On Vacuum Free Boundary Problem of the Spherically Symmetric Euler Equations with Damping and Solid Core

Yan-Lin Wang

Department of Applied Mathematics,
Zhejiang University of Technology
yanlin90118@163.com; yanlinwang.math@gmail.com

Abstract. The global existence of smooth solution and the long-time asymptotic stability of the equilibrium to vacuum free boundary problem of the spherically symmetric Euler equations with damping and solid core have been obtained for arbitrary finite positive gas constant \( A \) in the state equation \( p = A \rho^\gamma \) with \( p \) being the pressure and \( \rho \) the density, provided that \( \gamma > 4/3 \), initial perturbation is small and the radius of the equilibrium \( R \) is suitably larger than the radius of the solid core \( r_0 \). Moreover, we obtain the pointwise convergence from the smooth solution to the equilibrium in a surprisingly exponential time-decay rate. The proof is mainly based on weighted energy method in Lagrangian coordinate.

Keywords: Global solution, nonlinear asymptotic stability, damping, solid core, exponential decay
MSC: 35Q35; 35Q85; 76N10

1. Introduction

The motions of the damped atmosphere surrounding a heavy planet is governed by the following free boundary problem

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \quad &\text{in } \Omega(t), \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p &= -\rho \mathbf{u} - \rho \nabla \Phi_c \quad &\text{in } \Omega(t), \\
\rho &= 0 \quad &\text{in } \Omega(t), \\
\rho &= 0, \quad &\text{on } \Gamma(t) = \partial \Omega(t), \\
\mathbf{u}(\bar{\Gamma}, t) &= 0, \quad &\text{on } \Omega(0), \\
\mathbf{V}(\Gamma(t)) &= \mathbf{u} \cdot \mathbf{N}, \quad &\text{on } \Omega(0), \\
(\rho, \mathbf{u}) &= (\rho_0, \mathbf{u}_0)(x) \quad &\text{on } \Omega(0),
\end{align*}
\]

where \((t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^3\) and \( \rho, \mathbf{u}, \) and \( p \) denote the density, velocity, and pressure of gas atmosphere. \( \Omega(t) \) is the changing volume occupied by the gas at time \( t \), and \( \Gamma(t) \) represents the moving vacuum boundary of the gas atmosphere. Here \( \bar{\Gamma}, \mathbf{V}(\Gamma(t)), \mathbf{N} \) denote, respectively, the fixed boundary of the solid planet, the velocity of the moving boundary
\( \Gamma(t) \), and the exterior unit normal vector to \( \Gamma(t) \). \( \Phi_c \) is the gravitational potential of the solid planet satisfying
\[
-\Phi_c = 4\pi G_0 \rho_B \ast \frac{1}{|x|} = 4\pi G_0 \int_B \frac{\rho_B(y)}{|x-y|} \, dy,
\]
where \( \rho_B \) is the nonnegative density of the solid planet \( B \) and \( G_0 \) is the gravitational constant. The self-gravity of the atmosphere is neglected. That is, the total mass of the gas is relatively small compared with that of the inner solid planet. For more information on gaseous planets with solid core in astrophysics, one can refer to [1, 3, 30, 31].

In this paper, we assume that \( \rho_B \) is spherically symmetric and the solid planet is a ball with radius \( r_0 > 0 \). In spherically symmetric setting
\[
\rho(t,x) = \rho(t,r), \quad u(t,x) = u(t,r) \frac{x}{r}, \quad \text{with} \quad r = |x| \in (r_0, R(t)),
\]
we rewrite the system (1.1) into the following free boundary problem of the compressible Euler equations with damping and solid core:
\[
\begin{align*}
\rho_t + (\rho u)_r + \frac{2}{r} \rho u &= 0 \quad \text{in } (r_0, R(t)), \\
\rho (u_t + uu_r) + p_r &= -\frac{\rho g_0}{r^2} - \rho u \quad \text{in } (r_0, R(t)), \\
\rho > 0 &\quad \text{in } [r_0, R(t)), \\
\rho(R(t), t) = 0, \quad u(r_0, t) = 0, \\
\dot{R}(t) &= u(R(t), t), \quad R(0) = R_0, \\
(\rho, u)(r, t = 0) &= (\rho_0, u_0)(r) \quad \text{on } (r_0, R_0),
\end{align*}
\]
where \( r_0 \) is the radius of the central solid core, and \( g_0 = G_0 M_0 > 0 \), with \( M_0 = 4\pi \int_0^{r_0} \rho_B s^2 \, ds \) being the mass of the central core. The setup (1.3e) means that no vacuum exists between the solid core and the gas. In the sequel, we always assume that
\[
p = p(\rho) = A \rho^\gamma, \quad \gamma > 1,
\]
where \( A \) is a finite positive constant and \( \gamma \) is adiabatic exponent.

The equilibria \( (\bar{\rho}(r), 0) \) of problem (1.3) satisfies
\[
A(\bar{\rho}^\gamma)_r = -\frac{g_0 \bar{\rho}}{r^2}.
\]
More precisely, \( \bar{\rho}(r) \) is given by (cf. [15, 26])
\[
\bar{\rho}(r) = \begin{cases} 
\bar{A} \left( \frac{1}{r} - \frac{1}{R} \right)^{1/(\gamma-1)} & r_0 \leq r < R, \\
0 & r \geq R,
\end{cases}
\]
where \( R \) is an arbitrary number such that \( R > r_0 \) and \( \bar{A} \) is defined by
\[
\bar{A} = \left( \frac{(\gamma - 1) g_0}{\gamma \bar{A}} \right)^{1/(\gamma-1)}.
\]
Note that adiabatic exponent $\gamma$ is usually restricted to $\gamma \in (1, 2)$ for polytropic gas. For instance, $\gamma \to 1_+$ is for heavier molecules, and $\gamma = \frac{7}{5}$ for a monatomic gas and $\gamma = \frac{5}{3}$ for an atomic gas. However, for liquid (or compressed gas), the value of $\gamma$ may be extremely high such that $\gamma \geq 2$. (See [24], nonlinear stability for one dimensional Euler equations with damping and any given $\gamma > 1$.) Indeed, the solution (1.6) solves the equilibria equation (1.5) uniquely for any given $\gamma > 1$ mathematically.

The total mass $M$ of the equilibrium is given by

$$M = 4\pi \bar{A} \int_{r_0}^{R} \left( \frac{1}{r} - \frac{1}{R} \right)^{1/(\gamma-1)} r^2 dr,$$

which is an increasing function of $R$. It has been clarified in [15, 26] that given any total mass $M \in (0, \infty)$ for $\gamma \geq 4/3$, or $M \in (0, M^*)$ for $1 < \gamma < 4/3$, then the radius $R$ is uniquely determined. Here $M^*$ is given by

$$M^* = \frac{4\pi \bar{A}(\gamma - 1)}{4 - 3\gamma} r_0^{-(4-3\gamma)/(\gamma-1)}.$$

Without loss of generality, we assume $A = 1$ for simplicity of the symbol in the following parts.

Let $c(\rho) = \sqrt{\rho'(\rho)}$ be the sound speed, the condition

$$-\infty < \partial_r (c^2(\rho)) < 0 \quad \text{on} \quad R(t)$$

(1.9)

defines a physical vacuum boundary (cf. [5, 7, 16, 10, 17, 18]), which is also called a vacuum boundary with physical singularity. To capture this singularity, as in [24, 40, 42], we assume that the initial density satisfies that

$$\rho_0(r) > 0 \text{ for } r_0 \leq r < R_0, \quad \rho(R_0) = 0 \quad \text{and} \quad M = 4\pi \int_{r_0}^{R_0} \rho_0(r) r^2 dr,$$

(1.10a)

$$-\infty < \left( \rho_0^{\gamma-1}(r) \right)_r < 0 \quad \text{at} \quad r = R_0.$$

(1.10b)

Here (1.9) and (1.10b) indicate that the sound speed is $C^{1/2}$-Hölder continuous near the vacuum boundary.

Clearly, the stationary solution $\bar{\rho}(r)$ is defined on the domain $(r_0, R)$. We introduce a diffeomorphism $\eta_0 : (r_0, R) \to (r_0, R_0)$ satisfying

$$\int_{r_0}^{\eta_0(r)} r^2 \rho_0(r) dr = \int_{r_0}^{r} r^2 \bar{\rho}(\tau) d\tau, \quad r \in (r_0, R),$$

which implies

$$\eta_0^2(r) \rho_0(\eta_0(r)) \eta_0'(r) = r^2 \bar{\rho}(r), \quad r \in (r_0, R).$$

(1.11)

For $y \in (r_0, R)$, we define the Lagrangian variable $\eta(y, t)$ by

$$\eta(y, t) = u(\eta(y, t), t) \quad \text{for} \quad t > 0, \quad \text{and} \quad \eta(y, 0) = \eta_0(y).$$

(1.12)
Note that $\eta \geq r_0$ for all $t \geq 0$. Then the Lagrangian density and velocity are, respectively, defined by
\[ \rho(y, t) = \rho(\eta(y, t), t), \quad v(y, t) = u(\eta(y, t), t), \quad y \in (r_0, R). \quad (1.13) \]

Now, by means of the Lagrangian variables defined above, one can reformulate the system (1.3) on the reference domain $(r_0, R)$:
\[ \begin{align*}
    \left( \eta^2 \rho \right)_t + \eta^2 \left( \frac{\rho v_y}{\eta_y} \right)_y &= 0 & \text{in } (r_0, R) \times (0, \infty), \\
    \rho v_t + (\rho v)_y &= -\rho v - \frac{\rho g_0}{\eta^2} & \text{in } (r_0, R) \times (0, \infty), \\
    v(r_0, t) &= 0 & \text{on } (0, \infty), \\
    (\rho, v) &= (\rho_0(\eta_0), u_0(\eta_0)) & \text{on } (r_0, R) \times \{t = 0\}. 
\end{align*} \quad (1.14) \]

It follows from (1.11) (1.14a) that
\[ \begin{align*}
    \rho(y, t) \eta^2(y, t) \eta_y(y, t) &= \rho_0(\eta_0(y)) \eta^2_0(y) \eta_0(y) = y^2 \bar{\rho}(y), \quad y \in (r_0, R). 
\end{align*} \quad (1.15) \]

This, together with (1.11) (1.12) and (1.13), enables us to rewrite the equations (1.14) as following
\[ \begin{align*}
    \bar{\rho} \eta_{tt} + \bar{\rho} \eta_t + \left( \frac{\eta^2 \bar{\rho}}{\eta^2 \eta_y} \right)_{yy} + \frac{\rho g_0}{\eta^2} &= 0 & \text{in } (r_0, R) \times (0, \infty), \\
    \eta(r_0, t) &= r_0, \quad \eta_t(r_0, t) = 0, & \text{on } (0, \infty), \\
    (\eta, \eta_t) &= (\eta_0, u_0(\eta_0)) & \text{on } (r_0, R) \times \{t = 0\}. 
\end{align*} \quad (1.16) \]

A stationary solution to the equation (1.16a) reads
\[ \bar{\eta}(y, t) = y, \quad \bar{\eta}_t(y, t) = 0, \quad y \in (r_0, R), \quad (1.17) \]

which is just the equilibria (1.6) defined in Lagrangian version.

Let $y \zeta(y, t) := \eta(y, t) - \bar{\eta}(y, t) = \eta(y, t) - y$. Then, with the aid of (1.5), we can write the equation (1.16) into a perturbation form:
\[ \begin{align*}
    y \bar{\rho} \zeta_{tt} + y \bar{\rho} \zeta_t + (1 + \zeta)^2 \left[ \bar{\rho} \gamma (1 + \zeta)^{-2\gamma} (1 + \zeta + y \zeta_y)^{-\gamma} \right]_y - \frac{(\bar{\rho} \gamma)_y}{(1 + \zeta)^2} &= 0. 
\end{align*} \quad (1.18a) \]
\[ \begin{align*}
    \zeta(r_0, t) &= 0, \\
    (y \zeta, y \zeta_t)(y, 0) &= (\eta_0 - y, u_0(\eta_0)). 
\end{align*} \quad (1.18b) \]

The study of motions of the atmosphere surrounding a planet is very important in physics and mathematics (cf. [1, 3, 30, 31]). In 1990s, Okada M. proved the global existence of weak solution to 1-D compressible Navier-Stokes equations with a fixed boundary and a free boundary [32]. Then Matušů-Nečasová Š. Okada M. and Makino T. proved the existence and uniqueness of global weak solution for free boundary problem of the spherically symmetric Navier-Stokes equations with solid core in [33, 27], respectively. Later, they [28] verified the long-time asymptotic stability (in $L^2$) of the equilibrium
with respect to small perturbation in the regime of spherically symmetric Navier-Stokes equation with solid core in Lagrangian mass coordinate, provided that $\gamma > 4/3$ and the gas constant $A$ is sufficiently small (see [22] briefly). They pointed out that the assumption of smallness $A$ is a serious restriction but can not be removed in their scheme (cf. [28]). Then Lin S. verified the linear stability of the equilibrium (1.6) for the spherically symmetric Navier-Stokes-Poisson or Euler-Poisson (Euler) with solid core when $\gamma > 4/3$ ($\gamma > 1$) but without the assumption of smallness $A$, through spectral analysis of the linearized operators. Recently, Makino T. justified a long-time validity of the approximate solution constructed by the equilibria plus a time-periodic solution of the linearized equation around the equilibria for the spherically symmetric Euler equations with solid core [26]. However, the nonlinear asymptotic stability of the equilibrium for inviscid gaseous stars with solid core is still an open problem. In this paper, we consider the global existence of strong solution to the spherically symmetric Euler equations with damping and solid core. We will also verify the nonlinear stability of the equilibrium (1.6) under suitable assumption on the radius of the equilibrium but without the smallness of gas constant $A$.

If there is no solid core, some outstanding progress have been made on the global existence theory and long-time asymptotic stability for free boundary problem of viscous flow at present. For instance, when $4/3 < \gamma < 2$, Luo T. et al proved the long-time nonlinear asymptotic stability of Lane-Emden solution for the spherically symmetric Navier-Stokes-Poisson equations with constant viscosity [22, 20] and degenerate density-dependent viscosity [23]. They applied weighted energy methods to overcome the extreme difficulties caused by boundary singularity or coordinate singularity. For more global existence results of viscous flow with free boundary, we refer to [9, 21, 34, 35, 38, 39, 43] and references therein.

On the other hand, there are also some important progress on the global existence theory and nonlinear stability for free boundary problem of inviscid flow without solid core. When the inviscid gas flow is governed by Euler equations with damping, the long-time asymptotic stability of Barenblatt solution have been obtained by Luo and Zeng in [24] for one dimension and by Zeng in [40] for three dimensions in spherically symmetric setting (see also [41] for almost global existence in 3-D without symmetric assumption). Hadžić M. and Jang J [8], in 2019, obtained a remarkable result on a class of global solutions to free boundary problem of Euler-Poisson equations in three dimensions without any symmetric assumption for all $\gamma$ in the form $\gamma = 1 + \frac{1}{m}, m \in \mathbb{N} \setminus \{1\}$. The global solutions presented in [8] have initially small densities and compact supports, and they stay close to Sideris affine solutions of the Euler system. More recently, Zeng H. [42] proved the global existence for vacuum free boundary problem of Euler equations with damping and gravity in $\mathbb{T}^n, n = 1, 2, 3$, for $\gamma > 1$. Moreover, in [42], Zeng obtained a surprisingly exponential decay of smooth solution to the stationary solution $\bar{\rho}(x) = (\nu(h - x))^{1/(\gamma - 1)}$
$\nu, h$ are positive constants) when the initial perturbation is small. Here we give a brief comparison of the background solution mentioned above in Luo and Zeng’s work. The background solution (Barenblatt solution) in \cite{24,40} is determined by a porous media equation $\rho_t = \Delta p(\rho)$, which is obtained through simplifying the momentum equation by the Darcy’s law $\nabla p(\rho) = -\rho u$. The boundary of Barenblatt solution is expanding sublinearly in time (cf. \cite{2}). But the background solution for Euler equations with damping and gravity in \cite{42} is a stationary solution, the boundary of which is fixed.

However, there are very few results on global existence theory and nonlinear stability for free boundary problem of inviscid gas flow surrounding a solid core in mathematic literature. At this stage, it is meaningful to consider this issue for free boundary problem of Euler equations with damping and solid core in this paper. The approach proposed in this paper can also be applied to solve the global existence and asymptotic stability for free boundary problem of the spherically symmetric Euler-Poisson equations with damping and solid core with respect to small perturbation. The same problem for Euler (-Poisson) equations only with solid core are also open problems, but they are extremely challenging at present, due to the much less dissipation. If the readers are interested in the existence theory of stationary solution for inviscid rotating stars governed by Euler-Poisson equations with solid core, please refer to \cite{37} and references therein.

In this paper, we will apply weighted energy method, which is shown a powerful tool to handle vacuum free boundary problem with physical singularity (cf. \cite{10,11,20,22,23,24,40,41,42} and references therein), to show the global-in-time existence of smooth solution and to derive the nonlinear asymptotic stability with convergence rates, with the help of embedding of weighted Sobolev space. The main challenges in this paper are caused by the solid core and the vacuum boundary with physical singularity. More precisely, first, the solid core will bring a boundary effect, especially in the higher-order energy estimates, which lead to the loss of derivatives. Hence the approach in \cite{42} through taking spatial derivatives directly in energy estimates does not work smoothly in our situation, even though there is no coordinate singularity. To deal with this difficulty, we will apply elliptic estimates associated with suitable relation between the radius of the equilibrium and the radius of the solid core. Second, for the vacuum boundary with singularity, we overcome this difficulty by using weighted energy estimates together with Hardy inequality, which is mainly inspired by Luo and Zeng’s work \cite{24,40} on nonlinear asymptotic stability of Barenblatt solution for vacuum free boundary problem of Euler equations with damping. In this paper, the boundary of the stationary solution (1.6) for the spherically symmetric Euler equations with damping and solid core does not expand in time. This is at least one reason that we can hope a faster decay of the smooth solution with respect to a small perturbation around the equilibrium, as in \cite{42}.

The arrangement of the remaining parts of this paper is as following: Some crucial tools for our proof and main results of this paper will be stated in Section 2. Next, we will
prove our main results in Section 3 containing elliptic estimates and low-high weighted energy estimates.

2. Main results

Notations: In this paper, we use $C$ to denote a generic positive constant, which may depend on $M, A, g, r_0, R$, but independent of the time $t$ and the initial data. We use $a \sim b$ to denote $C^{-1}b \leq a \leqCb$, $a \preceq b$ to denote $a \leq Cb$, and $a \succeq b$ to denote $a \geq Cb$, for some generic positive constant $C$ defined above. We use $C(a)$ to represent that $C$ depends on $a$ additionally. For simplicity of symbol, in the rest of the paper, we will also use the following notations:

$$\int := \int_{r_0}^{R}, \| \cdot \| := \| \cdot \|_{L^2(r_0, R)}, \| \cdot \|_{L^\infty} := \| \cdot \|_{L^\infty(r_0, R)}.$$  

We set $\alpha = \frac{1}{\gamma - 1}$ and $\sigma(y) := \rho^{-1}(y) = A^{-1}(y R)^{-1}(R - y)$ for $y \in (r_0, R)$. So, $R^{-2}A^{-1}(R - y) \leq |\sigma(y)| \leq (r_0 R)^{-1}A^{-1}(R - y)$ and $\sigma(y) \sim d(y) = dist(y, \partial I)$. In some occasion, we may divide the domain $I := (r_0, R)$ into $I_t$ and $I_b$ with $I_t := (r_0, (R + r_0)/2)$ and $I_b := [(R + r_0)/2, (2, R)]$.

Near the boundary, we may use Hardy inequality (cf. [13]) to deal with the boundary singularity. That is, if $\int_{I_b} \sigma^k(y) (F^2 + F_y^2) dy < \infty$, then, for $k > 1$, it holds

$$\int_{I_b} \sigma^{k-2}(y) F^2 dy \leq C(R, r_0, \gamma, k) \int_{I_b} \sigma^k(y) (F^2 + F_y^2) dy. \quad (2.1)$$

We introduce the energy functionals $E_j(t)$ and $E_{j,i}(t)$ defined in the following form

$$E_j(t) = \int \left[ y^4 \sigma^\alpha |\partial_t^j (\zeta, \zeta_t)|^2 + y^2 \sigma^{\alpha+1} |\partial_t^{j+1} (\zeta, y \zeta_y)|^2 \right] (y, t) dy, \quad (2.2)$$

$$E_{j,i}(t) = \int \left[ y^2 \sigma^{\alpha+i+1} |\partial_t^i \partial_y^j \zeta|^2 + y^4 \sigma^{\alpha+i+1} |\partial_t^{i+1} \partial_y^j \zeta|^2 \right] (y, t) dy. \quad (2.3)$$

The dissipation $D_j(t)$ is defined by

$$D_j(t) = \int \left[ y^4 \sigma^\alpha |\partial_t^{j+1} \zeta|^2 + y^2 \sigma^{\alpha+1} |\partial_t^{j+1} (\zeta, y \zeta_y)|^2 \right] (y, t) dy. \quad (2.4)$$

Let us denote $n = 4 + [\alpha]$. The total energy functional is defined by

$$E(t) := \sum_{j=0}^{n} \left( E_j(t) + \sum_{i=1}^{n-j} E_{j,i}(t) \right). \quad (2.5)$$

The total dissipation $D(t)$ is denoted by

$$D(t) := \sum_{j=0}^{n} D_j(t). \quad (2.6)$$
In particular, one can derive from the verification of Lemma 3.7 in [40] that, for $n = 4 + [\alpha]$ and $E(t)$ defined in (2.3), it holds
\[
\sum_{j=0}^{3} \| \partial_t^j \zeta(y,t) \|_{L^\infty}^2 + \sum_{j=0}^{1} \| \partial_t^j \zeta_y(y,t) \|_{L^\infty}^2 + \sum_{0 \leq j + i \leq n} \| y^2 \sigma^{\max(0, \frac{2i-j}{2}-1)} \partial_t^i \partial_y^j \zeta(y,t) \|_{L^\infty}^2 \lesssim E(t), \quad y \in (r_0, R) \tag{2.7}
\]
provided that $E(t)$ is finite. In the derivation of (2.7), the following embedding of weighted Sobolev space play a crucial role:
\[
H^{a,b}(I) \hookrightarrow H^{b-a/2}(I), \quad \text{with} \quad \| F \|_{H^{b-a/2}} \leq C \| F \|_{H^{a,b}},
\]
where the weighted Sobolev space $H^{a,b}(I)$ is given by
\[
H^{a,b}(I) := \left\{ d^a(y) F \in L^2(I) : \int_I d^a(y) |\partial_y^k F|^2 dy < \infty, \quad 0 \leq k \leq b \right\},
\]
with the norm $\| F \|_{H^{a,b}(I)}^2 = \sum_{k=0}^{b} \int_I d^a(y) |\partial_y^k F|^2 dy$.

**A Priori Assumption.** Let $\eta(y,t) = y \zeta(y) + y$ be a solution to problem (1.16) satisfying the *a priori assumption*:
\[
E(t) \leq \epsilon_0^2, \quad t \in [0, T], \tag{2.9}
\]
for some suitably small fixed positive constant $\epsilon_0$ independent of $t$. Therefore, it follows from (2.7) and (2.9) that for $y \in (r_0, R), t \in [0, T],$
\[
\sum_{j=0}^{3} \| \partial_t^j \zeta(y,t) \|_{L^\infty}^2 + \sum_{j=0}^{1} \| \partial_t^j \zeta_y(y,t) \|_{L^\infty}^2 + \sum_{0 \leq j + i \leq 4 + \alpha} \| y^2 \sigma^{\max(0, \frac{2i-j}{2}-1)} \partial_t^i \partial_y^j \zeta(\cdot, t) \|_{L^\infty}^2 \lesssim E(t) \leq \epsilon_0^2. \tag{2.10}
\]

Notice that the coordinate singularity will not appear in this paper, due to (1.3c) and the inner solid core (cf. [25] [28]). Instead, we should pay attention to the boundary effect contacting the solid sphere. Now we state the main results as following.

**Theorem 2.1.** Suppose $\gamma > \frac{4}{3}$ in the barotropic gas pressure (1.4) and $r_0 < R \leq \frac{4}{3-\alpha} r_0 < \infty$ with $\alpha = \frac{1}{\gamma-1}$, where $R$ is the finite radius of the equilibria and $r_0$ is the radius of the solid core. There exists a positive constant $\epsilon \in (0, \epsilon_0]$ such that if $E(0) \leq \epsilon^2$, then problem (1.16) admits a global smooth solution in $(r_0, R) \times [0, \infty)$ satisfying
\[
E(t) \lesssim e^{-\delta t} E(0), \quad t \geq 0, \tag{2.11}
\]
for some suitably small positive constant $\delta$ independent of $t$.

**Remark 2.2.** As explained in last section, the radius $R$ is uniquely determined by the total mass $M = 4\pi \tilde{A} \int_{r_0}^{R} \left( \frac{1}{r} - \frac{1}{R} \right)^{(\frac{1}{\gamma}-1)} r^2 dr$ of the atmosphere surrounding planet. Hence, there always exists some given finite total mass $M > 0$ such that $r_0 < R \leq \frac{4}{3-\alpha} r_0 < \infty$.
holds. Here the upper bound of $R$ is merely a technical assumption in our paper, due to the jump discontinuity of the density $\rho(t, r)$ contacting the solid sphere (see elliptic estimates in section 3.1). When $\gamma > \frac{4}{3}$, it believes that Theorem 2.1 holds for any finite total mass $0 < M < +\infty$, i.e., $r_0 < R < +\infty$. Usually, we restrict $\gamma \in (1, 2)$ for isentropic polytropic gas, but Theorem 2.1 also holds for any given $\gamma \geq 2$ mathematically.

As a corollary of Theorem 2.1, we have the following convergence results of solutions to the original vacuum free boundary problem (1.3) concerning the vacuum free boundary $R(t)$, density $\rho$ and velocity $u$.

**Theorem 2.3.** Suppose $\gamma > \frac{4}{3}$ and $r_0 < R \leq \frac{4}{3-\alpha}r_0 < \infty$. There exists a positive constant $\epsilon \in (0, \epsilon_0]$ such that if $E(0) \leq \epsilon^2$, then problem (1.3) admits a global smooth solution $(\rho, u, R(t))$ for $t \in [0, \infty)$ satisfying

\[
|\rho(t, \eta(t, y)) - \bar{\rho}(y)| \lesssim \bar{A} \left( \frac{1}{y} - \frac{1}{R} \right)^{1/(\gamma-1)} \sqrt{e^{-\delta t}E(0)}, \tag{2.12}
\]

\[
|u(t, \eta(t, y))| + |R(t) - R| + \sum_{1 \leq i \leq 3} \left| \frac{d^i R(t)}{dt^i} \right| \lesssim \sqrt{e^{-\delta t}E(0)}, \tag{2.13}
\]

for all $y \in (r_0, R)$ and some suitably small positive constant $\delta$ independent of $t$, where $\bar{A}$ is a positive constant given in (1.7).

If the self-gravity of the atmosphere surrounding planet can not be neglected, then the spherically symmetric motion of the atmosphere is governed by the following system:

\[
\begin{align*}
\rho_t + (\rho u)_r + \frac{2}{r} \rho u_r &= 0 & \text{in } (r_0, R(t)), \tag{2.14a} \\
\rho (u_t + uu_r) + p_r &= -\rho u - \rho \frac{\dot{R}}{r^2} \left( g_0 + 4\pi G \int_{r_0}^r \rho(t, \tau) \tau^2 d\tau \right) & \text{in } (r_0, R(t)), \tag{2.14b} \\
\rho &> 0 & \text{in } [r_0, R(t)), \tag{2.14c} \\
\rho(R(t), t) &= 0, \ u(r_0, t) = 0, \tag{2.14d} \\
\dot{R}(t) &= u(R(t), t), \ R(0) = R_0, \tag{2.14e} \\
(\rho, u)(r, t = 0) &= (\rho_0, u_0)(r) & \text{on } (r_0, R_0), \tag{2.14f}
\end{align*}
\]

which is Euler-Poisson system with damping and solid core. Here $G$ is the gravitational constant for the atmosphere surrounding planet.

The equilibrium $(\bar{\rho}_*, 0)$ of (2.14) with $g_0 > 0$ satisfies

\[
A(\bar{\rho}_*)_r = -\frac{\bar{\rho}_*}{r^2} \left( g_0 + 4\pi G \int_{r_0}^r \bar{\rho}_*(t, \tau) \tau^2 d\tau \right), \tag{2.15}
\]

which has a unique solution if $\gamma \geq 4/3$, $g_0 > 0$ and the finite total mass of gas $M' = 4\pi \int_{r_0}^{R_G} \bar{\rho}_* r^2 dr > 0$ (cf. [13]). Here $R_G$ is the first zero of $\bar{\rho}_*(r)$. We assume $g_0 > 0$ for the case with solid core in this paper. However, if there is no solid core in (2.15), i.e. $g_0 = 0,$
then (2.15) is reduced to Lane-Emden equation, which also has a unique and compact supported solution provided that \( \gamma \in (4/3, 2) \) and \( 0 < M < +\infty \) (cf. [4, 14, 15, 19, 36]).

For system (2.14), similarly, we define the Lagrangian variable \( \eta(t, y) \) on the reference domain \( y \in (r_0, R_G) \) and obtain the following theorem.

**Theorem 2.4.** Suppose \( \gamma > \frac{4}{3} \) and \( r_0 < R_G \leq \frac{4}{3-\alpha}r_0 < \infty \). There exists a small spherically symmetric initial perturbation around the equilibrium \((\bar{\rho}_*, 0)\) defined in (2.15) such that the problem (2.14) admits a global smooth solution \((\rho, u, R(t))\) for \( t \in [0, \infty) \) satisfying

\[
|\rho(t, \eta(t, y)) - \bar{\rho}_*(y)| \lesssim (y - R_G)^{1/(\gamma-1)} e^{-\delta t} \mathcal{E}(0),
\]

\[
|u(t, \eta(t, y))| + |R(t) - R_G| + \sum_{1 \leq \iota \leq 3} \left| \frac{d\iota}{dt} R(t) \right| \lesssim e^{-\delta t} \mathcal{E}(0),
\]

for all \( y \in (r_0, R_G) \) and some suitably small positive constant \( \delta \) independent of \( t \), where \( \mathcal{E}(0) \) represents small initial perturbation around \((\bar{\rho}_*, 0)\) defined in the form (2.5).

**Remark 2.5.** We do not need the condition of smallness \( A \) assumed in [28] (which is for viscous case) in our setup. In [28], Matušů-Nečasová Š., Okada M. and Makino T. conjectured that the assumption \( \gamma > 4/3 \) can be removed for the asymptotic stability of the equilibrium for the spherically symmetric Navier-Stokes equations with solid core when the self-gravitation of gas is neglected. Similarly, relaxing the assumption \( \gamma > 4/3 \) in Theorem 2.1 to \( \gamma > 1 \) is also an unsolved problem. But, \( \gamma > 4/3 \) may be essential for the stability of the equilibrium of Euler-Poisson equations with damping and solid core (2.14) in Theorem 2.4 (cf. [12, 28]).

### 3. Proof of Theorem 2.1 – Theorem 2.4

The proof of Theorem 2.1 and Theorem 2.3 consists of elliptic estimates (controlling spatial derivatives by temporal derivatives) and weighted energy estimates (dealing with temporal derivatives). Indeed, it is shown this strategy works in dealing with the boundary effect near the solid core and singularity near the vacuum boundary in this paper. Theorem 2.4 can be proved similarly. The smallness of \( A \), which is assumed in nonlinear asymptotic stability of the equilibrium for the spherically symmetric motion of viscous gas surrounding a solid sphere (cf. [28]), is not required in this paper if the radius of the equilibrium is suitably larger than the radius of the inner solid ball.

#### 3.1. Elliptic Estimates

To begin with, we state the main result of this subsection as following.

**Proposition 3.1.** Suppose that (2.10) holds for a suitably small constant \( \epsilon_0 \in (0, 1) \). If \( \gamma > 4/3 \) and \( r_0 < R \leq \frac{1}{3-\alpha}r_0 < \infty \), then it holds that

\[
\mathcal{E}_{j,i}(t) \lesssim \sum_{\iota=0}^{i+j} \mathcal{E}_\iota(t), \quad \forall t \in [0, T],
\]
when \( j \geq 0, i \geq 0, i + j \leq n \). Moreover, the elliptic estimates (3.1) also hold for any \( \gamma \in (1, 4/3) \) and \( r_0 < R < \infty \).

The proof of the elliptic estimates in Proposition 3.1 consists of Lemma 3.2 and Lemma 3.3.

Note that \( \sigma_y(y) = -\bar{A}^{-1}/y^2, y \in (r_0, R) \). Dividing equation (1.18a) by \( \bar{\rho} \), we obtain

\[
y\zeta_t + y\zeta_x + \sigma(1 + \zeta)^2[(1 + \zeta)^{-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma}]_y + \frac{\gamma}{\gamma - 1}\sigma_y[(1 + \zeta)^{2-\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} - (1 + \zeta)^{-2}] = 0. \tag{3.2}
\]

Here we have

\[
(1 + \zeta)^2[(1 + \zeta)^{-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma}]_y = -\gamma(4\zeta_y + y\zeta_{yy}) + J_1,
\]

\[
(1 + \zeta)^{2-\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} - (1 + \zeta)^{-2}
\]

\[
= -\gamma y\zeta_y + (4 - 3\gamma)\zeta + J_2,
\]

where \( J_1, J_2 \) are given by

\[
J_1 := -2\gamma\zeta_y[(1 + \zeta)^{1-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} - 1]
\]

\[
- \gamma(2\zeta_y + y\zeta_{yy})[(1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma-1} - 1], \tag{3.3}
\]

\[
J_2 := (1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} - (1 + \zeta)^{-2}
\]

\[
+ \gamma y\zeta_y - (4 - 3\gamma)\zeta. \tag{3.4}
\]

This, together with the Taylor expansion and the smallness of \( \zeta \) and \( y\zeta_y \), implies that

\[
|J_1| \lesssim \epsilon_0(|y\zeta_{yy}| + |\zeta_y|), \tag{3.5}
\]

\[
|J_2| \lesssim \epsilon_0(|y\zeta_y| + |\zeta|). \tag{3.6}
\]

Then we can write (3.2) as following

\[
\gamma \left[ y\sigma_{yy} + 4\sigma\zeta_y + \frac{\gamma}{\gamma - 1}y\sigma_{y\zeta_y} \right] = y\zeta_t + y\zeta_x + \frac{\gamma(4 - 3\gamma)}{\gamma - 1}\sigma_y\zeta + \sigma J_1 + \frac{\gamma}{\gamma - 1}\sigma_y J_2. \tag{3.7}
\]

**Lemma 3.2** (Lower-Order Elliptic Estimates). Suppose that (2.10) holds for a suitably small constant \( \epsilon_0 \in (0, 1) \). If \( \gamma > 4/3 \) and \( r_0 < R \leq (1 + \frac{2}{3\alpha})r_0 < \infty \), then it holds that

\[
\mathcal{E}_{0,1}(t) \lesssim \sum_{i=0}^1 \mathcal{E}_i(t), \forall t \in [0, T]. \tag{3.8}
\]

**Proof.** Square the product of (3.7) and \( y\sigma^{\alpha/2} \) and then integrate the resulting formula with respect to spatial variable to obtain

\[
\left\| y^2\sigma^{1+\frac{\alpha}{2}}\zeta_{yy} + 4y\sigma^{1+\frac{\alpha}{2}}\zeta_y + (1 + \alpha)y^2\sigma^{\frac{\alpha}{2}}\sigma_y\zeta_y \right\|^2
\]
where the definition of \( \mathcal{E}_1 \) and the fact

\[
\bar{A}^{\gamma-1} / R^2 \lesssim |\sigma_y| \leq \bar{A}^{\gamma-1} / r_0^2
\]

have been used for the first inequality, and (3.5) (3.6) for the second inequality. The last term in (3.9) satisfies

\[
\|y \sigma_y^2 \zeta\|^2 = \int_{r_0}^{(R+r_0)/2} y^2 \sigma_y^2 \zeta^2 dy + \int_{(R+r_0)/2}^R y^2 \sigma_y^2 \zeta^2 dy
\]

\[
\lesssim \int_{r_0}^{(R+r_0)/2} y^2 \sigma_y^{1+\alpha} \zeta^2 dy + \int_{(R+r_0)/2}^R y^4 \sigma_y \zeta^2 dy \lesssim \mathcal{E}_0(t).
\]

Then it follows from (3.9) and (3.11) that

\[
\|y^2 \sigma_y^{1+\alpha} \zeta_{yy} + 4y \sigma_y^{1+\alpha} \zeta_y + (1+\alpha)y^2 \sigma_y^2 \sigma_y \zeta_y\|^2
\]

\[
\lesssim \frac{\bar{A}^{2(\gamma-1)}}{r_0^4} \mathcal{E}_0(t) + \mathcal{E}_1(t) + c_0^2 \left( \|y^2 \sigma_y^{1+\alpha} \zeta_{yy}\|^2 + \|y \sigma_y^{1+\alpha} \zeta_y\|^2 + \|y^2 \sigma_y^2 \sigma_y \zeta_y\|^2 \right). \tag{3.12}
\]

Now we analyze the left-hand side of (3.12), which reads

\[
\|y^2 \sigma_y^{1+\alpha} \zeta_{yy} + 4y \sigma_y^{1+\alpha} \zeta_y + (1+\alpha)y^2 \sigma_y^2 \sigma_y \zeta_y\|^2
\]

\[
= \|y^2 \sigma_y^{1+\alpha} \zeta_{yy}\|^2 + 16 \|y \sigma_y^{1+\alpha} \zeta_y\|^2 + (1+\alpha)^2 \|y^2 \sigma_y^2 \sigma_y \zeta_y\|^2
\]

\[
+ \int [4y^3 \sigma_y^{2+\alpha} + (1+\alpha)y^4 \sigma_y^{1+\alpha} \sigma_y] (\zeta^2_y) y dy + 8(1+\alpha) \int y^3 \sigma_y^{1+\alpha} \sigma_y \zeta^2_y dy. \tag{3.13}
\]

By virtue of \( \sigma_y(y) = -\bar{A}^{\gamma-1}/y^2, y \in (r_0, R) \), for the last term in (3.13), we obtain

\[
8(1+\alpha) \int y^3 \sigma_y^{1+\alpha} \sigma_y \zeta^2_y dy \lesssim \frac{\bar{A}^{\gamma-1}}{r_0^3} \int y^4 \sigma_y^{1+\alpha} \zeta^2_y dy \lesssim \frac{\bar{A}^{\gamma-1}}{r_0^3} \mathcal{E}_0(t). \tag{3.14}
\]

For the last second term in (3.13), with the aid of \( \bar{\rho}(R) = 0 \), integration by parts gives that

\[
\int [4y^3 \sigma_y^{2+\alpha} + (1+\alpha)y^4 \sigma_y^{1+\alpha} \sigma_y] (\zeta^2_y) y dy
\]

\[
= - \left\{ [4y^3 \sigma_y^{2+\alpha} + (1+\alpha)y^4 \sigma_y^{1+\alpha} \sigma_y] \zeta^2_y \right\}|_{y=r_0}
\]

\[
- \int [4y^3 \sigma_y^{2+\alpha} + (1+\alpha)y^4 \sigma_y^{1+\alpha} \sigma_y] \zeta^2_y dy
\]

\[
= : L_1 + L_2. \tag{3.15}
\]
If
\[ \gamma > \frac{4}{3} \quad (\text{i.e. } \alpha < 3) \quad \text{and} \quad r_0 < R \leq \frac{4}{3 - \alpha} r_0 = (1 + \frac{\gamma}{3\gamma - 4}) r_0, \]

then
\[ L_1 = -\sigma^{1+\alpha}(r_0)r_0^3A^{-1}\left[ \frac{3 - \alpha}{r_0} - \frac{4}{R} \right] \left( \zeta_y(r_0) \right)^2 \geq 0. \quad (3.16) \]

(When \( \alpha \geq 3 \), i.e. \( 1 < \gamma \leq 4/3 \), it also holds \( L_1 > 0 \). Here we only consider \( \gamma > 4/3 \), due to \( \gamma > 4/3 \) is essential for our energy estimates in next section.) Moreover, it holds
\[ L_2 \geq -12 \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 - (1 + \alpha)^2 \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 - C \left( A^{-1}, r_0^{-1} \right) E_0(t). \quad (3.17) \]

Then it follows from \((3.12)-(3.17)\) that
\[ \| y^2\sigma^{1+\frac{4}{3}} \zeta_{yy} \|^2 + 4 \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 \leq E_0 + E_1 + c_0^2 \left( \| y^2\sigma^{1+\frac{4}{3}} \zeta_{yy} \|^2 + \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 + \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 \right), \quad (3.18) \]

which, together with \((3.12)\), implies
\[ \| y^2\sigma^{1+\frac{4}{3}} \zeta_{yy} \|^2 + \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 + \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 \leq E_0 + E_1 + c_0^2 \left( \| y^2\sigma^{1+\frac{4}{3}} \zeta_{yy} \|^2 + \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 + \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 \right). \quad (3.19) \]

This, with the aid of the smallness of \( c_0 \), implies
\[ \| y^2\sigma^{1+\frac{4}{3}} \zeta_{yy} \|^2 + \| y\sigma^{1+\frac{4}{3}} \zeta_y \|^2 + \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 \leq E_0 + E_1. \quad (3.20) \]

Due to \( \sigma_y = -A^{-1}/y^2 \), \( y \in (r_0, R) \), we easily obtain
\[ \| y\sigma^{\frac{4}{3}} \zeta_y \|^2 \leq R^2 \| \sigma^{\frac{4}{3}} \zeta_y \|^2 \leq \frac{R^2}{A^{2\gamma-2}} \| y^2\sigma^{\frac{4}{3}} \sigma_y \zeta_y \|^2 \leq E_0 + E_1. \quad (3.21) \]

Therefore, \((3.8)\) follows from \((3.20)\) and \((3.21)\) and the definition of \( E_{0,1} \). The proof of this lemma is completed.

Next, we process the higher-order elliptic estimates. Let us denote \( C^j_m \) as the binomial coefficients:
\[ C^j_m = \frac{m!}{j!(m-j)!}, \quad 0 \leq j \leq m. \]

For \( i \geq 1 \) and \( j \geq 0 \), we act \( \partial^j_t \partial_y^{i-1} \) on \((3.7)\) to obtain that
\[ \gamma \left[ y\sigma \partial^i_t \partial_y^{i+1} \zeta + (i + 3)\sigma \partial^j_t \partial_y^j \zeta + (\alpha + i)y \sigma_y \partial^i_t \partial_y^j \zeta \right] = y \partial^{i+2} \partial_y^j \zeta + y \partial^i_t \partial_y^{i+1} \zeta + Q_1 + Q_2, \quad (3.22) \]
where
\[ Q_1 := -\gamma \partial^j_t \left[ \sum_{i=2}^{i-1} C^i_{i-1} \partial_y (y\sigma) \partial_y^{i+1-i} \zeta + 4 \sum_{i=1}^{i-1} C^i_{i-1} \partial_y^i \sigma \partial_y^{i-1} \zeta \right] \]
Indeed, (3.26) holds immediately for any $\gamma > 1$ and $i = 1$ and $i = 1, 2$, respectively.

Multiply the above equation (3.22) by $y \sigma^{\alpha + i + 1/2}$, and then integrate the quadratic form of the product with respect to spatial variable to obtain

$$
\|y^2 \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta + (i + 3) y \sigma^{\alpha + i + 1/2} \partial_{y}^{i} \partial_{y}^{j} \zeta + (\alpha + i) y^2 \sigma^{\alpha + i + 1} \sigma \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2}
\lesssim \|y^2 \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2} + \|y^2 \sigma^{\alpha + i + 1} \sigma \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2}
+ \|y \sigma^{\alpha + i + 1/2} Q_1\|^{-2} + \|y \sigma^{\alpha + i + 1} Q_2\|^{-2}.
$$

(3.25)

Similar to the derivation of (3.13)-(3.18), we obtain that the left-hand side of (3.25) satisfies

$$
\|y^2 \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta + (i + 3) y \sigma^{\alpha + i + 1/2} \partial_{y}^{i} \partial_{y}^{j} \zeta + (\alpha + i) y^2 \sigma^{\alpha + i + 1} \sigma \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2}
\gtrsim \mathcal{E}_{j,i}(t) - C \|y^2 \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2},
$$

(3.26)

if $\gamma > 4/3$ and $R \leq \frac{3 + i}{3 - \alpha} r_0 = \left(1 + \frac{i(\gamma - 1) + 1}{3\gamma - 4}\right) r_0$.

Indeed, (3.20) holds immediately for any $\gamma \in (1, 4/3]$ and $r_0 < R < \infty$. Then, for $r_0 < R \leq \frac{3 + i}{3 - \alpha} r_0, i = 1, 2, \ldots, n$, and $\gamma > 4/3$, we have

$$
\mathcal{E}_{j,i}(t) \lesssim \|y^2 \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2} + \|y^2 \sigma^{\alpha + i + 1} \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2}
+ \|y \sigma^{\alpha + i + 1/2} \partial_{y}^{i+1} \partial_{y}^{j} \zeta\|^{-2} + \|y \sigma^{\alpha + i + 1} Q_1\|^{-2} + \|y \sigma^{\alpha + i + 1} Q_2\|^{-2}.
$$

(3.27)

To end the proof of the Proposition 3.1 we will prove the following higher-order elliptic estimates.

**Lemma 3.3 (Higher-Order Elliptic Estimates).** Suppose that (2.10) holds for a suitably small constant $\epsilon_0 \in (0, 1)$. If $\gamma > 4/3$ and $r_0 < R \leq \frac{4}{3 - \alpha} r_0 < \infty$, then, for $j \geq 0, i \geq 0$ and $1 \leq i + j \leq n$, it holds that

$$
\mathcal{E}_{j,i}(t) \lesssim \sum_{i=0}^{i+j} \mathcal{E}_{i}(t), \quad \forall t \in [0, T].
$$

(3.28)

**Proof.** Inspired by the elliptic estimates in [24, 10], we plan to prove this lemma by mathematical induction. First, we notice that

$$
\mathcal{E}_{j,0}(t) = \int \left[y^2 \sigma^{\alpha - 1}(\partial_{t}^i \zeta)^2 + y^4 \sigma^{\alpha - 1}(\partial_{y}^i \partial_{y} \zeta)^2\right] (y, t) dy
$$
\[
\lesssim \int_{r_0}^{(r_0+R)/2} y^{4\sigma^\alpha} (\partial_t^i \zeta)^2 dy + \int_{(r_0+R)/2}^R y^{4\sigma^{\alpha+1}} (\partial_t^j \zeta_y)^2 dy + \mathcal{E}_j(t)
\]

\[
\lesssim \mathcal{E}_j(t), \quad j \geq 0.
\]

(3.29)

It has been verified in Lemma 3.2 that (3.28) holds for \(i + j = 1\). Then, for \(1 \leq k \leq n - 1\), we make the induction hypothesis that (3.28) holds for all \(j \geq 0, i \geq 0\) and \(i + j \leq k\), i.e.

\[
\mathcal{E}_{j,i}(t) \lesssim \sum_{i=0}^{i+j} \mathcal{E}_i(t), \quad j \geq 0, \quad i \geq 0, \quad i + j \leq k.
\]

(3.30)

Then it suffices to prove (3.28) for \(j \geq 0, i \geq 0\) and \(i + j = k + 1\). That is, we only need to prove

\[
\mathcal{E}_{k+1-i,i}(t) \lesssim \sum_{i=0}^{k+1} \mathcal{E}_i(t)
\]

from \(i = 0\) to \(i = k + 1\) step by step.

In what follows, we naturally assume \(j \geq 0, i \geq 1\) and \(i + j = k + 1 \leq n\) (for the case \(i = 0\), it has been shown in (3.29)). In view of the last two terms in (3.27), we have to estimate \(Q_1\) and \(Q_2\) to achieve our goal. For \(Q_1\), when \(i = 1\), we have \(Q_1 = 0\). When \(i \geq 2\), it follows from (3.10) that

\[
|Q_1| \lesssim \sum_{i=1}^{i-1} |\partial_t^i \partial_y^j \zeta| + (i - 1)(|\partial_t^{i+2} \partial_y^{-2} \zeta| + |\partial_t^{i+1} \partial_y^{-2} \zeta|).
\]

This implies that

\[
\|y^{\sigma^{\alpha+i-1}} Q_1\|^2 \lesssim \sum_{i=1}^{i-1} \|y^{\sigma^{\alpha+i-1}} \partial_t^i \partial_y^j \zeta\|^2 + (i - 1)^2 \sum_{i=j+1}^{j+2} \|y^{\sigma^{\alpha+i-1}} \partial_t^i \partial_y^{-2} \zeta\|^2. \tag{3.31}
\]

Hence we have

\[
\|y^{\sigma^{\alpha+i-1}} Q_1\|^2 \lesssim \sum_{i=1}^{i-1} \mathcal{E}_{j,i}(t) + \sum_{i=j+1}^{j+2} \mathcal{E}_{i,i-2}(t), \quad i \geq 2. \tag{3.32}
\]

For \(Q_2\), it follows from (2.10), (3.3), (3.4) and (3.10) that

\[
|Q_2| \lesssim \varepsilon_0 \sum_{i=0}^{i-1} \left( |\partial_t^{i} \partial_y^{i-1-i} (y^{\sigma} \zeta_y)| + |\partial_t^{i} \partial_y^{i-1-i} (\sigma_y)| \right)
\]

\[
\quad + |\partial_t^{i} \partial_y^{i-1-i} (y^{\sigma} \zeta_y)| + |\partial_t^{i} \partial_y^{i-1-i} (\sigma_y)| \right)
\]

\[
\lesssim \varepsilon_0 \sum_{i=0}^{i-1} \left( |y^{\sigma} \partial_t^{i} \partial_y^{i-1-i} \zeta| + \sum_{m=0}^{i-i} |\partial_t^{i} \partial_y^{i} \zeta| \right), \tag{3.33}
\]

which implies that

\[
\|y^{\sigma^{\alpha+i-1}} Q_2\|^2 \lesssim \varepsilon_0^2 \left( \mathcal{E}_{j,i}(t) + \sum_{i=0}^{i-1} \mathcal{E}_{j,i} \right). \tag{3.34}
\]
Here the fact $(n - [n/2] - \alpha)/2 > 1/2$ for $\gamma > 4/3$ have been used. So, for suitably small $\epsilon_0$, we deduce from (3.27) (3.32) (3.34) that
\[
\mathcal{E}_{j,i}(t) \lesssim \mathcal{E}_j(t) + \mathcal{E}_{j+1}(t), \quad i = 1,
\] (3.35)
and
\[
\mathcal{E}_{j,i}(t) \lesssim \mathcal{E}_{j,i-1}(t) + \mathcal{E}_{j+2,i-2}(t) + \mathcal{E}_{j+1,i-2}(t) + \sum_{\iota=0}^{i-1} \mathcal{E}_{j,\iota}(t), \quad i \geq 2.
\] (3.36)

With (3.35) and (3.36) in hand, together with the induction hypothesis (3.30), we are ready to show (3.28) holds for $j + i = k + 1$.

We first choose $j = k$ and $i = 1$. Then it follows from (3.35) that
\[
\mathcal{E}_{k,1}(t) \lesssim \mathcal{E}_k(t) + \mathcal{E}_{k+1}(t).
\] (3.37)

If we choose $j = k - 1$ and $i = 2$ in (3.36), with the aid of (3.29) and (3.30), we obtain
\[
\mathcal{E}_{k-1,2}(t) \lesssim \mathcal{E}_{k-1,1}(t) + \mathcal{E}_{k+1,0}(t) + \mathcal{E}_{k,0}(t) + \mathcal{E}_{k-1,0}(t) \lesssim \sum_{\iota=0}^{k+1} \mathcal{E}_{\iota}(t).
\] (3.38)

For other cases, $j = k - 2$ and $i = 3$, $j = k - 3$ and $i = 4$, and other pairs $(j, i)$ satisfying $j + i = k + 1$, with the help of (3.37) and (3.38), we can handle them similarly. Now we finish the proof of Lemma 3.3.

\section*{3.2. Basic Energy Estimate.}

\textbf{Lemma 3.4.} Assume that (2.10) holds for suitably small $\epsilon_0 \in (0, 1)$. If $\gamma > 4/3$, then it holds
\[
e^\delta t \mathcal{E}_0(t) + \int_0^t e^\delta s \mathcal{D}_0(s) ds \leq \mathcal{E}_0(0), \quad \forall t \in [0, T],
\] (3.39)
for some suitably small $\delta > 0$.

\textbf{Proof.} We integrate the product of (1.18) and $y^3 \zeta_t$ with respective to the space variable, by virtue of integration by parts, to obtain
\[
\frac{d}{dt} \int \frac{1}{2} y^4 \bar{\rho} \zeta_t^2 dy + \int y^4 \bar{\rho} \zeta_t^2 dy + \int \bar{\rho}^\gamma H_1 dy = 0,
\] (3.40)
where
\[
H_1 = -(1 + \zeta)^{-2\gamma} (1 + \zeta + y \zeta_y)^{-\gamma} \left( (1 + \zeta)^2 y^3 \zeta_t \right)_y + (1 + \zeta)^{-2} y^3 \zeta_t.
\]
Indeed, using a direct calculation, we have the following equality:
\[
\int \bar{\rho}^\gamma H_1 dy = \frac{d}{dt} \int y^2 \bar{\rho}^\gamma E_0 dy,
\] (3.41)
where
\[ E_0(y, t) = \frac{1}{\gamma - 1} [(1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{1-\gamma} - \frac{3(\gamma - 1)(1 + \zeta)^{-1} + (\gamma - 1)(1 + \zeta)^{-2}y\zeta_y + (3\gamma - 4)]]. \]

Then we can rewrite (3.40) as following
\[
\frac{d}{dt} \int \left[ \frac{1}{2} y^4 \bar{\rho} \zeta_t^2 + y^2 \bar{\rho}^\gamma E_0(y, t) \right] dy + \int y^4 \bar{\rho} \zeta_t^2 dy = 0. \tag{3.42}
\]

Taking advantage of the smallness of \( \zeta \) and \( y\zeta_y \), it follows from Taylor expansion that
\[
E_0 = \left[ \left( \frac{9}{2} \gamma - 6 \right) \zeta^2 + (3\gamma - 4)\zeta y\zeta_y + \frac{\gamma}{2} y\zeta_y^2 \right] + O(1)(|\zeta| + |y\zeta_y|)(\zeta^2 + y^2\zeta_y^2).
\]

Hence, if \( \gamma > 4/3 \), we have
\[
E_0 \sim (\zeta^2 + (y\zeta_y)^2) \tag{3.43}
\]
for suitably small \( \zeta \) and \( y\zeta_y \).

Multiplying (1.18a) by \( y^3\zeta \), and then integrating the product with respect to the space variable, we apply integration by parts to obtain
\[
\frac{1}{2} \frac{d}{dt} \int (2y^4 \bar{\rho} \zeta_t \zeta + y^4 \bar{\rho} ^\gamma) dy + \int \bar{\rho} ^\gamma H_2 dy = \int y^4 \bar{\rho} \zeta_t^2 dy, \tag{3.44}
\]
where
\[
H_2 := (1 + \zeta)^{-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma}((1 + \zeta)^2 y^3\zeta_y - (1 + \zeta)^{-2}y^3\zeta_y),
\]
due to \( \zeta(r_0, t) = 0, \sigma(R) = 0 \). It follows from Taylor expansion and the smallness of \( \zeta \) and \( y\zeta_y \) that
\[
H_2 \geq y^2[(9\gamma - 12)\zeta^2 + (6\gamma - 8)y\zeta_y \zeta + \gamma(y\zeta_y)^2] - C\epsilon_0 y^2(\zeta^2 + y^2\zeta_y^2)
\]
\[
\geq y^2(\zeta^2 + y^2\zeta_y^2), \tag{3.45}
\]
where the second inequality follows from \( \gamma > 4/3 \) and the smallness of \( \epsilon_0 \).

Calculate (3.44) + 2 × (3.42) to obtain
\[
\frac{d}{dt} \mathcal{E}_0(t) + \mathcal{D}_0(t) = 0, \tag{3.46}
\]
where
\[
\mathcal{E}_0(t) = \int (y^4 \bar{\rho} \zeta_t \zeta + \frac{1}{2} y^4 \bar{\rho} \zeta_t^2) dy + \int [y^4 \bar{\rho} \zeta_t^2 + 2y^2 \bar{\rho} ^\gamma E_0(y, t)] dy, \]
\[
\mathcal{D}_0(t) = \int (y^4 \bar{\rho} \zeta_t^2 + \bar{\rho} ^\gamma H_2) dy.
\]
Clearly, if \( \gamma > 4/3 \) and \( \epsilon_0 \) is suitably small, it follows from (3.43) and (3.45) that
\[
\mathcal{E}_0 \sim \mathcal{E}_0, \quad \mathcal{D}_0 \sim \mathcal{D}_0, \tag{3.47}
\]
and
\[ E_0(t) \lesssim D_0(t), \]  
(3.48)
due to
\[
\int y^4 \sigma^\alpha \zeta^2 dy \lesssim \int_{r_0}^{(r_0+R)/2} y^2 \sigma^{\alpha+1} \zeta^2 dy + \int_{(r_0+R)/2}^R \sigma^\alpha \zeta^2 dy \\
\lesssim D_0 + \int_{(r_0+R)/2}^R \sigma^{\alpha+2} (\zeta^2 + \zeta_y^2) dy \\
\lesssim D_0 + \int_{(r_0+R)/2}^R y^2 \sigma^{\alpha+2} (\zeta^2 + (y\zeta_y)^2) dy \\
\lesssim D_0.
\]

Then, with the aid of (3.47) and (3.48), we integrate the product of $e^{\delta t}$ and (3.46) with respect to time variable to obtain (3.39) for suitably small $\delta$. Hence we complete the proof of Lemma 3.4.

3.3. Higher-Order Energy Estimates. The equation (1.18a) can be wrote as
\[
y\bar{\rho}\zeta_{tt} + y\bar{\rho}\zeta_t + \{\bar{\rho}^\gamma[(1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} - (1 + \zeta)^{-2}]\}_y \\
- 2\bar{\rho}^\gamma[(1 + \zeta)^{1-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} + (1 + \zeta)^{-3}]\zeta_y = 0. \tag{3.49}
\]
The linearized equation of (3.49) reads
\[
y\bar{\rho}\zeta_{tt} + y\bar{\rho}\zeta_t + \{\bar{\rho}^\gamma[(4 - 3\gamma)\zeta - \gamma y\zeta_y]\}_y - 4\bar{\rho}^\gamma\zeta_y = 0.
\]
Let $j \geq 1$ and take the $j$-th time derivatives of the equation (3.49) to obtain
\[
y\bar{\rho}\partial_t^j \zeta_{tt} + y\bar{\rho}\partial_t^j \zeta_t + \{\bar{\rho}^\gamma \left( h_1 \partial_t^j \zeta + h_2 y\partial_t^j \zeta_y + R_1 \right) \}_y \\
+ \bar{\rho}^\gamma \left[ (3h_2 - h_1) \partial_t^j \zeta_y + R_2 \right] - 2\bar{\rho}^\gamma \left( h_3 \zeta_y \partial_t^j \zeta + R_3 \right) = 0, \tag{3.50}
\]
where
\[
h_1 = (2 - 2\gamma)(1 + \zeta)^{1-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} \\
- \gamma(1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma-1} + 2(1 + \zeta)^{-3},
\]
\[
h_2 = - \gamma(1 + \zeta)^{2-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma-1},
\]
\[
h_3 = (1 - 2\gamma)(1 + \zeta)^{-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma} \\
- \gamma(1 + \zeta)^{1-2\gamma}(1 + \zeta + y\zeta_y)^{-\gamma-1} - 3(1 + \zeta)^{-4},
\]
and
\[
R_1 = \partial_t^{j-1}(h_1\zeta_t + h_2 y\zeta_y) - (h_1 \partial_t^j \zeta + h_2 y\zeta \partial_t^j \zeta_y),
\]
\[
R_2 = \partial_t^{j-1}(3h_2 - h_1)\zeta_y] - (3h_2 - h_1) \partial_t^j \zeta_y,
\]
\[ R_3 = \partial_t^{j-1}(h_3 \zeta_y \zeta_t) - h_3 \zeta_y \partial_t^j \zeta. \]

Here \( R_1, R_2 \) and \( R_3 \) refer to lower-order terms involving \( \partial_t^j(\zeta, y\zeta_y) \) with \( j = 0, 1, \cdots, j - 1 \).

Taking advantage of the smallness of \( \zeta \) for \( h=1 \), \( \cdots, j - 1 \), we integrate the product of (3.50) and \( \partial_t^j(\zeta, y\zeta_y) \) with the aid of \( \zeta_1, \zeta_2, \cdots, \zeta_j \), to obtain

\[ h_1 = (4 - 3\gamma) + (9\gamma^2 - 9\gamma - 4)\zeta + \gamma(3\gamma - 1)y\zeta_y + \bar{h}_1, \]

\[ h_2 = -\gamma + \gamma(3\gamma - 1)\zeta + \gamma(\gamma + 1)y\zeta_y + \bar{h}_2, \]

\[ h_3 = -2 - 3\gamma + \bar{h}_3, \]  

(3.51)

where \( \bar{h}_i, i = 1, 2, 3, \) satisfy

\[ |\bar{h}_1| + |\bar{h}_2| \lesssim (\zeta^2 + (y\zeta_y)^2), \quad |\bar{h}_3| \lesssim (|\zeta| + |y\zeta_y|). \]  

(3.52)

**Lemma 3.5.** Assume that (2.10) holds for suitably small \( \epsilon_0 \in (0, 1) \). If \( \gamma > 4/3 \), then, for \( 1 \leq j \leq n \), it holds

\[ e^{\delta t}E_j(t) + \int_0^t e^{\delta s}D_j(s)ds \lesssim E(0), \quad \forall t \in [0, T]. \]  

(3.53)

**Proof.** We integrate the product of (3.50) and \( y^2\partial_t^j \zeta_t \) with respect to the spatial variable, with the aid of \( \partial_t^j(\zeta(r_0, t) = 0) \), to obtain

\[ \frac{d}{dt} \left[ \int \left( \frac{1}{2} y^4 \tilde{\rho} \left( \partial_t^j \zeta_t \right)^2 + y^2 \tilde{\rho}^7 E_j \right) dy + \tilde{R}_j^{-1} \right] + \int y^4 \tilde{\rho} \left( \partial_t^j \zeta_t \right)^2 dy = F_1^j + F_2^j, \]  

(3.54)

where

\[ E_j = -\frac{1}{2} \left[ (3h_1 + 2h_3y\zeta_y) \left( \partial_t^j \zeta \right)^2 + 2h_1(\partial_t^j \zeta_y) (y\partial_t^j \zeta_y) + h_2 (y\partial_t^j \zeta_y)^2 \right], \]

\[ \tilde{R}_j^{-1} = -\int y^2 \tilde{\rho} \left[ (3R_1 \partial_t^j \zeta + R_1 y\partial_t^j \zeta_y) - y(R_2 - 2R_3) \partial_t^j \zeta \right] dy, \]

\[ F_1^j = -\int \frac{1}{2} y^4 \tilde{\rho} \left[ (3h_1 + 2h_3y\zeta_y) \left( \partial_t^j \zeta \right)^2 + 2h_1(y\partial_t^j \zeta_y) \partial_t^j \zeta + h_2(y\partial_t^j \zeta_y)^2 \right] dy, \]

\[ F_2^j = -\int \tilde{\rho} y^2 \left[ 3R_1 \partial_t^j \zeta + R_1 y\partial_t^j \zeta_y - (R_2 - 2R_3) \partial_t^j \zeta \right] dy. \]

In particular, \( \tilde{R}_j^{-1} = 0 \) when \( j = 1 \), since \( R_1 = R_2 = R_3 = 0 \) when \( j = 1 \).

Similarly, we integrate the product of (3.50) and \( y^2\partial_t^j \zeta \) with respect to spatial variable to get

\[ \frac{d}{dt} \int y^4 \tilde{\rho} \left( \partial_t^j \zeta \partial_t^j \zeta_t + \frac{1}{2} \left( \partial_t^j \zeta \right)^2 \right) dy - \int y^4 \tilde{\rho} \left( \partial_t^j \zeta_t \right)^2 dy + \int y^2 \tilde{\rho}^7 D_j dy + \tilde{R}_j^{-1} = 0, \]  

(3.55)

where

\[ D_j = -(h_1 \partial_t^j \zeta + h_2 y\partial_t^j \zeta_y)(3\partial_t^j \zeta + y\partial_t^j \zeta_y) \]

\[ + (3h_2 - h_1)(y\partial_t^j \zeta_y) \partial_t^j \zeta - 2h_3(y\zeta_y) \left( \partial_t^j \zeta \right)^2. \]  

(3.56)
Here $D_j$ satisfies that
\[
D_j \geq 3(3\gamma - 4)(\partial_i^j \zeta)^2 + 2(3\gamma - 4)\partial_i^j \zeta \left( y\partial_i^j \zeta_y \right) + \gamma \left( y\partial_i^j \zeta_y \right)^2
- C(|\zeta| + |y\zeta_y|) \left[ (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right]
\geq \left[ (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right], \quad \text{if } \gamma > \frac{4}{3},
\] (3.57)
and
\[
D_j \lesssim \left[ (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right],
\] (3.58)
due to (3.51) (3.52) (2.10) and suitably small $\epsilon_0$.

Then, we calculate (3.55) $+ 2 \times$ (3.54) to obtain
\[
\frac{d}{dt} \left( \mathcal{E}_j + 2\tilde{R}_j^{i-1} \right) + \mathcal{D}_j = 2(F_j^1 + F_j^2) - \tilde{R}_j^{i-1},
\] (3.59)
where
\[
\mathcal{E}_j = \int \left\{ y^4 \bar{\rho} \left[ (\partial_i^j \zeta)^2 + \partial_i^j \zeta \partial_i^j \zeta_t + \frac{1}{2} \partial_i^j \zeta)^2 \right] + 2y^2 \bar{\rho}^\gamma E_j \right\} dy,
\] (3.60)
\[
\mathcal{D}_j = \int y^4 \bar{\rho} \left( \partial_i^j \zeta_t \right)^2 dy + \int y^2 \bar{\rho}^\gamma D^2 dy.
\] (3.61)

It follows from Taylor expansion and the smallness of $\zeta$ and $y\zeta_y$ that
\[
E_j = \frac{1}{2} \left[ 3(3\gamma - 4) \left( \partial_i^j \zeta \right)^2 + 2(3\gamma - 4)\partial_i^j \zeta \left( y\partial_i^j \zeta_y \right) + \gamma \left( y\partial_i^j \zeta_y \right)^2 \right]
+ O(1)(|\zeta| + |y\zeta_y|) \left( (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right)
\sim \left( (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right).
\] (3.62)

This, together with (3.57) (3.58) (3.60) and (3.61), implies that
\[
\mathcal{E}_j(t) \sim \mathcal{E}_j(t), \quad \text{and} \quad \mathcal{D}_j(t) \sim \mathcal{D}_j(t),
\] (3.63)
if $\gamma > 4/3$ and $\epsilon_0$ is suitably small.

Furthermore, similar to derive (3.48), it follows from the definition of $\mathcal{E}_j$, $\mathcal{D}_j$ and $\mathcal{D}$ that
\[
\mathcal{E}_j(t) \lesssim \mathcal{D}_j(t),
\] (3.64)
\[
\sum_{j=0}^{n} \mathcal{E}_j \lesssim \int y^4 \sigma^\alpha \zeta^2 dy + \mathcal{D}(t) \lesssim \mathcal{D}(t).
\] (3.65)

Thus, we can deduce from (3.65) and (3.1) and the definition of $\mathcal{E}(t)$ that
\[
\mathcal{E}(t) \lesssim \mathcal{D}(t).
\] (3.66)

Notice that
\[
|F_j^i| \lesssim \int y^2 \bar{\rho}^\gamma \left[ (|\zeta| + |y\zeta_y|) + (|\zeta| + |y\zeta_y|)(|\zeta_t| + |y\zeta_y|) \right] \left( (\partial_i^j \zeta)^2 + (y\partial_i^j \zeta_y)^2 \right) dy
\]
\[ \leq \epsilon_0 \int y^2 \bar{\rho} \left( (\partial_t \zeta)^2 + (y \partial_t \zeta_y)^2 \right) dy \leq \epsilon_0 \mathcal{E}_j, \] (3.67)

where Taylor expansion and (2.10) have been used.

**Claim:** for \( F_j^2 \) and \( \tilde{R}^{j-1} \), \( j \geq 1 \), it holds that

\[ |F_j^2| \lesssim \epsilon_0 \mathcal{E}_j + \epsilon_0 \sum_{i=0}^{j-1} \mathcal{E}_i, \] (3.68)

\[ |\tilde{R}^{j-1}| \lesssim \epsilon_0 \mathcal{E}_j + \epsilon_0 \sum_{i=0}^{j-1} \mathcal{E}_i, \] (3.69)

where the a priori estimate (2.10) has been used. We will prove this claim in the Appendix.

Once we have (3.68) and (3.69), together with (3.67), we multiply (3.59) by \( e^{\delta t} \) to obtain

\[
\frac{d}{dt} \left( e^{\delta t} (\mathcal{E}_j + 2 \tilde{R}^{j-1}) \right) - \delta e^{\delta t} (\mathcal{E}_j + 2 \tilde{R}^{j-1}) + e^{\delta t} D_j \\
\lesssim \epsilon_0 e^{\delta t} \mathcal{E}_j + \epsilon_0 e^{\delta t} \sum_{i=0}^{j-1} \mathcal{E}_i.
\] (3.70)

When \( j = 1 \), one has

\[
\frac{d}{dt} (e^{\delta t} \mathcal{E}_1) - \delta e^{\delta t} \mathcal{E}_1 + e^{\delta t} D_1 \lesssim \epsilon_0 e^{\delta t} \mathcal{E}_1 + \epsilon_0 e^{\delta t} \mathcal{E}_0,
\] (3.71)

which, together with (3.39) (3.48) (3.63), implies

\[
e^{\delta t} \mathcal{E}_1 + \int_0^t e^{\delta s} D_1 ds \lesssim \mathcal{E}_1(0),
\] (3.72)

for suitably small \( \delta \) and \( \epsilon_0 \).

When \( j = 2 \), it follows from (3.63) (3.64) (3.70) (3.68) and (3.69) that, for suitably small \( \delta \) and \( \epsilon_0 \),

\[
e^{\delta t} \mathcal{E}_2(t) + \int_0^t e^{\delta s} D_2(s) ds \\
\lesssim e^{\delta t} \sum_{i=0}^{1} \mathcal{E}_i(t) + \int_0^t \left( e^{\delta s} \sum_{i=0}^{1} \mathcal{E}_i(s) \right) ds \lesssim \mathcal{E}(0),
\] (3.73)

where (3.39) (3.64) and (3.72) have been used for the second inequality in (3.73). Similarly, for other cases \( k \geq 2 \), using bootstrap argument, one has

\[
e^{\delta t} \mathcal{E}_k(t) + \int_0^t e^{\delta s} D_k(s) ds \\
\lesssim e^{\delta t} \sum_{i=0}^{k-1} \mathcal{E}_i(t) + \int_0^t \left( e^{\delta s} \sum_{i=0}^{k-1} \mathcal{E}_i(s) \right) ds \lesssim \mathcal{E}(0),
\] (3.74)

for suitably small \( \delta \) and \( \epsilon_0 \). Thus we can conclude (3.53) by virtue of (3.39) and the definition of \( D_j(t) \) and \( \mathcal{E}_j(t) \). Now the proof of Lemma 3.5 is completed.

\[ \square \]
Proof of Theorem 2.1 and Theorem 2.3. It follows from (3.39) (3.53) and the elliptic estimates (3.1) and (3.65) (3.66) that
\[ e^{\delta t} \mathcal{E}(t) + \int_0^t e^{\delta s} \mathcal{D}(s) ds \lesssim \mathcal{E}(0), \quad \forall t \in [0, T], \] (3.75)
if \( \gamma > 4/3 \) and \( r_0 < R \leq \frac{4}{3-\alpha} r_0 < \infty \). Then we obtain (2.11) and finish the proof of Theorem 2.1 by a standard global existence argument.

Note that
\[
\rho(t, \eta(t, y)) - \bar{\rho}(y) = \frac{\bar{\rho}(y)}{(1 + \zeta)^2(1 + \zeta + y\zeta_y)} - \bar{\rho}(y) \\
= -\bar{\rho}(y) \frac{(\zeta + y\zeta_y) + \zeta(2 + \zeta)(1 + \zeta + y\zeta_y)}{(1 + \zeta)^2(1 + \zeta + y\zeta_y)}, \\
u(t, \eta(t, y)) = \eta_t(t, y) = y\zeta_t, \quad R(t) - R = R\zeta(R, t).
\]
This, together with (3.75) (2.7) and the fact \( H^{1/2+\varepsilon}(I) \hookrightarrow L^\infty(I) \) for \( \varepsilon > 0 \), implies (2.16) and (2.17). Then the proof of Theorem 2.3 is completed.

Proof of Theorem 2.4. Here we note that the equilibria equation (2.15) has a unique solution only if \( \gamma \geq 4/3 \). There are multiple solutions to equation (2.15) with \( 1 < \gamma < 4/3 \), which has been shown in [12]. The uniqueness of the equilibrium for Euler-Poisson equations with damping and solid core requires \( \gamma \geq 4/3 \), compared with that \( \gamma > 1 \) for Euler equations with damping and solid core.

Similar to the strategy in the proof of Theorem 2.1 and Theorem 2.3, by means of elliptic estimates, weighted temporal energy estimates and weighted Sobolev embedding theory, one can prove the global existence of smooth solution and stability property to Euler-Poisson equations with damping and solid core (2.14) for any given \( \gamma > 4/3 \) provided that the initial spherically symmetric perturbation around the equilibrium (2.15) is sufficiently small. We omit the repeat for simplicity.

Acknowledgements. This research was supported in part by China Postdoctoral Science Foundation grant 2021M691818; Natural Science Foundation of China (NSFC) Grants 12101350, 12171267, 12271284. The author is deeply grateful to the referee for his/her helpful suggestions which have helped to clarify some important points and improved the quality of the paper greatly.

Data Availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
**APPENDIX**

*Claim:* For $F_2^j$ and $\tilde{R}^{j-1}$, $j \geq 1$, it holds that

$$|F_2^j| \lesssim \epsilon_0 \mathcal{E}_j + \epsilon_0 \sum_{l=0}^{j-1} \mathcal{E}_l, \quad (3.76)$$

$$|\tilde{R}^{j-1}| \lesssim \epsilon_0 \mathcal{E}_j + \epsilon_0 \sum_{l=0}^{j-1} \mathcal{E}_l, \quad (3.77)$$

for a suitably small positive constant $\epsilon_0$.

*Proof of the Claim.* If $j = 1$, then $R_1 = R_2 = R_3 = \tilde{R}^0 = 0$, it is trivial.

For $F_2^j, 6 \geq n \geq j \geq 2$ (since $\alpha < 3$ for $\gamma > 4/3$), note that

$$R_{1t} = (j - 1) (h_{1t} \partial_t^1 \zeta + h_{2t} y \partial_t^j \zeta_y) + O(1) \sum_{l=2}^{j} |\partial_t^l (h_{1t}, h_{2t})| |\partial_t^{j+1-l} (\zeta, y \zeta_y)|,$$

$$y R_{2t} = (j - 1) (2h_2 - h_1) t (y \partial_t^j \zeta_y) + O(1) \sum_{l=2}^{j} |\partial_t^l (h_{1t}, h_{2t})| |\partial_t^{j+1-l} (y \zeta_y)|,$$

$$y R_{3t} = (j - 1) (h_{3ty} \zeta_y) t (\partial_t^j \zeta) + O(1) \sum_{l=2}^{j} |\partial_t^l (h_{3ty} \zeta_y)| |\partial_t^{j+1-l} \zeta|.$$

This, together with (3.51) (2.10), implies that

$$|F_2^j| \lesssim \epsilon_0 \int y^2 \sigma^{\alpha+1} |\partial_t^j (\zeta, y \zeta_y)|^2 \, dy + \int y^2 \sigma^{\alpha+1} (|\partial_t^j \zeta| + |y \partial_t^j \zeta_y|) \mathcal{R} \, dy, \quad (3.78)$$

where

$$\mathcal{R} = \sum_{l=2}^{j} |\partial_t^l (h_{1t}, h_{2t}, h_{3ty} \zeta_y)| |\partial_t^{j+1-l} (\zeta, y \zeta_y)|. \quad (3.79)$$

Here $\mathcal{R}$ satisfies that

$$\mathcal{R} \lesssim \epsilon_0 \sum_{l=1}^{j} |\partial_t^l (\zeta, y \zeta_y)| + \overline{\mathcal{R}}, \quad (3.80)$$

where

$$\overline{\mathcal{R}} = |\partial_t^2 (\zeta, y \zeta_y)| |\partial_t^j (\zeta, y \zeta_y)| + \left( |\partial_t^2 (\zeta, y \zeta_y)| + |\partial_t^3 (\zeta, y \zeta_y)| \right) |\partial_t^{j-2} (\zeta, y \zeta_y)|$$

$$+ \left( |\partial_t^2 (\zeta, y \zeta_y)| + |\partial_t^3 (\zeta, y \zeta_y)| + |\partial_t^4 (\zeta, y \zeta_y)| \right) |\partial_t^{j-3} (\zeta, y \zeta_y)|, \quad (3.81)$$

since $j \leq n \leq 6$. Then we use (2.10) to obtain

$$\overline{\mathcal{R}} \lesssim \epsilon_0 \sum_{l=1}^{[j-1]/2} \sigma^{-\frac{j}{2}} |\partial_t^{j-l} (\zeta, y \zeta_y)|. \quad (3.82)$$

Now we have

$$\int y^2 \sigma^{\alpha+1} (|\partial_t^j \zeta| + |y \partial_t^j \zeta_y|) \overline{\mathcal{R}} \, dy$$
\begin{align}
\lesssim \epsilon_0 \int y^2 \rho^\alpha | \partial_t^j (\zeta, y \zeta_y) |^2 dy + \epsilon_0 \sum_{i=1}^{[(j-1)/2]} \int y^2 \sigma^{\alpha+1-i} | \partial_t^{j-i} (\zeta, y \zeta_y) |^2 dy.
\end{align}

For \( \iota = 1, \cdots, [(j-1)/2] \), it follows from Hardy inequality \( (2.1) \) and the fact \( \alpha + 1 - \iota \geq \alpha + 1 - [(1 + [\alpha])/2] > 0 \) that
\begin{align}
\int y^2 \sigma^{\alpha+1-i} | \partial_t^{j-i} (\zeta, y \zeta_y) |^2 dy & \lesssim \int_{I_l} y^2 \sigma^{\alpha+1} | \partial_t^{j-i} (\zeta, y \zeta_y) |^2 dy + \int_{I_b} \sigma^{\alpha+1-i} | \partial_t^{j-i} (\zeta, y \zeta_y) |^2 dy \\
& \lesssim \mathcal{E}_j(t) + \int_{I_b} \sigma^{\alpha+3-i} | \partial_t^{j-i} (\zeta, \zeta_y \zeta_{yy}) |^2 dy \\
& \lesssim \mathcal{E}_j(t) + \cdots \\
& \lesssim \mathcal{E}_j(t) + \sum_{i=0}^{\iota+1} \int_{I_b} y^i \sigma^{\alpha+1+i} | \partial_t^{j-i} \partial_y^i \zeta |^2 dy \\
& \lesssim \mathcal{E}_j(t) + \sum_{i=1}^{\iota} \mathcal{E}_{j-i,i}(t) \\
& \lesssim \sum_{i=0}^{j} \mathcal{E}_i(t),
\end{align}

where the elliptic estimate \( (3.1) \) has been used in the last inequality. Then \( (3.76) \) follows from \( (3.78)-(3.84) \).

As for \( \tilde{R}^{j-1} \), \( j \geq 2 \), it can be handled by a similar but much easier way to dealing with \( F^j_2 \). Hence we omit the repeat for simplicity. Now we finish the proof of the claim stated in \( (3.68) \) and \( (3.69) \).

**References**

[1] Anderson K. R., Adams F. C., Effectsofcollisionswithrockyplanetsontheproperties of hot jupiters. Publ. Astron. Soc. Pac. 124(918), 809–822 (2012).

[2] Barenblatt G., On one class of solutions of the one-dimensional problem of nonstationary filtration of a gas in a porous medium. Prikl. Mat. i. Mekh. 17, 739–742 (1953).

[3] Burrows A., Hubeny I., Budaj J., Hubbard, W., Possible solutions to the radius anomalies of transiting giant planets. Astrophys. J. 661(1), 502 (2007).

[4] Chandrasekhar S., An Introduction to the Study of Stellar Structures. University of Chicago Press, 1939.

[5] Coutand D., Lindblad H., Shkoller S., A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum. Comm. Math. Phys. 296 (2010), no. 2, 559-587.

[6] Coutand D., Shkoller S., Well-posedness in smooth function spaces for moving-boundary 1-D compressible Euler equations in physical vacuum. Comm. Pure Appl. Math. 64 (2011), no. 3, 328-366.

[7] Coutand D., Shkoller S., Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum. Arch. Ration. Mech. Anal. 206 (2012), no. 2, 515–616.
[8] Hadžić M., Jang J., A class of global solutions to the Euler-Poisson system. Comm. Math. Phys. 370 (2019), no. 2, 475–505.

[9] Hong G., Luo T., Zhu C., Global solutions to physical vacuum problem of non-isentropic viscous gaseous stars and nonlinear asymptotic stability of stationary solutions. J. Differential Equations 265 (2018), no. 1, 177–236.

[10] Jang J., Masmoudi N., Well-posedness of compressible Euler equations in a physical vacuum, Comm. Pure Appl. Math. 68, 61–111 (2015).

[11] Jang J., Nonlinear instability theory of Lane-Emden stars. Comm. Pure Appl. Math. 67 (2014), no. 9, 1418–1465.

[12] Kuan W.-C., Lin S.-S., Numbers of equilibria for the equation of self-gravitating isentropic gas surrounding a solid ball. Japan J. Indust. Appl. Math. 13 (1996), no. 2, 311–331.

[13] Kufner A., Maligranda L., Persson L. E., The Hardy inequality, Vydavatelsky Servis, Plzen, 2007. About its history and some related results.

[14] Lieb E.H., Yau H.T., The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Commun. Math. Phys. 112(1), 147–174 (1987).

[15] Lin S.-S., Stability of gaseous stars in spherically symmetric motions. SIAM J. Math. Anal. 28 (1997), no. 3, 539–569.

[16] Liu T.-P., Compressible flow with damping and vacuum, Jpn. J. Appl. Math. 13, 25–32 (1996).

[17] Liu T.-P., Yang T., Compressible Euler equations with vacuum, J. Differ. Equ. 140, 223–237 (1997).

[18] Luo T., Xin Z., Yang T., Compressible flow with vacuum and physical singularity. Methods Appl. Anal. 7, 495–510 (2000).

[19] Luo T., Smoller J., Nonlinear dynamical stability of Newtonian rotating and non-rotating white dwarfs and rotating supermassive stars. Comm. Math. Phys. 284 (2008), no. 2, 425–457.

[20] Luo T., Wang Y.-L., Zeng H., Nonlinear Asymptomatic Stability of Gravitational Hydrostatic Equilibrium for Viscous White Dwarfs with Symmetric Perturbations, preprint, arXiv:2201.12742

[21] Luo T., Xin Z., Yang T., Interface behavior of compressible Navier-Stokes equations with vacuum. SIAM J. Math. Anal. 31, 1175–1191 (2000).

[22] Luo T., Xin Z., Zeng H., On nonlinear asymptotic stability of the Lane-Emden solutions for the viscous gaseous star problem. Adv. Math. 291 (2016), 90–182.

[23] Luo T., Xin Z., Zeng H., Nonlinear asymptotic stability of the Lane-Emden solutions for the viscous gaseous star problem with degenerate density dependent viscosities. Comm. Math. Phys. 347 (2016), no. 3, 657–702.

[24] Luo T., Zeng H., Global existence of smooth solutions and convergence to Barenblatt solutions for the physical vacuum free boundary problem of compressible Euler equations with damping. Comm. Pure Appl. Math. 69 (2016), no. 7, 1354–1396.

[25] Makino T., On the spherically symmetric motion of self-gravitating isentropic gas surrounding a solid ball. Nonlinear evolutionary partial differential equations (Beijing, 1993), 543–546, AMS/IP Stud. Adv. Math., 3, Amer. Math. Soc., Providence, RI, 1997.

[26] Makino T., On spherically symmetric motions of the atmosphere surrounding a planet governed by the compressible Euler equations. Funkcial. Ekvac. 58 (2015), no. 1, 43–85.

[27] Matušů-Nečasová Š., Okada M., Makino T., Free boundary problem for the equation of spherically symmetric motion of viscous gas. II. Japan J. Indust. Appl. Math. 12 (1995), no. 2, 195–203.

[28] Matušů-Nečasová Š., Okada M., Makino T., Free boundary problem for the equation of spherically symmetric motion of viscous gas. III. Japan J. Indust. Appl. Math. 14 (1997), no. 2, 199–213.
Review: Free boundary problem for the equation of spherically symmetric motion of viscous gas. III. (English summary) Japan J. Indust. Appl. Math. 14 (1997), no. 2, 199–213. MR1454522; (see also, MR1337205; MR1227730, reviewed by Guo Hang).

Militzer B., Hubbard W.B., Vorberger J., Tamblyn I., Bonev S., A massive core in Jupiter predicted from first-principles simulations. Astrophys. J. Lett. 688(1), L45 (2008).

Miller N., Fortney J. J., The heavy-element masses of extrasolar giant planets, revealed. Astrophys. J. Lett. 736(2), L29 (2011).

Okada M., Free boundary value problems for the equation of one-dimensional motion of viscous gas. Japan J. Appl. Math. 6, 161–177 (1989).

Okada M., Makino T., Free boundary problem for the equation of spherically symmetric motion of viscous gas. Japan J. Indust. Appl. Math. 10 (1993), no. 2, 219–235.

Ou Y., Low Mach and low Froude number limit for vacuum free boundary problem of all-time classical solutions of one-dimensional compressible Navier-Stokes equations. SIAM J. Math. Anal. 53 (2021), no. 3, 3265–3305.

Ou Y., Zeng H., Global strong solutions to the vacuum free boundary problem for compressible Navier-Stokes equations with degenerate viscosity and gravity force. J. Differential Equations 259 (2015), no. 11, 6803–6829.

Rein G., Non-linear stability of gaseous stars. Arch. Ration. Mech. Anal. 168(2), 115–130 (2003).

Wu Y., Existence of rotating planet solutions to the Euler-Poisson equations with an inner hard core. Arch. Ration. Mech. Anal. 219 (2016), no. 1, 1–26.

Yang T., Zhu C., Compressible Navier–Stokes Equations with Degenerate Viscosity Coefficient and Vacuum. Commun. Math. Phys. 230, 329–363 (2002).

Zeng H., Global-in-time smoothness of solutions to the vacuum free boundary problem for compressible isentropic Navier-Stokes equations. Nonlinearity 28 (2015), no. 2, 331–345.

Zeng H., Global resolution of the physical vacuum singularity for three-dimensional isentropic inviscid flows with damping in spherically symmetric motions. Arch. Ration. Mech. Anal. 226 (2017), no. 1, 33–82.

Zeng H., Almost global solutions to the three-dimensional isentropic inviscid flows with damping in a physical vacuum around Barenblatt solutions. Arch. Ration. Mech. Anal. 239 (2021), no. 1, 553–597.

Zeng H., Global Solution to the Vacuum Free Boundary Problem with Physical Singularity of Compressible Euler Equations with Damping and Gravity, 2021, arXiv:2110.14909

Zhu C., Asymptotic Behavior of Compressible Navier-Stokes Equations with Density-Dependent Viscosity and Vacuum. Commun. Math. Phys. 293, 279–299 (2010).

(Y.-L. Wang) DEPARTMENT OF APPLIED MATHEMATICS, ZHEJIANG UNIVERSITY OF TECHNOLOGY, HANGZHOU 310023, CHINA.

Email address: yanlin90118@163.com; yanlinwang.math@gmail.com