On the Generalization Properties of Minimum-norm Solutions for Over-parameterized Neural Network Models

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Abstract

We study the generalization properties of minimum-norm solutions for three over-parametrized machine learning models including the random feature model, the two-layer neural network model and the residual network model. We proved that for all three models, the generalization error for the minimum-norm solution is comparable to the Monte Carlo rate, up to some logarithmic terms, as long as the models are sufficiently over-parametrized.

Keywords: Generalization error, Minimum-norm solution, Random feature model, Two-layer neural network, Residual network

1. Introduction

Let \( \mathcal{F} \) and \( \{(x_i, y_i)\}_{i=1}^n \) denote the hypothesis space and training data, respectively. We will assume that \( \mathcal{F} \) is large enough to guarantee that interpolation is possible, i.e. there exists trial functions \( f \) in \( \mathcal{F} \) such that \( f(x_i) = y_i \) holds for all \( i = 1, \ldots, n \). We are interested in the generalization properties of the following minimum-norm interpolant:

\[
\hat{f} \in \text{argmin}_{f \in \mathcal{F}} \|f\|_F \quad \text{s.t.} \quad f(x_i) = y_i, \quad i = 1, \ldots, n.
\]  

Here \( \| \cdot \|_F \) is a norm imposed for the model, which is usually different for different models. This type of estimators are relevant for understanding various explicitly or implicitly regularized models and optimization methods. For example, it can be proved that for (generalized) linear models, gradient descent converges to minimum \( l_2 \)-norm solutions (Zhang et al., 2017), if we initialize all the coefficients from zero.

Recent literature on the mathematical analyses of the generalization properties of interpolated solutions includes work on the nearest neighbor scheme (Belkin et al., 2018a), linear regression (Bartlett et al., 2019; Hastie et al., 2019), kernel (ridgeless) regression (Belkin et al., 2018b; Liang and Rakhlin, 2018; Rakhlin and Zhai, 2019; Liang et al., 2019) and random feature model (Hastie et al., 2019). In this paper, we take a step further by considering neural network models.
Our contribution. We consider three models: the random feature model, the two-layer neural network model and the residual neural network model. We prove that in the absence of noise, the minimum-norm estimators can achieve the optimal rate, a rate that is comparable to the Monte Carlo rate up to logarithmic terms, as long as the target functions are in the right function spaces and the models are sufficiently over-parametrized. More precisely, we prove the following results.

- For the random feature model, the corresponding function space is the reproducing kernel Hilbert space (RKHS) associated with the corresponding kernel. Optimal rate for the generalization error is proved for the $l_2$ minimum-norm interpolated solution when the model is sufficiently over-parametrized.

- The same result is proved for two-layer neural network models. The corresponding function space for the two-layer neural network model is the Barron space define in (E et al., 2019b, 2018). Naturally the norm used in (1) is the Barron norm (note that the Barron norm is different from the Fourier transform-based norm used in Barron’s original paper (Barron, 1993)).

- The same result is also proved for deep residual network models for which the corresponding function space is the compositional function space defined in (E et al., 2019b), and the norm used in (1) is the compositional norm.

We remark that over-parametrization is a key for these results. This can be seen from the work (Belkin et al., 2019), which experimentally showed that minimum-norm interpolated solutions may generalize very badly if the model is not sufficiently over-parametrized. In contrast, the corresponding explicitly regularized models are always guaranteed to achieve the optimal rate (E et al., 2018; Caponnetto and De Vito, 2007; E et al., 2019b).

Notation. We use $\|v\|_q$ to denote the standard $\ell_q$ norm of a vector $v$, and $\|\cdot\|$ to denote the $l_2$ norm. For a matrix $A$, we use $\lambda_j(A)$ to denote the $j$-th largest eigenvalue of $A$ and we also define the norm $\|A\|_{1,1} = \sum_{i,j} |a_{i,j}|$. The spectral and Frobenius norms of a matrix are denoted by $\|\cdot\|$ and $\|\cdot\|_F$, respectively. We use $X \lesssim Y$ to mean that there exists a universal constant $C > 0$ such that $X \leq CY$. For any positive integer $d$, we let $\mathcal{S}^d := \{w|w \in \mathbb{R}^{d+1}, \|w\|_1 = 1\}$ and use $\pi_0$ to denote the uniform distribution over $\mathcal{S}^d$. For two matrices $A = (a_{i,j}), B = (b_{i,j})$ in $\mathbb{R}^{n \times m}$, if $a_{i,j} \leq b_{i,j}$ for any $i \in [n], j \in [m]$, then we write $A \preceq B$. For any positive integer $q$, we denote by $[q] := \{1, \ldots, q\}, 1_q = (1, \ldots, 1) \in \mathbb{R}^q$. For a scalar function $g : \mathbb{R} \rightarrow \mathbb{R}$ and matrix $A = (a_{i,j})$, we let $g(A) = (g(a_{i,j}))$.

2. Problem Setup

Assume that the training data $(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}, i = 1, \ldots, n$ are generated from the model

$$y_i = f^*(x_i), \quad i = 1, \ldots, n,$$

where $x_i \sim P_x$ and the random draws are independent. $f^*$ is the target function that we want to estimate from the $n$ training samples. In this paper, we will always assume that $\|x_i\|_\infty \leq 1, |y_i| \leq 1$. Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^{d \times n}$ and $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. 


Let $f(\cdot; \theta)$ denote the parametric model, which could be a random feature model, two-layer or residual neural network model in our subsequent analysis. We want to find $\theta$ that minimizes the generalization error (population risk)

$$R(\theta) := \mathbb{E}_{x,y}[\ell(f(x; \theta), y)].$$

Here $\ell(y, y') = \frac{1}{2}(y - y')^2$ is the loss function. But in practice, we can only deal with the risk defined on the training samples, the empirical risk:

$$\hat{R}_n(\theta) := \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i).$$

2.1. The models

The random feature model. The random feature model is given by

$$f_m(x; \alpha) := \frac{1}{m} \sum_{j=1}^{m} a_j \varphi(x; w_j), \quad (2)$$

where $\alpha = (a_1, \ldots, a_m)^T \in \mathbb{R}^m$ are the parameters to be learned from the data, and $\{w_j\}_{j=1}^{m}$ are i.i.d. random variables drawn from a fixed probability distribution $\mu$. For this model, there is a naturally related RKHS $H_k$ with the kernel defined by

$$k(x, x') := \mathbb{E}_{w \sim \mu}[\varphi(x; w)\varphi(x'; w)]. \quad (3)$$

For simplicity, we assume that $|\varphi(x; w)| \leq 1$.

Define two kernel matrices $K = (K_{i,j}), K^m = (K^m_{i,j}) \in \mathbb{R}^{n \times n}$ with

$$K_{i,j} = k(x_i, x_j), \quad K^m_{i,j} = \frac{1}{m} \sum_{s=1}^{m} \varphi(x_i; w_s)\varphi(x_j; w_s),$$

The latter is an approximation of the former.

The two-layer neural network model. A two-layer neural network is given by

$$f_m(x; \theta) = \frac{1}{m} \sum_{j=1}^{m} a_j \sigma(b_j \cdot x + c_j). \quad (4)$$

Here $\sigma(t) = \max(0, t)$ is the ReLU activation function. Let $\theta = \{(a_j, b_j, c_j)\}_{j=1}^{m}$ be all the parameters to be learned from the data.

If we define $\varphi(x; b, c) := \sigma(b \cdot x + c)$, the two-layer neural network is almost the same as the random feature model (2). The only difference is that $\{w_j\}_{j=1}^{m}$ in the random feature model is fixed during the training process, while the parameters $\{(b_j, c_j)\}_{j=1}^{m}$ in the two-layer neural network model are learned from the data.

Consider the case where $(b, c) \sim \pi_0$. We define $k_{\pi_0}(x, x') = \mathbb{E}_{\pi_0}[\sigma(b \cdot x + c)\sigma(b \cdot x' + c)]$ and the corresponding kernel matrix $K_{\pi_0} = (k_{\pi_0}(x_i, x_j)) \in \mathbb{R}^{n \times n}$. Let $\lambda_n = \lambda_n(K_{\pi_0})$, which will be used to bound the network width in our later analysis.
Residual neural networks. In this paper, we consider the following type of residual neural networks

\[ z_0(x) = V \tilde{x} \]

\[ z_{l+1}(x) = z_l(x) + \frac{1}{L} U_l \sigma(W_l z_l(x)), \quad l = 0, \ldots, L - 1 \]

\[ f_L(x; \theta) = \alpha^T z_L(x) \]

where \( \tilde{x} = (x^T, 1)^T \in \mathbb{R}^{d+1} \), \( W_l \in \mathbb{R}^{m \times D} \), \( U_l \in \mathbb{R}^{D \times m} \), \( \alpha \in \mathbb{R}^D \) and \( V = \left( I_{d+1} \ 0 \right) \in \mathbb{R}^{D \times (d+1)} \).

We use \( \theta = \{W_1, U_1, \ldots, W_L, U_L, \alpha\} \) to denote all the parameters to be learned from the training data. To explicitly show the dependence on the hyper-parameters, we call \( f_L(\cdot; \theta) \) a \((L, D, m)\) residual network.

3. Main results

3.1. The random feature model

Consider the minimum \(l_2\) norm solution defined by

\[ \hat{a}_n := \arg\min_{\hat{a}_n} ||\hat{a}_n\|_2. \]

About this estimator, we have the following theorem.

**Theorem 1** Assume that \( f^* \in \mathcal{H}_k \). For any \( \delta \in (0, 1) \), assume that \( m \geq \frac{8n^2 \ln(2n^2/\delta)}{\lambda_k^2(K)} \). Then with probability at least \( 1 - \delta \) over the random sampling of the data and the features, we have

\[ R(\hat{a}_n) \lesssim \frac{||f^*||^2_{\mathcal{H}_k}}{\sqrt{n}} + \frac{1}{\sqrt{n}} \left( 1 + \sqrt{\ln(2/\delta)} \right). \]

3.2. The two-layer neural network model

First we recall the definition of the Barron space (E et al., 2019b, 2018).

**Definition 2 (Barron space)** Let \( w = (b, c) \) and \( \tilde{x} = (x^T, 1)^T \). Consider functions that admit the following integral representation

\[ f(x) = \mathbb{E}_{w \sim \pi}[a(w)\sigma(w^T \tilde{x})], \]

where \( \pi \) is a probability measure over \( \mathbb{R}^d \) and \( a(\cdot) \) is a measurable function. Denote \( \Theta_f = \{(a, \pi)|f(x) = \mathbb{E}_{w \sim \pi}[a(w)\sigma(w^T \tilde{x})]\} \), the Barron norm is defined as follows

\[ ||f||_B := \inf_{(a, \pi) \in \Theta_f} \mathbb{E}_{w \sim \pi}|a(w)|. \]

The Barron space is defined as the set of continuous functions with a finite Barron norm, i.e.

\[ \mathcal{B} := \{f \mid ||f||_B < \infty\}. \]
Remark 3 Let $k_\pi(x, x') := \mathbb{E}_{\sim \pi}[\sigma(w \cdot \hat{x})\sigma(w \cdot \hat{x}')]$. In (E et al., 2019b), it is proved that $\mathcal{B} = \bigcup_{\pi \in P(S^d)} \mathcal{H}_k$, where $P(S^d)$ denote the set of Borel probability measures on $S^d$. Therefore the Barron space is much larger than the RKHS.

Let
\[
\hat{\theta}_n := \arg\min_{\theta} R_n(\theta) = 0 \|\theta\|_p,
\]
where $\|\cdot\|_p$ is the discrete analog of the Barron norm (also known as the path norm):
\[
\|\theta\|_p := \frac{1}{m} \sum_{j=1}^{m} |a_j| (\|b_j\|_1 + |c_j|).
\]

The generalization properties of the above estimator is given by the following theorem.

**Theorem 4** If $m \geq \frac{8n^2 \ln(2n^2)}{\lambda^2}$, then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the random choice of the training data, we have
\[
R(\hat{\theta}_n) \leq \frac{\|f^*\|_B^2}{\sqrt{n}} + \sqrt{\frac{\ln(2d) + \ln(2/\delta)}{n}}.
\]

### 3.3. The residual neural network models

First we recall the definition of the composition function space $\mathcal{D}_p$ (E et al., 2019b).

Let $\{\rho_t\}_{t \in [0, 1]}$ be a family of Borel probability measures over $\mathbb{R}^{D \times m} \times \mathbb{R}^m$. Consider functions $f_{\alpha, \{\rho_t\}}$ defined through the following ordinary differential equations (ODE),
\[
\begin{align*}
\dot{z}(x, 0) &= V \hat{x} \\
\frac{dz(x, t)}{dt} &= \mathbb{E}_{(U, W) \sim \rho_t}[U \sigma(Wz(x, t))] \\
f_{\alpha, \{\rho_t\}}(x) &= \alpha^T z(x, 1),
\end{align*}
\]

where $V \in \mathbb{R}^{D \times (d+1)}, U \in \mathbb{R}^{D \times m}, V \in \mathbb{R}^{m \times D}$ and $\alpha \in \mathbb{R}^D$. The ODE (9) can be viewed as the continuous limit of the residual network (5). To define the norm for controlling the complexity of the flow map of ODE (9), we need the following linear ODE
\[
\begin{align*}
n_p(0) &= |V|1_{d+1} \\
\frac{dn_p(t)}{dt} &= 3 (\mathbb{E}_{(U, W) \sim \rho_t}[(|U||W|)^p])^{1/p} n_p(t),
\end{align*}
\]

where $A^q = (a_{i,j}^q)$ for $A = (a_{i,j})$. Specifically, $p = 1, 2$ are used in this paper.

**Definition 5 (Compositional function space)** For a function $f$ that can be represented in the form (9), we define
\[
\|f\|_{\mathcal{D}_p} = \inf_{f = f_{\alpha, \{\rho_t\}}} |\alpha^T n_p(1) + \|n_p(1)\|_1 - D,
\]

\[
\text{to be the “$\mathcal{D}_p$ norm” of $f$. The space $\mathcal{D}_p$ is defined as the set of all functions that admit the representation $f_{V; \{\rho_t\}}$ with finite $\mathcal{D}_p$ norm.}
\]
Remark 6  It should be noted that the function space $\mathcal{D}_p$ actually depends on $D, m$. We use $\mathcal{D}_p^{D,m}$ to explicitly show this dependence when it is needed. In most cases, this dependence is omitted in the notation of $\mathcal{D}_p$ for simplicity.

In addition, we define the following norm to quantify the continuity of the sequence of probability measure $\{\rho_t\}_{t \in [0,1]}$.

Definition 7  Given a family of probability distribution $\{\rho_t\}_{t \in [0,1]}$, let $S(\{\rho_t\})$ denote the set of positive values $C > 0$ that satisfies

$$\left| \mathbb{E}_{\rho_t} U_\sigma(W z) - \mathbb{E}_{\rho_s} U_\sigma(W z) \right| \leq C |t - s| |z|,$$

and

$$\left| \left\| \mathbb{E}_{\rho_t} |U||W| \right\|_{1,1} - \left\| \mathbb{E}_{\rho_s} |U||W| \right\|_{1,1} \right| \leq C |t - s|,$$

for any $t, s \in [0,1]$ and $z \in \mathbb{R}^D$. We define the “Lipschitz norm” of $\{\rho_t\}_{t \in [0,1]}$ by

$$\mathbb{Lip}(\{\rho_t\}) = \left\| \mathbb{E}_{\rho_0} |U||W| \right\|_{1,1} + \inf_{C \in S(\{\rho_t\})} C.$$

To control the complexity of a residual network, we use the following compositional path norm defined in (E et al., 2019a), which can be viewed as a discrete analog of (10).

Definition 8  For any residual network $f_L(\cdot; \theta)$ given by (5), its compositional path norm is defined as,

$$\|\theta\|_C : = |\alpha| T \prod_{l=1}^{L} \left( I + \frac{3}{L} |U_l||W_l| |V| \right) 1_{d+1}.$$  

We can now define the minimum-norm estimator for residual neural networks:

$$\hat{\theta}_n : = \arg\min_{\theta} \mathbb{R}_{n(\theta) = 0} \|\theta\|_C,$$

Theorem 9  Assume that the target function $f^* \in \mathcal{D}_2^{D,m}$ and $c_0(f^*) := \inf_{f\in\mathcal{V},(\rho_t)=f^*} \|\{\rho_t\}\|_{\text{Lip}} < \infty$. If the model is a $(L, D + d + 2, m + 1)$ residual neural network with the depth satisfying

$$L \geq C \max \left( m^4 D^6 c_0^2(f^*) \|f^*\|_2^2 \frac{96nm^2}{\lambda_n}, \frac{n(1 + D)}{\lambda_n}, \frac{n^2 \ln(2n)}{\lambda_n^2} \right),$$

where $C$ is a universal constant. Then for any $\delta \in (0, 1),$ with probability $1 - \delta$ over the choice of the training data, we have

$$\mathbb{R}(\hat{\theta}_n) \lesssim \frac{\|f^*\|_2^2 + 1}{\sqrt{n}} \left( \sqrt{\ln(2d)} + \sqrt{\ln(2/\delta)} \right).$$
4. Proofs

**Definition 10 (Rademacher complexity)** Recall that $\mathcal{F}$ and $\{x_i\}_{i=1}^n$ denote the hypothesis space and the training data set respectively. The Rademacher complexity (Shalev-Shwartz and Ben-David, 2014) of $\mathcal{F}$ with respect to the data is defined by

$$\text{Rad}_n(\mathcal{F}) := \frac{1}{n} \mathbb{E}_{\xi_1, \ldots, \xi_n} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^n \xi_i f(x_i) \right],$$

where $\{\xi_i\}_{i=1}^n$ are i.i.d random variables with $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = \frac{1}{2}$.

We will use the following theorem to bound the generalization error.

**Theorem 11 (Theorem 26.5 of (Shalev-Shwartz and Ben-David, 2014))** Assume that the loss function $\ell(\cdot, y')$ is $Q$-Lipschitz continuous and bounded from above by $C$. For any $\delta \in (0, 1)$, with probability $1 - \delta$ over the random sampling of the training data, the following generalization bound hold for any $f \in \mathcal{F}$,

$$R(f) \leq \hat{R}_n(f) + 2Q\text{Rad}_n(\mathcal{F}) + 4C\sqrt{\frac{2\ln(2/\delta)}{n}}.$$ 

4.1. The random feature model

The following lemma says that the two kernel matrices are close when the random feature model is sufficiently over-parametrized.

**Lemma 12** For any $\delta \in (0, 1)$, with probability $1 - \delta$ over the random sampling of $\{w_j\}_{j=1}^m$, we have

$$\|K - K^m\| \leq \sqrt{\frac{n^2 \ln(2n^2/\delta)}{2m}}.$$ 

In particular, if $m \geq \frac{2n^2 \ln(2n^2/\delta)}{\lambda_n(K)}$, we have

$$\lambda_n(K^m) \geq \frac{\lambda_n(K)}{2}.$$ 

**Proof** According to the Hoeffding’s inequality, we have that for any $\delta' \in (0, 1)$, with probability $1 - \delta'$ the following holds for any specific $i, j \in [n]$,

$$|k(x_i, x_j) - \frac{1}{m} \sum_{j=1}^m \varphi(x_i; w_j)\varphi(x_j; w_j)| \leq \sqrt{\frac{\ln(2/\delta')}{2m}}.$$ 

Therefore, with probability $1 - n^2\delta'$, the above inequality holds for all $i, j \in [n]$. Let $\delta = n^2\delta'$, the above can be written as

$$|k(x_i, x_j) - \frac{1}{m} \sum_{k=1}^m \varphi(x_i; w_k)\varphi(x_j; w_k)| \leq \sqrt{\frac{\ln(2n^2/\delta)}{2m}}.$$ 


Thus we have
\[ \|K - K^m\| \leq \|K - K^m\|_F \leq \sqrt{n^2 \ln(2n^2/\delta)} / 2m. \]

Using Weyl’s inequality, we have
\[ \lambda_n(K^m) \geq \lambda_n(K) - \|K - K^m\| \geq \lambda_n(K) - \sqrt{n^2 \ln(2n^2/\delta)} / 2m. \]

When \( m \geq \frac{2n^2 \ln(2n^2/\delta)}{\lambda_n(K)} \), we have \( \lambda_n(K^m) \geq \frac{\lambda_n(K)}{2} \).

We first have the following estimate for kernel (ridgeless) regression.

**Lemma 13** \( y^T K^{-1} y \leq \|f^*\|_Y^2 \).

**Proof** Consider the following optimization problem
\[ \hat{h}_n = \arg \min_{R_n(h) = 0} \|h\|_Y^2. \]

According to the Representer theorem (see Theorem 16.1 of (Shalev-Shwartz and Ben-David, 2014)), we can write \( \hat{h}_n \) as follows
\[ \hat{h}_n = \sum_{i=1}^m \beta_i k(x_i, \cdot). \]

Plugging it into \( \hat{R}_n(\hat{h}_n) = 0 \) gives us that \( y = K\beta \), which leads to \( \beta = k^{-1}y \). According the Moore-Aronszajn theorem (Aronszajn, 1950), we have
\[ \|
\hat{h}_n\|_{\mathcal{H}_k}^2 = \beta^T K \beta = y^T K^{-1} y. \]

By definition \( \hat{h}_n \) is the minimum RKHS norm solutions and \( \hat{R}_n(f^*) = 0 \), it follows that \( \|
\hat{h}_n\|_{\mathcal{H}_k}^2 \leq \|f^*\|_{\mathcal{H}_k}^2 \). So we have \( y^T K^{-1} y \leq \|f^*\|_{\mathcal{H}_k}^2 \).

The following lemma provides an upper bound for the minimum-norm solution of the random feature model (6).

**Lemma 14** Assume that \( f^* \in \mathcal{H}_k \) with \( k(x, x') = \mathbb{E}_{uw}[\varphi(x; w) \varphi(x'; w)] \). Then the minimum-norm estimator satisfies
\[ \frac{1}{\sqrt{m}} \|\hat{a}_n\| \leq 2 \|f^*\|_{\mathcal{H}_k} \]

**Proof** Let \( \Phi = (\Phi_{i,j}) \in \mathbb{R}^{n \times m} \) with \( \Phi_{i,j} = \varphi(x_i; w_j) \). Then the solution of problem (6) is given by
\[ \hat{a}_n = m\Phi^T (\Phi \Phi^T)^{-1} y. \]

Obviously, \( K^m = \frac{1}{m} \Phi \Phi^T \). Therefore, we have
\[ \frac{1}{m} \|\hat{a}_n\|^2 = m y^T (\Phi \Phi^T)^{-1} \Phi \Phi^T (\Phi \Phi^T)^{-1} y = y^T (\frac{1}{m} \Phi \Phi^T)^{-1} y = y^T (K^m)^{-1} y \\
= y^T K^{-1} y + y^T ((K^m)^{-1} - K^{-1}) y \\
= y^T K^{-1} y + y^T (K)^{-1} (K^m - K) (K^m)^{-1} y \\
\leq y^T K^{-1} y + \|(K^m)^{-1/2} y\| \|(K^m)^{-1/2} (K - K^m) K^{-1/2} y\| \\
= y^T K^{-1} y + \|(K^m)^{-1/2} y\| \|(K^m)^{-1/2} (K - K^m) K^{-1/2} y\| \]
According to Lemma 13, we have $\mathbf{y}^T K^{-1} \mathbf{y} \leq \|f^*\|_{\mathcal{H}_k}^2$. Denote $t = \sqrt{\|\mathbf{a}_n\|^2/m} = \sqrt{\mathbf{y}^T (K^m)^{-1} \mathbf{y}}$, we have

$$t^2 \leq \|f^*\|_{\mathcal{H}_k}^2 + t\|f^*\|_{\mathcal{H}_k} \langle (K^m)^{-1/2} \|K - K^m\|_{K^{-1/2}} \|.$$

By Lemma 12, we have

$$t^2 \leq \|f^*\|_{\mathcal{H}_k}^2 + t\|f^*\|_{\mathcal{H}_k} \sqrt{\frac{n^2 \ln(2n^2/\delta)}{\lambda_n^2(K)m}}.$$  

(17)

Under the assumption that $m \geq \frac{2n^2 \ln(2n^2)}{\lambda_n^2(K)}$, we obtain

$$\frac{1}{\sqrt{m}}\|\mathbf{a}_n\| = t \leq 2\|f^*\|_{\mathcal{H}_k}.$$

**Proof of Theorem 1** Define $A_C = \{\mathbf{a} : \frac{1}{\sqrt{m}}\|\mathbf{a}\| \leq C\}$ and $\mathcal{F}_C = \{f_m(\cdot; \mathbf{a}) | \mathbf{a} \in A_C\}$. The Rademacher complexity of $\mathcal{F}_C$ satisfies,

$$\text{Rad}_n(\mathcal{F}_C) = \frac{1}{nm} \mathbb{E}_{\xi_i} \sup_{f \in \mathcal{F}_C} \sum_{i=1}^n \xi_i \frac{1}{m} \sum_{j=1}^m a_{ij} \varphi(x_i; \mathbf{w}_j) = \frac{1}{n} \mathbb{E}_{\xi_i} \sup_{f \in \mathcal{F}_C} \sum_{j=1}^m a_{ij} \sum_{i=1}^n \xi_i \varphi(x_i; \mathbf{w}_j)$$

$$\leq \frac{1}{nm} \mathbb{E}_{\xi_i} \sup_{f \in \mathcal{F}_C} \left[ \sum_{j=1}^m \left( \sum_{i=1}^n \xi_i \varphi(x_i; \mathbf{w}_j) \right)^2 \right]$$

$$\leq \frac{C}{n\sqrt{m}} \mathbb{E} \left[ \sum_{j=1}^m \left( \sum_{i=1}^n \xi_i \varphi(x_i; \mathbf{w}_j) \right)^2 \right]$$

$$\leq \frac{C}{n\sqrt{m}} \sum_{j=1}^m \sum_{i, i' = 1}^n \mathbb{E} [\xi_i \xi_{i'} \varphi(x_i; \mathbf{w}_j) \varphi(x_{i'}; \mathbf{w}_j) \leq \frac{C}{\sqrt{n}}.$$  

where (i) and (ii) follow from the Jensen’s inequality and $\mathbb{E} [\xi_i \xi_j] = \delta_{i,j}$, respectively. Moreover, for any $\mathbf{a} \in A_C$, we have $|f_m(x; \mathbf{a})| = \left| \frac{1}{m} \sum_{i=1}^m a_{ij} \varphi(x; \mathbf{w}_j) \right| \leq \frac{1}{m} \sqrt{\|\mathbf{a}\|^2 \sum_{j=1}^m \varphi(x; \mathbf{w}_j)^2} \leq C$. Thus, for any $f_m(\cdot; \mathbf{a}) \in \mathcal{F}_C$, the loss function $(f_m(x; \mathbf{a}) - f(x))^2/2$ is $(C + 1)$–Lipschitz continuous and bounded above by $(C + 1)^2/2$.

Take $C = 2\|f^*\|_{\mathcal{H}_k}$. We have $f_m(\cdot; \mathbf{a}_n) \in \mathcal{F}_C$. Thus, Theorem 11 implies

$$R(\mathbf{a}_n) \leq \hat{R}_n(\mathbf{a}_n) + 2(C + 1)\text{Rad}_n(\mathcal{F}_C) + \frac{4(C + 1)^2}{2} \sqrt{\frac{2\ln(2/\delta)}{n}}.$$  

(18)

$$\leq \frac{\|f^*\|_{\mathcal{H}_k}^2 + 1}{\sqrt{n}} \left( 1 + \sqrt{\ln(2/\delta)} \right).$$  

(19)

■
4.2. Two-layer neural networks

Before proving the main result, we need the following lemma.

**Lemma 15**  
For any \( r \in \mathbb{R}^n \), there exists a two-layer neural network \( f_m(\cdot; \theta) \) with \( m \geq \frac{2n^2 \ln(4n^2)}{\lambda_n^2} \), such that \( f_m(x_i; \theta) = r_i \) for any \( i \in [n] \) and \( \| \theta \|_p \leq \sqrt{\frac{2}{\lambda_n}} \| r \| \)

**Proof**  
Assume that \( \{ (b_j, c_j) \}_{j=1}^m \) are i.i.d. random variables drawn from \( \pi_0 \), the uniform distribution over the sphere \( S^d \). Recall that \( K^m := (K^m_{i,i'}) \in \mathbb{R}^{n \times n} \) with
\[
K^m_{i,i'} = \frac{1}{m} \sum_{j=1}^m \sigma(b_j \cdot x_i + c_j) \sigma(b_j \cdot x_{i'} + c_j)
\]
For any \( \delta \in (0, 1) \), if \( m \geq \frac{2n^2 \ln(2n^2/\delta)}{\lambda_n^2} \), Lemma 12 implies that the following hold with probability at least \( 1 - \delta \)
\[
\lambda_n(K^m) \geq \frac{1}{2} \lambda_n. \tag{20}
\]
Taking \( \delta = 1/2 \), the above inequality holds with probability 1/2. This means that there must exist \( \{ (\hat{b}_j, \hat{c}_j) \}_{j=1}^m \) such that (20) holds. Let \( \Psi \in \mathbb{R}^{n \times m} \), \( \Psi_{i,j} = \sigma(\hat{b}_j \cdot x_i + \hat{c}_j) \), denote the feature matrix. Then
\[
\sigma_n^2(\Psi) = \lambda_n(\Psi \Psi^T) = m \lambda_n(K^m) \geq \frac{1}{2} \lambda_n m. \tag{21}
\]
We next choose \( a \) as the solution of the following problem.
\[
\hat{a} = \arg\min_a \| a \| \quad \text{s.t.} \quad \frac{1}{\sqrt{m}} \Psi a = r.
\]
Then
\[
\| \hat{a} \| \leq \sigma_n^{-1}(\Psi) \| r \| \leq \sqrt{\frac{2}{\lambda_n}} \| r \|. \tag{22}
\]
Consider the two-layer neural network
\[
f_m(x; \hat{\theta}) = \frac{1}{m} \sum_{j=1}^m \hat{a}_j \sigma(\hat{b}_j \cdot x + \hat{c}_j).
\]
Then we have that \( f_m(x_j; \hat{\theta}) = r_j \) and \( \| \theta \|_p \leq \frac{1}{m} \sum_{j=1}^m |\hat{a}_j| \leq \| \hat{a} \| \leq \sqrt{\frac{2}{\lambda_n}} \| r \| \), where the last inequality follows from (22).

The following lemma provides an upper bound to the minimum path norm solutions.

**Lemma 16**  
Assume that \( f^* \in \mathcal{B} \) and \( m \geq \frac{6n^2 \ln(4n^2)}{\lambda_n^2} \), then the minimum path norm solution (7) satisfies
\[
\| \hat{\theta}_n \|_p \leq 3 \| f^* \|_B.
\]
Proof First by the approximation result of two-layer neural networks (See Proposition 2.1 in (E et al., 2018)), for any \( m > 0 \), there must exist a two-layer neural network \( f_{m_1}(\cdot; \theta^{(1)}) \) such that

\[
\hat{R}_n(\theta^{(1)}) = \| f_{m_1}(\cdot; \theta^{(1)}) - f^* \|^2 \leq \frac{3\| f^* \|^2_B}{m_1}, \tag{23}
\]

and

\[
\| \theta^{(1)} \|_P \leq 2\| f^* \|_B.
\]

where \( \hat{\rho}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta(x - x_i) \).

Let \( r = (y_1 - f_{m_1}(x_1; \theta^{(1)}), \ldots, y_n - f_{m_1}(x_n; \theta^{(1)})) \in \mathbb{R}^n \) to be the residual. Then \( \| r \| \leq \sqrt{\frac{3\| f^* \|_B}{m_1}} \). Applying Lemma 15, we know that there exists a two-layer neural network \( f_{m_2}(\cdot; \theta^{(2)}) \) with \( m_2 \geq \frac{2n^2 \ln(4n^2)}{\lambda_n} \) such that

\[
f_{m_2}(x_i; \theta^{(2)}) = r_i \tag{24}
\]

\[
\| \theta^{(2)} \|_P \leq \sqrt{\frac{2}{\lambda_n}} \| r \| \leq \| f^* \|_B, \tag{25}
\]

where the last inequality holds as long as \( m_1 \geq \frac{6n}{\lambda_n} \).

Putting \( f_{m_1}(\cdot; \theta^{(1)}), f_{m_2}(\cdot; \theta^{(2)}) \) together, let

\[
f_{m_1+m_2}(x; \theta) = f_{m_1}(x; \theta^{(1)}) + f_{m_2}(x; \theta^{(2)}),
\]

where \( \theta = \{ \theta^{(1)}, \theta^{(2)} \} \). It is obviously that

\[
\hat{R}_n(\theta) = 0, \quad \| \theta \|_P = \| \theta_1 \|_P + \| \theta_2 \|_P \leq 3\| f \|_B.
\]

Proof of Theorem 4 For any \( C > 0 \), let \( \mathcal{F}_C = \{ f_m(x; \theta) : \| \theta \|_P \leq C \} \). Using Lemma 4 of (E et al., 2018), we have \( \text{Rad}_n(\mathcal{F}_C) \leq 2C\sqrt{\frac{2\ln(2d)}{n}} \). By the definition of the minimum-norm solution and Lemma 16, we have

\[
\| \hat{\theta}_n \|_P \leq 3\| f^* \|_B.
\]

Taking \( C = 3\| f^* \|_B \), then we have \( f_m(\cdot; \hat{\theta}_n) \in \mathcal{F}_C \). Since the loss function is \((C + 1)\)-Lipschitz continuous and bounded from above by \((C + 1)^2/2\), by Theorem 11, for \( \delta \in (0, 1) \), the following holds with probability at least \( 1 - \delta \)

\[
R(\hat{\theta}_n) \leq R_n(\hat{\theta}_n) + 2(C + 1)\text{Rad}_n(\mathcal{F}_C) + \frac{4}{2}(C + 1)^2 \sqrt{\frac{2\ln(2/\delta)}{n}} \tag{26}
\]

\[
\lesssim \frac{\| f^* \|^2_B + 1}{\sqrt{n}} \left( \sqrt{\ln(2d)} + \sqrt{\ln(2/\delta)} \right) \tag{27}
\]
**4.3. Residual neural networks**

The following lemma shows that the addition of two residual networks can be represented by a wider residual network.

**Lemma 17** Suppose that $f(\cdot; \theta^{(1)})$ and $f(\cdot; \theta^{(2)})$ are $(L_1, D_1, m_1)$ and $(L_2, D_2, m_2)$ residual networks, respectively. Then $F := f(\cdot; \theta_1) + f(\cdot; \theta_2)$ can be represented as a $(\max(L_1, L_2), D_1 + D_2, m_1 + m_2)$ residual network $\theta^{(3)}$ and the compositional path norm satisfies

$$\|\theta^{(3)}\|_C = \|\theta^{(1)}\|_C + \|\theta^{(2)}\|_C.$$

**Proof** Without loss of generality, we assume $L_1 = L_2$. Otherwise, we can add extra identity layers without changing the represented function and the path norm. $f(\cdot; \theta^{(1)})$ can be written as

$$z_0^{(1)}(x) = V^{(1)}x$$

$$z_{l+1}^{(1)}(x) = z_l^{(1)}(x) + \frac{1}{L}U_l^{(1)}\sigma(W_l^{(1)}z_l^{(1)}(x)), \quad l = 0, \ldots, L - 1$$

where $U_l^{(1)} \in \mathbb{R}^{D_1 \times m_1}, W_l^{(1)} \in \mathbb{R}^{m_1 \times D_1}, V^{(1)} \in \mathbb{R}^{D_1 \times (d+1)}, \alpha^{(1)} \in \mathbb{R}^{D_1}$. Similarly, for $f(\cdot; \theta^{(2)})$, we have

$$z_0^{(2)}(x) = V^{(2)}x$$

$$z_{l+1}^{(2)}(x) = z_l^{(2)}(x) + \frac{1}{L}U_l^{(2)}\sigma(W_l^{(2)}z_l^{(2)}(x)), \quad l = 0, \ldots, L - 1$$

where $U_l^{(2)} \in \mathbb{R}^{D_2 \times m_2}, W_l^{(2)} \in \mathbb{R}^{m_2 \times D_2}, V^{(2)} \in \mathbb{R}^{D_2 \times (d+1)}, \alpha^{(2)} \in \mathbb{R}^{D_2}$.

Let

$$V = \begin{bmatrix} V^{(1)} \\ V^{(2)} \end{bmatrix}, U_l = \begin{bmatrix} U_l^{(1)} & 0 \\ 0 & U_l^{(2)} \end{bmatrix}, V_l = \begin{bmatrix} V_l^{(1)} & 0 \\ 0 & V_l^{(2)} \end{bmatrix}, \alpha = \begin{bmatrix} \alpha^{(1)} \\ \alpha^{(2)} \end{bmatrix}. \tag{28}$$

Consider the following residual network

$$z_0(x) = V\tilde{x}$$

$$z_{l+1}(x) = z_l(x) + \frac{1}{L}U_l\sigma(W_lz_l(x)), \quad l = 0, \ldots, L - 1$$

$$f(x; \theta^{(3)}) = \alpha^T z_L(x),$$

where $z_l(x) \in \mathbb{R}^{D_1 + D_2}$. Here $f(\cdot; \theta^{(3)})$ is a $(L, D_1 + D_2, m_1 + m_2)$ residual network and it is easy to show that

$$f(x; \theta^{(3)}) = f(x; \theta^{(1)}) + f(x; \theta^{(2)})$$

$$\|\theta^{(3)}\|_C = \|\theta^{(1)}\|_C + \|\theta^{(2)}\|_C.$$

The following lemma shows that any two-layer neural network can be converted to an residual network, without changing the norm too much.
Lemma 18  For any two layer neural network $f_m(\cdot; \theta)$ of width $m$. There exists a $(m, d + 2, 1)$ residual network $g_m(\cdot; \Theta)$ such that

$$g_m(x; \Theta) = f_m(x; \theta) \quad \forall x \in \mathbb{R}^d$$

$$\|\Theta\|_c = 3\|\theta\|_{\mathcal{P}}.$$

Proof  Assume the two-layer neural network is given by $f_m(x; \theta) = \frac{1}{m} \sum_{j=1}^{m} a_j \sigma(b_j \cdot x + c_j)$. Consider the following residual network,

$$z_0(x) = \begin{pmatrix} I_{d+1} \\ 0 \end{pmatrix} \tilde{x}$$

$$z_{j+1}(x) = z_j(x) + \frac{1}{m} \begin{pmatrix} 0_d \\ 0 \\ a_j \end{pmatrix} \sigma(\begin{pmatrix} b_j^T \\ c_j \end{pmatrix} z_j(x)), \quad j = 1, \ldots, m - 1$$

$$g_m(x; \Theta) = e_{d+2}^T z_m(x),$$

where $e_{d+2} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{d+2}$. Obviously, $g_m(x; \Theta) = f_m(x; \theta)$ for any $x \in \mathbb{R}^d$. Moreover, the compositional path norm satisfies

$$\|\Theta\|_c = e_{d+2}^T \left[ \prod_{l=1}^{M} \left( I + \frac{3}{m} \begin{pmatrix} 0_d \\ 0 \\ |a_l| \end{pmatrix} \left( |b_l|^T |c_l| 0 \right) \right] \left( I_{d+1} \\ 0 \right) 1_{d+1}$$

$$= \frac{3}{m} \sum_{j=1}^{M} |a_j| (\|b_j\|_1 + |c_j|) = \|\theta\|_{\mathcal{P}}. \quad (29)$$

Proof of Theorem 9  By the direct approximation theorem (Theorem 10 in (E et al., 2019b)), for any $\delta_0 \in (0, 1)$, there exists an $L_1 = (m^4 D^6 c_0^2 (f^*)^2 \|f^*\|_{\mathcal{D}_1}^2)^{3/\delta_0}$, such that for any $L \geq L_1$, there exists a $(L, D, m)$ residual network $f_L(\cdot; \theta^{(1)})$ such that

$$\hat{R}_n(\theta^{(1)}) = \|f^* - f_L(\cdot; \theta^{(1)})\|_{\mathcal{P}_n}^2 \leq \frac{24 m^2}{L^{1-2\delta_0/3}} \|f^*\|_{\mathcal{D}_1}^4 + \frac{3C}{L}(1 + D + \sqrt{\log L}) \|f^*\|_{\mathcal{D}_1}^2, \quad (30)$$

and

$$\|\theta^{(1)}\|_{\mathcal{P}} \leq 9 \|f^*\|_{\mathcal{D}_1}, \quad (31)$$

where $C$ is a universal constant.

Let $r = (y_1 - f_L(x_1; \theta^{(1)}), \ldots, y_n - f_L(x_n; \theta^{(1)}))^T \in \mathbb{R}^n$ to be the residual. $\|r\| = \sqrt{n \hat{R}_n(\theta^{(1)})}$. By Lemma 15, there exists a two-layer neural network $h_M(x; \theta) = \frac{1}{M} \sum_{j=1}^{M} a_j \sigma(b_j^T x + c_j)$ of $M = \frac{2m^2 \ln(4n^2)}{\lambda_n^2}$ such that $h_M(x_i; \theta) = r_i$ and

$$\frac{1}{M} \sum_{j=1}^{M} |a_j| (\|b_j\|_1 + |c_j|) \lesssim \sqrt{\frac{2}{\lambda_n}} \|r\|.$$
Inserting (30) gives us
\[
\frac{1}{M} \sum_{k=1}^{M} |a_k| (\|b_k\|_1 + |c_j|) \leq \frac{2n}{\lambda_n} \left( \frac{24m^2}{L^{1-2\alpha/3}} \|f^*\|_{D_1}^2 + \frac{3C}{L} (1 + D + \sqrt{\log L}) \right) \|f^*\|_{D_1}
\]
\[
\leq \|f^*\|_{D_1}, \tag{32}
\]
where the last inequality holds as long as \(L \geq \max((96m^2 n / \lambda_n)^{3/(3-2\alpha)}, 12C n (1 + D + \sqrt{\log L}) / \lambda_n)\).

By Lemma 18, there exists a \((M, d + 1, 1)\) residual network \(f_M(:, \theta(2))\) such that \(f_M(x_i; \theta(2)) = h_M(x_i; \theta)\) and
\[
\|\theta(2)\|_c = \frac{3}{M} \sum_{j=1}^{M} |a_j| (\|b_j\|_1 + |c_j|) \leq 3 \|f^*\|_{D_1}.
\]

Note that \(L \geq M\). Applying Lemma 17, we conclude that \(f_L(:, \theta(1)) + f_M(:, \theta(2))\) can be represented by a \((L, D + d + 2, m + 1)\) residual network \(f_L(:, \theta(3))\), which satisfies
\[
\hat{R}_n(\theta(3)) = 0,
\]
\[
\|\theta(3)\|_c = \|\theta(1)\|_c + \|\theta(2)\|_c \leq 12 \|f^*\|_{D_1},
\]
where the last inequality follows from (31) and (29). By the definition of the minimum-norm solutions (15), we have
\[
\|\theta_n\|_c \leq \|\theta_3\|_c \leq 12 \|f^*\|_{D_1}.
\]

Let \(\mathcal{F}_C = \{f_L(:, \theta) : \|\theta\|_c \leq C\}\) denote the set of \((L, D + d + 2, m + 1)\) residual network with the compositional path norm bounded from above by \(C\). Theorem 2.10 of (E et al., 2019a) states that
\[
\text{Rad}_n(\mathcal{F}_C) \leq 3C \sqrt{\frac{2 \log(2d)}{n}}.
\]

For any \(f \in \mathcal{F}_C\), we have \(|f| \leq C\), therefore the loss function is \((C + 1)\)-Lipschitz continuous and bounded by \((C + 1)^2 / 2\). Taking \(C = 12 \|f^*\|_{D_1}\), then we have \(f_L(:, \theta_n) \in \mathcal{F}_C\). Applying Theorem 11, we conclude that with probability at least \(1 - \delta\) over the sample of the training set, we have
\[
R(\hat{\theta}_n) \leq \hat{R}(\hat{\theta}_n) + 2(C + 1) \text{Rad}_n(\mathcal{F}_C) + \frac{4(C + 1)^2}{2} \sqrt{\frac{2 \ln(2/\delta)}{n}}
\]
\[
\leq \frac{\|f^*\|_{D_1}^2}{\sqrt{n}} + \left( \sqrt{\ln(2d)} + \sqrt{\ln(2/\delta)} \right),
\]
where the last inequality holds since \(C = 12 \|f^*\|_{D_1}\). Then taking \(\delta_0 = 1/2\), we complete the proof.

\section{Concluding Remarks}

In this work, we prove that learning with the minimum-norm interpolation scheme can reach the optimal rates for three models: the random feature model, the two-layer neural network model and the residual neural network model. The proofs rely on two assumptions: (1) the model is sufficiently
over-parametrized; (2) the labels are clean, i.e. \( y_i = f^*(x_i) \). The “double descent” phenomenon tells us the results are unlikely to be true when the models are not over-parametrized. For noisy labels, we also expect that the results deteriorate. However, recent work (Liang and Rakhlin, 2018; Liang et al., 2019; Zhang et al., 2017) showed that for kernel regression, the interpolation of noise does not hurt the generalization error too much, especially in the high-dimensional regime. It would be interesting to consider this issue for neural network models. We leave this to future work.

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