-1}(p) \cup c^{-1}(q)$ in $L(G)$. Since $G$ is a regular graph that belong to Class 1, $H_{p,q}$ is a spanning 2-regular subgraph of $L(G)$. Therefore every component of $H_{p,q}$ is an even cycle. Orient each of these cycles to obtain a directed cycle. We denote the resulting orientation by $D$. We claim that $D$ is a proper orientation of $L(G)$, with maximum indegree $3k$. Since $c$ is a proper edge coloring of $G$, it is sufficient to prove that for every vertex $e_i \in V(L(G))$, we have $d_D^-(e_i) = c(e_i) + (k-1)$. Let $e$ be a vertex of $L(G)$. First suppose that $c(e) < k$. For every number $i$, $i \in \{1, \ldots, 2k + 1\} \setminus \{c(e)\}$, the vertex $e$, has exactly two neighbors $e_i'$ and $e_i''$, with $c(e_i') = c(e_i'') = i$ in $L(G)$. In $D$ for every $i \neq c(e)$, $1 \leq i \leq c(e) + k$, $e$ has exactly one incoming edge from the set $\{e_i', e_i''\}$. Also for every $i$, $c(e) + k < i \leq 2k + 1$, $e$ doesn’t have any incoming edge from the set $\{e_i', e_i''\}$. Therefore $d_D^-(e_i) = c(e_i) + k - 1$. For every edge $e$ with the property $c(e) \geq k$, by a similar argument, we have $d_D^-(e_i) = c(e_i) + k - 1$. Consequently $\chi(L(G)) \leq 3k$ (in the following we see that if $\chi(L(G)) \leq 3k$, then $\chi(L(G)) = 3k$).

On the other hand, suppose that $G$ is a $(2k + 1)$-regular graph and $\chi(L(G)) \leq 3k$. Let $f$ be a proper orientation of $L(G)$ with the maximum indegree at most $3k$. Let $u$ be an arbitrary vertex of $G$ and $N_G(u) = \{w_1, \ldots, w_{2k+1}\}$. Since $f$ is a proper orientation of
$L(G)$ and $\{uw_1, \ldots, uw_{2k+1}\}$ is a clique in $L(G)$, so $d_f^{-}(uw_1), \ldots, d_f^{-}(uw_{2k+1})$ are distinct numbers. Consequently

$$\sum_{i=1}^{2k+1} d_f^{-}(uw_i) \leq (3k) + (3k-1) + \cdots + (3k-(2k)) = 2k(2k+1),$$

$$\sum_{v \in V(G)} \sum_{vw \in E(G)} d_f^{-}(vw) \leq 2k(2k+1)n.$$ 

On the other hand, by counting the number of edges in two ways, we have:

$$\sum_{v \in V(G)} \sum_{vw \in E(G)} d_f^{-}(vw) = 2\left( \sum_{e \in V(L(G))} d_f^{-}(e) \right) = 2|E(L(G))| = 2k(2k+1)n.$$ 

Thus, for every $v \in V(G)$ we have:

$$\sum_{vw \in E(G)} d_f^{-}(vw) = 2k(2k+1).$$

It means that for every $v \in V(G)$, $\{d_f^{-}(vw) | vw \in E(G)\} = \{k, k+1, \ldots, 3k\}$. Therefore $\chi'(L(G)) = 3k$ and also the function $d_f^{-}$ is a proper edge coloring of $G$ with $2k+1$ colors. It means that $G$ belongs to Class 1.

(ii) First suppose that $\chi'(G) = k+1$ through a proper orientation $D$. For every $i$, denote the number of vertices $v \in V(G)$ with the indegree $i$ by $n_i$. We have: $\sum_{i=0}^{k+1} n_i = n$ and $\sum_{i=0}^{k+1} i \cdot n_i = m = \frac{2k+1}{2}n$. So $n_{k+1} \geq m - kn = \frac{n}{2}$. The set of vertices $v$ with $d_D^{-}(v) = k + 1$, forms an independent set. Obviously every regular graph with $\alpha(G) \geq \frac{n}{2}$, is bipartite. Thus, $\chi(G) = 2$.

Next, let $G[X,Y]$ be a bipartite $(2k+1)$-regular graph. By Observation 1, $\chi'(G) \geq k+1$. By König’s theorem [23], $G$ has a decomposition into $2k+1$ perfect matchings. Orient the edges of $k+1$ perfect matchings from $X$ to $Y$, and other edges from $Y$ to $X$. It is easy to see that this is a proper orientation with maximum indegree $k+1$.

\[\square\]

Corollary 1 For every 3-regular graph $G$, other than $K_4$, $\chi'(G) = \chi(G)$.

Proof of Theorem 1. Clearly, the problem is in NP. It was shown that it is NP-hard to determine the edge chromatic number of a cubic graph [14]. By Lemma 1, for every cubic graph $G$, $\chi'(L(G)) = 3$, if and only if $G$ belongs to Class 1. So determining the
proper orientation number of a 4-regular graph is \( \text{NP} \)-hard. For the second part of the theorem, let \( G \) be a 3-regular graph, other than \( K_4 \), by Brooks’ Theorem [8] \( \chi(G) \leq 3 \). There is a polynomial-time algorithm for determining whether a given graph \( G \) has a chromatic number at most 2. So by Corollary 1, there is a polynomial-time algorithm for determining the proper orientation number of a given 3-regular graph.

□

It was shown that it is \( \text{NP} \)-hard to determine the edge chromatic number of an \( r \)-regular graph for any \( r \geq 3 \) [20]. So by Lemma 1 we have:

**Corollary 2** For any \( r \geq 1 \), the following problem is \( \text{NP} \)-hard: “Given a \( 4r \)-regular graph \( G \), determine \( \chi^-(G) \).”

Here, we present a simple \((2 - \frac{2}{r+2})\)-approximation algorithm for finding a proper orientation with the minimum of maximum indegree between the proper orientations of \( G \), for an \( r \)-regular graph \( G \). It is sufficient to start with a vertex \( v \) of maximum degree and for every edge \( e = vu \), simply orient \( e \) from \( u \) to \( v \) and next in \( G' = G \setminus \{v\} \) repeat the above procedure. By Observation 1 we have \( \chi^+ \geq \lceil \frac{r+1}{2} \rceil \), so the previous greedy algorithm is a \( \theta \)-approximation algorithm, where

\[
\theta = 2 - \frac{2}{r+2} \geq \frac{r}{\lceil \frac{r+1}{2} \rceil}.
\]

3 Planar graphs

Let \( \Phi \) be a 3-SAT formula with clauses \( C = \{c_1, \cdots, c_k\} \) and variables \( X = \{x_1, \cdots, x_n\} \). Let \( G(\Phi) \) be a graph with the vertices \( C \cup X \cup (\neg X) \), where \( \neg X = \{\neg x_1, \cdots, \neg x_n\} \), such that for each clause \( c_j = y \lor z \lor w \), \( c_j \) is adjacent to \( y, z \) and \( w \), also every \( x_i \in X \) is adjacent to \( \neg x_i \). \( \Phi \) is called planar 3-SAT(type 2) formula if \( G(\Phi) \) is a planar graph. It was shown that the problem of satisfiability of planar 3-SAT(type 2) is \( \text{NP} \)-complete [10]. In order to prove Theorem 2, we reduce the following problem to our problem.

**Problem:** Satisfiability of planar 3-SAT(type 2).

**Input:** A 3-SAT(type 2) formula \( \Phi \).

**Question:** Is there a truth assignment for \( \Phi \) that satisfies all the clauses?

**Proof of Theorem 2.** Consider an instance of planar 3-SAT(type 2) \( \Phi \) with variables \( X = \{x_1, \cdots, x_n\} \) and clauses \( C = \{c_1, \cdots, c_k\} \). We transform this into a planar graph
\(G(\Phi)\) such that \(\bar{\chi}(G(\Phi)) = 2\) if and only if \(\Phi\) is satisfiable. We use two auxiliary graphs \(H(x_i)\) and \(T(c_j)\) which are shown in Figure 1. We construct the planar graph \(\hat{G}(\Phi)\) from \(G(\Phi)\), for every \(x_i \in X\) replace the subgraph on \(\neg x_i\) and \(x_i\) by \(H(x_i)\), also replace every clause \(c_j\) of \(C\) by \(T(c_j)\). Furthermore, for every clause \(c_j\) with the literals \(x, y, z\) add the edges \(xy^j, yz^j\) and \(zx^j\). Call the resulting graph \(\hat{G}(\Phi)\). Clearly \(\bar{\chi}(\hat{G}(\Phi)) \geq 2\).

![Figure 1: The two auxiliary graphs \(H(x_i)\) and \(T(c_j)\)](image)

First suppose that \(\bar{\chi}(G(\Phi)) = 2\) and \(D\) is a proper orientation of \(G(\Phi)\) with maximum indegree 2. For each variables \(x_i\), the subgraph on \(x_i^1, x_i^2\) and \(x_i^3\) is a triangle, so \(\{d_D(x_i^1), d_D(x_i^2), d_D(x_i^3)\}\) \(=\) \(\{0, 1, 2\}\). Therefore the edges \(x_ix_i^1\) and \((\neg x_i)x_i^1\) were oriented from \(x_i^1\) to \(x_i\) and \(\neg x_i\). Consequently \(\{d_D(x_i), d_D(\neg x_i)\}\) \(=\) \(\{1, 2\}\). Let \(x_i\) and \(c_j\) be arbitrary variable and clause such that \(x_i\) or \(\neg x_i\) is used in \(c_j\). Since for every variable \(x_i\), \(\{d_D(x_i), d_D(\neg x_i)\}\) \(=\) \(\{1, 2\}\), therefore for every edge \(e\) between \(H(x_i)\) and \(T(c_j)\), \(e\) was oriented from \(H(x_i)\) to \(T(c_j)\). So we have the following.

**Fact 1.** For every edge \(e = v_is_j^t, (v_i \in \{x_i, \neg x_i\}, s_j^t \in \{s_j^1, s_j^2, s_j^3\})\), we have \(d^-_D(s_j^t) = 3 - d^+_D(v_i)\). \(\triangleleft\)

Therefore, for every \(t, 1 \leq t \leq 3\), two cases can be considered: \(d^-_D(s_j^t) = 1\) or \(d^-_D(s_j^t) = 2\).

- If \(d^-_D(s_j^t) = 1\), then \(s_j^{t+3}\) was oriented form \(s_j^t\) to \(s_j^{t+3}\) and so \(d^-_D(s_j^{t+3}) = 2\).
- If \(d^-_D(s_j^t) = 2\), then \(s_j^{t+3}\) was oriented form \(s_j^t\) to \(s_j^{t+3}\) and so \(d^-_D(s_j^{t+3}) \in \{0, 1\}\).

Let \(\Gamma : X \rightarrow \{True, False\}\) be a function such that if \(d_D(x_i) = 1\), then \(\Gamma(x_i) = True\) and if \(d_D(x_i) = 2\), then \(\Gamma(x_i) = False\). We show that \(\Gamma\) is a satisfying assignment for \(\Phi\). By Fact 1, it is enough to show that for every clause \(c_j\), there exists a \(s_j^t \in \{s_j^1, s_j^2, s_j^3\}\) such that \(d^-_D(s_j^t) = 2\). To the contrary suppose that \(d^-_D(s_j^1) = d^-_D(s_j^2) = d^-_D(s_j^3) = 1\), so, we have \(d^-_D(s_j^1) = d^-_D(s_j^2) = d^-_D(s_j^3) = 2\). Since \(D\) is a proper orientation, \(\{d^-_D(s_j^1), d^-_D(s_j^2), d^-_D(s_j^3)\}\) \(=\) \(\{0, 1\}\), so the edges \(s_j^1s_j^4\) and \(s_j^8s_j^5\) were oriented from \(s_j^7\) to \(s_j^4\) and \(s_j^8\) to \(s_j^5\), respectively. Similarly, \(d^-_D(s_j^{11}, d^-_D(s_j^{12})) = \{0, 1\}\) and the edge \(s_j^{12}s_j^4\) was oriented from \(s_j^{12}\) to \(s_j^4\). It means
Next, suppose that $\Phi$ is satisfiable with the satisfying assignment $\Gamma$. We present a proper orientation $D$ with maximum indegree 2. Let $x_i$ and $c_j$ be arbitrary variable and clause such that $x_i$ or $\neg x_i$ is used in $c_j$. For every edge $e = v_is^j_s$, ($v_i \in \{x_i, \neg x_i\}, s^j_s \in \{s^1_j, s^2_j, s^3_j\}$), orient $e$ from $v_i$ to $s^j_s$. For every $1 \leq i \leq n$, for each edge $e$ that is adjacent to $x_i^1$, orient $e$ from $x_i^1$ to the other end of $e$ and orient $x_i^2x_i^3$ from $x_i^2$ to $x_i^3$. Also if $\Gamma(x_i) = True$, orient $x_i\neg x_i$ from $x_i$ to $\neg x_i$, otherwise orient $x_i\neg x_i$ from $\neg x_i$ to $x_i$.

In order to orient the other edges, since $\Phi$ is satisfied by $\Gamma$ and by attention to the orientations of the edges $x_i\neg x_i$ ($1 \leq i \leq n$), for every clause $c_j$, at least one of its literals has indegree 1, therefore at least one of $s^1_j, s^2_j, s^3_j$ has indegree 2. With no loss of generality, three cases can be considered, the suitable orientations for these three cases are presented in Figure 2. This completes the proof.

Figure 2: The suitable orientations for three possible cases

Finding the optimal upper bound for $\chi'(G)$ seems to be an interesting intriguing problem. We state the following question for planar graphs.

**Problem 1.** Find the minimum number $k$ such that for every planar graph $G$, $\chi'(G) \leq k$.

4 Acknowledgment

We would like to thank Professor Douglas B. West and Professor Ebad S. Mahmoodian for their valuable answers to our questions about the definition of proper orientation number.
References

[1] L. Addario-Berry, R. E. L. Aldred, K. Dalal, and B. A. Reed. Vertex colouring edge partitions. *J. Combin. Theory Ser. B*, 94(2):237–244, 2005.

[2] L. Addario-Berry, K. Dalal, and B. A. Reed. Degree constrained subgraphs. *Discrete Appl. Math.*, 156(7):1168–1174, 2008.

[3] Yuichi Asahiro, Eiji Miyano, and Hirotaka Ono. Graph classes and the complexity of the graph orientation minimizing the maximum weighted outdegree. *Discrete Appl. Math.*, 159(7):498–508, 2011.

[4] Yuichi Asahiro, Eiji Miyano, Hirotaka Ono, and Kouhei Zenmyo. Graph orientation algorithms to minimize the maximum outdegree. *Internat. J. Found. Comput. Sci.*, 18(2):197–215, 2007.

[5] T. Bartnicki, J. Grytczuk, and S. Niwczyk. Weight choosability of graphs. *J. Graph Theory*, 60(3):242–256, 2009.

[6] J. A. Bondy and U. S. R. Murty. *Graph theory*. Graduate Texts in Mathematics 244, Springer, 2008 (2nd ed.).

[7] Mieczysław Borowiecki, Jarosław Grytczuk, and Monika Pilśniak. Coloring chip configurations on graphs and digraphs. *Inform. Process. Lett.*, 112(1-2):1–4, 2012.

[8] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[9] V. Chvátal and C. Thomassen. Distances in orientations of graphs. *J. Combinatorial Theory Ser. B*, 24(1):61–75, 1978.

[10] Ding-Zhu Du, Ker-K Ko, and J. Wang. *Introduction to Computational Complexity*. Higher Education Press, 2002.

[11] Andrzej Dudek and David Wajc. On the complexity of vertex-coloring edge-weightings. *Discrete Math. Theor. Comput. Sci.*, 13(3):45–50, 2011.

[12] Nicole Eggemann and Steven D. Noble. The complexity of two graph orientation problems. *Discrete Appl. Math.*, 160(4-5):513–517, 2012.

[13] T. Gallai. On directed paths and circuits. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 115–118. Academic Press, New York, 1968.
[14] Ian Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[15] Maciej Kalkowski, Michał Karoński, and Florian Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100(3):347–349, 2010.

[16] Michał Karoński, Tomasz Łuczak, and Andrew Thomason. Edge weights and vertex colours. *J. Combin. Theory Ser. B*, 91(1):151–157, 2004.

[17] Sanjeev Khanna, Joseph Naor, and F. Bruce Shepherd. Directed network design with orientation constraints. *SIAM J. Discrete Math.*, 19(1):245–257 (electronic), 2005.

[18] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, and B Stevens. Digraphs are 2-weight choosable. *Electron. J. Combin.*, 18(1):Paper 21,4, 2011.

[19] Łukasz Kowalik. Approximation scheme for lowest outdegree orientation and graph density measures. In *Algorithms and computation*, volume 4288 of *Lecture Notes in Comput. Sci.*, pages 557–566. Springer, Berlin, 2006.

[20] Daniel Leven and Zvi Galil. NP-completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4(1):35–44, 1983.

[21] V. Venkateswaran. Minimizing maximum indegree. *Discrete Appl. Math.*, 143(1-3):374–378, 2004.

[22] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.

[23] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.