DEFORMATION AND UNOBSERVEDNESS OF DETERMINANTAL SCHEMES

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ABSTRACT. A closed subscheme \( X \subset \mathbb{P}^n \) is said to be \textit{determinantal} if its homogeneous saturated ideal can be generated by the \( s \times s \) minors of a homogeneous \( p \times q \) matrix satisfying \( (p - s + 1)(q - s + 1) = n - \dim X \) and it is said to be \textit{standard determinantal} if, in addition, \( s = \min(p, q) \). Given integers \( a_1 \leq a_2 \leq \cdots \leq a_{t+c-1} \) and \( b_1 \leq b_2 \leq \cdots \leq b_t \) we consider \( t \times (t + c - 1) \) matrices \( \mathcal{A} = (f_{ij}) \) with entries homogeneous forms of degree \( a_j - b_i \) and we denote by \( W(b,a;r) \) the closure of the locus \( W(b,a;r) \subset \text{Hilb}^p(\mathbb{P}^n) \) of determinantal schemes defined by the vanishing of the \( (t - r + 1) \times (t - r + 1) \) minors of such \( \mathcal{A} \) for \( \max\{1, 2 - c\} \leq r < t \). \( W(b,a;r) \) is an irreducible algebraic set.

First of all, we compute an upper \( r \)-independent bound for the dimension of \( W(b,a;r) \) in terms of \( a_j \) and \( b_i \) which is sharp for \( r = 1 \). In the linear case \( (a_j = 1, b_i = 0) \) and cases sufficiently close, we conjecture and to a certain degree prove that this bound is achieved for all \( r \). Then, we study to what extent the family \( W(b,a;r) \) fills in a generically smooth open subset of the corresponding component of the Hilbert scheme \( \text{Hilb}^p(\mathbb{P}^n) \) of closed subschemes of \( \mathbb{P}^n \) with Hilbert polynomial \( p(t) \in \mathbb{Q}[t] \). Under some weak numerical assumptions on the integers \( a_j \) and \( b_i \) (or under some depth conditions) we conjecture and often prove that \( W(b,a;r) \) is a generically smooth component. Moreover, we also study the depth of the normal module of the homogeneous coordinate ring of \( (X) \in W(b,a;r) \) and of a closely related module. We conjecture, and in some cases prove, that their codepth is often \( 1 \) (resp. \( r \)). These results extend previous results on \textit{standard determinantal} schemes to \textit{determinantal} schemes; i.e. previous results of the authors on \( W(b,a;1) \) to \( W(b,a;r) \) with \( 1 \leq r < t \) and \( c \geq 2 - r \). Finally, deformations of exterior powers of the cokernel of the map determined by \( \mathcal{A} \) are studied and proven to be given as deformations of \( X \subset \mathbb{P}^n \) if \( \dim X \geq 3 \).

The work contains many examples which illustrate the results obtained and a considerable number of open problems; some of them are collected as conjectures in the final section.

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1. Introduction

In this monograph, we generalize previous results on deformation of standard determinantal schemes to cover deformations of determinantal schemes. Recall that a codimension $c$ subscheme $X \subset \mathbb{P}^n$ is called determinantal if its homogeneous ideal $I(X) \subset R := k[x_0, \ldots, x_n]$ can be generated by the $s \times s$ minors of a homogeneous $p \times q$ matrix $A$ and $c = (p - s + 1)(q - s + 1)$. $X$ is said to be standard determinantal if, in addition, $s = \min(p, q)$. We would like to parameterize all determinantal schemes $X$ by looking at $X$ as a point $(X)$ of a component of the Hilbert scheme $\text{Hilb}^{pX}(\mathbb{P}^n)$, to study to what extent the family $W(b; a; r)$ of determinantal schemes fills in this component and whether $\text{Hilb}^{pX}(\mathbb{P}^n)$ is generically smooth along $W(b; a; r)$. The locus of $\text{Hilb}^{pX}(\mathbb{P}^n)$ along standard determinantal schemes is quite well understood and in this work we study carefully the structure and dimension of $\text{Hilb}^{pX}(\mathbb{P}^n)$ along determinantal schemes.

A large and important class of classical varieties are cut by minors of a homogeneous matrix: Veronese varieties, Segre varieties, rational normal curves, Bordiga surface, Palatini scrolls, certain varieties of quasi-minimal degree, and even more any variety is isomorphic to a determinantal variety given by a matrix with linear entries (see [26, Pg. 112]). Determinantal schemes have been a central topic in both algebraic geometry and commutative algebra and their study has received considerable attention in the last decades. For instance, in [15], Eisenbud, Koh and Stillman proved that the homogeneous ideal of any curve $C \subset \mathbb{P}^r$ of degree $d$ and genus $g$ with $d \geq 4g + 2$ is generated by the $2 \times 2$ minors of a matrix with linear entries. A quite recent result which shows the importance of determinantal schemes is due to Sidman and Smith. In [48, Theorem 1.1], they prove that a sufficiently ample line bundle on a connected scheme $X$ is determinantly presented. Here a property for a sufficiently ample line bundle on $X$ holds if there is a line bundle $E$ such that the property holds for all $L \in \text{Pic}(X)$ for which $L \otimes E^{-1}$ is ample and, moreover, given a scheme $X$ embedded in $\mathbb{P}^n$ by a complete linear system $L$ we say that $L$ is determinantly presented if $I(X)$ can be generated by the $2 \times 2$ minors of a 1-generic matrix.

As we have just said the determinantal ideals $I_s(A)$ generated by the $s \times s$ minors of a $p \times q$ homogeneous matrix $A$ have been extensively studied by many people. One of the first important results is due to Eagon and Hochster who proved that $I_s(A)$ is a Cohen-Macaulay ideal [11]; hence $R/I_s(A)$ has a minimal free $R$-resolution of length equal to $n + 1 - \dim R/I_s(A)$. To find explicitly
such a minimal free resolution is a problem with a long history behind it. For \( s = 1 \), \( s = \min(p, q) \) or \( s = \min(p, q) - 1 \) such minimal free \( R \)-resolution is given by the Koszul complex, the Eagon-Northcott complex \([12]\) and the Akin-Buchsbaum-Weyman complex \([1]\), respectively; while the first in giving a minimal free \( R \)-resolution of \( I_s(A) \) for any \( 1 \leq s \leq \min(p, q) \) was Lascoux in \([22]\). All these results are crucial in our work.

Given integers \( a_1, a_2, \ldots, a_{t+c-1} \) and \( b_1, \ldots, b_t \) we denote by \( W(k; a; r) \subset \text{Hilb}^{p \times t}(\mathbb{P}^n) \) the locus of determinantal schemes \( X \subset \mathbb{P}^n \) defined by the \((t - r + 1) \times (t - r + 1)\) minors of a \( t \times (t + c - 1) \) matrix \( A = (f_{ij})_{i=1}^{t-c-1} \) where \( f_{ij} \in k[x_0, x_1, \ldots, x_n] \) is a homogeneous polynomial of degree \( a_j - b_i \). \( W(k; a; r) \) is an irreducible algebraic set by Lemma 6.2.

In this paper, we address the following four fundamental problems:

**Problem 1.1.**

1. To determine the dimension of \( W(k; a; r) \) in terms of \( a_j \) and \( b_i \) for all \( r \).
2. To determine whether the closure of \( W(k; a; r) \) is an irreducible component of \( \text{Hilb}^{p \times t}(\mathbb{P}^n) \).
3. To determine when \( \text{Hilb}^{p \times t}(\mathbb{P}^n) \) is generically smooth along \( W(k; a; r) \).
4. To determine whether any deformation of \( X \) with \( (X) \in W(k; a; r) \) comes from deforming its associated homogeneous matrix \( A \).

Due to Lemma 7.2 if Problem 1.1(4) holds, Problems 1.1(2) and (3) also hold.

The first important contribution to these problems was made in 1975 by Ellingsrud \([16]\). He proved that any arithmetically Cohen–Macaulay, closed subscheme \( X \subset \mathbb{P}^n \), \( n \geq 3 \), of codimension 2 is unobstructed and computed the dimension of the Hilbert scheme at \((X)\) (see \([19]\) for the case \( n = 2 \)). The purpose of this work is to extend Ellingsrud’s Theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary determinantal schemes and to show that, in general, the component of the Hilbert scheme which parameterizes determinantal schemes behaves well, contrary to what may happen more generally as predicted by Vakil’s Murphy’s law for singularities of the Hilbert scheme (see \([50]\)). The case of codimension \( c \) standard determinantal schemes (i.e. the case \( r = 1 \)) was mainly solved in \([39]\) for \( c = 3 \), in \([31]\) for \( c = 4 \) and partially for \( c = 5 \), and some cases when \( c \geq 6 \) and in \([32]\) for arbitrary \( c \) (See also \([18\), \([31\), \([33\), and \([36\) for more details). As our results and conjectures in this paper show, we think it is possible to solve the above four problems in full generality also for \( r > 1 \), provided \( \dim X > 2 \). In \([35\) we focused our attention on the first unsolved case and we dealt with codimension-4 determinantal schemes \( X \subset \mathbb{P}^n \) defined by the submaximal minors of a homogeneous square matrix. In this monograph, we will address the general case following the ideas developed by the authors in \([35\), see also \([31,36,39\). Indeed, we will prove our results by considering the smoothness of the Hilbert flag scheme of chains of closed subschemes obtained by deleting suitable columns and its natural projections into the usual Hilbert schemes. Then we recursively prove that the closure of \( W(k; a; r) \) is a generically smooth component of \( \text{Hilb}^{p \times t}(\mathbb{P}^n) \) under some mild numerical assumptions by starting the induction from the case where \( W(k; a; 1) \) parameterizes standard determinantal schemes (i.e. schemes defined by maximal minors). Our proof is based on the following key points: (i) A deep analysis of whether any deformation of a determinantal scheme \( X \) of \( W(k; a; r) \) comes from deforming its associated matrix, and (ii) the fact that any determinantal scheme \( X = \text{Proj}(A) \) is defined by a regular section of a...
"nice" sheaf $\tilde{N}$ on a determinantal scheme $Y = \text{Proj}(B)$ of lower codimension. More precisely let $\varphi : \bigoplus_{i=1}^t R(b_i) \rightarrow \bigoplus_{j=1}^{t+c-1} R(a_j)$ be the morphism of free graded $R$-modules induced by the transpose of $A$ and $\varphi_{t+c-2} : \bigoplus_{i=1}^t R(b_i) \rightarrow \bigoplus_{j=1}^{t+c-2} R(a_j)$ the morphism obtained deleting the last column of $A$. Set $A = R/I_{t+1-r}(\varphi^*), B = R/I_{t+1-r}(\varphi_{t+c-2}^*), MI = \text{coker}(\varphi^*)$ and $N = \text{coker}(\varphi_{t+c-2}^*)$. Then there is a regular section $\sigma^* : B(-a_{t+c-1}) \rightarrow N \otimes B$ fitting into an exact sequence

$$0 \rightarrow B(-a_{t+c-1}) \xrightarrow{\sigma^*} N \otimes B \rightarrow MI \otimes B \rightarrow 0,$$

such that $A = B/\text{im}(\sigma)$. Here $\sigma^*$ is locally given by all minors of size $t-r+1$ involving the deleted row. Such an exact sequence with properties as described in Theorem 4.3 has turned out to be very useful in the case of maximal minors, not only in several of our papers where we often consider deformation problems, but also for other purely algebraic classification problems (40). In 35 this was generalized to submaximal minors ($r = 2$) when $c = 1$, but now in full generality for minors of any size, only requiring $\dim Y \geq r$ and the codimension of the singular locus being at least 2. Our result extends to cover even the artinian case of $A$, as well as the case where the assumption $k$ a field, is replaced by $k$ a local artinian ring over which (the lifting of) $B$ is flat and allows a regular section of the corresponding $N \otimes B$ (see Theorem 5.5 and the end of its proof for details). This is sufficient for showing the just mentioned main result in (ii) above, namely that under some assumptions, any deformation of a determinantal scheme $X$ comes from deforming its associated matrix (Theorem 5.5). The key results (i) and (ii) together with important results of Bruns in 6 on the maximal Cohen-Macaulayness of $N \otimes B$ and its $B$-dual $M$ which we prove lead to the vanishing of several $\text{Ext}^1_B$-groups involving $N \otimes B$, $M$, $B$ and $I_{A/B} := \ker(B \rightarrow A)$ (Propositions 4.1 and 4.6), provide the basis upon which we are able to partially solve Problem 1.1. For instance the vanishing of these $\text{Ext}^1_B$-groups rather immediately solves Problems 1.1(1)-(3), without using (4), for the case of submaximal minors satisfying $\text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ (Proposition 4.1).

Next we outline the structure of the paper. Section 2 of the paper provides background and preliminary results needed later on. Section 3 is inserted both for sake of completeness and for including slightly new generalizations. It contains a summary of the main results on deformation and unobstructedness of standard determinantal schemes. In the next sections we will see that in many cases the same behaviour can be established for all determinantal schemes. The main result of section 4 states that any determinantal scheme $X = \text{Proj}(A)$ can be defined as the degeneracy locus of a regular section of a sheaf $\tilde{N}$ on a determinantal scheme $Y = \text{Proj}(B)$ of lower codimension (Theorem 4.3). In this section, we also show that if $\text{depth}_{I_A} A \geq 4$, $J_A := I_{t-r}(\varphi^*)$ then $\text{Hom}_B(I_{A/B}, A) \cong MI \otimes A(a_{t+c-1})$ and $\text{Ext}^1_A(I_{A/B}/I_A^2, A) = 0$ (Proposition 4.4). Since we recursively want to transfer properties (dimension and smoothness) of $\text{Hilb}^{p^v(t)}(\mathbb{P}^n)$ at $(Y)$ to $\text{Hilb}^{p^v(t)}(\mathbb{P}^n)$ at $(X)$ the above result is important because it implies that the 2nd projection,

$$p_2 : \text{Hilb}^{p^v(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}^{p^v(t)}(\mathbb{P}^n)$$

given by $X' \subset Y' \mapsto (Y')$, defined over the Hilbert-flag scheme, is smooth at $(X \subset Y)$ and with tangential fiber dimension $\dim(MI \otimes A)_{a_{t+c-1}}$. 
We start section 5 characterizing when any deformation of a determinantal ring $A$ comes from deforming its associated homogeneous matrix $A$ in terms of the surjectivity of its tangent map (Lemma 5.1). As an application we get that it holds for so-called generic determinantal rings $A(\lambda) := R/I_\lambda(A)$, $s = t + 1 - r$ where we have $R = k[x_{ij}], 1 \leq i \leq t, 1 \leq j \leq t + c - 1$, and $A = (x_{ij})$ the $t \times (t + c - 1)$ matrix of indeterminates of $R$. Indeed every deformation of $A(\lambda)$ comes from deforming $A = (x_{ij})$ provided $(s, c) \neq (t, 1)$ (Proposition 5.3). Then we prove the main result of this section, namely, Theorem 5.3 which more precisely states that if the following property: “any deformation of a determinantal ring comes from deforming its associated matrix”, holds for $A$ provided $\text{Ext}^2_B(I_B/I_B^2, I_{A/B}) = 0$. Thus it will be important to show the vanishing of $\text{Ext}^2_B(I_B/I_B^2, I_{A/B})$, also because one knows that its vanishing implies that the $1\text{st}$ projection

$$p_1 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}^{p_X(t)}(\mathbb{P}^n)$$

given by $(X' \subset Y') \mapsto (X')$

defined over the Hilbert-flag scheme, is smooth at $(X \subset Y)$ and with tangential fiber dimension $\dim \text{Hom}_B(I_B/I_B^2, I_{A/B})$.

In section 6, we address Problem 1.1(1). We fix integers $b = (b_1, ..., b_t), a = (a_1, a_2, ..., a_{t+c-1}), a' = (a_1, ..., a_{t+c-2})$ and $1 \leq r < t$ and we give an upper $r$-independent bound for $\dim W(b, a; r)$ in terms of $a_i$ and $b_i$ (Theorem 6.3). The bound is achieved for $r = 1$ provided $a_{t+1} > b_i$ for $2 \leq i \leq t$. We carefully analyze under which numerical hypothesis the bound is also achieved for $r > 1$. To do so, we need to compute $\dim (MI \otimes A)_{a_i+c-1}$ and $\dim \text{Hom}_B(I_B/I_B^2, I_{A/B})$. We compute the former under the assumption

$$(*) : a_{t+c-1} - b_1 < \sum_{i=1}^{t-r+1} (a_i - b_{r+i-1}),$$

which holds in the linear case and in cases sufficiently close. For the latter we succeed in showing the expected formula $\dim \text{Hom}_B(I_B/I_B^2, I_{A/B}) = \sum_{i=1}^{t+c-2} \left( a_j - a_{t+c-1} + n \right)$ under some assumptions (e.g. $b_t = b_1 < a_1$ and $a_i - r + 1 < a_{t+c-1}$) leading to the main result of this section (Theorem 6.14) which states that if also every deformation of $\text{Proj}(B)$ of $W(b, a'; r)$ comes from deforming its matrix, $\dim B > r + 2$, $\dim W(b, a'; r) = \lambda_{c-1}$ and $(*)$ holds, then $\dim W(b, a; r) = \lambda_c$ where

$$\lambda_c = \sum_{i,j} \left( a_i - b_j + \frac{n}{n} \right) + \sum_{i,j} \left( b_j - a_i + \frac{n}{n} \right) - \sum_{i,j} \left( a_i - a_j + \frac{n}{n} \right) - \sum_{i,j} \left( b_i - b_j + \frac{n}{n} \right) + 1.$$

The expected formula for $\dim \text{Hom}_B(I_B/I_B^2, I_{A/B})$, which leads to $\dim W(b, a; r) = \lambda_c$, remains conjectural if $\dim B > r + 2$, while for $\dim B = r + 2$ and $r = 2$, i.e. $\dim A = 2$ there are counterexamples for which $\dim W(b, a; r) = \lambda_{c-1}$, see Remark 6.20 and Example 7.14. Moreover we include a lot of examples to illustrate our results and based on examples and results we conjecture that $\dim W(b, a; r) = \lambda_c$ for $\dim A > 2$ (resp. $\dim A > 3$ if $c = 1$) if $(*)$ and $a_1 > b_t$ hold.

Section 7 is entirely devoted to solve Problems 1.1(2) and (3) by mainly using (4) to see when the closure of $W(b, a; r)$ fills in a generically smooth component of $\text{Hilb}^{p_X(t)}(\mathbb{P}^n)$. The main result of this section (Theorem 7.1) weakens the assumption in (4); every deformation of $A$ comes from deforming its associated matrix $A$, to the corresponding assumption on $B$, by in addition assuming
and the fiber dimension (tangential) of $p$ the tangential fiber dimension of the projection $B$ and letting columns from the right hand side we construct a flag of determinantal rings (see Proposition 5.3 and Corollary 7.6). Moreover by considering the degrees of the relations in the Lascoux-resolution of $B$ and using the inclusion $0 \Ext_B(I/I_2^2, I_A/B) \subset 0 \Ext_R(I/I_2^2, I_A/B)$, we always get $0 \Ext_B(I/I_2^2, I_A/B) = 0$ for $i = 0, 1$ if the increase in the sequence of numbers $a_t-a_{t+1} < a_t-a_{t+2} < a_t-a_{t+3}…$ is large enough (cf. Corollary 7.9). Based on our results and examples we conjecture that Problems 1.1(2) and (3) hold for generic determinantal schemes (see Proposition 5.3 and Theorem 7.15). Conjecture 7.15 is true for generic determinantal schemes for $c \geq 4 - r$, whence that Problems 1.1(2) and (3) hold. In the remaining case $c = 3 - r$, we have $0 \Ext_B(I/I_2^2, I_A/B) \neq 0$ even when $X$ is a generic determinantal scheme (see Example 7.5). Fortunately the condition $\gamma = 0$ of Theorem 7.1 seems to hold in the generic case.

In section 8, we weaken the assumptions in Theorems 6.14 and 7.1 by explicitly considering the recursively transfer of the property ”any deformation comes from deforming its associated homogeneous matrix” for the rings in a flag and thereby skipping this assumption on $B$ in Theorems 6.14 and 7.1 even though $0 \Ext_B(I/I_2^2, I_A/B) \neq 0$ for $c = 3 - r$ provide problems. Also weakening (*) in Theorem 6.14 and $\gamma = 0$ in Theorem 7.1 are considered. Indeed, by successively deleting columns from the right hand side we construct a flag of determinantal rings

$$A_{2-r} \rightarrow \cdots \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{c-1} \rightarrow A_c,$$

and letting $B := A_{c-1} \rightarrow A := A_c$, we show in Corollary 8.3 how $0 \Ext_B(I/I_2^2, I_A/B)$ is related to $0 \Ext_A(I_A/I_2^2, I_A/B)$ for $i = 0, 1$ and $j \in \{2 - r, 3 - r\}$. This has implications to the smoothness and the fiber dimension (tangential) of $p_1 : \Hilb^{pX(t), pY(t)}(\mathbb{P}^n) \rightarrow \Hilb^{pX(t)}(\mathbb{P}^n)$. We also consider the tangential fiber dimension of the projection $p_2 : \Hilb^{pX(t), pY(t)}(\mathbb{P}^n) \rightarrow \Hilb^{pY(t)}(\mathbb{P}^n)$. This leads to Theorem 8.7 for the case $c = 3 - r$ and to Theorems 8.10, 8.15 and 8.20 when $c \geq 4 - r$. We should have liked to recursively compute the dimension of $W(h; W, r)$ in terms of $a_i$ and $b_j$, starting the induction from the standard determinantal case ($j = 2 - r$), but we somehow have to start it from $j = 3 - r$ leaving the transfer from $j = 3 - r$ to $j = 2 - r$ as an assumption on $\gamma$ in Theorem 8.20. Similarly in Theorem 8.10 where the vanishing of two $\gamma$’s are assumptions. In Theorem 8.15 we manage to replace this vanishing of $\gamma$’s by a generalization of Theorems 6.14 and
In particular, for all $a$ in $\text{surjection } R$ proven results, or say we assume that Conjectures 6.19 and 7.11 hold for all the deformation functor of the exterior power $\wedge rM$ of $M$ and the deformation functor of the surjection $R \to R/t^rM(\varphi) := A_r$ (see Theorem 9.3). Moreover assuming (*) above, e.g. $A$ linear, we are in Theorem 9.6 and Remark 9.7 able to highlight important consequences. Indeed using proven results, or say we assume that Conjectures 6.19 and 7.11 hold for all $X_i := \text{Proj}(A_i)$ and that dim $A_r \geq 4$ and $c \geq 2$, then the local Hilbert functor $\text{Hilb}_{X_i}(\cdot)$ are for all $i$, $1 \leq i \leq r$ isomorphic to the local Hilbert functor $\text{Hilb}_{X_i}(\cdot)$ of deforming the scheme $X_1 \subset \mathbb{P}^n$ defined by maximal minors. In particular, for all $i$, $W(a \mid c; n) \subset \text{Hilb}^{X_i}(\cdot)(\mathbb{P}^n)$ is a generically smooth irreducible component of the same dimension $\lambda_c$ even though their Hilbert polynomials are very different.

Finally, in the last section, we collect some of the questions which naturally come up from our work, apart from the ones stemming from the conjectures posed along this monograph. In particular it concerns the problems which are not completely solved:

1. **Show**: $\dim \text{Hom}_B(I_B/I_B^2, I_{A/B}) = \sum_{j=1}^{t+c-2}(a_j - a_{t+c-1})$ if dim $A \geq 3$.
2. **Show**: $\text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ if dim $A \geq 3$ (resp. dim $A \geq 4$) for $c \neq 1$ (resp. $c = 1$).
3. **Determine** $\dim(MI \otimes A)_{a_{t+c-1}}$ in the case: $a_{t+c-1} - b_1 \geq \sum_{i=1}^{t-r+1}(a_i - b_{t+i-1})$.

The work contains many examples. In some of them we use Macaulay2 [23] and others follow from our results, but most of them can be computed with Macaulay2. In the Appendix we include the code that we have used to compute the examples not covered by our results, to control our results or to give new evidences to the conjectures stated along the monograph; and the reader can check them.

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Notation. Throughout this paper $k$ will be an algebraically closed field of characteristic zero, $R = k[x_0, x_1, \cdots, x_n]$, $m = (x_0, \ldots, x_n)$ and $\mathbb{P}^n = \text{Proj}(R)$. For any graded Cohen-Macaulay quotient $A$ of $R$ of codimension $c$, we let $I_A = \ker(R \to A)$ and $K_A = \text{Ext}^r_R(A, R)(-n-1)$ be its canonical module. If $M$ is a finitely generated graded $A$-module, let depth$_t M$ denote the length of a maximal $M$-sequence in a homogeneous ideal $J$ and let depth $M = \text{depth}_m M$. If $F$ and $G$ are two coherent $O_X$-modules, we denote the group of morphisms from $F$ to $G$ by $\text{Hom}_{O_X}(F, G)$ while $\text{Hom}_{O_X}(F, G)$ denotes the sheaf of local morphisms of $F$ into $G$. We often omit $O_X$ in $\text{Hom}_{O_X}(F, G)$ (resp. $\text{Hom}_{O_X}(F, G)$) if the underlying scheme $X$ is evident and we set $\dim(F, G) = \dim_k \text{Hom}(F, G)$.
where \( \dim_k \) denotes the dimension as \( k \)-vector space. For any closed subschemes \( X \subset Y \subset \mathbb{P}^n \) we denote by \( \mathcal{N}_Y \) the normal sheaf of \( Y \) and we write \( \mathcal{N}_{Y/X} := \Hom_{\mathcal{O}_Y}(\mathcal{I}_Y, \mathcal{O}_X) \).

Finally, we denote by \( \text{Hilb}^{p×q}(\mathbb{P}^n) \) the Hilbert scheme which parameterizes closed subschemes \( (X') \subset \mathbb{P}^n \) with Hilbert polynomial \( p_{X'} = p_X \) and we let \( \text{Hilb}^{p×q}(\mathbb{P}^n) \) be the Hilbert flag scheme parameterizing pairs of closed subschemes \( (X' \subset Y') \subset \mathbb{P}^n \) with Hilbert polynomials \( p_{X'} = p_X \) and \( p_{Y'} = p_Y \), respectively. By definition a closed subscheme \( X \subset \mathbb{P}^n \) is \textit{unobstructed} if \( \text{Hilb}^{p×q}(\mathbb{P}^n) \) is smooth at \( (X) \). When we define \( A \) by writing \( X = \text{Proj}(A) \) for a given closed subscheme \( X \subset \mathbb{P}^n \), we always take \( A = R/H^0_*(\mathcal{I}_X) \), so \( I_A = H^0_*(\mathcal{I}_X) \) and we also write \( I(X) = H^0_*(\mathcal{I}_X) \).

## 2. Preliminaries

In this section we fix the definitions, notation and some basic results that we are going to use in the sequel.

**Definition 2.1.** If \( A \) is a homogeneous \( p \times q \) matrix, we denote by \( I(A) \) the ideal of \( R \) generated by the maximal minors of \( A \) and by \( I_s(A) \) the ideal of \( R \) generated by the \( s \times s \) minors of \( A \). Assume \( p \leq q \). \( I(A) \) is called a \textit{standard determinantal} ideal if depth \( I(A) = q - p + 1 \), and \( I_s(A) \) and \( R/I_s(A) \) are said to be \textit{determinantal} if depth \( I_s(A) = (p - s + 1)(q - s + 1) \).

Let \( \varphi : F = \oplus_{t=1}^{q} R(b_t) \rightarrow G = \oplus_{j=1}^{t+c-1} R(a_j) \) be a morphism of free graded \( R \)-modules of rank \( t \) and \( t + c - 1 \), respectively, \( \varphi_{t+c-2} : F \rightarrow \oplus_{j=1}^{t+c-1} R(a_j) =: G_{t+c-2} \) the morphism obtained deleting the last row, \( \mathcal{A} \) the homogeneous matrix of \( \varphi^* \) and \( \mathcal{B} \) the homogeneous matrix of \( \varphi_{t+c-2}^* \) (i.e. \( \mathcal{B} \) is obtained deleting the last column of \( \mathcal{A} \)). For any integer \( r, 1 \leq r \leq t \), let \( \mathcal{B} = R/I_{t+1-r}(\varphi_{t+c-2}^*) \), \( \mathcal{A} = R/I_{t+1-r}(\varphi^*) \), \( N = \text{coker}(\varphi_{t+c-2}^*) \) and \( M = \text{coker}(\varphi^*) \). A codimension \( c \) subscheme \( X \subset \mathbb{P}^n \) is said to be \textit{standard determinantal} if its homogeneous saturated ideal \( I(X) = I_s(A) \) for some \( t \times (t + c - 1) \) homogeneous matrix \( A \) and it is said to be \textit{determinantal} if \( I(X) = I_s(A) \) for some \( p \times q \) homogeneous matrix \( A \) and \( c = (p - s + 1) \times (q - s + 1) \).

**Example 2.2.** (i) Complete intersections schemes \( X \subset \mathbb{P}^n \) of codimension \( c \geq 1 \) are examples of standard determinantal schemes.

(ii) Let \( X \subset \mathbb{P}^n \) be a rational normal curve defined as the image of the map
\[
v_n : \mathbb{P}^1 \longrightarrow \mathbb{P}^n
\]
\[
[a : b] \mapsto [a^n : a^{n-1}b : \cdots : ab^{n-1} : b^n].
\]
The homogeneous ideal \( I(X) \) of \( X \) is generated by the maximal minors of the homogeneous matrix
\[
\begin{pmatrix}
x_0 & x_1 & \cdots & x_{n-1} \\
x_1 & x_2 & \cdots & x_n
\end{pmatrix}.
\]
Therefore, \( X \) is a standard determinantal subscheme of \( \mathbb{P}^n \).

(iii) Let \( S \subset \mathbb{P}^8 \) be the 4-dimensional Segre variety defined as the image of the map
\[
s_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8
\]
Fix coordinates $x_0, x_1, \ldots, x_8$ in $\mathbb{P}^8$. The homogeneous ideal $I(S)$ of $S$ is generated by the $2 \times 2$ minors of the homogeneous matrix

$$
\begin{pmatrix}
x_0 & x_1 & x_2 \\
x_3 & x_4 & x_5 \\
x_6 & x_7 & x_8
\end{pmatrix}.
$$

Therefore, $S$ is a determinantal subscheme of $\mathbb{P}^8$.

Along this work, we assume $t \geq 2$ and $r < t$. The cases $t = 1$ or $t = r$ for determinantal ideals corresponds to the well known case of complete intersections. When $c = 1$ we also assume $r \geq 2$ (unless explicitly considering $(c, r) = (1, 1)$) since the case $(c, r) = (1, 1)$ corresponds to hypersurfaces (see, however, section 10 for some interesting problems about determinantal hypersurfaces).

**Definition 2.3.** We say that every deformation of $X = \text{Proj}(A)$ (or $A$) comes from deforming $\mathcal{A} = (f_{ij})$ if for every local artinian ring $T$ with residue field $k = T/\mathfrak{m}_T$ and every graded deformation $A_T$ of $A$ to $T$ there exists a homogeneous matrix $A_T$ with entries $f_{ij}^T \in R \otimes_k T := R_T$ that map to $f_{ij}$ via $T \to k$. Such matrix $A_T$ is called a lifting of $A$ to $T$. (By Remark 7.3 the definition in [31,32] which requires $T$ above to be a local ring, essentially of finite type over $k$, is equivalent to Definition 2.3.)

For $c = 2$ and $r = 1$, Ellingsrud proved in [16] that any deformation of $R/I_t(\varphi^*)$ comes from deforming the $t \times (t + 1)$ matrix $\mathcal{A}$ associated to $\varphi^*$. This was generalized in [32] Theorem 5.8] to cover any $c \geq 2$ when $r = 1$ and $n - c \geq 2$ but it is not always true when $r = 1$ and $n - c < 2$ (see, for instance, [31 Example 4.1]). One of the main goals of this monograph will be to generalize these results to $1 \leq r < t$ and $c \geq 2 - r$; and to study whether any deformation of $R/I_{t-r+1}(\varphi^*)$ comes from deforming the homogeneous matrix $\mathcal{A}$. We think that quite often it will be true that any deformation of a determinantal scheme comes from deforming its associated homogeneous matrix (cf. Conjecture 7.11). As in the case of standard determinantal schemes, we prove that it is indeed true in many cases (see Lemma 5.10 Theorem 5.15 Corollary 5.11 Theorem 7.1 and Corollary 8.22) but we have a few exceptions (see, for instance, Example 6.24).

Let $W(\underline{b}; \underline{a}; r) \subset \text{Hilb}^{p_X(t)}(\mathbb{P}^n)$ be the locus of determinantal schemes $X \subset \mathbb{P}^n$ of codimension $r(r+c-1)$ defined by the $(t-r+1) \times (t-r+1)$ minors of a $t \times (t+c-1)$ homogeneous matrix $\mathcal{A} = (f_{ij})_{i=1,\ldots,t; j=1,\ldots,t+c-1}$ where $f_{ij} \in k[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial of degree $a_j - b_i$. Correspondingly, set $\underline{a}' = (a_1, a_2, \ldots, a_{t+c-2})$ and let $W(\underline{b}; \underline{a}'; r) \subset \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$ be the locus of determinantal flags $X \subset Y \subset \mathbb{P}^n$ of the Hilbert-flag scheme with $X \in W(\underline{b}; \underline{a}; r)$ and $Y \in W(\underline{b}; \underline{a}'; r)$ defined by the $(t-r+1) \times (t-r+1)$ minors of the $t \times (t+c-2)$ matrix $\mathcal{B}$ that we obtain deleting the last column of $\mathcal{A}$. Note that the restriction of the natural projections

$$
p_1 : \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{p_X(t)}(\mathbb{P}^n)
$$

and

$$
p_2 : \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)
$$
to $W(\underline{b}; \underline{a}; r)$ maps into $W(b; \underline{a}; r)$ and $W(\underline{b}; a'; r)$, respectively. Since we will only consider the case $W(\underline{b}; a; 1) \neq \emptyset$, from now on we will assume (see [36] (2.2)):

\begin{equation}
(2.1) \quad a_1 \leq a_2 \leq \cdots \leq a_t+c-1, \quad b_1 \leq \cdots \leq b_t, \quad b_i \leq a_i \text{ for all } i \text{ and } b_{i_0} < a_{i_0} \text{ for some } i_0.
\end{equation}

**Definition 2.4.** Let $X \subset Y \subset \mathbb{P}^n$ be closed subschemes. We say that $X$ is $p_Y$-generic if there is an open subset of $\operatorname{Hilb}^{p_X(t)}(\mathbb{P}^n)$ containing $(X)$ whose members $(X')$ are subschemes of some closed scheme $Y'$ with $\operatorname{Hilb}^{p_Y}(\mathbb{P}^n)$.

In this paper we address the following fundamental problems:

**Problem 2.5.**  
(1) To determine the dimension of $W(\underline{b}; \underline{a}; r)$ in terms of $a_j$ and $b_i$;

(2) To determine whether the closure of $W(\underline{b}; \underline{a}; r)$ is an irreducible component of $\operatorname{Hilb}^{p_X(t)}(\mathbb{P}^n)$;

(3) To determine when $\operatorname{Hilb}^{p_X(t)}(\mathbb{P}^n)$ is generically smooth along $W(\underline{b}; \underline{a}; r)$, and;

(4) To determine whether any deformation of $X$, $(X) \in W(\underline{b}; \underline{a}; r)$ comes from deforming its associated homogeneous matrix $A$.

Due to Lemma 2.2 if Problem 2.5 (4) holds, Problems 2.5 (2) and (3) also hold.

The case $r = 1$ was treated in [16] and [19] for $c = 2$, [39] for $c = 3$ and [32] and [34] for arbitrary $c$ (See also [18], [31], [33], [36] and [45] for more details); while the case $r = 2$ and $c = 1$ was considered in [35]. In the next section of this work, for sake of completeness, we summarize and slightly generalize the case $r = 1$ and we devote the remaining part of the paper trying to generalize the mentioned results to $r > 1$ and to solve the above problems under weak numerical assumptions on the integers $a_i$ and $b_j$ and under some depth or other cohomological conditions.

In this paper we often switch between a subscheme $X \subset \mathbb{P}^n$, or a pair $X \subset Y \subset \mathbb{P}^n$ of subschemes, and the corresponding homogeneous coordinate rings and their surjections $R \rightarrow A$ and $R \rightarrow B \rightarrow A$. Indeed there is a scheme $\operatorname{GradAlg}(H_A)$ parameterizing graded surjections $R \rightarrow A$ of depth$_m A \geq 1$ which is the stratum of Grothendieck’s Hilbert scheme $\operatorname{Hilb}^{p_X(t)}(\mathbb{P}^n)$ consisting of points $(X = \operatorname{Proj}(A) \subset \mathbb{P}^n)$ with Hilbert function $H_X$ where $H_X(v) = p_X(v)$ for $v \gg 0$, cf. [30]. Note that we define the Hilbert function of $X$, or $A$, by $H_X(v) = H_A(v) := \dim A_v$. $\operatorname{GradAlg}(H_A)$ has a natural scheme structure whose tangent (resp. “obstruction”) space at $(X \subset \mathbb{P}^n)$ is $0\operatorname{Hom}_A(I_A/I^2_A; A) \simeq 0\operatorname{Hom}_R(I_A, A)$ (resp. $0\operatorname{Ext}_A^1(I_A/I^2_A, A)$ provided $R \rightarrow A$ is generically a complete intersection). $A$ is called unobstructed (as a graded $R$-algebra) if $\operatorname{GradAlg}(H_A)$ is smooth at $(R \rightarrow A)$. And we may use [16] to see that the open subscheme of $\operatorname{GradAlg}(H_A)$ consisting of graded Cohen-Macaulay quotients satisfying $\dim A \geq 2$ and the open subscheme of $\operatorname{Hilb}^{p_X}(\mathbb{P}^n)$ consisting of arithmetically Cohen-Macaulay (ACM) schemes satisfying $\dim X \geq 1$ where $X = \operatorname{Proj}(A)$ are isomorphic as schemes. Indeed if depth$_m A \geq 2$, we have

\begin{equation}
(2.2) \quad \operatorname{GradAlg}(H_A) \simeq \operatorname{Hilb}^{p_X}(\mathbb{P}^n) \quad \text{at} \quad (X = \operatorname{Proj}(A) \subset \mathbb{P}^n).
\end{equation}

In particular if $\dim X \geq 1$ we can replace the phrase “every graded deformation $A_T$ of $A$ to $T$” in Definition 2.3 by “every deformation $X_T \subset \operatorname{Proj}(R \otimes_k T)$ of $X \subset \mathbb{P}^n$ to $T$.”
Similarly there is a scheme \( \text{GradAlg}(H_B, H_A) \) representing the functor deforming flags (surjections) \( B \to A \) of graded quotients of \( R \) of positive depth at \( m \) and with Hilbert functions \( H_B \) and \( H_A \) respectively. Also in this case, if \( \text{depth}_m A \geq 2 \) and \( \text{depth}_m B \geq 2 \) we have

\[
\text{GradAlg}(H_B, H_A) \simeq \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \quad \text{at} \quad (X \subset Y \subset \mathbb{P}^n).
\]

In particular the open subscheme of \( \text{GradAlg}(H_B, H_A) \) consisting of Cohen-Macaulay pairs \( (B \to A) \) satisfying \( \text{dim} A \geq 2 \) are isomorphic (as schemes) to the corresponding open set of \( \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \). Thus restricting to these open subschemes, the projection

\[
p : \text{GradAlg}(H_B, H_A) \to \text{GradAlg}(H_A)
\]

induced by sending \( (B' \to A') \) onto \( (A') \) is isomorphic to the 1\(^{st} \) projection

\[
p_1 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \to \text{Hilb}^{p_X(t)}(\mathbb{P}^n)
\]

and \( q : \text{GradAlg}(H_B, H_A) \to \text{GradAlg}(H_B) \) to the 2\(^{nd} \) projection \( p_2 \) which maps into \( \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) \).

Since in this paper we almost always work under assumptions that imply \( \text{depth}_m A \geq 2 \) (and \( \text{depth}_m B \geq 2 \) if \( B \) is present), there is really no restriction to work with \( \text{GradAlg} \) instead of \( \text{Hilb} \).

We finish this section with the following useful comparison of cohomology groups. If \( Z \subset X \subset \mathbb{P}^n \) is a closed subset such that \( U = X \setminus Z \) is a local complete intersection, \( L \) and \( N \) are finitely generated \( R/I(X) \)-modules, \( \tilde{N} \) is locally free on \( U \) and \( \text{depth}_{I(Z)} L \geq r + 1 \), then the natural map

\[
\text{Ext}^i_{R/I(X)}(N, L) \to H^i(U, \text{Hom}_{O_X}(\tilde{N}, \tilde{L})) \cong \bigoplus_{\nu \in Z} \text{Ext}^i_{O_X}(\tilde{N}, \tilde{L}(\nu))
\]

is an isomorphism, (resp. an injection) for \( i < r \) (resp. \( i = r \)) cf. [24, exposé VI]. Recall that we interpret \( I(Z) \) as \( m \) if \( Z = \emptyset \).

### 3. Families of standard determinantal schemes

Until now all results we have proved about families of determinantal ideals deal with standard determinantal schemes i.e. schemes of codimension \( c \) in \( \mathbb{P}^n \) defined by the maximal minors of a \( t \times (t + c - 1) \) homogeneous matrix (according to our notation they correspond to the case \( r = 1 \)), except those in [35] where the authors considered ideals generated by the submaximal minors of a square homogeneous matrix (i.e. \( c = 1 \) and \( r = 2 \)). This last paper can be seen as a first attempt to address the general case (i.e. \( r \geq 1 \) and \( c \geq 2 - r \)). In this section, we state without proof what is known about families of standard determinantal schemes and we refer to [18, 31, 32, 33, 34, 36, 39 and 45] for a complete proof of these results. A few generalizations with proofs are, however, also included.

Given sequences of integers \( \underline{a} = (a_1, a_2, ..., a_{t+c-1}) \) and \( \underline{b} = (b_1, b_2, ..., b_t) \) satisfying (2.1) we denote by

\[
W(\underline{b}; \underline{a}) := W(\underline{b}; \underline{a}; 1) \subset \text{Hilb}^{p_X(t)}(\mathbb{P}^n)
\]
the locus of standard determinantal schemes. As always we assume \( t \geq 2 \). Using the convention \( \binom{m}{n} = 0 \) for \( m < n \) we define for any \( r \), max\{1, 2 − c\} \( \leq r \leq t \), the following invariants

\[
\begin{align*}
\ell_i & := \sum_{j=1}^{t+i-1} a_j - \sum_{k=1}^{t} b_k, \quad i \geq 1, \\
h_{i-3} & := 2a_{t+i-1} - \ell_i + n, \quad \text{for } 3 \leq i \leq c, \\
\lambda_c & := \sum_{i,j}(a_i - b_i + n) + \sum_{i,j}(b_i - a_i + n) - \sum_{i,j}(a_i - a_j + n) - \sum_{i,j}(b_i - b_j + n) + 1, \\
K_3 & := \binom{h_0}{n}, \\
K_4 & := \sum_{j=1}^{t+2}(h_1 + a_j) - \sum_{i=1}^{t}(h_1 + b_i), \quad \text{and,} \\
K_i & := \sum_{p+q-i-3}^{p+q-2} \sum_{p+q \geq 0}^{1 \leq i_1 < \cdots < i_p \leq t+i-2} (-1)^{i-1-p}(h_{i_1} + \cdots + a_{i_p} + b_{i_1} + \cdots + b_{i_p}) \quad \text{for } 3 \leq i \leq c.
\end{align*}
\]

Using the generalized Koszul complexes associated to a codimension \( c \) standard determinantal ideal \( I_t(A) \), one knows that the Eagon-Northcott complex (see [8] Theorem 2.20 or [13] Corollaries A2.12 and A2.13]) yields the following minimal free \( R \)-resolution of \( A := R/I_t(A) \)

\[
\begin{align*}
0 & \rightarrow \wedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \wedge^t F \rightarrow \wedge^{t+c-2} G^* \otimes S_{c-2}(F) \otimes \wedge^t F \rightarrow \ldots \\
& \rightarrow \wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \rightarrow R \rightarrow R/I_t(A) \rightarrow 0
\end{align*}
\]

and that the Buchsbaum-Rim complex yields a minimal free \( R \)-resolution of coker(\( \varphi^* \))

\[
\begin{align*}
0 & \rightarrow \wedge^{t+c-1} G^* \otimes S_{c-2}(F) \otimes \wedge^t F \rightarrow \wedge^{t+c-2} G^* \otimes S_{c-3}(F) \otimes \wedge^t F \rightarrow \ldots \\
& \rightarrow \wedge^{t+1} G^* \otimes S_0(F) \otimes \wedge^t F \rightarrow G^* \varphi^* F^* \rightarrow \text{coker}(\varphi^*) \rightarrow 0
\end{align*}
\]

Note that [32] shows that any standard determinantal scheme is arithmetically Cohen-Macaulay (ACM). Analogous result holds for determinantal schemes. Indeed, using the minimal free \( R \)-resolution of a determinantal ideal given by Lascoux in [12] Theorem 3.3 we get that any determinantal scheme is ACM.

Using essentially induction on the codimension by successively deleting columns from the right hand side (assuming, for instance, that the \( c \) columns that we delete do not contain units, to be sure that the rings we get by deleting columns are standard determinantal, cf. [7]), the Eagon-Northcott complex associated to a standard determinantal ideal \( I_t(A) \), the Buchsbaum-Rim complex, the theory of Hilbert flags schemes and the depth of certain mixed determinantal ideals, we can solve (for \( r = 1 \) Problem 2.5.1) under some weak numerical assumptions and prove:

**Theorem 3.1.** Fix integers \( t \geq 2, c \geq 2, \underline{a} = (a_1, a_2, \ldots, a_{t+c-1}) \) and \( \underline{b} = (b_1, b_2, \ldots, b_t) \) as above. Assume \( a_{i-1} \geq b_i \) for \( 2 \leq i \leq t \). It holds:

\[
\begin{align*}
&\text{i) } \dim W(\underline{b}; \underline{a}) \leq \lambda_c + K_3 + K_1 + \cdots + K_c. \\
&\text{ii) For } c = 2 \text{ and } n - c \geq 0, \dim W(\underline{b}; \underline{a}) = \lambda_c. \\
&\text{iii) Assume } c \geq 3. \text{ If } n - c > 0, \text{ or } n - c = 0, \text{ } a_1 > b_t \text{ and } a_{t+c-1} > a_{t-1}, \text{ then} \\
&\quad \dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c.
\end{align*}
\]

**Proof.** (i) See [31] Theorem 3.5] and Remark 3.12.3 below.

(ii) See [16] for the case \( n - c \geq 1, [31] \text{ Remark 4.6} \) (cf. [19] and [21] Appendix 3) for the case \( n - c = 0 \) and [32] Theorem 5.16] for a generalization.
(iii) See [31] Theorems 3.5, 4.5 and Corollaries 4.7, 4.10, 4.14 and 4.15 and [32] Corollary 5.6] for the case $n - c > 0$. For the case $n - c = 0$ the reader could look at [36] Theorem 3.2] for the case $3 \leq c \leq 5$ and [33] Theorem 6.1] for the case $c \geq 3$. (See also [39] for the cases $c = 3$). □

**Remark 3.2.** (1) For $n - c > 0$ Theorem 3.1 as well as [32] Corollary 5.6] or [18] prove Conjecture 4.1 stated in [36].

(2) For $c \geq 3$ and $n - c = 0$ the hypothesis $a_{t+c-1} > a_{t-1}$ in Theorem 3.1 cannot be deleted. In fact, we consider the case $t = 2$, $a_1 = \cdots = a_{c+1} = 1$ and $b_1 = b_2 = 0$, i.e., $A$ is a $2 \times (c + 1)$ matrix with entries general linear forms. The vanishing of the $2 \times 2$ minors of $A$ defines a reduced scheme $X \subset \mathbb{P}^c$ which consists in $c + 1$ different points in $\mathbb{P}^c$. So, $\dim_{(X)} \text{Hilb}^{P_{X}(f)}(\mathbb{P}^c) = c(c + 1)$ while $K_i = 0$ for $3 \leq i \leq c$ and $\lambda_c = c^2 + 2c - 2$. Hence, $\dim_{(X)} \text{Hilb}^{P_{X}(f)}(\mathbb{P}^c) < \lambda_c$ for $c \geq 3$, cf. [31] Example 3.3].

(3) Note that even though [34] assumes $A = (f_{ij})$ minimal, i.e. $f_{ij} = 0$ when $\deg f_{ij} = a_j - b_i = 0$, [34] Theorem 3.5] holds without this assumption owing to [2,1]! Indeed Proposition 3.12 and (3.3) of [34] yield Theorem 3.5. But [34] Proposition 3.12] is generalized in [33] Proposition 2.4] where we only assume the last $c - 2$ columns to be without units. Now as the degrees of the entries of any $A$ obey [2,1], we either have $a_i = b_i$ or $a_i > b_i$. Take the smallest $i$ (say $i = i_0$) with $a_i = b_i$. First if $f_{ii} = 0$ and $f_{ij} \neq 0$ for some $j > i$ (resp. $f_{ij} = 0$ for all $j > i$) with $a_j = a_i$, then interchange the columns $A^i$ and $A^j$ of $A$ (resp. leave $A$ unchanged). Next if $f_{ii}$ is (or has become by the operation above) a unit, then replace all columns $A^j$ with $j > i$ and $a_j = a_i$ by

$$A^j = \frac{f_{ij}}{f_{ii}} A^i$$

Then repeat the above taking the next smallest $i$ (say $i = i_1$) with $a_i = b_i$ and so on, and note that the column operations we perform when $i = i_1$ don’t change the $0$’s already created in the $i_0$-th row in the case $a_{i_1} = a_{i_0}$ and certainly not when $a_{i_1} > a_{i_0}$. Finally if $a_i > b_i$ for some $i := i_2$, then all $f_{ij}$ for $j > i_2$ are non-units (in the $i_2$-th row). Note that the above elementary column operations don’t change $I_t(A)$, cf. Fitting’s lemma. In addition the $A$ we create in this way is without units in the last $c - 1$ columns. Thus as [34] (3.3]) holds (this doesn’t dependent on a given $A$), [34] Theorem 3.5 and Proposition 12.12] hold under the assumptions of this paper.

(4) The following interesting observation also makes a correction to [33] Proposition 5.1]; note that the locus $W_s(b; a)$ inside GradAlg(H) of graded, possibly artinian, $A := R/I_t(A)$ of [33] equals the $W(b; a)$ of this paper via $(A) \mapsto (\text{Proj}(A))$ whenever $\dim A > 0$ (i.e. when $H(v) := \dim A_v \neq 0$ for all $v \gg 0$): Now suppose there exists $i, j'$ with $a_{j'} = b_i$. Then using [2,1] as above there’s a $j \leq i$ with $a_j = b_i$, namely $j := j'$ if $j' < i$ and $j := i$ if $j' \geq i$. Fix $i$ and $j \leq i$ with $a_j = b_i$. Repeatedly using the two column operations in (3) (also when $j' < j$), we get a matrix $A$ with $f_{ij}$ either 0 or a unit and with $f_{ij'} = 0$ for all $j' \neq j$ with $a_{j'} = a_j$.

In the case $f_{ij}$ is a unit we can further use the displayed formula in (3) to create a matrix with $f_{ij'} = 0$ for all $j' \neq j$. Then if $A'$ is the matrix obtained by deleting the $i$-th row and $j$-th column, $I_t(A) = I_{t-1}(A')$. Thus if $a'_t = a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{t+c-1}$ and $b'_t = b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_t$ and if all $A$ with $(R/I_t(A)) \in W_s(b; a)$ belongs to this case where $f_{ij}$ has become a unit, we get
\[ W_s(b'; a') = W_s(b; a) \] as conversely, given \( \mathcal{A}' \) we can create an \( \mathcal{A} \) by inserting a new \( i \)-th row and \( j \)-th column consisting only of 0's except at the \( i \)-th and \( j \)-th coordinate where we put the unit 1.

In the case \( f_{ij} \) of \( \mathcal{A} \) is zero and \( (R/I_1(\mathcal{A})) \in W_s(b; a) \), we consider the matrix \( \mathcal{A}_g \) obtained by letting \( f_{ij} = u \) with \( u \in k \) a unit and letting all the other \( (l, m) \) entries be those of \( \mathcal{A} \) plus a general enough \( g_{lm} \in R_{am-b_l} \). More precisely taking \( \{u_\nu\}_{\nu=1}^N = \dim \mathbb{H}om(F, G) \) as indeterminates with \( u_1 = u \) and all the other \( g_{lm} \) as a sum of different \( u_\nu \) times the monomials of \( R_{am-b_l} \), \( 2 \leq \nu \leq N \), then \( R[u_1, \ldots, u_N]/I_\mathcal{A}_g \) restricts by e.g. [33 Lemma 2.6] to a flat family over some open dense subset of \( \mathbb{A}_k^N \) containing \( (0, \ldots, 0) \), with dense image in \( W_s(b; a) \) (cf. the text after (6.1)) in which the zero \( N \)-tuple represents \( R/I_1(\mathcal{A}) \) and a general enough \( N \)-tuple represents a \( k \)-point, say, \( R/I_1(\mathcal{A}_1) \) contained in the subset \( W_s(b'; a'; a') \) of \( W_s(b; a) \) (much as in [33 Proposition 5.1, see third paragraph of its proof]; the argument above is more accurate concerning "general enough". As \( (R/I_1(\mathcal{A}_1)) \in W_s(b'; a'; a') \), the generalization \( R/I_1(\mathcal{A}_1) \) removes all ghost terms in the Eagon-Northcott resolution of \( R/I_1(\mathcal{A}) " \) arising from \( a_j = b_l \), e.g. see [33 Examples 5.3 and 5.4]).

Moreover \( W_s(b; a') \) is open in \( W_s(b; a) \) as they both are constructible by Chevalley's theorem (cf. (6.1)); moreover if we take any generalization \( R_{T'}/I_{T}(\mathcal{A}_{T'}) \) in \( W_s(b; a) \), \( T \) some local \( k \)-algebra with residue field \( k \) and \( \mathcal{A}_T := (f_{im,T}) \) a matrix with \( f_{im,T} \in R_T := R \otimes_k T \), \( \deg f_{im,T} = a_m - b_l \) satisfying \( f_{im,T} \otimes T \) as in [33 (2.14)], of an arbitrary \( R/I_1(\mathcal{A}_1) \) of \( W_s(b'; a'; a') \), then \( f_{ij,T} \) must be a unit as \( f_{ij} \) of \( \mathcal{A}_1 \) is, i.e. \( R_T/I_1(\mathcal{A}_T) \) is a \( T \)-point of \( W_s(b'; a'; a') \). So the latter is open in \( W_s(b; a) \); in fact with \( T \) as the local ring at \( x := (R/I_1(\mathcal{A}_1)) \) of an open affine Spec(\( V \)) \( W_s(b; a) \) (in the scheme structure), then there's an open dense subset \( U \) of Spec(\( V \)), \( x \in U \), in which all entries \( f_{im,T} \), thus the pullback of \( R_T/I_1(\mathcal{A}_T) \) to \( U \), are defined; whence \( U \subset W_s(b'; a'; a') \) as \( f_{ij,T} \) is a unit (so correct the 2nd sentence of [33 Proposition 5.1] "we have an inclusion \( W_s(b'; a'; a') \subset W_s(b; a) \) of open irreducible subsets of GradAlg(\( H \))" by deleting "open" in "open irreducible..." and replacing "an inclusion" by "an open inclusion"). Hence we always have \( \overline{W_s(b'; a')} = \overline{W_s(b; a)} \) when \( a_j = b_i \) for some \( i, j \). Thus we may prior to computing the invariants of (3.1), successively delete from \( b; a \) all pairs \( (a_j, b_i) \) with \( a_j = b_i \), i.e. changing \( b; a \) in this way will not change the invariants of (3.1).

Concerning Problems 2.25 (2) and (3) we also gave in [31 and 36 an affirmative answer to both questions in the range \( 2 \leq c \leq 4 \) and \( n-c \geq 2 \) and in the case \( c \geq 5 \) and \( n-c \geq 1 \) provided certain numerical assumptions are verified. More precisely, we have got:

**Theorem 3.3.** Fix integers \( t \geq 2, c \geq 2, a = (a_1, a_2, \ldots, a_t-c-1) \) and \( b = (b_1, b_2, \ldots, b_t) \) as above. Assume \( a_{t+1-min\{3, t\}} \geq b_t \) for \( min\{3, t\} \leq i \leq t \) and \( n-c \geq 1. \) It holds:

(i) If \( c = 2 \) then \( \overline{W(b; a)} \) is a generically smooth irreducible component of Hilb_{P_X(t)}(\mathbb{P}_n) \) of dimension \( \lambda_2 \). Moreover Hilb_{P_X(t)}(\mathbb{P}_n) \) is smooth at every point \( (X) \in \overline{W(b; a)} \).

(ii) Assume \( c \geq 3. \) If \( n-c > 1, or n-c = 1, a_1 > b_t \) and \( a_{t-c-1} > a_t + a_{t+1} - b_1 \), then \( \overline{W(b; a)} \) is a generically smooth irreducible component of Hilb_{P_X(t)}(\mathbb{P}_n) \) of dimension \( \lambda_c + K_3 + \cdots + K_c \).

**Proof.** (i) See [16 Théorème 2] and [32 Theorem 5.16] for a generalization.

(ii) Let \( n-c \geq 1. \) For \( c = 3 \) and 4 the reader can see [34 Corollary 5.10]. The case \( c \geq 5 \) was proved in [36 Corollary 3.8] under the hypothesis \( a_{t+4} > a_t + a_{t+1} - b_1 \) improving quite a lot the
previous results of the authors in [34] Corollary 5.9. If \( n - c > 1 \), these results are generalized in [32] Corollary 5.9 (which is a corollary of Theorem 3.6(ii) below), only assuming \( n - c > 1 \).

We will now prove (ii) in the case \( n - c = 1 \) and \( a_t + c - 1 > a_t + a_t + 1 - b_1 \), generalizing [36] Corollary 3.8 a bit. We get \( \dim W(b, a) \) from Theorem 3.1. Moreover by successively deleting columns from the right hand side of \( \mathcal{A} \), and taking maximal minors, one gets a flag of standard determinantal quotients

\[
A_2 \twoheadrightarrow A_3 \twoheadrightarrow \cdots \twoheadrightarrow A_c = A
\]

and we let \( I_{A_i} := \ker(R \to A_i) \) and \( I_i := I_{A_i+1}/I_{A_i} \). Then since \( \dim A_{c-1} = 3 \), we get that every deformation of \( \text{Proj}(A_{c-1}) \) comes from deforming its matrix \( A_{c-1} \) by [36, (3.3) and (2.17)] and by [36, (3.3) and (2.17)] and [36, Corollary 5.9] (which is a corollary of Theorem 3.7(ii) below), only assuming \( \mathcal{A} \) deformation of \( \text{Proj}(\mathcal{A}_{c-1}) \). Now, due to [31] Theorem 4.6 and Lemma 4.4 it suffices to show that \( \text{Ext}_{A_c-1}^1(I_{A_c-1}/I_{A_c-1}, I_{c-1}) = 0 \). Since \( \text{Ext}_{A_c-1}^1(I_{A_c-1}/I_{A_c-1}, I_{c-1}) \subset \text{Ext}_{A_c}^1(I_{A_c}/I_{A_c}, I_{c-1}) \) by [36] (3.3) and [36, Corollary 5.9], it suffices to show that \( \text{Ext}_{A_c}^1(I_{A_c}, I_{c-1}) = 0 \). By the Eagon-Northcott resolution [3.2] we see that the largest possible degree of a relation for \( I_{A_c} \) is \( \ell_2 - b_1 \) and the smallest possible degree of a generator of \( I_{c-1} \) is \( \ell_2 \). Since \( \ell_c = \ell_2 + \sum_{j=t+2}^{t+c-2} a_j \), we get \( \text{Ext}_{A}^1(I_{A_2}, I_{c-2}) = 0 \) from

\[
\ell_2 - b_1 < \ell_2 + \sum_{j=t+2}^{t+c-2} a_j - \sum_{j=t}^{t+c-1} a_j = \ell_2 - a_t - a_{t+1} + a_{t+c-1},
\]

i.e. from \( a_{t+c-1} > a_t + a_{t+1} - b_1 \) and we are done.

\( \square \)

Remark 3.4. (1) It is worthwhile to point out that Theorem 3.3 solves Conjecture 4.2 stated by the authors in [36]. Also [18] and [32] proves Conjecture 4.2.

(2) For \( n - c = 1 \) the hypothesis \( a_{t+c-1} > a_t + a_{t+1} - b_1 \) in Theorem 3.3 cannot be eliminated. In fact, as in [31] Example 4.1(ii)] we consider the case \( t = 2, a_1 = \cdots = a_c = 1, a_{c+1} = 2 \) and \( b_1 = b_2 = 0 \), i.e., \( \mathcal{A} \) is a \( 2 \times (c+1) \) matrix with entries general linear forms everywhere except in the last column where we have general quadratic forms. The vanishing of the \( 2 \times 2 \) minors of \( \mathcal{A} \) defines a smooth irreducible curve \( C \subset \mathbb{P}^{c+1} \) of degree \( \deg(C) = 2c + 1 \) and genus \( g(C) = c \). The degree and the genus of \( C \) can be computed using the resolution of \( I(C) \) given by the Eagon-Northcott complex [3.2]:

\[
0 \longrightarrow R(-c-2)^c \longrightarrow R(-c-1)^{c-1} \oplus R(-c-1)^{c-1} \longrightarrow \cdots \\
\longrightarrow R(-2)^{(2)} \oplus R(-3)^c \longrightarrow R \longrightarrow R/I(C) \longrightarrow 0.
\]

Since \( K_i = 0 \) for \( 3 \leq i \leq c \), we have

\[
\dim W(0, 3; 1, \cdots, 1, 2) = \lambda_c = c^2 + 7c + 2.
\]

On the other hand, \( \dim_{(C)} \text{Hilb}^{\mathbb{P}^c(t)}(\mathbb{P}^{c+1}) \) is at least

\[
(c + 2) \deg(C) + (c - 2)(1 - g(C)) = c^2 + 8c.
\]

Therefore, it follows that, for \( c \geq 3, W(0, 0; 1, \cdots, 1, 2) \) is not an irreducible component of \( \text{Hilb}^{\mathbb{P}^c(t)}(\mathbb{P}^{c+1}) \), whence not every deformation of \( C \) comes from deforming its associated matrix \( \mathcal{A} \).
(3) Let \( C \subseteq \mathbb{P}^6 \) be a smooth standard determinantal curve of degree 21 and arithmetic genus 15 defined by the maximal minors of a \( 3 \times 7 \) matrix \( A \) with linear entries. The closure of \( W(b; a) \) inside \( \text{Hilb}^{21t-14}(\mathbb{P}^6) \) is not an irreducible component. In fact, let \( H_{21,15} \subseteq \text{Hilb}^{21t-14}(\mathbb{P}^6) \) be the open subset parameterizing smooth connected curves of degree \( d = 21 \) and arithmetic genus \( g = 15 \). It is well known that any irreducible component of \( H_{21,15} \) has dimension \( \geq 7d + 3(1 - g) = 105 \); while by Theorem 3.1, \( \dim W(0,0,0;1,1,1,1,1,1,1) = 90 \). Therefore, there exist deformations of \( C \) which do not come from deforming \( A \).

To increase applications we will include a series of results that instead of considering a matrix with general entries we consider standard determinantal schemes defined by matrices with explicitly depth conditions. These results could be used to treat for instance Example 2.2(ii). In this example we have seen that the rational normal curve \( C \) in \( \mathbb{P}^n \) is a standard determinantal scheme whose associated matrix has no general linear entries but it is linear and the singular locus has enough depth. Many results in our earlier papers, as well as [18], deal with general matrices, while [32] proves most results for determinantal schemes with depth conditions on the locus of submaximal minors, allowing corollaries for determinantal schemes defined by general matrices. In fact we prove in [32] the results by directly compare deformations of \( R \to A \) to those of the \( R \)-module \( MI \) without deleting columns (except in [32] Theorem 4.5)), hence entries which are units are allowed. To introduce these results we first recall the definition of good determinantal scheme.

**Definition 3.5.** A codimension \( c \) standard determinantal scheme \( X \subseteq \mathbb{P}^n \) is called a **good determinantal** scheme if its associated \( t \times (t + c - 1) \) homogeneous matrix \( A \) contains a \( (t - 1) \times (t + c - 1) \) submatrix (allowing a change of basis if necessary) whose ideal of maximal minors defines a scheme of codimension \( c + 1 \).

**Remark 3.6.** It is well known that a good determinantal scheme \( X \subseteq \mathbb{P}^n \) is standard determinantal and the converse is true provided \( X \) is a generic complete intersection (i.e. \( \text{depth}_{\mathcal{I}}(A) A \geq 1 \)), cf. [10]. Thus good or standard (determinantal) are equivalent assumptions in Theorem 3.7 below.

In [32] the first author proved, for \( c \geq 2 \), the following theorem

**Theorem 3.7.** Let \( X = \text{Proj}(A) \subseteq \mathbb{P}^n \), \( A = R/\mathcal{I}(A) \), be a good determinantal scheme of \( W(b; a) \).

(i) If \( \text{depth}_{\mathcal{I}}(A) A \geq 3 \), or if \( n - c \geq 1 \) and we get a local complete intersection (e.g. a smooth) scheme by deleting some column of \( A \), then

\[
\dim W(b; a) = \lambda_e + K_3 + K_4 + \ldots + K_c .
\]

In particular this equality holds if \( n - c \geq 1 \) and \( a_{i-1} \geq b_i \) for \( 2 \leq i \leq t \).

(ii) If \( \text{depth}_{\mathcal{I}}(A) A \geq 4 \), or if \( n - c \geq 2 \) and we get a local complete intersection (e.g. a smooth) scheme by deleting some column of \( A \), then the Hilbert scheme \( \text{Hilb}^{\lambda_e}(\mathbb{P}^n) \) is smooth at \( (X) \), \( \dim_X \text{Hilb}^{\lambda_e}(\mathbb{P}^n) = \dim W(b; a) \), and every deformation of \( X \) comes from deforming \( A \).

**Proof.** See [32] Theorem 5.5, Corollary 5.6 and Theorem 5.8]. \( \square \)
Here we say that we get a local complete intersection (shortly, l.c.i.) scheme by deleting some column if
\[
(3.4) \quad m = \sqrt{I_{t-1}(B)},
\]
where \(B\) is the matrix obtained by deleting a column of \(A\). It implies \(\sqrt{I_{t-1}(B)} = \sqrt{I_{t-1}(A)}\) and depth\(I_{t-1}(A)_A = \dim A\). Using this notion we succeeded in [32], with a rather complicated proof (see [32, Theorem 4.5]), to weaken the assumption depth\(I_{t-1}(A)_A \geq j\) as above in several theorems. In this paper we generalize Theorem [32] further and by a much easier proof. Indeed, we will prove

**Theorem 3.8.** Let \(X = \text{Proj}(A) \subset \mathbb{P}^n\), \(A = R/I_t(A)\) be a standard determinantal scheme of \(W(k; a)\).

(i) If depth\(I_{t-1}(A)_A \geq 2\) and \(c \geq 2\), or depth\(I_{t-1}(A)_A \geq 3\) and \(c = 1\), then
\[
\dim W(k; a) = \lambda_c + K_3 + K_4 + \ldots + K_c .
\]

(ii) If depth\(I_{t-1}(A)_A \geq 3\) and \(c \geq 2\), then the Hilbert scheme \(\text{Hilb}^{pX(t)}(\mathbb{P}^n)\) is smooth at \((X)_t\)
\[
\dim_{(X)} \text{Hilb}^{pX(t)}(\mathbb{P}^n) = \dim W(k; a),
\]
and every deformation of \(X\) comes from deforming \(A\).

To show it we need to generalize the following result from [32] (avoiding [32, Theorem 4.5]) in a similar way, i.e. by replacing “depth\(I_{t-1}(A)_A \geq j\), or dim\(A \geq j - 1\)” provided we get an l.c.i. (e.g. a smooth) scheme by deleting some column of \(A\)” by “depth\(I_{t-1}(A)_A \geq j - 1\)” for \(j = 3, 4\).

**Theorem 3.9.** Let \(A = R/I_t(A)\) be a standard determinantal ring, let \(MI = \text{coker}(\varphi^*)\) and suppose

(i) either depth\(I_{t-1}(A)_A \geq 3\), or dim\(A \geq 2\) provided we get an l.c.i. (e.g. a smooth) scheme by deleting some column of \(A\). Then Hom\(_A(MI, MI) \simeq A\) and
\[
\text{Ext}^1_A(MI, MI) = 0 .
\]

(ii) either depth\(I_{t-1}(A)_A \geq 4\), or dim\(A \geq 3\) provided we get an l.c.i. (e.g. a smooth) scheme by deleting some column of \(A\). Then Hom\(_A(MI, MI) \simeq A\) and
\[
\text{Ext}^i_A(MI, MI) = 0 \quad \text{for } 1 \leq i \leq 2 .
\]

In particular \(\text{Ext}^1_R(MI, MI) \simeq \text{Hom}_R(I_t(A), A)\).

(iii) depth\(I_{t-1}(A)_A \geq 1\). Then the local deformation functor, Def\(_{MI/R}\), of \(MI\) as a graded \(R\)-module is formally smooth (i.e. \(MI\) is unobstructed), depth\(\text{Ext}^1_R(MI, MI) \geq \dim A - 1\), and
\[
\dim_0\text{Ext}^1_R(MI, MI) = \lambda_c + K_3 + K_4 + \ldots + K_c .
\]

**Proof.** See [32, Corollaries 4.7, 4.9 and Theorem 5.2] for (i) and (ii) and [32, Theorem 3.1] for (iii). Note that \(c \geq 2\) in [32], but the proof in [32, Theorem 3.1] easily applies to \(c = 1\), leading to
\[
(3.5) \quad \text{depth }\text{Ext}^1_R(MI, MI) \geq \dim A - 2 , \quad \text{and } \dim_0\text{Ext}^1_R(MI, MI) = \lambda_1 .
\]
when \( \text{depth}_{I_{t-1}(A)} A \geq 2 \) because then \( \text{Hom}_R(MI, MI) \simeq A \) by \([24]\) with \( Z = V(I_{t-1}(A)A) \). \(\square\)

**Remark 3.10.** (i) By deformation theory \( \partial \text{Ext}^1_R(MI, MI) \) is the tangent space of the local deformation functor \( \text{Def}_{MI/R} \). Moreover, using the Buchsbaum-Rim complex, we get that \( MI \) is unobstructed and that every deformation of \( MI \) comes from deforming \( A \) (see the 1st paragraph of the proof of \([32\), Theorem 3.1]). The unobstructedness of \( MI \) is also proved in Ile’s thesis (see \([28\), ch. 6).

(ii) By observing that the map
\[
d_1 : \wedge^{t+1} G^* \otimes S_0(F) \otimes \wedge^t F \to G^*
\]
appearing in the Buchsbaum-Rim complex \([33]\) satisfies \( d_1 \subset I_A \cdot G^* \) for \( c \geq 2 \) (\( d_1 = 0 \) if \( c = 1 \)), we get \( \partial \text{Hom}_R(d_1, MI) = 0 \) since \( MI \) is an \( A \)-module. Applying \( \text{Hom}_R(\cdot, MI) \) onto \([33]\), the definition of \( \partial \text{Ext}^1_R(MI, MI) \) yields, for \( c \geq 1 \), an exact sequence
\[
(3.6) \quad 0 \to \partial \text{Hom}_R(MI, MI) \to \partial \text{Hom}_R(F^*, MI) \to \partial \text{Hom}_R(G^*, MI) \to \partial \text{Ext}^1_R(MI, MI) \to 0 ,
\]
determining \( \text{dim} \partial \text{Ext}^1_R(MI, MI) \), and an injection
\[
(3.7) \quad \text{Ext}^2_R(MI, MI) \hookrightarrow \text{Hom}_R(\wedge^{t+1} G^* \otimes S_0(F) \otimes \wedge^t F, MI) .
\]

To see that the generalized version of Theorem \([3.9]\) leads to Theorem \([3.8]\) we use the exact sequence
\[
(3.8) \quad 0 \to \text{Ext}^1_A(MI, MI) \to \text{Ext}^1_R(MI, MI) \to E_2^{0,1} \to \text{Ext}^2_A(MI, MI) \to \text{Ext}^2_R(MI, MI) \to E_2^{1,1} \to \text{Ext}^2_R(MI, MI) \to 0,
\]
associated to the spectral sequence
\[
E_2^{p,q} := \text{Ext}^p_A(\text{Tor}^R_q(A, MI), MI) \Rightarrow \text{Ext}^{p+q}_R(MI, MI),
\]
noting that \( E_2^{p,0} \simeq \text{Ext}^p_A(MI, MI) \) and that \( \text{Tor}^R_q(A, MI) \simeq \text{Tor}^R_{q-1}(I(X), MI) \) for \( q \geq 1 \). We get
\[
E_2^{0,1} \simeq \text{Hom}_A(I(X) \otimes_R MI, MI) \simeq \text{Hom}_R(I(X), \text{Hom}_R(MI, MI)) \simeq \text{Hom}_R(I(X), A),
\]
and looking carefully (cf. \([33\), Lemma 3.3]) we may identify \( \text{Ext}^1_R(MI, MI) \to E_2^{0,1} \) with the tangent map from \( \text{Def}_{MI/R} \) into the functor of graded deformations of \( R \to A \) induced by Fitting’s lemma, cf. \([13\), Corollary 20.4]. Hence, it suffices to prove

**Theorem 3.11.** Let \( A = R/I_t(A) \) be a standard determinantal ring, let \( MI = \text{coker}(\varphi^*) \) and suppose \( c = 1 \) and \( \text{depth}_{I_{t-1}(A)A} A \geq 3 \), or \( c \geq 2 \) and \( \text{depth}_{I_{t-1}(A)A} A \geq 2 \) (resp. \( \text{depth}_{I_{t-1}(A)A} A \geq 3 \)). Then \( \text{Hom}_A(MI, MI) \simeq A \) and
\[
\text{Ext}^i_A(MI, MI) = 0 \quad \text{for} \quad i = 1 \quad \text{(resp.} \quad i = 1 \quad \text{and} \quad 2) .
\]
In particular, if \( \text{depth}_{I_{t-1}(A)A} A \geq 3 \) and \( c \geq 2 \), then \( \text{Ext}^1_R(MI, MI) \simeq \text{Hom}_R(I_t(A), A) \).

**Proof.** For \( c \geq 2 \), note that \( \text{Hom}_A(MI, MI) \simeq A \) by \([33\), Lemma 3.2], owing to \( \text{depth}_{I_{t-1}(A)A} A \geq 1 \). Let \( Z = V(I_{t-1}(A)A) \) and suppose \( \text{depth}_{I_{t-1}(A)A} A \geq 2 \) and \( c \geq 2 \). Using Theorem \([3.9\), iii) we get
\[
\text{depth}_{I(Z)} \text{Ext}^1_R(MI, MI) \geq \text{depth}_{I(Z)} A - 1 \geq 1 .
\]
Applying the local cohomology functor $H^0_{I(Z)}(-)$, which is left exact, onto (3.8) we therefore get $H^0_{I(Z)}(\text{Ext}^1_A(MI, MI)) \hookrightarrow H^0_{I(Z)}(\text{Ext}^1_R(MI, MI)) = 0$, whence $\text{Ext}^1_A(MI, MI) = 0$ because $\tilde{MI}$ is locally free of rank 1 over $\text{Spec}(A) \setminus Z$. Using (3.5) we get $\text{Ext}^1_A(MI, MI) = 0$ if $\text{depth}_{I_{t-1}(A)}A \geq 3$ and $c = 1$ by the same argument.

Suppose $\text{depth}_{I_{t-1}(A)}A \geq 3$ and $c \geq 2$ and define $C \hookrightarrow \text{Ext}^2_A(MI, MI)$ such that the sequence

$$0 \to \text{Ext}^1_R(MI, MI) \to \text{Hom}_R(I(X), A) \to C \to 0$$

obtained from (3.8) and $\text{Ext}^1_A(MI, MI) = 0$, is exact. Applying $H^0_{I(Z)}(-)$ to this sequence we get

$$H^0_{I(Z)}(C) \hookrightarrow H^1_{I(Z)}(\text{Ext}^1_R(MI, MI)) = 0.$$

But $C \subset \text{Ext}^2_A(MI, MI)$ and $\tilde{MI}$ is locally free over $\text{Spec}(A) \setminus Z$. So $\tilde{C} = 0$ over $\text{Spec}(A) \setminus Z$ and we get $C = 0$. Finally using (3.8) and (3.7) we get injections

$$\text{Ext}^2_A(MI, MI) \hookrightarrow \text{Ext}^2_R(MI, MI) \hookrightarrow \text{Hom}_R(\wedge^{t+1}G^* \otimes S_0(F) \otimes \wedge^t F, MI),$$

whence $H^0_{I(Z)}(\text{Ext}^2_A(MI, MI)) = 0$ since $H^0_{I(Z)}(MI) = 0$ and we have proved that $\text{Ext}^2_A(MI, MI) = 0$.

Finally remark how easily we get Theorem 3.5(ii) in the case $\text{depth}_{I_{t-1}(A)}A \geq 3$. Indeed by (3.8) $\text{Ext}^1_R(MI, MI) \cong \text{Hom}_R(I(X), A)$ are isomorphic and then Theorem 3.3(iii) concludes the proof.

To illustrate Theorems 3.1 and 3.3 and Remark 3.4 we consider the particular case of standard determinantal schemes defined by the maximal minors of a matrix with all entries of the same degree $d \geq 1$. We have

**Corollary 3.12.** Fix integers $d \geq 1$, $t \geq 2$, $c \geq 2$ and set $a_i = d$ for $1 \leq i \leq t + c - 1$ and $b_j = 0$ for $1 \leq j \leq t$. Assume $n - c \geq 2$. Then, $\overline{W(0; d)}$ is a generically smooth irreducible component of $\text{Hilb}^{\mathbb{P}^n}(-\mathbb{P}^n)$ of dimension

$$\lambda_c = t(t + c - 1)\left(\frac{n + d}{d}\right) - t^2 - (t + c - 1)^2 + 1.$$

As another example we have:

**Example 3.13.** Let $m \geq 1$ and $n \geq 4$ be integers. In the case $n = 4$ we also suppose $m \geq 3$. Let $X \subset \mathbb{P}^n$ be a standard determinantal scheme of codimension 3 given by the maximal minors of a $t \times (t + 2)$ matrix $[L, M]$ where $L$ is a matrix with linear entries and $M$ a column with entries homogeneous forms of degree $m$. Note that $X$ is a curve if $n = 4$, but Theorem 3.3(ii) applies to any $n \geq 4$, to get that $\overline{W(0; 1, \cdots, 1, m)}$ is a generically smooth irreducible component of $\text{Hilb}^{\mathbb{P}^n}(-\mathbb{P}^n)$ of dimension

$$\dim_{(X)} \text{Hilb}^{\mathbb{P}^n}(-\mathbb{P}^n) = \lambda_3 + K_3 = t(t + 2)(n + 1) - 2t^2 - 4t - 3 \quad \text{for} \quad m = 1,$$

and

$$\dim_{(X)} \text{Hilb}^{\mathbb{P}^n}(-\mathbb{P}^n) = \lambda_3 + K_3 = (\binom{m + n}{n}t + t(t + 1)(n - 1) - 1 - (t + 1)\binom{m + n - 1}{n}) + \binom{m + n - t - 1}{n}.$$  

Note that for $n = 4$ and $1 \leq m \leq 2$, $\overline{W(0; 1, \cdots, 1, m)}$ is not always a component, see Remark 3.4(2).
4. Unobstructedness of Quotients of Zero Sections

The aim of this section is to demonstrate the following key result in achieving our goals: Any determinantal subscheme \( X = \text{Proj}(A) \subset \mathbb{P}^n \), \( A = R/I_{t+1-r}(\varphi^*) \) is determined by a section

\[
\sigma^* : B \longrightarrow N \otimes B(a_{t+c-1}),
\]

(as its degeneracy locus) where \( B = R/I_{t+1-r}(\varphi^*_{t+c-2}) \) is the determinantal ring of \((t+1-r) \times (t+1-r)\) minors of the matrix obtained deleting the last column of \( \varphi^* \) and \( N = \text{coker}(\varphi^*_{t+c-2}) \). This result will allow us to obtain results for the Hilbert scheme \( \text{Hilb}^{\mathbb{P}X(t)}(\mathbb{P}^n) \) around \((X \subset \mathbb{P}^n)\) by using the deformation theory given in \([30]\) for deforming the embedded scheme \( Y = \text{Proj}(B) \subset \mathbb{P}^n \) together with deforming its regular section, i.e. for deforming the pair \(((N \otimes B)^*, \sigma), (-)^* = \text{Hom}_B(-, B)\).

To fix a general setting for deforming \( \text{Proj}(B) \subset \mathbb{P}^n \) and its regular section \( \sigma \), let \( B = R/I_B \) be any graded quotient, \( M \) a finitely generated graded \( B \)-module and \( p_Y(t) \in \mathbb{Q}[t] \) the Hilbert polynomial of \( Y := \text{Proj}(B) \subset \mathbb{P}^n \). Let \( Z \subset Y \) be a closed subset such that \( M|_U \) is locally free of rank \( r \) on \( U := Y \setminus Z \). Let \( \sigma_U \in H^0(U, \tilde{M}^*(s)) \) be a regular section on \( U \) inducing a section \( \sigma : M_1(-s) \longrightarrow B \) where \( M_1 = H^0(U, \wedge^r \tilde{M}) \). Let \( A = \text{coker}(\sigma), X := \text{Proj}(A) \) and \( I_{A/B} = \ker(B \rightarrow A) \). It holds

**Proposition 4.1.** Let \( r = 2 \). With notations and conditions as above, suppose

1. \( \text{depth}_{I(Z)} M_2 \geq 3, \text{depth}_{I(Z)} M \geq 2, \text{depth}_m M_2 \geq 4 \) and \( \text{depth}_m M \geq 3 \).
2. \( (-)^* \text{Ext}^2_B(M, M_2) = 0 \) and \( \text{Ext}^2_B(M, B) = 0 \).
3. \( \text{Ext}^2_B(M, M) = 0 \), or \( \text{Ext}^2_B(M, M) = 0 \) and \( M \) is “liftable” to any (graded) deformation of \( B \).

Then the 2\textsuperscript{nd} projection \( p_2 : \text{Hilb}^{\mathbb{P}X(t), p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) \) is smooth at \((X \subset Y)\). Moreover if

4. \( \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0 \),

then the 1\textsuperscript{st} projection \( p_1 : \text{Hilb}^{\mathbb{P}X(t), p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{\mathbb{P}X(t)}(\mathbb{P}^n) \) is smooth at \((X \subset Y)\). Hence the natural map \( H^0(X, \mathcal{N}_X) \longrightarrow H^0(X, \mathcal{N}_Y|_X) \) is surjective, \( X \) is \( p_Y \)-generic, and

\[
h^0 \mathcal{N}_X + \text{hom}_{\mathcal{O}^n}(\mathcal{I}_Y, \mathcal{I}_X|_Y) = h^0 \mathcal{N}_Y + \text{hom}_{\mathcal{O}_Y}(\mathcal{I}_X|_Y, \mathcal{O}_X),
\]

\[
\dim_X(\text{Hilb}^{\mathbb{P}X(t)}(\mathbb{P}^n)) + \text{hom}_{\mathcal{O}^n}(\mathcal{I}_Y, \mathcal{I}_X|_Y) = \dim_Y(\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)) + \text{hom}_{\mathcal{O}_Y}(\mathcal{I}_X|_Y, \mathcal{O}_X).
\]

In particular, \( Y \) is unobstructed if and only if \( X \) is unobstructed.

**Proof.** The proposition follows from \([30]\) Proposition 13 and Theorem 47] after having remarked some details. Indeed, since \( \sigma_U \) is a regular section on \( U \), we have the Koszul resolution

\[
0 \longrightarrow \wedge^2 \tilde{M}(-2s) \longrightarrow \tilde{M}(s) \xrightarrow{\sigma_U} \tilde{B} \longrightarrow \text{coker} \sigma_U \longrightarrow 0
\]
which is exact over $U$. Splitting the Koszul resolution into two short exact sequences and applying $H^0_s(U, -)$ we get that

\[(4.1) \quad 0 \to M_2(-2s) \to M_1(-s) \to H^0_s(U, \text{coker } \sigma_U) \]

is exact because $H^1_s(U, \wedge^2 \tilde{M}) = 0$ by the assumption (i). Since by definition $A = B/\text{im}(\sigma)$ we have proved the exactness of

\[0 \to M_2(-2s) \to M_1(-s) \to I_{A/B} \to 0.\]

If $\text{depth}_{I(Z)} M_2 \geq 4$ and $\text{depth}_{I(Z)} M_1 \geq 3$, we even get $H^1_s(U, \text{im}(\sigma_U)) = 0$, whence $A \cong H^0_s(U, \text{coker}(\sigma_U))$ and $\text{depth}_{I(Z)} A \geq 2$, while the weaker assumptions of (i) only imply $\text{depth}_m A \geq 2$. We also have $\text{depth}_B B \geq 2$ (since $\text{depth}_m M_1 \geq 3$), in which case the natural projection map

\[p_2 : \text{Hilb}_{\mathbb{P}^n}^{\mathbb{P}X(t), \mathbb{P}Y(t)}(\mathbb{P}^n) \to \text{Hilb}_{\mathbb{P}^n}^{\mathbb{P}Y(t)}(\mathbb{P}^n)\]

is isomorphic to the corresponding projection of the graded deformation functors at $(X \subset Y)$ by (2.2) and (2.3). Then [30, Theorem 47] (see its proof for the smoothness of $p_1$) applies since [30, Proposition 13] and (iii) above imply that the assumptions of [30, Theorem 47] are fulfilled.

The proof in [30, Theorem 47] is actually just a proof of the fact that any deformation $B_T \to A_T$, $T$ local artinian with residue field $k$, fits into a lifting of $M_1(-s) \to B$ in (4.1) to $T$ (so to $B_T$) and that this lifting $\sigma_T$ can be lifted further to any artinian local $T'$ with $T' \to T$.

**Remark 4.2.** (1) We have $M = M_1$ because $\text{depth}_{I(Z)} M \geq 2$. Similarly if we assume $2 \leq \text{depth}_{I(Z)}(\wedge^2 M)$ we get $M_2 \cong \wedge^2 M$ because $M_2 = H^0_s(U, \wedge^2 \tilde{M})$. In this case we may in Proposition 4.1(i) replace $M_2$ by $\wedge^2 M$.

(2) Due to the exact sequence

\[(4.2) \quad 0 \to M_2(-2s) \to M_1(-s) \to I_{A/B} \to 0,\]

we get that

\[(iv') \quad 0\text{Ext}^1_B(I_B/I_B^2, M_1(-s)) = 0 \quad \text{and} \quad 0\text{Ext}^2_B(I_B/I_B^2, M_2(-s)) = 0\]

imply (iv) of Proposition 4.1, i.e. $0\text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$. Using $(iv')$ instead of $(iv)$ in Proposition 4.1 we get a result with assumptions only on $B$ (and $M_i$, $i = 1, 2$).

Our next goal will be to analyze if the section $\sigma$ exists in our situation and whether the hypothesis (i)-(iv) of the above proposition are satisfied. There are quite a lot of Ext-groups that have to vanish, but fortunately our applications are for determinantal schemes and for determinantal schemes we will see that the assumptions are often fulfilled! Note that if $c = 1$, then [4.2] leads to

\[(4.3) \quad 0 \to K_B(t - 2s) \to N_B(-s) \to B \to A \to 0,\]

where $N_B := \text{Hom}_R(I_B, B)$ is the normal module, i.e. exactly to the case we studied in [35] in which many of the Ext-groups above vanish “almost for free” because $K_B$ is the canonical module.
We keep the notation introduced in section 2. So, we have a graded morphism

\[ \varphi : F := \bigoplus_{i=1}^{t} R(b_i) \longrightarrow \bigoplus_{j=1}^{t+c-1} R(a_j) =: G \]

and if \( c \geq 3 - r \), i.e. \( t + c - 2 \geq t + 1 - r \) we delete the last row to get

\[ \varphi_{t+c-2} : F \longrightarrow \bigoplus_{j=1}^{t+c-2} R(a_j) =: G_{t+c-2}. \]

Let \( B = R/I_{t+1-r}(\varphi^*_{t+c-2}) \), \( A = R/I_{t+1-r}(\varphi^*) \), \( N := \text{coker}(\varphi^*_{t+c-2}) \), \( M_I := \text{coker}(\varphi^*) \), \( X = \text{Proj}(A) \) and \( Y = \text{Proj}(B) \). We are going to prove the main result of this section, namely that \( X \) is determined by a (twisted) regular section \( \sigma^* \) of \( N \otimes B \) where \( \sigma^* = \text{Hom}_{B}(\sigma, B) \). This result generalizes [35, Proposition 4.3] from \( c = 1 \), \( r = 2 \) to \( c \geq 1 \), \( r \geq 1 \). With a slight change in the proof, mainly replacing \( = \text{Proj}(B) \) by \( = \text{Spec}(B) \), the result also holds for \( \dim B = r \). More precisely we have

**Theorem 4.3.** With \( B \twoheadrightarrow A \) and \( N \) as above (with \( B \) and \( A \) determinantal), we suppose \( \text{depth}_{J_B} B \geq 2 \), where \( J_B = I_{t-r}(\varphi^*_{t+c-2}) \) and \( \dim B > r \). Then the commutative diagram

\[
\begin{array}{cccccc}
0 & & & & & 0 \\
\downarrow & & & & & \downarrow \\
G^*_{t+c-2} \otimes B & \longrightarrow & F^* \otimes B & \longrightarrow & N \otimes B & \longrightarrow & 0 \\
\downarrow & & \| & & & \downarrow \\
G^* \otimes B & \longrightarrow & F^* \otimes B \\
\downarrow & & & & & \downarrow \\
B(-a_{t+c-1}) & & & & & 0 \\
\end{array}
\]

induces a homogeneous section \( \sigma^* : B \longrightarrow N \otimes B(a_{t+c-1}) \), regular on the open subset \( \text{Proj}(B) \setminus V(J_B) \) where \( N \otimes B \) is locally free of rank \( r \), whose zero locus precisely defines \( A \) as a quotient of \( B \), i.e. \( A = B/\text{im}(\sigma) \). Moreover a twist of \( \sigma^* \) fits into an exact sequence

\[ 0 \longrightarrow B(-a_{t+c-1}) \longrightarrow N \otimes B \longrightarrow M_I \otimes B \longrightarrow 0. \]

**Proof.** We will start proving the theorem for \( 2 \times 2 \) minors by describing the maps in the diagram

\[
\begin{array}{cccccc}
0 & & & & & 0 \\
\downarrow & & & & & \downarrow \\
B(-a_{t+c-1}) & \sigma^* \downarrow & & & & \\
\downarrow & & \| & & & \downarrow \\
G^*_{t+c-2} \otimes B & \varphi^*_{t+c-2} \otimes \text{id}_B & F^* \otimes B & \longrightarrow & N \otimes B & \longrightarrow & 0 \\
\end{array}
\]
over the open set $D(f_{11}) \subset \text{Proj}(B)$, letting the matrix of $\varphi^*$ be $A = (f_{ij})_{1 \leq i \leq t+e-1}$. We will use Gauss elimination as follows. Let

$$I(f_{11}) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-f_{21} & f_{11} & 0 & \cdots & 0 \\
-f_{31} & 0 & f_{11} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-f_{t1} & 0 & 0 & \cdots & f_{11}
\end{pmatrix}.$$  

Then

$$I(f_{11}) \cdot A = \begin{pmatrix}
f_{11} & f_{12} & f_{13} & \cdots & f_{1t+c-1} \\
0 & f_{22}^2 & f_{23}^2 & \cdots & f_{2t+c-1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & f_{t2}^2 & f_{t3}^2 & \cdots & f_{tt+c-1}^2
\end{pmatrix},$$

where $f_{ji}^2 = -f_{ji}f_{ii} + f_{11}f_{ji}$, $2 \leq i \leq t + c - 1$ and $2 \leq j \leq t$. These are all the $2 \times 2$ minors involving $f_{11}$. Moreover since $B = R/I_2(\varphi_{t+c-2}^*)$, all $2 \times 2$ minors above belong to $I_2(\varphi_{t+c-2}^*)$ and lead to a vanishing of a corresponding map in $[t+c-2]$, except those involving the last column. More precisely, $[t+c-2]$ extended to $G^*$ and restricted to the open set $D(f_{11}) \subset \text{Proj}(B)$ is given by

$$G^* \otimes B_{(f_{11})} \xrightarrow{\varphi^* \otimes \text{id}_{B_{(f_{11})}}} F^* \otimes B_{(f_{11})} \approx (\bigoplus_{i=1}^t R(-b_i)) \otimes B_{(f_{11})},$$

where $F^* \otimes B_{(f_{11})} \cong (\bigoplus_{i=1}^t R(-b_i)) \otimes B_{(f_{11})}$ is given by multiplication with $I(f_{11})$, and the morphism

$$G^* \otimes B_{(f_{11})} \longrightarrow (\bigoplus_{i=1}^t R(-b_i)) \otimes B_{(f_{11})}$$

is induced by

$$\begin{pmatrix}
f_{11} & f_{12} & \cdots & f_{1t+c-2} & f_{1t+c-1} \\
0 & 0 & \cdots & 0 & f_{2t+c-1}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & f_{tt+c-1}^2
\end{pmatrix}.$$  

So $G_{t+c-2}^* \otimes B_{(f_{11})} \xrightarrow{\bigoplus_{i=1}^t B(-b_i)_{(f_{11})}} \bigoplus_{i=2}^t B(-b_i)_{(f_{11})}$ and we have $N \otimes B_{(f_{11})} \cong \bigoplus_{i=2}^t B(-b_i)_{(f_{11})}$ since the ideal of $B_{(f_{11})}$ generated by $\{f_{11}, f_{12}, \cdots, f_{1t+c-2}\}$ contains $1 = f_{11}/f_{11}$. Moreover $\sigma^*$ restricted to the degree zero part of $B(-a_{t+c-1})_{(f_{11})}$ is given by mapping $1 \in B(-a_{t+c-1})_{(f_{11})}$ onto the transpose of the vector

$$[f_{2t+c-1}^2 f_{3t+c-1}^2 \cdots f_{tt+c-1}^2] := v_{11}^T.$$  

Note that this argument can be done for any entry $f_{ij}$, $j \neq t + c - 1$ of $A$, e.g. for $f_{1j}$, yielding a map $B(-a_{t+c-1})_{f_{1j}} \rightarrow \bigoplus_{i=2}^t B(-b_i)_{f_{1j}}$ taking $1 \in B(-a_{t+c-1})_{f_{1j}}$ onto the corresponding $v_{1j}$ which
for \( j = 2 \) is \([f_{2t+c-1}^2 f_{3t+c-1}^2 \cdots f_{tt+c-1}^2]^r\), where \( f_{ji}^2 = -f_{j1} f_{i1} + f_{j1} f_{i1} \). Since \( f_{ij} \cdot v_{11} = f_{ij} \cdot v_{1j} \) these local sections glue together.

The argument above extends also to \( f_{ij} \) for \( i > 1 \). Indeed, taking the \( i \)th row, we exchange it with the first row and proceed as above except for being a little more careful with the gluing. Say we exchange the 1st and 2nd row and we use Gauss elimination leaving the left upper corner, where now \( f_{21} \) sits, fixed. Then the section \( \sigma^* \) restricted to the degree zero part of \( B(-a_{t+c-1}) \) is given by mapping \( 1 \in B(-a_{t+c-1}) \) onto the transpose of \([f_{1t+c-1} f_{2t+c-1}^2 \cdots f_{tt+c-1}^2]^r\) = \( v_{21}^r \), where now \( f_{ji}^2 = -f_{j1} f_{i1} + f_{j1} f_{i1} \). Letting \( F(f_{21}) \) be the following \((t - 1) \times (t - 1)\) matrix:

\[
F(f_{21}) = \begin{pmatrix}
-f_{11} & 0 & 0 & \cdots & 0 \\
-f_{31} & f_{21} & 0 & \cdots & 0 \\
-f_{41} & 0 & f_{21} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-f_{t1} & 0 & 0 & \cdots & f_{21}
\end{pmatrix}
\]

the local sections glue because \( F(f_{21}) \cdot v_{11} = f_{11} \cdot v_{21} \). Thus we get \( \sigma^* \) well defined over

\[
\bigcup_{j \neq t+c-1} D(f_{ij}) = \text{Proj}(B) \setminus V(I_1(\varphi^*_{t+c-2}))
\]

and it extends to \( \text{Proj}(B) \) by the depth assumption. It follows that the description of \( \sigma \) in the Theorem holds noting that, with \( \{f_{ij}^2\} = \{f_{ij}^2\}_{2 \leq j \leq \min(t+c-2)} \), we have the following equalities of ideals

\[
I_2(\varphi^*_{t+c-2}) B(f_{i1}) = B(f_{i1})(\{f_{ij}^2\}),
\]

and \( I_2(\varphi^*_{t+c-2}) B(f_{21}) = B(f_{21})(\{f_{ij}^2\}) \) and correspondingly for all the \( B(f_{ij}) \) having \( j \neq t + c - 1 \).

Now we consider the case of \( 3 \times 3 \) minors and we will describe (4.5) in the open set \( D(f_{ij}^{12}) \) of \( \text{Proj}(B) \), where \( B = R/I_3(\varphi^*_{t+c-2}) \). To this end, we denote by \( f_{j_1j_2\cdots j_k}^{i_1i_2\cdots i_k} \) the determinant of the submatrix of \( A \) consisting of the columns \( i_1, i_2, \ldots, i_k \) and the rows \( j_1, j_2, \ldots, j_k \). As above, to describe (4.3) in the open set \( D(f_{ij}^{12}) \subset \text{Proj}(B) \) we use Gauss elimination. We multiply \( A \) by the \( t \times t \) matrix:

\[
I(f_{11}, f_{12}^{12}) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
-f_{21} & f_{11} & 0 & 0 & \cdots & 0 \\
f_{23}^{12} & -f_{13}^{12} & f_{12}^{12} & 0 & \cdots & 0 \\
f_{24}^{12} & -f_{14}^{12} & 0 & f_{12}^{12} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
f_{2t}^{12} & -f_{1t}^{12} & 0 & 0 & \cdots & f_{12}^{12}
\end{pmatrix}
\]
and we get, using determinantal expansion along columns, that

\[
I(f_{11}, f_{12}^3)A = \begin{pmatrix}
  f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & \cdots & f_{1t+c-1} \\
  0 & f_{12}^3 & f_{12}^3 & f_{12}^3 & f_{12}^3 & \cdots & f_{12t+c-1} \\
  0 & 0 & f_{12} & f_{12} & f_{12} & \cdots & f_{12t} \\
  0 & 0 & 0 & f_{12} & f_{12} & \cdots & f_{12t} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\
  0 & 0 & 0 & 0 & 0 & \cdots & f_{12} \\
\end{pmatrix},
\]

where all \(3 \times 3\) minors belong to \(I_3(\varphi^*_{t+c-2})\), except those involving the last column which define the section \(\sigma^*\) over the open \(D(f_{11}) \cap D(f_{12}^3) \subset \text{Proj}(B)\). Note that all \(3 \times 3\) minors of the form \(f_{12j}^{12i}, i, j \geq 3\) are obtained from the lower \((t-2) \times (t+c-3)\) block. And, moreover, if we start the Gauss elimination above by using the matrix \(I(f_{12}, f_{12}^3)\) defined to be equal to \(I(f_{11}, f_{12}^3)\) for all rows except the 2nd row where \([-f_{21} f_{11} 0 0 \cdots]\) is replaced by \([-f_{22} f_{12} 0 0 \cdots]\), we see that \(I(f_{12}, f_{12}^3)\) is invertible over \(D(f_{12}) \cap D(f_{12}^3)\) and that \(I(f_{12}, f_{12}^3)A\) is equal to \(I(f_{11}, f_{12}^3)A\) in all rows except the 2nd row where we get \([-f_{12}^3 f_{12}^3 f_{12}^3 \cdots f_{12}^3]\) for \(I(f_{12}, f_{12}^3)A\). This defines \(\sigma^*\) in the open set \(D(f_{12}) \cap D(f_{12}^3)\) and since \((f_{12}^3) \subset (f_{11}, f_{12})\) we have \(\sigma^*\) defined over

\[
(D(f_{12}) \cap D(f_{12}^3)) \cup (D(f_{11}) \cap D(f_{12}^3)) = D(f_{12}^3).
\]

Thus

\[
\sigma^*: B(-a_{t+c-1}(f_{12}^3)) \rightarrow N \otimes B(f_{12}^3) = \oplus_{i=3}^{t} B(-b_i(f_{12}^3))
\]

is given by the \(3 \times 3\) minors of the last column of \(I(f_{11}, f_{12}^3)A\), where \(\text{im}(\sigma) \otimes B(f_{12}^3)\) is the ideal of \(B(f_{12}^3)\) generated by \(\{f_{12}^{12t+c-2} f_{12}^{12t+c-2}, \cdots, f_{12}^{12t+c-2}\}\) and we conclude because a determinantal ideal does not change under elementary row/column operations.

It is now clear how to proceed to finish the proof. If \(B = R/I_4(\varphi^*_{t+c-2})\), we multiply \(A\) by

\[
I_1 = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  -f_{21} & f_{11} & 0 & 0 & 0 & \cdots & 0 \\
  f_{12} & -f_{12}^3 & f_{12}^3 & 0 & 0 & \cdots & 0 \\
  -f_{23} & f_{13}^3 & -f_{13}^3 & f_{13}^3 & 0 & \cdots & 0 \\
  -f_{12}^{123} & f_{12}^{123} & -f_{12}^{123} & f_{12}^{123} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\
  -f_{23}^{123} & f_{13}^{123} & -f_{12}^{123} & 0 & 0 & \cdots & f_{12}^{123} \\
\end{pmatrix}
\]

and determinantal expansions along columns imply that \(I_1 \cdot A\) is equal to \(I(f_{11}, f_{12}^3)A\) in the first 3 rows and that the last \(t-3\) rows of \(I_1 \cdot A\) will be of the form

\[
[0 | f_{12i}^{12j}], \quad 4 \leq i \leq t+c-1, \quad 4 \leq j \leq t
\]

and a \((t-3) \times 3\) matrix of zeros which lead to

\[
\text{im}(\sigma) \otimes \Gamma(U_i, \tilde{B}) \cong \Gamma(U_i, \tilde{I_{A/B}})
\]

where \(\{U_i\}\) is an open covering of \(\text{Proj}(B) \setminus V(I_3(\varphi^*_{t+c-2}))\).
More precisely
to get \[ -f_{22} \, f_{12} \, 0 \, 0 \, \cdots \]
proceeding as for \( 3 \times 3 \) minors above, the section \( \sigma^* \) will be defined over the open \( D(f_{12}^{132}) \cap D(f_{123}^{132}) \).
Then since
\[ f_{123}^{132} = f_{31} \, f_{12}^{132} - f_{32} \, f_{12}^{13} + f_{33} \, f_{12}^{12} \]
we replace the 3rd row of \( I_1 \) by \( \begin{bmatrix} f_{12}^{13} - f_{12}^{13} & f_{12}^{13} \, f_{12}^{1} \, 0 \, 0 \, \cdots \end{bmatrix} \) to get \( I_3 \), respectively \( \begin{bmatrix} f_{12}^{23} - f_{12}^{13} & f_{12}^{13} \, f_{12}^{1} \, 0 \, 0 \, \cdots \end{bmatrix} \) to get \( I_4 \).
Computing \( I_1 \cdot A \) for \( i = 3, 4 \), we obtain \( I_1 \cdot A \) except in the 3rd row where we obtain \( \begin{bmatrix} 0 & f_{123}^{123} & 0 \, f_{123}^{134} \, f_{123}^{135} \, \cdots \end{bmatrix} \) for \( I_3 \cdot A \); respectively, \( f_{123}^{13} \, 0 \, f_{123}^{23} \, f_{123}^{35} \, \cdots \) for \( I_4 \cdot A \).
Combining with replacing the 2nd row with \( [-f_{22} \, f_{12} \, 0 \, \cdots] \), cf. \( I_2 \) above, we get the section \( \sigma^* \) defined over \( D(f_{12}^{132}) \cap D(f_{123}^{132}) \), respectively \( D(f_{12}^{23}) \cap D(f_{123}^{123}) \).
Therefore, \( \sigma^* \) is defined over \( D(f_{123}^{132}) \) by \( (4.6) \).
Continuing this process we get in general the existence of the section \( \sigma^* \) whose zero locus defines \( A \) by \( A = B/ \text{im}(\sigma) \).
Finally note that \( \sigma^* \) is injective because it is regular on \( \text{Proj}(B) \setminus V(J_B) \) Thus the sequence
\[ 0 \to B(-a_{t+c-1}) \to N \otimes B \to MI \otimes B \to 0 \]
is exact, and we are done. \( \square \)

We will apply Proposition \( 4.4 \) to the section \( \sigma^* : B \to N \otimes B(a_{t+c-1}) \) given by Theorem \( 4.3 \). To do so letting \( M = (N \otimes B)^* := \text{Hom}_B(N \otimes B, B) \), we need to verify all assumptions of Proposition \( 4.4 \).

**Proposition 4.4.** Let \( M = (N \otimes B)^* \), \( J_B := I_{t-r}(\varphi^*_{t+c-2}) \) and assume \( c \geq 3 - r \) and \( r \geq 2 \).

It holds:

(i) Suppose \( \text{depth}_{J_B} B \geq 1 \) and \( J_B \neq R \). If \( c \geq 2 \) then \( M \) and \( N \otimes B \) are maximal Cohen-Macaulay \( B \)-modules. If \( c \leq 1 \), then \( M \) is a maximal Cohen-Macaulay \( B \)-module while \( N \otimes B \) has codepth 1.

(ii) If \( \text{depth}_{J_B} B \geq 3 \) and \( J_B \neq R \), then \( \text{Ext}^i_B(M, B) = 0 \) and \( \text{Ext}^i_B(N \otimes B, B) = 0 \) for \( i = 1, 2 \).

(iii) Suppose \( \text{depth}_{J_B} A \geq 2 \), \( J_A := I_{t-r}(\varphi^*) \neq R \) and if \( 3 - r \leq c \leq 0 \) we also suppose \( \text{depth}_{J_A} A \geq 3 \). Then \( \text{Ext}^1_B(M, I_{A/B}) = 0 \) and the normal module of \( B \to A \) is maximally Cohen-Macaulay (resp. of codepth 1) for \( c \geq 1 \) (resp. \( 3 - r \leq c \leq 0 \)) and satisfies
\[ \text{Hom}_B(I_{A/B}, A) \cong \text{Hom}_B(M, A)(a_{t+c-1}) \cong MI \otimes A(a_{t+c-1}) \).

(iv) Suppose \( \text{depth}_{J_B} A \geq 3 \), \( J_A \neq R \) and if \( 3 - r \leq c \leq 0 \) we also suppose \( \text{depth}_{J_A} A \geq 4 \). Then \( \text{Ext}^1_B(M, A) = \text{Ext}^1_A(I_{A/B}/I_{A/B}^2, A) = 0 \) and \( \text{Ext}^i_B(M, I_{A/B}) = 0 \) for \( i = 1, 2 \).

**Remark 4.5.** (1) Proposition \( 4.4 \) (i) and (ii) hold also for \( c \geq 2 - r \) and \( r = 1 \).

(2) Note that we have \( \dim B - \dim A = r \) and \( \dim B - \dim B/J_B = c + 2r - 1 \) if \( B/J_B \) is determinantal (allowing \( J_B := J_B \cdot B \)). Considering \( J_B \) as an ideal in \( A \), then \( A/J_B \cong B/J_B \) and we get:
\[ \text{depth}_{J_B} A = \dim A - \dim A/J_B = c + r - 1, \]
while \( \text{depth}_{J_B} A \leq c + r - 1 \) holds in general. So assuming \( \text{depth}_{J_B} A \geq 3 \) in Proposition \( 4.4 \) (iv) we implicitly assume \( c \geq 4 - r \). If the matrix \( A \) is defined by general homogeneous polynomials
and $b_i < a_1$, then we may replace depth$_J B \ A \geq 3$ by $c \geq 4 - r$ (and dim $A \geq 3$), and simultaneously depth$_J A \ A \geq 4$ by dim $A \geq 4$.

**Proof.** (i) It follows from [6, Theorems 1 and 2].

(ii) It follows from [6, Theorems 4 and 5].

(iii) Let $J = J_B$ and $U_B := \text{Proj}(B) \setminus V(J)$. Sheafifying the exact sequence

\[
\begin{array}{c}
0 \rightarrow B(-a_{t+c-1}) \xrightarrow{\sigma^*} N \otimes B \rightarrow MI \otimes B \rightarrow 0
\end{array}
\]

we see that $\tilde{\sigma}^*|_{U_B}$ is given by minors belonging to $\tilde{I}_{A/B}|_{U_B}$, whence $(\tilde{\sigma}^* \otimes \text{id}_A)|_{U} = 0$, where $U := U_B \cap \text{Proj}(A)$. It follows that $N \otimes A|_U \cong MI \otimes A|_U$.

Dualizing the exact sequence (4.7), we get the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & I_{A/B}(a_{t+c-1}) \\
\downarrow & & \downarrow \\
\text{Hom}_R(MI, B) & \rightarrow & \text{Hom}_R(N, B) \xrightarrow{\sigma} B(a_{t+c-1}).
\end{array}
\]

If we sheafify and restrict to $U_B$, we get again that $\tilde{\tilde{M}}|_{U_B} \xrightarrow{\tilde{\tilde{\sigma}}|_{U_B}} \tilde{I}_{A/B}(a_{t+c-1})|_{U_B}$ is generated by the minors of $\tilde{I}_{A/B}(a_{t+c-1})|_{U}$. Since $N \otimes B$ is locally free over $U_B$ we have

\[
\begin{array}{c}
\tilde{M} \otimes A|_U \cong \text{Hom}(\tilde{N} \otimes B, \tilde{A})|_{U_B} \\
\cong \text{Hom}(\tilde{N} \otimes A, \tilde{A})|_U \\
\cong \text{Hom}(MI \otimes A, \tilde{A})|_U.
\end{array}
\]

Let $c \geq 1$. Applying Hom$_R(\cdot, A)$ once more, and using depth$_J A = \text{depth}_J(MI \otimes A) \geq 2$ we get

\[
\text{Hom}_R(M, A) \cong H^0_* (U, \text{Hom}(\tilde{M}, \tilde{A})) \\
\cong H^0_* (U, M \otimes A) = MI \otimes A
\]

and, similarly

\[
\begin{array}{c}
\text{Hom}_R(M, A) \cong H^0_* (U, \text{Hom}(I_{A/B}(a_{t+c-1}) \otimes \tilde{A}, \tilde{A})) \\
\cong \text{Hom}_A(I_{A/B}(a_{t+c-1}) \otimes \tilde{A}, A)(-a_{t+c-1}).
\end{array}
\]

These are maximal Cohen-Macaulay $A$-modules because $MI \otimes A$ is a maximal Cohen-Macaulay $A$-module for $c \geq 1$ by (i). If, however, $3 - r \leq c \leq 0$, then the assumption depth$_J A \ A \geq 3$ implies depth$_J A(MI \otimes A) \geq 2$ and letting $U_A := \text{Proj}(A) \setminus V(J_A)$, we get

\[
\begin{array}{c}
MI \otimes A \cong H^0_* (U_A, MI \otimes A) \\
\cong H^0_* (U, MI \otimes A)
\end{array}
\]
because $MI \otimes A$ is locally free over $U_A \supset U$ and depth$_J A \geq 2$. Thus the two displayed formulas for $\text{Hom}_R(M, A)$ above hold in this case too. Moreover, since we have an exact sequence

$$0 \to \text{Hom}_B(M, I_{A/B}) \to \text{Hom}_B(M, B) \to \text{Hom}_B(M, A) \to \text{Ext}_B^1(M, I_{A/B}) \to \text{Ext}_B^2(M, B)$$

(4.8)

and $\text{Ext}_B^1(M, B) = 0$ by (ii), we get $\text{Ext}_B^1(M, I_{A/B}) = 0$.

(iv) Let $c \geq 1$. Since depth$_J A = \text{depth}_J(MI \otimes A) \geq 3$ and $M \otimes A|_U \cong I_{A/B} \otimes A(a_{t+c-1})|_U$ are locally free we get

$$\text{Ext}_B^1(M, A) \cong H_B^1(U_B, \text{Hom}(\tilde{M} \otimes \tilde{A}, \tilde{A})) \cong H_B^1(U_B, MI \otimes A) = H_B^3(MI \otimes A) = 0$$

and up to twist;

$$\text{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) \cong H_A^1(U, \text{Hom}(\tilde{I}_{A/B} \otimes \tilde{A}, \tilde{A})) \cong H_A^1(U_B, MI \otimes A) = H_A^3(MI \otimes A) = 0.$$

Then (4.8) leads to

$$\to \text{Ext}_B^1(M, I_{A/B}) \to \text{Ext}_B^1(M, B) \to \text{Ext}_B^1(M, A) \to \text{Ext}_B^2(M, I_{A/B}) \to$$

and using $\text{Ext}_B^1(M, B) = 0$ from (ii), we get $\text{Ext}_B^2(M, I_{A/B}) = 0$ which proves (iv) when $c \geq 1$.

Finally if $3 - r \leq c \leq 0$, then the assumption depth$_J A \geq 4$ implies $H_A^2(MI \otimes A) = 0$, and it suffices to show $H_A^1(U_B, MI \otimes A) \cong H_A^1(U_A, MI \otimes A)$. This, however, follows from $\text{depth}_J A \geq 3$ and the fact that $MI \otimes A$ is locally free over $U_A \supset U$. □

Now we restrict to $r = 2$, i.e. to $k$-algebras defined by submaximal minors. We have that

$$(4.9) \quad 0 \to M_2 \to M_1 \to B(a_{t+c-1}) \to A(a_{t+c-1}) \to 0$$

is exact, whence $M_2 = \text{Hom}_B(MI, B)$ is a maximal Cohen-Macaulay $B$-module. Indeed, since $M_1$ is a maximal Cohen-Macaulay $B$-module and $B$ and $A$ are determinantal, hence Cohen-Macaulay, the exact sequence (4.9) implies that $M_2$ is also a maximal Cohen-Macaulay $B$-module. In this case we have the vanishing of the following Ext-groups.

**Proposition 4.6.** Let $B = R/I_{t-1}(\varphi_{t+c-2}^*) \to A := R/I_{t-1}(\varphi^*)$ be determinantal rings and suppose $c \geq 1$ and $I_{t-2}(\varphi^*) \neq R$. Let $M = (N \otimes B)^*$ and $J_B = I_{t-2}(\varphi_{t+c-2}^*)$. Then, it holds:

(i) If $\text{depth}_J B \geq 3$ (resp. $\geq 4$), we have $\text{Ext}_B^i(M, M_2) = 0$ for $i = 1$ (resp. $i = 1, 2$).

(ii) Suppose $\text{depth}_J B \geq 2$ (resp. $\text{depth}_J A \geq 3$). Then we have

$$\text{Ext}_B^i(M, M) = 0 \text{ for } i = 1 \text{ (resp. } i = 1, 2).$$
Proof. (i) The exact sequence (4.9) is induced from the Koszul resolution of a regular sequence of 2 elements, whence $\tilde{M}_2|_{U_B} \cong \wedge^2 \tilde{M}(-a_{t+c-1})|_{U_B}$. Moreover since the rank of $M$ is 2, we have isomorphisms

$$\tilde{M}^* \otimes \wedge^2 \tilde{M}|_{U_B} \cong \tilde{M}|_{U_B} \cong \text{Hom}(\tilde{M}, \tilde{M}_2(a_{t+c-1}))|_{U_B}.$$ 

Using depth $J_B B = \text{depth}_B M = \text{depth}_B M_2 \geq 3$, we get

$$\text{Ext}_B^1(M, M_2) \cong \text{Ext}_{U_B}^1(\tilde{M}, \tilde{M}_2) \cong H_B^1(U_B, \text{Hom}(\tilde{M}, \tilde{M}_2)) \cong H_B^2(M(-a_{t+c-1})) = 0$$

and correspondingly, $\text{Ext}_B^2(M, M_2) \cong H_B^3(M(-a_{t+c-1})) = 0$ if depth $J_B B \geq 4$.

(ii) Using the exact sequence

$$0 \rightarrow M_2 \rightarrow M_1 = M \rightarrow I_{A/B}(a_{t+c-1}) \rightarrow 0,$$

we deduce an exact sequence

$$0 \rightarrow \text{Ext}_B^1(M, M_2) \rightarrow \text{Ext}_B^1(M, M) \rightarrow \text{Ext}_B^1(M, I_{A/B}(a_{t+c-1})) \rightarrow \text{Ext}_B^2(M, M_2) \rightarrow 0.$$ 

Since depth $J_B A \geq 2$ and depth $J_B B = \text{depth}_B A + 2 \geq 4$, we get $\text{Ext}_B^1(M, M) = 0$ by Proposition 4.4 (iii) and Proposition 4.6 (i).

Similarly, if depth $J_B A \geq 3$, we get

$$\text{Ext}_B^2(M, M) \cong \text{Ext}_B^2(M, I_{A/B}(a_{t+c-1})) = 0$$

by Proposition 4.4 (iv) and Proposition 4.6 (i), and we are done. \hfill \Box

Corollary 4.7. Let $B = R/I_{t-1}(\varphi_{t+c-2}^\ast) \rightarrow A := R/I_{t-1}(\varphi^\ast)$ be determinantal rings defined by submaximal minors, let $J_B := I_{t-2}(\varphi_{t+c-2}^\ast)$ and suppose $c \geq 1$ and $I_{t-2}(\varphi^\ast) \neq R$. Then:

(i) If depth $J_B A \geq 3$, then (i), (ii) and (iii) of Proposition 4.7 hold.

(ii) Suppose depth $J_B A = 2$ and that every deformation of $B$ comes from deforming its matrix, then (i), (ii) and (iii) of Proposition 4.7 hold.

Proof. Since depth $J_B B = \text{depth}_B A + 2 \geq 4$ and $M_2$ is maximally Cohen-Macaulay by (4.2) and we may take $Z = V(J_B)$, all conclusions follow from Propositions 4.4 and 4.6 except possibly $\text{Ext}_B^2(M, M) = 0$ which however is true if depth $J_B A \geq 3$. Only assuming depth $J_B A \geq 2$ we have at least $\text{Ext}_B^1(M, M) = 0$. Moreover in this case, $M$ is ”liftable” to any deformation of $B$. The argument for this is given right before (5.5) in the next section; the short version of that argument is that there is by assumption a matrix $B_T$ and a corresponding morphism $(\varphi_{t+c-2}^\ast)_T$ which defines any given deformation $B_T$ of $B$ to a local artinian $T$. Set $N_T = \text{coker}((\varphi_{t+c-2}^\ast)_T)$. Then $N_T$ is a deformation of $N$ and thus the $T$-flat $\text{Hom}(N_T \otimes B_T, B_T)$ lifts $M$ to $T$, and we are done. \hfill \Box

In [35] we studied the case $c = 1$ of Proposition 4.1. Now we take an example where $c = 2$. 


Example 4.8. (Determinantal quotients of $R = k[x_0, x_1, \cdots, x_n]$, using Proposition 4.1)

Let $A = [B, v]$ be a general $3 \times 4$ matrix with $B$ a linear matrix and $v$ a column with all entries of degree $m$. Thus the degree matrix of $A$ is $\begin{pmatrix} 1 & 1 & \cdots & 1 & m \\ 1 & 1 & \cdots & 1 & m \end{pmatrix}$ and set $b_i = 0$ for $1 \leq i \leq 3$. The vanishing of all $2 \times 2$ minors defines a determinantal ring that satisfies all conditions of Corollary 4.7 for all $m \geq 1$ provided $n \geq 8$. Indeed codim$_R B = 4$ and $r = 2$ (submaximal minors), so $n \geq 8$ is equivalent to dim $A \geq 2$. To avoid some details we suppose dim $A \geq 3$. So $n \geq 9$ and since $B$ is general, we may, after a linear coordinate change, suppose that $B$ is the “generic” linear matrix with entries $x_0, x_1, \ldots, x_8$. For such a matrix, by results of the next sections, one know that the closure of the determinantal locus $\overline{W(0, 0, 0; 1, 1, 1; 2)}$ containing $(\text{Proj}(B))$, is a generically smooth component of Hilb$(\mathbb{P}^n)$ by Proposition 5.3 and Corollary 5.4 and of dimension $64 + 9(n - 8) = 9n - 8$ (Example 6.18(i) and Corollary 6.15(ii)). In fact $\lambda_c$ defined in (3.1) for $c = 1$ is $9n - 8$.

We claim that $\overline{W(0, 0, 0; 1, 1, 1, m; 2)}$ is a generically smooth irreducible component of Hilb$(\mathbb{P}^n)$ for every $m \geq 3$ of dimension

$$\dim \overline{W(0, 0, 0; 1, 1, 1; 2)} + \dim (MI \otimes A)_m = 9n - 8 + \dim (MI \otimes A)_m.$$  

Indeed, all assumptions of Proposition 4.1 hold except possibly $\partial \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$. To show that it vanishes we consider the minimal free resolution of $R/I_B$ given by Lascoux in [32] or, even easier, the Gulliksen-Negaard free resolution (25) or you may for simplicity run Macaulay2 to see that the minimal free resolution is

$$0 \longrightarrow R(-6) \longrightarrow R(-4)^9 \longrightarrow R(-3)^{16} \longrightarrow R(-2)^9 \longrightarrow I_B \longrightarrow 0.$$  

(4.10)

The generators of $I_{A/B}$ are $2 \times 2$ minors involving the last column of $A$. So all generators of $I_{A/B}$ have degree $m + 1 \geq 4$. Applying $\partial \text{Hom}_R(-, I_{A/B})$ to (4.10) we get that all terms in

$$0 \longrightarrow \partial \text{Hom}_R(I_B, I_{A/B}) \longrightarrow \partial \text{Hom}_R(R(-2)^9, I_{A/B}) \longrightarrow \partial \text{Hom}_R(R(-3)^{16}, I_{A/B}) \longrightarrow$$

vanish. So, by definition of $\partial \text{Ext}^i_B(I_B, I_{A/B})$ for $i = 1, 2$, we have

$$\partial \text{Hom}_R(I_B, I_{A/B}) = \partial \text{Ext}^1_B(I_B, I_{A/B}) = 0.$$  

Finally, since $\partial \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) \subset \partial \text{Ext}^1_B(I_B, I_{A/B})$, it follows that $\partial \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$. Thus Proposition 4.1 applies and since smooth morphisms maps irreducible components to irreducible components we get the claim. Indeed, using also the exact sequences in Proposition 4.1 we get

$$\dim \overline{W(0, 0, 0; 1, 1, 1, m; 2)} = \dim \overline{W(0, 0, 0; 1, 1, 1; 2)} + \dim (MI \otimes A)_m$$

because Hom$_B(I_{A/B}, A) \cong MI \otimes A(a_t+c−1)$ by Proposition 4.4.

Example 4.8 illustrates how Corollary 4.7 “generalizes” the main results of [35], which only holds for submaximal minors with $c = 1$. Indeed, Corollary 4.7 also deals with submaximal minors but it allows any $c \geq 1$. However in [35] we also compute dim $\overline{W(b_a; r)}$ for $c = 1$, which really means that dim $(MI \otimes A)_{a_t+c−1}$ is computed, under quite weak assumption (which we generalize further in Theorem 6.22). For $c \geq 2$, we have not been able to compute dim $(MI \otimes A(a_t+c−1))_0$, in general, but as we will see in the next sections we succeed to compute it under some assumptions.
Remark 4.9. By Corollary 4.7, if \( \text{depth}_{J_B} A \geq 3 \) (resp. \( \text{depth}_{J_B} A \geq 2 \) under an additional assumption) and \( r = 2 \), then all assumptions of Proposition 4.1 are satisfied (except possibly (iv)), hence the first conclusion holds. The nice thing to remark now is that Proposition 4.4 holds for any \( r \geq 2 \). In particular, if \( \text{depth}_{J_B} A \geq 3 \), and \( \text{depth}_{J_A} A \geq 4 \) in the case \( c \leq 0 \), then

\[
\text{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) = 0
\]

for any \( r \geq 2 \). Since \( \text{depth}_{J_B} A \geq 3 \) implies

\[ H^2(B, A, A) \cong \text{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) \]

(because \( \text{Hom}(H_2(B, A, A), A) = 0 \), see [29, Section 1.1] and its references for background on algebra (co)homology) we get, also for \( r \geq 3 \), all conclusions of Proposition 4.1 using [29, Proposition 4(ii)].

More precisely, recalling \( J_A := J_{l-r}(\varphi^*) \), we have

Proposition 4.10. Let \( B \rightarrow A \) be as in Theorem 4.3 and suppose \( \text{depth}_{J_B} A \geq 3 \), \( J_A := J_{l-r}(\varphi^*) \neq R \), \( r \geq 2 \) and \( c \geq 3 - r \). Suppose also \( \text{depth}_{J_A} A \geq 4 \) if \( c \leq 0 \). Then \( \partial H^2(B, A, A) = 0 \), and the 2nd projection \( p_2 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) \) is smooth at \( (X \subset Y) \).

Moreover, if Proposition 4.1(iv) holds, or

\[ (\text{iv}^*) \quad \partial \text{Ext}_B^1(I_B/I_B^2, I_{A/B}) \leftrightarrow \partial \text{Ext}_B^1(I_B/I_B^2, B) \]

is injective and \( B \) unobstructed (as a graded algebra),

then the 1st projection \( p_1 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}^{p_X(t)}(\mathbb{P}^n) \) is smooth at \( (X \subset Y) \). Hence the natural map \( H^0(X, \mathcal{N}_X) \rightarrow H^0(X, \mathcal{N}_Y|_X) \) is surjective, \( X \) is \( p_Y \)-generic, and

\[
\dim(X) \text{Hilb}^{p_X(t)}(\mathbb{P}^n) + \dim(Y) \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) + \dim(\mathcal{X}) \text{Hilb}^{p_X(t)}(\mathbb{P}^n) + \dim(\mathcal{Y}) \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) = 0
\]

In particular, \( Y \) is unobstructed if and only if \( X \) is unobstructed.

Proof. By Remark 4.9 \( \partial H^2(B, A, A) = 0 \), whence we get the smoothness of the projection

\[ q : \text{GradAlg}(H_B, H_A) \rightarrow \text{GradAlg}(H_B) \]

given by \( q((B' \rightarrow A')) = (B') \) (cf. [30, p. 234]) which implies the surjectivity below

\[
\begin{align*}
\text{Hom}_B(I_{A/B}, A) & \leftrightarrow \text{Hom}_R(I_A, A) \rightarrow \partial \text{Hom}_R(I_B, A) \rightarrow \partial H^2(B, A, A) \\
\text{H}^0(X, \mathcal{N}_X) & \rightarrow \text{H}^0(X, \mathcal{H}(\mathcal{I}_Y, \mathcal{O}_X))
\end{align*}
\]

Indeed, the smoothness of

\[ q : \text{GradAlg}(H_B, H_A) \rightarrow \text{GradAlg}(H_B) \]
is a consequence of the fact that $\Omega^2(B,A,A)$ is the obstruction group of deforming $A_S$ to $B_T$ in the diagram

\[
\begin{array}{ccc}
B_T & \rightarrow & A_S \\
\downarrow & & \downarrow \\
B_S & \rightarrow & A_S \\
& & 0
\end{array}
\]

where $T \rightarrow S$ is a small artinian surjection and $B_T$, $B_S$ and $A_S$ are deformations of $B$ and $A$ (\cite{29} Remark 3)).

Finally either (iv): $0\Ext^1_B(I_B/I_B^2,I_{A/B}) = 0$ and \cite{29} Proposition 4(ii)], or the statement given in (iv” and \cite{29} Proposition 4(iii)], imply that the projection

\[ p : \text{GradAlg}(H_B,H_A) \rightarrow \text{GradAlg}(H_A) \]

given by $q((B' \rightarrow A')) = (A')$ is smooth at $(B \rightarrow A)$. The smoothness of $p$ and $q$ imply the smoothness of $p_1 : \text{Hilb}_{pX(t),pY(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}_{pX(t),pY(t)}(\mathbb{P}^n)$ and $p_2 : \text{Hilb}_{pX(t),pY(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}_{pY(t)}(\mathbb{P}^n)$ by (2.2) and (2.3) and hence all conclusions of Proposition 4.1.\hfill\Box

5. Deformation of minors

Our next goal is to analyze whether the deformation of a determinantal scheme comes from deforming the associated matrix. To this end, let $A$ (resp. $B$) be the homogeneous matrix representing $\varphi^*$ (resp. $\varphi^*_T$) if $c \geq 3 - r$). Let $\mathcal{L}$ be the category of local artinian $k$-algebras $T$ with residue field $k = T/m_T$ and with morphisms inducing the identity over $k$. Set $A = R/I_i(\varphi^*)$ where $i = t + 1 - r$ with $2 \leq i \leq t$ and $X = \text{Proj}(A)$.

Lemma 5.1. Let $\varphi^*_T$ be the map induced by a lifting $A_T$ of $A$ to $T \in \text{ob}(\mathcal{L})$, let $MI_T := \text{coker}(\varphi^*_T)$ and $A_T = R/I_i(\varphi^*_T)$ and suppose $A = R/I_i(\varphi^*)$ is determinantal. Then

(i) $MI_T$ (resp. $A_T$) are deformations of $MI$ (resp. $A$). In particular, there exists well defined maps

\[ \psi : 0\Ext^1_R(MI,MI) \rightarrow 0\Hom(I_i(\varphi^*),A) \]

and

\[ \text{Def}(\psi)(-) : \text{Def}_{MI}(-) \rightarrow \text{Def}_A(-), \]

the latter of local functors over $\mathcal{L}$, deforming $MI$ (resp. $A$) as an $R$-module (resp. $R$-quotient).

(ii) The following statements are equivalent for a fixed $i = t + 1 - r$.

(a) Every deformation of $A$ comes from deforming the matrix $A$ associated to $A$.

(b) The morphism $\text{Def}(\psi)$ of local graded deformation functors is smooth.

(c) The map $\psi$ is surjective.

Proof. (i) Let $A_T$ be a lifting of $A$ (see Definition 2.8). Recall that $A_T$ is flat over $T$ if not only generators of $I_i(\varphi^*)$ lift to polynomials in $I_i(\varphi^*_T) \subset R_T$, but also relations lift to relations of $I_i(\varphi^*_T)$. Hence we need to lift all relations since lifting of generators follows at once by taking...
appropriate minors of $A_T$. But the relations are given by the Lascoux resolution of $I_i(\varphi^*)$ (see \cite{[31]}, or see Lemma 7.7 for details of all syzygies). Indeed a relation is either given by computing the determinant of any $(i+1) \times (i+1)$ matrix arising from a submatrix of $A$ of size $i \times (i+1)$ (resp. $(i+1) \times i$) by repeating one of its rows (resp. columns) and expanding it along the mentioned row (resp. column), or by computing the determinant of an $(i+1) \times (i+1)$ submatrix of $A$ by column and row expansions and taking their differences. The latter gives also non-trivial relations. But we can do exactly the same computations using $A_T$ instead of $A$. Since the corresponding determinant (i.e. relation) or difference of determinants map via $T \mapsto k$ to the corresponding relation over $k$, we get that every relation lifts to a relation for $I_i(\varphi^*_T)$. This shows that $A_T$ is $T$-flat, and it follows that $A_T$ is a deformation of $A$.

For similar (and even easier) reasons $MI_T$ is $T$-flat. Indeed since $R/I_i(\varphi^*)$ is determinantal, it follows by \cite{[31]} Corollary 1 that $R/I_i(\varphi^*)$ is standard determinantal, whence that the Buchsbaum-Rim complex of $MI$, using $\mathbb{A}$ is exact. Moreover this resolution of $MI$ commutes with the Buchsbaum-Rim resolution of $MI_T$ using $A_T$, i.e. we can lift any relation of $MI$ to a relation of $MI_T$, whence $MI_T$ is a deformation of $MI$ (cf. \cite{[31]} Lemma 4.2)). Since $MI = \text{coker}(\varphi^*)$ it is clear that every element $(MI)_T \in \text{Def}_{MI}(T)$ is given by $MI_T = \text{coker}(\varphi^*_T)$ for some matrix $A_T$ that lifts $A$ to $T$ where $\varphi^*_T$ is the map induced by $A_T$. As proved above this matrix $A_T$ defines a deformation, i.e. an element $A_T \in \text{Def}_{A}(T)$. Since different matrices representing the same cokernel define the same determinantal ideal by e.g. Fitting’s lemma, $\text{Def}((\psi)(T))$ is well-defined. Then we define $\psi$ as the tangent map of $\text{Def}(\psi)$, i.e. we let $\psi = \text{Def}(\psi)(k[\varepsilon]/\varepsilon^2)$, and (i) is proved.

(ii) Since smooth maps are surjective on tangent spaces, the following implications are straightforward from Definition 2.3: (b) $\Rightarrow$ (a) $\Rightarrow$ (c). It only remains to prove (c) $\Rightarrow$ (b). To prove it, let us first describe $\psi$ more concretely. Take $\overline{\eta} \in \text{Ext}^1_R(MI, MI)$, let $\eta' \in \text{Hom}(G^*, MI)$ represent $\overline{\eta}$ and let $\eta \in \text{Hom}(G^*, F^*)$, with matrix $D$, map to $\eta'$. Let $D = k[\varepsilon]/(\varepsilon^2)$ and let $A_1 + \varepsilon D_1$ be some $i \times i$ submatrix of $A + \varepsilon D$ representing a corresponding composition

$$G^*_1 \hookrightarrow G^* \xrightarrow{\varphi^* + \xi \eta} F^* \rightarrow F^*_1 .$$

Then

$$\psi(\overline{\eta})(\text{det} A_1) = [\text{det}(A_1 + \varepsilon D_1) - \text{det}(A_1)]/\varepsilon = \text{det}(A_1^{\text{adj}} \cdot D_1) \otimes 1_A$$

is the image of $\text{det} A_1$ via $\psi(\overline{\eta}) \in \text{Hom}(I_i(\varphi^*), A)$ in $A$. If, say $i = 3$, $A_1 = [A^1 \ A^2 \ A^3]$ and $D_1 = [D^1 \ D^2 \ D^3]$, then

$$\psi(\overline{\eta})(\text{det} A_1) = \text{det}[D^1 \ A^2 \ A^3] + \text{det}[A^1 \ D^2 \ A^3] + \text{det}[A^1 \ A^2 \ D^3] .$$

To prove the smoothness, let $T := k[t]/(t^{n+1}) \rightarrow S := k[t]/(t^n)$ and consider the diagram

$$\begin{array}{ccc}
\text{Def}_{MI}(S) & \rightarrow & \text{Def}_{A}(S) \\
MI_S & \mapsto & A_S \\
\text{Def}_{A}(T).
\end{array}$$

Here $MI_S$ is a deformation of $MI$ given as the cokernel of $\varphi^*_S$ with matrix $A_S$ such that its $i$-minors define $A_S$. Moreover $A_T := R_T/I_T$ is an arbitrary deformation of $A_S$ to $T$. 


Let $\mathcal{A}_T$ be a lifting of the matrix $\mathcal{A}_S$ to $T$. Since $R_T \to R_S$ is surjective, it exists and defines a deformation $\mathcal{A}_T'$ (resp $MI_T'$) of $\mathcal{A}_S$ (resp. $MI_S$) to $T$. By deformation theory since $\mathcal{A}_T = R_T/I_T$ and $\mathcal{A}_T = R_T/I_T$ are deformations of the same algebra $A_S = R_S/I_S(\varphi^*_T)$ to $T$, then the difference of corresponding generators of $I_T$ and $I_I(\varphi^*_T)$ maps to zero via $(-) \otimes_T S$, i.e. these differences "belong" to

$$\text{Hom}_{R_S}(I_I(\varphi^*_S), A_S) \otimes_S (t^n)/(t^{n+1}) \cong \text{Hom}_{R}(I_I(\varphi^*_I), A) \otimes_k (t^n)/(t^{n+1}).$$

By the surjectivity assumption (b) and the description of the tangent map $\psi$ above there exists $\overrightarrow{\eta} \in \text{Ext}_R^1(MI, MI)$ and $\eta \in 0\text{Hom}(G^*, F^*)$, with matrix $\mathcal{D}$, that maps to $\overrightarrow{\eta}$ and such that

$$t^n \cdot tr(\mathcal{A}_1^{adj} \mathcal{D}_1) \otimes 1_A \mod (t^{n+1})$$

is equal to the difference of the corresponding generators of $I_T - I_I(\varphi^*_T)$ (and similarly for the other $i$-minors).

Now look at the matrix $\mathcal{A}_T + t^n \mathcal{D}$ (mod $(t^{n+1})$). By the same calculation as done for the map $\psi$, only replacing $\varepsilon$ by $t^n$, one shows that

$$[\det((\mathcal{A}_1)_T + t^n \mathcal{D}_1) - \det((\mathcal{A}_1)_T)] = t^n \det(\mathcal{A}_1^{adj} \mathcal{D}_1) \otimes 1_A.$$

It follows that the generators of $A_T$ are defined by the matrix $\mathcal{A}_T + t^n \mathcal{D}$, i.e. given by its $(t+1-r)$-minors. Since the cokernel of the map given by $\mathcal{A}_T + t^n \mathcal{D}$ defines a deformation $MI_T$ to $T$ that maps to $\mathcal{A}_T$ and reduces to $MI_S$ via $(-) \otimes_T S$, we have proved that Def$(\psi)(-)$ is (formally) smooth. \hfill $\square$

**Corollary 5.2.** Let $A = R/I_{t+1-r}(\mathcal{A})$ with $1 \leq r < t$ be a determinantal ring, let $X = \text{Proj}(A)$ and suppose that $A$ has the following property:

(*) every deformation of $A$ comes from deforming its matrix $\mathcal{A}$.

Then $A$ is unobstructed and the property (*) is an open property in GradAlg($H_A$). Hence if dim $X \geq 1$, then Hilb$^p_X(\mathbb{P}^n)$ is smooth at $(X)$ and the property (*) is open in Hilb$^p_X(\mathbb{P}^n)$.

**Proof.** To see that $A$ is unobstructed, let $T \to S$ be a surjection of artinian local rings whose kernel $a$ satisfies $a \cdot m_T = 0$, and let $A_S$ be a deformation of $A$ to $S$. By assumption, $A_S = R_S/I_{t+1-r}(\mathcal{A}_S)$ for some matrix $\mathcal{A}_S = (f_{ij,S})$. Since $T \to S$ is surjective, we can lift each $f_{ij,S}$ to a polynomial $f_{ij,T}$ with coefficients in $T$ such that $f_{ij,T} \otimes_T S = f_{ij,S}$. By Lemma 5.1 it follows that $A_T := R_T/I_{t+1-r}(\mathcal{A}_T)$ is flat over $T$. Since $A_T \otimes_T S = A_S$ we get the unobstructedness of $A$, and by (2.2) that Hilb$^p_X(\mathbb{P}^n)$ is smooth at $(X)$.

Alternatively one may prove that $MI$ is unobstructed as an $R$-module by a similar argument and then get that $A$ is unobstructed as a consequence of Def$(\psi)$ being smooth by Lemma 5.1. The smoothness also implies that the property (*) is open in GradAlg($H_A$), as well as in Hilb$^p_X(\mathbb{P}^n)$, cf. the text accompanying (2.2). \hfill $\square$

It is worthwhile to point out that it is not always true that every deformation of $A = R/I_I(\varphi^*)$ comes from deforming the homogeneous matrix $\mathcal{A}$ associated to $\varphi^*$ (see, for instance, Remark 5.3(2) and Examples 6.17 and 7.13).
We will now show for so-called *generic determinantal rings* $A_{(s)}$, that the tangent map $\psi$ in Lemma 5.1 is surjective. Thus, we can conclude that every deformation of $A_{(s)}$ comes from deforming its associated matrix $A$. Indeed, for $1 \leq i \leq t$, $1 \leq j \leq t + c - 1$, let $R = k[x_{ij}]$, $A = (x_{ij})$ be the $t \times (t + c - 1)$ matrix of indeterminates of $R$ and let $\varphi : F = R^{t} \to G = R(1)^{t+c-1}$ be the morphism induced by the transpose $A^{T}$ of $A$. Then $A_{(s)} := R/I_{s}(A)$, $s = t + 1 - r$, is called a *generic determinantal ring*. By [3] Theorem 15.15 $A_{(s)}$ is rigid for every $r$, $1 \leq r \leq t$, $(r, c) \neq (1, 1)$, i.e. the algebra cohomology group $H^{1}(k, A_{(s)}, A_{(s)}) = 0$ or, equivalently, the sequence

$$0 \to \text{Der}_{k}(A_{(s)}, A_{(s)}) \to \text{Der}_{k}(R, A_{(s)}) \to \gamma \text{Hom}_{R}(I_{s}(A), A_{(s)}) \to 0$$

is exact. Here $\text{Der}_{k}(S, L)$ is the set of $k$-derivations from a $k$-algebra $S$ to an $S$-module $L$.

**Proposition 5.3.** Let $A_{(s)}$ be a generic determinantal ring, $2 \leq s \leq t$. Then every deformation of $A_{(s)}$ comes from deforming the matrix $A$ above provided $(s, c) \neq (t, 1)$.

**Proof.** Let $MI = \text{coker} \varphi^{*}$. We claim that there exist morphisms fitting into the commutative diagram

$$\begin{array}{ccc}
\text{Hom}(G^{*} \otimes F, A_{(t)}) & \cong & \text{Hom}(G^{*}, F^{*} \otimes A_{(t)}) \\
\| & & \downarrow \psi \\
\text{Der}_{k}(R, A_{(t)}) & \to & \text{Der}_{k}(R, A_{(s)}) \to \gamma \text{Hom}_{R}(I_{s}(A), A_{(s)})
\end{array}$$

for every $2 \leq s \leq t$. By Lemma 5.1 this will prove the result because $\gamma$ is surjective by (5.1).

To see the upper horizontal surjection we combine morphisms appearing in the two exact sequences

$$G^{*} \otimes A_{(t)} \to F^{*} \otimes A_{(t)} \to MI \to 0, \text{ and}$$

$$\text{Hom}(F^{*}, MI) \to \text{Hom}(G^{*}, MI) \to \text{Ext}_{R}^{1}(MI, MI) \to 0$$

induced from $\cdots \to G^{*} \to F^{*} \to MI \to 0$ ($\epsilon$ is the splice map in the Buchsbaum-Rim resolution), recalling that $\text{Hom}(G^{*}, MI) \to \text{Ext}_{R}^{1}(MI, MI)$ is well-defined and surjective by [32] (3.1) (mainly because $\text{Hom}(\epsilon, MI) = 0$) and that $MI \otimes A_{(t)} \cong MI$. Moreover, the first lower surjection is induced from the natural surjection $A_{(t)} \to A_{(s)}$ since $I_{t}(A) \subset I_{s}(A)$. Finally the leftmost vertical isomorphism is a natural identification of $\text{Hom}(G^{*} \otimes F, A_{(t)})$ and $\text{Der}_{k}(R, A_{(t)})$ because the matrix of $G^{*} \to F^{*}$ is $A = (x_{ij})$ and $R = k[x_{ij}]$.

With this identification we now check that

$$\psi(\mathbf{7})(f) = \gamma(D)(f)$$

for every $\eta \in \text{Hom}(G^{*} \otimes F^{*}, A_{(t)})$ and every $s \times s$ minor $f$, where $\mathbf{7}$ (resp. $D$) is the image of $\eta$ in $\text{Ext}_{R}^{1}(MI, MI)$ (resp. $\text{Der}_{k}(R, A_{(s)})$). Running over the standard basis of the free $A_{(t)}$-module $\text{Hom}(G^{*} \otimes F, A_{(t)})$ it suffices to check (5.2) for each element of the basis. Indeed, we may just take $\eta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}$. For similar reasons we may let $f$ be the determinant of the $s \times s$ matrix.
appearing in the left-upper corner of $A$. Then we expand this minor along its first column and we get $f = x_{11}A_{11} - x_{21}A_{21} + \cdots$ where $A_{ij}$ is the $(s-1)$-minor corresponding to $x_{ij}$. We get
\[
\gamma(D)(f) = \sum \frac{\partial f}{\partial x_{ij}}D(x_{ij}) = \frac{\partial f}{\partial x_{11}} = A_{11}
\]
because $D \in \text{Der}_k(R,A_s)$, the derivation corresponding to $\eta$, is given by $D(x_{11}) = 1$ and $D(x_{ij}) = 0$ otherwise. Moreover, by the proof of Lemma 5.1, we get that every deformation $\phi$ satisfies $\partial f = \gamma(D)(f)$ and we are done.

**Corollary 5.4.** Let $A = R/I_s(A)$, $2 \leq s \leq t$ be a generic determinantal ring, or more generally a determinantal ring for which every deformation comes from deforming its matrix $A$. Let $S$ be a flat $R$-algebra and a polynomial ring over $k$, e.g. $S = R \otimes_k k[y]$ where $y_1, \cdots, y_e$ are indeterminates. Then the matrix $A = (f_{ij})$ induces a corresponding matrix $A_S$ whose entries consists of the images of $f_{ij}$ in $S$ and we let $A_S = S/I_s(A_S)$. Then every deformation of $A_S$ comes from deforming its matrix $A_S$.

**Proof.** Let $\varphi_S$ be the morphism corresponding to the transpose of $A_S$ and let $MI_S := \text{coker} \varphi_S$. Then $MI_S = MI \otimes_R S$ by the right-exactness of the tensor product. Moreover, if we apply $(-) \otimes_R S$ onto the first terms of the Lascoux resolution of $A$, the $R$-flatness of $S$ easily implies that $S/I := A \otimes_R S$ satisfies $I = I_s(A_S)$, and that we have a commutative diagram
\[
\begin{array}{ccc}
\text{Ext}_R^1(MI,MI) \otimes_R S & \xrightarrow{\psi_{\otimes S}} & \text{Hom}(I_s(A),A) \otimes_R S \\
\downarrow \cong & & \downarrow \cong \\
\text{Ext}_S^1(MI_S,MI_S) & \xrightarrow{\psi_S} & \text{Hom}(I_s(A_S),S/I)
\end{array}
\]
where $\psi_S$ is the map $\psi$ of Lemma 5.1 for the module $MI_S$. In particular, $\psi_S$ is surjective as $\psi$ is surjective by Proposition 5.3 or assumption. By Lemma 5.1 we get that every deformation of $S/I$ comes from deforming its matrix $A_S$. \qed

The main theorem of this section is:
Theorem 5.5. Let $B = R/I_B \rightarrow A$ be determinantal algebras defined by $t+1-r$ minors of matrices $A$ (resp. $B$) representing $\varphi^*$ (resp. $\varphi^*_{t+c-2}$), and suppose $r \geq 2$, $c \geq 3-r$, $J_A := I_{t-r}(A) \neq R$ and depth$_J B \geq r + 2$ where $J = I_{t-r}(B)$. If $c \leq 0$ we also suppose depth$_J A \geq 3$. Moreover, suppose
\begin{enumerate}[(i)]
  \item $\Ext^1_B(I_B/I_B^2, I_A/B) \rightarrow \Ext^1_B(I_B/I_B^2, B)$ is injective, and
  \item every deformation of $B$ comes from deforming $B$.
\end{enumerate}
Then every deformation of $A$ (or $\Proj(A)$) comes from deforming $A$.

Remark 5.6. Computations with Macaulay2 suggest for dim $A \geq 3$ (resp. dim $A \geq 4$), that $\Ext^1_B(I_B/I_B^2, I_A/B) = 0$ if $c \neq 1$ (resp. $c = 1$), but the dimension assumption seems important (see, for instance, Example 7.13). This vanishing of $\Ext^1_B(I_B/I_B^2, I_A/B)$ is also true under some restrictions on $a_i$ (cf. Corollary 7.9 and Proposition 7.17). So, quite often we can eliminate the hypothesis (i) in the above theorem.

Proof. Let $M = (N \otimes B)^*$ and note that depth$_J B \geq r + 2$ implies depth$_J A \geq 2$, and that we have
\begin{equation}
\Ext^1_B(N \otimes B, B) = \Ext^1_B(M, B) = \Ext^1_B(M, I_A/B) = 0
\end{equation}
by Proposition 4.2. Since it suffices by Lemma 5.1 to show that any deformation of $A$ to the dual numbers $T := k[\varepsilon]/(\varepsilon^2)$ comes from deforming $A$, we only consider deformations to $T$.

Let $A_T$ be any deformation of $A$ to $T$. By assumption (i) we have a surjection in the diagram
\begin{equation}
\begin{array}{cccc}
0\Hom(I_B, B) & \rightarrow & 0\Hom(I_B, A) & \leftarrow 0\Hom_R(I_A, A) \\
\exists(B_T) & \rightarrow & \bullet & \leftarrow (A_T)
\end{array}
\end{equation}
Hence there exists a deformation $B_T$ of $B$ to $T$ and a morphism $B_T \rightarrow A_T$ reducing to $B \rightarrow A$ via $(-) \otimes_T k$ (cf. Remark 5.7). Moreover by assumption (ii), there exists a $t \times (t+c-2)$-matrix $B_T$ such that $B_T = R_T/I_{r+1-r}(B_T)$, $R_T = R \otimes k T$. Let $N_T$ be the cokernel of the morphism determined by $B_T$. Since $B_T$ and $N_T$ are flat over $T$ by Lemma 5.1, we get that $N_T \otimes B_T$ is $T$-flat and $N_T \otimes B_T \otimes k = N \otimes B$. Then $M_T := \Hom_B(N_T \otimes B_T, B_T)$ is also $T$-flat because $\Ext^1_B(N \otimes B, B) = 0$ (see 9 Proposition (A1))). Moreover, $I_{A_T/B_T := \ker(B_T \rightarrow A_T)$ is $T$-flat, and hence we have deformations of $M$, $I_A/B$ and $B$ fitting into a diagram
\begin{equation}
\begin{array}{cccc}
M_T(-a_{t+c-1}) & \rightarrow & I_{A_T/B_T} & \rightarrow & B_T \\
\downarrow & & \downarrow & & \downarrow \\
M(-a_{t+c-1}) & \rightarrow & I_{A/B} & \rightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\end{equation}
where $i \sigma' = \sigma$ and where the dotted arrow exists due to the fact that $\Ext^1_B(M(-a_{t+c-1}), I_A/B) = 0$ and deformation theory. Note that $\sigma_T$ is surjective by Nakayama’s lemma. Dualizing (horizontal
compositions) once more, we get

\[
\begin{array}{ccc}
B_T(-a_{t+c-1}) & \xrightarrow{\sigma_T^*} & \text{Hom}_{B_T}(M_T, B_T) \\
\downarrow & & \downarrow \\
B(-a_{t+c-1}) & \xrightarrow{\sigma^*} & \text{Hom}_{B}(M, B) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

where \( \alpha \) must be an isomorphism since \( \text{Hom}_{B_T}(M_T, B_T) \) is a deformation of \( \text{Hom}_{B}(M, B) \), due to \( \text{Ext}^1_B(M, B) = 0 \). Indeed taking the cokernel \( C \) of \( \alpha \), we get \( C \otimes k = 0 \), i.e. \( C_\nu \otimes k = 0 \) where \( C = \oplus C_\nu \) and hence \( C_\nu = 0 \) by Nakayama’s lemma for all \( \nu \). In the same way the kernel of \( \alpha \) vanishes. Since \( \sigma^* \) defines the last column of \( A \) (restricted to \( B \)) we can use \( \sigma_T^* \) to define \( s_T \) in the diagram

\[
\begin{array}{ccc}
G_{t+c-2}^* \otimes B_T & \xrightarrow{\exists s_T} & F^* \otimes B_T \\
& \xrightarrow{\sigma_T^*} & N_T \otimes B_T \\
& \xrightarrow{\sigma_T} & 0
\end{array}
\]

and to lift the column \( s_T(1) \) to a column \( v_T \in (F^* \otimes R_T)_{(a_{t+c-1})} \) such that \( v_T \), by putting \( \varepsilon = 0 \), becomes exactly equal to the last column \( v \in \oplus_{i=1}^r R(a_{t+c-1} - b_i)_0 \) of \( A \), i.e. \( v = v_T \otimes k \). Set

\[
\text{A}_T = [B_T, v_T].
\]

It remains to see that the determinantal ring given by the \( t + 1 - r \) minors of \( A_T \) is \( A_T \) or, equivalently, that \( I_{A_T/B_T} \) is given by the \( t + 1 - r \) minors involving the last column since \( B_T \) is already given, i.e. it suffices to show

\[
I_{t+1-r}(A_T)/I_{t+1-r}(B_T) = I_{A_T/B_T}.
\]

But Theorem 4.3 and its proof imply that \( I_{t+1-r}(A_T)/I_{t+1-r}(B_T) = \text{im}(\sigma_T) \). Indeed in the proof of Theorem 4.3 we nowhere used that \( R \) was a polynomial ring over \( k \) as long as the invertible matrices used in the proof are still invertible with entries in \( R \otimes k T \), e.g. that they map to invertible matrices via \((-) \otimes k \). Since the diagram \( \text{(5.7)} \) shows that \( \text{im}(\sigma_T) = I_{A_T/B_T} \) we are done.

\[\square\]

**Remark 5.7.** The proof shows also that every deformation \( B_T \rightarrow A_T \) of \( B \rightarrow A \) to the dual numbers \( T \) is given by \((t + 1 - r)\)-minors of the matrix \( A_T = [B_T, v_T] \) of \( \text{(5.8)} \) where \( B_T \) defines \( B_T \), without supposing (i) of Theorem 4.3. And, moreover, fixing \( B_T \) and some matrix \( B_T \) defining \( B_T \) and \( N_T \), “the family of choices” of the last column \( v_T \) which via \( \text{(5.8)} \) leads to deformations of \( A \), corresponds precisely to the variation of \( \sigma_T^*(1)_{(a_{t+c-1})} \) in the diagram \( \text{(5.7)} \) whose dimension is \( \text{dim}(MI \otimes A)_{(a_{t+c-1})} \). Note that by Proposition 1.3 \( MI \otimes A \cong \text{Hom}_{B}(I_{A/B}(a_{t+c-1}), A) \). Thus, every deformation of the fiber of the natural projection

\[
p_2 : \text{Hilb}_{X(t), Y(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}_{Y(t)}(\mathbb{P}^n)
\]

comes from deforming (the last column of) the matrix \( A \).
Remark 5.8. It is possible to weaken the assumption (i) of Theorem 5.5 and still get the same conclusion. Indeed, the assumption (i) is equivalent to the surjectivity of the morphism

$$\text{pr}_1 : A^1_{(B \to A)} \to \text{Hom}_R(I_A, A)$$

in the diagram (5.9). We can enlarge this diagram and get the cartesian square (i.e. pullback diagram):

\[
\begin{array}{ccc}
A^1_{(B \to A)} & \stackrel{pr_2}{\longrightarrow} & \text{Hom}_R(I_B, B) \\
\downarrow \text{pr}_1 & & \downarrow p \\
\text{Hom}_R(I_A, A) & \longrightarrow & \text{Hom}_R(I_B, A)
\end{array}
\] (5.9)

where $A^1_{(B \to A)}$ is the tangent space of GradAlg$(H_B, H_A)$ at $(B \to A)$. If we instead of the assumption (i) in Theorem 5.5 assume the weaker assumption;

$$\text{pr}_1 : A^1_{(B \to A)} \to \text{Hom}_R(I_A, A)$$

is surjective, we still get the existence of $B_T \to A_T$ reducing to $B \to A$ via $(-) \otimes_T k$, and the rest of the proof holds. Note also that the surjectivity of $\text{pr}_1 : A^1_{(B \to A)} \to \text{Hom}_R(I_A, A)$ is equivalent to $\gamma = 0$ where $\gamma$ is the composition

$$\text{Hom}_R(I_A, A) \longrightarrow \text{Hom}_R(I_B, A) \longrightarrow \text{Ext}_B^1(I_B/I_B^2, I_{A/B})$$

of natural maps. This is easily seen since the map $p : \text{Hom}_R(I_B, B) \to \text{Hom}_R(I_B, A)$ in the diagram (5.9) is part of a long exact sequence

\[
\begin{array}{ccc}
\text{Hom}_R(I_B, B) & \stackrel{p}{\longrightarrow} & \text{Hom}_R(I_B, A) \\
& & \longrightarrow \text{Ext}_B^1(I_B/I_B^2, I_{A/B}) \\
& & \longrightarrow \text{Ext}_B^1(I_B/I_B^2, B)
\end{array}
\] (5.10)

Note that $\gamma = 0$ whenever $\text{Ext}_B^1(I_B/I_B^2, I_{A/B}) = 0$. See Remark 5.6 for the vanishing of this Ext group.

The notion “every deformation of $B$ (or Proj$(B)$) comes from deforming its associated matrix”, has another important consequence:

Proposition 5.9. Let $B = R/I_B \to A$ be determinantal algebras defined by $t + 1 - r$ minors of matrices $A$ (resp. $B$) representing $\varphi^*$ (resp. $\varphi^*_t + c - 2$) and let $X = \text{Proj}(A)$ and $Y = \text{Proj}(B)$, and suppose $r \geq 2$, $c \geq 3 - r$, $J_A := I_{t-r}(A) \neq R$ and $\text{depth}_J B \geq r + 2$ where $J = I_{t-r}(B)$. If $c \leq 0$ we also suppose $\text{depth}_J A \geq 3$. Moreover, suppose that every deformation of $Y$ comes from deforming $B$. Then the natural projection

$$p_2 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$$

given by $p_2((X' \subset Y')) = (Y')$, is smooth at $(X \subset Y)$.

Proof. To prove the smoothness of $p_2 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \longrightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$, or equivalently the smoothness at $(B \to A)$ of the projection

$GradAlg(H_B, H_A) \longrightarrow GradAlg^{HB}(R)$
Corollary 5.11. Let minors. But instead of only considering deformations to the dual numbers, we take a surjection of artinian local rings \((T, m_T) \rightarrow (S, m_S)\) whose kernel \(a\) satisfies \(a \cdot m_T = 0\). Let \(B_S \rightarrow A_S\) be a deformation of \(B \rightarrow A\) to \(S\) and let \(B_T\) be a deformation of \(B_S\) to \(T\). By definition of smoothness (see [49, Definition 2.2 and Remark 2.3]) it suffices to find a deformation \(A_T\) of \(A_S\) to \(T\) and a map \(B_T \rightarrow A_T\) reducing to \(B_S \rightarrow A_S\) via \((-) \otimes_T S\). Let \(I_{A_S/B_S} = \ker(B_S \rightarrow A_S)\). Since \(B_S\) is defined by a matrix \(B_S\) that lifts the matrix \(B\) by assumption, we can use the arguments leading to (5.5) and (5.6), which relies on the vanishing of three Ext\(^1\)\(-\)-groups, to get \(M_S\) and \(\sigma_S\) so that (5.5) and (5.6) hold, only replacing there \(T\) by \(S\). Using that \(B_T\) is defined by some matrix \(B_T\) that lifts \(B_S\) to \(T\) and letting \(N_T\) be the cokernel of the map induced by \(B_T\) and \(M_T := \text{Hom}_{B_T}(N_T, B_T)\), the vanishing of Ext\(^1\)\(-\)-groups yields a diagram as in (5.6), only indexing the lower line there by \(S\) (i.e. \(B_S, \sigma_S^*\) instead of \(B, \sigma^*\) etc.) and replacing \(\sigma_S^*\) by some deformation of \(\sigma_S^*\). Then we argue as in Theorem (5.5) to get the column \(v_T\), and we define the \(T\)-flat quotient \(A_T\) by the minors of the matrix in (5.8) of size \(t + 1 - r\). Thus we have found a deformation \(A_T\) of \(A_S\) and a map \(B_T \rightarrow A_T\) reducing to \(B_S \rightarrow A_S\) and we are done. 

\[\square\]

Remark 5.10. If we suppose \(\text{depth}_J A \geq 3\) (i.e. \(c \geq 4 - r\) for \(A\) general), and \(\text{depth}_{J_A} A \geq 4\) if \(c \leq 0\), then the smoothness of

\[p_2 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \rightarrow \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)\]

is much easier to prove because Proposition 4.4 implies that Ext\(^1\)\(_A(I_A/B, I_A^2/B)\) = 0, whence \(H^2(B, A, A) = 0\), see Proposition 4.10. It is important to point out that in Proposition 5.9 we “only” assume \(c \geq 3 - r\) because one of main theorems of this monograph recursively proves that the closure \(\overline{W(B, A; t)}\) is a generically smooth component of \(\text{Hilb}^{p_X(t)}(\mathbb{P}^n)\) under some assumptions by starting the induction from a case where \(c = 3 - r\), i.e. the case where \(B\) is defined by maximal minors.

Combining Lemma 5.1, Theorem 5.5 and Remark 5.8 we get

Corollary 5.11. Let \(B = R/I_B \rightarrow A\) be determinantal algebras defined by \(t + 1 - r\) minors of matrices \(A\) (resp. \(B\)) representing \(\varphi^*\) (resp. \(\varphi^*_{t+c-2}\)). Suppose \(\text{depth}_J B \geq r + 2\) where \(J = I_{t-r}(B)\), \(r \geq 2, c \geq 3 - r\) and \(J_A := I_{t-r}(A) \neq R\). Moreover suppose that every deformation of \(B\) comes from deforming \(B\). If \(c \leq 0\), suppose also \(\text{depth}_{J_A} A \geq 3\). Then the following statements are equivalent

(a) Every deformation of \(A\) comes from deforming the matrix \(A\).
(b) The map \(pr_1 : A^1_{(B \rightarrow A)} \rightarrow \text{Hom}_R(I_A, A)\) of (5.9) is surjective.
(c) The composition \(\gamma\) of natural maps in Remark 5.8 is zero.
(d) The map \(\psi_A : \text{Ext}^1_B(M, M) \rightarrow \text{Hom}_R(I_A, A)\) of Lemma 5.1 is surjective.

If furthermore \(\text{depth}_J A \geq 3\), and in case \(c \leq 0\); \(\text{depth}_{J_A} A \geq 4\), they are also equivalent to

(e) \(\text{Ext}^1_B(I_B/I_B^2, I_{A/B}) \rightarrow \text{Ext}^1_B(I_B/I_B^2, B)\) is injective.
Proof. Since \((a) \iff (d)\) by Lemma \ref{lem:5.1} and \((c) \iff (b) \Rightarrow (a)\) by Theorem \ref{thm:5.5} and Remark \ref{rem:5.8} we must show \((a) \Rightarrow (b)\). To prove it take any \((A_T) \in \Hom_R(I_A, A)\). By assumption \((a)\), \(A_T\) is of the form \(A_T = R_T/I_{t+r-1}(A_T)\) where \(T := k[\varepsilon]/(\varepsilon^2)\). Now if we delete the last column of \(A_T\), we get a matrix \(B_T\). By Lemma \ref{lem:5.1} \(B_T := R_T/I_{t+r-1}(B_T)\) is a deformation of \(B\) to \(T\). Hence we get an element \((B_T \to A_T) \in A^1_{\langle B \to A \rangle}\) that maps to \((A_T)\) via the map

\[
pr_1 : A^1_{\langle B \to A \rangle} \to \Hom_R(I_A, A)
\]

of diagram \ref{fig:5.9}, i.e. \((b)\) is proved.

Finally to see \((c) \iff (b)\) we notice that the lower horizontal map in the diagram \ref{fig:5.9} is surjective by Proposition \ref{prop:4.10} indeed \(H^2(B, A, A) = 0\). It follows that the maps

\[
p : \Hom_R(I_B, B) \to \Hom_R(I_B, A)
\]

and \(pr_1 : A^1_{\langle B \to A \rangle} \to \Hom_R(I_A, A)\) of \ref{fig:5.9} are surjective simultaneously and we conclude the argument by using the exact sequence \ref{fig:5.10}.

\[\square\]

6. The Dimension of the Determinantal Locus

Given integers \(\underline{a} = (a_1, a_2, \ldots, a_{t+c-1})\) and \(\underline{b} = (b_1, \ldots, b_t)\) recall that \(W(\underline{b}; \underline{a}; r) \subset \Hilb^{P \times \{t\}}(\mathbb{P}^n)\) is the locus parameterizing determinantal schemes \(X \subset \mathbb{P}^n\) of codimension \(r \cdot (c + r - 1)\) defined as the vanishing of the \((t - r + 1) \times (t - r + 1)\) minors of a \(t \times (t + c - 1)\) matrix \(A = (f_{ij})_{j=1}^{t+c-1}\) where \(f_{ij} \in k[x_0, x_1, \ldots, x_n]\) is a homogeneous polynomial of degree \(a_j - b_i\). Correspondingly, set \(\underline{a}' = (a_1, a_2, \ldots, a_{t+c-2})\) and let \(W(\underline{b}; \underline{a}'; r)\) be the locus of determinantal flags \(X \subset Y \subset \mathbb{P}^n\) of the Hilbert-flag scheme with \(X \in W(\underline{b}; \underline{a}'; r)\) and \(Y \in W(\underline{b}; \underline{a}'; r)\) defined by the \((t - r + 1) \times (t - r + 1)\) minors of the \(t \times (t + c - 2)\) matrix \(B\) that we obtain deleting the last column of \(A\).

As previously \(\varphi : F = \oplus_{i=1}^t R(b_i) \to G = \oplus_{j=1}^{t+c-1} R(a_j)\) is a graded morphism, \(\varphi_{t+c-2}\) is obtained deleting the last row, \(B := R/I_{t-r+1}(\varphi^{t+c-2}) \to A := R/I_{t-r+1}(\varphi^*)\), \(MI = \coker \varphi^*\) and \(N = \coker \varphi^*\). We also assume that the integers \(\{a_j\}\) and \(\{b_i\}\) satisfy the weak conditions \ref{eq:2.1}.

In this section, we address the first of the following three fundamental problems:

**Problem 6.1.**

1. To determine the dimension of \(W(\underline{b}; \underline{a}; r)\) in terms of \(a_j\) and \(b_i\).
2. To determine the unobstructedness of a generic point of \(W(\underline{b}; \underline{a}; r)\).
3. To determine whether \(W(\underline{b}; \underline{a}; r)\) fills in an open dense subset of the corresponding component of the Hilbert scheme.

Our proof will go by induction and use the following diagram:
\[ \text{Hom}(F, G) \]
\[ \gamma \hookrightarrow q \]
\[ W^{(b; a'; r)}_{(b; a; r)} \xrightarrow{p_{1W}} W(b; a'; r) \]
\[ \downarrow p_{1W} \]
\[ W(b; a; r) \]

where the maps \( p_{1W} \) is the restriction to \( W^{(b; a'; r)}_{(b; a; r)} \) of the projection \( p_i \) of the Hilbert-flag scheme into its Hilbert schemes. Since \( q \) is an algebraic rational morphism (rational as \( q \) is only defined for \( \varphi \) with \( \text{coker} \varphi^* \) and \( \text{coker} \varphi_{t+c-2}^* \) of maximal codimensions), there is an open subset of \( \text{Hom}(F, G) \) that via \( q \) (resp. \( p_{1W} \cdot q \)) maps surjectively onto an open dense subset of \( W^{(b; a'; r)}_{(b; a; r)} \) (resp. \( W(b; a; r) \)). Thus we have the first of the following lemmas:

**Lemma 6.2.** \( W(b; a; r) \) and \( W^{(b; a'; r)}_{(b; a; r)} \) are irreducible algebraic sets.

**Lemma 6.3.** With the above notation, if the last column of \( A \) does not contain any unit (e.g. if \( a_1 > b_t \) ), then \( p_{1W} \) is surjective.

**Proof.** The result immediately follows from the well-known fact that if the \( i \times i \) minors of an homogeneous matrix \( A \) defines a determinantal scheme \( X \) (i.e. \( X \) has the expected codimension) the same is true for the scheme defined by the \( i \times i \) minors of the matrix that we obtain deleting a column of \( A \) without units, see [7]. \( \square \)

Our first goal is to show that we have an upper bound for the dimension of \( W(b; a; r) \). Indeed if \( r = 1 \) we set \( W(b; a) := W(b; a; 1) \) and we have

\[ \dim W(b; a) = \lambda_c + K_3 + K_4 + \cdots + K_c \]

for \( n - c \geq 1 \) and \( c \geq 2 \), with \( \dim W(b; a) = \lambda_c \) for \( c = 2 \), provided \( a_i - 1 \geq b_i \) for \( i \geq 2 \) by Theorem [3.1] Here \( \lambda_c \) and the non-negative numbers \( K_i \) are defined in [3.1], and as long as \( c \geq 1 \), \( \text{i.e. } A \text{ is a homogeneous } t \times (t+c-1) \text{ matrix with } t \leq t+c-1 \text{ the expression } \lambda_c + K_3 + K_4 + \cdots + K_c \text{ turns out to be an upper bound for } \dim W(b; a; r) \text{ for every } r, 1 \leq r \leq t - 1. \text{ If } c \leq 1, \text{ we can find an upper bound for } \dim W(b; a; r) \text{ from the } c \geq 1 \text{ statement by transposing the matrix } A \text{ because } (A)^{tr} \text{ is a } (t+c-1) \times t \text{ matrix with } (t+c-1) \leq t. \text{ Thus, other non-negative numbers, } K'_i, \text{ given in our next lemma, come into play and are needed to bound } \dim W(b; a; r). \text{ These } K'_i \text{ are really the numbers } K_i \text{ defined by [3.1] using } (A)^{tr}, \text{ up to equivalent matrices, instead of } A. \text{ With more details we have}

**Lemma 6.4.** Let \( A \) be a homogeneous \( t \times (t+c-1) \) matrix with degree matrix \( D = (d_{ij})_{i=1,\ldots,t}^{j=1,\ldots,t+c-1} \) where \( d_{ij} = a_j - b_i \), and let \( \lambda(D) := \lambda_c \) be the associated number given by [3.1]. Then the transposed matrix \( (A)^{tr} \) with \( (t+c-1) \times t \) degree matrix \( D^{tr} \) and associated number \( \lambda(D^{tr}) := \lambda'_{2-c} \) given by [3.1] satisfies \( \lambda(D^{tr}) = \lambda(D), \) i.e. \( \lambda'_{2-c} = \lambda_c \). Moreover, for the loci of \( (t-r+1) \)-minors we have

\[ W(-a_{t+c-1}, -a_{t+c-2}, \ldots, -a_1; -b_t, \ldots, -b_1; c + r - 1) = W(b_1, \ldots, b_t; a_1, a_2, \ldots, a_{t+c-1}; r). \]
Finally, if \( c = 2 - r < 1 \), these loci are defined by maximal minors and if \( a_{i-r} \geq b_i \) for \( r + 1 \leq i \leq t \) we have

\[
\dim W(b; a; r) = \lambda_c + K'_3 + K'_4 + \cdots + K'_r \quad \text{for} \quad n - r \geq 1 \quad \text{where}
\]

\[
\ell'_i := \sum_{j=1}^{t-r+1} a_j - \sum_{k=r-i+1}^{i} b_k,
\]

\[
h'_i := -2b_{r-i+1} + \ell'_i + n, \quad \text{for} \quad 1 \leq i \leq c,
\]

(6.2)

\[
K'_3 := \left( \frac{h'_3}{n} \right),
\]

\[
K'_4 := \sum_{j=r-2}^{t} (h'_j - b_j) - \sum_{i=1}^{t-r+1} \left( \frac{h'_i - a_i}{n} \right), \quad \text{and, in general,}
\]

\[
K'_i := \sum_{x+y=i-3}^{x+y=0} \sum_{t-r+1}^{t-r+i} \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} (-1)^{i-1-x} \left( \frac{h'_{i-3} - b_{i_1} - \cdots - b_{i_r} - a_{j_1} - \cdots - a_{j_y}}{n} \right).
\]

**Proof.** In the definition (3.1) of \( \lambda(D) := \lambda_c \) we may put \( a_j - a_i = d_{ij} - d_{ii} \) for \( i, j \in \{ 1, \ldots, t + c - 1 \} \) and \( b_i - b_j = d_{i1} - d_{j1} \) for \( i, j \in \{ i, ..., t \} \). Applying (3.1) similarly onto \( \lambda(D^{tr}) \) with degree matrix given by \( d_{ij}^{tr} \) where \( d_{ij}^{tr} = d_{j,i} \), we get \( d_{ij}^{tr} - d_{ii}^{tr} = b_j - b_i \) and \( d_{i1}^{tr} - d_{j1}^{tr} = a_i - a_j \) and so we see that the sum of the negative terms in (3.1) defining \( \lambda \) is the same for \( \lambda(D^{tr}) \) and \( \lambda(D) \). The positive terms in the definition of \( \lambda \) in (3.1) are obviously equal, and we get \( \lambda(D^{tr}) = \lambda(D) \).

Moreover, since it is clear that the locus of the \((t-r+1) \times (t-r+1)\) minors of the \( t \times (t+c-1) \) matrix \( \mathcal{A} \) is exactly the same as the locus of \((t-r+1) \times (t-r+1)\) minors of the \((t+c-1) \times t\) transposed matrix \( \mathcal{A}^{tr} \) and \( t+c-1-(c+r-1)+1 = t-r+1 \), we get the equality of the loci, noticing that the inequalities

\[
b_1 \leq b_2 \leq \cdots \leq b_t, \quad a_1 \leq a_2 \leq \cdots \leq a_{t+c-1}
\]

of the numbers attached to \( \mathcal{A} \), are reversed for \( \mathcal{A}^{tr} \), i.e. they correspond to

\[
-b_1 \geq -b_2 \geq \cdots \geq -b_t, \quad -a_1 \geq -a_2 \geq \cdots \geq -a_{t+c-1}.
\]

Indeed, transforming \( \mathcal{A} \) by keeping the entry of degree \((a_i - b_i)\) in the lower left corner fixed (instead along the usual corner \((1, 1)\)) we get a \((t+c-1) \times t\) matrix \( \mathcal{A}' \) with degree matrix \((d_{i,j}^{tr})_{i=1, \ldots, t+c-1, j=1, \ldots, t}\),

(6.3) \[
d_{ji}^{tr} = a'_i - b'_j \text{ where } a'_i = -b_{t+1-i} \text{ for } i = 1, ..., t \text{ and } b'_j = -a_{t+c-j} \text{ for } j = 1, ..., t + c - 1
\]

which is equivalent to \( \mathcal{A}^{tr} \) (up to row- and column-equivalence). This \( \mathcal{A}' \) has the same property as \( \mathcal{A} \) with respect to inequalities between the entries, i.e. the one of the smallest degree sits in the lower left corner, and the degree of entries increases in all directions from there. Note also that we easily get \( \lambda(D^{tr}) = \lambda(D') \) by (6.3) and the first part of the proof.

In particular, for \( c = 2 - r \) then \( \mathcal{A}^{tr} \), or rather its equivalent matrix \( \mathcal{A}' \), is a \((t-r+1) \times t\) matrix and since the inequalities \( b_1 \leq b_2 \leq \cdots \leq b_t, a_1 \leq a_2 \leq \cdots \leq a_{t-r+1} \) attached to \( \mathcal{A} \) corresponds to \(-a_{t-r+1} \leq -a_{t-r} \leq \cdots \leq -a_1, -b_t \leq -b_{t-1} \leq \cdots \leq -b_1 \) attached to \( \mathcal{A}' \), we set \( b'_i = -a_{t-r+2-i} \) and \( a'_i = -b_{t+1-i} \) using primes ('') for \( \mathcal{A}' \). Then we get the lemma from (3.1) by substitution and the fact that

\[
\dim W(b'; a') = \lambda'_c + K'_3 + K'_4 + \cdots + K'_r
\]

for \( n - r \geq 1 \) provided \( a'_{i-1} \geq b'_i \) for \( 2 \leq i \leq t + 1 - r \) by Theorem 3.1. \( \square \)
Theorem 6.5.  
(i) Let $c \geq 1$ and suppose $a_{i-1} \geq b_i$ (and $a_{i-1} > b_i$ if $c = 1$) for $2 \leq i \leq t$. Then for any $r$, $1 \leq r < t$ we have

$$\dim W(b_1 ; p, a; r) \leq \lambda_c + K_3 + K_4 + \cdots + K_c$$

for $n - r(c + r - 1) \geq 1$.

In particular if $a_{t+c-1} < \ell_2 = \sum_{i=1}^{t+1} a_i - \sum_{i=1}^{t} b_i$, then $K_i = 0$ for $3 \leq i \leq c$, whence

$$\dim W(b_1 ; p, a; r) \leq \lambda_c.$$ 

(ii) Let $c < 1$ and suppose $a_{i-r} \geq b_i$ for $r + 1 \leq i \leq t$. Then for any $r$, $2 - c \leq r < t$, we have

$$\dim W(b_1 ; p, a; r) \leq \lambda_c + K'_3 + K'_4 + \cdots + K'_r$$

for $n - r(c + r - 1) \geq 1$.

In particular if $-b_1 < \ell'_2 = \sum_{j=1}^{t-r+1} a_j - \sum_{k=r-1}^{t} b_k$ (e.g. if $r = 2$), then $K'_i = 0$ for $3 \leq i \leq r$ and

$$\dim W(b_1 ; p, a; r) \leq \lambda_c.$$

(iii) If $r = 1$, $(r, c, n) \neq (1, 1, 2)$ in (i), resp. $r = 2 - c$ in (ii), then we have equalities in both displayed formulas of (i), resp. (ii).

Remark 6.6. Since the inequality $a_{t+c-1} < \ell_2$ in (i) is always satisfied for $c \in \{1, 2\}$ the second displayed formula holds for $c \in \{1, 2\}$ in (i), and similarly for $c = 0$ in (ii). The assumptions on $a_j$ and $b_i$ are at least needed in (iii). Moreover, since in this paper we almost always work with determinantal schemes $X \subset W(b_1 ; p, a; r)$ of dimension at least one and $r(c + r - 1)$ is the codimension of $X$ in $\mathbb{P}^n$, the assumption $n \geq r(c + r - 1) + 1$ in Theorem 6.5 is mostly a reminder of the settings in which we work.

Proof. (i) To prove the first inequality we observe that the morphism $p_{1W} : q$ which maps, say a sufficiently general $\varphi \in \text{Hom}(F, G)$, onto $(\text{Proj}(A)) \subset W(b_1 ; p, a; r)$, $A = R/I_{t+1-r}(A)$, factors via $MI : = \text{coker}(\varphi^*)$ by the Fitting’s lemma (Corollary-Definition 20.4]). Thus $p_{1W} : q$ factors through the moduli space of such $MI$, a moduli space which at least exits at a very local level, say at this $MI$. Since $MI$ is unobstructed by Theorem 3.1], the dimension of the moduli space at $(MI)$ is equal to the dimension of its tangent space $\text{Ext}_R^1(MI, MI)$. This argument implies

$$\dim W(b_1 ; p, a; r) \leq \text{Ext}_R^1(MI, MI),$$

and since we have

$$\text{Ext}_R^1(MI, MI) = \lambda_c + K_3 + K_4 + \cdots + K_c$$

by Theorem 3.1], we get the first inequality. Note that the part of the proof of Theorem 3.1 that concerns the unobstructedness of $MI$ and the dimension of $\text{Ext}_R^1(MI, MI)$ hold for $c = 1$ as well (with $E = 0$ there, whence all $K_i = 0$), cf. (3.5).

In Remark 3.4 we noticed that all the binomials in $K_i$ vanish under the numerical assumption $\ell_c > 2a_{t+c-1} + a_{t+c-2} + \cdots + a_{t+2}$ (when $c > 3$, cf. the published version). This assumption is equivalently to $a_{t+c-1} < \ell_2$. Indeed one may show that $K_i = 0$ using the minimal free resolution of $\varphi^*_{t+c-2}$ given by the Buchsbaum-Rim complex (see (3.3))

$$\wedge^{t+1} G^*_{t+c-2} \otimes \wedge^t F \rightarrow G^*_{t+c-2} \rightarrow F^* \rightarrow \text{coker}(\varphi^*_{t+c-2}) \rightarrow 0,$$
and recalling that \(K_c = \hom_0(B_{c-1}, R(a_{t+c-1}))\) where \(B_{c-1} = \coker(\varphi_{t+c-2})\), cf. \([34, (3.13)]\). Then we get that \(a_{t+c-1} < \ell_2\) implies \(K_c = 0\), also when \(c = 3\) (and all \(K_i = 0\)) because
\[
\hom_0(\wedge^{i+1} G_{t+c-2} \otimes \wedge^i F, R(a_{t+c-1})) = 0
\]
by only considering the degree of the generators of this Hom-group. This completes the proof when \(c \geq 1\).

(ii) If \(c < 1\) and we look at the locus of \((t-r+1) \times (t-r+1)\) minors of the \((t+c-1) \times t\) transposed matrix \(A^t\) (or rather its equivalent matrix \(A'\) to which \((6.3)\) belongs), we get a \(t' \times (t' + (2 - c) - 1)\) matrix with \(2 - c > 1\) and we have \(\dim W(b'_i; a'_i; r) \leq \dim W(b'_i; a'_i; 2 - c)\) by \((6.3)\) and the proven part of the theorem. Then we get the theorem from Lemma \([6.4]\) noticing that \(-b_1 < \ell'_2\) implies every \(K'_i = 0\) by the proven part and that \(r = 2\) implies \(-b_1 < \ell'_2\).

(iii) It follows from Lemma \([6.4]\) and Theorem \([5.1]\) \(\blacksquare\)

**Remark 6.7.** (Alternative proof of Theorem \([6.5](i)\))

The map \(q\) in diagram \((6.1)\) factors through the orbit set \(\Hom(F,G)\) since \(G := Aut(F) \times Aut(G)\) acts on \(\Hom(F,G)\) in a natural way such that every element of an orbit maps to the same determinantal scheme, cf. \([39, p. 97]\). Indeed the proof there works also for \(W(b, a; r)\) only replacing the Eagon-Northcott complex by the Lascoux resolution in the argument. Thus we obtain a dominating rational morphism \(\Hom(F,G) \to W(b, a; r)\) and \(\dim \Hom(F,G)\) is an upper bound for \(\dim W(b, a; r)\). In \([39, pp. 97-98]\) we compute \(\dim G\) (see Propositions 10.2 and 10.3) and get
\[
\dim W(b, a; r) \leq \lambda_c - 1 + \dim(B_c, B_c)
\]
where \(B_c = \coker(\varphi)\). Since we get \(\dim(B_c, B_c) = 1 + K_3 + K_4 + \cdots + K_c\) by \([34, Proposition 3.12]\) and Remark \([3.2](3)\), we have proved the first dimension formula.

Note that we have equalities in Theorem \([6.5]\) if \(r = 1\) and \(c > 1\), e.g. \(\dim W(b, a; r) = \lambda_c\) provided \(a_{t+c-1} < \ell_2\). One of the main goals in this work is to generalize this formula and to show \(\dim W(b, a; r) = \lambda_c\) for any \(r\) under some numerical assumptions. We will also analyze when the inequality \(\dim W(b, a; r) \leq \lambda_c + K_3 + \cdots + K_c\) (resp. \(\dim W(b, a; r) \leq \lambda_c + K'_3 + \cdots + K'_c\)) turns out to be an equality up to a correction term. To this end, we define
\[
s_r := \sum_{i=1}^{t-r+1} a_i - \sum_{i=1}^{t-r} b_{r+i},
\]
and note that \(s_0 = \ell_2\). Thus \(\dim W(b, a; r) = \lambda_c\) for \(r = 1\) and \(c > 1\) when \(a_{t+c-1} < s_0\) because
\[
(6.5) \quad K_i = 0 \quad \text{for all } \; 3 \leq i \leq c \quad \text{provided } \; a_{t+c-1} < s_0 := \ell_2
\]
by the proof of Theorem \([6.5]\). In what follows we will try to prove that \(\dim W(b, a; r) = \lambda_c\) provided \(a_{t+c-1} < s_r - b_r + b_1\) which for \(r = 1\) is just \(a_{t+c-1} < s_1\). This is a slightly stronger assumption that implies that all \(K_i = 0\) because, in general, we have \(b_j \leq a_j\) for \(1 \leq j \leq t, b_{j_0} < a_{j_0}\) for some \(j_0\) by \((2.1)\) which implies \(s_1 < s_0\), as well as
\[
(6.6) \quad s_r - b_r + b_1 < s_{r-1} - b_{r-1} + b_1 \leq s_1 = s_0 - a_{t+1} + b_1 \quad \text{for } 2 \leq r \leq t.
\]
Finally note that if we have $a_{t+c-1} \geq b_r - 1$ (a very weak assumption), we get that $a_{t+c-1} < s_r - b_r + b_1$ implies that all $K'_i = 0$ by Theorem 6.3 because $-b_1 < \ell'_2 = \sum_{j=1}^{t-r+1} a_j - \sum_{k=r-1}^{t} b_k = s_r - b_r - b_{r-1}$.

Now we consider the diagram of infinitesimal deformations that corresponds to diagram (6.1) at a given point $(X \subset Y) \in W^{(b_a)}(\mathbb{P})$:

\begin{equation}
\begin{array}{ccc}
A^1_{(B\to A)} & \xrightarrow{pr_2} & _0\text{Hom}_R(I_B, B) \\
\downarrow pr_1 & & \downarrow \\
_0\text{Hom}_R(I_A, A) & \rightarrow & _0\text{Hom}_R(I_B, A)
\end{array}
\end{equation}

where $X = \text{Proj}(A)$ and $Y = \text{Proj}(B)$. Here the tangent space of the fiber, $p_1^1((X))$, of

$p_1 : \text{Hilb}^p_{X(t), p_2(t)}(\mathbb{P}) \rightarrow \text{Hilb}^p_{X(t)}(\mathbb{P})$

at $(X \subset Y)$ corresponds to the kernel, $0\text{Hom}_R(I_B, I_A/B)$, of $pr_1$ and the fiber $p_2^{-1}((Y))$ of

$p_2 : \text{Hilb}^p_{X(t), p_2(t)}(\mathbb{P}) \rightarrow \text{Hilb}^p_{Y(t)}(\mathbb{P})$

to the kernel $0\text{Hom}_R(I_A/B, A) \cong (M I \otimes A)_{a_t+c-1}$ of $pr_2$ by Proposition 4.4(iii).

Let us compute the dimensions of these tangent spaces to fibers under appropriate assumptions, and we start by finding $\dim(M I \otimes A)_{a_{t+c-1}}$.

**Lemma 6.8.** Assume $r \geq 2$ and $\text{depth}_J A \geq 2$ where $J := I_{t-r}(\varphi_{t+c-2})$.

(i) If $c \geq 2 - r$ and $a_{t+c-1} + v < s_r - b_r + b_1$, then

$$\dim(M I \otimes A)_{a_{t+c-1}+v} = \dim(M I)_{a_{t+c-1}+v}.$$ 

(ii) Assume $c > 2 - r$. If $v$ is any integer such that $(M I \otimes I_A/B)_{a_{t+c-1}+v} = 0$ (e.g. $v < s_r - b_r - a_{t-r+1} + b_1$), then we have:

$$\dim(M I \otimes A)_{a_{t+c-1}+v} = \dim(N \otimes B)_{a_{t+c-1}+v} - \dim B_v.$$ 

Moreover in the case $v < s_r - b_r - a_{t-r+1} + b_1$, we also have $\dim B_v = \dim R_v$.

**Proof.** (i) Let $v' = a_{t+c-1} + v$. Since the smallest degree of a generator of $I_A := \ker(R \to A)$ is $s_r - b_r$ (cf. (6.8) below) and since $b_1 \leq a_1$ it follows that

$$(G^* \otimes I_A)(v'_0) = (F^* \otimes I_A)(v'_0) = 0$$

by the assumption $v' < s_r - b_r + b_1$. Considering the diagram

$$
\begin{array}{ccc}
G^* & \rightarrow & F^* \\
\downarrow & & \downarrow \\
G^* \otimes A & \rightarrow & F^* \otimes A
\end{array}
$$

in degree $v'$ we easily conclude the proof of (i).

(ii) Again set $v' = a_{t+c-1} + v$. We recall the exact sequence [4.4]:

$$0 \rightarrow B(-a_{t+c-1}) \rightarrow N \otimes B \rightarrow M I \otimes B \rightarrow 0.$$
Since by assumption we have \((MI \otimes I_{A/B})_{\nu'} = 0\) which implies \((MI \otimes B)_{\nu'} \cong (MI \otimes A)_{\nu'}\), we get
\[
\dim(MI \otimes A)_{\nu'} = \dim(N \otimes B)_{\nu'} - \dim B_{\nu'}.
\]
Moreover \(F^* \rightarrow M\) is surjective and we get \((MI \otimes I_{A/B})_{\nu'} = 0\) if we can show \((F^* \otimes I_{A/B})_{\nu'} = 0\).

To show it we consider the following diagram
\[
\begin{array}{ccc}
\land_{t-r+1}^* G_{t+c-2} \otimes \land_{t-r+1}^* F & \longrightarrow & R \\
\downarrow & & \downarrow \\
\land_{t-r+1} G^* \otimes \land_{t-r+1} F & \longrightarrow & R \\
\end{array}
\]
(6.8)
of exact horizontal sequences. The summand of \(\land_{t-r+1} G^* \otimes \land_{t-r+1} F\) of the smallest possible degree is \(\sum_{i=1}^{t-r+1} a_i - \sum_{i=0}^{t-r} b_{i+r} = s_r - b_r\) and correspondingly for \(\land_{t-r+1} G_{t+c-2}^* \otimes \land_{t-r+1} F\) while the smallest degree of a generator of \(I_{A/B}\) is \(s_r - b_r - a_{t-r+1} + a_{t+c-1}\) because it must involve \(a_{t+c-1}\).

Hence we get \((F^* \otimes I_{A/B})(a_{t+c-1} + v)_0 = 0\) because \(v < s_r - b_r - a_{t-r+1} + b_1\).

**Remark 6.9.** (1) In the linear case (i.e. \(b_i = 0\) and \(a_j = 1\) for all \(i, j\)) we have \(s_r = t - r + 1\), and Lemma 6.8 (i) and (ii) applies for \(v < t - r\).

(2) By \[\text{pgs 162-166}\] we have an exact sequence:
\[
\land_{t-r+1} G^* \otimes \land_{t-r} F \otimes A \rightarrow G^* \otimes A \rightarrow F^* \otimes A \rightarrow MI \otimes A \rightarrow 0.
\]

Since the smallest degree of a summand of \(\land_{t-r+1} G^* \otimes \land_{t-r} F\) is \(s_r\), we get the injectivity of the morphism \(G^* \otimes A \rightarrow F^* \otimes A\) in degree \(a_{t+c-1} + v\) only assuming \(v < s_r - a_{t+c-1}\). For such \(v\) we get
\[
\dim(MI \otimes A)_{a_{t+c-1} + v} = \dim(F^* \otimes A)_{a_{t+c-1} + v} - \dim(G^* \otimes A)_{a_{t+c-1} + v}
\]
which may be used to improve upon Lemma 6.8 in the case \(b_r > b_1\).

The problem of finding the dimension of the other fiber will mainly be considered in Section 8. There we successively delete \(c + r - 2\) columns from the right-hand side of the \(t \times (t + c - 1)\) homogeneous matrix \(A\), and taking the \((t - r + 1) \times (t - r + 1)\) minors, we get a flag of determinantal rings:
\[
A_{2-r} \rightarrow \cdots \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots \rightarrow A_c = A, \quad X_i = \text{Proj}(A_i)
\]
where e.g. \(A_1\) (resp. \(A_{2-r}\)) is Gorenstein (resp. standard determinantal) defined by the \((t - r + 1) \times (t - r + 1)\) minors of a \(t \times t\) (resp. \(t \times (t - r + 1)\)) matrix. Let \(I_i = I_{A_{i+1}/A_i}\) be the ideal defining \(A_{i+1}\) in \(A_i\) and let \(I_{A_i} = I_{x_{t-r+1}}(\varphi_{t-i-1}^*)\). We will prove that \(\dim W(b_r, a_r; r) = \lambda_c\) provided \(\dim W(b_r, a_r; r) = \lambda_{c-1}\) and to check it we will need
\[
\text{hom}_R(I_{A_{c-1}}, I_{c-1}) = \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}
\]
In Section 8, we will show the last equality under the numerical restrictions given in Corollary 8.3.

In particular, we prove there
**Corollary 6.10.** Let $A$ be a general homogeneous $t \times (t + c - 1)$ matrix with entries homogeneous forms of degree $a_j - b_i$. Let $c \geq 3 - r$ and suppose $a_1 > b_1$, $r \geq 2$ and $\dim A \geq 2$ if $c > 0$ and $\dim A \geq 3$ if $c \leq 0$. If $b_r - b_1 < \sum_{i=1}^{t-r}(a_i - b_{r+i}) + a_{t+c-1} - a_{t+c-2}$ and $a_{t-r+1} < a_{t+c-1} - \sum_{i=1}^{t-r+1}(b_{r+i-1} - b_i)$, then

$$0_{\hom_R(I_{A_{t-1}}, I_{c-1})} = \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}.$$  

Note that all the above inequalities (for $a_j, b_i$) hold if $b_1 = a_1$ and $a_{t-r+1} < a_{t+c-1}$ (and $r < t$).

Let us come back to the diagram of infinitesimal deformations at a point $(X \subset Y) \in W_{(b,a',r)}^{(k_{a'},r)}$:

$$
\begin{array}{ccc}
A^1_{(B \rightarrow A)} & \xrightarrow{pr_2} & 0_{\Hom_R(I_B, B)} \\
\downarrow pr_1 & & \downarrow \\
0_{\Hom_R(I_A, A)} & \longrightarrow & 0_{\Hom_R(I_B, B)}
\end{array}
$$

where $X = \Proj(A)$ and $Y = \Proj(B)$. By Proposition 4.10 if $\depth_{I_B} A \geq 3$, and $\depth_{I_A} A \geq 4$ for $c \leq 0$, then $\Ext^1(I_A/B; I_{A/B}, A) = 0$ which implies that

$$pr_2 : A^1_{(B \rightarrow A)} \longrightarrow 0_{\Hom_R(I_B, B)}$$

is surjective and the corresponding projection

$$p_2 : \Hilb^{pX(t),pY(t)}(\mathbb{P}^n) \longrightarrow \Hilb^{pY(t)}(\mathbb{P}^n)$$

is smooth at $(X \subset Y)$. Since in our next proposition we need the property “every deformation of $B$ comes from deforming its matrix” at several places in the proof, we rather use Proposition 5.9 which then implies that $p_2$ is smooth at $(X \subset Y)$ only assuming $\depth_{I_B} A \geq 2$ with $\depth_{I_A} A \geq 3$ if $c \leq 0$. The map called

$$p_{2W} : W_{(b,a',r)}^{(k_{a'},r)} \longrightarrow W_{(b,a',r)}$$

in diagram (6.1) is essentially the pullback of $p_2$ to $W_{(b,a',r)} \subset \Hilb^{pY(t)}(\mathbb{P}^n)$ and is therefore smooth at $(X \subset Y)$. Invoking also Remark 5.8 we get

**Proposition 6.11.** Let $c \geq 3 - r$, let $B \rightarrow A$ be determinantal algebras defined by $(t+1-r)$-minors of matrices $A$ and $B$ representing $\varphi^*$ and $\varphi^*_{t+r-2}$, respectively, and suppose $\depth_{I_B} B \geq r+2$, $J_B := I_{t-r}(B)$, $J_A := I_{t-r}(A) \neq R$ and that every deformation of $B$ comes from deforming $B$. If $c \leq 0$, we also suppose $\depth_{I_A} A \geq 3$. Letting $W := W_{(b,a',r)}^{(k_{a'},r)}$, then the closures $\overline{W} \subset \Hilb^{pX(t),pY(t)}(\mathbb{P}^n)$ and $\overline{W}_{(b,a',r)} \subset \Hilb^{pY(t)}(\mathbb{P}^n)$ are generically smooth irreducible components and the restriction $p_{2W}$ of the 2nd projection $p_2 : \Hilb^{pX(t),pY(t)}(\mathbb{P}^n) \longrightarrow \Hilb^{pY(t)}(\mathbb{P}^n)$ to $W$ is smooth at $(X \subset Y)$ with fiber of dimension $\dim(MI \otimes A)_{a_{t+c-1}}$. Supposing that the matrix $A$ is general, we have

$$\dim W_{(b,a',r)}^{(k_{a'},r)} = \dim W_{(b,a',r)}^{(k_{a'},r)} + \dim_k (MI \otimes A)_{a_{t+c-1}} - 0_{\hom_R(I_B, I_{A/B})}.$$  

First we prove the following
Lemma 6.12. Let $c$ and $B = R/I_{t+1-r}(B) \to A$ be as in Proposition 6.11 with $A$ not necessarily general. Let $X = \text{Proj}(A)$ and $Y = \text{Proj}(B)$. Then $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ is smooth at $(Y)$, and $\overline{W(b:a';r)}$ is an irreducible component of $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$. Moreover $\text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$ is smooth at $(X \subset Y)$, and $\overline{W}$ is an irreducible component of $\text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$.

Proof. By Corollary 5.2 we get that $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ is smooth at $(Y)$. By Proposition 6.11, $p_2 : \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n) \to \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ is smooth at $(X \subset Y)$, hence $\text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$, is smooth at $(X \subset Y)$.

To show that $\overline{W(b:a';r)}$ is an irreducible component of $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$, one may use Corollary 5.2 or argue as follows (we will need the argument later). Let $(T, \mathfrak{m}_T)$ be the local ring of $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ at $(Y)$ and let $B_{T_2}$ be the pullback of the universal object of $\text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ to Spec$(T_2)$ where $T_2 = T/m^2_T$. Since $T$ is a regular local ring, it suffices to show $\dim \overline{W(b:a';r)} = \dim \mathfrak{m}_T/m^2_T$, i.e. that the “universal object” $B_{T_2}$ is defined by some matrix $B_{T_2}; B_{T_2} = R_{T_2}/I_t(B_{T_2})$. This is, however, true by assumption, and even more is true (see Remark 7.3 for an extension).

The proof of $\overline{W}$ being an irreducible component of $\text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$ is very similar, only changing $(T, \mathfrak{m}_T)$ to be the local ring of $\text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$ at $(X \subset Y)$ and noticing that the “universal object” $B_{T_2} \to A_{T_2}$ is defined in terms of a matrix $A_{T_2} = [B_{T_2}, v_{T_2}]$ lifting $A = [B, v]$ to $T_2$ such that $A_{T_2}$ defines $A_{T_2}$ and $B_{T_2}$ defines $B_{T_2}$ by Remark 5.7.

Proof of Proposition 6.11. By Lemma 6.12 $\overline{W(b:a';r)} \subset \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ and $\overline{W} \subset \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n)$ are generically smooth irreducible components, and $p_2 : \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n) \to \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$, as well as its restriction $p_{2W} (W$ is dense in a component), is smooth at $(X \subset Y)$ by Proposition 5.9. Moreover the fiber $p_{2W}^{-1}(Y))$ of $p_{2W}$ is also smooth in some neighbourhood of $(X \subset Y)$, whence its dimension at $(X \subset Y)$ is given by the dimension of its tangent space which by Proposition 1.4 iii) is $\dim_{\text{hom}}(I_{A/B}, A) = \dim_k(MI \otimes A)_{(a_{t+e-1})}$.

It remains to see the dimension formula. To show it we endow $\overline{W(b:a';r)}$ with its reduced scheme structure, and we use generic smoothness (since char$k = 0$) onto the restriction map $p_{1W} : W \to \overline{W(b:a';r)}$ of the 1st projection $p_1 : \text{Hilb}^{p_X(t),p_Y(t)}(\mathbb{P}^n) \to \text{Hilb}^{p_Y(t)}(\mathbb{P}^n)$ to $W$. Since $p_{1W}$ is dominating, there is an open $U \subset \overline{W(b:a';r)}$ such that $p_1$ restricted to $p_{1W}^{-1}(U)$ is smooth, with fiber dimension $\dim_{\text{hom}}(I_{B}, I_{A/B})$. Hence we get

$$\dim \overline{W(b:a';r)} = \dim \overline{W(b:a';r)}$$

Then we conclude the proof by using the smoothness of $p_{2W}$ which implies

$$\dim \overline{W(b:a';r)} + \dim_k(MI \otimes A)_{(a_{t+e-1})} = \dim W.$$
Our goal is to compute \( \dim W(b; a; r) \) in terms of \( \dim W(k; a'; r) \). Due to Proposition 6.11 we need to find the difference, \( \dim(MI \otimes A)_{a_{t+c-1}} - \hom_R(I_B, I_{A/B}) \). Using the definition of \( \lambda_c \) and \( K_c \), and the exactness of the Buchsbaum-Rim complex for \( c \geq 2 \), it is not difficult to show that

\[
(6.9) \quad \dim(MI)_{a_{t+c-1}} - \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n} = \lambda_c - \lambda_{c-1} + K_c,
\]

cf. [34] proof of Theorem 4.5] for some details. Since, for \( 2 - r \leq c \leq 1 \) and \( A \) general, the sequence

\[
0 \to G^* \to F^* \to MI \to 0
\]
is exact, (6.9) also holds for \( 2 - r \leq c \leq 1 \) letting \( K_c := 0 \) for \( c \leq 2 \). Below \( A = R/I_{t-r+1}(A) \) and \( B = R/I_{t-r}(B) \) where \( B \) is obtained by deleting the last column of \( A \).

**Theorem 6.14.** Let \( \Proj(A) \in W(b; a; r) \) be general with \( \dim A \geq 2 \) and suppose \( 3 - r \leq c \), \( b_t < a_1 \) and that every deformation of \( \Proj(B) \in W(b; a'; r) \) comes from deforming its matrix. For \( c \leq 0 \), we also suppose \( \dim A \geq 3 \). Let \( \kappa \) be an integer satisfying \( \dim W(b; a; r) = \lambda_{c-1} - \kappa \) and suppose \( a_{t+c-1} \leq s_r - b_r + b_1 \). Moreover suppose

\[
(6.10) \quad \hom_R(I_B, I_{A/B}) \leq \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}
\]

(resp. with equality in (6.10), e.g. that \( b_t = b_1 \) and \( a_{t-r+1} < a_{t+c-1} \)). Then \( K_i = 0 \) for \( 3 \leq i \leq c \) and we have

\[
\dim W(b; a; r) \geq \lambda_c - \kappa \quad \text{(resp. } \dim W(b; a; r) = \lambda_c - \kappa \text{)}.
\]

In particular if \( \kappa = 0 \) and the inequality in (6.10) hold, then

\[
\dim W(b; a; r) = \lambda_c.
\]

**Proof.** Since we assume \( b_t < a_1 \) we have \( s_r \leq s_0 \) and hence \( K_i = 0 \) for \( 3 \leq i \leq c \) by (6.5). We also have depth\( I_{t-r}(B)A \geq \min\{\dim A, c + r - 1\} \geq 2 \) by Remark 4.3 and similarly depth\( I_{t-r}(A)A \geq \min\{\dim A, c + 2r\} \geq 3 \) for \( c \leq 0 \). Note that \( b_t < a_1 \) implies \( I_{t-r}(A) \neq R \). Since we by Lemma 6.8 and the assumption \( a_{t+c-1} < s_r - b_r + b_1 \) may replace \( \dim(MI)_{a_{t+c-1}} \) by \( \dim(MI \otimes A)_{a_{t+c-1}} \) in (6.9), it follows from (6.9), Proposition 6.11 and assumptions that we have the inequality

\[
\dim W(b; a; r) = \dim(MI \otimes A)_{a_{t+c-1}} - \hom_R(I_B, I_{A/B}) + \dim W(k; a'; r)
\]

\[
\geq \lambda_c - \lambda_{c-1} + K_c + \lambda_{c-1} - \kappa = \lambda_c - \kappa
\]

which, moreover, becomes an equality if we have equality in (6.10). Since the e.g.-assumptions imply \( \hom_R(I_B, I_{A/B}) = \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n} \) taking \( A_{c-1} = : B = R/I_B, A_c = : A = R/I_A \) and \( I_{A/B} = I_A/I_B \) in Corollary 6.10 we are done except for the final statement.

We have \( \dim W(b; a; r) \geq \lambda_c \) by the first part of the proof. Then we conclude the proof by

Theorem 6.5 because \( a_{t+c-1} < \ell_2 \) and \( -b_1 < \ell_2 \) holds by (6.6) and the text accompanying (6.6).

Below we denote \( \lambda_i \) and \( \lambda(I; R) \) by \( \lambda(R) \) and \( W(b; a; r, R) \) respectively since they obviously both depend on \( R \). Moreover, as always, \( B \) is obtained by deleting the last column of \( A \).
Corollary 6.15. (i) Let $\text{Proj}(A) \in W(k; \mathfrak{a}; r)$, $A := R/I_s(A)$ with $s = t + 1 - r$, be a general determinantal scheme and suppose $\dim A \geq 2$, $c \geq 3 - r$, $a_1 > b_t$ and that every deformation of $\text{Proj}(B) \in W(k; \mathfrak{a}'; r)$ comes from deforming its matrix $\mathcal{B}$. For $c \leq 0$, we also suppose $\dim A \geq 3$. Moreover suppose that $\dim W(k; \mathfrak{a}'; r, R) = \lambda_{c-1}(R)$ and $a_t + c - 1 < s_r - b_t + b_1$ and that $0\text{hom}_R(I_B, I_{A/B}) \leq \sum_{j=1}^{t+c-2} (a_j - a_{t+c-1} + n_j)$. Then

$$\dim W(k; \mathfrak{a}; r, R) = \lambda_c(R).$$

(ii) Suppose $a_1 = a_t + c - 2$ and $b_1 = b_t$ or more generally that $\lambda_{c-1}(R, I_{A/B}) = 0$ for all $v \leq -1$. Taking $s \times s$-minors of the matrices $\mathcal{A}$ and $\mathcal{B}$ considered belonging to a larger polynomial ring $R' := R[y]$ where $y := y_1, \ldots, y_e$ are $e$ indeterminates and supposing $\dim W(k; \mathfrak{a}'; r, R') = \lambda_{c-1}(R')$, then $A' := R'/I_s(A)$ and $B' := R'/I_s(B)$ satisfy all assumptions of (i), replacing $A, B, R$ there by $A', B', R'$, except that $A'$ is general. Indeed we have $\lambda_{c-1}(R, I_{A/B}) = \lambda_{c-1}(R, I_{A'/B'})$ and we get

$$\dim W(k; \mathfrak{a}; r, R') = \lambda_c(R').$$

Remark 6.16. (1) Note that in the case $c = 3 - r$ of Corollary 6.15, the assumptions “every deformation of $\text{Proj}(B) \in W(k; \mathfrak{a}'; r)$ comes from deforming $\mathcal{B}$”, “$\dim W(k; \mathfrak{a}'; r, R) = \lambda_{c-1}(R)$” and “$\dim W(k; \mathfrak{a}'; r, R') = \lambda_{c-1}(R')$” hold by Theorem 6.15(iii) and Theorem 3.8(ii).

(2) Since the smallest degree of a generator of $I_{A/B}$ is $s_r - b_r - a_{t-r+1} + a_{t+c-1}$, $s_r - b_r := \sum_{i=1}^{t-r+1} (a_i - b_{r-1+i})$, and the largest degree of a generator of $I_B$ is $\text{mdg}(I_B) := \sum_{j=r}^{t+c-2} a_j - \sum_{i=1}^{t+c-1} b_i$, we get $\lambda_{c-1}(R, I_{A/B}) = 0$ for all $t$ if $s_r - b_r - a_{t-r+1} + a_{t+c-1} \geq \text{mdg}(I_B)$, which holds if $a_1 = a_{t+c-2}$ and $b_1 = b_t$ because $a_{t+c-1} \geq a_t - r + 1$ always holds.

Proof. (i) This is the main statement of Theorem 6.14 because we only need the inequality in (6.11) to get $\dim W(k; \mathfrak{a}; r, R) = \lambda_c(R)$.

(ii) Suppose $\dim W(k; \mathfrak{a}; r, R') = \lambda_{c-1}(R')$. Then we claim that all assumptions of (i) holds for $A', B', R'$ instead of $A, B, R$. Indeed by Corollary 5.4 we get that $\text{Proj}(B') \in W(k; \mathfrak{a}'; r, R')$ comes from deforming its matrix $\mathcal{B}$. The other assumptions are straightforward provided we can prove $\lambda_{c-1}(R, I_{A/B}) = \lambda_{c-1}(R, I_{A'/B'})$. However, this will follow if we can show

$$\text{Hom}_R(I_B, I_{A/B}) \otimes_k k[y] \cong \text{Hom}_R(I_{A'/B'}, I_{A'/B'})$$

(6.11)

because we can then take the degree zero part to conclude using $\lambda_{c-1}(R, I_{A/B}) = 0$ for $v < 0$ and $\deg(y_i) = 1$ for all $i$. Hence, since $k[y]$ is obviously $k$-flat, it suffices to prove

$$I_B \otimes_k k[y] = I_{B'}$$

and

$$I_{A/B} \otimes_k k[y] = I_{A/B}.$$
difficult due to Corollary 5.2 and Remark 6.13 noting that the latter gives us exactly the inequality the right way to be consistent with (6.10). Thus we have proved \( \dim W(k; a; r, R') = \lambda_c(R') \). □

**Example 6.17.** (Determinantal quotients of \( R = k[x_0, x_1, \cdots, x_n] \), using Theorem 6.14 for \( r = 2 \))

Let \( \mathcal{A} = [\mathcal{B}, v] \) be a general \( 4 \times 4 \) matrix with linear (resp. quadratic) entries in the first, second and third (resp. fourth) column, i.e. where \( \mathcal{B} \) is a linear \( 4 \times 3 \) matrix. We claim the vanishing of all \( 3 \times 3 \) minors of \( \mathcal{A} \) defines a determinantal ring that satisfies the assumption of Theorem 6.14 for \( r = 2 \), \( c = 1 \) and \( n \geq 5 \). Indeed since the \( 3 \times 3 \) minors of \( \mathcal{B} \) are maximal minors defining \( \mathcal{B} \), one knows that every deformation of \( \mathcal{B} \) comes from deforming \( \mathcal{B} \) by Theorem 3.8 or more directly by \( 32 \) Theorem 5.16, cf. 16. By Theorem 6.14 we get that the dimension of

\[
\dim W(0^4; 1^3, 2; 2) := \dim W(0^4, 1^3, 2; 2) = 5.
\]

is \( \lambda_0 \). Since the assumption \( a_t < s_2 \) and the e.g. statement of Theorem 6.14 clearly hold we get the claim and hence that

\[
\dim W(0^4; 1^3, 2; 2) = \lambda_1 = 2n^2 + 15n - 12
\]

by Theorem 6.14. We have checked the answer by using Macaulay2 in the case \( \mathcal{B} \) is the generic linear matrix with entries \( x_0, x_1, \ldots, x_{11} \) and the transpose of \( v \) is \( x_0^2, x_1^2, x_2^2, x_3^2, x_4^2 \). We have got \( \text{Ext}^1_A(IA/I^2A, A) = 0 \) and that the dimension of the tangent space of \( \text{Hilb}^{\mathcal{B}(t)}(\mathbb{P}^{15}) \) at \( X := \text{Proj}(A) \) was 663, whence \( \dim(X) \text{Hilb}(\mathbb{P}^{15}) = 663 \), coinciding with our formula when \( n = 15 \).

In fact the Macaulay2 computation also implies that \( \dim W(0^4; 1^3, 2; 2) \) is a generically smooth irreducible component of \( \text{Hilb}^{\mathcal{B}(t)}(\mathbb{P}^{15}) \), a problem which we will study closely in Section 7 and 8. At this stage we remark that this is no longer true for small values of \( n \). Indeed in 35 Example 5.1 we considered an arithmetically Gorenstein scheme defined by the submaximal minors of a general \( 4 \times 4 \) matrix with \( \mathcal{A} = [\mathcal{B}, v] \) as above, and got that \( \dim(X) \text{Hilb}^{\mathcal{B}(t)}(\mathbb{P}^{5}) = 125 \) and

\[
\text{codim}_{\text{Hilb}^{\mathcal{B}(t)}(\mathbb{P}^{5})} W(0^4; 1^3, 2; 2) = 12
\]

by direct calculations using the theory developed there. Clearly the irreducible locus \( \dim W(0^4; 1^3, 2; 2) \) is not a component of \( \text{Hilb}(\mathbb{P}^{15}) \). Moreover note that we get

\[
\dim W(0^4; 1^3, 2; 2) = 113,
\]

confirming our formula above with \( n = 5 \).

**Example 6.18.** (Verifying the conditions of Theorem 6.14 by using Macaulay2)

In this example we consider determinantal rings where the entries \( x_{i,j} \) of the \( t \times (t + c - 1) \) matrix \( \mathcal{A} \) are the indeterminates of \( R \), i.e. so-called generic determinantal schemes \( X = \text{Proj}(A) \), and we use Macaulay2 to compute \( \text{hom}_R(I_B, IA/B) \) and check that

\[
(6.12) \quad \text{hom}_R(I_B, IA/B) = \sum_{j=1}^{t+c-2} (a_j - a_{t+c-1+j}) = t - c - 2
\]

where \( B = R/I_{t-r+1}(\mathcal{B}) \) and \( \mathcal{B} \) is obtained by deleting the last column of \( \mathcal{A} = [\mathcal{B}, v] \). Note that by Corollary 5.4 every deformation of \( \mathcal{B} \) comes from deforming \( \mathcal{B} \), whence all assumptions of
hold and we conclude that 
\[ \dim W(b; c; r) = \lambda_c . \]

(i) Submaximal minors, i.e. \( r = 2 \). Let \( t = 3 \), \( A = (x_{i,j}) \) the generic \( 3 \times (c + 2) \) matrix and let \( 1 \leq c \leq 7 \). The vanishing of all \( 2 \times 2 \) minors of \( A \) defines a determinantal ring. We successively show that all conditions of Theorem 6.14 are satisfied starting with the case \( c = 1 \). Due to Proposition 5.3 and Corollary 5.4 one knows that every deformation of \( B \) comes from deforming \( B \), and by Theorem 6.5(iii) that
\[ \dim W(0; 3; 1; c + 1; 2) = \lambda_{c+1} = 64 \]
and \( \dim W(0; 1; c + 2; 2) = \lambda_c \) by induction on \( c \geq 2 \). Indeed when we delete a column of the matrix \( A \) of indeterminates defining the ring \( R \), the resulting matrix \( B \) consists of indeterminates belonging to a ring which has more variables that those appearing in \( B \). So \( B \) is as \( A \) above and we get
\[ \dim W(0; 1; c + 3; 2) = \lambda_{c+2} = 8(c + 2)^2 - 8 \]
by Theorem 6.14 and in fact \( \dim W(0; 1; c + 2; 2) = \lambda_c (R') \) by Corollary 6.15(ii) and moreover that every deformation of \( R'/I_2(A) \) comes from deforming \( A \) by Corollary 5.4. Hence we conclude by induction. Finally it is worthwhile to point out that \( W(0; 1; c + 2; 2) \) is a generically smooth irreducible component of \( \text{Hilb}^{\text{px}(t)}(\mathbb{P}^{3c+5}) \) for \( 1 \leq c \leq 6 \) by Lemma 6.12 cf. Lemma 7.2 which also covers the case \( c = 7 \).

(ii) Submaximal minors, i.e. \( r = 2 \) with \( t = 4 \). Let \( A = (x_{i,j}) \) be the generic \( 4 \times (c + 3) \) matrix and let \( 1 \leq c \leq 3 \). The vanishing of all \( 3 \times 3 \) minors of \( A \) defines a determinantal ring and we verify (6.12) by using Macaulay2 for every \( c, 1 \leq c \leq 3 \). Then we can argue exactly as in (i) to get that
\[ \dim W(0; 4; 1; c + 3; 2) = \lambda_{c+3} = 15(c + 3)^2 - 15 . \]
By Lemma 6.12 \( W(0; 4; 1; c + 3; 2) \) is a generically smooth irreducible component of \( \text{Hilb}^{\text{px}(t)}(\mathbb{P}^{4c+11}) \) for \( 1 \leq c \leq 2 \).

(iii) “Subsubmaximal” minors, i.e. \( r = 3 \). Let \( t = 4 \), let \( A = (x_{i,j}) \) be the generic \( 4 \times (c + 3) \) matrix and let \( 1 \leq c \leq 4 \). The vanishing of all \( 2 \times 2 \) minors of \( A \) defines a determinantal ring \( A \) and
we verify (6.12) by Macaulay2 for every \( c, 1 \leq c \leq 4 \). Starting with \((A) \in W(0^4; 1^4; 3)\) where \( c = 1 \), then \( B \) belongs to \( W(0^4; 1^3; 3)\), a locus which we considered in (i) above. Indeed by Lemma 6.4 \( W(0^3; 1^4; 2) = W(-1^4; 0^3; 3) \) and obviously, \( W(-1^4; 0^3; 3) = W(0^4; 1^3; 3) \). Hence we get
\[
\dim W(0^4; 1^3; 3) = \lambda_0 = 120
\]
and that every deformation of \( B \) comes from deforming \( \mathcal{B} \). Similarly if \( R' := R[y_1, \cdots y_c] \) where the \( y_i \) are indeterminates, we get from (i) that \( \dim W(0^4; 1^3; 3; R') = \lambda_0(R') \) and that every deformation of \( R'/I_2(\mathcal{B}) \) comes from deforming \( \mathcal{B} \). It follows that \( \dim W(0^4; 1^4; 3) = \lambda_1 \) by Theorem 6.14 that \( \dim W(0^4; 1^4; 3; R') = \lambda_1(R') \) by Corollary 6.15(ii) and that every deformation of \( R'/I_2(A) \) comes from deforming \( A \) by Corollary 6.15. Then we can as previously use induction and Corollary 6.15 to get
\[
\dim W(0^4; 1^{c+3}; 3) = \lambda_c := 15(c + 3)^2 - 15.
\]
By Lemma 6.12 we get that \( W(0^4; 1^{c+3}; 3) \) for \( 1 \leq c \leq 3 \), is a generically smooth irreducible component of Hilb\( P^x(1, 3) \). Note that Macaulay2 is used to verify (6.12) for \( 1 \leq c \leq 4 \).

(iv) Let \( r = 3 \) (resp. \( r = 4 \)) and \( t = 5 \). The vanishing of all \( 3 \times 3 \) (resp. \( 2 \times 2 \)) minors of a generic \( 5 \times 5 \) matrix \( A \) defines a determinantal ring and we verify (6.12) by using Macaulay2. Moreover by Lemma 6.3 \( W(0^4; 1^5; 2) = W(-1^5; 0^4; 3) = W(0^5; 1^4; 3) \) (resp. \( W(0^4; 1^5; 3) = W(0^5; 1^4; 4) \)) and combining with (ii) (resp. (iii)) above, we get that a general \( B \) of the \( \lambda_0 \)-dimensional \( W(0^5; 1^4; 3) \) (resp. \( W(0^5; 1^4; 4) \)) comes from deforming \( B \). Then Theorem 6.14 imply that
\[
\dim W(0^5; 1^5; 3) = \dim W(0^5; 1^5; 4) = \lambda_1 = 576.
\]

Since we expect the condition on \( \mathfrak{Hom}_R(I_B, I_{A/B}) \) of Theorem 6.14 to be true in general provided \( \dim A \geq 3 \) (and not only under the e.g., assumptions there), we suggest the following:

Conjecture 6.19. Let \( A \) be a general homogeneous \( t \times (t + c - 1) \) matrix, let \( A = R/I_{t-r+1}(A) \) where \( 1 \leq r \leq t-1 \), \( 2 - r \leq c \) and suppose that \( \dim A \geq 2 \) for \( c \neq 1 \) and \( \dim A \geq 3 \) for \( c = 1 \). Moreover suppose \( \text{Proj}(A) \in W(k, A; r), a_1 > b_r \) and \( a_{t+c-1} < s_r - b_r + b_1 \). Then
\[
\dim W(k, A; r) = \lambda_c.
\]

Remark 6.20. The conjecture is true for \( r = 1 \) (and \( c = 2 - r \)) by Theorem 6.3(iii), (6.3) and the text accompanying (6.6). We have found examples in the case \( c = 1 \) and \( \dim A = 2 \) (and none when \( \dim A \geq 3 \)) where the conclusion of the conjecture is not true. Indeed, to support the conjecture we have for \( r \geq 2 \) and \( \dim A \geq 2 \) computed quite a lot of examples, mostly with linear matrices, to see if the assumption
\[
\mathfrak{Hom}_R(I_B, I_{A/B}) = \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}
\]
of Theorem 6.14 and hence Conjecture 6.19 hold, and we got \( \dim W(k, A; r) = \lambda_c \) except for one in Example 6.21(1) below where \( \dim A = 2 \). It may be true that the conclusion of Conjecture 6.19 even holds for \( \dim A = 2, c = 1, r \geq 2 \) and \( t \geq 4 \), see Example 6.21(2) and (3).
Example 6.21. (1) Let $R = k[x_0, x_1, \cdots, x_t], A = [B, v]$ a general linear $3 \times 3$ matrix, $v$ a column, and let $A$ and $B$ be the quotients of $R$ defined by their $2 \times 2$ minors. So $t = 3$, $r = 2$, $c = 1$ and $\dim A = 2$. A Macaulay2 computation shows that $\hom_R(I_B, I_{A/B}) = 3$ and not 2 as Theorem 6.14 assumes, and the proof of Theorem 6.14 yields

$$\dim W(0^3; 1^3; 2) = \lambda_c - 1.$$ 

So $\dim W(k; a; r) < \lambda_c$ may occur (even though it turns out that $W(0^3; 1^3; 2)$ above is an irreducible component, cf. Example 7.13)! The “same” example with one more variable in $R$ yields $\dim A = 3$, $\hom_R(I_B, I_{A/B}) = 2$ and using Remark 6.16(i) we get $\dim W(0^3; 1^3; 2) = \lambda_c$ by Theorems 6.14 or Corollary 6.15(i). Also the “same” example with two more variable in $R$, as well as the generic one where $\dim R = 9$, yields $\hom_R(I_B, I_{A/B}) = 2$ and hence $\dim W(0^3; 1^3; 2) = \lambda_c$. In fact since $\hom_R(I_B, I_{A/B}) = 2$ for $\dim R = 7$, we do not need to use Macaulay2 for further computations because Corollary 6.15(ii) and Remark 6.16(1) apply and we conclude that $\dim W(0^3; 1^3; 2) = \lambda_c$ for determinantal quotients $A$ of $R$ as above provided $\dim R \geq 7$.

(2) If $R = k[x_0, x_1, \cdots, x_n]$ and we take $A = [B, v]$ to be a general linear $4 \times 4$ matrix, so $t = 4$, $c = 1$ and we let $r = 3$, i.e. we define the determinantal rings $A$ and $B$ by the $2 \times 2$ minors of $A$ and $B$, then a Macaulay2 computation shows that $\hom_R(I_B, I_{A/B}) = 3$ in the case $n = 10$ (dim $A = 2$), as Theorem 6.14 or Corollary 6.15(i) requires to get $\dim W(0^4; 1^4; 3) = \lambda_1 = 145$. But we still have to show the other assumptions of Theorem 6.14. Therefore letting $B = [C, w]$ and $C$ be the ring defined by the $2 \times 2$ minors of $C$ we verify $\hom_R(I_C, I_{B/C}) = 2$ and $\hom_R(I_B, B) = 108$ by using Macaulay2. Hence Corollary 6.15(i) and Remark 6.16(1) applies and we get $\dim W(0^4; 1^4; 3) = \lambda_0$. And since $\lambda_0 = 108$ we get $\dim W(0^4; 1^4; 3) = \hom_R(I_B, B)$. It follows that every deformation of $B$ comes from deforming $B$ by using Theorem 5.5. Remark 5.8 for $C \to B$, instead of $B \to A$ (this is a consequence of Theorem 7.1 as explained in detail in Remark 7.4). Thus we have

$$\dim W(0^4; 1^4; 3) = \lambda_1 = 145 \quad \text{for} \quad n = 10 \quad (\dim A = 2).$$

Then using Corollary 6.15(ii) twice, first for $C \to B$ instead of $B \to A$, we get $\dim W(0^4; 1^3; 3) = \lambda_0$ for any $n \geq 10$, and then for $B \to A$, and we conclude that $\dim W(0^4; 1^4; 3) = \lambda_1$ for every $n \geq 10$ (dim $A \geq 2$).

(3) Let $R = k[x_0, x_1, \cdots, x_n]$ and let $A = [B, v]$ be a general linear $4 \times 4$ matrix, so $t = 4$, $c = 1$ but now we take $r = 2$. Then a Macaulay2 computation shows that $\hom_R(I_B, I_{A/B}) = 3$, as Theorem 6.14 requires, for every $n \in \{5, 6, 7\}$ and we get $\dim W(0^4; 1^4; 2) = \lambda_c$ in each of the cases dim $A = 2, 3, 4$. But again Macaulay2 is only needed in the case dim $A = 2$ ($n = 5$) because then Remark 6.16(1) allows to use Corollary 6.15(ii) to get $\dim W(0^4; 1^4; 2) = \lambda_c$ for every $n \geq 5$.

The case $r = 2$ and $c = 1$ of Theorem 6.14 is considered in [35], Theorem 4.6] where a correction term $\kappa$ to the dimension formula is introduced. Since [35, Theorem 4.6] assumes $a_t > a_{t-1} + a_{t-2} - b_1$ which is unnecessary for getting $\dim W(k; a; 2)$, we take the opportunity to generalize the dimension formula of $W(k; a; 2)$ given there. Recalling that we get $a'$ by deleting $a_t$ in $a := (a_1, ..., a_t)$ we have
**Theorem 6.22.** Suppose that Proj\((A) \in W(\mathfrak{b}; \mathfrak{a}; 2)\) is general with \(c = 1\), \(I_{t-2}(A) \neq R\), Proj\((B) \in W(\mathfrak{b}; \mathfrak{a'}; 2)\) and \(A \geq 2\). Moreover let \(s := \sum_{i=1}^{t} (a_i - b_i)\) and suppose \(a_i \geq b_{i+3}\) for \(1 \leq i \leq t - 3\) \((a_1 \geq b_1\) for \(t = 3)\) and that \(\varphi_{\text{Hom}}(I_B, I_{A/B}) = \sum_{j=1}^{t-1} (a_j - a_i + n)\) (e.g. \(t \geq 3\), \(b_t = b_1 < a_1\) and \(a_{t-1} < a_t\)). Then we have

\[
\begin{align*}
\dim W(\mathfrak{b}; \mathfrak{a}; 2) = &\lambda_1 - \kappa_1 \quad \text{where} \quad \\
\kappa_1 = &\sum_{1 \leq i \leq n} (a_t - s - b_i - b_k + a_j + n) - \sum_{1 \leq i \leq n} (a_t - s - b_i - a_k + a_j + n) \\
&+ \sum_{1 \leq i \leq n} (a_t - s - a_i - a_k + a_j + n) - \sum_{2 \leq i \leq n} (a_t - s + b_i - 2b_1 + n).
\end{align*}
\]

**Remark 6.23.** The first sentence implicitly implies that \(W(\mathfrak{b}; \mathfrak{a}; 2) \neq \emptyset\) and \(W(\mathfrak{b}; \mathfrak{a'}; 2) \neq \emptyset\), while the assumption \(a_i \geq b_{i+3}\) for \(1 \leq i \leq t - 3\) \((a_1 \geq b_1\) for \(t = 3)\) implies that \(\text{depth}_{I_{t-2}(B)} B \geq 4\) by [33, Remark 2.7]. Then \(I_{t-2}(A) \neq R\) is equivalent to \(a_i > b_{i+2}\) for some \(i, 1 \leq i \leq t - 2\).

**Proof.** Note that \(B\) is defined by the maximal minors of the matrix \(B\) associated to \(B\). By [16] it follows that every deformation of \(B\) comes from deforming \(B\) and that \(\dim W(\mathfrak{b}; \mathfrak{a'}; 2) = \dim(N_B)\) where \(N_B := \text{Hom}(I_B, B)\). We also have \(\text{depth}_{I_{t-2}(B)} B = \text{depth}_{I_B} A + 2 \geq 4\), \(J_B = I_{t-r}(B)A\) by Remark 6.23. Using Proposition 6.11 we get that \(pr_2\) is surjective and that

\[
\dim W(\mathfrak{b}; \mathfrak{a}; 2) = \dim(MI \otimes A)_{\mathfrak{a}_i} - \dim \varphi_{\text{Hom}}(I_B, I_{A/B}) + \dim W(\mathfrak{b}; \mathfrak{a'}; 2)
\]

and note that the assumptions of Proposition 4.4 (iii) \((c + r \geq 3, \text{depth}_{I_B} A \geq 2)\) hold, which implies \(MI \otimes A(a_{t+1}) \cong \text{Hom}(I_{A/B}, A)\). Applying [30, Lemma 28] in our situation where \(M = N_B\) and \(M^* = I_B/I_B^2\), cf. [31, (33) and Corollary 41] and using \(s := \sum_{i=1}^{t} (a_i - b_i) = s_0 - a_{t+1}\) we have

\[
\varphi_{\text{Hom}}(I_{A/B}, A) = \dim(I_B/I_B^2)_{s} - \varphi_{\text{Hom}}(I_B/I_B^2, I_B/I_B^2) + \dim(K_B)_{n+1-2s}
\]

because \(\text{Hom}(I_B/I_B^2, I_B/I_B^2) \cong \text{Hom}_{\mathfrak{O}_U}(\widetilde{I_B^2} | U, \widetilde{I_B^2} | U) \cong \text{Hom}_{\mathfrak{O}_B}(N_B, N_B)\) by (2.4) and \(\text{Ext}_B^1(N_B, B) = 0\) and \(\text{Ext}_B^2(N_B, N_B) = 0\) for similar reasons, cf. [30, proof of Corollary 41] for more details. By [30, Remark 35] or [35, section 2.3], the exact sequence

\[
0 \rightarrow \oplus_{j=1}^{t-1} R(-a_j + a_t - s) \rightarrow \oplus_{i=1}^{t} R(-b_i + a_t - s) \rightarrow I_B \rightarrow 0
\]

induces exact sequences

\[
0 \rightarrow R \rightarrow \oplus_i I_B(s - a_t + b_i) \rightarrow \oplus_j I_B(s - a_t + a_j) \rightarrow N_B \rightarrow 0,
\]

\[
0 \rightarrow \text{Hom}_B(I_B/I_B^2, I_B/I_B^2) \rightarrow \oplus_i I_B/I_B^2(s - a_t + b_i) \rightarrow \oplus_j I_B/I_B^2(s - a_t + a_j) \rightarrow N_B \rightarrow 0
\]

and

\[
\cdot \rightarrow \oplus_j B(s - a_t + a_j) \rightarrow K_B(n + 1) \rightarrow 0.
\]

Thus if \(\eta(v) = \dim(I_B/I_B^2)v\) we have

\[
\dim W(\mathfrak{b}; \mathfrak{a}; 2) = \eta(s) - \sum_{i=1}^{t-1} \eta(a_t - s - b_i) + \sum_{j=1}^{t} \eta(a_t - s - a_j) - \sum_{j=1}^{t} \left(\frac{a_j - a_t + n}{n}\right) + 1
\]
because 0 = B(s − a_t + a_j − 2s_0) → K_B(n + 1 − 2s_0) and \( 0_{\text{hom}}R(I_B, I_{A/B}) = \sum_{j=1}^{t-1} (a_j - a_t + n) \) by assumption. To compute \( \eta(v) \) we use the exact sequences

\[ 0 \to I_B^2 \to I_B \to I_B/I_B^2 \to 0 \]

and

\[ (6.13) \quad 0 \to \bigoplus_{1 \leq i < j \leq t-1} R(-a_i - a_j + 2a_t - 2s) \to \bigoplus_{1 \leq i \leq t} R(-b_i - a_j + 2a_t - 2s) \to \bigoplus_{1 \leq i \leq t} R(-b_i - b_j + 2a_t - 2s) \to I_B^2 \to 0 \]

and we refer to [35] to see that

\[
\eta(s) - \sum_{i=1}^{t-1} \eta(a_t - s - b_i) + \sum_{j=1}^{t} \eta(a_t - s - a_j) = \sum_{1 \leq i \leq t \atop 1 \leq j \leq t} \binom{a_j - b_i + n}{n} - \sum_{1 \leq i \leq t \atop 1 \leq j \leq t} \binom{b_i - a_j + n}{n} - \kappa_1,
\]

where

\[
\kappa_1 = \sum_{1 \leq i \leq t \atop 1 \leq j \leq k \leq t} \binom{a_t - s - b_i - b_k + a_j + n}{n} - \sum_{1 \leq i \leq t \atop 1 \leq k \leq t} \binom{a_t - s - b_i - a_k + a_j + n}{n} + \sum_{1 \leq i < k \leq t-1 \atop 1 \leq j \leq t} \binom{a_t - s - a_i - a_k + a_j + n}{n} - \sum_{2 \leq i \leq t} \binom{a_t - s + b_i - 2b_1 + n}{n}.
\]

To get \( \dim W(b; a; 2) \) we only need to subtract \( \sum_{j=1}^{t-1} (a_j - a_t + n) \) which amounts to change the indices of the second sum of binomials from 1 ≤ i ≤ t − 1, 1 ≤ j ≤ t to 1 ≤ i ≤ t, 1 ≤ j ≤ t. We can do exactly the same change for the indices of the fourth sum of binomials because the extra binomials we add are zero provided \( a_t > b_t \). Hence we get \( \dim W(b; a; 2) = \lambda_1 - \kappa_1 \) and we are done.

\[ \square \]

**Example 6.24.** (Determinantal quotients of \( R = k[x_0, x_1, \ldots, x_n] \), using Theorem 6.22 with \( r = 2 \))

Let \( A = [B, v] \) be a general \( 3 \times 3 \) matrix with linear (resp. quadratic) entries in the first and second (resp. third) column. The degree matrix of \( A \) is \( \left( \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{array} \right) \) and \( b_i = 0 \) for 1 ≤ i ≤ 3. The vanishing of all \( 2 \times 2 \) minors of \( A \) defines a determinantal ring that satisfies all conditions of Theorem 6.14 (with \( r = 2, t = 3, c = 1 \), noting that

\[ \dim W(0^3; 1^2; 2) = \lambda_0 \]

by Lemma 6.4), except \( a_t < s_2 = 2 \). However, the weak assumption of Theorem 6.22 \( a_{t-1} < a_t \) holds, and it follows that

\[ \dim W(0^3; 1^2; 2, 2) = \lambda_1 - \kappa_1 = (3n^2 + 17n - 24)/2 \]

by Theorem 6.22. Indeed \( \kappa_1 = 6 \). To check the answer using Macaulay2, let \( B \) be the generic linear matrix with entries \( x_0, x_1, \ldots, x_5 \) and let \( v^{tr} = (x_6^2, x_7^2, x_8^2) \). We get \( \text{Ext}_A^1(I_A/I_A^2, A) = 0 \) and \( \dim(X) \text{Hilb}^{px(t)}(\mathbb{P}^8) = 152 \) for \( X := \text{Proj}(A) \), coinciding with our formula for \( \dim W(0^3; 1^2; 2, 2) \) when \( n = 8 \). We have also checked the cases 5 ≤ n ≤ 7 using Macaulay2, and for \( n = 7 \) they
coincide while in the cases $5 \leq n \leq 6$, $\dim W(0^3; 1^2; 2; 2) < \dim (X)$ $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$, showing that $W(0^3; 1^2; 2; 2)$ is not an irreducible component of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ for $n < 7$, i.e. when $1 \leq \dim X \leq 2$, so deforming $X$ is not equivalent to deforming its associated homogeneous matrix. Note that the curve case ($n = 5$) of this example was thoroughly analysed in [30, Example 23].

7. Generically smooth components of the Hilbert scheme

The goal of this section is to address Problems 6.1 (2) and (3) and to examine when $W(k; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$. Letting $a' = a_1, a_2, \ldots, a_t+c-2$ as previously we have

**Theorem 7.1.** Let $c > 2 - r$, let $B = R/I_B \rightarrow A$ be determinantal algebras defined by $(t + 1 - r)$-minors of matrices $A$ and $B$ representing $\varphi^*$ and $\varphi^*_{t+c-2}$, respectively, and suppose that $X := \text{Proj}(A) \subset W(k; a; r)$ is general and satisfies $\dim A \geq 2$ and $a_1 > b_t$. If $c \leq 0$, we also suppose $\dim A \geq 3$. Moreover let $\gamma$ be the composition

$$0 \text{Hom}_R(I_A, A) \rightarrow 0 \text{Hom}_R(I_B, A) \rightarrow 0 \text{Ext}_B^1(I_B/I_B^2, I_A/I_B)$$

and suppose $\gamma = 0$ (e.g. $0 \text{Ext}_B^1(I_B/I_B^2, I_A/I_B) \rightarrow 0 \text{Ext}_B^1(I_B/I_B^2, B)$ is injective) and that every deformation of $B$ comes from deforming $A$. Then $W(k; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming $A$. Moreover

$$\dim W(k; a; r) = \dim W(k; a'; r) + \dim_k(MI \otimes A)_{(a_t+c-1)} = 0 \text{hom}_R(I_B, I_A/I_B).$$

Since we have $\text{depth}_{H_{t+1-r}(B)A} A \geq \min\{\dim A, c + r - 1\} \geq 2$ by Remark 5.5, the theorem is an immediate consequence of Theorem 5.5 Remark 5.8 Proposition 6.11 and the following

**Lemma 7.2.** Set $A = R/I_{t+1-r}(\varphi^*)$, let $X := \text{Proj}(A) \subset W(k; a; r)$, and suppose $\dim X \geq 1$ and that every deformation of $A$ comes from deforming $A$. Then $W(k; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$.

**Proof.** By Corollary 5.2 $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ is smooth at $(X)$. Then the proof in the 2nd paragraph of Lemma 6.12 with $A$ instead of $B$, applies, or see Remark 7.3 which proves even more. □

**Remark 7.3.** Let $(T, m_T)$ be the local ring of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ at $(X)$, $X = \text{Proj}(A)$ of dimension $\geq 1$ (cf. 2.2) and let $\text{Proj}(A_{T_2})$ be the pullback of the universal object of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ to $\text{Spec}(T_2)$ where $T_m = T/m^n_T$, $m \geq 2$ and $T_2 = k\{t_1, \ldots, t_k\}$. In the proof of Lemma 6.12 we observed that the “universal object” $A_{T_2}$ is defined by some matrix by assumption. Even more is true. In fact we can extend the pullback of the universal object of $\text{Hilb}^{pX(t)}(\mathbb{P}^n)$ to $\text{Spec}(\hat{T})$ where $\hat{T}$ is the completion of the regular local ring $T$ with respect to $m_T$. And since any deformation of the “universal quotient” $R \otimes_k T_2 \rightarrow A_{T_2}$ to $\hat{T}$ suffices to define the prrepresenting object by [43], proof of Theorem 4.2.4, up to isomorphism, we may take the matrix $A_{\hat{T}}$ of $A_{\hat{T}}$ as defined by some lifting (e.g. take the entries to be of degree one in the $t_i$) of $A_{T_2}$ to $\hat{T}$, whence the generators of $I_{t+1-r}(A_{\hat{T}})$ are polynomials, and not power series in $t_i$. Thus we can further extend the entries of $A_{\hat{T}}$ to polynomials $f_{ij,D}$ with
coefficients in $D$ where $\text{Spec}(D)$ is a small enough open set of $\text{Hilb}^{px}(\mathbb{P}^n)$ containing $(X)$ for which the Lascoux complex associated to the matrix $A_D = (f_{ij,D})$ is exact at any $(X') \in \text{Spec}(D)$.

**Remark 7.4.** (1) We often use Theorem 7.1 verifying $0\text{Ext}_B^1(I_B/I_B^2; I_{A/B}) = 0$, but the corresponding injectivity assumption in Theorem 7.1 is a priori weaker and, in fact, equivalent to the vanishing of the connecting map $0\text{Hom}_R(I_B, A) \to 0\text{Ext}_B^1(I_B/I_B^2; I_{A/B})$, so equivalent to the exactness of

\[(7.1) \quad 0 \to 0\text{Hom}_R(I_B, I_{A/B}) \to 0\text{Hom}_R(I_B, B) \to 0\text{Hom}_R(I_B, A) \to 0.
\]

Since the left-exactness always holds, the exactness may be verified by Macaulay2 by computing dimensions of these Hom-groups which may be faster than computing $0\text{Ext}_B^1(I_B/I_B^2, I_{A/B})$.

(2) Suppose all assumptions of Theorem 7.1, except $\gamma(1)$ is expected to be weaker only when $\dim A = 2$ and in the cases $(c, r) = (2, 2)$ and $r = 1$ because $0\text{Ext}_B^1(I_B/I_B^2; B) = 0$ is expected for $\dim B \geq 4$, $c \neq 2$ by Conjecture 7.15(i) and Proposition 7.17(i), see [32, Theorem 5.11] for $(c, r) = 0$ and only if $\gamma = 0$. Hence we get from diagram (5.9) that one may verify the condition $\gamma = 0$ by showing

\[(7.2) \quad 0\text{hom}_R(I_A, A) = 0\text{hom}_R(I_B, B) + \dim_k(MI \otimes A)_{(a_t+c-1)} - 0\text{hom}_R(I_B, I_{A/B}),
\]

and conversely that $\gamma = 0$ implies (7.2). Moreover $\gamma = 0$ is further equivalent to $W(b; a; r) \subset \text{Hilb}^{px}(\mathbb{P}^n)$ being a generically smooth irreducible component. Indeed we have one way by Theorem 7.1 and conversely, we use the dimension formula of Proposition 6.11 to see that (7.2) holds. Thus $\gamma = 0$ is also equivalent to $\dim W(b; a; r) = 0\text{hom}_R(I_A, A)$. Finally we notice that if also the assumptions of Theorem 6.14 hold with $\kappa = 0$, then (7.2) is equivalent to $0\text{hom}_R(I_A, A) = \lambda_c$.

(3) Suppose all assumptions of Theorem 7.1 except $\gamma = 0$, hold. Then the a priori weaker injectivity assumption of (1) is expected to be weaker only when $\dim A = 2$ and in the cases $(c, r) = (2, 2)$ and $r = 1$ because $0\text{Ext}_B^1(I_B/I_B^2; B) = 0$ is expected for $\dim B \geq 4$, $c \neq 2$ by Conjecture 7.15(i) and Proposition 7.17(i), see [32, Theorem 5.11] for $(c, r) = 0$. Moreover it is clear that (7.1) implies $\gamma = 0$ by the definition of $\gamma$. And conversely due to Remark 7.3 and (2) above we get that $\gamma = 0$ implies (7.1) provided $c \geq 4 - r$. However, if $c = 3 - r$ they are not equivalent, even in the generic case as Example 7.5 shows.

**Example 7.5.** Here is an example in the case $c = 3 - r$ where $0\text{Ext}_B^1(I_B/I_B^2, I_{A/B}) \neq 0$ and $0\text{Ext}_B^1(I_B/I_B^2, B) = 0$, but $\gamma = 0$. Indeed let $A$ be the generic determinantal ring defined by the ideal of all $2 \times 2$ minors of the $3 \times 3$ matrix $A = (x_{i,j})$ and let $B = R/I_2(B)$ where $A = [B, v]$ and $R = k[x_{i,j}]$. In this case $0\text{hom}_R(I_B, B) = 42$, $0\text{hom}_R(I_B, I_{A/B}) = 2$ and $0\text{hom}_R(I_A, A) = \lambda_1 = 64$ by Example 6.18(i) with $c = 1$. Using Macaulay2 we get $0\text{hom}_R(I_B, A) = 48$ and $\dim_k(MI \otimes A)_{(1)} = 24$, whence $\gamma = 0$, i.e. (7.2) holds while (7.1) is not true.

**Corollary 7.6.** Let $X = \text{Proj}(A) \in W(0^t; 1^{t+c-1}; r)$, $1 \leq r < t$, $(r, c) \neq (1, 1)$ be a generic determinantal scheme defined by $s \times s$-minors of a $t \times (t+c-1)$ matrix $A = (x_{ij})$ of indeterminates of the polynomial ring $R := k[x_{ij}]$, i.e. $A := R/I_s(A)$ with $s = t + 1 - r$. Then $W(0^t; 1^{t+c-1}; r)$ is a generically smooth irreducible component of $\text{Hilb}^{px}(\mathbb{P}^n)$, $n := t(t + c - 1) - 1$ and every deformation of $A$ comes from deforming $A$.

More generally taking $s \times s$-minors of a matrix $A = (x_{ij})$ of indeterminates belonging to a larger polynomial ring $R[x] := k[x_{ij}, y_k]$, $1 \leq k \leq e$ and letting $A' := R[y]/I_s(A)$ and $W(0^t; 1^{t+c-1}; r)$
Proof. This follows from Theorem 7.1, Remark 7.2, Corollary 5.11 and Proposition 5.3 (or more directly by combining Lemma 7.2 and Corollary 5.4) because (Proj(A')) ∈ \((0; 1^{t+c-1}; r; R[y])\). □

Since \(_0\text{Ext}^1_R(I_B/I^2_B, I_{A/B}) \subset \_0\text{Ext}^1_R(I_B, I_{A/B})\) we see that if the degree of all generators of \(I_{A/B}\) is larger than the maximum of the degree of the relations, \(mdr(I_B)\), of \(I_B\) appearing in the Lascoux resolution, the mentioned \(_0\text{Ext}^1\)-groups vanish. Here we define \(mdr(I)\) by \(mdr(I) = \max\{n_{2,j}\}\) and \(mdg(I)\) by \(mdg(I) = \max\{n_{1,i}\}\) where

\[
\oplus j R(-n_{2,j}) \rightarrow \oplus (n_{1,i}) \rightarrow I \rightarrow 0
\]

is a minimal presentation of a graded ideal \(I\). To apply Theorem 7.1 it is of interest to compute both \(mdg(I)\) and \(mdr(I)\). We have:

**Lemma 7.7.** Let \(A\) be a \(t \times (t + c - 1)\) matrix and let \(A = R/I_{t+1-r}(A)\) be determinantal. We have:

(i) \(mdg(I_{t+1-r}(A)) = \sum_{j=c+r-1}^{t+c-1} a_j - \sum_{i=1}^{t+1-r} b_i\).

(ii) \(mdr(I_{t+1-r}(A)) = \sum_{j=c+r-2}^{t+c-1} a_j - \sum_{i=1}^{t+2-r} b_i + m\) where \(m = \max\{b_{t+2-r} - b_1, a_{t+c-1} - a_{c+r-2}\}\).

**Proof.** (i) The generators \(J_{t+1-r}^{j_1\cdots j_{t+1-r}}(A)\) of \(I_{t+1-r}(A)\) are the determinant of the \((t+c-1)\) \((t+c-1)\) minors of size \((t+1-r) \times (t+1-r)\) that we obtain choosing \(t+1-r\) rows \((1 \leq i_1 < i_2 < \cdots < i_{t+1-r} \leq t+c-1)\) and \(t+1-r\) columns \((1 \leq j_1 < j_2 < \cdots < j_{t+1-r} \leq t)\) of \(A\). Therefore,

\[
mdg(I_{t+1-r}(A)) = \sum_{j=c+r-1}^{t+c-1} a_j - \sum_{i=1}^{t+1-r} b_i.
\]

(ii) According to [44] the first syzygies of \(I_{t+1-r}(A)\) are given by one of the following constructions:

(a) We choose \(t+1-r\) rows \(1 \leq i_1 < i_2 < \cdots < i_{t+1-r} \leq t+c-1\) of \(A\) and we construct a \((t+2-r) \times (t+c-1)\) matrix \(A_{i_1\cdots i_{t+1-r}1}^{1\cdots t+c-1}\) using all these rows \(i_v\) and repeating one of the \(i_v\). The determinant of the \((t+c-1)\) \((t+c-1)\) minors of size \((t+2-r) \times (t+2-r)\) of the matrix \(A_{i_1\cdots i_{t+1-r}1}^{1\cdots t+c-1}\) gives us a syzygy of \(I_{t+1-r}(A)\).

(b) We choose \(t+1-r\) columns \(1 \leq j_1 < j_2 < \cdots < j_{t+1-r} \leq t\) of \(A\) and we construct a \(t \times (t+2-r)\) matrix \(A_{j_1\cdots j_{t+1-r}v}^{t\cdots t+2-r}\) using all these columns \(j_v\) and repeating one of them. The determinant of the \((t+2-r)\) \((t+2-r)\) minors of size \((t+2-r) \times (t+2-r)\) of the matrix \(A_{j_1\cdots j_{t+1-r}v}^{t\cdots t+2-r}\) gives us a syzygy of \(I_{t+1-r}(A)\).

(c) We consider a \((t+2-r) \times (t+2-r)\) minor \(A_{i_1\cdots i_{t+2-r}}^{j_1\cdots j_{t+2-r}}\) of \(A\) choosing \(t+2-r\) rows \((1 \leq i_1 < i_2 < \cdots < i_{t+2-r} \leq t+c-1)\) and \(t+2-r\) columns \((1 \leq j_1 < j_2 < \cdots < j_{t+2-r} \leq t)\).
of $A$. Call $D_{j_1\ldots j_{t+2-r}}^{i_1\ldots i_{t+2-r}}(p,q)$ the relation that we get taking the difference of the expansion of the determinant of $A_{j_1\ldots j_{t+2-r}}^{i_1\ldots i_{t+2-r}}$ along the $p$-th row and along the $q$-th column. This also gives us a syzygy of $I_{t+1-r}(A)$.

From the above description of the first syzygies we immediately get that

$$mdr(I_{t+1-r}(A)) = \sum_{j=c+r-2}^{t+c-1} a_j - \sum_{i=1}^{t+2-r} b_i + m \text{ where } m = \max\{b_{t+2-r} - b_1, a_{t+c-1} - a_{c+r-2}\}.$$ 

\[\square\]

Due to results in Section 8 (cf. Theorem 8.10) it suffices to compute $mdr(I_B)$ when $I_B$ is defined by submaximal (or maximal) minors of a homogeneous matrix $A$. In these particular cases we have

**Corollary 7.8.** Let $A$ be a $t \times (t + c - 1)$ matrix and let $A = R/I_{t+1-r}(A)$ be determinantal.

(i) If $r = 2$, $c \geq 1$, then $mdr(I_A) = \sum_{i=t+1}^{t+c-1} a_i - \sum_{i=1}^{t} b_i + m$, where $m = \max\{a_{t+c-1} - a_c, b_t - b_1\}$.

(ii) If $r = 3 - r$ and $2 \leq r \leq t$ then $A$ is defined by submaximal minors and

$$mdr(I_A) = \sum_{i=1}^{t-r+2} a_i - \sum_{i=1}^{t-r+1} b_i + m \text{ where } m = \max\{a_{t-r+2} - a_1, b_{t-r+2} - b_1\}.$$ 

(iii) If $c = 2 - r$ and $2 \leq r \leq t - 1$ then $A$ is defined by maximal minors and

$$mdr(I_A) = a_{t-r+1} + \sum_{i=1}^{t-r+1} a_i - \sum_{i=1}^{t-r+2} b_i.$$ 

**Proof.** (i) It follows from Lemma 7.7 (ii).

(ii) If we transpose the matrix $A$ then we get a $(t - r + 2) \times t$ matrix which fits into the set-up of (i), but note that the inequalities $b_1 \leq b_2 \leq \cdots \leq b_t, a_1 \leq a_2 \leq \cdots \leq a_{t-r+2}$ attached to $A$ are reversed, i.e. we have the inequalities $-b_1 \geq -b_2 \geq \cdots \geq -b_t, -a_1 \geq -a_2 \geq \cdots \geq -a_{t-r+1}$, attached to $A^{tr}$. Thus if we for $A^{tr}$ also “transpose” both rows and columns or, more precisely, using primes (‘) for $A^{tr}$ we have $-b_i = a'_{t+1-i}$ and $-a_i = b'_{t-r+3-i}$. Then (i) implies

$$mdr(I_A) = \sum_{i=1}^{t-r+2} (-b_i) - \sum_{i=1}^{t-r+2} (-a_i) + m$$

where $m = \max\{-b_1 - (-b_{t-r+2}), -a_1 - (-a_{t-r+2})\}$.

(iii) This is rather immediate to see using the Eagon-Northcott resolution. \[\square\]

**Corollary 7.9.** Let $B = R/I_B \rightarrow A$ be as in (the first two sentences of) Theorem 7.1, and suppose $2 \leq r \leq t - 1$ and that every deformation of $B$ comes from deforming $B$. If

(i) $c = 3 - r$ and $a_{t-r+2} > 2a_{t-r+1} + \sum_{i=t}^{t} b_i - \sum_{i=1}^{t-r+2} b_i$, or

(ii) $c = 4 - r$ and $a_{t-r+3} > \sum_{i=t-r+1}^{t-r+2} a_i + \sum_{i=1}^{t-r+2} b_i - \sum_{i=1}^{t-r+2} b_i + \max\{a_{t-r+2} - a_1, b_{t-r+2} - b_1\}$

then $\Ext^i_R(I_B, I_A_B) = 0$ for $i = 0, 1$. In particular $W(b; a; r)$ is a generically smooth irreducible component of $\Hilb^p X(t)(\mathbb{G}^n)$ and every deformation of $A$ comes from deforming $A$. Furthermore

$$\dim W(b; a; r) = \dim W(b; a'; r) + \dim_k(MI \otimes A)_{(a_{t+c-1})}.$$
Proof. (i) It is easy to see that the smallest degree of the minimal generators of \( I_{A/B} \) is

\[
s(I_{A/B}) := a_{t-r+2} + \sum_{i=1}^{t-r} a_i - \sum_{i=r}^{t} b_i
\]

while the maximum degree of the relations, \( mdr(I_B) \), of \( I_B \) is

\[
mdr(I_B) = \sum_{i=1}^{t-r+1} a_i - \sum_{i=r}^{t-r+2} b_i + a_{t-r+1}
\]

by Corollary 7.8(iii). Indeed \( B \) is defined by maximal minors. Thus by the definition of \( \text{Ext}^1_R(I_B, I_{A/B}) \) this \( \text{Ext}^1_R \)-group (as well as \( \text{Hom}_R(I_B, I_{A/B}) \)) vanish if \( mdr(I_B) < s(I_{A/B}) \), which is equivalent to assumption (i) and Theorem 7.1 applies.

(ii) Now the smallest degree of a generator of \( I_{A/B} \) is

\[
s(I_{A/B}) := a_{t-r+3} + \sum_{i=1}^{t-r} a_i - \sum_{i=r}^{t} b_i.
\]

Since \( B \) is defined by submaximal minors, we have by Corollary 7.8(ii) a formula for the maximum degree of relations of \( I_B \). Again it is clear by definition of \( \text{Ext}^1_R(I_B, I_{A/B}) \) that this group vanishes if \( mdr(I_B) < s(I_{A/B}) \). Since this is equivalent to the assumption of (ii), we get \( \text{Ext}^1_R(I_B, I_{A/B}) = 0 \) for \( i = 1 \) and certainly also for \( i = 0 \), and we conclude the corollary by Theorem 7.1.

Example 7.10. (i) For any \( r, 2 \leq r \leq t-1 \), let \( A = [B, v] \) be a general \( t \times (t-r+2) \) matrix with \( B \) linear and \( v \) a column of cubic entries, let \( B = R/I_B \rightarrow A \) be defined by \( (t-r+1) \)-minors of \( B \), resp. \( A \), and suppose \( R \) is large enough so that \( \dim A \geq 3 \). Since \( B \) is defined by maximal minors and \( \dim B \geq 4 \) one know that every deformation of \( B \) comes from deforming \( B \) by Theorem 3.8. Since the numerical conditions of Corollary 7.9(i) are satisfied, \( \overline{W([k; A]; r)} \) is a generically smooth irreducible component of \( \text{Hilb}^{px(t)}(\mathbb{P}^n) \) and every deformation of \( A \) comes from deforming its associated matrix \( A \). Note that for \( r = 2 \), \( \dim \overline{W([k; A]; r)} \) is given by Theorem 6.22 and that this case was considered in \([35]\).

(ii) Let \( A = [B, w] \) be a general \( t \times (t-r+3) \) where the \( t \times (t-r+2) \) matrix \( B \) is exactly the matrix \( A \) in (i) above and \( w \) a column whose entries are of degree 7. Let \( B = R/I_B \rightarrow A \) be defined by \( (t-r+1) \)-minors of \( B \), resp. \( A \), and suppose that \( R \) is large enough so that \( \dim A \geq 3 \). Then every deformation of \( B \) comes from deforming \( B \) by (i) above and since \( B \) is defined by submaximal minors and the numerical conditions of Corollary 7.9(ii) are satisfied, \( \overline{W([k; A]; r)} \) is a generically smooth irreducible component of \( \text{Hilb}^{px(t)}(\mathbb{P}^n) \) and every deformation of \( A \) comes from deforming \( A \).

Examples seem to indicate that \( \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0 \) or more generally that \([7,2]\) hold for \( \dim A \) large enough, and further evidence to this observation is given in Section 8. We expect

Conjecture 7.11. Let \( r \geq 1, c \geq 2-r, (r,c) \neq (1,1) \) and let \( X = \text{Proj}(A) \in \overline{W([k; A]; r)} \) be defined by the vanishing of the \( (t-r+1) \times (t-r+1) \) minors of a general \( t \times (t+c-1) \) matrix \( A \) with...
$a_1 > b_t$ and suppose that $\dim A \geq 4$ for $c = 1$ and $\dim A \geq 3$ for $c \neq 1$. Then, $W(b; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming $A$.

**Remark 7.12.** The conjecture is true for $r = 1$ and $c = 2 - r$ by Theorem 6.8 and Lemma 6.1, and for the component $W(0; 1^{t+c-1}; r)$, $r \geq 1$ containing a generic determinantal scheme, whence for components $W(0; 1^{t+c-1}; r)$ with $\dim R \geq t(t+c-1)$, by Corollary 7.6. For $c \in \{0, 2\}$ the conclusion of Conjecture 7.11 may even be true for $\dim A \geq 2$, as it is for $(r, c) = (1, 2)$. We have considered many examples and used Macaulay2 to check if (7.2) or $\gamma = 0$ (or $0_{\text{ext}}^1(I_B/I_B^2, I_{A/B}) = 0$) of Theorem 7.1 hold, and it seems that only the Gorenstein case $(c = 1)$ requires $\dim A \geq 4$ while in all other cases, $\dim A \geq 3$ suffices. We now list some examples, mostly where the conclusion of Conjecture 7.11 fails.

**Example 7.13.** (i) Let $R = k[x_0, x_1, \ldots, x_0]$, let $A = [B, v]$, $v$ a column, be a general $3 \times 5$ matrix with linear entries, and let $A$ and $B$ be the quotients of $R$ defined by the $2 \times 2$ minors of $A$ and $B$, respectively. So $t = 3$, $r = 2$, $c = 3$ and $\dim A = 2$. A Macaulay2 computation shows that $\partial_{\text{hom}}(I_B, I_{A/B}) = 4$, $\partial_{\text{hom}}(I_B, B) = 96$, $\partial_{\text{hom}}(I_A, A) = 120$ and $\dim_k(M I \otimes A)_1 = 25$ and (7.2) is not satisfied. Let us check that every deformation of $B$ comes from deforming $B = [C, w]$. Indeed if $C$ is defined by the $2 \times 2$ minors of the $3 \times 3$ matrix $C$, then every deformation of $C$ comes from deforming $C$ owing to Remark 7.12 and $\dim R \geq 9$. Then $\partial_{\text{ext}}^1(C/I_C^2, I_{B/C}) = 0$ is easily checked using Macaulay2. Applying Theorem 7.1, replacing $B \to A$ there by $C \to B$, we get that every deformation of $B$ comes from deforming $B$. It follows from Remark 7.12 that $W(0^3; 1^5; 2)$ is not a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^9)$.

(ii) If we try to treat example (i) by deleting a row, one may transpose the matrix and instead delete a column, i.e. we look at a general $5 \times 3$ matrix $A = [B, v]$ with linear entries from $R = k[x_0, x_1, \ldots, x_9]$, $v$ a column, and we let $A$ and $B$ be the quotients of $R$ defined by their $2 \times 2$ minors, respectively. So $t = 5$, $r = 4$, $c = -1$ and $\dim A = 2$. The only problem with this approach is that Theorem 7.1 and its corollaries require $\dim A \geq 3$ when $c = -1$, i.e. they do not apply.

(iii) Let $R = k[x_0, x_1, \ldots, x_n]$ with $n = 6$, $A = [B, v]$ a general linear $3 \times 3$ matrix, $v$ a column, and let $A$ and $B$ be the quotients of $R$ defined by their $2 \times 2$ minors. So $t = 3$, $r = 2$, $c = 1$ and $\dim A = 3$. Using Macaulay2 we get that $\partial_{\text{hom}}(I_B, I_{A/B}) = 2$ and $\partial_{\text{hom}}(I_A, A) = 46$. Hence $\dim W(0^3; 1^3; 2) = \lambda_1$ by Theorem 6.14, and since $\lambda_1 = 46$, $W(0^3; 1^3; 2)$ is a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^6)$, cf. Remark 7.4(2). The “same” example with $n = 7$ yields $\partial_{\text{hom}}(I_B, I_{A/B}) = 2$ and $\partial_{\text{hom}}(I_A, A) = 55$, and arguing as above, we get that $W(0^3; 1^3; 2)$ is a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^7)$ of dimension $\lambda_1 = 55$. For $n \geq 8$ with $n = 8$ for the generic case, we do not need to use Macaulay2 because it follows from Corollary 7.6 that $W(0^3; 1^3; 2)$ is a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^n)$.

The “same” example with $n = 5$ yields $\partial_{\text{hom}}(I_B, I_{A/B}) = 3$, and

$$\dim W(0^3; 1^3; 2) = \lambda_c - 1 = 36,$$
as noticed in Example 6.24(1). Since we have \( \dim_{\mathcal{R}}(I_{A}, A) = 36 \) by Macaulay2, \( W(0^{4}; 1^{3}; 2) \subset \mathrm{Hilb}_{X}^{\alpha}(\mathbb{P}^{5}) \) is a generically smooth component of dimension \( \lambda_{c} - 1 \).

(iv) We repeat (iii) above with one change, namely we let \( \mathcal{A} = [B, v] \) be a general \( 3 \times 3 \) matrix, where \( B \) is linear while all entries of the column \( v \) are of degree 2. This example was considered in Example 6.24, where we computed \( \dim W(0^{3}; 1^{2}, 2; 2) \) in terms of \( n \). Using Macaulay2 we show \( \dim_{\mathcal{R}}(I_{A}, A) = 94 \) (resp. 71) for \( n = 6 \) (resp. \( n = 5 \)), which is different (for \( n \leq 6 \) only) from the values of \( \dim W(0^{3}; 1^{2}, 2; 2) \) we found in Example 6.24. This shows that \( W(0^{3}; 1^{2}, 2; 2) \) is not a generically smooth irreducible component of \( \mathrm{Hilb}_{X}^{\alpha}(\mathbb{P}^{5}) \). Note that \( c = 1 \) and \( \dim A = 3 \) (resp. \( \dim A = 2 \)) for \( n = 6 \) (resp. \( n = 5 \)), cf. Conjecture 7.11 and Remark 7.12.

(v) More “submaximal minors”: Let \( R = k[x_{0}, x_{1}, \ldots , x_{n}] \) with \( n = 6 \), let \( \mathcal{A} = [B, v] \) a general linear \( 4 \times 4 \) matrix, \( v \) a column, and let \( A \) and \( B \) be the quotients of \( R \) defined by their \( 3 \times 3 \) minors. So \( t = 4, r = 2, c = 1 \) and \( \dim A = 3 \). A Macaulay2 computation shows that \( \dim_{\mathcal{R}}(I_{B}, I_{A/B}) = 3 \), \( \dim_{\mathcal{R}}(I_{A}, A) = 88 \) and \( \dim_{\mathcal{R}}(M_{I} \otimes A)_{(1)} = 24 \) and since (7.2) is not satisfied, \( W(0^{4}; 1^{2}; 2) \) is not a generically smooth irreducible component of \( \mathrm{Hilb}_{X}^{\alpha}(\mathbb{P}^{6}) \) by Remark 7.12. Corresponding calculations of the “same” example, only changing \( n \) to \( n = 5 \), so \( \dim A = 2 \) yields \( \dim_{\mathcal{R}}(I_{B}, I_{A/B}) = 3 \), \( \dim W(0^{4}; 1^{4}; 2) = 65 \) and \( \dim_{\mathcal{R}}(I_{A}, A) = 80 \) and since (7.2) is not satisfied, \( W(0^{4}; 1^{4}; 2) \subset \mathrm{Hilb}_{X}^{\alpha}(\mathbb{P}^{11}) \) is not a generically smooth irreducible component, cf. [35] Example 5.1.

(vi) Finally we consider the case \( t = 4, r = 3, A \) Gorenstein with \( \dim A = 3 \) of “subspecifical minors”, i.e. let \( R = k[x_{0}, x_{1}, \cdots , x_{11}] \) and \( \mathcal{A} = [B, v] \) a general linear \( 4 \times 4 \) matrix, \( v \) a column, and let \( A \) and \( B \) be the quotients of \( R \) defined by the \( 2 \times 2 \) minors of \( \mathcal{A} \) and \( B \) respectively. By Example 6.24(2), \( \dim W(0^{4}; 1^{4}; 3) = \lambda_{c} = 161 \) while we have \( \dim_{\mathcal{R}}(I_{A}, A) = 162 \) by Macaulay2, whence \( W(0^{4}; 1^{4}; 4) \subset \mathrm{Hilb}_{X}^{\alpha}(\mathbb{P}^{11}) \) is not a generically smooth irreducible component.

**Example 7.14. Varieties of quasi-minimal degree.** It is a well known result in Algebraic Geometry that for any non-degenerate (reduced and irreducible) variety \( X \subset \mathbb{P}^{n} \) we have \( \deg(X) \geq \dim X + 1 \). The classification of varieties of minimal degree is well understood (see, for instance, [17]) and as an attempt to classify varieties of quasi-minimal degree (i.e. \( \deg(X) = \dim X + 2 \)) Hoa described their minimal graded free resolution and proved (cf. [27] Theorem 1):

Let \( X \subset \mathbb{P}^{n} \) be a non-degenerate ACM variety of \( \dim X \geq 1 \) and quasi-minimal degree, i.e. \( \deg X = \dim X + 2 \). Then, \( I(X) \) has a minimal free resolution of the following type:

\[
0 \longrightarrow R(-c - 2) \longrightarrow R(-c)^{\alpha_{c-1}} \longrightarrow \cdots \longrightarrow R(-2)^{\alpha_{1}} \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0
\]

where \( c = \dim X \) and

\[
\alpha_{i} = i \left( \frac{c + 1}{i + 1} \right) - \left( \frac{c}{i - 1} \right) \text{ for } 1 \leq i \leq c - 1.
\]

In particular, \( X \) is arithmetically Gorenstein and generated by hyperquadrics. To better understand the structure of \( X \) we can ask whether \( I(X) \) is generated by the \( 2 \times 2 \) minors of a \( t \times t \) matrix.
with linear entries. Looking at the graded Betti numbers this is possible if and only if \( t = 3 \). So, in these cases it is natural to ask whether any codimension 4 AG subscheme \( X \subset \mathbb{P}^n \) of degree 6 can be defined by the \( 2 \times 2 \) minors of a \( 3 \times 3 \) matrix with linear entries.

The answer is yes. Indeed, by Example 7.13(iii) we have \( \dim W(0^3; 1^3; 2) = \dim(X) \text{Hilb}^{px(t)}(\mathbb{P}^n) \) and we conclude that a general non-degenerate ACM variety \( X \subset \mathbb{P}^n \) of quasi-minimal degree is arithmetically Gorenstein, generated by 9 hyperquadrics and these hyperquadrics are the \( 2 \times 2 \) minors of a \( 3 \times 3 \) matrix with linear entries.

An algebraic related conjecture is concerned with the depth of the following “normal modules”:

**Conjecture 7.15.** Let \( r \geq 1 \), \( c \geq 2 - r \) and let \( A = R/I_A \) (resp. \( B = R/I_B \) if \( c > 2 - r \)) be defined by the vanishing of the \( (t - r + 1) \times (t - r + 1) \) minors of a general \( t \times (t + c - 1) \) matrix \( \mathcal{A} = [\mathcal{B}, v] \) with \( a_1 > b_i \), \( v \) a column. Let \( N_A := \text{Hom}_R(I_A, A) \) and suppose that \( \dim A \geq 3 \). Set \( I_{A/B} = I_A/I_B \).

(i) If \( c \notin \{0, 1, 2\} \) then the \( A \)-module \( N_A \) satisfies

\[
\text{codepth}(N_A) = 1.
\]

If \( c = 1 \) (resp. \( c = 0, 2 \)), then \( \text{codepth}(N_A) = \min\{2, \dim A - 2\} \) (resp. \( \text{codepth}(N_A) = 0 \)).

(ii) Let \( (r, c) \neq (1, 2) \). If \( c \geq 2 \) (resp. \( 3 - r \leq c \leq 1 \)) then the \( B \)-module \( \text{Hom}_R(I_B, I_{A/B}) \) satisfies

\[
\text{codepth}(\text{Hom}_R(I_B, I_{A/B})) = r \text{ (resp. } r + 1)\).
\]

**Remark 7.16.** If \( \dim A = 2 \) and otherwise with notations and assumptions as in Conjecture 7.15 it seems that we often have \( \text{codepth} \text{Hom}_R(I_B, I_{A/B}) = r \) and certainly \( \text{codepth}(N_A) = 0 \) because in general any \( \text{Hom}_A(-, A) \)-group has depth at least 2 if depth \( A \geq 2 \).

Conjecture 7.15(i) is related to the smoothness of \( \text{Hilb}^{px(t)}(\mathbb{P}^n) \) along \( W(b; c; r) \) while Conjecture 7.15(ii) is concerned with \( \text{Hilb}^{px(t)}(\mathbb{P}^n) \) as well as to the property “every deformation of \( A \) comes from deforming its matrix”. Moreover, under the assumptions of Theorem 7.11 we also get generically smoothness of \( \text{Hilb}^{px(t)}(\mathbb{P}^n) \) from Conjecture 7.15(ii), so that conjecture is really what we need in this paper.

To see the connection between Conjecture 7.15 and Conjecture 7.11, note the following

**Proposition 7.17.** Let \( d \) and \( e \geq -2 \) be integers, let \( A = R/I_A \) (resp. \( B = R/I_B \)) be defined by the vanishing of the \( (t - r + 1) \times (t - r + 1) \) minors of a general \( t \times (t + c - 1) \) matrix \( \mathcal{A} = [\mathcal{B}, v] \) with \( a_1 > b_i \) (resp. \( \mathcal{B} \)), \( v \) a column, and let \( I_A := \text{l}_{t-r}(\varphi^*) \) and \( J_B := \text{l}_{t-r}(\varphi^{*}_{t+c-2}) \).

(i) Let \( \dim A \geq c + 2r \). If \( \text{codepth}(N_A) = d \) then \( \text{Ext}^i_A(I_A/I_A^2, A) = 0 \) for \( 1 \leq i \leq c + 2r - d - 2 \). In particular if \( d \leq c + 2r - 3 \) (i.e. \( \text{depth}_{J_B} A \geq d + 3 \)), then \( \text{Ext}^1_A(I_A/I_A^2, A) = 0 \). Hence if Conjecture 7.15(i) holds and \( \dim A \geq 4 \) for \( c \neq 1 \) (resp. \( \dim A \geq 5 \) for \( c = 1 \)) then

\[
\text{Ext}^1_A(I_A/I_A^2, A) = 0.
\]

(ii) Let \( \dim A \geq c + r - 1 \). If \( \text{codepth}(\text{Hom}_R(I_B, I_{A/B})) = r + e \), then \( \text{Ext}^i_B(I_B/I_B^2, I_{A/B}) = 0 \) for \( 1 \leq i \leq c + r - e - 3 \). In particular, if \( e \leq c + r - 4 \) (i.e. \( \text{depth}_{J_B} A \geq e + 3 \)) then

\[
\text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0.
\]
\( \text{Ext}_B^1(I_B/I_B^2, I_{A/B}) = 0. \) Hence if Conjecture 7.15(ii) holds and \( c + r \geq 4 \) for \( c \neq 1 \) (resp. \( r \geq 4 \) for \( c = 1 \)) then
\[ \text{Ext}_B^1(I_B/I_B^2, I_{A/B}) = 0. \]

**Proof.** (i) Let \( X = \text{Proj}(A) \). Since \( A \) is general, it is well known that \( \text{depth}_{J_A} \ A = c + 2r \). It follows that \( \text{depth}_{J_A} \ N_A = c + 2r - d \) by assumption. Since \( U_A := X \setminus V(J_A) \) is a local complete intersection, \( \widetilde{N_A} \) is locally free on \( U_A \), and using \([2.4]\) we get natural isomorphisms
\[ \text{Ext}_A^i(I_A/I_A^3, A) \cong H_i^*(U_A, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{I_A/I_A^3}, \mathcal{O}_X)) \cong H^i_{J_A}(\text{Hom}_A(I_A/I_A^3, A)) \]
for \( 1 \leq i \leq c + 2r - 2 \). Since the latter group vanishes for \( i + 1 \leq c + 2r - d - 1 \), we get (i).

(ii) If \( Y = \text{Proj}(B) \), then depth_{J_B} B = c - 1 + 2r. Taking local cohomology \( H^i_{J_B}(-) \) of the sequence
\[ 0 \rightarrow I_{A/B} \rightarrow B \rightarrow A \rightarrow 0 \]
it follows that \( \text{codepth}(I_{A/B}) = r - 1 \) which implies \( \text{depth}_{J_B} (I_{A/B}) = c + r \). Using \([2.4]\) we get a natural isomorphism (resp. an injection)
\[ \text{Ext}_B^i(I_B/I_B^2, I_{A/B}) \rightarrow H^i_* (U_B, \mathcal{H}om_{\mathcal{O}_Y}(\widetilde{I_B/I_B^2}, \mathcal{O}_Y)) \cong H^i_{J_B}(\text{Hom}_B(I_B/I_B^2, I_{A/B})) \]
for \( 1 \leq i \leq c + r - 2 \) (resp. \( i = c + r - 1 \)). Since we by assumption have
\[ \text{depth}_{J_B} (\text{Hom}_B(I_B/I_B^2, I_{A/B})) = c - 1 + 2r - (r + e), \]
we get that the latter group vanishes for \( i + 1 \leq c + 1 + r - e - 1 \), i.e. for \( 1 \leq i \leq c + r - e - 3 \), cf. Remark 1.5. Since the other statements are straightforward we are done. \( \square \)

**Proposition 7.18.** Let \( A \) be a generic determinantal ring. Then Conjecture 7.15(i) holds. Moreover, Conjecture 7.15(ii) holds for \( c \geq 0 \), \( c 
eq 2 \) while for \( 4 - r \leq c \leq -1 \), resp. \( c = 2 \), we have
\[ \text{codepth} \text{Hom}_R(I_B, I_{A/B}) \leq r + 1 \) (resp. \( r \)).

**Proof.** Conjecture 7.15(i) holds by \([3]\) Theorem 15.10], see also Supplement to Theorem 15.10.

To see that Conjecture 7.15(ii) almost holds for \( c \geq 4 - r \), we use Proposition 4.3 (iv) which implies that \( \text{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) = 0. \) Thus there exists an exact sequence:
\[ (7.4) \quad 0 \rightarrow \text{Hom}_R(I_{A/B}, A) \rightarrow \text{Hom}_R(I_A, A) \rightarrow \text{Hom}_R(I_B, A) \rightarrow 0. \]

Moreover, by Proposition 5.3 and Corollary 5.11 it follows that \( pr_1 \) in the following diagram
\[ (7.5) \]
\[ A^1_{B \rightarrow A} \xymatrix{ \ar[r]^{pr_2} & \text{Hom}_R(I_B, B) \ar[d]^{p} & \text{Hom}_R(I_A, A) \ar[r]_{pr_1} & \text{Hom}_R(I_B, A) } \]
is surjective. Hence, taking degree zero in \((7.4)\), we get that \( p : \text{Hom}_R(I_B, B) \rightarrow \text{Hom}_R(I_B, A) \) is surjective. We can argue similarly for the surjectivity of the corresponding \( \tilde{p} : \text{Hom}_R(I_B, B) \rightarrow \text{Hom}_R(I_B, A) \) using the entire Hom-groups. Indeed, the proof of Proposition 5.3 shows that
Ext\textsuperscript{1}_{R}(MI, MI) \to \text{Hom}_{R}(I_A, A)$, and not only the degree zero part of this morphism, is surjective. Also the argument of Corollary 5.9 holds for non-graded deformations. Thus there exists a diagram similar to (7.5) where the lower index zero is removed and $A^{1}_{(B\to A)}$ correspondingly redefined. That diagram implies that $\tilde{p}$ is surjective. It follows that there is an exact sequence

\begin{equation}
0 \to \text{Hom}_{R}(I_B, I_{A/B}) \to \text{Hom}_{R}(I_B, B) \to \text{Hom}_{R}(I_B, A) \to 0.
\end{equation}

Using (7.4), (7.6), and say [13, Corollary 18.6] or rather the mapping cone construction, we can quite closely determine the depth of $H_1 := \text{Hom}_{R}(I_B, I_{A/B})$ by first using (7.4) for finding the depth of $H_2 := \text{Hom}_{R}(I_B, A)$. Indeed note that since $A$ is a generic determinantal ring, Conjecture 7.15(i) holds for the $A$-module $N_A := \text{Hom}_{R}(I_A, A)$, whence also applies for the $B$-module $N_B$, due to Corollary 5.4 and taking into account that $\dim B - \dim A = r$. Moreover, Proposition 4.5(iii) implies that $H_3 := \text{Hom}_{R}(I_{A/B}, A)$ is maximally CM for $c \geq 1$ and of codepth 1 if $c \leq 0$. Letting $\text{pd}(H)$ being the length of an free $R$-resolution of $H$, we can find $\text{pd}(H_1)$ by considering different cases of $c$ in the table:

| $c$  | $\text{pd}(H_3)$ | $\text{pd}(N_A)$ | $\text{pd}(H_2)$ | $\text{pd}(N_B)$ | $\text{pd}(H_1)$ | codepth $H_1$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|----------------|
| $\geq 3$ | $\ell$ | $\ell + 1$ | $\ell + 1$ | $\leq \ell - r + 1$ | $\ell$ | $r$ |
| 2 | $\ell$ | $\ell$ | $\leq \ell + 1$ | $\ell - r + 2$ | $\leq \ell$ | $\leq r$ |
| 1 | $\ell$ | $\ell + 2$ | $\ell + 2$ | $\ell - r$ | $\ell + 1$ | $r + 1$ |
| 0 | $\ell + 1$ | $\ell + 2$ | $\ell + 2$ | $\ell - r + 1$ | $\ell + 1$ | $r + 1$ |
| $\leq -1$ | $\ell + 1$ | $\ell + 1$ | $\leq \ell + 2$ | $\ell - r + 1$ | $\leq \ell + 1$ | $\leq r + 1$ |

where $\ell := \text{pd}(R)$ and the column of $\text{pd}(H_2)$, resp. $\text{pd}(H_1)$, is determined by 2 columns to the left of $\text{pd}(H_2)$, resp. $\text{pd}(H_1)$. Then the codepth column is just obtained by transferring the $\text{pd}(H_1)$ column to depth and then to codepth by using the Auslander-Buchsbaum formula, $\text{pd}(H_1) + \text{depth } H_1 = \dim R$, and we are done.

**Remark 7.19.** Conjecture 7.15(i) and (ii) essentially holds for $r = 1$, i.e. we have codepth $N_A \leq 1$ (resp. 0) for $c \geq 3$ (resp. $c = 2$) by Theorem 3.11 and Theorem 3.9(iii) (resp. e.g. [37 Corollary 3.7]). Moreover, codepth$(\text{Hom}_{R}(I_B, I_{A/B})) \leq 1$ by [37 Corollary 3.8 and Remark 3.6 of latest arXiv version], using these results for $B$ instead of $A$. We have for $r \geq 2$ considered several examples using Macaulay2 to check if the depth of the modules satisfies the conjectures. We list some of them below in Example 7.20. The computations with Macaulay2 were time-consuming (or aborted), and we should have liked to check more examples, e.g. in the range where $\dim A \in \{3, 4\}$ and especially for Conjecture 7.15(ii).

**Example 7.20.** (1) Submaximal minors, i.e. $r = 2$. Let $t = 3$. We have checked Conjecture 7.15(i) and (ii) for the following determinantal rings $R/I_{c-r+1}(A)$: For every $c, 1 \leq c \leq 4$ let $A = (x_{ij})$ by the generic $3 \times (c + 2)$ matrix, or a non-linear matrix of the same size whose entries are powers of the corresponding entries of the generic one (so $R$ contains at least $3(c + 2)$ variables and various such non-linear matrices are considered), and the conjectures hold. For $c = 1$ we have also checked...
non-generic (for every \( \dim A \in \{3, 4\} \)) as well as some non-linear determinantal rings and all satisfy Conjectures (7.15(i) and (ii)). The same pattern is valid for \( c = 2 \) for non-generic linear determinantal rings with \( \dim A \in \{3, 4, 5\} \).

(2) Let \( \mathcal{A} \) be the generic \( t \times (t - r + 2) \) matrix, i.e. \( A \) a generic determinantal ring and let \( t - r = 1 \). Then we have checked that Conjecture (7.15(ii)) holds for each \( t \in \{4, 5, 6\} \) confirming Proposition 7.18 and that we have equality in the codepth for mula there. For \( t = 4 \) we have checked non-generic linear determinantal rings for every \( \dim A \in \{3, 4\} \), and also some non-linear determinantal rings with \( \dim A = 6 \) and both conjectures hold.

(3) Let \( \mathcal{A} \) be the generic \( 4 \times 4 \) matrix, or some non-linear matrix whose entries are powers of the corresponding entries of the generic one (so \( R \) contains at least 16 variables and various such non-linear matrices are considered). Let \( r = 2 \), i.e. \( A \) is defined by submaximal minors. In this case Conjecture (7.15(ii)) and (ii) holds. Then we consider the “subsubmaximal case” of the same \( 4 \times 4 \) matrices, i.e. we let \( r = 3 \). Again both conjectures hold.

**Remark 7.21.** For a “generic” \( 5 \times 5 \) matrix \( A \), taking \( r = 3 \) we have checked using Macaulay2 that Conjecture (7.15(ii)) holds. In this case we have also got \( \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0 \) indicating that the case \( c = 1 \) of Proposition (7.17(ii)) may be improved to \( r \geq 3 \).

8. **Computing Dimensions by Deleting Columns**

Since Theorem 7.1 assumes \( 0 \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0 \) or the related assumption \( \gamma = 0 \) and Theorems 6.14 and 7.1 need the numbers \( 0 \text{hom}_B(I_B/I_B^2, I_{A/B}) \) and \( \dim(MI \otimes A)_{(a_{t+c-1})} \) to find \( \dim(W_{B,G,r}) \), the goal of this section is to compute the dimension of \( 0 \text{Ext}^i_B(I_B/I_B^2, I_{A/B}) \) for \( i = 0, 1 \) and \( (MI \otimes A)_{(a_{t+c-1})} \) more effectively. Indeed this \( 0 \text{Ext}^0_B - (\text{resp.} (MI \otimes A)_{(a_{t+c-1})}) \)-group is important because it is isomorphic to the tangent space of the fiber of the projection

\[
p_1 : \text{Hilb}^{px(t), pv(t)}(\mathbb{P}^n) \to \text{Hilb}^{px(t)}(\mathbb{P}^n) \quad \text{(resp. } p_2 : \text{Hilb}^{px(t), pv(t)}(\mathbb{P}^n) \to \text{Hilb}^{px(t)}(\mathbb{P}^n))
\]

at \( (B \to A) \), cf. (6.7) while the vanishing of \( 0 \text{Ext}^1_B(I_B/I_B^2, I_{A/B}) \) not only gives the smoothness of this fiber, but in fact the smoothness of \( p_1 \) at \( (B \to A) \).

To find \( \dim 0 \text{Ext}^i_B(I_B/I_B^2, I_{A/B}) \) for \( i = 0, 1 \), we consider a flag of determinantal rings:

\[
(8.1) \quad A_{2-r} \to \cdots \to A_0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_c = A, \quad B = A_{c-1}, \quad X_i = \text{Proj}(A_i)
\]

with corresponding cokernels

\[
N_{2-r} \to \cdots \to N_j \to \cdots \to N_c = MI
\]
of \( \varphi_{t+i-1}^* \), obtained by successively deleting \( c + r - 2 \) columns from the right-hand side of the \( t \times (t + c - 1) \) matrix \( A \), and letting the \( (t - r + 1) \times (t - r + 1) \) minors \( I_{t-r+1}(\varphi_{t+i-1}^*) \) define \( A_i \). Then e.g. \( A_1 \) (resp. \( A_{2-r} \)) is Gorenstein (resp. standard determinantal) defined by the \( (t - r + 1) \times (t - r + 1) \) minors of a \( t \times t \) (resp. \( t \times (t - r + 1) \)) matrix. Now recall that when we in Proposition 4.4(iii) proved

\[
\text{Hom}_B(M, A) \cong MI \otimes A \cong \text{Hom}(I_{A/B}(a_{t+c-1}), A)
\]
under some assumptions on depth $J_1A$ where $J_1 = I_{t+r} \varphi_{t+1}^*$, we first remarked that we have
\begin{equation}
\widetilde{M} \otimes A_U \cong I_{A/B}(a_{t+c-1}) \otimes A_U, \quad \text{and}
\end{equation}
\begin{equation}
\widetilde{N} \otimes A_U \cong M I \otimes A_U
\end{equation}
where $U := \text{Proj}(A) \setminus V(J_{c-1}A)$. These isomorphisms are also needed for describing the fiber of $p r_1 : A^t_{I_B - A} \rightarrow \mathfrak{h} \text{Hom}_R(I_A, A)$. Define
\begin{equation}
M_i := \text{Hom}_{A_i}(N_i \otimes R A, A_i) \quad \text{where} \quad N_i := \text{coker}(\varphi_{t+1}^*).
\end{equation}
Let $I_i = I_{A_{i+1}/A_i}$ be the ideal defining $A_{i+1}$ in $A_i$ and let $I_{A_i} = I_{t-r+1} (\varphi_{t+1}^*)$. Assuming $b_t < a_1$ and $A$ general; and using [36, Theorem 2.7 and Example 2.5], we obtain
\begin{equation}
\dim R/(J_i + I_{A_{c-1}}) = \dim \text{Ann}_{A_{c-1}} A_{c-1} = r + i \quad \text{for} \quad 2 - r \leq i \leq c - 2.
\end{equation}
Thus, we have
\[
\text{codim}_{A_{c-1}} R/(J_i + I_{A_{c-1}}) = \text{depth}_{A_{c-1}} A_{c-1} = r + i \quad \text{for} \quad 2 - r \leq i \leq c - 2
\]
and
\[
\text{codim}_{A_c} R/(J_i + I_{A_c}) = \text{depth}_{A_{c}} A_{c} = r + i
\]
for the same reason (provided $\text{dim} A_c \geq r + i$). Letting $U_i = X_i \setminus V(J_i A_i)$ and assuming $\text{dim} A_c \geq 2$, we get
\[
\text{Hom}_{A_c}(I_i, A_j) \cong \text{Hom}_{\mathcal{O}_{U_i}}(\widetilde{I}_i, \widetilde{A}_j) \quad \text{for} \quad c - 1 \leq j \leq c.
\]
Then the left exact sequence
\[
0 \rightarrow \text{Hom}_{A_i}(I_i, I_{c-1}) \rightarrow \text{Hom}_{A_i}(I_i, A_{c-1}) \rightarrow \text{Hom}_{A_c}(I_i, A_c)
\]
and the corresponding one for the global Hom of sheaves imply
\begin{equation}
\text{Hom}_{A_c}(I_i, I_{c-1}) \cong \text{Hom}_{\mathcal{O}_{U_i}}(\widetilde{I}_i, \widetilde{I}_{c-1}) \quad \text{for} \quad 2 - r \leq i \leq c - 1,
\end{equation}
and we get $\text{Hom}(M_i, I_{c-1}) \cong \text{Hom}_{\mathcal{O}_{U_i}}(\widetilde{M}_i, \widetilde{I}_{c-1})$ by the same argument. Continuing these left exact sequences by including $\text{Ext}^1(-, -)$ to the right and using the five-lemma we get that
\begin{equation}
\text{Ext}^1_{A_i}(M_i, I_{c-1}) \cong \text{Ext}^1_{\mathcal{O}_{U_i}}(\widetilde{M}_i, \widetilde{I}_{c-1}) \quad \text{for} \quad 2 - r < i \leq c - 1,
\end{equation}
and correspondingly for $\text{Ext}^2(-, -)$ if $3 - r < i \leq c - 1$. This leads to

**Proposition 8.1.** Let $A$ be general and suppose $a_1 > b_t$, $r \geq 2$ and $\text{dim} A \geq 2$. If $c \leq 0$ suppose also $\text{dim} A \geq 3$. Let $j$ be any integer satisfying $2 - r \leq j \leq c - 2$. Then $\text{Hom}_{A_j}(I_j, I_{c-1})$ is an $A_{c-1}$-module of codimension $r - 1$ and we have isomorphisms
\[
\text{Hom}_{A_j}(M_j, I_{c-1}) \cong \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1}), \quad \text{and}
\]
\[
\text{Hom}_{A_j}(M_j(-a_{t+j}), I_{c-1}) \cong \text{Hom}_{A_j}(I_j, I_{c-1}) \cong \text{Hom}_{A_{c-2}}(I_{c-2}, I_{c-1})(a_{t+j} - a_{t+c-2}).
\]
Moreover, \( \varphi_{\text{hom}}R(M_j, I_{c-1}) = \dim(N_{c-1} \otimes A_{c-1})_v - \dim(N_c \otimes A_c)_v \), for any integer \( v \), and if \( 2 - r < i \leq c - 2 \), then

\[
\text{Ext}^1_{A_i}(M_i, I_{c-1}) = \text{Ext}^1_{A_{c-1}}(M_{c-1}, I_{c-1}) = 0 \quad \text{and} \quad \text{Ext}^1_{A_{i+1}}(I_i/I_i^2, I_{c-1}) = 0.
\]

Furthermore, we have

\[
\mu\text{Hom}_{A_j}(I_j, I_{c-1}) \cong R(a_{t+j} - a_{t+c-1}) \quad \text{provided} \quad a_{t+j} - a_{t+c-1} + \mu < s_r - b_r - a_{t-r+1} + b_1.
\]

**Proof.** Using (8.2) - (8.6), letting \( U_i = X_i \setminus V(J_iA_i) \), we have isomorphisms

\[
\text{Hom}_{A_j}(I_j, I_{c-1}) \cong \text{Hom}_{O_U_j}(\overline{I_j}|U_j, \overline{I_{c-1}}|U_j) \quad \text{(by (8.6))}
\]

\[
\cong \text{Hom}_{O_U_j}(M_j(-a_{t+j}), \overline{I_{c-1}}) \quad \text{(by (8.2))}
\]

\[
\cong \text{Hom}_{O_U_j}(\text{Hom}(\overline{N_j}, \overline{A_{c-1}}) \otimes A_{c-1}, \overline{I_{c-1}}(a_{t+j})) \quad \text{(I_{c-1} an A_{c-1}-module)}
\]

\[
\cong \text{Hom}_{O_U_{c-1}}(M_{c-1}, \overline{I_{c-1}}(a_{t+j})) \quad \text{(by (8.3))}
\]

\[
\cong \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1})(a_{t+j}) \quad \text{(because depth}\ N_{c-1} \text{I}_{c-1} \geq 2)
\]

and also \( \text{Hom}_{A_j}(M_j, I_{c-1}) \cong \text{Hom}_{O_U_j}(\overline{M_j}, \overline{I_{c-1}}) \cong \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1}) \). Then the exact sequence

\[
0 \rightarrow I_{c-1} \rightarrow A_{c-1} \rightarrow A \rightarrow 0
\]

and Proposition 4.3(iii) yield the exact sequence

\[
0 \rightarrow \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1}) \rightarrow \text{Hom}_{A_{c-1}}(M_{c-1}, A_{c-1}) \cong N_{c-1} \otimes A_{c-1}
\]

\[
\rightarrow \text{Hom}_{A_{c-1}}(M_{c-1}, A) \cong N_c \otimes A_c.
\]

By Proposition 4.4(i), \( N_c \otimes A_c \) is a maximal Cohen-Macaulay \( A \)-module for \( c > 0 \) and of codepth 1 for \( c \leq 0 \), and similarly for \( N_{c-1} \otimes A_{c-1} \); and since \( N_{c-1} \otimes A_{c-1} \twoheadrightarrow N_c \otimes A_c \) is surjective, we get the formula for \( \varphi_{\text{hom}}R(M_j, I_{c-1}) \) and the codepth of \( A_{c-1} \)-module \( \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1}) \). Moreover, the dimension of ker \( p \) in degree \( v + a_{t+c-1} \) is determined in Lemma 6.8. Indeed, we have

\[
R_v \cong \text{Hom}_{A_{c-1}}(M_{c-1}, I_{c-1})(a_{t+c-1} + v) \cong \text{Hom}_{A_j}(I_j, I_{c-1})(a_{t+c-1} - a_{t+j})_v
\]

provided \( v < s_r - b_r - a_{t-r+1} + b_1 \), and we get the last statement of Proposition 8.1.

Finally let \( 2 - r < i \leq c - 1 \). Then we have depth\( J_i \)\( A_{c-1} \)\( A_{c-1} \geq 3 \) by (8.3). Applying \( H^0_*(U_j, -) \) to the exact sequence

\[
0 \rightarrow \text{Hom}_{O_U_j}(\overline{M_j}, \overline{I_{c-1}}) \rightarrow \overline{N_{c-1}} \otimes A_{c-1}|U_j \rightarrow \overline{N_c} \otimes A_i|U_j \rightarrow 0
\]

and noticing that \( H^0_*(U_j, \overline{N_i} \otimes A_i) \cong N_i \otimes A_i \) for \( i = c - 1, c \) and that \( p : N_{c-1} \otimes A_{c-1} \rightarrow N_c \otimes A_c \) is surjective, we get an injection

\[
H^1_*(U_j, \overline{\text{Hom}_{O_U_j}(\overline{M_j}, \overline{I_{c-1}})}) \hookrightarrow H^1_*(U_j, \overline{N_{c-1}} \otimes A_{c-1}) \cong H^2_{J_iA_{c-1}}(N_{c-1} \otimes A_{c-1}).
\]

If \( c > 1 \) (resp. \( j > 3 - r \)) the latter group vanishes because

\[
\text{depth}_{J_iA_{c-1}}(N_{c-1} \otimes A_{c-1}) = \text{depth}_{J_iA_{c-1}} A_{c-1} \geq 3
\]

(resp. \( \text{depth}_{J_iA_{c-1}}(N_{c-1} \otimes A_{c-1}) \geq \text{depth}_{J_iA_{c-1}} A_{c-1} - 1 \geq 4 - 1 = 3 \)

\[
\text{depth}_{J_iA_{c-1}}(N_{c-1} \otimes A_{c-1}) \geq 3
\]
by (8.5), noting that \( \dim A_{c-1} = \dim A_c + r \geq 4 \). Recalling that \( \widetilde{M}_j \) is a locally free \( \mathcal{O}_{U_j} \)-Module, we get \( \text{Ext}^1_{A_j}(M_j, I_{c-1}) = 0 \) by (8.7). For \( j < c - 2 \) we also get \( \text{Ext}^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) = 0 \) because \( I_j \cap A_{j+1} \cong I_j/I_j^2 \) is locally free over \( U_j \cap \text{Proj}(A_{j+1}) \) by (8.2).

It remains to consider the case \( c \leq 1 \) and \( j = 3 - r \leq c - 1 \) where we unfortunately only have \( \text{depth}_{j+1, A_{c-1}}(N_{c-1} \otimes A_{c-1}) \geq 3 \). If \( j < c - 2 \), we can, however, apply (8.5) onto the larger ideal \( J_{j+1}A_{c-1} \supset J_jA_{c-1} \). Indeed, we get \( \text{depth}_{j+1, A_{c-1}}(N_{c-1} \otimes A_{c-1}) \geq 3 \) by (8.5) which implies \( H^1_*(U_{j+1}, \overline{N_{c-1} \otimes A_{c-1}}) = 0 \). Observing that \( \text{Hom}_{\mathcal{O}_{U_j}}(\overline{M_{j+1}}, \overline{A_{c-1}}) \cong \overline{N_{c-1} \otimes A_{c-1}} \) and that \( \overline{M_{j+1}} \) is a locally free over \( U_{j+1} \), whence the depth of \( \overline{N_{c-1} \otimes A_{c-1}} \) and \( \overline{A_{c-1}} \) coincide at every point of \( U_{j+1} \), we get

\[
H^1_*(U_j, \overline{N_{c-1} \otimes A_{c-1}}) \cong H^1_*(U_{j+1}, \overline{N_{c-1} \otimes A_{c-1}}) = 0. 
\]

It follows that \( H^1_*(U_j, \text{Hom}_{\mathcal{O}_{U_j}}(\overline{M_{j+1}}, \overline{I_{c-1}})) = 0 \) and we get

\[
\text{Ext}^1_{A_j}(M_j, I_{c-1}) = \text{Ext}^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) = 0 
\]

by (8.7) and (8.2) as previously. Since, in the special case \( j = 3 - r = c - 1 \), we directly get \( \text{Ext}^1_{A_j}(M_j, I_{c-1}) = 0 \) from Proposition 1.4(iii), we are done. \( \square \)

**Remark 8.2.** Suppose \( \dim A \geq 3 \), and if \( c \leq 0 \) we also suppose \( \dim A \geq 4 \). Then using (8.5), we can mainly argue as in the last part of the proof above to get, for \( 3 - r < i \leq c - 2 \), that

\[
\text{Ext}^2_{A_i}(M_i, I_{c-1}) = \text{Ext}^2_{A_{i-1}}(M_{c-1}, I_{c-1}) = 0 \quad \text{and} \quad \text{Ext}^2_{A_{i+1}}(I_i/I_i^2, I_{c-1}) = 0. 
\]

**Corollary 8.3.** Let \( A \) be general, let \( c \geq 3 - r \) and suppose \( a_1 > b_t, r \geq 2, \dim A \geq 2 \) and \( a_{t-r+1} + a_{t+c-2} - a_{t+c-1} < s_r - b_r + b_1 \). If \( c \leq 0 \) suppose also \( \dim A \geq 3 \). Then it holds

(i) If \( c > 3 - r \), then

\[
0\hom_R(I_{A_{c-1}}, I_{c-1}) = 0\hom_R(I_{A_{t-r}}, I_{c-1}) + \sum_{j=t-r+3}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}
\]

and

\[
0\hom_R(I_{A_{2-r}}, I_{c-1}) \leq 0\hom_R(I_{A_{2-r}}, I_{c-1}) + \binom{a_{t-r+2} - a_{t+c-1} + n}{n}.
\]

Moreover, for \( 3 - r \leq j \leq c - 2 \), we have

\[
0\text{ext}^1_{A_{j+1}}(I_{A_j/I_j^2}, I_{c-1}) \leq 0\text{ext}^1_{A_j}(I_{A_j/I_j^2}, I_{c-1}).
\]

(ii) If \( 0\hom_R(I_{A_i}, I_{c-1}) = 0 \) for some \( i \), \( 2 - r < i \leq c - 1 \), or \( 0\hom_R(I_{A_{2-r}}, I_{c-1}) = 0 \) and \( a_{t-r+1} < a_{t+c-1} \), then

\[
0\hom_R(I_{A_{2-r}}, I_{c-1}) = \binom{a_{t-r+2} - a_{t+c-1} + n}{n} + \sum_{j=1}^{t-r+2} \binom{a_j - a_{t+c-1} + n}{n},
\]

whence

\[
0\hom_R(I_{A_{c-1}}, I_{c-1}) = \sum_{j=1}^{t+c-2} \binom{a_j - a_{t+c-1} + n}{n}.
\]
(iii) If \( a_{t-r+1} < a_{t+c-1} - \sum_{i=1}^{t-r} b_{r+i-1} + \sum_{i=1}^{t-r} b_i \) (e.g. \( a_{t-r+1} < a_{t+c-1} \) if \( b_1 = b_i \)), then
\[
\text{Hom}_R(I_{A_{t-r}}, I_{c-1}) = 0 \quad \text{and we have}
\]
\[
\text{Hom}_R(I_{A_{t-r}}, I_{c-1}) = \sum_{j=1}^{t+c-2} \left( a_j - a_{t+c-1} + n \right).
\]

Proof. (i) Since by Proposition 8.1 \( \text{Ext}^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) = 0 \) we get an exact sequence
\[
(8.8) \quad 0 \rightarrow \text{Hom}(I_j, I_{c-1}) \rightarrow \text{Hom}(I_{A_{j+1}}, I_{c-1}) \rightarrow \text{Hom}(I_{A_j}, I_{c-1}) \rightarrow 0
\]
induced by
\[
0 \rightarrow I_j \rightarrow I_{A_{j+1}} \rightarrow I_j \rightarrow 0.
\]

Indeed a long exact sequence of algebra cohomology, often called the Jacobi-Zariski sequence, implies (8.8), because we can continue to the right the left-exact part of (8.8) by
\[
\rightarrow \text{Ext}^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) \rightarrow \text{Ext}^1_{A_{j+1}}(I_{A_{j+1}}/I_{A_{j+1}}^2, I_{c-1}) \rightarrow \text{Ext}^1_{A_{j}}(I_j/I_j^2, I_{c-1}) \rightarrow .
\]

Note that e.g. (8.3) shows that the algebra cohomology groups are just these \( \text{Ext}^1 \)-groups above because their \( \text{Hom}(\cdot, I_{c-1}) \) terms in a well known spectral sequence relating algebra cohomology to algebra homology vanish. Since \( \text{Ext}^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) = 0 \), we also get the inequality
\[
\text{Ext}^1_{A_{j+1}}(I_{A_{j+1}}/I_{A_{j+1}}^2, I_{c-1}) \leq \text{Ext}^1_{A_{j}}(I_{A_{j}}/I_{A_{j}}^2, I_{c-1}) \quad \text{for} \quad 3 - r \leq j \leq c - 2.
\]

From (8.8) and Proposition 8.1 which implies \( \dim \text{Hom}(I_j, I_{c-1}) = (a_{t+j} - a_{t+c-1} + n) \) for \( 2 - r \leq j \leq c - 2 \), we get
\[
\text{Hom}_R(I_{A_{j+1}}, I_{c-1}) = \text{Hom}_R(I_{A_{j}}, I_{c-1}) + \left( a_{t+j} - a_{t+c-1} + n \right)
\]
and hence the equality of (i). Finally since \( \text{Hom}(\cdot, \cdot) \) is left-exact, (8.8) holds also for \( j = 2 - r \) except for the surjectivity to the right. Since Proposition 8.1 holds for \( j \) in the range \( 2 - r \leq j \leq c - 2 \) we get
\[
\text{Hom}_R(I_{A_{3-r}}, I_{c-1}) \leq \text{Hom}_R(I_{A_{2-r}}, I_{c-1}) + \left( a_{t-r+2} - a_{t+c-1} + n \right)
\]
and (i) is proved.

(ii) From the last inequality we get that \( a_{t-r+1} < a_{t+c-1} \) and \( \text{Hom}_R(I_{A_{2-r}}, I_{c-1}) = 0 \) imply
\[
\text{Hom}_R(I_{A_{3-r}}, I_{c-1}) \leq \left( a_{t-r+2} - a_{t+c-1} + n \right)
\]
while the left-exactness of \( \text{Hom}(\cdot, \cdot) \) in (8.8) for \( j = 2 - r \) implies equality. Moreover, for every \( j \) such that \( 2 - r < j < i \), the exact sequence (8.8) shows that if \( \text{Hom}_R(I_{A_{j+1}}, I_{c-1}) = 0 \) then we have \( \text{Hom}_R(I_{A_{j}}, I_{c-1}) = 0 \) as well as \( 0 = \text{Hom}_R(I_j, I_{c-1}) = (a_{t+j} - a_{t+c-1} + n) \), i.e. \( a_{t+j} < a_{t+c-1} \) and hence \( a_{t-r+2} < a_{t+c-1} \). Using this repeatedly we get \( \text{Hom}_R(I_{A_{3-r}}, I_{c-1}) = 0 \) and we are done since
\[
\sum_{j=1}^{t-r+2} \left( a_j - a_{t+c-1} + n \right) = 0.
\]

(iii) By (ii) it suffices to verify that \( \text{Hom}_R(I_{A_{2-r}}, I_{c-1}) = 0 \) and \( a_{t-r+1} < a_{t+c-1} \). Since
\[
-\sum_{i=1}^{t-r+1} b_{r+i-1} + \sum_{i=1}^{t-r+1} b_i \leq 0,
\]
the inequality follows from the assumption of (iii). To see that
$\text{Hom}_R(I_{A_2-r}, I_{c-1}) = 0$, we look at the degree of the minimal generators of $I_{A_2-r}$. By Proposition \[7.4\] the largest degree of the minimal generators of $I_{A_2-r}$ is

$$mdg(I_{A_2-r}) := \sum_{j=1}^{t-r+1} a_j - \sum_{i=1}^{t-r+1} b_i,$$

while the smallest degree of a generator of $I_{c-1}$ is just

$$s(I_{c-1}) := a_{t+c-1} + \sum_{i=1}^{t-r} a_i - \sum_{i=1}^{t-r+1} b_{r+i-1}.$$

Hence $mdg(I_{A_2-r}) < s(I_{c-1})$, i.e. the assumption of (iii), implies $\text{Hom}_R(I_{A_2-r}, I_{c-1}) = 0$. \[\square\]

**Remark 8.4.** (1) The general conditions of Corollary 8.3 are quite weak (and is not needed for sequence (8.8), nor for the inequality of $\text{Ext}^1$ in (i)). Indeed writing the condition on $s_r$ as

$$b_r - b_1 < (s_r - a_{t-r+1}) + (a_{t+c-1} - a_{t+c-2})$$

and noting that the right hand side is $\geq t - r$, due to $s_r - a_{t-r+1} = \sum_{i=1}^{t-r} (a_i - b_{r+i})$ and $a_1 > b_1$, we see that the condition is at least satisfied if $b_r - b_1 \leq t - r - 1$.

(2) Under the assumptions in Remark 8.2, most importantly assuming $j > 3 - r$, we have

$$\text{Hom}_{R}(I_{A_{j+1}}/I_{A_{j+1}}, I_{c-1}) = \text{Hom}_{A_{j}}(I_{A_{j}}/I_{A_{j}}, I_{c-1}).$$

So if we in Corollary 8.3 (i) increase the dimension assumptions by 1 it is only for $j = 3 - r$ where strict inequality in

$$\text{Hom}_{A_{3-r}}(I_{A_{3-r}}/I_{A_{3-r}}, I_{c-1}) \leq \text{Hom}_{A_{3-r}}(I_{A_{3-r}}/I_{A_{3-r}}, I_{c-1})$$

may occur. To give an example where this happens, let $A = R/I_A$ be defined by the $2 \times 2$ minors of a general linear $3 \times 5$ matrix $A = [B, v]$, let $B = R/I_B(= A_{3-r})$, resp. $C = R/I_C(= A_{3-r})$, be correspondingly defined by $B$, resp. $C$, where $B = [C, w]$ is a $3 \times 4$ matrix and $w$ a column. Set $I_{B/C} := I_B/I_C$. In this case we have used Macaulay2 to check that $\text{Ext}_C^1(I_C/I_A^2, I_{A/B}) \neq 0$ and even $\text{Hom}_{C}(I_{C}/I_A^2, I_{A/B}) \neq 0$, while we get

$$\text{Ext}_{C}^1(I_C/I_A^2, I_{B/C}) = \text{Ext}_{B}^1(I_B/I_A^2, I_{A/B}) = 0,$$

cf. Proposition \[7.17\] (ii). Deleting one more column of $C$ we get $A_{2-r}$, and letting $I_{D} := I_B(A_{2-r})$ then computations show $\text{Hom}_{B}^1(I_{D}/I_A^2, I_j) \neq 0$ for $2 - r \leq j \leq 4 - r$ $(r = 2)$, see Remark \[7.4\] (3).

To give further evidence to Conjecture 6.19 we remark that Corollary 8.3 reduces the computation of $\text{Hom}_R(I_{A_{c-1}}, I_{c-1})$ to computing $\text{Hom}_R(I_{A_{3-r}}, I_{c-1})$. Using Macaulay2 this at least allows a much faster verification of the conditions of Theorem 6.14 which imply Conjecture 6.19.

**Example 8.5.** In Example 6.18 we considered many generic determinantal schemes $X = \text{Proj}(A)$ for which Conjecture 6.19 holds. We now extend the result to cover many more cases by only
verifying

\( (8.9) \quad \text{ohom}_R(I_{A_{3-r}, I_{c-1}}) = \sum_{j=1}^{t-r+2} \left( a_j - a_{t+c-1} + \frac{n}{n} \right) = t - r + 2 \)

because then Corollary 8.3(i) implies that \( \text{ohom}_R(I_{A_{c-1}}, I_{c-1}) = \sum_{j=1}^{t+c-2} (a_j - a_{t+c-1} + n) \). Hence we can argue as in Examples 6.18 to get \( \dim W(b; a; r) = \lambda_c \) and \( \dim W(b; a; r; R') = \lambda_c(R') \) for \( R' = R[y] \), only using Corollary 6.15(ii). In (i) of Examples 6.18 we have considered the additional cases \( 8 \leq c \leq 18 \) of \( 2 \times 2 \) minors of a generic \( 3 \times (c + 2) \) matrix, in (ii) the cases \( 4 \leq c \leq 5 \) of \( 3 \times 3 \) minors of the generic \( 4 \times (c + 3) \) matrix, in (iii) the cases \( 5 \leq c \leq 9 \) of \( 2 \times 2 \) minors of the generic \( 4 \times (c + 3) \) matrix and in (iv) the cases of \( 2 \times 2 \) minors of the generic \( 5 \times (c + 4) \) matrix for \( c \in \{2, 3\} \), as well as the case of \( 3 \times 3 \) minors of the generic \( 5 \times 6 \) matrix, and everyone satisfies \( (8.9) \), whence Conjecture 6.19 holds in all these cases.

To describe the fiber at \((B \to A)\) of the other projection, \( p_2 : \text{Hilb}^{p_X(t), p_Y(t)}(\mathbb{P}^n) \to \text{Hilb}^{p_Y(t)}(\mathbb{P}^n) \) with tangent map \( p_2 : A_1(B \to A) \to \text{Hom}_R(I_B, B) \), cf. diagram (6.7), we consider the flag

\[ A_{2-r} \to \cdots \to A_0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_c = A, \quad B = A_{c-1}, \quad X_i = \text{Proj}(A_i) \]

of determinantal rings obtained by successively deleting columns from the right-hand side of a general matrix \( A \) with \( a_1 > b_t \). As usual

\[ N_{2-r} \to \cdots \to N_j \to \cdots \to N_c = MI \]

is the corresponding sequence of cokernels. Applying Lemma 6.8(ii) onto \( A_j \to A_{j+1} \) we get

\[ \dim (N_{j+1} \otimes A_{j+1})_{a_t+j+v} = \dim (N_j \otimes A_j)_{a_t+j+v} - \dim R_v \]

provided \( v < s_r - b_r - a_{t-r+1} + b_1 \) where \( s_r - a_{t-r+1} := \sum_{i=1}^{t-r} (a_i - b_{r+i}) \). Letting \( v = a_{t+k} - a_{t+j} \) where \( j \leq k \leq c - 1 \) and assuming \( a_{t+c-1} < s_r - b_r + a_{t+j} - a_{t-r+1} + b_1 \) we obtain

\[ (8.10) \quad \dim (N_{j+1} \otimes A_{j+1})_{a_t+k} = \dim (N_j \otimes A_j)_{a_t+k} - \left( a_{t+k} - a_{t+j} + \frac{n}{n} \right) \]

because \( a_{t+k} \leq a_{t+c-1} \). We also know that

\[ 0 \to D_j(-a_{t+j}) \to N_j \to N_{j+1} \to 0 \]

is exact where \( D_j = R/I_t(\varphi_{i+j-1}^*) \) for \( j > 0 \) (e.g. see the text after (3.1) of [39]) and \( D_j = R \) for \( j \leq 0 \) (because \( 0 \to G_{t+j}^* \to F^* \to N_j \to 0 \) is exact for \( j \leq 1 \)). Since \( I_t(\varphi_{i+j-1}^*)_v = 0 \) for \( v < s_1 - b_1 \) we get

\[ (8.11) \quad \dim (N_{j+1})_{a_{t+k}} = \dim (N_j)_{a_{t+k}} - \left( a_{t+k} - a_{t+j} + \frac{n}{n} \right) \]

provided \( a_{t+c-1} < s_1 - b_1 \). Hence assuming \( a_{t+c-1} < s_r - b_r + a_{t+j} - a_{t-r+1} + b_1 \), both \( (8.10) \) and \( (8.11) \) holds by (6.6). Note that if we apply Lemma 6.8(i) onto \( N_j \) and \( A_j \) and assume \( a_{t+c-1} < s_r - b_r + b_1 \) we get \( \dim (N_j \otimes A_j)_{a_{t+c-1}} = \dim (N_j)_{a_{t+c-1}} \), as previously. But the approach above using (ii) instead
of (i) in Lemma 6.8 leads to better results if we are able to find $R$-free resolutions of $N_j \otimes A_j$. For $r = 2$ and $c = 1$ this is somehow done in Theorem 8.7, where $N_0 \cong I_B(s - a_t)$ and

$$\dim(N_0)_{at+c-1} - \dim(N_0 \otimes B)_{at+c-1} = \dim \mathcal{I}_B^2(s - a_t + a_{t+c-1})_0,$$

which one may compute using the free resolution of $\mathcal{I}_B^2$ given in [6.13]. More generally for $c = 3-r$, $r \geq 2$ we have by (8.10) that

$$(8.12) \dim(N_{3-r} \otimes A_{3-r})_{at+r+2} = \dim(N_{2-r} \otimes A_{2-r})_{at+r+2} - 1$$

provided $b_r - b_1 < s_r - a_{t+1}$. Hence a method to compute $\dim(N_{2-r} \otimes A_{2-r})_{at+r+2}$ is needed.

**Lemma 8.6.** Let $r \geq 2$, $B_i := \text{coker } \varphi_{t+i-1}$, $J_{A_i} := I_{t-i}(\varphi_{t+i-1}^\ast) \neq R$ and suppose $\text{depth } J_{A_i} A_i \geq 1$ ($A$ not necessarily general). Then for $i \leq 1$ the following sequence of maximal Cohen-Macaulay $A_i$-modules

$$0 \rightarrow \text{Hom}_{A_i}(B_i \otimes A_i, A_i) \rightarrow G_{t+i-1}^t \otimes A_i \rightarrow F^* \otimes A_i \rightarrow N_i \otimes A_i \rightarrow 0$$

is exact. Moreover, if $i = 2 - r$ then $A_i$ is defined by maximal minors in which case we have the following minimal $R$-free resolution of $(B_i \otimes A_i)^*: = \text{Hom}_{A_i}(B_i \otimes A_i, A_i)$:

$$0 \rightarrow S_{t-1}(G_{t+i-1}^t) \rightarrow F^* \otimes S_{t-2}(G_{t+i-1}^t) \rightarrow \ldots$$

$$\rightarrow \wedge^{t-2} F^* \otimes G_{t+i-1}^t \rightarrow \wedge^{t-1} F^* \rightarrow (B_i \otimes A_i)^*(I_{2-r}) \rightarrow 0.$$

**Proof.** Note that Proposition 4.4 applies to $N_i = \text{coker } \varphi_{t+i-1}^\ast$ as well as to $B_i := \text{coker } \varphi_{t+i-1}$. In the latter case we get, for $i \leq 1$, that $B_i \otimes A_i$ and $\text{Hom}_{A_i}(B_i \otimes A_i, A_i)$ are maximal Cohen-Macaulay $A_i$-modules. Moreover applying $\text{Hom}_{A_i}(-, A_i)$ onto

$$\rightarrow F \otimes A_i \rightarrow G_{t+i-1} \otimes A_i \rightarrow B_i \otimes A_i \rightarrow 0$$

we get the leftmost part of the first exact sequence while applying $(-) \otimes A_i$ onto

$$\rightarrow G_{t+i-1}^t \rightarrow F^* \rightarrow N_i \rightarrow 0$$

takes care of the rightmost part. Finally in the case of maximal minors, the minimal $R$-free resolution of $B_{2-r}^t$ is well known (13 pg. 595), and that we may put it on the form above.

By applying Theorem 7.1 to every surjection $A_{i-1} \twoheadrightarrow A_i$, $i > 2 - r$ in the flag (8.1) and using Corollary 5.3 which implies that $0 \text{Ext}_B^1(I_B/I_B^2, I_A/B) = 0$ provided $0 \text{Ext}_{A_{3-r}}^1(I_{A_{3-r}}/I_{A_{3-r}}^2, I_{c-1}) = 0$ where $I_j = I_{A_{j+1}}/A_j$, $B = A_{3-1}$ and $A = A_{2-1}$, we are able to prove the main results of this section. First we consider the case $c = 3-r$, $B = A_{2-r}$ and $A = A_{3-r}$ which we need to start the induction and is of interest in itself. Here Lemma 8.6 allows us to compute the invariants involved in (ii).

**Theorem 8.7.** Suppose that $\text{Proj}(A) \in W[B, A; r]$ is general with $c = 3 - r$, $r \geq 2$ and $\dim A \geq 2$. If $c \leq 0$, we also suppose $\dim A \geq 3$. Moreover, let $\gamma$ be the composed map $\gamma: \text{Hom}_R(I_A, A) \rightarrow 0\text{Hom}_R(I_B, A) \rightarrow \text{Hom}^1_B(I_B/I_B^2, I_B/A_B)$ and suppose $a_1 > b_i$. 
(i) If $\gamma = 0$ or equivalently if \( \text{dim}(W(b; g; r)) \) holds, then \( W(b; g; r) \) is a generically smooth irreducible component of \( \text{Hilb}^{p_X(t)}(\mathbb{P}^n) \) and every deformation of \( A \) comes from deforming its matrix \( A \).

(ii) Let \( MI = \text{coker} \varphi^r_{t-r+2} \) and \( N = \text{coker} \varphi^r_{t-r+1} \). If \( \dim \text{hom}_R(I_B, I_{A/B}) = \sum_{i=1}^{t-r+1} (a_i-a_{t-r+2+n}) \) (e.g. \( b_l = b_1 \) and \( a_{t-r+1} < a_{t-r+2} \)), then
\[
\dim W(b; g; r) = \lambda + K'_3 + K'_4 + \cdots + K'_{r} - \kappa'
\]
where \( K'_i \) is defined in \( 6.2 \) and \( \kappa' \geq 0 \) is given by:
\[
\kappa' = \dim_k(\text{dim}(MI)(a_{t-r+2} + \dim_k(\text{dim}(MI \otimes A)(a_{t-r+2})).
\]
In particular, if also \( a_{t-r+2} < s_r - b_r + b_1 \) then every \( K'_i = 0, \kappa' = 0 \) and
\[
\dim W(b; g; r) = \lambda.
\]

More generally, if \( a_{t-r+1} < s_r - b_r + b_1 \) or, equivalently, \( b_r - b_1 < \sum_{i=1}^{t-r} (a_i - b_{r+i}) \) then \( K'_i = 0, \kappa' = \dim_k(N)(a_{t-r+2}) - \dim_k(N \otimes B)(a_{t-r+2}) \) and \( \dim W(b; g; r) = \lambda - \kappa' \) where \( \kappa' \) may be expressed in terms of binomials using Lemma \( 8.6 \) for \( i = 2 - r \), i.e. with \( N = N_{2-r} \) and \( B = A_{2-r} \).

Proof. Let \( B \) be the matrix of \( \varphi^r_{t+1-r} \), i.e. the matrix whose \((t+1-r)\)-minors define \( B \). Since we know that every deformation of \( B \) comes from deforming \( B \) by Theorem \( 3.8 \) and \( I_{t-r}(\varphi^*) \neq R \) by \( a_0 > b_1 \), we get (i) from Theorem \( 7.1 \) and Remark \( 7.4 \). Theorem \( 7.1 \) also implies that
\[
\dim W(b; g; r) = \dim W(b; g'; r) + \dim_k(\text{dim}(MI \otimes A)(a_{t+2-r}) - \text{dim}_R(I_B, I_{A/B})
\]
where \( MI = \text{coker}(\varphi^r_{t+2-r}), \varphi^r_{t+2-r} = \varphi \). By \( 6.9 \) and assumption we get
\[
\dim_k(\text{dim}(MI \otimes A)(a_{t+2-r}) - \text{dim}_R(I_B, I_{A/B}) = \lambda - \kappa' + \lambda_{3-r} - \lambda_{2-r}
\]
while Lemma \( 6.4 \) implies
\[
\dim W(b; g'; r) = \lambda_{2-r} + K'_3 + K'_4 + \cdots + K'_r,
\]
and we get the displayed dimension formula.

Finally if \( a_{t-r+2} < s_r - b_r + b_1 \), we get \( \kappa' = 0 \) by Lemma \( 6.8(i) \), and every \( K'_i = 0 \) by Lemma \( 6.4 \) because \( b_1 < b_2 = \sum_{j=1}^{t-r} a_j - \sum_{k=t-r-1}^{t-l} b_k = s_r - b_r - b_r - 1 \) and \( a_{t-r+2} \geq b_r - 1 \). Also the assumption \( a_{t-r+1} < s_r - b_r + b_1 \) implies that every \( K'_i = 0 \). Moreover since the map \( \varphi^r_{t+i-1} \) is injective for \( i \leq 1 \) we get an exact sequence
\[
0 \to B(-a_{t+2-r}) \to N_{3-r} \to N_{3-r} \to 0
\]
where \( N_{3-r} = MI, N_{2-r} = N \), by the snake lemma. Combining with \( 8.12 \), we see that
\[
\kappa' = \dim_k(N)(a_{t-r+2}) - \dim_k(N \otimes B)(a_{t-r+2})
\]
and we conclude the theorem by using Lemma \( 8.6 \).
Remark 8.8. If \( r = 2 \) in Theorem 8.7, we get \( \dim W(kA; r) = \lambda_{3-r} - \kappa' \). In this case we proved in Theorem 6.22 that \( \dim W(kA; r) = \lambda_1 - \kappa_1 \) under weaker assumptions. It follows that \( \kappa' = \kappa_1 \).

Example 8.9. (Determinantal quotients of \( R = k[x_0, x_1, \ldots, x_n] \), using Theorem 8.7(ii))

Let \( A = [B, v] \) be a general 4x3 matrix with linear (resp. quadratic) entries in the first and second (resp. third) column. The degree matrix of \( A \) is \( \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \) and \( b_i = 0 \) for \( 1 \leq i \leq 4 \). The vanishing of all 2 \times 2 minors of \( A \) (resp. \( B \)) defines a determinantal ring \( A \) (resp. \( B \)) with \( r = 3, t = 4, c = 0 \) (resp. \( c = -1 \)). Our goal is to find \( \dim W(0^4; 1^2, 2; 2) \). Since the condition \( a_{t-1} < s_3 = 2 \) of Theorem 6.14 does not hold, we will use the generalization in Theorem 8.7(ii) to find the dimension. Indeed since \( a_{t-2} < a_{t-1} \) and \( a_{t-2} < s_3 = 2 \) we have \( \dim W(0^4; 1^2, 2; 2) = \lambda_0 - \kappa' \) by Theorem 8.7(ii) where \( \kappa' = \dim_k(N(a_3)) - \dim_k(N \otimes B)(a_3) \) and \( N = \operatorname{coker}(G_2^* \to F^*) \), \( G_2^* = R(-1)^2 \) and \( F^* = R^4 \).

Moreover by Lemma 8.6 the sequences

\[ 0 \to (B_{-1} \otimes B)^* \to G_2^* \otimes B \to F^* \otimes B \to N \otimes B \to 0 \]

and

\[ 0 \to S_3(G_2^*) \to F^* \otimes S_2(G_2^*) \to \wedge^2 F^* \otimes G_2^* \to \wedge^3 F^* \to (B_{-1} \otimes B)^*(2) \to 0 \]

are exact. The first of these sequences together with the definition of \( N \) imply that \( \kappa' = 24 - \dim(B_{-1} \otimes B)^*_2 \), taking into account that \( I_B \) has 6 minimal generators of degree 2. Then the next displayed sequence implies that \( \dim(B_{-1} \otimes B)^*_2 = 4 \). It follows that

\[ \dim W(0^4; 1^2, 2; 2) = \lambda_0 - \kappa' = 8(n + 1) + 4(n^2 + 2) - 16 - (5 + 2(n + 1)) + 1 - \kappa' = 2n^2 + 12n - 30 \]

for \( n \geq 8 \) by definition of \( \lambda_0 \). To check the answer using Macaulay2, let \( B \) be the generic linear matrix with entries \( x_0, x_1, \ldots, x_7 \) and let \( v^r = (x_8^2, x_9^2, x_{10}^2, x_{11}^2) \). Computations show that \( \operatorname{Ext}^1_A(I_A/I_A^2, A) = 0 \) and \( \dim(X) \operatorname{Hilb}^x(t)(\mathbb{P}^{11}) = 344 \) at \( X = \operatorname{Proj}(A) \), coinciding with our formula for \( \dim W(0^4; 1^2, 2; 2) \) when \( n = 11 \). It also implies that \( \dim W(0^4; 1^2, 2; 2) \) is a generically smooth irreducible component of \( \operatorname{Hilb}^x(t)(\mathbb{P}^{11}) \). We also checked the case \( n = 8 \) where \( \dim A = 3 \) by using Macaulay2, and we have got that (7.2) holds with \( \operatorname{Hom}_R(I_A, A) = 194 \). Thus \( \dim W(0^4; 1^2, 2; 2) \) is a generically smooth irreducible component of \( \operatorname{Hilb}^x(t)(\mathbb{P}^{8}) \) of dimension 194, coinciding with our formula for \( \dim W(0^4; 1^2, 2; 2) \) and confirming Conjectures 6.19 and 7.11 in this case.

To state our next result, let us index the formulas, previously called just \( \gamma \), as follows:

\[ \gamma_{3, 2} : 0 \operatorname{Hom}_R(I_{A_{3-r}}, A_{3-r}) \to 0 \operatorname{Hom}_R(I_{A_{2-r}}, A_{3-r}) \to 0 \operatorname{Ext}^1_A(I_{A_{2-r}}/I_{A_{2-r}}^3, I_{2-r}) \text{, and} \]

\[ \gamma_{4, 3} : 0 \operatorname{Hom}_R(I_{A_{4-r}}, A_{4-r}) \to 0 \operatorname{Hom}_R(I_{A_{3-r}}, A_{4-r}) \to 0 \operatorname{Ext}^1_A(I_{A_{3-r}}/I_{A_{3-r}}^2, I_{3-r}) \text{ etc.} \]

Theorem 8.10. Suppose that \( \operatorname{Proj}(A) \subset W(kA; r) \) is general with \( c \geq 4 - r, r \geq 2 \) and \( \dim A \geq 2 \). If \( c \leq 0 \), we also suppose \( \dim A \geq 3 \). Moreover suppose that \( a_1 > b_1 \) and that the composed maps \( \gamma_{3, 2} \) and \( \gamma_{4, 3} \) are both zero. If \( 0 \operatorname{Ext}^1_A(I_{A_{4-r}}/I_{A_{4-r}}^3, I_j) = 0 \) for \( 4 - r \leq j \leq c - 1 \), then \( W(kA; r) \)
is a generically smooth irreducible component of $\text{Hilb}^{p_X(t)}(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming its matrix.

Proof. We will use induction on $c \geq 4 - r$. In the initial case $c = 4 - r$ we know that every deformation of $A_{3-r}$ comes from deforming its matrix by Theorem 8.7. Then we get the same conclusion for $A_{4-r}$, and also that $W(k; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{p_X(t)}(\mathbb{P}^n)$, by Theorem 7.1.

If $c > 4 - r$, we have by induction that every deformation of $A_{c-1}$ comes from deforming its matrix. Since, for $3 - r \leq j \leq c - 2$, we have

$$\mathfrak{o} \text{Ext}^1_{A_{j+1}}(I_{A_{j+1}}/I_{A_{j+1}}^2, I_{c-1}) \leq \mathfrak{o} \text{Ext}^1_{A_j}(I_{A_j}/I_{A_j}^2, I_{c-1})$$

by Corollary 8.3(i) and Remark 8.4 we get $\mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_{c-1}) = 0$.

Remark 8.11. (1) An analogous result with the same conclusion, where we assume $\gamma_{32} = 0$ and $\mathfrak{o} \text{Ext}^1_{A_{3-r}}(I_{A_{3-r}}/I_{A_{3-r}}^2, I_j) = 0$ for $3 - r \leq j \leq c - 1$, is true. The reason for not stating this result is seen in Remark 8.4(2) which implies

$$\mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_{c-1}) = \mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_{c-1})$$

for $\dim A > 3$. Since we expect the latter group to vanish by Proposition 7.17(2), it follows that $\mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_{c-1})$ should also vanish while $\mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_j)$ for $j > 3 - r$ and $\mathfrak{o} \text{Ext}^1_{A_{c-1}}(I_{A_{c-1}}/I_{A_{c-1}}^2, I_{c-1})$ may be non-vanishing as Remark 8.4(2) shows. The compositions $\gamma_{3,2}$ and $\gamma_{4,3}$ seem, however, to vanish. All this is supported by Macaulay2 computations. Moreover the vanishing above is indeed a main reason for expecting Conjecture 7.11 and to a certain degree Conjecture 7.13 to be true.

(2) Let $B = A_{c-1}$ and $A = A_c$ and suppose $\dim A \geq 3$, and $\dim A \geq 4$ in case $c \leq 0$. If $A$ is general and $c \geq 4 - r$ then by Proposition 4.14(iv), $\text{Ext}^1(A/B/I_{A/B}^2, A) = 0$, whence $\mathfrak{o} \text{Hom}_R(I_A, A) \to \mathfrak{o} \text{Hom}_R(I_B, A)$ is surjective by Remark 19 and 4.12. It follows that the assumption $\gamma_{c+r,c-r+1} = 0$ for $c \geq 4 - r$ is equivalent to the surjectivity of $\mathfrak{o} \text{Hom}_R(I_B, B) \to \mathfrak{o} \text{Hom}_R(I_B, A)$ by Remark 5.8 cf. Remark 7.4(3). In particular this applies to $\gamma_{43}$ of Theorem 8.10 but not to $\gamma_{32}$.

In the same way as the flag (6.1) allows us to simplify the calculation of $\dim \mathfrak{o} \text{Ext}^i_B(I_B/I_{B}^2, I_{A/B})$ for $i = 0, 1$, very similar arguments lead to a simplification of the normal modules.

Proposition 8.12. Let $A$ be general and suppose $a_1 > b_1$, $r \geq 2$ and $\dim A \geq 3$. If $c \leq 0$ suppose also $\dim A \geq 4$. Let $j$ be any integer satisfying $2 - r \leq j \leq c - 1$. Then $\text{Hom}_A(I_j, A)$ is a maximal Cohen-Macaulay $A$-module for $c > 0$ and of codepth 1 for $c \leq 0$, and we have isomorphisms

$$\text{Hom}_{A_c}(M_c, A) \cong \text{Hom}_{A_j}(M_j, A) \cong \text{Hom}_{A_j}(I_j(a_{t+j}), A) \cong N_c \otimes A_c.$$ 

In particular, for $c \geq 3 - r$ we have

$$\mathfrak{o} \text{Hom}_R(I_A, A) = \mathfrak{o} \text{Hom}_R(I_{A_{3-r}}, A) + \sum_{j=3-r}^{c-1} \dim(N_c \otimes A_c)(a_{t+j}).$$
Moreover for $3 - r \leq j \leq c - 1$, we have

$$\Ext^1_{A_j}(M_j, A) = \Ext^1_{A_{j+1}}(I_j/I_j^2, A) = \Ext^1_{A_{c}}(M_c, A) = 0$$

and

$$\oext^1_{A_{j+1}}(I_{j+1}/I_{j+1}^2, A) \leq \oext^1_{A_j}(I_j/I_j^2, A);$$

thus we get that $\oExt^1_{A_j}(I_j/I_j^2, A) = 0$ for some $j \geq 3 - r$ implies $\oExt^1(A/I_{A_j}^2, A) = 0$.

**Proof.** To use the proof of Proposition 8.1 effectively we remark that it suffices to prove Proposition 8.12 for $A_{c-1}$ (instead of $A_c$) under the assumption $2 - r \leq j \leq c - 2$ and the dimension assumptions: $\dim A_{c-1} \geq 3$, and $\dim A_{c-1} \geq 4$ if $c - 1 \leq 0$. Then we get the first displayed formula using the arguments in the very first part of Proposition 8.1, replacing $I_{c-1}$ there by $A_{c-1}$ and noticing that $\Hom_{A_{c-1}}(M_{c-1}, A_{c-1}) \cong N_{c-1} \otimes A_{c-1}$ and that $N_{c-1} \otimes A_{c-1}$ is a maximal Cohen-Macaulay $A_{c-1}$-module for $c > 1$ and of codepth 1 for $c \leq 1$ by Proposition 4.4. Moreover following the proof of Proposition 8.12 where we got $\Ext^1_{A_j}(M_j, I_{c-1}) = \Ext^1_{A_{j+1}}(I_j/I_j^2, I_{c-1}) = 0$ for $j \geq 3 - r$ by showing

$$H^1_*(U_j, \widetilde{N}_{c-1} \otimes \widetilde{A}_{c-1}) \cong H^1_*(U_j, \Hom_{O_{U_j}}(\widetilde{M}_j, \widetilde{A}_{c-1})) = 0,$$

we get the latter also now. We only need to be a little careful with the dimension of $A_{c-1}$ to get large enough depth, and one checks that our dimension assumptions suffice. Hence we get the vanishing of $\Ext^1_{A_j}(M_j, A_{c-1})$ and $\Ext^1_{A_{j+1}}(I_j/I_j^2, A_{c-1})$. We also get $\Ext^1_{A_{c}}(M_c, A) = 0$ by Proposition 4.4(ii).

Using $\Ext^1_{A_{j+1}}(I_j/I_j^2, A_{c-1})$ for $j \geq 3 - r$ we get an exact sequence

$$0 \to \oHom(I_j, A_{c-1}) \to \oHom(I_{j+1}, A_{c-1}) \to \oHom(I_j, A_{c-1}) \to 0$$

induced by

$$0 \to I_{A_j} \to I_{A_{j+1}} \to I_j \to 0.$$  

Indeed the long exact Jacobi-Zariski sequence implies (8.13) since we can continue (8.13) to the right by

$$0 = \oExt^1_{A_{j+1}}(I_j/I_j^2, A_{c-1}) \to \oExt^1_{A_{j+1}}(I_{j+1}/I_{j+1}^2, A_{c-1}) \to \oExt^1_{A_j}(I_j/I_j^2, A_{c-1}) \to .$$

The latter sequence shows the inequality of $\oext^1(-, -)$ of Proposition 8.12 while repeatedly using (8.13) and $\Hom(I_j, A_{c-1}) \cong N_{c-1} \otimes A_{c-1}(a_{t+j})$ for every $j$, $3 - r \leq j \leq c - 2$ we get a dimension formula for $\oHom_R(I_{A_{c-1}}, A_{c-1})$ which precisely corresponds to the displayed dimension formula for $\oHom_R(I_{A_c}, A)$ of Proposition 8.12 and we are done.

**Corollary 8.13.** Let $A$ be general and suppose $a_1 > b_t$, $r \geq 2$, $c \geq 4 - r$ and $\dim A \geq 3$. If $c \leq 0$ suppose also $\dim A \geq 4$. In the flag (8.1), let $B = A_{c-1}$, $A = A_c$ and $N_c = MI$. Then we have

$$\oHom_R(I_{A_{c-1}}, A_{c-1}) = \oHom_R(I_{A_{c-1}}, A_c) + \oHom_R(I_{A_{c-1}}, I_{c-1})$$

if and only if (7.2) holds, i.e.

$$\oHom_R(I, A) = \oHom_R(I, B) + \dim_k(MI \otimes A)(a_{t+c-1}) - \oHom_R(I, I_{A/B}).$$
Proof. Using Proposition 8.12 we get that

\[ 0\text{hom}_R(I_A, A) = 0\text{hom}_R(I_{A_3-r}, A_c) + \sum_{j=3-r}^{c-1} \dim(N_c \otimes A_c)_{(a_{t+j})}. \]

Since \( c - 1 \geq 3 - r \), Proposition 8.12 also implies

\[ 0\text{hom}_R(I_B, B) = 0\text{hom}_R(I_{A_3-r}, A_{c-1}) + \sum_{j=3-r}^{c-2} \dim(N_{c-1} \otimes A_{c-1})_{(a_{t+j})}. \]

Inserting these expressions into (7.2) and using MI \otimes A = N_c \otimes A_c we get that (7.2) is equivalent to

\[ 0\text{hom}_R(I_{A_3-r}, A_c) = 0\text{hom}_R(I_{A_3-r}, A_{c-1}) + \sum_{j=3-r}^{c-2} 0\text{hom}_R(I_j, I_{c-1}) - 0\text{hom}_R(I_B, I_{A/B}) \]

because we have

\[ 0\text{hom}_R(I_j, I_{c-1}) = \dim(N_{c-1} \otimes A_{c-1})_{(a_{t+j})} - \dim(N_c \otimes A_c)_{(a_{t+j})} \]

by Proposition 8.1. Now note that the exact sequence (8.8) holds for \( 3 - r \leq j \leq c - 2 \) by Proposition 8.1. Using (8.8) repeatedly for every such \( j \), we get

\[ \sum_{j=3-r}^{c-2} 0\text{hom}_R(I_j, I_{c-1}) = 0\text{hom}_R(I_{A_{c-1}}, I_{c-1}) - 0\text{hom}_R(I_{A_3-r}, I_{c-1}). \]

Since \( 0\text{hom}_R(I_B, I_{A/B}) = 0\text{hom}_R(I_{A_{c-1}}, I_{c-1}) \) we are done. \( \square \)

Remark 8.14. Let \( j_0 \) be some fixed integer satisfying \( c - 1 \geq j_0 \geq 3 - r \). Using the proof above replacing \( I_{A_{3-r}} \) by \( I_{A_{j_0}} \) and \( \sum_{j=3-r}^{c-2} \) by \( \sum_{j=j_0}^{c-2} \) several times we get that (7.2) is also equivalent to

\[ 0\text{hom}_R(I_{A_{j_0}}, A_c) = 0\text{hom}_R(I_{A_{j_0}}, A_{c-1}) + 0\text{hom}_R(I_{A_{j_0}}, I_{c-1}). \]

Theorem 8.15. Suppose that \( \text{Proj}(A) \in W(k; a'_1; r) \) is general with \( c \geq 4 - r, r \geq 2, a_1 > b_t \) and \( \dim A \geq 3 \). If \( c \leq 0 \), we also suppose \( \dim A \geq 4 \). Moreover in the flag (8.1), let \( B = A_{c-1} \) belong to \( W(k; a'_1; r) \) and suppose that every deformation of \( B \) comes from deforming its matrix \( B \).

(i) If (8.14) holds, then \( W(k; a'_1; r) \) is a generically smooth irreducible component of \( \text{Hilb}^{P_x(t)}(\mathbb{P}^n) \) and every deformation of \( A \) comes from deforming its matrix \( A \).

(ii) If \( 0\text{hom}_R(I_{A_{3-r}}, I_{A/B}) = \sum_{i=1}^{r-t+2} (a_i - a_{r+c-1+n}) \) (e.g., \( b_t = b_1 \) and \( a_{t-r+1} < a_{t-r+2}, a_{t+c-1} < s_r - b_r + b_1 \) (e.g., \( A \) linear) and \( \dim W(k; a'_1; r) = \lambda_{c-1} \), then

\[ \dim W(k; a'_1; r) = \lambda_c. \]

Remark 8.16. (1) In Remark 7.4 we noticed that the injectivity assumption \( 0\text{Ext}_B^1(I_B/I_B^2, I_{A/B}) \hookrightarrow 0\text{Ext}_B^1(I_B/I_B^2, B) \) in Theorem 7.1 is equivalent to the exactness of

\[ 0 \rightarrow 0\text{Hom}_R(I_B, I_{A/B}) \rightarrow 0\text{Hom}_R(I_B, B) \rightarrow 0\text{Hom}_R(I_B, A) \rightarrow 0, \]
which in the case \( c \geq 4 - r \) is equivalent to (7.2) (or to \( \gamma = 0 \)) by Proposition 4.14, as explained in Remark 8.11(ii). By Corollary 8.13 we now see that (7.2) is further equivalent to the exactness of

\[
0 \to \text{Hom}_R(I_{A_{3-r}}, I_{A/B}) \to \text{Hom}_R(I_{A_{3-r}}, B) \to \text{Hom}_R(I_{A_{3-r}}, A) \to 0.
\]

The exactness of the latter sequence is often faster to verify by Macaulay2 than using (7.2) by Theorem 8.15.

(2) Replacing \( a_{t+c-1} < s_r - b_r + b_1 \) in Theorem 8.15(ii) by the weaker assumption

\[
a_{t+c-1} < s_r - a_{t-r+1} + a_{t+2-r} - b_r + b_1
\]

one may use the proof of Theorem 8.20 to see that \( \dim W(b; g; r) = \lambda_c - \kappa_c \) where

\[
\kappa_c := \dim_k(N_{2-r}(a_{t+c-1}) - \dim_k(N_{2-r} \otimes A_{2-r})(a_{t+c-1})).
\]

Moreover \( \kappa_c \) may be expressed in terms of binomials using Lemma 8.6 for \( i = 2 - r \).

**Proof.** (of theorem 8.15) (i) This follows from Theorem 7.4, Remark 7.3, and Corollary 8.13.

(ii) Since every deformation of \( B \) comes from deforming \( B \) and \( a_{t+c-1} < s_r - b_r + b_1 \), we have

\[
\dim W(b; g; r) = \dim W(b; g'; r) + \dim_k(N_c)(a_{t+c-1}) - \text{Hom}_R(I_{A_{c-1}}, I_{c-1})
\]

by Proposition 6.11 and Lemma 6.8(i). Then by Corollary 8.3(i) and assumption we get that

\[
\text{Hom}_R(I_{A_{c-1}}, I_{c-1}) = \sum_{i=1}^{t+c-1} \left( \frac{a_i - a_{t+c-1} + n}{a_i} \right) + \text{kappa(hom(R(I_{A_{c-1}}, I_{c-1})))} = \lambda_c - \lambda_{c-1} + K_c.
\]

Since \( K_c = 0 \) by Theorem 6.5 and (6.6) and we are done.

To give more evidence to Conjectures 6.19 and 7.11, we have considered new examples using Theorem 8.15 and Macaulay2 because it is much faster to verify (8.14) than (7.2).

**Example 8.17.** (i) An aspect in the conjectures is how low we can take \( \dim A \) and still expect the conjectures to be true. Since the computations are time consuming in low dimensional cases, we have only checked a few examples in addition to those considered in Sections 6 and 7 (Remark 6.20, Example 6.21, Remark 7.12, and Example 7.13). Indeed we have checked that the quotient \( A \) of dimension 3 defined by the \( 2 \times 2 \) minors of a general \( 3 \times 6 \) matrix of linear entries in a polynomial ring \( R \) with 13 variables satisfies (8.14), as well as the assumptions of Theorem 8.15(ii), provided we can show \( \dim W(0^3; 1^5; 2) = \lambda_3 \) and that every deformation of \( B \) comes from deforming its matrix. \( B \) is, however, defined by the \( 2 \times 2 \) minors of a general \( 3 \times 5 \) matrix of entries in \( R \) and can be treated similarly, i.e. by deleting a column to get a quotient \( C \) defined in the same manner. Since we may suppose that the entries of the \( 3 \times 4 \) matrix of \( C \) are different indeterminates of \( R \), we can use Example 6.18 to get \( \dim W(0^3; 1^4; 2) = \lambda_2 \) and Remark 7.12 to see that every deformation of \( C \) comes from deforming its matrix. Since we have used Macaulay2 to check (8.14) and the first assumption of Theorem 8.15(ii) for \( B, C \), we conclude \( \dim W(0^3; 1^5; 2) = \lambda_3 \) and that every deformation of \( B \) comes from deforming its matrix. Then we get the corresponding conclusion for \( A \) by Theorem 8.15.
(ii) Finally we have checked the quotient \( A \) of dimension 3 defined by the \( 2 \times 2 \) minors of a general \( 3 \times 7 \) matrix of entries in a polynomial ring \( R \) with 15 variables. We delete one (resp. 2) columns to get \( B \) (resp. \( C \)). Using Macaulay2 we show \( [8.14] \) and that the first assumption of Theorem \( [8.15] \) hold for \( B, C \) as well as for \( A, B \). Now since \( C \) is defined by a general \( 3 \times 5 \) matrix of linear entries in a polynomial ring \( R \) with 15 variables, we may suppose that \( C \) is a generic determinantal ring, whence every deformation of \( C \) comes from deforming its matrix by Proposition \( [5.3] \) and \( \dim R \) matrix of linear entries in a polynomial ring \( B \). Theorem \( [8.15] \) hold for general \( 3 \times \) every deformation of \( A \) and suppose that every deformation of \( A \) is a generically smooth irreducible component of \( \text{Hilb}^{\lambda c}(\mathbb{P}^n) \) of dimension \( \lambda c \) and that the first assumption of \( [8.15] \) were satisfied. We could even have relaxed a little upon showing \( [8.14] \) each time, due to

\[
\text{Corollary 8.18. Let } \text{Proj}(A) \in W(k; a_i; r) \text{ be general with } c \geq 4 - r, r \geq 2, a_1 > b_1, a_{t+c-1} < s_r - b_r + b_1 \text{ (e.g. } A \text{ linear) and suppose that } \dim A \geq 3, \text{ and } \dim A \geq 4 \text{ if } c \leq 0. \text{ Let } A_{j_0} \to A_{j_0+1} \to \cdots \to A_c = A, j_0 \geq 3 - r \text{ be the flag we get by deleting columns from the right-hand side, and suppose that every deformation of } A_{j_0} \text{ comes from deforming its matrix. Moreover suppose that } \\
0\text{hom}_R(I_{A_{3-r}}, I_j) = \sum_{i=1}^{t-r+2} (a_i - a_{i+j+n}) \text{ for every } j, j_0 \leq j \leq c - 1, \text{ and that } \dim W(k; a_i^n; r) = \lambda_{j_0} \text{ where } \text{Proj}(A_{j_0}) \in W(k; a_i^n; r). \text{ If also} \\
(8.15) \quad 0\text{hom}_R(I_{A_{3-r}}, A_{j_0}) = 0\text{hom}_R(I_{A_{3-r}}, A_r) + \sum_{j=j_0}^{c-1} 0\text{hom}_R(I_{A_{3-r}}, I_j). \\
\text{then } W(k; a_i^n; r) \text{ is a generically smooth irreducible component of } \text{Hilb}^{\lambda n}(\mathbb{P}^n) \text{ of dimension } \lambda n \text{ and every deformation of } A \text{ (and of each } A_j) \text{ comes from deforming its matrix.}
\]

\text{Proof. By the left-exactness of } 0\text{Hom}_R(I_{A_{3-r}}, -) \text{ we always have an inequality} \\
0\text{hom}_R(I_{A_{3-r}}, A_j) \leq 0\text{hom}_R(I_{A_{3-r}}, A_{j+1}) + 0\text{hom}_R(I_{A_{3-r}}, I_j), \quad j_0 \leq j \leq c - 1 \text{ in } (8.14) \text{ that adds to a corresponding inequality the same way in } (8.15). \text{ Thus assuming equality} \\
in (8.15) \text{ we get equality in } (8.14) \text{ for every } j. \text{ Then we conclude by using Theorem } 8.15.\]

By applying Theorem \( 8.15 \) to every surjection \( A_{t-1} \to A_i, i > 3 - r \) in the flag \( 8.1 \) and Theorem \( 8.10 \) to start the induction, we are able to prove the following main results related to the flag \( 8.1 \). Note that using Remark \( 8.14 \) for \( j_0 = 4 - r \) we get that the following corollary to Theorem \( 8.15 \) generalizes Theorem \( 8.10 \) (provided we increase the dimension assumptions by 1).

\text{Corollary 8.19. Suppose that } \text{Proj}(A) \in W(k; a_i; r) \text{ is general with } c \geq 4 - r, r \geq 2, a_1 > b_1 \text{ and } \dim A \geq 3. \text{ If } c \leq 0, \text{ we also suppose } \dim A \geq 4. \text{ Moreover, suppose that the composition } \gamma_{32} \text{ of } \text{Theorem } 8.10 \text{ is zero. If, for every } j, \ 3 - r \leq j \leq c - 1, \text{ the maps} \\
0\text{Hom}_R(I_{A_{3-r}}, A_{j+1}) \to 0\text{Ext}^1_{A_{3-r}}(I_{A_{3-r}}, I_j) \\
\text{are zero, or equivalently if } (8.14) \text{ holds, or the maps} \\
0\text{Hom}_R(I_{A_{3-r}}, A_j) \to 0\text{Hom}_R(I_{A_{3-r}}, A_{j+1})
are surjective, then $W(b; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{p \times (t)}(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming its matrix.

Proof. We will use induction on $c \geq 4 - r$. The initial case $c = 4 - r$ follows from Theorem 8.10 and Remark 8.11(2).

If $c > 4 - r$, we have by induction that every deformation of $B := A_{c-1}$ comes from deforming $B$. Then we conclude the proof by Theorem 8.15.

Note that all assumptions in Corollary 8.19 are fulfilled if $0 \text{Ext}^1_{A_{2-r}}(I_{A_{2-r}}/I_{A_{2-r}}, J_{2-r}) = 0$ and $0 \text{Ext}^1_{A_{3-r}}(I_{A_{3-r}}/I_{A_{3-r}}, J_j) = 0$ for every $j$, $3 - r \leq j \leq c - 1$. Moreover since $0 \text{Ext}^1_{B}(I_{B}/I_{B}, I_{A/B}) \subset 0 \text{Ext}^1_{B}(I_{B}, I_{A/B})$ we see that if the degree of all generators of $I_{A/B}$ is larger than the maximum of the degree of the relations of $I_B$ appearing in the Lascoux resolution, both $0 \text{Ext}^1$-groups vanish. Also the corresponding $0 \text{Ext}^0$-groups vanish. The latter allows us to find $\dim W(b; a; r)$, and we get a result which extends the $c = 4 - r$ case of Corollary 7.19 substantially. But first we will prove our final theorem in this section which finds $\dim W(b; a; r)$ under some assumptions when a flag (8.1) is given. This somehow completes Theorem 8.15.

**Theorem 8.20.** Suppose that $\text{Proj}(A) \in W(b; a; r)$ is general with $c \geq 4 - r$, $r \geq 2$ and $\dim A \geq 2$. If $c \leq 0$, we also suppose $\dim A \geq 3$. Moreover suppose $a_1 > b_1$, $b_r - b_1 < s_r - a_{t-r+1}$ and that the composed map $\gamma_{32}$ of Theorem 8.17 is zero. If $c \geq 5 - r$ suppose also that the maps $0 \text{Hom}_R(I_{A_{3-r}}, A_{j+1}) \rightarrow 0 \text{Ext}^1_{A_{3-r}}(I_{A_{3-r}}/I_{A_{3-r}}, J_j)$ are zero for every $j$, $3 - r \leq j \leq c - 2$. If $b_t = b_1$ and $a_{t-r+1} < a_{t-r+2}$, or more generally, if

$$(*): 0 \text{hom}_R(I_{A_{2-r}}, I_{2-r}) = \sum_{i=1}^{t-r+1} \binom{a_i-a_{t-r+2}+n}{n} \text{ and } 0 \text{hom}_R(I_{A_{3-r}}, I_j) = \sum_{i=1}^{t-r+2} \binom{a_i-a_{t-r+1}+n}{n}$$

for every $j$, $3 - r \leq j \leq c - 1$, then

$$\dim W(b; a; r) = \lambda_c + K_3 + K_4 + \cdots + K_c - \kappa$$

where $\kappa$ is the following non-negative integer:

$$\kappa = \sum_{j=3-r}^{c} (\dim_k(N_j)(a_{t+j-1}) - \dim_k(N_j \otimes A_j)(a_{t+j-1})).$$

Moreover if $a_{t+c-1} < s_r - a_{t-r+1} + a_{t+2-r} - b_r + b_1$ then every $K_i = 0$ and $\dim W(b; a; r) = \lambda_c - \kappa$ where

$$\kappa = \sum_{j=3-r}^{c} (\dim_k(N_{2-r})(a_{t+j-1}) - \dim_k(N_{2-r} \otimes A_{2-r})(a_{t+j-1})).$$

whence $\kappa$ may be expressed in terms of binomials using Lemma 8.6 for $i = 2 - r$. In particular, if $a_{t+c-1} < s_r - b_r + b_1$ then every $K_i = 0$, $\kappa = 0$ and

$$\dim W(b; a; r) = \lambda_c.$$  

**Remark 8.21.** By Corollary 8.3, assumption (*) above holds in the following cases:

1. $a_{t-r+1} < a_{t-r+2} - \sum_{i=1}^{r-r+1} b_{r+i-1} + \sum_{i=1}^{r-r+1} b_i$.
2. $0 \text{hom}_R(I_{A_{2-r}}, I_j) = 0$ for $2 - r \leq j \leq c - 1$ and $a_{t-r+1} < a_{t-r+2}$.
Proof. Due to assumptions we know that every deformation of $B := A_{c-1}$ comes from deforming $B$ by Corollary 8.19. In Corollary 8.19 we used Theorem 7.1 at each step of the induction, but there is also a dimension formula attached to Theorem 7.1 (the same formula applies in Theorems 8.7 and 8.10) because these results are directly deduced from Theorem 7.1. Thus letting $a_{(j)} = a_1, \ldots, a_{t+j-1}$ and using that the Ext$^1$-vanishing for (ii) only holds for $3 - r \leq j \leq c - 2$, we get that

$$\dim W(b; a_{(j)}; r) = \dim W(b; a_{(j-1)}; r) + \dim_k(N_j \otimes A_j)_{(at_{+j-1})} - \text{hom}_R(I_{A_{j-1}}, I_{j-1}), \quad 4 - r \leq j \leq c$$

Since every deformation of $B$ comes from deforming its matrix, it follows from Proposition 6.11 that the above dimension formula holds for $j = c$ as well. From Theorem 8.7 we get that

$$\dim W(b; a_{(3-r)}; r) = \lambda_3 - r - \kappa'$$

because $a_{t-r+1} < s_r - b_r + b_1$ implies $-b_1 < \ell_2$, whence $K'_i = 0$ for all $i$ by Lemma 6.4. Summing all these dimensions we get

$$\dim W(b; a; r) = \lambda_3 - r - \kappa' + \sum_{j=4-r}^{c} \dim_k(N_j \otimes A_j)_{(at_{+j-1})} - \sum_{j=4-r}^{c} \text{hom}_R(I_{A_{j-1}}, I_{j-1}).$$

In this formula Corollary 8.3 and assumption (*) imply that $\text{hom}_R(I_{A_j}, I_j) = \sum_{i=1}^{t+j-1} (a_i - a_{i+j+n})$ for every $j$, $2 - r \leq j \leq c - 1$. Using (6.9) it follows that

$$\dim_k(N_j)_{(at_{+j-1})} - \text{hom}_R(I_{A_{j-1}}, I_{j-1}) = \lambda_j - \lambda_j - 1 + K_j, \quad \text{for} \quad 4 - r \leq j \leq c$$

where $K_j = 0$ for $j < 3$. Inserting this formula and recalling that

$$\kappa' = \dim_k(N_{3-r})_{(at_{-r+2})} - \dim_k(N_{3-r} \otimes A_{3-r})_{(at_{-r+2})}$$

by Theorem 8.7 we get the 1st displayed dimension formula of Theorem 8.10. Moreover if $a_{t+c-1} < s_r - a_{t-r+1} + a_{t+2-r} - b_r + b_1$, then (8.10) and (8.11) applies with $j = 2 - r$ and we get the last expression for $\kappa$ which we may compute using Lemma 8.6. We also get $K_i = 0$ for $3 \leq i \leq c$ by Theorem 6.5 and (6.6). Finally if $a_{t+c-1} < s_r - b_r + b_1$, we get $a_{t+j} < s_r - b_r + b_1$ for $j \leq c - 1$ and hence $\kappa = 0$ by Lemma 6.6(i) and we are done.

Corollary 8.22. Let $\text{Proj}(A) \in W(b; a; r)$ be general with $c \geq 4 - r$ and suppose $\dim A \geq 2$. If $c \leq 0$, we also suppose $\dim A \geq 3$. Set $b := \sum_{i=r}^{t} b_i - \sum_{i=1}^{r+2} b_i$ and suppose $a_1 > b_t$, $r \geq 2$ and $b_r - b_1 < s_r - a_{t-r+1} := \sum_{i=1}^{t-r} (a_i - b_{r+i})$. If

$$a_{t-r+2} > 2a_{t-r+1} + b \quad \text{and} \quad a_{t-r+3} > \sum_{i=t-r+1}^{t-r+2} a_i + b + \max\{a_{t-r+2} - a_1, b_{t-r+2} - b_1\}$$

then $W(b; a; r)$ is a generically smooth irreducible component of $\text{Hilb}^{P_X(t)}(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming $A$. Moreover

$$\dim W(b; a; r) = \lambda_c + K_3 + K_4 + \cdots + K_c - \kappa$$
where $\kappa$ is the following non-negative integer:

$$\kappa = \sum_{j=3-r}^{c} (\dim_k(N_j(a_{t+j-1}) - \dim_k(N_j \otimes A_j)(a_{t+j-1})).$$

In particular, if also $a_{t+c-1} < s_r - b_r + b_1$ then every $K_i = 0$, $\kappa = 0$, whence $\dim W(b; a; r) = \lambda_c$.

**Proof.** Consider the flag (8.1) where $I_j = I_{A_j+1}/A_j$. By Theorem 8.10 and Corollary 8.3(i), cf. Remark 8.4(1), it suffices to show $\Ext^0_{\mathcal{A}_{2-r}}(I_{A_{2-r}}/I_{A_{2-r}}^2, I_{2-r}) = 0$ for every $3 - r \leq j \leq c - 1$ under the given assumptions on $\dim A$ (which is one less than Corollary 8.19 requires). We have $\Ext^1_{\mathcal{A}_{2-r}}(I_{A_{2-r}}/I_{A_{2-r}}^2, I_{2-r}) = 0$ by Corollary 7.9 and assumption. Since the smallest degree of the minimal generators of $I_J$ is

$$s(I_j) := a_{t+j} + \sum_{i=1}^{t-r} a_i - \sum_{i=r}^{t} b_i$$

and the maximum degree of the relations of $I_{A_{3-r}}$ is

$$\text{mdr}(I_{A_{3-r}}) = \sum_{i=1}^{t-r+1} a_i - \sum_{i=1}^{t-r+2} b_i + a_{t-r+1}$$

by Corollary 7.8(iii), we get the vanishing of the $\Ext^0_{R}$-groups (as well as for the corresponding $\Hom$-groups which means that (*) of Theorem 8.20 holds) if $\text{mdr}(I_{A_{3-r}}) < s(I_j)$, or equivalently if $a_{t+j} > \sum_{i=t-r+1}^{t-r+2} a_i + b + \max\{a_{t-r+2} - a_1, b_{t-r+2} - b_1\}$ for every $j$. Since, in general, we assume $a_{t+j} \geq a_{t-r+3}$ for $j \geq 3 - r$, we are done by an assumption in Corollary 8.22.$\square$

Let us extend Example 7.10 a lot.

**Example 8.23.** For any $r$, $2 \leq r \leq t - 1$ and any $c \geq 4 - r$, let $\mathcal{A} = [\mathcal{B}, \mathcal{C}]$ be a general $t \times (t + c - 1)$ matrix where $\mathcal{B}$ is a $t \times (t - r + 2)$ matrix that is exactly equal to the matrix $\mathcal{A}$ in Example 7.10(i), i.e. all entries in the last column are of degree 3 and all other entries are of degree one (and take all $b_i = 0$, thus $b_r - b_1 < s_r - a_{t-r+1}$ holds). Moreover let all entries of the $t \times (c + r - 3)$ matrix $\mathcal{C}$ be of degree at least 7, and otherwise arbitrary, i.e. $7 \leq a_{t-r+3} \leq a_{t-r+4} \leq \cdots \leq a_{t+c-1}$, and suppose that $\dim R$ is large enough so that $\dim A \geq 3$. Then $W(b; a; r)$ is a generically smooth irreducible component of Hilb$^P_x(t)(\mathbb{P}^n)$ and every deformation of $A$ comes from deforming $\mathcal{A}$ by Corollary 8.22. Moreover dim $W(b; a; r) = \lambda_c$ holds if $t - r$ is large enough. More precisely, dim $W(b; a; r) = \lambda_c$ provided $a_{t+c-1} < s_r - b_r + b_1$, i.e. if $t - r \geq a_{t+c-1}$.

9. Deformations of exterior powers of modules over determinantal schemes

We keep the notation of the previous sections. So, $\varphi : F := \oplus_{i=1}^t R(b_i) \rightarrow G := \oplus_{j=1}^{t+c-1} R(a_j)$ is a graded $R$-morphism between two free $R$-modules of rank $t$ and $t+c-1$, respectively, and we suppose $a_1 > b_1$ in this section. Set $MI = \coker(\varphi^*)$, $I_A := I_{t-r+1}(\varphi^*)$, $A = R/I_A$ and $X := \Proj(A)$. If we simultaneously work with $A$ for different $r$, we use the notation $I_{A_r} = I_{t-r+1}(\varphi^*)$, $A_r = R/I_{A_r}$ and $X_r := \Proj(A_r)$. We assume that $A$ has codimension $r(r+c-1)$ in $R$, i.e. that $A$ is determinantal. In this section we also suppose $c \geq 1$ so that $MI$ is an $A_1$-module, i.e. supported
Remark 9.2. (1) The case \( J \) of deformations of \( \wedge^r MI \) in Theorem 3.11. Indeed the hypothesis \( \text{depth } J = 0 \) which induce an exact sequence \( \text{Tor}^R_1 \Rightarrow \text{Ext}^1_A(\wedge^r MI, \wedge^r MI) \Rightarrow 0 \) since \( \text{depth } A \) of \( \varphi^* \) is general.

Lemma 9.1. If \( \text{depth}_{J_A} A \geq j + 1 \) for some \( j \geq 1 \), then \( \text{Hom}_A(\wedge^r MI, \wedge^r MI) \cong A \) and

\[
\text{Ext}_A^i(\wedge^r MI, \wedge^r MI) = 0 \quad \text{for} \quad 1 \leq i \leq j - 1.
\]

In particular, if \( \text{depth}_{J_A} A \geq 4 \), then

\[
\text{Hom}_A(\wedge^r MI, \wedge^r MI) \cong A \quad \text{and} \quad \text{Ext}_A^i(\wedge^r MI, \wedge^r MI) = 0 \quad \text{for} \quad i = 1, 2.
\]

Remark 9.2. (1) The case \( r = 1 \) in Lemma 9.1 was considered in [32] and slightly generalized in Theorem 3.11. Indeed the hypothesis \( \text{depth}_{J_A} A \geq j + 1 \), for \( \varphi \) general, is weakened to \( \text{depth}_{J_A} A \geq j \) in Theorem 3.11 and Lemma 9.1 still works. Examples computed with Macaulay2 indicate that this also holds for \( r > 1 \).

(2) If \( c = 2 \), then the conclusions of Lemma 9.1 hold, without assuming \( \text{depth}_{J_A} A \geq 4 \). Indeed, \( \wedge^r MI \) is a twist of the canonical module \( K_A \) by [38] Proposition 3.5] and it is well known that \( A \cong \text{Hom}(K_A, K_A) \) and \( \text{Ext}_A^i(K_A, K_A) = 0 \) for \( i \geq 1 \) hold in general for Cohen-Macaulay rings.

Proof. Since \( \text{depth}_{J_A} A \geq 2 \), we have

\[
\text{Hom}_A(\wedge^r MI, \wedge^r MI) \cong H_0^0(X \setminus V(J_A), \mathcal{O}_X) \cong A
\]

and \( \text{Ext}_A^i(\wedge^r MI, \wedge^r MI) \cong \text{Ext}_{\mathcal{O}_X}^i(\wedge^r MI, \wedge^r MI) \) with \( X' := \text{Spec}(A) \setminus V(J_A) \).

Since \( \wedge^r MI \) is locally free of rank 1 on \( X' \), we get

\[
\text{Ext}_{\mathcal{O}_{X'}}^i(\wedge^r MI, \wedge^r MI) \cong H^i(X', \mathcal{H}_{\text{Hom}}(\wedge^r MI, \wedge^r MI)) \cong H^i(X', \mathcal{O}_{X'}) = 0
\]

(since \( H^i(X', \mathcal{O}_{X'}) \cong H^{i+1}_{J_A}(A) = 0 \) by \( \text{depth}_{J_A} A \geq j + 1 \) for \( 1 \leq i \leq j - 1 \)).

We suppose \( \text{depth}_{J_A} A \geq 4 \) and using Lemma 9.1 we will compare deformations of \( X \subset \mathbb{P}^n \) with deformations of \( \wedge^r MI \). Note that \( \text{Hom}_R(\wedge^r MI, \wedge^r MI) \cong \text{Hom}_A(\wedge^r MI \otimes A, \wedge^r MI) \cong A \) and recall that there is a spectral sequence

\[
\text{Ext}_A^i(\text{Tor}_j^R(\wedge^r MI, A), \wedge^r MI) \Rightarrow \text{Ext}_R^i(\wedge^r MI, \wedge^r MI)
\]

which induce an exact sequence

\[
0 \longrightarrow \text{Ext}_A^1(\wedge^r MI, \wedge^r MI) \longrightarrow \text{Ext}_R^1(\wedge^r MI, \wedge^r MI) \longrightarrow \text{Hom}_A(\text{Tor}_j^R(\wedge^r MI, A), \wedge^r MI) \longrightarrow \text{Ext}_A^2(\wedge^r MI, \wedge^r MI)
\]

\[
(9.1)\quad 0 \longrightarrow \text{Ext}_A^1(\wedge^r MI, \wedge^r MI) \longrightarrow \text{Ext}_R^1(\wedge^r MI, \wedge^r MI) \longrightarrow \text{Hom}_A(\text{Tor}_j^R(\wedge^r MI, A), \wedge^r MI) \longrightarrow \text{Ext}_A^2(\wedge^r MI, \wedge^r MI)
\]
where the $\text{Ext}_A^i$-groups for $i = 1, 2$ vanish. Now the exact sequence $0 \to I_A \to R \to A \to 0$ leads to
\[
\text{Tor}_1^R(\bigwedge^r MI, A) \cong \bigwedge^r MI \otimes I_A.
\]
Therefore, we have
\[
\text{Hom}_A(\text{Tor}_1^R(\bigwedge^r MI, A), \bigwedge^r MI) \cong \text{Hom}_A(\bigwedge^r MI \otimes I_A, \bigwedge^r MI) \\
\cong \text{Hom}_R(I_A, A)
\]
because $\text{Hom}_A(\bigwedge^r MI, \bigwedge^r MI) \cong A$. By (9.1) we get an isomorphism of the tangent spaces
\[
\text{Ext}_1^R(\bigwedge^r MI, \bigwedge^r MI) \cong \text{Hom}_R(I_A, A) \cong \text{Hom}_A(I_A/I_A^2, A)
\]
of the deformation functors we consider. Also note it is possible to continue (9.1) to see the injection
\[
\text{Ext}_2^R(\bigwedge^r MI, \bigwedge^r MI) \hookrightarrow \text{Ext}_2^R(\bigwedge^r MI, \bigwedge^r MI)
\]
of obstruction spaces of these functors. Thus we have the first isomorphism in

**Theorem 9.3.** Let $c \geq 1$, $(c, r) \neq (1, 1)$ and suppose $\text{depth}_{I_A} A \geq 4$ for $c \neq 2$. Then
\[
\text{Def}_{\bigwedge^r MI/R}(-) \cong \text{Def}_{A/R}(-) \cong \text{Hilb}_X(-)
\]
i.e. the deformation functors of $\bigwedge^r MI$ as an $R$ module, the deformation functor of the surjection $R \to A$ and the local Hilbert functor are isomorphic.

**Remark 9.4.** For $(c, r) = (1, 1)$, $\text{depth}_{I_A} A \geq 4$ is impossible. Note that the functors is defined on the category $\mathcal{E}$ described in section 5 and that some of the local functors in the theorem are considered in Lemma 5.1.

**Proof.** The first isomorphism follows from Lemma 9.1 and (9.1) by the arguments above while the second follows from (2.2). \qed

**Remark 9.5.** We have now the vertical isomorphism in
\[
\begin{array}{ccc}
\text{Def}_{MI/R}(-) & \cong & \text{Def}_{\bigwedge^r MI/R}(-) \\
\text{Hilb}_{X_1}(-) & \cong & \text{Hilb}_{X_r}(-)
\end{array}
\]
provided $\text{depth}_{I_A} A_r \geq 4$ and $\text{depth}_{I_{A_1}} A_1 \geq 3$ (Theorem 3.11). To define the dotted arrow, i.e. a morphism $\alpha(-) : \text{Def}_{MI/R}(-) \to \text{Def}_{\bigwedge^r MI/R}(-)$ of functors on $\mathcal{E}$, it suffices to observe that

if $MI_S$ is a deformation of $MI$, hence $S$-flat, then $MI_S^{\otimes r} := \bigotimes MI_S \otimes \cdots \otimes MI_S$ is obviously $S$-flat, and thinking of $\bigwedge^r MI_S$ as a direct summand of $MI_S^{\otimes r}$, we get that $\bigwedge^r MI_S$ is $S$-flat. Thus, we may define on $\mathcal{E}$
\[
\alpha(S) : \text{Def}_{MI/R}(S) \to \text{Def}_{\bigwedge^r MI/R}(S) \\
MI_S \mapsto \bigwedge^r MI_S.
\]
and use it to compare all four deformation functors above.
We will, however, treat this a little differently, using the theory we have developed. Indeed in the diagram of the remark above, there is a direct well-defined morphism
\[
\Def(\psi)(-) : \Def_{\mathcal{M}/R}(-) \longrightarrow \Def_{\mathcal{A}/R}(-) \cong \Hilb \mathcal{X}_r(-)
\]
of local functors over \( \ell \) defined in Lemma 5.1, with tangent map
\[
\psi : \Ext^1_R(\mathcal{M}, \mathcal{M}) \longrightarrow \Hom(I_{t-r+1}(\varphi^*), \mathcal{A})
\]
which, thanks to Lemma 5.1, is surjective if, and only if, every deformation of \( \mathcal{A} \) comes from deforming its matrix \( \mathcal{A} \). Thus redefining \( \alpha(-) \) to be \( \Def(\psi)(-) \) composed with the inverse of the isomorphism \( \Def \iso \Wedge^r \mathcal{M}/R(-) \longrightarrow \Hilb \mathcal{X}_r(-) \) we get

**Theorem 9.6.** Let \( c \geq 1, 1 \leq r < t, (c, r) \neq (1, 1) \), set \( \mathcal{A} := \mathcal{A}_r \) and suppose \( \dim \mathcal{A} \geq 2 \) and if \( c \neq 2 \) that \( \text{depth} \mathcal{J}_{\mathcal{A}} \mathcal{A} \geq 4 \) and \( \text{depth} \mathcal{J}_{\mathcal{A}_1} \mathcal{A}_1 \geq 3 \). Moreover, we suppose that every deformation of \( \mathcal{A} \) comes from deforming its matrix \( \mathcal{A} \) and that

\[
\hom(I_{t-r+1}(\varphi^*), \mathcal{A}_r) = \lambda_c + K_3 + K_4 + \cdots + K_c.
\]

If \( c \neq 1 \) then the following 4 functors are isomorphic and smooth:

\[
\begin{align*}
\Def_{\mathcal{M}/R}(-) & \cong \Def_{\Wedge^r \mathcal{M}/R}(-) \\
\Hilb \mathcal{X}_1(-) & \cong \Hilb \mathcal{X}_r(-)
\end{align*}
\]

In particular the dimension of their tangent spaces are all equal, and also equal to

\[
\lambda_c + K_3 + K_4 + \cdots + K_c = \dim \Wedge^c (b; g; 1) = \dim \Wedge^c (b; g; r).
\]

Moreover if \( c = 1 \) then the functors \( \Def_{\mathcal{M}/R}(-) \cong \Def_{\Wedge^r \mathcal{M}/R}(-) \cong \Hilb \mathcal{X}_r(-) \) are isomorphic and smooth, and the dimension of their tangent spaces are equal to \( \lambda_1 = \dim \Wedge^c (b; g; r) \).

**Proof.** By Theorem 3.9(iii), \( \dim \Ext^1_R(\mathcal{M}, \mathcal{M}) = \lambda_c + K_3 + K_4 + \cdots + K_c \). So \( \Def(\psi)(-) \) is an isomorphism on tangent spaces and since it is smooth by Lemma 5.1 \( \Def(\psi)(-) \) is an isomorphism. So for \( c \neq 1 \), all 4 functors are isomorphic by Theorems 9.3 cf. Remark 9.2 and smooth by Theorem 3.9(iii) or Corollary 5.2. Finally using Lemma 7.2 we get

\[
\dim \Wedge^c (b; g; r) = \hom(I_{t-r+1}(\varphi^*), \mathcal{A})
\]

which concludes the proof because for \( c = 1 \) similar arguments apply. \( \square \)

**Remark 9.7.** (1) A consequence of Theorem 9.6 is that we have

\[
\Ext^1_R(\mathcal{M}, \mathcal{M}) \cong \Ext^1_R(\bigwedge^c \mathcal{M}, \bigwedge^c \mathcal{M})
\]

under the assumptions of Theorem 9.6.

(2) As we have seen the assumption “every deformation of \( \mathcal{A} \) comes from deforming \( \mathcal{A} \)” often holds (Theorems 7.1 and 8.15) and is really the main reason that allows us to state Conjecture 7.11. The displayed assumption is then a main conclusion in Theorems 6.14 and 8.15 which, together with Macaulay2 computations, inspired Conjecture 6.19. Note that in these results we always
assume $a_{t+c-1} < s_r - b_r + b_1$, i.e. $a_{t+c-1} - b_1 < \sum_{i=1}^{r-1} (a_i - b_{r+i-1})$ which implies $K_i = 0$ by (6.5) and (6.6). In particular assuming $a_{t+c-1} < s_r - b_r + b_1$ the displayed assumption in Theorem 9.6 is plainly dim $W(\underline{a}; r) = \lambda_c$. In fact we are not aware of cases where the displayed assumption above holds unless $r = 1$, or $r > 1$ and $a_{t+c-1} < s_r - b_r + b_1$. But if the latter assumption holds, both these assumptions of Theorem 9.6 are expected if dim $A$ is large enough. More precisely if in addition dim $A \geq 4$ for $c \neq 2$ and dim $A \geq 2$ for $c = 2$ (cf. Conjecture 7.11 and Remark 7.12) the conclusions of Theorem 9.6 are expected.

(3) As a consequence of (2) and (6.6), if $a_{t+c-1} < s_r - b_r + b_1$ holds then $a_{t+c-1} < s_i - b_i + b_1$ hold for all $1 \leq i \leq r$. This implies that if the Conjectures 6.19 and 7.11 hold for all $A_i$, which is highly expected, and if say dim $A_r \geq 4$ and $c \geq 2$, then the three isomorphic deformation functors $\text{Def}_{A_i/R}(\cdot) \cong \text{Hilb}_{X_i}(\cdot) \cong \text{Def}_{\Lambda^r MI/R}(\cdot)$ are for all $i, 1 \leq i \leq r$ further isomorphic to e.g. the local Hilbert functor $\text{Hilb}_{X_i}(\cdot)$ of deforming the scheme $X_i \subset \text{Proj}(R)$ defined by maximal minors. In particular $W(\underline{b}; a; r)$ is for all $i$ a generically smooth irreducible component of dimension $\lambda_c$ of $\text{Hilb}^{pX_i(t)}(\mathbb{P}^n)$! Here the dimension of $X_i$ are very different and so are their Hilbert polynomials.

Let us finish this section by seeing how precisely the assumption $a_{t+c-1} < s_r - b_r + b_1$ lead to the isomorphisms given in Theorem 9.6. To illustrate the theorem we include also Macaulay2 computations of $\text{Ext}^1_{\Lambda^r MI/\Lambda^r MI}$ for various $r$.

**Example 9.8.** In all of this example $\mathcal{A}$ is a $4 \times 4$ matrix with 16 variable as in the generic case, and (i) is the generic case where all five deformation functors which we consider should be isomorphic due to Theorem 9.6. We only need to check that the tangent spaces are isomorphic since all functors are smooth. Below we let $M_r = \Lambda^r MI$. Since $c = 1$, Theorem 9.6 does not include $\text{Hilb}_{X_1}(\cdot)$.

(i) We let $\mathcal{A} = (x_{i,j})$ be the generic $4 \times 4$ linear matrix and using Macaulay2 we get

$$\text{Ext}^1_{\Lambda^r MI, \Lambda^r MI} = \text{Ext}^1_{M_2, M_2} = \text{Ext}^1_{M_3, M_3} = 225.$$  

Correspondingly for the local graded deformation functor, $\text{Hom}(I_2(\varphi^*), A_3) = \text{Hom}(I_3(\varphi^*), A_2) = 225$. Hence all five deformation functors are isomorphic. Note that $A_r$ and $I_{A_r}$ are defined by the $t - r + 1 = 5 - r$ minors, so e.g. for $r = 2$, $A_2 = R/I_{A_2}$ is defined by $3 \times 3$ minors (submaximal minors), so $I_{A_2} = I_3(\varphi^*)$ according to our notation.

(ii) In this case we consider $\mathcal{A} = (x_{i,j}^r)$ with $r_1 = r_2 = 1$ and $r_3 = r_4 = 2$, i.e. with the degree matrix $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$. This means that we can take all $b_i = 0$, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$. Then we get $s_2 = 4$ and $s_3 = 2$ by the definition of $s_r := \sum_{i=1}^{r-1} a_i$, whence $a_4 < s_2$ is satisfied while $a_4 < s_3$ is not. So according to Theorem 9.6 and Remark 9.7 the deformation functors of $A_3$, $M_3$ and $MI$ should be isomorphic. Moreover by Theorem 9.3 the deformation functors of $A_2$ and $M_2$ are isomorphic. Now

$$\text{Ext}^1_{\Lambda^r MI, \Lambda^r MI} = \text{Ext}^1_{M_2, M_2} = \text{Hom}(I_3(\varphi^*), A_2) = 1129$$

by using Macaulay2 while $\text{Hom}(I_2(\varphi^*), A_3) = \text{Ext}^1_{M_3, M_3} = 1089$, confirming what we expected by Theorems 9.6 and 9.3.
Grassmann proved that a general cubic form in $k^3$ for $r \times 3$.

Problem 10.1. Fix integers $n \geq 2$ and $d \geq 3$. To determine $r(n,d)$ or at least lower/upper bounds for $r(n,d)$.

10. Final comments and conjectures

This last section contains a collection of some natural open questions/conjectures arising from our work.

First of all, we would like to say few words about the (linear) representation of a hypersurface. The Hilbert scheme which parameterizes hypersurfaces $X \subset \mathbb{P}^n$ of fixed degree $d \geq 1$ is isomorphic to $\mathbb{P}^N$, $N := \binom{n+d}{n} - 1$ and there is an open dense subset $U \subset \mathbb{P}^N$ parameterizing smooth hypersurfaces. Inside $\mathbb{P}^N$ we have the determinantal locus $W(b_1, \cdots, b_t; a_1, \cdots, a_t; 1)$ which parameterizes hypersurfaces defined by a form $F \in k[x_0, \cdots, x_n]$ of degree $d$ which can be written as the determinant of a $t \times t$ matrix with entries homogeneous forms of degree $a_j - b_i$. We can ask, for instance, for codim$_{\mathbb{P}^N} W(b_1, \cdots, b_t; a_1, \cdots, a_t; 1)$. In fact, if $n \geq 3$, this codimension is $N - \lambda_1$ by Theorem 3.8. This leads us to a more basic problem: Given a general form $F \in k[x_0, \cdots, x_n]$ of fixed degree $d \geq 2$, we would like to determine the minimum integer $r = r(n,d,s)$ (resp. $p = p(n,d,s)$) such that $F^r$ (resp. $F^p$) can be written as the determinant of a matrix (resp. the pfaffian of a skew symmetric matrix) $A$ with entries homogeneous polynomials of degree $s \geq 1$. As a special case of great interest we highlight the case $s = 1$. In this case we are dealing with matrices with linear entries and, for simplicity, we set $r(n,d) := r(n,d;1)$.

It is a classical result that $r(2,d) = r(3,3) = 1$. Indeed, in [10], Dickson proved that a general form of degree $d$ in $k[x,y,z]$ is the determinant of a $d \times d$ matrix of linear forms and, in [22], Grassmann proved that a general cubic form in $k[x,y,z,t]$ can be written as the determinant of a $3 \times 3$ matrix with linear entries. More recently Beauville has proved that $r(3,d) = 2$ for $3 < d < 16$ and $r(3,d) > 2$ for $d > 15$ (see [3]). To our knowledge no other results are known concerning the (linear) representation of a general form $F \in k[x_0, \cdots, x_n]$ of degree $d > 2$ and this is a natural problem that we should address before studying the determinantal locus of hypersurfaces. So we propose the following to problems:

Problem 10.1. Fix integers $n \geq 2$ and $d \geq 3$. To determine $r(n,d)$ or at least lower/upper bounds for $r(n,d)$.
Fix \( n \geq 3 \) and \( d \geq 3 \). It follows from Theorem 3.8 that \( r(n, d) = 1 \) if and only if \( (n, d) = (3, 3) \). So we not only recover Grassmann’s result but we also prove that the converse holds.

Now, we come back to determinantal schemes with \( c \geq 2 \) and we collect the conjectures scattered throughout the work and introduce a new one.

**Conjecture 10.2.** Fix integers \( t \geq 2, a_1 \leq a_2 \leq \cdots \leq a_{t+c-1} \) and \( b_1 \leq b_2 \leq \cdots \leq b_t \). Let \( A \) be a homogeneous \( t \times (t + c - 1) \) matrix with entries forms of degree \( a_j - b_i \), let \( A = R/I_{t-r+1}(A) \) where \( 1 \leq r \leq t-1, 2 - r \leq c \) and suppose that \( \dim A \geq 2 \) for \( c \neq 1 \) and \( \dim A \geq 3 \) for \( c = 1 \). Moreover suppose \( \text{Proj}(A) \in W(b; a; r) \), \( a_1 > b_t \) and \( a_{t+c-1} - b_1 < \sum_{i=1}^{t-r+1}(a_i - b_{r+i-1}) \). Then

\[
\dim W(b; a; r) = \lambda_c.
\]

In particular, we would like to know if the above conjecture is at least true when the entries of \( \mathcal{A} \) are all of fixed degree \( e \geq 1 \). Note that the main assumption \( a_{t+c-1} - b_1 < \sum_{i=1}^{t-r+1}(a_i - b_{r+i-1}) \) holds in this case. More precisely,

**Conjecture 10.3.** Fix integers \( r \geq 1, c > 2 - r, t \geq 2 \) and \( d \geq 1 \). Let \( A \) be a homogeneous \( t \times (t + c - 1) \) matrix with entries forms of degree \( e \). Let \( A = R/I_{t-r+1}(A) \) and suppose \( \text{Proj}(A) \in W(\mathbf{b}; \mathbf{a}; r) \), \( \dim A \geq 2 \) for \( c \neq 1 \) and \( \dim A \geq 3 \) for \( c = 1 \). Then,

\[
\dim W(b; a; r) = t(t+c-1)\binom{e+n}{n} - t^2 - (t+c-1)^2 + 1.
\]

It will be also interesting to drop the assumption \( a_{t+c-1} - b_1 < \sum_{i=1}^{t-r+1}(a_i - b_{r+i-1}) \) and find \( \dim W(b; a; r) \) more generally. To give our best guess, we delete \( c-j \) columns from the right-hand side of the matrix \( \mathcal{A} \) of \( \varphi^* \) to define \( \varphi_{t+j-1}^* \), and we let \( A_j = R/I_{t-r+1}(\varphi_{t+j-1}^*) \) and \( N_j = \text{coker}(\varphi_{t+j-1}^*) \) for \( 2 - r \leq j \leq c \).

**Question 10.4.** Fix integers \( t \geq 2, a_1 \leq a_2 \leq \cdots \leq a_{t+c-1} \) and \( b_1 \leq b_2 \leq \cdots \leq b_t \). Let \( A \) be a homogeneous \( t \times (t + c - 1) \) matrix with entries forms of positive degree \( a_j - b_i \), let \( A = R/I_{t-r+1}(A) \) where \( 1 \leq r \leq t-1, 2 - r \leq c \). Suppose \( \text{Proj}(A) \in W(b; a; r) \) and \( \dim A \geq 2 \). If \( c = 1 \), we also suppose \( \dim A \geq 3 \). When is

\[
\dim W(b; a; r) = \lambda_c + K_3 + K_4 + \cdots + K_c - \kappa
\]

where \( \kappa \) is the following non-negative integer: \( \kappa = \sum_{j=3-r}^{c-r}(\dim_k(N_j)(a_{t+j-1}) - \dim_k(N_j \otimes A_j)(a_{t+j-1})) \)?

There is at least one case where this guess is correct, namely for \( r = 1 \). Indeed, then \( N_j \) is annihilated by \( I_i(\varphi_{t+j-1}^*) \), so \( N_j \otimes A_j \cong N_j \) and \( \kappa = 0 \) and we get \( \dim W(b; a; r) \) as above by Theorem 3.8. There is one more case where we expect the answer to be true, namely in the case where \( a_{t+c-1} - b_1 < \sum_{i=1}^{t-r+1}(a_i - b_{r+i-1}) \) because Lemma 6.8.1 for \( v = 0 \) and \( a_{t+j-1} \leq a_{t+c-1} \) show \( \dim(N_j \otimes A_j)(a_{t+j-1}) = \dim(N_j)(a_{t+j-1}) \), i.e. that \( \kappa = 0 \). Hence we get Conjecture 10.2. Unfortunately in all other cases we are aware of, \( \kappa \neq 0 \). So the correction term \( \kappa \) in Question 10.4 should not be skipped. Moreover due to Theorem 8.29 and the fact that the assumptions of Theorem 8.20 are designed to be true as far as our results and computations by Macaulay2 indicate, except for the inequality below, we expect
Conjecture 10.5. With notations and assumptions as in Question \[10.4\] if \( b_r - b_1 < s_r - a_{t-r+1} \), then

\[
\dim W(b; a; r) = \lambda_c + K_3 + K_4 + \cdots + K_c - \kappa.
\]

Unfortunately it seems difficult to compute \( \kappa \), but Theorems \[6.22\] and \[8.20\] give some answers under restrictive assumptions.

Conjecture 10.6. Fix integers \( r \geq 1, c \geq 2 - r, t \geq 2, a_1 \leq a_2 \leq \cdots \leq a_{t+c-1} \) and \( b_1 \leq b_2 \leq \cdots \leq b_t \). Let \( A \) be a homogeneous \( t \times (t + c - 1) \) matrix with entries forms of degree \( a_j - b_i \). Let \( A = R/I_{t-r+1}(A) \) and suppose \( \text{Proj}(A) \in W(b; a; r) \), \( \dim A \geq 4 \) for \( c = 1 \) and \( \dim A \geq 3 \) for \( c \neq 0 \). If \( a_1 > b_t \) then \( W(b; a; r) \) is a generically smooth irreducible component of \( \text{Hilb}^P(X, \mathbb{P}^n) \) and every deformation of \( A \) comes from deforming \( A \).

Related to the smoothness of the Hilbert scheme \( \text{Hilb}^P(X, \mathbb{P}^n) \) along \( W(b; a; r) \) and to whether any deformation of a determinantal scheme comes from deforming its associated homogeneous matrix, we have the following conjecture involving the codepth of the normal modules:

Conjecture 10.7. Let \( r \geq 1, c \geq 2 - r \) and let \( A = R/I_A \) (resp. \( B = R/I_B \) if \( c > 2 - r \)) be defined by the vanishing of the \((t-r+1) \times (t-r+1)\) minors of a general \( t \times (t + c - 1) \) matrix \( A \) with \( a_1 > b_r \) (resp. of \( B \) obtained by deleting a column of \( A \)). Let \( N_A := \text{Hom}_R(I_A, A) \) and suppose that \( \dim A \geq 3 \).

(i) If \( c \notin \{0, 1, 2\} \) then codepth\((N_A) = 1. If c = 1 \) (resp. \( c = 0, 2 \)), then codepth\((N_A) = \min\{2, \dim A - 2\} \) (resp. codepth\((N_A) = 0\).

(ii) Let \( (r, c) \neq (1, 2), c \geq 2 \) (resp. \( 3 - r \leq c \leq 1 \)) then codepth\((\text{Hom}_R(I_B, I_A/I_B)) = r \) (resp. \( r + 1 \)).

As we proved in Proposition \[7.18\] resp. in Remark \[7.19\] this last conjecture essentially holds for generic determinantal algebras, resp. for \( r = 1 \).

All results of this work deal with determinantal varieties of positive dimension and it will be very interesting generalize them to 0-dimensional schemes since so far only results for standard determinantal 0-dimensional schemes are known (see \[30\] and \[33\]).

We end this last chapter with three concrete problems that we have addressed in this work but unfortunately have not been able to completely solve. Their resolution will help to answer the above conjectures.

Problem 10.8.

1. Show that: \( \dim_0 \text{Hom}_B(I_B/I_B^2, I_A/B) = \sum_{j=1}^{t+c-1} (-a_j + a_{t+c-1} + n) \) if \( \dim A \geq 3 \).

2. Show that: \( _0 \text{Ext}^1_B(I_B/I_B^2, I_A/B) = 0 \) if \( \dim A \geq 3 \) (resp. \( \dim A \geq 4 \)) for \( c \neq 1 \) (resp. \( c = 1 \)).

3. Determine \( \dim_0 (MI \otimes A)}_{a_{t+c-1}} \) in the case: \( a_{t+c-1} - b_1 \geq \sum_{i=1}^{t-r+1} (a_i - b_{r+i-1}) \).
11. Appendix

Many examples included in this work have been or can be computed using Macaulay2 [23]. We include the code of some of these examples with the aim that the reader will be able to reproduce them.

**Example 11.1.** Set $R = k[x_0, \ldots, x_{15}]$ and let $\mathbb{P}^{15} = \text{Proj}(R)$. We consider a $4 \times 4$ matrix $\mathcal{A} = [\mathcal{B}, v]$ with linear (resp. quadratic) entries in the first, second, third (resp. fourth) column. As we saw in Example 6.17,

$$\dim W(0^4; 1^3, 2; 2) = 663 = \dim(\chi) \text{ Hilb}(\mathbb{P}^{15})$$

where $X = \text{Proj}(\mathcal{A})$ and $A = R/I_3(\mathcal{A})$. Let us check it with Macaulay2.

Macaulay2, version 1.9.2
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

```plaintext
i1 : kk=ZZ/3001;
i2 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15];
i3 : IA3=minors(3,matrix{{x0,x1,x2,x3^2},{x4,x5,x6,x7^2},{x8,x9,x10,x11^2},
{x12,x13,x14,x15^2}});
i4 : nA=Hom(IA3,R^1/IA3); apply(-3..5,i->hilbertFunction(i, nA))
o5 = (0, 4, 73, 663, 4087, 19418, 76153, 257315, 771393)
i6 : A=R/IA3;
i7 : IAIA=IA3*IA3; CoNA=IA3/IAIA; % conormal module of A
i9 : Ext^1(CoNA**A,A)==0
o9 = true
So the Hilbert scheme Hilb(\mathbb{P}^{15}) is smooth at (X) by o9 of dimension 663 by o5.
i10 : use R
i11 : IB3=minors(3,matrix{{x0,x1,x2},{x4,x5,x6},{x8,x9,x10},{x12,x13,x14}});
i12 : nB=Hom(IB3,R^1/IB3); apply(-3..5,i->hilbertFunction(i, nB))
o13 = (0, 0, 12, 168, 1260, 6720, 28560, 102816, 325584)
i14 : IAB=IA3/IB3;
i15 : Fibp2=Hom(IAB,R^1/IA3); apply(-3..5,i->hilbertFunction(i, Fibp2))
o16 = (0, 4, 61, 495, 2830, 12742, 47997, 157183, 459628)
i17 : Fibp1=Hom(IB3,IAB); apply(-3..5,i->hilbertFunction(i, Fibp1))
o18 = (0, 0, 0, 0, 3, 44, 404, 2684, 13819)
So dim W(0^4; 1^3, 2; 2) = 168 (by o13) + 495 (by o16) - 0 (by o18) = 663.
```
Example 11.2. As in Example 6.18(i) we consider a $3 \times (c + 2)$ generic matrix $A$, $1 \leq c \leq 7$ and the ideal generated by all $2 \times 2$ minors of $A$. Let us start with the case $c = 1$.

```plaintext
i55 : kk=ZZ/101; %COMPUTED ALSO OVER : kk=ZZ/701; AND : kk=ZZ/3001;
i56 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8]
i59 : IB2=minors(2,matrix{{x0,x1},{x3,x4},{x6,x7}});
i60 : IA2=minors(2,matrix{{x0,x1,x2},{x3,x4,x5},{x6,x7,x8}});
i62 : codim IA2
  o62 = 4
i63 : dim IA2
  o63 = 5
i64 : nB=Hom(IB2,R^1/IB2); apply(-3..5,i->hilbertFunction(i, nB))
o65 = (0, 0, 6, 42, 168, 504, 1260, 2772, 5544)
i68 : nA=Hom(IA2,R^1/IA2); apply(-3..5,i->hilbertFunction(i, nA))
o69 = (0, 0, 9, 64, 225, 576, 1225, 2304, 3969)
i70 : IAB=IA2/IB2;
i71 : Fibp1=Hom(IB2,IAB); apply(-3..5,i->hilbertFunction(i, Fibp1))
o72 = (0, 0, 0, 2, 33, 168, 560, 1476, 3339)
```

Only the value 2 in o72 is needed in Example 6.18 because then (6.12) holds. As we have seen in section 6, $\dim(nA)_0 = 8(c + 2)^2 - 8$, which is the number 64 in o69, so as a Macaulay2 check; $8(c + 2)^2 - 8 = 64$ for $c=1$. Similarly as a Macaulay2 check; we also have $\dim(nB)_0 = 42$ which agrees with the result obtained in the paper.

For $c = 2$, we have:

```plaintext
i1 : kk=ZZ/101; % COMPUTED ALSO OVER : kk=ZZ/701; AND : kk=ZZ/3001;
i8 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11]
i19 : IA2=minors(2,matrix{{x0,x1,x2,x9},{x3,x4,x5,x10},{x6,x7,x8,x11}});
i20 : IB2=minors(2,matrix{{x0,x1,x2},{x3,x4,x5},{x6,x7,x8}});
i27 : nA=Hom(IA2,R^1/IA2); apply(-3..5,i->hilbertFunction(i, nA))
o28 = (0, 0, 12, 120, 540, 1680, 4200, 9072, 17640)
i31 : IAB=IA2/IB2;
i32 : Fibp1=Hom(IB2,IAB); apply(-3..5,i->hilbertFunction(i, Fibp1))
o33 = (0, 0, 0, 3, 81, 525, 2103, 6450, 16614)
```

Again only the value 3 in o33 is needed in Example 6.18 because then (6.12) holds. We use Macaulay2 to check: $\dim(W(0^3; T^4; 2)) = 8(c + 2)^2 - 8 = 120$ (see value 120 in o28). Analogously, for $3 \leq c \leq 7$, we verify (not always over all three fields above, and sometimes $kk = ZZ/13$ is used) with the code above that (6.12) holds and then check $\dim(W(0^3; T^4; 2)) = 8(c + 2)^2 - 8 = 120$. All of Example 6.18 is verified using (6.12).

When the size of the matrices become large, it is much faster to verify formula (8.9) as we do in Example 11.6.
Example 11.3. In Example 6.24 we consider a general $3 \times 3$ matrix $A = [B, v]$ with degree matrix $egin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, and we let $A$ and $B$ be given by the ideal of all $2 \times 2$ minors of $A$ and $B$.

GENERAL DETERMINANTAL, $\dim A = 5$ ($A = R/IA^2$)

i1 : kk=ZZ/101;
i2 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8]
i3 : IA2=minors(2,matrix{{x0,x1,x2^2},{x3,x4,x5^2},{x6,x7,x8^2}});
i4 : IB2=minors(2,matrix{{x0,x1},{x3,x4},{x6,x7}});
i5 : nA=Hom(IA2,R^{-1}/IA2); apply(-3..4,i->hilbertFunction(i, nA))
o6 = (0, 3, 31, 152, 502, 1286, 2776, 5312)
i7 : IAIA=IA2*IA2; CoNA=IA2/IAIA; A=R/IA2;
i10 : Ext^1(CoNA**A,A)
o10 = 0
o10 : A-module

So the Hilbert scheme $\text{Hilb}(P^8)$ is smooth at $(X = \text{Proj}(A))$ by o10 of dimension 152 by o6.

i11 : use R
i12 : nB=Hom(IB2,R^{-1}/IB2); apply(-3..4,i->hilbertFunction(i, nB))
o13 = (0, 0, 6, 42, 168, 504, 1260, 2772)
i14 : IAB=IA2/IB2;
i15 : Fibp2=Hom(IAB,R^{-1}/IA2); apply(-3..4,i->hilbertFunction(i, Fibp2))
o16 = (0, 3, 25, 110, 336, 815, 1693, 3150)
i5 : MI=coker matrix{{x0,x1,x2^2},{x3,x4,x5^2},{x6,x7,x8^2}};
i6 : Fibp2t=MI**R^{-1}/IA2; apply(-3..5,i->hilbertFunction(i, Fibp2t))
o7 = (0, 0, 3, 25, 110, 336, 815, 1693)
i17 : Fibp1=Hom(IB2,IAB); apply(-3..4,i->hilbertFunction(i, Fibp1))
o18 = (0, 0, 0, 2, 33, 177, 610)

By Proposition 6.11, $\dim W(0^3;1^2,2;2) = 42$ (by o13) + 110 (by o16 or o7) - 0 (by o18) = 152, thus $\overline{W}(0^3;1^2,2;2)$ is a generically smooth component of $\text{Hilb}(P^8)$ by o6. The case $\dim A = 4$ is computed in the same way.

Now $\dim A = 3$:

i18 : R=kk[x0,x1,x2,x3,x4,x5,x6]
i19 : l0=random(1,R); l1=random(1,R);
i21 : m0=random(2,R); m1=random(2,R);
i23 : IA2=minors(2,matrix{{x0,x1,x2^2},{x3,x4,x5^2},{x6,10,m0}});
i24 : IB2=minors(2,matrix{{x0,x1},{x3,x4},{x6,10}});
i25 : nA=Hom(IA2,R^{-1}/IA2); apply(-3..4,i->hilbertFunction(i, nA))
o26 = (0, 3, 25, 94, 230, 434, 706, 1046)
i28 : nB=Hom(IB2,R^{-1}/IB2); apply(-3..4,i->hilbertFunction(i, nB))
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\[o_{29} = (0, 0, 6, 30, 90, 210, 420, 756)\]

\[i_{30} : \text{IAB=IA2/IB2;}\]

\[i_{31} : \text{Fibp2=Hom(IA2,R}^{-1}/\text{IB2); apply(-3..4,i->hilbertFunction(i, Fibp2))}\]

\[o_{32} = (0, 3, 19, 63, 141, 253, 399, 579)\]

\[i_{9} : \text{MI=coker matrix\{\{x0,x1,x2^2\},\{x3,x4,x5^2\},x6,10,m0\};}\]

\[i_{10} : \text{Fibp2t=MI**R}^{-1}/\text{IA2; apply(-3..5,i->hilbertFunction(i, Fibp2t))}\]

\[o_{11} = (0, 0, 0, 3, 19, 63, 141, 253, 399)\]

\[i_{33} : \text{Fibp1=Hom(IB2,IA2); apply(-3..4,i->hilbertFunction(i, Fibp1))}\]

\[o_{34} = (0, 0, 0, 2, 29, 113, 289)\]

\[i_{35} : \text{IAIA=IA2*IA2; CoNA=IA2/IAIA; A=R/IA2;}\]

\[i_{38} : \text{EA=Ext}^5(\text{CoNA},\text{R}); \quad \% \text{A is GORENSTEIN, so EA(-7)= Ext}^{-1}(\text{CoNA}^{\ast}A,A)\]

\[i_{39} : \text{apply(-11..-4,i->hilbertFunction(i, EA))}\]

\[o_{39} = (0, 2, 2, 0, 0, 0, 0, 0)\]

By Proposition 6.11 \[\dim W(0; 3; 1; 2, 2; 2) = 30\] (in \(o_{29}\)) + 63 (in \(o_{32}\) or \(o_{11}\)) = 93 while \(\dim_{[X]} \text{Hilb}(\mathbb{P}^6)\) = 94 by \(o_{26}\) and \(o_{39}\), thus \(W(0; 3; 1; 2; 2; 2)\) is of codimension 1 in \(\text{Hilb}(\mathbb{P}^6)\). Using the isomorphism in \(o_{38}\) finding \(\text{Ext}^1_{\Lambda}(I_{A}/I_{A}^3, A)\) allowed a much faster computation.

The case \(\dim A = 2\) is computed in the same way and shows \(\dim W(0; 1; 2; 2; 2; 2) = 3\) in \(\text{Hilb}(\mathbb{P}^5)\).

Example 11.4. In Example 7.13(i) \(\Lambda = [\mathcal{B}, v]\) is a general \(3 \times 5\) matrix with linear entries, and let \(A\) and \(B\) be the quotients of \(R\) defined by the \(2 \times 2\) minors of \(\Lambda\) and \(\mathcal{B}\), respectively. So \(\dim A = 2\). Let us check that \(W(0; 3; 1; 2; 2)\) is not a generically smooth component of \(\text{Hilb}(\mathbb{P}^9)\) by using (7.2).

GENERAL DETERMINANTAL, \(A\) defined by \(2 \times 2\) minors of a \(3 \times 5\) matrix. \(\dim A = 2\)

\[i_{1} : \text{kk=ZZ/13;}\]

\[i_{3} : \text{R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8,x9];}\]

\[i_{4} : \text{10=random(1,R); 11=random(1,R); 12=random(1,R);}\]

\[i_{7} : \text{13=random(1,R); 14=random(1,R);}\]

\[i_{12} : \text{IA2=minors(2,matrix\{\{x0,x1,x2,x3,10\},\{x4,x5,x6,14,11\},\{x7,x8,x9,13,12\}\);}\]

\[i_{13} : \text{IB2=minors(2,matrix\{\{x0,x1,x2,x3\},\{x4,x5,x6,14\},\{x7,x8,x9,13\}\);}\]

\[i_{14} : \text{dim IA2}\]

\[o_{14} = 2\]

\[i_{15} : \text{dim IB2}\]

\[o_{15} = 4\]

\[i_{16} : \text{codim IA2}\]

\[o_{16} = 8\]

\[i_{17} : \text{nB=Hom(IB2,R}^{-1}/\text{IB2); apply(-3..5,i->hilbertFunction(i, nB))}\]

\[o_{18} = (0, 0, 12, 96, 312, 720, 1380, 2352, 3696)\]

\[i_{25} : \text{IAB=IA2/IB2;}\]

\[i_{18} : \text{Fibp2=Hom(IA2,R}^{-1}/\text{IA2); apply(-3..5,i->hilbertFunction(i, Fibp2))}\]
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\[ o_{19} = (0, 0, 3, 25, 55, 85, 115, 145, 175) \]
\[ i_{26} : \text{Fibp1} = \text{Hom}(\text{IB2}, \text{IAB}); \text{apply}(-3..5, i->\text{hilbertFunction}(i, \text{Fibp1})) \]
\[ o_{27} = (0, 0, 0, 4, 127, 445, 1015, 1897, 3151) \]
\[ i_{28} : n_A = \text{Hom}(\text{IA2}, R^1/\text{IA2}); \text{apply}(-3..5, i->\text{hilbertFunction}(i, n_A)) \]

Aborted, but we try again.

\[ i_2 : \text{kk} = \mathbb{Z}/3; \]
\[ i_3 : R = \text{kk}[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \]
\[ i_4 : l_0 = \text{random}(1, R); l_1 = \text{random}(1, R); l_2 = \text{random}(1, R); \]
\[ i_7 : l_3 = \text{random}(1, R); l_4 = \text{random}(1, R); \]
\[ i_{10} : \text{IA2} = \text{minors}(2, \text{matrix}\{\{x_0, x_1, x_2, x_3, l_0\}, \{x_4, x_5, x_6, l_4, l_1\}, \{x_7, x_8, x_9, l_3, l_2\}\}); \]
\[ i_{11} : n_A = \text{Hom}(\text{IA2}, R^1/\text{IA2}); \text{apply}(-3..5, i->\text{hilbertFunction}(i, n_A)) \]
\[ o_{12} = (0, 0, 15, 120, 240, 360, 480, 600, 720) \]

Hence \( W(0^3; 1^{5}; 2) \) is not a generically smooth component because 120 (in o12) \( \neq 96 \) (in o18) + 25 (in o19) - 4 (in o27), i.e. (7.2) is not satisfied (because \( \dim(MI \otimes A)_1 = \dim(\text{Fibp2})_0 \)). But to apply Remark 7.4(2) we need to show that the conditions of Theorem 7.1 hold. Let us check that \( \text{Ext}^{1}_{C}(I_{C}/I_{C}^{2}, I_{B}/C) = 0 \) because then every deformation of \( B \) comes from deforming \( B \).

\[ i_{10} : \text{IB2} = \text{minors}(2, \text{matrix}\{\{x_0, x_1, x_2, x_3\}, \{x_4, x_5, x_6, 14, 11\}, \{x_7, x_8, x_9, 13, 12\}\}); \]
\[ i_{13} : \text{IC2} = \text{minors}(2, \text{matrix}\{\{x_0, x_1, x_2\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}\}); \]
\[ i_{14} : \text{IBC} = \text{IB2}/\text{IC2}; \]
\[ i_{15} : \text{ICIC} = \text{IC2*IC2}; \text{CoNC} = \text{IC2}/\text{ICIC}; C = R/\text{IC2}; \]
\[ i_{19} : \text{Ext}^{-1}(\text{CoNC}*C, \text{IBC}*C) == 0 \]
\[ o_{19} = \text{true} \]

The computations by Macaulay2 in Example 7.13 often check if (7.2) holds, or even simpler for matrices in the linear case where we only need to compute \( \dim(\text{Fibp1})_0 \) to see if \( \dim(W(0^t; 1^{t+c-1}; r)) = \lambda_c \) holds. If so then \( \lambda_c = \dim(nA)_0 \). If (7.2) holds and the other conditions of Theorem 7.1 hold, then \( W(0^t; 1^{t+c-1}; r) \) is a generically smooth component. And conversely for (7.2) by Remark 7.4(2), as in the example above. Also in the linear case verifying \( \dim(W(0^t; 1^{t+c-1}; r) = \lambda_c < \dim(nA)_0 \) and \( \text{Hilb}(\mathbb{P}^n) \) smooth at \( (X) \) imply a converse.

Example 11.5. In Example 7.20 we consider \( 3 \times (c + 2) \) linear and non-linear matrices \( A = [B, v] \), \( 1 \leq c \leq 4 \) with number of variables of \( R \) at least as in the generic case. The ideals of \( A \) and \( B \) are generated by all \( 2 \times 2 \) minors of \( A \) and \( B \).

\[ \text{GENERIC 3 by 4 matrix} \ cA = [\cB, v] \text{ defining } A \]
\[ i_1 : \text{kk} = \mathbb{Z}/101; \]
\[ i_2 : R = \text{kk}[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}] \]
\[ i_3 : \text{IA2} = \text{minors}(2, \text{matrix}\{\{x_0, x_1, x_2, x_3\}, \{x_4, x_5, x_6, x_7\}, \{x_8, x_9, x_{10}, x_{11}\}\}); \]
\begin{verbatim}
i4 : IB2=minors(2,matrix{{x0,x1,x2},{x4,x5,x6},{x8,x9,x10}});
i6 : IAB=IA2/IB2;
i7 : nB=Hom(IB2,R^{-1}/IB2); pdim nB
  o9 = 6
i10 : B=R^{-1}/IB2; pdim B
  o11 = 4
i12 : nA=Hom(IA2,R^{-1}/IA2); pdim nA
  o14 = 6
i15 : A=R^{-1}/IA2; pdim A
  o16 = 6
i62 : Fibp1=Hom(IB2,IA2/IB2); pdim Fibp1
  o63 = 6
\end{verbatim}

NON-LINEAR, thus NON-GENERIC $A$ and $B$.
\begin{verbatim}
i3 : IA2=minors(2,matrix{{x0,x1,x2^2,x3^3},{x4,x5,x6^2,x7^3},{x8,x9,x10^2,x11^3}});
i4 : IB2=minors(2,matrix{{x0,x1,x2^2},{x3,x4,x5^2},{x6,x7,x8^2}});
i5 : IAB=IA2/IB2;
i6 : nB=Hom(IB2,R^{-1}/IB2); pdim nB
  o7 = 6
i8 : B=R^{-1}/IB2; pdim B
  o9 = 4
i10 : nA=Hom(IA2,R^{-1}/IA2); pdim nA
  o11 = 6
i12 : A=R^{-1}/IA2; pdim A
  o13 = 6
i14 : Fibp1=Hom(IB2,IA2/IB2); pdim Fibp1
  o15 = 6
\end{verbatim}

LINEAR, NON-GENERIC $A$ of $\dim A = 3$.
\begin{verbatim}
i1 : kk=ZZ/101; % Computed also over kk=ZZ/11;
i2 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8]
i3 :  l0=random(1,R); l1=random(1,R); l2=random(1,R);
i6 : IB2=minors(2,matrix{{x0,x1,x2^2,x3^3},{x4,x5,x6^2,x7^3},{x8,x9,x10^2,x11^3}});
i7 : IA2=minors(2,matrix{{x0,x1,x2^2},{x3,x4,x5^2},{x6,x7,x8^2}});
i8 : IAB=IA2/IB2;
i9 : nB=Hom(IB2,R^{-1}/IB2); pdim nB
  o10 = 6
i11 : B=R^{-1}/IB2; pdim B
  o12 = 4
i13 : nA=Hom(IA2,R^{-1}/IA2); pdim nA
  o14 = 6
\end{verbatim}
From the projective dimensions and Auslander-Buchsbaum’s formula, we get that codepth \( n_B = 2 \) (\( B \) is Gorenstein) while codepth \( n_A = 0 \) (\( c = 2 \)) and codepth \( \text{Fibp1} = 2 \) (\( r = 2 \)) in the linear, as well as in the non-linear case, confirming Conjecture 7.15 and Proposition 7.18. All examples in Example 7.20 are computed in this way (or by using “betti res” in replacement of “pdim”).

**Example 11.6.** In Example 8.5, \( A = (x_{ij}) \) is a \( 3 \times (c + 2) \) generic matrix, \( 6 \leq c \leq 18 \) and \( I_A = \ker(R \to A) \) the ideal generated by all \( 2 \times 2 \) minors of \( A \). For \( c = 6 \) we get:

\[
\text{o64 : codim I_A} = 14
\]

\[
\text{o66 : IG2} = \text{minors}(2, \text{matrix}\{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, a, b, c, d, e, f\})
\]

\[
\text{o67 : IAB} = \text{IA}^2/\text{IB}^2
\]

\[
\text{o68 : nGIAB} = \text{Hom}(\text{IG}^2, \text{R}^1/\text{IA}^2); \text{apply}(-3..5, i->\text{hilbertFunction}(i, nGIAB))
\]

\[
\text{o75 : nGA} = \text{Hom}(\text{IG}^2, \text{R}^1/\text{IA}^2); \text{apply}(-3..5, i->\text{hilbertFunction}(i, nGA))
\]

Only the value 3 in o74 is needed in Example 8.5 to get \( \dim \overline{W}(0^3; 1^{c+2}; 2) = \lambda_c \) because then (8.9) holds. For a Macaulay2 check, to see if \( \overline{W}(0^3; 1^{c+2}; 2) \) is a generically smooth component (which we know is true by Proposition 7.18), note that (7.2) is equivalent to (8.14). To check (8.14) we subtract: 187 (in o70) - 184 (in o76) = 3 (in o74). Analogously we compute the cases \( 7 \leq c \leq 18 \) and the remainder of Example 8.5 by verifying (not always over all three fields above) that (8.9) holds. Thus we get \( \dim \overline{W}(0^3; 1^{c+2}; 2) = \lambda_c \). As a Macaulay2 check we also verify (8.14) which should hold by our results.

**Example 11.7.** In Example 8.17(i), \( A \) is a \( 3 \times 6 \) linear matrix, and we successively delete columns to get a flag of determinantal rings. In such a situation it is much faster to verify (8.14) than using (7.2) even though they are equivalent.
\textbf{Example 11.8.} In Example 9.8, \(A\) is a 4 \(\times\) 4 matrix, and five smooth deformation functors are considered. By Theorem 9.3 the deformation functors of \( M_r := \bigwedge^r M \) and of \( R/I_{t-r+1}(\varphi^*) \) are isomorphic for fixed \( r \). By Theorem 9.6 they should further be isomorphic to the deformation functor of \( MI \) (and if \( c > 1 \) also to the deformation functor of \( R/I_t(\varphi^*) \), but in our example \( c = 1 \). Theorem 9.6 are, however, true under the assumption: the compared functors have the same dimension of tangent spaces. Let us check this assumption by using Macaulay2.

\textbf{GENERIC DETERMINANTAL, five isomorphic deformation functors}

\begin{verbatim}
i1 : kk=ZZ/101;
i2 : R=kk[x0,x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12]
i3 : M1=coker matrix{{x0,x1,x2,x3},{x4,x5,x6,x7},{x8,x9,x10,x11}};
i4 : M2=exteriorPower(2,M1);
i5 : M3=exteriorPower(3,M2);
i6 : M4=exteriorPower(4,M3);
i7 : M5=exteriorPower(5,M4);
i8 : M6=exteriorPower(6,M5);\end{verbatim}

For the deformation of \( M_1 \) we use: \(\text{dim } H^1(M_1) = 3\), whence \(\text{dim } W(0^4;1^6;2) = 3\), and every deformation of \( M_1 \) comes from deforming its matrix by Theorem 8.15. Similarly to check (8.14) for \( B \rightarrow A \) we add: \(\text{dim } H^1(B^1) = 3\), whence \(\text{dim } W(0^4;1^6;2) = 3\), and \(W(0^4;1^6;2)\) is a generically smooth component of \(\text{Hilb}(\mathbb{P}^4)\) by Theorem 8.15.
So the five deformation functors we consider have all tangent space of dimension 225 (see o12, o14, o16, o21, o23), so their corresponding deformation functors are isomorphic.

In the non-linear case not all 5 deformation functors are necessarily isomorphic because when we take all \( b_i = 0, a_1 = a_2 = 1 \) and \( a_3 = a_4 = 2 \) we get \( s_2 = \sum_{i=1}^{5-2} a_i = 4 \) and \( s_3 = \sum_{i=1}^{5-3} a_i = 2 \), whence \( a_4 < s_2 \) is satisfied while \( a_4 < s_3 \) is not. So according to Theorem 9.6 and Remark 9.7 the deformation functors of \( A_2 = R/I_3(\varphi^*) \) and \( M_2 \) which are isomorphic by Theorem 9.3, should further be isomorphic the deformation functor of \( MI \), while those of \( A_3 = R/I_2(\varphi^*) \) and \( M_3 \) are isomorphic by Theorem 9.3. Let us check it.

NON-LINEAR: not all 5 deformation functors are isomorphic
So three of the deformation functors we consider have the same tangential dimension, namely equal to 1129 (see o26, o28, o40). Thus their corresponding deformation functors are isomorphic. Moreover, the tangent space dimensions in o30 and o38 coincide, so their deformation functors are isomorphic.

In the non-linear case, again 5 isomorphic deformation functors are expected in the case that all $b_i = 0$, $a_1 = 1$ and $a_2 = a_3 = a_4 = 2$. We have $s_2 = 5$ and $s_3 = 3$, whence both $a_4 < s_3$ and $a_4 < s_2$ are satisfied and Theorem 9.6 predicts that all five deformation functors are isomorphic. Let us check it.

NON-LINEAR, but again all 5 isomorphic deformation functors are isomorphic

% 5 functors with the same tangential dimension = 1623 (o49, o51, o53, o58, o60)
References

[1] K. Akin, D. Buchsbaum and J. Weyman, Resolutions of determinantal ideals: The submaximal minors, Advances in Mathematics 39 (1981), 1–30.

[2] K. Akin, D. Buchsbaum and J. Weyman, Schur functors and Schur complexes, Advances in Mathematics 44 (1982), 207–278.

[3] A. Beauville, Determinantal hypersurfaces, Michigan Mathematical Journal 48 (2000), 39–64.

[4] D. Buchsbaum and D. Eisenbud, Generic free resolutions and a family of generically perfect ideals, Advances in Mathematics 18 (1975), 245–301.

[5] L. Busé, Resultants of determinantal varieties, Journal of Pure and Applied Algebra 193 (2004), 71–79.

[6] W. Bruns, Generic maps and modules, Compositio Mathematica 47 (1982), no. 2, 171–193.

[7] W. Bruns, The Eisenbud-Evans generalized principal ideal theorem and determinantal ideals, Proceedings of the American Mathematical Society 83 (1981), 19-24.

[8] W. Bruns and U. Vetter, Determinantal rings, Springer-Verlag, Lectures Notes in Mathematics 1327, New York/Berlin, 1988.

[9] T. de Jong and D. van Straten, Deformations of normalization of hypersurfaces, Mathematische Annalen 288 (1990), 527–547.

[10] L.E. Dickson Determination of all general homogeneous polynomials expressible as determinants with linear elements, Transactions of the American Mathematical Society 22 (1921), 167—179.

[11] J.A. Eagon and M. Hochster, Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci, American Journal of Mathematics 93 (1971), 1020–1058.

[12] J.A. Eagon and D.G. Northcott, Ideals defined by matrices and a certain complex associated with them, Proceedings of the Royal Society of London 269 (1962), 188-204.

[13] D. Eisenbud, Commutative Algebra. With a view toward algebraic geometry, Springer-Verlag, Graduate Texts in Mathematics 150 (1995).

[14] D. Eisenbud and J. Harris, On Varieties of minimal degree, Proceedings of Symposia in Pure Mathematics 46 (1976), 3–24.

[15] D. Eisenbud, J. Koh and M. Stillman, Determinantal equations for curves of high degree, American Journal of Mathematics 110 (1988), 513–539.

[16] G. Ellingsrud, Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbb{P}^n$ a cône de Cohen-Macaulay, Annales Scientifiques de l’ École Normale Supérieure 8 (1975), no. 4, 423–432.

[17] D. Eisenbud and J. Herzog, The classification of homogeneous Cohen-Macaulay rings of finite representation type, Mathematische Annalen 280 (1988), no. 2, 347–352.

[18] D. Faenzi and L. Fania, On the Hilbert scheme of varieties defined by maximal minors, Mathematical Research Letters 21 (2014), no. 2, 297–311.

[19] J. Fogarty, Algebraic families on an algebraic surface, American Journal of Mathematics 90 (1968), 511-521.

[20] W. Fulton and J. Harris, Representation Theory A First Course, GTM 129 (1991), Springer-Verlag, New York.

[21] G. Gotzmann Topologische Eigenschaften von Hilbert-strata. Habilitationsschrift (1993).

[22] H. Grassmann, Die stereometrischen Gleichungen dritten Grades, und die dadurch erzeugten Oberflächen, Journal für die Reine und Angewandte Mathematik 49 (1855), 47–65.

[23] D. Grayson and M. Stillman, Macaulay 2-a software system for algebraic geometry and commutative algebra, available at [http://www.math.uiuc.edu/Macaulay2/].

[24] A. Grothendieck, Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux, North Holland, Amsterdam (1968).

[25] M. T. Guilliksen and O.G. Negård, Un complexe résolvant pour certains idéaux déterminants, C.R. Acad. Sc. Paris 274 (1972), 16-18.
[26] J. Harris, *Algebraic Geometry, A First Course*, GTM 133 (1995), Springer-Verlag, New York.
[27] L.T. Hoa, *On minimal free resolutions of projective varieties of degree = codimension + 2*, Journal of Pure and Applied Algebra 87 (1993), 241–250.
[28] R. Ile, *Obstructions to deforming modules*, PhD thesis, University of Oslo, 2001.
[29] J.O. Kleppe, *Maximal families of Gorenstein algebras*, Transactions of the American Mathematical Society 358 (2006), no. 7, 3133–3167.
[30] J.O. Kleppe, *Unobstructedness and dimension of families of Gorenstein algebras*, Collectanea Mathematica 58 (2007), no. 2, 199-238.
[31] J.O. Kleppe, *Families of low dimensional determinantal schemes*, Journal of Pure and Applied Algebra 215 (2011), 1711–1725.
[32] J.O. Kleppe, *Deformations of modules of maximal grade and the Hilbert scheme at determinantal schemes*, Journal of Algebra 338 (2014), 776–800.
[33] J.O. Kleppe, *Families of artinian and low dimensional determinantal rings*, Journal of Pure and Applied Algebra 222 (2018), 610–635.
[34] J.O. Kleppe and R.M. Miró-Roig, *Dimension of families of determinantal schemes*, Transactions of the American Mathematical Society 357 (2005), 2871-2907.
[35] J.O. Kleppe and R.M. Miró-Roig, *Ideals generated by submaximal minors*, Algebra Number Theory 3 (2009), no. 4, 367-392.
[36] J.O. Kleppe and R.M. Miró-Roig, *Families of determinantal schemes*, Proceedings of the American Mathematical Society 139 (2011), no. 11, 3831–3843.
[37] J.O. Kleppe and R.M. Miró-Roig, *On the normal sheaf of determinantal varieties*, Journal für die Reine und Angewandte Mathematik 719 (2016), 173–209. [http://dx.doi.org/10.1515/crelle-2014-0041](http://dx.doi.org/10.1515/crelle-2014-0041).
[38] J.O. Kleppe and R.M. Miró-Roig, *Schur powers of the cokernel of a graded morphism*, in preparation.
[39] J.O. Kleppe, J.C. Migliore, R.M. Miró-Roig, U. Nagel and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Memoirs of the American Mathematical Society 732, (2001).
[40] J. Kreuzer, J.C. Migliore, U. Nagel and C. Peterson, *Determinantal schemes and Buchsbaum-Rim sheaves*, Journal of Pure and Applied Algebra 150 (2000), 155-174.
[41] D. Laksov, *Deformation of determinantal schemes*, Compositio Mathematica 30 (1975), 273–292.
[42] A. Lascoux, *Syzygies des variétés déterminantales*, Advances in Mathematics 30 (1978), 202-237.
[43] O.A. Laudal, *Formal Moduli of Algebraic Structures*, Lecture Notes in Mathematics 754, Springer–Verlag, New York, 1979.
[44] Y. Ma, *The first syzygies of determinantal ideals*, Journal of Pure and Applied Algebra 85 (1993), 93–103.
[45] R.M. Miró-Roig, *Determinantal Ideals*, Birkhäuser, Progress in Mathematics 264 (2008).
[46] D.G. Northcott, *Finite free resolutions*, Cambridge Tracts in Mathematics 71 (1976), Cambridge University Press, Cambridge-New York-Melbourne, xii+271 pp.
[47] M. Schaps, *Versal determinantal deformations*, Pacific Journal of Mathematics 107 (1983), 213–221.
[48] J. Sidman and G. G. Smith, *Finite determinantal equations for all projective schemes*, Algebra and Number Theory 5 (2011), 1041–1061.
[49] M. Schlessinger, *Functors of artin rings*, Transactions of the American Mathematical Society, 130 (1968), no. 2, 208–222.
[50] R. Vakil, *Murphy’s law in algebraic geometry: Badly-behaved deformation spaces*, Inventiones Mathematicae 164 (2006), no. 3, 569 - 590
[51] U. Vetter, *Exterior powers of the cokernel of a generic map*, Commutative algebra (Trieste, 1992), 303-315, World Sci. Publ., River Edge, NJ, 1994.
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