Generalized Heisenberg Ferromagnet type
Equation and Modified Camassa-Holm
Equation: Geometric Formulation, Soliton
Solutions and Equivalence

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Abstract

We study the integrability and equivalence of a generalized Heisenberg ferromagnet-type equation (GHFE). The different forms of this equation as well as its reduction are presented. The Lax representation (LR) of the equation is obtained. We observe that the geometrical and
gauge equivalent counterpart of the GHFE is the modified Camassa-Holm equation (mCHE) with an arbitrary parameter $\kappa$. Finally, the
1-soliton solution of the GHFE is obtained.

1 Introduction

This work continues our research of Lax-integrable (i.e., admitting Lax pairs
with non-vanishing spectral parameter) generalized Heisenberg ferromagnet
type equations in 1+1 dimensions related with Camassa-Holm type equa-
tions (see, e.g., [1]-[4] and the references therein). In the theory of integrable

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systems (soliton theory) an important role plays the so-called gauge and geometrical equivalence between two integrable equations. The well-known classical example such equivalences is the (gauge and geometrical) equivalence between the Heisenberg ferromagnet equation (HFE)

\[ iA_t + \frac{1}{2}[A, A_{xx}] = 0 \]  

(1)

and the nonlinear Schrödinger equation (NLSE)

\[ iq_t + q_{xx} + 2|q|^2q = 0, \]  

(2)

where \( q \) is a complex function and

\[
A = \begin{pmatrix} A_3 & A^- \ 
A^+ & -A_3 \end{pmatrix}, \quad A^2 = I, \quad A = (A_1, A_2, A_3), \quad A^2 = 1.
\]  

(3)

In this paper, we study the Generalized Heisenberg ferromagnet - type equation (GHFE), namely, the so-called M-CXIIE equation (M-CXIIE ) and its relation with the modified Camassa-Holm equation (mCHE)

\[
m_t + (m(u^2 - u_x^2))_x + \kappa u_x = 0, \]  

(4)

\[
m - u + u_{xx} = 0, \]  

(5)

where \( u = u(x,t) \) is a real-valued function, \( \kappa = \text{const.} \). The mCHE (4)-(5) was proposed by Fuchssteiner \[\text{[5]}\] and Olver and Rosenau \[\text{[6]}\]. It was obtained from the two-dimensional Euler equations, where the variables \( u(x,t) \) and \( m(x,t) \) represent, respectively, the velocity of the fluid and its potential density \[\text{[7]}\]. Some properties of the mCHE and other related equations were studied in \[\text{[8]} - \text{[25]}\].

This paper is organized as follows. In Section 2, we give the M-CXIIE, its different forms, its Lax representation (LR) and a reduction. Geometric formulation of this equation in terms of space and plane curves is presented in Section 3. Using this geometric formalism, the geometrical equivalence between the M-CXIIE and the mCHE is established. Section 4 is devoted to the mCHE. Gauge equivalence between the M-CXIIE and the mCHE and the relation between their solutions we have established in Section 5. The 1-soliton solution of the M-CXIIE is obtained in Section 6. We discuss and conclude our results in Section 7.
2 The generalized Heisenberg ferromagnet type equation

2.1 Equation

There are exist several integrable and nonintegrable GHFE (see, e.g., [26]–[60]). In this paper, we consider one of such GHFE, namely, the so-called M-CXIIE. This equation has the form

\[
[A, A_{xt}] + (\phi[A, A_x])_x + \frac{\kappa u_x}{m}[A, A_x] + \frac{4\alpha_0}{\beta^2} A_x = 0, \tag{6}
\]

where \( \phi = u^2 - u_x^2 \) and

\[
A = \begin{pmatrix} A^3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I, \quad A = (A_1, A_2, A_3), \quad A^2 = 1. \tag{7}
\]

We can also write the M-CXIIE in the following form

\[
A_{xt} + \phi A_{xx} + v_1 A + v_2 A_x + \frac{\alpha_0}{\beta^2}[A, A_x] = 0, \tag{8}
\]

where

\[
v_1 = 0.5\{A_t, A_x\} + \phi A_x^2 = -2umI, \quad v_2 = \phi_x + \frac{\kappa u_x}{m}. \tag{9}
\]

Finally let us present the vector form of the M-CXIIE. It reads as

\[
A_{xt} + \phi A_{xx} + v_1 A + v_2 A_x + \frac{\alpha_0}{\beta^2} A \wedge A_x = 0. \tag{10}
\]

2.2 Lax representation

The M-CXIIE (6) is integrable. Its LR is given by

\[
\Phi_x = U_1 \Phi, \tag{11}
\]

\[
\Phi_t = V_1 \Phi, \tag{12}
\]

where

\[
U_1 = \frac{\alpha_0 - \alpha}{2} A + \frac{\lambda - \beta}{4\beta} [A, A_x], \tag{13}
\]

\[
V_1 = (\omega - \omega_0) A + \left\{ \frac{(\lambda - \beta)u}{2m\beta^2\lambda} - \frac{(\lambda - \beta)\phi}{4\beta} \right\}[A, A_x] + \frac{u_x(\alpha \beta - \alpha_0 \lambda)}{\beta^2 \lambda m}. \tag{14}
\]
or
\[ U_1 = \frac{\alpha_0 - \alpha}{2} A + \frac{\lambda - \beta}{4\beta} [A, A_x], \]
\[ V_1 = (\omega - \omega_0) A + \frac{(\lambda - \beta)}{4\beta} \left\{ \frac{2u}{m\beta\lambda} - \phi \right\} [A, A_x] + \frac{u_x}{\beta m} \left\{ \frac{\alpha}{\lambda} - \frac{\alpha_0}{\beta} \right\} A_x. \] (15)

Here \( \beta = \text{const}, \quad \alpha_0 = \alpha|_{\lambda=\beta}, \quad \alpha = \sqrt{1 - \frac{1}{2}\kappa\lambda^2}, \quad \omega_0 = \omega|_{\lambda=\beta}, \quad \omega = \alpha\lambda^{-2} + 0.5\alpha\phi, \quad u = \pm\beta^{-1}(1 - \partial_x^2)^{-1}\sqrt{0.5tr(A_x)^2} \] (17)

and
\[ \phi = u^2 - u_x^2, \quad m = \pm\beta^{-1}\sqrt{0.5tr(A_x)^2} = u - u_{xx} = (1 - \partial_x^2)u. \] (18)

### 2.3 Reductions

The M-CXIIE admits some reductions. For example in the case \( \kappa = 0 \), the M-CXIIE reduces to the so-called M-CXIE having the form [20]
\[ [A, A_{xt}] + (\phi [A, A_x])_x + \frac{4}{\beta^2} A_x = 0. \] (19)

This integrable GHFE was studied in [20]. Its LR is given by
\[ \Phi_x = U_2 \Phi, \]
\[ \Phi_t = V_2 \Phi, \] (20) (21)

where \( (z = \phi + \lambda^2, \ z_0 = \phi + \beta^{-2}) \) and
\[ U_2 = \frac{\lambda - \beta}{4\beta} [A, A_x], \] (22)
\[ V_2 = (z - z_0) A + \left\{ \frac{(\lambda - \beta)u}{2m\beta^2\lambda} - \frac{(\lambda - \beta)\phi}{4\beta} \right\} [A, A_x] + \frac{u_x(\beta - \lambda)}{\beta^2\lambda m} A_x. \] (23)

### 3 Integrable motion of curves induced by the M-CXIIE

The aim of this section is to present the geometric formulation of the M-CXIIE in terms of curves and to find its geometrical equivalent counterpart.
3.1 Integrable motion of space curves

We start from the differential geometry of space curves. In this subsection, we consider the integrable motion of space curves induced by the M-CXIIE. As usual, let us consider a smooth space curve \( \gamma(x, t) : [0, X] \times [0, T] \rightarrow \mathbb{R}^3 \) in \( \mathbb{R}^3 \). Let \( x \) is the arc length of the curve at each time \( t \). In differential language, such curve is given by the Frenet-Serret equation (FSE). The FSE and its temporal counterpart look like

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_x = C \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}, \quad \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_t = G \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix},
\]

(24)

where \( e_j \) are the unit tangent vector \((j = 1)\), principal normal vector \((j = 2)\) and binormal vector \((j = 3)\) which given by

\[
e_1 = \gamma_x, \quad e_2 = \frac{\gamma_{xx}}{\|\gamma_{xx}\|}, \quad e_3 = e_1 \wedge e_2,
\]

respectively. Here

\[
C = \begin{pmatrix}
0 & \kappa_1 & \kappa_2 \\
-\kappa_1 & 0 & \tau \\
-\kappa_2 & -\tau & 0
\end{pmatrix} = -\tau L_1 + \kappa_2 L_2 - \kappa_1 L_3 \in so(3),
\]

(25)

\[
G = \begin{pmatrix}
0 & \omega_3 & \omega_2 \\
-\omega_3 & 0 & \omega_1 \\
-\omega_2 & -\omega_1 & 0
\end{pmatrix} = -\omega_1 L_1 + \omega_2 L_2 - \omega_3 L_3 \in so(3),
\]

(26)

where \( \tau, \kappa_1, \kappa_2 \) are the "torsion", "geodesic curvature" and "normal curvature" of the curve, respectively; \( \omega_j \) are some functions. Note that \( L_j \) are basis elements of \( so(3) \) algebra and have the forms

\[
L_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad L_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad L_3 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(27)

They satisfy the following commutation relations

\[
[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.
\]

(28)

In the following, we need also in the basis elements of \( su(2) \) algebra. They have the forms

\[
e_1 = \frac{1}{2i} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad e_2 = \frac{1}{2i} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad e_3 = \frac{1}{2i} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

(29)

where the Pauli matrices have the form

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

(30)
These elements satisfy the following commutation relations
\[ [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \] (31)

Note that the Pauli matrices obey the following commutation relations
\[ \sigma_1 \sigma_2 = 2i \sigma_3, \quad \sigma_2 \sigma_3 = 2i \sigma_1, \quad \sigma_3 \sigma_1 = 2i \sigma_2 \] (32)
or
\[ [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k. \] (33)

The well-known isomorphism between the Lie algebras \( su(2) \) and \( so(3) \) means the following correspondence between their basis elements \( L_j \leftrightarrow e_j \). Using this isomorphism let us construct the following two matrices
\[
U = -\tau e_1 + \kappa_2 e_2 - \kappa_1 e_3 = -\frac{1}{2i} \begin{pmatrix} \kappa_1 & \tau + i\kappa_2 \\ \tau - i\kappa_2 & -\kappa_1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix},
\[
V = -\omega_1 e_1 + \omega_2 e_2 - \omega_3 e_3 = -\frac{1}{2i} \begin{pmatrix} \omega_3 & \omega_1 + i\omega_2 \\ \omega_1 - i\omega_2 & -\omega_3 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix}.
\] (34)

Hence we obtain
\[
\kappa_1 = -2iu_{11}, \quad \kappa_2 = -(u_{12} - u_{21}), \quad \tau = -i(u_{12} + u_{21}),
\]
\[
\omega_1 = -i(v_{12} + v_{21}), \quad \omega_2 = -(v_{12} - v_{21}), \quad \omega_3 = -2iv_{11}.
\] (36)

The compatibility condition of the equations (24) reads as
\[ C_t - G_x + [C, G] = U_t - V_x + [U, V] = 0 \] (38)
or in elements
\[
\kappa_{1t} - \omega_{3x} - \kappa_2 \omega_1 + \tau \omega_2 = 0, \quad (39)
\]
\[
\kappa_{2t} - \omega_{2x} + \kappa_1 \omega_1 - \tau \omega_3 = 0, \quad (40)
\]
\[
\tau_t - \omega_{1x} - \kappa_1 \omega_2 + \kappa_2 \omega_3 = 0. \quad (41)
\]

We now assume that
\[ \kappa_1 = i\alpha, \quad \kappa_2 = -\lambda m, \quad \tau = 0 \] (42)
and
\[
\omega_1 = -\frac{i\alpha u_x}{\lambda}, \quad (43)
\]
\[
\omega_2 = \frac{u}{\lambda} + \lambda m(u^2 - u_x^2), \quad (44)
\]
\[
\omega_3 = -2i\alpha[\lambda^{-2} + 0.5(u^2 - u_x^2)]. \quad (45)
\]
Then it is not difficult to verify that Eqs. (39)-(41) give us the following equations for $m, u$:

\[
m_t + (m(u^2 - u_x^2))_x + \kappa u_x = 0, \quad (46) \\
m - u + u_{xx} = 0. \quad (47)
\]

It is nothing but the mCHE. So, we have proved that the Lakshmanan (geometrical) equivalent of the M-CXIIE is the mCHE. Finally we note that as follows from (42), the corresponding space curve is with the zero torsion but with the constant geodesic curvature.

### 3.2 Integrable motions of plane curves

For the mCHE and that same for its equivalent counterpart - the M-CXIIE, more naturally corresponds the plane curves than the space curves. For that reason in this subsection, let us consider an integrable motions of plane curves induced by the M-CXIIE. In this case, Eqs. (24) take the following form

\[
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_x = C \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_t = G \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},
\]

where

\[
C = \begin{pmatrix} 0 & \kappa_1 \\ -\kappa_1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 \\ -\omega_3 & 0 \end{pmatrix}.
\]

At the same time, Eqs. (39)-(41) become

\[
\kappa_{1t} - \omega_{3x} = 0. \quad (50)
\]

We now assume that

\[
\kappa_1 = r, \quad \omega_3 = -r(u^2 - u_x^2), \quad (51)
\]

where

\[
r = \sqrt{(u - u_{xx})^2 + 0.5k}. \quad (52)
\]

Finally Eq.(50) gives

\[
r_t + [r(u^2 - u_x^2)]_x = 0. \quad (53)
\]

It is nothing but the mCHE in the conservation law form [21]-[22]. So, again we have proved that the equivalent counterpart of the M-CXIIE (6) is the mCHE (53). Note that in this case, the curve has the zero torsion and normal curvature.
4 Modified Camassa-Holm equation

In the previous section, we have proved that the geometrical equivalent of the M-CXIIE is the well-known mCHE. In this section, we give some main informations on the mCHE. The mCHE has the form

\[
m_t + (m(u^2 - u_x^2))_x + \kappa u_x = 0, \quad (54)
\]
\[
m - u + u_{xx} = 0. \quad (55)
\]

The mCHE can be rewritten in the conservation law form as [21]-[22]

\[
r_t + [r(u^2 - u_x^2)]_x = 0, \quad (56)
\]
\[
r - \sqrt{(u - u_{xx})^2 + \sigma^2} = 0, \quad (57)
\]

where \(2\sigma^2 = \kappa\). As well-known, the mCHE is an integrable nonlinear partial differential equation. Its LR read as [15]-[16]

\[
\Psi_x = U_3 \Psi, \quad (58)
\]
\[
\Psi_t = V_3 \Psi, \quad (59)
\]

where

\[
U_3 = \frac{1}{2} \left( -\frac{\alpha}{\lambda} - \frac{\lambda m(x,t)}{\alpha} \right), \quad (60)
\]
\[
V_3 = \left( \frac{\alpha}{\lambda} + \frac{\lambda (u^2 - u_x^2)}{2\lambda(u^2 - u_x^2)} - \frac{u - \alpha u_x}{\lambda} - \frac{1}{2} \lambda (u^2 - u_x^2) m \right) \quad (61)
\]

with

\[
\alpha = \sqrt{1 - \frac{1}{2} \kappa \lambda^2}, \quad \phi = u^2 - u_x^2. \quad (62)
\]

Note that

\[
V_3 = -\phi U_3 + V'_2 = -\phi U + \left( \frac{\alpha}{\lambda} - \frac{u - \alpha u_x}{\lambda} \right). \quad (63)
\]

The compatibility condition

\[
U_{3t} - V_{3x} + [U_3, V_3] = 0 \quad (64)
\]

gives the mCHE (54)-(55). In fact, we have

\[
\lambda : \quad m_t + (m\phi)_x + \kappa u_x = 0, \quad (65)
\]
\[
\frac{\alpha}{\lambda} : \quad m = u - u_{xx}. \quad (66)
\]
Note that the mCHE admits at least one reduction. Let \( \kappa = 0 \), then the mCHE takes the form
\[
\begin{align*}
t &+ (m(u^2 - u_x^2))_x = 0, \\
m - u + u_{xx} = 0.
\end{align*}
\]
(67) (68)
Its LR is given by \([15]-[16]\)
\[
\begin{align*}
Z_x & = U_4 Z, \\
Z_t & = V_4 Z,
\end{align*}
\]
(69) (70)
where
\[
\begin{align*}
U_4 & = \frac{1}{2} \begin{pmatrix} -1 & \lambda m(x,t) \\
-\lambda m(x,t) & 1 \end{pmatrix}, \\
V_4 & = \begin{pmatrix} \frac{1}{\lambda} + \frac{1}{2}(u^2 - u_x^2) & \frac{-u - u_x}{\lambda} - \frac{1}{2}\lambda(u^2 - u_x^2)m \\
\frac{u + u_x}{\lambda} + \frac{1}{2}\lambda(u^2 - u_x^2)m & \frac{-1}{\lambda^2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}.
\end{align*}
\]
(71) (72)

5 \quad \text{Gauge equivalence between the mCHE and the M-CXIIE}

The M-CXIIE (6) is gauge equivalent to the mCHE (54)-(55). In fact, let us consider the gauge transformation
\[
\Phi = g^{-1}\Psi,
\]
(73)
where \( g = \Psi|_{\lambda=\beta} \). Then the relation between the Lax pairs \( U_1 - V_1 \) and \( U_3 - V_3 \) is given by
\[
U_1 = g^{-1}U_3 g - g^{-1}g_x, \quad V_1 = g^{-1}V_3 g - g^{-1}g_t.
\]
(74)
In the case \( \kappa = 0 \), the gauge equivalence between the M-CXIIE and the mCHE was established in \([20]\). The gauge equivalence between the mCHE and the M-CXIIE induces some important relations between solutions of these equations. Here we present some of them. For example, it can be shown that the solutions \( A \) and \( m \) is related by the formula
\[
tr(A_x)^2 = 2A_x^2 = 2(A_{1x}^2 + A_{2x}^2 + A_{3x}^2) = 2\beta^2m^2
\]
(75)
or
\[
A_x^2 = A_{1x}^2 + A_{2x}^2 + A_{3x}^2 = \beta^2m^2.
\]
(76)
Consider the angle parametrization of the spin vector
\[ A^\perp = \sin \theta e^{i \varphi}, \quad A_3 = \cos \theta. \tag{77} \]

Then from (25) we obtain
\[ A_x^2 = \theta^2_x + \varphi^2_x \sin^2 \theta = \beta^2 m^2. \tag{78} \]

We can consider the following two particular cases: \( \theta = \text{const} \) and \( \varphi = \text{const} \).
In this paper we consider the case \( \varphi = \text{const} \) and assume that \( \beta \in \mathbb{R} \). Then Eq.(76) takes the form
\[ A_x^2 = \theta^2_x = \beta^2 m^2 \tag{79} \]
so that
\[ \theta_x = \pm \beta m. \tag{80} \]

We now return to the mCHE (54)-(55). In terms of \( \theta \) it takes the form
\[ \begin{align*}
\theta_{xt} + ((u^2 - u_x^2)\theta_x)_x &\pm \beta \kappa u_x = 0, \\
\frac{m - u + u_{xx}}{\Delta} & = 0
\end{align*} \tag{81, 82} \]
or
\[ \begin{align*}
\theta_t + (u^2 - u_x^2)\theta_x &\pm \beta \kappa u = c, \\
\theta_x \mp \beta (u - u_{xx}) & = 0.
\end{align*} \tag{83, 84} \]

### 6 Soliton solutions of the M-CXIIE

As the integrable equation, the M-CXIIE has all ingredients of integrable systems like LR, conservation laws, bi-Hamiltonian structure, soliton solutions and so on. In particular, it admits the peakon solutions. Here let us present a one peakon solution of the M-CXIIE. To construct this 1-peakon solution, we use the corresponding 1-peakon solution of the mCHE \[?\].

\[ A = g^{-1} \sigma_3 g = \begin{pmatrix} A_3 & A^- \\ A^\perp & -A_3 \end{pmatrix}, \tag{85} \]
where
\[ g = \begin{pmatrix} g_1 & -\bar{g}_2 \\ g_2 & \bar{g}_1 \end{pmatrix}, \quad g^{-1} = \frac{1}{\Delta} \begin{pmatrix} \bar{g}_1 & \bar{g}_2 \\ -g_2 & g_1 \end{pmatrix}, \quad \Delta = |g_1|^2 + |g_2|^2. \tag{86} \]
As a result we obtain the following formulas for the components of the spin matrix:

\[ A^+ = -\frac{2g_1g_2}{|g_1|^2 + |g_2|^2}, \quad A_3 = \frac{|g_1|^2 - |g_2|^2}{|g_1|^2 + |g_2|^2}. \] (87)

To construct the 1-soliton solution of the M-CXIIE, we consider the following seed solution of the mCHE

\[ u = 0. \] (88)

Then the equations (58)-(59) have the solutions

\[ g_1 = a_1 e^{-\theta}, \quad g_2 = a_2 e^\theta, \] (89)

where \( a_j \) are complex constants and

\[ \theta = \frac{\alpha_0}{2} x - \frac{\alpha_0}{\beta^2}, \quad a_j = |a_j| e^{i\delta_j}, \quad \delta_j = \text{consts.} \] (90)

Thus the 1-soliton solution of the M-CXIIE has the form

\[ A^+ = -\frac{e^{i(\delta_1 + \delta_2)}}{\cosh(\delta - \theta)}, \quad A_3 = \tanh(\delta - \theta), \] (91)

where \( \delta = \ln |\frac{a_1}{a_2}|. \)

7 Conclusion

In the paper, one of the GHFE, namely, the M-CXIIE is investigated. The different forms of this equation and its reduction are given. Its LR is presented. The geometric formulation of this equation is presented. It is also shown that this equation is geometrical and gauge equivalent to the mCHE. Finally we note that it is interesting to investigate the surface geometry of the M-CXXIIE and the mCHE [26]-[62]. The M-CXIIE seemingly admits not only soliton but also peakon type solutions. It will be interesting to study the properties of the M-CXIIE and other integrable GHFE related with Camassa-Holm type equations in more detail elsewhere.

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References

[1] Assem Mussatayeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Kuralay Yesmakhanova, Ratbay Myrzakulov. Integrable Motion of Curves, Spin Equation and Camassa-Holm Equation, [arXiv:1907.10910]

[2] Bayan Kutum, Gulgassyl Nugmanova, Tolkynay Myrzakul, Kuralay Yesmakhanova, Ratbay Myrzakulov. Integrable Deformation of Space Curves, Generalized Heisenberg Ferromagnet Equation and Two-Component Modified Camassa-Holm Equation, [arXiv:1908.01371]

[3] Gulmira Yergaliyeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Kuralay Yesmakhanova, Ratbay Myrzakulov. Camassa-Holm and M-CIV equations with self-consistent sources: geometry and peakon solutions, [arXiv:1909.10606]

[4] Aigul Taishiyeva, Tolkynay Myrzakul, Gulgassyl Nugmanova, Shynaray Myrzakul, Kuralay Yesmakhanova, Ratbay Myrzakulov. Geometric Flows of Curves, Two-Component Camassa-Holm Equation and Generalized Heisenberg Ferromagnet Equation, [arXiv:1910.13281]

[5] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the CamassaCHolm equation, Physica D, 95 (1996), 229-243.

[6] P. J. Plver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E, 53 (1996), 1900-1906.

[7] Z. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, J. Math. Phys., 47 (2006), 112701.

[8] Gui, G., Liu, Y., Olver, P.J., and Qu, C., Wave-breaking and peakons for a modified Camassa-Holm equation, Commun. Math. Phys., 319 (2013), 731-759.

[9] Y. Matsuno, Smooth and singular multisoliton solutions of a modified Camassa-Holm equation with cubic nonlinearity and linear dispersion, J. Phys. A: Math. Theor., 47(2014) 125203 (25pp).

[10] Z. Qiao and X. Q. Li, An integrable equation with nonsmooth solitons, Theor. Math. Phys., 267(2011), 584-589.

[11] J. Xu, Long time asymptotics for the short pulse equation, J. Differential Equations, 265(2018), 3494-3532.
[12] Q.T. Zhang, Global wellposedness of cubic Camassa-Holm equations, Nonlinear Analysis, 133(2016), 61C73.

[13] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71:11 (1993), 1661C1664.

[14] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their B?cklund transformations and hereditary symmetries, Physica D, 4 (1981), 47-66.

[15] Anne Boutet de Monvel, Irina Karpenko and Dmitry Shepelsky. A Riemann-Hilbert approach to the modified Camassa-Holm equation with nonzero boundary conditions, [arXiv:1911.07263]

[16] Jian Xu, Engui Fan. Long-time asymptotics behavior for the integrable modified Camassa-Holm equation with cubic nonlinearity, [arXiv:1911.12554]

[17] Li-Yuan Ma, Shou-Feng Shen, Zuo-Nong Zhu. Integrable nonlocal complex mKdV equation: soliton solution and gauge equivalence, [arXiv:1612.06723]

[18] Li-Yuan Ma, Zuo-Nong Zhu. Nonlocal nonlinear Schr?dinger equation and its discrete version: soliton solutions and gauge equivalence, [arXiv:1503.06909]

[19] Julia Cen, Francisco Correa and Andreas Fring. Nonlocal gauge equivalence: Hirota versus extended continuous Heisenberg and Landau-Lifschitz equation, [arXiv:1910.07272]

[20] Z. Umurzakhova et al. Gauge equivalence between the Myrzakulov-CXI and modified Camassa-Holm equations [in preparation].

[21] Y. Matsuno. Smooth and singular multisoliton solutions of a modified Camassa-Holm equation with cubic nonlinearity and linear dispersion, [arXiv:1310.4011]

[22] Baoqiang Xia, Ruguang Zhou, Zhijun Qiao. Darboux transformation and multi-soliton solutions of the Camassa-Holm equation and modified Camassa-Holm equation, [arXiv:1506.08630]

[23] Chang X-K., Hu X-B., Szmigiels J. Multipeakons of a two-component modified Camassa-Holm equation and the relation with the finite Kac-Van Moerbeke lattice, [arXiv:1512.08300]
[24] Ivanov R.I. *Two component integrable systems modelling shallow water waves: the constant vorticity case*, [arXiv:0906.0780](http://arxiv.org/abs/0906.0780).

[25] Hone A.N.W., Wang J.P. *Integrable peakon equations with cubic nonlinearity*, [arXiv:0805.4310](http://arxiv.org/abs/0805.4310).

[26] G. Nugmanova, Z. Zhunussova, K. Yesmakhanova, G. Mamyrbekova, R. Myrzakulov. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering*, 9, N8, 328-331 (2015).

[27] J.-S. He, Y. Cheng, Y.-S. Li. *Commun. Theor. Phys.*, 38, 493-496 (2002).

[28] U. Saleem, M. Hasan. *J. Phys. A: Math. Theor.*, 43, 045204 (2010).

[29] M. Lakshmanan, *Phil. Trans. R. Soc. A*, 369, 1280-1300 (2011).

[30] M. Lakshmanan, *Phys. Lett. A*, 64, 53-54 (1977).

[31] R. Myrzakulov, S. Vijayalakshmi, G. Nugmanova, M. Lakshmanan *Physics Letters A*, 233, 14-6, 391-396 (1997).

[32] R. Myrzakulov, S. Vijayalakshmi, R. Syzdykova, M. Lakshmanan, *J. Math. Phys.*, 39, 2122-2139 (1998).

[33] R. Myrzakulov, M. Lakshmanan, S. Vijayalakshmi, A. Danlybaeva, *J. Math. Phys.*, 39, 3765-3771 (1998).

[34] Myrzakulov R, Danlybaeva A.K, Nugmanova G.N. *Theoretical and Mathematical Physics*, V.118, 13, P. 441-451 (1999).

[35] Myrzakulov R., Nugmanova G., Syzdykova R. *Journal of Physics A: Mathematical & Theoretical*, V.31, 147, P.9535-9545 (1998).

[36] Myrzakulov R., Daniel M., Amuda R. *Physica A.*, V.234, 13-4, P.715-724 (1997).

[37] Myrzakulov R., Makhankov V.G., Pashaev O.?., *Letters in Mathematical Physics*, V.16, N1, P.83-92 (1989).

[38] Myrzakulov R., Makhankov V.G., Makhankov A. *Physica Scripta*, V.35, N3, P. 233-237 (1987)

[39] Myrzakulov R., Pashaev O.?., Kholmurodov Kh. *Physica Scripta*, V.33, N4, P. 378-384 (1986)
[40] Anco S.C., Myrzakulov R. Journal of Geometry and Physics, v.60, 1576-1603 (2010)

[41] Myrzakulov R., Rahimov F.K., Myrzakul K., Serikbaev N.S. On the geometry of stationary Heisenberg ferromagnets. In: "Non-linear waves: Classical and Quantum Aspects", Kluwer Academic Publishers, Dordrecht, Netherlands, P. 543-549 (2004)

[42] Myrzakulov R., Serikbaev N.S., Myrzakul Kur., Rahimov F.K. On continuous limits of some generalized compressible Heisenberg spin chains. Journal of NATO Science Series II. Mathematics, Physics and Chemistry, V 153, P. 535-542 (2004)

[43] R.Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, M. Lakshmanan. Symmetry, 7(3), 1352-1375 (2015). [arXiv:1305.0098]

[44] R.Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, K. Yesmakhanova, M. Lakshmanan. Physics Letters A, 378, N30-31, 2118-2123 (2014). [arXiv:1404.2088]

[45] Myrzakulov R., Martina L., Kozhamkulov T.A., Myrzakul Kur. Integrable Heisenberg ferromagnets and soliton geometry of curves and surfaces. In book: "Nonlinear Physics: Theory and Experiment. II". World Scientific, London, P. 248-253 (2003)

[46] Myrzakulov R. Integrability of the Gauss-Codazzi-Mainardi equation in 2+1 dimensions. In "Mathematical Problems of Nonlinear Dynamics", Proc. of the Int. Conf. "Progress in Nonlinear sciences", Nizhny Novgorod, Russia, July 2-6, 2001, V.1, P.314-319 (2001)

[47] Chen Chi, Zhou Zi-Xiang. Darboux Transformation and Exact Solutions of the Myrzakulov-I Equations. Chin. Phys. Lett., 26, N8, 080504 (2009)

[48] Chen Hai, Zhou Zi-Xiang. Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation. Chin. Phys. Lett., 31, N12, 120504 (2014)

[49] Zhao-Wen Yan, Min-Ru Chen, Ke Wu, Wei-Zhong Zhao. J. Phys. Soc. Jpn., 81, 094006 (2012)

[50] Yan Zhao-Wen, Chen Min-Ru, Wu Ke, Zhao Wei-Zhong. Commun. Theor. Phys., 58, 463-468 (2012)

[51] K.R. Ysmakhanova, G.N. Nugmanova, Wei-Zhong Zhao, Ke Wu. Integrable inhomogeneous Lakshmanan-Myrzakulov equation. [nlin/0604034]
[52] Zhen-Huan Zhang, Ming Deng, Wei-Zhong Zhao, Ke Wu. *On the integrable inhomogeneous Myrzakulov-I equation*, [arXiv: nlin/0603069](https://arxiv.org/abs/nlin/0603069)

[53] Martina L, Myrzakul Kur., Myrzakulov R, Soliani G. Journal of Mathematical Physics, V.42, 13, P.1397-1417 (2001).

[54] Xiao-Yu Wu, Bo Tian, Hui-Ling Zhen, Wen-Rong Sun and Ya Sun. Journal of Modern Optics, 2015.

[55] Z.S. Yersultanova, M. Zhassybayeva, K. Yesmakhanova, G. Nugmanova, R. Myrzakulov. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1404.2270](https://arxiv.org/abs/1404.2270)

[56] Myrzakul Akbota and Myrzakulov Ratbay. *Integrable Motion of Two Interacting Curves and Heisenberg Ferromagnetic Equations*, Abstracts of XVIII-th Intern. Conference ”Geometry, Integrability and Quantization”, June 3-8, 2016, Bulgaria.

[57] Myrzakul Akbota and Myrzakulov Ratbay. *Integrable motion of two interacting curves, spin systems and the Manakov system*. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1606.06598](https://arxiv.org/abs/1606.06598)

[58] Myrzakul Akbota and Myrzakulov Ratbay. *Darboux transformations and exact soliton solutions of integrable coupled spin systems related with the Manakov system*, [arXiv:1607.08151](https://arxiv.org/abs/1607.08151)

[59] Myrzakul Akbota and Myrzakulov Ratbay. *Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrodinger equation*. International Journal of Geometric Methods in Modern Physics, 13, N1, 1550134 (2016). [arXiv:1608.08553](https://arxiv.org/abs/1608.08553)

[60] Myrzakulova Zh., Myrzakul A., Nugmanova G., MyrzakulovR. *Notes on Integrable Motion of Two Interacting Curves and Two-layer Generalized Heisenberg Ferromagnet Equations*, [arXiv:1811.12216](https://arxiv.org/abs/1811.12216)

[61] Hussien R.A., Mohamed S.G. *Generated Surfaces via Inextensible Flows of Curves in $R^3$*. Journal of Applied Mathematics, v.2016, Article ID 6178961 (2016).

[62] C. Qu, J. Song and R. Yao. *Multi-Component Integrable Systems and Invariant Curve Flows in Certain Geometries*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 001, 19 pages [arXiv:1301.0180](https://arxiv.org/abs/1301.0180)