Cosmological Analysis of Scalar Field Models in $f(R,T)$ Gravity

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Abstract

This paper determines the existence of Noether symmetry in non-minimally coupled $f(R,T)$ gravity admitting minimal coupling with scalar field models. We consider a generalized spacetime which corresponds to different anisotropic and homogeneous universe models. We formulate symmetry generators along with conserved quantities through Noether symmetry technique for direct and indirect curvature-matter coupling. For dust and perfect fluids, we evaluate exact solutions and construct their cosmological analysis through some cosmological parameters. We conclude that decelerated expansion is obtained for quintessence model with dust distribution while perfect fluid with dominating potential energy over kinetic energy leads to current cosmic expansion for both phantom as well as quintessence models.

Keywords: Noether symmetry; Conserved quantity; $f(R,T)$ gravity.
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1 Introduction

The generic function in $f(R)$ gravity is a coupling-free function which helps to resolve many cosmological issues. Nojiri and Odintsov [1] proposed the
concept of non-minimal curvature-matter coupling which introduced a fresh insight among researchers. This coupling successfully incorporates clusters of galaxies or dark matter in galaxies, yielding natural preheating conditions corresponding to inflationary models as well as introduces the idea of traversable wormholes in the absence of any exotic matter. Harko et al. proposed a new version of modified theory whose generic function incorporates curvature as well as matter known as \( f(R, T) \) gravity (\( T \) is the trace of energy-momentum tensor). This function induces strong interactions of gravity and matter which play a dynamical role to analyze current cosmic expansion. Sharif and Zubair investigated some cosmic issues like energy conditions, thermodynamics, anisotropic exact solutions, reconstruction of some dark energy models and also studied stability issue in this gravity.

The curiosity for exact solutions of higher order non-linear differential equations keep motivating the researchers as these are extensively used to investigate different cosmic aspects. Harko and Lake discussed exact solutions of cylindrical spacetime in the presence of non-minimal coupling between \( R \) and matter Lagrangian density (\( \mathcal{L}_m \)). The higher order non-linear differential equations of \( f(R, T) \) gravity attract many researchers to construct cosmological analysis via exact solutions of the field equations. Sharif and Zubair considered exponential and power-law expansions to evaluate some exact solutions and kinematical quantities of Bianchi type I (BI) model in this gravity. Shamir and Raza formulated exact solutions corresponding to cosmic string as well as non-null electromagnetic field. Shamir found exact solutions of locally rotationally symmetric BI model and studied their physical behavior through cosmological parameters.

In mathematical physics and theoretical cosmology, continuous symmetry reduces the complexity of non-linear system which successfully yields exact solutions. In a dynamical system, Noether symmetry proposes a correspondence between infinitesimal symmetry generator and conserved quantity. Capozziello et al. used this approach to find exact solutions of spherically symmetric spacetime in \( f(R) \) gravity. Hussain et al. investigated the existence of Noether symmetry of power-law \( f(R) \) model and found boundary term to be vanished for flat FRW universe model but Shamir et al. obtained non-zero boundary term of the same model. Momeni et al. explored Noether point symmetry of isotropic universe in mimetic \( f(R) \) and \( f(R, T) \) gravity theories. Shamir and Ahmad constructed exact solutions in \( f(\mathcal{G}, T) \) gravity (\( \mathcal{G} \) denotes Gauss-Bonnet term).

Sanyal determined exact solutions of Kantowski-Sachs (KS) universe
model through Noether symmetry technique in non-minimally coupled gravity with scalar field. Camci and Kucukakca [16] extended this work by adding BI as well as BIII universe models and formulated explicit forms of scalar field. Kucukakca et al. [17] discussed the presence of Noether symmetry to formulate exact solutions of locally rotationally symmetric BI universe. Camci et al. [18] generalized this work for anisotropic universe models such as BI, BIII and KS. We have obtained exact solutions of $f(R)$ power-law model [19] as well as $f(R,T)$ model admitting indirect non-minimal curvature-matter coupling [20].

In non-minimally coupled gravitational theory, the Noether symmetry approach is extensively used to study different cosmological models and the dynamical role of various scalar field models [21]. Vakili [22] identified the existence of Noether point symmetry along with conserved quantity for flat FRW universe and studied the behavior of effective equation of state (EoS) parameter for quintessence model in $f(R)$ gravity. Zhang et al. [23] explored multiple scalar field scenario and formulated a relationship of potential function with quintessence and phantom models. Jamil et al. [24] ensured the presence of Noether symmetry with conservation law for $f(R)$ tachyon model. Sharif and Shafique [25] obtained exact solutions of isotropic and anisotropic universe models in scalar-tensor theory non-minimally coupled with torsion scalar.

In this paper, we discuss the existence of Noether symmetries of non-minimally coupled $f(R,T)$ gravity interacting with generalized scalar field model. The format of the paper is as follows. Section 2 introduces some basic aspects of this gravity. In section 3, we discuss all possible Noether symmetries with associated conserved quantities for two particular models of this theory. We also formulate exact solutions for dust as well as perfect fluid distribution and study their physical behavior through some cosmological parameters. In the last section, we present final remarks.

2 Some Basics of $f(R,T)$ Gravity

We consider the action incorporating gravity, matter and scalar field as

$$\mathcal{I} = \int d^4x \sqrt{-g} [\mathcal{L}_g + \mathcal{L}_m + \mathcal{L}_\phi],$$

(1)

where $g$ denotes determinant of the metric tensor, $\mathcal{L}_g$ and $\mathcal{L}_\phi$ represent gravity and scalar field Lagrangian densities. For non-minimal coupling, the
gravitational Lagrangian is considered to be a generic function $f(R, T)$ admitting minimal coupling only with $L_m$ and $L_\phi$ \[3\]. In this case, the metric variation of $L_g$ and $L_m$ yields

$$f_R(R, T) R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu) f_R(R, T) + f_T(R, T) T_{\mu\nu}$$

$$+ f_T(R, T) (g_{\mu\nu} L_m - 2T_{\mu\nu} - 2g_{\mu\nu} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\mu\nu}}) = \kappa^2 T_{\mu\nu},$$

where subscripts $R$ and $T$ describe corresponding partial derivatives of $f$, $\nabla_\mu$ indicates covariant derivative and $T_{\mu\nu}$ represents energy-momentum tensor.

The divergence of energy-momentum tensor leads to

$$\nabla^\mu T_{\mu\nu} = \frac{f_T}{\kappa^2 - f_T} \left[ (T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu \ln f_T + \nabla^\mu \Theta_{\mu\nu} - \frac{g_{\mu\nu} \nabla^\mu \nabla^\nu T}{2} \right].$$

In non-minimally coupled modified gravity, the energy-momentum tensor remains no more conserved. This non-zero divergence introduces an extra force in equation of motion which is responsible for deviation of massive test particles from geodesic trajectories.

A generalization of some anisotropic and homogeneous universe models is given as \[26\]

$$ds^2 = -dt^2 + a(t)^2 (d\theta^2 + \sin^2 \theta d\phi^2) + b(t)^2 (d\theta^2 + \sin \theta d\phi^2),$$

where $a$ and $b$ are scale factors and $\zeta(\theta) = \theta$, $\sin h\theta$, $\sin \theta$ identify BI, BIII and KS models with the following relationship

$$\frac{1}{\zeta} \frac{d^2 \zeta}{d\theta^2} = -\xi.$$

For $\xi = 0, -1, 1$, the spacetime \[2\] corresponds to BI, BIII and KS universe models, respectively. For perfect fluid, the energy-momentum tensor is

$$T_{\mu\nu} = (\rho_m + p_m) u_\mu u_\nu + p_m g_{\mu\nu},$$

where $p_m$ and $\rho_m$ define pressure and energy density, respectively whereas $u$ represents four-velocity of the fluid. For the action \[1\], the Lagrangian density of matter and scalar fields are defined as \[27\]

$$L_m = p_m(a, b), \quad L_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),$$

(3)
where $V(\phi)$ denotes potential energy of the scalar field and $\epsilon = 1, -1$ indicate scalar field models, i.e., quintessence and phantom models.

Phantom model suffers with number of troubles like violation of dominant energy condition, the entropy of phantom-dominated universe is negative and consequently, black holes disappear. Such a universe ends up with a finite time future singularity dubbed as big-rip singularity [28]. Different ideas are proposed to cure the troubles of this singularity such as considering phantom acceleration as transient phenomenon with different scalar potentials or to modify the gravity, couple dark energy with dark matter or to use particular forms of EoS for dark energy taking into account some quantum effects (giving rise to the second quantum gravity era) which may delay/stop the singularity occurrence [29]. Inserting Eq. (3) into (1), we obtain

$$I = \int d^4 x \sqrt{-g} \left[ \frac{f(R, T)}{2\kappa^2} + p_m(a, b) + \frac{\epsilon}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (4)$$

where

$$R = \frac{2}{ab^2} \left( \ddot{a}b^2 + 2ab\ddot{b} + 2\dot{a}\dot{b} + 2\xi \right), \quad T = 3p_m(a, b) - \rho_m(a, b).$$

To evaluate Lagrangian corresponding to the action (4) for configuration space $Q = \{a, b, R, T, \phi\}$, we use Lagrange multiplier approach which yields

$$\mathcal{L} = ab^2 \left[ f(R, T) - Rf_R(R, T) + f_T(R, T)(3p_m(a, b) - \rho_m(a, b) - T) - \frac{\epsilon}{2} \dot{\phi}^2 \right] + p_m(a, b) - V(\phi) \right) - (4\dot{a}\dot{b} + 2ab^2 - 2\xi) \right) f_R(R, T) - (2b^2 \ddot{a}R + 4ab\dot{b}\dot{R}) \times f_{RR}(R, T) - (2b^2 \ddot{a}T + 4ab\dot{b}\dot{T}) f_{RT}(R, T). \quad (5)$$

In a dynamical system, the Euler-Lagrange equation, Hamiltonian ($\mathcal{H}$) and conjugate momenta ($p_i$) play a significant role to determine basic features of the system, defined as

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{dp_i}{dt} = 0, \quad \mathcal{H} = \sum_i \dot{q}^i p_i - \mathcal{L}, \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i},$$

where $q^i$ shows $n$ coordinates of the system. For Lagrangian (5), the conjugate momenta take the following form

$$p_a = -4bbf_R - 2b^2(\dot{R}f_{RR} + \dot{T}f_{RT}), \quad p_\phi = -ab^2 \epsilon \dot{\phi},$$

5
\[ p_b = -4f_R(ab + b\dot{a}) - 4ab(\dot{R}f_{RR} + \ddot{T}f_{RT}), \]
\[ p_R = -(4ab + 2b^2\dot{a})f_{RR}, \quad p_T = -(4ab + 2b^2\dot{a})f_{RT}. \]

The dynamical equations of the system are

\[ 2f_R(R, T) \left( \frac{\dot{b}^2}{b^2} + \frac{2\dot{b}}{b^2} + \frac{2\xi}{b^2} \right) + f - Rf_R + f_T(3p_m(a, b) - \rho_m(a, b) - T) \]
\[ + \quad p_m(a, b) - \frac{\dot{\phi}^2}{2} - V(\phi) + a\{f_T(3p_{m_a} - \rho_{m_a}) + p_{m_{a}}\} + 4b^{-1}b\ddot{R}f_{RR} \]
\[ + \quad 4b^{-1}b\dot{T}f_{RT} + 2\dot{R}f_{RR} + 2\dot{R}^2f_{RRR} + 4\dot{R}\ddot{T}f_{RRT} + 2\dot{T}f_{RT} + 2\dot{T}^2f_{RTT} = 0, \quad (6) \]
\[ 2f_R \left( \frac{\dot{a}}{a} + \frac{\dot{ab}}{ab} + \frac{\dot{b}}{b} \right) + f - Rf_R + f_T(3p_m(a, b) - \rho_m(a, b) - T) + p_m(a, b) \]
\[ - \quad \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{b}{2}\{f_T(3p_{m_b} - \rho_{m_b})\} + p_{m_{b}}\} + 2(a^{-1}\ddot{a}R + \ddot{R})f_{RR} + 2\dot{R}^2 \]
\[ \times \quad f_{RRR} + 2(a^{-1}\ddot{a}\ddot{T} + \dddot{T})f_{RT} + 2(b^{-1}\ddot{b}\ddot{R} + 2\dot{R}\dddot{T} + \dot{T}^2)f_{RRT} + 2b^{-1}b\dddot{T}f_{RTT} = 0, \quad (7) \]
\[ f_{RT}(3p_m(a, b) - \rho_m(a, b) - T) = 0, \quad f_{TT}(3p_m(a, b) - \rho_m(a, b) - T) = 0, \]
\[ \dot{\phi} + 2\epsilon b^{-1}b\ddot{\phi} + \epsilon a^{-1}\ddot{a}\phi - V'(\phi) = 0. \quad (8) \]

In order to evaluate total energy of the dynamical system, we formulate Hamiltonian as

\[ \mathcal{H} = 2f_R \left( \frac{\dot{b}^2}{b^2} + \frac{2\dot{ab}}{ab} \right) + 2 \left( \frac{2\dot{b}}{b} + \frac{\dot{a}}{a} \right) \dot{R}f_{RR} + 2 \left( \frac{2\dot{b}}{b} + \frac{\dot{a}}{a} \right) \dot{T}f_{RT} + f - Rf_R \]
\[ + \quad f_T(3p_m(a, b) - \rho_m(a, b) - T) + p_m(a, b) + \frac{\epsilon \dot{\phi}^2}{2} - V(\phi) + \frac{2\xi f_R}{b^2}. \quad (9) \]

The Hamiltonian constraint \( \mathcal{H} = 0 \) yields total pressure of the dynamical system.

### 3 Noether Symmetry and Conserved Quantities

The Noether symmetry approach helps to solve complicated non-linear system of partial differential equations yielding exact solutions at theoretical
grounds of physics and cosmology. Noether theorem states that if Lagrangian of a dynamical system remains invariant under a continuous group then group generator leads to the associated conserved quantity. The conservation of energy and linear momentum appears for translational invariant Lagrangian in time and position, respectively whereas the angular momentum is conserved for rotationally symmetric Lagrangian \[30\]. In gravitational theories, the presence of conserved quantities also enhances physical interpretation of theory but if it does not appreciate the existence of any conserved quantity, then the theory will be abandoned due to its non-physical features.

To investigate the existence of Noether symmetry and associated conserved quantity in non-minimally coupled gravitational theory, we consider the first order prolongation \(K^{[1]}\) of continuous group defined as

\[
K^{[1]} = K + (\varphi^{j, t} + \varphi^{j, \dot{q}^i} - \partial_{\dot{q}^i} \varphi^{j, \dot{q}^i}) \frac{\partial}{\partial \dot{q}^j},
\]

where cosmic time \(t\) is considered to be an affine parameter and \(K\) represents symmetry generator given by

\[
K = \vartheta(t, q^i) \frac{\partial}{\partial t} + \varphi^j(t, q^i) \frac{\partial}{\partial q^j}.
\]

Here \(\vartheta\) and \(\varphi^j\) are unknown coefficients of the generator. The existence of Noether symmetry is ensured when \(K\) follows the invariance condition as

\[
K^{[1]} \mathcal{L} + (D\vartheta)\mathcal{L} = DB(t, q^i), \quad D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i},
\]

where \(D\) is the total derivative while \(B\) represents boundary term of \(K\). When the symmetry generator becomes independent of affine parameter then boundary term along with first order prolongation vanishes yielding

\[
K = g^i(q^i) \frac{\partial}{\partial q^i} + \left[ \frac{d}{dt}(g^i(q^i)) \right] \frac{\partial}{\partial \dot{q}^i}, \quad L_K \mathcal{L} = 0,
\]

where \(L\) identifies Lie derivative. The symmetries coming from symmetry generators \(10\) and \(13\) lead to corresponding conservation law through the first integral defined as

\[
\Sigma = B - \vartheta \mathcal{L} - (\varphi^j - \dot{q}^i \vartheta) \frac{\partial \mathcal{L}}{\partial \dot{q}^i}, \quad \Sigma = -\eta^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j}.
\]
For $Q = \{t, a, b, R, T, \phi\}$, the infinitesimal symmetry generator and corresponding first order prolongation take the form

$$K = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} + \gamma \frac{\partial}{\partial R} + \delta \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \phi}, \quad K[1] = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} + \gamma \frac{\partial}{\partial R} + \delta \frac{\partial}{\partial T} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{b}} + \dot{\gamma} \frac{\partial}{\partial \dot{R}} + \dot{\delta} \frac{\partial}{\partial \dot{T}} + \dot{\eta} \frac{\partial}{\partial \dot{\phi}},$$

where time derivative of unknown coefficients $\tau, \alpha, \beta, \gamma, \delta$ and $\eta$ are

$$\dot{\sigma}_l = D\sigma_l - q^l D\tau, \quad l = 1...5,$$

Here $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\sigma_5$ correspond to $\alpha, \beta, \gamma, \delta$ and $\eta$, respectively.

In order to discuss the presence of Noether symmetry generator and relative conserved quantity of the model [2], we insert the first order prolongation [11] along with [11] in [12], it follows a system of equations given in Appendix A. From Eq. (A7), we have either $f_R, f_{RR}, f_{RT} = 0$ with $\tau_a, \tau_b, \tau_R, \tau_T \neq 0$ or vice versa. For non-trivial solution, we consider second possibility $(\tau_a, \tau_b, \tau_R, \tau_T = 0)$ as the first choice yields trivial solution. We investigate the existence of symmetry generators, relative conserved quantities for the following two models [3]

- $f(R, T) = R + 2g(T),$
- $f(R, T) = F(R) + h(R)g(T).$

We also formulate corresponding exact solutions to analyze cosmological picture of these two models.

### 3.1 $f(R, T) = R + 2g(T)$

This model incorporates an indirect non-minimal curvature-matter coupling and also admits a correspondence with standard cosmological constant cold dark matter (ΛCDM) model if it comprises a trace dependent cosmological constant defined as

$$f(R, T) = R + 2\Lambda(T) + g(T).$$

To evaluate the coefficients of symmetry generator [11], we solve the system [A1]–[A22] via separation of variables method which gives

$$\alpha = \alpha_1(t)\alpha_2(a)\alpha_3(b)\alpha_4(R)\alpha_5(T)\alpha_6(\phi), \quad \delta = \delta_1(t)\delta_2(a)\delta_3(b)\delta_4(R)\delta_5(T)\delta_6(\phi),$$
\[ \gamma = \gamma_1(t)\gamma_2(a)\gamma_3(b)\gamma_4(R)\gamma_5(T)\gamma_6(\phi), \quad \eta = \eta_1(t)\eta_2(a)\eta_3(b)\eta_4(R)\eta_5(T)\eta_6(\phi), \]
\[ \beta = \beta_1(t)\beta_2(a)\beta_3(b)\beta_4(R)\beta_5(T)\beta_6(\phi), \quad \tau = \tau_1(t), \]
\[ B = B_1(t)B_2(a)B_3(b)B_4(R)B_5(T)B_6(\phi). \]

For these coefficients, the system (A1)-(A22) yields

\[ \alpha = -2ac_1, \quad \beta = c_1b, \quad \gamma = 0, \quad \delta = 0, \quad \eta = c_4, \]
\[ B = c_4t + c_3, \quad \tau = c_5, \quad V(\phi) = c_6\phi + c_7, \]
\[ p_m(a, b) = -\frac{c_4c_6}{2c_1} \ln a + 2c_1a^\frac{1}{2}b - \frac{2\xi}{b^2} - \frac{c_2 \ln a}{2c_1ab^2}, \]
\[ \rho_m(a, b) = -\frac{3c_4c_6}{2c_1} \ln a + 2c_1a^\frac{1}{2}b - 6\xi - \frac{3c_2 \ln a}{2c_1ab^2}, \]

where \( c_i \)'s \( (i = 1...7) \) denotes arbitrary constants. For these coefficients, we split the symmetry generator and corresponding first integral into the following form

\[ K_1 = \frac{\partial}{\partial t}, \quad \Sigma_1 = -ab^2 \{ f - Rf_R + f_T(3p_m - \rho_m - T) + p_m - c_6\phi - c_7 \} + 2a\xi f_R - 4ab\dot{f}_R - 2ab^2 f_R - \frac{\epsilon\dot{\phi}^2ab^2}{2}, \]
\[ K_2 = -2a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}, \quad \Sigma_2 = -4abf_R + 4b^2\dot{a}f_R, \]
\[ K_3 = \frac{\partial}{\partial \phi}, \quad \Sigma_3 = \epsilon ab^2\dot{\phi}. \]

For the model (17), the system (A1)-(A22) yields three symmetry generators and associated conserved quantities. In this case, the symmetry generator \( K_1 \) leads to energy conservation while \( K_2 \) represents scaling symmetry corresponding to conservation of linear momentum.

Next, we explore the presence of Noether symmetry in the absence of affine parameter and boundary term of extended symmetry which leads to establish corresponding conservation law. In this case, the infinitesimal generator of continuous group for \( Q = \{ a, b, R, T, \phi \} \) turns out to be

\[ K = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} + \gamma \frac{\partial}{\partial R} + \delta \frac{\partial}{\partial T} + \eta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{b}} + \dot{\gamma} \frac{\partial}{\partial \dot{R}} + \dot{\delta} \frac{\partial}{\partial \dot{T}} + \dot{\eta} \frac{\partial}{\partial \dot{\phi}}, \]

(20)
where \( \dot{\alpha} = \dot{q} \frac{\partial \alpha}{\partial q} \), \( \dot{\beta} = \dot{q} \frac{\partial \beta}{\partial q} \), \( \dot{\gamma} = \dot{q} \frac{\partial \gamma}{\partial q} \), \( \dot{\delta} = \dot{q} \frac{\partial \delta}{\partial q} \) and \( \dot{\eta} = \dot{q} \frac{\partial \eta}{\partial q} \). Due to the absence of affine parameter, the separation of variables method yields

\[
\begin{align*}
\alpha &= \alpha_1(a)\alpha_2(b)\alpha_3(R)\alpha_4(T)\alpha_5(\phi), & \beta &= \beta_1(a)\beta_2(b)\beta_3(R)\beta_4(T)\beta_5(\phi), \\
\gamma &= \gamma_1(a)\gamma_2(b)\gamma_3(R)\gamma_5(\phi), & \delta &= \delta_1(a)\delta_2(b)\delta_3(R)\delta_4(T)\delta_5(\phi), \\
\eta &= \eta_1(a)\eta_2(b)\eta_3(R)\eta_4(T)\eta_5(\phi).
\end{align*}
\]

In order to explore the consequences of indirect non-minimal curvature-matter coupling, we evaluate symmetry generators with corresponding conservation laws for non-existing boundary term. We also establish cosmological analysis through exact solutions for both dust and perfect fluid distributions.

**Dust Case**

Dust fluid investigates matter contents of the universe when the existence of radiations is not so worthy and the formation of massive stars is possible only if dust particles interact with radiations. Here we consider \( T_{\mu\nu} = \rho_m u_\mu u_\nu \) and solve the system for (20) via separation of variables which yields

\[
\begin{align*}
\alpha &= -2ac_1', & \beta = c_1'b, & \gamma = 0, & \delta = 0, & \eta = 0, \\
\rho_m(a, b) &= \frac{\xi}{b^2c_2'} + a^\frac{b}{2}b, & \Lambda(T) &= -\frac{g(T)}{2} + c_4'T + c_3'.
\end{align*}
\]

where \( c_j' \)’s (\( j = 1...3 \)) represents arbitrary constants. The corresponding symmetry generator and associated conserved quantity are

\[
K = -2ac_1' \frac{\partial}{\partial a} + c_1'b \frac{\partial}{\partial b}, \quad \Sigma = 4c_1'abbf_R - 4c_1'b^2af_R.
\]

For dust fluid, there exists only scaling symmetry in the absence of affine parameter as well as boundary term of extended symmetry and the model (17) reduces to

\[
f(R, T) = R + 2c_4'T + 2c_3'. \quad (21)
\]

For exact solution of equations of motion, we insert density of dust fluid and model (21) in Eqs.(6) and (7) yielding

\[
a(t) = \frac{(40c_2't + 40c_3')^{\frac{1}{2}}}{16}, \quad b(t) = \frac{c_1'(40c_2't + 40c_3')^{\frac{1}{2}}}{4}.
\]
This leads to expansion of the universe whether it is accelerated or deaccelerated. The power-law scale factor \((a(t) = t^\lambda)\) identifies both expansions as for \(\lambda > 1\), it measures accelerated expansion while it corresponds to decelerated expansion for \(\lambda < 1\). When \(\lambda = \frac{1}{2}\) and \(\lambda = \frac{2}{3}\), we have radiation and matter dominated eras of the universe.

To analyze the behavior of power-law type exact solution, we construct cosmological analysis through some cosmological parameters such as Hubble, deceleration, \(r - s\) and EoS. These parameters are useful to study current expansion as well as different eras of the universe. The Hubble parameter \((H)\) determines the rate of expansion while deceleration parameter \((q)\) evaluates the nature of cosmic expansion whether decelerated \((q > 0)\), accelerated \((q < 0)\) or constant \((q = 0)\), respectively. In case of anisotropic universe models, these parameters turn out to be

\[
H = \frac{64c_2'(40c_2't + 40c_3')^{-1}}{3}, \quad q = \frac{7}{8}.
\]

The remarkable pair of \(r - s\) parameters explores the characteristics of dark energy candidates by establishing a correspondence between constructed and standard cosmic models. When the pair lies in \((r, s) = (1, 0)\) region, this corresponds to standard ΛCDM model while the trajectories with \(s > 0\) and \(r < 1\) correspond to quintessence and phantom phases of dark energy. In the present case, we obtain \(r = 0\) with \(s = -\frac{8}{9}\) indicating that the constructed model does not correspond to any standard dark energy universe model. The EoS parameter \((\omega)\) investigates different cosmic eras such as it identifies radiation and matter dominated eras for \(\omega = \frac{1}{3}\) and \(\omega = 0\), respectively. This parameter specifies dark energy era \((\omega = -1)\) into quintessence and phantom phases when \(-1 < \omega \leq -1/3\) and \(\omega < -1\), respectively. The corresponding effective EoS parameter is

\[
\omega_{\text{eff}} = \frac{128c_2' + (40c_2't + 40c_3')\left(5c_2'^2c_1't^2 + 10c_2'c_3'c_4't + 5c_1'^2c_2'^2t^2 + 50(40c_2't + 40c_3')\right)}{128c_2'}. 
\]

The potential and kinetic energies of the scalar field play a dynamical role to study cosmic expansion. For accelerated expansion, the field \(\phi\) evolves negatively and potential dominates over the kinetic energy \((\frac{\dot{\phi}^2}{2} < V(\phi))\) whereas negative potential follows the kinetic energy for decelerated expansion of the universe \((\frac{\dot{\phi}^2}{2} > -V(\phi))\). Using Eq. (5), we obtain

\[
\phi = \int \frac{1}{20\epsilon(c_2't + c_3')} \left( -c_2' \left( 25c_2'^2(40c_2't + 40c_3')c_1't^2 + 50(40c_2't + 40c_3') \right) \right)
\]
Figure 1: Plots of scale factors \( a(t) \) (left) and \( b(t) \) (right) versus cosmic time \( t \) for \( c'_1 = 0.24 \), \( c'_2 = 0.45 \) and \( c'_3 = 5.5 \).

\[
V(\phi) = \frac{1}{800 \left( c'_2 t^2 + 2 c'_2 c'_3 t + c'_3^2 \right)} \left[ 25 c'_1 \left( 5 c'_2^2 t^2 + 5 c'_2 c'_3^2 + 10 c'_2 c'_3 t \right) (40 c'_2 t \\
+ 40 c'_3) \right] + 8 c'_2 c'_3 \left( -200 c'_2 t^2 + 400 c'_3 t \right) - 8 \left( 48 c'_2^2 + 200 c'_3^3 \right). 
\]

Figure 2: Plots of Hubble \( H(t) \) (left) and EoS parameters \( \omega_{\text{eff}} \) (right) versus cosmic time \( t \).

Figure 1 shows the graphical analysis of the scale factors for the dust case. The scale factor \( a(t) \) indicates large cosmic expansion in \( x \)-direction but \( b(t) \) represents that the universe is expanding very slowly in \( y \) and \( z \)-directions. Figure 2 (left plot) indicates that Hubble parameter is decreasing with the passage of time. In the right plot of Figure 2, the effective EoS parameter identifies that initially, the universe appreciates radiation dominated era and after sometime, it corresponds to dark energy era by crossing matter
Figure 3: Plots of scalar field $\phi(t)$ (left) versus cosmic time $t$ and potential energy $V(\phi)$ versus kinetic energy $\frac{\dot{\phi}^2}{2}$ (right) for $\epsilon = -1$.

Figure 4: Plots of scalar field $\phi(t)$ (left) versus cosmic time $t$ and potential energy $V(\phi)$ versus kinetic energy $\frac{\dot{\phi}^2}{2}$ (right) for $\epsilon = 1$.

dominated phase.

Figures 3 and 4 analyze the behavior of scalar field and cosmic expansion via phantom and quintessence models. The left plot of Figure 3 shows that the scalar field is positive initially yielding decelerated expansion but gradually, it starts increasing negatively which describes accelerated expansion. In case of quintessence model, the scalar field grows from negative to positive indicating decelerated expansion of the universe. The right plots of 3 and 4 satisfy $\frac{\dot{\phi}^2}{2} < V(\phi)$ and $\frac{\dot{\phi}^2}{2} > -V(\phi)$ implying that phantom model yields accelerated expansion while quintessence model corresponds to decelerated expansion.

To analyze a big-rip free model, the key point is that if EoS parameter
rapidly approaches to -1 and Hubble rate tends to be constant (asymptotically de Sitter universe), then it is possible to have a model in which the time required for singularity is infinite, i.e., the singularity effectively does not occur [31]. The occurrence of maximum potential of a phantom scalar field is another evident fact to avoid this singularity [32]. The graphical behavior of EoS parameter represents that $\omega_{\text{eff}}$ rapidly approaches to -1 and Hubble rate is decreasing but potential is not maximum. We may avoid the big-rip singularity in the present case if we choose $c_2'$ to be negatively large that yields asymptotic behavior of Hubble rate.

**Non-Dust Case**

At large scales, perfect fluid successfully illustrates cosmic matter distribution in the presence of radiations. In the absence of boundary term and affine parameter, the coefficients of symmetry generator (20) corresponding to $a, b, R, T, \phi$ remain the same as in the presence of boundary term of extended symmetry. Thus, generator of Noether symmetry and associated first integrals reduce to

\[
K = -2ac_1 \frac{\partial}{\partial a} + c_1b \frac{\partial}{\partial b} + c_2 \frac{\partial}{\partial \phi},
\]

\[
\Sigma = -4c_1 ab f_R + 4c_1 b^2 \dot{a} f_R + \epsilon c_2 a b^2 \dot{\phi}.
\]

In order to formulate exact solution of dynamical equations for perfect fluid distribution, we insert Eqs.(18) and (19) into (6) and (7) yielding

\[
a(t) = \left( \frac{5}{c_{10}} \right)^{\frac{1}{5}} \left( c_2 \sin(c_{10} t) + c_3 \cos(c_{10} t) \right)^{\frac{1}{5}},
\]

\[
b(t) = c_4 \left( \frac{5}{c_{10}} \right)^{\frac{1}{5}} \left( c_2 \sin(c_{10} t) + c_3 \cos(c_{10} t) \right)^{\frac{1}{5}}.
\]

This describes oscillatory solution of $f(R, T)$ model admitting indirect non-minimal curvature-matter coupling. To study the cosmological behavior of this solution, we consider cosmological parameters as follows

\[
H = \frac{8c_{10} (c_2 \sin(c_{10} t) + c_3 \cos(c_{10} t))}{15(c_2 \sin(c_{10} t) + c_3 \cos(c_{10} t))^{\frac{2}{5}}}.
\]
Figure 5: Plots of scale factor $a(t)$ (left) and $b(t)$ (right) versus cosmic time $t$ for $c_2 = c_3 = c_9 = 5.5$ and $c_{10} = 0.005$.

Figure 6: Plots of $H(t)$ (left) and $q(t)$ (right) versus cosmic time $t$.

$$q = \frac{-8c_2^2 \cos^2(c_{10}t) + 7c_3^2 + 8c_3^2 \cos^2(c_{10}t) + 15c_2^2 + 16c_2c_3 \cos(c_{10}t) \sin(c_{10}t)}{8(c_2 \sin(c_{10}t) + c_3 \cos(c_{10}t))^2},$$

$$s = \left( -45((4c_2^4 - 4c_3^4) \cos^2(c_{10}t) - c_3^4 - 6c_2^2c_3 - 5c_2^4 - (8c_2^3c_3 + 8c_2c_3^3) \right) \times \cos(c_{10}t) \sin(c_{10}t))/256((c_2^4 + c_3^4 - 6c_2^2c_3^2) \cos^4(c_{10}t) + (6c_2^2c_3^2 - 2c_3^4) \times \cos^2(c_{10}t) + (-4c_2^3c_3 + 4c_2c_3^3) \sin(c_{10}t) \cos^3(c_{10}t) - 4c_2^3c_3^3 \cos(c_{10}t) \times \sin(c_{10}t) + c_3^4),$$

$$\omega_{\text{eff}} = \frac{\chi(3p_m - \rho_m) + p_m - \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{2\xi}{\rho} + a(3p_m; a - \rho_m; a) + \rho_m; a}{\chi(3p_m - \rho_m) + p_m + \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{2\xi}{\rho}}.$$

The scalar field as well as corresponding kinetic and potential energies identify the early as well as current cosmic expansion and also characterize decelerated expansion of the universe when kinetic energy dominates negative
Figure 7: Plot of $\omega_{\text{eff}}$ and $r - s$ parameters versus cosmic time $t$ for $c_2 = c_3 = 5.5$ and $c_{10} = 0.005$.

Figure 8: Plots of scalar field $\phi(t)$ (left) versus cosmic time $t$ and potential energy $V(\phi)$ versus kinetic energy $\dot{\phi}^2$ (right) for $c_2 = 5.5$, $c_4 = -10^3$, $c_6 = 0.5$ and $c_{10} = 0.005$.

potential. In this case, Eq.(8) yields

$$\phi = \int \frac{5c_{10}c_2^2 \cos(2c_{10}t) \left[-22_{1}^{2F1} \left(\frac{3}{4}, \frac{1}{2} \sin \left[\frac{2}{5} + c_{10}t\right]\right)^2 + \sqrt{2} \sin [2c_{10}t] \right]}{16c_{10} \cos \left[\frac{2}{5} + c_{10}t\right] \epsilon (c_2 \cos \left[c_{10}t\right] + c_2 \sin \left[c_{10}t\right])^{8/5}} \, dt,$$

where ${}_{2}F_{1}$ represents hypergeometric function.

In Figure 5, the right plot shows that the universe experiences immense amount of expansion in $y$ and $z$-directions whereas the left plot yields a small amount of expansion in $x$-direction. Figure 6 provides information about increasing rate of expansion through Hubble parameter while negatively increasing deceleration parameter assures accelerated cosmic expansion. The
left plot of Figure 7 characterizes quintessence phase of dark energy era while the right plot identifies the $r - s$ parameters trajectories in quintessence and phantom phases as $s > 0$ when $r < 1$. Both plots of Figure 8 verify the current cosmic expansion for quintessence as well as phantom models as $\phi$ continuously increasing negatively and potential energy of the field is dominating over kinetic energy. The graphical interpretation of EoS parameter yields $\omega_{\text{eff}} < -1$ which is not a sufficient condition for the existence of singularity as potential turns out to be maximum with the passage of time. Thus, we may avoid big-rip singularity if Hubble rate decreases asymptotically in the presence of minimal coupling of $f(R, T)$ gravity with scalar field.

### 3.2 $f(R, T) = F(R) + h(R)g(T)$

To analyze the effect of direct non-minimal curvature-matter coupling, we consider this model and evaluate symmetry generators as well as associated conservation laws. Inserting the model in Eqs. (A2)-(A4), (A10), (A11) and (A15) and using separation of variables approach, we obtain

$$\beta = -\frac{b\alpha}{2a} + \phi Y_1(t, a, b) + Y_2(t, a, b),$$

$$F(R) = \frac{\epsilon}{4d_3} (-d_3 Y_{12}(R) + d_2 Y_9(R)) + d_5 R + d_6,$$

$$h(R) = -\frac{\epsilon}{4d_3} (-d_3 Y_9(R), R + d_1 R) + d_4,$$

$$g(T) = d_2 + d_3 Y_{10}(T),$$

$$\eta = \frac{1}{b} \left[ Y_1(t, a, b)(Y_{10}(T)(d_1 + Y_9(R), R) - \phi^2 + Y_{12}(R), R) + b\phi \tau - 2\phi \right.\left. \times Y_2(t, a, b) + bY_{14}(t, a, b) \right],$$

where $d_i$'s ($i = 1...7$) denote constants. We substitute these values in Eqs. (A1), (A8) and (A9) which yield

$$\tau = \int -\frac{Y_{23}(t)}{\epsilon} dt + d_3 t + d_9, \quad B = \frac{1}{6d_4} \left[ 6ab(1 + Y_{19}(T)d_1 + d_4 \phi^2 + Y_{19}(R)d_4 e^{-R} + Y_{19}(T) \right.\left. \times d_1 Y_{16}(t, b), +3d_4 (2Y_{22}(t, a, b) + 2\phi Y_{21}(t, a, b), +ab^2 \phi^2 Y_{23}(t, b)) \right],$$

$$Y_1(t, a, b) = Y_{16}(t, b) + Y_{15}(a, b) \quad Y_{10}(T) = -\frac{Y_{19}(T)d_3 + \epsilon d_2 d_4}{\epsilon d_3 d_4}.$$
\[ Y_{12}(R) = -\frac{d_2d_4e^{-R}}{d_3} + d_6R + d_7, \quad Y_6(R) = -e^{-R}d_4 - 2d_1R + d_2, \]
\[ Y_{14}(t, a, b) = -\frac{Y_{21}(t, a, b)}{b^2a\epsilon} - \frac{bad_2e_{d_1}Y_{16}(t, b) + d_6ebad_3Y_{16}(t, b)}{eb^2ad_3} + Y_{24}(b, a). \]

To evaluate remaining unknown functions, we insert the above functions into \( \beta, \eta, F, g, h \) and solve Eqs. (A5)-(A7) with (A12)-(A14) and (A16)-(A21) leading to

\[ \begin{align*}
Y_{21}(t, a, b) &= Y_{26}(a, b), \quad Y_{22}(t, a, b) = d_10t, \quad Y_{16}(t, b) = -d_{12}b, \\
Y_{15}(a, b) &= d_{12}b, \quad Y_{24}(b, a) = 0, \quad Y_2(t, a, b) = d_9b, \\
Y_{23}(t) &= \epsilon(-2d_9 + e^{-R}d_4d_3d_{11}e^R + d_8), \quad \delta = 0, \quad \gamma = \frac{d_{11}e^R}{d_3}, \\
&\times (e^{-R}d_4T_d_{13}d_3 - d_1d_{13}d_5T + (2((-2d_5 + \frac{1}{2}(d_6)d_3 + d_1d_8)e))d_4).
\end{align*} \]

Using these solutions in Eq. (A22) with \( d_{11}=0 \) and \( d_6 = \frac{d_{10}d_1}{d_3} \), it follows that

\[ \begin{align*}
\tau &= 3d_9, \quad \alpha = d_{10}a, \quad \beta = b(d_9 - \frac{d_{10}}{2}), \quad \delta = 0, \quad \gamma = 0, \\
B &= d_{10}t, \quad \eta = -\frac{d_1}{\epsilon} + 2d_{12}d_6, \quad F(R) = d_6 + d_5R - \frac{3d_6\epsilon R}{4}, \\
h(R) &= d_4 - \frac{\epsilon}{4d_3}(d_4e^{-R} + d_1R - 2d_1), \quad g(T) = d_2 - \frac{d_2d_4\epsilon - d_3d_{13}T}{d_4\epsilon}.
\end{align*} \]

Inserting \( F, h \) and \( g \), the \( f(R, T) \) model becomes

\[ f(R, T) = -\frac{3d_6\epsilon R}{4} + d_5R + d_6 + (d_4 - \frac{\epsilon}{4d_3}(d_4e^{-R} + d_1R - 2d_1))(\frac{d_3d_{13}T}{d_4\epsilon}). \]

Thus, the constructed model also experiences a direct coupling between curvature and matter parts. In this case, the symmetry generators and associated conserved quantities are

\[ K_1 = 3\frac{\partial}{\partial t} + b\frac{\partial}{\partial b}; \quad \Sigma_1 = \frac{1}{4d_3\epsilon}(-4ab^2e^2d_3\phi^2 + 4d_{10}d_3\epsilon t + 3tab^2d_1RTd_3\epsilon \\
- 9tab^2d_1R_{\rho\rho}d_3\epsilon + 3tab^2d_1R_{\rho\rho}d_3\epsilon - 12td_1T_{\hat{a}b\hat{b}}d_3\epsilon - 4b^2d_4T \\
\times \hat{a}e^{-R}d_3\epsilon - 4b^2aT_{\hat{a}b}d_4e^{-R}d_3\epsilon - 9tab^2d_4\rho d_3\epsilon + 3tab^2d_4\rho d_3 \\\n\times e^{-R}d_3\epsilon + 6td_1T_{\hat{a}b}e^{-R}d_3\epsilon + 6td_1T_{\hat{a}b}e^{-R}d_3\epsilon - 4bd_4T_{\hat{a}b}e^{-R}d_3\epsilon \]

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We see that scaling symmetry appears through generator $K_2$ with the first integral $\Sigma_2$ leading to conserved linear momentum.

Now we investigate the existence of Noether symmetry in the absence of affine parameter and boundary term of the extended symmetry and also study the effect of direct curvature-matter coupling on conservation laws. For this purpose, we solve Eqs. (A5), (A6), (A9) and (A12)-(A21) which give

$$V(\phi) = k_{10} \phi + k_{11}, \quad Y_{10}(a, R, T, b) = k_8 a, \quad Y_4(a, b) = 0,$$

$$Y_{12}(a, R, T, b) = -\frac{k_8}{2k_2} \left(\epsilon(k_6 T + k_7) + 2k_4 k_2 + \epsilon k_1\right), \quad Y_5(a, b) = -\frac{k_6 k_8 \epsilon}{2},$$

where $k_l$'s ($l = 1...5$) are arbitrary constants. Inserting these solutions into the remaining equations of the system, we obtain

$$\delta = -\frac{a}{2Y_9(T)} (\frac{1}{3} Y'_4(a, b), Y_9(T) + 2Y'_4(a, b), Y_5(b) \phi + \phi^3 Y_5(a, b), \phi + 2Y_4(a, b), Y_5(b) \phi + Y_{12}(a, R, T, b), \quad F(R) = k_4 R + k_5,$$

$$\beta = -\frac{b}{2a} (Y_10(a, R, T, b) + a Y_5(a, b)), \quad g(T) = k_1 + Y_9(T) k_2$$

$$\eta = \frac{1}{2} (\phi^2 + 2Y_6(T) + 2Y_8(b)) Y_4(a, b) + Y_5(a, b) \phi + Y_7(b, a),$$

$$h(R) = \frac{\epsilon R}{2(k_2 + k_3)}, \quad \gamma = Y_{11}(a, b, R, T),$$

$$\alpha = -Y_4(a, b) a \phi + Y_{10}(a, b, R, T),$$

We see that scaling symmetry appears through generator $K_2$ with the first integral $\Sigma_2$ leading to conserved linear momentum.
\[ Y_7(b, a) = k_9, \quad Y_9(T) = k_6 T + k_7, \]
\[ p_m = \frac{2k_9 k_{10}}{\epsilon k_8 k_6} - k_5 + k_{11} + \frac{2k_2 k_4 k_3}{\epsilon} + a \frac{k_{12}}{\epsilon} ba \epsilon^2 - \frac{a^2 \epsilon}{4}, \]
\[ \rho_m = \frac{k_7}{k_6} + \frac{6k_2 k_4 k_3}{\epsilon} - 3k_5 + 3k_{11} + \frac{2k_4}{k_6 \epsilon} + \frac{6k_9 k_{10}}{k_8 k_6 \epsilon} + \frac{k_1}{k_2 k_6} \]
\[ + a \frac{k_{12}}{\epsilon} ba \epsilon^2 - \frac{a^2 \epsilon}{4}. \]

The corresponding Noether symmetry generator with associated first integral take the form
\[ K_1 = a \frac{\partial}{\partial a} - \frac{b}{2} \left(1 - \frac{k_6 \epsilon}{2}\right) \frac{\partial}{\partial b} + R \frac{\partial}{\partial R} - (k_1 \epsilon + (k_6 T + k_7) + 2k_4) \]
\[ \times \frac{1}{2k_2} \frac{\partial}{\partial T} - \frac{k_6 \epsilon \phi}{2} \frac{\partial}{\partial \phi}, \quad \Sigma_1 = ab \epsilon k_6 T - \frac{b^2 \epsilon k_1 \dot{a}}{k_2} - b^2 \epsilon k_6 T \dot{a} - ba \epsilon k_6 k_4 \dot{b} \]
\[ - \frac{ba \epsilon^2 k_4^2 T \dot{b}}{2} - \frac{ba \epsilon^2 k_6 k_7 \dot{b}}{2} + ab \epsilon k_1 \frac{k_2}{k_2} + ab \epsilon k_7 - \frac{ba \epsilon^2 k_6 k_1 \dot{b}}{2k_2} - \frac{ab^2 \epsilon \dot{\phi} k_6 \dot{\phi}}{2} \]
\[ + 2abk_4 - 2b^2 \epsilon k_4 \dot{a} - b^2 \epsilon k_7 \dot{a} + \frac{b^2 a \epsilon^2 k_6^2 T}{2}, \]
\[ K_2 = \frac{\partial}{\partial \phi}, \quad \Sigma_2 = ab \epsilon \dot{\phi} k_9. \]

Here the symmetry generator \( K_1 \) yields scaling symmetry.

4 Final Remarks

In this paper, we have analyzed the existence of Noether symmetry in a non-minimally coupled \( f(R, T) \) gravity interacting with scalar field model for anisotropic homogeneous universe models like BI, BIII and KS models. Using Noether symmetry approach, we have found conserved quantities associated with symmetry generators and studied the contribution of direct as well as indirect curvature-matter coupling through two \( f(R, T) \) models. We have also formulated exact solutions for dust and perfect fluid distributions whose cosmological analysis is discussed through cosmological parameters.

For \( f(R, T) \) model admitting indirect curvature-matter coupling, we have found three symmetry generators in the presence of affine parameter and boundary term. The first generator of translational symmetry in time yields energy conservation law whereas the second generates scaling symmetry. For
the second model, we have formulated four conserved quantities associated with symmetry generators but only one generator provides scaling symmetry leading to the conservation of linear momentum. In the absence of boundary term of extended symmetry and affine parameter, the symmetry generator of first model assures the existence of scaling symmetry for dust as well as perfect fluid while we have found two symmetry generators for the second model.

For the first model, we have evaluated exact solutions without considering boundary term. For dust distribution, we have found power-law solution. The graphical analysis of scale factors and cosmological parameters lead to decelerating phase of the universe. The positively increasing scalar field and dominating kinetic energy over potential energy ensure the decelerating behavior of cosmos for quintessence model. In case of phantom model, the scalar field rolls down positively and tends to increase negatively while kinetic energy dominates over potential energy for $t \in [0.8, 1.6]$. The graphical behavior of effective EoS parameter reveals that the universe experiences a phase transition from radiation dominated era to dark energy era by crossing matter dominated phase. For perfect fluid, we have determined an oscillatory solution with increasing rate of Hubble parameter, negative deceleration parameter and $\omega_{eff} < -1$. The trajectories of $r - s$ parameters identify quintessence and phantom phases as $s > 0$ when $r < 1$. For quintessence and phantom models, the scalar field continuously increasing negatively and potential energy of the field is dominating over kinetic energy. This analysis indicates that an epoch of accelerated expansion is achieved for non-dust distribution.

Shamir [9] investigated exact solution of BI model without using Noether symmetry approach in $f(R, T)$ gravity. For indirect curvature-matter coupling, the exact solution is determined using a relationship between expansion and shear scalars. The study of corresponding cosmological parameters yield positive deceleration parameter, $\omega_{eff} = 1$, volume and average scale factor turn out to be zero at $t = 0$. Thus, the analysis of this exact solution yields a decelerating epoch for $R + 2f(T)$ model. For $f_1(R) + f_2(T)$ model, a power-law form of $f_1(R)$ is considered that gives exponential and power-law solutions for different choices of $f_2(T)$. For exponential solution, the average Hubble parameter becomes zero leading to Einstein universe. Camci et al. [18] formulated exact solutions of these anisotropic models via Noether symmetry approach in non-minimally scalar coupled gravity. The scale factors are found to be proportional to inverse of the scalar field whose
explicit form is not determined for any anisotropic model. Consequently, the cosmological analysis of these exact solutions is not established. In the present paper, we have found two exact solutions, power-law and oscillatory solutions via Noether symmetry approach that correspond to decelerating as well as current accelerating universe for dust and non-dust distribution.

We conclude that the constructed $f(R, T)$ models admit direct as well as indirect curvature-matter coupling. The existence of symmetry generators and associated conserved quantities is assured for both $f(R, T)$ models. It is worthwhile to mention here that we have found maximum symmetry generators along with conserved quantities for the second $f(R, T)$ model in the presence of boundary term. This indicates that the model appreciating direct curvature-matter coupling leads to more physical results relative to the first model while the exact solutions analyze cosmic evolution.

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Appendix A

For invariance condition (12), the system of equations is

$$
\epsilon ab^2 \eta_{\tau} = -B_{\gamma}, \quad (A1)
$$

$$
b \alpha + 2a \beta + 2ab \eta_{\alpha} - ab \tau = 0, \quad (A2)
$$

$$
2b \alpha \gamma_{\alpha} f_{\alpha R} + 4a b \gamma_{\alpha} f_{\alpha R} + ab \eta_{\alpha} f_{\alpha R} = 0, \quad (A3)
$$

$$
2b \alpha \gamma_{\alpha} f_{\alpha T} + 2a b \eta_{\alpha} f_{\alpha T} + ab \eta_{\alpha} f_{\alpha T} = 0, \quad (A4)
$$

$$
4 \beta_{\gamma} f_{\gamma R} + 2b \gamma_{\gamma} f_{\gamma R} + 2b \delta_{\gamma} f_{\gamma T} + ab \eta_{\gamma} = 0, \quad (A5)
$$

$$
4 \beta_{\gamma} f_{\gamma R} + 4 \alpha \beta_{\gamma} f_{\gamma R} + 4ab \gamma_{\gamma} f_{\gamma R} + 4ab \delta_{\gamma} f_{\gamma T} + ab^2 \epsilon \eta_{\alpha} = 0, \quad (A6)
$$

$$
\tau_{\alpha} f_{\alpha R} = 0, \quad \sigma_{\beta} f_{\beta T} = 0, \quad \tau_{\gamma} f_{\gamma R} = 0, \quad \tau_{\delta} f_{\delta T} = 0, \quad \tau_{\epsilon} = 0, \quad (A7)
$$

$$
2b^2 \alpha_{\gamma} f_{\gamma R} + 4ab \beta_{\gamma} f_{\gamma R} = -B_{\gamma}, \quad (A8)
$$

$$
2b^2 \alpha_{\gamma} f_{\gamma T} + 4ab \beta_{\gamma} f_{\gamma T} = -B_{\gamma T}, \quad (A9)
$$

$$
b \alpha_{\gamma} f_{\gamma R} + 2ab \beta_{\gamma} f_{\gamma R} = 0, \quad (A10)
$$

$$
b \alpha_{\gamma} f_{\gamma T} + 2ab \beta_{\gamma} f_{\gamma T} = 0, \quad (A11)
$$

$$
2 \beta_{\gamma} f_{\gamma R} + b \gamma_{\gamma} f_{\gamma R} + b \delta_{\gamma} f_{\gamma T} = 0, \quad (A12)
$$
$4b\beta, f_R + 2b^2\gamma, f_{RR} + 2b^2\delta, f_{RT} = -B, \quad (A13)$

$4b\gamma, f_R + 2b^2\gamma, f_{RR} + 4ab\gamma, f_{RT} = -B, \quad (A14)$

$4b\alpha, f_{RT} + 2a\beta, f_{RR} + 2a\beta, f_{RT} = 0, \quad (A15)$

$\alpha f_R + a\gamma f_{RR} + a\delta f_{RT} + 2b\alpha, f_R + 2a\beta, f_R + 2ab\gamma, f_{RR} + 2ab\delta, f_{RT}$

$-a\tau, f_R = 0, \quad (A16)$

$2\beta f_R + b\gamma f_{RR} + b\delta f_{RT} + 2b\alpha, f_R + 2\beta, f_R + 2b\gamma, f_{RR}$

$+b\delta, f_{RT} - b\tau, f_{RR} = 0, \quad (A17)$

$2\beta f_R + b\gamma f_{RR} + b\delta f_{RT} + b\alpha, f_R + 2\beta, f_R + 2b\gamma, f_{RR}$

$+b\tau, f_{RT} - b\tau, f_{RR} = 0, \quad (A18)$

$2\beta f_R + b\gamma f_{RR} + b\delta f_{RT} + 2b\alpha, f_R + 2\beta, f_R + 2b\gamma, f_{RR}$

$+b\gamma, f_{RR} + 2ab\delta, f_{RT} + b^2\delta, f_{RT} - 2b\tau, f_R = 0, \quad (A19)$

$2\beta f_R + 2b\gamma f_{RR} + 2b\delta f_{RT} + 2b\alpha, f_R + 2\beta, f_R + 2b\gamma, f_{RR}$

$+2b\gamma, f_{RR} + 2ab\delta, f_{RT} + b^2\delta, f_{RT} - 2b\tau, f_R = 0, \quad (A20)$

$2\beta f_R + 2\beta, f_{RR} + 2a\beta, f_R + 2ab\gamma, f_{RR} + 2ab\delta, f_{RT} - 2ab\tau, f_R = 0, \quad (A21)$

$2\beta f_R + 2\beta, f_{RR} + 2a\beta, f_R + 2ab\gamma, f_{RR} + 2ab\delta, f_{RT} + 2ab\tau, f_R = 0, \quad (A22)$

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