2D Topological $p + ip$ Superconductivity in Doped Graphenelike BC$_3$

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We theoretically study exotic superconducting phases with enhanced transition temperature in a 2D hexagonal system doped to near its type-II Van Hove singularity whose saddle point momenta are not time-reversal-invariant. From renormalization group analysis and random phase approximation calculations we show that the dominant superconducting instability induced by weak repulsive interactions is in the triplet channel (either $p + ip$ or f-wave) because of strong ferromagnetic fluctuations. Interestingly, the graphenelike one-atom thick material BC$_3$ can realize such type-II Van Hove singularity by doping approximately 1/8 electrons per site, where we show that the leading instability is superconductivity with time-reversal-invariant $p + ip$ pairing which carries nontrivial $Z_2$ topological invariant. It provides a promising route to realize a genuine 2D helical $p + ip$ superconductor in Nature.

Introduction: Many exciting discoveries of topological quantum states of matter have been made in the past several years, including 2D and 3D topological insulators protected by time reversal symmetry [11, 2], quantum Hall effect in graphene [3, 4], and quantum anomalous Hall effect [5]. Nonetheless, a class of intrinsic superconductors with fully-gapped bulk excitations but robust gapless boundary excitations dubbed as “topological superconductors” (TSC) have not been unambiguously identified in Nature even though enormous efforts have been devoted into their discoveries. Among them, a 2D chiral $p + ip$ superconductor is of special importance partly because its magnetic vortices support Majorana zero modes [9–11], which obey non-Abelian statistics [11, 12] and which are believed to be a promising tool for topological quantum computation [13, 14]. A close cousin of chiral $p + ip$ superconductors in 2D is a time-reversal preserving helical $p + ip$ superconductor [6, 15, 16].

The quasi-2D material Sr$_2$RuO$_4$ was proposed to be a candidate of realizing intrinsic chiral $p + ip$ superconductivity (SC) [17–20]; but experimental evidence for it is still not definitive [21]. As a genuine two-dimensional (2D) electronic system on a honeycomb lattice, graphene [22–24] has attracted special attention as candidate materials harboring unconventional SC induced by electronic interactions [25, 26] when doped away from half-filling. In particular, at about 1/4 electron or hole-doping, the Fermi level is at Van Hove singularity (VHS) where density of states (DOS) is logarithmically divergent and where it was proposed that SC with chiral $d + id$ pairing and relatively high transition temperature may be induced by repulsive interactions [27, 28]. Singlet $d + id$ SC was obtained at the VHS of 1/4-doped graphene in previous theoretical analysis; the arguably more interesting triplet $p + ip$ pairing was not seen there. It was pointed out recently [30] that the absence of $p + ip$ triplet pairing in graphene at the VHS is mainly due to the fact that its saddle point momenta $K$ are time-reversal-invariant (TRI), namely $K = -K$. Such Van Hove saddle points are called “type-I” [30]. For systems exactly at type-I VHS, triplet pairing potential at saddle points must vanish due to Pauli exclusion principle; consequently, triplet pairing in systems at type-I VHS is normally suppressed. The concept of type-II VHS was introduced in Ref. [30]; for type-II VHS, Van Hove saddle point momenta $K$ are not TRI, namely $K \neq -K$. It was shown by renormalization group (RG) analysis that systems with type-II VHS are promising arena to look for $p + ip$ triplet pairings [30].

We show that type-II VHS can be realized by doping BC$_3$, which is a graphenelike genuine 2D material as shown in Fig. 1(a) and which was successfully fabricated in experiments about one decade ago [31]. Undoped BC$_3$ is a band insulator with band gap of about 0.5eV, as shown in Fig. 1(b); its conduction band mainly consists of $p_z$ orbitals of boron atoms with $\pi$-hopping [32]. When doped with about 1/8 electrons per site, its Fermi surfaces go through a Lifshitz transition at which it has six saddle points inside its Brillouin zone [33], as shown in Fig. 1(c). This is exactly type-II VHS in a hexagonal system. As far as we know, this is the first 2D material which realizes hexagonal type-II VHS. First principle calculations were performed to investigate its possible magnetic instability induced by this fermiology: it was shown that the system is in the vicinity of magnetic ordering. Both ferromagnetic and certain spin density wave fluctuations are strong, which could mediate unconventional singlet or triplet SC [34]. An important question to ask is what kind of broken symmetry phase emerges in the ground state.

In this paper, we construct an effective model of doped BC$_3$ and perform RG analysis and random phase approximation (RPA) calculations to investigate competing orders in BC$_3$ doped to near its type-II VHS. We show that the dominant instability is SC when considering weak re-
Electrons can be doped into BC$_3$ by various methods, including electric gating and doping adatoms. It was shown that doping may be achieved through chemical absorption with alkali adatoms\cite{22}. For instance, Li adatoms can be employed to form Li$_x$BC$_3$. With small $x$ the Fermi surface (FS) consists three electrons pockets around M points. At the critical doping $x_c$, the FS crosses six Van Hove saddle points, as shown in Fig. 1(c). The corresponding FS is calculated via density functional method and is shown in Fig. 1(c). The six saddle points lie on $ \Gamma$-K or $ \Gamma$-K’ lines. In the limit of weak interactions, low energy physics are dominated by electrons near the FS. Moreover, in a 2D system at VHS, electrons near those saddle points dominate the log-divergent DOS. Consequently, we take the patch approach, namely consider only the fermions near the saddle points. With this patch approximation, the low energy behavior of the system can be described by the following effective action in the continuum limit:

$$S = \int d\tau d^2r \sum_{\alpha=1}^{6} \sum_{\sigma=1}^{2} \psi_{\alpha\sigma}^\dagger \left[\partial_\tau - \varepsilon_\alpha(\delta k_x, \delta k_y) - \mu\right] \psi_{\alpha\sigma} + \sum_{\alpha, \beta \sigma, \sigma'} \frac{\mathcal{g}_{\alpha, \beta \gamma}}{2} \psi_{\alpha\sigma}^\dagger \psi_{\beta\sigma}^\dagger \psi_{\gamma\sigma'} \psi_{\eta\sigma'},$$

where $\psi_{\alpha\sigma}^\dagger$ creates an electron in patch $\alpha = 1, \cdots, 6$ with spin polarization $\sigma = \uparrow, \downarrow$ and $\varepsilon_\alpha(\delta k_x, \delta k_y) = \delta k_x \delta k_y/(2m_\alpha) + O(\delta k^3)$ represents electronic dispersions in the patch $\alpha$ with $\delta \vec{k} = \vec{k} - \vec{P}_\alpha$ [$\vec{P}_\alpha$ denotes the saddle point momentum in patch $\alpha$, as shown in Fig. 1(c)]. Here $\mu = 0$ describes the system exactly at the Van Hove filling. We label eigenvalues of the mass matrix $\delta m_\alpha$ as $m_1$ and $m_2$ ($m_1 \approx 1.6 eV$ and $m_2 \approx 1.2 eV$ for BC$_3$ at type-II VHS).
Note that we neglect the spin-orbit coupling here because it is expected to be weak in material consisting of such light atoms as borons and carbons; but we shall consider it when we dope alkali atoms into the system. Here \( g_{\alpha\beta\gamma} \) describes various interactions and \( \eta \) in the above equation is implicitly determined by the momentum conservation \( \tilde{P}_\alpha + \tilde{P}_\beta - \tilde{P}_\gamma = 0 \). Since symmetry-related \( g_{\alpha\beta\gamma} \) must be equal, there are only nine inequivalent interactions: \( g_{111}, g_{121}, g_{122}, g_{131}, g_{133}, g_{141}, g_{142}, g_{143}, \) and \( g_{114} \). Hereafter, we introduce dimensionless interaction parameters \( g_{\alpha\beta\gamma} \to v_0 g_{\alpha\beta\gamma} \) where \( v_0 = \sqrt{m_1 m_2}/(4\pi^2) \).

The DOS per patch \( \rho(\omega) \) around VHS diverges logarithmically: \( \rho(\omega) \approx 2\nu_0 \log(\Lambda/\omega) \) where \( \Lambda \) is order of band width, and \( \omega \) is the energy away from VHS. For BC3 at its type-II VHS, we obtain \( v_0 \approx 0.012/\text{eV} \).

**RG analysis:** To investigate possible phase transitions as temperature decreases, we study how interactions flow using RG equations derived from gradually integrating out electrons between a decreasing \( \omega \) and the ultraviolet cutoff \( \Lambda \). Due to the quadratic nature of electronic dispersion around Van Hove saddle points, the interaction parameters \( g_{\alpha\beta\gamma} \) are marginal at the tree level. We then go to one-loop level and derive the following flow equation of interactions \( g_{\alpha\beta\gamma} \), [27][30]:

\[
\frac{\partial g_{\alpha\beta\gamma}}{\partial y} = \frac{d_{pp}^{\alpha\beta\gamma}}{2} \left( g_{\alpha\beta\gamma} + d_{pp}^{\alpha\gamma} \Psi_2 + d_{pp}^{\beta\gamma} \Phi_3 \right),
\]

where \( y = \log^2(\Lambda/\omega) \) is the flow parameter and \( \Phi_{\beta\gamma}^{\alpha\beta\gamma}(g) \) are quadratic functions of \( g \), given explicitly in the Supplemental Material. Here \( d_{pp}^{\alpha\beta\gamma} = \frac{2}{v_0} \frac{\partial \chi_{\alpha\beta\gamma}^{pp}}{\partial y} \), \( d_{pp}^{\alpha\gamma} = \frac{2}{v_0} \frac{\partial \chi_{\alpha\gamma}^{pp}}{\partial y} \), and \( d_{pp}^{\beta\gamma} = \frac{2}{v_0} \frac{\partial \chi_{\beta\gamma}^{pp}}{\partial y} \), in which \( \chi_{\alpha\beta\gamma}^{pp} = \chi_{pp}(\tilde{P}_\alpha + \tilde{P}_\beta) \) and \( \chi_{\alpha\beta}^{pp} = \chi_{pp}(\tilde{P}_\alpha - \tilde{P}_\beta) \) are the susceptibilities of noninteracting electrons in the electron-electron and electron-hole channels, respectively. For \( \omega < \Lambda \), they are given by

\[
\chi_{pp}(\tilde{Q}) \simeq \frac{v_0}{2} \log^2(\Lambda/\omega), \quad \chi_{pp}(\bar{Q}) \simeq v_0 \log(\Lambda/\omega),
\]

\[
\chi_{pp}(\tilde{Q}_1) \simeq a_v \nu_0 \log(\Lambda/\omega), \quad \chi_{pp}(\tilde{Q}_2) \simeq a_v \nu_0 \log(\Lambda/\omega), \quad \chi_{pp}(\tilde{Q}_3) \simeq a_3 \nu_0 \log(\Lambda/\omega),
\]

where \( \tilde{Q}_i = \tilde{P}_i - \tilde{P}_1 \). Here \( a \) and \( a_3 \) are functions of the mass ratio \( \kappa = m_1/m_2 \), while \( 0 < a_3 < 1 \) depends on the details of dispersions around Van Hove saddle points. Note that \( \chi_{pp}(\tilde{Q}_1) \) and \( \chi_{pp}(\tilde{Q}_2) \) have identical leading logarithmic divergent behaviors, which is required by the lattice symmetry of the hexagonal system we consider. Similarly, \( \chi_{ph}(\tilde{Q}_1) \) and \( \chi_{ph}(\tilde{Q}_2) \) have identical leading logarithmic divergent behaviors.

The detailed behavior of \( d_{pp}^{\alpha\beta\gamma} \) and \( d_{ph}^{\alpha\beta\gamma} \) depends on specifics of the band structure. Nonetheless, they have the asymptotic forms: as \( y \to \infty \), \( d_{pp}(\bar{Q}) \to 0 \), \( d_{pp}(\tilde{Q}_1) \to a_v \sqrt{\gamma} \), \( d_{pp}(\tilde{Q}_2) \to a_3 \sqrt{\gamma} \), \( d_{pp}(\tilde{Q}_3) \to a_3 \sqrt{\gamma} \), \( d_{pp}(\tilde{Q}_4) \to a_3 \sqrt{\gamma} \), \( d_{pp}(\tilde{Q}_5) \to a_3 \sqrt{\gamma} \).

**FIG. 2:** (a) and (b): The phase diagrams as a function of \( a \) and \( b \), with \( b_1 + b_2 = 0.1 \). The results are calculated for weak interactions \( a_0 = 0.01 \) and \( b_0 = 0.02 \); (c)~(e): The evolution of susceptibility exponents of various types of broken symmetries as a function of \( g_1 \) for three different choices of nesting parameters: (i) \( a = 2.0, b_1 = 0.025, b_2 = 0.075 \); (ii) \( a = 2.0, b_1 = 0.075, b_2 = 0.025 \); and (iii) \( a = 0.0, b_1 = 0.025 \) and \( b_2 = 0.075 \).
From the RG flow equations, we obtain that \( g_{\alpha \beta \gamma}(y) \) flow to the infinity as \( y \to y_c \) with the asymptotic behavior \( g_{\alpha \beta \gamma} \sim \frac{G_0}{y - y_c} \). Such divergences indicate that certain instability occurs at the corresponding energy scale \( \omega_c \sim \Lambda \exp(-1/\sqrt{|\kappa|}) \), which is order of the transition temperature. In order to obtain the most favored broken symmetry, we can compare susceptibilities of various broken symmetries of the interacting electrons, \( \chi \sim (y - y_c)^{-\gamma} \), and identify the one which diverge fastest, namely find the exponent \( \gamma \) which is most positive. We consider weak interactions and are interested in superconducting instabilities in various channels. For the hexagonal system which we concern in this paper, SC can occur in \( s \), \( p_x \), \( p_y \), \( d_{x^2-y^2} \), and \( d_{xy} \), and \( f \) channels. Here \( p_x \) and \( p_y \) form a two-dimensional irreducible representation of the hexagonal system; consequently they have degenerate pairing instability. The same is true for \( d_{x^2-y^2} \) and \( d_{xy} \) pairings. For such degenerate pairing channels, their interplay below the transition temperature can be analyzed from the quartic terms in the Ginzburg-Landau free energy. In the Supplemental Material, we show that the \( p+ip \) pairing (\( d+id \) pairing) always has lower energy than nodal \( p \)-wave pairing (\( d \)-wave pairing) because the FS can be fully gapped by it. There are two types of triplet \( p+ip \) pairing, namely chiral and helical \( p+ip \) pairings, which have degenerate energy when spin-orbit coupling is absent. Nonetheless, there is only one type of singlet \( d+id \) pairing which is always chiral. The susceptibility \( \gamma \) in all channels are given by \( \gamma_n = -2 \sum_{\alpha=1}^6 G_{3\alpha 6\alpha} \exp(na\pi i/3) \), where \( n = 0, 1, 2, 3 \) represent \( s \), \( p+ip \), \( d+id \), and \( f \)-wave pairing, respectively. The most positive \( \gamma \) tells us the leading superconducting instability. The definition of exponents for other particle-particle as well as particle-particle channels are given in the Supplemental Material.

**Phase diagram:** We derive phase diagram by solving the RG flow equations (see the Supplemental Material). The RG flows depend on the bare interaction parameters \( g_{\alpha \beta \gamma}(y = 0) \) as well as those particle-particle and particle-hole susceptibilities of non-interacting electrons, which are functions of \( y \). The bare values of \( g_{\alpha \beta \gamma} \) are obtained by projecting the microscopic interactions into interactions between Van Hove electrons. For the Hubbard interactions, we obtain \( g_{\alpha \beta \gamma}(y = 0) \equiv U \tau_0 \). For weak \( g_0 \), the flow of interaction parameters always favors superconducting phases, as expected and as shown in Fig. 2(c–e). Whether the flow favors singlet or triplet pairings depends on how nested the FS is by the vectors \( \vec{Q}_1 \) and \( \vec{Q}_2 \), namely depends on the values of \( a \), \( b_1 \), and \( b_2 \), which we dub as “nesting parameters”. Perfect nesting is represented by \( a \to \infty \). It turns out the triplet pairing is the leading instability for a FS not close to perfect nesting. But whether \( p+ip \) or \( f \)-wave triplet pairing is more favored depends on the difference of sub-logarithmical behaviors between \( d_{ph}(\vec{Q}_1) \) and \( d_{ph}(\vec{Q}_2) \) at finite \( y \) even though \( d_{ph}(\vec{Q}_1) \) and \( d_{ph}(\vec{Q}_2) \) have identical asymptotic behaviors at \( y \to \infty \) in a hexagonal system. Their sub-logarithmical difference is modeled by \( b_1 \neq b_2 \). When \( b_1 < b_2 \), namely fluctuations at \( \vec{Q}_1 \) is stronger, \( p+ip \) pairing is more favored; otherwise, \( f \)-wave pairing is more favored.

We obtain various susceptibility exponents as a function of \( g_0 \) in Fig. 2(c–e) for three different set of nesting parameters: (i) \( a = 2.0, b_1 = 0.025 \), and \( b_2 = 0.075 \); (ii) \( a = 2.0, b_1 = 0.075 \), and \( b_2 = 0.025 \); and (iii) \( a = 9.0, b_1 = 0.025 \) and \( b_2 = 0.075 \). The leading instability is determined by the order parameter with the most positive susceptibility exponent. In cases (i) and (ii) with \( a = 2.0 \), the FS is not close to perfect nesting. The ferromagnetism is the leading instability for \( g_0 \gtrsim 0.04 \), which is in contrast to systems at type-I VHS where spin density waves at wave vectors with finite momentum are usually more favorable than ferromagnetism. This is mainly due to the difference in the number of saddle points. Normally, systems at type-II VHS has twice saddle points as those systems at type-I VHS with the same lattice symmetry. As a consequence, \( \chi_{ph}(\vec{Q}) \) is enhanced. So is the instability to ferromagnetism. When \( g_0 \lesssim 0.04 \), the full-gapped \( p+ip \) or nodal \( f \)-wave triplet pairing becomes the leading instability, depending on whether \( b_1 \) is larger or smaller than \( b_2 \). It is clear that triplet SC occurs in the vicinity of ferromagnetism and it is believed to be induced by ferromagnetic fluctuations[17, 41, 42]. For the case (iii) with \( a = 9.0 \), the FS is relatively close to perfect nesting. When \( g_0 \gtrsim 0.05 \), the leading instability is still ferromagnetism. When \( g_0 \) is weaken such that \( 0.01 \lesssim g_0 \lesssim 0.05 \), triplet pairing is the most favored instability. Again, triplet pairing appears in the vicinity of ferromagnetism. For even weaker interactions \( g_0 \lesssim 0.01 \), RG flow diverges only at extremely low energy where the system’s instability is dominated by the fact that FS is close to perfect nesting. Consequently, the leading instability is the \( d+id \) pairing, as expected. For a fixed weak interaction \( g_0 = 0.01 \), we obtain the phase diagram as a function of nesting parameters \( a \) and \( b \equiv b_1 - b_2 \), as shown in Fig. 2(a). For stronger interactions \( g_0 = 0.02 \), as shown in Fig. 2(b), it is clear that the region of triplet pairings is enlarged, consistent with the heuristic argument we discuss above.

**Real material BC3:** Type-II VHS is realized in BC3 when it is doped with about 1/8 electron per site and the corresponding FS is shown in Fig. 1(c). From its tight-binding band structure, we obtain the nesting parameter \( a \approx 1.3 \), which indicates that the FS of BC3 is not close to perfect nesting at all. Since its FS is not sufficiently close to perfect nesting and since the interactions in BC3 are relatively weak, we expect that the triplet pairing is the most leading instability. Indeed, from first principles calculations, we estimate that \( U \sim 0.7 \text{eV} \). This corresponds to \( g_0(y = 0) \sim 0.01 \) in the RG flows. In other words, the phase diagram in Fig. 2(a) is expected to be relevant for the BC3 doped to its VHS. Moreover, the
magnetic fluctuations at \( \vec{Q}_1 \) is stronger than the ones at \( \vec{Q}_2 \), as can be seen clearly from the shape of FS, we obtain \( b_1 \approx b_2 \). Consequently, the triplet \( p+ip \) pairing is more favored than \( f \)-wave pairing and is the leading instability in BC\(_3\) when doped to its type-II VHS.

**Finite doping regime:** The RG analysis above shows that the leading instability in the BC\(_3\) doped exactly to its type-II VHS is the \( p+ip \) triplet pairing. To investigate its broken symmetry phases away from but near to the type-II VHS at \( x_c \approx 0.127 \), we have also performed a RPA based study for the pairing symmetries of the system near \( x_c \) for supplement. Standard multi-orbital RPA approach\([34, 33]\) is adopted in our study for the doping regime \( x \in (0.11, 0.135) \) with \( U = 0.8t_1 \). For such weak interactions, the spin and charge susceptibilities are obtained via RPA; the susceptibilities peak at the \( \Gamma \)-point in the Brillouin-Zone, suggesting that the dominating spin fluctuations in the system are ferromagnetic. Via exchanging spin fluctuations, electrons near the FS acquire an effective pairing interaction \( V_{\text{eff}} \), from which one obtains the linearized gap equation which has solutions in various pairing channels. The leading instability occurs in the pairing channel with the largest eigenvalue \( r \) of the linearized gap equation with \( T_c \sim t_1e^{-1/r} \). The doping dependence of \( r \) in various pairing channels near VHS of \( x_c \) in this system is shown in Fig. 3(a), which shows that the odd parity \( p+ip \) and \( f \)-wave pairings are the leading and subleading pairing symmetries, respectively. This is consistent with the RG analysis above.

**Spin orbit coupling:** The leading triplet \( p+ip \) pairing obtained in BC\(_3\) is characterized by its \( \vec{d}_k \) defined through \( \langle \psi_{ks}^\dagger \psi_{-k_s'} \rangle \propto (\vec{d}_k \cdot \vec{\sigma} \delta_{ss'})_{ss'} \). Without spin-orbit coupling (SOC), chiral \( p+ip \) pairing is degenerate with helical \( p+ip \) pairing besides the degeneracy generated by global rotations of \( \vec{d}_k \), as shown in the Supplemental Material. A finite SOC can lift the degeneracy between the helical and chiral \( p+ip \) pairings. For simplicity, we consider atomic Kane-Mele SOC, whose strength is parameterized by \( \lambda \), in the tight-binding Hamiltonian and then perform RPA calculations to obtain the leading pairing symmetry for weak interactions \( g_0 \). From RPA calculations, triplet \( p+ip \) pairing is the leading instability when \( \lambda = 0 \). For weak but finite SOC, we find that the helical \( p+ip \) pairing is always more favored than chiral \( p+ip \) pairing. For instance, with extremely weak SOC (\( \lambda = 0.05t_1 \)), the helical \( p+ip \) pairing is slightly more favored than the chiral one, as shown in Fig. 3(b).

**Concluding remarks:** We have performed RG analysis and RPA calculations for a hexagonal system near its type-II Van Hove singularity and show that triplet pairing (either \( p+ip \) or \( f \)-wave pairing) is the dominant instability for weak repulsive interactions. The competition between \( p+ip \) and \( f \)-wave SC depends on the competition between spin fluctuations at \( \vec{Q}_1 \) and \( \vec{Q}_2 \). Spin fluctuations at \( \vec{Q}_1 \) favor \( p+ip \) pairing while the ones at \( \vec{Q}_2 \) favor \( f \)-pairing. More interestingly, a graphene-like atom-thick material BC\(_3\) can be tuned to its type-II Van Hove singularity by doping approximately \( 1/8 \) electrons per site. At this Van Hove filling, we show that a helical \( p+ip \) triplet SC is the leading instability with a relatively-high transition temperature enhanced by the VHS, where the triplet SC are related to strong ferromagnetic fluctuations in doped BC\(_3\) at the Van Hove filling and the helical nature of pairing is pinned by a weak spin-orbit coupling. Such helical \( p+ip \) SC respects time reversal symmetry with the hallmark that it carries helical gapless Majorana edge states which are robust against disorder as long as the time reversal symmetry is preserved and which should be detectable in STM measurements. Besides being a promising material to look for a genuine 2D helical \( p+ip \) SC with nontrivial \( \mathbb{Z}_2 \) topological invariant\([11, 15]\), doped BC\(_3\) might have potential applications in areas such as topological quantum computations in the future.

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SUPPLEMENTAL MATERIAL

A. The RG flow equations for hexagonal systems at the type-II Van Hove singularity

The interaction parameters $g_{\alpha\beta\gamma\delta} = g_{\alpha\beta\gamma}$, where $\delta$ is implicitly determined from momentum conservation, satisfy the flow equations:

$$
\frac{dg_{\alpha\beta\gamma\delta}}{dy} = -d_{pp}^{\alpha\beta}\sum_{\eta=1}^{6}g(\vec{P}_\alpha, \vec{P}_\beta, \vec{P}_\gamma, \vec{P}_\delta) - d_{ph}^{\alpha\beta}\sum_{\eta=1}^{6}\gamma(\vec{P}_\alpha, \vec{P}_\beta, \vec{P}_\gamma, \vec{P}_\delta) \cdot g(\vec{P}_\eta, \vec{P}_\alpha, \vec{P}_\beta - \vec{P}_\eta, \vec{P}_\gamma, \vec{P}_\delta)
$$

in which $\gamma_{\alpha\beta\gamma\delta} = g_{\alpha\beta\gamma\delta} - g_{\alpha\beta\gamma}$, $d_{pp}^{\alpha\beta} = d_{pp}(\vec{P}_\alpha + \vec{P}_\beta)$ and $d_{ph}^{\alpha\beta} = d_{ph}(\vec{P}_\alpha - \vec{P}_\gamma)$. Because of the lattice symmetries of the hexagonal system BC$_3$, there are totally only nine inequivalent interaction parameters: $g_1 = g_{1441}$, $g_2 = g_{1436}$, $g_3 = g_{1425}$, $g_4 = g_{1414}$, $g_5 = g_{1313}$, $g_6 = g_{1331}$, $g_7 = g_{1212}$, $g_8 = g_{1221}$, $g_9 = g_{1111}$. We obtain their RG flow equations as follows:

$$
\begin{align*}
\frac{dg_1}{dy} &= -d_{pp}^{14}(g_1^2 + 2g_5^2 + g_6^2 + g_7^2) + d_{ph}^{14}(g_4g_9 + 2g_5g_7 + 2g_5g_6 - g_1g_4 - 4g_6g_9), \\
\frac{dg_2}{dy} &= -2d_{pp}^{14}(g_1g_2 + g_2g_3 + g_3g_4) + 2d_{ph}^{13}(g_2g_6 + g_3g_7 - g_2g_7), \\
\frac{dg_3}{dy} &= -d_{pp}^{14}(g_1g_3 + 2g_2g_3 + g_2^2 + g_3^2) + 2d_{ph}^{12}(g_1g_6 + 2g_3g_6 + 2g_5g_6 + 2g_5g_6 - g_3g_6), \\
\frac{dg_4}{dy} &= -2d_{pp}^{14}(g_1g_4 + 2g_2g_4) + 2d_{ph}^{11}(g_4g_7 + 2g_5g_7) + 2d_{ph}^{14}(g_4g_1 - g_4^2), \\
\frac{dg_5}{dy} &= -2d_{pp}^{3}(g_5g_6 + g_5^2 + 2g_5g_7 + g_7^2 + g_5^2) + 2d_{ph}^{13}(g_5g_9 + g_6g_9 + 2g_5g_6 - g_2g_6 - g_5g_9 - 2g_1g_9), \\
\frac{dg_6}{dy} &= -d_{pp}^{13}(g_5^2 + g_6^2) + 2d_{ph}^{12}(g_1g_6 + g_2g_6 + g_5^2 + g_6^2) + 2d_{ph}^{13}(g_5g_9 + g_6g_9 + 2g_5g_6 + g_1g_7 - g_2^2 - g_5g_9 - 2g_1g_9), \\
\frac{dg_7}{dy} &= -2d_{pp}^{12}(g_7g_8 + 2g_5g_7 + g_5g_7 + g_4g_5) + 2d_{ph}^{12}(g_7g_8 + 2g_5g_7 + g_5g_7 - g_2^2), \\
\frac{dg_8}{dy} &= -d_{pp}^{12}(g_7^2 + g_8^2) + 2d_{ph}^{12}(g_3g_6 + g_4g_6 + g_5g_6 + g_6g_7 + g_7g_9 - 2g_1g_9 - 2g_5g_6 - g_6g_9), \\
\frac{dg_9}{dy} &= -d_{pp}^{11}(g_9^2 + 2g_5^2 + 2g_7^2 + g_5g_7 + g_7^2) + 2d_{ph}^{11}(g_3g_4 + 2g_5g_6 + 2g_7g_8 - g_1^2 - g_2^2 - g_5^2),
\end{align*}
$$

where $c_{pp}^{\alpha\beta}$ and $d_{ph}^{\alpha\beta}$ are defined in the main text.

B. Susceptibility exponents for various types of broken symmetries

The interactions $g_1, \ldots, g_9$ defined above have the asymptotic behavior $g_i \sim \frac{C_i}{y - y_c}$ as $y \rightarrow y_c$. The susceptibility exponents for various types of broken symmetries can be expressed as the linear combination of $G_i$. The susceptibility
exponents for s-wave paring, p-wave paring, d-wave paring, f-wave paring, SDW($\vec{Q}_1$) (or SDW1), SDW($\vec{Q}_2$) (or SDW2), ferromagnetism, CDW($\vec{Q}_1$) (or CDW1), and CDW($\vec{Q}_2$) (or CDW2) are given as:

\[
\begin{align*}
\gamma_s &= -2(G_1 + 2G_2 + 2G_3 + G_4) \\
\gamma_p &= -2(G_1 + G_2 - G_3 - G_4) \\
\gamma_d &= -2(G_1 - G_2 - G_3 + G_4) \\
\gamma_f &= -2(G_1 - 2G_2 + 2G_3 - G_4) \\
\gamma_{SDW1} &= 2(G_3 + G_8) \cdot d_{ph}(\vec{Q}_1, y_c) \\
\gamma_{SDW2} &= 2(G_2 + G_6) \cdot d_{ph}(\vec{Q}_2, y_c) \\
\gamma_{FM} &= 2(G_2 + G_5 + G_7 + G_9) \cdot d_{ph}(\vec{0}, y_c) \\
\gamma_{CDW1} &= 2(G_3 + G_8 - 2G_2 - 2G_7) \cdot d_{ph}(\vec{Q}_1, y_c) \\
\gamma_{CDW2} &= 2(G_2 + G_6 - 2G_3 - 2G_5) \cdot d_{ph}(\vec{Q}_2, y_c)
\end{align*}
\]

C. The Ginzburge-Landau free energy of $p+ip$ superconductors

To derive the Landau-Ginzburg free energy, we start with the partition function $Z = \int D[\tilde{\psi}\psi] \exp(-\int \mathcal{L}(\tilde{\psi}\psi))$ where

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = \sum_{\alpha=1}^{6} \sum_{\sigma} \tilde{\psi}_{\alpha\sigma} [\partial_\tau + \varepsilon_\alpha(\delta \vec{k}) - \mu] \psi_{\alpha\sigma} + \frac{1}{2} \sum_{\alpha\beta\sigma\sigma'} g_{\alpha\beta\beta\sigma} \tilde{\psi}_{\alpha\sigma\psi_{\beta\sigma'}} \psi_{\beta\sigma'}
\]

Here $\vec{P}_\alpha = -\vec{P}_\alpha$. Notice that $g_1 = g_{1441}, g_2 = g_{1436}, g_3 = g_{1425}, g_4 = g_{1414}$, we can write $\mathcal{L}_{int}$ in the matrix form as:

\[
\mathcal{L}_{int} = \frac{1}{2} \begin{bmatrix}
\tilde{\psi}_{1\sigma} \psi_{4\sigma'} \\
\tilde{\psi}_{2\sigma} \psi_{5\sigma'} \\
\tilde{\psi}_{3\sigma} \psi_{6\sigma'} \\
\tilde{\psi}_{4\sigma} \psi_{1\sigma'} \\
\tilde{\psi}_{5\sigma} \psi_{2\sigma'} \\
\tilde{\psi}_{6\sigma} \psi_{3\sigma'}
\end{bmatrix}^T \begin{bmatrix}
g_1 & g_2 & g_3 & g_4 & g_2 \\
g_2 & g_1 & g_2 & g_3 & g_4 \\
g_3 & g_2 & g_1 & g_2 & g_3 \\
g_4 & g_3 & g_2 & g_1 & g_2 \\
g_2 & g_3 & g_4 & g_2 & g_1
\end{bmatrix} \begin{bmatrix}
\psi_{4\sigma'} \psi_{1\sigma} \\
\psi_{5\sigma'} \psi_{2\sigma} \\
\psi_{6\sigma'} \psi_{3\sigma} \\
\psi_{1\sigma'} \psi_{4\sigma} \\
\psi_{2\sigma'} \psi_{5\sigma} \\
\psi_{3\sigma'} \psi_{6\sigma}
\end{bmatrix}
\]

The eigenvectors of the interaction matrix are:

\[
\begin{align*}
\Delta_s &= \frac{\Delta}{\sqrt{6}} (1, 1, 1, 1, 1, 1) \\
\Delta_{p_x} &= \frac{\Delta}{\sqrt{4}} (0, 1, 1, 0, -1, -1) \\
\Delta_{p_y} &= \frac{\Delta}{\sqrt{12}} (2, 1, -1, -2, -1, 1) \\
\Delta_{d_{x^2-y^2}} &= \frac{\Delta}{\sqrt{4}} (0, 1, -1, 0, 1, -1) \\
\Delta_{d_{xy}} &= \frac{\Delta}{\sqrt{12}} (2, -1, -1, 2, -1, 1) \\
\Delta_f &= \frac{\Delta}{\sqrt{6}} (1, -1, 1, -1, 1, -1)
\end{align*}
\]

which represent $s, p_x, p_y, d_{x^2-y^2}, d_{xy}$, and f-wave paring symmetries, respectively. The corresponding eigenvalues:

\[
\begin{align*}
\lambda_s &= g_1 + 2g_2 + 2g_3 + g_4 \\
\lambda_p &= g_1 + g_2 - g_3 - g_4 \\
\lambda_d &= g_1 - g_2 - g_3 + g_4 \\
\lambda_f &= g_1 - 2g_2 + 2g_3 - g_4
\end{align*}
\]
describe the paring strength between the electrons. We focus on the \( p \)-wave superconductors which are proved to be the leading instability in the main text. To decouple the quartic interactions, we introduce the order parameter in the patch space:

\[
\Delta = [i(\vec{\Delta}_1 \cdot \vec{\sigma})\sigma_y] \otimes [P_x] + [i(\vec{\Delta}_2 \cdot \vec{\sigma})\sigma_y] \otimes [P_y]
\]  

(11)

in which \( \vec{\Delta}_i \) (\( i = 1, 2 \)) are complex vectors and \( \vec{\sigma} \) are Pauli matrix. The matrix \( P_x \) and \( P_y \) are:

\[
P_x = \frac{1}{2} \text{diag} \left( 0, 1, 1 \right)
\]

\[
P_y = \frac{1}{12} \text{diag} \left( 2, 1, -1 \right)
\]

(12)

Using the Hubbard-Stratonovich transisdormation, we obtain:

\[
\mathcal{L}' = \Psi \begin{pmatrix} G_{-\Delta}^{-1} & \Delta \\ \Delta^\dagger & G_{-\Delta}^{-1} \end{pmatrix} \Psi + \frac{|\Delta_1|^2 + |\Delta_2|^2}{|\lambda_p|}
\]

(13)

Here \( \Psi = \left( \psi_{1\uparrow} \ \psi_{1\downarrow} \ \psi_{2\uparrow} \ \psi_{2\downarrow} \ \psi_{3\uparrow} \ \psi_{3\downarrow} \ \psi_{4\uparrow} \ \psi_{4\downarrow} \ \psi_{5\uparrow} \ \psi_{5\downarrow} \ \psi_{6\uparrow} \ \psi_{6\downarrow} \right)^T \). \( G_+ \) and \( G_- \) are particle and hole propagators with the form \( G_{\pm} = -i\omega_n \pm \epsilon(\vec{k}) - \mu \). They are diagonal in the patch space. By integrating out the fermion operators, we get the effective action:

\[
\mathcal{L}'' = -\text{Tr} \ln \begin{pmatrix} G_{-\Delta}^{-1} & \Delta \\ \Delta^\dagger & G_{-\Delta}^{-1} \end{pmatrix} + \frac{|\Delta_1|^2 + |\Delta_2|^2}{|\lambda_p|}
\]

(14)

From expanding the first term in \( \mathcal{L}'' \) to the quartic term in \( \Delta \), we get:

\[
\text{Tr} \ln \begin{pmatrix} G_{-\Delta}^{-1} & \Delta \\ \Delta^\dagger & G_{-\Delta}^{-1} \end{pmatrix} \approx -\text{Tr}[G_+ G_- G_- G_+] - \frac{1}{2} \text{Tr}[G_+ G_- G_- G_+ G_+ G_- G_- G_+] - \frac{1}{2} \text{Tr}[G_+ G_- G_- G_- G_+ G_+ G_+ G_-]
\]

(15)

The trace above means integration over \( \vec{k} \). Due to the rotational symmetry, \( \text{Tr}[G_+ G_-] \) is identical for all the patches and could be factored out of the trace over the patch space. Using the identity \( \text{Tr}[P_x P_y] = 1/2 \), \( \text{Tr}[P_y P_y] = 1/2 \), \( \text{Tr}[P_x P_y] = 0 \), \( \text{Tr}[P_x^2] = 1/8 \), \( \text{Tr}[P_y^2] = 1/8 \), \( \text{Tr}[P_x P_y] = 1/24 \) and \( [i(\Delta_1 \cdot \vec{\sigma})\sigma_y][i(\Delta_1 \cdot \vec{\sigma})\sigma_y]^\dagger = |\Delta_1|^2 \mathbf{1} + i(\vec{\Delta}_1 \cdot \vec{\sigma}) \), the trace over \( \Delta \Delta^\dagger \) and \( \Delta \Delta^\dagger \Delta \Delta^\dagger \) are:

\[
\text{Tr}[\Delta \Delta^\dagger] = \frac{1}{2} \text{Tr}[|\Delta_1|^2 \mathbf{1} + i(\vec{\Delta}_1 \cdot \vec{\sigma}) \cdot \vec{\sigma}] = |\Delta_1|^2 + |\Delta_2|^2
\]

(16)

\[
\text{Tr}[\Delta \Delta^\dagger \Delta \Delta^\dagger] = \frac{1}{8} \left( (|\Delta_1|^2 - (\vec{\Delta}_1 \cdot \vec{\sigma})^2 + |\Delta_2|^2 - (\vec{\Delta}_2 \cdot \vec{\sigma})^2) + \frac{1}{6} \left( |\Delta_1|^2 |\Delta_2|^2 - (\vec{\Delta}_1 \cdot \vec{\sigma})(\vec{\Delta}_2 \cdot \vec{\sigma}) \right) \right)
\]

(17)

Since \( \vec{\Delta}_1 \times \vec{\Delta}_1^\dagger = -\vec{\Delta}_1 \cdot \vec{\Delta}_1^\dagger \), \( \vec{\Delta}_2 \times \vec{\Delta}_2^\dagger \) is pure imaginary. To minimize \( \text{Tr}[\Delta \Delta^\dagger \Delta \Delta^\dagger] \), we have \( \vec{\Delta}_1 \times \vec{\Delta}_1^\dagger = \vec{\Delta}_2 \times \vec{\Delta}_2^\dagger = 0 \). This implies \( \vec{\Delta}_1 = \vec{d}_1 \exp(i\theta_1) \) and \( \vec{\Delta}_2 = \vec{d}_2 \exp(i\theta_2) \) in which \( \vec{d}_1 \) and \( \vec{d}_2 \) are real vectors. Then, \( \text{Tr}[\Delta \Delta^\dagger \Delta \Delta^\dagger] \) becomes:

\[
\text{Tr}[\Delta \Delta^\dagger \Delta \Delta^\dagger] = \frac{1}{8} \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 + \frac{4}{3} |\vec{d}_1|^2 |\vec{d}_2|^2 \right) + \frac{1}{12} \cos(2(\theta_1 - \theta_2))(\vec{d}_1 \cdot \vec{d}_2)^2 - |\vec{d}_1 \times \vec{d}_2|^2
\]

(18)

The further minimization requires \( \cos(2(\theta_1 - \theta_2)) = 1 \). This constrains \( \theta_1 - \theta_2 = \theta \), which leads to \( \vec{\Delta}_1 = \vec{d}_1 \) and \( \vec{\Delta}_2 = \vec{d}_2 \exp(-i\theta) \). When \( \cos(2\theta) = 1 \), we have \( \vec{d}_1 \perp \vec{d}_2 \). When \( \cos(2\theta) = -1 \), we have \( \vec{d}_1 \parallel \vec{d}_2 \). In both cases, the effective action \( \mathcal{L}'' \) becomes:

\[
\mathcal{L}'' = \frac{1}{|\lambda_p|} \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 \right) + \frac{1}{8} \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 \right) \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 + \frac{2}{3} |\vec{d}_1|^2 |\vec{d}_2|^2 \right)
\]

\[
= \frac{1}{|\lambda_p|} \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 \right) \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 \right) - \frac{4}{3} |\vec{d}_1|^2 |\vec{d}_2|^2
\]

(19)
This action is minimized when $|\vec{d}_1|^2 = |\vec{d}_2|^2$. But $\vec{d}_1$ can rotate freely.

Consider the case with $\cos(2\theta) = -1$ and $\vec{d}_1 \parallel \vec{d}_2$. It is straightforward to obtain

$$\Delta_2 = \pm i\Delta_1 = (\pm id_{1x}, \pm id_{1y}, \pm id_{1z}) .$$

Then the order parameter is:

$$\Delta = [i(\vec{\Delta}_1 \cdot \vec{\sigma})\sigma_y] \otimes [P_x] + [i(\vec{\Delta}_2 \cdot \vec{\sigma})\sigma_y] \otimes [P_y]$$

$$= \left( -d_{1x} + id_{1y} \quad d_{1z} \\ d_{1z} \quad d_{1x} + id_{1y} \right) \otimes [P_x \pm iP_y]$$

which corresponds to chiral $p + ip$ superconductor.

When $\cos(2\theta) = 1$, we have $\vec{d}_1 \perp \vec{d}_2$ and $\exp(-i\theta) = \pm 1$. Since $\vec{d}_1$ can rotate freely, we take $d_{1z} = 0$ for simplicity. $\vec{d}_1 \perp \vec{d}_2$ could be fulfilled if we take $d_{2x} = d_{1y}$ and $d_{2y} = -d_{1x}$. To satisfy $\exp(-i\theta) = \pm 1$, it is clear that $\vec{\Delta}_1 = \vec{d}_1$ and $\vec{\Delta}_2 = \pm \vec{d}_2$. Both $\vec{\Delta}_1$, $\vec{\Delta}_2$ are real. For illustration, we take $\vec{\Delta}_2 = \vec{d}_2$. The order parameter is:

$$\Delta = [i(\vec{\Delta}_1 \cdot \vec{\sigma})\sigma_y] \otimes [P_x] + [i(\vec{\Delta}_2 \cdot \vec{\sigma})\sigma_y] \otimes [P_y]$$

$$= \left( -d_{1x} + id_{1y} \quad 0 \\ 0 \quad d_{1x} + id_{1y} \right) \otimes [P_x + iP_y] + \left( -d_{1y} - id_{1x} \quad 0 \\ 0 \quad d_{1y} - id_{1x} \right) \otimes [P_x - iP_y]$$

The total order parameter preserves the time reversal symmetry since both $\vec{\Delta}_1$ and $\vec{\Delta}_2$ are real. This state corresponds to a helical $p + ip$ superconductor.