Black hole entropy from KMS-states of quantum isolated horizons

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By reintroducing Lorentz invariance via a complex connection formulation in canonical loop quantum gravity, we define a geometrical notion of temperature for quantum isolated horizons. Upon imposition of the reality conditions in the form of the linear simplicity constraints for an imaginary Barbero-Immirzi parameter, the exact formula for the temperature can be derived by demanding that the horizon state satisfying the boundary conditions be a KMS-state. In this way, our analysis reveals the connection between the passage to the Ashtekar self-dual variables and the thermality of the horizon. The horizon equilibrium state can then be used to compute both the von Neumann and the Boltzmann entropies. By means of a natural cut-off introduced by the topological theory on the boundary, we show that the two provide the same finite answer which allows us to recover the Bekenstein-Hawking formula in the semi-classical limit. The connection with Connes-Rovelli thermal time proposal for a general relativistic statistical mechanics is worked out.

I. INTRODUCTION

The analogy between black holes and thermodynamical systems is now well established. In particular, one can associate to a black hole a notion of entropy and temperature [1, 2]. However, the statistical mechanics understanding of these thermodynamical properties is not clear yet, as this would require most likely a quantum gravity treatment of the horizon microscopic degrees of freedom (dof). Loop quantum gravity (LQG) has identified the entropy dof as the quantum fluctuations of the polymer-like geometry of the horizon [3, 4] (see [5] for a review). While the result of the entropy counting showed a linear behavior with the horizon area (and the appearance of sub-leading logarithmic corrections) [7], the exact numerical match with the Bekenstein-Hawking formula \( \mathcal{S} = \frac{A}{4\ell_\text{P}^2} \) required to fix the Barbero-Immirzi parameter \( \gamma \) (entering the spectrum of the area operator) to a particular real value. The need to eliminate a purely quantum ambiguity represented by \( \gamma \) through the request of agreement with a semiclassical result may be seen as a not very natural, let alone elegant, passage.

Indeed, this feature of the LQG calculation has received quite some attention during the years and some proposals for its removal have appeared (see, e.g., [6, 9]). However, only recently a key observation was made: in all of the state counting techniques developed in the literature the notion of temperature was never explicitly used. This observation led to the local stationary observer description of the horizon properties introduced in [10, 11], which allowed to single out a physical notion of local horizon energy. Such an ingredient, together with the introduction by hand of the Unruh temperature [12] for the thermal atmosphere around the horizon, provides a leading term for the state counting independent on the Barbero-Immirzi parameter and in agreement with the Bekenstein-Hawking entropy [10].

At the same time, the discovery of the important role played by the local description of the horizon physics and by the Unruh temperature led to the application of some techniques of states construction developed in the spin foam formalism [13] to the definition of a quantum Rindler horizon [14]. Here, in analogy with the semi-classical analysis, the boost operator of the Lorentz algebra is used to define an Hamiltonian generating a quantum Rindler horizon by ‘evolving’ \( SL(2, \mathbb{C}) \) spin network states and defining a notion of energy. By coupling a two-levels detector to derive the temperature of the quantum Rindler horizon, application of the Clausius relation yields the Bekenstein-Hawking formula for the entropy variation. At the statistical mechanical level, the result of [14] is interpreted as an entanglement entropy between the microscopic dof living on the two sides of the spin network links piercing the horizon [15]. However, taking into account just a single puncture, the analysis of [14] does not derive the whole horizon entropy.

It might seem quite surprising that the role played by the horizon temperature in recovering the semiclassical result was realized only several years after the first ideas about black hole entropy in LQG appeared. However, it should be noted that the notion of Unruh temperature is intimately related to the local Lorentz invariance in LQG appeared. On the other hand, in canonical LQG one uses the \( SU(2) \) Ashtekar-Barbero connection [18] obtained from the Ashtekar’s new complex variables [19] by means of canonical transformations with real values of \( \gamma \). Differently from the \( SL(2, \mathbb{C}) \) Ashtekar connection, which can be derived from a manifestly covariant action [20] maintaining full local Lorentz invariance, the Ashtekar-Barbero connection cannot be interpreted as a space-time connection, since it does not transform correctly under space-time diffeomorphisms [21, 22].

Therefore, it seems necessary to start with a covariant approach to LQG in order to introduce the ingredient of local Lorentz invariance and being able to derive a notion of temperature for the horizon. We believe this is the source of confusion which affected the standard LQG black hole entropy calculation in the past.

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In this direction it could be understood the recent proposal of [23] of an analytic continuation of the Chern-Simons Hilbert space formula for real values of the level \( k \) to imaginary values, corresponding to taking a purely imaginary Barbero-Immirzi parameter. Such a mathematical passage is interpreted as an analytic continuation from the compact gauge group \( SU(2) \) to the non-compact group \( SL(2, \mathbb{C}) \) and it is argued to provide a result for the counting which is compatible with the Bekenstein-Hawking entropy, once the large-spin limit for a fixed number of punctures is taken. However, working in the micro-canonical ensemble, the analysis of [23] never refers to an explicit notion of temperature and the passage to \( \gamma = i \) doesn’t have a physical motivation. Moreover, the issue of the reality of the area spectrum raised by the passage to the complex Ashtekar variables is left open.

In this letter we show that indeed, once Lorentz symmetry is introduced in the framework of quantum isolated horizons (QIH) [4, 5], a geometric notion of temperature and the passage to \( \gamma = i \) is in fact well known (see, for instance, [28, 29]) that the thermal asymptotic behavior of a black hole [30] is analogous to the thermal-structure everywhere (at each node). In section III we are going to use an analogy with the Euclidean path integral derivation of black hole entropy and the relation (1) to argue that one can associate a local temperature to the conical singularities in the quantum horizon geometry induced by the punctures. This notion of temperature is then derived from the construction of a KMS-state [25] for the geometry of QIH associated to a proper sub-algebra of the LQG holonomy-flux \( \star \)-algebra. Such a construction is possible via the imposition of the horizon boundary conditions [24] and the analytic continuation of the simplicity constraints [11] to \( \gamma = i \). In this way, we show how the passage to the Ashtekar self-dual variables at the end of the analysis is related to the definition of the QIH temperature, shedding light on the calculation of [23].

In section III the KMS-state will be used to compute both the von Neumann and the Boltzmann entropies of QIH, showing how the two provide the same finite answer which allows us to recover the Bekenstein-Hawking formula in the semi-classical limit. In particular, despite the imaginary spectrum of the area operator associated to the analytic continuation to \( \gamma = i \), we show that the large area limit of the entropy is real.

Interpretation of our result and comments will be presented in the final section IV.

In appendix A the relation with the Tomita-Takesaki modular theory [26] and the thermal time hypothesis of Connes and Rovelli [27] is derived.

## II. GEOMETRIC TEMPERATURE AND KMS-STATE OF QIH

In order to define a KMS-state associated to the geometry of QIH, let us start by recalling the Euclidean path integral representation for the vacuum of a relativistic quantum field restricted to a Rindler wedge. It is in fact well known (see, for instance, [28, 29]) that the thermal asymptotic behavior of a black hole [30] is analogous to the thermal-structure everywhere (at each node). More precisely, by means of an Euclidean path integral representation, the ground state wave functional of a field \( \phi \) can be written as a reduced density matrix for the state restricted to the Rindler wedge. When expressed in terms of eigenstates \( |n\rangle \) of the boost Hamiltonian \( K_\eta \) generating the Rindler horizon in terms of shift of the hyperbolic angle \( \eta \), the vacuum of a relativistic quantum field takes the form of the canonical thermal state

\[
\rho_\eta = \text{tr}_L |0\rangle \langle 0 | = \sum_n e^{-2\pi E_n} |n\rangle_{R,L} \langle n| ,
\]

where \( E_n \) are the eigenvalues of \( K_\eta \) and \( R, L \) refer to the modes on the right and left wedges. In this description one defines the horizon temperature as the period of the rotation angle corresponding to the euclidean continuation of \( \eta \). Such a notion of temperature as the period of the Euclidean time coordinate in a set covering only part of the manifold can easily be extended to any static spacetime with a bifurcation horizon, and in particular to the Schwarzschild space-time. This is possible due to the fact that a wide class of metrics with horizons can be mapped to the Rindler form near the horizon. Taking the trace of the reduced density matrix [26] one obtains the partition function \( Z \). In order to use this \( Z \) to compute the entropy then, one has to go off-shell [25]. More precisely, the thermodynamic entropy is given by

\[
F^i(A) = \frac{\pi(1 - \gamma^2)}{a_\mu} L^i \quad ,
\]

1 KMS-states correspond to the correct physical extension of Gibbs equilibrium thermal states to infinite dimensional quantum systems [25].
\[ S = -\left( \beta \frac{\partial}{\partial \beta} - 1 \right) \ln Z; \] 

therefore, in order to be able to derive with respect to \( \beta \), \( Z \) can be computed by evaluating the Euclidean functional integral over fields periodic in the Euclidean time under \( t_E \rightarrow t_E + \beta \). This is achieved by extending the classical black hole action to conical geometries; namely, one introduces a conical singularity in the \( (r,t) \)-plane at the point where the horizon is located. One can show that the deficit angle of the cusp \( \delta = 2\pi - \beta \) and the horizon area are canonically conjugate. By means of the effective action of a quantum field on the background metric with a conical singularity at the entangling surface, the expression \( \text{(20)} \), when evaluated for standard black hole solutions (i.e. \( \delta = 0 \)), reproduces the familiar Bekenstein-Hawking entropy \( \text{(3)} \).

The entanglement entropy interpretation of this black hole entropy derivation, originally proposed in \( \text{(34)} \), is clear in the geometric formulation of \( \text{(35)} \) (see \( \text{(36)} \) for a review of these ideas). Here the trace of the partition function \( \text{(3)} \) is obtained by equating the values of the field \( \phi \eta \) above and below the cut \( \eta = 0 \) and doing the unrestricted Euclidean path integral. The introduction of the conical singularity corresponds to computing the trace of the \( n \)-th power of the density matrix by evaluating the Euclidean functional integral over fields defined on an \( n \)-sheeted covering of the cut spacetime. In fact, the \( n \)-fold cover turns the \( (r,t) \) plane into a flat cone with deficit angle \( \delta = 2\pi(1 - n) \) at the origin. By means of the rotational symmetry with respect to the Killing vector \( \partial_\eta \) (which generates rotations in the \( 2 \)-plane orthogonal to the entangling surface), this construction can then be analytically continued to non-integer values of \( n \). In this way, one can define a partition function

\[ Z_\delta = \text{tr}(\rho^n) = \text{tr} e^{-\left(2\pi - \delta\right)K_\eta} \]  

and use the "replica trick" to rewrite the entropy \( \text{(20)} \) as

\[ S = -\text{tr}(\hat{\rho} \ln \hat{\rho}) = (2\pi \frac{\partial}{\partial \delta} + 1) \ln Z_{\delta=0}, \]  

where \( \hat{\rho} \) is the normalized density matrix.

We will exploit this key role played by the introduction of a conical singularity in the underlying geometry (when deriving the black hole entropy via Euclidean path integral techniques) in the context of QIH geometry. In order to do so, we now want to reintroduce Lorentz invariance in the canonical framework and use it to define a notion of temperature for the horizon.

A possible way to do this is to follow the approach of \( \text{(24)} \), where a program of canonical quantization with respect to complex variables, following standard LQG techniques, has been initiated. From this analysis, the simplicity constraints \( \text{(1)} \) naturally appear as reality conditions on the momentum variable. Borrowing techniques developed in the spin foam context in order to embed the \( SU(2) \) kinematical Hilbert space via the notion of projected spin networks \( \text{(31)} \), \( \text{(24)} \) shows how the weak imposition of \( \text{(1)} \) provides a limiting procedure in which one can recover the usual kinematical Hilbert space of the real valued Ashtekar-Barbero connection.

At this point some comments are in order. First of all, the linear simplicity constraints will always be used inside expectation values in the following. Thus, only their weak imposition will be necessary, consistently with the "new" spin foam models. Secondly, since the bulk dynamics is frozen for a QIH (the lapse function vanishes on the horizon \( \text{(25)} \)), what we are interested in here is just the restriction to the \( SU(2) \) kinematical Hilbert space and not a given spin foam vertex amplitude. In particular, we do not need to choose a specific map implementing \( \text{(1)} \) weakly on the labelings of the unitary representations of \( SL(2,\mathbb{C}) \) at each puncture. In this way, one may consider a bigger space of solutions than the one defined by the EPRL model \( \text{(41)} \), like for instance in \( \text{(32)} \).

Henceforth, relying on the techniques developed in \( \text{(24, 31)} \), we can use the structures of the \( SU(2) \) Hilbert space and at the same time the reality conditions \( \text{(1)} \) to relate the (quantum) geometry in the \( (r,t) \)-plane to the one in the \( (\theta,\phi) \)-plane \( \text{(34)} \). Such duality represents the key mechanism to encode the information about the extrinsic curvature enclosed in the gauge connection in the flux-Hamiltonian \( L \), via eq. \( \text{(2)} \). In this way, imposition of the horizon boundary conditions in the quantum theory provides a natural notion of local temperature associated to the quantum geometry of the horizon, which allows us to construct the analog of the KMS-state \( \text{(3)} \) for QIH.

### A. Temperature

After imposition of \( \text{(2)} \) the QIH state is flat except at the punctures \( \{p\} \), intersecting the surfaces over which the flux-Hamiltonian \( L \) is discretized, where it has conical singularities with quantized deficit angles \( \delta = 2\pi/k \) \( \text{(4)} \). Therefore, in analogy with the Euclidean path integral approach, one can define a temperature at each puncture associated to the period of the rotational symmetry encoded in the action of the flux operator, namely

\[ \eta = 2\pi(1 - \frac{1}{k}). \]  

Hence, such analogy, possible by means of the local observer perspective of \( \text{(11)} \) and the simplicity constraints \( \text{(1)} \), suggests that one can introduce a notion of QIH temperature associated to the quantum geometry deficit angle at the conical singularities in the boundary curvature induced by the bulk geometry. Since the r.h.s. of the

\[ \ldots \]

\[ \ldots \]
boundary conditions \(^2\) can be derived by means of the Gauss-Bonnet theorem \(^3\), such a notion of temperature has a geometric connotation analogue to the derivation of \(^3\). Notice that the (dimensionless) Urruh temperature can be recovered by going to the large area semi-classical limit; in fact, in the large Chern-Simons level limit one gets immediately \(\beta_U = \lim_{k \to \infty} \eta = 2\pi \). In the next section we are going to make this analogy with the euclidean path integral rigorous. More precisely, by defining a QIH state as a solution to the boundary condition \(^2\), we show that this satisfies the KMS-condition defining thermal states for a value of the temperature corresponding exactly to \(^7\), once the value of the Barbero-Immirzi parameter is set to \(\gamma = i\).

**B. KMS-state**

The starting point for the construction of a KMS-state is a \(C^*\)-algebra. As pointed out above, the imposition of the linear simplicity constraints \(^1\) allows us to use the real \(SU(2)\) kinematical Hilbert space. Therefore, we are going to define our \(C^*\)-algebra as a sub-algebra \(A^0_{\Gamma}\) of the holonomy-flux \(\star\)-algebra of LQG \(^3\) localized on the horizon \(H\). For the construction of \(A^0_{\Gamma}\) we follow the analysis of \(^3\), where the authors study a modification of the Ashtekar-Lewandowski measure on the space of generalized connections and look for a representation of this algebra containing states that solve the quantum analog of the flux-holonomy algebra of LQG. While the construction of \(^3\) is not fully worked out in the \(SU(2)\) case for all relevant observables, for the definition of \(A^0_{\Gamma}\) we only use a subset of observables, which are not affected by quantization ambiguities.

Let us consider a given boundary graph \(\Gamma\), that is a set of \(N\) edges piercing the horizon. As shown in \(^3\), holonomies on all contractible loops over the horizon are fixed and there are no local gauge invariant dof associated to them, in accordance with the expected topological nature of the boundary theory. Therefore, we do not consider them in the algebra \(A^0_{\Gamma}\). Holonomies on loops around punctures in \(\Gamma\) are fixed by the modified measure on \(H\) implementing the geometric condition \(^2\).

On the other hand, in the case \(H\) being a topological sphere, holonomies that run between punctures (and the conjugate fluxes) represent the only dof lying on the horizon and independent states on the horizon are labelled by a single intertwiner between them \(^3\). In this way one recovers the \(SU(2)\) intertwiner model of \(^3\). Therefore, these holonomies in \(H\) connecting the bulk links piercing the horizon in a single intertwiner can be seen as an extension of the edges in \(\Gamma\). In other words, a puncture coming from the bulk and piercing the horizon can either extend in the interior of the horizon or along its surface. In either case, all these extensions form an intertwiner representing dof not accessible to the exterior observer and we are thus going to trace over them. In the next section we will see how the information encoded in the intertwiner structure has an imprint in the thermal correlations of the resulting density matrix.

Then the horizon fluxes are defined on a discrete set of surfaces formed only by those branes \(\{p_1, \ldots, p_n\}\) which intersect one of the edges in \(\Gamma\) \(^3\). On each of these branes then the fluxes are quantized as

\[
\hat{L}_p = \bigoplus_p \hat{L}_i.
\]

The last fundamental ingredient comes from the TQFT structure of the solutions to the condition \(^2\) found in \(^3\). Namely, assuming the convergence between Chern-Simons and LQG isolated horizons quantization (as suggested also by the results of \(^3\)) we take the level \(k\) as a cut-off for the \(SU(2)\) irreps labeling the punctures Hilbert space. In this way the boundary operators are bounded. This completes the definition of the \(C^*\)-algebra \(A^0_{\Gamma}\).

Following the Euclidean path integral approach, we can write the QIH state as a pure state by considering for each puncture the tensor product between the associated inside and outside (of the horizon) states. Here by inside state we refer to the extension of the bulk link that lies either in the interior of the black hole or along the horizon, as explained above (in both cases we use the suffix \(I\) for these parts).

FIG. 1. Gauge invariant QIH state.

Due to the entanglement between these two components then, one can obtain a mixed state when restricting to the exterior. Explicitly, let us divide the boundary punctures Hilbert space in its internal and external parts by inserting a bivalent intertwiner at the point where the

\[^3\] The usual formula \(2\pi/\kappa\) for the Unruh temperature is recovered once we relate the flux-Hamiltonian to the horizon local energy of \(^1\), that is as we express the horizon evolution in terms of the static observer local time \(t = \eta\ell\), where \(\ell = \kappa^{-1}\) is the observer proper distance from the horizon. (Unfortunately, the standard notation might be confusing in this case: \(k\) = Chern-Simons level, \(\kappa\) = horizon surface gravity).

\[^4\] This restriction guarantees that the density matrix defined in \(^2\) can be represented as a cyclic and separating vector on a Hilbert space (see Appendix \(^4\)).
edge pierces the horizon (see FIG. 1) and write the state associated to each puncture in the spin representation as

\[ |j_p⟩ = \sum_{m_p = -j_p}^{+j_p} |j_p, m_p⟩, \quad (9) \]

then, the horizon ‘vacuum’ state can be written as

\[ Ω = \bigotimes_p |j_p⟩⟨j_p|. \quad (10) \]

Since the state (10) has to satisfy the quantum version of the boundary conditions (2) (which here we assume to be imposed like in [8] by means of a modified measure on the horizon Hilbert space), in particular it has to be gauge invariant. This means that we can act with SU(2) group elements \( g_p \) on each puncture and the state is left invariant, namely

\[ Ω = \bigotimes_p g_p |j_p⟩⟨j_p|. \quad (11) \]

If we now trace over the degrees of freedom \( |j, m⟩ \), inside the horizon and use the (exponentiated) boundary com (2) (by computing the scalar product on the internal states \( I \) by means of the modified measure defined in [8]), the quantum statistical state is given by the density matrix

\[ \hat{ρ} = \frac{1}{Z} \bigotimes_{p=1}^{N} P_{O,p}^{j} e^{i(\frac{π}{2} - 2\pi) L_p} P_{O,p}^{j}, \quad (12) \]

where the projector \( P_{O,p}^{j} \) is given by

\[ P_{O,p}^{j} = \sum_{m_p = -j_p}^{+j_p} |j_p, m_p⟩⟨j_p, m_p|. \quad (13) \]

and

\[ Z = \text{tr} \left( \bigotimes_{p=1}^{N} P_{O,p}^{j} e^{i(\frac{π}{2} - 2\pi) L_p} P_{O,p}^{j} \right). \quad (14) \]

The last passage in (12) follows from the periodicity of the action of the exponential of the flux operator [5].

We can thus see how the information contained in the boundary conditions [2] is encoded in a Boltzmann-like factor on each puncture associated to the flux-Hamiltonian, in analogy with the Unruh vacuum [3], which emerges as a consequence of tracing over the inside states. In other words, the ‘Boltzmann’ operator is not introduced by hand in the horizon state, but its appearance is a consequence of the QIH definition. Then, as it will be clearer in a moment, the thermality of the density matrix associated to the horizon quantum state originates from the entanglement between internal and external horizon dof, consistently with the intuition of [15].

The trace-class operator \( \hat{ρ} \) defines a positive linear functional (i.e. a state) over \( A^H_t \) that can be represented as

\[ \hat{ρ}(A) = \text{tr}(A\hat{ρ}) \quad (15) \]

for every \( A ∈ A^H_t \). Now remember that the state [3] represent an example of KMS-state for the automorphism generated by the boost Hamiltonian over the algebra defined by functional calculi of the fields \( φ \) with support on the Rindler wedge [10]. Therefore, based on the Bisognano-Wichmann theorem [10] and the proposal of [14], on each brane we can take the boost operator \( K \) as the local horizon generator and consider the one-parameter automorphism group \( α_t \) it generates on the \( C^* \)-algebra \( A^H_t \) localized on the horizon defined by

\[ α_t(A) = \bigotimes_{p=1}^{N} Pe^{iKt} P A \bigotimes_{p' = 1}^{N} P' e^{-iKt} P', \quad A ∈ A^H_t, \quad t ∈ \mathbb{R}, \quad (16) \]

where from now on we adopt the notation \( P ≡ P_{O,p}^{j} \), \( P' ≡ P_{O,p'}^{j} \). Using the GNS-construction to define a representation associated to the state (12), \( α_t \) can be shown to be the modular automorphism group on \( A^H_t \) associated to the state (12) (see Appendix A). In this way then our analysis can be embedded in the formalism defined in [15] for a proposal of general relativistic statistical mechanics.

From this point of view, the geometrical interpretation of the thermal time [2] flow generated by the statistical state (12) is encoded in the boundary condition [2].

If we now define the complex correlation function \( f_{AB}(z) \) as

\[ f_{AB}(z) = \hat{ρ}(α_z (A)B) \quad \text{with } z ∈ \mathbb{C}, \quad (17) \]

then the QIH state (12) satisfies the KMS-condition [17]

\[ f_{AB}(-iβ) = \hat{ρ}(α_{-iβ} (A)B) = \hat{ρ}(Bα_{iβ}(A)) = f_{BA}(0) \quad (18) \]

with temperature \( β = η \) given by (7), once the relation (11) is implemented weakly on each brane with \( γ = i \).

**Proof:**

\[ f_{AB}(-iβ) = \text{tr} \left( \bigotimes_{p'} P' e^{i(\frac{π}{2} - 2\pi) L_{p'}} P A \bigotimes_{p} P e^{i(\frac{π}{2} - 2\pi) L_{p}} \frac{Pe^{i2π(\frac{1}{4} - 1)L_{p}P}}{Z} \right) \]

\[ = \text{tr} \left( B \bigotimes_{p'} Pe^{i2π(\frac{1}{4} - 1)L_{p'}} P A \bigotimes_{p} Pe^{i2π(\frac{1}{4} - 1)L_{p}} P' \frac{P'e^{-i2π(\frac{1}{4} - 1)L_{p'}P}}{Z} \right) \]

\[ = \text{tr} \left( B \bigotimes_{p} P e^{i2π(\frac{1}{4} - 1)L_{p}} P A \bigotimes_{p'} P e^{i2π(\frac{1}{4} - 1)L_{p'}} P' \frac{P' e^{-i2π(\frac{1}{4} - 1)L_{p'}}}{Z} \right) = f_{BA}(0), \]

where the projectors relation
\[ \otimes P \otimes P' \]
\[ = \otimes \sum_{p} \sum_{m_p} |j_p, m_p \rangle \langle j_p, m_p| \otimes \sum_{p'} \sum_{m_p'} |j'_p, m'_p \rangle \langle j'_p, m'_p| \]
\[ = \otimes \sum_{p} \sum_{m_p} |j_p, m_p \rangle \langle j_p, m_p| = \otimes P \]

has been applied; in the second to last line we have used (1) and in the last one set
\[ \gamma = i \quad \text{and} \quad \beta = 2\pi(1 - \frac{1}{k}) \]. ☐

**III. ENTROPY**

We can now use the thermal equilibrium state \( \hat{\rho} \) to compute the von Neumann and the Boltzmann entropies of QIH. Given the expression for the entanglement entropy of a reduced density matrix
\[ S_{\text{ent}} = -\text{tr}(\hat{\rho} \ln \hat{\rho}) \],
(19)
from the state (12) and by means of the Chern-Simons formula of the LQG area operator, shown to be related where
\[ \gamma = i \quad \text{and} \quad \beta = 2\pi(1 - \frac{1}{k}) \].

\[ S_{\text{BH}} = S_{\text{ent}} = S_{\text{Bol}} \]
(23)

once the modification of the first law of QIH mechanics proposed in (10) is taken into account—such a modification encodes the quantum hair structure associated to the QIH geometry, as investigated more in detail in (42).

This extra term is present in the derivation of (14) as well and its appearance can be better understood once taking into account also matter dof living on the horizon punctures. In fact, from the local observer perspective and the thermality of QIH derived here, it is natural to expect a distribution of matter fields quanta in a thermal bath near the horizon. Such a ‘gas’ will have its own entropy contribution to be added to the geometrical one (15). At large scales this quantum gravity local correction is expected to disappear due to the temperature red-shift (and more in general the IR regime). A mechanism for such a disappearance has been proposed in (43) using the renormalization argument of (5). However, the behavior of this chemical potential correction at low energies and the implications of an imaginary \( \gamma \) for it require further investigation.

Nevertheless, even though the introduction of a notion of temperature for QIH requires the passage to the Ashtekar self-dual variables and leads to an imaginary area spectrum, the large area limit of the entropy is real and can be made consistent with the semiclassical result.

**IV. DISCUSSION**

We have seen that reintroducing Lorentz invariance in the canonical formalism of LQG and imposing the reality conditions in the form of the linear simplicity constraints, one can define a notion of temperature for QIH upon passage to the Ashtekar self-dual connection. Such microscopic derivation of the horizon temperature shows an intrinsically geometrical nature related to the topological aspects of the boundary theory, namely the Chern-Simons theory, and is thus a direct consequence of the imposition of the boundary conditions (2) together with the trace over the horizon internal states. Moreover, it reveals the connection between the analytic continuation to an imaginary Barbero-Immirzi parameter and the thermality of the horizon, clarifying the proposal of (23) (and the related (50, 51)).

The expression of the QIH temperature (7) presents a (complex) quantum correction 1\( /k \) which disappears in the semi-classical limit, when the Unruh formula is recovered. However, such correction encodes a sort of

\[ ^5 \text{For } \gamma = i \text{ the level } k \text{ becomes purely imaginary (23).} \]
back-reaction in the temperature formula and it is expected to become relevant for microscopic black holes. Investigation of its effect on the Hawking radiation spectrum and implications for the information paradox along the lines of the analysis performed in \[42, 44\] is left for future work.

Furthermore, the Chern-Simons level \(k\) plays a fundamental role in the entropy calculation. In fact, it provides a natural IR cut-off \(\gamma\) allowing us to perform the calculation and obtain a finite answer. Moreover, the \(k\)-dependence is such that the entropy calculation performed here seems compatible with the semi-classical analysis of \[33, 45\] without need to fine-tune the cut-off.

More precisely, the extra term appearing in both entropies \(20\) and \(22\) represents a quantum gravity correction; however, this is not a novel feature of LQG. In fact, it appears also in the statistical mechanical approach of \[38\], where the idea to link the entropy with the surface dof which give rise to 'off-shell' conical singularities at the horizon for Euclidean black-hole geometries has inspired our new analysis of the LQG horizon description. As shown in \[45\], when determining the available phase space for the surface fields accounting for the entropy, the requirement of general covariance introduces a single dimensionless parameter \(\mathcal{N}\) fixing the shape of the region of horizon phase-space available to the black hole. This \(\mathcal{N}\), which enters the definition of the statistical ensemble, is thus associated to the volume of the horizon 2-sphere diffeomorphisms group and it yields a \(\log \mathcal{N}\) correction to the Bekenstein-Hawking formula. The possibility to associate a new thermodynamic parameter for black holes to this quantum mechanical correction was underlined already in \[45\].

Besides the puzzle of the meaning of an imaginary \(\gamma\) in the entropy calculation, in this letter we have also solved the issue of which microscopic interpretation to assign to black hole entropy. More precisely, as shown in section \[1\], once identified the QIH state with a KMS-state over a proper restriction of the LQG holonomy-flux \(*\)-algebra to the horizon, both the von Neumann and the Boltzmann entropies can be computed from it. The results \(20\) and \(22\) show how the two coincide.

This equality might not be surprising since the partition function used to compute the Boltzmann entropy \(21\) was derived from the density matrix \(12\). However, the interesting part is the fact that the resulting partition function \(14\), obtained by tracing over the internal states dof, has the same form of the one used in the canonical ensemble calculation corresponding to the counting of the Chern-Simons Hilbert space dimension \(11\). This is the origin of the equivalence between entanglement entropy and state-counting, which is not built-in a priori in our analysis. From a physical point of view, this correspondence can be understood as a relation between the quantum geometry dof correlations across the horizon encoded in \(\rho\) and the number of horizon 'quantum shapes' encoded in the intertwiner space. The duality between these two descriptions of the horizon quantum geometry originates from the fact that both the thermality of the reduced density matrix \(12\) and the intertwiner structure of the boundary Hilbert space are consequences of the imposition of the boundary com \(2\).

In this way, our analysis reveals and explains the equivalence between the near-horizon quantum fields entanglement proposal of \[34, 44\], applied in the context of LQG in \[14, 47, 49\], with the original state-counting interpretation of black hole entropy \[3, 4\].

To conclude, let us point out how our analysis is based on a nice interplay between techniques developed both in the canonical and covariant formalisms of the theory. In this way, black hole entropy calculation in LQG provides a clear example of how insights gained in one formulation can be fruitfully applied to the dual one.

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Appendix A: modular automorphism group

The Tomita-Takesaki modular theory \[26\] (see \[52\] for a detailed technical description and \[53\] for an application as a possible approach to quantum gravity) provides a beautiful mathematical characterization of equilibrium states in statistical mechanics. Here we give a brief introduction and show its relation with the automorphism \(10\) on the algebra \(A^\gamma\).

First of all, let us recall that, given an abstract \(C^*\)-algebra \(A\) and a state \(\omega\) over \(A\), the GNS-construction provides us with a Hilbert space \(\mathcal{H}\) with a preferred cyclic vector \(|\Psi_0\rangle\) and a representation \(\pi_\omega\) of \(A\) as a concrete algebra of operators on \(\mathcal{H}\) such that
\[
\omega(A) = \langle \Psi_0 | \pi_\omega(A) | \Psi_0 \rangle. \tag{A1}
\]

Now, let us consider a von Neumann algebra \(\mathcal{R}\) acting on a Hilbert space \(\mathcal{H}\) possessing a cyclic \((\mathcal{A}|\Omega\rangle\) dense in \(\mathcal{H}\)) and separating \((\mathcal{A}|\Omega\rangle = 0 \rightarrow A = 0)\) vector \(|\Omega\rangle\). Consider the conjugate linear operator \(S\) defined by:
\[
S|\Omega\rangle = A^*|\Omega\rangle \quad \forall A \in \mathcal{R}. \tag{A2}
\]

One can show that \(S\) admits a polar decomposition
\[
S = J\Delta^{1/2}, \tag{A3}
\]
where \(J\) is the modular conjugation and \(\Delta\) is the modular operator associated with \(\omega\). The choice \(\rho = J|\Omega\rangle\langle\Omega|\) as a density matrix gives rise to a modular automorphism group \(\mathcal{G}_\omega\) with modular Hamiltonian \(H_\omega\) and modular state \(\omega_\Omega\) such that \(\omega_\Omega = \omega\).
where $\Delta$ is a self-adjoint positive operator and $J$ an anti
unitary operator. The Tomita-Takesaki theorem states that the map $\sigma_t : \mathcal{R} \rightarrow \mathcal{R}$ defined by
$$\sigma_t(A) = \Delta^{it}A\Delta^{-it} \quad A \in \mathcal{R} \quad (A4)$$
defines a 1-parameter group of automorphisms of the algebra $\mathcal{R}$, called the group of modular automorphisms of the state $\omega$ on the algebra $\mathcal{R}$. Correspondingly, $J$ is called the modular conjugation and $\Delta$ the modular operator of $(\mathcal{R}, \Omega)$. Then
$$JRJ = \mathcal{R}' \quad \Delta^{it}\mathcal{R}\Delta^{-it} = \mathcal{R},$$
where $\mathcal{R}'$ is the set of all bounded linear operators on $\mathcal{H}$ which commute with all elements of $\mathcal{R}$. It follows that the state $\omega$ is invariant under $\sigma_t$, i.e. $\omega(\sigma_t(A)) = \omega(A)$ for all $A \in \mathcal{R}$ and $t \in \mathbb{R}$. Moreover, one can show that $\omega$ satisfies the KMS-condition with respect to the automorphism group $\sigma_t$ for the inverse temperature value $\beta = 1$. Therefore, an equilibrium state with inverse temperature $\beta$ may be characterized as a faithful state over the observables algebra whose modular automorphism group $\sigma_t$ (where $\tau = t/\beta$) is the time translation group. This is the key observation at the base of the thermal time hypothesis. Notice that, since a von Neumann algebra is also a concrete $C^*$-algebra, the Tomita-Takesaki theorem applies also to an arbitrary faithful state $\omega$ over an abstract $C^*$-algebra $\mathcal{A}$. This is due to the fact that $\omega$ defines a representation of $\mathcal{A}$ in terms of bounded linear operators on a Hilbert space via the GNS-construction, as recalled above.

Following and the idea of, we are now going to use the GNS-construction to define a representation of $\mathcal{A}'$ associated to the state in which the map $\alpha_t$ given by (10) can be shown to be the modular automorphism group on it. That the state $\hat{\rho}$ is cyclic can be easily seen from the properties of the holonomy-flux algebra. In order to be separating as well, we need to restrict the discretization of horizon fluxes to a set of surfaces intersecting once any of the boundary punctures. Moreover, notice that all the links in the end on these surfaces due to the trace over the interior dof necessary to derive the density matrix; in this way, possible ambiguities and difficulties related to the non-commutative $SU(2)$ structure in the algebra disappear.

In order to derive the modular operator $\Delta$ from the polar decomposition, we need first to construct a new representation in which $\hat{\rho}$ is given by a vector $|\Omega\rangle$ in the new Hilbert space. This can be done by considering the set of Hilbert-Schmidt density matrices on the boundary Hilbert space $\mathcal{H}_B$, that is the set $\{\kappa : \text{tr}(\kappa^*\kappa) < \infty, \kappa \in B(\mathcal{H}_B)\}$, where $B(\mathcal{H}_B)$ is the set of all bounded, linear operators acting in $\mathcal{H}_B$. This set forms a Hilbert space $\mathcal{K}$ with respect to the scalar product
$$\langle \kappa | \kappa' \rangle = \text{tr}(\kappa^* \kappa). \quad (A5)$$

Since the thermal state $\hat{\rho}$ is positive, the operator
$$\kappa_0 = \hat{\rho}^{1/2} \quad (A6)$$
can be seen as a pure vector in $\mathcal{K}$, which we denote $|\kappa_0\rangle$.

We can now consider the following representation of $\mathcal{A}'_\tau \subset B(\mathcal{H}_B)$ by operators acting on $\mathcal{K}$
$$\pi(A)|\kappa\rangle = |A\kappa\rangle : \kappa \in \mathcal{K}, \quad A \in \mathcal{A}'_\tau. \quad (A7)$$

Due to the intersection property introduced above, $|\kappa_0\rangle$ is a cyclic and separating vector for the representation $\pi$. Moreover, one has
$$\langle \kappa_0 | \pi(A) | \kappa_0 \rangle = \hat{\rho}(A), \quad (A8)$$

that is $|\kappa_0\rangle$ plays the role of the ‘vacuum state’ vector in $\pi$ corresponding to the density operator $\hat{\rho}$. Furthermore, the state $|\kappa_0\rangle$ is time invariant in the sense that, by representing the time flow generated by the automorphism as

$$U(t)|\kappa\rangle = |\kappa_0\rangle \quad (A9)$$

We are now ready to identify the modular conjugation $J$ and operator $\Delta$ in the polar decomposition such that
$$SA|\kappa_0\rangle = A^*|\kappa_0\rangle = |A^*\kappa_0\rangle, \quad (A11)$$

namely
$$J|\kappa\rangle = |\kappa^*\rangle \quad (A12)$$

and
$$\Delta^{1/2}|\kappa\rangle = \left| \bigotimes_{p=1}^{N} P e^{\frac{i}{2}\hat{K}_p P} \kappa \bigotimes_{p'=1}^{N} P' e^{\frac{i}{2}\hat{K}_{p'} P'} \right). \quad (A13)$$

Let us show that with this choice of $J$ and $\Delta$ (A11) is satisfied

$$SA|\kappa_0\rangle = J\Delta^{1/2}|A^*\kappa_0\rangle \frac{1}{\sqrt{Z}} \bigotimes_{p} P e^{-\frac{i}{2}\hat{L}_p P}$$

$$= J \bigotimes_{p'} P' e^{\frac{i}{2}\hat{K}_{p'} P'} A \bigotimes_{p} P e^{-\frac{i}{2}\hat{L}_p P} \bigotimes_{p''} P'' e^{\frac{i}{2}\hat{K}_{p''} P''}$$

$$= J \bigotimes_{p'} P' e^{\frac{i}{2}\hat{K}_{p'} P'} A \frac{1}{\sqrt{Z}} \bigotimes_{p} P$$

$$= A^* \frac{1}{\sqrt{Z}} \bigotimes_{p} P e^{-\frac{i}{2}\hat{L}_p P}$$

$$= A^*|\kappa_0\rangle,$$

where we have used analytic continuation of the the simplicity constraint to $\gamma = i$ and the value $\beta = \eta$ for the temperature, as found in section (11). Finally, a similar calculation shows that the time evolution automorphism is related to the modular group $(A3)$ by
$$\sigma_t = \alpha_{\beta t}. \quad (A14)$$
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