Asymptotics of Solutions of Some Integral Equations Connected with Differential Systems with a Singularity

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Our studies concern some aspects of scattering theory of the singular differential systems
\[ y' - x^{-1}Ay - q(x)y = \rho By, \quad x > 0 \]
with \( n \times n \) matrices \( A, B, q(x) \), \( x \in (0, \infty) \), where \( A, B \) are constant and \( \rho \) is a spectral parameter. We concentrate on investigation of certain Volterra integral equations with respect to tensor-valued functions. The solutions of these integral equations play a central role in construction of the so-called Weyl-type solutions for the original differential system. Actually, the integral equations provide a method for investigation of the analytical and asymptotical properties of the Weyl-type solutions while the classical methods fail because of the presence of the singularity. In the paper, we consider the important special case when \( q \) is smooth and \( q(0) = 0 \) and obtain the classical-type asymptotical expansions for the solutions of the considered integral equations as \( \rho \to \infty \) with \( o\left(\rho^{-1}\right) \) rate remainder estimate. The result allows one to obtain analogous asymptotics for the Weyl-type solutions that play in turn an important role in the inverse scattering theory.

Keywords: differential systems, singularity, integral equations, asymptotical expansions.

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INTRODUCTION

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with \( n \times n \) matrices \( A, B, q(x) \), \( x \in (0, \infty) \), where \( A, B \) are constant and \( \rho \) is a spectral parameter.

Differential equations with coefficients having non-integrable singularities at the end or inside the interval often appear in various areas of natural sciences and engineering. For \( n = 2 \), there exists an extensive literature devoted to different aspects of spectral theory of the radial Dirac operators, see, for instance \([1–5]\).

Systems of the form (1) with \( n > 2 \) and arbitrary complex eigenvalues of the matrix \( B \) appear to be considerably more difficult for investigation even in the “regular” case \( A = 0 \) \([6]\). Some difficulties of principal matter also appear due to the presence of the singularity. Whereas the “regular” case \( A = 0 \) has been studied fairly completely to date \([6–8]\), for the system (1) with \( A \neq 0 \) there are no similar general results.

The important role in scattering theory is played by a certain distinguished basis of generalized eigenfunctions for (1) (the so-called Weyl-type solutions, see, for instance \([9]\)). In the presence of the singularity construction and investigation of this basis encounters some difficulties which do not appear in the “regular” case \( A = 0 \). In particular, one can not use the auxiliary Cauchy problems with the initial conditions
at \( x = 0 \). The approach presented in [10] (see also [11] and references therein) for the scalar differential operators

\[
\ell y = y^{(n)} + \sum_{j=0}^{n-2} \left( \frac{\nu_j}{x^{n-j}} + q_j(x) \right) y^{(j)}
\]

is based on using some special solutions of the equation \( \ell y = \lambda y \) that also satisfy certain Volterra integral equations. This approach assumes some additional decay condition for the coefficients \( q_j(x) \) as \( x \to 0 \), moreover, the required decay rate depends on eigenvalues of the matrix \( A \). In this paper, we do not impose any additional restrictions of such a type. Instead, we use a modification of the approach first presented in [12] for the higher-order differential operators with regular coefficients on the whole line and recently adapted for differential systems of the form (1) on the semi-axis in [9].

In brief outline the approach can be described as follows. We consider some auxiliary systems with respect to the functions with values in the exterior algebra \( \Lambda \mathbb{C}^n \). Our study of these auxiliary systems centers on two families of their solutions that also satisfy some asymptotical conditions as \( x \to 0 \) and \( x \to \infty \) respectively, and can be constructed as solutions of certain Volterra integral equations. As in [12] we call these distinguished tensor solutions the fundamental tensors. The main difference from the above-mentioned method used in [10] is that we use the integral equations to construct the fundamental tensors rather than the solutions for the original system. Since each of the fundamental tensors has minimal growth (as \( x \to 0 \) or \( x \to \infty \)) among solutions of the same auxiliary system, this step does not require any decay of \( q(x) \) as \( x \to 0 \).

Construction and properties of the fundamental tensors were considered in details in our paper [9] provided that \( q(\cdot) \) is absolutely continuous and both \( q, q' \) are integrable on the semi-axis \((0, \infty)\). In this paper, we consider the important special case \( q(0) = 0 \) and obtain the classical-type asymptotical expansions for the fundamental tensors as \( \rho \to \infty \) with \( o(\rho^{-1}) \) rate remainder estimate.

1. **ASSUMPTIONS AND NOTATIONS. FORMULATIONS OF THE RESULTS**

We are to discuss first the unperturbed system:

\[
y' - x^{-1} Ay = \rho By
\]

and its particular case corresponding to the value \( \rho = 1 \) of the spectral parameter

\[
y' - x^{-1} Ay = By
\]

but to complex (in general) values of \( x \).

**Assumption 1.** Matrix \( A \) is off-diagonal. The eigenvalues \( \{\mu_j\}^n_{j=1} \) of the matrix \( A \) are distinct and such that \( \mu_j - \mu_k \notin \mathbb{Z} \) for \( j \neq k \), moreover, \( \text{Re}\mu_1 < \text{Re}\mu_2 < \cdots < \text{Re}\mu_n \), \( \text{Re}\mu_k \neq 0, k = 1, n \).

**Assumption 2.** \( B = \text{diag}(b_1, \ldots, b_n) \), the entries \( b_1, \ldots, b_n \) are nonzero distinct points on the complex plane such that \( \sum_{j=1}^n b_j = 0 \) and such that any 3 points are noncolinear.

Under Assumption 1 system (4) has the fundamental matrix \( c(x) = (c_1(x), \ldots, c_n(x)) \), where

\[
c_k(x) = x^{\mu_k} \hat{c}_k(x),
\]

\( \det c(x) \equiv 1 \) and all \( \hat{c}_k(\cdot) \) are entire functions, \( \hat{c}_k(0) = \h_k \), \( \h_k \) is an eigenvector of the matrix \( A \) corresponding to the eigenvalue \( \mu_k \). We define \( C_k(x, \rho) := c_k(\rho x), x \in (0, \infty), \rho \in \mathbb{C} \).
\[ \rho \in \mathbb{C}. \] 

We note that the matrix \( C(x, \rho) \) is a solution of the unperturbed system (3) (with respect to \( x \) for the given spectral parameter \( \rho \)).

Let \( \Sigma \) be the following union of lines through the origin in \( \mathbb{C} \):

\[ \Sigma = \bigcup_{(k,j): j \neq k} \{ z : \text{Re}(zb_j) = \text{Re}(zb_k) \}. \]

By virtue of Assumption 2 for any \( z \in \mathbb{C} \setminus \Sigma \) there exists the ordering \( R_1, \ldots, R_n \) of the numbers \( b_1, \ldots, b_n \) such that \( \text{Re}(R_1z) < \text{Re}(R_2z) \cdots < \text{Re}(R_nz) \). Let \( \mathcal{S} \) be a sector \{ \( z = r \exp(i\gamma), r \in (0, \infty), \gamma \in (\gamma_1, \gamma_2) \} \) lying in \( \mathbb{C} \setminus \Sigma \). Then [13] the system (4) has the fundamental matrix \( e(x) = (e_1(x), \ldots, e_n(x)) \) which is analytic in \( \mathcal{S} \), continuous in \( \mathcal{S} \setminus \{0\} \) and admits the asymptotics:

\[ e_k(x) = e^{xR_k}(f_k + x^{-1}\eta_k(x)), \quad \eta_k(x) = O(1), \quad x \to \infty, \quad x \in \mathcal{S}, \]

where \( (f_1, \ldots, f_n) = \tilde{f} \) is a permutation matrix such that \( (R_1, \ldots, R_n) = (b_1, \ldots, b_n)\tilde{f}. \) We define \( E(x, \rho) := e(\rho x) \).

Everywhere below we assume that the following additional condition is satisfied.

**Condition 1.** For all \( k = 2, \ldots, n \) the numbers

\[ \Delta_k^0 := \det(e_1(x), \ldots, e_{k-1}(x), e_k(x), \ldots, e_n(x)) \]

are not equal to 0.

Under Condition 1 the system (4) has the fundamental matrix \( \psi^0(x) = (\psi_1^0(x), \ldots, \psi_n^0(x)) \) which is analytic in \( \mathcal{S} \), continuous in \( \mathcal{S} \setminus \{0\} \) and admits the asymptotics:

\[ \psi_k^0(xt) = \exp(xtR_k)(f_k + o(1)), \quad t \to \infty, \quad x \in \mathcal{S}, \quad \psi_k^0(x) = O(x^{h_k}), \quad x \to 0. \]

We define \( \Psi^0(x, \rho) := \psi^0(\rho x) \). As above, we note that the matrices \( E(x, \rho), \Psi^0(x, \rho) \) solve (3).

In the sequel we use the following notations:

- \( \{e_k\}_{k=1}^n \) is the standard basis in \( \mathbb{C}^n \);
- \( \mathcal{A}_m \) is the set of all ordered multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_m), \alpha_1 < \alpha_2 < \cdots < \alpha_m, \alpha_j \in \{1,2,\ldots,n\}; \)
- for a sequence \( \{u_j\} \) of vectors and a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \) we define \( u_\alpha := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m} \);
- for a numerical sequence \( \{a_j\} \) and a multi-index \( \alpha \) we define \( a_\alpha := \sum_{j \in \alpha} a_j \), \( a^\alpha := \prod_{j \in \alpha} a_j \);
- for a multi-index \( \alpha \) the symbol \( \alpha' \) denotes the ordered multi-index that complements \( \alpha \) to \( (1,2,\ldots,n) \);
- for \( k = 1, \ldots, n \) we denote

\[ \alpha^k := \sum_{j=1}^k a_j, \quad \alpha_k := \sum_{j=k}^n a_j, \quad \alpha^k := \prod_{j=1}^k a_j, \quad \alpha_k := \prod_{j=k}^n a_j. \]

We note that Assumptions 1, 2 imply, in particular, \( \sum_{k=1}^n \mu_k = \sum_{k=1}^n R_k = 0 \) and therefore for any multi-index \( \alpha \) one has \( R_{\alpha'} = -R_\alpha \) and \( \mu_{\alpha'} = -\mu_\alpha \);
the symbol $V^{(m)}$, where $V$ is $n \times n$ matrix, denotes the operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors $u_1, \ldots, u_m$ the following identity holds:

$$V^{(m)}(u_1 \wedge u_2 \wedge \cdots \wedge u_m) = \sum_{j=1}^{m} u_1 \wedge u_2 \wedge \cdots \wedge u_j \wedge V u_j \wedge u_{j+1} \wedge \cdots \wedge u_m;$$

- if $h \in \wedge^n \mathbb{C}^n$ then $|h|$ is a number such that $h = |h| e_1 \wedge e_2 \wedge \cdots \wedge e_n$;
- for $h \in \wedge^m \mathbb{C}^n$ we set $\|h\| := \sum_{\alpha \in \mathcal{A}_m} |h_{\alpha}|$, where $\{h_{\alpha}\}$ are the coefficients from the expansion $h = \sum_{\alpha \in \mathcal{A}_m} h_{\alpha} e_\alpha$.

We use the same notation $L_\rho(a, b)$ for all the spaces of the form $L_\rho((a, b), \mathcal{E})$, where $\mathcal{E}$ is a finite-dimensional space. The notation $C[a, b]$ for the spaces of continuous functions will be used in a similar way.

Everywhere below the symbol $\mathcal{S}$ denotes some (arbitrary) open sector with the vertex at the origin lying in $\mathbb{C} \setminus \Sigma$.

For each fixed $\rho \in \mathcal{S} \setminus \{0\} =: \mathcal{S}'$ we consider the following Volterra integral equations ($k = 1, n$):

$$Y(x) = T^0_k(x, \rho) + \int_0^x G_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t)Y(t)) \, dt, \quad (5)$$

$$Y(x) = F^0_k(x, \rho) - \int_x^\infty G_k(x, t, \rho) (q^k(t)Y(t)) \, dt, \quad (6)$$

where

$$T^0_k(x, \rho) := C_k(x, \rho) \wedge \cdots \wedge C_n(x, \rho),$$

$$F^0_k(x, \rho) := E_1(x, \rho) \wedge \cdots \wedge E_k(x, \rho) = \Psi^0_1(x, \rho) \wedge \cdots \wedge \Psi^0_k(x, \rho)$$

and $G_m(x, t, \rho)$ is an operator acting in $\wedge^m \mathbb{C}^n$ as follows:

$$G_m(x, t, \rho)f = \sum_{\alpha \in \mathcal{A}_m} \sigma_{\alpha} |f \wedge C_{\alpha'}(t, \rho)| C_{\alpha}(x, \rho).$$

Here and below $\sigma_{\alpha} := |h_{\alpha} \wedge h_{\alpha'}|$.

For any $\rho \in \mathcal{S}'$ equations (5) and (6) were shown to have the unique solutions $T_k(x, \rho)$ and $F_k(x, \rho)$ respectively such that (see [9] for details):

$$\|T_k(x, \rho)\| \leq M \left\{ \begin{array}{ll} (\rho x)^{\frac{m}{M}} & |\rho x| \leq 1, \\
exp(|\rho x R_k|) & |\rho x| > 1, \end{array} \right.$$ 

$$\|F_k(x, \rho)\| \leq M \left\{ \begin{array}{ll} (\rho x)^{\frac{m}{M}} & |\rho x| \leq 1, \\
exp(|\rho x R_k|) & |\rho x| > 1. \end{array} \right.$$

We call the functions $F_k(x, \rho)$, $T_k(x, \rho)$ the fundamental tensors. Note that the fundamental tensors solve the auxiliary systems

$$Y' = Q^{(m)}(x, \rho)Y, \quad Q(x, \rho) := x^{-1} A + \rho B + q(x)$$

with $m = k$ and $m = n - k + 1$. 

We note that the tensors \( \{ E_\alpha(x, \rho) \}_{\alpha \in \mathcal{A}_m} \) form the fundamental system of solutions for the system (10) in the “unperturbed” case. Therefore, the following representation holds:

\[
T_k^0(x, \rho) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} T_{k\alpha}^0 E_\alpha(x, \rho)
\]

(11)

with \( x \)-independent coefficients \( T_{k\alpha}^0 \). Taking into account the special construction of the fundamental matrices \( C(x, \rho) \), \( E(x, \rho) \) one can conclude that the coefficients \( T_{k\alpha}^0 \) do not depend on \( \rho \) as well.

The \( G_m(x, t, \rho) \) terms in equations (5), (6) are actually the Green operator functions for the nonhomogeneous systems:

\[
Y' = Q^{(m)}(x, \rho) Y + f(x).
\]

In order to construct them one can use various fundamental systems of solutions of the unperturbed system (3). In particular the following representations hold:

\[
G_m(x, t, \rho) f = \sum_{\alpha \in \mathcal{A}_m} \chi_\alpha \left| f \wedge \Psi_\alpha^0(t, \rho) \right| \Psi_\alpha^0(x, \rho) = \sum_{\alpha \in \mathcal{A}_m} \chi_\alpha \left| f \wedge E_\alpha(t, \rho) \right| E_\alpha(x, \rho).
\]

(12)

Here and below \( \chi_\alpha := |f_\alpha \wedge f_{\alpha'}| \).

In the paper, we study the asymptotical behavior of the fundamental tensors for \( \rho \to \infty \). In [9] the following expansions were obtained:

\[
T_k(x, \rho) = T_k^0(x, \rho) + O \left( \rho^{-\varepsilon} \exp \left( \rho x \overline{R}_k \right) \right), \quad \varepsilon \in (0, 1),
\]

\[
F_k(x, \rho) = F_k^0(x, \rho) + O \left( \rho^{-1} \exp \left( \rho x \overline{R}_k \right) \right)
\]

for any fixed \( x \in (0, \infty) \) and \( \rho \to \infty \), \( \rho \in \mathcal{P}' \). We show that under the additional condition \( q(0) = 0 \) more detailed expansion can be obtained.

Let \( W_0(\xi) \) be the function defined as follows:

\[
W_0(\xi) = (1 - |\xi|)\xi + |\xi|^2, \quad |\xi| \leq 1, \quad W_0(\xi) := (W_0(\xi^{-1}))^{-1}, \quad |\xi| > 1.
\]

Notice that \( W_0(\xi) \) is continuous in \( \xi \in \mathbb{C} \), never vanishes for nonzero \( \xi \) and admits the estimate:

\[
M_1|\xi| \leq |W_0(\xi)| \leq M_2|\xi|
\]

for all \( \xi \in \mathbb{C} \). Moreover, we have \( W_0(\xi) = 1 \) if \( |\xi| = 1 \) and the asymptotics \( W_0(\xi) = \xi(1 + o(1)) \) hold as \( \xi \to 0 \) and \( \xi \to \infty \).

We introduce the following weight functions:

\[
W_k(\xi) := \begin{cases}
W_0(\xi^k) \exp(R_k \xi), & |\xi| \leq 1, \\
\exp(R_k \xi), & |\xi| > 1.
\end{cases}
\]

From the definition and the above-mentioned properties of \( W_0(\cdot) \) it follows that the weight functions \( W_k(\cdot), k = 1, n \) are all continuous in \( \mathcal{P}' \), never vanish and admit the asymptotics \( W_k(\xi) = \xi^k(1 + o(1)) \) as \( \xi \to 0 \). We define

\[
\tilde{F}_k(x, \rho) := (\tilde{W}_k(\rho x))^{-1} F_k(x, \rho), \quad \tilde{T}_k(x, \rho) := (\tilde{W}_k(\rho x))^{-1} T_k(x, \rho).
\]
Theorem 1. Suppose that $q(\cdot)$ is an absolutely continuous off-diagonal matrix function such that $q(0) = 0$. Denote by $\tilde{q}_o(\cdot)$ the off-diagonal matrix function such that $[B, \tilde{q}_o(x)] = -\tilde{q}(x)$ for all $x > 0$ (here $[\cdot, \cdot]$ denotes the matrix commutator). Define the diagonal matrix $d(x) = \text{diag}(d_1(x), \ldots, d_n(x))$, where

$$d_k(x) := \int_{x}^{\infty} t^{-1} ([\tilde{q}_o(t), A])_{kk} \, dt$$

and set $\tilde{q}(x) := \tilde{q}_o(x) + d(x)$.

Suppose that all the functions $q_{ij}(\cdot), q_{ij}^o(\cdot)$ and $\tilde{q}_{ij}(\cdot)$, where $\tilde{q}(x) := \tilde{q}'(x) + x^{-1}[\tilde{q}(x), A]$ are from $X_p := L_1(0, \infty) \cap L_p(0, \infty)$, $p > 2$.

Then for each fixed $x > 0$ and $\rho \to \infty$, $\rho \in \mathcal{S}'$ the following asymptotics hold:

$$\rho(\tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)) = d_{0k} \tilde{T}^0_k(x, \rho) + \sum_{\alpha, \beta \in \mathcal{A}_{n-k-1}} T_{k\beta}^0 g_{\alpha\beta}(x) \exp(\rho x (R_{\beta} - \tilde{R}_k)) f_\alpha + o(1),$$

$$\rho \left( \tilde{F}_k(x, \rho) - \tilde{F}_k^0(x, \rho) \right) = \sum_{\alpha \in \mathcal{A}_k} f_{\alpha}(x) f_\alpha + o(1).$$

Here

$$d_{0k} = -\sigma_\alpha(k) \left| (d(n-k+1)(0) h_{\alpha^*}(k)) \wedge h_{(\alpha^*)'}(k) \right|,$$

$\alpha^*(k) := (k, \ldots, n)$ and the coefficients in the representations are defined as follows:

$$f_{\alpha}(x) = \chi_\alpha \left| (\tilde{q}_k(x) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right|$$

for $\alpha \neq \alpha^*(k) := (1, \ldots, k)$,

$$f_{\alpha, \alpha^*}(x) = -\sum_{\alpha \in \mathcal{A}_k} \int_0^{\infty} \chi_\alpha \left| (q_k(t) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right| \chi_\alpha \left| (\tilde{q}_k(t) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right| \, dt;$$

$$g_{\alpha\beta}(x) = \chi_\alpha \left| (q(n-k+1)(x) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right|$$

for $\beta \neq \alpha$,

$$g_{\beta\beta}(x) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} \int_0^{x} \chi_\beta \left| (q(n-k+1)(t) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right| \chi_\alpha \left| (q(n-k+1)(t) f_{\alpha^*}(k)) \wedge f_{\alpha^*} \right| \, dt.$$

2. PROOF OF THEOREM 1

We consider in details the function $T_k(x, \rho)$, for the function $F_k(x, \rho)$ similar arguments are valid.

For the function $\tilde{T}_k(x, \rho) := \tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)$ we have the representation

$$\tilde{T}_k(\cdot, \rho) = (I - \mathcal{K}(\rho))^{-1} v_k(\cdot, \rho),$$

where $\mathcal{K}(\rho)$ is an operator of the form:

$$(\mathcal{K}(\rho) f)(x) := \int_0^{x} \mathcal{G}_{n-k+1}(x, t, \rho) (q(n-k+1)(t) f(t)) \, dt$$

acting in $L_\infty(0, T), \; T \in (0, \infty)$ is arbitrary. Here and below

$$\mathcal{G}_{n-k+1}(x, t, \rho) := \frac{\tilde{W}_k(\rho t)}{W_k(\rho x)} G_{n-k+1}(x, t, \rho),$$
$$v_k(x, \rho) = \int_0^x G_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt.$$ 

Let us consider first the function $v_k(x, \rho)$. From the identity:

$$\rho \left( q^{(n-k+1)}(t)T_k^0(t, \rho) \right) \wedge E_{\alpha'}(t, \rho) = \frac{d}{dt} \left( (q^{(n-k+1)}(t)T_k^0(t, \rho)) \wedge E_{\alpha'}(t, \rho) \right) - (q^{(n-k+1)}(t)T_k^0(t, \rho)) \wedge E_{\alpha'}(t, \rho),$$

where $\alpha \in \mathcal{A}_{n-k+1}$ is arbitrary it follows the relation:

$$\rho \int_0^x G_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt =$$

$$= G_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)T_k^0(t, \rho) \right) \bigg|_{t=x}^{t=x_0} - \int_0^x G_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt.$$

Passing to the limits as $x_0 \to 0$ and taking into account that $\tilde{q}_0(0) = 0$ we arrive at the relation:

$$\rho v_k(x, \rho) = \frac{d}{dt} \left( \tilde{q}^{(n-k+1)}(t)T_k^0(t, \rho) \right) \bigg|_{t=0}^{t=x} - \int_0^x \frac{d}{dt} \left( \tilde{q}^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt.$$

Since $\tilde{q}_{ij} = 0$, $j = \overline{1, n}$, from (13) and [14] we obtain (in particular) the estimate:

$$\|v_k(\cdot, \rho)\|_{BC[0, \infty)} = O(\rho^{-1}), \quad \rho \in \mathcal{I}'.$$  \hspace{1cm} (14)

In what follows if $V = V(x, \rho)$ is some matrix function then $\tilde{V}$ denotes the matrix function $\tilde{V}(x, \rho) := V(x, \rho)(W(\rho x))^{-1}$, where $W = \text{diag}(W_1, \ldots, W_n)$. Since $\tilde{V}(x, \rho)$ is continuous and bounded in $[0, \infty) \times \mathcal{I}'$ we have:

$$\|G_{n-k+1}(x, t, \rho)\| \leq M, \quad 0 < t < \infty, \quad \rho \in \mathcal{I}'$$  \hspace{1cm} (15)

with some absolute constant $M$.

Using the boundedness of $G_{n-k+1}(x, t, \rho)$ one can obtain the estimate (see also the proof of [9, Theorem 3.1]):

$$\|X(\rho)\| \leq M_0 \frac{M_1}{r!} \left( \int_0^T \|q(t)\| dt \right)^r,$$

where the norm $\|X(\rho)\|$ assumes the norm of the operator acting in $L_\infty(0, T)$ for arbitrary $T > 0$ and the constants $M_0, M_1$ do not depend on $T$. This yields the estimate $\|\left(I = X(\rho)\right)^{-1}\| = O(1)$ uniformly in $\rho \in \mathcal{I}'$. Thus (with taking into account (14)), we obtain the auxiliary prior estimate for $T_k$:

$$\|T_k(\cdot, \rho)\|_{L_\infty(0, T)} = O(\rho^{-1}), \quad \rho \in \mathcal{I}'$$  \hspace{1cm} (16)
for any $T > 0$.

In order to make a more detailed study we represent the operator $\mathcal{K}(\rho)$ in the form $\mathcal{K}(\rho) = \mathcal{K}_0(\rho) + \mathcal{K}_1(\rho)$, where:

$$\mathcal{K}_0(\rho)f(x) := \theta^+(|\rho x| - 1) \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \int_0^x \exp(\rho(x - t)(R_\alpha - \mathcal{R}_k)) \left| (q^{(n-k+1)}(t)f(t)) \wedge f_{\alpha'} \right| f_\alpha \, dt.$$  

Here and below the symbols $\theta^\pm(\cdot)$ denote the Heaviside step functions:

$$\theta^+(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & \xi > 0, \end{cases} \quad \theta^-(\xi) = \begin{cases} 1, & \xi \leq 0, \\ 0, & \xi > 0 = 1 - \theta^+(\xi). \end{cases}$$

**Lemma 1.** Under the conditions of Theorem 1 one has the estimate $||\mathcal{K}_1(\rho)|| = O(\rho^{-1})$.

**Proof.** We split the operator as follows: $\mathcal{K}_1 = \mathcal{K}_0^{(1)} + \mathcal{K}_1^{(1)} + \mathcal{K}_2^{(1)}$, where:

- $(\mathcal{K}_0^{(1)} f)(x) = \theta^-(|\rho x| - 1) \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)f(t) \right) \, dt$,
- $(\mathcal{K}_1^{(1)} f)(x) = \theta^+(|\rho x| - 1) \int_0^{|\rho|^{-1}} \mathcal{G}_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)f(t) \right) \, dt$.

By virtue of (15) we have:

$$||\mathcal{K}_1^{(1)}|| \leq M ||f|| \cdot \int_0^{|\rho|^{-1}} ||q(t)|| \, dt \leq M|\rho|^{-1}||f|| \cdot ||q(\cdot)||_{L_\infty(0, T)}.$$  

Proceeding in a similar way and taking into account that $(\mathcal{K}_0^{(1)} f)(x) \neq 0$ only if $|\rho x| \leq 1$ one can obtain the similar estimate for $||\mathcal{K}_0^{(1)}||$.

Let us consider $\mathcal{K}_2^{(1)}$. Using the representation (9) for $\mathcal{G}_{n-k+1}(x, t, \rho)$, the asymptotics

$$E_\alpha(x, \rho) = \exp(\rho x R_\alpha)(f_\alpha + O((\rho x)^{-1})),$$

which is uniform in $|\rho x| \geq 1$ and taking into account that $\text{Re}(\rho(x - t)(R_\alpha - \mathcal{R}_k)) \leq 0$ for any $0 \leq t \leq x$, $\rho \in \mathcal{A}', \alpha \in \mathcal{A}_{n-k+1}$ we obtain the estimate:

$$\theta^+(|\rho x| - 1)\theta^+(|\rho t| - 1) \theta^+(x - t) \left| \mathcal{G}_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)f(t) \right) \right| - \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \exp(\rho(x - t)(R_\alpha - \mathcal{R}_k)) \left| \left( q^{(n-k+1)}(t)f(t) \right) \wedge f_{\alpha'} \right| f_\alpha \left| f_\alpha \right| \leq \frac{M}{|\rho t|} ||q(t)||$$

with some absolute constant $M$. Since under the conditions of Theorem 1 $t^{-1}q(t) \in L_1(0, \infty)$ the estimate above yields

$$||\mathcal{K}_2^{(1)}|| \leq M|\rho|^{-1}||f|| \cdot \int_0^{\infty} t^{-1}||q(t)|| \, dt$$

and therefore $||\mathcal{K}_2^{(1)}|| = O(\rho^{-1})$.  

□
\textbf{Lemma 2.} Under the conditions of Theorem 1 one has the estimate $\|\mathcal{K}^2_0(\rho)\| = O(\rho^{-1})$.

\textbf{Proof.} We have:

\[
(\mathcal{K}^2_0 f)(x) = \theta^+(|\rho x| - 1) \sum_{\alpha \in \mathcal{A}_{n-k+1}} \int_{|\rho|^{-1}}^x \exp(\rho(x - t)(R_\alpha - \bar{R}_k)) \chi_\alpha \times
\]

\[
\times \left| (q^{(n-k+1)}(t)(\mathcal{K} f)(t)) \wedge \hat{f}_\alpha^t \right| f_\alpha dt,
\]

\[
\chi_\alpha \left| (q^{(n-k+1)}(t)(\mathcal{K} f)(t)) \wedge \hat{f}_\alpha^t \right| = \theta^+(|\rho t| - 1) \times
\]

\[
\times \sum_{\beta \in \mathcal{A}_{n-k+1}} \chi_\beta \int_{|\rho|^{-1}}^x \exp(\rho(t - \tau)(R_\beta - \bar{R}_k)) \left| (q^{(n-k+1)}(\tau) f(\tau)) \wedge \hat{f}_\beta^t \right| Q_{\alpha\beta}(t) d\tau,
\]

where $Q_{\alpha\beta}(t) := \chi_\alpha (q^{(n-k+1)}(t)\hat{f}_\beta) \wedge \hat{f}_\alpha^t$.

Thus, we can rewrite:

\[
(\mathcal{K}^2_0 f)(x) = \theta^+(|\rho x| - 1) \sum_{\alpha \beta \in \mathcal{A}_{n-k+1}} \int_{|\rho|^{-1}}^x \left| (q^{(n-k+1)}(\tau) f(\tau)) \wedge \hat{f}_\beta^t \right| H_{\alpha\beta}(x, \tau, \rho) d\tau,
\]

where:

\[
H_{\alpha\beta}(x, \tau, \rho) = \int^x_{\tau} Q_{\alpha\beta}(t) \exp(\rho(x - t)(R_\alpha - \bar{R}_k) + \rho(t - \tau)(R_\beta - \bar{R}_k)) \hat{f}_\alpha^t dt.
\]

We notice again that $\Re(\rho(x - t)(R_\alpha - \bar{R}_k) + \rho(t - \tau)(R_\beta - \bar{R}_k)) \leq 0$ for any $0 \leq \tau \leq x, \rho \in \mathcal{S}'$, $\alpha, \beta \in \mathcal{A}_{n-k+1}$. Moreover, under the conditions of Theorem 1 $Q_{\alpha\beta}(\cdot)$ are absolutely continuous and $Q_{\alpha\beta}(t) \equiv 0$ if $\alpha = \beta$. This yields the estimate

\[
\theta^+(|\rho \tau| - 1) H_{\alpha\beta}(x, \tau, \rho) = O(\rho^{-1}),
\]

which is uniform in $0 \leq \tau \leq x, \rho \in \mathcal{S}'$. The estimate implies the required assertion. \qed

\textbf{Proof of Theorem 1.} We have $\hat{T}_k(\cdot, \rho) = v_k(\cdot, \rho) + \mathcal{K}(\rho)v_k(\cdot, \rho) + \mathcal{K}^2(\rho)\hat{T}_k(\cdot, \rho)$.

We note that

\[
(\mathcal{K}(\rho)\hat{T}_k^0(\cdot, \rho))(x) = \int^x_0 \mathcal{A}_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)\hat{T}_k^0(t, \rho) \right) = v_k(x, \rho) = O(\rho^{-1})
\]

uniformly for $\rho \in \mathcal{S}', x \in (0, T)$.

This, prior estimate (16), (14) and Lemmas 1, 2 yield:

\[
\hat{T}_k(\cdot, \rho) = v_k(\cdot, \rho) + \mathcal{K}(\rho)\omega_k(\cdot, \rho) + O(\rho^{-2}),
\]

where

\[
\rho \omega_k(x, \rho) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \left| \left( q^{(n-k+1)}(x)\hat{T}_k^0(x, \rho) \right) \wedge E_\alpha(x, \rho) \right| E_\alpha(x, \rho) -
\]

\[
- \int^x_0 \mathcal{A}_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)\hat{T}_k^0(t, \rho) \right) dt.
\]
From [14, Theorem 1] and (18) we have:

\[
\rho \omega_k(x, \rho) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_{\alpha} \left| (\hat{q}^{(n-k+1)}(x)\tilde{T}_k(x, \rho)) \right| E_{\alpha}(x, \rho) + o(1),
\]

that yields:

\[
\theta^+(|\rho t| - 1)\rho \omega_k(t, \rho) = \\
\theta^+(|\rho t| - 1) \sum_{\alpha, \beta \in \mathcal{A}_{n-k+1}} T^0_{k\beta} \exp(\rho t(\beta - \tilde{R}_{k})) \hat{Q}_{\alpha\beta}(t) f_\alpha + \rho^{-1} \omega_k(t, \rho) + o(1),
\]

where \( \hat{Q}_{\alpha\beta}(t) = \chi_{\alpha} \left| (\hat{q}^{(n-k+1)}(t)f_{\beta}) \right| f_{\alpha} \beta \), the \( o(\cdot) \) term assumes an estimate in \( L_\infty(0, T) \) norm and \( t \omega_k(t, \rho) \) is uniformly bounded in \( \{|\rho t| \geq 1\} \).

Under the conditions of Theorem 1 we have \( t^{-1}q(t) \in L_1(0, \infty) \). This yields \( \mathcal{K}_0(\rho)\omega_k(\cdot, \rho) = O(1) \) and thus from the representation above we obtain:

\[
(\mathcal{K}_0(\rho)\omega_k(\cdot, \rho))(x) = \rho^{-1}\theta^+(|\rho x| - 1) \sum_{\alpha, \beta, \gamma \in \mathcal{A}_{n-k+1}} \chi_{\gamma} T^0_{k\beta} \int_{|\rho|^{-1}} \exp(\rho(x - t)(\gamma - \tilde{R}_{k})) + \\
+ \rho t(R_\beta - \tilde{R}_{k}) \hat{Q}_{\alpha\beta}(t) \left| (\hat{q}^{(n-k+1)}(t)f_{\alpha}) \right| f_{\gamma} dt + o(\rho^{-1}) = \rho^{-1}\theta^+(|\rho x| - 1) \times
\]

\[
\times \sum_{\beta, \gamma \in \mathcal{A}_{n-k+1}} T^0_{k\beta} \int_{|\rho|^{-1}} \exp(\rho(x - t)(\gamma - \tilde{R}_{k}) + \rho t(R_\beta - \tilde{R}_{k})) \hat{Q}_{\gamma\beta}(t) f_{\gamma} dt + o(\rho^{-1}),
\]

where:

\[
\hat{Q}_{\gamma\beta}(t) := \sum_{\alpha \in \mathcal{A}_{n-k+1}} \hat{Q}_{\gamma\alpha}(t) \hat{Q}_{\alpha\beta}(t), \quad Q_{\gamma\alpha}(t) = \chi_{\gamma} \left| (\hat{q}^{(n-k+1)}(t)f_{\alpha}) \right| f_{\alpha} \gamma
\]

and the \( o(\cdot) \) term assumes an estimate in \( L_\infty(0, T) \). Under the conditions of Theorem 1 the functions \( Q_{\alpha\beta} \) и \( \hat{Q}_{\alpha\beta} \) (for any pair of multi-indices \( \alpha, \beta \)) are absolutely continuous. Therefore, we have for \( \gamma \neq \beta \):

\[
\int_{|\rho|^{-1}} \exp(\rho(x - t)(\gamma - \tilde{R}_{k}) + \rho t(R_\beta - \tilde{R}_{k})) \hat{Q}_{\gamma\beta}(t) dt = O(\rho^{-1}),
\]

that yields:

\[
(\mathcal{K}_0(\rho)\omega_k(q, \cdot, \rho))(x) = \\
= \rho^{-1}\theta^+(|\rho x| - 1) \sum_{\beta \in \mathcal{A}_{n-k+1}} T^0_{k\beta} \exp(\rho x(R_\beta - \tilde{R}_{k})) \int_{|\rho|^{-1}} \hat{Q}_{\beta\beta}(t) dt f_\beta + o(\rho^{-1}).
\]

Substituting the obtained asymptotics to the representation (17) we arrive at:

\[
\hat{T}_k(x, \rho) = v_k(x, \rho) + \rho^{-1}\theta^+(|\rho x| - 1) \times
\]

\[
\times \sum_{\beta \in \mathcal{A}_{n-k+1}} T^0_{k\beta} \exp(\rho x(R_\beta - \tilde{R}_{k})) \int_{|\rho|^{-1}} \hat{Q}_{\beta\beta}(t) dt f_\beta + o(\rho^{-1}). \quad (19)
\]
Here, as above, the $o(\cdot)$ term assumes an estimate in $L_\infty(0,T)$ norm. But all the terms in (19) are actually continuous with respect to $x \in (|\rho^{-1}|,T)$. This means that the expansion can be considered in point-wise sense as $\rho \to \infty$ while $x > 0$ is arbitrary fixed.

Now we notice that

$$\int_{|\rho|^{-1}}^{x} \tilde{Q}_{\beta\beta}(t) \, dt \to \int_{0}^{x} \tilde{Q}_{\beta\beta}(t) \, dt = g_{k\beta\beta}(x)$$

as $\rho \to \infty$. Then we use the representation (13) for $v_k(x,\rho)$ and thus we obtain the required asymptotics. □

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Асимптотики решений некоторых интегральных уравнений, связанных с дифференциальными системами с особенностью

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В работе изучаются некоторые аспекты теории рассеяния для сингулярных систем дифференциальных уравнений

\[ y' - x^{-1} A y - q(x) y = \rho B y, \quad x > 0 \]

со спектральным параметром \( \rho \), где \( A, B, q(x), x \in (0, \infty) - n \times n \) матрицы, причем матрицы \( A, B \) постоянны. Основным предметом исследования являются некоторые волнтерровские интегральные уравнения относительно тензорно-значных функций. Решения этих уравнений играют центральную роль в построении так называемых решений типа Вейля для исходной системы дифференциальных уравнений. Поскольку классические методы при наличии особенности оказываются неприменимыми, изучение рассматриваемых интегральных уравнений становится в этом случае ключевым этапом исследования аналитических и асимптотических свойств решений типа Вейля. В данной работе мы рассматриваем важный частный случай, когда матрица-функция \( q(\cdot) \) является гладкой и \( q(0) = 0 \). В этом случае для решений рассматриваемых интегральных уравнений удается получить асимптотические разложения при \( \rho \to \infty \) с оценкой остаточного члена \( o(\rho^{-1}) \). Полученный результат позволяет получить асимптотики для решений типа Вейля, играющие, в свою очередь, важную роль при исследовании обратной задачи рассеяния.

Ключевые слова: дифференциальные системы, особенности, интегральные уравнения, асимптотические разложения.

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