GENERALIZED TURÁN PROBLEMS FOR EVEN CYCLES

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Abstract. Given a graph $H$ and a set of graphs $\mathcal{F}$, let $\text{ex}(n, H, \mathcal{F})$ denote the maximum possible number of copies of $H$ in an $\mathcal{F}$-free graph on $n$ vertices. We investigate the function $\text{ex}(n, H, \mathcal{F})$, when $H$ and members of $\mathcal{F}$ are cycles. Let $C_k$ denote the cycle of length $k$ and let $\mathcal{F}_k = \{C_3, C_4, \ldots, C_k\}$. We highlight the main results below.

(i) We show that $\text{ex}(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $l, k \geq 2$. Moreover, in some cases we determine it asymptotically.

(ii) Erdős’s Girth Conjecture states that for any positive integer $k$, there exist a constant $c > 0$ depending only on $k$, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ with girth more than $2k$.

Solymosi and Wong proved that if this conjecture holds, then for any $l \geq 3$ we have $\text{ex}(n, C_{2l}, C_{2l-1}) = \Theta(n^{2l/(l-1)})$. We prove that their result is sharp in the sense that forbidding any other even cycle decreases the number of $C_{2l}$’s significantly.

(iii) We prove $\text{ex}(n, C_{2l+1}, C_{2l}) = \Theta(n^{2+1/l})$, provided a stronger version of Erdős’s Girth Conjecture holds (which is known to be true when $l = 2, 3, 5$). This result is also sharp in the sense that forbidding one more cycle decreases the number of $C_{2l+1}$’s significantly.

1. Introduction

The Turán problem for a set of graphs $\mathcal{F}$ asks the following. What is the maximum number $\text{ex}(n, \mathcal{F})$ of edges that a graph on $n$ vertices can have without containing any $F \in \mathcal{F}$ as a subgraph? When $\mathcal{F}$ contains a single graph $F$, we simply write $\text{ex}(n, F)$. This function has been intensively studied, starting with Mantel [17] who determined $\text{ex}(n, K_3)$ and with Turán [21] who determined $\text{ex}(n, K_r)$ for every $r$, where $K_r$ denotes the complete graph on $r$ vertices with $r \geq 3$. See [9] for surveys on this topic.

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For some integer $k$ let $C_k$ denote a cycle on $k$ vertices and let $\mathcal{C}_k$ denote the set $\{C_3, C_4, \ldots, C_k\}$. For even cycles $C_{2k}$, Bondy and Simonovits [4] proved the following upper bound.

**Theorem 1** (Bondy, Simonovits [4]). For $k \geq 2$ we have 
$$\text{ex}(n, C_{2k}) = O(n^{1+1/k}).$$

The order of magnitude in the above theorem is known to be sharp only for $k = 2, 3, 5$. If all the cycles in $\mathcal{C}_k$ are forbidden, then Alon, Hoory and Linial [1] proved the following.

**Theorem 2** (Alon, Hoory, Linial [1]). For any $k \geq 2$ we have

(i) $\text{ex}(n, \mathcal{C}_{2k}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$, 

(ii) $\text{ex}(n, \mathcal{C}_{2k+1}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$.

For more information on the Turán number of cycles one can consult the survey [22].

1.1. Generalized Turán problems

For two graphs $H$ and $G$, let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. Given a graph $H$ and a set of graphs $\mathcal{F}$, let 

$$\text{ex}(n, H, \mathcal{F}) = \max_G \{\mathcal{N}(H, G) : G \text{ is an } \mathcal{F}-\text{free graph on } n \text{ vertices}\}.$$ 

If $\mathcal{F} = \{F\}$, we simply denote it by $\text{ex}(n, H, F)$. This problem was initiated by Erdős [6], who determined $\text{ex}(n, K_s, K_t)$ exactly. Concerning cycles, Bollobás and Győri [3] proved that 

$$(1 + o(1)) \frac{1}{3\sqrt{3}} n^{3/2} \leq \text{ex}(n, C_3, C_5) \leq (1 + o(1)) \frac{5}{4} n^{3/2}$$

and this result was extended by Győri and Li [14] for $\text{ex}(n, C_3, C_{2k+1})$ ($k > 2$). Other improvements can be found in [8].

Another notable result is to determine the value of $\text{ex}(n, C_5, C_3)$ by Hatami, Hladký, Královic, Norine, and Razborov [15] and independently by Grzesik [12], where they showed that it is equal to $(\frac{2}{3})^5$. Very recently, the asymptotic value of $\text{ex}(n, C_k, C_{k-2})$ was determined for every odd $k$ by Grzesik and Kielak in [13].

1.2. Forbidding a set of cycles

The famous Girth Conjecture of Erdős [5] asserts the following.

**Conjecture 3** (Erdős’s Girth Conjecture [5]). For any positive integer $k$, there exist a constant $c > 0$ depending only on $k$, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ and the girth of $G_n$ is more than $2k$.

This conjecture has been verified for $k = 2, 3, 5$, see [2, 23]. For a general $k$, Sudakov and Verstraëte [20] showed that if such graphs exist, then they contain a $C_{2l}$ for any $l$ with $k < l \leq Cn$, for some constant $C > 0$. More recently, Solymosi and Wong [19] proved that if such graphs exist, then in fact, they contain many $C_{2l}$’s for any fixed $l > k$. More precisely they proved:
Theorem 4 (Solymosi, Wong [19]). If Erdős’s Girth Conjecture holds for k, then for every $l > k$ we have

$$\text{ex}(n, C_{2l}, \mathcal{C}_{2k}) = \Omega(n^{2l/k}).$$

Remark 1. It is easy to see that if $k + 1$ divides $2l$, then $\text{ex}(n, C_{2l}, \mathcal{C}_{2k}) = O(n^{2l/k})$. Indeed, let us associate to each $C_{2l}$, one fixed ordered list of $2l/(k + 1)$ edges $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, where $e_1$ appears as the first edge (chosen arbitrarily) on the $C_{2l}$, $e_{k+1}$ as the $(k + 1)$-th edge, $e_{2k+1}$ as the $(2k + 1)$-th edge and so on. Note that at most one $C_{2l}$ is associated to an ordered tuple $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, because there is at most one path of length $k - 1$ connecting the endpoints of any two edges (as all the short cycles are forbidden). Since there are at most $O(n^{1+1/k})$ ways to select each edge, this shows the number of $C_{2l}$’s is at most $O((n^{1+1/k})^{2l/(k+1)}) = O(n^{2l/k})$, showing that the bound in Theorem 4 is sharp when $k + 1$ divides $2l$.

2. Our results

Note that all the proofs of the results (and even more results) can be found in [10], the article version of this extended abstract. For any two positive integers $n$ and $l$, let $(n)_l$ denote the product $n(n - 1)(n - 2)\ldots(n - (l - 1))$.

2.1. Forbidding a cycle of given length

We determine the order of magnitude of $\text{ex}(n, C_{2l}, C_{2k})$ below.

Theorem 5.

- For any $l \geq 3$ and $k \geq 2$ we have $\text{ex}(n, C_{2l}, C_{2k}) \leq (1 + o(1))\frac{2^{k - 2}(k - 1)^l}{2^l} n^l$.
- For any $k > l \geq 2$ we have $\text{ex}(n, C_{2l}, C_{2k}) \geq (1 + o(1))\frac{(k - 1)^l}{2^l} n^l$.
- For any $l > k \geq 3$ we have $\text{ex}(n, C_{2l}, C_{2k}) \geq (1 + o(1))\frac{2}{l} n^l$.

Theorem 5 and Theorem 6 (stated below) show that $\text{ex}(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $k, l \geq 2$, except for the lower bound in the case $k = 2$, which can be easily shown by counting cycles in the orthogonal polarity graph of the classical projective plane constructed by Erdős and Rényi [7].

We note that Theorem 5 has been proven independently by Gishboliner and Shapira [11] and recently extended by Morrison, Roberts and Scott in [18].

Solymosi and Wong [19] asked whether a similar lower bound (to that of Theorem 4) on the number of $C_{2l}$’s holds, if just $C_{2k}$ is forbidden instead of forbidding $\mathcal{C}_{2k}$. Theorem 5 answers this question in the negative.

Asymptotic results. We determine $\text{ex}(n, C_4, C_{2k})$ asymptotically.

Theorem 6. For $k \geq 2$ we have

$$\text{ex}(n, C_4, C_{2k}) = (1 + o(1))\frac{(k - 1)(k - 2)}{4} n^2.$$
vertices. Our methods give sharper bounds for $\text{ex}_{\text{bip}}(n, C_{2l}, C_{2k})$ compared to the bounds in Theorem 5 and in the case $l = 3$, $k = 4$ we can even determine the asymptotics.

**Theorem 7.** We have

$$\text{ex}_{\text{bip}}(n, C_6, C_8) = n^3 + O(n^{5/2}).$$

### 2.2. Forbidding a set of cycles

It is easy to see that when counting copies of an even cycle, forbidding an odd cycle does not change the order of magnitude. Therefore by Theorem 4 and Remark 1 we have

**Corollary 8.** Suppose $l \geq 3$ and Erdős’s Girth Conjecture is true for $l - 1$. Then we have

$$\text{ex}(n, C_{2l}, C_{2l-1}) = \Theta(n^{2l/(l-1)}).$$

So the maximum number of $C_{2l}$’s in a graph of girth $2l$ is $\Theta(n^{2l/(l-1)})$. We prove that the previous statement is sharp in the sense that forbidding one more even cycle decreases the order of magnitude significantly: More generally, we show the following.

**Theorem 9.** For any $k > l \geq 3$ and $m \geq 2$ such that $2k \neq ml$ we have

$$\text{ex}(n, C_{ml}, C_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

It is easy to see that forbidding even more cycles does not decrease the order of magnitude, as long as we do not forbid $C_{2l}$ itself as shown by ($l$, $[n/l]$)-theta-graph and some isolated vertices, where for $l, t \geq 1$ the ($l$, $t$)-theta-graph with endpoints $x$ and $y$ is the graph obtained by joining two vertices $x$ and $y$, by $t$ internally disjoint paths of length $l$.

Corollary 8 determines the order of magnitude of maximum number of $C_{2l}$’s in a graph of girth $2l$. It is then very natural to consider the analogous question for odd cycles: What is the maximum number of $C_{2k}$’s in a graph of girth $2k + 1$? Before answering this question, we state a strong form of Erdős’s Girth Conjecture that is known to be true for small values of $k$.

A graph $G$ on $n$ vertices, with average degree $d$, is called *almost-regular* if the degree of every vertex of $G$ is $d + O(1)$.

**Conjecture 10** (Strong form of Erdős’s Girth Conjecture). For any positive integer $k$, there exists a family of almost-regular graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq \frac{n^{2+1/k}}{2}$ and $G_n$ is $\{C_4, C_6, \ldots, C_{2k}\}$-free.

Lazebnik, Ustimenko and Woldar [16] showed Conjecture 10 is true when $k \in \{2, 3, 5\}$ using the existence of polarities of generalized polygons. We show the following that can be seen as the ‘odd cycle analogue’ of Theorem 4.

**Theorem 11.** Suppose $k \geq 2$ and Conjecture 10 is true for $k$. Then we have

$$\text{ex}(n, C_{2k+1}, C_{2k}) = (1 + o(1))\frac{n^{2+\frac{1}{k}}}{4k+2}.$$
To show that Theorem 11 is sharp in the same sense that Theorem 9 is (in the case of $m = 2$) for odd cycles, we prove that if we forbid one more even cycle, then the order of magnitude goes down significantly:

**Theorem 12.** For any integers $k > l \geq 2$, we have

$$\Omega(n^{1 + \frac{1}{2k+1}}) = \text{ex}(n, C_{2l+1} \cup \{C_{2k}\}) = O(n^{1 + \frac{1}{2l+1}}).$$

However, if the additional forbidden cycle is of odd length, we can only prove a quadratic upper bound. We conjecture that the truth is also sub-quadratic here.

**Theorem 13.** For any integers $k > l \geq 2$, we have

$$\Omega(n^{1 + \frac{1}{2k+2}}) = \text{ex}(n, C_{2l+1} \cup \{C_{2k+1}\}) = O(n^2).$$

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