Classifying directional Gaussian quantum entanglement, EPR steering and discord

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Quantum discord quantifies how much Alice’s system is disrupted after a measurement is performed on Bob’s. Conceptually, this behavior acts the same way as quantum steering and we find that the discord grows with better steering from Bob to Alice. Using Venn diagrams, the relations between the different classes of Gaussian continuous variable entanglement and the links to discord for the squeezed-thermal states are established. We identify a directional quantum teleportation task for each class of squeezed-thermal state entanglement, and establish a unified signature for quantum steering, entanglement and discord beyond entanglement. Quantum steering and discord are promising candidates to quantify the potential of the directional quantum tasks where Alice and Bob possess asymmetrically noisy channels.

The topic of quantum correlations has received much attention in modern physics \textsuperscript{1-2}. Entanglement is a distinctive feature of quantum correlations \textsuperscript{3} and it is considered that all entangled states are useful for quantum information processing (QIP) \textsuperscript{4}. Einstein-Podolsky-Rosen (EPR) correlations enable error-free predictions for the position and the momentum of one particle given some type of measurement on another. EPR correlations are especially useful \textsuperscript{5}. As one example, the fidelity of the quantum teleportation (QT) of a coherent state is directly related to the strength of EPR correlation available in the quantum resource \textsuperscript{6}.

Very recently, there has been an appreciation of the importance of asymmetry and direction in quantum correlations \textsuperscript{7-11}. Entanglement is a property shared between two parties, and measures of it have not been sensitive to differences between the quantum parties involved \textsuperscript{12}. Yet, the original EPR argument was expressed asymmetrically between the two systems. The analysis by Schrodinger introduced the asymmetric term “steering” to describe the EPR idea of one party apparently adjusting the state of another by way of local measurements \textsuperscript{13}. This aspect has been beautifully captured in two recent alternative definitions for quantum correlations: quantum discord \textsuperscript{7,8} and EPR steering \textsuperscript{9,10}. Besides being of intrinsic fundamental interest, these asymmetrical nonlocalities are attracting a great deal of attention \textsuperscript{13,17} for special tasks in QIP e.g. cloning of correlations \textsuperscript{18}, quantum metrology \textsuperscript{19}, quantum state merging \textsuperscript{20}, remote state preparation \textsuperscript{21} and one-sided device-independent quantum key distribution \textsuperscript{22}. Surprisingly, for mixed states, quantum discord can emerge without entanglement and recent experiments \textsuperscript{23} have used discord to distribute entanglement using separable states only \textsuperscript{24}. Despite the potential value of directional quantum correlation, relatively little is known about the quantitative link between discord and steering, and methodologies to characterise quantum states for their asymmetrical correlation.

Our aim in this Letter is to provide such a characterisation and to explain the link between discord and steering for the purpose of continuous variable (CV) QIP. We focus on the subclass of bipartite quantum systems called Gaussian states \textsuperscript{9,26} which have enabled experimental milestones such as deterministic QT \textsuperscript{27}. Asymmetrical Gaussian entanglement and its application to QIP is not fully understood. To illustrate, it is often interpreted that CV quantum teleportation (QT) requires a resource with the “Duan” \textsuperscript{31} symmetric form of entanglement, for which the measures of EPR steering and discord are largely unaltered if the roles of the two parties are exchanged \textsuperscript{16,28,32}.

Here, we address this gap in knowledge by introducing a classification of the space of Gaussian states into distinct sets of directional entanglement classes. We establish the strict relations between the classes and the links to quantum discord, for the experimentally relevant subclass of squeezed-thermal (STS) states. Moreover, we relate each of these classes to a special directional QT task, showing that the whole subclass of STS Gaussian entangled states including those with asymmetric quantum correlation can be used for QT. By introducing an EPR steering parameter, we establish an experimental signature to distinguish the states of different classes, whether EPR steering, entanglement, or discord beyond entanglement. Finally, we show how one can manipulate the two-mode squeezed EPR state to cross between the different classes of quantum correlation, by adding asymmetric amounts of thermal noise to each sub-system.

Our method connects three Gaussian measures of quantum correlation: Simon’s positive partial transpose (PPT) condition for entanglement \textsuperscript{29}, the criterion of Ref. \textsuperscript{30} for EPR steering, and the measure of Giorda and Paris for discord \textsuperscript{25}. We explain how the PPT condition is equivalent to a condition on an EPR-type variance. The condition works efficiently for all Gaussian states due to the introduction of a gain factor \( g_{\text{sym}} \) which we show gives information about the symmetry of
We then show that the steering from the two systems. In fact, steering can be created “one-way” using thermal manipulation. We consider the different types of quantum correlation for the subclass of Gaussian states. The larger blue circle III contains states satisfying the Duan criterion for entanglement $\Delta_{ent} < 1$. The inner blue circle I contains states with the symmetric EPR steering correlation given by $\Delta_{ent} < 0.5$. The set of all entangled states quantified by the PPT criterion $\text{Ent}_{\text{PPT}} < 0$ are contained in the larger green ellipse VI. The smaller orange IV and yellow V ellipses enclose states that display one-way steering $E_{A/B} < 1$ and $E_{B/A} < 1$, respectively. Their intersection (colored yellow) is the set of two-way steerable states, which is a strict subset of the states in I. All two-way steerable states are a subset of the entangled states quantified by the Duan condition $\Delta_{ent} < 1$. One-way steering states are a strict subset of the PPT entangled states, and are strictly not contained in the Duan circle $\Delta_{ent} < 1$. The outer ellipse III contains the set of Gaussian states with non-zero quantum A and B discord. All Gaussian states except product states are contained in III, which is a strict superset of all Gaussian entangled states [20].

the quantum correlation, and how the resource can be utilised for QT. Entanglement can be quantified by the steering measure for each party, and by $g_{sym}$. Interestingly, we find that “quantum A(B) discord” grows with better steering from Bob (Alice) to Alice (Bob). We will see that the steering from B to A and quantum A (B) discord are asymmetrically sensitive to the thermal noise on the two systems. In fact, steering can be created “one-way” using thermal manipulation. We then show that while resources with symmetric quantum correlation are useful for QT via traditional protocols, those with asymmetric correlation require asymmetric protocols.

All Gaussian properties can be determined from the symplectic form of the covariance matrix (CM) defined as $C_{ij} = \langle (X_i X_j + X_j X_i) \rangle / 2 - \langle X_i \rangle \langle X_j \rangle$ where $X \equiv (X_A, P_A, X_B, P_B)$ is the vector of the field quadratures:

$$C = \begin{pmatrix} n & 0 & c_1 & 0 \\ 0 & n & 0 & c_2 \\ c_1 & 0 & m & 0 \\ 0 & c_2 & 0 & m \end{pmatrix}$$ (1)

The symplectic invariants are defined by $I_1 = n^2$, $I_2 = m^2$, $I_3 = c_1 c_2$, $I_4 \equiv \text{det}(C) = (nm - c_1^2)(nm - c_2^2)$, and the symplectic eigenvalues $d_{\pm} = \sqrt{(\Delta \pm \sqrt{\Delta^2 - 4\text{det}(C)})} / 2$ with $\Delta = I_1 + I_2 + 2I_3$.

Our classification will be exemplified by the Gaussian two-mode squeezed thermal state (STS) for which $c_1 = -c_2 = c$. We thus follow [26] and focus on this subclass of Gaussian states for the remainder of the paper. The covariance matrix elements in the STS case are $n = (2n_A + 1)\cos^2(r) + (2n_B + 1)\sin^2(r)$, $m = (2n_B + 1)\cos^2(r) + (2n_A + 1)\sin^2(r)$, $c = (n_A + n_B + 1)\sinh(2r)$, where $n_A$, $n_B$ are the average number of thermal photons for each system and $r$ denotes the squeezing parameter. Here, we normalise the vacuum fluctuations so that $\Delta X \Delta P \geq 1$. We can specify Simon’s PPT criterion for entanglement as [32]

$$\text{Ent}_{\text{PPT}} = (nm - c^2) + 1 - (n^2 + m^2 + 2c^2) < 0,$$ (2)

which becomes a necessary and sufficient condition for Gaussian states [29]. According to the PPT criterion [2], a two-mode STS is entangled iff $r$ exceeds the following threshold value: $\cos^2(r_{ent}) = (n_A + 1/n_B + 1/3)$. The complete set of PPT entangled states is depicted as contained within the green ellipse of Fig. 1. This set is not exhaustive for Gaussian states as seen by the values for $\text{Ent}_{\text{PPT}}$ versus the thermal noises $n_A$ and $n_B$ shown in Fig. 2a [20].

Entanglement can also be determined using an EPR-type correlation [31, 35]. On defining the weighted difference variance $\Delta^2(X_A - gX_B) = n - 2g + g^2m = \Delta^2(P_A + P_B)$, entanglement between modes A and B is confirmed when

$$\text{Ent}_{\text{g}}^{A|B} = \Delta^2(X_A - gX_B)/(1 + g^2) < 1.$$ (3)

Here $g$ is an arbitrary real constant but which can be optimally chosen to minimise the value of $\text{Ent}_{\text{g}}^{A|B}$. For the restricted subclass of Gaussian EPR resources, there is symmetry between the X and P moments so that a single $g$ suffices. With the optimal choice of $g = g_{sym} \equiv \sqrt{(n - m + \sqrt{(n - m)^2 + 4c^2})} / 2c$, it is straightforward to show that the entanglement bounds of $\text{Ent}_{\text{g}}^{A|B}$ and $\text{Ent}_{\text{PPT}} < 0$ ($d_+ = 1$, obtained by replacing $I_3 \to -I_3$ in the formula for $d_-$) are equivalent. Note that the entanglement between modes A and B can be also confirmed when $\text{Ent}_{\text{g}}^{g|A} = \Delta^2(X_B - g'X_A)/(1 + g'^2) < 1$, which is the same threshold as for $\text{Ent}_{\text{g}}^{A|B}$ but with $g' = g_{sym}^{B|A} \equiv (m - n + \sqrt{(m - n)^2 + 4c^2}) / 2c = 1/g_{sym}^{A|B}.$

This is to be expected: Entanglement is by definition a quantity shared between two systems, and its PPT measure does not account for the directional properties associated with quantum correlation.

Where one has complete symmetry between the systems, $n = m$ and $g_{sym}^{A|B} = 1$. The PPT criterion [3] for entanglement then reduces to the measure of “Duan entanglement” [31, 32]

$$\Delta_{ent} = \left| \Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \right| / 4 < 1.$$ (4)
Resources with the property \([4]\) are required for the CV quantum teleportation (QT) of a coherent state, as achieved using the standard protocol of Braunstein and Kimble \([4]\). The STS squeezing threshold for Duan entanglement is \(r > r_{QT,\text{duan}} = \ln\sqrt{n_A + n_B + 1}\). These states are depicted as enclosed within the dark blue circle \(II\) of Fig. 1. Sufficiently asymmetric systems (where \(n \gg m\)) may arise for example when coupling massive objects to laser pulses, and may require the full PPT entanglement test (outside the blue circle \(II\), but within the green ellipse) as illustrated in Fig. 1 \([36]\).

Quantum discord is by definition a measure of asymmetric quantum correlation between the two subsystems \([2]\). The “quantum A discord” that considers the conditional information for Alice’s system \(A\) based on measurements on system \(B\) by Bob, has been derived for a Gaussian state by Giorda and Paris as \([20]\)

\[
D_{A|B} = f(m) - f(d_+) - f(d_-) + f(z),
\]

where \(z = \frac{n+mn-2d^2}{m+1}\) and \(f(x) = (\frac{x+1}{x+1})\ln(\frac{x+1}{x+1}) - (\frac{x+1}{x+1})\ln(\frac{x+1}{x+1})\). With the exchanging \(m \leftrightarrow n\) and hence \(I_1 \leftrightarrow I_2\), we obtain the result for the B discord \(D_{B|A}\). Quantum \(A\) discord is obtained for all bipartite Gaussian states that are not product states, although there are non-entangled states that have nonzero discord \([20]\). The quantum discord is the difference between two classically-equivalent definitions of conditional entropy \([7, 8, 26]\). Denoting the von Neumann entropy of the quantum state \(\rho\) by \(S(\rho)\), the first \(S(\rho_{A|B}) \sim f(d_+) + f(d_-) - f(m)\) arises from using the definition of mutual information based on the bipartite state \(\rho_{AB}\). The second arises from quantisation of the expressions for the conditional entropy: \(H(\rho_{A|B}) = \sum_k p_B(k)S(\rho_{A|k}) \sim f(\sqrt{z})\) where \(p_B(k)\) is the probability of result \(k\) for a measurement at \(B\), and \(S(\rho_{A|k}) = \sum_i p(i|k)S(\rho_{i|k})\) where \(p(i|k)\) is the conditional probability of outcome \(i\) at \(A\) given the result \(k\) at \(B\). The discord \([5]\) is obtained by minimising the mismatch over all Gaussian measurements. The terms in the quantum discord \(H\) quantify the available information for the conditional state of \(A\) after measurement on \(B\), and also reflect uncertainty in measurements of Alice when Bob’s outcome \(k\) is known.

Interestingly, this reminds us of the other asymmetric nonlocality, EPR steering from \(B\) to \(A\) \([11, 9, 10]\), which is realized for Gaussian systems iff \([9, 30]\)

\[
E_{A|B} = \Delta_{inf} X_{A|B} \Delta_{inf} P_{A|B} < 1
\]

Here \(\Delta_{inf} X_{A|B} = \sum_k p_B(k)\Delta^2(X_A|k)\) where \(\Delta^2(X_A|k)\) is the variance of the conditional distribution for Alice’s “position” \(X_A\) conditional on the result \(k\). The measurement at \(B\) is selected to minimise the quantity \(\Delta_{inf} X_{A|B}\). The \(\Delta_{inf} P_{A|B} = \sum_k p_B(k')\Delta^2(P_A|k')\) is defined similarly, for the momentum \(P_A\). The states with the property \([6]\) are depicted by the small orange ellipse of Fig. 1. For Gaussian states, we can write \(\Delta_{inf} X_{A|B} = \Delta^2(X_A - gX_B)\) and \(\Delta_{inf} P_{A|B} = \Delta^2(P_A + gP_B)\) where \(g\) is a real constant \([30, 33]\), noting that for the restricted subclass \(E_{A|B}(g) = \Delta_{inf} X_{A|B} = \Delta_{inf} P_{A|B} = n + g^2 m - 2gc\). The optimal measurement is defined by the optimal \(g\). The quantity \(E_{A|B}(g)\) is minimized to \(E_{A|B} = n - c^2 / m\) by the optimal factor \(g = c / m\) \([30, 32]\), and its smallness gives a measure of the degree of nonlocal correlations. Ideally, it becomes zero in the limit of large \(r\). As with discord, we obtain the result for the steering from \(A\) to \(B\) by interchanging parameters: \(E_{B|A}(g') = \Delta^2(X_B - g'X_A)\) and \(\Delta_{inf} P_{B|A} = \Delta^2(P_B + g'P_A)\) where \(g'\) is a real constant \([30, 33]\).
The presence of asymmetric noises creates the possibility of asymmetric steering/discard, making steering/disruption from A to B more difficult than that from B to A. Entanglement is absent for $E_{\text{PT}} \geq 0$, the region above the green curve in Fig. 2a. All regions show “quantum discord”, given by $D_{\text{sym}} > 0$ (Fig. 2b). Thermal noises tend to suppress entanglement, for which the dependence on $n_A$ and $n_B$ is symmetric. However, the effect on the discord is more complex and asymmetrical. We can see that $D_{\text{sym}}$ is maximised when most of thermal noise is placed on the unmeasured system A.

The behavior of discord is strongly related to steering (Fig. 2). We note the similarity between the conditional entropy $\mathcal{H}(\rho_{AB})$ (Fig. 2b-1) and $E_{\text{PT}}$ (Fig. 2a). As steering increases (so that $E_{\text{PT}} \rightarrow 0$) the variances of the conditional distribution are reduced. We find that for better steering of Alice by Bob, quantum discord becomes larger (see Supplemental Materials [34]). This is consistent with the picture that more of the EPR-type disturbances happen to Alice’s system because of Bob’s measurements.

Finally, we emphasize potential applications of asymmetric correlation. We show in the Supplementary Materials [34] that the directional entangled states are useful as a resource for the quantum teleportation of a coherent state from Alice to Bob (if $g_{\text{sym}} \leq 1$), or from Bob to Al-
ice (if $g_{sym} \geq 1$). This is achieved using the asymmetric protocol of Fig. 3. We leave open the question of whether the asymmetric value of discord may also produce directional quantum tasks only successful either from Alice to Bob, or Bob to Alice.

In conclusion, we have established classes of CV Gaussian quantum correlation, determined how to signify and generate states of a given class, and shown how the states of each entanglement class can be utilised for a quantum teleportation task. We explored the relation between two asymmetric nonclassical correlations, steering and discord. Our results suggest asymmetric correlations such as EPR steering and discord to be promising candidates for quantum tasks requiring a directional operation.

We thank the Australian Research Council for funding via Discovery and DECRA grants. Q. Y. H thanks support from the National Natural Science Foundation of China under grants No.11121091 No.11274025.

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