RIGIDITY RESULTS FOR HERMITIAN-EINSTEIN MANIFOLDS

STUART J. HALL AND THOMAS MURPHY

Abstract. This article draws attention to a differential operator introduced by A. Gray on the unit sphere bundle of a Kähler-Einstein manifold. We study this operator for various families of almost-Hermitian Einstein manifolds.

One application of our techniques is a classification of closed cscK surfaces with non-negative bisectional curvature. This gives a simplified proof of the generalized Frankel conjecture for surfaces using Kähler-Ricci flow, following work of Chen and Tian. We prove a rigidity theorem classifying complex space forms amongst compact Hermitian surfaces. Following Sekigawa and Sato we extend the study of Gray’s operator to nearly Kähler manifolds, obtaining results characterising the nearly Kähler six sphere. We also use Gray’s operator to estimate the first eigenvalue of the Laplacian for the Sasaki metric on the unit sphere bundle of a Kähler-Einstein manifold and give an application of this estimate to spectral geometry.

AMS Subject Classification: 53C55, 53C25.

1. INTRODUCTION

1.1. Results for the Kähler case. Rigidity theorems in Kähler geometry aim to show that the only Kähler manifolds satisfying some natural curvature condition are the known examples. A famed example is the following. Complex projective space $\mathbb{C}P^N$ equipped with the standard Fubini-Study metric $g_{FS}$ has positive sectional curvature, and hence positive bisectional curvature. This is immediate from the fact that the bisectional curvature can be written as

$$B_{ij} = R_{ii^*jj^*} = \sec_{ij} + \sec_{ij^*}$$

with respect to a complex orthonormal basis $\{e_1, e_1^* = Je_1, \ldots\}$ on a Kähler manifold. The Frankel conjecture then states that every compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbb{C}P^N$. This was established independently by Mori [20], and Siu-Yau [24]. An interesting new proof was given by Chen and Tian [8] using the Kähler-Ricci flow. Positive bisectional curvature is preserved under this flow, and they proved exponentially fast convergence to a Kähler-Einstein metric. Then one simply applies the following classical result of Berger to deduce the Frankel conjecture.
Theorem 1.1. (Berger [2], Goldberg-Kobayashi [14]). A closed cscK manifold has positive bisectional curvature if, and only if, it is isometric to $(\mathbb{C}P^N, g_{FS})$.

This is proven via relatively straightforward maximum principle arguments using the ordinary Laplacian.

A second example of a rigidity theorem comes from considering Hermitian locally symmetric spaces of compact type. These were classified by Cartan, and arise as certain quotients of compact Lie groups equipped with biinvariant metrics. Thus they always have non-negative sectional curvature by O’Neill’s formula, and hence non-negative bisectional curvature. Moreover, products of such manifolds also have non-negative bisectional curvature. The converse statement is the generalized Frankel conjecture, established by Mok [17]. It is natural to attempt to use the Kähler-Ricci flow to prove this result, as the condition of nonnegative bisectional curvature is also preserved under the Kähler-Ricci flow. The results of Chen and Tian (c.f. Remark 9.1 of [8]) hold in this setting too, so one obtains exponentially fast convergence to a Kähler-Einstein metric with $B_{ij} \geq 0$. This leads to the natural question:

Which closed cscK manifolds have non-negative bisectional curvature?

This was asked by Siu in his ICM address [23] and answered by Mok and Zhong [19]. Chen and Tian then appeal to their work to answer this question and give a new proof of the generalized Frankel conjecture.

Historically however, the first attempt to answer this question came from the following theorem, which is the main inspiration for this work.

Theorem 1.2. (Gray [10]) Let $(M, g)$ be a closed cscK manifold. If $M$ has non-negative sectional curvature, then $(M, g)$ is isometric to a locally symmetric space of compact type.

Gray also gives two new proofs of Theorem [11] as a consequence of his method. Mok and Zhong state that there are apparently serious difficulties to adapting Gray’s method to the case where the bisectional curvature is non-negative. One application of our work is to show that a modification of Gray’s approach does work for surfaces, and in fact gives a markedly simpler proof which is similar to the proof of Theorem [11]. Mok and Zhong’s approach uses, amongst other things, theorems of Cartan-Oka concerning coherent sheaves, a fourth-order maximum principle and an involved argument involving parallel translation of certain subspaces in the tangent bundle which satisfy special curvature relations.

The method Gray uses to prove this theorem is extremely interesting. He introduces a linear differential operator $L$ on the unit sphere bundle $S(M)$ of $M$, whose coefficients are determined by the sectional curvatures of $M$. Thus, for example, when $M$ is positively curved $L$ turns out to be elliptic. However $L$ is not as well-behaved if one assumes $M$ has nonnegative...
bisectional curvature. Gray states in [10] that he expects the method to have further applications; we aim to realize his vision.

Throughout the paper, we will assume all manifolds are smooth, connected, and without boundary. For any manifold \((M, g)\), let \(S(M)\) denote the unit sphere bundle of \((M, g)\), with fibre \(S_p(M)\) over a point \(p \in M\). Equip \(S(M)\) with the Sasaki metric \(g_{\text{sas}}\). An almost complex structure will always be denoted by \(J\). When \((M, g)\) has an almost-complex structure \(J\), we will be concerned with studying the holomorphic sectional curvature

\[
H(x) = B_{xx} = R_{xJxxJx}.
\]

This is closely related to studying \(S(M)\) because, as Berger [2] noticed, \(H\) can be viewed as a smooth function on \((S(M), g_{\text{sas}})\).

The main result for Kähler surfaces is as follows:

**Theorem 1.3.** A closed cscK surface has non-negative bisectional holomorphic curvature if, and only if, it is isometric to a locally symmetric space of compact type.

In particular such a surface is isometric to either \((\mathbb{C}P^2, g_{FS})\) or products of \((\mathbb{C}P^1, g_{FS})\) and flat complex tori, and this gives a direct proof of the generalized Frankel conjecture for surfaces. The idea is to apply the maximum principle to the horizontal part of Gray’s \(L\) operator. Any attempts to directly mimic Berger’s proof will not work as one cannot give a sign to several of the terms if the bisectional curvature is only assumed to be nonnegative.

### 1.2. Results for nearly Kähler metrics.

In the third section of the paper, we generalize this approach to study the differential operator \(L\) on nearly Kähler manifolds. This approach was first taken in Sekigawa-Sato [22], whose work heavily influenced us. Recall that a nearly Kähler manifold is defined to be an almost-Hermitian manifold \((M, g, J)\) such that

\[
(\nabla_X J)X = 0
\]

for all \(X \in \Gamma(TM)\). These objects, brought to the fore by Gray’s work on weak holonomy, are fascinating as they lie at the crossroads of many important paths in Riemannian geometry. Some examples: they are the fundamental supersymmetric solutions in type IIB string theory, and occur as critical points of the Hitchin functional. All six-dimensional strict (i.e. not Kähler) examples admit Killing spinors, and so are Einstein. They furnish examples of non-Kähler metrics with vanishing first Chern class. As a final example, a closed Riemannian six-manifold \((M, g)\) is nearly-Kähler if, and only if, the cone metric on \(\mathbb{R} \times M\) has holonomy \(G_2\) (with one exception), and in this way their study is intimately connected with the study of conical singularities of \(G_2\) metrics [16].

Known examples of strict nearly Kähler manifolds arise via algebraic methods. In particular, every known closed example is a 3-symmetric space. Via the well-known work of Nagy [21], every nearly Kähler manifold is six-dimensional, up to finitely many exceptions. Following the convention in
the literature, unless we state otherwise we will assume throughout that a nearly Kähler manifold is strict, six-dimensional and is scaled to have scalar curvature \( s = 30 \). These are also called Gray manifolds. The easiest nontrivial example is \( S^6 = G_2/SU(3) \), equipped with the standard round metric of sectional curvature 1. As Verbitsky [26] points out, the outstanding question is to determine if there are examples beyond these. Locally, Bryant [6] has calculated that nearly Kähler structures on six manifolds depend upon two functions of five variables, but nobody has constructed a global deformation of a 3-symmetric space to produce new closed examples. Taking the sine-cone over a five-dimensional Sasaki-Einstein metric gives many explicit local examples of nearly Kähler six-manifolds.

To state our main result about nearly Kähler metrics, the first step is the observation that Gray’s method can be rephrased if we assume the metric is Kähler-Einstein. Let us reformulate his theorem from our perspective to make this clear.

Consider the sphere bundle \( \mathcal{E} \to S(M) \), with the fibre at \((p, x) \in S(M)\) defined by

\[
\mathcal{E}_x := \{y \in T_p M : g(x, y) = 0, \|y\| = 1\}.
\]

We have \( \mathcal{E}_x = S^{n-2} \). This is an alternative description of the two-plane Grassmannian, and so the sectional curvature is a function \( \sec : \mathcal{E} \to \mathbb{R} \). Theorem 1.2 states that if \( \sec \) is a nonnegative function on \( \mathcal{E} \) then \((M, g)\) is locally symmetric. Let \( \Gamma(\mathcal{E}) \) denote the space of smooth sections of \( \mathcal{E} \).

Then the same proof as Theorem 1.2 yields:

**Theorem 1.4.** (Gray [10]) Let \((M, g)\) be a closed Kähler-Einstein manifold. Then there exists \( G \in \Gamma(\mathcal{E}) \) such that \( (\sec)|_G \geq 0 \) if, and only if, \((M, g)\) is locally isometric to a symmetric space of compact type. In particular, if \( (\sec)|_G > 0 \), then \( M \) is isometric to \( \mathbb{C}P^N \).

The section \( G \) is described via an orthonormal framing of \( M \). This result states that if the function \( \sec \) is nonnegative on this distinguished section \( G \) of \( \mathcal{E} \) (which we will call the Gray section) the assumption that the underlying metric is Kähler-Einstein is strong enough to force \( \sec \) to be nonnegative everywhere on \( \mathcal{E} \). From our perspective, this is notable as one obtains global information about the behavior of a function on \( \mathcal{E} \) from just knowing it on one section, namely \( G \). A priori, anything could happen away from \( G \). The study of the sectional curvature of the Gray section is one of the key techniques we exploit to prove our results throughout this paper.

We then prove the following, building on ideas in the proof of Theorem 4.2 in [22]:

**Theorem 1.5.** Let \((M^6, g)\) be a closed nearly Kähler manifold. If \( (\sec)|_G \geq \frac{3}{4} \), then \((M, g)\) is isometric to \( S^6 \) equipped with the round metric of sectional curvature 1.
Just as in the Kähler case, the point is that if the metric looks like one of the canonical known examples on the Gray section $G \in \Gamma(\mathcal{E})$ then it must be isometric to the canonical example.

**Corollary 1.6.** Let $(M^6, g)$ denote a closed nearly Kähler manifold. Then it has holomorphic sectional curvature $H \geq \frac{2}{3}$ if, and only if, $H = 1$ and $(M^6, g)$ is isometric to $S^6$ equipped with the round metric of sectional curvature 1.

### 1.3. The spectral geometry of the Sasaki metric.

The first non-zero eigenvalue of the scalar Laplacian is one of the most important quantities associated to any metric. There is a famous bound due to Lichnerowicz for the first non-zero eigenvalue of the Laplacian of a closed manifold with positive Ricci curvature. It is of great interest to see if one can get similar bounds for other naturally occurring families of metrics. Throughout this section let $(M, g)$ be any closed Kähler-Einstein metric of real dimension $n = 2N$. We adopt the convention that the Laplacian has nonpositive eigenvalues. Our goal is to derive a universal upper bound for $\lambda_1$ of the Sasaki metric $g_{sas}$ on $S(M)$ using Gray’s differential operator.

$(M, g)$ is normalized when

$$\max_{x,y \in TM}\{|\text{sec}(x,y)| = 1\}.$$ 

This is just to factor out rescaling the metric by homothety, and can always be done if $M$ is not isometric to a flat torus.

**Theorem 1.7.** Let $(M, g)$ be a closed normalized Kähler-Einstein manifold which is not isometric to a complex space form. Then

$$\lambda_1(S(M), g_{sas}) \geq -6(n + 2).$$

**Remark.** In the special cases that $(M, g)$ has positive Einstein constant (i.e. $M$ is a Fano manifold), and in addition admits holomorphic vector fields, the bound given by the above theorem is not optimal. One could pull back an eigenfunction for the first eigenvalue of $(M, g)$ and obtain a better lower bound on $\lambda_1(S(M), g_{sas})$. Theorem 1.7 does however yield information for the general Fano case, as well as for the Ricci-flat and negative cases. The striking thing about the result is that it is completely independent of the volume, diameter and a lower bound for $\text{Ric}$. Rescaling has been ruled out by the assumption $g$ is normalized. In particular, if one has a family of normalized Kähler-Einstein manifolds $M^n_i$ with $\text{Vol}_i \to 0$, then $\text{Vol}(S(M), g_i)$ also goes to zero but $\lambda_1(g_i)$ remains bounded. A family of such metrics are given by the following construction. Let $E = X \times \mathbb{T}^n$ where $(X, g_X)$ is a normalized K3 surface and $(\mathbb{T}^n, g_0)$ is a flat metric. Equip $E$ with the metric $g_i = g_X + ig_0$, and let $i \to 0$.

**Corollary 1.8.** The unit sphere bundle over a normalized family $(E, g_i)$ satisfies $\lambda_1 \geq -6(n + 2)$ for all $i$. 

Spectral geometers have long been interested in constructing metrics with such counterintuitive properties, motivated by a question of Berger [4]. Such metrics exist on any manifold due to work of Colbois and Dodziuk [9], yet it is notable that one always has such behavior for such a naturally occurring family of metrics.

1.4. Results for Hermitian-Einstein surfaces. A second motivating result in writing this work is the following classical result of Berger (which was instrumental in his proof of Theorem 1.2 for the case of positive sectional curvature):

\textbf{Theorem 1.9.} [2] Let \((M, g)\) be a Kähler manifold of dimension real dimension \(n = 2N\). Then

\[ \int_{S_p(M)} H\omega_2 = \frac{s}{(N)(N + 1)} \text{Vol}(S^{n-1}) \]

for all \(p \in M\).

The term \(\omega_2\) in the above result comes from the splitting of the of canonical volume form of the Sasaki metric on \(S(M)\) \(\omega =\omega_1 \wedge \omega_2\), where \(\omega_1\) is the volume form for \((M, g)\) and \(\omega_2\) is the standard volume form for the sphere with radius 1. Finally \(s\) denotes the scalar curvature of \(g\). The classification of Kähler-Einstein manifolds with non-zero holomorphic sectional curvature is a notable open problem; for example \(\mathbb{C}P^N \times \mathbb{C}P^N\) has positive holomorphic sectional curvature, but so too does the complex quadric \(Q^3\) (all spaces here are equipped with their standard symmetric-space metrics). To our knowledge this theorem of Berger gives us the best-known rigidity result in this direction, namely it follows immediately from his result that a Kähler-Einstein manifold has \(H \leq \frac{s}{(N)(N + 1)}\) (or \(\geq\)) if, and only if \(H = \frac{s}{(N)(N + 1)}\) and consequently \((M, g)\) is a complex space form.

We can extend this to any almost-Hermitian manifold as follows:

\textbf{Theorem 1.10.} Let \((M, g, J)\) be an almost-Hermitian manifold of real dimension \(n = 2N\). Then

\[ \int_{S_p(M)} H\omega_2 = \frac{3s^* + s}{4(N)(N + 1)} \text{Vol}(S^{n-1}) \]

for all \(p \in M\).

We define \(R^*(x, y)\), the \(\ast\)-Ricci curvature, in the following way. Then

\[ R^*_{ij} := \sum_{a=1}^{n} R_{ai^* ja^*} \]

is the star Ricci tensor, and its trace \(\sum_i R^*_{ii} = \sum_i R_{ai^* i}\) is the \(\ast\)-scalar curvature \(s^*\). We caution the reader that this definition is slightly different to the usual one in the literature: our definition agrees with the usual definition in the Kähler and nearly Kähler cases. We use this convention as it is more
convenient to express our results and it gives the correct generalization of Berger’s result. We have to use very different ideas to Berger, who heavily relies on the Kähler identities and local calculations in the curvature tensor to prove his result. Of course in the Kähler case $s^* = s$ and so we recover his result. The technique we use to calculate this identity arose from Sekigawa and Sato’s work [22] extending the study of Gray’s differential operator to the nearly Kähler case. The main application of this estimate is that, combined with the work of Apostolov, Davidov and Muskarov [1] we obtain the following rigidity result for closed Hermitian surfaces;

**Corollary 1.11.** Let $(M^4, g, J)$ be a closed Hermitian surface Then

$$H \leq \frac{3s^* + s}{24}$$

(or $\geq$) if, and only if, we have equality and $(M, g, J)$ is isometric to $\mathbb{C}P^2$, $\mathbb{C}^2/\Gamma$ or $\mathbb{C}H^2/\Gamma$ equipped with their standard symmetric space metrics.

---

2. **Gray’s differential operator**

In this section we review the techniques Gray used to prove Theorem 1.2, remarking that his entire construction generalizes to the case of $J$ being an almost-complex structure. We caution the reader that there are some problems with Lemma 5.3 of [10], which we correct here. As Gray’s paper is tersely written with many nontrivial steps omitted we record here some of the arguments for the convenience of the reader. We follow the convention that $X, Y, Z \in \Gamma(TM)$ are smooth vector fields, and $x, y, z$ denote tangent vectors at $T_pM$.

We define the Riemannian curvature tensor as

$$R_{WXYZ} = g(\nabla_{[W,X]}Y - [\nabla_W, \nabla_X]Y, Z),$$

and set

$$\text{sec}(x, y) = \frac{R(x, y, x, y)}{||x||^2||y||^2 - g(x, y)}$$

to be the sectional curvature of the plane spanned by $x, y \in T_pM$. We will occasionally write $R(W; X, Y, Z)$ for $R_{WXYZ}$ to make calculations easier to read. For $x \in S_p(M)$ take an orthonormal bases $\{e_1, \ldots, e_n\}$ of $T_pM$ with the convention that $x = e_1$. Then $S_p$ is the fibre of the sphere bundle $S(M) \to M$ over $p \in M$. Equip $S(M)$ with the Sasaki metric, and for $X \in \Gamma(TM)$ denote by $X^h$ (resp. $X^v$) the horizontal (resp. vertical) lift. Then $\{e^h_1, e^v_1\}$ form an orthonormal basis of $T_{(p,x)}S(M)$. Denote by $(y_2, \ldots, y_n)$ the corresponding system of normal coordinates defined on a neighbourhood of $x$ in the sphere $S_p$, and let $(x_1, \ldots, x_n)$ denote the normal coordinates corresponding to $(e^h_1)$. Set $y_0(x) = 0$. 

2.1. **The operator $\Box_\psi$.** Given any Riemannian manifold $(M, g)$ and a section $\psi \in \Gamma(\text{End}(TM))$, Gray defines the following second-order differential operator $\Box_\psi$.

**Definition 2.1.** For $f \in C^\infty(M)$, set

$$\Box_\psi f = \text{tr}(\psi \circ \nabla \text{grad}(f)) = \text{tr}(\psi \circ \nabla^2 f).$$

The principal symbol of $\Box_\psi$ is easily seen to be $\psi^*$. We have the following well-known yet important lemma;

**Lemma 2.2** (Lemma 3.1 in [10]). Let $(M^n, g)$ be a closed orientable manifold, $\psi \in \Gamma(\text{End}(TM))$ be a symmetric section of the endomorphism bundle with $\text{div}(\psi) = 0$. Then $\Box_\psi$ is self-adjoint.

**Proof.** We first note the following identities:

$$\text{div}(\psi(X)) = \sum_{i=1}^n g(\nabla_{E_i} \psi(X), E_i) = \sum_{i=1}^n \nabla_{E_i} g(\psi(X), E_i)$$

and

$$g(\text{div} \psi, X) = \sum_{i=1}^n g(\nabla_{E_i} \psi)(E_i), X) = \sum_{i=1}^n (\nabla_{E_i} g(\psi(X), E_i) - g(\psi(\nabla_{E_i} X), E_i)),$$

where $X \in \Gamma(TM)$ and $\{E_i\}$ is a normal orthonormal frame. Hence we can write

$$\text{tr}(\psi \circ \nabla X) = \text{div}(\psi(X)) - g(\text{div} \psi, X)$$

and so if $\text{div} \psi = 0$,

$$\Box_\psi f = \text{div}(\psi(\text{grad}(f))).$$

Hence for two functions $f_1, f_2 \in C^2(M)$ we have

$$\int_M (\Box_\psi f_1) \cdot f_2 dV_g = \int_M \text{div}(\psi(\text{grad}(f_1))) \cdot f_2 dV_g$$

$$= \int_M g(\psi(\text{grad}(f_1)), \text{grad}(f_2)) dV_g$$

$$= \int_M f_1 \cdot \text{div}(\psi(\text{grad}(f_2))) dV_g$$

$$= \int_M f_1 \cdot (\Box_\psi f_2) dV_g.$$
2.2. The Horizontal Laplacian. Given any connection on $TM$ (in particular the Levi-Civita connection) we get a splitting of $TTM$ into

$$TTM = \mathcal{H} \oplus \mathcal{V}$$

where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical subspaces respectively. This allows one to define two projections in the obvious manner,

$$Hor : TTM \to \mathcal{H} \quad \text{and} \quad Ver : TTM \to \mathcal{V}$$

Given a vector field $X \in \Gamma(TTM)$ we will write $X^h = Hor(X)$ and $X^v = Ver(X)$. As $\mathcal{H}$ and $\mathcal{V}$ are orthonormal with respect to the Sasaki metric on $TM$, the maps $Hor$ and $Ver$ are symmetric as sections of $End(TTM)$. They both descend to symmetric sections of $End(TS(M))$ and in particular we have the following:

Lemma 2.3 (Lemma 6.1 in [10]). Let $Hor : TS(M) \to \mathcal{H}$ be defined as above. Then $\text{div}(Hor) = 0$.

Proof. This is easily seen by choosing a normal orthonormal frame about any point $(p, x) \in S(M)$. □

Using Lemmas 2.2 and 2.3 we see that the operator $\Box_{Hor}$ is self adjoint. The operator $\Box_{Hor}$ is called the horizontal Laplacian and is denoted by $\Delta^h$.

2.3. An operator on $S(M)$. We consider an tensor field $\phi \in End(T(T_pM))$ defined by

$$g_{sas}(\phi(u), v)(p, x) = R_{xuxv}(p)$$

where $u, v \in T_x(T_pM)$. The vectors in the righthand side are identified with vectors in $T_pM$ via parallel transport. The symmetries of the curvature operator make it clear that $\phi$ is a symmetric endomorphism. This tensor field restricts to a tensor field on the unit sphere in $T_pM$. The divergence of $\phi$ (calculated with respect to the Sasaki metric on the sphere bundle $S(M)$ restricted to the fibre over $p$) is a vector field on $S_p$. Gray calculates the following pointwise identity:

Lemma 2.4 (Lemma 4.3 in [10]). Let $(M, g)$ be a closed manifold and let $\phi$ be defined as above. Then

$$g_{sas}(\text{div}(\phi), v)(p, x) = -\text{Ric}(x, v)(p)$$

for any $v \in T_x(T_pM)$.

This immediately yields that $\text{div}(\phi) = 0$ for an Einstein manifold.

We can extend this to a section of $End(TS(M))$ using the $Ver$ map from the previous section. Specifically, we have

$$g_{sas}(\phi(u), v)(p, x) = R(x, u^v, x, v^v)(p)$$

where the vectors on the righthand side are identified with those in $T_pM$ via parallel transport.
Lemma 2.5 (Lemma 6.1 in [10]). Let $(M, g)$ be a Riemannian manifold and let $\phi \in \text{End}(TS(M))$ be defined as above. Then if $(M, g)$ is Einstein $\text{div}(\phi) = 0$ and so $\square_{\phi}$ is a self-adjoint operator on $S(M)$.

2.4. Gray’s $L$ operator. For each $y \in T_pM$ the tangent space $T_y(T_pM)$ is identified with $T_pM$ by means of parallel translation. Under this identification we write $\frac{\partial}{\partial u^i}$ to correspond to $e_i$. Set $h_{ij} = R_{ixjx}$. We now define the operator $L$ on $S(M)$.

Definition 2.6. Let $(M, g)$ be a Riemannian manifold and let $\phi \in \text{End}(TS(M))$ be defined as above. Then the operator $L$ is defined by

$$L = \Delta^h + \frac{1}{2} \square_{\phi}.$$ 

If $(M, g)$ is Einstein then this operator is self-adjoint by Lemma 2.5. This operator can be calculated in the local normal coordinates $\{x^i, y_\alpha\}$ by

$$L(p, x) := \left\{ \sum_{i=1}^n \frac{\partial^2}{\partial x^2_i} + \frac{1}{2} \sum_{i,j=2}^n h_{ij} \frac{\partial^2}{\partial y_i \partial y_j} \right\}_{(p, x)}.$$ 

Viewing $H(x)$ as a function on $S(M)$, we wish to compute $L(H)$. The first step is to calculate the gradient of $H$.

Lemma 2.7 (Lemma 4.1 in [10]). Let $(M, g)$ be Riemannian manifold and let $p \in M$. If $F : S_p \to \mathbb{R}$ is a $C^1$ function then

$$\frac{\partial F}{\partial y_i} = \frac{\partial}{\partial u^i} F \left( \cos(r)x + \frac{\sin(r)}{r} \sum_{\gamma=2}^n u_\gamma e_\gamma \right),$$

where $r^2 = \sum_{\gamma=2}^n u_\gamma^2$.

This yields:

Lemma 2.8. Let $(M, g, J)$ be a Kähler manifold and for $x \in S_p$ let $H(x)$ be the holomorphic sectional curvature. If $\{e_1, ..., e_n\}$ is a normal orthonormal frame in a neighbourhood of $p \in M$ with $e_1(p) = x$. Then

1. $\text{grad}^h(H)(x) = \sum_{i=1}^n g(\nabla_{e_i} R)(x, Jx, x, Jx)e_i^h$

2. $\text{grad}^v(H)(x) = 4 \sum_{i=2}^n g(R(x, Jx)x, Je_i)e_i^v$,

where $e_i^h$ and $e_i^v$ are the horizontal and vertical lifts of $e_i$ respectively.

Proof. Identity (1) follows from

$$\text{grad}^h(H)(x) = \sum_{i=1}^n \nabla_{e_i}^h R(x, Jx, x, Jx)e_i^h = \sum_{i=1}^n g(\nabla_{e_i} R)(x, Jx)x, Jx)e_i^h.$$
For (2) we take the normal coordinates corresponding to \(\{e^h_i, e^v_i\}\), denoted \((x_1, \ldots, x_n, y_2, \ldots, y_n)\). It follows from Lemma 2.7 that
\[
\frac{\partial H}{\partial y_i}(p, x) = \left. \frac{\partial}{\partial u_i} H \left( \cos(r)x + \frac{\sin(r)}{r} \sum_{\gamma > 1} u_\gamma e_\gamma \right) \right|_{u=0}.
\]
Expanding and using the \(J\)-invariance of the curvature tensor we see that
\[
H \left( \cos(r)x + \frac{\sin(r)}{r} \sum_{\gamma > 1} u_\gamma e_\gamma \right) = \cos(r)H(x) + \frac{\cos^3(r)\sin(r)}{r} \sum_{\gamma > 1} u_\gamma 4R_{x,Jx,Je_\gamma} + O(u^2).
\]
Taking the derivative and evaluating at \(u = 0\) the result follows. \(\square\)

The following lemma is the crucial reason why Gray’s method is so powerful in the Kähler-Einstein setting.

**Lemma 2.9** (Lemma 5.1 in [10]). If \((M, g)\) is a Kähler-Einstein manifold, then \(L(H) = 0\).

This allows us to reprove Berger’s classical result following Gray’s proof [10].

**Theorem 2.10.** A closed Kähler-Einstein manifold has positive sectional curvature if, and only if, it is isometric to \((\mathbb{C}P^n, g_{FS})\).

**Proof.** If \(\text{sec} > 0\), then locally \(L\) has the form
\[
L = A_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i}
\]
with \(A_{ij}(x)\) positive definite. If \(L(H) = 0\) then the Hopf maximum principle for \(L\) yields that \(H\) is constant. The result then follows. \(\square\)

A vertical vector \(v \in T_x(S_p(M))\) is written as \(\sum_{i=2}^n v_i e^v_i\). Then there is a corresponding horizontal vector \(\sum_{i=2}^n v_i e^h_i\). Thus we get a map from \(v\) to an element of \(S_x(M)\), which we denote as \(v \rightarrow \eta(v)\).

The following is the key calculation in Gray’s paper:

**Lemma 2.11** (Lemma 5.3 and 7.1 in [10]). Let \((M, g, J)\) be an almost-complex Einstein manifold. Then the following holds:

1. \(L(H^2) = 2\|\text{grad}^h H\|^2_{(p,x)} + R_{x,\eta(\text{grad}^v H(x)),x,\eta(\text{grad}^v H(x))}\).

2. \(\int_{S(M)} L(H^2) \omega = 0\).

**Remark.** Note that (1) above corrects mistakes in Lemma 5.3 of [10]. The first is a slight abuse of notation: the vector \(\text{grad}^v H\) is identified with the vector in \(T_pM\) that lifts to \(\text{grad}^v H\). More importantly, the curvature term should actually be
\[
R_{x,\eta(\text{grad}^v H(x)),x,\eta(\text{grad}^v H(x))} = \text{sec}(x,\eta(\text{grad}^v H(x)))\|\eta(\text{grad}^v H(x))\|^2.
\]
The term Gray missed is positive everywhere which means that the same arguments to prove Theorem 1.2 can be adapted in the Kähler case.

Finally, we note that it is well-known that for a nearly Kähler manifold (not necessarily strict) \( \text{grad}^v H = 0 \) if, and only if, \( M \) has constant holomorphic sectional curvature [25].

Now we record Gray’s proof of Theorem 1.4 to highlight the role of the Gray section.

**Proof.** Suppose \((M,g)\) is Kähler-Einstein. We define the section \( G \) as the map
\[
G : x \to \eta(\text{grad}^v H(x))
\]
From Lemma 2.11 we have
\[
0 = \int_{S(M)} L(H^2) \omega = \int_{S(M)} 2\|\text{grad}^h H\|^2 \omega + \int_{S(M)} R_{xG(x)G(x)} \omega
\]
The second term can be rewritten as \( \sec(x,G(x))\|G(x)\|^2 \), which is nonnegative because by assumption the sectional curvature restricted to the section \( G \) is nonnegative. Hence \( \|\text{grad}^h H\|^2 = 0 \), whence
\[
\nabla_x R_{y,G^y} \omega(p) = 0.
\]
From here, a computation using the Kähler identities finishes the proof. A similar computation can be found in [11].

Note that in general we cannot deduce global results by just knowing the sectional curvature on one section of \( E \). This can be seen by considering the section \( J \) which is defined by \( J(x) = Jx \). Obviously \( \sec_{J}(x) = H(x) \), but \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) with the standard product metric gives an example of a Kähler-Einstein manifold with \( H > 0 \). Surprisingly, it remains an interesting open problem to classify the closed Kähler-Einstein metrics with positive holomorphic sectional curvature.

### 3. Nearly Kähler manifolds

The proof of Theorem 1.5 is essentially follows ideas of [22], which was a major inspiration to us in writing this paper. Lemma 2.11 still holds, but crucially it is no longer true that \( L(H) = 0 \). Once one has decomposed \( L \) in the same manner following [22], the proof will be along the same lines as their Theorem 4.2. However, this proof has some errors in the calculations, so we explain a corrected approach here. We also prove that one only has to impose the condition on the sectional curvatures along the Gray section. Throughout this section, we will assume \((M,g)\) is a strict nearly Kähler six-manifold. This implies \( M \) is Einstein.

Following the approach in [22], define
\[
f_1(p,x) = \sum_{i,j=1}^{6} R_{iJ^iJ^j} g((\nabla_{e_i} J)x, (\nabla_{e_j} J)x),
\]
\[ f_2(p, x) = \sum_{i,j=1}^{6} g((\nabla_{e_i} R)(x,Jx)x, (\nabla_{e_j} J) x) , \]

\[ f_3(p, x) = \sum_{i,j=1}^{6} g(R(x, (\nabla_{e_i} J)x)x, (\nabla_{e_j} J)x) , \]

and

\[ f_4(p, x) = \sum_{i=1}^{6} g(R(x, Jx)x, (\nabla_{e_i}^2 J)x) . \]

Then one calculates that

\[ L(H)(p, x) = 6f_1(p, x) + 4f_2(p, x) + 2f_3(p, x) + 2f_4(p, x) , \]

for all \((p, x) \in S(M)\). As \(M\) is Einstein, \(L\) is self-adjoint and so, as in Lemma 2.11

\[ 0 = \int_{S(M)} L(H^2) \omega \]
\[ = \int_{S(M)} 2HL(H) + 2\|\text{grad}^h H\|^2 + R_{xG(x)xG(x)} \omega . \]

Now we have to evaluate the integral \(\int_{S(M)} \|\text{grad}^h H\|^2\), again following [22]. This involves defining the following functions

\[ g_1(p, x) = \sum_{i=1}^{6} g((\nabla_{e_i} R)(x,Jx)x, Jx)^2 , \]
\[ g_2(p, x) = \sum_{i=1}^{6} g((\nabla_{e_i} R)(x,Jx)x, Jx) \cdot g(R(x, Jx,x, (\nabla_{e_j} J)x) , \]
\[ g_3(p, x) = \sum_{i=1}^{6} g(R(x, Jx)x, (\nabla_{e_i} J)x)^2 . \]

Then

\[ \int_{S(M)} \|\text{grad}^h H\|^2 \omega = \int_{S(M)} (g_1 + 4g_2 + 4g_3) \omega . \]

This allows \(\int_{S(M)} L(H^2) = 0\) to be rewritten as

\[ 0 = \int_{S(M)} 2g_1 + 8g_2 + 8g_3 + 12H f_1 + 8H f_2 + 4H f_3 + 4H f_4 \]
\[ + R_{xG(x)xG(x)} \omega . \]
We note without proof that, for a nearly Kähler manifold, \( g_1 = 0 \) if, and only if, \((M, g, J)\) is a 3-symmetric space \([12]\). Then one calculates \([25]\) that

\[
\begin{align*}
f_1 &= \frac{s}{30} \left( \frac{s}{30} - H(x) \right), \\
f_3 &= -\frac{s}{30} \left( H(x) - \frac{s}{6} \right), \\
f_4 &= -\frac{2s}{15}.
\end{align*}
\]

To work out \(L(H)\), one needs the following lemma:

**Lemma 3.1.** (See \([22]\)) We have the following identities, where \(\omega_2\) denotes the standard volume form on \(S_p\):

\[
\begin{align*}
(1) \quad \int_{S_p} H \omega_2 &= \frac{s}{30} \text{Vol}(S^5), \\
(2) \quad \int_{S_p} H^2 \omega_2 &= \frac{1}{16} \int_{S_p} \|\text{grad}^v H\|^2 \omega_2 + \left( \frac{s^2}{900} \right) \omega_2.
\end{align*}
\]

We now give the proof of Theorem 1.5.

**Proof.** Scale \((M, g)\) so that \(s = 30\). Expanding out using Lemma 3.1 and the facts that

\[
\begin{align*}
\int_{S_p} g_3 \omega_2 &= \frac{1}{16} \int_{S_p} \|\text{grad}^v H\|^2 \omega_2 \\
\int_{S(M)} g_2 \omega &= -2 \int_{S(M)} g_3 \omega - \int_{S(M)} (H f_2 + H f_3 + H f_4) \omega,
\end{align*}
\]

one gets finally

\[
0 = \int_{S(M)} \left( g_1 - 4g_3 + (6H f_1 - 2H f_3 - 2H f_4) + \frac{1}{2} R_{xG(x)xG(x)} \right) \omega.
\]

Equation 3.2 simplifies, using Lemma 3.1 to

\[
0 = \int_{S(M)} g_1 \omega + \int_{S(M)} \left( \frac{\text{sec}(x, G(x))}{2} - \frac{3}{8} \right) \|\text{grad}^v H\|^2 \omega,
\]

and the result follows as \(g_1 \geq 0\). Hence if \(\text{sec}(x, G(x)) > \frac{4}{7}\), then \(\text{grad}^v H \equiv 0\) and \(g_1 = 0\) so applying work of Tanno \([25]\) we see \(M\) has constant sectional curvature 1. Finally, Gray \([12]\) proved that a nearly Kähler manifold with positive holomorphic sectional curvature is simply connected and the proof is completed. \(\square\)

**Remark.** As mentioned, our equation 3.2 and thus our result differ from Equation (4.7) in \([22]\). Firstly, it seems likely to us that the \(H\) term in front of \((6f_1 - 2f_3 - 2f_4)\) was dropped in their calculation. Then one would get...
Equation (4.7), except that the coefficient $\frac{s}{120}$ in front of $R_{xG(x)xG(x)}$ should be $\frac{s}{60}$. This is because
\[
\int_{S(M)} (6f_1 - 2f_3 - 2f_4) \omega = 0.
\]
The factor of 2 in the coefficient of $R_{xG(x)xG(x)}$ was also dropped by mistake.

Similarly, one must adapt the proof of Theorem 4.3 of [22] to get

**Theorem 3.2.** Let $(M^6, g, J)$ be a closed nearly Kähler manifold with $s = 30$. If $M$ satisfies $H \geq 2/3$, it is isometric to $S^6$ equipped with the round metric of sectional curvature 1.

4. **Proof of Theorem 1.3**

Firstly, recall that there is a standard correspondence in Kähler geometry between $T^{1,0}(M)$ and the real tangent bundle $TM$. We pass from a real frame \(\{e_i, e^*_i = Je_i\}, i = 1, \ldots, N = n\) to a complex frame \(\{z_i, \bar{z}_i = \bar{z}_i\}, i = 1, \ldots, N\) (where $Jz_i = \sqrt{-1}z_i$), via the formulae
\[
z_i = \frac{x_i - \sqrt{-1}Jx_1}{\sqrt{2}}, x_i = \frac{z_i + \bar{z}_i}{\sqrt{2}}, i = 1, \ldots, N.
\]
Complexifying the curvature operator we obtain $R^C$ and it is standard to see that
\[
B^C_{ij} = R^C_{ij \bar{j} \bar{i}} = R_{i\bar{i}j\bar{j}^*} = B_{ij}.
\]
Now we give the proof of Theorem 1.3.

**Proof.** Firstly we may assume $M$ is irreducible, as otherwise we are done (if the metric splits, the curves will have nonnegative bisectional curvature precisely when it is $\mathbb{C}P^1$ or a flat torus). A standard argument using Weitzenböck techniques implies that $(M, g)$ is Kähler-Einstein (see [10]). Since $M$ is closed, so is $S(M)$ and so $H$ attains a maximum at some point $(p, x)$. Choose a frame $e_i$ as above such that $e_1(p) = x$ and the frame is normal at $p$. Then the function $f = R_{11^*11^*}$ attains a maximum at $p$, so
\[
0 \geq \Delta f(p) = \sum_{i=1}^{N} \nabla^2_{ii} f = \sum_{i=1}^{N} \nabla^2_{ii} R_{11^*11^*}(p)
= \sum_{i=1}^{N} (\nabla^2_{ii} R)_{11^*11^*}(p) = \Delta^b(H)(p, x).
\]
Thus, via Equation (5.3) in [10]
\[
0 \geq \Delta^b(H)(p, x) = (H_1 - B_{12})B_{12} - 4R_{1212}R_{12^*12^*} + 4R^2_{1212^*}.
\]
Here the fact that $R_{1211^*}$ and $R_{22^*12^*}$ vanish at $p$ is freely used. This follows from the fact that sectional curvatures of orthogonal planes are equal for an Einstein four manifold and the formula for $\text{grad}^v H(p, x)$. 

Now we claim that $B_{12} = 0$. Otherwise, when we complexify the curvature, we get $B^C_{12} = 0$. To see this, assume to the contrary it is not zero. Then it is positive, and one follows the calculation in Lemma 8.18 [8] that $M$ is isometric to $(\mathbb{C}P^2, g_{FS})$. The next step is to notice that, from this formula $R_{1212}$ and $R_{12 \cdot 12^*}$ have the same sign if they are nonzero. But $B_{12} = R_{1212} + R_{12 \cdot 12^*}$. Hence they are both zero, and so therefore is also $R_{1212^*}$. Set $H^{max} = H(e_1) = H_1$.

We now claim that there exists an $r > 0$ such that for any $q \in B(p, r)$ we have an element $y \in S_q(M)$ with $H(y) = H^{max}$. Pick a local orthonormal complex frame $\{1, 1^*, 2, 2^*\}$ which agrees with the above frame at $(p, x)$. Choose $r$ small enough so $H_1 = H_2$ is always positive in the ball. The equality in the last equation is again just the fact that sectional curvatures on orthogonal planes are always equal in an Einstein four-manifold. Then the claim is that $B_{12}(q) = 0$. This is true as otherwise $B_{12}(q) > 0$, but then $B^C_{12} > 0$ at $q$. In such a situation we could apply the calculation in [8] and conclude $M$ is $(\mathbb{C}P^2, g_{FS})$. But from the equation

$$H_1 + B_{12} = \Lambda,$$

where $\Lambda$ is the Einstein constant, we conclude $H_1 = H_2 = H^{max}$ at $q$. Repeating the above argument at every point, we see that on $B(q, r)$ we have

$$R_{1212} = R_{12 \cdot 12^*} = R_{1211^*} = R_{22^*21} = 0.$$

Let us denote by $f_i$ the usual basis of $T_q(\mathbb{C}P^1 \times \mathbb{C}P^1)$ equipped with the inner product induced from the standard metric (so $f_1$ and $f_1^*$ span the first factor) and consider the map $\Phi : e_i \rightarrow f_i$ identifying the tangent space $T_q M$ with $T_q(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

Now pick any $y_1, y_2 \in T_q M$. Then via polarization one may express $sec(y_1, y_2)$ in terms of holomorphic sectional curvatures. But the holomorphic sectional curvature of a tangent vector $\zeta$, expressed in terms of the basis $e_i$, are given by the same calculation as calculating $H(\Phi(\zeta))$ in the $f_i$ basis. Setting $\zeta = \sum a_i e_i = \sum a_i f_i$, this can be seen from the calculation

$$H(\zeta) = R \sum a_i e_{i,j}(\sum a_i e_i) \sum a_i e_{i,j} J(\sum a_i e_i)$$

$$= \sum_{i,j,k,l} a_ia_ja_ka_l R_{e_i e_j e_k e_l^*}$$

$$= (a_1^4 + a_1^4) H_1 + (a_2^4 + a_2^4) H_2$$

$$= H(\Phi(\zeta)).$$

Therefore $sec(y_1, y_2)$ is non-negative, because $\mathbb{C}P^1 \times \mathbb{C}P^1$ with its standard metric has nonnegative sectional curvature. This is independent of the point $q \in M$. Hence $M$ has non-negative sectional curvatures, and then Theorem 1 implies the result. $\square$
5. Proof of Theorems 1.7 and 1.10

The following fact is well-known.

**Proposition 5.1.** Let \( \mathbb{R}^n \) denote Euclidean space and \( f \) a homogeneous polynomial of degree \( r \geq 1 \) on \( \mathbb{R}^n \). Then
\[
\int_{S^{n-1}} (Df) \omega_2 = r(n + r - 2) \int_{S^{n-1}} f |_{S^{n-1}} \omega_2.
\]
where \( D \) is the Laplace-Beltrami operator of \( \mathbb{R}^n \) and \( \omega_2 \) denotes the volume element of the round sphere \( S^{n-1} \) with sectional curvature 1.

Now we give the proof of Theorem 1.7.

**Proof.** Let \( \Lambda \) denote the Einstein constant of \((M, g)\). Now Lemma 2.11 and the pointwise estimate on the curvature norm together imply
\[
2 \int_{S(M)} \| \text{grad}^h H \|^2_{(m,x)} \omega = - \int_{S(M)} \text{sec}(x, G(x)) \| G(x) \|^2 \omega 
\leq \int_{S(M)} \| G(x) \|^2 \omega
\]
where in the last line we use the fact \((M, g)\) is normalized.

This gives the bound
\[
\int_{S(M)} \| \text{grad}(H) \|^2 \omega \leq \frac{3}{2} \int_{S(M)} \| \text{grad}^p(H)(x) \|^2 \omega.
\]

The idea is to use the Raleigh quotient with \( H \) as a test function to estimate \( \lambda_1(S(M), g_{sas}) \). \( H \) is never constant by assumption, so we can always normalize and use \( H \) as a test function on the sphere bundle. It remains to estimate
\[
\int_{S(M)} (H - H_{av})^2 \omega = \int_{S(M)} (H^2) \omega - Vol(S(M)) \int_{S(M)} (H_{av}^2) \omega.
\]

Let us define the functions \( F, f \) on \( T_p M \), for \( p \) fixed, by setting
\[
F(v) = R(v, Jv, v, Jv), \quad f = F^2
\]
for \( v \in T_p M \). Writing \( v = \sum v_i e_i \),
\[
(5.1) \quad F(v) = \sum_{i, j, k, l \geq 1} R_{ij} v_i v_j v_k v_l.
\]

By definition of \( F \) and \( f \), \( F|_{S_p} = H \) and \( f|_{S_p} = H^2 \). This gives
\[
\text{grad}(F) = 4 \sum_{i, j, k, l \geq 1} R_{ij} v_i v_j v_l R_{ij} v_k v_l.
\]

Next we compute
\[
\mathbb{D} H(v) = 4 \sum_{ij} (3R_{ij} + R_{ji}) v_i v_j = 16 \Lambda \| v \|^2
\]
for \( v \in T_p M \). In particular, for \((p, x) \in S(M)\) we obtain
\[
\mathcal{D}H(p, x) = 16\Lambda.
\]

In a similar fashion
\[
\text{grad}(F)(p, x) = 4 \sum_{l > 1} R_{xJxx}e_i e_i^x + 4H(x) e_i^x
\]
\[
= \text{grad}^v(H)(x)(p, x) + 4H(x) e_i^x.
\]

Applying Proposition 5.1 to \( f \) yields that
\[
\int_{S^{n-1}} (\mathcal{D}f) \omega_2 = 8(n + 6) \int_{S^{n-1}} H^2 \omega_2.
\]

But we have that
\[
\mathcal{D}f|_{S_p(M)} = 2\left( \|\text{grad}^v H\|^2 + 16H^2 + H\mathcal{D}H \right)
\]
\[
= 2\left( \|\text{grad}^v H\|^2 + 16H^2 + 16\Lambda H \right).
\]

Rearranging, this yields
\[
\int_{S_p(M)} (H - H_{av})^2 \omega_2 = \frac{1}{4(n + 2)} \int_{S_p(M)} \|\text{grad}^v (H)\|^2
\]
\[
+ \left( \frac{4}{n + 2} \Lambda - H_{av} \right) H_{av} Vol(S_p) \omega_2.
\]

But, again using Proposition 5.1, \( H_{av} = \frac{4}{n + 2} \Lambda \), so
\[
\int_{S_p(M)} (H - H_{av})^2 \omega_2 = \frac{1}{4(n + 2)} \int_{S_p(M)} \|\text{grad}^v (H)\|^2 \omega_2.
\]

The result follows.

Remark. In many known Ricci-flat examples, Hans-Joachim Hein has pointed out to us that on a noncompact component of the moduli space of Ricci flat metrics can admit deformations through Ricci-flat metrics which allow one to get a lower estimate for the first non-zero eigenvalue of the Laplacian of \((M, g)\), and thus pulling the corresponding eigenfunctions back to \(S(M)\) a better estimate than our work. Of course, such deformations do not generally exist. Our bound, in contrast, holds uniformly with a precise constant over the whole moduli space.

Finally we prove Theorem 1.10.

Proof. This follows from a similar calculation to the proof of Theorem 1.7; take again the function \( F \) defined on \( T_p M \) for some \( p \in M \). Then from
Equation (5.1)

\[ \mathbb{D}H(v) = 4 \sum_{a,i,j=1}^{n} \left( R_{ai}^{*}a^{*} + R_{ai}^{*}a^{*} + R_{ai}^{*}a^{*} \right) v_i v_j \]

But

\[ \int_{S^{n-1}} (v_i v_j) \omega_2 = 0 \]

if \( i \neq j \). Similarly, it is easy to see that

\[ \int_{S^{n-1}} (v_i^2) \omega_2 = \frac{1}{n} Vol(S^{n-1}) \]

for all \( i \). Then from Proposition 5.1,

\[ \frac{(n)(n+2)}{Vol(S^{n-1})} \int_{S^{n-1}} H \omega_2 = \sum_{a,i,j=1}^{n} \left( R_{ai}^{*}a^{*} + R_{ai}^{*}a^{*} + R_{ai}^{*}a^{*} \right) \bigg|_{i=j} \]

\[ = \sum_{i,j=1}^{n} \left( R_{i}^{*}i^{*} + R_{i}^{*}j^{*} + R_{i}^{*}i^{*}j^{*} + R_{i}^{*}j^{*} \right) \bigg|_{i=j} \]

\[ = \sum_{i=1}^{n} R_{i}^{*}i^{*} + 2R_{ii}^{*} + R_{i}^{*}i^{*}. \]

But \( \sum_{i=1}^{n} R_{i}^{*}i^{*} = \sum_{i=1}^{n} R_{ii} = s \) and \( \sum_{i=1}^{n} R_{i}^{*}i^{*} = \sum_{i=1}^{n} R_{ii}^{*} = s^{*} \) and the result follows. \( \square \)

The corollary then follows immediately from work of Apostolov et. al. \( \square \).

5.1. **Acknowledgements.** We are greatly indebted to Koeui Sekigawa, who very kindly explained his work to us in detail, encouraged our research and gave us constructive feedback. T.M. thanks McKenzie Wang for his support. We thank Bruno Colbois for useful communications, as well as Ha-Jo Hein and Joel Fine for their comments concerning the examples in Corollary 1.8. T. M. was supported by an A.R.C. grant whilst this research began, and later by an NSERC grant.

**REFERENCES**

[1] Apostolov, V., Davidov, J. and Muskarov, O. Compact self-dual Hermitian surfaces, Trans. Amer. Math. Soc. 348 (1996), no. 8, 3051–3063.

[2] Berger, M. Sur les variétés d’Einstein compactes, Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d’Expression Latine (Namur, 1965) pp. 35–55.

[3] Berger, M A panoramic view of Riemannian geometry, Springer-Verlag, Berlin, 2003.

[4] Berger, M. Sur les premières valeurs propres des variétés riemanniennes, Compositio Math. 26 (1973), 129–149.

[5] Berger, M. A panoramic view of Riemannian geometry, Springer-Verlag, Berlin, 2003.

[6] Bryant, R.L. On the geometry of almost complex 6-manifolds, Asian J. Math. 10 (2006), no. 3, 561–605.
[7] Chavel, I. The Laplacian in Riemannian geometry, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.
[8] Chen, X.X. and Tian, G. Ricci flow on Kähler-Einstein manifolds, Duke Math. Journal 131 (2006), No. 1, 17–73.
[9] Colbois, B. and Dodziuk, J. Riemannian manifolds with large $\lambda_1$, Proc. Amer. Math. Soc., 122 (1994) no. 3, 905–906.
[10] Gray, A. Kähler manifolds with non-negative sectional curvature, Inventiones Math. 41 (1977), 33–43.
[11] Gray, A. Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geometry 7 (1972), 343–369.
[12] Gray, A. Nearly Kähler manifolds, J. Differential Geometry, 4 (1970) 283—309.
[13] Gray, A. Tubes, Progress in Mathematics, 221. Birkhäuser Verlag, 2004.
[14] Goldberg, S. and Kobayashi, S. Holomorphic Bisectional Curvature, J. Differential Geom., 1 (1967), 225–233.
[15] Gu, H-L. A new proof of Mok’s generalized Frankel conjecture theorem, Proc. A.M.S. 137 (2009), No. 3 1065–1068.
[16] Lotay, J. and Karigiannis, S. Deformation Theory of G2 Conifolds, 2012, arXiv:1212.6457.
[17] Mok, N. The uniformization theorem for compact Kähler manifolds of nonnegative bisectional curvature, J. Diff. Geom. 27, (1988), 179-214.
[18] Mok, N. Metric rigidity results on Hermitian locally symmetric spaces, Series in Pure Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.
[19] Mok, N. and Zhong, J.Q. Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature, Ann. of Math. (2) 129 (1989) no 3, 427–470.
[20] Mori S. Projective manifolds with ample tangent bundles, Ann. of Math., 110 (1979), 593–606.
[21] Nagy, P.A. Nearly Kähler manifolds and complex Riemannian foliations, Asian J. Math. 6 (2002), no. 3, 481–504.
[22] Sekigawa, K. and Sato, T. Nearly Kähler manifolds with positive holomorphic sectional curvature, Kodai J. Math., 8 (1985), 139–156.
[23] Siu, Y.T. Some recent developments in complex differential geometry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 287–297, PWN, Warsaw, 1984.
[24] Siu, Y.T. and Yau, S. T. Compact Kähler manifolds of positive bisectional curvature, Invent. Math. 59 (1980), no. 2, 189- 204.
[25] Tanno, S. Constancy of holomorphic sectional curvature in almost Hermitian manifolds, Kodai Math. Sem. Rep., 25(1973), 190–201.
[26] Verbitsky, M. Hodge theory on nearly Kähler manifolds, Geom. Topol. 15 (2011), no. 4, 2111–2133.