Commuting Magic Square Matrices
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Abstract. We review a known method of compounding two magic square matrices of order \( m \) and \( n \) with the all-ones matrix to form two magic square matrices of order \( mn \). We show that these compounded matrices commute. Simple formulas are derived for their Jordan form and singular value decomposition. We verify that regular (associative) and pandiagonal commuting magic squares can be constructed by compounding. In a special case the compounded matrices are similar. Generalization of compounding to a wider class of commuting magic squares is considered. Three numerical examples illustrate our theoretical results.

1 Introduction

The construction of magic squares by compounding smaller ones arose over 1000 years ago and has attracted interest ever since as reviewed by Pickover [11], Chan and Loly [1], and Rogers, et. al [13]. In the present article we begin with a formulation of the compounding construction given by Eggermont [2] and extended in [13]. Two basic magic square matrices of order \( m \) and \( n \) \((m, n \geq 3)\) are compounded with the all-ones matrix by means of the tensor (Kronecker) product to form two unnatural magic square matrices of order \( mn \) that are shown to commute. These two matrices are combined to form two natural magic square matrices of order \( mn \) that commute. When \( m = n \) and the two basic magic squares are identical, we verify a result in [13] that the two compounded magic square matrices are related by a row/column permutation (shuffle) that we express in matrix form which shows that they are similar matrices. We verify a known result [2] that regular (associative) and pandiagonal special properties are preserved by compounding. Simple formulas are derived for the matrices in the Jordan form and singular value decomposition of the compounded matrices in terms of those of the basic matrices. Generalization to a wider class of commuting magic squares is considered but our formulas for the Jordan form and SVD do not apply to them. Examples are given of commuting magic square matrices of orders 9, 12, and 16, all with special properties.

2 Construction

Magic Squares. To begin, let \( E_n \) denote the order-\( n \) square matrix with all elements one and let \( R_n \) denote the order-\( n \) square matrix with ones on the cross diagonal and all other elements zero. Let \( M_n \) be an order-\( n \) magic square matrix whose rows, columns and two main diagonals add to the summation index \( \mu_n \), i.e.

\[
M_n E_n = E_n M_n = \mu_n E_n,
\]

\[
\text{tr} [M_n] = \text{tr} [R_n M_n] = \mu_n. \tag{1}
\]

We further require that \( M_n \) be a natural magic square with elements \( 0, 1, \ldots, n^2 - 1 \), whence

\[
\mu_n = \frac{n}{2} (n^2 - 1). \tag{2}
\]

From two magic squares \( M_m \) and \( M_n \), two order- \( mn \) matrices can be formed by compounding as follows [2] [13]:

\[
A_{mn} = E_m \otimes M_n, \quad B_{mn} = M_m \otimes E_n, \tag{3}
\]
where the symbol "\( \otimes \)" denotes the tensor (Kronecker) product \[^3\] \[^6\]. On noting that
\[
E_m \otimes E_n = E_n \otimes E_m = E_{mn}, \quad E^2_n = nE_n, \quad \text{tr} [E_n] = n,
\]
\[
R_m \otimes R_n = R_n \otimes R_m = R_{mn}, \quad R^2_n = I_n, \quad R_nE_n = E_nR_n = E_n,
\]
and using formulas for the tensor product \[^6\], from (1), (3), and (4), we find that
\[
m\mu_n
\]
and similarly for \( B \) and using formulas for the tensor product \[^6\], from (1), (3), and (4), we find that
\[
E_{mn}A_{mn} = (E_m \otimes E_n)(E_m \otimes M_n) = E_{mn}^2 \otimes (E_nM_n) = mE_m \otimes (\mu_nE_n) = m\mu_nE_{mn},
\]
\[
\text{tr} [A_{mn}] = \text{tr} [E_m \otimes M_n] = \text{tr} [E_m] \text{tr} [M_n] = m\mu_n,
\]
\[
\text{tr} [R_{mn}A_{mn}] = \text{tr} [(R_m \otimes R_n)(E_m \otimes M_n)] = \text{tr} [(R_mE_m) \otimes (R_nM_n)] = m\mu_n,
\]
and similarly for \( B_{mn} \). Thus, \( A_{mn} \) and \( B_{mn} \) are unnatural magic squares with summation indices \( m\mu_n \) and \( n\mu_m \), respectively. Furthermore, we find that \( A_{mn} \) and \( B_{mn} \) commute since
\[
A_{mn}B_{mn} = (E_m \otimes M_n)(M_m \otimes E_n) = (E_mE_m) \otimes (M_nE_n) = m\mu_nE_{mn},
\]
\[
B_{mn}A_{mn} = (M_m \otimes E_n)(E_m \otimes M_n) = (M_mE_m) \otimes (E_nM_n) = m\mu_mE_{mn}.
\]
In addition, it can be shown that each of the eight phases \[^4\] \[^7\] of \( A_{mn} \) commutes with the corresponding phase of \( B_{mn} \).

We note that \( A_{mn} \) and \( B_{mn} \) are an orthogonal pair since the \( n \) by \( n \) subsquares of \( A_{mn} \) contain \( n^2 \) replicas of \( M_n \), whereas the elements of the \( n \) by \( n \) subsquares of \( B_{mn} \) all are the same number as that of the corresponding element of \( M_m \) (as seen in the examples below). Thus, all combinations of two numbers from \( A_{mn} \) and \( B_{mn} \) occur once and only once, i.e. \( A_{mn} \) and \( B_{mn} \) are orthogonal. Therefore, we can form two natural magic squares by the Euler composition formula \[^10\] \[^2\] as follows:
\[
M^A_{mn} = A_{mn} + n^2B_{mn}, \quad M^B_{mn} = B_{mn} + m^2A_{mn}
\]
which can be verified to satisfy the magic square conditions \[^7\], e.g.
\[
E_{mn}M^A_{mn} = E_{mn}A_{mn} + n^2E_{mn}B_{mn} = (m\mu_n + n^2\mu_m)E_{mn}
\]
\[
= \frac{1}{2}mn(m^2n^2 - 1)E_{mn} = \mu_{mn}E_{mn},
\]
\[
\text{tr} M^A_{mn} = \text{tr} A_{mn} + n^2 \text{tr} B_{mn} = m\mu_n + n^3\mu_m = \mu_{mn}.
\]
We note that \( M^A_{mn} \) and \( M^B_{mn} \) commute since, by \[^7\] and \[^6\]
\[
M^A_{mn}M^B_{mn} = (A_{mn} + n^2B_{mn})(B_{mn} + m^2A_{mn})
\]
\[
= A_{mn}B_{mn} + m^2A_{mn}A_{mn} + n^2B_{mn}B_{mn} + m^2n^2B_{mn}A_{mn},
\]
\[
M^B_{mn}M^A_{mn} = (B_{mn} + m^2A_{mn})(A_{mn} + n^2B_{mn})
\]
\[
= B_{mn}A_{mn} + n^2B_{mn}B_{mn} + m^2A_{mn}A_{mn} + m^2n^2A_{mn}B_{mn},
\]
\[
M^A_{mn}M^B_{mn} = M^B_{mn}M^A_{mn}.
\]
The foregoing compounding construction can be repeated using \( M^A_{mn} \) or \( M^B_{mn} \) in \[^3\] to produce higher order magic squares which again commute. Gigantic commuting magic squares can be produced by repeated compounding as done by Chan and Loly \[^1\].

**Regular Magic Squares.** In a regular (associative) matrix any two elements that are symmetric about the center element add to the same number and in an odd-order regular matrix the center element is one-half this number. The regularity condition on \( M_n \) can be expressed as
\[
M_n + R_nM_nR_n = \frac{2\mu_n}{n}E_n = (n^2 - 1)E_n,
\]
\[^2\]Pasles \[^10\] suggests that Benjamin Franklin used this formula prior to Euler.
where the factor $2\mu_n/n$ can be verified by taking the trace of this equation. We wish to show that if $M_m$ and $M_n$ are regular, then so are $A_{mn}, B_{mn}, M^A_{mn},$ and $M^B_{mn}$ as noted by Eggermont [2] for $M^A_{mn}$. From [1], [3], and [10], we find that

$$
R_{mn}A_{mn}R_{mn} + A_{mn} = (R_m \otimes R_n) (E_m \otimes M_n) (R_m \otimes R_n) + A_{mn}
$$

$$
= (R_m E_m R_m) \otimes (R_n M_n R_n) + A_{mn}
$$

$$
= E_m \otimes ((n^2 - 1) E_n - M_n) + E_m \otimes M_n
$$

$$
= (n^2 - 1) E_{mn}
$$

(11)

and similarly

$$
R_{mn}B_{mn}R_{mn} + B_{mn} = (m^2 - 1) E_{mn}
$$

(12)

which are the regularity conditions for $A_{mn}$ and $B_{mn}$. From their definitions [7], $M^A_{mn}$ and $M^B_{mn}$ are regular when $A_{mn}$ and $B_{mn}$ are regular as seen from

$$
M^A_{mn} + R_{mn} M^A_{mn} R_{mn} = (A_{mn} + n^2 B_{mn}) + R_{mn} (A_{mn} + n^2 B_{mn}) R_{mn}
$$

$$
= A_{mn} + n^2 B_{mn} + (n^2 - 1) E_{mn} - A_{mn} + n^2 (m^2 - 1) E_{mn} - n^2 B_{mn}
$$

$$
= (m^2 n^2 - 1) E_{mn}
$$

(13)

and similarly for $M^B_{mn}$.

In addition, for a regular magic square $M^{(1)}_n$ and its $180^\circ$ rotation $M^{(2)}_n$, given by

$$
M^{(2)}_n = R_n M^{(1)}_n R_n,
$$

(14)

it follows from [10] that $M^{(1)}_n$ and $M^{(2)}_n$ commute and similarly for $M^{(1)}_m$ and $M^{(2)}_m$. These two commuting duos can be used in equations of the form [3] and [7] to form a quartet of mutually commuting regular magic squares. Repeated compounding of these squares leads in an immense number of commuting regular magic squares of increasing order. A class of pandiagonal squares given in [7] also can be used to form commuting duos.

**Pandiagonal Magic Squares.** In a *pandiagonal* magic square of order $n$, all $2n$ diagonals, including broken ones in both directions, sum to $\mu_n$. It is known that a regular magic square $M_{Rn}$ of doubly-even order ($n = 4k, k = 1, 2, \ldots$) can be transformed to a pandiagonal magic square $M_{Pn}$ by the Planck transformation [12, 7]. Thus, if $M_m$ and $M_n$ are regular and $mn$ is doubly-even, then $M^A_{mn}$ and $M^B_{mn}$ are regular (as shown above) and doubly-even order. Therefore, they can be transformed to pandiagonal magic squares which can be shown to commute.

It also is possible to compound commuting pandiagonal magic squares $M^A_{mn}$ and $M^B_{mn}$ directly from [3] and [7] starting with pandiagonal magic squares $M_m$ and $M_n$ as noted by Eggermont [2] and carried out by Chan and Loly [1] for $M^B_{36}$. The pandiagonality of the compounded matrix is established by them and is verified in an example below for $M^A_{12}$ and $M^B_{12}$. Magic squares that are both regular and pandiagonal are called *ultra-magic* squares, with 5 being the lowest order for their existence, leading to order-25 commuting ultra-magic squares by the compounding construction.

**Special Case - Permutation.** Simplification is possible by taking $M_m = M_n (m = n)$ in [3]. In this case, as noted by Rogers, et. al [13], interchange (shuffling) of rows and columns of $A_{nn}$ leads to $B_{nn}$ and vice versa. We find that this interchange can be expressed as

$$
B_{nn} = P_{nn} A_{nn} P_{nn}, \quad A_{nn} = P_{nn} B_{nn} P_{nn},
$$

(15)

where $P_{nn}$ is a symmetric permutation matrix that can be written in block form as

$$
P_{nn} = \begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1n} \\
P_{21} & P_{22} & \cdots & P_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} & P_{n2} & \cdots & P_{nn}
\end{bmatrix}
$$

(16)
in which \( p_{ij} \) are order-\( n \) matrices with element \( p_{ij} = 1 \) and all other elements zero. It can be shown that
\[
R_{nn}P_{nn}R_{nn} = P_{nn}, \quad P_{nn}^T = P_{nn}^{-1} = P_{nn}.
\]
(17)
Thus, (15) is a similarity transformation \( \tilde{A} \) and \( A_{nn} \) and \( B_{nn} \) are similar matrices. The formulas (15), (10) and (17) will be verified in the examples below. From (15) and (7), similarity transformation shown that
\[
in which \( p \)
\[
\lambda
\]
(6). Thus, commuting \( \tilde{M} \) with zero elements except for eigenvalues \( E \) is shown to be a similarity transformation. For a generalized eigenvector \( \tilde{M} \) where
\[
S
\]
s for a simple eigenvector \( \lambda \) of algebraic multiplicity \( k \), (18) leads to
\[
(M - \lambda I)^{k} s_k^{(i)} = (M - \lambda I)^{k-1} s_{k-1}^{(i)} = \ldots = (M - \lambda I) s_1^{(i)} = 0.
\]
(20)
For a simple eigenvector \( s_i \) with eigenvalue \( \lambda_i \) of algebraic multiplicity 1, (19) gives
\[
(M - \lambda I) s_i = 0.
\]
(21)
If all the eigenvectors of \( M \) are simple, then \( J = D \) is diagonal.

The Jordan form of magic square matrices is studied extensively in [4, 5, 7, 8, 13]. When \( M_n \) is a magic square matrix, according to (1), it has an all-ones eigenvector \( s_1 \) with eigenvalue \( \lambda_1 = \mu_n \). On applying \( E_n \) to (20), we find that
\[
(\mu_n - \lambda_i) E_n s_k^{(i)} = (\mu_n - \lambda_i) E_n s_{k-1}^{(i)} = \ldots = (\mu_n - \lambda_i) E_n s_1^{(i)} = 0.
\]
(22)
Since it is known [5] that \( |\lambda_i| < \mu_n \) (\( i \geq 2 \)), it follows that
\[
E_n s_1 = n \mu_n s_1, \quad E_n s_i = 0, \quad i = 2, 3, \ldots.
\]
(23)
The eigenvalues of \( E_n \) are \( n, 0, 0, \ldots, 0 \) as shown in [3]. Thus, from (19) for \( E_n \), using the \( S_n \) matrix of \( E_n \), we have
\[
E_n S_n = S_n D E_n
\]
(24)

3 Jordan Form

We derive formulas for the Jordan-form matrices of \( A_{nn}, B_{nn}, M_{nn}^A, \) and \( M_{nn}^B \) from those of \( E_n, E_n, M_n, \) and \( M_n \). To review from [3, 6], the Jordan form of a square matrix \( M \) is given by
\[
M = SJS^{-1}, \quad MS = SJ,
\]
(19)
where \( S \) is a matrix whose columns are the simple or generalized eigenvectors \( s_i \) of \( M \) and \( J \) is the matrix with zero elements except for eigenvalues \( \lambda_i \) on the main diagonal and ones on the diagonal above it corresponding to generalized eigenvectors. For a generalized eigenvector \( s_k^{(i)} \) with eigenvalue \( \lambda_i \) of algebraic multiplicity \( k \), (19) leads to
\[
(M - \lambda_i I)^{k} s_k^{(i)} = (M - \lambda_i I)^{k-1} s_{k-1}^{(i)} = \ldots = (M - \lambda_i I) s_1^{(i)} = 0.
\]
(20)
For a simple eigenvector \( s_i \) with eigenvalue \( \lambda_i \) of algebraic multiplicity 1, (19) gives
\[
(M - \lambda_i I) s_i = 0.
\]
(21)
If all the eigenvectors of \( M \) are simple, then \( J = D \) is diagonal.
which is correct, whence $S_n$ is the eigenvector matrix for both $M_n$ and $E_n$ and their Jordan forms read

$$M_n = S_n J_{Mn} S_n^{-1}, \quad E_n = S_n D_{En} S_n^{-1}. \quad (25)$$

Since $E_n$ is symmetric, it also has an orthogonal eigenvector matrix which is not used here. When all the eigenvectors of $M_n$ are simple, its eigenvalue matrix can be written as

$$J_{Mn} = \text{diag} (\mu_n, \lambda_{n2}, \lambda_{n3}, \ldots, \lambda_{nn}) \quad (26)$$

and for generalized eigenvectors of $M_n$ there are ones on the diagonal above the main diagonal for their corresponding eigenvalues in $J_{Mn}$. Equations of the same form as the foregoing ones apply to $M_m$ and $E_m$.

Using a compounding technique given by Nordgren [5], from (25) and (3), we find that

$$A_{mn} = E_m \otimes M_n = (S_m D_{Em} S_m^{-1}) \otimes (S_n J_{Mn} S_n^{-1})$$
$$= (S_m \otimes S_n) (D_{Em} \otimes J_{Mn}) (S_m \otimes S_n)^{-1} = S_{mn} J_{mn}^A S_{mn}^{-1},$$

$$B_{mn} = M_m \otimes E_n = (S_m J_{Mm} S_m^{-1}) \otimes (S_n D_{En} S_n^{-1})$$
$$= (S_{nn} \otimes S_m) (J_{Mm} \otimes D_{En}) (S_n \otimes S_m)^{-1} = S_{mn} J_{mn}^B S_{mn}^{-1}, \quad (27)$$

where

$$S_{mn} = S_m \otimes S_n, \quad J_{mn}^A = D_{Em} \otimes J_{Mn}, \quad J_{mn}^B = J_{Mm} \otimes D_{En}. \quad (28)$$

Then, with (26), it follows that the nonzero eigenvalues of $A_{mn}$ and $B_{mn}$ are

$$J_{mn}^A : n \mu_n, n \lambda_{n2}, n \lambda_{n3}, \ldots, n \lambda_{nn},$$

$$J_{mn}^B : n \mu_n, n \lambda_{n2}, n \lambda_{n3}, \ldots, n \lambda_{nn}. \quad (29)$$

When $M_m$ and/or $M_n$ have generalized eigenvectors, $J_{mn}^A$ and $J_{mn}^B$ from (28) are not in standard form but they can be brought there by modifying $S_{mn}$ as indicated in the example below for $mn = 12$. Furthermore, (7) with (27) gives the Jordan form of $M_{mn}^A$ and $M_{mn}^B$ as

$$M_{mn}^A = S_{mn} J_{Mmn}^A S_{mn}^{-1}, \quad M_{mn}^B = S_{mn} J_{Mmn}^B S_{mn}^{-1}, \quad (30)$$

where

$$J_{Mmn}^A = J_{mn}^A + n^2 J_{mn}^B, \quad J_{Mmn}^B = J_{mn}^B + n^2 J_{mn}^A. \quad (31)$$

and their eigenvalues can be expressed using (29).

**Special Case.** In the special case where $M_m = M_n$ ($m = n$), according to (28), $A_{nn}$ and $B_{nn}$ have the same nonzero eigenvalues from (29), namely

$$n \mu_n, n \lambda_2, n \lambda_3, \ldots, n \lambda_n, \quad (32)$$

but they appear in a different order in $J_{nn}^A$ and $J_{nn}^B$. To see this, by (13), (28), and (27), we form

$$A_{nn} = (P_{nn} S_{nn} P_{nn}) (P_{nn} J_{nn}^A P_{nn}) (P_{nn} S_{nn} P_{nn})^{-1} = S_{nn} J_{nn}^A S_{nn}^{-1}, \quad (33)$$

and similarly for $B_{nn}$, whence

$$P_{nn} S_{nn} P_{nn} = S_{nn}, \quad J_{nn}^A = P_{nn} J_{nn}^A P_{nn}, \quad J_{nn}^B = P_{nn} J_{nn}^B P_{nn} \quad (34)$$

which confirms that $J_{nn}^A$ and $J_{nn}^B$ contain the same eigenvalues and indicates their reordering. Furthermore, $M_{nn}^A$ and $M_{nn}^B$ also have the same nonzero eigenvalues, namely

$$n \left(1 + n^2 \right) \mu_n, n \lambda_2, n \lambda_3, \ldots, n \lambda_n, n^3 \lambda_2, n^3 \lambda_3, \ldots, n^3 \lambda_n \quad (35)$$

and equations of the form (34) apply to $J_{Mnn}^A$ and $J_{Mnn}^B$.

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Rogers, et. al [13] derive similar formulas for eigenvalues and eigenvectors in a somewhat different manner.
4 Singular Value Decomposition

We derive formulas for the matrices in the singular value decomposition (SVD) of $A_{mn}$, $B_{mn}$, $M_A^{mn}$, and $M_B^{mn}$ in terms of those of $E_m$, $E_n$, $M_m$, and $M_n$. To review [3, 6], the SVD of any real square matrix $M$ is expressed as

$$M = U\Sigma V^T,$$

where $U$ and $V$ are orthogonal matrices, and $\Sigma$ is a diagonal matrix with non-negative real numbers (the singular values) on the diagonal. It follows from (36) that

$$MM^T = U\Sigma^2 U^T, \quad M^TM = V\Sigma^2 V^T$$

(37)

which are Jordan forms of the symmetric, positive semi-definite matrices $MM^T$ and $M^TM$. In particular, the Jordan form of $MM^T$ can be used to determine $U$ and $\Sigma$ after which $V$ can be determined from (36). If $\Sigma$ is nonsingular, then (36) leads to

$$V = M^T U\Sigma^{-1}.$$  

(38)

If $\Sigma$ is singular, then an alternate approach given by Meyer [6] applies.

The SVD of magic square matrices is studied in [4, 8, 13]. When $M_n$ is a magic square matrix, the $U_n$ and $V_n$ matrices for $M_n$ also apply to $E_n$ as we show next. In view of (1), $M_n$ and $M_n^T$ have an eigenvalue $\mu_n$ with eigenvector $s_1$ composed of constant elements $c$, therefore

$$M_n s_1 = M_n^T s_1 = \mu_n s_1,$$

$$M_n M_n^T s_1 = \mu_n^2 s_1,$$

$$s_1 = c e_n, \quad E_n s_1 = nce_n,$$

(39)

where $e_n$ is the order-$n$ (column) vector with all elements one. From (37) and (39), we see that $s_1$ also is an eigenvector in $U_n$ for the singular value $\mu$. Since $U_n$ is orthogonal, $u_1 = s_1$ must be a unit vector, hence

$$u_1^T u_1 = c^2 e_n^T e_n = c^2 n = 1, \quad u_1 = \frac{\sqrt{n}}{n} e_n, \quad E_n u_1 = \sqrt{n} e_n.$$  

(40)

The remaining eigenvectors in $U_n$ namely $u_2, u_3, \ldots, u_n$ for singular values $\sigma_2, \sigma_3, \ldots, \sigma_n$ in $\Sigma_{M_n}$, according to (37), must satisfy

$$(M_n M_n^T - \sigma_i^2 I_n) u_i = 0, \quad i = 2, 3, \ldots, n.$$  

(41)

Application of $E_n$ to this equation results in

$$(\mu^2 - \sigma_i^2) E_n u_i = 0, \quad i = 2, 3, \ldots, n$$  

(42)

and, since it is known [5] that $\sigma_i^2 < \mu^2$ ($i \geq 2$), it follows that

$$E_n u_i = 0, \quad i = 2, 3, \ldots, n.$$  

(43)

A similar argument holds for the singular vectors $v_i$ of $V_n$ and we may write (in block form)

$$U_n = \left[ \frac{\sqrt{\pi}}{n} e_n \quad u_2 \quad \ldots \quad u_n \right], \quad V_n = \left[ \frac{\sqrt{\pi}}{n} e_n \quad v_2 \quad \ldots \quad v_n \right],$$

$$E_n U_n = E_n V_n = \left[ \sqrt{n} e_n \quad 0 \quad \ldots \quad 0 \right],$$

(44)

4The SVD also applies to complex matrices and rectangular matrices which are not considered here.
where $v_2, v_3, \ldots v_n$ remain to be determined from (50) as already noted. From the SVD for $E_n$ with $U_n$ and $V_n$ from (44), we have

$$\Sigma E_n = U_n^T E_n V_n = \begin{bmatrix} \sqrt{n} u_1^T \\ \vdots \\ \sqrt{n} u_n^T \end{bmatrix} \begin{bmatrix} n & 0 & \ldots & 0 \end{bmatrix} = \text{diag} \{n, 0, 0, \ldots, 0\}$$ (45)

which is correct. Therefore

$$E_n = U_n \Sigma E_n V_n^T, \quad \Sigma E_n = \text{diag} \{n, 0, 0, \ldots, 0\},$$ (46)

i.e. $E_n$ has the same singular-value matrices $U_n$ and $V_n$ as $M_n$.

Using a compounding technique given by Nordgren [8], by (43), (45), and (46), we have

$$A_{mn} = E_m \otimes M_n = (U_m \Sigma E_m V_m^T) \otimes (U_n \Sigma M_n V_n^T)$$

$$= (U_m \otimes U_n) (\Sigma E_m \otimes \Sigma M_n) (V_m \otimes V_n)^T = U_{mn} \Sigma_{mn}^A V_{mn}^T,$$

$$B_{mn} = M_m \otimes E_n = (U_m \Sigma M_m V_m^T) \otimes (U_n \Sigma E_n V_n^T)$$

$$= (U_m \otimes U_n) (\Sigma M_m \otimes \Sigma E_n) (V_m \otimes V_n)^T = U_{mn} \Sigma_{mn}^B V_{mn}^T,$$ (47)

where

$$U_{mn} = U_m \otimes U_n, \quad V_{mn} = V_m \otimes V_n,$$

$$\Sigma_{mn}^A = \Sigma E_m \otimes \Sigma M_n = m \text{ diag} \{\mu_n, \sigma_n, \ldots, \sigma_n, 0, \ldots, 0\},$$ (48)

$$\Sigma_{mn}^B = \Sigma M_m \otimes \Sigma E_n = n \text{ diag} \{\mu_m, 0, \ldots, 0, \sigma_m, 0, \ldots, 0, \sigma_m, 0, \ldots, 0, \sigma_m, 0, \ldots, 0\}.$$

Thus, $A_{mn}$ and $B_{mn}$ have the same $U_{mn}$ and $V_{mn}$ and their SVD’s are given by (47). Furthermore, it follows from (48) and (47) that the SVD’s of $M_{mn}^A$ and $M_{mn}^B$ are

$$M_{mn}^A = U_{mn} \Sigma_{mn}^A V_{mn}^T,$$

$$M_{mn}^B = U_{mn} \Sigma_{mn}^B V_{mn}^T,$$ (49)

where

$$\Sigma_{mn}^A = \Sigma_{mn}^A + m^2 \Sigma_{mn}^B,$$

$$\Sigma_{mn}^B = \Sigma_{mn}^B + n^2 \Sigma_{mn}^A.$$ (50)

**Special Case.** In the special case where $M_m = M_n$ ($m = n$), according to (48), $A_{nn}$ and $B_{nn}$ have the same singular values but they are in a different order in $\Sigma_{nn}^A$ and $\Sigma_{nn}^B$. To examine this, by (47) and (48), we form

$$A_{nn} = (P_{nn} U_{nn} P_{nn}) (P_{nn} \Sigma_{nn}^B P_{nn}) (P_{nn} V_{nn}^T P_{nn})^{-1} = U_{nn} \Sigma_{nn}^A V_{nn}^T,$$

$$B_{nn} = (P_{nn} U_{nn} P_{nn}) (P_{nn} \Sigma_{nn}^A P_{nn}) (P_{nn} V_{nn}^T P_{nn})^{-1} = U_{nn} \Sigma_{nn}^B V_{nn}^T,$$ (51)

whence

$$P_{nn} U_{nn} P_{nn} = U_{nn}, \quad P_{nn} V_{nn} P_{nn} = V_{nn},$$

$$\Sigma_{nn}^A = P_{nn} \Sigma_{nn}^B P_{nn}, \quad \Sigma_{nn}^B = P_{nn} \Sigma_{nn}^A P_{nn}$$ (52)

which indicates the reordering of the same singular values in $\Sigma_{nn}^A$ and $\Sigma_{nn}^B$. The SVD’s of $M_{nn}^A$ and $M_{nn}^B$ are still given by (44) with singular value matrices from (50). By (50) and (52), we have

$$\Sigma_{Mnn}^A = P_{nn} \Sigma_{Mnn}^B P_{nn}, \quad \Sigma_{Mnn}^B = P_{nn} \Sigma_{Mnn}^A P_{nn}$$ (53)

which confirms that $\Sigma_{Mnn}^A$ and $\Sigma_{Mnn}^B$ contain the same singular values and indicates their reordering.

Next, numerical examples are given that illustrate and confirm the foregoing theoretical results.

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*Rogers, et. al.* [13] also give formulas for the SVD of compound matrices.
5 Examples

Order 9. We construct two order-9, commuting, regular, magic squares by compounding. Following Rogers, et.al [13], we start with the order-3 Lo-Shu regular magic square $M_3$ and the all-ones square $E_3$, namely

$$M_3 = \begin{bmatrix} 3 & 8 & 1 \\ 2 & 4 & 6 \\ 7 & 0 & 5 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (54)$$

We compound these matrices according to (3) to form

$$A_9 = \begin{bmatrix} 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \\ 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \\ 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \end{bmatrix}, \quad B_9 = \begin{bmatrix} 3 & 3 & 8 & 3 & 8 & 1 & 1 & 1 & 1 \\ 3 & 3 & 8 & 3 & 8 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 & 4 & 4 & 6 & 6 & 6 \\ 7 & 7 & 7 & 0 & 0 & 0 & 5 & 5 & 5 \\ 2 & 2 & 2 & 4 & 4 & 4 & 6 & 6 & 6 \\ 7 & 7 & 7 & 0 & 0 & 0 & 5 & 5 & 5 \\ 3 & 3 & 8 & 3 & 8 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 & 4 & 4 & 6 & 6 & 6 \\ 7 & 7 & 7 & 0 & 0 & 0 & 5 & 5 & 5 \end{bmatrix}. \quad (55)$$

which are unnatural, regular, magic squares that commute. Since $A_9$ and $B_9$ are an orthogonal pair, two commuting, regular, magic squares can be formed from $A_9$ and $B_9$ according to (7) as

$$M_9^A = \begin{bmatrix} 30 & 35 & 28 & 75 & 80 & 73 & 12 & 17 & 10 \\ 29 & 31 & 33 & 74 & 76 & 78 & 11 & 13 & 15 \\ 34 & 27 & 32 & 79 & 72 & 77 & 16 & 9 & 14 \\ 21 & 26 & 19 & 39 & 44 & 37 & 57 & 62 & 55 \\ 20 & 22 & 24 & 38 & 40 & 42 & 56 & 58 & 60 \\ 25 & 18 & 23 & 43 & 36 & 41 & 61 & 54 & 59 \\ 66 & 71 & 64 & 3 & 8 & 1 & 48 & 53 & 46 \\ 65 & 67 & 69 & 2 & 4 & 6 & 47 & 49 & 51 \\ 70 & 63 & 68 & 7 & 0 & 5 & 52 & 45 & 50 \end{bmatrix}, \quad M_9^B = \begin{bmatrix} 30 & 75 & 12 & 35 & 80 & 17 & 28 & 73 & 10 \\ 21 & 39 & 57 & 26 & 44 & 62 & 19 & 37 & 55 \\ 66 & 3 & 48 & 71 & 8 & 53 & 64 & 1 & 46 \\ 29 & 74 & 11 & 31 & 76 & 13 & 33 & 78 & 15 \\ 20 & 38 & 56 & 22 & 40 & 58 & 24 & 42 & 60 \\ 65 & 2 & 47 & 67 & 4 & 49 & 69 & 6 & 51 \\ 34 & 79 & 16 & 27 & 72 & 9 & 32 & 77 & 14 \\ 25 & 43 & 61 & 18 & 36 & 54 & 23 & 41 & 59 \\ 70 & 7 & 52 & 63 & 0 & 45 & 68 & 5 & 50 \end{bmatrix}. \quad (56)$$

As noted in [13], $M_9^A$ was known prior to 1000 AD and $M_9^B$ dates to 1275 AD.

In addition, as noted in Section 2, a huge number of pairs of commuting magic squares $\tilde{M}_9^A$ and $\tilde{M}_9^B$ can be constructed according to (7) using various combinations of the eight phases of $M_3$ as the nine subsquares of generalized $\tilde{A}_9$ and any phase of $M_3$ as the basis for $B_9$ in (3). A regular $\tilde{A}_9$ results from using any five phases of $M_3$ as the nine subsquares of $\tilde{A}_9$ placed in a regular block pattern, e.g.

$$\tilde{A}_9 = \begin{bmatrix} 5 & 6 & 1 & 3 & 2 & 7 & 3 & 8 & 1 \\ 0 & 4 & 8 & 8 & 4 & 0 & 2 & 4 & 6 \\ 7 & 2 & 3 & 1 & 6 & 5 & 7 & 0 & 5 \\ 1 & 6 & 5 & 7 & 2 & 3 & 1 & 6 & 5 \\ 8 & 4 & 0 & 0 & 4 & 8 & 8 & 4 & 0 \\ 3 & 2 & 7 & 5 & 6 & 1 & 3 & 2 & 7 \\ 3 & 8 & 1 & 3 & 2 & 7 & 5 & 6 & 1 \\ 2 & 4 & 6 & 8 & 4 & 0 & 0 & 4 & 8 \\ 7 & 0 & 5 & 1 & 6 & 5 & 7 & 2 & 3 \end{bmatrix}. \quad (57)$$

It is easy to see that $\tilde{A}_9$ and $B_9$ are orthogonal and they commute since

$$\tilde{A}_9 B_9 = B_9 \tilde{A}_9 = 144 E_9 = (\mu_9)^2 E_9. \quad (58)$$
Thus, as noted in Section 2, commuting regular $\tilde{M}_3^A$ and $\tilde{M}_3^B$ can be formed from them using (7).

The permutation matrix $P_9$ that connects $A_9$ and $B_9$ according to (13) is given by (16) as

$$P_9 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$  \hspace{1cm} (59)

It can be verified that $P_9$ satisfies (17) and connects $A_9$ to $B_9$ and $M_9^A$ to $M_9^B$ according to (15) and (18). Thus, they are pairs of similar matrices as noted in Section 2.

In order to see why $P_9$ works and can be generalized to higher order $nn$, we consider a general $M_3$ compounded with $E_3$ written in block form as

$$B_9 = M_3 \otimes E_3 = \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix},$$  \hspace{1cm} (60)

where $b_{ij}$ is a block that has all elements $m_{ij}$. Then, the permutation (15) of $B_9$ can be written as

$$P_9B_9P_9 = \begin{bmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{bmatrix} \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix} = \begin{bmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{bmatrix},$$  \hspace{1cm} (61)

and on carrying out the matrix multiplication we find that

$$P_9B_9P_9 = \begin{bmatrix}
M_3 & M_3 & M_3 \\
M_3 & M_3 & M_3 \\
M_3 & M_3 & M_3
\end{bmatrix} = E_3 \otimes M_3 = A_9.$$  \hspace{1cm} (62)

As an example of this matrix multiplication, the element in the first row, second column of $P_9B_9P_9$ is given by

$$p_{11}b_{11}p_{12} + p_{12}b_{21}p_{12} + p_{13}b_{31}p_{12} + p_{11}b_{12}p_{22} + p_{12}b_{22}p_{22} + p_{13}b_{32}p_{22} + p_{11}b_{13}p_{32} + p_{12}b_{23}p_{32} + p_{13}b_{33}p_{32}$$

$$= \begin{bmatrix}
m_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & m_{12} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & m_{13} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & m_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & m_{23} & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
m_{31} & m_{32} & m_{33}
\end{bmatrix} = M_3.$$  \hspace{1cm} (63)

In view of (62), we have verified (15) for $n = 3$. It should be clear that a similar verification of (15) applies for higher orders. Also, (17) can be verified in a similar manner.
Next, we construct the Jordan-form matrices of $A_9$, $B_9$, $M_9^A$, and $M_9^B$ from the following Jordan-form matrices of $M_3$ and $E_3$:

$$S_3 = \begin{bmatrix}
1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} \\
1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} \\
1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6}
\end{bmatrix}, \quad (64)$$

$$D_{M_3} = \text{diag} \begin{bmatrix} 12, 2i\sqrt{6}, -2i\sqrt{6} \end{bmatrix},$$

$$D_{E_3} = \text{diag} [3, 0, 0],$$

where all the eigenvectors in $S_3$ are simple. As noted in Section 2, the eigenvector matrix $S_3$ for $M_3$ is also an eigenvector matrix for $E_3$. Also, the eigenvalues of the regular matrix $M_3$ are $\mu_3$ and a $\pm$ pair. By (64) and (28), we obtain the following Jordan-form matrices for $A_9$ and $B_9$:

$$S_9 = \begin{bmatrix}
1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & 8 + i\sqrt{6} & 58 + 16i\sqrt{6} \\
1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & 8 + i\sqrt{6} & -44 + 12i\sqrt{6} \\
1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & 8 + i\sqrt{6} & -14 - 28i\sqrt{6} \\
1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & -4 + 2i\sqrt{6} & -44 + 12i\sqrt{6} \\
1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & -4 + 2i\sqrt{6} & -8 - 16i\sqrt{6} \\
1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & -4 + 2i\sqrt{6} & 52 + 4i\sqrt{6} \\
1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & -4 - 3i\sqrt{6} & -14 - 28i\sqrt{6} \\
1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & -4 - 3i\sqrt{6} & 52 + 4i\sqrt{6} \\
1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & -4 - 3i\sqrt{6} & -38 + 24i\sqrt{6}
\end{bmatrix}, \quad (65)$$

$$D_9^A = \text{diag} \begin{bmatrix} 36, 6i\sqrt{6}, -6i\sqrt{6}, 0, 0, 0, 0, 0 \end{bmatrix},$$

$$D_9^B = \text{diag} \begin{bmatrix} 36, 0, 0, 6i\sqrt{6}, 0, 0, -6i\sqrt{6}, 0, 0 \end{bmatrix}, \quad (66)$$

Again, the eigenvalues of the regular matrices $A_9$ and $B_9$ are $\mu_9$ and $\pm$ pairs but in a different order as related by (64) with (66). Also, (64) for $S_9$ can be verified. According to (30) and (31), $M_9^A$ and $M_9^B$ have the eigenvector matrix $S_9$ and eigenvalues

$$360, 6i\sqrt{6}, -6i\sqrt{6}, 54i\sqrt{6}, -54i\sqrt{6}, 0, 0, 0, 0,$$  

(67)

as can be verified directly. These eigenvalues agree with those given by Rogers, et al. [13].
Next, we construct the SVD matrices of $A_9$, $B_9$, $M^A_9$, and $M^B_9$ from the following SVD matrices of $M_3$ and $E_3$:

$$U_3 = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{2} & \frac{1}{3}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{2} & \frac{1}{3}\sqrt{6} \end{bmatrix},$$

$$V_3 = \begin{bmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{6} & -\frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} & 0 \end{bmatrix},$$

$$\Sigma_{M3} = \text{diag} \begin{bmatrix} 12, 4\sqrt{3}, 2\sqrt{3} \end{bmatrix},$$

$$\Sigma_{E3} = \text{diag} [3, 0, 0].$$

The SVD matrices for $A_9$, $B_9$, $M^A_9$, and $M^B_9$ are obtained from (68) and (69) as

$$U_9 = \frac{1}{6} \begin{bmatrix} 2 & \sqrt{2} & \sqrt{6} & 3 & \sqrt{3} & \sqrt{2} & \sqrt{3} & \sqrt{6} & 1 \\ 2 & 0 & -2\sqrt{2} & 0 & -2\sqrt{3} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V_9 = \frac{1}{6} \begin{bmatrix} 2 & \sqrt{2} & \sqrt{6} & 3 & -\sqrt{3} & \sqrt{2} & -\sqrt{3} & 1 \\ 2 & -\sqrt{2} & \sqrt{6} & -3 & \sqrt{3} & \sqrt{2} & -\sqrt{3} & 1 \\ 2 & \sqrt{2} & \sqrt{6} & 0 & 0 & 0 & -2\sqrt{2} & -2\sqrt{3} & -2 \\ 2 & 0 & -2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} & 0 & 4 \\ 2 & 0 & \sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} & 2\sqrt{3} & -2 \\ 2 & -\sqrt{2} & -\sqrt{6} & 0 & 2\sqrt{3} & \sqrt{2} & 0 & -2 \\ 2 & \sqrt{2} & -\sqrt{6} & 0 & -2\sqrt{3} & \sqrt{2} & -\sqrt{3} & 1 \end{bmatrix},$$

$$\Sigma^A_9 = \text{diag} \begin{bmatrix} 36, 12\sqrt{3}, 6\sqrt{3}, 0, 0, 0, 0, 0, 0 \end{bmatrix},$$

$$\Sigma^B_9 = \text{diag} \begin{bmatrix} 36, 0, 0, 0, 0, 6\sqrt{3}, 0, 0 \end{bmatrix},$$

$$\Sigma^A_{M9} = \text{diag} \begin{bmatrix} 360, 12\sqrt{3}, 6\sqrt{3}, 108, 0, 0, 54\sqrt{3}, 0, 0 \end{bmatrix},$$

$$\Sigma^B_{M9} = \text{diag} \begin{bmatrix} 360, 0, 0, 0, 0, 0, 6\sqrt{3}, 0, 0 \end{bmatrix},$$

which can be verified directly from their SVD definitions (68). Also, (52) can be verified. The singular values for $M^A_9$ and $M^B_9$ agree with those given in [13].

11
Order 12. We start with the regular magic squares used by Rogers, et. al [13], namely

\[ M_n = M_3 = \begin{bmatrix} 3 & 8 & 1 \\ 2 & 4 & 6 \\ 7 & 0 & 5 \end{bmatrix}, \quad M_m = M_4 = \begin{bmatrix} 4 & 3 & 15 & 8 \\ 10 & 13 & 1 & 6 \\ 9 & 14 & 2 & 5 \\ 7 & 0 & 12 & 11 \end{bmatrix}. \] (72)

From (3) we form the order-12, commuting regular, unnatural, magic squares

\[ A_{12} = \begin{bmatrix} 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \\ 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \\ 3 & 8 & 1 & 3 & 8 & 1 & 3 & 8 & 1 \\ 2 & 4 & 6 & 2 & 4 & 6 & 2 & 4 & 6 \\ 7 & 0 & 5 & 7 & 0 & 5 & 7 & 0 & 5 \end{bmatrix}, \] (73)

\[ B_{12} = \begin{bmatrix} 4 & 4 & 4 & 3 & 3 & 3 & 15 & 15 & 15 & 8 & 8 & 8 \\ 4 & 4 & 4 & 3 & 3 & 3 & 15 & 15 & 15 & 8 & 8 & 8 \\ 4 & 4 & 4 & 3 & 3 & 3 & 15 & 15 & 15 & 8 & 8 & 8 \\ 10 & 10 & 10 & 13 & 13 & 13 & 1 & 1 & 1 & 6 & 6 & 6 \\ 10 & 10 & 10 & 13 & 13 & 13 & 1 & 1 & 1 & 6 & 6 & 6 \\ 10 & 10 & 10 & 13 & 13 & 13 & 1 & 1 & 1 & 6 & 6 & 6 \\ 9 & 9 & 9 & 14 & 14 & 14 & 2 & 2 & 2 & 5 & 5 & 5 \\ 9 & 9 & 9 & 14 & 14 & 14 & 2 & 2 & 2 & 5 & 5 & 5 \\ 9 & 9 & 9 & 14 & 14 & 14 & 2 & 2 & 2 & 5 & 5 & 5 \\ 7 & 7 & 7 & 0 & 0 & 0 & 12 & 12 & 12 & 11 & 11 & 11 \\ 7 & 7 & 7 & 0 & 0 & 0 & 12 & 12 & 12 & 11 & 11 & 11 \\ 7 & 7 & 7 & 0 & 0 & 0 & 12 & 12 & 12 & 11 & 11 & 11 \end{bmatrix}. \] (74)

Then, (7) gives the commuting, regular, magic squares

\[ M^A_{12} = \begin{bmatrix} 39 & 44 & 37 & 30 & 35 & 28 & 138 & 143 & 136 & 75 & 80 & 73 \\ 38 & 40 & 42 & 29 & 31 & 33 & 137 & 139 & 141 & 74 & 76 & 78 \\ 43 & 36 & 41 & 34 & 27 & 32 & 142 & 135 & 140 & 79 & 72 & 77 \\ 93 & 98 & 91 & 120 & 125 & 118 & 12 & 17 & 10 & 57 & 62 & 55 \\ 92 & 94 & 96 & 119 & 121 & 123 & 11 & 13 & 15 & 56 & 58 & 60 \\ 97 & 90 & 95 & 124 & 117 & 122 & 16 & 9 & 14 & 61 & 54 & 59 \\ 84 & 89 & 82 & 129 & 134 & 127 & 21 & 26 & 19 & 48 & 53 & 46 \\ 83 & 85 & 87 & 128 & 130 & 132 & 20 & 22 & 24 & 47 & 49 & 51 \\ 88 & 81 & 86 & 133 & 126 & 131 & 25 & 18 & 23 & 52 & 45 & 50 \\ 66 & 71 & 64 & 3 & 8 & 1 & 111 & 116 & 109 & 102 & 107 & 100 \\ 65 & 67 & 69 & 2 & 4 & 6 & 110 & 112 & 114 & 101 & 103 & 105 \\ 70 & 63 & 68 & 7 & 0 & 5 & 115 & 108 & 113 & 106 & 99 & 104 \end{bmatrix}. \] (75)
The Jordan-form matrices of $M_4$ and $E_3$ are given by (64) and those of $M_4$ are

$$S_4 = \begin{bmatrix} 1 & 48 & 14 & 3 \\ 1 & -16 & 10 & -1 \\ 1 & 16 & 6 & -1 \\ 1 & -48 & -2 & -1 \end{bmatrix}, \quad J_4 = \begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (77)$$

where $S_4$ has a generalized eigenvector. Then, (28) results in

$$S_{12} = \begin{bmatrix} 1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & 48 & 384 + 48i\sqrt{6} & 384 - 48i\sqrt{6} \\ 1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & 48 & -192 + 96i\sqrt{6} & -192 - 96i\sqrt{6} \\ 1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & 48 & -192 - 144i\sqrt{6} & -192 + 144i\sqrt{6} \\ 1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & -16 & -128 - 16i\sqrt{6} & -128 + 16i\sqrt{6} \\ 1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & -16 & 64 - 32i\sqrt{6} & 64 + 32i\sqrt{6} \\ 1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & -16 & 64 + 48i\sqrt{6} & 64 - 48i\sqrt{6} \\ 1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & 16 & 128 + 16i\sqrt{6} & 128 - 16i\sqrt{6} \\ 1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & 16 & -64 - 32i\sqrt{6} & -64 + 32i\sqrt{6} \\ 1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & 16 & -64 - 48i\sqrt{6} & -64 + 48i\sqrt{6} \\ 1 & 8 + i\sqrt{6} & 8 - i\sqrt{6} & -48 & -384 - 128i\sqrt{6} & -384 + 128i\sqrt{6} \\ 1 & -4 - 3i\sqrt{6} & -4 + 3i\sqrt{6} & -48 & 192 - 96i\sqrt{6} & 192 + 96i\sqrt{6} \\ 1 & -4 + 2i\sqrt{6} & -4 - 2i\sqrt{6} & -48 & 192 - 144i\sqrt{6} & 192 + 144i\sqrt{6} \end{bmatrix}, \quad (78)$$
\[ J_{A12} = \text{diag} \left[ 12, 8i\sqrt{6}, -8i\sqrt{6}, 0, \ldots, 0 \right], \]

\[ J_{B12} = \begin{bmatrix}
90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \quad (79) \]

Although \( J_{B12} \) is not in standard form, it can be brought there by rearranging and scaling the generalized eigenvectors in \( S_{12} \) without affecting the eigenvalues in \( J_{A12} \). From (31), the nonzero eigenvalues of \( M_{A12} \) and \( M_{B12} \) are found to be

\[ M_{A12} : 858, 8i\sqrt{6}, -8i\sqrt{6}, \quad M_{B12} : 858, 128i\sqrt{6}, -128i\sqrt{6}. \quad (80) \]

The nonzero eigenvalues of \( M_{A12} \) agree with those given by Rogers, et. al [13] who do not construct \( M_{B12} \).

The SVD matrices of \( M_3 \) are given by (68) and those of \( M_4 \) are

\[ U_4 = \frac{1}{10} \begin{bmatrix}
5 & -5 & 3\sqrt{5} & \sqrt{5} \\
5 & 5 & -\sqrt{5} & 3\sqrt{5} \\
5 & -5 & \sqrt{5} & -3\sqrt{5} \\
5 & -5 & -3\sqrt{5} & -\sqrt{5} \\
\end{bmatrix}, \]

\[ V_4 = \frac{1}{10} \begin{bmatrix}
5 & \sqrt{5} & -5 & -3\sqrt{5} \\
5 & 3\sqrt{5} & 5 & \sqrt{5} \\
5 & -3\sqrt{5} & 5 & -\sqrt{5} \\
5 & -\sqrt{5} & -5 & 3\sqrt{5} \\
\end{bmatrix}, \quad (81) \]

\[ \Sigma_4 = \text{diag} \left[ 30, 8\sqrt{5}, 2\sqrt{5}, 0 \right]. \]

The \( U_{12} \) and \( V_{12} \) matrices for \( A_{12}, B_{12}, M_{A12}, \) and \( M_{B12} \) can be obtained from (48) with (68) and (81). The singular value matrices from (48), (68), (81) and (50) are

\[ \Sigma_{A12} = \text{diag} \left[ 48, 16\sqrt{3}, 8\sqrt{3}, 0, 0, \ldots, 0 \right], \]

\[ \Sigma_{B12} = \text{diag} \left[ 90, 0, 0, 24\sqrt{5}, 0, 0, 6\sqrt{5}, 0, 0, \ldots, 0 \right], \]

\[ \Sigma_{A_{M12}} = \text{diag} \left[ 858, 16\sqrt{3}, 8\sqrt{3}, 216\sqrt{5}, 0, 0, 54\sqrt{5}, 0, 0, \ldots, 0 \right], \quad (82) \]

\[ \Sigma_{B_{M12}} = \text{diag} \left[ 858, 256\sqrt{3}, 128\sqrt{3}, 24\sqrt{5}, 0, 0, 6\sqrt{5}, 0, 0, \ldots, 0 \right]. \]

\[ \text{6The singular values obtained for } M_{A12} \text{ (verified by MAPLE).agree with those of Rogers, et.al [13] except for their } 24\sqrt{5} \text{ and } 6\sqrt{5} \text{ instead of our } 216\sqrt{5} \text{ and } 54\sqrt{5}. \]
In addition, as noted by Rogers, et. al [13], a different pair of $M_A^{12}$ and $M_B^{12}$ can be constructed by interchanging $M_m$ and $M_n$ in (72), i.e.

$$
M_n = M_4 = \begin{bmatrix}
4 & 3 & 15 & 8 \\
10 & 13 & 1 & 6 \\
9 & 14 & 2 & 5 \\
7 & 0 & 12 & 11 \\
\end{bmatrix}, \quad M_m = M_3 = \begin{bmatrix}
3 & 8 & 1 \\
2 & 4 & 6 \\
7 & 0 & 5 \\
\end{bmatrix},
$$

(83)

leading to the commuting pair of regular magic squares

$$
\hat{M}_A^{12} = \begin{bmatrix}
52 & 51 & 63 & 56 & 132 & 131 & 143 & 136 & 20 & 19 & 31 & 24 \\
58 & 61 & 49 & 54 & 138 & 141 & 129 & 134 & 26 & 29 & 17 & 22 \\
57 & 62 & 50 & 53 & 137 & 142 & 130 & 133 & 25 & 30 & 18 & 21 \\
55 & 48 & 60 & 59 & 135 & 128 & 140 & 139 & 23 & 16 & 28 & 27 \\
36 & 35 & 47 & 40 & 68 & 67 & 79 & 72 & 100 & 99 & 111 & 104 \\
42 & 45 & 33 & 38 & 74 & 77 & 65 & 70 & 106 & 109 & 97 & 102 \\
41 & 46 & 34 & 37 & 73 & 78 & 66 & 69 & 105 & 110 & 98 & 101 \\
39 & 32 & 44 & 43 & 71 & 64 & 76 & 75 & 103 & 96 & 108 & 107 \\
116 & 115 & 127 & 120 & 4 & 3 & 15 & 8 & 84 & 83 & 95 & 88 \\
122 & 125 & 113 & 118 & 10 & 13 & 1 & 6 & 90 & 93 & 81 & 86 \\
121 & 126 & 114 & 117 & 9 & 14 & 2 & 5 & 89 & 94 & 82 & 85 \\
119 & 112 & 124 & 123 & 7 & 0 & 12 & 11 & 87 & 80 & 92 & 91 \\
\end{bmatrix},
$$

(84)

$$
\hat{M}_B^{12} = \begin{bmatrix}
39 & 30 & 138 & 75 & 44 & 35 & 143 & 80 & 37 & 28 & 136 & 73 \\
93 & 120 & 12 & 57 & 98 & 125 & 17 & 62 & 91 & 118 & 10 & 55 \\
84 & 129 & 21 & 48 & 89 & 134 & 26 & 53 & 82 & 127 & 19 & 46 \\
66 & 3 & 111 & 102 & 71 & 8 & 116 & 107 & 64 & 1 & 109 & 100 \\
38 & 29 & 137 & 74 & 40 & 31 & 139 & 76 & 42 & 33 & 141 & 78 \\
92 & 119 & 11 & 56 & 94 & 121 & 13 & 58 & 96 & 123 & 15 & 60 \\
83 & 128 & 20 & 47 & 85 & 130 & 22 & 49 & 87 & 132 & 24 & 51 \\
65 & 2 & 110 & 101 & 67 & 4 & 112 & 103 & 69 & 6 & 114 & 105 \\
43 & 34 & 142 & 79 & 36 & 27 & 135 & 72 & 41 & 32 & 140 & 77 \\
97 & 124 & 16 & 61 & 90 & 117 & 9 & 54 & 95 & 122 & 14 & 59 \\
88 & 133 & 25 & 52 & 81 & 126 & 18 & 45 & 86 & 131 & 23 & 50 \\
70 & 7 & 115 & 106 & 63 & 0 & 108 & 99 & 68 & 5 & 113 & 104 \\
\end{bmatrix},
$$

(85)

The matrices in the Jordan form and SVD of $\hat{M}_A^{12}$ and $\hat{M}_B^{12}$ can be constructed as before. It follows from formulas in Sections 3 and 4 that the eigenvalues and singular values for $\hat{M}_A^{12}$ are the same as those for $M_B^{12}$ and the eigenvalues and singular values for $\hat{M}_B^{12}$ are the same as those for $M_A^{12}$ (verified by MAPLE). However, their respective eigenvalue matrices are different and they do not commute.

Again, as noted in Section 2, a huge number of pairs of commuting magic squares $M_A^{12}$ and $M_B^{12}$ can be constructed according to (7) using various combinations of phases of $M_3$ as subsquares in generalized $\tilde{A}_{12}$ and any $M_4$ as the basis for $B_{12}$ in (3). This same construction applies to $\hat{M}_A^{12}$ and $\hat{M}_B^{12}$ using various combinations of phases of $M_4$ as subsquares in generalized $\tilde{A}_{12}$ and any $M_3$ as the basis for $B_{12}$ in (3).
Order 16. We start with the order-4, pandiagonal, magic square equivalent to the one used by Chan and Loly [1], namely

\[ M_4 = \begin{bmatrix}
13 & 6 & 11 & 0 \\
10 & 1 & 12 & 7 \\
4 & 15 & 2 & 9 \\
3 & 8 & 5 & 14
\end{bmatrix}. \] (86)

By compounding according to (3) with \( M_m = M_n = M_4 \), we form the following order-16, commuting, pandiagonal, unnatural magic squares

\[ A_{16} = \begin{bmatrix}
13 & 6 & 11 & 0 & 13 & 6 & 11 & 0 & 13 & 6 & 11 & 0 \\
10 & 1 & 12 & 7 & 10 & 1 & 12 & 7 & 10 & 1 & 12 & 7 \\
4 & 15 & 2 & 9 & 4 & 15 & 2 & 9 & 4 & 15 & 2 & 9 \\
3 & 8 & 5 & 14 & 3 & 8 & 5 & 14 & 3 & 8 & 5 & 14
\end{bmatrix}, \] (87)

\[ B_{16} = \begin{bmatrix}
13 & 13 & 13 & 13 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 \\
13 & 13 & 13 & 13 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 \\
13 & 13 & 13 & 13 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 \\
13 & 13 & 13 & 13 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 \\
10 & 10 & 10 & 10 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 7 & 7 & 7 & 7 \\
10 & 10 & 10 & 10 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 7 & 7 & 7 & 7 \\
10 & 10 & 10 & 10 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 7 & 7 & 7 & 7 \\
10 & 10 & 10 & 10 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 7 & 7 & 7 & 7 \\
4 & 4 & 4 & 4 & 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 & 9 & 9 & 9 & 9 \\
4 & 4 & 4 & 4 & 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 & 9 & 9 & 9 & 9 \\
4 & 4 & 4 & 4 & 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 & 9 & 9 & 9 & 9 \\
4 & 4 & 4 & 4 & 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 & 9 & 9 & 9 & 9 \\
3 & 3 & 3 & 3 & 8 & 8 & 8 & 8 & 5 & 5 & 5 & 5 & 14 & 14 & 14 & 14 \\
3 & 3 & 3 & 3 & 8 & 8 & 8 & 8 & 5 & 5 & 5 & 5 & 14 & 14 & 14 & 14 \\
3 & 3 & 3 & 3 & 8 & 8 & 8 & 8 & 5 & 5 & 5 & 5 & 14 & 14 & 14 & 14 \\
3 & 3 & 3 & 3 & 8 & 8 & 8 & 8 & 5 & 5 & 5 & 5 & 14 & 14 & 14 & 14
\end{bmatrix}. \] (88)

It is not difficult to verify that these two matrices are pandiagonal by comparing their diagonals with those of \( M_4 \) as done by Eggermont [2]. Two such diagonals of \( A_{16} \) and \( B_{16} \) are shown in bold. The elements on the diagonals of \( A_{16} \) are simply four replications of the diagonals of \( M_4 \). The elements on the diagonals of \( B_{16} \) are weighted combinations of two adjacent diagonals of \( M_4 \). It should be clear that a similar argument applies to other cases of compounded pandiagonal magic squares.\(^7\)

\(^7\) A lengthy formal proof of the pandiagonality of general \( A_{mn} \) and \( B_{mn} \) is possible but it is not given here.
essentially agrees with the matrix constructed in [1]. The similar matrices
the Jordan form and SVD of these order-16 matrices from our formulas for them.

\[
\tilde{A} \quad \text{and transforming these}
\]

\[
\tilde{M}_{16} = \\
\begin{bmatrix}
221 & 214 & 219 & 208 & 109 & 102 & 107 & 96 & 189 & 182 & 187 & 176 & 13 & 6 & 11 & 0 \\
218 & 209 & 220 & 215 & 106 & 97 & 108 & 103 & 186 & 177 & 188 & 183 & 10 & 1 & 12 & 7 \\
212 & 233 & 210 & 217 & 100 & 111 & 98 & 105 & 180 & 191 & 178 & 185 & 4 & 15 & 2 & 9 \\
211 & 216 & 213 & 222 & 99 & 104 & 101 & 110 & 179 & 184 & 181 & 190 & 3 & 8 & 5 & 14 \\
173 & 166 & 171 & 160 & 29 & 22 & 27 & 16 & 205 & 198 & 203 & 192 & 125 & 118 & 123 & 112 \\
170 & 161 & 172 & 167 & 26 & 17 & 28 & 23 & 202 & 193 & 204 & 199 & 122 & 113 & 124 & 119 \\
164 & 175 & 162 & 169 & 20 & 31 & 18 & 25 & 196 & 207 & 194 & 201 & 216 & 112 & 127 & 114 & 121 \\
163 & 168 & 165 & 174 & 19 & 24 & 21 & 30 & 195 & 200 & 197 & 206 & 115 & 120 & 117 & 126 \\
77 & 70 & 75 & 64 & 253 & 246 & 251 & 240 & 45 & 38 & 43 & 32 & 157 & 150 & 155 & 144 \\
74 & 65 & 76 & 71 & 250 & 241 & 252 & 247 & 42 & 33 & 44 & 39 & 154 & 145 & 156 & 151 \\
68 & 79 & 66 & 73 & 244 & 255 & 242 & 249 & 36 & 47 & 34 & 41 & 148 & 159 & 146 & 153 \\
67 & 72 & 69 & 78 & 243 & 248 & 245 & 254 & 35 & 40 & 37 & 46 & 147 & 152 & 149 & 158 \\
61 & 54 & 59 & 48 & 141 & 134 & 139 & 128 & 93 & 86 & 91 & 80 & 237 & 230 & 235 & 224 \\
58 & 49 & 60 & 55 & 138 & 129 & 140 & 135 & 90 & 81 & 92 & 87 & 234 & 225 & 236 & 231 \\
52 & 63 & 50 & 57 & 132 & 143 & 130 & 137 & 84 & 95 & 82 & 89 & 228 & 239 & 226 & 233 \\
51 & 56 & 53 & 62 & 131 & 136 & 133 & 142 & 83 & 88 & 85 & 94 & 227 & 232 & 229 & 238
\end{bmatrix}
\tag{89}
\]

\[
M_{16}^B = \\
\begin{bmatrix}
221 & 109 & 189 & 13 & 214 & 102 & 182 & 6 & 219 & 107 & 187 & 11 & 208 & 96 & 176 & 0 \\
173 & 29 & 205 & 125 & 156 & 22 & 198 & 118 & 171 & 27 & 203 & 123 & 160 & 16 & 192 & 112 \\
77 & 253 & 45 & 157 & 70 & 246 & 38 & 150 & 75 & 251 & 43 & 155 & 64 & 240 & 32 & 144 \\
61 & 141 & 93 & 237 & 54 & 134 & 86 & 230 & 59 & 139 & 91 & 235 & 48 & 128 & 80 & 224 \\
218 & 106 & 186 & 10 & 209 & 97 & 177 & 1 & 220 & 108 & 188 & 12 & 215 & 103 & 183 & 7 \\
170 & 26 & 202 & 122 & 161 & 17 & 193 & 113 & 172 & 28 & 204 & 124 & 167 & 23 & 199 & 119 \\
74 & 250 & 42 & 154 & 65 & 241 & 33 & 145 & 76 & 252 & 44 & 156 & 71 & 247 & 39 & 151 \\
58 & 138 & 90 & 234 & 49 & 129 & 81 & 225 & 60 & 140 & 92 & 236 & 55 & 135 & 87 & 231 \\
212 & 100 & 180 & 4 & 223 & 111 & 191 & 15 & 210 & 98 & 178 & 2 & 217 & 105 & 185 & 9 \\
164 & 20 & 196 & 116 & 175 & 31 & 207 & 127 & 162 & 18 & 194 & 114 & 169 & 25 & 201 & 121 \\
68 & 244 & 36 & 148 & 79 & 255 & 47 & 159 & 66 & 242 & 34 & 146 & 73 & 249 & 41 & 153 \\
52 & 132 & 84 & 228 & 63 & 143 & 95 & 239 & 50 & 130 & 82 & 226 & 57 & 137 & 89 & 233 \\
211 & 99 & 179 & 3 & 216 & 104 & 184 & 8 & 213 & 101 & 181 & 5 & 222 & 110 & 190 & 14 \\
163 & 19 & 195 & 115 & 168 & 24 & 200 & 120 & 165 & 21 & 197 & 117 & 174 & 30 & 206 & 126 \\
67 & 243 & 35 & 147 & 72 & 248 & 40 & 152 & 69 & 245 & 37 & 149 & 78 & 254 & 46 & 158 \\
51 & 131 & 83 & 227 & 56 & 136 & 88 & 232 & 53 & 133 & 85 & 229 & 62 & 142 & 94 & 238
\end{bmatrix}
\tag{90}
\]

where pandiagonality follows from (7) and the pandiagonality of \(A_{16}\) and \(B_{16}\). The matrix \(M^A_{16}\) essentially agrees with the matrix constructed in [1]. The similar matrices \(M^A_{16}\) and \(M^B_{16}\) are related by (18) with \(P_{16}\) constructed from (16). We leave to the reader the pleasure (or pain) of constructing the Jordan form and SVD of these order-16 matrices from our formulas for them.

Again, as noted in Section 2, a huge number of pairs of commuting magic squares \(\tilde{M}_{16}^A\) and \(\tilde{M}_{16}^B\) can be constructed according to (7) using various 16 combinations of the known 880 \(M_4\) magic squares, [11] as the 16 subsquares of generalized \(\tilde{A}_{16}\) and any \(M_4\) as the basis for \(B_{16}\) in (3). Pandiagonal \(M^A_{16}\) and \(M^B_{16}\) of this form are not possible in general except by first constructing regular \(M^A_{16}\) and \(M^B_{16}\) using any eight regular \(M_4\) as subsquares of \(\tilde{A}_{16}\) arranged in a regular block pattern and transforming these \(M^A_{16}\) and \(M^B_{16}\) to pandiagonal magic squares by the Planck transformation [12, 7].
6 Conclusion

Commuting magic square matrices can be formed by compounding two magic square matrices with the all-ones matrix in a known manner. We verify that the compounded matrices retain the regular (associative) and pandiagonality properties of the original magic squares as noted by Eggermont [2]. In the case where a single matrix is compounded with the all-ones matrix in two ways, the compounded matrices are related by a row/column permutation (shuffle) of their elements as noted by Rogers, et. al [13] and expressed here in a matrix form which shows that they are similar. We derive simple formulas for the Jordan-form matrices and SVD matrices of the compounded magic square matrices in terms of those of the original magic squares. A wider class of commuting magic squares is considered but our formulas for the Jordan form and SVD do not apply to them. Three examples illustrate the constructions and validate our formulas. The methods presented here should apply to other compound matrix constructions, such as the additional ones given in [13].

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