ANALOGUES OF SOME FUNDAMENTAL THEOREMS OF SUMMABILITY THEORY

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Abstract. In 1911, Steinhaus presented the following theorem: if $A$ is a regular matrix then there exists a sequence of 0’s and 1’s which is not $A$-summable. In 1943, R. C. Buck characterized convergent sequences as follows: a sequence $x$ is convergent if and only if there exists a regular matrix $A$ which sums every subsequence of $x$. In this paper, definitions for “subsequences of a double sequence” and “Pringsheim limit points” of a double sequence are introduced. In addition, multidimensional analogues of Steinhaus’ and Buck’s theorems are proved.

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1. Introduction. In [2, 3, 4, 5, 8], the 4-dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=0}^{\infty} a_{m,n,k,l}x_{k,l}$ is studied extensively by Robison and Hamilton. Here we define new double sequence spaces and consider the behavior of 4-dimensional matrix transformations on our new spaces. Such a 4-dimensional matrix $A$ is said to be RH-regular if it maps every bounded P-convergent sequence (defined below) into a P-convergent sequence with the same P-limit. In [9] Steinhaus proved the following theorem: if $A$ is a regular matrix then there exists a sequence of 0’s and 1’s which is not $A$-summable. This implies that $A$ cannot sum every bounded sequence. In this paper, we prove a theorem for double sequences and 4-dimensional RH-regular matrices that is analogous to Steinhaus’ theorem. One of the fundamental facts of sequence analysis is that if a sequence is convergent to $L$, then all of its subsequences are convergent to $L$. In a similar manner, R. C. Buck [1] characterized convergent sequences by: a sequence $x$ is convergent if and only if there exists a regular matrix $A$ which sums every subsequence of $x$. We characterize P-convergent double sequences as follows: first, we prove that a double sequence $x$ is P-convergent to $L$ if all of its subsequences are P-convergent to $L$; then we prove that a double sequence $x$ is P-convergent if there exists an RH-regular matrix $A$ which sums every subsequence of $x$. In addition, we provide definitions for “subsequences” and “Pringsheim limit points” of double sequences and for divergent double sequence.

2. Definitions, notations, and preliminary results

**Definition 2.1** (Pringsheim, 1900). A double sequence $x = [x_{k,l}]$ has Pringsheim limit $L$ (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that
$|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We describe such an $x$ more briefly as “P-convergent.”

**Definition 2.2** (Pringsheim, 1900). A double sequence $x$ is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers $n_1$ and $n_2$ such that $|x_{n_k}| > G$ for $n \geq n_1$, $k \geq n_2$.

**Definition 2.3.** The sequence $y$ is a subsequence of the double sequence $x$ provided that there exist two increasing double index sequences $\{n_{i}^j\}$ and $\{k_{i}^j\}$ such that

$n_0^1 = k_0^1 = n_0^{0_1} = k_0^{0_1} = 0$ and

$n_i^1$ and $k_i^1$ are both chosen such that $\max\{n_{2i-3}^{i-1}, k_{2i-3}^{i-1}\} < n_i^1, k_i^1$,

$n_i^2$ and $k_i^2$ are both chosen such that $\max\{n_{2i}^i, k_{2i}^i\} < n_i^2, k_i^2$,

$n_i^3$ and $k_i^3$ are both chosen such that $\max\{n_{2i+1}^i, k_{2i+1}^i\} < n_i^3, k_i^3$,

$\vdots$

$n_{2i-1}^i$ and $k_{2i-1}^i$ are both chosen such that $\max\{n_{2i-2}^{i-1}, k_{2i-2}^{i-1}\} < n_{2i-1}^i, k_{2i-1}^i$, with

$y_{1,i} = x_{n_1^i, k_1^i}$, $y_{2,i} = x_{n_2^i, k_2^i}$, $y_{3,i} = x_{n_3^i, k_3^i}$,

$\vdots$

$y_{1,i} = x_{n_{i-1}^i, k_{i-1}^i}$, $y_{i,i-1} = x_{n_{i+1}^i, k_{i+1}^i}$,

$\vdots$

$y_{1,2i-1} = x_{n_{2i-1}^i, k_{2i-1}^i}$

for $i = 1, 2, 3, \ldots$.

A double sequence $x$ is bounded if and only if there exists a positive number $M$ such that $|x_{k,l}| < M$ for all $k$ and $l$. Define

$$S''\{x\} = \{\text{all subsequences of } x\};$$

$$C'' = \{\text{all bounded P-convergent sequences}\};$$

$$C''_{A} = \left\{ x_{k,l} : (Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \text{ is P-convergent} \right\}.$$

(2.1)

See Figure 1 for an illustration of the procedure for selecting terms of a subsequence. A 2-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [6] characterizes the regularity of 2-dimensional matrix transformations. In 1926, Robison presented a 4-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of the regularity for 4-dimensional matrices will be stated below, with the Robison-Hamilton characterization of the regularity of 4-dimensional matrices.
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The selection process of terms for subsequence \( y \) of \( x \), where

\[
x[n(i,j),k(i,j)] = x_{n^i_j,k_j}, \quad n(i,j) = n^i_j, \quad k(i,j) = k^i_j.
\]

**Figure 1.** The selection process of terms for subsequence \( y \) of \( x \), where \( x[n(i,j),k(i,j)] = x_{n^i_j,k_j} \), \( n(i,j) = n^i_j \), \( k(i,j) = k^i_j \).

**Definition 2.4.** The 4-dimensional matrix \( A \) is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

**Theorem 2.1** (Hamilton [2], Robison [8]). The 4-dimensional matrix \( A \) is RH-regular if and only if

- RH1: \( P\lim_{m,n}a_{m,n,k,l} = 0 \) for each \( k \) and \( l \);
- RH2: \( P\lim_{m,n} \sum_{k+l=0}^{\infty} a_{m,n,k,l} = 1 \);
- RH3: \( P\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0 \) for each \( l \);
- RH4: \( P\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0 \) for each \( k \);
- RH5: \( \sum_{k,l=0}^{\infty} |a_{m,n,k,l}| \) is P-convergent;
- RH6: there exist finite positive integers \( A \) and \( B \) such that \( \sum_{k,l=B}^{\infty} |a_{m,n,k,l}| < A \).

**Remark 2.1.** The definition of a Pringsheim limit point can also be stated as follows: \( \beta \) is a Pringsheim limit point of \( x \) provided that there exist two increasing index sequences \( \{n_i\} \) and \( \{k_i\} \) such that \( \lim_{i} x_{n_i,k_i} = \beta \).

**Definition 2.5.** A double sequence \( x \) is divergent in the Pringsheim sense (P-divergent) provided that \( x \) does not converge in the Pringsheim sense (P-convergent).
Remark 2.2. Definition 2.5 can also be stated as follows: a double sequence \( x \) is P-divergent provided that either \( x \) contains at least two subsequences with distinct finite limit points or \( x \) contains an unbounded subsequence. Also note that, if \( x \) contains an unbounded subsequence then \( x \) also contains a definite divergent subsequence.

Remark 2.3. For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the 2-dimensional plane, as illustrated by the following example.

Example 2.1. The sequence

\[
x_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
1, & \text{if } n = 0, k = 1, \\
1, & \text{if } n = 1, k = 0, \\
0, & \text{otherwise}
\end{cases}
\]  

contains only two subsequences, namely, \([y_{n,k}] = 0 \) for each \( n \) and \( k \), and

\[
z_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
0, & \text{otherwise};
\end{cases}
\]  

neither subsequence is \( x \).

The following proposition is easily verified, and is worth stating since each single-dimensional sequence is a subsequence of itself. However, this is not the case for double-dimensional sequences.

Proposition 2.1. The double sequence \( x \) is P-convergent to \( L \) if and only if every subsequence of \( x \) is P-convergent to \( L \).

3. Main results. The next result is a “Steinhaus-type” theorem, so named because of its similarity to the Steinhaus theorem in [9] quoted in the introduction.

Theorem 3.1. If \( A \) is an RH-regular matrix, then there exists a bounded double sequence \( x \) consisting only of 0’s and 1’s which is not \( A \)-summable.

Proof. Let \( m_i, n_j, k_i, \) and \( l_j \) be increasing index sequences which we define as follows:

Let \( k_0 := l_0 := -1 \) and choose \( m_0 \) and \( n_0 \) such that \( m_0, n_0 > B \), where \( B \) is defined by RH6 and RH2 to imply

\[
\left| \sum_{k,l=0}^{\infty,\infty} a_{m_0,n_0,k,l} \right| > \frac{1}{4},
\]  

whenever \( m_0, n_0 > B \).

Also, by RH1, RH3, RH4, and RH5 we choose \( k_1 > k_0 \) and \( l_1 > l_0 \) such that
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\[
\left| \sum_{k < k_1, l < l_1} a_{m_0, n_0, k, l} \right| > 1 - \frac{1}{4},
\]

\[
\sum_{k > k_1, l > l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4},
\]

\[
\sum_{k < k_1, l > l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4},
\]

\[
\sum_{k > k_1, l < l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4}.
\]

(3.2)

Next use RH$_1$, RH$_2$, RH$_3$, and RH$_4$ to choose $m_1 > m_0$ and $n_1 > n_0$ such that

\[
\sum_{k < k_1, l < l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k < k_1, l > l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k > k_1, l > l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k > k_1, l < l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\left| \sum_{k, l = 0}^{\infty, \infty} a_{m_1, n_1, k, l} \right| > 1 - \frac{1}{9}.
\]

(3.3)

These inequalities imply

\[
\sum_{k > k_1, l > l_1} |a_{m_1, n_1, k, l}| > 1 - \frac{4}{9},
\]

(3.4)

because

\[
\left| \sum_{k > k_1, l > l_1} a_{m_1, n_1, k, l} \right| \geq - \sum_{k < k_1, l < l_1} |a_{m_1, n_1, k, l}| + 1 - \frac{1}{9}
\]

\[
- \sum_{k > k_1, l > l_1} |a_{m_1, n_1, k, l}|
\]

\[
- \sum_{k < k_1, l < l_1} |a_{m_1, n_1, k, l}|.
\]

(3.5)

We now choose $k_2 > k_1$ and $l_2 > l_1$ such that

\[
\left| \sum_{k_1 < k < k_2, l_1 < l < l_2} a_{m_1, n_1, k, l} \right| > 1 - \frac{4}{9},
\]

\[
\sum_{k > k_2, l > l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k < k_2, l > l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k > k_2, l < l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9},
\]

\[
\sum_{k < k_2, l < l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9}.
\]

(3.6)
In general, having
\[ m_0 < \cdots < m_{i-1}, \quad k_0 < \cdots < k_{i-1} < k_i, \]
\[ n_0 < \cdots < n_{j-1}, \quad l_0 < \cdots < l_{j-1} < l_j, \] (3.7)
we choose \( m_i > m_{i-1} \) and \( n_j > n_{j-1} \) such that by RH1
\[ \sum_{k \leq k_i, l \leq l_j} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}, \] (3.8)
and by RH3, RH4
\[ \sum_{k \leq k_i, l > l_j} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}, \]
\[ \sum_{k \geq k_i, l \leq l_j} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}. \] (3.9)
In addition, by RH2
\[ \left| \sum_{k,l=0}^{\infty, \infty} a_{m_i, n_j, k, l} \right| > 1 - \frac{1}{(i+2)(j+2)}, \] (3.10)
so
\[ \sum_{k > k_i, l > l_j} |a_{m_i, n_j, k, l}| > 1 - \frac{4}{(i+2)(j+2)}. \] (3.11)
We now choose \( k_{i+1} > k_i \) and \( l_{j+1} > l_j \) such that
\[ \left| \sum_{k_i < k < k_{i+1}, l_i < l < l_{i+1}} a_{m_i, n_j, k, l} \right| > 1 - \frac{4}{(i+2)(j+2)}, \]
\[ \sum_{k \geq k_{i+1}, l \geq l_{j+1}} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}, \]
\[ \sum_{k_i < k < k_{i+1}, l_i < l_i} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}, \]
\[ \sum_{k \geq k_{i+1}, l < l_{j+1}} |a_{m_i, n_j, k, l}| < \frac{1}{(i+2)(j+2)}. \] (3.12)
Define \( x \) as follows:
\[ x_{k,l} = \begin{cases} 1, & \text{if } k_{2p} < k < k_{2p+1} \text{ and } l_{2t} < l < l_{2t+1} \text{ for } p, t = 0, 1, 2, \ldots, \\ 0, & \text{otherwise}. \end{cases} \] (3.13)
Let us label and partition \((AX)_{mi, nj}\) as follows:

\[
(AX)\left_{\begin{array}{c} 0 \leq k \leq ki, 0 \leq l \leq lj \\ 0 \leq k \leq k_{i+1}, 0 \leq l \leq l_{j+1} \\ 0 \leq l \leq l_{j+1}, k_{i+1} \leq k \leq \infty \\ k_{i+1} \leq k \leq ki, 0 \leq l \leq lj \\ k_{i+1} \leq k \leq ki, l_{j+1} \leq l \leq \infty \\ l_{j+1} \leq l \leq \infty, k_{i+1} \leq k \leq ki \\ l_{j+1} \leq l \leq \infty, 0 \leq k \leq k_{i+1} \\ 0 \leq k \leq \infty, 0 \leq l \leq lj \\ 0 \leq k \leq \infty, l_{j+1} \leq l \leq \infty \\ 0 \leq l \leq \infty, 0 \leq k \leq \infty \end{array}} + \sum_{0 \leq k \leq k_{i+1}, 0 \leq l \leq l_{j+1}} a_{mi, nj, k, l}x_{k, l}, \tag{3.14}
\]

where the general term \(a_{mi, nj, k, l}x_{k, l}\) is the same for each of the nine sums. Note that,

\[
|\alpha_4 + \alpha_5| \leq \frac{1}{(i + 2)(j + 2)},
\]

\[
|\alpha_2 + \alpha_6| \leq \frac{1}{(i + 2)(j + 2)}. \tag{3.15}
\]

**CASE 1.** If \(i\) and \(j\) are even, then

\[
\left| (AX)\left_{mi, nj} \right| > 1 - \frac{1}{(i + 2)(j + 2)} - |\alpha_1| - \cdots - |\alpha_8| > 1 - \frac{7}{(i + 2)(j + 2)}, \tag{3.16}
\]

and the last expression has P-limit 1.

**CASE 2.** If at least one of \(i\) and \(j\) is odd, then \(\alpha_9 = 0\) and

\[
\left| (AX)\left_{mi, nj} \right| \leq |\alpha_1| + |\alpha_2| + \cdots + |\alpha_8| \leq \frac{6}{(i + 2)(j + 2)}, \tag{3.17}
\]

and the last expression of (3.17) has P-limit 0. Thus the P-limit of \((AX)\left_{m,n}\) does not exist, and we have shown that an RH-regular matrix \(A\) cannot sum every double sequence, of 0’s and 1’s.

As with the original Steinhaus Theorem [9], we can state the following as an immediate consequence of Theorem 3.1.

**Corollary 3.1.** If \(A\) is an RH-regular matrix, then \(A\) cannot sum every bounded double sequence.

The next result is a “Buck-type” theorem.

**Theorem 3.2.** The bounded double complex sequence \(x\) is P-convergent if and only if there exists an RH-regular matrix \(A\) such that \(A\) sums every subsequence of \(x\).

**Proof.** Since every subsequence of a P-convergent sequence \(x\) is bounded and P-convergent, and \(A\) is an RH-regular matrix, then for such an \(x\) there exists an RH-regular matrix \(A\) such that \(\hat{S}'\{x\} \subseteq C''\).

Conversely, we use an adaptation of Buck’s proof [1] to show that if \(A\) is any
RH-regular matrix and \( x \notin C'' \) then there exists a subsequence \( y \in S'' \{x\} \) such that \( Ay \notin C'' \).

Note that every subsequence of \( x \) is bounded and \( x \notin C'' \), which implies that \( x \) has at least two distinct Pringsheim limit points, say \( \alpha \) and \( \beta \). Thus there exist increasing index sequences \( \{n_j\} \) and \( \{k_i\} \) such that \( \limsup x_{n_j,k_i} = \alpha \) and \( \liminf x_{n_j,k_i} = \beta \) with \( \alpha \neq \beta \).

Now define
\[
y = \frac{x - \beta}{\alpha - \beta}
\]
which yields \( \limsup y_{n_j,k_i} = 1 \) and \( \liminf y_{n_j,k_i} = 0 \). As a result there exist two disjoint pairs of index sequences \( \{\hat{n}_j, k_j\} \) and \( \{v_j, k_j\} \) such that the sequences \( \hat{y}_1 \) and \( \hat{y}_2 \) constructed using \( \{\hat{n}_j, k_j\} \) and \( \{v_j, k_j\} \), respectively, have \( P \)-limits 1 and 0, respectively. Let
\[
y_{n,k}^* := \begin{cases} 1, & \text{if } n = \hat{n}_j, k = \hat{k}_j, \\ 0, & \text{if } n = v_j, k = k_j, \\ y, & \text{otherwise}. \end{cases}
\]
Hence, \( \{y_{n,k}^*\} \) contains a subsequence \( \{\hat{y}_{n,k}^*\} \) with infinitely many 0’s and 1’s, along its diagonal. This implies that \( S'' \{\hat{y}^*\} \) contains all sequences of 0’s and 1’s. Thus by Theorem 3.1, there exists \( \hat{y}^* \in S'' \{\hat{y}^*\} \) such that \( A\hat{y}^* \notin C'' \). Also, \( P\lim (y - y^*)_{i,j} = 0 \). We now select a subsequence \( \{\hat{y}_{i,j}\} \) of \( \{y_{i,j}\} \) with terms satisfying \( \limsup y_{n_j,k_i} = 1 \) and \( \liminf y_{n_j,k_i} = 0 \) corresponding to the 0’s and 1’s, respectively of \( \{\hat{y}_{n,k}^*\} \). Therefore \( P\lim (\hat{y} - \hat{y}^*)_{i,j} = 0 \) and \( \hat{y}_{i,j} - \hat{y}_{i,j}^* \) is bounded. By the linearity and regularity of \( A, A(\hat{y} - \hat{y}^*)_{i,j} = (A\hat{y})_{i,j} - (A\hat{y}^*)_{i,j} \) and \( P\lim A(\hat{y} - \hat{y}^*)_{i,j} = 0 \). Now since \( A\hat{y}^* \notin C'' \), it follows that \( A\hat{y} \notin C'' \); and since \( \hat{y} = \frac{x - \beta}{\alpha - \beta} \), we have \( Ax \notin C'' \).

\( \square \)

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