On modeling of undular bores based on the second approximation of the shallow water theory

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Abstract. It is studied the possibility of modeling of undular bores on the basis of the second approximation of the shallow water theory. The classical differential Green-Naghdi model cannot be used for correct numerical simulation of wave flows with undular bores. The reason for this is that this model is derived within the framework of the long-wave approximation, by virtue of which the characteristic depth of the stream is much less than the characteristic length of the surface waves, which is not performed in the undular bore front. An integro-differential Green-Naghdi model is proposed for numerical simulation of undular bores. In this model we used the divergent differential form of the continuity equation and the integral conservation law of horizontal momentum. This model is derived from two-dimensional integral conservation laws of mass and momentum, describing a plane-parallel flow of an ideal incompressible fluid over a horizontal bottom. The basis of this conclusion is the concept of a local hydrostatic approximation, which generalizes the concept of the long-wave approximation and is used to justify the applicability of shallow water models to describe wave flows with the hydraulic bores.

1. Introduction

The system of differential Green-Naghdi equations [1–3] of the second approximation of the shallow water theory [4, 5] is derived from continuity and Euler equations by expanding on a small long-wave approximation parameter

$$\varepsilon = \frac{H^2}{L^2} \ll 1,$$

where $H$ and $L$ is the characteristic depth of flow and the characteristic length of surface waves respectively. As shown in [5], the consequence of inequality (1) is that the spatial derivative of the depth $h(x, t)$ of a plane-parallel flow must be rather small, i.e. it must satisfies the condition

$$\left| \frac{\partial h}{\partial x} \right| \leq O(\sqrt{\varepsilon}).$$

However, field observations and laboratory modeling show [5, 8] that the transverse dimensions $l$ of real hydraulic bores are comparable with the characteristic depths of the streams, i.e. $l = O(H)$, and so the corresponding transition regions do not satisfy the long-wave approximation condition $H \ll l$. At the same time, when hydraulic bores is calculated by
the help of hyperbolic models [4, 9] of the first approximation of the shallow water theory we have at the shock $\partial h/\partial x = \infty$, and when it is calculated by the help of the dispersion Green-Naghdi model [3] we have at the bores front $\partial h/\partial x \geq O(1)$, which contradicts the condition (2). As a result, numerical calculations of the dam-break problem using the differential equations of the Green-Naghdi model do not effectively reproduce the sequence of damped surface waves behind the front of the undular bore [10, 11]. The amplitude and length of these waves, obtained in numerical calculations, significantly different from those observed in laboratory experiments [6, 7].

Applicability of the first shallow water approximation for modeling wave flows with hydraulic bores in the spatially one-dimensional case is given in [12, 13] and in the spatially two-dimensional case in [14]. Justification of such applicability is obtained by deriving the basic conservation laws of this approximation from two- and three-dimensional integral conservation laws of mass and momentum describing a plane-parallel flow of an ideal incompressible fluid above a horizontal bottom. Wherein the concept of a hydrostatic approximation was used [13, 14], which is local and generalizes the classical concept of long-wave approximation. In the present paper, the local hydrostatic approximation is used to derive the integral conservation laws for the Green-Naghdi model and to analyze the applicability of this model for the simulation of wave flows with undular bores.

2. Mass conservation law

Let us consider a plane-parallel potential flow of an ideal incompressible fluid in a rectangular channel with a horizontal bottom. The $x$-axis of the Cartesian coordinate system $\{x, y, z\}$ is placed at the bottom of the channel, parallel to its side walls, and the $z$-axis is directed vertically up, perpendicular to the bottom of the channel. In such a coordinate system, the flow is described by the depth of the fluid $h(x, t)$, as well as its horizontal $v(x, z, t)$ and vertical $w(x, z, t)$ velocities, directed respectively along the axes $x$ and $z$. Fluid flow occurs in the field of gravity, creating the acceleration $g = (0, 0, -g)$, directed vertically downwards.

Set the space-time area

$$G = \{(x, z, t) : a(t) \leq x \leq b(t), \ 0 \leq z \leq h(x, t), \ t_1 \leq t \leq t_2\},$$

which in the time interval $[t_1, t_2]$ is occupied by the fluid in the channel between the moving vertical planes $x = a(t)$ and $x = b(t)$, where $a(t) < b(t)$. The integral continuity equation (mass conservation law) in the area (3) is

$$\int_a^b h(x, t) dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \int_0^{h(x, t)} v(x, z, t) dz \right) dt \bigg|_a^b = 0. \quad (4)$$

We introduce the following notation

$$V(x, t) = \frac{1}{h(x, t)} \int_0^{h(x, t)} v(x, z, t) dz$$

for the depth-averaged horizontal fluid velocity.

Using notation (5), we represent integral equation (4) as follows

$$\int_a^b h dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} hV dt \bigg|_a^b = 0. \quad (6)$$
Equation (6) is an integral mass conservation law for shallow water model. Note that in the derivation of this conservation law, we did not used any restrictions on the flow parameters. If the functions of depth $h(x,t)$ and averaged velocity $V(x,t)$ are smooth enough, from integral equation (6) follows differential form of the mass conservation law

$$ \frac{\partial h}{\partial t} + \frac{\partial (hV)}{\partial x} = 0. \tag{7} $$

3. Momentum conservation law

The conservation law of the fluid horizontal momentum in region (3), with taking into account formula (5), has the form

$$ \int_{h}^{b} hV dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \int_{h}^{0} \left( v^2 + p \right) dz dt \right) = 0, \tag{8} $$

where $p = p(x, z, t)$ is the specific pressure. Let us transform in formula (8) the integrals over the vertical coordinate $z$. For this we introduce the following designation $\tilde{v} = v - V$ for the deviation of the horizontal fluid velocity $v$ from its average vertical value $V$. Using this designation we find

$$ \int_{0}^{h} \tilde{v}^2 dz = hV^2 + \int_{0}^{h} \tilde{v}^2 dz. \tag{9} $$

Given the potentiality of the flow $\frac{\partial v}{\partial z} - \frac{\partial w}{\partial x} = 0$ we have

$$ \tilde{v} = v - V = \int_{\xi(V)}^{\xi} \frac{\partial w}{\partial x} dz, \tag{10} $$

where $\xi(V) = \xi(V(x,t))$ is a root of the equation $v(x, \xi(V(x,t)), t) = V(x,t)$. Substituting expression (10) in the right side integral of the formula (9) we find

$$ \int_{0}^{h} \tilde{v}^2 dz = \int_{0}^{\xi \left( \frac{\partial w}{\partial x} dz \right)^2 d\xi. \tag{11} $$

Taking into account the Euler equation for the vertical fluid velocity

$$ \frac{dw}{dt} + \frac{\partial p}{\partial z} + g = 0, $$

where

$$ \frac{dw}{dt} = \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z}, $$

we have

$$ \int_{0}^{h} pdz = \frac{gh^2}{2} + \int_{0}^{\xi} \int_{0}^{h} \frac{dw}{dt} dz d\xi. \tag{12} $$
It can be shown that using differential form of the continuity equation \( \partial v/\partial x + \partial w/\partial z = 0 \) and non-divergent form of equation (7)

\[
\frac{dh}{dt} + h \frac{\partial V}{\partial x} = 0, \quad \frac{dh}{dt} = \frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x},
\]

we can derive the following formula

\[
\int_0^h \int_\xi^h \frac{dw}{dt} dz d\xi = \frac{h^2}{3} \frac{d^2 h}{dt^2} + \int_0^h \int_\xi^h \frac{d\psi}{dt} dz d\xi,
\]

where

\[
\frac{d^2 h}{dt^2} = \frac{\partial}{\partial t} \left( \frac{dh}{dt} \right) + V \frac{\partial}{\partial x} \left( \frac{dh}{dt} \right), \quad \frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial x} + w \frac{\partial \psi}{\partial z},
\]

\[
\psi = \int_0^z \left( \frac{\partial}{\partial x} \int_\xi^\xi \frac{\partial w}{\partial x} dz \right) d\xi.
\]

4. Local hydrostatic approximation in the region containing an undular bore

Suppose that the flow under consideration contains an isolated undular bore propagating in the positive direction of the \( x \)-axis (Figure 1) and the front of this bore at the time \( t \) occupies the region

\[
W = \left\{ (x, z, t) : x_1(t) < x < x_2(t), \, \tilde{h}(x, t) < z < h(x, t) \right\},
\]

where \( \tilde{h}(x, t) > 0, \, \tilde{h}(x_i(t), t) = h(x_i(t), t) \), \( x_i'(t) > 0 \). As the characteristic depth of the flow, we choose the value \( H = (H_1 + H_2)/2 \), where \( H_1 \) is characteristic depth before the bore front and \( H_2 \) is ones behind the bore front. Since inequality (2) is not fulfilled at the bore front, it is impossible to determine correctly the characteristic length \( L \) of the surface waves, which is used in the condition (1) of the long-wave approximation. Therefore, to describe our flow, we apply the concept of local hydrostatic approximation introduced in [13, 14]. According to this concept the flow at the point \( A = (x, z, t) \) locally satisfies the hydrostatic approximation with a small parameter \( \delta \) if the following inequality holds

\[
w^2(A)/C^2 < \delta \ll 1,
\]

where \( C = \sqrt{gH} \) is characteristic speed of a small disturbances propagation in the shallow water.

Flux perturbations arising at the front of the undular bore form a sequence of damped surface waves propagating behind the bore front. If the characteristic length \( l \) of these waves is comparable to the characteristic depth of the flow, i.e. \( l = O(H) \), then a surface liquid layer is formed behind the bore front, inside which the condition (17) is not satisfied. The Green-Naghdi model does not effectively reproduce the structure of these undular waves, whereby such flows are more accurately approximated by the basic conservation laws of the first approximation of the shallow water theory [13]. Therefore, we will assume below that the characteristic length of undular waves \( l \gg H \) and condition (17) of the local hydrostatic approximation is satisfied at all fluid points \( A \notin W \).

For an approximate simulation of the considered flow, we (following [13, 14]) introduce the dimensionless variables

\[
x^* = \frac{\sqrt{\delta}}{H} x, \quad z^* = \frac{z}{H}, \quad t^* = \frac{\sqrt{\delta} C}{H} t, \quad h^* = \frac{h}{H}, \quad v^* = \frac{v}{C}, \quad p^* = \frac{p}{C^2}, \quad w^* = \frac{w}{\sqrt{\delta} C},
\]

\[
(18)
\]
Figure 1. Ondular bore propagating in the positive direction of the $x$-axis.

which, as a result of the substitution of $\delta = H^2/L^2$, transform into dimensionless variables of long-wave approximation (1), where the value of $L$ can be considered as a characteristic wavelength in the areas before and behind the front of the undular bore. Further, along with (18), we will use the notation

$$V^* = \frac{1}{h^*} \int_0^{h^*} v^* dz^* = \frac{1}{C} \left( \frac{1}{h} \int_0^{h} v dz \right) = \frac{V}{C} \quad (19)$$

for a dimensionless horizontal velocity averaged over the fluid depth.

It can be shown that the conservation laws (4), (6)–(8) and equation (9) have the same form when they are written in dimensionless variables (18), (19), but equations (11) and (12) take the form

$$\int_0^{h} \tilde{v}^2 \, dz = \delta^2 \int_0^{h} \left( \int_0^{\xi(V)} \frac{\partial w}{\partial x} \, dz \right)^2 \, d\xi, \quad \int_0^{h} p \, dz = \frac{h^2}{2} + \delta \int_0^{h} \int_0^{h} \frac{d\tilde{w}}{dt} \, dz \, d\xi, \quad (20)$$

where the superscript asterisk here and everywhere further is omitted for brevity. In this variables the equation (14) with taking into account formula (15) takes the form

$$\int_0^{h} \int_0^{h} \frac{dw}{dt} \, dz \, d\xi = \frac{h^2}{3} \frac{d^2h}{dt^2} + \delta \int_0^{h} \frac{dw}{dt} \left( \int_0^{Z(V)} \frac{\partial w}{\partial x} \int_0^{\xi} \frac{\partial w}{\partial x} \, dz \right) \, d\xi \, dz \, d\xi. \quad (21)$$

It follows from the formulas (20) and (21) that integral equation (8) with an accuracy of $O(\delta^2)$ is transformed into the integral conservation law of the total momentum for the Green-Naghdi model

$$\int_a^b hV \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_0^{h} \left( hV^2 + \frac{h^2}{2} + \delta \frac{h^2}{3} \frac{d^2h}{dt^2} \right) \, dz \, dt \bigg|_a^b = 0. \quad (22)$$
5. On the numerical simulation of the undular bore

If the lateral boundaries \( x = a(t) \) and \( x = b(t) \) of the integration region (3) do not intersect with the domain (16) in which the bore front is located, i.e. they satisfy one of the three conditions

1) \( b(t) < x_1(t) \),  
2) \( x_2(t) < a(t) \),  
3) \( a(t) < x_1(t) < x_2(t) < b(t) \),

then integral Green-Naghdi equation (22) approximates by a small parameter \( \delta \) the integral conservation law of horizontal momentum (8). If one of this lateral boundaries intersects with the domain (16), i.e. it satisfies one of the two conditions

1) \( x_1(t) < a(t) < x_2(t) \),  
2) \( x_1(t) < b(t) < x_2(t) \),

then there is no such approximation in general. In this case, in order for equation (22) to approximate equation (8), an additional assumption is needed about the linear dimensions of the integration region (3). If under one of the conditions (24) the length \( h = O(H) \) of the integration segment for \( z \)-axis is much less than the length \((b - a)\) of the integration segment for \( x \)-axis, i.e. the inequality

\[
\Delta = \frac{H}{b-a} \ll 1
\]

is satisfied, then integral equation (22) approximates the space two-dimensional integral equation (8) with relative error \( O(\Delta) \).

The consequence of the integral equation (22) in the regions \( x < x_1(t) \) and \( x > x_2(t) \) outside the bore front (16) is the differential form of the total-momentum conservation law

\[
\frac{\partial (hU)}{\partial t} + \frac{\partial}{\partial x} \left( hV^2 + \frac{h^2}{2} + \frac{\delta h^2}{3} \frac{d^2 h}{dt^2} \right) = 0,
\]

where taking into account formulas (13) we have

\[
\frac{d^2 h}{dt^2} = h \left( \left( \frac{\partial V}{\partial x} \right)^2 - \frac{\partial^2 V}{\partial x \partial t} - V \frac{\partial^2 V}{\partial x^2} \right).
\]

From equation (25) with the help of the differential form of the mass conservation law (7) we receive the equation describing the change of the fluid velocity in the Green-Naghdi model

\[
\frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left( \frac{V^2}{2} + h \right) + \frac{\delta}{3h} \frac{\partial}{\partial x} \left( h^2 \frac{d^2 h}{dt^2} \right) = 0.
\]

The system of equations (7), (26) was obtained in [1–3] from the continuity and Euler equations, taking into account the assumption of a linear dependence of the vertical fluid velocity \( w \) on the coordinate \( z \). The disadvantage of this system is that the finite-difference schemes constructed on its basis (see, for example, [10]) generally non-divergently approximate the differential form of total-momentum conservation law (25). This disadvantage is absent in conservative finite-difference schemes [11], which directly approximate divergent equations (7), (25). In many papers, for example [10, 11], the systems (7), (25) and (7), (26) are used for the numerical simulation of undular bores, arising, in particular, when calculating the dam-break problem. However, on the interval \((x_1(t), x_2(t))\), over which the bore front is located (16), equations (25) and (26) are not a consequence of the integral conservation law of total momentum (22), by virtue of which these equations do not describe the real flow on this interval. As a result, finite-difference schemes based on them do not allow to reproduce correctly the ondulations appearing behind the bore front.
Numerical algorithms designed to simulate undular bores need to be construct in some $(x, t)$-plane neighborhood

$$W_{\varepsilon} = \{(x, t) : x_1(t) - \varepsilon < x < x_2(t) + \varepsilon\}, \quad (27)$$

of the bore front (16) by approximating the integral conservation law of total momentum (22) written relative to regions (3) for which $b - a > x_2 - x_1$ at the third condition (23), $b - a \gg H$ at one of the conditions (24) and $t_1, t_2$ are successive layers of the difference grid. If integration region (3) contains the bore front, i.e. satisfies the third condition (23), the corresponding conservation law (22), with an accuracy of $O(\delta^2)$, coordinates the flow parameters before and behind the bore front, thus representing an analogue of the Rankine-Hugoniot conditions in the hyperbolic model of the first approximation of the shallow water theory. If the right boundary $x = b(t)$ of the integration region (3) lies inside the bore front and the inequality $b - a \gg H$ fulfilled then the corresponding conservation law (22), with an accuracy of $O(\Delta)$, should facilitate the reproduction of a sequence of damped undular waves behind the bore front.

Since the integral mass conservation law (6) in the shallow water model is derived without any restrictions on the flow parameters, its differential form (7) is performed in the entire flow region, including the bore front (16). Therefore, for the mass conservation law we can use a conservative finite-difference approximation of the divergent differential equation (7). The numerical algorithm obtained in this way at the stage of approximation of the integral momentum conservation law (22) will be implicit and for its implementation it is necessary to use multipoint grid sweeps with nonlinearity iterations on the upper time layer. In the numerical simulation of complex fluid flows with undular bores, we can use the method of combined finite-difference schemes [15, 16] in which a finite-difference approximation of the differential Green-Naghdi model (7), (25) is used in an external region outside the $\varepsilon$-neighborhood (27) of the bore front, and a corresponding approximation of the integro-differential version of this model (7), (22) is used inside the $\varepsilon$-neighborhood (27).

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