Spacetime structure of massive gravitino

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Abstract

We present reasons as to why an ab initio analysis of the spacetime structure of massive gravitino is necessary. Afterwards, we construct the relevant representation space, and finally, give a new physical interpretation of massive gravitino.

Key words: Massive gravitino, Spacetime symmetries, Rarita-Schwinger framework.

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1 Motivation

A massive gravitino is described by $\psi^\mu$. As far as its spacetime properties are concerned, it transforms as a finite dimensional non-unitary representation of the Lorentz group,

$$\psi^\mu : \left[ \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \right] \otimes \left( \frac{1}{2}, \frac{1}{2} \right).$$

(1)

The unitarily transforming physical states are built upon this structure [1].

We enumerate two circumstances that motivate us to take an ab initio look at this representation space.

(1) For the vector sector, it has recently been called to attention that the Proca description of the (1/2, 1/2) representation space is incomplete

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An \textit{ab initio} construction of this sector reveals that the St"uckelberg contribution to the propagator, so important for the renormalization of the gauge theories with massive bosons \cite{3}, is found to naturally reside in the $(1/2, 1/2)$ representation space.

At the same time, the properties of the $(1/2, 1/2)$, along with that of the $(1/2, 0) \oplus (0, 1/2)$, representation space determine the structure of $\psi^\mu$. In order to impose a single-spin, i.e., spin $3/2$, interpretation on the latter, the lower spin-$1/2^+$ and spin-$1/2^-$ components of $\psi^\mu$ are considered as redundant, unphysical, states that are claimed to be excluded from consideration by means of the two supplementary conditions: $\gamma^\mu \psi^\mu(x) = 0$, and $\partial^\mu \psi^\mu(x) = 0$, respectively. However, this time-honored framework was questioned by a recent empirical observation regarding the $N$ and $\Delta$ resonances \cite{4}. The available data on high-spin resonances reveal an unexpected and systematic clustering in terms of the $(j/2, j/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ representations with $j = 1, 3$ and $5$ \textit{without} imposition of the supplementary conditions. For the $N$ and $\Delta$ resonances these results are summarized in Fig. 1. For example, in the standard theoretical framework $N(1440), N(1535), \Delta(1620)$ and $\Delta(1750)$ should have been absent. Experimental data shows them to be present at statistically significant level\cite{7}.

In regard to the latter of the two enumerated circumstances, we take the position that any solution of the QCD Lagrangian for particle resonances must carry well-defined transformation properties when looked upon from different inertial frames. This forces these resonances to belong to one, or the other, of various representation spaces of the Lorentz group. For this reason the data on particle resonances may furnish hints on physical interpretation of various Lorentz group representations that one needs in gauge theories, or theories of supergravity.

For exploring the spacetime structure of massive gravitinos the charge conjugation properties play an important role. Under the operation of charge conjugation, one may choose the spinor sector to behave as a Dirac object, and implement the Majorana nature of the massive gravitino at the level of the Fock space. This is standard, see, e.g., Ref. \cite{7}. Or, from the very beginning choose the spinor sector to behave as a Majorana object. Since we wish to stress certain non-trivial aspects of massive gravitino that do not – at least qualitatively – depend on this choice, we shall here treat the spinor sector to be of Dirac type.

Very nature of our \textit{ab initio} look at the representation space defined in Eq. (1), obliges us to present sufficient pedagogic details so that by the end of the

\footnote{The $N(1440), N(1535), \Delta(1620)$ carry four star status, while at present $\Delta(1750)$ simply has a one star significance \cite{6}.}
paper much that is needed to form an opinion on the arrived results is readily
available. At the same time, Letter nature of this manuscript would prevent us
from delving into subtle details which are, for present, of secondary importance
(but have been studied and are planned to be presented elsewhere).

We shall work in the momentum space. The notation will be essentially that
introduced in Ref. [2].

2 Construction of the Spinor and Vector Sectors

We now wish to construct the primitive objects that span the representation
space defined in Eq. (1) for an arbitrary \( \vec{p} \). We first construct the spinor and
vector sectors in the rest frame, and then boost them using the following boosts:

\[
\kappa^{(\frac{1}{2},0)} \oplus (0,\frac{1}{2}) = \kappa^{(\frac{1}{2},0)} \oplus \kappa^{(0,\frac{1}{2})}, \quad \kappa^{(\frac{1}{2},\frac{1}{2})} = \kappa^{(\frac{1}{2},0)} \otimes \kappa^{(0,\frac{1}{2})},
\]

with

\[
\kappa^{(\frac{1}{2},0)} = \frac{1}{\sqrt{2m(E + m)}} [(E + m)I_2 + \vec{\sigma} \cdot \vec{p}],
\]

\[
\kappa^{(0,\frac{1}{2})} = \frac{1}{\sqrt{2m(E + m)}} [(E + m)I_2 - \vec{\sigma} \cdot \vec{p}] .
\]

where \( I_n \) stands for an \( n \times n \) identity matrix, while the remaining symbols
carry their usual contextual meaning. We define the spin-1/2 helicity operator:
\( \Sigma = (\vec{\sigma}/2) \cdot \vec{p} \), where \( \vec{p} = \vec{p}/|\vec{p}| \), and \( \vec{p} = |\vec{p}|(\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)) \). Keeping full freedom in the choice of phases, its positive and negative helicity
states are:

\[
h^+ = N \exp(i\vartheta_1) \begin{pmatrix} \cos(\theta/2) \exp(-i\phi/2) \\ \sin(\theta/2) \exp(i\phi/2) \end{pmatrix},
\]

\[
h^- = N \exp(i\vartheta_2) \begin{pmatrix} \sin(\theta/2) \exp(-i\phi/2) \\ -\cos(\theta/2) \exp(i\phi/2) \end{pmatrix} .
\]
2.1 \((1/2, 0) \oplus (0, 1/2)\) Representation space

In this subsection we present the kinematic structure of the \((1/2, 0) \oplus (0, 1/2)\) representation space in such a manner that its extension to the \((1/2, 1/2)\) representation space becomes transparent. The familiarity of the equation presented shall not, we hope, mar the procedure adopted. The entire construct, in essence, relies on nothing more than the boost operators.

The rest-frame \((1/2, 0) \oplus (0, 1/2)\) spinors are then chosen to be:

\[
\begin{align*}
  u_{+1/2}(\vec{0}) &= \begin{pmatrix} h^+ \\ h^+ \end{pmatrix}, & u_{-1/2}(\vec{0}) &= \begin{pmatrix} h^- \\ h^- \end{pmatrix}, \\
  v_{+1/2}(\vec{0}) &= \begin{pmatrix} h^+ \\ -h^+ \end{pmatrix}, & v_{-1/2}(\vec{0}) &= \begin{pmatrix} h^- \\ -h^- \end{pmatrix}.
\end{align*}
\]

The choice of the phases made in writing down these spinors has been determined by the demand of parity covariance.

The boosted spinors, \(u_{\pm 1/2}(\vec{p})\) and \(v_{\pm 1/2}(\vec{p})\) are obtained by applying the boost operator \(\kappa_{(1/2, 0) \oplus (0, 1/2)}\) to the above spinors, yielding:

\[
\begin{align*}
  u_{+1/2}(\vec{p}) &= \frac{N \exp(i\vartheta_1)}{\sqrt{2m(m+E)}} \begin{pmatrix}
    \exp(-i\varphi/2)(m + |\vec{p}| + E) \cos(\theta/2) \\
    \exp(i\varphi/2)(m + |\vec{p}| + E) \sin(\theta/2) \\
    \exp(-i\varphi/2)(m - |\vec{p}| + E) \cos(\theta/2) \\
    \exp(i\varphi/2)(m - |\vec{p}| + E) \sin(\theta/2)
  \end{pmatrix}, \\
  u_{-1/2}(\vec{p}) &= \frac{N \exp(i\vartheta_2)}{\sqrt{2m(m+E)}} \begin{pmatrix}
    \exp(-i\varphi/2)(m - |\vec{p}| + E) \sin(\theta/2) \\
    -\exp(i\varphi/2)(m - |\vec{p}| + E) \cos(\theta/2) \\
    \exp(-i\varphi/2)(m + |\vec{p}| + E) \sin(\theta/2) \\
    -\exp(i\varphi/2)(m + |\vec{p}| + E) \cos(\theta/2)
  \end{pmatrix}, \\
  v_{+1/2}(\vec{p}) &= \frac{N \exp(i\vartheta_1)}{\sqrt{2m(m+E)}} \begin{pmatrix}
    \exp(-i\varphi/2)(m + |\vec{p}| + E) \cos(\theta/2) \\
    \exp(i\varphi/2)(m + |\vec{p}| + E) \sin(\theta/2) \\
    -\exp(-i\varphi/2)(m - |\vec{p}| + E) \cos(\theta/2) \\
    -\exp(i\varphi/2)(m - |\vec{p}| + E) \sin(\theta/2)
  \end{pmatrix}, \\
  v_{-1/2}(\vec{p}) &= \frac{N \exp(i\vartheta_2)}{\sqrt{2m(m+E)}} \begin{pmatrix}
    \exp(-i\varphi/2)(m - |\vec{p}| + E) \sin(\theta/2) \\
    -\exp(i\varphi/2)(m - |\vec{p}| + E) \cos(\theta/2) \\
    \exp(-i\varphi/2)(m + |\vec{p}| + E) \sin(\theta/2) \\
    -\exp(i\varphi/2)(m + |\vec{p}| + E) \cos(\theta/2)
  \end{pmatrix}.
\end{align*}
\]
\[
\psi_{-1/2}(\vec{p}) = \frac{N \exp(i\vartheta)}{\sqrt{2m(m+E)}} \begin{pmatrix}
\exp(-i\phi/2)(m-|\vec{p}|+E)\sin(\theta/2) \\
-\exp(i\phi/2)(m-|\vec{p}|+E)\cos(\theta/2) \\
-\exp(-i\phi/2)(m+|\vec{p}|+E)\sin(\theta/2) \\
\exp(i\phi/2)(m+|\vec{p}|+E)\cos(\theta/2)
\end{pmatrix}.
\]

In the standard notation, these satisfy the orthonormality and completeness relations, along with the Dirac equation:

\[
\overline{u}_h(\vec{p}) u_{h'}(\vec{p}) = +2N^2 \delta_{hh'}, \quad \overline{v}_h(\vec{p}) v_{h'}(\vec{p}) = -2N^2 \delta_{hh'}, \quad \gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.
\]

The wave equation satisfied by the \(u_h(\vec{p})\) and \(v_h(\vec{p})\) spinors follows if we note:

\[
\frac{1}{2N^2} \sum_{h=\pm 1/2} u_h(\vec{p}) \overline{u}_h(\vec{p}) + \sum_{h=\pm 1/2} v_h(\vec{p}) \overline{v}_h(\vec{p}) \equiv \gamma \mu P^\mu + mI_4.
\]

where we defined,

\[
\gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.
\]

The 0\(_n\) are \(n \times n\) null matrices. Adding/subtracting Eqs. (9) and (10), yields:

\[
\frac{1}{2N^2} \sum_{h=\pm 1/2} u_h(\vec{p}) \overline{u}_h(\vec{p}) = \frac{\gamma_\mu P^\mu + mI_4}{2m}, \quad \frac{1}{2N^2} \sum_{h=\pm 1/2} v_h(\vec{p}) \overline{v}_h(\vec{p}) = \frac{\gamma_\mu P^\mu - mI_4}{2m}.
\]

Multiplying Eq. (12) from the right by \(u_{h'}(\vec{p})\), and Eq. (13) by \(v_{h'}(\vec{p})\), and using Eqs. (8), immediately yield the momentum-space wave equation for the \((1/2,0) \oplus (0,1/2)\) representation space,

\[
(\gamma_\mu P^\mu \pm mI_4) \psi_h(\vec{p}) = 0.
\]

\(^2\) This part of the calculation is best done if one starts with the momenta in the Cartesian co-ordinates, and takes the “axis of spin quantization” to be the z-axis.
In Eq. (14), the minus sign is to be taken for, \( \psi_h(\vec{p}) = u_h(\vec{p}) \), and the plus sign for, \( \psi_h(\vec{p}) = v_h(\vec{p}) \).

The essential element to note in regard to Eq. (14) is that it follows directly from the explicit expressions for the \( u_h(\vec{p}) \) and \( v_h(\vec{p}) \).

2.2 \((1/2, 1/2)\) Representation space – An ab initio construct

Next, we introduce the rest-frame vectors for the \((1/2, 1/2)\) representation space,

\[
\begin{align*}
\xi_1(\vec{0}) &= h^+ \otimes h^+, \\
\xi_2(\vec{0}) &= \frac{1}{\sqrt{2}} \left( h^+ \otimes h^- + h^- \otimes h^+ \right), \\
\xi_3(\vec{0}) &= h^- \otimes h^-, \\
\xi_4(\vec{0}) &= \frac{1}{\sqrt{2}} \left( h^+ \otimes h^- - h^- \otimes h^+ \right).
\end{align*}
\]

The boosted vectors are thus: \( \xi_\zeta(\vec{p}) = \kappa^{(1/2, 1/2)} \xi_\zeta(\vec{0}) \), \( \zeta = 1, 2, 3, 4 \),

\[
\begin{align*}
\xi_1(\vec{p}) &= \frac{\exp(i 2 \phi_1) N^2}{2 \sqrt{2} m} \begin{pmatrix} 2 \exp(-i\phi) \cos^2(\theta/2) \\ \sin(\theta) \\ \sin(\theta) \\ 2 \exp(i\phi) \sin^2(\theta/2) \end{pmatrix}, \\
\xi_2(\vec{p}) &= \frac{\exp(i 2 \phi_1) N^2}{2 \sqrt{2} m} \begin{pmatrix} \exp(-i\phi) E \sin(\theta) \\ -(|\vec{p}| + E \cos(\theta)) \\ |\vec{p}| - E \cos(\theta) \\ -\exp(i\phi) E \sin(\theta) \end{pmatrix}, \\
\xi_3(\vec{p}) &= \frac{\exp(i 2 \phi_1) N^2}{2} \begin{pmatrix} 2 \exp(-i\phi) \sin^2(\theta/2) \\ -\sin(\theta) \\ -\sin(\theta) \\ 2 \exp(i\phi) \cos^2(\theta/2) \end{pmatrix}.
\end{align*}
\]
\[\xi_4(\vec{p}) = \frac{\exp(i2\theta_1)N^2}{\sqrt{2m}} \begin{pmatrix} \exp(-i\phi)|\vec{p}| \sin(\theta) \\ - (E + |\vec{p}| \cos(\theta)) \\ E - |\vec{p}| \cos(\theta) \\ - \exp(i\phi)|\vec{p}| \sin(\theta) \end{pmatrix}. \tag{19}\]

In the notation of Ref. [2], these satisfy the orthonormality and completeness relations, along with a new wave equation. The orthonormality and completeness relations are:

\[
\bar{\xi}_\zeta(\vec{p})\xi_{\zeta'}(\vec{p}) = -N^4 \delta_{\zeta\zeta'}, \quad \zeta = 1, 2, 3, \\
\bar{\xi}_4(\vec{p})\xi_{\zeta'}(\vec{p}) = +N^4 \delta_{4\zeta'}, \quad \zeta = 4, \\
\frac{1}{N^4} \left[ \xi_4(\vec{p})\bar{\xi}_4(\vec{p}) - \sum_{\zeta=1,2,3} \xi_\zeta(\vec{p})\bar{\xi}_\zeta(\vec{p}) \right] = I_4, \tag{20}\]

where

\[
\bar{\xi}_\zeta(\vec{p}) \equiv \xi_\zeta(\vec{p})^\dagger \lambda_{00}, \tag{21}\]

with

\[
\lambda_{00} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{22}\]

The parity operator for the (1/2, 1/2) representation space is:

\[
P = \lambda_{00} \exp[i\alpha] \mathcal{R}, \quad \mathcal{R} : \{\theta \rightarrow \pi - \theta, \phi \rightarrow \pi + \phi\} \\
\alpha = \text{a real number}, \tag{23}\]

while the helicity operator for this space is, \(\vec{J} \cdot \vec{p}\), with \(\vec{J}\) given by:

\[\text{In the cited work we presented the (1/2, 1/2) representation space in its parity realization. Here, the presentation is in terms of helicity realization. The two descriptions have mathematically similar but physically distinct structures, which, e.g., show up in their different behavior under the operation of Parity.}\]
We now must take a small definitional detour towards the notion of the dragged Casimirs for spacetime symmetries. It arises in the following fashion. The second Casimir operator, \(C_2\), of the Poincaré group is defined as the square of the Pauli-Lubanski pseudovector:

\[
\mathcal{W}^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_{\sigma}, \quad (25)
\]

where \(\epsilon^{\mu\nu\rho\sigma}\) is the standard Levi-Civita symbol in four dimensions, while \(M_{\mu\nu}\) denote generators of the Lorentz group,

\[
M_{0i} = K_i, \quad M_{ij} = \epsilon_{ijk} J^k, \quad (26)
\]

where each of the \(i, j, k\) runs over 1, 2, 3. The \(P_{\mu}\) are generators of the spacetime translations. In general, these have non-vanishing commutators with \(M_{\mu\nu}\),

\[
[P_\mu, M_{\rho\sigma}] = i (\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho). \quad (27)
\]

On using Eq. (27), we rewrite \(C_2\) as

\[
C_2 = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\lambda\kappa\zeta} M_{\nu\rho} M^{\lambda\kappa} P_\sigma P_\zeta
\]

\[
+ \left[ i \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\lambda\kappa\zeta} M_{\nu\rho} \eta_\lambda P_\kappa P_\zeta + i \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\rho\lambda\kappa} M_{\nu\sigma} \eta_\kappa P_\lambda P_\zeta \right]. \quad (28)
\]

The squared brackets vanishes due to antisymmetry of the Levi-Civita symbol. As such, the space-time translation operators entering the definition of \(C_2\) can be moved to the very right.

This observation allows for introducing the dragged Casimir \(\bar{C}_2\) as an operator with the same form as \(C_2\) – the difference being that the commutator in Eq. (27) is now set to zero \(\text{as is appropriate for finite dimensional}\).
\[ SU_R(2) \otimes SU_L(2) \text{ representations} \]. Consequently, while \( C_2 \) and \( \tilde{C}_2 \) carry same \[2\] invariant eigenvalues when acting upon momentum eigenstates, their commutators with the Lorentz group generators are no longer identical. For the \((1/2, 0) \oplus (0, 1/2)\) representation space, \([\tilde{C}_2, \hat{J}^2]\) vanishes. For the \((1/2, 1/2)\) representation space, \([\tilde{C}_2, \hat{J}^2]\) does not vanish (except when acting upon rest states), and equals \(-4E \vec{P} \cdot \vec{K}\). This leads to the fact that while the former representation space is endowed with a well-defined spin, the latter is not:

As an immediate application, \( \tilde{C}_2 \) for the \((1/2, 1/2)\) representation space bifurcates this space into two sectors. The three states \( \xi_{\zeta}(\vec{p}) \) with \( \zeta = 1, 2, 3 \) are associated with the \( \tilde{C}_2 \) eigenvalue, \(-2m^2\); while the, \( \zeta = 4 \), corresponds to eigenvalue zero.

Thus, all the \( \xi_{\zeta}(\vec{p}) \), except for the rest frame, cease to be eigenstates of the \((1/2, 1/2)\)'s \( \hat{J}^2 \) and do not carry definite spins. This contrasts with the situation for the \((1/2, 0) \oplus (0, 1/2)\) representation space, where the \( \psi_{\zeta}(\vec{p}) \) are eigenstates of the corresponding \( \hat{J}^2 \).

Now in order that the \( \xi_{\zeta}(\vec{p}) \) carry the standard contravariant Lorentz index, we introduce a rotation in the \((1/2, 1/2)\) representation space via [2]:

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & i & -i & 0 \\
-1 & 0 & 0 & i \\
1 & 0 & 0 & 1 \\
0 & i & i & 0
\end{pmatrix}
\]

Then, the \((1/2, 1/2)\) representation space is spanned by four Lorentz vectors:

\[
\mathcal{A}_\zeta^{\mu}(\vec{p}) = S^{\mu\alpha} [\xi_{\zeta}(\vec{p})]_\alpha, \quad \zeta = 1, 2, 3, 4,
\]

and the superscript \( \mu \) is the standard Lorentz index. Following the procedure established in Sec. 2.1, they can be shown to satisfy a new wave equation [2],

\[
\left(\Lambda_{\mu\nu} p^\mu p^\nu \pm m^2 I_4\right) \mathcal{A}_\zeta^{\mu}(\vec{p}) = 0,
\]

where the plus sign is to be taken for, \( \zeta = 1, 2, 3 \), while the minus sign belongs to, \( \zeta = 4 \). The \( \Lambda_{\mu\nu} \) matrices are: \( \Lambda_{00} = \text{diag}(1, -1, -1, -1) \), \( \Lambda_{11} = \text{diag}(1, -1, 1, 1) \), \( \Lambda_{22} = \text{diag}(1, 1, -1, -1) \), \( \Lambda_{33} = \text{diag}(1, 1, 1, -1) \), and

\[4\] To avoid confusion, note that \( \tilde{C}_2 \) is defined in the \( SU_R(2) \otimes SU_L(2) \); while \( C_2 \) is defined in the Poincaré group.
The remaining $\Lambda_{\mu\nu}$ are obtained from the above expressions by noting: $\Lambda_{\mu\nu} = \Lambda_{\nu\mu}$. Parenthetically, we note that the S-transformed $\lambda_{00}$ equals $\Lambda_{00}$ and is nothing but the standard spacetime metric (for flat spacetime).

It can also be seen that $\xi_\zeta(\vec{p})$, for $\zeta = 1, 2, 3$, coincide with the solutions of Proca framework (and are divergence-less); whereas $\xi_4(\vec{p})$, that gives the St"uckelberg contribution to the propagator, lies outside the Proca framework:

\begin{array}{cccc}
\zeta & p_\mu A^\mu(\vec{p}) & \overline{\tilde{W}}_\mu^{(1/2,1/2)} A^\mu(\vec{p}) & \lambda_c \\
1, 2, 3 & 0 & \neq 0 & 2 \text{ Proca Sector} \\
4 & \neq 0 & = 0 & 0 \text{ St"uckelberg Sector} \\
\end{array}

In the above table we have introduced the $\lambda_c$ via the equation:

$$\tilde{C}_2^{(1/2,1/2)} A(\vec{p}) = -\lambda_c m^2 A(\vec{p}) .$$ (33)

3 Construction of spinor vector: The massive gravitino

We now wish to present the basis vectors for the representation space defined by Eq. (1) in a language which is widely used [7]. This would allow the present analysis to be more readily available, and also bring out the relevant similarities and differences with the framework of Rarita and Schwinger [8].

\[5\] We define the dragged Pauli-Lubanski pseudovector, $\overline{\tilde{W}}_\mu$, in a manner parallel to the introduction of the dragged second Casimir operator.
In writing down the basis spinor-vectors, we will use the fact that in the
(1/2, 1/2) representation space the charge conjugation is implemented by

\[ \mathcal{C}^{(1/2,1/2)} : \ A(\vec{p}) \rightarrow [A(\vec{p})]^* . \]  

(34)

In the (1/2, 0) ⊕ (0, 1/2) representation space the charge conjugation operator
is \( \mathcal{C}^{(1/2,0)⊕(0,1/2)} : i \gamma^2 K \), where \( K \) complex conjugates the spinor to its right.
Then, with the uniform choice,

\[ \vartheta_1 = -\vartheta_2, \]  

(35)

we obtain:

\[ \mathcal{C}^{(1/2,0)⊕(0,1/2)} : \begin{cases} 
  u_{+1/2}(\vec{p}) \rightarrow -v_{-1/2}(\vec{p}) , u_{-1/2}(\vec{p}) \rightarrow v_{+1/2}(\vec{p}) , \\
v_{+1/2}(\vec{p}) \rightarrow u_{-1/2}(\vec{p}) , v_{-1/2}(\vec{p}) \rightarrow -u_{+1/2}(\vec{p}) . 
\end{cases} \]  

(36)

In the spirit outlined, the massive gravitino lives in a space spanned by sixteen
spinor-vectors defined in items A, B, C below:

A. Of these, eight spinor-vectors have \( \bar{C}_2 \) – but not \( \vec{J}^2 \) – eigenvalues, \( -\frac{15}{4} m^2 \).
These can be further subdivided into particle,

\[ \psi^\mu_a(\vec{p}) : \begin{cases} 
  \psi^{\mu}_{1}(\vec{p}) = u_{+1/2}(\vec{p}) \otimes A^{\mu}_1(\vec{p}) , \\
  \psi^{\mu}_{2}(\vec{p}) = \sqrt{\frac{2}{3}} u_{+1/2}(\vec{p}) \otimes A^{\mu}_2(\vec{p}) + \sqrt{\frac{1}{3}} u_{-1/2}(\vec{p}) \otimes A^{\mu}_3(\vec{p}) , \\
  \psi^{\mu}_{3}(\vec{p}) = \sqrt{\frac{2}{3}} u_{+1/2}(\vec{p}) \otimes A^{\mu}_3(\vec{p}) + \sqrt{\frac{1}{3}} u_{-1/2}(\vec{p}) \otimes A^{\mu}_3(\vec{p}) , \\
  \psi^{\mu}_{4}(\vec{p}) = u_{-1/2}(\vec{p}) \otimes A^{\mu}_3(\vec{p}) , 
\end{cases} \]

and antiparticle sectors:

\[ [\psi^\mu(\vec{p})]^C : \begin{cases} 
  \psi^\mu_{5}(\vec{p}) = -v_{-1/2}(\vec{p}) \otimes [A^{\mu}_1(\vec{p})]^* , \\
  \psi^\mu_{6}(\vec{p}) = -\sqrt{\frac{2}{3}} v_{-1/2}(\vec{p}) \otimes [A^{\mu}_2(\vec{p})]^* + \sqrt{\frac{1}{3}} v_{+1/2}(\vec{p}) \otimes [A^{\mu}_3(\vec{p})]^* , \\
  \psi^\mu_{7}(\vec{p}) = -\sqrt{\frac{2}{3}} v_{-1/2}(\vec{p}) \otimes [A^{\mu}_3(\vec{p})]^* + \sqrt{\frac{1}{3}} v_{+1/2}(\vec{p}) \otimes [A^{\mu}_3(\vec{p})]^* , \\
  \psi^\mu_{8}(\vec{p}) = v_{+1/2}(\vec{p}) \otimes [A^{\mu}_3(\vec{p})]^* . 
\end{cases} \]

Here, \( [\psi^\mu(\vec{p})]^C = \mathcal{C}^{(1/2,0)⊕(0,1/2)} \otimes \mathcal{C}^{(1/2,1/2)} \psi^\mu(\vec{p}) \), \( \tau = a, b, c \).

B. Four spinor-vectors have \( \bar{C}_2 \) – but not \( \vec{J}^2 \) – eigenvalues, \( -\frac{3}{4} m^2 \):

\[ \psi^\mu_{9}(\vec{p}) : \begin{cases} 
  \psi^\mu_{9}(\vec{p}) = \sqrt{\frac{2}{3}} u_{-1/2}(\vec{p}) \otimes A^{\mu}_1(\vec{p}) - \sqrt{\frac{1}{3}} u_{+1/2}(\vec{p}) \otimes A^{\mu}_2(\vec{p}) , \\
  \psi^\mu_{10}(\vec{p}) = \sqrt{\frac{2}{3}} u_{-1/2}(\vec{p}) \otimes A^{\mu}_2(\vec{p}) - \sqrt{\frac{1}{3}} u_{+1/2}(\vec{p}) \otimes A^{\mu}_2(\vec{p}) , 
\end{cases} \]
\[ \psi^\mu_b(\vec{p}) \] C: \begin{cases} 
\psi_{11}^\mu(\vec{p}) = \sqrt{\frac{2}{3}} v_{+1/2}(\vec{p}) \otimes [A_4^\mu(\vec{p})]^* + \sqrt{\frac{1}{3}} v_{-1/2}(\vec{p}) \otimes [A_5^\mu(\vec{p})]^*, \\
\psi_{12}^\mu(\vec{p}) = \sqrt{\frac{1}{3}} v_{+1/2}(\vec{p}) \otimes [A_4^\mu(\vec{p})]^* + \sqrt{\frac{2}{3}} v_{-1/2}(\vec{p}) \otimes [A_5^\mu(\vec{p})]^*. 
\end{cases} 

C. Another set of four spinor-vectors with \( C_2 \) but not \( \vec{J}_2 \) eigenvalues, \(-\frac{2}{3} m^2\):

\[ \psi^\mu_c(\vec{p}) \] C: \begin{cases} 
\psi_{13}^\mu(\vec{p}) = u_{+1/2}(\vec{p}) \otimes A_4^\mu(\vec{p}), \\
\psi_{14}^\mu(\vec{p}) = u_{-1/2}(\vec{p}) \otimes A_4^\mu(\vec{p}), \\
\psi_{15}^\mu(\vec{p}) = -v_{-1/2}(\vec{p}) \otimes [A_4^\mu(\vec{p})]^*, \\
\psi_{16}^\mu(\vec{p}) = v_{+1/2}(\vec{p}) \otimes [A_4^\mu(\vec{p})]^*. 
\end{cases} 

We have evaluated \( \gamma_\mu \psi^\mu(\vec{p}), p_\mu \psi^\mu(\vec{p}), \text{ and } \tilde{W}_\mu(1/2,1/2) \psi^\mu(\vec{p}), \) for all of the above sixteen spinor vectors. The \( p_\mu \psi^\mu(\vec{p}), \) when transformed to the configuration space, tests the divergence of \( \psi^\mu(x) \).

For \( \eta = 1, 4, 5, 8, \gamma_\mu \psi^\mu(\vec{p}) \) identically vanishes. Requiring it to vanish for \( \eta = 2, 3, 6, 7 \) results in:

1. \( \vartheta_1 = \vartheta_2 \). When combined with Eq. (35) this implies, \( \vartheta_1 = 0 = \vartheta_2 \). As a consequence, the global phase factors \( \vartheta_1 \) and \( \vartheta_2 \) that appear in Eqs. (5), (7), and (19) are not entirely free.

2. \( E^2 = |\vec{p}|^2 + m^2 \)

The \( \tau = b, c \) sectors, if (wrongly) imposed with the vanishing of, \( \gamma_\mu \psi^\mu(\vec{p}) \) and \( p_\mu \psi^\mu(\vec{p}), \) results in kinematically acausal dispersion relation (i.e., in \( E^2 \neq |\vec{p}|^2 + m^2 \)). This could be the source of the well-known problems of the Rarita-Schwinger framework as noted in works of Johnson and Sudarshan [9], and those of Velo and Zwanziger [10]. In this context one may wish to recall that interactions can induce transitions between different \( \tau \) sectors.

The analysis for all the \( \tau \) sectors of the \( \psi^\mu(\vec{p}) \) can be summarized in the following table:

| \( \tau \) | \( p_\mu \psi^\mu(\vec{p}) \) | \( \gamma_\mu \psi^\mu(\vec{p}) \) | \( \tilde{W}_\mu(1/2,1/2) \psi^\mu(\vec{p}) \) | \( \lambda_c \) | Remarks |
|---|---|---|---|---|---|
| a | 0 | 0 | \( \neq 0 \) | \( \frac{15}{4} \) | Rarita-Schwinger Sector |
| b | 0 | \( \neq 0 \) | \( \neq 0 \) | \( \frac{3}{4} \) | |
| c | \( \neq 0 \) | \( \neq 0 \) | = 0 | \( \frac{4}{3} \) | |

The table clearly illustrates that there is no particular reason – except (the unjustified) insistence that each particle of nature be associated with a definite spin – to favor one \( \tau \) sector over the other. Each of the \( \tau \) sectors is
endowed with specific properties. The Rarita-Schwinger sector has no more, or no less, physical significance than the other two sectors. While, for instance, the Rarita-Schwinger sector can be characterized by vanishing of the $p_\mu \psi_\mu(\vec{p})$ and $\gamma_\mu \psi_\mu(\vec{p})$; the $\tau = c$ sector is uniquely characterized by vanishing of $\tilde{W}_\mu^{(1/2,1/2)}(\vec{p})$. The $\tau = b$ sector allows for vanishing of $p_\mu \psi_\mu(\vec{p})$ only.

Except for the rest frame, the $\psi_\mu(\vec{p})$, in general, are not eigenstates of the $\vec{J}^2$ for representation space (1). Instead, the three $\tau$ sectors of the representation space under consideration correspond to the following inertial-frame independent values of the associated dragged second Casimir invariant:

$$\tilde{C}_2^{[(1/2,0)\oplus(0,1/2)]\otimes(1/2,1/2)} \psi_\tau^\mu(\vec{p}) = -m^2 \lambda_c \psi_\tau^\mu(\vec{p}).$$

(37)

For each of the $\tau$ sectors, the $\lambda_c$ are given in the table above. Stated differently, the $\tau = b,c$ sectors do not carry spin one half. Similarly, the, $\tau = a$, sector is not a spin three half sector. The consequence is that the $\tau = b,c$ sector, in particular, should not be treated as a Dirac representation space. The correct wave equation for $\psi_\tau^\mu(\vec{p})$ is:

$$\left[ (\Lambda_\mu p^\mu p^\nu \pm m^2 I_4) \otimes (\gamma_\mu p^\mu \pm m I_4) \right] \psi_\tau^\epsilon(\vec{p}) = 0.$$

(38)

In the standard Rarita-Schwinger framework $\partial_\mu \psi_\mu(x)$ and $\gamma_5 \gamma_\mu \psi_\mu(x)$ do indeed behave as Dirac spinors, and do indeed satisfy the Dirac equation. However, they are not identical to the $\tau = b,c$ sectors (which do not carry a characterization in terms of spin one half). If one (mistakenly) makes this identification, and sets $\partial_\mu \psi_\mu(x)$ and $\gamma_5 \gamma_\mu \psi_\mu(x)$ to zero, one introduces an element of kinematic acausality.

4 Interpretation

If one is to respect the mathematical completeness of the spinor-vector representation space associated with $\psi_\mu(x)$, the Rarita-Schwinger framework cannot be considered to describe the full physical content of the representation space associated with a massive gravitino. This circumstance is akin to Dirac’s observation that a part of a representation space [which would have violated the completeness of the $(1/2,0)\oplus(0,1/2)$] cannot be “projected out” without introducing certain mathematical inconsistencies, and loosing its physical content (i.e. antiparticle, or particles). Further, the same qualitative remarks apply

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6 Similar remarks apply to Majorana construct in the $(1/2,0) \oplus (0,1/2)$ representation [11].
to the \((1/2, 1/2)\) representation space when in the Proca framework one only confines to the divergence-less vectors. The “projecting out” of the divergence-full vector, throws away the Stückelberg contribution to the propagator, and in addition leaves the \((1/2, 1/2)\) representation space mathematically incomplete. Now, we suggest that for the representation space defined by Eq. (1), one needs to consider all three \(\tau\) sectors of \(\psi^\mu(x)\) as physical, and necessary for its mathematical consistency. The suggested framework already carries consistency with the known data on the \(N\) and \(\Delta\) resonances, and asks that massive gravitino be considered as an object that is better described by the eigenvalues of the dragged second Casimir operator. In its rest frame it is endowed with a spin three half, and two spin half, components. A spin measurement for unpolarized ensemble of massive gravitinos at rest would yield the results \(3/2\) with probability one half, and \(1/2\) with probability one half. The latter probability is distributed uniformly, i.e. as one quarter, over each of the, \(\tau = b\), and, \(\tau = c\), sectors.
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Fig. 1. Summary of the data on \(N\) and \(\Delta\) resonances. The breaking of the mass degeneracy for each of the clusters at about 5% may in fact be an artifact of the data analysis, as has been suggested by Höhler [5]. The filled circles represent known resonances, while the sole empty circle corresponds to a prediction.