Phase Space and Quantization of 2D BF Theory Coupled to 1D Quantum Mechanics

Seyed Faroogh Moosavian\textsuperscript{1} and Yehao Zhou\textsuperscript{2,3}

\textsuperscript{1}Department of Physics, McGill University, Montréal QC H3A 2T8, Canada
\textsuperscript{2}Perimeter Institute for Theoretical Physics, Waterloo ON N2L 2Y5, Canada
\textsuperscript{3}University of Waterloo, Waterloo ON N2L 3G1, Canada

Abstract

We study ring of functions on the (classical and quantized) phase space of 2-dimensional BF theory with the gauge group $\text{GL}_N$ coupled to a 1-dimensional quantum mechanics with global symmetry $\text{GL}_K$. These functions are gauge-invariant local observables of the coupled system. We first construct the classical phase space of this system and describe its ring of functions and their large-$N$ limit. We next compute the Hilbert series of these algebras for finite-$N$ and also in the large-$N$ limit. We then study the quantization of this phase space and the deformation quantization of its ring of functions, elaborate its relation to the Yangian, and construct its coproduct. Finally, we identify these quantized algebras with the quantized Coulomb branch algebras of certain 3-dimensional $\mathcal{N} = 4$ quiver gauge theories.

Contents

1 Introduction 2

2 Geometry of the Phase Space $\mathcal{M}(N, K)$ 5

2.1 Generators of $\mathbb{C}[\mathcal{M}(N, K)]$ 5

2.2 Factorization 6

2.3 Singularities and resolution 6

2.4 Poisson structure 8

2.5 Multiplication morphism 8

2.6 Embedding $\mathcal{M}(N, K) \hookrightarrow \mathcal{M}(N', K)$ 9

2.7 $\mathcal{M}(N, K)$ as a Hamiltonian reduction 10

3 Large-$N$ Limit 10

4 Hilbert Series of $\mathbb{C}[\mathcal{M}(N, K)]$ 13

4.1 Reduction steps 14

4.2 Calculation on affine Grassmannian 16

4.3 $N \to \infty$ limit 17

5 Quantization of $\mathcal{M}(N, K)$ 18

5.1 Another map from $Y_\lambda(\mathfrak{gl}_K) \otimes \Lambda$ to $\mathbb{C}_\hbar[\mathcal{M}(N, K)]$ 20

5.2 Kernel of $\rho_N$ 23

5.3 Quantized coproduct 25

5.4 Quantized phase space and Coulomb branch algebra 27

\*sfmoosavian@gmail.com
\textsuperscript{1}yehaozhou1994@gmail.com
1 Introduction

Holography is one of the main active area of research in finding a theory of quantum gravity [1, 2]. The prime example of this concept is the AdS/CFT Correspondence [3, 4].

Recently, a twisted version of the AdS/CFT Correspondence has been formulated by Costello and Li [5, 6]. According to this framework, the holography can be understood as certain algebraic relation, known as Koszul duality, between the algebra of operators in the two sides of the correspondence (see [7] for an earlier example and also [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] for follow-up and related works). For a recent and very readable review of Koszul duality aimed at physicists, we refer the reader to [24]. An instance of this twisted version has been studied in [8], where it was shown that the algebra of local operators in 2d BF theory with gauge group GL_N coupled to a 1d fermionic quantum mechanics with global symmetry GL_K (the boundary side) and the algebra of scattering states computed using Witten diagrams of 4d Chern-Simons theory with gauge group GL_K (the bulk side) match and in the large-N limit approach the Yangian (see [8, Theorem 1]).

In this paper, we study a closely-related system, which is 2d BF theory with gauge group GL_N coupled to a 1d bosonic quantum mechanics with global symmetry GL_K. We will show that the quantized algebra of gauge-invariant local operators in this coupled theory is a certain truncation of the Yangian of g_dK, and it approaches Y_{h}(g_dK) in the N → ∞ limit, see Theorem 1.1 below.

The action functional of this system is given by

\[ S = S_{BF} + S_{QM} = \int_{\mathbb{R}^2} \text{Tr}(BF_A) + \int_{\mathbb{R}^2 \times \{w=0\}} \sum_{a=1}^{K} I_a(\partial_w + A)J_a. \] (1.1)

Here A is the GL_N connection on \( \mathbb{R}^2 \), \( F_A \) is the curvature form associated to A, B is a scalar field on \( \mathbb{R}^2 \) valued in \( \text{End}(\mathbb{C}^N) \), \( J_a \) is a scalar field on \( \mathbb{R}^2 \times \{w=0\} \) valued in \( (\mathbb{C}^N)^* \), and \( I_a \) is a scalar field on \( \mathbb{R}^2 \times \{w=0\} \) valued in \( \mathbb{C}^N \). We would like to analyze the geometry of the phase space of the system (1.1) and its quantization. One of the reasons we are studying the problem from the phase space perspective is that geometric tools allows us to make statements at finite \( N \).

We choose the gauge \( A = 0 \). Then the equations of motion are

\[ \partial_x J_a = \partial_x I_a = 0, \quad \partial_x B = 0, \quad \partial_w B - \sum_{a=1}^{K} J_a I_a \delta_{w=0} = 0. \] (1.2)

The solution is that \( J_a \) and \( I_a \) are constant along the line defect, \( B \) is constant on the regions \( w < 0 \) and \( w > 0 \), and

\[ B_{w>0} - B_{w<0} = \sum_{a=1}^{K} J_a I_a. \]

So the phase space, denoted by \( \mathcal{M}(N, K) \), is parametrized by \( B_{w>0}, J_a \) and \( I_a \), modulo the GL_N action. This is the quiver variety associated to the framed quiver in Figure 1 i.e.

\[ \mathcal{M}(N, K) \cong \text{Rep}(N, K) \sslash \text{GL}_N, \] (1.3)

where \( \text{Rep}(N, K) := \text{End}(\mathbb{C}^N) \oplus \text{Hom}(\mathbb{C}^N, \mathbb{C}^K) \oplus \text{Hom}(\mathbb{C}^K, \mathbb{C}^N) \) is the linear space of representations of quiver in Figure 1. We study the geometry of \( \mathcal{M}(N, K) \) in Section 2.

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1In this paper we consider bosonic quantum mechanics and the resulting operator algebra in the large-N limit is the same as fermionic quantum mechanics. However, the brane configuration which leads to the fermionic vs bosonic quantum mechanics would be different [25, 26].
Figure 1: The quiver description of the fields contents.

Summary of the results

The basic logic of the paper is to first study the classical phase space $\mathcal{M}(N, K)$ and its ring of functions $\mathbb{C}[\mathcal{M}(N, K)]$, and finally their large-$N$ limit. We then study modules for these algebras. Then, we considered the quantization of the classical phase space and the deformation quantization $\mathbb{C}_\hbar[\mathcal{M}(N, K)]$ of its ring of functions, which leads to the algebra. We study its structure, especially its coproduct and its identification with the Coulomb branch algebra of 3d $\mathcal{N} = 4$ theories.

Our main result is the following.

**Theorem 1.1** (Corollary 5.1, Proposition 5.2). There is an isomorphism between Poisson varieties

$$\mathcal{M}(N, K) \cong \text{Coulomb branch associated to the quiver in Figure 2 with flavor symmetry}$$

$$\cong \text{Beilinson-Drinfeld slice } \mathcal{W}_0^\lambda \text{ for } \text{GL}_K,$$

where $\lambda = N\omega_1 + N\omega_{K-1}$, and $\omega_i$ is the $i$-th fundamental coweight for $\text{GL}_K$. There is an isomorphism between $\mathbb{C}[\hbar]$ algebras

$$\mathbb{C}_\hbar[\mathcal{M}(N, K)] \cong \text{quantized Coulomb branch algebra associated to the quiver in Figure 2}$$

with flavor symmetry

$$\cong \text{truncated Yangian } Y_0^\lambda$$

Moreover, there exists a surjective $\mathbb{C}[\hbar]$ algebra homomorphism

$$\rho_N : Y_h(\mathfrak{g}_K) \otimes \Lambda \rightarrow \mathbb{C}_\hbar[\mathcal{M}(N, K)]$$

such that $\bigcap_{N=1}^\infty \ker(\rho_N) = 0$. Here $\Lambda := \mathbb{C}[p_i : i \in \mathbb{Z}_{\geq 1}]$ is the polynomial ring of countably many generators, which is naturally identified with ring of symmetric polynomials with $p_i$ being the $i$-th power sum function.

We emphasize that the isomorphism between truncated Yangian $Y_0^\lambda$ and quantized ring $\mathbb{C}_\hbar[\mathcal{M}(N, K)]$ is explicit, see the proof of Theorem 5.1. We also provide a more conceptual (non-explicit) proof of the isomorphism $Y_0^\lambda \cong \mathbb{C}_\hbar[\mathcal{M}(N, K)]$ using the tool of ring objects in the derived category of constructible sheaves on affine Grassmannian [27], see Remark 5.7.

Structure of the paper

This paper is organized as follows.

In Section 2, we investigate the geometry of the phase space of BF theory coupled to our quantum-mechanical system and study the algebra of functions in this phase space. The main results of this section are the following

1. The first result is concerned with the structure of the phase space; we show that $\mathcal{M}(N, K)$ is a normal affine variety of dimension $2NK$. This is shown in Proposition 2.3
Proposition 4.3. The main results of this section are as follows.

1. We show that $\bigcup_{n \leq 0} M(n,K)$ is Zariski-dense in $L^-GL_K \times L^-GL_1$, where $L^-G$ is the negative loop group associated to an algebraic group $G$ defined in (1.1). This result is the content of Theorem 3.1.

2. Using this result, we then show that $\mathcal{M}(\infty,K) \cong L^-GL_K \times L^-GL_1$. This in turn would imply that

$$\mathbb{C}[\mathcal{M}(\infty,K)] \cong \mathbb{C}[L^-GL_K] \otimes \mathbb{C}[L^-GL_1].$$

(1.7)

Section 3 is devoted to study of Hilbert series of modules for $\mathbb{C}[\mathcal{M}(N,K)]$. The main result of this section is the computation of Hilbert series for $\mathbb{C}[\mathcal{M}(N,K)]$ in Theorem 4.2 and its large-$N$ limit $\mathbb{C}[\mathcal{M}(\infty,K)]$ in Proposition 4.3.

In Section 5 we study the quantization $\mathbb{C}_h[\mathcal{M}(N,K)]$ of the ring of functions $\mathbb{C}[\mathcal{M}(N,K)]$ on the phase space. Quantization amounts to replace the Poisson brackets (1.4) with commutators and studying the resulting algebras. The main results of this section are

1. We first prove the commutator of $T_{ab}^{(n)}$:

$$[T_{ab}^{(p)}, T_{cd}^{(q)}] = \hbar \sum_{i=-1}^{\min(p,q)-1} \left( T_{cb}^{(p+q-1-i)} T_{ad}^{(i)} - T_{cb}^{(p+q-1-i)} T_{ad}^{(i)} \right).$$

(1.8)
This is equivalent to the RTT relation if one defines the generating functions $T_{ab}(z)$ (the RTT generators) of $T^{(n)}_{ab}$ by the following power-series expansion at $z \to \infty$

$$T_{ab}(z) := \sum_{n \geq 1} T^{(n)}_{ab} z^{-n-1} = \delta_{ab} + \frac{1}{z - B} J_b.$$

2. We next construct two different surjective algebra maps from $Y_h(\mathfrak{gl}_K) \otimes \Lambda$ to $\mathbb{C}_h[M(N, K)]$, the first one denoted $\rho_N$ is constructed in Proposition 5.2 and the second one denoted $\tilde{\rho}_N$ is constructed Section 5.1. Then we prove the key technical result Theorem 5.1 which characterizes the kernel of $\rho_N$. The proof of Theorem 5.1 leads to our main result Corollary 5.1.

3. In Section 5.3 we construct the coproducts of $Y_h(\mathfrak{gl}_K) \otimes \Lambda$ and $\mathbb{C}_h[M(N, K)]$, and they are compatible in the sense that the following diagram

$$
\begin{array}{ccc}
Y_h(\mathfrak{gl}_K) \otimes \Lambda & \xrightarrow{\Delta} & (Y_h(\mathfrak{gl}_K) \otimes \Lambda) \otimes (Y_h(\mathfrak{gl}_K) \otimes \Lambda) \\
\rho_{n_1+n_2} \downarrow & & \downarrow \rho_{n_1} \otimes \rho_{n_2} \\
\mathbb{C}_h[M(n_1 + n_2, K)] & \xrightarrow{\rho_{n_1} \otimes \rho_{n_2}} & \mathbb{C}_h[M(n_1, K)] \otimes \mathbb{C}_h[M(n_2, K)]
\end{array}
$$

is commutative.

4. Finally, we explain the identification between the quantized ring of functions on the phase space $\mathbb{C}_h[M(N, K)]$ and the Coulomb branch algebra of certain 3d $\mathcal{N} = 4$ quiver gauge theories. This leads to a conceptual proof of Corollary 5.1 see Remark 5.7.

Some details are relegated to the appendices. The Hall-Littlewood polynomial has been reviewed in Appendix A, Geometrization of the Jing operators, which are used in giving a vertex-algebra definition of the Hall-Littlewood polynomials is explained in Appendix B.

## 2 Geometry of the Phase Space $\mathcal{M}(N, K)$

### 2.1 Generators of $\mathbb{C}[\mathcal{M}(N, K)]$

By invariant theory, the algebra of functions on $\mathcal{M}(N, K)$, denoted by $\mathbb{C}[\mathcal{M}(N, K)]$, is generated by

$$\text{Tr}(B^n) = B_{i_1 i_2} B_{i_2 i_3} \cdots B_{i_n i_1}, \quad \text{and} \quad I_a B^m J_b = I_{a i_1} B_{i_1 i_2} \cdots B_{i_m i_{m+1}} J_{i_{m+1} b}.$$

(2.1)

where $n \geq 1$, $m \geq 0$, and $1 \leq a, b \leq K$. In fact, the trace relations 2 guarantee that the subset $\{\text{Tr}(B^n), I_a B^m J_b : 1 \leq n \leq N, 0 \leq m \leq N - 1, 1 \leq a, b \leq K\}$ of generators (2.1) is enough to generates $\mathbb{C}[\mathcal{M}(N, K)]$.

When $K = 1$, it turns out $\{\text{Tr}(B^n), I B^m J : 1 \leq n \leq N, 0 \leq m \leq N - 1\}$ freely generates $\mathbb{C}[\mathcal{M}(N, 1)]$ so that we have $\mathcal{M}(N, 1) \cong \mathbb{A}^{2N}$. In fact, we will see shortly in Proposition 2.3 that $\dim \mathcal{M}(N, 1) = 2N$. Since the map $\mathcal{M}(N, 1) \to \mathbb{A}^{2N}$ induced by the generators $\{\text{Tr}(B^n), I B^m J : 1 \leq n \leq N, 0 \leq m \leq N - 1\}$ is closed embedding, this map must be an isomorphism by dimensional reason.

For general $K$, let us fix a pair of integers $a, b$, then the functions $\text{Tr}(B^n), I_a B^m J_b$ give rise to a morphism $\eta_{ab} : \mathcal{M}(N, K) \to \mathcal{M}(1, K)$ sending a triple $(B, J, I)$ to $(B, J_b, I_a)$. From the above discussions, we have the following result.

---

2The identity $\det(\text{Id} - t \cdot B) = \exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n!} \text{Tr}(B^n) \right)$ implies that the right-hand-side is a polynomial of degree $N$, thus $\text{Tr}(B^n)$ ($r > N$) can be expressed as polynomials of $\{\text{Tr}(B^n) : 1 \leq n \leq N\}$. The Cayley-Hamilton identity $\det(x \cdot \text{Id} - B)|_{x=\rho} = 0$ shows that $B^s$ ($s \geq N$) can be expressed as polynomials of $\{B^n : 0 \leq m \leq N - 1\}$ and $\{\text{Tr}(B^n) : 1 \leq n \leq N\}$. 

---
Proposition 2.1. The product of $\eta_{ab}$ is a closed embedding

$$\eta := \prod_{1 \leq a, b \leq K} \eta_{ab} : \mathcal{M}(N, K) \hookrightarrow \mathcal{M}(N, 1) \times_{\mathbb{A}(N)} \cdots \times_{\mathbb{A}(N)} \mathcal{M}(N, 1),$$

where the right hand side has $K^2$ copies of $\mathcal{M}(N, 1)$.

2.2 Factorization

There is an obvious morphism:

$$f_{N_1, N_2} : \mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K) \rightarrow \mathcal{M}(N_1 + N_2, K),$$

$$(B^{(1)}, J^{(1)}, I^{(1)}) \times (B^{(2)}, J^{(2)}, I^{(2)}) \mapsto \begin{pmatrix} [B^{(1)}] & 0 & [J^{(1)}] & [I^{(1)}] \\ 0 & B^{(2)} & J^{(2)} & I^{(2)} \end{pmatrix}.$$

(2.4)

Consider the natural projection

$$\Phi_N : \mathcal{M}(N, K) \rightarrow \mathbb{A}^N.$$

(2.5)

Here $\Phi_N$ maps a triple $(B, J, I)$ to the coefficients of the characteristic polynomial of $B$, and $\mathbb{A}^N$ is the $N$-th symmetric product of affine line $\mathbb{A}^1$, which parametrizes coefficients of the characteristic polynomial of $B$. Denote by $(\mathbb{A}^{N_1} \times \mathbb{A}^{N_2})_{\text{disj}}$ the open subset of $\mathbb{A}^{N_1} \times \mathbb{A}^{N_2}$ such that eigenvalues of $B^{(1)}$ is disjoint from eigenvalues of $B^{(2)}$. Analogous to the $K = 1$ case discussed in [28], we have the following factorization isomorphism

Proposition 2.2. The restriction of $f_{N_1, N_2}$ on $(\mathbb{A}^{N_1} \times \mathbb{A}^{N_2})_{\text{disj}}$ is isomorphism:

$$f_{N_1, N_2} : (\mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K))_{\text{disj}} \cong \mathcal{M}(N_1 + N_2, K) \times_{\mathbb{A}^{N_1 + N_2}} (\mathbb{A}^{N_1} \times \mathbb{A}^{N_2})_{\text{disj}}.$$

Here $(\mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K))_{\text{disj}}$ is the restriction of $\mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K)$ on $(\mathbb{A}^{N_1} \times \mathbb{A}^{N_2})_{\text{disj}}$.

Remark 2.1. The factorization isomorphism $f_{N_1, N_2}$ is compatible with the embedding $\eta$ in (2.2).

Proposition 2.3. $\mathcal{M}(N, K)$ is a normal affine variety of dimension $2NK$.

Proof. $\mathcal{M}(N, K)$ is normal and affine since is the quotient of an affine space by $\text{GL}_N$, we only need to show that its dimension is $2NK$. By the factorization isomorphism, it suffices to show that dim $\mathcal{M}(1, K) = 2K$. Note that $\mathcal{M}(1, K)$ is isomorphic to the $\mathbb{A}^1$ times the variety of $K \times K$ matrices with rank $\leq 1$, and it is known that the latter has dimension $2K - 1$ [29].

Proposition 2.4. The morphism $\Phi_N : \mathcal{M}(N, K) \rightarrow \mathbb{A}^N$ is flat.

Proof. Since the map $\text{gl}_N \rightarrow \mathbb{A}^N$ sending a matrix $B$ to the coefficients of the characteristic polynomial of $B$ is flat [30 §3.2], Therefore the morphism $\text{Rep}(N, K) \rightarrow \mathbb{A}^N$ sending $(B, J, I)$ to the coefficients of the characteristic polynomial of $B$ is flat. Since $\text{GL}_N$ is reductive, the $\text{GL}_N$-invariant subspace $\mathbb{C}[\mathcal{M}(N, K)]$ is a direct summand of $\mathbb{C}[	ext{Rep}(N, K)]$ as $\mathbb{C}[\mathbb{A}^N]$-module. Thus $\Phi_N : \mathcal{M}(N, K) \rightarrow \mathbb{A}^N$ is flat.

2.3 Singularities and resolution

Let us pick a character $\zeta : \text{GL}_N \rightarrow \mathbb{C}^\times$ such that $\zeta(g) = \det(g)$. According to [31] Proposition 3.1, a point in $\text{Rep}(N, K)$ is $\zeta$-semistable if and only if the following condition is satisfied:

$$\text{if } S \subseteq \mathbb{C}^N \text{ is a linear subspace such that } B(S) \subseteq S, \text{ and } \text{im}(J) \subseteq S, \text{ then } S = \mathbb{C}^N.$$
It is straightforward to see that a point in \( \text{Rep}(N, K) \) is \( \zeta \)-semistable if and only if it is \( \zeta \)-stable. The \( \zeta \)-stable locus in \( \text{Rep}(N, K) \) is denoted \( \text{Rep}(N, K)^{\zeta-ss} \). It is easy to see that \( \text{Rep}(N, K)^{\zeta-ss} \) is nonempty. Define the GIT quotient
\[
\mathcal{M}^\zeta(N, K) := \text{Rep}(N, K) \sslash \zeta \text{ GL}_N = \text{Rep}(N, K)^{\zeta-ss} \sslash \text{ GL}_N.
\] (2.6)

Then \( \mathcal{M}^\zeta(N, K) \) is a smooth variety of dimension \( 2NK \), since \( \text{ GL}_N \) acts on \( \text{Rep}(N, K)^{\zeta-ss} \) freely. According to [31] §2 there is a natural projective morphism
\[
f : \mathcal{M}^\zeta(N, K) \longrightarrow \mathcal{M}(N, K).
\] (2.7)

**Proposition 2.5.** The projective morphism \( f : \mathcal{M}^\zeta(N, K) \rightarrow \mathcal{M}(N, K) \) is a resolution of singularities. Moreover, we have \( \mathcal{O}_{\mathcal{M}(N,K)} \cong Rf_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)}, \) i.e.
\begin{enumerate}
  \item \( R^i f_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)} = 0 \) for \( i > 0 \),
  \item the natural homomorphism \( \mathcal{O}_{\mathcal{M}(N,K)} \rightarrow f_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)} \) is an isomorphism.
\end{enumerate}

**Proof.** Let us prove the second statement first. Since \( \mathcal{M}(N, K) \) is affine, it suffices to show that
\[
H^i(\mathcal{M}^\zeta(N, K), \mathcal{O}_{\mathcal{M}^\zeta(N,K)}) = \begin{cases} 
  \mathbb{C}[\mathcal{M}(N, K)] & \text{if } i = 0, \\
  0 & \text{if } i > 0.
\end{cases}
\] (2.8)

Then (2.8) is a special case of [32] Theorem 3.29] by taking \( F^* = G^* = \mathcal{O}_{\text{Rep}(N,K)/\text{GL}_N} \).

To show that \( f : \mathcal{M}^\zeta(N, K) \rightarrow \mathcal{M}(N, K) \) is a resolution of singularities, it is enough to show that \( f \) is birational, i.e. there exists an open subscheme \( U \subset \mathcal{M}(N, K) \) such that \( f \) induces an isomorphism \( f^{-1}(U) \cong U \). As we have shown that the natural homomorphism \( \mathcal{O}_{\mathcal{M}(N,K)} \rightarrow f_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)} \) is an isomorphism, \( f \) must be a dominant morphism, i.e. the image of \( f \) is dense in \( \mathcal{M}(N, K) \). Since \( f \) is proper, it follows that \( f \) is surjective. Then there exists an open subscheme \( U \subset \mathcal{M}(N, K) \) such that \( f^{-1}(U) \rightarrow U \) is finite, because \( \dim \mathcal{M}^\zeta(N, K) = 2NK = \dim \mathcal{M}(N, K) \). The isomorphism \( \mathcal{O}_{\mathcal{M}(N,K)} \cong f_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)} \) implies that \( f^{-1}(U) \rightarrow U \) is an isomorphism. \( \square \)

Recall that an algebraic variety \( X \) is said to have rational singularities if for every resolution of singularities \( \pi : Y \rightarrow X \) we have \( \mathcal{O}_X \cong R\pi_* \mathcal{O}_Y \). An algebraic variety \( X \) is said to have Gorenstein singularities if the dualizing complex \( \omega_X \) is a locally free sheaf.

**Corollary 2.1.** \( \mathcal{M}(N, K) \) has rational and Gorenstein singularities.

**Proof.** By [33] Theorem 5.10], an algebraic variety \( X \) has rational singularities if and only if there exists a resolution of singularities \( \pi : Y \rightarrow X \) such that \( \mathcal{O}_X \cong R\pi_* \mathcal{O}_Y \). In particular, \( \mathcal{M}(N, K) \) has rational singularities by Proposition 2.5. By Lemma 2.1 below, the canonical line bundle on \( \mathcal{M}^\zeta(N, K) \) is trivial, i.e. \( \mathcal{K}_{\mathcal{M}^\zeta(N,K)} \cong \mathcal{O}_{\mathcal{M}^\zeta(N,K)} \). Then the dualizing sheaf \( \omega_{\mathcal{M}(N,K)} \) is computed by
\[
\omega_{\mathcal{M}(N,K)} \cong Rf_* \mathcal{K}_{\mathcal{M}^\zeta(N,K)} \cong Rf_* \mathcal{O}_{\mathcal{M}^\zeta(N,K)} \cong \mathcal{O}_{\mathcal{M}(N,K)},
\]
which is a line bundle. Thus \( \mathcal{M}(N, K) \) has Gorenstein singularities. \( \square \)

**Lemma 2.1.** The canonical line bundle on \( \mathcal{M}^\zeta(N, K) \) is trivial.

**Proof.** Denote by \( \mathcal{V} \) the tautological sheaf on \( \mathcal{M}^\zeta(N, K) \), which is the descent of \( \mathcal{C}^N \) along the quotient \( \text{Rep}(N, K)^{\zeta-ss} \rightarrow \mathcal{M}^\zeta(N, K) \), and denote by \( W \) the framing vector space, then there is a short exact sequence
\[
0 \longrightarrow \text{End}(\mathcal{V}) \longrightarrow \text{End}(\mathcal{V}) \oplus W \otimes \mathcal{V}^* \oplus W^* \otimes \mathcal{V} \longrightarrow T_{\mathcal{M}^\zeta(N,K)} \longrightarrow 0.
\] (2.9)

Here \( T_{\mathcal{M}^\zeta(N,K)} \) is the tangent sheaf of \( \mathcal{M}^\zeta(N, K) \). From this short exact sequence we get
\[
\mathcal{K}_{\mathcal{M}^\zeta(N,K)} = \text{det} \ T_{\mathcal{M}^\zeta(N,K)} \cong \text{det}(W \otimes \mathcal{V}^*) \otimes \text{det}(W^* \otimes \mathcal{V}) \cong \mathcal{O}_{\mathcal{M}^\zeta(N,K)}.
\] \( \square \)
2.4 Poisson structure

Let us introduce a Poisson structure on the space of $(B, J, I)$ as following

$$\{J_{ia}, I_{bj}\} = \delta_{ab}\delta_{ij}, \quad \{B_{mn}, B_{pq}\} = \delta_{pn}B_{mq} - \delta_{mq}B_{pn}, \quad \{B_{mn}, I_{bj}\} = \{B_{mn}, J_{ia}\} = 0. \tag{2.10}$$

Here we treat $J, I$ as usual bosonic variables, i.e. commute instead of anti-commute with each other. This Poisson structure comes from the classical limit of $U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_{NK}^{\otimes N}$, where $\text{Weyl}_{NK}^{\otimes N}$ is the Weyl algebra generated by $J, I$. It is easy to see that the Poisson structure is equivariant under the $\text{GL}_N$ action, so it descends to $\mathcal{M}(N, K)$.

Remark 2.2. This is not the Poisson structure for the Zastava space. In fact, when $K = 1$, this Poisson structure on $\mathbb{C}[\mathcal{M}(N, 1)]$ is trivial, see the Theorem 2.6 below.

Define $T_{ab}^{(n)} = I_{a} B^{n} J_{b}$, and we use the convention $T_{ab}^{(-1)} = \delta_{ab}$, then denote by $T_{ab}(z)$ the power series expanded at $z \to \infty$:

$$T_{ab}(z) = \sum_{n \geq 1} T_{ab}^{(n)} z^{-n-1} = \delta_{ab} + I_{a} \frac{1}{z - B} J_{b}. \tag{2.11}$$

Proposition 2.6. The Poisson brackets between $T_{ab}^{(k)}$ are:

$$\{T_{ab}^{(p)}, T_{cd}^{(q)}\} = \sum_{i = 1}^{\min(p, q) - 1} \left( T_{cb}^{(p+q-1-i)} T_{ad}^{(i)} - T_{cb}^{(i)} T_{ad}^{(p+q-1-i)} \right). \tag{2.12}$$

And for all $n \geq 1$, $\text{Tr}(B^n)$ is Poisson central.

Proof. This is the classical limit of (5.3), which will be proven independently. \hfill \Box

Remark 2.3. In [8], another presentation of Poisson structure is obtained:

$$\{J_{ia}, I_{bj}\} = \delta_{ab}\delta_{ij}, \quad \{B_{mn}, B_{pq}\} = \delta_{np} \sum_{a} I_{aq} J_{ma} - \delta_{mq} \sum_{a} I_{an} J_{pa},$$

$$\{B_{mn}, I_{bj}\} = \{B_{mn}, J_{ia}\} = 0, \tag{2.13}$$

2.5 Multiplication morphism

Apart from the obvious factorization map (2.2), there is another map

$$\mathfrak{m}_{N_1, N_2} : \mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K) \longrightarrow \mathcal{M}(N_1 + N_2, K), \tag{2.14}$$

$$(B^{(1)}, J^{(1)}, I^{(1)}) \times (B^{(2)}, J^{(2)}, I^{(2)}) \mapsto \begin{pmatrix} B^{(1)} & J^{(1)} I^{(2)} \\ 0 & B^{(2)} \end{pmatrix}, \begin{pmatrix} J^{(1)} \\ J^{(2)} \end{pmatrix}, \begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix}. \tag{2.15}$$

We have the following elementary property of the multiplication morphism.

Proposition 2.7. The multiplication morphism $\mathfrak{m}_{N_1, N_2}$ is dominant.

Proof. It suffices to prove that the composition $\mathfrak{m}_{N_1, N_2}^{-1} \mathfrak{m}_{N_1, N_2}$ is dominant when restricted on $(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)})_{\text{disj}}$.

First of all, we construct a $\text{GL}_{N_1} \times \text{GL}_{N_2}$ equivariant map

$$\tilde{\mathfrak{m}}_{N_1, N_2} : (\text{Rep}(N_1, K) \times \text{Rep}(N_2, K))_{\text{disj}} \longrightarrow (\text{Rep}(N_1, K) \times \text{Rep}(N_2, K))_{\text{disj}}, \tag{2.16}$$

such that $\tilde{\mathfrak{m}}_{N_1, N_2}$ descends to $\mathfrak{m}_{N_1, N_2}^{-1} \mathfrak{m}_{N_1, N_2}$ after taking the quotient by $\text{GL}_{N_1} \times \text{GL}_{N_2}$. The construction is as follows. If the spectra of $B_1$ and $B_2$ are disjoint from each other, then linear map $\text{Mat}(N_1, N_2) \to \text{Mat}(N_1, N_2), X \mapsto B_1 X - X B_2$ is an isomorphism. Let $A$ be the unique $N_1 \times N_2$ matrix such that

$$B^{(1)} A - A B^{(2)} = J^{(1)} I^{(2)} \tag{2.17}$$

holds.
Then we can use the matrix
\[
\begin{bmatrix}
  \text{Id} & A \\
  0 & \text{Id}
\end{bmatrix}
\]
to diagonalize
\[
\begin{bmatrix}
  B^{(1)} & J^{(1)}I^{(2)} \\
  0 & B^{(2)}
\end{bmatrix}
\]
and it accordingly maps \([I^{(1)}, I^{(2)}]\) to \([I^{(1)}, I^{(2)} - I^{(1)} A]\) and \([J^{(1)}, J^{(2)}]^t\) to \([J^{(1)} + AJ^{(2)}, J^{(2)}]^t\). Hence we define \(\tilde{m}_{N_1, N_2}\) as
\[
(B^{(1)}, J^{(1)}, I^{(1)}) \times (B^{(2)}, J^{(2)}, I^{(2)}) \mapsto (B^{(1)}, J^{(1)} + AJ^{(2)}, I^{(1)}) \times (B^{(2)}, J^{(2)}, I^{(2)} - I^{(1)} A).
\]
(2.15)
Notice that the tangent map \(d\tilde{m}_{N_1, N_2}\) is an isomorphism at any point \((B^{(1)}, 0, I^{(1)}) \times (B^{(2)}, J^{(2)}, 0)\), so \(\tilde{m}_{N_1, N_2}\) is generically étale thus it is dominant. Then it follows that \(m_{N_1, N_2}\) is dominant. □

**Proposition 2.8.** The multiplication morphism \(m_{N_1, N_2}\) has following properties

1. \(m_{N_1, N_2}\) is Poisson,

2. \(m_{N_1+N_2, N_3} \circ (m_{N_1, N_2} \times \text{Id}) = m_{N_1, N_2+N_3} \circ (\text{Id} \times m_{N_2, N_3})\), i.e. multiplication is associative.

The proposition will be evident once we make connection to the multiplication map on the loop group in the next section. Note that the factorization map \(f_{N_1, N_2}\) is not Poisson in general.

### 2.6 Embedding \(\mathcal{M}(N, K) \hookrightarrow \mathcal{M}(N', K)\)

Suppose that \(N < N'\), then we have a morphism
\[
\iota_{N,N'} : \mathcal{M}(N, K) \longrightarrow \mathcal{M}(N', K),
\]
(2.16)
(\(B, J, I\) \mapsto \([B \ 0], [J \ 0], [I \ 0]\)).
\[
\iota_{N,N'}^*(\text{Tr}(B^n)) = \text{Tr}(B^n), \quad \iota_{N,N'}^*(T^{(m)}_{ab}) = T^{(m)}_{ab},
\]
(2.17)
so \(\iota_{N,N'}^*\) is surjective, thus \(\iota_{N,N'}\) is a closed embedding.

**Proposition 2.9.** The embedding \(\iota_{N,N'}\) has following properties

1. \(\iota_{N',N''} \circ \iota_{N,N'} = \iota_{N,N''}\),

2. \(\iota_{N,N'}\) is Poisson,

3. \(m_{N_1', N_2'} \circ (\iota_{N_1, N_1'} \times \iota_{N_2, N_2'}) = m_{N_1+N_2, N_1'+N_2'} \circ m_{N_1, N_2}\).

**Proof.** Property (1) is obvious from definition of \(\iota_{N,N'}\), (2) is a corollary of Proposition 2.6 only (3) needs explanation. Using (property 1), the proof of (3) reduces to the cases of either \(N_1' = N_1, N_2' = N_2 + 1\) or \(N_1' = N_1 + 1, N_2' = N_2\). The first case is obvious from the definition of embedding and multiplication morphism, so we only need to consider the case when \(N_1' = N_1 + 1, N_2' = N_2\). It amounts to showing that
\[
\begin{bmatrix}
  B^{(1)} & 0 & J^{(1)}I^{(2)} \\
  0 & 0 & 0 \\
  0 & 0 & B^{(2)}
\end{bmatrix}
\]
is equivalent to
\[
\begin{bmatrix}
  B^{(1)} & J^{(1)}I^{(2)} & 0 \\
  0 & B^{(2)} & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]
under the action of some matrix \(W \in \text{GL}_{N_1+N_2+1}\). It is elementary to check that
\[
W = \begin{bmatrix}
  \text{Id}_{N_1} & 0 \\
  0 & w_{N_2}w_{N_2-1} \cdots w_1
\end{bmatrix}
\]
does the job, where \(w_i \in \text{GL}_{N_2+1}\) switches row \(i\) and row \(i+1\). □
2.7 \( \mathcal{M}(N, K) \) as a Hamiltonian reduction

The phase space \( \mathcal{M}(N, K) \) can be equivalently described by a Hamiltonian reduction. Namely, we consider an enlarged space of representations

\[
\tilde{\text{Rep}}(N, K) := \text{Rep}(N, K) \oplus \text{End}(\mathbb{C}^N),
\]

where the \( N \times N \) matrices in the newly-added component \( \text{End}(\mathbb{C}^N) \) will be denoted \( \tilde{B} \). \( \text{End}(\mathbb{C}^N) \) is endowed with the Poisson structure

\[
\{ \tilde{B}_{mn}, \tilde{B}_{pq} \} = \delta_{pn} \tilde{B}_{mq} - \delta_{mq} \tilde{B}_{pn}.
\]

The action of gauge group \( \text{GL}_N \) on \( \tilde{\text{Rep}}(N, K) \) is Hamiltonian with the moment map

\[
\mu : \tilde{\text{Rep}}(N, K) \rightarrow \mathfrak{gl}_N^*, \quad \mu(B, J, I, \tilde{B}) = B + \tilde{B} - JI.
\]

We note that the equation \( \mu = 0 \) solves \( \tilde{B} \) in terms on \( (B, J, I) \), so \( \text{Rep}(N, K) \cong \mu^{-1}(0) \). Therefore, \( \mathcal{M}(N, K) \) is isomorphic to the Hamiltonian reduction \( \mu^{-1}(0) \sslash \text{GL}_N \).

3 Large-\( N \) Limit

In this section we use the embeddings \( \iota_{N,N'} : \mathcal{M}(N, K) \hookrightarrow \mathcal{M}(N', K) \) constructed in the previous section to define the large-\( N \) limit of the family \( \mathcal{M}(N, K) \) as the spectrum of \( \mathbb{C}^\times \)-finite elements in the inverse limit of algebras \( \mathbb{C}[\mathcal{M}(N, K)] \), and show that the large-\( N \) limit is isomorphic to the Poisson group \( L^-(\text{GL}_K \times \text{GL}_1) \), defined below. It is known that \( L^-(\text{GL}_K \times \text{GL}_1) \) quantizes to the Yangian \( Y_h(\mathfrak{gl}_K) \otimes \Lambda \), and we will explore the quantized version of the large-\( N \) limit in the next section.

**Definition 3.1.** Define \( \mathbb{C}[\mathcal{M}(\infty, K)] \) to be the subalgebra of \( \lim_{N \rightarrow \infty} \mathbb{C}[\mathcal{M}(N, K)] \) generated by \( \{ T^{(n)}_{ab}, \text{Tr}(B^m) : n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{>1}, 1 \leq a, b \leq K \} \). Then define \( \mathcal{M}(\infty, K) := \text{Spec} \mathbb{C}[\mathcal{M}(\infty, K)] \).

Define \( L^-\text{GL}_K \) to be the group of power series

\[
\text{Id} + \sum_{i=1}^{\infty} g_i z^{-i}, \quad g_i \in \mathfrak{gl}_K.
\]

Here the group structure on \( L^-\text{GL}_K \) is the multiplication of power series in matrices. \( L^-\text{GL}_K \) is endowed with scheme structure of an infinite dimensional affine space. Consider the morphism

\[
i_N = (\pi_N, \varphi_N) : \mathcal{M}(N, K) \rightarrow L^-\text{GL}_K \times L^-\text{GL}_1, \quad (B, J, I) \mapsto \left( \text{Id} + I \frac{1}{z-B} J, \frac{1}{z^N \det(z-B)} \right)
\]

which is a closed embedding because \( T^{(n)}_{ab} \) and \( \text{Tr}(B^m) \) generate \( \mathbb{C}[\mathcal{M}(N, K)] \). Here \( (z-B)^{-1} \) is expanded as a power series of matrices in \( z^{-1} \). It is known that \( L^-\text{GL}_K \) is a Poisson-Lie group scheme whose Poisson structure comes from the Manin triple [34, §2.3]

\[
(\mathfrak{gl}_K(\langle z^{-1} \rangle), z^{-1} \mathfrak{gl}_K(\langle z^{-1} \rangle), \mathfrak{gl}_K(z)).
\]

Explicitly, let \( T^{(n)}_{ab}, n \geq -1 \) be the function on \( L^-\text{GL}_K \) that takes the value of \( ab \) component of \( g_{n+1} \) and we use the convention that \( T^{(-1)}_{ab} = \delta_{ab} \), then the Poisson structure on \( L^-\text{GL}_K \) is determined by the equation

\[
(u - v)\{T_{ab}(u), T_{cd}(v)\} = T_{ad}(v)T_{cb}(u) - T_{ad}(u)T_{cb}(v), \quad \text{where } T_{ab}(u) = \sum_{i=-1}^{\infty} T_{ab}^{(i)} u^{-i-1}.
\]

Compare equation (3.4) with equation (2.11), and we have
Proposition 3.1. The morphism \( i_N : \mathcal{M}(N, K) \rightarrow L^\infty GL_K \times L^\infty GL_1 \) is Poisson.

Proposition 3.2. \( i_N \) is compatible with embedding \( \iota_{N,N'} \) and multiplication \( m_{N_1, N_2} \), i.e.

\[
\begin{align*}
(1) \quad i_{N'} \circ \iota_{N,N'} &= i_N, \\
(2) \quad i_{N_1+N_2} \circ m_{N_1,N_2} &= m \circ (i_{N_1} \times i_{N_2}).
\end{align*}
\]

Here \( m : L^\infty (GL_K \times GL_1) \times L^\infty (GL_K \times GL_1) \rightarrow L^\infty (GL_K \times GL_1) \) is the multiplication map of the group \( L^\infty (GL_K \times GL_1) \).

Proof. (1) is obvious from definition. (2) can be shown by direct computation. If \((B^{(1)}, J^{(1)}, I^{(1)})\) is a point in \( \mathcal{M}(N_1, K) \) and \((B^{(2)}, J^{(2)}, I^{(2)})\) is a point in \( \mathcal{M}(N_2, K) \), then \( \pi_{N_1+N_2} \circ m_{N_1,N_2} \) maps this pair of representations to

\[
\begin{align*}
\text{Id} + [I^{(1)} \quad I^{(2)}] \left( z - \begin{bmatrix} B^{(1)} & J^{(1)} I^{(2)} \\ 0 & B^{(2)} \end{bmatrix} \right)^{-1} \begin{bmatrix} J^{(1)} \\ J^{(2)} \end{bmatrix} \\
= \text{Id} + [I^{(1)} \quad I^{(2)}] \left( z - \begin{bmatrix} B^{(1)} & J^{(1)} I^{(2)} \\ 0 & B^{(2)} \end{bmatrix} \right)^{-1} \begin{bmatrix} J^{(1)} \\ J^{(2)} \end{bmatrix} \\
= \text{Id} + I^{(1)} \frac{1}{z - B^{(1)}} J^{(1)} + I^{(2)} \frac{1}{z - B^{(2)}} J^{(2)} \\
+ \sum_{i,j=0}^{\infty} I^{(1)} \left( B^{(1)} \right)^i J^{(1)} I^{(2)} \left( B^{(2)} \right)^j J^{(2)} z^{-i-j-2} \\
= \left( \text{Id} + I^{(1)} \frac{1}{z - B^{(1)}} J^{(1)} \right) \left( 1 + I^{(2)} \frac{1}{z - B^{(2)}} J^{(2)} \right).
\end{align*}
\]

And we also have \( \varphi_{N_1+N_2} \circ m_{N_1,N_2} = m \circ (\varphi_{N_1} \times \varphi_{N_2}) \) since determinant of a block diagonal matrix is the product of determinants of each block.

Proof of Proposition 2.8. (1) follows from Proposition 3.1 and the fact that the Poisson structure on \( L^\infty (GL_K \times GL_1) \) makes it a Poisson-Lie group, i.e. \( m \) is Poisson. (2) is a direct consequence of Proposition 3.2.

Since \( i_N \) is compatible with \( \iota_{N,N'} \) and the generators of \( \mathbb{C}[L^\infty GL_K \times L^\infty GL_1] \) are mapped to polynomials in \( \{ T_{ab}^{(n)} \text{, } \text{Tr}(B^m) : n, m \in \mathbb{Z}_{\geq 0}, 1 \leq a, b \leq K \} \), we obtain a morphism

\[
i_\infty : \mathcal{M}(\infty, K) \rightarrow L^\infty GL_K \times L^\infty GL_1. \tag{3.5}\]

Theorem 3.1. \( \bigcup_N \mathcal{M}(N, K) \) is Zariski-dense in \( L^\infty GL_K \times L^\infty GL_1 \).

Proof. Define \( L^\infty GL_K \) to be the closed subscheme of \( L^\infty GL_K \) consisting of Laurent polynomials of the form

\[
\text{Id} + \sum_{i=1}^{N} g_i z^{-i}, \quad g_i \in \mathfrak{gl}_K. \tag{3.6}
\]

It is easy to see that \( \bigcup_N L^\infty GL_K \times L^\infty GL_1 \) is Zariski-dense in \( L^\infty GL_K \times L^\infty GL_1 \). To prove the theorem, it suffices to show that for every \( N \), there exists \( N' \) such that \( L^\infty GL_K \times L^\infty GL_1 \) is a closed subscheme of \( \mathcal{M}(N', K) \). Denote by \( m = m_{L^\infty GL_K} \times m_{L^\infty GL_1} \) the multiplication map on \( L^\infty GL_K \times L^\infty GL_1 \). We make two observations.
Corollary 3.1. $\mathcal{M}(\infty, K) \cong L^- GL_K \times L^- GL_1$, i.e.

$$\mathbb{C}[\mathcal{M}(\infty, K)] \cong \mathbb{C}[L^- GL_K] \otimes \mathbb{C}[L^- GL_1].$$

Proof. The algebra map $\mathbb{C}[L^- GL_K] \otimes \mathbb{C}[L^- GL_1] \to \mathbb{C}[\mathcal{M}(\infty, K)]$ is surjective because all the generators $\{T_{ab}^{(n)}, \text{Tr}(B_m^n) : n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq 1}, 1 \leq a, b \leq K\}$ are in the image. This map is also injective because the algebra map $\mathbb{C}[L^- GL_K] \otimes \mathbb{C}[L^- GL_1] \to \varinjlim N \mathbb{C}[\mathcal{M}(N, K)]$ is injective as $\bigcup N \mathcal{M}(N, K)$ is Zariski-dense in $L^- GL_K \times L^- GL_1$. 

4 Hilbert Series of $\mathbb{C}[\mathcal{M}(N,K)]$

Recall that we have a resolution of singularities $f : \mathcal{M}^\zeta(N,K) \to \mathcal{M}(N,K)$ by Proposition 2.5, where $\mathcal{M}^\zeta(N,K)$ is the moduli space of $\zeta$-stable representations of the quiver in the Figure 1. The action of gauge group $\text{GL}_N$ on the space of $\zeta$-stable representations $\text{Rep}(N,K)^{\zeta-\text{stable}}$ is free, so the quotient map $\text{Rep}(N,K)^{\zeta-\text{stable}} \to \mathcal{M}^\zeta(N,K)$ is a principal $\text{GL}_N$-bundle. The gauge node vector space $\mathbb{C}^N$ is a trivial bundle on $\text{Rep}(N,K)$ which is endowed with a non-trivial equivariant structure under the action of $\text{GL}_N$, then it descend to a locally free sheaf $\mathcal{V}$ on $\mathcal{M}^\zeta(N,K)$ since the $\text{GL}_N$ action on the stable locus is free. $\mathcal{V}$ is called the \textit{tautological sheaf}, and its determinant line bundle is called the \textit{tautological line bundle} which will be denoted $\text{Det}$.

By the geometric invariant theory, $\mathcal{M}^\zeta(N,K)$ can be presented as a projective spectrum [31 §2]

$$\mathcal{M}^\zeta(N,K) = \text{Proj} \left( \bigoplus_{m=0}^{\infty} \mathbb{C}[\text{Rep}(N,K)]^{\text{GL}_N,\chi^n} \right),$$

where $\mathbb{C}[\text{Rep}(N,K)]^{\text{GL}_N,\chi^n}$ is the subspace of $\mathbb{C}[\text{Rep}(N,K)]$ on which $\text{GL}_N$ acts via the character $\chi^n$. Then $\text{Det}$ is identified with $O(1)$ of the above projective spectrum. In particular $\text{Det}$ is an ample line bundle on $\mathcal{M}^\zeta(N,K)$.

Recall the following Grauert-Riemenschneider vanishing theorem.

\textbf{Theorem 4.1} ([33 Corollary 2.68]). Let $h : X \to Y$ be a resolution of singularities in characteristic zero, and let $\mathcal{L}$ be an ample line bundle on $X$, then $R^i h_*(K_X \otimes \mathcal{L}) = 0$ for $i > 0$. Here $K_X$ is the canonical line bundle of $X$.

Applying the above theorem to the resolution of singularities $f : \mathcal{M}^\zeta(N,K) \to \mathcal{M}(N,K)$ and the ample line bundle $\text{Det}$, we get the following cohomology vanishing result.

\textbf{Lemma 4.1.}

$$H^i(\mathcal{M}^\zeta(N,K), \text{Det}^{\otimes n}) = 0, \text{ for all } i > 0 \text{ and } n \geq 0. \quad (4.1)$$

\textit{Proof.} By Lemma 2.1, $K_{\mathcal{M}^\zeta(N,K)} \cong O_{\mathcal{M}^\zeta(N,K)}$. Then the lemma follows from Theorem 4.1. \qed

\textbf{Definition 4.1.} For a nonnegative integer $n$, define the $\mathbb{C}[\mathcal{M}(N,K)]$ module of level $n$, denoted $\Gamma(N,K,n)$, to be the global section of $n$-th power of tautological line bundle, i.e.

$$\Gamma(N,K,n) := \Gamma(\mathcal{M}^\zeta(N,K),\text{Det}^{\otimes n}). \quad (4.2)$$

In this section, we compute the Hilbert series of $\mathbb{C}[\mathcal{M}(N,K)]$ and $\Gamma(N,K,n)$. Before starting, let us introduce some notations and explain what we are going to compute.

The quiver in Figure 1 admits an action of $\text{GL}_K \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times$, where $\text{GL}_K$ is the flavour symmetry which acts on the framing vector space, $\mathbb{C}_q^\times$ scales $B$ by $B \mapsto q^{-1}B$, and $\mathbb{C}_t^\times$ scales $I$ by $I \mapsto t^{-1}I$. The convention of the inverse $q^{-1}$ and $t^{-1}$ is such that the functions $\text{Tr}(B^n)$ and $I_nB_mJ_b$ scales by $q^n$ and $q^mt$ respectively (since functions are dual to the space). Although $\mathbb{C}[\mathcal{M}(N,K)]$ is infinite dimensional, every $\mathbb{C}_q^\times \times \mathbb{C}_t^\times$-weight space of $\mathbb{C}[\mathcal{M}(N,K)]$ is finite dimensional, thus it makes sense to regard $\mathbb{C}[\mathcal{M}(N,K)]$ as an element in $K_{\text{GL}_K}(\text{pt})[q,t]$. Similarly, the same properties hold for $\Gamma(N,K,n)$. The goal of this section is to compute these elements.

\textbf{Definition 4.2.} Denote the complexified $\text{GL}_K$-equivariant K-theory of a point by

$$K_{\text{GL}_K}(\text{pt}) = \mathbb{C}[x_1^\pm, \cdots, x_K^\pm]^{\mathfrak{S}_K},$$

where $\mathfrak{S}_K$ is the permutation group acting on $x_1, \cdots, x_K$. We use shorthand notation $f(x)$ for a function of $x_1, \cdots, x_K$, and $f(x^{-1}) = f(x_1^{-1}, \cdots, x_K^{-1})$. Denote by $Z_{N,K}(x;q,t)$ the element of $\mathbb{C}[\mathcal{M}(N,K)]$ in $K_{\text{GL}_K}(\text{pt})[q,t]$, and denote by $Z_{N,K}^{(n)}(x;q,t)$ the element of $\Gamma(N,K,n)$ in $K_{\text{GL}_K}(\text{pt})[q,t]$. 

13
By Lemma 4.1 we have $Z_{N,K}^{(n)}(x; q, t) = \chi(\mathcal{M}\varepsilon(N, K), \det^{\otimes n})$. The case $K = 1$ is trivial: The functions $\text{Tr}(B), \ldots, \text{Tr}(B^N), J, IBJ, \ldots, IB^{N-1}J$ give rise to an isomorphism $\mathcal{M}(N, 1) \cong \mathbb{A}^{2N}$. The Lemma 4.2 below, together with the fact that the Hilbert-Chow map for Hilbert scheme of points on smooth curve is isomorphism, implies that $\mathcal{M}\varepsilon(N, K) \cong \mathcal{M}(N, K)$. In fact, $\det$ in this case is a trivial bundle, with $\mathbb{C}_q^\times \mathbb{C}_t^\times$-weight $(1, 0)$, thus

$$Z_{N,1}^{(n)}(x; q, t) = q^n Z_{N,1}(x; q, t) = \dfrac{q^n}{(q; q)_N(t; q)_N}. \tag{4.3}$$

Here we use the $q$-Pochhammer symbol

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

The case when $K > 1$ is harder. In principal, one can use the localization technique to get a formula for $\chi(\mathcal{M}\varepsilon(N, K), \det^{\otimes n})$ in terms of summation over fixed points, but it involves complicated denominators that are hard to extract the power series in $q$ and $t$ explicitly. Our strategy is to reduce the computation to Euler character of vector bundles on Quot scheme, which is related to the affine Grassmannian of $GL_K$, and apply the results on the geometry of the affine Grassmannian of $GL_K$ to finish the calculation. We present the result here first and explain the calculation in steps afterwards.

**Theorem 4.2.** The Hilbert series of $\Gamma(N, K, n) = \Gamma(\mathcal{M}\varepsilon(N, K), \det^{\otimes n})$ is

$$Z_{N,K}^{(n)}(x; q, t) = \frac{1}{(q; q)_N} \sum_{\mu} \ell[\mu] H_{\mu+(nN)}(x; q) h_{\mu_1}(x^{-1}) \cdots h_{\mu_N}(x^{-1}). \tag{4.4}$$

Here the summation is over arrays $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{Z}^{\geq 0}_N$, $(nN)$ is the array consisting of $N$ copies of $n$, i.e. $(nN) = (n, n, \ldots, n)$, and $|\mu| := \sum_{i=1}^N \mu_i$, and $h_k(x)$ is the complete homogeneous symmetric polynomials of degree $k$, and $H_\lambda(x; q)$ is the generalized transformed Hall-Littlewood polynomial of the array $\lambda$, defined in [A.6].

### 4.1 Reduction steps

Recall that the stability condition in the definition of $\mathcal{M}\varepsilon(N, K)$ is that if $V \subseteq \mathbb{C}^N$, $B(V) \subseteq V$ and $\im(J) \subseteq V$ then $V = \mathbb{C}^N$, in particular the sub-quiver consisting of arrows $(B, J)$ is stable under the same stability condition, so we have:

**Lemma 4.2.** The moduli of stable representations $\mathcal{M}\varepsilon(N, K)$ is a vector bundle over the Quot scheme of $\mathbb{A}^1$ which parametrizes length $N$ quotients of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$, denoted by $\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$:

$$\mathcal{M}\varepsilon(N, K) = \mathbb{V}(V \otimes W^*) \xrightarrow{p} \text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$$

Here $V$ is the tautological sheaf on $\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$, and $W$ is the framing vector space.

**Proof.** Consider a point $(B, J, I) \in \mathcal{M}\varepsilon(N, K)$, the action $B$ on $\mathbb{C}^N$ makes it into a $\mathbb{C}[z]$-module such that $z$ acts as $B$. The stability on $(B, J)$ is equivalent to that $\mathbb{C}^N$ is a quotient module of a free module of rank $K$. This gives rise to a morphism $p : \mathcal{M}\varepsilon(N, K) \to \text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$, and the extra information in $\mathcal{M}\varepsilon(N, K)$ compared to the Quot scheme is a homomorphism from the universal quotient $V$ to the framing vector space $W$, so $\mathcal{M}\varepsilon(N, K)$ is represented by $\mathbb{V}(V \otimes W^*)$. \qed
Lemma 4.2 implies that

$$\chi(\mathcal{M}(N, K), \text{Det}^\otimes n) = \sum_{m=0}^{\infty} t^m \chi(\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \text{Sym}^m(\mathcal{V} \otimes \mathcal{W}^*) \otimes \text{Det}^\otimes n).$$  (4.5)

Here in each summand, $\chi(\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \text{Sym}^m(\mathcal{V} \otimes \mathcal{W}^*))$ is in $K_{\text{GL}_K}(pt)[[q]]$. So the computation of $\mathbb{C}[\mathcal{M}(N, K)]$ boils down to the computation of equivariant Euler characters of sheaves on the Quot scheme.

The Quot scheme has a nice structure: there is morphism $h : \text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) \to \mathbb{A}^N$ where in the last equation we use the proper base change (since $h$ is the Hilbert-Chow morphism for the Quot scheme). In the language of quivers, $h$ maps $(B, J)$ to the set of eigenvalues of $B$ (counted with multiplicities), regarded as a divisor of degree $N$ in $\mathbb{A}^1$.

**Lemma 4.3.** The central fiber $h^{-1}(0)$ of the Hilbert-Chow morphism $h : \text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) \to \mathbb{A}^N$, endowed with reduced scheme structure, is isomorphic to the affine Grassmannian Schubert variety $\text{Gr}_{\text{GL}_K}^{N\omega_1}$. Here $\omega_1 = (1, 0, \cdots, 0)$ is the first fundamental coweight of $\text{GL}_K$.

**Proof.** The central fiber $h^{-1}(0)$ is the moduli space of submodules of $\mathbb{C}[z]^{\oplus K}$ whose cokernels are finite of length $N$ and are supported at $0$. Since finite quotients of $\mathbb{C}[z]^{\oplus K}$ are equivalent to finite quotients of $\mathbb{C}[z]^N$, $h^{-1}(0)$ is the moduli space of submodules of $\mathbb{C}[z]^{\oplus K}$ whose cokernels are finite of length $N$. This implies that $h^{-1}(0)$ is isomorphic to a closed subscheme of $\text{Gr}_{\text{GL}_K}$ which is set-theoretically identified with $\text{Gr}_{\text{GL}_K}^{N\omega_1}$, thus $h^{-1}(0)_{\text{red}} \cong \text{Gr}_{\text{GL}_K}^{N\omega_1}$. \(\square\)

**Proposition 4.1.** The Hilbert-Chow morphism $h : \text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) \to \mathbb{A}^N$ is flat.

**Proof.** By the deformation theory, $\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ is smooth of dimension $NK$. $h^{-1}(0)_{\text{red}} \cong \text{Gr}_{\text{GL}_K}^{N\omega_1}$ has dimension $(K-1)N$, which equals to $\dim(\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) - \dim \mathbb{A}^N)$, thus $h$ is flat along $h^{-1}(0)$ by miracle flatness theorem.\[35\] Tag 00R4. Since flatness is an open condition, $h$ is flat in an open neighborhood of $h^{-1}(0)$. Since Hilbert-Chow morphism $h$ is proper, there is an open neighborhood $U$ of $0 \in \mathbb{A}^N$ such that $h|_{h^{-1}(U)}$ is flat. Finally $h$ is equivariant under the $\mathbb{C}^\times$ action on $\mathbb{A}^1$ which contracts $\mathbb{A}^N$ to $0$, so the flatness is transported from $U$ to the whole $\mathbb{A}^N$. \(\square\)

Proposition 4.1 provides a tool that reduces the computation of Euler character to the central fiber. In effect, to compute $\chi(\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \mathcal{F})$ for a locally free sheaf $\mathcal{F}$, we can apply $\mathbb{C}^\times$-localization to its derived pushforward $Rh_* (\mathcal{F})$:

$$\chi(\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \mathcal{F}) = \chi(\mathbb{A}^N, Rh_* (\mathcal{F})) = \frac{\chi(h^{-1}(0), \mathcal{F}|_{h^{-1}(0)})}{\prod_{i=1}^N (1 - q^i)},$$  (4.6)

where in the last equation we use the proper base change (since $\mathcal{F}$ is flat over $\mathbb{A}^N$ by Proposition 4.1), and the denominator comes from the tangent space of $\mathbb{A}^N$ at $0$ which has $\mathbb{C}^*$-weights $-1, \cdots, -N$.

**Proposition 4.2.** The central fiber $h^{-1}(0)$ is isomorphic to $\text{Gr}_{\text{GL}_K}^{N\omega_1}$ as a scheme.

**Proof.** In view of Lemma 4.3 the proposition is equivalent to that $h^{-1}(0)$ is reduced. Since $h$ is flat with domain and codomain being smooth, $h^{-1}(0)$ is a Cohen-Macaulay scheme, therefore it is enough to show that $h^{-1}(0)$ is generically reduced. We claim that $h$ is smooth at the point $z^{N\omega_1}$. Assume that the claim is true, then $h$ is smooth in an open neighborhood of $z^{N\omega_1}$, thus $h^{-1}(0)$ is generically reduced.

The claim follows from the deformation theory of $\text{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$. Namely, if $e_1, \cdots, e_K$ is the basis of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$, then $z^{N\omega_1}$ corresponds to short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{A}^1}^{\oplus K} \to Q \to 0$$

\[3\]Here we endow all affine Grassmannian Schubert varieties with reduced scheme structures.
such that $\mathcal{E}$ is the subsheaf of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$ generated by $z^N e_1, e_2, \ldots, e_K$. Then the tangent space of Quot$_N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ at $z^{N\omega_1}$ is

$$\text{Hom}_{\mathcal{O}_{\mathbb{A}^1}}(\mathcal{E}, Q).$$

In particular, the tangent space contains $\text{Hom}_{\mathcal{O}_{\mathbb{A}^1}}(z^N \mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1}/z^N \mathcal{O}_{\mathbb{A}^1})$ as a subspace, and the latter projects isomorphically onto the tangent space of $\mathbb{A}^{(N)}$ at 0. In particular, the tangent map at $z^{N\omega_1}$ is surjective, thus $h$ is smooth at $z^{N\omega_1}$.

By the Lemma 4.1 and localization formula (4.6), we reduce the calculation to

$$Z_{N,K}^{(n)}(x; q, t) = \chi(\mathcal{M}^s(N, K), \text{Det}^{\otimes n}) = \frac{1}{(q; q)_N} \sum_{m=0}^{\infty} t^m \chi(\mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}, \text{Sym}^m(\mathcal{V} \otimes W^*) \otimes \mathcal{O}(n)), \quad (4.7)$$

where $\mathcal{V}$ is the restriction of the universal quotient sheaf to $\mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}$.

4.2 Calculation on affine Grassmannian

It remains to do the calculation on affine Grassmannian for

$$\sum_{m=0}^{\infty} t^m \chi(\mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}, \text{Sym}^m(\mathcal{V} \otimes W^*) \otimes \mathcal{O}(n)) = \chi(\mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}, S^*_t(\mathcal{V} \otimes W^*) \otimes \mathcal{O}(n)).$$

Here we use the notation $S^*_t(\mathcal{V} \otimes W^*) = \bigoplus_{m \geq 0} t^m \text{Sym}^m(\mathcal{V} \otimes W^*)$. To start with, note that there is a convolution map on $\mathcal{Gr}_{\mathbb{A}^1}$:

$$m : \mathcal{Gr}_{\mathbb{A}^1}^{\omega_1} \times \mathcal{Gr}_{\mathbb{A}^1}^{(N-1)\omega_1} \to \mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1},$$

see appendix B.2 for definition of the convolution product. The key property of the convolution product is that

$$\mathcal{O} \cong Rm_* \mathcal{O}. \quad (4.8)$$

See the proof of appendix B.3 for an explanation of this isomorphism. Here $\mathcal{O}$ is the structure sheaves, we omit the subscripts labelling the domain and codomain, since the meaning of the homomorphism is clear. In view of (4.8), we have

$$\chi(\mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}, S^*_t(\mathcal{V} \otimes W^*) \otimes \mathcal{O}(n)) = \chi(\mathcal{Gr}_{\mathbb{A}^1}^{\omega_1} \times \mathcal{Gr}_{\mathbb{A}^1}^{(N-1)\omega_1}, S^*_t(m^* \mathcal{V} \otimes W^*) \otimes \mathcal{O}(n)).$$

Let us write $\mathcal{V}_N$ for $\mathcal{V}$ to indicate the rank of the gauge group.

**Lemma 4.4. There is a short exact sequence**

$$0 \to \mathcal{V}_{N-1} \to m^* \mathcal{V}_N \to p^* \mathcal{V}_1 \to 0,$$

where $p : \mathcal{Gr}_{\mathbb{A}^1}^{\omega_1} \times \mathcal{Gr}_{\mathbb{A}^1}^{(N-1)\omega_1} \to \mathcal{Gr}_{\mathbb{A}^1}^{N\omega_1}$ is the projection to the first component map, and $\mathcal{V}_{N-1}$ is the sheaf $\mathcal{GL} \left( \mathcal{X} \right)_{\mathbb{A}^1} \times \mathcal{V}_{N-1}$.

**Proof.** $\mathcal{V}_N$ is the universal quotient of $\mathbb{C}[z]^{\oplus K}$. Denote the kernel by $L_N$. Then the pullback of $\mathcal{V}_N$ to the twisted product $\mathcal{Gr}_{\mathbb{A}^1}^{\omega_1} \times \mathcal{Gr}_{\mathbb{A}^1}^{(N-1)\omega_1}$ is by definition the extension of $\mathcal{V}_1$ by $\mathcal{V}_{N-1}$, except that in the definition of $\mathcal{V}_{N-1}$ the free module $\mathbb{C}[z]^{\oplus K}$ is replaced by $L_1$ (this is the meaning of twist). \[\square\]
Note that $\mathcal{V}_i$ is of rank one, so it is by definition the determinant line bundle $\mathcal{O}(1)$ on the affine Grassmannian $\text{Gr}_{GL_K}$ restricted on $\text{Gr}^{\omega_1}_{GL_K}$. The convolution map easily generalizes to multiple copies of $\text{Gr}_{GL_K}$:

$$m : \text{Gr}^{\omega_1}_{GL_K} \times \cdots \times \text{Gr}^{\omega_1}_{GL_K} \longrightarrow \overline{\text{Gr}}^{\omega_1}_{GL_K},$$

and we can apply Lemma 4.4 recursively and see that $m^* \mathcal{V}_i$ is a consecutive extension of (twisted) $\mathcal{O}(1)$. Since we only care about the Euler character, we can forget about the extension structure and focus on the $K$-theory class, in other words, we have:

$$\chi_{\overline{\text{Gr}}^{\omega_1}_{GL_K}}, S_i^*(\mathcal{V} \boxtimes W^*) \otimes \mathcal{O}(n))$$

$$= \chi(\text{Gr}^{\omega_1}_{GL_K} \times \cdots \times \text{Gr}^{\omega_1}_{GL_K}, S_i^*((\mathcal{O}(1) + \mathcal{O}(1) + \cdots + \mathcal{O}(1)) \otimes (\mathcal{O}(n) \boxtimes \cdots \boxtimes \mathcal{O}(n))))$$

$$= \sum_{\mu} t^{\mu} \chi(\text{Gr}^{\omega_1}_{GL_K} \times \cdots \times \text{Gr}^{\omega_1}_{GL_K}, \mathcal{O}(\mu_1 + n) \boxtimes \cdots \boxtimes \mathcal{O}(\mu_N + n)) \chi(S^{\mu_1}(W^*)) \cdots \chi(S^{\mu_N}(W^*)) (4.9)$$

Here the summation is over arrays $\mu = (\mu_1, \cdots, \mu_N) \in \mathbb{Z}^N_{geq 0}$, $|\mu| = \sum_{i=1}^{N} \mu_i$, and $\chi(S^{k}(W^*))$ is the GL$_K$-equivariant $K$-theory class of the $k$-th symmetric tensor product of $W^*$, where $W$ is the fundamental representation of GL$_K$. It is well-known that $\chi(S^{k}(W^*)) = h_k(x^{-1})$, where $h_k(x)$ is the complete homogeneous symmetric polynomial of degree $k$. Finally, the remaining part of the computation, which is the character of $\mathcal{O}(\mu_1 + n) \boxtimes \cdots \boxtimes \mathcal{O}(\mu_N + n)$, is related to a well-understood family of symmetric functions, the transformed Hall-Littlewood polynomial. In fact we have

$$\chi(\text{Gr}^{\omega_1}_{GL_K} \times \cdots \times \text{Gr}^{\omega_1}_{GL_K}, \mathcal{O}(\mu_1 + n) \boxtimes \cdots \boxtimes \mathcal{O}(\mu_N + n)) = H_{\mu+(nN)}(x; q). (4.10)$$

where $H_i(x; q)$ is the generalized transformed Hall-Littlewood polynomial of the array $\mu + (nN)$, see [A.6] for definition. For the derivation of this formula, see Corollary 4.5.2 in the appendix.

4.3 $N \to \infty$ limit

Recall that $\mathbb{C}[\mathcal{M}(\infty, K)]$ is the subalgebra of $\lim_{N} \mathbb{C}[\mathcal{M}(N, K)]$ generated by $T_{ab}^{(n)}$ and $\text{Tr}(B^m)$, for all $n, m \in \mathbb{Z}_{geq 0}$ and $1 \leq a, b \leq K$ (Definition 3.1).

**Lemma 4.5.** $\mathbb{C}[\mathcal{M}(\infty, K)]$ contains all $T \times \mathbb{C}_q^{x} \times C_{\overline{t}}^{x}$ eigenvectors in $\lim_{N} \mathbb{C}[\mathcal{M}(N, K)]$, where $T \subset GL_K$ is the maximal torus.

**Proof.** We claim that for fixed $n \in \mathbb{Z}_{geq 0}$, the dimension of $\mathbb{C}_q^{x}$-weight $n$ space of $\mathbb{C}[\mathcal{M}(N, K)]$ stabilizes when $N \gg 0$, more precisely there exists $N$ such that for all $N' > N$ the kernel of $\mathbb{C}[\mathcal{M}(N', K)] \to \mathbb{C}[\mathcal{M}(N, K)]$ has $\mathbb{C}_q^{x}$-weights > $n$. To see why this is true, we take $N$ such that $L_n^{-1}GL_K \times L_n^{-1}GL_1 \subseteq \mathcal{M}(N, K)$ ($N$ can be $(n+1)K$ according to the proof of Theorem 3.1), then $\ker(\mathbb{C}[\mathcal{M}(N', K)] \to \mathbb{C}[\mathcal{M}(N, K)])$ is a subquotient of $\ker(\mathbb{C}[L^{-1}GL_K \times L^{-1}GL_1] \to \mathbb{C}[L^{-1}GL_K \times L^{-1}GL_1])$, and the latter is an ideal generated by elements of $\mathbb{C}_q^{x}$-weights greater than $n$.

Now assume that $a \in \lim_{N} \mathbb{C}[\mathcal{M}(N, K)]$ is a $T \times \mathbb{C}_q^{x} \times C_{\overline{t}}^{x}$ eigenvector, and let its $\mathbb{C}_q^{x}$ be $n$. Then there exists $N$ such that for all $N' > N$ the kernel of $\mathbb{C}[\mathcal{M}(N', K)] \to \mathbb{C}[\mathcal{M}(N, K)]$ has $\mathbb{C}_q^{x}$-weights greater than $n$. Consider the image of $a$ in $\mathbb{C}[\mathcal{M}(N, K)]$, denoted by $\overline{a}$, and take a $T \times \mathbb{C}_q^{x} \times C_{\overline{t}}^{x}$-equivariant lift of $\overline{a}$ along the projection $\mathbb{C}[L^{-1}GL_K \times L^{-1}GL_1] \to \mathbb{C}[\mathcal{M}(N, K)]$, and we denote the lift by $a'$, then $a - a'$ has $\mathbb{C}_q^{x}$-weight $n$ and is zero in $\mathbb{C}[\mathcal{M}(N, K)]$, thus $a - a'$ is in the kernel of $\mathbb{C}[\mathcal{M}(N', K)] \to \mathbb{C}[\mathcal{M}(N, K)]$ for all $N' > N$, which forces $a = a'$ in $\mathbb{C}[\mathcal{M}(N', K)]$ because of weight consideration, therefore $a = a'$ in $\lim_{N} \mathbb{C}[\mathcal{M}(N, K)]$. $\square$
Proposition 4.3. The Hilbert series of \( \mathbb{C}[\mathcal{M}(\infty, K)] \) equals to the \( N \to \infty \) limit of Hilbert series of \( \mathbb{C}[\mathcal{M}(N, K)] \), i.e.

\[
\mathbb{C}[\mathcal{M}(\infty, K)] = \frac{1}{(q;q)_\infty} \lim_{N \to \infty} \sum_{\mu} e^{[\mu]} H_\mu(x;q) h_{\mu_1}(x^{-1}) \cdots h_{\mu_N}(x^{-1})
\] (4.11)

Proof. The \( N \to \infty \) limit of Hilbert series of \( \mathbb{C}[\mathcal{M}(N, K)] \) enumerates \( T \times \mathbb{C}_q^r \times \mathbb{C}_t^r \) eigenvectors in \( \lim_{N} \mathbb{C}[\mathcal{M}(N, K)] \), which is the same as \( T \times \mathbb{C}_q^r \times \mathbb{C}_t^r \) eigenvectors in \( \mathbb{C}[\mathcal{M}(\infty, K)] \), by Lemma 4.5. □

On the other hand, \( \mathbb{C}[\mathcal{M}(\infty, K)] \) is freely generated by \( I_d B^n J_b, \text{Tr}(B^m) \), which makes its Hilbert series easily computed by

\[
\text{PE}((t + tq + tq^2 + \cdots) \chi(g_{K})) \text{PE}(q + q^2 + \cdots).
\] (4.12)

Here \( \chi(g_{K}) \) is the character of the adjoint representation of \( GL_K \), and \( \text{PE} \) is the plethestic exponential. Note that \( \chi(g_{K}) \) can be written as a symmetric function \( 1 + \frac{s_{\lambda_{ad}}(x)}{m_K(x)} \), where \( \lambda_{ad} \) is the Young tableaux corresponding to the adjoint representation of \( SL_K \), and \( s_{\lambda_{ad}}(x) \) is the Schur function associated to \( \lambda_{ad} \), and \( m_K(x) = x_1 x_2 \cdots x_K \). Moreover,

\[
\text{PE}(q + q^2 + \cdots) = \prod_{i=1}^{\infty} \frac{1}{1 - q^i} = \frac{1}{(q;q)_\infty}.
\]

Compare equation (4.11) with (4.12) we get the following interesting equation, which we do not know other way to prove.

Corollary 4.1.

\[
\lim_{N \to \infty} \sum_{\mu} e^{[\mu]} H_\mu(x;q) h_{\mu_1}(x^{-1}) \cdots h_{\mu_N}(x^{-1}) = \text{PE} \left( \frac{t}{1 - q} \left( 1 + \frac{s_{\lambda_{ad}}(x)}{m_K(x)} \right) \right).
\] (4.13)

Here \( \lambda_{ad} \) is the Young tableaux corresponding to the adjoint representation of \( SL_K \), and \( s_{\lambda_{ad}}(x) \) is the Schur function associated to \( \lambda_{ad} \), \( m_K(x) = x_1 x_2 \cdots x_K \), and \( \text{PE} \) is the plethestic exponential.

5 Quantization of \( \mathcal{M}(N, K) \)

In this section we study the quantization of \( \mathcal{M}(N, K) \), namely we quantizes the Poisson structure (2.10) to the commutation relation:

\[
[J_{ia}, I_{bj}] = h \delta_{ab} \delta_{ij}, \quad [B_{mn}, B_{pq}] = h(\delta_{pm} B_{mq} - \delta_{mq} B_{pm}), \quad [B_{mn}, I_{bj}] = [B_{mn}, J_{ia}] = 0.
\] (5.1)

This is the algebra \( U_h(\mathfrak{g}_{\mathcal{N}}) \otimes \text{Weyl}^{\otimes NK}_h \), and we define the quantized ring of functions on the phase space \( \mathcal{C}_h[\mathcal{M}(N, K)] \) by the invariant part \( (U_h(\mathfrak{g}_{\mathcal{N}}) \otimes \text{Weyl}^{\otimes NK}_h)^{\text{GL}_{\mathcal{N}}} \). Since \( \text{GL}_{\mathcal{N}} \) is reductive, we have \( \mathcal{C}_h[\mathcal{M}(N, K)]/(h) = \mathcal{C}[\mathcal{M}(N, K)] \). Note that \( \mathcal{C}_h[\mathcal{M}(N, K)] \) possesses a natural grading by setting

\[
\deg(J) = 0, \deg(I) = 1, \deg(B) = 1, \deg(h) = 1.
\] (5.2)

Lemma 5.1. \( \mathcal{C}_h[\mathcal{M}(N, K)] \) is flat over \( \mathbb{C}[h] \).

Proof. Since \( U_h(\mathfrak{g}_{\mathcal{N}}) \otimes \text{Weyl}^{\otimes NK}_h \) is flat over \( \mathbb{C}[h] \), the subalgebra \( \mathcal{C}_h[\mathcal{M}(N, K)] \) is \( h \)-torsion free, thus it is also flat over \( \mathbb{C}[h] \). □
Remark 5.1. On the moduli of $\zeta$-stable representations $\mathcal{M}^\zeta(N, K)$, there is a notion of quantized structure sheaf. Namely, consider the completion of $U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\mathbb{Z}[N, K]}$ in the $h$-adic topology, then localize this algebra in the Zariski topology of the affine space $\text{Rep}(N, K)$, and by taking $\text{GL}_N$-invariant on the open locus of stable representations $\text{Rep}(N, K)^{\zeta-s}$, we get a sheaf of flat $\mathbb{C}[h]$-algebras on $\mathcal{M}^\zeta(N, K)$, denoted by $\hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)}$. By construction we have $\hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)}/(h) = \mathcal{O}_{\mathcal{M}^\zeta(N, K)}$. This sheaf is related to $\mathbb{C}_h[\mathcal{M}(N, K)]$ as follows. By construction we have a natural homomorphism of algebras $\mathbb{C}_h[\mathcal{M}(N, K)] \rightarrow \Gamma(\mathcal{M}^\zeta(N, K), \hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)})$, which preserves the grading $[5.2]$. On the other hand, by Proposition 2.5 we have

- $H^i(\mathcal{M}^\zeta(N, K), \hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)}) = 0$, for $i > 0$.
- $\Gamma(\mathcal{M}^\zeta(N, K), \hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)})$ is a flat $\mathbb{C}[h]$-algebra, and $\Gamma(\mathcal{M}^\zeta(N, K), \hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)})/(h) = \mathbb{C}[\mathcal{M}(N, K)]$.

Let us give a $\mathbb{Z}_{\geq 0}$-grading to $U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\mathbb{Z}[N, K]}$ by setting

$$\deg I = \deg J = 1, \quad \deg B = \deg h = 2.$$  

Since $\mathbb{C}_h[\mathcal{M}(N, K)]$ is generated by positive degree elements, we conclude that $\mathbb{C}_h[\mathcal{M}(N, K)]$ is naturally identified with the $\mathbb{C}[h]$-subalgebra of $\Gamma(\mathcal{M}^\zeta(N, K), \hat{\mathcal{O}}_{\mathcal{M}^\zeta(N, K)})$ generated by homogeneous elements.

The set of elements

$$\{ \text{Tr}(B^n), T_{ab}^{(m)} : 1 \leq n \leq N, 0 \leq m \leq N - 1, 1 \leq a, b \leq K \}$$

generates $\mathbb{C}[\mathcal{M}(N, K)]$ (Section 2.1), so it also generate $\mathbb{C}_h[\mathcal{M}(N, K)]$. It is a straightforward computation to see that $\text{Tr}(B^k)$ commutes with all elements in $U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\mathbb{Z}[N, K]}$, therefore $\text{Tr}(B^k)$ is central in $\mathbb{C}_h[\mathcal{M}(N, K)]$. We use $T_{ab}(z)$ to denote the following power series:

$$T_{ab}(z) := \sum_{n \geq 1} T_{ab}^{(n)} z^{-n-1} = \delta_{ab} + \frac{1}{z - B} J_b.$$  

Proposition 5.1. The commutators between $T_{ab}^{(k)}$ are:

$$[T_{ab}^{(p)}, T_{cd}^{(q)}] = h \sum_{i = -1}^{\min(p, q) - 1} \left( T_{cb}^{(i)} T_{ad}^{(p+q-1-i)} - T_{cb}^{(p+q-1-i)} T_{ad}^{(i)} \right).$$  

(5.3)

Proof. It is easy to see that (5.3) is equivalent to

$$[T_{ab}^{(p+1)}, T_{cd}^{(q)}] - [T_{ab}^{(p)}, T_{cd}^{(q+1)}] = h \left( T_{cb}^{(p)} T_{ad}^{(q)} - T_{cb}^{(q)} T_{ad}^{(p)} \right).$$

We compute the left-hand-side of the above equation:

$$[T_{ab}^{(p+1)}, T_{cd}^{(q)}] - [T_{ab}^{(p)}, T_{cd}^{(q+1)}] = I_{am} I_{cr} \left( [(B^{p+1})_{mn}, (B^q)_{rs}] - [(B^p)_{mn}, (B^{q+1})_{rs}] \right) J_{bn} J_{ds}$$

$$= h I_{am} I_{cr} \sum_{i=1}^{\min(p, q) - 1} \left( (B^{i-1})_{rn} (B^{p+1+q-i})_{ms} - (B^{i+p})_{rn} (B^{q-i})_{ms} \right) J_{bn} J_{ds}$$

$$- h I_{am} I_{cr} \sum_{i=1}^{\min(p, q) - 1} \left( (B^{i-1})_{rn} (B^{p+1+q-i})_{ms} - (B^{i+p-1})_{rn} (B^{q+1-i})_{ms} \right) J_{bn} J_{ds}$$

$$= h I_{am} I_{cr} \left( (B^p)_{rn} (B^q)_{ms} - (B^q)_{rn} (B^p)_{ms} \right) J_{bn} J_{ds},$$

which equals to the right-hand-side. 

\[ \square \]
Remark 5.2. The commutators \([I_{ab}, I_{cd}]\) is equivalent to the RTT equation

\[
[I_{ab}(u), I_{cd}(v)] = \frac{\hbar}{u-v} (I_{cb}(u)I_{ad}(v) - I_{cb}(v)I_{ad}(u)).
\]  

(5.4)

The classical embedding \(\mathcal{M}(L, K) \hookrightarrow \mathcal{M}(N, K)\) for \(L < N\) can be quantized as follows. Consider the left ideal of \(U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK}\) generated by \(B_{ij}\) and \(J_{ia}\) for all \(L < i,j \leq N\) and \(1 \leq a \leq K\), denote it by \(I_{L,N}^0\); then \(I_{L,N} := \left( U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK} \right)^{GL_{N-L}} \cap I_{L,N}^0\) is a two-sided ideal in \(\left( U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK} \right)^{GL_{N-L}}\), where \(GL_{N-L}\) acts on indices \(L < i,j \leq N\). It is easy to see that

\[
\left( U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK} \right)^{GL_{N-L}} = (U_h(\mathfrak{gl}_L) \otimes \text{Weyl}_h^{\otimes LK}) \oplus I_{L,N}
\]

as vector spaces, thus \(U_h(\mathfrak{gl}_L) \otimes \text{Weyl}_h^{\otimes LK} = \left( U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK} \right)^{GL_{N-L}} / I_{L,N}\). Restricting to the subalgebra \(\mathbb{C}[\mathcal{M}(N, K)] = \left( U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK} \right)^{GL_N}\), we get a map between graded \(\mathbb{C}[\hbar]\) algebras

\[
\tau^N_L : \mathbb{C}_h[\mathcal{M}(N, K)] \rightarrow \mathbb{C}_h[\mathcal{M}(L, K)],
\]

\[
\tau^N_L(T(u)) = T(u), \quad \tau^N_L(\text{Tr}(B^n)) = \text{Tr}(B^n).
\]

(5.5)

which quantizes the embedding \(\mathcal{M}(L, K) \hookrightarrow \mathcal{M}(N, K)\). We note that \(\tau^N_L\) is surjective.

Definition 5.1. Let \(Y_h(\mathfrak{gl}_K)\) be the Yangian of \(\mathfrak{gl}_K\) with generators \(\{ \Upsilon_{ab}(u) := \delta_{ab} + \sum_{n=0}^{\infty} \Upsilon^{(n)}_{ab} u^{-n-1} : 1 \leq a, b \leq K\}\) subject to the RTT relations

\[
[\Upsilon_{ab}(u), \Upsilon_{cd}(v)] = \frac{\hbar}{u-v} (\Upsilon_{cb}(u)\Upsilon_{ad}(v) - \Upsilon_{cb}(v)\Upsilon_{ad}(u)).
\]

Define \(\Lambda\) as the polynomial ring of countably many generators \(\mathbb{C}[p_i : i \in \mathbb{Z}_{\geq 1}]\), which is naturally identified with the ring of symmetric polynomials with \(p_i\) being the \(i\)-th power sum function.

Proposition 5.2. For every \(N \in \mathbb{Z}_{>0}\) there is a surjective map of \(\mathbb{C}[\hbar]\) algebras

\[
\rho_N : Y_h(\mathfrak{gl}_K) \otimes \Lambda \rightarrow \mathbb{C}_h[\mathcal{M}(N, K)],
\]

\[
\rho_N(\Upsilon(u)) = T(u), \quad \rho_N(p_n) = \text{Tr}(B^n).
\]

(5.6)

These maps are compatible in the sense that for all \(N > L\) the diagram

\[
Y_h(\mathfrak{gl}_K) \otimes \Lambda \xrightarrow{\rho_N} \mathbb{C}_h[\mathcal{M}(N, K)] \xrightarrow{\tau^N_L} \mathbb{C}_h[\mathcal{M}(L, K)]
\]

is commutative. Moreover, \(\bigcap_{N=1}^{\infty} \ker(\rho_N) = 0\).

Proof. According to Proposition 5.1 \(\rho_N\) is a \(\mathbb{C}[\hbar]\) algebra homomorphism. \(\rho_N\) is surjective since it is surjective modulo \(\hbar\). The compatibility with \(\tau^N_L\) is clear from construction. The intersection of kernels is zero because \(\mathbb{C}_h[\mathcal{M}(N, K)]\) is flat over \(\mathbb{C}[\hbar]\) and the intersection of kernels modulo \(\hbar\) is zero. \(\square\)

5.1 Another map from \(Y_h(\mathfrak{gl}_K) \otimes \Lambda\) to \(\mathbb{C}_h[\mathcal{M}(N, K)]\)

Recall that \(\mathcal{M}(N, K)\) is isomorphic to the Hamiltonian reduction description \(\text{Rep}(N, K) / / GL_N\), see Section 2.7. This admits quantization. Namely, consider the algebra \(U_h(\mathfrak{gl}_N) \otimes U_h(\mathfrak{gl}_N) \otimes \text{Weyl}_h^{\otimes NK}\) whose generators will be denoted as \((B, \bar{B}, J, I)\) with commutation relations (5.1) and

\[
[\bar{B}_{ij}, \bar{B}_{kl}] = \hbar (\delta_{ik} \bar{B}_{j} - \delta_{kj} \bar{B}_{i}).
\]

(5.7)
Definition 5.2. The quantum moment map \( \mu : \mathfrak{gl}_N \to U_h(\mathfrak{gl}_N) \otimes U_h(\mathfrak{gl}_N) \otimes \text{Weyl}^\otimes_{N,K} \) is

\[
\mu(E_{ij}) = B_{ij} + \tilde{B}_{ij} - \sum_{a=1}^{K} J_{ia}I_{aj} + \hbar N \delta_{ij}. \tag{5.8}
\]

And the quantum Hamiltonian reduction \((U_h(\mathfrak{gl}_N) \otimes U_h(\mathfrak{gl}_N) \otimes \text{Weyl}^\otimes_{N,K})//\text{GL}_N\) is defined as the \(\text{GL}_N\) invariant of \(U_h(\mathfrak{gl}_N) \otimes U_h(\mathfrak{gl}_N) \otimes \text{Weyl}^\otimes_{N,K}\) quotient by the left ideal generated by \(\mu(\mathfrak{gl}_N)\). Denote the quantum Hamiltonian reduction by \(A_{N,K}\).

Obviously there are two isomorphisms between \(\mathbb{C}_h[\mathcal{M}(N,K)]\) and \(A_{N,K}\), corresponding to two set of generators which are packaged in the generating functions

\[
T_{ab}(u) = \delta_{ab} + I_a \frac{1}{u - B} J_b, \quad Z(u) = 1 - \text{Tr} \left( \frac{\hbar}{u - B} \right),
\]

\[
\tilde{T}_{ab}(u) = \delta_{ab} + I_a \frac{1}{u + B} J_b, \quad \tilde{Z}(u) = 1 - \text{Tr} \left( \frac{\hbar}{u + B} \right). \tag{5.9}
\]

Then analogous to Proposition 5.2 for every \(N \in \mathbb{Z}_{>0}\) there is a surjective map of \(\mathbb{C}[\hbar]\) algebras

\[
\tilde{\rho}_N : Y_h(\mathfrak{gl}_K) \otimes \Lambda \to \mathbb{C}_h[\mathcal{M}(N,K)]. \tag{5.10}
\]

These maps are compatible in the sense that for \(N > L\) the diagram

\[
\begin{array}{ccc}
Y_h(\mathfrak{gl}_K) \otimes \Lambda & \xrightarrow{\tilde{\rho}_N} & \mathbb{C}_h[\mathcal{M}(N,K)] \\
\tilde{T}_{ab}(u) & \xrightarrow{\tau^N_L} & \mathbb{C}_h[\mathcal{M}(L,K)]
\end{array}
\]

is commutative, where \(\tau^N_L : \mathbb{C}_h[\mathcal{M}(N,K)] \to \mathbb{C}_h[\mathcal{M}(L,K)]\) is defined similar to that of \(\tau^N_L\) in (5.5), with \(B\) replaced by \(\tilde{B}\). Moreover, \(\bigcap_{N=1}^\infty \ker (\tilde{\rho}_N) = 0\).

There are nontrivial relations between generators in (5.9), which are originally proven in [36, Appendix A]. We collect these results and reproduce the proof in the notation of this paper for the convenience of readers.

Lemma 5.2. We have the following identities:

\[
T_{ab}(u)\tilde{T}_{bc}(-u) = \delta_{ac},
\]

\[
T_{ab}(u)\tilde{T}_{ba}(-u + Kh) = KZ(u)\tilde{Z}(-u + Kh). \tag{5.12}
\]

Proof. First of all, we compute

\[
T_{ab}(u)\tilde{T}_{bc}(w) = \delta_{ac} + I_a \frac{1}{u - B} J_c + I_a \frac{1}{w - B} J_c + I_a \frac{1}{u - B} J_b I_b \frac{1}{w - B} J_c
\]

\[
= \delta_{ac} + I_a \frac{1}{u - B} J_c + I_a \frac{1}{w - B} J_c + I_a \frac{1}{u - B} (B + \tilde{B}) \frac{1}{w - B} J_c
\]

\[
= \delta_{ac} + (u + w) I_a \frac{1}{u - B} \frac{1}{w - B} J_c.
\]

Taking \(w = -u\), we get \(T_{ab}(u)\tilde{T}_{bc}(-u) = \delta_{ac}\). Contracting with \(\delta_{ac}\), we get

\[
T_{ab}(u)\tilde{T}_{ba}(w) = K + (u + w) \text{Tr} \left( \frac{1}{w - B} J I \frac{1}{u - B} \right) - \hbar (u + w) \text{Tr} \left( \frac{1}{w - B} \frac{1}{u - B} \right)
\]

\[
= K + (u + w) \text{Tr} \left( \frac{1}{w - B} (B + \tilde{B}) \frac{1}{u - B} \right) + \hbar (u + w) \text{Tr} \left( \frac{1}{w - B} \right) \text{Tr} \left( \frac{1}{u - B} \right) - \hbar (u + w) \text{Tr} \left( \frac{1}{w - B} \frac{1}{u - B} \right).
\]
Here the second equality follows from moment map condition. Taking \( w = -u + K\hbar \), we get

\[
T_{ab}(u)\tilde{T}_{ba}(-u + K\hbar) = K \left( 1 - \text{Tr} \left( \frac{\hbar}{u - B} \right) \right) \left( 1 - \text{Tr} \left( \frac{\hbar}{-u + K\hbar - B} \right) \right) = K Z(u) \tilde{Z}(-u + K\hbar).
\]

\( \square \)

Recall that the quantum determinant of \( T(u) \) is defined as \([37, \S 2.6]\)

\[
\text{qdet} \ T(u) = \sum_{\sigma \in \mathfrak{S}_K} \text{sgn}(\sigma) T_{\sigma(1),1}(u) \cdots T_{\sigma(K),K}(u - (K - 1)\hbar).
\]

(5.13)

It is proposed in \([36]\) that quantum determinant of \( T(u) \) should be related to Capelli’s determinant of \( B \) and \( \tilde{B} \). We prove their proposal in the next proposition.

**Proposition 5.3.** Let \( C(u) \) be the Capelli’s determinant of \( B \)

\[
C(u) = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma)(u - (N - 1)\hbar - B)_{\sigma(1),1} \cdots (u - B)_{\sigma(N),N},
\]

(5.14)

and similarly let \( \tilde{C}(u) \) be the the Capelli’s determinant of \( \tilde{B} \). Then

\[
\text{qdet} \ T(u) = \frac{\tilde{C}(-u + (K - 1)\hbar)}{(-1)^N \cdot C(u)}.
\]

(5.15)

**Proof.** According to the quantum Newton’s formula \([37, \text{Theorem 4.1}]\), we have

\[
Z(u) = \frac{C(u - \hbar)}{C(u)}, \quad \tilde{Z}(u) = \frac{\tilde{C}(u - \hbar)}{\tilde{C}(u)}.
\]

(5.16)

Define a rational function

\[
f(u) := \text{qdet} \ T(u) \frac{C(u)}{C(-u + (K - 1)\hbar)}.
\]

Compare the quantum Liouville formula \([37, \text{Theorem 2.28}]\):

\[
\frac{1}{K} T_{ab}(u)\tilde{T}_{ba}(-u + K\hbar) = \frac{\text{qdet} \ T(u - \hbar)}{\text{qdet} \ T(u)},
\]

(5.17)

with (5.12) and (5.16), we get \( f(u)/f(u - \hbar) = 1 \). It follows that \( f(u) \) does not depend on \( u \). In particular, \( f(u) = \lim_{u \to \infty} f(u) = (-1)^N \), i.e. \( \text{qdet} \ T(u) = (-1)^N \tilde{C}(-u + (K - 1)\hbar)/C(u) \). \( \square \)

**Remark 5.3.** Now we have RTT generator \( T(u) \) and its inverse \( \tilde{T}(-u) \), then the \( J \)-generators of the Yangian for \( \mathfrak{sl}_K \) can be obtained from them, in fact one define \( B_{\text{av}} = \frac{1}{2} (B - \tilde{B}) \), and

\[
J_{ab}^{(n)} = I_n B_{\text{av}}^n J_b,
\]

(5.18)

then \( J_{ab}^{(0)} \) are generators of \( \mathfrak{gl}_K \) and they act on \( J_{ab}^{(1)} \) as adjoint representation, and

\[
[J_{ab}^{(1)}, J_{cd}^{(1)}] = \hbar (\delta_{bc} J_{ad}^{(2)} - \delta_{ad} J_{cb}^{(2)}) + \frac{\hbar}{4} (J_{cd}^{(0)} J_{ab}^{(0)} - J_{ab}^{(0)} J_{cd}^{(0)} - J_{ec}^{(0)} J_{ab}^{(0)} + J_{ab}^{(0)} J_{ce}^{(0)}).
\]

(5.19)

The above commutation relation shows that \( J_{ab}^{(0)} = J_{ab}^{(0)} - \frac{1}{K} \delta_{ab} \sum_{c=1}^K J_{cc}^{(0)} \) and \( J_{ab}^{(1)} = J_{ab}^{(1)} - \frac{1}{K} \delta_{ab} \sum_{c=1}^K J_{cc}^{(1)} \) generate the subalgebra \( Y_{\hbar}(\mathfrak{sl}_K) \subset Y_{\hbar}(\mathfrak{gl}_K) \otimes \Lambda \).
5.2 Kernel of $\rho_N$

In this subsection we characterize the kernel of the quotient map $\rho_N : Y_h(\mathfrak{g}_K) \otimes \Lambda \rightarrow \mathcal{C}_h[M(N, K)]$.

**Definition 5.3.** Fix $N$, define a power series $\mathcal{C}(u) = u^N + \sum_{n>0} \mathcal{C}_n u^{N-n}$ with coefficients $\mathcal{C}_n \in Y_h(\mathfrak{g}_1)$ which is uniquely determined by

$$1 - \frac{Nh}{u} - h \sum_{n>0} \frac{p_n}{u^{n+1}} = \frac{\mathcal{C}(u-h)}{\mathcal{C}(u)}. \quad (5.20)$$

Here $p_n$ are the power sum generators of $Y_h(\mathfrak{g}_1)$. Let RTT generator of $Y_h(\mathfrak{g}_K)$ be

$$\mathcal{F}(u) = 1 + \sum_{n \geq 0} \mathcal{F}^{(n)} u^{-n-1},$$

then we define a power series $\widetilde{\mathcal{C}}(u) = u^N + \sum_{n>0} \widetilde{\mathcal{C}}_n u^{N-n}$ by

$$\widetilde{\mathcal{C}}(-u + (K-1)h) = (-1)^N \mathcal{C}(u) \text{qdet} \mathcal{F}(u). \quad (5.21)$$

Write the quantum minor of $\mathcal{F}(u)$ for row indices $\underline{a} = (a_1 < \cdots < a_i)$ and column indices $\underline{b} = (b_1 < \cdots < b_i)$ as

$$\mathcal{F}_{\underline{a}\underline{b}}(u) = \sum_{\sigma \in \mathfrak{S}_i} \text{sgn}(\sigma) \mathcal{F}_{\sigma(a_1), b_1} \cdots \mathcal{F}_{\sigma(a_i), b_i} (u - (i-1)h). \quad (5.22)$$

**Remark 5.4.** Let $C(u)$ be the Capelli’s determinant of $B$, then by the quantum Newton’s formula [37], we have

$$1 - \text{Tr} \left( \frac{h}{u-B} \right) = \frac{C(u-h)}{C(u)}, \quad (5.23)$$

therefore the image of $\mathcal{C}(u)$ in $\mathcal{C}_h[M(N, K)]$ is $C(u)$. In the classical limit $h \rightarrow 0$, $C(u)$ equals to $\text{det}(u-B)$, and $C_n \equiv (-1)^n m_n \mod h$, where $m_n$ are the generators of $\mathbb{C}[L^{-}\text{GL}_1]$ that take the value of $a_n$ in the power series $1 + \sum_{n \geq 1} a_n z^{-n} \in L^{-}\text{GL}_1$.

**Theorem 5.1.** The kernel of $\rho_N : Y_h(\mathfrak{g}_K) \otimes \Lambda \rightarrow \mathcal{C}_h[M(N, K)]$ is generated by all coefficients for negative powers in $u$ in the power series

$$\mathcal{C}(u), \quad \widetilde{\mathcal{C}}(u), \quad \mathcal{C}(u) \mathcal{F}_{\underline{a}\underline{b}}(u), \quad (5.24)$$

for all $\underline{a} = (a_1 < \cdots < a_i), \underline{b} = (b_1 < \cdots < b_i)$ and all $1 \leq i \leq K-1$.

**Proof.** First of all, we show that (5.24) are mapped to polynomials. For $\mathcal{C}(u)$, its image is the Capelli’s determinant $C(u)$ of $B$, which is a polynomial. Note that $C(u)$ is known to be noncommutative version of characteristic polynomial in the sense that $C(B) = 0$ [37], thus we have recursion relations: $T^{(m)}_{ab} + \sum_{n=1}^{N} C_n T^{(m-n)}_{ab} = 0$ for all $m \geq N$, which is equivalent to that $C(u)T_{ab}(u)$ is a polynomial. It follows from (5.15) that $C(u)\text{qdet} T(u)$ is a polynomial. Next we consider the embedding $\mathcal{C}_h[M(N, i)] \hookrightarrow \mathcal{C}_h[M(N, K)]$ by $B \mapsto B$ and $J_{is} \mapsto J_{ias}$ and $I_{si} \mapsto I_{ais}$. This implies that $C(u)T_{\underline{a}\underline{b}}(u)$ are polynomials for all $\underline{a} = (a_1 < \cdots < a_i)$ and all $1 \leq i \leq K$. After taking commutators with $T_{rs}$ for various indices $r$ and $s$, we see that all coefficients for negative powers in $u$ in the power series $C(u)T_{\underline{a}\underline{b}}(u)$ are zero. In particular $\mathcal{C}(u)$ (5.21) is mapped to a polynomial. Thus we see that (5.24) are mapped to polynomials.

Next we show that the kernel of $Y_h(\mathfrak{g}_K)\otimes \Lambda \rightarrow \mathcal{C}_h[M(N, K)]$ is generated by all coefficients for negative powers in $u$ in the power series (5.24). By the flatness over $\mathbb{C}[h]$ (Lemma 5.1), it suffices to show that they generate the ideal modulo $h$. Let us define the closed subscheme $X$ in $L^{-}\text{GL}_K \times L^{-}\text{GL}_1$ by vanishing of all coefficients for negative powers in $u$ in the power series (5.24) modulo $h$. Then $\mathcal{M}(N, K)$ is a closed
subscheme of $X$. In view of Proposition 2.3, it is then enough to show that $X$ is reduced and irreducible of dimension $2NK$.

Consider the Drinfeld’s generators $\{H_i(u), E_i(u), F_i(u) : 1 \leq i \leq K - 1\}$ of the subalgebra $Y_h(\mathfrak{gl}_K) \subset Y_h(\mathfrak{gl}_K) \otimes \Lambda$, they are related to generators $\{\mathfrak{c}(u), \mathfrak{f}(u)\}$ by the following [35] Theorem 12.1.4:

$$
H_i(u) = \mathfrak{c}(u)^{b_i} \cdot \frac{A_{i-1}(u + \frac{1}{2})A_{i+1}(u + \frac{1}{2})}{A_{i}(u)A_{i}(u + \hbar)},
$$

$$
E_i(u) = \mathfrak{c}(u + \frac{i-1}{2}\hbar)\mathfrak{z}_{i}^{-1}(u + \frac{i-1}{2}\hbar)A_{i}(u)^{-1},
$$

$$
F_i(u) = \mathfrak{c}(u + \frac{i-1}{2}\hbar)A_{i}(u)^{-1}\mathfrak{z}_{i}^{-1}(u + \frac{i-1}{2}\hbar),
$$

where $1 \leq i \leq K - 1$, and $\iota = (1 < \cdots < i)$, and $\iota^{\perp} = (1 < \cdots < i - 1 < i + 1)$, and $A_0(u) = 1$, and

$$
A_i(u) = u^N + \sum_{p=1}^{\infty} A_i^{(p)} u^{N-p} := \mathfrak{c}(u + \frac{i-1}{2}\hbar)\mathfrak{z}_{i}^{-1}(u + \frac{i-1}{2}\hbar),
$$

By [37] Theorem 2.20, we have $Y_h(\mathfrak{gl}_K) \cong Y_h(\mathfrak{sl}_K) \otimes Z(Y_h(\mathfrak{gl}_K))$, where $Z(Y_h(\mathfrak{gl}_K))$ is a commutative $\mathbb{C}[\hbar]$ algebra freely generated by coefficients of power series $\text{qdet} \ \mathfrak{f}(u)$. According to [39, (5.21)], $Z(Y_h(\mathfrak{gl}_K)) \otimes \Lambda$ is a commutative $\mathbb{C}[\hbar]$ algebra freely generated by $\{\mathfrak{c}_n, \mathfrak{c}_n : n \in \mathbb{Z}_{>0}\}$. Consider the algebra

$$
Y_h(\mathfrak{sl}_K)[z_{1,1}, \cdots, z_{1,N}, z_{K-1,1}, \cdots, z_{K-1,N}] := Y_h(\mathfrak{sl}_K) \otimes \mathbb{C}[z_{1,1}, \cdots, z_{1,N}, z_{K-1,1}, \cdots, z_{K-1,N}],
$$

and we construct an algebra homomorphism

$$
Y_h(\mathfrak{sl}_K)[z_{1,1}, \cdots, z_{1,N}, z_{K-1,1}, \cdots, z_{K-1,N}] \longrightarrow Y_h(\mathfrak{gl}_K) \otimes \Lambda/(\mathfrak{c}_n, \mathfrak{c}_n : n > N),
$$

which is defined by the identity map on $Y_h(\mathfrak{sl}_K)$ and

$$
\prod_{i=1}^{N} (u - z_{1,i}) = \mathfrak{c}\left(u - \frac{1}{2}\hbar\right), \prod_{i=1}^{N} (u - z_{K-1,i}) = (-1)^N \mathfrak{c}\left(-u + \frac{K}{2}\hbar\right).
$$

By the discussions above, (5.26) is an isomorphism. Therefore we have an isomorphism between quotient algebras

$$
\frac{Y_h(\mathfrak{gl}_K) \otimes \Lambda}{(\text{coeff. of negative powers in (5.24))}} \cong \frac{Y_h(\mathfrak{sl}_K)[z_{1,1}, \cdots, z_{1,N}, z_{K-1,1}, \cdots, z_{K-1,N}]}{(A_i^{(p)} : 1 \leq i \leq K - 1, p > N)}.
$$

According to [39, §1.2.1] and [31] Theorem 4.10], the RHS of (5.28) is isomorphic to the quantized Coulomb branch algebra $\mathcal{A}_h$ associated to the quiver in Figure 3 with gauge symmetry. This implies that the

![Figure 3: The quiver diagram for the Coulomb branch description.](image)

scheme $X$ whose ring of functions is the LHS of (5.28) modulo $\hbar$ is isomorphic to the Coulomb branch $\mathcal{M}_C$ associated to the quiver in Figure 3 with flavor symmetry. Then $X$ is reduced and irreducible of dimension $2NK$ by [40] Theorem 3.20]. This concludes the proof. \qed
Corollary 5.1. \( M(N, K) \) is isomorphic to the Coulomb branch associated to the quiver in Figure 3 with flavor symmetry. Therefore according to [37, Theorem 3.20], \( M(N, K) \) is isomorphic to the Beilinson-Drinfeld slice \( \mathcal{W}_0 \) for GL\(_K\), where \( \lambda = N\omega_1 + N\omega_{K-1} \), \( \omega_i \) is the \( i \)-th fundamental coweight for GL\(_K\). \( C_h[M(N, K)] \) is isomorphic to the quantized Coulomb branch algebra associated to the quiver in Figure 3 with flavor symmetry, which is the truncated Yangian \( \mathcal{Y}_0 \).

5.3 Quantized coproduct

In this subsection, we shall use the notation \( C_N(u) \) to denote the power series define in Definition 5.3.

Let us define a coproduct

\[
\Delta_{n_1, n_2} : Y_h(\mathfrak{gl}_K) \otimes \Lambda \to (Y_h(\mathfrak{gl}_K) \otimes \Lambda) \otimes (Y_h(\mathfrak{gl}_K) \otimes \Lambda),
\]

\[
\Delta_{n_1, n_2} (\mathcal{T}_{ab}(u)) = \sum_{c=1}^K \mathcal{T}_{ac}(u) \otimes \mathcal{T}_{cb}(u), \quad \Delta_{n_1, n_2} (\mathcal{C}_{n_1+n_2}(u)) = \mathcal{C}_{n_1}(u) \otimes \mathcal{C}_{n_2}(u). \tag{5.29}
\]

We note that the restriction of \( \Delta_{n_1, n_2} \) to the subalgebra \( Y_h(\mathfrak{gl}_K) \) agrees with the ordinary coproduct on \( Y_h(\mathfrak{gl}_K) [37, \S 2.5] \). The coproduct [5,29] is coassociative in the sense that

\[
\left( \Delta_{n_1, n_2} \otimes \text{Id} \right) \circ \Delta_{n_1+n_2, n_3} = \left( \text{Id} \otimes \Delta_{n_2, n_3} \right) \circ \Delta_{n_1, n_2+n_3} \tag{5.30}
\]

Proposition 5.4. There exists a unique \( \mathbb{C}[\hbar] \) algebra homomorphism

\[
\Delta_{n_1, n_2} : C_h[M(n_1 + n_2, K)] \to C_h[M(n_1, K)] \otimes C_h[M(n_2, K)]
\]

which makes the following diagram

\[
\begin{array}{ccc}
Y_h(\mathfrak{gl}_K) \otimes \Lambda & \xrightarrow{\rho_{n_1+n_2}} & (Y_h(\mathfrak{gl}_K) \otimes \Lambda) \otimes (Y_h(\mathfrak{gl}_K) \otimes \Lambda) \\
\Delta & \downarrow & \rho_{n_1} \otimes \rho_{n_2} \\
\mathbb{C}_h[M(n_1 + n_2, K)] & \xrightarrow{\Delta} & \mathbb{C}_h[M(n_1, K)] \otimes \mathbb{C}_h[M(n_2, K)]
\end{array}
\tag{5.31}
\]

commutative. Here the vertical maps are constructed in Proposition [5,2], and they are explicitly given by \( \rho_N(\mathcal{T}(u)) = T(u) \) and \( \rho_N(\mathcal{C}_N(u)) = C_N(u) \), where \( C_N(u) \) is the Capelli determinant of \( B \) [5,14].

Proof. We note that \( \rho_{n_1+n_2} \) is surjective by Proposition 5.2, therefore it is enough to show that

\[
\Delta_{n_1, n_2} (\ker(\rho_{n_1+n_2})) \subseteq \ker(\rho_{n_1}) \otimes \ker(\rho_{n_2}).
\]

By Theorem 5.1 we are left with showing that

\[
(\rho_{n_1} \otimes \rho_{n_2}) \circ \Delta_{n_1, n_2} (\mathcal{C}_{n_1+n_2}(u)) \quad \text{and} \quad (\rho_{n_1} \otimes \rho_{n_1}) \circ \Delta_{n_1, n_2} (\mathcal{C}_{n_1+n_2}(u) \mathcal{T}_{ab}(u))
\]

are polynomials in \( u \), for all \( \mathcal{a} = (a_1 < \cdots < a_i), \mathcal{b} = (b_1 < \cdots < b_i) \) and all \( 1 \leq i \leq K \). By the definition of \( \Delta_{n_1, n_2} \), we have

\[
(\rho_{n_1} \otimes \rho_{n_1}) \circ \Delta_{n_1, n_2} (\mathcal{C}_{n_1+n_2}(u)) = \rho_{n_1} (\mathcal{C}_{n_1}(u)) \otimes \rho_{n_2} (\mathcal{C}_{n_2}(u)) = C_{n_1}(u) \otimes C_{n_2}(u),
\]

which is a polynomial in \( u \). By [37, Proposition 2.14], we have

\[
\Delta_{n_1, n_2} (\mathcal{T}_{ab}(u)) = \sum_{\mathcal{a}} \mathcal{T}_{ab}(u) \otimes \mathcal{T}_{ab}(u),
\]

25
To write down the antipode map \( p \) we define the map \( \delta \) which plays the role of treating \( n \) because each summand is a polynomial.

The coproduct \( \Delta \) is explicitly given by

\[
\Delta_{n_1,n_2} (T_{ab}(u)) = T_{ac}(u) \otimes T_{cb}(u), \quad \Delta_{n_1,n_2} (C_{n_1+n_2}(u)) = C_{n_1}(u) \otimes C_{n_2}(u). \tag{5.32}
\]

Equivalently, one can write down \( \Delta \) using the second line of (5.34):

\[
\Delta_{n_1,n_2} (\tilde{T}_{ab}(u)) = \tilde{T}_{cb}(u) \otimes \tilde{T}_{ac}(u), \quad \Delta_{n_1,n_2} (\tilde{C}_{n_1+n_2}(u)) = \tilde{C}_{n_1}(u) \otimes \tilde{C}_{n_2}(u). \tag{5.33}
\]

We can group the family of coproducts \( \Delta_{n_1,n_2} \) into a single one, at the cost of introducing a variable which plays the role of treating \( n_1, n_2 \) as undetermined variables.

**Definition 5.4.** We define the \( \mathbb{C}[\hbar] \)-Hopf algebra \( Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta \) as the algebra \( Y_h(\mathfrak{gl}_K) \otimes \Lambda[\delta] \), equipped with the coproduct

\[
\Delta(\mathcal{X}_{ab}(u)) = \mathcal{X}_{ac}(u) \otimes \mathcal{X}_{cb}(u), \quad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta, \\
\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n - \hbar \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i},
\]

where we set \( p_0 := \delta \). The counit \( \epsilon : Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta \to \mathbb{C}[\hbar] \) is defined by

\[
\epsilon(\mathcal{X}_{ab}(u)) = \delta_{ab}, \quad \epsilon(p_n) = \epsilon(\delta) = 0. \tag{5.35}
\]

To write down the antipode map \( S \), we define power series \( \mathcal{A}(u) = 1 + \sum_{n=0}^\infty \mathcal{A}_n u^{-n}, \mathcal{A}_n \in Y_h(\mathfrak{sl}_1) \) by

\[
\left( 1 - \frac{\hbar}{u} \right)^{-\delta} \left( 1 - \hbar \sum_{n=0}^\infty \frac{p_n}{u^{n+1}} \right) = \frac{\mathcal{A}(u - \hbar)}{\mathcal{A}(u)}, \tag{5.36}
\]

then the antipode \( S \) is given by

\[
S(\mathcal{X}(u)) = \mathcal{X}(u)^{-1}, \quad S(\delta) = -\delta, \quad S(\mathcal{A}(u)) = \mathcal{A}(u)^{-1}. \tag{5.37}
\]

The quotient map \( \rho_N \) in Proposition 5.2 naturally extends to \( \rho'_N : Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta \to \mathbb{C}_h[\mathcal{M}(N, K)] \) which is given by

\[
\rho'_N(\mathcal{X}(u)) = T(u), \quad \rho'_N(\mathcal{A}(u)) = u^{-N} C_N(u), \quad \rho'_N(\delta) = N.
\]

Moreover the following diagram

\[
\begin{array}{ccc}
Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta & \xrightarrow{\Delta} & (Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta) \otimes (Y_h(\mathfrak{gl}_K) \otimes \Lambda_\delta) \\
\rho'_{n_1+n_2} \downarrow & & \downarrow \rho'_{n_1} \otimes \rho'_{n_2} \\
\mathbb{C}_h[\mathcal{M}(n_1 + n_2, K)] & \xrightarrow{\Delta} & \mathbb{C}_h[\mathcal{M}(n_1, K)] \otimes \mathbb{C}_h[\mathcal{M}(n_2, K)]
\end{array}
\]

is commutative.

**Remark 5.5.** The second line of (5.31) can be written in the following compact form

\[
\Delta(\mathcal{A}(u)) = \mathcal{A}(u) \otimes \mathcal{A}(u). \tag{5.39}
\]
5.4 Quantized phase space and Coulomb branch algebra

In this subsection we give a conceptual understanding of the identification between the quantized phase space $\mathbb{C}_h[\mathcal{M}(N,K)]$ and Coulomb branch algebra associated to the quiver in Figure $3$.

Given a quiver $Q$, we denote by $A_h(Q)$ the quantum Coulomb branch algebra associated to the quiver $Q$ with flavor symmetry, i.e.

$$A_h(Q) := H^\text{GL(V)}_\ast \times \text{GL(W)}_\ast \times \mathbb{C}_\times (\mathcal{R}),$$

where $\mathcal{R}$ is the BFN's space of triples associated to the quiver $Q$, see [10] Appendix A(ii) for details.

Example 5.1. The $3d, N = 4$ gauge theory associated to the following quiver is known as $T[\text{SU}(N)]$, its generators of $U_h(\mathfrak{gl}_N)$ satisfying relations $[E_{ij}, E_{kl}] = h(\delta_{jk}E_{il} - \delta_{il}E_{kj})$. Define

$$A_n(u) = u^{[n]} \delta_{ij} (u + n - 1, 2), \quad u^{[n]} := (u + \frac{n - 1}{2}h)(u + \frac{n - 3}{2}h) \cdots (u - \frac{n - 1}{2}h),$$

where $\delta_{ij}$ is the quantum determinant of the submatrix of $\mathcal{D}(u)$ consisting of first $n$ rows and first $n$ columns. Write $A_n(u) = u^n + \sum_{i>0} A_n^{(i)} u^{n-i}$, then the kernel of $Y_h(\mathfrak{gl}_N) \rightarrow U_h(\mathfrak{gl}_N)$ contains $A_n^{(p)}$ for all $p > n$ and for all $1 \leq n \leq N$. In the Drinfeld generators, we have

$$H_n(u) = \frac{A_{n-1}(u + \frac{n}{2})A_{n+1}(u + \frac{n}{2})}{A_n(u)A_{n}(u + h)}, \quad 1 \leq n \leq N - 1, \quad A_0(u) = 1.$$

Compare with [10] Corollary B.17 we conclude that the quotient of $Y_h(\mathfrak{gl}_N)$ by the ideal generated by $A_n^{(p)}$ for all $p > n$ and for all $1 \leq n \leq N$ is the truncated Yangian $Y^{N\omega_N-1}_0$ for $\mathfrak{sl}_N$, where $\omega_N$ is the $(N-1)$-th fundamental coweight of $\mathfrak{sl}_N$, and the generating function of mass parameters $Z_{N-1}(u) = \prod_{n=1}^N (u - z_{N-1,i})$ is identified as $Z_{N-1}(u) = A_N(u + h)$. The classical limit of $Y^{N\omega_N-1}_0$ is the function ring of $\overline{W^{N\omega_N-1}_{0,\text{SL}_N}}$ (see [10] §3(v)). It is known that $\overline{W^{N\omega_N-1}_{0,\text{SL}_N}}$ is reduced and irreducible of dimension $N^2$. On the other hand, the classical limit of $U_h(\mathfrak{gl}_N)$ is the function ring of $\mathfrak{gl}_N^*$, which has dimension $N^2$ and embeds into $\overline{W^{N\omega_N-1}_{0,\text{SL}_N}}$ as a closed subscheme, then the same argument as the proof of Theorem $5.1$ shows that

$$U_h(\mathfrak{gl}_N) \cong A_h associated to the quiver in the Figure $3$ with flavor symmetry.

Recall that balanced subquiver $Q^\text{bal} \subseteq Q$ is formed by those edge-loop-free nodes $i \in Q_0$ such that $2 \dim V_i = \dim W_i + \sum_j a_{ij} \dim V_j$ where $a_{ij}$ is the number of edges between $i$ and $j$. It is well-known that $Q^\text{bal}$ is a union of finite ADE quivers, unless $Q^\text{bal}$ is a union of connected components of $Q$ of affine type with zero framing on them [27] Appendix A]. It is shown in [27] Appendix A] that if it is not the latter case then the corresponding ADE group, denoted by $I^\text{bal}$, acts on the Coulomb branch algebra $A_h(Q)$, such that the infinitesimal action is generated by $\frac{1}{h}[F_i^{(1)} \cdot \bullet], \frac{1}{h}[E_i^{(1)} \cdot \bullet], \frac{1}{h}[F_i^{(1)} \cdot \bullet]$ for those $i \in Q^\text{bal}_0$. 

27
**Example 5.2.** In the case that \( Q \) is of ADE type with gauge dimension vector \( \mathbf{v} \) and flavour dimension vector \( \mathbf{w} \), the classical Coulomb branch \( \mathcal{M}_C(Q) \) is the Poisson variety \( \overline{W}_\mu^\lambda \), where \( \lambda = \sum_{i \in Q_0} w_i \lambda_i, \mu = \lambda - \sum_{i \in Q_0} v_i \alpha_i, \lambda^* = -w_0(\lambda), \) \( \lambda_i \) are fundamental coweights and \( \alpha_i \) are fundamental coroots and \( w_0 \) is the longest element in the Weyl group of \( G \). It is shown in \cite{27}, Example A.5] that \( L_{\text{bal}} \) action can be identified with the natural action of \( \text{Stab}_G(\mu^*) \) on \( \overline{W}_\mu^\lambda \) when \( \mu \) is dominant. This holds for general \( \mu \). In fact we can take a dominant \( \nu \) such that \( (\nu, \alpha_i) = 0, \forall i \in Q^\text{bal}_0 \) and \( \mu + \nu \) is dominant, then the shift map \( i_{0,\nu^*} : \mathbb{C}[\overline{W}_{\mu^*+\nu^*}^\lambda] \to \mathbb{C}[\overline{W}_{\mu^*}^\lambda] \) commutes with the action of \( \text{Stab}_G(\mu^*) \subseteq \text{Stab}_G(\mu^* + \nu^*) \). Since \( i_{0,\nu^*} \) is Poisson and preserves \( E_1^{(1)}, F_1^{(1)}, H_1^{(1)} \) for \( i \in Q^\text{bal}_0 \), it follows that the action of \( L_{\text{bal}} \) constructed in \cite{27}, Proposition A.3] commutes with the shift map. Since the action of \( L_{\text{bal}} \) agrees with the natural one for \( \text{Stab}_G(\mu^*) \) on \( \overline{W}_{\mu^*+\nu^*}^\lambda \), and the shift map is birational and equivariant for both of actions, these two actions agree on \( \overline{W}_{\mu^*}^\lambda \) as well.

**Remark 5.6.** Suppose that there is another action of \( L_{\text{bal}} \) on \( \mathcal{A}_h(Q) \) which acts trivially on \( h \) (not necessarily the one constructed in \cite{27}, Appendix A]), such that these two actions agree after modulo \( h \) and mass parameters (generators of \( H^*_\text{GL}(W) (\text{pt}) \)), then these two actions must agree on \( \mathcal{A}_h(Q) \). In fact \( \mathcal{A}_h(Q) \) is a flat deformation of \( \mathcal{A}_h(Q)/(h, \text{mass}) \) and the deformation spaces of modules for reductive group are trivial.

**Example 5.3.** The Coulomb branch associated to the quiver of Figure 5 is isomorphic to generalized transverse slice \( \overline{W}_{w_0(\lambda_N)}^\lambda \) \cite{40}, Theorem 3.10], where \( \lambda_N \) is the \( N \)-th fundamental coweight of \( \text{GL}_N(K+N) \) and \( w_0 \) is the longest element of the Weyl group of \( \text{GL}_N(K+N) \) and \( \lambda_N = -w_0(\lambda_N) \). The projection \( \text{GL}_N(K+N)(z) \to \text{Gr}_{\text{GL}_N(K+N)} \) identifies \( \overline{W}_{w_0(\lambda_N)}^\lambda \) with the cotangent bundle of the orbit \( U_{\lambda_N} \cdot z^{-\lambda_N} \) as a Poisson variety, where \( U_{\lambda_N} \) is the unipotent group whose Lie algebra is the eigenspace of \( \lambda_N \) with eigenvalue \(-1\), see \cite{41], Theorem 4.8]. \( U_{\lambda_N} \) can be naturally identified with the lower-triangular block in the following block-decomposition:

\[
\begin{pmatrix}
\text{SL}_N & 0 \\
U_{\lambda_N} & \text{SL}_{N+K}
\end{pmatrix}.
\]

\( L_{\text{bal}} = \text{Stab}_{\text{SL}_N(K+N)}(w_0(\lambda_N)) = \text{SL}_N \times \text{SL}_{N+K} \) acts on \( U_{\lambda_N} \) naturally, and the action of \( L_{\text{bal}} \) on the Coulomb branch \( \overline{W}_{w_0(\lambda_N)}^\lambda \cong T^*U_{\lambda_N} \) is induced from its action on \( U_{\lambda_N} \) by extending to the cotangent bundle. The half of cohomological grading \( \text{deg}_h \) is such that degree of linear coordinates on \( U_{\lambda_N} \) is zero, and degree of linear coordinates on cotangent fiber is 1. The quantized Coulomb branch algebra \( \mathcal{A}_h \) is an \( L_{\text{bal}} \)-equivariant and \( \text{deg}_h \)-graded quantization of the Poisson variety \( T^*U_{\lambda_N} \). There exists a unique such quantization, which is the ring of \( h \)-differential operators \( \text{Diff}_h(U_{\lambda_N}) \). \( \text{Diff}_h(U_{\lambda_N}) \) is isomorphic to the Weyl algebra \( \text{Weyl}_{h}^\otimes N(K+N) \), therefore we conclude that

\[
\text{Weyl}_{h}^\otimes N(K+N) \cong \mathcal{A}_h \text{ associated to the quiver in the Figure 5}.
\]
These quantized Coulomb branches also appear in the context of quantized phase spaces of minuscule 't Hooft operators in 4d Chern-Simons theories, see [42].

Consider a quiver $Q$ containing following part

![Figure 6: The quiver $Q$.](image)

Then $\mathcal{A}_h(Q)$ admits an action of $\text{SL}_N$, and also a grading ($\mathbb{C}^\times$ action) coming from $\pi_0(\text{Gr}_{\text{GL}_N})$ which commutes with the $\text{SL}_N$ action, thus $\mathcal{A}_h(Q)$ admits an action of $\text{GL}_N$. Denote the following quiver by $Q'$

![Figure 7: The quiver $Q'$.](image)

then we have

**Lemma 5.3.** $\mathcal{A}_h(Q') \cong \mathcal{A}_h(Q)^{\text{GL}_N}$.

**Proof.** Consider the affine Grassmannian $\text{Gr}_{\text{GL}_N}$ and denote by $\mathcal{A}_Q$ (resp. $\mathcal{A}_{Q'}$) the ring object in $D_{\text{GL}_n(\mathcal{O}) \rtimes C^\times(\text{Gr}_{\text{GL}_N})}$ coming from pushing forward of the dualing complex on the BFN space of triples corresponding to quiver gauge theory $Q$ (resp. $Q'$), see [27]. Then we have $\mathcal{A}_Q \cong \mathcal{A}_R \otimes \mathcal{A}_{Q'}$ [27], where $\mathcal{A}_R$ is the regular ring object with a natural $\text{GL}_N$ action (which is called the right action in [27]). Therefore we have

$$
\mathcal{A}_h(Q)^{\text{GL}_N} = H^*_\text{Gr}_{\text{GL}_N(\mathcal{O}) \times C^\times}(\text{Gr}_{\text{GL}_N}, \mathcal{A}_R \otimes \mathcal{A}_{Q'})^{\text{GL}_N} = H^*_\text{Gr}_{\text{GL}_N(\mathcal{O}) \times C^\times}(\text{IC}_0 \otimes \mathcal{A}_Q') = \mathcal{A}_h(Q').
$$

\[\square\]

**Remark 5.7.** Apply Lemma 5.3 to the quiver in the Figure 5 with $K = 0$, and we see that $U_h(\mathfrak{gl}_N) \cong \left(\text{Weyl}_h^{\otimes N^2}\right)^{\text{GL}_N}$. It follows that $C_h[\mathcal{M}(N, K)] \cong \left(\text{Weyl}_h^{\otimes N(K+N)}\right)^{\text{GL}_N \times \text{GL}_N}$ where the action comes from the restriction of $\text{GL}_N \times \text{GL}_{N+K}$ to $\text{GL}_N \times \text{GL}_N$. Apply Lemma 5.3 to the quiver in the Figure 5 with nonzero $K$, followed by removing the edge between flavor vector spaces as it has no effect on Coulomb branch, we see that

$$
C_h[\mathcal{M}(N, K)] \cong \mathcal{A}_h \text{ associated to the quiver in the Figure 5 with flavor symmetry.}
$$

**Remark 5.8.** Apply Lemma 5.3 to the quiver in the Figure 4 with $N$ replaced by $N + K$, and we see that

$$
U_h(\mathfrak{gl}_{N+K})^{\text{GL}_N} \cong Y^\lambda_{0},
$$

where the right-hand-side is a truncated Yangian for $\mathfrak{sl}_K$ and $\lambda = N\omega_1 + (N+K)\omega_{K-1}$, where $\omega_i$ is the $i$-th fundamental coweight of $\mathfrak{sl}_K$, see [40, Appendix B(viii)]. This is known as the centralizer construction of Yangian in the literature [37, §2.13].

29
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A Hall-Littlewood Polynomials

In this appendix we review some background on symmetric functions, following [43, §3].

Definition A.1. For a tuple of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{N}^n$, we denote its associated partition by $(\alpha_1, \alpha_2, \cdots)$, then the Hall-Littlewood polynomial $P_{\lambda}(x; q)$ in the variables $x = (x_1, \cdots, x_n)$ and $q$ is defined by the formula

$$P_{\lambda}(x; q) := \frac{1}{\prod_{i \geq 0} [\alpha_i]_q!} \sum_{w \in \mathfrak{S}_n} w \left( x^\lambda \prod_{i < j} \frac{1 - qx_j/x_i}{1 - x_j/x_i} \right). \tag{A.1}$$

Here $\alpha_0 = n - \sum_{i \geq 1} \alpha_i$, and $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, and we use the following $q$-number notation

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q \quad \text{and} \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The Hall-Littlewood polynomial $P_{\lambda}(x; q)$ is an interpolation between Schur symmetric functions $s_\lambda(x)$ and monomial symmetric functions $m_\lambda(x)$, in fact we have

$$P_{\lambda}(x; 0) = s_\lambda(x), \quad P_{\lambda}(x; 1) = m_\lambda(x). \quad \tag{A.2}$$

Definition A.2. The Kostka-Foulkes functions are coefficients of the expansion

$$s_\lambda(x) = \sum_{\mu} K_{\lambda\mu}(q) P_{\mu}(x; q). \tag{A.3}$$

In particular, by (A.2) we have

$$K_{\lambda\mu}(0) = \delta_{\lambda\mu}.$$

A.1 Jing operators and transformed Hall-Littlewood polynomials

N. Jing found a definition of Hall-Littlewood polynomials using vertex algebra [44]. Before giving his definition, we recall some plethystic notations.

The ring of symmetric functions, denoted by $\Lambda$, is freely generated by power sum functions $p_k$, that is

$$\Lambda = \mathbb{C}[p_k : k \in \mathbb{Z}_{\geq 1}].$$

Let $A$ be a formal Laurent series in indeterminates $a_1, a_2, \cdots$, we define $p_k[A]$ to be the result of replacing each indeterminate $a_i$ in $A$ by $a_i^k$. Then for any $f \in \Lambda$, the plethystic substitution of $A$ into $f$, denoted $f[A]$, is the image of $f$ under the homomorphism sending $p_k$ to $p_k[A]$.

Example A.1. We list some special cases here.
• Let $A = a_1 + \cdots + a_n$, then $p_k[A] = a_1^k + \cdots + a_n^k = p_k(a_1, \cdots, a_n)$, and thus for any $f \in \Lambda$, we have $f[A] = f(a_1, \cdots, a_n)$.

• Let $A, B$ be formal Laurent series, then $p_k[A \pm B] = p_k[A] \pm p_k[B]$.

• Let $\text{PE} = \exp \left( \sum_{k=1}^{\infty} p_k/k \right)$, then we have

$$\text{PE}[A + B] = \text{PE}[A]\text{PE}[B], \quad \text{PE}[A - B] = \text{PE}[A]/\text{PE}[B].$$

For a single variable $x$, we have $\text{PE}(x) = \frac{1}{1-x}$, thus for a summation $X = x_1 + x_2 + \cdots$,

$$\text{PE}(X) = \prod_{i \geq 1} \frac{1}{1-x_i}, \quad \text{PE}(-X) = \prod_{i \geq 1} (1-x_i).$$

For the rest of this section, we fix the notation $X = x_1 + x_2 + \cdots$.

**Definition A.3.** The Jing operators are the coefficients $S^q(u) = \sum_{m \in \mathbb{Z}} S^q_m u^m$ of the operator generating function $S^q(u)$ defined by

$$S^q(u)f = f[X + (q-1)u^{-1}]\text{PE}[uX]. \quad (A.4)$$

**Proposition A.1 ([33] Proposition 2.12]).** Jing operators $S^q_m$ satisfy relations [1]:

$$S^q_n S^q_{m+1} - q S^q_{m+1} S^q_n = q S^q_{n+1} S^q_m - S^q_{m} S^q_{n+1}. \quad (A.5)$$

**Definition A.4.** Let $\mu = (\mu_1 \geq \cdots \geq \mu_l) \in \mathbb{Z}_{\geq 0}^l$ be a tuple of non-increasing integers, define the transformed Hall-Littlewood polynomial by

$$H_\mu(x; q) = S^q_{\mu_1} S^q_{\mu_2} \cdots S^q_{\mu_l}(1). \quad (A.6)$$

For a general array $\mu = (\mu_1, \cdots, \mu_l) \in \mathbb{Z}_{\geq 0}^l$, we define the generalized transformed Hall-Littlewood polynomial by the same formula above.

Using relations (A.5) recursively, we can bring a product of operators $S^q_{\mu_1} \cdots S^q_{\mu_l}$ for an array $\mu = (\mu_1, \cdots, \mu_l) \in \mathbb{Z}_{\geq 0}^l$ into a linear combination of operators $S^q_{\mu_1'} \cdots S^q_{\mu_l'}$ such that $\mu_1' \geq \cdots \geq \mu_l'$, in other words, a generalized transformed Hall-Littlewood polynomial can be written as linear combination of usual transformed Hall-Littlewood polynomials.

The following proposition summarizes the fundamental properties of transformed Hall-Littlewood polynomials.

**Proposition A.2 ([33] 3.4.3]).** The transformed Hall-Littlewood polynomials $H_\mu$ are related to the classical Hall-Littlewood polynomials $P_\mu$ by

$$H_\mu[(1-q)X; q] = (1-q)^{l(\mu)} \prod_{i=1}^{\mu_1} [a_i(\mu)]_q! P_\mu(x; q). \quad (A.7)$$

They are uniquely characterized by the following properties.

(i) $H_\mu(x; q) \in s_\mu(x) + \mathbb{Z}[q] \cdot \{s_\lambda(x) : \lambda > \mu\},$

(ii) $H_\mu[(1-q)x; q] \in \mathbb{Z}[q] \cdot \{s_\lambda(x) : \lambda \leq \mu\}.$

---

*We note that our $q$ is denoted by $t$ there and our $S^q_m$ is denoted by $H_m$ there.*
And $H_\mu$ is related to Schur functions by

$$H_\mu(x; q) = \sum_\lambda K_{\lambda\mu}(q)s_\lambda(x). \quad \text{(A.8)}$$

We can write down the action of Jing operators $S_m^q$ explicitly as in the following lemma.

**Lemma A.1.** For an $n$-variable function $f \in \mathbb{C}[p_1, \cdots, p_n](q)$, where $p_k(x) = x_1^k + \cdots + x_n^k$, Jing operator $S_m^q$ acts on it as

$$(S_m^q f)(x; q) = \sum_{i=1}^{n} f(x_1, \cdots, qx_i, \cdots, x_n; q) \frac{x_i^m}{\prod_{j \neq i}(1 - x_j/x_i)}. \quad \text{(A.9)}$$

**Proof.** Notice that

$$PE(uX) = \prod_{i=1}^{n} \frac{1}{1 - ux_i} = \sum_{i=1}^{n} \frac{1}{1 - ux_i} \prod_{j \neq i} \frac{1}{1 - x_j/x_i}. \quad \text{(A.10)}$$

Without loss of generality, we assume that $f = p_{k_1} \cdots p_{k_s}$, then by definition, $S_m^q$ is the coefficient of $u^m$ in the series expansion

$$(p_{k_1} + (q^{k_1} - 1)u^{-k_1}) \cdots (p_{k_s} + (q^{k_s} - 1)u^{-k_s}) \sum_{i=1}^{n} \frac{1}{1 - ux_i} \prod_{j \neq i} \frac{1}{1 - x_j/x_i}.
$$

For the $i$-th summand, its $u^m$ coefficient is

$$= \left(x_1^{k_1} + \cdots + q^{k_1}x_1^{k_1} + \cdots + x_n^{k_1}\right) \cdots \left(x_s^{k_s} + \cdots + q^{k_s}x_1^{k_s} + \cdots + x_n^{k_s}\right) \prod_{j \neq i} \frac{x_i^m}{(1 - x_j/x_i)}
$$

Summing over $i$ gives the desired formula \text{(A.9)}.

\[\square\]

**B. Affine Grassmannians and Geometrization of Jing Operators**

In this section we give a geometric definition of Jing operators $S_m^q$. Recall that

$$K_{GL_n \times \mathbb{C}^\times}(pt) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x_1^\pm, \cdots, x_n^\pm, q^\pm]^{S_n}. \quad \text{(B.1)}$$

To simplify notation, we will abbreviate $K_{GL_n \times \mathbb{C}^\times}(pt) \otimes_{\mathbb{Z}} \mathbb{C}$ to $K_{GL_n \times \mathbb{C}^\times}(pt)$. Notice that there is a subalgebra $\mathbb{C}[p_1, \cdots, p_n, q^\pm] \subset K_{GL_n \times \mathbb{C}^\times}(pt)$, where $p_i = x_1^i + \cdots + x_n^i$ is the $i$-th power sum polynomial.

Consider the affine Grassmannian $Gr_{GL_n} = GL_n(\mathcal{X})/GL_n(\mathcal{O})$, and let $\omega = (1, 0, \cdots, 0)$ be the first fundamental coweight of $GL_n$, then the $GL_n(\omega)$-orbit $Gr_{\omega}$ is isomorphic to $\mathbb{P}^{n-1}$ and it is fixed by the $\mathbb{C}^\times$-rotation.

Consider the $GL_n(\omega) \times \mathbb{C}^\times$-equivariant bounded derive category of coherent sheaves on $Gr_{GL_n}$, denoted by $D^b_{GL_n(\omega) \times \mathbb{C}^\times}(Gr_{GL_n})$. Here coherent sheaves on ind-scheme like $Gr_{GL_n}$ are defined to have finite type support. In particular, for any $\mathcal{F} \in D^b_{GL_n(\omega) \times \mathbb{C}^\times}(Gr_{GL_n})$, we have $\chi(\mathcal{F}) \in K_{GL_n(\omega) \times \mathbb{C}^\times}(pt) = K_{GL_n \times \mathbb{C}^\times}(pt)$.

There is a convolution product on affine Grassmannian, defined as:

$$m : Gr_{GL_n} \times Gr_{GL_n} = GL_n(\mathcal{X}) \times GL_n(\mathcal{X})/GL_n(\mathcal{O}) \rightarrow GL_n(\mathcal{X})/GL_n(\mathcal{O}). \quad \text{(B.2)}$$
Here the map sends \((g_1, g_2)\) to \(g_1g_2\). The convolution map of \(\text{Gr}_{GL_n}\) induces a functor \(* : D^b_{\text{GL}_n(\mathcal{O})} \times D^b_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times} \to D^b_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times}\) defined as
\[
\mathcal{F} * \mathcal{G} = \text{R}m_*(\mathcal{F} \boxtimes \mathcal{G}).
\]
Passing to the \(K\)-theory, we obtain an map
\[
* : K_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times} \to K_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times}. \tag{B.3}
\]
It is known that the \(*\)-product on \(K_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times}\) is associative and its classical limit \(K_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times}\) is commutative, this is an example of \(K\)-theoretic Coulomb branch in the sense of [40].

Using the convolution algebra \(K_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times}\) we can realize Jing operators \(S_m^n\) geometrically as follows. The determinant line bundle \(\mathcal{O}(1)\) on \(\text{Gr}_{GL_n}\) is \(\mathbb{C}^\times\)-equivariant construction. Let us use \(\mathcal{O}_{\text{Gr}^1}(m)\) to denote \(i_* i^* \mathcal{O}(1)^{\otimes m}\) where \(i : \text{Gr}^1 \hookrightarrow \text{Gr}_{GL_n}\) is the natural embedding. Since \(i\) is \(\mathbb{C}^\times\)-equivariant, \(\mathcal{O}_{\text{Gr}^1}(m)\) is also \(\mathbb{C}^\times\)-equivariant.

**Proposition B.1.** For \(\mathcal{F} \in D^b_{\text{GL}_n(\mathcal{O}) \times \mathbb{C}^\times} \), let \(\chi = \chi(\mathcal{F}) = \chi([x_1^+, \cdots, x_n^+, q^\pm]) \mathbb{S}^n\) be the equivariant Euler characteristic of \(\mathcal{F}\), similarily let \(\tilde{\chi} = \chi(\mathcal{O}_{\text{Gr}^1}(m) * \mathcal{F})\).

Then
\[
\tilde{\chi}(x; q) = \sum_{i=1}^{n} \chi(x_1, \cdots, qx_i, \cdots, x_n; q) \prod_{j \neq i} (1 - x_j/x_i). \tag{B.4}
\]

**Proof.** Let \(p : \text{Gr}_{GL_n} \times \text{Gr}_{GL_n} \to \text{Gr}_{GL_n}\) be the projection to the first component map, i.e. \(p(g_1, g_2) = g_1\), this is a fibration with fibers isomorphic to \(\text{Gr}_{GL_n}\). Then by the projection formula we have
\[
\chi(\mathcal{O}_{\text{Gr}^1}(m) * \mathcal{F}) = \chi(\mathbb{P}^{n-1}, \mathcal{O}(m) \otimes L^{*} R^{*} \mathcal{F}). \tag{B.5}
\]
Here \(\tilde{\mathcal{F}} = \mathcal{O}(\mathcal{F})\) is the twist of \(\mathcal{F}\) on \(\text{Gr}_{GL_n} \times \text{Gr}_{GL_n}\). We use the localization on \(\mathbb{P}^{n-1}\) to compute the right hand side of (B.5) as following. Let the maximal torus of \(GL_n\) be \(T\), then \(T\)-fixed points of \(\mathbb{P}^{n-1}\) are \([1, 0, \cdots, 0], \cdots, [0, \cdots, 1, 0], \cdots, [0, \cdots, 1]\) (in homogeneous coordinates of \(\mathbb{P}^{n-1}\)), label these points by \(e_1, \cdots, e_n\). Observe that

1. The fiber of determinant line bundle \(\mathcal{O}(1)\) at \(e_i\) has \(T\)-weight \(x_i\),
2. The tangent space at \(e_i\) has \(T\)-weights \(x_i/x_j, j \in \{1, \cdots, n\} \setminus \{i\}\),
3. The fiber of \(L^{*} R^{*} \mathcal{F}\) at \(e_i\) has the same \(T\)-weights as \(\chi(\mathcal{F})\), but the \(\mathbb{C}^\times\)-action is different, because the fiber \(p^{-1}(e_i)\) is is identified with \(\text{Gr}_{GL_n}\) via a translation \(g \mapsto z^{\omega_i - \omega_i - 1} g\) and the new \(\mathbb{C}^\times\) acts through the diagonal of \(\mathbb{C}^\times_{\text{rotation}} \times T_i\), where \(T_i\) is the \(i\)'th \(\mathbb{C}^\times\)-component of \(T\). In other word, the fiber of \(L^{*} R^{*} \mathcal{F}\) at \(e_i\) has the \(T \times \mathbb{C}^\times\)-weights
\[
\chi(\mathcal{F})(x_1, \cdots, qx_i, \cdots, x_n; q).
\]

Then (B.4) follows from applying localization to \(\mathcal{O}(m) \otimes L^{*} R^{*} \mathcal{F}\) using three observations made above.

Comparing (B.1) and (A.9), we have the following

**Corollary B.1.** If \(\chi(\mathcal{F}) \in \mathbb{C}[p_1, \cdots, p_n, q^\pm] \subset K_{\text{GL}_n \times \mathbb{C}^\times}(pt)\), then
\[
\chi(\mathcal{O}_{\text{Gr}^1}(m) * \mathcal{F}) = S_m^q \chi(\mathcal{F}). \tag{B.6}
\]

From this corollary we see that the operator \(\mathcal{O}_{\text{Gr}^1}(m) * (-)\) is a geometrization of the Jing operator \(S_m^q\). In fact, it extends the domain of \(S_m^q\) to \(K_{\text{GL}_n \times \mathbb{C}^\times}(pt)\), and negative \(m\) is also allowed.
Corollary B.2. Let $\mu = (\mu_1, \ldots, \mu_l)$ be an array of nonnegative integers, then
\[ H_\mu(x; q) = \chi(\text{Gr}_{GL_n}, \mathcal{O}_{G^\omega_1}(\mu_1) \ast \cdots \ast \mathcal{O}_{G^\omega_1}(\mu_l)). \] (B.7)

Proof. Combine (B.6) with the definition of $H_\mu$ in terms of iterative action of $S^\mu_n$ (A.6).

Corollary B.3. Let $\text{Gr}^{N_\omega_1}$ be the closure of the $\text{GL}_n(\mathcal{O})$-orbit through $z^{N_\omega_1}$, then
\[ \chi(\text{Gr}^{N_\omega_1}, \mathcal{O}(k)) = H_{(k^N)}(x; q). \] (B.8)

Here $(k^N)$ is the short-hand notation for $N$-tuples of $k$, i.e. $(k^N) := (k, k, \ldots, k)$.

Proof. Let $m : \text{Gr}_{GL_n} \times \text{Gr}_{GL_n} \times \cdots \times \text{Gr}_{GL_n} \to \text{Gr}_{GL_n}$ be the convolution map of $N$-copies of $\text{Gr}_{GL_n}$, it follows from the definition of determinate line bundle that there is a $\text{GL}_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant isomorphism
\[ m^*\mathcal{O}(1) \cong \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1). \] (B.9)

It is known that $m(\text{Gr}^{\omega_1} \times \cdots \times \text{Gr}^{\omega_1}) = \text{Gr}^{N_\omega_1}$, and it is birational, thus $m$ is a resolution of singularities. It is also known that $\text{Gr}^{N_\omega_1}$ has rational singularities (this is true for all $G(\mathcal{O})$-orbit closure on affine Grassmannian of any reductive group $G$, see [34, Theorem 2.7]), therefore $Rm_*\mathcal{O} \cong \mathcal{O}$ and $Rm_*m^*\mathcal{O}(k) \cong \mathcal{O}(k)$, thus
\[ \chi(\text{Gr}^{N_\omega_1}, \mathcal{O}(k)) = \chi(\text{Gr}^{\omega_1} \times \cdots \times \text{Gr}^{\omega_1}, m^*\mathcal{O}(k)) \]
by (B.9) \[ = \chi(\text{Gr}_{GL_n}, \mathcal{O}_{G^\omega_1}(k) \ast \cdots \ast \mathcal{O}_{G^\omega_1}(k)) \]
by (B.6) \[ = H_{(k^N)}(x; q). \]

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