Review of Born–Infeld electrodynamics

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Born–Infeld electrodynamics is motivated by the infinite self-energy of the point charge in Maxwell electrodynamics. In BI electrodynamics, an upper bound $b$ is imposed on the electric field, thus limiting the self-energy of the point charge. This is a review paper in which we motivate the BI Lagrangian and from it derive the field equations. We find the stress–energy tensor in BI. We calculate the potential due to the point charge in BI. We find order $b^{-2}$ wave solutions to BI in $1 + 1$ dimensions. We examine BI plane waves normally incident on a mirror.

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1 Motivation

In [1], Born and Infeld describe their theory. In Maxwell electrodynamics, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (E^2 - B^2). \quad (1)$$

The field due to a point charge is given by Coulomb’s law which implies $E \to \infty$ at the position of the particle. This leads to an infinite Lagrangian and self-energy for the point particle. In order to eliminate this infinity the following replacement is proposed.

$$\mathcal{L} = b^2 \left( 1 - \sqrt{1 - \frac{F_{\mu\nu} F^{\mu\nu}}{2b^2}} \right)$$

$$= b^2 \left( 1 - \sqrt{1 - \frac{E^2 - B^2}{b^2}} \right) \quad (2)$$
where $b$ is some large constant. This is analogous to, in special relativity, replacing $mv^2/2$ with $mv^2(1 - \sqrt{1 - u^2/c^2})$. The immediate effect is to set an upper bound $b$ on $E$ just as $c$ is an upper bound on velocity. (1) is recovered from (2) by Taylor expansion for $b^2 \gg F_{\mu \nu} F^{\mu \nu}$. (2) is only an ansatz, the Born–Infeld Lagrangian (in Minkowski space) is

$$\mathcal{L} = b^2 \left( 1 - \sqrt{-\det \left( \eta_{\mu \nu} + \frac{1}{b} F_{\mu \nu} \right)} \right) = b^2 \left( 1 - \sqrt{1 - \frac{E^2 - B^2}{b^2} - \frac{(E \cdot B)^2}{b^4}} \right).$$

(3)

(2) and (3) are equivalent for the static case. (3) is the simplest linear combination of $[- \det (\eta_{\mu \nu} + F_{\mu \nu}/b)]^{1/2}$, $[- \det (\eta_{\mu \nu})]^{1/2}$, and $[- \det (F_{\mu \nu}/b)]^{1/2}$ (these expressions are all generally covariant) from which (1) is recovered upon Taylor expansion for $F_{\mu \nu} \ll b$. This is discussed further in section 2.

The infinite self-energy caused problems in the formulation of a quantum theory of electrodynamics. Born–Infeld is a 1930s attempt to unify quantum mechanics and field electrodynamics [2]. The theory does not achieve this goal (it is not the basis of quantum electrodynamics) but is interesting nonetheless. In the 1970s, it was discovered that Born-Infeld is (except for a physically uninteresting case) the only nonlinear theory which does not predict vacuum birefringence and has correct shock wave characteristics [3, 4]. In the 1980s, string theorists found that the Born–Infeld action is also the action for an open superstring when the derivatives of the spacetime fields can be neglected; that is, for slowly varying fields [5, 6].

In this paper, we review basic aspects of Born–Infeld electrodynamics, including the field equations [1], the stress–energy tensor [1], the electrostatic field due to a point charge [1], the wave equation in 1 + 1 dimensions [7], standing wave solutions [7], free wave solutions [8], and the particular standing waves which arise from plane waves normally incident on a mirror [8].

## 2 Lagrangian and equations of motion

The principle underlying the Lagrangian postulated is that the action integral must be generally invariant. [1] has the goal of finding the conditions that satisfy this postulate and under which [1] is recovered for small fields. The Lagrangian must have the form

$$\mathcal{L} = \sqrt{\det (a_{\mu \nu})}$$

(4)

where $a_{\mu \nu}$ is some covariant tensor. $a_{\mu \nu}$ can be split into an anti-symmetric tensor—we can use the electromagnetic tensor $F_{\mu \nu}$—and symmetric tensor—we can use the metrical tensor $g_{\mu \nu}$—such that $a_{\mu \nu} = g_{\mu \nu} + F_{\mu \nu}$. As the tensor is second order, $\mathcal{L}$ can be any homogeneous function of the determinants of the tensors to the power of $1/2$. Therefore, following [1] we
consider the Lagrangian
\[ \mathcal{L} = \sqrt{-\det (g_{\mu\nu} + F_{\mu\nu})} + A \sqrt{-\det (g_{\mu\nu})} + B \sqrt{-\det (F_{\mu\nu})} \] (5)
we have set \( b = 1 \) to simplify the notation, \( A \) and \( B \) are some constants, and the negative
signs have been added to keep \( \mathcal{L} \) real. The last term is a total derivative that drops out of
the action \( S = \int d^4x \mathcal{L} \) since
\[ \sqrt{\det (F_{\mu\nu})} = \frac{1}{4} \sqrt{-g} F_{\mu\nu} = \frac{1}{4} \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}) , \] (6)
where \( g = \det g_{\mu\nu}, \tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \), and \( \epsilon^{\mu\nu\rho\sigma} \) is the Levi-Civita symbol with \( \epsilon^{0123} = 1 \).
Thus one can set \( B = 0 \). Furthermore, requiring that the Lagrangian gives the the classical
Maxwell Lagrangian, for small values of \( F_{\mu\nu} \), we must set \( A = -1 \). This gives
\[ \mathcal{L} = \sqrt{-\det (g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-g} \]
\[ = -g \left( \sqrt{1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - g^{-1} \det (F_{\mu\nu})} - 1 \right) \] (7)
where \( F_{\mu\nu} F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \). This can be derived by writing
\[ \det (g_{\mu\nu} + F_{\mu\nu}) = \det \left[ g_{\mu\rho} (\delta_\nu^\rho + F_\nu^\rho) \right] = g \det (\delta_\nu^\rho + F_\nu^\rho) = g \det (\delta_\nu^\rho + g^{\rho\sigma} F_{\mu\nu}) \] (8)
and using the general formula valid for antisymmetric 4 \( \times \) 4 matrix \( X \) given by
\[ \det(1 + X) = 1 - \frac{1}{2} \text{tr} X^2 + \det X . \] (9)
This, in turn, can be derived from the general formula \( \det Y = \exp \left[ \text{tr} (\log Y) \right] \) for a general
matrix \( Y \), easily seen by diagonalizing the matrix \( Y \).

To find the field equations, we minimize the action by varying the potential as follows.
\[ \delta S = \int d^4x \delta_A \left[ \sqrt{-g} \left( \sqrt{1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - g^{-1} \det (F_{\mu\nu})} - 1 \right) \right] \]
\[ = \int d^4x \left[ \frac{\sqrt{-g}}{2\sqrt{\Pi}} \left( \frac{\partial (\frac{1}{2} F_{\mu\nu} F^{\mu\nu})}{\partial F_{\mu\nu}} \delta F_{\mu\nu} - \frac{\partial (g^{-1} \det (F_{\mu\nu}))}{\partial F_{\mu\nu}} \delta F_{\mu\nu} \right) \right] \]
\[ = \int d^4x \left[ \frac{\sqrt{-g}}{\sqrt{\Pi}} \left( \frac{1}{2} F^{\mu\nu} - 2 \left( \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \right) \right] \delta A_\nu \] (10)
where \( \Pi = 1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - g^{-1} \det (F_{\mu\nu}) \). Integrating by parts and dropping the surface term
we find
\[ \delta S = \int d^4x \partial_\mu \left[ \frac{\sqrt{-g}}{\sqrt{\Pi}} \left( \frac{1}{2} F^{\mu\nu} - \frac{1}{2} \left( \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) \right) \right] \delta A_\nu , \] (11)
\[ ^1 \text{Note that } \det (F_{\mu\nu}) = (B \cdot E)^2 \text{ and } \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = (B^2 - E^2). \]
Thus, the variational principle, $\delta S = 0$ gives the field equation

$$\nabla_\mu \left[ \frac{1}{\sqrt{\Pi}} \left( F^{\mu\nu} - \left( \frac{1}{4} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \tilde{F}^{\mu\nu} \right) \right] = 0 \quad (12)$$

where the covariant derivative $\nabla_\mu$ is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho,$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho,$$

where $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^\rho_{\sigma} \left[ \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right]$. Thus, we have

$$\nabla_\mu V^{\nu\rho} = \partial_\mu V^{\nu\rho} - \Gamma_{\nu\sigma}^{\nu\rho} V^\sigma,$$

Reintroducing the parameter $b$, the Born–Infeld action takes the form

$$\mathcal{L} = \sqrt{-g} \left( \sqrt{1 + \frac{1}{b^2} \left( \frac{1}{2} F^{\mu\nu} - \frac{1}{4} b^2 \frac{1}{2} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \tilde{F}^{\mu\nu} \right) b^2 \quad (15)$$

and the field equations are given by

$$\nabla_\mu \left[ \frac{1}{\sqrt{\Pi}} \left( F^{\mu\nu} - \frac{1}{4} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \tilde{F}^{\mu\nu} \right] = 0, \quad (16)$$

where

$$\Pi = 1 + \frac{1}{b^2} \left( \frac{1}{2} F^{\mu\nu} - \frac{1}{4} b^2 \frac{1}{2} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \tilde{F}^{\mu\nu} - \frac{1}{g b^4} \det (F_{\mu\nu}) - 1.$$  

The electric displacement field given by $D = b^2 \frac{\partial \mathcal{L}}{\partial E}$ and $H$-field $H = b^2 \frac{\partial \mathcal{L}}{\partial B}$ are

$$D = \frac{1}{\sqrt{\Pi}} \left( E - \left( \frac{1}{4} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \frac{b^2}{2} B \right) \quad \text{and} \quad H = \frac{1}{\sqrt{\Pi}} \left( B - \left( \frac{1}{4} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right) \frac{b^2}{2} E \right)$$

or

$$D = \varepsilon E - \nu B \quad \text{and} \quad H = \frac{1}{\mu} B - \nu E$$

with $\varepsilon = \frac{1}{\sqrt{\Pi}}$, $\mu = \sqrt{\Pi}$, $\nu = \frac{\left( \frac{1}{4} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \right)}{b^2 \sqrt{\Pi}}.$

These yield a nonlinear version of the Maxwell equations. From the equations above and the Bianchi identity [1]

$$\nabla \cdot D = 0, \quad \frac{\partial D}{\partial t} = \nabla \times H,$$

$$\nabla \cdot B = 0, \quad -\frac{\partial B}{\partial t} = \nabla \times E.$$  

2 Lagrangian and equations of motion
3 Stress–energy tensor

In order to find the stress–energy tensor, we must vary the action $S$ with respect to the metric $g_{\mu\nu}$. This will give us the following relation.

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$$  \hspace{1cm} (20)

where $g^{\mu\nu}$ is the inverse metric.

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\rho$$

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}$$  \hspace{1cm} (21)

Note that $g^{\mu\nu} \delta g_{\mu\nu} = - (\delta g^{\mu\nu}) g_{\mu\nu}$. Starting from (10),

$$\delta S = \int d^4x \delta g \left[ \sqrt{-g} \left( \sqrt{1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - g^{-1} \det (F_{\mu\nu}) - 1} \right) \right]$$

$$= \int d^4x \left[ (\delta g \sqrt{-g}) L + \sqrt{-g} \delta g L \right]$$  \hspace{1cm} (22)

where $L = \sqrt{1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - g^{-1} \det (F_{\mu\nu}) - 1}$. Variation of $\sqrt{-g}$ is given by (21). As for the variation of $L$, first note

$$\frac{\delta}{\delta g^{\mu\nu}} \left( \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) = \frac{\delta}{\delta g^{\mu\nu}} \left( \frac{1}{4} F_{\mu\nu} \frac{1}{2 \sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_\rho F_\sigma \right) = \frac{1}{2} \left( \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) g_{\mu\nu}$$  \hspace{1cm} (23)

$$\frac{\delta}{\delta g^{\mu\nu}} \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right) = \frac{\delta}{\delta g^{\mu\nu}} \left( \frac{1}{2} F_{\mu\nu} g^{\mu\nu} F_{\nu\alpha} g^{\rho\sigma} \right) = 2 \left( \frac{1}{2} F_{\mu\nu} g^{\rho\sigma} F_{\nu\alpha} \right)$$  \hspace{1cm} (24)

as, by definition, $F^{\mu\nu}$ is independent of the metric. From here, it is easy to see

$$\delta g L = \frac{1}{2 \sqrt{\Pi}} \left( F_{\mu\rho} g^{\rho\sigma} F_{\nu\sigma} - \left( \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right)^2 g_{\mu\nu} \right) \delta g^{\mu\nu}.$$  \hspace{1cm} (25)

So (22) can be rewritten

$$\delta S = \int d^4x \left[ \left( -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right) L ight.$$  

$$+ \sqrt{-g} \frac{\delta g^{\mu\nu}}{2 \sqrt{\Pi}} \left( F_{\mu\rho} g^{\rho\sigma} F_{\nu\sigma} - \left( \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right)^2 g_{\mu\nu} \right) \right].$$  \hspace{1cm} (26)

From this, (20), and the necessity that the functional derivative must coincide with $-\sqrt{-g}/2T_{\mu\nu}$ we obtain

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g} \delta g^{\mu\nu}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$= g_{\mu\nu} L - \frac{1}{\sqrt{\Pi}} \left( F_{\mu\rho} F_{\nu}^\rho - \left( \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right)^2 g_{\mu\nu} \right).$$  \hspace{1cm} (27)
4 Point charge

The electric field due to a point charge in Born–Infeld electrodynamics was derived in [1]. Following [1] we start by considering a stationary point charge \( q \) at the origin. The problem is static (\( \mathbf{B} = \mathbf{H} = 0 \) and nothing depends on \( t \)) so \((19)\) becomes

\[
\nabla \times \mathbf{E} = 0, \\
\nabla \cdot \mathbf{D} = 0.
\]

(28)

And \((18)\) is

\[
\mathbf{D} = \frac{\mathbf{E}}{\sqrt{1 - \frac{E^2}{b^2}}}. \tag{29}
\]

Also, since \( \mathbf{E} \) is curl free, \( \mathbf{E} = -\nabla \Phi \) where \( \Phi \) is some scalar potential.

The problem is spherically symmetric and static so the fields must depend only on \( r \) and be only in the \( \mathbf{\hat{r}} \) direction. So \((28)\) is nothing but

\[
\frac{d}{dr} r^2 D_r = 0 \quad \Rightarrow \quad D_r = \frac{C}{r^2} \tag{30}
\]

where \( C \) is some constant which we can determine by applying Gauss’s law to a sphere centered at the origin.

\[
\int \mathbf{D} \cdot d\mathbf{a} = q \\
4\pi C = q. \tag{31}
\]

So \( D_r = (q)/(4\pi r^2) \). Now from this result, \((29)\), and \( E_r = -\partial \Phi / \partial r = -d\Phi / dr \) we have the following equation.

\[
\frac{q}{4\pi r^2} = -\frac{d\Phi / dr}{\sqrt{1 - \frac{1}{b^2} (d\Phi / dr)^2}}. \tag{32}
\]

This is separable.

\[
(d\Phi)^2 = \left( \frac{q}{4\pi r_0^2} \right)^2 \left( \frac{dr}{1 + (r/r_0)^4} \right)^2 \\
d\Phi = -\frac{q}{4\pi r_0^2} \frac{dr}{\sqrt{1 + (r/r_0)^4}} \tag{33}
\]

where the substitution \( r_0^2 := |q|/(4\pi b) \) has been made and the overall sign is determined by the physical condition that the signs of \( d\Phi / dr \) and \( q \) differ. We take the reference point to be
Let $r \to \infty$ and integrate (33) from the reference point to $r$, which, along with the substitution $u := r/r_0$, yields

$$
\Phi = \frac{q}{4\pi r_0} f\left(\frac{r}{r_0}\right) \quad \text{where} \quad f(x) := \int_x^\infty \frac{du}{\sqrt{1 + u^4}}.
$$

(34)

This is the potential of the point charge in Born–Infeld. For $x \gg 1$ (so for large $b$ or large $r$), Coulomb’s law is recovered.

Making the substitution $u = \tan \beta/2$ leads to [1]

$$
f(x) = \frac{1}{2} \int_\beta^\pi \frac{d\beta}{\sqrt{1 - \frac{1}{2} \sin^2 \beta}} = \frac{1}{2} \int_0^\pi \frac{d\beta}{\sqrt{1 - \frac{1}{2} \sin^2 \beta}} - \frac{1}{2} \int_0^\beta \frac{d\tilde{\beta}}{\sqrt{1 - \frac{1}{2} \sin^2 \tilde{\beta}}}
$$

$$
= f(0) - \frac{1}{2} F\left(\beta', \frac{1}{\sqrt{2}}\right)
$$

(35)

where $\beta' = 2 \arctan x$ and $F(\beta', 1/\sqrt{2})$ is the elliptic integral of the first kind for coefficient $1/2$ and upper bound $\beta'$.

To find $f(0)$ we evaluate the elliptic integral

$$
f(0) = \frac{1}{2} \int_0^\pi \frac{d\beta}{\sqrt{1 - \frac{1}{2} \sin^2 \beta}} = \frac{1}{2} F\left(\pi, \frac{1}{\sqrt{2}}\right) = 1.85.
$$

(36)

This is the maximum value of $f$ and so the maximum value of $\Phi$ is $(1.85 \times q)/(4\pi r_0)$ at $r = 0$.

The Born-Infeld and Maxwell (Coulomb) potentials for a point charge are compared in figure [1].

From (33)

$$
E = -\hat{\mathbf{r}} \frac{d\Phi}{dr} = \frac{q}{4\pi r_0^2} \frac{\hat{\mathbf{r}}}{\sqrt{1 + (r/r_0)^4}}.
$$

(37)

So in Born–Infeld electrodynamics, the potential and the electric field of the point charge are finite everywhere and have a discontinuity at the position of the point charge.

5 Wave equation in 1 + 1 dimensions and its solutions

In this section, following [7], we will review the derivation of a 1 + 1 dimensional wave equation for the electromagnetic field in Born–Infeld electrodynamics. We will use the Poincaré-Lindstedt method to find a solution for a standing wave in 1 + 1 dimensions in between two infinite plates. We will also look at a free wave solution in 1 + 1 dimensions to $\mathcal{O}(b^{-4})$. 

5 Wave equation in 1 + 1 dimensions and its solutions
Figure 1 The potentials of the point charge in Born–Infeld (solid) and Maxwell electrodynamics (dashed).
5.1 Wave equation

Let us consider an electromagnetic wave propagating in the \( x \) direction and polarized in the \( y \) direction. To this end, and choosing the gauge \( A_z = 0 \), we consider following ansatz

\[
A_y (r, t) = u(x, t), \quad \Phi = A_x = A_z = 0
\]

which gives

\[
F_{02} = E_y = -\partial_t A_y = -u_t \quad \text{(39)}
\]

and

\[
F_{12} = -B_z = -\partial_x A_y = -u_x \quad \text{(40)}
\]

where we adopt the convention \( u_x := \partial u / \partial x \). A defense of why this is a good choice of gauge can be found in [7]. We can put (39) and (40) into (12) with the condition of flat spacetime. It can be seen that, in flat spacetime, \( \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} = 0 \) which simplifies (12) to

\[
\partial_u \left( \frac{1}{\sqrt{\Pi}} F^{\mu \nu} \right) = 0 \quad \text{(41)}
\]

\[
\partial_t \left( \frac{1}{\sqrt{\Pi}} F^{02} \right) + \partial_x \left( \frac{1}{\sqrt{\Pi}} F^{12} \right) = 0. \quad \text{(42)}
\]

It is useful to note

\[
\Pi = 1 + b^{-2} \left( \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \right) = 1 + b^{-2} (u_x^2 - u_t^2). \quad \text{(43)}
\]

The equations of motion simplify to (by a hearty application of the chain rule)

\[
\left( 1 - \frac{1}{b^2} u_t^2 \right) u_{xx} - \left( 1 + \frac{1}{b^2} u_x^2 \right) u_{tt} + \frac{2}{b^2} u_x u_t u_{xt} = 0. \quad \text{(44)}
\]

Apart from \( 1 + 1 \), it is easy to convert (44) to 2 dimensional form for \( B = 0 \) with the substitution \( t \to iy \) to obtain the equation [7]

\[
\left( 1 - \frac{1}{b^2} u_y^2 \right) u_{xx} - \left( 1 - \frac{1}{b^2} u_x^2 \right) u_{yy} + \frac{2}{b^2} u_x u_y u_{yx} = 0. \quad \text{(45)}
\]

(44) is the equation of motion for an electromagnetic wave. In the next section, we will review a solution that describes standing wave between two parallel conducting plates. We will also review an approximate wave solution.

5.2 Standing wave solution

In this section, following [7], we describe standing wave solution between two infinite parallel conducting plates. Let us take these plates to be in the \( y-z \) plane at \( x = 0 \) and \( x = L \) at
which the electric field must vanish. The other condition (which ensures the solutions are oscillatory) is

\[ 1 + b^{-2} (u_x^2 - u_t^2) > 0. \] (46)

We will use the iterative Poincaré-Lindstedt method to solve (44) up to higher and higher orders of \( b^{-1} \). (44) can be rewritten

\[ u_{xx} - u_{tt} - b^{-2} (u_x^2 u_{xx} + u_t^2 u_{tt} - 2u_x u_t u_{xt}) = 0. \] (47)

We neglect \( O(b^{-2}) \) terms and get \( u_{xx} - u_{tt} = 0 \) which has the satisfactory solution

\[ u^{(0)} = A \sin kx \cos \omega t = \frac{A}{2} [\sin (kx + \omega t) + \sin (kx - \omega t)]. \] (48)

We define

\[ s_{nm} := \sin(nkx + m\omega t) + \sin(nkx - m\omega t) \] (49)

so that \( u^{(0)} = \frac{A}{2} s_{11} \). We evaluate the LHS of (47) with \( u = u^{(0)} \) and get

\[ \frac{A}{2} (\omega^2 - k^2) s_{11} - \frac{A^3 \omega^2 k^2}{8b^2} (s_{13} - s_{31} - 2s_{11}). \] (50)

We want this expression to be 0. We'll look at each coefficient on each \( s_{nm} \). First, for the coefficient on \( s_{11} \) to be 0, we can impose the following condition on the frequency.

\[ \omega^2 - k^2 + \frac{\omega^2 \epsilon^2}{2} = 0 \quad \text{where} \quad \epsilon := \frac{Ak}{b}. \] (51)

We solve this for the frequency and Taylor expand in \( \epsilon^2 \) to get

\[ \frac{\omega^2}{k^2} = 1 - \frac{\epsilon^2}{2} + O(\epsilon^4). \] (52)

Now what about the coefficients on \( s_{13} \) and \( s_{31} \)? Since

\[ (\partial_x^2 - \partial_t^2) s_{nm} = (m^2 \omega^2 - n^2 k^2) s_{nm}, \] (53)

we can eliminate those coefficients by adding

\[ u^{(1)} := \frac{A \epsilon^2}{8} \left( \frac{s_{13}}{9\omega^2 - k^2} - \frac{s_{31}}{\omega^2 - 9k^2} \right) \] (54)

to \( u^{(0)} \). Since (52), we can Taylor expand (54) in \( \epsilon \) to get

\[ u^{(1)} = \frac{A \epsilon^2}{64} (s_{13} + s_{31}) + O(\epsilon^4). \] (55)

To go to the next order we again evaluate the LHS of (47) but with \( u = u^{(0)} + u^{(1)} \), and to \( O(\epsilon^4) \). We write the coefficient on \( s_{11} \), say it must be 0, solve for \( \omega^2 / k^2 \), and Taylor expand to \( O(\epsilon^4) \). We look at all the \( s_{n \neq m} \) terms and find \( u^{(3)} \).

This can be repeated for higher orders of \( \epsilon \). It is not known whether the series converge.
5.3 Free wave solution

Next, following [8], we describe free wave solution of (44). It is convenient to work in the complex plane such that

\[ dx^2 - dt^2 = |dz|^2 = dz \, d\bar{z} \]  

(56)

and the metric tensor will have the components

\[ g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}. \]  

(57)

It is then possible to replace \( u_x^2 + u_y^2 = g^{\mu\nu} u_\mu u_\nu = 4u_z u_{\bar{z}} \) which leads to a reformulation of (44) as

\[ u_{zz} + \frac{1}{b^2} u_z^2 u_{zz} - \frac{2}{b^2} u_z u_{\bar{z}} u_{zz} + \frac{1}{b^2} u_{\bar{z}}^2 u_{z\bar{z}} = 0. \]  

(58)

We can use the equipotential spacetime lines \( u(x, t) = \text{constant} \) as a non-Cartesian basis and change coordinates \((z, \bar{z}) \rightarrow (w, \bar{w})\) where \( w = u + iv \) and \((u, v) \in \mathbb{R}^2\). The Jacobian matrix is

\[
\begin{pmatrix}
  u_z & u_{\bar{z}} \\
  v_z & v_{\bar{z}}
\end{pmatrix} = 
\begin{pmatrix}
  \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\
  \frac{\partial \bar{z}}{\partial u} & \frac{\partial \bar{z}}{\partial v}
\end{pmatrix} = 
\begin{pmatrix}
  (z_w + z_{\bar{w}}) & i(z_w + z_{\bar{w}}) \\
  (\bar{z}_w + \bar{z}_{\bar{w}}) & i(\bar{z}_w + \bar{z}_{\bar{w}})
\end{pmatrix}^{-1}.
\]  

(59)

At this point, it is worth revisiting the action. From (7) we have

\[ S[u] = \frac{b^2}{8\pi i} \int dz \, d\bar{z} \sqrt{1 - 4b^{-2} u_z u_{\bar{z}}}. \]  

(60)

Note that (58) can be obtained from (60). It is easier to solve (58) for \( z(w, \bar{w}) \) and not \( u(z, \bar{z}) \) as it is possible to choose a convenient coordinate \( v \). Rewriting the new action in terms of \( z(w, \bar{w}) \), we have

\[ dz \, d\bar{z} = (z_w z_{\bar{w}} - z_{\bar{w}} z_w) dw \, d\bar{w} \]  

(61)

and we can solve for \( u_z \) to get

\[ u_z = \frac{1}{2} \frac{\bar{z}_w - \bar{z}_{\bar{w}}}{z_{\bar{w}} z_w - \bar{z}_w \bar{z}_{\bar{w}}}. \]  

(62)

So (60) can be rewritten

\[
dz \, d\bar{z} \sqrt{1 - 4b^{-2} u_z u_{\bar{z}}} \\
= dw \, d\bar{w} \sqrt{(z_w \bar{z}_w - z_{\bar{w}} \bar{z}_{\bar{w}})^2 - b^{-2} (z_w - \bar{z}_{\bar{w}})(\bar{z}_w - z_{\bar{w}})}.
\]  

(63)

To simplify (63), we choose \( w = u(z, \bar{z}) + iv(z, \bar{z}) \) such that the vector \( \frac{\partial}{\partial w} \) is in the form

\[ 2b \frac{\partial}{\partial w} = \xi \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{z}} \]  

(64)
for some function $\xi$. Therefore

$$\frac{\partial z}{\partial w} = \frac{\xi}{2b}, \quad \frac{\partial z}{\partial \overline{w}} = \frac{1}{2b\overline{x}i}. \quad (65)$$

This implies $(u, v)$ are orthogonal. $(63)$ becomes

$$\frac{dw \wedge d\overline{w}}{4b^2} \left( \frac{1}{|\xi|^2} - 2 + |\xi|^2 \right) = dw \wedge d\overline{w} \left( \left| \frac{\partial z}{\partial \overline{w}} \right|^2 - 2 + \left| \frac{\partial z}{\partial w} \right|^2 \right) \quad (66)$$

which leads to the equations of motion

$$\frac{\partial^2 z}{\partial w \partial \overline{w}} = 0. \quad (67)$$

Note that although $b$ is absent in $(67)$, the derivatives of $z(w, \overline{w})$ are connected to $(65)$.

This gives the solution (with a factor $4b^2$ for convenience) of

$$z = f(w) + \frac{g(\overline{w})}{4b^2} \quad (68)$$

and where $f$ and $g$ are linked by

$$f'(w) g'(\overline{w}) = 1. \quad (69)$$

Our task then is to invert $(68)$ to find $w(z, \overline{z})$ of which the real part is the potential $u(z, \overline{z})$. We do this perturbatively.

To begin, we differentiate $(68)$ with respect to $z$ and then $\overline{z}$ to get

$$1 = f'(w) \frac{\partial w}{\partial z} + \frac{g'(\overline{w})}{4b^2} \frac{\partial \overline{w}}{\partial z}, \quad (70)$$

$$0 = f'(w) \frac{\partial w}{\partial \overline{z}} + \frac{g'(\overline{w})}{4b^2} \frac{\partial \overline{w}}{\partial \overline{z}} \quad (71)$$

respectively. Solve $(71)$ for $\partial w/\partial z$, conjugate to get $\partial \overline{w}/\partial z$, and substitute into $(70)$. This yields

$$1 = f'(w) \frac{\partial w}{\partial z} - \frac{1}{16b^4} |g'(\overline{w})|^2 \frac{\partial w}{\partial z} = f'(w) \frac{\partial w}{\partial z}. \quad (72)$$

at the first order in $b^{-2}$. Furthermore, with this $\partial \overline{w}/\partial z$, the $z$ derivative of $(68)$ becomes

$$1 = f'(w) \frac{\partial w}{\partial z} + O(b^{-2}x^2)$$

$$f'(w) = \frac{\partial z}{\partial w}. \quad (73)$$

Substituting this into $(72)$ yields

$$\frac{\partial z}{\partial w} \frac{\partial w}{\partial z} = 1. \quad (74)$$

5 Wave equation in $1 + 1$ dimensions and its solutions
This is trivial only in Maxwell electrodynamics where \( w \) is analytic in \( z \). In bielectronrodynamics, \( w \) is a function of \( z \) and \( \bar{z} \). But (74) says what is trivial in Maxwell is true in bi; at least, at the first order. From (69) and (72) we find

\[ g'(w) = \frac{1}{f'(w)} = \frac{\partial w}{\partial \bar{z}}. \]  

(75)

Substitute this and (73) into (71) to get

\[ \frac{\partial w}{\partial \bar{z}} + \frac{1}{4b^2 \partial z} \left( \frac{\partial w}{\partial \bar{z}} \right)^2 = 0. \]  

(76)

Now we have only to solve (76) for \( w \) and take the real part. Drawing inspiration from the Maxwell solution \( w_M(z) = D \cos k z \), let us take as ansatz

\[ w(z, \bar{z}) = D \cos \left[ k z + \Psi(z) \right] \quad k \in \mathbb{R}. \]  

(77)

Now substitute this into (76) but in the term which is diminished by \( b^{-2} \), use \( w_M \) instead.

\[ \Psi'(z) + \frac{k^3 D^2}{4b^2} \sin^2 k z = 0 \]  

(78)

from which we find

\[ \Psi(z) = -\frac{k^3 D^2}{8b^2} z + \frac{k^2 D^2}{16b^2} \sin 2k z. \]  

(79)

So then the ansatz is now

\[ w(z, \bar{z}) = D \cos \left( k z - \frac{k^3 D^2}{8b^2} z + \frac{k^2 D^2}{16b^2} \sin 2k z \right) \]

\[ = D \cos \left( k z - \frac{k^3 D^2}{8b^2} z - \frac{k^2 D^2}{16b^2} \sin (2k z) \sin \left( k z - \frac{k^3 D^2}{8b^2} z \right) \right). \]  

(80)

Note that in the limit \( k^2 D^2 \ll b^2 \), \( w \rightarrow w_M \). But this solution is problematic; when the LHS of (76) is evaluated with this solution, we get a second order term linear in \( z \). So (80) is only a small \(|z|\) approximation. We can fix this by replacing

\[ z \rightarrow z + \frac{\alpha}{b^2} z. \]  

(81)

This is permissible since \( \partial w/\partial z \) only appears in (76) in the term diminished by \( b^{-2} \). With \( \alpha = -D^2k^2/8 \) the solution becomes

\[ w(z, \bar{z}) = D \cos \left[ \left( 1 + \frac{k^4|D|^4}{64b^4} \right) k z - \frac{k^3 D^2}{8b^2} z \right] \]

\[ - \frac{k^2 D^2}{16b^2} \sin \left[ 2k \left( z - \frac{k^2 D^2}{8b^2} z \right) \right] \sin \left[ k z - \frac{k^3 D^2}{8b^2} z \right]. \]  

(82)
This solution is satisfactory; that is, the $b^{-4}$ terms in the LHS of (76) for (82) are all bounded. Now we have only to take the real part. Let $D = D_1 + iD_2$. Consider that, with $z = x + iy$

$$\text{Re}(w_M) = u_M = D_1 \cos kx \cos kx + iD_2 \sin kx \sin iy.$$  \hspace{1cm} (83)

If we make the replacement $y \to -it$ and set $D_1 = a_1 + a_2$ and $D_2 = i(a_2 - a_1)$ we get two waves of different amplitudes propagating in opposite directions. Let us do the same for (82). The result is

$$u = \left\{a_1 \cos \left[\left(1 - \frac{a_2^2k^2}{2b^2}\right) kx - \left(1 + \frac{a_2^2k^2}{2b^2}\right) kt\right] + a_2 \cos \left[\left(1 - \frac{a_2^2k^2}{2b^2}\right) kx + \left(1 + \frac{a_2^2k^2}{2b^2}\right) kt\right]\right\}$$

$$\times \left\{1 - \frac{a_1a_2k^2}{2b^2} \sin \left[\left(1 - \frac{a_2^2k^2}{2b^2}\right) kx - \left(1 + \frac{a_2^2k^2}{2b^2}\right) kt\right]\right\}$$

$$\times \sin \left[\left(1 - \frac{a_1^2k^2}{2b^2}\right) kx + \left(1 + \frac{a_2^2k^2}{2b^2}\right) kt\right].$$ \hspace{1cm} (85)

When the LHS of (44) is evaluated with (85), all the $b^{-4}$ order terms are bounded. This is the wave solution.

### 5.4 Mirror

Suppose a mirror is placed in the $y-z$ plane. Then $E_y = \partial u/\partial t$ must be zero at the mirror which is at $x = 0$. This is the case for $a_1 - a_2 =: a$. (85) then becomes

$$u = 2a \sin \vartheta \sin \varphi \left[1 + \frac{a^2k^2}{4b^2} (\cos 2\vartheta - \cos 2\varphi)\right]$$

$$= 2a \left\{\sin \vartheta \sin \varphi + \frac{a^2k^2}{8b^2} \left[\sin \varphi (\sin 3\vartheta - \sin \vartheta) - \sin \varphi (\sin 3\varphi - \sin \varphi)\right]\right\}$$ \hspace{1cm} (86)

where

$$\vartheta := \left(1 + \frac{a^2k^2}{2b^2}\right) kt \quad \text{and} \quad \varphi := \left(1 - \frac{a^2k^2}{2b^2}\right) kx.$$ \hspace{1cm} (87)

Thus, the solution is a stationary wave solution given by $E_y = 0$ and $u$ given in (86), with the stationary wave nodes at $x_n$, derived from $\sin \varphi|_{x_n} = 0$, given by

$$\left(1 - \frac{a^2k^2}{2b^2}\right) kx_n = n\pi.$$ \hspace{1cm} (88)

So, at the order $b^{-2}$, the nodes depend on the amplitude and the third harmonic is present. Both the amplitude dependence and the third harmonic are diminished by a factor of $b^{-2}$.
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