ON THE LORENZ ’96 MODEL AND SOME GENERALIZATIONS

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Abstract. In 1996, Edward Lorenz introduced a system of ordinary differential equations that describes a scalar quantity evolving on a circular array of sites, undergoing forcing, dissipation, and rotation invariant advection. Lorenz constructed the system as a test problem for numerical weather prediction. Since then, the system has also found use as a test case in data assimilation. Mathematically, this is a dynamical system with a single bifurcation parameter (rescaled forcing) that undergoes multiple bifurcations and exhibits chaotic behavior for large forcing. In this paper, the main characteristics of the advection term in the model are identified and used to describe and classify possible generalizations of the system. A graphical method to study the bifurcation behavior of constant solutions is introduced, and it is shown how to use the rotation invariance to compute normal forms of the system analytically. Problems with site-dependent forcing, dissipation, or advection are considered and basic existence and stability results are proved for these extensions. We address some related topics in the appendices, wherein the Lorenz ’96 system in Fourier space is considered, explicit solutions for some advection-only systems are found, and it is demonstrated how to use advection-only systems to assess numerical schemes.

1. Introduction. The Lorenz ’96 (L96) system was introduced by Edward Lorenz in 1996 [18] as a toy model for studying predictability, especially in weather and atmospheric systems. It is most often used as a test case for new data assimilation and ensemble forecasting techniques [19], but has also been used to study some general aspects of chaos, turbulence, and linear response theory [21, 10]. For these applications, the system has been considered mostly far into the chaotic regime in order to replicate properties of turbulent flow. However, there is a rich bifurcation structure as the system approaches chaos. Until recent years, there was little in the literature related directly to the mathematics of the model itself [12]. As recognized by Lorenz and subsequent researchers, the L96 system lends itself to a number of modifications [20, 26, 10]. The main purpose of this paper is to introduce and study some new modifications motivated by fundamental properties of the system. This will lead to broad classes of L96-like systems for which we present some analytical and numerical tools.

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1.1. The L96 system. The model is a system of coupled ordinary differential equations, which describes the time evolution of some scalar atmospheric quantity. It includes three characteristic processes of atmospheric systems: advection, dissipation, and external forcing. The model is defined as follows. Consider a circular array of \( N \) sites that are labeled \( i = 0, 1, 2, \ldots, N - 1 \). The labels \( i \leq -1 \) and \( i \geq N \) are extended periodically, identifying site 0 with site \( N \), site \( -1 \) with site \( N - 1 \), and so on. Associated with site \( i \) at time \( t \) is a quantity \( X_i(T) \). These quantities are governed by the system of differential equations

\[
\frac{d}{dT} X_i(T) = \alpha X_{i-1}(T)(X_{i+1}(T) - X_{i-2}(T)) - \beta X_i(T) + \gamma
\]  

(1)

where \( \alpha, \beta, \gamma \) are fixed positive constants. In the standard interpretation of the model, the sites are imagined to be equally spaced points circling the globe latitudinally. The quantity in question is added to each site at the rate \( \gamma \) per time unit, via the forcing term. During short time intervals of length \( \Delta T \), an approximate fraction \( \beta \Delta T \) of the quantity present at a given site is destroyed or dissipated, via the dissipation term, and an amount that is proportional to \( \alpha \Delta T \) enters site \( i \) and an amount proportional to \( \Delta T \) leaves site \( i \), with common proportionality factor \( \alpha \), via the advection term.

The effective number of parameters may be reduced by rescaling in the following way. Let

\[
x_i(T) = \lambda X_i(T), \quad t = \sigma T.
\]  

(2)

Then by choosing \( \sigma = \beta, \lambda = \frac{\alpha}{\beta} \) and setting \( F = \frac{\alpha\gamma}{\beta^2} \) we arrive at the system of equations

\[
\frac{d}{dt} x_i(t) = x_{i-1}(t)(x_{i+1}(t) - x_{i-2}(t)) - x_i(t) + F.
\]  

(3)

This is the original system considered in [18]. Writing \( \mathbf{x} = (x_0, x_1, \ldots, x_{N-1}) \), considered as a column vector, and \( \mathbf{e} = (1, 1, \ldots, 1) \), the system may be written as

\[
\dot{\mathbf{x}} = G_L(\mathbf{x}) - \mathbf{x} + F \mathbf{e}
\]  

(4)

where the dot denotes differentiation with respect to \( t \) and the mapping \( G_L : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is defined by

\[
G_L(\mathbf{x})_i = x_{i-1}(x_{i+1} - x_{i-2}).
\]  

(5)

It is easy to see that constant functions \( \mathbf{x}(t) = F \mathbf{e} \) always solve Eq. (4), and it is known that these solutions are stable for \(-\frac{3}{2} < F < \frac{5}{2}\). As \( F \) increases, solutions with spatial and temporal periodicity appear. This is illustrated for \( F = 2 \) in Fig. 1. Figure 1a shows the solution for \( N = 36 \) at \( t = 500 \), starting from random initial data and resulting in a periodic spatial pattern with spatial period 9. Fig. 1b of \( t \mapsto x_0(t) \) shows that the solution is in fact periodic in time with period near 4. The right panel is a Hovmoeller plot of the solution for 500 \( \leq t \leq 510 \), where sites are plotted horizontally, time increases in the vertical direction, and the solution value at each site and time is shown according to the color key on the far right. The solution is interpolated with a cubic spline. The plot shows that spatial and temporal periodicity come from regularly spaced waves that move to the left (“westward”) at a speed of about 1.2 sites per time unit.

As \( F \) increases further, the system becomes chaotic. The waves eventually break up and the behavior at any given site remains irregular even after a very long time. This is illustrated for \( F = 8 \) in Fig. 2, well into the chaotic regime. Figure 2a
1.2. Overview of the paper. In this paper, we generalize the L96 system in two ways, first by considering new advection terms, then by allowing for site-dependent parameters. We provide some broadly applicable tools to study the bifurcations of equilibria and the limit cycles of such systems. Some specific systems, including the original L96 system, are considered in more detail.

The paper is organized as follows. In Section 2, we discuss general classes of quadratic advection terms that share important properties with the term $G_L$ defined in Eq. (5). We characterize all such terms and compute their linearizations about constant solutions for later use. In Section 3, we show how to describe bifurcations from such constant solutions, and we prove that the first Hopf bifurcation for the L96 system is always supercritical, using a general approach for computing normal form coefficients. Section 4 discusses multiple stable limit cycles of periodic solutions. Following the approach in [13], we show how this can be explained by embedding the system in a suitable two parameter family of systems. As our main contribution, we demonstrate how all normal form coefficients of the system for any number of sites may be computed exactly, which reveals the presence of a Neimark-Sacker (N-S) bifurcation. In Section 5, we turn to systems with site-dependent advection, dissipation, and forcing. We give general existence proofs for static and dynamic problems, show that for a large class of such problems the constant solution is
always globally asymptotically stable for small forcing, and give some numerical results for the cases of site-dependent advection and dissipation. In Section 6 we give conclusions and list some open questions.

Finally, we include a few appendices. Appendix A summarizes aspects of L96-like systems in Fourier space, which are useful for the normal form analysis. Appendix B is devoted to the special case of problems that have only advection, but no dissipation or forcing. Such systems are of interest because they have more conserved quantities than the full L96 system. For a system with symmetrized advection we find closed form solutions for $N = 4$ and $N = 6$. In Appendix C we compare the behavior of mid accuracy ODE solvers (e.g., Runge-Kutta) to that of high accuracy solvers with step size control by examining the computed energy of solutions of systems that have only advection.

2. Advection terms. In this section, $G : \mathbb{R}^N \to \mathbb{R}^N$ denotes an arbitrary continuously differentiable mapping, while $G_L : \mathbb{R}^N \to \mathbb{R}^N$ denotes the Lorenz advection term that is defined in Eq. (5). We identify and examine some properties of $G_L$ and identify other mappings $G$ that share these properties, described in (i)-(iv) in Section 2.1 below.

2.1. Properties of suitable advection terms. (i) The mapping $G_L$ is quadratic. A mapping $G$ has this property if it may be written as

$$G(x) = \frac{1}{2}B(x, x)$$

where $B : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is symmetric and bilinear. The bilinear map $B$ may be recovered from $G$ by the formula

$$B(x, y) = G(x + y) - G(x) - G(y).$$

Specifically, in the L96 case the bilinear map is given by

$$B_L(x, y)_j = x_{j-1}(y_{j+1} - y_{j-2}) + y_{j-1}(x_{j+1} - x_{j-2}).$$

Consequently, the Taylor expansion of a quadratic mapping $G$ about any $x_0 \in \mathbb{R}^N$ has the form

$$G(x_0 + y) = G(x_0) + A[x_0]y + G(y)$$

where now $A$ is a linear map from $\mathbb{R}^N$ to the set of linear self-maps of $\mathbb{R}^N$, defined by

$$A[x] = B(x, \cdot).$$

(ii) Further, the mapping $G_L$ is energy-preserving. A mapping $G$ has this property if for all $x$

$$x^T G(x) = 0.$$  

This follows for $G_L$ by a direct calculation. We define the energy as $E = x^T x$.

(iii) In addition, the mapping $G_L$ is equivariant with respect to the group of coordinate rotations acting on $\mathbb{R}^N$. This group is generated by the rotation $\rho : \mathbb{R}^N \to \mathbb{R}^N$, i.e.

$$\rho(x) = (x_1, x_2, \ldots, x_{N-1}, x_0).$$

A mapping $G$ from a suitable domain to itself is said to be equivariant with respect to a group $\mathcal{H}$ of transformations of its domain if for all group elements $h \in \mathcal{H}$ and all $x$ in the domain

$$G(h(x)) = h(G(x)).$$
We call the mapping \( G \) \((\rho )\)-equivariant. Clearly, a quadratic mapping \( G \) is \((\rho )\)-equivariant if and only if the associated bilinear map \( B \) from Eq. (6) satisfies \( \rho B(x, y) = B(\rho x, \rho y) \) for all \( x, y \). From now on, we use “equivariant” to mean “\((\rho )\)-equivariant”.

An equivariant mapping is defined by its behavior on a single element of each orbit of the group. Here, such a mapping may be described by its behavior at a single component. In particular, if \( G \) is quadratic, then we may write

\[
G(x)_0 = \frac{1}{2} B(x, x)_0 = \frac{1}{2} \sum_{i,j} Q_{ij} x_i x_j = \frac{1}{2} x^T Q x, \quad B(x, y)_0 = x^T Q y
\] (13)

where \( Q \) is a real symmetric matrix since \( B \) is symmetric. Identifying \( \rho \) with its matrix representation, the equivariance of \( G \) gives

\[
G(x)_m = \frac{1}{2} (\rho^m x)^T Q \rho^m x = \frac{1}{2} x^T (\rho^T)^m Q \rho^m x.
\] (14)

When applied on the left of any matrix \( M \), \( \rho \), resp. \( \rho^T \), shifts the rows of \( M \) up, resp. down, by one row, circulanty. When applied on the right, it shifts the columns of \( M \) right, resp. left, by one column, circulanty. Therefore \( (\rho^T)^m Q \rho^m \) is \( Q \) shifted down and to the right, circulantly, by \( m \). Since \( Q \) is symmetric, \( (\rho^T)^m Q \rho^m \) is also symmetric.

If \( G \) is equivariant and energy-preserving, then it must vanish on all constant input vectors \( \lambda e \), since \( G(\lambda e) \) must be a multiple of \( e \) and \( e^T G(\lambda e) = 0 \). Additionally, its linearization at a constant input vector \( \lambda e \) is also equivariant, meaning it may be described by a circulant matrix.

(iv) Finally, each component of \( G_L \) depends only on a few neighboring components of \( x \). We say that the mapping \( G \) is \( k \)-localized if for each \( i \), \( G(x)_i \) depends only on \( x_{i-k}, \ldots, x_{i+k} \), where indices are taken modulo \( N \). For example, \( G_L \) is 2-localized.

An equivariant, quadratic mapping \( G \) may be described by its behavior at the 0-th component via Eq. (13). If the mapping is \( k \)-localized, then shifting \( Q \) by \( k \) columns to the right and \( k \) rows down, circulanty, results in a matrix that is zero outside its top left \((2k+1) \times (2k+1)\) block.

For later use, we define the coordinate reflection map \( \tau \) defined by

\[
\tau(x) = (x_{N-1}, x_{N-2}, \ldots, x_1, x_0).
\] (15)

Then if a mapping \( G \) is quadratic, energy-preserving, equivariant, or \( k \)-localized, then so is the mapping \( \tilde{G} \), defined by \( \tilde{G}(x) = \tau \circ G(\tau(x)) \). When \( G \) is replaced with \( \tilde{G} \), the direction of the advection term is simply reversed everywhere.

2.2. Classification of maps. We now give a complete classification of all quadratic, energy-preserving, equivariant, 3-localized mappings \( G \).

Consider first the set of all quadratic equivariant maps \( G \), which forms a vector space of dimension \( \frac{N(N+1)}{2} \), as Eq. (13) shows. If \( G \) is also energy-preserving, then

\[
0 = \sum_{k} x_k x^T (\rho^T)^k Q \rho^k x
\]

\[
= \sum_{i,j,k} x_k x_i [(\rho^T)^k Q \rho^k]_{ij} x_j
\]

\[
= \sum_{i,j,k} Q_{(i-k)(j-k)} x_i x_j x_k
\]
where \(i, j, k\) run from 0 to \(N - 1\) and indices are taken modulo \(N\). Since \(Q\) is symmetric, this condition is equivalent to

\[
Q_{(i-k)(j-k)} + Q_{(i-j)(k-j)} + Q_{(j-i)(k-i)} = 0 \quad \forall i, j, k \in \{0, \ldots, N - 1\}. \tag{16}
\]

Taking \(k = 0\) corresponds to \(G(x)_0\); that is, \(Q_{ij}\) is the coefficient of \(x_ix_j\), giving

\[
Q_{ij} + Q_{(i-j)(-j)} + Q_{(j-i)(-i)} = 0. \tag{17}
\]

Next consider quadratic, equivariant maps \(G\) that are also \(k\)-localized. Thus each component of \(G(x)\) depends on up to \(2k + 1\) sites. These sites should all be different, therefore we require \(N \geq 2k + 2\). For \(k = 1\), \(N \geq 4\), this is a 6-dimensional space, for \(k = 2\), \(N \geq 6\) it is a 15-dimensional space, and for \(k = 3\), \(N \geq 8\) it is a 28-dimensional space. If now \(G\) is also energy-preserving, then Eq. (17) restricts the set of possible mappings further. The following result allows one to list all such maps for \(k \leq 3\). Clearly, any equivariant, quadratic, energy-preserving map must have at least two quadratic terms in each component. The mappings that are listed below are among the simplest possible in that they each use exactly two terms.

**Theorem 2.1.** Consider the set of all quadratic, energy-preserving, equivariant, \(k\)-localized maps \(G : \mathbb{R}^N \to \mathbb{R}^N\).

a) If \(k = 1\) and \(N \geq 4\), this is a two-dimensional space, \(G_1\). A basis is given by \(G_1, \tilde{G}_1\), defined by

\[
G_1(x)_0 = x_1^2 - x_0x_1, \quad \tilde{G}_1 = \tau \circ G_1 \circ \tau, \quad \tilde{G}_1(x)_0 = x_1^2 - x_0x_1. \tag{18}
\]

b) If \(k = 2\) and \(N \geq 6\), this is a six-dimensional space, \(G_2\). A basis is given by \(G_1, \tilde{G}_1, G_2, \tilde{G}_2, G_3, \tilde{G}_3\), defined by

\[
G_2(x)_0 = x_2^2 - x_0x_2, \quad \tilde{G}_2 = \tau \circ G_2 \circ \tau
\]

\[
G_3(x)_0 = x_1x_2 - x_2x_1, \quad \tilde{G}_3 = \tau \circ G_3 \circ \tau. \tag{19}
\]

c) If \(k = 3\) and \(N \geq 8\), this is a 12-dimensional space, \(G_3\). A basis is given by \(G_1, \tilde{G}_1, \ldots, \tilde{G}_3, G_4, \tilde{G}_4, \ldots, \tilde{G}_6\), defined by

\[
G_4(x)_0 = x_3^2 - x_0x_3, \quad \tilde{G}_4 = \tau \circ G_4 \circ \tau
\]

\[
G_5(x)_0 = x_2x_3 - x_3x_2, \quad \tilde{G}_5 = \tau \circ G_5 \circ \tau
\]

\[
G_6(x)_0 = x_1x_3 - x_3x_1, \quad \tilde{G}_6 = \tau \circ G_6 \circ \tau. \tag{20}
\]

**Proof.** Start by noting that Eq. (17) implies \(Q_{00} = 0\) and \(Q_{rs} = 0\) if \(|s - r| > k\) and \(N \geq 2k + 2\).

For \(k = 1\) this leaves the four possible nonzero coefficients \(Q_{-1,-1}, Q_{-1,0} = Q_{0,-1}, Q_{0,1} = Q_{1,0}, Q_{1,1}\) which must satisfy the equations \(Q_{1,1} + 2Q_{-1,0} = 0\) and \(Q_{-1,-1} + 2Q_{0,1} = 0\). The maps \(G_1\) and \(\tilde{G}_1\) correspond to two independent solutions of this homogeneous system.

For \(k = 2\) the same reasoning results in 11 nonzero coefficients and three homogeneous equations in addition to the two equations for \(k = 1\). The maps \(G_1, \ldots, \tilde{G}_3\) correspond to six independent solutions of this homogeneous system.

The same technique may be applied to the case \(k = 3\). \(\square\)

Henceforth, we call any quadratic, energy-preserving, equivariant map from \(\mathbb{R}^N\) to itself a \(G\)-map. A map in \(G_k\) as described in Theorem 2.1 is a \(k\)-localized \(G\)-map. In the following sections, when it is clear that we are talking about a \(G\)-map we sometimes use “advection term” interchangeably.
The map \( G_3 \) is the L96 term \( G_L \). Also, \( G_2 \) is essentially the same as \( G_1 \) with interacting sites always at a distance of \( k = 2 \) instead of \( k = 1 \), and \( G_4 \) is essentially the same as \( G_1 \) with interaction at a distance of \( k = 3 \). Some additional \( G \)-maps that have only two quadratic terms may be generated from the list above. The main examples are

\[
G_7 = G_3 - \tilde{G}_3, \quad G_7(x)_0 = x_1 x_2 - x_{-1} x_{-2} \tag{21}
\]

\[
G_8 = G_5 - \tilde{G}_6, \quad G_8(x)_0 = x_2 x_3 - x_{-1} x_{-3}, \quad \tilde{G}_8 = \tau \circ G_8 \circ \tau. \tag{22}
\]

Note that \( \tau \circ G_7 \circ \tau = -G_7 \) and therefore there is no reason to consider \( \tilde{G}_7 \) separately.

There is no 1-localized \( G \)-map with the Liouville property \( \nabla \cdot G(x) = 0 \) as the theorem shows. The 2-localized \( G \)-maps with this property are precisely the linear combinations of \( G_3 \) and \( G_3 \). Equations with nonlinearity from this set, called the Orszag-McLaughlin family after [22] where a special case was introduced, were studied in [3].

3. Linearization and eigenvalue curves. Recall the definition of the left rotation \( \rho \) in Eq. (11), identified with its matrix representation. This is a unitary circulant matrix, i.e. \( \rho^{-1} = \rho^T \).

\[
\rho = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\tag{23}
\]

Matrix rows and columns are labeled from 0 to \( N - 1 \). The eigenvalues of \( \rho \) are \( \omega_N^j, \ j = 0, \ldots, N - 1 \), where \( \omega_N = e^{2\pi i/N} \) is a primitive \( N \)-th unit root. The left and right eigenvectors are the rows and columns of the Fourier matrix given in Eq. (47) in Appendix A.

Throughout the section, advection terms \( G \) will be assumed to be \( G \)-maps. Let \( F \in \mathbb{R} \). We are interested in the general system

\[
\dot{x} = G(x) - x + Fe. \tag{24}
\]

Applying Eq. (8), it follows that the linearization of this system (24) about the constant solution \( x_F = Fe \) is the system

\[
\dot{y} = (FA - I)y \tag{25}
\]

where \( A \) is the linearization of \( G \) about the constant vector \( e \). The elements of \( A \) are \( A_{ij} = \sum_{k=0}^{N-1} Q_{(j-i)(k-i)} \), with \( Q \) defined by Eq. (13).

Since \( G \) is equivariant, \( A \) is circulant and can be written as \( A = \sum_{j=0}^{N-1} A_{0j} \rho^j \), where \( \rho \) is as in Eq. (23). Its eigenvalues are therefore given by \( \lambda_\ell = \sum_{j=0}^{N-1} A_{0j} \omega_N^j \), \( \ell = 0, \ldots, N - 1 \), i.e. they are the values of the polynomial with coefficients in the first row of \( A \) at the \( N \)-th unit roots. We now replace positive powers \( z^j \) of such unit roots with negative powers \( z^{j-N} \) if \( j \geq \frac{N}{2} \), resulting in the Laurent polynomial

\[
p_A(z) = \sum_{0 \leq j < N/2} A_{0j} z^j + \sum_{N/2 \leq j < N} A_{0j} z^{j-N}. \tag{26}
\]

The eigenvalues of \( A \) therefore can also be written as \( \lambda_\ell = p_A(\omega_N^\ell) \). Now if \( G \) is \( k \)-localized, then \( A_{0j} = 0 \) for \( k < j < N - k \). For \( N \geq 2k \) the Laurent polynomial \( p_A \) then only contains powers \( z^j \) with \( |j| \leq k \) and does not change with \( N \). For such \( N \),
all eigenvalues of $A$ therefore lie on the curve that is the image of the complex unit circle $S^1$ under the map $z \mapsto p_A(z)$. We therefore call such an image an eigenvalue curve.

These eigenvalue curves are symmetric about the real axis, since $p_A$ has real coefficients. If a map $G$ is replaced by $\tilde{G} = \tau \circ G \circ \tau$, then $A$ is replaced by $A^T$ and thus $p_{A^T}(z) = \overline{p_A(z)}$. Since $p_A(z^{-1}) = \overline{p_A(z)}$ for $|z| = 1$, the image of the unit circle does not change, but the curve is traversed in the opposite direction.

If $G$ is replaced with $-G$, the image of the curve is reflected about the imaginary axis. Since $p_A(1) = 0$ for all such $A$ (a consequence of the assumption that $G$ is energy preserving), such eigenvalue curves pass through the origin.

For a concrete system, the eigenvalues of $A$ are a discrete set of points. For the case of the L96 system with $G = G_L = G_3$, we have $p_A(z) = z - z^{-2}$. In the left panel of Fig. 3, we show the eigenvalue curve and the eigenvalues $\lambda_j = \omega_{36}^j - \omega_{36}^{2j}$, $\omega_{36} = e^{\pi i/18}$ of $A$ in the case $N = 36$. In this case the single eigenvalue with smallest real part is $\lambda_{18} = -2$. The pair of eigenvalues with largest real part is $(\lambda_8, \lambda_{28})$ with $\Re \lambda_8 = \Re \lambda_{28} = \cos \frac{\pi}{9} + \sin \frac{\pi}{18} \approx 1.11334$. The pair $(\lambda_7, \lambda_{29})$ with $\Re \lambda_7 = \Re \lambda_{29} = \cos \frac{2\pi}{9} + \sin \frac{\pi}{18} \approx 1.10806$ is close behind. There are four eigenvalues with real part 1, namely $\lambda_6 = 1 + \sqrt{3} \cdot i$, $\lambda_9 = 1 + i$, and their conjugates $\lambda_{27}$ and $\lambda_{30}$.

Consider now the matrix $FA - I$ which arises in the linearization in Eq. (25). Its eigenvalues lie on the curve $Fp_A(S^1) - 1$ that is obtained by stretching $p_A(S^1)$...
by the factor $F$ and shifting it to the left by one unit. For small $|F|$, the curve is close to $-1$ and therefore entirely in the left half plane. More precisely there is an open interval containing 0, possibly unbounded, such that the constant solution $F e$ is globally asymptotically stable for all $F$ in this interval. In particular, $F e$ is the only stationary solution for such $F$.

**Proposition 1.** Consider the problem (24) with a $G$-map $G$ that is $k$-localized, on $N \geq 2k$ sites. Let $A$ be the linearization of $G$ about $e$ and let $p_A(z)$ be the associated Laurent polynomial. Let $p_+ = \max_j \Re p_A(\omega_N^j)$, $p_- = \min_j \Re p_A(\omega_N^j)$. If $F p_+ < 1$ and $F p_- < 1$, then the constant solution $F e$ of this problem is globally asymptotically stable.

**Proof.** Note that $p_+ \geq 0 \geq p_-$, since the eigenvalue curve of $A$ passes through the origin. Thus the condition $F p_+ < 1$ restricts only positive $F$ and $F p_- < 1$ is a restriction for negative $F$. Let $t \mapsto x(t)$ be a solution of Eq. (24). Set $w(t) = x(t) - F e$. Since $G$ is quadratic, $w$ satisfies

$$\dot{w} = G(w) + (FA - I)w.$$ 

Multiply this equation with $w^T$. Since $G$ is energy preserving, the result is

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 = w^T (FA - I)w = w^T (FA + A^T/2 - I)w.$$ 

Since $A$ and $A^T$ are both circulant and therefore have the same eigenvector systems, it follows that $(A + A^T)/2$ has the eigenvalues $\Re p_A(\omega_N^j)$. By the assumption for $F$, the matrix $(FA + A^T)/2 - I$ is negative definite and there is a $\gamma > 0$ such that $\frac{d}{dt} \|w\|^2 \leq -\gamma \|w\|^2$. This implies that $\|w(t)\| = O(e^{-\gamma t/2})$ and the statement follows. \qed

As $F$ increases in magnitude, the curve $F p_A(S^1) - 1$ expands and some of its portions including eigenvalues of $FA - I$ may cross the imaginary axis. If a single real eigenvalue crosses into the right half plane, stationary solutions bifurcate. The first such bifurcation cannot be a transcritical or saddle-node bifurcation, since stationary solutions exists for all $F$ and are unique on one side of the bifurcation value, by the above proposition. Therefore a supercritical pitchfork bifurcation is expected in all such cases. If at the first bifurcation a complex pair of eigenvalues crosses the imaginary axis, a Hopf bifurcation is expected to occur.

This is illustrated in the right panel of Fig. 3 for the case of the L96 system. In this case, $\min_{z \in S^1} \Re p_A(z) = -2$ and $\max_{z \in S^1} \Re p_A(z) = 9/8$. As $F$ increases past 8/9, a pair of complex eigenvalues crosses the imaginary axis. The exact $F$-value for this depends on the location of the eigenvalues of $A$ and thus on $N$. For $N = 36$, this happens as $F$ increases beyond $p_+ = (\cos \frac{2\pi}{9} + \sin \frac{\pi}{3})^{-1} \approx 0.902474$. When $F$ decreases below $-\frac{1}{2}$, a pitchfork bifurcation occurs for even $N$ at $F = -1/2$ (since non-constant stationary solutions come in pairs and no other bifurcations are possible). For odd $N$, $-2$ cannot be an eigenvalue of $A$ and therefore a pair of complex eigenvalues of $FA - I$ crosses the imaginary axis at a bifurcation value $<-1/2$. Its precise value depends again on $N$.

### 3.1. Characterizing eigenvalue curves.

We now give a table of the Laurent polynomials $p_A$ for the maps $G_1, \ldots, G_8$ identified in Theorem 2.1 and also describe the shape of the image $p_A(S^1)$. These images are plotted in Fig. 4 for some of the Laurent polynomials. We also list the type of bifurcations corresponding to each eigenvalue curve.
\(G\)-map | Laurent polynomial \(p_A(z)\) | Shape of \(p_A(S^1)\) | \(F > 0\) | \(F < 0\)
---|---|---|---|---
\(G_1\) | \(-z^4 - 1 + 2z\) | ellipse | none | pitchfork/Hopf
\(G_2\) | \(-z^2 - 1 + 2z^2\) | ellipse | none | pitchfork/Hopf
\(G_3\) | \(-z^2 + z\) | trefoil | Hopf | pitchfork/Hopf
\(G_4\) | \(-z^3 - 1 + 2z^3\) | ellipse | none | pitchfork/Hopf
\(G_5\) | \(-z^2 - z + z^2 + z^3\) | butterfly | Hopf | Hopf
\(G_6\) | \(-z^1 + z - z^2 + z^3\) | kidney | Hopf | pitchfork/Hopf
\(G_7\) | \(-z^2 - z^{-1} + z + z^2\) | vertical line | none | none
\(G_8\) | \(-z^3 - z^{-1} + z^2 + z^3\) | bee | pitchfork/Hopf | Hopf

**Table 1.** Description of the eigenvalue curves of the eight simplest 3-localized \(G\)-maps identified in Section 2.2. The two rightmost columns give the types of the first expected bifurcation for \(F > 0\) and \(F < 0\) as the magnitude of \(F\) increases. Asterisks indicate exceptions for certain site numbers.

Whenever a pair of complex eigenvalues crosses the imaginary axis due to increased forcing \(F > 0\), a Hopf bifurcation is expected to occur off the branch of constant solutions. As Fig. 4 shows, such Hopf bifurcations are therefore also expected to occur for \(G_5\) and \(G_6\). The rightmost two columns in Table 1 list the types of the first expected bifurcation for \(G_1, \ldots, G_8\) for increasing \(F > 0\) or decreasing \(F < 0\). Explanations are given below.

Whenever a bifurcation occurs for a \(G\)-map, it also occurs for its tilde conjugate, since the shape of the curve does not change. When such a map is placed by its negative, the eigenvalue curve is flipped about the imaginary axis.

Some comments are in order.

(a) For \(G_1, G_2, G_4\), no eigenvalue of \(FA - I\) can move into the right half plane if \(F > 0\). This is because no eigenvalues of \(A\) are in the right half plane. In the notation of Proposition 1, \(p_+ = 0\). Therefore no bifurcation can occur for \(F > 0\). For \(G_7\), all eigenvalues of \(FA - I\) have real part \(-1\) and no bifurcation can occur for any \(F\). The constant solution \(Fe\) is globally asymptotically stable for all \(F\).

(b) An entry “pitchfork/Hopf” in one of the last two columns means that there is a pitchfork bifurcation if \(N\) is even and a Hopf bifurcation if \(N\) is odd. Exceptions are indicated by asterisks. If \(N\) is even, there is a real eigenvalue \(F p_\omega(N^{1/2}) - 1 = F p_\omega(-1) - 1\) of \(FA - I\) which may cross into the right half plane for suitable \(F\). The result is a pitchfork bifurcation in which two stationary solutions are born which are shifted versions of each other. If \(N\) is odd, such a real eigenvalue does not exist, but there is a pair of complex conjugate eigenvalues nearby, resulting in a Hopf bifurcation. This occurs for \(G_1, G_3,\) and \(G_6\) for negative \(F\) and for \(G_8\) for positive \(F\). A more detailed discussion may be found in subsection 3.4.

(c) Here is an explanation of the entries with asterisks for \(G_2\) and \(G_4\). The eigenvalue curves are the same as for \(G_1\) but are traversed twice or three times as \(z\) traverses \(S^1\). In the case of \(G_2\), this results in a Hopf bifurcation for negative \(F\), if \(N\) is odd. If \(N = 2K\) is even, the system decouples completely into two systems with \(K\) sites each, one at the odd numbered sites and one at the even numbered sites and each governed by \(G_1\). Then each system undergoes a pitchfork or a Hopf bifurcation, depending on the parity of \(K\). In the case of \(G_4\), if \(N\) is not divisible by 3, a pitchfork bifurcation occurs for \(F < 0\) if \(N\) is even and a Hopf bifurcation occurs if \(N\) is odd. But if \(N = 3K\), the system decouples completely into three
systems, each governed by $G_1$, and their bifurcation behavior depends on the parity of $K$ as before.

(d) For $G_8$ and $F < 0$, only a Hopf bifurcation can occur. This is because the leftmost intersection of the curve $p_A(S^1)$ with the real axis occurs for $z^* = \exp(i(k\pi \pm \arctan(\sqrt{15})))$ with $\lambda^* = p_A(z^*) = -9/8$. But $\arctan(\sqrt{15})$ is not a rational multiple of $\pi$ and thus $\lambda^*$ cannot be an eigenvalue of $A$ for any $N$. Instead, there is a pair of complex eigenvalues of $A$ near $\lambda^*$ and a Hopf bifurcation must occur for $F \approx -8/9$ independent of the parity of $N$.

![Images of the complex unit circle under the Laurent polynomials given in Table 1.](image)

3.2. **Normal form at the first Hopf bifurcation.** We now present an approach to find the first Lyapunov coefficient for any system of the form Eq. (24) that undergoes a Hopf bifurcation, where $G$ is a $G$-map, and apply it to the L96 system.

Let $A$ be a general circulant matrix with a pair of purely imaginary eigenvalues, say $\pm i\tau_0$. We first consider the general system

$$\dot{y} = Ay + G(y) = Ay + \frac{1}{2}B(y,y)$$

where $B$ is related to $G$ as in Eq. (7). To compute the first Lyapunov exponent, we use the approach described in [15] and [16]. This requires the computation of normal form coefficients for the cubic approximation of this system, which explicitly given above.
Let $p_A$ be the Laurent polynomial for $A$. As summarized in Appendix A, the
eigenvectors of $A$ are the columns $\mathbf{q}_j$ of the normalized $N \times N$ Fourier matrix, and
the eigenvalues are $p_A(\omega_N^j)$. Let $\mathbf{q}_\ell$ be the right eigenvector of $A$ for the eigenvalue
$i \tau_0$, let $\mathbf{q}_\ell^* = \mathbf{q}_{N-\ell}$ be its complex conjugate, and let $\mathbf{p}_\ell = \mathbf{q}_\ell$ be the eigenvector
of $A^T$ for the eigenvalue $-i \tau_0 = p_A(\omega_N^\ell)$. All eigenvectors are normalized to unit
length. According to [15] and [16], section 5.4, the first Lyapunov coefficient then is

$$I_1 = \frac{1}{2\tau_0} \Re \left( -2\langle \mathbf{p}_\ell, \mathcal{B}(\mathbf{q}_\ell, A^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) \rangle + \langle \mathbf{p}_\ell, \mathcal{B}(\mathbf{q}_\ell, (2i \tau_0 I - A)^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) \rangle \right),$$

where $\langle \ , \rangle$ is the usual scalar product in $\mathbb{C}^N$ (linear in the second argument,
conjugate-linear in the first). This quantity can be now computed in closed form,
because the eigenvectors of a circulant matrix are columns of the Fourier matrix
and $\mathcal{B}$ maps pairs of such vectors to multiples of these eigenvectors (by adding their
indices).

**Lemma 3.1.** Consider system (27) with the notation and assumptions as above.
Define $P_B$ as in Eq. (48). Assume that $A$ has a pair of purely imaginary eigenvalue
$i \tau_0 = p_A(\omega_N^\ell), -i \tau_0$, and set $z_1 = \omega_N^\ell$. Then the first Lyapunov exponent of the
system (27) is

$$I_1 = \frac{1}{2\tau_0 N} \left( -2P_B(z_1, \bar{z}_1) \Re \frac{P_B(z_1, 1)}{p_A(1)} + \Re \frac{P_B(z_1, z_1)P_B(z_1^2, \bar{z}_1)}{2i \tau_0 - p_A(z_1^2)} \right).$$

**Proof.** We obtain for the first term inside the parentheses in Eq. (28)

$$\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell) = \frac{P_B(z_1, z_1)}{\sqrt{N}} \mathbf{q}_0 = \frac{P_B(z_1, \bar{z}_1)}{\sqrt{N}} \mathbf{e}$$

$$A^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell) = \frac{1}{p_A(1)} \frac{P_B(z_1, \bar{z}_1)}{\sqrt{N}} \mathbf{e}$$

$$\mathcal{B}(\mathbf{q}_\ell, A^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) = \frac{P_B(z_1, \bar{z}_1)P_B(z_1, 1)}{p_A(1)N} \mathbf{q}_\ell$$

$$\langle \mathbf{q}_\ell, \mathcal{B}(\mathbf{q}_\ell, A^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) \rangle = \frac{P_B(z_1, \bar{z}_1)P_B(z_1, 1)}{p_A(1)N}.$$

Note that $P_B(z_1, \bar{z}_1) \in \mathbb{R}$. For the second term we obtain similarly

$$\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell) = \frac{P_B(z_1, z_1)}{\sqrt{N}} \mathbf{q}_{2\ell}$$

$$(2i \tau_0 I - A)^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell) = \frac{1}{2i \tau_0 - p_A(z_1^2)} \frac{P_B(z_1, z_1)}{\sqrt{N}} \mathbf{q}_{2\ell}$$

$$\mathcal{B}(\mathbf{q}_\ell, (2i \tau_0 I - A)^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) = \frac{1}{2i \tau_0 - p_A(z_1^2)} \frac{P_B(z_1, z_1)P_B(z_1^2, \bar{z}_1)}{N} \mathbf{q}_\ell$$

$$\langle \mathbf{p}_\ell, \mathcal{B}(\mathbf{q}_\ell, (2i \tau_0 I - A)^{-1}\mathcal{B}(\mathbf{q}_\ell, \mathbf{q}_\ell)) \rangle = \frac{P_B(z_1, z_1)P_B(z_1^2, \bar{z}_1)}{(2i \tau_0 - p_A(z_1^2))N}.$$

Combining these two computations, we obtain Eq. (29). \hfill \Box

We now apply this approach to the L96 system.
Theorem 3.2. Consider the L96 system for $N \geq 4$ sites. Let $F_1$ be the smallest positive forcing for which the linearization $FA - I$ has eigenvalues $\pm i\tau_0$ on the imaginary axis. Then the first Lyapunov exponent is negative, i.e. the resulting Hopf bifurcation is supercritical.

Proof. Set $y(t) = x(t) - Fe$. Then $y$ satisfies the system (27) with $A = F_1A - I$. Since $p_A(1) = 0$, therefore $p_A(1) = -1$. Next, recall that $P_B(z, w) = P_L(z, w) = (z - z^{-2})w^{-1} + (w - w^{-2})z^{-1}$, see Eq. (50).

The first Lyapunov exponent can now be computed as in Lemma 3.1. We must examine the terms in Eq. (29). Let $t_1 = 2\ell \pi/N$ such that $z_1 = e^{it_1}$ and $F_1p_A(z_1) - 1 = i\tau_0$. Then we can compute $P_L(z_1, z_1) = 2(\cos 2t_1 - \cos t_1)$ and $P_L(z_1, 1) = z - z^{-2} = p_A(z)$ and thus $\Re p_A(z_1) = \cos t_1 - \cos 2t_1$. Recalling $p_A(1) = -1$, the first term in the parentheses in Eq. (29) therefore becomes $-4(\cos t_1 - \cos 2t_1)^2$.

For the second term in the parentheses, we first observe that $P_L(z_1, z_1) = 2(z_1^2 - 2 + z_1^{-2}) = 4(\cos 3t_1 - 1)$ which is also a real number. Therefore Eq. (29) now takes the form

$$I_1 = \frac{1}{2\tau_0 N} \left(-4(\cos t_1 - \cos 2t_1)^2 + 4(\cos 3t_1 - 1)\Re \frac{1}{2\tau_0 + 1 - F_1p_A(z_1^2)} \right).$$ (30)

Here $\frac{2\tau_0}{\pi} \leq t_1 \leq \frac{\pi}{2}$, where the lower bound is attained for $N = 7$ and the upper bound for $N = 12$. Now $-(\cos t - \cos 2t)^2$ and $\cos 3t_1 - 1$ are both strictly negative on $(0, 2\pi/3)$. Also clearly $F_1p_A(z_1^2) - 1$ cannot have positive real part, since this number is an eigenvalue of $F_1A - I$ and none of these eigenvalues are in the right half plane. Consequently, $\Re \frac{1}{2\tau_0 + 1 - F_1p_A(z_1^2)}$ is non-negative. All this implies that $I_1 < 0$ for all possible $t_1$ and therefore the first Hopf bifurcation is always supercritical. \qed

We note that the same approach may be used to show that the first Hopf bifurcations for $F > 0$ for the advection terms $G_5$ and $G_6$ defined in Eq. (20), are also always supercritical.

3.3. Determining advection terms from the linearization. We now discuss whether it is possible to retrieve advection terms given just the eigenvalue curve. Consider the spaces $G_1, G_2, G_3$ identified in Theorem 2.1. Linearizing a $G$-map about the constant vector $e$ results in a circulant matrix $A$ whose top row gives the coefficients of an associated Laurent polynomial $p_A$. This is a clearly a linear operation. A $k$-localized $G$-map has a Laurent polynomial of the form $d_0z^{-k} + \cdots + d_kz^k$ whose coefficients $d_j = A_0j = \sum_{i} Q_{ij}$ must add to zero, as discussed earlier.

Observe first that for maps in $G_1$ this defines a bijection between two two-dimensional vector spaces. Thus for any Laurent polynomial $p_A(z) = d_{-1}z^{-1} + d_0 + d_1z$ with $d_{-1} + d_0 + d_1 = 0$ there exists a unique 1-localized $G$-map.

Consider next the set of maps in $G_2$ and the associated Laurent polynomials, which have the form

$$p_A(z) = d_{-2}z^{-2} + d_{-1}z^{-1} + d_0 + d_1z + d_2z^2.$$ (31)

The space $G_2$ is six-dimensional, while the associated Laurent polynomials belong to a four-dimensional space, due to the condition $\sum_{|j| \leq 2} d_j = 0$. Indeed, it is easy to verify that any four of the Laurent polynomials of the form Eq. (31) in Table 1 along with their tilde conjugate polynomials $p_A(z^{-1})$ span this space. Consequently there
is a two-dimensional null space of such maps whose associated Laurent polynomial vanishes. One can verify that this null space is spanned by the map

\[ G_0 = G_3 - 2 \tilde{G}_3 + \tilde{G}_1 - G_2 \]  

(32)

together with the associated \( \tilde{G}_0 \). A Laurent polynomial of the form Eq. (31) determines a 2-localized \( \mathcal{G} \)-map up to multiples of \( G_0 \) and \( \tilde{G}_0 \). In particular, for systems of the form Eq. (24) where \( G \) is a linear combination of \( G_0 \) and \( \tilde{G}_0 \), the linearization about any stationary solution \( \mathbf{y} \mathcal{F} = \mathbf{F} \) always has the form \( \dot{\mathbf{y}} = -\mathbf{y} \). Such stationary solution are therefore globally asymptotically stable for the full system and must be unique.

The space \( \mathcal{G}_3 \) is 12-dimensional, while the space of associated Laurent polynomials is 6-dimensional, resulting in a 6-dimensional null space of 3-localized \( \mathcal{G} \)-maps whose linearizations about \( \mathbf{y} = 0 \) vanish.

3.4. **Eigenvalue curves resulting in a Hopf bifurcation.** We wish to identify conditions such that the resulting complex curve has a pair of lobes, symmetric about the real axis, on the right, which will therefore cross the imaginary axis simultaneously and with positive velocity as \( F \) increases. As a pair of eigenvalues on these lobes crosses the imaginary axis, the constant stationary solution \( \mathbf{F} \mathcal{F} \) will lose its stability in a Hopf bifurcation.

We will first consider the simplest case of a 2-localized eigenvalue curve. Take \( G \in \mathcal{G}_2 \) and let \( p_A \) be the associated Laurent polynomial of the linearization of \( G \) about \( \mathbf{e} \). Given \( F \in \mathbb{R} \), the real and imaginary parts of the eigenvalues of \( FA - I \) are of the form \( FP_A(e^{2\pi is}) - 1 \) where \( p_A \) is of the form Eq. (31). The real and imaginary parts of \( p_A(e^{2\pi is}) \) can be written as

\[ \Lambda_R(s) := \Re p_A(e^{2\pi is}) = R_0 + R_1 \cos(2\pi s) + R_2 \cos(4\pi s) \]
\[ \Lambda_I(s) := \Im p_A(e^{2\pi is}) = I_1 \sin(2\pi s) + I_2 \sin(4\pi s). \]

Here \( R_j = d_j + d_{-j} \), \( I_j = d_j - d_{-j} \) for \( j = 1, 2 \) and \( R_0 = -R_1 - R_2 \).

Note first that \( \Lambda_R \) is even about \( s = 1/2 \) while \( \Lambda_I \) is odd about this point. A calculus argument shows that \( \Lambda_R \) has critical points at \( s = 0 \) and \( s = 1/2 \). If \( R_2 \neq 0 \) and \(-1 < \frac{R_1}{4R_2} < 1 \), there are two additional critical points

\[ s_1 = \frac{1}{2\pi} \cos^{-1} \left( \frac{|R_1|}{4R_2^2} \right), \quad s_2 = 1 - s_1. \]

(34)

These are positive maxima if \( 4R_2 < R_1 < -4R_2 \), i.e. \( R_2 < 0 \), and negative minima otherwise. Therefore in the case \( R_2 < 0 \) a Hopf bifurcation will occur for \( F > 0 \), while in the case \( R_2 > 0 \) a Hopf bifurcation will occur for \( F < 0 \).

Since \( \Lambda_I(s) \) is odd about \( s = 1/2 \), it follows that \( \Lambda_I(s_1) = -\Lambda_I(s_2) \). Requiring that these two values are distinct is equivalent to the constraint \( I_2R_1 + 2I_1R_2 \neq 0 \).

In summary, in order for a L96-like system with a 2-localized advection term to undergo a Hopf bifurcation for some value of the forcing parameter \( F \), it is sufficient that its Laurent polynomial \( p_A \) satisfy the two following conditions:

1. \(|d_{-1} + d_1| < 4|d_{-2} + d_2|\)
2. \((d_{-2} - d_2)(d_{-1} + d_1) \neq 2(d_1 - d_{-1})(d_{-2} + d_2)\)

Then, for large \( N \), the first Hopf bifurcation will occur very near \( F = F_1 = \Lambda_R(s_1)^{-1} \).

There is another case which can lead to a Hopf bifurcation. Note that when \( N \) is even, the critical point \( s = 1/2 \) of \( \Lambda_R \) corresponds always to a real eigenvalue of \( A \), and then the stationary solution \( \mathbf{F} \mathcal{F} \) may lose its stability in a pitchfork bifurcation.
But if $N$ is odd, there is no eigenvalue corresponding to $s = 1/2$ and there are two complex conjugate eigenvalues at $s = \frac{1}{2} \pm \frac{1}{2N}$. These may cross the imaginary axis as the magnitude of $F$ increases, resulting in a Hopf bifurcation. If $s = 0$ and $s = 1/2$ are the only critical points of $\Lambda_R$, i.e. $|R_1| \geq 4|R_2|$, this always happens. If the critical points of $\Lambda_R$ in Eq. (34) exist, i.e. $|R_1| < 4|R_2|$, then this may happen for sufficiently large $N$. In the remaining cases, $\Lambda_R$ is non-positive everywhere and no bifurcation off the constant stationary solution is possible. Details are left to the reader.

For $k$-localized advection terms with $k \geq 3$, Eq. (33) may be extended in the obvious way. The problem of finding the nontrivial critical points of $\Lambda_R$ may always be reduced to finding the roots of a $(k-1)$-th degree polynomial in $\cos(2\pi s)$. For each real root $r$ satisfying $-1 < r < 1$, we get a pair of solutions of the form Eq. (34) with $R_1/4R_2$ replaced by $r$. These are symmetric about $s = 1/2$ with $0 < s_1 < 1/4$ and thus $|s_2 - s_1| > 1/2$.

In Figure 5 are plotted some examples of eigenvalue curves, multiplied by $F$ and shifted by one to the left. The eigenvalue curve in Fig. 5c results from a $G$-map that is a linear combination of the $G$-maps that produce the eigenvalue curves in Fig. 5a and Fig. 5b. A Hopf bifurcation occurs for positive $F$ in Fig. 5b and Fig. 5c, but no bifurcation occurs for positive $F$ in Fig. 5a.

**Figure 5.** Eigenvalue curves of advection terms in $G_2$ for various values of $F$. (a) $G_1$, no bifurcation occurs for positive $F$. (b) $G_5$, a supercritical Hopf bifurcation occurs for positive $F$. (c) $-G_1 + \frac{1}{2}G_5$, a supercritical Hopf bifurcation occurs for positive $F$.

### 3.5. Waves resulting from the Hopf bifurcation.

In this section we describe some properties of the stable limit cycle following the first Hopf bifurcation for L96-like systems.

Recall that the $j$-th eigenvector-eigenvalue pair of a $N \times N$ circulant matrix with elements $A_{mn}$ is (see also Eq. (47))

$$v_j = (v_j^k)_{0 \leq k < N} = (e^{2\pi ijk/N})_{0 \leq k < N}, \quad \lambda_j = \sum_{\ell} A_{0\ell} e^{2\pi i\ell/N}. \tag{35}$$

Therefore small perturbations $y(t) = (y_0(t), \ldots, y_{N-1}(t))^T$ from the stationary solution of Eq. (24) with linearization Eq. (25) evolve over a short time approximately as

$$y_k(t) = \sum_j b_j v_j^k e^{(F\lambda_j - 1)t} = \sum_j b_j e^{F(\Re\lambda_j - 1)t} e^{2\pi ijk/N + iF\Im\lambda_j t}$$

for some coefficients $b_j$ determined by the initial perturbation.
Assume $N$ is large so that $j/N$ can be approximated by $s \in [0,1)$ and suppose $F$ is slightly greater than its first Hopf bifurcation value so that only one pair of eigenvalues has positive real part. Then the $s_1$ mode will travel with some amplitude $|W|$ according to

$$y_k(t) = We^{i(2\pi s_1 k + FA_1(s_1)t)}.$$ \hspace{1cm} (36)

where $A_1$ is as in Eq. (33). This mode has wavelength $s_1^{-1} > 4$ sites since $0 < s_1 < 1/4$. The $s_2$ mode gives the same result (up to an irrelevant overall sign in the exponent) since $k$ is an integer, and therefore $\exp(2\pi i(1-s_1)k) = \exp(-2\pi is_1k)$.

As $F$ increases, other pairs of eigenvalues will cross the imaginary axis, so more modes will grow. Since the $s_1$ mode grows fastest, it is expected to dominate. We can approximate the phase velocities and group velocity of the resulting wave as

$$c_p(s) = -\frac{FA_1(s)}{2\pi s}, \quad c_g = \frac{FA_1'(s_1)}{2\pi}$$ \hspace{1cm} (37)

where the prime denotes a derivative with respect to $s$. Note that $\text{sgn}(c_p(s))$ is the same for all $s$ due to the symmetry of the eigenvalue curve. If we replace $G$ by $\tilde{G}$ in our system, then the sign of the phase velocity is reversed since the eigenvalue curve is traced in the opposite direction.

For 2-localized advection terms, some algebra shows that $c_p$ and $c_g$ have the same sign if and only if the coefficients $I_1, I_2$ in Eq. (33) satisfy

$$I_2 I_1 + 2s_1 I_2 > \frac{s_1}{2(1-s_1^2)}$$

and opposite sign if the inequality is reversed. The reverse inequality always holds if $I_2$ is negative. For the L96 system, the phase velocity is negative and the group velocity is positive with roughly the same magnitude, so disturbances travel in the opposite direction as the wave train, as can be seen in Fig. 2.

4. Coexistence of stable limit cycles. Consider again the L96 system on $N$ sites. As $F$ increases past the first Hopf bifurcation value $F_1$, a stable limit cycle always appears. A second unstable limit cycle appears at the second Hopf bifurcation point $F_2 > F_1$. There is abundant numerical evidence that as $F$ increases further past some value $F_3 > F_2$, there exist two stable limit cycles. In fact, for $N=12$, two pairs of complex eigenvalues cross the imaginary axis simultaneously (a Hopf-Hopf bifurcation) as $F$ increases past $F_1 = F_2 = 1$, and two stable limit cycles appear immediately.

An explanation for this phenomenon was given in [13]. The authors construct an embedding of the L96 system (3) into a two-parameter system by adding a linear term $\alpha Cx$ to the right hand side of Eq. (3), where $C$ is a suitably chosen circulant matrix. For some small positive $\alpha_0$ that depends on $N$, as $F$ increases, two pairs of eigenvalues then cross the imaginary axis simultaneously at some $F = \tilde{F}$. By the analysis in [16], under suitable conditions there exist two stable limit cycles for $F > \tilde{F}$, $\alpha = \alpha_0$. These conditions may be expressed in terms of normal form coefficients for the system. Then if $\alpha_0$ is sufficiently small, it is plausible that this occurs also for $\alpha$ near $\alpha_0$, e.g. for $\alpha = 0$. In [13], this is carried out largely numerically.

Our contribution to this question consists in the following modifications and extensions of the approach in [13].

1. We use a new choice of perturbation matrix $C$, which simplifies the analysis.
2. We show how to compute all normal form coefficients analytically, using Proposition 5 in Appendix A.
3. We approximate $F_3$ using normal form coefficients and compare this approximation to numerical experiments.

4.1. Perturbation near Hopf-Hopf bifurcation. For $0 \leq j < N$, let $\lambda_j(F) = Fp_A(\omega_j^F) - 1$ be the $j$-th eigenvalue of the linearization $FA - I$ about the constant state $F_0 e$. Let $\lambda_k(F), \lambda_{N-k}(F)$ be the first pair of complex eigenvalues that crosses the imaginary axis at $F = F_1$, i.e. $\lambda_k(F_1) = i\tau_1$, $\lambda_{N-k}(F_1) = -i\tau_1$. Similarly let $\lambda_\ell(F), \lambda_{N-\ell}(F)$ be the second pair that crosses the imaginary axis at $F = F_2 > F_1$, with $\lambda_\ell(F_2) = i\tau_2$, $\lambda_{N-\ell}(F_2) = -i\tau_2$. Typically, $|\ell - \ell| = 1$. Consider now the Fourier matrix $F_N$, defined in Eq. (47), with columns $q_j$, $0 \leq j < N$. Define the matrix $C_\ell = q_\ell q_{N-\ell}^T + q_{N-\ell} q_\ell^T$. This is a real valued rank 2 circulant matrix with eigenvectors $q_\ell$ and $q_{N-\ell}$, both with eigenvalue 1. All other columns $q_j$ are in the null space of $C_\ell$.

Now consider the perturbed system

$$\dot{x} = G_L(x) - x + \alpha C_\ell x + Fe$$

for $\alpha \geq 0$. Its linearization about the constant state $F_0 e$ is $\dot{y} = (FA - I + \alpha C_\ell)y$ and its eigenvalues are $\lambda_j(F) = Fp_A(\omega_j^F) - 1 + \alpha(\delta_\ell + \delta_{N-\ell})$, where $\delta_\ell$ is the Kronecker delta. Thus for any fixed $\alpha \geq 0$, $(\lambda_k(F), \lambda_{N-k}(F))$ is still a pair of eigenvalues of the perturbed linearization that crosses the imaginary axis at $F = F_1$. The pair of eigenvalues $(\lambda_\ell(F), \lambda_{N-\ell}(F)) = (\lambda_\ell(F) + \alpha, \lambda_{N-\ell}(F) + \alpha)$ of $FA - I + \alpha C_\ell$ crosses the imaginary axis at $F = F_2(1 - \alpha)$. For $\alpha = \alpha_0 := \frac{F_2 - F_1}{F_2 F_1}$, both pairs of eigenvalues cross the imaginary axis simultaneously at $F = F_1$.

This is illustrated in Fig. 6. The left panel shows the eigenvalues of $F_1A - I$ (black and red circles) and the eigenvalues of $F_1A - I + \alpha_0 C_\ell$ (black and green circles), for $N = 14$ sites. The right panel shows curves in the $(F, \alpha)$ plane where bifurcations occur. The values $F = F_1$ (blue circle) and $F = F_2$ (green circle) are marked on the axis $\{\alpha = 0\}$ (black). There is a Hopf bifurcation for system (38) for all $\alpha \geq 0$ at $F = F_1$ (blue line), when $\lambda_k(F)$ and its complex conjugate cross the imaginary axis. There is also a Hopf bifurcation for all $F = F_2(1 - \alpha)$ (green line), when $\lambda_\ell(F)$ and its complex conjugate cross the imaginary axis. For $F = F_1$, $\alpha = \alpha_0$ (red triangle), two pairs of eigenvalues are on the imaginary axis. The remaining features in the right panel will be explained below.

We note that all Hopf bifurcations described here are supercritical. Their first Lyapunov coefficients can be expressed in ways that are completely analogous to Eq. (30), with $i\tau_0$ replaced by some other imaginary number and $F_1$ replaced by $F = F_2(1 - \alpha)$. We leave the details to the reader.

4.2. Normal form at Hopf-Hopf bifurcation. We now examine the perturbed system (38) for $\alpha = \alpha_0$, $F = F_1$. At this set of parameters, two eigenvalues $\lambda_k(F_1) = i\tau_1 = F_1 p_A(z_1) - 1$ and $\lambda_\ell(F_1) = i\tau_2 = F_1 p_A(z_2) - 1 + \alpha_0$ together with their conjugates are on the imaginary axis. Here $z_1 = \omega_N^F$ and $z_2 = \omega_N^F$.

The corresponding right eigenvector pairs of the linearization are $q_k, q_{N-k}$ for the first pair of eigenvalues and $q_\ell, q_{N-\ell}$ for the second pair. The left eigenvectors of the linearization for all these eigenvalues are the conjugate transposes of the right eigenvectors, i.e. $q_{N-k}, q_k$ for the first pair and $q_{N-\ell}, q_\ell$ for the second pair.

To analyze the behavior of the full system near $\alpha = \alpha_0, F = F_1$, we use the approach described in [15] and [16], section 8.6. This requires the computation of normal form coefficients for the cubic approximation of the system that is satisfied
by \( y = x - F_1 e \). As before, this approximation is exact and has the form Eq. (27), where \( B = B_A \) is as in Eq. (7) and \( A = F_1 A + \alpha_0 C_\ell - I \).

To study the behavior of the system near \( F = F_1, \alpha = \alpha_0 \), four normal coefficients \( p_{rs}, r, s = 1, 2 \) must be computed. These all have the same general form as the first Lyapunov coefficient defined in Eq. (28). For example,

\[
p_{12} = \Re \left( \langle q_k, B_L (i(\tau_1 + \tau_2)I - A)^{-1} B_L (q_k, q_l, q_{N-\ell}) \rangle + \langle q_k, B_L (i(\tau_1 - \tau_2)I - A)^{-1} B_L (q_k, q_{N-\ell}, q_l) \rangle 
+ \langle q_k, B_L (-A^{-1} B_L (q_l, q_{N-\ell}), q_k) \rangle \right).
\]

Since \( B_L \) maps pairs of eigenvectors of \( A \) to multiples of eigenvectors (see Proposition 5 in Appendix A), these terms can all be evaluated in closed form, in the same manner as in Lemma 3.1 and Theorem 3.2. For example, writing \( z_1 = \omega_N^k, z_2 = \omega_N^l \), one obtains

\[
p_{12} = \frac{1}{N} \Re \left( \frac{P_L(z_1, z_2)P_L(z_1 \bar{z}_2, \bar{z}_2)}{\Re (\tau_1 + \tau_2) + 1 - F_1 p_A(z_1 \bar{z}_2)} + \frac{P_L(z_1, \bar{z}_2)P_L(z_1 \bar{z}_2, z_2)}{\Re (\tau_1 - \tau_2) + 1 - F_1 p_A(z_1 \bar{z}_2)} - P_L(z_2, \bar{z}_2)P_L(1, z_1) \right). \tag{39}
\]

The other three normal form coefficients \( p_{rs} \) can be expressed similarly in terms of the Laurent polynomials \( p_A \) and \( P_L \) and can therefore be evaluated numerically. The details of this calculation are left to the reader.

4.3. Neimark-Sacker bifurcation. Consider now the perturbed system in Eq. (38) for \( \alpha = \alpha_0 = 1 - F_1/F_2 \). As \( F \) increases past \( F_1 \), two eigenvalue pairs cross the

![Image of a figure illustrating eigenvalues and bifurcations](image_url)
imaginary axis simultaneously, resulting in a Hopf-Hopf bifurcation. The bifurcation picture now depends on the normal form coefficients $p_{ij}$, $i, j = 1, 2$ which may be computed as in the previous section. A complete description is given in [16], section 8.6.

In all cases of the L96 system that we examined, it turns out that $p_{11}p_{22} > 0$ (the “simple case” in the terminology of [16]) and $p_{11}p_{22} - p_{12}p_{21} < 0$ (“type I” in that reference). By the results in [16], there exist smooth curves $\alpha = \gamma_1(F)$ (thick red in Fig. 6) and $\alpha = \gamma_2(F)$ (thin red in Fig. 6), both passing through the point $(\alpha_0, F_1)$, such that for $F > F_1$ and close to $F_1$ and for $\gamma_1(F) < \alpha < \gamma_2(F)$ there exist two stable limit cycles for the system (38) (red stipples in Fig. 6). For $1 - F/F_2 < \alpha < \gamma_1(F)$ a stable limit cycle and an unstable limit cycle coexist (green stipples in Fig. 6). At the transition curve $\alpha = \gamma_1(F)$, a subcritical Neimark-Sacker (N-S) bifurcation occurs in which a two-dimensional unstable torus bifurcates from the unstable limit cycle. As a result, this limit cycle now becomes stable. Recall that an $N$-S bifurcation occurs in which a two-dimensional unstable torus bifurcates from the unstable limit cycle.

For $\alpha = 0$, it is therefore plausible that two stable limit cycles can coexist for $F > F_3$, $\gamma_1(F_3) = 0$, assuming that this value is defined. It is usually impossible to obtain detailed analytical information about $\gamma_1$ and therefore about $F_3$. The authors in [13] use numerical methods to trace these curves. Alternatively, using the approach outlined here it is straightforward to compute $\gamma_1(F_1)$ from the known normal form coefficients (see Section 4.2). The tangent line approximation of $\gamma_1$ at $\alpha = \alpha_0$ (magenta line in Fig. 6) can thus be found and $F_3$ can be approximated by $F_3^* = F_1 - F_1/\gamma_1'(F_1)$, shown as a magenta circle in that figure. The approximation is expected to be more accurate if $F_1$ and $F_2$ are relatively close, such that $\alpha_0 = (F_2 - F_1)/F_2$ is small.

4.4. Numerical results. We have carried out numerical searches for coexisting stable attractors. For various site numbers $N$, the exact Hopf bifurcation values $F_1, F_2$ were computed as well as the approximate N-S bifurcation value $F_3^*$, using a linear approximation. Starting with a value $F$ that is somewhat larger than $F_3^*$, we then computed 100 numerical solutions with initial data equal to $Fe$ plus a random normally distributed perturbation, for $0 \leq t \leq T = 1000$. At this $T$, solutions typically settled into one of several (usually two) stable limit cycles, which could be characterized by their spatial period. Lowering $F$ and following these stable limit cycles, approximate values $F_3 \approx F_3^*$ for the N-S bifurcations were found, and by increasing $F$, an approximate value $F_4$ was obtained up to which two stable limit cycles can be usually be observed.

In Table 2, we show results of these experiments for various site numbers $N$. In addition to the bifurcation parameters $F_i$, etc. we also give the spatial periods $m_1, m_2$ for the two limit cycles which appear as $F$ increases past $F_3$.

The results show that if $F_2$ is close to $F_1$ (i.e. if $\alpha_0$ is small), then the approximation $F_3^*$ is also close to $F_3$ and two stable coexisting limit cycles can be observed $(N = 12, 22, 28, 36)$. This happens generally whenever $N$ is sufficiently large, since then the eigenvalues of $FA - I$ are more closely spaced. On the other hand, for relatively small $N$ ($N = 14, 18$), we find that $F_2$ is substantially larger than $F_1$, therefore $\alpha_0$ is relatively large, and it is not clear if two stable limit cycles coexist for some $F > F_2$. 

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If $N$ is sufficiently large and has many small divisors, then more than two stable limit cycles may be observed for moderate $F$. For example, if $N = 144$ and $F = 2$, one can observe four different stable limit cycles, with spatial periods $m \in \{9, 24, 36, 144\}$. In fact, for any site number $N$ and $F$ near 1, there must be limit cycles with spatial periods for all divisors of $N$, since L96-like systems with site numbers that divide $N$ can be embedded periodically into a system with site number $N$.

5. **General existence and stability.** We now move on to a second type of generalization of the L96 system. One can modify the system to include site-specific dissipation, advection, and time-dependent forcing terms in order to consider, for example, changes in atmospheric dynamics and predictability over ocean and land. We therefore consider the general system

$$\dot{x}(t) = CG(x(t)) - Bx(t) + F(t),$$  \hspace{1cm} (40)

where $G$ is quadratic, $C$, $B$ are matrices, and $F(\cdot)$ is now a vector valued function. Problems of this form occur generally when spectral expansions of atmospheric flow models are truncated after second order interaction terms, leading to systems of ordinary differential equations with quadratic terms that often turn out to be energy-preserving; see e.g. [8], [6], and of course [17]. Another generalization of the L96 model, where forcing is time-periodic and possibly state dependent, was explored in [21].

5.1. **Stationary solutions.** We begin by looking at time-independent solutions of Eq. (40), that is solutions of the system

$$0 = CG(x) - Bx + F$$

where $F$ is a constant vector and $G$ is a quadratic energy preserving map. We assume that $C$ is invertible and therefore we may replace $C$ with $I$ and $B$ with $C^{-1}B$.

**Proposition 2.** Let $G$ be quadratic and energy preserving. Set $B_0 = (B + B^T)/2$ and assume that $B_0$ is positive definite. Then for any $F \in \mathbb{R}^N$ there exists a solution $x$ of

$$0 = G(x) - Bx + F$$  \hspace{1cm} (41)

and all such solutions satisfy

$$\|B_0^{1/2}x\| \leq \|B_0^{-1/2}F\|.$$  \hspace{1cm} (42)
Proof. Let \( x \) be such a solution. Taking the scalar product with \( x \), using the fact that \( G \) is energy-preserving, and applying the Cauchy-Schwarz inequality results in

\[
0 = -x^T B_0 x + x^T F = -\|B_0^{1/2} x\|^2 + x^T F \\
\leq -\|B_0^{1/2} x\|^2 + \|B_0^{1/2} x\| \|B_0^{-1/2} F\|.
\]

Rearranging this implies the estimate Eq. (42).

To show existence, we use a degree argument for continuous maps; see e.g. [4]. Consider the function

\[
\Phi : [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \Phi(t, z) = G(z) - Bz + t \cdot F.
\]

It is jointly continuous in \( t \) and \( z \). Let \( K = \{ z : \|B_0^{1/2} z\| \leq \|B_0^{-1/2} F\| + 1 \} \subset \mathbb{R}^N \).
Then Eq. (42) implies that the equation \( \Phi(t, x) = 0 \) does not have any solutions on the boundary \( \partial K \) for \( 0 \leq t \leq 1 \). Therefore the mapping degree \( \deg(\Phi(t, \cdot), K, 0) \) is constant for \( 0 \leq t \leq 1 \). Since for \( t = 0 \) the only solution is \( x = 0 \), due to estimate Eq. (42), \( \deg(\Phi(0, \cdot), K, 0) = \deg(\Phi(1, \cdot), K, 0) = 1 \), and the equation \( \Phi(1, x) = G(x) - Bx + F = 0 \) also has a solution in \( K \). \( \square \)

Figure 7. Stationary solutions of Eq. (41) with \( G = G_L, C = B = I \) and inhomogeneous forcing \( F \) for \( N = 120 \) sites. Here \( F_i = 1 \) for \( 0 \leq i < N/2 \) and \( F_i = M \) for \( i \geq N/2 \).

A similar result was proved with the same argument in [11]. The assumption on \( B_0 \) holds if both \( C \) and \( B \) are diagonal with positive entries, but not necessarily if \( B \) and \( C \) are merely positive definite. It should also be noted that solutions of Eq. (41) need not be unique. Indeed, it is known that for the L96 system there may be many different solutions for sufficiently large negative \( F \), assuming that \( N \) is divisible by a large power of 2 ([14]). These solutions appear in a cascade of
positive diagonal matrices and if $G$ is quadratic, a straightforward exercise shows

Solutions of Eq. (41) may be computed by choosing a sequence $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ and finding solutions $x_i$ of $\Phi(t_i, x_i) = 0$, with $\Phi$ as in Eq. (43). Each solution $x_i$ is computed using a Newton iteration with $x_{i-1}$ as a starting value. A thorough discussion of such homotopy methods may be found in [4].

This is illustrated in Fig. 7. Here $N = 120$ and $F$ is non-constant, with $F_i = 1$ for $i < N/2$ and $F_i = M$ for $i \geq N/2$, for five different choices of $M$. Solutions were found by choosing $t_i = i \delta$, with $\delta = 10^{-1}$ for $M = 2$ and $\delta = 10^{-3}$ for $M = 24$

For small and moderate $M$, the solutions satisfy $x_i \approx F_i$ except to the left of the transition points $i = 0, i = N/2$. For all $M$, solutions are maximal near $i = N/2$ and minimal near $i = 0$, with roughly monotone transitions and superposed oscillations of period $3$. It also appears that the maxima of these solutions are close to $M$ for small and moderate $M$, but not for large $M$. It is unclear how to explain this behavior. Note however that this is only one stationary solution for each $M$ out of possibly very many others. We did not investigate the dynamic stability of these stationary solutions, but suspect that they become unstable as $M$ increases.

5.2. Dynamic problem. An existence result for solutions of the time-dependent problem Eq. (40) can be shown under fairly general conditions for arbitrary time-dependent forcing $F(t)$.

Proposition 3. Let $C$ be a symmetric positive definite $N \times N$ matrix, $B$ an arbitrary $N \times N$ matrix, and assume that $G$ is an energy preserving mapping. Then for any $x_0 \in \mathbb{R}^N$ and any continuous curve $\mathbb{R} \ni t \mapsto F(t)$ there exists a global solution $\mathbb{R} \ni t \mapsto x(t)$ of Eq. (40) with $x(0) = x_0$.

Proof. By standard existence and uniqueness results, there exists a unique solution of Eq. (40) in a maximal open interval $I = (t_0, t_1)$ that contains $t = 0$. To prove that the solution exists on $\mathbb{R}$, it is sufficient to show that it remains bounded on any such open interval $I$. Consider first the set $[0, t_1)$ and assume that $t_1$ is finite.

Take the scalar product of Eq. (40) with $C^{-1} x(t)$, use the assumption that $G$ is energy-preserving, and integrate over $[0, t] \subset [0, t_1)$. The result is the identity

$$\frac{1}{2} (x(t)^T C^{-1} x(t) - x_0^T C^{-1} x_0) = \int_0^t (-x^T(s) C^{-1} B x(s) + x^T(s) C^{-1} F(s)) \, ds .$$

Since $C$ is positive definite, this implies after some standard estimates

$$\|x(t)\|^2 \leq c_0 + \int_0^t c_1 \|x(s)\|^2 \, ds + \int_0^t c_2 \|x(s)\| \, ds \leq c_0 + \int_0^t \left( c_1 + \frac{c_2}{2} \right) \|x(s)\|^2 \, ds + \frac{c_2 t_1}{2}$$

where the $c_i$ are suitable constants and in particular $c_2$ is a multiple of $\max_{0 \leq s \leq t} \|F(s)\|$. By the Gronwall lemma, $\|x(t)\| \leq \text{const.}$ for all $t < t_1$, with the constant independent of $t$. The solution therefore may be continued for all positive $t$. To show that the solution also may be continued for all negative $t$, replace $t$ with $-t$, $B$ with $-B$, and $G$ with $-G$ (which is also energy-preserving), and apply the same argument. $\square$

5.3. Global stability for small forcing. Here, we consider the non-stationary inhomogeneous problem Eq. (40) for constant forcing terms $F$. If e.g. $B$ and $C$ are positive diagonal matrices and if $G$ is quadratic, a straightforward exercise shows
that the zero solution is asymptotically stable for zero forcing. In that case, the mapping \( x \mapsto CG(x) - Bx \) is locally invertible near the origin, and therefore, by a standard perturbation argument, small solutions are also asymptotically stable for appropriately small forcing. Here we show that if \( G \) is in addition energy-preserving, then solutions for sufficiently small forcing are actually globally asymptotically stable and in particular unique.

**Proposition 4.** Let \( G \) be a quadratic and energy-preserving mapping. Let \( C \) be positive definite symmetric and assume that the symmetric part \( B_0 = (C^{-1}B + (C^{-1}B)^T)/2 \) is positive definite, Then there exists \( \varepsilon > 0 \) such that for all constant \( F \) with \( \|F\| < \varepsilon \) and all \( x_0 \) the solution of Eq. (40) with \( x(0) = x_0 \) converges to a unique solution \( x_\infty \) of the stationary problem Eq. (41).

**Proof.** Let \( x_\infty \) be any solution of Eq. (41). Set \( y(t) = x(t) - x_\infty \). Since \( G \) is quadratic, then due to Eq. (8) the function \( y(\cdot) \) satisfies

\[
C^{-1}\dot{y}(t) = G(y(t)) + A|x_\infty|y(t) - C^{-1}By(t).
\]

Here \( A \) is a linear matrix valued map defined on \( \mathbb{R}^N \) that depends on \( G \). By assumption the symmetric part of \( -C^{-1}B \) is negative definite. Now choose \( \delta > 0 \) such that the symmetric part of \( -C^{-1}B + A|x| \) is still negative definite for all \( \|x\| < \delta \). Multiply Eq. (44) with \( y^T(t) \) and use the assumption that \( G \) is energy-preserving. It follows that if \( \|x_\infty\| < \delta \), then for some \( \lambda > 0 \)

\[
\frac{1}{3} \frac{d}{dt}y^T(t)C^{-1}y(t) = y^T(t)(A|x_\infty| - C^{-1}B)y(t) - \lambda y^T(t)C^{-1}y(t).
\]

This inequality of course implies that \( y(t) = x(t) - x_\infty \to 0 \) and that \( x_\infty \) is unique. By Eq. (42), there is a constant \( c_3 \) such that \( \|x\| \leq c_3\|F\| \) for all solutions of the stationary problem. Thus if \( \|F\| < \varepsilon = \frac{\lambda}{c_3} \), then \( \|x_\infty\| < \delta \) and therefore \( x(t) \to x_\infty \) no matter what \( x_0 \) is.

\( \square \)

5.4. **Numerical examples.** Recall how to rescale the system (2) so that it has the form of Eq. (4). If \( \tilde{x}_i(t) \) is the solution of Eq. (2) with initial state \( \tilde{x}_0 \), and \( x_i(t) \) is the solution of Eq. (4) with \( F = \alpha\gamma/\beta^2 \) and initial state \( \alpha\beta^{-1}x_0 \), then \( \tilde{x}_i(t) = \alpha\beta^{-1}x_i(\beta^{-1}t) \).

We now consider inhomogeneous systems of the form

\[
\dot{x}_i = \alpha_i x_{i-1}(x_{i+1} - x_{i-2}) - \beta_i x_i + \gamma_i
\]

with

\[
(\alpha_i, \beta_i, \gamma_i) = \begin{cases} 
(1, 1, 2), & \text{for } i \leq N/2 \\
(\alpha, \beta, \gamma), & \text{for } i > N/2 
\end{cases}
\]

This means that at sites \( i < \frac{N}{2} \) ("left half"), the system is expected to behave as shown in Fig. 1. At the remaining sites ("right half") the scaling argument shows that, qualitatively, the dynamics should be approximately the same as the dynamics of the standard system with \( F = \alpha\gamma/\beta^2 \), multiplied by \( \beta/\alpha \) and sped up by a factor \( \beta^{-1} \). Although this scaling argument cannot be applied to sites very near the boundaries between the regions, some insight into the dynamics at the boundaries can be gained by considering the behavior on either side.

Figure 8 shows two Hovmoeller plots for systems with \( N = 100 \) and \( 500 \leq t \leq 510 \), with random initial values at \( t = 0 \). Note first that the parameters for the left halves of each panel are the same, but the solution behavior is clearly quite different,
Figure 8. Hovmoeller plots showing inhomogeneous advection and dissipation as described in Eq. (46), for $N = 100$. Both panels use the same parameters $(1, 1, 2)$ in the left half, but solutions have very different behavior. **Left:** Sites in the right half have parameters $(0.5, 1, 1)$. Smaller advection in the right half leads to smaller spatial amplitudes. Perturbations are seen to travel to the right. **Right:** Sites in the right half have parameters $(1, 1.5, 1)$. Larger dissipation in the right half leads to nearly constant solutions over a substantial range of sites.

showing a slowly moving solution with spatial period near $m = 4$ in the left panel and a faster moving solution with period near $m = 8$ in the right panel. The left panel shows the effect of a lower advection in the right half while the right panel shows the effect of increased dissipation in the right half. Some expected effects can be noticed right away. For example, the left panel shows waves of larger amplitude with midpoint shifted upward in the left half, which leads to large fluctuations at its downstream boundary $i = 50$ around $t = 503$ and $t = 506$ due to the large positive advection term there. In both halves of the left panel, perturbations at the downstream boundary are seen to travel to the right, as predicted by the discussion in Section 3.5. The right panel shows waves travelling to the left in the left half, while in the right half the effective $F = \alpha \gamma / \beta^2 = 8/9$ now is less than the Hopf bifurcation value and the solution is close to constant for a substantial range of sites.

We also noticed some unanticipated effects, especially for larger forcing. For example, a very small parameter difference between the regions may completely change which attractor the system reaches.

6. **Conclusions and open questions.** In this paper we have described several analytical and numerical tools for the study of the L96 system. These tools are equally useful for several possible generalizations of the L96 system, which are also proposed here. Specifically, we have exploited the fact that linearizations of the system about constant states are given by circulant matrices to introduce a graphical approach for the study of its spectrum and in order to demonstrate that normal forms of the system at bifurcation points may be computed analytically. This allows one to compute, for example, the first Lyapunov exponent for the L96 system, proving that the Hopf bifurcation must be supercritical.

Further, we have classified all advection terms that share essential features with the original Lorenz term. Our analysis identifies the original L96 advection term $G_L$ as the simplest such term that produces the signature complex wave behavior. While this version of the system appears to have received the most attention in the
literature so far, we have identified and classified a whole class of related advection
terms that can be expected to lead to similar dynamics.

We have also considered possible modifications of the L96 system by allowing
site-dependent advection, dissipation, and forcing. We have presented basic results
for the existence and stability of solutions for time-independent and time-dependent
problems. We have also explored examples numerically and explained some of the
new phenomena using site-dependent scaling arguments.

The possible modifications presented in this paper open up many avenues of
investigation. It seems natural to consider systems with alternative advection terms
for large forcing in order to discover whether, and in which cases, the rich dynamic
behavior of the L96 system still results. A concrete problem in this domain is to
compare two systems having different advection terms but the same linearization
and thus the same eigenvalue curve, and to study how they differ with increasing
forcing.

There is also a variety of questions related to bifurcations to explore. Do multiple
stable limit cycles appear also for other advection terms that admit Hopf bifurca-
tions, followed by chaotic behavior for larger forcing? Do these phenomena persist
universally in spite of the large flexibility in the set of quadratic, equivariant, energy-
preserving, localized advection terms that has been identified here? It was observed
numerically that in the case of large site numbers with many small divisors, more
than two stable limit cycles may exist for $F$ near 1. Is there a rigorous explanation
for this phenomenon?

A broader range of questions can be asked for inhomogeneous systems. In the
case of inhomogeneous advection terms, constant equilibrium solutions can still be
found, but the application analysis becomes more involved because linearizations no
longer result in circulant matrices. One should ask about the stability of stationary
solutions in the case of inhomogeneous forcing such as e.g. in Fig. 7. How do
inhomogeneous advection/dissipation/forcing affect attractors and their basins of
attraction? And finally, are there better ways of characterizing and identifying
attractors than have been found numerically?

The Lorenz ’96 system exhibits a wide range of complex dynamical behavior.
Together with its many possible modifications, it is a showcase for a variety of
interesting phenomena and a rich source of mathematical challenges. It is our
hope that this paper will ultimately contribute to better attention to this subject,
especially from the mathematics community.

Appendix A: Discrete fourier transform. Here we collect some basic facts
relating the L96 system to the discrete Fourier transform. The same arguments can
be applied to any $G$-map. Let

$$
F_N = \frac{1}{\sqrt{N}} \left( e^{2\pi i k t / N} \right)_{0 \leq k, t < N} = \frac{1}{\sqrt{N}} \left( \omega_N^{k t} \right)_{0 \leq k, t < N}
$$

be the normalized symmetric Fourier matrix, where as before $\omega_N = e^{2\pi i / N}$. We
denote the columns of $F_N$ by $q_t$ (see also Eq. (35)). The conjugate matrix $F_N^*$
has columns $q_{N-t}$ and is the inverse of $F_N$.

Let $A$ be any real circulant $N \times N$ matrix, with top row $(c_0, c_1, \ldots, c_{N-1})$. Let
$p_A(z) = \sum c_j z^j$ be the associated Laurent polynomial, where the summation
extends over $-N/2 < j \leq N/2$. Then $A$ has eigenvalues $p_A(\omega_N^t)$ with right eigenvectors $q_t$. The left eigenvector of $A$ for the eigenvalue $p_A(\omega_N^t)$ is the $t$-th row of $F_N^*$
or $q_{N-t}^T$. The matrix $F_N^* A F_N$ is diagonal with entries $p_A(\omega_N^t)$.
We now describe the action of a bilinear \( \langle \rho \rangle \)-equivariant map \( B \), defined in Eq. (6), in terms of the Fourier matrix. As before, all indices are understood modulo \( N \). Let \( Q \) be the associated symmetric matrix, defined in Eq. (13), extended \( N \)-periodically for negative indices.

**Proposition 5.** For \( z, w \in \mathbb{C} \), \( z \neq 0 \neq w \) let

\[ P_B(z, w) = \sum_{-N/2 < r, s \leq N/2} z^r Q_{rs} w^s. \]  

(a) Let \( 0 \leq k, \ell < N \), then

\[ B(q_k, q_\ell) = P_B(\omega_N^k, \omega_N^\ell) q_{k+\ell} \quad \text{and} \quad G(q_k) = \frac{P_B(\omega_N^k, \omega_N^k)}{2\sqrt{N}} q_{2k}. \]  

(b) In particular, for the L96 case,

\[ P_L(z, w) = z^{-1}(w - w^{-2}) + w^{-1}(z - z^{-2}) \]  

and

\[ B_L(q_k, q_\ell) = P_L(\omega_N^k, \omega_N^\ell) q_{k+\ell}, \quad G_L(q_k) = \frac{1 - \omega_N^{-3k}}{\sqrt{N}} q_{2k}. \]  

(c) Let \( A = B(\cdot, e) \) be the linearization of a quadratic map \( G \) about the unit vector \( e \) and let \( p_A \) be the associated Laurent polynomial with coefficients coming from the top row of \( A \). Then for all non-zero \( z \in \mathbb{C} \)

\[ p_A(z) = P_B(z, 1). \]  

(d) Let \( y \in \mathbb{R}^N \). Then

\[ (F_N^* G(F_N y))_t = \sum_{j=0}^{N-1} \frac{P_B(\omega_N^j, \omega_N^{t-j})}{2\sqrt{N}} y_j y_{t-j}. \]  

These formulæ can be proved by direct computation.

Finally it should be noted that if \( N \) is a multiple of 3 and \( k \) is a multiple of \( N/3 \), then \( G_L(q_k) = 0 \), since then \( \omega_N^{-3k} = 1 \). More generally it can be shown that \( P_L(z, w) = 0 \) on the torus where \( |z| = |w| = 1 \) if and only if \( z \) and \( w \) are of the form \( e^{\pm 2\pi i/3} \).

**Appendix B: Nondissipative systems.** In this section we consider versions of Eq. (24) that contain only advection terms; that is,

\[ \dot{x} = G(x) \]  

where \( G \) is a \( \mathcal{G} \)-map. The system (54) appears as a formal limit for \( F \to \infty \), if \( t \) and \( x \) are rescaled as \( \tau = F^\gamma t, y(\tau) = F^{-\gamma} x(\tau) \) with \( \gamma > \frac{1}{2} \). For \( \gamma = \frac{2}{3} \) this rescaling was proposed in [20]. The limiting behavior as \( F \) becomes large was studied systematically in [10]. Rescaling results in the system

\[ \frac{d}{d\tau} y = G(y) - F^{-\gamma} y + F^{1-2\gamma}. \]  

The system (54) resulting from sending \( F \to \infty \) is also known as inviscid Lorenz 96 system and was studied in [1]. In [22], the inviscid system (54) was introduced for the symmetric case \( G = G_3 + G_5 \). Another rescaling was proposed in [5], with the goal of deriving and investigating a continuous long wave approximation,
which is effectively a quasilinear second order partial differential equation without dissipation and with conserved energy in the case of zero forcing.

Here are some observations for the cases $G = G_3$ (the original Lorenz system) and the symmetric version $G = G_3 - \tilde{G}_3$.

If $G = G_3$ or $G = G_3 - G_3$ and if the number of sites $N$ is a multiple of 3, initial data that are periodic with period 3 result in constant solutions. If $G = G_3 - \tilde{G}_3$ and the number of sites is even, initial data that are periodic with period 2 also result in constant solutions. For $G = G_3 - \tilde{G}_3$ in the case where $N$ is a multiple of 6, there is therefore a four-dimensional variety of initial data with period 6 that result in constant solutions.

In the case $G = G_3 - \tilde{G}_3$, the sum over all sites $\sum_j x_j(t)$ is constant along solutions. If $N$ is even, then the energy $\sum_j x_{2j}(t)^2$ at the even-numbered sites and the energy at the odd-numbered sites are conserved separately. If $N$ is a multiple of 3, then the sums over the sites that are three spaces apart, i.e. $\sum_j x_{3j+k}(t)$ for $k = 0, 1, 2$ are separately conserved.

For the case $G = G_3 - \tilde{G}_3$, if $N = 4$, the system is Hamiltonian. In this case there are the two conserved quantities $x_0(t)^2 + x_2(t)^2 = \rho_0^2$ and $x_1(t)^2 + x_3(t)^2 = \rho_1^2$. Introducing polar coordinates $x_0(t) + i x_2(t) = \rho_0 e^{i \alpha_0(t)}$ and $x_1(t) + i x_3(t) = \rho_1 e^{i \alpha_1(t)}$, one obtains after some algebra the system

$$a_0'(t) = -\rho_1 \sqrt{2} \cos \left( \alpha_1(t) + \frac{\pi}{4} \right)$$

$$a_1'(t) = \rho_0 \sqrt{2} \cos \left( \alpha_0(t) + \frac{\pi}{4} \right).$$

This is a Hamiltonian system with $H(a_0, a_1) = \sqrt{2} \left( \rho_0 \sin(a_0 + \frac{\pi}{4}) + \rho_1 \sin(a_1 + \frac{\pi}{4}) \right)$.

For the case $G = G_3 - \tilde{G}_3$ and $N = 6$, all solutions may be written in the form $x_j(t) = c_j + \frac{1}{3}(-1)^j y_j(t)$ for $j = 0, 1, 2$ and $x_j(t) = c_{j-3} + \frac{1}{3}(-1)^j y_{j-3}(t)$, for $j = 3, 4, 5$, for suitable functions $y_0, y_1, y_2$ and constants $c_j$. Let $c = (c_0, c_1, c_2)^T$ and $y(t) = (y_0(t), y_1(t), y_2(t))^T$. Then a direct calculation shows that $y(t)$ satisfies the linear system $\ddot{y}(t) = c \times \dot{y}(t)$.

Appendix C: Numerical considerations. The papers [18] and [19] used the fourth order Runge-Kutta scheme (RK4) with fixed temporal step width $\Delta t = 0.05$ to solve Eq. (3) numerically. While this method is straightforward to implement and makes all computations replicable, it only has mid-level accuracy.

In this paper, we use a high accuracy ODE solver with variable step width instead, described in the article [23] and implemented as lsoda in the R package deSolve (see [24]). To test the accuracy of the solver, we solved the advection-only version Eq. (54) with the L96 advection term $G_L$. Then the energy $E(t) = \sum_j x_j^2(t)$ is exactly conserved. By rescaling, if $t \mapsto x(t)$ is a solution, then so is $t \mapsto \lambda x(\lambda t)$; that is, for larger energy value the system has essentially the same trajectories on a faster time scale. The performance of the RK4 solver with a fixed time step can then be expected to deteriorate, while the performance of the lsoda solver should be less affected, since it has automatic stepsize control and is designed to handle stiff equations.

As expected, for both solvers the numerical energy is not exactly constant. For small values of the energy, e.g. near 1, both solvers nearly conserve energy for long time intervals. If the energy of the initial value becomes larger, solutions lose energy for both solvers, but solutions found with LSODA lose energy at a much slower rate. The papers [1, 2, 3, 10] use a variety of numerical solvers and emphasize the need
to use high order adaptive methods, as measured e.g. by observed energy loss for inviscid systems.

As an example, Fig. 9 plots the relative energy loss $\frac{\Delta E(t)}{E(0)}$ of numerical solutions with initial data $x_j(0) = c(1 + \sin 2\pi \frac{j}{N})$ for $0 \leq j < N = 36$, where $c$ is chosen such that $E(0) = 400$. The solution computed with lsoda loses about $2 \cdot 10^{-5}\%$ of its energy per unit time while the solution from RK4 with $\Delta t = 0.05$ loses between $0.03\%$ and $0.25\%$ of the energy per unit time.

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