Analysis of difficulties for pre-service mathematics teacher in problem solving of division and divisibility based on theory of action, process, object, and schemes

Y Fitrianti¹*, D Suryadi², dan Kusnandi²

¹Program Studi Pendidikan Matematika, Universitas Islam Negeri Raden Fatah, Jl. Zainal Abidin Fikri, Palembang 30126, Indonesia
²Sekolah Pascasarjana, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi No.299, Bandung 40154, Indonesia

*Corresponding author: yulifitrianti_uin@radenfatah.ac.id

Abstract. Action, process, object, and schemes is an acronym of APOS. The difficulty in the perspective of APOS theory is the inability of students to access specific schemes in all situations, thereby inhibit the construction of other concept. Knowing student’s difficulties in using the concept of division and divisibility is necessary to design genetic decomposition of that concept. This article is a report from qualitative research that discusses conceptual issues about the idea of division and divisibility given by prospective mathematics teacher students who use the APOS theory. The data for the study were collected from an interview. The researcher took three interview subjects from 30 pre-service mathematics teachers who have attended and completed a test on the division and divisibility in an elementary number theory course. These participants were selected based on their performance in the test. However, the interview conducted showed that there are five difficulties involved and they include solving the divisibility problem on integers, algebraic manipulation and recognize structures in algebraic expressions that correspond to the division algorithm or the definition of divisibility, distinguishing between variables and parameters on division problem presented in algebraic expression, associating between division algorithm and definition, and application of divisibility properties and theorems.

1. Introduction
The concept of division and divisibility is a fundamental concept in the theory of numbers. That concept underlies the construction of most thoughts in the Theory of Numbers. In some literature, the division concept discussed in the Theory of Numbers is the integer division or is often called the division algorithm or theorem [1,2]. Whereas the divisibility concept is a particular case of the division concept where the remainder is zero. Both ideas are interrelated with each other.

Knowing students’ difficulties in using the concept of division and divisibility is necessary to design genetic decomposition of that concept. It is because of genetic decomposition that can be developed from data (for example, interviews) by finding students’ difficulties in completing assignments that might indicate mental structures that students might not construct.

Genetic decomposition is a description of some individuals' possible or proprietary mental construction [3]. Genetic decomposition also informs how the learning design of certain mathematical concepts [4]. Additionally, genetic decomposition is not unique and is revised. Conceptual difficulties,
therefore, needed to be explored not only through the study results that were made, but also in the field needed for data on the difficulties experienced by students. It is because the knowledge everyone has is different even though they construct the same concept, and the problem may also vary.

Action, process, object, and schemes is an acronym of APOS. In APOS perspective, mathematical knowledge guides how a person responds to the problem in a particular precise situation that he or she faces by reflecting his problems and solutions in a mathematical action, process, and object and organizes them in schemes to deal with a particular situation [5]. Actions, processes, objects, and schemes define mental construction of a person. Actions are formed by the manipulation of a previous mental or physical object. The interiorization of these actions will build a process later encapsulated into objects. Objects can be de-encapsulation back into the process. Eventually, the action, process and object will be organized in the scheme.

Specific schemes may be inaccessible to students in a mathematical situation. It is causing difficulties for students. Also, besides ability of student to reflect their thoughts for construction or the construction of mathematical concepts can be an obstacle on their own. Difficulty with the idea of division and divisibility, therefore, can be analyzed on issues of the concept's mental structures. Exploration of various conceptual problems on mathematical concepts using the theory is done in among the concepts of calculus, expression of algebra, fractions, rules of chains, and the vector room [3,6,7,8,9].

The division and divisibility concept taught at university levels uses an axiomatic approach [10]. Conceptual patterns constitute the laws of thought that have universal applications [2], so division and divisibility issues are largely represented in algebra. This representation includes an algebraic study as the relationship between quantity [11]. In the concept of division and divisibility, the quantity referred to is a division, a dividend, quotient and a remainder.

Studies reveal student difficulties in understanding the concept of division and divisibility. Among other things: understanding the Theory of Numbers because students tendency is to think of integer division in the term rational number or real, taking into account the structure in solving problems of divisibility and resolving the divisibility issues of refutation in the form of refuting false statements [2,12,13]. The study of these difficulties is a literature review, but researchers still need exploration of the conceptual problems in the concept of division and divisibility as empirical evidence in designing early genetic decomposition.

Genetic decomposition is designed based on the experience of researchers as lecturer of the theory of numbers, a text of the materials, and research data. A theoretical construct concept of division that students may have had is described in 4 of the following steps.

- The division action is carried out according to the mathematical situation to get a quotient and the remainder.
- Interiorization of actions for the process is recognizing and being able to determine the quotient and remainder of integer division in a specific mathematical situation without doing explicit division.
- Encapsulation of the process to the object is to recognize the relationship between the divisor, dividend, quotient, and reminder, and the inherent qualities in that relationship.
- Thematizing object to schemes is awareness and can use a link between division algorithms and divisibility in solving problems.
Student conceptual exploration of the concept of division and divisibility follows the genetic decomposition.

2. Methods
This study was intended to explore the conceptual difficulties of pre-service mathematics teacher at the conceptual notion of division and divisibility based on the APOS theory perspective. From that goal, it would include a type of exploratory research with a qualitative approach [14]. Division and divisibility problems design on a test sheet. Thirty pre-service mathematics teacher who have completed college the basic number theory is selected to take a test. There are two problems in the test, namely (see Table 1):

| Concept   | Problem                                                                 |
|-----------|-------------------------------------------------------------------------|
| Division  | Use the algorithm division to establish that, the square of any integer is either of the form \(3k\) or \(3k + 1\). |
| Divisibility | For \(n \in \mathbb{Z}\), prove that \(\frac{n(n+1)(2n+1)}{6}\) is an integer. |

The selected interview subjects represent the group type of student's performance obtained after analyzing the test results. Interview questions adapted to the responses students gave in the test. Thus, each student got an interview question that may differ from each other.

3. Result and Discussion
There are 30 students who took the concept of division and divisibility tests, none of the students could answer the problems appropriately given. The performance of the students shown on the tests is as follows:

- The performance group of division concepts is: 1) not answer or answer but not responding to problem, 2) give examples of numbers that satisfy statements in the right way, 3) give examples of numbers that answer statements in the wrong way, and 4) give an example of numbers that fail to respond to statements because they misunderstanding a mathematical situation given.

- The performance group of divisibility concepts is: 1) not answer or answer but not related to problems, 2) chose several whole numbers to be adapted on appropriate form of algebra to show true statements, 3) use mathematical induction to show that the statement is true but the induction is not completed, 4) use mathematical induction to show that the statement is true and the induction step is complete.

The difficulties experienced by students in the 1.1 performance group (type-1 division concept) and 2.1 (type – 1 divisibility concept) cannot be investigated for not answering or answering but not according to the problem so that it cannot be analyzed using the APOS theory. For other performance groups are explored through interviews of subject representatives of S1 is 1st subject in performance groups of 1.3 and 2.2, S2 is the 2nd subject in performance groups 1.4 and 2.3, S3 is the 3rd subject in performance groups 1.2 and 2.4.

3.1. Student's Difficulties on The Division Concept
There Student difficulties in the division idea were explored through problem 1. Through these problems, Students are expected to be able to use the division algorithm to show that the square of any integer is either of the form \(3k\) or \(3k + 1\). But based on the interviews, it is known that students were not taught the division concept or algorithm. So, the discussion was focused on how students chose numbers that fulfilled the statements given.

S1 gives an example of numbers that satisfy to statements but in the wrong way. He was selecting some \(k\) integer to be substitution on \(3k\) or \(3k + 1\) then showing that the result is squared numbers. But with in-depth exploration, it turned out that S1 way of determining the number that satisfy the statement indicates that he had a conception of function. He states that "initially I took the value, but I did not get
the squared numbers. I take value then I am squared. I can get the value of $3k$ or $3k + 1$. That is easier to do because if you choose a value first, it can result in a value that cannot be squared."

S2 gives examples of numbers in the wrong sense of pronouncements, besides that, she does not know the concept of a function and so has difficulty obtaining the appropriate number. He stated that "I think of a number that can produce squared numbers if it is substitution into $3k$ and $3k + 1$. [pause] If $3.1 + 1$ it could be a square, so $k = 1$ [writing]. But $3(1) = 3$ is not squared. If $3.2 + 1$ and $3.2$ are not the squared. If $3.3$ is the same as before. For addition, $3.5 + 11$ also the square numbers [writing], but $3.5$ is not the squared numbers. So, it cannot be indicated that a squared number is informed $3k$ and $3k + 1". From this explanation, it saw that S2 does the procedural activity internally. It uses multiplication and sums procedures to get the appropriate numbers. She substitutes one number at a time from $3k$ and $3k + 1$.

S3 gives an example of a number that fills the statement by taking some an integer and then indicates that the square of the number is $3k$ or $3k + 1$. Her ability to translate division issue and think of numbers that fulfil statements mean the conception of function.

From the test and interview, all subjects are at the action and process stage. They tend to work through procedurally. Although S2 as able to carry out procedural activity internally, she needed a complicated process to obtain some number satisfy with $3k$ and $3k + 1$. That’s because she doesn’t have conceptual knowledge about functions. Someone who is only able to connect the two variables involved in an expression explicitly means that there is at the conception stage of the action of a function [15]. It is this kind of inaccessible experience that causes difficulties when constructing another concept in the new situation [4]. The idea of the function contained in the algebraic expression on the issue of division has the meaning of the basic object of algebra as a variable and the parameters exist [11]. Different conditions happened to S1. Although he is at the stage of a division action, but his way of thinking of selecting the appropriate number suggests that he has conceptual knowledge about functions. Someone who can see functional relations in an algebraic expression is said to have conceptual knowledge about functions [4]. In the meantime, S3 had no problem giving an example of the number that satisfies the statement. This means that students who can’t tell the difference between the variables and the parameters will have difficulty interpreting the problem of division in algebra.

3.2. Student’s Difficulties on The Divisibility Concept

Student's struggles on the divisibility concept are being explored using question 2. From these problems, students are expected to be able to make a connection between division and divisibility and to use properties of divisibility.

From the previous, it has appeared that students did not learn division concept, so they could not make a connection between the concept of division and divisibility. So during the discussion of interviews, researchers will only focus on the subject’s performance in using the properties of divisibility.

S1’s performance showed that she was at the divisibility action stage. She responds only to the problem by deciding on divisibility in specific numbers as the statement that $\frac{n(n+1)}{6}$ is an integer.

S2 respond to underlying issues using mathematical induction but incomplete. She does not perform binomial multiplication and does not recognize structures in algebraic expressions that correspond to the definition of divisibility. Just like S2, S3 responds to underlying issues using mathematical induction and can complete induction steps. From the proving step, S3 has the manipulation of algebra and structure sense skills. She could describe $(k + 1)((k + 1) + 1)(2(k + 1) + 1)$ being $2k^2 + 9k^2 + 13k + 6$ who demonstrated his skills in algebra manipulation. His structure sense skills were demonstrated by her ability to transform expression into another. Also, S3 is able to recognize the properties in divisibility when he chooses steps $(2k^2 + 3k^2 + k) + 1k + 6 = \frac{2k^2 + 3k^2 + k}{6} + \frac{1k}{6} + \frac{6}{6}$.

The Theory of Numbers is part of algebra, hence much of the problem in the Theory of Numbers represented in algebra [2]. The basic algebra concept of pre-service mathematics teacher has profoundly influenced his performance to solve the problems of divisibility. The performance he had with the S2
showed that he had difficulty answering the issue of divisibility because of his limited ability to manipulate algebra. Jojo in his research state many students found that difficult to manipulate basic algebra while constructing the concept of chain rules [3]. In addition to algebra manipulation skills, S2 has difficulty recognizing in algebraic expressions that correspond to the definition of divisibility and making it difficult for her to choose appropriate manipulation. Skills in identifying the structure are referred to by Dreyfus and Hoch as the structure sense [16]. In other studies, Novatna and Hoch state that structure sense became a challenge not only for low-skilled students but also for highly skilled students [17]. Zazkis also found out that students did not consider the structure on the matter of divisibility [12]. In another hand, S3 showed that algebraic manipulation skills, structure sense and the ability to use the properties of divisibility had been a great help in solving the problem of divisibility represented in algebra.

4. Conclusion
Division problems represented in an algebraic expression requires a student knowledge of function, so they have a variable and parameters understanding. With that understanding, they can more easily translate the problem by presenting an example of numbers that is satisfy some mathematical situation and realizing functional relationships between divisor, dividend, quotient, and a reminder.

Students who are not taught the concept of division will have difficulty solving the problems of divisibility in integer. They have no way to solve it because of the contextual constraints on the application of divisibility concept or division algorithm. Also, students have difficulty using the properties of divisibility because their understanding is still in the action and process stage.

Mistake displayed by S2 and the successful S3 in completing his mathematical induction on the divisibility issues indicates that manipulation and structure sense skills are needed in resolving the divisibility issues. Taking into consideration that the Theory of Numbers is part of algebra, so these skills must be in students’ hands to succeed in completing the divisibility task.

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