TOPICS IN HIDDEN SYMMETRIES. V.

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Abstract. This note being devoted to some aspects of the inverse problem of representation theory contains a new insight into it illustrated by two topics. The attention is concentrated on the manner of representation of abstract objects by the concrete ones as well as on the abstract objects themselves.

This paper being a continuation of the previous four parts [1] explicates novel features of the general ideology presented in the review [2] (see also the research article [3], where a wide interpretation of the inverse problem of representation theory was originally proposed). The attention is concentrated on the manner of concrete representations of abstract objects as well as on the least themselves.

The concrete lines of topics have their origins in the author’s papers [4,5:App.A], where some objects, which gave start to the constructions below, appeared.

1. Topic Nine: Lie composites and their representation.
Composed representations of Lie algebras

This topic may be considered as a variation of a theme briefly discussed at the end of the article [6] (the so–called isotopic composites and their representations). Though the particular case considered below looses some interesting combinatorial features of the general one, e.g. the graph–representations, it is nevertheless rather interesting.

Definition 1.
A. A linear space v is called a Lie composite iff there are fixed its subspaces v_1, . . . , v_n (dim v_i > 1) supplied by the compatible structures of Lie algebras. Compatibility means that the structures of the Lie algebras induced in v_i ∩ v_j from v_i and v_j are the same. The Lie composite is called dense iff v_1 ⊔ . . . ⊔ v_n = v (here ⊔
denotes the sum of linear spaces). The Lie composite is called connected iff for all \( i \) and \( j \) there exists a sequence \( k_1, \ldots, k_m \) \((k_1 = i, k_m = j)\) such that \( \mathfrak{v}_{k_i} \cap \mathfrak{v}_{k_{i+1}} \neq \emptyset \).

**B.** A representation of the Lie composite \( \mathfrak{v} \) in the space \( H \) is the linear mapping \( T : \mathfrak{v} \mapsto \text{End}(H) \) such that \( T|_{\mathfrak{v}_i} \) is a representation of the Lie algebra \( \mathfrak{v}_i \) for all \( i \).

**C.** Let \( g \) be a Lie algebra. A linear mapping \( T : g \mapsto \text{End}(H) \) is called the composed representation of \( g \) in the linear space \( H \) iff there exists a set \( g_1, \ldots, g_n \) of the Lie subalgebras of \( g \), which form a dense connected composite and \( T \) is its representation.

Reducibility and irreducibility of representations of the Lie composites are defined in the same manner as for Lie algebras. One may also formulate a superanalog of the Definition 1. The set of representations of the fixed Lie composite is closed under the tensor product and, therefore, may be supplied by the structure of a tensor category.

**Example 1 (The Octahedron Lie Composite).** Let us consider an octahedron with the vertices \( A, B, C, D, E, F \), the edges \((AB), (AC), (AD), (AE), (BC), (BF), (CD), (CF), (DE), (DF), (EF)\), and the faces \((ABC), (ACD), (ADE), (AEB), (BCF), (CDF), (DEF), (EFB)\). Let \( \mathfrak{v} \) be a six–dimensional linear space with the basis labelled by the vertices of the octahedron, \( \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \) be four three–dimensional subspaces in \( \mathfrak{v} \) corresponded to the faces \((ABC), (ADE), (CDF), (EFB)\). All subspaces \( \mathfrak{v}_i \) are supplied by the structures of the Lie algebras isomorphic to \( \mathfrak{so}(3) \) (such structures are compatible to the orientations on the faces). The pentuple \((\mathfrak{v}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)\) is a dense connected Lie composite.

**Proposition.** Let \( T \) be an arbitrary representation of the Lie composite \((\mathfrak{v}; \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)\) in the finite–dimensional linear space \( H \), then \( H \) admits a representation of the Lie algebra \( \mathfrak{so}(4) \). If \( T \) is an irreducible representation then there exist the real numbers \( \lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E, \lambda_F \) such that the operators \( T(A) - \lambda_A 1, T(B) - \lambda_B 1, T(C) - \lambda_C 1, T(D) - \lambda_D 1, T(E) - \lambda_E 1, T(F) - \lambda_F 1 \) form an irreducible representation of \( \mathfrak{so}(4) \).

**Proof.** First, note that the commutator of operators corresponded to the opposite vertices commute with operators corresponded to other four vertices. It commutes with all six operators because they may be expressed as commutators of the least four operators. So the commutator of operators corresponded to the opposite vertices belongs to the center of the Lie algebra generated by the all six operators. Let us factorize this Lie algebra \( g \) by the center. Such quotient is isomorphic to \( \mathfrak{so}(4) \) (one uses the fact that formulas for commutators of all six operators are known up to the center of \( g \)). The statement of the theorem is a consequence of this result and the fact that any central extension of the semisimple Lie algebra is trivial (i.e. may be splitted – see f.e.[7]) \( \square \)

The construction of the Octahedron Lie composite may be generalized on the certain class of polyhedra. However, any analogs of the Proposition are not known for such general case.

**Example 2 (The Witt composite).** Let \( \mathfrak{w} \) be the so–called Witt algebra, which is a subalgebra of the complexification \( \mathbb{C} \text{Vect}(S^1) \) of the Lie algebra \( \text{Vect}(S^1) \) of the smooth vector fields on a circle \( S^1 \) [8]. The Witt algebra \( \mathfrak{w} \) consists of all polynomial vector fields and admits a basis \( e_k \ (k \in \mathbb{Z}) \) with commutation relations \([e_i, e_j] = (i - j)e_{i+j}\).
Let us consider two subalgebras \( p_+ \) of \( \mathfrak{w} \) generated by \( e_i \) with \( i \geq -1 \) and \( i \leq 1 \); note that \( p_+ \cap p_- = \mathfrak{sl}(2, \mathbb{C}) \). The triple \( (\mathfrak{w}; p_+, p_-) \) is a dense connected Lie composite.

**Theorem 1.** The action of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) in any Verma module \( V_h \) (\( h \) is the highest weight) may be extended to the representation of the Lie composite \( (\mathfrak{w}; p_+, p_-) \) and, hence, to the composed representation of the Witt algebra \( \mathfrak{w} \).

*Proof.* The proof of the theorem is based on the following Lemma.

**Lemma [4:§2.2;5:App.A3].** The generators of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) in the Verma module \( V_h \) may be included into the infinite family of tensor operators (of the so-called \( q_R \)-conformal symmetries). If the Verma module \( V_h \) is realized in the space of polynomials of one variable \( z \) and the generators of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) have the form

\[
L_1 = (\xi + 2h)\partial_z, \quad L_0 = \xi + h, \quad L_{-1} = z,
\]

where \( \xi = z\partial_z \) and \( [L_i, L_j] = (i - j)L_{i+j} \), then the tensor operators are of the form

\[
L_k = (\xi + (k+1)h)\partial_z^k \quad (k \geq 0), \quad L_{-k} = z^k \frac{\xi + (k+1)h}{(\xi + 2h)\cdots(\xi + 2h + k - 1)} \quad (k \geq 1).
\]

It follows from the explicit formulas for the tensor operators that \( [L_i, L_j] = (i - j)L_{i+j} \) for all \( i, j \geq -1 \) or \( i, j \leq 1 \). □

*Comment.* The \( \mathfrak{w} \)-representations of Theorem 1 in the Verma modules over \( \mathfrak{sl}(2, \mathbb{C}) \) generate a tensor category of the category of all representations of \( \mathfrak{w} \).

**Remark 1.** The construction of the Witt composite may be generalized on the Riemann surfaces of higher genus in lines of I.M.Krichever and S.P.Novikov [9].

**Remark 2.** Generalizing the terminology of [1,2] one may say that the tensor operators (of spin 2, i.e. the \( q_R \)-conformal symmetries [4:§2.2;5:App.A3]) in the Verma modules \( V_h \) over the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) form the set of hidden symmetries, whose algebraic structure is one of the Witt composite.

Note that if the hidden symmetries realize a representation of the Lie composite they should not be unpacked (a similar situation appears also in the case of the isocommutator algebras of hidden symmetries and the related Lie \( \mathfrak{g} \)-bunches [1:Topic 3.2;§2.1]).

**Example 3.** Let \( \mathfrak{w} \) be the Witt algebra and \( (\mathfrak{w}; p_{\pm}) \) be the Witt composite. Let us consider the abelian extension \( \mathfrak{w}^c \) of the Witt algebra by the generators \( f_i \) \((i \in \mathbb{Z})\) such that \([e_i, f_j] = jf_j \). The subalgebras \( p_{\pm} \) of \( \mathfrak{w} \) may be extended to the subalgebras \( p_{\pm}^c \) of \( \mathfrak{w}^c \) by the generators \( f_i \), where \( i \geq 0 \) and \( i \leq 0 \), respectively. The triple \( (\mathfrak{w}^c; p_{\pm}^c) \) form the extended Witt composite.

**Theorem 2.** The representation of the Witt composite in any Verma module \( V_h \) (\( h \) is the highest weight) over the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) may be extended to the representation of the Lie composite \( (\mathfrak{w}^c; p_{\pm}^c) \) and, hence, the composed representation of \( \mathfrak{w} \) in \( V_h \) may be extended to the composed representation of \( \mathfrak{w}^c \).

*Proof.* The additional generators \( f_i \) are represented by the tensor operators of spin 1, namely, \( f_i \mapsto \partial_z^i \quad (i \geq 0) \), \( f_{-i} \mapsto z^i \frac{1}{(\xi + 2h)\cdots(\xi + 2h + i - 1)} \quad (i \geq 1) \). □
Comment. The $\mathfrak{w}^e$–representations of Theorem 2 in the Verma modules over $\mathfrak{sl}(2, \mathbb{C})$ generate a tensor subcategory of the category of all representations of $\mathfrak{w}^e$.

Remark 3. It is very interesting to consider the composed representations of the real semisimple Lie algebras $\mathfrak{g}$, which unduce representations of some natural subalgebras (for instance, of two opposite maximal parabolic subalgebras or two opposite Borel subalgebras, perhaps plus some $\mathfrak{sl}(2, \mathbb{C})$ imbed into $\mathfrak{g}$, etc.).

2. Topic Ten: $\mathfrak{A}$–projective representations of Lie algebras

The theme of this topics is $\mathfrak{A}$–projective representations ($\mathfrak{A}$ is an associative algebra), which are certain generalizations of the ordinary projective representations (see [9]).

Definition 2A. Let $\mathfrak{A}$ be an arbitrary associative algebra represented in the linear space $H$ and $\mathfrak{g}$ be a Lie algebra. The linear mapping $T : \mathfrak{g} \mapsto \text{End}(H)$ is called $\mathfrak{A}$–projective representation iff for all $X$ and $Y$ from $\mathfrak{g}$ there exists an element of $\mathfrak{A}$ represented by the operator $A_{XY}$ such that

$$[T(X), T(Y)] - T([X, Y]) = A_{XY}.$$ 

If $H$ is infinite dimensional the representation may be realized by the unbounded operators.

Remark 4. The definition may be generalized on any anticommutative algebras. In this situation it is closely related to the constructions of representations of the anticommutative algebras $\mathfrak{jl}(2, \mathbb{C})$ and $\mathfrak{sl}^\ast(2, \mathbb{C})$ in [10;11;§2]. In general, it should be viewed in the context of the old ideas of A.I.Maltsev on the representations of arbitrary nonassociative algebras [12]. The general anticommutative algebras and the constructions of their $\mathfrak{A}$–projective representations is of an interest in the context of the quasi–Hopf algebras, whose coalgebraic structure is not associative, Lie (Jacobian) and co–Jacobian quasi–bialgebras and related structures [13] (see also [14]).

Example. If $(\mathfrak{g}, \mathfrak{h})$ is the reductive pair then any representation of $\mathfrak{g}$ is an $\mathcal{U}(\mathfrak{h})$–projective representation of the binary anticommutative algebra $\mathfrak{p}$ ($\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, the operation in $\mathfrak{p}$ is of the standard form: $[X, Y]_\mathfrak{p} = \pi([X, Y])$, where $[\cdot, \cdot]$ is the commutator in $\mathfrak{g}$ and $\pi$ is the projector of $\mathfrak{g}$ onto $\mathfrak{p}$ along $\mathfrak{h}$ [15]).

Remark 5. The standard projective representation appears if $\mathfrak{A}$ is one–dimensional algebra acting by scalar matrices.

Remark 6. If $H$ is the Hilbert (or pre–Hilbert) space then one may consider the algebra $\mathcal{HS}$ of all Hilbert–Schmidt operators as $\mathfrak{A}$. In this case our construction is deeply related to ideas of A.I.Shtern on the almost representations and to some aspects of the pseudodifferential calculus [16] (see also [14]). One may also consider the algebra $\mathcal{TC}$ of all trace class operators and the algebra $\mathcal{B}$ of all bounded operators. Any $\mathcal{HS}$–projective representation by unbounded operators is also a $\mathcal{B}$–projective representation.

Definition 2B. Let $\mathfrak{A}$ be an associative algebra with an involution $\ast$ symmetrically represented in the Hilbert space $H$. If $\mathfrak{g}$ is a Lie algebra with an involution $\ast$ then its $\mathfrak{A}$–projective representation $T$ in the space $H$ is called symmetric iff for all elements
Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra ($\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$) with an involution $\ast$ such that $\mathfrak{g}_{-n}^* = \mathfrak{g}_{-n}$ and the involution is identical on the subalgebra $\mathfrak{g}_0$.

Let us extend the $\mathbb{Z}$-grading and the involution $\ast$ from $\mathfrak{g}$ to the tensor algebra $T(\mathfrak{g})$.

The symmetric $\mathfrak{A}$–projective representation of $\mathfrak{g}$ is called absolutely symmetric iff for any element $a$ of $T(\mathfrak{g})$ such that $\deg(a) = 0$ the equality $T(a) = T^*(a)$ holds (here the representation $T$ of $\mathfrak{g}$ in $H$ is extended to the mapping from $T(\mathfrak{g})$ to $\mathrm{End}(H)$).

Note that the Witt algebra $\mathfrak{w}$ has the natural $\mathbb{Z}$-grading and involution $\ast$.

**Theorem 3.** The composed representation of the Witt algebra $\mathfrak{w}$ in the unitizable Verma module $V_\hbar$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is the absolutely symmetric $\mathcal{HS}$–projective representation of $\mathfrak{w}$.

The statement of the theorem follows from the explicit formulas for the tensor operators of $q_R$–conformal symmetries in the Verma module over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

It is very interesting to describe all highest weight composed representations of the Witt composite, which are $\mathcal{HS}$–projective representations of the Witt algebra or its central Gelfand–Fuchs extension (the Virasoro algebra) [8].

**Remark 7.** The Definition 2 may be generalized from the Lie algebras to their numerous nonlinear analogs such as quantum groups, Sklyanin algebra, Racah–Wigner algebras, mho–algebras, NWSO–algebras (see [1,2]), etc.

It is very interesting to consider the composed representations of the semisimple Lie algebras mentioned in Remark 3, which are the $\mathcal{HS}$–projective representations of the algebras. The importance of such combination is illustrated by the following example.

**Example.** Let $\mathfrak{b}_\pm$ be two opposite Borel subalgebras of $\mathfrak{sl}(2, \mathbb{C})$, then any representation of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ may be considered as a representation of the Lie composite $(\mathfrak{sl}(2, \mathbb{C}); \mathfrak{b}_+, \mathfrak{b}_-)$, however, none of the infinite–dimensional $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$–modules of the category $\mathcal{O}$ [17:§9.2] defines the $\mathcal{HS}$–projective representation of $\mathfrak{sl}(2, \mathbb{C})$.

**Remark 8.** The sets of the irreducible infinite–dimensional $\mathcal{HS}$–projective highest weight representations for two nonlinear $\mathfrak{sl}_2$ [18] with commutation relations $[e_0, e_pm] = \pm e_\pm$ and $[e_+, e_-] = R_{1,2}(e_0)$ coincides if and only if for all $h \in \mathbb{R}$ $\sum_{j=0}^\infty |R_1(h + j) - R_2(h + j)|^2 < \infty$.

**Definition 2C.** The $\mathfrak{A}$–projective representation $T$ of the Lie algebra $\mathfrak{g}$ in the linear space $H$ will be called almost absolutely closed iff for any $n \geq 1$ and for any elements $X_0, X_1, X_2, \ldots X_{n+1}$ of $\mathfrak{g}$ there exists an element $\varphi(X_0, X_1, X_2, \ldots X_{n+1})$ of $\mathfrak{g}$ such that

$$[\ldots [T(X_0), T(X_1)], T(X_2)], \ldots , T(X_{n+1})] \equiv T(\varphi(X_0, X_1, X_2, \ldots X_{n+1})) \pmod{\mathfrak{A}},$$

here $\mathfrak{A}$ is considered as being mapped into $\mathrm{End}(H)$. The almost absolutely closed $\mathfrak{A}$–projective representation $T$ of the Lie algebra $\mathfrak{g}$ in the linear space $H$ will be called absolutely closed iff $\varphi(\cdot, \ldots, \cdot) \equiv 0$.

The mappings $(X_0, X_1, X_2, \ldots X_{n+1}) \mapsto \varphi(X_0, X_1, X_2, \ldots X_{n+1})$ associated with any almost absolutely closed $\mathfrak{A}$–projective representation of the Lie algebra $\mathfrak{g}$ define the higher brackets in the Lie algebra $\mathfrak{g}$. The objects with higher brackets systematically appears in many branches of mathematics and mathematical physics (see,
Remark 9. The HS–projective representations of the Witt algebra in the Verma modules over $\mathfrak{sl}(2,\mathbb{C})$ are absolutely closed.

Note that the extended Witt algebra $\mathfrak{w}^e$ has the natural $\mathbb{Z}$–grading and involution $\ast$.

Theorem 4. The composed representation of the extended Witt algebra $\mathfrak{w}^e$ in the unitarizable Verma module $V^h$ over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the absolutely closed and absolutely symmetric HS–projective representation of $\mathfrak{w}^e$.

The statement follows from the explicit formulas for the tensor operators of spins 1 and 2 in the Verma modules over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Remark 10 (perhaps, very crucial). The HS–projective (and, simultaneously, composed) representations of the Lie algebras $\mathfrak{w}$ and $\mathfrak{w}^e$ from Theorems 3,4 realize generators of the algebras as infinite–dimensional hidden symmetries of the Verma modules over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. It seems that many other objects of the representation theory of the reductive Lie algebras (Verma modules, their models and skladens – see e.g.[4]*, general constructive and Harish–Chandra modules, etc. – see e.g.[20,17]) possess analogous infinite–dimensional hidden symmetries, which may constitute as familiar as novel abstract algebraic structures.

Problems:

– What algebraic structure is represented by the tensor operators $W_n$ ($n \in \mathbb{Z}$) of the spin 3 ($q_R$–$W_3$–symmetries) in the Verma module $V^h$ over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$? The generators $W_i$ ($i = -2,-1,0,1,2$) together with the $\mathfrak{sl}(2,\mathbb{C})$–generators form the Racah–Wigner algebra $\mathcal{RW}(\mathfrak{sl}(2,\mathbb{C}))$ for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ [2:§§1.1,1.2;4:§2.2]. The conformally invariant analogue of the requested structure is the Zamolodchikov $W_3$–algebra [21]. Some aspects of the problem were discussed in the author’s article [22:§4].

– What algebraic structure is represented by the tensor operators $U_n$ ($n \in \mathbb{Z}$) of the spin 4 in the Verma module $V^h$ over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$? The generators $U_i$ ($i = -3,-2,-1,0,1,2,3$) together with the $\mathfrak{sl}(2,\mathbb{C})$–generators form the mho–algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}),\pi_3)$ over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ [1:Topic 2:2:§1.4]. The operators $U_i$ ($i = -3,-2,-1,0,1,2,3$), $W_i$ ($i = -2,-1,0,1,2$) and the $\mathfrak{sl}(2,\mathbb{C})$–generators form the higher Racah–Wigner algebra $\mathcal{RW}_3(\mathfrak{sl}(2,\mathbb{C}))$ [2:§1.2] (see also [23:§1.2]).

– To describe the tensor category of representations of the Witt composite generated by its representations in the Verma modules over $\mathfrak{sl}(2,\mathbb{C})$ and by the Verma modules over the Gelfand–Fuchs central extension of the Witt algebra (Verma modules over the Virasoro algebra [24]).

Conclusions

Even the sketchy discussion of two topics above allows to state that the actual richness and attractiveness of the inverse problem of representation theory are based

*The originally Russian term “skladen” was misleadingly translated from Russian into English as “collection” by a translator of the article [4], a possible correct translation is “unfolding” but the direct transliteration is preferable.
not only on a large scope of various interesting abstract algebraic structures, which may be concretely represented and somehow unravelled, but also on the diversity of the manners of representation, which may produce very intriguing unexpected and nontrivial effects under the intent look even in the simple and almost hackneyed situations.

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P.S. The reader interested in the further elaboration of the ideas of the articles [4,5] (and also of [11]) should be addressed to the report:

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