Time correlation functions and Fisher zeros for q-deformed Bose gas

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Abstract – The time-dependent correlation functions of the q-deformed Bose gas are studied. We find the relation of zeros of the correlation functions with the Fisher zeros of the partition function of the system. The complex temperature appears as a result of q-deformation and evolution of the correlation function. A particular case of q-deformed Bose particles on two levels is examined and the zeros of correlation functions and Fisher zeros of the partition function are analyzed.

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Introduction. – Studies of the partition function zeros are fundamentally important. Due to papers [1–3] the studies become widely used in examinations of the thermodynamical properties of many-body systems, in considerations of phase transitions in various physical systems [4]. Also, the zeros of the partition function fully determine the analytic properties of the free energy of a system.

In contrast to the case of the Hamiltonian of a system with real parameters, in the case in which the parameters in the Hamiltonian are allowed to be complex, the partition function may have zeros which after the works of Lee and Yang [1,2] are called Lee-Yang zeros.

In paper [2] Lee and Yang considered the ferromagnetic Ising model with complex magnetic field and found the zeros of the partition function of the system. The authors proved the theorem that all zeros are purely imaginary. In [5] Lieb and Sokal proved that the Lee-Yang theorem holds for any Ising-like model with ferromagnetic interaction (see also [6–8]). The results of Lee and Yang were generalized to the case of complex temperature by Fisher in his paper [3].

For real temperature the partition function of a physical system cannot be equal to zero since the Boltzmann factor is always positive. The situation changes if the analytic continuation of the inverse temperature to the complex plane is made. In this case the partition function may have zeros called Fisher zeros. In [9] the authors presented a method to determine the order and strength of phase transitions on the basis of the analysis of the density of the Fisher zeros of the partition function. This method had success in statistical physics and lattice field theory [10,11]. The Fisher zeros were studied for different spin systems [9–16], Bose systems [17,18], polymers [19].

There are difficulties with the realization of many-body system with complex parameters in experiments. Therefore, for a long time the Lee-Yang zeros were only theoretically studied. The experimental access to the statistical theory of Lee and Yang was provided in [20] and the density function of zeros on the Lee-Yang circle was determined for a ferromagnet. Then five years ago it was shown that it is possible to observe the Lee-Yang zeros of the partition function of a spin system in experiments [21,22]. In paper [23] the authors reported the experimental observation of Lee-Yang zeros.

Within the framework of studies of the Lee-Yang zeros the analysis of the dynamical phase transitions was done [24]. In a recent paper [25] the experimental determination of the dynamical Lee-Yang zeros was reported.

We would like to mention that there are many papers devoted to studies of the zeros of the partition function for spin systems (see, for instance, [21,23,26,27] and references therein). At the same time the zeros of the partition function of the Bose system and Fermi system are not widely studied (see, for instance, [28–32]).
with experimentally observable quantities, namely, with zeros of two-time correlation functions. This relation in principle allows the experimental observation of the zeros of the partition function of the Bose system.

In the present paper we study the correlation functions of the \( q \)-deformed Bose gas which is based on the \( q \)-deformation of the canonical commutation relation that plays an important role in different branches of physics and mathematics (see, for instance, [34–42] and references therein).

The paper is organized as follows. In the next section we give a preliminary information on the \( q \)-deformed Bose gas. In the third section we find the relation of the zeros of the correlation function of the Bose system. Conclusions are presented in the last section.

**\( q \)-deformed Bose gas.** – We consider a system of \( N \) \( q \)-deformed Bose particles placed on \( s \) levels \( \epsilon_i \) \( (i = 1, 2, \ldots, s) \) and described by the following Hamiltonian (see, for instance, [43–45]):

\[
H = \sum_{i=1}^{s} \epsilon_i \hat{a}_i^+ \hat{a}_i,
\]

where the creation and annihilation operators of a boson on the \( i \)-th level \( \hat{a}_i^+ \), \( \hat{a}_i \) satisfy the \( q \)-deformed commutation relations

\[
[\hat{a}_i, \hat{a}_j^+]_q = \hat{a}_i \hat{a}_j^+ - q \hat{a}_j^+ \hat{a}_i = 1,
\]

where \( q > 0 \) is the real parameter of deformation. It is worth mentioning that the deformation of the commutation relations (2) with \( q > 1 \) is related with the quantized space with nonzero minimal uncertainties in both position and momentum [34,35]. Analysis and comparison of the cases with 0 < \( q < 1 \) and \( q > 1 \) can be found in [46]. It is also worth noting that the commutation relation (2) corresponds to the Ariki-Coon oscillator algebra [47].

Operators, corresponding to different levels, commute

\[
[\hat{a}_i, \hat{a}_j^+] = 0, \quad i \neq j.
\]

The operator of number of particles \( \hat{n}_i \) on the \( i \)-th level satisfies the following commutation relations:

\[
[\hat{a}_i, \hat{n}_i] = \hat{a}_i, \quad [\hat{a}_i^+, \hat{n}_i] = -\hat{a}_i^+.
\]

These relations are in fact the definition of the occupation number operator for the \( q \)-deformed case (see, for instance, [36]).

The operators \( \hat{a}_i, \hat{a}_i^+ \) and \( \hat{n}_i \) can be represented as (see, for instance, [34])

\[
\hat{a}_i = \hat{b}_i f(\hat{b}_i^+ \hat{b}_i), \quad \hat{a}_i^+ = f(\hat{b}_i^+ \hat{b}_i) \hat{b}_i^+, \quad \hat{n}_i = \hat{b}_i^+ \hat{b}_i,
\]

where operators \( \hat{b}_i, \hat{b}_i^+ \) and the occupation number operator \( \hat{n}_i \) satisfy the ordinary commutation relations

\[
[\hat{b}_i, \hat{b}_j^+] = \delta_{ij}, \quad [\hat{b}_i, \hat{n}_i] = \hat{b}_i, \quad [\hat{b}_i^+, \hat{n}_i] = -\hat{b}_i^+.
\]

The function \( f(x) \) is defined as

\[
f(x) = \frac{\sqrt{|x|_q}}{x}, \quad |x|_q = \frac{q^x - 1}{q - 1}.
\]

Here \( |x|_q \) denotes the so-called \( q \)-deformed numbers. Direct verification shows that \( \hat{a}_i \) and \( \hat{a}_i^+ \) defined in (5) really satisfy the deformed commutation relation (2).

The eigenvalues of the number operator \( \hat{n}_i \) are occupation numbers \( n_i = 0, 1, 2 \ldots \). Note also that the eigenvalues of the operator \( \hat{a}_i^+ \hat{a}_i \) are \( q \)-deformed numbers,

\[
n_i|_q = \frac{q^{n_i} - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n_i-1}.
\]

One can easily find eigenstates of the Hamiltonian (1)

\[
|n_1, n_2, \ldots, n_s\rangle
\]

and corresponding energy levels

\[
E_{n_1, n_2, \ldots, n_s} = \sum_{i=1}^{s} \epsilon_i |n_i|_q,
\]

where \( n_i \) are occupation numbers, \( n_i = 0, 1, 2, \ldots \), and \( |n_i|_q \) is the \( q \)-deformed number which is defined in (8).

Note that we consider the canonical ensemble with fixed number of Bose particles \( N \), therefore the occupation numbers \( n_i \) satisfy the following condition:

\[
\sum_{i=1}^{s} n_i = N.
\]

In the next section we consider the time-dependent correlation functions of the \( q \)-deformed Bose gas and find the relation of the zeros of the functions with the Fisher zeros of the partition function.

**Correlation functions and Fisher zeros.** – Let us consider the following correlation functions of the \( q \)-deformed Bose system:

\[
\left< \prod_{j=1}^{s} \hat{a}_j^+ (t_1) \hat{a}_j (t_2) \right> = \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} \hat{a}_j^+ (t_1) \hat{a}_j (t_2) \ldots \hat{a}_j^+ (t_1) \hat{a}_j (t_2),
\]

where \( Z(\beta) \) is the partition function

\[
Z(\beta) = \text{Tr} e^{-\beta H},
\]

here \( \beta = 1/kT \) is the inverse temperature and

\[
\hat{a}_j (t) = e^{iHt/\hbar} \hat{a}_j e^{-iHt/\hbar}.
\]
Substituting (1) into (14), we obtain
\[
\hat{a}_j(t) = e^{i\epsilon_j t} \hat{a}_j / \hat{a}_j e^{-i\epsilon_j \hat{a}_j / \hbar}.
\] (15)

In order to rewrite \(\hat{a}_j(t)\) in the form which is convenient for calculation of the correlation functions, we use the following identities:
\[
\hat{a}_j \phi(\hat{a}_j^+ \hat{a}_j) = \phi(\hat{a}_j^+ \hat{a}_j + 1) \hat{a}_j,
\]
\[
\phi(\hat{a}_j^+ \hat{a}_j) = \phi(\hat{a}_j^+ \hat{a}_j / q - 1/q) \hat{a}_j = \hat{a}_j \phi(\hat{a}_j^+ \hat{a}_j / q - 1/q).
\] (16)

Here \(\phi\) is an arbitrary function for which the Taylor expansion exists. Rewriting the \(n\)-th term in Taylor expansion in the left-hand side of (16) as follows:
\[
\hat{a}_j(\hat{a}_j^+ \hat{a}_j)^n = \hat{a}_j(\hat{a}_j^+ \hat{a}_j)(\hat{a}_j^+ \hat{a}_j) \ldots (\hat{a}_j^+ \hat{a}_j) = (\hat{a}_j \hat{a}_j^+)(\hat{a}_j \hat{a}_j^+) \ldots (\hat{a}_j \hat{a}_j^+) \hat{a}_j = (\hat{a}_j \hat{a}_j^+)^n \hat{a}_j
\] (18)

and using the deformed commutation relation (2) we obtain the identity (16). Similarly we get the identity (17).

Using identity (17), we can rewrite (15) as follows:
\[
\hat{a}_j(t) = \hat{a}_j e^{-i\epsilon_j \tau / q} e^{-(1 - 1/q)\epsilon_j \hat{a}_j^+ \hat{a}_j / \hbar}.
\] (19)

The conjugated operator to \(\hat{a}_j(t)\) reads
\[
\hat{a}_j^+(t) = e^{i\epsilon_j \tau / q} e^{(1 - 1/q)\epsilon_j \hat{a}_j^+ \hat{a}_j / \hbar} \hat{a}_j^+.
\] (20)

Substituting (19) and (20) into (12), we find
\[
\bigg| \prod_{j=1}^s \hat{a}_j^+(t_1) \hat{a}_j(t_2) \bigg| = e^{i \sum_j \epsilon_j \tau / q \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} \prod_j \hat{a}_j^+ \hat{a}_j}.
\] (21)

where \(\tau = t_1 - t_2\). Taking into account (1), the expression for the correlation function can be rewritten in the following form:
\[
\bigg| \prod_{j=1}^s \hat{a}_j^+(t_1) \hat{a}_j(t_2) \bigg| = e^{i \sum_j \epsilon_j \tau / q \beta \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} \prod_j \hat{a}_j^+ \hat{a}_j},
\] (22)

where we introduce the complex temperature
\[
\bar{\beta} = \beta - i(1 - 1/q)\tau / \hbar = \beta + i\beta_1.
\] (23)

We would like to stress that the imaginary part \(\beta_1\) of the complex temperature is caused by the \(q\)-deformation and is related with the time of evolution. Note that in the case in which deformation is absent, namely \(q = 1\), the imaginary part of the complex temperature is equal to zero.

This result can be rewritten as
\[
\bigg| \prod_{j=1}^s \hat{a}_j^+(t_1) \hat{a}_j(t_2) \bigg| = e^{i \sum_j \epsilon_j \tau / q \beta \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H}}.
\] (24)

Taking Tr over the eigenstates (9) of the Hamiltonian (1), we find
\[
\frac{\beta}{\beta(\beta)} = \frac{\beta(\beta)}{\beta(\beta)} = \sum_{n_1=0}^N e^{-\beta \epsilon_1 n_1 + \epsilon_2 n_2 + \cdots + \epsilon_s n_s}.
\] (25)

Here the occupation numbers \(n_i\) satisfy condition (11). Therefore, the sum over the occupation numbers in (25) cannot be factorized.

It is important to note that the partition function (25) contains the complex temperature. Thus, we find the relation of the correlation function with the partition function containing the complex temperature (24).

**\(q\)-deformed Bose particles on two levels.** Let us study a particular case of two-level system of \(q\)-deformed Bose particles which is described by the Hamiltonian (1) with \(s = 2\). In this case, taking into account condition (11), we have \(n_2 = N - n_1\). So, the partition function (25) can be reduced to the following expression:
\[
\beta(\beta) = \sum_{n_1=0}^N e^{-\beta \epsilon_1 n_1 + \epsilon_2 [N - n_1]}.
\] (26)

For correlation function (22) in this case we have
\[
\bigg| \prod_{j=1}^s \hat{a}_j^+(t_1) \hat{a}_j^+(t_2) \hat{a}_j(t_2) \bigg| = e^{i \sum_j \epsilon_j \tau / q \beta \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H}} \cdot \sum_{n_1=0}^N e^{-\beta \epsilon_1 n_1 + \epsilon_2 [N - n_1]}[N - n_1] = e^{i \sum_j \epsilon_j \tau / q \beta \frac{1}{Z(\beta)}} Z_\epsilon(\beta),
\] (27)

where we introduce the notation
\[
Z_\epsilon(\beta) = \sum_{n_1=0}^N e^{-\beta \epsilon_1 n_1 + \epsilon_2 [N - n_1]}[N - n_1].
\] (28)

Note that the zeros of the correlation function correspond to the zeros of \(Z_\epsilon(\beta)\).

The result (27) can be derived also from (24). Indeed, using (24) for \(s = 2\) we have
\[
\bigg| \prod_{j=1}^s \hat{a}_j^+(t_1) \hat{a}_j^+(t_2) \hat{a}_j(t_2) \bigg| = \frac{1}{Z(\beta)} \frac{1}{Z(\beta)} = e^{i \sum_j \epsilon_j \tau / q \beta \frac{1}{Z(\beta)}} \frac{1}{Z(\beta)} \frac{1}{Z(\beta)} \frac{1}{Z(\beta)} \frac{1}{Z(\beta)} Z(\beta),
\] (29)
where $Z(\tilde{\beta})$ is given by (26). Substituting (26) into (29) and taking derivatives over $\epsilon_1$ and $\epsilon_2$ we find (27).

The Fisher zeros of the partition function and the zeros of the correlation function can be found analytically for a small number of particles. In the case of $N = 1$ the correlation function is zero and the partition function reads

$$Z(\tilde{\beta}) = e^{-\tilde{\beta}\epsilon_1} \left( 1 + e^{-\tilde{\beta}\Delta\epsilon} \right),$$

(30)

where $\Delta\epsilon = \epsilon_2 - \epsilon_1$. In this case the equation for the zeros of the partition function $Z(\tilde{\beta}) = 0$ has the following solutions:

$$\beta = 0, \quad \beta_1 \Delta\epsilon = \pi(2n + 1), \quad n = 0, \pm 1, \pm 2, \ldots$$

(31)

Here $\beta$ and $\beta_1$ are the real and imaginary parts of $\tilde{\beta}$. It is convenient to introduce a new variable $z = e^{-\tilde{\beta}\epsilon_1}$. Then the possible zeros lay on the circle of the unit radius in the $z$-plane. Note that according to (23) at $q = 1$ the imaginary part of the temperature reads $\beta_1 = 0$. Therefore, in the case in which $q = 1$ we have no solutions for the zeros of the partition function.

In the case of a system of two particles one also has a trivial result for the correlation function, it is equal to zero. In the case in which $N = 3$ the zeros of the correlation function can be found analytically and they are nontrivial. We have

$$Z_c(\tilde{\beta}) = (1 + q)e^{-\tilde{\beta}(\epsilon_1(q + 1) + \epsilon_2)}(1 + e^{-\tilde{\beta}q\Delta\epsilon}).$$

(32)

The zeros of the correlation function in this case are achieved at

$$\beta = 0, \quad \beta q\Delta\epsilon = \pi(2n + 1), \quad n = 0, \pm 1, \pm 2, \ldots$$

(33)

From (23) and (33) we find that the correlation function in this case has zeros at times

$$\tau = \frac{\hbar}{(q - 1)\Delta\epsilon} \pi(2n + 1), \quad n = 0, \pm 1, \pm 2, \ldots$$

(34)

The zeros for the correlation function in this case lie on the circle of the unit radius in the $z$-plane. For $q = 1$ the correlation function does not have finite zeros, namely, the zeros of the correlation function tend to infinity at $q \to 1$.

We would like to note that in the general case of an arbitrary number of particles $N$ the zeros of the partition function and the zeros of the correlation function determined on the $z$-plane ($z = e^{-\tilde{\beta}\epsilon_1}$) are roots of polynomials with real powers,

$$Z = \sum_{n_1=0}^{N} z^{[n_1]_q + [N-n_1]_q \epsilon_2 / \epsilon_1},$$

(35)

$$Z_c = \sum_{n_1=0}^{N} z^{[n_1]_q + [N-n_1]_q \epsilon_2 / \epsilon_1}[n_1]_q[N-n_1]_q.$$  

(36)

In the particular case in which the parameter of deformation reads $q = 2$ we have that the $q$-numbers

$$[n_1]_q = 2^{n_1} - 1, \quad [N-n_1]_q = 2^{N-n_1} - 1$$

(37)

are integer. So, when in addition $\epsilon_2 / \epsilon_1$ is integer, the expressions (35) and (36) can be rewritten in the form of polynomial over $z$ with integer powers

$$Z = \sum_{n_1=0}^{N} z^{(2^{n_1} - 1) + (2^{N-n_1} - 1) \epsilon_2 / \epsilon_1},$$

(38)

$$Z_c = \sum_{n_1=0}^{N} z^{(2^{n_1} - 1) + (2^{N-n_1} - 1) \epsilon_2 / \epsilon_1}(2^{n_1} - 1)(2^{N-n_1} - 1).$$

(39)

The Fisher zeros of the partition function and correlation function in this case are presented in fig. 1. Note that the number of the Fisher zeros exceed the number of particles. This is the result of the exponential dependence of energy on the number of $q$-deformed Bose particles on a given level.

**Conclusion.** In this paper the time-dependent correlation functions of the $q$-deformed Bose gas have been...
studied. Our main result is the relation of the correlation function of the $q$-deformed Bose system with the partition function which depends on the complex temperature. Namely, we have found that correlation functions can be represented as (24). So, there is the relation of the zeros of the correlation function of the $q$-deformed Bose gas with the Fisher zeros of the partition function. It is important to note that the complex temperature is caused by $q$-deformation and evolution of the system. If deformation is absent, namely if $q = 1$, the imaginary part of the temperature is equal to zero.

The particular case of a system of $q$-deformed Bose particles on two levels has been examined. We have found analytically that the zeros of the partition function for one particle $N = 1$ and the zeros of the correlation function for $N = 3$ lie on the unit circle in the $z$-plane which corresponds to purely imaginary zeros in the $\beta$-plane. In this case the zeros of the correlation function are achieved during the evolution in the times which are given by (34).

We have also considered the particular case in which $q = 2$ and the ratio of energies of two levels $\epsilon_2/\epsilon_1$ is integer. In this case the zeros of the partition function and the zeros of the correlation function correspond to the roots of a polynomial with integer powers. The Fisher zeros of the partition function and correlation function for the $q$-deformed Bose gas in the case of $q = 2$ for particular numbers of particles of the system are presented in fig. 1. Note that, as is shown in the figure, the zeros lie close to the unit circle but not exactly on the circle.

Note that in contrast to the ordinary Bose gas where the interaction is responsible for the Lee-Yang zeros (as was shown in [33]), the $q$-deformation leads to the Fisher zeros.

Finally, we would like to note the following important result of this paper. We find that the time-dependent correlation function of the $q$-deformed Bose gas is related with the partition function with the complex temperature. We show that the complex parameter effectively appears because of the $q$-deformation. So, we obtain the relation of the zeros of the correlation function with the Fisher zeros. This relation in principle gives additional possibilities to examine the Fisher zeros which have fundamental importance.

We would like also to mention that it is known that there are difficulties with the direct realization of a many-body system with complex parameters in experiments. Therefore, the experimental observation of the Fisher or Lee-Yang zeros is not a simple problem. It is the important to note that time-dependent correlation functions are in principle experimentally observable quantities. Thus, in this paper we relate the Fisher zeros with observable quantities. It is worth mentioning that a $q$-deformed oscillator recently was realized using the RLC circuit [48]. So, we hope that the relation of the Fisher zeros with the zeros of time-dependent correlation functions of the $q$-deformed Bose gas will allow the experimental observation of the Fisher zeros for the system proposed in [48]. In addition, our result can be also useful for the description of the Bose gas in quantized space, namely in space with minimal length and minimal momenta, where $q$-deformation appears on the physical level [34,35].

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REFERENCES

[1] Yang C. N. and Lee T. D., Phys. Rev., 87 (1952) 404.
[2] Lee T. D. and Yang C. N., Phys. Rev., 87 (1952) 410.
[3] Fisher M. E., in Lectures in Theoretical Physics, edited by Brittin W. E., Vol. 7c (University of Colorado Press, Boulder, Col.) 1965, p. 1.
[4] Wu F. Y., Int. J. Mod. Phys. B, 22 (2008) 1899.
[5] Lieb E. H. and Sokal A. D., Commun. Math. Phys., 80 (1981) 153.
[6] Kozitsky Yu. V., J. Stat. Phys., 87 (1997) 799.
[7] Kozitsky Yu., Appl. Math. Comput., 141 (2003) 103.
[8] Kozitsky Yu. and Wołowski L., Complex Variables Theory Appl.: Int. J., 44 (2001) 225.
[9] Janke W. and Kenna R., J. Stat. Phys., 102 (2001) 1211.
[10] Janke W. and Kenna R., Comput. Phys. Commun., 147 (2002) 443.
[11] Janke W. and Kenna R., Nucl. Phys. B Proc. Suppl., 106 (2002) 905.
[12] De Szeze L. and Itzykson C., J. Stat. Phys., 33 (1983) 3.
[13] Dolan B. P., Johnstone D. A. and Stathakopoulos M., J. Phys. A: Math. Gen., 34 (2001) 6211.
[14] Janke W., Johnstone D. A. and Stathakopoulos M., Nucl. Phys. B, 614 (2001) 494.
[15] Ghulghazaryan R. G. and Ananikian N. S., J. Phys. A: Math. Gen., 36 (2003) 6297.
[16] Hovhannisyan V. V., Ananikian N. S. and Kenna R., Physica A, 453 (2016) 116.
[17] Mulken O., Bormann P., Harting J. and Stamm-Johanns H., Phys. Rev. A, 64 (2001) 013611.
[18] van Dyk W., Lobo C., MacDonald A. and Bhadur R. K., Can. J. Phys., 93 (2015) 830.
[19] Rocha J. C. S., Schnibel S., Landau D. P. and Bachmann M., Phys. Rev. E, 90 (2014) 022601.
[20] Binke C., Phys. Rev. Lett., 81 (1998) 5644.
[21] Wei Bo-Bo and Liu Ren-Bao, Phys. Rev. Lett., 109 (2012) 185701.
[22] Wei Bo-Bo, Chen Shao-Wen, Po Hoi-Chun and Liu Ren-Bao, Sci. Rep., 4 (2014) 5202.
[23] Peng Xinhua, Zhou Hui, Wei Bo-Bo, Cui Jiangyu, Du Jiangfeng and Liu Ren-Bao, Phys. Rev. Lett., 114 (2015) 010601.
[24] Flindt Christian and Garrahan Juan P., Phys. Rev. Lett., 110 (2013) 050601.
[25] Brandner Kay, Maisi Ville F., Pekola Jukka P., Garrahan Juan P. and Flindt Christian, Phys. Rev. Lett., 118 (2017) 180601.
[26] Krasnytska M., Berche B., Holovatch Yu. and Kenna R., EPL, 111 (2015) 60009.
[27] Krasnytska M., Berche B., Holovatch Yu. and Kenna R., J. Phys. A, 49 (2016) 135001.
[28] Mulken O., Borrmann P., Harting J. and Stammerjohanns H., Phys. Rev. A, 64 (2001) 013611.
[29] van Dijk W., Lobo C., MacDonald A. and Bhaduri R. K., Can. J. Phys., 93 (2015) 830.
[30] Borrmann P., Mulken O. and Harting J., Phys. Rev. Lett., 84 (2000) 3511.
[31] Bhaduri R. K., MacDonald A. and van Dijk W., EPL, 96 (2011) 56003.
[32] Zvyagin A. A., Phys. Rev. B, 95 (2017) 075122.
[33] Gnatenko Kh. P., Kargol A. and Tkachuk V. M., Phys. Rev. E, 96 (2017) 032116.
[34] Quesne C., Pensn K. A. and Tkachuk V. M., Phys. Lett. A, 313 (2003) 29.
[35] Quesne C. and Tkachuk V. M., J. Phys. A: Math. Gen., 36 (2003) 10373.
[36] Burban I. M., Phys. Lett. A, 366 (2007) 308.
[37] Borowiec A., Lukierski J. and Tolstoy V. N., Eur. Phys. J. C, 57 (2008) 601.
[38] Dey S., Fring A., Gouba L. and Castro P. G., Phys. Rev. D, 87 (2013) 084033.
[39] Dey S., Phys. Rev. D, 91 (2015) 044024.
[40] Gavriliuk A. M. and Kachurik I. I., Mod. Phys. Lett. A, 31 (2016) 1650024.
[41] Dey S. and Hussin V., Phys. Rev. A, 93 (2016) 053824.
[42] Jayakrishnan M. P., Dey S. and Sudheesh M. F. C., Ann. Phys., 385 (2017) 584.
[43] Su Gang and Ge Mo-lin, Phys. Lett. A, 173 (1993) 17.
[44] Manko V. I. Marmon G., Solimeno S. and Zaccaria F., Phys. Lett. A, 176 (1993) 173.
[45] Chang Zhe and Chen Shao-xia, J. Phys. A: Math. Gen., 35 (2002) 9731.
[46] Quesne C., Pensn K. A. and Tkachuk V. M., Phys. Lett. A, 313 (2003) 29, 322 (2004) 402.
[47] Arik M. and Coon D. D., J. Math. Phys., 17 (1976) 524.
[48] Batouli J., El Baz M. and Maouni A., Phys. Lett. A, 379 (2015) 1619.