Numerical homogenization for poroelasticity problem in heterogeneous media

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Abstract. We consider the poroelasticity problems in heterogeneous porous media. Mathematical model contains coupled system of the equations for pressure and displacements. We construct coarse grid solver using numerical homogenization technique, where we calculate effective permeability and elastic properties of the media by solution of the local problems. We present numerical results for the model problem in two and three-dimensional formulations. The results show that method can provide good accuracy and computationally effective.

1. Introduction
We consider a poroelasticity problem in heterogeneous media. The process is described by the coupled system of equations for pressure and displacements [5, 6, 7]. The computational algorithm for the numerical solution is based on the finite-element method using linear basis functions. For the applicability and convergence of these methods, the size of the computational grids should be small enough to resolve small scale heterogeneities. For effective numerical solution of such problems different multiscale or homogenization techniques are developed [8, 9, 10, 3, 1].

In this paper, we present a numerical homogenization technique for poroelasticity problem in non-periodic heterogeneous media [2, 4]. In the proposed method, the macroscopic coefficients are computed by the solution of the local problems for each coarse cell. Using the macroscopic tensor coefficients, we solve the global coarse-grid problem.

The work is organized as follows. In Section 2, we provide the mathematical model of the poroelasticity problem in heterogeneous media and construct fine grid approximation using the finite element method. In Section 3, we present coarse grid approximation using numerical homogenization technique, where we calculate effective permeability and elastic properties. Finally, numerical results are presented in Section 4 for two- and three-dimensional model problems. We compare results with the reference fine-scale solution. Our numerical results show a good agreement and shows that numerical homogenization techniques can be effectively used for representing porous media heterogeneities.

2. Problem formulation and fine grid approximation
We consider linear poroelasticity problem in domain \( \Omega \)
\[- \text{div} \sigma(u) + \alpha \text{grad} \ p = 0, \quad x \in \Omega, \]
\[
\frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial \text{div} u}{\partial t} - \text{div} \left( \frac{k}{v} \text{grad} \ p \right) = f, \quad x \in \Omega, \tag{1}
\]

where \( p \) and \( u \) are the pressure and displacement, \( \sigma \) is the stress tensor, \( k \) is the permeability, \( v \) is the fluid viscosity, \( f \) is the source term, \( M \) to be the Biot modulus and \( \alpha \) is the Biot-Willis fluid-solid coupling coefficient.

We have following linear elastic stress-strain constitutive relation

\[ \sigma(u) = 2\mu \varepsilon(u) + \lambda \text{div} \ (u), \quad \varepsilon(u) = \frac{1}{2} (\text{grad} \ u + \text{grad} \ u^T), \]

where \( \mu, \lambda \) are Lame coefficients, \( I \) is the identity tensor.

In this work, we consider problem (1) with heterogeneous coefficients

\[ k = k(x), \quad E = E(x), \quad x \ \text{in} \]

and

\[
\mu = \frac{E}{2(1 + \eta)}, \quad \lambda = \frac{E\eta}{(1 + \eta)(1 - 2\eta)},
\]

where \( \eta \) is the Poisson’s ratio and \( E \) is the elastic modulus.

For numerical solution of the poroelasticity problem on fine grid, we use a Galerkin finite element method. We have following variational formulation of the problem: find \((u, p) \in V \times Q (V = [H^1(\Omega)]^d \text{ and } Q = H^1(\Omega))\) such that

\[ a(u, v) - b(v, p) = 0, \quad \forall v \in V, \]
\[ b \left( \frac{du}{dt}, q \right) + c \left( \frac{dp}{dt}, q \right) + s(p, q) = l(q), \quad \forall q \in Q, \tag{2} \]

where

\[ a(u, v) = \int_{\Omega} \sigma(u) \cdot \varepsilon(v) \, dx, \quad s(p, q) = \int_{\Omega} \frac{k}{v} \text{grad} \ p \cdot \text{grad} \ q \, dx, \quad c(p, q) = \int_{\Omega} \frac{1}{M} \ p \ q \, dx, \]

and

\[ b(u, p) = \int_{\Omega} \alpha \text{div} \ u \ p \, dx, \quad l(q) = \int_{\Omega} f \ q \, dx. \]

We will use fine grid formulation for reference solution and error calculations in Section 4.

3. Numerical homogenization

Let \( \mathcal{H} \) is the coarse grid and

\[ \mathcal{H} = \bigcup_{i=1}^{N} K_i, \quad i = 1, N, \]

where \( N \) is the number of the coarse grid cells and \( j \) is the coarse grid cell index.
For construction of the coarse grid approximation, we calculate effective permeability and elastic coefficient

\[
k^* = \begin{bmatrix} k_{11}^* & k_{12}^* \\ k_{21}^* & k_{22}^* \end{bmatrix}, \quad C^* = \begin{bmatrix} C_{11}^* & C_{12}^* & C_{13}^* \\ C_{12}^* & C_{22}^* & C_{23}^* \\ C_{13}^* & C_{23}^* & C_{33}^* \end{bmatrix}, \tag{3}
\]

for 2D case and

\[
k^* = \begin{bmatrix} k_{11}^* & k_{12}^* & k_{13}^* \\ k_{12}^* & k_{22}^* & k_{23}^* \\ k_{13}^* & k_{23}^* & k_{33}^* \end{bmatrix}, \quad C^* = \begin{bmatrix} C_{11}^* & C_{12}^* & C_{13}^* & C_{14}^* & C_{15}^* & C_{16}^* \\ C_{21}^* & C_{22}^* & C_{23}^* & C_{24}^* & C_{25}^* & C_{26}^* \\ C_{31}^* & C_{32}^* & C_{33}^* & C_{34}^* & C_{35}^* & C_{36}^* \\ C_{41}^* & C_{42}^* & C_{43}^* & C_{44}^* & C_{45}^* & C_{46}^* \\ C_{51}^* & C_{52}^* & C_{53}^* & C_{54}^* & C_{55}^* & C_{56}^* \\ C_{61}^* & C_{62}^* & C_{63}^* & C_{64}^* & C_{65}^* & C_{66}^* \end{bmatrix} \tag{4}
\]

for 3D case. We calculate the effective coefficients for non-periodic heterogeneous media in each coarse cell, \(\{C^{*,K_i}\}_{i=1..N}\) and \(\{k^{*,K_i}\}_{i=1..N}\) for \(K_i \in T_H\).

For calculation of the effective permeability, we solve following local problem in \(K_i\)

\[
-k \nabla \cdot (k^K_i \nabla \psi^K_i) = 0, \quad x \text{ in } K_i, \\
\psi^K_i = x_j, \quad x \text{ on } \partial K_i, \tag{5}
\]

where \(k^K_i = k^K_i(x)\) is the restriction of the heterogeneous coefficient \(k\) to local domain \(K_i, x = (x_1, x_2)\) for 2D case and \(x = (x_1, x_2, x_3)\) for 3D case. Therefore, for 2D problem, we solve two local problems and for 3D problem, we have three local problems. Note that, another boundary conditions can be applied for local problem.

Next, we can find elements of the effective permeability tensor for current \(K_i\)

\[
k^{*,K_i}_{ij} = \frac{1}{|K_i|} \int_{K_i} k^K_i(x) \frac{\partial \psi^K_i}{\partial x_j} dx, \quad l, j = 1..d, \tag{6}
\]

where \(d\) is problem dimension, \(d = 2\) or 3.

For effective elastic modulus, we apply similar algorithm and solve following local problem in \(K_i\)

\[
-k \nabla \cdot \sigma(\phi) = 0, \quad x \text{ in } K_i, \\
\phi^K_i = \Lambda^{(rs)} x, \quad x \text{ on } \partial K_i, \tag{7}
\]

where \(\phi = (\phi_1, \phi_2)\) for \(d = 2, \phi = (\phi_1, \phi_2, \phi_3)\) for \(d = 3\) and

\[
\Lambda^{(rs)}_{ij} = \frac{1}{2} (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}), \quad r, s = 1..d.
\]

Here

\[
\sigma(\phi) = C : \varepsilon(\phi),
\]

or
\[ C^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_{12})^T \) for 2D and

\[ C^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_{23}, \sigma_{13}, \sigma_{12})^T \) for 3D.

The elements of the effective elastic modulus are calculated as follows

\[ C^{*,K_i}_{rs,pq} = \frac{1}{|K_i|} \int_{K_i} C_{ijkl} \varepsilon \left( \phi^{(rs)}_{ij} \right) : \varepsilon \left( \phi^{(pq)}_{kn} \right) dx, \quad r, s, p, q = 1..d. \quad (8) \]

Finally, we solve the poroelastic problem on a coarse mesh with effective permeability and elastic modulus

\[ - \text{div} \left( C^* : \varepsilon (u) \right) + \alpha \text{grad} \, p = 0, \quad \frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial}{\partial t} \text{div} u - \text{div} \left( \frac{k^*}{\nu} \text{grad} \, p \right), \quad (9) \]

Using Galerkin finite element method.

4. Numerical results

In this section, we present numerical results of the poroelasticity problems in heterogeneous media.

We consider two test problems:

- Two-dimensional problem.
- Three-dimensional problem.

For poroelastic problem, we set \( \alpha = 1 \) and \( M = 1 \). As initial conditions, we set \( p_0 = 0 \) and perform calculations for \( T_{\text{max}} = 0.01 \) with 10-time steps for 2D and 20-time steps for 3D. We perform calculations using structured coarse and fine grids.

4.1 Two-dimensional problem

We solve poroelastic problem in \( \Omega = [0,1] \times [0,1] \). We set following boundary conditions

\[ u_1 = 0, \quad \sigma_2 = 0, \quad x \text{ on } \Gamma_L \]
\[ \sigma_1 = 0, \quad u_2 = 0, \quad x \text{ on } \Gamma_B \]
\[ p = 1, \quad x \text{ on } \Gamma_T. \]

where \( \Gamma_L \) and \( \Gamma_R \) are the left and right boundaries, \( \Gamma_B \) and \( \Gamma_T \) are the bottom and top boundaries, \( \partial \Omega = \Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T \).
Figure 1: heterogeneous elasticity modulus and permeability for two-dimensional problem. Cases 1, 2 and 3 (from left to right)

Table 1: Relative errors for displacement and pressure between coarse grid and reference solution for two-dimensional problem. Cases 1, 2 and 3

| Case | $L^p_2$ (%) | $H^p_1$ (%) | $L^u_2$ (%) | $H^u_1$ (%) |
|------|-------------|-------------|-------------|-------------|
| 1    | 2.726       | 12.433      | 7.698       | 15.867      |
| 2    | 2.158       | 14.092      | 9.338       | 14.479      |
| 3    | 2.448       | 9.772       | 5.584       | 13.764      |

We consider three test cases for different heterogeneities (see Figure 1). Coarse grid is $10 \times 10$ and fine grid is $320 \times 320$. In Figure 3, we present the distribution of pressure and displacement along the X and Y directions at the last moment of time for the coarse and fine grids. In Table 1, we present the weighted $L_2$ and $H_1$ errors for multiscale solver. Relative errors vs time are presented in Figure 2. The task execution time is 2.0304 seconds for the coarse grid and 173.62 for the fine grid. Fine grid consists of 102400 cells and coarse grid contains 100 cells.

Figure 2: Relative errors vs time for two-dimensional problem. For displacement and pressure between coarse grid and fine grid solution. Case 1
3. Distribution of pressure, displacement along $X$ and $Y$ directions at the last moment of time for fine grid (top) and coarse grid using numerical homogenization (bottom) for two-dimensional problem. Case 1

4.2 Three-dimensional problem
Next, we solve model problem in $\Omega = [0,1] \times [0,1] \times [0,1]$. Coarse grid is $5 \times 5 \times 5$ and fine grid is $20 \times 20 \times 20$.

We set following boundary conditions

$$
\begin{align*}
\sigma_1 &= 0, & \sigma_2 &= 0, & \sigma_3 &= 0, & x & \text{ on } \Gamma_L \\
\sigma_1 &= 0, & \sigma_2 &= 0, & \sigma_3 &= 0, & x & \text{ on } \Gamma_B \\
p &= 1, & x & \text{ on } \Gamma_T.
\end{align*}
$$

where $\Gamma_L$ and $\Gamma_R$ are the left and right boundaries, $\Gamma_B$ and $\Gamma_T$ are the bottom and top boundaries, $\Gamma_F$ and $\Gamma_W$ are the forward and backward boundaries $\partial \Omega = \Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T \cup \Gamma_F \cup \Gamma_W$.

We consider three test cases for different heterogeneities (see Figure 4). In Figure 5, we present the distribution of pressure and displacement along the $X$, $Y$ and $Z$ directions at the last moment of time for the coarse and fine grids. In Table 2, we present the weighted $L2$ and $H1$ errors for multiscale solver. The task execution time is 2.4078 seconds for the coarse grid and 757.52 for the fine grid. Fine grid consists 8000 cells and coarse grid contains 125 cells. We observe good accuracy for both 2D and 3D test problems with different heterogeneities.
Figure 4: Heterogeneous elasticity modulus and permeability for *three-dimensional problem*. Cases 1, 2 and 3 (from left to right)

Table 2: Relative errors for displacement and pressure between coarse grid and reference solution for *three-dimensional problem*. Cases 1, 2 and 3

| Case | $L_2^p$ (%) | $H_1^p$ (%) | $L_2^d$ (%) | $H_1^d$ (%) |
|------|-------------|-------------|-------------|-------------|
| 1    | 7.923       | 20.993      | 8.607       | 24.203      |
| 2    | 7.385       | 21.724      | 7.475       | 19.563      |
| 3    | 7.223       | 21.603      | 5.399       | 21.777      |

Figure 5: Distribution of pressure, displacement along $X$, $Y$ and $Z$ directions at the last moment of time for fine grid (top) and coarse grid using numerical homogenization (bottom) for *three-dimensional problem*. Case 1
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