COMBINATORICS OF REGULAR PARTITIONS

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Abstract. Two partition identities are given, which are concerned with $r$-regular partitions and $r$-class regular partitions. Together with some discussions on the Hall-Littlewood and the Kostka-Foulkes polynomials, these formulas lead to the computation of the $r$-regular character table of the symmetric group.

1. Introduction

Let $r$ be an integer greater than 1. A partition is said to be $r$-regular if no part is repeated $r$ or more times, and is said to be $r$-class regular if no part is divisible by $r$. In this short note we present an “$r$-congruence property” of $r$-regular / $r$-class regular partitions (Theorem 2.1), and an “$r$-adic property” of $r$-class regular partitions (Theorem 3.1). We will prove these formulas by manipulation of generating functions.

Next we consider the ordinary character table of the symmetric group $S_n$. More precisely we compute the determinant of the “regular character table”, which consists of the values of irreducible characters corresponding to the $r$-regular partitions on $r$-regular conjugacy classes. Determinant of the regular character table has been computed by Olsson [6] (see also [1]). We will give another proof of Olsson’s result, by examining the transition matrices of the Hall-Littlewood symmetric functions and the Schur functions. When $r$ is prime, it is known that the determinant of the regular character table equals that of the $r$-Brauer character table.

2. Regular partitions

Throughout this note, we fix a positive integer $r \geq 2$. For $n \geq 0$, let $P_n$ be the set of partitions of $n$. We denote by $m_i(\lambda)$ the multiplicity of $i$ in $\lambda \in P_n$. A partition $\lambda$ is said to be an $r$-regular if $m_i(\lambda) < r$ holds for any $i \geq 1$. A partition $\rho$ is said to be an $r$-class regular if all parts are not divisible by $r$. Let $RP_{r,n}$ (resp. $CP_{r,n}$) be the set of $r$-regular (resp. $r$-class regular) partitions. A classical result says that $RP_{r,n}$ and $CP_{r,n}$ have the

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same cardinality. The generating function
\[ \Phi_r(q) = \sum_{n \geq 0} |RP_{r,n}| q^n = \sum_{n \geq 0} |CP_{r,n}| q^n \]
has a product expression
\[ \Phi_r(q) = \prod_{k \geq 1} \frac{1 - q^{rk}}{1 - q^k}. \]

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( j \in \{1, 2, \ldots, r - 1\} \), we put
\[ x_{r,j}(\lambda) = |\{i \mid \lambda_i \equiv j \pmod{r}\}| \quad \text{and} \quad y_{r,j}(\lambda) = |\{i \mid m_i(\lambda) \geq j\}|. \]

We define
\[ X_{r,j,n} = \sum_{\rho \in CP_{r,n}} x_{r,j}(\rho) \quad \text{and} \quad Y_{r,j,n} = \sum_{\lambda \in RP_{r,n}} y_{r,j}(\lambda). \]

**Theorem 2.1.** The number \( X_{r,j,n} - Y_{r,j,n} \) is a non-negative integer independent for \( j = 1, 2, \ldots, r - 1 \), and equal to \( c_{r,n} \), the coefficient of \( q^n \) in
\[ \Phi_r(q) \sum_{k \geq 1} \frac{q^{rk}}{1 - q^{rk}}. \]

Before proving this theorem, we give an example.

**Example 2.2.** We take \( r = 3 \) and \( n = 7 \). The following table lists the 3-class regular partitions of 7:

| \( \rho \) | 7 | 52 | 51^2 | 421 | 41^3 | 2^3 | 2^2 | 3 | 1^7 | total |
|----------|---|----|------|-----|------|-----|-----|---|----|-------|
| \( x_{3,1}(\rho) \) | 1 | 0 | 2 | 2 | 4 | 1 | 3 | 5 | 7 | 25 |
| \( x_{3,2}(\rho) \) | 0 | 2 | 1 | 1 | 0 | 3 | 2 | 1 | 0 | 10 |

From the table above, we have \( X_{3,1,7} = 25 \) and \( X_{3,2,7} = 10 \). As for the 3-regular partitions of 7, we have

| \( \lambda \) | 7 | 61 | 52 | 51^2 | 52 | 421 | 3^2 | 32^2 | 321^2 | total |
|-------------|---|----|----|------|----|-----|-----|------|-------|-------|
| \( \prod_{i \geq 1} m_i(\lambda)! \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 21 | 1 | 1 | 1 | 1 | 121 | \( - \) |
| \( y_{3,1}(\lambda) \) | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 2 | 3 | 19 |
| \( y_{3,2}(\lambda) \) | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 4 |

From the second table, we have \( Y_{3,1,7} = 19 \) and \( Y_{3,2,7} = 4 \). Thus we see
\[ X_{3,1,7} - Y_{3,1,7} = X_{3,2,7} - Y_{3,2,7} = 6. \]

On the other hand we have
\[ \Phi_3(q) \sum_{k \geq 1} \frac{q^{3k}}{1 - q^{3k}} = q^3 + q^4 + 2q^5 + 4q^6 + 6q^7 + 9q^8 + 13q^9 + 19q^{10} + \cdots. \]
Proof. First we will compute the generating function of $X_{r,j,n}$. For $i \not\equiv 0 \pmod{r}$, we have

$$\Phi_r(q) \frac{1 - q^i}{1 - tq^i} = \sum_{n \geq 0} \left( t^{\sum_{\rho \in CP_{r,n}} m_i(\rho)} \right) q^n. \tag{2.1}$$

Taking the $t$-derivative at $t = 1$, we obtain

$$\Phi_r(q) \frac{q^i}{1 - q^i} = \sum_{n \geq 0} \left( \sum_{\rho \in CP_{r,n}} m_i(\rho) \right) q^n. \tag{2.1}$$

Since $x_{r,j}(\rho) = \sum_{k \geq 0} m_{kr+j}(\rho)$, we have the following generating function $X_{r,j}(q)$ of $X_{r,j,n}$.

$$X_{r,j}(q) = \sum_{n \geq 0} X_{r,j,n} q^n = \Phi_r(q) \sum_{k \geq 0} \frac{q^{rk+j}}{1 - q^{rk+j}}. \tag{2.2}$$

Second, we consider the $r$-regular partitions and the generating function of $Y_{r,j,n}$. We put

$$\Phi_{r,j}(q,t) = \prod_{k \geq 1} (1 + q^k + q^{2k} + \ldots + q^{(j-1)k} + tq^{j+k} + tq^{(j+1)k} + \ldots + tq^{(r-1)k}). \tag{2.2}$$

Immediately we have

$$\Phi_{r,j}(q,t) = \sum_{n \geq 0} \left( \sum_{\lambda \in RP_{r,n}} y_{r,j}(\lambda) \right) q^n. \tag{2.3}$$

Taking the $t$-derivative at $t = 1$, we obtain

$$\left. \frac{d}{dt} \Phi_{r,j}(q,t) \right|_{t=1} = \sum_{n \geq 0} \left( \sum_{\lambda \in RP_{r,n}} y_{r,j}(\lambda) \right) q^n = \sum_{n \geq 0} Y_{r,j,n} q^n. \tag{2.4}$$

As for the equation (2.2), we have

$$\left. \frac{d}{dt} \Phi_{r,j}(q,t) \right|_{t=1} = \Phi_{r,j}(q,t) \sum_{k \geq 1} \frac{q^{jk} - q^{rk}}{1 - q^{rk}}. \tag{2.5}$$

The generating function $Y_{r,j}(q)$ of $Y_{r,j,n}$ reads

$$Y_{r,j}(q) = \sum_{n \geq 0} Y_{r,j,n} q^n = \Phi_r(q) \sum_{k \geq 1} \frac{q^{jk} - q^{rk}}{1 - q^{rk}}. \tag{2.6}$$
To complete the proof, we compute
\[ X_{r,j}(q) - Y_{r,j}(q) = \Phi_r(q) \left( \sum_{k \geq 0} (q^{rk+j} + q^{2(rk+j)} + q^{3(rk+j)} + \cdots) \right) \]
\[ - \sum_{m \geq 1} (q^{jm} + q^{(r+j)m} + q^{(2r+j)m} + \cdots) + \sum_{k \geq 1} \frac{q^r}{1 - q^{rk}} \]
\[ = \Phi_r(q) \left( \sum_{k \geq 0} \sum_{m \geq 1} q^{m(rk+j)} - \sum_{m \geq 1} \sum_{k \geq 0} q^{(kr+j)m} + \sum_{k \geq 1} \frac{q^r}{1 - q^{rk}} \right) \]
\[ = \Phi_r(q) \sum_{k \geq 1} \frac{q^r}{1 - q^{rk}} = \sum_{n \geq 0} c_{r,n} q^n. \]

The concluding \(q\)-series is independent of \(j\). Therefore \(X_{r,j,n} - Y_{r,j,n}\) does not depend on the choice of \(j\). \(\square\)

3. Class regular partitions

We define, for \(j \geq 1\),
\[ V_{r,j,n} = \sum_{\rho \in CP_{r,n}} m_j(\rho) \text{ and } W_{r,j,n} = \sum_{\rho \in CP_{r,n}} y_{r,j}(\rho). \]

Note that the sum is taken over \(r\)-class regular partitions of \(n\) both in \(V_{r,j,n}\) and \(W_{r,j,n}\).

**Theorem 3.1.** For \(j \not\equiv 0 \pmod{r}\), we have
\[ V_{r,j,n} = \sum_{i \geq 0} W_{r,ri,j,n}. \]

Before proving this theorem, we give an example.

**Example 3.2.** There are sixteen 4-class regular partitions of \(n = 8\). The following table lists \(V_{r,j,n}\) and \(W_{r,j,n}\) of them:

| \(j\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| \(V_{4,j,8}\) | 38 | 16 | 8 | 0 | 3 | 2 | 1 | 0 |
| \(W_{4,j,8}\) | 33 | 15 | 8 | 5 | 3 | 2 | 1 | 1 |

We have \(V_{4,1,8} = W_{4,1,8} + W_{4,4,8}\) and \(V_{4,2,8} = W_{4,2,8} + W_{4,8,8}\).

**Proof.** From (2.1), we have the generating function of \(V_{r,j,n}\):
\[ \sum_{n \geq 0} V_{r,j,n} q^n = \Phi_r(q) \frac{q^j}{1 - q^j}. \]
Let $k \not\equiv 0 \pmod{r}$. We have
\[
\sum_{n \geq 0} \{ \rho \in CP_{r,n} \mid m_k(\rho) \geq j \} \cdot q^n = \Phi_r(q)(1 - q^k)(q^{jk} + q^{(j+1)k} + q^{(j+2)k} + \ldots)
\]
\[= \Phi_r(q)q^{jk}.
\]
Thus we have the generating function of $W_{r,j,n}$:
\[
\sum_{n \geq 0} W_{r,j,n} q^n = \Phi_r(q) \sum_{k \not\equiv 0 \pmod{r}} q^{jk} = \Phi_r(q) \sum_{k \geq 1} (q^{jk} - q^{rkj})
\]
\[= \Phi_r(q) \left( \frac{q^j}{1 - q^j} - \frac{q^{rj}}{1 - q^{rj}} \right).
\]
Hence the generating function of RHS of the claim reads
\[
\sum_{n \geq 0} \left( \sum_{i \geq 0} W_{r,r^i,j,n} \right) q^n = \Phi_r(q) \sum_{i \geq 0} \left( \sum_{n \geq 0} \Phi_r(q) \left( \frac{q^{r^i j}}{1 - q^{r^i j}} - \frac{q^{r^{i+1} j}}{1 - q^{r^{i+1} j}} \right) \right)
\]
\[= \Phi_r(q) \frac{q^j}{1 - q^j}.
\] (3.1)
as desired. \hfill \Box

Theorem 3.1 implies the following formulas.

**Corollary 3.3.**

1. $c_{r,n} = \sum_{j \not\equiv 0 \pmod{r}} \sum_{i \geq 1} iW_{r,r^i,j,n}$.

2. \[
\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} m_i(\rho)! = \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i.
\]

**Proof.** Since $V_{r,j,n} = 0$ for any $j \not\equiv 0 \pmod{r}$, we have
\[
\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i = \prod_{j \geq 1} j^{V_{r,j,n}} = \prod_{j \not\equiv 0 \pmod{r}} j^{V_{r,j,n}}.
\]
On the other hand we compute
\[
\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} m_i(\rho)! = \prod_{j \geq 1} j^{W_{r,j,n}} = \prod_{j \not\equiv 0 \pmod{r}} \prod_{i \geq 0} j^{W_{r,r^i,j,n}}
\]
\[= \left( \prod_{j \not\equiv 0 \pmod{r}} \prod_{i \geq 0} r^i j^{W_{r,r^i,j,n}} \right) \times \left( \prod_{j \not\equiv 0 \pmod{r}} \prod_{i \geq 0} j^{W_{r,r^i,j,n}} \right)
\]
\[= r^{d_{r,n}} \prod_{j \not\equiv 0 \pmod{r}} j^{\sum_{i \geq 0} W_{r,r^i,j,n}}
\]
\[= r^{d_{r,n}} \prod_{j \not\equiv 0 \pmod{r}} j^{V_{r,j,n}},
\] as desired. \hfill \Box
where we have put

\[ d_{r,n} = \sum_{j \not\equiv 0 \pmod{r}} \sum_{i \geq 1} iW_{r,r^i,j,n}. \]

We verify that \( d_{r,n} = c_{r,n} \) as follows.

\[
\sum_{n \geq 0} \left( \sum_{i \geq 0} iW_{r,r^i,j,n} \right) q^n = \sum_{i \geq 0} i \left( \sum_{n \geq 0} W_{r,r^i,j,n} q^n \right) = \sum_{i \geq 0} i \Phi_r(q) \left( \frac{q^{r^i j}}{1 - q^{r^i j}} - \frac{q^{r^{i+1} j}}{1 - q^{r^{i+1} j}} \right) = \Phi_r(q) \sum_{i \geq 1} \frac{q^{r^i j}}{1 - q^{r^i j}}.
\]

Now we take sum over \( j \not\equiv 0 \pmod{r} \) to have the generating function of \( d_{r,n} \).

\[
\sum_{n \geq 0} d_{r,n} q^n = \sum_{j \not\equiv 0 \pmod{r}} \Phi_r(q) \left( \sum_{i \geq 1} \frac{q^{r^i j}}{1 - q^{r^i j}} \right) = \Phi_r(q) \sum_{n \geq 1} \frac{q^{r^n}}{1 - q^{r^n}} = \sum_{n \geq 0} c_{r,n} q^n.
\]

The formula (2) in Corollary 3.3 is due to Olsson [6]. In the case of prime \( r \), arithmetic of \( r \)-regular / \( r \)-class regular partitions is closely studied by Bessenrodt and Olsson [1] in connection with the modular representations of the symmetric groups.

4. Regular partitions and Hall-Littlewood symmetric functions

In this section, we apply Theorem 2.1 to computations of some minor determinants of transition matrices and the character tables of the symmetric groups.

4.1. Hall-Littlewood symmetric functions at root of unity. The Hall-Littlewood \( P \)- and \( Q \)-symmetric functions (4) are a one parameter family of symmetric functions satisfying the orthogonality relation:

\[
\langle P_{\lambda}(x; t), Q_{\mu}(x; t) \rangle_t = \delta_{\lambda\mu},
\]

where the inner product \( \langle \cdot, \cdot \rangle_t \) is defined by \( \langle p_{\lambda}(x), p_{\mu}(x) \rangle_t = z_{\lambda}(t) \delta_{\lambda\mu} \) with \( z_{\lambda}(t) = z_{\lambda} \prod_{i \geq 1} (1 - t^{\lambda_i})^{-1} \). Let \( (a; t)_n \) be a \( t \)-shifted factorial:

\[
(a; t)_n = \begin{cases} (1 - a)(1 - at) \cdots (1 - at^{n-1}) & (n \geq 1) \\ 1 & (n = 0). \end{cases}
\]
The relation between $P$- and $Q$- functions is described as

$$Q_\lambda(x) = b_\lambda(t)P_\lambda(x),$$

where $b_\lambda(t) = \prod_{i \geq 1}(t; t)_{m_i(\lambda)}$.

4.2. $Q'$-functions. In this note, we are interested in the case that parameter $t$ is a primitive $r$-th root of unity $\zeta$. The Hall-Littlewood symmetric functions at root of unity is studied at the first time by [5]. We remark that $\{Q_\lambda(x; \zeta) \mid \lambda \in RP_{r,n}\}$ is a $Q(\zeta)$-basis for the subspace $\Lambda^{(r)} = Q(\zeta)[p_s(x) \mid s \neq 0 \pmod{r}]$ of the symmetric function ring $\Lambda = Q(\zeta)[p_s(x) \mid s = 1, 2, \ldots]$. This can be shown along the arguments in [4, Chap. 3-8], where the case $r = 2$ is discussed. In [3], Lascoux, Leclerc and Thibon consider the dual basis $(Q'_\lambda)$ of $P$-functions, relative to the inner product at $t = 0$. Namely $P$- and $Q'$-functions satisfy the Cauchy identity:

$$\sum_\lambda P_\lambda(x; t)Q'_\lambda(y; t) = \prod_{i, j}(1 - x_iy_j)^{-1}.$$  

When $t = \zeta$, the $Q'$-functions have the following nice factorization property.

**Proposition 4.1** ([3]). Let $\zeta$ be a primitive $r$-th root of unity. If a partition $\lambda$ satisfies $m_i(\lambda) \geq r$, then we have

$$Q'_\lambda(x; \zeta) = (-1)^{i(r-1)}Q'_{\lambda \setminus (i^r)}(x; \zeta)h_i(x^r).$$

Here $h_i(x^r) = h_i(x_1^r, x_2^r, \ldots)$ and $\lambda \setminus (i^r)$ is a partition obtained by removing the rectangle $(r^i)$ from the Young diagram $\lambda$.

We define an $r$-reduction for a symmetric function $f(x)$ by

$$f^{(r)}(x) = f(x)|_{p_r(x)=p_{2r}(x)=p_{3r}(x)=\ldots=0}.$$  

Proposition 4.1 leads us to the following lemma.

**Lemma 4.2.** $Q_\lambda^{(r)}(x; \zeta) = 0$ unless $\lambda$ is an $r$-regular partition.

We set

$$Q_\lambda(x; \zeta) = \sum_{\rho \in CP_{r,n}} Q_\rho^\lambda P_\rho(x) \quad \text{and} \quad Q^{(r)}_\lambda(x; \zeta) = \sum_{\rho \in CP_{r,n}} Q^\lambda P_\rho(x).$$  

**Proposition 4.3.** Let $\lambda \in RP_{r,n}$ and $\rho \in CP_{r,n}$. We have

$$Q^{(r)}_\rho = \prod_{i \geq 1}(1 - \zeta^i)^{-1}Q^\lambda_\rho.$$
Proof. We compute inner products at $t = \zeta$ and $t = 0$ for $r$-regular partitions $\lambda$ and $\mu$. Namely, we see

$$\delta_{\lambda \mu} = \langle P_\lambda(x; \zeta), Q_\mu(x; \zeta) \rangle_\zeta = b_\lambda(\zeta)^{-1} \sum_{\rho \in CP_{r,n}} Q_\rho Q_\rho^\mu z_\rho(\zeta)$$

and

$$\delta_{\lambda \mu} = \langle P_\lambda(x; \zeta), Q_\mu^{(r)}(x; \zeta) \rangle_0 = b_\lambda(\zeta)^{-1} \sum_{\rho \in CP_{r,n}} Q_\rho Q_\rho^\mu z_\rho.$$

Since $\{P_\lambda(x; \zeta) \mid \lambda \in RP_{r,n}\}$ is also a basis of $\Lambda^{(r)}$, we have the claim. \hfill \Box

We define $L_{\lambda \mu}(t)$ by

$$s_\lambda(x) = \sum_{\mu \in P_n} L_{\lambda \mu}(t)Q_\mu(x; t),$$

where $s_\lambda(x)$ denotes the Schur function. Let $K_{\lambda \mu}(t)$ be the Kostka-Foulkes polynomial \cite{[4]}. In other words, the matrix $K(t) = (K_{\lambda \mu}(t))_{\lambda, \mu \in P_n}$ is the transition matrix $M(s, P)$ from the Schur functions to the Hall-Littlewood $P$-functions. It is known that $K(t)$ is an upper unitriangular matrix.

**Lemma 4.4.** For partitions $\lambda$ and $\mu$, we have $L_{\lambda \mu}(t) = K_{\mu \lambda}^{(-1)}(t)$, the $(\lambda, \mu)$-entry of the matrix $K(t)^{-1}$.

**Proof.**

$$L_{\lambda \mu}(t) = \langle s_\lambda(x), P_\mu(x; t) \rangle_0 = \langle s_\lambda(x), \sum_{\nu \in P_n} K_{\mu \nu}^{(-1)} s_\nu \rangle_0 = K_{\mu \lambda}^{(-1)}(t).$$

\hfill \Box

**Example 4.5.** We take $\zeta = -1$ and $n = 4$. Then we have

$$s_4(x) = Q_4'(x; -1),$$
$$s_{31}(x) = Q_{31}'(x; -1) + Q_4'(x; -1),$$
$$s_{22}(x) = Q_{22}'(x; -1) + Q_{31}'(x; -1),$$
$$s_{211}(x) = Q_{211}'(x; -1) + Q_{22}'(x; -1) + Q_{31}'(x; -1) + Q_4'(x; -1),$$
$$s_{1111}(x) = Q_{1111}'(x; -1) + Q_{211}'(x; -1) - Q_{22}'(x; -1) + Q_4'(x; -1).$$

Lemma \cite{[42]} and \cite{[44]} give the following expansion formula.

**Proposition 4.6.** Let $\lambda \in P_n$ and $\mu \in RP_{r,n}$. We have

$$s_\lambda^{(r)}(x) = \sum_{\mu \in RP_{r,n}} K_{\mu \lambda}^{(-1)}(\zeta) Q_\mu^{(r)}(x; \zeta).$$

In particular, $(L_{\lambda \mu}(\zeta))_{\lambda, \mu \in RP_{r,n}}$ is a lower unitriangular matrix.
Example 4.7. By Proposition 4.6, we immediately see

\[
\begin{align*}
    s_{1}^{(2)}(x) &= Q_{4}^{(2)}(x; -1), \\
    s_{31}^{(2)}(x) &= Q_{31}^{(2)}(x; -1) + Q_{4}^{(2)}(x; -1), \\
    s_{22}^{(2)}(x) &= Q_{31}^{(2)}(x; -1), \\
    s_{211}^{(2)}(x) &= Q_{31}^{(2)}(x; -1) + Q_{4}^{(2)}(x; -1), \\
    s_{1111}^{(2)}(x) &= Q_{4}^{(2)}(x; -1).
\end{align*}
\]

From the first two equations above, we have

\[
(L_{\lambda \mu}(-1))_{\lambda, \mu \in \text{RP}_{2,4}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

We set

\[
    s^{(r)} = \{ s_{\lambda}^{(r)}(x) \mid \lambda \in \text{RP}_{r,n} \},
    Q^{(r)} = \{ Q_{\lambda}^{(r)}(x) \mid \lambda \in \text{RP}_{r,n} \}
\]

and

\[
P^{(r)} = \{ p_{\lambda}(x) \mid \lambda \in \text{CP}_{r,n} \}.
\]

For \( u, v \in \{ s^{(r)}, Q^{(r)}, p^{(r)} \} \), we denote by \( M(u, v) \) the transition matrix from \( u \) to \( v \). By Lemma 4.2, we have that \( M(s^{(r)}, Q^{(r)}) \) is obtained by removing non-\( r \)-regular rows and columns from the transposed inverse of \( K(t) \).

**Theorem 4.8.** We have \( \det M(Q^{(r)}, p^{(r)}) \in \mathbb{R} \) and

\[
\det M(Q^{(r)}, p^{(r)}) = \pm \frac{1}{r^{\text{cr}_{r,n}} \prod_{\rho \in \text{CP}_{r,n}} \prod_{i \geq 1} \rho_{i}}.
\]

**Proof.** The orthogonality relation of \( P_{\lambda}(x; \zeta) \) and \( Q_{\mu}^{(r)}(x; \zeta) \):

\[
\delta_{\lambda \mu} = \langle P_{\lambda}(x; \zeta), Q_{\mu}^{(r)}(x; \zeta) \rangle_{0} = b_{\lambda}(\zeta)^{-1} \sum_{\rho \in \text{CP}_{r,n}} Q_{\rho}^{\lambda} Q_{\rho}^{(r)} z_{\rho},
\]
and Proposition 4.3 give

\[
\det M(Q^{(r)}, p^{(r)})^2 = \left( \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \frac{1}{1 - \zeta^i} \right)^2 \frac{\prod_{\lambda \in RP_{r,n}} b_{\lambda}(\zeta)}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\zeta)}
\]

\[
= \frac{\prod_{\lambda \in RP_{r,n}} b_{\lambda}(\zeta)}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i))}
\]

\[
= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i)^{m_{r,n}(\lambda)}}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i))}
\]

\[
= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i)^{r_{r,n}(\lambda)}}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i))}
\]

\[
= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i)^{Y_{r,j,n}}}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i))}
\]

\[
= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i)^{X_{r,j,n}}}{\prod_{\rho \in CP_{r,n}} z_{\rho}(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^i))}
\]

We apply Theorem 2.1 to the last equality above. By noticing \(\prod_{i=1}^{r}(1 - \zeta^i) = r\) and using Corollary 3.3 (2), we obtain the formula. \(\square\)

4.3. Regular character tables of the symmetric groups. Let \(T_n = (\chi_{\lambda}^\rho)_{\lambda, \rho \in P_n}\) be the ordinary character table of the symmetric group \(S_n\). The orthogonality relation of the characters implies

\[
(\det T_n)^2 = \prod_{\rho \in P_n} z_{\rho}.
\]

From James’s book [2, Corollary 6.5], this formula can be simplified as

\[
(\det T_n)^2 = \prod_{\rho \in P_n} \rho_i^2.
\]

Olsson considers the \(r\)-regular character table \(T_n^{(r)} = (\chi_{\lambda}^\rho)_{\lambda \in RP_{r,n}, \rho \in CP_{r,n}}\) and computes its determinant. He proves the following theorem.

Theorem 4.9 ([6]).

\[
\det T_n^{(r)} = \pm \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i.
\]
Proof. Theorem 4.8 and Proposition 4.6 enable us to compute the determinant of the regular character table as follows:
\[
det M(s^{(r)}, p^{(r)})^2 = \det M(s^{(r)}, Q^{(r)})^2 \det M(Q^{(r)}, p^{(r)})^2 = 1 \times \frac{1}{r^{2e_{r,n}} \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i^2}
\]

Since
\[
det M(s^{(r)}, p^{(r)})^2 = (\det T_n^{(r)})^2 \times \prod_{\rho \in CP_{r,n}} z_{\rho}^{-2} = (\det T_n^{(r)})^2 \times r^{-2e_{r,n}} \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i^{-4},
\]
we have
\[
(\det T_n^{(r)})^2 = \left( \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i \right)^2.
\]

□

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