Classification of static and homogeneous solutions in exactly solvable models of two-dimensional dilaton gravity

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We give the full list of types of static (homogeneous) solutions within a wide family of exactly solvable 2D dilaton gravities with backreaction of conformal fields. It includes previously known solutions as particular cases. Several concrete examples are considered for illustration. They contain a black hole and cosmological horizon in thermal equilibrium, extremal and ultraextremal horizons, etc. In particular, we demonstrate that adS and dS geometries can be exact solutions of semiclassical field equations for a non-constant dilaton field.

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I. INTRODUCTION

Semiclassical physics of black holes, combining issues of space-time, thermodynamics and quantum theory is the one of the most fascinating areas in physics. In our real four-dimensional world high mathematical complexities obscure the analysis of interplay between these aspects. This explains why the two-dimensional (2D) black hole physics (and, in more general settings, 2D dilaton gravity theories) became so popular during last decade. The powerful incentive was given due to Callan, Giddins, Harvey and Strominger (CGHS) work on evaporation of two-dimensional black holes. Meanwhile, the exact solutions for the metric and dilaton discussed in [1] were pure classical. The situation becomes much more complex if backreaction is taken into account. Then even within the set of two-dimensional
theories it is not a simple task to solve and analyze semiclassical field equations. As a consequence, self-consistent generalization of the CGHS theory turned out to be a non-trivial problem. To this end, a series of particular exactly solvable models were suggested and analyzed [2] - [6]. In [7] there has been suggested an unified approach based on symmetries of the non-linear sigma model to which the gravitation-dilaton action is related. This enabled to embrace previously known exactly solvable models within an unified scheme, the condition of exact solvability representing some relation between coefficients which enter the form of the action. This condition was independently refound in [8] in a more direct way, starting from the gravitation-dilaton action itself. It turned out that rather wide classes of solutions in such theories shares common properties (thermodynamics, space-time structure, etc.) which were discussed in [8], [9]. Meanwhile, these classes of solutions do not exhaust all the possibilities and, in some respect, are not applicable in some physically interesting situations. For instance, black holes considered in [8], are always non-extreme. The solutions for the extreme case can be obtained explicitly on the pure classical level (see, e.g., the recent paper [10]), but this problem becomes much more complex, when quantum backreaction is taken into account. For exactly solvable models of 2D dilaton gravity extremality can be achieved by special “tuning” asymptotic behavior of some action coefficients near the horizon [11], [12] for rather special families of solutions in which quantum stresses diverge on the horizon, the geometry remaining regular there. Being interesting on its own, such kinds of solutions do not represent, however, zero temperature black holes in the Hartle-Hawking state. Meanwhile, it was shown recently [13] that the latter type of solutions does appear in the exactly solvable models but only in some degenerate cases.

Thus, different classes of the same exactly solvable models may exhibit quite different properties and this motivates constructing the general scheme which would include all kinds of solutions. This is just the main purpose of our work. Such classification is a necessary step for better understanding the structure of exactly solvable models in dilaton gravity. It may also serve as a basis for diverse set of physical applications, that are contained in 2D dilaton gravity (see, e.g., recent reviews [14], [15]).
II. BASIC EQUATIONS

Hereafter we restrict ourselves to semiclassical dilaton gravity with backreaction of conformal fields only\textsuperscript{1}. Consider the action

\[ I = I_0 + I_{PL}, \]  

where

\[ I_0 = \frac{1}{2\pi} \int_M d^2 x \sqrt{-g} [F(\phi)R + V(\phi)(\nabla \phi)^2 + U(\phi)] \]  

and the Polyakov-Liouville action \textsuperscript{22}

\[ I_{PL} = -\frac{\kappa}{2\pi} \int_M d^2 x \sqrt{-g} \left[ \frac{\nabla \psi}{2} + \psi R \right] \]

is responsible for backreaction. Here the function $\psi$ obeys the equation

\[ \Box \psi = R, \]

where $\Box = \nabla_\mu \nabla^\mu$, $\kappa = N/24$ is the quantum coupling parameter, $N$ is number of scalar massless fields, $R$ is a Riemann curvature. We omit the boundary terms in the action as we are interested only in field equations and their solutions.

From eqs. \textsuperscript{11} - \textsuperscript{14} one can infer field equations (see below) which are valid for any gravitation-dilaton system of the kind under discussion. Meanwhile, our main goal is to analyze possible exactly solvable cases. The typical representative of the corresponding family reads

\[ F = \exp(-2\phi) + 2\kappa (d-1)\phi, \quad V = 4\exp(-2\phi) + 2(1-2d)\kappa + 4C(e^{-2\phi} - \kappa d)^2, \quad U = 4\lambda^2 \exp(-2\phi), \]

\textsuperscript{1}We do not consider additional scalar, Yang-Mills or fermion fields \textsuperscript{16} - \textsuperscript{20}, theories nonlinear with respect to curvature \textsuperscript{21}, etc, where, however, exact integrability is achieved for the classical case only.
λ and d are constants. If C = 0, this model turns to that suggested in [23]. In turn, it includes different particular known models. For example, in the case d = 0 one obtains the model suggested in [24], if d = 1/2 it coincides with the RST model [4]. Meanwhile, the family of exactly solvable models under discussion in our paper is wider than (5), including it only as a particular class.

Let us return to the issue of field equations in the generic case. Varying the action with respect to a metric gives us ($T_{\mu\nu} = 2\frac{\delta I}{\delta g_{\mu\nu}}$):

$$T_{\mu\nu} \equiv T_{\mu\nu}^{(0)} - T_{\mu\nu}^{(PL)} = 0,$$  \hspace{1cm} (6)

where

$$T_{\mu\nu}^{(0)} = \frac{1}{2\pi} \{2(g_{\mu\nu} \Box F - \nabla_{\mu} \nabla_{\nu} F) - U g_{\mu\nu} + 2V \nabla_{\mu} \phi \nabla_{\nu} \phi - g_{\mu\nu} V (\nabla \phi)^2\},$$  \hspace{1cm} (7)

$$T_{\mu\nu}^{(PL)} = \frac{\kappa}{2\pi} \{\partial_{\mu} \psi \partial_{\nu} \psi - 2\nabla_{\mu} \nabla_{\nu} \psi + g_{\mu\nu} [2R - \frac{1}{2} (\nabla \psi)^2]\}$$  \hspace{1cm} (8)

Variation of the action with respect to φ gives rise to the equation

$$R \frac{dF}{d\phi} + \frac{dU}{d\phi} = 2V \Box \phi + \frac{dV}{d\phi} (\nabla \phi)^2.$$  \hspace{1cm} (9)

In general, field equations cannot be solved exactly and the function ψ, the dilaton φ and metric depend on both time-like (t) and space-like (σ) coordinates: $\psi = \psi(t, \sigma), \phi = \phi(t, \sigma)$. In what follows we restrict ourselves to such kind of solutions that ψ can be expressed in terms of φ only: $\psi = \psi(\phi)$. We will see that this leads to the existence of the Killing vector. On the other hand, as all static or homogeneous solutions depend on one variable, one may exclude it and express ψ in terms of φ. Thus, the assumption $\psi = \psi(\phi)$ turns out to be equivalent to the static or homogeneous character of solutions.

Let us take the trace of eqs. (3)-(8) and eq. (3). Denoting

$$\tilde{F} \equiv F - \kappa \psi, U \equiv \Lambda e^\int d\phi \omega,$$  \hspace{1cm} (10)

we get
\[ U = \Box \tilde{F} \] (11)

\[ A_1 \Box \phi + A_2 (\nabla \phi)^2 = 0, \]
\[ A_1 = (u - \kappa \omega) \psi' + \omega u - 2V, \] (12)
\[ A_2 = (u - \kappa \omega) \psi'' + \omega u' - V', \]

where \( u \equiv F' \) and prime throughout the paper denotes differentiation with respect to \( \phi \).

For arbitrary coefficients \( A_1(\phi), A_2(\phi) \) eq.(12) cannot be solved exactly. This can be done, however, under some restrictions on the form of the coefficients \( A_1, A_2 \). Let us demand that

\[ A_1 = (u - \kappa \omega) \chi', \quad A_2 = (u - \kappa \omega) \chi'' \]
(13)

where \( \chi = \chi(\phi) \) and \( \Box \chi = 0 \). Then it follows that \( \psi = \psi_0 + \chi \), where

\[ \psi_0' = \frac{2V - \omega u}{u - \kappa \omega}, \] (14)

which enables us to find at once \( \psi_0 \) in terms of known functions \( u, V, \omega \) by direct integration. Demanding that both equations in (13) be consistent with each other, we obtain the restriction on the action coefficients

\[ u'(2V - \omega u) + u(\omega' - V') + \kappa (\omega V' - 2V \omega') = 0 \] (15)

This equation can be solved:

\[ V = \omega (u - \frac{\kappa \omega}{2}) + C(u - \kappa \omega)^2, \] (16)

where \( C \) is a constant.

The fact that the function \( \psi \) is defined up to the function whose Laplacian vanishes is explained by eq. (4) which is, in fact, the definition of \( \psi \). The presence of \( \chi \) reveals itself in the nature of quantum state (see below). Eq. (13) is just the condition obtained in [8], so account for \( \chi \) does not generate new types of exactly solvable models but extends the set of solutions within these models.
With eq. (11) taken into account, the field equations (6) - (8) can be rewritten in the form

\[
[\xi_1 \Box + \xi_2 (\nabla \phi)^2] g_{\mu\nu} = 2(\xi_1 \nabla_\mu \nabla_\nu \phi + \xi_2 \nabla_\mu \phi \nabla_\nu \phi) \tag{17}
\]

where \( \xi_1 = \frac{d\tilde{E}}{d\sigma} \), \( \xi_2 = \frac{d^2 \tilde{E}}{d\sigma^2} - \tilde{V} \), \( \tilde{V} = V - \frac{1}{2} \left( \frac{d\psi}{d\phi} \right)^2 \). Let us multiply this equation by the factor \( \zeta \) chosen in such a way that \( \xi_2 \zeta = \frac{d}{d\phi} (\xi_1 \zeta) \). Then eq. (17) turns into

\[
g_{\mu\nu} \Box \mu = 2\nabla_\mu \nabla_\nu \mu, \tag{18}
\]

where by definition \( \mu' = \xi_1 \zeta \). This equation takes the same form as eq.(2.24) from [26] and entails the same general conclusion about the existence of the Killing vector \( l_\alpha = \varepsilon^\alpha_{\beta} \mu_{\beta} \). In the present paper we consider the case when the Killing vector is time-like everywhere that gives rise to static solutions and mainly concentrate on black hole ones.

It is convenient to work in the conformal gauge

\[
d s^2 = g(-d t^2 + d \sigma^2), \tag{19}
\]

where, in accordance with the choice of the Killing vector, \( g = g(\sigma) \) and does not depend on a time-like coordinate \( \sigma \). In the gauge (19) the curvature

\[
R = -g^{-1} \frac{\partial^2 \ln g}{\partial \sigma^2}. \tag{20}
\]

Eq. (11) takes the form

\[
\Lambda e^\eta = \frac{\partial^2 \tilde{F}}{\partial \sigma^2} g^{-1}, \eta = \int d\phi \omega. \tag{21}
\]

Now for any function \( f(\sigma) \) we have \( \Box f = g^{-1} \frac{\partial^2 f}{\partial \sigma^2} \) whence it is clear that \( \chi = \gamma \sigma \), where \( \gamma \) is a constant. Thus, we have

\[
\psi = \psi_0 + \gamma \sigma, \tag{22}
\]

where \( \psi_0 \) is defined according to (14). It follows from (14) that

\[
g = e^{-\psi - a \sigma} = e^{-\psi_0 - b \sigma}, \tag{23}
\]

\]
where \( a \) is a constant, \( \delta = \gamma + a \). After simple rearrangement the (00) and (11) field equations (1), (17) with the metric in the conformal gauge (19) are reduced to one equation

\[
\xi_1 \frac{d^2 \phi}{d\sigma^2} + \xi_2 \left( \frac{d\phi}{d\sigma} \right)^2 - \xi_1 g^{-1} \frac{dg}{d\phi} \frac{d\phi}{d\sigma} = 0. \tag{24}
\]

It is convenient to split coefficients in eq. (24) into two parts singling out the term which is built up with the help of \( \psi_0 \):

\[
\xi_1 = \xi_1^{(0)} - \kappa \gamma \frac{d\sigma}{d\phi}, \quad \xi_2 = \xi_2^{(0)} - \kappa \gamma \frac{d^2 \sigma}{d\phi^2} + \kappa \left[ \frac{dn}{d\phi} \frac{d\sigma}{d\phi} + \frac{1}{2} (\gamma \frac{d\sigma}{d\phi})^2 \right],
\]

\[
\xi_1^{(0)} = \frac{d\tilde{F}(0)}{d\phi}, \quad \xi_2^{(0)} = \frac{d^2 \tilde{F}(0)}{d\phi^2} - \tilde{V}(0), \quad \tilde{F}(0) = F - \kappa \psi_0, \quad \tilde{V}(0) = V - \frac{\kappa}{2} \left( \frac{d\psi_0}{d\phi} \right)^2. \tag{25}
\]

Then eq. (24) takes the form

\[
\xi_1^{(0)} \frac{d^2 \phi}{d\sigma^2} + \xi_2^{(0)} \left( \frac{d\phi}{d\sigma} \right)^2 + \xi_1^{(0)} \frac{d\phi}{d\sigma} \frac{d\psi_0}{d\sigma} + \delta = \kappa \gamma (\delta - \frac{\gamma}{2}). \tag{26}
\]

Let us multiply this equation by the factor \( s \) such that \( \xi_2^{(0)} s = \frac{d(\xi_1^{(0)} s)}{d\phi} \), \( s = \exp \left[ \frac{\xi_2^{(0)}}{\xi_1^{(0)}} + \delta \right] = \exp \left[ -\tilde{V}(0)/\xi_1^{(0)} \right] \).

Then eq. (26) can be cast into the form

\[
\frac{dz}{d\sigma} + z \left( \frac{d\psi_0}{d\sigma} + \delta \right) = \kappa \gamma (\delta - \frac{\gamma}{2}) s, \tag{27}
\]

where \( z = s \xi_1^{(0)} \frac{d\phi}{d\sigma} = s \frac{d\tilde{F}(0)}{d\sigma} \). It follows from (14) and (19) that

\[
\psi_0 = \eta + 2CH, \tag{28}
\]

\[
g = e^{-\eta - 2CH - \delta \sigma}, \tag{29}
\]

and

\[
\tilde{F}(0) = H (1 - 2\kappa C), \tag{30}
\]

where \( H = F - \kappa \eta \). If \( \Lambda \neq 0 \), the metric function (up to the constant factor) is equal to

\[
g = \frac{e^{-\delta \sigma}}{U}, \tag{31}
\]

\[
H = F - \kappa \ln U + \text{const}. \tag{32}
\]
We obtain from (16), (25), (28), (30)

\[ \tilde{V}^{(0)} = (1 - 2\kappa C)H'(\omega + CH'), \quad \xi_1^{(0)} = (1 - 2\kappa C)H', \]  

whence

\[ s = e^{-\eta - CH}. \]  

Then after simple rearrangement eq. (27) gives rise to

\[ \frac{d^2H}{d\sigma^2} + C \left( \frac{dH}{d\sigma} \right)^2 + \delta \frac{dH}{d\sigma} = \alpha, \]  

where

\[ \alpha = \kappa \gamma \left( \delta - \frac{\gamma}{2} \right)/(1 - 2\kappa C). \]  

It is convenient to introduce a new variable \( \rho \), where \( |\rho| = e^{CH} \). Then we have the linear equation

\[ \frac{d^2\rho}{d\sigma^2} + \delta \frac{d\rho}{d\sigma} = \alpha C \rho \]  

One can seek a solution in the form \( \rho \sim e^{\beta \sigma} \), whence we obtain

\[ \beta^2 + \delta \beta - \alpha C = 0 \]  

This equation is quadratic and has two roots \( \beta_1, \beta_2 \). Depending on their properties, one can classify all possible types of solutions and describe their properties. In a natural way, the solutions fall into three different classes: I (both \( \beta_1, \beta_2 \) are real, \( \beta_1 \neq \beta_2 \)); II (\( \beta_1, \beta_2 \) are real, \( \beta_1 = \beta_2 \)), III (roots are complex, \( \beta_1 = \beta_2^* \)). We describe the results below.

III. GENERAL CASE, TYPES OF SOLUTIONS

It is convenient to cast the solutions of eq. (35) into uniform formulas:

\[ CH = CH_0 - \frac{\delta \sigma}{2} + \ln |f|, \]
where the function obeys the equation
\[
\frac{d^2 f}{d\sigma^2} = f \varepsilon^2, \quad \varepsilon^2 = \frac{\delta^2}{4} + \alpha C. \tag{40}
\]

We get the following different cases.

I\(_a\): \(\varepsilon^2 > 0\), \(f = \frac{\sin\varepsilon\sigma}{\varepsilon}\);

I\(_b\): \(f = \frac{\cosh\varepsilon\sigma}{\varepsilon}\);

II\(_a\): \(\varepsilon = 0\), \(f = \sigma\);

II\(_b\): \(f = 1\);

III: \(\varepsilon \equiv -\kappa^2 < 0\), \(f = \frac{\sin\kappa\sigma}{\kappa}\).

It follows from (21) that
\[
\frac{\Lambda C}{1 - 2\kappa C} = e^{2CH_0 z}, \tag{41}
\]

where \(z = 1\) for the I\(_b\) case, \(z = 0 = \Lambda\) for II\(_b\) and \(z = -1\) in cases I\(_a\), II\(_a\), III.

The Riemann curvature reads the following.

I\(_a\), II\(_a\), III:
\[
R = \frac{UC}{1 - 2\kappa C} [2 + \frac{\omega}{CH'} - \frac{1}{C^2 H'} \left(\frac{\omega}{H'}\right)' q^2]. \tag{42}
\]

I\(_b\):
\[
R = \frac{UC}{1 - 2\kappa C} [2 + \frac{\omega}{CH'} + \frac{1}{C^2 H'} \left(\frac{\omega}{H'}\right)' q^2] \tag{43}
\]

II\(_b\):
\[
R = e^{\eta + 2CH_0} \frac{1}{1 - 2\kappa C} C^2 H' \left(\frac{\omega}{H'}\right)' \frac{\delta^2}{4}. \tag{44}
\]

Here \(q = (\frac{df}{d\sigma} - \frac{\delta}{2} f)\).

In a similar way, we get the general structure of the expression for quantum stresses.

Two nonzero components of quantum stresses are connected for conformal fields by the well known relationship \(T_0^{0(PL)} + T_1^{1(PL)} = \frac{\kappa R}{\pi}\) (see eq. (8)). Here we list the component \(T_1^{1(PL)}\) only. One obtains from (8), (28), (29), (36):
\[
T_1^{1} = \frac{1}{4\pi g} \left[ \kappa \left( \frac{\partial \psi_0}{\partial \sigma} + 2\delta \right) \frac{\partial \psi_0}{\partial \sigma} + 2\alpha (1 - 2\kappa C) \right], \tag{45}
\]

\[
\frac{\partial \psi_0}{\partial \sigma} = \frac{\omega}{CH'} + 2 \frac{q}{f}. \tag{f}
\]
whence

\[ T_1^{(PL)} = -\frac{\kappa}{4\pi} \frac{|UC|}{1 - 2\kappa C} Z, \]  

(46)

\[ Z = \left( \frac{q\omega}{C\mathcal{H}} + 2f' \right)^2 - (\delta - \gamma)^2 f^2, \]  

(47)

except the case II_b, when

\[ T_1^{(PL)} = -\frac{\kappa}{4\pi} e^{2C\mathcal{H}_0 + \eta} Z, \]  

(48)

\[ Z = \frac{\delta^2}{4} \left( \frac{\omega}{C\mathcal{H}'} \right)^2 - (\delta - \gamma)^2. \]

IV. PARTICULAR CASES AND LIMITING TRANSITIONS

The solutions obtained depend on several parameters. In what follows it is assumed that the dilaton is not identically constant. The quantities \( \Lambda \) and \( C \) enter the definition of the action coefficients: \( \Lambda \) is the ”amplitude” of the potential \( U \) of a generic model according to eq. (10), while the parameter \( C \) defines the coefficient \( V \) of an exactly solvable one (16). Meanwhile, the quantities \( \delta \) and \( \alpha \) are the parameters of the solutions of field equations, they do not enter the action but characterize the different solutions for the same model.

Let us denote the symbolically \( [C, \Lambda](\delta, \alpha) \) the solutions with given parameters for a given action, where it is supposed that the values of parameters differ from zero, unless stated explicitly. Different limiting cases can be described on the basis of eqs. (35)-(38) and eq. (21). Consider first the case

A. \( C = 0 \)

Now

\[ \tilde{F}^{(0)} = H, \psi_0 = \eta, g = e^{-\eta - \delta \sigma}. \]  

(49)
Then it follows from (21) that \( \frac{d^2\rho}{d\sigma^2} = 0 \).

In the cases [0, 0](0, \alpha) and [0, 0](\delta, 0) equations (35) and (21) are mutually inconsistent, so these cases cannot be realized.

[0, 0](0, 0)

\[
H = A\sigma, \quad g = e^{-\eta}, \quad R = \frac{A^2}{H'} \left( \frac{\omega}{H'} \right)' e^\eta, \quad T_1^{(PL)} = -\frac{\kappa}{4\pi} A^2 \omega^2 e^\eta \frac{H'}{H'^2} < 0.
\]

Here \( A \) is an arbitrary constant.

[0, 0](\delta, \alpha)

\[
H = \frac{\alpha}{\delta} \sigma, \quad R = \frac{\alpha^2}{\delta^2} \frac{1}{H'} \left( \frac{\omega}{H} \right) ' \exp(\eta + \frac{\delta^2}{\alpha} H), \quad T_1^{(PL)} = -\frac{\exp(\eta + \frac{\delta^2}{\alpha} H)}{4\pi} \alpha \left[ \kappa \left( \frac{\alpha \omega}{\delta H'} + 2\delta \right) \frac{\omega}{\delta H'} + 2 \right].
\]

If \( \alpha = A\delta \) and \( \delta \to 0 \), while \( A \) is kept fixed, (51) turns into (50).

2. Case [0, \Lambda]

[0, \Lambda](\delta, \alpha)

\[
H = H_0 + \frac{\alpha}{\delta} \sigma + De^{-\delta \sigma}, \quad R = \frac{e^\eta}{H'} \left[ \omega + \omega \frac{\omega}{H'} \right] ' \left( \frac{\alpha^2}{\delta^2} e^{\delta \sigma} - 2\alpha D + D^2 \delta^2 e^{-\delta \sigma} \right), \quad T_1^{(PL)} = -\frac{1}{4\pi} e^{\eta + \delta \sigma} \left\{ 2\alpha + \kappa \left( \frac{\alpha}{\delta} - D\delta e^{-\delta \sigma} \right) \omega H'^{-1} [2\delta + \left( \frac{\alpha}{\delta} - D\delta e^{-\delta \sigma} \right) \omega H'^{-1}] \right\}, \quad D\delta^2 = \Lambda.
\]

[0, \Lambda](0, 0)

\[
H = H_0 + De^{-\delta \sigma}, \quad g = e^{-\eta (H - H_0)} D^{-1}, \quad R = \frac{U}{H'} \left[ \omega + (H - H_0) \left( \frac{\omega}{H'} \right) ' \right], \quad T_1^{(PL)} = \frac{\kappa \omega U}{4\pi H'} [2 - \frac{\omega (H - H_0)}{H'}].
\]
It is seen from (35) and (21) that the solution \([0, \Lambda] (0, 0)\) is impossible.

\[H = \frac{\alpha \sigma^2}{2} + H_0, \; g = e^{-\eta}, \; \Lambda = \alpha = -\frac{\kappa \gamma^2}{2} < 0,\]  

\[R = U [\frac{\omega}{H'} + 2 \left( \frac{\omega}{H'} \right)' \frac{H - H_0}{H'}],\]

\[T_1^{(PL)} = -\frac{U}{2\pi} \left[1 + \left( \frac{\omega}{H'} \right)^2 \kappa(H - H_0) \right].\]

The above formulae exhaust all the possibilities for \(C = 0\).

**B. \(C \neq 0\)**

Let now \(C \neq 0\).

1. **Case \([C, \Lambda]\)**

\([C, \Lambda](0, 0)\). Then it follows directly from eq. (35) and eq. (11) that

\[H = C^{-1} \ln \left| \frac{\sigma}{\sigma_0} \right|, \; g = e^{-\eta} \left( \frac{\sigma_0}{\sigma} \right)^2,\]

\[R = \frac{U}{1 - 2\kappa C} \left[\frac{\omega}{H'} + 2C - \left( \frac{\omega}{H'} \right)' \frac{1}{H' C} \right],\]

\[\Lambda C = -(1 - 2\kappa C)\sigma_0^{-2} < 0,\]

\[T_1^{(PL)} = \frac{\kappa UC}{4\pi (1 - 2\kappa C)} (2 + \frac{\omega}{CH'})^2 < 0,\]

where \(\sigma_0\) is a constant.

\([C, \Lambda](\delta, 0)\): this case can be obtained by putting \(\alpha = 0\) directly in the formulas for the case I. However, we list equations explicitly since this value is singled out from the physical viewpoint, giving a typical black hole in the Hartle-Hawking state. Here \(D\) is a constant, the factor \(C\) is singled out for convenience.

\[e^{CH} = e^{CH_0} \left|1 + DC e^{-\delta \sigma}\right|, \; g = e^{-\eta} \frac{e^{C(H - H_0)\nu} - 1}{DC} e^{-2CH},\]

\[R = \frac{U}{(1 - 2\kappa C) \left(2C + \frac{\omega}{H'} + \frac{1}{CH'} \left( \frac{\omega}{H'} \right)' \left[ ne^{C(H - H_0)} - 1 \right] \right)},\]
\[ \Lambda = e^{2C\theta_0} (1 - 2\kappa C) D \delta^2, \]
\[ T_1^{(PL)} = \frac{UC}{4\pi} \frac{\kappa}{1 - 2\kappa C} \left( \frac{\omega}{CH'} + 2 \right) \{2 + \frac{\omega}{CH'} [1 - \nu e^{C(H - H_0)}]\}, \]
\[ \nu = \text{sign}(1 + D C e^{-\delta \sigma}). \] If \( \nu > 0 \), our solution, written in the conformal gauge, corresponds to eq. (25) of Ref. \[8\], where the Schwarzschild gauge was used.

\([C, \Lambda](0, \alpha)\). It can be obtained directly from types I or III by putting \( \delta = 0 \).

2. Case \([C, 0]\)

\([C, 0](\delta, \alpha)\):

\[ CH = CH_0 + \beta_{\pm} \sigma, \quad g = \exp(-\eta - 2CH_0 \mp 2\varepsilon CH \beta_{\pm}^{-1}), \quad R = \frac{\beta_{\pm}^2}{C^2 H'} \left( \frac{\omega}{H'} \right)' e^{\eta + 2CH_0 \pm 2\varepsilon CH \beta_{\pm}^{-1}}, \]
\[ T_1^{(PL)} = -\frac{1}{4\pi} \exp(\eta + 2CH_0 \pm 2\varepsilon \frac{CH}{\beta_{\pm}}) \{ \kappa \beta_{\pm} (2 + \frac{\omega}{CH'}) [2(\beta_{\pm} \pm \delta) + \frac{\beta_{\pm} \omega}{CH'}] \} + 2\alpha (1 - 2\kappa C), \]
\[ \beta_{\pm} \] are the roots of eq. (38). The solution \([C, 0](0, \alpha)\) does not bring any qualitative new features and can be obtained directly from (58) by putting \( \delta = 0 \).

\([C, 0](\delta_0, \alpha)\):

\[ H = H_0 - \frac{\delta \sigma}{2C}, \quad g = e^{\eta - 2CH_0}, \]
\[ R = -\frac{\alpha}{CH'} \left( \frac{\omega}{H'} \right)' e^{\eta + 2CH_0}, \]
\[ T_1^{(PL)} = \frac{\alpha}{4\pi} \left( \frac{\kappa \omega^2}{CH'^2} - 2 \right) e^{\eta + 2CH_0}. \]

The solution \([C, 0](\delta, 0)\) can be obtained from (58) by putting \( \alpha = 0 \).

The solution \([C, 0](0, 0)\) does not exist.

The solutions with \( C = 0 \) and \( C \neq 0 \) are described by qualitatively different formulas, so one may ask in what way the first class can be obtained from the second one by limiting transition \( C \to 0 \). In such a transition one should carefully take into account not only terms with \( C = 0 \) but also terms linear in \( C \) and, if necessary, make a shift in the coordinate. For example, compare the cases I and \([0, \Lambda](\delta, \alpha)\). In the formulas for roots of eq. (39) we have
in the limit under consideration (let for definiteness $\delta > 0$): $\varepsilon = \frac{\delta}{2} + \frac{\alpha C}{\delta}$. Introducing a new variable according to $\sigma = \sigma' + \sigma_0$ and choosing $\sigma_0$ to make the right hand side of eq. (39) of the first order in $C$, we put $\exp(-\varepsilon \sigma_0) = CD$, where $D$ does not contain $C$. Then after simple rearrangement we obtain $H = \frac{\alpha}{\delta} \sigma + De^{-\delta \sigma}$ that agrees with (52). On the other hand, there also exist solutions with $C \neq 0$ (for instance, $[C, \Lambda](0, 0)$) which have no analogues among those with $C = 0$.

Thus, we obtained the following qualitatively different cases.

Generic types: $I_a$, $I_b$, $II_a$, $II_b$, $III$.

Particular ones:
- $[0, 0](0, 0)$;
- $[0, 0](\delta, \alpha)$;
- $[0, \Lambda](\delta, \alpha)$;
- $[0, \Lambda](\delta, 0)$;
- $[0, \Lambda](0, \alpha)$;
- $[C, \Lambda](0, 0)$;
- $[C, \Lambda](\delta, 0)$;
- $[C, \Lambda](\delta, \alpha)$;
- $[C, 0](\delta, \alpha)$.

The case $II_a$ is equivalent to $[C, \Lambda](\delta_0, \alpha)$ and $II_b$ is equivalent to $[C, 0](\delta_0, \alpha)$.

All other particular cases either are impossible or can be obtained directly by letting the parameters their particular values.

**V. EXAMPLES**

In this section we restrict ourselves to examples that possess properties, missed or overlooked in previously known exactly solvable models. Consider, for example, the case $[0, \Lambda](\delta, 0)$. It is convenient to introduce a Schwarzschild coordinate $x$ according to $dx = d\sigma g$.

Then it follows from (54) that

$$g = a \frac{H - H_0}{U},$$

$$\frac{dx}{d\phi} = B^{-1} \frac{H'}{U},$$

where the constant $B$ obeys the relationships $a = \Lambda/D$, $B = \delta/a$, $aB^2 = 1$. 

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If $H(\phi) = H_0$ at some $\phi = \phi_0$, we have a horizon. Meanwhile, an additional horizon may appear at $H \to \infty$. As a result, we may obtain black hole and cosmological horizons. Indeed, consider the model for which at $\phi \to \infty$

$$U \sim e^{\phi m}(1 + U_1 e^{-\phi}), \quad H \sim e^{\phi n}(1 + H_1 e^{-\phi}), \quad n, m > 0.$$ (64)

Then direct implication of eq. (54) shows that $g \sim \exp[\phi(n-m)] - x - x_h$ ($x_h$ is the horizon value of $x$),

$$R \sim \exp[\phi(m-n-1)],$$ (65)

$$T_{1}^{(PL)} \sim (2n-m) \exp[\phi(m-n)] + \text{const} \exp[\phi(m-n-1)] + ..., \quad (66)$$

the Hawking temperature $T_h^{(1)} = \frac{|\delta|}{4\pi}$ at the black hole horizon at $\phi = \phi_0$ and

$$T_h^{(2)} = \frac{|\delta|}{4\pi} \frac{(m-n)}{n} \quad (67)$$

at the cosmological horizon $\phi = \infty$.

The value $\phi = \infty$ is indeed the horizon provided $n < m$, the condition of regularity of the cosmological horizon reads $m \leq n + 1$, the finiteness of quantum stresses on this horizon occurs if $2n - m = 0$. All three criteria are met for $m = 2n, n \leq 1$, in which case $T_h^{(1)} = T_h^{(2)}$ and we obtain two horizons at thermal equilibrium, quantum stresses being finite on them.

On the other hand, if $n < m \leq n + 1$, $m \neq 2n$, the cosmological horizon is regular but quantum stresses diverge on it.

One can observe that the solutions $[C, \Lambda](0, 0)$ and $[C, \Lambda](\delta, 0)$ in the case $U = \text{const}$ ($\omega = 0$) give the constant curvature solutions $R = -2\sigma_0^2 < 0$ (2d adS metric) in the first case and $R = e^{2H_0 D\delta^2}$ in the second one (2D dS metric, if $DC > 0$). It was shown earlier that dS and adS metric appear in 2D dilaton theories for constant dilaton solutions, $(\nabla \phi)^2 = 0$ [26], [27]. However, we see that the reverse is not necessarily true: we obtained the constant curvature solutions with essentially inhomogeneous dilaton field. This is due to $C \neq 0$, so these solutions could not appear in previous studies of exactly solvable models [6], [9].

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In the case \([C, \Lambda](0, 0)\) with \(U = U_0 = \text{const}\), the curvature \(R = \frac{2U_0 C}{1-2\kappa C} = \text{const} < 0\) (\(\Lambda\) and \(C\) have different signs according to (56)) that is nothing else than the usual two-dimensional adS space-time with an acceleration horizon. If the potential is not constant identically but \(U \to U_0 = \text{const}\) asymptotically, we get a black hole extremal horizon.

If, say, at \(\phi \to \infty\) \(H \sim H_1 \phi\) and \(U = \Lambda e^{-\phi}\), we have for the same type of solutions \(g \sim \sigma^{m-2}\), \(m = (C H_1)^{-1}\). If \(0 < m < 1\), \(g \sim (x - x_h)^n\) with \(n = \frac{2-m}{1-m} > 2\). In this sense a horizon is ultraextremal.

Let at \(\phi \to \infty\) \(\eta \sim 2\phi, H \sim e^\phi\). Then \(g \sim \sigma^{-2}\) and we again obtain an extremal horizon. In fact, as in this example the potential \(U = 0\), we can choose the function \(\eta\) at our will. For instance, let \(\eta = \ln \cosh \phi, H = \sinh \phi\). Then the solution is symmetric with respect to reflection \(\phi \to -\phi\), \(g = (\cosh \phi)^{-1}\) and at both infinities we have extremal horizons in equilibrium. In so doing, they are ”ultracold”:\(x \sim \phi\) and \(g \sim (\cosh x)^{-1}\), so not only the metric function but also their derivatives vanish at infinity.

Consider the case \(H = e^{-2\phi} - \kappa \phi, \omega = -2\). It corresponds to the solutions found in \([25]\) in the cosmological context. It is convenient to introduce the coordinate \(\rho\) according to \(d\rho = \sqrt{g}d\sigma\), the proper length \(l = |\rho|\). Then after some algebraic manipulations it follows from (50) that \(g = a^2\),

\[
a = \frac{\sqrt{b^2 \rho^2 + 2\kappa} + b \rho}{\kappa},
\]

\(b\) is a constant. If \(\kappa \to 0, b\rho < 0\), we obtain \(a \to |b| l^{-1}\). The region \(b\rho > 0\) does not have a classical counterpart. The curvature

\[
R = -\frac{4\kappa b^2}{(b^2 \rho^2 + 2\kappa)^{3/2} \left(b \rho + \sqrt{b^2 \rho^2 + 2\kappa}\right)}
\]

is everywhere finite, including infinity. In the limit \(\rho b \to +\infty\) the metric function \(g \sim l^2\), the curvature \(R \sim l^{-4} \to 0\). If \(b\rho \to -\infty, g \sim l^{-2}, R \sim l^{-2} \to 0\). Thus, we have a nonextreme horizon at one infinity and the Rindler metric at the other one.

The solution of \([0, 0](\delta, \alpha)\) type corresponds to the static analogue of what is called ”the second branch” in the cosmological context \([23]\).
It is worth noting correspondence between some types of exact solutions that follows directly from the explicit formulas. For the solutions \([0, 0](\delta, \alpha)\) and \([C, \Lambda](0, 0)\) the dependence of the metric function on dilaton \(g(\phi)\) coincide provided \(C = \delta^2/2\alpha\); for the solutions \([0, 0](0, 0)\) and \([C, \Lambda](0, 0)\) the spatial dependence of the dilaton on the proper length coincide, provided the constant \(|A| = |C\sigma_0|\).

One can observe that the solutions \([C, \Lambda](0, 0)\) and \([C, \Lambda](\delta, 0)\) in the case \(U = \text{const} (\omega = 0)\) give the constant curvature solutions. It was shown earlier that dS and adS metric appear in 2d dilaton theories for constant dilaton solutions, \((\nabla\phi)^2 = 0\) [26], [27]. However, we see that the reverse is not necessarily true: we obtained the constant curvature solutions with essentially inhomogeneous dilaton field. This is due to \(C \neq 0\), so these solutions could not appear in previous studies of exactly solvable models [8], [9].

VI. SUMMARY

Thus, we considered a rather wide family of exactly solvable models of 2D dilaton gravity with backreaction of conformal fields, which includes previously known particular models of this kind, and enumerated all possible types of static solutions which appear in this family. In so doing, the explicit results were listed for static solutions. However, if a time and space variable are interchanged, we obtain (with signs properly reversed) exact solutions for string-inspired cosmology that gives potential set for the choice of everywhere regular space-times, detailed description of inflation, etc. The list of solutions given above may also describe an initial and final configurations in the problems of black hole formation and evaporation. Further applications for black hole physics and cosmology depend strongly on the concrete choice of the models.

[1] C. G. Callan, S. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D 45 (1992) R1005.
[2] A. Bilal and C. G. Callan, Nucl. Phys. B 394, 73 (1993).

[3] S. P. de Alwis, Phys. Rev. D 46, 5429 (1992).

[4] J. G. Russo, L. Susskind, and L. Thorlacius, Phys. Rev. D 46 (1992) 3444; Phys. Rev. D 47 (1992) 533.

[5] G. Michaud and R. C. Myers, Two-Dimensional Dilaton Black Holes, gr-qc/9508063.

[6] A. Fabbri and J. G. Russo, Phys. Rev. D 53, 6995 (1995).

[7] Y. Kazama, Y. Satoh, and A. Tsuichiya, Phys. Rev. D 51, 4265 (1995).

[8] O. B. Zaslavskii, Phys. Rev. D 59, 084013 (1999).

[9] O. B. Zaslavskii, Phys. Lett. B 459, 105 (1999).

[10] N. Berkovits, S. Gukov and B.C. Vallilo, Nucl.Phys. B 614, 195 (2001).

[11] O.B. Zaslavskii, Phys. Lett. B 475, 33 (1999).

[12] O. B. Zaslavskii, Mod. Phys. Lett. A 17, 1175 (2002).

[13] O. B. Zaslavskii, Exactly solvable models in 2D semiclassical dilaton gravity and extremal black holes, hep-th/0211207 (To appear in Class.Quant. Grav.).

[14] D. Grumiller, W. Kummer and D. V. Vasilevich, Phys.Rept. 369 (2002) 327.

[15] S.Nojiri and S. Odintsov, Int. J. Mod. Phys. A 16, 1015 (2001).

[16] A. T. Filippov and V. G. Ivanov, Phys.Atom.Nucl. 61, 1639 (1998) [hep-th/9803059].

[17] A. T. Filippov, Mod.Phys.Lett. A 11, 1691 (1996).

[18] E. Elizalde and S.D. Odintsov, Nucl. Phys. B 399, 581 (1993).

[19] T. Kloesch and T. Strobl, Class.Quant.Grav. 13, 965 (1996); Erratum-ibid. 14, 825 (1997).

[20] H Pelzer and T. Strobl, Class.Quant.Grav. 15, 3803 (1998).
[21] E. Elizalde, P. Fosalba-Vela, S. Naftulin, S. D. Odintsov, Phys. Lett. B 352, 235 (1995).

[22] A. M. Polyakov, Phys. Lett. B 103, 207 (1981).

[23] J. Cruz and J. Navarro-Salas, Phys. Lett. B 375, 47 (1996).

[24] S. Bose, L. Parker, and Y. Peleg, Phys. Rev. D 52, 3512 (1995).

[25] S. Bose and S. Kar, Phys. Rev. D 56, 4444 (1997).

[26] S. N. Solodukhin, Phys. Rev. D 53, 824 (1996).

[27] O. B. Zaslavskii, Phys. Lett. B 424, 271 (1998).