A NOTE ON THE PAPER: ON ITERATIONS FOR FAMILIES OF ASYMPOTOTICALLY PSEUDOCONTRACTIVE MAPPINGS.

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Abstract. It is our aim in this note to give a counter example to an argument used in the proof of the main theorem of the paper: On iterations for families of asymptotically pseudocontractive mappings, Applied Mathematics Letters, 24 (2011), 33-38 by A. Rafiq [4], and give an alternative condition to correct the anomaly.

1. Introduction.
This work is motivated by the recent paper of A. Rafiq [4]. Careful reading of Rafiq’s work shows that there is a serious gap in the proof of Theorem 5 of [4], which happens to be main theorem of the paper.

It is our aim to give a counter example to the argument used in the proof of Theorem 5 of [4] and suggest an alternative condition in order to close the observed gap.

2. Preliminary.
Let E be a real Banach space with dual E∗ and let ⟨., .⟩ be the duality pairing between members of E and E∗. The mapping J : E → 2E∗ defined by

\[ J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = ||x||^2; ||f^*|| = ||x||, x \in E \}, \]

is called the normalized duality mapping. We note that in a Hilbert space H, J is the identity operator. The single valued normalized duality mapping is denoted by j.

A mapping T : D(T) ⊂ E → E is said to be L-Lipschitzian if there exists L > 0 such that

\[ \| Tx - Ty \| \leq L \| x - y \| \forall x, y \in D(T); \]

and T is said to be uniformly L-Lipschitzian if there exists L > 0 such that

\[ ||T^n x - T^n y|| \leq L ||x - y|| \forall x, y \in D(T), \forall n \geq 1, \]

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where $D(T)$ denotes the domain of $T$. It is well known that the class of uniformly $L$-Lipschitzian mappings is a proper subclass of the class of $L$-Lipschitzian mappings.

The mapping $T$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n \geq 1} \subset [1, +\infty)$ with $\lim_{n \to \infty} k_n = 1$ and for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$
\left\langle T^n x - T^n y, j(x - y) \right\rangle \leq k_n \|x - y\|^2 \forall x, y \in D(T), \forall n \geq 1.
$$

In [4], A. Rafiq studied the strong convergence of the sequence $\{x_n\}_{n \geq 1}$ defined by

$$
x_1 \in K,
$$
$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n_1 y^n_1
$$
$$
y^n_n = (1 - \beta^n_1)x_n + \beta^n_1 T^n_1 y^n_1
$$
$$
\vdots
$$
$$
y^n_{p-1} = (1 - \beta^{p-1}_n)x_n + \beta^{p-1}_n T^n_1 y^n_1, n \geq 1, \tag{1.4}
$$

for approximation of common fixed point of finite family of asymptotically pseudocontractive mappings in real Banach space. He proved the following theorem.

**Theorem 2.1.** (See Theorem 5 of [4]) Let $K$ be a nonempty closed convex subset of a real Banach space $E$ and $T_l : K \to K$, $l = 1, 2, \ldots, p; p \geq 2$ be $p$ asymptotically pseudocontractive mappings with $T_1$ and $T_2$ having bounded ranges and a sequence $\{k_n\}_{n \geq 1} \subset [1, +\infty)$, $\lim_{n \to \infty} k_n = 1$ such that $x^* \in \bigcap_{l=1}^{p} F(T_l) = \{ x \in K : T_1 x = x = T_2 x = \ldots = T_p x \}$. Further, let $T_1$ be uniformly continuous and $\{\alpha_n\}_{n \geq 1}$, $\{\beta_n\}_{n \geq 1}$, $\{\beta^{p-1}_n\}_{n \geq 1}$ be sequences in $[0, 1]$, $i = 1, 2, \ldots, p; p \geq 2$ such that

(i) $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n$;

(ii) $\sum_{n \geq 1} \alpha_n = \infty$.

For arbitrary $x_1 \in K$, let $\{x_n\}_{n \geq 1}$ be iteratively defined by (1.4). Suppose that for any $x^* \in \bigcap_{l=1}^{p} F(T_l)$, there exists a strictly increasing function $\Psi : [0, +\infty) \to [0, +\infty)$, $\Psi(0) = 0$ such that

$$
\left\langle T^n l x - x^*, j(x - x^*) \right\rangle \leq k_n \|x_n - x^*\|^2 - \Psi(\|x - x^*\|), \text{ for all } x \in K, l = 1, 2, \ldots, p; p \geq 2.
$$

Then $\{x_n\}_{n \geq 1}$ converges strongly to $x^* \in \bigcap_{l=1}^{p} F(T_l)$.

**Remark 2.2.** There are a lot to say about this result but let us first and foremost address the major issue arising from the proof of this theorem.

On page 37 of [4], immediately after inequality (2.7), the author wrote:

"From the condition (i) and (2.7), we obtain

$$
\lim_{n \to \infty} \|y^n_n - x_{n+1}\| = 0,
$$

"
and the uniform continuity of $T_1$ leads to
\[ \lim_{n \to \infty} \| T_1^n y_n^1 - T_1^n x_{n+1} \| = 0. \]

This claim is, however, not true. To see this, we consider the following example:

**Example 2.3.** Let $\mathbb{R}$ denote the set of real numbers endowed with usual topology. Define $T : \mathbb{R} \to \mathbb{R}$ by $T x = 2x \forall x \in \mathbb{R}$, then
\[ |T x - T y| = 2|x - y| \forall x, y \in \mathbb{R}. \]

This implies that $T$ is a Lipschitz mapping with Lipschitz constant $L = 2$. Thus, $T$ is uniformly continuous since every Lipschitz map is uniformly continuous. Now, suppose $y_n^1 = 1 + \frac{1}{n}$ and $x_{n+1} = 1 - \frac{1}{n}$ for all $n \geq 1$, then
\[ |y_n^1 - x_{n+1}| = \left| \left( 1 + \frac{1}{n} \right) - \left( 1 - \frac{1}{n} \right) \right| = \frac{2}{n} \to 0 \text{ as } n \to \infty. \]

We now show that
\[ \lim_{n \to \infty} |T_n y_n^1 - T_n x_{n+1}| \neq 0. \]

Observe that
\[
\begin{align*}
T y_n^1 &= 2y_n^1 = 2 \left( 1 + \frac{1}{n} \right) = 2 + \frac{2}{n} \\
T^2 y_n^1 &= T(T y_n^1) = 2 \left( 2 + \frac{2}{n} \right) = 2^2 + \frac{2^2}{n} \\
T^3 y_n^1 &= T(T^2 y_n^1) = 2 \left( 2^2 + \frac{2^2}{n} \right) = 2^3 + \frac{2^3}{n} \\
&\vdots \\
T^n y_n^1 &= 2^n + \frac{2^n}{n} \text{ for all } n \geq 1.
\end{align*}
\]

Similar computation gives $T^n x_{n+1} = 2^n - \frac{2^n}{n}$ for all $n \geq 1$.

Thus,
\[ |T^n y_n^1 - T^n x_{n+1}| = \left| \left( 2^n + \frac{2^n}{n} \right) - \left( 2^n - \frac{2^n}{n} \right) \right| = \frac{2^{n+1}}{n} \forall n \geq 1. \]

It is easy to see (using mathematical induction) that $2^{n+1} \geq n \forall n \geq 1$. So,
\[ |T^n y_n^1 - T^n x_{n+1}| = \frac{2^{n+1}}{n} \geq 1 \forall n \geq 1. \]

Hence,
\[ \lim_{n \to \infty} |T^n y_n^1 - T^n x_{n+1}| \neq 0. \]

This contradicts the claim of A. Rafiq [4].

To correct the error in the result of A. Rafiq, we shall rather assume that $T_1$ is uniformly $L$-Lipschitzian so that
\[ d_n = M\| T_1^n y_n^1 - T_1^n x_{n+1} \| \leq ML\| y_n^1 - x_{n+1} \| \to 0 \text{ as } n \to \infty. \]

The rest of the result follows as in [4].
Remark 2.4. In as much as the error in the proof of Theorem 5 of [4] has been pointed out and corrected, it is not clear what the author really want to achieve by constructing such a complicated scheme given by (1.4). If a clear study of the proof of [4] is made, one will easily observe that the mappings \( T_l, 3 \leq l \leq p \) played no role at all. This suggests that the scheme will only make sense if only two operators \( T_1 \) and \( T_2 \) are considered. Besides, it is not specified in Theorem 5 of [4] which of the operators the sequence \( \{k_n\}_{n \geq 1} \) is associated with. Meanwhile, condition (*) guarantees that the fixed point \( x^* \) of these operators is unique. This thus reduces the entire problem to what has been studied in [1] and [3]. We note that the result of Chidume and Chidume [2] and Ofoedu [3] remain correct if it were further assumed that the mapping \( \phi : [0, +\infty) \to [0, +\infty) \) in their results is onto.

References

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