Current Algebra in Three Dimensions

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Abstract

We study a three dimensional analogue of the Wess–Zumino–Witten model, which describes the Goldstone bosons of three dimensional Quantum Chromodynamics. The topologically non–trivial term of the action can also be viewed as a nonlinear realization of Chern–Simons form. We obtain the current algebra of this model by canonical methods. This is a three dimensional generalization of the Kac–Moody algebra.

PACS: 11.40.F, 11.10.Lm, 02.40.+m
Three dimensional field theories with topological terms in the action have been studied recently in many physical contexts such as Chern–Simons theories [1], models for antiferromagnets [2] and anyon statistics [3]. (See [4] for an overview). It is known that topological terms affect the statistics of solitons [3], [5], and can change the canonical commutation relation of observables [6], [7], [8]. In three dimensions, this issue was studied in refs. [9], [10]. From another direction, three dimensional nonlinear models have been shown to be renormalizable [11] in the $1/N$ expansion, although they are not so by power counting.

In this paper we study a three dimensional nonlinear sigma model on a coset space with a topological term. It can be viewed as a three dimensional analogue of the Wess–Zumino–Witten [5] model or as a nonlinear Chern–Simons theory[1]. The model arises as the low energy limit of three dimensional Quantum Chromodynamics (QCD) [12]. It is also related to certain models for anti–ferromagnetism that arise in attempts to explain high $T_c$ superconductivity[2]. There are solitons in this model whose statistics is determined by the topological term. Furthermore, the theory should be renormalizable in the $1/N$ expansion. The focus of the present paper is the canonical formalism of this model. We discover by this method a three dimensional generalization of the Kac–Moody algebra; it is a non–trivial abelian extension of the naive current algebra. Like the Kac–Moody algebra, this can be further extended to a semi–direct product with the algebra of vector fields.

The Wess–Zumino–Witten (WZW) model [5] in an even dimensional space–time describes anomalous global symmetries. The field variable $g$ takes values in a compact Lie group $G$ (typically $SU(N)$). It satisfies the classical equation of motion (for two dimensions):

$$\partial_\mu (g^{-1} \partial^\mu g) + \lambda \epsilon^{\mu\nu}(g^{-1} \partial_\mu gg^{-1} \partial_\nu g) = 0.$$  

If $\lambda = 0$, the equation is invariant under two discrete symmetries, $P_1 : g \rightarrow g^{-1}$, and $P_2 : g(t, x) \rightarrow g(t, -x)$. The WZW term breaks the symmetry down to the product $P_1P_2$. 

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Thus, the WZW term forces the fundamental field of the theory to be a pseudo–scalar. It is possible to formulate [5],[6], [13], this theory in a canonical formalism entirely in terms of currents. The classical Poisson brackets of the currents then define an infinite dimensional Lie algebra, the current algebra. In the absence of the WZW term the current algebra is just the set of maps from space to the Lie algebra of $G$, with the point–wise bracket [6].

The current algebra is modified by the WZW term. In general, it provides an extension of the current algebra by an abelian algebra[7],[8]. In particular, in two dimensional space–time, the current algebra is a central extension of the loop algebra, the well-known affine Kac–Moody algebra. The representation theory of this algebra is well–understood. The relation to WZW model has clarified the representation theory by relating it to conformal field theory.

Much less is known about the representation theory of current algebras in higher dimensions. Some progress has been made in this direction [14], although a complete understanding is still not available. This motivated us to look for an analogue of the WZW model in $2 + 1$ dimensions. This would be a way to study current algebras and Bose–Fermi equivalence in a context simpler than $3 + 1$ dimensions, yet more general than $1 + 1$ dimensions.

However, there are no anomalies for continuous symmetries in odd dimensional space–time. This is related to the fact that $H^4(G)$ vanishes for the classical Lie groups. We can find an analogue by looking for a nonlinear sigma model on a target space with $H^4$ non–zero. Futhermore the additional term must preserve parity if the field variable is a pseudo–scalar. The answer[12] is the target space $Gr_{n,N} = U(N)/U(n) \times U(N − n)$, the Grassmannian. We can parametrize $Gr_{n,N}$ by an $N \times N$ matrix $\Phi$:

$$Gr_{n,N} = \{\Phi | \Phi^\dagger = \Phi; \Phi^2 = 1; \text{tr} \Phi = N − 2n\}.$$  

The nonlinear model with this target space has the field equation $[\Phi, \partial_\mu \partial^\mu \Phi] = 0$. The cohomology group $H^4(Gr_{n,N}) = Z \oplus Z$, (for $N \geq 4, n \geq 2$) is generated by $\omega_4$ and $\omega_2 \wedge \omega_2$, 

$$3$$
where $\omega_{2k} = \text{tr}\Phi(d\Phi)^{2k}$. Of the two generators, only $\omega_4$ is odd under the transformation $\Phi \rightarrow -\Phi$. Thus we arrive at a generalization of the WZW model to $2 + 1$ dimensions,

$$F_\pi[\Phi, \partial_\mu \partial^\mu \Phi] + \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \partial_\mu \Phi \partial_\nu \Phi \partial_\rho \Phi = 0. \tag{1}$$

The coupling constant $F_\pi$ has dimension of inverse length in the classical theory. This equation of motion follows from the multivalued action

$$S[\Phi] = \frac{F_\pi}{2} \int_M \text{tr}d\Phi \ast d\Phi + \frac{k}{64\pi} \int_{M_4} \text{tr}\Phi(d\Phi)^4. \tag{2}$$

Here, $M_4$ is a four dimensional manifold whose boundary is space–time $M$. As in the WZW model, in order that $\exp(iS[\Phi])$ be independent of the continuation into the fourth dimension, $k$ must be an integer. This theory is invariant under parity (with $\Phi$ transforming as a pseudo–scalar) if $N = 2n$. In this case it is the low energy limit of three dimensional QCD with an even number $2n$ of flavors and $k$ colors [12]. But we will study this theory for general $N$ and $n$.

We will now present a canonical formulation of this theory, in terms of a set of Poisson brackets for the basic observables, a set of first class constraints and a Hamiltonian. The Poisson brackets of the theory will be a generalization of the Kac–Moody algebra to $2 + 1$ dimensions. The canonical formalism is in terms of a decomposition of space–time $M = \Sigma \times R$, $\Sigma$ being the two dimensional surface at fixed time. It is possible to derive this formalism from an action principle, but the appropriate one is not the multi–valued action (2). Define a new variable $g$ valued in $G \equiv U(N)$. One can always write $\Phi = ge^g$ with $\epsilon = \text{diag}\{1, \cdots, 1, -1, \cdots, -1\}$, and $\text{tr}\epsilon = N - 2n$. Then $\Phi$ is invariant under right multiplication of $g$ by elements that commute with $\epsilon$; i.e., under $g \mapsto gh, h \in H \equiv U(n) \times U(N - n)$. These transformations are therefore like gauge transformations and we can write an action for the theory in terms of $g$ that respects this gauge invariance:

$$S[g, A] = -2F_\pi \int_M \text{tr}\left((g^\dagger dg - A) \ast (g^\dagger dg - A)\right) - \frac{k}{8\pi} \int_M \text{tr}\left(\epsilon(g^\dagger dg)^3 - \frac{1}{3}(g^\dagger dg\epsilon)^3\right). \tag{3}$$
The one form $A$ is an auxiliary gauge field valued in $H$, the Lie algebra of $H$. Its purpose is to remove the unwanted degrees of freedom. In this form of the action, the topologically non-trivial term is a nonlinear realization of the Chern–Simons term. Unlike (2), action (3) is a local integral on space–time but it is only gauge invariant up to a multiple of $2\pi$.

It is now possible to derive Poisson brackets and constraints from this action by a conventional procedure. The Poisson brackets so obtained at first will involve nonlinear (cubic) terms in $\Phi$. However we can remove these by appropriate redefinition of the generators. The variables $\Phi, J$ that satisfy simple commutation relations are, in this language, $\Phi = gg^\dagger$ and

$$J = g \left( F_\pi R + \frac{k}{16\pi} \epsilon^{ij}(g^\dagger \partial_i gg^\dagger \partial_j g\epsilon + \epsilon g^\dagger \partial_i gg^\dagger \partial_j g - \epsilon g^\dagger \partial_i g \epsilon g^\dagger \partial_j g\epsilon) \right) g^\dagger.$$  

Here, $R$ is the projection of $g^\dagger \dot{g}$ on the orthogonal complement of the gauge group and must satisfy the constraint $[R, \epsilon]_+ = 0$. (The symbol $[\ ,\ ]_+$ will denote the anti-commutator throughout the paper). Our conventions are that $\Phi$ is hermitian and $J$ anti–hermitian.

We present only the results of the canonical analysis, leaving the details for a longer publication. The basic observables of the theory are $\Phi$ and $J$, (which is essentially the time component of the current), specified on $\Sigma$. It is natural to think of $\Phi$ as a scalar on $\Sigma$ and to $J$ as a scalar density (two–form). Let us also introduce the test functions $\lambda : \Sigma \to G$ scalar, and $\xi : \Sigma \to G$ scalar density. ($G$ denotes the Lie algebra of $G$). We give the Poisson brackets in terms of the dimensionless quantities $\Phi(\xi) \equiv \int_\Sigma \text{tr}(\Phi\xi)d^2x$ and $J(\lambda) \equiv \int_\Sigma \text{tr}(J\lambda)d^2x$. They are

$$\{\Phi(\xi), \Phi(\xi')\} = 0 , \quad \{J(\lambda), \Phi(\xi)\} = \Phi([\lambda, \xi]) , \quad \text{and}$$

$$\{J(\lambda), J(\lambda')\} = J([\lambda, \lambda']) + k\Phi(\omega(\lambda, \lambda')).$$  

(4)

In (4), $\omega$ is defined as $\omega(\lambda, \lambda') = (1/16\pi) \epsilon^{ij} [\partial_i \lambda, \partial_j \lambda']_+$. If the space $\Sigma$ is a torus, we can
write these relations more explicitly in a plane wave basis:

\[
\{ \Phi^a_m, \Phi^b_n \} = 0 , \quad \{ J^a_m, \Phi^b_n \} = f^{abc} \Phi^c_{m+n} , \quad \text{and} \\
\{ J^a_m, J^b_n \} = f^{abc} J^c_{m+n} - \frac{k}{16\pi} d^{abc} \epsilon^{ij} m_i n_j \Phi^c_{m+n} .
\] (5)

In (5), \( m, n \) are two-dimensional vectors with integer components. Also, \( d^{abc} \) is the usual symmetric cubic invariant of \( U(N) \) and \( f^{abc} \) the structure constants.

The algebra has to be supplemented by two constraints

\[
\Phi^2 = 1 \quad \text{and} \quad [J, \Phi]_+ + \frac{k}{16\pi} \epsilon^{ij} (\partial_i \Phi \partial_j \Phi) = 0 .
\] (6)

One can verify that these are first class constraints. This is one major difference between our treatment of the problem and the usual canonical formalism for similar models [9]. With our method, second class constraints never arise and there is no need to introduce Dirac brackets. Actually, our constraints (6) satisfy even stronger relations that the conditions for being first class. It can be easily checked from (4) that the Poisson brackets of \( \Phi \) and \( J \) with (6) vanish weakly. This means that every two weakly equivalent observables \( A \approx B \) of the theory will have weakly equivalent Poisson brackets with any third observable \( C \): \( \{ A, C \} \approx \{ B, C \} \). Both constraints and Poisson brackets are also invariant under diffeomorphisms of \( \Sigma \).

The canonical formalism is completed by the Hamiltonian

\[
H = \frac{1}{2} \int_{\Sigma} \text{tr} \left( -\frac{1}{F^{2} \sqrt{g}} (J + \frac{k}{32\pi} \epsilon^{ij} \partial_i \Phi \partial_j \Phi)^2 + \frac{F^{2} \sqrt{g}}{4} g^{ij} \partial_i \Phi \partial_j \Phi \right) d^2 x .
\] (7)

Of course, the hamiltonian does depend on the metric \( g_{ij} \) on \( \Sigma \). The equations of motion that follow are

\[
\dot{\Phi} = \frac{1}{F^{2} \sqrt{g}} [J, \Phi] \\
\dot{J} = -\frac{1}{4} F^{2} \sqrt{g} g^{ij} [\Phi, \partial_i \partial_j \Phi] + \frac{k}{32\pi F^{2} \sqrt{g}} \epsilon^{ij} \left( [\partial_i J, \partial_j \Phi]_+ - \partial_i (\Phi J \Phi) \partial_j \Phi - \partial_j \Phi \partial_i (\Phi J \Phi) \right) .
\] (8)
The equation (1) for \( \Phi \) then follows (in flat space) from this system of first order equations and the constraints. This completes our discussion of the canonical formalism.

Equations (4), (or (5)), define our current algebra \( \mathcal{G}_k \) (more exactly a current–field algebra). Notice that the Poisson brackets yield linear relations in \( \Phi \) and \( J \), so (4) defines a Lie algebra. If we set \( k = 0 \), the \( J \)'s alone form a subalgebra \( \mathcal{J} \). The \( \Phi \)'s generate an abelian sub–algebra \( \mathcal{V} \) of \( \mathcal{G} \)-valued densities. The vector space \( \mathcal{V} \) can be identified with dual of the Lie algebra \( \mathcal{J} \) by the natural pairing \(< \lambda, \xi > = \int_\Sigma \text{tr}(\lambda \xi) d^2x\), so that it carries the co–adjoint representation of \( \mathcal{J} \). When \( k = 0 \), our algebra \( \mathcal{G}_0 \) reduces to the semi–direct product of \( \mathcal{J} \) with its co–adjoint representation.

When \( k \neq 0 \), \( \mathcal{G}_k \) is an abelian extension of the map algebra \( \mathcal{J} \) by its co–adjoint representation. The Jacobi identity of \( \mathcal{G}_k \) is equivalent to the statement that \( \omega : \mathcal{J} \wedge \mathcal{J} \to \mathcal{V} \) is a two–cocycle of the Lie algebra cohomology:

\[
\partial \omega(\lambda_1, \lambda_2, \lambda_3) \equiv [\lambda_1, \omega(\lambda_2, \lambda_3)] + \omega(\lambda_1, [\lambda_2, \lambda_3]) + \text{cyclic} = 0,
\]

which can be verified by direct computation. If \( \omega \) had been exact there would have been a linear function \( \mu : \mathcal{J} \to \mathcal{V} \) such that

\[
\omega(\lambda, \lambda') = \partial \mu(\lambda, \lambda') \equiv -\mu([\lambda, \lambda']) + [\lambda, \mu(\lambda')] - [\lambda', \mu(\lambda)].
\]

There is no such \( \mu \), so that we cannot reduce our algebra to a semi–direct product by a change of basis; \( \mathcal{G}_k \) is a non–trivial abelian extension of \( \mathcal{J} \) by \( \mathcal{V} \).

It is useful to note that the above extension can be ‘exponentiated’ to an extension of the group of maps \( g : \Sigma \to G \) by the vector space \( \mathcal{V} \), (thought of as an abelian group). The multiplication law \( \circ \) is

\[
(g_1, \xi_1) \circ (g_2, \xi_2) = (g_1g_2, \xi_1 + g_1\xi_2g_1^{-1} + k\Omega(g_1, g_2)),
\]
where \( \Omega(g_1, g_2) = (1/16\pi) \epsilon^{ij} \partial_i g_1 \partial_j g_2 g_2^{-1} g_1^{-1} \). The associativity of the group multiplication \( \circ \) requires that \( \Omega \) be a group 2-co-cycle: [15]

\[
\partial \Omega(g_1, g_2, g_3) \equiv -\Omega(g_1 g_2, g_3) + \Omega(g_1, g_2 g_3) - \Omega(g_1, g_2) + g_1 \Omega(g_2, g_3) g_1^{-1} = 0.
\] (12)

This identity can be proved by direct computation. It is possible to understand the constraints (6) as describing a co-adjoint orbit of the above group. The symplectic form and hence the part of the action that is linear in time derivatives can be understood from Kirillov’s method of orbits applied to this case. In fact, it turns out that the Poisson bracket structure derived in this way coincides with (4).

Finally, recall that the Kac–Moody algebra is invariant under the action of the Virasoro algebra. In fact the generators of the Virasoro algebra can be written in terms of the currents. The analogue of the Virasoro algebra in our case is the algebra of vector fields on \( \Sigma \). The Lie algebra \( \mathcal{G}_k \) is invariant under diffeomorphisms, so that it can be extended as a semi–direct product with the algebra of vector fields on \( \Sigma \). If \( u \) and \( v \) are such vector fields, and \( L \) is the generator associated to them, satisfying \( \{L(u), L(v)\} = L([u, v]) \), then,

\[
\{L(u), J(\lambda)\} = J(u^i \partial_i \lambda) \quad \text{and} \quad \{L(u), \Phi(\xi)\} = \Phi(\partial_i (u^i \xi)).
\] (13)

It would be interesting to develop a representation theory for the algebra (5). Physically, that would correspond to quantizing the above field theory. This might look impossible at first because the theory is not renormalizable by power counting. However, as remarked in [12], the theory is renormalizable in the \( 1/N \) expansion, provided one allows for massive vector fields to be dynamically generated. We would first consider the limit \( N \to \infty \) (keeping \( n \) and \( k \) fixed) that is solvable by the saddle point method. This model has a non–trivial UV fixed point. This is the analogue of the UV fixed point of the WZW model in 1+1 dimensions (although in that case the UV fixed point is trivial). The WZW model also has a non–trivial IR stable fixed point. It is possible that there is an analogous (non–trivial) IR fixed point in our theory as well.
Another interesting issue is that of Fermi–Bose correspondence. In two dimensions the WZW model, at the IR fixed point, corresponds to a free Fermi theory. There has already been an attempt to prove such an equivalence for the $CP^1$ model [16]. However there is, at present, no reliable approximation method to study this issue, because the correspondence breaks down for $CP^N$ with $N > 1$. Our model should have Fermionic equivalents for any $N, n$, so that the issue can be studied within the $1/N$ expansion. We will report on work in this direction in a later publication.

We would like to thank A.P. Balachandran, P. Teotonio and Z. Yang for discussions. This work was supported in part by the US Department of Energy, Contract No. DE-AC02-76ER13065.
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