Nonlinear wave solutions of the Kudryashov–Sinelshchikov dynamical equation in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity

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Abstract
In this research, we constructed the exact travelling and solitary wave solutions of the Kudryashov–Sinelshchikov (KS) equation by implementing the modified mathematical method. The KS equation describe the phenomena of pressure waves in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity. Our new obtained solutions in the shape of hyperbolic, trigonometric, elliptic functions including dark, bright, singular, combined, kink wave solitons, travelling wave, solitary wave and periodic wave. We showed the physical interpretation of obtained solutions by three-dimensional graphically. These new constructed solutions play vital role in mathematical physics, optical fiber, plasma physics and other various branches of applied sciences.

1. Introduction
Kudryashov and Sinelshchikov (KS) in (2010) first time introduced a nonlinear evolution equation with the help of theoretical and experimental work [1]. The KS equation describe the phenomena of pressure waves in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity. The KS equation given [1], as

\[ u_t + \gamma u u_x + u_{xxx} - \varepsilon (u u_{xx})_x - \kappa u_x u_{xx} - \nu u_{xxx} - \delta (u u_x)_x = 0. \] \tag{1}

In Equation (1), \( u(x, t) \) is a function which denote to density, heat transfer and viscosity models, \( \gamma, \varepsilon, \kappa, \nu, \delta \) are real constant parameters. If \( \kappa = \varepsilon = \delta = \nu = 0 \), then Equation (1), change into Korteweg-de Vries equation [2], as

\[ u_t + \gamma u u_x + u_{xxx} = 0. \] \tag{2}

If \( \kappa = \varepsilon = \delta = 0 \), then Equation (1), reduced into Korteweg-de Vries Burgers equation [3], as

\[ u_t + \gamma u u_x + u_{xxx} - \nu u_{xx} = 0. \] \tag{3}

The Equation (1), is the general form of KdV equation and KdVB equation. When \( \gamma = \varepsilon = 1, \nu = \delta = 0 \), then Equation (1), obtain [4], as

\[ u_t + u u_x + u_{xxx} - (u u_{xx})_x - \kappa u_x u_{xx} = 0. \] \tag{4}

The author using the modification form of truncated expansion technique to investigate the solitary wave solutions of Equation (4), under the following condition \( \kappa = -3, \nu = -4 \), [4]. Recently, many researchers found different types of solutions of KS equation by applying different techniques at different conditions of parameters, as in these references [5–16].

In the last few decades, a lots of research have been done to determine the exact solutions of nonlinear evolution equations (NLEEs). The investigations of exact solutions of NLEEs have a great deal to know the structure, provide better information and its applications. Therefore, to calculate the exact and solitary solutions of nonlinear partial differential equations (NLPDEs), many researchers and mathematician introduced a lot of methods. Such as, Hirota bilinear method, Exp-function technique, Jacobian elliptic method, Trial equation method, extended simple equation method, F-expansion method [17–29] and many researcher in auxiliary equation mapping method, improved extended Fan-Subequation method, Darboux transformation, sinh-cosh method, sech-tanh method, extended direct algebraic method, extended auxiliary equation mapping method, improved F-expansion method [17–29] and many researcher in applications of mathematical methods [30–42].

In this current work, we investigated the exact travelling and solitary solutions of nonlinear KS equation by implementing the modified mathematical method [43–50].
2. Description of proposed method

Here we describe the main future of the modified mathematical technique for finding the solutions of nonlinear partial differential equations (PDEs). We consider the general form of nonlinear PDEs as

\[ Q(u, ut, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0. \]  

Where \( Q \) denote to polynomial function of \( u(x, t) \) and their derivatives. We explain the features of modified mathematical technique as

**Step 1.** We consider the transformations of travelling wave as

\[ u(x, t) = U(\zeta), \quad \zeta = (x - \lambda t). \]  

In Equation (6), \( \lambda \) is the wave frequency. We obtain the ODE of Equation (5), as

\[ R(U, U', U'', \ldots) = 0. \]  

In Equation (7), \( R \) denote to polynomial function in \( U(\zeta) \) and their derivative.

**Step 2.** We consider the trial solution of Equation (7), as

\[ U(\zeta) = \sum_{i=0}^{N} a_i \Psi(\zeta)^i + \sum_{i=1}^{N} b_i - i \Psi(\zeta)^i \]

\[ + \sum_{i=2}^{N} c_i \Psi(\zeta)^{i-2} \Psi'(\zeta) + \sum_{i=1}^{N} d_i \left( \frac{\Psi'(\zeta)}{\Psi(\zeta)} \right)^i. \]  

Here \( (a_i, b_i, c_i, d_i) \) are constants which we calculate later, the derivatives of \( \Psi(\zeta) \) satisfy the following auxiliary equation

\[ (\Psi'(\zeta))^2 = \beta_1 \Psi^2(\zeta) + \beta_2 \Psi^3(\zeta) + \beta_3 \Psi^4(\zeta); \]

\[ \Psi''(\zeta) = \beta_1 \Psi(\zeta) + \frac{3}{2} \beta_2 \Psi^2(\zeta) + 2 \beta_3 \Psi^3(\zeta). \]  

In Equation (9), \( \beta_1, \beta_2, \beta_3 \) are real constants which are found later.

**Step 3.** We balance the terms of nonlinear and derivative of higher order in Equation (7), determined \( N \) of Equation (8).

**Step 4.** Putting Equation (8) in Equation (9) and collecting every coefficients of \( \Psi^{i}(\zeta) \Psi^{j}(\zeta) \) \((i = 1, 2, 3, \ldots ; j = 0, 1)\), then every coefficients make zero and get a system of equation, solve these system of equations using any computer software, the values of these parameters \( (a_i, b_i, c_i, d_i) \), are found.

**Step 5.** Substituting parameters values which are obtained and \( \Psi(\zeta) \) in Equation (9), then we obtain the required solutions of Equation (5).

3. KS equation

Here we apply the described technique to construct the solitary wave solutions for the KS equation.

\[ u_t + \gamma uu_x + u_{xxx} - \varepsilon (uu_{xx})_x - \kappa uu_{xx} 
- \nu u_{xx} - \delta (uu_x)_x = 0. \]  

We apply transformation of the wave

\[ u(x, t) = U(\zeta), \quad \zeta = (x - \lambda t). \]  

Substituting Equation (11) in Equation (10) and integrating once w.r to \( \zeta \) with zero integration constant, we obtain the following:

\[ -\lambda U + \frac{\gamma}{2} U^2 + U'' - \varepsilon UU'' - \kappa UU_x - \nu U - \delta UU_x = 0. \]  

We balance the term of nonlinear and derivative of higher order in Equation (12), we get \( N = 2 \). Trial solution of Equation (12), take as

\[ U(\zeta) = a_0 + a_1 \Psi(\zeta) + a_2 \Psi^2(\zeta) + \frac{b_1}{\Psi'(\zeta)} + \frac{b_2}{\Psi(\zeta)^2} + c_2 \Psi'(\zeta) + d_1 \frac{\Psi'(\zeta)}{\Psi(\zeta)} + d_2 \left( \frac{\Psi'(\zeta)}{\Psi(\zeta)} \right)^2. \]  

Substituting Equation (13) in Equation (12) and collect every coefficients of \( \Psi^{i}(\zeta) \Psi^{j}(\zeta) \) \((i = 1, 2, 3, \ldots ; j = 0, 1)\), compare every coefficients to zero. We obtain a system of equations. These system of equations solve by using the computer software Mathematica, the values of constants obtained are as follows:

**Case-I**

\[ a_0 = \frac{\sqrt{b_1^2 b_2^2 \kappa^2 + 4 \beta_1 (\beta_1 - b_1^2 b_3 \kappa^2)}}{2 \beta_1 \kappa}, \]

\[ a_1 = -\beta_2 d_2, \quad a_2 = -\beta_3 d_2, \]

\[ b_1 = b_1, b_2 = c_2 = d_1 = 0, \quad d_2 = d_2, \]

\[ \lambda = \frac{(\varepsilon + \kappa) \sqrt{b_1^2 b_2^2 \kappa^2 + 4 \beta_1 (\beta_1 - b_1^2 b_3 \kappa^2) + 2 \beta_1 \epsilon}}{2 \kappa}, \]

\[ \gamma = \beta_1 (2 \epsilon + \kappa), \quad \delta = \nu = 0. \]
Substituting Equation (14) in Equation (13), we get the solutions of Equation (10) as

\[
\begin{align*}
\mathbf{u}_1(x, t) &= \beta_1 d_2 \left( e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) + \frac{\beta_1 d_2 e^2 \csc \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{4 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2} \\
&\quad - \frac{b_1 \beta_1 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\beta_2} - \frac{\beta_2 \beta_3 d_2 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2}{\beta_2} \\
&\quad - \sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - b_1 \beta_2 k + 2b_1 (\beta_1 d_2 + 1) \\
&\quad - \frac{\sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - b_1 \beta_2 k + 2b_1 (\beta_1 d_2 + 1)}{2\beta_1 k} .
\end{align*}
\]

(15)

\[
\begin{align*}
\mathbf{u}_2(x, t) &= \frac{1}{4} \left( 4b_1 d_2 (\eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + e)^2 \\
&\quad - 2b_1 \sqrt{\beta_1} \left( \frac{e \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right) - \beta_1 d_2 \left( \frac{e \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right)^2 \\
&\quad + 2\beta_2 \sqrt{\beta_1} d_2 \left( \frac{e \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right) \\
&\quad - 2 \left( \sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - b_1 \beta_2 k + 2b_1 (\beta_1 d_2 + 1) \right) \\
&\quad - \frac{2 \left( \sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - b_1 \beta_2 k + 2b_1 (\beta_1 d_2 + 1) \right)}{2\beta_1 k} .
\end{align*}
\]

(16)

\[
\begin{align*}
\mathbf{u}_3(x, t) &= b_1 \left( - \frac{\epsilon (\eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right) + \left( \frac{\epsilon^2 (\eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right) \\
&\quad - \frac{\beta_2 d_2 \left( - \frac{\epsilon (\eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right)}{2\beta_1 k} \\
&\quad - \frac{\beta_3 d_2 \left( - \frac{\epsilon (\eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right)^2}{2\beta_1 k} \\
&\quad - \frac{\sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - b_1 \beta_2 k + 2b_1 (\beta_1 d_2 + 1)}{2\beta_1 k} .
\end{align*}
\]

(17)

Case-II

\[
a_0 = \frac{\sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} + b_1 \beta_2 k - 2b_1 (\beta_1 d_2 + 1)}{2\beta_1 k} , a_1 = -\beta_2 d_2 , a_2 = -\beta_3 d_2 , b_1 = b_1 ,
\]

\[
b_2 = c_2 = d_1 = 0 , d_2 = d_2 , \lambda = \frac{(e + \kappa) \sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} - 2b_1 \epsilon}{2\kappa} ,
\]

(18)

\[
\gamma = \beta_1 (2e + \kappa) , \delta = \nu = 0 .
\]

Substituting Equation (18) in Equation (13), we obtain the solutions of Equation (10) as

\[
\begin{align*}
\mathbf{u}_4(x, t) &= \beta_1 d_2 \left( e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) + \frac{\beta_1 d_2 e^2 \csc \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{4 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2} \\
&\quad - \frac{b_1 \beta_1 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)}{\beta_2} - \frac{\beta_2 \beta_3 d_2 (e \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2}{\beta_2} \\
&\quad + \frac{\sqrt{b_1^2 \beta_2^2 k^2 + 4b_1 (\beta_1 - b_1 \beta_3 k^2)} + b_1 \beta_2 k - 2b_1 (\beta_1 d_2 + 1)}{2\beta_1 k} .
\end{align*}
\]

(19)
\[
\begin{align*}
\frac{\partial u_5(x,t)}{\partial t} &= \frac{1}{4} \left( \frac{4\beta_1 d_2 (\eta e \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)] + e^2) + \beta_1 d_2 \left( \left( \frac{e \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]}{\eta + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} + 1 \right)^2 \right) - 2\beta_1 \left( \frac{\beta_1}{\beta_3} \frac{e \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]}{\eta + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} + 1 \right) + 2\beta_2 \sqrt{\beta_1} d_2 \left( \frac{e \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]}{\eta + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} + 1 \right) + 2 \left( \sqrt{\beta_1^2 \beta_2^2 \kappa^2 - 4\beta_1 \beta_2 \beta_3 \kappa^2 + 4\beta_1^2 + b_1 \beta_2 - 2\beta_1 (\beta_1 d_2 + 1) \right) }{\beta_1 \kappa} \right) \\
\frac{\partial u_6(x,t)}{\partial t} &= b_1 \left( \frac{- e \left( p + \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)] \right)}{e \sqrt{p^2 + 1} + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} \right) + \left( \frac{e^2}{\eta \sqrt{p^2 + 1} + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} + 1 \right) \frac{- p \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]}{\eta \sqrt{p^2 + 1} + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} \frac{- \beta_1 d_2}{\left( \frac{e \left( p + \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)] \right)}{e \sqrt{p^2 + 1} + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} \right)} - \beta_2 d_2 + \frac{\beta_3 d_2}{\left( \frac{e \left( p + \sinh [\sqrt{\beta_1} (x - \lambda t + \xi_0)] \right)}{e \sqrt{p^2 + 1} + \cosh [\sqrt{\beta_1} (x - \lambda t + \xi_0)]} \right)}^2 + \frac{\sqrt{b_1^2 \beta_2^2 \kappa^2 + 4\beta_1 (b_1 - b_2 \beta_3 \kappa^2 + b_1 \beta_2 \kappa - 2\beta_1 (\beta_1 d_2 + 1))}{\beta_1 \kappa} \right) \\
\end{align*}
\]

**Case-III**

\[
\begin{align*}
\alpha_0 &= A \beta_2 \beta_1 \beta_3 \kappa \left( \epsilon - 2\kappa \right) - 4\beta_3 \beta_1 \beta_3 \beta_2 \kappa + \beta_1 \beta_3 \kappa \left( 4\beta_3 \beta_3 \beta_2 \kappa - 2\beta_1 \beta_3 \kappa \right) + \beta_1 \beta_3 \kappa \left( 4\beta_3 \beta_3 \beta_2 \kappa - 2\beta_1 \beta_3 \kappa \right) + \beta_1 \beta_3 \kappa \left( 4\beta_3 \beta_3 \beta_2 \kappa - 2\beta_1 \beta_3 \kappa \right) \\
\alpha_1 &= -\beta_2 d_2, a_2 = -\beta_3 d_2, b_1 = \frac{2(A - 2\beta_1 \beta_2 \kappa (\epsilon + \kappa))}{\kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)}, b_2 = c_2 = d_1 = 0, \\
d_2 &= d_2, \lambda = \frac{-2\beta_2 \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)}{\beta_2 \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)} + \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right), \gamma = \beta_1 (\beta_3 \kappa), \delta = \nu = 0, \\
\end{align*}
\]

where

\[
A = \sqrt{\beta_1 \beta_2 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)} - 4\beta_1 \beta_3 \kappa. 
\]

Substituting Equation (22) in Equation (13), we get the solutions of Equation (10) as

\[
\begin{align*}
\frac{\partial u_7(x,t)}{\partial t} &= \frac{\beta_1 d_2 e^2 \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right]^4 + \beta_1 d_2 \left( \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + 1 \right)^2}{4 \left( \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + 1 \right)^2} + 2\beta_1 (A - 2\beta_1 \beta_2 \kappa (\epsilon + \kappa)) \left( \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + 1 \right) \\
&= \frac{\beta_1 d_2 \beta_2 \kappa \left( \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + 1 \right)^2}{2 \beta_2 \kappa} - \frac{\beta_3 \beta_1 \beta_2 \kappa \left( \cosh \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + 1 \right)^2}{\beta_2 ^2} + \frac{-A \beta_2 \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)}{\beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right)} + \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right) \left( \beta_2 \beta_1 \beta_3 \kappa \right) + \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right) \left( \beta_2 \beta_1 \beta_3 \kappa \right) + \beta_1 \beta_3 \kappa \left( \beta_2 \beta_1 \beta_3 \kappa \right) \left( \beta_2 \beta_1 \beta_3 \kappa \right) \\
&= \frac{\beta_1 d_2 \left( \eta e \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + \epsilon \right)^2}{\eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \xi_0) \right]^2 \left( \eta \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] + \epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \xi_0) \right] \right)^2} \\
\end{align*}
\]
\[
\begin{align*}
\frac{1}{4} \beta_1 d_2 & \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right) + \frac{1}{2} \beta_2 \sqrt{\beta_1} d_2 \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right) \\
& - \frac{\sqrt{\beta_1}}{\beta_3} (A - 2 \beta_1 \beta_2 \kappa (\epsilon + \kappa) \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right) \\
& + \frac{1}{\kappa} \left( \frac{\beta_2 (e^2 - 4 \kappa^2) - 4 \beta_1 \beta_2 \epsilon^2}{\beta_3} \right) \right) \\
& + \frac{-A \beta_2 (e - 2 \kappa) - 4 \beta_1 \beta_3 \epsilon^2 \left( \beta_1 d_2 \epsilon - 1 \right) + \beta_2 \beta_1 \epsilon \kappa (e - 2 \kappa) (\beta_1 d_2 (e + 2 \kappa) + 1)}{\beta_1 \epsilon \kappa \left( \beta_2^2 (e^2 - 4 \kappa^2) + 4 \beta_1 \beta_3 \epsilon^2 \right)}. \tag{24}
\end{align*}
\]

\[
\begin{align*}
u_9(x, t) &= \beta_1 d_2 \epsilon^2 \left( \eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - p \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) \\
& \left( \left( \eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right) \right)^2 \\
& - \eta \sqrt{p^2 + 1} + p \epsilon + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]^2 \\
& - \beta_2 d_2 \left( \frac{\epsilon \left( p + \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right) \\
& - \beta_3 d_2 \left( \frac{\epsilon \left( p + \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right)^2 \\
& \frac{2 (A - 2 \beta_1 \beta_2 \kappa (\epsilon + \kappa) \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta \sqrt{p^2 + 1} + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right)}{\kappa \left( \beta_2^2 (e^2 - 4 \kappa^2) - 4 \beta_1 \beta_2 \epsilon^2 \right)} \\
& + \frac{-A \beta_2 (e - 2 \kappa) - 4 \beta_1 \beta_3 \epsilon^2 \left( \beta_1 d_2 \epsilon - 1 \right) + \beta_2 \beta_1 \epsilon \kappa (e - 2 \kappa) (\beta_1 d_2 (e + 2 \kappa) + 1)}{\beta_1 \epsilon \kappa \left( \beta_2^2 (e^2 - 4 \kappa^2) + 4 \beta_1 \beta_3 \epsilon^2 \right)}. \tag{25}
\end{align*}
\]

**Case-IV**

\[
a_0 = \sqrt{b_1^2 \beta_2^2 e^2 + 4 \beta_1^2 \beta_3 \epsilon^2 + b_1 \beta_2 \kappa - 2 \beta_1 \left( \beta_1 d_2 + 1 \right)} / \beta_2, \quad a_1 = -\beta_2 d_2, \quad b_1 = b_2, \quad a_2 = b_2 = c_2 = d_1 = 0,
\]

\[
d_2 = d_2, \quad \lambda = \frac{(\epsilon + \kappa) \sqrt{b_1^2 \beta_2^2 e^2 + 4 \beta_1^2 - 2 \beta_1 \epsilon}}{2 \kappa}, \quad \gamma = \beta_1 (2 \epsilon + \kappa), \quad \delta = \nu = 0. \tag{26}
\]

Substituting Equation (26) in Equation (13), the solutions of Equation (10) are given as

\[
\begin{align*}
u_{10}(x, t) &= \frac{1}{4} \beta_1 d_2 \epsilon \left( 4 \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]^4}{\left( \epsilon \coth \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right)^4} \right) \\
& - \beta_2 \left( \beta_1^2 \beta_2^2 e^2 + 4 \beta_1^2 \beta_3 \epsilon^2 - 2 \beta_1 \right) + b_1 \kappa \left( \beta_2^2 - 2 \beta_1 \left( \epsilon \coth \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) \right) \\
& + \frac{2 \beta_1 \beta_2 \kappa}{2 \beta_1 \beta_2 \kappa}. \tag{27}
\end{align*}
\]

\[
\begin{align*}
u_{11}(x, t) &= \frac{1}{2} \left( \frac{2 \beta_1 d_2 \epsilon \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon^2}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right) \\
& + \sqrt{b_1^2 \beta_2^2 e^2 + 4 \beta_1^2 \beta_3 \epsilon^2 + b_1 \beta_2 \kappa - 2 \beta_1 \left( \beta_1 d_2 \kappa + 1 \right)} - b_1 \sqrt{\beta_1 \kappa} \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right) \\
& + \left( \frac{\epsilon \sinh \sqrt{\beta_1} (x - \lambda t + \zeta_0)}{\eta + \cosh \sqrt{\beta_1} (x - \lambda t + \zeta_0)} + 1 \right) \beta_2 \frac{\sqrt{\beta_1} d_2}{\beta_3}. \tag{28}
\end{align*}
\]

\[
\begin{align*}
u_{12}(x, t) &= b_1 \left( -\frac{\epsilon \left( p + \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right)}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - 1} \right) + \epsilon^2 \left( \eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) \\
& - \eta \sqrt{p^2 + 1} + p \epsilon + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right) \right)^2 \beta_1 d_2 / \left( \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right)^2. \tag{29}
\end{align*}
\]
\[
\left( \eta \sqrt{p^2 + 1} + \beta \right) \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + \epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right]^2 \\
- \frac{\beta^2 \beta_x^2 (2e + 5k) + 2 \beta \beta_x (\beta_1 + 2k) + 1)}{2 \beta_1} + \beta_2 d_2 \left( - \frac{\epsilon (\eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right])}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]} - 1 \right).
\]

\textbf{Case-V}

\[a_0 = -\sqrt{\beta_1} \delta \left( 2e + 5k \right) + \beta \beta_x (\beta_1 + 2k) + 1, \quad a_1 = -\beta_2 d_2,\]

\[b_1 = \frac{2 \left( \sqrt{\beta_1} \beta_x^2 (2e + 5k) + 2 \beta \beta_x (\epsilon + 2k) \right)}{\beta_1 \beta_x (\epsilon + 2k)}, \quad b_2 = b_2 = c_2 = d_2 = d.\]

\[\lambda = \frac{\beta_1 \beta_x^2 (2e + 5k) + 2 (\epsilon + 2k) \beta_1 \beta_x (\epsilon + 2k)}{\beta_2 \delta (\epsilon^2 - 4k^2)}, \quad \gamma = \beta_1 (2e + k), \quad \delta = \eta = 0.\]

Substituting Equation (30) in Equation (13), we obtain the solutions of Equation (10) as

\[u_{13}(x, t) = \beta_1 d_2 \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + 1 \right) + \frac{\beta_1 d_2 \epsilon^2 \cosh \left[ \frac{1}{2} \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + 1}{4 (\epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + 1)^2} \]

\[2 \beta_1 \left( \sqrt{\beta_1} \beta_x^2 (2e + 5k) + 2 \beta \beta_x (\epsilon + 2k) \right) \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + 1 \right)
- \frac{\beta^2 \beta_x^2 (4k^3 - \epsilon^2 k)}{\beta_1 \beta_x (\epsilon + 2k)}.
\]

\[u_{14}(x, t) = \beta_1 d_2 \left( \eta \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + \epsilon \right) \]

\[\left( \eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right) \left( \eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + \epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right)
- \frac{\beta^2 \beta_x^2 (2e + 5k) + 2 \beta \beta_x (\beta_1 + 2k) + 1)}{\beta_1 \beta_x (\epsilon + 2k)}
+ \frac{1}{2} \frac{\beta_1}{\beta_x} \left[ \frac{\epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]}{\eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]} + 1 \right]
- \frac{\beta^2 \beta_x^2 (4k^3 - \epsilon^2 k)}{\beta_1 \beta_x (\epsilon + 2k)}.
\]

\[u_{15}(x, t) = \beta_1 d_2 \epsilon^2 \left( \eta \sqrt{p^2 + 1} \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] - \epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + 1 \right)^2 \]

\[\left( \left( \eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right) \left( \eta \sqrt{p^2 + 1} + \epsilon \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right)
+ \epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right)^2 \beta_2 d_2 \left( - \frac{\epsilon (\eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right])}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]} - 1 \right)
\]

\[2 \beta_1 \left( \sqrt{\beta_1} \beta_x^2 (2e + 5k) + 2 \beta \beta_x (\epsilon + 2k) \right) \left( - \frac{\epsilon (\eta + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right])}{\eta \sqrt{p^2 + 1} + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]} - 1 \right)
+ \frac{\beta^2 \beta_x^2 (4k^3 - \epsilon^2 k)}{\beta_1 \beta_x (\epsilon + 2k)}.
\]

\textbf{Case-VI}

\[a_0 = a_0, \quad a_1 = -\beta_2 - d_2, \quad a_2 = -\beta_2 d_2, \quad b_1 = b_2 = c_2 = d_2 = d = 0.
\]
\[ d_2 = d_2, \lambda = \frac{1}{2} \gamma (a_0 + \beta_1 d_2). \]

Substituting Equation (34) in Equation (13), the solutions of Equation (10) are given as

\[ u_{16}(x, t) = a_0 + \frac{\beta_1 d_2 \varepsilon \csc h^4 \left( \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right)}{4 (\epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2} + \beta_1 d_2 \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) \]

\[ - \frac{\beta_2^2 \beta_3 d_2 \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right)^2}{\beta_2^2}. \]

\[ u_{17}(x, t) = a_0 + \frac{\beta_1 d_2 (\eta \epsilon \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon)^2}{(\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right])^2 (\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right])^2} \]

\[ - \frac{1}{4} \beta_1 d_2 \left( \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]} + 1 \right)^2 + \frac{1}{2} \beta_2 \sqrt{\frac{\beta_1}{\beta_3}} \left( \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]} + 1 \right). \]

\[ u_{18}(x, t) = a_0 + \beta_1 d_2 \epsilon^2 \left( \eta \sqrt{\beta_1^2 + 1} \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] - p \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right)^2 \]

\[ \left( \left( \eta \sqrt{\beta_1^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right)^2 \left( \eta \sqrt{\beta_1^2 + 1} + \epsilon \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right) \right) \]

\[ - \beta_2 d_2 \left( - \frac{\epsilon \left( p + \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right]}{\eta \sqrt{\beta_1^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right)^2 \]

\[ - \beta_3 d_2 \left( \frac{\epsilon \left( p + \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] \right]}{\eta \sqrt{\beta_1^2 + 1} + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1} \right)^2. \]

\[ \text{Case-VII} \]

\[ a_0 = \frac{1}{4} \left( -4 \beta_1 d_2 - \frac{3}{\epsilon + \kappa} + \frac{2}{\kappa} \right), a_1 = -\beta_2 d_2, a_2 = -\frac{\beta_2^2 d_2 \left( \epsilon^2 - 4 \kappa^2 \right)}{4 \beta_1 \epsilon^2}, \]

\[ b_1 = \frac{\beta_1 (2 \epsilon + \kappa)}{2 \beta_2 \kappa (\epsilon + \kappa)}, b_2 = c_2 = d_1 = 0, d_2 = d_2, \lambda = \frac{1}{2} \beta_1 \left( \kappa \frac{2}{\epsilon} + 1 \right), \]

\[ \gamma = \beta_1 (2 \epsilon + \kappa), \delta = \nu = 0, \beta_3 = \frac{\beta_2^2 \left( \epsilon^2 - 4 \kappa^2 \right)}{4 \beta_1 \epsilon^2}. \]

Substituting Equation (38) in Equation (13), the solutions of Equation (10) are obtained as

\[ u_{19}(x, t) = \frac{1}{4} \left( \frac{3}{\epsilon + \kappa} + \frac{2}{\kappa} - \frac{3}{\epsilon + \kappa} - 4 \beta_1 d_2 + 4 \beta_1 d_2 \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right) \right) \]

\[ - \beta_1 d_2 \left( \epsilon^2 - 4 \kappa^2 \right) \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right)^2 \]

\[ + \frac{\beta_1 d_2 \epsilon^2 \csc h^4 \left( \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right)^4}{(\epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1)^2} - \frac{2 \beta_1^2 (2 \epsilon + \kappa) \left( \epsilon \coth \left[ \frac{1}{2} \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + 1 \right)}{\beta_2^2 (2 \epsilon + \kappa)}. \]

\[ u_{20}(x, t) = \frac{1}{16} \left( \frac{16 \beta_1 d_2 (\eta \epsilon \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon)^2}{(\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right])^2 (\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right] + \epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right])^2} \right) + \frac{4 \left( -4 \beta_1 d_2 + \frac{3}{\epsilon + \kappa} + \frac{2}{\kappa} \right) - \frac{4 \beta_1 \sqrt{\frac{\beta_1}{\beta_3}} (2 \epsilon + \kappa) \left( \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]} + 1 \right)}{\beta_2 \kappa (\epsilon + \kappa)} \right) \]

\[ + 8 \beta_2 \sqrt{\frac{\beta_1}{\beta_3}} \left( \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]} + 1 \right) - \frac{\beta_2^2 d_2 \left( \epsilon^2 - 4 \kappa^2 \right) \left( \frac{\epsilon \sinh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]}{\eta + \cosh \left[ \sqrt{\beta_1} (x - \lambda t + \zeta_0) \right]} + 1 \right)^2}{\beta_3 \epsilon^2}. \]
$u_{21}(x, t) = \frac{1}{4} \left( \frac{2}{\varepsilon + \kappa} - \frac{3}{\varepsilon + \kappa} - 4\beta_1 d_2 + (4\varepsilon^2 (\eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + 1)
- p \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right)^2 \frac{\beta_1 d_2}{\left( \eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + 1 \right)^2}
+ \left( \eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + p \epsilon + \cosh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] + \epsilon \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right] \right)^2\right)
+ 2\beta_1 (2\varepsilon + \kappa) \left( \frac{\epsilon (p + \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]) - 1}{\eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + 1} \right) - 4 \left( \frac{\epsilon (p + \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]) - 1}{\eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + 1} \right)
\frac{\beta_2^2 d_2 \left( \varepsilon^2 - 4\kappa^2 \right) \left( \frac{\epsilon (p + \sinh \left[ \sqrt{\beta_1 (x - \lambda t + \zeta_0)} \right]) + 1}{\eta \sqrt{\beta_1 (x - \lambda t + \zeta_0)} + 1} \right)^2}{\beta_1 \epsilon^2}.
(41)

4. Results and discussion

Many researchers determined different types of solutions of KS equation by applying different techniques. In this recent work, we have found new and more general exact travelling and solitary wave solutions, the important thing in this study is the trial solution of Equation (8) uses the range of four parameters that have different structures. The values of constant parameters \(a_i, b_i, c_i, d_i\) are collected by applying the Mathematica, then Equation (9) has different types of solutions which are hyperbolic, rational and trigonometric functions. As a result, are obtained new families of exact travelling and solitary wave solutions with the help of this powerful technique. Now we discuss the differences and similarities of our new obtained solutions with already that have been found in the past literature by using different techniques.

In previous literature, many authors have been determined various types of solutions of KS equation such as elliptic, trigonometric, hyperbolic, rational functions including dark, bright, kink, anti-kink, periodic wave solutions with the help of modified truncated expansion method, dynamical system, bifurcation technique, backlund transformation, lie symmetry analysis, F-expansion method, improved subequation method, the \((G'/G)\)-expansion method [5–16]. But our obtained solutions have different structures in the shape of dark, bright, singular, combined, kink wave solitons, periodic solitary wave, traveling wave (see Figures 1–10).

In Figure 1(a), dark soliton solution for Equation (15) and (b) bright soliton solution for Equation (16), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 8, \eta = 2, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.4, b_1 = 1.5, d_2 = 2\).
In Figure 2, 3D (a) bright soliton solution for Equation (17), and (b) dark soliton solution for Equation (19), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.4, b_1 = 1.5, d_2 = 2, p = 2\) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 1.4, b_1 = 1.5, d_2 = 2, p = 2\), respectively. In Figure 3(a), combined dark–bright soliton solutions for Equation (20) and (b) bright soliton solutions for Equation (21), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 8, \eta = 2, \zeta_0 = 0.3, \lambda = 4, \kappa = 1.4, b_1 = 1.5, d_2 = 2\) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.4, b_1 = 1.5, d_2 = 2, p = 0.5\), respectively. In Figure 4(a), dark soliton solution for Equation (23) and (b) Kink wave soliton solution for Equation (24), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 8, \eta = 2, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.5, \beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2, A = 2, \) respectively. In Figure 5(a,b), solitary wave solutions for Equation (25) and Equation (27), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2\). In Figure 6(a,b), solitary traveling wave solutions for Equation (28) and Equation (29), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2, p = 2\) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = 8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2, p = 2\), respectively. In Figure 7(a), solitary wave solution for Equation (31) and (b) Solitary traveling wave solution for Equation (32), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2.5, b_1 = 1.5, d_2 = 2, \) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -8, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2, p = 2\), respectively. In Figure 8(a), solitary traveling wave solution for Equation (33) and (b) bright soliton solution for Equation (36), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -6, \zeta_0 = 0.3, \lambda = 4, \kappa = 2, b_1 = 1.5, d_2 = 2, p = 2\) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -8, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.4, b_0 = 2, d_2 = 1.5, \) respectively. In Figure 9(a), solitary wave solutions for Equation (37) and (b) bright soliton solution for Equation (39), with these parameters values \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -8, \zeta_0 = 0.3, \lambda = 4, \kappa = 1.4, b_1 = 1.5, d_2 = 2, p = 2\) and \(\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -8, \zeta_0 = 0.3, \lambda = 4, \kappa = 0.4, b_0 = 2, d_2 = 1.5, \) respectively.
0.3, $\lambda = 4$, $a_0 = 2$, $d_2 = 1.5$, $p = 0.5$ and $\beta_1 = 2$, $\beta_2 = 4$, $\epsilon = 18$, $\eta = -8$, $\zeta_0 = 0.3$, $\lambda = 4$, $\epsilon = 1.5$, $\kappa = 1.4$, $d_2 = 2$, respectively. In Figure 10(a,b), solitary wave solutions for Equation (40) and Equation (41), with these parameters values $\beta_1 = 2$, $\beta_2 = 4$, $\epsilon = 18$, $\eta = -8$, $\zeta_0 = 0.3$, $\lambda = 4$, $\epsilon = 1.5$, $\kappa = 2.4$, $d_2 = 2$ and $\beta_1 = 2$, $\beta_2 = 4$, $\beta_3 = 2$, $\epsilon = 18$, $\eta = -8$, $\zeta_0 = 0.3$, $\lambda = 4$, $\epsilon = 1.5$, $\kappa = 2.4$, $p = 1$, $d_2 = 2$, respectively.

We can conclude that from the above comparison and detailed discussion, our obtained solutions are new and more generally which have not formulated before by other techniques. It is proved that our modified technique is fruitful, reliable, straight forward and effective to investigate other nonlinear evolution equations.

5. Conclusion
We successfully constructed some new exact travelling and solitary wave solutions of nonlinear KS equation by applying the modified mathematical technique. Our solutions are different and new from other researcher
Figure 4. (a) Dark soliton solution for Equation (23) and (b) Kink wave soliton solution for Equation (24).

Figure 5. (a,b) Solitary wave solutions for Equation (25) and Equation (27).

Figure 6. (a,b) Solitary traveling wave solutions for Equation (28) and Equation (29).

Figure 7. (a) Solitary wave solution for Equation (31) and (b) solitary traveling wave solution for Equation (32).
found by using the different techniques before this work. Our new exact solutions obtained in the shape of dark solitons, bright solitons, travelling wave, solitary wave and periodic wave. These new solutions are more useful in the study of quantum plasma, optical fibres, dynamics of solitons, dynamics of fluid, problems of biomedical, mathematical physics, engineering and many other branches. The physical structure of new solutions shows the effectiveness and power of this technique. This research work completed by using the Mathematica. We can also apply this technique on other nonlinear evolution equations involves in optical fibre, Geo physics, mathematical physics, plasma physics, fluid dynamics, hydrodynamics, mechanics, mathematical biology, field of engineering and many other applied sciences.

Disclosure statement
No potential conflict of interest was reported by the authors.

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