The Magnus force in superfluids and superconductors

E.B. Sonin

Low Temperature Laboratory, Helsinki University of Technology, 02150 Espoo, Finland
and
Ioffe Physical Technical Institute, St. Petersburg 194021, Russia

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The forces on the vortex, transverse to its velocity, are considered. In addition to the superfluid Magnus force from the condensate (superfluid component), there are transverse forces from thermal quasiparticles and external fields violating the translational invariance. The forces between quasiparticles and the vortex originate from interference of quasiparticles with trajectories on the left and on the right from the vortex like similar forces for electrons interacting with the thin magnetic-flux tube (the Aharonov-Bohm effect). These forces are derived in the Born approximation for phonons from the equations of superfluid hydrodynamics, and for BCS quasiparticles from the Bogolyubov-de Gennes equations.

The effect of external fields violating translational invariance is analyzed for vortices in the two-dimensional Josephson junction array. The symmetry analysis of the classical equations for the array shows that the total transverse force on the vortex vanishes. Therefore the Hall effect which is linear in the transverse force is absent also. This means that the Magnus force from the superfluid component exactly cancels with the transverse force from the external fields.

The results of other approaches are also brought together for discussion.

I. INTRODUCTION

The Magnus force on a vortex has long been known in classical hydrodynamics. This force appears if the vortex moves with respect to a liquid. The force is normal to the relative vortex velocity and therefore is reactive and does not produce a work. In general, such a force arises always when a body with a flow circulation around it moves through a liquid or a gas (the Kutta-Joukowski theorem). The most important example is the lifting force on a wing of an aeroplane which keeps the aeroplane in the air.

The key role of the Magnus force in vortex dynamics has become clear from the very beginning of studying superfluid hydrodynamics. The superfluid Magnus force was defined as a force between a vortex and a superfluid and therefore was proportional to the superfluid density $\rho_s$. But in the two-fluid hydrodynamics the superfluid Magnus force is not the only force on the vortex transverse to its velocity: there was also a transverse force between the vortex and quasiparticles moving with respect to the vortex. The transverse force from rotons was found by Lifshitz and Pitaevskii from the quasiclassical scattering theory. Later Iordanskii revealed the transverse force from phonons which was equal in magnitude and opposite in sign with the quasiclassical force of Lifshitz and Pitaevskii. From the very beginning the Iordanskii force was a controversial matter. Iordanskii suggested that his force and the Lifshitz-Pitaevskii force were of different origins and for rotons they should be summed. As a result, he concluded that the transverse force from rotons vanished. But the analysis done in Ref. demonstrated that the Iordanskii force for rotons is identical to the Lifshitz-Pitaevskii force and they must not be added. In addition, the Lifshitz-Pitaevskii force from rotons was calculated in the original paper with a wrong sign. After its correction the transverse force on the vortex had a same sign and a value both for rotons (the Lifshitz-Pitaevskii force) and for phonons (the Iordanskii force). In the same paper it was pointed out that the Iordanskii force for phonons and rotons results from interference between quasiparticles which move past the vortex on the left and on the right sides with different phase shifts, like in the Aharonov-Bohm effect.

In the theory of superconductivity the Magnus force appeared first in the paper by Nozières and Vinen. In clean superconductors the BCS quasiparticles produce an additional transverse force on the vortex analogous to the Iordanskii force in superfluids. The total transverse force is responsible for the Hall effect in the mixed state of a superconductor. But the Hall effect was rather weak in classical superconductors. An explanation of it was suggested by Kopnin and Kravtsov (see also Ref.): impurities interact with quasiparticles bound in the vortex core and this interaction produces an additional transverse force on the vortex. In contrast with the quasiparticle transverse force which increases the total transverse force, the impurity force decreases it and in a dirty superconductor completely cancels the Magnus force. As a result, the strong Hall effect is possible only in superclean superconductors. The transverse force from the bound states in the core has been recently rephrased in terms of the spectral flow through the quasiparticle bound states.
A new wave of interest to the Magnus force came with discovery of high-\(T_c\) superconductivity. A few reasons of this interest might be mentioned: (i) The so-called Hall anomaly was observed [14]; near \(T_c\) the sign of the Hall voltage is opposite to that expected from the standard vortex dynamics. (ii) It has become possible to obtain superclean single crystals with large Hall angle as predicted in the superclean limit of the theory of Kopnin and Kravtsov. This made possible to observe magnetoresonances in the a.c. response of the superconducting single crystals connected with waves propagating along vortices [13,16]. (iii) The effective Magnus force governs quantum vortex nucleation in clean high-\(T_c\) superconductors intensively discussed now [17,18].

Despite a lot of work done to understand and calculate the Magnus force, it still remains to be a matter of controversy with a number conflicting points of view on it. A new discussion has been launched by the paper of Ao and Thouless [13]. The main claim of Ao and Thouless is that there is an universal \textit{exact} expression for the total transverse force on the vortex (the effective Magnus force) which does not depend on the presence of quasiparticles or impurities. This force derived from the concept of the geometrical phase (the Berry phase) coincides with the superfluid Magnus force and therefore is proportional to the superfluid density. According to Ao and Thouless, there is no transverse force on the vortex from quasiparticles and impurities, though they might influence the value of the superfluid density and thereby influence the amplitude of the Magnus force.

The Ao-Thouless theory is in an evident disagreement with the previous calculations of the transverse force on the vortex (the effective Magnus force) in superfluids and superconductors reviewed above. It attracted a great attention and has been supported in a number of recent publications of other theorists (see e.g. Refs. [20–22]). If the Ao-Thouless were true, it would be necessary to revise the whole basis of the vortex dynamics. For example, on the basis of the Ao-Thouless theory Šimanek [21] suggested that quantum vortex tunnelling is governed by the Magnus force obtained from the Berry phase approach, i.e., proportional to the superfluid density, in contradiction to the previous theory [17,18]. Therefore it is important to understand what the Magnus force is and whether the Ao-Thouless theory is true or not.

The present paper is to analyze this controversy. Among the sources of controversy there is semantics. Therefore it is important to define force terminology from the very beginning. The word \textit{force} itself is only a label to describe a transfer of momentum between two objects. Before using these labels one must to analyze the momentum balance for any object and only then to label various contributions to these balances as forces. Keeping this in mind, the forces under discussions may be defined as the following:

- There is a momentum transfer between a superfluid and a vortex. It is revealed analyzing the momentum balance for the superfluid moving with the velocity \(\vec{v}_s\) whereas the vortex moves through the superfluid with a different velocity \(\vec{v}_L\). This momentum transfer is the \textit{superfluid Magnus force}. It is proportional to the superfluid density \(\rho_s\) and transverse to the relative velocity \(\vec{v}_L - \vec{v}_s\).
- Analyzing the momentum balance for the \textit{whole} translationally invariant liquid (including the superfluid and normal parts of it) around the vortex one may reveal a contribution presenting the momentum transfer between the vortex moving with the velocity \(\vec{v}_L\) and the normal fluid (the gas of quasiparticles) moving with the velocity \(\vec{v}_n\). This force is proportional to the relative velocity \(\vec{v}_L - \vec{v}_n\) and has the components longitudinal and transverse to \(\vec{v}_L - \vec{v}_n\). The transverse component is the \textit{Iordanskii force}.
- If there is no translational invariance, as in a dirty superconductor, the momentum balance for the whole liquid must include also forces external for the liquid, namely, the momentum transfer to the impurities rigidly connected with the crystal lattice. When the latter is at rest, this momentum transfer, or the force from impurities, is proportional to the absolute value of the vortex velocity \(\vec{v}_L\). Its component transverse to \(\vec{v}_L\) is the Kopnin-Kravtsov force.
- The momentum balance for the whole liquid around the vortex is at the same time an equation from which one must find the vortex velocity \(\vec{v}_L\). Therefore it is useful to collect all the terms proportional to \(\vec{v}_L\) together. After it the term uniting all contributions transverse to \(\vec{v}_L\) is the total transverse force on the vortex, or the \textit{effective Magnus force}.

Thus in general three forces contribute to the effective Magnus force: the superfluid Magnus force, the Iordanskii force, and the Kopnin-Kravtsov force. Experimentally they can determine the effective Magnus force, but not the \textit{bare} superfluid Magnus force. In rotating superfluids the effective Magnus force determines the mutual friction. One can find the latest reviews of the experiment and the theory on mutual friction in \(^3\text{He}\) in Refs. [23,24]. In superconductors the Hall conductivity [14,23] and the acoustic Faraday effect for the transverse ultrasound wave propagating along vortices [26] are linear in the effective Magnus force. The process of vortex quantum tunnelling is also influenced by the effective, but not the superfluid Magnus force. So the final outcome of the theory must be the amplitude of the effective Magnus force. Its presentation as a combination of three forces is an intermediate stage of
the theory. In fact, this presentation is valid only if (i) the number of quasiparticles is not too large and their mutual interaction is not strong; (ii) external fields violating translational invariance are not too strong. The first condition is violated close to $T_c$, where other approaches based on the Ginzburg-Landau theory (or its analogue for superfluids, the Ginzburg-Pitaevskii theory) must be used \cite{27,28}. The second condition doesn’t hold in the Josephson junction array considered in the present paper (see below). In these cases the theory deals directly with the effective Magnus force in the equation of vortex motion: its decomposition on the “bare” superfluid Magnus force and the forces from quasiparticles or impurities becomes conventional and of a little physical sense.

Whereas there is a consensus among theorists on the superfluid Magnus force, the Ao-Thouless theory rejects the Iordanskii force from quasiparticles and the Kopnin-Kravtsov force from impurities claiming that amplitudes of the effective and the superfluid Magnus forces are exactly equal. Therefore the present paper considers the effect on the force balance from (i) quasiparticles, and (ii) from external fields violating the translational invariance of the superfluid.

In my analysis of the quasiparticle effect I chose the phonon-vortex interaction which may be described by the nonlinear Schrödinger equation long ago suggested for a weakly nonideal Bose-gas (the Gross-Pitaevskii theory \cite{23}). It is well known that the nonlinear Schrödinger equation yields the usual superfluid hydrodynamics. It is a good starting point for further discussion, which one may expect a consensus of all parties on. Indeed, according to Aitchison, Ao, and Thouless \cite{30}, the nonlinear Schrödinger equation, and the superfluid hydrodynamics derived from it, is a reliable model to describe both Bose- and Fermi-superfluids near $T = 0$. The next step is to analyze scattering of the sound wave (phonon) by the vortex in hydrodynamics. Just at this stage a disagreement appears. Ao and Thouless believe that this scattering can produce only a dissipative force on the vortex, but not a transverse one. Recently Demircan, Ao, and Niu \cite{32} tried to prove it using the Born approximation. But they ignored peculiarities of the phonon Born scattering at small angles which resulted in the Iordanskii force. It is important to note that the controversy arises not from a difference in ideology: anyone is free to choose a language to derive the Magnus force at $T = 0$: either the standard hydrodynamics, or newest topological concepts of the geometrical phase. But there is a disagreement in calculation of integrals describing the phonon scattering in the first order of the perturbation theory. We hope to show in this paper at which point Demircan, Ao, and Niu \cite{32} missed to take into account the Aharonov-Bohm interference of phonons which was ignored by the Ao-Thouless theory.

Now discussions around the transverse force on the vortex in the presence of impurities violating the translational invariance concentrate mostly on the contribution of the core bound states in Fermi superfluids. This requires a rather sophisticated analysis (see \cite{13} and references therein). In the present paper I chose another example when the translational invariance is absent: the two-dimensional Josephson junction array (JJA). This is a regular lattice of nodes with the Josephson coupling between them. Experimentally, any node corresponds to a superconducting island in an artificially prepared JJA, or to a grain in a granular superconducting film. The behavior of the JJA in an external magnetic field is usually described in the picture of moving vortices similar to the mixed state of type II superconductors. The dynamics of vortices in JJA attracts a great interest of experimentalists \cite{33,34} and theorists \cite{35–44}. There is an intrinsic pinning of vortices at the JJA cells, and vortices can move only if the driving supercurrent is more than the critical value. But when they start to move, in many cases a good approximation is to replace the lattice by a continuous superconducting film. However, the hydrodynamic derivation of the Magnus force is not valid since it assumes the momentum conservation law and the translational invariance. In the present paper it will be shown that the Hall effect is exactly absent in the classical theory of JJA which neglects the charge quantization. Since the Hall effect is linear in the amplitude of the effective Magnus force, the latter also vanishes in the classical JJA. This statement directly follows from the symmetry of the dynamic equations. At the same time the superfluid density is finite in the continuum limit of JJA and therefore the superfluid Magnus force doesn’t vanish. Therefore the theory based on the Berry phase approach \cite{23} predicted a finite effective Magnus force and the Hall effect for JJA. Our result might be interpreted as that the superfluid Magnus force is compensated by some force external for the liquid, like the Kopnin-Kravtsov force in a dirty superconductor. But as mentioned above, JJA is a system with a strong violation of translational invariance, for which this interpretation is purely formal. Only the resultant effective Magnus force has a physical meaning.

We start from Sec. II which shows how the Magnus force appears in the phenomenological theory of neutral and charged superfluids. The force terminology is also introduced explaining to which term and in which equation any force under discussion corresponds. Section III is devoted to the transverse force between quasiparticles and the vortex (the Iordanskii force) and its connection with the Aharonov-Bohm effect. Section III A reminds connection between the nonlinear Schrödinger equation for the condensate (the Gross-Pitaevskii theory) and superfluid hydrodynamics and phonons. Scattering of the sound wave (phonon) by the vortex in hydrodynamics is analyzed in Sec. III B using the Born approximation and the effective cross-sections. It is shown that the standard scattering-theory approach fails to reveal the transverse Iordanskii force because of the divergence of the scattering amplitude at small angles of scattering. The analysis of the small-angle scattering is presented in Sec. III C. It reveals the interference between quasiparticles with trajectories on the left and on the right from the vortex. In Sec. III D the same results is rederived using the
II. WHERE AND HOW THE MAGNUS FORCE APPEARS

A. The Magnus force in classical hydrodynamics

As mentioned in Introduction, the Ao-Thouless theory connects the Magnus force with the concept of the geometrical phase. But for better understanding of the Magnus force origin it is worth to remind how the Magnus force arises in classical hydrodynamics.

Let us consider an isolated straight vortex line in an incompressible inviscid liquid. The line along the axis $z$ induces the velocity field

$$\vec{v}_v(\vec{r}) = \frac{\vec{\kappa} \times \vec{r}}{2\pi r^2}.$$  \hspace{1cm} (1)

Here $\vec{r}$ is the position vector in the plane $xy$, and $\vec{\kappa}$ is the circulation vector directed along the axis $z$. The circulation, given by

$$\kappa = \oint \vec{v} \cdot d\vec{l},$$  \hspace{1cm} (2)

may have arbitrary values in classical hydrodynamics. In addition, there is a fluid current past the vortex line with a velocity $\vec{v}_0$. Then the net velocity field around the line is

$$\vec{v}(\vec{r}) = \vec{v}_v(\vec{r}) + \vec{v}_0.$$  \hspace{1cm} (3)

The Euler equation for the liquid is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P + \frac{\vec{F}}{\rho} \delta_2(\vec{r}).$$  \hspace{1cm} (4)

Here $\rho$ is the liquid density and $P$ is the pressure. This equation suggests that an external $\delta$-function force $\vec{F}$ is applied to the liquid at the vortex line.

Assuming that the vortex line moves with the constant velocity $\vec{v}_L$, one obtains that

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v}_L \cdot \vec{\nabla}) \vec{v}. $$  \hspace{1cm} (5)

Then the Euler equation (4) yields the Bernoulli law for the pressure:

$$P = P_0 - \frac{1}{2} \rho [\vec{v}(\vec{r}) - \vec{v}_L]_2^2 = P'_0 - \frac{1}{2} \rho \vec{v}_v(\vec{r})^2 - \rho \vec{v}_v(\vec{r}) \cdot (\vec{v}_0 - \vec{v}_L).$$  \hspace{1cm} (6)

Here $P_0$ and $P'_0 = P_0 - \frac{1}{2} \rho (\vec{v}_0 - \vec{v}_L)^2$ are constants which are of no importance for the following derivation.

Next one should consider the momentum balance for a cylindrical region of a radius $r_0$ around the vortex line. The momentum-flux tensor is given by

$$\Pi_{ij} = P \delta_{ij} + \rho v_i(\vec{r}) v_j(\vec{r}),$$  \hspace{1cm} (7)

or in the reference frame moving with the vortex velocity $\vec{v}_L$:

$$\Pi'_{ij} = P \delta_{ij} + \rho (v_i - v_{Li})(v_j - v_{Lj}).$$  \hspace{1cm} (8)
The momentum conservation law requires that the external force $\vec{F}$ on the vortex line must be equal to the momentum flux through the entire cylindrical boundary in the reference frame moving with the vortex velocity $\vec{v}_L$. The latter is given by the integral $\int dS_j \Pi_i^j$, where $dS_j$ are the components of the vector $d\vec{S}$ directed along the outer normal to the boundary of the cylindrical region and equal to the elementary area of the boundary in magnitude. Then using Eqs. (1), (6), and (8), the momentum balance yields the following relation:

$$\rho\left((\vec{v}_L - \vec{v}_0) \times \vec{r}\right) = \vec{F}. \quad (9)$$

On the left-hand side of this equation one can see the Magnus force as it comes in the classical hydrodynamics. A half of this force is due to the Bernoulli contribution to the pressure, Eq. (6); another half is due to the convection term $\propto v_i v_j$ in the momentum flux. The Magnus force balances the resultant of all external forces applied to the liquid at the vortex line (the force $\vec{F}$). In the absence of external forces the vortex moves with the velocity of the liquid: $\vec{v}_L = \vec{v}_0$ (the Helmholtz theorem).

This derivation demonstrates the classical origin of the Magnus force: quantization of circulation is not necessary for its existence. During the derivation we referred to the hydrodynamic equations only at large distances from the vortex line. It might seem as if the fluid in the vortex core did not matter at all. However, the derivation is based on the assumption that the momentum is a well-defined conserved quantity everywhere even inside of the vortex core where the hydrodynamic theory does not hold.

**B. The superfluid Magnus force**

In the superfluid hydrodynamics one can refer this derivation to the superfluid component with density $\rho_s$. The Euler equation for the superfluid component [10] after adding the external $\delta$-function force $\vec{F}$ applied at the vortex line is

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla)\vec{v}_s = -\nabla \mu + \frac{\vec{F}}{\rho_s} \delta_2(\vec{r}), \quad (10)$$

where $\mu$ is the chemical potential.

For charged superfluids (superconductors) the Euler equation should include also the electromagnetic forces. In particular, the chemical potential must be replaced by the electrochemical potential. But except for vortex cores one may use the quasineutrality condition that the total electron charge is approximately equal to the background ion charge. This allows to neglect the chemical potential gradient. Finally the Euler equation may be written as

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla)\vec{v}_s = \frac{e}{m} \left(\vec{E} + \frac{1}{c} [\vec{v}_s \times \vec{H}]\right) + \frac{\vec{F}}{\rho_s} \delta_2(\vec{r}), \quad (11)$$

where $\vec{E}$ and $\vec{H}$ are the electric and the magnetic fields.

Let us consider a vortex line in a neutral or charged superfluids with the velocity field Eq. (1). Now circulation is quantized, and the circulation quantum is $\kappa = h/m$ in a Bose superfluid and $\kappa = h/2m$ in a Fermi superfluid. One can repeat the analysis of the momentum balance for a cylindrical region around the vortex line. Now the velocity $\vec{v}_0$ is the superfluid velocity $\vec{v}_s$ far from the vortex line and the Bernoulli law is used for variation of the chemical potential or the electric potential in the neutral and the charged superfluids respectively. But the momentum of the superfluid component is not conserved because of interactions with quasiparticles (and impurities in the case of superconductors) in the vicinity of the vortex line. One may assume that all these interactions are incorporated by the external force $\vec{F}$, localized at the vortex line. Then one has instead of Eq. (1):

$$\rho_s\left((\vec{v}_L - \vec{v}_s) \times \vec{r}\right) = \vec{F}. \quad (12)$$

The force $\vec{F}$, which enters the theory as a $\delta$-function force, is distributed over a small vicinity of the vortex line in reality. The dimension of this vicinity is not necessary to be of the vortex-core size, but must be smaller than all relevant hydrodynamic scales (e.g., the intervortex distance, or the vortex line curvature radius).

Replacing the external force by the Magnus force, the Euler equations for the neutral and the charged superfluids are

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla)\vec{v}_s = -\nabla \mu + \left[(\vec{v}_L - \vec{v}_s) \times \vec{r}\right] \delta_2(\vec{r}), \quad (13)$$
\[
\frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_s \cdot \nabla) \vec{v}_s = \frac{e}{m} \left( \vec{E} + \frac{1}{c} [\vec{v}_s \times \vec{B}] \right) + \left[ (\vec{v}_L - \vec{v}_n) \times \kappa \right] \delta_2(\vec{r}) . \tag{14}
\]

One can transform the Euler equation using the vector identity
\[
(\vec{v}_s \cdot \nabla) \vec{v}_s = \nabla \frac{\vec{v}_s^2}{2} - \vec{v}_s \times [\nabla \times \vec{v}_s] . \tag{15}
\]
In a neutral superfluid vorticity is concentrated on the vortex line: \[\nabla \times \vec{v}_s = \kappa \delta(\vec{r})\]. But in a superconductor \[\nabla \times \vec{v}_s = \kappa \delta(\vec{r}) - \frac{e}{mc} \vec{H} \].

Then the Euler equations are
\[
\frac{\partial \vec{v}_L}{\partial t} = -\nabla \left( \mu + \frac{\vec{v}_s^2}{2} \right) + [\vec{v}_L \times \kappa] \delta_2(\vec{r}) , \tag{16}
\]
\[
\frac{\partial \vec{v}_n}{\partial t} = \frac{e}{m} \vec{E} - \nabla \left( \frac{\vec{v}_s^2}{2} \right) + [\vec{v}_L \times \kappa] \delta_2(\vec{r}) . \tag{17}
\]

This analysis demonstrates that the total external force on the superfluid in the vicinity of the vortex line is exactly balanced by the superfluid Magnus force \(\rho_s[(\vec{v}_L - \vec{v}_n) \times \kappa]\). In fact, the term \(\times \vec{v}_L\) in the Euler equation may be received from a pure kinematics: it presents the flow of the vortex lines across the line between two points which changes the phase difference between them (the phase slip). After replacing the external force by the Magnus force in the Euler equation, the latter does not contain any information on the nature and the magnitude of the external force. But the Euler equation is not sufficient for description of superfluid motion: an additional equation for the vortex velocity \(\vec{v}_L\) is necessary. In order to derive it, one should specify the force \(\vec{F}\). This may be done by considering the momentum balance of the whole liquid, but not only its superfluid component.

C. Equation of vortex motion and effective Magnus force

The momentum balance for the whole liquid in a cylindrical region around the vortex line has been studied in Ref. [6] using the collisionless kinetic equation for quasiparticles and assuming translational invariance. The balance yielded the equation of vortex motion which is a linear relation imposed on three velocities \(\vec{v}_s, \vec{v}_n, \) and \(\vec{v}_L\). Since translational invariance was assumed, this equation depended only on velocity differences, but not their absolute values. However, one can also include into this balance some interactions with external fields, e.g., with impurities in superconductors. Assuming the axial symmetry in the plane normal to the vortex line, the momentum balance yields the following equation of vortex motion:
\[
\rho_s[(\vec{v}_L - \vec{v}_n) \times \kappa] = -D(\vec{v}_L - \vec{v}_n) - D'(\hat{z} \times (\vec{v}_L - \vec{v}_n)) - d\vec{v}_L - d'[\hat{z} \times \vec{v}_L] . \tag{18}
\]
Comparing it with Eq. (12), one obtains the expression for \(\vec{F}\):
\[
\vec{F} = -D(\vec{v}_L - \vec{v}_n) - D'[\hat{z} \times (\vec{v}_L - \vec{v}_n)] - d\vec{v}_L - d'[\hat{z} \times \vec{v}_L] . \tag{19}
\]
The forces proportional to \(D\) and \(D'\) are due to scattering of free quasiparticles by the vortex, therefore they are proportional to the difference between the drift velocity of quasiparticles (the normal velocity \(\vec{v}_n\)) and the vortex velocity \(\vec{v}_L\). The forces proportional to \(d\) and \(d'\) are due to interaction between the vortex line and impurities which are frozen into the crystal and therefore do not move if the crystal is at rest (The case when the crystal is not at rest is discussed in Ref. [24]). These forces include also the interaction of impurities with the quasiparticles bound in the vortex core, and therefore moving with \(\vec{v}_L\), but not with \(\vec{v}_n\).

One can rewrite the equation (13) of vortex motion collecting together the terms proportional to the velocity \(\vec{v}_L\):
\[
\rho_M[\vec{v}_L \times \kappa] + \eta \vec{v}_L = \rho_s[\vec{v}_s \times \kappa] + D\vec{v}_n + D'[\hat{z} \times \vec{v}_n] . \tag{20}
\]
Here \(\rho_M = \rho_s - (D' + d')/\kappa\) and \(\eta = D + d\). The forces on the right-hand side are driving forces produced by the superfluid and normal flows. In the theory of superconductivity the force \(\vec{F}_L = \rho_s[\vec{v}_s \times \kappa] = (1/c)[\vec{j}_s \times \Phi_0]\), proportional to the superfluid velocity \(\vec{v}_s\) (or to the supercurrent \(j_s = en_s\vec{v}_s\)), is called the Lorentz force. Here \(\Phi_0 = hc/2e\) is the magnetic-flux quantum and the vector \(\Phi_0\) is parallel to \(\kappa\). There are also forces on the vortex produced by the normal
current \( \tilde{j}_n = e n_s \tilde{v}_n \). One can find discussion of the effect of the normal-current force on electrodynamics of a type II superconductors in Ref. [46].

The left-hand side of Eq. (20) presents the response of the vortex to these driving forces. The factor \( \rho_M \), which determines the amplitude of the effective Magnus force on the vortex is not equal to the superfluid density \( \rho_s \) in general: it may be more or less than \( \rho_s \). Note that the Hall conductivity is governed by \( \rho_M \), but not by \( \rho_s \). At low magnetic fields the normal current is small compared to the supercurrent, i.e. the total current \( \tilde{j} \approx \tilde{j}_n = e n_s \tilde{v}_s \) and one may neglects the terms \( \times \tilde{v}_n \) on the right-hand side of Eq. (20). On the other hand, the electric field is connected with the vortex velocity by the Josephson relation \( \tilde{E} = \frac{1}{2} \tilde{H} \times \tilde{v}_L \). Then the equation of vortex motion is equivalent to the Ohm law connecting the current and the electric field. One can easily check that the Hall component of the conductivity is linear in \( \rho_M \).

In the superfluidity theory they usually present the equation of vortex motion using the mutual friction parameters \( B \) and \( B' \) introduced by Hall and Vinen [3]. Because of translational invariance \( d = d' = 0 \) for superfluids, and neglecting the normal motion \( \tilde{v}_n = 0 \) the equation is

\[
\tilde{v}_L = \left( 1 - \frac{\rho_n}{2\rho} B' \right) \tilde{v}_s + \frac{\rho_n}{2\rho} B \tilde{z} \times \tilde{v}_s = \frac{\rho_s \rho_M \kappa^2}{D^2 + (\rho_M \kappa)^2} \tilde{v}_s + \frac{\rho_s \kappa D}{D^2 + (\rho_M \kappa)^2} [\tilde{z} \times \tilde{v}_s].
\]

(21)

In the next section we shall calculate the amplitude \( D' \) of the Iordanskii force analyzing the interaction of the vortex with phonons in the Born approximation.

### III. IORDANSKII FORCE AND AHARONOV-BOHM EFFECT

#### A. Nonlinear Schrödinger equation and superfluid hydrodynamics

The Gross-Pitaevskii theory [24] has suggested the nonlinear Schrödinger equation to describe a weakly nonideal Bose gas:

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V|\psi|^2 \psi.
\]

(22)

Here \( \psi = a \exp(i\phi) \) is the condensate wave function and \( V \) is the amplitude of two-particle interaction. Using the Madelung transformation [17], this equation for a complex function may be transformed into two real equations for the liquid density \( \rho = ma^2 \) and the liquid velocity \( \tilde{v} = \frac{\hbar}{m} \nabla \phi \) where \( \kappa = \hbar/m \) is the circulation quantum. Far from the vortex line these equations are hydrodynamic equations for an ideal inviscid liquid:

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \tilde{v}) = 0,
\]

(23)

\[
\frac{\partial \tilde{v}}{\partial t} + (\tilde{v} \cdot \nabla) \tilde{v} = -\nabla \mu.
\]

(24)

Here \( \mu = V a^2/m \) is the chemical potential and \( c \) is the sound velocity.

Suppose that a plane sound wave propagates in the liquid generating the phase variation \( \phi(\tilde{r}, t) = \phi_0 \exp(i\tilde{k} \cdot \tilde{r} - i\omega t) \). Then the liquid density and velocity are functions of the time \( t \) and the position vector \( \tilde{r} \) in the plane \( xy \) and can be written in the form

\[
\rho(\tilde{r}, t) = \rho_0 + \rho_1(\tilde{r}, t), \quad \tilde{v}(\tilde{r}, t) = \tilde{v}_0 + \tilde{v}_1(\tilde{r}, t),
\]

(25)

where \( \rho_0 \) and \( \tilde{v}_0 \) are the constant density and velocity in the liquid without the sound wave, \( \rho_1(\tilde{r}, t) \) and \( \tilde{v}_1(\tilde{r}, t) = \frac{\hbar}{m} \nabla \phi \) are the changes in the density and the velocity due to the sound wave. They should be determined from the hydrodynamic equations (24) and (24) after their linearization. In particular, Eq. (24) gives the relation between the density variation and the phase \( \phi \):

\[
\rho_1 = \rho_0 \frac{\rho_0}{c^2} \mu(1) = \frac{\rho_0}{c^2} \frac{\kappa}{2\pi} \left\{ \frac{\partial \phi}{\partial t} + \tilde{v}_0 \cdot \nabla \phi(\tilde{r}) \right\}.
\]

(26)

Substitution of this expression into Eq. (23) yields the wave equation for a moving liquid. The sound wave has the spectrum \( \omega = ck + \tilde{k} \cdot \tilde{v}_0 \). The sound propagation is accompanied with the transport of mass. In the reference frame
moving with the average velocity $\bar{v}_0$ of the liquid the mass transport is determined by the mass current $\bar{j}^{ph}$ which is of the second order with respect to the wave amplitude. Averaging over the wave period, one obtains:

$$\bar{j}^{ph}(\bar{k}) = \langle \rho_0(\bar{v}_{(1)}) \rangle = \rho_0 \phi_0^2 \frac{k^2 \bar{k}}{8\pi^2 c} n(\bar{p})\bar{p}.$$  \hfill (27)

This expression supposes that the plane sound wave corresponds to a number $n(\bar{p})$ of phonons with the momentum $\bar{p} = \hbar \bar{k}$ and the energy $E = \varepsilon(\bar{p}) + \bar{p} \cdot \bar{v}_0$ where $\varepsilon(\bar{p}) = cp$ is the energy in the reference frame moving with the liquid velocity $\bar{v}_0$. Then the total mass flow is

$$j = \rho_0 \bar{v}_0 + \frac{1}{\hbar^2} \int d_3 \bar{p} n(\bar{p})\bar{p}.$$  \hfill (28)

In the thermal equilibrium at $T > 0$, the phonon numbers are given by the Planck distribution $n(\bar{p}) = n_0(E, \bar{v}_n)$ with the drift velocity $\bar{v}_n$ of quasiparticles:

$$n_0(E, \bar{v}_n) = \frac{1}{\exp \frac{E_0(\bar{p}) - \bar{p} \cdot \bar{v}_0}{kT} - 1} = \frac{1}{\exp \frac{\varepsilon(\bar{p}) + \bar{p} \cdot (\bar{v}_0 - \bar{v}_n)}{kT} - 1}. \hfill (29)$$

Linearizing Eq. (23) with respect to the relative velocity $\bar{v}_0 - \bar{v}_n$ one can see that this expression is equivalent to the two-fluid expression $j = \rho_0 \bar{v}_s + \rho_n(\bar{v}_n - \bar{v}_s) = \rho_s \bar{v}_s + \rho_n \bar{v}_n$ assuming that $p = \rho_0 = \rho_s + \rho_n$, $\bar{v}_0 = \bar{v}_s$, and the normal density is given by the usual two-fluid expression:

$$\rho_n = -\frac{1}{3\hbar^2} \int \frac{\partial n_0(\varepsilon, 0)}{\partial E} p^2 d_3 \bar{p}.$$ \hfill (30)

This simple analysis demonstrates that two-fluid hydrodynamics with phonon quasiparticles is identical to the nonlinear Schrödinger equation with thermally excited sound waves. A next step is to analyze scattering of phonons by the vortex in hydrodynamics of an ideal liquid.

**B. Scattering of phonons by the vortex in hydrodynamics**

The phonon scattering by a vortex line was studied beginning from the works by Pitaevskii and Fetter. Interaction is weak and one may use the perturbation theory (the Born approximation).

Let us consider a sound wave $\phi(\vec{r}, t) = \phi_0 \exp(i\vec{k} \cdot \vec{r} - i\omega t)$ propagating in the plane $xy$ normal to a vortex line along the axis $z$. Then in linearized hydrodynamic equations of the previous section the fluid velocity $\vec{v}_0$ should be replaced by the velocity $\vec{v}_n(\vec{r}, t)$ around the vortex line which now depends not only on the position vector $\vec{r}$, but also on time, since the vortex line is not at rest when the sound wave propagates. Thus it means that $\vec{\nabla} \mu$ in Eq. (1) must be replaced by $\vec{\nabla} - \vec{\nabla} L \partial t$ and $\partial \vec{v}_n / \partial t = -(\vec{v}_L \cdot \vec{\nabla}) \bar{v}_n - \vec{\nabla}(\vec{v}_L \cdot \bar{v}_n)$. Since there is no external force on the liquid, the vortex moves with the velocity in the sound field: $\vec{v}_L = \vec{v}_n(0, t)$. In the presence of the vortex line linearized hydrodynamic equations are:

$$\frac{\partial \rho_0(1)}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_n(1) + \bar{v}_n \cdot \vec{\nabla} \rho_0(1) = 0,$$  \hfill (31)

$$\frac{\partial \vec{v}_n(1)}{\partial t} = -\vec{\nabla} \mu - \vec{\nabla}(\vec{v}_L \cdot \vec{v}_n) - \vec{\nabla} \mu.$$  \hfill (32)

Since $\vec{\nabla} \mu = \frac{\varepsilon^2}{\rho_0} \vec{\nabla} \rho_0$, Eq. (12) yields:

$$\rho_0(1) = -\frac{\rho_0}{c^2} \frac{\kappa}{2\pi} \left\{ \frac{\partial \phi}{\partial t} + \bar{v}_n \cdot [\vec{\nabla} \phi(\vec{r}) - \vec{\nabla} \phi(0)] \right\}. \hfill (33)$$

Finally the linear equation for the phonon-induced phase is

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \vec{\nabla}^2 \phi = -2 \bar{v}_n(\vec{r}) \cdot \vec{\nabla} \frac{\partial}{\partial t} \left[ \phi(\vec{r}) - \frac{1}{2} \phi(0) \right]. \hfill (34)$$
In the Born approximation one treats interaction with the vortex velocity field [the right-hand side of Eq. (34)] as a perturbation. Then

$$\phi = \phi_0 \exp(-i\omega t) \left\{ \exp(ik \cdot \vec{r}) + \frac{ik}{4c} \int d_2 \vec{r}_1 H_0^{(1)}(k|\vec{r} - \vec{r}_1|)\vec{k} \cdot \vec{v}(\vec{r}_1)|2\exp(ik \cdot \vec{r}_1) - 1 \right\} \quad (35)$$

Here $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind and $\frac{4}{\pi} H_0^{(1)}(k|\vec{r} - \vec{r}_1|)$ is the Green function for the 2D wave equation. i.e. satisfies to the equation

$$(k^2 - \vec{\nabla}^2)\phi(\vec{r}) = \delta_2(\vec{r} - \vec{r}_1). \quad (36)$$

In our problem the small perturbation parameter is $\kappa k/c$ which is on the order of the ratio of the wavelength $2\pi/k$ to the vortex core radius $r_c \sim \kappa/c$.

The standard procedure in the scattering theory is the following. One uses the asymptotic expression for the Hankel function at large values of the argument:

$$\lim_{z \to \infty} H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i(z - \pi/4)}. \quad (37)$$

Then it is assumed that the perturbation is confined to a finite vicinity of the line, where $r_1 \ll r$, and

$$|\vec{r} - \vec{r}_1| \approx r - \frac{(\vec{r}_1 \cdot \vec{r})}{r}. \quad (38)$$

Finally the phase field at large values of $kr$ may be presented as a superposition of the plane wave $\propto \exp(ik \cdot \vec{r})$ and the scattered wave $\propto \exp(ikr)$:

$$\phi = \phi_0 \exp(-i\omega t) \left[ \exp(ik \cdot \vec{r}) + \frac{ia(\varphi)}{\sqrt{r}} \exp(ikr) \right]. \quad (39)$$

Here $a(\varphi)$ is the scattering amplitude which is a function of the angle $\varphi$ between the initial wave vector $\vec{k}$ and the wave vector $\vec{k}' = k\vec{r}/r$ after scattering. For scattering of phonons by the vortex the Born approximation yields that

$$a(\varphi) = -\sqrt{\frac{k}{2\pi c}}e^{i\varphi}[\vec{k} \times \vec{k}'] \cdot \vec{k} \left( \frac{1}{q^2} - \frac{1}{2k^2} \right) = \frac{1}{2} \sqrt{\frac{k}{2\pi c}}e^{i\varphi} \sin \varphi \cos \varphi, \quad (40)$$

where $q = \vec{q} - \vec{k}'$ is the momentum transferred by the scattered phonon to the vortex, and $q = 2k\sin(\varphi/2)$. The second therm $1/2k^2$ in parenthesis is due to the vortex line motion.

In order to find a force on the vortex from the sound wave, one must determine a contribution of the wave to the average total momentum flux $F_i^{ph} = \int dS_j \Pi_i^{ph}$ over the cylindrical surface around the vortex line where

$$\Pi_i^{ph} = \langle P_{(2)} \rangle \delta_{ij} + \langle (\rho_{(1)}(v_{(1)})_i)v_{v_j} + (\rho_{(1)}(v_{(1)})_j)v_{v_i} + \rho_0((v_{(1)})_i(v_{(1)})_j) \rangle \quad (41)$$

Here $P_{(2)} = \rho_0 \mu_{(2)} + \frac{\partial \rho_0}{\partial \mu} \frac{\rho_0^2}{2}$ is the second-order phonon contribution to the pressure where $\frac{\partial \mu}{\partial \rho_0} = \rho_0/c^2$. According to the Euler equation (4) the second-order contribution to the chemical potential is $\mu_{(2)} = -\frac{\nu^2_{(1)}}{2}$. Then

$$\langle P_{(2)} \rangle = \frac{\rho_0^2 \langle \rho_{(1)}^2 \rangle}{\rho_0} - \frac{\rho_0 \langle \nu_{(1)}^2 \rangle}{2}. \quad (42)$$

In Appendix A it is shown that if the perturbation of the sound wave is confined to a finite vicinity of the line, the force on the line from the sound wave,

$$\vec{F}^{ph} = \sigma || \vec{c}^{ph} - \sigma_\perp c [\hat{z} \times \vec{j}^{ph}] \quad (43)$$

is determined by two effective cross-sections: the transport cross-section for the dissipative force component,

$$\sigma_\parallel = \int \sigma(\varphi)(1 - \cos \varphi)d\varphi, \quad (44)$$
and the transverse cross-section for the transverse force component,

$$\sigma = \int \sigma(\varphi) \sin \varphi d\varphi.$$  \hfill (45)

The differential cross-section $$\sigma(\varphi) = |a(\varphi)|^2$$ in these expressions is known due to Eq. (40) for the scattering amplitude $$a(\varphi)$$. It is quite natural that in the Born approximation the transverse cross-section vanishes since the differential cross-section is quadratic in the circulation $$\kappa$$.

However, the standard scattering-theory approach fails to describe the phonon scattering at small angles $$\varphi$$. Indeed, the velocity $$v_\varphi$$ induced around the vortex is decreasing very slowly, as $$1/r$$. Therefore the terms $$\propto v_\varphi$$ in the phonon momentum flux, Eq. (41), are important in the momentum balance. In addition, the scattering amplitude is divergent at $$\varphi \to 0$$:

$$\lim_{\varphi \to 0} a(\varphi) = \sqrt{\frac{k^2}{2\pi c}} \frac{1}{\varphi}.$$  \hfill (46)

This divergence is integrable in the integral for the transport cross-section, Eq. (44). So the calculation of the transport cross-section is reliable. Contrary to it, the integrand in Eq. (45) for the transverse cross-section has a pole at $$\varphi = 0$$, and the contribution of this pole requires an additional analysis. A proper analysis of the phonon small-angle scattering was fulfilled in Refs. [5,6]. However, in the recent publication Demircan, Ao, and Niu [32] considered the phonon scattering by the vortex ignoring special features of the small-angle scattering. This is the reason why they could not find the transverse force from phonons on the vortex.

C. Small-angle phonon scattering in the Born approximation and the Iordanskii force

At small scattering angles $$\varphi \lesssim 1/\sqrt{kr}$$ the asymptotic expansion given by Eq. (39) does not hold. The accurate calculation of the integral in Eq. (35) for small angles was done in Ref. [6]. A simplified version of this calculation is presented in Appendix B. It yields that at $$\varphi \ll 1$$

$$\varphi = \varphi_0 \exp(-i\omega t) \left[ 1 + \frac{i\kappa k}{2c} \sqrt{\frac{kr}{2\varphi}} \right].$$  \hfill (47)

Using an asymptotic expression for the error integral

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad \xrightarrow{z \to \infty} \frac{z}{|z|} + \sqrt{\frac{2}{\pi z}} \exp(-z^2)$$  \hfill (48)

at large $$z$$, one obtains for angles $$1 \gg \varphi \gg 1/\sqrt{kr}$$:

$$\varphi = \varphi_0 \exp(-i\omega t) \left[ \exp(i\kappa \cdot \vec{r}) \left( 1 + \frac{i\kappa k}{2c} \frac{\varphi}{|\varphi|} \right) + \frac{i\kappa}{c} \sqrt{\frac{k}{2\pi r \varphi}} \exp\left(ikr + i\frac{\pi}{4}\right) \right].$$  \hfill (49)

The second term in brackets coincides with scattering wave at small angles $$\varphi \ll 1$$ when the scattering amplitude is given by Eq. (46). But now one can see that the standard scattering theory misses to reveal a very important non-analytical correction to the incidental plane wave. We shall see in Sec. II E that the factor $$\pm \kappa k/c$$ which determines this correction is exactly the phase shift of the sound wave along the quasiclassical trajectories past the vortex on the right and left sides. This is a manifestation of the Aharonov-Bohm effect [7]: the sound wave after its interaction with the vortex velocity field has different phases on the left and on the right of the vortex line, and this phase difference results in an interference.

In the interference region the velocity induced by the sound wave is obtained by taking the gradient of the phase given by Eq. (17). The velocity component normal to the wave vector $$\vec{k}$$,

$$v_{(1)\perp} = \frac{\kappa}{2\pi r \varphi} = \varphi_0 \exp(-i\omega t + ikr) \frac{k^2 k}{2\pi c} \sqrt{\frac{k}{2\pi r}},$$  \hfill (50)
\[
\int dS_j \rho_0 (\langle v_1 \rangle \perp \langle v_1 \rangle_j) = \int \rho_0 (\langle v_1 \rangle \perp \langle v_1 \rangle_j) r d\varphi = \frac{1}{8\pi} \sqrt{\frac{\rho_0 \phi_0^2 k^2}{c}} \sqrt{k r} \int d\varphi \cos \left(\frac{1}{2} k r \varphi^2 \right) = \frac{1}{8\pi} \rho_0 \phi_0^2 \frac{k^3 k^2}{c}. \tag{51}
\]

Note, that this force contribution arises from the interference region with the transverse dimension \(d_{\text{int}} \sim \sqrt{r_0/k}\). Here \(r_0\) is the large distance from the vortex line where the momentum balance is considered. But the interference region corresponds to very small scattering angles \(\sim d_{\text{int}}/r_0 = 1/\sqrt{kr_0}\). Thus an infinitesimally small angle interval yields a finite contribution to the transverse force. One couldn’t reveal such a contribution from the standard scattering theory using the differential cross-section.

Exactly the same contribution to the transverse force, as in Eq. (51), arises from the term \(v_\nu \langle \rho_1 \rangle (\langle v_1 \rangle_j)\) in the momentum-flux tensor, Eq. (41). In this term the mass flow \(\langle \rho_1 \rangle (\langle v_1 \rangle_j)\) for the plane wave may be used [see Eq. (27)], since we take into account only terms which are of the first-order in the Born parameter \(\kappa k/c\). Finally this yields the transverse force given by the transverse cross-section \(\sigma_\perp = \kappa/c\) which is linear in the circulation quantum \(\kappa\) and therefore cannot be obtained from the differential cross-section quadratic in \(\kappa\). This is the Iordanskii force. We shall rederive it in the next section using expansion in partial waves in order to demonstrate the analogy with the Aharonov-Bohm effect.

D. Partial-waves analysis and the Aharonov-Bohm effect

Neglecting the terms of the second order in \(v_\nu\), the equation

\[
k^2 \phi - \left(-i \vec{\nabla} + \frac{k}{c} \vec{v}_\nu\right)^2 \phi = 0 \tag{52}
\]

is equivalent to Eq. (24) written for the harmonic sound wave with frequency \(\omega = ck\). The contribution from the vortex-line motion \(\vec{v}_\nu\) on the right-hand side of Eq. (54) is now neglected as unimportant for the transverse force. Equation (52) is analogous to an equation which describes interaction of an electron with the electromagnetic vector is connected with the magnetic flux \(\Phi\) confined to a thin tube (the Aharonov-Bohm effect [4]):

\[
E \psi(\vec{r}) = \frac{1}{2m} \left(-\hbar \vec{\nabla} - \frac{e}{c} \vec{A}\right)^2 \psi(\vec{r}). \tag{53}
\]

Here \(\psi\) is the electron wave function with energy \(E\) and the electromagnetic vector is connected with the magnetic flux by the relation similar to that for the velocity \(\vec{v}_\nu\) around the vortex line [Eq. (14)]:

\[
\vec{A} = \frac{\Phi}{2\pi} \frac{\vec{z} \times \vec{r}}{2\pi r^2}. \tag{54}
\]

Let us look for a solution of this equation as a superposition of the partial cylindrical waves using the cylindrical system of coordinates \((r, \varphi)\):

\[
\psi = \sum_l \psi_l(r) \exp(il\varphi). \tag{55}
\]

The partial-wave amplitudes should satisfy equations

\[
\frac{d^2 \psi_l}{dr^2} + \frac{1}{r} \frac{d\psi_l}{dr} - \frac{(l-\gamma)^2}{r^2} \psi_l + k^2 \psi_l = 0. \tag{56}
\]

Here \(k\) is the wave number of the electron far from the vortex so that \(E = \hbar^2 k^2/2m\) and \(\gamma = \Phi/\Phi_1\) where \(\Phi_1 = \hbar c/e\) is the magnetic-flux quantum for one electron (two times larger than the magnetic-flux quantum \(\Phi_0 = \hbar c/2e\) for a Cooper pair). A solution of this equation is the Bessel function \(J_{l-\gamma}(kr)\) with the following asymptotics at large arguments:

\[
J_{l-\gamma}(kr) \rightarrow \sqrt{\frac{2}{\pi kr}} \cos(kr - \frac{\pi}{2} |l - \gamma| - \frac{\pi}{4}). \tag{57}
\]

On the other hand, the expansion of the plane wave in the partial cylindrical waves is

\[
\exp(i \vec{k} \cdot \vec{r}) = \exp(i k r \cos \varphi) = \sum_l J_l(kr) \exp[i(l(\varphi + \pi/2))], \tag{58}
\]
or at large \( kr \):

\[
\exp(\mathbf{i}\mathbf{k} \cdot \mathbf{r}) = \sqrt{\frac{2}{\pi kr}} \sum_{l} \cos \left( kr - \frac{\pi}{2}l - \frac{\pi}{4} \right) \exp[il(\varphi + \pi/2)] .
\] (59)

In order to obtain the solution “the incoming plane wave + the scattered cylindrical wave” like Eq. (39), one should determine the partial waves \( \psi_l \) from the condition that the incoming components \( \propto \exp(-ikr) \) in the plane wave and in the solution Eq. (55) coincide. This yields that

\[
\psi_l = \sqrt{\frac{2}{\pi kr}} \exp \left[ \mathbf{i} \frac{\pi}{2} (l - |l - \gamma|) \right] \cos(kr - \frac{\pi}{2}|l - \gamma| - \frac{\pi}{4}) .
\] (60)

Then the solution of the Aharonov-Bohm problem takes the form of Eq. (39) with the scattering amplitude

\[
a(\varphi) = \sqrt{\frac{1}{2\pi k}} \exp \left( \mathbf{i} \frac{\pi}{4} \right) \sum_{l} [1 - \exp(2\mathbf{i}\delta_l)] \exp(\mathbf{i}l\varphi) ,
\] (61)

where

\[
\delta_l = (l - |l - \gamma|)\pi/2
\] (62)
is the partial-wave phase shift.

Equation (45) for the transverse cross-section may be rewritten as an expansion in partial waves [51]:

\[
\sigma_\perp = \int |a(\varphi)|^2 \sin \varphi d\varphi = \frac{1}{k} \sum_{l} \sin 2(\delta_l - \delta_{l+1}) .
\] (63)

Using the phase shift values for the Aharonov-Bohm effect, Eq. (62), the transverse cross-section is

\[
\sigma_\perp = -\frac{1}{k} \sin 2\pi\gamma .
\] (64)

One can obtain the cross-section for phonon scattering from this expression assuming that \( \gamma = -\kappa k/2\pi c \) and expanding the sine-function in small \( \gamma \).

The cross-section for the Aharonov-Bohm effect is periodic in the magnetic flux with the period equal to the one-electron flux quantum \( \Phi_1 \). If the electron is scattered by the Cooper-pair flux quantum \( \Phi_0 = \Phi_1/2 \) the transverse cross-section vanishes. But presented analysis of the Aharonov-Bohm effect is based on the assumption that the total magnetic flux is concentrated in a very thin tube. Namely, the radius of the tube must be much less than the electron wavelength. This condition certainly doesn’t hold in superconductors, where the electron wavelength is on the order of the interatomic distance and the effective radius of the magnetic-flux tube is the London penetration depth. Therefore one should consider scattering of electrons by a thick magnetic-flux tube [51]. Scattering of the BCS quasiparticles by the magnetic field of the vortex is contributes to the transverse force on the vortex, but this contribution cancels with the bulk electromagnetic force on the superfluid electrons [9].

In their original paper [4] Aharonov and Bohm considered only the effect of the magnetic-flux tube on the electron wave. The force from the electrons on the fluxon was considered later by Aharonov and Casher [52] and therefore is called the Aharonov-Casher effect. So for our problem of interaction between quasiparticles and the vortex both effects, Aharonov-Bohm and Aharonov-Casher, are important. But these effects are so close one to another that throughout the paper we use only the name “Aharonov-Bohm effect”.

**E. The transverse force in the quasiclassical theory**

Now we shall show that the Iordanskii force follows also from the quasiclassical theory of scattering by the vortex, despite one cannot use the quasiclassical theory for phonons. But the quasiclassical theory is valid to describe the roton contribution to the transverse force.

Let us consider a quasiparticle with an arbitrary spectrum \( \varepsilon(p) \). If the quasiparticle moves in the velocity field induced by the vortex, its energy is \( E(p) = \varepsilon(p) + \mathbf{p} \cdot \mathbf{v} \), which may be treated as a Hamiltonian to write the classic equation of motion:
\[
\frac{d\vec{r}}{dt} = \frac{\partial E}{\partial \vec{p}} = \nu_G \frac{\vec{p}}{p} + \vec{v}_s, \\
\frac{d\vec{p}}{dt} = -\frac{\partial E}{\partial \vec{r}} = -\frac{\partial}{\partial \vec{r}}(\vec{p} \cdot \vec{v}_s),
\]

(65)

where \( \nu_G(p) = d\varepsilon / dp \) is the quasiparticle group velocity. As well as in phonon scattering, we assume that the roton moves in the plane \( xy \) normal to the vortex. From this equations one can find the classical trajectory of the quasiparticle moving past the vortex line. Usually it is close to a straight line, and a distance of the straight trajectory from the vortex line is the impact parameter \( b \). We are looking for the transverse force only, then we need only the variation \( \delta p_\perp \) of the momentum component normal to the initial momentum \( \vec{p} \) which results from quasiparticle motion past the vortex. In the first order with respect to \( \vec{v}_s \) Eqs. (65) yield that

\[
\delta p_\perp(b) = -\frac{\partial}{\partial b} \int_{-\infty}^{\infty} dl \frac{p}{2\pi \nu_G} \frac{\kappa b}{b^2 + l^2}.
\]

(66)

where \( l \) is the coordinate along the trajectory. On the other hand, the momentum of the quasiparticle is connected to the classical action: \( \vec{p} = \partial S/\partial \vec{r} \). Then \( \delta p_\perp(b) = \partial \delta S(b)/\partial b \) where the total variation of the classical action along the trajectory is a function of the impact parameter \( b \):

\[
\delta S(b) = -\frac{p}{2\pi \nu_G} \int_{-\infty}^{\infty} dl \frac{\kappa b}{b^2 + l^2}.
\]

(67)

The scattering angle of the quasiparticle is \( \varphi \approx -\delta p_\perp(b)/p \) and the transverse cross-section is

\[
\sigma_\perp = \int_{-\infty}^{\infty} d\varphi \sin \varphi \approx \int_{-\infty}^{\infty} d\varphi(b) \frac{\delta S(-\infty) - \delta S(+\infty)}{p} = \frac{1}{2\pi \nu_G} \int_{-\infty}^{\infty} db \frac{\partial}{\partial b} \int_{-\infty}^{\infty} dl \frac{\kappa b}{b^2 + l^2} = \frac{\kappa}{\nu_G}.
\]

(68)

For phonons the group velocity \( \nu_G \) is equal to the sound velocity and Eq. (68) yields the correct value of \( \sigma_\perp \) even though the quasiclassical theory is not valid for phonons: there is no well-defined classical trajectories for phonons except for large impact parameters \( b \) at which the scattering angle \( \varphi \) is negligible. But in order to obtain the transverse cross-section \( \sigma_\perp \), one should know only the action variation at large impact parameters whereas the derivative \( \partial \delta S(b)/\partial b \) of the action variation, which determines the scattering angle, is not essential.

It is important to note that the double integral of Eq. (68) is improper: its value depends on what integration is done first. The correct procedure which was justified in Ref. [6] is to integrate along the trajectory first, and to integrate over the impact parameters afterwards. A way to check it is the following. We choose some finite limits in the double integral of Eq. (68) which means that the integration is restricted by some area around the vortex line. The integral depends on the shape of this area. For example, a circular border of the area yields \( \sigma_\perp \) by a factor 2 less than that of Eq. (68). But the full solution of the collisionless kinetic equation for the quasiparticles made in Ref. [6] showed that other terms contribute to the momentum balance, also originating from the slow decrease of the velocity field. Taking into account all of them, we arrive again to the expression for the force via the cross-section given by Eq. (68). The order of integrations in Eq. (68) assumes that the integration area has a shape of a rectangular with a long side along the quasiparticle trajectory. For such a shape all other contributions to the transverse force exactly cancel.

The transverse cross-section \( \sigma_\perp \) determines the amplitude \( D' \) of the transverse force on the vortex in Eqs. (19) and (21). For quasiparticles moving at an arbitrary angle to the vortex line \( \nu_G \) in Eq. (68) is the component of the group velocity in the plane normal to the vortex line. The amplitude \( D' \) of the transverse force must be determined using the Planck distribution for phonons with the drift velocity \( \vec{v}_s \). This distribution should be linearized with respect to the relative velocity \( \vec{v}_n = (\vec{v}_s - \vec{v}_L) + (\vec{v}_L - \vec{v}_n) \). But the velocity \( \vec{v}_n - \vec{v}_L \) enters the energy \( E(\vec{p}) = \varepsilon(\vec{p}) + \vec{p} \cdot (\vec{v}_n - \vec{v}_L) \) of the quasiparticle in the reference frame connected with the vortex. Using the condition of the detailed balance one can show that the quasiparticle distribution function which depends only on the energy cannot produce a force. Thus the only part of the distribution function which contributes to the force is that linear in the relative drift velocity \( \vec{v}_n - \vec{v}_L \). Finally the amplitude of the transverse force is

\[
D' = \frac{1}{3h^2} \int \frac{\partial n_0(\varepsilon, 0)}{\partial E} p^2 \sigma_\perp \nu_G d\delta \vec{p}.
\]

(69)
Using the cross-section $\sigma_\perp$ from Eq. (68), one obtains that $D' = -\kappa \rho_s$. This means that in the translationally invariant superfluid without scattering by impurities the amplitude $\rho_M$ of the effective Magnus force on the vortex in Eq. (24) is equal to the total density $\rho$, but not to the superfluid density $\rho_s$. Then neglecting the viscous forces $[D = d = d' = 0$ in Eq. (24)] the vortex moves with the center-of-mass velocity.

A rather simple and universal expression $D' = -\kappa \rho_s$ for the Iordanskii force amplitude tempts to claim its universal topological origin, since $\kappa$ in this expression is a topological charge. However, in the next section we shall see that the expression is not universal, in fact. For quasiparticles in a BCS superconductor with energy much exceeding the gap an additional small factor should be put in this expression.

IV. IORDANSKII FORCE FOR QUASIPARTICLES IN BCS SUPERCONDUCTORS

The wave-function of quasiparticles in the BCS theory has two components,

$$\psi(\vec{r}) = \left( \begin{array}{c} u(\vec{r}) \\ v(\vec{r}) \end{array} \right) ,$$

which are determined from the Bogolyubov-de Gennes equations:

$$-\frac{\hbar^2}{2m} \left( \vec{\nabla}^2 + k_F^2 \right) u(\vec{r}) + \Delta \exp[i\theta(\vec{r})] v(\vec{r}) = Eu(\vec{r}) ,$$

$$\frac{\hbar^2}{2m} \left( \vec{\nabla}^2 + k_F^2 \right) v(\vec{r}) + \Delta \exp[-i\theta(\vec{r})] u(\vec{r}) = Ev(\vec{r}) .$$

Here $k_F$ is the Fermi wave number. We neglect the magnetic field effect which is weak if the London penetration depth is large compared to other relevant scales. Without the vortex the order parameter phase $\theta$ is a constant and the equations yield the well-known BCS quasiparticle spectrum $E = \sqrt{\xi^2 + \Delta^2}$, where $\xi = (\hbar^2/2m)(k^2 - k_F^2)$ is the quasiparticle energy in the normal Fermi-liquid.

In Refs. 9,10 the Bogolyubov-de Gennes equations for a quasiparticle passing a vortex were solved with help of the partial-wave expansion, earlier used also in Ref. 51. Quasiparticles with the energy close to the gap $(\xi \ll \Delta)$ behave as rotons and the transverse cross-section for them is given by Eq. (68) in which the group velocity for the BCS quasiparticles is $v \sim \frac{\kappa}{\Delta}$.

The partial-wave expansion, earlier used also in Ref. [51]. Quasiparticles with the energy close to the gap $(\xi \ll \Delta)$ behave as rotons and the transverse cross-section for them is given by Eq. (68) in which the group velocity for the BCS quasiparticles is $v \sim \frac{\kappa}{\Delta}$. This equation is similar to equation (34) for the sound wave and using this analogy one easily obtains the expression for the transverse cross-section:

$$\sigma_\perp = \frac{\Delta^2 \pi}{2\xi^2 k_F} = \frac{\Delta^2 \kappa}{2\xi^2 v_F} .$$

Since the group velocity of quasiparticles with $E \gg \Delta$ is about the Fermi velocity $v_F$, this expression differs from Eq. (68) by the factor $\Delta^2/2\xi^2$. Thus the Iordanskii force becomes small close to $T_c$: it decreases proportionally to $\rho_s \sim \Delta^2$, like the superfluid Magnus force.
V. MAGNUS FORCE IN THE JOSEPHSON JUNCTION ARRAY

In the continuum limit, the equation of motion for a vortex has been derived in Ref. [30]. This derivation has not revealed any force normal to the vortex velocity. Absence of the Magnus force suggests that the vortices move parallel to the driving force, i.e., normally to the current, and there is no Hall effect. Then in the limit of weak dissipation the ballistic vortex motion is possible, which is a free vortex motion without friction and the driving force. The Hall effect and the ballistic motion are incompatible, since the latter is a regime with a finite electrical field and no external current which is impossible for a finite Hall resistance. Though there have been experimental evidences of the ballistic vortex motion [34], one may suspect that a more sophisticated theory would reveal the Hall resistance, however small. In the present section it will be shown that the Hall effect is exactly absent in the classical limit for the JJA. It directly follows from the symmetry of the dynamic equations.

Let us consider a conductor in a magnetic field \( \vec{H} \). When its symmetry is not less than the three-fold (which includes a triangular and square lattices), the Ohm law is:

\[
\vec{E} = \rho_L \vec{I} + \rho_H \vec{n} \times \vec{I},
\]

where \( \vec{n} = \vec{H}/H \), \( \vec{E} \) is the electrical field, \( \rho_L \) is the longitudinal resistance and \( \rho_H \) is the Hall resistance in the magnetic field \( \vec{H} \). Now let us consider the transformation in which the directions of the fields \( \vec{E} \) and \( \vec{H} \) and the current \( \vec{I} \) are reversed:

\[
\vec{E} \rightarrow -\vec{E}, \quad \vec{n} \rightarrow -\vec{n}(H \rightarrow -H), \quad \vec{l} \rightarrow -\vec{l}.
\]

The Ohm law Eq. (76) is invariant with respect to this field-current inversion only for a system without the Hall effect (\( \rho_H = 0 \)). On the microscopical level the field-current-inversion invariance is a direct result of the particle-hole symmetry which was shown to forbid the Hall effect in the Ginzburg-Landau theory (see [28] and the references therein).

Next we consider the JJA with the energy

\[
\mathcal{E} = \frac{1}{2} \sum_{I,J} \left[ Q_I C^{-1}_{I,J} Q_J - E_J \sin(\phi_I - \phi_J) \right],
\]

and the equations of motion

\[
V_{lI} = \frac{\hbar}{2e} \frac{d\phi_{lI}}{dt}, \quad \sum_{\mu} C_{lI+l\mu} \frac{dV_{lI+l\mu}}{dt} - I_C \sum_{\mu} \sin(\phi_{lI} - \phi_{lI+l\mu}) + \sum_{\mu} \sigma_{lI+l\mu} V_{lI+l\mu} = 0.
\]

Here \( V_{lI} \) is the electric potential, \( Q_I = \sum_{\mu} C_{lI+l\mu} V_{lI+l\mu} \) is the electric charge, and \( \phi_{lI}(t) \) is the gauge invariant phase at the node specified by the discrete two-dimensional position vector \( \vec{lI} \), \( E_J \) is the Josephson coupling energy, \( I_C = 2eE_J/\hbar \) is the critical current, \( C_{lI,\mu} \) and \( \sigma_{lI,\mu} \) are the capacity and the conductance matrices respectively. In the external magnetic field \( \vec{H} = \vec{\nabla} \times \vec{A} \), the gauge invariant phase \( \phi_{lI} \) is not single-valued; in fact only its difference between neighboring nodes is well-defined:

\[
\phi_{lI+l\mu} - \phi_{lI} = \varphi_{lI+l\mu} - \varphi_{lI} = \frac{2\pi}{\Phi_0} \int_{\vec{lI}}^{\vec{lI+l\mu}} A \cdot d\vec{l}.
\]

Here \( \varphi_{lI} \) is the canonical phase at the node \( \vec{lI} \), and the integral over \( \vec{A} \) is taken between the centers of the two neighboring nodes. The canonical phase is not single-valued too, but its circulation along any closed path through the nodes of JJA is always an integer number of \( 2\pi \), while the circulation of the gauge invariant phase may be any number depending on the magnetic field.

When both the external current \( \vec{lI} \) and the magnetic field \( \vec{H} \) are applied to the JJA, the gauge-invariant phase can be presented as \( \phi_{lI} = \phi_{lI}^H + \phi_{lI}^I \). Here \( \phi_{lI}^H \) is the time-independent phase in the equilibrium state without an external current, and \( \phi_{lI}^I \) is the time-dependent contribution to the phase from the external current \( \vec{lI} \). The multi-valuedness of the phase related to the magnetic field is present only in the static phase \( \phi_{lI}^H \); the time-dependent

\[
\phi_{lI}^H = \frac{\hbar}{2e} \int_{\vec{lI}}^{\vec{lI+l\mu}} A \cdot d\vec{l}, \quad \phi_{lI}^I = \frac{2\pi}{\Phi_0} \int_{\vec{lI}}^{\vec{lI+l\mu}} A \cdot d\vec{l}.
\]
dynamical contribution $\phi_l^{(l)}$ to the phase is single-valued. One sees then that the field-current inversion [Eq. (77)] simply corresponds to the change of signs of all phases, and the equations of motion, Eqs. (79) and (80), are invariant with respect to this transformation. It proves that the Hall effect does not exist in the JJA, i.e., the effective Magnus force vanishes.

The crucial point of this very simple derivation is that one can use the static vortex solution $\phi_l^{(l)}$ for the dynamical problem. This assumes that singularities of the phase distribution related to the presence of vortices are kept at rest despite the vortices themselves are driven by the Lorentz force. For continuous superconductors this approach is invalid and our derivation does not work. So there is a fundamental difference between vortices in a lattice and vortices in a continuous superconductor. Indeed, in the lattice there are no singular vortex lines. They appear only in the continuum limit. At best, one can define the lattice cell containing the vortex center. This definition has been borrowed from the continuous theory: it is the cell, around which the circulation of the phase $\varphi$ is equal to $2\pi$. However, in the lattice the circulation around a closed path is not well-defined. Let us consider some closed path through a discrete number of nodes with the phase circulation $2\pi$. One may change the phase difference by $-2\pi$ between any two neighboring nodes on the path without any effect on observed physical parameters (currents, voltages and so on). Then the circulation vanishes along the path considered, but must appear along a path over other nodes. Thus one cannot locate the position of the phase singularity. In order to avoid this ambiguity in the JJA model, a special rule has been formulated: the phase difference between two neighboring nodes must not exceed $\pi$. When for some bond the phase difference achieves the value $\pi$, one must redefine the phases; as a result, the vortex center is put into another cell. This procedure is usual for numerical studies of the vortex motion in JJA [39]. However, this rule is not obligatory for the dynamic theory of JJA. Instead, one may keep $2\pi$ circulations of the phase $\varphi$ at fixed cells during the dynamic process without worrying where the vortex center (defined according to the aforementioned rule) is really located.

It is important to stress, that one cannot derive the effective Magnus force in JJA using the continuous approach. Let us discuss this in more details. In the continuum limit the set of discrete vectors $\vec{l}$ is replaced with the continuum space of $\vec{l}$. One can define the field of the canonical (but not gauge invariant!) phase $\varphi(\vec{l})$ in this space everywhere except for the singular points which are the centers of the vortices with the phase circulation $2\pi$. However, in the lattice the circulation around a closed path is not well-defined. Let us consider some closed path through a discrete number of nodes with the phase circulation $2\pi$. One may change the phase difference by $-2\pi$ between any two neighboring nodes on the path without any effect on observed physical parameters (currents, voltages and so on). Then the circulation vanishes along the path considered, but must appear along a path over other nodes. Thus one cannot locate the position of the phase singularity. In order to avoid this ambiguity in the JJA model, a special rule has been formulated: the phase difference between two neighboring nodes must not exceed $\pi$. When for some bond the phase difference achieves the value $\pi$, one must redefine the phases; as a result, the vortex center is put into another cell. This procedure is usual for numerical studies of the vortex motion in JJA [39]. However, this rule is not obligatory for the dynamic theory of JJA. Instead, one may keep $2\pi$ circulations of the phase $\varphi$ at fixed cells during the dynamic process without worrying where the vortex center (defined according to the aforementioned rule) is really located.

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factor $q$ should be proportional to some electric charge, since the charge is a variable conjugate to the canonical phase $\varphi$. But it remains unclear what is this charge: either the background charge determined by the whole Fermi-see of the superconducting island, or an external charge induced outside as suggested in Ref. [43]. Thus in the continuum limit the problem of the effective Magnus force and the Hall effect in the JJA remains unresolved. It must not be a surprise since in the continuum limit the JJA model becomes translationally invariant and “forgets” that originally it had been a lattice model without translational invariance. Meanwhile, the latter is crucial for the amplitude of the vortex velocity. This principle is provided by our symmetry analysis. According to it $q = 0$ and one should not include the Wess-Zumino term into the field Lagrangian.

The presence of the external charge has no effect on our symmetry analysis. In order to take into account the external charge, one should use the Gibbs potential $G = E - V^{ex} \sum \bar{Q}_F$, where $E$ is given by Eq. (78) and $V^{ex}$ is the electric potential which creates the external charge $Q^{ex} = V^{ex} \sum \bar{Q}_M \bar{C}_{\bar{M},\bar{H}}$. Then introducing the charge deviation $Q^I = Q_I - Q^{ex}$, one returns back to the energy $E$ with $Q^I$ instead of $Q_F$. These arguments show that the external charge cannot lead to the Hall effect: its effect is restricted with the shift of the Fermi level, but the particle-hole symmetry is restored with respect to the new Fermi level. However, the external electric charge may produce the Magnus force in the quantum theory of JJA which takes into account the electron charge quantization [44]. Then the Magnus force and the Hall conductivity are periodic in the electron charge.

VI. SUMMARY AND DISCUSSION

We have shown how the Magnus force appears in the equation of motion for a superfluid component (the superfluid Magnus force) and the equation of motion for a vortex (the effective Magnus force). Whereas the superfluid Magnus force proportional to the superfluid density is known exactly (from classical hydrodynamics, or from the Berry phase approach), there is no general expression for the effective Magnus force: it depends on interaction of the vortex with quasiparticles and with the external fields, like those from impurities in a dirty superconductors. Meanwhile, it is mostly the effective Magnus force which determines the observable effects: the mutual friction in superfluids, the Hall effect and the acoustic Faraday effect in superconductors, vortex quantum tunnelling.

We have calculated the contribution of quasiparticles to the effective Magnus force for phonons in a superfluid and for high-energy quasiparticles in a BCS superconductor using the Born approximation. The transverse force from quasiparticles on the vortex (the Iordanskii force) originates from interference between quasiparticles passing on different sides of the vortex (the Aharonov-Bohm effect).

Our symmetry analysis of the classical Josephson junction array has demonstrated that the effective Magnus force exactly vanishes and there is no Hall effect despite the finite superfluid density. One may formally interpret this result that the force from external fields violating translational invariance exactly compensates the superfluid Magnus force, though the analysis is not able to reveal these two forces separately, but only their joint outcome, namely, the effective Magnus force.

The Ao-Thouless theory yields only the superfluid Magnus force which appears in the momentum balance of the superfluid component (the condensate). Indeed, Gaitan [22] derived the Ao-Thouless result for a charged superfluid, analyzing the momentum balance for the condensate. In order to derive the effective Magnus force (the total transverse force on the vortex), one must consider the momentum balance for the whole system.

Ao, Niu, and Thouless [54] stated that the Iordanskii force did not appear in their Berry phase approach. But the recent paper on the Born quasiparticle scattering by Demircan, Ao, and Niu [22] demonstrated that they admitted the force from quasiparticle scattering in their approach, but concluded that this didn’t yield any transverse force on the vortex. This conclusion was based on a wrong analysis of the Born phonon scattering missing the contribution from the Aharonov-Bohm interference. Thus the source of controversy is not in a difference of approaches, but in the problem how to calculate integrals for the Born phonon scattering.

The Ao-Thouless theory rejects also any force on the vortex from the external fields, like the Kopnin-Kravtsov force in a dirty superconductor. On the basis of this theory Gaitan and Shenoy [22] predicted the finite effective Magnus force and the Hall effect for the Josephson-junction array. This prediction contradicts to our symmetry analysis and to the experiment. Gaitan and Shenoy [22] used in their analysis the Wess-Zumino term in the Lagrangian for the continuum limit of JJA. We have shown in Sec. V why this approach is not reliable. On the other hand, Zhu, Tan, and Ao [22] try to conciliate the negligible Hall effect in JJA with the Ao-Thouless theory supposing that the vortices cross JJA between superconducting islands where the superfluid order parameter vanishes and therefore the Magnus force is negligible. But the superfluid order parameter does not vanish between the islands even though it is much smaller than in the islands. In fact, this small order parameter determines a finite superfluid density in the continuum limit.
Thus in JJA the Hall effect vanishes despite a finite superfluid density, contrary to the prediction of the Ao-Thouless theory.

Makhlin and Volovik suggested that the superfluid Magnus force in JJA is nearly compensated by the force from the bound states in the junctions (the spectral flow of bound states). But they did not conclude that the compensation is complete, and assumed the Fermi superfluid in islands and the SNS Josephson junctions. Our analysis shows that the Magnus force exactly vanishes for JJA independently on microscopic nature of the superconducting islands and the junctions. This shows that the core bound states and the spectral flow are not the only explanation for compensation of the Magnus force in the systems without translational invariance.

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APPENDIX A: THE FORCE ON THE LINE SCATTERING THE SOUND WAVE

First we derive the analogue of the optical theorem for the sound wave. For the latter we use the asymptotic representation, Eq. (39), in which the scattering amplitude $a(\varphi)$ is not necessarily obtained in the Born approximation. But in general $a(\varphi)$ should satisfy the condition that the total mass flow through the cylindrical surface surrounding the scattering line vanishes.

An asymptotic expression for the average mass flow from the sound wave is

$$j^{ph} = \langle \rho(1) \tilde{v}(1) \rangle = \rho_0 \phi_0^2 \frac{k^2}{8 \pi^2 e} \left\{ \tilde{k} + \frac{|a|^2}{r} \tilde{k}' - (\tilde{k} + \tilde{k}') \frac{1}{\sqrt{r}} [\text{Im} \{a\} \cos(\text{kr} - \tilde{k} \cdot \tilde{r})] + \text{Re} \{a\} \sin(\text{kr} - \tilde{k} \cdot \tilde{r}) \right\} . \quad (A.1)$$

The condition that the total flow through the cylindrical surface around the scattering line vanishes is

$$\int j^{ph}_i dS_i = \int \langle \rho(1) \tilde{v}(1)_i \rangle dS_i = \rho_0 \phi_0^2 \frac{k^2}{8 \pi^2 e} \int \left\{ \cos \varphi + \frac{|a(\varphi)|^2}{r} - \frac{\text{Im} \{a(\varphi)\}}{\sqrt{r}} (1 + \cos \varphi) \cos(\text{kr}(1 - \cos \varphi)) \right\} d\varphi = 0 . \quad (A.2)$$

The integral over the term $\propto \text{Im} \{a(\varphi)\}$ expands only over the region of small angles since $\text{kr} \gg 1$. Finally this condition yields

$$-2 \sqrt{\pi} k \text{Im} \{a(0)\} + \int |a(\varphi)|^2 d\varphi = 0 . \quad (A.3)$$

Next let us consider the momentum balance which determines the force on the scattering line from the sound wave

$$F^{ph}_{ij} = -\int dS_i \Pi^{ph}_{ij}$$

where:

$$\Pi^{ph}_{ij} = \left( \frac{\rho^2}{\rho_0} \frac{\langle \rho^2(1) \rangle}{2} - \rho_0 \frac{\langle v^2(1) \rangle}{2} \right) \delta_{ij} + \rho_0 \langle (v(1)_i v(1)_j) \rangle . \quad (A.4)$$

The pressure term vanishes after averaging, but the convection term is essential and yields for the force on the vortex:

$$F^{ph} = -\rho_0 \phi_0^2 \frac{k^2}{8 \pi^2} \int \left\{ \tilde{k} \cos \varphi + \frac{|a(\varphi)|^2}{r^2} \tilde{k}' - \frac{\text{Im} \{a(\varphi)\}}{\sqrt{r}} \cos(\text{kr}(1 - \cos \varphi)) \left( \tilde{k} + \tilde{k}' \right) \right\} \ d\varphi$$

$$\approx -\rho_0 \phi_0^2 \frac{k^2}{8 \pi^2} \int \left\{ \tilde{k} \cos \varphi + \frac{|a(\varphi)|^2}{r} \tilde{k}' - \frac{\text{Im} \{a(0)\}}{\sqrt{r}} \cos \left( \frac{1}{2} \left( k r \varphi^2 \right) \right) \frac{2k}{\tilde{k}} \right\} \ d\varphi$$

$$= -\rho_0 \phi_0^2 \frac{k^2}{8 \pi^2} \int \frac{|a(\varphi)|^2}{r} \tilde{k}' d\varphi - 2 \sqrt{\frac{\pi}{k}} \text{Im} \{a(0)\} \tilde{k} . \quad (A.5)$$

With help of the optical theorem Eq. (A.3) one obtains the expression Eq. (43) with the effective cross-sections determined by Eqs. (13) and (45).
APPENDIX B: SMALL-ANGLE SCATTERED SOUND WAVE

Using the asymptotics of the Hankel function, Eq. (35) can be rewritten as

\[
\phi = \phi_0 \exp(-i\omega t) \left\{ \exp(ik \cdot \vec{r}) + \frac{\kappa k}{c} \left( \frac{i}{2\pi r} \int d_2 \vec{r}_1 \exp(ik \cdot \vec{r}_1 + ik|\vec{r} - \vec{r}_1|) \frac{\vec{k} \cdot [\hat{z} \times \vec{r}_1]}{r_1^2} \right) \right\}. \tag{B.1}
\]

Here the effect of the vortex-line motion was neglected as irrelevant for small-angle scattering. Expansion Eq. (38) is not accurate enough and next terms of the expansion must be kept:

\[
|\vec{r} - \vec{r}_1| \approx r - \frac{(\vec{r}_1 \cdot \vec{r})}{r} + \frac{r_1^2}{2r} - \frac{(\vec{r}_1 \cdot \vec{r})^2}{2r^2}. \tag{B.2}
\]

The terms of the second order in \( r_1 \) are important since the perturbation is not well localized near the vortex line, but decreasing slowly when \( r_1 \) is increasing. Using the Cartesian coordinates of the position vector \( \vec{r}_1(x, y) \) and the inequality \( \varphi \ll 1 \), one obtains

\[
\phi = \phi_0 \exp(-i\omega t) \left\{ \exp(ik \cdot \vec{r}) + \frac{\kappa k^2}{c} \sqrt{\frac{i}{2\pi r_1^2}} \int \int dx \, dy \, \exp \left[ ik \left( r - y \varphi + \frac{y^2}{2r} \right) \right] \frac{y}{x^2 + y^2} \right\}. \tag{B.3}
\]

The double integral in this expression may be transformed into the error integral:

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp \left[ ik \left( r - y \varphi + \frac{y^2}{2r} \right) \right] \frac{y}{x^2 + y^2} = -\pi \sqrt{\frac{2\pi r}{k}} \exp \left[ ikr \left( 1 - \frac{\varphi^2}{2} \right) \right] \Phi \left( \frac{\varphi \sqrt{kr}}{2i} \right).
\]

Then Eq. (B.3) coincides with (47).

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