Moment properties for two-type continuous-state branching processes in Lévy random environments

Shukai Chen and Xiangqi Zheng*
School of Mathematics and Statistics, Fujian Normal University
Fuzhou, 350007, People’s Republic of China
School of Mathematics, East China University of Science and Technology,
Shanghai, 200237, People’s Republic of China
E-mail: skchen@mail.bnu.edu.cn and zhengxq@ecust.edu.cn

Abstract. We first derive the recursions for integer moments of two-type continuous-state branching processes in Lévy random environments. We show that the $n$th moment of the process is a polynomial of the initial value of the process with at most $n$ degree. Meanwhile, the criteria for the existence of $f$-moment of the process is also established under some natural conditions.

Key words and Phrases: Moment property, continuous-state branching process, random environment.

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1 Introduction and main results

The branching model in continuous time and state called continuous state branching process (CB-process) was first introduced by Jiřina [5]. This model can be regarded as the scaling limits of classical Galton-Watson branching processes, see [3] [10] [11]. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Given $\sigma \geq 0, b \in \mathbb{R}$ constants. Let $m(dz)$ be a $\sigma$-finite measure on $[0, \infty)$ satisfying $\int_0^\infty (z \land z^2) m(dz) < \infty$. Suppose that $\{B(t)\}$ is a standard $(\mathcal{F}_t)$-Brownian motion and $\{M(ds, dz, du)\}$ is a $(\mathcal{F}_t)$-Poisson random measure with intensity $dsm(dz)du$ and $\tilde{M}(ds, dz, du) = M(ds, dz, du) - dsm(dz)du$ is the compensated measure. We assume those two noises are independent. By Theorem 3.1 in [4], there is a unique positive strong solution to

$$X(t) = X(0) + \int_0^t \sqrt{2\sigma X(s)}dB(s) + \int_0^t \int_0^{X(s-)} z \tilde{M}(ds, dz, du) - \int_0^t bX(s)ds + \int_0^t \int_1^{X(s-)} z M(ds, dz, du) \quad (1.1)$$

and the solution $\{X(t) : t \geq 0\}$ is a CB-process with branching mechanism $\phi$ satisfying a Lévy-Khintchine’s type

$$\phi(\lambda) = b\lambda + \sigma\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z 1_{\{z \leq 1\}}) m(dz), \quad \lambda \geq 0.$$
Based on [11], [6] studied the moment property of CB-processes. Indeed, suppose that \( f \) is a positive continuous function on \([0, \infty)\) satisfying

**Condition A.** There exist constants \( c \geq 0 \) and \( K > 0 \) such that

(A1) \( f \) is convex on \([c, \infty)\);

(A2) \( f(xy) \leq K f(x)f(y) \) for all \( x, y \in [c, \infty) \).

If \( \{X(t) : t \geq 0\} \) is the strong solution of equation (1.1) with \( P(X(0) > 0) > 0 \), then for any \( t > 0 \), the equivalent condition of \( Ef(X(t)) < \infty \) is \( Ef(X(0)) < \infty \) and \( \int_1^\infty f(z)m(dz) < \infty \). Later, [7] generalised the result to continuous-state branching processes in Lévy random environments (CBRE-processes). A CBRE-process \( \{Y(t) : t \geq 0\} \) can be seen as the unique strong solution of

\[
Y(t) = Y(0) - \int_0^t bY(s)ds + \int_0^t \sqrt{2\sigma Y(s)}dB(s) + \int_0^t \int_1^\infty \int_0^1 Y(s-)zM(ds, dz, du)
+ \int_0^t \int_0^1 \int_0^\infty z\tilde{M}(ds, dz, du) + \int_0^t Y(s-)dL(s),
\]

(1.2)

where \( b, \sigma, B, M \) are the same as that in (1.1), \( \{L(t) : t \geq 0\} \) is a \((\mathcal{F}_t)\)-Lévy process defined by

\[
L(t) = \beta t + \sigma_1 B(t) + \int_0^t \int_{\mathbb{D}_1} (e^z - 1)\tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{D}_1^c} (e^z - 1)N(ds, dz),
\]

(1.3)

where \( \mathbb{D}_1 = [-1, 1], \beta \in \mathbb{R}, \sigma_1 \geq 0, \{B(t)\} \) is a standard \((\mathcal{F}_t)\)-Brownian motion and \( \{N(ds, dz)\} \) is a \((\mathcal{F}_t)\)-Poisson random measure on \((0, \infty) \times \mathbb{R} \) with intensity \( ds\nu(dz) \), \( \nu \) is a \( \sigma \)-finite measure satisfying \( \int_0^\infty (1 \wedge z^2)\nu(dz) < \infty \). In particular, when \( L(t) \equiv 0 \) for all \( t \geq 0 \), the CBRE-process reduces to a CB-process with branching mechanism \( \phi \). An associated Lévy process \( \{\xi(t) : t \geq 0\} \) is defined by

\[
\xi(t) = at + \sigma_1 B(t) + \int_0^t \int_{\mathbb{D}_1} z\tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{D}_1^c} zN(ds, dz),
\]

(1.4)

where \( a = \beta - \frac{1}{2}\sigma_1^2 - \int_{\mathbb{D}_1} (e^z - 1 - z)\nu(dz) \). Clearly, the two processes \( \{\xi(t) : t \geq 0\} \) and \( \{L(t) : t \geq 0\} \) generate the same filtration. We refer to [4, 14] for more details of CBRE-processes. Under the hypothesis \( P(Y(0) > 0) > 0 \), [7] showed that \( Ef(Y(t)) < \infty \) for any \( t > 0 \) if and only if \( Ef(Y(0)) < \infty \), \( \int_1^\infty f(z)m(dz) < \infty \) and \( \int_1^\infty f(e^z)\nu(dz) < \infty \), here the function \( f \) satisfies **Condition A**. A recursion formula of the \( n \)-moment of such processes is also established in [7].

A two-type continuous-state branching process in Lévy random environment (two-type CBRE-process) with a slightly stronger moment condition on the branching mechanism was constructed by [13], where the environment process is still defined by (1.3) or (1.4), and the branching mechanism \( \phi = (\phi_1, \phi_2) \) is a function from \( \mathbb{R}_+^2 \) to itself with the following representations,

\[
\phi_1(\lambda) = b_{11}\lambda_1 + b_{12}\lambda_2 + c_1\lambda_1^2 + \int_{\mathbb{R}_+^2} (e^{-\langle \Lambda, z \rangle} - 1 + \lambda_1 z_1)m_1(dz),
\]

\[
\phi_2(\lambda) = b_{21}\lambda_1 + b_{22}\lambda_2 + c_2\lambda_2^2 + \int_{\mathbb{R}_+^2} (e^{-\langle \Lambda, z \rangle} - 1 + \lambda_2 z_2)m_2(dz).
\]
Here, \((b_{ij})\) is a \((2 \times 2)\)-matrix with \(b_{12}, b_{21} \leq 0, c_1, c_2 \geq 0, m_1, m_2\) are \(\sigma\)-finite measures on \(\mathbb{R}^2_+\) supported by \(\mathbb{R}^2_+ \setminus \{0\}\), satisfying

\[
\int_{\mathbb{R}^2_+ \setminus \{0\}} (z_1 \wedge z_1^2 + z_2) m_1(dz) + \int_{\mathbb{R}^2_+ \setminus \{0\}} (z_2 \wedge z_2^2 + z_1) m_2(dz) < \infty.
\]

A two-dimensional Markov process \(\{X(t) = (X_1(t), X_2(t)) : t \geq 0\}\) is a CBRE-process if its transition semigroup \(\{P_t : t \geq 0\}\) is determined by

\[
\int_{\mathbb{R}^2_+} e^{-(\lambda, y)} P_t(x, dy) = E \left[ \exp \left\{ -\langle x, \phi_{0,t}(\xi, \lambda) \rangle \right\} \right],
\]

where \(r \mapsto \phi_{r,t}(\xi, \lambda) = (\phi_{r,t}^{(1)}(\xi, \lambda), \phi_{r,t}^{(2)}(\xi, \lambda))\) is the unique solution to

\[
\phi_{r,t}^{(i)}(\xi, \lambda) = e^{\xi(t) - \xi(r)} \lambda_i - \int_r^t e^{\xi(s) - \xi(r)} \phi_i(\phi_{s,t}(\xi, \lambda)) \, ds, \quad i = 1, 2, \quad r \leq t \in \mathbb{R}
\]

and we take \(r = 0\) in (1.3).

Let \(\{B_1(t)\}\) and \(\{B_2(t)\}\) be two standard \((\mathcal{F}_t)\)-Brownian motions, \(\{M_1(ds, du, dz)\}\) and \(\{M_2(ds, du, dz)\}\) be two \((\mathcal{F}_t)\)-Poisson random measures on \((0, \infty)^4\) with characteristic measures \(m_1\) and \(m_2\), respectively, \(\{M_1(ds, du, dz)\}\) and \(\{M_2(ds, du, dz)\}\) are associated compensated measures. We assume those random elements are mutually independent. It was proved in [13] that a two-type CBRE-process with branching mechanism \(\phi = (\phi_1, \phi_2)\) and environment \(\{\xi(t) : t \geq 0\}\) or \(\{L(t) : t \geq 0\}\) can also be seen as the unique non-negative strong solution to

\[
X_1(t) = X_1(0) - \int_0^t \left( b_{11}X_1(s) + b_{21}X_2(s) \right) \, ds + \int_0^t \sqrt{2c_1X_1(s)} \, dB_1(s) \\
+ \int_0^t \int_0^s \int_{\mathbb{R}^2_+ \setminus \{0\}} z_1 M_1(ds, du, dz) + \int_0^t X_1(s-) \, dL(s) \\
+ \int_0^t \int_0^s \int_{\mathbb{R}^2_+ \setminus \{0\}} z_1 M_2(ds, du, dz), \quad (1.6)
\]

\[
X_2(t) = X_2(0) - \int_0^t \left( b_{12}X_1(s) + b_{22}X_2(s) \right) \, ds + \int_0^t \sqrt{2c_2X_2(s)} \, dB_2(s) \\
+ \int_0^t \int_0^s \int_{\mathbb{R}^2_+ \setminus \{0\}} z_2 M_1(ds, du, dz) + \int_0^t X_2(s-) \, dL(s) \\
+ \int_0^t \int_0^s \int_{\mathbb{R}^2_+ \setminus \{0\}} z_2 M_2(ds, du, dz). \quad (1.7)
\]

In particular, when \(L(t) \equiv 0\) for all \(t \geq 0\), the two-type CBRE-process reduces to a two-type CB-process with branching mechanism \(\phi\). One can refer to [2], [12], [13] for more details on the multi-type continuous-state branching processes. Moreover, using the tool of stochastic differential equations, Barczy, Li and Pap [3] calculate the \(n\)-moment of multi-type continuous-state branching process. Our result is an extension of [3]. Let \(\{X(t) = (X_1(t), X_2(t)) : t \geq 0\}\) be the strong solution of the stochastic equation system (1.6)–(1.7). We calculate the integer moment recursions of \(\{X(t) = (X_1(t), X_2(t)) : t \geq 0\}\) and the equivalent condition for the existence of \(f\)-moment.
Theorem 1.1. (n-moment)

Suppose that there exists an integer \( n \geq 2 \) such that
\[
E\|X(0)\|^n < \infty, \quad \int_{\mathbb{R}_+^2 \setminus \{0\}} \|z\|^n(m_1 + m_2)(dz) < \infty, \quad \int_1^\infty e^nz(dz) < \infty.
\]
Then,
\[
E[X_1(t)]^n = E[X_1(0)]^n e^{n\bar{\xi}_1(t)} + \sum_{j=0}^{n-2} A_{n,j}^1 \int_0^t E\xi_1(s-j)^{j+1}Ee^{n\xi_1(t)-n\bar{\xi}_1(s)}ds
\]
\[
+ \sum_{j=0}^{n-1} B_{n,j}^1 \int_0^t E\xi_1(s-j)X_2(s-E)Ee^{n\xi_1(t)-n\bar{\xi}_1(s)}ds
\]
and
\[
E[X_2(t)]^n = E[X_2(0)]^n e^{n\bar{\xi}_2(t)} + \sum_{j=0}^{n-2} A_{n,j}^2 \int_0^t E\xi_2(s-j)^{j+1}Ee^{n\xi_2(t)-n\bar{\xi}_2(s)}ds
\]
\[
+ \sum_{j=0}^{n-1} B_{n,j}^2 \int_0^t E\xi_2(s-j)X_1(s-E)Ee^{n\xi_2(t)-n\bar{\xi}_2(s)}ds,
\]
where \( \bar{\xi}_1(t) = \xi(t) - b_1 t, \ \bar{\xi}_2(t) = \xi(t) - b_2 t, \)
\[
A_{n,j}^1 = \binom{n}{j} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^{n-j}m_1(dz), \quad 0 \leq j < n-2,
\]
\[
A_{n,j}^2 = \binom{n}{n-2} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^{n-j}m_1(dz) + c_1 n(n-1), \quad j = n-2,
\]
\[
B_{n,j}^1 = \binom{n}{j} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^{n-j}m_2(dz), \quad j < n-1,
\]
\[
B_{n,j}^2 = \binom{n}{n-1} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^{n-j}m_2(dz) - b_{21} n, \quad j = n-1,
\]
\[
A_{n,j}^2 = \binom{n}{j} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_2^{n-j}m_2(dz), \quad j < n-2,
\]
\[
A_{n,j}^2 = \binom{n}{n-2} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_2^{n-j}m_2(dz) + c_2 n(n-1), \quad j = n-2,
\]
\[
B_{n,j}^2 = \binom{n}{j} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_2^{n-j}m_1(dz), \quad j < n-1,
\]
\[
B_{n,j}^2 = \binom{n}{n-1} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_2^{n-j}m_1(dz) - b_{12} n, \quad j = n-1.
\]
Moreover, for each \( t \geq 0, k = 1, 2, \cdots, n \) and \( i = 1, 2 \), there exists a polynomial function \( Q_{i,t,k} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) having degree at most \( k \) such that,
\[
E[X_i(t)]^k = E[Q_{i,t,k}(X(0))].
\]
Theorem 1.2. (\(f\)-moment)

Suppose that \(\mathbf{P}(\|\mathbf{X}(0)\| > 0) > 0\). Suppose further that \(f\) satisfies Condition A. Then for any \(t > 0\), \(\mathbf{E}f(\|\mathbf{X}(t)\|) < \infty\) if and only if

\[
\mathbf{E}f(\|\mathbf{X}(0)\|) < \infty, \quad \int_{\{\|z\| \geq 1\}} f(\|z\|)(m_1 + m_2)(dz) < \infty, \quad \int_1^{\infty} f(\varepsilon^2)\nu(dz) < \infty.
\]

The proofs will be given in the next two sections.

**Notation.** For any \(\{x(t) : t \geq 0\}\) taking values in \(\mathcal{S}\), the space of two-dimensional functions with cádlág paths, write \(x(t) = (x_1(t), x_2(t))\). For any \(x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2_+,\) write \(y \geq x\) if \(y_1 \geq x_1\) and \(y_2 \geq x_2\). Throughout this paper, we make the conventions

\[
\int_{a}^{b} = \int_{[a, b]} \quad \text{and} \quad \int_{a}^{\infty} = \int_{(a, \infty)}
\]

for any \(b \geq a \geq 0\). Given a function \(f\) defined on a subset of \(\mathbb{R}\), we write

\[
\Delta_z f(x) = f(x + z) - f(x) \quad \text{and} \quad D_z f(x) = \Delta_z f(x) - f'(x)z
\]

for \(x, z \in \mathbb{R}\) if the right-hand side is meaningful.

## 2 \(n\)-moment of two-type CBRE processes

In this section, we derive the recursions of integer moments of the process with the help of truncated processes \(\{\mathbf{X}^{(k)}(t) : t \geq 0\}\) \((k = 1, 2, \cdots)\) defined as the unique non-negative strong solution of the following equation system

\[
X_1^{(k)}(t) = X_1^{(k)}(0) + \int_0^t \sqrt{2c_1 X_1^{(k)}(s)} dB_1(s) - \int_0^t \left(b_{11} X_1^{(k)}(s) + b_{21} X_2^{(k)}(s)\right) ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^2_+ \setminus \{0\}} z_1 1_{\{\|z\| \leq k\}} M_1(ds, du, dz) + \int_0^t X_1^{(k)}(s-)dL^{(k)}(s)
\]

\[
+ \int_0^t \int_{\mathbb{R}^2_+ \setminus \{0\}} z_1 1_{\{\|z\| \leq k\}} M_2(ds, du, dz),
\]

\[
X_2^{(k)}(t) = X_2^{(k)}(0) + \int_0^t \sqrt{2c_2 X_2^{(k)}(s)} dB_2(s) - \int_0^t \left(b_{12} X_1^{(k)}(s) + b_{22} X_2^{(k)}(s)\right) ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^2_+ \setminus \{0\}} z_2 1_{\{\|z\| \leq k\}} M_1(ds, du, dz) + \int_0^t X_2^{(k)}(s-)dL^{(k)}(s)
\]

\[
+ \int_0^t \int_{\mathbb{R}^2_+ \setminus \{0\}} z_2 1_{\{\|z\| \leq k\}} M_2(ds, du, dz),
\]

where \(\{L^{(k)}(t) : t \geq 0\}\) is a Lévy processes with the following Lévy-Itô decomposition,

\[
L^{(k)}(t) = \beta t + \sigma_1 W(t) + \int_0^t \int_{\mathbb{D}_1} (\varepsilon^2 - 1) \tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{D}_1} (\varepsilon^2 1_{\{z \leq k\}} - 1) N(ds, dz).
\]
The associated Lévy process \( \{ \xi^{(k)}(t) : t \geq 0 \} \) is defined by

\[
\xi^{(k)}(t) = at + \sigma_1 W(t) + \int_0^t \int_{D_1} z \tilde{N}(ds, dz) + \int_0^t \int_{D_i^c} z 1_{\{z \leq k\}} N(ds, dz).
\]

As \( k \to \infty \), the truncated sequence \( \{X^{(k)}(t) : t \geq 0\}_{k \geq 1} \) converges increasingly to \( \{X(t) : t \geq 0\} \). And we will prove it through the following two propositions.

**Proposition 2.1.** For all \( n_2 \geq n_1 > 1 \),

\[
P\left( X^{(n_1)}(t) \leq X^{(n_2)}(t) \text{ for all } t \geq 0 \right) = 1.
\]

**Proof.** Fix \( n_1, n_2 \) with \( n_2 \geq n_1 > 1 \). Define \( \zeta(t) = X^{(n_1)}(t) - X^{(n_2)}(t) \).

\[
\zeta_1(t) \leq \zeta_1(0) + \int_0^t \left( \sqrt{2c_1 X_1^{(n_1)}(s)} - \sqrt{2c_1 X_1^{(n_2)}(s)} \right) dB_1(s)
\]

\[
+ \int_0^t [-b_{11} \zeta_1(s) - b_{21} \zeta_2(s)] ds
\]

\[
+ \int_0^t \int_{\mathbb{R}_2^+ \backslash \{0\}} \int_{\mathbb{R}_2^+ \{0\}} \int_{\mathbb{R}_2^+ \{0\}} z_1 1_{\{\|z\| \leq n_2\}} M_1(ds, du, dz)
\]

\[
+ \int_0^t \int_{\mathbb{R}_2^+ \{0\}} \int_{\mathbb{R}_2^+ \{0\}} z_1 1_{\{\|z\| \leq n_2\}} M_2(ds, du, dz)
\]

\[
+ \int_0^t \zeta_1(s-) dL^{(n_2)}(s).
\]

For \( m \geq 1 \), define \( \tau_m := \inf\{t \geq 0 : X_1^{(n_1)}(t) \lor X_1^{(n_2)}(t) \lor X_2^{(n_1)}(t) \lor X_2^{(n_2)}(t) \geq m\} \). For \( t \geq 0 \), choose a decreasing sequence \( \{a_k\} \) such that \( a_0 = 1 \), \( a_k \to 0 \), \( \int_{a_k}^{a_{k-1}} z^{-1} dz = k \) for \( k \geq 1 \). Let \( x \mapsto \psi_k(x) \) be non-negative functions on \( \mathbb{R} \) supported by \( (a_k, a_{k-1}) \), and satisfying \( \int_{a_k}^{a_{k-1}} \psi_k(x) dx = 1 \), \( 0 \leq \psi_k(x) \leq 2 \) for \( k \geq 1 \), define a twice-differentiable non-negative function

\[
\varphi_k(z) = \int_0^z dy \int_0^y \psi_k(x) dx, \quad z \in \mathbb{R}.
\]

By Itô’s formula,

\[
\varphi_k[\zeta_1(t \land \tau_m)]
\]

\[
\leq \varphi_k[\zeta_1(0)] + \int_0^{t \land \tau_m} \varphi_k'[\zeta_1(s)] [-b_{11} \zeta_1(s) - b_{21} \zeta_2(s)] ds
\]

\[
+ \int_0^{t \land \tau_m} \varphi_k''[\zeta_1(s)] c_1(\sqrt{X_1^{(n_1)}(s)} - \sqrt{X_1^{(n_2)}(s)})^2 ds + \frac{1}{2} \int_0^{t \land \tau_m} \varphi_k'(\zeta_1(s)) \sigma^2 \zeta_1^2(s) ds
\]

\[
+ \int_0^{t \land \tau_m} \int_{\mathbb{R}_2^+ \backslash \{0\}} \Delta_{z_1} \varphi_k(\zeta_1(s-)) \zeta_2(s-) 1_{\{\|z\| \leq n_2\}} 1_{\{\zeta_2(s-) > 0\}} m_2(dz) ds
\]

\[
- \int_0^{t \land \tau_m} \int_{\mathbb{R}_2^+ \backslash \{0\}} \Delta(-z_1) \varphi_k(\zeta_1(s-)) \zeta_2(s-) 1_{\{\|z\| \leq n_2\}} 1_{\{\zeta_2(s-) \leq 0\}} m_2(dz) ds
\]
By Taylor's expansion, when \( \zeta \):

\[
\int_0^{t \wedge \tau_m} \int_{\mathbb{R}_2^+ \setminus \{0\}} D_{z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2 \}} \mathbf{1}_{\{\zeta_1(s-) > 0 \}} m_1(dz) ds
\]

\[
- \int_0^{t \wedge \tau_m} \int_{\mathbb{R}_2^+ \setminus \{0\}} D_{-z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2 \}} \mathbf{1}_{\{\zeta_1(s-) \leq 0 \}} m_1(dz) ds
\]

\[
+ \int_0^{t \wedge \tau_m} \mathbb{E}_{1} D_{\zeta_1(s-)(e^z-1)} \varphi_k(\zeta_1(s-)) \nu(dz) ds
\]

\[
+ \int_0^{t \wedge \tau_m} \mathbb{E}_{1} \Delta_{\zeta_1(s-)(e^z-1)} \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2 \}} \nu(dz) ds + \text{mart.}
\]

It is not hard to see that, as \( k \to \infty \),

\[
\varphi''(\zeta_1(s))(\zeta_1(s))^2 \to 0,
\]

\[
\varphi''(\zeta_1(s)) \left( \sqrt{X_1^{(n_1)}(s)} - \sqrt{X_1^{(n_2)}(s)} \right)^2 \to 0.
\]

Moreover,

\[
0 \leq \Delta_{z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{\zeta_2(s-) \geq 0 \}} \leq z_1 \zeta_2(s-) +.
\]

Since \( z \mapsto \varphi(z) \) is nondecreasing,

\[
\Delta_{-z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{\zeta_2(s-) \leq 0 \}} \geq 0.
\]

By Taylor's expansion, when \( \zeta_1(s-) > 0 \),

\[
D_{z_1} \varphi_k(\zeta_1(s-)) = z_1^2 \int_0^1 \varphi''(\zeta_1(s-)) + tz_1)(1-t) dt
\]

\[
= z_1^2 \int_0^1 \psi_k(\zeta_1(s-)) + tz_1)(1-t) dt
\]

\[
\leq z_1^2 \int_0^1 \frac{2}{k(\zeta_1(s-)) + tz_1)(1-t) dt
\]

\[
\leq \frac{z_1^2}{k \zeta_1(s-)}.
\]

Meanwhile,

\[
D_{z_1} \varphi_k(\zeta_1(s-)) \leq \varphi_k(\zeta_1(s-)) + z_1) - \varphi_k(\zeta_1(s-)) \leq z_1.
\]

Thus,

\[
\int_{\mathbb{R}_2^+ \setminus \{0\}} D_{z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2 \}} \mathbf{1}_{\{\zeta_1(s-) > 0 \}} m_1(dz)
\]

\[
= \int_{\mathbb{R}_2^+ \setminus \{0\}} D_{z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2, z_1 > 1 \}} \mathbf{1}_{\{\zeta_1(s-) > 0 \}} m_1(dz)
\]

\[
+ \int_{\mathbb{R}_2^+ \setminus \{0\}} D_{z_1} \varphi_k(\zeta_1(s-)) \varphi_k(\zeta_1(s-)) \mathbf{1}_{\{||z|| \leq n_2, z_1 \leq 1 \}} \mathbf{1}_{\{\zeta_1(s-) > 0 \}} m_1(dz)
\]

\[
\leq \int_{\mathbb{R}_2^+ \setminus \{0\}} z_1 \zeta_1(s-) \mathbf{1}_{\{||z|| \leq n_2 \}} \mathbf{1}_{\{\zeta_1(s-) > 0 \}} \mathbf{1}_{\{z_1 > 1 \}} m_1(dz)
\]
Hence,

\[\int_{\mathbb{R}_+^2 \setminus \{0\}} \frac{z_1^2}{k\zeta_1(s^-)^2} \zeta_1(s^-) 1_{\{\|z\| \leq n_2\}} 1_{\{\zeta_1(s^-) > 0\}} 1_{\{z_1 \leq 1\}} m_1(dz)\]

\[= \zeta_1(s^-)^+ \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 1_{\{\|z\| \leq n_2, z_1 > 1\}} m_1(dz) + \frac{1}{k} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^2 1_{\{\|z\| \leq n_2, z_1 \leq 1\}} m_1(dz).\]

Similarly, when \(z \in \mathbb{D}_1\), \(\zeta_1(s^-) > 0\),

\[D_{\zeta_1(s^-)(e^z - 1)} \varphi_k(\zeta_1(s^-))\]

\[= \zeta_1^2(s^-)(e^z - 1)^2 \int_0^1 \psi_k[\zeta_1(s^-)(t(e^z - 1) + 1)](1 - t) dt\]

\[\leq \zeta_1^2(s^-)(e^z - 1)^2 \int_0^1 \frac{2(1 - t)}{\zeta_1(s^-)[t(e^z - 1) + 1]} dt\]

\[\leq \zeta_1(s^-)(e^z - 1)^2.\]

Hence,

\[\int_0^{t \wedge \tau_m} \int_{\mathbb{D}_1} D_{\zeta_1(s^-)(e^z - 1)} \varphi_k(\zeta_1(s^-)) \nu(dz) ds\]

\[\leq \int_0^{t \wedge \tau_m} \int_{\mathbb{D}_1} \zeta_1(s^-)(e^z - 1)^2 \nu(dz) ds.\]

Then,

\[\varphi_k[\zeta_1(t \wedge \tau_m)] \leq \varphi_k(\zeta_1(0)) + \int_0^{t \wedge \tau_m} \varphi_k'[\zeta_1(s)][-b_{11}\zeta_1(s) - b_{21}\zeta_2(s)] ds\]

\[+ \int_0^{t \wedge \tau_m} \varphi_k''(\zeta_1(s)) c_1(\sqrt{X_1^{(n_1)}(s)} - \sqrt{X_1^{(n_2)}(s)})^2 ds\]

\[+ \frac{1}{2} \int_0^{t \wedge \tau_m} \varphi_k''(\zeta_1(s)) \sigma^2 \zeta_1^2(s) ds\]

\[+ \int_0^{t \wedge \tau_m} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 \zeta_2(s^-) 1_{\{\|z\| \leq n_2\}} m_2(dz) ds\]

\[+ \int_0^{t \wedge \tau_m} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 \zeta_1(s^-) 1_{\{\|z\| \leq n_2\}} 1_{\{z_1 > 1\}} m_1(dz) ds\]

\[+ \frac{1}{k} \int_0^{t \wedge \tau_m} \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1^2 1_{\{\|z\| \leq n_2\}} 1_{\{z_1 \leq 1\}} m_1(dz) ds\]

\[+ \int_0^{t \wedge \tau_m} \int_{\mathbb{D}_1} \zeta_1(s^-)(e^z - 1) 1_{\{z \leq n_2\}} \nu(dz) ds\]

\[+ \int_0^{t \wedge \tau_m} \int_{\mathbb{D}_1} \zeta_1(s^-)(e^z - 1)^2 \nu(dz) ds + mart.\]

Taking expectations on both sides, and letting \(k \to \infty\), we can find constant \(C_1\) large enough such that,

\[E[\zeta_1(t \wedge \tau_m)^+] \leq C_1 \int_0^t E[\zeta_1(s \wedge \tau_m)^+] + \zeta_2(s \wedge \tau_m)^+] ds.\]
Symmetrically, there exists $C_2$ such that,

$$
E[\zeta_2(t \wedge \tau_m)^+] \leq C_2 \int_0^t E[\zeta_1(s \wedge \tau_m)^+ + \zeta_2(s \wedge \tau_m)^+]ds.
$$

Let $C = C_1 + C_2$,

$$
E[\zeta_1(t \wedge \tau_m)^+ + \zeta_2(t \wedge \tau_m)^+] \leq C \int_0^t E[\zeta_1(s \wedge \tau_m)^+ + \zeta_2(s \wedge \tau_m)^+]ds.
$$

By Gronwall’s inequality,

$$
E[\zeta_1(t \wedge \tau_m)^+ + \zeta_2(t \wedge \tau_m)^+] = 0, \forall t \geq 0.
$$

Since $\{X^{(n_1)}(t) : t \geq 0\}$ and $\{X^{(n_2)}(t) : t \geq 0\}$ are càdlàg, $\tau_m \to \infty$, as $m \to \infty$. Thus $P\{X^{(n_1)}(t) \leq X^{(n_2)}(t), \forall t \geq 0\} = 1$. 

**Proposition 2.2.** $E(||X(t) - X^{(k)}(t)||) \to 0$ and $X^{(k)}(t) \uparrow X(t)$ $P$-a.s. as $k \to \infty$ for all $t \geq 0$. Moreover, If $E||X(t)|| < \infty$, then $\int_1^\infty e^{z\nu}(dz) < \infty$.

**Proof.** Define $\xi^{(k)}(t) := \xi(k)(t) - b_i t$, $i = 1, 2$. Applying Itô’s formula to $e^{\xi^{(k)}(t)} X^{(k)}(t)e^{-\xi^{(k)}(t)}$ and $e^{\xi^{(k)}(t)} X^{(k)}(t)e^{-\xi^{(k)}(t)}$, independently, where $e_1 = (1, 0)$, one can see that

$$
X^{(k)}_1(t) = X_1(0)e^{\xi^{(k)}(t)} + \int_0^t e^{\xi^{(k)}(t) - \xi^{(k)}(s)} \sqrt{2c_1 X^{(k)}_1(s)} dB_1(s)
$$

and

$$
X_1(t) = X_1(0)e^{\xi^{(k)}(t)} + \int_0^t e^{\xi^{(k)}(t) - \xi^{(k)}(s)} \sqrt{2c_1 X_1(s)} dB_1(s)
$$

Hence,

$$
X_1(t) \leq X^{(k)}_1(t) + \int_0^t e^{\xi^{(k)}(t) - \xi^{(k)}(s)} \sqrt{2c_1 [X_1(s) - X^{(k)}_1(s)]} dB_1(s)
$$
where

\[ \text{Moment properties for two-type CBRE-processes} \]

\[ \int_0^t b_2 [X_2(s) - X_2^{(k)}(s)] e^{\xi_2^{(k)}(s)} \xi_2^{(k)}(s) \, ds \]

\[ + \int_0^t \int_{X_1^{(k)}(s)} X_1^{(k)}(s) \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 e^{\xi_1^{(k)}(t)} - \xi_1^{(k)}(s) M_1(ds, du, dz) \]

\[ + \int_0^t \int_{X_2^{(k)}(s)} X_2^{(k)}(s) \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 e^{\xi_1^{(k)}(t)} - \xi_1^{(k)}(s) M_2(ds, du, dz) \]

\[ + \int_0^t \int_{\mathbb{R}_+} X_1(s) e^{\xi_1^{(k)}(t)} - \xi_1^{(k)}(s) (e^{\xi_1^{(k)}(|s| > k)} - 1) N(ds, dz). \]

Notice that \( t \mapsto E e^{m \xi_1^{(k)}(t)} \) is locally bounded for every \( m \geq 0 \). If \( E \|X(t)\| < \infty \), then

\[ \left\{ \int_0^t e^{-\xi_1^{(k)}(s)} \sqrt{2c_1[X_1(s) - X_1^{(k)}(s)]} dB_1(s) : t \geq 0 \right\} \]

and

\[ \left\{ \int_0^t X_1^{(k)}(s) \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 e^{-\xi_1^{(k)}(s)} M_1(ds, du, dz) : t \geq 0 \right\} \]

are martingales w.r.t. the filtration \( \mathcal{F}_t \). Hence

\[ E \int_0^t e^{\xi_1^{(k)}(t) - \xi_1^{(k)}(s)} \sqrt{2c_1[X_1(s) - X_1^{(k)}(s)]} dB_1(s) \]

\[ = E \left[ e^{\xi_1^{(k)}(t)} \right] E^\xi \left( \left\{ \int_0^t e^{-\xi_1^{(k)}(s)} \sqrt{2c_1[X_1(s) - X_1^{(k)}(s)]} dB_1(s) \right\} \right) \]

\[ = 0, \]

where \( E^\xi \) is the quenched law given \( \{\xi(t) : t \geq 0\} \) or \( \{L(t) : t \geq 0\} \). Similarly,

\[ E \int_0^t X_1^{(k)}(s) \int_{\mathbb{R}_+^2 \setminus \{0\}} z_1 e^{-\xi_1^{(k)}(s)} M_1(ds, du, dz) = 0. \]

Taking expectations on both sides,

\[ E[X_1(t) - X_1^{(k)}(t)] \]

\[ \leq \int_0^t \left[ \int_{\mathbb{R}_+^2} z_1 m_2(dz) - b_2 \right] E[X_2(s) - X_2^{(k)}(s)] E e^{\xi_2^{(k)}(t) - \xi_2^{(k)}(s)} ds \]

\[ + \int_{\mathbb{R}} (e^{\xi_1^{(k)}(|s| > k)} - 1) \nu(dz) \int_0^t E X_1(s) e^{\xi_1^{(k)}(t) - \xi_1^{(k)}(s)} ds. \]

Symmetrically,

\[ E[X_2(t) - X_2^{(k)}(t)] \]

\[ \leq \int_0^t \left[ \int_{\mathbb{R}_+^2} z_2 m_1(dz) - b_1 \right] E[X_1(s) - X_1^{(k)}(s)] E e^{\xi_1^{(k)}(t) - \xi_1^{(k)}(s)} ds \]
\[ + \int_{\mathbb{R}} (e^{\xi_1(z)} - 1) \nu(dz) \int_{0}^{t} \mathbb{E}X_2(s^-)\mathbb{E}e^{\xi_1(t) - \xi_1^*(t)}Y_1(dz). \]

Since \( E\|X(t)\| < \infty \), \( \int_{\mathbb{R}} (e^{\xi_1(z)} - 1) \nu(dz) < \infty \). Thus,

\[ \int_{\mathbb{R}} (e^{\xi^*_{1}(z)}) - 1) \nu(dz) = \int_{\mathbb{R}} (e^{\xi^*} - 1) 1_{1(|z| > k)} \nu(dz) \]

tends to 0 as \( k \to \infty \). Then by taking limits of \( k \to \infty \) and Gronwall’s inequality, we get the conclusion. \( \square \)

**Proof of Theorem 1.1:**

**Step 1.** Denote \( \tilde{\beta} = a + \frac{1}{2} \sigma_1^2 + \int_{D_1} (e^z - 1 - z) \nu(dz) + \int_{D_1^*} (e^z - 1) \nu(dz), \)** By Lemma 3.2 in \([7]\),

\[ Ee^{\xi(t)} = e^{\tilde{\beta} t}, \forall t \geq 0. \]

According to the formula (40) in \([15]\),

\[ E^F[\exp\{-\langle \lambda, X(t) \rangle \}|\mathcal{F}_t] = \exp\{-\langle \lambda, X(r), \nu_r(t)(\xi, \lambda) \rangle \}, \quad t \geq r \geq 0. \]

Taking derivatives with respect to \( \lambda = 0^+ \) on both sides, we have,

\[ E^\xi(X(t)|\mathcal{F}_r) = e^{\xi(t) - \xi(r)}e^{(r-t)b^\top}X(r). \]

Hence for any \( \mathcal{F}_r \)-measurable function \( F \),

\[ E\left(Fe^{-\tilde{\beta} t}e^{b^\top}X(t)\right) = E\left(Fe^{-\tilde{\beta} t}e^{b^\top}E^\xi(X(t)|\mathcal{F}_r)\right) \]
\[ = e^{-\tilde{\beta} t}e^{b^\top}E\left(Fe^{(r-t)b^\top}X(r)\right) \]
\[ = e^{-\tilde{\beta} t}e^{b^\top}e^{\tilde{\beta} - \tilde{\beta} t}E\left(Fe^{(r-t)b^\top}X(r)\right) \]
\[ = E\left(Fe^{-\tilde{\beta} r}e^{b^\top}X(r)\right), \]

where \( b \) is a 2 \times 2 matrix with \( \bar{b}_{11} = b_{11}, \bar{b}_{22} = b_{22}, \bar{b}_{12} = b_{12} - \int_{\mathbb{R}^2_+} z_2 m_1(dz), \bar{b}_{21} = b_{21} - \int_{\mathbb{R}^2_+} z_1 m_2(dz). \) Thus, \( \mathcal{M}(t) = e^{-\tilde{\beta} t}e^{b^\top}X(t) \) is a two-dimensional martingale, and

\[ E\mathbb{X}(t) = \left[ E\frac{\nu_0(t)(\xi, \lambda)}{d\lambda} \right]|_{\lambda = 0^+}. \]

**Step 2.** Denote

\[ \beta(n) = an + \frac{\sigma^2n^2}{2} + \int_{D_1} (e^z - 1 - nz) \nu(dz) + \int_{D_1^*} (e^z - 1) \nu(dz). \]

Since \( \int_{1}^{\infty} e^z \nu(dz) < \infty \), by similar arguments of Lemma 3.2 in \([7]\), for any integer \( m \leq n \), we have

\[ Ee^{n\xi_1(s)} \leq C_m(t) := \exp\{(\beta(m) - b_{11}m)\mathbf{t}) \vee 0\} < \infty, \quad s \in [0, t], \]

where \( \xi_1(s) = \xi(s) - b_{11} s. \) Applying Itô’s formula to \([X_1(t) e^{-\xi(t)}]^n\),
Moment properties for two-type CBRE-processes

\[ [X_1(t)e^{-\xi_1(t)}]^n \]
\[ = \sum_{j=0}^{n-2} A_{n,j}^1 \int_0^t [X_1(s)]^{j+1}e^{-n\xi_1(s)}ds - \int_0^t b_{21}n[X_1(s)]^{n-1}X_2(s)e^{-n\xi_1(s)}ds \]
\[ + n \int_0^t [X_1(s)]^{n-1}e^{-n\xi_1(s)}\sqrt{2c_1X_1(s)}dB_1(s) + [X_1(0)]^n \]
\[ + \sum_{j=0}^{n-1} (n \int_0^t \int_1^{X_1(s)} \int_{\mathbb{R}_1^+ \setminus \{0\}} X_1(s-)^j z_1^{-n-j}e^{-n\xi_1(s)}M_1(ds, du, dz) \]
\[ + \int_0^t \int_0^1 \int_{\mathbb{R}_1^+ \setminus \{0\}} \left( [X_1(s) + z_1]^n - [X_1(s)]^n \right) e^{-n\xi_1(s)}M_2(ds, du, dz), \]

By a standard stopping time argument, we can see that
\[ \int_0^t [X_1(s)]^{n-1}e^{-n\xi_1(s)}\sqrt{2c_1X_1(s)}dB_1(s) \]
and
\[ \int_0^t \int_0^1 \int_{\mathbb{R}_1^+ \setminus \{0\}} X_1(s-)^j z_1^{-n-j}e^{-n\xi_1(s)}M_1(ds, du, dz) \]
are local martingales. Therefore, there exists a sequence of stopping times \( \tau_k \) such that
\[ \mathbb{E}^\xi[X_1(t \land \tau_k)e^{-\xi_1(t \land \tau_k)}]^n = \mathbb{E}^\xi[X_1(0)]^n + \sum_{j=0}^{n-2} A_{n,j}^1 \mathbb{E}^\xi \int_0^{t\land \tau_k} [X_1(s)]^{j+1}e^{-n\xi_1(s)}ds \]
\[ + \sum_{j=0}^{n-1} B_{n,j}^1 \mathbb{E}^\xi \int_0^{t\land \tau_k} [X_1(s)]^{j}X_2(s)e^{-n\xi_1(s)}ds, \]

Therefore,
\[ \mathbb{E}[X_1(t \land \tau_k)]^n \]
\[ = \mathbb{E}[X_1(0)]^n \mathbb{E}e^{n\xi_1(t \land \tau_k)} + \sum_{j=0}^{n-2} A_{n,j}^1 \mathbb{E}(e^{n\xi_1(t \land \tau_k)} \int_0^{t\land \tau_k} [X_1(s)]^{j+1}e^{-n\xi_1(s)}ds) \]
\[ + \sum_{j=0}^{n-1} B_{n,j}^1 \mathbb{E}(e^{n\xi_1(t \land \tau_k)} \int_0^{t\land \tau_k} [X_1(s)]^{j}X_2(s)e^{-n\xi_1(s)}ds). \] (2.2)

By Fubini’s Theorem,
\[ \mathbb{E}(e^{n\xi_1(t \land \tau_k)} \int_0^{t\land \tau_k} [X_1(s)]^{j}X_2(s)e^{-n\xi_1(s)}ds) \]
\[ \leq \int_0^t \mathbb{E}[X_1(s \land \tau_k)]^{j}X_2(s \land \tau_k) \mathbb{E}e^{n\xi_1(s \land \tau_k) - \xi(s \land \tau_k)} \] (2.3)
and
\[ \mathbb{E}(e^{n\xi_1(t \land \tau_k)} \int_0^{t\land \tau_k} [X_1(s)]^{j+1}e^{-n\xi_1(s)}ds) \]
and

\[ \mathcal{M}(t) = e^{-\beta t} e^{ib^T X(t)} \text{ is a martingale.} \]

Hence, for any positive integer \( m \) and \( s \in [0,t] \),

\[ E[X_1(s)]^m = e^{m \beta s} E[e^{s \beta^T \mathcal{M}(s)}] \]

\[ \leq e^{m \beta s} E[E[e^{-s \beta^T \mathcal{M}(s)}]^{m}] \]

\[ \leq e^{m \beta s} ||e^{-s \beta^T}||^m |\mathcal{M}(s)|^m ] \]

\[ \leq e^{m \beta s} ||e^{-s \beta^T}||^m (\sqrt{2})^m E[(\mathcal{M}(0))^{m}(\mathcal{M}(t))^{m}] \]

\[ \leq e^{m \beta s} ||e^{-s \beta^T}||^m (\sqrt{2})^m E|\mathcal{M}(t)|^m \]

\[ \leq e^{m \beta s} ||e^{-s \beta^T}||^m |\mathcal{M}(t)|^m \]

\[ \leq O(t, m) E \| X(t) \|^m \]

(2.5)

where \( O(t, m) := \sup_{n \leq m \ s \in [0,t]} e^{m \beta s} ||e^{-s \beta^T}||^m |\mathcal{M}(t)|^m \). Symmetrically,

\[ E[X_2(t \wedge \tau_k)]^n \leq C_n(t) \left[ E[X_2(0)]^n + \sum_{j=0}^{n-2} A_{n,j}^2 \int_0^t E[X_2(t \wedge \tau_k)]^{j+1} ds \right. \]

\[ + \sum_{j=0}^{n-1} B_{n,j}^1 \int_0^t E[X_2(t \wedge \tau_k)]^{j+1} ds \left. \right] \]

and

\[ E[X_2(s)]^m \leq O(t, m) E \| X(t) \|^m, \quad s \in [0,t], \]

(2.6)

where \( C_n(t) := \exp\{[(\beta m) - b_{22} m] t/4\} \). Define \( \tilde{C}_n(t) := C_n(t) \vee C_n'(t), A_{n,j} := A_{n,j}^1 \vee A_{n,j}^2, B_{n,j} := B_{n,j}^1 \vee B_{n,j}^2 \). From above, it is obvious that for \( i = 1, 2 \),

\[ E[X_i(t \wedge \tau_k)]^n \leq \tilde{C}_n(t) \left[ E[X_i(0)]^n + t \sum_{j=0}^{n-2} A_{n,j} O(t, n) E \| X(t) \|^{j+1} ds \right. \]

\[ + t \sum_{j=0}^{n-1} B_{n,j} O(t, n) E \| X(t) \|^{j+1} ds \left. \right]. \]

By (2.5) and (2.4), \( t \mapsto E \| X(t) \|^m \) is locally bounded under the assumption that \( E \| X(t) \|^m < \infty \). By induction one can find a finite function \( g_n(t) \) s.t.

\[ g_n(t) \geq \tilde{C}_n(t) \left[ E[X_i(0)]^n + t \sum_{j=0}^{n-2} A_{n,j} O(t, n) E \| X(t) \|^{j+1} ds \right. \]

(2.4)
Thus, by Fatou’s Lemma, for \( i = 1, 2, \)

\[
E[X_i(t)]^n \leq \lim \inf_{k \to \infty} E[X_i(t \land \tau_k)]^n \leq g_n(t) < \infty,
\]

which implies \( E\|X(t)\|^n < \infty. \)

**Step 3.** Applying Itô’s formula to \([e^{1}X^{(k)}(t) \exp\{-\bar{\xi}_1^{(k)}(t)\}]^n,\)

\[
[X_1^{(k)}(t)e^{-\bar{\xi}_1^{(k)}(t)}]^n = [X_1(0)]^n - \int_0^t b_{21}n[X_1^{(k)}(s)]^{n-1}X_2^{(k)}(s)e^{-n\bar{\xi}_1^{(k)}(s)}ds
\]

\[
+ n \int_0^t [X_1^{(k)}(s)]^{n-1}e^{-n\bar{\xi}_1^{(k)}(s)}\sqrt{2c_1X_1^{(k)}(s)}dB_1(s)
\]

\[
+ \int_0^t n(n-1)c_1[X_1^{(k)}(s)]^{n-1}e^{-n\bar{\xi}_1^{(k)}(s-)}ds
\]

\[
- \int_0^t \int_{\mathbb{R}_+^2 \setminus \{0\}} n[X_1^{(k)}(s)]^{n}e^{-n\bar{\xi}_1^{(k)}(s)}z_11_{\{\|z\| \leq k\}}m_1(dz)ds
\]

\[
+ \sum_{j=0}^{n-1} \binom{n}{j} \int_0^t \int_{\mathbb{R}_+^2 \setminus \{0\}} X_1^{(k)}(s-)^jz_1^{n-j}1_{\{\|z\| \leq k\}}e^{-n\bar{\xi}_1^{(k)}(s)}M_1(ds, du, dz)
\]

\[
+ \sum_{j=0}^{n-1} \binom{n}{j} \int_0^t \int_{\mathbb{R}_+^2 \setminus \{0\}} X_2^{(k)}(s-)^jz_1^{n-j}1_{\{\|z\| \leq k\}}e^{-n\bar{\xi}_1^{(k)}(s)}M_2(ds, du, dz).
\]

Since \( E\|X^{(k)}(t)\| < \infty \) and \( Ee^{n\bar{\xi}_1^{(k)}(t)} < \infty \) for each positive integer \( n, \) and \( X^{(k)}(t) \uparrow X(t), \)

\( \bar{\xi}_1^{(k)}(t) \uparrow \bar{\xi}_1(t), \)

almost surely for \( P \) as \( k \to \infty, \) we have

\[
[X_1^{(k)}(t)]^n = [X_1^{(k)}(0)]^n e^{n\bar{\xi}_1(t)} - \int_0^t b_{21}n[X_1^{(k)}(s)]^{n-1}X_2^{(k)}(s)e^{n(\bar{\xi}_1^{(k)}(t)-\bar{\xi}_1^{(k)}(s))}ds
\]

\[
+ \int_0^t n(n-1)c_1[X_1^{(k)}(s)]^{n-1}e^{n(\bar{\xi}_1^{(k)}(t)-\bar{\xi}_1^{(k)}(s-))}ds
\]

\[
+ \sum_{j=0}^{n-2} \binom{n}{j} \int_0^t \int_{\mathbb{R}_+^2 \setminus \{0\}} X_1^{(k)}(s-)^jz_1^{n-j}1_{\{\|z\| \leq k\}}e^{n(\bar{\xi}_1^{(k)}(t)-\bar{\xi}_1^{(k)}(s))}m_1(dz)ds
\]

\[
+ \sum_{j=0}^{n-1} \binom{n}{j} \int_0^t \int_{\mathbb{R}_+^2 \setminus \{0\}} X_1^{(k)}(s-)^jX_2^{(k)}(s-)^{n-j}z_1^{n-j}1_{\{\|z\| \leq k\}}e^{n(\bar{\xi}_1^{(k)}(t)-\bar{\xi}_1^{(k)}(s))}m_2(dz)ds + \text{mart.}
\]

Taking expectation on both sides and letting \( k \to \infty, \) we get the desired result by monotone convergence theorem. \( \square \)
3 \( f \)-moment of two-type CBRE processes

In this section, we present the equivalent condition for the existence of \( \mathbb{E}f(\|X(t)\|) \), where \( f \) is a nonnegative continuous function \([0, \infty)\) satisfying \textbf{Condition A}. By discussions in [I], pp.154, this condition can be changed by

\textbf{Condition B}. There exists constants \( K > 0 \) such that

\begin{enumerate}[(B1)]  
  \item \( f \) is convex and nondecreasing on \([0, \infty)\);  
  \item For all \( x, y \in [0, \infty) \), \( f(xy) \leq Kf(x)f(y) \);  
  \item For all \( x \in [0, \infty) \), \( f(x) > 1 \).
\end{enumerate}

\textbf{Proposition 3.1}. Suppose that \( f \) satisfies \textbf{Condition B}. Then for any \( t \geq 0 \) and \( y \geq x \in \mathbb{R}^2_+ \), \( \mathbb{E}f(\|X(t, y)\|) < \infty \) if and only if \( \mathbb{E}f(\|X(t, x)\|) < \infty \), where \( \{X(t, x) : t \geq 0\} \) is a unique strong solution on \( \mathbb{R}^2_+ \) to (1.7) starting from any fixed point \( x \in \mathbb{R}^2_+ \).

\textbf{Proof}. Denote \( X^k(t, x) = X(t, kx) - X(t, (k-1)x) \). By the quenched branching property, \( \{X^k(t, x) : t \geq 0\}, \ k = 1, 2, \ldots \) are independently identically distributed. For \( z \in [0, \infty) \), denote \([z]\) as the integer part of \( z \). For any \( y \geq x \in \mathbb{R}^2_+ \), define \( r(x, y) := \lfloor y_1/x_1 \rfloor \lor \lfloor y_2/x_2 \rfloor \). It is clear that \( r(x, y) \leq \|y_1/x_1 \lor y_2/x_2\| \). By \textbf{Condition B},

\[
\mathbb{E}^\xi[f(\|X(t, y)\|)1_{\{t < \tau_0(y)\}}] = \mathbb{E}^\xi[f(\|\sum_{k=1}^{\tau_0(y)} X^k(t, x) + X(t, y) - X(t, r(x, y)x)\|)1_{\{t < \tau_0(y)\}}] \\
\leq Kf(1 + ||y_1/x_1, y_2/x_2||)E^\xi[f(\|X(t, x)\|)1_{\{t < \tau_0(x)\}}],
\]

where \( \tau_0(x) = 0, \tau_0(x) represents the \( n \)th jumping time that the jump size of \( \{X(t, x) : t \geq 0\} \) falls into \( \mathbb{D}_2 := \mathbb{R}^2_+ \setminus \{0, 1\}^2 \). Taking expectation on both sides and letting \( n \to \infty \),

\[
\mathbb{E}[f(\|X(t, y)\|)] \leq Kf(1 + ||y_1/x_1, y_2/x_2||)\mathbb{E}[f(\|X(t, x)\|)].
\]

The desired result follows. \( \square \)

\textbf{Proposition 3.2}. Suppose that \( f \) satisfies \textbf{Condition B}, and \( \mathbb{E}f(\|X(t, x)\|) < \infty \) for some \( x \in \mathbb{R}^2_+ \) and some \( t \geq 0 \). Then \( \mathbb{E}f(\|X(t)\|) < \infty \) if and only if \( \mathbb{E}f(\|X(0)\|) < \infty \).

\textbf{Proof}. Some simple calculations lead to,

\[
\mathbb{E}f(\|X(t)\|) \leq \frac{1}{2}K^2f(2)[f(1) + \mathbb{E}f(\|X(0)\|)]\mathbb{E}[f(\|X(t, 1)\|)].
\]

By Proposition 3.1 we have \( \mathbb{E}f(\|X(t, 1)\|) < \infty \). Then \( \mathbb{E}f(\|X(0)\|) < \infty \) leads to \( \mathbb{E}f(\|X(t)\|) < \infty \). Conversely, we suppose that \( \mathbb{E}f(\|X(t)\|) < \infty \). According to Step 1, \( \mathcal{M}(t) = e^{-\beta t}e^{\beta t}X(t) \) is a two-dimensional martingale, then

\[
\mathbb{E}f(\|\mathcal{M}(t)\|) \leq K^2f(e^{\beta t})f(||e^b||)\mathbb{E}f(\|X(t)\|) < \infty.
\]

Moreover, by the convexity of \( f \),

\[
\mathbb{E}f(\|X(0)\|) = \mathbb{E}f(\|\mathcal{M}(0)\|) \leq \mathbb{E}f(\mathcal{M}_1(0) + \mathcal{M}_2(0))
\]
Therefore
\[
\frac{1}{2}K f(2)\left[\mathbb{E}f(M_1(0)) + \mathbb{E}f(M_2(0))\right]
\]
\[
\leq \frac{1}{2}K f(2)\left[\mathbb{E}f(M_1(t)|F_0) + \mathbb{E}f(M_2(t)|F_0)\right]
\]
\[
\leq K f(2)\mathbb{E}\left[\mathbb{E}[\|M(t)\|]|F_0]\right]
\]
\[
\leq K f(2)\mathbb{E}\left[f(\|M(t)\|)|F_0]\right]
\]
\[
= K f(2)\mathbb{E}f(\|M(t)\|).
\]

Finally, the desired result follows. □

Lemma 3.3. Suppose that $f$ satisfies Condition B, and for any $n \geq 1$, $\int_{\|z\| \geq 1} \|z\|^n (m_1 + m_2)(dz)$ and $\int_1^\infty e^{nz}\nu(dz)$ are finite. Then for any $x \in \mathbb{R}_+^2$, $t \mapsto \mathbb{E}f(\|X(t, x)\|)$ is locally bounded on $[0, \infty)$.

Proof. From the previous section, we have $\mathbb{E}f(\|X(t, x)\|)^n < \infty$. In view of pp.160 in [16] and the proof of Lemma 3.7 in [6], there exist a constant $c > 0$ and a positive integer $n$ such that for all $z \geq 1$, $f(e^{nz}) \leq ce^{nz}$. Then
\[
\mathbb{E}f(\|X(t, x)\|) \leq (f(1) + c\mathbb{E}[\|X(t, x)\]|^n) < \infty.
\]

Since for $i = 1, 2$, $f(M_i(t, x))$ is an $\mathcal{F}_t$-submartingale, for $t \in [0, T]$,
\[
\mathbb{E}f(\|M(t, x)\|) \leq \mathbb{E}f(M_1(t, x) + M_2(t, x))
\]
\[
\leq \frac{1}{2}K f(2)\left[\mathbb{E}f(M_1(t, x)) + \mathbb{E}f(M_2(t, x))\right]
\]
\[
\leq \frac{1}{2}K f(2)\left[\mathbb{E}f(M_1(T, x)) + \mathbb{E}f(M_2(T, x))\right]
\]
\[
\leq K f(2)\mathbb{E}f(\|\mathcal{M}(T, x)\|)
\]
\[
\leq K f(2)\mathbb{E}f(\|\mathcal{M}(T, x)\|).
\]

Hence,
\[
\mathbb{E}f(\|X(t, x)\|) = \mathbb{E}f([\beta_t e^{-\beta T} + \beta_t e^{\mu}\mathbb{E}X(t, x)]) \leq K f(2)\left[\|\beta_t e^{-\beta T}\|f(||e^{\mu}\| \vee 1)\mathbb{E}(\|X(T, x)\|)\right].
\]

Therefore $t \mapsto \mathbb{E}f(\|X(t, x)\|)$ is locally bounded on $[0, \infty)$. □

For $x \in \mathbb{R}_+^2$, let $\theta_0(x) = 0$ and $\theta_n(x) = \theta_n'(x) \wedge \theta_n''(x)$ for $n \geq 1$, where
\[
\theta_n'(x) = \inf \{t > \theta_{n-1}(x) : \xi_t - \xi_{t-} > 1\},
\]
\[
\theta_n''(x) = \inf \{t > \theta_{n-1}(x) : [X_1(t, x) - X_1(t-, x)] \wedge [X_2(t, x) - X_2(t-, x)] > 1, \xi_t = \xi_{t-}\}.
\]

Let $J(dt)$ be the distribution of $\theta_1(1)$, that is $J(dt) = \mathbb{E}(\theta_1(1) \in dt)$. Define
\[
\mu_n(t) := \mathbb{E}[f(\|X(t, 1)\|); t < \theta_1(1)].
\]

Notice that $\mu_0(t) = 0$.

Proposition 3.4. Suppose that $f$ satisfies Condition B and
\[
\int_{\|z\| \geq 1} f(\|z\|)(m_1 + m_2)(dz) < \infty, \quad \int_1^\infty f(e^z)\nu(dz) < \infty.
\]

Then, for any $x \in \mathbb{R}_+^2$, $t \mapsto \mathbb{E}f(\|X(t, x)\|)$ is locally bounded on $[0, \infty)$. 

Proof. Recall that \( \{X(t, 1) : t \geq 0\} \) is the strong solution with initial value \( 1 = (1, 1) \). On the same probability space, define \( R(t, x) \) to the strong solution of the following equation system:

\[
R_1(t) = x_1 + c_1 \int_0^t \int_0^{R_1(s)} W_1(ds, du) + \int_0^t (-b_1 R_1(s) - b_2 R_2(s))ds \\
+ \int_0^t \int_0^{R_1(s)} z_1 \tilde{M}_1(ds, du, dz) \\
+ \int_0^t \int_0^{R_2(s)} z_2 M_2(ds, du, dz) \\
+ \int_0^t \int_{D_1} R_1(s)(e^z - 1) \tilde{N}(ds, dz) \\
+ \int_0^t \int_{-\infty}^{R_1(s)} R_2(s)(e^z - 1) N(ds, dz) \tag{3.2}
\]

and

\[
R_2(t) = x_2 + c_2 \int_0^t \int_0^{R_2(s)} W_2(ds, du) + \int_0^t (-b_2 R_1(s) - b_2 R_2(s))ds \\
+ \int_0^t \int_0^{R_1(s)} z_1 M_1(ds, du, dz) \\
+ \int_0^t \int_0^{R_2(s)} z_2 \tilde{M}_2(ds, du, dz) \\
+ \int_0^t \int_{D_1} R_2(s)(e^z - 1) \tilde{N}(ds, dz) \\
+ \int_0^t \int_{-\infty}^{R_2(s)} R_2(s)(e^z - 1) N(ds, dz). \tag{3.3}
\]

Let \( W' \) be the space consisting of all càdlàg paths \( t \mapsto x(t) \) from \([0, \infty)\) to \( \mathbb{R}^2 \) with Skorokhod topology. Let \( \mathfrak{G} = \sigma\{x(s) : s \geq 0\}, \mathfrak{G}_t = \sigma\{x(s) : 0 \leq s \leq t\}, t \geq 0 \) be natural filtrations on \( W' \). Denote \( \mathbb{P}_x \) the distribution of \( \{X(t, x) : t \geq 0\} \) on \( W' \), then \( (W', \mathfrak{G}, \mathfrak{G}_t, \mathbb{P}_x) \) is a canonical realization of the two-dimensional CBRE process with branching mechanism \( \phi \). Denote \( \sigma_n \) the stopping time of \( \{x(t) : t \geq 0\} \) corresponding to the stopping time \( \theta_n(x) \) of \( \{X(t, x) : t \geq 0\} \). And \( \mathbb{E}_x \) stands for the mathematical expectation with respect to \( \mathbb{P}_x \). Then,

\[
\mu_n(t) = \mathbb{E}[f(\|X(t, 1)\|)1_{\{t < \theta_n(1)\}}] + \mathbb{E}[f(\|X(t, 1)\|)1_{\{\theta_n(1) \leq t < \theta_n(1)\}}] \\
= \mathbb{E}[f(\|R(t, 1)\|)] + \mathbb{E}\{1_{\{\theta_n(1) \leq t\}} \mathbb{E}[f(\|X(t, 1)\|)1_{\{t < \theta_n(1)\}}] \mathbb{E}_x(\phi_1(1))\} \\
\leq \mathbb{E}[f(\|R(t, 1)\|)] + \mathbb{E}\{1_{\{\theta_n(1) \leq t\}} \mathbb{E}_x(\phi_1(1))f(\|x(t - \theta_1(1))\|)1_{\{t < \theta_n(1) < \sigma_n - 1\}}\} \\
\leq \mathbb{E}[f(\|R(t, 1)\|)] \\
+ \mathbb{E}\{1_{\{\theta_n(1) \leq t\}} \mathbb{E}_x(R(\theta_1(1) + \Delta x(\theta_1(1)))f(\|x(t - \theta_1(1))\|)1_{\{t - \theta_1(1) < \sigma_n - 1\}}\})].
\]

Without loss of generalization, suppose \( m_1(D_2), m_2(D_2) \) and \( \nu(1, \infty) \) are positive. Denote

\[
\tilde{m}_1(dz) = \frac{1_{\{z \in D_2\}}}{m_1(D_2)}m_1(dz),
\]
\[ \hat{m}_2(\mathrm{d}z) = \frac{1_{\{z \in \mathbb{D}_2\}}}{m_2(\mathbb{D}_2)} m_2(\mathrm{d}z), \]
\[ \nu(\mathrm{d}z) = \frac{1_{\{z \in (1, \infty)\}}}{\nu(1, \infty)} \nu(\mathrm{d}z). \]

By assumption, \( M_1(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z) \), \( M_2(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z) \) and \( N(\mathrm{d}s, \mathrm{d}z) \) are mutually independent, \[
\mathbb{E}\{1_{\{\theta(1) \leq t\}} \mathbb{E}(R(\theta(1)) + \Delta X(\theta(1)))|f(\|x(t - \theta(1))\|)|1_{\{t - \theta(1) < \sigma_{n-1}\}}\} \]
\[ \leq \int_0^t J(\mathrm{d}s) \int_{\mathbb{D}_2} \int_1^\infty \mathbb{E}\{e^s R(\mathrm{d}s, 1) + u + v|f(\|x(t - s)\|)|1_{\{t - s < \sigma_{n-1}\}}\} \hat{m}_1(\mathrm{d}u) \hat{m}_2(\mathrm{d}v) \nu(\mathrm{d}z). \]

According to (3.1), the above is no more than \[
K \int_0^t \mu_{n-1}(t - s) J(\mathrm{d}s) \int_{\mathbb{D}_2} \int_1^\infty \mathbb{E}(e^s R(t, 1) + u + v + 1) \hat{m}_1(\mathrm{d}u) \hat{m}_2(\mathrm{d}v) \nu(\mathrm{d}z). \]

On the other hand,
\[
\int_{\mathbb{D}_2} \int_1^\infty \mathbb{E}(e^s R(s, 1) + u + v + 1) \hat{m}_1(\mathrm{d}u) \hat{m}_2(\mathrm{d}v) \nu(\mathrm{d}z)
\leq K f(4) \int_{\mathbb{D}_2} \int_1^\infty \mathbb{E}(1 + e^s R(s, 1) + u + v + 1) \hat{m}_1(\mathrm{d}u) \hat{m}_2(\mathrm{d}v) \nu(\mathrm{d}z)
\leq \frac{1}{4} K f(4) \left[ K \int_1^\infty f(e^z) \nu(\mathrm{d}z) \mathbb{E}(R(s, 1)) + \int_{\mathbb{D}_2} f(\|z\|) \hat{m}_1(\mathrm{d}z) \right.
\left. + \int_{\mathbb{D}_2} f(\|z\|)(\hat{m}_1 + \hat{m}_2)(\mathrm{d}z) + f(1) \right]
\leq \frac{1}{4} K f(4) \left[ \sup_{0 \leq t \leq T} \mathbb{E}(R(t, 1)) \right] K \int_1^\infty f(e^z) \nu(\mathrm{d}z)
\leq \int_{\mathbb{D}_2} f(\|z\|)(\hat{m}_1 + \hat{m}_2)(\mathrm{d}z) + f(1) \right].

Let \( O_1(T) = \sup_{0 \leq t \leq T} \mathbb{E}(R(t, 1)) \),
\[
O_2(T) := \frac{1}{4} K f(4) \left[ O_1(T) K \int_1^\infty f(e^z) \nu(\mathrm{d}z) + \int_{\mathbb{D}_2} f(\|z\|)(\hat{m}_1 + \hat{m}_2)(\mathrm{d}z) + f(1) \right].
\]

The finiteness of \( O_1(T) \) and \( O_2(T) \) follow from Lemma 3.3, \( \int_{\|z\| \geq 1} f(\|z\|)(m_1 + m_2)(\mathrm{d}z) < \infty \) and \( \int_1^\infty f(e^z) \nu(\mathrm{d}z) < \infty \). Then we have for any \( t \in [0, T] \),
\[
\mu_n(t) \leq O_1(T) + O_2(T) \int_0^t \mu_{n-1}(t - u) J_1(\mathrm{d}u).
\]

On the other hand, according to Lemma 2 on pp.145 of [1], there exists a positive function \( t \mapsto \mu(t) \) bounded on \([0, T]\) s.t.
\[
\mu(t) = O_1(T) + O_2(T) \int_0^t \mu(t - u) J_1(\mathrm{d}u).
\]
Recall that $\mu_0(t) = 0$. It is clear that $\mu_n(t) \leq \mu(t)$ for all $n = 0, 1, 2, \ldots$ and $0 \leq t \leq T$. As $n$ tends to infinity, we have

$$\mathbb{E}f(||X(t, x)||) \leq \lim_{n \to \infty} \mu_n(t) \leq \mu(t).$$

Using Proposition 3.1, it is obvious that for any $x \in \mathbb{R}^2_+$, $t \mapsto \mathbb{E}f(||X(t, x)||)$ is locally bounded on $[0, \infty)$. \hfill \Box

Proof of Theorem 1.2:

The sufficiency comes from Proposition 3.2 and Proposition 3.4 Conversely, if for some $t > 0$, $\mathbb{E}f(||X(t)||) < \infty$. Let $J_1(dt) = \mathbb{E}(\rho_1 \in dt)$, where

$$\rho_1 = \inf \{ t > 0 : [X_1(t) - X_1(t-)] \wedge [X_2(t) - X_2(t-)] > 1, \xi_t = \xi_{t-} \}.$$ 

Clearly, $\mathbb{E}\{1_{\rho_1 < t}\mathbb{E}[f(||X(t)||)]\mathcal{F}_{\rho_1}\} \leq \mathbb{E}f(||X(t)||)$. By the strong Markov property,

\[
\begin{align*}
&\mathbb{E}\{1_{\rho_1 < t}\mathbb{E}[f(||X(t)||)]\mathcal{F}_{\rho_1}\} \\
&= \mathbb{E}\{1_{\rho_1 < t}\mathbb{E}_X(\rho_1)[f(||x(t - \rho_1)||)]\} \\
&\geq \mathbb{E}\{1_{\rho_1 < t}\mathbb{E}_\Delta X(\rho_1)[f(||x(t - \rho_1)||)]\} \\
&\geq \mathbb{E}\{1_{\rho_1 < t}J_1(ds) \int_{\mathbb{R}^2_+ \setminus \{0\}} \mathbb{E}_z f(||x(t - s)||) \bar{m}_1(dz)\} \\
&= \mathbb{E}\{1_{\rho_1 < t}J_1(ds) \int_{\mathbb{R}^2_+ \setminus \{0\}} \mathbb{E}_z f(||X(t - s, z)||) \bar{m}_1(dz)\} \\
&= \mathbb{E}\{1_{\rho_1 < t}J_1(ds) \int_{\mathbb{R}^2_+ \setminus \{0\}} \mathbb{E}_z f(||R(t - s, z)||) \bar{m}_1(dz)\},
\end{align*}
\]

In the above equations, $\{R(t, z) : t \geq 0\}$ is the strong solution of (3.2) with initial value $z$. Since $t \mapsto J_1(0, t]$ is strictly increasing, there exists some $s \in (0, t]$, such that

$$\int_{\mathbb{R}^2_+ \setminus \{0\}} f(||R(t - s, z)||) \bar{m}_1(dz) < \infty.$$ 

By Proposition 3.2, $\int_{\mathbb{R}^2} f(||z||) \bar{m}_1(dz) < \infty$. Then $\int_{\mathbb{R}^2} f(||z||) m_1(dz) < \infty$. Similarly, $\int_{\mathbb{R}^2} f(||z||) m_2(dz) < \infty$. In conclusion, $\int_{||z|| \geq 1} f(||z||)(m_1 + m_2)(dz) < \infty$. Similarly, define $\rho_2 = \inf \{ t > 0 : \xi(t) - \xi(t-1) > 1 \}$, $J_2(dt) = \mathbb{P}(\rho_2 \in dt)$. By the strong Markov property,

\[
\begin{align*}
&\mathbb{E}\{1_{\rho_2 < t}\mathbb{E}[f(||X(t)||)]\mathcal{F}_{\rho_2}\} \\
&= \mathbb{E}\{1_{\rho_2 < t}\mathbb{E}_X(\rho_2)[f(||x(t - \rho_2)||)]\} \\
&\geq \int_{t}^\infty \mathbb{E}\{1_{\rho_2 < t}\mathbb{E}_x(\rho_2)[f(||x(t - \rho_2)||)]\} \hat{\nu}(dz) \\
&\geq \int_{0}^t \mathbb{E}\{1_{\rho_2 > \epsilon}J_2(ds) \int_{t}^\infty \mathbb{E}_z f(||x(t - s)||) \hat{\nu}(dz)\} \\
&= \int_{0}^t \mathbb{E}\{1_{\rho_2 > \epsilon}J_2(ds) \int_{t}^\infty \mathbb{E}f(||X(t - s, e^z\epsilon)||) \hat{\nu}(dz)\} \\
&\geq \int_{0}^t \mathbb{E}\{1_{\rho_2 > \epsilon}J_2(ds) \int_{t}^\infty \mathbb{E}f(||R(t - s, e^z\epsilon)||) \hat{\nu}(dz)\},
\end{align*}
\]
where \( \{R(t, e^\varepsilon) : t \geq 0\} \) is the strong solution of (3.2)–(3.3) starting from \( e^\varepsilon \) with \( \varepsilon \in \mathbb{R}_+^2 \setminus \{0\} \) s.t. \( P(R(t-s) > \varepsilon) > 0 \). Thus \( t \mapsto J_2(0,t] \) is strictly increasing. Therefore, there exists \( s \in (0,t] \) such that

\[
\int_1^\infty Ef\left(\|R(t-s, e^\varepsilon)\|\right)\hat{\nu}(dz) < \infty.
\]

By Proposition 3.2,

\[
\int_1^\infty f(e^\varepsilon)\hat{\nu}(dz) \leq Kf(\|\varepsilon\|^{-1}) \int_1^\infty f(\|e^\varepsilon\|)\hat{\nu}(dz).
\]

Hence, \( \int_1^\infty f(e^\varepsilon)\nu(dz) < \infty \) and \( Ef(\|X(0)\|) < \infty \).}

### 4 Future research

In this paper we calculate the integer moments and \( f \)-moment for fixed \( t \geq 0 \), where \( f \) is a function on \( [0, \infty) \). The case when \( f \) is a bivariate function and the moment behavior as \( t \to \infty \) is still to be explored. Also, some probability distributions can be uniquely determined by the integer moments, see [17] and [18] for instance. And we are interested in the moment determinacy of CB-processes for fixed \( t \geq 0 \) and also the limit as \( t \to \infty \). Furthermore, since we are discussing two-type CBRE-process here, another direction for the future is to allow the number of types going up to countable infinity, like the authors in [9] did.

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### Declarations

- The authors have no conflicts of interest to declare. All co-authors have seen and agree with the contents of the manuscript.
- The data that support the findings of this study are available on request from the corresponding author.

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