OPTIMAL COMPARISON OF $P$-NORMS OF DIRICHLET POLYNOMIALS

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ABSTRACT. Let $1 \leq p < q < \infty$. We show that

$$\sup \|D\|_{H_q} \|D\|_{H_p} = \exp \left( \frac{\log x}{\log \log x} \left( \log \sqrt{\frac{q}{p}} + O \left( \frac{\log \log \log x}{\log \log x} \right) \right) \right),$$

where the supremum is taken over all non-zero Dirichlet polynomials of the form $D(s) = \sum_{n \leq x} a_n n^{-s}$. An application is given to the study of multipliers between Hardy spaces of Dirichlet series.

1. INTRODUCTION

Let $1 \leq p < \infty$. Given a Dirichlet polynomial $D(s) = \sum_n a_n n^{-s}$, its $p$-norm is defined as

$$\|D\|_{H_p} := \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |D(it)|^p dt \right)^{1/p}. \quad (1)$$

The fact that the previous limit exists, can be argued by means of Bohr’s one-to-one correspondence between Dirichlet series and (formal) power series in infinitely many variables [3]. Using that every $n \in \mathbb{N}$ has a unique prime number decomposition $n = p^{\alpha} := p_1^{\alpha_1} p_2^{\alpha_2} \ldots$, where $p = (p_n)_{n \in \mathbb{N}}$ is the ordered sequence of primes and $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, the set of eventually null sequences in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Following Bohr [3] we can identify every Dirichlet series $D = \sum_n a_n n^{-s}$ with the (formal) power series

$$LD \equiv \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{p^\alpha} z^\alpha,$$

the so-called Bohr lift. In case $D(s)$ is a Dirichlet polynomial, $LD$ is then a trigonometric polynomial. And if $d\omega$ denotes the Haar measure on the infinite-dimensional torus $\mathbb{T}^\mathbb{N}$, Birkhoff Ergodic Theorem implies that the limit in (1) exists, being $\|D\|_{H_p} = \|LD\|_{L_2(\mathbb{T}^\mathbb{N})}$ (see [3] for the details). This shows in particular that $\| \cdot \|_{H_p}$ is a norm on the space of Dirichlet polynomials; and moreover, its completion $\mathcal{H}_p$ can be seen as a Banach space of Dirichlet series isometric to the Hardy space $H_p(\mathbb{T}^\mathbb{N})$ (defined as in [5]) through Bohr’s identification. The

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systematic study of the Banach spaces $\mathcal{H}_p$ started in [11] and [9]. Recall that in the
setting of almost periodic functions, this type of limit was firstly considered by
Besicovitch [2].

Given $1 \leq p, q < \infty$ we define

$$\mathfrak{U}(q, p, x) := \sup \|D\|_{\mathcal{H}_q} \text{ taken over all } 0 \neq D(s) = \sum_{n \leq x} a_n s^n.$$ 

Along the paper we will always assume that $1 \leq p < q < \infty$, since this is the
interesting case.

Let us introduce some notation: given $x > 0$ big enough, we define recursively

$$\log_1 x := \log x$$
$$\log_k x := \log_{k-1} x \text{ for } k > 1.$$ 

The main result of the paper

reads as follows:

**Theorem 1.1.** For every $1 \leq p < q < \infty$

$$\mathfrak{U}(q, p, x) = \exp \left( \frac{\log x}{\log_2 x} \left( \log \sqrt{\frac{q}{p}} + O \left( \frac{\log_3 x}{\log_2 x} \right) \right) \right).$$ 

There already exist inequalities comparing the $p$-norms of certain type of trigono-
metric polynomials on $\mathbb{T}^N$

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha.$$ 

Recall that $P(z)$ is said to be $m$-homogeneous (for some $m \in \mathbb{N}$) if $c_\alpha = 0$
whenever $|\alpha| := \alpha_1 + \alpha_2 + \ldots \neq m$. Let us denote

$$H_{m}^{q,p} := \sup \|P\|_{L_q(\mathbb{T}^N)} / \|P\|_{L_p(\mathbb{T}^N)},$$

where the supremum is taken over all $m$-homogeneous polynomials $P(z) \neq 0$.
Basing on Weissler’ result [14] about hypercontractive estimates for the Poisson
semigroup, Bayart [11 Theorem 9] proved that

$$H_{m}^{q,p} \leq \left( \frac{\sqrt{q}}{p} \right)^m.$$ 

Recently, it has been shown that the best constant $C > 0$ such that $H_{m}^{q,p} \leq C^m$, is
precisely $C = \sqrt{q/p}$ (see [7]). We can deduce from the previous estimation that
every polynomial $P(z)$ as above satisfies

$$\|P\|_{L_q(\mathbb{T}^N)} \leq H_{\deg(P)}^{q,p} \|P\|_{L_p(\mathbb{T}^N)},$$

where $\deg(P) := \max \{|\alpha| : c_\alpha \neq 0\}$.

Indeed, the rotation invariance of the Haar measure yields that

$$\|P\|_{L_r(\mathbb{T}^N)} = \|\hat{P}\|_{L_r(\mathbb{T} \times \mathbb{T}^N)} \text{ for each } 1 \leq r < \infty$$

where $\hat{P}$ is the trigonometric polynomial on $\mathbb{T} \times \mathbb{T}^N \equiv \mathbb{T}^N$ given by

$$\hat{P}(z, \omega) = z^{\deg(P)} P(\omega_1 z^{-1}, \omega_2 z^{-1}, \ldots), \quad (z, \omega) \in \mathbb{T} \times \mathbb{T}^N.$$
But \( \tilde{P} \) is an \( m \)-homogeneous polynomial with \( m = \deg(P) \), so can apply (2) to \( \tilde{P} \) and use (4) to conclude that (3) holds. Using Bohr’s lift, we can reformulate this last inequality in terms of Dirichlet polynomials \( D(s) = \sum_n a_n n^{-s} \) as

\[
\|D\|_{H_q^m} \leq H_{q,p}^m \|D\|_{H_q^p}, \quad \text{where } m = \max \{ \Omega(n) : a_n \neq 0 \}.
\]

Recall that \( \Omega(n) = \Omega(p^\alpha) = |\alpha| \) is the function which counts the number of prime divisors of \( n \) (with multiplicity). It satisfies \( \Omega(n) \leq \log n / \log 2 \), which let us deduce that

\[
\mathcal{U}(q,p,x) \leq \exp \left( \frac{\log x}{\log 2} \sqrt{\log x} \frac{\log \sqrt{\frac{q}{p}}}{\log \sqrt{\frac{p}{q}}} \right).
\]

Nevertheless, this upper bound is far from being optimal: A well-known inequality due to Helson [10] together with an old estimation of \( \max \{ d(n) : n \leq x \} \) in terms of \( x \) due to Wigert [15], gives that

\[
\sum_{n \leq x} a_n \nu_n \leq \exp \left( \frac{\log x}{\log 2} \left( \log \sqrt{\frac{q}{p}} + O \left( \frac{\log q}{\log 2} \right) \right) \right).
\]

This is the upper estimate for the special case \( \mathcal{U}(2,1,x) \) given in Theorem 1.1, and hence in this case it remains to prove the lower estimate. But in the general case the estimate for \( \mathcal{U}(q,p,x) \) needs a more delicate argument which is carried out in Section 2. It relies on a decomposition method inspired by [11], in combination with (2) and a deep number theoretical result of Bruijn. Section 3 deals with the construction of a suitable family of Dirichlet polynomials to obtain the lower estimate for \( \mathcal{U}(q,p,x) \). We follow an argument based on the Central Limit Theorem, which was used in [12] to give optimal bounds for the constants in the Khintchine-Steinhaus inequality. To adapt this idea to our problem, we have to develop a quantitative result concerning the convergence of the \( p \)-moments for the special sequence of random variables we handle (Lemma 3.1 and Theorem 3.2).

2. Estimation from Above

Here we prove the upper estimate from Theorem 1.1

\[
\mathcal{U}(q,p,x) \leq \exp \left( \frac{\log x}{\log 2} \left( \log \sqrt{\frac{q}{p}} + O \left( \frac{\log q}{\log 2} \right) \right) \right)
\]

\textbf{Proof.} Fix \( 2 \leq y \leq x \) and denote

\[
S(x,y) = \{ n \leq x : p_t | n \Rightarrow p_t \leq y \},
\]

\[
L(x,y) = \{ n \leq x : p_t | n \Rightarrow p_t > y \}.
\]

Let \( D(s) = \sum_{n \leq x} a_n n^{-s} \) be a Dirichlet polynomial. Since each \( 1 \leq n \leq x \) can be uniquely decomposed as a product \( n = jk \) for some \( j \in S(x,y) \) and \( k \in L(x,y) \),
We choose a proper value of uniformly for \( p \geq 1 \). To prove it, we will use Bohr’s lift and translate the previous elements into trigonometric polynomials. Let \( P = \mathcal{L} D \) be the trigonometric polynomial

\[
P(\omega) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \omega^\alpha.
\]

If \( \lambda := \pi(y) \), each \( \alpha \in \mathbb{N}_0^n \) has the form \( \alpha = (\gamma, \beta) \) where \( \gamma \in \mathbb{N}_0^\lambda \), \( \beta \in \mathbb{N}_0^{\mathbb{N}} \). Hence, for \( \omega = (u, v) \in \mathbb{T}^\lambda \times \mathbb{T}^\gamma = \mathbb{T}^\gamma \)

\[
P(u, v) = \sum_{\gamma \in \mathbb{N}_0^\lambda} P_\gamma(v) u^\gamma \quad \text{where} \quad P_\gamma(v) = \sum_{\beta \in \mathbb{N}_0^{\mathbb{N}}} c_{(\gamma, \beta)} v^\beta.
\]

For each \( j \in S(x, y) \) we have that \( \mathcal{L} D_j = P_\gamma \) whether \( j = p^{(\gamma,0)}. \) Hence

\[
\| D_j \|^p_{\mathcal{H}_p} = \| P_\gamma \|^p_{L_p^{\mathbb{N} \setminus \gamma}} = \int_{\mathbb{T}^{\mathbb{N} \setminus \gamma}} \left| \int_{\mathbb{T}^\gamma} P(u, v) u^{-\gamma} du \right|^p dv \leq \int_{\mathbb{T}^{\mathbb{N} \setminus \gamma}} \left| \int_{\mathbb{T}^\gamma} |P(u, v)|^p dv \right| du = \| P \|^p_{L_p^{\mathbb{N} \setminus \gamma}} = \| D \|^p_{\mathcal{H}_p}.
\]

This proves the claim. Notice that every \( k \in L(x, y) \) satisfies \( x \geq k \geq y^{\Omega(k)}. \) Combining this inequality with (5) and (2), for each \( j \in S(x, y) \) we have that

\[
\| D_j \|^p_{\mathcal{H}_q} \leq \exp \left( \frac{\log x}{\log y} \log \sqrt{\frac{q}{p}} \right) \| D \|^p_{\mathcal{H}_p}.
\]

Applying this to (7), we get

\[
\| D \|^p_{\mathcal{H}_q} \leq \sum_{k \in S(x, y)} \| D_j \|^p_{\mathcal{H}_q} \leq |S(x, y)| \exp \left( \frac{\log x}{\log y} \log \sqrt{\frac{q}{p}} \right) \| D \|^p_{\mathcal{H}_p}.
\]

A deep result due to Bruijn [13, p. 359, Theorem 2] states that

\[
\log |S(x, y)| = Z \left( 1 + O \left( \frac{1}{\log y} + \frac{1}{\log_2 x} \right) \right)
\]

uniformly for \( 2 \leq y \leq x \), where

\[
Z = Z(x, y) := \frac{\log x}{\log y} \log \left( 1 + \frac{y}{\log x} \right) + \frac{y}{\log y} \log \left( 1 + \frac{\log x}{y} \right).
\]

We choose a proper value of \( y \) to minimize the constant in (8).

\[
y := \exp \left( \frac{(\log x)^2}{\log_2 x + \log_3 x} \right) = \log x \exp \left( \frac{(\log x)^2}{\log_2 x + \log_3 x} \right),
\]

Notice that

\[
\frac{y}{\log y} = \frac{\log x}{\log_2 x} O \left( \frac{1}{\log_2 x} \right) \quad \text{and} \quad \log \left( 1 + \frac{\log x}{y} \right) = O(\log_3 x).
\]
Using that \( \log (1 + t) \leq t \) for each \( t > 0 \), we can bound
\[
Z \leq \frac{y}{\log y} \left( 1 + \log \left( 1 + \frac{\log x}{y} \right) \right) \quad \text{and so} \quad Z = \frac{\log x}{\log_2 x} O \left( \frac{\log_3 x}{\log_2 x} \right).
\]
Using this estimation in (9), we get that
\[
(10) \quad \log |S(x, y)| = \frac{\log x}{\log_2 x} O \left( \frac{\log_3 x}{\log_2 x} \right).
\]
On the other hand, for the taken value of \( y \)
\[
(11) \quad \exp \left( \frac{\log x}{\log y} \log \sqrt{\frac{q}{p}} \right) = \exp \left( \frac{\log x}{\log_2 x} \left( \log \sqrt{\frac{q}{p}} + \frac{\log_3 x}{\log_2 x} \right) \right).
\]
Replacing estimations (10) and (11) in (8), we conclude the result. \( \square \)

3. Estimation from below

Along this section we will denote for every \( n \in \mathbb{N} \) and \( z \in \mathbb{C}^n \)
\[
Q_n(z) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} z_j.
\]
A special case of the Khinchine-Steinhaus inequality given in [12, Theorem 2] states that for every \( r \geq 1 \) and every \( n \)
\[
\int_{\mathbb{T}^n} |Q_n(z)|^{2r} \, dz \leq \Gamma(r + 1).
\]
Let us point out that here the constant on the right side of this inequality is independent of \( n \), and even optimal since by the central limit theorem
\[
\lim_{n \to \infty} \int_{\mathbb{T}^n} |Q_n(z)|^{2r} \, dz = \Gamma(r + 1).
\]
Hence by Stirling’s formula for every \( r \geq 1 \)
\[
(12) \quad \int_{\mathbb{T}^n} |Q_n(z)|^{2r} \, dz \leq \sqrt{2\pi r} \left( \frac{r}{e} \right)^r e^{\frac{4m^2}{e}}.
\]
We will need a similar lower estimate.

**Lemma 3.1.** For \( m, n \in \mathbb{N} \) with every \( n > m + 1 \) we have
\[
\int_{\mathbb{T}^n} |Q_n(z)|^{2m} \, dz \geq \frac{1}{n^m} \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} \frac{m!}{\alpha!} \frac{z^{\alpha}}{\sqrt{n}^m},
\]
(as usual we here write \( \alpha! = \prod_j \alpha_j! \)), hence by integration (and the orthogonality of the monomials on \( \mathbb{T}^n \))
\[
\int_{\mathbb{T}^n} |Q_n(z)|^{2m} \, dz = \frac{1}{n^m} \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} \left( \frac{m!}{\alpha!} \right)^2.
\]
Now we make use of the Cauchy-Schwartz inequality and again the multinomial formula to deduce that

\[
\left( \sum_{\alpha \in \mathbb{N}_0^{|\alpha|=m}} \frac{m!}{\alpha!} \right)^{1/2} \left( \sum_{\alpha \in \mathbb{N}_0^{|\alpha|=m}} 1 \right)^{1/2} \geq \sum_{\alpha \in \mathbb{N}_0^{|\alpha|=m}} \frac{m!}{\alpha!} = n^m,
\]

and since

\[
\sum_{\alpha \in \mathbb{N}_0^{|\alpha|=m}} 1 = \binom{m+n-1}{m},
\]

we arrive at

\[
(13) \quad \int_{\mathbb{T}^n} |Q_n(z)|^{2m} \, dz \geq n^m \binom{m+n-1}{m}^{-1}.
\]

In order to be able to handle the binomial coefficient we need the following estimate

\[
(14) \quad \binom{m+n-1}{m} \leq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{m+n-1}{n-1}} \frac{(m+n-1)^{m+n-1}}{(n-1)^{n-1} m^m};
\]

indeed, by Stirling’s formula for every \( k \)

\[
\sqrt{2\pi} \left( \frac{k}{e} \right)^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k} \left( \frac{k}{e} \right)^k e^{\frac{1}{12k}}
\]

which gives

\[
\frac{(m+n-1)!}{(n-1)! m!} \leq \frac{\sqrt{2\pi(m+n-1)}}{\sqrt{2\pi(n-1)}} \left( \frac{n-1}{m} \right)^{n-1} e^{\frac{1}{12(n-1)+1}} \frac{e^{\frac{1}{12m+1}}}{2\pi m} \left( \frac{m}{n} \right)^m e^{\frac{1}{12m+1}},
\]

and consequently (14). We combine now (13) and (14) to obtain

\[
\int_{\mathbb{T}^n} |Q_n(z)|^{2m} \, dz \geq \sqrt{2\pi} n^m \sqrt{\frac{m(n-1)}{m+n-1}} \frac{(n-1)^{n-1} m^m}{m+n-1} \frac{m^m}{(1 + \frac{m}{n-1})^{n-1}} \sqrt{\frac{n-1}{m+n-1}} \left( \frac{n}{m+n-1} \right)^m
\]

\[
\geq \sqrt{2\pi} m \left( \frac{m}{e} \right)^m \left( \frac{n-1}{m+n-1} \right)^{m+1/2};
\]

for the last estimate we use that \((1 + 1/x)^x \leq e\) for \( x \geq 1 \). To bound the last factor, we use that \((1 - 1/x)^x > e^{-2}\) for \( x > 2 \), so that

\[
\left( \frac{n-1}{m+n-1} \right)^{m+1/2} \geq e^{-2 \frac{m(n+1/2)}{m+n-1}} \geq e^{-4m^2/n}.
\]

This completes the argument.

To simplify the notation, from now on given two functions \( f, g \) depending on \( p, q \) and probably other variables, we will write \( f \gg g \) when \( f \geq c g \) for some constant \( c = c(p, q) \) depending on \( p \) and \( q \) but independent of the rest of variables.
Theorem 3.2. Let \( n, k \in \mathbb{N} \) with \( n > \lfloor kq/2 \rfloor + 1 > \lfloor kp/2 \rfloor + 1 > 1 \). Then

\[
\frac{\|Q^k_n\|_q}{\|Q^k_n\|_p} \gg k^{\frac{k}{2p} - \frac{1}{p}} \left( \frac{q}{p} \right)^{k/2} e^{-\frac{ak^2}{n}}.
\]

Proof. Since \( kp \geq 2 \) by hypothesis, we can use (12) bound

\[
\|Q^k_n\|_p \leq (\pi kp)^{\frac{1}{2p}} \left( \frac{k}{2e} \right)^{\frac{k}{2}} e^{\frac{1}{4k^2}}.
\]

On the other hand, we want to give a lower bound of \( \|Q^k_n\|_q \). Let \( m := \lfloor kq/2 \rfloor \geq 1 \).

Since \( kq \geq 2m \), we can write

\[
\|Q^k_n\|_q = \int_{\mathbb{T}^n} |Q_n(\omega)|^{kq} \, d\omega \geq \left( \int_{\mathbb{T}^n} |Q_n(\omega)|^{2m} \, d\omega \right)^{\frac{kq}{2m}}.
\]

We can then use the lower bound of Lemma 3.1 in (16) to deduce that

\[
\|Q^k_n\|_q \geq (\sqrt{2\pi m})^{k/2m} \left( \frac{m}{e} \right)^{k/2} e^{-\frac{2km}{n}}.
\]

Combining (17) and (15) we arrive to

\[
\frac{\|Q^k_n\|_q}{\|Q^k_n\|_p} \geq (\sqrt{2\pi m})^{\frac{k}{2m} - \frac{1}{p}} \cdot \frac{m^{\frac{k}{2m}}}{(\frac{k}{2e})^{\frac{k}{2}}} \cdot \frac{m^{\frac{k}{2}}}{(\frac{k}{2})^{\frac{k}{2}}} \cdot e^{\frac{1}{4k^2} e^{-\frac{2km}{n}}}.
\]

Using again \( kq/2 \geq m \geq kq/2 - 1 \), we get that

\[
\frac{m^{\frac{k}{2m}}}{(\frac{k}{2})^{\frac{k}{2}}} \gg k^{\frac{k}{2p} - \frac{1}{p}} \quad \frac{m^{\frac{k}{2m}}}{(\frac{k}{2e})^{\frac{k}{2}}} \geq k^{\frac{k}{2p} - \frac{1}{p}} \quad \frac{m^{\frac{k}{2}}}{(\frac{k}{2})^{\frac{k}{2}}} \gg k^{\frac{k}{2p} - \frac{1}{p}}
\]

(19)

\[
\frac{m^{\frac{k}{2}}}{(\frac{k}{2})^{\frac{k}{2}}} \gg \left( \frac{q}{p} \right)^{\frac{k}{2}} \quad \frac{m^{\frac{k}{2m}}}{(\frac{k}{2e})^{\frac{k}{2}}} \gg \left( \frac{q}{p} \right)^{\frac{k}{2}}.
\]

(20)

Applying (19) and (20) to (18) we can conclude that

\[
\frac{\|Q^k_n\|_q}{\|Q^k_n\|_p} \gg k^{\frac{k}{2p} - \frac{1}{p}} \left( \frac{q}{p} \right)^{\frac{k}{2}} e^{-2km/n} \geq k^{\frac{k}{2p} - \frac{1}{p}} \left( \frac{q}{p} \right)^{\frac{k}{2}} e^{-\frac{ak^2}{n}},
\]

which is what we wanted. \( \square \)

The trigonometric polynomial \( Q^k_n = \sum_{\alpha} c_\alpha z^\alpha \) satisfies that \( c_\alpha \neq 0 \) if and only if \( \alpha \in \mathbb{N}_0^k \) with \( |\alpha| = k \). Let us fix a real number \( x > e^{e^x} \) and consider the values

\[
k(x) := \left\lceil \frac{\log x}{\log_2 x + \log_3 x} \right\rceil \quad \text{and} \quad n(x) := \pi(x^{1/k(x)}).
\]

The correspondent Dirichlet series via Bohr transform is then of the form

\[
D_x(s) = \mathcal{L}^{-1} Q^k_{n(x)} = \left( \sum_{i=1}^{n(x)} \frac{1}{\sqrt{n}p_i^n} \right)^{k(x)} = \sum_{m \leq x} a_m m^{-s}.
\]
Theorem 3.3. For each $1 \leq p < q < \infty$

$$\frac{\|D_{x}\|_{q}}{\|D_{x}\|_{p}} \geq \exp \left( \frac{\log x}{\log_{2} x} \left( \log \sqrt{\frac{q}{p}} + O \left( \frac{\log_{3} x}{\log_{2} x} \right) \right) \right).$$

Proof. Using the prime number theorem, and more specifically a bound due to Dusart [8, Theorem 1.10], we have that for $x^{1/k(x)} \geq 599$

$$\pi(x^{1/k(x)}) \geq \frac{k(x)x^{1/k(x)}}{\log x} \left( 1 + \frac{k(x)}{\log x} \right).$$

Therefore

$$\frac{k(x)}{n(x)} \leq \frac{\log x}{x^{1/k(x)}} = \exp \left( \log_{2} x - \frac{\log x}{k(x)} \right) \leq \exp \left( - \frac{\log_{3} x}{\log_{2} x} \right) = \frac{1}{\log_{2} x}.$$ 

Note that $n(x)/k(x)$ tends to infinity when $x$ does, so for $x$ big enough the hypothesis of Theorem 3.2 are satisfied. This means that we can bound

$$\frac{\|D_{x}\|_{q}}{\|D_{x}\|_{p}} \gg k(x) \frac{1}{x^{1/k(x)}} \left( \frac{q}{p} \right)^{k(x)/2} e^{-\frac{k(x)^{2}q}{n(x)}} = \exp \left( k(x) \left( \log \sqrt{\frac{q}{p}} + f(x) \right) \right)$$

where

$$f(x) = \left( \frac{1}{2q} - \frac{1}{2p} \right) \log k(x) - \frac{k(x)}{n(x)} = O \left( \frac{1}{\log_{2} x} \right).$$

Finally observe that

$$k(x) = \frac{\log x}{\log_{2} x} \left( 1 + O \left( \frac{\log_{3} x}{\log_{2} x} \right) \right),$$

which completes the proof. □

4. Application to multipliers

Recall that a sequence of real numbers $(\lambda_{n})_{n \in \mathbb{N}}$ is said to be a multiplier from $\mathcal{H}_{p}$ to $\mathcal{H}_{q}$, if for every Dirichlet series $\sum_{n} a_{n} n^{-s}$ in $\mathcal{H}_{p}$ we have that $\sum_{n} \lambda_{n} a_{n} n^{-s}$ belongs to $\mathcal{H}_{q}$. In [1], Bayart makes use of Weissler result [14] to obtain sufficient conditions for a multiplicative sequence $(\lambda_{n})$ (i.e., $\lambda_{nm} = \lambda_{n} \lambda_{m}$ for all $m, n$) to be a multiplier from $\mathcal{H}_{p}$ to $\mathcal{H}_{q}$. Here we use Theorem [1] to give a sufficient condition for a not necessarily multiplicative sequence of positive real numbers to be a multiplier.

Theorem 4.1. Given $1 \leq p < q < \infty$, let $(\lambda)_{n \in \mathbb{N}}$ be a decreasing sequence of positive real numbers satisfying

$$\sum_{n} \frac{\lambda_{n}}{n \log \log n} \left( \sqrt{\frac{q}{p}} + \varepsilon \right)^{\frac{\log n}{\log \log n}} < \infty \quad \text{for some } \varepsilon > 0.$$

Then $(\lambda_{n})_{n \in \mathbb{N}}$ is a multiplier from $\mathcal{H}_{p}$ to $\mathcal{H}_{q}$. 
Proof. Let us denote
\[ g(x) := \exp\left(\frac{\log x}{\log_2 x}\right) A \] where \( A := \log \sqrt{\frac{q}{p}} + \varepsilon. \)

Recall that there exists \( C > 0 \) such that for every \( x > 1 \), the partial sum operator \( S_x(\sum_n a_n n^{-s}) = \sum_{n \leq x} a_n n^{-s} \) has norm \( \|S_x\|_{\mathcal{H}_p \to \mathcal{H}_p} \leq C \log x \). Given \( D = \sum_n a_n n^{-s} \) in \( \mathcal{H}_p \), we then have
\[
\left\| \sum_{n \leq x} \frac{a_n}{n^s} \right\|_{\mathcal{H}_q} \leq \mathcal{O}(q, p, x) \left\| \sum_{n \leq x} \frac{a_n}{n^s} \right\|_{\mathcal{H}_p} \leq \mathcal{O}(q, p, x) C \log x \|D\|_{\mathcal{H}_p}.
\]

By Theorem [1.1] we deduce that when \( x \) is big enough
\[
(21) \quad \left\| \sum_{n \leq x} \frac{a_n}{n^s} \right\|_{\mathcal{H}_q} \leq g(x) \|D\|_{\mathcal{H}_p}.
\]

Moreover, also if \( x \) tends to infinity we have that
\[
0 \leq g'(x) \leq \frac{A g(x)}{x \log_2 x} \quad \text{and so} \quad \frac{d}{dx} \left( \frac{g(x)}{x \log_2 x} \right) \leq g(x) \left( A - \log_2 x \right) < 0.
\]

This means that for \( n \) big enough,
\[
(22) \quad g(n + 1) - g(n) = \int_n^{n+1} g'(x) \, dx \leq \frac{A g(n)}{n \log_2 n}.
\]

Let \( 0 < m < M \) be natural numbers. Using Abel’s summation formula
\[
\sum_{n=m}^{M} \frac{\lambda_n a_n}{n^s} = \sum_{n=m}^{M-1} \left( \sum_{k=1}^{n} \frac{a_k}{k^s} \right) (\lambda_n - \lambda_{n+1}) - \left( \sum_{k=1}^{M} \frac{a_k}{k^s} \right) \lambda_m + \left( \sum_{k=1}^{M} \frac{a_k}{k^s} \right) \lambda_M.
\]

Therefore, taking \( m \) big enough and using (21) and (22)
\[
\left\| \sum_{n=m}^{M} \frac{\lambda_n a_n}{n^s} \right\|_q \leq \|D\|_p \left( \sum_{n=m}^{M-1} g(n) (\lambda_n - \lambda_{n+1}) + g(m-1)\lambda_m + g(M)\lambda_M \right)
\]
\[
\leq \|D\|_p \left( 2\lambda_m g(m) + \sum_{n=m}^{M-1} \lambda_n (g(n+1) - g(n)) \right)
\]
\[
\leq \|D\|_p \left( 2\lambda_m g(m) + A \sum_{n=m}^{M-1} \frac{\lambda_n g(n)}{n \log_2 n} \right).
\]

The series \( \sum_{n} \frac{\lambda_n g(n)}{n \log_2 n} \) converges by hypothesis. On the other hand, it also follows from this fact that there is an increasing sequence \((N_k)_{k \in \mathbb{N}}\) of natural numbers such that \( \lim_k \lambda_{N_k} g(N_k) = 0 \). Hence, the inequality above leads to the existence of a subsequence of the partial sums of \( \sum_n \lambda_n a_n n^{-s} \) converging in \( \mathcal{H}_q \), which in particular means that \( \sum_n \lambda_n a_n n^{-s} \in \mathcal{H}_q \). \( \square \)
5. Remarks

One of the main tools in the proof of Theorem 1.1 has been (2). This estimation is also valid when we deal with $m$-homogenous polynomials with coefficients in an arbitrary (complex) Banach space (see [4]). This means that the argument in the proof of Theorem 6 also works for Dirichlet polynomials with coefficients in some complex Banach space.

Although probably without leading to a better estimate in Theorem 1.1, we strongly believe that the inequality from (2) can be improved in the following way:

**Conjecture 5.1.** For every $1 \leq p < q < \infty$ and $m \in \mathbb{N}$ we have that

$$H_{m}^{q,p} \leq m^{\frac{1}{p} - \frac{1}{q}} \left( \frac{q}{p} \right)^{m}.$$

Indeed, for the case in which $p < q$ are powers of two, we can use elementary methods to show that this conjecture is true. We sketch here the proof of the case $p = 2$ and $q = 4$:

Let $P = \sum_{|\alpha|=m} c_{\alpha} \omega^{\alpha}$ be an $m$-homogeneous polynomial. Given $\alpha, \gamma \in \mathbb{N}_{0}^{(N)}$ we write $\alpha \leq \gamma$ whenever $\alpha_{n} \leq \gamma_{n}$ for each $n \in \mathbb{N}$. We this notation

$$\|P\|_{2}^{4} = \left( \sum_{|\alpha|=m} |c_{\alpha}|^{2} \right)^{2} = \sum_{|\gamma|=m} \left( \sum_{|\alpha|=m, \alpha \leq \gamma} |c_{\alpha}|^{2} |c_{\gamma - \alpha}|^{2} \right).$$

$$\|P\|_{4}^{4} = \sum_{|\gamma|=2m} \sum_{|\alpha|=m} c_{\alpha} c_{\gamma - \alpha} \leq \sum_{|\gamma|=2m} \left( \sum_{|\alpha|=m, \alpha \leq \gamma} |c_{\alpha}|^{2} |c_{\gamma - \alpha}|^{2} \right) \kappa(\gamma, m).$$

where $\kappa(\gamma, m) = |\{ \alpha : |\alpha| = m, \alpha \leq \gamma \}|$. Among all $\gamma \in \mathbb{N}_{0}^{(N)}$ with $|\gamma| = 2m$, we have that the maximum value of $\kappa(\gamma, m)$ is attained whenever the entries of $\gamma$ are all either one or zero. In this case, we can calculate explicitly $\kappa(\gamma, m)$ in terms of a combinatorial number that can be estimated by means of Lemma 14 as

$$\kappa(\gamma, m) \leq \binom{2m}{m} \leq \frac{4^{m}}{\sqrt{\pi} m}.$$ 

We then conclude that

$$\|P\|_{4}^{4} \leq \binom{2m}{m} \leq \frac{4^{m}}{\sqrt{\pi} m} \|P\|_{2}^{4}$$

which gives the desired result.

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