On Simple Transitive 2-representations of Bimodules over the Dual Numbers

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Abstract
We study the problem of classification of simple transitive 2-representations for the (non-finitary) 2-category of bimodules over the dual numbers. We show that simple transitive 2-representations with finitary apex are necessarily of rank 1 or 2, and those of rank 2 are exactly the cell 2-representations. For 2-representations of rank 1, we show that they cannot be constructed using the approach of (co)algebra 1-morphisms. We also propose an alternative definition of (co-)Duflo 1-morphisms for finitary 2-categories and describe them in the case of bimodules over the dual numbers.

Keywords Bimodule · Cell 2-representation · Left cell · Quiver · Simple transitive 2-representation

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1 Motivation, Introduction and Description of the Results
Classification problems form an important and intensively studied class of questions in modern representation theory. One of the natural examples of these kinds of problems is the problem of classification of all “simple” representations of a given mathematical object. During the last 20 years, the study of representations of tensor categories and 2-categories has attracted a lot of attention, see [3, 12] and references therein. In particular, there are by now a number of interesting tensor categories (and 2-categories) for which the structure of “simple” representations is well-understood. To the best of our knowledge, the first deep results of this kind can be found in [24, 25], we refer to [3] for more details.

Around 2010, Mazorchuk and Miemietz started a systematic study of representation theory of finitary 2-categories, see the original series [14–19] of papers by these authors. Finitary 2-categories can be considered as natural 2-analogues of finite-dimensional algebras, in particular, they have various finiteness properties, analogous to those of the category

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of projective modules over a finite-dimensional algebra. The paper [18] introduces the notion of simple transitive 2-representations of finitary 2-categories and provides convincing arguments, including an adaptation of the Jordan-Hölder theorem, of why these 2-representations are a natural 2-analogue for the notion of a simple module over an associative algebra. This motivated the natural problem of classification of simple transitive 2-representations for various classes of finitary 2-categories. This problem was considered and solved in a number of special cases, see [6–11, 18, 20–23, 29–31] and also [13] for a slightly outdated overview on the status of that problem.

Arguably, one of the most natural examples of a finitary 2-category is the 2-category $\mathcal{C}_A$ of projective bimodules over a finite-dimensional associative algebra $A$, introduced in [14, Subsection 7.3]. The classification of simple transitive 2-representations of $\mathcal{C}_A$ is given in [21], with the special case of a self-injective $A$ treated already in [18, 19]. The reason to restrict to projective bimodules is that the monoidal category $A\text{-mod-}A$ of all finite-dimensional $A$-$A$-bimodules is not finitary in general, because it has infinitely many indecomposable objects. The only basic connected algebras $A$, for which $A\text{-mod-}A$ is finitary, are the radical square zero quotients of the path algebras of uniformly oriented type $A$ Dynkin quivers, see [23]. Moreover, for almost all $A$, the category $A\text{-mod-}A$ is wild, that is the associative algebra $A \otimes_k A^{\text{op}}$, whose module category is equivalent to $A\text{-mod-}A$, has wild representation type, and hence indecomposable objects of $A\text{-mod-}A$ are unknown (and, perhaps, will never be known).

The smallest example of the algebra $A$ for which the category $A\text{-mod-}A$ is not finitary, but is, at least, tame, is the algebra $D := k[x]/(x^2)$ of dual numbers. The combinatorics of tensor product of indecomposable objects in $D\text{-mod-}D$ is described in [4, 5]. In particular, although not being finitary itself, $D\text{-mod-}D$ has a lot of finitary subcategories and subquotients. The main motivation for the present paper is to understand simple transitive 2-representations of $D\text{-mod-}D$ which correspond to simple transitive 2-representations of its finitary subquotients.

Our main result is Theorem 1, which can be found in Section 3.2. It asserts that simple transitive 2-representations of $D\text{-mod-}D$ with finitary apex are necessarily of rank 1 or 2 and, in the latter case, each such 2-representation is necessarily equivalent to a so-called cell 2-representation, which is an especially nice sort of 2-representation. Unfortunately, at this stage we are not able to classify (or, for that matter, even to construct, with one exception) rank 1 simple transitive 2-representations. One possible reason for that is given in Theorem 21 in Section 8.4 which asserts that potential simple transitive 2-representations of rank 1 cannot be constructed using the approach of (co)algebra 1-morphisms, developed in [8] for the so-called fiat 2-categories, that is finitary 2-categories with a weak involution and adjunction morphism. Needless to say, neither $D\text{-mod-}D$ nor any of its finitary subquotients is fiat.

Sections 7 and 8 summarize, in some sense, the outcome of our failed attempt to adjust the approach of [8] for the construction of simple transitive 2-representations of $D\text{-mod-}D$. Due to the fact that $D\text{-mod-}D$ is not fiat, several classical notions for fiat 2-categories require non-trivial adaptation to the more general setup of $D\text{-mod-}D$. One of these, discussed in detail in Section 7, is that of a Duflo 1-morphism. Originally, it is defined in [14] in the fiat setup and slightly adjusted in [28] to a more general finitary setting. Here we propose yet another alternative definition of Duflo 1-morphisms (and the dual notion of co-Duflo 1-morphisms) using certain universal properties, see Sections 7.3 and 7.5. We show in Proposition 13 that our notion agrees with the notion of Duflo 1-morphisms from [14] in the fiat case. We show that some left cells in $D\text{-mod-}D$ have a Duflo 1-morphism and that some other left cells have a co-Duflo 1-morphism, see Sections 7.4 and 7.5. In Section 8,
we further show that these Duflo and co-Duflo 1-morphisms admit the natural structure of coalgebra and algebra 1-morphisms, respectively.

All necessary preliminaries are collected in Section 2. Our main Theorem 1 has four statements. The first one is proved in Section 3.3. The other three are proved in Sections 4, 5 and 6, respectively.

2 Preliminaries

2.1 2-categories

A 2-category $\mathcal{C}$ consists of

- objects $i, j, k, \ldots$,
- for each pair of objects $i, j$, a small category $\mathcal{C}(i, j)$ of morphisms from $i$ to $j$, objects of $\mathcal{C}(i, j)$ are called 1-morphisms $F, G, H, \ldots$, and morphisms of $\mathcal{C}(i, j)$ are called 2-morphisms $\alpha, \beta, \ldots$,
- for each object $i$, an identity 1-morphism $1_i$;
- bifunctorial composition $\circ: \mathcal{C}(j, k) \times \mathcal{C}(i, j) \to \mathcal{C}(i, k)$.

This datum is supposed to satisfy the obvious set of strict axioms. The internal composition of 2-morphisms in $\mathcal{C}(i, j)$ is called vertical and denoted by $\circ_v$. The composition of 2-morphisms induced by $\circ$ is called horizontal and denoted $\circ_h$.

Let $k$ be a field. A 2-category $\mathcal{C}$ is called $k$-linear if all morphism categories $\mathcal{C}(i, j)$ are $k$-linear, and horizontal composition is $k$-bilinear. Recall that a category $\mathcal{C}$ is finitary $k$-linear if it is equivalent to $A$-proj for some finite-dimensional $k$-algebra $A$. A 2-category $\mathcal{C}$ is called finitary (over $k$) if it has finitely many objects, $\mathcal{C}(i, j)$ is a finitary $k$-linear category for any objects $i, j$ in $\mathcal{C}$, and compositions are biadditive and $k$-bilinear whenever applicable.

Important examples of 2-categories are

- $\textbf{Cat}$, the 2-category whose objects are small categories, 1-morphisms are functors, and 2-morphisms are natural transformations of functors;
- $\mathfrak{A}_{k}^f$, the 2-category whose objects are finitary $k$-linear categories, 1-morphisms are additive $k$-linear functors, and 2-morphisms are natural transformations of functors;
- $\mathfrak{G}_{k}$, the 2-category of finitary $k$-linear abelian categories, whose objects are small categories equivalent to module categories of finite-dimensional associative $k$-algebras, 1-morphisms are right exact additive $k$-linear functors, and 2-morphisms are natural transformations of functors.

2.2 2-representations

Let $\mathcal{C}$ be a 2-category. A 2-representation of $\mathcal{C}$ is a strict 2-functor $M: \mathcal{C} \to \textbf{Cat}$.

For example, given an object $i$ in $\mathcal{C}$, we can define the principal representation $P_i = \mathcal{C}(i, -)$.

Assume that $\mathcal{C}$ is additive, $k$-linear and Krull-Schmidt. Assume further that for each object $i$ of $\mathcal{C}$, the identity 1-morphism $1_i$ is indecomposable. Then a finitary 2-representation of $\mathcal{C}$ is a strict $k$-linear 2-functor $M: \mathcal{C} \to \mathfrak{A}_{k}^f$, cf. [16] for the setting when $\mathcal{C}$ itself is finitary.
A finitary 2-representation $\mathcal{M}$ of $\mathcal{C}$ is called *transitive* if, for any indecomposable object $X \in \mathcal{M}(\mathfrak{i})$ and $Y \in \mathcal{M}(\mathfrak{j})$, there is a 1-morphism $U$ in $\mathcal{D}$ such that $Y$ is isomorphic to a direct summand of $\mathcal{M}(U)X$. We, further, say that $\mathcal{M}$ is *simple* if it has no proper nonzero $\mathcal{C}$-stable ideals. While simplicity implies transitivity, we follow [18] and speak of simple transitive 2-representations to emphasize the two levels (objects and morphisms) of the involved structure.

All 2-representations of $\mathcal{C}$ form a 2-category, see [16, Subsection 2.3] for details. In particular, two 2-representations $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{C}$ are equivalent if there is a 2-natural transformation $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ which restricts to an equivalence $\mathcal{M}(\mathfrak{i}) \rightarrow \mathcal{N}(\mathfrak{i})$ for every object $\mathfrak{i} \in \mathcal{C}$.

If $\mathcal{C}$ has only one object $\mathfrak{i}$, we say that a finitary 2-representation $\mathcal{M}$ of $\mathcal{C}$ has *rank* $r$ if the category $\mathcal{M}(\mathfrak{i})$ has exactly $r$ isomorphism classes of indecomposable objects.

### 2.3 Abelianization

For every finitary 2-representation $\mathcal{M}$ of $\mathcal{C}$, we can consider its (projective) abelianization $\overline{\mathcal{M}}$ as defined in [8, Section 3]. Then $\overline{\mathcal{M}}$ is a 2-functor from $\mathcal{C}$ to $\mathcal{R}_k$ and, up to equivalence, $\mathcal{M}$ is recovered by restricting $\overline{\mathcal{M}}$ to the subcategories of projective objects in the underlying abelian categories of the abelian 2-representation $\overline{\mathcal{M}}$.

There is also the dual notion of (injective) abelianization $\mathcal{M}$.

### 2.4 Cells and Cell 2-representations

On the set of isomorphism classes of indecomposable 1-morphisms in $\mathcal{C}$, define the *left preorder* $\leq_L$ by $F \leq_L G$ if there is some $H$ such that $G$ is a direct summand of $H \circ F$. The induced equivalence relation $\sim_L$ is called *left equivalence*, and the equivalence classes are called *left cells*. Similarly, we can define the right preorder $\leq_R$ by composing with $H$ from the right, and two-sided preorder $\leq_J$ by composing with $H_1$ and $H_2$ from both sides. Right and two-sided equivalence and cells are also defined analogously.

For any transitive 2-representation $\mathcal{M}$ of $\mathcal{C}$, there is, by [2], a unique two-sided cell, maximal with respect to the two-sided preorder, which is not annihilated by $\mathcal{M}$. This two-sided cell is called the *apex* of $\mathcal{M}$.

A two-sided cell $\mathcal{J}$ is called *idempotent* if it contains $F$, $G$ and $H$ such that $H$ is isomorphic to a direct summand of $F \circ G$. The apex of a 2-representation is necessarily idempotent.

Let $\mathcal{L}$ be a left cell in $\mathcal{C}$ and let $\mathfrak{i} = \mathfrak{i}_L$ be the object such that all 1-morphisms in $\mathcal{L}$ start at $\mathfrak{i}$. Then the principal representation $\mathcal{P}_{\mathfrak{i}}$ has a subrepresentation given by the additive closure of all 1-morphisms $F$ such that $F \geq_L \mathcal{L}$. This, in turn, has a unique simple transitive quotient which we call the *cell 2-representation* associated to $\mathcal{L}$ and denote by $\mathcal{C}_L$. We refer to [14, 15] for more details.

### 2.5 Action Matrices

Let $\mathcal{M}$ be a finitary 2-representation of $\mathcal{C}$ and $F$ a 1-morphism in $\mathcal{C}(\mathfrak{i}, \mathfrak{j})$. Let $X_1, X_2, \ldots, X_k$ be a complete list of representatives of isomorphism classes of indecomposable objects in $\mathcal{M}(\mathfrak{i})$ and $Y_1, Y_2, \ldots, Y_m$ be a complete list of representatives of isomorphism classes of indecomposable objects in $\mathcal{M}(\mathfrak{j})$. Then we can define the *action matrix* $[F]$ of $F$ as the integral $m \times k$-matrix $(r_{ij})_{i=1,\ldots,m}^{j=1,\ldots,k}$ where $r_{ij}$ is the multiplicity of $Y_i$ as a direct summand of $\mathcal{M}(F)X_j$. Clearly, we have $[FG] = [F][G]$. 

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If \( \mathcal{C} \) has only one object, then \( \mathbf{M} \) is transitive if and only if all coefficients of \( [F] \) are positive, where \( F \) is such that it contains, as direct summands, all indecomposable 1-morphisms in the apex of \( \mathbf{M} \).

If \( \overline{\mathbf{M}}(F) \) is exact, then we can also consider the matrix \([F]\) which bookkeeps the composition multiplicities of the values of \( \overline{\mathbf{M}}(F) \) on simple objects in \( \mathcal{M}(\pm) \). If \((G, F)\) is an adjoint pair of 1-morphisms in \( \mathcal{C} \), then the functor \( \overline{\mathbf{M}}(F) \) is exact, and we have the relation \([F] = [G]^{tr} \).

3 Bimodules over the Dual Numbers and the Main Result

3.1 The 2-category of Bimodules over the Dual Numbers

In the remainder of the paper, we work over an algebraically closed field \( \mathbb{k} \) of characteristic 0. Denote by \( D = \mathbb{k}[x]/(x^2) \) the dual numbers. Fix a small category \( \mathcal{C} \) equivalent to \( D\text{-mod}. \) Let \( \mathcal{D} \) be the 2-category which has

- one object \( \mathbf{1} \) (which we identify with \( \mathcal{C} \)),
- as 1-morphisms, all endofunctors of \( \mathcal{C} \) isomorphic to tensoring with finite-dimensional \( D\text{-}D \)-bimodules,
- as 2-morphisms, all natural transformations of functors (these are given by homomorphisms of the corresponding \( D\text{-}D \)-bimodules).

Indecomposable \( D\text{-}D \)-bimodules can be classified, up to isomorphism, following \([1, 27]\). Using the notation from \([5]\), they are the following.

- The (unique) projective-injective bimodule \( D \otimes_{\mathbb{k}} D \).
- The band bimodules \( B_k(\lambda) \), indexed by \( k \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{k}\setminus\{0\} \). As vector space, \( B_k(\lambda) \) is \( \mathbb{k}^{2k} \). The left and right \( x \)-actions are given by

\[
\begin{bmatrix}
0 & 0 \\
I_k & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 \\
Q_k(\lambda) & 0 \\
\end{bmatrix}
\]

respectively. Here \( I_k \) denotes the \( k \times k \)-identity matrix, and \( Q_k(\lambda) \) the Jordan block of size \( k \) with eigenvalue \( \lambda \).

The bimodule \( B_k(\lambda) \) can also be depicted as follows.

\[
s \cdot \mathbb{i} = \mathbb{i} \quad \text{and} \quad s \cdot Q_k(\lambda) \quad \text{for} \quad s \in \mathbb{k}^k
\]

In particular, the regular bimodule \( D \text{-} D \) is isomorphic to the band bimodule \( B_1(1) \).

- String bimodules of four shapes \( W, S, N \) and \( M \) indexed by \( k \in \mathbb{Z}_{\geq 0} \). For a string bimodule \( U \), the integer \( k \) is the number of valleys in the graph representing this bimodule, alternatively, \( k = \dim(DU \cap UD) \). The graphs representing the bimodules \( W_1, S_1, N_1 \) and \( M_1 \) look, respectively, as follows (here vertices \( \bullet \) and \( \circ \) represent a fixed basis with \( \circ \) depicting the valley, the non-zero right action of \( x \in D \) is depicted by horizontal
arrows, the non-zero left action of \( x \in D \) is depicted by vertical arrows and all non-zero coefficients of both actions are equal to 1):

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\downarrow & \downarrow & \leftarrow & \leftarrow \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

The dimensions of the string bimodules are as follows:

- \( \dim(W_k) = 2k + 1 \),
- \( \dim(S_k) = \dim(N_k) = 2k + 2 \),
- \( \dim(M_k) = 2k + 3 \).

Except for \( W_0, S_0 \) and \( N_0 \), described below, we can characterise the string bimodules by the first and last arrows in the corresponding graphs, reading from top to bottom, and from left to right:

- shape \( W \) begins by \( \downarrow \) and ends by \( \leftarrow \),
- shape \( S \) begins and ends by \( \downarrow \),
- shape \( N \) begins and ends by \( \leftarrow \),
- shape \( M \) begins by \( \leftarrow \) and ends by \( \downarrow \).

An indecomposable bimodule is called \( k \)-split if it is of the form \( U \otimes_k V \) for indecomposables \( U \in D\text{-}\text{mod} \) and \( V \in \text{mod-}D \). The \( k \)-split bimodules \( D \otimes D, \ W_0 \simeq k \otimes k, S_0 \simeq D \otimes k \) and \( N_0 \simeq k \otimes D \) form the unique maximal two-sided cell \( J_{k\text{-split}} \), with left cells inside it indexed by indecomposable right \( D \)-modules and right cells inside it indexed by indecomposable left \( D \)-modules, cf. [20].

As was shown in [5], band bimodules form one cell (both left, right and two-sided), which we denote \( J_{\text{band}} \). Moreover, for each positive integer \( k \), the four string bimodules with \( k \) valleys form a two sided cell \( J_k \), see Section 3.4 for more details. The string bimodule \( M_0 \) forms its own two-sided cell \( J_{M_0} \). The two-sided cells are linearly ordered as follows:

\[
J_{k\text{-split}} > J_{M_0} > J_1 > J_2 > \ldots > J_{\text{band}}.
\]

All two-sided cells except \( J_{M_0} \) are idempotent. Note also that all two-sided cells except the minimal cell \( J_{\text{band}} \) are finite.

3.2 The Main Result

The following theorem is the main result of this paper.

**Theorem 1**

(i) Any simple transitive 2-representation of \( \mathcal{D} \) with apex \( J_{k\text{-split}} \) is equivalent to a cell 2-representation.

(ii) Any simple transitive 2-representation of \( \mathcal{D} \) with apex \( J_k \), where \( k \geq 1 \), has rank 1 or rank 2.

(iii) Any simple transitive 2-representation of \( \mathcal{D} \) with apex \( J_k \), where \( k \geq 1 \), of rank 2 is equivalent to the cell 2-representation \( \mathcal{C}_\mathcal{L} \), where \( \mathcal{L} = \{M_k, N_k\} \) (or, equivalently, \( \mathcal{L} = \{W_k, S_k\} \)).

(iv) There exists a simple transitive 2-representation of \( \mathcal{D} \) with apex \( J_1 \) which has rank 1.

Taking Theorem 1 into account, the following conjecture seems very natural.
Conjecture 2 For each $k \geq 1$, there exists a unique, up to equivalence, simple transitive 2-representation of $\mathcal{D}$ of rank 1 with apex $\mathcal{J}_k$.

3.3 Proof of Theorem 1(i)

For an arbitrary indecomposable $\mathbb{k}$-split $D$-$D$-bimodule $U \otimes_k V$, using adjunction and projectivity of both $V$ and $\text{End}_{D\cdot}(U)$ as $k$-modules, we have

$$\text{End}_{D\cdot}(U \otimes_k V) \cong \text{Hom}_{D\cdot}(U \otimes_k V, U \otimes_k V) \cong \text{Hom}_{D\cdot}(V, \text{Hom}_{D\cdot}(U, U \otimes_k V))$$

$$\cong \text{Hom}_k(\mathbb{k}, \text{Hom}_{D\cdot}(V, \text{Hom}_{D\cdot}(U, U \otimes_k V))) \cong \text{Hom}_k(\mathbb{k}, \text{Hom}_{D\cdot}(U, U) \otimes_k \text{Hom}_{D\cdot}(V, V))$$

$$\cong \text{End}_{D\cdot}(U) \otimes_k \text{End}_{D\cdot}(V).$$

(1)

Consider the finite-dimensional algebra $A = \text{End}_{D\cdot}(D \oplus D \mathbb{k})$ (note that it can be described as the path algebra of the quiver

![Quiver](quiver_image)

modulo the relation $\alpha \beta = 0$). Then we have the 2-category $\mathcal{C}_A$ of projective $A$-$A$-bimodules. By [21, Theorem 12], any simple transitive 2-representation of $\mathcal{C}_A$ is equivalent to a cell 2-representation.

Let $M$ be a simple transitive 2-representation of $D$ with apex $\mathcal{J}_k$-split. Then the restriction of $M$ to $A$ is also simple transitive and hence this restriction is equivalent to a cell 2-representation of $\mathcal{A}$ by the previous paragraph. Now, the arguments similar to the ones in [18, Theorem 18] imply that $M$ is equivalent to a cell 2-representation of $\mathcal{D}$. This proves Theorem 1(i).

3.4 The Two-sided Cell $\mathcal{J}_k$, where $k \geq 1$

Fix a positive integer $k$. Recall from [5] that the two-sided cell $\mathcal{J}_k$ has the following egg-box diagram in which columns are left cells and rows are right cells.

![Egg-box](egg-box_image)

Modulo the two-sided cells that are strictly larger with respect to the two sided order, the multiplication table of $\mathcal{J}_k$ is as follows.

$$\begin{array}{c|c|c|c|c}
\otimes_D & W_k & S_k & N_k & M_k \\
W_k & W_k & W_k & N_k & N_k \\
S_k & S_k & S_k & M_k & M_k \\
N_k & W_k & W_k & N_k & N_k \\
M_k & S_k & S_k & M_k & M_k \\
\end{array}$$

(2)
Lemma 3  For any \( k \geq 0 \), the pair \( (S_k \otimes_D -, \ N_k \otimes_D -) \) is an adjoint pair of endofunctors of \( D\text{-mod} \).

Proof  By [23, Lemma 13], it is enough to show that \( S_k \) is projective as a left \( D \)-module, and that \( \text{Hom}_{D-}(S_k, D) \cong N_k \) as \( D\text{-}\text{bimodules} \). As a left module, \( S_k \) is a direct sum of \( k + 1 \) copies of the left regular module \( D \). This also implies that \( \text{Hom}_{D-}(S_k, D) \) is projective as a right module. Moreover

\[
\dim \text{Hom}_{D-}(S_k, D) = \dim \text{Hom}_{D-}(D^{\otimes k+1}, D) = (k + 1) \dim \text{End}_{D-}(D) = 2(k + 1).
\]

Note that \( D \) is a symmetric algebra and thus \( DD \cong D^* \). Hence, by adjunction, we get

\[
\text{Hom}_{D-}(S_k, D) \cong \text{Hom}_{D-}(S_k, \text{Hom}_k(D, k)) \cong \text{Hom}_k(D \otimes_D S_k, k) \cong \text{Hom}_k(S_k, k)
\]

so that \( \text{Hom}_{D-}(S_k, D) \) and \( S_k^* \) are isomorphic as \( D\text{-}\text{bimodules} \). Since \( S_k \) is indecomposable as a \( D\text{-}\text{bimodule} \), so is \( \text{Hom}_{D-}(S_k, D) \).

The indecomposable, right projective, \( 2(k + 1) \)-dimensional \( D\text{-}\text{bimodules} \) are the following.

- \( N_k \),
- \( B_{k+1}(\lambda) \),
- \( D \otimes_k D \) (in the case \( k = 1 \)).

To show that \( \text{Hom}_D(S_k, D) \cong N_k \), note first that

\[
\text{Hom}_{D-}(S_0, D) = \text{Hom}_{D-}(D \otimes_k k, D) \cong \text{Hom}_k(k, \text{Hom}_D(D, D)) \cong \text{Hom}_k(k, D),
\]

so it is clear that \( \text{Hom}_{D-}(S_0, D) \cong N_0 = k \otimes_k D \), as \( D\text{-}\text{bimodules} \). Now, for any \( k \geq 1 \), there is a short exact sequence of \( D\text{-}\text{bimodules} \)

\[
0 \to S_{k-1} \to S_k \to S_0 \to 0.
\]

Apply the functor \( \text{Hom}_{D-}(-, D) \) to this sequence. As the regular \( D\text{-}\text{bimodule} \) is injective as a left module, this functor is exact. Therefore we get a short exact sequence of \( D\text{-}\text{bimodules} \)

\[
0 \to \text{Hom}_{D-}(S_0, D) \to \text{Hom}_{D-}(S_k, D) \to \text{Hom}_{D-}(S_{k-1}, D) \to 0. \tag{3}
\]

Hence \( \text{Hom}_{D-}(S_0, D) \cong N_0 \) is a submodule of any \( \text{Hom}_{D-}(S_k, D) \), implying that \( \text{Hom}_{D-}(S_k, D) \) is not a band bimodule. This proves the statement for \( k \neq 1 \). Moreover, by setting \( k = 2 \) in Eq. 3, we see that \( \text{Hom}_{D-}(S_1, D) \) is the quotient of \( \text{Hom}_{D-}(S_2, D) \cong N_2 \) by \( \text{Hom}_{D-}(S_0, D) \cong N_0 \), that is

\[
\text{Hom}_{D-}(S_1, D) \cong N_2/N_0 \cong N_1.
\]

This concludes the proof. \( \square \)

The following statement is an adjustment of [31, Theorem 3.1] to a slightly more general setting, into which simple transitive 2-representations of \( \mathcal{D} \) with apex \( \mathcal{J}_k \), where \( k \geq 1 \), fit.

Theorem 4  Let \( \mathcal{C} \) be an additive, \( \mathbb{K}\text{-linear} \), Krull-Schmidt 2-category with finitely many objects, such that for any \( \mathfrak{a}, \mathfrak{j} \in \text{ob} \mathcal{C} \), \( \mathcal{C}(\mathfrak{a}, \mathfrak{j}) \) is idempotent split and has finite-dimensional spaces of 2-morphisms. Let \( \mathcal{M} \) be a finitary simple transitive 2-representation of \( \mathcal{C} \) such that the apex \( \mathcal{J} \) of \( \mathcal{M} \) is finite. Assume that \( F \in \mathcal{J} \). Then the following holds.

(i)  For every object \( X \) in any \( \mathcal{M}(\mathfrak{a}) \), the object \( \mathcal{M}(F)X \) is projective.

(ii)  If \( \mathcal{M}(F) \) is left exact, then \( \mathcal{M}(F) \) is a projective functor.

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Proof We can restrict to the finitary 2-subcategory of $\mathcal{C}$ given by the identities and the apex and then apply [31, Theorem 3.1].

Corollary 5 Let $M$ be a simple transitive 2-representation of $\mathcal{D}$ with apex $J_k$, where $k \geq 1$. Then the functor $\overline{M}(N_k)$ is a projective functor (in the sense that it is given by tensoring with a projective bimodule over the underlying algebra of the 2-representation).

Proof From Lemma 3 it follows that $\overline{M}(N_k)$ is left exact. Therefore we may apply Theorem 4 and the claim follows.

4 Combinatorial Results

Fix a simple transitive 2-representation $M$ of $\mathcal{D}$ with apex $J_k$, where $k \geq 1$. Let $B$ be a basic associative $k$-algebra for which $M(i)$ is equivalent to $B$-proj. Let $1 = \varepsilon_1 + \ldots + \varepsilon_r$ be a decomposition of the identity in $B$ into a sum of pairwise orthogonal primitive idempotents. Denote by $P_i$ the $i$'th indecomposable projective left $B$-module $B\varepsilon_i$, and denote by $L_i$ its simple top.

The aim of this section is to prove the following.

Proposition 6 Let $M$ be a simple transitive 2-representation of $\mathcal{D}$ with apex $J_k$, where $k \geq 1$. Then the action matrices of indecomposable 1-morphisms in $J_k$ are, up to renumbering of projective objects in $M(i)$, either all equal to $[4]$ or

\[ N_k = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_k = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \]

In particular, Proposition 6 implies Theorem 1(ii). The remainder of this section is devoted to the proof of Proposition 6.

Lemma 7 (i) If the matrix $[N_k]$ has a zero column, then the corresponding row in $[S_k]$ must be zero.

(ii) If the matrix $[S_k]$ has a zero column, then the corresponding row in $[N_k]$ must be zero.

Proof By Lemma 3, the functor $N_k$ is exact. By [18, Lemma 10], we have $[[N_k]] = [S_k]^t$. If column $i$ in the matrix $[N_k]$ is zero, then $N_k P_i = 0$. As $L_i$ is the top of $P_i$, the object $N_k L_i$ must be zero as well. This proves (i). On the other hand, if column $i$ of $[S_k]$ is zero, then row $i$ of $[S_k]^t = [[N_k]]$ is zero. This means that nothing in the image of $N_k$ can have $L_i$ as a simple subquotient. In particular, $P_i$ cannot occur in the image of $N_k$, and so row $i$ of $[N_k]$ must be zero. This proves (ii).

Note that $W_k$, $S_k$, $N_k$ and $M_k$ are all idempotent modulo strictly greater two-sided cells. Setting $F = W_k + S_k + N_k + M_k$ yields $F \otimes_D F = F^{\oplus 4}$. Hence the action matrix of $F$ must be an irreducible positive integer matrix satisfying $[F]^2 = 4[F]$. Such matrices are classified in [26]. They are, up to permutations of rows and columns, the following.

\[
\begin{bmatrix} 4 \\ 2 & 2 \\ 2 & 2 \\ 3 & 3 \\ 3 & 1 \\ 3 & 1 \\ 2 & 4 \\ 2 & 1 \\ 4 & 2 \\ 4 & 2 \end{bmatrix}
\]
Since the functors $M(W_k), M(S_k), M(N_k)$ and $M(M_k)$ are all idempotent, their action matrices are idempotent as well. The rank of an idempotent matrix equals its trace. The trace of $[F]$ is 4, so the action matrices $[W_k], [S_k], [N_k], [M_k]$ must all have trace and rank 1. The action matrices also inherit left, right and two-sided preorders and equivalences, so we speak of these notions for 1-morphisms and action matrices interchangeably. Directly from the multiplication table we can also conclude the following about the action matrices.

- If $A \sim_r B$, then $AB = B$ and $BA = A$. This also implies
  \[
  \text{im}(B) = \text{im}(AB) \subseteq \text{im}(A)
  \]
  \[
  \text{im}(A) = \text{im}(BA) \subseteq \text{im}(B),
  \]
  so that $\text{im } A = \text{im } B$. For matrices of rank 1 this means that all nonzero columns of $A$ and $B$ are linearly dependent.

- If $A \sim_l B$, then $AB = A$ and $BA = B$. This also implies
  \[
  \ker(A) = \ker(AB) \supseteq \ker(B)
  \]
  \[
  \ker(B) = \ker(BA) \supseteq \ker(A),
  \]
  so that $\ker A = \ker B$. Hence $A$ and $B$ have the same zero columns.

**Lemma 8**

(i) $[W_k] = [S_k]$ if and only if $[N_k] = [M_k]$.

(ii) $[W_k] = [N_k]$ if and only if $[S_k] = [M_k]$.

(iii) If $[W_k] = [M_k]$ or $[S_k] = [N_k]$, then $[W_k] = [S_k] = [N_k] = [M_k]$.

**Proof** If $[W_k] = [S_k]$, then

\[
[N_k] = [W_k][N_k] = [S_k][N_k] = [M_k].
\]

On the other hand, if $[N_k] = [M_k]$, then

\[
[W_k] = [N_k][W_k] = [M_k][W_k] = [S_k].
\]

This proves claim (i); claim (ii) is similar. Finally, if $[W_k] = [M_k]$, then

\[
[W_k] = [W_k][W_k] = [W_k][M_k] = [N_k]
\]

and

\[
[W_k] = [W_k][W_k] = [M_k][W_k] = [S_k].
\]

This proves one of the implications in (iii), the other is similar.

In particular, Lemma 8 implies that, if the matrix $[F]$ has 1 as an entry, then the matrices $[W_k], [S_k], [N_k]$ and $[M_k]$ are all different.

We now do a case-by-case analysis depending on the rank of the 2-representation (i.e. the size of action matrices).

### 4.1 Rank 1

If $F = [4]$, then $[W_k] = [S_k] = [N_k] = [M_k] = [1]$. 
4.2 Rank 2

Consider first the case $F \in \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$. Since $F$ has entries equal to 1, the action matrices of $W_k$, $S_k$, $N_k$ and $M_k$ must all be different. They all have trace 1 and their sum has diagonal $(3, 1)$, so we must have four different matrices with non-negative integer entries:

$$A = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & c \\ d & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & e \\ f & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & g \\ h & 1 \end{bmatrix}.$$

Two of those with diagonal $(1, 0)$, say $A$ and $B$, must belong to the same left cell. Then $AB = A$, i.e.

$$A = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix} = AB = \begin{bmatrix} 1 + ad & c \\ d & bc \end{bmatrix}$$

which implies $a = c$ and $b = d$, so that $A = B$, a contradiction.

Assume now $F \in \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Then $W_k$, $S_k$, $N_k$ and $M_k$ will be given by the following matrices:

$$A = \begin{bmatrix} 1 & * \\ * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & * \\ * & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}.$$

We see that $AB, BA \in \{A, B\}$. This implies that either $A \sim_L B$ or $A \sim_R B$.

If $A \sim_L B$, then $A \not\sim_R B$, so we can assume $A \sim_R C$ and $B \sim_R G$. By comparing images, and using that all ranks are 1, we get

$$A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = G = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

A $\sim_L B$ and $C \sim_L G$ tells us that left equivalent functors are represented by the same matrix. By symmetry we can set

$$[N_k] = [M_k] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[S_k] = [W_k] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

However, now the second column of $[N_k]$ is zero, but the second row of $[S_k]$ is nonzero. This contradicts Lemma 7(i), so we discard this case.

If, instead, $A \sim_R B$ and $C \sim_R G$, we can assume $A \sim_L C$ and $B \sim_L G$. Since the first column of $A$ is nonzero, so is the first column of $C$. At the same time, the second column of $C$ is nonzero, so the second column of $A$ is as well. Together with ranks being 1 and right equivalences, this yields

$$A = B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = G = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

By symmetry we can set

$$[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

4.3 Rank 3

$$F \in \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. $$

Either choice of the matrix of $F$ has 1 as entry, so all of $[W_k], [S_k], [N_k]$ and $[M_k]$ have to be different. As the diagonal of $F$ is $(2, 1, 1)$, they must
be represented by idempotent matrices $A, B, C, G$, all of rank 1, as follows.

$$A = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & * \\ 1 & 0 \\ * & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & * \\ 0 & 0 \\ * & 1 \end{bmatrix}. $$

Note that $AB, BA \in \{A, B\}$, so $A$ and $B$ are either left or right equivalent. We consider these two cases separately.

Assume first $A \sim_L B, C \sim_L G$, so that $C$ and $G$ have the same kernel. Hence the third column of $C$ and the second of $G$ are nonzero. Since the ranks are 1 we get

$$C = \begin{bmatrix} 0 & * \\ 1 & 1 \\ * & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & * \\ 0 & 0 \\ * & 1 \end{bmatrix}. $$

Taking into account that the lower right submatrix of $F$ has all entries 1, this implies

$$A = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}. $$

Since $A \sim_L B$, we have $A \not\sim_R B$. We can thus assume $A \sim_R C$ and $B \sim_R G$. Then $A$ and $C$ have the same image, and $B$ and $G$ have the same image, so that

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}. $$

Then $\{S, N\}$ is either $\{A, G\}$ or $\{B, C\}$. Any such choice contradicts Lemma 7(ii).

Assume now $A \sim_R B$ and $C \sim_R G$, so that $C$ and $G$ have the same image. Then

$$C = \begin{bmatrix} 0 & * \\ 1 & 0 \\ * & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & * \\ 1 & 1 \\ * & 0 \end{bmatrix}. $$

By considering the lower right $2 \times 2$-submatrix of $F$, we conclude

$$A = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}. $$

$A \sim_R B$ implies $A \not\sim_L B$, so we can assume $A \sim_L C$ and $B \sim_L G$. Using now that left equivalence means common kernel, together with all ranks being 1, we get

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. $$

Then $\{S, N\}$ is either $\{A, G\}$ or $\{B, C\}$. Any such choice contradicts Lemma 7(i).

### 4.4 Rank 4

Assume that $W_k, S_k, N_k$ and $M_k$ are given by

$$A = \begin{bmatrix} 1 & * \\ 0 & 0 \\ * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & * \\ 1 & 0 \\ * & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & * \\ 0 & 1 \\ * & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & * \\ 0 & 0 \\ * & 1 \end{bmatrix}. $$
As all entries of \( F \) are 1, we have that, for each position \((i, j)\), one of \( A, B, C, G \) has entry 1 at this position, while the others have entry 0 at this position.

We can, without loss of generality, assume that \( A \sim_R B, C \sim_R G, A \sim_L C \) and \( B \sim_L G \). This gives us immediately

\[
A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.
\]

Then \( \{S_k, N_k\} \) is either \( \{A, G\} \) or \( \{B, C\} \). Any such choice contradicts Lemma 7. This completes the proof of Proposition 6.

5 Each Simple Transitive 2-representation of Rank 2 is Cell

Fix a simple transitive 2-representation \( M \) of \( \mathcal{D} \) with apex \( J_k \), where \( k \geq 1 \).

Let \( L \) be the left cell \( \{N_k, M_k\} \). As seen in Proposition 6, the action matrices of \( M(U_k) \), where \( U_k \in J_k \), are as follows:

\[
[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.
\]

Let us see what this says about the basic algebra \( B \) underlying \( \overline{M}(\pm) \). The rank is two, so we have a decomposition \( 1 = \varepsilon_1 + \varepsilon_2 \) of the identity in \( B \) into primitive orthogonal idempotents. Denote by \( P_1 = B\varepsilon_1 \) and \( P_2 = B\varepsilon_2 \) the indecomposable projective left \( B \)-modules, and by \( L_1, L_2 \) their respective simple tops. Then, for \( i = 1, 2 \), we have

\[
M(N_k)P_i \simeq M(W_k)P_i \simeq P_1 \quad \text{and} \quad M(S_k)P_i \simeq M(M_k)P_i \simeq P_2.
\]

Moreover, since

\[
[[N_k]] = [[S_k]] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},
\]

we have \( \overline{M}(N_k)L_1 = 0 \) and \( \overline{M}(N_k)L_2 \) has simple subquotients \( L_1, L_2 \). As seen in Corollary 5, the functor \( \overline{M}(N_k) \) is given by tensoring by a projective \( B \)-\( B \)-bimodule. Hence

\[
\overline{M}(N_k) = \left( \bigoplus_{i,j = 1,2} (B\varepsilon_i \otimes \varepsilon_j B)^{a_{ij}} \right) \otimes_B \overline{M}(N_k)P_1 \simeq \overline{M}(N_k)(P_1 \otimes B).
\]

for some nonnegative integers \( a_{ij} \). As established, \( \overline{M}(N_k)L_1 = 0 \), which implies \( a_{11} = a_{21} = 0 \). Moreover, \( \overline{M}(N_k)P_2 = P_1 \), implying \( a_{22} = 0 \). Now, finally, the relation \( \overline{M}(N_k)\overline{M}(N_k) = \overline{M}(N_k) \) implies \( a_{12} = 1 \). We conclude

\[
\overline{M}(N_k) = B\varepsilon_1 \otimes \varepsilon_2 B \otimes_B \overline{M}(N_k).
\]

It now follows that \( \overline{M}(N_k)L_2 \) must be isomorphic to the direct sum of a number of copies of \( P_1 \). Therefore we see that \( \overline{M}(N_k)L_2 \simeq P_1 \), and \( P_1 \) has length 2 with socle \( L_2 \). In the underlying quiver of \( B \) this means that we have exactly one arrow \( \alpha \) from 1 to 2, and no
loops at 1. If there is an arrow $\beta$ from 2 to 1 then $\beta \alpha = 0$. Moreover, if there is a loop $\gamma$ at 2 then $\gamma \alpha = 0$.

$$\begin{array}{c}
\circlearrowleft[\text{1} \\
\text{2} \\
\circlearrowright[\text{3}]
\end{array}$$

(4)

This also yields

$$\dim(\varepsilon_1 B \varepsilon_1) = \dim \text{Hom}_B(P_1, P_1) = 1$$

$$\dim(\varepsilon_2 B \varepsilon_1) = \dim \text{Hom}_B(P_2, P_1) = 1.$$  

As seen in Lemma 3, $(S_k, N_k)$ is an adjoint pair, so this also gives

$$M(S_k) \approx B \varepsilon_2 \otimes (B \varepsilon_1)^* \otimes B - ,$$

cf. [14, Subsection 7.3]. Again, using that $(S_k, N_k)$ is an adjoint pair, yields

$$\dim(\varepsilon_2 B \varepsilon_2) = \dim \text{Hom}_B(P_2, P_2) =$$

$$= \dim \text{Hom}_B(M(S_k) P_1, P_2) =$$

$$= \dim \text{Hom}_B(P_1, M(N_k) P_2) =$$

$$= \dim \text{Hom}_B(P_1, P_1) =$$

$$= 1.$$  

In the quiver Eq. 4, this rules out loops at 2. Moreover, it implies

$$M(W_k) \approx M(N_k) M(S_k) \approx B \varepsilon_1 \otimes (B \varepsilon_1)^* \otimes B - .$$

Because $W_k$ is idempotent, $\dim ((B \varepsilon_1)^* \otimes_B B \varepsilon_1) = 1$. Hence, it follows that

$$M(M_k) = M(S_k) M(N_k) = B \varepsilon_2 \otimes \varepsilon_2 B \otimes B - .$$

Consider now $(B \varepsilon_1)^*$. As seen above, $P_1 = B \varepsilon_1$ has Jordan-Hölder series $L_1, L_2$, so $(B \varepsilon_1)^*$ has top $L_2^*$ and socle $L_1^*$ (these are simple right $B$-modules). As $\dim(\varepsilon_2 B \varepsilon_1) = 1 = \dim(\varepsilon_2 B \varepsilon_2)$, this implies that $(B \varepsilon_1)^*$ is exactly the projective right module $\varepsilon_2 B$. Hence we conclude

$$M(N_k) \approx M(W_k) \approx B \varepsilon_1 \otimes \varepsilon_2 B \otimes B -$$

$$M(S_k) \approx M(M_k) \approx B \varepsilon_2 \otimes \varepsilon_2 B \otimes B - .$$

Thus the Cartan matrix of $M$ is

$$\begin{bmatrix}
1 & c \\
1 & 1
\end{bmatrix}$$

where $c = \dim \text{Hom}_B(P_1, P_2)$ remains unknown.

Since $\dim \text{Hom}_B(P_2, P_2) = 1$, and $P_1$ has Jordan Hölder series $L_1, L_2$, we must have a short exact sequence

$$L_1^{\oplus c} \overset{g}{\rightarrow} P_2 \rightarrow L_2.$$

In the quiver Eq. 4, this corresponds to the fact that we have exactly $c$ arrows $\beta_1, \ldots, \beta_c : 2 \rightarrow 1$ and the relations

$$\alpha \beta_i = 0 = \beta_i \alpha.$$  

Let us sum up what we know so far:

- $P_1$ has basis $\{\varepsilon_1, \alpha\}$,
• $P_2$ has basis $\{\varepsilon_2, \beta_1, \ldots, \beta_c\}$,
• $\text{Hom}_B(P_1, P_2)$ has a basis $\{f_1, \ldots, f_c\}$ where $f_i(\alpha) = 0$ and $f_i(\varepsilon_1) = \beta_i$.

However, all functors above are of the form $M(U) = B\varepsilon_i \otimes \varepsilon_2 B \otimes_B -$.

The module $\varepsilon_2 B$ has basis $\{\varepsilon_2, \alpha\}$, and, as seen above, we have $\alpha \beta_i = 0 = \varepsilon_2 \beta_i$.

Thus, for $U \in \mathcal{J}_k$, we have

$$\overline{M}(U)(f_i)(\varepsilon_1) = 0,$$

so that $\overline{M}(U)(f_i) = 0$. But then, for $i = 1, \ldots, c$, the $f_i$ generate a proper $\mathcal{D}$-invariant ideal in $M(\mathfrak{1})$. By simplicity of $M$, this ideal is $\{0\}$. Thus $c = 0$ and the Cartan matrix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The rest of the proof now goes as in e.g. [18, Proposition 9] or [7, Subsection 4.9]. Consider the principal 2-representation $P_\mathfrak{1}$ and the subrepresentation $N$ with $N(\mathfrak{1}) = \text{add}\{F \mid F \geq_L \mathcal{L}\}$. Recall that there is a unique maximal ideal $I$ in $N$ such that $N/I \simeq C\mathcal{L}$. The map

$$\Phi : P_\mathfrak{1} \rightarrow \overline{M}$$

$$\mathfrak{1}_\mathfrak{1} \mapsto L_2$$

extends to a 2-natural transformation by the Yoneda Lemma, [15, Lemma 9]. Since

$$\overline{M}(N_k)L_2 = P_1 \quad \text{and} \quad \overline{M}(M_k)L_2 = P_2,$$

$\Phi$ induces a 2-natural transformation $\Psi : N \rightarrow \overline{M}_{\text{proj}}$. Note that $\overline{M}_{\text{proj}}$ is equivalent to $M$. By uniqueness of the maximal ideal $I$ the kernel of $\Psi$ is contained in $I$, so $\Psi$ factors through $C\mathcal{L}$. On the other hand, the Cartan matrices of $M$ and $C\mathcal{L}$ coincide. Consequently, $\Psi$ induces an equivalence of 2-representations between $C\mathcal{L}$ and $M$.

This proves Theorem 1(iii).

6 A Simple Transitive 2-representation of Rank 1 with Apex $\mathcal{J}_1$

Recall that we have the two-sided cell $\mathcal{J}_0$ containing only the 1-morphism $M_0$. We have $\mathcal{J}_{k\text{-split}} \succeq \mathcal{J}_0 \succeq \mathcal{J}_1$. The cell $\mathcal{J}_0$ is not idempotent, since $M_0 \otimes_D M_0 \simeq D \otimes D \oplus k$.

However, for all $U \in \mathcal{J}_1$, we have

$$U \otimes_D M_0 \simeq M_0 \oplus V,$$

where all indecomposable direct summands of $V$ are $k$-split. Since $\mathcal{J}_0$ contains only one element, it is also a left cell. Therefore the cell 2-representation $C_{\mathcal{J}_0}$ is a simple transitive 2-representation of $\mathcal{D}$ with apex $\mathcal{J}_1$. Note that the matrix describing the action of each 1-morphism in $\mathcal{J}_1$ is $[1]$, agreeing with Proposition 6.

This proves Theorem 1(iv) and thus completes the proof of Theorem 1.
7 (Co-) Duflo 1-morphisms

7.1 2-morphisms to and from $\mathbb{1}_i$

7.1.1 String Bimodules

Recall that $W_0$ is the unique simple $D$-$D$-bimodule (up to isomorphism). Given a string bimodule $U$, there are of course $D$-$D$-bimodule morphisms

$$U \to \text{top}(U) \to W_0 \sim \text{soc}(D) \leftarrow D$$

$$D \to \text{top}(D) \sim W_0 \hookrightarrow \text{soc}(U) \hookrightarrow U.$$ 

For our purposes, morphisms $U \to D$ and $D \to U$, which do not admit such factorization via the simple $D$-$D$-bimodule $W_0$, are of interest. Note that, for any nonsimple string bimodule $U$, $\text{soc}(U) = \text{rad}(U)$.

In what follows we will need a more detailed description of string bimodules. We will index the basis elements of $M_k$ and $W_k$ as follows:

$$
\begin{array}{cccccccc}
& m_1 & \leftarrow & m_2 & \leftarrow & \cdots & \leftarrow & m_k \\
\downarrow & & & & & & & \\
& m_{2k+1} & \leftarrow & m_{2k+2} & \leftarrow & \cdots & \leftarrow & m_{2k+3} \\
\downarrow & & & & & & & \\
& & w_1 & \leftarrow & w_2 & \leftarrow & w_3 & \leftarrow & w_k \\
\downarrow & & & & & & & \\
& & & & & & & w_{2k} \leftarrow w_{2k+1} \\
\end{array}
$$

With this convention, we have $N_k \simeq M_k/\text{span}(m_{2k+3})$, $S_k \simeq M_k/\text{span}(m_1)$ and $W_k \simeq M_k/\text{span}(m_1, m_{2k+3})$.

**Lemma 9** Let $k$ be a positive integer.

(i) The only element of $J_k$ admitting a $D$-$D$-bimodule morphism to $\mathbb{1}_i$ which does not factor through the simple bimodule is $M_k$.

(ii) The only element of $J_k$ admitting a $D$-$D$-bimodule morphism $\mathbb{1}_i \to U_k$ which does not factor through the simple bimodule is $W_k$.

**Proof** The regular bimodule $\mathbb{1}_i \simeq D_D$ has standard basis $\{1, x\}$.

There is a $D$-$D$-bimodule morphism $\varphi_k : M_k \to \mathbb{1}_i$ given by

$$\varphi_k(m_j) = \begin{cases} 
1, & j \text{ even} \\
x, & j \text{ odd}
\end{cases}.$$ 

That is, $\varphi_k$ maps standard basis elements from $\text{rad}(M_k)$ to $x \in D_D$, and the rest of the standard basis elements to 1. We prove that any $D$-$D$-bimodule morphism $\varphi : W_k \to \mathbb{1}_i$ factors through the simple bimodule, and similar arguments for $S_k$ and $N_k$ complete the proof of part (i). Assume that $\varphi : W_k \to \mathbb{1}_i$ is a $D$-$D$-bimodule morphism. Consider the standard basis vector $w_1$. Since $w_1x = 0$ we must have $\varphi(w_1) \in \text{span}(x)$. Thus

$$\varphi(w_2) = \varphi(xw_1) = x\varphi(w_1) = 0.$$
As \( w_2 = w_3 x \) this, in turn, implies \( \varphi(w_3) \in \text{span}[x] \) and so on. We will have \( \varphi(w_{2j}) = 0 \) for all \( j \), i.e. \( \varphi \) annihilates \( \text{rad}(W_k) \). Thus \( \varphi \) factors through the simple bimodule.

For part (ii), it is straightforward to check that \( \psi_k : 1_k \rightarrow W_k \) given by

\[
\psi_k(1) = w_1 + w_3 + \ldots + w_{2k+1}, \\
\psi_k(x) = w_2 + w_4 + \ldots + w_{2k},
\]

is a homomorphism of \( D-D \)-bimodules. If \( \eta : 1_k \rightarrow M_k \) is a bimodule morphism, then

\[
\eta(1) = \sum_{j=1}^{2k+3} \lambda_j m_j,
\]

for some \( \lambda_j \in \k \). Then

\[
\eta(x) = \eta(1)x = \sum_{j=1}^{k+1} \lambda_2 jm_{2j-1} = x\eta(1) = \sum_{j=1}^{k+1} \lambda_2 jm_{2j+1}.
\]

Comparing the coefficients, we conclude that \( \lambda_j = 0 \), for \( j = 1, \ldots, k+1 \), so that \( \eta(1) \in \text{rad}(M_k) \). Thus \( \varphi \) factors through the simple bimodule. For \( S_k \) and \( N_k \), the proof is similar. \( \square \)

Note that \( \varphi_0 : M_0 \rightarrow 1_k \) is also defined. If we fix integers \( l \leq k \), then \( \varphi_l \) factors through \( \varphi_k \), and \( \psi_l \) factors through \( \psi_k \). Indeed, \( M_k \) has a subbimodule isomorphic to \( M_l \) spanned by \( \{ m_j \mid j = 1, \ldots, 2l+3 \} \). Letting \( u_{l,k} : M_l \rightarrow M_k \) be the inclusion of \( M_l \) into \( M_k \), it is clear that \( \varphi_l = \varphi_k \circ u_{l,k} \). Similarly, denote by \( \pi_{k,l} : W_k \rightarrow W_l \) the projection whose kernel is spanned by \( \{ w_j \mid j \geq 2l+2 \} \). Then \( \psi_l = \pi_{k,l} \circ \psi_k \).

Let us now address the problem of uniqueness of \( \varphi_k \) and \( \psi_k \). For a non-negative integer \( k \), denote by \( \hat{V}_k \) the subspace of \( \text{Hom}_{D-D}(M_k, D) \) consisting of all homomorphisms which factor through the simple \( D-D \)-bimodule. For a positive integer \( k \), denote by \( \hat{V}_k \) the subspace of \( \text{Hom}_{D-D}(D, W_k) \) consisting of all homomorphisms which factor through the simple \( D-D \)-bimodule.

**Corollary 10**

(i) For any non-negative integer \( k \), we have \( \dim \text{Hom}_{D-D}(M_k, D) / V_k = 1 \).

(ii) For any positive integer \( k \), we have \( \dim \text{Hom}_{D-D}(D, W_k) / \hat{V}_k = 1 \).

**Proof** Assume that \( \varphi \in \text{Hom}_{D-D}(M_k, D) \setminus V_k \). Then \( \varphi(m_2) \in D \setminus \k(x) \), in particular, \( x\varphi(m_2) = \varphi(xm_2) = \varphi(m_3) \neq 0 \). Using the right action of \( x \), we have \( \varphi(m_3) = \varphi(m_4)x = \varphi(m_4)x \), which uniquely determines the image of \( \varphi(m_4) \) in \( D/\k(x) \). Similarly, the image of each \( \varphi(m_i) \), where \( i \) is even, in \( D/\k(x) \) is uniquely determined. As \( \k(x) \subset D \) is a simple \( D-D \)-bimodule, claim (i) follows. Claim (ii) is proved similarly. \( \square \)

### 7.1.2 Band Bimodules

From the definition of band bimodules, it follows directly that, for all \( n \geq 2 \), there are short exact sequences of \( D-D \)-bimodules

\[
0 \rightarrow B_1(1) \xrightarrow{\alpha_n} B_n(1) \rightarrow B_{n-1}(1) \rightarrow 0 
\]

and

\[
0 \rightarrow B_{n-1}(1) \rightarrow B_n(1) \xrightarrow{\beta_n} B_1(1) \rightarrow 0.
\]
It is a technical but not difficult exercise to verify that, for any \( n \) and \( k \), the morphism \( \varphi_k \) factors through \( \beta_n \), and the morphism \( \psi_k \) factors through \( \alpha_n \).

These factorizations are by no means unique. We shall here briefly indicate how they can be determined. Consider the specialization to \( \lambda = 1 \) of the description of \( B_n(\lambda) \) in Section 3.1, and assume that the matrices are given with respect to some basis \( \{y_1, \ldots, y_n, z_1, \ldots, z_n\} \). We can then define a 2-morphism \( \eta_{k,n} : M_k \to B_n(1) \) uniquely by requiring

\[
\eta_{k,n}(m_{2k+2}) = y_n \\
\eta_{k,n}(\text{span} \{m_2, \ldots, m_{2k+2}\}) \subseteq \text{span} \{y_1, \ldots, y_n\}.
\]

This will indeed give the factorization \( \varphi_k = \beta_n \circ \eta_{n,k} \).

Similarly, there is a unique 2-morphism \( \tau_{n,k} : B_n(1) \to W_k \) satisfying

\[
\tau_{n,k}(y_1) = w_1 + w_3 + \ldots + w_{2k+1}, \\
\tau_{n,k}(\text{span} \{y_2, \ldots, y_n\}) \subseteq \text{span} \{w_1, w_3, \ldots, w_{2k-1}\},
\]

and this will yield \( \psi_k = \tau_{n,k} \circ \alpha_n \).

### 7.2 Duflo 1-morphisms in Fiat 2-categories

Following [14], recall that a finitary 2-category \( \mathcal{C} \) is called fiat if it has a weak involution \( \ast \) such that each pair \( (F, F^\ast) \) of 1-morphisms is an adjoint pair via some choice of adjunctions morphisms between the compositions \( FF^\ast, F^\ast F \) and the relevant identities.

Let \( \mathcal{C} \) be a fiat 2-category and \( \mathcal{L} \) a left cell in \( \mathcal{C} \). Let \( \mathfrak{i} = \mathfrak{i}_L \) be the object such that all 1-morphisms in \( \mathcal{L} \) start in \( \mathfrak{i} \). A 1-morphism \( G \in \mathcal{L} \) is called a Duflo 1-morphism for \( \mathcal{L} \), cf. [14, Subsection 4.5], if the indecomposable projective module \( P_{\mathfrak{i}} \) in \( \underline{P}_{\mathfrak{i}}(\mathfrak{i}) \) has a submodule \( K \) such that

1. \( P_{\mathfrak{i}} / K \) is annihilated by all \( F \in \mathcal{L} \),
2. there is a surjective morphism \( P_G \to K \).

By [14, Proposition 17], any left cell in a fiat 2-category \( \mathcal{C} \) has a unique Duflo 1-morphism. These Duflo 1-morphisms play a major role in the construction of cell 2-representations, cf. [14].

### 7.3 Duflo 1-morphisms for Finitary 2-categories

The paper [28] gives a different definition of Duflo 1-morphisms in finitary 2-categories. One significant difference with [14] is that, in the general case, Duflo 1-morphisms in the sense of [28] do not have to exist, and if they exist, they do not have to belong to the left cell they are associated to. Below we propose yet another alternative.

Let \( \mathcal{C} \) be a finitary 2-category, \( \mathcal{L} \) a left cell in \( \mathcal{C} \) and \( \mathfrak{i} = \mathfrak{i}_L \) the object such that all 1-morphisms in \( \mathcal{L} \) start at \( \mathfrak{i} \).

**Definition 11**

(i) A 1-morphism \( G \) in \( \mathcal{C} \) is good for \( \mathcal{L} \) if there is a 2-morphism \( \varphi : G \to \mathfrak{i} \) such that \( F \varphi : FG \to F \) is right split, for any \( F \in \mathcal{L} \) (i.e. there is \( \xi : F \to FG \) such that \( F \varphi \circ \nu \xi = \text{id}_F \)).

(ii) A 1-morphism \( G \) in \( \mathcal{C} \) is great for \( \mathcal{L} \) if it is good for \( \mathcal{L} \), and, for any \( G' \) with \( \varphi' : G' \to \mathfrak{i} \) which is also good for \( \mathcal{L} \), there is a 2-morphism \( \beta : G \to G' \) such that \( \varphi = \varphi' \circ \beta \).

**Remark 12** That \( \mathcal{C} \) is finitary is not necessary for Definition 11.
For fiat 2-categories, the following proposition relates the latter notion to that of Duflo 1-morphisms.

**Proposition 13** Let \( \mathcal{C} \) be a fiat 2-category and \( \mathcal{L} \) a left cell in \( \mathcal{C} \). Then \( G \in \mathcal{L} \) is great for \( \mathcal{L} \) if and only if \( G \) is the Duflo 1-morphism of \( L \).

**Proof** The proof goes as follows: we first prove that the Duflo 1-morphism of \( \mathcal{L} \) is good for \( \mathcal{L} \). Then we prove that if \( G \) is great for \( \mathcal{L} \), then \( G \) is the Duflo 1-morphism for \( \mathcal{L} \). Finally, we prove that the Duflo 1-morphism is great for \( \mathcal{L} \).

Assume first that \( G \) is the Duflo 1-morphism of \( \mathcal{L} \). Let \( K \subseteq P_{1,1} \) be the submodule from the definition and \( \alpha : PG \to K \) a surjective morphism. Let \( f : PG \to P_{1,1} \) be the composition \( PG \xrightarrow{\alpha} K \xleftarrow{\iota} P_{1,1} \). The morphism \( f \) is given by a morphism \( \varphi : G \to 1_{1,1} \) as represented in the commutative diagram

\[
P_G \xrightarrow{f} P_{1,1} = 0 \quad \xrightarrow{\varphi} \quad G = 0 \quad \xrightarrow{\varphi} \quad 1_{1,1}.
\]

Consider the short exact sequences

\[
\ker \xrightarrow{\iota} PG \xrightarrow{\alpha} K \\
K \xleftarrow{\iota} P_{1,1} \xrightarrow{\iota} P_{1,1}/K.
\]

As \( \mathcal{C} \) is fiat, each 1-morphism of \( \mathcal{C} \) acts as an exact functor on each abelian 2-representation of \( \mathcal{C} \). Therefore applying \( F \in \mathcal{L} \) yields short exact sequences

\[
FK \xrightarrow{F\iota} FP_{1,1} \xrightarrow{F\alpha} F(K).
\]

By assumption \( F(P_{1,1}/K) = 0 \), so \( F\iota : FK \to FP_{1,1} \) is an isomorphism, in particular, it is surjective. Thus \( Ff = F\iota \circ F\alpha : FP_G \to F1_{1,1} \) is also surjective, implying that it is right split.

By considering the right column of the diagram

\[
0 \xrightarrow{\varphi} FG \\
\varphi \downarrow \quad \downarrow F\varphi \\
0 \xrightarrow{\iota} F,
\]

we see that \( F\varphi \) is right split. Therefore \( G \) is good for \( \mathcal{L} \). This completes the first step of our proof.

To prove the second step, assume that \( G \) is great for \( \mathcal{L} \). Let \( \varphi : G \to 1_{1,1} \) be the corresponding 2-morphism from the definition. This extends to a morphism \( PG \to P_{1,1} \) in \( \mathcal{L} \) as
in Eq. 5 and the submodule $K$ of $P_{\mathbb{1}_i}$ is the image of this morphism. We now have a short exact sequence

$$0 \rightarrow K \xrightarrow{\mathcal{T}} P_{\mathbb{1}_i} \xrightarrow{g} P_{\mathbb{1}_i}/K \rightarrow 0.$$  

Applying $F \in \mathcal{L}$, which is exact, we get a short exact sequence

$$0 \rightarrow FK \xrightarrow{F\mathcal{T}} PF \xrightarrow{Fg} F(P_{\mathbb{1}_i}/K) \rightarrow 0.$$  

Note that, since $F\phi$ is right split, the induced morphism $K \rightarrow PF$ in $P_{\mathbb{1}_i}$ is also right split and therefore surjective. Hence $F\mathcal{T} : FK \rightarrow PF$ is an isomorphism. By exactness, we obtain $F(P_{\mathbb{1}_i}/K) = 0$.

To conclude that $G$ is the Duflo 1-morphism in $\mathcal{L}$, it remains to show that $G \in \mathcal{L}$. Assume that $H$ is the Duflo 1-morphism of $\mathcal{L}$, and that $K_H \subseteq P_{\mathbb{1}_i}$ is the submodule from the definition. We shall prove that $G = H$. By the above, $H$ is good for $\mathcal{L}$, with the corresponding morphism $H \rightarrow \mathbb{1}_i$, so there is a morphism $\alpha : G \rightarrow H$ making the following diagram commutative:

$$\begin{array}{ccc}
H & \xrightarrow{\phi} & \mathbb{1}_i \\
\alpha \downarrow & & \downarrow \\
G & & \\
\end{array}$$

Therefore $K \subseteq K_H \subseteq P_{\mathbb{1}_i}$. Note that $K_H$ has simple top $L_H$. By [14, Proposition 17(b)], for all $F \in \mathcal{L}$, the object $FL_H$ has simple top $L_F$, in particular $FL_H \neq 0$. Since $F(P_{\mathbb{1}_i}/K) = 0$, for all $F \in \mathcal{L}$, we conclude that $K_H \subseteq K$. Thus $K_H = K$. But $K_H$ has simple top $L_H$ and $K$ has simple top $L_G$, so $H = G$ is the Duflo 1-morphism of $\mathcal{L}$. This completes the second step of our proof.

Finally, let $G$ be the Duflo 1-morphism of $\mathcal{L}$. We have already seen that $G$ is good for $\mathcal{L}$, it remains to prove that it is great. Assume that $H$ is also good for $\mathcal{L}$, with $\psi : H \rightarrow \mathbb{1}_i$ being the morphism such that $F\psi$ is right split, for all $F \in \mathcal{L}$.

As above, $\text{im } \phi$ and $\text{im } \psi$ give submodules $K_G$ and $K_H$ of $P_{\mathbb{1}_i}$ with $F(P_{\mathbb{1}_i}/K_G) = 0$ and $F(P_{\mathbb{1}_i}/K_H) = 0$, for all $F \in \mathcal{L}$. Since the top of $K_G$ is $L_G$ and $L_G$ is not annihilated by $F \in \mathcal{L}$, there is a nonzero morphism $K_G \rightarrow K_H$ such that the diagram

$$\begin{array}{ccc}
P_G & \xrightarrow{\mathcal{T}} & K_G \\
\downarrow & & \downarrow \\
P_H & \xrightarrow{\mathcal{T}} & K_H \\
\end{array}$$

commutes. Since $P_G$ is projective, there is a morphism $\alpha : P_G \rightarrow P_H$ making the left square commute. Thus the whole diagram commutes and we obtain a factorization

$$\begin{array}{ccc}
P_G & \xrightarrow{\alpha} & P_{\mathbb{1}_i} \\
\downarrow & & \downarrow \\
P_H & & \\
\end{array}$$

implying that $G$ is great for $\mathcal{L}$. 

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7.4 Duflo 1-morphisms in $\mathcal{D}$

For a positive integer $m$, we denote by $\mathcal{D}^{(m)}$ the 2-full 2-subcategory of $\mathcal{D}$ given by the additive closure of all 1-morphisms in all two-sided cells $\mathcal{J}$ such that $\mathcal{J} \geq \mathcal{J}_m$, together with $\mathbb{1}_\mathcal{J}$. Note that $\mathcal{D}^{(m)}$ is a finitary 2-category.

The following proposition suggests that $M_k$ is a very good candidate for being called a Duflo 1-morphism in its left cell in $\mathcal{D}^{(k)}$.

**Proposition 14** For any $m \geq k \geq 1$, the 1-morphism $M_k$ of the finitary 2-category $\mathcal{D}^{(m)}$ is great for $\mathcal{L} = \{N_k, M_k\}$.

**Proof** Let us first establish that $M_k$ is good for $\mathcal{L}$. It is easy to check, by a direct computation (see Section 8.2), that the composition $M_k \otimes_D M_k$ has a direct summand isomorphic to $M_k$ spanned by $\{m_2 \otimes m_1, m_j \otimes m_j, m_{j+1} \otimes m_j \mid j = 2, 4, \ldots, 2k + 2\}$, and that the projection onto this summand is a right inverse to $M_k \varphi_k$. Using $N_k \cong M_k/\text{span}\{m_{2k+3}\}$, gives also that $N_k \varphi_k$ is right split. Therefore $M_k$ is good for $\mathcal{L}$ with respect to the morphism $\varphi_k : M_k \rightarrow \mathbb{1}_\mathcal{J}$. Note also that, by Corollary 10, the choice of $\varphi_k$ is unique up to a non-zero scalar and up to homomorphisms which factor through the simple $D-D$-bimodule.

Let now $F$ be a 1-morphism in $\mathcal{D}^{(k)}$ which is good for $\mathcal{L}$ via the map $\alpha : F \rightarrow D$. To start with, we argue that $\alpha$ does not factor through the simple $D-D$-bimodule. Indeed, if $\alpha$ does factor through the simple $D-D$-bimodule, it is not surjective as a map of $D-D$-bimodules. Applying the right exact functor $M_k \otimes_D -$ to the exact sequence

$$F \xrightarrow{\alpha} D \xrightarrow{\text{Coker}} 0,$$

we get the exact sequence

$$M_k \otimes_D F \xrightarrow{M_k \otimes_D \alpha} M_k \otimes_D \text{Coker} \rightarrow 0.$$

Note that Coker is the simple $D-D$-bimodule and that $M_k \otimes_D \text{Coker} \neq 0$. Therefore $M_k \otimes_D \alpha$ is not right split. This implies that $\alpha$ is surjective as a map of $D-D$-bimodules.

Now we show that if $F$ has an indecomposable direct summand $G \in \mathcal{J}_l$, $k < l \leq m$, such that the restriction of $\alpha$ to $G$ does not factor through the simple $D-D$-bimodule, then $\varphi_k$ factors through $\alpha$. Indeed, by Lemma 9 the only such possibility is $G \cong M_l$, and by Corollary 10 the restriction of $\alpha$ to this summand is a scalar multiple of $\varphi_l$. As noted in Section 7.1.1, $\varphi_k$ factors via $\varphi_l$ for $k \leq l$, so this provides a factorization of $\varphi_k$ through $\alpha$.

As the next step, we show that if the condition of the previous paragraph is not satisfied, then $F$ contains a summand isomorphic to either $D$ or $M_k$ such that the restriction of $\alpha$ to this summand does not factor through the simple $D-D$-bimodule. Indeed, assume that this is not the case. Then, by Lemma 9, the only possible indecomposable summands $G$ of $F$ for which the restriction of $\alpha$ does not factor through the simple $D-D$-bimodule come from two-sided cells $\mathcal{J}$ such that $\mathcal{J} > \mathcal{J}_k$. However, for such $G$, the composition $M_k G$ cannot have any summands in $\mathcal{J}_k$ since $\mathcal{J} > \mathcal{J}_k$. Since $M_k$ is indecomposable, it follows that any
morphism $M_k G \to M_k$ is a radical morphism. That $M_k G \to M_k$ is a radical morphism, for any summand $G$ isomorphic to $D$ or $M_k$, follows from our assumption by the arguments in the previous paragraph. Therefore $M_k \otimes_D \alpha$ is a radical morphism and hence not right split, as $M_k$ is indecomposable, a contradiction.

Because of the previous paragraph, there is a direct summand $G$ of $F$ isomorphic to either $M_k$ or $D$ such that the restriction of $\alpha$ to $G$ does not factor through the simple $D$-$D$-bimodule. If $G \cong D$, then the restriction of $\alpha$ to it is an isomorphism. We can pull back $\varphi_k$ via this isomorphism and define the map from $M_k$ to all other summands of $F$ as zero. This provides the necessary factorization of $\varphi_k$ via $\alpha$.

If $G \cong M_k$, we can pull back $\varphi_k$ using first Corollary 10 and then correction via morphisms from $M_k$ to the socle of $G$ (such morphisms factor through the simple $D$-$D$-bimodule). In any case, the constructed factorization implies that $M_k$ is great for $L$ and completes the proof of our proposition.

7.5 Co-Duflo 1-morphisms in $\mathcal{D}$

We can dualize Definition 11. Given a 2-category $\mathcal{C}$ and a left cell $L$ in $\mathcal{C}$ with $i = i L$, we say that a 1-morphism $H$ in $\mathcal{C}$ is co-good for $L$ if there is a 2-morphism $\psi : \mathbb{1} \to H$ such that $F \psi$ is left split, for all $F \in L$. Moreover, we say that $H$ is co-great for $L$ if $H$ is co-good for $L$ and, for any $H'$ which is co-good for $L$ with $\psi' : \mathbb{1} \to H'$, there is a 2-morphism $\gamma : H' \to H$ such that $\psi = \gamma \circ \psi'$.

The following proposition suggests that $W_k$ is a very good candidate for being called a co-Duflo 1-morphism in its left cell in $\mathcal{D}^{(k)}$.

**Proposition 15** For any $m \geq k \geq 1$, the 1-morphism $W_k$ of the finitary 2-category $\mathcal{D}^{(m)}$ is co-great for the left cell $L = \{W_k, S_k\}$.

**Proof** Consider the 2-morphism $\psi_k : \mathbb{1} \to W_k$. By a direct calculation, it is easy to check that $W_k \otimes_D W_k$ has a unique direct summand isomorphic to $W_k$ and that $S_k \otimes_D W_k$ has a unique direct summand isomorphic to $S_k$. The projections onto these summands provide left inverses for $W_k \psi_k$ and $S_k \psi_k$, respectively. This implies that $W_k$ is co-good for $L$ via $\psi_k$.

Assume now that $F$ is co-good for $L$ via some $\alpha : \mathbb{1} \to F$. We need to construct a factorization $F \to W_k$. Since multiplication with $x$ is a nilpotent endomorphism of $D$, the endomorphism $W_k \otimes_D x$ is a nilpotent endomorphism of $W_k$. In particular, this endomorphism is a radical map. By a direct computation, one can check that, for any $\beta : D \to W_k$ which factors through the simple $D$-$D$-bimodule, the endomorphism $W_k \otimes_D \beta$ is not injective, in particular, it is a radical map.

Now, using arguments similar to the ones in the proof of Proposition 14, one shows that there must exist a summand $G$ of $F$, the restriction of $\alpha$ to which does not factor through the simple $D$-$D$-bimodule and that this summand must be isomorphic to either $D$ or $W_l$ for some $l \geq k$. In the former case, the restriction of $\alpha$ to $G$ is an isomorphism and the necessary factorization $F \to W_k$ is constructed via $G \to W_k$ using this isomorphism. In the latter case, the necessary factorization is constructed via $G \to W_k$ using Corollary 10 and the observation that $\varphi_l$ factors via $\varphi_k$ for $k < l$, and then correction via morphisms from $G$ to $W_k$ which factor through the simple $D$-$D$-bimodule.
8 Some Algebra and Coalgebra 1-morphisms in $\mathcal{D}$

8.1 Algebra and Coalgebra 1-morphisms

Let $\mathcal{C}$ be a 2-category. Recall that an algebra structure on a 1-morphism $A \in \mathcal{C}(i, i)$ is a pair $(\mu, \eta)$ of morphisms $\mu : AA \to A$ and $\eta : 1_i \to A$ which satisfy the usual associativity and unitality axioms

$$\mu \circ_v (\mu \circ_h \text{id}) = \mu \circ_v (\text{id} \circ_h \mu), \quad \text{id} = \mu \circ_v (\text{id} \circ_h \eta), \quad \text{id} = \mu \circ_v (\eta \circ_h \text{id}).$$

Similarly, a coalgebra structure on a 1-morphism $C \in \mathcal{C}(i, i)$ is a pair $(\delta, \varepsilon)$ of morphisms $\delta : C \to CC$ and $\varepsilon : C \to 1_i$ which satisfy the usual coassociativity and counitality axioms

$$(\delta \circ_h \text{id}) \circ_v \delta = (\text{id} \circ_h \delta) \circ_v \delta, \quad \text{id} = (\text{id} \circ_h \varepsilon) \circ_v \delta, \quad \text{id} = (\varepsilon \circ_h \text{id}) \circ_v \delta.$$

In the case of fiat 2-categories, it is observed in [8, Section 6] that a Duflo 1-morphism often has the structure of a coalgebra 1-morphism (as suggested by the existence of a map from the identity to a Duflo 1-morphism) This is particularly interesting as it is shown in [8] that any simple transitive 2-representation of a fiat 2-category can be constructed using categories of certain comodules over coalgebra 1-morphisms.

Let $(A, \mu, \eta)$ be an algebra 1-morphism in $\mathcal{C}$. A right module over $A$ is a pair $(M, \rho)$, where $M$ is a 1-morphism in $\mathcal{C}$ and $\rho : MA \to M$ is such that the usual associativity and unitality axioms are satisfied:

$$\rho \circ_v (\rho \circ_h \text{id}) = \rho \circ_v (\text{id} \circ_h \mu), \quad \text{id} = \rho \circ_v (\text{id} \circ_h \eta).$$

Dually, one defines the notion of a comodule over a coalgebra. Morphisms between (co)modules are defined in the obvious way. We denote by $\text{mod}_{\mathcal{C}}(A)$ the category of all right $A$-modules in $\mathcal{C}$, and by $\text{comod}_{\mathcal{C}}(C)$ the category of all right $C$-comodules in $\mathcal{C}$.

8.2 Coalgebra Structure on Duflo 1-morphisms

Given the results from the previous section, it is natural to ask whether $M_k$ is a coalgebra 1-morphism in $\mathcal{D}$.

**Proposition 16** For a positive integer $k$, the 1-morphism $M_k$ has the structure of a coalgebra 1-morphism in $\mathcal{D}$. Moreover, the 1-morphism $N_k$ has the structure of a right $M_k$-module.

**Proof** Recall the standard basis of the bimodule $M_k$ from Section 7.1. The tensor product $M_k \otimes_D M_k$ has a unique direct summand isomorphic to $M_k$ with a basis given by

$$m_2 \otimes m_1 \leftarrow m_2 \otimes m_2 \quad \leftarrow m_3 \otimes m_2 \quad \leftarrow \cdots \quad m_{2k+1} \otimes m_{2k} \leftarrow m_{2k+2} \otimes m_{2k+2} \quad \leftarrow m_{2k+3} \otimes m_{2k+2}.$$
Moreover, we have $m_{2j+1} \otimes m_{2j} = m_{2j+2} \otimes m_{2j+1}$, for $j = 1, \ldots, k$. We define the comultiplication $\delta : M_k \to M_k \otimes_D M_k$ explicitly as follows:

$$\begin{align*}
\delta(m_{2j}) &= m_{2j} \otimes m_{2j}, & 1 \leq j \leq k + 1 \\
\delta(m_{2j+1}) &= m_{2j+1} \otimes m_{2j} = m_{2j+2} \otimes m_{2j+1}, & 1 \leq j \leq k + 1 \\
\delta(m_1) &= m_2 \otimes m_1 \\
\delta(m_{2k+3}) &= m_{2k+3} \otimes m_{2k+2}
\end{align*}$$

As a counit, we take the morphism $\varphi_k$ from Section 7.1. The counitality and comultiplication axioms are now checked by a lengthy but straightforward computation.

To prove that $N_k$ is a right $M_k$-comodule, we recall that $N_k \simeq M_k / \text{span}\{m_{2k+3}\}$. Let $\pi : M_k \to N_k$ be the canonical projection. Then $\rho = \pi \circ h \mu$ makes $N_k$ a right $M_k$-comodule.

Indeed, all necessary properties for $\rho$ follow directly from the corresponding properties for $\mu$. \hfill \square

**Corollary 17** The 2-representation $\mathcal{C} M_k \subset \text{comod}_\mathcal{C}(\mathcal{C})$ of $\mathcal{C}$ has a unique simple transitive quotient, moreover, this quotient is equivalent to the cell 2-representation $\mathcal{C} \mathcal{L}$, where $\mathcal{L} = \{M_k, N_k\}$.

**Proof** As $M_k$ is indecomposable, the unique simple transitive quotient $M$ of $\mathcal{C} M_k$ is the quotient of $\mathcal{C} M_k$ by the sum of all $\mathcal{C}$-stable ideals in $\mathcal{C} M_k$ which do not contain $\text{id}_{M_k}$. Clearly, $M_k$ does not annihilate $M_k$. At the same time, for any $F > J M_k$, we have that $\mathcal{C} F M_k$ does not contain $\text{id}_{M_k}$. Therefore any such $F$ is killed by $M$. This means that $M$ has apex $J_k$.

Further, $N_k M_k$ does not have any copy of $M_k$ as a direct summand. Therefore the rank of $M$ is at least 2. Now the claim of our corollary follows from Theorem 1(iii). \hfill \square

### 8.3 Algebra Structure on Co-Duflo Algebra 1-morphisms

Similarly to the previous section, it is natural to ask whether $W_k$ is an algebra 1-morphism in $\mathcal{D}$.

**Proposition 18** For a positive integer $k$, the 1-morphism $W_k$ has the structure of an algebra 1-morphism in $\mathcal{D}$. Moreover, the 1-morphism $S_k$ has the structure of a right $W_k$-module.

**Proof** The tensor product $W_k \otimes_D W_k$ has a unique direct summand isomorphic to $W_k$, namely, the direct summand with the basis

$$\begin{align*}
w_1 \otimes w_1 \\
w_2 \otimes w_1 &\leftarrow w_3 \otimes w_3 \\
\vdots &\leftarrow w_{2k-1} \otimes w_{2k-1} \\
w_{2k} \otimes w_{2k-1} &\leftarrow w_{2k+1} \otimes w_{2k+1}
\end{align*}$$
moreover, \( w_{2j} \otimes w_{2j-1} = w_{2j+1} \otimes w_{2j} \), for \( 1 \leq j \leq k \). This allows us to define multiplication \( \mu \) as the projection onto this direct summand. As the unit morphism, we take \( \psi_k \) from Section 7.1. All necessary axioms are checked by a straightforward computation.

The projection onto the unique summand of \( S_k \otimes D W_k \) isomorphic to \( S_k \) provides \( S_k \) with the structure of a right \( W_k \)-module. Note that letting \( \theta : M_k \to S_k \) and \( \zeta : M_k \to W_k \) be the canonical projections (see Section 7.1.1), and \( \pi_{M_k} : M_k \otimes D M_k \to M_k \) the projection as in the proof of Proposition 14, the projection \( S_k \otimes D W_k \to W_k \) makes the following diagram commute.

\[
\begin{array}{ccc}
M_k \otimes D M_k & \xrightarrow{\pi_{M_k}} & M_k \\
\theta \otimes \zeta \downarrow & & \downarrow \theta \\
S_k \otimes D W_k & \to & S_k
\end{array}
\]

Verifying that this gives \( S_k \) the structure of a right \( W_k \)-module is done by straightforward computation.

Corollary 19 The 2-representation \( C W_k \subset \text{comod}(C) \) of \( C \) has a unique simple transitive quotient, moreover, this quotient is equivalent to the cell 2-representation \( C_\mathcal{L} \), where \( \mathcal{L} = \{ W_k, S_k \} \).

\[ \text{Proof} \quad \text{Mutatis mutandis the proof of Corollary 17.} \]

8.4 Rank 1 Representations are Non-constructible

In this last subsection we would like to emphasize one major difference between the 2-representation theory of \( \mathcal{D} \) and that of fiat 2-categories.

Definition 20 Let \( C \) be a (finitary) 2-category and let \( \mathcal{B} \in \{ C, \overline{C}, \overline{\overline{C}} \} \). A 2-representation \( M \) of \( C \) is called \( \mathcal{B} \)-constructible if there is a (co)algebra 1-morphism \( C \) in \( \mathcal{B} \), a \( C \)-stable subcategory \( \mathcal{X} \) of the category of right \( C \)-(co)modules, and a \( C \)-stable ideal \( \mathcal{I} \) in \( \mathcal{X} \) such that \( M \) is equivalent to \( \mathcal{X}/\mathcal{I} \).

If \( C \) is fiat, then any simple transitive 2-representation of \( C \) is both \( \overline{\overline{C}} \)- and \( \overline{C} \)-constructible by [8]. From [9, Section 3] it follows that faithful simple transitive 2-representation of \( \mathcal{J} \)-simple fiat 2-categories are even \( C \)-constructible.

Corollary 17 implies that, for each \( k \geq 1 \), the cell 2-representation \( C_\mathcal{L} \) of \( \mathcal{D}^{(k)} \), where \( \mathcal{L} = \{ M_k, N_k \} \), is \( \mathcal{D}^{(k)} \)-constructible.

The following statement, in some sense, explains why the statement of Theorem 1(iv) is as it is.

Theorem 21 Let \( k \) and \( m \) be positive integers such that \( 2 \leq k \leq m \). Let \( M \) be a rank 1 simple transitive 2-representation of \( \mathcal{D}^{(m)} \) with apex \( \mathcal{J}_k \). Then \( M \) is not \( \mathcal{D}^{(m)} \)-constructible.

\[ \text{Proof} \quad \text{Assume towards contradiction that} \ M \text{ is} \mathcal{D} \text{-constructible. Let} \ \mathcal{X} \text{ be as in Definition 20 and consider some object} \ X \in \mathcal{X} \text{ which is nonzero in the quotient by} \mathcal{I}. \text{Then, for each} \ U \in \mathcal{J}_k, \ \text{we must have} \ U X \simeq X + \mathcal{I}. \]
If \( X \in \text{add}\{J | J > J_k\} \), then the action of \( J_{k-1} \) is nonzero on \( X \), implying that \( J_k \) is not the apex of the representation (note that \( k > 1 \)). This means that all indecomposable summands of \( X \) which matter for the computations in \( X/I \) are in \( J_k \).

From Eq. 2, we obtain that, modulo higher two-sided cells, \( N_k X_m \in \text{add}\{N_k \oplus W_k\} \) while \( M_k X_m \in \text{add}\{M_k \oplus S_k\} \). Since \( J_k \) is the apex of \( M \), both \( N_k X_m \) and \( M_k X_m \) are non-zero. This contradicts the assumption that \( M \) has rank 1.

\[ \square \]

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