Exterior spacetime for stellar models in 5-dimensional Kaluza-Klein gravity

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Abstract

It is well-known that Birkhoff’s theorem is no longer valid in theories with more than four dimensions. Thus, in these theories the effective four-dimensional picture allows the existence of different possible, non-Schwarzschild, scenarios for the description of the spacetime outside of a spherical star, contrary to four-dimensional general relativity. We investigate the exterior spacetime of a spherically symmetric star in the context of Kaluza-Klein gravity. We take a well-known family of static spherically symmetric solutions of the Einstein equations in an empty five-dimensional universe, and analyze possible stellar exteriors that are conformal to the metric induced on four-dimensional hypersurfaces orthogonal to the extra dimension. We show that all these non-Schwarzschild exteriors can continuously be matched with the interior of the star, indicating that the matching conditions at the boundary of a star do not require an unique exterior. Then, without making any assumptions about the interior solution, we prove the following statement: the condition that in the weak-field limit we recover the usual Newtonian physics singles out an unique exterior. This exterior is “similar” to Scharzschild vacuum in the sense that it has no effect on gravitational interactions. However, it is more realistic because instead of being absolutely empty, it is consistent with the existence of quantum zero-point fields (P.S. Wesson, Phys. Essays, Orion 5, 591(1992)). We also examine the question of how would the deviation from the Schwarzschild vacuum exterior affect the parameters of a neutron star. In the context of a model star of uniform density, we show that the general relativity upper limit $M/R < 4/9$ is significantly increased as we go away from the Schwarzschild vacuum exterior. We find that, in principle, the compactness limit of a star can be larger than 1/2, without being a black hole. The generality of our approach is also discussed.

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1 Introduction

In Newtonian and relativistic theories of gravity the simplest models of isolated stars are provided by static spherically symmetric distributions of matter surrounded by empty space. In general relativity, the interior of a star is modeled by a solution of the Einstein field equations with some energy-momentum tensor satisfying physical conditions. The exterior of an isolated star is described by the Schwarzschild metric, which according to Birkhoff’s theorem is the unique asymptotically flat vacuum solution with spatial spherical symmetry. At the boundary of the star, which is a spherical surface of zero-pressure, both solutions have to be matched following a standard procedure in general relativity. These models are used to study the physical features of compact objects such as white dwarfs and neutron stars.

Today, there are a number of theories that envision our four-dimensional world as embedded in a larger universe with more than four dimensions. In these theories it is natural to look for stellar models, similar to those of general relativity, and study the consequences of the existence of extra dimensions on the structure of neutron stars. However, there is an important aspect that complicates the construction of such models in these theories. Namely that Birkhoff’s theorem is no longer valid in more than four dimensions, i.e., there is no an unique asymptotically flat vacuum solution with spatial spherical symmetry. As a consequence, the effective theory in 4D allows the existence of different possible, non-Schwarzschild, scenarios for the description of the spacetime outside of a spherical star, contrary to four-dimensional general relativity.

In braneworld models, Germani and Maartens [1] have found two exact vacuum solutions, both asymptotically Schwarzschild, which can be used to represent the exterior of an uniform-density star on the brane. The analysis of the gravitational collapse and black hole formation on the brane is even more complicated; Bruni, Germani and Maartens [2] have shown that the vacuum exterior of a collapsing dust cloud cannot be static, although, on general physical grounds it is expected that the non-static behavior will be transient, so that the exterior tends to a static form. Similar results, for the collapse of a dust cloud, have been derived by Kofinas and Papantonopoulos [3] in the context of various braneworld models with curvature corrections.

In Kaluza-Klein theories a milder version of Birkhoff’s theorem is true. Namely, there is only one family of spherically symmetric exact solutions of the field equations $R_{AB} = 0$ which are asymptotically flat, static and independent of the “extra” coordinates. In five-dimensions, in the form given by Davidson and Owen [4], it is described by the line element

$$dS^2 = \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma k} dt^2 - \frac{1}{a^4 r^4} \left(\frac{ar + 2\sigma(k+1)+1}{ar - 1}\right)^{2\sigma} (dr^2 + r^2 d\Omega^2) \pm \left(\frac{ar + 1}{ar - 1}\right)^{2\sigma} dy^2, \quad (1)$$

where $a$ is a constant with dimensions of $L^{-1}$; and $\sigma$ as well as $k$ are parameters that obey the constraint

$$\sigma^2(k^2 - k + 1) = 1. \quad (2)$$

This family of solutions has been rediscovered in different forms by Kramer [5] and, although in a different context, by Gross and Perry [6] (for a recent discussion see [7] and references therein). Its importance resides in the fact that in the limit $k \to \infty$ ($\sigma k \to 1$), it yields

$$dS^2 = \left(\frac{1 - 1/ar}{1 + 1/ar}\right)^2 dt^2 - \left(1 + \frac{1}{ar}\right)^4 [dr^2 + r^2 d\Omega^2] \pm dy^2, \quad (3)$$

which on every subspace $y =$ constant reduces to the Schwarzschild metric, in isotropic coordinates, for a central mass $M = 2/a$. Namely,

$$ds^2 = \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2 d\Omega^2]. \quad (4)$$

In five-dimensional Kaluza-Klein theory, these solutions play a central role in the discussion of many important observational problems, which include the classical tests of relativity, as well as the geodesic precession of a gyroscope.

1 This metric is known as the Schwarzschild black string.
and possible departures from the equivalence principle \[8\]. In the context of the induced-matter approach, the metric \[1\] is interpreted as describing extended spherical objects called solitons \[9\]. In this interpretation the matter distribution contains a lightlike singularity at \(r = 1/a\) \[7\] \[10\], which, in principle, can be visible to an external observer. Only in the black string limit \(k \to \infty\) (\(\sigma k \to 1\)) the 5D metric \(1\) possesses an event horizon; for any finite value of \(k\) the horizon is reduced to a singular point.

However, the presence of naked singularities makes everybody uncomfortable. In order to avoid such singularities, we have to exclude the central region and require \(r > 1/a\). What this suggests is that the asymptotically flat metric \(1\) should be interpreted as an “exterior” solution, describing the gravitational field outside of the core of a spherical matter distribution. In this interpretation, the effective exterior is not empty because there are nonlocal stresses induced from the Weyl curvature in 5D, which in 4D behave like radiation.\[2\] The “interior” region \(r \leq 1/a\) has to be described by some another solution of the field equations, which must be regular at the origin and not necessarily asymptotically flat. The aim of this work is to investigate this interpretation and its consequences on stellar models.

In this context, the simplest candidate to describe the exterior of a spherical star is given by the spacetime part of the 5D vacuum solution \(1\), which is the metric induced on every hypersurface \(y = \text{constant}\). Nevertheless, we show here that this simple exterior leads to stellar models that have no Newtonian limit. On the other hand, according to the correspondence principle, we should require that our stellar models be consistent not only with general relativity but also with Newtonian models. Therefore, in this work we consider a more general scenario. The paper is organized as follows.

In section 2 we use the five-dimensional line element \(1\) to generate a family of asymptotically flat metrics in 4D which are conformal to the metric induced on \(y = \text{constant}\) hypersurfaces. We examine this family as a possible choice for describing the four-dimensional spacetime outside of a spherical star of radius \(r_b > 1/a\).

In section 3, from the boundary conditions, we find that there is only one member of this family compatible with the weak-field limit. In other words, the requirement that in the weak-field limit we recover the usual Newtonian physics singles out an unique exterior. This exterior depends on one parameter, which we call \(\varepsilon\), and includes the vacuum Schwarzschild exterior for \(\varepsilon = 1\). The intriguing feature of this metric is that it is “similar” to the Schwarzschild vacuum in the sense that it has no effect on gravitational interactions. However, it is more realistic because instead of being absolutely empty, it is consistent with the existence of quantum zero-point fields \[11\].

In section 4, in order to illustrate our results we examine in detail the particular model where the central core is represented by a perfect fluid sphere of homogeneous incompressible fluid. We calculate the physical parameters of a neutron star for distinct exteriors and different degrees of deviation from the Schwarzschild metric. We show that the general relativity compactness limit \(M/R < 4/9\) is significantly increased as we go away from the Schwarzschild vacuum exterior. In fact, we find that in principle \(M/R\) can be larger than \(1/2\), without being a black hole. Finally, in section 5 we give a summary of our work and discuss the generality of our results.

2 The model

In this section we present the main steps of our work. First, we address the question of how to identify the effective four-dimensional spacetime. The correct answer to this question is crucial in order to be able to predict effects from an extra dimension. Then, from \(1\), we construct the “appropriate” set of possible exterior metrics. Next, we describe the model for the central core and examine the boundary conditions, which ensure that the interior and exterior metric join continuously.

2.1 The effective spacetime

In five-dimensional models, the effective four-dimensional picture\[3\] is usually recovered, or constructed, from \(g_{\mu\nu}^{\text{ind}}\) the four-dimensional metric induced on 4D hypersurfaces that are orthogonal to what is considered the “extra”

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\[1\]: It is sometimes called “black” or “Weyl” radiation, because in this case the induced energy-momentum tensor \(T_{\mu\nu}\) can be expressed as \(T_{\mu\nu} = -\epsilon E_{\mu\nu}\), where \(E_{\mu\nu}\) represents the spacetime projection of the five-dimensional Weyl tensor, which is traceless.

\[2\]: This is the picture registered by an observer who is not aware of the existence of an extra dimension.
dimension [12]. For a given line element in 5D

\[ ds^2 = g_{\mu\nu}(x^\rho, y) dx^\mu dx^\nu + \epsilon\Phi^2(x^\rho, y) dy^2, \quad \epsilon = \pm 1, \]

the induced metric, on any hypersurface \( y = \text{constant} \), is just \( g_{\mu\nu} \), i.e., \( g^{\text{ind}}_{\mu\nu} = g_{\mu\nu} \). The question is how to obtain our physical spacetime, with metric \( g^{\text{eff}}_{\mu\nu} \), from the induced metric \( g^{\text{ind}}_{\mu\nu} \).

Unfortunately, there is no a consensus answer to this question. In fact, there are different competing approaches which are not equivalent. Today, in braneworld models, as well as in space-time-matter (STM) theory, the standard technique is to identify the metric of our spacetime with the induced one, i.e., \( g^{\text{eff}}_{\mu\nu} = g^{\text{ind}}_{\mu\nu} \). However, there are a number of theories and approaches where \( g^{\text{eff}}_{\mu\nu} \) is constructed in a different ways.

The simplest example is given by a metric which has been termed canonical [13], where the \( g^{\text{eff}}_{\mu\nu} \) is conformal to \( g^{\text{ind}}_{\mu\nu} \), with a conformal factor depending only on the extra coordinate. Also, a number of authors [8] [14] [15] [16] [17] [18] have considered models where \( g^{\text{eff}}_{\mu\nu} \) is conformal to \( g^{\text{ind}}_{\mu\nu} \), with a conformal factor that is a “function” of the scalar field \( \Phi \).

In this work we examine the consequences of 4D models obtained from a dimensional reduction of the metric in 5D, on a hypersurface \( y = \text{constant} \), with a conformal rescaling

\[ g^{\text{eff}}_{\mu\nu} = \Phi^N g^{\text{ind}}_{\mu\nu}, \]

where \( N \) is an arbitrary real constant. This \textit{ansatz} consolidates various approaches in the literature. In the context of Kaluza-Klein gravity and STM, it has been considered by Wesson [8], Kokarev [13]-[15], Sajko [16], and the present author [17]. For \( N = 0 \), it reproduces the usual interpretation of braneworld and STM theories. For \( N = -2 \) the interpretation is similar to the one provided by the canonical metric. For \( N = 1 \) it yields the classical Davidson-Owen [4] and Dolan-Duff [19] interpretation.

### 2.2 Possible non-Schwarzschild stellar exteriors

The effective four-dimensional spacetime [5], corresponding to the \textit{empty} five-dimensional universe described by metric [1], can be written as

\[ ds^2 = \left( \frac{ar - 1}{ar + 1} \right)^{2\epsilon} dt^2 - \frac{(ar + 1)^2(1 - ar)}{a^4 r^4} \left( \frac{ar + 1}{ar - 1} \right)^{2(\epsilon + \sigma(N-1))} [dr^2 + r^2 d\Omega^2], \]

where \( \epsilon = \sigma(k - N/2) \) and\(^4\)

\[ \sigma = \begin{cases} 
2 \left[ \epsilon(1 - N) + \sqrt{(N^2 - 2N + 4) - 3\epsilon^2} \right], & \text{for } N \geq 1; \\
2 \left[ \epsilon(1 - N) - \sqrt{(N^2 - 2N + 4) - 3\epsilon^2} \right], & \text{for } N < 1.
\]

This parameterization is useful because the “deviation” from the exterior Schwarzschild metric can be measured by \( |\epsilon - 1| \). Since the exterior spacetime is not in general empty, from a physical point of view taking \( \epsilon \neq 1 \) is equivalent to giving the star an “atmosphere” extending all the way to infinity with a matter density decreasing as \( \rho \sim 1/a^2 r^4 \). The special case where the star has a sharp surface corresponds to \( \epsilon = 1 \), where we recover the exterior Schwarzschild metric, for any value of \( N \). In Appendix A we present the equation of state and gravitational mass, of such an atmosphere. Thus, the questions of interest here are the following:

1. How does the existence of stars restrict the possible values of \( N \)?

\(^4\)We note that for \( N = 1 \) the choice of \( \sigma \) is immaterial, because the metric depends only on \( \epsilon \). However, we have selected a positive \( \sigma \) in [5] for consistency with our previous work [7].
2. How does a possible deviation from the Schwarzschild vacuum exterior, i.e., going away from $\varepsilon = 1$, can affect the star parameters?

The total gravitational mass, measured at infinity, is given by

$$ M = \frac{2\varepsilon}{a}. \quad (9) $$

Thus, in the Schwarzschild limit $M = 2/a$. Therefore, to ensure the positivity of $M$, we will choose $a > 0$ and $\varepsilon > 0$ everywhere.

We note that $\sigma(N - 1)$ is invariant under the transformation $N \to (2 - N)$. This means that the effective spacetime for $N = 0$ is identical to the one for $N = 2$. The same for $N = 3$ and $N = -1$; $N = 4$ and $N = -2$; and so on. In other words: the effective spacetimes with $N > 1$ duplicate the ones with $N < 1$. Only the one with $N = 1$ remains the same.

### 2.3 The interior solution and matching conditions

In this work we restrict our discussion to the case where the interior solution is time independent. The matching conditions are easier to derive if we take the line element inside the fluid sphere with the same symmetry properties as the exterior metric (7). Namely, the interior static will be taken as

$$ ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (10) $$

The three-dimensional surface $\Sigma$, defined by the equation

$$ r - r_b = 0, \quad (11) $$

where $r_b$ is a constant, separates the spacetime into two regions: $r \geq r_b$ and $r \leq r_b$, described by the metrics (7) and (10), respectively.

The problem of joining two distinct metrics across surfaces of discontinuity has been discussed by a number of authors (see [20] and references therein). Robson [20] showed the complete equivalence of the junction conditions proposed by Lichnerowicz and by O’Brien and Synge at non-null hypersurfaces. The latter are analogous to the requirement that the first and second fundamental forms be continuous at $\Sigma$.

For the line elements under consideration, the first fundamental form at $\Sigma$, becomes

$$ ds^2_{(\Sigma:r=r_b)} = e^{\nu(r_b)}dt^2 - e^{\lambda(r_b)}r_b^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (12) $$

Therefore, its continuity across $\Sigma$ requires

$$ e^{\nu(r_b)} = e^{(ar_b - 1)^2[\frac{1}{a}r_b - 1]} \quad (13) $$

Besides, the “physical” radius $R_b$ of the sphere with coordinate radius $r_b$ is given by

$$ R_b = r_b e^{\lambda(r_b)/2} = \frac{(ar_b + 1)^2(1 - ar_b)^2}{a^2r_b^2} \left(\frac{ar_b + 1}{ar_b - 1}\right)^{[\varepsilon + \sigma(N-1)]}. \quad (14) $$

The second fundamental form, say $dK^2$, at $\Sigma$ is given by

$$ dK^2 = n_{\mu\nu}dx^\mu dx^\nu, \quad (15) $$

These are just examples, we are not assuming that $N$ is an integer number.
where \( n^\mu \) is the unit vector orthogonal to \( \Sigma \). In the present case, if we take
\[
n_\mu = \delta^1_\mu e^{\lambda/2},
\]
then we find
\[
dR^2_{(\Sigma:r=r_b)} = -e^{-\lambda(r_b)/2} \left[ \frac{\nu'(r_b)}{2} \left( e^{\nu(r_b)/2} dt \right)^2 - \left( \frac{1}{r_b} + \frac{\lambda'(r_b)}{2} \right) r_b^2 e^{\lambda(r_b)} \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right].
\]
Consequently, in isotropic coordinates, the requirement of continuity of the second fundamental form across \( \Sigma \) is equivalent to demand that the first derivatives of the metric be continuous across \( \Sigma \).

For the exterior metric (17), the continuity of \( \nu' \) and \( \lambda' \) yields
\[
\nu'(r_b) = \frac{4 \alpha \epsilon}{(a^2 r_b^2 - 1)},
\]
\[
\lambda'(r_b) = \frac{4 \left( 1 - [\epsilon + \sigma(N-1)] a r_b \right)}{r_b (a^2 r_b^2 - 1)}.
\]

From a physical point of view, the continuity of \( \nu' \) is equivalent to the continuity of the gravitational mass (A-11), in such a way that the force on test particles located at the surface will be the same whether we calculate it from inside or outside. While, the continuity of \( \lambda' \) means that the radial pressure (i.e. \(-T^1_1\)) of the fluid at the surface joins continuously with the pressure of space outside the sphere.

Instead of (10), we may consider the most general static interior metric, which in curvature coordinates can be written as
\[
ds^2 = e^{\omega(R)} dt^2 - e^{\mu(R)} dR^2 - R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) = 0.
\]
Now the continuity of the second fundamental form does not require the continuity of \( d\mu/dR \). However, by virtue of the equivalence between Lichnerowicz’s and O’Brien-Synge’s junction conditions, this and the exterior metric match if the mass and radial pressure are continuous at \( \Sigma \).

**Embedding the interior solution in 5D:** The interior solution (10), as well as (19), can be embedded in the five-dimensional world by foliating the 5-dimensional manifold as a set of 4-dimensional hypersurfaces (or slices), in such a way that the effective metric in 4D coincides with the induced one on every \( y = \text{constant} \) hypersurface. The embedding can be taken as
\[
ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \pm dy^2.
\]
which physically corresponds to a star carrying no-scalar charge [21], [22]. The corresponding field equations in 5D are
\[
G_{AB} = k^2_{(5)} (5) T_{AB},
\]
where \( k^2_{(5)} \) is a constant introduced for dimensional considerations, \((5) T_{AB}\) is the energy-momentum tensor in 5D and the non-vanishing components of the Einstein tensor \( G_{AB} \) are
\[
G^0_0 = -e^{-\lambda} \left( \frac{\lambda'' + \frac{\lambda^2}{4} + 2 \frac{\lambda'}{r}}{4} \right),
\]
\[
G^1_1 = -e^{-\lambda} \left( \frac{\lambda^2}{4} + \frac{\lambda' \nu'}{2} + \frac{\lambda' + \nu'}{r} \right),
\]
\[
G^2_2 = -e^{-\lambda} \left( \frac{\lambda'' + \nu''}{2} + \frac{\nu^2}{4} + \frac{\lambda' + \nu'}{2r} \right),
\]
\[\text{where} \quad k^2_{(5)} \text{ is a constant introduced for dimensional considerations.}
\]

Thus, in isotropic coordinates the “weaker” junction conditions of O’Brien and Synge are equivalent to those of Lichnerowicz.
\[ G_3^3 = G_2^2, \text{ and} \]
\[ G_4^4 = -e^{-\lambda} \left( \lambda'' + \frac{\lambda'^2}{4} + 2 \frac{\lambda'}{r} + \frac{\nu'}{4} + \frac{\nu'}{r} + \frac{\nu'^2}{4} + \frac{\nu''}{2} \right). \quad (23) \]

We note the relationship \( G_0^0 + G_1^1 + 2G_2^2 = 2G_4^4. \) Thus, taking \((5)T^A_B = \text{diag}(\rho, -p_r, -p_\perp, -p_\perp, -p_5)\) (see equation \( (A-5) \)) we find that the matter in 5D satisfies the equation of state
\[ \rho = p_r + 2p_\perp - 2p_5, \quad (24) \]
which is the counterpart of the equation of state \( (A-8) \) for the exterior of a spherical star. A four-dimensional observer living in a subspace \( y = \text{constant} \) is not directly aware of \((23)\) the “pressure” along the extra dimension, and therefore will not be able to find such a simple equation of state between the density and pressures. However, this does not mean that she/he loses all information about the 5th dimension; it is consolidated in the nonlocal stresses induced in 4D from the Weyl curvature in 5D. As a consequence, even in the absence of matter, the exterior of a spherical star is not in general an empty Schwarzschild spacetime. We note that in the above discussion the signature of the extra dimension is irrelevant.

An observer in 4D interprets the spacetime part of \((5)T_{AB}\) as the energy-momentum in 4D. More precisely, as
\[ 8\pi T_{\alpha\beta} = G_{\alpha\beta}. \quad (25) \]

It is clear that the five-dimensional conservation equation \((5)T_{A;B} = 0\), in the present case yields the conservation equation \( T_{\mu\nu} = 0 \) in 4D, which is equivalent to the “generalized” Tolman-Oppenheimer-Volkov equation of hydrostatic equilibrium
\[ (\rho + p_r)\frac{M}{r^2}e^{-(\nu + \lambda)/2} = -\frac{dp_r}{dr} + \left(\frac{2}{r} + \lambda'\right) (p_\perp - p_r). \quad (26) \]

Finally, we note that for metrics \((10) \text{ and } (20)\) a motion along a geodesic in 4D is also geodesic in 5D. Therefore, a particle located on a hypersurface \( y = \text{constant} \), with zero momentum along the extra dimension, will stay on that hypersurface.

### 3 General interior and the weak-field limit

The main question in this paper is whether the existence of stars leads to some constraints on \( N \). In this section we give a positive answer to this question. We use the boundary conditions, as well as the weak-field limit, to show that the exterior of a spherical star is given by the metric \( (7) \) with \( N = 1 \); for any other \( N \neq 1 \) there are problems in the Newtonian limit.

First of all, let us introduce the notation
\[ x = ar_b, \quad (27) \]
in terms of which the continuity equations \((18)\) become
\[ \frac{\nu'(r_b)}{4a} = \frac{\varepsilon}{x^2 - 1}, \quad x > 1, \]
\[ \frac{\lambda'(r_b)}{4a} = \frac{\{1 - [\varepsilon + \sigma(N - 1)] x\}}{x (x^2 - 1)}. \quad (28) \]

These can be used to get an equation for \( x \). Namely,
\[ x = \left\{ \varepsilon \left[ 1 + \frac{\lambda'(r_b)}{\nu'(r_b)} \right] - \sigma(1 - N) \right\}^{-1}. \quad (29) \]

\(^7\)Here \( \rho, p_r \) and \( p_\perp \) represent the energy density, radial and tangential “pressure”, respectively; \( p_5 \) is the pressure along the extra dimension.
In the limit $\varepsilon \to 1$, the radial pressure vanishes at the boundary surface, i.e., we recover the usual matching conditions for the Schwarzschild exterior. However, we note that in the weak-field limit (29) violates the positivity of $x$. Indeed, in the Newtonian limit (see equation (33) below)

$$\frac{\lambda'}{\nu'} \sim -1.$$  

(30)

On the other hand $\sigma(1 - N) > 0$ for any $N \neq 1$ and $\varepsilon \neq 1$. Therefore, $\lambda'/\nu'$ cannot approximate $-1$ otherwise $x < 0$. This shows that, given the standard matching conditions at the boundary, the effective exterior of a spherical star must be described either by the vacuum Schwarzschild spacetime or by the line element (7) with $N = 1$, otherwise the model star is inconsistent with a Newtonian limit.

**Newtonian limit:** In the weak field-limit we set $e^\nu = 1 + \xi f(r)$ and $e^\lambda = 1 + \xi h(r)$, where $\xi$ is a “small” dimensionless parameter, i.e., $|\xi| \ll 1$. In this limit $\rho \gg p$.

Therefore, from (A-10) it follows that

$$M(r) = 4\pi \int_0^r \bar{r}^2 \rho(\bar{r}) d\bar{r} + O(\xi^2).$$  

(31)

In this limit the energy density is given by

$$8\pi \rho = -\xi \left( h'' + \frac{2h'}{r} \right).$$  

(32)

From which we get $M = -\xi r^2 h'/2$. On the other hand, in this limit, (A-11) yields $M = \xi r^2 f'/2$. Consequently,

$$f'(r) = -h'(r), \quad \frac{\lambda'}{\nu'} = -1 + \xi(h - f).$$  

(33)

Then, the pressures are of second order in $\xi$. Namely,

$$8\pi p_r = -\frac{\xi^2}{4r} (8h'h + rh'^2) + \frac{\xi^2 h'}{r}(f + h),$$

$$8\pi p_\perp = -\frac{\xi^2}{4r} (3rh'^2 + 4rh'' + 4hh') + \frac{\xi^2 (rh'' + h')}{2r} (f + h).$$  

(34)

We note that

$$8\pi \left[ -\frac{dp_r}{dr} + \frac{2}{r}(p_\perp - p_r) \right] = \xi^2 \left( \frac{h'^2}{r} + \frac{h'h''}{2r} \right).$$  

(35)

The r.h.s. can be written as $-(\xi h'/2)(8\pi \rho) = 8\pi \rho M/r^2$. Thus, we recover the Newtonian equation of hydrostatic equilibrium, viz.,

$$\frac{M}{r^2} = -\frac{dp_r}{dr} + \frac{2}{r}(p_\perp - p_r),$$  

(36)

which is nothing but the non-relativistic limit of the TOV equation (26).

The non-Schwarzschild stellar exterior:  The conclusion from this section is that, in Kaluza-Klein gravity the only exterior spacetime that is consistent with the existence of a Newtonian limit is the one given by (7) with $N = 1$. In this case, the exterior of a spherical star takes a particular simple form. Namely,

$$ds^2 = \left( \frac{ar - 1}{ar + 1} \right)^{2\varepsilon} d\bar{r}^2 - \frac{1}{a^4 r^4} \left( \frac{ar + 1}{ar - 1} \right)^{2[\varepsilon + 1]} [dr^2 + r^2 d\Omega^2].$$  

(37)

In addition to the equation of state $\rho = -(p_r + 2p_\perp)$ mentioned in (A-9), we find that the external field obeys a second equation of state, which is

$$p_\perp = -p_r.$$  

(38)
In other words; \( p_r = \rho \) and \( p_\perp = -\rho \), where the energy density is given by

\[
8\pi \rho = \frac{4\alpha a^4(1-\varepsilon^2)}{(ar+1)^3(ar-1)^4} \left( \frac{ar-1}{ar+1} \right)^{2\varepsilon},
\]

and the gravitational mass (A-12) is constant outside the fluid sphere. Namely,

\[
M = \frac{2\varepsilon}{a}.
\]

This means that the (total) gravitational mass resides entirely inside the fluid sphere; the external cloud has no influence on the gravitational mass. This is due to the equation of state \( \rho = -(p_r + 2p_\perp) \), which describes matter with no effect on gravitational interactions.

Finally, we note that the conditions \( \rho \geq 0 \) and \( M > 0 \) demand \( 0 < \varepsilon \leq 1 \). For \( \varepsilon = 1 \) we recover the Schwarzschild exterior.

4 Model for the stellar interior

In order to illustrate the above results we need to assume some interior model. Here we consider a model star in the shape of a homogeneous sphere of incompressible fluid with radius \( r_b \). Thus, inside the sphere the metric will be the interior Schwarzschild metric, and outside \( r_b \) it will join continuously to the exterior metric (7). Despite their simplicity, uniform-density models give insight about much wider classes of stellar models [26].

The line element describing the interior of a fluid sphere with constant energy density \( \rho_0 \) and isotropic pressures can be written as

\[
ds^2 = A^2 \left( \frac{1 + Br^2}{1 + Cr^2} \right)^2 dt^2 - \frac{D^2}{(1 + Cr^2)^2} (dr^2 + r^2 d\Omega^2).\]

The matter quantities are

\[
8\pi \rho_0 = \frac{12C}{D^2}, \quad C > 0,
\]

\[
8\pi p(r) = \frac{4 \left( (B - 2C) + Cr^2(C - 2B) \right)}{D^2(1 + Br^2)}.
\]

In order to simplify these expressions we evaluate the pressure and the equation of state at the center. Namely,

\[
8\pi p_c = \frac{4(B - 2C)}{D^2}, \quad p_c = \gamma \rho_0,
\]

where \( p_c = p(0) \) and \( \gamma \) is a constant. In terms of these quantities we find: \( B = C(2 + 3\gamma) \); \( B - 2C = 3C\gamma \) and \( (C - 2B) = -3C(1 + 2\gamma) \). Consequently, the appropriate line element and pressure are

\[
ds^2 = A^2 \left( \frac{1 + (2 + 3\gamma)Cr^2}{1 + Cr^2} \right)^2 dt^2 - \frac{3C}{2\pi \rho_0(1 + Cr^2)^2} (dr^2 + r^2 d\Omega^2),
\]

\[
p = \rho_0 \frac{[\gamma - (1 + 2\gamma)Cr^2]}{[1 + (2 + 3\gamma)Cr^2]}.
\]

Since the density and pressure are positive, it follows that

\[
\gamma > 0.
\]

We note that, in principle, \( \gamma \) can take arbitrarily small positive values; there is no other restriction.

\[8\text{This equation of state is the generalization of } \rho = -3p, \text{ which has been considered in different contexts by several authors. Notably, in discussions of cosmic strings by Gott and Rees [23], by Kolb [24], as well as in certain sources (called “limiting configurations”) for the Reissner-Norström field by the present author [25]. Besides, it is the only equation of state consistent with the existence of quantum zero-point fields by Wesson [11].}\]
For completeness, we note that although not directly observed in 4D, the pressure along the extra dimension \( p_5 \), can be obtained from the equation of state (23) as

\[
p_5 = \frac{1}{2}(3p - \rho).
\]

(48)

Then from (46) we find

\[
p_5 = \rho_0 \frac{[(3\gamma - 1) - (5 + 9\gamma)C_{r^2}]}{2[1 + (2 + 3\gamma)C_{r^2}]}.
\]

(49)

4.1 Boundary conditions for the Schwarzschild interior

In terms of the dimensionless quantities \( \zeta, x \) and \( Y \) defined as

\[
\zeta = C/a^2, \quad x = ar_b, \quad Y = aR_b,
\]

(50)

we have \( C_{r^2} = \zeta x^2 \) and

\[
Y = \frac{1}{x} \frac{(x + 1)[\varepsilon + \sigma(N-1)]}{(x - 1)[\varepsilon - \sigma(N-1)]}.
\]

(51)

With this notation the continuity of the first fundamental form gives

\[
A = \frac{(1 + \zeta x^2)}{[1 + (2 + 3\gamma)\zeta x^2]} \left[ \frac{x^2 - 1}{xY} \right]^{\varepsilon/[\varepsilon + \sigma(N-1)]},
\]

(52)

and

\[
Y = \frac{\sqrt{3\zeta x}}{\sqrt{2\pi(\rho_0/a^2)(1 + \zeta x^2)}}.
\]

(53)

From the continuity of the second fundamental form we get

\[
\frac{(3\gamma + 1)\zeta x}{(1 + \zeta x^2)[1 + (2 + 3\gamma)\zeta x^2]} = \frac{\varepsilon}{(x^2 - 1)},
\]

(54)

and

\[
\frac{\zeta x^2}{1 + \zeta x^2} = \frac{[\varepsilon + \sigma(N-1)]x - 1}{(x^2 - 1)}.
\]

(55)

These equations require \( x > 1 \) and \( |\varepsilon + \sigma(N-1)|x \neq 1 \), otherwise \( C = 0 \), for all values of \( \varepsilon \) and \( N \).

4.2 Solution of the boundary conditions

Dividing (54) by (55) we obtain

\[
\frac{3\gamma + 1}{[1 + (2 + 3\gamma)\zeta x^2]} = \frac{\varepsilon x}{[\varepsilon + \sigma(N-1)]x - 1},
\]

(56)

from which we get

\[
(\zeta x^2) = \frac{[(N - 1)(3\gamma + 1)\sigma + 3\varepsilon\gamma] x - (1 + 3\gamma)}{\varepsilon x(2 + 3\gamma)}.
\]

(57)

On the other hand, from (55)

\[
(\zeta x^2) = \frac{[\varepsilon + \sigma(N-1)]x - 1}{x[x - \varepsilon - \sigma(N-1)]}.
\]

(58)

We note that \( x \neq \varepsilon + \sigma(N-1) \), otherwise for \( \varepsilon = 1 \) one would have \( x = 1 \).

Equating (57) and (58) we get the desired equation for \( x \). Namely,

\[
[(1 + 3\gamma)(N - 1)\sigma + 3\varepsilon\gamma] x^2 - (1 + 3\gamma)[(N - 1)^2\sigma^2 + 3(N - 1)\varepsilon\sigma + 2\varepsilon^2 + 1] x + [(1 + 3\gamma)(N - 1)\sigma + 3\varepsilon(1 + 2\gamma)] = 0.
\]

(59)
**Physical conditions:** The solutions of the above equations have to satisfy physical requirements. Positivity of mass requires $\varepsilon > 0$; positivity of $\rho_0$ requires $C > 0$, which translates into $\zeta > 0$. Consequently, from (57) we get

$$\rho_0 > 0 \implies C_1 := [(N - 1)(3\gamma + 1)\sigma + 3\varepsilon \gamma] > 0,$$

which also implies that the positivity of the coefficient in front of $x^2$. On the other hand, from (A-1) it follows that

$$\rho \geq 0 \implies C_2 := [3N(2 - N)\sigma^2 + 4\varepsilon(1 - N)\sigma] \geq 0.$$

**4.2.1 Solution for $\varepsilon = 1$**

In order to facilitate the discussion, we first discuss the solution of the boundary conditions for the Schwarzschild exterior metric. This will allow us to compare and contrast the parameters of our model star obtained under different conditions.

In this case $\sigma = 0$, and the above equation for $x$ reduces to

$$\gamma x^2 - (1 + 3\gamma)x + (1 + 2\gamma) = 0,$$

whose solution is

$$x = \frac{1 + 2\gamma}{\gamma}.$$

The other “solution” $x = 1$ is excluded. Substituting this expression into (65) we find

$$(\zeta x^2) = \frac{\gamma}{1 + 2\gamma},$$

and from (51) we obtain

$$Y = \frac{(3\gamma + 1)^2}{\gamma(1 + 2\gamma)}.$$

Now, the original constants can be obtained from (50) and (52). Namely,

$$A = \frac{1}{1 + 3\gamma}, \quad \frac{3C}{2\pi \rho_0} = \left(\frac{1 + 3\gamma}{1 + 2\gamma}\right)^6, \quad C_{\gamma^2} = \frac{\gamma}{1 + 2\gamma} \left(\frac{r}{r_b}\right)^2.$$

In addition, we find

$$r_b = \frac{M(1 + 2\gamma)}{2\gamma}, \quad \sqrt{2\pi \rho_0 M} = \frac{2\sqrt{3}\gamma^{3/2}(1 + 2\gamma)^{3/2}}{(1 + 3\gamma)^3},$$

where the total mass has been obtained from (53). Thus, the boundary conditions allow us to obtain the value of all the constants in the solution in terms of the equation of the state at the center.

**Buchdahl limit:** In the present case, from (69) it follows that $a = 2/M$. Consequently, using (65) we obtain

$$\frac{M}{R_b} = \frac{2\gamma(1 + 2\gamma)}{(3\gamma + 1)^2}.$$

In the limit $\gamma \to \infty$ we get the famous absolute upper limit

$$\left(\frac{M}{R_b}\right)_{max} = \frac{4}{9} \approx 0.444,$$

discovered by Buchdahl [27], for all static fluid spheres whose (i) energy density does not increase outward and (ii) the material of the sphere is locally isotropic.
Newtonian limit: In the limit $\gamma \to 0$, from (60) we see that $A \to 1$ and $(3C/2\pi \rho_0) \to 1$. Also, since $0 \leq (r/r_b) \leq 1$, it follows that $Cr \to 0$. Consequently, $e^\nu \to 1$ and $e^\lambda \to 1$. Therefore, in this limit (20) reduces to

$$\frac{\rho_0 M}{r^2} = - \frac{dp}{dr}, \quad (70)$$

which is the usual balance equation in Newtonian physics.

4.2.2 General solution: $\varepsilon \neq 1$

The general solution of (59) can be written as

$$x = \frac{[(N - 1)^2 \sigma^2 + 3\varepsilon \sigma (N - 1) + (1 + 2\varepsilon^2)] + \sqrt{F}}{2(N - 1)\sigma + 6\gamma \varepsilon / (1 + 3\gamma)}, \quad (71)$$

where

$$F = (N - 1)^4 \sigma^4 + 6(N - 1)^3 \varepsilon \sigma^3 + (N - 1)^2 (13\varepsilon^2 - 2) \sigma^2 + 6(N - 1)(2\varepsilon^2 - 1) \varepsilon \sigma + (2\varepsilon^2 - 1)^2 + \frac{4(3\gamma + 2)\varepsilon^2}{(3\gamma + 1)^2}. \quad (72)$$

The sign of the root in (71) has been chosen positive to make sure that in the limit $\varepsilon \to 1$ we recover the solution discussed in the preceding subsection.

Thus, for any given $(N, \varepsilon, \gamma)$ we get $x$ from (71). Substituting the result into (53) we find $(\zeta x^2)$. Using them in (50) we obtain the constants $C$, $r_b$ and $A$. Equation (60) epitomizes the fact that the first two constants depend on the total mass. In the present case the total mass of the configuration, i.e., the mass of the fluid sphere added to the mass of the external cloud, is obtained from (51) and (63). The result is

$$\sqrt{2\pi \rho_0 M} = \frac{2\sqrt{3(\zeta x^2)}}{(1 + \zeta x^2)(x + 1)} \varepsilon x(x - 1)^{(\varepsilon + \sigma(N - 1) - 1)} / (1 + (\zeta x^2) x + 1)^{(\varepsilon + \sigma(N - 1) + 1)}. \quad (73)$$

Warning $M \neq \varepsilon M_{Schw}$: At this point, in order to avoid a misunderstanding, the following comment is in order. From (51), one could conclude that, since for $\varepsilon = 1$ (which corresponds to the Schwarzschild exterior metric) the total mass is $M_{Schw} = 2/a$, then the total mass for any $\varepsilon \neq 1$ is just $M = \varepsilon M_{Schw}$. However, this is not true; it is easy to see that (73) $\neq \varepsilon \times$ (77). This is because $a$ is not a fixed constant but it is given by the boundary condition (63). What is true is that for $\varepsilon = 1$, and the corresponding values of $x$ and $(\zeta x^2)$ given by (63) and (64), we recover the mass in the Schwarzschild case.

Coming back to our discussion we calculate the mass of the star, say $M_b$. By virtue of the continuity of the second fundamental form, it can be easily obtained from (A-12). Namely,

$$M_b = M \left( \frac{x - 1}{x + 1} \right)^{(1 - N)} \sigma^N. \quad (74)$$

We note that $M_b = M$ only in two cases: for the Schwarzschild exterior metric $(\sigma = 0)$, and for $N = 1$.

Now, the gravitational potential at the surface of the star is given by

$$\frac{M_b}{R_b} = \frac{2\varepsilon x(x - 1)^{(\varepsilon - 1)}}{(x + 1)^{(\varepsilon + 1)}}. \quad (75)$$

For practical reasons, it is useful to express the mass of the star in terms of the solar mass $M_\odot$, and its radius in kilometers (km). They are given by

$$M_b = \frac{1.089 \times 10^9 \sqrt{\zeta x^2}}{\sqrt{\rho_0} [(x + 1)^{(\varepsilon + 1)(1 + (\zeta x^2)])}} \varepsilon x(x - 1)^{(\varepsilon - 1)} \times M_\odot, \quad (76)$$

$$R_b = \frac{0.803 \times 10^9 \sqrt{\zeta x^2}}{\sqrt{\rho_0} [1 + (\zeta x^2)])} \text{[km]}, \quad \text{(76)}$$
where \( \rho_0 \) denotes the numerical value of the energy density in \((gr/cm^3)\).

For \( \varepsilon = 1 \) the star is surrounded by empty space with \( \rho = p_r = p_\perp = 0 \) for \( r > r_b \). However, for any \( \varepsilon \neq 1 \) it does have an atmosphere, whose matter density decreases as \( \rho \sim 1/a^2 r^4 \) from its maximum value, \( \rho(\Sigma) \), which is just outside \( \Sigma : r = r_b \), to zero at infinity. For the case under consideration, from (53), (A-1) and using (58) we find

\[
\frac{\rho(\Sigma)}{\rho_0} = \frac{x\sigma^2[3N(2 - N) + 4\varepsilon(1 - N)/\sigma]}{12 [x - \varepsilon - \sigma(N - 1)[x(\varepsilon + \sigma(N - 1)) - 1]].
\] (77)

In view of the complexity of the general solution, we will analyze it for different values of \( \varepsilon \) and \( N \) separately.

### 4.3 Factorization with \( N = 1 \)

In this case the above expressions are specially simple. As in the Schwarzschild case, we find that for any fixed \( \varepsilon \neq 1 \) the radius and the mass of the fluid sphere are very sensitive to the equation of state \( \gamma \). This is because \( x \) decreases very fast, as \( x \sim 1/\gamma \), for small \( \gamma \), to a constant value for relatively large values of \( \gamma \).

#### 4.3.1 Limiting configurations

Let us first examine the solution of (71) in the limit \( \gamma \to \infty \). In this case from (72) we get \( F = (2\varepsilon^2 - 1)^2 \). Since we are looking for solutions that are continuously connected to the Schwarzschild one, we take \( \varepsilon^2 > 1/2 \). With this choice, from (71) we obtain

\[
x = 2\varepsilon.
\] (78)

Then from (58) we find

\[
(\zeta x^2) = \frac{2\varepsilon^2 - 1}{2\varepsilon^2}.
\] (79)

For \( \varepsilon = 1 \) we recover \( x = 2 \) and \( (\zeta x^2) = 1/2 \), which are the corresponding limiting values in the Schwarzschild case.

Using (78) in (50) and (53) we find the constants

\[
C = \frac{2\varepsilon^2 - 1}{2\varepsilon^2 M^2}, \quad A \to \frac{(2\varepsilon - 1)(\varepsilon + 1)}{3\gamma(2\varepsilon^2 - 1)(2\varepsilon + 1)(\varepsilon - 1)};
\] (80)

as well as the mass \( M \), as a function of \( \varepsilon \). Namely,

\[
M = \frac{4\varepsilon^3 \sqrt{[3(2\varepsilon^2 - 1)/\pi]}}{\sqrt{\rho_0}(2\varepsilon + 1)(\varepsilon + 2)}.
\] (81)

From (51) and (40) we get the upper limit of the surface gravitational potential

\[
\left(\frac{M}{R_b}\right)_{\text{max}} = 4\varepsilon^3 (2\varepsilon - 1)(\varepsilon - 1)(2\varepsilon + 1)(\varepsilon + 1).
\] (82)

**Buchdahl limit for different \( \varepsilon \):** For \( \varepsilon = (1.00; 0.80; 0.75) \) we find

\[
\sqrt{\rho_0}M = (0.145; 0.135; 0.112),
\sqrt{\rho_0}R_b = (0.326; 0.265; 0.207),
\left(\frac{M}{R_b}\right) = (4/9; 0.508; 0.538),
\] (83)

respectively. Thus, with the decrease of \( \varepsilon \) both the mass and the radius of the star decrease, in such a way that its surface gravitational potential increases. We note that \( M/R_b > 1/2 \), for certain values of \( \varepsilon \).

The red-shift \( Z(r) \) of the light emitted from a point \( r \) in the sphere to infinity is given by

\[
Z(r) = 1/\sqrt{\rho_0(r)} - 1.
\] (84)
In the present case, the red-shift of the light emitted from the boundary surface is given by

\[
Z_b = \left(\frac{2\varepsilon + 1}{2\varepsilon - 1}\right)^\varepsilon - 1. \tag{85}
\]

Since, in the present limit, \(\varepsilon^2 > 1/2\) it follows that the maximum surface red-shift is \(Z_b = 2.478\). This indicates that, although the surface gravitational potential is \(M/R_b > 1/2\) (instead of the usual \(M/R_b < 1/2\)), the star is not a black hole.

**Newtonian limit for different \(\varepsilon\):** For \(\gamma \to 0\), from (71) with \(N = 1\), we get

\[
x \to \frac{2\varepsilon^2 + 1}{3\varepsilon\gamma}. \tag{86}
\]

Substituting this into (51), or (52), we find

\[
(\zeta x^2) \to \frac{3\varepsilon^2\gamma}{2\varepsilon^2 + 1}. \tag{87}
\]

For \(\varepsilon = 1\) these quantities become \(x \to 1/\gamma\), \((\zeta x^2) \to \gamma\), which are identical to their Newtonian values in the Schwarzschild case. In this limit, (51) yields \(Y \sim x\) Substituting these expressions into (52) and (53) we get

\[
A \to 1, \quad \frac{3C}{2\pi\rho_0} \to 1, \quad C\gamma^2 \to \frac{3\varepsilon\gamma}{2\varepsilon^2 + 1} \left(\frac{r}{r_b}\right)^2. \tag{88}
\]

Therefore, for \(N = 1\) in the limit \(\gamma \to 0\) we recover flat spacetime as well as the Newtonian balance equation (70).

### 4.3.2 Effects of \(\varepsilon \neq 1\) on neutron star parameters

The second important question in this work is how going away from \(\varepsilon = 1\) may affect the star parameters. In order to study this question, we have calculated the surface gravitational potential \(M/R_b\), the red-shifts at the center \(Z_c\) and at the boundary \(Z_b\) for different values of \(\gamma\). The mass and radius of the fluid sphere are given by (76). To obtain some numerical values for \(M\) and \(R_b\) we have taken \(\rho = 2 \times 10^{14}\text{gr/cm}^3\) which has been used in different models of neutron stars [28], [29]. Table 1 shows the results for the Schwarzschild solution (\(\varepsilon = 1\)), which we use to compare and contrast with the results of Tables 2 and 3 for \(\varepsilon = 0.9\) and \(\varepsilon = 0.8\), respectively.

| \(\gamma = p_c/p_e\) | \(M/R_b\) | \(Z_c\) | \(Z_b\) | \(R_b(\text{km})\) | \(M_g(M_\odot)\) |
|-------------------|----------|--------|--------|----------------|----------------|
| 1/10              | 0.14     | 0.30   | 0.18   | 15.13          | 1.46           |
| 1/3               | 0.28     | 1      | 0.50   | 21.16          | 3.99           |
| 1                 | 0.38     | 3.00   | 1.00   | 24.59          | 6.25           |
| 3                 | 0.42     | 9.00   | 1.50   | 26.02          | 7.41           |
| 10                | 0.44     | 30     | 1.82   | 26.54          | 7.87           |
| \(\infty\)        | 4/9      | \(\infty\) | 2.00   | 26.77          | 8.08           |

Table 1 shows how the physical features of a neutron star, with a vacuum Schwarzschild exterior, depend on the equation of state at the center.

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\(^9\)In order to avoid misunderstanding, it should be noted that for finite values of \(\gamma\), like \(\gamma = 1/3\) or \(\gamma = 1\), the parameter \(\varepsilon\) is *not* restricted by this condition.
4.4 Factorization with $N \neq 1$ and $\varepsilon \neq 1$

The energy condition \[60\] requires the denominator in \[71\] to be positive. However, it vanishes at $\gamma = \bar{\gamma}$ given by

$$\bar{\gamma} = \frac{(1 - N)\sigma}{3[\varepsilon + (1 - N)\sigma]}.$$  \hspace{1cm} \text{(89)}

We note that $\bar{\gamma} = 0$ only for $N = 1$ and the Schwarzschild exterior. Since $\sigma$ is positive for $N < 1$, and negative for $N > 1$, it follows that $(1 - N)\sigma > 0$. Consequently, for any other $N$ and $\varepsilon$ we have $\bar{\gamma} > 0$.

On the other hand, since $x$ is positive for $\gamma > \bar{\gamma}$ and negative for $\gamma < \bar{\gamma}$, it follows that $\gamma$ has a positive lower bound, i.e.,

$$\gamma > \bar{\gamma}.$$  \hspace{1cm} \text{(90)}

Thus, in accordance with our discussion in section 3, we conclude that the factorization with $N \neq 1$, and $\varepsilon \neq 1$, is incompatible with the Newtonian limit, which requires $\gamma \to 0$. Let us consider some particular cases with more detail.

4.4.1 Factorization with $N = 0$

This corresponds to the case where the metric of the physical spacetime is identified with the metric induced in 4$D$, which is the typical approach in induced-matter and braneworld theories. In this case the exterior metric \[7\] can be written as

$$ds^2 = \left(\frac{ar - 1}{ar + 1}\right)^{2\varepsilon} dt^2 - \frac{(ar + 1)^2(a_0 - 1)^2}{a^4 r^4} \left(\frac{ar + 1}{ar - 1}\right)^{\varepsilon + \sqrt{4 - 3\varepsilon^2}} [dr^2 + r^2 d\Omega^2],$$  \hspace{1cm} \text{(91)}

where we have used that, from \[8\], in the present case

$$\sigma = \frac{\varepsilon - \sqrt{4 - 3\varepsilon^2}}{2},$$  \hspace{1cm} \text{(92)}

which requires $\varepsilon \leq 2/\sqrt{3}$.
Now, from (60) and (61) we get the constraints

\[ C_1 := -(3\gamma + 1)\sigma + 3\varepsilon\gamma > 0, \]
\[ C_2 := 4\varepsilon\sigma \geq 0. \]

From which we find that the positivity of the energy density inside of the fluid sphere requires

\[ \gamma > \bar{\gamma} = \varepsilon - \sqrt{4 - 3\varepsilon^2} \frac{3}{2(\varepsilon + \sqrt{4 - 3\varepsilon^2})}. \]

We note that \( x \) is positive for \( \gamma > \bar{\gamma} \) and negative for \( \gamma < \bar{\gamma} \). In order to get a better understanding of these constraints, let us take \( \varepsilon = (0.8, 0.9, 1, 1.1, 1.15, 2/\sqrt{3}) \). We find

\[ \gamma > (-0.09, -0.05, 0, 0.09, 0.23, 1/3), \]
\[ \sigma = (-0.32, -0.18, 0, 0.25, 0.48, 1/\sqrt{3}), \]

respectively. Thus, the positivity of the energy density outside of the fluid sphere requires \( \varepsilon \geq 1 \). For \( \varepsilon = 1 \) we recover the usual Schwarzschild picture, as expected. However, for \( \varepsilon > 1 \), \( \gamma \) has a lower positive bound, e.g., \( \bar{\gamma} = 0.23 \) for \( \varepsilon = 1.15 \), which increases with the increase of \( \varepsilon \). Certainly these models have no Newtonian limit.

### 4.4.2 Factorization with \( N = 2 \)

In this case

\[ \sigma = -\varepsilon + \sqrt{4 - 3\varepsilon^2} \frac{2}{3}. \]

Substituting this into (7) we find that the line element is identical to (91), which is consistent with the symmetry under the change \( N \to (2 - N) \) noted in subsection 2.2. Also, both factorizations, \( N = 0 \) and \( N = 2 \), yield the same equation of state (A-6), and have the same lower limit on \( \gamma \).

The discussion in this section epitomizes a situation frequently found in physics. Namely, that the analysis of some particular “simple” example can be much more involved, in technical details, than the general case. However, the effort is worthwhile: the main conclusions from our uniform-density model are consistent with the general results presented in section 3. Specifically, (i) Only the factorization with \( N = 1 \) produces stars models that have a Newtonian limit; (ii) the factorizations with \( N \) and \( (2 - N) \) are physically indistinguishable.

### 5 Summary and concluding remarks

For theories in more than four-dimensions the crucial question is, how to recover the physics of our 4-dimensional world. The answer to this question is important in order to be able to calculate and predict specific effects from extra dimensions. For example, the possible existence of extra “non-gravitational” forces on test particles and their effect on the classical tests of relativity and the principle of equivalence [8], [30].

In this paper, we have considered this question from the point of view of the stellar structure. Namely, since Birkhoff’s theorem is no longer valid in more than four-dimensions, there is no an unique asymptotically flat vacuum solution with spatial spherical symmetry. In other words, regardless of whether we choose to work in Kaluza-Klein gravity [9], STM [8], braneworld models [11, 12] or others theories with curvature corrections [9], the fact is that the effective four-dimensional theory allows the existence of different possible, non-Schwarzschild, scenarios for the description of the spacetime outside of a spherical star, contrary to four-dimensional general relativity.

In five-dimensional Kaluza-Klein gravity the well-known solutions due to Kramer-Davidson-Owen-Gross-Perry seem to represent the natural generalization of the Schwarzschild spacetime [9-12]. Even in this simple case of spherical symmetry, in ordinary three space, there are various possible options leading to different exteriors for a spherical star. A popular assumption is that we recover our 4D spacetime by going onto a hypersurface \( y = \) constant. With this assumption, the information about the fifth dimension is consolidated in the nonlocal stresses induced in
4D from the Weyl curvature in 5D. As a consequence, even in the absence of matter, the exterior of a spherical star, like our sun, is not in general an empty Schwarzschild spacetime.

However, early in our research we found that this simple approach leads to contradictions in the Newtonian limit. Therefore, in section 2 we generalized our discussion by considering, as a possible exterior for a static spherical star, the family of asymptotically flat metrics represented by (7). These are conformal to the induced metric (6) and constitute a generalization to a number of approaches in the literature (see for example [13]-[19]).

The main question under scrutiny here was whether the existence of stars can be used to constraint the number of possible candidates for exteriors. Using the junctions conditions at the boundary of the star, and without making any assumptions about the interior metric, we have showed that (37) is the only exterior metric compatible with both; the weak-field Newtonian limit as well as the general relativistic Schwarzschild limit. All the other metrics with $N \neq 1$ do not have a Newtonian counterpart, although they are consistent with the Schwarzschild limit.

It should be noted that the resulting effective matter (39), outside of a spherical star, is “similar” to Schwarzschild vacuum in the sense that it has no effect on gravitational interactions. In fact, it satisfies the equation of state $p = -\rho c^2$. However, early in our research we found that this simple approach leads to contradictions in the Newtonian limit. One can speculate that this exterior (with $N = 1$) is more realistic than the Schwarzschild one, because instead of being absolutely empty ($p = \rho = 0$), it is consistent with the existence of quantum zero-point fields [11].

The question may arise about the generality of our result. Namely, what would happen if we assume that the effective four-dimensional metric is like

$$g^\text{eff}_{\mu\nu} = F(\Phi)g^{\text{ind}}_{\mu\nu},$$

where $F$ is some “function” of $\Phi$? It is not difficult to show, using the boundary conditions, that the only function compatible with a Newtonian limit is $F \sim N$, i.e., the factorization with $N = 1$.

Another important question here is what are the consequences of going away from the Schwarzschild exterior metric. Using a uniform-density model for the interior of the star, in Tables 1-3 we show how some physical quantities as the mass, radius and surface gravitational potential of a star are affected by an extra dimension. For the same equation of state at the center, we see that both the mass and the radius of a neutron star are reduced as we separate from the Schwarzschild vacuum exterior. The most striking feature is that, in principle, as a consequence of the extra dimension, stars may exist with gravitational potentials larger than 1/2 without being black holes. Certainly, this can provide some clues for the observational test of the theory.

Regarding the classical tests of general relativity and other experimental observations, within the context of Kaluza-Klein gravity [8], they have been calculated under the assumption that the physical spacetime outside an astrophysical object, like our sun, was described by the spacetime part of (1). Our results here suggest to consider another important question here is what are the consequences of going away from the Schwarzschild exterior metric. Using a uniform-density model for the interior of the star, in Tables 1-3 we show how some physical quantities as the mass, radius and surface gravitational potential of a star are affected by an extra dimension. For the same equation of state at the center, we see that both the mass and the radius of a neutron star are reduced as we separate from the Schwarzschild vacuum exterior. The most striking feature is that, in principle, as a consequence of the extra dimension, stars may exist with gravitational potentials larger than 1/2 without being black holes. Certainly, this can provide some clues for the observational test of the theory.

As far as we know this line element has never been used in this kind of discussions.

**Appendix A: Possible non-vacuum exteriors of a spherical star.**

An observer in 4D, who is not directly aware of the existence of an extra dimension, will interpret the metric (7) as if it were governed by an effective energy-momentum tensor.

$$8\pi T_0^0 = \frac{a^6r^4\sigma^2[3N(2 - N) + 4\varepsilon(1 - N)/\sigma]}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2[\varepsilon + \sigma(N - 1)]},$$

and

$$8\pi T_1^1 = -\frac{4a^6r^4\sigma^2 \{[1 - N](ar[4\sigma - 2\varepsilon - N\sigma]/[2\sigma] + [a^2r^2 + 1]/\sigma] + (3/4)N^2ar\}}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2[\varepsilon + \sigma(N - 1)]}.$$
\[8\pi T_2^2 = -\frac{2a^5 r^3 \sigma^2 \left\{ 2a \left[ (2\varepsilon + N\sigma)(1 - N)/(2\sigma) - 1 \right] - (1 - N)(a^2 r^2 + 1)/\sigma \right\} + (1/2)N^2 a r \right\} \left( \frac{ar - 1}{ar + 1} \right)^{2[\varepsilon + \sigma(N - 1)]},\]  
(A-3)

and \(T_3^3 = T_2^2\), as expected by virtue of the spherical symmetry. These equations show that \(T_2^2 \neq T_1^1\) for any value of \(N\) and \(\varepsilon \neq 1\). Thus, the effective matter behaves as an anisotropic fluid, which can be described by an effective energy-momentum tensor of the form

\[T_{\mu\nu} = (\rho + p_\perp)u_\mu u_\nu - p_\perp g_{\mu\nu} + (p_r - p_\perp)\chi_\mu \chi_\nu,\]  
(A-4)

where \(u^\mu\) is the four-velocity; \(\chi^\mu\) is a unit spacelike vector orthogonal to \(u_\mu\); \(\rho\) is the energy density; \(p_r\) is the pressure in the direction of \(\chi_\mu\), and \(p_\perp\) is the pressure on the two-space orthogonal to \(\chi_\mu\). Since \(T_{01} = 0\), it follows that the fluid is at rest in the system of reference of (7). Therefore, we can always choose \(u_\mu = \delta_\mu^0 / \sqrt{g_{00}}\) and \(\chi_\mu = \delta_1^\mu / \sqrt{(-g_{11})}\). Thus,

\[T_0^0 = \rho, \quad T_1^1 = -p_r, \quad \text{and} \quad T_2^2 = T_3^3 = -p_\perp.\]  
(A-5)

**Equation of state:** Now, from (A-1)-(A-3), we find the equation of state of the effective matter. It is

\[\rho = f(N, \sigma)(p_r + 2p_\perp),\]  
(A-6)

with

\[f(N, \sigma) = \frac{4\varepsilon(1 - N) - 3N\sigma(N - 2)}{4\varepsilon(1 - N) + 3N\sigma(N - 2)},\]  
(A-7)

where \(\sigma\) is given by (8). Note that \(f = 1\) for \(N = 0\) and \(N = 2\) the equation of state becomes

\[\rho = p_r + 2p_\perp.\]  
(A-8)

For \(N = 1\) we get

\[\rho = -(p_r + 2p_\perp).\]  
(A-9)

In general, for any value of \(N\) the equation of state approximates (A-8), for “small” values of \(\sigma\), i.e., for \(\varepsilon \approx 1\).

**Gravitational mass:** In 4D, the gravitational mass inside a 3D volume \(V_3\) is given by the Tolman-Whittaker formula viz.,

\[M(r) = \int \left( T_0^0 - T_1^1 - T_2^2 - T_3^3 \right) \sqrt{-g_3} dV_3.\]  
(A-10)

In isotropic coordinates, with \(g_{00} = e^{\nu(r)}\) and \(g_{11} = -e^{\lambda(r)}\), this simplifies to

\[M(r) = \frac{1}{2} r^2 e^{(\nu + \lambda)/2} \nu',\]  
(A-11)

where the prime \(\prime\) denotes derivative with respect to \(r\). Using this equation, for the metric (7) we obtain

\[M(r) = \frac{2\varepsilon}{a} \left( \frac{ar - 1}{ar + 1} \right)^{\sigma(1-N)}.\]  
(A-12)

Here, we require \(\varepsilon > 0\) in order to ensure the positivity of \(M\).
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