KÄHLER-EINSTEIN METRICS ON
PASQUIER’S TWO-ORBITS VARIETIES

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Abstract. We show that there exist Kähler-Einstein metrics on two exceptional Pasquier’s two-orbits varieties. As an application, we will provide a new example of K-unstable Fano manifold with Picard number one.

INTRODUCTION

0.1. Pasquier’s two-orbits varieties. In 2009, as an application of his study on horospherical varieties, Pasquier classified a special kind of two-orbits varieties that satisfies the following condition [Pas09]:

Condition 0.1.

- $X$ is a Fano manifold with Picard number one.
- The connected automorphism group $G := \text{Aut}^0(X)$ acts on $X$ with two orbits $X^0 \amalg Z$, where $X^0$ is the open orbit and $Z$ is the closed orbit.
- The blow-up $\text{Bl}_{Z} X$ is again a two-orbits variety $X^0 \amalg E$ with respect to $G$, where $E$ is the exceptional divisor of the blow-up.

In summary, he showed that each Fano manifold with the above condition satisfies one of the following:

1. $X$ is a horospherical variety;
2. $X$ is isomorphic to a 23-dimensional Fano manifold $\mathcal{P}_{F_4}$ with $G = F_4$;
3. $X$ is isomorphic to an 8-dimensional Fano manifold $\mathcal{P}_{A_1 \times G_2}$ with $G = A_1 \times G_2$.

In the course of his study, he also determined the group $G$ for case (1); in each of horospherical cases, $G$ is not reductive, and hence there are no Kähler-Einstein metrics on $X$ [Mat57].

In a previous article [Kan20], the author studied stability of tangent bundles for varieties satisfying Condition 0.1. Therein we have provided examples of Fano manifolds with Picard number one whose tangent bundles are unstable, which disproved a folklore conjecture. The present article is a continuation of this previous paper [Kan20]; the purpose of this article is to show the existence of Kähler-Einstein metrics in the two remaining cases $\mathcal{P}_{F_4}$ and $\mathcal{P}_{A_1 \times G_2}$:

Theorem 0.2. There are Kähler-Einstein metrics on $\mathcal{P}_{F_4}$ and $\mathcal{P}_{A_1 \times G_2}$.

Remark 0.3. In [Del20], Theorem 4.1, Delcric also obtained the proof of the above result independently. The method used in his proof is different from our proof; Delcric’s proof relies on a criterion of K-stability for spherical varieties, while our proof does not use the fact $\mathcal{P}_{F_4}$ and $\mathcal{P}_{A_1 \times G_2}$ are spherical, but relies on the $G$-equivariant valuative criterion of K-stability.

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Remark 0.4. More precisely, we will show that \( \mathcal{P}_F \) and \( \mathcal{P}_{A_1 \times G_2} \) are K-polystable, which is equivalent to the existence of Kähler-Einstein metrics by virtue of [Tia97, Don02, Ber16, CDS15a, CDS15b, CDS15c, Tia15].

The variety \( \mathcal{P}_{A_1 \times G_2} \) is a Mukai 8-fold of genus 7 (see [BFM20, Section 4.2] or [Kan20, Remark 4.2]). Recall that, by [Kuz18, Section 6], there are two isomorphic classes \( X_{sp} \) and \( X_{gen} \) of Mukai 8-folds of genus 7, and \( X_{gen} \) degenerates to \( X_{sp} \). One can prove that \( \mathcal{P}_{A_1 \times G_2} \approx X_{gen} \) (cf. [Kan20, Remark 4.2]). See Section 4 for a proof of this fact and also [BFM20, Proposition 4.8, Remark 4.10, Remark 5.4] for proofs. Thus, by the general properties of K-moduli, we have:

**Corollary 0.5.** \( X_{sp} \) is not K-semistable. In particular, there are no Kähler-Einstein metrics on \( X_{sp} \).

Remark 0.6. The above corollary provides a new counter-example of a conjecture due to Odaka and Okada [OO13, Conjecture 5.1]. See [Fuj17] and [Del20a] for other examples.

0.2. Organization of the paper. The article is organized as follows: Section 1 provides preliminaries on \( G \)-equivariant valuative criterion of K-stability. The main ingredient of our proof is Theorem 1.5, which provides a criterion of K-stability by means of valuations. In Subsection 1.4 we provide the outline of the proof.

In Section 2 we briefly recall the geometry of \( \mathcal{P}_F \) and \( \mathcal{P}_{A_1 \times G_2} \). Then, in Section 3 we complete our proof of Theorem 1.2. In the last section (=Section 4), we prove Corollary 0.5. We also give a remark on the relation between Kähler-Einstein metrics and foliations.

**Convention 0.7.** We work over the complex number field \( \mathbb{C} \). For a vector space \( V \), \( P_{\text{sub}}(V) \) denotes the parameter space of 1-dimensional subspaces in \( V \).

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1. Preliminaries: \( G \)-equivariant valuative criterion of K-stability

Here we briefly recall the definition of K-stability and its criterion. For simplicity, we assume that \( X \) is a smooth Fano variety and the polarization is given by \( L := -K_X \).

1.1. K-stability. K-stability is defined by using test configurations and their Donaldson-Futaki invariants, which encode informations about degenerations of the variety \( X \) in question. We briefly recall the definitions.

**Definition 1.1 ([Tia97, Don02]).**

(1) A test configuration of the pair \((X, L)\) is the following data:

- a normal variety \( X \);
- a proper flat morphism \( p: X \to \mathbb{A}^1 \);
- a \( p \)-ample \( Q \)-line bundle \( L \) on \( X \);
- a \( \mathbb{G}_m \)-action on \((X, L)\) which makes the morphism \( p \) \( \mathbb{G}_m \)-equivariant with respect to the natural action \( \mathbb{G}_m \to \mathbb{A}^1, (g, x) \mapsto gx \);
- a \( \mathbb{G}_m \)-equivariant isomorphism \((X \setminus \{\lambda_0\}, L|_{X \setminus \{\lambda_0\}}) \cong (X \times (\mathbb{A}^1 \setminus \{0\}, \text{pr}_1^* L)) \).
(2) A test configuration is called a product test configuration if \((X', L') \simeq (X \times A^1, pr_1^* L)\).

Let \((\overline{X}, \overline{L})/\mathbb{P}^1\) be the compactification of \((X, L)/A^1\) obtained by gluing \((X, L)/A^1\) and \((X \times \mathbb{P}^1 \setminus 0, pr_1^* L)\). We denote by \(\overline{\pi}: \overline{X} \to \mathbb{P}^1\) the natural projection.

**Definition 1.2** ([Tia97], [Don02], [Wan12], [Oda13]). For a test configuration of \((X, L)\), the Donaldson-Futaki invariant is defined as follows:

\[
\text{DF}(X, L) := \frac{n+1}{n+1} \cdot \overline{L}^{n+1} L^n + \frac{\overline{L}^n}{L^n} K_{\overline{\pi}/\mathbb{P}^1},
\]

where \(K_{\overline{\pi}/\mathbb{P}^1} := K_{\overline{X}} - \overline{\pi}^* K_{\mathbb{P}^1}\).

**Definition 1.3.**

- The pair \((X, L)\) is called K-semistable if \(\text{DF}(X, L) \geq 0\) for any test configuration.
- The pair \((X, L)\) is called K-polystable if it is K-semistable, and moreover \(\text{DF}(X, L) = 0\) only when \((X, L)\) is a product test configuration.

1.2. Valuative criterion. Here we recall the \((G\text{-equivariant})\) valuative criterion of K-stability [Fuj19], [Li17], [Zhu20]. A prime divisor over \(X\) is a prime divisor \(F\) on a resolution \(\sigma: Y \to X\).

**Definition 1.4.**

(1) The log discrepancy of \(X\) along \(F\) is defined as follows:

\[
A(F) := 1 + \text{ord}_F(K_Y - \sigma^* K_X).
\]

(2) ([BJ20]) \(S(F)\) is defined as follows:

\[
S(F) := \frac{1}{L^n} \int_0^{\infty} \text{vol}(L - xF)dx,
\]

where \(\text{vol}(L - xF) := \text{vol}_Y(\sigma^* L - xF)\).

(3) ([Fuj19], [Li17]) Then the \(\beta\)-invariant of \(F\) is defined as follows:

\[
\beta(F) := (L^n)(A(F) - S(F)).
\]

**Theorem 1.5** ([Zhu20] Corollary 4.14). Assume that a reductive algebraic group \(G\) acts on \((X, L)\). If \(\beta(F) > 0\) for any \(G\)-invariant prime divisor \(F\) over \(X\), then \(X\) is K-polystable.

1.3. \(\xi\)-invariant. By using formula (1.8.1) below, we replace the calculation of \(\beta\)-invariant with that of \(\xi\)-invariant [Fuj15], which corresponds to the slope stability of a subvariety in the sense of Ross-Thomas [RT07].

**Definition 1.6.** Let \(X\) be a Fano manifold and \(Z\) a smooth subvariety of codimension \(r\). Denote by \(\varphi: \tilde{X} \to X\) the blow-up of \(X\) along \(Z\). Then the Seshadri constant of \(\epsilon(Z)\) is defined as follows:

\[
\epsilon(Z) := \max\{t \in \mathbb{R}_{>0} \mid \varphi^* L - tE \text{ is nef on } \tilde{X}\}.
\]

**Proposition 1.7** ([Fuj15] Proposition 3.2]). Set

\[
\xi(Z) := r \text{vol}_X(-K_X) + (\epsilon(Z) - r) \text{vol}_{\tilde{X}}(\varphi^*(-K_X) - \epsilon(Z) E) - \int_0^{\epsilon(Z)} \text{vol}_{\tilde{X}}(\varphi^*(-K_X) - xE)dx.
\]

Then

\[
\xi(Z) = n \int_0^{\epsilon(Z)} (r - x)(E \cdot (\varphi^*(-K_X) - xE)^{n-1})dx.
\]
Remark 1.8. Note that, if $\varphi^*(-K_X) - xE$ is not big for $x \geq \epsilon(Z)$, then 
$$\text{vol}_{\tilde{X}}(\varphi^*(-K_X) - xE) = 0$$
for $x \geq \epsilon(Z)$. Thus 
$$\xi(Z) = r \text{vol}_{X}(-K_X) - \int_{0}^{\epsilon(Z)} \text{vol}_{\tilde{X}}(\varphi^*(-K_X) - xE)dx$$
(1.8.1)
$$= r \text{vol}_{X}(-K_X) - \int_{0}^{\infty} \text{vol}_{\tilde{X}}(\varphi^*(-K_X) - xE)dx$$
$$= (A(E) - S(E))L^n$$
$$= \beta(E),$$
where $E$ is the exceptional divisor of the blow-up $\tilde{X} \to X$.

1.4. Outline of the proof. Here we briefly sketch the outline of our proof. In 
the case of Pasquier’s two-orbits varieties $X = \mathcal{P}_{F_4}$ or $\mathcal{P}_{A_1 \times G_2}$, the blow-up $\tilde{X}$ of 
$X$ along $Z$ admits a smooth fibration $\pi: \tilde{X} \to Y$ and hence we have the following 
diagram (see [Kan20, Propositions 2.5 and 2.7]):

$$\begin{array}{ccc}
E & \xrightarrow{\varphi|_E} & Z \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\varphi} & X \\
\downarrow & \pi & \downarrow \\
Y & & 
\end{array}$$

(1.8.2)

Note that $\xi(Z) = \beta(E)$ by (1.8.1). By Condition 0.1, $E$ is the unique $G$-invariant 
prime divisor over $X$. Note also that $\text{Aut}_0(X)$ is reductive. Thus 
$X$ is K-polystable $\iff \beta(E) > 0 \iff \xi(Z) > 0$.

Thus it is enough to show $\xi(Z) > 0$. By virtue of Proposition 1.7, we can calculate 
$\xi(Z)$ by computing intersection numbers on a homogeneous space $E$, and we will 
show $\xi(Z) > 0$, which completes the proof.

2. Geometry of Pasquier’s two-orbits varieties

Here we recall brief descriptions of Pasquier’s two varieties $\mathcal{P}_{F_4}$ and $\mathcal{P}_{A_1 \times G_2}$. For 
detailed descriptions and the original descriptions, we refer the reader to [Kan20], [Pas99].

Proposition 2.1 ([Kan20, Proposition 2.5 and 2.7]). Consider diagram 1.8.2. 
Let $H_X$ and $H_Y$ be the ample generator of $\text{Pic}(X)$ and $\text{Pic}(Y)$ respectively.

(1) If $X = \mathcal{P}_{F_4}$, then we have the following:
• $\text{dim} X = 23$ and $\text{dim} Y = 20$;
• $-K_X = 8H_X$;
• $E = -\pi^*H_Y + \varphi^*H_X$.
In particular, $\epsilon(Z) = 8$ and 
$$\xi(Z) = 23 \int_{0}^{8} (3-x)(E \cdot (\varphi^*(-K_X) - xE)^2)dx.$$ 

(2) If $X = \mathcal{P}_{A_1 \times G_2}$, then we have the following:
• $\text{dim} X = 8$ and $\text{dim} Y = 5$;
• $-K_X = 6H_X$;
• $E = -\pi^*H_Y + 2\varphi^*H_X$. 

In particular, \( \epsilon(Z) = 3 \) and
\[
\xi(Z) = 8 \int_0^3 (2 - x)(E \cdot (\varphi^*(-K_X) - xE))^7 dx.
\]

2.1. Geometry of the exceptional divisor. Here we recall the description of the exceptional divisor \( E \). In the following, we denote by \( p_Y : E \to Y \) and \( p_Z : E \to Z \) the natural projections respectively.

**Definition 2.2** (\cite[Definition 2.1]{Kan20}). The associated triples \((D, \omega_Y, \omega_Z)\) for \( \mathcal{P}_{F_4} \) and \( \mathcal{P}_{A_1 \times G_2} \) are defined as follows:
- \((D, \omega_Y, \omega_Z) : = (F_4, \omega_1, \omega_1)\); 
- \((D, \omega_Y, \omega_Z) : = (A_1 \times G_2, \omega_2, \omega_1 + \omega_1)\).

Here \((D, \omega_Y, \omega_Z)\) consists of a Dynkin diagram \( D \) and weights \( \omega_Y \) and \( \omega_Z \). For the descriptions of the root systems \( F_4 \) and \( A_1 \times G_2 \), see Section 3 below.

**Notation 2.3.** In the following \( G \) is the simply connected semisimple algebraic group whose Dynkin diagram is \( D \).
- \( V_Y \) and \( V_Z \) are the irreducible \( G \)-representations with highest weights \( \omega_Y \) and \( \omega_Z \) respectively.
- \( \nu_Y \) and \( \nu_Z \) are the corresponding highest weight vectors.
- \([\nu_Y] \in \mathcal{P}_{\text{sub}}(V_Y) \) and \([\nu_Z] \in \mathcal{P}_{\text{sub}}(V_Z) \) are the corresponding points.
- \( P_Y \) and \( P_Z \) are the stabilizer groups of \([\nu_Y] \) and \([\nu_Z] \) respectively, which are parabolic subgroups.
- \( P_{Y,Z} := P_Y \cap P_Z \).

Then \( \mathcal{P}_{\text{sub}}(V_Y) \supset G \cdot [\nu_Y] = G/P_Y \) and \( \mathcal{P}_{\text{sub}}(V_Z) \supset G \cdot [\nu_Z] = G/P_Z \). The polarizations of these two homogeneous spaces are given by the divisors corresponding to the weights \( \omega_Y \) and \( \omega_Z \).

**Proposition 2.4** (\cite[Sections 2.3, 2.4]{Kan20}). Then \( E \simeq G/P_{Y,Z} \), \( Y \simeq G/P_Y \) and \( Z \simeq G/P_Z \). The morphisms \( p_Y : E \to Y \) and \( p_Z : E \to Z \) are the quotient maps.

Moreover \( H_Y \) is the divisor corresponding to \( \omega_Y \) and \( H_X|_Z \) is the divisor corresponding to \( \omega_Z \).

**Corollary 2.5.**
1. If \( X = \mathcal{P}_{F_4} \), then
\[
\xi(Z) = 23 \int_0^8 (3 - x)(xp_Y^*H_Y + (8 - x)p_Z^*(H_X|_Z))^{22}
\]
2. If \( X = \mathcal{P}_{A_1 \times G_2} \), then
\[
\xi(Z) = 8 \int_0^3 (2 - x)(xp_Y^*H_Y + (6 - 2x)p_Z^*(H_X|_Z))^7.
\]

Moreover \( p_Y^*H_Y \) and \( p_Z^*(H_X|_Z) \) are the divisors on \( E = G/P_{Y,Z} \) corresponding to the weights \( \omega_Y \) and \( \omega_Z \) respectively.

3. Proof of Theorem 0.2 for \( \mathcal{P}_{F_4} \). First we briefly recall the description of the root system \( F_4 \). We follow the description in \cite{Bou92}.

Let \( V = \mathbb{R}^4 \) be the Euclidian space of dimension 4 and \( e_i \) be the \( i \)-th fundamental vector. Then the root system \( F_4 \) consists of the following 48 roots:
- \( \pm e_i \) (\( 1 \leq i \leq 4 \));
- \( \pm e_i \pm e_j \) (\( 1 \leq i \leq j \leq 4 \));
- \( \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \).

Then the following four roots define a basis \( \Delta \) of the root system:
\[ \begin{align*}
\bullet & \ \alpha_1 := e_2 - e_3; \\
\bullet & \ \alpha_2 := e_3 - e_4; \\
\bullet & \ \alpha_3 := e_4; \\
\bullet & \ \alpha_4 := \frac{1}{2}(e_1 - e_2 - e_3 - e_4).
\end{align*} \]

The Dynkin diagram associated to this root system (with the above basis) is as follows:

```
1 -- 2 -- 3 -- 4
```

and the Cartan matrix is

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]

With respect to this choice of basis \( \Delta \), the set \( \Phi^+ \) of positive roots consists of the following vectors: \((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (0, 1, 1, 1), (0, 1, 2, 0), (1, 1, 2, 0), (0, 1, 2, 1), (1, 2, 2, 0), (1, 1, 2, 1), (0, 1, 2, 2), (1, 2, 2, 1), (1, 2, 3, 1), (1, 2, 2, 2), (1, 2, 3, 2), (1, 2, 4, 2), (1, 3, 4, 2), (2, 3, 4, 2), \).\( \langle a, b, c, d \rangle \) means the root \( a \alpha_1 + b \alpha_2 + c \alpha_3 + d \alpha_4 \).

The fundamental weights are as follows:

\[
\begin{align*}
\omega_1 & := (2, 3, 4, 2), \\
\omega_2 & := (3, 6, 8, 4), \\
\omega_3 & := (2, 4, 6, 3), \\
\omega_4 & := (1, 2, 3, 2),
\end{align*}
\]

and the half-sum of positive roots is

\( \rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 = (8, 15, 21, 11) \).

In the following we denote by \( H_\omega \) the divisor corresponding to a weight \( \omega \). Then, by [BH59] Theorem 24.10, the degree of \( H_\omega \) on \( E \) is as follows:

\[
\deg H_\omega = (\dim E)! \prod_{\gamma \in C} \frac{\langle \omega, \gamma \rangle}{\langle \rho, \gamma \rangle}.
\]

where the set \( C \) of complementary roots to the subset \( \{ \alpha_1, \alpha_3 \} \) is, by definition, the set of positive roots each of which is not a linear combination of roots in \( \Delta \setminus \{ \alpha_1, \alpha_3 \} \). By the description of positive roots, we see that \( C \) is \( \Phi^+ \setminus \{ \alpha_2, \alpha_4 \} \), which consists of 22 vectors. Therefore

\[
\xi(Z) = 23 \int_0^8 (3 - x)(xp_1^* H_Y + (8 - x)p_2^* H_Z)^{22} dx \\
= 23 \int_0^8 (3 - x) (22!) \prod_{\gamma \in C} \frac{\langle x \omega_1 + (8 - x) \omega_3, \gamma \rangle}{\langle \rho, \gamma \rangle} dx.
\]

Note that the matrix of products \( \langle \omega_1, \alpha_j \rangle \) are as follows:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1/2
\end{pmatrix}
\]

Thus the products with complementary roots are given as follows:

\[
\begin{pmatrix}
(x \omega_1 + (8 - x) \omega_3, -) & (\rho, -)
\end{pmatrix}
\]
Thus, we have
\[
\prod_{\gamma \in C} (\rho, \gamma) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot 1 \cdot \frac{5}{2} \cdot 2 \cdot 3 \cdot 3 \cdot \frac{5}{2} \cdot 4 \cdot \frac{7}{2} \cdot 3 \cdot \frac{9}{2} \cdot 4 \cdot 5 \cdot 5 \cdot \frac{11}{2} \cdot 6 \cdot 7 \cdot 8
\]
\[= 2^4 \cdot 3^7 \cdot 5^4 \cdot 7^2 \cdot 11\]
and
\[
\prod_{\gamma \in C} (x\omega_1 + (8 - x)\omega_3, \gamma) = -2^{14} x^2 (x - 8)^7 (x + 8)^2 (x - 24)^2 (x - 16)^2.
\]
Therefore
\[
\xi(Z) = \frac{23 \cdot (22!) \cdot 2^{14}}{24 \cdot 3^7 \cdot 5^4 \cdot 7^2 \cdot 11} \int_0^8 x^2 (x - 3)(x - 8)^7 (x + 8)^2 (x - 24)^2 (x - 16)^2 dx
\]
\[= 2^{73} \cdot 19 \cdot 23 \cdot 199 \cdot 1049 > 0.
\]
Hence $\mathcal{P}_{F_4}$ is K-polystable.

### 3.2. Proof of Theorem 0.2 for $\mathcal{P}_{A_1 \times G_2}$

First we recall the root system $G_2$.

Consider two vectors $\alpha_1$ and $\alpha_2$ in the Euclidian vector space $V = \mathbb{R}^2$, where
- $||\alpha_1||^2 = 2$;
- $||\alpha_2||^2 = 6$;
- $(\alpha_1, \alpha_2) = -3$. 

| $\gamma$ | $x$ | 1 |
|---------|-----|---|
| $(1, 0, 0, 0)$ | $x$ | 1 |
| $(0, 0, 1, 0)$ | $(8 - x)/2$ | 1/2 |
| $(1, 1, 0, 0)$ | $x$ | 2 |
| $(0, 1, 1, 0)$ | $(8 - x)/2$ | 3/2 |
| $(0, 0, 1, 1)$ | $(8 - x)/2$ | 1 |
| $(1, 1, 1, 0)$ | $(8 + x)/2$ | 5/2 |
| $(0, 1, 1, 1)$ | $(8 + x)/2$ | 3 |
| $(0, 1, 2, 0)$ | $8 - x$ | 2 |
| $(1, 1, 2, 0)$ | $8$ | 3 |
| $(0, 1, 2, 1)$ | $8 - x$ | 5/2 |
| $(1, 2, 2, 0)$ | $8$ | 4 |
| $(1, 1, 2, 1)$ | $8$ | 7/2 |
| $(0, 1, 2, 2)$ | $8 - x$ | 3 |
| $(1, 2, 2, 1)$ | $8$ | 9/2 |
| $(1, 1, 2, 2)$ | $8$ | 4 |
| $(1, 2, 3, 1)$ | $(24 - x)/2$ | 5 |
| $(1, 2, 2, 2)$ | $8$ | 5 |
| $(1, 2, 3, 2)$ | $(24 - x)/2$ | 11/2 |
| $(1, 2, 4, 2)$ | $16 - x$ | 6 |
| $(1, 3, 4, 2)$ | $16 - x$ | 7 |
| $(2, 3, 4, 2)$ | $16$ | 8 |
Then the set of the following vectors form the root system $G_2$: $\pm\alpha_1$, $\pm\alpha_2$, $\pm(\alpha_1 + \alpha_2)$, $\pm(2\alpha_1 + \alpha_2)$, $\pm(3\alpha_1 + \alpha_2)$, $\pm(3\alpha_1 + 2\alpha_2)$.

Then the roots $\alpha_1$ and $\alpha_2$ define a basis of $G_2$ and the Dynkin diagram is as follows:

Then the fundamental weights are as follows:
- $\omega_1 = 2\alpha_1 + \alpha_2$
- $\omega_2 = 3\alpha_1 + 2\alpha_2$

and the half sum of positive roots is $\rho = \omega_1 + \omega_2 = 5\alpha_1 + 3\alpha_2$.

By considering the direct sum with the root system $A_1$, we have the root system of $A_1 \times G_2$. In the following, we denote the positive root of $A_1$ by $\alpha_0$ and thus the Dynkin diagram is as follows:

Then the half sum of positive roots of $A_1 \times G_2$ is $\rho = \omega_0 + \omega_1 + \omega_2 = \alpha_0 + 5\alpha_1 + 3\alpha_2$.

Similarly to the case $X = \mathcal{P}_{F_4}$, we have

$$
\xi(Z) = 8 \int_0^3 (2 - x)(xp_Y H_Y + (6 - 2x)p_Z H_Z)^7 dx \\
= 8 \int_0^3 (2 - x)(7!) \prod_{\gamma \in C} \frac{(x\omega_Y + (6 - 2x)\omega_Z, \gamma)}{(\rho, \gamma)} dx,
$$

where the set $C$ of complementary roots to the subset $\{\alpha_0, \alpha_1, \alpha_2\}$ is, by definition, the set of positive roots each of which is not a linear combination of roots in $\Delta \setminus \{\alpha_0, \alpha_1, \alpha_2\} = \emptyset$. Thus $C$ is the set of positive roots $\Phi^+$, which consists of 7 vectors.

Note that $\omega_Y = \omega_2$ and $\omega_Z = \omega_0 + \omega_1$. Thus the products with complementary roots are given as follows:

| $x\omega_Y + (6 - 2x)\omega_Z, -$ | $\rho, -$ |
|-----------------------------|-------------|
| $\alpha_0$                 | $6 - 2x$    | 1           |
| $\alpha_1$                 | $6 - 2x$    | 1           |
| $\alpha_2$                 | $3x$        | 3           |
| $\alpha_1 + \alpha_2$      | $6 + x$     | 4           |
| $2\alpha_1 + \alpha_2$     | $12 - x$    | 5           |
\[
\begin{array}{c|c|c}
3\alpha_1 + \alpha_2 & 18 - 3x & 6 \\
3\alpha_1 + 2\alpha_2 & 18 & 9 \\
\end{array}
\]

Therefore
\[
\prod_{\gamma \in C} (\rho, \gamma) = 1 \cdot 1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 9 \\
= 2^3 \cdot 3^3 \cdot 5
\]

and
\[
\prod_{\gamma \in C} (x\omega_Y + (6 - 2x)\omega_Z, \gamma) = 2^3 \cdot 3^4 \cdot x(x - 3)^2(x - 6)(x + 6)(x - 12).
\]

Hence we have
\[
\zeta(Z) = \frac{-8 \cdot (7!) \cdot 2^3 \cdot 3^4}{2^3 \cdot 3^4 \cdot 5} \int_0^3 x(x - 2)(x - 3)^2(x - 6)(x + 6)(x - 12)dx \\
= 2^4 \cdot 3^3 \cdot 5 \cdot 11 \\
> 0.
\]

Then \(\mathcal{P}_{A_1 \times G_2}\) is K-polystable.

4. Remarks

4.1. Remark on specialization of \(\mathcal{P}_{A_1 \times G_2}\). As is mentioned in [Kan20, Remark 4.2], \(X = \mathcal{P}_{A_1 \times G_2}\) is a Mukai 8-fold of genus 7, i.e., \(-K_X = 6H_X\) and \(H_X^5 = 12\). By [Kuz18], it is known that there are two isomorphic classes \(X_{\text{gen}}\) and \(X_{\text{sp}}\) of Mukai 8-fold of genus 7. Moreover \(X_{\text{gen}}\) degenerates isotrivially to \(X_{\text{sp}}\).

The following proposition can be found in [BFM20, Proposition 4.8, Remark 4.10, Remark 5.4]:

**Proposition 4.1.** \(\mathcal{P}_{A_1 \times G_2} \simeq X_{\text{gen}}\).

**Proof.** Set \(X = \mathcal{P}_{A_1 \times G_2}\). Note that \(\text{Aut}^0(X) = A_1 \times G_2\) and thus
\[
\dim H^0(T_X) = 17.
\]

On the other hand we know that \(\chi(T_X) = 17\). By the Akizuki-Nakano vanishing, we have
\[
\dim H^{\geq 2}(T_X) = 0.
\]

Thus \(\dim H^1(T_X) = 0\) and thus \(X\) is rigid. \(\square\)

**Proof of Corollary [7.5]** Assume to that contrary that \(X_{\text{sp}}\) is K-semistable. Then, \(X_{\text{sp}}\) degenerates to a K-polystable Fano variety [CDS15c], [LWX18], and we obtain a contradiction from the description of K-moduli [BX19], [LWX19], [LWX18], [Oda15], [SSY16]. For instance, by [BX19, Theorem 1.1], \(X_{\text{gen}}\) and \(X_{\text{sp}}\) are \(S\)-equivalent. By [LWX18], \(X_{\text{gen}}\) and \(X_{\text{sp}}\) have a common K-polystable degeneration, which should be \(X_{\text{gen}}\). This contradicts to the fact that \(X_{\text{gen}}\) is rigid. \(\square\)

**Remark 4.2.** In fact, by carefully checking constructions in [Kuz18], we have a test configuration which degenerates \(X_{\text{gen}}\) to \(X_{\text{sp}}\). Together with the K-polystability of \(X_{\text{gen}}\), this implies the K-unstability of \(X_{\text{sp}}\).

**Remark 4.3.** In a response to the first version of this paper, Baohua Fu has pointed out that the automorphism group of \(X_{\text{sp}}\) is not reductive as follows: By [FH18, Remark 2.13] (see also [BFM20, Section 4.2]), \(X_{\text{sp}}\) is an equivariant compactification of \(\mathbb{C}^8\). Then, it follows from [AP14, Proposition 1] that the automorphism group of \(X_{\text{sp}}\) is not reductive, since \(X_{\text{sp}}\) is not homogeneous.
This implies that there are no Kähler-Einstein metrics on $X_{sp}$. Note that the non-reductivity of the automorphism group is not enough to conclude the K-unstability of $X_{sp}$.

4.2. Remark on foliations. Let $X$ be a Fano manifold satisfying Condition 0.1. Then $X$ is horospherical or isomorphic to $\mathcal{P}_{F_4}$ or $\mathcal{P}_{A_1 \times G_2}$. In each case, the blow-up along the closed orbit admits a similar diagram as in (1.8.2), and the smooth morphism $\pi$ defines a $G$-invariant foliation $\mathcal{F}$ on $X$, which is called the canonical foliation on $X$ [Kan20, Section 3]. The first Chern class of $\mathcal{F}$ and the existence of Kähler-Einstein metrics are summarized as follows (see also [Kan20, Proposition 3.5]):

| $X$          | $c_1(\mathcal{F})$ | KE or not |
|--------------|---------------------|-----------|
| horospherical| $> 0$               | Not KE    |
| $\mathcal{P}_{A_1 \times G_2}$ | 0                    | KE        | $\mathcal{P}_{F_4}$ |

A naive question asks whether or not there is a relation between $G$-invariant foliations with positive first Chern class and Kähler-Einstein metrics.

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