Approximation by Lipschitz functions
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Abstract

On any metric space, I provide an intrinsic characterization for the uniform closure of the set of all complex-valued Lipschitz functions. There are applications to function theory on complete Riemannian manifolds and, in particular, on bounded symmetric domains (BSD) in $\mathbb{C}^n$.

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1. Introduction. On any metric space $(X, \beta(\cdot, \cdot))$, we say a complex-valued function $f$ is uniformly continuous if, for arbitrary real $\epsilon > 0$, and $x, y$ in $X$, there is a real $\delta = \delta(\epsilon) > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $\beta(x, y) < \delta$. The set of all uniformly continuous functions on $(X, \beta)$ is denoted by $UC(X)$. The Lipschitz functions $Lip(X)$ are the subset of $UC(X)$ with the property that, for all $x, y$ in $X$ and $f$ in $Lip(X)$, $|f(x) - f(y)| \leq C\beta(x, y)$, for some positive constant $C = C(f)$.

We will be concerned with an intermediate set of functions, the uniform closure of $Lip(X)$, which I denote by $Lip_c(X)$. I will show that $Lip_c(X)$ consists precisely of those functions $f$ for which, given any $\epsilon > 0$, there is a constant $C = C(\epsilon)$ so that

\[(*) \quad |f(x) - f(y)| < \epsilon + C(\epsilon)\beta(x, y),\]

for all $x, y$ in $X$.

As an application of this result, I give a concise proof of the known equivalence $UC(X) \equiv Lip_c(X)$ for the special case of complete (connected) Riemannian manifolds $X$, with metric the usual Riemannian distance function induced by the infinitesimal Riemannian metric. The prototypical complete Riemannian manifold is real $n$-dimensional space $\mathbb{R}^n$ and for $x, y$ in $\mathbb{R}^n$, we have the usual norm $|x|$ and the Riemannian distance function is just $\beta(x, y) = |x - y|$. This application holds, in particular, for all bounded symmetric domains (BSD) $\Omega$ in $\mathbb{C}^n$. In this case, stronger results (with a more complicated proof) are known [1]: the real-analytic Lipschitz functions are uniformly dense in $UC(\Omega)$.  

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For a definitive, fairly recent, treatment of approximation by Lipschitz functions, see [6]. The result characterizing $\text{Lip}_c(X)$ does not seem to be in the literature. While evidently not as useful as the notion of "Lipschitz in the small," it still seems to be worth some attention.

2. A characterization of $\text{Lip}_c(X)$. For $(X, \beta(\cdot, \cdot))$ any metric space, we recall the extension result due to E. J. McShane [8] for real-valued Lipschitz functions:

**Proposition 1.** For any non-empty subset $S$ of $X$ and any real-valued function $f$ in $\text{Lip}(S)$, there is a real-valued function $F$ in $\text{Lip}(X)$ with $F|_S = f$ and with $F$ having the same Lipschitz constant as $f$.

**Proof.** Suppose that $|f(s) - f(t)| \leq C\beta(s, t)$ for all $s, t$ in $S$. For any $x$ in $X$ we define

$$F(x) = \inf \{ f(s) + C\beta(x, s) : s \in S \}.$$  

To see that $F(x)$ is finite for every $x$ in $X$, fix $s_0$ in $S$. Then we can check that for any $s$ in $S$

$$f(s) + C\beta(s, x) \geq f(s_0) - C\beta(s, s_0) + C\beta(s, x) \geq f(s_0) - C\beta(x, s_0).$$

For $x$ in $S$, $F(x) \leq f(x)$. But, for all $s$ in $S$, $f(x) \leq f(s) + C\beta(x, s)$ so $f(x) \leq F(x)$.

Finally, we check that $F$ is Lipschitz on $X$. For $x, y$ in $X$, note that

$$F(x) = \inf \{ f(s) + C\beta(s, x) : s \in S \}$$

$$\leq \inf \{ f(s) + C\beta(s, y) + C\beta(y, x) : s \in S \}$$

$$\leq \inf \{ f(s) + C\beta(s, y) ; s \in S \} + C\beta(y, x)$$

$$\leq F(y) + C\beta(x, y)$$

so $|F(x) - F(y)| \leq C\beta(x, y)$. 

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Corollary. For $S$ any non-empty subset of $X$ and $f$ any complex-valued function in $Lip(S)$ with $|f(x) - f(y)| \leq C \beta(x, y)$, there is a complex-valued function in $Lip(X)$ with $F|_S = f$ and $|F(x) - F(y)| \leq 2C \beta(x, y)$ for all $x, y$ in $X$.

Proof. We first check that the real and imaginary parts of $f$ are in $Lip(S)$ with the same Lipschitz constant $C$ as $f$. By Proposition 1, there are real-valued $U, V$ in $Lip(X)$ with the same Lipschitz constant $C$ and such that $U|_S = Re(f), V|_S = Im(f)$. Taking $F = U + iV$ gives the desired result.

Let $t_0 = \sup \{ \beta(x, y) : x, y \in X \}$. If $\beta$ is unbounded, take $t_0 = \infty$.

Assume that $t_0 > 0$. I can now prove the main result.

Theorem 1. On any metric space $(X, \beta)$, a complex-valued function $f$ is in $Lip_c(X)$ if and only if for every $\epsilon > 0$, there is a $C = C(\epsilon) > 0$ so that

\[(*) \quad |f(x) - f(y)| < \epsilon + C(\epsilon) \beta(x, y)\]

for all $x, y$ in $X$.

Proof. If $g$ is Lipschitz, with $|f(x) - g(x)| < \epsilon/2$ for all $x$ in $X$, then

\[|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)|\]

\[< \epsilon + C \beta(x, y),\]

where $C$ is a Lipschitz constant for $g$.

For the converse, suppose $f$ satisfies $(*)$ for every $\epsilon > 0$. Without loss of generality, we may choose $C(\epsilon)$ in $(*)$ with $C(\epsilon) > \epsilon/t_0$ so there are $x_1, x_2 \in X$ with $\beta(x_1, x_2) \geq \epsilon/C(\epsilon)$. By Zorn’s Lemma, with $t = \epsilon/C(\epsilon)$ there is a maximal $t$-separated subset of $X$, $S = S_\epsilon$, which contains $x_1, x_2$. For any $x, y$ in $S$ with $x \neq y$, we have $\beta(x, y) \geq t$ and by $(*)$,

\[
\frac{|f(x) - f(y)|}{\beta(x, y)} < \frac{\epsilon}{\beta(x, y)} + C(\epsilon)
\leq \epsilon t^{-1} + C(\epsilon)
\leq 2C(\epsilon),
\]

so $f|_S$ is Lipschitz with Lipschitz constant $2C(\epsilon)$. 3
By the Corollary to Proposition 1, \( f|_S \) extends to a function \( F \) which is in \( \text{Lip}(X) \) and has Lipschitz constant \( 4C(\epsilon) \). For any \( x \) in \( X \), by maximality of \( S \), we may choose a \( y \) in \( S \) with \( \beta(x,y) < t \). Now \( F(y) = f(y) \) so

\[
|F(x) - f(x)| \leq |F(x) - F(y)| + |f(y) - f(x)| \\
\leq 4C(\epsilon)\beta(x,y) + C(\epsilon)\beta(x,y) + \epsilon \\
< 6\epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, the proof is complete.

**Corollary.** \( \text{Lip}_c(X) \subset UC(X) \).

**Proof.** Trivial.

**Remarks.** The proof of Theorem 1 benefitted from a reading of [4, Proposition 2.1]. Functions in \( \text{Lip}_c(X) \) can grow no faster than \( f(x) = \beta(a,x) \) for any fixed \( a \) in \( X \). In the next sections, I discuss some known examples where \( \text{Lip}_c(X) = UC(X) \).

### 3. Complete Riemannian manifolds.

I give a concise proof, using Theorem 1, of a known result [6, pp.286,289]. The key property of metric distance functions of complete (connected) Riemannian manifolds used here is:

geodesic completeness–between every two points \( a, b \) there is a geodesic arc \( \gamma \) of length \( \beta(a,b) \).

**Proposition 2.** For any complete (connected) Riemannian manifold \( (X, \beta) \), \( \text{Lip}_c(X) = UC(X) \).

**Proof.** For any \( a \neq b \) in \( X \), there is a geodesic segment \( \gamma \) of length \( \beta(a,b) \) joining \( a \) to \( b \). For \( f \) in \( UC(X) \) and \( \epsilon, \delta(\epsilon) \) as in the definition of uniform continuity above, let \( N \) be the integer such that

\[
N \leq \beta(a,b)\delta(\epsilon)^{-1} < N + 1.
\]

Divide \( \gamma \) into \( N + 1 \) equal-length segments, each of length less than \( \delta(\epsilon) \).
The triangle inequality then shows that

\[
|f(a) - f(b)| < (N + 1)\epsilon \\
\leq \beta(a, b)\delta(\epsilon)^{-1}\epsilon + \epsilon \\
\leq C(\epsilon)\beta(a, b) + \epsilon,
\]

where \(C(\epsilon) = \epsilon\delta(\epsilon)^{-1}\). Thus, (*) holds.

**Remark.** This idea was used in [1, Lemma 2.1] when we considered the special case of bounded symmetric domains (BSD) \(\Omega\) and obtained real-analytic Lipschitz approximants for all functions in \(UC(\Omega)\).

### 4. Bounded symmetric domains

The bounded symmetric domains (BSD) in complex n-space \(C^n\) play a significant role in geometry and in representation theory [7]. These domains are all bounded open convex sets in \(C^n\) which carry intrinsic complete Riemannian (Bergman) metrics. The prototype is just the hyperbolic metric on the open disc. Using the results in [1], boundedness and compactness were determined for Toeplitz operators with uniformly continuous symbols on BSD’s in [2].

There are two quite different natural metrics on BSD \(\Omega\): the Bergman metric, with distance function \(\beta(\cdot, \cdot)\) and the restricted Euclidean metric from \(C^n\). The two different corresponding notions of uniform continuity are related by the fact [5, p. 1167] that \(|x - y| \leq C_\Omega \beta(x, y)\) so that \(UC(\Omega)_{|\cdot|} \subset UC(\Omega)_\beta\). This provides a useful source of bounded functions in \(UC(\Omega)_\beta\), which also includes unbounded functions like \(f(z) = \beta(a, z)\) for any fixed \(a\) in \(\Omega\). It follows easily from equation (*) of Theorem 1 that functions in \(UC(\Omega)_\beta\) grow no faster than \(\beta(a, z)\). Finally, we observe that [3, Theorem E] \(\beta(a, z)\) is of slow growth near the boundary of \(\Omega\): it is in \(L^p(\Omega, dv)\) for all \(p > 0\).
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