On Cannon-Thurston maps for relatively hyperbolic groups

Yoshifumi Matsuda∗† and Shin-ichi Oguni‡§

Abstract

Baker and Riley proved that a free group of rank 3 can be contained in a hyperbolic group as a subgroup for which the Cannon-Thurston map is not well-defined. By using their result, we show that the phenomenon occurs for not only a free group of rank 3 but also every non-elementary hyperbolic group. In fact it is shown that a similar phenomenon occurs for every non-elementary relatively hyperbolic group.

Keywords: Cannon-Thurston maps; relatively hyperbolic groups; geometrically finite convergence actions; convergence actions

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1 Introduction

Given an injective group homomorphism from a hyperbolic group to another hyperbolic group, whether the map can be continuously extended on the Gromov boundaries is an interesting question by Mitra (see [13, Section 1]). If such an extension is well-defined, the induced map on the Gromov boundaries is called the Cannon-Thurston map. The first non-trivial example was known by Cannon and Thurston in the 1980’s (see [6]). Indeed their main theorem implies that for a closed hyperbolic 3-dimensional manifold $M$ which fibers over the circle with fiber a closed hyperbolic surface $S$, when we consider the induced injective group homomorphism between fundamental groups of $S$ and $M$, the Cannon-Thurston map is well-defined. Also more examples for which the Cannon-Thurston maps are well-defined can be recognized by Mitra’s results.

∗Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan, ymatsuda@ms.u-tokyo.ac.jp
†The author is supported by the Global COE Program at Graduate School of Mathematical Sciences, the University of Tokyo, and Grant-in-Aid for Scientific Researches for Young Scientists (B) (No. 22740034), Japan Society of Promotion of Science.
‡Department of Mathematics, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama, Ehime, 790-8577 Japan, oguni@math.sci.ehime-u.ac.jp
§The author is supported by Grant-in-Aid for Scientific Researches for Young Scientists (B) (No. 24740045), Japan Society of Promotion of Science.
At the present time, there are many works related to well-definedness of the Cannon-Thurston maps. Nevertheless Baker and Riley gave a negative answer ([2, Theorem 1]). Indeed they showed that a free group of rank 3 can be contained in a hyperbolic group as a subgroup for which the Cannon-Thurston map is not well-defined. In this paper we show that the phenomenon occurs for not only a free group of rank 3 but also every non-elementary hyperbolic group. In fact it is shown that a similar phenomenon occurs for every non-elementary relatively hyperbolic group.

Throughout this paper, every countable group is endowed with the discrete topology. We use a definition of relative hyperbolicity for groups from a dynamical viewpoint (see [5, Definition 1], [19, Theorem 0.1] and [9, Definition 3.1]). Also we use a definition of relative quasicovexity for subgroups of relatively hyperbolic groups from a dynamical viewpoint (see [7, Definition 1.6]). Refer to [9, Section 3 and Section 6] for other several equivalent definitions of those. Also see [17], [4] and [5] for some definitions and properties related to convergence actions.

Let $G$ be a non-elementary countable group and $\mathcal{H}$ be a conjugacy invariant collection of proper infinite subgroups of $G$. Suppose that $G$ is hyperbolic relative to $\mathcal{H}$, that is, there exists a compact metrizable space endowed with a geometrically finite convergence action of $G$ such that $\mathcal{H}$ is the set of all maximal parabolic subgroups of $G$. Such a space is unique up to $G$-equivariant homeomorphisms and called the Bowditch boundary of $(G, \mathcal{H})$. In this paper we denote it by $\partial(G, \mathcal{H})$. We remark that the set of conjugacy classes of elements of $\mathcal{H}$ is automatically finite by [18, Theorem 1B]. When the group $G$ is hyperbolic, it is hyperbolic relative to the empty collection $\emptyset$ and the Bowditch boundary $\partial(G, \emptyset)$ is nothing but the Gromov boundary $\partial G$.

We consider another non-elementary countable group $G'$ which is hyperbolic relative to a conjugacy invariant collection $\mathcal{H}'$ of proper infinite subgroups of $G'$. Suppose that $G$ is a subgroup of $G'$. Then we can consider the restricted action of $G$ on $\partial(G', \mathcal{H}')$ and the limit set $\Lambda(G, \partial(G', \mathcal{H}'))$. If there exists a $G$-equivariant continuous map from $\partial(G, \mathcal{H})$ to $\partial(G', \mathcal{H}')$, then it is unique and the image is equal to $\Lambda(G, \partial(G', \mathcal{H}'))$ (see for example [11, Lemma 2.3 (1), (2)]). When the map exists, it is also called the Cannon-Thurston map. If the Cannon-Thurston map is well-defined, then any $H \in \mathcal{H}$ is contained in some $H' \in \mathcal{H}'$ (see for example [11, Lemma 2.3 (5)]). In general the converse is not true by [2, Theorem 1] (see also Lemma 2.1). Our main theorem claims that the converse is also not true for the case where the pair of $G$ and $\mathcal{H}$ is not necessarily the pair of a free group of rank 3 and $\emptyset$. More precisely we have the following:

**Theorem 1.1.** Let $G$ be a non-elementary countable group which is hyperbolic relative to a conjugacy invariant collection $\mathcal{H}$ of proper infinite subgroups of $G$. Then there exist a countable group $G'$ containing $G$ as a subgroup and a conjugacy invariant collection $\mathcal{H}'$ of proper infinite subgroups of $G'$ satisfying the following:

(i) the group $G'$ is hyperbolic relative to $\mathcal{H}'$;
(ii) every \( H \in \mathcal{H} \) belongs to \( \mathcal{H}' \) and each \( H' \in \mathcal{H}' \) is conjugate to some \( H \in \mathcal{H} \) in \( G' \);

(iii) there exists no \( G \)-equivariant continuous map from \( \partial(G, \mathcal{H}) \) to \( \partial(G', \mathcal{H}') \);

(iv) the group \( G \) is not quasiconvex relative to \( \mathcal{H}' \) in \( G' \).

If we apply Theorem 1.1 for the case where \( G \) is hyperbolic and \( \mathcal{H} = \emptyset \), then \( G' \) is hyperbolic and \( \mathcal{H}' = \emptyset \). We remark that our proof uses \([2, \text{Theorem 1}]\).

**Remark 1.2.** Theorem 1.1 (i), (ii) and (iv) can be considered as a generalization of \([10, \text{Theorem A}]\) for relatively hyperbolic groups.

Let \( G \) be a countable group and \( X \) be a compact metrizable space endowed with a minimal non-elementary convergence action of \( G \). We denote by \( \mathcal{H}(G, X) \) the set of all maximal parabolic subgroups with respect to the action of \( G \) on \( X \) and call it the peripheral structure with respect to the action of \( G \) on \( X \). Let us consider another compact metrizable space \( Y \) endowed with a minimal non-elementary convergence action of \( G \). When there exists a \( G \)-equivariant continuous map from \( X \) to \( Y \), we say that \( X \) is a blow-up of \( Y \) and that \( Y \) is a blow-down of \( X \). Suppose that the action of \( G \) on \( X \) is geometrically finite. \([11, \text{Proposition 1.6}]\) claims that \( X \) has no proper blow-ups with the same peripheral structure. \([11, \text{Theorem 1.4}]\) gives a family of uncountably infinitely many blow-downs of \( X \) with the same peripheral structure. On the other hand Theorem 1.1 implies that there exists a compact metrizable space endowed with a minimal non-elementary convergence action of \( G \) such that the peripheral structure is equal to \( \mathcal{H}(G, X) \) and it is not a blow-down of \( X \). In fact the following is shown:

**Corollary 1.3.** Let \( G \) be a countable group. Let \( X \) be a compact metrizable space endowed with a geometrically finite convergence action of \( G \). Then there exists a compact metrizable space \( Y \) endowed with a minimal non-elementary convergence action of \( G \) satisfying the following

(i) \( \mathcal{H}(G, X) = \mathcal{H}(G, Y) \);

(ii) the spaces \( X \) and \( Y \) has no common blow-ups. In particular \( Y \) is not a blow-down of \( X \).

**Remark 1.4.** For every non-elementary relatively hyperbolic group (resp. every non-elementary hyperbolic group), the second question (resp. the first question) in \([14, \text{Section 1}]\) has a negative answer by this corollary. Also this corollary implies that every non-elementary relatively hyperbolic group does not have the universal convergence action which is defined by Gerasimov \([8, \text{Subsection 2.4}]\).

## 2 Proof of Theorem 1.1

Before we show Theorem 1.1 we fix some notations. Let a countable group \( G \) act on a compact metrizable space \( X \). Suppose that the action is a minimal
non-elementary convergence action. Then $X$ can be regarded as a boundary of $G$. In fact $G \cup X$ has the unique topology such that this is a compactification of $G$ and the natural action on $G \cup X$ is a convergence action whose limit set is $X$ (see for example [11, Lemma 2.1]). Let $L$ be a subgroup of $G$. Then the restricted action of $L$ on $X$ is a convergence group action. We denote by $\Lambda(L, X)$ the limit set. If $L$ is neither virtually cyclic nor parabolic with respect to the action on $X$, then the induced action of $L$ on $\Lambda(L, X)$ is also a minimal non-elementary convergence action.

We need the following lemma in order to show Theorem 1.1.

**Lemma 2.1.** Let $G'$ be a countable group and have a subgroup $G$. Let $X$ an $X'$ be compact metrizable spaces endowed with minimal non-elementary convergence actions of $G$ and $G'$, respectively. Then the following is equivalent:

(i) there exists a $G$-equivariant continuous map from $X$ to $X'$;

(ii) there exists a $G$-equivariant continuous map from $X$ to $\Lambda(G, X')$;

(iii) the injection $G \to G'$ is continuously extended to a map from $G \cup X$ to $G' \cup X'$.

**Proof.** The implication from (iii) to (i) (resp. from (i) to (ii)) is trivial. We show that (ii) implies (iii). Suppose that we have a $G$-equivariant continuous map $\phi$ from $X$ to $\Lambda(G, X')$. Then this is extended to a continuous map $id_G \cup \phi : G \cup X \to G \cup \Lambda(G, X')$ (see [11, Lemma 2.3]). Since $G \cup \Lambda(G, X')$ is regarded as the closure of $G$-orbit of the unit element of $G'$ in $G' \cup X'$, the injection $\iota : G \cup \Lambda(G, X') \to G' \cup X'$ is continuous. Then $\iota \circ (id_G \cup \phi) : G \cup X \to G' \cup X'$ is a continuous extension of the injection $G \to G'$.

**Proof of Theorem 1.1.** Since $G$ has the maximal finite normal subgroup by [11, Lemma 3.3], we denote it by $M(G)$. By using [11, Theorem B.1], we take a subgroup $F' = F \times M(G)$ of $G$ such that $F$ is a free group of rank 3 and $G$ is hyperbolic relative to

$$\mathcal{H} \cup \{K \subset G \mid K = gF'g^{-1} \text{ for some } g \in G\}.$$ Take a hyperbolic group $L$ containing $F$ as a subgroup such that the injection $F \to L$ can not continuously extend on the Gromov-boundaries by [2] Theorem 1. We remark that there exists no $F$-equivariant continuous map from $\partial F$ to $\partial L$ by Lemma 2.4. We put $L' := L \times M(G)$, $G' := G \rtimes F$, $L'$ and

$$\mathcal{H}' := \{H' \subset G' \mid H' = \gamma H \gamma^{-1} \text{ for some } H \in \mathcal{H} \text{ and for some } \gamma \in G'\}.$$ By the construction, we have the condition (ii). Also it follows from [7, Theorem 0.1 (2)] that $G'$ is hyperbolic relative to

$$\mathcal{H}' \cup \{K \subset G' \mid K = \gamma' H \gamma'^{-1} \text{ for some } \gamma' \in G'\}.$$ Since $L'$ is hyperbolic, we have the condition (i) by [15, Theorem 2.40].
Now we show the condition (iii). Assume that there exists a $G$-equivariant continuous map $\phi : \partial(G, \mathcal{Y}) \to \partial(G', \mathcal{Y}')$. The map $\phi$ implies $F$-equivariant continuous map $\Lambda(F, \partial(G, \mathcal{Y})) \to \Lambda(L, \partial(G', \mathcal{Y}'))$. Since $F'$ (resp. $L'$) is hyperbolically embedded into $G$ (resp. $G'$) relative to $\mathcal{Y}$ (resp. $\mathcal{Y}'$) in the sense of [15] Definition 1.4, $F$ (resp. $L$) is strongly quasiconvex relative to $\mathcal{Y}$ (resp. $\mathcal{Y}'$) in $G$ (resp. $G'$) in the sense of [15] Definition 4.11 by [16] Theorem 1.5 and [15] Theorem 4.13. Then the action on $F$ (resp. $L$) on $\Lambda(F, \partial(G, \mathcal{Y}))$ (resp. $\Lambda(L, \partial(G', \mathcal{Y}'))$) is $F$-equivariant (resp. $L$-equivariant) homeomorphic to the Gromov boundary $\partial F$ (resp. $\partial L$) by [3] Theorem 0.1 and [18] Theorem 1A. Hence $\phi$ gives an $F$-equivariant continuous map from $\partial F$ to $\partial L$. This contradicts the fact that there exists no such maps.

Finally we show the condition (iv). Assume that $G$ is quasiconvex relative to $\mathcal{Y}'$ in $G'$. The peripheral structure with respect to the action of $G$ on $\partial(G', \mathcal{Y}')$ is

$$\mathcal{Y}' := \{ P \subset G \mid P = G \cap H' \text{ for some } H' \in \mathcal{Y}' \}.$$  

By [11] Definition 1.6, $\partial(G, \mathcal{Y})$ is $G$-equivariant homeomorphic to $\Lambda(G, \partial(G', \mathcal{Y}'))$. Since we have $\mathcal{Y} \subset \mathcal{Y}'$ by the condition (ii), there exists a $G$-equivariant continuous map from $\partial(G, \mathcal{Y})$ to $\partial(G', \mathcal{Y}')$ by [11] Theorem 1.1. Thus we have a $G$-equivariant continuous map from $\partial(G, \mathcal{Y})$ to $\partial(G', \mathcal{Y}')$. This contradicts the condition (iii).

**Proof of Corollary 1.3.** For $G$ and $\mathcal{Y} := \mathcal{Y}(G, X)$, we take $G'$ and $\mathcal{Y}'$ in Theorem 1.1 and put $Y := \Lambda(G, \partial(G', \mathcal{Y}'))$. Note that $X = \partial(G, \mathcal{Y})$ and $\mathcal{Y} = \mathcal{Y}(G, Y)$. Assume that there exists a common blow-up of $X$ and $Y$. Since $\mathcal{Y}(G, X) = \mathcal{Y}(G, Y)$, we have a compact metrizable space $Z$ endowed with a minimal non-elementary convergence action of $G$ which is a common blow-up of $X$ and $Y$ such that $\mathcal{Y}(G, Z) = \mathcal{Y}(G, X) = \mathcal{Y}(G, Y)$ by [11] Lemma 2.6. Then the $G$-equivariant continuous map from $Z$ to $X$ is a $G$-equivariant homeomorphism by [11] Proposition 1.6. Hence $Y$ is a blow-down of $X$ and thus we have a $G$-equivariant continuous map from $\partial(G, \mathcal{Y})$ to $\partial(G', \mathcal{Y}')$ by Lemma 2.1. This contradicts the condition (iii) in Theorem 1.1.

**Remark 2.2.** The space $Y$ in the above proof cannot be written as inverse limit of any inverse system of compact metrizable spaces endowed with geometrically finite convergence actions of $G$ (compare with [11] Theorem 1.4). Indeed assume that $Y$ is inverse limit of an inverse system of compact metrizable spaces $X_i$ ($i \in I$) endowed with geometrically finite convergence actions of $G$. Since every element $H \in \mathcal{Y}$ is parabolic with respect to the action on $Y$ and thus on $X_i$ for each $i \in I$, there exists a unique $G$-equivariant continuous map from $\partial(G, \mathcal{Y})$ to $X_i$ for each $i \in I$ by [11] Theorem 1.1. Hence we have a $G$-equivariant continuous map from $\partial(G, \mathcal{Y})$ to $Y$. This contradicts Corollary 1.3.

It may be interesting to ask whether a given compact metrizable space endowed with a geometrically infinite convergence action of $G$ can be written as...
inverse limit of some inverse system of compact metrizable spaces endowed with geometrically finite convergence actions of $G$.

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