Euclidean Quantum Gravity in Light of Spectral Geometry

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Abstract. A proper understanding of boundary-value problems is essential in the attempt of developing a quantum theory of gravity and of the birth of the universe. The present paper reviews these topics in light of recent developments in spectral geometry, i.e. heat-kernel asymptotics for the Laplacian in the presence of Dirichlet, or Robin, or mixed boundary conditions; completely gauge-invariant boundary conditions in Euclidean quantum gravity; local vs. non-local boundary-value problems in one-loop Euclidean quantum theory via path integrals.
1. Introduction

The aim of theoretical physics is to provide a clear conceptual framework for the wide variety of natural phenomena, so that not only are we able to make accurate predictions to be checked against observations, but the underlying mathematical structures of the world we live in can be thoroughly understood by the scientific community. What are therefore the key elements of a mathematical description of the physical world? Can we derive all basic equations of theoretical physics from a few symmetry principles? What do they tell us about the origin and evolution of the universe? Why is gravitation so peculiar with respect to all other fundamental interactions?

The above questions have received careful consideration over the last decades, and have led, in particular, to several approaches [1] to a theory aiming at achieving a synthesis of quantum physics on the one hand, and general relativity on the other hand. This remains, possibly, the most important task of theoretical physics. The need for a quantum theory of gravity is already suggested from singularity theorems in classical cosmology [2]. Such theorems prove that the Einstein theory of general relativity leads to the occurrence of space-time singularities in a generic way. At first sight one might be tempted to conclude that a breakdown of all physical laws occurred in the past, or that general relativity is severely incomplete, being unable to predict what came out of a singularity. It has been therefore pointed out that all these pathological features result from the attempt of using the Einstein theory well beyond its limit of validity, i.e. at energy scales where the fundamental theory is definitely more involved. General relativity might be therefore viewed as a low-energy limit of a richer theory [3], which achieves the synthesis of both the basic principles of modern physics and the fundamental interactions in the form presently known.

Within the framework just outlined it remains however true that the various approaches to quantum gravity developed so far suffer from mathematical inconsistencies, or incompleteness in their ability of accounting for some basic features of the laws of nature. From the point of view of general principles, the space-time approach to quantum
mechanics [4] and quantum field theory [5], and its application to the quantization of gravitational interactions [6–9], remains indeed of fundamental importance. When one tries to implement the Feynman sum over histories one discovers that, already at the level of non-relativistic quantum mechanics, a well defined mathematical formulation is only obtained upon considering a heat-equation problem. The measure occurring in the Feynman representation of the Green kernel is then meaningful, and the propagation amplitude of quantum mechanics in flat Minkowski space-time is obtained by analytic continuation. This is a clear indication that quantum-mechanical problems via path integrals are well understood only if the heat-equation counterpart is mathematically well posed. In quantum field theory one then deals with the Euclidean approach, and its application to quantum gravity [10] relies heavily on the theory of elliptic operators on Riemannian manifolds [11–14]. To obtain a complete picture one has then to specify the boundary conditions of the theory, i.e. the class of Riemannian geometries with their topologies involved in the sum, and the form of boundary data assigned on the bounding surfaces [15–29].

In particular, recent work [30,31] has shown that the only set of local boundary conditions on metric perturbations which are completely invariant under infinitesimal diffeomorphisms is incompatible with a good elliptic theory. More precisely, while the resulting operator on metric perturbations can be made of Laplace type and elliptic in the interior of the Riemannian manifold under consideration, the property of strong ellipticity [12–14] is violated. This is a precise mathematical expression of the requirement that a unique smooth solution of the boundary-value problem should exist which vanishes at infinite geodesic distance from the boundary. This opens deep interpretive issues, since only for gravity does the requirement of complete gauge invariance of the boundary conditions turn out to be incompatible with a good elliptic theory. It is then impossible to make sense even just of the one-loop semiclassical approximation, because the functional trace of the heat operator is found to diverge (cf. section 6).

We have been therefore led to consider, in our more recent research, non-local boundary conditions for the quantized gravitational field at one-loop level [32–35]. On the one hand, such a scheme already arises in simpler problems, i.e. the quantum theory of a free particle subject to non-local boundary data on a circle [36]. One then finds two families
of eigenfunctions of the Hamiltonian: surface states which decrease exponentially as one moves away from the boundary, and bulk states which remain instead smooth and non-vanishing. The generalization to an Abelian gauge theory such as Maxwell theory can fulfill non-locality, ellipticity and complete gauge invariance of boundary conditions providing one learns to work with pseudo-differential operators in one-loop quantum theory [37]. On the other hand, in the application to quantum gravity, since the boundary operator acquires new kernels responsible for the pseudo-differential nature of the boundary-value problem, one might hope to be able to recover a good elliptic theory under a wider variety of conditions [32–35].

Boundary field theory is therefore highly relevant for understanding quantum cosmology, quantum gravity and the foundations of quantized gauge theories, and it has deep roots in global analysis and spectral geometry, since, after imposing a supplementary condition, the gauge-field operator in the path integral for such theories [5,9] is either of Laplace type or non-minimal (on working with positive-definite metrics). In agreement with our pedagogical aims, we first review operators of Laplace type in section 2, the functorial method for heat-kernel asymptotics in section 3 and the topic of mixed boundary conditions in section 4. Section 5 is then devoted to gauge-invariant boundary conditions for the quantized gravitational field, while recent developments and open problems are outlined in section 6.

2. Asymptotics of the Laplacian on manifolds with boundary

Following Branson and Gilkey [38], we are interested in a second-order differential operator, say $P$, with leading symbol given by the metric tensor (more precisely, with scalar leading symbol) on a compact $m$-dimensional Riemannian manifold $M$ with boundary $\partial M$. Denoting by $\nabla$ the connection on the vector bundle $V$ over $M$, our assumption implies that $P$, called an operator of Laplace type, reads

$$P = -g^{ab}\nabla_a \nabla_b - E,$$

(2.1)
where $E$ is an endomorphism of $V$ (i.e. the ‘potential’ term in the physics-oriented literature). The heat equation for the operator $P$ is

$$\left( \frac{\partial}{\partial t} + P \right) \varphi = 0. \quad (2.2)$$

By definition, the heat kernel is the solution, for $t > 0$, of the equation

$$\left( \frac{\partial}{\partial t} + P \right) U(x, x'; t) = 0, \quad x \text{ and } x' \in M, \quad (2.3)$$

subject to the boundary condition

$$\left[ B U(x, x'; t) \right]_{\partial M} = 0, \quad (2.4)$$

jointly with the (initial) condition

$$\lim_{t \to 0^+} \int_M U(x, x'; t) \rho(x') dx' = \rho(x), \quad (2.5)$$

which is a rigorous mathematical expression for the Dirac delta behaviour as $t \to 0^+$. The heat kernel can be written as

$$U(x, x'; t) = \sum_{(n)} \varphi_{(n)}(x) \varphi_{(n)}(x') e^{-\lambda_{(n)} t}, \quad (2.6)$$

where $\{ \varphi_{(n)}(x) \}$ is a complete orthonormal set of eigenfunctions of $P$ with eigenvalues $\lambda_{(n)}$. The index $n$ is enclosed in round brackets, to emphasize that, in general, a finite collection of integer labels occurs therein.

Since, by construction, the heat kernel behaves as a distribution in the neighbourhood of the boundary, it is convenient to introduce a smooth function, say $f \in C^\infty(M)$, and consider a slight generalization of the trace function (or integrated heat kernel), i.e. $\text{Tr}_{L^2} \left( f e^{-tP} \right)$. It is precisely the consideration of $f$ that makes it possible to deal properly with the distributional behaviour of the heat kernel near $\partial M$. A key idea is therefore to
work with arbitrary $f$, and then set $f = 1$ only when all coefficients in the asymptotic expansion

$$\text{Tr}_{L^2} \left( f e^{-tP} \right) \equiv \int_M \text{Tr} \left[ fU(x, x; t) \right]$$

$$\sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} t^{n/2} a_{n/2}(f, P, \mathcal{B}) \quad (2.7)$$

have been evaluated. The term $U(x, x; t)$ is called the heat-kernel diagonal. By virtue of Greiner’s result [39], the heat-kernel coefficients $a_{n/2}(f, P, \mathcal{B})$, which are said to describe the global asymptotics, are obtained by integrating local formulae. More precisely, they admit a split into integrals over $M$ (interior terms) and over $\partial M$ (boundary terms). In such formulae, the integrands are linear combinations of all geometric invariants of the appropriate dimension (see below) which result from the Riemann curvature $R^a_{bcd}$ of the background, the extrinsic curvature of the boundary, the differential operator $P$ (through the endomorphism $E$), and the boundary operator $\mathcal{B}$ (through the endomorphisms, or projection operators, or more general matrices occurring in it). With our notation, the indices $a, b, \ldots$ range from 1 through $m$ and index a local orthonormal frame for the tangent bundle of $M$, $TM$, while the indices $i, j, \ldots$ range from 1 through $m - 1$ and index the orthonormal frame for the tangent bundle of the boundary, $T(\partial M)$. The boundary is defined by the equations

$$\partial M : \quad y^a = y^a(x), \quad (2.8)$$

in terms of the functions $y^a(x), x^i$ being the coordinates on $\partial M$, and the $y^a$ those on $M$. Thus, the intrinsic metric, $\gamma_{ij}$, on the boundary hypersurface $\partial M$, is given in terms of the metric $g_{ab}$ on $M$ by

$$\gamma_{ij} = g_{ab} y^a_{,i} y^b_{,j}. \quad (2.9)$$

On inverting this equation one finds (here, $n^a = N^a$ is the inward-pointing normal)

$$g^{ab} = q^{ab} + n^a n^b, \quad (2.10)$$

where

$$q^{ab} = y^a_{,i} y^b_{,j} \gamma^{ij}. \quad (2.11)$$
The tensor \( q^{ab} \) is equivalent to \( \gamma^{ij} \) and may be viewed as the \textit{induced metric} on \( \partial M \), in its contravariant form. The tensor \( q^a_b \) is a projection operator, in that

\[
q^a_b q^b_c = q^a_c, \quad (2.12)
\]

\[
q^a_b n^b = 0. \quad (2.13)
\]

The extrinsic-curvature tensor \( K_{ab} \) (or second fundamental form of \( \partial M \)) is here defined by the projection of the covariant derivative of an \textit{extension} of the inward, normal vector field \( n \):

\[
K_{ab} \equiv n_{cd} q^c_{\cdot a} q^d_{\cdot b}, \quad (2.14)
\]

and is symmetric if the metric-compatible connection on \( M \) is torsion-free. Only its spatial components, \( K_{ij} \), are non-vanishing.

The semicolon \( ; \) denotes multiple covariant differentiation with respect to the Levi–Civita connection \( \nabla_M \) of \( M \), while the stroke \( | \) denotes multiple covariant differentiation tangentially with respect to the Levi–Civita connection \( \nabla_{\partial M} \) of the boundary. When sections of bundles built from \( V \) are involved, the semicolon means

\[
\nabla_M \otimes 1 + 1 \otimes \nabla,
\]

and the stroke means

\[
\nabla_{\partial M} \otimes 1 + 1 \otimes \nabla.
\]

The curvature of the connection \( \nabla \) on \( V \) is denoted by \( \Omega \).

When Dirichlet or Robin boundary conditions are imposed on sections of \( V \):

\[
[\phi]_{\partial M} = 0, \quad (2.15)
\]

or

\[
\left[ (n^a \nabla_a + S)\phi \right]_{\partial M} = 0, \quad (2.16)
\]

the global asymptotics in (2.7) is expressed through some \textit{universal constants}

\[
\{ \alpha_i, b_i, c_i, d_i, e_i \}
\]
such that (here \( R_{ab} \equiv R^c_{\ abc} \) is the Ricci tensor, \( R \equiv R^a_a \), and \( \nabla^a \nabla_a = g^{ab} \nabla_a \nabla_b \))

\[
a_0(f, P, B) = \int_M \text{Tr}(f),
\]

\[
a_{1/2}(f, P, B) = \delta(4\pi)^{1/2} \int_{\partial M} \text{Tr}(f),
\]

\[
a_1(f, P, B) = \frac{1}{6} \int_M \text{Tr} \left[ \alpha_1 f E + \alpha_2 f R \right] + \frac{1}{6} \int_{\partial M} \text{Tr} \left[ b_0 f (\text{tr} K) + b_1 f_i N + b_2 f S \right],
\]

\[
a_{3/2}(f, P, B) = \frac{\delta}{96} (4\pi)^{1/2} \int_{\partial M} \text{Tr} \left[ f(c_0 E + c_1 R + c_2 R^i_{\ NiN} + c_3 (\text{tr} K)^2 + c_4 K_{ij} K^{ij} + c_7 S(\text{tr} K) + c_8 S^2) \right] + f_i N \left( c_5 (\text{tr} K) + c_9 S \right) + c_6 f_i ;N N,
\]

\[
a_2(f, P, B) = \frac{1}{360} \int_M \text{Tr} \left[ f \left( \alpha_3 \nabla^a \nabla_a E + \alpha_4 R E + \alpha_5 E^2 \right. \right.
\]

\[
\left. + \frac{1}{360} \int_{\partial M} \text{Tr} \left[ f \left( d_1 E_{;N} + d_2 R_{;N} + d_3 (\text{tr} K)_{;i} \right. \right.
\]

\[
\left. + d_5 E(\text{tr} K) + d_6 R(\text{tr} K) + d_7 R^i_{\ NiN}(\text{tr} K) + d_8 R_{iNJN} K^{ij} \right.
\]

\[
+ d_9 R^i_{\ ilj} K^{ij} + d_{10} (\text{tr} K)^3 + d_{11} K_{ij} K^{ij}(\text{tr} K) + d_{12} K^{j} \left. K_{i} \right| i + d_{13} \Omega_{iN} ;i + d_{14} S E
\]

\[
+ d_{15} S R + d_{16} S R^i_{\ NiN} + d_{17} S(\text{tr} K)^2 + d_{18} S K_{ij} K^{ij} \right.
\]

\[
+ d_{19} S^2(\text{tr} K) + d_{20} S^3 + d_{21} S^{;i} \right. \left. + f_i N \left( e_1 E + e_2 R \right. \right.
\]

\[
\left. + e_3 R^i_{\ NiN} + e_4 (\text{tr} K)^2 + e_5 K_{ij} K^{ij} + e_8 S(\text{tr} K) + e_9 S^2 \right)
\]

\[
+ f_N N \left( e_6 (\text{tr} K) + e_{10} S \right) + e_7 f_a a N \right].
\]
These formulae may seem to be very complicated, but there is indeed a systematic way to write them down and then compute the universal constants. To begin, note that, if $k$ is odd, $a_{k/2}(f, P, B)$ receives contributions from boundary terms only, whereas both interior terms and boundary terms contribute to $a_{k/2}(f, P, B)$, if $k$ is even and positive. In the $a_1$ coefficient, the integrand in the interior term must be linear in the curvature, and hence it can only be a linear combination of the trace of the Ricci tensor, and of the endomorphism $E$ in the differential operator. In the $a_2$ coefficient, the integrand in the interior term must be quadratic in the curvature, and hence one needs a linear combination of the eight geometric invariants 

$$\square E, \ RE, \ E^2, \ R, \ R^2, \ R_{ab}R^{ab}, \ R_{abcd}R^{abcd}, \ \Omega_{ab}\Omega^{ab}. $$

In the $a_1$ coefficient, the integrand in the boundary term is a local expression given by a linear combination of all invariants linear (or of the same dimension as terms linear) in the extrinsic curvature: $\text{tr}K, S$ and $f_{iN}$. In the $a_{3/2}$ coefficient, the integrand in the boundary term must be quadratic in the extrinsic curvature. Thus, bearing in mind the Gauss–Codazzi equations, one finds the general result (2.20). Last, in the $a_2$ coefficient, the integrand in the boundary term must be cubic in the extrinsic curvature. This leads to the boundary integral in (2.21), bearing in mind that $f_{iN}$ is on the same ground of a term linear in $K_{ij}$, while $f_{iNN}$ is on the same ground of a term quadratic in $K_{ij}$. Note that the interior invariants are built universally and polynomially from the metric tensor, its inverse, and the covariant derivatives of $R_{abcd}, \Omega_{ab}$ and $E$. By virtue of Weyl’s work on the invariants of the orthogonal group [12,38], these polynomials can be formed using only tensor products and contraction of tensor arguments. Here, the structure group is $O(m)$. However, when a boundary occurs, the boundary structure group is $O(m-1)$. Weyl’s theorem is used again to construct invariants as in the previous equations [38].

3. Functorial method

Let $T$ be a map which carries finite-dimensional vector spaces into finite-dimensional vector spaces. Thus, to every vector space $V$ one has an associated vector space $T(V)$. 9
The map $T$ is said to be a **continuous functor** if, for all $V$ and $W$, the map

$$T : \text{Hom}(V,W) \longrightarrow \text{Hom}(T(V),T(W))$$

is continuous.

In the theory of heat kernels, an application of the functorial method is the analysis of heat-equation asymptotics with respect to conformal variations. Indeed, the behaviour of classical and quantum field theories under conformal rescalings of the metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad (3.1)$$

with $\Omega$ a smooth function, is at the heart of many deep properties: light-cone structure, conformal curvature (i.e. the Weyl tensor), conformal-infinity techniques, massless free-field equations, twistor equation, twistor spaces, Hodge-star operator in four dimensions, conformal anomalies [26]. In the functorial method, one chooses $\Omega$ in the form

$$\Omega = e^{\varepsilon f}, \quad (3.2)$$

where $\varepsilon$ is a real-valued parameter, and $f \in C^\infty(M)$ is the smooth function considered in section 2. One then deals with a one-parameter family of differential operators

$$P(\varepsilon) = e^{-2\varepsilon f} P(0), \quad (3.3)$$

boundary operators

$$B(\varepsilon) = e^{-\varepsilon f} B(0), \quad (3.4)$$

connections $\nabla^\varepsilon$ on $V$, endomorphisms $E(\varepsilon)$ of $V$, and metrics

$$g_{ab}(\varepsilon) = e^{2\varepsilon f} g_{ab}(0). \quad (3.5)$$

For example, the form (3.2) of the conformal factor should be inserted into the general formulae which describe the transformation of Christoffel symbols under conformal rescalings:

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \Omega^{-1} \left( \delta^a_b \Omega,^c + \delta^a_c \Omega,^b - g_{bc} g^{ad} \Omega,^d \right). \quad (3.6)$$
This makes it possible to obtain the conformal-variation formulae for the Riemann tensor \( R^a_{bcd} \) and for all tensors involving the effect of Christoffel symbols. For the extrinsic-curvature tensor defined in Eq. (2.14) one finds

\[
\hat{K}_{ab} = \Omega K_{ab} - n_a \nabla_b \Omega + g_{ab} \nabla_{(n)} \Omega,
\]

which implies

\[
K_{ij}(\varepsilon) = e^{\varepsilon f} K_{ij}(0) - \varepsilon g_{ij} e^{\varepsilon f} f_{;N}.
\]

The application of these methods to heat-kernel asymptotics relies on the work by Branson and Gilkey [38]. Within this framework, a crucial role is played by the following ‘functorial’ formulae (\( F \) being another smooth function):

\[
\left[ \frac{d}{d\varepsilon} a_{n/2} \left( 1, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon=0} = (m - n) a_{n/2}(f, P(0)),
\]

\[
\left[ \frac{d}{d\varepsilon} a_{n/2} \left( 1, P(0) - \varepsilon F \right) \right]_{\varepsilon=0} = a_{n/2-1}(F, P(0)),
\]

\[
\left[ \frac{d}{d\varepsilon} a_{n/2} \left( e^{-2\varepsilon f} F, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon=0} = 0.
\]

Equation (3.11) is obtained when \( m = n + 2 \). These properties can be proved by (formal) differentiation, as follows.

If the conformal variation of an operator of Laplace type reads

\[
P(\varepsilon) = e^{-2\varepsilon f} P(0) - \varepsilon F,
\]

one finds

\[
\left[ \frac{d}{d\varepsilon} \text{Tr}_{L^2} \left( e^{-tP(\varepsilon)} \right) \right]_{\varepsilon=0} = \text{Tr}_{L^2} \left[ \left( 2tfP(0) + tF \right) e^{-tP(0)} \right]
\]

\[
= -2t \frac{\partial}{\partial t} \text{Tr}_{L^2} \left( f e^{-tP(0)} \right) + t \text{Tr}_{L^2} \left( F e^{-tP(0)} \right).
\]
following Branson and Gilkey [38] one can, instead, absorb such factors into the definition of the coefficients $a_{n/2}(f, P)$

$$\frac{\partial}{\partial t} \text{Tr}_{L^2}(f e^{-tP(0)}) \sim - \frac{1}{2} \sum_{n=0}^{\infty} (m - n) t^{\frac{n}{2} - \frac{m}{2} - 1} a_{n/2}(f, P(0)). \quad (3.14)$$

Thus, if $F$ vanishes, Eqs. (3.13) and (3.14) lead to the result (3.9). By contrast, if $f$ is set to zero, one has $P(\varepsilon) = P(0) - \varepsilon F$, which implies

$$\left[ \frac{d}{d\varepsilon} \text{Tr}_{L^2}(e^{-tP(\varepsilon)}) \right]_{\varepsilon=0} \sim t^{-m/2} \sum_{n=0}^{\infty} t^{n/2+1} a_{n/2}(F, P(0))$$

$$= t^{-m/2} \sum_{l=2}^{\infty} t^{l/2} a_{l/2-1}(F, P(0)), \quad (3.15)$$

which leads in turn to Eq. (3.10). Last, to obtain the result (3.11), one considers the two-parameter conformal variation

$$P(\varepsilon, \gamma) = e^{-2\varepsilon f} P(0) - \gamma e^{-2\varepsilon f} F. \quad (3.16)$$

Now in Eq. (3.9) we first replace $n$ by $n + 2$, and then set $m = n + 2$. One then has, from Eq. (3.16):

$$\frac{\partial}{\partial \varepsilon} a_{n/2+1}(1, P(\varepsilon, \gamma)) = 0. \quad (3.17)$$

Equation (3.17) can be differentiated with respect to $\gamma$, i.e. (see (3.10))

$$0 = \frac{\partial^2}{\partial \gamma \partial \varepsilon} a_{n/2+1}(1, P(\varepsilon, \gamma)) = \frac{\partial}{\partial \varepsilon} \frac{\partial}{\partial \gamma} a_{n/2+1}(1, e^{-2\varepsilon f}(P(0) - \gamma F))$$

$$= \frac{\partial}{\partial \varepsilon} a_{n/2}(e^{-2\varepsilon f} F, e^{-2\varepsilon f} P(0)), \quad (3.18)$$

and hence Eq. (3.11) is proved.

To deal with Robin boundary conditions one needs another lemma, which is proved following again Branson and Gilkey. What we obtain is indeed a particular case of a more general property, which is proved in section 6 of Ref. [28]. Our starting point is
$M$, a compact, connected one-dimensional Riemannian manifold with boundary. In other words, one deals with the circle or with a closed interval. If

$$b : C^\infty(M) \rightarrow \mathbb{R}$$

is a smooth, real-valued function, one can form the first-order operator

$$A \equiv \frac{d}{dx} - b,$$  \hspace{1cm} (3.19)

and its (formal) adjoint

$$A^\dagger \equiv -\frac{d}{dx} - b.$$  \hspace{1cm} (3.20)

From these operators, one can form the second-order operators

$$D_1 \equiv A^\dagger A = -\left[\frac{d^2}{dx^2} - b_x - b^2\right],$$  \hspace{1cm} (3.21)

$$D_2 \equiv AA^\dagger = -\left[\frac{d^2}{dx^2} + b_x - b^2\right],$$  \hspace{1cm} (3.22)

where $b_x \equiv \frac{db}{dx}$. For $D_1$, Dirichlet boundary conditions are taken, while Robin boundary conditions are assumed for $D_2$. On defining

$$f_x \equiv \frac{df}{dx}, \quad f_{xx} \equiv \frac{d^2f}{dx^2},$$

one then finds the result

$$(n - 1) \left[a_{n/2}(f, D_1) - a_{n/2}(f, D_2)\right] = a_{n/2-1}(f_{xx} + 2bf_x, D_1).$$  \hspace{1cm} (3.23)

As a first step in the proof of (3.23), one takes a spectral resolution for $D_1$, say $\{\theta_\nu, \lambda_\nu\}$, where $\theta_\nu$ is the eigenfunction of $D_1$ belonging to the eigenvalue $\lambda_\nu$:

$$D_1 \theta_\nu = \lambda_\nu \theta_\nu.$$  \hspace{1cm} (3.24)
Thus, differentiation with respect to $t$ of the heat-kernel diagonal:

$$U(D_1, x, x; t) = \sum_{\nu} e^{-t\lambda_{\nu}} \theta_{\nu}^2(x), \quad (3.25)$$

yields

$$\frac{\partial}{\partial t} U(D_1, x, x; t) = - \sum_{\nu} \lambda_{\nu} e^{-t\lambda_{\nu}} \theta_{\nu}^2(x) = - \sum_{\nu} e^{-t\lambda_{\nu}} (D_1 \theta_{\nu}) \theta_{\nu}. \quad (3.26)$$

Moreover, for any $\lambda_{\nu} \neq 0$, the set

$$\left\{ \frac{A_{\theta_{\nu}}}{\sqrt{\lambda_{\nu}}} \lambda_{\nu} \right\}$$

provides a spectral resolution of $D_2$ on $\text{Ker}(D_2)\perp$, and one finds

$$\frac{\partial}{\partial t} U(D_2, x, x; t) = - \sum_{\lambda_{\nu} \neq 0} \lambda_{\nu} e^{-t\lambda_{\nu}} \theta_{\nu}^2(x)$$

$$= - \sum_{\lambda_{\nu} \neq 0} e^{-t\lambda_{\nu}} \left( \sqrt{\lambda_{\nu}} \theta_{\nu} \right) \left( \sqrt{\lambda_{\nu}} \theta_{\nu} \right)$$

$$= - \sum_{\lambda_{\nu} \neq 0} e^{-t\lambda_{\nu}} (A_{\theta_{\nu}})(A_{\theta_{\nu}}). \quad (3.27)$$

Bearing in mind that $A_{\theta_{\nu}} = 0$ if $\lambda_{\nu} = 0$, summation may be performed over all values of $\nu$, to find

$$2 \frac{\partial}{\partial t} \left[ U(D_1, x, x; t) - U(D_2, x, x; t) \right]$$

$$= 2 \sum_{\nu} e^{-t\lambda_{\nu}} \left[ \left( \theta''_{\nu} \theta_{\nu} - b' \theta_{\nu}^2 - b^2 \theta_{\nu}^2 \right) + (\theta'_{\nu} - b \theta_{\nu})(\theta'_{\nu} - b \theta_{\nu}) \right]$$

$$= 2 \sum_{\nu} e^{-t\lambda_{\nu}} \left[ \theta''_{\nu} \theta_{\nu} - b' \theta_{\nu}^2 + (\theta'_{\nu})^2 - 2b \theta'_{\nu} \theta_{\nu} \right]. \quad (3.28)$$

On the other hand, differentiation with respect to $x$ yields

$$\left( \frac{\partial}{\partial x} - 2b \right) U(D_1, x, x; t) = \sum_{\nu} e^{-t\lambda_{\nu}} \left( 2 \theta'_{\nu} \theta_{\nu} - 2b \theta_{\nu}^2 \right). \quad (3.29)$$
which implies
\[
    f \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - 2b \right) U(D_1, x, x; t)
    = 2f \frac{\partial}{\partial t} \left[ U(D_1, x, x; t) - U(D_2, x, x; t) \right].
\] (3.30)

We now integrate this formula over \( M \) and use the boundary conditions described before, jointly with integration by parts. All boundary terms are found to vanish, so that
\[
    \int_M f \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - 2b \right) U(D_1, x, x; t) dx
    = \int_M \left( \frac{\partial^2 f}{\partial x^2} + 2b \frac{\partial f}{\partial x} \right) U(D_1, x, x; t) dx
    = \int_M 2f \frac{\partial}{\partial t} \left[ U(D_1, x, x; t) - U(D_2, x, x; t) \right].
\] (3.31)

Bearing in mind the standard notation for heat-kernel traces, Eq. (3.31) may be re-expressed as
\[
    2 \frac{\partial}{\partial t} \left[ \operatorname{Tr}_{L^2} \left( f e^{-tD_1} \right) - \operatorname{Tr}_{L^2} \left( f e^{-tD_2} \right) \right]
    = \operatorname{Tr}_{L^2} \left[ (f_{xx} + 2bf_x)e^{-tD_1} \right],
\] (3.32)

which leads to Eq. (3.23) by virtue of the asymptotic expansion (2.7).

The algorithm resulting from Eq. (3.10) is sufficient to determine almost all interior terms in heat-kernel asymptotics. To appreciate this, notice that one is dealing with conformal variations which only affect the endomorphism of the operator \( P \) in (2.1). For example, on setting \( n = 2 \) in Eq. (3.10), one ends up by studying (the tilde symbol is now used for interior terms)
\[
    \tilde{a}_1(1, P) \equiv \frac{1}{6} \int_M \operatorname{Tr} \left[ \alpha_1 E + \alpha_2 R \right] = \tilde{a}_1(E, R),
\] (3.33)
which implies
\[
\tilde{a}_1(1, P(0) - \varepsilon F) = \tilde{a}_1(E, R) - \tilde{a}_1(E - \varepsilon F, R) \\
= \frac{1}{6} \int_M \text{Tr} \left[ \alpha_1(E - (E - \varepsilon F)) + \alpha_2(R - R) \right] \\
= \frac{1}{6} \int_M \text{Tr}(\alpha_1 \varepsilon F),
\]
(3.34)

and hence
\[
\left[ \frac{d}{d\varepsilon} \tilde{a}_1(1, P(0) - \varepsilon F) \right]_{\varepsilon=0} = \frac{1}{6} \int_M \text{Tr}(\alpha_1 F) = \tilde{a}_0(F, P(0)) = \int_M \text{Tr}(F).
\]
(3.35)

By comparison, Eq. (3.35) shows that
\[
\alpha_1 = 6.
\]
(3.36)

An analogous procedure leads to (see (2.21))
\[
\tilde{a}_2(1, P(0) - \varepsilon F) = \tilde{a}_2\left( E, R, \text{Ric}, \text{Riem}, \Omega \right) \\
- \tilde{a}_2\left( E - \varepsilon F, R, \text{Ric}, \text{Riem}, \Omega \right) \\
= \frac{1}{360} \int_M \text{Tr} \left[ \alpha_4 R \varepsilon F + \alpha_5(E^2 - (E - \varepsilon F)^2) \right] \\
= \frac{1}{360} \int_M \text{Tr} \left[ \alpha_4 R \varepsilon F + \alpha_5 \left( - \varepsilon^2 F^2 + 2\varepsilon EF \right) \right].
\]
(3.37)

Now one can apply Eq. (3.10) when \( n = 4 \), to find
\[
\left[ \frac{d}{d\varepsilon} \tilde{a}_2(1, P(0) - \varepsilon F) \right]_{\varepsilon=0} = \frac{1}{360} \int_M \text{Tr} \left[ \alpha_4 FR + 2\alpha_5 FE \right] \\
= \tilde{a}_1(F, P(0)) = \frac{1}{6} \int_M \text{Tr} \left[ \alpha_1 FE + \alpha_2 FR \right].
\]
(3.38)

Equating the coefficients of the invariants occurring in the equation (3.38) one finds
\[
\alpha_2 = \frac{1}{60} \alpha_4,
\]
(3.39)
\[ \alpha_5 = 30\alpha_1 = 180. \] (3.40)

Furthermore, the consideration of Eq. (3.11) when \( n = 2 \) yields

\[
\left[ \frac{d}{d\varepsilon} \tilde{a}_1 \left( e^{-2\varepsilon f} F, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon = 0} = \frac{1}{6} \int_M \left\{ \left[ \frac{d}{d\varepsilon} \text{Tr}(\alpha_1 FE) \right]_{\varepsilon = 0} + 2f\text{Tr}(\alpha_1 FE) \right. \\
+ \left[ \frac{d}{d\varepsilon} \text{Tr}(\alpha_2 FR) \right]_{\varepsilon = 0} + 2f\text{Tr}(\alpha_2 FR) \right\}. \quad (3.41)
\]

At this stage, we need the conformal-variation formulae

\[
\left[ \frac{d}{d\varepsilon} E(\varepsilon) \right]_{\varepsilon = 0} = -2fE + \frac{1}{2}(m-2) \Box f, \quad (3.42)
\]

\[
\left[ \frac{d}{d\varepsilon} R(\varepsilon) \right]_{\varepsilon = 0} = -2fR - 2(m-1) \Box f. \quad (3.43)
\]

Since we are studying the case \( m = n + 2 = 4 \), we find

\[
\left[ \frac{d}{d\varepsilon} \tilde{a}_1 \left( e^{-2\varepsilon f} F, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon = 0} = \frac{1}{6} \int_M \text{Tr} \left[ (\alpha_1 - 6\alpha_2) F \Box f \right] = 0, \quad (3.44)
\]

which implies

\[ \alpha_2 = \frac{1}{6} \alpha_1 = 1, \quad (3.45) \]

\[ \alpha_4 = 60\alpha_2 = 60. \quad (3.46) \]

After considering the Laplacian acting on functions for a product manifold \( M = M_1 \times M_2 \) (this is another application of functorial methods), the complete set of coefficients for interior terms can be determined [38]:

\[ \alpha_3 = 60, \; \alpha_6 = 12, \; \alpha_7 = 5, \; \alpha_8 = -2, \; \alpha_9 = 2, \; \alpha_{10} = 30. \quad (3.47) \]
As far as interior terms are concerned, one has to use Eqs. (3.9) and (3.11), jointly with two conformal-variation formulae which provide divergence terms that play an important role [38]:

\[
\left[ \frac{d}{d\varepsilon} a_1 \left( F, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon=0} - (m-2)a_1(fF, P(0))
\]

\[= \frac{1}{6} (m-4) \int_M \text{Tr} \left( F \Box f \right), \quad (3.48)\]

\[
\left[ \frac{d}{d\varepsilon} a_2 \left( F, e^{-2\varepsilon f} P(0) \right) \right]_{\varepsilon=0} - (m-4)a_2(fF, P(0))
\]

\[= \frac{1}{360} (m-6) \int_M F \text{Tr} \left[ 6f^{;b} a^{;b} + 10f^{;a} R 
+ 60f^{;a} E + 4f^{;b} R^{ab} \right]_{;a}. \quad (3.49)\]

Equations (3.48) and (3.49) are proved for manifolds without boundary in Lemma 4.2 of Ref. [38]. They imply that, for manifolds with boundary, the right-hand side of Eq. (3.48), evaluated for \( F = 1 \), should be added to the left-hand side of Eq. (3.9) when \( n = 2 \). Similarly, the right-hand side of Eq. (3.49), evaluated for \( F = 1 \), should be added to the left-hand side of Eq. (3.9) when \( n = 4 \). Other useful formulae involving boundary effects are [38]

\[
\int_M \left( f^{;b} R^{ab} \right)_{;a} = \int_{\partial M} \left[ f^{;ji} \left( K^{ij}_{\mid i} - (\text{tr}K)^{ij} \right) + f^{;iN} R^{iN}_{\mid iN} \right], \quad (3.50)
\]

\[
f^{;i}_{;i} = f^{;i}_{\mid i} - (\text{tr}K)f^{;iN}, \quad (3.51)\]

\[
\int_M \Box f = - \int_{\partial M} f^{;iN}, \quad (3.52)\]

\[
\int_{\partial M} \Box f = \int_{\partial M} \left[ f^{;iNN} - (\text{tr}K)f^{;iN} \right]. \quad (3.53)\]

On taking into account Eqs. (3.48)–(3.53), the application of Eq. (3.9) when \( n = 2, 3, 4 \) leads to 18 equations which are obtained by setting to zero the coefficients multiplying

\( f^{;iN} \) (when \( n = 2 \),

18
\[ f_{i:NN}, f_{i:N}(\text{tr}K), f_{i:N}S \text{ (when } n = 3), \]

and

\[ f_{i:a}^a_N, f_{i:N}E, f_{i:N}R, f_{i:N}R^{i}_{N}\iota_N, \]

\[ f_{i:NN}(\text{tr}K), f_{i:N}(\text{tr}K)^2, f_{i:N}K_{ij}K^{ij}, f_{i:(\text{tr}K)^i}^i, \]

\[ f_{i}^i K^{ij}_{\iota_j}, f_{i:\iota}^i N^i_N, f_{i:N}(\text{tr}K), f_{i:N}S^2, \]

\[ f_{i:N}S, f_{i}^{|i}_i S \text{ (when } n = 4). \]

The integrals of these 18 terms have a deep geometric nature in that they form a basis for the integral invariants. The resulting 18 equations are

\[ -b_0(m - 1) - b_1(m - 2) + \frac{1}{2} b_2(m - 2) - (m - 4) = 0, \quad (3.54) \]

\[ \frac{1}{2} c_0(m - 2) - 2c_1(m - 1) + c_2(m - 1) - c_6(m - 3) = 0, \quad (3.55) \]

\[ -\frac{1}{2} c_0(m - 2) + 2c_1(m - 1) - c_2 - 2c_3(m - 1) \]

\[ -2c_4 - c_5(m - 3) + \frac{1}{2} c_7(m - 2) = 0, \quad (3.56) \]

\[ -c_7(m - 1) + c_8(m - 2) - c_9(m - 3) = 0, \quad (3.57) \]

\[ -6(m - 6) + \frac{1}{2} d_1(m - 2) - 2d_2(m - 1) - e_7(m - 4) = 0, \quad (3.58) \]

\[ -60(m - 6) - 2d_1 - d_5(m - 1) + \frac{1}{2} d_{14}(m - 2) - e_1(m - 4) = 0, \quad (3.59) \]

\[ -10(m - 6) - 2d_2 - d_6(m - 1) + d_9 + \frac{1}{2} d_{15}(m - 2) - e_2(m - 4) = 0, \quad (3.60) \]

\[ -d_7(m - 1) - d_8 + 2d_9 - e_3(m - 4) + \frac{1}{2} d_{16}(m - 2) + 4(m - 6) = 0, \quad (3.61) \]

\[ \frac{1}{2} d_5(m - 2) - 2d_6(m - 1) + d_7(m - 1) + d_8 - e_6(m - 4) = 0, \quad (3.62) \]
\[- \frac{1}{2} d_5(m - 2) + 2d_6(m - 1) - d_7 - d_9 - 3d_{10}(m - 1) \]
\[- 2d_{11} + \frac{1}{2} d_{17}(m - 2) - e_4(m - 4) = 0, \quad (3.63)\]
\[- d_8 - d_9(m - 3) - d_{11}(m - 1) - 3d_{12} \]
\[+ \frac{1}{2} d_{18}(m - 2) - e_5(m - 4) = 0, \quad (3.64)\]
\[d_3(m - 4) - \frac{1}{2} d_5(m - 2) + 2d_6(m - 1) - d_7 - d_9 - 4(m - 6) = 0, \quad (3.65)\]
\[d_4(m - 4) - d_8 - d_9(m - 3) + 4(m - 6) = 0, \quad (3.66)\]
\[(m - 4)d_{13} = 0, \quad (3.67)\]
\[- \frac{1}{2} d_{14}(m - 2) + 2d_{15}(m - 1) - d_{16} - 2d_{17}(m - 1) \]
\[- 2d_{18} + d_{19}(m - 2) - e_8(m - 4) = 0, \quad (3.68)\]
\[-d_{19}(m - 1) + \frac{3}{2} d_{20}(m - 2) - e_9(m - 4) = 0, \quad (3.69)\]
\[\frac{1}{2} d_{14}(m - 2) - 2d_{15}(m - 1) + d_{16}(m - 1) - e_{10}(m - 4) = 0, \quad (3.70)\]
\[\frac{1}{2} d_{14}(m - 2) - 2d_{15}(m - 1) + d_{16} - d_{21}(m - 4) = 0. \quad (3.71)\]

This set of algebraic equations among universal constants holds independently of the choice of Dirichlet or Robin boundary conditions. Another set of equations which hold for either Dirichlet or Robin boundary conditions is obtained by applying Eq. (3.11) when \(n = 3, 4\). One then obtains five equations which result from setting to zero the coefficients multiplying

\[F_i N f_i N \] (when \(n = 3\)),
\[f_i N F_i NN, \quad f_i NN F_i N, \quad f_i N F_i N(\text{tr} K), \quad f_i N F_i N S \] (when \(n = 4\)).
The explicit form of these equations is [38]

\[-4c_5 - 5c_6 + \frac{3}{2}c_9 = 0, \quad (3.72)\]

\[-5c_6 - 4c_7 + 2c_{10} = 0, \quad (3.73)\]

\[2e_1 - 10e_2 + 5e_3 - 2e_7 = 0, \quad (3.74)\]

\[-2e_1 + 10e_2 - e_3 - 10e_4 - 2e_5 - 5e_6 + 6e_7 + 2e_8 = 0, \quad (3.75)\]

\[-5e_8 + 4e_9 - 5e_{10} = 0. \quad (3.76)\]

Last, one has to use the Lemma expressed by Eq. (3.23) when \(n = 2, 3, 4\), bearing in mind that

\[E_1 \equiv E(D_1) = -b_x - b^2, \quad (3.77)\]

\[E_2 \equiv E(D_2) = b_x - b^2. \quad (3.78)\]

For example, when \(n = 2\), one finds

\[a_1(f, D_1) - a_1(f, D_2) - a_0\left(f_{xx} + 2bf_x, D_1\right) = 0, \quad (3.79)\]

which implies (with Dirichlet conditions for \(D_1\) and Robin conditions for \(D_2\))

\[\int_M \left[6f(E_1 - E_2) - 6f_{xx} - 12Sf_x\right] + \int_{\partial M} \left[-b_2fS - (3 + b_1)f_N\right] = 0. \quad (3.80)\]

The integrand of the interior term in Eq. (3.80) may be re-expressed as a total divergence, and hence one gets

\[\int_{\partial M} \left[(12 - b_2)fS + (3 - b_1)f_N\right] = 0, \quad (3.81)\]

which leads to

\[b_1 = 3, \quad b_2 = 12. \quad (3.82)\]

Further details concern only the repeated application of all these algorithms, and hence we refer the reader to Branson and Gilkey [38]. We should emphasize, however, that no proof exists, so far, that functorial methods lead to the complete calculation of all heat-kernel
coefficients. For the time being one can only say that, when Dirichlet or Robin boundary conditions, or a mixture of the two (see section 4) are imposed, functorial methods have been completely successful up to the evaluation of the $a_{5/2}$ coefficient (see Ref. [27] and references therein).

4. Mixed boundary conditions

Mixed boundary conditions are found to occur naturally in the theory of fermionic fields, gauge fields and gravitation, in that some components of the field obey one set of boundary conditions, and the remaining part of the field obeys a complementary set of boundary conditions [25]. Here, we focus on some mathematical aspects of the problem. 

The framework of our investigation consists, as in section 3, of a compact Riemannian manifold, say $M$, with smooth boundary $\partial M$. Given a vector bundle $V$ over $M$, we assume that $V$ can be decomposed as the direct sum

$$V = V_n \oplus V_d,$$

near $\partial M$. The corresponding projection operators are denoted by $\Pi_n$ and $\Pi_d$, respectively. On $V_n$ one takes Neumann boundary conditions modified by some endomorphism, say $S$, of $V_n$ (see (2.16)), while Dirichlet boundary conditions hold on $V_d$. The (total) boundary operator reads therefore [12]

$$Bf \equiv \left[ (\Pi_n f) \right]_{\partial M} + S\Pi_n f \oplus \Pi_d f \right]_{\partial M} .$$

On defining

$$\psi \equiv \Pi_n - \Pi_d,$$

seven new universal constants are found to contribute to heat-kernel asymptotics for an operator of Laplace type, say $P$. In other words, the linear combination of projectors considered in Eq. (4.3) gives rise to seven new invariants in the calculation of heat-kernel coefficients up to $a_2$: one invariant contributes to $\tilde{a}_{3/2}(f,P,B)$, whereas the other six contribute to $\tilde{a}_2(f,P,B)$ (of course, the number of invariants is continuously increasing as one
considers higher-order heat-kernel coefficients). The dependence on the boundary operator is emphasized by including it explicitly into the arguments of heat-kernel coefficients. One can thus write the general formulae (cf. (2.20) and (2.21))

\[
\tilde{a}_{3/2}(f, P, B) = \frac{\delta}{96}(4\pi)^{1/2} \int_{\partial M} \text{Tr} \left[ \beta_1 f \psi_{|i} \psi^{|i} \right] + a_{3/2}(f, P, B),
\]

\[
\tilde{a}_2(f, P, B) = \frac{1}{360} \int_{\partial M} \text{Tr} \left[ \beta_2 f \psi_{|i} \Omega^i_N + \beta_3 f \psi_{|i} \psi^{|i}(\text{tr}K) + \beta_4 f \psi_{|i} K^{ij} + \beta_5 f \psi_{|i} \psi^{il}S + \beta_6 f \psi_{|i} \psi^{li} + \beta_7 f \psi_{|i} \Omega^i_N \right] + a_2(f, P, B),
\]

where \(a_{3/2}(f, P, B)\) is formally analogous to Eq. (2.20), but with some universal constants replaced by linear functions of \(\psi, \Pi_n, \Pi_d\), and similarly for \(a_2(f, P, B)\) and Eq. (2.21). The work in Refs. [12,40] has fixed the following values for the universal constants \(\{\beta_i\}\) occurring in Eqs. (4.4) and (4.5):

\[
\beta_1 = -12, \ \beta_2 = 60, \ \beta_3 = -12, \ \beta_4 = -24, \ \beta_5 = -120, \ \beta_6 = -18, \ \beta_7 = 0.
\]

To obtain this result, it is crucial to bear in mind that the correct functorial formula for the endomorphism \(S\) is

\[
\left[ \frac{d}{d\varepsilon} S(\varepsilon) \right]_{\varepsilon=0} = -fS + \frac{1}{2} (m-2)f_{;N} \Pi_n.
\]

This result was first obtained in Ref. [40], where the author pointed out that \(\Pi_n\) should be included, since the variation of \(S\) should compensate another suitable variation only on the subspace \(V_n\). The unfortunate omission of \(\Pi_n\) led to incorrect results in physical applications, which were later corrected by Moss and Poletti [41], hence confirming the analytic results in Refs. [18,21].
5. Gauge-invariant boundary conditions for the gravitational field

For gauge fields and gravitation, the boundary conditions are mixed, in that some components of the field (more precisely, a one-form or a two-form) obey a set of boundary conditions, and the remaining part of the field obeys another set of boundary conditions. Moreover, the boundary conditions are invariant under local gauge transformations provided that suitable boundary conditions are imposed on the corresponding ghost zero-form or one-form.

We are here interested in the derivation of mixed boundary conditions for Euclidean quantum gravity. The knowledge of the classical variational problem, and the principle of gauge invariance, are enough to lead to a highly non-trivial boundary-value problem. Indeed, it is by now well-known that, if one fixes the three-metric at the boundary in general relativity, the corresponding variational problem is well-posed and leads to the Einstein equations, providing the Einstein-Hilbert action is supplemented by a boundary term whose integrand is proportional to the trace of the second fundamental form [42]. In the corresponding ‘quantum’ boundary-value problem, which is relevant for the one-loop approximation in quantum gravity, the perturbations \( h_{ij} \) of the induced three-metric are set to zero at the boundary. Moreover, the whole set of metric perturbations \( h_{\mu\nu} \) are subject to the so-called infinitesimal gauge transformations

\[
\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla(\mu \varphi_\nu),
\]

where \( \nabla \) is the Levi-Civita connection of the background four-geometry with metric \( g \), and \( \varphi_\nu dx^\nu \) is the ghost one-form. In geometric language, the difference between \( \hat{h}_{\mu\nu} \) and \( h_{\mu\nu} \) is given by the Lie derivative along \( \varphi \) of the four-metric \( g \).

For problems with boundary, Eq. (5.1) implies that

\[
\hat{h}_{ij} = h_{ij} + \varphi\left(dx_i\right) + K_{ij}\varphi_0,
\]

where the stroke denotes, as usual, three-dimensional covariant differentiation tangentially with respect to the intrinsic Levi-Civita connection of the boundary, while \( K_{ij} \) is the extrinsic-curvature tensor of the boundary. Of course, \( \varphi_0 \) and \( \varphi_i \) are the normal and
tangential components of the ghost one-form, respectively. Note that boundaries make it necessary to perform a 3+1 split of space-time geometry and physical fields. As such, they introduce non-covariant elements in the analysis of problems relevant for quantum gravity. This seems to be an unavoidable feature, although the boundary conditions may be written in tensor language.

In the light of (5.2), the boundary conditions

\[
\left[h_{ij}\right]_{\partial M} = 0 \tag{5.3a}
\]

are gauge invariant, i.e.

\[
\left[h_{ij}\right]_{\partial M} = 0, \tag{5.3b}
\]

if and only if the whole ghost one-form obeys homogeneous Dirichlet conditions, so that

\[
\left[\varphi_0\right]_{\partial M} = 0, \tag{5.4}
\]

\[
\left[\varphi_i\right]_{\partial M} = 0. \tag{5.5}
\]

The conditions (5.4) and (5.5) are necessary and sufficient since \(\varphi_0\) and \(\varphi_i\) are independent, and three-dimensional covariant differentiation commutes with the operation of restriction at the boundary. Indeed, we are assuming that the boundary is smooth and not totally geodesic, i.e. \(K_{ij} \neq 0\). However, at those points of \(\partial M\) where the extrinsic-curvature tensor vanishes, the condition (5.4) is no longer necessary.

The problem now arises to impose boundary conditions on the remaining set of metric perturbations. The key point is to make sure that the invariance of such boundary conditions under the infinitesimal transformations (5.1) is again guaranteed by (5.4) and (5.5), since otherwise one would obtain incompatible sets of boundary conditions on the ghost one-form. Indeed, on using the Faddeev-Popov formalism for the amplitudes of quantum gravity, it is necessary to use a gauge-averaging term in the Euclidean action, of the form

\[
I_{g.a.} = \frac{1}{32\pi G\alpha} \int_M \Phi_\nu \Phi^\nu \sqrt{\det g} \, d^4x, \tag{5.6}
\]
where \( \Phi_\nu \) is any gauge-averaging functional which leads to self-adjoint elliptic operators on metric and ghost perturbations. One then finds that

\[
\delta \Phi_\mu(h) \equiv \Phi_\mu(h) - \Phi_\mu(\hat{h}) = \mathcal{F}_\mu^\nu \varphi_\nu,
\]

where \( \mathcal{F}_\mu^\nu \) is an elliptic operator that acts linearly on the ghost one-form. Thus, if one imposes the boundary conditions

\[
\left[ \Phi_\mu(h) \right]_{\partial M} = 0,
\]

and if one assumes that the ghost field can be expanded in a complete orthonormal set of eigenfunctions \( u_\nu^{(\lambda)} \) of \( \mathcal{F}_\mu^\nu \) which vanish at the boundary, i.e.

\[
\mathcal{F}_\mu^\nu u_\nu^{(\lambda)} = \lambda u_\mu^{(\lambda)},
\]

\[
\varphi_\nu = \sum_\lambda C_\lambda u_\nu^{(\lambda)},
\]

\[
\left[ u_\mu^{(\lambda)} \right]_{\partial M} = 0,
\]

the boundary conditions (5.8) are automatically gauge-invariant under the Dirichlet conditions (5.4) and (5.5) on the ghost.

At a deeper level, the boundary conditions (5.3)–(5.5) and (5.8) are invariant under Becchi–Rouet–Stora–Tyutin transformations [43], but their consideration has been abandoned after the proof of resulting lack of strong ellipticity in Refs. [30,31].

6. Recent developments and open problems

The boundary conditions involving tangential derivatives of the field have been studied in detail not only in Refs. [25,28,30,31] but also in Refs. [44–46]. In particular, the work in Refs. [45,46] has produced valuable results in heat-kernel asymptotics which are very important to get a geometric understanding of one-loop divergences in quantum field theory.
As far as Euclidean quantum gravity is concerned, we can see three main alternatives if it is approached from the point of view of spectral geometry (for a modern perspective on yet other issues the reader is referred to the paper by Vassilevich in this volume):

(i) If one insists on using the completely gauge-invariant boundary conditions of section 5 when a supplementary (i.e. gauge-fixing) condition of de Donder type is used [47], one should prove that, since the $L^2$ trace of the heat semigroup can be then split into a regular part and a singular part [31], a regularized $\zeta(0)$ value can be defined (cf. Ref. [48]) which is only affected by the regular part of $\text{Tr}_{L^2}(e^{-tP})$.

(ii) One can instead choose local and strongly elliptic boundary conditions for quantum gravity, which are, however, only partially gauge-invariant [49].

(iii) Last, but not least, one can resort to non-local field theory by considering a non-local gauge-fixing functional in the path integral. If one could prove that the resulting set of symbols for the gauge-field operator and boundary operator which are compatible with strong ellipticity of the boundary-value problem [32–35] is non-empty, one would understand from first principles why the universe starts in a quantum state but eventually reaches a classical regime, since non-local boundary conditions may lead to ‘surface states’ having precisely this property [36].

Moreover, it remains to be seen whether such investigations have an impact on current developments in string and brane theory [50].

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