Analog of Superradiance effect in BEC

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Abstract

We investigate the scattering of phase oscillation of Bose-Einstein Condensate by a 'draining of bathtub' type fluid motion. We derive a relation between the reflection and transmission coefficients which exhibits existence of analog of 'Superradiance effect' in BEC vortex with sink.

1 Introduction

It has been a hundred years since Einstein’s discovery of general relativity which has been experimentally verified to a high degree of accuracy. However, on the semiclassical front the absence of direct experimental observation of the phenomenon of particle emission predicted by Hawking [11] in 1975, still stands out as a problem. In 1981 though, Unruh [2, 3] showed that if a fluid is barotropic and inviscid, and the flow of the fluid is irrotational, the equation of motion that fluctuation of the velocity potential of acoustic disturbance obeys, is identical to that of a minimally coupled massless scalar field propagating in an effective curved space-time Lorentzian geometry. One can simulate artificial black hole [2, 3] in this system by trapping the sound wave in a compact region using radial velocity of the fluid flow. Introduction of a rotational component [4, 5, 6] to that radial velocity will make the artificial black hole an analog of rotating black hole in physical gravitational
system. He also pointed out that there is possibility of experimental observation of the acoustic analog of Hawking radiation from regions of flow of inviscid and barotropic fluids behaving as outer trapped surfaces (‘acoustic event horizons’). This model is sufficiently rich to enable probing of almost all the kinematic aspects of general relativity but not its dynamics, because dynamics of a fluid system is governed by Euler equation and continuity equation but dynamics of a gravitational system is governed by Einstein’s equations.

In our earlier work [7, 8], using the analogy between a shrinking fluid vortex (‘drain bathtub’) modelled as a (2+1) dimensional fluid flow with a sink at the origin and a rotating (2+1) dimensional black hole with an ergosphere, it has been shown that a scalar sound wave is reflected from such a vortex with an amplification for a specific range of frequencies of the incident wave, depending on the angular velocity of rotation of the vortex. This is analogous to superradiant scattering [9, 10, 11] which occurs in the case of rotating black holes in asymptotically flat spacetime. A more detailed analysis this phenomenon in the low frequency range has also been done in the next paper [8]. The reflection coefficient exhibit its frequency dependence which is consistent with the results in [8]. We have also discussed the possibility of observation of this phenomenon especially for inviscid fluids like He-II, where vortices with quantized angular momentum may occur.

Although we have demonstrated the analog of superradiance in the superfluid helium, BEC is a far cleaner system which can be manipulated to produce the kind of metric we have (DB) in a much easier manner. That is why BEC seems to be better suited for this purpose. Recent experimental investigation of Bose-Einstein condensation [12, 13, 14] has further improved the prospects of experimental realization of the analog of kinematic effects of gravity.

In this paper we investigate the possibility of the acoustic analog of superradiance scattering from hydrodynamic vortex with a sink in Bose-Einstein condensate. Linear acoustic perturbations in such a system are shown to scatter from the ergoregion with an enhancement in amplitude, for a restricted range of frequencies of the incoming wave.

The paper is organised follows. First, in section(2), we briefly discuss how effective Lorentzian geometry emerges from the propagation of phase oscillation in BEC. In section(3) we briefly describe the properties of effective geometry that emerges from propagation of phase oscillations through BEC. In section(4) we demonstrate analog of superradiance effect in BEC.
for position dependent background configuration. We conclude with some remarks in section (5).

2 Effective Geometry in BEC

A Bose-Einstein condensate [15, 16] is a collection of indistinguishable bosons such that all of them are in same single-particle quantum state $\phi(r_i)$. If there is no interaction between the bosons, then the second quantized Hamiltonian for $N$ such bosons is given by,

$$ H = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + V(r_i) \right) $$

and the corresponding wave function of this system is,

$$ \Phi(r_1, r_2, \ldots, r_N) = \phi(r_1) \phi(r_2), \ldots, \phi(r_N) $$

Even there are very week interaction between the atoms, at 0°k almost all the atoms in the condensate remains in the same single-particle state. If the number of boson is large enough then the time evolution of such a condensate is determined by Gross-Pitaevskii equation,

$$ i\hbar \partial_t \Phi = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + 4\pi a\hbar^2 - \frac{1}{m} |\Phi|^2 \right) \Phi, $$

where $m$ is the mass of the atoms, $a$ is the scattering length, and we normalise to the total number of atoms $\int d^3x |\Phi(x, t)|^2 = N$.

Expressing $\Phi(r, t)$ in terms of its amplitude $\sqrt{\rho(r, t)}$ and phase $\Psi(r, t)$, the above wave equation is converted into two first order partial differential equation,

$$ \partial_t \Psi = -\frac{1}{2m} (\bar{\nabla} \Psi, \bar{\nabla} \Psi) - V - \frac{4\pi a}{m} \rho + \frac{\hbar^2}{2m} \sqrt{\rho} \nabla^2 \sqrt{\rho}, $$

$$ \partial_t \rho = -\frac{1}{m} (\bar{\nabla} \rho, \bar{\nabla} \Psi + \rho \nabla^2 \Psi) $$

Now linearising $\rho$ and $\Psi$ around there background values $\rho_0$ and $\Psi_0$ respectively,
∂Ψ_1 = − \frac{1}{m} \vec{\nabla} \Psi_0 \cdot \vec{\nabla} \Psi_1 - \frac{4 \pi a \hbar}{m \rho} + \frac{\hbar^2}{2m} \left\{ \frac{1}{2 \sqrt{\rho_0}} \nabla^2 \left( \frac{\rho_1}{\sqrt{\rho_0}} \right) - \rho_1 \frac{\nabla^2 \sqrt{\rho_0}}{2 \rho_0^2} \right\}

∂t \rho_1 = - \frac{1}{m} \vec{\nabla} (\rho_0 \vec{\nabla} \Psi_1) - \frac{1}{m} \vec{\nabla} (\rho_1 \vec{\nabla} \Psi_0)

Here ρ_1 and Ψ_1 are the perturbed values of ρ and Ψ respectively.

If we neglect the quantum pressure term, then the combination of above equations (5) results the following covariant differential equation,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Psi) = 0$$

where,

$$g^{\mu \nu}(t, \vec{r}) = \frac{1}{\rho_0 c} \begin{pmatrix} -1 & \ddots & -v_0^i \\ \vdots & \ddots & \vdots \\ -v_0^i & \vdots & (c_s^2 \delta^{ij} - v_0^i v_0^j) \end{pmatrix}.$$  

The speed of sound (c) and the background velocity (v_0) are defined as,

$$c(r) = \frac{\hbar}{m \sqrt{4 \pi a \rho_0(r)}} \quad v_0^i = \frac{1}{m} \partial_i \Psi_0$$

Therefore the propagation of the phase oscillations in Bose-Einstein condensate is same as the propagation of a minimally coupled massless scalar field in an effective Lorentzian geometry which is determined by the background velocity and background density of the condensate and also by the speed of sound in the condensate [17, 18, 19].

Since in this paper, we investigate the possibility of the acoustic analog of superradiance (a phenomenon that we call ‘superresonance’), i.e., the amplification of a sound wave by reflection from the ergo-region of a rotating acoustic black hole, we choose the so-called ‘draining bathtub’ type of fluid flow [5], which is basically a (2+1) dimensional flow with a sink at the origin. A two surface in this flow, on which the fluid velocity is everywhere pointing towards the sink, and the radial velocity component exceeds the local sound velocity everywhere, behaves as an outer trapped surface in this ‘acoustic’ spacetime, and is identified with the (future) event horizon of the black hole analog. Thus, the velocity potential for the flow has the form [5] (in polar coordinates on the plane),

$$\psi(r, \phi) = A \log r + B \phi,$$  

4
where, $A$ and $B$ are real constants. This leads to the velocity profile,

$$\vec{v} = -\frac{A}{r} \hat{r} + \frac{B}{r} \hat{\phi}. \quad (9)$$

For the velocity potential given in equation (8), the analog black hole metric is (2+1) dimensional with Lorentzian signature, and is given by,

$$ds^2 = \left(\frac{\rho_0}{c}\right)^2 \left[ -\left(c^2 - \frac{A^2 + B^2}{r^2}\right) dt^2 - \frac{2A}{r} dr dt - 2B d\phi dt + dr^2 + r^2 d\phi^2 \right] \quad (10)$$

where $c$ is the velocity of sound. In our earlier work we had neglected the conformal factor in the effective metric because the background density and hence the background pressure of fluid were assumed to be constant. In this paper we have relaxed this condition by considering position dependence of the density profile, which leads to the position dependence of the speed of sound also.

Recently Slatyer and Savage [20] have suggested the following density profile [16] which could be realized (at least nearly) in future BEC experiments,

$$\rho_0(r) = \rho_\infty \frac{(r - r_0)^2}{(r - r_0)^2 + 2\sigma^2} \quad (11)$$

where $\sigma$ is the spatial scale of variance, the distance over which the wave function of the Bose-Einstein Condensate tends to it bulk values, when the wave function is subjected to a localized perturbation. For analytical simplicity we have considered $r_0 = 0$ case in this paper. The sound speed $c(r)$ corresponding to this density profile is,

$$c(r) = c_\infty \sqrt{\frac{r^2}{r^2 + 2\sigma^2}} \quad (12)$$

where,

$$c_\infty = \frac{\hbar}{m} \sqrt{\frac{4\pi a\rho_\infty}{c}} \quad (13)$$

Considering the above velocity profile (9) and new density profile yields new locations of analog of event horizon and ergosphere. As for the Kerr black hole [21] in general relativity, the radius of the ergosphere is given by the vanishing of $g_{00}$, i.e., at

$$r_e = \left[ \frac{(A^2 + B^2)}{2c_\infty^2} \left\{ 1 + \left(1 + \frac{8c_\infty^2 \sigma^2}{A^2 + B^2}\right)^{\frac{1}{2}} \right\} \right]^{\frac{1}{2}} \quad (14)$$
The metric has a (coordinate) singularity at,

\[ r_h = \left[ \frac{A^2}{2 c^2} \left\{ 1 + \left( 1 + \frac{8 c^2 \sigma^2}{A^2} \right)^{\frac{1}{2}} \right\} \right]^{\frac{1}{2}} \]  \tag{15} \]

which signifies the horizon, i.e., the boundary of the outer trapped surface.

It should be noticed that both the radius of the horizon and the ergosphere are larger than what we found in our earlier work assuming constant density profile. This result is expected because in this case the speed of sound is decreasing towards the centre of the vortex and hence it becomes lesser than fluid velocity at larger value of radial coordinate.

3 Superresonance Scattering

From the components of the draining vortex metric it is clear that the (2+1)-dimensional curved spacetime possesses isometries that correspond to time translations and rotations on the plane. The solution of the massless Klein Gordon equation\(^5\) can therefore be written as,

\[ \Psi(t, r, \phi) = Re\{ R(r) e^{-i \omega t} e^{i m \phi} \} \]  \tag{16} \]

where \( \omega \) and \( m \) are real and positive. In order to make \( \Psi(t, r, \phi) \) single valued, \( m \) should take integer values.

Then the radial function \( R(r) \) satisfies,

\[ \frac{d^2 R(r)}{dr^2} + P_1(r) \frac{dR(r)}{dr} + Q_1(r) R(r) = 0 , \]  \tag{17} \]

where,

\[ P_1(r) = \frac{r^2 + 2 \sigma^2}{c_\infty r \left\{ r^4 (r_h^2 + 2 \sigma^2) - r_h^4 (r^2 + 2 \sigma^2) \right\}} P_2(r) \]

\[ P_2(r) = c_\infty r_h^4 - 2 i W r^2 r_h^2 \sqrt{r_h^2 + 2 \sigma^2} + \frac{c_\infty r^4 (r_h^2 + 2 \sigma^2) (r^2 + 6 \sigma^2)}{(r^2 + 2 \sigma^2)^2} \]

\[ Q_1(r) = -\left( \frac{\sqrt{r_h^2 + 2 \sigma^2}}{c_\infty r^2 (r^2 - r_h^2) (r_h^2 + 2 (r^2 + r_h^2) \sigma^2)} \right) Q_2(r) \]
\[ Q_2(r) = 2iBc_\infty m r_h^2 \left( r^2 + 2\sigma^2 \right) + r^4 \sqrt{r_h^2 + 2\sigma^2} \left\{ c_\infty^2 m^2 - W^2 \left( r^2 + 2\sigma^2 \right) \right\} \]

and,

\[ W = \omega - \frac{B m}{r^2} \]  \hspace{1cm} (18)

Now in order to eliminate the imaginary part from the \( P_1(r) \) and \( Q_1(r) \) we made the following substitution,

\[ R(r) = e^{i \omega f_1(r)} e^{-i m f_2(r)} \ G(r) \]  \hspace{1cm} (19)

\[ f_1(r) = \frac{\sqrt{r_h^2 + 2\sigma^2}}{2 c_\infty \left( r_h^4 + 6 r_h^2 \sigma^2 + 8 \sigma^4 \right)} \ f_3(r) \]

\[ f_2(r) = \frac{B}{2 c_\infty r_h^2 \left( r_h^2 + 4 \sigma^2 \right)} \ f_4(r) \]

\[ f_3(r) = \left( r_h^2 + 2\sigma^2 \right)^2 \log \left( r^2 - r_h^2 \right) - 4 \sigma^4 \log \left\{ r^2 r_h^2 + 2 \left( r^2 + r_h^2 \right) \sigma^2 \right\} \]

\[ f_4(r) = \left( r_h^2 + 2\sigma^2 \right) \log \left( \frac{r^2}{r_h^2} \right) + 2 \sigma^2 \log \left( \frac{r^2 r_h^2 + 2 \left( r^2 + r_h^2 \right) \sigma^2}{r^2} \right) \]

In terms of new radial function \( G(r) \) the radial equation (17) takes the following form,

\[ \frac{d^2 G(r)}{dr^2} + S_1(r) \frac{dG(r)}{dr} + S_2(r) \ G(r) = 0 \]  \hspace{1cm} (20)

where,

\[ S_1(r) = \frac{3 r^4 + r_h^4}{r \left( r^4 - r_h^4 \right)} - \frac{2r}{r^2 + 2\sigma^2} + \frac{2r r_h^4}{\left( r^2 + r_h^2 \right) \left( r^2 r_h^2 + 2 \left( r^2 + r_h^2 \right) \sigma^2 \right)} \]

\[ S_2(r) = \frac{r^2 \left( r_h^2 + 2\sigma^2 \right)}{c_\infty^2 \left( r^2 - r_h^2 \right)^2 \left( r^2 r_h^2 + 2r^2 \sigma^2 + 2r_h^2 \sigma^2 \right) S_3(r) \]

7
\[ S_3(r) = W^2 r^4 \left( r^2 + 2 \sigma^2 \right) \left( r_h^2 + 2 \sigma^2 \right) + c^2_{\infty} m^2 \left\{ r_h^4 \left( r^2 + 2 \sigma^2 \right) - r^4 \left( r_h^2 + 2 \sigma^2 \right) \right\} \]

Now we introduce tortoise coordinate \( r^* \) through the equation,

\[
\frac{d}{dr^*} = \left( 1 - \frac{A^2}{r^2 c^2} \right) \frac{d}{dr} \tag{21}
\]

which implies that,

\[
r^* = r - \frac{2 r_h \sigma^3}{\sqrt{\frac{r^2}{2} + \sigma^2 \left( r_h^2 + 4 \sigma^2 \right)}} \cot^{-1} \left( \frac{\sqrt{2} r_h \sigma}{r \sqrt{r_h^2 + 2 \sigma^2}} \right) - \frac{r_h \left( r_h^2 + 2 \sigma^2 \right) \tanh^{-1} \left( \frac{r}{r_h} \right)}{r_h^2 + 4 \sigma^2}
\]

This tortoise coordinate spans the entire real line as opposed to \( r \) which spans only the half-line. The horizon at \( r = r_h \) maps to \( r^* \to -\infty \), while \( r \to \infty \) corresponds to \( r^* \to +\infty \).

Let us now define a new radial function \( G(r) \) as,

\[ G(r) = \frac{\sqrt{r^2 + 2 \sigma^2}}{r^{\frac{3}{2}}} H(r) \tag{22} \]

Substituting this in equation (21) we observe that \( H(r) \) satisfies the differential equation,

\[
P(r) \frac{d^2 H(r)}{dr^2} + Q(r) \frac{dH(r)}{dr} + V(r) H(r) = 0 \tag{23}
\]

where,

\[
P(r) = \left\{ 1 - \frac{r_h^4 \left( r^2 + 2 \sigma^2 \right)}{r^4 \left( r_h^2 + 2 \sigma^2 \right)} \right\}^2
\]

\[
Q(r) = \frac{2 r_h^4 \left( r^2 + 4 \sigma^2 \right)}{r^5 \left( r_h^2 + 2 \sigma^2 \right)} \left\{ 1 - \frac{r_h^4 \left( r^2 + 2 \sigma^2 \right)}{r^4 \left( r_h^2 + 2 \sigma^2 \right)} \right\}
\]

\[
V(r) = \frac{(r^2 + 2 \sigma^2)}{c^2_{\infty} r^2} W^2 + \frac{\left( r^2 - r_h^2 \right) \left( r_h^2 \sigma^2 + 2 r_h^2 \sigma^2 + 2 r_h^2 \sigma^2 \right)}{(r^2 + 2 \sigma^2)^2 \left( r_h^2 + 2 \sigma^2 \right)^2} S(r)
\]

and,

8
$S(r) = -\frac{51 r_h^4 \sigma^4}{r^8} - \frac{42 r_h^4 \sigma^6}{r^{10}} - \frac{\sigma^2 \left\{ 35 r_h^4 + 2 \sigma^2 (3 + 4 m^2) (r_h^2 + 2 \sigma^2) \right\}}{2 r^6}$

$+ \frac{(1 - 4 m^2) (r_h^2 + 2 \sigma^2)}{4 r^2} - \frac{5 r_h^4 - \sigma^2 (5 - 4 m^2) (r_h^2 + 2 \sigma^2)}{4 r^4}$

Now in terms of tortoise coordinate one obtains the modified differential equation as,

$$\frac{d^2 H(r^*)}{dr^{*2}} + V(r)H(r^*) = 0 \quad (24)$$

We analyse this differential equation in two distinct radial regions, viz., near the sonic horizon, i.e., at $r^* \rightarrow -\infty$ and at asymptopia, i.e., at $r^* \rightarrow +\infty$.

Our choice of coordinates and substitutions are with a view to transform differential equation (17) into a form (24) which is simple enough to identify ingoing and outgoing modes both in asymptotic and near horizon regions. In addition we have managed to eliminate the first derivative term from the differential equation which leads to constancy of Wronskian of the differential equation. This will facilitate the calculations of the transmission and reflection coefficients. Since the second term of $V(r)$ in equation (24) tends to zero both in asymptotic region and near the horizon, the first term in $V(r)$ is the only important term.

In the asymptotic region, the above differential equation (24) can be written approximately as,

$$\frac{d^2 H(r^*)}{dr^{*2}} + \frac{\omega^2}{c_\infty^2} H(r^*) = 0 \quad (25)$$

This can be solved trivially,

$$H(r^*) = R \omega m \exp \left( i \frac{\omega}{c_\infty} r^* \right) + \exp \left( -i \frac{\omega}{c_\infty} r^* \right) \quad (26)$$

The first term in equation (25) corresponds to reflected wave and the second term to the incident wave, so that $R$ is the reflection coefficient in the sense of potential scattering. It is not difficult to calculate the Wronskian of the solutions (26); one obtains,

$$W(+\infty) = 2 i \frac{\omega}{c_\infty} \left( 1 - |R \omega m|^2 \right) \quad (27)$$

Similarly, near the horizon the above differential equation can be written approximately as,
\[ \frac{d^2 H(r^*)}{dr^*} + \left[ \frac{r_h^2 + 2 \sigma^2}{r_h^2} \left( \frac{\omega - m \Omega_H}{c_\infty} \right)^2 \right] H(r^*) = 0 \]  \hspace{1cm} (28)

where \( \Omega_H \) is the angular velocity of the sonic horizon. We impose the physical boundary condition that of the two solutions of this equation, only the ingoing one is physical, so that one has,

\[ H(r^*) = T_{\omega m} \exp \left\{ -i \sqrt{\frac{(r_h^2 + 2 \sigma^2)}{r_h^2}} \left( \frac{\omega - m \Omega_H}{c_\infty} \right) r^* \right\} \]  \hspace{1cm} (29)

Once again, it is easy to calculate the Wronskian of this solution; one obtains,

\[ W(-\infty) = 2 i \left( \frac{\omega - m \Omega_H}{c_h} \right) |T_{\omega m}|^2. \]  \hspace{1cm} (30)

Now using these approximate solutions (26,29) of the above differential equation together with their complex conjugates and recalling the fact two linearly independent solutions of this differential equation (24) must lead to a constant Wronskian, it is easy to show that,

\[ 1 - \left| R_{\omega m} \right|^2 = \left( 1 - \frac{m \Omega_H}{\omega} \right) \frac{c_\infty}{c_h} \left| T_{\omega m} \right|^2 \]  \hspace{1cm} (31)

where,

\[ c_h = \left( \frac{r_h^2}{r_h^2 + 2 \sigma^2} \right)^{\frac{1}{2}} c_\infty \]  \hspace{1cm} (32)

Here \( R_{\omega m} \) and \( T_{\omega m} \) are the amplitudes of the reflection and transmission coefficients of the scattered wave, respectively. It is obvious from equation (31) that, for frequencies in the range \( 0 < \omega < m \Omega_H \), the reflection coefficient has a magnitude larger than unity. This is similar to the amplification relation that emerges in our earlier analysis of superresonance [7, 8]. The only difference here is the appearance of a factor \( c_\infty/c_h \). This is due to the fact that speed of sound in Bose-Einstein condensate depends on radial coordinate.

4 Conclusion

In this paper we have shown that when acoustic disturbances propagate through a Bose-Einstein condensate, they get scattered by the flow of the condensate and
in the particular frequency range acoustic disturbances are reflected with an amplitude greater than the amplitude of the incident wave. The frequency range is determined by angular momentum of the incident wave and angular velocity of the horizon. The extra energy that reflected waves carry is taken from the rotational energy of the condensate. As a result vortex motion is slowed down and eventually stops when all the rotational energy of the condensate is extracted out of it. At other frequencies acoustic disturbances transmit through the condensate and ultimately get trapped inside the Dumb hole. This is analogous to the superradiance effect that occurs exclusively in the case of rotating black holes in physical gravitational systems. Because of the discreteness of the parameter B of rotational component of the velocity profile (as evident in BEC), the angular momentum of the acoustic black hole will also be proportional to an integer and hence causes the energy flux to change discretely making observation of the phenomenon easier than that in actual black holes. This also manifests itself in the reflection coefficient whose spectrum has equally spaced peaks of varying strengths all of which are multiples of the minimum strength. These interesting aspects are worth being studied quantitatively in greater detail.

Unlike our earlier work we have considered, following the suggestion of Slatyer and Savage [20], a particular type of spatial dependence of the background density of the condensate. Presence of spatial scale of variation of the condensate wave function has nontrivial effect on the artificial Lorentzian geometry that emerges from BEC. Due to spatial dependence of the background density of the fluid, speed of sound in such a system varies when it propagates though the fluid. This spatial dependence modifies the relation between reflection and transmission coefficients of the scattered waves by a factor $c_\infty/c_H$, but does not affect the cutoff frequency of the superradiance effect. In the limit $\sigma$ tends to zero the background density and hence the speed of sound in the fluid become constant. As a check we note that in this limit we recover the results of our earlier works.

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