$\mathcal{N} = 2$ Instanton Effective action in $\Omega$-background and D3/D($-1$)-brane System in R-R background

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Abstract. We study the relation between the ADHM construction of instantons in the $\Omega$-background and the fractional D3/D($-1$)-branes at the orbifold singularity of $\mathbb{C}^2/\mathbb{Z}_2$ in Ramond-Ramond (R-R) 3-form field strength background. We calculate disk amplitudes of open strings connecting the D3/D($-1$)-branes in certain R-R background to obtain the D($-1$)-brane effective action deformed by the R-R background. We show that the deformed D($-1$)-brane effective action agrees with the instanton effective action in the $\Omega$-background.

1. Introduction

On the study of the instanton effects for the gauge theory, it is shown in [1] that the deformations of the theory are very useful in the calculation of the instanton partition function for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory using the localization technique, which is called the Nekrasov’s formula. The noncommutativity of the gauge theory resolves the the small instanton singularity of the instanton moduli space [2], and the $\Omega$-background isolates the fixed points of BRST nilpotent operator, which is obtained by the topological twist of the theory [3].

In order to apply Nekrasov’s formula to various gauge theories, it would be important to realize the $\Omega$-background in superstring theory. Recently, the low-energy effective theories on D-branes in the constant R-R 3-form field strength background have been attracted much attentions in study of non-perturbative effects in supersymmetric gauge theories. For example, the field theory on D-branes with NS-NS B-field background is described by the noncommutative gauge theory [4, 5]. The noncommutativity of the gauge theory resolves the small instanton singularity of the instanton moduli space [2], which helps the calculation of the instanton partition function via localization technique [1]. In [6], the authors studied the low-energy effective action of D($-1$)-branes in the D3-D($-1$) system located at the singularity of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ in some constant self-dual R-R 3-form background and found that the effective action of the D($-1$)-branes coincides with the effective action for the instanton moduli, which is called the ”instanton effective action”, of $\mathcal{N} = 2$ super Yang-Mills theory in the self-dual $\Omega$-background.

Following these results, in this work [7] we will study the instanton effective action in general (non-(anti-)self-dual) $\Omega$-background and its relations to the D3/D($-1$)-brane system in certain R-R 3-form background. First we we introduce four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory in the $\Omega$-background and study the ADHM construction of instantons. We compute...
the instanton effective action in the Ω-background and show that the action is exact under the deformed supersymmetry. Next we investigate general R-R background and calculate the disk amplitudes in the presence of the R-R 3-form background. Based on the amplitudes, we construct the effective action of D(-1)-branes in the fractional D3/D(-1)-brane system.

2. \( \mathcal{N} = 2 \) instanton effective action in Ω-background

The Lagrangian for \( \mathcal{N} = 2 \) super Yang-Mills theory is given by

\[
\mathcal{L}_0 = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F_{mn} F^{mn} - \frac{i g^2}{32\pi^2} F_{mn} \tilde{F}^{mn} + \Lambda^I \sigma^m D_m \bar{\Lambda}_I + (D_m \varphi) D^m \bar{\varphi} 
- i \frac{g}{\sqrt{2}} \Lambda^I [\bar{\varphi}, \Lambda_I] + i \frac{g}{\sqrt{2}} \bar{\Lambda}_I [\varphi, \bar{\Lambda}^I] + \frac{g^2}{2} [\varphi, \bar{\varphi}]^2 \right],
\]

where \( F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n] \) is the gauge field strength of \( U(N) \) gauge field \( A_m \) \((m = 1, 2, 3, 4)\), \( \Lambda^I, \bar{\Lambda}_I \) are Weyl fermions, \( \varphi \) and \( \bar{\varphi} \) are complex scalars, \( g \) is the gauge coupling constant and \( D_m = \partial_m + ig[A_m, \cdot] \) is the gauge covariant derivative. We also define the Dirac matrices \( \sigma_m = (\tau^1, i\tau^2, i\tau^3, 1) \) and \( \bar{\sigma}_m = (-i\tau^1, -i\tau^2, -i\tau^3, 1) \), where \( \tau^c \) \((c = 1, 2, 3)\) are the Pauli matrices. \( \sigma^m \) and \( \bar{\sigma}^m \) are the Lorentz generators defined by

\[
(\sigma^m)_\alpha^\beta = \frac{1}{4} (\sigma^m_{\alpha\alpha} \sigma^{\alpha\beta} - \sigma^m_{\alpha\beta} \sigma^{\alpha\alpha} - \sigma^{m\alpha} \sigma_{\alpha\beta} + \sigma^{m\beta} \sigma_{\alpha\alpha}), \quad (\bar{\sigma}^m)_\alpha^\beta = \frac{1}{4} (\bar{\sigma}^{m\alpha} \sigma_{\alpha\beta} - \bar{\sigma}^{m\beta} \sigma_{\alpha\alpha}).
\]

\( \theta \) is the theta-angle and \( \tilde{F}_{mn} = \frac{1}{2} \epsilon_{mpnq} F^{pq} \) is the dual of \( F_{mn} \). The projection of the field strength into the (anti-)self-dual part is given by \( F^{\pm}_{mn} = \frac{1}{2} (F_{mn} \pm \tilde{F}_{mn}) \). The second term of (1) is topological and the instanton number is defined by

\[
k = \int d^4x \frac{1}{\kappa} \text{Tr} \left[ \frac{g^2}{32\pi^2} F_{mn} \tilde{F}^{mn} \right].
\]

Four-dimensional \( \mathcal{N} = 2 \) super Yang-Mills theory in \( \Omega \)-background is obtained by dimensional reduction of six-dimensional \( \mathcal{N} = 1 \) super Yang-Mills theory. The \( \Omega \)-background \([8, 9, 1, 10, 11]\) is defined by a nontrivial fibration of \( \mathbb{R}^4 \) over \( T^2 \). We denote the complex coordinates of a two-dimensional torus \( T^2 \) by \((z, \bar{z})\). The six-dimensional metric is given by

\[
ds_{6\Omega}^2 = 2d\bar{z}dz + (dx^a + \Omega^a d\bar{z} + \bar{\Omega}^a dz)^2,
\]

where \( \Omega^a = \Omega^{mn} x_n, \bar{\Omega}^a = \bar{\Omega}^{mn} x_n \). \( \Omega^a \) and \( \bar{\Omega}^a \) are constant antisymmetric matrices. We require that the metric (4) is flat. This leads to the condition that \( \Omega^a \) and \( \bar{\Omega}^a \) commute with each other:

\[
\Omega^{mn} \bar{\Omega}_{np} - \bar{\Omega}^{mn} \Omega_{np} = 0.
\]

We also introduce the R-symmetry Wilson line by gauging \( SU(2)_I \) R-symmetry as

\[
A^I_{\ j} = A^I_{\ j}d\bar{z} + i\bar{A}^I_{\ j}dz,
\]

where we consider the case such that the R-symmetry gauge fields \( A^I_{\ j}, \bar{A}^I_{\ j} \) are constant.

The instanton effective action for the action (1) in the \( \Omega \)-background is obtained by deriving the instanton solution for the equations of motion for the action using the ADHM construction \([12, 13]\) and substituting the solution to the action. In order to solve the equations of motion,
we developed the deformed ADHM construction of the instanton solution. The result of the
instanton effective action is

\[ S^{(0)}_{\text{eff}} = \frac{2\pi^2}{\kappa} \text{tr} \left[ -2 \left( [\bar{\chi}, a'_m] + i\bar{\Omega}_{mn}a'_n \right) \left( [\chi, a^m] - i\Omega^{mn}a'_p \right) + (\bar{\chi}\bar{w}^\alpha - \bar{w}^\alpha\phi^0) (w_\alpha\chi - \phi^0 w_\alpha) + (\chi\bar{w}^\alpha - \bar{w}^\alpha\phi^0) (w_\alpha\bar{\chi} - \phi^0 w_\alpha) \right] \]

\[ + \frac{i}{2\sqrt{2}} \mathcal{M}^{\alpha IJ} \left( [\bar{\chi}, \mathcal{M}_\alpha^I] - \frac{i}{2} \bar{\Omega}^+_{mn} (\sigma^{mn})_{\alpha\beta} \mathcal{M}_{\beta J}^I \right) \]

\[ - \frac{i}{2\sqrt{2}} \bar{\mu} I_1 \mu J + \frac{1}{2} \bar{\mathcal{A}} I J \left( \bar{\mu} I_1 \mu J + \frac{1}{2} \mathcal{M}^{\alpha I} \mathcal{M}_{\alpha J} \right) \]

\[ + i\Omega^-_{mn} (\bar{\sigma}^{mn})_{\bar{\alpha}\bar{\beta}} \bar{w}^{\bar{\alpha}} \phi^0 w_\bar{\alpha} + i\Omega^-_{mn} (\bar{\sigma}^{mn})_{\bar{\alpha}\bar{\beta}} \bar{w}^{\bar{\alpha}} \phi^0 w_\bar{\alpha} \]

\[ - 2i\Omega^-_{mn} \bar{\chi} [a'_m, a'_n] - 2i\Omega^-_{mn} \chi [a'_m, a'_n] + \frac{1}{2} \Omega^-_{mn} \bar{\Omega}^-_{mn} w_\bar{\alpha} w_\bar{\alpha} \]

\[ - i\bar{\psi} I (\bar{\mu} I_1 w_\bar{\alpha} + \bar{w}_\alpha \mu J + [\mathcal{M}^{\alpha I}, a'_\alpha a'_\beta]) \]

\[ + iD^c(\tau^c)^{\bar{\alpha}}_{\bar{\beta}} (\bar{w}^{\bar{\beta}} w_\alpha + \bar{a}^{\bar{\beta}\alpha} a'_{\alpha\beta}) \] \ (7)

This instanton effective action has an equivariantly nilpotent operator, which is needed to obtain
the instanton partition function using the localization formula.

3. Effective action of D3/D(-1)-brane system in R-R background
In the context of the string theory, the instanton effective action of 4d \( N = 2 \) U(N) super Yang-
Mills theory is equivalent to the effective action of D3/D(-1)-brane system [14, 15] located at
the fixed point of the orbifold \( C \times C^2/Z_2 \) [6]. In order to consider deformations to the D(-1)
effective action, we incorporate R-R 3-form flux. In [6], the case of the constant self-dual R-R
3-form background \( F_{mn} \) (\( m, n = 1, \cdots, 4 \), \( a = 5, 6 \)) in type IIB string theory has been studied.
By identifying the self-dual R-R backgrounds with the self-dual \( \Omega \)-background parameters with
\( \Omega_{mn} = \Omega^+_mn, \bar{\Omega}^-_{mn} = \bar{\Omega}^-_{mn} \), the authors showed that the deformed low-energy effective action of
D3/D(-1)-branes coincides with the instanton effective action in \( N = 2 \) super Yang-Mills theory
in the self-dual \( \Omega \)-background. In order to realize the deformed instanton effective action in the
general \( \Omega \)-background, we need to turn on non-(anti)-self-dual R-R 3-form flux with indices \( F_{mn} \)
and \( F_{abc} \). Computing the disk amplitudes of strings connecting the branes and collecting the
non-vanishing amplitudes under the field theory limit, we obtain the following effective action :

\[ S_{\text{str}}(C, \bar{C}, m) = \frac{2\pi^2}{\kappa} \text{tr} \left[ -2 \left( [\bar{\chi}, a'_m] + C_{mn}a'_n \right) \left( [\chi, a^m] + C^{mk}a'_k \right) \right] \]

\[ + \frac{i}{2\sqrt{2}} \mathcal{M}^{\alpha IJ} \left( [\bar{\chi}, \mathcal{M}_\alpha^I] + \frac{1}{2} C^{+mn} (\sigma_{mn})_{\alpha\beta} \mathcal{M}_{\beta J} + \frac{1}{2} m_{IJ} \mathcal{M}^{\alpha I} \right) \]

\[ + \left( \bar{\chi}\bar{w}^\alpha - \bar{w}^\alpha\phi^0 + \frac{1}{2} C^{-mn} (\bar{\sigma}_{mn})^{\bar{\alpha}\bar{\beta}} \bar{w}^{\bar{\alpha}} \right) \left( w_\alpha\chi - \phi^0 w_\alpha + \frac{1}{2} C^{-mn} (\bar{\sigma}_{mn})^{\bar{\alpha}\bar{\gamma}} w_\bar{\gamma} \right) \]

\[ + \left( \chi\bar{w}^\alpha - \bar{w}^\alpha\phi^0 + \frac{1}{2} C^{-mn} (\sigma_{mn})^{\bar{\alpha}\bar{\beta}} \bar{w}^{\bar{\beta}} \right) \left( w_\alpha\bar{\chi} - \phi^0 w_\alpha + \frac{1}{2} C^{-mn} (\bar{\sigma}_{mn})^{\bar{\alpha}\bar{\gamma}} w_\bar{\gamma} \right) \]

\[ + \frac{i}{2\sqrt{2}} \bar{\mu} I_1 \mu J + \frac{1}{2} \bar{\mathcal{A}} I J \left( \bar{\mu} I_1 \mu J + \frac{1}{2} \mathcal{M}^{\alpha I} \mathcal{M}_{\alpha J} \right) \]

\[ - i\bar{\psi} I (\bar{\mu} I_1 w_\bar{\alpha} + \bar{w}_\alpha \mu J + [\mathcal{M}^{\alpha I}, a'_\alpha a'_\beta]) \]

\[ + iD^c(\tau^c)^{\bar{\alpha}}_{\bar{\beta}} (\bar{w}^{\bar{\beta}} w_\alpha + \bar{a}^{\bar{\beta}\alpha} a'_{\alpha\beta}) \] \ (8)
where we have defined $C_{mn} = C_{mn}^+ + C_{mn}^-$, $\bar{C}_{mn} = \bar{C}_{mn}^+ + \bar{C}_{mn}^-$ and

$$C_{mn}^+ = -2\pi \sqrt{2}(2\pi \alpha')^{1/2} \epsilon_{\beta\gamma}(\sigma_{mn})^\alpha_\gamma \epsilon_{IJ}\mathcal{F}(\alpha\beta)[IJ],$$

$$C_{mn}^- = -2\pi \sqrt{2}(2\pi \alpha')^{1/2} \epsilon_{\beta\gamma}(\sigma_{mn})^\alpha_\gamma \epsilon_{I'J'}\mathcal{F}(\alpha\beta)[I'J'],$$

$$\bar{C}_{mn}^+ = -2\pi \sqrt{2}(2\pi \alpha')^{1/2} \epsilon_{\beta\gamma}(\sigma_{mn})^\alpha_\gamma \epsilon_{IJ}\mathcal{F}(\alpha\beta)[IJ],$$

$$\bar{C}_{mn}^- = -2\pi \sqrt{2}(2\pi \alpha')^{1/2} \epsilon_{\beta\gamma}(\sigma_{mn})^\alpha_\gamma \epsilon_{I'J'}\mathcal{F}(\alpha\beta)[I'J'],$$

$$m_{(IJ)} = \pi i (2\pi \alpha')^{1/2} \mathcal{F}(\alpha\beta)[IJ] \epsilon^\alpha\beta.$$  (9)

Comparing the actions (8) with (7), they coincide if we identify $C_{mn} = -i\Omega_{mn}$, $\bar{C}_{mn} = -i\bar{\Omega}_{mn}$ and

$$m_{(IJ)} = -\frac{1}{2\sqrt{2}} \epsilon_{JK} \tilde{A}^K J.$$  (10)

then redefine $D^c$ in (7) as

$$D^c \rightarrow D^c - \frac{i}{2} \eta_{mn}(\tilde{\Omega}^{-mn}\chi + \Omega^{-mn}\tilde{\chi}).$$  (11)

### 4. Conclusion

In this paper, we have studied general (non-(anti-)self-dual) $\Omega$-background deformation of the instanton effective action of $\mathcal{N} = 2$ super Yang-Mills theory and the R-R 3-form background deformation of the D(-1)-brane effective action. We have found that these two effective actions coincide under the appropriate identification of the deformation parameters.

**Acknowledgments**

The work of H. N. is supported by Mid-career Researcher Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(No. 2009-0084601). The work of T. S. is supported by the Global Center of Excellence Program by MEXT, Japan through the "Nanoscience and Quantum Physics" Project of the Tokyo Institute of Technology, and by Iwami Fujikai Foundation. The work of S. S. is supported by the Japan Society for the Promotion of Science (JSPS) Research Fellowship.

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