ON THE BRAUER $p$-DIMENSION OF HENSELIAN DISCRETE VALUED FIELDS OF RESIDUAL CHARACTERISTIC $p > 0$

IVAN D. CHIPCHAKOV

Abstract. Let $(K, v)$ be a Henselian discrete valued field with residue field $\hat{K}$ of characteristic $p > 0$, and $\text{Br}_d(K)$ be the Brauer $p$-dimension of $K$. This paper shows that $\text{Br}_d(K) \geq n$ if $[\hat{K} : \hat{K}^p] = p^n$, for some $n \in \mathbb{N}$. It proves that $\text{Br}_d(K) = \infty$ if and only if $[\hat{K} : \hat{K}^p] = \infty$.

1. Introduction

Let $E$ be a field, $\text{Br}(E)$ its Brauer group, $s(E)$ the class of associative finite-dimensional central simple algebras over $E$, and $d(E)$ the subclass of division algebras $D \in s(E)$. For each $A \in s(E)$, let $[A]$ be the equivalence class of $A$ in $\text{Br}(E)$, and let $\deg(A)$, $\text{ind}(A)$, $\text{exp}(A)$ be the degree, the Schur index and the exponent of $A$, respectively. It is well-known (cf. [26], Sect. 14.4) that $\text{exp}(A)$ divides $\text{ind}(A)$ and shares with it the same set of prime divisors; also, $\text{ind}(A) | \deg(A)$, and $\deg(A) = \text{ind}(A)$ if and only if $A \in d(E)$. Note that if $B_1, B_2 \in s(E)$ and $\text{g.c.d.}\{\text{ind}(B_1), \text{ind}(B_2)\} = 1$, then $\text{ind}(B_1 \otimes_E B_2) = \text{ind}(B_1)\text{ind}(B_2)$; equivalently, if $B'_j \in d(E)$, $j = 1, 2$, and $\text{g.c.d.}\{\deg(B'_1), \deg(B'_2)\} = 1$, then $B'_1 \otimes_E B'_2 \in d(E)$ (see [26], Sect. 13.4).

Since $\text{Br}(E)$ is an abelian torsion group and $\text{ind}(A)$, $\text{exp}(A)$ are invariants both of $A$ and $[A]$, these results show that the study of the restrictions on the pairs $\text{ind}(A)$, $\text{exp}(A)$, $A \in s(E)$, reduces to the special case of $p$-primary pairs, for an arbitrary prime $p$. The Brauer $p$-dimensions $\text{Br}_d(E)$, $p \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers, are defined as in [4], and contain essential information on these restrictions. We say that $\text{Br}_d(E) = n < \infty$, for a given $p \in \mathbb{P}$, if $n$ is the least integer $\geq 0$, for which $\text{ind}(P) | \text{exp}(P)^n$ whenever $P \in s(E)$ and $[P]$ lies in the $p$-component $\text{Br}(E)_p$ of $\text{Br}(E)$; if no such $n$ exists, we put $\text{Br}_d(E) = \infty$. For instance, $\text{Br}_d(E) \leq 1$, for all $p \in \mathbb{P}$, if and only if $E$ is a stable field, i.e. $\text{deg}(D) = \text{exp}(D)$, for each $D \in d(E)$; $\text{Br}_d(E) = 0$, for some $p' \in \mathbb{P}$, if and only if the $p'$-component $\text{Br}(E)_{p'}$ of $\text{Br}(E)$ is trivial.

The absolute Brauer $p$-dimension $\text{abrd}_d(E)$ of $E$ is defined to be the supremum of $\text{Br}_d(R)$: $R \in \text{Fe}(E)$, where $\text{Fe}(E)$ is the set of finite extensions of $E$ in a separable closure $E_{\text{exp}}$. This trivially implies $\text{abrd}_d(E) \geq \text{Br}_d(E)$, for each $p$. We have $\text{abrd}_d(E) \leq 1$, $p \in \mathbb{P}$, if $E$ is an absolutely stable field, i.e. its finite extensions are stable fields. Class field theory gives examples

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of such fields: it shows that \( \text{Brd}_p(\Phi) = \text{abrd}_p(\Phi) = 1 \), \( p \in \mathbb{P} \), if \( \Phi \) is a global or local field (see, e.g., [27], (31.4) and (32.19)). The same equalities hold, if \( \Phi = \Phi_0((X))((Y)) \) is an iterated formal Laurent power series field in 2 variables over a quasifinite field \( \Phi_0 \) (see [3], Corollary 4.5 (ii)).

The knowledge of the sequence \( \text{Brd}_p(E), \text{abrd}_p(E) : p \in \mathbb{P} \), is helpful for better understanding the behaviour of index-exponent relations over finitely-generated transcendental extensions of \( E \) [3]. This is demonstrated by the description in [39] of the set of sequences \( \text{Brd}_p(K_q), \text{abrd}_p(K_q), p \in \mathbb{P}, p \neq q \), where \( K_q \) runs across the class of fields with Henselian valuations \( v_q \) whose residue fields \( \hat{K}_q \) are perfect of characteristic \( q \geq 0 \), such that their absolute Galois groups \( \mathcal{G}_{\hat{K}_q} = \mathcal{G}(\hat{K}_q,\text{sep}/\hat{K}_q) \) are projective profinite groups, in the sense of [29]. The description relies on formulae for \( \text{Brd}_p(K_q), p \neq q \), which depend only on whether \( \hat{K}_q \) contains a primitive \( p \)-th root of unity. Thus \( \text{Brd}_p(K_q) \) is determined, for each \( p \neq q \), by two invariants: one of the value group \( v_q(\hat{K}_q) \), and one of the Galois group \( \mathcal{G}((\hat{K}_q(p)/\hat{K}_q)) \) of the maximal \( p \)-extension \( \hat{K}_q(p) \) of \( \hat{K}_q \) in \( \hat{K}_q \),sep.

A formula for \( \text{Brd}_q(K_q) \) in terms of invariants of \( \hat{K}_q \) and \( v_q(K_q) \) has also been found when \( \text{char}(K_q) = q > 0 \), \( \hat{K}_q \) is perfect and \( (K_q,v_q) \) is a maximally complete field (see [10], Proposition 3.5). By definition, the imposed restriction on \( (K_q,v_q) \) means that it does not admit immediate proper extensions, i.e. valued extensions \( (K'_q,v'_q) \neq (K_q,v_q) \) with \( \hat{K}_q = \hat{K}_{q'} \) and \( v'_q(K'_q) = v_q(K_q) \). The considered fields are singled out by the fact (established by Krull, see [32], Theorem 31.24 and page 483) that every valued field \( (L,q) \) has an immediate extension \( (L_1,\lambda_1) \) that is a maximally complete field. Note here that no formula for \( \text{Brd}_q(K_q) \) as above exists if \( (K_q,v_q) \) is only Henselian. More precisely, one can show using suitably chosen valued subfields of maximally complete fields that if \( (K,v) \) runs across the class of Henselian fields of characteristic \( q \), then \( \text{Brd}_q(K) \) does not depend only on \( \hat{K} \) and \( v(K) \). Specifically, it has been proved (see [10], Example 3.7) that for any integer \( t \geq 2 \), the iterated formal Laurent power series field \( Y = \mathbb{F}_q((T_1)) \ldots ((T_t)) \) in \( t \) variables over the field \( \mathbb{F}_q \) with \( q \) elements possesses subfields \( K_\infty \) and \( K_n, n \in \mathbb{N} \), such that:

\[
(1.1) \quad \begin{align*}
(a) & \quad \text{Brd}_q(K_\infty) = \infty; \quad n + t - 1 \leq \text{Brd}_q(K_n) \leq n + t, \text{ for each } n \in \mathbb{N}; \\
(b) & \quad \text{The valuations } v_m \text{ of } K_m, m \leq \infty, \text{ induced by the standard } \mathbb{Z}^t\text{-valued valuation of } Y \text{ are Henselian with } \hat{K}_m = \mathbb{F}_q \text{ and } v_m(K_m) = \mathbb{Z}^t; \text{ here } \mathbb{Z}^t \text{ is viewed as an abelian group endowed with the inverse-lexicographic ordering.}
\end{align*}
\]

Statement (1.1) attracts interest in the study of Brauer \( p \)-dimensions of Henselian fields of residual characteristic \( p > 0 \) from suitably chosen special classes. This paper considers \( \text{Brd}_p(K) \), for a Henselian discrete valued field (abbr., an HDV-field) \( (K,v) \) with \( \text{char}(\hat{K}) = p \). Our research is related to the problem of describing index-exponent relations over finitely-generated field extensions. It proves the right-to-left implication in the equivalence

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\text{Brd}_p(K) = \infty \iff \text{the degree } [\hat{K} : \hat{K}^p] \text{ is infinite (in case char}(K) = 0, \text{ the inverse implication is a consequence of } [25], \text{ Corollary 2.5, see also Fact } 3.5), \hat{K}^p \text{ being the subfield } \{u^p : u \in \hat{K}\}. \text{ When } [\hat{K} : \hat{K}^p] < \infty, \text{ we prove the
lower bound in the following conjecture (stated by Bhaskhar and Haase [5] for complete discrete valued fields):

**Conjecture 1.1.** If \((K,v)\) is an HDV-field with \(\text{char}(\hat{K}) = p > 0\) and \([\hat{K}: K^p] = p^n\), for some \(n \in \mathbb{N}\), then \(n \leq \text{abrd}_p(K) \leq n + 1\).

Conjecture 1.1 has been stated at the end of [5], under the extra hypothesis that \(\text{char}(K) = 0\) and \(\text{char}(\hat{K}) = p\). This restriction is not emphasized in the present paper, as we prove Conjecture 1.1 in case \(\text{char}(K) = p\) (see Proposition 7.1). Note also that the class of HDV-fields \((K,v)\) with \(\text{char}(\hat{K}) = p > 0\) and \([\hat{K}: K^p] = p^n\) is closed under taking finite extensions (cf. [14], Corollary 14.2.2, and [5], Lemma 2.12). It is therefore clear that the upper bound in Conjecture 1.1 will follow, if the inequality \(\text{Brd}_p(K) \leq n + 1\) holds, for an arbitrary HDV-field \((K,v)\) with \(\hat{K}\) as above. The inequality \(\text{Brd}_p(K) \geq n\) implies trivially the lower bound \(n \leq \text{abrd}_p(K)\) in Conjecture 1.1. It attracts interest in finding formulae for \(\text{Brd}_p(K)\), for example, when \((K,v)\) belongs to some basic classes of HDV-fields of residual characteristic \(p\) (see Conjecture 7.3 and Problem 7.4).

**Basic notation and abbreviations used in the paper**

- \(\mathbb{P}\) - the set of prime numbers; \(\mathbb{N}\) - the set of positive integers; \(\mathbb{Z}\) - the set (additive group, ring) of integers;
- \(\mathbb{Q}\) and \(\mathbb{R}\) the additive groups (the fields) of rational numbers and of real numbers, respectively;
- Abbreviations: HDV - Henselian discrete valued; TR - totally rami-
fied;
- For any field \(E\), we use the following notation:
  - \(E^*\) is the multiplicative group of \(E\); \(E^{*n}\) is the subgroup of \(n\)-th powers \(E^{*n} = \{\alpha^n : \alpha \in E^*\}\), for each \(n \in \mathbb{N}\);
  - \(s(E)\) - the class of associative finite-dimensional central simple \(E\)-
algebras, \(d(E)\) - the subclass of division algebras \(D \in s(E), \text{Br}(E)\) -
  the Brauer group of \(E\);
  - \(E_{\text{sep}}\) is a separable closure of \(E\); \(\text{Fe}(E)\) is the set of finite extensions of \(E\) in \(E_{\text{sep}}\); \(G_E := G(E_{\text{sep}}/E)\) is the absolute Galois group of \(E\); \(N(E_1/E)\) denotes the norm group of the extension \(E_1/E\), for any \(E_1 \in \text{Fe}(E)\);
  - For each \(p \in \mathbb{P}\), \(p\text{Br}(E) = \{b_p \in \text{Br}(E) : \ p b_p = 0\}\) is the maximal subgroup of \(\text{Br}(E)\) of period dividing \(p\), \(\text{Br}(E)_p\) - the \(p\)-component of \(\text{Br}(E)\), \(\text{Brd}_p(E)\) - the Brauer \(p\)-dimension of \(E\), \(\text{abrd}_p(E)\) - the absolute Brauer \(p\)-dimension of \(E\); also, \(E(p)\) is the maximal \(p\)-extension of \(E\) (in \(E_{\text{sep}}\)), and \(\text{cd}_p(G_E)\) - the cohomological \(p\)-dimensions of \(G_E\), in the sense of [29];
- For any field extension \(E'/E\), \(I(E'/E)\) denotes the set of intermediate fields of \(E'/E\), and \(\text{Br}(E'/E)\) is the relative Brauer group of \(E'/E\);

\(^1\)The Brauer \(p\)-dimension, in the sense of [29] and [5], means the same as the absolute Brauer \(p\)-dimension in the present paper.
Algebraic structures attached to a field $K$ with a nontrivial Krull valuation $v$: $O_v(K) = \{ a \in K : v(a) \geq 0 \}$ - the valuation ring of $(K,v)$; $M_v(K) = \{ \mu \in K : v(\mu) > 0 \}$ - the maximal ideal of $O_v(K)$; $O_v(K)^* = \{ u \in K : v(u) = 0 \}$ - the multiplicative group of $O_v(K)$; $v(K)$ - the value group of $(K,v)$; $v(\hat{K})$ - a divisible hull of $v(K)$; for each $\gamma \in v(\hat{K})$, $\gamma \geq 0$, $\nabla_\gamma(K)$ denotes the set $\{ \lambda \in K : v(\lambda - 1) > \gamma \}$.

$\hat{K} = O_v(K)/M_v(K)$ is the residue field of $(K,v)$, and for any $\lambda \in O_v(K)$, $\hat{\lambda}$ is the residue class $\lambda + M_v(K)$; $(K,v)$ is said to be of mixed characteristic $(0,p)$ if $\text{char}(K) = 0$ and $\text{char}(\hat{K}) = p > 0$.

When $(K,v)$ is a real-valued field, $K_v$ stands for the completion of $K$ with respect to the topology induced by $v$, and $\bar{v}$ is the valuation of $K_v$ continuously extending $v$.

Given an HDV-field $(K,v)$ of mixed characteristic $(0,p)$, and a primitive $p$-th root of unity $\varepsilon \in K_{\text{sep}}$, we write $\beta \approx \beta'$ for some $\beta, \beta' \in K(\varepsilon)^*$, if $v(\beta - \beta') > p\kappa$, where $\kappa = v(p)/(p-1)$; given an element $\pi \in K$ with $0 < v(\pi) < p\kappa$, we write $\beta \sim \beta'$ if $v(\beta - \beta') > p\kappa - v(\pi) = v((1-\varepsilon)^p\pi^{-1})$.

2. Statement of the main result

Let $(K,v)$ be an HDV-field with $\text{char}(\hat{K}) = p > 0$. As shown in [23], if $\text{char}(K) = 0$ and $[\hat{K} : \hat{K}^p] = p^n$, for some $n \in \mathbb{N}$, then $[n/2] \leq \text{abrd}_p(K) \leq 2n$; $\text{abrd}_p(K) = \infty$ if and only if $[\hat{K} : \hat{K}^p] = \infty$ (this is contained in [24], Corollary 2.5 and Lemma 2.6). When $[\hat{K} : \hat{K}^p] = p^n$ and $n$ is odd, it has been proved in [5] that $\text{abrd}_p(K) \geq 1 + [n/2]$. The proofs of these results show their validity for $\text{Brd}_p(K)$ if $K$ contains a primitive $p$-th root of unity (see Remark [6.2]).

The purpose of the present paper, in the first place, is to prove the inequality $\text{Brd}_p(K) \geq n$ in general, and thereby, to obtain the inequality $\text{abrd}_p(K) \geq n$ in Conjecture [1.1]. Also, its major objective is to give an optimal infinitude criterion for $\text{Brd}_p(K)$. Our main result can be stated as follows:

**Theorem 2.1.** Let $(K,v)$ be an HDV-field with $\text{char}(\hat{K}) = p > 0$. Then:

(a) $\text{Brd}_p(K)$ is infinite if and only if $\hat{K}/\hat{K}^p$ is an infinite extension;

(b) There exists $D \in d(K)$ with $\exp(D) = p$ and $\deg(D) = p^n$, provided that $[\hat{K} : \hat{K}^p] = p^n$, for some $n \in \mathbb{N}$; in particular, $\text{Brd}_p(K) \geq n$.

Theorem 2.1 (b) and the right-to-left implication in Theorem 2.1 (a) are proved in Section 6. For our proof, we construct in Section 3 an algebra $D \in d(K)$ with $\exp(D) = p$ and $\deg(D) = p^n$, assuming that $K$ has a TR and Galois extension $M_\mu$ of degree $[M_\mu : K] = p^n \leq [\hat{K} : \hat{K}^p]$ with an abelian Galois group $G(M_\mu/K)$ of period $p$. By a TR-extension of $K$, we mean here a finite extension $M/K$ with $M = \hat{K}$. This agrees, by Lemma 3.2 (b), with the definition of a TR-extension over any valued field, given before the statement of Lemma 5.3 (for the case of a discrete valued field, see the paragraph before the statement of Lemma 5.2). The existence of
$M_\mu$ is a consequence of the following result, which is of independent interest when $(K,v)$ is of mixed characteristic $(0,p)$:

**Lemma 2.2.** Let $(K,v)$ be an HDV-field with $\text{char}(\hat{K}) = p > 0$ and $\hat{K}$ infinite. Then $K$ has TR extensions $M_\mu$, $\mu \in \mathbb{N}$, such that $[M_\mu : K] = p^\mu$, $M_\mu/K$ is a Galois extension and the group $G(M_\mu/K)$ is abelian of period $p$, for each $\mu$.

Theorem 2.1 and Lemma 2.2 have already been proved in case $\text{char}(K) = p$ (cf. [8], Lemma 4.2). Moreover, it follows that, in the setting of the lemma, if $\text{char}(\hat{K}) = p$, then each finite $p$-group $G$ is isomorphic to $G(M_G/K)$, for some TR and Galois extension $M_G$ of $K$ (see [10], Lemma 2.3). When $\text{char}(K) = 0$ and $(K,v)$ is an HDV-field of type II, in the sense of Kurihara, this is not true, for any cyclic $p$-group $G$ of sufficiently large order [18], 12.2, Theorem (b).

Lemma 2.2 is proved in Sections 5 and 6. Section 3 contains valuation-theoretic preliminaries used in the sequel. We also show there how Theorem 2.1 can be deduced from Lemma 2.2 (see Lemma 3.6). For reasons noted above, here we focus our attention on the mixed characteristic case $(0,p)$. For the proof of Lemma 2.2, we take into consideration whether or not $v(p) \in pv(K)$ (see Lemmas 4.8, 6.1 and Lemma 5.2 (b), respectively). Section 4 is devoted to the technical preparation for the proof of Lemma 2.2. As noted above, in Section 7, we prove Conjecture 1.1 for an HDV-field of characteristic $p$. Open questions concerning $\text{Brd}_p(K)$ are also posed in two frequently considered special cases.

The basic notation, terminology and conventions kept in this paper are standard and essentially the same as in [21], [30] and [8]. Missing definitions concerning central simple algebras can be found in [26]. Throughout, Brauer and value groups are written additively, Galois groups are viewed as profinite under the Krull topology, and by a profinite group homomorphism, we mean a continuous one. For any discrete valued field $(K,v)$, we suppose that $v(K)$ is chosen to be a subgroup of the additive group $\mathbb{Q}$ of rational numbers. By an $n$-dimensional local field, for some $n \in \mathbb{N}$, we mean a complete $n$-discretely valued field $K_n$, in the sense of [15] (see also [33]), with a quasifinite $n$-th residue field $K_0$.

### 3. Preliminaries

Let $K$ be a field with a (nontrivial) Krull valuation $v$. We say that $v$ is Henselian, if it extends uniquely, up-to equivalence, to a valuation $v_L$ on each algebraic extension $L$ of $K$. This holds, if $K = K_v$ and $(K,v)$ is a real-valued field, i.e. $v(K)$ is isomorphic to an ordered subgroup of the additive group $\mathbb{R}$ of real numbers (cf. [21], Ch. XII). Maximally complete fields are also Henselian, since Henselizations of valued fields are their immediate extensions (see [14], Theorem 15.3.5). The valuation $v$ is Henselian if and only if any of the following two equivalent conditions holds (cf. [14], Sect. 18.1, and [32], Theorem 32.19):
(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where $f'$ is the formal derivative of $f$, there is a zero $c \in O_v(K)$ of $f$ satisfying the equality $v(c - a) = v(f(a)/f'(a))$.

(b) For each normal extension $\Omega/K$, $v'(v(\mu)) = v(\mu)$ whenever $\mu \in \Omega$, $v'$ is a valuation of $\Omega$ extending $v$, and $\tau$ is a $K$-automorphism of $\Omega$.

When $(K,v)$ is real-valued, it is Henselian if and only if $K$ is (relatively) separably closed in $K_v$ (cf. [14], Theorems 15.3.5, 17.1.5). The following lemma allows to extend to the Henselian case results on complete real-valued fields (e.g., the Grunwald-Wang theorem, see [22] and Remark 5.3).

Lemma 3.1. Let $(K,v)$ be a real-valued field, $\bar{v}$ the continuous prolongation of $v$ on $K_v$, and $(K,v')$ an intermediate valued field of $(K_v,\bar{v})/(K,v)$. Suppose that $(K,v')$ is Henselian, identify $K_{v,\text{sep}}$ with its $K$-isomorphic copy in $K_{v,\text{sep}}$, and let $f$ be the mapping $\text{Fe}(\bar{K}) \to \text{Fe}(K_v)$, by the rule $\mathcal{N} \to \mathcal{N}K_v$.

Then:

(a) $K_{v,\text{sep}} \cap K_v = K$, and each $\lambda \in \text{Fe}(K_v)$ contains a primitive element $\lambda \in K_{v,\text{sep}}$ over $K_v$; such that $[K_v(\lambda):K_v] = [K(\lambda):K]$;

(b) $K_{\text{sep}}K_v = K_{v,\text{sep}}$ and $\mathcal{G}_K \cong \mathcal{G}_{K_v}$;

(c) The correspondence $f$ is bijective and degree-preserving; moreover, $f$ and the inverse mapping $f^{-1}: \text{Fe}(K_v) \to \text{Fe}(K)$, preserve the Galois property and the isomorphism class of the corresponding Galois groups;

(d) For each $\nu \in \mathbb{N}$ not divisible by $\text{char}(K)$, $K_{v^\nu}^c \cap K^c = K_{v^\nu}$.

Proof. The conditions on $(K,v)$ and $(K,v')$ ensure that $K_{v,\text{sep}} \cap K_v = K$. The latter part of Lemma 3.1 (a) can be deduced from Krasner’s lemma (see [20], Ch. II, Propositions 3, 4). Lemma 3.1 (c) follows from Lemma 3.1 (a) and Galois theory (cf. [21], Ch. VI, Theorem 1.12), and Lemma 3.1 (b) - from Lemma 3.1 (a), (c) and the definition of the Krull topology on $\mathcal{G}_K$ and $\mathcal{G}_{K_v}$. Lemma 3.1 (d) is implied by the density of $K$ in $K_v$, and by the fact that the set $\nabla_{\gamma}(K_v) = \{\alpha \in K_v: \bar{v}(\alpha - 1) > \gamma\}$ is an open subgroup of $K_{v^\nu}^c$, provided $\gamma \in \mathbb{R}$ is sufficiently large (one may put $\gamma = 0$ if $\text{char}(K) \nmid \nu$).

When $v$ is Henselian, so is $v_L$, for any algebraic field extension $L/K$; in this case, $\bar{L}/\bar{K}$ is algebraic as well. We write $v$ instead of $v_L$ and view $v(L)$ as an ordered subgroup of a fixed divisible hull $\bar{v}(K)$. This is allowed, since $v(K)$ is an ordered subgroup of $v(L)$, such that $v(L)/v(K)$ is a torsion group; hence, $v(L)$ embeds in $\bar{v}(K)$ as an ordered subgroup. These facts follow from Ostrowski’s theorem (see [14], Theorem 17.2.1), namely, the assertion that if $[L:K]$ is finite, then $[\bar{L}:\bar{K}]e(L/K)$ divides $[L:K]$ and $[L:K][\bar{L}:\bar{K}]^{-1}e(L/K)^{-1}$ has no divisor in $\mathbb{P}$, $p \neq \text{char}(\bar{K})$; here $e(L/K)$ denotes the ramification index of $L/K$ (the index $[v(L):v(K)]$ of $v(K)$ in $v(L)$). We state below several known criteria that $[L:K] = [\bar{L}:\bar{K}]e(L/K)$:

Lemma 3.2. Let $(K,v)$ be a Henselian field and $L/K$ a finite extension. Then $[L:K] = [\bar{L}:\bar{K}]e(L/K)$ in the following cases:

(a) If $\text{char}(\bar{K}) \nmid [L:K]$ (apply Ostrowski’s theorem);

(b) If $(K,v)$ is HDV and $L/K$ is separable (see [14], Sect. 17.4);

(c) When $(K,v)$ is maximally complete (cf. [32], Theorem 31.21).
Under the hypotheses of (c), if char$(K) = p > 0$, then $K^p$ is maximally complete (relative to the valuation induced by $v$) with a residue field $\hat{K}^p$ and a value group $\text{pv}(K)$; this ensures that $[K : K^p]$ is finite if and only if so are $[\hat{K} : \hat{K}^p]$ and the quotient group $v(K)/pv(K)$.

Assume that $(K, v)$ is a nontrivially valued field. A finite extension $R$ of $K$ is said to be inertial with respect to $v$, if $R$ has a unique (up-to equivalence) valuation $v_R$ extending $v$, the residue field $\hat{R}$ of $(R, v_R)$ is separable over $\hat{K}$, and $[R : K] = [\hat{R} : \hat{K}]$; $R/K$ is called a TR-extension with respect to $v$, if $v$ has a unique prolongation $v_R$ on $R$, and the index $[v_R(R) : v(K)]$ equals $[R : K]$. When $v$ is Henselian, $R/K$ is TR if and only if $e(R/K) = [R : K]$.

Inertial extensions of Henselian fields have useful properties, some of which are presented by the following lemma (for a proof, see [30], Theorem A.23):

**Lemma 3.3.** Let $(K, v)$ be a Henselian field. Then:

(a) An inertial extension $R'/K$ is Galois if and only if $\hat{R}'/\hat{K}$ is Galois. When this holds, $\mathcal{G}(R'/K)$ and $\mathcal{G}(\hat{R}'/\hat{K})$ are canonically isomorphic.

(b) The compositum $K_{ur}$ of inertial extensions of $K$ in $K_{sep}$ is a Galois extension of $K$ with $\mathcal{G}(K_{ur}/K) \cong \mathcal{G}_R$.

(c) Finite extensions of $K$ in $K_{ur}$ are inertial, and the natural mapping of $I(K_{ur}/K)$ into $I(\hat{K}_{sep}/\hat{K})$ is bijective.

It is known (cf. [28], Ch. 2, Sect. 7, and [30], Sect. 1.2.2) that if $(K, v)$ is Henselian, then $v$ extends on each $D \in d(K)$ to a unique valuation $v_D$, up-to equivalence. Put $v(D) = v_D(D)$ and denote by $\hat{D}$ the residue division ring of $(D, v_D)$. Note that $\hat{D}$ is a division $\hat{K}$-algebra with $[\hat{D} : \hat{K}] < \infty$, and $v(D)$ is an ordered abelian group including $v(K)$ is an ordered subgroup of finite index $e(D/K)$. In addition, the following holds, by [31], Proposition 2.2:

**Lemma 3.4.** If $(K, v)$ is an HDV-field, then $[D : K] = [\hat{D} : \hat{K}]v(D/K)$, for every $D \in d(K)$.

Next we state results on any HDV-field $(K, v)$ that are used in Section 7 for proving Conjecture [14] in the case of char$(K) = p$. They reduce the proof of the upper bound in this conjecture to considering only the case where $(K, v)$ is a complete discrete valued field (which allows to apply results of [25] and [5]):

**Fact 3.5.** (a) The scalar extension map $\text{Br}(K) \rightarrow \text{Br}(K_v)$ is an injective homomorphism which preserves Schur indices and exponents (cf. [12], Theorem 1, and [28], Ch. 2, Theorem 9); hence, $\text{Brd}_{p'}(K) \leq \text{Brd}_{p'}(K_v)$, for every $p' \in \mathbb{P}$;

(b) The valued field $(K_v, \bar{v})$ (see page 4) is maximally complete (cf. [28], Ch. 2, Theorem 8, or [30], Example 3.11); in addition, $(K_v, \bar{v})/(K, v)$ is an immediate extension (cf. [14], Theorem 9.3.2, or [21], Ch. XII, Sect. 5).

Let now $(K, v)$ be an HDV-field with char$(\hat{K}) = p$. Suppose that there exists a Galois extension $M/K$ with $\mathcal{G}(M/K)$ abelian of period $p$ and order $p^\mu$, for some $\mu \in \mathbb{N}$. Then, by Galois theory, $M$ equals the compositum
Let $L_1 \ldots L_\mu$ of degree $p$ (Galois) extensions $L_j$ of $K$ in $M$, $j = 1, \ldots, \mu$. This enables one to construct various algebras of degree $p^\mu$ presentable as tensor products of cyclic $K$-algebras of degree $p$ (concerning cyclic algebras in general, see, e.g., [26], Sect. 15). When $M/K$ is a TR-extension and $p^\mu \leq [\hat{K} : \hat{K}^p]$, our next lemma provides a criterion for an algebra of this type to lie in $d(K)$, which is used for proving Theorem 2.1. Before stating it, note that a finite system $\Theta$ of $m$ elements of a field $E$ with char$(E) = p$ is called $p$-independent over $E^p$, if $[E^p(\Theta) : E^p] = p^m$.

**Lemma 3.6.** Let $(K, v)$ be an HDV-field with char$(\hat{K}) = p > 0$, and let $M/K$ be a TR and Galois extension with $G(M/K)$ abelian of period $p$ and finite order $p^\mu \leq [\hat{K} : \hat{K}^p]$. Fix a presentation $M = L_1 \ldots L_\mu$ as a compositum of degree $p$ extensions of $K$ in $M$, take a generator $\sigma_j$ of $G(L_j/K)$, for each index $j$, and choose elements $a_j \in O_v(K)$, $j = 1, \ldots, \mu$, so that the system $\hat{a}_j \in \hat{K}$, $j = 1, \ldots, \mu$, be $p$-independent over $\hat{K}^p$. Then the tensor product $D_\mu = \otimes_{j=1}^\mu \Delta_j$ of the cyclic $K$-algebra $\Delta_j = (L_j/K, \sigma_j, a_j)$, $j = 1, \ldots, \mu$, lies in $d(K)$, where $\otimes = \otimes_K$. Moreover, $v(D_\mu) = v(M)$ and $D_\mu$ is a root field over $\hat{K}$ of the binomials $X^p - \hat{a}_j$, $j = 1, \ldots, \mu$, so $[D_\mu : \hat{K}] = p^\mu$.

The proof of Lemma 3.6 is done by induction on $\mu$, by the method of proving [8], Lemma 4.2 (b) (which covers the case of $p = \text{char}(K)$). For convenience of the reader, we outline its main steps. In fact, it suffices to prove that $D_\mu \in d(K)$; then the rest of the lemma can be deduced from Lemma 3.4, the equality $[D_\mu : K] = p^\mu$, and the existence of $K$-subalgebras $\Theta_\mu$ and $W_\mu$ of $D_\mu$, such that $\Theta_\mu \cong M$ and $W_\mu$ is a root field over $K$ of the binomials $X^p - a_j$, $j = 1, \ldots, \mu$. If $\mu = 1$, then $\hat{a}_1 \notin \hat{K}^p = \hat{D}_1^p$, which implies $a_1 \notin N(L_1/K)$; hence, by [26], Proposition 15.1 b, $D_1 \in d(K)$. When $\mu \geq 2$, it suffices to show that $D_\mu \in d(K)$, under the extra hypothesis that the centralizer $C = C_{D_\mu}(L_\mu)$ lies in $d(L_\mu)$. As $C = D_{\mu-1} \otimes_K L_\mu$, where $D_{\mu-1} = \otimes_{j=1}^{\mu-1} \Delta_j$, it is easy to see that $v_C(C) = v(M)$ and $\hat{C}$ equals the (commutative) field $\hat{K}(\sqrt{a_1}, \ldots, \sqrt{a_{\mu-1}})$; in particular, $\hat{C}$ does not possess nontrivial $K$-automorphisms. Observing that $D_\mu \in s(K)$, consider the $K$-automorphism $\varphi$ of $C$ which induces the identity on $D_{\mu-1}$ and the automorphism $\sigma_\mu$ of $L_\mu$. It follows from the Skolem-Noether theorem (see [26], Sect. 12.6) that $\varphi$ is induced by the inner automorphism of $D_\mu$ defined by conjugation by an element $x_\mu \in \Delta_\mu$ that induces $\sigma_\mu$ on $L_\mu$, satisfies $x_\mu^p = a_\mu$, and generates $D_\mu$ over $C$. Thus, $D_\mu$ is a cyclic generalized crossed product over $C$ as described in [2], Ch. XI, Theorems 10, 11. In view of (3.1) (b), it is easily verified that the composition $v_C \circ \varphi$ is a valuation of $C$ extending the prolongation of $v$ on $L_\mu$. As $v$ is Henselian, this means that $v_C \circ \varphi = v_C$ which implies $v_C(d) = 0$ and $\hat{d} = \hat{d}^p \in \hat{C}^p$, provided that $d = \prod_{i=0}^{p-1} \varphi^i(d')$, for some $d' \in C$ with $v_C(d') = 0$. Since $\hat{a}_\mu \notin \hat{C}^p$ (and $v_C(d) \neq 0$ if $v_C(d') \neq 0$), one thereby concludes that $\prod_{i=0}^{p-1} \varphi^i(d) \neq a_\mu$, for any $d \in C$. Hence, by the equality $x_\mu^p = a_\mu$ and the hypothesis that $C \in d(L_\mu)$, the assertion that $D_\mu \in d(K)$ can be obtained from [2], Ch. XI, Theorem 12, so Lemma 3.6 is proved.
Theorem 2.1 is implied by Lemmas 3.6 and 2.2, so our main goal in the rest of the paper is to prove Lemma 2.2. As noted in Section 2, one may consider only the case of char$(K) = 0$. Our next lemma is used in Section 5 for proving Lemma 2.2, under the extra hypothesis that $v(p) \notin pv(K)$.

**Lemma 3.7.** Let $(K,v)/(\Phi,\omega)$ be a valued field extension, such that the index $|v(K): \omega(\Phi)|$ of $\omega(\Phi)$ in $v(K)$ is finite, and let $\Psi$ be an extension of $\Phi$ in $K_{se}p$ of degree $p^\mu$, for some $p \in \mathbb{P}$, $\mu \in \mathbb{N}$. Suppose that $\Psi$ is TR over $\Phi$ relative to $\omega$, and $p \nmid |v(K): \omega(\Phi)|$. Then $\Psi K/K$ is TR relative to $v$ and $[\Psi K: K] = p^\mu$.

**Proof.** In view of [14], Theorem 15.3.5, and our assumptions, one may suppose, for the proof, that the value groups of all valuations of $\Psi K$ extending $\omega$ are ordered subgroups of $v(K)$. Let $v'$ be any valuation of $\Psi K$ extending $v$. By the Fundamental Inequality (cf. [14], Theorem 17.1.5),

$$(3.2) \frac{|v'(\Psi K): v(K)|}{|v(\Psi K): v(K)|} \leq \frac{[\Psi K: K]}{[\Psi: \Phi]} = p^\mu.$$ 

As $\Psi/\Phi$ is TR relative to $\omega$, $\Psi$ has a unique valuation $\omega'$ extending $\omega$. This shows that $\omega'$ equals the valuation of $\Psi$ induced by $v'$. Note further that

$$p^\mu = |\omega'(\Psi): \omega(\Phi)|,$$

$$|\omega'(\Psi): \omega(\Phi)| \mid |v'(\Psi K): \omega(\Phi)|$$

and

$$|v'(\Psi K): \omega(\Phi)| = |v(\Psi K): v(K)|.$$ 

Since $p \nmid |v(K): \omega(\Phi)|$ by hypothesis, it follows that $p^\mu \mid |v'(\Psi K): v(K)|$, which implies the inequalities in (3.2) must be equalities. Hence, $[\Psi K: K] = p^\mu$, and by the Fundamental Inequality, it turns out that $v'$ is the unique valuation of $\Psi K$ extending $v$ and, moreover, $\Psi K/K$ is TR relative to $v$, as required.  

The next lemma presents well-known properties of binomial extensions of prime degree, and of cyclotomic extensions. They are often used without an explicit reference (for a proof of the lemma, see [21], Ch. VI, Sects. 3, 9).

**Lemma 3.8.** Let $E$ be a field and $p \in \mathbb{P}$. Then:

(a) For any $\theta \in E^*$, the polynomial $X^p - \theta$ is irreducible over $E$ if and only if it has no root in $E$.

(b) If $L/E$ is a finite extension, such that $p \nmid [L: E]$, then $L^{p^\mu} \cap E^* = E^{p^\mu}$.

(c) If $p \neq \text{char}(E)$ and $\varepsilon$ is a primitive $p$-th root of unity in $E_{\text{sep}}$, then $E(\varepsilon)/E$ is a Galois extension with $G(E(\varepsilon)/E)$ cyclic and $[E(\varepsilon): E] = p - 1$; in particular, $E(\varepsilon)^{p^\mu} \cap E^* = E^{p^\mu}$.

At the end of this section we recall some known properties of cyclotomic fields that are used in the sequel.

**Lemma 3.9.** Let $(K,v)$ be a valued field of mixed characteristic $(0,p)$ containing a primitive $p$-th root of unity $\varepsilon$. Then:

(a) $v(1 - \varepsilon) = v(p)/(p-1)$;

(b) $v(-i + \sum_{j=0}^{i-1} \varepsilon^j) \geq v(1 - \varepsilon)$, for each $i \in \mathbb{N}$ not divisible by $p$;

(c) $v((1 - \varepsilon)^{p^\mu} + p) \geq v((1 - \varepsilon)^p) = pv(p)/(p-1)$. 


Proof. The assumption on $\text{char}(\hat{K})$ ensures that $v(p) > 0$, $\varepsilon \in O_v(K)$ and the residue class $\hat{e}$ equals the unit of $\hat{K}$. Therefore, $v(1 - \varepsilon) > 0$, and by the proof of Proposition 4.1.2 (i) of [13], $v(p) = (p - 1)v(1 - \varepsilon)$, as claimed by Lemma 3.9 (a). Also, the inequality $v(1 - \varepsilon) > 0$ implies $\hat{e}_i = i \neq 0$, for each $i \in \mathbb{N}$ not divisible by $p$. Let $\eta = \varepsilon - (\varepsilon^p - 1, p)$. From the fact that $\mathbb{Z}[\varepsilon] \subset O_v(K)$ and $\varepsilon - 1$ divides (in the ring $\mathbb{Z}[\varepsilon]$) the elements $e_i - i = \sum_{j=0}^{p-1}(\varepsilon^j - 1)$, $i = 1, \ldots, p - 1$. Clearly, Lemma 3.9 (b) shows that $v((p - 1)! - \prod_{i=1}^{p-1} e_i) \geq v(1 - \varepsilon)$, which implies together with the equalities

$$\Phi_p(1) = \prod_{i=1}^{p-1} (1 - \varepsilon^i) = p = (1 - \varepsilon)^{p-1} \prod_{i=1}^{p-1} e_i,$$

where $\Phi_p(X) = \sum_{j=0}^{p-1} X^j$ is the $p$-th cyclotomic polynomial, that

$$v((p - 1)! (1 - \varepsilon)^{p-1} - p) \geq v((1 - \varepsilon)^p) = pv(p)/(p - 1).$$

As $(p - 1)! \equiv -1 \pmod{p}$ (Wilson’s theorem), this proves Lemma 3.9 (c). \hfill \Box

4. Normal elements and radical degree $p$ extensions of HDV-fields

Our goal in this section is to prepare technically the proof of Lemma 2.2. In order to achieve it, we need information on the algebraic properties of $p$-th roots of elements of $\nabla_0(F)$, for a valued field $(F, v)$ of mixed characteristic $(0, p)$. A part of this information is contained in the following two lemmas.

Lemma 4.1. Let $(F, v)$ be a valued field of mixed characteristic $(0, p)$, and let $\alpha \in F$, $\beta \in F^*$ be elements, such that $(1 + \beta)^p = 1 + \alpha$ and $v(\alpha) > 0$. Put $\eta = \alpha - \beta^p - p \beta$ and $\kappa = v(p)/(p - 1)$. Then $v(\eta) \geq v(p) + 2v(\beta)$. Moreover,

(a) $v(\alpha) < p \kappa$ if and only if $v(\beta) < \kappa$; when this holds, $v(\beta) = v(\alpha)/p$ and $v(\beta^p - \alpha) > v(\alpha) = v(\beta^p)$.

(b) If $v(\alpha) = p \kappa$, then $v(\beta) = \kappa$.

Proof. By Newton’s binomial formula, one has

$$(1 + \beta)^p = 1 + \alpha = 1 + \beta^p + \sum_{i=1}^{p-1} \binom{p}{i} \beta^i.$$

Since $v(\alpha) > 0$ and $\text{char}(\hat{F}) = p$, this ensures that $v(\beta) > 0$. The binomial formula also shows that $\eta = 0$ if $p = 2$, and $\eta = \sum_{i=2}^{p-1} \binom{p}{i} \beta^i$ if $p > 2$. Note further that $v(\binom{p}{i} \beta^i) = v(p)$, for all $i < p$, which implies that, in case $p > 2$, the sequence of values $v(\binom{p}{i} \beta^i)$, $i = 1, \ldots, p - 1$, strictly increases. These facts prove that $v(\eta) \geq v(p) + 2v(\beta)$. The obtained inequality has the following consequences, which in turn imply statements (a) and (b) of Lemma 4.1.

(i) If $v(\beta) < \kappa$, then $v(\alpha) = v(\beta^p) < pv(\beta) < p \kappa$ and $v(\alpha - \beta^p) = v(p \beta + \eta) = v(p \beta) > v(\beta^p)$;

(ii) If $v(\beta) > \kappa$, then $v(\beta^p) > v(p \beta)$, $v(\alpha - p \beta) = v(\beta^p + \eta) > v(p \beta)$, and $v(\alpha) = v(p \beta) = v(p) + v(\beta) > p \kappa$;

(iii) If $v(\beta) = \kappa$, then $v(\beta^p) = v(p \beta) = p \kappa < v(\eta)$, whence, $v(\alpha) \geq p \kappa$. \hfill \Box
Lemma 4.2. Let \((F, v)\) be a valued field of mixed characteristic \((0, p)\), and there exist \(\gamma \in F^*\) with \(v(\gamma) = v(p)/(p-1) := \kappa\). Assume that \(\alpha \in F\) and \(\beta \in F^*\) satisfy \(v(\alpha) \geq \kappa p\) and \((1 + \beta)^p = 1 + \alpha\), put \(\delta = \beta/\gamma\), and denote by \(g\) the polynomial \(g(X) = \gamma^{-p}((1 + \gamma X)^p - (1 + \alpha)) \in F[X]\). Then:

(a) \(g\) is monic of degree \(p\), \(g(\delta) = 0\) and \(g \in O_v(F)[X]\);

(b) The reduction \(\hat{g} \in \hat{F}[X]\) of \(g\) modulo \(M_v(F)\) equals \(X^p + \hat{c}X - \hat{d}\), where \(c = p/\gamma^{p-1}\), \(\hat{c} \neq 0\) and \(d = \gamma^{-p}\alpha\); also, \(\hat{d} \neq 0\) if and only if \(v(\alpha) = p\kappa\).

Proof. (a): Evidently, \(1 + \beta\) is a root of the binomial \(X^p - (1 + \alpha)\), so \(h(\beta) = 0\), where \(h(X) = (X+1)^p - 1 - \alpha = X^p + (\sum_{i=1}^{p-1} \binom{p}{i} X^{p-i}) - \alpha\). Observing also that \(g(X) = \gamma^{-p}h(\gamma X) = X^p + \sum_{i=1}^{p-1} \binom{p}{i} \gamma^i X^{p-i} - (\alpha/\gamma^p)\), one obtains that \(g(X)\) is monic of degree \(p\) and \(g(\delta) = 0\), as required by Lemma 4.2 (a). Since \(v(\alpha) \geq \kappa p\), \(v(\gamma) = \kappa\), and \(p \mid \binom{p}{i}\), \(i = 1, \ldots, p-1\), it is easily verified that \(v(\alpha/\gamma^p) \geq 0\) and \(v(\binom{p}{i})/\gamma^i) = (p - i - 1)\kappa \geq 0\), for \(i = 1, \ldots, p-1\), proving that \(g(X) \in O_v(F)[X]\).

(b): The preceding calculations show that \(v(\binom{p}{p-1}/\gamma^{p-1}) = 0\), and in case \(p > 2\), they yield \(v(\binom{p}{i}/\gamma^i) > 0\), \(i = 1, \ldots, p-2\). Also, by the assumptions on \(v(\gamma)\) and \(v(\alpha)\), there exist \(c \in O_v(F)^*\) and \(d \in O_v(F)\), such that \(p = \gamma^{p-1}c\) and \(\alpha = \gamma^p d\). These observations show that \(\hat{g}(X) = X^p + \hat{c}X - \hat{d} \in \hat{F}[X]\) and \(\hat{c} \neq 0\). They also prove that \(\hat{d} \neq 0\) if and only if \(v(\alpha) = p\kappa\), as claimed. □

Our approach to the proof of Lemma 4.2 in the case where \(v(p) \in pv(K)\) relies on the following lemma (which is an extended version of [31], Lemma 2.1).

Lemma 4.3. Let \((K, v)\) be a Henselian field of mixed characteristic \((0, p)\), \(\varepsilon\) be a primitive \(p\)-th root of unity in \(K_{sep}\), and \(\kappa = v(p)/(p-1)\). Then:

(a) The polynomial \(g_\lambda(X) = (1 - \varepsilon)^{-p}[(1 - \varepsilon)X + 1]^p - \lambda\) lies in \(O_v(K(\varepsilon))[X]\) and has a root in \(K(\varepsilon)\), for each \(\lambda \in \nabla_{pv}(K(\varepsilon))\); in particular, \(\lambda \in K(\varepsilon)^{sp}\).

(b) \(\nabla_{pv}(K) \subset K^{sp}\), in case \(\kappa' \in v(K)\) and \(\kappa' \geq \kappa\).

(c) For any pair \(\lambda_1 \in \nabla_0(K), \lambda_2 \in K\), such that \(v(\lambda_1 - \lambda_2) > \kappa\), the elements \(\lambda_2\) and \(\lambda_2 \lambda_1^{-1}\) lie in \(\nabla_0(K)\) and \(K^{sp}\), respectively.

Proof. (a): We have \(v(1 - \varepsilon) = \kappa\) and \(v(\lambda - 1) > p\kappa\), whence, Lemma 4.2 applies to \(g_\lambda(X)\) and yields \(g_\lambda(X) \in O_v(K(\varepsilon))[X]\). Denote by \(\hat{K}_\varepsilon\) the residue field of \((K(\varepsilon), v)\), Lemma 4.2, combined with Lemma 3.3 (c), shows that the reduction \(\hat{g}_\lambda(X) \in \hat{K}_\varepsilon[X]\) of \(g_\lambda(X)\) modulo \(M_v(K(\varepsilon))\) equals the binomial \(X^p - X\) \((\hat{g}_\lambda(0) = 0\), since \(v((\lambda - 1)/(1 - \varepsilon)) > 0\)). This implies \(\hat{g}_\lambda(X)\) has a simple zero in \(\hat{K}_\varepsilon\), so it follows from (3.1) (a) that \(g_\lambda(X)\) has a zero in \(O_v(K(\varepsilon))\); hence, \(\lambda \in K(\varepsilon)^{sp}\).

(b): Lemmas 3.3 (c) and 4.2 (a) imply \(\nabla_{pv}(K) \subset (K(\varepsilon)^{sp} \cap K^*) = K^{sp}\).

(c): Clearly, \(\nabla_0(K)\) contains \(\lambda_2\) and \(\lambda_2^{-1}\), and \(\lambda_2 \lambda_1^{-1} = 1 + (\lambda_2 - \lambda_1)\lambda_1^{-1}\) lies in \(\nabla_{pv}(K(\varepsilon))\), whence, \(\lambda_2 \lambda_1^{-1} \in (K(\varepsilon)^{sp} \cap K^*) = K^{sp}\), as claimed. □

Definition 1. An element \(\lambda \in \nabla_0(K)\), where \((K, v)\) is an HDV-field of mixed characteristic \((0, p)\), is called normal over \(K\) (or \(K\)-normal), if \(\lambda \notin K^{sp}\) and \(v(\lambda - 1) \geq v(\lambda' - 1)\), for each element \(\lambda'\) of the coset \(\lambda K^{sp}\).
When $\lambda \notin K^{*p}$, $\lambda K^{*p}$ contains $K$-normal elements, as Lemma 4.3(b) and the cyclicity of $v(K)$ show that the system $v(\lambda - 1)$, $\lambda' \in \lambda K^{*p}$, contains a maximal element $v(\xi - 1)$ (and $\xi$ is $K$-normal). Our next lemma characterizes $K$-normal elements. Its conclusions follow from Lemma 3.1 and 17. Lemma (2-16), if $K$ contains a primitive $p$-th root of unity. Stating the lemma, we use the implication $pv(p)/(p-1) \in v(K) \Rightarrow v(p)/(p-1) \in v(K)$.

**Lemma 4.4.** Let $(K,v)$ be an HDV-field of mixed characteristic $(0,p)$, and let $v$ be a primitive $p$-th root of unity in $K_{sep}$. Suppose that $\lambda \in \nabla_0(K)$, put $\pi = \lambda - 1$, $\kappa = v(p)/(p-1)$, and let $K'$ be an extension of $K$ in $K_{sep}$ obtained by adjunction of a $p$-th root $\lambda'$ of $\lambda$. Then $\lambda$ is $K$-normal if and only if one of the following three conditions is fulfilled:

(a) $v(\pi) < \kappa \pi$ and $v(\pi) \notin pv(K)$; when this holds, $K'/K$ is TR;
(b) $v(\pi) < \kappa \pi$ and $\pi = \pi_1^p a$, for some $\pi_1 \in K$, $a \in O_v(K)^*$ with $\hat{a} \notin \hat{K}^{*p}$; in this case, $\hat{a} \in \hat{K}^{*p}$ and $K'/K$ is purely inseparable of degree $p$;
(c) $v(\pi) = \kappa \pi$, and for any $\pi_1 \in K$ with $v(\pi_1) = \kappa$, the polynomial $X^p + bX - \hat{d} \in \hat{K}[X]$ is irreducible over $\hat{K}$, $\hat{b}$ and $\hat{d}$ being the residue classes of the elements $b = p/\pi_1^{(p-1)}$ and $d = \pi/\pi_1^p$, respectively; when this holds, $K'/K$ is inertial and $K(\sqrt[p]{b}) = K(\pi)$.

**Proof.** Put $\pi' = \lambda' - 1$. The conditions of the lemma show that $v(\pi) > 0$ and $\lambda' \in \nabla_0(K')$, i.e. $v(\pi') > 0$. In view of Lemma 4.3(b), one may assume, for the proof, that $v(\pi) \leq \kappa \pi$. Hence, by Lemma 3.1(a) and (b) (applied to $(1 + \pi')^p = 1 + \pi'$), $v(\pi') \leq \kappa$, where equality holds only in case $v(\pi) = \kappa \pi$. Our proof proceeds in three steps.

Step 1. Let $v(\pi) < \kappa \pi$ and $\pi$ violate both conditions (a) and (b). Then $\lambda = 1 + \pi_0^p a_0 + \pi_0$, for some $a_0 \in O_v(K)^*$ and $\pi_0, \pi_1 \in K$, such that $v(\pi_0^p) = v(\pi) < v(\pi_0)$. Therefore, applying Lemma 4.4 to $(1 - \pi_0^p a_0)^p - 1 > v(\pi) = v(\lambda - 1)$; hence, $\lambda$ is $K$-normal.

Step 2. Assume now that $\pi$ satisfies condition (a) or (b) of Lemma 4.3 Then, for each $\lambda \in \nabla_0(K)$ with $v(\lambda - 1) > v(\pi)$, the element $\lambda \lambda - 1$ has value $v(\lambda \lambda - 1) = v(\pi)$ and satisfies the same condition as $\pi$. Moreover, under condition (b), $(\lambda \lambda - 1)/\pi_1^p$ lies in $O_v(K)^*$ and its residue class equals $\hat{a}$. Observing also that $\hat{\lambda}^{-1} \in \nabla_0(K)$ and $v(\hat{\lambda}^{-1} - 1) = v(\lambda - 1)$, one concludes that the $K$-normality of $\lambda$ will be proved, if we show that $\lambda \notin K^{*p}$. The equality $(1 + \pi)^p = 1 + \pi = \lambda$ and Lemma 3.1(a) imply $v(\pi - \pi) > v(\pi) = v(\pi^p) = pv(\pi)$, proving that $v(\pi) \in pv(K)$. When $v(\pi) \notin pv(K)$, this means that $K'/K$ is TR, $[K': K] = p$ and $\lambda \notin K^{*p}$. Similarly, it follows from Lemma 3.1(a) that if $\pi = \pi_1^p a$, where $\pi_1 \in K$ and $a \in O_v(K)^*$, then $\pi' = \pi_1 a_1$, for some $a_1 \in O_v(K)^*$ with $v(a - a_1) > 0$; hence, $\hat{a}_1 = \hat{a}$, proving that $\hat{a} \in \hat{K}^{*p}$. This shows that if $\hat{a} \notin \hat{K}^{*p}$, then $[K': K] = [\hat{K}': \hat{K}] = p$, $\hat{K}'/\hat{K}$ is purely inseparable and $\lambda \notin K^{*p}$. Thus our assumptions on $\pi$ guarantee that, in both cases, $\lambda \lambda^{-1} \lambda \notin K^{*p}$, for any $\lambda \in \nabla_0(K)$ with $v(\lambda - 1) > v(\pi)$, which implies $\lambda$ is $K$-normal.

Step 3. Suppose that $v(\pi) = \kappa \pi$, take $\pi_1 \in K$ so that $v(\pi_1) = \kappa$, define $b$ and $d$ as in Lemma 4.3(c), and put $g(X) = \pi_1^{-p}[(1 + \pi_1 X)^p - \lambda]$. It is easily verified that $v(b) = v(d) = 0$, $g(\pi') = 0$, and $g(X) \in K[X]$ is monic; also,
Definition 2. It follows from Lemma 3.3 (a) that \( g(X) \) is irreducible over \( K \) if and only if \( \lambda \notin K^{*p} \). At the same time, Lemma 4.3 (b) implies \( \lambda \notin K^{*p} \) if and only if \( \lambda \) is \( K \)-normal. Note further that, by Lemma 4.2, \( g(X) \in O_v(K)[X] \) and its reduction \( \tilde{g}(X) \) modulo \( M_v(K) \) equals the trinomial \( X^p + bX - \tilde{d} \in \widehat{K}[X] \). In addition, the equality \( v(b) = 0 \) shows that \( \tilde{g}(X) \) is separable. Using (3.1)(a), one also proves that \( \tilde{g}(X) \) is irreducible over \( \widehat{K} \) if and only if \( \lambda \notin K^{*p} \). It is now easy to see that \( \lambda \) is \( K \)-normal if and only if \( K' / K \) is inertial with \( [K' : K] = p \).

For the rest of the proof of Lemma 4.4 (c), we assume that \( \lambda \notin K^{*p} \), fix a root \( \xi \in K_{\text{sep}} \) of the binomial \( b(x) = X^{p-1} + b \), and put \( B = K(\xi) \). We first show that \( [B : K] \mid p - 1 \). As \( \text{char}(\widehat{K}) = p \), \( \widehat{K} \) contains a primitive \((p - 1)\)-th root of unity \( \hat{\rho} \), and since \( v \) is Henselian, (3.1)(a), applied to the binomial \( X^{p-1} - 1 \), shows that \( \hat{\rho} \) can be lifted to such a root \( \rho \in K \). Hence, the fact that \( [B : K] \mid p - 1 \) follows from Galois theory (cf. [21], Ch. VI, Theorem 6.2).

Finally, we prove that \( \overline{B} = K(\varepsilon) \). It is easily verified that \( \pi' / \pi_1 \) is a root of the monic polynomial \( h(X) = \xi^{-p}g(\xi X) \). Observing that \( v(\xi) = 0 \), one obtains from the already noted properties of \( g(X) \) that \( h(X) \in O_v(B)[X] \) and the reduction \( \hat{h}(X) \in \widehat{B}[X] \) of \( h(X) \) modulo \( M_v(B) \) is an Artin-Schreier trinomial. Moreover, it becomes clear that \( \hat{h}(X) = \xi^{-p}\hat{\xi}(\hat{\xi}X) \), which implies in conjunction with Lemma 3.3 (b) (and the divisibility of \( p - 1 \) by \( [B : K] \)) that \( \hat{g}(X) \) and \( \hat{h}(X) \) are irreducible over \( \widehat{B} \). Hence, by Lemma 3.3 and the Artin-Schreier theorem (cf. [21], Ch. VI, Sect. 6), applied to \( \hat{h}(X) \), \( K' \overline{B} / \overline{B} \) is an inertial Galois extension of degree \( p \). In view of the definition of \( K' \), this proves that \( \varepsilon \in B \). Let now \( \hat{K}_\varepsilon \) be the residue field of \( (K(\varepsilon), v) \), and set \( g_0(X) = (1 - \varepsilon)^{-p}[1 + (1 - \varepsilon)X^p - \lambda] \). Then \( g_0(X) \) is monic, and it follows from Lemmas 4.2 (a), 3.9 (a) that \( g_0(\pi' / (1 - \varepsilon)) = 0 \) and \( g_0(X) \in O_v(K(\varepsilon))[X] \). Moreover, Lemmas 3.3 (a) and 3.9 (c) imply the reduction \( \hat{g}_0(X) \in \hat{K}_\varepsilon[X] \) is an Artin-Schreier trinomial irreducible over \( \hat{K}_\varepsilon \). Lemma 4.2 (b), applied to \( g(X) \) and \( g_0(X) \), further indicates that if \( c = (1 - \varepsilon) / \pi_1 \), then \( v(c) = 0 \) and \( c^{p-1} = -\tilde{b} \in \hat{K}_\varepsilon \). Hence, by (3.1)(a), \( b(x) \) has a root in \( K(\varepsilon) \). As \( K \) contains a primitive \((p - 1)\)-th root of unity, this means that all roots of \( b(X) \) in \( K_{\text{sep}} \) in fact lie in \( K(\varepsilon) \). It is now obvious that \( B = K(\varepsilon) \), so Lemma 4.4 is proved. □

It follows from Lemmas 4.2 (b) and 4.4 that if \( \alpha \in K \) is normal over \( K \), then it is normal over any finite extension of \( K \) of prime-to \( p \) degree.

**Definition 2.** In the setting of Lemma 4.4, an element \( \lambda \in \nabla_0(K) \) is called \((u)\)-normal over \( K \), where \( (u) \in \{(a), (b), (c)\} \), if it satisfies condition \((u)\).

Next we present Albert’s characterization [11], Ch. IX, Theorem 6, of Galois extensions of prime degree different from the characteristic of the ground field. The characterization is based on Lemma 3.3 (c).

**Lemma 4.5.** Assume that \( K \) is an arbitrary field, \( \varepsilon \) is a primitive \( p \)-th root of unity in \( K_{\text{sep}} \), for some \( p \in \mathbb{P} \setminus \{\text{char}(K)\} \), and \( \varphi \) a generator of \( \mathcal{G}(K(\varepsilon)/K) \). Fix an integer \( s > 0 \) satisfying \( \varphi(\varepsilon) = \varepsilon^s \), and let \( \lambda \) be an element of \( K(\varepsilon)^{**} \). Then the following conditions are equivalent:
(a) \( \lambda \notin K(\varepsilon)^p \) and \( \varphi(\lambda)\lambda^{-s} \in K(\varepsilon)^p \);
(b) If \( L_\lambda^p = K(\varepsilon)^p \), then \( L_\lambda^p \) contains as a subfield a Galois extension \( L_\lambda \) of \( K \) of degree \( p \) (equivalently, the extension \( L_\lambda^p/K \) is Galois with \( G(L_\lambda^p/K) \) cyclic and \( [L_\lambda^p : K] = p[K(\varepsilon) : K] \)).

Denote by \( K(p, 1) \) the compositum of the extensions of \( K \) in \( K(p) \) of degree \( p \), put \( K_\varphi = \{ \alpha \in K(\varepsilon)^p : \varphi(\alpha)\alpha^{-s} \in K(\varepsilon)^p \} \), and fix \( \ell \in \mathbb{N} \) so that \( s\ell \equiv 1(\text{mod } p) \). Obviously, \( K_\varphi \) is a subgroup of \( K(\varepsilon)^p \) including \( K(\varepsilon)^p \). Note also that \( K(p, 1)/K \) is a Galois extension with \( G(K(p, 1)/K) \) abelian of period \( p \); this can be deduced from Galois theory and the normality of maximal subgroups of nontrivial finite \( p \)-groups (see [21], Ch. I, Sect. 6; Ch. VI, Theorem 1.14). With this notation, Lemma 4.5 can be supplemented as follows:

**Lemma 4.6.** (a) There is a bijection \( \varrho \) of the set \( \Sigma_p \) of finite extensions of \( K \) in \( K(p, 1) \) upon the set \( \mathcal{G}_p \) of finite subgroups of \( K_\varphi \), such that \( \varrho(\Lambda) \cong G(\Lambda/K) \cong G(\Lambda(\varepsilon)/K(\varepsilon)) \), for each \( \Lambda \in \Sigma_p \);

(b) For each \( \lambda \in K(\varepsilon)^* \), the product \( \Omega(\lambda) = \prod_{j=0}^{m-1} \varphi^j(\lambda)\ell(j) \) lies in \( K_\varphi \), where \( m = [K(\varepsilon) : K] \) and \( \ell(j) = \ell, \ j = 0, \ldots, m - 1 \).

**Proof.** It follows from Lemma 3.8 (c) and Galois theory (cf. [21], Ch. VI, Theorem 1.12) that the mapping \( \sigma \) of \( \Sigma_p \) into the set \( \Sigma_p' \) of finite extensions of \( K(\varepsilon) \) in \( K(p, 1) \), by the rule \( \Lambda \to \Lambda(\varepsilon) \), is bijective with \( G(\Lambda/K) \cong G(\Lambda(\varepsilon)/K(\varepsilon)) \), for each \( \Lambda \in \Sigma_p \). Moreover, by Kummer theory and Lemma 4.5 there is a bijection \( \varrho' : \Sigma_p' \to \mathcal{G}_p \), such that \( \varrho'(\Lambda') \cong G(\Lambda'/K(\varepsilon)) \), for each \( \Lambda' \in \Sigma_p' \). Therefore, the composition \( \varrho = \varrho' \circ \sigma \) has the properties required by Lemma 4.6 (a).

We prove Lemma 4.6 (b). If \( \varepsilon \in K \), then the assertion is obvious, so we assume that \( \varepsilon \notin K \), i.e. \( m \geq 2 \). It is easily verified that

\[
\varphi(\Omega(\lambda)) = \prod_{j=0}^{m-1} \varphi^{j+1}(\lambda)\ell(j) = \prod_{j=1}^{m} \varphi^j(\lambda)\ell(j-1) = \lambda^{\ell(m-1)} \prod_{j=1}^{m-1} \varphi^j(\lambda)\ell(j-1),
\]

and \( \Omega(\lambda)^s = \Omega(\lambda^s) = \prod_{j=0}^{m-1} \varphi^j(\lambda^s)\ell(j) \), for each \( \lambda \in K(\varepsilon)^* \). Since \( s^m \equiv s\ell \equiv 1(\text{mod } p) \), it follows that \( \ell^m \equiv 1(\text{mod } p) \), \( s \equiv \ell^m - 1(\text{mod } p) \), and \( s\ell(j) \equiv \ell(j-1)(\text{mod } p), j = 1, \ldots, m - 1 \), so our calculations prove that \( \varphi(\Omega(\lambda)), \Omega(\lambda)^{-s} \in K(\varepsilon)^p \), as claimed. \( \square \)

**Remark 4.7.** Let \( (K, v) \) be an HDV-field of mixed characteristic \((0, p)\), and let \( \varepsilon \) be a primitive \( p \)-th root of unity in \( K_{\text{ur}} \). Then:

(a) The existence of a \((c)\)-normal element over \( K \) ensures that \( \varepsilon \in K_{\text{ur}} \).
(b) It can be deduced from Lemma 4.5 that if \( K(\varepsilon)/K \) is TR and \( \varepsilon \notin K \) (this holds, for example, if \( v(p) \) generates \( v(K) \)) then each Galois extension \( L \) of \( K \) of degree \( p \) is \( K \)-isomorphic to \( L_{\lambda(L)} \), for some \( \lambda(L) \in K_\varphi \cap \mathcal{V}_0(K(\varepsilon)) \).
(c) When \( (v(p)) = v(K) \), we have \( (v(1 - \varepsilon)) = v(K(\varepsilon)) \), which enables one to obtain from Lemma 4.5 the preceding observation and Lemma 4.4 (applied over \( K(\varepsilon) \)) that a Galois extension of \( K \) of degree \( p \) is either inertial
or TR (this is a special case of Miki’s theorem, see [18], 12.2). Moreover, it turns out that degree $p$ extensions of $K_{ur}$ in $K_{ur}(p)$ are TR (whereas finite extensions of $K_{ur}$ in $K_{ur}(p)$ need not be TR unless $\bar{K}$ is perfect, see Lemmas 5.3 and 5.2 (b)).

We conclude this section with the following lemma. As demonstrated in Section 6, it makes it possible to turn Lemmas 4.3, 4.4 and 4.6 (a) into the tools we need for the proof of Lemma 2.2 in the case where $v(p) \in p\nu(K)$.

**Lemma 4.8.** Let $(K,v)$ be an HDV-field of mixed characteristic $(0,p)$. Fix a primitive $p$-th root of unity $\varepsilon \in K_{sep}$, a generator $\varphi$ of $G(K(\varepsilon)/K)$, and some $s \in \mathbb{N}$ so that $\varphi(\varepsilon) = \varepsilon^s$. Take any $\alpha \in K(\varepsilon)$ with $v(\alpha) > v(p)$, and put $\lambda = 1 + \alpha$. Then $\varphi(\lambda)\lambda^{-s} \in K(\varepsilon)^{p}$ in case $v(\varphi(\alpha) - s\alpha) > p\nu(p)/(p-1)$. This holds, if $\alpha = p(1-\varepsilon)\xi^{-1}$, where $\xi \in K^*$ with $v(\xi) < v(p)/(p-1)$.

**Proof.** Put $\kappa = v(p)/(p-1)$, and use the relation $\approx$ introduced on page 3. Since for $j \geq 2$, $v(\alpha^j) > 2v(p) \geq v(p)$, Newton’s binomial formula shows that $\lambda^s \approx 1 + sa$; hence, $\lambda^{-s} \approx 1 - sa$. Note also that $v(\varphi(\alpha)) = v(\alpha)$ because $v$ is Henselian (apply (3.1) (b)). Thus,

$$\varphi(\lambda)\lambda^{-s} \approx (1 + \varphi(\alpha))(1 - sa) \approx (1 + \varphi(\alpha) - sa) \approx 1.$$

Hence, $\varphi(\lambda)\lambda^{-s} \in K(\varepsilon)^p$, by Lemma 4.3 (a).

Let now $\alpha = p(1-\varepsilon)\xi^{-1}$, where $\xi \in K^*$ with $0 < v(\xi) < \kappa$. Then Lemma 3.1 (b) implies the following, for each $t \in \mathbb{N}$ not divisible by $p$:

$$v(1-\varepsilon^t - t(1-\varepsilon)) = v((1-\varepsilon)\sum_{j=0}^{t-1}(\varepsilon^j - 1)) \geq 2\kappa.$$

Therefore, $v(\alpha) = v(p) + \kappa - v(\xi) > v(p)$ and

$$v(\varphi(\alpha) - sa) = v(p)((1-\varepsilon^s) - s(1-\varepsilon)\xi^{-1}) \geq v(p) + 2\kappa - v(\xi) > p\kappa. \quad \Box$$

**5. Proof of Lemma 2.2 in case char$(K) = 0$ and $v(p) \notin p\nu(K)$**

In this section, we consider degree $p$ cyclic extensions related to Lemma 4.1 (a) and (b), which allows to prove Lemma 2.2 and Theorem 2.1 in the case where char$(K) = 0$ and $v(p) \notin p\nu(K)$. Our starting point is the following lemma.

**Lemma 5.1.** Let $(K,v)$ be an HDV-field of mixed characteristic $(0,p)$, and let $\varepsilon \in K_{sep}$ be a primitive $p$-th root of unity, $\varphi$ a generator of $G(K(\varepsilon)/K)$, $s$ and $l$ positive integers, such that $\varphi(\varepsilon) = \varepsilon^s$ and $sl \equiv 1 \pmod{p}$. Assume that $|K(\varepsilon)/K| = m$, and $\lambda = 1+(1-\varepsilon)^p\pi^{-1}$, for some $\pi \in K$ with $0 < v(\pi) < p\kappa$, where $\kappa = v(p)/(p-1)$. Denote by $\lambda$ the element $\Omega(\lambda)$ defined in Lemma 4.6 (b), and let $L_{\lambda}$ be the extension of $K$ in $K_{sep}$ associated with $\bar{\lambda}$ in accordance with Lemma 3.9 (b). Then:

(a) If $v(\pi) \notin p\nu(K)$, then $\lambda$ and $\bar{\lambda}$ are $(a)$-normal over $K(\varepsilon)$; in addition, $[L_{\lambda}: K] = p$, and $L_{\lambda}/K$ is both Galois and TR;
(b) If \( \pi = \pi_1^p a \), where \( \pi \in K \), \( a \in O_v(K)^* \) and \( \hat{a} \notin \hat{K}^p \), then \( \lambda \) and \( \hat{\lambda} \) are (b)-normal over \( K(\varepsilon) \); also, \( L_{\hat{\lambda}}/K \) is Galois, \([L_{\hat{\lambda}}; K] = p \) and \( \hat{L}_{\hat{\lambda}} = \hat{K}(\sqrt[p]{a}) \).

Proof. Our assumptions and Lemma 3.8 (c) imply \( v(\pi) \in pv(K) \) if and only if \( v(\pi) \in pv(K(\varepsilon)) \), and \( v(\lambda - 1) \in pv(K(\varepsilon)) \) if and only if \( v(\pi) \in pv(K) \).

They prove that \( \hat{K}_{\varepsilon}^p \cap \hat{K} = \hat{K}^p \), \( \hat{K}_{\varepsilon} \) being the residue field of \( (K(\varepsilon), v) \).

Putting \( e_n = \sum_{\nu=1}^{\nu=n} \varepsilon^\nu \), for each \( n \in \mathbb{N} \), one obtains from Lemma 3.9 (a), (b) that \( v(\nu - \varepsilon^u.e_n) \geq v(1 - \varepsilon) \), for any pair \( u, n \in \mathbb{N} \) with \( p \not| n \). Since \( p \not| n^p - n \) (by Fermat’s little theorem), \( v(1 - \varepsilon) = \kappa \), and \( n^p - e_n = \prod_{u=0}^{p-1}(n - \varepsilon^u.e_n) \), this shows that \( v(e_n^p - n) \geq v(p) \), which implies the following:

\[
(5.1) \ v((1 - \varepsilon^u)^p - n(1 - \varepsilon)^p) \geq v((1 - \varepsilon)^p) + v(p) > pk.
\]

Our proof of Lemma 5.1 also relies on the following facts:

\[
(5.2) \ (a) \ v(\tilde{\lambda} - (1 + m(1 - \varepsilon)^p\pi^{-1})) > v((1 - \varepsilon)^p\pi^{-1});
\]

\[
(b) \ v(\tilde{\lambda} - 1) = v(m(1 - \varepsilon)^p\pi^{-1}) = pk - v(\pi).
\]

The equalities in (5.2) (b) follow from (5.2) (a) (and the equality \( v(m) = 0 \) implied by Lemma 3.8 (c)). To prove (5.2) (a) we use the relation \( \sim \) defined on page 4 (\( \sim \) depends on \( j \)). As \( st \equiv 1(\bmod p) \), the relations below, where \( s(j) = s^j \) and \( \ell(j) = \ell^j \), include the content of (5.2) (a) (and forms of (5.1)):

\[
\begin{align*}
\tilde{\lambda} &= \prod_{j=0}^{m-1} [1 + (1 - \varepsilon^{s(j)}\pi^{-1}]\ell(j) \sim 1 + \sum_{j=0}^{m-1} \ell(j)(1 - \varepsilon^{s(j)}\pi^{-1} \\
&\sim 1 + \sum_{j=0}^{m-1} \ell(j)s(j)(1 - \varepsilon)^p\pi^{-1} \sim 1 + m(1 - \varepsilon)^p\pi^{-1}.
\end{align*}
\]

Statements (5.2) and observations at the beginning of our proof imply the former parts of Lemma 5.1 (a) and (b), so we assume further that either \( v(\pi) \notin pv(K) \) or \( \pi = \pi_1^p a \), for some \( \pi_1 \in K \) and \( a \in O_v(K)^* \) with \( \hat{a} \notin \hat{K}^p \). In the former case, \( \lambda \) and \( \hat{\lambda} \) are (a)-normal (over \( K(\varepsilon) \)), and in the latter one, they are (b)-normal. Let \( L'_{\lambda} = K(\varepsilon, \hat{\lambda}) \), where \( \hat{\lambda} \in K_{\text{sep}} \) and \( \hat{\lambda}^p = \hat{\lambda} \).

The normality of \( \hat{\lambda} \) over \( K(\varepsilon) \) ensures that \([L'_{\lambda}; K(\varepsilon)] = p \). Using Lemma 4.3 one obtains that: if \( \tilde{\lambda} \) is (a)-normal, then \( L'_{\lambda}/K(\varepsilon) \) is TR; when \( \tilde{\lambda} \) is (b)-normal, \( \hat{L}_{\lambda}/\hat{K}_{\varepsilon} \) is inseparable of degree \( p \) with \( \hat{a} \in \hat{L}_{\lambda}^p \). Also, it follows from Lemmas 4.5 4.6 (b) and the \( K(\varepsilon) \)-normality of \( \tilde{\lambda} \) that \( L'_{\lambda} = \hat{L}_{\lambda}(\varepsilon) \), and the extension \( \hat{L}_{\lambda} \) of \( K \) in \( L'_{\lambda} \) pointed out in the statement of Lemma 5.1 is Galois with \([L'_{\lambda}; K] = p \). As \([L'_{\lambda}; L_{\lambda}] = m \) and \( m \mid p - 1 \), these observations prove the following: \( \hat{L}_{\lambda}/K \) is TR if and only if so is \( L_{\lambda}/K(\varepsilon) \); \( \hat{L}_{\lambda}/\hat{K}_{\varepsilon} \) is inseparable of degree \( p \) if and only if so is \( \hat{L}_{\lambda}/\hat{K}_{\varepsilon} \). Note finally that \( \hat{L}_{\lambda}/\hat{L}_{\lambda} \) | \([L'_{\lambda}; L_{\lambda}] \). This implies together with Lemma 3.8 (b) that if \( \hat{\lambda} \) is (b)-normal, then \( \hat{a} \in \hat{L}_{\lambda}^p \), which completes our proof. \( \Box \)

Lemma 3.6 and our next lemma prove Theorem 2.1 in case \( \text{char}(K) = 0 \) and \( v(p) \notin pv(K) \). In this situation, our proof of the lemma relies on the
fact (see [16], Ch. 2, (3.6), and [14], Theorem 15.3.5) that a finite extension $E'$ of a discrete valued field $(E, w)$ is TR relative to $w$ if and only if $E'/E$ has a primitive element $\theta$ whose minimal polynomial $f$ over $E$ is Eisenstein at $w$, i.e. $f$ is monic, all of its coefficients but the leading one lie in $M_w(E)$, and the free coefficient of $f$ generates $M_w(E)$ as an ideal of $O_w(E)$.

**Lemma 5.2.** Let $(K, v)$ be an HDV-field of mixed characteristic $(0, p)$. Suppose that one of the following two conditions is satisfied:

(a) $\widehat{K}$ is an infinite perfect field;
(b) $\widehat{K}$ is imperfect and $v(p) \not\in pv(K)$.

Then there exist TR and Galois extensions $M_\mu/K$, $\mu \in \mathbb{N}$, such that $[M_\mu: K] = p^\mu$ and $\mathcal{G}(M_\mu/K)$ is abelian of period $p$, for each $\mu$.

**Proof.** We assume, in agreement with conditions (a) and (b), that $\widehat{K}$ is infinite. Since the prime subfield, say $\mathbb{F}$, of $\widehat{K}$ is finite, this ensures that $\widehat{K}/\mathbb{F}$ is an infinite extension, whence, there is a sequence $\bar{b} = b_\mu \in O_v(K)^*$, $\mu \in \mathbb{N}$, such that the system $\bar{b} = \hat{b}_\mu \in \widehat{K}$, $\mu \in \mathbb{N}$, is linearly independent over $\mathbb{F}$. Denote by $V$ the $\mathbb{F}$-linear span of the set $\{b_\mu : \mu \in \mathbb{N}\}$ and fix a primitive $p$-th root of unity $\varepsilon \in K_{sep}$, a generator $\varphi$ of $\mathcal{G}(K(\varepsilon)/K)$, and integers $s$, $t$ as in Lemma 5.1. Define $K_\varphi$ and $\Omega : K(\varepsilon)^* \rightarrow K_\varphi$ as in Lemma 4.6 and put $m = [K(\varepsilon): K]$ and $\lambda_\mu = \Omega(1 + (1 - \varepsilon)p^{\pi - 1}b_\mu)$, $\mu \in \mathbb{N}$, where $\pi \in K$ is fixed so that $v(\pi) \not\in pv(K)$ and $0 < v(\pi) \leq v(p)$. Take a $p$-th root $\eta_\mu \in K_{sep}$ of $\lambda_\mu$, for each $\mu \in \mathbb{N}$, and consider the fields $L_\mu' = K(\varepsilon, \eta_\mu)$, $\mu \in \mathbb{N}$, Lemmas 4.4 and 4.5 show that $[L_\mu': K(\varepsilon)] = p$ and there is a unique Galois extension $L_\mu$ of $K$ in $L_\mu'$ of degree $[L_\mu : K] = p$. Let $L_\infty'$ be the compositum of the fields $L_\mu'$, $\mu \in \mathbb{N}$, and $\Lambda$ be the subgroup of $K(\varepsilon)^*$ generated by the set $K(\varepsilon)^* \cup \{\lambda_\mu : \mu \in \mathbb{N}\}$. Obviously, $\Lambda$ is a subgroup of $K_\varphi$ including $K(\varepsilon)^p$. It follows from the assumption on the sequence $\bar{b}$ that, for each $h \in \Lambda \setminus K(\varepsilon)^p$, the coset $hK(\varepsilon)^p$ contains an element of the form $\lambda(h) = 1 + m(1 - \varepsilon)p^{\pi - 1}\beta_h + \pi(h)$, where $\pi(h) \in K(\varepsilon)$, $v(\pi(h)) > v(m(1 - \varepsilon)p^{\pi - 1})$, $\beta_h \in O_v(K)^*$ and $\hat{\beta}_h \in V$. Therefore, by the assumptions on $\pi$, $\lambda(h)$ is (a)-normal over $K(\varepsilon)$ (so Lemma 5.1 (a) applies to it). This implies $\hat{\beta}_h$ is uniquely determined by $h$ and $\pi$, and does not depend on the choice of $\lambda(h)$ (see Step 2 of the proof of Lemma 4.4). More precisely, if $h = \lambda^{k_1}_{\mu_1} \ldots \lambda^{k_y}_{\mu_y}$, for some $y \in \mathbb{N}$, and $k_1, \ldots, k_y \in \mathbb{N}$, with $p \nmid k_j$, for at least one index $j'$, then $h \notin K(\varepsilon)^p$, so one may put $\lambda(h) = h$ and $\hat{\beta}_h = \sum_{j=1}^y k_j \hat{b}_{\mu_j}$. These observations prove that

$$\{\lambda_\mu K(\varepsilon)^p : \mu \in \mathbb{N}\} \text{ is a minimal generating set of } \Lambda/K(\varepsilon)^p, \text{ and there is a unique isomorphism } \rho \text{ of } \Lambda/K(\varepsilon)^p \text{ upon the additive group of } V, \text{ which maps the coset } \lambda_\mu K(\varepsilon)^p \text{ into } \hat{b}_{\mu}, \text{ for each } \mu \in \mathbb{N}. $$

Statement (5.3), the argument proving it, and Lemmas 4.6 and 5.1 (a) imply that the fields $L_\infty'$ and $L_\mu$, $\mu \in \mathbb{N}$, satisfy the following:

(5.4) (a) $[L_1 \ldots L_\mu : K] = [L'_1 \ldots L'_\mu : K(\varepsilon)] = p^\mu$, for each $\mu$;

(b) The compositum $L_\infty$ of all $L_\mu$, $\mu \in \mathbb{N}$, is an infinite Galois extension of $K$ with $L_\infty(\varepsilon) = L'_\infty$ and $\mathcal{G}(L_\infty/K)$ abelian of period $p$.
(c) Every extension of $K$ in $L_\infty$ of degree $p$ is Galois and TR over $K$.

Suppose now that $\hat{K}$ is perfect. Then every $R \in Fe(K)$ contains as a subfield an inertial extension $R_0$ of $K$ with $\hat{R}_0 = \hat{R}$ (cf. [30], Proposition A.17). In view of Lemmas 3.2 and 3.3 (c), this allows to deduce from (5.4) (b), (c) and Galois theory that finite extensions of $K$ in $L_\infty$ are TR. Thus the fields $M_\mu = L_1 \ldots L_\mu$, $\mu \in \mathbb{N}$, have the properties claimed by Lemma 5.2.

It remains for us to prove Lemma 5.2 (b). The idea of our proof has been borrowed from [23], 2.2.1. Identifying $\mathbb{Q}$ with the prime subfield of $K$, put $E_0 = \mathbb{Q}(t_0)$, where $t_0 \in \mathcal{O}_v(K)^*$ is chosen so that $\hat{t}_0 \notin \hat{K}^p$ (whence, $t_0$ is transcendental over $\mathbb{F}$). Denote by $\omega$ and $v_0$ the valuations induced by $v$ upon $\mathbb{Q}$ and $E_0$, respectively, and fix a system $t_\mu \in K_{\text{sep}}$, $\mu \in \mathbb{N}$, such that $t_\mu^p = t_\mu - 1$, for each $\mu > 0$. It is easy to see that $\mathbb{F}$ equals the residue field of $(\mathbb{Q}, \omega)$, and the fields $E_\mu = \mathbb{Q}(t_\mu)$, $\mu \in \mathbb{N}$, are purely transcendental extensions of $\mathbb{Q}$. Let $v_\mu$ be the restricted Gauss valuation of $E_\mu$ extending $\omega$, in the sense of [14], for each $\mu \in \mathbb{N}$. Clearly, for any pair of indices $\nu, \mu$ with $0 < \nu \leq \mu$, $E_{\nu-1}$ is a subfield of $E_\mu$ and $v_\mu$ is the unique prolongation of $v_{\nu-1}$ on $E_\mu$. Hence, the union $E_\infty = \cup_{\mu=0}E_\mu$ is a field with a unique valuation $v_\infty$ extending $v_\mu$, for every $\mu < \infty$. Denote by $\hat{E}_\mu$ the residue field of $(E_\mu, v_\mu)$, for each $\mu \in \mathbb{N} \cup \{0, \infty\}$. The Gaussian property of $v_\mu$, $\mu < \infty$, guarantees that $v_\mu(E_\mu) = \omega(\mathbb{Q})$, $v_\mu(t_\mu) = 0$, $\hat{t}_\mu$ is a transcendental element over $\mathbb{F}$ and $\hat{E}_\mu = \mathbb{F}(\hat{t}_\mu)$ (see [14], Examples 4.3.2 and 4.3.3). Observing also that $\hat{t}_\mu^p = \hat{t}_\mu - 1$, $\mu \in \mathbb{N}$, $\hat{E}_\infty = \cup_{\mu=1}\hat{E}_\mu$ and $\mathbb{F}^p = \mathbb{F}$, one concludes that $\hat{E}_\infty$ is infinite and perfect. It is therefore clear from Lemma 5.2 (a) and Grunwald-Wang’s theorem (see Remark 5.3), that if $(E'_\infty, v'_\infty)$ is a Henselization of $(E_\infty, v_\infty)$ with $E'_\infty \subset K_{\text{sep}}$, then there exist TR and Galois extensions $T'_\mu/E'_\infty$ and $T_\mu/E_\infty$, $\mu \in \mathbb{N}$, such that $[T'_\mu : E'_\infty] = [T_\mu : E_\infty] = p^\nu$, $T'_\mu = T_\mu E'_\infty$, $\mathcal{G}(T'_\mu/E'_\infty)$ is abelian of period $p$, and $\mathcal{G}(T_\mu/E_\infty) \cong \mathcal{G}(T'_\mu/E'_\infty)$, for every $\mu$. Now fix an arbitrary index $\mu$, choose $\theta \in T_\mu$, so that the minimal polynomial $f(X)$ of $\theta$ over $E_\infty$ be Eisenstein at $v_\mu$, and take a sufficiently large index $k \geq \mu$ such that $f(X) \in E_k[X]$ and $f(X)$ splits over $E_k(\theta)$. Then the extension $E_k(\theta)/E_k$ is both TR and Galois with $\mathcal{G}(E_k(\theta)/E_k) \cong \mathcal{G}(T_\mu/E_\infty)$, and $f(X)$ is Eisenstein at $v_k$. Let $\psi$ be the isomorphism $E_k \rightarrow E_0$ mapping $t_k$ into $t_0$. Then $\psi$ extends uniquely to a degree-preserving isomorphism $\psi : E_k[X] \rightarrow E_0[X]$ of polynomial rings, such that $\psi'((X)) = X$; also, $\psi'$ maps $O_{v_0}(E_k)[X]$ into $O_{v_0}(E_0)[X]$. Note that, for each $g(X) \in E_k[X]$, $\psi'$ induces canonically a ring isomorphism $\psi'_g : R_k \rightarrow R_0$ extending $\psi$, where $R_k = E_k[X]/(g(X))$ and $R_0 = E_0[X]/(\psi'(g(X)))$. Clearly, $\psi'_g$ maps bijectively the set of roots of $g(X)$ in $R_k$ on the set of roots of $\psi'(g(X))$ in $R_0$. One also sees that $g(X)$ is irreducible over $E_k$ if and only if so is $\psi'(g(X))$ over $E_0$. Therefore, $R_k/E_k$ is a field extension if and only if so is $R_0/E_0$; when this occurs, $[R_k : E_k] = [R_0 : E_0] = \deg(g)$. Moreover, it follows that $R_k/E_k$ is Galois if and only if so is $R_0/E_0$ (and this holds if and only if $g(X)$ is irreducible over $E_k$ and $R_k$ is a root field of $g(X)$ over $E_k$). Suppose now that $R_k/E_k$ is Galois. Then, for each $\sigma \in \mathcal{G}(R_k/E_k)$, there is a unique $\sigma' \in \mathcal{G}(R_0/E_0)$, such that $\sigma'(\psi'_g(r_k)) = \psi'_g(\sigma'(r_k))$, for every $r_k \in R_k$; in addition, the mapping of $\mathcal{G}(R_k/E_k)$ into $\mathcal{G}(R_0/E_0)$, by the rule $\sigma \rightarrow \sigma'$, is an isomorphism.
Note finally that \( v_k(e_k) = v_0(\psi(e_k)) \), for every \( e_k \in E_k \), which implies \( g(X) \) is Eisenstein at \( v_k \) if and only if so is \( \psi'(g(X)) \) at \( v_0 \); hence, \( R_k/E_k \) is TR relative to \( v_k \) if and only if so is \( R_0/E_0 \) relative to \( v_0 \). When \( g(X) = f(X) \), these observations show that \( \psi \) extends to an isomorphism of \( E_k(\theta) \) on the root field \( R \in \text{Fe}(E_0) \) of \( \psi'(g(X)) \) over \( E_0 \), and that \( R/E_0 \) is TR (relative to \( v_0 \)) and Galois with \( \mathcal{G}(R/E_0) \cong \mathcal{G}(E_k(\theta)/E_k) \). As \( v(p) \notin v(K) \), one obtains from Lemma 3.7 and the described properties of \( R/E_0 \) (regarding \( E_{0,\text{sep}} \) as an \( E_0 \)-subalgebra of \( K_{\text{sep}} \) that \( RK/K \) is TR and Galois, \( [RK: K] = p^\mu \) and \( \mathcal{G}(RK/K) \cong \mathcal{G}(R/E_0) \) is abelian of period \( p \). Because of the arbitrary choice of the index \( \mu \), this proves Lemma 5.2 (b).

\[ \square \]

Remark 5.3. Lemma 3.1 shows that given a field \( L \) with nonequivalent real-valued valuations \( \bar{w}_1, \ldots, \bar{w}_n \), for some \( n \in \mathbb{N} \), Grunwald-Wang’s theorem holds, if applied to a Henselization of \( (L, \bar{w}_i) \) (instead of \( (L_{w_i}, \bar{w}_i) \)), for \( i = 1, \ldots, n \).

Lemma 5.4. Let \( (K, v) \) be an HDV-field of mixed characteristic \((0, p)\) with \( v(p) \in pv(K) \) and \( \bar{K} \neq \bar{K}^p \). Let \( \bar{\Lambda}/\bar{K} \) be an inseparable extension of degree \( p \). Then there exists \( \Lambda \in I(K(p)/K) \) with [\( \Lambda : K \) = \( p \)] and \( \bar{\Lambda} \cong \bar{\Lambda} \) over \( \bar{K} \).

Proof. The condition that \( v(p) \in pv(K) \) means that there is \( \pi_1 \in K \) with \( v(\pi_1) = v(p)/p \); so our conclusion follows at once from Lemma 5.1 (b).

Proposition 5.5. Let \( (K, v) \) be an HDV-field of mixed characteristic \((0, p)\) and with \( \bar{K} \neq \bar{K}^p \). Suppose that \( v(p) \in pv(K) \) or \( K \) contains a primitive \( p \)-th root of unity \( \varepsilon \). Then each proper extension \( L \) of \( \bar{K} \) satisfying the inclusion \( \bar{L}^p \subseteq \bar{K} \) is \( \bar{K} \)-isomorphic to \( \bar{L} \), for some Galois extension \( L \) of \( K \), such that \( v(L) = v(K) \) and \( \mathcal{G}(L/K) \) is an abelian group of period \( p \).

Proof. If \( v(p) \in pv(K) \), then our assertion follows from Lemmas 3.2, 5.4 and Galois theory; when \( \varepsilon \in K \), it can be deduced from Kummer theory.

Remark 5.6. Let \( (K, v) \), \( p \) and \( \varepsilon \in K_{\text{sep}} \) satisfy the conditions of Lemma 4.4 and let \( \bar{K} \neq \bar{K}^p \). Take \( c \in O_v(K) \) with \( \hat{c} \notin \bar{K}^p \), and suppose that \( K \) has a degree \( p \) extension \( C \) in \( K(p) \), such that \( \hat{c} \in \hat{C}^p \). By Lemma 5.3 an extension of this kind exists if \( v(p) \notin pv(K) \) or \( \varepsilon \in K \) (this need not hold in general, see Remark 4.4 (c)). It is easily verified that \( v(C) = v(K) \), \( v(z) \in pv(K) \), for each \( z \in N(C/K) \), and \( \hat{z} \in \hat{C}^p \) in case \( v(z) = 0 \). Therefore, if [\( \bar{K} : \bar{K}^p \) \( ) \geq p^2 \), \( \hat{c}, \hat{b} \in \bar{K} \) are \( p \)-independent over \( \bar{K}^p \), and \( \bar{b} \in O_v(K) \) is a pre-image of \( \hat{b} \), then \( \bar{b} \notin N(C/K) \) and (by 29, Proposition 15.1 b) the cyclic \( K \)-algebra \( V = (C/K, \tau, b) \), \( b \) of degree \( p \) lies in \( d(K) \), \( \tau \) being a generator of \( \mathcal{G}(C/K) \). Since \( v_C(\tau(\alpha) - \alpha) > v_C(\alpha) \), for any \( \alpha \in C^* \), this implies \( \hat{V} \) contains commuting \( p \)-th roots \( \hat{\eta}_c = \sqrt[p]{c} \) and \( \hat{\eta}_b = \sqrt[p]{b} \). Hence, by Lemma 3.4 \( v(V) = v(K) \) and \( \hat{V} \) equals the field \( \hat{K}(\hat{\eta}_c, \hat{\eta}_b) \). Also, it follows from Kummer theory that \( V \) is a symbol \( K \)-algebra, in the sense, e.g. of 23, if and only if \( \varepsilon \in K \).
Lemma 6.1. Let \( L \) be an \( \text{TR} \) field. Lemmas 3.2 (b) and 3.8 imply an integer \( \mu > K \) is infinite and \( L \) linearly independent over \( F \). Since, by Galois theory, the existence of an element \( \pi \in K \) containing a primitive \( \epsilon > v(\pi) < pv(p)/(p-1) \). We first show that one may consider only the special case where \( \mu \) is replaced by \( \hat{K} \). Using a standard inductive argument, one may assume for the rest of our proof relies on the fact that, by Lemma 3.8 (a), \( L_1/K \) is TR and \([L_1: K] = p\), which means that \( M/K \) is TR, provided so is \( \hat{K} \). Henceforth, we assume that \( \epsilon \in K \). Then the concluding assertion of Lemma 6.1 is implied by Kummer theory and the definition of \( M \). Thus, the assertion of Lemma 6.1 holds, for any HDV-field \((K', v')\) of mixed characteristic \((0, p)\) with \( \hat{K}' \) infinite and \( v'(p) \in pv'(K') \). Then the assertion that \( M/L_1 \) is TR of degree \( p^{\mu-1} \) can be deduced from the existence of elements \( \pi_1 \) and \( \lambda_{1,j} \in L_1^{*} \), \( \alpha_{1,j} \in O_{v_1}(L_1^*) \), \( j = 2, \ldots, \mu \), such that:

(6.1) \( \alpha_{1,2}, \ldots, \alpha_{1,\mu} \) are linearly independent over \( F \); \( v(\pi_1) = pk - (\gamma/p) \) (whence, \( v(\pi_1) \notin pv(L_1) \)) \( \lambda_{1,j} = 1 + \pi_1 \alpha_{1,j}^{p^{\mu-1}} \) and
\[ \lambda_j L_1^p = \lambda_j L_1^{np}, j = 2, \ldots, \mu. \]

Since the elements \( \hat{\alpha}_j \hat{\alpha}_1^{-1}, j = 1, \ldots, \mu, \) are linearly independent over \( \mathbb{F} \), it suffices to prove the existence of elements satisfying the conditions of (6.1) only in the special case where \( \alpha_1 = 1 \) (considering \( \pi \alpha_1^{\nu} \) and \( \alpha_2 \alpha_1^{-1}, \ldots, \alpha_\mu \alpha_1^{-1} \) instead of \( \pi \) and \( \alpha_2, \ldots, \alpha_\mu \), respectively). Putting \( \eta_1 = \lambda_1^p - 1 \), we show that, in this case, \( \pi_1 \) and \( \alpha_1, \lambda_{1,j}, j = 2, \ldots, \mu, \) can be chosen as follows:

\[(6.2) \pi_1 = \eta_1, \alpha_{1,j} = \alpha_j - \alpha_j^p, \text{ and } \lambda_{1,j} = 1 - \eta_1. \]

In the rest of the proof, we use the relation \( \approx \) introduced on page 2. As \( (1 + \eta_1)^p = 1 + \pi \) (and \( p \geq 2 \)), Lemma 4.1(a) shows that

\[ v(\eta_1) = v(\pi) < v(p)/p, \text{ so } v(\eta_1^p) > (p + 2)v(\eta_1) \geq pv. \]

hence; by the former conclusion of Lemma 4.1, \( \pi \approx \eta_1^p + \pi \eta_1 \). At the same time, the equality \( \lambda_j = 1 + \pi \alpha_j^{\nu} \) implies \( \lambda_j^{-1} = 1 - \pi \alpha_j^{\nu} \). Likewise, from \( \lambda_{1,j} = 1 - \eta_1 \), one obtains that \( \lambda_{1,j}^{-1} = 1 + \eta_1 \). Let \( \Omega_j = 1 + \eta_1 \alpha_j^{\nu}. \) Then

\[ \Omega_j^{\nu} \approx 1 + \eta_1^p \alpha_j^{\nu} + \eta_1 \alpha_j^{\nu} = \left[ 1 + (\eta_1^p + \eta_1) \alpha_j^{\nu} \right] + [\eta_1(\alpha_j^{\nu} - \alpha_j^p)] \]

\[ \approx \lambda_j + [\eta_1(\alpha_j - \alpha_j^p)] = \lambda_j + \eta_1 = \lambda_j^p. \] 

Hence, by Lemma 4.3(c), \( \lambda_j \lambda_{1,j}^{-1} \in L_1^{sp}. \)

We are now in a position to prove Lemma 6.1. As already shown, \( v(p) < v(\pi_1) = v(\eta_1^p) = v(p) + v(\eta_1) = pv(\gamma/p) \) and \( pv_1(L) = v(K) \), which implies \( v(\eta_1) \notin pv(L_1) \). Observing that \( \alpha_\gamma = 1 \), the field \( \mathbb{F} \) equals the set \( \{ \hat{y} \in \hat{K} : \hat{y}^p = \hat{y} \} \), and \( \alpha_{1,j} = \alpha_j - \alpha_j^p, j = 2, \ldots, \mu, \) are elements of \( O_v(L_1)^* \), such that \( \hat{\alpha}_1, \ldots, \hat{\alpha}_\mu \) are linearly independent (over \( \mathbb{F} \)), one concludes that \( \hat{\alpha}_1, \ldots, \hat{\alpha}_\mu \) are linearly independent as well. Thus the field \( M \) and the elements \( \pi_1 = \eta_1, \alpha_{1,j}, \lambda_{1,j}, j = 2, \ldots, \mu, \) defined in (6.2) satisfy the conditions of Lemma 6.1(over \( L_1 \)), and by the inductive hypothesis, \( M/L_1 \) is a TR-extension of degree \( p^{\mu-1} \), so Lemma 6.1 is proved.

We can now take the final step towards the proof of Lemma 2.2 (and Theorem 2.1) in general. In view of Lemma 2.6 and 8, Lemma 4.2, one may consider only the case of mixed characteristic \((0,p)\). We also assume that \( v(p) \in pv(K) \) and \( \hat{K} \neq \hat{K}^p \), which is allowed by Lemma 5.2. As \( v(K) \) is cyclic, the condition on \( v(p) \) ensures that there is \( \xi \in \hat{K} \) with \( 0 < v(\xi) \leq v(p)/p \) and \( v(\xi) \notin pv(K) \). Since \( \hat{K} \) is finite, there are \( \alpha_\nu \in O_v(K)^*, \nu \in \mathbb{N}, \) such that the system \( \hat{\alpha}_\nu \in \hat{K}, \nu \in \mathbb{N}, \) is linearly independent over the prime subfield of \( \hat{K} \). Take a primitive \( p \)-th root of unity \( \varepsilon \in K_{sep}, \) a generator \( \varphi \) of \( G(K(\varepsilon)/K) \), and \( s \in \mathbb{N} \) so that \( \varphi(\varepsilon) = \varepsilon^s \). Fix any \( \mu \in \mathbb{N}, \) put \( \lambda_j = 1 + (1 - \varepsilon^{s-1}) \alpha_j^{\nu}, \) for \( j = 1, \ldots, \mu, \) and denote by \( M'_\mu \) the extension of \( K(\varepsilon) \) generated by the set \( \{ \lambda'_j : j = 1, \ldots, \mu \} \), where \( \lambda'_j \in K_{sep} \) and \( \lambda_j^p = \lambda_j, \) for any index \( j. \) It follows from Lemma 5.1 that \( M'_\mu/K(\varepsilon) \) is TR and Galois of degree \( p^\mu \) with \( G(M'_\mu/K(\varepsilon)) \) abelian of period \( p. \) Furthermore, Lemma 4.8 and the conditions on \( \xi \) and \( \alpha_1, \ldots, \alpha_\mu \) show that
Proposition 7.1. If \( \varphi(\lambda_j) \lambda_j^{-s} \in K(\varepsilon)^{p^n}, j = 1, \ldots, \mu. \) Therefore, Lemmas 4.5 and 4.6 (a) yield \( M'_\mu = M_\mu(\varepsilon), \) for some Galois extension \( M_\mu \) of \( K \) in \( K(p), \) such that \( \mathcal{G}(M_\mu/K) \cong \mathcal{G}(M'_\mu/K(\varepsilon)); \) hence, \( [M_\mu : K] = p^n \) and \([M'_\mu : M_\mu] = [K(\varepsilon) : K].\)

As \( p \nmid [K(\varepsilon) : K] \) and \( M'_\mu/K(\varepsilon) \) is TR, it is now easy to see that \( M'_\mu/K \) is also TR. Because of the arbitrary choice of \( \mu, \) this proves Lemma 2.2 Theorem 2.1 (b) and the right-to-left implication in Theorem 2.1 (a). Finally, by Fact 3.5, the converse implication follows from [25], Corollary 2.5, so Theorem 2.1 is proved.

Remark 6.2. It should be pointed out that in case \( (K, v) \) is an HDV-field containing a primitive \( p \)-th root unity \( \varepsilon, \) the right-to-left-implication in Theorem 2.1 (a) becomes obvious as a result of the proof of the lower bound for \( \text{abrd}_p(K) \) in [25], Lemma 2.6. The conditions of the cited lemma do not require that \( \varepsilon \in K. \) However, the assumption that \( \varepsilon \in K \) is necessary to define over \( K \) tensor products of symbol algebras like those used in the proof of [25], Lemma 2.6. This allows to show easily that if \( \varepsilon \in K, \) then the lower bound in the cited lemma is also such a bound for \( \text{Brd}_p(K), \) which proves the right-to-left-implication in Theorem 2.1 (a).

To end the present Section, we note that Theorem 2 of [25] and the conclusion of Theorem 2.1 (b) in case \( \text{char}(K) = 0 \) leave open the question of whether \( \text{abrd}_p(E) > 2\text{Brd}_p(E) + 1, \) for any field \( E \) with a primitive \( p \)-th root of unity and \( \text{Brd}_p(E) < \infty. \) Moreover, it seems to be unknown whether \( \text{abrd}_p(E) = \infty. \)

7. Open problems and further results

We begin this section with a proof of Conjecture 1.1 in case \( \text{char}(K) = p. \)

Proposition 7.1. If \( (K, v) \) is an HDV-field with \( \text{char}(K) = p > 0, \) then:

(a) \( \text{Brd}_p(K) = \infty \) if \( [\hat{K} : \hat{K}^p] = \infty; \) when \( (K, v) \) is complete, the equality \( [\hat{K} : \hat{K}^p] = \infty \) holds if and only if \( [K : K^p] = \infty; \)

(b) \( n \leq \text{Brd}_p(K) \leq n + 1, \) provided that \( n < \infty \) and \( [\hat{K} : \hat{K}^p] = p^n; \)

(c) If \( (K, v) \) is complete, \( [\hat{K} : \hat{K}^p] = p^n \) and \( K'/K \) is a finite field extension, then \( [K' : K^p] = p^{n+1}. \)

Proof. The former part of Proposition 7.1 (a) and the lower bound on \( \text{Brd}_p(K) \) in Proposition 7.1 (b) are implied by [8], Lemma 4.2 (b). Proposition 7.1 (c) and the latter part of Proposition 7.1 (a) follow from Fact 3.5 (b), Lemma 3.2, and the equality \( [L : L^p] = [K : K^p], \) for every finite extension \( L/K \) (cf. [4], Lemma 2.12). It remains to prove the upper bound in Proposition 7.1 (b). Let \( \overline{K} \) be an algebraic closure of \( K. \) In view of [7], Lemma 4.1, it suffices to show that, for any finite extension \( K'/K \) in \( \overline{K}, \) we have \( \deg(D') \mid p^{n+1} \) whenever \( D' \in d(K') \) and \( \exp(D') = p. \)

In addition, Fact 3.5 (a) allows us to consider only the case of \( K = K_v. \) Let \( K_1 = \{ \lambda \in \overline{K} : \lambda^p \in K' \}. \) Then \( K_1 \in I(\overline{K}/K'), K_1^p = K', \) and by Proposition 7.1 (c), \( [K_1' : K'] = p^{n+1}. \) Since, by Albert’s theorem, \( p\overline{\text{Br}}(K') \)
Proposition 7.2. Assume that \((K, v)\) is an HDV-field, such that \(\hat{K}\) is an \(n\)-dimensional local field with \(\text{char}(\hat{K}) = p\). Then \(\text{Brd}_p(K) \geq n\). Moreover, if the \(n\)-th residue field is finite, then \(\text{abrd}_p(K) \leq n + 1\).

Proof. As \([\hat{K} : \hat{K}^p] = p^n\), Theorem 2.1 (b) yields \(\text{Brd}_p(K) \geq n\), so it suffices to prove that if \(\hat{K}_0\) is finite, then \(\text{abrd}_p(K) \leq n + 1\). In view of Proposition 7.1 (b) and Fact 3.5 (a), one may consider only the case of char \((K) = 0\), whereas the formula in Conjecture 7.3, this can be obtained by using Theorem 2.1 (b), [5], Theorem 3.3. As to Conjecture 7.3, it need not be true if \((K, v)\) is merely HDV with char \((\hat{K}) = p\) and \([\hat{K} : \hat{K}^p] < \infty\). One may take as a counter-example the iterated formal power series field \(K = \hat{K}_0((X_1)) \ldots ((X_n))((Y))\) in a system of variables \(X_1, \ldots, X_n, Y\) over a quasifinite field \(\hat{K}_0\) with char \((\hat{K}_0) = p\). Then \(\text{Brd}_p(K) = n\), by [10], Proposition 3.5 (implied by [7], Lemma 4.3 (b), [5], Lemma 4.2 and [3], Theorem 3.3), whereas the formula in Conjecture 7.3 requires \(\text{Brd}_p(K) = n + 1\) (the standard discrete valuation on \(K\) is Henselian.

Our next result proves Conjecture 1.1 in the special case where \(K\) is a subgroup of \(\text{Br}(K'/K')\) (cf. [2], Ch. VII, Theorem 28), that this yields \(\deg(D') \mid p^{n+1}\) (see [20], Sect. 13.4), so Proposition 7.1 is proved. □
with \( \hat{K} = \hat{K}_0((X_1)) \ldots ((X_n)) \), whence, \( [\hat{K}: \hat{K}^p] = p^n \) and \( \text{cd}_p(\hat{K}_0) = 1 \).

This example as well as Proposition 7.2 draw one’s attention to the following problem:

**Problem 7.4.** Let \((K, v)\) be an HDV-field with \( \text{char}(\hat{K}) = p > 0 \). Suppose that \( \hat{K} \) is an \( n \)-dimensional local field, for some \( n \in \mathbb{N} \), with an \( n \)-th residue field \( \hat{K}_0 \). Find whether \( \text{Brd}_p(K) = n \).

The conditions of Problem 7.4 show that \( K_v \) is an \((n + 1)\)-dimensional local field with last residue field \( \hat{K}_0 \) (and \( \hat{K} \) is isomorphic to an iterated formal power series field in \( n \) variables over the quasifinite field \( \hat{K}_0 \), see [14], 2.5.2). Therefore, in case \( \text{char}(K) = p \), Fact 3.5(a) and [10], Proposition 3.5, give an affirmative answer to Problem 7.4. When \( n = 1 \), such an answer is contained in the following result of [11], obtained as a final step towards a full characterization of stable HDV-fields by properties of their residue fields:

**Proposition 7.5.** Let \((K, v)\) be an HDV-field with \( \text{char}(\hat{K}) = p > 0 \). Then \( \text{Brd}_p(K) \leq 1 \) if and only if the following condition is fulfilled:

\[ [K: \hat{K}^p] \leq p, \text{ and in case } \text{Brd}_p(K) \neq 0, \text{ every degree } p \text{ extension of } \hat{K} \text{ in } \hat{K}(p) \text{ is embeddable as a } \hat{K} \text{-subalgebra in each } D_p \in d(\hat{K}) \text{ of degree } p. \]

The equality \( \text{Brd}_p(K) = 0 \) holds if and only if \( \hat{K} \) is perfect and \( \hat{K}(p) = \hat{K} \).

**Remark 7.6.** The inequalities \( n \leq \text{Brd}_p(K) \leq n + 1 \) hold, for any HDV-field \((K, v)\), such that \( \hat{K} \) is an \( n \)-dimensional local field with a finite \( n \)-th residue field and with \( \text{char}(\hat{K}_1) = p \), \( \hat{K}_1 \) being the \((n - 1)\)-th residue field of \( \hat{K} \). Proposition 7.2 reduce the proofs to the case of \( \text{char}(\hat{K}) = 0 \) (and \( n \geq 3 \), in view of Proposition 7.3). Then the stated inequalities are contained in [10], Proposition 4.4.

Note finally that the interest in the question of whether \( \text{Brd}_p(K) = n \), if \((K, v)\) is an HDV-field, \( \text{char}(\hat{K}) = p > 0 \), \( \hat{K}_{\text{sep}} = \hat{K} \) and \( [\hat{K}: \hat{K}^p] = p^n \), for some \( n \in \mathbb{N} \), is motivated not only by Theorem 2.1(b) and [4], Theorem 4.16, but also by the following well-known conjecture (see, e.g., [4], Sect. 4):

**Conjecture 7.7.** Assume that \( F \) is a field of type \( C_{uv} \), i.e. each homogeneous polynomial \( f(X_1, \ldots, X_m) \in F[X_1, \ldots, X_m] \) of degree \( d \) with \( 0 < d'' < m \), has a nontrivial zero over \( F \). Then \( a \text{brd}_p(F) < v \).

To show how Conjecture 7.7 is related to the noted question, fix an HDV-field \((E, \omega)\) so that \( \text{char}(\hat{E}) = p > 0 \), \( \hat{E} \) be algebraically closed, and when \( \text{char}(E) = p \), \( E = E_{\omega} \). Consider a finitely-generated extension \( F/E \) of transcendence degree \( n \). By Lang’s theorem [10], \( E \) is of type \( C_1 \), whence, by the Lang-Nagata-Tsen theorem [24], \( F \) is of type \( C_{n+1} \). The assumptions on \( F \) and \( E \) also imply the existence of a discrete valuation \( \omega' \) of \( F \) extending \( \omega \), such that \( \hat{F}/\hat{E} \) is a finitely-generated extension of transcendence degree \( n \) (when \( F/E \) is purely transcendental, one may take as \( \omega' \) the restricted Gauss prolongation of \( \omega \) on \( F \)). Thus it follows that \( [\hat{F}' : \hat{F}^p] = p^n \), for every finite
extension \(F'/F\). This enables one to deduce (e.g., from [8], Lemmas 3.1 and 4.3) that if \((L, w)\) is a Henselization of \((F, \omega')\), then \(\text{abrd}_p(L) \leq \text{Br}_p(F)\). Hence, Conjecture 7.7 and the \(C_{n+1}\) type of \(F\) require that \(\text{abrd}_p(L) \leq n\).

On the other hand, \((L, w)/(F, \omega')\) is immediate, so \([\hat{L} : \hat{F}] = p^n\), and by Theorem 2.1 (b), \(\text{Br}_p(L) \geq n\). Thus the assertion that \(\text{Br}_p(L) = n\) can be viewed as a special case of Conjecture 7.7.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria: E-mail address: chipchak@math.bas.bg