Some thin film models based on Shikhmurzaev’s approach

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Abstract

We derive several lubrication approximation models, using Shikhmurzaev’s approach to the contact line problem, obtained in [7]. The first two lubrication models describe thin film flow of incompressible fluids on solid substrates, based on different orders of magnitude of the slip length parameter. The third lubrication model describes a meniscus formation where a wedge-shaped solid immerses in a thin film of fluid.

1 Introduction

In this article we derive different lubrication models using Shikhmurzaev’s approach to the contact line problem. The process of displacement of one fluid by another immiscible fluid from a solid surface is described as the (moving) contact line phenomena (also known as dynamic wetting). Each of the two phases (fluid/liquid/solid) are separated by interfaces, and the three interfaces meet on a region where the molecules are subject to the action of intermolecular forces from three bulk phases. In the continuum approximation, this region becomes a one-dimensional three-phase contact line. Here we restrict ourselves to the case of liquid/gas/solid case. With time, the fluid particles which are initially located on the liquid-gas interface, arrive at the contact line and then move along the solid surface away from it. Thus one needs to take into account the disappearance of the liquid-gas interface and the formation of the liquid-solid one simultaneously. It should be noted, however, that the liquid-gas interface is just one source of material for building the new liquid-solid interface, and the rest of it is provided by the bulk, which makes the interface formation dependent on the bulk flow. As a result, the contact line moves along the solid. The problem is to model and develop a mathematical theory of this interface formation/disappearance in the framework of continuum mechanics, in order to describe the dynamics of the system and in particular the contact line. The situation where a liquid displaces a gas from a solid

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surface is known as the advancing motion; On the other hand, kinematics of the receding contact line motion where a liquid is displaced by a gas is somewhat different than the previous case. Zero contact line velocity marks the transition from the advancing to the receding fluid motion. We discuss below the advancing motion only. Note that, one might consider other similar situations in the same spirit, for example fluid/fluid/solid system, or fluid/solid/solid system.

Among several other approaches (cf. [17, Section 3.4]), Y. Shikhmurzaev [15] proposed such a model, based on the work of Bedeaux, Albano and Mazur [1]. The core of his approach is based on the idea that the interfaces which are considered as mathematical surfaces of thickness zero, in the continuum scale, possess their own physical properties, such as surface tension. Hence, these are regarded as two-dimensional continua, in other words, as surface phases. Then one can formulate boundary conditions for the equations describing the bulk phases, using the theory of irreversible thermodynamics.

**Contact line model of Shikhmurzaev**

Let us first describe briefly the model based on Shikhmurzaev’s approach. We consider a liquid-gas-solid system as in Figure 1. The liquid is described as a continuum medium with constant density $\rho_L$, velocity $v_L$, pressure $p_L$ and dynamic viscosity $\mu$. The solid is assumed to be smooth, chemically homogeneous and union of a finite number of rigid pieces having velocity $v_S$, while the gas is assumed to be ideal with pressure $p_G$. The interaction between these phases takes place through the liquid-gas (free surface), liquid-solid and gas-solid interfaces, denoted by $\Sigma_i$, $i = 1, 2, 3$ respectively. The interfaces are assumed to have their own characteristic properties, such as the surface densities $\rho_s$, velocities $v_s$ and surface tensions $p_s$. The contact line $C$ where the three interfaces meet has velocity $v_c$. We denote by $\kappa_i$ the curvature of the interfaces and $n_i$, $i = 1, 2$ the unit normal vectors to $\Sigma_i$, inward with respect to the liquid domain. The following notation is used later: for any vector $w$ defined over a surface with the unit normal vector $n$, it can be decomposed into its normal and tangential components as,

$$w = w_n n + w_\tau,$$

where $w_n = w \cdot n$ and $w_\tau = (I - n \otimes n) \cdot w$.

Note that, according to the above notation, $w_n$ is a scalar quantity whereas $w_\tau$ is a vector. Further, $V_n$ denotes the normal velocity of the free interface $\Sigma_1$ and $\nu_i, i = 1, 2$ are the unit vectors, orthogonal to the contact line, but tangent to the interfaces $\Sigma_i$, respectively. The angle between the free surface and the solid boundary, measured through the liquid, is the dynamic contact angle, denoted by $\theta$. Also we require to introduce some differential quantities, such as the deformation tensor, denoted by $D_v := \frac{1}{2} (\nabla_v + \nabla_v^T)$ (also known as the symmetric gradient), the normal derivative over a surface $\partial_{\Sigma}^n$, the tangential gradient $\nabla_\tau$ and the surface divergence div$\Sigma$. For precise definitions, we refer to [7, Appendix]. The quantities $g_i^s, i = 1, 2$ are the surface densities corresponding to zero surface pressure and $\tau_i, i = 1, 2$ are the relaxation time, i.e. the time taken by the surface properties to reach their equilibrium values. As understood from the above notations, the indices $s$ and $i$ over a quantity means it is defined over the interface $\Sigma_i$.

Then, Shikhmurzaev’s model describing the dynamics of the contact line reads as follows.
The fluid motion is governed by the incompressible Navier-Stokes equations,

\[
div_x v^L = 0, \quad \rho^L ( \partial_t v^L + v^L \cdot \nabla_x v^L ) = \mu \Delta v^L - \nabla_x p^L,
\]

which satisfy the boundary conditions at the liquid-gas interface,

\[
\begin{align*}
V_n &= v^L_1 \cdot n_1, \\
\partial_t \theta^L_1 + \text{div}_\Sigma ( \theta^L_1 v^L_1, n_1 ) + \kappa_\Sigma \theta^L_1 v^L_1, n_1 &= - \frac{\theta^L_1 - \theta^L_1^c}{\tau_1}, \\
\rho^L_1 ( v^L_1 - v^L_1, n_1 ) &= \frac{\theta^L_1 - \theta^L_1^c}{\tau_1}, \\
(p_G - p^L) + 2 \mu [ \nabla v^L_1 \cdot n_1 ] \cdot n_1 + p_1^i \kappa_\Sigma &= 0, \\
2 \mu [ \nabla v^L_1 \cdot n_1 ] - \nabla_P p^L_1 &= 0, \\
-(1 + 4 \alpha_1 \beta_1) \nabla_P p^L_1 &= 4 \beta_1 ( v^L_1, \tau - v^L_1, \tau ) , \\
p^L_1 &= \gamma_1 ( \theta^L_1 - \theta^L_1^c ) ,
\end{align*}
\]

whereas the boundary conditions at the liquid-solid interface are,

\[
\begin{align*}
\partial_t \theta^L_2 + \text{div}_\Sigma ( \theta^L_2 v^L_2, n_2 ) + \kappa_\Sigma \theta^L_2 v^L_2, n_2 &= - \frac{\theta^L_2 - \theta^L_2^c}{\tau_2}, \\
\rho^L_2 ( v^L_2 - v^L_2, n_2 ) &= \frac{\theta^L_2 - \theta^L_2^c}{\tau_2}, \\
2 \mu [ \nabla v^L_2 \cdot n_2 ] - \frac{1}{2} \nabla_P p^L_2 &= \beta_2 ( v^L_2 - v^L_2^c ) , \\
(v^L_2 - v^L_2^c, n_2) &= 0, \\
v^L_2, \tau &= \frac{1}{2} ( v^L_1 + v^L_2^c ) - \alpha_2 \nabla_P p^L_2 , \\
p^L_2 &= \gamma_2 ( \theta^L_2 - \theta^L_2^c ) ,
\end{align*}
\]
together with the conditions at the triple junction,

\[
\begin{align*}
\rho_1^s (v_{1,\tau} - v^c) \cdot \nu_1 + \rho_2^s (v_{2,\tau} - v^c) \cdot \nu_2 &= 0, \\
p_1^s \cos \theta &= p_3^s - p_2^s, \\
m (\rho_1^s - \rho_1^{s\text{ie}}) - (\rho_2^s - \rho_2^{s\text{ie}}) &= 0,
\end{align*}
\]

and initial condition. In addition to the kinematic condition \((1.2a)\) and conditions on the normal and tangential stress \((1.2d), (1.2e)\), the model accounts for the mass exchange between the bulk and the surface phase \((1.2b), (1.2c)\) as the interface relaxes towards its equilibrium state with the surface density \(\rho_i^s\) tending to its equilibrium value \(\rho_i^{s\text{ie}}\). With the presence of a free surface, its surface tension plays an important role for the process to be described and the capillarity equation \((1.2d)\) determines the free surface shape. The appearance of gradient of the surface tension in \((1.2e)\) influences the flow via the tangential stress condition (Marangoni effect), which makes in turn the tangential surface velocity \(v_{i,\tau}\) deviate from the corresponding component of the bulk velocity \(v_{i\tau}\), evaluated at the interface, expressed by \((1.2\text{f})\). The equation of state in the surface phase \((1.2g)\) expresses a simple barotropic form, approximating the general equation of state \(p_i^s = f(\rho_i^s, T)\) where \(T\) is the absolute temperature, for the process of interface formation. The concept of the surface density as a parameter of state was first introduced by Gibbs and since then, used in different forms, together with the notion of surface velocity, by many authors \([4]\).

The liquid-solid interface can be modelled in a similar way taking into account the mass exchange between the liquid and the interface via equations \((1.3a), (1.3b)\), while equation \((1.3d)\) expresses the assumption that the solid is impermeable. Equation \((1.3c)\) takes into account the possible slip of the fluid over the solid boundary. Then the tangential surface velocity can be obtained from equation \((1.3e)\) and the equation of state is given by condition \((1.3f)\) as for the liquid-gas interface.

The equation \((1.2\text{a})\) expressing the free surface curvature as a function of the free surface position being elliptic, one needs one boundary condition at each point of the contact line, to specify the free surface shape. Hence, from the balance of forces acting on the contact line, one obtains the Young equation \((1.4b)\), relating the contact angle with the surface tensions acting on the contact line. The other two conditions \((1.4a)\) and \((1.4c)\) express the mass and energy balance equations at the contact line. Note that the equation \((1.4c)\) is a condition which is missing in the original work \([15]\), but later it has been shown in \([2]\) to be necessary for well-posedness of the contact line problem. The parameter \(\beta_2\) is the slip coefficient on the liquid-solid interface whereas the other parameters \(\alpha_i, \beta_1, \gamma_i, m\) are some phenomenological non-negative constants. For a detailed derivation of the model, we refer to \([7]\).

The main novelty of Shikhmurzaev’s approach is that it can be used to describe a large number of situations which involve interfaces. Hence, we would like to study the lubrication approximation of the above model for different configurations. Our motivation is to obtain a simpler reduced model than the original one, in which the differences between this approach and the other solution models of the contact line problem (for example, Navier
boundary condition, disjoining pressure, precursor film etc) can be compared and to under-
stand whether one obtains better description of the region near the contact line, captured
by this approach.

The thin film equation has been discussed in many works, both experimentally and the-
oretically, taking into account different theories and approaches. Derivation of the classical
thin film equation in case of a droplet can be found in, for example [10, 11]. Also in [8], the
authors have considered thin film equation introducing the slip parameter in the model. On
the other hand, there are other lubrication models, such as of a thin fluid in the Hele-Shaw
cell [3], of Darcy flow [9], rupture of thin film subject to van der Waals forces [5], consider-
ing different orders of magnitude of the slip-length parameter [13], for an active gel [12], for
surfactant driven viscous film [6]. We further refer to the work [14] which discusses many
such models considering different particular cases, from a physical point of view. In the
following, we examine the asymptotic limit of Shikhmurzaev’s model depending on different
geometries. In [16], the author has discussed a rupture phenomenon in the framework of the
thin film approximation.

2 Lubrication approximation

The general three-dimensional problem can be considerably simplified in the thin film
approximation where the ratio of the characteristic length scales in the vertical and hori-
zontal direction to the film is considered as a small parameter. Below we derive different
asymptotic limits depending on different orders of magnitude for the slip length parameter
and also for different models.

2.1 Thin film model for small slip length

First we consider the simple case of spreading of a viscous, incompressible droplet of
liquid over a smooth solid surface where the displaced medium is an inviscid gas, in two
dimension.

![Figure 2: thin film model for droplet](image)

The liquid-gas interface is given by \( y = h(x, t) \) which is the free surface and the liquid-
solid interface is \( y = 0 \). The contact points where these interfaces meet are \( s_i(t) = (\Lambda_i(t), 0) \).
We denote by \( L \) the characteristic length scale of the droplet, \( H \) the characteristic thickness
of the film with \( H \ll L \) and \( \varepsilon = \frac{H}{L} \). We are interested in obtaining the limiting model as \( \varepsilon \to 0 \).

In Shikhmurzaev’s model (1.1)-(1.4b), we use the notation \( \mathbf{v}^L = (u, v), \mathbf{v}^S = (u^S, v^S), i = 1, 2 \) and we assume \( \mathbf{v}^S = 0 \), i.e. the fluid droplet is on a plane at rest. We omit the index for the pressure of the liquid and just denote it by \( p \). Let us then consider the following scales,

\[
\begin{align*}
\tau &= \frac{x}{L}, \\
\eta &= \frac{y}{H}, \\
\Lambda_i &= \frac{1}{L} \Lambda_i, \\
\mathcal{H} &= \frac{h}{H}, \\
\bar{t} &= \frac{\varepsilon^3 \sigma_{1e} t}{L \mu}, \\
\bar{u} &= \frac{\mu}{\varepsilon^4 \sigma_{1e}} u, \\
\bar{v} &= \frac{\mu}{\varepsilon^4 \sigma_{1e}} v, \\
\bar{p} &= \frac{L}{\varepsilon \sigma_{1e}} p, \\
\bar{u}^S &= \frac{\mu}{\varepsilon^3 \sigma_{1e}} u^S, \\
\bar{v}^S &= \frac{\mu}{\varepsilon^3 \sigma_{1e}} v^S, \\
\bar{\rho}^S &= \frac{\varepsilon^2 \sigma_{1e}}{\gamma_1} \zeta_i,
\end{align*}
\]

Here \( \sigma_{1e} := -\gamma(\rho^S_1) = -\gamma_1(\rho^S_{1e} - \rho^S_{1,0}) > 0 \). The time scale is chosen in such a way in order to make the characteristic time of spreading of the droplet of order one. With the time and length scales fixed, it is then standard to re-scale the velocities accordingly. The pressure is scaled as in the standard thin film approximation, in order not to make the first component of the Navier-Stokes equation trivial (cf. (2.13)). Also, the scaling for difference of the surface densities is made due to the relation with the surface tensions (1.2g) or, (1.3f).

### 2.1.1 Reduced model

We show below that equations (1.1)-(1.4c) reduce, to leading order terms in \( \varepsilon \), to the following system:

in \((\tau, \eta) \in (\Lambda_1, \Lambda_2) \times (0, \mathcal{H})\):

\[
\partial_\tau \bar{u} + \partial_\eta \bar{v} = 0, \quad \partial_\tau^2 \bar{u} = \partial_\tau \bar{p}, \quad \partial_\eta \bar{p} = 0; \tag{2.1}
\]

on \( \eta = \mathcal{H} \):

\[
\begin{align*}
\partial_\tau \bar{H} + \bar{u} \partial_\tau \bar{H} &= \bar{v}, \tag{2.2a} \\
\partial_\tau^2 \bar{H} &= -\frac{1}{\lambda_1} \zeta_1, \tag{2.2b} \\
\bar{u} \partial_\tau \bar{H} - \bar{v} \partial_\tau \bar{H} &= \bar{\pi} \partial_\tau \bar{H} - \bar{v}, \tag{2.2c} \\
p &= -\partial_\eta^2 \bar{H}, \tag{2.2d} \\
\partial_\eta \bar{p} &= -\partial_\tau \zeta_1, \tag{2.2e} \\
(\bar{u} - \bar{v}) &= -d_1 \partial_\tau \zeta_1; \tag{2.2f}
\end{align*}
\]
on \( \tilde{y} = 0 \):

\[
\begin{align*}
\partial_x \pi_2^x &= -\frac{1}{\lambda_2} \zeta_2, \quad (2.3a) \\
\nu &= \nu_2^x, \quad (2.3b) \\
\partial_x \tilde{u} &= \frac{1}{2} a \partial_x \zeta_2 + \beta \tilde{u}, \quad (2.3c) \\
\nu_2^x &= 0, \quad (2.3d) \\
\nu_2^x &= \frac{1}{2} \tilde{u} - d_2 \partial_x \zeta_2; \quad (2.3e)
\end{align*}
\]

at \((\tilde{x}, \tilde{y}) = (\Lambda_i, 0), i = 1, 2\):

\[
\begin{align*}
\sigma_{1x} (\pi_1^x - \partial_y \tilde{\Lambda}_1) + \sigma_{2x} (\pi_2^x - \partial_y \tilde{\Lambda}_2) &= 0, \quad (2.4a) \\
\zeta_1 + a \zeta_2 + \frac{1}{2} |\partial_x \tilde{h}|^2 &= b, \quad (2.4b) \\
m \zeta_1 &= \zeta_2. \quad (2.4c)
\end{align*}
\]

In the above equations, 
\( d_1 = \frac{\mu}{L} \frac{(1 + 4 \alpha \beta_1)}{4 \beta_1} \),  \( \lambda_1 = \frac{L \mu}{\varepsilon \sigma_{1x} \gamma_1 \tau_1} \),  \( \lambda_2 = \frac{L \mu}{\varepsilon \sigma_{2x} \gamma_1 \tau_2} \),  \( a = \frac{\tau_2}{\gamma_1} \),  \( \beta = \varepsilon \beta_2 L \mu \),  \( d_2 = \alpha_2 \varepsilon \gamma_2 \mu \),  \( b = -\frac{\alpha_2}{\varepsilon \gamma_2} \sigma_3 \) and all these non-negative constants are \( O(1) \) and dimensionless. The derivation of the above system is sketched below.

To derive (2.1)\,-\,(2.4c), let us first compute some related terms in rescaled variables. The free surface, i.e. the fluid-gas interface is now given by \( \tilde{y} = \tilde{h}(\tilde{x}, \tilde{t}) \), and the unit normal vector, outward with respect to the gas domain and the corresponding tangent vector are given by

\[
\begin{align*}
\mathbf{n}_1 &= \frac{(\varepsilon \partial_x \tilde{h}, -1)}{\sqrt{1 + \varepsilon^2 |\partial_x \tilde{h}|^2}}, \quad \tau = \frac{(1, \varepsilon \partial_x \tilde{h})}{\sqrt{1 + \varepsilon^2 |\partial_x \tilde{h}|^2}}, \\
\end{align*}
\]

and the curvature by

\[
\kappa_{\Sigma} = \frac{\varepsilon \partial_x^2 \tilde{h}}{L (1 + \varepsilon^2 |\partial_x \tilde{h}|^2)^{3/2}}. \quad (2.6)
\]

We compute the rate of strain tensor

\[
\mathbb{D} \mathbf{v}^L = \frac{1}{2} \begin{bmatrix} 2 \partial_x u & \partial_y u + \partial_x v \\ \partial_y u + \partial_x v & 2 \partial_y v \end{bmatrix} = \frac{1}{2} \varepsilon^3 \sigma_{1x} \frac{1}{L \mu} \begin{bmatrix} 2 \varepsilon \partial_x \tilde{u} & \frac{1}{\varepsilon} \partial_y \tilde{u} + \varepsilon \partial_x \tilde{u} \\ \frac{1}{\varepsilon} \partial_y \tilde{u} + \varepsilon \partial_x \tilde{u} & 2 \varepsilon \partial_y \tilde{u} \end{bmatrix}.
\]

Therefore, the normal and the tangential stress on the surface \( \tilde{y} = \tilde{h}(\tilde{x}, \tilde{t}) \) are given by,

\[
\mathbb{D} \mathbf{v}^L \cdot \mathbf{n}_1 \cdot \mathbf{n}_1 = \frac{\varepsilon^3 \sigma_{1x}}{L \mu} \frac{1}{1 + \varepsilon^2 |\partial_x \tilde{h}|^2} \left( -\partial_y \tilde{u} \partial_x \tilde{h} + \partial_y \tilde{u} + O(\varepsilon^2) \right) \quad (2.7)
\]

and

\[
\mathbb{D} \mathbf{v}^L \cdot \mathbf{n}_1 \cdot \tau = \frac{\varepsilon^3 \sigma_{1x}}{L \mu} \frac{1}{2} \frac{1}{1 + \varepsilon^2 |\partial_x \tilde{h}|^2} \left( -\frac{1}{\varepsilon} \partial_y \tilde{u} + O(\varepsilon) \right). \quad (2.8)
\]
Similarly, the tangential gradient on $\mathbf{y} = \mathbf{h}(\mathbf{x}, t)$ becomes (cf. (A.6)),
\[
\nabla_x p_1^s = \frac{1}{\sqrt{1 + \varepsilon^2 |\partial_x \mathbf{h}|^2}} \partial_x \bar{p}_1^s. \tag{2.9}
\]

For the differential terms defined along the surface such as $\partial_i^S$, $\text{div}_\Sigma$, we can compute them using the explicit forms in Appendix A. Also we will not distinguish below between $\bar{p}_1^s$ and $p_1^s$, if not necessary.

On the liquid-solid surface $\mathbf{y} = 0$, corresponding quantities are simpler. The outward unit normal vector with respect to the solid surface is $\mathbf{n}_2 = (0, 1)$, whereas $\tau = (1, 0)$, $\kappa_\Sigma = 0$. Thus, $[\mathbf{D} \mathbf{v}^L \cdot \mathbf{n}_2] \cdot \tau = \frac{1}{2} \frac{\varepsilon^2 \sigma_{ie}}{\mu} \left( \frac{1}{\varepsilon^2} \partial_{\mathbf{h}} \mathbf{u} + \varepsilon \partial_x \mathbf{u} \right)$, and $\nabla_x p_2^s = \frac{1}{L} \partial_x \tilde{p}_2^s$.

We also use the linear approximation for the surface pressures (1.2g) and (1.3f) in the following form involving the equilibrium density functions $\varrho_{ie}^0$,
\[
\sigma_i^s \equiv -p_i^s = \sigma_{ie} + \gamma_i (\varrho_{ie}^0 - \varrho_{ie}^s), \quad i = 1, 2, \tag{2.10}
\]
where $\sigma_{ie} := \sigma_i^0(\varrho_{ie}^0) = -\gamma_i (\varrho_{ie}^0 - \varrho_{ie,0}^0) > 0$.

Further, the following assumptions are made,
\[
\tau_1, \tau_2 \ll \frac{L \mu}{\varepsilon^3 \sigma_{1e}} \quad \text{and} \quad \frac{\varepsilon^2 \sigma_{1e}}{\gamma_1} \ll \varrho_{1e}^s, \varrho_{2e}^s, \tag{2.11}
\]
where the first condition means that the relaxation times $\tau_i$ are very small compared to the macroscopic time scale and the second condition says the deviation of $\varrho_i^s$ from its equilibrium value $\varrho_{ie}^0$, on both interfaces, is also small.

**Fluid equation:** Under the scaling, the incompressibility condition (1.1)$_1$ remains same in the new variables, leading to (2.1)$_1$. The Navier-Stokes equation (1.1)$_2$ becomes in the first component
\[
\varrho^e \varepsilon^3 \sigma_{1e} \mu \frac{L}{\mu} \left( \partial_x \mathbf{u} + \nu \partial_x \mathbf{u} + \nu \partial_x \mathbf{u} \right) = \left( \partial_{\mathbf{h}} \mathbf{u} + \frac{1}{\varepsilon^2} \partial_{\mathbf{h}} \mathbf{u} \right) - \frac{1}{\varepsilon^2} \partial_x \mathbf{h}.
\]
Thus in the limit $\varepsilon \to 0$, both the time derivative and the non-linear term vanish compared to the viscous term when Reynolds number satisfies $\varepsilon^2 \text{Re} \ll 1$ and one obtains (2.1)$_2$. Similarly the second component of the Navier-Stokes equation reduces to (2.1)$_3$.

**Boundary conditions:** The kinematic condition (1.2a) becomes in the limit (2.2a), using the fact that the new free boundary is $\mathbf{y} = \mathbf{h}(\mathbf{x}, t)$ (cf. (A.2)). For the other boundary conditions, writing $\varrho_i^s = \varrho_{ie}^0 + \frac{\varepsilon^2 \sigma_{1e}}{\gamma_1} \zeta_i, i = 1, 2$, one can compute the terms of equations (1.2b) and (1.3a) (cf. Appendix A) which yields,
\[
\frac{\varepsilon^3 \sigma_{1e} \varepsilon^2 \sigma_{1e}}{L \mu} \frac{\varepsilon^3 \sigma_{1e} \partial_x \zeta_i}{\gamma_1} + \frac{1}{L} \frac{\varepsilon^3 \sigma_{1e}}{\mu} \partial_x \left( \tilde{p}_i^s \left( \varrho_{ie}^0 + \frac{\varepsilon^2 \sigma_{1e}}{\gamma_1} \zeta_i \right) \right) = -\frac{1}{\tau_i} \frac{\varepsilon^3 \sigma_{1e} \zeta_i}{\gamma_1}. 
\]
Taking into account assumptions (2.11), we obtain the limit equations (2.2b) and (2.3a), where the time derivative vanishes and we assume $\frac{1}{\lambda} := \frac{Lp}{\varepsilon \gamma_1 \sigma_i} = O(1)$. Equation (1.2c) (similarly (1.3b)) reduces to,

$$
(\eta - \eta_i^s) \partial_\eta \overline{\eta} = (\eta - \eta_i) + \frac{\mu}{\varepsilon^2 \gamma_1 \sigma_i \rho L} \zeta_1.
$$

(2.12)

At this point, note that, as the thickness of the interface $\delta$ is assumed to be very small compared to the height of the fluid, i.e. $\delta \ll \varepsilon L$, and we also assume that the surface density $\sigma_i^s$ is of the same order as the bulk fluid density $\rho L^2$, i.e. $\sigma_i^s L^2 \sim \delta L^2 \rho L$, we have, $\sigma_i^s \ll \varepsilon L \rho L$. With the assumption that $\frac{Lp}{\varepsilon \gamma_1 \sigma_i} = O(1)$, the last term of equation (2.12) then vanishes in the limit and we obtain (2.2c) (similarly (2.3b)). To rewrite equations (1.2d), (1.2f), (1.3c), (1.3e), one needs to use approximation (1.2g) and (1.3f), or equivalently (2.10), which says

$$
p_i^s = \sigma_{1e} (\varepsilon^2 \zeta_1 - 1) \quad \text{and} \quad p_2^s = \varepsilon^2 \sigma_{1e} \frac{\gamma_2}{\gamma_1} \zeta_2 - \sigma_{2e}.
$$

(2.13)

Then equation (1.2d) becomes, with the help of (2.7), (2.13) and (2.6),

$$
p^G - \frac{\sigma_{1e}}{L} \rho + 2 \varepsilon^3 \sigma_{1e} \frac{1}{(1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2)} \left( - \partial_\eta \overline{\eta} \partial_\eta \overline{\eta} + \partial_\eta \overline{\eta} + O(\varepsilon^2) \right)
$$

$$
+ \sigma_{1e} \left( \varepsilon^2 \zeta_1 - 1 \right) \frac{\varepsilon \partial_\eta \overline{\eta}}{L (1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2)^{3/2}} = 0,
$$

which one can write in the following form,

$$
L \varepsilon^2 p^G - \varepsilon^3 \sigma_{1e} \rho + 2 \varepsilon^3 \sigma_{1e} \frac{1}{(1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2)} \left( - \partial_\eta \overline{\eta} \partial_\eta \overline{\eta} + \partial_\eta \overline{\eta} + O(\varepsilon^2) \right)
$$

$$
+ \left( \varepsilon^3 \sigma_{1e} \zeta_1 - \varepsilon^3 \sigma_{1e} \right) \frac{\partial_\eta \overline{\eta}}{(1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2)^{3/2}} = 0.
$$

Now, assuming that $\varepsilon^3 \sigma_{1e} = O(1)$, one obtains in the limit (2.2d). For equation (1.2e) (similarly (1.3c)), plugging in the expressions (2.8), (2.9) and (2.13), one gets

$$
\varepsilon^2 \sigma_{1e} \frac{1}{L (1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2)} \left( - \frac{1}{\varepsilon} \partial_\eta \overline{\eta} + O(\varepsilon) \right) = \frac{\varepsilon^2 \sigma_{1e}}{L \sqrt{1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2}} \partial_\eta \zeta_1,
$$

which, taking into account the scaling $\varepsilon^3 \sigma_{1e} = O(1)$, simplifies to (2.2c) to leading orders. Also, equation (1.2f) (similarly (1.3e)) for the tangential surface velocity becomes in the new variables, combining (2.9) and (2.13),

$$
- \frac{(1 + 4 \alpha_1 \beta_1)}{4 \beta_1} \frac{\varepsilon^2 \sigma_{1e}}{L \sqrt{1 + \varepsilon^2 |\partial_\eta \overline{\eta}|^2}} \partial_\eta \zeta_1 = \frac{\varepsilon^3 \sigma_{1e}}{\mu} \left( \frac{\overline{u}_i}{\zeta_1} - \overline{u} \right) + \varepsilon^2 \partial_\eta \overline{\eta} \left( \frac{\overline{v}_i}{\zeta_1} - \overline{v} \right),
$$

which reduces to (2.2f), where we make the assumption that $\frac{\mu L}{(1 + 4 \alpha_1 \beta_1)} = O(1)$. Lastly, equation (1.3d) can be computed easily in the same manner.
Contact line condition: For the equations at the contact line, the velocity of the contact point can be calculated as $v^c = \partial_t s_i = \frac{\epsilon^2 \sigma_{1e}}{\gamma_1} (\partial_\tau \Lambda_i (t), 0)$, $i = 1, 2$. Also, the contact angle being very small, the normal vectors at the contact point satisfy $\nu_1 = \nu_2 = (1, 0)$. Therefore, the continuity equation (1.4a) reads as

$$
\left( \phi_1^e + \frac{\epsilon^2 \sigma_{1e}}{\gamma_1} \phi_1^e e_\zeta \right) \frac{\epsilon^3 \sigma_{1e}}{\mu} (\partial^\tau_1 - \partial_\tau \Lambda_1) + \left( \phi_2^e + \frac{\epsilon^2 \sigma_{1e}}{\gamma_1} \phi_2^e e_\zeta \right) \frac{\epsilon^3 \sigma_{1e}}{\mu} (\partial^\tau_2 - \partial_\tau \Lambda_2) = 0,
$$

which simplifies to (2.4a), taking into account assumption (2.11). Also the contact angle being small, we approximate $\cos \theta \approx 1 - \frac{\theta^2}{2}$. Further, let us assume that $p_3^s = \sigma_{3e} (1 - \epsilon^2 \sigma_3) \geq 0$ is a constant. Note that the surface tension on the solid-gas interface works in opposite direction to that of the liquid-solid surface. Therefore, as the surface tensions at equilibrium are balanced, i.e.

$$\sigma_{1e} + \sigma_{2e} + \sigma_{3e} = 0,$$

the Young’s equation (1.4b) can be written as, using (2.13),

$$\sigma_{1e} (\epsilon^2 \zeta_1 - 1) \left( 1 - \frac{\epsilon^2}{2} |\partial_\tau \Lambda|^2 \right) = \sigma_{3e} (1 - \epsilon^2 \sigma_3) - \left( \epsilon^2 \sigma_{1e} \frac{\gamma_2}{\gamma_1} \zeta_2 - \sigma_{2e} \right).$$

Using the equilibrium condition (2.14), the above contact angle condition simplifies to (2.4b). Finally, the missing condition (1.4c) becomes (2.4c) with standard computation. From now on, we omit the bars for convenience.

2.1.2 Derivation of thin film equation

Next, we would like to derive the lubrication equation for the profile $h(x, t)$ from equations (2.1)-(2.4c). We show below that with $g = \frac{\phi_x^e - \phi_x^s}{\phi_x^e - \phi_x^s}, b_1 = a \left( \frac{1}{4} + \alpha_2 \beta_2 \right), b_2 = a \left( \frac{1}{4} - \alpha_2 \beta_2 \right), c_1 = (1 - 2\beta d_0 q), c_2 = 2a \left( \frac{1}{4} - \alpha_2 \beta_2 \right)$, for $t > 0$, one obtains,

$$
\begin{align*}
\partial_t h + \partial_x \left( \frac{h}{3} + \frac{1}{\beta} \right) h^2 \partial_x^3 h - \frac{1}{2} \partial_x \zeta_1 h^2 - \frac{1}{2} (\partial_x \zeta_1 + \frac{a}{2} \partial_x \zeta_2) h &= 0, \\
\lambda_1 \partial_x \left( (h + \frac{1}{\beta} + d_1) \partial_x \zeta_1 + \frac{a}{2\beta} \partial_x \zeta_2 - \left( \frac{h}{2} + \frac{1}{\beta} \right) h \partial_x^3 h \right) &= \zeta_1, \quad x \in (\Lambda_1, \Lambda_2); \\
\frac{g \lambda_1}{\beta} \partial_x \left( b_1 \partial_x \zeta_2 + \frac{1}{2} \left( \partial_x \zeta_1 - h \partial_x^3 h \right) \right) &= \zeta_2,
\end{align*}
$$

(2.15)

$$
\begin{align*}
-h \partial_x^3 h + c_1 \partial_x \zeta_1 + c_2 \partial_x \zeta_2 &= 0, \\
\zeta_1 + a \zeta_2 + \frac{1}{2} (\partial_x h)^2 &= b, \\
m \zeta_1 &= \zeta_2, \\
h &= 0,
\end{align*}
$$

(2.16)

where $\Lambda_1$ and $\Lambda_2$ are determined by (2.11), (2.14).
Indeed, from (2.13) and (2.2d), we can first conclude that \( p \) is independent of \( y \). Thus, relation (2.12) gives
\[
u(x, y) = \frac{1}{2} \partial_x p (y - 2h) y + A(x) y + B(x) \quad \text{in} \quad (\Lambda_1, \Lambda_2) \times (0, h), \quad (2.17)
\]
which yields, due to relations (2.2e) and (2.3c) that
\[A(x) = -\partial_x \zeta_1 \quad \text{and} \quad B(x) = -\frac{1}{\beta} \left( \partial_x \zeta_1 + \partial_x p \ h + \frac{a}{2} \partial_x \zeta_2 \right) \].

Thus, (2.17) becomes, together with (2.2d),
\[
u(x, y) = \frac{1}{2} \partial_x^3 h (2h - y) y - \partial_x \zeta_1 y - \frac{1}{\beta} \left( \partial_x \zeta_1 - h \partial_x^3 h + \frac{a}{2} \partial_x \zeta_2 \right) \quad \text{in} \quad (\Lambda_1, \Lambda_2) \times (0, h). \quad (2.18)
\]

Now, since due to (2.3d) and (2.3b), we have \( v|_{y=0} = 0 \), one gets from the divergence condition (2.1),
\[
v|_{y=h} = -\int_0^h \partial_x u \, dy = -\partial_x \left( \int_0^h u \, dy \right) + u \partial_x h.
\]

Combining this relation with (2.2a) and (2.2c), we can write the equation for the normal velocity of the free surface in the following conservation form,
\[
\partial_t h = -\partial_x \left( \int_0^h u \, dy \right). \quad (2.19)
\]

Therefore, with the help of (2.18), we obtain the thin film equation (2.15). Furthermore, plugging the value of \( u_1^s \) from (2.21) into the equation (2.2b), one obtains (2.15), using (2.18) at \( y = h \). Similarly, equation (2.3a) yields (2.15), with the help of (2.3e) and (2.18), as
\[
\zeta_2 = g \lambda_1 \partial_x \left( -\frac{1}{2\beta} \left( \partial_x \zeta_1 - \partial_x^3 h \ h + \frac{a}{2} \partial_x \zeta_2 \right) - d_2 \partial_x \zeta_2 \right)
\]
\[
= -g \lambda_1 \beta \partial_x \left( \frac{1}{2} \left( \partial_x \zeta_1 - \partial_x^3 h \ h + b_1 \partial_x \zeta_2 \right) \right) \quad \text{on} \quad y = 0.
\]

Finally, the mass balance condition at the triple junction (2.4a) gives, with the help of the equations (2.21) and (2.3e) for \( u_1^s \), together with the expression (2.18) for \( u \),
\[
-(1+q) \beta \partial_t \Lambda + q \left[ \partial_x^3 h \ h - (1+\beta d_1) \partial_x \zeta_1 - \frac{a}{2} \partial_x \zeta_2 \right]
\]
\[
= \frac{1}{2} \left( \partial_x \zeta_1 - \partial_x^3 h \ h \right) + \left( \frac{a}{4} + \beta d_2 \right) \partial_x \zeta_2 \quad \text{at} \quad y = h = 0. \quad (2.20)
\]
Note that for the equilibrium surface density in the liquid-solid surface, it holds \( \rho_{2e} > \rho_{1e} \) which means \( q < 1 \). Also noting that the velocity of the contact point \( s_i(t) \) is given by, from the thin film equation (2.15),

\[
\partial_t \Lambda_i = \frac{1}{\beta} \left( h \partial_x^3 h - \frac{a}{2} \partial_x \zeta_1 \right) \bigg|_{x=\Lambda_i},
\]
we may further simplify the above boundary condition (2.20) to (2.16).

Observe that, in the final thin film model (2.15)-(2.16), we have only two parameters: \( \frac{1}{\beta} \) which represents the ratio of the slip-length and the film thickness and \( \lambda_1 \) which is of order 1 or small.

**Remark 1** In the case when the slip length is small compared to the film height, i.e. \( \frac{1}{\beta} \ll 1 \), we obtain from (2.15),

\[
\partial_t h + \partial_x \left( \frac{1}{3} h^3 \partial_x^3 h - \frac{1}{2} h \partial_x \zeta_1 h \right) = 0,
\]
which can be seen as the counterpart of the classical thin film equation with no-slip boundary condition

\[
\partial_t h + \partial_x (h^3 \partial_x^2 h) = 0.
\]

### 2.2 Thin film model for large slip length

If we assume that the slip length is large compared to the film thickness i.e. \( \frac{1}{\beta} \gg 1 \), one obtains from (2.15)-(2.16), with the new time scale \( \tilde{t} = \frac{\sigma_1 \epsilon^2}{\beta \sigma_1} t \) and the variable \( \tilde{\lambda}_1 = \frac{\lambda_1}{\beta} = O(1) \), the following thin film model, for \( t > 0 \), (removing the tildas once again)

\[
\begin{align*}
\partial_t h + \partial_x \left( h^2 \partial_x^3 h - \left( \partial_x \zeta_1 + \frac{a}{2} \partial_x \zeta_2 \right) h \right) &= 0, \\
\lambda_1 \partial_x \left( \partial_x \zeta_1 + \frac{a}{2} \partial_x \zeta_2 - h \partial_x^2 h \right) &= \zeta_1, \\
g \lambda_1 \partial_x \left( \frac{a}{4} \partial_x \zeta_2 + \frac{1}{2} \left( \partial_x \zeta_1 - h \partial_x^2 h \right) \right) &= \zeta_2, \\
-h \partial_x^3 h + \partial_x \zeta_1 + \frac{a}{2} \partial_x \zeta_2 &= 0, \\
\zeta_1 + a \zeta_2 + \frac{1}{2} \partial_x h^2 &= b, \\
m \zeta_1 = \zeta_2, \\
\text{and} \quad h &= 0.
\end{align*}
\]

**Remark 2** The above thin film equation can be compared to the standard one with slip boundary condition,

\[
\partial_t h + \partial_x (h^2 \partial_x^2 h) = 0.
\]
2.3 Thin film model of meniscus formation

Figure 3: General meniscus formation

In this subsection, we would like to consider a different thin film model, where the liquid is touched by a wedge-shaped solid (cf. Figure 3). As the initial state is not in equilibrium, the contact line starts moving, forming a meniscus. We first describe in Section 2.3.1, the thin film model in this case under the standard approach i.e. with slip boundary condition at the liquid-solid interface, as we could not find any rigorous mathematical derivation regarding such a model. The approach of Shikhmurzaev is discussed in Section 2.3.2.

Let us consider a two-dimensional (horizontal) viscous, incompressible Newtonian fluid film of thickness $H$ at the initial time and the wedge-shaped solid which is vertically symmetric and makes an angle $\delta$ with the liquid (cf. Figure 4a). As before, let $\mathbf{v}^L = (u,v)$ be the velocity of the fluid, with constant density $\rho^L$, viscosity $\mu$ and pressure $p^L$, governed by the Navier-Stokes equations, while the solid moves vertically downward only, with velocity $\mathbf{v}^S(t)$. The bottom part of the fluid domain is given by $y = 0$. The contact point moves and makes an angle $(\pi - \theta)$ between the liquid-solid and the liquid-gas interface. The liquid-gas interface is described by $y = h(x,t)$ and the liquid-solid rigid boundary is given by $y = \tilde{h}(t) + \delta|x|$, where $\tilde{h}(t)$ is the distance between the tip of the wedge and the bottom of the liquid domain. The free interface makes an angle $\tilde{\theta}$ with the horizontal plane (cf. Figure 4b). Therefore, $\theta = \tilde{\theta} + \delta$ and $\tan \tilde{\theta} = -\partial_x h$. To obtain the lubrication approximation, we need to assume that the two angles $\theta$ and $\delta$ are very small and of the same order, which means that the contact angle $(\pi - \theta)$ is close to $\pi$ in this setting. Note that the thin film approximation would not be valid if $\delta$ is large. The contact points are given by $s_i = (\Lambda_i(t), y)$. Also we assume $\tilde{h}(t) = H \left( 1 - \left( \frac{t}{t_0} \right)^\eta \right)$ with $\eta > 0$. Here $t_0$ is the time scale that characterises the motion of the solid wedge moving down. The particular case of constant velocity of the wedge corresponds to $\eta = 1$, while the case of constant acceleration corresponds to $\eta = 2$. One may consider more general profile as well. Finally, we assume that the far field condition $h \to H$ as $|x| \to \infty$ holds.

Since we are in a symmetric configuration with respect to the $y$-axis, it is enough to analyse the situation $x > 0$ only. Therefore, in the following, we call the contact point $\Lambda$ instead of $\Lambda_1$.

2.3.1 Classical approach

In classical fluid mechanics, conditions at contact lines are considered only at equilibrium, hence the contact angle is prescribed. Therefore, the contact line problem of a meniscus
formation in the classical approach, with the slip condition at the fluid-solid interface, can be described as follows: (for the notations below, refer to the introduction),

**In the liquid:**

\[
\text{div}_x \mathbf{v}^L = 0; \quad \rho^L \left( \partial_t \mathbf{v}^L + \mathbf{v}^L \cdot \nabla_x \mathbf{v}^L \right) = \mu \Delta \mathbf{v}^L - \nabla p^L. \tag{2.21}
\]

**At the liquid-gas interface** \( y = h(x,t), x > \Lambda(t) \):

\[
V_n = \mathbf{v}^L \cdot \mathbf{n}_1, \tag{2.22a}
\]

\[
(p^G - p^L) + 2\mu \left[ \nabla \mathbf{v}^L \cdot \mathbf{n}_1 \right] \cdot \mathbf{n}_1 + p^s \kappa_\Sigma = 0, \tag{2.22b}
\]

\[
\left[ \nabla \mathbf{v}^L \cdot \mathbf{n}_1 \right]_{\tau} = 0. \tag{2.22c}
\]

**At the liquid-solid interface** \( \{ y = 0 \} \cup \{ y = \tilde{h}(t) + \delta |x|, 0 < x < \Lambda(t) \} \):

\[
\left( \mathbf{v}^L - \mathbf{v}^S \right) \cdot \mathbf{n}_2 = 0, \tag{2.23a}
\]

\[
2\mu \left[ \nabla \mathbf{v}^L \cdot \mathbf{n}_2 \right]_{\tau} = \beta_2 \left( \mathbf{v}^L - \mathbf{v}^S \right). \tag{2.23b}
\]

**At the triple junction** \( x = \Lambda(t) \):

\[
h(x,t)|_{x=\Lambda} = \tilde{h}(t) + \delta |x|_{x=\Lambda}, \tag{2.24a}
\]

\[
\partial_x h|_{x=\Lambda} = \delta - \theta. \tag{2.24b}
\]

Here \( p^s \) is the (constant) surface tension, \( \kappa_\Sigma \) is the curvature and \( p^G \) is the constant pressure of the gas. Also at the liquid-solid interface, the slip boundary condition is assumed, although one may consider no-slip condition at the bottom part as well. We do not distinguish here between the two slip-coefficients appearing at the lower and upper fluid-solid interface and denote both of them by \( \beta_2 \) only. We further assume for simplicity that the lower boundary \( y = 0 \) is fixed, i.e. \( \mathbf{v}^S = 0 \) at \( y = 0 \).

We also suppose

\[
h(x,0) = H \quad \text{at} \quad t = 0. \tag{2.25}
\]

Furthermore, due to the symmetry assumption of the considered configuration, one has,

\[
u = 0 \quad \text{on} \quad x = 0. \tag{2.26}
\]
Now let us introduce the length scale $L = \frac{H}{\varepsilon}$ where $\varepsilon > 0$ is small. Then the usual scalings of the variables are given by,

$$\begin{align*}
(x, y) &= \left( \frac{x}{L}, \frac{y}{H} \right), \\
\Lambda_i &= \frac{1}{L} \Lambda_i, \\
\bar{h} &= \frac{h}{H}, \quad \bar{\dot{h}} = \frac{\dot{h}}{H}, \\
\bar{\tau} = \sigma \frac{t}{L^\mu}, \quad \bar{\tau}_0 = \frac{\sigma}{L^\mu} t_0, \\
(\bar{u}, \bar{v}) &= \left( \frac{u}{\Lambda^{\mu}}, \frac{v}{\Lambda^{\mu}} \right), \\
p &= \frac{\varepsilon^2 L}{\sigma} p, \\
\delta &= k_1 \varepsilon, \quad \theta = k_2 \varepsilon
\end{align*}$$

with $\sigma = -\varepsilon^3 p^s$ is the scaled surface tension.

**Rescaled system:** Under the scaling, we obtain the following approximated system of equations from (2.21)-(2.24b),

$$\begin{align*}
\partial_x u + \partial_y v &= 0, \\
\partial_x^2 u &= \partial_x \bar{p}, \quad \partial_y \bar{p} = 0, \\
\bar{\tau} &= 0, \\
\partial_x \bar{p} &= \beta \bar{\tau}, \quad \text{on } y = 0, \\
k_1 \bar{u} &= \bar{v} - \partial_t \bar{h}, \\
\partial_x \bar{u} + \beta \bar{u} &= 0, \quad \text{on } \bar{x} < \bar{\Lambda}, \quad \bar{y} = \bar{h} + k_1 \bar{x}, \\
\partial_t \bar{h} + \bar{\tau} \partial_x \bar{h} - \bar{v} &= 0, \quad \partial_x \bar{p} = 0, \quad p = -\partial_x^2 \bar{h}, \quad \text{on } \bar{x} > \bar{\Lambda}, \quad \bar{y} = \bar{h}, \\
\bar{h} &= \left( 1 - \left( \frac{y}{\bar{t}_0} \right)^\eta \right) + k_1 \bar{\Lambda}, \quad \partial_x \bar{h} = k_1 - k_2, \quad \text{at } \bar{x} = \bar{\Lambda},
\end{align*}$$

where $\beta = \frac{\varepsilon \beta L}{\mu} = O(1)$. The system is complemented by equations (2.25), (2.26).

Indeed, the time-dependent incompressible Navier-Stokes equations (2.21) take the non-dimensional form (2.27a), to the leading order in $\varepsilon$, assuming that $\varepsilon^2 \text{Re} \ll 1$ where $\text{Re} = \frac{\rho L v L}{\mu}$ is the Reynolds number.

Next we discuss the boundary conditions. At the liquid-gas interface $\bar{y} = \bar{h}(\bar{x}, \bar{t})$, $\bar{x} > \bar{\Lambda}(\bar{t})$, with the expressions (2.5), (2.6) for the unit normal and tangent vectors and the curvature, the kinematic boundary condition (2.22a) and the tangential and normal component of the capillary force (2.22b), (2.22c) reduce to the non-dimensional form (2.27b) (cf. (A.2)).

At the liquid-solid interface, with $\mathbf{n}_2 = (0, 1)$, $\tau = (1, 0)$ at the bottom part $\bar{y} = 0$, the conditions (2.23a), (2.23b) read as

$$\begin{align*}
v &= 0, \quad \mu (\partial_y u + \partial_x v) = \beta_2 u \quad \text{at } y = 0,
\end{align*}$$

which then convert into (2.27b). Similarly, at the upper fluid-solid interface, the normal and tangent vectors being,

$$\mathbf{n}_2 = \frac{(\delta, -1)}{\sqrt{1 + \delta^2}}, \quad \tau = \frac{(1, \delta)}{\sqrt{1 + \delta^2}},$$

and $\mathbf{v}_S = (0, \partial_t \bar{h})$, the boundary conditions (2.23a), (2.23b) transform into the non-dimensional form (2.27c).

The conditions at the contact point (2.24a), (2.24b) transform into (2.27c).
Thin film equations: The thin film model can now be derived from (2.27a)-(2.27e) together with the initial data (2.25). As deduced later, we obtain

\[ \frac{\partial h}{\partial t} + \partial_x \left( \left( \frac{h}{3} + \frac{1}{\beta} \right) h^2 \partial^3_x h \right) = 0, \quad \text{in } x > \Lambda, \quad (2.28) \]

\[ h = \left( 1 - \left( \frac{t}{t_0} \right)^\eta \right) + k_1 \Lambda, \quad \partial_x h = k_1 - k_2, \quad \text{at } x = \Lambda, \quad (2.29) \]

\[ h = 1, \quad \text{at } t = 0. \quad (2.30) \]

Indeed, dropping the bars from now on and integrating (2.27a) twice gives the profile of the horizontal velocity,

\[ u(x, y) = \frac{1}{2} \partial_x p \, y^2 + A(x, t) y + B(x, t), \quad x > 0, \quad (2.31) \]

where \( A \) and \( B \) are to be determined. The condition at the lower boundary (2.27b) implies,

\[ A = \beta B, \quad x > 0. \quad (2.32) \]

Also the condition at the free boundary (2.27d) yields,

\[ A = -\partial_x p \, h \quad \text{and therefore} \quad B = -\frac{1}{\beta} \partial_x p \, h, \quad x > \Lambda. \quad (2.33) \]

Further, as the pressure is independent of \( y \) due to (2.27a), the horizontal velocity (2.31) becomes, together with (2.33) and (2.27d),

\[ u = \frac{1}{2} \partial^3_x h \left( 2h - y + \frac{2}{\beta} h \right), \quad x > \Lambda. \quad (2.34) \]

Exactly as in the previous section, we derive from (2.27a), (2.27b) and (2.27d), equation (2.19) for \( x > \Lambda \). Substituting (2.34) then gives the thin film equation (2.28). The contact line condition (2.29) is nothing but (2.27e) and the initial data (2.30) follows from (2.25).

Inner region: In this case, in order to determine the dynamics of the contact point fully, we need to prescribe sufficient conditions in the inner region \( x < \Lambda \) as well. To do so, the condition at the fluid-solid interface (2.27c) together with the condition (2.32), for \( x < \Lambda \), \( y = \tilde{h} + k_1 x \), which is

\[ \left[ \partial_x p \left( \tilde{h} + k_1 |x| \right) + A \right] + \beta \left[ \frac{1}{2} \partial_x p \left( \tilde{h} + k_1 |x| \right)^2 + A \left( \tilde{h} + k_1 |x| \right) + \frac{A}{\beta} \right] = 0, \]

gives a relation between \( \partial_x p \) and \( A \) for \( x \in (0, \Lambda) \), that is

\[ \partial_x p = -\frac{2A}{\tilde{h} + k_1 x}. \quad (2.35) \]
The incompressibility condition \((2.27a)\) implies together with \((2.27b)\) for \(x \in (0, \Lambda)\), that

\[
v|_{y=\bar{h}+k_1x} = - \int_0^{\bar{h}+k_1x} \partial_x u \, dy
\]

\[
= - \left( \bar{h} + k_1x \right) \left[ 1/6 \partial_x^2 p \left( \bar{h} + k_1x \right)^2 + 1/2 \partial_x A \left( \bar{h} + k_1x \right) + 1/\beta \partial_x A \right].
\]

Furthermore, condition \((2.27c)\) implies,

\[
k_1 \left[ \frac{1}{2} \partial_x p \left( \bar{h} + k_1x \right)^2 + A \left( \bar{h} + k_1x + \frac{1}{\beta} \right) \right] + \partial_t \bar{h}
\]

\[
+ \left( \bar{h} + k_1x \right) \left[ \frac{1}{6} \partial_x^2 p \left( \bar{h} + k_1x \right)^2 + 1/2 \partial_x A \left( \bar{h} + k_1x \right) + 1/\beta \partial_x A \right] = 0.
\]

This above equation together with the relation \((2.35)\) yields an equation for \(A\) with rational coefficients,

\[
\partial_x A + r_1(x, t) A + r_2(x, t) = 0, \quad x < \Lambda,
\]

where

\[
r_1(x, t) = \frac{k_1 \left( \frac{1}{\gamma} + \frac{1}{5} \left( \bar{h} + k_1x \right) \right)}{\left( \bar{h} + k_1x \right) \left( \frac{1}{\gamma} + \frac{1}{6} \left( \bar{h} + k_1x \right) \right)}, \quad r_2(x, t) = \frac{\partial \bar{h}}{\left( \bar{h} + k_1x \right) \left( \frac{1}{\gamma} + \frac{1}{6} \left( \bar{h} + k_1x \right) \right)}.
\]

Also, due to the symmetry assumption \((2.26)\), we have the boundary condition

\[
A(x, t) = 0 \quad \text{at} \quad x = 0.
\]

Therefore, equation \((2.36)\) determines \(A\) and in turn \(\partial_x p\) in the inner region \(x \in (0, \Lambda)\). Here we assume that the horizontal velocity \(u\) is continuous, thus the relation \((2.31)\) holds at \(x = 0\) as well. Moreover, the continuity assumption of the velocity \(u\) further imposes the condition

\[
A = h \partial_x^3 h \quad \text{at} \quad x = \Lambda.
\]

This above relation \((2.37)\), together with the thin film equations \((2.28)-(2.30)\) determines fully the dynamics and the position of the contact point.

### 2.3.2 Shikhmurzaev’s approach

Let us now consider Shikhmurzaev’s approach \((1.1)-(1.4c)\), together with \((2.24a)\), to describe the above thin film model of meniscus formation in Figure 4a, i.e. we now take into account the surface quantities for this model. We use the same scaling as in Section 2.1, only with the adaptation for the quantities \(\bar{h}, t_0, \text{ and } \delta\), that is,

\[
\bar{h} = \frac{\h}{H}, \quad \bar{t}_0 = \frac{\varepsilon^3 \sigma_1 e t_0}{L \mu}, \quad \delta = k_1 \varepsilon.
\]
The difference with the thin film model described in Section 2.1 or 2.2 is that, the equations on the liquid-gas interface (1.2a)-(1.2g) are valid for \( y = h(x, t), x > \Lambda(t) \), while the liquid-solid interface conditions (1.3a)-(1.3f) are valid for \( y = \tilde{h}(t) + \delta x, x \in (0, \Lambda(t)) \), with \( \mathbf{v}^S = (0, \partial_t \tilde{h}) \) (at the upper boundary) and for \( y = 0 \) with \( \mathbf{v}^S = 0 \) (at the lower boundary). Using the same scaling as in Section 2.1, we only indicate here the differences from the previous case. Essentially, it is enough to calculate the boundary conditions at the liquid-solid interface \( y = \tilde{h}(t) + \delta x, x \in (0, \Lambda(t)) \) together with the contact line conditions (1.4a)-(1.4c).

**Rescaled system:** The rescaled equations for this contact line problem become, (omitting the bar, but all the following quantities should be understood after the rescaling), just (2.1)-(2.3e), while on \( x \in (0, \Lambda) \), \( y = \tilde{h} + k_1 x \), we obtain,

\[
\begin{align*}
\partial_x u_2^s &= -\frac{1}{\lambda_2} \zeta_2, \\
k_1 (u - u_2^s) - (v - v_2^s) &= 0, \\
\partial_y u + \frac{1}{2} a \partial_x \zeta_2 + \beta u &= 0, \\
k_1 u_2^s - (v_2^s - \partial_t \tilde{h}) &= 0, \\
u_2^s &= \frac{1}{2} u - d_2 \partial_x \zeta_2;
\end{align*}
\]  

and at \( x = \Lambda \),

\[
\begin{align*}
g_1 \zeta_1 + g_2 \zeta_2 &= 0, \\
\zeta_1 + a \zeta_2 + \frac{1}{2} |\partial_x h|^2 &= b, \\
m \zeta_1 &= \zeta_2.
\end{align*}
\]  

Recall that the constants appearing in the above system, defined after (2.4c), are non-negative, dimensionless and of order one. The derivation is a straight-forward calculation.

**Thin film model:** We obtain as before, equations (2.15) in \( t > 0, x > \Lambda \). On \( t > 0 \) and \( x = \Lambda \), we obtain the following boundary conditions (cf. (2.16) where (2.16) is replaced by (2.29)),

\[
\begin{align*}
-\frac{1}{2} h \partial_x^2 h + c_1 \partial_x \zeta_1 + c_2 \partial_x \zeta_2 &= 0, \\
\zeta_1 + a \zeta_2 + \frac{1}{2} |\partial_x h|^2 &= b, \\
m \zeta_1 &= \zeta_2, \\
h &= \left( 1 - \left( \frac{t}{t_0} \right)^\nu \right) + k_1 \Lambda.
\end{align*}
\]
**Inner region:** Next we need to determine the dynamics of the internal region \( x < \Lambda \), as in the previous case in Subsection 2.3.1. For that, using the expression for the horizontal fluid velocity

\[
u(x, y) = \frac{1}{2} \partial_x p \, y^2 + A(x, t) y + B(x, t), \quad x > 0,
\]

(2.41)

and the boundary condition (2.3c), one obtains the following relation

\[
B = \frac{1}{\beta} \left( A - \frac{a}{2} \partial_x \zeta_2 \right), \quad x > 0.
\]

Next from the boundary condition (2.38)3, plugging in the above relations, one gets (2.35). Furthermore, we derive from (2.1)1, combining (2.3b), (2.3d), for \( x < \Lambda \) that

\[
v|_{y=\tilde{h}+k_1 x} = - \left( \tilde{h} + k_1 x \right) \left[ \frac{1}{6} \partial_x^2 p \left( \tilde{h} + k_1 x \right)^2 + \frac{1}{2} \partial_x A \left( \tilde{h} + k_1 x \right) + \frac{1}{3} \left( \partial_x A - a \partial_x^2 \zeta_2 \right) \right].
\]

Combining the two conditions (2.38)2 and (2.38)4, one obtains equation (2.27c1). This yields, with the help of the above expressions for \( u, v \) and the relation (2.35), an equation for \( A \) of the following form,

\[
\tilde{r}_1(x, t) \partial_x A + \tilde{r}_2(x, t) A + F \left( \partial_x^2 \zeta_2, \partial_x \zeta_2 \right) + \partial_t \tilde{h} = 0, \quad x \in (0, \Lambda),
\]

(2.42)

where

\[
\tilde{r}_1(x, t) = \left( \tilde{h} + k_1 x \right) \left( \frac{1}{\beta} + \frac{1}{6} \left( \tilde{h} + k_1 x \right) \right), \quad \tilde{r}_2(x, t) = k_1 \left( \frac{1}{\beta} + \frac{1}{3} \left( \tilde{h} + k_1 x \right) \right),
\]

and

\[
F = - \frac{a}{2\beta} \left( k_1 \partial_x \zeta_2 + \left( \tilde{h} + k_1 x \right) \partial_x^2 \zeta_2 \right).
\]

Recall from (2.18) that the horizontal fluid velocity in the outer region takes the form,

\[
u(x, y) = \frac{1}{2} \partial_x^2 h (2h - y) y - \partial_x \zeta_1 y - \frac{1}{\beta} \left( \partial_x \zeta_1 - h \partial_x^2 \zeta_2 + \frac{a}{2} \partial_x \zeta_2 \right), \quad x > \Lambda.
\]

(2.43)

As before, due to the assumption of continuity of velocity \( u \) across the contact point and at the origin, we obtain two conditions from (2.43) and (2.41),

\[
A = h \partial_x^2 h - \partial_x \zeta_1 \quad \text{at} \quad x = \Lambda,
\]

(2.44)

and since \( u = 0 \) at \( x = 0 \) (cf. (2.26)),

\[
A = 0, \quad \partial_x \zeta_2 = 0 \quad \text{at} \quad x = 0.
\]

(2.45)

Now for the surface density \( \zeta_2 \) in the inner region, we use (2.38)1 and (2.38)5 to deduce,

\[
- \zeta_2 = \lambda_2 \partial_x \left[ \frac{1}{4} \left( \tilde{h} + k_1 x \right)^2 \partial_x p + \left( \tilde{h} + k_1 x + \frac{1}{\beta} \right) A - \left( a \frac{2}{2\beta} + d_2 \right) \partial_x \zeta_2 \right], \quad x \in (0, \Lambda).
\]

(2.46)
Note that in (2.46), $\partial_x \rho$ can be written in terms of $A$ and $\partial_x \zeta_2$ from (2.35). Therefore, equations (2.42) and (2.46) are a coupled system with boundary conditions (2.45). Solving this would provide $A$ at $x = \Lambda$ which in turn gives us the necessary condition (2.44) in order to determine the position of the contact point.

Summing up, we obtain the thin film equation (2.15), (2.40) together with the condition (2.44).

2.3.3 Conclusion

From the above discussion in Section 2.3.1 and Section 2.3.2 on the thin film model of a solid wedge moving downward into a fluid, with two different approaches, we conclude that the classical model says that the contact angle is created instantaneously (cf. equation (2.29), while Shikhmurzaev’s approach says that the contact angle is created in a continuous manner (cf. equation (2.40)). This above model of meniscus formation as a general phenomena (not the thin film approximation) has been mentioned in [18, Fig 6].

A Appendix

Here we write down explicitly several differential terms over the surface, in dimension two, for clarity.

Let us consider an interface $y = h(x, t)$ with velocity $v^s = (v_{x1}^s, v_{x2}^s)$. The unit outward normal vector at the surface is then given by

$$\mathbf{n} = \frac{(-\partial_x h, 1)}{\sqrt{1 + |\partial_x h|^2}}. \quad (A.1)$$

The shape of the surface can be characterized in terms of its normal velocity (cf. (1.2a)),

$$\partial_t h + v_1 \partial_x h - v_2 = 0. \quad (A.2)$$

Indeed, at the free interface $f \equiv h(x, t) - y = 0$, the normal is given by $\mathbf{n}_1 = \frac{\nabla_x f}{|\nabla_x f|}$ and hence, equation (1.2a) means precisely the convective derivative is zero, i.e.

$$\partial_t f + v^s \cdot \nabla_x f = 0.$$

In order to compute $\partial_t^\Sigma$, we choose a suitable normal parametrization of the surface,

$$\gamma : \zeta \in \mathbb{R} \to \gamma(\zeta, t) \in \mathbb{R}^2 \text{ such that } \frac{\partial \gamma}{\partial t} \parallel \mathbf{n}.$$ 

We would like to relate $\gamma$ with the known parametrization $h$ (which is not necessarily a normal parametrization) and then express $\partial_t^\Sigma$ in terms of $h$. Let us introduce the following notations, for any generic function $w$ and a fixed point $(\overline{x}, \overline{y})$ on the interface $y = h(x, 0),$

$$W(\overline{x}, t) := w(\gamma(\overline{x}, t), t) = w(\gamma_1(\overline{x}, t), \gamma_2(\overline{x}, t), t), \quad \overline{y} = h(\overline{x}, 0)$$

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and
\[ \tilde{w}(x, t) := w(x, h(x, t), t). \]

If we write the following Taylor expansion,
\[ \gamma(x, t) = \left[ \begin{array}{c} \gamma_1(x, t) \\ \gamma_2(x, t) \end{array} \right] = \left[ \begin{array}{c} x \\ y \end{array} \right] + t v_n(x, 0) n(x, t) + O(t^2), \tag{A.3} \]
then
\[ \frac{\partial \gamma}{\partial t} = v_n(x, 0) n(x, t) + O(t) \]
as desired. Further notice that
\[ \gamma_2(x, t) = h(\gamma_1(x, t)). \]
Indeed, from (A.3), we have, combining (A.1) and (A.2),
\[ h(\gamma_1(x, t)) = h \left( \frac{x}{\sqrt{1 + |\partial_x h|^2}} \partial_x h(x, 0) v_n(x, 0) \right) \]
\[ = h(x, 0) - \partial_x h(x, 0) \frac{t}{\sqrt{1 + |\partial_x h|^2}} \partial_x h(x, 0) v_n(x, 0) + t \partial_t h(x, 0) \]
\[ = y - \frac{t}{\sqrt{1 + h^2}} (\partial_x h(x, 0))^2 v_n(x, 0) + \sqrt{1 + |\partial_x h|^2} v_n(x, 0) t \]
\[ = y + \frac{t}{\sqrt{1 + |\partial_x h|^2}} v_n(x, 0) = \gamma_2(x, t). \]
Therefore, we obtain
\[ W(x, t) = \tilde{w}(\gamma_1(x, t), t) = \tilde{w} \left( \frac{x}{\sqrt{1 + |\partial_x h|^2}} \partial_x h(x, 0) v_n(x, 0), t \right), \]
and then by definition,
\[ \partial_t \Sigma w = \frac{dW}{dt} \bigg|_{t=0} = - \partial_x h(x, 0) v_n(x, 0) \frac{\partial \tilde{w}}{\partial x} + \frac{\partial \tilde{w}}{\partial t} \tag{A.4} \]

Next, for any tangential vector over the surface \( A = a(s) \tau = a(x, h(x, t)) \tau \) where \( s \) is the arc length, defined by,
\[ s = \int_0^x \sqrt{1 + |\partial_x h(\zeta)|^2} \, d\zeta, \quad \text{hence,} \quad ds = \sqrt{1 + |\partial_x h|^2} \, dx, \]
the surface divergence reduces to,
\[ \int_S \text{div}_\Sigma A \varphi \, ds = - \int_S A \cdot \nabla_\tau \varphi \, ds = - \int_S a \tau \cdot \frac{d\varphi}{ds} \, ds = - \int_S \frac{a}{ds} \frac{d\varphi}{ds} \, ds \]

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which gives,

$$\text{div}_\Sigma A = \frac{da}{ds} = \frac{1}{\sqrt{1 + |\partial_x h|^2}} \frac{d}{dx}(\tilde{a}(x))$$  \hspace{1cm} (A.5)

where we have denoted \( \tilde{a}(x) := a(s) \).

We can further obtain from (A.5), the following formula for tangential gradient, for a function \( f \) on the surface \( y = h(x,t) \),

$$\nabla_\tau f \cdot \tau = \frac{1}{\sqrt{1 + |\partial_x h|^2}} \frac{df}{dx},$$  \hspace{1cm} (A.6)

where \( \tilde{f}(x) := f(x,h(x,t)) \). One can also obtain the above formula by considering any extension \( \tilde{f} \) in \( \mathbb{R}^2 \) and then computing \( \nabla \tilde{f} \cdot \tau \), since the final result does not depend on the choice of extension.

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