On the Generalized Fibonacci-circulant-Hurwitz numbers

Ömür Deveci¹, Zafer Adıgüzel² and Taha Doğan³

¹ Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100, Turkey
e-mail: odeveci36@hotmail.com

² Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100, Turkey
e-mail: zafer-adiguzel36@hotmail.com

³ Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100, Turkey
e-mail: tahadogan8636@gmail.com

Received: 28 May 2019 Revised: 22 December 2019 Accepted: 30 December 2019

Abstract: The theory of Fibonacci-circulant numbers was introduced by Deveci et al. (see [5]). In this paper, we define the Fibonacci-circulant-Hurwitz sequence of the second kind by Hurwitz matrix of the generating function of the Fibonacci-circulant sequence of the second kind and give a fair generalization of the sequence defined, which we call the generalized Fibonacci-circulant-Hurwitz sequence. First, we derive relationships between the generalized Fibonacci-circulant-Hurwitz numbers and the generating matrices for these numbers. Also, we give miscellaneous properties of the generalized Fibonacci-circulant-Hurwitz numbers such as the Binet formula, the combinatorial, permanental, determinantal representations, the generating function, the exponential representation and the sums.

Keywords: Fibonacci-circulant-Hurwitz Sequence, Circulant matrix, Hurwitz matrix, Representation.

2010 Mathematics Subject Classification: 11K31, 11B50, 11C20, 20D60.
1 Introduction

The $k$-step Fibonacci sequence $\{F^k_n\}$ is defined by initial values $F^k_0 = F^k_1 = F^k_{k-2} = 0$, $F^k_{k-1} = 1$ and recurrence relation

$$F^k_{n+k} = F^k_{n+k-1} + F^k_{n+k-2} + \cdots + F^k_n$$ for $n \geq 0$.

For detailed information about the $k$-step Fibonacci sequence, see [9, 21].

In [5], Deveci et al. defined the Fibonacci-circulant sequence of the second kind as shown:

$$x^2_1 = \cdots = x^2_4 = 0, x^2_5 = 1$$ and

$$x^2_n = -x^2_{n-3} + x^2_{n-4} - x^2_{n-5}$$ for $n \geq 6$.

Note that the characteristic polynomial of the Fibonacci-circulant sequence of the second kind is as follows:

$$f(x) = -x^5 + x^2 + x - 1.$$

Let an $n$-th degree real polynomial $f$ be given by

$$f^2(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n.$$

In [8], the Hurwitz matrix $H_n = [h_{i,j}]_{n \times n}$ associated to the polynomial $f$ was defined as shown:

$$H_n = \begin{bmatrix}
c_1 & c_3 & c_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
c_0 & c_2 & c_4 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & c_1 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & c_0 & c_2 & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\
\vdots & 0 & c_1 & \ddots & \ddots & \ddots & c_n & \vdots & \vdots \\
\vdots & \vdots & c_0 & \ddots & \ddots & \ddots & c_{n-1} & 0 & \vdots \\
\vdots & \vdots & 0 & \cdots & \cdots & \cdots & c_{n-2} & c_n & \vdots \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & c_{n-3} & c_{n-1} & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & c_{n-4} & c_{n-2} & c_n
\end{bmatrix}$$

Consider the $k$-step homogeneous linear recurrence sequence $\{a_n\}$,

$$a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [9], Kalman derived a number of closed-form formulas for the sequence $\{a_n\}$ by matrix method as follows:

$$A^n \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{k-1}
\end{bmatrix} = \begin{bmatrix}
a_n \\
a_{n+1} \\
\vdots \\
a_{n+k-1}
\end{bmatrix}$$
where

\[ A = [a_{i,j}]_{k \times k} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & c_{k-2} & c_{k-1}
\end{bmatrix}. \]

Number theoretic properties such as these obtained from Fibonacci numbers relevant to this paper have been studied by many authors [1, 4, 7, 11, 12, 20, 23, 27, 28]. Now we define the generalized Fibonacci-circulant-Hurwitz numbers and then, we obtain their miscellaneous properties using the generating matrix and the generating function of these numbers.

2 Significance

As it is well-known that recurrence sequences, circulant matrix and Hurwitz matrix appear in modern research in many fields from mathematics, physics, computer science, architecture to nature and art (see, for example, [6, 10, 13, 14, 17, 18, 22, 24, 25, 26]). This paper is expanded the concept to the generalized Fibonacci-circulant-Hurwitz sequence which is defined by using circulant and Hurwitz matrices.

3 The main results

By the polynomial \( f^2(x) \), we can write the following Hurwitz matrix:

\[ M^2 = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}. \]

Using the matrix \( M^2 \), we define the Fibonacci-circulant-Hurwitz sequence of the second kind as shown:

\[ a_1^2 = \cdots = a_4^2 = 0, \quad a_5^2 = 1 \quad \text{and} \quad a_{n+1}^2 = -a_n^2 + a_{n-1}^2 + a_{n-2}^2 + a_{n-4}^2 \quad \text{for} \quad n \geq 5. \]

Now we consider a new sequence which is a generalized form of the the Fibonacci-circulant-Hurwitz sequence of the second kind and is called the generalized Fibonacci-circulant-Hurwitz sequence. The sequence is defined by integer constants \( a_1^k = \cdots = a_{k-1}^k = 0, \quad a_k^k = 1 \) and the recurrence relation

\[ a_{n+1}^k = -a_n^k + a_{n-1}^k + \cdots + a_{n-k+3}^k + a_{n-k+1}^k \quad \text{(1)} \]

for \( n \geq k \), where \( k \) is a positive integer such that \( k \geq 4 \).
From (1), we may write the following matrix:

\[
M_k = [m_{i,j}]_{k \times k} = 
\begin{bmatrix}
-1 & 1 & 1 & \cdots & 1 & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}.
\]

(2)

The matrix \(M_k\) is called the generalized Fibonacci-circulant-Hurwitz matrix.

Note that \(\det(M_k) = (-1)^{k+1}\) for \(k \geq 4\).

By induction on \(n\), we get

\[
(M_4)^n = 
\begin{bmatrix}
4^{4+4} & 4^{4+3} + 4^{4+1} & 4^{4+2} & 4^{4+3} \\
4^{4+3} & 4^{4+2} + 4^{4} & 4^{4+1} & 4^{4} \\
4^{4+2} & 4^{4+1} + 4^4 & 4^4 & 4^4 \\
4^{4+1} & 4^4 + 4^4 & 4^4 & 4^4 \\
\end{bmatrix}
\]

\[
(M_5)^n = 
\begin{bmatrix}
5^{5+5} & 5^{5+4} + 5^{5+3} & 5^{5+2} + 5^{5+1} & 5^{5+1} & 5^{5} \\
5^{5+4} & 5^{5+3} + 5^{5+2} & 5^{5+1} & 5^{5} & 5^{5} \\
5^{5+3} & 5^{5+2} + 5^{5+1} & 5^{5+1} & 5^{5} & 5^{5} \\
5^{5+2} & 5^{5+1} + 5^{5} & 5^{5} & 5^{5} & 5^{5} \\
\end{bmatrix}
\]

and

\[
(M_k)^n = 
\begin{bmatrix}
\begin{array}{cccc}
a_{n+k} & a_{n+k+1} + a_{n+k} & a_{n+k-1} + a_{n+k-3} & a_{n+k-2} + a_{n+k-1} \\
a_{n+k-2} & a_{n+k-1} + a_{n+k-3} & a_{n+k-2} + a_{n+k-4} & a_{n+k-3} \\
\vdots & \vdots & \vdots & \vdots \\
a_{n+1} & a_{n+2} + a_{n+1} & a_n + a_{n-2} & a_{n-1} + a_n \\
\end{array}
\end{bmatrix}
\]

(3)

for \(k \geq 6\), where \((M_k)^*\) is a matrix with \(k\) row and \(k-5\) column given below:

\[
\begin{bmatrix}
a_{n+k-1} + \cdots + a_{n+4} + a_{n+2} & a_{n+k-1} + \cdots + a_{n+5} + a_{n+3} & \cdots & a_{n+k-1} + a_{n+k-3} + a_{n+k-2} + a_{n+k-4} \\
a_{n+k-2} + \cdots + a_{n+3} + a_{n+1} & a_{n+k-2} + \cdots + a_{n+4} + a_{n+2} & \cdots & a_{n+k-2} + a_{n+k-3} + a_{n+k-5} \\
\vdots & \vdots & \ddots & \vdots \\
a_n + \cdots + a_{n-k+4} + a_{n-k+2} & a_n + \cdots + a_{n-k+5} + a_{n-k+3} & \cdots & a_n + a_{n-1} + a_{n-3} \\
\end{bmatrix}.
\]

Lemma 3.1. The characteristic equation of all the generalized Fibonacci-circulant-Hurwitz numbers \(x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1 = 0\) does not have multiple roots for \(k \geq 4\).
Theorem 3.1. Let \( f(x) = x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1 \). We easily see that \( f(1) \neq 1 \). Consider \( h(x) = (x - 1) f(x) \). Since \( f(1) \neq 1 \), \( 1 \) is root but not a multiple root of \( h(x) \). Assume that \( u \) is a multiple root of \( h(x) \). Then \( h(u) = 0 \) and \( h'(u) = 0 \). So we get

\[
(1 - k) u^k + ku^3 + (k - 7) u^2 + (4 - 2k) u + 2 (k - 1) = 0.
\]

Using appropriate softwares such as Wolfram Mathematica 10.0 [29], one can see that this last equation does not have a solution which is a contradiction. This contradiction proves that the equation \( f(x) \) does not have multiple roots. \( \square \)

If \( x_1, x_2, \ldots, x_k \) are the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix \( M_k \), then by Lemma 3.1, it is known that \( x_1, x_2, \ldots, x_k \) are distinct. Let a \( k \times k \) Vandermonde matrix \( V^k \) be given by

\[
V^k = \begin{bmatrix}
(x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_k)^{k-1} \\
(x_1)^{k-2} & (x_2)^{k-2} & \cdots & (x_k)^{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & \cdots & x_k \\
1 & 1 & \cdots & 1
\end{bmatrix}.
\]

Now assume that \( W^k(i) \) is a \((p + 2) \times 1\) matrix as shown:

\[
W^k(i) = \begin{bmatrix}
(x_1)^{n+k-i} \\
(x_2)^{n+k-i} \\
\vdots \\
(x_{p+2})^{n+k-i}
\end{bmatrix}
\]

and \( V^k(i,j) \) is a \( k \times k \) matrix derived from the Vandermonde matrix \( V^k \) by replacing the \( j \)-th column of \( V^k \) by matrix \( W^k(i) \).

Now we give the Binet formulas for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorem.

**Theorem 3.1.** Let \( k \) be a positive integer such that \( k \geq 4 \) and let \( (M_k)^{\alpha} = \begin{bmatrix} m_{i,j}^{(\alpha)} \end{bmatrix} \) for \( \alpha \geq 1 \), then

\[
m_{i,j}^{(\alpha)} = \frac{\det V^k(i,j)}{V^k}.
\]

**Proof.** Since the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix \( M_k \) are distinct, \( M_k \) is diagonalizable. Then, we may write \( M_k V^k = V^k D_k \), where \( D_k = \text{diag}(x_1, x_2, \ldots, x_k) \). Since \( \det V^k \neq 0 \), we get

\[
(V^k)^{-1} M_k V^k = D_k.
\]

It will thus be seen that the matrices \( M_k \) and \( D_k \) are similar. Then we can write the matrix equation \( (M_k)^{\alpha} V^k = V^k (D_k)^{\alpha} \) for \( \alpha \geq 1 \). Since \( (M_k)^{\alpha} = \begin{bmatrix} m_{i,j}^{(\alpha)} \end{bmatrix} \), we get

\[
\begin{cases}
m_{i,1}^{(\alpha)} (x_1)^{k-1} + m_{i,2}^{(\alpha)} (x_1)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_1)^{\alpha+k-i} \\
m_{i,1}^{(\alpha)} (x_2)^{k-1} + m_{i,2}^{(\alpha)} (x_2)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_2)^{\alpha+k-i} \\
\vdots \\
m_{i,1}^{(\alpha)} (x_k)^{k-1} + m_{i,2}^{(\alpha)} (x_k)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_k)^{\alpha+k-i}
\end{cases}
\]

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So we conclude that

\[ m_{i,j}^{(\alpha)} = \frac{\det V^k(i,j)}{V^k} \]

for each \( i, j = 1, 2, \ldots, k \). \( \square \)

Thus by Theorem 3.1 and the matrix \((M_k)^n\), we have the following useful results.

**Corollary 3.1.** Let \( a_k^n \) be the \( n \)-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then

\[ a_k^n = \frac{\det V^k(k,k)}{V^k} = \frac{\det V^k(k-1,k-1)}{V^k} \]

for \( k \geq 4 \).

Now we consider the combinatorial representations for all the generalized Fibonacci-circulant-Hurwitz numbers.

Let a \( k \times k \) companion matrix \( C(c_1, c_2, \ldots, c_k) \) be given by

\[
C(c_1, c_2, \ldots, c_k) = \begin{bmatrix}
    c_1 & c_2 & \cdots & c_k \\
    1 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 1 & 0
\end{bmatrix}.
\]

For more details on the companion type matrices, see [15, 16].

**Theorem 3.2** (Chen and Louck [3]). The \((i,j)\) entry \( c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k) \) in the matrix \( C^{\alpha}(c_1, c_2, \ldots, c_k) \) is given by the following formula:

\[
c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} c_1^{t_1} \cdots c_k^{t_k} \tag{4}
\]

where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + kt_k = \alpha - i + j \), \( \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} \) is a multinomial coefficient, and the coefficients in (4) are defined to be 1 if \( \alpha = i - j \).

**Corollary 3.2.** Let \( k \) be a positive integer such that \( k \geq 4 \) and let \( a_k^n \) be the \( n \)-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then

\[
a_k^n = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} = \sum_{(t_1, t_2, \ldots, t_{p+2})} \frac{t_{k-1} + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k}
\]

where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + kt_k = n \).

**Proof.** In Theorem 3.2, if we choose \( i = j = k \) and \( i = j = k - 1 \), then the proof is immediately seen from (3). \( \square \)
Definition 3.1. An \( u \times v \) real matrix \( A = [a_{i,j}] \) is called a contractible matrix in the \( n \)-th column (resp. row) if the \( n \)-th column (resp. row) contains exactly two non-zero entries.

Let \( x_1, x_2, \ldots, x_u \) be row vectors of the matrix \( A \). If \( A \) is contractible in the \( n \)-th column such that \( a_{\tau,n} \neq 0, a_{\sigma,n} \neq 0 \) and \( \tau \neq \sigma \), then the \( (u-1) \times (v-1) \) matrix \( A_{\tau,\sigma,n} \) obtained from \( A \) by replacing the \( \tau \)-th row with \( a_{\tau,n} x_\sigma + a_{\sigma,n} x_\tau \) and deleting the \( \sigma \)-th row. We call the \( n \)-th column the contraction in the \( n \)-th column relative to the \( \tau \)-th row and the \( \sigma \)-th row.

In [2], it was shown that \( \text{per}(A) = \text{per}(B) \) if \( A \) is a real matrix of order \( u > 1 \) and the matrix \( B \) is a contraction of \( A \).

Let \( u \geq k \) and let a \( u \times u \) super-diagonal matrix \( N^k_u = [n^k_{i,j}] \) be given by

\[
n^k_{i,j} = \begin{cases} 
1 & \text{if } i = s \text{ and } j = s + 1 \text{ for } 1 \leq s \leq u - 1, \\
1 & i = s \text{ and } j = s + 2 \text{ for } 1 \leq s \leq u - 2, \\
\vdots & \vdots \\
1 & i = s \text{ and } j = s + k - 3 \text{ for } 1 \leq s \leq u - k + 3, \\
i = s \text{ and } j = s + k - 1 \text{ for } 1 \leq s \leq u - k + 1 & \text{and} \\
\vdots & \vdots \\
i = s + 1 \text{ and } j = s \text{ for } 1 \leq s \leq u - 1, \\
0 & \text{otherwise}, \\
-1 & \text{if } i = s \text{ and } j = s \text{ for } 1 \leq s \leq u,
\end{cases}
\]

where \( k \geq 4 \).

Now we give the permanental representations for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorems.

**Theorem 3.3.** Let \( a_n \) be the \( n \)-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then

\[
\text{per}(N^k_u) = a^k_{u+k}
\]

for \( u \geq k \).

**Proof.** The assertion may be proved by induction on \( u \). Assume that the result hold for any integer greater than or equal to \( k \). Then we show the equation holds for \( u + 1 \). Expanding the \( \text{per}(N^k_u) \) by the Laplace expansion of permanent according to the first row gives us

\[
\text{per}(N^k_{u+1}) = -\text{per}(N^k_u) + \text{per}(N^k_{u-1}) + \cdots + \text{per}(N^k_{u-k+3}) + \text{per}(N^k_{u-k+1}).
\]

Since

\[
\text{per}(N^k_u) = a^k_{u+k}, \quad \text{per}(N^k_{u-1}) = a^k_{u+k-1}, \ldots, \quad \text{per}(N^k_{u-k+3}) = a^k_{u+3}, \quad \text{per}(N^k_{u-k+1}) = a^k_{u+1},
\]

by using the recurrence relation of the generalized Fibonacci circumulant-Hurwitz numbers, we obtain \( \text{per}(N^k_{u+1}) = a^k_{u+k+1}. \)
Suppose that $u > k$ and the $u \times u$ matrices $H_u^k = [h_{i,j}^k]$ and $T_u^k = [t_{i,j}^k]$ are defined by

$$h_{i,j}^k = \begin{cases} 1 & \text{if } i = s \text{ and } j = s + \rho \text{ for } 1 \leq s \leq u - k + 2, \\ 1 & \text{if } i = s \text{ and } j = s + k - 1 \text{ for } 1 \leq s \leq u - k + 1, \\ -1 & \text{if } i = s + 1 \text{ and } j = s \text{ for } 1 \leq s \leq u - l, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_u^k = \begin{bmatrix} (u - k) \cdot \text{th} \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & H_{u-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$k \geq 4$.

Using the matrices $H_u^k = [h_{i,j}^k]$ and $T_u^k = [t_{i,j}^k]$ and the above results we can give more general permanental representations.

**Theorem 3.4.** For $u > k$,

$$\per(H_u^k) = a_u^k,$$

and

$$\per(T_u^k) = \sum_{\tau=0}^{u-1} a_\tau^k.$$

**Proof.** Consider the first part of the theorem. We prove this by the induction method. Suppose that the equation holds for $u > k$, then we show that the equation holds for $u + 1$. If we expand the $\per(H_u^k)$ by the Laplace expansion of permanent according to the first row, then we get

$$\per(H_{u+1}^k) = -\per(H_u^k) + \per(H_{u-1}^k) + \cdots + \per(H_{u-k+3}^k) + \per(H_{u-k+1}^k)$$

$$= -a_u^k + a_{u-1}^k + \cdots + a_{u-k+3}^k + a_{u-k+1}^k$$

$$= a_{u+1}^k.$$

Prove the second part of the theorem: Expanding the $\per(T_u^k)$ with respect to the first row, we can write

$$\per(T_u^k) = \per(T_{u-1}^k) + \per(H_{u-1}^p).$$

Thus, by the results and an inductive argument, the proof is easily seen. \qed
Using the definition of the generalized Fibonacci-circulant-Hurwitz numbers we find the generating function $g(x)$ as shown

$$g(x) = \frac{x^k}{1 + x - x^2 - \ldots - x^{k-2} - x^k}$$

where $k \geq 4$.

Now we investigate an exponential representation for the generalized Fibonacci-circulant-Hurwitz numbers.

**Theorem 3.5.** For $k \geq 4$, the generalized Fibonacci-circulant-Hurwitz numbers have the following exponential representation:

$$g(x) = x^k \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} (-1 + x + \ldots + x^{k-3} + x^{k-1})^n \right).$$

**Proof.** We consider the generating function $g(x) = \frac{x^k}{1 + x - x^2 - \ldots - x^{k-2} - x^k}$. Since

$$\ln g(x) = \ln \left( \frac{x^k}{1 + x - x^2 - \ldots - x^{k-2} - x^k} \right),$$

$$\ln g(x) = \ln x^k - \ln (1 + x - x^2 - \ldots - x^{k-2} - x^k)$$

and

$$\ln (1 + x - x^2 - \ldots - x^{k-2} - x^k) = -[x (-1 + x + x^2 + \ldots + x^{k-3} + x^{k-1}) + \frac{1}{2} x^2 (-1 + x + x^2 + \ldots + x^{k-3} + x^{k-1})^2 + \ldots + \frac{1}{i} x^i (-1 + x + x^2 + \ldots + x^{k-3} + x^{k-1})^i + \ldots],$$

it is clear that

$$\ln \frac{g(x)}{x^k} = \sum_{n=1}^{\infty} \frac{x^n}{n} (-1 + x + \ldots + x^{k-3} + x^{k-1})^n. \quad \square$$

Now we consider the sums of all the generalized Fibonacci-circulant-Hurwitz numbers. Let the $k \times k$ matrix $M_k$ be as in (2) and let the sums of the generalized Fibonacci-circulant-Hurwitz numbers from 1 to $n$, $(n > 1)$ be denoted by $S_n$, that is,

$$S_n = \sum_{i=1}^{n} a_i^k.$$ 

If we define the $(k+1) \times (k+1)$ matrix $Z_k$ as in the following form:

$$Z_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \vdots \\ \vdots & & & M_k \\ 0 & & & \end{bmatrix},$$

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then by using induction on $n$, we may write

$$ (Z_k)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+k-1} & S_{n+k-1} & \cdots & (M_k)^n \\ \vdots \\ S_n \end{bmatrix} $$

Acknowledgements

This Project was supported by the Commission for the Scientific Research Projects of Kafkas University, Project number 2017-FM-65.

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