Correlation functions on a curved background

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We investigate gravitational correlation functions in a curved background with the help of non-perturbative renormalization group methods. Beta functions for eleven couplings are derived, two of which correspond to running gauge parameters. A unique ultraviolet fixed point is found, suitable for a UV completion in the sense of Asymptotic Safety. To arrive at a well-behaved flow in a curved background, the regularization must be chosen carefully. We provide two admissible choices to solve this issue in the present approximation. We further demonstrate by an explicit calculation that the Landau limit is a fixed point also for quantum gravity, and additionally show that in this limit, the gauge parameter $\beta$ does not flow.

I. INTRODUCTION

For several decades, Einstein’s general relativity has successfully precluded any attempt of quantization. At the heart of the problem lies the negative mass dimension of Newton’s constant, which implies the failure of standard perturbation theory. An alternative route was suggested by Weinberg [1]. He proposed that gravity might be interacting in the far ultraviolet, but controlled by a fixed point of its renormalization group (RG) flow. Such an interacting, or nontrivial fixed point is termed asymptotically safe, in contrast to an asymptotically free fixed point, where the couplings vanish.

With the advent of modern functional RG equations [2–4], the Asymptotic Safety scenario received growing attention. Starting with the seminal paper [5], in which the nonperturbative beta functions for Newton’s constant and the cosmological constant were derived for the first time, approximations were successively improved. This includes aspects of the Einstein-Hilbert approximation [6–25], higher derivative terms [26–34], $f(R)$ gravity [35–52], the resolution of the fate of the perturbative counterterm [53], the quantization of ADM variables [54–56], the inclusion of torsion and nonmetricity [57] and progress on unitarity [58]. Recently, there is growing interest in the study of gravity-matter systems [55, 59–80]. All studies come to the conclusion that there is indeed a suitable fixed point which facilitates an ultraviolet (UV) completion of gravity. Phenomenologically, black holes [81–86], cosmological aspects [87–97], the Unruh effect [98], the C-function [99] and the dispersion of different modes [54] have been investigated. Assuming that there is an asymptotically safe fixed point, a precise prediction of the Higgs mass was made in [100] before its measurement at the LHC [101, 102].

For technical reasons, the background field method is indispensable in these calculations. In this, the metric is split into a background and a (not necessarily small) fluctuation field. It was soon realized that the disentanglement of these two quantities is central to obtain reliable results. In [63, 103] it was shown that an improper treatment of this difference can alter universal one-loop beta functions, and even destroy asymptotic freedom in Yang-Mills theory. Similarly, the well-known Wilson-Fisher fixed point can disappear [104]. To solve this problem, one has to deal either with two fields [34, 72, 73, 75, 94, 105–113], or solve the corresponding split Ward identities [51, 104, 114–121]. Closely related are geometric quantization schemes [43, 116, 122, 123].

In the study of fluctuation correlation functions, so far the analysis was restricted to a flat background. This bears technical advantages, e.g., the full momentum dependence can be resolved [109]. However, it is the functional dependence of the effective action on the background field which is necessary for the calculation of observables [124]. Hence, the introduction of a generic background is unavoidable and the study of correlation functions including the background curvature is important. As a first step, we resolve the curvature dependence of the graviton propagator to linear order in the curvature within a derivative expansion. This is a further step in the systematic exploration of correlation functions in quantum gravity, enabling us to assess in what way quantum effects in the UV shift the dependence of the propagator on the curvature compared to the classical expectation from the Einstein-Hilbert action.

We show within our truncation that the Landau gauge is a fixed point of the RG flow of both gauge parameters. This may seem to be a trivial statement, but it turns out that a careful choice of the regulator is necessary to obtain a finite right-hand side of the flow equation [6]. It is in general difficult to find a valid choice on a curved background geometry, since functions of the Laplacean don’t commute with covariant derivatives. We provide two different admissible regulators to linear order in the background curvature. One of the choices does not involve the gauge parameter $\alpha$, and thus is technically superior as it allows to take the Landau limit at the level.

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of the propagator. Furthermore, it is straightforward to extend this regulator to higher order truncations. The other regulator is the curved version of the regulator as employed in [34, 108, 109, 112, 113]. For fluctuation flows on flat background in Landau limit, both regulators give the same result. To judge the quality of our approximation, we study the beta functions for general gauge parameters, similar to our earlier work within the background field approach [6].

In Landau gauge, we find one fixed point which still depends somewhat on the remaining gauge parameter $\beta$. For some choices, this fixed point is UV-repulsive, where the leading critical exponents have large imaginary parts, which we take as a hint that the inclusion of higher order correlation functions is necessary to ultimately fix the critical quantities.

This work is structured as follows: in section II, we give the basic notions of our RG setup, together with the employed truncation in subsection II A, the regularization in subsection II B and the projection scheme, together with a discussion of the Landau gauge in subsection II C. We go on with the discussion of the results in section III, where we first consider the 0th order curvature couplings in subsection III A, then the 1st order curvature couplings in subsection III B. We end with a conclusion in section IV. The appendices collect some technical information. In appendix A, we give a basis for a set of correlation functions, whereas in appendix B, we give some helpful relations concerning the propagator functions for a symmetric spin 2 field. In appendix C, we give explicit fixed point values for the two different regulators.

II. NONPERTURBATIVE CORRELATION FUNCTIONS IN QUANTUM GRAVITY

A theory is completely fixed if a complete set of correlation functions is given, as any observable can be constructed from these basic building blocks. These correlation functions are generated by the effective action at vanishing fluctuation field. To study the effective action nonperturbatively, we use the formulation of the functional RG by Wetterich [2]. For this, a fiducial scale $k$ is introduced, and momentum shells are integrated out at this scale successively in a Wilsonian sense. The $k$-dependent, so-called effective average action, $\Gamma$, fulfills the RG equation

$$\dot{\Gamma} \equiv k \partial_k \Gamma = \frac{1}{2} \text{STr} \left[ (\Gamma^{(2)} + \mathcal{R} )^{-1} k \partial_k \mathcal{R} \right].$$

In this equation, $\text{STr}$ indicates a supertrace, which includes summation over discrete and integration over continuous indices as well as a minus sign for Grassmann-valued fields, and $\mathcal{R}$ is a regulator, which effectively behaves like a momentum-dependent mass term. Reviews of the functional RG in gravity can be found in [14, 18, 115, 125–127].

A. Truncation

In the following we present our truncation scheme to solve (1). As a starting point, we take the Einstein-Hilbert action,

$$S_{\text{cl}} = \frac{1}{16\pi G_N} \int \sqrt{\bar{g}} (-R + 2\Lambda),$$

where $G_N$ is the classical Newton’s constant, $\Lambda$ the cosmological constant and $R$ the Ricci scalar of the metric $g$. We implement the background field method by a linear split,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}.$$  

Other choices are also possible, see e.g. [6, 23, 24, 32, 47, 48, 50, 70, 73, 74, 120, 128, 129]. Our goal is to resolve the flow of the two-point-correlator $\Gamma^{(2)}$, including up to two derivatives or one curvature, and parts of the three-graviton vertex $\Gamma^{(3)}$ similar to [112]. Let us start by parameterizing the inverse propagator. For this, we amend the quadratic part of the classical action with a gauge fixing and further couplings, which allows us to go beyond the background field approximation. The constant part can be spanned by two gaps, $\Lambda_{TL}$ and $\Lambda_T$, corresponding to the traceless and the trace sector of the fluctuation, respectively. To linear order in the background curvature, five independent tensor structures appear, and we supply each of them with a unique coupling $Z_\alpha$. These couplings are introduced in such a way that their classical value is zero. Finally, we introduce a uniform wave function renormalization $Z_\chi$.

The gauge fixing is given by

$$F_\mu = \left( \bar{g}^{(2)}(D) - \frac{1 + \beta}{4} \bar{g}^{(4)}(D) \right) g_{\alpha\beta}.$$

Here, $D$ denotes the covariant derivative with respect to the background metric $\bar{g}$, whereas $\bar{D}$ in the following corresponds to the covariant derivative with respect to the full metric $g$. The gauge fixing is implemented in a standard way by the Faddeev-Popov construction,

$$\Gamma_{g_\ell} = \frac{1}{16\pi G_N} \int \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu,$$

which gives also rise to the ghost action

$$\Gamma_{g_\chi} = -\int \sqrt{\bar{g}} \bar{g}_{\mu} \left[ 2 \bar{g}^{\mu(\alpha} D^{\beta)} - \frac{1 + \beta}{2} \bar{g}^{\alpha\beta} D^2 \right] D_\alpha c_\beta.$$

Our ansatz for the quadratic part of the effective action thus amounts to
\[ \Gamma_{\text{quad}} = \frac{Z_h}{64\pi} \int \sqrt{g} \, h_{\mu\nu} \left[ \Pi_{\text{TL},\rho\sigma}^{\mu\nu} \left( \Delta + \frac{3}{2} (1 + 3R_{\text{RTL}}) \widehat{R} - 2\Lambda_{\text{TL}} \right) + 2 (R_C - 1) \bar{C}^\mu \_\nu^\sigma + 2 R_{\text{STL}} \Pi_{\text{TL},\alpha\beta}^{\mu\nu} \bar{S}_{\alpha}^{\mu} \Pi_{\text{TL},\rho\sigma}^{\beta \gamma} \right. \\
+ \frac{2(\alpha - 1)}{\alpha} \Pi_{\text{TL},\alpha\beta}^{\mu\nu} \bar{D}_\alpha \bar{D}_\beta \Pi_{\text{TL},\rho\sigma}^{\gamma \delta} + \frac{\beta - \alpha}{\alpha} g^{\mu\nu} \bar{D}_\alpha \bar{D}_\beta \Pi_{\text{TL},\rho\sigma}^{\gamma \delta} + 4 R_{\text{STL}} g^{\mu\nu} \bar{S}_{\alpha\beta} \Pi_{\text{TL},\rho\sigma}^{\gamma \delta} \left. \right] h_{\rho\sigma} \] 

where we introduced the background Laplacian \( \Delta = -D^\mu D_\mu \). Everything is spanned in a traceless decomposition. In particular, we use the Weyl tensor and the Ricci tensor, 

\[ R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + g_{\mu(\rho} S_{\sigma)\nu} + g_{\nu(\rho} S_{\sigma)\mu} + \frac{1}{6} R g_{\mu(\rho} g_{\sigma)\nu} , \] 

and the Ricci tensor,

\[ R_{\mu\nu} = S_{\mu\nu} + \frac{1}{4} R g_{\mu\nu} . \]

The projectors \( \Pi_{\text{TL}} \) and \( \Pi_{\text{Tr}} \) are defined as

\[ \Pi_{\text{TL}}^{\mu\nu\rho\sigma} = 1^{\mu\nu\rho\sigma} - \Pi_{\text{Tr}}^{\mu\nu\rho\sigma} , \]

\[ \Pi_{\text{Tr}}^{\mu\nu\rho\sigma} = \frac{1}{4} g^{\mu\nu} g_{\rho\sigma} , \]

\[ 1^{\mu\nu\rho\sigma} = \frac{1}{2} \left( \delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\nu \delta_\sigma^\mu \right) . \]

It is useful to introduce a redefined gap parameter instead of \( \Lambda_{\text{Tr}} \), which accounts for the fact that the two (off-shell) scalar degrees of freedom of the graviton mix, depending on the gauge fixing. Defining

\[ \bar{\Lambda} = \frac{6\Lambda_{\text{Tr}} - 2\beta^2 \Lambda_{\text{TL}}}{(\beta - 3)^2} , \]

all propagators in the Landau limit have the denominator structure \( \Delta - 2\Lambda_2 \), with \( \Lambda_2 \) either being \( \Lambda_{\text{TL}} \) or \( \bar{\Lambda} \). In the results below one can see that this parameterization is reasonable since fixed point values of these quantities change only mildly under the variation of the gauge parameter \( \beta \), whereas \( \Lambda_{\text{Tr}} \) shows a strong gauge dependence.

For the higher order correlation functions, we use the classical tensor structure as a model, in complete analogy to \([112]\). In particular, this entails

\[ \Gamma_{\text{cub}} = Z_h^{3/2} G_3^{1/2} G_N S_{\text{cub}} \bigg|_{\Lambda_{\text{Tr}} = \Lambda_3} , \]

\[ \Gamma_{\text{quat}} = Z_h^2 G_4 G_N S_{\text{cub}} \bigg|_{\Lambda_{\text{Tr}} = \Lambda_4} , \]

\[ \Gamma_{\text{quint}} = Z_h^{5/2} G_5^{3/2} G_N S_{\text{cub}} \bigg|_{\Lambda_{\text{Tr}} = \Lambda_5} , \]

for the terms cubic, quartic and quintic in the fluctuation field \( h \), respectively. The \( G_n \) uniformly parameterize the interaction strength of the \( n \)-graviton vertex, while the \( \Lambda_n \) characterize their constant parts. In order to close the flow equation, we identify the couplings of the four- and five-point correlators with those of the three-point correlator, i.e.

\[ G_5 = G_4 = G_3 , \quad \Lambda_5 = \Lambda_4 = \Lambda_3 . \]

### B. Regularization

On a general background, it is nontrivial to find a regulator such that also the curvature part of the flow stays finite, since there is a subtle interplay between the placing of the shape functions and the operators to be regularized. As a validity criterion for a given regulator, we demand that the Landau limit of the regularized propagator exists, since this is also the case for the unregularized propagator, and seems to be a natural requirement.

In appendix B we derive the general propagator of a symmetric spin 2 field on a flat background. Taking this as a starting point, we observe that for small momenta \( (i.e. \text{ small eigenvalues of the Laplacian}) \) only the prefactors of \( \Pi_{\text{TL}} \) and \( \Pi_{\text{Tr}} \) appear in denominators, and at least they need regularization. This inspires the following choice for the regulator:

\[ \Delta S_h = \frac{Z_h}{64\pi} \int \sqrt{g} \, h_{\mu\nu} \left[ \Pi_{\text{TL},\rho\sigma}^{\mu\nu} - \frac{5 + \beta(\beta - 2)}{2} \Pi_{\text{Tr},\rho\sigma}^{\mu\nu} \right] R(\Delta) h^{\rho\sigma} . \]

The argument is similar for the ghost field, which is a vector: only the prefactor of the identity appears in denominators, thus we can regularize

\[ \Delta S_e = \int \sqrt{g} \, \bar{c}_\mu R(\Delta) e^\mu . \]

This regulator is well-behaved in the Landau limit by construction, and the prefactors are arranged such that in this limit, the denominators have the canonical form \( \Delta + R(\Delta) = 2\Lambda_2 \) for the graviton, and \( \Delta + R(\Delta) \) for the ghost. As a downside, it doesn’t regularize the pure gauge modes at all. This poses no obstruction for the calculation of the flow of the fluctuation couplings. As it turns out, their contribution to the flow of fluctuation couplings drops out in the Landau gauge even for more general regulators which regularize all modes (see below). Note however that this is not true for the flow of the
background couplings. There the above regulator is ill-behaving since it destroys the on-shell cancellation of the Faddeev-Popov ghosts and the gauge modes, i.e. one expects for the background cosmological constant a positive contribution from five transverse traceless (TT) modes, three transverse vector modes and two scalar modes, and a negative contribution from the eight ghost modes, leading to two physical modes for the graviton. However, this cancellation in the background sector is broken for the above regulator. On the other hand, for fluctuation flows, this regulator has several virtues. First of all, it doesn’t involve the gauge parameter \( \alpha \), thus it allows for technical simplifications in the Landau limit, which can be taken already after the propagator has been calculated. Secondly, this regulator can be trivially extended to higher derivative theories.

A second possibility to regularize the propagator is to generalize the regulator of \([34, 108, 109, 112, 113]\) to curved space. For this to give well-defined flows, the ordering of the derivatives and the shape function is crucial. To see this, first note that \([f(\Delta), \bar{D}_\mu] \neq 0\), since we are considering a curved space. Second, there is a subtle interplay of taking the Landau limit and the propagator becoming degenerate in the scalar sector of the TT-decomposition. These two properties can lead to divergences, cf. equation (25) of \([6]\). It turns out that the following ordering is well-defined, at least to linear order in the background curvature that we consider here:

\[
\Delta S_c = -\int \sqrt{\tilde{g}} \tilde{c}_\mu \left[ \delta^\mu_\rho \bar{D}_\rho \tau(\Delta) \bar{D}_\mu + \frac{1 - \frac{\beta}{2}}{2} \left( \bar{g}^{\rho\sigma} \bar{D}(\rho) \tau(\Delta) \bar{D}_\sigma + \bar{g}^{\rho\sigma} \bar{D}(\rho) \tau(\Delta) \bar{D}_\sigma \right) \right] c^\nu.
\]

In this, we substituted \( \Theta(\Delta) \) by \( \Delta \tau(\Delta) \). In a similar way, for the ghosts we choose

\[
\Delta S_c = -\int \sqrt{\tilde{g}} \tilde{c}_\mu \left[ \delta^\mu_\rho \bar{D}_\rho \tau(\Delta) \bar{D}_\mu + \frac{1 - \frac{\beta}{2}}{2} \left( \bar{g}^{\rho\sigma} \bar{D}(\rho) \tau(\Delta) \bar{D}_\sigma + \bar{g}^{\rho\sigma} \bar{D}(\rho) \tau(\Delta) \bar{D}_\sigma \right) \right] c^\nu.
\]

It is this regulator that we use in presenting numerical results. Still, as already stressed earlier, for the fluctuation flows on flat background, both regulators give the same result for the graviton contribution to the flow, which is a highly nontrivial result. Although small, a natural difference appears due to the different regularizations of the ghost modes. For the 1st order curvature couplings, also only minor differences in the flow induced by the gravitons arise. For the actual shape function, we take the Litim regulator \([130]\),

\[
\tau(\Delta) = \left( \frac{k^2}{\Delta} - 1 \right) \theta \left( 1 - \frac{\Delta}{k^2} \right),
\]

where \( \theta \) is the Heaviside theta function.

### C. Projection of flow equations and Landau limit

We now give some information on the projection scheme to extract beta functions, and comment on the Landau gauge. For the two-point function, all terms appearing on the right-hand side are sorted such that they have a similar form as (7). From this we immediately get the flows of the first order curvature couplings and the gaps. We can also extract the flow of the gauge parameters in this way, but it turns out that in the Landau limit no nontrivial contribution from their flow enters the flow equation of any other coupling. To make this precise, let us introduce the anomalous dimension

\[
\eta = -k \partial_k \ln Z_h,
\]

and “gauge anomalous dimensions”, cf. equation (3.18) of \([131]\),

\[
\eta_\xi = -k \partial_k \ln Z_\xi, \quad \eta_\xi = \frac{Z_h}{\alpha}, \quad \eta_\sigma = -k \partial_k \ln Z_\sigma, \quad Z_\sigma = \frac{Z_h (3 - \beta)^2}{16 \alpha}.
\]

By direct evaluation, we get for the left-hand side of the flow equation,

\[
k \partial_k Z_\xi = -\frac{Z_h}{\alpha} \eta - \frac{Z_h}{\alpha^2} \dot{\alpha},
\]

\[
k \partial_k Z_\sigma = -\frac{(3 - \beta)^2 Z_h}{16 \alpha} \eta - \frac{(3 - \beta)^2 Z_h}{16 \alpha^2} \dot{\alpha} - \frac{(3 - \beta) Z_h}{8 \alpha} \dot{\beta}.
\]

On the other hand, the right-hand side of the flow equation evaluates to

\[
k \partial_k Z_\xi = C_1 \dot{\alpha} + C_2 \dot{\beta} + C_3,
\]

\[
k \partial_k Z_\sigma = C_4 \dot{\alpha} + C_5 \dot{\beta} + C_6,
\]

where the \( C_i \) are functions of the couplings and gauge parameters. It is important to note that these stay finite in the Landau limit only if a well-behaved regulator is chosen. This is the case for both (14) and (16), within our truncation.

Now we can equate both sides of the flow equation, i.e.
we take (22) and subtract from this (21), to get
\[ 0 = \left( \frac{Z_h}{\alpha^2} + C_1 \right) \dot{\alpha} + \frac{Z_h}{\alpha} \eta + C_2 \dot{\beta} + C_3, \]
\[ 0 = \left( \frac{(3-\beta)^2 Z_h}{16\alpha^2} + C_4 \right) \dot{\alpha} + \frac{(3-\beta)^2 Z_h}{16\alpha} \eta \]
\[ + \left( \frac{(3-\beta) Z_h}{8\alpha} + C_5 \right) \dot{\beta} + C_6. \]  
(23)

These equations can now be easily solved for \( \dot{\alpha} \) and \( \dot{\beta} \), but already by inspection of the leading \( \alpha \to 0 \) divergences we find
\[ \dot{\alpha} = -\alpha \eta + \mathcal{O}(\alpha^2), \]
\[ \dot{\beta} = -\frac{16 C_6 - C_3 (3-\beta)^2}{2 Z_h (3-\beta)} + \mathcal{O}(\alpha^2). \]  
(24)

Hence, we immediately infer that the Landau gauge is a fixed point for both gauge parameters with arbitrary \( \beta < 3 \). Equally well we can formulate this in terms of the gauge anomalous dimensions, \( \eta_\xi \) and \( \eta_\sigma \),
\[ \eta_\xi = \frac{\dot{\alpha}}{\alpha} + \eta = \mathcal{O}(\alpha), \quad \eta_\sigma = \frac{\dot{\beta}}{3-\beta} + \frac{3}{2} \eta_\xi = \mathcal{O}(\alpha). \]  
(25)

These results suggest that the gauge modes should not be rescaled by the wave function renormalization \( Z_h \). This removes the appearances of \( Z_h \) and \( \eta \) in the above equations, and implies that both gauge parameters are exactly marginal, since the stability matrix evaluated in the Landau limit has zero eigenvalues,
\[ \left( \begin{array}{cc}
\frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial \beta} \\
\frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial \beta}
\end{array} \right) = \lim_{\alpha \to 0} \frac{3-\beta}{2\alpha} \begin{pmatrix} 0 & 0 \\ \eta_\sigma - \eta_\xi & 0 \end{pmatrix} + \mathcal{O}(\alpha). \]  
(26)

The limit is finite since both gauge anomalous dimensions are of \( \mathcal{O}(\alpha) \), see (25).

In the remainder of this work, we approximate \( \eta = 0 \), which was shown to be a very good effective approximation of the fully momentum-dependent anomalous dimension \[109].

Finally, we have to specify the projection scheme for the couplings \( G_3 \) and \( \Lambda_3 \). A full characterization of the three-point function seems difficult at present due to the high number of different operators, see appendix A. To enable checks with previous works, we choose the projection as in \[112]. In the language of appendix A, this amounts to projecting on the following linear combinations:
\[ \tilde{G}_3 \sim -\frac{2}{3} c_{17} + \frac{1}{21} c_{13} - \frac{6}{21} c_{12}, \quad \tilde{\Lambda}_3 \sim c_3. \]  
(27)

Note that the corresponding operators are exactly the ones containing the TT-mode of the fluctuation only. We verified that we get the same flow equations for \( G_3 \) and \( \Lambda_3 \) as given in \[112] if we choose \( \beta = 1 \) and identify \( \Lambda_{TL} = \tilde{\Lambda}_{TL} = \Lambda_2 \). The Landau limit flow equations for all couplings are given in the supplemented notebook. To derive the flow equations, we used the Mathematica suite \textit{xAct} \[132–137\], and to calculate the traces we used covariant heat kernel techniques \[29, 138–141\].

### III. FIXED POINT ANALYSIS

We can now discuss the fixed point structure of our system. For this, we introduce dimensionless couplings in the following way:
\[ g = G_3 k^2, \quad \lambda_3 = \Lambda_3 / k^2, \quad \lambda_{TL} = \tilde{\Lambda}_{TL} / k^2, \quad \tilde{\lambda} = \tilde{\Lambda} / k^2. \]  
(28)

The 1st order curvature couplings are already dimensionless.

First we discuss the flow of the couplings which also exist on a flat background, \((g, \lambda_{TL}, \tilde{\lambda}, \lambda_3)\), in the Landau limit, for arbitrary \( \beta \). If these couplings don’t show a fixed point, the full system cannot show it, since the flow of these couplings by construction doesn’t depend on the 1st order curvature couplings. Afterwards, we discuss the fate of the latter. For definiteness, we only discuss the results obtained with the regulator (16). The results for the other regulator (14) are quantitatively very similar.

#### A. 0th order curvature couplings

It turns out that we find a single fixed point which is rather stable under variation of the gauge parameter \( \beta \). The fixed point values of the 0th order curvature couplings in dependence on \( \beta \) are shown in Figure 1. It can be seen that all couplings depend mildly on \( \beta \).
particular, both gaps behave very similarly, and are effectively only shifted by a constant. Figure 2 shows the critical exponents of these couplings, being minus the eigenvalues of the stability matrix. We generically find one complex conjugate pair, and two real repulsive exponents. The complex pair corresponds to relevant operators for $\beta \gtrsim -0.98$, and to irrelevant operators for values of $\beta$ less than this. They have their main direction along the $(g, \lambda_{TL})$-plane. One should however note that in any case, the imaginary part dominates and the absolute value seems to be rather large, indicating that further operators might be necessary to pin down the relevance of these operators. The generically irrelevant operators are $\lambda_3$ and $\bar{\lambda}$, where the latter is more strongly irrelevant.

Typically, for a given value of $\beta$, other fixed points exist in the physical regime. As an example, for both choices $\beta = 1$ and $\beta = -1$, we find a fixed point with two or one relevant direction, respectively. Nevertheless, changing $\beta$ reveals that these fixed points depend strongly on the gauge. This emphasizes the need to check gauge dependence if one wants to reliably select a suitable fixed point for the UV completion of quantum gravity in an RG setup.

Let us also note that in the limit $\beta \to -\infty$, which was preferred in the background field approximation [6] due to its weak gauge dependence, we don’t find a physically interesting fixed point. This means that all fixed points that are found have either a negative $g$ or are behind the singularities at $\lambda_{TL} = 1/2$ and $\lambda_{TL} = -1/4$. The latter pole comes from the fact that the gap of the scalar modes, $\Delta$, given by (11), doesn’t include $\lambda_{TL}$ in that limit,

$$ \Delta \begin{array}{c} \to \infty \\ \to -2\lambda_{TL} \end{array} ,$$

and thus there is a second pole induced solely by $\lambda_{TL}$.

**B. 1st order curvature couplings**

Let us now turn our attention to the 1st order curvature couplings. The fixed point values are shown in Figure 3. One can see a much stronger gauge dependence than in the case of the 0th order curvature couplings. This has mainly two reasons. On the one hand, many further operators contribute to their flow equation, which however are higher order in our ordering scheme, e.g. $h^{\mu\nu} \bar{R} \Delta h_{\mu\nu}$. On the other hand, the regulator choice, in particular the inclusion of endomorphisms, decides how modes are integrated out, and thus it also has a leading order effect on the flow of the 1st order curvature couplings. It is hence not that surprising that some couplings even show divergences for specific values of $\beta$, indicating the breakdown of the truncation or the regulator.

![Figure 2](image1.png)  
**Figure 2.** Dependence of the critical exponents of the 0th order curvature couplings on the gauge parameter $\beta$. The complex pair of critical exponents mainly corresponds to the couplings $g$ and $\lambda_{TL}$, the third mainly to $\lambda_3$, and the most irrelevant one to $\bar{\lambda}$.

![Figure 3](image2.png)  
**Figure 3.** Dependence of the fixed point values of the 1st order curvature couplings on the gauge parameter $\beta$. Some couplings show divergences for specific values of $\beta$, indicating the breakdown of the truncation or the regulator.
TT legs, whereas $\mathcal{R}_{\text{STL}}$ couples a TT leg and a Tr leg. Naively, this suggests that $\mathcal{R}_{\text{STL}}$ should be more relevant than $\mathcal{R}_{\text{STT}}$. By contrast, this expected ordering of relevance emerges for the operators involving the Ricci scalar.

The general trend is that two to three critical exponents are positive and thus correspond to relevant operators. A comparatively strong gauge dependence indicates that either further operators need to be included, or endomorphisms have to be added in the regularization, to make conclusive statements. Nevertheless, also irrelevant operators appear, which is encouraging for the Asymptotic Safety scenario.

IV. CONCLUSIONS

In this work we made progress on several frontiers of the Asymptotic Safety program for quantum gravity. Above all, for the first time we resolved fluctuation correlation functions on a generically curved background. For this, we studied the propagator to linear order in the background curvature. We further disentangled the flow of the two gaps of the graviton propagator. Together with a gauge-dependent redefinition of the scalar gap, we found a UV fixed point suitable for Asymptotic Safety, where couplings vary only mildly for different curvature, and distinguished by different tensor structures. Both regulators lead to agreeing results for the graviton contribution to the flow of the 0th order curvature couplings.

Future work can go in several directions. For once, higher order correlation functions, as in [113], should be included to stabilize the system. As a long term goal, all operators of the basis presented in the appendix A should be resolved. Compared to the present setting, this would include about 25 more couplings. One might also expect that momentum dependencies will become more important if the coupling to the background curvature is more explicitly related.

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Appendix A: Basis for correlation functions

Here, we specify a basis for all correlation functions with up to three gravitons, including up to two derivatives or one background curvature. The one-point correlator has 3 independent structures,

$$\Gamma^{(1)} \sim a_1 h + a_2 h \bar{R} + a_3 h_{\mu\nu} \bar{g}^{\mu\nu}. \quad (A1)$$

For the two-point function, there are 2 invariants without derivatives or curvature,

$$\Gamma^{(2)}_\lambda \sim b_1 h^2 + b_2 h_{\mu\nu} h^{\mu\nu} \quad (A2)$$

4 invariants with 2 derivatives,

$$\Gamma^{(2)}_D \sim b_3 h \Delta h + b_4 h_{\mu\nu} \Delta h^{\mu\nu} + b_5 D^\mu D^\nu h_{\mu\nu} + b_6 D^\mu h_{\mu\rho} D^\nu h_{\nu\rho}, \quad (A3)$$

and 5 invariants with a background curvature,

$$\Gamma^{(2)}_R \sim b_7 h_{\mu\nu} \bar{C}^{\mu\nu\rho\sigma} h_{\rho\sigma} + b_8 h_{\mu\nu} S^{\mu\nu\rho} h_{\rho} + b_9 h_{\mu\nu} \bar{g}^{\mu\nu} h + b_{10} h_{\mu\nu} \bar{R} h_{\mu\nu} + b_{11} h \bar{R} h. \quad (A4)$$
Finally, the three-point correlator can be spanned by 3 terms without derivatives or curvature,
\[ \Gamma^{(3)}_\lambda \sim c_1 h^3 + c_2 h_{\mu\nu} h^{\mu\nu} h + c_3 h_{\mu}^{\nu} h_{\nu}^{\rho} h_{\rho}^{\mu}, \quad (A5) \]

\[ \Gamma^{(3)}_D \sim c_4 h^2 \Delta h + c_5 h^2 \tilde{D}_\mu \tilde{D}_\nu h^{\mu\nu} + c_6 h \mu^{\rho} \tilde{D}_\mu \tilde{D}_\nu h^{\mu\nu} + c_7 h_{\mu}^{\rho} h^{\mu\sigma} \tilde{D}_\rho \tilde{D}_\sigma h + c_8 h (\tilde{D}_\rho h_{\mu}^{\rho}) \tilde{D}_\sigma h^{\mu\sigma} \]
\[ + c_9 h_{\mu}^{\rho} \tilde{D}_\rho h_{\nu}^{\sigma} \tilde{D}_\mu \tilde{D}_\sigma h^{\mu\nu} + c_{10} h_{\mu}^{\rho} h_{\nu}^{\rho} h_{\rho}^{\sigma} \tilde{D}_\mu \tilde{D}_\nu \tilde{D}_\rho h^{\mu\nu \sigma} \]
\[ + c_{11} h_{\mu}^{\rho} h_{\nu}^{\sigma} \tilde{D}_\mu \tilde{D}_\nu h^{\rho\sigma} + c_{12} h_{\mu}^{\rho} h_{\nu}^{\sigma} \tilde{D}_\mu \tilde{D}_\nu h^{\rho\sigma} + c_{13} h_{\mu}^{\rho} h_{\nu}^{\sigma} \tilde{D}_\mu \tilde{D}_\nu h^{\rho\sigma} \tilde{D}_\rho \tilde{D}_\sigma h^{\mu\nu \sigma}, \quad (A6) \]

and 9 invariants with a background curvature,
\[ \Gamma^{(3)}_R \sim c_{14} C_{\mu\nu\rho\sigma} h_{\mu\nu} h_{\rho\sigma} + c_{15} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{16} C_{\mu\nu\rho\sigma} h_{\mu\sigma} h_{\nu\rho} + c_{17} C_{\mu\nu\rho\sigma} h_{\mu\sigma} h_{\nu\rho} + \]
\[ + c_{18} C_{\mu\nu\rho\sigma} h_{\nu\mu} h_{\rho\sigma} + c_{19} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{20} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{21} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} \]
\[ + c_{22} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{23} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{24} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{25} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + c_{26} C_{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} \]. \quad (A7)

### Appendix B: General formula for the propagator on flat background

In this section, we show how to get the propagator on a flat background, for a general ansatz independent of the truncation. The most general form of a symmetric rank (2,2) tensor $T$ on a flat background depending on a single momentum vector $p$ reads
\[ T^{\mu\nu}_{\rho\sigma} = A_1 \Pi_{TL}^{\mu\nu}_{\rho\sigma} + A_2 \Pi_{Tr}^{\mu\nu}_{\rho\sigma} + A_3 p^{\mu} \delta_{\nu\rho}^{(p)} p_{\sigma} \]
\[ + \frac{A_4}{2} (p^\lambda p^\nu \tilde{g}_{\rho\sigma} + \tilde{g}^{\mu\nu} p_{\rho} p_{\sigma}) + A_5 p^\mu p^\nu p_{\rho} p_{\sigma}. \quad (B1) \]

Both the second variation of the action and the propagator are of this form. It is straightforward to calculate the inverse of this tensor,
\[ T^{\mu\nu}_{\rho\sigma} (T^{-1})^{\rho\sigma}_{\mu\nu} = \delta^{\mu\nu}_{\alpha\beta}, \quad (B2) \]

by explicit insertion of the ansatz. The coefficients $B_i$ of the inverse of the tensor with coefficients $A_i$ read
\[ B_1 = 1 \]
\[ B_2 = \frac{4A_2^2 + (A_1(5A_3 + 4A_4) - A_2A_3)p^2 + ((5A_1 - A_2)A_5 + A_4^2)p^4}{4A_1^2 A_2 + A_1 c_1 p^2 + A_1 c_2 p^4}, \]
\[ B_3 = -\frac{2A_1}{2A_1^2 + A_1 c_3 p^2}, \]
\[ B_4 = \frac{2(A_1(-A_3 - 2A_4) + A_2A_3) - 2((A_1 - A_2)A_5 + A_4^2)p^2}{4A_1^2 A_2 + A_1 c_1 p^2 + A_1 c_2 p^4}, \]
\[ B_5 = \frac{2(A_1((A_3 + 2A_4)^2 - 4A_2A_5 + A_2^2) + 2A_3((A_1 + A_2)A_5 - A_4^2)p^2}{8A_1^2 A_2 + 2A_1^2 (c_1 + 2A_2 A_3)p^2 + A_1(2A_1 c_2 + A_3 c_1)p^4 + A_1 A_3 c_2 p^6}, \]

where $p^2 = p_\mu p^\mu$, $c_1 = A_1(A_3 + 4A_4) + 3A_2 A_3$ and $c_2 = (A_1 + 3A_2)A_5 - 3A_4^2$. For small $p^2$, only $A_1$ and $A_2$ appear in the denominators, thus any regulator has to regularize at least these two structures. For computational reasons, we often need derivatives of the $B_i$ w.r.t. $p^2$. As the expressions (B3) are rational functions, their derivatives are rather lengthy. Here it helps to note, that for a simple function of the form $g(x) = \frac{1}{f(x)}$ the derivative can be written as $g'(x) = -g^2(x)f'(x)$. Similar expressions for the derivatives of the $B_i$ can be found if we rescale the $A_i$ and the $B_i$ in such a way that no explicit $p^2$ appears in their relation,
\[ \tilde{A}_1 = A_1, \quad \tilde{A}_2 = A_2, \quad \tilde{A}_3 = p^2 A_3, \quad \tilde{A}_4 = p^2 A_4, \quad \tilde{A}_5 = p^4 A_5, \]
\[ \tilde{B}_1 = B_1, \quad \tilde{B}_2 = B_2, \quad \tilde{B}_3 = p^2 B_3, \quad \tilde{B}_4 = p^2 B_4, \quad \tilde{B}_5 = p^4 B_5. \quad (B4) \]
Then the $\tilde{B}_i$ as a function of the $\tilde{A}_i$ are given by equations (B3), replacing $A_i \rightarrow \tilde{A}_i$, $B_i \rightarrow \tilde{B}_i$ and $\rho^2 \rightarrow 1$. Then one can check that the derivatives of the $\tilde{B}_i$ are given by

\begin{align*}
\tilde{B}_1' &= -\tilde{B}_1 \cdot \tilde{A}_1' , \\
\tilde{B}_2' &= -\frac{1}{2}(4\tilde{B}_1 + 3\tilde{B}_4)\tilde{A}_1 - \frac{1}{2}(2\tilde{B}_2 + \tilde{B}_4)^2 \cdot \tilde{A}_2 - \frac{1}{4}\tilde{d}_1' \cdot \tilde{A}_3 - \frac{1}{2}\tilde{d}_1(2\tilde{B}_2 + \tilde{B}_4) \cdot \tilde{A}_4 - \frac{1}{4}\tilde{d}_2' \cdot \tilde{A}_5', \\
\tilde{B}_3' &= -\frac{1}{2}(4\tilde{B}_1 + \tilde{B}_3)\tilde{A}_1 - \frac{1}{2}(2\tilde{B}_1 + \tilde{B}_3)^2 \cdot \tilde{A}_5', \\
\tilde{B}_4' &= -\frac{1}{4}(4\tilde{B}_1\tilde{A}_1 - (2\tilde{B}_1 - 3\tilde{B}_4)(\tilde{B}_3 + \tilde{B}_5)) \cdot \tilde{A}_1' - \frac{1}{4}(2\tilde{B}_2 + \tilde{B}_4)(\tilde{B}_3 + 2\tilde{B}_4 + \tilde{B}_5) \cdot \tilde{A}_2' \\
&\quad - \frac{1}{4}\tilde{d}_1d_2 \cdot \tilde{A}_3' - \frac{1}{4}((-\tilde{B}_1 + 5\tilde{B}_2 + 4\tilde{B}_4)(\tilde{B}_3 + \tilde{B}_5) + 4\tilde{B}_1\tilde{B}_2 + 5\tilde{B}_1^2 + 4\tilde{B}_2\tilde{B}_4) \cdot \tilde{A}_4' - \frac{1}{4}\tilde{d}_1d_2 \cdot \tilde{A}_5', \\
\tilde{B}_5' &= -\frac{1}{4}(8\tilde{B}_1 + 3\tilde{B}_5)\tilde{B}_3 + 6\tilde{B}_3\tilde{B}_5 + \tilde{B}_3^2 \cdot \tilde{A}_1' - \frac{1}{4}(\tilde{B}_3 + 2\tilde{B}_4 + \tilde{B}_5)^2 \cdot \tilde{A}_2' \\
&\quad - \frac{1}{4}(\tilde{B}_3 + \tilde{B}_4 + 2\tilde{B}_5)(4\tilde{B}_1 + 3\tilde{B}_3 + \tilde{B}_4 + 2\tilde{B}_5) \cdot \tilde{A}_3' - \frac{1}{2}\tilde{d}_2(\tilde{B}_3 + 2\tilde{B}_4 + \tilde{B}_5) \cdot \tilde{A}_4' - \frac{1}{4}\tilde{d}_2^2 \cdot \tilde{A}_5',
\end{align*}

where $d_1 = -\tilde{B}_1 + \tilde{B}_2 + 2\tilde{B}_4$ and $d_2 = 2\tilde{B}_1 + 2\tilde{B}_3 + \tilde{B}_4 + 2\tilde{B}_5$.

**Appendix C: Regulator comparison**

In this appendix, we present a comparison of fixed point values and critical exponents for both regulators. In particular, we choose $\beta = 1$, since close to this value all fixed point quantities seem to show a weak $\beta$-dependence, see the figures in the main text. Specifically, the most irrelevant critical exponents, $\theta_4$ and $\theta_{S_2}$, which in general strongly depend on $\beta$, show a local maximum near to this value of $\beta$. Moreover, for this choice, both regulators agree completely on a flat background, and only differ to linear order in the background curvature. For the 0th order curvature couplings, we obtain

\begin{align}
\nu = 0.196, \quad \lambda_3 = -0.00807, \\
\lambda_{\text{TTL}} = 0.197. \quad \tilde{\lambda} = 0.399,
\end{align}

**(C1)**

together with the critical exponents

\begin{align}
\theta_{1,2} = 1.65 \pm 3.70i, \quad \theta_3 = -5.43, \quad \theta_4 = -28.6.
\end{align}

**(C2)**

For the 1st order curvature couplings, we find that only some of them have differing fixed point values due to the different regularization. With the minimal regulator (14), we obtain

\begin{align}
R_C = 0.164, &\quad \theta_C = 1.39, \\
R_{\text{TTL}} = 1.13, &\quad \theta_{S_1} = 1.24, \\
R_{\text{STL}} = 0.476, &\quad \theta_{S_2} = -1.44, \\
R_{\text{RTL}} = 0.453, &\quad \theta_{R_1} = 0.607, \\
R_{\text{RTL}} = -0.252, &\quad \theta_{R_2} = -31.5.
\end{align}

**(C3)**

Employing the regulator (16), we find

\begin{align}
R_C = 0.164, &\quad \theta_C = 1.39, \\
R_{\text{STL}} = 1.39, &\quad \theta_{S_1} = 1.24, \\
R_{\text{STL}} = 0.348, &\quad \theta_{S_2} = -1.44, \\
R_{\text{RTL}} = 0.447, &\quad \theta_{R_1} = 0.607, \\
R_{\text{RTL}} = -0.169, &\quad \theta_{R_2} = -31.5.
\end{align}

**(C4)**

It can be seen that the fixed point value of $R_C$ is the same for both regulators. Even more surprisingly, the critical exponents do not depend at all on the regulator for this choice of the gauge fixing. For general choices of $\beta$, there is a small difference between the two regulators in all couplings and critical exponents, typically on the percent level.

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