The main cuboid

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Abstract

The connectedness locus in the parameter space of quadratic polynomials is called the Mandelbrot set. A good combinatorial model of this set is due to Thurston. By definition, the principal hyperbolic domain of the Mandelbrot set consists of parameter values, for which the corresponding quadratic polynomials have an attracting fixed point. The closure of the principal hyperbolic domain of the Mandelbrot set is called the main cardioid. Its topology is completely described by Thurston’s model. Less is known about the connectedness locus in the parameter space of cubic polynomials. In this paper, we discuss cubic analogues of the main cardioid and establish relationships between them.

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1. Introduction

Studying parameter spaces of polynomials is a major task of complex dynamics. The main achievements here concern quadratic polynomials. One of them is Thurston’s beautiful description of a combinatorial model for the parameter space of quadratic polynomials. Motivated by it, we want to study the structure of the closure of the principal hyperbolic domain in the parameter space of cubic polynomials. To this end we study topological dynamics of cubic polynomials, building our investigation on a partial similarity with the dynamics of quadratic polynomials.
For a complex polynomial \( f \), let \( K(f) \) be the filled Julia set of \( f \) consisting of all complex numbers \( z \), whose \( f \)-orbits do not escape to infinity. We write \( J(f) \) for the Julia set of \( f \), which is the boundary of \( K(f) \). An equivalent definition of the Julia set is that it consists of points \( z \) such that the forward \( f \)-orbits fail to be Lyapunov stable in a neighbourhood of \( z \) (equivalently, the sequence of iterates \( f^\circ n \) fails to be equicontinuous in a neighbourhood of \( z \)).

Quadratic polynomials depend essentially on one complex parameter \( c \), since every quadratic polynomial is affinely conjugate to a polynomial of the form \( f_c(z) = z^2 + c \). The Mandelbrot set \( M_2 \) is a subset of the parameter plane, the \( c \)-plane, consisting of all values \( c \in \mathbb{C} \), for which \( K(f_c) \) is connected. The geometric shape of \( M_2 \) gives a very informative overview of how the dynamics of \( f_c \) depends on \( c \). In particular, as \( c \) varies within one interior component of \( M_2 \), the topological dynamics of the map \( f_c : J(f_c) \to J(f_c) \) does not change, i.e. the maps \( f_c \) and \( f_c' \) are topologically conjugate on their Julia sets if \( c \) and \( c' \) belong to the same interior component of \( M_2 \); see [MSS83]. As \( c \) crosses the boundary of \( M_2 \), bifurcations happen. For example, going from one interior component of \( M_2 \) to an adjacent component corresponds to a bifurcation, during which a periodic cycle changes its period.

The central part of \( M_2 \) is the so-called main cardioid consisting of all parameter values \( c \), for which \( f_c \) has an attracting or neutral fixed point. It can also be defined as the closure of the principal hyperbolic domain \( \text{PHD}_2 \), the interior component of the Mandelbrot set such that for all \( c \in \text{PHD}_2 \), the Julia set \( J(f_c) \) is homeomorphic to the circle, and the dynamics of \( f_c : J(f_c) \to J(f_c) \) is topologically conjugate to the dynamics of the angle doubling map on \( \mathbb{R}/\mathbb{Z} \). It is easy to see that polynomials in \( \text{PHD}_2 \) have no periodic cutpoints in \( J(f) \) except for at most one neutral fixed point. Recall that a cutpoint of a topological space \( X \) is a point \( \alpha \in X \) such that \( X \setminus \{\alpha\} \) is disconnected.

As follows from the Douady–Hubbard–Sullivan–Yoccoz landing theorem [DH84, Hub93], the Mandelbrot set itself can be thought of as the union of the main cardioid and limbs (connected components of \( M_2 \setminus \text{PHD}_2 \)) parametrized by reduced rational fractions \( p/q \in (0, 1) \). The limb corresponding to \( p/q \) consists of \( c \) such that \( K(f_c) \) is connected, and there is a repelling fixed cutpoint \( \alpha \) of \( K_c \) of combinatorial rotation number \( p/q \) (this description of limbs is given in [Mil00a]). The point \( \alpha \) being repelling means that \( |f_c'(\alpha)| > 1 \), then indeed \( \alpha \) repels all nearby points. Finally, having combinatorial rotation number \( p/q \) means the following: the set \( K_c \setminus \{\alpha\} \) has \( q \) connected components, whose germs at \( \alpha \) are mapped to each other like rotation by \( p/q \), i.e., for every component \( X \) of \( K_c \setminus \{\alpha\} \), there is a small neighbourhood \( U \) of \( \alpha \) with the property that \( f_c(X \cap U) \) is mapped to the \( p \)th component from \( X \) in the counterclockwise order.

This description of the Mandelbrot set serves as a motivation for our study of higher degree analogues of \( \text{PHD}_2 \). By classes of degree \( d \) polynomials, we mean affine conjugacy classes. Denote by \([f]\) the class of a polynomial \( f \). Note that, for a quadratic polynomial \( f_c \), the class \([f_c]\) can be identified with \( c \). The degree \( d \) connectedness locus \( M_d \) (the higher degree generalization of the Mandelbrot set) is the set of classes of degree \( d \) polynomials \( f \) with connected \( K(f) \) (equivalently, \([f] \in M_d \) if all critical points of \( f \) belong to \( K(f) \)). A polynomial of any degree is said to be hyperbolic if the orbits of all its critical points converge to attracting cycles. The set of all classes \([f] \in M_d \) such that \( f \) is hyperbolic splits into the so-called hyperbolic components of \( M_d \). The degree \( d \) principal hyperbolic domain \( \text{PHD}_d \) is the hyperbolic component of \( M_d \) consisting of classes \([f]\) such that \( K(f) \) is a Jordan disc. Equivalently, the class \([f]\) of a degree \( d \) polynomial \( f \) belongs to \( \text{PHD}_d \) if all critical points of \( f \) are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. See figure 1 for pictures of parameter spaces of quadratic and cubic polynomials.

Thus, the study of \( \text{PHD}_d \) in the context of complex dynamics is the next logical step. Working in this direction and similar to the quadratic case, we establish properties of
polynomials in $\text{PHD}_d$ in theorem A which lists necessary conditions for $[f]$ to belong to $\text{PHD}_d$.

**Theorem A.** Let $f$ be a polynomial, whose class belongs to $\text{PHD}_d$. Then $f$ has a fixed non-repelling point and no repelling periodic cutpoints in the Julia set of $f$. Moreover, all non-repelling periodic cutpoints, except at most one fixed point, have multiplier 1.

This motivates the following definition of a cubic analogue of the main cardioid.

**Definition 1.1 (main cuboid).** The **main cuboid** is the set $\text{CU}$ of classes of cubic polynomials $f$ with connected $J(f)$ such that:

1. the polynomial $f$ has at least one non-repelling fixed point,
2. there are no repelling periodic cutpoints in $J(f)$ and
3. all non-repelling periodic cutpoints of $J(f)$, except at most one fixed point, have multiplier 1.

The following corollary is immediate from theorem A.

**Corollary 1.2.** $\text{PHD}_3 \subset \text{CU}$.

If $[f] \in \text{PHD}_d$, then $f$ cannot have two attracting periodic points as otherwise any small perturbation of $f$ will also have two attracting periodic points while there exist polynomials with classes from $\text{PHD}_d$ (and hence with only one periodic attracting point) arbitrarily close to $f$. A part of theorem A extends this observation to non-repelling periodic points. Observe that, by definition, if $J(f)$ is disconnected, then $[f] \not\in \text{CU}$. Also, by the Fatou–Shishikura inequality [Fat20, Shi87], a cubic polynomial $f$ has at most two non-repelling cycles.

Let $F_\lambda$ be the space of all cubic polynomials of the form

$$f_{\lambda, b}(z) = \lambda z + b z^2 + z^3, \quad b \in \mathbb{C}.$$ 

This space maps onto the space of classes of all cubic polynomials with a fixed point of multiplier $\lambda$ as a finite branched covering. This branched covering is equivalent to the map $b \mapsto a = b^2$, i.e. classes of polynomials $f_{\lambda, b} \in F_\lambda$ are in one-to-one correspondence with the values of $a$. Thus, if we talk about, say, points $[f]$ of $\mathcal{M}_3$, then it suffices to take $f \in F_\lambda$ for some $\lambda$. A polynomial $f \in F_\lambda$ is called **stable with respect to 0** if its Julia set admits an...
equivariant holomorphic motion over some neighbourhood of \( f \) in \( \mathcal{F}_\lambda \). Intuitively, this means that no bifurcations happen near \( f \) in \( \mathcal{F}_\lambda \) (precise definitions will be given later). The \((\lambda-)\) stable set \( S_\lambda \subset \mathbb{C} \) is the set of all \( b \in \mathbb{C} \) such that \( f_{\lambda,n} \) is stable with respect to 0. A \((\lambda-)\) stable domain is a component of \( S_\lambda \). A polynomial \( g \) is said to be stable if \( g \in \{f\} \) for some polynomial \( f \in \mathcal{F}_\lambda \) that is stable with respect to 0.

**Definition 1.3.** The set \( \text{PHD}_3 \) is the union of \( \text{PHD}_1 \) and classes of all polynomials from all \( \lambda \)-stable domains \( \Lambda \) with \(|\lambda| \leq 1\) such that for all \( b \in \text{Bd}(\Lambda) \), we have \( \{f_{\lambda,n}\} \in \text{PHD}_3 \).

We conjecture that \( \text{PHD}_3 = \text{CU} = \text{PHD}_1 \). To support this conjecture, we prove theorem B. Let \( \mathcal{L} \) be the set of classes of all polynomials with locally connected Julia set.

**Theorem B.** We have \( \text{PHD}_3 \subset \text{CU} \) and \( \mathcal{L} \cap \text{CU} = \mathcal{L} \cap \text{PHD}_3 \).

To state theorem C, we need the language of laminations. Laminations have been introduced by Thurston [Thu85] to study topological models of polynomials with locally connected Julia sets. Consider a degree \( d \) polynomial \( f \) such that \( K(f) \) is connected. Let \( S^1 = \mathbb{R}/\mathbb{Z} \) be the unit circle and \( \sigma_d : S^1 \to S^1 \) be the map defined by \( \sigma_d(x) = \frac{dx}{d} \mod 1 \).

If \( K(f) \) is locally connected then there exists an equivalence relation \( \sim_f \) on \( S^1 \) such that \( \sigma_d \) maps \( \sim_f \)-classes to \( \sim_f \)-classes, \( S^1 / \sim_f \) is homeomorphic to \( J(f) \), and \( f|_{\sim_f} \) is topologically conjugate to the self-mapping of \( S^1 / \sim_f \) induced by \( \sigma_d \). By [Kiw04, BCO11], the definition of \( \sim_f \) can be extended to the case where \( K(f) \) is not locally connected. Then the quotient of \( S^1 \) by \( \sim_f \) is the biggest quotient of \( J(f) \) that is a locally connected continuum, and, as before, \( \sigma_d \) maps \( \sim_f \)-classes to \( \sim_f \)-classes.

If \( K(f) \) (equivalently, \( J(f) \)) is not locally connected, then \( \sim_f \) may have infinite classes. The purpose of laminations is to describe the combinatorial structure of \( f|_{\sim_f} \).

**Definition 1.4.** A curve \( \Gamma \) in the dynamic plane of \( f \) consisting of (pre)periodic dynamic external rays \( R_f(\theta_1), R_f(\theta_2) \) and their common landing point \( x \in J(f) \) is called a (pre)periodic cut with vertex \( x \). The set \( G(x) \) then is defined as the convex hull of the arguments of all dynamic external rays landing at \( x \).

Define the set \( \mathcal{L}_f \) of chords in \( \mathcal{D} \) consisting of all edges of convex hulls of all \( \sim_f \)-classes, all edges of polygons \( G(x) \) associated with vertices \( x \) of (pre)periodic cuts, and all limits of these edges. A chord \( ab \) in \( \mathcal{L}_f \) is a leaf of \( \mathcal{L}_f \); set \( \sigma_3(ab) = \sigma_3(a)\sigma_3(b) \). The closure of a complementary domain to \( \mathcal{L}_f \) in \( \mathcal{D} \) is a gap of \( \mathcal{L}_f \); we write \( \sigma_3(G) \) for the convex hull of \( \sigma_3(G \cap S^1) \). The pair \( (\sim_f, \mathcal{L}_f) \) is called the laminational pair associated with \( f \). A rotational set \( H \) of \( \sim_f \) is a \( k \)-periodic leaf or a gap of \( \mathcal{L}_f \) such that the map \( \sigma_3^k : H \cap S^1 \to H \cap S^1 \) extends to a monotone map of \( S^1 \) topologically semi-conjugate to a non-trivial rotation of \( S^1 \).

**Theorem C.** If \( \{f\} \in \text{CU} \) then \( \mathcal{L}_f \) coincides with the set of leaves of \( \sim_f \), and the following properties hold:

1. Each periodic leaf of \( \sim_f \) has attached to it gap \( G \) such that \( G \cap S^1 \) is infinite and is not one \( \sim_f \)-class;
2. \( \sim_f \) has at most one rotational set.

**Notation and terminology.** We write \( \overline{A} \) for the closure of a set \( A \) in a topological space and \( \text{Bd}(A) \) for the boundary of \( A \); the \( n \)th iterate of a map \( f \) is denoted by \( f^n \). For \( A \subset \mathbb{C} \) let \( \text{CH}(A) \) be the convex hull of \( A \) in \( \mathbb{C} \). If it does not cause ambiguity, we speak of cutpoints meaning cutpoints of the appropriate Julia sets. We will consistently identify angles, i.e. elements of \( \mathbb{R}/\mathbb{Z} \), with points of the unit circle \( S^1 \subset \mathbb{C} \). If \( G \) is the convex hull of some closed subset \( G' \subset S^1 \), then we call \( G' \) the basis of \( G \). A gap \( G \) is said to be finite or infinite according to whether \( G' \) is finite or infinite.
2. Proof of theorem A

We first recall some terminology and notation.

2.1. Dynamic rays

Let $f(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$ be a monic degree $d$ polynomial. The Green function $G_f$ is defined by the formula

$$G_f(z) = \lim_{n \to \infty} \frac{\log |f^n(z)|}{d^n},$$

where $\log r$ equals $\log r$ if $r > 0$ and $0$ otherwise. The function $G_f$ is harmonic on the complement of the filled Julia set $K(f)$ of $f$ and is equal to $0$ on $K(f)$. Define dynamic rays as unbounded trajectories of the gradient flow for $G_f$ (in the introduction, dynamic rays have been defined in the case where $K(f)$ is connected; this new definition matches the old one in this case). Let $V(f)$ be the union of all dynamic rays of $f$. Then $V(f)$ is a forward-invariant open set, and there is a conformal isomorphism $\phi_f$ between $V(f)$ and some open subset of the set $\{|z| > 1\}$ with the following properties:

$$\phi_f(f(z)) = \phi_f(z)^d, \quad G_f(z) = \log |\phi_f(z)|.$$

These properties define the map $\phi_f$ almost uniquely: the only way to change the map $\phi_f$ without violating the two properties is to post-compose it with multiplication by a $(d - 1)$-st root of unity.

The map $\phi_f$ is called a Böttcher coordinate. It is used to parametrize dynamic rays of $f$. Every dynamic ray is the preimage of a straight radial ray $\{re^{2\pi i \theta} | r > r_0\}, r_0 \geq 1$, under the map $\phi_f$ (in the case where $K(f)$ is disconnected, we may have $r_0 > 1$ if the dynamic ray contains a pre-critical point in its closure). We will write $R_f(\theta)$ for this ray, and call it the dynamic ray of argument $\theta$. If $f$ is a degree $d$ polynomial, not necessarily monic, then we can make it monic by a complex linear change of variables. Thus, it still makes sense to talk about dynamic rays of $f$.

However, arguments of dynamic rays are not well-defined, since they depend on the choice of a Böttcher coordinate. Hence every time we consider rays in dynamic planes of different polynomials, we must resolve the issue of choosing the arguments consistently. For example, if a sequence $f_n$ of degree $d$ polynomials converges to a degree $d$ polynomial $f$, then we can choose any Böttcher coordinate for $f$, and then, for $f_n$ sufficiently close to $f$, choose the Böttcher coordinate for $f_n$ that is close to the chosen Böttcher coordinate for $f$.

Suppose that the Julia set $J(f)$ is connected. Consider a periodic repelling cutpoint $\alpha$ of $f$, and let $r$ be its minimal period. Then, by the Douady–Hubbard–Sullivan–Yoccoz landing theorem [DH84, Hub93], there are finitely many dynamic rays landing at $\alpha$; we will assume that the choice of their arguments is fixed, and denote the set of the arguments by $A_{\alpha}(\alpha)$. Every wedge between consecutive rays landing at $\alpha$ contains exactly one component of $J(f) \setminus \{\alpha\}$. The dynamic rays landing at $\alpha$ may form one or more orbits under the map $f^r$. Choose one of the orbits, and let $\theta_0, \theta_1, \ldots, \theta_{q-1}$ denote the arguments of all rays in this orbit labelled in counterclockwise order. Suppose that $d' \cdot \theta_i = \theta_{i+p} \pmod{q}$, where $d'$ is the degree of the polynomial $f$. In this case, we say that $\alpha$ has combinatorial rotation number $p/q$. It is easy to see that the combinatorial rotation number is well defined, i.e. does not depend on the choice of the orbit. Observe that $p$, $q$ are coprime except for the case when $p/q = 0$ and all rays landing at $\alpha$ are invariant. Every repelling fixed point has a well-defined combinatorial rotation number (thus, $p/q$ above does not depend on the choice of $\theta_0$).
2.2. Polynomials in $\text{PHD}_d$

We now recall lemma B.1 from [GM93] that goes back to Douady and Hubbard [DH8485].

**Lemma 2.1.** Let $f$ be a polynomial, and $z$ be a repelling periodic point of $f$. If a periodic ray $R_f(\theta)$ lands at $z$, then, for every polynomial $g$ sufficiently close to $f$, the ray $R_g(\theta)$ lands at a repelling periodic point $w$ close to $z$. Moreover, $w$ depends holomorphically on $g$.

Choose a polynomial $f$ with $[f] \in \text{PHD}_d$. We want to prove the following statements:

1. the map $f$ has no repelling periodic cutpoints and
2. the map $f$ has at most one non-repelling periodic point of multiplier different from 1.

Statement (1) follows from lemma 2.1. Indeed, suppose that $\alpha$ is a repelling periodic cutpoint of $f$. Let $\theta_0, \ldots, \theta_{q-1}$ be the arguments of dynamic rays landing at $\alpha$. Since $\alpha$ is a cutpoint, we have $q > 1$. Lemma 2.1 says that, for $g$ sufficiently close to $f$, the dynamic rays with arguments $\theta_0, \ldots, \theta_{q-1}$ in the dynamic plane of $g$ land at the same periodic point that is obtained from $\alpha$ by analytic continuation. We get a contradiction if we choose $g$ such that $[g] \in \text{PHD}_d$.

The second statement is a consequence of the Pommerenke–Levin–Yoccoz inequality, see, e.g., [Hub93]. It follows from the Pommerenke–Levin–Yoccoz inequality that, for any sequence of polynomials $f_n$ with repelling fixed points $\alpha_n \to \alpha$ whose multipliers are different from 1, the combinatorial rotation numbers of $f_n$ at $\alpha_n$ converge to $\rho$. We can now prove lemma 2.2.

**Lemma 2.2.** Consider a polynomial $f$, whose class belongs to $\text{PHD}_d$, and a sequence of polynomials $f_n$ converging to $f$, whose classes belong to $\text{PHD}_d$. If $f$ has a non-repelling fixed point $\alpha$, whose multiplier is different from 1, then $\alpha$ is the limit of the attracting fixed points of $f_n$.

**Proof.** Let $\alpha$ be neutral and $\alpha_n$ be a fixed point of $f_n$ such that $\alpha_n \to \alpha$ as $n \to \infty$. If $\alpha_n$ are attracting for arbitrarily large $n$, then we are done. Otherwise assume that for all large $n$, the points $\alpha_n$ are repelling fixed points. Then their combinatorial rotation numbers equal $0$. By the assumptions, $f'(\alpha) = e^{2\pi i \rho}$, where $\rho \not\equiv 0 \pmod{2\pi}$. On the other hand, by the Pommerenke–Levin–Yoccoz inequality, the combinatorial rotation numbers of $f_n$ at $\alpha_n$ (which are all equal to 0) must converge to $\rho$, a contradiction.

We can now complete the proof of theorem A.

**Proof of theorem A.** Observe that if $[f] \in \text{PHD}_d$, then one of the non-repelling cycles of $f$ must be a fixed point (indeed, as we approximate $f$ with polynomials $g$, whose classes belong to $\text{PHD}_d$, the attracting fixed points of $g$ converge to a non-repelling fixed point of $f$). Let $[f] \in \text{PHD}_d$. By way of contradiction, suppose that $\alpha$ and $\beta$ are two non-repelling periodic points, whose multipliers are different from one. Replacing $f$ with a suitable iterate, we may assume that $\alpha$ is fixed but has multiplier different from 1. By lemma 2.2, $\alpha$ is the limit of the attracting fixed points of polynomials $g$ with $[g] \in \text{PHD}_d$ approximating the polynomial $f$. However, the same must be true for point $\beta$, a contradiction. □
3. Proof of the first part of theorem B

We will write $F$ for the union of all $\mathcal{F}_\lambda$ and $\mathcal{F}_{nr}$ for the union of all $\mathcal{F}_v$ with $|\lambda| \leq 1$. Fix $\lambda$ with $|\lambda| \leq 1$. Let $g_{\lambda,b}$ be the Green function for $K(f_{\lambda,b})$. Let $V_{\lambda,b}$ be the union of all unbounded trajectories of the gradient flow generated by $g_{\lambda,b}$. The Böttcher coordinate is an analytic map $\Phi_{\lambda,b} : V_{\lambda,b} \to \mathbb{C}$ with $\Phi_{\lambda,b} \circ f_{\lambda,b} = \Phi_{\lambda,b}$ and $\Phi_{\lambda,b}(z) = z + o(z)$ as $z \to \infty$. The following theorem is an easy consequence of the analytic dependence of the Böttcher coordinate on parameters [DH84, BrHu88].

**Theorem 3.1** ([BuHe01], proposition 2). Let $V_{\lambda}$ be the union of $[b] \times V_{\lambda,b}$ for all $b \in \mathbb{C}$. This set is open in $\mathbb{C}$. The map $F_{\lambda} : V_{\lambda} \to \mathbb{C}^2$ given by the formula $F_{\lambda}(b,z) = (b, \Phi_{\lambda,b}(z))$ is an analytic embedding of $V_{\lambda}$ into $\mathbb{C}^2$.

We will write $R_{\lambda,b}(\theta)$ for the dynamic ray $R_{f_{\lambda,b}}(\theta)$.

3.1. Polynomials with parabolic points and their petals

Let $g$ be a polynomial of arbitrary degree such that 0 is a fixed parabolic point of $g$ of multiplier 1. Suppose that $g(z) = z + a_1z^{q+1} + o(z^{q+1})$, where $q$ is a positive integer. Below, we recall some basic terminology, which is used when working with parabolic points; a general reference is [Mil06]. An attracting vector for $g$ is defined as a vector (= complex number) $v$ such that $avq$ is a negative real number, i.e. $v$ and $avq$ have opposite directions. Clearly, there are $q$ straight rays consisting of attracting vectors, which divide the plane of complex numbers into $q$ repelling sectors.

Consider a repelling sector $S$. Note that the set $S^{-q} = \{z \in \mathbb{C} \mid z^{-q} \in S\}$ is the complement of the ray $\{t a \mid t > 0\}$ in $\mathbb{C}$. We will write $F : S^{-q} \to \mathbb{C}$ for the composition of the function $w \mapsto w^{-1/q}$ mapping $S^{-q}$ onto $S$, the function $g$ mapping $S$ onto $g(S)$, and the function $z \mapsto z^{-q}$ mapping $g(S)$ to $\mathbb{C}$. We have $F(w) = w - qa + \alpha(w)$, where $\alpha(w)$ denotes a power series in $w^{-1/q}$ that converges in a neighbourhood of infinity, and whose free term is zero (note that the function $w \mapsto w^{-1/q}$ is single valued and holomorphic on $S^{-q}$). It follows that there exists a positive real number $r$ with the property that $|\alpha(w)| < \frac{|w|}{2}$ whenever $|w| \geq r|a|$. Consider the half-plane $\Pi$ given by the inequality $\text{Re}(w/a) \geq r$. We include the point $\infty$ into $\Pi$ so that $\Pi$ is a compact subset of the Riemann sphere. Since $w \in \Pi$ implies that $|w| \geq r|a|$, we have $F(\Pi) \supset \Pi$, and also that the shortest Euclidean distance from a finite point on the boundary of $\Pi$ to a point on the boundary of $F(\Pi)$ is at least $(q - \frac{1}{2})|a|$. The preimage $RP$ of the half-plane $\Pi$ under the map $z \mapsto z^{-q}$ from $S \cup \{\infty\}$ to $S^{-q} \cup \{0\}$ is called a (closed) repelling petal of $g$.

We choose one repelling petal in every repelling sector; thus, our polynomial $g(z) = z + a_1z^{q+1} + o(z^{q+1})$ has $q$ repelling petals. A repelling petal $RP$ of $g$ is such that $g(RP) \supset RP$. Let us discuss the dependence of the repelling petals on parameters. The argument of the following lemma is rather standard, see, e.g., the proof of lemma 5 in [BuHe01].

**Lemma 3.2.** Let $g_t(z) = z + a_t z^{q+1} + o(z^{q+1})$ be a continuous family of polynomials, in which $a_t$ never vanishes, and the parameter $t$ runs through some locally compact metric space $I$ that is a union of countably many compact spaces. Then all $q$ repelling petals of $g_t$ can be chosen to vary continuously with respect to the parameter.

**Proof.** First note that, since the attracting vectors depend continuously on the parameter $t$, so do the repelling sectors (to be more precise, the intersections of the repelling sectors with a finite disc about 0 depend continuously on $t$ with respect to the Hausdorff metric). We will choose $S_i$ to be a continuous family of repelling sectors for $g_i$. 1885
Consider some particular parameter value $t$. The map

$$F_t(w) = w - qa_t + a_t(w)$$

of $S_t^{-q}$ to $\mathbb{C}$ is obtained from the map $g_t : S_t \to g_t(S_t)$ by the conjugation $z \mapsto z^{-q}$, as above.

Let $r_t$ be a positive real number with the property that $|a_t(w)| < \frac{|a_t|}{r_t}$ whenever $|w| \geq r_t |a_t|$. Set $\Pi_t(a)$ to be the half plane given by the inequality $\Re(w/a) \geq r$. We include the point $\infty$ to $\Pi_t(a)$. Thus $\Pi_t(a)$ is compact. Note that $a_t(w')$ is a continuous function of $(t', w')$ defined at least for $w' \in \Pi_t(a_t)$ and $t'$ sufficiently close to $t$. It follows that there is some open neighbourhood $E_t$ of $t$ such that $|a_t(w)| < \frac{|a_t|}{2}$ for all $t' \in E_t$ and all $w \in \Pi_t(a_t)$ (we use the compactness of $\Pi_t(a)$ and its continuous dependence on $a$ in the Hausdorff metric associated with the spherical metric).

The family of neighbourhoods $E_t$ is an open covering of $I$. Choose a countable and locally finite subcovering $E_n$. Let $\varphi_n : I \to \mathbb{R}$ be the associated partition of unity, so that the support of the continuous function $\varphi_n \geq 0$ is contained in $E_n$, and $\sum_n \varphi_n = 1$. Set $R_t = \sum_n \varphi_n(t) r_n$. Then $R_t$ depends continuously on $t$, and we have $|a_t(w)| < \frac{|a_t|}{2}$ for all $w \in \Pi_t(a_t)$ and $t \in I$.

The half-plane $\Pi_t(a_t)$ depends continuously on $t$ in the Hausdorff metric. Then the image of $\Pi_t(a_t)$ under the map $w \mapsto w^{-1/q}$ from $S_t^{-q}$ to $S_t$ is a repelling petal depending continuously on $t$.\]

### 3.2. Stability of rays and their perturbations

Throughout this subsection, we fix that $\lambda$ is a root of unity, i.e. $\lambda = \exp(2\pi i p/q)$ for some relatively prime $p$ and $q$. Since $\lambda$ is fixed, we will omit $\lambda$ from the notation $f_b$, $R_b(\theta)$, etc. We discuss conditions that guarantee that a dynamical ray $R_b(\theta)$ landing at $0$ is stable, i.e. for $b'$ sufficiently close to $b$, the ray $R_{b'}(\theta)$ also lands at $0$.

**Proposition 3.3.** We have $f_b^{-\infty}(z) = z + T_{p/q}(b)z^{q+1} + o(z^{q+1})$, where $T_{p/q}(b)$ is a non-zero polynomial in $b$.

**Proof.** By the petal theorem [Bea91, theorem 6.5.10], we have

$$f_b^{-\infty}(z) = z + T_{p/q}(b)z^{q+1} + o(z^{q+1}).$$

It remains to prove that the polynomial $T_{p/q}(b)$ cannot be identically equal to zero. For any $b$ such that $T_{p/q}(b) = 0$, the polynomial $f_b$ has at least two cycles of attracting petals at $0$. Each of the associated cycles of Fatou domains must contain a critical point of $f_b$. Thus both critical orbits of $f_b$ converge to $0$. However, for large $b$, one of the critical points escapes. Therefore, for such $b$, we have $T_{p/q}(b) \neq 0$.\]

**Proposition 3.4.** Suppose that a dynamic ray $R_{b_\star}(\theta)$ with periodic $\theta$ lands at $0$, and $T_{p/q}(b_\star) \neq 0$. Then, for all $b$ sufficiently close to $b_\star$, the ray $R_b(\theta)$ lands at $0$.

**Proof.** By lemma 10.1 of [Mil06], the ray $R_{b_\star}(\theta)$ must be tangent to some repelling vector of $f_{b_\star}$ at $0$. Let $R_{b_\star}$ be the corresponding repelling petal of $f_{b_\star}^{-q}$. The period of $\theta$ is equal to $q$ by theorem 18.13 of [Mil06]. There are two points $z_\star$ and $f_{b_\star}^{-1}(z_\star)$ in $R_{b_\star}(\theta)$ that lie in the interior of $R_{b_\star}$. By lemma 3.2 for all $b$ sufficiently close to $b_\star$, we can define a repelling petal $R_b$ of $f_b^{-q}$ that is close to $R_{b_\star}$ in the Hausdorff metric.

Let $L_\varepsilon$ denote the subray of the ray $R_{b_\star}(\theta)$ from $z_\star$ to infinity. By theorem 3.1, for every $\varepsilon > 0$, we can choose a neighbourhood $U$ of $b_\star$ such that, for all $b \in U$, the corresponding piece $L$ of $R_b$ is $\varepsilon$-close to $L_\varepsilon$ in the Hausdorff metric. The number $\varepsilon$ can be chosen so that this
implies that \( L \) enters the corresponding petal \( RP_b \). Dynamics inside \( RP_b \) implies that \( R_b(\theta) \) lands at 0.

3.3. The proof of inclusion \( \text{PHD}_3 \subset CU \)

We first recall the necessary terminology. Let \( \Lambda \) be a Riemann surface and \( Z \subset C \) any (!) subset. A holomorphic motion of the set \( Z \) is a map \( \mu : Z \times \Lambda \rightarrow C \) with the following properties:

- for every \( z \in Z \), the map \( \mu(z, \cdot) : \Lambda \rightarrow C \) is holomorphic;
- for \( z \neq z' \) and every \( \nu \in \Lambda \), we have \( \mu(z, \nu) \neq \mu(z', \nu) \);
- there is a point \( v_0 \) such that \( \mu(z, v_0) = z \) for all \( z \in Z \).

A crucial result about holomorphic motions is the \( \lambda \)-lemma of Mané et al [MSS83]: a holomorphic motion of a set \( Z \) extends to a unique holomorphic motion of the closure \( \overline{Z} \); moreover, this extension is a continuous function in two variables. Suppose that for each \( \nu \in \Lambda \) a map \( h_\nu : Z \rightarrow C \) is given. A holomorphic motion \( \mu : Z \times \Lambda \rightarrow C \) is called equivariant (with respect to the family of maps \( h_\nu \)) if for every \( \nu \in \Lambda \) and every \( z \in Z \) with \( h_\nu(z) \in Z \) we have \( h_\nu(\mu(z, \nu)) = \mu(h_\nu(z), \nu) \). In the introduction, we have defined the \( \lambda \)-stable set \( S_\lambda \) consisting of parameter values \( b_\lambda \) such that \( J(f_{\lambda,b}) \) moves holomorphically with respect to \( b \) in some neighbourhood of \( b_\lambda \), and the corresponding holomorphic motion is equivariant. We have also defined \( \lambda \)-stable domains as components of \( S_\lambda \). Often, by stable domains we also mean the corresponding subsets of \( F_\lambda \).

By theorem A, we have \( \text{PHD}_3 \subset CU \). Consider a stable domain \( \mathcal{U} \subset F_\lambda \) such that for all \( f \in \text{Bd}(\mathcal{U}) \), we have \( [f] \in \text{PHD}_3 \). We need to prove that classes of all polynomials in \( \mathcal{U} \) belong to \( CU \). Suppose that \( f_\circ = f_{\lambda,b} \in \mathcal{U} \), and show that then \( f_\circ \) has properties (1)–(3) from definition 1.1.

**Property 1.** Clearly, \( f_\circ \) has a fixed non-repelling point 0, thus property (1) is fulfilled.

**Property 2.** Let us prove that \( f_\circ \) has no repelling cutpoints. Assume that \( f_\circ \) has a repelling periodic cutpoint \( z_{b_\circ} \). The set \( \text{Arg}(z_{b_\circ}) \) of arguments of external rays of \( f_\circ \) landing at \( z_{b_\circ} \) consists of at least two angles. Since all maps in \( \mathcal{U} \) are quasi-symmetrically conjugate, it is easy to see (e.g., by lemma 3.5 [BOPT13b]) that all maps \( f_{\lambda,b} \in \mathcal{U} \) have repelling periodic cutpoints \( z_b \) corresponding to \( z_{b_\circ} \). By lemma 2.1 \( \text{Arg}(z_b) = \text{Arg}(z_{b_\circ}) \). Suppose that \( [\alpha, \beta] \subset \text{Arg}(z_{b_\circ}) \).

Let \( \Lambda \) be the set of all parameter values \( b \) with \( f_{\lambda,b} \in \mathcal{U} \), and choose a sequence \( b_{\circ} \rightarrow b' \in \text{Bd}(\Lambda) \). We may assume that \( z_{b_\circ} \rightarrow z_{b'} \), where \( z_{b'} \) is a non-attracting periodic point of \( f_{\lambda,b'} \). If both rays \( R_b(\alpha), R_b(\beta) \) land at repelling periodic points, then these landing points must coincide as otherwise by lemma 2.1 we obtain a contradiction with the fact that \( R_b(\alpha), R_b(\beta) \) land at \( z_{b_\circ} \) and \( z_{b_\circ} \rightarrow z_{b'} \). However, by theorem A, the map \( f_{\lambda,b'} \) does not have repelling periodic cutpoints. Hence one of the rays \( R_b(\alpha), R_b(\beta) \) lands at a parabolic periodic point. Clearly, for at most finitely many parameter values \( b' \in \text{Bd}(\Lambda) \) the rays \( R_b(\alpha) \) or \( R_b(\beta) \) land at a parabolic point distinct from 0. Assume that for infinitely many \( b' \in \text{Bd}(\Lambda) \) the rays \( R_b(\alpha) \) land at 0, which is a parabolic fixed point: \( \lambda = \exp(2\pi ip/q) \) for some relatively prime \( p \) and \( q \).

Let us show that in the above case \( T_{p/q}(b') = 0 \). Indeed, arbitrarily close to \( b' \), there are parameter values \( b \), for which \( R_b(\alpha) \) does not land at 0. It follows from proposition 3.4 that \( T_{p/q}(b') = 0 \). However, the polynomial \( T_{p/q} \) has only finitely many roots, a contradiction.

**Property 3.** Suppose that \( f_\circ \) has a non-repelling \( n \)-periodic point \( z_{b_\circ} \neq 0 \) with multiplier not equal to 1. Since \( f_\circ \) is stable, the corresponding periodic point \( z_{b_\circ} \neq 0 \) of \( f_{b_\circ}, b_\circ \in \Lambda \), is
Definition 4.1 (Laminations). A lamination \( \sim \) on \( \mathbb{C} \) is called a laminations if either \( \mathbb{S}^1 \) is one \( \sim \)-class (such laminations are called degenerate), or the following hold:

(E1) the graph of \( \sim \) is a closed subset in \( \mathbb{S}^1 \times \mathbb{S}^1 \);

(E2) if \( t_1 \sim t_2 \in \mathbb{S}^1 \) and \( t_3 \sim t_4 \in \mathbb{S}^1 \), but \( t_2 \not\sim t_3 \), then the open straight line segments in \( \mathbb{C} \) with endpoints \( t_1, t_2 \) and \( t_3, t_4 \) are disjoint;

(E3) each equivalence class of \( \sim \) is totally disconnected.

A lamination \( \sim \) admits a canonical extension onto \( \mathbb{C} \): its classes are either convex hulls of classes of \( \sim \), or points which do not belong to such convex hulls. By Moore’s theorem the space \( \mathbb{C}/\sim \) is homeomorphic to \( \mathbb{C} \). The quotient map \( p_{\sim} : \mathbb{S}^1 \to \mathbb{S}^1 / \sim \) extends to the plane with the only non-trivial point-preimages (fibres) being the convex hulls of \( \sim \)-classes. From now on we will always consider such extensions of the quotient map.

Definition 4.2 (Laminations and dynamics). A lamination \( \sim \) is called \((\sigma, g)\)-invariant if:

(D1) \( \sim \) is forward invariant: for a \( \sim \)-class \( g \), the set \( \sigma_d(g) \) is a \( \sim \)-class;
(D2) for any $\sim$-class $g$, the map $\sigma_d : g \mapsto \sigma_d(g)$ extends to $S^1$ as an orientation preserving covering map such that $g$ is the full preimage of $\sigma_d(g)$ under this covering map.

For a $\sigma_d$-invariant lamination $\sim$ consider the topological Julia set $\mathbb{S}^1/\sim = J_\sim$ and the topological polynomial $f_\sim : J_\sim \to J_\sim$ induced by $\sigma_d$. One can extend $f_\sim$ to a branched-covering map $f_\sim : \mathbb{C} \to \mathbb{C}$ of degree $d$ called a topological polynomial too. The map $\rho_\sim$ semi-conjugates $\sigma_d$ with $f_\sim$, at least on the unit circle and all leaves of $\sim$. Unlike complex polynomials, topological polynomials can have periodic critical points in their topological Julia sets. The complement $K_\sim$ of the unique unbounded component $U_\infty(J_\sim)$ of $\mathbb{C} \setminus J_\sim$ is called the filled topological Julia set. For $a, b \in \mathbb{S}^1$, let $\overline{ab}$ be the chord endpoints $a$ and $b$. If $A \subset \mathbb{S}^1$ is closed, boundary chords of the convex hull $\text{CH}(A)$ of $A$ are called edges of $\text{CH}(A)$.

**Definition 4.3 (Leaves and gaps).** If $A$ is a $\sim$-class, call an edge $\overline{ab}$ of $\text{Bd}(\text{CH}(A))$ a leaf. All points of $\mathbb{S}^1$ are also called (degenerate) leaves. The family $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the geometric lamination (geolamination) generated by $\sim$. Let $\mathcal{L}_\sim^+$ be the union of all leaves of $\mathcal{L}_\sim$. The closure of a non-empty component of $\mathbb{D} \setminus \mathcal{L}_\sim^+$ is called a gap of $\sim$. Leaves and gaps of $\mathcal{L}_\sim$ are called $\mathcal{L}_\sim$-sets; a leaf which is not an edge of a finite gap is called independent. If $G$ is a gap or leaf, we call the set $G' = \mathbb{S}^1 \cap G$ the basis of $G$.

Extend $\sigma_d$ (keeping the notation) linearly over all individual chords in $\overline{\mathbb{D}}$ (e.g., over leaves of $\mathcal{L}_\sim$); even though the extended $\sigma_d$ is not well-defined on the entire disc, it is well-defined on $\mathcal{L}_\sim^+$. A gap or leaf $U$ is said to be (pre)periodic if $\sigma_d^m(U') = \sigma_d^n(U')$ for some $m, n \geq 0$, $k > 0$. If $m$ above can be chosen to be 0, then $U$ is called periodic; the minimal number $k$ above is called the period of $U$. If $U$ is (pre)periodic but not periodic then it is called preperiodic.

**Definition 4.4 (Rotational sets and numbers).** If $g$ is a periodic non-degenerate finite $\sim$-class of period $n$, the map $\sigma_d^n|_g$ is conjugate (by a conjugacy that preserves the cyclic order) to a rigid rotation $R_\rho$ by a rational angle $\rho$ on a finite $R_\rho$-invariant subset of $\mathbb{S}^1$. The number $\rho$ is then called the rotation number of $g$. A gap $G$ such that its basis $G'$ is infinite is called a Fatou gap. A periodic Fatou gap $G$ of period $n$ such that $f_\sim^n|_{\text{Bd}(p_\sim(G))}$ is conjugate to an irrational rotation by an angle $\rho$, is called a Siegel gap while $\rho$ is called the rotation number of $G$. Otherwise $f_\sim^n|_{\text{Bd}(p_\sim(G))}$ is conjugate to a map $\sigma_k$ with some $k > 1$ and $G$ is called a Fatou gap of degree $k$. Siegel gaps and finite $\sim$-classes with non-zero rotation number are called rotational sets.

Let $X \subset \mathbb{C}$ be a continuum, and let $U_\infty(X)$ be the unbounded component of $\mathbb{C} \setminus X$. If $X = \text{Bd}(\mathcal{U}_\infty(X))$, we call $X$ unshielded. A continuous map $\varphi : Y \to Z$ is monotone if all fibres are continua. Let $A$ be a continuum. A monotone onto map $\varphi : A \to Y_{\varphi,A}$ with locally connected $Y_{\varphi,A}$ is called a finest (monotone) map if for any monotone map $\psi : A \to L$ onto a locally connected continuum $L$ there is a map $h : Y_{\varphi,A} \to L$ with $\psi = h \circ \varphi$ (then $h$ is monotone because for $x \in L$, we have $h^{-1}(x) = \varphi(\psi^{-1}(x))$). If $\varphi : A \to B, \varphi' : A \to B'$ are two finest maps, then the map associating points $\varphi(x) \in B$ and $\varphi'(x) \in B'$ for every $x \in A$ is a homeomorphism between $B$ and $B'$. Hence all sets $Y_{\varphi,A}$ are homeomorphic, all finest maps $\varphi$ are the same up to a homeomorphism, and we can call talk of the finest model $Y_{A} = Y$ of $A$ and the finest map $\varphi_A = \varphi$ of $A$ onto $Y$.

**Theorem 4.5 (Theorem 1 [BCO11]).** Let $Q$ be an unshielded continuum. Then there exist the finest map $\varphi$ and the finest model $Y$ of $Q$ given by a lamination $\sim_Q$ on $\mathbb{S}^1$ so that $Y = \mathbb{S}^1/\sim_Q$; moreover, $\varphi$ can be extended to a map $\varphi : \mathbb{C} \to \mathbb{C}$, which collapses only those complementary domains to $Q$ whose boundaries are collapsed by $\varphi$, and is a homeomorphism elsewhere in $\mathbb{C} \setminus Q$. For $y \in Y$ the fibre $\varphi^{-1}(y)$ coincides with the topological hull of the union of impressions of all external to $Q$ rays with arguments from the set $p_\varphi^{-1}(y)$. 1889
Here $p_{\sim}$ is the quotient map of $S^1$ onto $Y$; we call $\sim$ the finest lamination. By a finest map we mean any extension of the finest map of $Q$ over $\mathbb{C}$.

**Definition 4.6 (Critical leaves and gaps).** A leaf of a lamination $\sim$ is called critical if its endpoints have the same image. A gap $G$ is said to be critical if $\sigma_{d|G}$ is at least $k$-to-1 for some $k > 1$.

Lemma 4.7 is well known; we state it here without a proof.

**Lemma 4.7.** An edge of a periodic gap is either (pre)critical or (pre)periodic.

In [BCO11], theorem 4.5 is applied to polynomials with connected Julia set which yields theorem 4.8 (a similar earlier result is due to Kiwi [Kiw04]).

**Theorem 4.8 ([BCO11], theorem 2).** Let $f$ be a complex polynomial with connected Julia set and finest lamination $\sim_f = \sim_{J(f)}$. Then there exists a topological polynomial $f_{\sim_f}: \mathbb{C} \to \mathbb{C}$ that semi-conjugates $f$ and $f_{\sim_f}$. If $x \in J_{\sim_f}$ corresponds to a finite periodic $\sim_{f_{\sim_f}}$-class $p^{-1}_{\sim_f}(x)$ then the fibre $\phi_{\sim_f}^{-1}(x)$ is a point. No periodic Fatou domain of $f$ of degree greater than 1 is collapsed by $\phi_f$.

We need the following definition.

**Definition 4.9.** Call gaps finite or infinite if their bases are finite or infinite; infinite $\sim$-classes have infinite gaps as their convex hulls (such gaps will be called infinite gap-classes). By [BL02] all such gaps are (pre)periodic, and periodic infinite gap-classes are Fatou gaps of degree greater than 1. Call the corresponding fibres CS-fibres. Thus, if $x \in J_{\sim_f}$ corresponds to an infinite gap-class $p^{-1}_{\sim_f}(x)$ then the $x$-fibre $\phi_{\sim_f}^{-1}(x)$ is said to be a CS-fibre.

The following lemma explains the terminology.

**Lemma 4.10 ([BOPT13b], proposition 4.4).** A periodic CS-fibre contains either a Cremer point or a Siegel point.

The drawback of using the lamination $\sim_f$ to model the dynamics of $f$ is that $\sim_f$ may incompletely reflect the properties of repelling periodic cutpoints of $J(f)$. In a lot of cases this does not happen. Indeed, if all $\sim_f$-classes are finite then by theorem 4.8 there is a one-to-one correspondence between repelling and parabolic periodic cutpoints of $f$ and their preimages on the one hand and the (pre)periodic non-degenerate classes of $\sim_f$. However, in the case when some $\sim_f$-classes are infinite this may no longer be the case.

For example, suppose that a cubic polynomial $f$ has a fixed repelling point $0$ at which $R_f(0)$ and $R_f(\frac{1}{2})$ land, and no more repelling periodic cutpoints. Moreover, suppose that in each ‘half-plane’ created by the cut $R_f(0) \cup \{0\} \cup R_f(\frac{1}{2})$ there is a fixed Cremer point. Denote these fixed Cremer points by $a$ and $b$. Then the thickening construction of Douady and Hubbard [DH85] (see also [Mil00b] and, more specifically, [EY99]) shows that there are two quadratic-like Julia sets in $J(f)$, namely $J_a$ (containing $a$) and $J_b$ (containing $b$). Each of them corresponds to a quadratic Julia set with a Cremer fixed point, and by [BO06] the only monotone map of $J_a$ (or $J_b$) onto a locally connected continuum is a collapse to one point. It follows that the only monotone map of $J$ onto a locally connected continuum is a collapse to one point. Hence the lamination $\sim_f$ identifies all points of the circle and misses the fact that $f$ has a fixed repelling cutpoint 0.

Let $F$ be a fibre associated with an infinite $\sim_f$-class. We saw that $F$ may contain periodic repelling points cutting $F$ such that the corresponding leaves are not included in $L_{\sim_f}$. By proposition 40 [BCO11] there are at most finitely many repelling or parabolic cutpoints in $F$. 

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With each such point \( x \) we associate the convex hull of the set \( \mathcal{A}_f(x) \). We add the edges of such convex hulls to \( \mathcal{L}_\sim \). Then we add to \( \mathcal{L}_\sim \) the edges of gaps corresponding to preimages of such points. Finally, we take the limit leaves of this family of leaves and add them to \( \mathcal{L}_\sim \). This creates a new geolamination \( \mathcal{L}_f \) called the geolamination generated by \( f \). In this way we combine \( \mathcal{L}_\sim \) with the rational lamination defined by Kiwi in [Kiw04]. In \( \mathcal{L}_f \) we will distinguish between Fatou gaps corresponding to non-degenerate Fatou domains of \( f \), infinite gaps of \( \mathcal{L}_f \) that are gap-classes of \( \mathcal{L}_\sim \), and infinite gaps \( H \) of \( \sim_f \) such that \( H \cap \mathbb{S}^1 \) is one \( \sim_f \)-class subdivided by finitely many finite gaps or leaves and their preimages as in the definition of \( \mathcal{L}_f \).

Definition 4.11. A laminational pair is a pair \( \{ \sim, \mathcal{L} \} \) where \( \mathcal{L} \supset \mathcal{L}_\sim \) is a geolamination obtained by adding to \( \mathcal{L}_\sim \) finitely many finite periodic gaps or leaves inside the convex hull of each infinite \( \sim \)-class as well as all their pullbacks and limits so that \( \mathcal{L} \) is a geolamination.

Definition 4.12. A cubic laminational pair \( \{ \sim, \mathcal{L} \} \) is cuboidal if \( \mathcal{L} \) has at most one rotational set and each periodic non-degenerate leaf of \( \mathcal{L} \) has an attached to it Fatou gap whose basis is not contained in one \( \sim \)-class.

There are two extreme cases for \( \{ \sim_f, \mathcal{L}_f \} \). First, \( \sim_f \) may identify no two points. Then \( J(f) \) is a Jordan curve, \( \mathcal{L}_f \) has no leaves, and \([f]\) \( \in \) CU. We call such a laminational pair empty. Second, \( \sim_f \) may identify all points of \( \mathbb{S}^1 \) while \( \mathcal{L}_f \) contains no leaves. By our lemma 5.1 then again \([f]\) \( \in \) CU. We call such a laminational pair degenerate. The degenerate and the empty laminational pairs share the same geolamination, are cuboidal, and correspond to polynomials \( f \) with \([f]\) \( \in \) CU, yet correspond to two very different types of dynamics. In all other cases \( \mathcal{L}_f \) includes some non-degenerate leaves.

5. Proof of theorem C

Lemma 5.1 deals with the case when \( \mathcal{L}_f \) has only degenerate leaves. We will assume that 0 is a fixed point of \( f \).

Lemma 5.1. If all leaves of \( \mathcal{L}_f \) are degenerate then \([f]\) \( \in \) CU, and 0 is the unique non-repelling periodic point of \( f \). Moreover, if \( \sim_f \) consists of one class, then 0 is a Cremer or Siegel fixed point.

Proof. We may assume that \( \sim_f \) consists of one class coinciding with \( \mathbb{S}^1 \). By definition of \( \mathcal{L}_f \) the map \( f \) has no repelling periodic cutpoints, and, by theorem 4.8, the polynomial \( f \) has no attracting or parabolic periodic points. By lemma 4.10, the point 0 is a fixed Cremer or Siegel point. Suppose that there is a non-repelling periodic point \( x \neq 0 \) of \( f \). Similar to the above \( x \) is also a Cremer or a Siegel periodic point. Then by [Kiw00] there exists a repelling periodic point separating \( x \) and 0, a contradiction.

Lemma 5.2. If \([f]\) \( \in \) CU then \( \mathcal{L}_\sim \mathcal{L}_f \).

Before we prove lemma 5.2, we need to recall a description of quadratic (i.e., degree 2) invariant gaps given in [BOP13a]. Let \( G \) be a quadratic invariant gap of some \( \sigma_3 \)-invariant lamination. Then there is a unique edge \( M \) of \( G \) (the major of \( G \)) separating the circle into two arcs, one of which contains all vertices of \( G \) and is of length at most \( \frac{2}{3} \); the leaf \( M \) must be critical or periodic. Moreover, all edges of \( G \) are iterated \( \sigma_3 \)-preimages of \( M \). Suppose that a quadratic invariant gap \( G \) is a gap of \( \sim_f \). By [BOP13b, theorem 7.7], if \( M = \theta_1 \theta_2 \) is a periodic major of \( G \), then the external rays \( R_f(\theta_1), R_f(\theta_2) \) land at the same point. This
implies that if $\alpha \beta$ is a (pre)periodic edge of $G$ then the external rays $R_f(\alpha)$, $R_f(\beta)$ land at the same point.

**Proof of lemma 5.2.** Suppose that $[f] \in \text{CU}$ and $\mathcal{L}_f \neq \mathcal{L}_G$. Then $f$ has a periodic CS-fibre $F$. By lemma 4.10, there exists a Cremer or Siegel periodic point $y \in F$. If $F$ is not invariant then $y$ is not fixed contradicting definition 1.1. Thus $F$ is invariant, and we may assume that $y = 0 \in F$ is a fixed Cremer or Siegel point. As above, the corresponding to $F$ invariant Fatou gap $G$ is of degree greater than 1. If $G$ is of degree 3 then $G^* = S^3$. Since $[f] \in \text{CU}$, the map $f$ does not have repelling periodic cutpoints. Since $G^* = S^3$, hence $F = J(f)$, by theorem 4.8, the map $f$ cannot have parabolic periodic points. Hence by definition in this case all leaves of $\mathcal{L}_f = \mathcal{L}_G$ are degenerate. Assume that $G$ is of degree 2. Since $[f] \in \text{CU}$ has no repelling cutpoints, then $L_{\infty} \neq \mathcal{L}_f$ implies that there is a parabolic periodic cutpoint $x$ of $F$. Since $f$ is cubic, by the Fatou–Shishikura inequality, the union of the orbit of $x$ and the point 0 is the set of all non-repelling periodic points of $f$. In particular, there are no other periodic cutpoints of $F$.

Let $\tilde{X}$ be the union of all rays landing at $x$ and $\{x\}$ itself. Some edges of the convex hull $X$ of $\text{Ar}_f(x)$ are contained inside $G$ (otherwise $x$ would not be a cutpoint of $F$). Apply the map $\psi_G$ which collapses all edges of $G$ to points. It semi-conjugates $\sigma G |_{\text{bd}(G)}$ to $\sigma_G$ so that the restriction of $\mathcal{L}_f$ onto $G$ induces a $\sigma_2$-invariant geolamination $\mathcal{L}_G^2$ which contains, by the above, some periodic leaves. By proposition II.6.10b of [Thu85], the lamination $\mathcal{L}_G^2$ has an invariant gap $H$ of non-zero rotation number or the leaf $H = \frac{1}{3}$. Theorem II.5.3 of [Thu85] shows that if $H$ is a gap then either $H$ is a Siegel gap, or it is a gap with countably many vertices, or it is a finite gap. However, in the first two cases it follows that the lamination $\mathcal{L}_G^2$ contains an isolated critical leaf. On the other hand, the construction of $\mathcal{L}_G^2$ implies that all non-degenerate leaves of $\mathcal{L}_G^2$ are either (pre)periodic with non-degenerate images, or limits of (pre)periodic, a contradiction. Thus, either $H = \frac{1}{3}$, or $H$ is a finite gap of rational rotation number.

Consider the convex hull $H_1$ of $\psi_G^{-1}(H')$. Then $H_1$ has either the same number of vertices as $H$, or twice as many vertices as $H$ (if vertices of $H$ are $\psi_G$-images of edges of $G$). We want to prove that there is an $f$-fixed point associated with $H_1$ such that external rays of $f$ whose arguments are vertices of $H_1$ land at that point. Indeed, suppose otherwise. Then by definition of our laminations we may assume that there are $\sigma_2$-pullbacks of $\psi_G(X)$ accumulating on each edge of $H$. Let $\ell = \overline{ab}$ be an edge of $H$. Then the corresponding $\sigma_2$-pullbacks of $X$ will accumulate on the corresponding edge of $\overline{a \sigma_2 b}$ of $H_1$. The corresponding cuts of $F$ formed by the corresponding pullbacks of $\tilde{X}$ can be chosen so that their ‘vertices’ (i.e., corresponding pullbacks of $x$) converge to a point $y_1 \in H_1$ belonging to the impression of $R_f(a_1)$ and the impression of $R_f(b_1)$. Thus, impressions of $R_f(a_1)$ and $R_f(b_1)$ are non-disjoint.

If $H_1$ and $H$ have the same number of vertices, it follows that the union $K$ of all impressions of angles with arguments that are vertices of $H_1$ is a continuum. If $H_1$ has twice as many vertices as $H$, for every vertex $l$ of $H$ there is an edge $\ell = \overline{uv}$ of $H_1$ such that $\psi_G(\ell) = l$. By [BOPT13b, theorem 7.7], the external rays $R_f(u), R_f(v)$ land at the same point. Hence in that case the union $K$ of impressions of angles that are vertices of $H_1$ is a continuum too. Clearly, $K$ is invariant and separated from impressions of rays with arguments which are not vertices of $H_1$ (either by the just discussed pullbacks of $\tilde{X}$, or by the appropriate fibres approaching $F$). By [BCO11, lemma 37] then $K$ is a fixed repelling or parabolic point. Since $[f] \in \text{CU}$, $K$ is parabolic. Since $H$, and hence $H_1$, are of non-zero rotation number, the multiplier at $K$ is not one. On the other hand, $x$ is a Cremer or Siegel point. Thus, $f$ has at
least two periodic points of multiplier not equal to 1, a contradiction with \([ f ] \in \text{CU}\). This shows that \(L_{\sim y} = L_f\).

Proof of theorem C. In view of lemma 5.2, it remains to prove that, for \([ f ] \in \text{CU}\), the lamination \(L_f = L_{\sim y}\) is cuboidal. Let us prove that \(L_f\) has at most one rotational set \(G\), and \(G\) is invariant. Suppose that \(G'\) is a finite \(\sim f\)-class. Then, by theorem 4.8, it corresponds to a periodic repelling or parabolic cutpoint \(y(G) = y\) of \(J(f)\). Since \([ f ] \in \text{CU}\), then, by definition 1.1(2), the point \(y\) is parabolic and by definition 1.1(3) \(y = 0\). Hence \(L_f\) cannot have two finite rotational classes. Now, if \(G\) is a Siegel gap of \(\sim f\) then there must exist a Siegel periodic point \(y\) of \(f\) inside \(\varphi_f^{-1} - p_{-y}(G)\); thus, \(y = 0\). Hence \(L_f\) has at most one rotational set \(G\), and \(G\) is invariant.

Again, let \(G\) be a finite rotational \(\sim f\)-class. Since \(y(G) = y\) is a cutpoint of \(J(f)\), by definition 1.1(2), the point \(y\) is parabolic. Hence there are parabolic domains attached to \(y\). By theorem 4.8 they are not collapsed by \(\varphi_f\). Hence along at least one cycle of edges of \(G\) such that the period of the endpoints of these edges is, say, \(m\), there are Fatou gaps of period \(m\) attached to \(G\) and which do not correspond to one \(\sim f\)-class as required in definition 4.12. By [BLOPT13a, corollary 5.5] this implies that for every periodic leaf \(\ell\) of \(L_f\) whose endpoints are of period \(t\) there exists a Fatou gap of \(L_f\) of period \(t\) attached to \(\ell\). It remains to prove that such gaps cannot be contained in convex hulls of \(\sim f\)-classes.

By the above the only hypothetical situation which we need to consider is as follows: there is a periodic finite gap or leaf \(G\) of \(L_f\) with two cycles of edges on its boundary such that Fatou gaps which are not convex hulls of a single \(\sim f\)-class are attached to one of these cycles of edges while Fatou gaps which are convex hulls of a single \(\sim f\)-class are attached to the other cycle of edges. Denote by \(H\) a Fatou gap which is one \(\sim f\)-class attached to an edge of \(G\); let \(F\) be the corresponding CS-fibre. By lemma 4.10 there is a Cremer or Siegel point \(x \in F\). Since \([ f ] \in \text{CU}\), the point \(x = 0\) is fixed and so \(H\) is invariant. Clearly, the only way it can happen is when \(G = \overline{0_{\mathbb{Z}^2}}\), a contradiction since if \(\overline{0_{\mathbb{Z}^2}}\) is a leaf of \(L_f\) then from at least one side it has an attached Fatou gap that does not coincide with the convex hull of a \(\sim f\)-class as desired (so that \(L_f\) is a CU-lamination).

We can partially reverse theorem C. First we prove lemma 5.3.

**Lemma 5.3.** If a cubic polynomial \(f\) has no repelling cutpoints then it has a non-repelling fixed point.

**Proof.** Consider fixed external rays \(R_f(0)\) and \(R_f(1/2)\). If they land at the same point \(w\) then by the assumptions \(w\) is non-repelling as desired. Suppose that the ray \(R_f(0)\) lands at \(z\), the ray \(R_f(1/2)\) lands at \(y\), and \(z \neq y\) are repelling. By [GM93] there exists either an invariant Fatou domain \(U\) or a fixed point \(x \in J(f) \setminus \{y, z\}\). In the first case \(f\) has either an attracting or a Siegel fixed point, and we are done. In the second case there are two possibilities. First, a periodic ray \(R\) may land at \(x\). By the assumption about \(R_f(0), R_f(1/2)\) the ray \(R\) is not invariant, hence \(x\) is a cutpoint. Since \(f\) does not have repelling periodic cutpoints, \(x\) is parabolic and we are done. Second, suppose that no periodic ray lands at \(x\). Then \(x\) is a Cremer fixed point, and we are done.

**Lemma 5.4.** Suppose that \((\sim f, L_f)\) is a cuboidal laminational pair, \(f\) has no repelling periodic cutpoints and at most one periodic attracting point. Then \([ f ] \in \text{CU}\).
Proof. By lemma 5.3 we may assume that 0 is an \( f \)-fixed point, \(|f'(0)| \leq 1\), and if there is a fixed non-repelling point with multiplier not equal to 1 then \( f'(0) \neq 1 \). By definition 4.12, \( L_{\sim f} = L_f \). By definition 1.1 we need to show that all non-repelling periodic points \( f \) but perhaps 0 have multiplier 1. Assume the contrary: \( f \) has a periodic non-repelling point \( x \neq 0 \), whose multiplier is different from 1.

We need an observation concerning any parabolic point \( y \) of \( f \). By [Kiw02] either there is one cycle of rays landing at \( y \), or there are two cycles of rays landing at \( y \). In the first case inside each wedge at \( y \) there is a parabolic Fatou domain attached to \( y \). In the second case \textit{a priori} it may happen that there is one cycle of Fatou domains attached to \( y \) inside one cycle of wedges at \( y \), and the other cycle of wedges at \( y \) contains no Fatou domains attached to \( y \) inside them. However, since \((\sim f, L_f)\) is cuboidal, it follows that if there are two cycles of rays (and hence wedges) at \( y \), then there are two cycles of Fatou domains at \( y \). Now we can consider several cases.

(1) Assume that 0 is attracting. Then there is an invariant Fatou domain \( U \) containing 0. If \( x \) is attracting, Cremer or Siegel then by [Kiw00, lemma 3.1] there exists a repelling periodic cutpoint, a contradiction. Assume that \( x \) is parabolic. Then the fact that the multiplier at \( x \) is not 1 implies that \( x \) cannot be a boundary point of \( U \). By the above there are two cases. First, there may be one cycle of rays and one cycle of Fatou domains at \( x \). Clearly, then we can find a point from the orbit of \( x \) and a Fatou domain attached to it which can only be separated from 0 by a repelling periodic cutpoint, a contradiction. Second, there may be two cycles of Fatou domains at \( x \). Together with \( U \) they will form three cycles of Fatou domains of a cubic polynomial \( f \), a contradiction.

(2) Assume that 0 is Cremer or Siegel. If \( x \) is attracting, Cremer or Siegel then by [Kiw00, lemma 3.1] there exists a repelling periodic cutpoint separating 0 and \( x \) in \( J(f) \), a contradiction. Suppose that \( x \) is parabolic. As in (1), the fact that \( f \) is cubic implies that there is exactly one cycle of Fatou domains at \( x \). However, this implies that there will be one Fatou domain at one of the points of the orbit of \( x \), which can only be separated from 0 by a repelling periodic cutpoint as in [Kiw00, lemma 3.1], a contradiction.

(3) Assume that 0 is parabolic. By the above there are two subcases here. First, assume that there are two cycles of Fatou domains at 0. Let \( G \) be the convex hull of \( A_r f (0) \). If \( G \) is a gap, then each cycle of Fatou domains at 0 consists of at least two domains. If one of them is a cycle of attracting Fatou domains, then we have at least two attracting periodic points of \( f \), a contradiction. If both are cycles of parabolic domains then we cannot have a non-repelling periodic point \( x \neq 0 \) by the Fatou–Shishikura inequality. Thus, we may assume that \( G \) is a leaf. Then having two cycles of Fatou domains at 0 (actually, each cycle in this case consists of just one Fatou domain) means having two cycles of Fatou gaps attached to \( G \), which implies that \( G = 0 \frac{1}{2} \). If both Fatou domains at 0 are parabolic, we cannot have a non-repelling periodic point \( x \neq 0 \). Hence one of the Fatou domains at 0 is attracting and the other one is parabolic. However, in that case by our choice of \( f \) we should have moved the attracting fixed point to 0, a contradiction.

Second, assume that there is one cycle of Fatou domains and one cycle of rays landing at 0. Then it is easy to see (similar to the arguments above) that there must exist a repelling cutpoint separating one of these Fatou domains at 0 from a specifically chosen Fatou domain at one of the points from the orbit of \( x \). In any case, we get a contradiction with the assumption that \( f \) has no repelling periodic cutpoints. \( \square \)
6. Proof of the second part of theorem B

We need to prove that \( \mathcal{LC} \cap \mathcal{CU} = \mathcal{LC} \cap \mathcal{PHD}_3 \) (\( \mathcal{LC} \) is the set of classes of polynomials with locally connected Julia sets). By the first part of theorem B, we have \( \mathcal{PHD}_3 \subset \mathcal{CU} \). Hence we have to consider cubic polynomials \( f \) such that \([f] \in \mathcal{CU} \setminus \mathcal{PHD}_3 \). By theorem C, the laminational pair \((\sim_f, \mathcal{L}_f)\) is cubioidal. We may assume that \( f \in \mathcal{F}_{ur} \).

6.1. Main analytic tools

According to [BOPT13b], there is a well-defined principal critical point \( \omega_1(f) \) of \( f \) that depends holomorphically on \( f \) at least in a small neighbourhood of \( f \) in \( \mathcal{F}_{ur} \). If \( \lambda = f'(0) \) is a root of unity, then \( \omega_1 \) is in a parabolic domain attached to \( 0 \), in particular, the orbit of \( \omega_1(f) \) converges to 0.

**Theorem 6.1 ([BOPT13b], theorem B).** If \( f \in \mathcal{F}_{ur} \) and \([f] \notin \mathcal{PHD}_3 \) then there are Jordan domains \( U^* \) and \( V^* \) such that \( f : U^* \to V^* \) is a quadratic-like map hybrid equivalent to \( z^3 + c \) with \( c \in \mathbb{PHD}_2 \).

We will write \( J(f) \) for the Julia set of the quadratic-like map \( f : U^* \to V^* \), and \( K^* \) for the filled Julia set of this map. Theorem 6.1 implies that in the case \( J(f) \) is locally connected we may assume that \( L_{\sim_f} \) has some non-degenerate leaves.

A stand-alone quadratic invariant gap \( U \) is a quadratic invariant gap \( U \) of some lamination considered by itself (without the lamination). We say that \( U \) is of regular critical type if the major \( M = \overline{\theta_1 \theta_2} \) of \( U \) is critical. If a gap \( U \) is of regular critical type, then there exists a unique lamination such that \( U \) is its gap. Basically, this lamination is obtained by taking pullbacks of \( U \). This lamination is called the canonical lamination of the gap \( U \) [BOPT13a].

We say that \( U \) is of periodic type if its major \( M = \overline{\theta_1 \theta_2} \) is periodic of some period \( k \). Call such \( M \) a major (leaf) of periodic type.

6.2. The proof of the second part of theorem B

By way of contradiction, assume that \( f \) is a polynomial with a locally connected Julia set \( J(f) \) such that \([f] \in \mathcal{CU} \setminus \mathcal{PHD}_3 \). Recall that there are Jordan domains \( U^* \) and \( V^* \) such that \( f : U^* \to V^* \) is a quadratic-like map with a connected filled Julia set \( K^* \). Define a subset \( G' \subset \mathcal{S}^1 \) as the set of arguments of all external rays of \( f \) landing in \( K^* \); and set \( G \) to be the convex hull of \( G' \).

Since \( J(f) \) is locally connected, there is an invariant lamination \( \sim_f \) and a monotone map \( p : \mathbb{C} \to \mathbb{C} \), whose restriction to \( \mathcal{S}^1 \) semi-conjugates \( \sigma_3 \) with \( f|_{J(f)} \), and whose fibres are points, leaves or finite gaps of \( \sim_f \).

**Lemma 6.2.** Consider a complementary component \((a, b)\) of \( G' \) in \( \mathcal{S}^1 \). Then the rays \( R_f(a) \) and \( R_f(b) \) land at the same point.

**Proof.** Let us first prove that the chord \( \overline{ab} \) cannot cross a leaf \( \overline{xy} \) of \( \sim_f \). Assume the contrary: \( \overline{xy} \in L_{\sim_f} \), where \( x \in (a, b) \) and \( y \in \mathcal{S}^1 \setminus [a, b] \). Then the union of \( K^* \cup R_f(a) \cup R_f(b) \) separates \( R_f(x) \) from \( R_f(y) \). Since the landing points of \( R_f(x) \) and \( R_f(y) \) coincide, this common landing point must belong to \( K^* \), a contradiction with \( x \in (a, b) \).

It follows that the chord \( \overline{ab} \) is either a leaf of \( \sim_f \) or is contained in an infinite gap \( H \) of \( \sim_f \). Consider the latter case. Then \( p(\text{Bd}(H)) \) is the boundary of some Fatou component \( W \) of \( f \). Consider \( X = K^* \cap \text{Bd}(W) \). Clearly, \( X \) is connected, as if \( X \) is disconnected, then so is \( K^* \), a contradiction. On the other hand, no ray with the argument in \((a, b)\) lands in \( K^* \).
Hence, $X$ is a closed subarc of $\text{Bd}(W)$ with endpoints $p(a)$ and $p(b)$. By the properties of the locally connected Julia sets for some numbers $n < m$ the union $\bigcup_{i=n}^{m} f^i(X)$ is the boundary of a periodic Fatou domain $Q$ of $f$. Hence $X$ is a subarc of the appropriate pullback $S \subset J^*$ of $\text{Bd}(Q)$, which is impossible. □

**Lemma 6.3.** The set $G$ is a stand-alone quadratic invariant gap.

**Proof.** With every complementary component $(a, b)$ of $G'$, we associate the corresponding $G$-cut $\Gamma(a, b) = \Gamma$ consisting of the rays $R_f(a), R_f(b)$, and their common landing point (called the vertex of $\Gamma$). We claim that the $f$-image of a $G$-cut is a $G$-cut. If $v$ is not critical then $f|_{K^*}$ is a local homeomorphism near $v$. Hence, if $K^*$ is not locally separated by $\Gamma$, then $K^* = f(K^*)$ cannot be locally separated by $f(\Gamma)$. Assume now that $v$ is critical and $f(\Gamma)$ is not a $G$-cut. Denote by $W$ the wedge of $\mathbb{C}$ with boundary $G$ not containing points of $K^*$.

Consider two cases. First, let $\sigma_3(a) = \sigma_3(b)$. Then $W$ contains points mapped to $K^*$ and located arbitrarily close to $v$. This contradicts the definition of polynomial-like map. Now, let $\sigma_3(a) \neq \sigma_3(b)$. Since $f(\Gamma)$ is not a $G$-cut, both components of $f(\Gamma)$ contain points of $K^*$. Hence, again, $W$ contains points mapped to $K^*$ and located arbitrarily close to $v$, a contradiction as above. □

Let $M = \overline{ab}$ be the major of $G$. If $M$ is of regular critical type, then both critical points of $f$ are contained in $K^*$, a contradiction. Thus $M$ is of periodic type. The point $p(M)$ is a periodic cutpoint of $J(f)$. Since $[f] \in CU$, this point cannot be repelling. Therefore, $p(M)$ is parabolic of multiplier 1. The Jordan domain $U^*$ from the definition of quadratic-like map intersects the immediate parabolic basin of $p(M)$. Since in an arbitrarily small neighbourhood of $p(M)$, there are points of the immediate parabolic basin that stay in this neighbourhood, these points must lie in $K^*$. By the properties of Julia sets this implies that the entire immediate parabolic basin of $p(M)$ is a subset of $K^*$. It follows that $K^*$ contains both critical points of $f$, a contradiction.

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