Stochastic Gradient Descent for Stochastic Doubly-Nonconvex Composite Optimization

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Abstract

The stochastic gradient descent has been widely used for solving composite optimization problems in big data analyses. Many algorithms and convergence properties have been developed. The composite functions were convex primarily and gradually nonconvex composite functions have been adopted to obtain more desirable properties. The convergence properties have been investigated, but only when either of composite functions is nonconvex. There is no convergence property when both composite functions are nonconvex, which is named the doubly-nonconvex case. To overcome this difficulty, we assume a simple and weak condition that the penalty function is quasiconvex and then we obtain convergence properties for the stochastic doubly-nonconvex composite optimization problem. The convergence rate obtained here is of the same order as the existing work. We deeply analyze the convergence rate with the constant step size and mini-batch size and give the optimal convergence rate with appropriate sizes, which is superior to the existing work. Experimental results illustrate that our method is superior to existing methods.

1 Introduction

Many optimization problems in machine learning can be written as the following composite optimization problem:

$$\arg\min_{\theta \in \mathbb{R}^p} \Psi(\theta) := f(\theta) + h(\theta).$$

(1)

Typically, $f(\theta)$ is a convex loss function (e.g., least squared loss) and $h(\theta)$ is a convex and possibly nonsmooth regularization function (e.g., $L_1$ penalty [1]). However, it is known that the resulting estimate has a bias. To overcome this problem, nonconvex regularizations such as SCAD [2] and MCP [3] are becoming popular. Nonconvex loss functions are also becoming popular. One of the most successful nonconvex machine learning applications is a deep learning [4–6]. For matrix completion, both loss and regularization functions have been extend to nonconvex cases [7,8]. In robust statistics, it is known that nonconvex loss functions can show more desirable robust properties than convex loss functions [9].

Many algorithms have been developed for the composite optimization problem. To dramatically reduce the computational cost in big data analyses, the stochastic gradient descent (SGD) and its variants have been adopted [10–14]. Convergence properties of SGD have been intensively studied, but only when $f(\theta)$ is nonconvex and $h(\theta)$ is convex. There is no theoretical property when both composite functions are nonconvex, which is named the doubly-nonconvex case. To overcome this
difficulty, we assume that $h(\theta)$ is quasiconvex in addition to well-used conditions and then we obtain convergence properties for the stochastic doubly-nonconvex composite optimization problem.

Here, we note that most of the existing works focus on the empirical loss $f(\theta) = 1/n \sum_{i=1}^{n} f_i(\theta)$ \cite{b25,b26} This case is called the finite-sum stochastic composite optimization problem. On the other hand, the purely stochastic composite optimization problem focuses on the expected loss $f(\theta) = E_X [f(\theta; X)]$. These two problems are quite different \cite{b15}. This paper mainly focuses on the purely stochastic composite optimization problem, but we also have some results for the finite-sum stochastic composite optimization problem, using the theoretical results obtained in the purely stochastic composite optimization problem.

**Related Work:** An asymptotic convergence of the SGD was proved in the seminal work \cite{b16}. This work was extended to non-asymptotic convergence rates for stochastic convex composite optimization problems including the case $h(\theta) \equiv 0$ \cite{b17,b21}. In particular, \cite{b22,b24} focused on the stochastic mirror descent method which included the SGD as a special case and succeeded to obtain the non-asymptotic optimal convergence rate in a wider class of algorithm than the SGD. However, most of the results hold only when $f(\theta)$ is convex and $h(\theta)$ is convex or $h(\theta) \equiv 0$. Recently, the SGD and its variants for nonconvex composite optimization problems have been intensively studied. For the finite-sum stochastic setting, variance reduction techniques were proposed and convergence rates were shown \cite{b25,b27}. For the purely stochastic setting, \cite{b28} investigated theoretical properties when $f(\theta)$ was nonconvex and $h(\theta) \equiv 0$. They adopted a new output selection scheme, named random selection, and succeeded to give a non-asymptotic convergence rate for the stochastic mirror descent method. \cite{b29,b30} adopted the mini-batch scheme and extended the previous work \cite{b28} to the composite case where $f(\theta)$ was nonconvex and $h(\theta)$ was convex. In particular, \cite{b30} obtained a faster rate than that of \cite{b29} by virtue of a acceleration technique. \cite{b31} adopted a different mini-batch scheme, named minibatch-prox, and succeeded to prove a convergence property.

Note that $h(\theta)$ is convex or $h(\theta) \equiv 0$ in the existing works. This paper considers the case where $h(\theta)$ is nonconvex as well as $f(\theta)$. Here we reconsider an advantage of the convexity of $h(\theta)$. It enables us to regard an update rule as a projection onto a convex set which is derived from a sublevel set of $h(\theta)$, even when $f(\theta)$ is nonconvex and $h(\theta)$ is nonsmooth. Therefore, we assume that $h(\theta)$ is quasiconvex which implies that the sublevel set of $h(\theta)$ is convex, instead of the convexity of $h(\theta)$. It is a broad class and includes many nonconvex penalties. Under the condition of quasiconvexicity of $h(\theta)$, we show theoretical properties of SGD for the stochastic doubly-nonconvex composite optimization problem.

**Our Contribution:**

- We show that the SGD converges for the stochastic doubly-nonconvex composite optimization problem under the simple and weak condition, quasiconvexity, and achieves the same convergence rate as the existing work \cite{b29} except for a constant factor. To the best of our knowledge, our paper is the first work for proving the convergence of the SGD for the stochastic doubly-nonconvex composite optimization problem.
- Our problem formulation is the purely stochastic setting. However, our theoretical results can be easily applied to the finite-sum stochastic setting.
- We deeply analyze the convergence rate with the constant step size and mini-batch size and give the optimal convergence rate with appropriate sizes, which is superior to the existing work \cite{b29}.

## 2 Preliminary

### 2.1 Notations and Definitions

We present some notations which are used in this paper. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^p$ and $\|\cdot\|$ be the Euclidean norm. For any $\gamma \geq 0$, $\|\cdot\|_\gamma$ denotes the $l_\gamma$ norm. For any real number $r$, $\lfloor r \rfloor$ and $\lceil r \rceil$ denote the floor function and the ceiling function, respectively.

**Definition 2.1. (Lipschitz smooth)** A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be $L$-Lipschitz smooth for some $L > 0$ if

$$\|\nabla f(\theta_1) - \nabla f(\theta_2)\| \leq L\|\theta_1 - \theta_2\| \quad \text{for any } \theta_1, \theta_2 \in \text{dom}(f).$$

(2)
From (2), the following inequality can be derived:

\[ |f(\theta_1) - f(\theta_2) - \langle \nabla f(\theta_2), \theta_1 - \theta_2 \rangle| \leq \frac{L}{2} \|\theta_1 - \theta_2\|^2 \quad \text{for any } \theta_1, \theta_2 \in \text{dom}(f). \] (3)

Next, we define a quasiconvex function, which plays a key role in this paper.

**Definition 2.2. (Quasiconvex) A function \( h(\theta) : \mathbb{R}^p \rightarrow \mathbb{R} \) is said to be quasiconvex, if its sublevel set \( S_\alpha(h) := \{ \theta \in \text{dom}(h) | h(\theta) \leq \alpha \} \) is convex for any \( \alpha \in \mathbb{R} \).**

### 2.2 Problem Formulation

In problem (1), we suppose the following assumptions:

**Assumption 1.** The objective function \( \Psi(\theta) = f(\theta) + h(\theta) \) is bounded below; \( \Psi(\theta) \geq \Psi^* \).

**Assumption 2.** The function \( f \) is \( L \)-Lipschitz smooth (possibly nonconvex).

**Assumption 3.** Let \( X_1, X_2, \ldots \) be the i.i.d. random variables. For any \( t \geq 1 \), instead of the full gradient \( \nabla f(\theta^{(t)}) \), we only have access to a noisy gradient \( G_{X_t}(\theta^{(t)}) \), which satisfies

\[
E_{X_t} \left[ G_{X_t}(\theta^{(t)}) \right] = \nabla f(\theta^{(t)}),
\] (4)

\[
E_{X_t} \left[ \| G_{X_t}(\theta^{(t)}) - \nabla f(\theta^{(t)}) \|^2 \right] \leq \sigma^2,
\] (5)

where \( \theta^{(t)} \) is the \( t \)-th iterate parameter and \( \sigma \) is a positive parameter.

**Assumption 4.** We assume either (i) or (ii):

(i) The function \( h(\theta) \) is separable w.r.t. the parameter \( \theta \) with non-negative weights, more precisely,

\[ h(\theta) = \sum_{j=1}^{p} \lambda_j h_j(\theta_j), \quad \theta = (\theta_1, \ldots, \theta_p)^T, \quad \lambda_j \geq 0, \]

where the function \( h_j(\theta_j) : \mathbb{R} \rightarrow \mathbb{R} \) is proper lower semi-continuous (possibly nonsmooth) and quasiconvex.

(ii) The function \( h(\theta) \) is proper lower semi-continuous (possibly nonsmooth) and quasiconvex.

**Discussion of Assumptions:** Assumptions 1 and 2 are commonly used in the first-order non-stochastic and stochastic optimization literatures; Non-stochastic: FISTA [32], GIST [33], mAPG [34], PALM [35], Stochastic: SAG [36, 37], SDCA [38], SVRG [25, 27], SAGA [13], SCSG [39], Katyusha [14], RSG [28], RSHP [29], RSAG [30]. Instead of Assumption 1, most of these methods assume that a global minimizer exists, \( \theta^* = \text{argmin}_{\theta} f(\theta) + h(\theta) \), which is stronger than Assumption 1. Assumption 3 is a general assumption in the first-order stochastic optimization literatures; RSG [28], RSHP [29], RSAG [30], MP [31]. In particular, (1) is known for a first-order stochastic oracle. Assumption 4 is satisfied for well-known nonconvex examples of \( h(\theta) \), as will be shown later. It may seem that Assumption 4(i) implies Assumption 4(ii), but this is not true: e.g., \( h(\theta) = \theta_1^3 + \theta_2^3 \). The assumptions except for Assumption 4 are the same as in [29].

These assumptions cover many applications in machine learning, signal processing and computer vision.

**Examples of \( f(\theta) \):** (Convex) Linear/Logistic regression [40], t-distribution [41], Tukey’s biweight [41], Matrix completion [42], Total variation model [43].

**Examples of \( h(\theta) \):** (Convex) Ridge [44], l_1 [1], elasticnet [45], Adaptive lasso [46]}, (Nonconvex) SCAD [4], MCP [4], Log-sum penalty [47, 48], Capped-l_1 penalty [49], \( l_\gamma(0 \leq \gamma < 1) \) penalty [50, 47].

In what follows, we discuss our method under Assumption 4(i) instead of Assumption 4(ii), because most of the examples can be found under Assumption 4(i). The theoretical properties obtained in this paper can also be proved under Assumption 4(ii) in a similar way.
3 SGD for Stochastic Doubly-Nonconvex Composite Optimization

3.1 Algorithm

Update Rule: We consider the following standard update rule of the mini-batch SGD:

\[
\theta^{(t+1)} = \arg\min_{\theta \in \mathbb{R}^p} \left \langle G(\theta^{(t)}), u \right \rangle + \sum_{j=1}^{p} \lambda_j h_j(u_j) + \frac{1}{2\eta_t} \|u - \theta^{(t)}\|^2,
\]

where \(G(\theta^{(t)}) = \frac{1}{m_t} \sum_{i=1}^{m_t} G_{X_i,\theta}(\theta^{(t)})\), \(m_t\) is the size of the mini-batch at the \(t\)-th iteration, \(u_j\) is the \(j\)-th elements of \(u\) and \(\eta_t > 0\) is the step size at the \(t\)-th iteration. Then, the update rule \((6)\) can be reduced to the coordinate-wise update rule as follows:

\[
\theta_j^{(t+1)} = \arg\min_{\tilde{u} \in \mathbb{R}} G(\theta^{(t)})_j \tilde{u} + \lambda_j h_j(\tilde{u}) + \frac{1}{2\eta_t} (\tilde{u} - \theta_j^{(t)})^2 \quad \text{for } j = 1, \ldots, p,
\]

where \(G(\theta^{(t)})_j\) and \(\theta_j^{(t)}\) are the \(j\)-th elements of \(G(\theta^{(t)})\) and \(\theta^{(t)}\), respectively.

Proximal Operator: The update rule \((7)\) is equivalent to the proximal operator problem given by:

\[
\theta_j^{(t+1)} = \arg\min_{\tilde{u} \in \mathbb{R}} \eta_t \lambda_j h_j(\tilde{u}) + \frac{1}{2} (\tilde{u} + \eta_t G(\theta^{(t)})_j - \theta_j^{(t)})^2.
\]

This problem is nonconvex and a minimizer does not always exist. However, [35] pointed out that a proximal operator problem with a proper lower semi-continuous function always has a well-defined solution set, i.e., a non-empty and compact solution set. Therefore, \(\theta_j^{(t+1)}\) exists in all iterative steps. Moreover, some important nonconvex examples, e.g., SCAD, MCP, Log-sum penalty, Capped-\(l_1\) penalty, have closed-form solutions [33]. In addition, \(l_0\) penalty has the closed-form solution known as Hard-Thresholding. We illustrate some examples and corresponding closed-form solutions in Appendix A1.

Output Selection: The SGD for a convex objective function generally uses the average of iterates as an output. However, for a nonconvex objective function, iterates are not always gathered around a local minimum and the average of the iterates does not work well in a similar way to in a convex case. Therefore, following existing methods such as [28–31], we adopt randomized selection, i.e., we select an output randomly from iterates according to a probability mass function \(P_R\). Our method randomly selects the only one output according to \(P_R\). In order to decrease a large deviation of output, [29] proposed the two-phase scheme which randomly selects multiple outputs, and validates them, and then chooses the final output from the validated outputs. In our experiments, we adopted this two-phase scheme.

Finally, we give the pseudocodes of our methods by Algorithm 1 and 2.

**Algorithm 1**

**Input:** The initial point \(\theta^{(1)}\), the step size \(\eta_t\), the mini-batch size \(m_t\), the iteration limit \(T\) and the probability mass function \(P_R\) supported on \(\{1, \ldots, T\}\).

Let \(R\) be a random variable generated by a probability mass function \(P_R\).

for \(t = 1, \ldots, R\) do

\[
\theta_j^{(t+1)} = \arg\min_{\tilde{u} \in \mathbb{R}} G(\theta^{(t)})_j \tilde{u} + \lambda_j h_j(\tilde{u}) + \frac{1}{2\eta_t} (\tilde{u} - \theta_j^{(t)})^2 \quad \text{for } j = 1, \ldots, p.
\]

end for

**Output:** \(\theta^{(R)}\).

3.2 Characterization of Update Rule

The update rule \((7)\) can be seen as a Lagrangian relaxation problem (also called Lagrange form) with the Lagrange multiplier \(\eta_t \lambda_j \geq 0\) and \(K_j \in \mathbb{R}\):

\[
\min_{\tilde{u} \in \mathbb{R}} \frac{1}{2} (\tilde{u} + \eta_t G(\theta^{(t)})_j - \theta_j^{(t)})^2 + \eta_t \lambda_j (h_j(\tilde{u}) - K_j),
\]

(8)
Then, the original problem (also called constrained form) of (8) may be given by
\[
\begin{align*}
\text{minimize } & \frac{1}{2} (\hat{u} + \eta_t G(\theta^{(t)}) - \theta_j^{(t)})^2 \\
\text{subject to } & h_j(\hat{u}) \leq K_j,
\end{align*}
\]
(9)
If the objective function in the Lagrangian relaxation problem (8) is convex, a minimizer in (8) also minimizes the original problem (9) under some regularity conditions. It is known as a sufficient optimality condition for convex programming problem (51). However, it does not generally hold for nonconvex cases. For a global minimizer in (8), we provide the following sufficient optimality condition.

**Proposition 3.1.** If a global minimizer \( \theta_j^{(t+1)} \) in (8) satisfies
\[
\begin{align*}
\text{Primal feasibility: } & h_j(\theta_j^{(t+1)}) \leq K_j, \\
\text{Complementary slackness: } & \eta_t \lambda_j (h_j(\theta_j^{(t+1)}) - K_j) = 0,
\end{align*}
\]
(10, 11)
then \( \theta_j^{(t+1)} \) is the unique global minimizer in (9).

**Proof.** Since \( \theta_j^{(t+1)} \) is a global minimizer of \( L(\hat{u}) \), we have \( L(\theta_j^{(t+1)}) \leq L(\theta_j^{(t)}) \). After simple calculation with (11) and \( \eta_t \lambda_j \geq 0 \), we have
\[
(\theta_j^{(t+1)} + \eta_t G(\theta^{(t)}) - \theta_j^{(t)})^2/2 \leq (\hat{u} + \eta_t G(\theta^{(t)}) - \theta_j^{(t)})^2/2 \text{ for any } h_j(\hat{u}) \leq K_j.
\]
Therefore, \( \theta_j^{(t+1)} \) is the unique global minimizer in (9), because (10) and \((\hat{u} + \eta_t G(\theta^{(t)}) - \theta_j^{(t)})^2/2 \) is a strongly convex function.

**Remark on Proposition 3.1:** We can show that the constraint \( h_j(\hat{u}) \leq K_j \) is a non-empty closed convex set, because \( h_j(\hat{u}) \) is a proper lower semi-continuous quasiconvex function, so that the update rule (7) is regarded as the Euclidean projection onto the convex set from the point of view of the original problem (9). Actually, Proposition 3.1 holds for any nonconvex function \( h_j(\hat{u}) \), but the corresponding original problem can not be generally regarded as the Euclidean projection onto a convex set, unless \( h_j(\hat{u}) \) is quasiconvex. Recall that important examples of \( h_j(\hat{u}) \), e.g., SCAD, MCP, Log-sum penalty, Capped-l1 penalty and \( \ell_p \) penalty, have closed-form solutions, and satisfy the assumptions in Proposition 3.1 when \( K_j \) is set to \( h_j(\theta_j^{(t+1)}) \).

Another sufficient optimality condition, which focuses on a nonsmooth quasiconvex function, can be found in (52), (52) uses a variant of directional derivative to characterize a nonsmooth stationary condition. In particular, even if \( \theta_j^{(t+1)} \) is a local minimizer, the sufficient optimality condition in (52) holds. Our Lagrangian relaxation problem (8) and the corresponding original problem (9) satisfy the assumptions supposed by (52). Therefore, we can adopt the sufficient optimality condition in (52), which is deeply discussed in Appendix A2.

**Relation between Full Gradient and Stochastic Gradient:** The following lemma shows that the Euclidean projection is a non-expansive mapping.

**Lemma 3.1.** Let \( C \subseteq \mathbb{R}^p \) be a convex set. The Euclidean projection onto the convex set \( C \) is defined by \( \hat{x}(\hat{y}) = \text{argmin}_{u \in C} \|u - x(\hat{y})\|^2 \). Then, we have \( \|\hat{x} - \hat{y}\| \leq \|x - y\| \).

This is a classical well-known Lemma (see, Corollary 12.20 in (53)). Let the update rule based on the full gradient be defined by
\[
\theta_j^{(t+1)} = \text{argmin}_{u \in \mathbb{R}^p} \left\langle \nabla f(\theta_j^{(t)}), u \right\rangle + \sum_{j=1}^{p} \lambda_j h_j(u_j) + \frac{1}{2\eta_t} \|u - \theta_j^{(t)}\|^2.
\]
(12)

We provide the following proposition.

**Proposition 3.2.** Let \( \theta_j^{(t+1)} \) and \( \nabla f(\theta_j^{(t+1)}) \) be the \( j \)-th elements of \( \theta_j^{(t+1)} \) and \( \nabla f(\theta_j^{(t+1)}) \), respectively. Then, we have
\[
|\theta_j^{(t+1)} - \theta_j^{(t+1)}| = \eta_t |G(\theta_j^{(t)}) - \nabla f(\theta_j^{(t)})| \text{ for } j = 1, \ldots, p.
\]
(13)
Proof. We see from Remark on Proposition 3.1 that $\theta^{(t+1)}$ is the Euclidean projection onto a convex set. The updated rule (12) can be reduced to a coordinate-wise update rule and then it is regarded as the Euclidean projection onto a convex set in a similar manner to $\theta^{(t+1)}$. Replacing $x$ and $y$ by $\theta_j^{(t)} - \eta_t G(\theta^{(t)})_j$ and $\theta_j^{(t)} - \eta_t \nabla f(\theta^{(t)})_j$, respectively, in Lemma 3.1 then we have (13). □

Let the objective function in (12) be denoted by $M(u)$. Since $\theta^{(t+1)}_{full}$ is a global minimizer of $M(u)$, we have $M(\theta^{(t+1)}_{full}) \leq M(\theta^{(t+1)}_{full})$. Then, it follows from this inequality and (3) that the target function $f(\theta) + h(\theta)$ decreases as $\theta$ is changed from $\theta^{(t)}_{full}$ to $\theta^{(t+1)}_{full}$ under $\eta_t \leq 1/L$. The update rule (12) for the stochastic doubly-nonconvex composite optimization problem does not have such a desirable property. However, Proposition 3.2 implies that such a desirable property approximately holds, although it depends on the accuracy of the approximation of the noisy gradient to the full gradient. In addition, Proposition 3.2 plays a key role in the proof of convergence property.

3.3 Convergence Analysis

Convergence Property: Let us define

$$\hat{P}_{X,R} = (\theta^{(R)} - \theta^{(R+1)})/\eta_t, \quad P_{X,R} = (\theta^{(R)} - \theta^{(R+1)})/\eta_t.$$ (14)

We obtain the following convergence property. The proof is in Appendix A3.

**Theorem 3.1.** Suppose that the step sizes $\eta_t$’s are chosen such that $0 < \eta_t \leq 1/L$ with $\eta_t < 1/L$ for at least one $t$, and the probability mass function $P_{R}$ is chosen such that for $t = 1, \ldots, T$,

$$P_{R}(t) := \text{Prob} \{ R = t \} \geq \frac{\eta_t - L\eta^2_t}{\sum_{t=1}^{T}(\eta_t - L\eta^2_t)}.$$ (15)

Then, we have

$$E\left[ \|\hat{P}_{X,R}\|^2 \right] \leq \frac{2LD^2\Psi + 2\sigma^2 \sum_{t=1}^{T}(\eta_t/m_t)}{\sum_{t=1}^{T}(\eta_t - L\eta^2_t)}.$$ (16)

where the expectation is taken with respect to $R$ and $X_{t,i}$’s, and $D_{\Psi} = \left(\Psi(\Psi(1)) - \Psi^*)/L\right)^{\frac{1}{2}}$.

**Remark on Theorem 3.1:** [29] studied the convergence rate in the case of a nonconvex loss function $f(\theta)$ with a convex penalty $h(\theta)$. Even when $h(\theta)$ is nonconvex, we have succeeded to attain the same convergence rate as in Theorem 2(a) of [29] except for a constant factor. Actually, we can obtain a better convergence rate than that of [29] under a specific setting of some parameters, as will be shown later. Moreover, we can obtain the same convergence rate even under the finite-sum stochastic setting in a similar manner.

Here, we deeply consider the step size and mini-batch size. For a stochastic convex optimization problem, a decreasing step size, e.g., $\eta_t = 1/L$, is generally used, which can guarantee the convergence in expectation (see, e.g., Chapter 6 in [54]). However, a decreasing step size is not suitable for our method, because the selecting probability (15) with a decreasing step size implies that early iterates tend to be selected, although later iterates are expected to be more proper than early iterates. Therefore we consider the constant step size. The mini-batch size is closely related to the accuracy of the approximation of the noisy gradient to the full gradient. This accuracy is important, as seen in Proposition 3.2. The increasing/decreasing mini-batch size gives a better approximation at later/early iterates. It is not clear which idea is better, because it depends on the problem. Therefore we consider the constant mini-batch size. In the constant size case, we can obtain the following theorem. The proof is in Appendix A4.

**Theorem 3.2.** Suppose that the step sizes and mini-batch sizes are constant, i.e., $\eta_t = 1/2L$ and $m_t = m$ ($\geq 1$) for all $t = 1, \ldots, T$, and the probability mass function $P_{R}$ is chosen as (17). Then, we have

$$E\left[ \|\hat{P}_{X,R}\|^2 \right] \leq \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m},$$

$$E\left[ \|P_{X,R}\|^2 \right] \leq \left(1 + \frac{1}{c}\right)\frac{8L^2D^2_{\Psi}}{T} + \left(5 + \frac{4}{c}\right)\frac{\sigma^2}{m},$$ (17)

where $c > 0$. 

How to Select Mini-batch Size: The bound (17) depends on the iteration limit $T$ and the mini-batch size $m$. These are closely related to the total number of the sequences $X_t$, say $N$. Here we consider the case $T = N/2m$ for simplicity, because $T$ is at most $\lfloor N/m \rfloor$. In this case, the bound (17) is minimized at $m = (\sigma/4LD\Psi\sqrt{(c + 4)N}$. If we know $\sigma$, $L$ and $R_\Psi$, then we can use this optimal value of $m$ to attain the optimal convergence rate. Two values $\sigma$ and $L$ can be estimated, but it is difficult to estimate $D_\Psi$, so that $D_\Psi$ is often replaced by an appropriate value $\tilde{D}$ we can set (see [29] for details). The resulting value of $m$ may not be a positive integer and not be smaller than $N$. Therefore, we propose

$$m = \min \left\{ \max \left\{ 1, \frac{\sigma}{4LD\sqrt{(c + 4)N}} \right\}, N \right\} \text{ for some } \tilde{D} > 0,$$

and then we obtain the following theorem. The proof is in Appendix A5.

Theorem 3.3. Assume the conditions in Theorem 3.2. Let $m$ be the value defined by (18). Then we have

$$E \left[ \|P_{X,R}\|^2 \right] \leq \left( 1 + \frac{1}{c} \right) \frac{16L^2D^2_\Psi}{N} + 4L\sigma \left( 1 + \frac{1}{c} \right) \sqrt{\frac{c + 4}{N}} \left( \frac{D^2_\Psi}{\tilde{D}} + \tilde{D} \max \left\{ 1, \frac{\sigma}{4L\tilde{D}\sqrt{4N}} \right\} \right).$$

(19)

Remark on Theorem 3.3: Suppose that $N$ is relatively large. When $\tilde{D}$ is the ideal value $D_\Psi$, (19) reduces to

$$E \left[ \|P_{X,R}\|^2 \right] \leq \left( 1 + \frac{1}{c} \right) \frac{16L^2D^2_\Psi}{N} + 8LD_\Psi\sigma \left( 1 + \frac{1}{c} \right) \sqrt{\frac{c + 4}{N}}.$$

(20)

In the view of the convergence speed on $N$, we focus on the dominant term in the bound (20), i.e., the second term of the right-hand side in (20). We can easily show that this term attains the minimum when $c = (1 + \sqrt{33})/2$. Here we compare two convergence rates:

SGD case of (4.23) in [29]: $E \left[ \|P_{X,R}\|^2 \right] \leq \frac{16L^2D^2_\Psi}{N} + \frac{8L\sqrt{6D_\Psi}\sigma}{\sqrt{N}}$,

(20) with $c = (1 + \sqrt{33})/2$; $E \left[ \|P_{X,R}\|^2 \right] \leq \left( \frac{15 + \sqrt{33}}{16} \right) \left\{ \frac{16L^2D^2_\Psi}{N} + \sqrt{\frac{9 + \sqrt{33}}{2L}} \frac{LD_\Psi\sigma}{\sqrt{N}} \right\}.$

Focusing on the dominant term in terms of the convergence speed on $N$, we can easily see that the latter bound is smaller than the former bound, i.e., our convergence rate with $c = (1 + \sqrt{33})/2$ is better than that in the SGD case of (4.23) in [29].

4 Experiments

We present numerical experiment results on representable machine learning task: Classification. All results were obtained in R version 3.3.0 with Intel Core i7-4790K machine.

Problem Formulation: We consider the following regularized logistic regression problem:

$$f(\theta) = E_{(x,y)} \left[ \log(1 + \exp(-yx^T\theta)) \right] = E_{(x,y)} \left[ l((x,y); \theta) \right],$$

where $x \in \mathbb{R}^p$ and $y \in \{-1, 1\}$ represents a class label. For $h(\theta)$, we use SCAD, Capped $l_1$ penalty and $l_0$ penalty.

Dataset: We used several real-world datasets, which were available at the UCI Machine Learning Repository [55]. Table 1 shows the detail of datasets. All datasets were normalized and divided into training and test in advance.

Parameter Setting: The initial point was set to be random and then we generated 30 different initial points randomly. We estimated $L$ and $\sigma^2$ by using relatively small size of subsamples, which were drawn from training data. For estimating $L$ and $\sigma^2$, we followed the way to in Sect. 6 in [29]. This subsamples were only used for estimating $L$ and $\sigma^2$ and the size of this subsamples was set to 200. For the two phase scheme, we used $\lfloor 0.1n_{\text{train}} \rfloor$ samples for the validation and randomly selected
Table 1: Detail of datasets.

| Dataset                                      | Training Size $n_{\text{train}}$ | Test Size $n_{\text{test}}$ | Features |
|----------------------------------------------|----------------------------------|------------------------------|----------|
| MAGIC Gamma Telescope (MGT)                  | 15000                            | 4020                         | 11       |
| MiniBooNE particle identification (MBpi)      | 70000                            | 60064                        | 50       |

10 outputs. The step size was set to be $\eta_t = 1/2L$. The mini-batch size $m_t$ was set to be the same as in Corollary 3.3. The tuning parameter $D$ was set to $D_\psi = \left[\psi(\theta^{(1)})/2L\right]^{1/2}$, because the objective function is non-negative, $f(\theta) + h(\theta) \geq 0$, and then $\psi^* \geq 0$. The tuning parameter $c$ was set to $(1 + \sqrt{33})/2$. The tuning parameter $\lambda_j$ of $h(\theta) = \sum_{j=1}^p \lambda_j h_j(\theta_j)$ was set to be $\lambda_j = \lambda$ for $j = 1, \ldots, p$, and $\lambda$ was set to $10^{-2}, 10^{-1}, 10^0$.

**Convergence Criterion:** To verify the convergence, we used the modified $\|P_{X,R}\|$ whose full gradient $\nabla f(\theta^{(R)})$ was approximated by using test data, i.e., $1/n_{\text{test}} \sum_{i=1}^{n_{\text{test}}} \nabla l(x_i, y_i; \theta^{(R)})$. For comparative methods, $\theta^{(R)}$ was replaced by the final output $\hat{\theta}$.

**Comparative Method:** There is no existing SGD method which guarantees the convergence for stochastic doubly-nonconvex composite optimization. Therefore, we compared our method with the averaging SGD (ASGD) and polynomial-decay averaging SGD (PDSGD), which guarantee the convergence for the stochastic convex composite optimization problem. ASGD and PDSGD use the averages $\bar{\theta}_A = \frac{1}{T} \sum_{t=1}^T \theta^{(t)}$ and $\bar{\theta}^{(T)}_{PD} = \frac{t \bar{\theta}_P^{(T)}}{t+2} + \frac{2 \theta^{(t+1)}}{t+2}$, respectively. The final output $\hat{\theta}$ was set to be $\bar{\theta}_A$ and $\bar{\theta}^{(T)}_{PD}$ for ASGD and PDSGD, respectively. Moreover, we incorporated the mini-batch scheme into the comparative methods. The mini-batch size was set to be the same as that in our setting.

**Result:** Table 2 shows the average of the convergence criterion with 30 different initial points. Our method outperformed the comparative methods. As the tuning parameter $\lambda$ was larger, our method tended to show smaller values, but the comparative methods rather larger. When $\lambda = 10^0$, our method was much better than the comparative methods. This would be because a nonconvex effect is larger as $\lambda$ is larger. The sample size of the MBpi dataset is larger than that of the MGT dataset, although the number of features of the former is also larger than that of the latter. Our method gave much smaller values for the MBpi dataset than the MGT dataset. The comparative methods did not show such a behavior and rather presented worse behaviors at some cases.

Table 2: Average of convergence criterion $\|P_{X,R}\|$ with 30 different initial points.

| Dataset | Tuning parameter $\lambda$ | Methods       | SCAD | Capped $l_1$ penalty | $l_0$ penalty |
|---------|---------------------------|---------------|------|----------------------|---------------|
| MGT     | $10^{-2}$                 | Our method    | 0.204| 0.188                | 0.173         |
|         |                           | ASGD          | 0.338| 0.343                | 0.644         |
|         |                           | PDSGD         | 0.271| 0.276                | 0.844         |
|         | $10^{-1}$                 | Our method    | 0.184| 0.267                | 0.174         |
|         |                           | ASGD          | 0.338| 0.475                | 2.79          |
|         |                           | PDSGD         | 0.271| 0.406                | 2.22          |
|         | $10^0$                    | Our method    | 0.173| 0.125                | 0.173         |
|         |                           | ASGD          | 0.512| 2.8                  | 9.85          |
|         |                           | PDSGD         | 0.575| 2.14                 | 5.46          |
| MBpi    | $10^{-2}$                 | Our method    | 0.0297| 0.0487             | 0.0224        |
|         |                           | ASGD          | 0.151| 0.168                | 1.12          |
|         |                           | PDSGD         | 0.055| 0.0808               | 1.05          |
|         | $10^{-1}$                 | Our method    | 0.0259| 0.0281             | 0.0221        |
|         |                           | ASGD          | 0.154| 0.648                | 3.87          |
|         |                           | PDSGD         | 0.0579| 0.557              | 3.1           |
|         | $10^0$                    | Our method    | 0.022| 0.00781             | 0.02          |
|         |                           | ASGD          | 1.05 | 4.32                 | 15.1          |
|         |                           | PDSGD         | 0.841| 1.8                  | 11.9          |
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Appendix

A1: Nonconvex Examples of $h_j(\theta_j)$

We show nonconvex examples $h_j(\theta_j)$ and closed-form solutions given by [33]. For the ease of notation, we remove subscripts $\lambda_j h_j(\theta_j)$ and denote the corresponding proximal operator problem as $\hat{\theta} = \arg\min_{\theta \in \mathbb{R}} L(\theta) := \frac{1}{2}(\theta - \alpha)^2 + \lambda h(\theta)$. The tuning parameters $c_1 > 2$ and $c_2 > 0$.

**SCAD:** It has the following form.

$$h(\theta) = \begin{cases} 
\lambda |\theta| & \text{if } |\theta| \leq \lambda, \\
\frac{-\theta^2 + 2c_1 \lambda |\theta| - \lambda^2}{2(c_1 - 1)} & \text{if } \lambda_j < |\theta_j| \leq c_1 \lambda, \\
(c_1 + 1)\lambda^2 / 2 & \text{if } |\theta| > c_1 \lambda.
\end{cases}$$

The closed-form solution is given by

$$\hat{\theta} = \arg\min_{\theta \in \{\hat{\theta}, \check{\theta}, \hat{\theta}\}} L(\theta),$$

where $\hat{\theta} = \text{sign}(\alpha) \min(\lambda, \max(0, |\alpha| - \lambda))$, $\check{\theta} = \text{sign}(\alpha) \min\left(c_1 \lambda, \max\left(\lambda, \frac{|\alpha|(c_1 - 1) - c_1 \lambda}{(c_1 - 2)}\right)\right)$ and $\hat{\theta} = \text{sign}(\alpha) \max(c_1 \lambda, |\alpha|)$. 

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We define the Bouligand tangent cone of the set $S$ as the smallest set containing all tangent directions at a point in $S$. This cone is used to study the local behavior of a function near a point.

We denote the sublevel set of the function $f$ at a point $x$ as $\{ y \in S | f(y) \leq f(x) \}$. This set is important in optimization since it represents the set of points where the function does not exceed a certain value.

The closed-form solution is given by

$$\hat{\theta} = \arg\min_{\theta \in \{ \hat{\theta}, \hat{\theta}, \hat{\theta} \}} L(\theta),$$

where $\hat{\theta} = 0$, $\hat{\theta} = \text{sign}(\alpha)c_2\lambda$, $\hat{\theta} = \text{sign}(\alpha)|\alpha|$ and $\hat{\theta} = \text{sign}(\alpha)\frac{c_2(|\alpha| - \lambda)}{(c_2 - 1)}$ if $c_2 \neq 1$.

**Log-sum penalty:** It has the following form

$$h(\theta) = \lambda \log(1 + |\theta|/c_2).$$

The closed-form solution is given by

$$\hat{\theta} = \arg\min_{\theta \in \{ \hat{\theta}, \hat{\theta}, \hat{\theta} \}} L(\theta),$$

where $\hat{\theta} = 0$, $\hat{\theta} = \left[ \frac{(|\alpha| - c_2) - \sqrt{(|\alpha| - c_2)^2 - 4(|\alpha| - c_2)}}{2} \right]_+$, $\hat{\theta} = \left[ \frac{(|\alpha| - c_2) + \sqrt{(|\alpha| - c_2)^2 - 4(|\alpha| - c_2)}}{2} \right]_+$

and $[\cdot]_+ = \max(0, \cdot)$.

**Capped $l_1$ penalty:** It has the following form

$$h(\theta) = \lambda \min(|\theta|, c_2).$$

The closed-form solution is given by

$$\hat{\theta} = \arg\min_{\theta \in \{ \hat{\theta}, \hat{\theta} \}} L(\theta),$$

where $\hat{\theta} = \text{sign}(\alpha)\max(c_2, |\alpha|)$ and $\hat{\theta} = \text{sign}(\alpha)\min(c_2, \max(0, |\alpha| - \lambda))$.

**l_0 penalty:** It has the following form

$$h(\theta) = \lambda \|\theta\|_0.$$

The closed-form solution given by

$$\hat{\theta} = \arg\min_{\theta \in \{ \hat{\theta}, \hat{\theta} \}} L(\theta),$$

where $\hat{\theta} = 0$ and $\hat{\theta} = \alpha$.

**A2: Sufficient Optimality Condition in [52]**

In this section, we modify and show the sufficient optimality condition in [52] in order to apply our method. We follow many notations given by [52]. For more detailed descriptions and proofs, we refer to [52].

Let $S$ be a set in the Euclidean space $\mathbb{R}^n$. Let $cl(S)$ denote closed hull of the set $S$.

We denote the $r$-neighbourhood by

$$B(x, r) := \{ y \in \mathbb{R}^n | \| y - x \| \leq r \}.$$

We denote the sublevel set of the function $f : S \rightarrow \mathbb{R}$ at $x \in S$ by

$$L(f; x; S) := \{ y \in S | f(y) \leq f(x) \}.$$

We define the Bouligand tangent cone of the set $S$ at $x \in cl(S)$ by

**Definition 4.1. [Bouligand tangent cone]**

$$T(S, x) := \{ u \in \mathbb{R}^n | \exists \{ t_k \}, t_k \rightarrow +0, \exists \{ u_k \} \subset \mathbb{R}^n, u_k \rightarrow u \text{ such that } x + t_ku_k \in S \text{ for all } k \geq 1 \}.$$
We define the lower Hadamard directional derivative of the function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in direction $u \in \mathbb{R}^n$ by

**Definition 4.2. [Lower Hadamard directional derivative]**

$$f^L_H(x; u) := \liminf_{(t,u') \to (0,u)} \frac{f(x + tu') - f(x)}{t}.$$  

If the function $f$ is differentiable, it reduces to $\langle \nabla f, u' \rangle$.

We define the strongly pseudoconvex sublevel set by

**Definition 4.3. [Strongly pseudoconvex sublevel set]** Let $f : X \to \mathbb{R}$ and $\bar{x} \in X$. If for all $x \in L(f; \bar{x}; X)$ and $x \neq \bar{x}$, there is a number $\epsilon > 0$ and sequences $t_k \to +0$, $u_k \to (x - \bar{x})$ such that

$$f(\bar{x} + t_k u_k) \leq f(\bar{x}) - \epsilon t_k.$$  

The sublevel set $L(f; \bar{x}; X)$ is said to be strongly pseudoconvex.

Actually, any differentiable strictly convex function satisfy this definition. For example, $f(x) = \|x\|^2$ satisfies this definition because it is differentiable strong convex.

We consider the following optimization problem (P):

$$\min_{x \in \mathbb{R}^n} f(x)$$  
$$\text{subject to } g(x) \leq 0.$$  

Then, we show the following modified sufficient optimality condition in [52].

**Theorem 4.1. [The modified Sufficient Optimality Condition of Theorem 10 in [52]]** Let $\bar{x}$ be a feasible point of (P). Assume that $f(x)$ is differentiable, that its sublevel set is strongly pseudoconvex, that $g(x)$ is quasiconvex, that $g(\bar{x}) = 0$, and that there exists

$$(\lambda, \mu) \in [0, +\infty) \times [0, +\infty) \setminus \{(0, 0)\}.$$  

If the following condition for $u \in T(L(f; \bar{x}; X), \bar{x}) \cap T(L(g; \bar{x}; X), \bar{x})$ is satisfied,

$$\langle \lambda f + \mu g \rangle^L_H(x; u) \geq 0,$$

then, $\bar{x}$ is unique global minimizer of (P).

In our problem formulation, $f(x)$ corresponds to $x + \eta_t G(\theta^{(t)}) - \theta_j^{(t)}^2/2$ and $g(x)$ corresponds to $h_j(x) - K_j$ and $\mu = \eta_t \lambda_j \geq 0$ and $\lambda = 1$. When we set $K_j$ to $h_j(x)$, the assumption $g(\bar{x}) = 0$ is satisfied. As showed before, $x + \eta_t G(\theta^{(t)}) - \theta_j^{(t)}$ has the strongly pseudoconvex sublevel set. If $\bar{x}$ is a local minimizer of $f(x) + \mu g(x)$, it follows that $f(\bar{x}) + \mu g(\bar{x}) \leq f(x) + \mu g(x)$ with $x \in B(\bar{x}, r)$ for some $r > 0$. Then, we see that $f + \mu g(\bar{x})^L_H(x; u) \geq 0$ for any direction $u \in \mathbb{R}^n$.

Therefore, our problem formulation with a local minimizer $\bar{x}$ satisfies the conditions of the modified Sufficient Optimality Condition of Theorem 10 in [52].
A3: Proof of Theorem 3.1

Proof. Replacing $\theta^{(t+1)}$ and $\theta^{(t)}$ by $\theta_1$ and $\theta_2$, respectively, in (3). Then, we see from (14) that

$$f(\theta^{(t+1)}) \leq f(\theta^{(t)}) + \langle \nabla f(\theta^{(t)}), \theta^{(t+1)} - \theta^{(t)} \rangle + \frac{L}{2} \| \theta^{(t+1)} - \theta^{(t)} \|^2$$

$$= f(\theta^{(t)}) + \langle \nabla f(\theta^{(t)}), \theta^{(t+1)} - \theta^{(t)} \rangle + \frac{L\eta^2}{2} \| \hat{P}_{X,t} \|^2$$

$$= f(\theta^{(t)}) - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), \theta^{(t+1)} - \theta^{(t)} \rangle + \frac{L\eta^2}{2} \| \hat{P}_{X,t} \|^2 + \langle G(\theta^{(t)}), \theta^{(t+1)} - \theta^{(t)} \rangle$$

$$= f(\theta^{(t)}) - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), \theta^{(t)} - \theta^{(t+1)} \rangle - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle$$

$$\quad + \frac{L\eta^2}{2} \| \hat{P}_{X,t} \|^2 + \sum_{j=1}^{p} G(\theta^{(t)})_{j}(\theta_{j}^{(t+1)} - \theta_{j}^{(t)})$$

$$= f(\theta^{(t)}) - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), \theta^{(t+1)} - \theta^{(t+1)} \rangle - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle$$

$$\quad + \frac{L\eta^2}{2} \| \hat{P}_{X,t} \|^2 + \sum_{j=1}^{p} G(\theta^{(t)})_{j}(\theta_{j}^{(t+1)} - \theta_{j}^{(t)}). \quad (21)$$

Let the objective function in (7) be denoted by $L(\tilde{u})$. Since $\theta_{j}^{(t+1)}$ is a global minimizer of $L(\tilde{u})$, we have $L(\theta_{j}^{(t+1)}) \leq L(\theta_{j}^{(t)})$. After simple calculation, we obtain

$$\frac{1}{2\eta} (\theta_{j}^{(t+1)} - \theta_{j}^{(t)})^2 + \lambda_j h(\theta_{j}^{(t+1)}) - \lambda_j h(\theta_{j}^{(t)}) \leq G(\theta^{(t)})_{j}(\theta_{j}^{(t)} - \theta_{j}^{(t+1)}). \quad (22)$$

Hence, it follows from (1), (21) and (22) that

$$f(\theta^{(t+1)}) \leq f(\theta^{(t)}) - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), \theta_{j}^{(t+1)\text{full}} - \theta^{(t+1)} \rangle - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle$$

$$+ \frac{L\eta^2}{2} \| \hat{P}_{X,t} \|^2 - \sum_{j=1}^{p} \left\{ \frac{1}{2\eta}(\theta_{j}^{(t+1)} - \theta_{j}^{(t)})^2 - \lambda_j h(\theta_{j}^{(t+1)}) + \lambda_j h(\theta_{j}^{(t)}) \right\}$$

$$\Psi(\theta^{(t+1)}) \leq \Psi(\theta^{(t)}) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \| \hat{P}_{X,t} \|^2$$

$$\quad - \sum_{j=1}^{p} \langle \nabla f(\theta^{(t)})_{j} - G(\theta^{(t)})_{j}(\theta_{j}^{(t+1)\text{full}} - \theta_{j}^{(t+1)}), P_{X,t} \rangle$$

$$\quad \leq \Psi(\theta^{(t)}) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \| \hat{P}_{X,t} \|^2$$

$$\quad + \sum_{j=1}^{p} \langle \nabla f(\theta^{(t)})_{j} - G(\theta^{(t)})_{j}(\theta_{j}^{(t+1)} - \theta_{j}^{(t+1)\text{full}}), P_{X,t} \rangle.$$
Therefore, it follows from Proposition 3.2 that

\[
\Psi(\theta^{(t+1)}) \leq \Psi(\theta^{(t)}) - \left( \frac{\eta_t}{2} - \frac{L\eta_t^2}{2} \right) \| \tilde{P}_{X,t} \|^2 \\
+ \sum_{j=1}^{P} |\nabla f(\theta^{(t)})_j - G(\theta^{(t)})_j\|_{\theta^{(t+1)}}^2 - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \\
\leq \Psi(\theta^{(t)}) - \left( \frac{\eta_t}{2} - \frac{L\eta_t^2}{2} \right) \| \tilde{P}_{X,t} \|^2 \\
+ \eta_t \left\{ \nabla f(\theta^{(t)}) - G(\theta^{(t)}) \right\}^2 - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \\
\leq \Psi(\theta^{(t)}) - \left( \frac{\eta_t}{2} - \frac{L\eta_t^2}{2} \right) \| \tilde{P}_{X,t} \|^2 \\
+ \eta_t \left\{ \nabla f(\theta^{(t)}) - G(\theta^{(t)}) \right\}^2 - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle.
\]

Summing up the above inequalities for \( t = 1, \ldots, T \). Hence, we see from assumption 1 that

\[
\sum_{t=1}^{T} \left\{ \left( \frac{\eta_t}{2} - \frac{L\eta_t^2}{2} \right) \| \tilde{P}_{X,t} \|^2 \right\} \leq \Psi(\theta^{(1)}) - \Psi(\theta^{(T)}) \\
+ \sum_{t=1}^{T} \left\{ \eta_t \| \nabla f(\theta^{(t)}) - G(\theta^{(t)}) \|^2 - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \right\}
\]

\[
\sum_{t=1}^{T} \left\{ \left( \frac{\eta_t}{2} - \frac{L\eta_t^2}{2} \right) \| \tilde{P}_{X,t} \|^2 \right\} \leq \Psi(\theta^{(1)}) - \Psi^* \\
+ \sum_{t=1}^{T} \left\{ \eta_t \| G(\theta^{(t)}) - \nabla f(\theta^{(t)}) \|^2 - \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \right\}.
\]

(23)

It holds from (4) that

\[
E \left[ \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \right] = 0.
\]

Moreover, we see from (4) and (5) that

\[
E \left[ \| G(\theta^{(t)}) - \nabla f(\theta^{(t)}) \|^2 \right] \\
= E \left[ \left\| \frac{1}{m_t} \sum_{i=1}^{m_t} G_{X_{t,i}}(\theta^{(t)}) - \nabla f(\theta^{(t)}) \right\|^2 \right] \\
= E \left[ \sum_{i=1}^{m_t} \left\| \frac{1}{m_t} G_{X_{t,i}}(\theta^{(t)}) - \frac{1}{m_t} \nabla f(\theta^{(t)}) \right\|^2 \right] \\
+ 2 \sum_{1 \leq j \neq k \leq m_t} \left\| \frac{1}{m_t} G_{X_{t,j}}(\theta^{(t)}) - \frac{1}{m_t} \nabla f(\theta^{(t)}) \right\| \left\| \frac{1}{m_t} G_{X_{t,k}}(\theta^{(t)}) - \frac{1}{m_t} \nabla f(\theta^{(t)}) \right\| \\
= E \left[ \sum_{i=1}^{m_t} \left\| \frac{1}{m_t} G_{X_{t,i}}(\theta^{(t)}) - \frac{1}{m_t} \nabla f(\theta^{(t)}) \right\|^2 \right] \\
\leq \frac{\sigma^2}{m_t}.
\]
Taking expectation with respect to $X_t, \ldots, X_T$ on both sides of (23). Hence, we obtain

$$
\sum_{t=1}^{T} \left\{ \left( \frac{\eta_t}{2} - \frac{L \eta_t^2}{2} \right) E \left[ \left\| \tilde{P}_{X,t} \right\|^2 \right] \right\} \leq \Psi(\theta^{(1)}) - \Psi^* 
+ \sum_{t=1}^{T} \left\{ \eta_t E \left[ \left\| G(\theta^{(t)}) - \nabla f(\theta^{(t)}) \right\| \right] 
- E \left[ \langle \nabla f(\theta^{(t)}) - G(\theta^{(t)}), P_{X,t} \rangle \right] \right\}
$$

$$
\sum_{t=1}^{T} \left\{ \left( \frac{\eta_t}{2} - \frac{L \eta_t^2}{2} \right) E \left[ \left\| \tilde{P}_{X,t} \right\|^2 \right] \right\} \leq \Psi(\theta^{(1)}) - \Psi^* + \sigma^2 \sum_{t=1}^{T} \frac{\eta_t}{m_t}.
$$

Furthermore, diving both sides of the above inequality by $\sum_{t=1}^{T} \left( \eta_t/2 - L \eta_t^2/2 \right)$ under the condition that $\eta_t \leq 1/L$ with $\eta_t < 1/L$ for at least one $t$.

$$
\frac{\sum_{t=1}^{T} \left\{ \left( \frac{\eta_t}{2} - \frac{L \eta_t^2}{2} \right) E \left[ \left\| \tilde{P}_{X,t} \right\|^2 \right] \right\}}{\sum_{t=1}^{T} \left( \frac{\eta_t}{2} - \frac{L \eta_t^2}{2} \right)} \leq \frac{\Psi(\theta^{(1)}) - \Psi^* + \sigma^2 \sum_{t=1}^{T} \frac{\eta_t}{m_t}}{\sum_{t=1}^{T} \left( \frac{\eta_t}{2} - \frac{L \eta_t^2}{2} \right)}.
$$

\[ \square \]

A4: Proof of Theorem 3.2

Proof. Substituting $\eta_t = 1/2L$ and $m_t = m$ into (16). Then, we have

$$
E \left[ \left\| \tilde{P}_{X,R} \right\|^2 \right] \leq 2 \frac{LD_\psi^2 + \sigma^2 \sum_{t=1}^{T} (1/2mL)}{\sum_{t=1}^{T} (1/2L - 1/4L)} = \frac{8L^2D_\psi^2}{T} + \frac{4\sigma^2}{m}.
$$

We see from the above inequality and $0 \leq \left( \sqrt{c}x - \frac{y}{\sqrt{c}} \right)^2$ for any $c \geq 0$ that

$$
E \left[ \left\| P_{X,R} \right\|^2 \right] = E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} + \tilde{P}_{X,R} \right\|^2 \right]
= E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\|^2 \right] + E \left[ \left\| \tilde{P}_{X,R} \right\|^2 \right] + 2E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\| \left\| \tilde{P}_{X,R} \right\| \right]
\leq E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\|^2 \right] + E \left[ \left\| \tilde{P}_{X,R} \right\|^2 \right] + cE \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\|^2 \right]
+ \frac{1}{c} E \left[ \left\| \tilde{P}_{X,R} \right\|^2 \right]
= (1 + c)E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\|^2 \right] + \left( 1 + \frac{1}{c} \right) E \left[ \left\| \tilde{P}_{X,R} \right\|^2 \right]
\leq (1 + c)E \left[ \left\| P_{X,R} - \tilde{P}_{X,R} \right\|^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D_\psi^2}{T} + \frac{4\sigma^2}{m} \right).
$$
It follows from assumption 3, (13) and (14) that

\[
(1 + c)E \left[ \| P_{X,R} - \tilde{P}_{X,R} \|^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m} \right)
\]

\[
= \frac{(1 + c)}{\eta^2_R} E \left[ \| \theta_{f \alpha}^{(R+1)} - \theta^{(R+1)} \|^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m} \right)
\]

\[
= \frac{(1 + c)}{\eta^2_R} E \left[ \sum_{j=1}^{p} (\theta_{f \alpha \gamma,j}^{(R+1)} - \theta_{j}^{(R+1)})^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m} \right)
\]

\[
\leq (1 + c) E \left[ \| G(\theta^{(R)}) - \nabla f(\theta^{(R)}) \|^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m} \right)
\]

\[
\leq (1 + c) E \left[ \| G(\theta^{(R)}) - \nabla f(\theta^{(R)}) \|^2 \right] + \left( 1 + \frac{1}{c} \right) \left( \frac{8L^2D^2_{\Psi}}{T} + \frac{4\sigma^2}{m} \right)
\]

\[
= \left( 1 + \frac{1}{c} \right) \frac{8L^2D^2_{\Psi}}{T} + \left( 5c + \frac{4}{c} \right) \frac{\sigma^2}{m}.
\]

\[\square\]

**A5: Proof of Theorem 3.3**

**Proof.** The iteration limit $T$ is at most $\lfloor N/m \rfloor$, then we have $T \geq N/2m$. Moreover, we see from (18) that

\[
m \leq \left( 1 + \frac{\sigma}{4LD} \sqrt{(c + 4)N} \right),
\]

\[
\frac{1}{m} \leq \max \left( \frac{4LD \tilde{D}}{\sigma \sqrt{(c + 4)N}}, \frac{1}{N} \right).
\]

Hence, it follows from (17) and these inequalities that

\[
E \left[ \| P_{X,R} \|^2 \right] \leq \left( 1 + \frac{1}{c} \right) \frac{8L^2D^2_{\Psi}}{T} + \left( 5c + \frac{4}{c} \right) \frac{\sigma^2}{m}
\]

\[
\leq \left( 1 + \frac{1}{c} \right) \frac{16mL^2D^2_{\Psi}}{N} + \left( 5c + \frac{4}{c} \right) \frac{\sigma^2}{m}
\]

\[
\leq \left( 1 + \frac{1}{c} \right) \frac{16L^2D^2_{\Psi}}{N} \left( 1 + \frac{\sigma}{4LD} \sqrt{(c + 4)N} \right) + \left( 5c + \frac{4}{c} \right) \max \left( \frac{4LD \tilde{D} \sigma}{\sqrt{(c + 4)N}}, \frac{\sigma^2}{N} \right)
\]

\[
= \left( 1 + \frac{1}{c} \right) \frac{16L^2D^2_{\Psi}}{N}
\]

\[
+ \left( 1 + \frac{1}{c} \right) \frac{4LD^2_{\Psi} \sigma}{D} \sqrt{\frac{c + 4}{N}} + \left( 1 + \frac{1}{c} \right) 4L \tilde{D} \sigma \sqrt{\frac{c + 4}{N}} \max \left( 1, \frac{\sigma}{4LD} \sqrt{\frac{c + 4}{N}} \right)
\]

\[
= \left( 1 + \frac{1}{c} \right) \frac{16L^2D^2_{\Psi}}{N} + 4L \sigma \left( 1 + \frac{1}{c} \right) \sqrt{\frac{c + 4}{N}} \left( \frac{D^2_{\Psi}}{D} + \tilde{D} \max \left( 1, \frac{\sigma}{4LD} \sqrt{\frac{c + 4}{N}} \right) \right).
\]

\[\square\]