Projective Dimension and Regularity of Powers of Edge Ideals of Vertex-Weighted Rooted Forests

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Abstract
In this paper, we provide some exact formulas for the projective dimension and regularity of powers of edge ideals of some vertex-weighted rooted forests. These formulas are functions of the weight of vertices and the number of edges.

Keywords Projective dimension · Regularity · Edge ideal · Powers of the edge ideal · Vertex-weighted rooted forest

Mathematics Subject Classification Primary 13C10 · 13D02; Secondary 05E40 · 05C20 · 05C22

1 Introduction

The study of the minimal free resolutions of homogenous ideals and their powers is an interesting topic in commutative algebra. Two invariants which measure the complexity of the minimal free resolutions are the projective dimension and the Castelnuovo–Mumford regularity (or simply regularity) of the ideal. In general, it is a difficult task to compute or bound these two invariants. However, in the case of monomial ideals, one can apply combinatorial and topological techniques to char-
acterize them. Among them, a widely studied class of monomial ideals is quadratic squarefree monomial ideals; they are associated to simple graphs and are also called edge ideals. Many authors have studied their regularity, projective dimension and Betti numbers, e.g. [1,6,22,24,36,37].

For any homogeneous ideal $I$, it is well known that the regularity of $I^t$ is asymptotically a linear function in $t$, that is, there exist constants $a$ and $b$ such that $\text{reg} (I^t) = at + b$ for all $t \gg 0$ (see [10,25,34]). Generally, the problem of finding this exact linear form $at + b$ and the smallest value $t_0$ such that $\text{reg} (I^t) = at + b$ for all $t \geq t_0$ has proved to be very difficult. On the other hand, Brodmann in [7] showed that depth $(S/I^t)$ is a constant for $t \gg 0$, and this constant is bounded above by $n - \ell(I)$, where $\ell(I)$ is the analytic spread of $I$. It is shown that depth $(S/I^t)$ is a nonincreasing function of $t$ if all powers of $I$ have linear resolutions in [19, Theorem 1.2]. This means that the projective dimension $\text{pd} (S/I^t)$ is a constant for $t \gg 0$ by Auslander–Buchsbaum formula. In this regard, there has been an interest in determining the smallest value $t_0$ such that $\text{pd} (S/I^t)$ is a constant for all $t \geq t_0$ (see [14,19,23,29]). To the best of our knowledge, a few papers consider how to compute $\text{pd} (S/I^t)$ for a homogeneous ideal $I$.

A directed graph or digraph $D$ consists of a finite set $V(D)$ of vertices, together with a collection $E(D)$ of ordered pairs of distinct points called edges or arrows. A vertex-weighted (or simply weighted) directed graph is a triplet $D = (V(D), E(D), w)$, where $w$ is a weight function $w : V(D) \to \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$. Sometimes for short we denote the vertex set $V(D)$ and edge set $E(D)$ by $V$ and $E$, respectively. The weight of $x_i \in V$ is $w(x_i)$, denoted by $w_i$ or $w_{x_i}$.

As a generalization of edge ideals of simple graphs, one may consider ideals associated to some weighted oriented graphs. More precisely, let $D = (V, E, w)$ be a weighted digraph with $V = \{x_1, \ldots, x_n\}$. We consider the polynomial ring $S = k[x_1, \ldots, x_n]$ in $n$ variables over a field $k$. The edge ideal of $D$, denoted by $I(D)$, is the ideal of $S$ given by

$$I(D) = \left\langle x_ix_j^{w_{ij}} \mid x_i, x_j \in E \right\rangle.$$

According to the above definition, the edge ideal $I(D)$ of $D$ is independent of the set of its isolated vertices and the weight of a source vertex $x_i$ (i.e. has only arrows leaving $x_i$). Therefore, we shall always assume $D$ has no isolated vertices and $w_i = 1$ for any source vertex $x_i$ throughout this paper. If $w_j = 1$ for all vertices $j$, then $I(D)$ is the edge ideal of an unweighted unoriented graph $G(D)$ of $D$ which was introduced by Villarreal in [35] and has been studied extensively. Many researchers have tried to find $a$, $b$ and $t_0$ such that $\text{reg} (I(G(D))^t) = at + b$ for all $t \geq t_0$ (see [2–4,28,33]) and provide some bounds for depth $(S/I(G(D))^t)$ or $\text{pd} (I(G(D))^t)$ (see [14,19,22,29]) for edge ideals of some special graphs. To the best of our knowledge, the study of edge ideals of weighted oriented graphs and their powers is much more recent and consequently there are fewer results in this direction. In [40], the first three authors derive some exact formulas for the projective dimension and regularity of edge ideals of some weighted rooted forests and oriented cycles. In [38], we provide some exact formulas for the regularity and projective dimension of edge ideals of some weighted
oriented $m$-partite graphs. In [39], we present some precise formulas for regularity and give the upper and lower bounds of projective dimension of powers of edge ideal of some weighted oriented gap free bipartite graphs.

Edge ideals of weighted digraphs arose in the theory of Reed–Muller codes as initial ideals of vanishing ideals of projective spaces over finite fields [27,31]. In this article, we study the regularity and projective dimension of powers of edge ideals of weighted oriented graphs with the goal of characterizing these algebraic invariants in terms of the combinatorial data of weighted oriented graphs. To describe the generators of the edge ideal of a weighted oriented graph, one needs to consider the structure of its underlying graph, the orientation of its edges and the weight function. Thus, it is a quite difficult problem to incorporate all of these data and provide general formulas for the regularity and projective dimension of an arbitrary weighted oriented graph. Hence, we restrict our attention to some weighted rooted forests such that $w(x) \geq 2$ if $\deg(x) \neq 1$ and derive some exact formulas for the projective dimension and regularity of powers of their edge ideals by using the approaches of Betti splitting and polarization. The results are as follows:

Theorem 1.1 Let $t$ be a positive integer and $D = (V, E, w)$ a weighted rooted forest such that $w(x) \geq 2$ if $\deg(x) \neq 1$. Then,

1. $\text{reg} \left( I(D)^t \right) = \sum_{x \in V} w(x) - |E| + 1 + (t - 1)(w + 1)$

   where $w = \max \{w(x) \mid x \in V\}$,

2. $\text{pd} \left( I(D)^t \right) = |E| - 1$.

Our paper is organized as follows. In Sect. 2, we recall some definitions and basic facts used in the following sections. In Sect. 3, we consider the regularity of powers of edge ideal of a weighted rooted tree with the property that the distance between the root and every leaf is at most 2. In Sect. 4, we give some exact formulas for the projective dimension and regularity of powers of edge ideals of general weighted rooted forests.

For all unexplained terminology and additional information, we refer to [21] (for the theory of digraphs), [5] (for graph theory), and [8,20] (for the theory of edge ideals of graphs and monomial ideals). We gratefully acknowledge the use of the computer algebra system CoCoA [9] for our experiments.

2 Preliminaries

In this section, we gather together the needed definitions and basic facts, which will be used throughout this paper. However, for more details, we refer the reader to [1,2,5,15,17,20,21,27,32,38,40].

Given a weighted digraph $D = (V(D), E(D))$, we write $uv$ for $\{u, v\}$ if $\{u, v\} \in E(D)$ is a directed edge from $u$ to $v$. Meanwhile, we can associate a graph $G$ on the same vertex set simply by replacing each directed edge by an edge with the same ends. This graph is called the underlying graph of $D$, denoted by $G(D)$. Conversely, any graph $G$ can be regarded as a digraph, by replacing each of its edges by just one of the two oppositely oriented edge with the same ends. Such a direction is called
an orientation of $G$. A simple graph whose every edge has an orientation is called a simple digraph.

Every concept that is valid for graphs automatically applies to digraphs too. For example, the degree of a vertex $x$ in a digraph $D$, denoted by $\deg(x)$, is simply the degree of $x$ in $G(D)$. The distance between two vertices is the minimum length of the path connecting them. Likewise, a digraph is said to be connected if its underlying graph is connected. An oriented path or oriented cycle is a path or cycle with an orientation where each vertex dominates its successor in the sequence. An oriented acyclic graph is a simple digraph without oriented cycles. An oriented tree is an oriented acyclic graph formed by orienting edges of undirected acyclic graphs. A rooted tree is an oriented tree in which all edges are oriented either away from or towards its root. Unless specifically stated, a rooted tree in this article is an oriented tree in which all edges are oriented away from its root. An oriented forest is a disjoint union of some oriented trees. A rooted forest is a disjoint union of some rooted trees.

For any homogeneous ideal $I$ in the polynomial ring $S = k[x_1, \ldots, x_n]$, there exists a finite minimal graded free resolution

$$0 \to \bigoplus_j S(-j)^{\beta_{p,j}(I)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_0,j(I)} \to I \to 0,$$

where $p \leq n$ and $S(-j)$ is the $S$-module obtained by shifting degrees of $S$ by $j$. The number $\beta_{i,j}(I)$, the $(i,j)$-th graded Betti number of $I$, is an invariant of $I$ that equals the number of minimal generators of degree $j$ in the $i$th syzygy module of $I$. Of particular interest is the following invariants which measure the size of the minimal graded free resolution of $I$. The projective dimension of $I$, denoted by $\text{pd}(I)$, is defined to be

$$\text{pd}(I) := \max \{i \mid \beta_{i,j}(I) \neq 0\}.$$

The regularity of $I$, denoted by $\text{reg}(I)$, is defined to be

$$\text{reg}(I) := \max \{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

Calculating or even estimating the regularity or projective dimension for a general ideal is a difficult problem. We will provide some formulas for $\text{pd}(I)$ and $\text{reg}(I)$ in some special cases by using some tools developed in [15]. Let $G(I)$ denote the unique minimal set of monomial generators of a monomial ideal $I \subset S$ and let $u \in S$ be a monomial. We set $\text{supp}(u) = \{x_i : x_i | u\}$. If $G(I) = \{u_1, \ldots, u_m\}$, we set $\text{supp}(I) = \bigcup_{i=1}^m \text{supp}(u_i)$.

**Definition 2.1** Let $I$ be a monomial ideal, and suppose that there exist monomial ideals $J$ and $K$ such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then, $I = J + K$ is Betti splitting if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \quad \text{for all} \quad i, j \geq 0,$$

where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.  

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This formula was first obtained for the total Betti numbers by Eliahou and Kervaire [12] and extended to the graded case by Fatabbi [13]. In [15], the authors describe some sufficient conditions for an ideal $I$ to be Betti splitting. We need the following lemma.

**Lemma 2.2** [15, Corollary 2.7] Suppose that $I = J + K$ where $G(J)$ consists of all the elements of $G(I)$ divisible by some variable $x_i$ and $G(K)$ is a nonempty set consisting of the remaining elements of $G(I)$. If $J$ has a linear resolution, then $I = J + K$ is Betti splitting. Hence, Definition 2.1 implies the following results:

1. \[ \text{reg} (I) = \max \{ \text{reg} (J), \text{reg} (K), \text{reg} (J \cap K) - 1 \} \]
2. \[ \text{pd} (I) = \max \{ \text{pd} (J), \text{pd} (K), \text{pd} (J \cap K) + 1 \} \]

**Definition 2.3** Suppose that $u = x_1^{d_1} \cdots x_n^{d_n}$ is a monomial in $S$. We define the polarization of $u$ to be the squarefree monomial

$$P(u) = x_1^{d_1} x_2^{d_2} \cdots x_1^{d_1} x_2 \cdots x_n$$

in the polynomial ring $k[x_{ij} \mid 1 \leq j \leq a_i, 1 \leq i \leq n]$. If $I \subset S$ is a monomial ideal with $G(I) = \{ u_1, \ldots, u_m \}$ and $u_i = \prod_{j=1}^{a_i} x_j^{a_{ij}}$ where each $a_{ij} \geq 0$ for $i = 1, \ldots, m$. The polarization of $I$, denoted by $I^P$, is defined as:

$$I^P = (P(u_1), \ldots, P(u_m)),$$

which is a squarefree monomial ideal in the polynomial ring $S^P = k[x_{j1}, x_{j2}, \ldots, x_{j a_j} \mid j = 1, \ldots, n] \text{ where } a_j = \max \{ a_{ij} \mid i = 1, \ldots, m \} \text{ for any } 1 \leq j \leq n$.

Here is an example of how polarization works.

**Example 2.4** Let $I(D) = (x_1 x_3, x_2 x_3, x_3 x_4^2, x_4 x_5^5)$ be the edge ideal of a weighted rooted tree $D$, then its polarization is the ideal

$$I(D)^P = (x_1 x_3 x_2 x_4 x_5, x_2 x_3, x_3 x_4 x_5 x_6, x_4 x_5 x_6 x_7 x_8 x_9 x_{10}).$$

A monomial ideal $I$ and its polarization $I^P$ share many homological and algebraic properties. The following results are some very useful properties of polarization.

**Lemma 2.5** [20, Corollary 1.6.3] Let $I \subset S$ be a monomial ideal and $I^P \subset S^P$ its polarization. Then,

1. \[ \beta_{i,j} (I) = \beta_{i,j} (I^P) \text{ for all } i \text{ and } j, \]
2. \[ \text{reg} (I) = \text{reg} (I^P), \]
3. \[ \text{pd} (I) = \text{pd} (I^P). \]

The following four lemmas are often used for computing the projective dimension and regularity of an ideal.

**Lemma 2.6** [18, Lemma 1.3] Let $I$ be a proper nonzero homogeneous ideal in $S$. Then,
(1) \( \text{reg}(I) = \text{reg}(S/I) + 1 \),
(2) \( \text{pd}(I) = \text{pd}(S/I) - 1 \).

**Lemma 2.7** [17, Lemmas 2.2 and 3.2], [20, Theorem 1.3.3] Let \( S_1 = k[x_1, \ldots, x_m] \) and \( S_2 = k[x_{m+1}, \ldots, x_n] \) be two polynomial rings, \( I \subset S_1 \) and \( J \subset S_2 \) be two nonzero homogeneous ideals. Then,

(1) \( \text{reg}(I + J) = \text{reg}(I) + \text{reg}(J) - 1 \),
(2) \( \text{pd}(I + J) = \text{pd}(I) + \text{pd}(J) + 1 \).

In particular, if \( J = (u) \), where \( u \) is a monomial of degree \( m \) such that \( \text{supp}(u) \cap \text{supp}(I) = \emptyset \), then \( \text{reg}(J) = m \) and \( \text{reg}(JI) = \text{reg}(I) + m \).

**Lemma 2.8** [30, Theorem 1.1] Let \( S_1 = k[x_1, \ldots, x_m] \) and \( S_2 = k[x_{m+1}, \ldots, x_n] \) be two polynomial rings, and let \( I \subset S_1 \) and \( J \subset S_2 \) be two monomial ideals. Then,

\[
\text{reg}((I + J)^t) = \max_{i \in [t-1]} \left\{ \text{reg}(I^{t-i}) + \text{reg}(J^i), \text{reg}(I^{t-j+1}) + \text{reg}(J^j) - 1 \right\}
\]

for any \( t \geq 1 \).

**Lemma 2.9** [18, Lemmas 1.1 and 1.2] Let \( 0 \to A \to B \to C \to 0 \) be a short exact sequence of finitely generated graded \( S \)-modules. Then,

(1) \( \text{reg}(B) \leq \max \{ \text{reg}(A), \text{reg}(C) \} \), the equality holds if \( \text{reg}(A) - 1 \neq \text{reg}(C) \),
(2) \( \text{pd}(B) \leq \max \{ \text{pd}(A), \text{pd}(C) \} \), the equality holds if \( \text{pd}(C) \neq \text{pd}(A) + 1 \).

### 3 Regularity of Powers of Edge Ideals of Weighted Rooted Trees: A Special Case

In this section, by using approaches of Betti splitting and polarization, we will provide some formulas for the regularity of powers of edge ideals of weighted rooted trees with the property that the distance between the root and every leaf is at most 2. We start with the definition of hypergraphs.

A hypergraph \( H = (X, E) \) consists of \( X \) and a collection \( E \) of nonempty subsets of \( X \), which are called to be the edges of \( H \). Let \( Y \subseteq X \). The induced subhypergraph of \( H \) on \( Y \), denoted by \( H[Y] \), is a hypergraph with the vertex set \( Y \) and edge set \( \{ E \in E \mid E \subseteq Y \} \). A hypergraph \( H \) is simple if there is no nontrivial containment between any pair of its edges.

We need the following lemma.

**Lemma 3.1** [16, Lemma 3.1] Let \( H \) be a simple hypergraph and \( H' \) its induced subhypergraph. Then,

\[
\text{reg}(H') \leq \text{reg}(H).
\]
Let $D = (V(D), E(D), w)$ be a weighted oriented graph. For $x \in V(D)$, we call $N_D^+(x) = \{y : xy \in E(D)\}$ and $N_D^-(x) = \{y : yx \in E(D)\}$ to be the out-neighbourhood and in-neighbourhood of $x$, respectively. The neighbourhood of $x$ is the set $N_D(x) = N_D^+(x) \cup N_D^-(x)$. A weighted oriented graph $H = (V(H), E(H), w')$ is called to be the induced subgraph of $D$ if $V(H) \subseteq V(D)$, for any $x, y \in V(H)$, $xy$ is an oriented edge in $H$ if and only if $xy$ is an oriented edge in $D$, and the weight function $w'$ satisfies $w'(x) = 1$ if $x$ is a source in $H$, otherwise, $w'(x) = w(x)$. For $P \subseteq V(D)$, we denote $D \setminus P$ to be the induced subgraph of $D$ obtained by removing vertices in $P$ and edges incident to these vertices. If $P = \{x\}$, then we write $D \setminus x$ for $D \setminus \{x\}$. For $W \subseteq E(D)$, we define $D \setminus W$ to be the subgraph of $D$ with all edges in $W$ deleted (but its vertices remained). When $W = \{e\}$, we write $D \setminus e$ instead of $D \setminus \{e\}$.

The following proposition is needed to facilitate calculating the projective dimension and regularity of powers of an edge through induction on powers.

**Proposition 3.2** Let $t \geq 2$ be an integer and $D = (V(D), E(D), w)$ a weighted oriented graph. Let $x_n$ be a leaf with $N_D^-(x_n) = \{x_{n-1}\}$. Then,

1. $(I(D)^t, x_n^{w_n}) = (I(D \setminus x_n)^t, x_n^{w_n})$,
2. $(I(D)^t : x_n^{-1}x_n^{w_n}) = (I(D)^{t-1})$,
3. $((I(D)^t : x_n^{w_n}), x_{n-1}) = ((I(D \setminus x_{n-1})^t : x_n^{w_n}), x_{n-1}) = (I(D \setminus x_{n-1})^t, x_{n-1})$.

**Proof** (1) It is clear that $(I(D \setminus x_n)^t, x_n^{w_n}) \subseteq (I(D)^t, x_n^{w_n})$. For any monomial $f \in G(I(D)^t) \setminus G(I(D \setminus x_n)^t)$, we get that $x_n$ may divides $f$. It follows that $x_{n-1}x_n^{w_n}$ may also divides $f$ because of $N_D^-(x_n) = \{x_{n-1}\}$. This implies $f \in (x_n^{w_n})$.

(2) Consider any monomial $f \in G(I(D)^t : x_n^{-1}x_n^{w_n})$. Then, $fx_n^{-1}x_n^{w_n} \in I(D)^t$. We can write $fx_n^{-1}x_n^{w_n} = e_i e_{i+1} \ldots e_t f$ for some monomial $h$, where $e_i = x_{j_i} x_{j_i}$ such that $x_{j_i} x_{j_i} \in E(D)$. If there exists some $j_i \in \{1, \ldots, t\}$ such that $x_n^{w_n}$ divides $e_i$, then $e_i = x_n^{-1} x_{n-1}$ because of $N_D^-(x_{n-1}) = \{x_{n-1}\}$. This implies that $f \in I(D)^{t-1}$. If $x_n^{w_n}$ does not divide $e_i$ for any $j_i \in \{1, \ldots, t\}$, then $x_n^{w_n}$ divides $h$. Thus, $f \in I(D)^t$.

(3) $((I(D \setminus x_{n-1})^t : x_n^{w_n}), x_{n-1}) = (I(D \setminus x_{n-1})^t, x_{n-1})$ can be shown by a similar argument as the proof of (2). Now assume that a monomial $f \in G((I(D)^t : x_n^{w_n}), x_{n-1})$ and $x_n$ does not divide $f$. Then, $f \in (I(D)^t : x_n^{w_n})$. It follows that $fx_n^{w_n} \in I(D \setminus x_{n-1})^t$. This means that $f \in (I(D \setminus x_{n-1})^t : x_n^{w_n})$. Hence, $f \in ((I(D \setminus x_{n-1})^t : x_n^{w_n}), x_{n-1})$.

\[ \Box \]

**Theorem 3.3** Let $t$ be a positive integer and $D = (V, E, w)$ a weighted oriented star graph. Assume that the edge set of $D$ is one of the following two cases $\{x_1 x_2, x_1 x_3, \ldots, x_1 x_n\}$ and $\{x_2 x_1, x_3 x_1, \ldots, x_n x_1\}$. Then,

\[ \text{reg} (I(D)^t) = \sum_{i=1}^{n} w_i - |E| + 1 + (t-1)(w+1) \]

where $w = \max \{w_i \mid 1 \leq i \leq n\}$.
\textbf{Proof} The cases \(E = \{x_1, x_2, \ldots, x_n\}\) and \(E = \{x_1, x_3, \ldots, x_n, x_1\}\) can be shown by a similar argument. Hence, we only consider \(E = \{x_1, x_2, x_3, \ldots, x_n\}\). In this case, we have
\[
I(D)^t = (x_1^t)(x_2^{w_2}, x_3^{w_3}, \ldots, x_n^{w_n})^t.
\]
From Lemmas 2.7 (2), 2.8, it follows that
\[
\text{reg} \,(I(D)^t) = \text{reg} \,( (x_1^t)) + \text{reg} \,( (x_2^{w_2}, \ldots, x_n^{w_n})^t) = t + \sum_{i=2}^{n} w_i - (n - 2) + (t - 1)w
\]
\[
= \sum_{i=1}^{n} w_i - |E| + 1 + (t - 1)(w + 1).
\]

\(\square\)

\textbf{Lemma 3.4} Let \(t\) be a positive integer and \(P_3\) a weighted oriented line graph with edge ideal \(I(P_3) = (x_1^{w_2}, x_2^{w_3})\) where \(w_2 \geq 2\). Then,
\[
\text{reg} \,(I(P_3)^t) = \sum_{i=1}^{3} w_i - 1 + (t - 1)(w + 1)
\]
where \(w = \max \{w_i \mid 1 \leq i \leq 3\}\).

\textbf{Proof} We apply induction on \(t\). The case \(t = 1\) follows from [40, Theorem 3.1]. Now assume that \(t \geq 2\). Consider the following short exact sequences
\[
0 \longrightarrow \frac{S}{I(P_3)^t : x_3^{w_3}}(-w_3) \xrightarrow{x_3^{-w_3}} \frac{S}{I(P_3)^t} \xrightarrow{S} \frac{S}{(I(P_3)^t, \, x_3^{w_3})} \longrightarrow 0 \quad (\dagger)
\]
\[
0 \longrightarrow \frac{S}{I(P_3)^t : x_2 x_3^{w_3}}(-1) \xrightarrow{x_2} \frac{S}{I(P_3)^t : x_3^{w_3}} \xrightarrow{S} \frac{S}{((I(P_3)^t : x_3^{w_3}), \, x_2)} \longrightarrow 0. \quad (\dagger\dagger)
\]
We distinguish into the following two steps:

Step 1: We will compute \(\text{reg} \,(I(P_3)^t : x_2 x_3^{w_3})\), \(\text{reg} \,(((I(P_3)^t : x_3^{w_3}), \, x_2))\) and \(\text{reg} \,(((I(P_3)^t, \, x_3^{w_3})))\).

By Proposition 3.2, we obtain \(I(P_3)^t : x_2 x_3^{w_3} = I(P_3)^{t-1}, ((I(P_3)^t : x_3^{w_3}), \, x_2) = (x_2)\) and \((I(P_3)^t, \, x_3^{w_3}) = ((x_1 x_2^{w_2})^t, \, x_3^{w_3})\). It follows from Lemma 2.7 (1) and induction hypothesis on \(t\) that
\[
\text{reg} \,(I(P_3)^t : x_2 x_3^{w_3}) = \text{reg} \,((I(P_3)^t)^{t-1}) = \sum_{i=1}^{3} w_i - 1 + (t - 2)(w + 1)
\]
\[
\leq \sum_{i=1}^{3} w_i - 1 + (t - 1)(w + 1) - w_3 - 1, \quad (1)
\]
\[ \text{reg} \left( \left( I(P_3)^t : x_3^{w_3} \right), x_2 \right) = \text{reg} \left( (x_2) \right) = 1 < \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1) - w_3, \]  \hspace{1cm} (2)

\[ \text{reg} \left( \left( I(P_3)^t, x_3^{w_3} \right) \right) = \text{reg} \left( \left( (x_1x_2^{w_2})^t, x_3^{w_3} \right) \right) = t(w_2 + 1) + w_3 - 1 = \sum_{i=1}^{3} w_i - 1 + (t-1)(w_2 + 1), \]  \hspace{1cm} (3)

where the first inequality holds because of \( w \geq w_3 \).

Step 2: We will prove \( \text{reg} \left( I(P_3)^t \right) = \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1) \).

Using Lemmas 2.2.1, 2.9 (1), formulas (1), (2) and (3) on the exact sequences (\( \dagger \dagger \)) and (\( \dagger \)), we obtain

\[ \text{reg} \left( I(P_3)^t \right) = \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1) \text{ if } w_2 = w, \]

and

\[ \text{reg} \left( I(P_3)^t \right) \leq \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1) \text{ if } w_2 < w. \]  \hspace{1cm} (4)

It is enough to prove \( \text{reg} \left( I(P_3)^t \right) = \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1) \) if \( w_2 < w \).

In this case, we get \( w = w_3 \). We may write \( I(P_3)^t \) as \( I(P_3)^t = J + K \), where \( G(K) = \{ x_1^j, x_2^{t(w_3)} \} \) and \( G(J) = G(I(P_3)^t) \setminus G(K) \). Let \( J^P, K^P \) and \( I(P_3)^t)^P \) be the polarizations of \( J, K \) and \( I(P_3)^t \) respectively. Then, we obtain

\[ (I(P_3)^t)^P = J^P + K^P, \quad J^P \cap K^P = \left( x_{11}^{w_2 + t - 1} \prod_{j=1}^{w_3} x_{2j} \prod_{j=1}^{w_3} x_{3j} \right). \]

Let \( H = (V(H), \mathcal{E}(H)) \) and \( H' = (V(H'), \mathcal{E}(H')) \) be hypergraphs associated to \( G(I(P_3)^t)^P \) and \( G(J^P) \), respectively. Then, \( H' \) is an induced subhypergraph of \( H \). By Lemmas 2.5 (2), 3.1 and formula (4), we obtain

\[ \text{reg} \left( J^P \right) \leq \text{reg} \left( (I(P_3)^t)^P \right) = \text{reg} \left( I(P_3)^t \right) \leq \sum_{i=1}^{3} w_i - 1 + (t-1)(w+1). \]  \hspace{1cm} (5)

Note that \( K^P \) has a linear resolution and the variable \( x_{3, t w_3} \) in \( K^P \) cannot divide any element in \( G(J^P) \), it follows that \( (I(P_3)^t)^P = J^P + K^P \) is Betti splitting by Lemma 2.2. From Lemmas 2.5 (2), 2.2 (1) and formula (5), we obtain

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reg \((I(P_3)^t) = \operatorname{reg} \big( (I(P_3)^t)^P \big) = \max \{ \operatorname{reg}(J^P), \operatorname{reg}(K^P), \operatorname{reg}(J^P \cap K^P) - 1 \} \)

\[= \max \{ \operatorname{reg}(J^P), t(w + 1), 1 + (t - 1 + w_2) + tw_3 - 1 \} \]

\[= \sum_{i=1}^{3} w_i - 1 + (t - 1)(w + 1). \]

This proof is completed. \(\square\)

**Theorem 3.5** Let \(t\) be a positive integer and \(D = (V(D), E(D), w)\) a weighted rooted tree whose maximum distance between the root and leaves is 2. Assume that \(w(x) \geq 2\) if \(\deg(x) \neq 1\), then

\[
\operatorname{reg} (I(D)^t) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)
\]

where \(w = \max \{ w(x) \mid x \in V(D) \} \).

**Proof** We apply induction on \(t\) and \(|E(D)|\).

The cases \(t = 1, t \geq 2\) and \(|E(D)| = 2\) follow from [40, Theorem 3.5] and Lemma 3.4, respectively. Now assume that \(t \geq 2\) and \(|E(D)| \geq 3\). Since the maximum distance between the root and leaves is 2, there are at least two leaves in \(D\). Let \(x_n\) and \(x_{n-1}\) be leaves with \(w_n \leq w_{n-1}\) and \(N_D(x_n) = \{x_m\}\). Consider the following short exact sequences

\[
\begin{align*}
0 \to & \frac{S}{(I(D)^t : x_n^{w_n}) : x_m} \to \frac{S}{I(D)^t} \to \frac{S}{(I(D)^t : x_n^{w_n})} \to 0 \\
0 \to & \frac{S}{(I(D)^t : x_n^{w_n}) : x_m} \to \frac{S}{I(D)^t} \to \frac{S}{((I(D)^t : x_n^{w_n}), x_m)} \to 0.
\end{align*}
\]

By Proposition 3.2 (1), Lemma 2.7 (1) and induction hypothesis on \(|E(D)|\), we have

\[
\begin{align*}
\operatorname{reg} \left( (I(D)^t, x_n^{w_n}) \right) &= \operatorname{reg} \left( (I(D) \setminus x_n)^t, x_n^{w_n} \right) = \operatorname{reg} \left( (I(D) \setminus x_n)^t \right) + w_n - 1 \\
&= \left( \sum_{x \in V(D) \setminus x_n} w(x) - |E(D) \setminus x_n| + 1 + (t - 1)(w' + 1) \right) + w_n - 1 \\
&= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1). \\
\end{align*}
\]

where \(w' = \max \{ w(x) \mid x \in V(D) \setminus x_n \}\) and the last equality holds because of \(w' = w\).

Using Proposition 3.2 (2) and induction hypothesis on \(t\), we get

\[
\begin{align*}
\operatorname{reg} \left( (I(D)^t : x_n^{w_n}) \right) &= \operatorname{reg} \left( (I(D)^{t-1}) \right) \\
&= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 2)(w + 1).
\end{align*}
\]
Let \( x_1 \) be the root of \( D \). We distinguish into the following three cases:

1. If \( x_m \neq x_1 \) and \( \deg(x_1) = 1 \), then

\[
((I(D)^t : x^{w_n}_m), x_m) = (I(D \setminus x_m)^t, x_m) = (x_m)
\]

by Proposition 3.2 (3). It follows that

\[
\begin{align*}
\text{reg} \left( ((I(D)^t : x^{w_n}_m), x_m) \right) &= \text{reg} \left( (I(D \setminus x_m)^t, x_m) \right) \\
&\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) - w_n.
\end{align*}
\]

(1)

2. If \( x_m \neq x_1 \) and \( \deg(x_1) \geq 2 \), then by Proposition 3.2 (3), Lemma 2.7 (1) and induction hypothesis on \( |E(D)| \), we have

\[
\begin{align*}
\text{reg} \left( ((I(D)^t : x^{w_n}_m), x_m) \right) &= \text{reg} \left( (I(D \setminus x_m)^t, x_m) \right) \\
&= \sum_{x \in V(D \setminus x_m)} w(x) - |E(D \setminus x_m)| + 1 + (t - 1)(w'' + 1) = A + B - w_n
\end{align*}
\]

where \( w'' = \max \{ w(x) \mid x \in V(D \setminus x_m) \} \), \( A = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w'' + 1), B = |N_D(x_m)| - \sum_{x \in N_D(x_m) \setminus x_n} w(x) - w_m \), the inequality holds because of \( w'' \leq w \) and \( w_m \geq 2 \).

3. If \( x_m = x_1 \). Let \( N^+_D(x_1) = \{ x_2, \ldots, x_\ell, x_{\ell + 1}, \ldots, x_p, x_n \} \) where \( x_2, \ldots, x_\ell, x_n \) are leaves of \( D \) and \( x_{\ell + 1}, \ldots, x_p \) are not leaves of \( D \). By Proposition 3.2 (3), Lemmas 2.7 (1), 2.8 and induction hypothesis on \( |E(D)| \), we have

\[
\begin{align*}
\text{reg} \left( ((I(D)^t : x^{w_n}_m), x_m) \right) &= \text{reg} \left( (I(D \setminus x_1)^t, x_1) \right) = \text{reg} \left( (D \setminus x_1)^t \right) \\
&= \sum_{x \in V(D \setminus x_1)} w(x) - |E(D \setminus x_1)| + 1 + (t - 1)(w'' + 1) = A' + B' - w_n
\end{align*}
\]

where \( w'' = \max \{ w(x) \mid x \in V(D \setminus x_1) \} \), \( A' = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w'' + 1), B' = p - \sum_{i=1}^{\ell} w_i - \sum_{i=\ell+1}^{p} (w_i - 1), \) the fourth equality holds because \( D \setminus x_1 \) is an induced subgraph of \( D \) on the subset \( V(D \setminus \{ x_1, \ldots, x_\ell, x_n \} \) and the last inequality holds because of \( w'' \leq w \) and \( w_i \geq 2 \) for any \( \ell + 1 \leq i \leq p \).

In short, in the above three cases, we obtain

\[
\begin{align*}
\text{reg} \left( ((I(D)^t : x^{w_n}_m), x_m) \right) &\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) - w_n.
\end{align*}
\]

(3)
Using Lemmas 2.6 (1), 2.9 (1), formulas (1), (2) and (3) and the exact sequences (**) and (**), we obtain

\[
\text{reg} \left( I(D)^t \right) = \text{reg} \left( \left( I(D)^t, x_n^{w_n} \right) \right) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1).
\]

The proof is completed. \(\square\)

The following example shows that the assumption \(w(x) \geq 2\) if \(\deg(x) \neq 1\) cannot be dropped in Theorem 3.5.

**Example 3.6** Let \(I(D) = (x_1x_2^2, x_1x_3, x_3x_4^2)\) be the edge ideal of a weighted rooted tree \(D = (V, E, w)\), its weight function is \(w_1 = w_3 = 1\) and \(w_2 = w_4 = 2\). By using CoCoA, we obtain \(\text{reg}(I(D)^2) = 6\) which is not equal to the value provided by Theorem 3.5.

### 4 Projective Dimension and Regularity of Powers of Edge Ideals of Weighted Rooted Forests

In this section, we will give some formulas for the regularity and projective dimension of powers of edge ideals of weighted rooted forests.

**Theorem 4.1** Let \(t\) be a positive integer and \(D = (V(D), E(D), w)\) a weighted rooted forest such that \(w(x) \geq 2\) if \(\deg(x) \neq 1\). Then,

\[
\text{reg} \left( I(D)^t \right) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)
\]

where \(w = \max \{w(x) \mid x \in V(D)\}\).

**Proof** Let \(D_1, D_2, \ldots, D_s\) be all the connected components of \(D\). Then, each \(D_i\) is a rooted tree. By Lemma 2.8, we only need to prove that for any \(1 \leq i \leq s\),

\[
\text{reg} \left( I(D_i)^t \right) = \sum_{x \in V(D_i)} w(x) - |E(D_i)| + 1 + (t - 1)(\tilde{w}_i + 1),
\]

where \(\tilde{w}_i = \max \{w(x) \mid x \in V(D_i)\}\). Hence, we may assume that \(D\) is a weighted rooted tree.

We apply induction on \(t\) and \(|E(D)|\). The case \(t = 1\) follows from [40, Theorem 3.5]. The case \(t \geq 2\) and \(|E(D)| = 1\) is clear. Now we assume that \(t \geq 2\) and \(|E(D)| \geq 2\). Let \(x_n\) be a leaf of \(D\) whose distance from the root is maximum, say \(d\), and let \(N_D^{-1}(x_n) = \{x_{n-1}\}\). The cases \(d = 1\) and \(d = 2\) follow from Theorems 3.3 and 3.5, respectively. Now we assume that \(d \geq 3\). This implies the distance between the root and \(x_{n-1}\) is at least 2.
Consider the short exact sequences

\[
\begin{align*}
0 & \rightarrow \frac{S}{I(D)^t} : \frac{w_n}{x_n^{w_n}}(-w_n) \rightarrow \frac{S}{I(D)^t} : \frac{x_n^{w_n}}{x_n} \rightarrow \frac{S}{(I(D)^t, x_n^{w_n})} \rightarrow 0 \quad (\dagger) \\
0 & \rightarrow \frac{S}{I(D)^t} : \frac{x_{n-1}^{w_n}}{x_n^{w_n}}(-1) \rightarrow \frac{S}{I(D)^t} : \frac{x_n^{w_n}}{x_{n-1}} \rightarrow \frac{S}{(I(D)^t : x_n^{w_n}, x_{n-1})} \rightarrow 0. \quad (\ddagger)
\end{align*}
\]

First, we will compute \( \text{reg} \((I(D)^t, x_n^{w_n})\), \text{reg} \((I(D)^t : x_{n-1}^{w_n})\) and \( \text{reg} \(((I(D)^t : x_n^{w_n}, x_{n-1})\))\).

By Proposition 3.2, Lemma 2.7 (1) and induction hypotheses on \(|E(D)|\) and \(t\), we obtain

\[
\text{reg} \((I(D)^t, x_n^{w_n})\) = \text{reg} \(((I(D) \setminus x_n)^t, x_n^{w_n})\) = \text{reg} \(((I(D) \setminus x_n)^t)\) + w_n - 1
\]

\[
= \sum_{x \in V(D \setminus x_n)} w(x) - |E(D \setminus x_n)| + 1 + (t - 1)(w' + 1) + w_n - 1
\]

\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w' + 1), \quad (1)
\]

where \(w' = \max\{w(x) \mid x \in V(D \setminus x_n)\}\),

\[
\text{reg} \((I(D)^t : x_{n-1}^{w_n})\) = \text{reg} \((I(D)^{t-1})\) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 2)(w + 1)
\]

\[
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) - w_n - 1 \quad (2)
\]

and

\[
\text{reg} \(((I(D)^t : x_n^{w_n}, x_{n-1})\) = \text{reg} \(((I(D) \setminus x_{n-1})^t, x_{n-1})\) = \text{reg} \(((I(D) \setminus x_{n-1})^t)\)
\]

\[
= \sum_{x \in V(D \setminus x_{n-1})} w(x) - |E(D \setminus x_{n-1})| + 1 + (t - 1)(w'' + 1)
\]

\[
= A + \left(|N_D(x_{n-1})| - \sum_{x_i \in N_D^+(x_{n-1}) \setminus \{x_n\}} w_i - w_{n-1}\right) - w_n
\]

\[
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w'' + 1) - w_n, \quad (3)
\]

where \(A = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w'' + 1)\), \(w'' = \max\{w(x) \mid x \in V(D \setminus x_{n-1})\}\) and the above inequality holds because of \(w_{n-1} \geq 2\) and \(w \geq w''\).
Using Lemmas 2.6 (1), 2.9 (1), formulas (1), (2), (3) and the exact sequences (††) and (†), we obtain

$$\text{reg } (I(D))' = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \text{ if } w' = w,$$

and

$$\text{reg } (I(D))' \leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \text{ if } w' < w. \quad (4)$$

It is enough to prove $\text{reg } (I(D))' = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)$ if $w' < w$. In this case we have $w = w_n$. Let $N_D(x_{n-1}) = \{x_{n-2}\}$. Write $I(D)'$ as $I(D)' = J + K$, where $K = (x_{n-1}^t x_{n-2}^w)$. Let $I(D)' \cap J = G(K)$, and $J = \mathcal{G}(D) \setminus \mathcal{G}(K)$. Let $J^P$, $K^P$, and $I(D)'^P$ be the polarizations of $J$, $K$ and $I(D)'$, respectively. Note that $w_{n-1} \geq 2$ by the hypothesis, we obtain

$$I(D)'^P = J^P + K^P \text{ and } K^P \cap J^P = K^P L,$$

where $L = I(D) \setminus x_{n-1})^P + (x_{n-2,1} \prod_{j=t+1}^{n} x_{n-1,j}) + \sum_{x_i \in P} (\prod_{j=1}^{n} x_{ij})$ and $P = N_D^+(x_{n-1}) \setminus \{x_n\}$.

We may write $L = L' + J'$, where

$$L' = I(D) \setminus x_{n-1})^P + \left( x_{n-2,1} \prod_{j=t+1}^{n} x_{n-1,j} \right) \text{ and } J' = \sum_{x_i \in P} \left( \prod_{j=1}^{n} x_{ij} \right).$$

Let $D' = (V', E', w')$ be a rooted tree with the vertex set $V(D) \setminus N_D^+(x_{n-1})$, the edge set $E' = E(D) \setminus x_{n-1}) \cup \{x_{n-2} x_{n-1}\}$ and the weight function $w'(x_i) = w(x_i)$ for $x_i \in V(D) \setminus (N_D(x_{n-1}) \cup \{x_{n-1}\})$ and $w'(x_{n-1}) = w_{n-1} - 1$. Then, $L'$ and $I(D')$ have the same polarizations (up to a relabelling vertices), and thus, they have the same regularities, i.e. $\text{reg } (L') = \text{reg } (I(D'))$. Note that $\text{supp}(I(D')) \cap \text{supp}(J') = \emptyset$. Thus, by Lemmas 2.7 (1), 2.5 (2) and [40, Theorem 3.5], we obtain

$$\text{reg } (L) = \text{reg } (L' + J') = \text{reg } (I(D')) + \text{reg } (J') - 1$$

$$= \left( \sum_{x \in V(D')} w(x) - |E(D')| + 1 \right) + \sum_{x_i \in P} w_i - (|P| - 1) - 1$$

$$= \left[ \sum_{x \in V(D')} w(x) + \sum_{x_i \in P} w_i + w_n + 1 \right] - (|E(D')| + |P| + 1) - w_n + 1$$

$$= \sum_{x \in V(D)} w(x) - |E(D)| - w_n + 1. \quad (6)$$
Using Lemma 2.7 (2) and formula (6), we have
\[
\text{reg} (K^P \cap J^P) = \text{reg} (K^P L) = \text{reg} (K^P) + \text{reg} (L)
\]
\[
= t(w + 1) + \sum_{x \in V(D)} w(x) - w_n - |E(D)| + 1
\]
\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1). \quad (7)
\]

Let \( H = (V(H), E(H)) \) and \( H' = (V(H'), E(H')) \) be hypergraphs associated to \( G((I(D))^t P) \) and \( G(J^P) \), respectively. Then, \( H' \) is an induced subhypergraph of \( H \). Thus, by Lemmas 2.5 (2), 3.1 and the formula (5), we obtain
\[
\text{reg} (J^P) \leq \text{reg} ((I(D))^t P) = \text{reg} (I(D)) + \text{reg} (J^P \cap K^P) - 1
\]
\[
= \max \left\{ \text{reg} (J^P), t(w + 1), \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \right\}
\]
\[
= \sum_{x \in V(D)} w(x) - 1 + (t - 1)(w + 1).
\]

This proof is completed. \( \square \)

As a consequence of the above theorem, we have

**Corollary 4.2** Let \( t \) be a positive integer and \( D = (V(D), E(D), w) \) a weighted rooted forest as Theorem 4.1. Then,
\[
\text{reg} (I(D)^t) = \text{reg} (I(D)) + (t - 1)(w + 1)
\]
where \( w = \max \{ w(x) \mid x \in V(D) \} \).

**Proof** This is a direct consequence of the above theorem and [40, Theorem 3.5]. \( \square \)

**Theorem 4.3** Let \( t \) be a positive integer and \( D = (V(D), E(D), w) \) a weighted rooted forest such that \( w(x) \geq 2 \) if \( \deg(x) \neq 1 \). Then,
\[
\text{pd} (I(D)^t) = |E(D)| - 1.
\]
We apply induction on $|E(D)|$ and $t$. The case $|E(D)| = 1$ is clear. The case $t = 1$ follows from [40, Theorem 3.3]. Now we assume that $|E(D)| \geq 2$ and $t \geq 2$.

Let $x_n$ be a leaf of $D$ and $N_D^{-}(x_n) = \{x_{n-1}\}$.

Consider the following short exact sequences

\[
0 \rightarrow \frac{S}{I(D)^t : x_n^{w_n}(-w_n)} \rightarrow \frac{S}{I(D)^t} \rightarrow \frac{S}{(I(D)^t : x_n^{w_n})} \rightarrow 0
\]

\[
0 \rightarrow \frac{S}{I(D)^t : x_{n-1}^{w_n}(-1)} \rightarrow \frac{S}{I(D)^t : x_n^{w_n}} \rightarrow \frac{S}{(I(D)^t : x_n^{w_n}, x_{n-1})} \rightarrow 0
\]

using Proposition 3.2, Lemma 2.7 (3) and induction hypotheses on $|E(D)|$ and $t$, we obtain

\[
\text{pd}((I(D)^t : x_n^{w_n})) = \text{pd}((I(D \setminus x_n)^t, x_n^{w_n})) = \text{pd}((I(D \setminus x_n)^t) + \text{pd}(x_n^{w_n})) + 1
\]

\[
= |E(D \setminus x_n)| - 1 + 1 = |E(D)| - |N_D(x_{n-1})|
\]

\[
\leq |E(D)| - 1,
\]

(1)

\[
\text{pd}(I(D)^t : x_{n-1}^{w_n}) = \text{pd}(I(D)^{t-1}) = |E(D)| - 1.
\]

(2)

\[
\text{pd}(((I(D)^t : x_n^{w_n}), x_{n-1})) = \text{pd}((I(D \setminus x_{n-1})^t, x_{n-1})) = \text{pd}(I(D \setminus x_{n-1})^t) + 1
\]

\[
= |E(D \setminus x_{n-1})| - 1 + 1
\]

\[
\leq |E(D)| - 1.
\]

(3)

Using Lemmas 2.6 (2), 2.9 (2) on the exact sequences (‡) and (‡‡), and formulas (1)~(3), we get

\[
\text{pd}(I(D)^t) = |E(D)| - 1.
\]

An immediate consequence of the above theorem is the following corollary.

**Corollary 4.4** Let $t$ be a positive integer and $D = (V(D), E(D), w)$ a weighted rooted forest as in Theorem 4.3. Then, depth $(I(D)^t) = |V(D)| - |E(D)| + 1.$

**Proof** By Auslander–Buchsbaum formula (see Theorem 1.3.3 of [8]), it follows that

\[
\text{depth}(I(D)^t) = |V(D)| - \text{pd}(I(D)^t) = |V(D)| - |E(D)| + 1.
\]

The following example shows that the assumption that $w(x) \geq 2$ if $\deg(x) \neq 1$ cannot be dropped in Theorems 4.1 and 4.3.
Example 4.5 Let $I(D) = (x_1x_2^2, x_2x_3, x_3x_4^2, x_5x_6^2, x_6x_7, x_7x_8^2)$ be the edge ideal of a weighted rooted forest $D = (V, E, w)$, its weight function is $w_1 = w_3 = w_5 = w_7 = 1$ and $w_2 = w_4 = w_6 = w_8 = 2$. By using CoCoA, we obtain $pd(I(D)^2) = 4$ and $\text{reg}(I(D)^2) = 8$, which are not equal to the values provided Theorems 4.3 and 4.1, respectively.

The following two examples show that the projective dimension and regularity of powers of edge ideals of weighted directed forests are related to the orientation of their edges.

Example 4.6 Let $G = (V(G), E(G))$ be a weighted tree with vertex set $V(G) = \{x_1, \ldots, x_8\}$ and edge set $E(G) = \{x_1x_2, x_2x_3, x_2x_4, x_1x_5, x_3x_6, x_1x_7, x_7x_8\}$ and the weight function is $w_1 = 1$ and $w_2 = \cdots = w_8 = 2$. Let $D_1$, $D_2$, and $D_3$ be three weighted directed trees with the same underlying graph $G$, and their edge sets $E(D_1) = \{x_1x_2, x_2x_3, x_2x_4, x_1x_5, x_3x_6, x_1x_7, x_7x_8\}$, $E(D_2) = \{x_1x_2, x_3x_2, x_2x_4, x_1x_5, x_3x_6, x_7x_1, x_8x_7\}$ and $E(D_3) = \{x_2x_1, x_3x_2, x_4x_2, x_5x_1, x_5x_6, x_7x_1, x_8x_7\}$, respectively. Then, $D_1$ is a weighted rooted tree satisfying its maximum distance from the root to leaves is 2, both $D_2$ and $D_3$ are weighted directed trees. Their edge ideals are $I(D_1) = (x_1x_2^2, x_2x_3^2, x_2x_4^2, x_1x_5^2, x_3x_6^2, x_1x_7^2, x_7x_8^2)$, $I(D_2) = (x_1x_2^2, x_3x_2^2, x_2x_4^2, x_1x_5^2, x_3x_6^2, x_7x_1^2, x_8x_7^2)$ and $I(D_3) = (x_2x_1^2, x_3x_2^2, x_4x_2^2, x_5x_1^2, x_5x_6^2, x_7x_1^2, x_8x_7^2)$, respectively. By using CoCoA, we obtain $\text{reg}(I(D_1)^2) = 12$, $\text{reg}(I(D_2)^2) = 11$, $\text{reg}(I(D_3)^2) = 11$, $pd(I(D_1)^2) = 6$, $pd(I(D_2)^2) = 5$ and $pd(I(D_3)^2) = 4$. 

Example 4.7 Let $G = (V(G), E(G))$ be a weighted tree with vertex set $V(G) = \{x_1, \ldots, x_8\}$ and edge set $E(G) = \{x_1x_2, x_2x_3, x_2x_4, x_3x_5, x_3x_6, x_5x_7, x_5x_8\}$ and the weight function is $w_1 = 1$ and $w_2 = \cdots = w_8 = 2$. Let $D_1$, $D_2$, and $D_3$ be three weighted directed trees with the same underlying graph $G$, and their edge sets $E(D_1) = \{x_1x_2, x_2x_3, x_2x_4, x_3x_5, x_3x_6, x_5x_7, x_5x_8\}$, $E(D_2) = \{x_1x_2, x_2x_3, x_2x_4, x_3x_5, x_3x_6, x_7x_1, x_8x_7\}$ and $E(D_3) = \{x_1x_2, x_2x_3, x_4x_2, x_5x_3, x_6x_3, x_5x_7, x_5x_8\}$, respectively. Then, $D_1$ is a rooted tree, both $D_2$ and $D_3$ are weighted directed trees. Their edge ideals are $I(D_1) = (x_1x_2^2, x_2x_3^2, x_2x_4^2, x_3x_5^2, x_3x_6^2, x_5x_7^2, x_5x_8^2)$, $I(D_2) = (x_1x_2^2, x_2x_3^2, x_2x_4^2, x_3x_5^2, x_3x_6^2, x_7x_1^2, x_8x_7^2)$ and $I(D_3) = (x_1x_2^2, x_2x_3^2, x_4x_2^2, x_5x_3^2, x_6x_3^2, x_5x_7^2, x_5x_8^2)$, respectively. By using CoCoA, we obtain $\text{reg}(I(D_1)^2) = 12$, $\text{reg}(I(D_2)^2) = 11$, $\text{reg}(I(D_3)^2) = 10$, $pd(I(D_1)^2) = 6$, $pd(I(D_2)^2) = 5$ and $pd(I(D_3)^2) = 4$.

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