Edge Contraction and Forbidden Induced Subgraphs

Hany Ibrahim Peter Tittmann
University of Applied Sciences Mittweida

Abstract

Given a family of graphs $\mathcal{H}$, a graph $G$ is $\mathcal{H}$-free if any subset of $V(G)$ does not induce a subgraph of $G$ that is isomorphic to any graph in $\mathcal{H}$. We present sufficient and necessary conditions for a graph $G$ such that $G/e$ is $\mathcal{H}$-free for any edge $e$ in $E(G)$. Thereafter, we use these conditions to characterize claw-free, $2K_2$-free, $P_4$-free, $C_4$-free, $C_5$-free, split, pseudo-split, and threshold graphs.

1 Introduction

A graph $G$ is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set of vertices and $E(G)$ is a set of 2-elements subsets of $V(G)$ called edges. The set of all graphs is $\mathcal{G}$. The degree of a vertex $v$, denoted by $\text{deg}(v)$, is the number of edges incident to $v$. We denote the maximum degree of a vertex in a graph $G$ by $\Delta(G)$. We call two vertices adjacent if there is an edge between them, otherwise, we call them nonadjacent. Moreover, the set of all vertices adjacent to a vertex $v$ is called the neighborhood of $v$, which we denote by $N(v)$. On the other hand, the closed neighborhood of $v$, denoted by $N[v]$, is $N(v) \cup \{v\}$. Generalizing this to a set of vertices $S$, the neighborhood of $S$, denoted by $N(S)$, is defined by $N(S) := \bigcup_{v \in S} N(v) - S$. Similarly the closed neighborhood of $S$, denoted by $N[S]$, is $N(S) \cup S$. Moreover, for a subset of vertices $S$, we denote the set of vertices in $S$ that are adjacent to $v$ by $N_S(v)$. Furthermore, we write $v$ is adjacent to $S$ to mean that $S \subseteq N(v)$ and $v$ is adjacent to exactly $S$ to mean that $S = N(v)$.

A set of vertices $S$ is independent if there is no edge between any two vertices in $S$. We call a set $S$ dominating if $N[S] = V(G)$. A subgraph $H$ of a graph $G$ is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced graph $G[S]$ for a given set $S \subseteq V$, is a subgraph of $G$ with vertex set $S$ and two vertices in $G[S]$ are adjacent if and only if they are adjacent in $G$. Two graphs $G, H$ are isomorphic if there is a bijective mapping $f : V(G) \to V(H)$ where $u, v \in V(G)$ are adjacent if and only if $f(u), f(v)$ are adjacent in $H$. In this case we call the mapping $f$ an isomorphism. Two graphs that are not isomorphic are called non-isomorphic. In particular, an isomorphism from a graph to itself is called
automorphism. Furthermore, two vertices $u, v$ are similar in a graph $G$ if there is an automorphism that maps $u$ to $v$. The set of all automorphisms of a graph $G$ forms a group called the automorphism group of $G$, denoted by $\text{Aut}(G)$. The complement of a graph $G$, denoted by $\overline{G}$, is a graph with the same vertex set as $V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are nonadjacent in $G$.

The independence number of a graph $G$, denoted by $\alpha(G)$, is the largest cardinality of an independent set in $G$. In this thesis, we write singletons $\{x\}$ just as $x$ whenever the meaning is clear from the context. A vertex $u$ is a corner dominated by $v$ if $N[u] \subseteq N[v]$. Let $\mathcal{H}$ be a set of graphs. A graph $G$ is called $\mathcal{H}$-free if there is no induced subgraph of $G$ that is isomorphic to any graph in $\mathcal{H}$, otherwise, we say $G$ is $\mathcal{H}$-exist.

By contracting the edge between $u$ and $v$, we mean the graph constructed from $G$ by adding a vertex $w$ with edges from $w$ to the union of the neighborhoods of $u$ and $v$, followed by removing $u$ and $v$. We denote the graph obtained from contracting $uv$ by $G/uv$. If $e$ is the edge between $u$ and $v$, then we also denote the graph $G/uv$ by $G/e$. Further, we call $G/e$ a $G$-contraction. Finally, any graph in this paper is simple. For notions not defined, please consult [3]. Additionally, we divide longer proofs into smaller claims, and we prove them only if their proofs are not apparent.

For a graph invariant $c$, a graph $G$, and a $G$-contraction $H$, the question of how $c(G)$ differs from $c(H)$ is investigated for different graph invariants. For instance, how contracting an edge in a graph affects its $k$-connectivity. Hence, the intensively investigated ([18]) notion of $k$-contractible edges in a $k$-connected graph $G$ is defined as the edge whose contraction yields a $k$-connected graph. Another example is in the game Cops and Robber where a policeman and a robber are placed on two vertices of a graph in which they take turns to move to a neighboring vertex. For any graph $G$, if the policeman can always end in the same vertex as the robber, we call $G$ cop-win. However, $G$ is CECC if it is not cop-win, but any $G$-contraction is cop-win. The characteristics of a CECC graph are studied in [6].

A further example is the investigation of the so-called contraction critical, with respect to independence number, that is, an edge $e$ in a graph $G$ where $\alpha(G/e) \leq \alpha(G)$, studied in [24]. Furthermore, the case where $c$ is the chromatic and clique number, respectively, has been investigated in [11, 22, 23].

In this article, we investigate the graph invariant $H$-free for a given set of graphs $\mathcal{H}$. In particular, we present sufficient and necessary conditions for a graph $G$ such that any $G$-contraction is $H$-free.

Let $\mathcal{H}$ be a set of graphs. The set of elementary (minimal) graphs in $\mathcal{H}$, denoted by $\text{elm}(\mathcal{H})$, is defined as $\{H \in \mathcal{H} : \text{if } G \in \mathcal{H} \text{ and } H \text{ is } G\text{-exist, then } G \text{ is isomorphic to } H\}$. From the previous definition, we can directly obtain the following.

**Proposition 1.** Let $\mathcal{H}$ be a set of graphs. Graph $G$ is $\mathcal{H}$-free if and only if $G$ is $\text{elm}(\mathcal{H})$-free.

We call an $\mathcal{H}$-free graph $G$, strongly $\mathcal{H}$-free if any $G$-contraction is $\mathcal{H}$-free.
Furthermore, an $\mathcal{H}$-exist graph $G$ is a critically $\mathcal{H}$-exist if any $G$-contraction is $\mathcal{H}$-free. If we add any number of isolated vertices to a strongly $\mathcal{H}$-free or critically $\mathcal{H}$-exist graph, we obtain a graph with same property. Thus, from this section and forward, we exclude graphs having isolated vertices unless otherwise stated.

We conclude directly the following.

**Proposition 2.** Let $\mathcal{H}$ be a set of graphs and $G$ be a graph where $G$ is neither critically $\mathcal{H}$-exist nor $\mathcal{H}$-free but not strongly $\mathcal{H}$-free. The graph $G$ is $\mathcal{H}$-free if and only if any $G$-contraction is $\mathcal{H}$-free.

Given a graph $G$ and a set of graphs $\mathcal{H}$, we call $G$ $\mathcal{H}$-split if there is a $G$-contraction isomorphic to a graph in $\mathcal{H}$. Furthermore, $G$ is $\mathcal{H}$-free-split if $G$ is $\mathcal{H}$-split and $\mathcal{H}$-free. Moreover, the set of all $\mathcal{H}$-free-split graphs, for a given $\mathcal{H}$, is denoted by $\text{fs}(\mathcal{H})$.

**Proposition 3.** Let $\mathcal{H}$ be a set of graphs and $G$ be a $\mathcal{H}$-free graph. Then $G$ is strongly $\mathcal{H}$-free if and only if $G$ is $\text{fs}(\mathcal{H})$-free.

**Proof.** Assume for the sake of contradiction that there exists a strongly $\mathcal{H}$-free graph $G$ with an induced $\mathcal{H}$-free-split subgraph $J$. Consequently, there is an edge $e$ in $J$ such that $J/e$ induces a graph in $\mathcal{H}$. As a result, $G/e$ is $\mathcal{H}$-exist, which contradicts the fact that $G$ is strongly $\mathcal{H}$-free.

In contrast, if $G$ is an $\mathcal{H}$-free but not a strongly $\mathcal{H}$-free, then there is a set $U \subseteq V(G)$ such that there is an edge $e \in E(G[U])$ where $G/e$ is $\mathcal{H}$-exist. Let $U$ be a minimum set with such a property. Thus $G[U]$ is $\mathcal{H}$-free-split. 

From Propositions 2 and 3, we deduce the following.

**Theorem 4.** Let $\mathcal{H}$ be a set of graphs and $G$ be a $\text{fs}(\mathcal{H})$-free graph where $G$ is not critically $\mathcal{H}$-exist. The graph $G$ is $\mathcal{H}$-free if and only if any $G$-contraction is $\mathcal{H}$-free.

Theorem 4 provides a sufficient and necessary condition that answers the question we investigate in this thesis, however, it translates the problem to determining characterizations for critically $\mathcal{H}$-exist and $\mathcal{H}$-free-split graphs for a set of graphs $\mathcal{H}$. In Sections 1.1 and 1.2, we present some properties for these families of graphs.

### 1.1 The $\mathcal{H}$-Split Graphs

Let $H$ be a graph with $v \in V(H)$ and $N_H(v) = U \cup W$. The splitting$(H, v, U, W)$ is the graph obtained from $H$ by removing $v$ and adding two vertices $u$ and $w$ where $N_H(u) = U \cup \{w\}$ and $N_H(w) = W \cup \{u\}$. Furthermore, splitting$(H, v)$ is the set of all graphs for any possible $U$ and $W$. Moreover, splitting$(H)$ is the union of the splitting$(H, v)$ for any vertex $v \in V(H)$. Given a set of graphs $\mathcal{H}$, splitting$(\mathcal{H})$ is the union of the splittings of every graph in $\mathcal{H}$.

**Theorem 5.** For a graph $G$ and a set of graphs $\mathcal{H}$, $G$ is an $\mathcal{H}$-split if and only if $G \in \text{splitting}(\mathcal{H})$. 

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Proof. Let \( G \) be an \( H \)-split. Hence there is a graph \( H \in \mathcal{H} \) such that \( G \) is \( H \)-split. Thus, there are two vertices \( u, w \in V(G) \) such that \( G/uv \) is isomorphic to \( H \). Let \( x := V(G/uv) - V(G) \), then \( N_{G/uv}(x) = (N_G(u) \cup N_G(w)) - \{u, w\} \). As a result, \( G \in \text{splitting}(H, x) \). Consequently, \( G \in \text{splitting}(H) \).

Conversely, let \( G \in \text{splitting}(H) \). Hence there is a graph \( H \in \mathcal{H} \) such that \( G \in \text{splitting}(H) \). Thus, there are two adjacent vertices \( u, w \in V(G) \) such that \( G/uv \cong H \). Thus, \( G \) is \( H \)-split.

For a set of graphs \( \mathcal{H} \) and using Theorem 5, we can use \( \text{splitting}(\mathcal{H}) \) to construct all \( \mathcal{H} \)-split graphs, consequently \( \mathcal{H} \)-free-split graphs.

**Proposition 6.** In a graph \( G \), let \( u, v \in V(G) \). If \( u \) is similar to \( v \), then \( \text{splitting}(G, u) = \text{splitting}(G, v) \).

By the previous proposition, for a graph \( H \), the steps to construct the \( H \)-free-split graphs are:

- Let \( \pi \) be the partition of \( V(H) \) induced by the orbits generated from \( \text{Aut}(H) \);
- for every orbit \( o \in \pi \), we choose a vertex \( v \in o \); and
- construct \( \text{splitting}(H, v) \).

**Proposition 7.** Let \( G \) be a graph, \( v \) a vertex in \( V(G) \) where \( N_G(v) = U \cup W \). If \( U = N_G(v) \) or \( W = N_G(v) \), then \( \text{splitting}(G, v, U, W) \) is not \( G \)-free-split.

**Proposition 8.** Let \( G \) be a graph and \( v \) a vertex in \( V(G) \). If \( \deg(v) = 1 \), then \( \text{splitting}(G, v) \) contains no \( G \)-free-split graph.

**Proposition 9.** If \( G \) is a path, then \( \text{splitting}(G) \) contains no \( G \)-free-split graph.

**Proposition 10.** If \( G \) is a \( C_n \) for an integer \( n \geq 3 \), then the \( G \)-free-split is \( C_{n+1} \).

### 1.2 Critically \( \mathcal{H} \)-Exist Graphs

**Theorem 11.** Let \( G \) be a graph and \( \mathcal{H} \) be a set of graphs. If \( G \) is a critically \( \mathcal{H} \)-exist, then for any \( S \subseteq V(G) \) such that \( G[S] \) is isomorphic to a graph in \( \mathcal{H} \), the followings properties hold:

1. \( V(G) - S \) is independent and
2. there is no corner in \( V(G) - S \) that is dominated by a vertex in \( S \).

**Proof.**

1. For the sake of contradiction, assume there is a \( S \subseteq V(G) \) such that \( G[S] \) is isomorphic to a graph \( H \in \mathcal{H} \) but \( V(G) - S \) is not independent. Hence, there are two vertices \( u, v \in V(G) - S \) where \( u \) and \( v \) are adjacent. Consequently, \( G/uv[S] \) is isomorphic to \( H \), which contradicts the fact that \( G \) is a critically \( \mathcal{H} \)-exist.
2. Since \( V(G) - S \) is independent, the neighborhood of any vertex in \( V(G) - S \) is a subset of \( S \). For the sake of contradiction, assume that there is a corner \( u \in V(G) - S \) that is dominated by \( v \in S \). However, \( G/uv[S] \) is isomorphic to a graph \( H \in \mathcal{H} \), which contradicts the fact that \( G \) is a critically \( \mathcal{H} \)-exist.

\[ \square \]

**Corollary 12.** Let \( G \) be a critically \( \mathcal{H} \)-exist graph for a set of graphs \( \mathcal{H} \). If \( S \) is a vertex set that induces a graph in \( \mathcal{H} \), then no vertex in \( V(G) - S \) is adjacent to exactly one vertex, two adjacent vertices, three vertices that induce either \( P_3 \) or \( C_3 \), or a vertex with degree \( |V(G)| - 1 \).

Let \( G \) be a graph with adjacent vertices \( u, v \), and \( \{w\} := V(G/uv) - V(G) \).

We define the mapping \( f : 2^{V(G)} \rightarrow 2^{V(G/uv)} \) as follows:

\[
f(S) = \begin{cases} 
S & \text{if } u, v \notin S, \\
(S \cup \{w\}) - \{u, v\} & \text{otherwise.}
\end{cases}
\]

Let \( S \) be a vertex set such that \( G[S] \) is isomorphic to a given graph \( H \). We call an edge \( uv \), \( H \)-critical for \( S \) if \( G/uv[f(S)] \) is non-isomorphic to \( H \). Furthermore, we call the edge \( uv \) \( H \)-critical in \( G \) if for any vertex subset \( S \) that induces \( H \), \( uv \) is \( H \)-critical for \( S \).

**Theorem 13.** Let \( G \) be a graph and \( S \subseteq V(G) \) where \( H \) is the graph induced by \( S \) in \( G \). For any edge \( uv \in E(G) \), \( uv \) is \( H \)-critical for \( S \) if and only if

1. \( u, v \in S \) or
2. \( u \in V(G) - S \), \( v \in S \), and \( u \) is not a corner dominated by \( v \) in the subgraph \( G[S \cup \{u\}] \).

**Proof.**

1. If \( u, v \in S \), then \( |f(S)| < |S| \). Thus, \( G/uv[f(S)] \) is non-isomorphic to \( H \).

2. Let \( u \in V(G) - S \), \( v \in S \), and \( u \) is not a corner dominated by \( v \) in the subgraph \( G[S \cup \{u\}] \). Additionally, let \( w \in N_S(u) \) but \( w \notin N_S(v) \). In \( G/uv \), let \( x := V(G/uv) - V(G) \). Clearly, \( x \) is adjacent to any vertex in \( N_S(v) \cup \{w\} \). Hence, the size of \( G/uv[f(S)] \) is larger than that of \( G[S] \). Thus, \( G/uv[f(S)] \) is non-isomorphic to \( H \). Conversely, if none of the conditions in the theorem hold, then one of the following holds:

1. both \( u \) and \( v \) are not in \( S \), or
2. \( u \in V(G) - S \), \( v \in S \), and \( u \) is a corner dominated by \( v \) in the subgraph \( G[S \cup \{u\}] \).

In both cases, \( G[S] \cong G/uv[f(S)] \cong H \). Consequently, \( uv \) is not \( H \)-critical for \( S \). \( \square \)
2 Special graphs

Proposition 14. If a graph $G$ is $C_n$-exist, where $n \geq 4$, then there is a $G$-contraction that is $C_{n-1}$-exist.

Proposition 15. The only critical $C_3$-exist graph is $C_3$.

3 Claw-Free Graphs

There are several graph families that are subfamilies of claw-free graphs, for instance, line graphs and complements of triangle-free graphs. For more graph families and results about claw-free graphs, please consult [13]. Additionally, for more structural results about claw-free graphs, please consult [7, 8]. In the following, we call the graph $H_5$ in Figure 1 bull.

Proposition 16. The graphs in Figure 1 are the only claw-split graphs.

Corollary 17. Bull is the only claw-free-split graph.

![Figure 1: Claw-split graphs](image)

Proposition 18. The graphs in Figure 2 are the only critically claw-exist graphs.

Proof. Through this proof, we assume that $G$ is a critically claw-exist graph with $S := \{r, s, t, u\}$, where $G[S]$ is isomorphic to a claw and $u$ is its center. By Theorem 11, $V(G) - S$ is independent. Thus, any vertex in $V(G) - S$ is adjacent to vertices only in $S$. By Corollary 12, if $v \in V(G) - S$, then neither $|N(v)| = 1$ nor $v$ is adjacent to $u$.

Let $v, w \in V(G) - S$ such that $N(v) = N(w)$ where $|N(v)| = 2$. W.l.o.g., assume that $N(v) = \{r, s\}$, however, in $G/tu$, $f(\{v, u, v, w\})$ induces a claw,
which contradicts the fact that $G$ is a critically claw-exist. Thus, if $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$, then $N(v) \neq N(w)$.

Let $v, w, x \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$. No two vertices of $v, w,$ and $x$ (if $|N(x)| = 2$) are adjacent to the same vertices in $S$. W.l.o.g., assume that $N(v) = \{r, s\}$, $N(w) = \{r, t\}$, and $\{s, t\} \subseteq N(x)$. In $G/tx$, $f(\{r, u, v, w\})$ induces a claw, which contradicts the fact that $G$ is a critically claw-exist. Thus, if $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$, then $G$ is isomorphic to $H_5$.

Let $v, w, x \in V(G) - S$ such that $|N(v)| = |N(w)| = 3$. W.l.o.g., assume that $s$ is adjacent to $x$. In $G/sx$, $f(\{r, u, v, w\})$ induces a claw, which contradicts the fact that $G$ is a critically claw-exist. Thus, if $v, w \in V(G) - S$ such that $N(v) = N(w)$ and $|N(v)| = 3$, then $G$ is isomorphic to $H_3$.

Consequently, the possible critically claw-exist graphs are those presented in Figure 2. To complete the proof, we have to show that all these graphs are critically claw-exist, which is straightforward in each case. □

By Theorem 4, Corollary 17, and Proposition 18, we obtain the following result.

**Theorem 19.** Let $G$ be a bull-free graph that is non-isomorphic to any graph in Figure 2. The graph $G$ is claw-free if and only if any $G$-contraction is claw-free.

## 4 The $2K_2$-Free Graphs

Different graphs families are $2K_2$-free graphs; for instance split, pseudo-split, threshold, and co-chordal graphs. Various graph invariants were studied for $2K_2$-free graphs, please consult [4, 5, 9, 12, 15]. The class of $2K_2$-free graphs has been characterized in different ways, see [20, 25]. We call the graph $H_5$ in Figure 1 Bull.

We call an edge $uv$ in a graph $G$ almost-dominating if $V(G) - N\{u, v\}$ induces edgeless graph.
Proposition 20. A graph $G$ is $2K_2$-free if and only if any edge in $E(G)$ is almost-dominating.

Lemma 21. Let $G$ be a graph with a unique subset $S \subseteq V(G)$ such that $G[S]$ induces $2K_2$. If every edge $e$ in $E(G)$ is $e$ is $2K_2$-critical for $S$, then $G$ is a critically $2K_2$-exist.

Proof. Let $H$ be a $G$-contraction. Every edge $e$ in $E(G)$ is $2K_2$-critical for $S$, then $V(G) - S$ is independent set. Furthermore, every vertex in $V(G) - S$ is adjacent to at least two nonadjacent vertices in $S$. In $H$, let $u \in V(G) - f(S)$ and $v \in f(S)$. If $u, v$ are adjacent, then $uv$ is almost-dominating. Let $u \in f(S)$, then $uv$ is almost-dominating. Hence, every edge in $H$ is almost-dominating. Thus, $G$ is a critically $2K_2$-exist. \qed

Proposition 22. The graphs $P_2 \cup C_3$ and $P_2 \cup P_3$ are the only $2K_2$-split graphs.

Corollary 23. There is no $2K_2$-free-split graph.

Proposition 24. The graphs in Figure 3 are the only critically $2K_2$-exist graphs.

Proof. Through this proof, we assume that $G$ is a critically $2K_2$-exist graph with $S = \{r, s, t, u\}$ such that $G[S]$ is isomorphic to $2K_2$, where $rs$ and $tu$ are edges in $G$. By Theorem 11, we note that $V(G) - S$ is independent. Thus, any vertex in $V(G) - S$ is adjacent to vertices only in $S$. By Corollary 12, if $v \in V(G) - S$, then neither $|N(v)| = 1$ nor $v$ is adjacent to exactly two adjacent vertices.

Figure 3: Critically $2K_2$-exist graphs
Claim 24.1. If $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$ while $N(v) \cap N(w) = \phi$, then $G$ is isomorphic to $H_1$.

Proof. W.l.o.g., let $N(v) = \{r, u\}$ and $N(w) = \{s, t\}$. We will show that $V(G) = S \cup \{v, w\}$. For the sake of contradiction, assume that there is a vertex $x \in V(G) - S$. Thus, $x$ is adjacent to at least one vertex in $S$. W.l.o.g., let $x$ be adjacent to $r$. In $G/rx$, $f(\{s, u, v, w\})$ induces $2K_2$, which contradicts the fact that $G$ is a critically $2K_2$-exist.

Claim 24.2. If $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$, $|N(x)| = 3$ while $N(v) = N(w)$, then $N(v) \subset N(x)$.

Proof. W.l.o.g., let $N(v) = N(w) = \{r, u\}$. For the sake of contradiction and w.l.o.g., assume $N(x) = \{r, s, t\}$. In $G/rw$, $f(\{s, u, v, x\})$ induces $2K_2$, which contradicts the fact that $G$ is a critically $2K_2$-exist.

Claim 24.3. If $v, w, x \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$ while $|N(v) \cap N(w)| = 1$ and $|N(x)| = 3$ where $N(v) \cap N(w) \cap N(x) = \phi$, then $G$ is isomorphic to $H_4$.

Proof. W.l.o.g., let $N(v) = \{r, u\}$, $N(w) = \{r, t\}$, and $N(x) = \{s, t, u\}$. For the sake of contradiction, assume that there is a vertex $y \in V(G) - S$. Hence, $y$ is adjacent to at least one vertex in $S$. If $y$ is adjacent to $s$ (or $u$), then $f(\{r, t, v, x\})$ induces $2K_2$ in $G/sy$ (or $G/uy$), which contradicts the fact that $G$ is a critically $2K_2$-exist. Moreover, if $y$ is adjacent to $t$, then $f(\{r, u, w, x\})$ induces $2K_2$ in $G/ty$, which contradicts the fact that $G$ is a critically $2K_2$-exist.

Claim 24.4. If $v, w, x \in V(G) - S$ such that $|N(v)| = |N(w)| = 2$ while $|N(v) \cap N(w)| = 1$ and $|N(x)| = 3$, then either $N(v) \cup N(w) = N(x)$ or $N(v) \cap N(w) \cap N(x) = \phi$ and $G$ is isomorphic to $H_4$.

Proof. By Claim 24.3, if $N(v) \cap N(w) \cap N(x) = \phi$, then $G$ is isomorphic to $H_4$. If $N(v) \cup N(w) = N(x)$, then we are done. As a result, and w.l.o.g., let $N(v) = \{r, u\}$ and $N(w) = \{r, t\}$. Assume for the sake of contradiction that $N(x) = \{r, s, t\}$. However, $f(\{s, u, v, x\})$ induces $2K_2$ in $G/rw$, which contradicts the fact that $G$ is a critically $2K_2$-exist.

Claim 24.5. If $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 3$, while $N(v) \cap N(w)$ consists of two nonadjacent vertices in $S$, then $G$ is isomorphic to $H_3$.

Proof. W.l.o.g., let $N(v) = \{r, t, u\}$ and $N(w) = \{r, s, t\}$. For the sake of contradiction, assume that there is a vertex $x \in V(G) - S$. If $x$ is adjacent to $r$ (or $t$), then $f(\{s, u, v, w\})$ induces $2K_2$ in $G/rx$ (or $G/tx$), which contradicts the fact that $G$ is a critically $2K_2$-exist graph. Thus, $N(x) = \{s, u\}$, however, $f(\{s, t, v, x\})$ induces $2K_2$ in $G/rw$, which contradicts the fact that $G$ is a critically $2K_2$-exist.

Claim 24.6. If $v, w \in V(G) - S$ such that $|N(v)| = |N(w)| = 3$, while $N(v) \cap N(w)$ is two adjacent vertices in $S$, then $|N(x)| \neq 2$. 

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Proof. W.l.o.g., Let \( N(v) = \{r, t, u\} \) and \( N(w) = \{s, t, u\} \). W.l.o.g and for the sake of contradiction, assume that \( N(x) = \{r, u\} \). However, \( f(\{r, t, w, x\}) \) induces \( 2K_2 \) in \( G/uv \), which contradicts the fact that \( G \) is a critically \( 2K_2 \)-exist.

By Claims 24.1 to 24.6, the possible critically \( 2K_2 \)-exist graphs are those presented in Figure 3 whose proofs of being critically \( 2K_2 \)-exist for \( H_1 \), \( H_2 \), \( H_3 \), and \( H_4 \) are straightforward.

Claim 24.7. The graph \( H_5 \) in Figure 3 is a critically \( 2K_2 \)-exist.

Proof. Graph \( H_5 \) in Figure 3 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u\} \), where \( G[S] \) is isomorphic to a \( 2K_2 \) and \( rs, tu \in E(G) \). Moreover, \( V(G) = S \cup W \cup X \cup Y \), such that \( N(w \in W) = \{r, s, t\} \), \( N(x \in X) = \{r, s, u\} \), \( N(y \in Y) = \{r, s, t, u\} \), and \( |W|, |X|, |Y| \geq 0 \).

We note that \( S \) is the only vertex set inducing \( 2K_2 \) in \( G \). Moreover, every edge in \( E(G) \) is \( G[S] \)-critical for \( S \). Thus, and by Lemma 21, \( H_5 \) in Figure 3 is a critically \( 2K_2 \)-exist. \(\Box\)

Claim 24.8. Graph \( H_6 \) in Figure 3 is a critically \( 2K_2 \)-exist.

Proof. Graph \( H_6 \) in Figure 3 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u\} \) where \( G[S] \) is isomorphic to a \( 2K_2 \) and \( rs, tu \in E(G) \). Moreover, \( V(G) = S \cup W \cup X \cup Y \cup Z \), such that \( N(w \in W) = \{s, t\} \), \( N(x \in X) = \{s, u\} \), \( N(y) = \{s, t, u\} \), \( N(z) = \{r, s, t, u\} \), and \( |W|, |X|, |Y|, |Z| \geq 0 \).

We note that \( S \) is the only vertex set inducing \( 2K_2 \) in \( G \). Moreover, every edge in \( E(G) \) is \( G[S] \)-critical for \( S \). Thus, and by Lemma 21, \( H_6 \) in Figure 3 is a critically \( 2K_2 \)-exist. \(\Box\)

By Claims 24.7 and 24.8, the proof is complete. \(\Box\)

By Theorem 4, Corollary 23, and Proposition 24, we obtain the following.

Theorem 25. Let \( G \) be a graph that is non-isomorphic to any graph in Figure 3. The graph \( G \) is \( 2K_2 \)-free if and only if any \( G \)-contraction is \( 2K_2 \)-free.

5 The \( P_4 \)-Free Graphs

By Proposition 9, the following result follows.

Corollary 26. There is no \( P_4 \)-free-split graph.

Proposition 27. The graphs in Figure 4 are the only critically \( P_4 \)-exist graphs.
Proof. Through this proof, we assume that $G$ is a critically $P_4$-exist graph with $S = \{r, s, t, u\}$ such that $G[S]$ is isomorphic to $P_4$ where $rs, st, tu \in E(G)$. By Theorem 11, $V(G) - S$ is independent. Thus, any vertex in $V(G) - S$ is adjacent to vertices only in $S$. By Corollary 12, if $v \in V(G) - S$, then $v$ is nonadjacent to exactly: one vertex, two adjacent vertices, or three vertices inducing a path.

Let $v, w \in V(G) - S$ such that $v$ is adjacent to exactly the leaves of $S$. If $w$ is adjacent to $r$ (or $u$), then in $G/rw$, $f(\{s, t, u, v\})$ induces $P_4$, which contradicts the fact that $G$ is a critically $P_4$-exist. If $w$ is adjacent to $s$ (or $t$), then in $G/sw$, $f(\{r, t, u, v\})$ induces $P_4$, which contradicts the fact that $G$ is a critically $P_4$-exist. Thus, if there is a vertex $v \in V(G) - S$ such that $v$ is adjacent to exactly the leaves of the path induced by $S$, then $G$ is isomorphic to $H_1$.

Let $v \in V(G) - S$ such that $|N(v)| = 2$. If $v$ is adjacent to exactly the leaves of the path induced by $S$, then $G$ is isomorphic to $H_1$. Assume that $v$ is nonadjacent to exactly the leaves of the path induced by $S$. Hence, $v$ is exactly adjacent to $\{r, t\}$ (or $\{s, u\}$). Assume that there is a vertex $w \in V(G) - S$ that is not exactly adjacent to $\{r, t\}$. Hence, $w$ is nonadjacent to exactly $\{r, u\}$. If $w$ is adjacent to $s$, then in $G/sw$, $f(\{r, t, u, v\})$ induces $P_4$, which contradicts the fact that $G$ is a critically $P_4$-exist. Hence, $w$ is adjacent to exactly $\{r, t, u\}$, however, in $G/st$, $f(\{r, u, v, w\})$ induces $P_4$, which contradicts the fact that $G$ is a critically $P_4$-exist. Thus, if $v \in V(G) - S$ such that $|N(v)| = 2$, then $G$ is isomorphic to either $H_1$ or a graph in $H_4$.

Figure 4: The critically $P_4$-exist graphs
Figure 4 whose proofs, of being critically $P_4$-exist, are straightforward, which completes the proof. □

By Theorem 4, Corollary 26, and Proposition 27, we obtain the following.

**Theorem 28.** Let $G$ be a graph that is non-isomorphic to any graph in Figure 4. The graph $G$ is $P_4$-free if and only if any $G$-contraction is $P_4$-free.

### 6 The $C_4$-Free Graphs

**Proposition 29.** The graphs in Figure 5 are the only $C_4$-split graphs.

**Corollary 30.** $C_5$ is the only $C_4$-free-split graph.

![Figure 5: $C_4$-split graphs](image)

**Proposition 31.** The graphs in Figure 6 are the only critically $C_4$-exist graphs.

![Figure 6: Critically $C_4$-exist graphs](image)

*Proof.* Through this proof, we assume that $G$ is a critically $C_4$-exist graph with $S = \{r, s, t, u\}$ such that $G[S]$ is isomorphic to $C_4$ where $r$ and $t$ are adjacent to both $s$ and $u$. By Theorem 11, we note that $V(G) - S$ is independent. Thus, any vertex in $V(G) - S$ is adjacent to vertices only in $S$. By Corollary 12, if $v \in V(G) - S$, then $v$ is nonadjacent to exactly: one vertex, two adjacent vertices, or three vertices.

Let $v \in V(G) - S$ such that $|N(v)| = 2$. W.l.o.g, assume that $N(v) = \{r, t\}$. Let $w \in V(G) - S$, however, if $w$ is adjacent to $s$ (or $u$), then in $G/sw$, $f(\{r, t, u, v\})$ induces $C_4$, which contradicts the fact that $G$ is a critically $C_4$-exist. Thus, if $v \in V(G) - S$ such that $|N(v)| = 2$, then $G$ is isomorphic to a graph in $H_1$. 

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Let $v, w, x \in V(G) - S$ such that $|N(v)| = |N(w)| = 4$. Hence, $|N(x)| = 4$, however, in $G/rv$, $f(\{s, u, w, x\})$ induces $C_4$, which contradicts the fact that $G$ is a critically $C_4$-exist. Thus, if there is a vertex outside $S$ that is adjacent to every vertex in $S$ then $G$ is isomorphic to either $H_2$ or $H_3$.

Consequently, the possible critically $C_4$-exist graphs are those presented in Figure 6 whose proofs, of being critically $C_4$-exist, are straightforward, which completes the proof. □

By Theorem 4, Corollary 30, and Proposition 31, we obtain the following.

**Theorem 32.** Let $G$ be a $C_5$-free graph that is non-isomorphic to any graph in Figure 6. The graph $G$ is $C_4$-free if and only if any $G$-contraction is $C_4$-free.

### 7 The $C_5$-Free Graphs

**Proposition 33.** The graphs in Figure 7 are the only $C_5$-split graphs.

![Figure 7: $C_5$-split graphs](image)

**Corollary 34.** $C_6$ is the only $C_5$-free-split graph.

**Proposition 35.** The graphs in Figure 8 are the only critically $C_5$-exist graphs.

**Proof.** Through this proof, we assume that $G$ is a critically $C_5$-exist graph with $S = \{r, s, t, u, v\}$ such that $G[S]$ is isomorphic to a $C_5$ where $rs, st, tu, uv$, and $vr$ are the edges in $E(G[S])$. By Theorem 11, we note that $V(G) - S$ is independent. Thus, any vertex in $V(G) - S$ is adjacent to vertices only in $S$. By Corollary 12, if $w \in V(G) - S$, then the neighborhood of $w$ is not exactly: one vertex, two adjacent vertices, or three vertices that induce a path.

**Claim 35.1.** If $w, x \in V(G) - S$ such that $|N(w)| = 2$, then for any vertex $y$ in $N(x)$, $N(w) \cup \{y\}$ do not induce $P_3$.

**Proof.** For the sake of contradiction, and w.l.o.g., assume that there are vertices $w, x \in V(G) - S$ such that $N(w) = \{r, t\}$ and there is a vertex $y \in |N_S(x)|$, where $N(w) \cup \{y\}$ induces $P_3$. Consequently, $y = s$. In $G/sx$, the set $f(\{r, t, u, v, w\})$ induces $C_5$, which contradicts the fact that $G$ is a critically $C_5$-exist. □

**Claim 35.2.** If $w, x, y \in V(G) - S$ such that $|N(w)| = |N(x)| = 3$ where $N(w) \cup N(x) = S$, then $y$ is nonadjacent to any vertex in $S$ that is adjacent to $N(w) \cap N(x)$. 

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Proof. W.l.o.g., assume that there are vertices $w, x, y \in V(G) - S$ such that $N(w) = \{r, t, u\}$ and $N(w) \cup N(x) = S$. As a result, $N(x) = \{s, t, v\}$ or $\{s, u, v\}$. Because of symmetry, we assume that $N(x) = \{s, t, v\}$. For the sake of contradiction, we assume that $y$ is adjacent to $s$ (or $u$). However, in $G/sy$ (or $G/uy$), the set $f(\{r, t, v, w, x\})$ induces $C_5$ which contradicts the fact that $G$ is a critically $C_5$-exist. \(\square\)
Claim 35.3. If \( w, x, y \in V(G) - S \) such that \( |N(w)| = |N(x)| = 4 \), where \( N(w) \cap N(x) \) induces \( P_3 \), then \( y \) is nonadjacent to the leaves in the path induced by \( N(w) \cap N(x) \).

Proof. W.l.o.g., assume that there are vertices \( w, x, y \in V(G) - S \) such that \( N(w) = \{r, s, t, u\} \), while \( N(w) \cap N(x) \) induces \( P_3 \). As a result, \( N(x) = \{s, t, u, v\} \) or \( \{r, t, u, v\} \). Because of symmetry, we assume that \( N(x) = \{s, t, u, v\} \). For the sake of contradiction, we assume that \( y \) is adjacent to \( s \) (or \( u \)). However, in \( G/\text{sy} \) (or \( G/\text{uy} \)), the set \( f(\{r, t, v, w, x\}) \) induces \( C_5 \), which contradicts the fact that \( G \) is a critically \( C_5 \)-exist. \( \Box \)

Claim 35.4. If \( w, x \in V(G) - S \) such that \( |N(w)| = 3 \) and \( |N(x)| = 4 \) where neither of the leaves of the \( P_4 \) induced by \( N(x) \) are in \( N(w) \), then \( G \) is isomorphic to \( H_1 \).

Proof. W.l.o.g., assume vertices \( w, x \in V(G) - S \) such that \( N(w) = \{r, t, u\} \) and \( N(x) = \{s, t, u, v\} \). For the sake of contradiction, we assume that there is a vertex \( y \in V(G) - S \). If \( y \) is adjacent to \( s \) (or \( u \)), then in \( G/\text{sy} \) (or \( G/\text{uy} \)), the set \( f(\{r, t, v, w, x\}) \) induces \( C_5 \), which contradicts the fact that \( G \) is a critically \( C_5 \)-exist. In contrast, if \( y \) is adjacent to \( t \) (or \( v \)), then in \( G/\text{ty} \) (or \( G/\text{vy} \)), the set \( f(\{r, s, u, w, x\}) \) induces \( C_5 \), which contradicts the fact that \( G \) is a critically \( C_5 \)-exist. \( \Box \)

By Claims 35.1 to 35.4, we deduce that the possible critically \( C_5 \)-exist graphs are the ones presented in Figure 8. To complete the proof, we demonstrate that every graph in Figure 8 is a critically \( C_5 \)-exist.

Claim 35.5. There is no \( C_6 \)-exist graph in Figure 8.

Proof. Assume for the sake of contradiction, there is a graph \( G \) in in Figure 8 that is \( C_6 \)-exist. Moreover, \( T \subseteq V(G) \) where \( G[T] \) induces \( C_6 \). Let \( S \subseteq V(G) \) such that \( G[S] \) induces \( C_5 \). No more than two vertices in \( S \) can form an independent set. Hence \( T \) can contain at most two vertices from \( S \). Consequently, \( T \) has four vertices from \( V(G) - S \), however, two of such four vertices are adjacent, which contradicts the fact that \( V(G) - S \) is independent set. \( \Box \)

All graphs in Figure 8 are \( C_6 \)-free. Consequently, any contraction of a graph of Figure 8 has an induced \( C_5 \), then this \( C_5 \) must be an induced subgraph of the original graph too. Thus, it would be sufficient to prove that in every graph \( G \) in Figure 8, the edges of \( G \) are critical for every induced \( C_5 \) in \( G \).

Claim 35.6. The graph \( H_1 \) in Figure 8 is a critically \( C_5 \)-exist.

Proof. The graph \( H_1 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, uv, vr \in E(G) \). Moreover, \( V(G) = S \cup \{w, x\} \) such that \( N(w) = \{r, t, u\} \) and \( N(x) = \{s, t, u, v\} \). Clearly, the vertex subsets of \( V(G) \) that induce \( C_5 \) are \( S, \{r, s, u, w, x\} \), and \( \{r, t, v, w, x\} \). Furthermore, it is straightforward that every edge in \( E(G) \) is
\[ C_5 \text{-critical for the previous three vertex subsets. Thus, } G \text{ is a critically } C_5\text{-exist.} \]

\textbf{Claim 35.7.} The graph \( H_2 \) in Figure 8 is a critically \( C_5\)-exist.

\textbf{Proof.} The graph \( H_2 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, uw, vr \in E(G) \). Moreover, \( V(G) = S \cup \{w, x\} \cup Y \), such that \( N(w) = \{r, t, u\} \), \( N(x) = \{s, u, v\} \), \( N(y \in Y) = \{r, s, u\} \), and \( |Y| \geq 0 \).

We prove that \( G \) contains only two induced \( C_5 \). For any \( i, j \) where \( 1 \leq i < j \leq |Y| \), if both \( y_i \) and \( y_j \) are in a vertex subset that induces a cycle in \( G \), then this vertex subset would be \( \{r, u, y_i, y_j\} \) and \( \{s, u, y_i, y_j\} \). Thus, no induced \( C_5 \) in \( G \) contains both \( y_i \) and \( y_j \). Moreover, the vertex subsets that induce a cycle in \( G \) that contain \( y_i \) but not \( y_j \) are \( \{r, s, y_i\} \), \( \{r, u, v, y_i\} \), \( \{s, t, u, y_i\} \), and \( \{s, u, x, y_i\} \). Clearly, none of them induces \( C_5 \). Thus, no induced \( C_5 \) in \( G \) contain a vertex \( y_i \) for any \( i \leq l \). Furthermore, no four vertices from \( S \) with either \( w \) or \( x \) induce a \( C_5 \). Consequently, no vertex subset that induces \( C_5 \) in \( G \) contains either \( w \) or \( x \). As a result, a cycle \( C_5 \) is induced in \( G \) only by \( S \) or \( \{r, s, u, w, x\} \). We note that any edge in \( E(G) \) is a critical edge for all subsets that induce \( C_5 \). Thus, \( G \) is a critically \( C_5\)-exist. \( \Box \)

The proof of the following claim can be performed in a similar way as that one of Claim 35.7. We will just explain the structure of the graph \( H_3 \), the remaining part is left to the interested reader.

\textbf{Claim 35.8.} The graph \( H_3 \) in Figure 8 is a critically \( C_5\)-exist.

\textbf{Proof.} The graph \( H_3 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, uw, vr \in E(G) \). Moreover, \( V(G) = S \cup \{w, x\} \cup Y \), such that \( N(w) = \{r, s, t, u\} \), \( N(x) = \{s, t, u, v\} \), \( N(y \in Y) = \{r, t, v\} \), and \( |Y| \geq 0 \). \( \Box \)

\textbf{Claim 35.9.} The graph \( H_4 \) in Figure 8 is a critically \( C_5\)-exist.

\textbf{Proof.} The graph \( H_4 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, uw, vr \in E(G) \). Moreover, \( V(G) = S \cup W \cup X \cup Y \), such that \( N(w \in W) = \{r, t\} \), \( N(x \in X) = \{r, u\} \), \( N(y \in Y) = \{r, t, u\} \), and \( |W|, |X|, |Y| \geq 0 \).

For any \( k, p \) where \( 1 \leq k < p \leq |Y| \), if both \( y_k \) and \( y_p \) are in a vertex subset that induces cycle in \( G \), then this vertex subset would be \( \{r, u, y_k, y_p\} \) or \( \{r, t, y_k, y_p\} \). Thus, no induced \( C_5 \) in \( G \) contains both \( y_k \) and \( y_p \). Additionally, the induced cycles in \( G \) that contain \( y_k \) but not \( y_p \) (or vice versa) are of length less than five. Thus, no induced \( C_5 \) in \( G \) contains one vertex \( y_k \) for any \( k \leq n \). Indeed, a vertex subset in \( G \) that induces a \( C_5 \) is composed of \( \{r, t, u\} \), one vertex from \( W \cup \{s\} \), and one vertex from \( X \cup \{v\} \). We conclude that any edge in \( E(G) \) is a critical edge for all subsets that induce \( C_5 \). Thus, \( G \) is a critically \( C_5\)-exist. \( \Box \)

\textbf{Claim 35.10.} The graph \( H_5 \) in Figure 8 is a critically \( C_5\)-exist.
Proof. The graph \( H_5 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, wv, vr \in E(G) \). Moreover, \( V(G) = S \cup W \cup X \cup Y \cup Z \), such that \( N(w \in W) = \{r, t\}, N(x \in X) = \{r, t, u\}, N(y \in Y) = \{r, t, v\}, N(z \in Z) = \{r, t, u, v\} \), and \( |W|, |X|, |Y|, |Z| \geq 0 \).

It is clear that no vertex subset in \( G \) that induces \( C_5 \) contains two vertices from \( V(G) - S \). Moreover, we can prove that no vertex subset in \( G \) that induces \( C_5 \) contains one vertex from \( V(G) - (S \cup W) \). As a result, the vertex subsets that induce a \( C_5 \) in \( G \) are either \( S \) or \( \{r, t, u, v\} \) together with one vertex from \( \{w_1, w_2, \ldots, w_l\} \). Any edge in \( E(G) \) is a critical for all subsets that induce \( C_5 \). Thus, \( G \) is a critically \( C_5 \)-exist.

Being similar to the proof of Claim 35.10, the proof of Claim 35.11 is left for the interested reader; however, we explain the structure of \( H_6 \) in Figure 8 in the proof.

Claim 35.11. The graph \( H_6 \) in Figure 8 is a critically \( C_5 \)-exist.

Proof. The graph \( H_6 \) in Figure 8 is isomorphic to a graph \( G \) that contains a vertex subset \( S = \{r, s, t, u, v\} \) that induces \( C_5 \), where \( rs, st, tu, wv, vr \in E(G) \). Moreover, \( V(G) = S \cup W \cup X \cup Y \cup Z \cup A \), such \( N(w \in W) = \{r, t, u\}, N(x \in X) = \{r, t, v\}, N(y \in Y) = \{r, s, t, u\}, N(z \in Z) = \{r, t, u, v\} \), \( N(a \in A) = S \), and \( |W|, |X|, |Y|, |Z|, |A| \geq 0 \).

By Claims 35.5 to 35.11, the proof is complete. \( \square \)

By Theorem 4, Corollary 34, and Proposition 35, we obtain the following.

Theorem 36. Let \( G \) be a \( C_6 \)-free graph that is non-isomorphic to any graph in Figure 8. The graph \( G \) is \( C_5 \)-free if and only if any \( G \)-contraction is \( C_5 \)-free.

8 Split Graphs

Split graphs were introduced in [14] and were characterized as follows:

Theorem 37. [14] A graph \( G \) is split if and only if \( G\) is \( \{2K_2, C_4, C_5\}\)-free.

Thus, we call a graph that is \( \{2K_2, C_4, C_5\}\)-exist non-split graph. Additionally, split graphs have been characterized in [14] as chordal graphs whose complements are also chordal. Furthermore, it was characterized by its degree sequences in [16]. Moreover, further properties of split graphs are studied in [1, 17, 21].

By Theorems 4, 25, 32, and 36 and Propositions 10, 14, and 15, we obtain:

Theorem 38. Let \( G \) be a graph that is non-isomorphic to any graph in Figure 9. The graph \( G \) is split if and only if any \( G \)-contraction is split.

The class of split graphs is a closed class under edge contraction. The definition of a split graph implies that by contraction of an arbitrary edge in a split graph leads to another split graph. So the contribution of Theorem 38 is in listing the critically non-split graphs.
9 Pseudo-Split Graphs

In [2], the family of \{2K_2, C_4\}-free graphs was investigated and later referred to as pseudo-split graphs in [19]. Thus, we call a graph that is \{2K_2, C_4\}-exist non-pseudo-split graph.

By Theorems 4, 25 and 32 and Propositions 10, 14, and 15, we obtain:

**Theorem 39.** Let \( G \) be a \( C_5 \)-free graph that is non-isomorphic to any graph in Figure 10. The graph \( G \) is Pseudo-split if and only if any \( G \)-contraction is Pseudo-split.

10 Threshold Graphs

Threshold graphs were characterized as follows:
Theorem 40. [10] A given a graph $G$ is threshold if and only if $G$ is $\{2K_2, P_4, C_4\}$-free.

Thus, we call a graph that is $\{2K_2, P_4, C_4\}$-exist non-threshold graph.

By Theorems 4, 25, 28, and 32 and Propositions 10, 14, and 15, we obtain:

Theorem 41. Let $G$ be a $C_5$-free graph that is non-isomorphic to any graph in Figure 9. The graph $G$ is threshold if and only if any $G$-contraction is threshold.

![Critically non-threshold graphs](image)

Figure 11: The critically non-threshold graphs

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