R.K. RAINA
R.K. LADDA

A new family of functional series relations involving digamma functions

Annales mathématiques Blaise Pascal, tome 3, nº 2 (1996), p. 189-198

<http://www.numdam.org/item?id=AMBP_1996__3_2_189_0>
A NEW FAMILY OF FUNCTIONAL SERIES
RELATIONS INVOLVING DIGAMMA FUNCTIONS

par R.K. RAINA and R.K. LADDAA

Abstract:

The purpose of the present investigation is to obtain certain new families of functional series relations involving the digamma functions. Several types of summation formulae are deducible from the main results. Extension of the results associating the familiar \( H \)-functions are also studied.

1. Introduction and preliminaries.

For any real or complex \( \lambda \), we denote by \( (\lambda)_n \) the Pochhammer symbol defined by

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1 & \text{if } n = 0, \\
\lambda(\lambda + 1)...(\lambda + n - 1) & \forall \ n \in \mathbb{N}.
\end{cases}
\]  

(1.1)

Using techniques of fractional calculus, Kalla and Ross [7] obtained a summation formula (due to Nörlund [9 a, p. 168]).

\[
\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} = \Psi(\lambda) - \Psi(\lambda - \alpha),
\]  

(1.2)

\( \text{Re } (\lambda - \alpha) > 0, \) \( (\lambda \neq 0, -1, -2...) \)

involving the digamma (or \( \Psi \)) function [12]

\[
\Psi(z) = \frac{d}{dz}(\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}.
\]  

(1.3)

The interesting summation formula (1.2), in fact, was motivated by Ross's formula [11]:

\[
\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{n.2^n.n!} = \ln 4,
\]  

(1.4)

which emerges from (1.2) in the special case when \( \alpha = \frac{1}{2}, \lambda = 1. \)

The result (1.2) and its special cases were revived and studied in recent years as worthwhile illustrations emphasizing the fruitfulness of the approach of fractional
calculus in evaluating infinite sums. One may refer to the papers [1], [4], [7], [8] and [12] on the subject.

Alternative proof of (1.2) was furnished by using l'Hospital's theorem on limit by Kalla and Al-Saqabi [6]. Also, direct simple proofs to some of the generalizations of (1.2) have been given by Nishimoto and Srivastava [9] and Srivastava and Nishimoto [13].

This paper is devoted to obtaining a new family of functional series relations involving the digamma functions without using the techniques of fractional calculus. The approach of derivations is direct and based upon simple series rearrangement methods. The paper is arranged as follows:

Section 2 gives an interesting lemma which is used in the derivation of the main results [Eqns. (2.7) and (2.8) below]. In section 3, a number of examples are deduced, and in the concluding section 4, we study a useful application in associating the well known $H$-function [3] in functional series relations.

The advantages of our results are that several known and new results on series summations can be deduced from them, and some of these are invariably pointed out in the paper.

In the sequel we use various notations and symbols. The set of positive integers is denoted by $N$, and $N_0 = NU(0)$. $C$ means the complex number field and $R_+ = (0, \infty)$. As usual $(a_p)$ stands for the array of parameters $a_1, ..., a_p$.

2. Functional series relations.

Before stating our main results giving series relations involving the digamma functions, we require the following result:

**Lemma.**

Let $\alpha, \lambda, \mu \in C$ and $\nu \in N_0$ such that $Re(\lambda - \alpha) > 0, Re(\mu) > 0$, then

$$
\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{(\alpha + n)_k(\lambda - \mu)_k(-\nu)_k}{(\alpha)_k(\lambda + n)_k k!} = \frac{\Gamma(\lambda)\Gamma(\mu + \nu)}{\Gamma(\mu)\Gamma(\lambda + \nu)} \left[ \Psi(\mu) + \Psi(\lambda + \nu) - \Psi(\lambda - \alpha) - \Psi(\mu + \nu) \right],
$$

(2.1)

provided that both $\lambda, \nu + \lambda \neq 0, -1, -2, ...$; and $\Psi(z)$ is defined by (1.3).
Proof.
Let
\[ u(n, k) = \frac{(\alpha)_n (\alpha + n)_k (\lambda - \mu)_k (-\nu)_k}{n(\lambda)_n (\alpha)_k (\lambda + n)_k k!} \tag{2.2} \]
be the general term of the double series on the L.H.S. of (2.1). Also, put
\[ v(n, k) = \frac{(\lambda - \mu)_k}{(\lambda)_k k!} \left[ \frac{(\alpha + k)_n}{n(\lambda + k)_n} - \frac{(\lambda - \mu + k)_n}{n(\lambda + k)_n} \right], \tag{2.3} \]
and
\[ w(n, k) = \frac{(\lambda - \mu)_n (\lambda - \mu + n)_k (-\nu)_k}{n(\lambda)_n (\lambda + n)_k k!}. \tag{2.4} \]

By using the elementary identity \((\alpha)_n (\alpha + n)_k = (\alpha)_{n+k} = (\alpha)_k (\alpha + k)_n\), then we observe that (2.2), (2.3) and (2.4) satisfy the following relation:
\[ u(n, k) = v(n, k) + w(n, k). \tag{2.5} \]

Now summing separately the \(v\)-series and \(w\)-series by virtue of Gauss’s summation theorem [14, p. 18] and the formula (1.2), we find that
\[ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} v(n, k) = \frac{\Gamma(\lambda)\Gamma(\mu + \nu)}{\Gamma(\mu)\Gamma(\lambda + \nu)} [\Psi(\mu) - \Psi(\lambda - \alpha)], \tag{2.6} \]
and
\[ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} w(n, k) = \frac{\Gamma(\lambda)\Gamma(\mu + \nu)}{\Gamma(\mu)\Gamma(\lambda + \nu)} [\Psi(\lambda - \nu) - \Psi(\mu - \nu)]. \tag{2.6a} \]

The result (2.1) follows thus from (2.2), (2.6) and (2.6a).

Remark 1.

The formula (2.1) would also follow from a recent known result due to Raina [10, p. 173, Eqn. (1.5)] in which the method of derivation is based on techniques of fractional calculus. The alternative proof given above is a simple and direct one. It may be observed that the summation formula (1.2) is recoverable from (2.1) either when \(\nu = 0\) or \(\lambda = \mu\).

We state now the main results giving a family of functional series relations involving digamma functions which are contained in the following:
Theorem.

Let $A(r)$ be an arbitrary bounded sequence of complex numbers, and the parametric restrictions be same as stated in the hypothesis of (2.1). Then

$$
\sum_{n=1}^{\infty} \frac{(\alpha)_{n}}{n(\lambda)_{n}} \sum_{k=0}^{\infty} \frac{(\alpha + n)_{k}(\lambda - \mu)_{k}(-\nu)_{k}}{(\alpha)_{k}(\lambda + n)_{k}k!} \cdot \sum_{r=0}^{\infty} \frac{A(r)}{(\lambda + n + k)_{r}}
$$

$$
= \sum_{r=0}^{\infty} A(r) \cdot \frac{\Gamma(\lambda)\Gamma(\mu + \nu + r)}{\Gamma(\mu + r)\Gamma(\lambda + \nu + r)} \cdot [\Psi(\mu + r) + \Psi(\lambda + \nu + r) - \Psi(\mu + \nu + r) - \Psi(\lambda - \alpha)]
$$

and

$$
\sum_{n=1}^{\infty} \frac{(\alpha)_{n}}{n(\lambda)_{n}} \sum_{k=0}^{\infty} \frac{(\alpha + n)_{k}(\lambda - \mu)_{k}(-\nu)_{k}}{(\alpha)_{k}(\lambda + n)_{k}k!} \cdot \sum_{r=0}^{\infty} \frac{A(r)}{(\alpha + k)_{r} (\lambda + n + k)_{r}}
$$

$$
= \sum_{r=0}^{\infty} A(r) \cdot \frac{\Gamma(\lambda)\Gamma(\mu + \nu + r)}{\Gamma(\mu + r)\Gamma(\lambda + \nu + r)} \cdot [\Psi(\mu + r) + \Psi(\lambda + \nu + r) - \Psi(\mu + \nu + r) - \Psi(\lambda - \alpha)].
$$

Proof.

To prove (2.7), we have by (1.1) and the rearrangement of involved series (which is justified under the stated conditions):

L.H.S. of (2.7) = \[ \sum_{r=0}^{\infty} \frac{A(r)}{(\lambda)_{r}} \left( \sum_{k=0}^{\infty} \frac{(\alpha)_{n}}{n(\lambda + r)_{n}} \cdot \sum_{k=0}^{\infty} \frac{(\alpha + n)_{k}(\lambda - \mu)_{k}(-\nu)_{k}}{(\alpha)_{k}(\lambda + r + n)_{k}k!} \right) \]

Applying appropriately (2.1) (i.e., replacing $\lambda$ by $\lambda + r$ and $\mu$ by $\mu + r$) for getting the closed form expression for the inner double summation, we arrive at the desired R.H.S. of Eqn. (2.7), under of course, the restrictions stated with the theorem.

The result (2.8) can similarly be established.

Remark 2.

When $\lambda = \mu$ or $\nu = 0$, Eqns. (2.7) and (2.8) yield the functional series relations involving digamma functions given by Nishimoto and Srivastava [9, p. 104, Eqns. (4.5) and (4.6)].
3. DEDUCTIONS.

Due to the generality of the functional series relations involving the arbitrary coefficients \( A(r) \), several known and new series summations involving the digamma functions can be derived from (2.7) and (2.8). To illustrate, we derive the following summation formula:

Let us put

\[
A(r) = z^r \left( \prod_{i=1}^{p} (a_i)_r \right) \left( \prod_{i=1}^{q} (b_i)_r \right)^{-1} (p, q \in \mathbb{N}), \quad (3.1)
\]

in (2.7), then in terms of the triple hypergeometric series \( F^3[x, y, z] \) (for details: see [14, p. 44]) , we get the following:

Example 1.

\[
\sum_{r=0}^{\infty} \prod_{i=1}^{p} (a_i)_r \sum_{r=0}^{q} \prod_{i=1}^{q} (b_i)_r \frac{z^r}{r!} \frac{\Gamma(\mu + \nu + r)}{\Gamma(\mu + r)\Gamma(\lambda + \nu + r)} \left[ \Psi(\mu + r) + \Psi(\lambda + \nu + r) - \Psi(\mu + \nu + r) - \Psi(\lambda - \alpha + r) \right]
\]

\[
= \frac{\alpha}{\Gamma(\lambda + 1)} F^3 \left[ - : 1 + \alpha ; - ; - ; 1, 1 : \lambda - \mu, -\nu : (a_p) ; 1, 1, z \right], \quad (3.2)
\]

where \( b_i \neq 0, -1, -2, \ldots (i = 1, \ldots, q) \), \( Re(\lambda) > Re(\alpha) > 0 \), \( Re(\mu) > 0 \), and \( q > p - 2, |z| < \infty \)

Setting \( \lambda = \mu \) in (3.2), we have the simple relation:

Example 2.

\[
\sum_{r=0}^{\infty} \prod_{i=1}^{p} (a_i)_r \sum_{r=0}^{q} \prod_{i=1}^{q} (b_i)_r \frac{z^r}{r!} \frac{1}{\Gamma(\lambda + r)} \left[ \Psi(\lambda + r) - \Psi(\lambda - \alpha + r) \right]
\]

\[
= \frac{\alpha}{\Gamma(\lambda + 1)} F^{0,3}_{1,1} \left[ - : 1 + \alpha, 1, 1 : (a_p) ; 1, 2 : (b_q) ; 1, z \right], \quad (3.3)
\]

provided that the conditions stated with (3.2) hold true, where the \( F \)-function on the right side of (3.3) is the Kampé de Fériet hypergeometric function of two variables, [14, p. 26].

Again, with the substitution of (3.1) in Eqn (2.8), we have

Example 3.
\[ \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_r}{\prod_{i=1}^{q} (b_i)_r} \cdot \frac{z^r}{r!} \frac{\Gamma(\mu + \nu + r)}{\Gamma(\mu + r) \Gamma(\lambda + \nu + r)} \]  

under the conditions similar to those given in Example 1. Setting \( A = - \) in (3.4), we have

Example 4.

\[ \frac{\alpha}{\Gamma(\lambda + 1)} \, \text{F}_{1}^{3} \left[ \begin{array}{c} 1 + \alpha ; -; -; 1 + \lambda ; -; \alpha; -; 1; 1, z \end{array} \right| 1, 1, 1 - \mu, -\nu ; (a_p); 1, 1, z \right], \]

under the conditions similar to those given in Example 1. Setting \( \lambda = \mu \) in (3.4), we have

Example 4.

\[ \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_r}{\prod_{i=1}^{q} (b_i)_r} \cdot \frac{z^r}{r!} \frac{1}{\Gamma(\lambda + r)} \left[ \Psi(\lambda - r) - \Psi(\lambda - \alpha) \right] \]

\[ = \frac{\alpha}{\Gamma(\lambda + 1)} \, \text{F}_{1}^{3} \left[ \begin{array}{c} \alpha + 1 ; 1 + \lambda ; 1 + 1 - \mu, -\nu ; (b_q); 1; 1, z \end{array} \right], \]

under the conditions similar to those given with (3.2).

4. APPLICATIONS TO H-FUNCTION.

It would be interesting to apply our main result (2.7) in deriving a summation formula involving the well known H-function of Fox [3]. In this direction we invoke the following H-function representation due to Braaksma [2], see also [15, p. 12].

\[ H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right] = \sum_{h=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s \theta(\sigma)s^s}{s! B_h}, \]

where

\[ \sigma = \sigma(h, s) = \frac{s + b_h}{B_h}; h = 1, \ldots, m; s \in N \]

and

\[ \theta(\sigma) = \prod_{j=m+1}^{j=n+1} \frac{\Gamma(1 - B_j \sigma)}{\Gamma(1 - B_j \sigma) \prod_{j=n+1}^{p} \Gamma(a_j - A_j \sigma)} \prod_{j=1}^{j=q} \Gamma(b_j - B_j \sigma) \prod_{j=1}^{j=p} \Gamma(1 - a_j + A_j \sigma). \]
Empty product is interpreted as 1, \( m \in N \); \( n, p, q \in N_0 \) such that \( 0 \leq n \leq p \) and \( 1 \leq m \leq q \),

with \( a_j \in C \), \( A_j \in R_+ \) \( (j = 1, \ldots, p) \),
\( b_j \in C \), \( B_j \in R_+ \) \( (j = 1, \ldots, q) \).

Also, we have (for convergence) either

\[
\Omega = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0, \quad 0 < |x| < \infty
\]  

(4.4)

or

\[
\Omega = 0, \quad 0 < |x| < \Delta = \sum_{j=1}^{p} A_j^{-A_j} \sum_{j=1}^{q} B_j^{-B_j}.
\]  

(4.5)

To make our analysis relatively simple, we prefer a limiting case (2.1) of (2.7) which is attained when \( A(r) \to (r \geq 1) \) and \( A(0) = 1 \) in Eqn. (2.7)). Replacing \( \lambda \) in Eqn. (2.1) by \( \lambda + c \sigma \) \((c > 0)\), and then multiplying both sides by

\[
\frac{(-1)^{q} \theta(\sigma) x^{\sigma}}{s! B_{h}}
\]

where \( \sigma = \sigma(h, s) \) and \( \theta(\sigma) \) are given by (4.2) and (4.3), respectively. Now summing the resulting equation so obtained from (2.1) first w.r.t. \( h \) from \( h = 1 \) to \( h = m \), and then w.r.t. \( s \) from \( s = 0 \) to \( s = \infty \). Inverting the order of summation (which is permissible in view of the conditions stated with (2.1) and the conditions (4.4) and (4.5)), then after a little simplification we are lead to the following results:

\[
\sum_{n=1}^{\infty} \frac{(\alpha)_{n}}{n} \sum_{k=0}^{\infty} \frac{(\alpha + n)_{k}(-\nu)_{k}}{(\alpha)_{k} k!} . H_{p+2,q+2}^{m,n+2} \left[ x \left| (1 - \lambda, c), (\lambda - \mu + k, c), (a_1, A_1), \ldots, (a_p, A_p), (b_1, B_1), \ldots, (b_q, B_q), (\lambda - \mu, c), (1 - \lambda - n - k, c) \right. \right]
\]

(4.6)

provided that \( c > 0, Re(\mu) + \nu > 0, Re(\lambda - \alpha) > 0 \), and
\( \nu, \lambda + \nu \) (both) \( \not\equiv 0, -1, -2, \ldots \)

and the convergence conditions (4.4) and (4.5) are satisfied.

It may be observed that the class of summation formula (4.6) includes the results of Nishimoto and Saxena [8 ] and Nishimoto and Srivastava [9 ] . To deduce a simpler summation formula from (4.6) in terms of the Wright's generalised hypergeometric function [14, p. 21 ]
where \( \alpha_j \in \mathbb{R}_+ \) \( (j = 1, \ldots, p) \), \( B_j \in \mathbb{R}_+ \) \( (j = 1, \ldots, q) \), \( 1 + \sum_{j=1}^{q} B_j = \sum_{j=1}^{p} A_j \geq 0 \), we obtain the following special case of (4.6):

\[
\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n} \sum_{k=0}^{\infty} \frac{(\alpha + n)_k (-\nu)_n}{(\alpha)_k k!} \prod_{j=1}^{p} \Gamma(a_j + A_j n) = \sum_{n=0}^{\infty} \prod_{j=1}^{q} \Gamma(B_j + B_j n)
\]

where \( \alpha_j \in \mathbb{R}_+ \) \( (j = 1, \ldots, p) \), \( B_j \in \mathbb{R}_+ \) \( (j = 1, \ldots, q) \), \( 1 + \sum_{j=1}^{q} B_j = \sum_{j=1}^{p} A_j \geq 0 \), we obtain the following special case of (4.6):

\[
\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n} \sum_{k=0}^{\infty} \frac{(\alpha + n)_k (-\nu)_n}{(\alpha)_k k!} \prod_{j=1}^{p} \Gamma(a_j + A_j n) = \sum_{n=0}^{\infty} \prod_{j=1}^{q} \Gamma(B_j + B_j n)
\]

\[
\prod_{j=1}^{p} \Gamma(a_j + A_j n)
\]

\[
\prod_{j=1}^{q} \Gamma(B_j + B_j n)
\]

\[
[\Psi(\mu) + \Psi(\lambda + \nu + cs) - \Psi(\lambda - \alpha + cs) - \Psi(\mu + \nu)]
\]

\[
(c > 0, Re(\lambda - \alpha) > 0, \lambda \neq 0, -1, -2, \ldots)
\]

provided that each member of (4.8) exists.

**ACKNOWLEDGEMENT**: 

The authors thank the referee for useful suggestions.
REFERENCES

[1 ] B. N. Al-Saqabi, S.L. Kalla and H.M. Srivastava,
A certain family of infinite series associated with digamma functions, J. Math. Anal. Appl. 159 (1991), 361-372.

[2 ] B. L. J. Braaksma,
Asymptotic expansions and analytical continuations for a class of Barnes - integral, Compositio Math. 15 (1964), 239-341.

[3 ] C. Fox,
The G and H functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

[4 ] S.L. Kalla,
Functional relations by means of Riemann - Liouville operator, Serdica 13 (1987),
170-173.

[5 ] S.L. Kalla and B. Al - Saqabi,
A functional relation involving Ψ-function, Rev. Tech. Fac. Ingr. Univ. Zulia 8 (1985), 31-35.

[6 ] S.L. Kalla and B. Al-Saqabi,
Summation of certain infinite series with digamma functions, C.R. Acad. Bulgare Sci. 41 (11) (1988), 15-17.

[7 ] S.L. Kalla and B. Ross,
The development of functional relations by means of fractional calculus, in Fractional Calculus, Pitman Ad. Publishing Program Boston (1985), 32-43.

[8 ] K. Nishimoto and R.K. Saxena,
An application of Riemann - Liouville operator in the unification of certain functional relations, J. College Engrg. Nihon Univ. 32 (1991), 133-139.

[9 ] K. Nishimoto and H.M. Srivastava,
Certain class of infinite series summable by means of fractional calculus, J. College Engrg. Nihon Univ. 30 (1989), 97-106.

[9 a ] N.-E. Nörlund,
Leçons sur les séries d'interpolation, Paris, Gauthier-Villars, 1926.

[10 ] R.K. Raina,
A series relation by means of certain fractional calculus operators, Indian J. Pure Appld. Math. 21(2) (1990), 172-175.
[11] B. Ross,  
Serendipidity in maths or how one is led to discover that  
\[ \sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{n.2^n.n!} = \frac{1}{2} + \frac{3}{16} + \frac{15}{144} + ... = \ln 4, \]  
Amer. Math. Monthly 90 (1983) 562-566.

[12] B. Ross,  
Methods of Summation, Descrates Press, Koriyama (1987).

[13] H.M. Srivastava and K. Nishimoto,  
An elementary proof of a generalization of a certain functional relations derived by means of fractional calculus, J. Fractional calculus 1 (1992), 69-74.

[14] H.M. Srivastava and P.W. Karlsson,  
Multiple Gaussian Hypergeometric Series, Ellis Horwood Publication (1985).

[15] H.M. Srivastava, K.C. Gupta and S.P. Goyal,  
The H functions of One and Two Variables with Applications, South Asian Publishers, New Delhi and Madras (1982).