Local approach to thermodynamics of black holes

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Abstract

Hamiltonian description of gravitational field contained in a spacetime region with boundary $S$ being a null-like hypersurface (a wave front) is discussed. Complete generating formula for the Hamiltonian dynamics (with no surface integrals neglected) is presented. A quasi-local proof of the 1-st law of black holes thermodynamics is obtained as a consequence, in case when $S$ is a non-expanding horizon. The 0-th law and Penrose inequalities are discussed from this point of view.

1 Introduction

Evolution of gravitational field within a finite tube with a time-like boundary was considered in [1] and then reformulated in [6]. Here, we extend this description to the case of a wave front (a three-dimensional submanifold whose internal metric is degenerate). Restricting our result to the special case of wave fronts, namely to non-expanding horizons, we obtain the 1-st law of thermodynamics of black holes as a simple consequence.

Hamiltonian formulation of any field theory needs always an appropriate control of boundary data. To illustrate this point, consider as an example the linear theory of an elastic string. Field configuration of a string is

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described by its displacement function: \( \mathbb{R} \times [a, b] \ni (t, x) \rightarrow \varphi(t, x) \in \mathbb{R} \), fulfilling the wave equation, where velocity “\( c \)” is a combination of the string’s proper density (per unit length) and its elasticity coefficient:

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \frac{\partial^2}{\partial x^2} \varphi .
\]  

Passing to appropriate time and length units, we may always put \( c = 1 \).

The dynamics of the system may be derived from the Lagrangian density

\[
L = -\frac{1}{2} \sqrt{\det g} \ g^\mu\nu (\partial_\mu \varphi)(\partial_\nu \varphi) = \frac{1}{2} \sqrt{\det g} \ (\dot{\varphi}^2 - (\varphi')^2) ,
\]  

where \( \mu, \nu = 0, 1 \) and \((x^0, x^1) = (t, x)\), \( g_{\mu\nu} = \text{diag}(-1, +1)\), “dot” denotes the time derivative and “prime” denotes the space derivative. Entire information about the dynamics of the string may be encoded in the form of the following Lagrangian generating formula (see [7] or [8] for its correct symplectic interpretation):

\[
\delta L(\varphi, \partial_\nu \varphi) = \partial^\mu (p^\mu \delta \varphi) = (\partial^\mu p^\mu) \delta \varphi + p^\mu \delta (\partial_\mu \varphi) .
\]  

It contains the definition of the canonical momenta: 1) the kinetic momentum

\[
\pi := p^0 = \frac{\partial L}{\partial (\partial_0 \varphi)} = \partial_0 \varphi = \dot{\varphi} ,
\]

and 2) the stress density

\[
\pi^\perp := p^1 = \frac{\partial L}{\partial (\partial_1 \varphi)} = -\partial_1 \varphi = -\varphi' ,
\]

together with the Euler-Lagrange equation, obviously equivalent to (1):

\[
\partial\mu p^\mu = \frac{\partial L}{\partial \varphi} .
\]

Integrating infinitesimal generating formula (3) over the entire string \([a, b]\) we obtain the finite generating formula:

\[
\delta \int_a^b L = \int_a^b \left( \pi^\perp \delta \varphi + \pi \delta \dot{\varphi} \right) + \left[ \pi^\perp \delta \varphi \right]_a^b .
\]  

Hamiltonian description of the same dynamics is obtained via Legendre transformation between \( \pi \) and \( \dot{\varphi} \), putting \( \pi \delta \dot{\varphi} = \delta (\pi \dot{\varphi}) - \dot{\varphi} \delta \pi \):

\[
-\delta H = \int_a^b \left( \pi \delta \dot{\varphi} - \dot{\varphi} \delta \pi \right) + \left[ \pi^\perp \delta \varphi \right]_a^b ,
\]

with

\[
H := \int_a^b (\pi \dot{\varphi} - L) = \frac{1}{2} \int_a^b (\pi^2 + (\varphi')^2) .
\]  

This formal expression acquires a precise, infinitely-dimensional, Hamiltonian meaning:

\[
\dot{\pi} = -\frac{\delta H}{\delta \varphi} , \quad \dot{\varphi} = \frac{\delta H}{\delta \pi} ,
\]  

(7)
as soon as the boundary terms in (5) are killed by imposing e.g. the Dirichlet boundary conditions, i.e. by restricting ourselves to an infinitely dimensional phase space of initial data \((\varphi, \pi)\), defined on \([a, b]\) and fulfilling conditions: \(\varphi(a) \equiv A, \varphi(b) \equiv B\). Within this phase space we have \(\delta \varphi(a) = \delta \varphi(b) = 0\) and equations (4) hold.

Consider now the subspace of static solutions: \(\dot{\pi} = 0 = \dot{\varphi}\). Due to (7), these are points where the derivative of the functional \(H\) vanishes and the Hamiltonian formula (5) reduces to the formula for virtual work

\[
\delta H = -\left[\pi^\perp \delta \varphi\right]^b_a,
\]

But, due to (6), \(H\) is manifestly convex. This implies that every static solution gives the minimal value of the Hamiltonian in the corresponding phase space. Due to equation (11) and to boundary conditions, such a solution is given by: \(\pi \equiv 0\) and \(\varphi(x) = A + (x - a) \frac{B - A}{b - a}\). Inserting this value into (6) we obtain the following “Penrose-like inequality”:

\[
\frac{(B - A)^2}{b - a} \leq H,
\]

analogous to the gravitational Penrose inequality relating the energy carried by Cauchy data outside of a horizon \(S\) and the energy of a black hole corresponding to the same value of appropriate boundary data on \(S\).

Instead of controlling the string configuration at the boundary, we may control e.g. its stress by applying an appropriate force \(F\). This leads to the Neumann control mode \(\pi^\perp(a) = F_{\text{left}}, \pi^\perp(b) = F_{\text{right}}\), which is again a legitimate Hamiltonian system

\[
-\delta \tilde{H} = \int_a^b (\dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi) - \left[\varphi \delta \pi^\perp\right]^b_a,
\]

where \(\tilde{H}\), obtained via Legendre transformation between \(\varphi\) and \(\pi^\perp\) at the boundary, plays role of a free energy:

\[
\tilde{H} := H + \left[\varphi \pi^\perp\right]^b_a = H - \left[\varphi \varphi'\right]^b_a = \frac{1}{2} \int_a^b (\pi^2 - (\varphi')^2 - 2 \varphi \varphi'').
\]

Again, the boundary term in (10) vanishes due to Neumann conditions and the field dynamics reduces to (7). However, the new Hamiltonian (11) is obviously non-convex. There is, therefore, no “Penrose-like” inequality in this mode: static solutions corresponding to stationary points of the Hamiltonian \(\tilde{H}\) are not minimal points of such a “free energy”.

In [1] the dynamics of the gravitational field within a time-like world tube \(S\) was analyzed in a similar way. For this purpose the so called “affine variational principle” was used, where the Lagrangian function depends on the Ricci tensor of a spacetime connection \(\Gamma\). In this picture, the metric tensor \(g\) arises only in the Hamiltonian formulation as the momentum canonically conjugate to \(\Gamma\). Later, it was proved in [6] that the Hamiltonian dynamics obtained this way is universal and does not depend upon a specific variational formulation we start with (actually, it can be derived from field equations only, without any use of variational principles,
the existence of them being a consequence of the “reciprocity” of Einstein equations – see [7] and [8]). On the contrary, the Hamiltonian picture is very sensitive to the method of controlling the boundary data. A list of natural control modes, leading to different “quasilocal Hamiltonians”, is given in [6]. A conjecture about the “true mass”, based on an analysis of the linearized theory [4], is also formulated there.

The aim of the present paper is to give a generalization of the above results to the case when the boundary \( S \) is a three-dimensional submanifold of spacetime \( M \) whose internal three-metric \( g_{ab} \) is degenerate.

\section{Dynamics of the gravitational field inside a null hypersurface}

Consider gravitational field dynamics inside a null hypersurface \( S \):

Parameter \( s = \pm 1 \) labels two possible situations: an expanding or a shrinking wave front (if \( S \) is a horizon, these correspond to a black hole or a white hole case). To simplify notation we use coordinates \( x^\mu, \mu = 0, 1, 2, 3 \), adapted to the above situation: \( x^0 = t \) is constant on a chosen family of Cauchy surfaces whereas \( x^3 \) is constant on the boundary \( S \) (this does not mean that \( x^3 \) is null-like everywhere, but only on \( S \)). Coordinates \( x^A, A = 1, 2, \) are “angular” coordinates on the 2-surface \( \partial V = V \cap S \) whose topology is assumed to be that of a 2-sphere. Finally, \( x^k, k = 1, 2, 3, \) are spatial coordinates on the Cauchy surfaces \( \{ x^0 = \text{const.} \} \) and \( x^a, a = 0, 1, 2, \) are coordinates on the boundary \( S \).

In paper [9] we derive the following formula:

\begin{equation}
-\delta \mathcal{H} = \frac{1}{16\pi} \int_V \left( \mathcal{P}^{kl} \delta g_{kl} - \mathcal{Q}_{kl} \delta \mathcal{P}^{kl} \right) + \frac{s}{8\pi} \int_{\partial V} (\lambda \delta a - \tilde{a} \delta \lambda)
+ \frac{s}{16\pi} \int_{\partial V} \left( \lambda \delta \Lambda^0 - 2 \left( w_0 \delta \Lambda^0 - \Lambda^0 \delta w_0 \right) \right),
\end{equation}

where

\begin{equation}
\mathcal{H} = \frac{1}{8\pi} \int_V \mathcal{G}^0 + \frac{s}{8\pi} \int_{\partial V} \lambda \mathcal{L} \equiv \frac{s}{8\pi} \int_{\partial V} \lambda \mathcal{L},
\end{equation}

and \( \mathcal{P}^{kl} \) denotes external curvature of the Cauchy surface, written in ADM form. Moreover, \( \lambda = \sqrt{\det g_{AB}} \) is a two-dimensional volume form and \( a = -\frac{1}{2} \log |g^{00}| \). The remaining objects are constructed from a null field \( K \) tangent to \( S \). It is not unique, since \( fK \) is also a null field for any function \( f \) on \( S \). For purposes of the Hamiltonian formula (12) we
always use the normalization compatible with the (3+1)-decomposition used here: $\langle K, dx^0 \rangle = 1$. Hence, $K = \partial_0 - n^a \partial_A$. The vector-density $\Lambda^a = \lambda K^a = (\lambda, -\lambda n^A)$ is uniquely defined on $S$. Now we define

$$l_{ab} := -g(\partial_b, \nabla_a K) = -\frac{1}{2} \mathcal{L}_K g_{ab},$$  \hspace{1cm} \text{(14)}$$

$$w_a := -\langle \nabla_a K, dx^0 \rangle,$$  \hspace{1cm} \text{(15)}$$

where $g_{ab}$ is the induced (degenerate) metric on $S$. Denoting by $g^{AB}$ the inverse two-metric, we define the null mean curvature: $\kappa = \tilde{g}^{AB} l_{AB}$ (often denoted by $\theta$—see \cite{5}).

The volume term in \text{(13)} vanishes due to constraint equations $G_{0\nu} = 0^2$. $G_{0\nu}$ is often denoted by $NH + N^k H_k$ (see e.g. \cite{5}), where $H$ is the scalar (“Hamiltonian”) constraint and $H_k$ are the vector (“momentum”) constraints, $N$ and $N^k$ are the lapse and the shift functions. Constraint equations $H = 0$ and $H_k = 0$ imply vanishing of $G_{0\nu}$.

In \cite{6} we give two independent proofs of the formula \text{(12)}. The first one is analogous to the transition from formula \text{(3)} to formul a \text{(5)}. For this purpose we use Einstein equations written analogously to \text{(1)}:

$$\delta L = \partial_\kappa \left( \pi^{\mu\nu} \delta A^\mu_{\nu} \right),$$  \hspace{1cm} \text{(16)}$$

where $\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu}$, and $A^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \Gamma^\kappa_{\nu)\kappa}$.

Integrating \text{(16)} over a volume $V$ and using metric constraints for the connection $\Gamma$, we directly prove \text{(12)}. However, an indirect proof is also provided, based on a limiting procedure, when a family $S_\epsilon$ of time-like surfaces tends to a light-like surface $S$. It is shown that the non-degenerate formula derived in \cite{1} and \cite{6} gives \text{(12)} as a limiting case for $\epsilon \to 0$.

The last term in \text{(12)} may be written in the following way

$$-\Lambda^A \delta w_A = \lambda n^A \delta w_A = n^A \delta W_A - n^A w_A \delta \lambda,$$

where $W_A := \lambda w_A$ and $n^A := \tilde{g}^{AB} g_{0B}$. Denoting $\kappa := n^A w_A - w_0 = -K^a w_a$, we finally obtain the following generating formula:

$$-\delta H = \frac{1}{16\pi} \int_V \left( \dot{\rho}^k \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right) + \frac{s}{8\pi} \int_{\partial V} \left( \dot{\lambda} \delta a - \dot{a} \delta \lambda \right)$$  \hspace{1cm} \text{(17)}$$

$$+ \frac{s}{16\pi} \int_{\partial V} \left( \dot{\lambda}^A \delta g_{AB} + 2 \left( \kappa \delta \lambda - n^A \delta W_A \right) \right).$$  \hspace{1cm} \text{(18)}$$

Quantity $\kappa$ fulfilling: $K^a \nabla_a K = \kappa K$, is traditionally called a “surface gravity” on $S$. Its value is not an intrinsic property of the surface itself, but depends upon a choice of the null field $K$ on $S$ (i.e. the (3+1)-decomposition of spacetime). In black hole thermodynamics there is a privileged time, compatible with the Killing field of stationary solution and normalized to unity at infinity. In this case the above formula provides, as will be seen, the so called first law of black hole thermodynamics.

\text{\textsuperscript{1}}This is the (2+1)-decomposition of the extrinsic curvature $Q^a_{\nu}(K)$ defined in \text{\cite{5}}.

\text{\textsuperscript{2}}In the presence of matter the volume term equals $G_{00}^\nu - 8\pi T_{00}$ and also vanishes due to constraint equations.
We stress that the symplectic structure of Cauchy data, given by two integrals in (17), is invariant with respect to spacetime diffeomorphisms (see [6]). Neglecting the last, surface integral and defining symplectic form only by the volume integral destroys this gauge invariance.

3 Dynamics of gravitational field outside the null surface

Consider now dynamics of the gravitational field outside a wave front $S^-$. We first add an external, timelike (non-degenerate) boundary $S^+$ and the situation is illustrated by the following figure:

\[ s<0 \quad s>0 \quad s>0 \quad s<0 \]

\[ \partial V^- \quad \partial V^+ \quad \partial V^- \quad \partial V^+ \]

where $\partial V^+ = V \cap S^+$, and $\partial V^- = V \cap S^-$. Because $\partial V^-$ enters with negative orientation, we have: $\int_{\partial V} = \int_{\partial V^+} - \int_{\partial V^-}$. Integrating again Einstein equations written in the form (16) over $V$, using techniques derived in [1] and [6] to handle surface integrals over $\partial V^+$ and formula (12) to handle the surface integrals over $\partial V^-$, we obtain:

\[-\delta H = -\delta H^+ - \delta H^- = \frac{1}{16\pi} \int_V \left( \dot{P}_{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right)\]

\[+ \frac{1}{8\pi} \int_{\partial V^+} \left( \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda \right) + \frac{s}{8\pi} \int_{\partial V^-} \left( \dot{\lambda} \delta a - \dot{a} \delta \lambda \right) - \frac{1}{16\pi} \int_{\partial V^+} Q^{ab} \delta g_{ab}\]

\[+ \frac{s}{16\pi} \int_{\partial V^-} \left( \lambda l_{AB} \delta g_{AB} - 2 \left( w_0 \delta A^0 - \Lambda^A \delta w_A \right) \right), \quad (19)\]

where $\alpha$ is the “hyperbolic angle” between $V$ and $S^+$, whereas $Q^{ab}$ is the external curvature of $S^+$ written in the ADM form (cf. [6]). The contribution $H^+$ to the total Hamiltonian from the external boundary is written here in the form of a “free energy” proposed in [6]:

\[H^+ = -\frac{1}{8\pi} \int_{\partial V^+} Q^0_0 - E_0, \quad (20)\]

where the additive gauge $E_0$ is chosen in such a way that the entire quantity vanishes if $\partial V^+$ is a round sphere in a flat space. The internal contribution to the energy is given by formula (13) with $\partial V$ replaced by $\partial V^-$. Shifting the external boundary to space infinity: $\partial V^+ \to \infty$, the external energy $H^+$ gives the ADM mass, which we denote by $M$, whereas the remaining surface integrals over $\partial V^+$ vanish. This way we obtain the following generating formula for the field dynamics outside of an arbitrary
wave front $S^-$ in an asymptotically flat spacetime:

$$\begin{align}
-\delta M - \delta \mathcal{H}^- &= \frac{1}{16\pi} \int_V \left( \dot{F}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta F^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^-} \left( \lambda \delta a - \dot{a} \delta \lambda \right) \\
&\quad + \frac{s}{16\pi} \int_{\partial V^-} \left( \Lambda^{AB} \delta g_{AB} + 2 \left( \kappa \delta \lambda - n^A \delta W_A \right) \right) .
\end{align}$$

(21)

4 Black hole thermodynamics

In this Section we apply the above result to the situation, when the wave front $S^-$ is a non-expanding horizon, i.e. $l = 0$ (see [5]). In this case the “internal energy” $\mathcal{H}^-$ vanishes. Moreover, Einstein equations imply $\Lambda^{AB} = 0$ (see [2]) and the definition of $\omega$ reduces to:

$$\nabla_a K = -\omega_a K .$$

We obtain the following generating formula for the black hole dynamics

$$\begin{align}
-\delta M &= \frac{1}{16\pi} \int_V \left( \dot{F}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta F^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^-} \left( \lambda \delta a - \dot{a} \delta \lambda \right) \\
&\quad + \frac{s}{8\pi} \int_{\partial V^-} \left( \kappa \delta \lambda - n^A \delta W_A \right) ,
\end{align}$$

(22)

where $s = 1$ for a white hole, and $s = -1$ for a black hole.

The so called “black hole thermodynamics” consists in analyzing possible stationary situations. By stationarity we understand the existence of a timelike symmetry (Killing) vector field. If such a field exists, we may always choose a coordinate system such that the Killing field becomes $\frac{\partial}{\partial x^0}$ and all the time derivatives (dots) vanish. Hence, formula (22) reduces to:

$$\delta M = -\frac{s}{8\pi} \int_{\partial V^-} \left( \kappa \delta \lambda - n^A \delta W_A \right) .$$

(23)

We assumed here that $\frac{\partial}{\partial x^0}$ is tangent to $S$. If this is not the case, we would have a one-parameter family of horizons. Such phenomenon corresponds to the Kundt’s class of metrics (see e.g. [11]). The known metrics of this class are not asymptotically flat. We do not know whether or not this is a universal property and we exclude such a pathology by the above assumption.

We have shown in [2] that there is a canonical affine fibration $\pi : S \to B$ over a base manifold $B$, whose topology is assumed to be that of a sphere $S^2$. The affine structure of the fibers is implied by the fact that they are null-geodesic lines in $M$. Identity $-2\eta_{ab} = \mathcal{L}_K g_{ab} = 0$ implies that the metrics $g$ on $S$ may be projected onto the base manifold $B$, which acquires a Riemannian two-metric tensor $h_{AB}$. The degenerate metric $g_{ab}$ on a manifold $S$ is simply the pull back of $h_{AB}$ from $B$ to $S$: $g = \pi^* h$.

The quantity $\omega_a$ is not an intrinsic property of the surface itself, but depends upon a choice of the null field $K$ on $S$. Indeed, if $\dot{K} = \exp(-\gamma)K$ then $\dot{\omega}_a = \omega_a + \partial_a \gamma$. In particular, there are on $S$ vector fields $K$ such that $K^a \nabla_a K = 0$ and, consequently, $\kappa = 0$. These are null geodesic fields tangent to fibers of $\pi : S \to B$.

In case of a black hole, there is a privileged field $K$, compatible with the time-like symmetry of the solution, which is normalized to unity at infinity. This way the quantities $\kappa$ and $w_A$ in formula (23) become uniquely defined.
We have, therefore, two symmetry fields of the metric $g_{ab}$ on $S$: $\partial_0$ and $K$. Due to normalization chosen above, we have $<\partial_0 - K, dx^b> = 0$. Hence, the field $\vec{n} := \partial_0 - K = n^A \partial_A$ is purely space-like and projects on $B$. Moreover, it is a symmetry field of the Riemannian two-metric $h_{AB}$.

Because the conformal structure of $h_{AB}$ is always isomorphic to the conformal structure of the unit sphere $S^2$, we are free to choose a coordinate system in which $h_{AB} = f \hat{h}_{AB}$ (and $\hat{h}_{AB}$ denotes the standard unit 2-sphere metrics). The field $\vec{n}$ is, therefore, the symmetry field of this conformal structure. Consequently, $\vec{n}$ belongs to the six-dimensional space of conformal fields on the 2-sphere. Using remaining gauge freedom, we may choose angular coordinates $(x^A) = (\theta, \phi)$ in such a way that $\vec{n}$ becomes a rotation field on the 2-sphere. This means (cf. [10] or [9]) that there exists a coordinate system in which the following holds:

$$\vec{n} = -\Omega^k \epsilon_{klm} y^l \partial_m.$$  \hspace{1cm} (24)

Here, $\Omega^k$ are components of a three-dimensional vector called angular velocity of the black hole, and $y^k$ are functions on $S^2$ created by restricting Cartesian coordinates on $\mathbb{R}^3$ to a unit 2-sphere. We can also set $z$-coordinate axis parallelly to angular velocity vector field. After a suitable rotation we have: $(\Omega^k) = (0, 0, \Omega)$, $z = y^3 = \cos \theta$, and:

$$\vec{n} = -\Omega \frac{\partial}{\partial \phi}.$$  \hspace{1cm} (25)

Inserting this into (25) we obtain

$$-\frac{1}{8\pi} \int_{\partial V^-} n^A \delta W_A = \Omega \delta J,$$  \hspace{1cm} (26)

where

$$J \equiv J_z := \frac{1}{8\pi} \int_{\partial V^-} W_\phi,$$  \hspace{1cm} (27)

is the $z$-component of the black hole angular momentum.

Up to now we have used only the symmetry of conformal structure carried by $h_{AB}$. The symmetry of the metric itself implies that the conformal factor $f$ is constant along the field $\vec{n}$. This follows from the observation that the trace of the Killing equation implies vanishing of divergence of the field $\vec{n}$:

$$0 = \partial_A (\sqrt{\det h_{CD}} n^A) = n^A \sqrt{\det h_{CD}} \partial_A f,$$  \hspace{1cm} (28)

where the fact that $\vec{n}$ is the symmetry field of the metric $\hat{h}$ has been used. Formula (28) implies that $\partial_\phi f = 0$ and the conformal factor $f$ must be a function of the variable $\theta$ only.

It turns out that also its canonical conjugate $\kappa$ may be gauged in such a way that it is constant along the field $\vec{n}$ (see [9] for a proof).\(^3\)

This result was obtained locally, or rather quasi-locally – i.e. from the analysis of the field on the horizon itself. However, the global theorems

\(^3\)In case $\Omega = 0$, quantities $\kappa$ and $f$ are arbitrary functions on $S^2$.\)
on the existence of stationary solutions possessing a horizon, imply the so called 0-th law of thermodynamics of black holes (see \[12\]), according to which the surface gravity \(\kappa\) must be constant along the horizon. But \(A := \int_{S^2} \lambda\) is the area of the horizon \(S\). Taking this into account and using \(26\), we derive from \(23\) the “first law of black holes thermodynamics”:

\[
-\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J.
\]  \(29\)

Contrary to the theory proposed by Wald and Iyer in \[15\], the 1-st law \(29\) is, in our approach, a simple consequence of the complete Hamiltonian formula \(22\), restricted to the stationary case. As illustrated by an example of the string dynamics, where formula \(3\) for virtual work was a consequence of the Hamiltonian formula \(8\), a similar “thermodynamics of boundary data” may be expected in any Hamiltonian field theory (see e.g. \[6\] for the corresponding analysis of the Maxwell electrodynamics). Also a “Penrose-like” inequality (analogous to \(9\) in the string theory) is satisfied as soon as the Hamiltonian is convex. We very much hope that the gravitational Penrose inequality can be proved along these lines. Preliminary results in this direction, based on the analysis of the field Hamiltonian in linearized gravity (see \[4\]), are promising.

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