On the intersection of unknotting tunnels and the decomposing annulus in connected sums

Yoav Moriah

Abstract

Given \((V_1, V_2)\) a Heegaard splitting of the complement of a composite knot \(K = K_1 \# K_2\) in \(S^3\), where \(K_i, i = 1, 2\) are prime knots, we have a unique, up to isotopy, decomposing annulus \(A\). When the intersection of \(A\) and \(V_1\) is a minimal collection of disks we study the components of \(V_1 - N(A)\) and show that at most one component is a 3-ball meeting \(A\) in two disks. This is a crucial step in proving the conjecture that a necessary and sufficient condition for the tunnel number of a connected sum to be less than or equal to the sum of the tunnel numbers is that one of the knots has a Heegaard splitting in which a meridian curve is primitive.

1 Introduction

The way in which the tunnel number \(t(K)\) of a knot \(K = K_1 \# K_2\) relates to \(t(K_1)\) and \(t(K_2)\) is a long standing question. It was long known that \(t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1\). That this inequality is best possible was proved by the second author and Rubinstein in [MR] and by Morimoto, Sakuma and Yokota in [MSY]. However it is not yet understood when this phenomenon occurs. We state the following conjecture and note that it was proved to be true by Morimoto, for the special class of knots which do not contain essential meridional surfaces, also known as smallish knots (see [Mo3]):

*Supported by The Fund for Promoting Research at the Technion, grant 100-127 and the Technion VRP fund, grant 100-127.
Conjecture 1.1. The knots $K_1 \subset S^3, K_2 \subset S^3$ and $K_1 \# K_2$ satisfy the inequality $t(K_1 \# K_2) \leq t(K_1) + t(K_2)$ if and only if either $E(K_1)$ or $E(K_2)$, say $E(K_1)$, has a minimal genus Heegaard splitting $(V_1, V_2)$ in which $\partial E(K_1) \subset V_1$ and a meridian curve $\mu \subset \partial E(K_1)$ is isotopic to a curve $\mu^*$ on the Heegaard surface $\partial V_2$ so that $\mu^*$ intersects an essential disk $D \subset V_2$ in a single point.

A meridian which has the property above will be called primitive.

The “if” part of the statement is a well known result proved in Lemma 2.1, however the “only if ” part turned out to be much more difficult.

When attempting to prove the conjecture one needs to deal with the following stitution:

We are given a minimal genus Heegaard splitting $(V_1, V_2)$ for $E(K_1 \# K_2)$ and we need to find minimal genus Heegaard splittings for $E(K_1)$ or $E(K_2)$ in which a meridian is primitive. We may assume that $V_1$ is a small neighborhood of the spine i.e., $V_1$ is a regular neighborhood of the boundary torus union a regular neighborhood of the tunnels, and we can choose a decomposing annulus $A$ which intersects $V_1$ in a minimal number of disks. However it is exactly here that a major problem arises as it is conceivable, a priori, that the interior of a single tunnel will intersect $A$ in a large number of points. If this occurs, the decomposing annulus $A$ will cut $\partial V_1$ into pieces, a lot of which will be annuli. Since the Euler characteristic of an annulus is zero we can have arbitrarily many of them. It follows that the induced (as in Section 2) Heegaard splittings on $E(K_1)$ and $E(K_2)$ will be of very high genus and it is easy to generate such Heegaard splittings where the meridian curve $\mu$ is primitive. However these high genus Heegaard splittings will give us no information about a minimal genus one.

In this paper we show that this situation cannot happen. Let us call a closure of a component of $V_1 - A$ which intersects $A$ in $n$ disks an $n$-float. With this terminology we show:

Theorem 4.1. Let $K_i \subset S^3$ be prime knots and $K = K_1 \# K_2$ be their connected sum. Let $(V_1, V_2)$ be a Heegaard splitting of $E(K)$ and $A$ a decomposing annulus which intersects $V_1$ in a minimal number of disks. Then: $V_1 - N(A)$ has at most one component which is a 2-float of genus zero.
For the proof we need the following result which is of independent interest:

**Theorem 3.3.** Let $K_1, K_2 \subset S^3$ be prime knots. Then every Heegaard splitting $(V_1, V_2)$ for $E(K) = E(K_1 \# K_2)$ has a spine which contains at least one cycle disjoint from a decomposing annulus for $K$ which minimizes the number of intersections with $V_1$.

Note that if the meridian $\mu$ of $K_1$ or $K_2$ is primitive it does not follow that the Heegaard splitting of $E(K_1 \# K_2)$ induced, as in Lemma 2.1, by the Heegaard splittings of $E(K_1)$ and $E(K_2)$ is of minimal genus. It was shown by Morimoto in [Mo2] that there are prime knots for which $t(K_1 \# K_2) < t(K_1) + t(K_2)$. In further work Kobayashi showed that if one allows the knots to be non-prime then the degeneration of tunnel number can be arbitrary i.e, for each $n$ there are knots $K_1^n$ and $K_2^n$ so that $t(K_1^n \# K_2^n) < t(K_1^n) + t(K_2^n) - n$. These results are characterized by an attempt to get upper bounds on $t(K)$ in terms of $t(K_1) + t(K_2)$.

Results in the opposite direction were obtained by M. Scharlemann and J. Schultens. They analyze essential annuli coming from the intersection of a decomposing annulus with a strongly irreducible Heegaard splitting to get results generalizing those of Kwong (see [SS] and [Kw]). In particular they show that $t(K) \geq \frac{5}{4}(t(K_1) + t(K_2))$.

For definitions of the above terminology see Section 2.

**Acknowledgments:** We would like to thank Bronek Wajnryb for many conversations and Ying-Qing Wu for suggesting a somewhat shorter version of the proof of Theorem 4.1. Also thanks to Anna Klebanov for an argument for Case 1 in the proof of Theorem 4.1.
2 Preliminaries

In this section we define some of the notions and state some of the results needed for the proof of the main theorem.

Throughout the paper $K_1$ and $K_2$ will be knots in $S^3$ and $K = K_1 \# K_2$ will denote the connected sum of $K_1$ and $K_2$. The knots $K_i$ will be called the \textit{summands} of the composite knot $K$. Let $N()$ denote an open regular neighborhood in $S^3$.

Recall that $(S^3, K)$ is obtained by removing from each space $(S^3, K_i), i = 1, 2$, a small 3-ball intersecting $K_i$ in a short unknotted arc and gluing the two remaining 3-balls along the 2-sphere boundary so that the pair of points of $K_1$ on the 2-sphere are identified with the pair of points of $K_2$. If we denote $S^3 - N(K)$ by $E(K)$ then $E(K)$ is obtained from $E(K_i), i = 1, 2$, by identifying a meridional annulus $A_1$ on $\partial E(K_1)$ with a meridional annulus $A_2$ on $\partial E(K_2)$. A knot $K \subset S^3$ is prime if it is not a connected sum of two non-trivial knots. The annulus $A_1 = A_2$ will be denoted by $A$ and called the \textit{decomposing annulus}. If both knots $K_1, K_2$ are prime then the decomposing annulus is unique up to isotopy.

A \textit{tunnel system} for an arbitrary knot $K \subset S^3$ is a collection of properly embedded arcs $\{t_1, \ldots, t_n\}$ in $S^3 - N(K)$ so that $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ is a handlebody.

Given a tunnel system for a knot $K \subset S^3$ note that the closure of $N(K \cup t_1 \cup \cdots \cup t_n)$ is always a handlebody denoted by $V_1$ and the handlebody $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ will be denoted by $V_2$. For a given knot $K \subset S^3$ the smallest cardinality of any tunnel system is called the \textit{tunnel number} of $K$ and is denoted by $t(K)$.

A compression body $V$ is a compact orientable and connected 3-manifold with a preferred boundary component $\partial_+ V$ that is obtained from a collar of $\partial_+ V$ by attaching 2-handles and 3-handles, so that the connected components of $\partial_- V = \partial V - \partial_+ V$ are all distinct from $S^2$. The extreme cases, where $V$ is a handlebody i.e., $\partial_- V = \emptyset$, or where $V = \partial_+ V \times I$, are allowed. Alternatively we can think of $V$ as obtained from $(\partial_- V) \times I$ and 3-balls by attaching 1-handles to $(\partial_- V) \times \{1\}$ and the 3-balls. An essential annulus in a compression body will be called a \textit{vertical (or a spanning) annulus} if it has its boundary components on both of $(\partial_- V)$ and $(\partial_+ V)$.

A Heegaard splitting for a manifold $M$ is a decomposition $M = V_1 \cup V_2$, so that and $V_1 \cap V_2 = \Sigma$, where $V_i, i = 1, 2$ are compression bodies and $\Sigma = \partial_+ V_1 = \partial_+ V_2$ is the Heegaard splitting surface.
Given a knot $K \subset S^3$ a Heegaard splitting for $E(K)$ is a decomposition of $E(K)$ into a compression body $V_1$ containing $\partial E(K)$ and a handlebody $S^3 - \text{int}(V_1)$. Hence, a tunnel system $\{t_1, \ldots, t_n\}$ in $S^3 - N(K)$ for $K$ determines a Heegaard splitting of genus $n + 1$ for $E(K)$. Conversely given a Heegaard splitting $(V_1, V_2)$ for $E(K)$ any minimal complete disk system $D = \{D_1, \ldots, D_n\}$ so that $V_1 - N(D) = T^2 \times I$ determines a tunnel system by taking $t_i$ to be the cocore arc $\{0\} \times I$ of $\partial(\text{int}(D_i)) = D_i \times I$.

Given a Heegaard splitting $(V_1, V_2)$ for $S^3 - N(K_1 \# K_2)$ we will choose a decomposing annulus $A$ which intersects the compression body $V_1$ in two \textit{vertical annuli} $A_1^*, A_2^*$ i.e., annuli with one boundary component on $\partial_+$ and one on $\partial_-$, and in a \textit{minimal} collection of disks $D = \{D_1, \ldots, D_d\}$. Note also that $A$ intersects $V_2$ in a connected incompressible planar surface.

Let $\mathcal{E} = \{E_1, \ldots, E_{t(K)+1}\}$ be a complete meridian disk system for $V_2$, chosen to minimize the intersection with $A$. Since $V_2$ is a handlebody it is irreducible and we can assume that no component of $\mathcal{E} \cap A$ is a simple closed curve.

When we cut $E(K)$ along a decomposing annulus $A$ any Heegaard splitting $(V_1, V_2)$ of $E(K)$ induces Heegaard splittings on both of $E(K_1)$ and $E(K_2)$, as follows: Set $V_i^1 = (V_1 \cap E(K_i)) \cup_{D \cup A_1^* \cup A_2^*} \text{N}(A)$; it is a compression body as it is a union of an \textit{annulus} $\times I$ and some 1-handles, glued along the two vertical annuli and a collection of disks. Now set $V_i^2 = V_2 - \text{N}(A)$; it is a handlebody since the annulus $A$ meets $V_2$ in an incompressible connected planar surface $P$ which separates $V_2$ into two components each of which is a handlebody. Hence the pair $(V_i^1, V_i^2)$ is a Heegaard splitting for $E(K_i)$ and will be referred to as the \textit{induced Heegaard splitting} of $E(K_i)$.

In the other direction we have:

\textbf{Lemma 2.1.} Assume that either $E(K_1)$ or $E(K_2)$, say $E(K_1)$, has a minimal genus Heegaard splitting $(V_1^1, V_2^1)$ in which $\partial E(K_1) \subset V_1^1$ and a meridian curve $\mu \subset \partial E(K_1)$ is isotopic to a curve $\mu^*$ on the Heegaard surface $\partial V_2^1$ so that $\mu^*$ intersects an essential disk $D \subset V_2^1$ in a single point. Then the knots $K_1 \subset S^3, K_2 \subset S^3$ and $K_1 \# K_2$ satisfy $t(K_1 \# K_2) \leq t(K_1) + t(K_2)$.

\textit{Proof.} Choose a meridional annulus $A_1$ in $E(K_1)$ so that $A_1 \cap V_1^1$ are two vertical annuli $A_1^*, A_2^*$ and $A_1 \cap V_2^1$ is a regular neighborhood of $\mu^*$. Let $(V_1^2, V_2^2)$ be any Heegaard splitting of $E(K_2)$. When we glue $E(K_1)$ to $E(K_2)$ by identifying the meridional annulus $A_1$ with a meridional annulus $A_2 \subset \partial E(K_2)$ we glue $V_1^1$ to $V_1^2$ along two vertical annuli, so that the result is
clearly a compression body $V_1$ with $\partial V_1 = T^2$. We also need to glue $V_2^1$ to $V_2^2$ by identifying $A_1$ with an annulus on $V_2^2$. Since $A_1$ intersects an essential disk $D$ in $V_2^1$ in a single arc the resulting manifold will be a handlebody $V_2$ of genus $g(V_2^1) + g(V_2^2) - 1 = t(K_1) + t(K_2)$. So we have obtained a Heegaard splitting $(V_1, V_2)$ of $E(K_1\#K_2)$ of genus $t(K_1) + t(K_2) + 1$ and hence $t(K_1\#K_2) \leq t(K_1) + t(K_2)$.

Following the notation of Morimoto (see [Mo1]) we consider now the planar surface $P = A \cap V_2$. It has two distinguished boundary components coming from the vertical annuli $A_1^*, A_2^*$ and denoted by $C_1^*, C_2^*$ respectively. There are exactly $d$ other boundary components of $P$ which we denote by $C_1, \ldots, C_d$. With this notation we have $\partial D_i = C_i$. The arcs of $E \cap A$ are contained in $P$ and can be divided into three types:

1. An arc $\alpha$ of Type I is an arc connecting two different boundary components of $P$.

2. An arc $\alpha$ of Type II is an arc connecting a single boundary component of $P$ to itself so that the arc does not separate the boundary components $C_1^*, C_2^*$.

3. An arc $\alpha$ of Type III is an arc connecting a single boundary component of $P$ to itself with the additional property that the arc does separate the boundary components $C_1^*, C_2^*$.

Since the annulus $A$ was chosen to minimize the number of disks in $A \cap V_1$ the planar surface $P$ is incompressible in the handlebody $V_2$. Hence there is a sequence of boundary compressions of $P$ along disjoint arcs $\alpha_i$ using sub-disks of $E$ so that the end result is a collection of disks. Any such sequence defines an order on the arcs $\alpha_i$.

**Definition 2.2.** Let $\alpha_i$ be an arc of intersection of $P \cap E$. We call $\alpha$ a $d-arc$ if there is some compression order so that $\alpha_i$ is of type I and there is some component $C$ of $\partial P - (C_1^* \cup C_2^*)$ which meets $\alpha_i$ and does not meet any $\alpha_j$ for any $j < i$. If $\alpha_i$ is of type I and connects $C_1^*$ to $C_2^*$ it is called an $e-arc$.

Any outermost arc $\alpha_i$ determines a sub-disk $\Delta$ on some $E_i$ where $\partial \Delta = \alpha_i \cup \beta$ and $\beta$ is an arc on $\partial V_1 = \partial V_2$. When we perform an isotopy of type A i.e., pushing $P$ through $\Delta$ as in [Ja], we produce a band $b$ with core $\beta$ on $\partial V_1 = \partial V_2$. The following result is proved in [Mo1] pp. 41- 42, and [Oc]:

6
Theorem 2.3 (Morimoto). If the decomposing annulus is chosen to minimize the number of components of $V_1 \cap A$ and $V_1 \cap A \neq A_1^* \cup A_2^*$ then in $V_2 \cap E = P \cap E$:

(a) there are no $d$-arcs.

(b) there are no $e$-arcs.

(c) there are no arcs of type II.

(d) each component $C \subset \partial P$ has an arc $\alpha$ of type III with end points on $C$.

Remark: It follows from the above theorem that every component $C$ of $\partial P$ has arcs of type I and each component $C$ of $\partial P - (C_1^* \cup C_2^*)$ has arcs of type III. Thus there is an order on the disk components of $A \cap V_1$ starting from $A_1^*$ to $A_2^*$ or vice versa.

3 Interior tunnels

Consider now a Heegaard splitting $(V_1, V_2)$ for $E(K)$ the exterior of $K = K_1 \# K_2$, where $\partial E(K) \subset V_1$ and the decomposing annulus $A$ meets $V_1$ in disks and two vertical annuli. Since the annulus $A$ meets $V_2$ in a connected planar surface $P$ it separates $V_2$ into two components each of which is a handlebody. We will denote the handlebodies $cl(V_2 - A) \cap E(K_i)$ by $V_i^2$ respectively. However $V_1 - A$ might have many components.

Definition 3.1. A component of $cl(V_1 - A)$ which is disjoint from $\partial E(K_i)$ and intersects $A$ in $n$ disks will be called an $n$-float. An $n$-float is either a 3-ball or a handlebody of some genus $g$ if its spine is not a tree. In this case we say that the $n$-float is of genus $g$.

Remark: Note that there are always exactly two components of $cl(V_1 - A)$ not disjoint from $\partial E(K_i)$ (one in each of $E(K_1)$ and $E(K_2)$) and each one is a handlebody of genus at least one as $V_1$ is a compression body with a $T^2$ boundary. We denote these special components by $N_1$ and $N_2$ depending on whether they are contained in $E(K_1)$ or $E(K_2)$ respectively.

Consider now $E_i \subset \mathcal{E}$ any one of the meridian disks of $V_2$. On $E_i$ we have a collection of arcs corresponding to the intersection with the decomposing
annulus. These arcs, as indicated in Fig. 1, separate $E_i$ into sub-disks where disks on opposite sides of arcs are contained in opposite sides of $A$ i.e., in $E(K_1)$ or $E(K_2)$ respectively. So each sub-disk is contained in either $E(K_1)$ or $E(K_2)$. The boundary of these sub-disks is a collection of alternating arcs $\cup(\alpha_i \cup \beta_i)$ where $\alpha_i$ are arcs on $A$ and $\beta_i$ are arcs on some component of $cl(V_1 - A)$.

---

**Proposition 3.2.** Let $K_1$ and $K_2$ be knots in $S^3$ and let $K, A, E$ be the connected sum, a minimal intersection decomposing annulus and a meridional system for some Heegaard splitting of $E(K)$ as above. Then

(a) the $\beta$ arc part of the boundary of an outermost sub-disk in $E$ cannot be contained in a $n$-float of genus 0.

(b) if the $\beta$ arc part of the boundary of an outermost sub-disk in $E$ is contained in an $N_i$ component $i = 1$ or 2 and $K_i$ is prime the genus of $N_i$ is greater than one.

**Proof.** Denote an outermost sub-disk of some $E_j$ by $\Delta$ and suppose it is cut off by an arc $\alpha$ on $A$ with end points on a disk $D_i$ which belongs to some $n$-float of genus 0. Since $\Delta$ is an outermost disk, by Theorem 2.3, the arc $\alpha$ must be of type III. Further assume $\partial \Delta = \alpha \cup \beta$ where $\beta$ is an arc on the $n$-float meeting $D_i$ in exactly two points $\partial \beta = \partial \alpha$. On $\partial D_i$ there is a small
arc \(\gamma\) so that \(\gamma \cup \beta\) is a simple closed curve on the \(n\)-float bounding a disk \(D\) there, since the \(n\)-float has no genus (see Fig. 2 below). Furthermore \(\gamma \cup \alpha\) is a simple closed loop on \(A\) which together with a boundary component of \(A\) bounds a sub-annulus of \(A\). Hence \(\gamma \cup \alpha\) bounds a disk \(D'\) on the decomposing 2-sphere of \(K\) intersecting \(K\) in a single point. Thus we obtain a 2-sphere \(D \cup \Delta \cup D'\) which intersects the knot \(K\) in a single point. This is a contradiction which finishes case (a).

For case (b), assume that the outermost disk \(\Delta\) is contained in \(N_1\), say, and that genus \(N_1\) is one. As before we have \(\partial \Delta = \alpha \cup \beta\) where \(\beta\) is an arc on \(N_1\) and a small arc \(\gamma\) so that \(\gamma \cup \beta\) is a simple closed curve on \(N_1\). If \(\gamma \cup \beta\) bounds a disk in \(N_1\) we have the same proof as in case (a). If \(\gamma \cup \beta\) does not bound a disk on \(N_1\) we consider small sub-arcs \(\beta_1\) and \(\beta_2\) of \(\beta\) which are respective closed neighborhoods of \(\partial \beta\). These arcs together with a small arc \(\delta\) on \(\partial N_1 - \partial E(K_1)\) and \(\gamma\) bound a small band \(b\) on \(\partial N_1\). Notice that \(b \cup \beta_1, \beta_2, \Delta\) is an annulus \(A'\). The annulus \(A'\) together with the sub-annulus \(A''\) of \(A\) cut off by \(\alpha \cup \gamma\) defines an annulus \(A' \cup \alpha \cup \gamma\) \(A''\) which determines an isotopy of a meridian curve \(C_1\) to a simple closed curve \(\lambda\) on \(\partial N_1\). Note that \(N_1\) is a solid torus and \(\pi_1(N_1) = \mathbb{Z}\) which is generated by a meridian \(\mu\) of \(E(K_1)\). Hence \([\lambda] = [C_1] = \mu \in \pi_1(N_1)\) (see Fig. 3).
Now we can consider the annulus \((A - A'') \cup A')\). If it is non-boundary parallel then since both knots \(K_1, K_2\) are prime it must be a decomposing annulus which has at least one less disk component intersection than \(A\) in contradiction to the choice of \(A\). If it is boundary parallel, then as above, we have \(A'' \cup A'\) as a decomposing annulus with a smaller number of disks. Again in contradiction to the choice of \(A\). So genus \(N_1\) cannot be one and this finishes case (b).

As a corollary we obtain:

**Theorem 3.3.** Let \(K_1, K_2 \subset S^3\) be prime knots. Then every Heegaard splitting \((V_1, V_2)\) for \(E(K) = E(K_1 \# K_2)\) has a spine which contains at least one cycle disjoint from a decomposing annulus for \(K\) which minimizes the number of intersections with \(V_1\).

*Proof.* Since the \(\beta\) part of an outer-most disk must be contained in a float of genus greater than one we must have a 1–*handle* on the float to create the genus. The core arc of the 1-handle which is disjoint from the decomposing annulus \(A\) determines the cycle.
As a side benefit of these considerations we have the following:

**Remark 3.4.** Given a minimal genus Heegaard splitting \((V_1, V_2)\) for \(E(K) = E(K_1 \# K_2)\), where \(K_i, i = 1, 2\) are prime knots in \(S^3\), assume that the decomposing annulus with minimal intersection with the compression body \(V_1\) has \(d\) disk components and two vertical annuli. Recall that if \(\text{genus}(E(K)) = g\) then \(g = t(K) + 1\) and assume also that \(\text{cl}(\partial_+ V_1 - A)\) has \(k\) components \(\Sigma_i\) each of which is a surface of genus \(g_i\) with \(c_i\) boundary components. Since we can obtain \(\Sigma_i\) from \(\Sigma = \partial_+ V_1\) by cutting \(\Sigma\) along annuli we have the following formula:

\[
\begin{align*}
(a) & \quad -\chi(\Sigma) = 2g - 2 = 2t(K) \\
(b) & \quad 2(2 + d) = \sum_{i=1}^{k} c_i \\
(c) & \quad \sum_{i=1}^{k} c_i = 2t(K) + \sum_{i=1}^{k} 2 - 2g_i \quad \text{Thus:} \\
(d) & \quad 2(2 + d) = 2t(K) + \sum_{i=1}^{k} 2 - 2g_i.
\end{align*}
\]

Given an \(n_i\)-float \(F_i\) contained in \(E(K_j)\) it determines \(n_i - 1\) tunnels which intersect \(A\) and an additional \(g_i\) interior tunnels which do not meet \(A\). Recall that since we have two components which intersect \(A\) in two vertical annuli each the total number of floats is \(k - 2\). Further note that each disk in \(A \cap V_1\) corresponds to tunnels on both sides of \(A\). Putting all of the above together we see that the total number of the tunnels in both induced Heegaard splittings on \(E(K_1)\) and \(E(K_2)\) is:

\[
t^*(K_1) + t^*(K_2) = 2d - (k - 2) + \sum_{i=1}^{k} g_i
\]

Where \(t^*(K_i)\) is the tunnel number of the induced Heegaard splitting on \(E(K_i)\). Combined with the above we get:

\[
t^*(K_1) + t^*(K_2) = 2t(K) + (k - 2) - \sum_{i=1}^{k} g_i
\]

In the next section we show that by ruling out 2-floats of genus zero we can give a global upper bound on the number \((k - 2) - \sum_{i=1}^{k} g_i\), and hence a meaningful upper bound on \(t^*(K_1) + t^*(K_2)\) in terms of \(t(K)\) (see Theorem 4.4).
4 Ruling out genus zero 2-floats

We have the following theorem:

**Theorem 4.1.** Let $K_i \subset S^3$ be prime knots and $K = K_1 \# K_2$ be their connected sum. Let $(V_1, V_2)$ be a Heegaard splitting of $E(K)$ and $A$ a decomposing annulus which intersects $V_1$ in a minimal number of disks. Then: $V_1 - N(A)$ has at most one component which is a 2-float of genus zero.

**Remark:** If we generalize the definition of tunnels to a neighborhood of a complete system of meridian disks for $V_1$ we have: With the above assumptions at most one tunnel of $K$ has a U-turn. i.e., at most one tunnel pierces $A$ in one direction and then pierces it again in the opposite direction without meeting $V_1$. Furthermore this phenomenon occurs at most once.

**Proof.** Assume that some 2-float is a 3-ball meeting $A$ in exactly two disks. Denote these disks by $D_1$ and $D_2$ but note that these indices do not necessarily agree with the natural order defined on the disks $D_i$ by Theorem 2.3. We first need the following lemma:

**Lemma 4.2.** There is some $E_j \subset \mathcal{E}$ and a sub-disk $\Delta \subset E_j$ so that $\partial \Delta \subset 2$-float $\cup A$. Hence $\partial \Delta = \cup (\alpha_r \cup \beta_s)$, where the $\beta_s$ arcs are contained in the 2-float and the $\alpha_r$ are arcs on $A$.

**Proof.** (of Lemma) By Theorem 2.3 there is some arc $\alpha$ in some $E_j$ with end points on $D_1$. Consider the disk $\Delta$ of $E_j - A$ adjacent to $\alpha$ on the same side of $A$ as the 2-float. We can assume that $\Delta$ is to the right of $\alpha$ on $E_j$ and it cannot be outer-most as the 2-float has no genus. Hence there are more arcs of $E_j \cap A$ further to the right of $\alpha$. If all such arcs of $\Delta$ which are not on $\partial E_j,$ are of type III the disk $\Delta$ satisfies the conclusion of the lemma and we are done: Since all arcs of type III have end points on $D_1$ or $D_2$ the $\beta$ arcs must be contained in the 2-float.

So we assume that some arc to the right of $\alpha$ is of type I. If the lemma fails there are at least two arcs of type I with one end point not on $D_1$ or $D_2.$ Consider such an arc $\rho$ of type I, it cannot be outer-most by Theorem 2.3. Hence further to the right there is some arc $\alpha'$ of type III with end points on $D_1$ or $D_2.$ One of the two disks adjacent to $\alpha'$ is on the 2-float. Assume that it is $\Delta'$ and that it is on the right of $\alpha'$. We now start the argument again with $\alpha'$. This procedure must end since the intersection is finite.
Assume therefore that $\Delta'$ is to the left of $\alpha'$. If all arcs in $\partial\Delta' - \partial E_j$ have end points on $D_1$ or $D_2$ we are done as before. If there is an arc with no end points on $D_1$ or $D_2$ then there is an arc $\rho'$ of type I with exactly one end point on $D_1$ or $D_2$. It cannot be outermost as before so farther out there is an arc of type III with end points on $D_1$ or $D_2$. So we can start the argument with $\rho'$. However the procedure must terminate as the intersection is finite. Hence at some stage we obtain a disk $\Delta$ with $\partial\Delta - \partial E_j$ consisting of arcs of type I or type III all of which have end points on $D_1$ or $D_2$. Hence all the $\beta$ arcs are on the 2-float. (see Fig. 1)

\[ \square \]

**Corollary 4.3.** The disk $\Delta$ contains at most one arc $\alpha$ of type I on its boundary.

\[ \square \]

Consider now an essential sub-annulus $A'$ of $A$ containing the disks $D_1$ and $D_2$. It is a meridional annulus in $(S^3, K)$ so we can cap off $A'$ by two meridian disks $D_1^*$ and $D_2^*$ in $(S^3, K)$ to obtain a 2-sphere intersecting $K$ in exactly two points in $D_1^*$ and $D_2^*$. If we attach the boundary of the 2-float to this 2-sphere along $D_1$ and $D_2$ we get a 2-torus $T$. By the above lemma $\partial \Delta$ is contained in $T$.

**Case 1:** Assume that $\partial \Delta$ is an inessential curve on $T$ and bounds a disk $\Delta'$ there which does not contain the disks $D_1^*$ and $D_2^*$. The intersection of $\partial \Delta$ with a core curve of the meridional annulus $A'$ is even. Similarly the intersection of $\partial \Delta$ with the boundary of a cocore disk of the 2-float is even. Hence the number of arcs of type I is even and so is the number of $\beta$ arcs (these are the arcs which intersect the boundary of a cocore disk of the 2-float). Hence the number of arcs of type III (the $\alpha$ arcs) is also even. As a consequence the disk $\Delta'$ is a union of bands glued together to each other at their ends. The bands correspond to the areas in $A$ between the arcs of type I and between the arcs of type III and also on the 2-float between the $\beta$ arcs (see Fig. 4).

Since the bands are glued to each other along small arcs on both ends, the number of gluing arcs is equal to the number of bands. So an Euler characteristic argument shows that

\[
\chi(\Delta') = \sum \chi(\text{bands}) - \sum \chi(\text{gluing arcs}) = 0
\]

13
But this is obviously a contradiction and hence \( \partial \Delta \) is essential in \( T \) or bounds a disk \( \Delta' \) containing one or both of \( D_1^* \) and \( D_2^* \).

**Case 2:** If \( \Delta' \) contains only one of \( D_1^* \) or \( D_2^* \) then the 2-sphere \( \Delta' \cup \Delta \) intersects \( K \) in a single point in contradiction. Assume therefore, that \( \Delta' \) contains both of \( D_1^* \) and \( D_2^* \). Consider the annulus \( A'' = \text{cl}(\Delta' \cup \Delta - (D_1^* \cup D_2^*)) \). We can push \( A'' \) by a small ambient isotopy relative to its boundary \( \partial D_1^* \cup \partial D_2^* \) in the direction away from the 2-float so that after the isotopy \( A'' \) does not intersect the disks \( D_1 \) and \( D_2 \). If the sub-annuls \( A - A' \) contains disks of intersection with \( V_1 \) then the annulus \( A'' \) is not parallel into \( A - A' \) and the annulus \( (A - A') \cup A'' \) is a new decomposing annulus intersecting \( V_1 \) in fewer disks than \( A \), in contradiction to the choice of \( A \).

**Case 3:** If \( \partial \Delta \) is an essential curve on \( T \) we do 2-surgery on \( \partial \Delta \) by removing an annulus neighborhood of \( \partial \Delta \) and gluing two copies of \( \Delta \). By an Euler characteristic argument we obtain a 2-sphere intersecting \( K \) in two points on \( D_1^* \) and \( D_2^* \). If we remove \( D_1^* \) and \( D_2^* \) we obtain an annulus \( A'' \). As above can now replace the annulus \( A' \) by the annulus \( A'' \) and get a new annulus \( (A - A') \cup A'' \) which does not intersect the disks \( D_1 \) and \( D_2 \). The annulus \( A'' \) cannot be parallel into \( A' \) as this would imply that \( (A - A') \cup A'' \) is parallel into \( A \) in contradiction to the choice of \( A \) as minimizing the intersection with \( V_1 \). If the sub-annuls \( A - A' \) contains disks of intersection with \( V_1 \) then the annulus \( A'' \) is not parallel into \( A - A' \) and the annulus \( (A - A') \cup A'' \) is a new decomposing annulus intersecting \( V_1 \) in fewer disks than \( A \), in contradiction to the choice of \( A \).
Hence we conclude that the disks $D_1$ and $D_2$ must be the first and last disks in $A \cap V_1$, i.e., the two disks closest to $A_1^*$ and $A_2^*$. This implies that each knot complement $E(K_1)$ and $E(K_2)$ contains at most one component of $V_1 - A$ which is a 2-float of genus zero. Since $V_1$ is connected we cannot have two 2-floats of genus zero one on each side both intersecting $A$ in the disks $D_1$ and $D_2$.

As an immediate application of Theorem 4.1 we have the following bound on the tunnel number of $K = K_1 \# K_2$ in terms of the tunnel numbers of $K_1$ and $K_2$. Note that in [SS] Schultens and Scharlemann have obtained a better bound. However we bring it here as it is an immediate corollary of our main theorem:

By the above theorem at most one component of $V_1 - A$ is a 2-float. Hence for all other components $C_i$ we have $\partial C_i - A = \Sigma_i$ is a surface so that $-\chi(\Sigma_i) \geq 1$, as $\Sigma_i$ is either a punctured 2-sphere with at least three punctures or a surface of positive genus with at least one puncture. Since we also have, by Proposition 3.2, at least one component with positive genus we have as a worst case the following situation:

One component $C_1$ of $V_1 - A$ which intersects the vertical annuli and $\partial C_1 - \partial E(K)$ is an annulus. Since $V_1$ is connected the other component $C_1$ of $V_1 - A$ which intersects the vertical annuli must have at least one other disk of intersection with $A$. Furthermore we have one component which is a 2-float, one component of genus one and all other components are 3-floats of genus zero. For a closed surface of genus $h$ the latter type of components can number at most $2h - 2$ so $k = 2(g - 2) - 2 + 4 = 2g - 2$ and $\sum_{i=1}^{k} g_i - (k - 2) \geq 1 - (2g - 2) + 2 = 3 - 2t(K)$. Hence by Remark 3.4:

**Theorem 4.4.** For all prime knots $K_1$ and $K_2$ in $S^3$ and $K = K_1 \# K_2$ we have:

$$t^*(K_1) + t^*(K_2) \leq 4t(K) - 3$$

The above theorem suggest that by ruling out $n$-floats of genus zero with $n = 3, 4, \ldots$ one can get better bounds which would converge to approximately $t^*(K_1) + t^*(K_2) \leq 2t(K)$. 

15
5 References

[Ja] W. Jaco; Lectures on three manifold topology CBMS Regional Conference Series in Mathematics 43 1977.

[Ko] T. Kobayashi; A construction of arbitrarily high degeneration of tunnel number of knots under connected sum, J. Knot Theory and its Ramifications 3 (1994), 179 - 186 .

[Kw] H.-Z. Kwong; Straightening tori in Heegaard splittings, Ph.D Thesis U.C. Santa Barbara 1994 .

[MR] Y. Moriah, H. Rubinstein; Heegaard structures of negatively curved 3-manifolds Comm. in Anal. and Geom. 5 (1997), 375 - 412 .

[Mo1] K. Morimoto; On the additivity of tunnel number of knots, Topology and its Applications 53 (1993) 37 - 66.

[Mo2] K. Morimoto; There are knots whose tunnel numbers go down under connected sum, Proc. Amer. Math. Soc. 123 (1995) 3527 - 3532.

[Mo3] K. Morimoto; On the super additivity of tunnel number of knots, preprint

[MSY] K. Morimoto, M. Sakuma, Y. Yokota; Examples of tunnel number one knots which have the property that “1 + 1 = 3”, Math. Proc. Camb. Phil. Soc., 119 (1996) 113 - 118 .

[Oc] M. Ochiai; On Haken’s theorem and its extension, Osaka J. of Math. 20 (1983) 461 - 468.

[SS] M. Scharlemann, J. Schultens; Annuli in generalized Heegaard splitting and degeneration of tunnel number; Preprint.
